AN ARITHMETIC REGULARITY LEMMA, AN ASSOCIATED COUNTING LEMMA, AND APPLICATIONS

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Abstract. Szemerédi’s regularity lemma can be viewed as a rough structure theorem for arbitrary dense graphs, decomposing such graphs into a structured piece (a partition into cells with edge densities), a small error (corresponding to irregular cells), and a uniform piece (the pseudorandom deviations from the edge densities). We establish an arithmetic regularity lemma that similarly decomposes bounded functions $f : [N] \to \mathbb{C}$, into a (well-equidistributed, virtual) $s$-step nilsequence, an error which is small in $L^2$ and a further error which is minuscule in the Gowers $U^{s+1}$-norm, where $s \geq 1$ is a parameter. We then establish a complementary arithmetic counting lemma that counts arithmetic patterns in the nilsequence component of $f$.

We provide a number of applications of these lemmas: a proof of Szemerédi’s theorem on arithmetic progressions, a proof of a conjecture of Bergelson, Host and Kra, and a generalisation of certain results of Gowers and Wolf.

Our result is dependent on the inverse conjecture for the Gowers $U^{s+1}$ norm, recently established for general $s$ by the authors and T. Ziegler.

To Endre Szemerédi on the occasion of his 70th birthday.

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1. Introduction

Szemerédi’s celebrated regularity lemma \[46, 47\] is a fundamental tool in graph theory; see for instance \[34\] for a survey of some of its many applications. It is often described as a structure theorem for graphs \(G = (V, E)\), but one may also view it as a decomposition for arbitrary functions \(f : V \times V \to [0, 1]\). For instance, one can recast the regularity lemma in the following “analytic” form. Define a growth function to be any monotone increasing function \(F : \mathbb{R}^+ \to \mathbb{R}^+\) with \(F(M) \geq M\) for all \(M\).

Lemma 1.1 (Szemerédi regularity lemma, analytic form). Let \(V\) be a finite vertex set, let \(f : V \times V \to [0, 1]\) be a function, let \(\varepsilon > 0\), and let \(F : \mathbb{R}^+ \to \mathbb{R}^+\) be a growth function. Then there exists a positive integer\(^1\) \(M = O_{\varepsilon, F}(1)\) and a decomposition

\[ f = f_{\text{str}} + f_{\text{sml}} + f_{\text{unf}} \tag{1.1} \]

of \(f\) into functions \(f_{\text{str}}, f_{\text{sml}}, f_{\text{unf}} : V \times V \to [-1, 1]\) such that:

- \((f_{\text{str}}\text{ structured})\) \(V\) can be partitioned into \(M\) cells \(V_1, \ldots, V_M\), such that \(f_{\text{str}}\) is constant on \(V_i \times V_j\) for all \(i, j\) with \(1 \leq i, j \leq M\);

- \((f_{\text{sml}}\text{ small})\) The quantity \(\|f_{\text{sml}}\|_{L^2(V \times V)} := \left(\mathbb{E}_{v, w \in V} |f_{\text{sml}}(v, w)|^2\right)^{1/2}\) is at most \(\varepsilon\).

- \((f_{\text{unf}}\text{ very uniform})\) The box norm \(\|f_{\text{unf}}\|_{\square^2(V \times V)}\), defined to be the quantity

\[ \left(\mathbb{E}_{v_1, v_2, w_1, w_2 \in V} f_{\text{unf}}(v_1, w_1)f_{\text{unf}}(v_1, w_2)f_{\text{unf}}(v_2, w_1)f_{\text{unf}}(v_2, w_2)\right)^{1/4}, \]

is at most \(1/F(M)\).

- \((\text{Nonnegativity})\) \(f_{\text{str}}\) and \(f_{\text{str}} + f_{\text{sml}}\) take values in \([0, 1]\).

Informally, this regularity lemma decomposes any bounded function into a structured part, a small error, and an extremely uniform error. While this formulation does not, at first sight, look much like the usual regularity lemma, it easily implies that result: see \[51\]. The idea of formulating the regularity lemma with an arbitrary growth function \(F\) first appears in \[1\], and is also very useful for generalisations of the regularity lemma to hypergraphs. See, for example, \[50\]. The bound on \(M\) turns out to essentially be an iterated version of the growth function \(F\), with the number of iterations being polynomial in \(1/\varepsilon\). In applications, one usually selects the growth function to be exponential in nature, which then makes \(M\) essentially tower-exponential in \(1/\varepsilon\). See \[49, 52\] for a general discussion of these sorts of structure theorems and their applications in combinatorics. See also \[40\] for a related analytical perspective on the regularity lemma.

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\(^1\)As usual, we use \(O(X)\) to denote a quantity bounded in magnitude by \(CX\) for some absolute constant \(X\); if we need \(C\) to depend on various parameters, we will indicate this by subscripts. Thus for instance \(O_{\varepsilon, F}(1)\) is a quantity bounded in magnitude by some expression \(C_{\varepsilon, F}\) depending on \(\varepsilon, F\).

\(^2\)We use here the expectation notation \(\mathbb{E}_{a \in A} f(a) := \frac{1}{|A|} \sum_{a \in A} f(a)\) for any finite nonempty set \(A\), where \(|A|\) denotes the cardinality of \(A\).
In applications the regularity lemma is often paired with a counting lemma that allows one to control various expressions involving the function $f$. For example, one might consider the expression

$$E_{u,v,w} \in V f(u,v)f(v,w)f(w,u),$$

which counts triangles in $V$ weighted by $f$. Applying the decomposition (1.1) splits expressions such as (1.2) into multiple terms (in this instance, 27 of them). The key fact, which is a slightly non-trivial application of the Cauchy-Schwarz inequality, is that the terms involving the box-norm-uniform error $f_{\text{unf}}$ are negligible if the growth function $F$ is chosen rapidly enough. The terms involving the small error $f_{\text{sml}}$ are somewhat small, but one often has to carefully compare those errors against the main term (which only involves $f_{\text{str}}$) in order to get a non-trivial bound on the final expression (1.2). In particular, one often needs to exploit the positivity of $f_{\text{str}}$ and $f_{\text{str}} + f_{\text{sml}}$ to first localise expressions such as (1.2) to a small region (such as the portion of a graph between a “good” triple $V_i, V_j, V_k$ of cells in the partition of $V$ associated to $f_{\text{str}}$) before one can obtain a useful estimate.

The graph regularity and counting lemmas can be viewed as the first non-trivial member of a hierarchy of hypergraph regularity and counting lemmas, see e.g. [8, 17, 18, 41, 42, 50]. The formulation in [50] is particularly close to the formulation given in Theorem 1.1. These lemmas are suitable for controlling higher order expressions such as

$$E_{u,v,w,x} \in V f(u,v,w)f(v,w,x)f(w,x,u)f(x,u,v).$$

Our objective in this paper is to introduce an analogous hierarchy of such regularity and counting lemmas (one for each integer $s \geq 1$), in arithmetic situations. Here, the aim is to decompose a function $f: [N] \rightarrow [0, 1]$ defined on an arithmetic progression $[N] := \{1, \ldots, N\}$ instead of a graph. One is interested in counting averages such as

$$E_{n,r} \in [N] f(n)f(n+r)f(n+2r),$$

which counts 3-term arithmetic progressions weighted by $f$, as well as higher order expressions such as

$$E_{n,r} \in [N] f(n)f(n+r)f(n+2r)f(n+3r).$$

As it turns out, the former average will be best controlled using the $s = 1$ regularity and counting lemmas, while the latter requires the $s = 2$ versions of these lemmas. In this paper we shall see several examples of these types of applications of the two lemmas.

The arithmetic regularity lemma. We begin with by formulating our regularity lemma. Following the statement we explain the terms used here.

**Theorem 1.2** (Arithmetic regularity lemma). Let $f: [N] \rightarrow [0, 1]$ be a function, let $s \geq 1$ be an integer, let $\varepsilon > 0$, and let $F: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a growth
function. Then there exists a quantity \( M = O_{s, \varepsilon, F}(1) \) and a decomposition
\[
f = f_{\text{nil}} + f_{\text{sml}} + f_{\text{unf}}
\]
of \( f \) into functions \( f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}} : [N] \to [-1, 1] \) of the following form:
- \( f_{\text{nil}} \) (structured) has an \( \mathcal{F}(M, N) \)-irrational virtual nilsequence of degree \( \leq s \), complexity \( \leq M \), and scale \( N \);
- \( f_{\text{sml}} \) (small) has an \( L^2[N] \) norm of at most \( \varepsilon \);
- \( f_{\text{unf}} \) (very uniform) has a \( U^{s+1}[N] \) norm of at most \( 1/\mathcal{F}(M) \);
- (Nonnegativity) \( f_{\text{nil}} \) and \( f_{\text{nil}} + f_{\text{sml}} \) take values in \([0, 1]\).

Remark. This result easily implies the recently proven inverse conjecture for the Gowers norms (Theorem [2,1]). Conversely, this inverse conjecture, together with the equidistribution theory of nilsequences, will be the main ingredient used to prove Theorem [1,2].

We prove this theorem in [2]. We turn now to a discussion of the various concepts used in the above statement. Readers who are interested in applications may skip ahead to the end of the section.

The \( L^2[N] \) norm, used to control \( f_{\text{sml}} \), is simply
\[
\| f \|_{L^2[N]} := (\mathbb{E}_{n \in [N]} |f(n)|^2)^{1/2}.
\]

We turn next to the Gowers uniformity norm \( U^{s+1}[N] \), used to control \( f_{\text{unf}} \). If \( f : G \to \mathbb{C} \) is a function on a finite additive group \( G \), and \( k \geq 1 \) is an integer, then the Gowers uniformity norm \( \| f \|_{U^k(G)} \) is defined by the formula
\[
\| f \|_{U^k(G)} := \left( \mathbb{E}_{x, h_1, \ldots, h_k \in G} \Delta_{h_1} \cdots \Delta_{h_k} f(x) \right)^{1/2^k},
\]
where \( \Delta_h f : G \to \mathbb{C} \) is the multiplicative derivative of \( f \) in the direction \( h \), defined by the formula
\[
\Delta_h f(x) := f(x + h) \overline{f(x)}.
\]

In this paper we will be concerned with functions on \([N]\), which is not quite a group. To define the Gowers norms of a function \( f : [N] \to \mathbb{C} \), set \( G := \mathbb{Z}/\tilde{N} \mathbb{Z} \) for some integer \( \tilde{N} \geq 2^k N \), define a function \( \tilde{f} : G \to \mathbb{C} \) by \( \tilde{f}(x) = f(x) \) for \( x = 1, \ldots, N \) and \( \tilde{f}(x) = 0 \) otherwise, and set \( \| f \|_{U^k[N]} := \| \tilde{f} \|_{U^k(G)} / \| 1_N \|_{U^k(G)} \), where \( 1_N \) is the indicator function of \([N]\). It is easy to see that this definition is independent of the choice of \( \tilde{N} \), and so for definiteness one could take \( \tilde{N} := 2^k N \). Henceforth we shall write simply \( \| f \|_{U^k} \), rather than \( \| f \|_{U^k[N]} \), since all Gowers norms will be on \([N]\). One can show that \( \| : \|_{U^k} \) is indeed a norm for any \( k \geq 2 \), though we shall not need this here; see [16]. For further discussion of the Gowers norms and their relevance to counting additive patterns see [16, 25, §5] or [53, §11].

Finally, we turn to the notion of an irrational virtual nilsequence, which is the concept that defines the structural component \( f_{\text{nil}} \). This is the most complicated concept, and requires a certain number of preliminary definitions. We first need the notion of a filtered nilmanifold. The first two sections of [28] may be consulted for a more detailed discussion.
**Definition 1.3** (Filtered nilmanifold). Let $s \geq 1$ be an integer. A *filtered nilmanifold* $G/\Gamma = (G/\Gamma, G_\bullet)$ of degree $\leq s$ consists of the following data:

- A connected, simply-connected nilpotent Lie group $G$;
- A discrete, cocompact subgroup $\Gamma$ of $G$ (thus the quotient space $G/\Gamma$ is a compact manifold, known as a *nilmanifold*);
- A filtration $G_\bullet = (G_i)_{i=0}^\infty$ of closed connected subgroups $G = G(0) \geq G(1) \geq \ldots$ of $G$, which are *rational* in the sense that the subgroups $\Gamma_i := \Gamma \cap G_i$ are cocompact in $G_i$, such that $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j \geq 0$, and such that $G_i = \{\text{id}\}$ whenever $i > s$;
- A Mal’cev basis $\mathcal{X} = (X_1, \ldots, X_{\dim(G)})$ adapted to $G_\bullet$.

Once a Mal’cev basis has been specified, notions such as the rationality of subgroups may be quantified in terms of it. Furthermore one may use a Mal’cev basis to define a metric $d_{G/\Gamma}$ on the nilmanifold $G/\Gamma$. The results of this paper are rather insensitive to the precise metric that one takes, but one may proceed for example as in [28, Definition 2.2]. We encourage the reader not to think too carefully about the precise definition (or about Mal’cev bases in general), but it is certainly important to have some definite metric in mind so that one can make sense of notions such as that of a Lipschitz function on $G/\Gamma$.

Observe that every filtered nilmanifold $G/\Gamma$ comes with a canonical *probability Haar measure* $\mu_{G/\Gamma}$, defined as the unique Borel probability measure on $G/\Gamma$ that is invariant under the left action of $G$. We abbreviate $\int_{G/\Gamma} F(x) \, d\mu_{G/\Gamma}(x)$ as $\int_{G/\Gamma} F$.

We will need a quantitative notion of *complexity* for filtered nilmanifolds, though once again, the precise definition is somewhat unimportant.

**Definition 1.4** (Complexity). Let $M \geq 1$. We say that a filtered nilmanifold $G/\Gamma = (G/\Gamma, G_\bullet)$ has complexity $\leq M$ if the dimension of $G$, the degree of $G_\bullet$, and the rationality of the Mal’cev basis $\mathcal{X}$ (cf. [28, Definition 2.4]) are bounded by $M$.

The model example of a degree $\leq 2$ filtered nilmanifold is the *Heisenberg nilmanifold*

$$G/\Gamma := \left( \begin{array}{ccc} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{array} \right) / \left( \begin{array}{ccc} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{array} \right)$$

with the lower central series $G(0) = G(1) = G$ and $G(2) = [G, G] = \left( \begin{array}{ccc} 1 & \mathbb{R} & 0 \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{array} \right)$.
with Mal’cev basis $\mathcal{X} = \{X_1, X_2, X_3\}$ consisting of the matrices

$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

With the definition of filtered nilmanifold in place, the next thing we need is the idea of a polynomial sequence. The basic theory of such sequences was laid out in Leibman [35], and was extended slightly to general filtrations in [28]. An extensive discussion may be found in Section 6 of that paper.

**Definition 1.5 (Polynomial sequence).** Let $(G/\Gamma, G_\bullet)$ be a filtered nilmanifold, with filtration $G_\bullet = (G_{(i)})_{i=0}^{\infty}$. A (multidimensional) polynomial sequence adapted to this filtered nilmanifold is a sequence $g: \mathbb{Z}^D \to G$ for some $D \geq 1$ with the property that

$$\partial h_1 \ldots \partial h_i g(n) \in G_{(i)}$$

for all $i \geq 0$ and $h_1, \ldots, h_i, n \in \mathbb{Z}^D$, where $\partial_h g(n) := g(n + h)g(n)^{-1}$ is the derivative of $g$ with respect to the shift $h$. The space of all such polynomial sequences will be denoted $\text{poly}(\mathbb{Z}^D, G_\bullet)$. The space of polynomial sequences taking values in $\Gamma$ will be denoted $\text{poly}(\mathbb{Z}^D, \Gamma_\bullet)$. When $D = 1$, we refer to multidimensional polynomial sequences simply as polynomial sequences.

**Remark.** We will be primarily interested in the one-dimensional case $D = 1$, but will need the higher $D$ case in order to establish the counting lemma, Theorem 1.11.

One of the main reasons why we work with polynomial sequences, instead of just linear sequences such as $n \mapsto g_0 g_1^n$, is that the former forms a group.

**Theorem 1.6 (Lazard-Leibman).** If $(G/\Gamma, G_\bullet)$ is a filtered nilmanifold and $D \geq 1$ is an integer, then $\text{poly}(\mathbb{Z}^D, G_\bullet)$ is a group (and $\text{poly}(\mathbb{Z}^D, \Gamma_\bullet)$) is a subgroup.

**Proof.** See [36] or [28, Proposition 6.2].

With the concept of a polynomial sequence in hand, it is easy to define a polynomial orbit.

**Definition 1.7 (Orbits).** Let $D, s \geq 1$ be integers, and $M, A > 0$ be parameters. A (multidimensional) polynomial orbit of degree $\leq s$ and complexity $\leq M$ is any function $n \mapsto g(n)\Gamma$ from $\mathbb{Z}^D \to G/\Gamma$, where $(G/\Gamma, G_\bullet)$ is a filtered nilmanifold of complexity $\leq M$, and $g \in \text{poly}(\mathbb{Z}^D, G_\bullet)$ is a (multidimensional) polynomial sequence.

Using the concept of polynomial orbit, we can define the notion of a (polynomial) nilsequence, as well as a generalisation which we call a virtual nilsequence, in analogy with virtually nilpotent groups (groups with a finite index nilpotent subgroup).

With the following additions:

4Strictly speaking, the orbit is the tuple of data $(G, \Gamma, G/\Gamma, G_\bullet, n \mapsto g(n)\Gamma)$, rather than just the sequence $n \mapsto g(n)\Gamma$, but we shall abuse notation and use the sequence as a metonym for the whole orbit.
**Definition 1.8** (Nilsequences). A (multidimensional, polynomial) nilsequence of degree \( \leq s \) and complexity \( \leq M \) is any function \( f : \mathbb{Z}^D \to \mathbb{C} \) of the form \( f(n) = F(g(n)\Gamma) \), where \( n \mapsto g(n)\Gamma \) is a polynomial orbit of degree \( \leq s \) and complexity \( \leq M \), and \( F : G/\Gamma \to \mathbb{C} \) is a function of Lipschitz norm \( \leq M \).

**Definition 1.9** (Virtual nilsequences). Let \( N \geq 1 \). A virtual nilsequence of degree \( \leq s \) and complexity \( \leq M \) at scale \( N \) is any function \( f : [N] \to \mathbb{C} \) of the form \( f(n) = F(g(n)\Gamma, n(\mod q), n/N) \), where \( 1 \leq q \leq M \) is an integer, \( n \mapsto g(n)\Gamma \) is a polynomial orbit of degree \( \leq s \) and complexity \( \leq M \), and \( F : G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R} \to \mathbb{C} \) is a function of Lipschitz norm at most \( M \). (Here we place a metric on \( G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R} \) in some arbitrary fashion, e.g. by embedding \( \mathbb{Z}/q\mathbb{Z} \) in \( \mathbb{R}/\mathbb{Z} \) and taking the direct sum of the metrics on the three factors.)

One concept that featured in Theorem 1.2 remains to be defined: that of an irrational orbit. The definition is a little technical and takes some setting up, and so we defer it and the discussion of some motivating examples to Appendix A. Very roughly speaking, an irrational orbit is one that is equidistributed and for which the filtration \( G^* \) is as small as possible.

This concludes our attempt to discuss all the concepts involved in the arithmetic regularity lemma, Theorem 1.2; we turn now to a statement and discussion of the counting lemma.

**Counting lemma.** In applications of the arithmetic regularity lemma, we will be interested in counting additive patterns such as arithmetic progressions or parallelepipeds. To understand the phenomena properly it is advantageous to work in a somewhat general setting similar to that taken in \([20, 21, 22, 29]\). In the latter paper one works with a family \( \Psi = (\psi_1, \ldots, \psi_t) \) of integer-coefficient linear forms (or equivalently, group homomorphisms) \( \psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z} \), and consider expressions such as

\[
\mathbb{E}_{n \in \mathbb{Z}^D \cap P} f(\psi_1(n)) \cdots f(\psi_t(n))
\]

where \( P \) is a convex subset of \( \mathbb{R}^D \). Thus, for instance, if counting arithmetic progressions, one might use the linear forms

\[
\psi_i(n_1, n_2) := n_1 + (i - 1)n_2; \quad i = 1, \ldots, k
\]

whilst for counting parallelepipeds one might instead use the linear forms

\[
\psi_{\omega_1, \ldots, \omega_k}(n_0, n_1, \ldots, n_k) := n_0 + \omega_1 n_1 + \ldots + \omega_k n_k; \quad \omega_1, \ldots, \omega_k \in \{0, 1\},
\]

The (inhomogeneous) Lipschitz norm \( \|F\|_{\text{lip}} \) of a function \( F : X \to \mathbb{C} \) on a metric space \( X = (X, d) \) is defined as

\[
\|F\|_{\text{lip}} := \sup_{x \in X} |F(x)| + \sup_{x, y \in X, x \neq y} \frac{|F(x) - F(y)|}{d(x, y)}.
\]
In order to understand the contribution to (1.3) coming from the structured part $f_{\text{nil}}$ of $f$, one is soon faced with the question of understanding the equidistribution of the orbit

$$(g(\psi_1(n)), \ldots, g(\psi_t(n)) \Gamma)$$

inside $(G/\Gamma)^t$, where $n = (n_1, \ldots, n_D)$ ranges over $\mathbb{Z}^D \cap P$. We abbreviate this orbit as $g^\Psi(n) \Gamma$, where $g^\Psi : \mathbb{Z}^D \to G^t$ is the polynomial sequence

$$g^\Psi(n) := (g(\psi_1(n)), \ldots, g(\psi_t(n))).$$

A very useful model for this question, in which infinite orbits were considered in the “linear” case $g(n) = g^n x$, was studied by Leibman [39]. His work leads one to the following definition.

