The Non-Perturbative $\mathcal{O}(g^6)$ Contribution to the Free Energy of Hot SU(N) Gauge Theory

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The non-perturbative input necessary for the determination of the $\mathcal{O}(g^6)$ part of the weak coupling expansion of the free energy density for SU(2) and SU(3) gauge theories is estimated. Although the perturbative information completing the contribution to this order is missing, we give arguments that the magnetic fluctuations are dominated by screened elementary magnetic gluons.

1. Motivation & Introduction

Perturbation theory in its original formulation fails beyond $\mathcal{O}(g^5)$ in the calculation of the free energy density of non-Abelian gauge theories [1]. Very recently Braaten [2] has pointed out that the coefficients of the higher powers in the weak coupling expansion can be determined by invoking non-perturbative information from the effective theory of three-dimensional static magnetic fluctuations.

In Ref. [3] a systematic two-step separation of the perturbatively treatable fluctuations from the static magnetic sector has been proposed. In the first step the full (electric and magnetic) static sector is represented by an effective three-dimensional theory. In this theory (called EQCD) a massive $m_E$ adjoint scalar field stands for the screened electric fluctuations. In the second step this theory is matched onto an effective magnetic theory (MQCD) with a separation cut-off $\Lambda_M$. While the contribution of the non-static and static electric modes to the free energy can be safely calculated perturbatively (expansion parameters are $g(T)$ and $m_E/T$, respectively), the magnetic sector is still non-perturbative and should be investigated by numerical methods. This was done by simulating SU(3) pure gauge theory in 3 dimensions which is just the MQCD.

We also calculated the string tension in this theory and compared it to the 4 dimensional spatial string tension and to a possible magnetic mass.

2. Part A: Free Energy

According to the two step reduction program the high temperature free energy of the full theory can be written as

$$f_{QCD}(T >> T_c) = f_{NS} + f_E + f_M. \quad (1)$$

The first term contains all non-static diagrams in the full 4-d theory, while the second term contains the contribution from EQCD. It can be calculated safely in an expansion in $m_E/T$.

The non-perturbative information contained in the third term comes from the minimal MQCD theory. The purpose of our paper is to present first results of a lattice analysis of MQCD, i.e. the 3-dimensional SU(N) gauge theory. This does provide the non-perturbative input needed to evaluate the $\mathcal{O}(g^6)$ part of $f_M$, denoted in the following by $f_M^{(0)}$.

With the Lagrangian $L_{MQCD}^{(0)} = \frac{1}{4} G_{ij}^a G_{ij}^a$, one has the following cut-off dependence:

$$f_M^{(0)} = T \cdot \left[ (\text{div.terms}) + (a_4 + a'_4 \log \frac{\Lambda_M}{g_3^2}) g_3^6 \right] + \text{(higher ord.)}$$

The first terms are divergent with some power of $\Lambda_M$, but they cancel against analogous terms in the electric sector. So the numbers $a_4$ and $a'_4$ carry the interesting finite information and are to be determined in lattice calculations.
On the lattice it is easier to use the internal energy $\epsilon$

$$
\epsilon = \frac{T^2}{V} \frac{d}{dT} \log Z = -\frac{T^2}{V} \frac{dg_2^2}{dT} \epsilon_3
$$

$$
\epsilon_3 = -\frac{1}{V} \frac{d}{dg_3^2} \log Z
= -3\Lambda_3^3 \langle P \rangle \frac{\beta_3}{g_3^2}, \beta_3 = \frac{2N}{ag_3^2}
$$

because it is related to the Plaquette expectation value $\langle P \rangle$. We expand $\langle P \rangle$ to identify the relevant term ($\sim \beta_3^{-4}$) using the ansatz $\langle P \rangle = \sum_{n=1}^{\infty} c_n \beta_3^{-n}$ where the coefficients $c_1$ and $c_2$ are known from lattice perturbation theory. Further we define

$$
\Delta = \beta_3^3 (\langle P \rangle - c_1 \beta_3^{-1} - c_2 \beta_3^{-2}).
$$

Simulations were performed in 3 dimensional SU(2) and SU(3) on $16 \times 16 \times 64$ resp. $32^3$ lattices with Plaquette measurements for $\beta_3 \in [6.0; 14.0]$ resp. $\beta_3 \in [12.0; 30.0]$. The results for $\Delta$ are shown in figures 1 and 2.

