LOWER BOUNDS FOR THE SPECTRUM OF THE LAPLACE AND
STOKES OPERATORS

ALEXEI A. ILYIN

Abstract. We prove Berezin–Li–Yau-type lower bounds with additional term for the
eigenvalues of the Stokes operator and improve the previously known estimates for the
Laplace operator. Generalizations to higher-order operators are given.

Dedicated to Professor R. Temam on the occasion of his 70th birthday

1. Introduction

Sharp lower bounds for the sums of the first \(m\) eigenvalues of the Dirichlet Laplacian
\[-\Delta \varphi_k = \mu_k \varphi_k, \quad \varphi_k |_{\partial \Omega} = 0\]
were obtained in [10]:
\[
\sum_{k=1}^{m} \mu_k \geq \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n}.
\]
Here \(|\Omega| < \infty\) denotes the volume of a domain \(\Omega \subset \mathbb{R}^n\) and \(\omega_n\) denotes the volume of the
unit ball in \(\mathbb{R}^n\). It was shown in [9] that the estimate (1.1) is equivalent by means of the
Legendre transform to an earlier result of Berezin [3].

In view of the classical H. Weyl asymptotic formula
\[
\mu_k \sim \left( \frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} k^{2/n} \quad \text{as} \ k \to \infty,
\]
the coefficient of \(m^{1+2/n}\) in (1.1) is sharp. However, an improvement of the Li–Yau bound
with additional term that is linear in \(m\) was obtained in [11]:
\[
\sum_{k=1}^{m} \mu_k \geq \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n} + c_n \frac{|\Omega|}{I} m,
\]
where
\[
I = \int_{\Omega} x^2 dx,
\]
and the constant \(c_n\) depends only on the dimension: \(c_n = c/(n + 2)\) with \(c\) being an
absolute constant (in fact, (1.2) holds with \(c = 1/24\)). Of course \(I\) can be replaced by
\(I = \min_{a \in \mathbb{R}^n} \int_{\Omega} (x - a)^2 dx\).

In the theory of the attractors for the Navier–Stokes equations (see, for example, [2, 4, 15]
and the references therein) lower bounds for the sums of the eigenvalues \(\{\lambda_k\}_{k=1}^\infty\) of the
1991 Mathematics Subject Classification. 35P15, 35Q30.

Key words and phrases. Stokes operator, Dirichlet Laplacian, lower bounds, Navier–Stokes equations.

This work was supported in part by the Russian Foundation for Basic Research, grant nos. 09-01-00288
and 08-01-00099, and by the the RAS Programme no.1 ‘Modern problems of theoretical mathematics’. 
Stokes operator are very important. In the case of a smooth domain the eigenvalue problem for the Stokes operator reads:

\[- \Delta u_k + \nabla p_k = \lambda_k v_k, \quad \text{div } u_k = 0, \quad u_k|_{\partial \Omega} = 0. \] (1.4)

Li–Yau-type lower bounds for the spectrum of the Stokes operator were obtained in [6]:

\[\sum_{k=1}^{m} \lambda_k \geq \frac{n}{2 + n} \left(\frac{(2\pi)^n}{\omega_n(n-1)|\Omega|}\right)^{2/n} n^{1+2/n}. \] (1.5)

The coefficient of \(n^{1+2/n}\) here is also sharp in view of the asymptotic formula ([11] (\(n = 3\), [12] (\(n \geq 2\))):

\[\lambda_k \sim \left(\frac{(2\pi)^n}{\omega_n(n-1)|\Omega|}\right)^{2/n} k^{2/n} \text{ as } k \to \infty. \] (1.6)

The main results of this paper are twofold. First, we extend the approach of [11] to the case of the Laplacian and, secondly, we obtain the exact solution of the corresponding minimization problem, thereby giving a much better value of the constant \(c_n\) in [11,2] (in fact, the sharp value in the framework of the approach of [11]).

To describe the minimization problem we consider in the case of the Laplacian an \(L_2\)-orthonormal family of functions \(\{\varphi_k\}_{k=1}^{m} \in H^1_0(\Omega)\). Then the function \(F(\xi)\)

\[F(\xi) = \sum_{k=1}^{m} |\tilde{\varphi}_k(\xi)|^2, \quad \tilde{\varphi}_k(\xi) = (2\pi)^{-n/2} \int_{\Omega} \varphi_k(x)e^{-i\xi x} \, dx\]

satisfies \(F(\xi) \leq M = (2\pi)^{-n}|\Omega|\) (see [10]) and the additional regularity property which was found and used in [11]: \(|\nabla F(\xi)| \leq L = 2(2\pi)^{-n}\sqrt{|\Omega|}\). Here and in what follows \(I\) is defined in (1.3).

For the Stokes operator we consider an \(L_2\)-orthonormal family of vector functions \(\{u_k\}_{k=1}^{m} \in H^1_0(\Omega)\), with \(\text{div } u_k = 0\). Then as we show in [2] the corresponding function \(F(\xi) = \sum_{k=1}^{m} |\tilde{u}_k(\xi)|^2\) satisfies the conditions \(F(\xi) \leq M = (2\pi)^{-n}|(n-1)|\) and \(|\nabla F(\xi)| \leq L = 2(2\pi)^{-n}(n(n-1))^{1/2}\sqrt{|\Omega|}\).

By orthonormality we always have \(\int_{\mathbb{R}^n} F(\xi) d\xi = m\), and taking the first \(m\) eigenfunctions of the Laplace (or Stokes) operator for the \(\varphi_k\) (or the \(u_k\), respectively) we get

\[\int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi = \sum_{k=1}^{m} \mu_k = \sum_{k=1}^{m} \lambda_k, \quad \text{and } \sum_{k=1}^{m} \mu_k \geq \Sigma_M(m) \text{ (respectively, } \sum_{k=1}^{m} \lambda_k \geq \Sigma_M(m)\),

where \(\Sigma_M(m)\) is the solution of the minimization problem: find \(\Sigma_M(m)\)

\[\int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi \rightarrow \inf := \Sigma_M(m), \quad \text{under the conditions}

\[0 \leq F(\xi) \leq M, \quad \int_{\mathbb{R}^n} F(\xi) d\xi = m. \] (1.7)

It was shown in [10] that the minimizer \(F_\ast\) is radial and has the form shown in Fig. 1 where \(r_\ast\) is defined by the condition \(\int_{\mathbb{R}^n} F_\ast(|\xi|) d\xi = m:\)

\[\int_{\mathbb{R}^n} F_\ast(|\xi|) d\xi = \sigma_n \int_0^{r_\ast} r^{n-1} F_\ast(r) dr = \omega_n M r_\ast^n = m. \]

Then

\[\Sigma_M(m) = \int_{\mathbb{R}^n} |\xi|^2 F_\ast(\xi) d\xi = \sigma_n M \int_0^{r_\ast} r^{n+1} dr = \frac{n}{n+2} \left(\frac{1}{\omega_n M}\right)^{2/n} m^{1+2/n} \]
giving (1.1) upon substituting $M = (2\pi)^{-n} |\Omega|$ for the Laplacian and giving (1.3) upon substituting $M = (2\pi)^{-n(n-1)} |\Omega|$ for the Stokes operator [6].

