Quantization of noncompact coverings and its physical applications

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Abstract. A rigorous algebraic definition of noncommutative coverings is developed. In the case of commutative algebras this definition is equivalent to the classical definition of topological coverings of locally compact spaces. The theory has following nontrivial applications:

• Coverings of continuous trace algebras,
• Coverings of noncommutative tori,
• Coverings of the quantum SU(2) group,
• Coverings of foliations,
• Coverings of isospectral deformations of Spin – manifolds.

The theory supplies the rigorous definition of noncommutative Wilson lines.

1. Motivation. Preliminaries

Gelfand-Naïmark theorem [2] states the correspondence between locally compact Hausdorff topological spaces and commutative C*-algebras.

Theorem 1.1. [2] (Gelfand-Naïmark). Let A be a commutative C*-algebra and let $X$ be the spectrum of A. There is the natural *-isomorphism $\gamma : A \to C_0(X)$.

So any (noncommutative) C*-algebra may be regarded as a generalized (noncommutative) locally compact Hausdorff topological space. But *-homomorphisms are not good analogs of continuous maps, because there is no a *-homomorphism $C_0(\mathcal{X}) \to C_0(\tilde{\mathcal{X}})$ which corresponds to a continuous map from a non-compact topological space $\mathcal{X}$ to a compact one $\tilde{\mathcal{X}}$. However there is the natural *-homomorphism $C_0(\mathcal{X}) \to C_b(\tilde{\mathcal{X}})$. Following theorem gives a pure algebraic description of finite-fold coverings of compact spaces.

Theorem 1.2. [9] Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact Hausdorff connected spaces and $p : \mathcal{Y} \to \mathcal{X}$ is a continuous surjection. If $C(\mathcal{Y})$ is a projective finitely generated Hilbert module over $C(\mathcal{X})$ with respect to the action

$$(f\xi)(y) = f(y)\xi(p(y)), \ f \in C(\mathcal{Y}), \ \xi \in C(\mathcal{X}),$$

then $p$ is a finite-fold covering.
This article contains pure algebraic generalizations of following topological objects:

- Coverings of noncompact spaces,
- Infinite coverings.

The words "set", "family" and "collection" are synonyms.

Following table contains special symbols.

| Symbol | Meaning |
|--------|---------|
| \( A_+ \) | Cone of positive elements of \( C^* \) algebra, i.e. \( A_+ = \{ a \in A \mid a \geq 0 \} \) |
| \( A^G \) | Algebra of \( G \) - invariants, i.e. \( A^G = \{ a \in A \mid ga = a, \forall g \in G \} \) |
| \( \text{Aut}(A) \) | Group of * - automorphisms of \( C^* \) algebra \( A \) |
| \( B(\mathcal{H}) \) | Algebra of bounded operators on a Hilbert space \( \mathcal{H} \) |
| \( \mathbb{C} \) (resp. \( \mathbb{R} \)) | Field of complex (resp. real) numbers |
| \( \mathcal{C}(\mathcal{X}) \) | \( C^* \) - algebra of continuous complex valued functions on a compact space \( \mathcal{X} \) |
| \( \mathcal{C}_0(\mathcal{X}) \) | \( C^* \) - algebra of continuous complex valued functions on a locally compact topological space \( \mathcal{X} \) equal to 0 at infinity |
| \( \mathcal{C}_c(\mathcal{X}) \) | Algebra of continuous complex valued functions on a topological space \( \mathcal{X} \) with compact support |
| \( \mathcal{C}_b(\mathcal{X}) \) | \( C^* \) - algebra of bounded continuous complex valued functions on a locally compact topological space \( \mathcal{X} \) |
| \( G(\tilde{\mathcal{X}} \mid \mathcal{X}) \) | Group of covering transformations of covering \( \tilde{\mathcal{X}} \to \mathcal{X} \) [10] |
| \( \delta_{ij} \) | Delta symbol. If \( i = j \) then \( \delta_{ij} = 1 \). If \( i \neq j \) then \( \delta_{ij} = 0 \) |
| \( \mathcal{H} \) | Hilbert space |
| \( \mathcal{K} = \mathcal{K}(\mathcal{H}) \) | \( C^* \) - algebra of compact operators on the separable Hilbert space \( \mathcal{H} \) |
| \( \lim \) | Direct limit |
| \( \lim \) | Inverse limit |
| \( \mathcal{M}(A) \) | A multiplier algebra of \( C^* \)-algebra \( A \) |
| \( \mathcal{M}_n(A) \) | The \( n \times n \) matrix algebra over \( C^* \) - algebra \( A \) |
| \( \mathbb{N} \) | A set of positive integer numbers |
| \( \mathbb{N}^0 \) | A set of nonnegative integer numbers |
| \( U(A) \subset A \) | Group of unitary operators of algebra \( A \) |
| \( \mathbb{Z} \) | Ring of integers |
| \( \mathbb{Z}_n \) | Ring of integers modulo \( n \) |
| \( \tilde{k} \in \mathbb{Z}_n \) | An element in \( \mathbb{Z}_n \) represented by \( k \in \mathbb{Z} \) |
| \( X \setminus A \) | Difference of sets \( X \setminus A = \{ x \in X \mid x \notin A \} \) |
| \( |X| \) | Cardinal number of a finite set \( X \) |
| \( |x| \) | The range projection of element \( x \) of a von Neumann algebra. |
| \( f\big|_{A'} \) | Restriction of a map \( f : A \to B \) to \( A' \subset A \), i.e. \( f\big|_{A'} : A' \to B \) |
2. Prototype. Inverse limits of coverings in topology

