Entropy Problem in the AdS$_3$/CFT correspondence

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Abstract

We resolve the entropy problem in the AdS$_3$/CFT correspondence by introducing both the normalizable and non-normalizable bulk modes. On the boundary, the normalizable Liouville states gives us $c = 1$ conformal field theory (CFT), whereas the non-normalizable Liouville states provide $c = \frac{3\ell^2}{27G}$ CFT. Such (boundary) non-normalizable modes come from non-normalizable bulk modes which serve as classical, non-fluctuating bulk background and encode the choice of local operator insertion on the boundary. Since the non-normalizable bulk modes can transfer information from bulk to boundary, it suggests that counting of non-normalizable states on the boundary at infinity leads to the entropy ($S = \frac{2\pi r_+}{4G}$) of (2+1)-dimensional gravity with $\Lambda = -1/\ell^2$. 
Recently the AdS/CFT correspondence has attracted much interest. This is based on the duality relation between the string theory (bulk theory) on AdS\(_{d+1}\) and a conformal field theory (CFT) as its d-dimensional boundary theory \([1–3]\). The calculation of greybody factor for the BTZ black hole on AdS\(_3\) is in agreement with the CFT calculation on the boundary \([4–6]\). Here one introduced a set of test fields to connect the bulk to the boundary. These couple to conformal operators on the boundary: the tachyon couples to (1/2,1/2) operator, while the dilaton couples to (2,2) operator. Actually we calculated the greybody factor of test fields by using their non-normalizable modes and AdS\(_3\)/CFT correspondence. The normalizable and non-normalizable modes emerge naturally from a direct solution of the wave equation (\(\nabla^2 \Phi - m^2 \Phi = 0\)) in the BTZ background \((r, t, \varphi)\) as

\[
\Phi = \Phi^{\text{nor}} + \Phi^{\text{non-nor}} = c_1 x^{-(1+\sqrt{1+m^2\ell^2})} + c_2 x^{(-1+\sqrt{1+m^2\ell^2})}
\]

when \(x = r/\ell\) is large. Here \(m^2\ell^2 = (h + \bar{h})(h + \bar{h} - 2)\) and \(c_1, c_2\) are constants. The first (second) terms with \(m^2\ell^2 > 0\) converges (diverges) at \(x = \infty\) and thus turn out to be the normalizable (non-normalizable) modes. This means that \(\Phi^{\text{nor}}\) satisfies the Dirichlet boundary condition at spatial infinity, but \(\Phi^{\text{non-nor}}\) does not. In the case of \(m^2\ell^2 = 8\) (dilaton), we find both the normalizable (\(x^4\)) and non-normalizable (\(x^2\)) modes. For \(m^2\ell^2 = 0\) (a free scalar) one has a constant at \(x = \infty\) and for \(m^2\ell^2 = -1\) (tachyon) one has only normalizable modes. \(\Phi^{\text{nor}}\) propagates in the bulk and corresponds to physical state, while \(\Phi^{\text{non-nor}}\) plays a role of classical, non-fluctuating background. We emphasize that the non-normalizable modes encode the local operator insertions on the boundary and thus correspond to introducing the particular boundary condition. Hence, on the basis of \(\Phi^{\text{non-nor}}\), one can calculate the greybody factor of relevant fields \([5,6]\). The result is consistent with that of dual CFT on the boundary \([8]\). This proves the \(AdS_3\)/CFT correspondence concerning the greybody factor (dynamical property). Here the non-normalizable modes plays a role of messenger to transfer information from bulk to boundary.

On the other hand, nowadays it seems to be a discrepancy for counting the entropy (static property) of the BTZ black hole in relation to the AdS/CFT correspondence. Here
two important questions arise: 1) what fields provide the relevant degrees of freedom? 2) where do these excitations live? An answer is that the freedom at infinity is relevant to counting of the entropy. We agree that the freedom at infinity is given by a single Liouville field. But there are two central charges ($c = 1$ \cite{12} and $c = \frac{3\ell}{2G}$ \cite{13,18}) for the Liouville model. The problem is which one describes the boundary states correctly. Hence we have to understand a discrepancy between $c = 1$ and $c = \frac{3\ell}{2G}$.

In this paper, we wish to resolve this urgent problem. Aside from test of three and four point functions \cite{19,21}, the correct counting of entropy in both bulk and boundary will prove the validity of AdS$_3$/CFT correspondence.