The additional regularity property of $F(\xi)$: $|\nabla F(\xi)| \leq L$ gives a better lower bound [11]:

$$\sum_{k=1}^{m} \mu_k \geq \Sigma_{M,L}(m),$$

where $\Sigma_{M,L}(m)$ is the solution of the minimization problem

$$\int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi \to \inf =: \Sigma_{M,L}(m) \quad \text{under the conditions},$$

$$0 \leq F(\xi) \leq M, \quad \int_{\mathbb{R}^n} F(\xi) d\xi = m, \quad |\nabla F(\xi)| \leq L.$$  \hspace{1cm} (1.8)

Clearly $\Sigma_{M,L}(m) \geq \Sigma_M(m)$ and Lemma 1 in [11] (in the notation our paper) reads:

$$\Sigma_{M,L}(m) \geq \frac{n}{n + 2} \left( \frac{1}{\omega_n M} \right)^{2/n} m^{1+2/n} + \frac{1}{6(n + 2)} \frac{M^2}{L^2} m,$$  \hspace{1cm} (1.9)

giving (1.2) with $c_n = 1/(24(n+2))$ by substituting $M = (2\pi)^{-n} |\Omega|$ and $L = 2(2\pi)^{-n} \sqrt{|\Omega|/T}$.

In [3] we find the exact solution of the minimization problem (1.8):

$$\Sigma_{M,L}(m) = \frac{\sigma_n M^{n+3}}{(n + 2)(n + 3) L^{n+2}} \left( (t(m) + 1)^{n+3} - t(m)^{n+3} \right),$$

where $t(m)$ is the unique positive root of the equation

$$(t + 1)^{n+1} - t^{n+1} = m, \quad m = m \frac{(n + 1)L}{\omega_n M^{n+1}}.$$

We also find the first three terms of the asymptotic expansion of the solution $\Sigma_{M,L}(m)$ in the following descending powers of $m$: $m^{1+2/n}, m, m^{1-2/n}, m^{1-4/n}, \ldots$. Namely,

$$\Sigma_{M,L}(m) = \Sigma_0(m) + O(m^{1-4/n}),$$

$$\Sigma_0(m) = \frac{n}{n + 2} \left( \frac{1}{\omega_n M} \right)^{2/n} m^{1+2/n} + \frac{n}{12} \frac{M^2}{L^2} m - \frac{n(n-1)(3n+2)}{1440} \frac{M^4(M\omega_n)^{2/n}}{L^4} m^{1-2/n},$$

which shows that the second term is for all $n$ linear with respect to $m$ and positive with coefficient that is $n(n+2)/2$ times greater than that in (1.9), while the third term is always negative.

Dropping the third term and using the expressions for $M$ and $L$ we obtain the following asymptotic lower bounds. Accordingly, for large $m$ the coefficient of $m$ in the second term on the right-hand side in (1.11) is $n(n+2)/2$ times greater than that in (1.2).
Theorem 1.1. The eigenvalues \( \{\mu_k\}_{k=1}^{\infty} \) and \( \{\lambda_k\}_{k=1}^{\infty} \) of the Laplace and Stokes operators satisfy the following lower bounds:

\[
\sum_{k=1}^{m} \mu_k \geq \frac{n}{n+2} \left( \frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{n}{48} \beta_n \frac{|\Omega|}{I} m (1 - \varepsilon_n(m)), \tag{1.11}
\]

\[
\sum_{k=1}^{m} \lambda_k \geq \frac{n}{n+2} \left( \frac{(2\pi)^n}{\omega_n (n-1) |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{(n-1)}{48} \beta_n \frac{|\Omega|}{I} m (1 - \varepsilon_n(m)), \tag{1.12}
\]

where \( \varepsilon_n(m) \geq 0, \varepsilon_n(m) = O(m^{-2/n}). \)

Then in [4] we turn to the analysis of the particular cases \( n = 2, 3, 4 \). The main result consists in the explicit formulas for \( \Sigma_{M,L}(m) \). The case \( n = 2 \) is the simplest and we find (see Lemma [3]) the explicit formula for the exact solution which coincides with the first three terms of its asymptotic expansion

\[
\Sigma_{M,L}(m) = \Sigma_0(m) = \frac{1}{2\pi M} m^2 + \frac{M^2}{6L^2} m - \frac{\pi M^5}{90 L^4}.
\]

For \( n = 3, 4 \) by means of the explicit formulas in Lemmas [4, 3] and [4, 2] we show that

\[
\Sigma_{M,L}(m) > \Sigma_0(m).
\]

(The inequality \( \Sigma_{M,L}(m) \geq \Sigma_0(m) \) probably holds for any \( n \), not only for \( n = 2, 3, 4 \).) Then the negative contribution from the third term in \( \Sigma_{M,L}(m) \) is compensated by a \((1 - \beta)\)-part of the positive second term (where \( 0 < \beta < 1 \) and \( \beta \) is sufficiently close to 1) and we obtain the following theorem.

Theorem 1.2. The eigenvalues \( \{\mu_k\}_{k=1}^{\infty} \) and \( \{\lambda_k\}_{k=1}^{\infty} \) for \( n = 2, 3, 4 \) satisfy:

\[
\sum_{k=1}^{m} \mu_k \geq \frac{n}{n+2} \left( \frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{n}{48} \beta_n \frac{|\Omega|}{I} m, \tag{1.13}
\]

\[
\sum_{k=1}^{m} \lambda_k \geq \frac{n}{n+2} \left( \frac{(2\pi)^n}{\omega_n (n-1) |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{(n-1)}{48} \beta_n \frac{|\Omega|}{I} m, \tag{1.14}
\]

where in the two-dimensional case \( \beta_2^L = \frac{119}{120}, \beta_2^S = \frac{230}{240} \), while for \( n = 3, 4 \) it suffices to take \( \beta_3^L = 0.986, \beta_3^S = 0.986 \) and \( \beta_4^L = 0.983, \beta_4^S = 0.978 \).

Finally, in [5] we prove two-term lower bounds for the Dirichlet bi-Laplacian.

