Research Article

Some Properties and Inequalities for the \((h, s)\)-Nonconvex Functions

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The purpose of this paper is to introduce the notion of strongly \((h, s)\)-nonconvex functions and to present some basic properties of this class of functions. We present Schur inequality, Jensen inequality, Hermite–Hadamard inequality, and weighted version of the Hermite–Hadamard inequality.

1. Introduction

Convexity, the study of convex functions, has scope in various fields of science and pure mathematics, as well as applied mathematics. From the last few decades, many extensions and generalizations of convexity have been expressed to support different research ideas in mathematics, see, for instance, [1–12].

Suppose \(f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R}\) be a convex function and \(\mu_1 < \mu_2\), where \(\mu_1, \mu_2 \in I = [\mu_1, \mu_2]\), then the inequality
\[
 f\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} f(u) \, du \leq \frac{f(\mu_1) + f(\mu_2)}{2},
\]
is named as the Hermite–Hadamard equality. The above inequality is a necessary condition for a function to be convex.

The generalization of inequality (1) was given by Fejér as
\[
 f\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} f(u) \, du \leq \frac{f(\mu_1) + f(\mu_2)}{2},
\]
and then
\[
 Obeidat [13] generalized the class of \((h, s)\)-convex functions, which is called strongly \((h, s)\)-convex functions. In this article, we generalized the concept of \((h, s)\)-convex functions and defined strongly \((h, s)\)-convex functions. Moreover, we present some basic properties and results of strongly \((h, s)\)-nonconvex functions, and Jensen inequality, Schur inequality, Hermite–Hadamard inequality, and weighted version of Hermite–Hadamard-type inequalities are obtained for this class of functions.

2. Preliminaries and Some Properties

In this section, we recall some definitions from the literature which are helpful for the study of strongly \((h, s)\)-nonconvex functions.
Definition 1 (convex function). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is said to be a convex function if
\[
f(\xi u + (1 - \xi)v) \leq \xi f(u) + (1 - \xi)f(v),
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \).

Definition 2 (h-convex function, see [2]). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is called a h-convex function if
\[
f(\xi u + (1 - \xi)v) \leq h(\xi)f(u) + h(1 - \xi)f(v),
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \).

Definition 3 (strongly convex function, see [14]). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is said to be a strongly convex function with modulus \( h \geq 0 \) if
\[
f(\xi u + (1 - \xi)v) \leq \xi f(u) + (1 - \xi)f(v) - h(1 - \xi)(u - v)^2,
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \).

Definition 4 (strongly h-convex function, see [15]). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is said to be a strongly h-convex function with modulus \( h \geq 0 \) if
\[
f(\xi u + (1 - \xi)v) \leq h(\xi)f(u) + h(1 - \xi)f(v) - h(1 - \xi)(u - v)^2,
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \).

Definition 5 (s-convex function). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is said to be a s-convex function in the second sense if
\[
f((1 - \xi)u + \xi v) \leq (1 - \xi)^s f(v) + (\xi^s) f(u),
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \), where \( I \subset \mathbb{R}_+ \).

Definition 6 ((h,s)-convex function, see [1]). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is said to be a (h,s)-convex function if
\[
f((1 - \xi)u + \xi v) \leq h(1 - \xi)^s f(u) + h(\xi^s) f(v),
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \).

Now, we give our definition of a strongly (h,s)-convex function.

Definition 7 (strongly (h,s)-convex function, see [13]). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is said to be a strongly (h,s)-convex function with modulus \( h \geq 0 \) if
\[
f((1 - \xi)u + \xi v) \leq h(1 - \xi)^s f(u) + h(\xi^s) f(v) - h(1 - \xi)(u - v)^2,
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \).

Definition 8 (nonconvex set). A set \( I \) is a \( p \)-convex set if \( \{\lambda u^p + (1 - \lambda)v^p \}_{\lambda \in [0, 1]} \) is a \( p \)-convex set for all \( u, v \in I \), \( p = 2k + 1 \) (or \( p = (2s + 1)/2t + 1 \)), and \( \lambda \in [0, 1] \), where \( k, t \in \mathbb{N} \).

Definition 9 (nonconvex function). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is called a nonconvex function if
\[
f(\xi u^p + (1 - \xi)v^p)^{(1/p)} \leq \xi f(u) + (1 - \xi)f(v),
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \), where \( I \) is a \( p \)-convex set.