**Definition 1.10 (The Leibman group).** Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$. For any $i \geq 1$, define $\Psi[i]$ to be the linear subspace of $\mathbb{R}^k$ spanned by the vectors $(\psi_{1j}(n), \ldots, \psi_{ij}(n))$ for $1 \leq j \leq i$ and $n \in \mathbb{Z}^D$. Given a filtered nilmanifold $(G/\Gamma, G_\bullet)$, we define the Leibman group $G^\Psi \triangleleft G^t$ to be the Lie subgroup of $G^t$ generated by the elements $g_i^{\vec{v}_i}$ for $i \geq 1$, $g_i \in G(i)$, and $\vec{v}_i \in \Psi[i]$, with the convention that

$$g^{(v_1, \ldots, v_t)} := (g^{v_1}, \ldots, g^{v_t})$$

for each $g \in G$. Note that $G^\Psi$ is normal in $G^t$ because $G(i)$ is normal in $G$. We will show in §3 that $G^\Psi$ is also a rational subgroup of $G^t$, thus $\Gamma^\Psi := \Gamma^t \cap G^\Psi$ is a discrete cocompact subgroup of $G^\Psi$.

**Examples.** Two particular instances of this construction correspond to the two lattices (1.4) and (1.5) above. In the case of arithmetic progressions, where $\Psi$ is as in (1.4), the Leibman group $G^\Psi$ is sometimes referred to as the *Hall-Petresco group* $\text{HP}^k(G_\bullet)$ and has the particularly simple alternative description

$$\text{HP}^k(G_\bullet) = G^\Psi = \{(g(0), \ldots, g(k - 1)) : g \in \text{poly}(G_\bullet)\},$$

We will prove this fact in §3. In the case of parallelepipeds, where $\Psi$ as in (1.5), the Leibman group $G^\Psi$ has been referred to as the *Host-Kra cube group* [29] and it too has an alternative description. See [29, Appendix E] for more information: we will not be making use of this particular group here.

Let $g \in \text{poly}(\mathbb{Z}, G_\bullet)$ be a polynomial sequence, and let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^d \to \mathbb{Z}$. It turns out (see Lemma 3.2) that the sequence $g^\Psi$ takes values in $G^\Psi$. More remarkably, the orbit (1.6) is in fact *totally equidistributed* on $G^\Psi/\Gamma^\Psi$ if $g$ is sufficiently irrational. It is this result that we refer to as our counting lemma.

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6We define $g^v$ for real $v$ by the formula $g^v := \exp(v \log(g))$, where $\exp : g \to G$ is the usual exponential map from the Lie algebra $g$ to $G$ (this is a homeomorphism since $G$ is nilpotent, connected, and simply connected).
Theorem 1.11 (Counting lemma). Let $M, D, t, s$ be integers with $1 \leq D, t, s \leq M$, let $(G/\Gamma, G_\bullet)$ be a degree $\leq s$ filtered nilmanifold of complexity $\leq M$, let $g : \mathbb{Z} \to G$ be an $(A, N)$-irrational polynomial sequence adapted to $G_\bullet$, let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$ with coefficients of magnitude at most $M$, and let $P$ be a convex subset of $[-N, N]^D$. Then for any Lipschitz function $F : (G/\Gamma)^t \to \mathbb{C}$ of Lipschitz norm at most $M$, one has

$$
\sum_{n \in \mathbb{Z}^D \cap P} F(g^\Psi(n)\Gamma^t) = \text{vol}(P) \int_{g(0)^\Delta G/\Gamma^\Psi} F + o_{A \to \infty; M}(N^D) + o_{N \to \infty; M}(N^D),
$$

where $g(0)^\Delta := (g(0), \ldots, g(0)) \in G^t$ and the integral is with respect to the probability Haar measure on the coset $g(0)^\Delta G/\Gamma^\Psi$, viewed as a sub-nilmanifold of $(G/\Gamma)^t$, and $\text{vol}(P)$ is the Lebesgue measure of $P$ in $\mathbb{R}^D$.

More generally, whenever $\Lambda \leq \mathbb{Z}^D$ is a sublattice of index $[\mathbb{Z}^D : \Lambda] \leq M$, and $n_0 \in \mathbb{Z}^D$, one has

$$
\sum_{n \in (n_0 + \Lambda) \cap P} F(g^\Psi(n)\Gamma^t) = \frac{\text{vol}(P)}{[\mathbb{Z}^D : \Lambda]} \int_{g(0)^\Delta G/\Gamma^\Psi} F + o_{A \to \infty; M}(N^D) + o_{N \to \infty; M}(N^D).
$$

The counting lemma is, of course, best understood by seeing it in action as we shall do several times later on. The errors $o_{A \to \infty; M}(N^D)$ and $o_{N \to \infty; M}(N^D)$ are negligible in most applications, as $A$ will typically be a huge function $F(M)$ of $M$, and $N$ can also be taken to be arbitrarily large compared to $M$.

We remark that one could easily extend the above lemma to control averages of virtual irrational nilsequences, rather than just irrational sequences, by introducing some additional integrations over the local factors $\mathbb{Z}/q\mathbb{Z}$ and $\mathbb{R}$, but this would require even more notation than is currently being used and so we do not describe such an extension here.

Applications. The proofs of the regularity and counting lemmas occupy about half the paper. In the remaining half, we give a number of applications of these results to problems in additive combinatorics. The scheme of the arguments in all of these cases is similar. First, one applies the arithmetic regularity lemma to decompose the relevant function $f$ into structured, small, and (very) uniform components $f = f_{\text{nil}} + f_{\text{sml}} + f_{\text{unif}}$. Very roughly speaking, these are analysed as follows:

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We use $o_{A \to \infty; M}(X)$ to denote a quantity bounded in magnitude by $c_M(A)X$, where $c_M(A) \to 0$ as $A \to \infty$ for fixed $M$. Similarly for other choices of subscripts.
$f_{nil}$ is studied using algebraic properties of nilsequences, particularly the
counting lemma;

$f_{sml}$ is shown to be negligible, though often (unfortunately) some addi-
tional algebraic input is required to ensure that this error does not conspire
to destroy the contribution from $f_{nil}$;

$f_{unf}$ is easily shown to be negligible using results of “generalised von
Neumann” type as discussed in §4.

As we shall see, dealing with the error $f_{sml}$ can cause a certain amount
of pain. To show that this error is truly negligible, one often has to prove
that patterns guaranteed by $f_{nil}$ (such as arithmetic progressions) do not
concentrate on some small set which might be contained in the support of
$f_{sml}$.

We now give specific examples of this paradigm. In §6 we give a “new”
proof of Szemerédi’s famous theorem on arithmetic progres-
sions. This is
hardly exciting nowadays, with at least 14 proofs already in the literature
[2, 3, 9, 12, 16, 17, 42, 43, 48, 50] as well as (slightly implicit-
ly) in [4, 33, 55]. However this proof makes the point that for a certain class of
problems it suffices to “check the result for nilsequences”, and in so doing
one really sees the structure of the problem. Just as random and structured
graphs are two obvious classes to test conjectures against in graph theory,
we would like to raise awareness of nilsequences as potential (and, in certain
cases such as this one, the only) sources of counterexamples.

The second application, proven in §5, is to establish a conjecture of Bergel-
son, Host and Kra [4]. Here and in the sequel we use the notatio-

$X \ll_{\alpha, \varepsilon} Y$ or $Y \gg_{\alpha, \varepsilon} X$ synonymously with $X = O_{\alpha, \varepsilon}(Y)$, and similarly for other choice
of subscripts.

**Theorem 1.12 (Bergelson-Host-Kra conjecture).** Let $k = 1, 2, 3$ or 4, and
suppose that $0 < \alpha < 1$ and $\varepsilon > 0$. Then for any $N \geq 1$ and any subset
$A \subseteq [N]$ of density $|A| \geq \alpha N$, one can find $\gg_{\alpha, \varepsilon} N$ values of $d \in [-N, N]$
such that there are at least $(\alpha^k - \varepsilon)N$ $k$-term arithmetic progressions in $A$
with common difference $d$.

**Remarks.** The claim is trivial for $k = 1$, and follows from an easy aver-
ing argument when $k = 2$. This theorem was established in the case
$k = 3$ by the first author in [23]; we give a new proof of this result which
may be of independent interest. The case $k = 4$ is new, although a finite
field analogue of this result previously appeared in lecture notes of the first
author [21] (reporting on joint work). A counterexample example of Ruzsa
in the appendix to [4] shows that Theorem 1.12 fails when $k \geq 5$.

Finally, in §7, we establish a generalisation of a recent result of Gowers
and Wolf [20, 21, 22] regarding the “true” complexity of a system of linear
forms.

**Theorem 1.13.** Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms from
$\mathbb{Z}^D \to \mathbb{Z}$, and let $s \geq 1$ be an integer such that the polynomials $\psi_1^{s+1}, \ldots, \psi_t^{s+1}$
are linearly independent. Then for any function $f : [N] \to \mathbb{C}$ bounded in
magnitude by 1 (and defined to be zero outside of $[N]$) obeying the bound $\|f\|_{U^{s+1}[N]} \leq \delta$ for some $\delta > 0$, one has

$$E_{n \in [N]} \prod_{i=1}^{t} f(\psi_i(n)) = o_{\delta \to 0; s,D,t,\Psi}(1).$$

Remarks. This result was conjectured in [20], where it was shown that the linear independence hypothesis was necessary. The programme in [20, 21, 22] gives an alternate approach to this result that avoids explicit mention of nilsequences, and in particular establishes the counterpart to Theorem 1.13 in finite characteristic; their work also gives a proof of this theorem in the case when the Cauchy-Schwarz complexity of the system (see Theorem 4.1) is at most two, and with better bounds that our result, which is all but ineffective. It is worth mentioning that the arguments in [20, 21, 22] also develop several structural decomposition theorems along the lines of Theorem 1.2, but using the language of locally polynomial phases rather than nilsequences.

Relation to previous work. A result closely related to Theorem 1.2 in the case $s = 1$ was proved by Bourgain as long ago as 1989 [6]. In that paper, the decomposition was applied to give a different proof of Roth’s theorem, that is to say Szemerédi’s theorem for 3-term progressions. A different take on this result was supplied by the first author in [23], where the application to the case $k = 3$ of the Bergelson-Host-Kra conjecture was noted. In that same paper a construction of Gowers [14] was modified to show that any application of the arithmetic regularity lemma must lead to awful (tower-type) bounds; the same kind of construction would show that the cases $s \geq 2$ of Theorem 1.2 lead to tower-type bounds as well. In [24] the analogue of the case $s = 2$ of Theorem 1.2 in a finite field setting was stated, proved, and used to deduce the finite field analogue of the Bergelson-Host-Kra conjecture in the case $k = 4$. In that same paper the present work was promised (as reference [22]) at “some future juncture”. Four years later we have reached that juncture and we apologise for the delay. We note, however, that until the very recent resolution of the inverse conjectures for the Gowers norms [31, 32] many of our results would have been conditional; furthermore, we are heavily dependent on our work [28], which had not been envisaged when the earlier promise was made.

In the meantime a greater general understanding of decomposition theorems of this type has developed through the work of Gowers [19], Reingold-Trevisan-Tulsiani-Vadhan [44], and Gowers-Wolf [20, 21, 22]; see also the survey [52] of the second author. While Theorem 1.2 is related to several of these general decomposition theorems, it also relies upon specific structure.

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8The relevant part of these lecture notes by the first author reported on joint work of the two of us.
of nilmanifolds. In any case it seems appropriate, in this volume, to give a proof using the “energy increment argument” pioneered by Szemerédi.

ACkNOWLEDGMENTS. BG was, while this work was being carried out, a fellow at the Radcliffe Institute at Harvard. He is very happy to thank the Institute for proving excellent working conditions. TT is supported by a grant from the MacArthur Foundation, by NSF grant DMS-0649473, and by the NSF Waterman award.

2. PROOF OF THE ARITHMETIC REGULARITY LEMMA

We now prove Theorem 1.2. The proof proceeds in two main stages.

Firstly, we establish a “non-irrational regularity lemma”, which establishes a weaker version of Theorem 1.2 in which the structured component $f_{nil}$ is a polynomial nilsequence, but one which is not assumed to be irrational. The main tool here is the inverse conjecture GI$(s)$ for the Gowers norms [32], combined with the energy incrementation argument that appears in proofs of the graph regularity lemma. In the second stage, we upgrade this weaker regularity lemma to the full regularity lemma by converting the nilsequence to a irrational nilsequence. The main tool here is a dimension reduction argument and a factorisation of nilsequences similar to that appearing in [28].

The non-irrational regularity lemma. We begin the first stage of the argument. As mentioned above, the key ingredient is the following result.

**Theorem 2.1 (GI$(s)$).** Let $s \geq 1$, and suppose that $f : [N] \to \mathbb{C}$ is a function bounded in magnitude by 1 such that $\|f\|_{U^{s+1}[N]} \geq \delta$ for some $\delta > 0$. Then there is a degree $\leq s$ polynomial nilsequence $\psi : \mathbb{Z} \to \mathbb{C}$ of complexity $O_{s,\delta}(1)$ such that $|\langle f, \psi \rangle_{L^2[N]}| \gg_{s,\delta} 1$, where

$$\langle f, \psi \rangle_{L^2[N]} := \mathbb{E}_{n \in [N]} f(n) \overline{\psi(n)}$$

is the usual inner product.

Remark. The difficulty of this conjecture increases with $s$. The case $s = 1$ easily follows from classical harmonic analysis. The case $s = 2$ was established by the authors in [26], building upon the breakthrough paper of Gowers [15]. The case $s = 3$ was recently established by the authors and Ziegler in [31], and the general case will appear in the forthcoming paper [32] by the authors and Ziegler.

For technical reasons, it is convenient to replace the notion of a degree $\leq s$ polynomial nilsequence by a slightly different concept. The following definition is not required beyond the end of the proof of Proposition 2.7.

**Definition 2.2 (s-measurability).** Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a growth function and $s \geq 1$. A subset $E \subseteq [N]$ is said to be $s$-measurable with growth function $\Phi$ if for every $M \geq 1$, there exists a degree $\leq s$ polynomial nilsequence...
\[ \psi : \mathbb{Z} \to [0,1] \text{ of complexity } \leq \Phi(M) \text{ such that} \]
\[ \|\psi - 1_E\|_{L^2[N]} \leq 1/M. \]

An example of a 1-measurable function would be a regular Bohr set, as introduced in [7] and discussed further in [26, §2]. We will not need Bohr sets elsewhere in this paper, so we shall not dwell any longer on this example. However the reader will see ideas related to the basic theory of those sets in the proof of Corollary 2.3 below.

We make the simple but crucial observation that if \( E, F \) are \( s \)-measurable with some growth functions \( \Phi, \Phi' \) respectively, then boolean combinations of \( E, F \) such as \( E \cap F, E \cup F, \) or \( [N] \setminus E \) are also \( s \)-measurable with some growth function depending on \( \Phi, \Phi' \). Underlying this, of course, is that fact that the product and sum of two nilsequences is also a nilsequence, and hence the set of nilsequences form a kind of algebra (graded by complexity). The role of algebraic structure of this kind was brought to the fore in the work of Gowers [19] cited above.

Theorem 2.1 then implies

**Corollary 2.3 (Alternate formulation of GI(s)).** Let \( s \geq 1 \), and suppose that \( f : [N] \to [-1,1] \) is such that \( \|f\|_{U^{s+1}[N]} \geq \delta \) for some \( \delta > 0 \). Then there exists a growth function \( \Phi_{s,\delta} \) depending only on \( s, \delta \), and an \( s \)-measurable set \( E \subset N \) with growth function \( \Phi_{s,\delta} \), such that
\[ |E_{\{n \in [N] : \psi(n) \geq t\}}| \gg s,\delta. \]

**Proof.** We allow implied constants to depend on \( s, \delta \). By Theorem 2.1, there exists a degree \( \leq s \) polynomial nilsequence \( \psi \) of complexity \( O(1) \) such that
\[ |\mathbb{E}_{n \in [N]} f(n) 1_E(n)| \gg_{s,\delta} 1. \]

By taking real and imaginary parts of \( \psi \), and then positive and negative parts, and rescaling, we may assume without loss of generality that \( \psi \) takes values in \([0,1]\). By Fubini’s theorem, we then have
\[ |\int_0^1 \mathbb{E}_{n \in [N]} f(n) 1_{E_t}(n) \, dt| \gg 1 \]

where \( E_t := \{ n \in [N] : \psi(n) \geq t \} \). We thus see that there is a subset \( \Omega \subset [0,1] \) of Lebesgue measure \( |\Omega| \gg 1 \) such that
\[ |\mathbb{E}_{n \in [N]} f(n) 1_{E_t}(n)| \gg 1 \]

uniformly for all \( t \in \Omega \).

It remains to show that at least one of the \( E_t \) is \( s \)-measurable with respect to a suitable growth function. For any \( t \in \mathbb{R} \), we consider the maximal function
\[ M(t) := \sup_{r > 0} \frac{1}{2r} \frac{1}{N} |\{ n \in [N] : |\psi(n) - t| \leq r \}|. \]

\footnote{Here we are, in some sense, finding a “regular” nil-Bohr set \( \{ n \in [N] : \psi(n) \geq t \}, \)
that is to say one rather insensitive to small changes in the value of \( t \). A similar idea also appears in [44 Claim 2.2].}
From the Hardy-Littlewood maximal inequality or the Besicovitch covering lemma we have that the set \( \{ t \in \mathbb{R} : M(t) \geq \lambda \} \) has Lebesgue measure \( O(1/\lambda) \) for any \( \lambda > 0 \). Thus, we can find \( t \in \Omega \) such that \( M(t) = O(1) \). Fixing such a \( t \), we then see that

\[
|\{ n \in [N] : |\psi(n) - t| \leq r \}| \ll rN
\]

for all \( r > 0 \). As a consequence, for any \( r > 0 \), one can then approximate \( 1_{E_t} \) to within \( O(\sqrt{r}) \) in \( L^2[N] \) norm by a Lipschitz function of \( \psi \) with Lipschitz norm \( O(1/r) \). This implies that \( 1_{E_t} \) is \( s \)-measurable with some growth function \( \Phi \) depending only on \( s, \delta \), and the claim follows.

We rephrase this fact in terms of conditional expectations. The following definition, like Definition 2.2, will only be needed until the end of the proof of Proposition 2.7.

**Definition 2.4 (s-factors).** An \( s \)-factor \( B \) of complexity \( \leq M \) and growth function \( \Phi \) is a partition of \([N]\) into at most \( M \) sets (or cells) \( E_1, \ldots, E_m \) which are \( s \)-measurable of growth function \( \Phi \). Given an \( s \)-factor \( B \) and a function \( f : [N] \to \mathbb{C} \), we define the conditional expectation \( \mathbb{E}(f|B) : [N] \to \mathbb{C} \) of \( f \) with respect to the \( s \)-factor to be the function which equals \( \mathbb{E}_{n \in E_i} f(n) \) on each cell of the partition. We define the index or energy \( \mathcal{E}(B) \) of the \( s \)-factor \( B \) relative to \( f \) to be the quantity \( \| \mathbb{E}(f|B) \|_{L^2[N]}^2 \).

An \( s \)-factor \( B' \) is said to refine another \( B \) if every cell of \( B' \) is contained in a cell of \( B \).

**Corollary 2.5 (Lack of uniformity implies energy increment).** Let \( s \geq 1 \), let \( B \) be an \( s \)-factor of complexity \( \leq M \) and some growth function \( \Phi \), and suppose that \( f : [N] \to [0,1] \) is such that \( \| f - \mathbb{E}(f|B) \|_{U^{s+1}[N]} \geq \delta \) for some \( \delta > 0 \). Then there exists a refinement \( B' \) of \( B \) of complexity \( \leq 2M \) and some growth function depending on \( s, \delta, M, \Phi \), such that

\[
\mathcal{E}(B') - \mathcal{E}(B) \gg_{s, \delta} 1.
\]

**Proof.** By Corollary 2.3, we can find an \( s \)-measurable set \( E \) with a growth function depending on \( s, \delta \) such that

\[
|\langle f - \mathbb{E}(f|B), 1_E \rangle_{L^2[N]}| \gg_{s, \delta} 1 \tag{2.1}
\]

Now let \( B' \) be the partition generated by \( B \) and \( E \); then \( B' \) clearly has complexity \( \leq 2M \) and a growth function depending on \( s, \delta, M, \Phi \). Since \( 1_E \) is measurable with respect to the partition \( B' \) (that is to say it is constant on each cell of this partition), we can rewrite the left-hand side of (2.1) as

\[
|\langle \mathbb{E}(f|B'), \mathbb{E}(f|B), 1_E \rangle_{L^2[N]}|
\]

and hence by the Cauchy-Schwarz inequality

\[
\| \mathbb{E}(f|B') - \mathbb{E}(f|B) \|_{L^2[N]} \gg_{s, \delta} 1.
\]

The claim then follows from Pythagoras’ theorem. \( \square \)
We can iterate this to obtain a weak regularity lemma, analogous to the weak graph regularity lemma of Frieze and Kannan [13].

**Corollary 2.6.** Let \( s \geq 1 \), let \( B \) be an \( s \)-factor of complexity \( \leq M \) and some growth function \( \Phi \), let \( f : [N] \to [0,1] \), and let \( \varepsilon > 0 \). Then there exists a refinement \( B' \) of \( B \) of complexity \( O_{s,M,\varepsilon}(1) \) and some growth function depending on \( s, \varepsilon, M, \Phi \), such that

\[
\| f - \mathbb{E}(f|B') \|_{U^{s+1}[N]} \leq \varepsilon. \tag{2.2}
\]

**Proof.** We define a sequence of successively more refined factors \( B' \), starting with \( B' := B \). If (2.2) already holds then we are done, so suppose that this is not the case. Then by Corollary 2.5, we can find a refinement \( B'' \) of complexity \( O_{s,M,\varepsilon}(1) \) and some growth function depending on \( s, \varepsilon, M, \Phi \) whose energy is larger than that of \( B' \) by a factor \( \gg_{s,\varepsilon} 1 \). On the other hand, the energy clearly ranges between 0 and 1. Thus after replacing \( B' \) with \( B'' \) and iterating this algorithm at most \( O_{s,\varepsilon}(1) \) times we obtain the claim. \( \square \)

One final iteration then gives the full non-irrational regularity lemma.