Remark 1.1. Two term lower bounds for the 2D Laplacian with the second term of growth higher than linear in \( m \) were obtained in [7]. They depend on the shape of \( \partial \Omega \).

2. Estimates for orthonormal vector functions

Throughout \( \Omega \) is an open subset of \( \mathbb{R}^n \) with finite \( n \)-dimensional Lebesgue measure \( |\Omega| \):

\[
\Omega \subset \mathbb{R}^n, \ n \geq 2, \quad |\Omega| < \infty.
\]

We recall the functional definition of the Stokes operator [4, 8, 14]: \( \mathcal{V} \) denotes the set of smooth divergence-free vector functions with compact supports

\[
\mathcal{V} = \{ u : \Omega \to \mathbb{R}^n, \ u \in C_0^\infty(\Omega), \ \text{div} u = 0 \}.
\]
and $H$ and $V$ are the the closures of $\mathcal{V}$ in $L_2(\Omega)$ and $H^1(\Omega)$, respectively. The Helmholtz–Leary orthogonal projection $P$ maps $L_2(\Omega)$ onto $H$, $P : L_2(\Omega) \to H$. We have (see [14])

$$L_2(\Omega) = H \oplus H^\perp, \quad H^\perp = \{u \in L_2(\Omega), u = \nabla p, p \in L_2^\text{loc}(\Omega)\},$$

(2.1)

where $H^\perp$ is the orthogonal complement of $H$. The last inclusion becomes equality for a bounded $\Omega$ with Lipschitz boundary. The Stokes operator $A$ is defined by the relation

$$(Au, v) = (\nabla u, \nabla v) \quad \text{for all } u, v \in V$$

(2.2)

and is an isomorphism between $V$ and $V'$. For a sufficiently smooth $u$

$$Au = -P\Delta u.$$ The Stokes operator $A$ is an unbounded self-adjoint positive operator in $H$ with discrete spectrum $\{\lambda_k\}_{k=1}^\infty$, $\lambda_k \to \infty$ as $k \to \infty$:

$$Av_k = \lambda_k v_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots,$$

(2.3)

where $\{v_k\}_{k=1}^\infty \subseteq V$ are the corresponding orthonormal eigenvectors. Taking the scalar product with $v_k$ we have by orthonormality and (2.2) that

$$\lambda_k = \|\nabla v_k\|^2.$$ (2.4)

In case when $\Omega$ is a bounded domain with smooth boundary the eigenvalue problem (2.3) goes over to (1.4).

We recall that a family $\{\varphi_i\}_{i=1}^m \in L_2(\Omega)$ is called suborthonormal [5] if for any $\zeta \in \mathbb{C}^m$

$$\sum_{i,j=1}^m \zeta_i \overline{\zeta_j} (\varphi_i, \varphi_j) \leq \sum_{j=1}^m |\zeta_j|^2.$$ (2.5)

**Lemma 2.1.** Any suborthonormal family $\{\varphi_i\}_{i=1}^m$ satisfies Bessel’s inequality:

$$\sum_{k=1}^m c_k^2 \leq \|f\|^2_{L_2(\Omega)}, \quad \text{where} \quad c_k = (\varphi_k, f).$$ (2.6)

**Proof.** Given an suborthonormal system $\{\varphi_i\}_{i=1}^m$ (with supports in $\Omega$), we build it up to a orthonormal system $\{\psi_i\}_{i=1}^m \in L_2(\mathbb{R}^n)$ of the form

$$\psi_k = \varphi_k + \chi_k, \quad \chi_k(x) = \sum_{j=1}^m a_{kj} \omega_j(x),$$

where $\{\omega_i\}_{i=1}^m$ is an arbitrary orthonormal system with supports in $\mathbb{R}^n \setminus \Omega$. The condition $(\psi_k, \psi_l) = \delta_{kl}$ is satisfied if we chose for the matrix $a = a_{ij}$ the symmetric non-negative matrix $a = b^{1/2}$, where $b_{ij} = \delta_{ij} - (\varphi_i, \varphi_j)$ (in view of (2.5), $b$ is non-negative).

The system $\{\psi_i\}_{i=1}^m$ classically satisfies Bessel’s inequality, and since $(\psi_k, f) = (\varphi_k, f)$, this gives (2.6). \qed

Suborthonormal families typically arise as a result of the action of an orthogonal projection [3].

**Lemma 2.2.** If $\{\varphi_k\}_{k=1}^m$ is orthonormal and $P$ is an orthogonal projection, then both families $\eta_k = P\varphi_k$ and $\xi_k = (I - P)\varphi_k$ are suborthonormal.

We now obtain some estimates for the Fourier transforms for (sub)orthonormal families.

**Lemma 2.3.** If $\{\varphi_k\}_{k=1}^m$ is suborthonormal, then

$$\sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2 \leq (2\pi)^{-n}|\Omega|.$$ (2.7)

**Proof.** This follows from (2.6) with $f(x) = f_k(x) = (2\pi)^{-n/2}e^{-i\xi x}$. \qed
Corollary 2.1. If the family of vector functions \( \{u_k\}_{k=1}^m \) is orthonormal in \( L_2(\Omega) \), then
\[
\sum_{k=1}^m |\tilde{u}_k(\xi)|^2 \leq (2\pi)^{-n} n|\Omega|.
\] (2.8)

Proof. By Lemma 2.2 for each \( j = 1, \ldots, n \) the family \( \{u_k^j\}_{k=1}^m \) is suborthonormal and (2.8) follows from Lemma 2.3.

The next lemma [6] is essential for the Li–Yau bounds for the Stokes operator and says that under the additional condition \( \text{div} u_k = 0 \) the factor \( n \) in the previous estimate is replaced by \( n - 1 \).

Lemma 2.4. If the family of vector functions \( \{u_k\}_{k=1}^m \) is orthonormal and \( u_k \in H \), then
\[
\sum_{k=1}^m |\tilde{u}_k(\xi)|^2 \leq (2\pi)^{-n} (n - 1)|\Omega|.
\] (2.9)

Proof. First we observe that \( \xi \cdot \tilde{u}_k(\xi) = (2\pi)^{-n/2} i \int u_k \cdot \nabla_x e^{-i\xi x} \, dx = 0 \) for all \( \xi \in \mathbb{R}^n \) since the \( u_k \)'s are orthogonal to gradients (see (2.1)). Let \( \xi_0 \neq 0 \) be of the form:
\[
\xi_0 = (a, 0, \ldots, 0), \quad a \neq 0.
\] (2.10)

Since \( \xi_0 \cdot \tilde{u}_k(\xi_0) = 0 \), it follows that \( \tilde{u}_k^j(\xi_0) = 0 \) for \( k = 1, \ldots, m \). Hence, by (2.7)
\[
\sum_{k=1}^m |\tilde{u}_k(\xi_0)|^2 = \sum_{j=1}^m \sum_{k=1}^m |\tilde{u}_k^j(\xi_0)|^2 \leq (2\pi)^{-n} (n - 1)|\Omega|.
\]