This subsection is concerned with a topological construction of the inverse limit in the category of coverings.

**Definition 2.1.** [10] Let \( \tilde{\pi} : \tilde{\mathcal{X}} \to \mathcal{X} \) be a continuous map. An open subset \( U \subset \mathcal{X} \) is said to be evenly covered by \( \tilde{\pi} \) if \( \tilde{\pi}^{-1}(U) \) is the disjoint union of open subsets of \( \tilde{\mathcal{X}} \) each of which is mapped homeomorphically onto \( U \) by \( \tilde{\pi} \). A continuous map \( \tilde{\pi} : \tilde{\mathcal{X}} \to \mathcal{X} \) is called a covering projection if each point \( x \in \mathcal{X} \) has an open neighborhood evenly covered by \( \tilde{\pi} \). \( \tilde{\mathcal{X}} \) is called the covering space and \( \mathcal{X} \) the base space of the covering.

**Definition 2.2.** [10] A fibration \( p : \tilde{\mathcal{X}} \to \mathcal{X} \) with unique path lifting is said to be regular if, given any closed path \( \omega \) in \( \mathcal{X} \), either every lifting of \( \omega \) is closed or none is closed.

**Definition 2.3.** [10] A topological space \( \mathcal{X} \) is said to be locally path-connected if the path components of open sets are open.

Denote by \( \pi_1 \) the functor of fundamental group [10].

**Theorem 2.4.** [10] Let \( p : \tilde{\mathcal{X}} \to \mathcal{X} \) be a fibration with unique path lifting and assume that a nonempty \( \tilde{\mathcal{X}} \) is a locally path-connected space. Then \( p \) is regular if and only if for some \( \tilde{x}_0 \in \tilde{\mathcal{X}} \), \( \pi_1(\mathcal{X})(\pi_1(\tilde{\mathcal{X}}, \tilde{x}_0)) \) is a normal subgroup of \( \pi_1(\mathcal{X}, p(\tilde{x}_0)) \).

**Definition 2.5.** [10] Let \( p : \tilde{\mathcal{X}} \to \mathcal{X} \) be a covering. A self-equivalence is a homeomorphism \( f : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}} \) such that \( p \circ f = p \). This group of such homeomorphisms is said to be the group of covering transformations of \( p \) or the covering group. Denote by \( G(\tilde{\mathcal{X}} | \mathcal{X}) \) this group.