On the dual CFT side, one finds the asymptotic density of states at infinity (by Cardy)

$$\ln \rho(\Delta, \bar{\Delta}) \sim 2\pi \sqrt{\frac{c_R \Delta}{6}} + 2\pi \sqrt{\frac{c_L \bar{\Delta}}{6}}$$

with central charges $c_L = c_R = \frac{3\ell}{2G}$ for two copies of Virasoro algebra \cite{13,15}. Here $\Delta, \bar{\Delta}$ are the eigenvalues of two Virasoro generators $L_0, \bar{L}_0$. Considering the BTZ black hole with mass $M = (L_0 + \bar{L}_0)/\ell$ and angular momentum $J = L_0 - \bar{L}_0$, we obtain the correct Bekenstein-Hawking entropy

$$S_{B-H} = \frac{2\pi r_+}{4G}.$$  \hfill (3)

Here we introduce a set of test fields($\{\Phi_i\}$) with normalizable and non-normalizable modes to resolve a discrepancy. This corresponds to introducing a Liouville field with normalizable and non-normalizable modes on the boundary. First we wish to discuss the normalizable modes. These are quantized in the bulk and correspond to states in the boundary Hilbert space. In other words, the normalizable bulk modes can be realized as normalizable Liouville modes on the boundary, whose lowest Virasoro eigenvalue is given by $\Delta = (c_{\text{Liou}} - 1)/24$. How can we realize normalizable bulk modes as normalizable Liouville modes on the boundary? This can be understood from a mechanism of conformal anomaly inflow onto the boundary without non-normalizable modes \cite{22}. Thus, instead of $c_R$ in Eq. (2), one has to use the effective central charge ($c_{\text{eff}} = c_{\text{Liou}} - 24\Delta = 1$) \cite{14}. Recently
this mechanism can be further clarified by Martinec [12]. He showed that (2+1)-dimensional gravity with $\Lambda = -1/\ell^2$ is a pure gauge theory and thus it examines only its macroscopic properties (thermodynamics) by a set of Noether charges. Hence the boundary theory of (2+1)-dimensional gravity appears as a collective field excitation of the microscopic dual CFT. Its effective bulk/boundary theory turns out to be $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ Chern-Simons/$c = 1$ Liouville theory. Also this can be confirmed by analogy with quantum Hall effect, whose effective theory is $U(1)$ Chern-Simons/$c = 1$ chiral boson [23]. The boundary theory of (2+1) gravity with $\Lambda = -1/\ell^2 (c = 1$ normalizable Liouville modes) cannot be used for the correct counting of microscopic states of the BTZ black hole.

Now we are in a position to remark the non-normalizable modes. In order to construct the microscopic structure on the boundary, we need a set of test fields ($\{\Phi_i\}$) with non-normalizable modes. These can be used for the insertion of conformal operators on the boundary and thus make a microscopic structure on the boundary. In turn, these non-normalizable modes can be expressed by non-normalizable Liouville modes with $\Delta = 0$ [11]. This is so because the Liouville model can be considered as a $\sigma$-model description for a set of test fields (for example, tachyon($T$), dilaton($\phi$), · · · )

$$S_{\sigma} = \frac{k - 2}{4\pi} \int_{\partial M_{\infty}} \left( \partial_u \lambda \partial_v \lambda + T(\lambda) + QR^{(2)} \phi(\lambda) \right),$$

(4)

where $k = \ell/4G$, and $R^{(2)}$ is the Ricci curvature of boundary world sheet ($u = t - \varphi, v = t + \varphi$) at infinity [17]. The Liouville field ($\lambda = \ln x$) corresponds to the radial coordinate of the target space. As a result, the non-normalizable Liouville modes correspond to operator insertions and thus account for the microscopic structure of the boundary CFT. On the other hand, we remind the reader that the gauge theory of branes (dual CFT) is a tool to investigate its microscopic features. And thus we have to investigate operator insertions to count the boundary states rather than considering the Hilbert space of normalizable modes. Here instead of operator insertions (non-normalizable bulk modes), one can use Liouville modes to obtain the central charge. In this case one finds it by calculating the dilaton $\beta$-function as
\[ \tilde{c}_\text{Liou} = 1 + 6(k-2)Q^2 = \frac{3k}{k-2} - 2 + 6k, \]  

(5)

where \( Q \) is a background coupling charge(\( Q = (1 - k)/(k - 2) \)). It is worth noting that the tachyon contains only normalizable modes. From Eqs.(4) and (5), if the dilaton is absent, \( \tilde{c}_\text{Liou} \) leads to 1 \cite{[10]}. This is just the central charge for normalizable Liouville modes. It accounts for normalizable tachyonic bulk modes. If the dilaton is present, one can lead to \( c = \frac{3\ell}{2G} \) CFT. This is so because the dilaton has non-normalizable bulk modes and thus couples to (2,2) operator on the boundary. On the boundary the Liouville modes represent these non-normalizable modes.

The central charge \( \tilde{c}_\text{Liou} \) is determined by the coupling constants and can be chosen to be arbitrarily large. In the classical limit of \( k \to \infty \), the last term(6k) is relevant and this leads to \( c = \frac{3\ell}{2G} \).

In conclusion, \( c = 1 \) Liouville theory is the effective boundary theory for normalizable bulk modes, whereas \( c = \frac{3\ell}{2G} \) Liouville model is the effective boundary description for non-normalizable bulk modes. Since the non-normalizable modes connect the bulk information to boundary correctly, the relevant one to the AdS\(_3\)/CFT correspondence is just \( c = \frac{3\ell}{2G} \) Liouville model. Then one finds the same entropy as in Eq.(3) by the boundary theory. This confirms the AdS\(_3\)/CFT correspondence in relation to the entropy of the BTZ black hole. Finally it is noted that we introduce only two test fields(tachyon and dilaton). In order to understand this picture fully, one needs a complete set of test fields.

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