The general case reduces to the case (2.10) by the corresponding rotation of \( \mathbb{R}^n \) about the origin represented by the orthogonal \( (n \times n) \)-matrix \( \rho \). Given a vector function \( u(x) = (u^1(x), \ldots, u^n(x)) \) we consider the vector function
\[
u_{\rho}(x) = \rho u(\rho^{-1} x), \quad x \in \rho \Omega.
\]
A straightforward calculation gives that \( \text{div} \, \nu_{\rho}(x) = \text{div} \, u(y) \), where \( \rho^{-1} x = y \). In addition, \( (u_{\rho}, v_{\rho}) = (u, v) \). Combining this we obtain that the family \( \{(u_{\rho})_k\}_{k=1}^m \) is orthonormal and belongs to \( H(\rho \Omega) \).

Next we calculate \( \tilde{\nu}_\rho \) and see that \( \tilde{\nu}_\rho(\xi) = \rho \tilde{\nu}(\rho^{-1} \xi) \). We now fix an arbitrary \( \xi \in \mathbb{R}^n \), \( \xi \neq 0 \) and set \( \xi_0 = (|\xi|, 0, \ldots, 0) \). Let \( \rho \) be the rotation such that \( \xi = \rho^{-1} \xi_0 \). Then we have
\[
\sum_{k=1}^m |\tilde{\nu}_k(\xi)|^2 = \sum_{k=1}^m |\tilde{\nu}_k(\rho^{-1} \xi_0)|^2 = \sum_{k=1}^m |\rho^{-1}(\tilde{u}_k)_\rho(\xi_0)|^2 = \sum_{k=1}^m |(\tilde{u}_k)_\rho(\xi_0)|^2 \leq (2\pi)^{-n} (n - 1)|\Omega|,
\]
where we have used that inequality (2.9) has been proved for \( \xi \) of the form (2.10) for any orthonormal family of divergence-free vector functions. Finally, the estimate (2.9) is extended to \( \xi = 0 \) by continuity.

For the orthonormal family \( \{u_k\}_{k=1}^m \in H \) we set
\[
F_S(\xi) = \sum_{k=1}^m |\tilde{u}_k(\xi)|^2.
\] (2.11)

Lemma 2.5. The following inequality holds:
\[
|\nabla F_S(\xi)| \leq 2(2\pi)^{-n}(n(n - 1))^{1/2} \sqrt{|\Omega|/7}.
\] (2.12)
Proof. The proof is similar to that in [11] for the Laplacian. We have
\[ \partial_j \hat{u}_k^j(\xi) = -(2\pi)^{-n/2}i \int_{\Omega} u_k^j(x) x_j e^{-i\xi x} \, dx. \]
Since the family \( \{u_k^j\}_{k=1}^m \) is subobthonormal, by Lemma 2.3 we have
\[ \sum_{k=1}^m |\partial_j \hat{u}_k^j(\xi)|^2 \leq (2\pi)^{-n} \int_{\Omega} x_j^2 \, dx, \]
and
\[ \sum_{k=1}^m |\nabla \hat{u}_k(\xi)|^2 \leq (2\pi)^{-n} \int_{\Omega} x^2 \, dx = (2\pi)^{-n} nI(\Omega), \quad I(\Omega) = \int_{\Omega} x^2 \, dx. \]
Next, using (2.9) we obtain
\[ |\nabla F_S(\xi)| \leq 2 \left( \sum_{k=1}^m |\hat{u}_k(\xi)|^2 \right)^{1/2} \left( \sum_{k=1}^m |\nabla \hat{u}_k(\xi)|^2 \right)^{1/2} \leq 2(2\pi)^{-n} (n(1-n))^{1/2} \sqrt{|\Omega|I}. \]

If, in addition, the orthonormal family \( \{u_k\}_{k=1}^m \) belongs to \( V \subseteq \{u \in H_0^1(\Omega), \, \text{div} \, u = 0\} \), then, by the Plancherel theorem, the function \( F_S(\xi) \) defined in (2.11) satisfies
\[ 0 \leq F_S(\xi) \leq M_S = (2\pi)^{-n}(n-1)|\Omega|; \]
\[ |\nabla F_S(\xi)| \leq L_S = 2(2\pi)^{-n}(n(n-1))^{1/2} \sqrt{|\Omega|I}; \]
\[ \int F_S(\xi) \, d\xi = m; \quad (2.13) \]
\[ \int |\xi|^2 F_S(\xi) \, d\xi = \sum_{k=1}^m \|\nabla u_k\|^2. \]

In the case of the Laplace operator, that is, for an orthonormal family \( \{\varphi_k\}_{k=1}^m \in H_0^1(\Omega) \) the corresponding function \( F_L(\xi) = \sum_{k=1}^m |\varphi_k(\xi)|^2 \) satisfies [10], [11]
\[ 0 \leq F_L(\xi) \leq M_L = (2\pi)^{-n}|\Omega|; \]
\[ |\nabla F_L(\xi)| \leq L_L = 2(2\pi)^{-n} \sqrt{|\Omega|I}; \]
\[ \int F_L(\xi) \, d\xi = m; \quad (2.14) \]
\[ \int |\xi|^2 F_L(\xi) \, d\xi = \sum_{k=1}^m \|\nabla \varphi_k\|^2. \]

3. Minimization problem

There is not much difference now between the Laplace and the Stokes operators, and the problem of lower bounds for the eigenvalues reduces to the problem of finding \( \Sigma_{M,L}(m) \) defined in the minimization problem (1.8).

We consider the symmetric-decreasing rearrangement \( F^*(\xi) \) of the \( F(\xi) \). It is well known (see, for example, [13]) that \( 0 \leq F^*(\xi) \leq M, \int F^*(\xi) \, d\xi = \int F(\xi) \, d\xi = m \) and, in addition, \( |\nabla F^*(\xi)| \leq \text{ess sup} |\nabla F(\xi)| \). Also,
\[ \int |\xi|^2 F(\xi) \, d\xi \geq \int |\xi|^2 F^*(\xi) \, d\xi. \quad (3.1) \]
This inequality follows from the Hardy–Littlewood inequality
\[
\int G(\xi)F(\xi)d\xi \leq \int G^*(\xi)F^*(\xi)d\xi,
\]
where \(G(\xi) = G^*(\xi) = R^2 - |\xi|^2\) and without loss of generality we assume that the ball \(B_R\) contains the supports of \(F\) and \(F^*\).