**Proposition 2.6.** [10] If \( p : \tilde{\mathcal{X}} \to \mathcal{X} \) is a regular covering and \( \tilde{\mathcal{X}} \) is connected and locally path connected, then \( \mathcal{X} \) is homeomorphic to space of orbits of \( G(\tilde{\mathcal{X}} | \mathcal{X}) \), i.e. \( \mathcal{X} \approx \tilde{\mathcal{X}} / G(\tilde{\mathcal{X}} | \mathcal{X}) \). So \( p \) is a principal bundle.

**Corollary 2.7.** [10] Let \( p : \tilde{\mathcal{X}} \to \mathcal{X} \) be a fibration with a unique path lifting. If \( \tilde{\mathcal{X}} \) is connected and locally path-connected and \( \tilde{x}_0 \in \tilde{\mathcal{X}} \) then \( p \) is regular if and only if \( G(\tilde{\mathcal{X}} | \mathcal{X}) \) transitively acts on each fiber of \( p \), in which case

\[
\psi : G(\tilde{\mathcal{X}} | \mathcal{X}) \approx \pi_1(\mathcal{X}, p(\tilde{x}_0)) / \pi_1(p)(\tilde{\mathcal{X}}, \tilde{x}_0).
\]

**Remark 2.8.** Above results are copied from [10]. Below the covering projection word is replaced with covering.

**Definition 2.9.** [7] A compactification of a space \( \mathcal{X} \) is a compact Hausdorff space \( \mathcal{Y} \) containing \( \mathcal{X} \) as a subspace and the closure \( \overline{\mathcal{X}} \) of \( \mathcal{X} \) is \( \mathcal{Y} \), i.e. \( \overline{\mathcal{X}} = \mathcal{Y} \).

**Remark 2.10.** In the following text we denote compactifications by \( \mathcal{X} \hookrightarrow \mathcal{Y} \).

The algebraic construction requires a following definition.

**Definition 2.11.** [5] A covering \( \pi : \tilde{\mathcal{X}} \to \mathcal{X} \) is said to be a covering with compactification if there are compactifications \( \mathcal{X} \hookrightarrow \mathcal{Y} \) and \( \tilde{\mathcal{X}} \hookrightarrow \tilde{\mathcal{Y}} \) (cf. Definition 2.9) such that:

- There is a covering \( \tilde{\pi} : \tilde{\mathcal{Y}} \to \mathcal{Y} \),
- The covering \( \pi \) is the restriction of \( \tilde{\pi} \), i.e. \( \pi = \tilde{\pi}|_{\tilde{\mathcal{X}}} \) and \( \tilde{\pi}(\tilde{\mathcal{X}}) = \mathcal{X} \).
Definition 2.12. The sequence of regular finite-fold coverings with compactification
\[ X = X_0 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots \]
is said to be a (topological) finite covering sequence if following conditions hold:
- The space \( X_n \) is a second-countable [7] locally compact connected Hausdorff space for any \( n \in \mathbb{N}^0 \),
- If \( k < l < m \) are any nonnegative integer numbers then there is the natural exact sequence
  \[ \{e\} \to G(X_m | X_l) \to G(X_m | X_k) \to G(X_l | X_k) \to \{e\} \].

For any finite covering sequence we will use a following notation
\[ S = \{X = X_0 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots\} = \{X_0 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots\}, \quad S \in \text{FinTop}. \]

Definition 2.13. Let \( \{X = X_0 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots\} \in \text{FinTop} \), and let \( \hat{X} = \lim \leftarrow X_n \) be the inverse limit in the category of topological spaces and continuous maps (See [10]). If \( \hat{\pi}_0 : \hat{X} \to X_0 \) is the natural continuous map then homeomorphism \( g \) of the space \( \hat{X} \) is said to be a covering transformation if the following condition holds
\[ \hat{\pi}_0 = \hat{\pi}_0 \circ g. \]

The group \( \hat{G} \) of covering homeomorphisms is said to be the group of covering transformations of \( S \). Denote by \( G(\hat{X} | X) \) \( \text{def} \ G \).