![Figure 1. $\Delta$ versus $1/\beta_3$ in SU(2). The solid (dashed) line is the case I (case II) fit both with $c_5 = 0$.](image1)

![Figure 2. $\Delta$ versus $1/\beta_3$ in SU(3). The curve shows the case I and case II ($c_5 = 0$) fits, which lie on top of each other.](image2)

With the fit-functions

$$
\Delta = \begin{cases} 
  c_3 + \frac{c_4}{\beta_3^2} + \frac{c_5}{\beta_3^3}, & \text{case I} \\
  c_3 + (c_4' \log \beta_3 + c_4) \frac{1}{\beta_3^2} + \frac{c_5}{\beta_3^3}, & \text{case II}
\end{cases}
$$

we got the following parameters:

| SU(2) | case I       | case II      |
|-------|--------------|--------------|
| $c_3$ | 0.351(5)    | 0.45(5)     |
| $c_4$ | 0.635(37)   | 1.4(4)      |
| $c_4'$| -0.75(35)   | -0.75(35)   |
| $c_5$ | -           | -           |
| $a_4$ | -0.010(1)   | -0.002(16)  |

| SU(3) | case I       | case II      |
|-------|--------------|--------------|
| $c_3$ | 7.30(11)    | 8.11(20)    |
| $c_4$ | 15.2(3.6)   | 99(6)       |
| $c_4'$| -29.2(3.4)  | -29.2(3.4)  |
| $c_5$ | 252(29)     | -           |
| $a_4$ | -0.07(2)    | -0.17(6)    |

In case II it was not possible to fit the logarithmic correction and the quadratic term at the same time, so we set $c_5 = 0$. Although the SU(2) data indicate a quadratic behaviour the case I fits were unstable with a free $c_5$. From the coefficients $c_1$ one can then extract $a_4$ by comparing the expansions of $\epsilon_3$ and $f^{(0)}_M$.

3. Conclusions, Part A

The analysis presented above gives the framework to determine the $O(g^6)$ contribution to the free energy from lattice input. At present the data are not accurate enough to be sensitive for logarithmic corrections. A calculation of $c_3$ in lattice perturbation theory, which is in principle pos-
sible, would, however, improve the analysis. The $O(g^6)$ contribution is about 10% of the Stefan-Boltzmann value for $g^2 \simeq 1$, which is in magnitude compatible with the known perturbative contributions.

The contribution of the magnetic sector can be explained with the existence of a pseudo particles with mass $m_M \sim g^2 T$ and degeneracy $N_D = 2(N^2 - 1)$. It would give the following contribution to the energy density:

$$\epsilon = T^2 \frac{d g_3^2 (T)}{dT} \cdot \frac{N_D m_M^3}{4 \pi g_3^2}.$$  

Using our fit results we obtain the following gluon masses:

$$m_{\text{gluon}} = \begin{cases} 
(0.397 \pm 0.008) g_3^2, & \text{SU(2), I} \\
(0.55 \pm 0.04) g_3^2, & \text{SU(3), I} \\
(0.78 \pm 0.29) g_3^2, & \text{SU(3), II}
\end{cases}$$

This may be compared with calculations of the magnetic mass from the gluon propagator in Landau gauge [3], $m_{\text{gluon}} = 0.371(27) g_3^2$ for SU(2).

4. Part B: Spatial String Tension

The motivation to study 3-d string tension and 4-d spatial string tension is twofold: First one wants to know how good the reduction to MQCD works. And the second question is, to what extent is the non-vanishing spatial string tension related to the magnetic mass? This determination of $\sigma_s$ has already been done for SU(2) and is now repeated for SU(3).

In 4 dimensions a pseudo potential can be extracted from space-like Wilson loops. Also above $T_c$ this has a non vanishing linear coefficient, $\sigma_s$. We analysed 10 $\beta$ values between 6.1 (1.1$T_c$) and 7.2 (4.5$T_c$) on a $32^3 \times 8$ lattice. More details about the determination of $\sigma_s$ can be found in [4].

At high temperature an identification with $\sigma_3$ from MQCD is natural.

$$\sqrt{\sigma_3} \sim g_3^2, \quad g_3^2 = g^2(T) T, \quad \sqrt{\sigma_s} \sim g^2(T) T$$

A 2-loop fit yields $\sqrt{\sigma_s(T)} = c \cdot g^2(T) \cdot T$ with $c = 0.566(13)$. The spatial string tensions and the fit are shown in figure 3.

From a corresponding analysis in the three dimensional SU(3) theory [5] we get $\sqrt{\sigma_3} = 0.544(4) \cdot g_3^2$.

5. Conclusions, Part B

Our data confirm a $g^2(T) T$ behaviour of $\sqrt{\sigma_s}$ above 1.5$T_c$. There is a 90 % coincidence with the numerical value for $\sqrt{\sigma_3}$ from MQCD. The numbers are also very similar to the estimated magnetic masses from part A, $m_{\text{gluon}} \approx (0.5 - 0.8) \cdot g_3^2$.

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