Thus, we obtain a one-dimensional problem equivalent to Lemma 3.1:
\[
\sigma_n \int_0^\infty r^{n+1}F(r)dr \rightarrow \inf =: \Sigma_{M,L}(m),
\]
where \(F\) is absolutely continuous function \(\Phi\).

\[
0 \leq F(r) \leq M, \quad \sigma_n \int_0^\infty r^{n-1}F(r)dr = m, \quad -F'(r) \leq L,
\]
where \(F(r)\) is decreasing and without loss of generality we assume that \(F\) is absolutely continuous.

We consider the function \(\Phi_s(r)\) shown in Fig. 2
\[
\Phi_s(r) = \begin{cases} 
M, & \text{for } 0 \leq r \leq s; \\
M - Lr, & \text{for } s \leq r \leq s + M/L; \\
0, & \text{for } s + M/L \leq r.
\end{cases}
\]

**Lemma 3.1.** Suppose that \(\int_0^\infty r^\alpha \Phi_s(r)dr = M^*\) and \(\beta \geq \alpha\). Then for any decreasing absolutely continuous function \(F\) satisfying the conditions
\[
0 \leq F \leq M, \quad \int_0^\infty r^\alpha F(r)dr = M^*, \quad -F' \leq L,
\]
the following inequality holds:
\[
\int_0^\infty r^\beta F(r)dr \geq \int_0^\infty r^\beta \Phi_s(r)dr. \tag{3.4}
\]

**Proof.** If \(F\) is an admissible function and \(F(s) = \Phi_s(s) (= M)\), then \(F = \Phi_s\). Hence for any admissible function \(F\) such that \(F \neq \Phi_s\) (and, hence, \(F(r_0) < M = \Phi_s(r_0)\) at some point \(r_0, 0 \leq r_0 < s\)), the graph of \(F\) intersects the graph of \(\Phi_s\) to the right of \(r_0\) at exactly one point with \(r\)-coordinate \(a\), where \(a\) is in the region \(s < a < s + M/L\). In other words, \(F(r) \leq \Phi_s(r)\) for \(0 \leq r \leq a\) and \(F(r) \geq \Phi_s(r)\) for \(a \leq r < \infty\). Therefore
\[
\int_0^a r^\beta (\Phi_s(r) - F(r))dr \leq a^{\beta-\alpha} \int_0^a r^\alpha (\Phi_s(r) - F(r))dr =
\]
\[
a^{\beta-\alpha} \int_a^\infty r^\alpha (F(r) - \Phi_s(r))dr \leq \int_0^\infty r^\beta (F(r) - \Phi_s(r))dr,
\]
where the functions under the integral signs are non-negative. \(\square\)

**Lemma 3.2.** By a straight forward calculation
\[
\int_0^\infty r^\gamma \Phi_s(r)dr = \frac{M^{\gamma+2}}{(\gamma + 1)(\gamma + 2)L^{\gamma+1}}((t + 1)^{\gamma+2} - t^{\gamma+2}), \quad s = \frac{tM}{L}. \tag{3.5}
\]

Combining the above results we see that the minimizing function is given by (3.3) and the second condition in (3.2) becomes \(\sigma_n \int_0^\infty r^{n-1}\Phi_s(r)dr = m\), which in view of (3.5) gives the equation for \(t\) (and \(s\)):
\[
(t + 1)^{n+1} - t^{n+1} = m \frac{n(n + 1)L^n}{\sigma_n M^{n+1}} = m \frac{(n + 1)L^n}{\omega_n M^{n+1}} =: m_s. \tag{3.6}
\]
It will be shown (see (3.11)) that for \( m \geq 1 \) the right-hand side in (3.6) is greater than 1. Since the left-hand side is a polynomial of order \( n \) (with positive coefficients) monotonically increasing from 1 to \( \infty \) on \( \mathbb{R}^+ \), the equation (3.6) has a unique solution \( t = t(m_*) \geq 0 \). Using (3.5) this time with \( \gamma = n + 1 \) we find the solution of (1.8), that is, \( \Sigma_{M,L}(m) \). In other words, we have just proved the following result.

**Proposition 3.1.** The solution of the minimization problem (1.8) is given by

\[
\Sigma_{M,L}(m) = \frac{\sigma_n M^{n+3}}{(n+2)(n+3)L^{n+2}} \left( (t(m_*)+1)^{n+3} - t(m_*)^{n+3} \right),
\]  

(3.7)

where \( t(m_*) \) is the unique positive root of the equation (3.6).

**Remark 3.1.** The shape of the minimizer (3.3) was found in [7]. We use it here to find the exact solution (3.7) of the minimization problem (1.8).

We give explicit expressions for \( \Sigma_{M,L}(m) \) (and thereby explicit lower bounds for sums of eigenvalues of the Laplace and Stokes operators) for the dimension \( n = 2, 3, 4 \) in §4. Meanwhile we obtain the asymptotic expansion for \( \Sigma_{M,L}(m) \) valid for all dimensions \( n \).

First, it is convenient to write the right-hand side in (3.6) in the form

\[
(t+1)^{n+1} - t^{n+1} = (\eta + 1/2)^{n+1} - (\eta - 1/2)^{n+1}, \quad \eta = t + 1/2,
\]  

(3.8)

since this substitution kills half of the coefficients in the explicit expression for the polynomial. Then the equation (3.8) takes the form

\[
(n+1) \left( \eta^n + \frac{n(n-1)}{24} \eta^{-2} + \frac{n(n-1)(n-2)(n-3)}{1920} \eta^{-4} + \ldots \right) = m_*.
\]  

(3.9)

The unique positive root \( \eta(m_*) \) of this equation has the asymptotic expansion

\[
\eta(m_*) = \left( \frac{m_*}{n+1} \right)^{1/n} - \frac{n-1}{24} \left( \frac{m_*}{n+1} \right)^{-1/n} + \frac{(n-1)(n-3)(2n-1)}{5760} \left( \frac{m_*}{n+1} \right)^{-3/n} + \ldots.
\]  

(3.10)

The first term here is obvious, the second and the third terms can be found in the standard way. Therefore substituting (3.9) into the second factor in (3.7) we obtain

\[
(t(m_*)+1)^{n+3} - t(m_*)^{n+3} = (\eta(m_*) + 1/2)^{n+3} - (\eta(m_*) - 1/2)^{n+3} =
\]  

\[
(n+3) \left[ \left( \frac{m_*}{n+1} \right)^{1+2/n} + \frac{(n+2)}{12} \frac{m_*}{n+1} - \frac{(n-1)(n+2)(3n+2)}{1440} \left( \frac{m_*}{n+1} \right)^{1-2/n} + \ldots \right],
\]  