**Proposition 2.7.** Let \( f : [N] \to [0,1] \), let \( s \geq 1 \), let \( \varepsilon > 0 \), and let \( F : \mathbb{R}^+ \to \mathbb{R}^+ \) be a growth function. Then there exists a quantity \( M = O_{s,\varepsilon,F}(1) \) and a decomposition

\[ f = f_{nil} + f_{sml} + f_{unf} \]

of \( f \) into functions \( f_{nil}, f_{sml}, f_{unf} : [N] \to [-1,1] \) such that:

1. (\( f_{nil} \) structured) \( f_{nil} \) equals a degree \( \leq s \) polynomial nilsequence of complexity \( \leq M \).
2. (\( f_{sml} \) small) \( \| f_{sml} \|_{L^2[N]} \leq \varepsilon \).
3. (\( f_{unf} \) very uniform) \( \| f_{nil} \|_{U^{s+1}[N]} \leq 1/F(M) \).
4. (Nonnegativity) \( f_{nil} \) and \( f_{nil} + f_{sml} \) take values in \([0,1]\).

**Proof.** We need a growth function \( \tilde{F} : \mathbb{R}^+ \to \mathbb{R}^+ \), somewhat more rapidly growing than \( F \) in manner that depends on \( F, s, \varepsilon \). We will specify the exact requirements we have of it later. We then define a sequence \( 1 = M_0 \leq M_1 \leq \ldots \) by setting \( M_0 := 1 \) and \( M_{i+1} := \tilde{F}(M_i) \).

Applying Corollary 2.6 repeatedly, we may find for each \( i \geq 0 \) an \( s \)-factor \( B_i \) of complexity \( O_{s,M_i}(1) \) and a growth function depending on \( s, M_i \), such that each \( B_i \) refines \( B_{i-1} \), and such that

\[ \| f - \mathbb{E}(f|B_i) \|_{U^{s+1}[N]} \leq 1/M_i \]

for all \( i \geq 0 \).

By Pythagoras’ theorem, the energies \( \mathcal{E}(B_i) \) are non-decreasing, and also range between 0 and 1. Thus by the pigeonhole principle, one can find \( i = O_{\varepsilon}(1) \) such that

\[ \mathcal{E}(B_{i+1}) - \mathcal{E}(B_i) \leq \varepsilon^2/4, \]

which by Pythagoras’ theorem again is equivalent to

\[ \| \mathbb{E}(f|B_{i+1}) - \mathbb{E}(f|B_i) \|_{L^2[N]} \leq \varepsilon/2. \]
Meanwhile, as $B_i$ is an $s$-factor and $f$ is bounded, we can find a degree $\leq s$ polynomial nilsequence $f_{\text{nil}} : [N] \to \mathbb{R}$ of complexity $O_{s,M_0}(1)$ such that

$$\|E(f|B_i) - f_{\text{nil}}\|_{L^2[N]} \leq \varepsilon / 2.$$

Since $E(f|B_i)$ ranges in $[0, 1]$, we may retract $f_{\text{nil}}$ to $[0, 1]$ also (note that this does not increase the complexity of $f_{\text{nil}}$). If we then set $f_{\text{unf}} := f - E(f|B_{i+1})$ and $f_{\text{sm}} := E(f|B_{i+1}) - f_{\text{nil}}$, we obtain the claim. □

**Remark.** The application of the Hardy-Littlewood maximal inequality in the proof of Corollary 2.3 makes for a reasonably tidy argument. A more direct approach would be to carve up $[N]$ into approximate level sets of nilsequences, and then to approximate the projections onto the factors thus defined by nilsequences using the Weierstrass approximation theorem. There are a number of technicalities involved in this approach, chiefly involving the need to choose the approximate level sets randomly. This kind of argument was employed, in a closely related context, in [25, Chapter 7]. One can also use utilise arguments based on the Hahn-Banach theorem instead; see [19], [44], and [20, 21, 22].

**Obtaining irrationality.** Our task now is to replace the nilsequence $f_{\text{nil}}$ appearing in Proposition 2.7 with a highly “irrational” nilsequence as advertised in the statement of our main theorem, Theorem 1.2. It turns out to be sufficient to establish the following claim.

**Proposition 2.8.** Let $s, M_0 \geq 1$, let $F$ be a growth function, and let $f : \mathbb{Z} \to [0, 1]$ be a degree $\leq s$ nilsequence of complexity $\leq M_0$. Then there exists an $M = O_{s,M_0,F}(1)$, such that $f$ (when restricted to $[N]$) is also a $(F(M),N)$-irrational degree $\leq s$ virtual nilsequence of complexity $\leq M$ at scale $N$.

To establish Theorem 1.2 from this and Proposition 2.7, one first applies the latter result with $F$ replaced by a much more rapid growth function $F'$, and then one applies Proposition 2.8 to the structured component $f_{\text{nil}}$ obtained in Theorem 2.6.

It remains to prove Proposition 2.8. Let $s, M_0, F, \psi$ be as in that proposition. By definition, we have $\psi = F_0(g_0(n)\Gamma)$ for some degree $\leq s$ filtered nilmanifold $(G/\Gamma, G_\bullet)$ of complexity $\leq M_0$, a polynomial sequence $g_0 \in \text{poly}(\mathbb{Z}, G_\bullet)$, and a function $F_0 : G/\Gamma \to \mathbb{C}$ which has a Lipschitz norm of at most $M_0$. Since $\psi$ takes values in $[0, 1]$, we may assume without loss of generality that $F_0$ is real, and by replacing $F_0$ with the retraction $\max(\min(F_0,1),0)$ to $[0, 1]$ if necessary, we may assume that $F_0$ also takes values in $[0, 1]$. Henceforth $(G/\Gamma, G_\bullet)$, $g_0$, and $F_0$ are fixed.

**Factorisation results.** One of the main results of our paper [28] was a decomposition of an arbitrary polynomial nilsequence $g$ on $G/\Gamma$ into a
product \( \beta g' \gamma \), where \( \beta \) is “smooth”, \( \gamma \) is “rational”, and \( g'(n) \Gamma \) is equidistributed inside some possibly smaller nilmanifold \( G'/\Gamma' \). We need a similar result here, but with \( g' \) having the somewhat stronger property of being irrational that we mentioned in the introduction. The notion of irrationality is discussed in more detail in Appendix A.

We will be also using the notions of smooth and rational polynomial sequences from [28]. Again, the basic definitions and properties of these concepts are recalled in Appendix A.

Define a complexity \( \leq M \) subnilmanifold of \( (G/\Gamma, G_\bullet) \) to be a degree \( \leq s \) filtered nilmanifold \( (G'/\Gamma', G'_\bullet) \) of complexity \( \leq M \), where each subgroup \( G'_{(i)} \) in the filtration \( G'_\bullet \) is a rational subgroup of the associated subgroup \( G_{(i)} \) of complexity \( \leq M \), \( \Gamma' = G' \cap \Gamma \), and each element of the Mal’cev basis of \( (G'/\Gamma', G'_\bullet) \) is a rational linear combination of the Mal’cev basis of \( (G/\Gamma, G_\bullet) \), where the coefficients all have height \( \leq M \). We define the total dimension of such a nilmanifold to be the quantity \( \sum_{i=0}^{s} \dim(G'_{(i)}) \); this is also the dimension of poly \( (\mathbb{Z}, G_\bullet) \) (thanks to the Taylor series expansion, Lemma A.1).

We make the easy remark that if \( (G'/\Gamma', G'_\bullet) \) is a complexity \( \leq M \) subnilmanifold of \( (G/\Gamma, G_\bullet) \) for some \( M \geq M_0 \), and \( (G''/\Gamma'', G''\bullet) \) is a complexity \( \leq M \) subnilmanifold of \( (G'/\Gamma', G'_\bullet) \), then \( (G''/\Gamma'', G''\bullet) \) is a complexity \( O_M(1) \) subnilmanifold of \( (G/\Gamma, G_\bullet) \).

Our first lemma is very similar in form to [28, Lemma 7.9].

**Lemma 2.9 (Initial factorisation).** Let \( (G'/\Gamma', G'_\bullet) \) be a complexity \( \leq M \) subnilmanifold of \( (G/\Gamma, G_\bullet) \) for some \( M \geq M_0 \), let \( g' \in \text{poly}(\mathbb{Z}, G'_\bullet) \), and let \( A > 0 \) and \( N \geq 1 \). Then at least one of the following statements hold:

- **(Irrationality)** \( g' \) is \( (A, N) \)-irrational in \( (G'/\Gamma', G'_\bullet) \).
- **(Dimension reduction)** There exists a factorisation

\[
g' = \beta g'' \gamma
\]

where \( \beta \in \text{poly}(\mathbb{Z}, G'_\bullet) \) is \( (O_{M,A}(1), N) \)-smooth, \( g'' \in \text{poly}(\mathbb{Z}, G''_\bullet) \) takes values in a subnilmanifold \( (G''/\Gamma'', G''_\bullet) \) of \( (G'/\Gamma', G'_\bullet) \) of strictly smaller total dimension and of complexity \( O_{M,A}(1) \), and \( \gamma \in \text{poly}(\mathbb{Z}, G'_\bullet) \) is \( O_{M,A}(1) \)-rational.

**Proof.** To make this proof a little more readable, we drop one dash from every expression. Thus \( g' \) becomes \( g \), \( G'' \) becomes \( G''_\bullet \), and so on. Suppose that \( g \) is not \( (A, N) \)-irrational. Recall (see Lemma A.1) that \( g \) has a Taylor expansion that we may write in the form

\[
g(n) = g_0 g_1^{(1)} g_2^{(2)} \cdots g_s^{(s)},
\]

where \( g_i \in G_{(i)} \) for each \( i \). It follows from Lemma A.7 that for some \( i \), \( 1 \leq i \leq s \), we can factorise

\[
g_i = \beta_i g_i' \gamma_i,
\]

where \( g_i' \) becomes \( g_i' \).

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10In our paper [28] the letter \( \varepsilon \) was used for a smooth nilsequence, but we use \( \beta \) here to avoid conflict with various uses of \( \varepsilon \) to denote a small positive real number.
where \( g'_i \in G_{(i)} \) lies in the kernel of some horizontal character \( \xi_i : G_{(i)} \to \mathbb{R} \) of complexity \( O_{A,M}(1) \), \( \gamma_i \in G_{(i)} \) is \( O_{A,M}(1) \)-rational in the sense that \( \gamma_i^m \in \Gamma_{(i)} \) for some \( m = O_{A,M}(1) \), and \( \beta_i \in G_{(i)} \) has distance \( O_{A,M}(1/N^i) \) from the origin.

We now divide into two cases, depending on whether \( i > 1 \) or \( i = 1 \). First suppose that \( i > 1 \). Then the Taylor expansion of \( g \) reads, with an obvious notation,

\[
g(n) = g_{<i}(n)(\beta_i \gamma_i^m)g_{>i}(n).
\]

By commutating all the \( \beta_i \)'s to the left and all the \( \gamma_i \)'s to the right, and using the group properties of polynomial sequences (Theorem 1.6), one can rewrite this as

\[
g(n) = \beta_i^{(n)} g'(n) \gamma_i^{(n)}
\]

where

\[
g'(n) := g_{<i}(n)g_i^{(n)} g_{>i}(n)
\]

and \( g_{>i}(n) \) is another polynomial sequence taking values in \( G_{(i+1)} \). Observe that \( g' \) is then a polynomial sequence adapted to the sub-nilmanifold \( (G'/\Gamma', G'_{(i)}) \), where \( G'/\Gamma' = G/\Gamma \) and \( G'_{(j)} = G_{(j)} \) for \( j \neq i \), but \( G'_{(i)} = \ker(\xi'_i) \). This is indeed a sub-nilmanifold, with complexity \( O_{A,M}(1) \); note that \( (G'_i)_{i=0}^\infty \) is a filtration, thanks to our insistence in the definition of \( i \)-horizontal character (cf. Definition A.6) that \([G_{(j)}, G_{(i-j)}] \subseteq \ker(\xi'_i)\) for all \( 0 \leq j \leq i \). Meanwhile, \( \beta_i^{(n)} \) is a \( O_{A,M}(1), N \)-smooth sequence and \( \gamma_i^{(n)} \) is an \( O_{A,M}(1) \)-rational sequence, so we have the desired factorisation in the \( i > 1 \) case.

When \( i = 1 \), the above argument does not quite work, because \( G'_{(1)} \) would be distinct from \( G'_{(0)} \) and would thus not qualify as a filtration. But this can be easily remedied by performing an additional factorisation

\[
g_0 = \tilde{g}_{>1}(n)
\]

where \( \beta_0 \in G' \) is a distance \( O_{A,M}(1) \) from the identity, and \( g_0' \) lies in the kernel of \( \xi'_1 \). This leads to a factorisation of the form

\[
g(n) = \beta_0 \beta_i^{(n)} g'(n) \gamma_1^{(n)}
\]

where

\[
g'(n) = g_0' g_1^{(n)} g_{>1}(n)
\]

and \( g_{>1} \) is a polynomial sequence taking values in \( G'_{(2)} \). One then argues as before, but now one sets both \( G''_0 \) and \( G''_1 \) equal to the kernel of \( \xi'_1 \). \( \square \)

We can iterate the above lemma to obtain the following result, which is analogous to [38, Theorem 1.19]. Apart from dealing with irrationality rather than equidistribution, the following result is somewhat different to that just cited in that one requires an arbitrary (rather than polynomial) growth function, but one does not (of course) need polynomial complexity bounds. A variant of [38, Theorem 1.19] was also given in [31, Theorem 4.2].
Lemma 2.10 (Complete factorisation). Let \((G/\Gamma, G_\bullet)\) be a degree \(\leq s\) filtered nilmanifold of complexity \(\leq M_0\), and let \(g \in \text{poly}(\mathbb{Z}, G_\bullet)\). For any growth function \(F'\), we can find a quantity \(M_0 \leq M \leq O_M, F'(1)\) and a factorisation \(g = \beta g' \gamma\) where:
- \(\beta \in \text{poly}(\mathbb{Z}, G_\bullet)\) is \((O_M(1), N)\)-smooth;
- \(g' \in \text{poly}(\mathbb{Z}, G_\bullet)\) is \((F'(M), N)\)-irrational in a subnilmanifold \((G'/\Gamma', G'_\bullet)\) of \((G/\Gamma, G_\bullet)\) of complexity \(O_M(1)\), and
- \(\gamma \in \text{poly}(\mathbb{Z}, G_\bullet)\) is \(O_M(1)\)-periodic.

Proof. We use an iterative argument, setting \(\beta = \gamma = \text{id}\), \(g' = g\), \(M = M_0\), and \((G'/\Gamma', G'_\bullet) = (G/\Gamma, G_\bullet)\) to begin with. In particular, \((G'/\Gamma', G'_\bullet)\) is initially a subnilmanifold of \((G/\Gamma, G_\bullet)\) of complexity \(O_M(1)\). If \(g'\) is \(F'(M)\)-equidistributed in \((G'/\Gamma', G'_\bullet)\) then we are done; otherwise, by Lemma 2.9 we may factorise \(g' = \beta g'' \gamma'\) where \(\gamma' = O_M(1)\)-periodic, and \(g''\) now takes values in a subnilmanifold \((G''/\Gamma'', G''_\bullet)\) of \((G'/\Gamma', G'_\bullet)\) of complexity \(O_M(1)\) and smaller total dimension than \((G'/\Gamma', G'_\bullet)\). We then replace \(\beta\) by \(\beta \beta'\), \(\gamma\) by \(\gamma' \gamma\), \(g'\) by \(g''\), \((G'/\Gamma', G'_\bullet)\) by \((G''/\Gamma'', G''_\bullet)\), and increase \(M\) to a quantity of the form \(O_M(1)\), using Lemma A.4 to conclude that the new \(\beta\) is smooth and the new \(\gamma\) is rational. We then iterate this process. Since the total dimension of \((G/\Gamma, G_\bullet)\) is initially \(O_M(1)\), this process can iterate at most \(O_M(1)\) times, and the claim follows. \(\square\)

With this lemma we can now establish Proposition 2.8 and hence Theorem 1.2. Let \(F'\) be a rapid growth function (depending on \(\varepsilon, M_0, F\)) to be chosen later. We apply Lemma 2.10, obtaining some \(M\) with \(M_0 \leq M \leq O_M, F'(1)\) and a factorisation

\[
\psi(n) = F(\beta(n)g'(n)\gamma(n)\Gamma)
\]

with \(\beta, g'\) and \(\gamma\) having the properties described in that lemma.

The sequence \(\gamma\) is \(O_M(1)\)-rational and so, by Lemma A.4, the orbit \(n \mapsto \gamma(n)\Gamma\) is periodic with some period \(q = O_M(1)\), and thus \(\gamma(n)\Gamma\) depends only on \(n \mod q\).

For each \(n\), the rationality of \(\gamma(n)\) ensures that \(\gamma(n)\Gamma\) intersects \(\Gamma\) in a subgroup of \(\Gamma\) of index \(O_M(1)\). Since there are only \(O_M(1)\) different possible values of \(\gamma(n)\Gamma\), we may thus find a subgroup \(\Gamma'\) of \(\Gamma\) of index \(O_M(1)\) such that \(\Gamma' \subseteq \gamma(n)\Gamma\) for all \(n\).

We can thus express \(\psi\) as a virtual nilsequence

\[
\psi(n) = \hat{F}(g'(n)\Gamma', n \mod q, n/N)
\]

where \(\hat{F} : G/\Gamma' \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R}\) is defined by the formula

\[
\hat{F}(x, a, y) := F(\beta(Ny)\hat{x}\gamma(\hat{a})\Gamma)
\]

whenever \(y \in \frac{1}{N}\mathbb{Z}\) and by Lipschitz extension to all \(y \in \mathbb{R}\). where \(\hat{a}\) is any integer with \(\hat{a} = a \mod q\), and \(\hat{x}\) is any element of \(G\) such that \(\hat{x}\Gamma' = x\).

One easily verifies that \(\hat{F}\) is well-defined and has a Lipschitz norm of \(O_M(1)\).

Also, since \(g'\) was already \((F(M), N)\)-irrational in \(G/\Gamma\), and \(\Gamma'\) has index \(O_M(1)\) in \(\Gamma\), we see that \(g'\) is \((\gg_M, F(M), N)\)-irrational in \(G/\Gamma'\). Proposition
2.8 now follows by replacing $M$ by a suitable quantity of the form $O_M(1)$, and choosing $F'$ sufficiently rapidly growing depending on $F$.

3. Proof of the counting lemma

The purpose of this section is to prove the counting lemma, Theorem 1.11.

We begin by recalling from the introduction the definition of the Leibman group $G^\Psi$.

**Definition 3.1** (The Leibman group). Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$. For any $i \geq 1$, define $\Psi[i]$ to be the linear subspace of $\mathbb{R}^t$ spanned by the vectors $(\psi_j^1(n), \ldots, \psi_j^j(n))$ for $1 \leq j \leq i$ and $n \in \mathbb{Z}^D$. Given a filtered nilmanifold $(G/\Gamma, G^\bullet)$, we define the *Leibman group* $G^\Psi < G^t$ to be the Lie subgroup of $G^t$ generated by the elements $g_i^{\vec{v}}$ for $i \geq 1$, $g_i \in G(i)$, and $\vec{v} \in \Psi[i]$, with the convention that if $\vec{v} = (v_1, \ldots, v_t)$ then

$$g^{\vec{v}} := (g^{v_1}, \ldots, g^{v_t}).$$

Now might be a good time to remark explicitly that we have introduced a slightly vulgar convention that we hope will help the reader follow this section and other parts of the paper. Bold font letters such as $n \in \mathbb{R}^D$ denote $D$-dimensional vectors, whilst arrows such as $\vec{v} \in \mathbb{R}^t$ denote $t$-vectors. Occasionally we shall write $m_i := \dim(\Psi[i]).$

When reading this section, it might be found helpful to have a running example in mind. We will take as an illustrative example the case $D = 2$, $t = 4$ and $\Psi = (\psi_1, \ldots, \psi_4)$, where $\psi_i(n) = n_1 + in_2$ for $i = 0, 1, 2, 3$. The system $\Psi$, of course, defines a 4-term arithmetic progression. As we remarked in the introduction the corresponding Leibman group $G^\Psi$ is also known as the *Hall-Petresco group* $\text{HP}^4(G)$. The reader will easily confirm that in this case we have

$$\Psi[1] = \mathbb{R}(1, 1, 1, 1) \oplus \mathbb{R}(0, 1, 2, 3)$$

and

$$\Psi[2] = \mathbb{R}(1, 1, 1, 1) \oplus \mathbb{R}(0, 1, 2, 3) \oplus \mathbb{R}(0, 0, 1, 3)$$

and

$$\Psi[3] = \mathbb{R}(1, 1, 1, 1) \oplus \mathbb{R}(0, 1, 2, 3) \oplus \mathbb{R}(0, 0, 1, 3) + \mathbb{R}(0, 0, 0, 1) = \mathbb{R}^4.$$

Some work must be done before we can describe $G^\Psi = \text{HP}^4(G)$ in a pleasant way. However we can already establish the following lemma, whose statement and proof go some way towards explaining the introduction of the Leibman group.

**Lemma 3.2.** Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$. Suppose that $(G/\Gamma, G^\bullet)$ is a filtered nilmanifold and that $g \in \text{poly}(\mathbb{Z}, G^\bullet)$ is a polynomial sequence. Then the sequence $g^\Psi : \mathbb{Z}^D \to G^t$ defined by $g^\Psi(n) := (g(\psi_1(n)), \ldots, g(\psi_t(n)))$ takes values in $G^\Psi$. 
Proof. The sequence $g(n)$ has a (unique) Taylor expansion

$$g(n) = g_0 g_1^{(n)} \ldots g_s^{(n)}$$

with $g_i \in G(i)$ for all $i$ (see Lemma A.1). Substituting in, it follows that

$$g^\Psi(n) = \prod_{i=0}^{s} g_i^{(\psi_1(n), \ldots, \psi_t(n))},$$

and it is immediate from the definition that each element in this product lies in $G^\Psi$. □

The counting lemma, whose proof is the main objective of this section, was stated as Theorem 1.11. Essentially, it states that $g^\Psi(n)G^\Psi$ is equidistributed in $G^\Psi/G^\Psi$ as $n$ ranges over “nice” subsets of “big” lattices, provided that the original sequence $g$ is suitably irrational. We will recall what that means in due course, but our first task is to develop the basic theory of the Leibman group $G^\Psi$. At the moment, for example, we have not established that $G^\Psi$ is a connected Lie subgroup of $G^t$ or that $G^\Psi/G^\Psi$ has the structure of a filtered nilmanifold. Nor have we developed tools for calculating inside this group.