Definition 2.14. Let \( S = \{X_0 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots\} \) be a finite covering sequence. The pair \( (Y, \{\pi^Y_n\}_{n \in \mathbb{N}}) \) of a (discrete) set \( Y \) with and surjective maps \( \pi^Y_n : Y \to X_n \) is said to be a coherent system if for any \( n \in \mathbb{N}^0 \) following diagram
\[ \begin{tikzcd}
X_n & Y \\
\pi^Y_n & \pi^Y_{n-1} \\
\end{tikzcd} \]
is commutative.

Definition 2.15. Let \( S = \{X_0 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots\} \) be a topological finite covering sequence. A coherent system \( (Y, \{\pi^Y_n\}_{n \in \mathbb{N}}) \) is said to be a connected covering of \( S \) if \( Y \) is a connected topological space and \( \pi^Y_n \) is a regular covering for any \( n \in \mathbb{N} \). We will use following notation \( (Y, \{\pi^Y_n\}) \downarrow S \) or simply \( Y \downarrow S \).

Definition 2.16. Let \( (Y, \{\pi^Y_n\}) \) be a coherent system of \( S \) and \( y \in Y \). A subset \( V \subset Y \) is said to be special if \( \pi^Y_0(V) \) is evenly covered by \( X_1 \to X_0 \) and for any \( n \in \mathbb{N}^0 \) following conditions hold:
- \( \pi^Y_0(V) \subset X_n \) is an open connected set,
- The restriction \( \pi^Y_n|_V : V \to \pi^Y_0(V) \) is a bijection.

Remark 2.17. If \( (Y, \{\pi^Y_n\}) \) is a covering of \( S \) then the topology of \( Y \) is generated by special sets.
**Definition 2.18.** Let us consider the situation of the Definition 2.15. A morphism from \((\mathcal{Y}', \{\pi^{\mathcal{Y}'}_n\}) \downarrow \mathcal{G}\) to \((\mathcal{Y}'', \{\pi^{\mathcal{Y}''}_n\}) \downarrow \mathcal{G}\) is a covering \(f : \mathcal{Y}' \to \mathcal{Y}''\) such that
\[
\pi^{\mathcal{Y}''}_n \circ f = \pi^{\mathcal{Y}'}_n
\]
for any \(n \in \mathbb{N}\).

**2.19.** There is a category with objects and morphisms described by Definitions 2.15, 2.18. Denote by \(\downarrow \mathcal{G}\) this category.

**Lemma 2.20.** [5] There is the final object of the category \(\downarrow \mathcal{G}\) described in 2.19.

**Remark 2.21.** In [5] it is proven that if \(\tilde{X}\) is the final object of the category \(\downarrow \mathcal{G}\) then following condition holds
\[
X = \bigsqcup_{g \in \hat{G}/G} g \tilde{X}
\]  
(2.1)

where

- \(\mathcal{X}\) is a topological space which coincides with \(\lim_{\leftarrow} X_n\) as a set and topology of \(\mathcal{X}\) is generated by special sets,
- \(\hat{G} = \lim_{\leftarrow} G_n\),
- \(G = G(\tilde{X}, \mathcal{X})\).
- \(\hat{G}/G\) is a set of representatives of \(\hat{G}/G\).

**Definition 2.22.** The final object \((\tilde{X}, \{\pi^{\tilde{X}}_n\})\) of the category \(\downarrow \mathcal{G}\) is said to be the (topological) inverse limit of \(\downarrow \mathcal{G}\). The notation \((\tilde{X}, \{\pi^{\tilde{X}}_n\}) = \lim_{\leftarrow} \downarrow \mathcal{G}\) or simply \(\tilde{X} = \lim_{\leftarrow} \mathcal{G}\) will be used.

The space \(\mathcal{X}\) from the proof of the Lemma 2.20 is said to be the disconnected inverse limit of \(\mathcal{G}\).

3. Noncommutative finite-fold coverings

**Definition 3.1.** If \(A\) is a C*-algebra then an action of a group \(G\) is said to be involutive if \(ga^* = (ga)^*\) for any \(a \in A\) and \(g \in G\). The action is said to be non-degenerated if for any nontrivial \(g \in G\) there is \(a \in A\) such that \(ga \neq a\).