(3.10)

and then (3.7) along with the expression for \( m_* \) in (3.6) finally gives (1.10).
Proof of Theorem 1.1. The difference between the Laplace and Stokes operators is now only in the definition of $M$ and $L$ and we consider the case of the Stokes operator. Since

$$\sum_{k=1}^{m} \|\nabla u_k\|^2 = \int |\xi|^2 F_S(\xi) \, d\xi \geq \Sigma_{M,L}(m),$$

it remains to substitute into (1.10) $M_S$ and $L_S$ from (2.13). This gives that $\sum_{k=1}^{m} \|\nabla u_k\|^2 \geq$ r. h. s of (1.12) and inequality (1.12) follows by taking the first normalized eigenvectors of the Stokes problem for the $u_k$’s. The proof of (1.11) is totally similar. $\square$

We conclude this section by checking that both for the Laplace and Stokes operators $m^* \geq 1$, that is,

$$\frac{(n + 1)L^n}{\omega_n M^{n+1}} \geq 1.$$ (3.11)

(Geometrically this means that $\Phi_s$ always has a horizontal part.) This follows from the inequality

$$I = \int_{\Omega} |x|^2 \, dx \geq \frac{n |\Omega|^{1+2/n}}{(n + 2) \omega_n^{2/n}},$$ (3.12)

which, in turn, is (3.1) with $F$ being the characteristic function of $\Omega$. In fact, (3.12) and the formulas for $M$ and $L$ give much more than (3.11):

$$m^* \geq m_0^L = \frac{(n + 1)(4\pi)^n}{\omega_n^2} \left(\frac{n}{n + 2}\right)^{n/2}, \quad m^* \geq m_0^S = \frac{(n + 1)(4\pi)^n}{(n - 1) \omega_n^2} \left(\frac{n^2}{(n - 1)(n + 2)}\right)^{n/2}$$ (3.13)

for the Laplace and Stokes operators, respectively, in the sense that the right-hand sides in (3.13) tend to infinity as $n \to \infty$.

4. Lower bounds for the Laplace and Stokes operators for $n = 2, 3, 4$

The case $n = 2$. The two-dimensional case is the simplest and the results are the most complete.

Lemma 4.1. In the two-dimensional case

$$\Sigma_{M,L}(m) = \frac{1}{2\pi M} m^2 + \frac{M^2}{6L^2} m - \frac{\pi M^5}{90L^4}.$$ (4.1)

Proof. In view of (3.7) we only need to calculate the last factor there. The positive root $t(m_*)$ of the equation (3.6) $n=2$, which is the quadratic equation $(t + 1)^3 - t^3 = m_*$, is

$$t(m_*) = \sqrt{\frac{m_*}{3} - \frac{1}{12} - \frac{1}{2}}$$ (4.2)

and using (3.8) we obtain

$$(t(m_*) + 1)^5 - t(m_*)^5 = \frac{5}{9} m_*^2 + \frac{5}{9} m_* - \frac{1}{9}.$$

The rest is a direct substitution. We note that $\Sigma_{M,L}(m) = \Sigma_0(m)_{n=2}$, see (1.10). $\square$

Theorem 4.1. For $n = 2$ the eigenvalues of the Laplace and Stokes operators satisfy

$$\sum_{k=1}^{m} \mu_k \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{24} \frac{|\Omega|}{T} m \left(1 - \frac{1}{120m}\right) \geq \frac{2\pi}{|\Omega|} m^2 + \frac{119}{24} \frac{|\Omega|}{120T} m,$$ (4.3)

$$\sum_{k=1}^{m} \lambda_k \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{48} \frac{|\Omega|}{T} m \left(1 - \frac{1}{240m}\right) \geq \frac{2\pi}{|\Omega|} m^2 + \frac{239}{48} \frac{|\Omega|}{240T} m.$$ (4.4)
Proof. We consider (1.3). In view of (2.14) we have \( M = M_L = (2\pi)^{-2}|\Omega| \) and \( L = L_L = 2(2\pi)^{-2}\sqrt{|\Omega|}/I \), therefore (1.1) gives for the Laplacian
\[
\sum_{k=1}^{\lambda} \mu_k \geq \Sigma_{M,L}(m) = \frac{2\pi}{|\Omega|} m^2 + \frac{1}{24} \frac{|\Omega|}{I} m - \frac{1}{90 \cdot 2^6 \pi I^2} \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{24} \frac{|\Omega|}{I} m - \frac{1}{90 \cdot 2^5 \pi I},
\]
where the last inequality follows from (3.12): \( |\Omega|^2/I \leq 2\pi \). The proof (4.4) is similar: \( M = M_S = (2\pi)^{-2}|\Omega| \), \( L = L_S = 2(2\pi)^{-2}\sqrt{2|\Omega|}/I \) and by (1.1)
\[
\sum_{k=1}^{\lambda} \lambda_k \geq \Sigma_{M,L}(m) = \frac{2\pi}{|\Omega|} m^2 + \frac{1}{48} \frac{|\Omega|}{I} m - \frac{1}{90 \cdot 2^8 \pi I^2} \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{48} \frac{|\Omega|}{I} m - \frac{1}{90 \cdot 2^7 \pi I}.
\]
The proof of this theorem (which is Theorem (1.2)_{n=2}) is complete. \( \square \)

The case \( n = 4 \).

Lemma 4.2. In the four-dimensional case
\[
\Sigma_{M,L}(m) \geq \frac{8\sqrt{2}}{3\pi M^{1/2}} m^{3/2} + \frac{1}{3} \cdot \beta \frac{M^2}{L^2} m,
\]
where \( \beta = \beta_L^4 = 0.983 \) for the Laplace operator and \( \beta = \beta_S^4 = 0.978 \) for the Stokes operator.

Proof. The positive root \( t(m) \) of the equation (3.6)_{n=4} (which is biquadratic with respect to \( \eta = t + 1/2 \)) is
\[
t(m) = \sqrt{20m + 5/10 - 1/4 - 1/2}
\]
and with the help of (3.8) we find that
\[
\sigma(m) := (t(m) + 1)^7 - t(m)^7 = (7/50)(m\sqrt{20m + 5} + 5m - \sqrt{20m + 5} + 15/7) \geq \frac{7}{50} \left(2\sqrt{5}m^{3/2} + 5m - \frac{7\sqrt{5}}{4}m^{1/2} + \frac{15}{7} - \frac{17\sqrt{5}}{64}m^{-1/2}\right) > \frac{7\sqrt{5}}{25}m^{3/2} + \frac{7}{10}m - \frac{49\sqrt{5}}{200}m^{1/2},
\]
where we used the inequality \( 1 + x/2 - x^2/8 < \sqrt{1 + x} < 1 + x/2 \) and the fact that \( m \geq 1 \). We observe that the three terms on the right here are as in (3.10)_{n=4} so that \( \Sigma_{M,L}(m) > \Sigma_0(m)_{n=4} \), see (1.10).