**Basic facts about the Leibman group and nilmanifold.** We can endow $\mathbb{R}^t$ with the structure of a commutative algebra over $\mathbb{R}$ by using the pointwise product

$$\vec{x} \cdot \vec{y} = (x_1 y_1, \ldots, x_t y_t)$$

and setting $\vec{1} = (1, \ldots, 1)$ to be the multiplicative identity. With this algebra structure, one can view the spaces $\Psi[i]$ defined in Definition 1.10 as the span of the powers $\Psi(n)^j$ for $n \in \mathbb{Z}^D$ and $1 \leq j \leq i$, where we view $\Psi$ as a homomorphism from $\mathbb{Z}^D$ to $\mathbb{Z}^t$. We have the following alternate definition of the $\Psi[i]$.

**Lemma 3.3 (Depolarisation).** $\Psi[i]$ is the span of the products

$$\Psi(n_1) \ldots \Psi(n_j),$$

where $1 \leq j \leq i$ and $n_1, \ldots, n_j \in \mathbb{Z}^D$.

Proof. Clearly $\Psi[i]$ is contained in this span. To establish the reverse containment, we observe the elementary depolarisation identity

$$\Psi(n_1) \ldots \Psi(n_j) = \frac{(-1)^j}{j!} \sum_{\omega \in \{0, 1\}^j} (-1)^{|\omega|} \Psi(\omega_1 n_1 + \ldots + \omega_j n_j)^j,$$

where $\omega = (\omega_1, \ldots, \omega_j)$ and $|\omega| := \omega_1 + \ldots + \omega_j$, and the claim follows. □

As an immediate consequence we have

**Corollary 3.4 (Filtration property).** For any $i, j \geq 0$, we have $\Psi[i] \cdot \Psi[j] \subseteq \Psi[i+j]$. 
Let \((G/T, G_*)\) be a degree \(\leq s\) filtered nilmanifold. From Definition \[1.10\], the Leibman group \(G^Ψ\) is the subgroup of \(G^t\) generated by the group elements \(g_i^{v_i}\) for \(i \geq 1\), \(v_i \in Ψ[i]\), and \(g_i \in G(i)\). For any \(i_0 \geq 1\), let \(G^Ψ_{(i_0)}\) be the subgroup of \(G^Ψ\) generated by those \(g_i^{v_i}\) with \(i \geq i_0\), \(v_i \in Ψ[i]\), \(g_i \in G(i)\), with the convention that \(G^Ψ_{(0)} := G^Ψ\).

**Lemma 3.5** (Filtration property for \(G^Ψ\)). \(G^Ψ := (G^Ψ_{(i)})_{i=0}^{∞}\) is a filtration on \(G^Ψ\). In other words, the \(G^Ψ_{(i)}\) are nested with \([G^Ψ_{(i)}, G^Ψ_{(j)}] \subset G^Ψ_{(i+j)}\) for all \(i, j \geq 0\).

**Proof.** It suffices to check that if \(g_i \in G(i), g_j \in G(j), \overset{i}{\vvec} = (v_1, \ldots, v_t) \in Ψ[i]\) and \(\overset{j}{\vvec} = (v_{j1}, \ldots, v_{jt}) \in Ψ[j]\) then \([g_i^{\overset{i}{\vvec}}, g_j^{\overset{j}{\vvec}}] \in G_{(i+j)}\). But this follows from the Baker-Campbell-Hausdorff formula (see \((C.2)\)), the filtration property of \(G_{(i)}\) and Corollary \[3.3\].

The spaces \(Ψ[i]\) form a flag

\[
0 \leq Ψ[1] \leq \ldots \leq Ψ[s] \leq \mathbb{R}^t
\]

of subspaces which are rational (i.e. they can be defined over \(\mathbb{Q}\)). From a greedy algorithm (and clearing denominators) we may thus find a basis \(\overset{1}{\vvec}, \ldots, \overset{s}{\vvec}_{m_s} \in Ψ[s]\) with the following properties:

- **(Integrality)** \(\overset{i}{\vvec}, \ldots, \overset{s}{\vvec}_{m_s}\) all lie in \(\mathbb{Z}^t\);
- **(Partial span)** For every \(1 \leq i \leq s\), \(\overset{i}{\vvec}, \ldots, \overset{s}{\vvec}_{m_s}\) span \(Ψ[i]\);
- **(Row echelon form)** For each \(1 \leq j \leq m_s\), there exists \(l_j, 1 \leq l_j \leq t, \) such that \(\overset{j}{\vvec}\) has a non-zero \(l_j\) coordinate, but such that \(\overset{j'}{\vvec}\) has a zero \(l_j\) coordinate for all \(j < j' \leq m_s\).

For instance, the basis

\[
\overset{1}{\vvec} := (1, 1, 1, 1); \; \overset{2}{\vvec} := (0, 1, 2, 3); \; \overset{3}{\vvec} := (0, 0, 1, 3); \; \overset{4}{\vvec} := (0, 0, 0, 1)
\]

we implicitly gave above for our running example is already in this form.

Fix such a basis. For each basis element \(\overset{j}{\vvec}\), we can define the **degree** \(\deg(\overset{j}{\vvec})\) of that element to be the first \(i\) for which \(j \leq m_s\), thus \(\deg(\overset{j}{\vvec})\) is an integer between 1 and \(s\), and \(\overset{j}{\vvec} \in Ψ[\deg(\overset{j}{\vvec})]\).

Observe that an arbitrary element of \(G^Ψ\) can be expressed as a product of finitely many elements of the form \(g_j^{\overset{j}{\vvec}}\) for \(0 \leq j \leq m_s\) and \(g_j \in G(\deg(\overset{j}{\vvec}))\). By many applications\[11\] of the Baker-Campbell-Hausdorff formula (see \((C.1)\)) and Lemma \[3.3\], we can now express any element of \(G^Ψ\) in the form

\[
\prod_{j=1}^{m_s} g_j^{\overset{j}{\vvec}}
\]

\[11\]Indeed, one uses \((C.1)\) and Lemma \[3.3\] to extract out and collects all terms with degree \(\deg(\overset{j}{\vvec}) = 1\), leaving only terms with base \(g_j\) in \(G(2)\). Then one extracts out those terms with degree 2 (merging them with the \(i = 1\) terms as necessary), leaving only terms with base in \(G(3)\). Continuing this process gives the desired factorisation.
where \( g_j \in G_{(\deg(\vec{v}_j))} \) for all \( 1 \leq j \leq m_s \).

Thus, in our running example, we have the explicit description of \( G^\Psi = HP^4(G) \) as

\[
\{(g_0, g_0 g_1, g_0 g_1^2 g_2, g_0 g_1^3 g_3) : g_0 \in G_{(0)}, g_1 \in G_{(1)}, g_2 \in G_{(2)}, g_3 \in G_{(3)}\}\.
\]

Note that from results on the Taylor expansion (see Lemma A.1) this group may also be identified as

\[
\{ (g(0), g(1), g(2), g(3)) : g \in \text{poly}(\mathbb{Z}, G_*). \}
\]

The group nature of \( HP^4(G) \) is then easily deduced from Theorem 1.6, but this presentation is somewhat specific to the Hall-Petresco case and we shall not require it further.

From the row-echelon form one can verify inductively that the representation (3.1) is unique (this can be seen clearly by working with the Hall-Petresco example presented above). This gives \( G^\Psi \) the structure of a connected, simply connected Lie group, with dimension

\[
\dim(G^\Psi) = \sum_{i=1}^{s} \dim(G^{(i)})(\dim(\Psi[i]) - \dim(\Psi[i-1]))
\]  

(with the convention that \( \Psi[0] \) is trivial). A similar argument also shows that every element of \( G^\Psi_{(i)} \) can be expressed uniquely in the form (3.1), where now \( g_j \) is constrained to lie in \( G_{(\max(\deg(\vec{v}_j), i_0))} \) rather than \( G_{(\deg(\vec{v}_j))} \).

In particular, by reading off the coefficients \( g_j \) one at a time, this implies the pleasant identity

\[
G^\Psi_{(i)} = G^\Psi \cap (G_{(i)})^k.
\]

**Remark.** From Taylor expansion (see Lemma A.1) we see that the sequence \( g^\Psi \) in (1.7) lies in \( \text{poly}(\mathbb{Z}, G^\Psi_*). \) While we do not directly use this fact here, it may help explain why the filtration \( G^\Psi_* \) plays a prominent role in the proof of the counting lemma that we will shortly come to.

Recall that we normalised the basis vectors \( \vec{v}_j \in \mathbb{Z}^t \) to have integer coefficients. As a consequence, we see that if the \( g_j \) are in \( \Gamma \), then the expression (3.1) lies in \( \Gamma^k \). From this (and many applications of Lemma 3.5) we see that \( \Gamma^\Psi_{(i)} := \Gamma^k \cap G^\Psi_{(i)} \) is cocompact in \( G^\Psi_{(i)} \) for each \( i \), and so \( (G^\Psi/\Gamma^\Psi, G^\Psi_*) \) is a filtered nilmanifold. Furthermore, the same argument shows that the \( G^\Psi_{(i)} \) are rational subgroups of \( G^k \) and so \( (G^\Psi/\Gamma^\Psi, G^\Psi_*) \) is a subnilmanifold of \( (G^k/\Gamma^k, G^k_*) \).

**The Counting Lemma: Preliminary Manoeuvres.** Now that we have verified that \( G^\Psi/\Gamma^\Psi \) is indeed a nilmanifold, we can begin the proof of Theorem 1.11.

We begin with some easy reductions. First, observe that for fixed \( M \), there are only finitely many possibilities for \( s, D, t, \Psi \), and (up to isomorphism) there are only finitely many possibilities for \( (G/\Gamma, G_*) \) and \( \Gamma \). Thus it will suffice to establish the result for a single choice of \( s, D, t, \Psi, (G/\Gamma, G_*) \), with
the bounds depending on these quantities. Hence, we fix these quantities and allow all implicit constants to depend on these quantities (thus, in this section, we will not explicitly subscript out $O(1)$ quantities).

Similarly, because the space of Lipschitz functions with Lipschitz norm $O(1)$ is precompact in the uniform topology (by the Arzelá-Ascoli theorem), it suffices to prove the desired bound for each fixed $F$, as the uniformity in $F$ then follows from an easy approximation argument. Thus we fix $F$ and allow all quantities to depend on $F$.

Next, we observe that we may normalise $g(0) = \text{id}$. Indeed, we may factorise $g(0) = c_0 \gamma_0$ where $d_G(c_0, \text{id}) = O(1)$ and $\gamma_0 \in \Gamma$. Factorising, we obtain $g(n) = c_0 g'(n) \gamma_0$ where $g'(n) := c_0 \gamma_0^{-1} g(n) \gamma_0$. Note that $g'(0) = \text{id}$ and that Taylor coefficients of $g'$ are given by $g'_i = \gamma_0^{-1} g_i \gamma_0$, and so $g'$ is also $(A,N)$-irrational.

It is then an easy matter to see that Theorem 1.11 for $g$ and $F$ follows from Theorem 1.11 for $g'$ and for the shifted function $F'(x) := F(c_0 x)$, which is still Lipschitz with norm $O(1)$.

Note that we may assume that $A$ and $N$ are large, as the claim is trivial otherwise.

Equidistribution in the Leibman group. Let us recall what we are trying to prove. In the counting lemma, Theorem 1.11, our aim is to show that if $g(n)$ is suitably irrational then the orbit $(g^\psi(n))_{n \in (n_0 + \Lambda) \cap P}$ is equidistributed on the Leibman nilmanifold $G^\psi/\Gamma^\psi$. We shall proceed by contradiction, supposing this orbit is not equidistributed and deducing that $g(n)$ could not have been irrational. The reader should recall the definition of irrational in this context: it is given in Definition A.6.

Our main tool will be a mild generalisation of the “multiparameter Leibman criterion”, which is [28, Theorem 8.6]. Here is the statement we shall use.

**Theorem 3.6.** Suppose that $(G/\Gamma, G_\bullet)$ is a filtered nilmanifold of complexity $\leq M$ and that $g \in \text{poly}(\mathbb{Z}^D, G_\bullet)$ is a polynomial sequence for some $D \leq M$. Suppose that $\Lambda \subseteq \mathbb{Z}^D$ is a lattice of index $\leq M$, that $\mathbf{n}_0 \in \mathbb{Z}^D$ has magnitude $\leq M$, and that $P \subseteq [-N,N]^D$ is a convex body. Suppose that $\delta > 0$, and that

$$\left| \sum_{\mathbf{n} \in (\mathbf{n}_0 + \Lambda) \cap P} F(g(\mathbf{n})\Gamma) - \frac{\text{vol}(P)}{[\mathbb{Z}^D : \Lambda]} \int_{G/\Gamma} F \right| > \delta N^D \|F\|_{\text{Lip}}$$

for some Lipschitz function $F : G/\Gamma \to \mathbb{C}$. Then there is a nontrivial homomorphism $\eta : G \to \mathbb{R}$ which vanishes on $\Gamma$, has complexity $O_M(1)$ and such that

$$\|\eta \circ g\|_{C^\infty([-N]_D^D)} = O_{\delta, M}(1).$$

**Remarks.** This differs from [28, Theorem 8.6] in several insubstantial ways. On the one hand we have no concern here with the polynomial bounds
that were important in that setting. However, we are dealing here with a
sublattice \( \Lambda \subseteq \mathbb{Z}^D \) rather than \( \mathbb{Z}^D \) itself, and with an arbitrary convex body
\( P \) rather than the box \([N]^D\). This more general result can be deduced from
[28, Theorem 8.6] in a somewhat routine, though slightly tedious, manner.
We sketch the details in Appendix B. The notation \( C^\infty([N]^D) \) is recalled
both in the appendix and later in this section.

Later on, the notation will get a little complicated. Let us, then, first
apply Theorem 3.6 to establish the following very simple special case of the
counting lemma (it is, of course, the special case in which \( \Psi \) consists of the
single form \( \psi_1(n) = n^1 \)).

**Lemma 3.7 (Irrational implies equidistributed).** Suppose that
\((G/\Gamma, G_\bullet)\) is
a filtered nilmanifold of complexity at most \( M \) and that \( g : \mathbb{Z} \to G \) is an
\((A,N)\)-irrational polynomial sequence. Then we have the equidistribution
property
\[
\mathbb{E}_{n \in [N]} F(g(n)\Gamma) = \int_{G/\Gamma} F + O_M\left(A^{-cM} \|F\|_{\text{Lip}}\right)
\]
for all Lipschitz \( F : G/\Gamma \to \mathbb{C} \) and some \( c_M > 0 \).

**Proof.** Suppose the conclusion is false. Then by[12] Theorem 3.6 there is some
continuous homomorphism \( \eta : G \to \mathbb{R} \) which vanishes on \([G,G]\) and \( \Gamma \), has
complexity \( O_\delta(1) \), and for which \( \|\eta \circ g\|_{C^\infty([N])} \leq \delta^{-\Theta(1)} \). Recall (cf. [28,
Definition 2.7]) what this means: in the Taylor expansion
\[
\eta \circ g(n) = \alpha_0 + \alpha_1 (n^1) + \cdots + \alpha_s (n^s),
\]
the \( j \)th coefficient \( \alpha_j \) satisfies \( \|\alpha_j\|_{\mathbb{R}/\mathbb{Z}} \leq \delta^{-\Theta(1)}/N^j \) for \( j = 1, \ldots, s \). If the
sequence \( g \) is developed as a Taylor expansion
\[
g(n) = g_0 g_1 (n^1) \cdots g_s (n^s)
\]
then we of course have \( \alpha_j = \eta(g_j) \). Choose \( i \) maximal so that the restriction
\( \eta|_{G(i)} \) is nontrivial. Then certainly \( \|\eta(g_i)\|_{\mathbb{R}/\mathbb{Z}} \leq \delta^{-\Theta(1)}/N^i \). We claim that
\( \eta \) is an \( i \)-horizontal character in the sense of Definition A.5, a state-
ment which will clearly contradict the supposed \((A,N)\)-irrationality of \( g \) if \( \delta \) is
a sufficiently small power of \( 1/A \). To this end all we need do is confirm
that \( \eta \) vanishes on \( G(i+1), \Gamma(i) \) and on \([G(j), G(i-j)]\) for \( 0 \leq j \leq i \). The
first of these follows from the maximality of \( i \), whilst the second and third
follow immediately from the properties of \( \eta \) stated at the beginning of the
proof. \( \Box \)

Let us turn now to the more notationally intensive general case. Now,
we apply Theorem 3.6 to \( G^\Psi/\Gamma^\Psi \) to conclude that there is a non-trivial
continuous homomorphism \( \eta : G^\Psi \to \mathbb{R} \) which maps \( \Gamma^\Psi \) to \( \mathbb{Z} \), has complexity
\( O_\delta(1) \), and satisfies
\[
\|\eta \circ g^\Psi\|_{C^\infty([N]^D)} = O_\delta(1).
\]
Much as in the proof of Lemma 3.7, what this means is that if $\eta \circ g^\Psi(n)$ is developed as a Taylor series in multi-binomial coefficients $\binom{n}{j} = \binom{n_1}{j_1} \cdots \binom{n_D}{j_D}$ (see Lemma A.31), the coefficient $\alpha_1$ satisfies $\|\alpha_j\|_{\mathbb{R}/\mathbb{Z}} \ll \delta N^{-|j|}$. Our aim is to use this information to contradict the assumption that $g(n)$ is $(A,N)$-irrational.

Let us once again take $i$ maximal such that $\eta|_{\mathcal{C}_j}$ is nontrivial. Considering again the Taylor expansion of $g(n)$, we have

$$
(\eta \circ g^\Psi)(n) = \sum_{j=1}^i \eta(g_j^{(\psi_1(n))}, \ldots, g_j^{(\psi_t(n))}).
$$

(3.5)

Take the basis $\vec{v}_1, \vec{v}_2, \ldots$ for $\Psi[i]$ described earlier. Then, since the vector

$$
\left(\left(\begin{array}{c}
\psi_1(n) \\
\vdots \\
\psi_t(n)
\end{array}\right)\right)
$$

lies in $\Psi[j]$, there is an expansion

$$
\left(\left(\begin{array}{c}
\psi_1(n) \\
\vdots \\
\psi_t(n)
\end{array}\right)\right) = P_{j,1}(n)\vec{v}_1 + \cdots + P_{j,m_j}(n)\vec{v}_{m_j}
$$

(3.6)

for $j = 1, \ldots, i$, where the $P_{j,k}: \mathbb{Z}^D \to \mathbb{R}$ are polynomials of degree at most $j$, recalling that $m_j := \dim(\Psi[j])$. Comparing with (3.5), we obtain

$$
(\eta \circ g^\Psi)(n) = \sum_{j=1}^i \sum_{k=1}^{m_j} P_{j,k}(n)\eta(g_j^{\vec{v}_k}).
$$

(3.7)

We are going to look at the coefficients $\alpha_i$ of (3.7) for the monomial $n^i := n_1^{i_1} \cdots n_D^{i_D}$, where $\mathbf{i} = (i_1, \ldots, i_D)$ and $|\mathbf{i}| := |i_1| + \cdots + |i_D| = i$. We are assuming that every such coefficient satisfies $\|\alpha_i\|_{\mathbb{R}/\mathbb{Z}} \ll \delta N^{-i}$. Note also that

$$
\alpha_i = \sum_{k=1}^{m_i} (P_{i,k})_i \eta(g_j^{\vec{v}_k}),
$$

(3.8)

where $(P_{i,k})_i$ is the $n^i$ coefficient of $P_{i,k}(n)$; this is because terms of total degree $i$ cannot arise from the terms $j = 1, \ldots, i - 1$ in the sum on the right hand side of (3.7).

On the other hand by taking $j = i$ in (3.6) we have

$$
(P_{i,1}(n))_1 \vec{v}_1 + \cdots + (P_{i,m_i}(n))_1 \vec{v}_{m_i}
$$

$$
= \frac{1}{i_1! \cdots i_D!} (\varphi_1(e_1)^{i_1} \cdots \varphi_1(e_D)^{i_D}, \ldots, \varphi_t(e_1)^{i_1} \cdots \varphi_t(e_D)^{i_D})
$$

$$
= \frac{1}{i_1! \cdots i_D!} \Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D},
$$

(3.9)

where $e_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^D$, the $1$ being in the $j$th position, and $\Psi(e_j) := (\psi_1(e_j), \ldots, \psi_t(e_j)) \in \mathbb{R}^t$. 

Comparing (3.8) and (3.9) and using the fact that $\eta$ is a homomorphism on $G^\Psi$, we obtain
\[ \alpha_i = \frac{1}{i_1! \cdots i_D!} \eta(g^{\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D}}). \]
Thus, for each $i$ with $|i| = |i_1| + \cdots + |i_D| = i$, we have
\[ \|\eta(g^{\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D}})\|_{R/Z} \ll_{\delta} N^{-i}. \tag{3.10} \]
To obtain the desired contradiction with the $(A, N)$-irrationality hypothesis and thus complete the proof, it suffices (after taking $A$ sufficiently large depending on $\delta$) to establish that for at least one choice of $i$, the map $\xi_i : G_{(i)} \to \mathbb{R}$ defined by
\[ \xi_i(g) := \eta(g^{\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D}}) \]
is a nontrivial horizontal $i$-character of complexity $O_{\delta}(1)$.

The complexity bound follows from the fact that the coefficients of the forms $\psi_i$ are integers of size $O(1)$ and the Baker-Campbell-Hausdorff formula (Appendix C). That at least one of these maps is nontrivial follows from that fact that $\eta$ is nontrivial on $G_{(i)}^\Psi$ and the fact that the vectors $\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D}$, $i_1 + \cdots + i_D = i$, span $\Psi[i]$ (a consequence of Lemma 3.3).