Definition 10 ((h,s)-nonconvex function). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is called a (h,s)-nonconvex function if
\[
f(\xi u^p + (1 - \xi)v^p)^{(1/p)} \leq h(\xi^s)f(u) + h(1 - \xi)^s f(v),
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \).

Definition 11 (strongly (h,s)-nonconvex function). A function \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is called a strongly (h,s)-nonconvex function with modulus \( h \geq 0 \) if
\[
f(\xi u^p + (1 - \xi)v^p)^{(1/p)} \leq h(\xi^s)f(u) + h(1 - \xi)^s f(v) - h(1 - \xi)(u^p - v^p)^2,
\]
for all \( u, v \in I = [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \).

Clearly, a strongly (h,s)-nonconvex function covers \((h,s)\)-convex function, \((h,s)\)-nonconvex function, \( h \)-convex function, and \( s \)-convex function.

Remark 1. Allowing \( p = 1 \), strong \((h,s)\)-convexity is obtained, and with s = 1 and \( p = 1 \), strong \( h \)-convexity is obtained. Similarly, replacing \( h(\xi) = 1 \), \( s = 1 \), \( p = 1 \), and \( \delta = 0 \), the classical convexity is retracted.

In the following, we present some properties of strongly \((h,s)\)-nonconvex functions.

Proposition 1. Let \( h_1 \) and \( h_2 \) be nonnegative functions defined on interval \( I = [\mu_1, \mu_2] \) such that \( h_2(\xi^s) \leq h_1(\xi^s) \) where \( \xi \in [0, 1] \) and \( s \in [0, 1] \). If \( f \) is a strongly \((h_2 - s)\)-nonconvex function, then \( f \) is a strongly \((h_1 - s)\)-nonconvex function.

Proof. If \( f \) is a strongly \((h_2 - s)\)-nonconvex function, then for every \( u, v \in [\mu_1, \mu_2] \) and \( \xi \in [0, 1] \), we have
Proof. We start the proof with

\[
\lambda f \left[ (\xi u^p + (1 - \xi) v^p)^{(1/p)} \right] \leq \lambda h(\xi) f (u) + \lambda h(1 - \xi) f (v) - 2\lambda(1 - \xi) (u^p - v^p)^2
\]

\[
- \theta \mu_1 t (1 - \xi) (u^p - v^p)^2,
\]

where \( \theta \mu_1 = \lambda \theta \).

\( \square \)

**Theorem 1.** Suppose \( h(\xi) \geq \xi \) for each \( \xi \) in \([0, 1]\), then \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is a strongly \((h, s)\)-nonconvex function with modulus \( \theta \), and \( g : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is defined as

\[
g(u) = f(u) - \theta (u^p)^2,
\]

then \( g \) is a \((h, s)\)-nonconvex function.

Proof. We start the proof with

\[
g \left[ (\xi v^p + (1 - \xi) u^p)^{(1/p)} \right] = f \left[ (\xi v^p + (1 - \xi) u^p)^{(1/p)} \right] - \theta (\xi v^p + (1 - \xi) u^p)^2
\]

\[
\leq h(\xi) f (v) + h(1 - \xi) f (u) - \theta (1 - \xi) (u^p - v^p)^2
\]

\[
- \theta (\xi v^p + (1 - \xi) u^p)^2
\]

\[
= h(\xi) g (v) + h(1 - \xi) g (u) + \theta h(1 - \xi) (u^p)^2 + \theta h(\xi) (v^p)^2
\]

\[
- \theta (1 - \xi) (u^p - v^p)^2 - \theta (\xi v^p + (1 - \xi) u^p)^2
\]

\[
\leq h(\xi) g (u) + h(1 - \xi) g (v) + \theta (h(1 - \xi) - (1 - \xi)) (u^p)^2
\]

\[
+ \theta (h(\xi) - t) (v^p)^2.
\]

As \( h(\xi) \leq \xi \) and \( h(1 - \xi) \leq (1 - \xi) \), then we obtain

\[
g \left[ (\xi v^p + (1 - \xi) u^p)^{(1/p)} \right] \leq h(\xi) g (v) + h(1 - \xi) g (u).
\]

\( \square \)

**Remark 2.** Theorem 1 is not true for arbitrary \( h \), see Example 1 in [13].

**Theorem 2.** Suppose \( h(\xi) \geq \xi \) for each \( \xi \) in \([0, 1]\), then \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is a \((h, s)\)-nonconvex function, and \( g : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) is defined as

\[
g(u) = f(u) - \theta (u^p)^2,
\]

then \( g \) is a strongly \((h, s)\)-nonconvex function.