We now take advantage of the fact that \( m \) is large, namely, \( m \geq m_0^L = (5/9)2^{12} = 2275.5 \ldots \) and \( m \geq m_0^S = 5 \cdot 2^{16}/3^5 = 1348.7 \ldots \), respectively, (see (3.13)). The smallest constant \( \alpha > 0 \) such that
\[
\alpha m \geq \frac{49\sqrt{5}}{200}m^{1/2}, \quad m \in [m_0, \infty)
\]
clearly is \( \alpha_0 = (49\sqrt{5}/200)m_0^{-1/2} \). For the Laplace operator \( \alpha_L^0 = (49\sqrt{5}/200)(m_0^L)^{-1/2} = 0.01148 \ldots \), while for the Stokes operator \( \alpha_S^0 = 0.01491 \ldots \). Hence
\[
\sigma(m) > \frac{7\sqrt{5}}{25}m^{3/2} + \frac{7}{10}m \geq \frac{7\sqrt{5}}{10}m, \quad \beta = 1 - \frac{10}{7}\alpha,
\]
where \( \beta_L = 0.9835 \ldots \) and \( \beta_S = 0.9786 \ldots \), respectively, and (1.5) follows by going over from \( m \) to \( m \) (see (3.6), (3.7), (1.10)). \( \square \)

Proof of Theorem (1.2)_{n=4}. We substitute the expressions for \( M \) and \( L \) into (1.5) and get the result. \( \square \)
The case $n = 3$.

**Lemma 4.3.** In the tree-dimensional case

$$
    \Sigma_{M,L}(m) \geq \frac{3}{5} \left( \frac{3}{4\pi M} \right)^{2/3} m^{5/3} + \frac{1}{4} \cdot \beta \frac{M^2}{L^2} m, \tag{4.6}
$$

where $\beta = \beta_3^L \approx 0.9869$ and $\beta = \beta_3^S \approx 0.9861$ for the Laplace and Stokes operators, respectively.

**Proof.** The unique positive root $t(m_*)$ of the cubic equation (3.6) for $n = 3$ is given by Cardano’s formula (in which all the roots are taken positive)

$$
    t(m_*) = \frac{1}{2} \left( m_* + \sqrt{m_*^2 + \frac{1}{27}} \right)^{1/3} - \frac{1}{2} \left( -m_* + \sqrt{m_*^2 + \frac{1}{27}} \right)^{1/3} - \frac{1}{2}.
$$

By a direct substitution using (3.8) we have

$$
\sigma(m_*) := (t(m_*) + 1)^6 - t(m_*)^6 =
\frac{1}{48} \left( 3\sqrt{3 + 81m_*^2} + 27m_* \right)^{2/3} (11m_* - \sqrt{3 + 81m_*^2}) + 
\frac{5}{8} m_* + 
\frac{1}{48} \left( 3\sqrt{3 + 81m_*^2} - 27m_* \right)^{2/3} (11m_* + \sqrt{3 + 81m_*^2}) - 7 \left( 3\sqrt{3 + 81m_*^2} + 27m_* \right)^{1/3} + 
\frac{7}{48} \left( 3\sqrt{3 + 81m_*^2} - 27m_* \right)^{1/3},
$$

where the four terms above are written in the order $m_*^{5/3}$, $m_*$, $m_*^{1/3}$, $m_*^{-1/3}$. We now obtain a lower bound for $\sigma(m_*)$. Using the inequality $\sqrt{1 + x} < 1 + x/2$ below we get that the first term is greater than

$$
\frac{3 \cdot 2^{2/3}}{8} m_*^{5/3} - \frac{2^{2/3}}{32} m_*^{-1/3}.
$$

The third term is equal to

$$
- \frac{90m_* + 12\sqrt{3 + 81m_*^2}}{48(3\sqrt{3 + 81m_*^2} + 27m_*)^{2/3}} > - \frac{198m_* + 2/m_*}{48(54m_*^2)^{2/3}} = - \frac{11 \cdot 2^{1/3}}{48} m_*^{1/3} - \frac{2^{1/3}}{48 \cdot 9} m_*^{-5/3}.
$$

The fourth term is equal to

$$
\frac{7}{16(3\sqrt{3 + 81m_*^2} + 27m_*)^{1/3}} > \frac{7m_*^{-1/3}}{48 \cdot 2^{1/3}} \left( 1 + \frac{1}{27m_*^2} \right)^{-1/3} > \frac{7 \cdot 2^{1/3}}{96} m_*^{-1/3},
$$

since $m_* \geq 1$. Collecting these estimates we obtain

$$
\sigma(m_*) > \frac{3 \cdot 2^{2/3}}{8} m_*^{5/3} + \frac{5}{8} m_* - \frac{11 \cdot 2^{1/3}}{48} m_*^{1/3}, \tag{4.7}
$$

so that as for $n = 4$ we have $\Sigma_{M,L}(m) \geq \Sigma_0(m)_{n=3}$, see (1.10).

As in Lemma 4.2 we have from (3.13) that $m_* \geq m^*_0 = (16 \cdot 27\pi/5)(3/5)^{1/2} = 210.2\ldots$ and $m_* \geq m^*_8 = 72\pi(9/10)^{3/2} = 193.1\ldots$ for the Laplace and Stokes operators, respectively. Therefore the inequality

$$
\alpha m_* - \frac{11 \cdot 2^{1/3}}{48} m_*^{1/3} \geq 0, \quad m_* \in [m_0, \infty)
$$
is satisfied for all $\alpha \geq \alpha_0 = \frac{11}{48} \frac{2^{2/3}}{m_0^{2/3}}$. Hence for the Laplace operator $\alpha_0^L = 0.008165 \ldots$, while for the Stokes operator $\alpha_0^S = 0.008641 \ldots$. Hence

$$\sigma(m_*) > \frac{3 \cdot 2^{2/3}}{8} m_*^{5/3} + \frac{5}{8} \beta m_*, \quad \beta = 1 - \frac{8}{5} \alpha,$$

where $\beta^L = 0.9869 \ldots$ and $\beta^S = 0.9861 \ldots$, respectively, which proves (4.6) (see 1.10).