**Definition 3.2.** Let \(A \hookrightarrow \tilde{A}\) be an injective *-homomorphism of unital C*-algebras. Suppose that there is a non-degenerated involutive action \(G \times \tilde{A} \to \tilde{A}\) of a finite group \(G\), such that \(A = \tilde{A}^G \stackrel{\text{def}}{=} \{a \in \tilde{A} | a = ga; \forall g \in G\}\). There is an \(A\)-valued product on \(\tilde{A}\) given by
\[
\langle a, b \rangle_{\tilde{A}} = \sum_{g \in G} g (a^* b)
\]
(3.1)

and \(\tilde{A}\) is an \(A\)-Hilbert module. We say that a triple \((A, \tilde{A}, G)\) is an unital noncommutative finite-fold covering if \(\tilde{A}\) is a finitely generated projective \(A\)-Hilbert module.

**Remark 3.3.** Above definition is motivated by the Theorem 1.2.

**Definition 3.4.** Let \(A, \tilde{A}\) be C*-algebras such that following conditions hold:
We shall consider $\widetilde{A}$ the noncommutative finite-fold covering $A$, which follows from (3.4) is said to be a noncommutative finite-fold covering.

**Definition 3.8.** Let $\left(A, \widetilde{A}, G\right)$ be a noncommutative finite-fold covering. The algebra $\widetilde{A}$ is a Hilbert $A$-module with an $A$-valued product given by

$$\langle a, b \rangle_{\widetilde{A}} = \sum_{g \in G} g(a^*b); \ a, b \in \widetilde{A}. \quad (3.5)$$

We say that this structure of Hilbert $A$-module is induced by the covering $\left(A, \widetilde{A}, G\right)$. Henceforth we shall consider $\widetilde{A}$ as a right $A$-module, so we will write $\widetilde{A}_A$.

### 4. Noncommutative infinite coverings

This section contains a noncommutative generalization of infinite coverings.

**Definition 4.1.** Let

$$S = \left\{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_n} A_{n+1} \xrightarrow{\pi_{n+1}} \ldots \right\}$$

be a sequence of $C^*$-algebras and noncommutative finite-fold coverings such that:

(a) There are unital $C^*$-algebras $B, \tilde{B}$ and inclusions $A \subset B, \widetilde{A} \subset \tilde{B}$ such that $A$ (resp. $\tilde{A}$) is an essential ideal of $B$ (resp. $\tilde{B}$),

(b) There is an unital noncommutative finite-fold covering $\left(B, \tilde{B}, G\right)$,

(c) $\widetilde{A} = \left\{ a \in \tilde{B} \mid \langle \tilde{B}, a \rangle_{\tilde{B}} \in A \right\}. \quad (3.2)$

The triple $\left(A, \widetilde{A}, G\right)$ is said to be a noncommutative finite-fold covering. The group $G$ is said to be the covering transformation group (of $\left(A, \widetilde{A}, G\right)$) and we use the following notation

$$G \left(\widetilde{A} \mid A\right) \overset{\text{def}}{=} G. \quad (3.3)$$

**Lemma 3.5.** [5] Let us consider the situation of the Definition 3.4. Following conditions hold:

(i) From (3.2) it turns out that $\widetilde{A}$ is a closed two sided ideal of $\tilde{B}$,

(ii) The action of $G$ on $\tilde{B}$ is such that $G\tilde{A} = \widetilde{A}$, i.e. there is the natural action of $G$ on $\widetilde{A}$,

(iii) $A \cong \widetilde{A}^G = \left\{ a \in \widetilde{A} \mid a = ga; \forall g \in G \right\}. \quad (3.4)$

**Remark 3.6.** The Definition 3.5 is motivated by the Theorem 5.2.

**Definition 3.7.** The injective $*$-homomorphism $A \hookrightarrow \widetilde{A}$, which follows from (3.4) is said to be a noncommutative finite-fold covering.