Proof. We start the proof with

\[
g \left[ (\xi v^p + (1 - \xi) u^p)^{(1/p)} \right] = f \left[ (\xi v^p + (1 - \xi) u^p)^{(1/p)} \right] - \theta (\xi v^p + (1 - \xi) u^p)^2
\]

\[
\leq h(\xi) f (v) + h(1 - \xi) f (u) - \theta (1 - \xi) (u^p - v^p)^2
\]

\[
- \theta (\xi v^p + (1 - \xi) u^p)^2
\]

\[
= h(\xi) g (v) + h(1 - \xi) g (u) + \theta h(1 - \xi) (u^p)^2 + \theta h(\xi) (v^p)^2
\]

\[
- \theta (1 - \xi) (u^p - v^p)^2 - \theta (\xi v^p + (1 - \xi) u^p)^2
\]

\[
\leq h(\xi) g (u) + h(1 - \xi) g (v) + \theta (h(1 - \xi) - (1 - \xi)) (u^p)^2
\]

\[
+ \theta (h(\xi) - t) (v^p)^2.
\]

\( \square \)
As \( h(\xi^t) \geq \xi^t \geq \xi \) and \( h(1 - \xi)^t \geq (1 - \xi^t) \geq (1 - \xi) \), then the above inequality yields
\[
\alpha \left[ (\xi^{1/p} + (1 - \xi)u^p) - \frac{1}{p} \left(\sum_{i=1}^{n} h(\xi_i)u_i^p\right) \right] \leq h(\xi^t)g(v) + h(1 - \xi^t)g(u) - 8\xi(1 - \xi)(u^{p} - v^p)^2.
\] (21)

This completes the proof. \( \square \)

Remark 3. Theorem 2 is not true for arbitrary \( h \), see Example 2 in [13].

3. Main Results

In this section, we intend to make the reformulations of the Jensen-type inequality, Hermite–Hadamard type inequalities, Fejer type inequality, and Schur type inequality for strongly \((h,s)\)-nonconvex functions.

**Theorem 3** (Jensen inequality). Suppose \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) be a strongly \((h,s)\)-nonconvex function, then the inequality
\[
f \left[ \frac{1}{\alpha} \left( \sum_{i=1}^{n} h(\xi_i^t)u_i^p \right)^{(1/p)} \right] \leq \sum_{i=1}^{n} h(u_i) f(u_i^p) - \frac{1}{p} \sum_{i=1}^{n} h(\xi_i^t)(u_i^p - \bar{v}^p)^2,
\] (22)
holds for all \( u_1^p, \ldots, u_n^p \in I = [\mu_1, \mu_2] \) and \( h(\xi_1^t), \ldots, h(\xi_n^t) > 0 \) with \( h(\xi_1^t) + \ldots + h(\xi_n^t) = 1 \) and \( \bar{v}^p = h(\xi_1^t)u_1^p + \ldots + h(\xi_n^t)u_n^p \).

**Proof.** Take \( u_1, \ldots, u_n \in I = [\mu_1, \mu_2] \) and \( h(\xi_1^t), \ldots, h(\xi_n^t) > 0 \), where \( \xi_1, \ldots, \xi_n > 0 \), such that \( h(\xi_1^t) + \ldots + h(\xi_n^t) = 1 \).

Set \( \bar{v}^p = h(\xi_1^t)u_1^p + \ldots + h(\xi_n^t)u_n^p \) and take a function
\[
g(u) = u(x - \bar{v}^p)^2 + a(x - \bar{v}^p)^2 + f(\bar{v}^p),
\] (23)
supporting \( f \) at \( \bar{v}^p \), that is, \( g(\bar{v}^p) = f(\bar{v}^p) \) and \( g(u) \leq f(u), u \in I = [\mu_1, \mu_2] \).