Furthermore $\xi_i$ always annihilates $\Gamma_{(i)}^\Psi$ and $G_{(i+1)}^\Psi$ (by the asserted maximality of $i$). To qualify as an $i$-horizontal character we must also show that it vanishes on $[G_{(j)}^\Psi, G_{(i-j)}^\Psi]$ for each $0 \leq j \leq i$. To this end, note that we may factor
\[ \Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D} = ww', \]
where $w \in \Psi[j]$ and $w' \in \Psi[i-j]$. Indeed, we may take
\[ w = \Psi(e_1)^{j_1} \cdots \Psi(e_D)^{j_D}, \quad w' = \Psi(e_1)^{i_1-j_1} \cdots \Psi(e_D)^{i_D-j_D} \]
for any indices $j_1, \ldots, j_D$ with $j_l \leq i_l$ and $j_1 + \cdots + j_D = j$, whereupon the relevant containments follow from Lemma 3.3. Now if $g \in G_{(j)}^\Psi$ and $g' \in G_{(i-j)}^\Psi$ are arbitrary then we have
\[ [g^w, g'^{w'}] \equiv [g, g']^{ww'} \mod G_{(i+1)}^\Psi \]
by the Baker-Campbell-Hausdorff formula (B.2). Applying $\eta$, which is trivial on $G_{(i+1)}^\Psi$ by assumption, we obtain
\[ \xi_i([g, g']) = \eta([g, g']^{ww'}) = \eta([g^w, g'^{w'}]) = 0, \]
the last step being a consequence of the fact that $\eta$ has abelian image and hence vanishes on $[G_{(j)}^\Psi, G_{(i)}^\Psi]$. This concludes the proof of the counting lemma, Theorem 1.11.
4. Generalised von Neumann type theorems

In this section we recall a number of results asserting the connection between Gowers norms and various types of linear configuration. These results are collectively known in the literature as “generalised von Neumann theorems”. The connection between Gowers norms (not called by that name, of course) and linear configurations was first made in [15]. A fairly general result of this type, which appears in [29], is the following.

**Theorem 4.1 (Generalised von Neumann Theorem).** Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$ for some $t, D \geq 1$, any two of which are linearly independent. Then there exists an integer $s = s(\Psi)$ with the property that one has the inequality

$$|E_{n \in [N]} \prod_{i=1}^t f_i(\psi_i(n))| \ll t, D, \Psi \inf_{1 \leq i \leq m} \|f_i\|_{U^{s+1}[N]}$$

for all $N \geq 1$ and all $f_1, \ldots, f_m : [N] \to \mathbb{C}$ bounded in magnitude by 1.

**Remarks.** A natural value of $s(\Psi)$ comes from the proof in [29], which proceeds via $s$ applications of the Cauchy-Schwarz inequality. For this reason Gowers and Wolf [20] call $s(\Psi)$ the Cauchy-Schwarz complexity of the system $\Psi$. There is a linear-algebra recipe for computing $s(\Psi)$ which is not especially enlightening but sufficiently simple that we can give it here (see the introduction to [29] for more details). If $1 \leq i \leq t$ and $s \geq 0$ then we say that $\Psi$ has $i$-complexity at most $s$ if one can cover the $t - 1$ forms $\{\psi_j : j \in [t] \setminus \{i\}\}$ by $s+1$ classes, such that $\psi_i$ does not lie in the linear span of the forms in any one of these classes. Then $s(\Psi)$ is the smallest $s$ for which the system has $i$-complexity at most $s$ for all $1 \leq i \leq t$. Note, then, that the Cauchy-Schwarz complexity of the system $\Psi = \{n_1, n_1 + n_2, \ldots, n_1 + (k-1)n_2\}$ corresponding to a $k$-term arithmetic progression is $k - 2$. As a final remark, let us note that Theorem 4.1, as proved in [29, Appendix C], is regrettably somewhat difficult to understand as we had to establish a more general result in which the functions $f_i$ were bounded by an arbitrary pseudorandom measure, and this is notationally heavy. For a gentle explanation of the special case $\Psi = \{n_1, n_1 + n_2, n_1 + 2n_2, n_1 + 3n_2\}$ (where $s = 2$) the reader may consult [24, Proposition 1.11]. A sketch of the proof of Theorem 4.1 is also given in [20, §2]. See also [5] for a variant of these notions of complexity in the ergodic setting, and for polynomial forms instead of linear ones.

We will need a twisted version of the Generalised von Neumann inequality, in which an additional nilsequence of lower degree is inserted. We shall not need it for general linear forms, so we formulate just the special case we need.

**Lemma 4.2 (Twisted generalised von Neumann theorem).** Let $k \geq 3$, let $f_0, \ldots, f_{k-1} : [N] \to \mathbb{C}$ be bounded in magnitude by 1, let $c_0, \ldots, c_{k-1}$ be distinct integers, and let $F(g(n)\Gamma)$ be a degree $\leq (k - 2)$ nilsequence of
complexity at most $M$. Then

$$|E_{n \in [N], d \in [-N, N]} F(g(d)G) \prod_{i=0}^{k-1} f_i(n + c_i d)| \ll_{k, M, c_0, \ldots, c_{k-1}} \inf_{0 \leq i < k-1} \|f_i\|_{U^{k-1}[N]}.$$ 

Proof. We induct on $k$, starting with the case $k = 3$. The underlying nilmanifold $G/\Gamma$ is then a torus $(\mathbb{R}/\mathbb{Z})^m$ with $m = O_M(1)$, and $g(n) = \theta n + \theta_0$ may be taken to be linear. By a standard Fourier decomposition we may assume that $F(x) = e(\xi \cdot x)$ for some $\xi \in \mathbb{Z}^m$ with $|\xi| = O_M(1)$, in which case we may rewrite the estimate to be proven as

$$|E_{n \in [N], d \in [-N, N]} f_0(n + c_0 d) f'_1(n + c_1 d) f'_2(n + c_2 d)| \ll_{k, M} \inf_{i=0,1,2} \|f_i\|_{U^2[N]},$$

where $f'_1(n) = f_1(n) e(-(c_2 - c_1)^{-1} \xi \cdot \theta n)$ and $f'_2(n) = f_2(n) e((c_2 - c_1)^{-1} \xi \cdot \theta n)$. However it is easy to establish the invariance properties $\|f_1\|_{U^2} = \|f'_1\|_{U^2}$ and $\|f_2\|_{U^2} = \|f'_2\|_{U^2}$, and so the result follows immediately from Theorem 4.1.

Now suppose that $k \geq 4$ and the claim has already been proven for smaller $k$. By permuting indices and then translating $n$, it suffices to show that

$$|E_{n \in [N], d \in [-N, N]} F(g(d)G) \prod_{i=0}^{k-1} f_i(n + c_i d)| \ll_{k, M, c_0, \ldots, c_{k-1}} \|f_{k-1}\|_{U^{k-1}[N]} \tag{4.2}$$

under the assumption that $c_0 = 0$.

Recall from [28] that we define a vertical character to be a continuous homomorphism $\xi : G_{(k-2)}/(G_{(k-2)} \cap \Gamma) \to \mathbb{R}/\mathbb{Z}$. We say that $F$ has vertical frequency $\xi$ if one has $F(g_{k-2} x) = e(\xi(g_{k-2})) F(x)$ for all $x \in G/\Gamma$ and $g_{k-2} \in G_{(k-2)}$. By a standard Fourier decomposition in the vertical direction (e.g. by arguing exactly as in [28] Lemma 3.7) we may assume without loss of generality that $F$ has a vertical frequency $\xi$.

Applying the Cauchy-Schwarz inequality, we can bound the left-hand side of (4.2) by

$$\ll |E_{n \in [N], h, d \in [-N, N]} F(g(d+h)G) \overline{F(g(d)G)} \prod_{i=0}^{k-1} f_i(n + c_i d + c_i h) f_i(n + c_i d)|^{1/2}.$$ 

Because $F$ has a vertical frequency, $F(g(d+h)) \overline{F(g(d)G)}$ is a degree $\leq (k-3)$ nilsequence of complexity $O_M(1)$ (see [28] Proposition 7.2). Applying the induction hypothesis, we may thus bound the above expression by

$$\ll_{M, k, c_0, \ldots, c_{k-1}} \left( |E_{h \in [-N, N]} \|\Delta_{c_i h} f_i\|_{U^{k-2}[N]}^{2k-2} \right)^{1/2}$$

which by Hölder’s inequality can be bounded by

$$\ll_{M, k, c_0, \ldots, c_{k-1}} \left( |E_{h \in [-|c_i| N, |c_i| N]} \|\Delta_{c_i h} f_i\|_{U^{k-2}[N]}^{2k-2} \right)^{1/2k-2}$$

and the claim follows from the recursive definition of the Gowers norms. □
Remark. The above argument is very similar to the short proof presented in [31, Appendix G] that $s$-step nilsequences obstruct uniformity in the $U^{s+1}$-norm (that is, the inverse conjecture GI(s) is an if-and-only-if statement).

5. ON A CONJECTURE OF BERGELSON, HOST, AND KRA

We now apply the arithmetic regularity and counting lemmas to establish Theorem 1.12, the proof of the conjecture of Bergelson, Host and Kra. It will suffice to prove the following claim.

**Theorem 5.1.** Let $k = 1, 2, 3$ or $4$, and suppose that $0 < \alpha < 1$ and $\varepsilon > 0$. Then for any $N \geq 1$ and any subset $A \subseteq [N]$ of density $|A| \geq \alpha N$, one can find a function $\mu : \mathbb{Z} \to \mathbb{R}^+$ such that

$$E_{d \in [-N, N]} \mu(d) = 1 + O(\varepsilon) \quad (5.1)$$

and

$$\sup_{d \in [-N, N]} \mu(d) \ll_{\alpha, \varepsilon} 1 \quad (5.2)$$

such that

$$E_{n \in [N]} 1_A(n) 1_A(n + d) \ldots 1_A(n + (k - 1)d) \mu(d) \geq \alpha^k - O(\varepsilon). \quad (5.3)$$

Indeed, from (5.1), (5.3), we see that we have

$$E_{n \in [N]} 1_A(n + d) \ldots 1_A(n + (k - 1)d) \geq \alpha^k - O(\varepsilon)$$

for all $d$ in a subset $E$ of $[-N, N]$ with $E_{d \in [-N, N]} E(d) \mu(d) \gg_{\alpha, \varepsilon} N$. From (5.2) we conclude that $|E| \gg_{\alpha, \varepsilon} N$, and Theorem 1.12 follows (after shrinking $\varepsilon$ by an absolute constant). Conversely, it is not difficult to deduce Theorem 1.12 from Theorem 5.1.

It remains to establish Theorem 5.1. We may assume that $N$ is large depending on $\alpha, \varepsilon$ as the claim is trivial otherwise (just take $\mu$ to be the Kronecker delta function at 0).

For $k = 1$ one can simply take $\mu \equiv 1$. For $k = 2$, we first observe that

$$E_{n \in [N]} E_{h \in [-\varepsilon N, \varepsilon N]} 1_A(n + h) = \alpha + O(\varepsilon);$$

applying Cauchy-Schwarz we conclude that

$$E_{h, h' \in [-\varepsilon N, \varepsilon N]} E_{n \in [N]} 1_A(n + h) 1_A(n + h') \geq \alpha^2 - O(\varepsilon).$$

The claim then follows, with $\mu$ being the probability density function of $h - h'$ as $h, h'$ range uniformly in $[-\varepsilon N, \varepsilon N]$.

Now we turn to the cases $k = 3, 4$. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a sufficiently rapidly growing function depending on $\alpha, \varepsilon$ in a manner to be specified later. We apply Theorem 1.12 with $s := k - 2$ to obtain a quantity $M = O_{\varepsilon, F}(1)$ and a decomposition

$$1_A(n) = f_{\text{nil}}(n) + f_{\text{sm}}(n) + f_{\text{unf}}(n) \quad (5.4)$$

such that

(i) $f_{\text{nil}}(n)$ is a $(F(M), N)$-irrational degree $\leq k - 2$ virtual nilsequence of complexity at most $M$ and scale $N$;
(ii) $f_{\text{sml}}$ has an $L^2[N]$ norm of at most $\varepsilon/100$;
(iii) $f_{\text{unf}}$ has an $U^{k-1}[N]$ norm of at most $1/F(M)$;
(iv) $f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}}$ are all bounded in magnitude by 1; and
(v) $f_{\text{nil}}$ and $f_{\text{nil}} + f_{\text{sml}}$ are non-negative.

It is clear that $|E_n \in [N] f_{\text{sml}}(n)| = O(\varepsilon)$, and furthermore, by Theorem 4.1 (setting all but one of the functions equal to 1) we also have $|E_n \in [N] f_{\text{unf}}(n)| = O(\varepsilon)$ if $F$ grows rapidly enough. Therefore

$$E_n \in [N] f_{\text{nil}}(n) \geq \alpha - O(\varepsilon). \quad (5.5)$$

The heart of the matter is the following proposition.

**Proposition 5.2** (Bergelson-Host-Kra for $f_{\text{nil}}$). Let $k = 3, 4$. Then there exists a non-negative $(k-2)$-step nilsequence $\mu : \mathbb{Z} \to \mathbb{R}^+$ of complexity $O_{\alpha,\varepsilon,M}(1)$ obeying the normalisation

$$E_{d \in [N]} \mu(d) = 1 + O(\varepsilon) \quad (5.6)$$

and such that

$$E_{n, d \in [N]} f_{\text{nil}}(n) f_{\text{nil}}(n + d) \ldots f_{\text{nil}}(n + (k-1)d) \mu(d) \geq \alpha^k - O(\varepsilon). \quad (5.7)$$

**Deduction of Theorem 5.1 from Proposition 5.2.** Using (5.4), one can expand the left-hand side of (5.3) into $3^k$ terms, one of which is (5.7). As for the other terms, any term involving at least one copy of $f_{\text{unf}}$ is of size $O_{\alpha,\varepsilon,M}(1/F(M))$ by Lemma 4.2 and the $U^{k-1}$ norm bound on $f_{\text{unf}}$. Finally, consider a term that involves at least one copy of $f_{\text{sml}}$. Suppose first that we have a term that involves $f_{\text{sml}}(n)$. Then after performing the average in $d$ using (5.6), we see that this term is $O(E_n \in [N] |f_{\text{sml}}(n)|)$, which is $O(\varepsilon)$ by the $L^2[N]$ bound on $f_{\text{sml}}$ and the Cauchy-Schwarz inequality. Similarly for any term that involves $f_{\text{sml}}(n + id)$, after making a change of variables $(n', d) := (n + id, d)$. Putting all this together we obtain the result. □

It remains, of course, to establish Proposition 5.2. We may assume that $N$ is sufficiently large depending on $\alpha, \varepsilon, M$, as the claim is trivial otherwise by taking $\mu$ to be a delta function.

We first establish the proposition in the easier of the two cases, namely the case $k = 3$. This was previously considered in [23]. In this case it is actually easier to work with the (easier) weak regularity lemma, Proposition 2.7, in which the degree 1 polynomial sequence $g(n)$ is not required to be irrational. Note that we have not made any use of irrationality so far, though we shall do so later when discussing the case $k = 4$. We may identify $G/\Gamma$ with $(\mathbb{R}/\mathbb{Z})^m$ for some $m = O_M(1)$ and, by modulating $F$ if necessary, we may suppose that $g(n) = \theta n$ is linear with no constant term, where $\theta \in \mathbb{R}^m$. Then

$$f_{\text{nil}}(n) = F(n\theta),$$

where $F : (\mathbb{R}/\mathbb{Z})^m \to \mathbb{C}$ has Lipschitz norm $O_M(1)$.
Let \( \varepsilon' > 0 \) be a small number depending on \( \varepsilon \) and \( M \) to be chosen later, and let \( B_1, B_2 \subseteq [-N, N] \) denote the two Bohr sets

\[
B_1 := \{ d \in [-\varepsilon'N, \varepsilon'N] : \text{dist}_{(\mathbb{R}/\mathbb{Z})^m}(\theta d, 0) \leq \varepsilon' \}
\]

and

\[
B_2 := \{ d \in [-\varepsilon'N, \varepsilon'N] : \text{dist}_{(\mathbb{R}/\mathbb{Z})^m}(\theta d, 0) \leq \varepsilon'/2 \}.
\]

By the usual Dirichlet pigeonhole argument we see that \(|B_2| \gg \varepsilon' M N \). Also, from the Lipschitz nature of \( F \), we see that

\[
f_{\text{nil}}(n + d) = f_{\text{nil}}(n) + O_M(\varepsilon')
\]

whenever \( d \in B_1 \) and \( n \in [-1 - \varepsilon', 1 - \varepsilon']N \). As a consequence, it follows that

\[
\mathbb{E}_{n \in [N]} f_{\text{nil}}(n) f_{\text{nil}}(n + 2d) = \mathbb{E}_{n \in [N]} f_{\text{nil}}(n)^3 + O_M(\varepsilon')
\]

for such \( d \). However from (5.5) and Hölder’s inequality one has

\[
\mathbb{E}_{n \in [N]} f_{\text{nil}}(n)^3 \geq \alpha^3 - O(\varepsilon).
\]

Proposition 5.2 (in the case \( k = 3 \)) now follows by taking \( \mu(d) = c \psi(\theta d) \), where \( \psi : (\mathbb{R}/\mathbb{Z})^m \to [0, 1] \) is an \( O_M, \varepsilon'(1) \)-Lipschitz function which is 1 on \( B_2 \) and 0 outside \( B_1 \), \( c = O_M, \varepsilon'(1) \) is a suitable normalisation constant, and by taking \( \varepsilon' \) to be suitably small.

We now turn to the \( k = 4 \) case of Proposition 5.2. For simplicity let us first consider the model case when \( f_{\text{nil}} \) is a genuine nilsequence and not just a virtual nilsequence, that is to say

\[
f_{\text{nil}}(n) = F(g(n) \Gamma)
\]

where \( (G/\Gamma, G_*) \) is a degree \( \leq 2 \) filtered nilmanifold of complexity \( O_M(1) \), and \( g \in \text{poly}(\mathbb{Z}, G_*) \) is \( (\mathcal{F}(M), N) \)-irrational. By Taylor expansion (see Appendix A), we have

\[
g(n) = g_0 g_1^n g_2^n
\]

for some \( g_0, g_1 \in G \) and \( g_2 \in G_2 \). The \( (\mathcal{F}(M), N) \)-irrationality of \( g \) ensures certain irrationality properties on \( g_1 \) and \( g_2 \), though we will not need these properties explicitly here, as we will only be using them through the counting lemma (Theorem 11.11), which we shall be using as a black box.

Let \( \pi : G \to T_1 \) be the projection homomorphism to the torus \( T_1 := G/(G(2) \Gamma) \). Then

\[
\pi(g(n)) = \pi(g_0 \pi(g_1)^n).
\]

Let \( \varepsilon' > 0 \) be a small quantity depending on \( \varepsilon, M \) to be chosen later. We set

\[
\mu(d) := c \chi_{[-\varepsilon', \varepsilon']N}(d) \phi(\pi(g_1)^d),
\]

where, much as in the analysis of the case \( k = 3 \), \( \phi : T_1 \to \mathbb{R}^+ \) is a smooth non-negative cutoff to the ball of radius \( \varepsilon' \) centered at the origin that is

\textsuperscript{13}Note this is not quite the same thing as the horizontal torus, which is so important in [23], which is \((G/\Gamma)_{\text{ab}} := G/[G, G]/\Gamma\).
not identically zero, and $c$ is a normalisation constant to be chosen shortly. From Theorem 1.11 one has

$$\mathbb{E}_{d \in [-\varepsilon', N, \varepsilon']} \phi(g_1)^d = \int_{T_1} \phi + o_{\mathcal{F}(M) \rightarrow \infty; \varepsilon', M}(1) + o_{N \rightarrow \infty; \varepsilon', M}(1).$$

Thus if we set

$$c := \frac{1}{\int_{T_1} \phi} = O_{\varepsilon', M}(1) \quad (5.9)$$

then we have the normalisation (5.6), if $\mathcal{F}$ is sufficiently rapid, depending on the way in which $\varepsilon'$ depends on $\varepsilon, M$, and $N$ is sufficiently large depending on $\varepsilon, \varepsilon', M$. From the bound on $c$ we see that $\mu$ is a degree $\leq 1$ (and hence also degree $\leq 2$) nilsequence of complexity $O_{\varepsilon', M}(1)$.

We now apply the counting lemma, Theorem 1.11, to conclude that

$$\mathbb{E}_{n, d \in [N]} f_{\text{nil}}(n) f_{\text{nil}}(n + d) f_{\text{nil}}(n + 2d) f_{\text{nil}}(n + 3d) \mu(d)$$

$$= \int_{G^\Psi / \Gamma^\Psi} \tilde{F} + o_{\mathcal{F}(M) \rightarrow \infty; \varepsilon', M}(1) + o_{N \rightarrow \infty; \varepsilon', M}(1) \quad (5.10)$$

where $G^\Psi \subseteq G^4$ is the Liebman group associated to the collection $\Psi = (\psi_0, \psi_1, \psi_2, \psi_3) : \mathbb{Z}^2 \rightarrow \mathbb{Z}^4$ of linear forms $\psi_i(n) := n_1 + in_2, i = 0, 1, 2, 3$, that is to say the Hall-Petresco group $H^4(G)$, and $\tilde{F} : G^\Psi \rightarrow \mathbb{C}$ is the function

$$\tilde{F}(x_0, x_1, x_2, x_3) := c\phi(\pi(x_1)\pi(x_0)^{-1}) F(x_0) F(x_1) F(x_2) F(x_3)$$

(here we use the identity $\pi(g(n + d))^{-1} \pi(g(n)) = \pi(g_1)^d$, immediately verified from the Taylor expansion).