We shall consider $\widetilde{A}$ as a right $A$-module, so we will write $\widetilde{A}_A$.
The sequence $\mathcal{S}$ is said to be an \textit{(algebraical) finite covering sequence}. For any finite covering sequence we will use the notation $\mathcal{S} \in \mathfrak{FinAlg}$.\footnote{\textbf{Definition 4.2.} Let $\hat{A} = \lim A_n$ be the $C^*$-inductive limit \cite{8}, and suppose that $\hat{G} = \lim G (A_n \mid A)$ is the projective limit of groups \cite{10}. There is the natural action of $\hat{G}$ on $\hat{A}$. A non-degenerate faithful representation $\hat{A} \rightarrow B (\mathcal{H})$ is said to be \textit{equivariant} if there is an action of $\hat{G}$ on $\mathcal{H}$ such that for any $\xi \in \mathcal{H}$ and $g \in \hat{G}$ following condition holds

$$\left( g a \right) \xi = g \left( a \left( g^{-1} \xi \right) \right).$$

(4.1)\footnote{\textbf{Definition 4.3.} Let $\pi : \hat{A} \rightarrow B (\mathcal{H})$ be an equivariant representation. A positive element $a \in B (\mathcal{H})_+$ is said to be \textit{special} (with respect to $\mathcal{S}$) if following conditions hold:

(a) For any $n \in \mathbb{N}^0$ the following series

$$a_n = \sum_{g \in \ker(\hat{G} \rightarrow G (A_n \mid A))} g \pi$$

is strongly convergent and the sum lies in $A_n$, i.e. $a_n \in A_n$;

(b) If $f_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$f_\varepsilon (x) = \begin{cases} 0 & x \leq \varepsilon \\ x - \varepsilon & x > \varepsilon \end{cases}$$

(4.2) then for any $n \in \mathbb{N}^0$ and for any $z \in A$ following series

$$b_n = \sum_{g \in \ker(\hat{G} \rightarrow G (A_n \mid A))} g (z \pi z^*),$$

$$c_n = \sum_{g \in \ker(\hat{G} \rightarrow G (A_n \mid A))} g (z \pi z^*)^2,$$

$$d_n = \sum_{g \in \ker(\hat{G} \rightarrow G (A_n \mid A))} g f_\varepsilon (z \pi z^*)$$

are strongly convergent and the sums lie in $A_n$, i.e. $b_n, c_n, d_n \in A_n$;

(c) For any $\varepsilon > 0$ there is $N \in \mathbb{N}$ (which depends on $\pi$ and $z$) such that for any $n \geq N$ a following condition holds

$$\|b_n^2 - c_n\| < \varepsilon.$$ \quad (4.3)

An element $\pi' \in B (\mathcal{H})$ is said to be \textit{weakly special} if

$$\pi' = x \pi y; \text{ where } x, y \in \hat{A}, \text{ and } \pi \in B (\mathcal{H}) \text{ is special.}$$

\textbf{Lemma 4.4.} \cite{5} If $\pi \in B (\mathcal{H})_+$ is a special element and $G_n = \ker \left( \hat{G} \rightarrow G (A_n \mid A) \right)$ then from

$$a_n = \sum_{g \in G_n} g \pi,$$

it follows that $\pi = \lim_{n \rightarrow \infty} a_n$ in the sense of the strong convergence. Moreover one has $\pi = \inf_{n \in \mathbb{N}} a_n$.}
Corollary 4.5. [5] Any weakly special element lies in the enveloping von Neumann algebra \( \hat{A}'' \) of \( \hat{A} \). If \( \hat{A}_\pi \subset B(\mathcal{H}) \) is the \( C^* \)-norm completion of an algebra generated by weakly special elements then \( \hat{A}_\pi \subset \hat{A}'' \).

Lemma 4.6. [5] If \( \pi \in B(\mathcal{H}) \) is special, (resp. \( \pi' \in B(\mathcal{H}) \) weakly special) then for any \( g \in \hat{G} \) the element \( g\pi \) is special, (resp. \( g\pi' \) is weakly special).

Corollary 4.7. [5] If \( \hat{A}_\pi \subset B(\mathcal{H}) \) is the \( C^* \)-norm completion of algebra generated by weakly special elements, then there is a natural action of \( G \) on \( \hat{A}_\pi \).