Then, we have
\[
f(u_i^p) \geq g(u_i^p) = u_i^p(\bar{v}^p - u_i^p)^2 + a(\bar{v}^p - u_i^p)^2 + f(\bar{v}^p),
\] (24)
for every \( \xi \in [1,2,\ldots,n] \).

Multiplying on both sides of the above inequality by \( h(\xi^t) \) and summing up, we obtain
\[
\sum_{i=1}^{n} h(\xi_i^t) f(u_i^p) \geq \sum_{i=1}^{n} g(u_i^p) = \sum_{i=1}^{n} h(\xi_i^t)(u_i^p - \bar{v}^p)^2 + a \sum_{i=1}^{n} h(\xi_i^t)(u_i^p - \bar{v}^p)^2 + f(\bar{v}^p).
\] (25)

Since \( \sum_{i=1}^{n} h(\xi_i^t)(u_i^p - \bar{v}^p)^2 = 0 \), we have
\[
f(\bar{v}^p) \leq \sum_{i=1}^{n} h(\xi_i^t) f(u_i^p) - \frac{1}{p} \sum_{i=1}^{n} h(\xi_i^t)(u_i^p - \bar{v}^p)^2,
\] (26)
which completes the proof. \( \square \)

Now, the following results are obtained.

**Corollary 1.** Suppose \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) be a strongly \((s,p)\)-convex function, then the inequality
\[
f \left[ \left( \sum_{i=1}^{n} \xi_i^t u_i^p \right)^{(1/p)} \right] \leq \sum_{i=1}^{n} \xi_i^t f(u_i^p) - \frac{1}{p} \sum_{i=1}^{n} \xi_i^t(u_i^p - \bar{v}^p)^2,
\] (27)
holds for all \( u_1^p, \ldots, u_n^p \in I = [\mu_1, \mu_2], \xi_1^t, \ldots, \xi_n^t > 0, \) and \( s \in [0,1] \) with \( \xi_1^t + \ldots + \xi_n^t = 1 \) and \( \bar{v}^p = \xi_1^tu_1^p + \ldots + \xi_n^tu_n^p \).

**Corollary 2.** Suppose \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) be a strongly \((h,s)\)-convex function, then the inequality
\[
f \left[ \left( \sum_{i=1}^{n} \xi_i^t u_i^p \right)^{(1/p)} \right] \leq \sum_{i=1}^{n} \xi_i^t f(u_i^p) - \frac{1}{p} \sum_{i=1}^{n} \xi_i^t(u_i^p - \bar{v}^p)^2,
\] (28)
holds for all \( u_1, \ldots, u_n \in I = [\mu_1, \mu_2], \xi_1^t, \ldots, \xi_n^t > 0, \) and \( s \in [0,1] \) with \( \xi_1^t + \ldots + \xi_n^t = 1 \) and \( \bar{v}^p = \xi_1^tu_1 + \ldots + \xi_n^tu_n \).

Fixing \( h(\xi^t) = \xi^t \) in inequality (22), we obtain Corollary 1. Similarly, for \( p = 1 \), the inequality (22) yields Corollary 2. If we impose the above conditions with \( s = 1 \) on inequality (22), we obtain the Jensen inequality for a strongly convex function.

**Theorem 4** (Schur inequality). Assume that \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) be a strongly \((h,s)\)-nonconvex function with modulus \( \bar{\theta} \geq 0 \), then for all \( u_1^p, u_2^p, u_3^p \in I = [\mu_1, \mu_2] \) such that \( u_1^p < u_2^p < u_3^p \) and \( u_2^p - u_1^p, u_3^p - u_2^p, u_2^p - u_1^p \in (0,1) \), the inequality
\[
h(u_3^p - u_2^p)^s f(u_1) + h(u_2^p - u_1^p)^s f(u_2) + h(u_3^p - u_1^p)^s f(u_3) - u_2^p - u_1^p)(u_2^p - u_1^p)^s f(u_3) \geq 0,
\] (29)
holds for \( s \in [0,1] \).

**Proof.** The proof starts with the assumption that \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) be a strongly \((h,s)\)-nonconvex function with modulus \( \bar{\theta} \geq 0 \) and \( u_1^p, u_2^p, u_3^p \in I = [\mu_1, \mu_2] \), then \( (u_2^p - u_1^p)/u_2^p - u_1^p), (u_3^p - u_2^p)/u_2^p - u_1^p, (u_2^p - u_1^p)/u_2^p - u_1^p \in (0,1) \), and \( (u_3^p - u_2^p)/u_2^p - u_1^p) + (u_2^p - u_1^p)/u_2^p - u_1^p = 1 \).