We now do some calculations in the Hall-Petresco group very similar to those in [3]. We saw in [3] that

$$G^\Psi = \{(g_0, g_0 g_1, g_0 g_1^2 g_2, g_0 g_1 g_2^3) : g_0, g_1 \in G, g_2, g_3 \in G(2)\}$$

(note, of course, that $G(3) = \text{id}$ in the case we are considering). For our calculations it is convenient to use the following obviously equivalent representation:

$$G^\Psi = \{(g_0 g_2, 0, g_0 g_1 g_2, g_0 g_1 g_2^2, g_0 g_1 g_2^3) : g_0, g_1 \in G; g_2, g_3 \in G(2); g_2, g_3, g_0 g_2 g_2^3 g_2^3 g_2^3 g_2^3 = \text{id}\}.$$  

Here we have taken note of the fact that

$$\Psi^{[2]} = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 - 3x_1 + 3x_3 - x_3 = 0\}.$$  

This last equation is quite special in that it exhibits a certain “positivity”, as we shall see later; this is key to our argument. The lattice $\Gamma^\Psi$ can be similarly described by requiring $g_0, g_1, g_2, \ldots, g_2, g_3$ to also lie in $\Gamma$. As a consequence of this, an arbitrary point of the nilmanifold $G^\Psi / \Gamma^\Psi$ can be parameterised uniquely as

$$(g_0 g_2, 0, g_0 g_1 g_2, g_0 g_1 g_2^2, g_0 g_1 g_2^3) \Gamma^\Psi$$  

(5.11)
where \( g_0, g_1 \) lie in a fundamental domain \( \Sigma_1 \subset G \) of the horizontal torus \( T_1 \) (i.e. a smooth manifold with boundary on which \( \pi \) is a bijection from \( \Sigma_1 \) to \( T_1 \)), and \( g_{2,0}, \ldots, g_{2,3} \) lie in a fundamental domain \( \Sigma_2 \subset \Gamma(2) \) of the vertical torus \( T_2 := G(2)/\Gamma(2) \) subject to the constraint \( g_{2,0}g_{2,1}^3g_{2,2}g_{2,3}^{-1} \in \Gamma(2) \). For such a point \((5.11)\), the function \( \tilde{F} \) takes the value

\[
c\phi(\pi(g_1)) \prod_{j=0}^{3} F(g_0g_1^jg_{2,j}\Gamma).
\]

On the support of \( \phi \), \( g_1 \) is a distance \( O_M(\varepsilon') \) from the identity (if the fundamental domain \( \Sigma_1 \) was chosen in a suitably smooth fashion), and so by the Lipschitz nature of \( F \) and the boundedness of \( g_0 \) we have

\[
F(g_0g_1^jg_{2,j}) = F(g_0g_{2,j}\Gamma) + O_M(\varepsilon').
\]

As a consequence, the integral \( \int_{G^\psi/\Gamma^\psi} \tilde{F} \) can be expressed as

\[
c \int_{g_0 \in \Sigma_1} \int_{g_1 \in \Sigma_1} \phi(\pi(g_1)) \left( \int_{g_{2,0}, \ldots, g_{2,3} \in T_2} \prod_{j=0}^{3} F(g_0g_{2,j}\Gamma) + O_M(\varepsilon') \right) \tag{5.12}
\]

where all integrals are with respect to Haar measure.

Let \( \xi \in T_2 \) be a vertical character, i.e. a continuous homomorphism from \( T_2 \) to \( \mathbb{R}/\mathbb{Z} \). For any \( x \in G/\Gamma \), we can define the vertical Fourier transform \( \hat{F}(x, \xi) \) to be the quantity

\[
\hat{F}(x, \xi) := \int_{g_2 \in T_2} e(-\xi(g_2))F(g_2x).
\]

From the Fourier inversion formula we have

\[
\int_{g_{2,0}, \ldots, g_{2,3} \in T_2} \prod_{j=0}^{3} F(g_0g_{2,j}\Gamma) = \sum_{\xi \in T_2} |\hat{F}(g_0, \xi)|^2|\hat{F}(g_0, 3\xi)|^2.
\]

In particular, we have\(^{14}\)

\[
\int_{g_{2,0}, \ldots, g_{2,3} \in T_2} \prod_{j=0}^{3} F(g_0g_{2,j}\Gamma) \geq |\hat{F}(g_0, 0)|^4.
\]

Inserting this bound and \((5.9)\) into \((5.12)\), we conclude that

\[
\int_{G^\psi/\Gamma^\psi} \tilde{F} \geq \int_{g_0 \in \Sigma_1} |\hat{F}(g_0\Gamma, 0)|^4 - O_M(\varepsilon') - o_{F(M) \to \infty, \varepsilon', M}(1).
\]

From Fubini’s theorem we have

\[
\int_{g_0 \in \Sigma_1} \hat{F}(g_0\Gamma, 0) = \int_{G/\Gamma} \hat{F}.
\]

---

\(^{14}\)This is the “positivity” alluded to earlier. The argument is essentially that used in \([4]\) and it is special to the \( k = 4 \) case, which is of course consistent with the failure of Theorem 5.1 to extend to \( k \geq 5 \).
and from Theorem \[1.11\] \[5.8\] and \[5.5\] we have
\[\int_{G/\Gamma} F = \alpha + O(\varepsilon) + o_{F(M) \to \infty; \varepsilon', M}(1) + o_{N \to \infty; \varepsilon', M}(1).\]

Applying Hölder’s inequality, we conclude that
\[\int_{G^\Psi/\Gamma^\Psi} \tilde{F} \geq \alpha^4 - O(\varepsilon) - O_{M(\varepsilon')} - o_{F(M) \to \infty; \varepsilon', M}(1) - o_{N \to \infty; \varepsilon', M}(1),\]

and so \[\text{(5.7)}\] follows from \[\text{(5.10)}\], if \(\varepsilon'\) is sufficiently small depending on \(\varepsilon, M\), \(F\) is sufficiently rapid depending on \(\varepsilon\), and \(N\) is sufficiently large depending on \(\varepsilon', M\).

This concludes the proof of the \(k = 4\) case of Proposition \[5.2\] in the special case when \(f_{\text{nil}}(n) = F(g(n)\Gamma)\) with \(g\) irrational. Unfortunately Theorem \[1.2\] requires us to deal with the somewhat more general setting of virtual nilsequences, in which there is dependence on \(n \mod q\) or \(n/N\). The extra details required are fairly routine but notationally irritating. Let us now suppose, then, that
\[f_{\text{nil}}(n) = F(g(n)\Gamma, n \mod q, n/N).\]

We let \(\varepsilon'\) be as before, but modify \(\mu\) to now be given by
\[\mu(d) := q1_{[\varepsilon'N, \varepsilon'N]}(d)\phi(\pi(g_1)^d),\]
with \(c\) still chosen by \[5.9\]. As before, one can use Theorem \[1.11\] to establish \[5.6\].

Now consider the left-hand side of the expression \[5.7\] we are to bound in Proposition \[5.2\] that is to say
\[\mathbb{E}_{n, d \in [N]} f_{\text{nil}}(n)f_{\text{nil}}(n + d)f_{\text{nil}}(n + 2d)f_{\text{nil}}(n + 3d)\mu(d).\]

Splitting into residue classes modulo \(q\), we can express this as
\[c\mathbb{E}_{r \in [q]} \mathbb{E}_{n \in [N/q]} \mathbb{E}_{d \in [-\varepsilon'N/q, \varepsilon'N/q]} 3 \prod_{i=0}^{3} F(g(qn + qid + r)\Gamma, r, q(n + ir)/N)\phi(\pi(g_1)^{qd}) + O_{N \to \infty; \varepsilon', M}(1).\]

We partition \([N/q]\) into intervals \(P\) of length \(|\varepsilon'N|\) (plus a remainder of cardinality \(O(\varepsilon'N))\). We can then rewrite the above expression as
\[c\mathbb{E}_{P} \mathbb{E}_{r \in [q]} \mathbb{E}_{n \in P} \mathbb{E}_{d \in [-\varepsilon'N/q, \varepsilon'N/q]} 3 \prod_{i=0}^{3} F(g(qn + qid + r)\Gamma, r, q(n + ir)/N)\phi(\pi(g_1)^{qd}) + O(\varepsilon') + O_{N \to \infty; \varepsilon', M}(1).\]

For each such expression, we can use the Lipschitz nature of \(F\) to replace \(q(n + ir)/N\) by \(qn_P/N\), where \(n_P\) is an arbitrary element of \(P\), losing only
an error of $O_M(\varepsilon')$. The above expression thus becomes
\[
c\mathbb{E}_{\Gamma} \mathbb{E}_{r \in [q]} \mathbb{E}_{n \in P} \mathbb{E}_{d \in [-\varepsilon', M(1) + \varepsilon]} \prod_{i=0}^{3} F(g(qn + qid + r)\Gamma, r, qn_P/N) \phi(\pi(g_1)^{qd}) + O_M(\varepsilon') + O_{N \to \infty; \varepsilon', M(1)}.
\]
Because the orbit $n \mapsto g(n)\Gamma$ is $(\mathcal{F}(M), N)$-irrational, we see from Lemma \ref{lemma:irrational} that shifted translate $n \mapsto g(q(n + np) + r)\Gamma$ is $(\gg_M \mathcal{F}(M), N)$-irrational. We may then argue as in the previous case and bound the above average below by
\[
\geq \mathbb{E}_{\Gamma} \mathbb{E}_{r \in [q]} \int_{G/\Gamma} F(\cdot, r, qn_P/N)^4 - O(\varepsilon) - O_M(\varepsilon') - o_{\mathcal{F}(M) \to \infty; \varepsilon', M(1)} - o_{N \to \infty; \varepsilon', M(1)}.
\]
Using Theorem \ref{theorem:szemeredi} again, we have
\[
\mathbb{E}_{n \in P} \phi_{\text{full}}(qn + r) = \int_{G/\Gamma} F(\cdot, r, qn_P/N) + o_{\mathcal{F}(M) \to \infty; \varepsilon', M(1)} + o_{N \to \infty; \varepsilon', M(1)}
\]
and so \eqref{eq:main} is at least
\[
\geq \mathbb{E}_{\Gamma} \mathbb{E}_{r \in [q]} \mathbb{E}_{n \in P} \phi_{\text{full}}(qn + r)^4 - O(\varepsilon) - O_M(\varepsilon') - o_{\mathcal{F}(M) \to \infty; \varepsilon', M(1)} - o_{N \to \infty; \varepsilon', M(1)}.
\]
Now from \eqref{eq:average} and double-counting one has
\[
\mathbb{E}_{\Gamma} \mathbb{E}_{r \in [q]} \mathbb{E}_{n \in P} \phi_{\text{full}}(qn + r) = \alpha + O(\varepsilon)
\]
and so, from Hölder’s inequality, we deduce that \eqref{eq:main} is
\[
\geq \alpha^4 - O(\varepsilon) - O_M(\varepsilon') - o_{\mathcal{F}(M) \to \infty; \varepsilon', M(1)} - o_{N \to \infty; \varepsilon', M(1)}.
\]
Proposition \ref{prop:szemeredi} now follows by once again choosing $\varepsilon'$ small enough depending on $\varepsilon, M$, and choosing $\mathcal{F}$ rapid enough depending on $\varepsilon$, and $N$ sufficiently large depending on $\varepsilon, \varepsilon', M$.

6. Proof of Szemerédi’s theorem

We turn now to the proof of Szemerédi’s theorem. We deemed this result too famous to state in the introduction but, for the sake of fixing notation, we recall it here now. It is most natural to establish what might be called the “functional” form of the theorem which is \textit{a priori} a stronger statement (though quite easily shown to be equivalent to the standard formulation by an argument of Varnavides \cite{varnavides}).

**Theorem 6.1** (Szemerédi’s theorem). Let $0 < \alpha \leq 1$, let $k \geq 3$, and let $N \geq 1$. If $f : [N] \to [0, 1]$ is a function with $\mathbb{E}_{n \in [N]} f(n) \geq \alpha$ then
\[
\Lambda_k(f, f, \ldots, f) \gg_k \alpha 1,
\]
where
\[
\Lambda_k(f_1, \ldots, f_k) := \mathbb{E}_{n \in [N]} \prod_{d \in [-N, N]} f_1(n) f_2(n + d) \ldots f_k(n + (k - 1)d).
\]
is the multilinear operator counting arithmetic progressions.

We now prove this theorem. We fix $k, \alpha$, and allow implied constants to depend on these quantities.

As usual, we begin by applying the regularity lemma, Theorem 1.2. In view of the generalised von Neumann theorem, Theorem 4.1, it is natural to apply this theorem with $s = k - 2$ (which, as remarked in §4, is the Cauchy-Schwarz complexity $s = s(\Psi)$ of the system $\Psi$ of linear forms $n_1, n_1 + n_2, \ldots, n_1 + (k-1)n_2$). If we do so, with a small parameter $\varepsilon > 0$ depending on $\alpha, k$ to be chosen later, and a growth function $F$ depending on $\alpha, k, \varepsilon$ to be specified later, we obtain a decomposition

$$f(n) = f_{\text{nil}}(n) + f_{\text{sml}}(n) + f_{\text{unf}}(n)$$

(6.1)

where

(i) $f_{\text{nil}}$ is a $(F(M), N)$-irrational degree $\leq k - 2$ virtual nilsequence of complexity $\leq M$ and scale $N$;
(ii) $f_{\text{sml}}$ has an $L^2[N]$ norm of at most $\varepsilon$;
(iii) $f_{\text{unf}}$ has an $U^{k-1}[N]$ norm of at most $1/F(M)$;
(iv) $f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}}$ are all bounded in magnitude by 1; and
(v) $f_{\text{nil}}$ and $f_{\text{nil}} + f_{\text{sml}}$ are non-negative.

As we shall soon see, the contribution of $f_{\text{unf}}$ can be quickly discarded using the generalised von Neumann theorem. If one could also easily discard the contribution of the small term $f_{\text{sml}}$, then matters would simply reduce to verifying that the contribution of $f_{\text{nil}}$ is bounded away from zero, which would be an easy consequence of the counting lemma. Unfortunately the small term $f_{\text{sml}}$ is only moderately small (of size $O(\varepsilon)$) rather than incredibly small (e.g. of size $O(1/F(M))$), and so one has to take a certain amount of care in dealing with this term, which makes the analysis significantly more delicate.

We turn to the details. Much as the key to proving Theorem 1.12 was to establish Proposition 5.2, the key to establishing Szemerédi’s theorem is the following proposition.

**Proposition 6.2** (Szemerédi for $f_{\text{nil}}$). Let $f_{\text{nil}}$ be as above, and let $\varepsilon > 0$. Then there exists a function $\mu : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^+$ supported on the set

$$\{(n,d) \in \mathbb{Z} \times \mathbb{Z} : d \in [-\varepsilon N, \varepsilon N]; n + id \in [N] \text{ for all } i = 0, \ldots, k-1\}$$

(6.2)

with

$$E_{n \in [N]; d \in [-\varepsilon N, \varepsilon N]} \mu(n, d) = 1 + O(\varepsilon)$$

(6.3)

and with $\mu$ bounded in magnitude by $O_{M, \varepsilon}(1)$, such that

$$f_{\text{nil}}(n + id) = f_{\text{nil}}(n) + O(\varepsilon)$$

(6.4)

\footnote{In the language of ergodic theory, the problem here is that the characteristic factor is not necessarily a nilsystem, but may merely be a pro-nilystem - an inverse limit of nilsystems.}
whenever $0 \leq i \leq k - 1$ and $\mu(n,d) \neq 0$, and such that one has the equidistribution property
\begin{equation}
\mathbb{E}_{n \in [N]} |\mathbb{E}_{d \in [-\varepsilon N,\varepsilon N]} \mu(n - id, d)|^2 = 1 + O(\varepsilon) \tag{6.5}
\end{equation}
for all $0 \leq i \leq k - 1$.

The crucial feature of Proposition 6.2 for us is that, with the exception of the uniform bound on $\mu$, the error terms here decay as $\varepsilon \to 0$, even if the complexity bound $M$ on $f_{\text{nil}}$ is extremely large compared to $1/\varepsilon$.

The reader may benefit from a few words about the role of the function $\mu$. Supposing that $f_{\text{nil}}(n) = F(g(n)\Gamma)$ is a genuine nilsequence, this function acts like a kind of “weight” on progressions $(n, n + d, \ldots, n + (k - 1)d)$ which are “almost diagonal” in the sense that $g(n) \Gamma \approx \cdots \approx g(n + (k - 1)d)\Gamma$ in $G/\Gamma$. The condition (6.5) reflects the fact that the weighted number of almost diagonal progressions whose $i$th point is $n$ is roughly independent of $n$. This “non-concentration” of almost diagonal progressions ultimately means that the error $f_{\text{sm}}$ cannot destroy too many of these progressions, a fact that is crucial for our argument.

Let us assume Proposition 6.2 for now and see how it implies Theorem 6.1. We use (6.1) to expand out the form $\Lambda_k(f, \ldots, f)$ into $3^k$ terms. By Theorem 4.2 any term that involves $f_{\text{sunf}}$ will be of size $O(1/\mathcal{F}(M))$, thus
\begin{equation}
\Lambda_k(f, \ldots, f) = \Lambda_k(f_{\text{nil}} + f_{\text{sm}}, \ldots, f_{\text{nil}} + f_{\text{sm}}) + O(1/\mathcal{F}(M)). \tag{6.6}
\end{equation}
Next, we use the weight $\mu$ arising from Proposition 6.2 and the non-negativity of $f_{\text{nil}} + f_{\text{sm}}$ guaranteed by Theorem 1.2 to write
\begin{align*}
\Lambda_k(f_{\text{nil}} + f_{\text{sm}}, \ldots, f_{\text{nil}} + f_{\text{sm}}) \\
&\gg M, \varepsilon \mathbb{E}_{n \in [N]} |d \in [-\varepsilon N, \varepsilon N]| (f_{\text{nil}} + f_{\text{sm}})(n) \cdots (f_{\text{nil}} + f_{\text{sm}})(n + (k - 1)d) \mu(n,d).
\end{align*}

We then expand this latter average into the sum of $2^k$ terms. The main term is
\begin{equation}
\mathbb{E}_{n \in [N]} |d \in [-\varepsilon N, \varepsilon N]| f_{\text{nil}}(n) \cdots f_{\text{nil}}(n + (k - 1)d) \mu(n,d), \tag{6.7}
\end{equation}
and the other terms are error terms, involving at least one factor of $f_{\text{sm}}$.

Consider one of the error terms, involving the factor $f_{\text{sm}}(n + id)$ (say) for some $0 \leq i \leq k - 1$. We can bound the contribution of this term by
\begin{equation}
\mathbb{E}_{n \in [N]} |d \in [-\varepsilon N, \varepsilon N]| f_{\text{sm}}(n + id) \mu(n,d),
\end{equation}
which by a change of variables $n \mapsto n - id$ we can write as
\begin{equation}
\mathbb{E}_{n \in [N]} |f_{\text{sm}}(n)| \mathbb{E}_{d \in [-\varepsilon N, \varepsilon N]} \mu(n - id, d).
\end{equation}
By Cauchy-Schwarz, (6.5), and the $L^2[N]$ bound on $f_{\text{sm}}$, this is $O(\varepsilon)$.

Finally, we look at the main term (6.7). Using (6.4) we can approximate
\begin{equation}
f_{\text{nil}}(n) \cdots f_{\text{nil}}(n + (k - 1)d) = f_{\text{nil}}(n)^k + O(\varepsilon)
\end{equation}
and so (using (6.3)) we can write (6.7) as
\begin{equation}
\mathbb{E}_{n \in [N]} f_{\text{nil}}(n)^k \mathbb{E}_{d \in [-\varepsilon N, \varepsilon N]} \mu(n,d) + O(\varepsilon).
\end{equation}
Now, from (6.3) one has
\[ E_{n \in [N]} E_{d \in [-\varepsilon N, \varepsilon N]} \mu(n, d) = 1 + O(\varepsilon) \]
and hence by (6.5)
\[ E_{n \in [N]} |E_{d \in [-\varepsilon N, \varepsilon N]} \mu(n, d) - 1|^2 = O(\varepsilon). \]
In particular, by Chebyshev’s inequality, we have
\[ E_{d \in [-\varepsilon N, \varepsilon N]} \mu(n, d) = 1 + O(\varepsilon^{1/3}) \]
for all \( n \in E \), where \( E \subseteq [N] \) has cardinality \( |E| \geq (1 - O(\varepsilon^{1/3}))N \). Thus, for \( \varepsilon \) small enough, we can bound (6.7) from below by
\[ \gg E_{n \in [N]} 1_E(n) f_{\text{nil}}(n)^k - O(\varepsilon^{1/3}). \]
Now from hypothesis we have \( E_{n \in [N]} f(n) \gg 1 \). From Cauchy-Schwarz we have
\[ E_{n \in [N]} f_{\text{sml}}(n) = O(\varepsilon), \]
and from Theorem 4.1 we also have
\[ E_{n \in [N]} f_{\text{unf}}(n) = O(\varepsilon) \]
if \( F \) is rapid enough. Thus if \( \varepsilon \) is small enough we have \( E_{n \in [N]} f_{\text{nil}}(n) \gg 1 \), which implies that \( E_{n \in [N]} 1_E(n) f_{\text{nil}}(n) \gg 1 \), and hence by Hölder’s inequality that \( E_{n \in [N]} 1_E(n) f_{\text{nil}}^k(n) \gg 1 \). Putting all this together, we conclude that (6.7) is \( \gg 1 \) if \( \varepsilon \) is small enough, and thus
\[ \Lambda_k(f_{\text{nil}} + f_{\text{sml}}, \ldots, f_{\text{nil}} + f_{\text{sml}}) \gg M, \varepsilon \]
Inserting this bound into (6.6) we obtain the claim, completing the proof of Szemerédi’s theorem, if \( F \) is chosen sufficiently rapid.

Proof of Proposition 6.2. Let us first establish this in the easy case \( k = 3 \). In this case, \( f_{\text{nil}} \) is essentially quasiperiodic, which will allow us to take \( \mu(n, d) \) to be of the form
\[ \mu(n, d) = 1_{[2\varepsilon N, (1-2\varepsilon)N]}(n) \mu(d) \]
with \( \mu(d) \) normalised by requiring
\[ E_{d \in [-\varepsilon N, \varepsilon N]} \mu(d) = 1 + O(\varepsilon). \]
It is then easy to verify that both (6.3) and (6.5) follow from this normalisation. To establish the remaining claims in Proposition 6.2 we use the degree \( \leq 1 \) nature of the orbit \( n \mapsto g(n) \Gamma \) as in Section 5 to write \( f_{\text{nil}} \) as
\[ f_{\text{nil}}(n) = F(n\theta) \]
for some \( \theta \in (\mathbb{R}/\mathbb{Z})^D \) with \( D = O_M(1) \) and some \( F : (\mathbb{R}/\mathbb{Z})^D \to \mathbb{C} \) of Lipschitz constant \( O_M(1) \). If one then sets \( \mu \) to equal
\[ \mu(d) := \frac{[\varepsilon N, \varepsilon N]}{|B|} 1_B(d) \]
where $B$ is the Bohr set
\[ \{ d \in [-\varepsilon N, \varepsilon N] : d_{(R/Z)^0} (d\theta, 0) \leq \delta \} \]
and $\delta > 0$ is sufficiently small depending on $\varepsilon, M$, one easily verifies all the required claims.