Definition 4.8. Let \( \mathcal{S} = \left\{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_n} A_n \xrightarrow{\pi_{n+1}} \ldots \right\} \) be an algebraical finite covering sequence. Let \( \pi : \hat{A} \rightarrow B(\mathcal{H}) \) be an equivariant representation. Let \( \hat{A}_\pi \subset B(\mathcal{H}) \) be the \( C^* \)-norm completion of algebra generated by weakly special elements. We say that \( \hat{A}_\pi \) is the disconnected inverse noncommutative limit of \( \mathcal{S} \) (with respect to \( \pi \)). The triple \( (A,\hat{A}_\pi,G(\hat{A}_\pi \mid A) \) \) is said to be the disconnected infinite noncommutative covering of \( \mathcal{S} \) (with respect to \( \pi \)). If \( \pi \) is the universal representation then "with respect to \( \pi'' \) is dropped and we will write \( (A,\hat{A},G(\hat{A} \mid A) \).

Definition 4.9. A maximal irreducible subalgebra \( \tilde{A}_\pi \subset \hat{A}_\pi \) is said to be a connected component of \( \mathcal{S} \) (with respect to \( \pi \)). The maximal subgroup \( G_\pi \subset G(\hat{A}_\pi \mid A) \) among subgroups \( G \subset G(\hat{A}_\pi \mid A) \) such that \( G\hat{A}_\pi = \tilde{A}_\pi \) is said to be the \( \tilde{A}_\pi \)-invariant group of \( \mathcal{S} \). If \( \pi \) is the universal representation then "with respect to \( \pi'' \) is dropped.

Remark 4.10. From the Definition 4.9 it follows that \( G_\pi \subset G(\hat{A}_\pi \mid A) \) is a normal subgroup.

Definition 4.11. Let 
\[
\mathcal{S} = \left\{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_n} A_n \xrightarrow{\pi_{n+1}} \ldots \right\} \in \mathfrak{FinAlg},
\]
and let \( (A,\tilde{A}_\pi,G(\tilde{A}_\pi \mid A)) \) be a disconnected infinite noncommutative covering of \( \mathcal{S} \) with respect to an equivariant representation \( \pi : \lim A_n \rightarrow B(\mathcal{H}) \). Let \( \tilde{A}_\pi \subset \hat{A}_\pi \) be a connected component of \( \mathcal{S} \) with respect to \( \pi \), and let \( G_\pi \subset G(\hat{A}_\pi \mid A) \) be the \( \tilde{A}_\pi \)-invariant group of \( \mathcal{S} \). Let \( h_n : G(\hat{A}_\pi \mid A) \rightarrow G(A_n \mid A) \) be the natural surjective homomorphism. The representation \( \pi : \lim A_n \rightarrow B(\mathcal{H}) \) is said to be good if it satisfies to following conditions:

(a) The natural \(*\)-homomorphism \( \lim A_n \rightarrow M(\tilde{A}_\pi) \) is injective,

(b) If \( J \subset G(\hat{A}_\pi \mid A) \) is a set of representatives of \( G(\hat{A}_\pi \mid A) / G_\pi \), then the algebraic direct sum
\[
\bigoplus_{g \in J} g\tilde{A}_\pi
\]
is a dense subalgebra of \( \tilde{A}_\pi \),

(c) For any \( n \in \mathbb{N} \) the restriction \( h_n|_{G_\pi} \) is an epimorphism, i.e. \( h_n(G_\pi) = G(A_n \mid A) \).

If \( \pi \) is the universal representation we say that \( \mathcal{S} \) is good.