Substituting \( \xi = (u_3^p - u_2^p)/u_2^p - u_1^p), u_3^p = u_1^p, \) and \( v^p = u_3^p \) in inequality (12) yields
Corollary 3. Assume that \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) be a strongly \((h, s)\)-convex function with modulus \( \vartheta \geq 0 \), then for all \( \mu_1, \mu_2, \mu_3 \in I = [\mu_1, \mu_2] \) such that \( \mu_1 < \mu_2 < \mu_3 \) and \( \mu_1 < \mu_2, \mu_2 < \mu_3, \mu_1 < \mu_3 \in (0, 1) \), the inequality
\[
h(\mu_3 - \mu_1)f(\mu_3) + h(\mu_2 - \mu_1)f(\mu_2) + h(\mu_3 - \mu_2)f(\mu_3) - u(\mu_3 - \mu_2)(\mu_2 - \mu_1) \geq 0,
\]
holds for \( s \in [0, 1] \).

Theorem 5 (Hermite–Hadamard inequality). Let \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) be a strongly \((h, s)\)-nonconvex function with modulus \( \vartheta \geq 0 \) and \( h(1/2) > 0 \), then the inequality
\[
\frac{1}{2h(1/2)} \left[ f\left( \frac{\mu_1^p + \mu_2^p}{2} \right) \right]^{(1/p)} + \frac{\vartheta(\mu_2^p - \mu_1^p)^2}{12}
\leq \frac{p}{\mu_2^p - \mu_1^p} \int_{\mu_1}^{\mu_2} u^{p-1} f(u) \, du
\leq [f(\mu_1) + f(\mu_2)] \int_0^1 h(\xi^p) \, d\xi - \frac{\vartheta(\mu_2^p - \mu_1^p)^2}{6},
\]
holds.

Proof. We begin the proof by using the definition of the strongly \((h, s)\)-nonconvex function:
\[
f\left( \xi \mu_1^p + (1 - \xi) \mu_2^p \right)^{(1/p)} \leq h(\xi^p) f(\mu_1) + h(1 - \xi)^p f(\mu_2) - \vartheta(1 - \xi)(\mu_2^p - \mu_1^p)^2,
\]
for all \( \xi \in [0, 1] \) and \( s \in [0, 1] \).

Integrating the above inequality with respect to “\( t \)” over \([0, 1]\) yields
\[
\int_0^1 f\left( \xi \mu_1^p + (1 - \xi) \mu_2^p \right)^{(1/p)} d\xi
\leq f(\mu_1) \int_0^1 h(\xi^p) d\xi + f(\mu_2) \int_0^1 h(1 - \xi)^p d\xi
- \vartheta(\mu_2^p - \mu_1^p)^2 \int_0^1 t(1 - \xi) d\xi
\leq f(\mu_1) \int_{\mu_1}^{\mu_2} u^{p-1} f(u) \, du
\leq \frac{p}{\mu_2^p - \mu_1^p} \int_{\mu_1}^{\mu_2} u^{p-1} f(u) \, du
\leq \frac{p}{\mu_2^p - \mu_1^p} \int_{\mu_1}^{\mu_2} u^{p-1} f(u) \, du
\leq \frac{p}{\mu_2^p - \mu_1^p} \int_{\mu_1}^{\mu_2} u^{p-1} f(u) \, du,
\]
which completes the proof.

Remark 4. If we set \( p = 1 \) in (33), we obtain the Hermite–Hadamard inequality for a strongly \((h, s)\) convex
function, see [13]. And, allowing \( s = 1, h(\xi') = \xi' \), and \( \theta = 0 \), it yields a classical Hermite–Hadamard inequality.