We now turn to the case $k > 3$, which is harder because $f_{\text{nil}}$ is no longer quasiperiodic, and so $\mu(n, d)$ will have to depend more heavily on $n$ and not just on $d$. By arguing as in the previous section we can normalise $g(0)$ to equal id. We may also assume $N$ is sufficiently large depending on $\varepsilon, M$, since otherwise we may simply take $\mu(n, d) = 1_{[N]}(n)\delta_0(d)$ where $\delta_0$ is the Kronecker delta function at 0. We may of course also assume that $\varepsilon < 1$.

We take an $O_M(1)$-rational Mal’cev basis $X_1, \ldots, X_{\dim(G)}$ for the Lie algebra $g = \log G$ adapted to the filtration $G_\bullet$ as described in [28, Appendix A]. For any radius $r > 0$, we define the “ball” $B_r$ in $G$ to be the set of all group elements of the form
\[ \exp(\sum_{j=1}^{\dim(G)} t_j X_j) \] (6.8)
where the $t_j$ are real numbers with $t_j \leq r^{s+1-i}$ whenever $1 \leq i \leq s$ and $j \leq \dim(G) - \dim(G_{(i)})$. Thus, when $r$ is small, $B_r$ is quite “narrow” (of diameter comparable to $r^s$) when projected down to $G/G_{(2)}$, but is relatively large when restricted to the top order component $G_{(s)}$ (of diameter comparable to $r$). This type of eccentricity is necessary in order to make $B_r$ approximately “normal” with respect to conjugations. Indeed, we have

**Lemma 6.3 (Approximate normality).** Let $A, \delta > 0$, and let $g \in G$ be such that $d_G(g, \text{id}) \leq A$. Then we have the containments
\[ B_{(1-\delta)r} \subseteq gB_rg^{-1} \subseteq B_{(1+\delta)r}. \] (6.9)
whenever $r > 0$ is sufficiently small depending on $A, \delta, M$.

**Proof.** We prove the second inclusion only, as the first is similar (and can also be deduced from the second). The conjugation action $h \mapsto ghg^{-1}$ on $G$ induces a Lie algebra automorphism $\exp(\text{ad}(\log g)) : g \to g$. If we conjugate the group element (6.8) by $g$, we thus obtain
\[ \exp(\sum_{j=1}^{\dim(G)} t_j \exp(\text{ad}(\log g))(X_j)). \]
But if $1 \leq i \leq s$ and $j \leq \dim(G) - \dim(G_{(i)})$, we see from the Baker-Campbell-Hausdorff formula (C.2) that
\[ \exp(\text{ad}(\log g))(X_j) = X_j + \sum_{j' = \dim(G) - \dim(G_{(i)})+1}^{\dim(G)} c_{j,j'} X_{j'} \]
for some coefficients \( c_{j,j'} \) of size \( O_{A,M}(r^{s+1-i}) \). Collecting all the coefficients together, we obtain the claim for \( r \) small enough. \( \square \)

Let \( 0 < \delta < 1/10 \) be a small quantity (depending on \( \varepsilon, M \)), let \( R \) be a large quantity depending on the same parameters, and let \( r_0 > 0 \) be an even smaller\(^{16} \) quantity than \( \delta \) (depending on \( \varepsilon, M, \delta, R \)) to be chosen later. For each \( r \) with \( 0 < r < r_0 \) take a Lipschitz function \( \phi_r : G \to \mathbb{R}^+ \) of Lipschitz norm \( O_{M,r,\delta}(1) \) which is supported on \( B_r \) and equals one on \( B(1-\delta)r \), and choose these functions so that \( \phi_r \leq \phi'_r \) pointwise whenever \( 0 < r < r' < r_0 \).

For each such \( r \), let \( \Phi_r : G/\Gamma \times G/\Gamma \to \mathbb{R}^+ \) be the induced function

\[
\Phi_r(x, x') := \sum_{g \in G : gx = x'} \phi_r(g).
\]

This function \( \Phi_r \) is supported near the diagonal of \( G/\Gamma \times G/\Gamma \); indeed, \( \Phi_r(x, x') \) is only non-zero when \( x' \in B_r x \), and furthermore if \( x' \in B(1-\delta)r \) then \( \Phi_r(x, x') = 1 \). If \( r_0 \) is chosen sufficiently small depending on \( M, \delta \), we conclude from Lemma 6.3 that we have the approximate shift-invariance

\[
\Phi_{(1-3\delta)r}(x, x') \leq \Phi_r(gx, gx') \leq \Phi_{(1+3\delta)r}(x, x') \quad (6.10)
\]

whenever \( x, x' \in G/\Gamma \) and \( g \in G \) is such that \( d_G(g, id) \leq R \) (say).

We now define our cutoff function \( \mu = \mu_r \) by

\[
\mu_r(n, d) := c_r \prod_{i=1}^{k-1} \Phi_r(g(n)\Gamma, g(n+id)\Gamma)
\]

(6.11)

where \( c_r > 0 \) is a normalisation constant to be chosen later. This function, as discussed immediately following the statement of Proposition 6.2, is a smooth cutoff to the set of “almost-diagonal” progressions in \( G/\Gamma \). Specifically, \( \mu_r \) is supported in \( \{\delta\} \), and also in the region where \( g(n+id)\Gamma \in B_r g(n)\Gamma \), \( |d| \leq \delta N \), and \( q|d \) for \( i = 0, \ldots, k-1 \). From the Lipschitz nature of \( F \) we thus have

\[
F(g(n+id)\Gamma, (n+id)(\text{mod } q), (n+id)/N) = F(g(n)\Gamma, n(\text{mod } q), n/N) + O_M(r_0)
\]

for \( (n, d) \) in the support of \( \mu_r \), which gives \( (6.14) \) for \( \mu_r \) if \( r_0 \) is sufficiently small depending on \( \varepsilon, M \).

\(^{16}\)Readers may find it helpful to keep the hierarchy of scales

\[
1 \sim 1/k, \alpha \gg \varepsilon \gg 1/M \gg \delta \gg 1/R \gg r_0 \gg r \gg 1/F(M) \gg 1/N > 0
\]

in mind.
Next, we compute the expectation of $\mu_r(n,d)$, in order to work out what the normalisation constant $c_r$ should be. Observe that

$$\mathbb{E}_{n \in [N], d \in [-\varepsilon N, \varepsilon N]} \mu_r(n,d) = \frac{\delta}{q^\varepsilon} (1 + O(\varepsilon)) c_r \times$$

$$\times \mathbb{E}_{n \in [k \varepsilon N, (1-k) \varepsilon N]; d \in [-\delta N, \delta N]; q|d} \tilde{\Phi}_r(g(n)\Gamma, \ldots, g(n+(k-1)d)\Gamma),$$

where $\tilde{\Phi}_r : (G/\Gamma)^k \rightarrow \mathbb{R}$ is the function

$$\tilde{\Phi}_r(x_0, \ldots, x_{d-1}) := \prod_{i=1}^{k-1} \Phi_r(x_0, x_i). \quad (6.13)$$

Observe that $\tilde{\Phi}$ has a Lipschitz norm of $O_{M,r,\delta}(1)$. Applying Theorem 1.11, we can express (6.12) as

$$\frac{\delta}{q^\varepsilon} (1 + O(\varepsilon)) c_r \int_{G^\Psi/\Gamma^\Psi} \tilde{\Phi}_r + o_F(M) \rightarrow \infty; M,r,\delta(1) + o_N \rightarrow \infty; M,r,\delta(1)),$$

where $G^\Psi \subseteq G^k$ is the $k^{th}$ Hall-Petresco group, that is to say the Leibman group associated to the collection $\Psi = (\psi_0, \ldots, \psi_{k-1})$ of linear forms $\Psi^{(i)} := (n, d) \mapsto n + id$ for $i = 0, \ldots, k-1$.

The group $G^\Psi$ is a $O_M(1)$-rational subgroup of $G^k$, which itself has complexity $O_M(1)$. Meanwhile, the function $\tilde{\Phi}_r$ equals 1 on a ball of radius $r^{O_M(1)}$ centred at the identity, and is bounded by 1 throughout. We conclude that the quantity

$$v_r := \int_{G^\Psi/\Gamma^\Psi} \tilde{\Phi}_r$$

obeys the bounds

$$r^{O_M(1)} \ll_M v_r \leq 1.$$

Furthermore, from the properties of the functions $\phi_r$, we have the monotonicity property

$$v_{r(1-\delta)r} \leq v_r$$

for any $0 < r < r_0$. Applying the pigeonhole principle (using the fact that polynomial growth is always slower than exponential growth), and choosing $\delta \gg \varepsilon, M$ sufficiently small depending on $\varepsilon, M$, one can thus find a radius

$$r_0 > r \gg r_0, \varepsilon, \delta, M$$

such that we have the regularity property

$$(1 - O(\varepsilon)) v_r \leq v_{r(1-3\delta)r} \leq v_{r(1+3\delta)r} \leq (1 + O(\varepsilon)) v_r. \quad (6.14)$$

Note that this idea of picking a “regular” radius originates, in additive combinatorics, in Bourgain’s paper [7]. Fix from now on a value of $r$ with this property. If we then set

$$c_r := \frac{q^\varepsilon}{\delta v_r} \quad (6.15)$$
we conclude that
\[ c_r \ll_{M,r_0,\varepsilon} 1 \]  
(6.16)

and
\[ \mathbb{E}_{n \in [N], d \in [-\varepsilon N,\varepsilon N]} \mu_r(n,d) = 1 + O(\varepsilon) + o_{\mathcal{F}(M) \to \infty; M,\varepsilon,r_0}(1) + o_{N \to \infty; M,\varepsilon,r_0}(1). \]

This will give (6.3) provided that \( r_0 \) is chosen to depend on \( M,\varepsilon,\delta \), that \( \mathcal{F} \) is sufficiently rapid depending on \( \varepsilon \), and \( N \) is sufficiently large depending on \( M,\varepsilon \).

Our remaining task, and the most difficult one, is to study the expression in (6.5). That is to say, we fix \( 0 \leq i \leq k-1 \) and consider
\[ \mathbb{E}_{n \in [N], d \in [-\varepsilon N,\varepsilon N]} \mu_r(n-id,d) = 1 + O(\varepsilon) + o_{F(M) \to \infty; M,\varepsilon,r_0}(1) + o_{N \to \infty; M,\varepsilon,r_0}(1). \]

(6.17)

Using (6.11), we can write this expression as
\[ (1 + O(\varepsilon))(\frac{\varepsilon}{q\delta}c_r)^2 \mathbb{E}_{n \in [k \varepsilon N, (1-k)\varepsilon N]} \mathbb{E}_{d,d' \in [-\delta N,\delta N], |q|d,d'} \tilde{\Phi} \otimes^2_r (g(n-id)\Gamma, \ldots, g(n+(k-1-i)d)\Gamma, g(n-id')\Gamma, \ldots, g(n+(k-1-i)d')\Gamma) \]

where \( \tilde{\Phi} \otimes^2_r : (G/\Gamma)^k \times (G/\Gamma)^k \to \mathbb{R}^+ \) is the tensor square
\[ \tilde{\Phi} \otimes^2_r(x,x') := \tilde{\Phi}_r(x)\tilde{\Phi}_r(x'). \]

Applying Theorem 1.11, we can thus express (6.17) as
\[ (1 + O(\varepsilon))(\frac{\varepsilon}{q\delta}c_r)^2 \left( \int_{G^{\Psi(i)} / \Gamma^{\Psi(i)}} \tilde{\Phi} \otimes^2_r + o_{\mathcal{F}(M) \to \infty; M,\varepsilon,r_0}(1) + o_{N \to \infty; M,\varepsilon,r_0}(1) \right) \]

(6.18)

where \( G^{\Psi(i)} \subset G^{2k} \) is the Leibman group associated to the collection
\[ \Psi(i) := (\psi_{0,i}, \ldots, \psi_{k-1,i}, \psi'_{0,i}, \ldots, \psi'_{k-1,i}) \]

of linear forms
\[ \psi_{j,i} : (n,d,d') \mapsto n + (j-i)d \]

and
\[ \psi'_{j,i} : (n,d,d') \mapsto n + (j-i)d' \]

for \( j = 0,\ldots,k-1 \).

We will be establishing the following claim.

**Claim 6.4 (Approximate factorisation).** We have
\[ \int_{G^{\Psi(i)} / \Gamma^{\Psi(i)}} \tilde{\Phi} \otimes^2_r = (1 + O(\varepsilon))v_r^2. \]

(6.19)

**Proof of Proposition 6.2 assuming Claim 6.4.** Substitute back into (6.18) and use (6.15), (6.16) to conclude that
\[ (6.17) = 1 + O(\varepsilon) + o_{\mathcal{F}(M) \to \infty; M,\varepsilon,r_0}(1) + o_{N \to \infty; M,\varepsilon,r_0}(1). \]
This gives the result upon choosing \( r_0 \) sufficiently small depending on \( \varepsilon, M, \delta, \mathcal{F} \) sufficiently rapid depending on \( \varepsilon \), and \( N \) sufficiently large depending on \( \varepsilon, M \).

It remains to establish Claim 6.4. For notational simplicity we establish only the claim \( i = 0 \) (the others being very similar). The intuition behind this claim (and behind the key assertion that the number of almost-diagonal progressions whose \( i \)-th term is \( n \) does not depend on \( n \)) is that the linear forms \( (\psi_0, \ldots, \psi_{k-1}, 0) \) and \( (\psi'_0, \ldots, \psi'_{k-1}, 0) \) are almost independent of each other, except for the fact that they are coupled via the obvious identity \( \psi_0 = \psi'_0 \).

One way to encode this formally is to note that the Leibman group \( G^{\Psi(0)} \) is given by

\[
H := \{ (x, x') \in G^{\Psi} \times G^{\Psi} : x_0 = x'_0 \},
\]

a product of two copies of the Hall-Petresco group \( G^{\Psi} = HP^k(G) \) fibred over the zeroth coordinate. To prove this, one may note that the containment \( G^{\Psi(0)} \subseteq H \) is obvious. On the other hand, one may compute directly using the dimension formula (3.1) that

\[
\dim(G^{\Psi}) = \dim(G) + \sum_{i=1}^{k-2} \dim(G^{(i)}),
\]

and

\[
\dim(G^{\Psi(0)}) = \dim(G) + 2 \sum_{i=1}^{k-2} \dim(G^{(i)})
\]

and thus

\[
\dim(G^{\Psi(0)}) = 2 \dim(G^{\Psi}) - \dim(G) = \dim(H),
\]

and so since both sides are connected, simply-connected nilpotent Lie groups (and so both are homeomorphic to their Lie algebras) we have \( G^{\Psi(0)} = H \).

Write \( J_r \) for the integral appearing in (6.19), that is to say

\[
J_r := \int_{(x, x') \in G^{\Psi} \times G^{\Psi} : x_0 = x'_0} \tilde{\Phi}_r^\otimes 2(x, x').
\]

Let \( R \) be some quantity, and suppose that \( \text{dist}_G(g, \text{id}) \leq R \). Then by the almost-invariance property (6.10) we have

\[
\int_{(x, x') \in G^{\Psi} \times G^{\Psi} : x_0 = gx'_0} \tilde{\Phi}_r^\otimes 2(x, x') \geq J_r.
\]

Integrate this over the ball \( B_R := \{ g \in G : \text{dist}_G(g, \text{id}) \leq R \} \). Then we obtain

\[
\int_{(x, x') \in (G^{\Psi} / \Gamma^{\Psi})^2} \lambda(x, x') \tilde{\Phi}_r^\otimes 2_{(1+3\delta)}(x, x') \geq \text{vol}(B_R) J_r,
\]

where \( \lambda(x, x') \) is the number of \( g \in B_R \) for which \( x_0 = gx'_0(\text{mod} \Gamma) \), or equivalently

\[
\lambda(x, x') := |\Gamma \cap x^{-1}_0 B_R x'_0|.
\]
Choose representatives $x_0, x'_0$ in some fundamental domain with $x_0, x'_0 = O_M(1)$. By a volume-packing argument and simple geometry we then have

$$\lambda(x, x') = \text{vol}(B_R)(1 + o_{R \to \infty}; M(1)).$$

Comparing with the above we have

$$\lambda(x, x') = \frac{\text{vol}(B_R)}{2}(1 + o_{R \to \infty}; M(1)).$$

This gives the upper bound for Claim 6.4. The lower bound is proven similarly. This concludes the proof of Proposition 6.2 and thus Theorem 6.1.

7. On a theorem of Gowers and Wolf

Our aim in this section is to prove Theorem 1.13, whose statement we recall now.

**Theorem 7.1 (Theorem 1.13).** Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$, and let $s \geq 1$ be an integer such that the polynomials $\psi_1^{s+1}, \ldots, \psi_t^{s+1}$ are linearly independent. Then for any function $f : [N] \to \mathbb{C}$ bounded in magnitude by 1 (and defined to be zero outside of $[N]$) obeying the bound $\|f\|_{U_{s+1}[N]} \leq \delta$ for some $\delta > 0$, one has

$$E_n \in [N] \prod_{i=1}^t f(\psi_i(n)) = o_{\delta \to 0; s, D, t, \Psi}(1).$$

Henceforth we allow all implied constants to depend on $d, t, s, \Psi$ without indicating this explicitly. Let $s' = s'(\Psi)$ be the Cauchy-Schwarz complexity of the linear forms $\Psi$, as defined in Theorem 4.1. We may of course assume that $s' > s$, as Theorem 1.13 is immediate otherwise. We may also assume that $N$ is large depending on $\delta$, since otherwise the claim is trivial from a compactness argument.

Let $\varepsilon > 0$ be a small number depending on $\delta$ to be chosen later, and let $F$ be a growth function depending on $\varepsilon$ to be chosen later. Applying Theorem 1.2 at degree $s'$ (after first decomposing $f$ as a linear combination of $O(1)$ functions taking values in $[0, 1]$), we can find a positive quantity $M = O_{\varepsilon, F}(1)$ and a decomposition

$$f = f_{\text{nil}} + f_{\text{sml}} + f_{\text{unf}} \quad (7.1)$$

where:

- $f_{\text{nil}}$ is a $(F(M), N)$-irrational virtual nilsequence of degree $\leq s'$, complexity $\leq M$, and scale $N$;
- $f_{\text{sml}}$ has $L^2[N]$ norm at most $\varepsilon$;
- $f_{\text{unf}}$ has $U_{s'+1}[N]$ at most $1/F(M)$;
- All functions $f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}}$ are bounded in magnitude by $O(1)$. 


We apply this decomposition to split the expression

$$\mathbb{E}_{n \in [N]^D} \prod_{i=1}^{t} f(\psi_i(n))$$

as the sum of $3^t$ terms, in which each copy of $f$ has been replaced with either $f_{\text{nil}}$, $f_{\text{sml}}$, or $f_{\text{unf}}$.

Any term involving at least one factor of $f_{\text{sml}}$ can be easily seen to be of size $O(\varepsilon)$ by crudely estimating all other factors by 1. By (4.1), any term involving at least one factor of $f_{\text{unf}}$ is of size $O(1/F(M))$, which is also of size $O(\varepsilon)$ if $F$ is chosen to be sufficiently rapidly growing depending on $\varepsilon$.

We can therefore express (7.2) as

$$\mathbb{E}_{n \in [N]^D} \prod_{i=1}^{t} f_{\text{nil}}(\psi_i(n)) + O(\varepsilon).$$

By hypothesis, we can write

$$f_{\text{nil}}(n) = F(g(n)\Gamma, n(\text{mod } q), n/N)$$

for some $q$ with $1 \leq q \leq M$, some degree $\leq s$, $(F(M), N)$-irrational, orbit $n \mapsto g(n)\Gamma$ of complexity $\leq M$ and some Lipschitz function $F : G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R}$ of norm at most $M$. The mod $q$ and Archimedean behaviour in $f_{\text{nil}}$ are nothing more than technical annoyances, and we set about eliminating them now. We encourage the reader to work through the heart of the argument, starting at (7.3) below, in the model case $f_{\text{nil}} = F(g(n)\Gamma)$. Let $\varepsilon'$ be a small quantity depending on $\varepsilon, M$ to be chosen later.

We partition $[N]$ into progressions $P$ of spacing $q$ and length $\varepsilon'N$, plus a remainder set of size $O(M)$. We can then rewrite the above expression as

$$\mathbb{E}_{P_1, \ldots, P_D} \mathbb{E}_{n \in P_1 \times \ldots \times P_D} \prod_{i=1}^{t} f_{\text{nil}}(\psi_i(n)) + O(\varepsilon).$$

We abbreviate $P_1 \times \ldots \times P_D$ as $P$. For a given $P$, observe that as $n$ ranges in $P$, the residue class of $\psi_i(n)$ modulo $q$ is equal to a fixed class $a_{P,i}$, and the value of $\psi_i(P)/N$ differs by at most $O_M(\varepsilon')$ from a fixed number $x_{P,i}$. We may assume that $x_{P,i} \in [0,1]$ for each $i$, otherwise the inner expectation is zero (except for a few “boundary” values of $P$ which give a net contribution of $O_M(\varepsilon')$).

If $\varepsilon'$ is small enough depending on $\varepsilon, M$, the $O_M(\varepsilon')$ error in the above discussion can be absorbed in the $O(\varepsilon)$ error, and so we have

$$\mathbb{E}_{n \in [N]^D} \prod_{i=1}^{t} f(\psi_i(n)) = \mathbb{E}_{P} \mathbb{E}_{n \in P} \prod_{i=1}^{t} F(g(\psi_i(n))\Gamma, a_{P,i}, x_{P,i}) + O(\varepsilon).$$

Readers may find it helpful to keep the hierarchy of scales

$$1 \gg \varepsilon \gg 1/M, 1/q \gg \varepsilon' \gg 1/F(M) \gg \delta \gg 1/N > 0$$

in mind.
We now apply Theorem 1.11, which tells us that the right-hand side here is
\[
\mathbb{E}_P \int_{G^\Psi/\Gamma^\Psi} \tilde{F}_P + O(\varepsilon) + o_{T(M)\to\infty; M, \varepsilon, \varepsilon'}(1) + o_{N\to\infty; M, \varepsilon, \varepsilon'}(1),
\]
(7.3)
where as usual \(G^\Psi \leq G^t\) is the Leibman group associated to the system of forms \(\Psi = \{\psi_1, \ldots, \psi_t\}\), and here \(\tilde{F}_P : G^\Psi/\Gamma^\Psi \to \mathbb{C}\) is the function
\[
\tilde{F}_P((g_1, \ldots, g_t)\Gamma^\Psi) := \prod_{i=1}^t F(g_i\Gamma, a_{P,i}, x_{P,i}).
\]

The heart of the matter is to obtain an upper bound on the quantity \(\mathbb{E}_P \int_{G^\Psi/\Gamma^\Psi} \tilde{F}_P\) appearing in (7.3). To do this, of course, we need to make use of the assumption on the forms \(\psi_1, \ldots, \psi_t\), as well as the fact that \(\|f\|_{U^{s+1}} \leq \delta\).