Definition 4.12. Let \( \mathcal{S} = \{ A = A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n \rightarrow \ldots \} \in \mathfrak{FinAlg} \) be an algebraical finite covering sequence. Let \( \pi : \hat{A} \rightarrow B(\mathcal{H}) \) be a good representation. A connected component \( \tilde{A}_\pi \subset \hat{A}_\pi \) is said to be the inverse noncommutative limit of \( \mathcal{S} \) (with respect to \( \pi \)). The \( \tilde{A}_\pi \)-invariant group \( G_\pi \) is said to be the covering transformation group of \( \mathcal{S} \) (with respect to \( \pi \)). The
triple \((A, \tilde{A}, G)\) is said to be the infinite noncommutative covering of \(\mathcal{S}\) (with respect to \(\pi\)). We will use the following notation

\[
\lim_{\pi} \downarrow \mathcal{S} \overset{\text{def}}{=} \tilde{A}_\pi, \\
G \left( \tilde{A}_\pi \mid A \right) \overset{\text{def}}{=} G_\pi.
\]

If \(\pi\) is the universal representation then “with respect to \(\pi\)” is dropped and we will write \((A, \tilde{A}, G), \lim_{\pi} \downarrow \mathcal{S} \overset{\text{def}}{=} \tilde{A}\) and \(G \left( \tilde{A} \mid A \right) \overset{\text{def}}{=} G\).

5. Examples

5.1. Quantization of topological coverings

Following theorems state the equivalence between algebraical and topological definitions of coverings.

**Theorem 5.2.** [5] If \(X, \tilde{X}\) are locally compact spaces, and \(\pi : \tilde{X} \to X\) is a surjective continuous map, then following conditions are equivalent:

(i) The map \(\pi : \tilde{X} \to X\) is a finite-fold covering with a compactification,

(ii) There is a natural noncommutative finite-fold covering \((C_0(X), C_0(\tilde{X}), G)\).

**Theorem 5.3.** [5] If \(\mathcal{S}_X = \{X = X_0 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots\} \in \mathfrak{FinTop}\) and \(\mathcal{S}_{C_0(X)} = \{C_0(X) = C_0(X_0) \to \ldots \to C_0(X_n) \to \ldots\} \in \mathfrak{FinAlg}\)

is an algebraical finite covering sequence then following conditions hold:

(i) \(\mathcal{S}_{C_0(X)}\) is good.

(ii) There are isomorphisms:

\[
\lim_{\pi} \downarrow \mathcal{S}_{C_0(X)} \approx C_0 \left( \lim_{\pi} \downarrow \mathcal{S}_X \right);
\]

\[
G \left( \lim_{\pi} \downarrow \mathcal{S}_{C_0(X)} \mid C_0(X) \right) \approx G \left( \lim_{\pi} \downarrow \mathcal{S}_X \mid X \right).
\]

5.4. Coverings of noncommutative C*-algebras

There are known finite-fold and infinite coverings of the following C*-algebras:

- Continuous trace algebras,
- Noncommutative torus,
- Quantum \(SU_q(2)\) group,
- Coverings of foliations,
- Isospectral deformations of spectral triples.

These coverings are described in [5,6].
6. Coverings and fundamental groups in physics

The math definition of fundamental group is presented in [10]. Below there is a physical explanation. The fundamental group \( \pi_1(X) \), which counts how many ways a loop can be mapped into a path-connected space \( X \). More precisely, we define \( \pi_1(X, x) \) to be the set of all homotopy classes of parameterized loop mappings that begin and end at a base-point \( x \). For a path-connected space we can add a path from \( x \) to any other point and back as part of the loop, so that \( \pi_1(X, x) \) is independent of \( x \) and is written as \( \pi_1(X) \). There is a connection between fundamental group and coverings, i.e. a fundamental group is a group of self-equivalences of coverings. The rigorous math foundation of the above statement is described in [10]. This connection is implicitly described in [3] (Section 16.3) where it is noted that there are two equivalent descriptions of Wilson lines. First description is based on representation of the fundamental group, second one uses the group of self-equivalences of coverings. The noncommutative geometry has no points, so it has not paths. It turns out that applications of fundamental group is impossible in noncommutative physics. However we can use group of self-equivalences of noncommutative coverings. The physical explanation of this fact is noted in [3]. The heuristic physical description of noncommutative coverings is already described in [1] where authors calculate Wilson lines on noncommutative tori. However [1] does not contain the rigorous math foundation. The rigorous foundation of noncommutative Wilson lines is described in [4].

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