**Theorem 6** (Fejér inequality). Let \( f : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) be a strongly \((h, s)\)-nonconvex function, \( h(1/2') > 0 \), and \( w : I = [\mu_1, \mu_2] \rightarrow \mathbb{R} \) be a nonnegative symmetric function about \((\mu_1^p + \mu_2^p)/2\), then

\[
\frac{p}{2h(1/2')(\mu_2^p - \mu_1^p)} \left[ f\left(\frac{\mu_1^p + \mu_2^p}{2}\right) \right] \int_{\mu_1}^{\mu_2} w(u)du \\
+ \frac{\theta}{4} \int_{\mu_1}^{\mu_2} (\mu_2^p + \mu_1^p - 2u^p)^2 w(u)du \\
\leq \frac{p}{\mu_2^p - \mu_1^p} \int_{\mu_1}^{\mu_2} u^{p-1} w(u)f(u)du
\]

(40)

**Proof.** Suppose \( f \) is a strongly \((h, s)\)-nonconvex function, then for \( u = \left[ (\xi \mu_2^p + (1 - \xi)\mu_1^p) \right] \in I = [\mu_1, \mu_2] \), we have

\[
f\left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \leq h(\xi') f(\mu_2) + h(1 - \xi') f(\mu_1) - u \xi (1 - \xi)(\mu_2^p - \mu_1^p)^2,
\]

(41)

where \( \xi \in [0, 1] \). Since \( w \) is a nonnegative symmetric function about \((\mu_1^p + \mu_2^p)/2\), then we have

\[
w\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \right] f\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \right] \\
\leq \left[ h(\xi') f(\mu_2) + h(1 - \xi') f(\mu_1) - u \xi (1 - \xi)(\mu_2^p - \mu_1^p)^2 \right] \\
w\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) + u \xi (1 - \xi)(\mu_2^p - \mu_1^p)^2 \right].
\]

(42)

Integrating the above inequality with respect to “\( \xi \)” over \([0, 1]\) yields

\[
\int_0^1 w\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \right] f\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \right] d\xi \\
\leq \int_0^1 h(\xi') f(\mu_2) + h(1 - \xi') f(\mu_1) - u \xi (1 - \xi)(\mu_2^p - \mu_1^p)^2 \\
w\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \right] d\xi \\
- \int_0^1 u \xi (1 - \xi)(\mu_2^p - \mu_1^p)^2 w\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \right] d\xi.
\]

(43)

and since \( w \) is symmetric about \((\mu_1^p + \mu_2^p)/2\), we obtain

\[
\int_0^1 \left[ (\xi \mu_2^p + (1 - \xi)\mu_1^p) \right] f\left[ (\xi \mu_2^p + (1 - \xi)\mu_1^p) \right] d\xi \\
= \frac{p}{\mu_2^p - \mu_1^p} \int_{\mu_1}^{\mu_2} u^{p-1} f(u)w(u)du
\]

(44)

Hence, we obtain

\[
\frac{p}{\mu_2^p - \mu_1^p} \int_{\mu_1}^{\mu_2} u^{p-1} w(u)f(u)du \leq p \left[ f(\mu_1) + f(\mu_2) \right]
\]

(45)

\[
\int_{\mu_1}^{\mu_2} u^{p-1} h\left(\frac{\mu_2^p - u^p}{\mu_2^p - \mu_1^p}\right) w(u)du \\
- \frac{u \xi (1 - \xi)(\mu_2^p - \mu_1^p)^2 w\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \right] d\xi.
\]

(46)

For the left-hand side of inequality (40), we observe that for all \( \xi \in [0, 1] \), we have

\[
f\left[ \left(\frac{\mu_1^p + \mu_2^p}{2}\right) \right] = f\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p + \xi \mu_1^p + (1 - \xi)\mu_2^p\right) \right] \\
\leq h(1/2) \left[ f\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \right] + f\left[ \left(\xi \mu_1^p + (1 - \xi)\mu_2^p\right) \right] \right] \\
- \frac{u \xi (1 - 2\xi)^2 w\left[ \left(\xi \mu_2^p + (1 - \xi)\mu_1^p\right) \right] d\xi.
\]

(47)