Proof of Theorem 1.2. The proof immediately follows from (4.6). The proof of Theorem 1.2 is complete. \qed

5. Further Examples. Dirichlet bi-Laplacian

Other elliptic equations and systems with constant coefficients and Dirichlet boundary conditions can be treated quite similarly. We restrict ourselves to the Dirichlet bi-Laplacian:

$$\Delta^2 \varphi_k = \nu_k \varphi_k, \quad \varphi_k|_{\partial \Omega} = 0, \quad \frac{\partial \varphi_k}{\partial n}|_{\partial \Omega} = 0. \quad (5.1)$$

We consider the $L_2$-orthonormal family of eigenfunctions $\{\varphi_k\}_{k=1}^m \in H_0^2(\Omega)$. Then the function $F(\xi) = \sum_{k=1}^m |\tilde{\varphi}_k(\xi)|^2$ satisfies the same three conditions:

1) $0 \leq F(\xi) \leq M$,  2) $|\nabla F(\xi)| \leq L$,  3) $\int_{\mathbb{R}^2} F(\xi) \, d\xi = m$, \quad (5.2)

where as before $M = (2\pi)^{-n} |\Omega|$ and $L = 2(2\pi)^{-n} \sqrt{\text{Vol}(\Omega)}$. Since $\sum_{k=1}^m \nu_k = \int_{\mathbb{R}^n} |\xi|^4 F(\xi) \, d\xi$, we have to find the solution $\Sigma_{M,L}^4(m)$ of the minimization problem

$$\int_{\mathbb{R}^2} |\xi|^4 f(\xi) d\xi \rightarrow \inf =: \Sigma_{M,L}^4(m) \quad \text{under conditions (5.2)}, \quad (5.3)$$

whose solution is found similarly to Proposition 3.1.

Proposition 5.1. The solution of the minimization problem (5.3) is given by

$$\Sigma_{M,L}^4(m) = \frac{\sigma_n M^{n+5}}{(n + 4)(n + 5)L^{n+4}} \left( (t(m_*) + 1)^{n+5} - t(m_*)^{n+5} \right), \quad (5.4)$$

where $t(m_*)$ is the unique positive root of the equation (3.6).

Proof. The minimizer (5.3) and the equation for $s$ (3.6) are the same as before. It remains to calculate the integral $\int_{\mathbb{R}^n} |\xi|^4 \Phi_s(|\xi|) \, d\xi$ based on Lemma 3.2 \qed

We restrict ourselves to the least technical two-dimensional case.

Lemma 5.1. For $n = 2$ the exact solution $\Sigma_{M,L}^4(m)$ can be found explicitly:

$$\Sigma_{M,L}^4(m) = \frac{1}{3\pi^2 M^2} m^3 + \frac{M}{3\pi L^2} m^2 - \frac{\pi M^7}{7 \cdot 3^4 L^6}.$$

Proof. As before the unique positive root $t(m_*)$ of the equation $(t + 1)^3 - t^3 = m_*$ is given by (4.2): $t(m_*) = \sqrt{m_*}/3 - 1/2 - 1/2$, and a direct substitution gives

$$(t(m_*) + 1)^3 - t(m_*)^3 = \frac{7}{27} m_*^3 + \frac{7}{9} m_*^2 - \frac{1}{27}. \quad (5.5)$$

It remains to substitute (5.5) into (5.4) with

$$m_* = m \frac{(n + 1)\omega_n}{M^{n+1}} \big|_{n=2} = m \frac{3L^2}{\pi M^2}.$$
Theorem 5.1. For \( n = 2 \) the eigenvalues of the Dirichlet bi-Laplacian satisfy

\[
\sum_{k=1}^{m} \nu_k \geq \frac{16\pi^2}{3|\Omega|^2} m^3 + \frac{\pi}{3} m^2 \left( 1 - \frac{1}{7 \cdot 3^3 \cdot 2^6 m^2} \right) \geq \frac{16\pi^2}{3|\Omega|^2} m^3 + \frac{\pi}{3} \frac{12095}{12096} m^2. 
\] (5.6)

Proof. Similar to Theorem 4.1. \( \Box \)

Remark 5.1. The coefficient of the leading term \( m^3 \) in (5.6) is sharp.

References

1. Babenko K.I. On the asymptotic behavior of the eigenvalues of linearized Navier–Stokes equations. *Dokl. Akad. Nauk SSSR* 263 (1982), 521–525. English transl. *Soviet Math. Dokl.* 25 (1982), 359–364.
2. Babin A.V. and Vishik M.I. *Attractors of Evolution Equations*. Nauka, Moscow, 1988; English transl. North-Holland, Amsterdam, 1992.
3. Berezin F.A. Covariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR* 37 (1972), 1134–1167; English transl. in *Math. USSR Izv.* 6 (1972).
4. Constantin P. and Foias C. *Navier-Stokes Equations*. The University of Chicago Press, 1988.
5. Ghidaglia J. M., Marion M. and Temam R. Generalization of the Sobolev–Lieb–Thirring inequalities and applications to the dimension of attractors. *Differential and Integral Equations* 1 (1988), 1–21.
6. Ilyin A.A. On the spectrum of the Stokes operator. *Funkts. Analiz i ego Prilozh.* (2009) to appear; English transl. in *Func. Anal. Appl.*; arXiv:0802.4358v1 [math.AP]
7. Kovarík H., Vugalter S. and Weidl T. Two dimensional Berezin–Li–Yau inequalities with a correction term. (2008): arXiv:0802.2792v1 [math.SP].
8. Ladyzhenskaya O.A. *The Mathematical Theory of Viscous Incompressible Flow*, Nauka, Moscow 1970; English transl. Gordon and Breach, New York 1969.
9. Laptev A. and Weidl T. Recent results on Lieb-Thirring inequalities. In *Journées Équations aux Dérivées Partielles (La Chapelle sur Erdre, 2000)*, pages Exp. No. XX, 14. Univ. Nantes, Nantes, 2000.
10. Li P. and Yau S.-T. On the Schrödinger equation and the eigenvalue problem. *Commun. Math. Phys.* 8 (1983), 309–318.
11. Melas A. A lower bound for sums of eigenvalues of the Laplacian. *Proc. Amer. Math. Soc.* 131 (2002), 631–636.
12. Metivier G. Valeurs propres des opérateurs définis sur la restriction de systèmes variationnels à des sous-espaces. *J. Math. Pures Appl.* 57 (1978), 133–156.
13. Talenti G. Inequalities in rearrangement-invariant function spaces, Nonlinear Analysis, Function Spaces and Applications, Vol. 5, 177–230, Prague, Prometheus, 1995.
14. Temam R. *Navier–Stokes Equations. Theory and Numerical Analysis*, Amsterdam, North-Holland, 1984.
15. Temam R. *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd Edition. New York, Springer-Verlag, 1997.

KELDYSH INSTITUTE OF APPLIED MATHEMATICS, MOSCOW

E-mail address: ilyin@keldysh.ru