After integrating the above inequality with respect to “\( \xi \)” over \([0, 1]\) and multiplying both sides by \( w \), which is a nonnegative and symmetric function about \((\mu_1^p + \mu_2^p)/2\), we obtain
\[
f\left(\left(\frac{\mu_i^p + \mu_i^s}{2}\right)^{1/p}\right)\int_0^1 \omega \left[\left(\xi \mu_i^p + (1 - \xi) \mu_i^s\right)^{(1/p)}\right] d\xi
\]

\[
\leq h\left(\frac{1}{2^p}\right)\int_0^1 \left[f(\xi \mu_i^p + (1 - \xi) \mu_i^s) + f(\xi \mu_i^p + (1 - \xi) \mu_i^s)\right] d\xi
\]

\[
-\omega\left[\left(\xi \mu_i^p + (1 - \xi) \mu_i^s\right)^{(1/p)}\right] d\xi
\]

\[
-\frac{u(\mu_i^p - \mu_i^s)^2}{4} \int_0^1 (1 - 2\xi)^2 \omega \left[\left(\xi \mu_i^p + (1 - \xi) \mu_i^s\right)^{(1/p)}\right] d\xi,
\]

\[
\int_0^1 f\left[\left(\xi \mu_i^p + (1 - \xi) \mu_i^s\right)^{(1/p)}\right] \omega \left[\left(\xi \mu_i^p + (1 - \xi) \mu_i^s\right)^{(1/p)}\right] d\xi
\]

\[
= \frac{p}{\mu_i^p - \mu_i^s} \int_{\mu_i^p}^{\mu_i^s} u^{p-1} \omega(u) du
\]

\[
= \frac{p}{\mu_i^p - \mu_i^s} \int_{\mu_i^p}^{\mu_i^s} u^{p-1} (\mu_i^p + \mu_i^s - u^p) du
\]

\[
= \int_0^1 f\left[\left(\xi \mu_i^p + (1 - \xi) \mu_i^s\right)^{(1/p)}\right] \omega \left[\left(\xi \mu_i^p + (1 - \xi) \mu_i^s\right)^{(1/p)}\right] d\xi.
\]

Thus, we obtain

\[
\frac{p}{\mu_i^p - \mu_i^s} \int_{\mu_i^p}^{\mu_i^s} u^{p-1} \omega(u) f(u) du
\]

\[
\leq \frac{2ph(1/2^p)}{\mu_i^p - \mu_i^s} \int_{\mu_i^p}^{\mu_i^s} u^{p-1} f(u) \omega(u) du
\]

\[
-\frac{up}{4(\mu_i^p - \mu_i^s)} \int_{\mu_i^p}^{\mu_i^s} u^{p-1} \left(\mu_i^p + \mu_i^s - 2u^p\right)^2 \omega(u) du,
\]

since \( h(1/2^2) > 0 \), which yields

\[
\frac{p}{\mu_i^p - \mu_i^s} \int_{\mu_i^p}^{\mu_i^s} u^{p-1} \omega(u) f(u) du
\]

\[
\geq \frac{p}{2h(1/2^2)(\mu_i^p - \mu_i^s)} \int_{\mu_i^p}^{\mu_i^s} \left(f\left(\left(\frac{\mu_i^p + \mu_i^s}{2}\right)^{(1/p)}\right)\right) \omega(u) du
\]

\[
\int_{\mu_i^p}^{\mu_i^s} \omega(u) du + \frac{8}{4} \int_{\mu_i^p}^{\mu_i^s} \left(\mu_i^p + \mu_i^s - 2u^p\right)^2 \omega(u) du du.
\]

(47)

This completes the proof. \( \Box \)

**Remark 5.** If we allow \( p = 1 \) in Theorem 6, then it reduces to the Fejér-type inequality for a strongly \((h, s)\)-convex function, and for \( p = 1 \) and \( \vartheta = 0 \), we obtain the inequalities generalized in [1].

### 4. Conclusion

In this paper, we have introduced a strongly \((h, s)\)-nonconvex function, which is the generalization of many existing definitions. We also proved several inequalities, for example, Schur inequality, Jensen inequality, and Hermite–Hadamard inequality, for a newly defined strongly \((h, s)\)-nonconvex function. This definition can also be used to develop inequalities presented in [16–18] and references therein.

### 5. Future Directions

In future, we are interested to work on generalizations of stochastic \( h \)-convex processes and stochastic \((h, s)\) processes. We will develop Schur inequality, Jensen inequality, and Hermite–Hadamard inequality for these generalizations.

### Data Availability

All data required for this paper are included within this paper.

### Conflicts of Interest

The authors do not have any conflicts of interest.

### Authors’ Contributions

M. S. Saleem proposed the problem. H. Rehman proved the results. M. Imran wrote the first version of the paper. C. Wang analysed the results, revised the paper, and proposed future directions.

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