The aforementioned assumption, namely that \(\psi_1^{s+1}, \ldots, \psi_t^{s+1}\) are linearly independent, implies that \(\Psi^{[s+1]}\) is the whole of \(\mathbb{R}^t\) which, in view of the definition of the Leibman group \(G^\Psi\), implies that \(G^t_{(s+1)} \leq G^\Psi\). By Fubini’s theorem, we thus have
\[
\int_{G^\Psi/\Gamma^\Psi} \tilde{F}_P = \int_{G^\Psi/\Gamma^\Psi} \tilde{F}_{P, \leq s}
\]
where
\[
\tilde{F}_{P, \leq s}((g_1, \ldots, g_t)\Gamma^\Psi) := \prod_{i=1}^t F_{\leq s}(g_i\Gamma, a_{P,i}, x_{P,i})
\]
(7.4)
and \(F_{\leq s}\) is defined by averaging over cosets of \(G_{(s+1)}\), specifically
\[
F_{\leq s}(g\Gamma, a, x) := \int_{G_{(s+1)}/\Gamma_{(s+1)}} F(gg_{s+1}\Gamma, a, x) \, dg_{s+1}.
\]

Since \(F\) was Lipschitz with norm \(O_M(1)\), we see that \(F_{\leq s}\) is Lipschitz with norm \(O_M(1)\) also. Also, since \(F\) is bounded in magnitude by \(O(1)\), so is \(F_{\leq s}\).

As the forms \(\psi_1^{s+1}, \ldots, \psi_t^{s+1}\) are independent, we see in particular that \(\psi_1\) is non-zero. This implies that the projection of \(G^\Psi\) to the first coordinate \(G\) is surjective. Meanwhile, from (7.4) and the boundedness of \(F_{\leq s}\) we have the crude upper bound
\[
|\tilde{F}_{P, \leq s}((g_1, \ldots, g_t)\Gamma)| \ll |F_{\leq s}(g_1\Gamma, a_{P,1}, x_{P,1})|.
\]
From Fubini’s theorem, we obtain the bound
\[
|\int_{G^\Psi/\Gamma^\Psi} \tilde{F}_P| \ll \int_{G/\Gamma} |F_{\leq s}(\cdot, a_{P,1}, x_{P,1})|.
\]
(7.5)
To proceed further, we need a crucial smallness estimate on \(F_{\leq s}\).
Proposition 7.2 (\(F_{\leq s}\) small in \(L^2\)). For any \(a \in \mathbb{Z}/q\mathbb{Z}\) and \(x \in [0, 1]\), one has

\[
\int_{G/\Gamma} |F_{\leq s}(\cdot, a, x)|^2 \ll O(\varepsilon) + O_M(\varepsilon') + o_{\delta \to \infty; M, \varepsilon, \varepsilon'}(1) + o_{F(M) \to \infty; M, \varepsilon, \varepsilon'}(1) + o_{N \to \infty; M, \varepsilon, \varepsilon'}(1).
\]

Proof. By reflection symmetry we may assume that \(x \leq 1/2\). We may also round \(x\) so that \(x = \frac{qn_0}{N}\) for some \(n_0 \in [N/2q]\), as the error in doing so can be easily absorbed by the Lipschitz properties of \(F_{\leq s}\).

By construction, \(F_{\leq s}\) is invariant on \(G_{(s+1)}\)-cosets, while \(F - F_{\leq s}\) integrates to zero on any such coset. In particular, \(F_{\leq s}(\cdot, a, x)\) and \(F - F_{\leq s}(\cdot, a, x)\) are orthogonal, and thus

\[
\int_{G/\Gamma} |F_{\leq s}(\cdot, a, x)|^2 = \int_{G/\Gamma} F_{\leq s}(\cdot, a, x).
\]

Applying Theorem 1.1 (really just the special case of this result asserting that \((g(n)\Gamma)\) is equidistributed, cf. Lemma 3.7) and the Lipschitz nature of \(F_{\leq s}\), the right-hand side can be written as

\[
\mathbb{E}_{n \in [\varepsilon'N]} F_{\leq s}(g(qn + qn_0 + a)\Gamma, a, x) + o_{F(M) \to \infty; M, \varepsilon, \varepsilon'}(1) + o_{N \to \infty; M, \varepsilon, \varepsilon'}(1).
\]

Let \(P\) be the progression \(\{qn + qn_0 + a : n \in [\varepsilon'N]\}\). Then by a further use of the Lipschitz properties of \(F\), we can rewrite the above expression as

\[
\mathbb{E}_{n \in P} F(g(n)\Gamma, n \mod q, n/N)\psi(n) + O_M(\varepsilon') + o_{F(M) \to \infty; M, \varepsilon, \varepsilon'}(1) + o_{N \to \infty; M, \varepsilon, \varepsilon'}(1) \quad (7.6)
\]

where

\[
\psi(n) := F_{\leq s}(g(n)\Gamma, a, x).
\]

Note that, as a consequence of the \(G_{(s+1)}\)-invariance of \(F_{\leq s}\), \(\psi(n)\) is a degree \(\leq s\) nilsequence of complexity \(O_M(1)\). Now by (7.1) we have

\[
F(g(n)\Gamma, n \mod q, n/N) = f(n) - f_{\text{unif}}(n) - f_{\text{smil}}(n).
\]

The contribution of \(f_{\text{smil}}(n)\) to (7.6) is \(O(\varepsilon)\) by the Cauchy-Schwarz inequality. Now consider the contribution of \(f\). Observe that because \(F_{\leq s}\) is \(G_{(s+1)}\)-invariant, \(\psi\) is a degree \(\leq s\) nilsequence of complexity \(O_M(1)\). Meanwhile, \(\|f\|_{U^{s+1}[N]} \leq \delta\) by hypothesis. Applying the converse to the inverse conjecture for the Gowers norms (first established in [26]), though for a simple proof see [31, Appendix G]), we see that

\[
\mathbb{E}_{n \in P} f(n)\psi(n) = o_{\delta \to 0; M, \varepsilon, \varepsilon'}(1).
\]

Similarly, since \(\|f_{\text{unif}}\|_{U^{s'+1}[N]} \leq 1/F(M)\) and \(s' \geq s\), we have

\[
\mathbb{E}_{n \in P} f(n)\psi(n) = o_{F(M) \to 0; M, \varepsilon, \varepsilon'}(1).
\]

Putting all of these estimates together, we obtain the claim. □
Applying this bound and (7.5), we can thus bound (7.3) in magnitude by

\[ O(\varepsilon) + O_M(\varepsilon') + o_{\mathcal{F}(M) \to \infty; M, \varepsilon, \varepsilon'}(1) + o_{N \to \infty; M, \varepsilon, \varepsilon'}(1). \]

Choosing \( \varepsilon' \) sufficiently small depending on \( M \) and \( \varepsilon \), and choosing \( \mathcal{F} \) sufficiently rapidly growing depending on \( \varepsilon \), and then using the bound \( M = O_{\varepsilon, \mathcal{F}}(1) \) (and recalling that \( N \) can be chosen large depending on \( \delta \)), we conclude that

\[ |E_{n \in [N]^D} \prod_{i=1}^t f(\psi_i(n))| \ll \varepsilon \]

whenever \( \delta \) is sufficiently small depending on \( \varepsilon \). Theorem 1.13 follows.

Remark. It seems certain that one can extend this result to the case when one has \( t \) distinct functions \( f_1, \ldots, f_t : [N] \to \mathbb{C} \) rather than a single function \( f : [N] \to \mathbb{C} \). The main change in the argument would be to use a version of the regularity lemma (Theorem 1.2) valid for several functions simultaneously, in which one regularises the \( f_1, \ldots, f_t \) using the same data \( M, q, (G/\Gamma, G_\bullet), g() \) (but allows each function \( f_i \) to be given a separate Lipschitz function \( F_i : G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R} \to \mathbb{C} \)). Such a result could be obtained by straightforward modifications to the proof of Theorem 1.2, but we do not pursue this matter here.

Appendix A. Properties of polynomial sequences

In this appendix we collect a variety of facts and definitions concerning polynomial sequences in nilpotent groups, all of which were required at some point in the paper proper. We take for granted the definition of filtration \( G_\bullet \) and of the group \( \text{poly}(\mathbb{Z}^d, G_\bullet) \) of polynomial sequences \( g : \mathbb{Z}^d \to G \) adapted to \( G_\bullet \); these notions were recalled in the introduction.

Taylor expansions. Polynomial sequences may be described in terms of so-called Taylor expansions. In the lemma that follows we make use of the generalised binomial coefficients \( \binom{n}{i} \) are the generalised binomial coefficients

\[
\binom{n_1, \ldots, n_D}{i_1, \ldots, i_D} := \frac{n_1}{i_1} \cdots \frac{n_D}{i_D},
\]

where

\[
\binom{n}{i} := \frac{n(n-1) \cdots (n-i+1)}{i!}.
\]

If \( i = (i_1, \ldots, i_d) \in \mathbb{N}^D \) is a \( D \)-tuple of non-negative integers we define the degree \( |i| := i_1 + \ldots + i_D \). Choose an arbitrary ordering on \( \mathbb{N}^D \) with the property that \( |i| \geq |j| \) whenever \( i \geq j \).

Lemma A.1 (Taylor expansions). Suppose that \( g \in \text{poly}(\mathbb{Z}^d, G_\bullet) \). Then there are unique Taylor coefficients \( g_i \in G_{|i|} \) with the property that

\[
g(n) = \prod_{i \in \mathbb{N}^d} g_i^{\binom{n}{i}}
\]
for all \( n \in \mathbb{Z}^D \). Conversely, every Taylor expansion of this type gives rise to a polynomial sequence \( g \in \text{poly}(\mathbb{Z}^D, G\cdot) \).

Remarks. This is proven in [28, Lemma 6.7]. Note that, since \( G \) is nilpotent, this is a finite expansion. In the case \( D = 1 \) (which will feature most prominently in the paper) it takes the form

\[ g(n) = g_0 g_1^{(n)} \cdots g_s^{(n)}. \]

Note how, from the presentation of polynomial sequences as Taylor expansions, it is by no means clear (and somewhat remarkable) that they form a group under pointwise multiplication (Theorem 1.6).

Polynomial sequences that vary slowly, in a certain sense, are called smooth. We employ the following definition, which is the same as the one given in the introduction to [28].

**Definition A.2** (Smooth sequences). Let \( A \) be a positive parameter and let \( N \geq 1 \) be an integer. Let \( \beta \in \text{poly}(\mathbb{Z}, G\cdot) \). We say that \( \beta \) is \((A,N)\)-smooth if we have \( d_G(\beta(n), \text{id}) \leq A \) and \( d_G(\beta(n), \beta(n+1)) \leq A/N \) for all \( n \in [N] \).

Here \( d_G \) is a metric on the group \( G \) constructed using the Mal’cev basis, see [28, Definition 2.2]. The precise definition of this metric is not terribly important for our analysis.

In counterpoint to the notion of a smooth sequence is that of a rational sequence.

**Definition A.3** (Rational sequences). Let \( A \geq 1 \) be an integer, and let \((G/\Gamma, G\cdot)\) be a filtered nilmanifold. Then an element \( g \in G \) is \( A \)-rational if there is some \( q, 1 \leq q \leq A \), such that \( g^q \in \Gamma \). If \( \gamma \in \text{poly}(\mathbb{Z}, G\cdot) \) is a polynomial sequence then we say that it is \( A \)-rational if \( \gamma(n) \) is \( A \)-rational for every integer \( n \).

We have the following basic facts about smooth and rational sequences:

**Lemma A.4** (Basic facts). Let \((G/\Gamma, G\cdot)\) be a filtered nilmanifold of complexity \( \leq M_0 \). By a “sequence”, we mean an element of \( \text{poly}(\mathbb{Z}, G\cdot) \). Then:

(i) The product of two \((A,N)\)-smooth sequences is \( O_{M_0,A}(1) \)-smooth;

(ii) The product of two \( A \)-rational sequences is \( O_{M_0,A}(1) \)-rational;

(iii) Any \( A \)-rational sequence is periodic with period \( O_{M_0,A}(1) \).

**Proof.** For (i), see [28, Lemma 10.1]; for (ii), see [28, Lemma A.11 (v)]; and for (iii), see [28, Lemma A.12 (ii)]. In fact these results hold in the multiparameter setting, with polynomially effective bounds, but we will not need these facts here. \( \square \)

We turn now to an important new definition for this paper, that of an irrational polynomial sequence. In [28], much emphasis was placed on the

\[ \text{One could take an “adelic” perspective here and view smooth sequences as those that are local to the Archimedean place } \infty, \text{ while rational sequences are those that are local to finite places } p. \]
notion of an equidistributed polynomial sequence \( g : \mathbb{Z} \to G \): one for which the orbit \( \{g(n)\Gamma\}_{n \in \mathbb{N}} \) is close to equidistributed on \( G/\Gamma \). The notion of an irrational sequence implies equidistribution (see Lemma 3.7, which is also a special case of Theorem 1.11), but also encodes an assertion that the filtration \( G_\bullet \) is in some sense “minimal” for the sequence. To illustrate the difference, let us think about a simple abelian case in which \( G/\Gamma \) is just the unit circle \( \mathbb{R}/\mathbb{Z} \) (written additively), and \( g : \mathbb{Z} \to \mathbb{R} \) is a polynomial

\[
g(n) = \alpha_0 + \alpha_1 \binom{n}{1} + \ldots + \alpha_s \binom{n}{s}.
\]

This sequence is adapted to the filtration in which \( G_{(i)} = \mathbb{R} \) for \( i \leq s \) and \( G_{(i)} = \{0\} \) for \( i > s \). Qualitatively speaking, \( g \) is equidistributed if at least one of \( \alpha_1, \ldots, \alpha_s \) is irrational; in contrast, \( g \) is irrational with respect to this filtration if it is \( \alpha_s \) which is irrational. Note that if \( s > 1 \) and \( \alpha_s \) is rational, then (after removing the periodic component \( \alpha_s n^s \) from \( g \)) \( g \) is now adapted to the filtration \( G'_\bullet \) in which \( G'_{(i)} = \mathbb{R} \) for \( i \leq s - 1 \) and \( G'_{(i)} = \{0\} \) for \( i > s - 1 \), which has a strictly smaller total dimension. This basic example is the model for the more sophisticated result in Lemma 2.9.

Let us turn now to the precise definition in the more general setting of Lie group-valued polynomial sequences, in which the role of the \( \alpha_i \) is played by the Taylor coefficients of \( g \). We need a preliminary definition.

**Definition A.5** (i-horizontal characters). Let \((G/\Gamma, G_\bullet)\) be a filtered nilmanifold of degree \( \leq s \) with filtration \( G_\bullet = (G_{(i)})_{i=0}^s \). Then by an \( i \)-horizontal character we mean a continuous homomorphism from \( \xi_i : G_{(i)} \to \mathbb{R} \) which vanishes on \( G_{(i+1)}, \Gamma_{(i)} \) and on \([G_{(j)}, G_{(i-j)}]\) for any \( 0 \leq j \leq i \). We say that such a character is non-trivial if it is not constant. We can assign a notion of complexity by taking a Mal’cev basis adapted to \( G_\bullet \), whereupon one has a natural isomorphism \( G_{(i)}/G_{(i+1)} \cong \mathbb{R}^k \). Writing \( \psi(g_i) \) for the coordinates of \( g_i (\text{mod } G_{(i+1)}) \), any \( i \)-horizontal character has the form \( \xi_i(g_i) = \bar{m}.\psi(g_i) \), for some vector \( \bar{m} = (m_1, \ldots, m_k) \) of integers. We may then define the complexity of \( \xi_i \) to be \( |m_1| + \cdots + |m_k| \).

The list of subgroups on which \( \xi_i \) is required to vanish looks rather restrictive and slightly unnatural at first sight. Roughly speaking, this list is intended to isolate that behaviour which genuinely “belongs” to the degree \( i \) portion of the filtered nilmanifold, as opposed to arising from those terms of higher or lower degree, or which disappear after quotienting out by the lattice \( \Gamma \).

**Definition A.6** (Irrationality). Let \((G/\Gamma, G_\bullet)\) be a filtered nilmanifold of degree \( \leq s \) with filtration \( G_\bullet = (G_{(i)})_{i=0}^s \). Let \( g_i \in G_{(i)} \). Let \( A, N > 0 \). Then we say that \( g_i \) is \((A, N)\)-irrational in \( G_{(i)} \) if for every non-trivial \( i \)-horizontal character \( \xi_i : G_{(i)} \to \mathbb{R} \) of complexity \( \leq A \) one has \( \|\xi_i(g_i)\|_{\mathbb{R}/\mathbb{Z}} \geq A/N^s \). We say that the sequence \( g(n) \) is \((A, N)\)-irrational if its \( i \)th Taylor coefficient \( g_i \) is \((A, N)\)-irrational in \( G_{(i)} \) for each \( i, 1 \leq i \leq s \).
To understand this definition, it is helpful to consider examples. We leave it as an exercise to check that in the abelian case (A.1) this amounts to stipulating that the top coefficient of \( g \) is poorly approximated by rationals, thus \( \| qα_0 \|_{\mathbb{R}/\mathbb{Z}} \geq A'/N^s \) whenever \( 1 \leq q \leq A' \).

A second interesting case to examine is in which \( g(n) = g^n \) is a linear polynomial sequence adapted to the lower central series filtration \( (G_i)_{i=0}^{\infty} \). For the lower central series filtration there are no nontrivial \( i \)-horizontal characters when \( i \geq 2 \), and 1-horizontal characters are the same thing as horizontal characters in the sense of [28] Definition 1.5. It follows from this and [28] Theorem 1.16 that \( g(n) \) is irrational if and only if \( (g(n)\Gamma)_{n \in [N]} \) is equidistributed. Now polynomial sequences that are not linear do not arise naturally in ergodic-theoretic settings such as those considered in [4, 39], and thus the equivalence of the notions of “irrational” and “equidistributed” in this setting explains why the former concept has not appeared in the literature before. The need for it is a new feature of the quantitative world, as is the need for polynomial nilsequences themselves, for reasons explained on [28] §1.

The following third example is also edifying. Take \( g(n) \) to be any polynomial sequence on the Heisenberg group, for example \( g(n) = \begin{pmatrix} 1 & \alpha n & \gamma n^2 \\ 0 & 1 & \beta n \\ 0 & 0 & 1 \end{pmatrix} \).

This sequence is a polynomial sequence adapted to the lower central series filtration \( G_0 = G_1 = G, G_2 = [G, G], G_3 = \{\text{id}\} \), and it will be equidistributed in that setting for generic \( \alpha, \beta, \gamma \). However \( g \) is also a polynomial sequence with respect to some much flabbier filtrations, for example the one in which \( G(0) = G(1) = G(2) = \cdots = G(10) = G, G(11) = \cdots = G(100) = [G, G] \) and \( G(i) = \{\text{id}\} \) for \( i \geq 101 \). It is easy to check that \( g \) is not irrational in this setting, and indeed irrationality is somehow detecting the fact that a given filtration \( G_\bullet \) is minimal for \( g \). This point is quite clear in the proof of Lemma 2.9 (which itself depends on Lemma A.7 below), where the failure of a sequence to be irrational is used to create a coarser filtration for a polynomial sequence related to \( g \).

**Lemma A.7.** Suppose that \( (G/\Gamma, G_\bullet) \) is a filtered nilmanifold of degree \( \leq s \) with filtration \( G_\bullet = (G(i))_{i=0}^{\infty} \). Suppose that \( g \) is not \((A, N)\)-irrational. Then there is an index \( i, 1 \leq i \leq s \) such that the \( i \)-th Taylor coefficient \( g_i \) factors as \( \beta_i \gamma_i \), where \( \beta_i, g'_i, \gamma_i \in G(i) \), \( g'_i \) lies in the kernel of some \( i \)-horizontal character \( \xi_i : G(i) \to \mathbb{R} \) of complexity at most \( A \), \( d_G(\beta_i, \text{id}) = O_{A,M}(N^{-1}) \) and \( \gamma_i \) is \( O_{A,M}(1) \)-rational.

**Proof.** The proof is (unsurprisingly) extremely similar to that of [28] Lemma 7.9. Reversing the definition of irrational polynomial sequence, we see that there is an index \( i \) together with an \( i \)-horizontal character \( \xi_i : G(i) \to \mathbb{R} \) such that \( \| \xi_i(g_i) \|_{\mathbb{R}/\mathbb{Z}} \leq A/N^i \). It is convenient at this point to work in a Mal’cev coordinate system adapted to \( G_\bullet \), whereby \( G(i)/G(i+1) \) may be identified with \( \mathbb{R}^k \) and \( \Gamma(i)/G(i+1) \) with \( \mathbb{Z}^k \). If \( g_i \in G(i) \) then, as above,
we write $\psi(g) \in \mathbb{R}^k$ for the corresponding coordinates. Then $\xi_i$ has the form $\xi_i(g_i) = \bar{m}_i \psi(g_i)$ for some vector $\bar{m}_i = (m_1, \ldots, m_k)$ of integers with $|m_1| + \cdots + |m_k| \leq A$. Now by assumption we have $\|\bar{m}_i \psi(g_i)\|_{\mathbb{R}/\mathbb{Z}} \leq A/N^i$, and therefore $\bar{m}_i \psi(g_i) = r + O(A/N^i)$ for some integer $r$. It follows from simple linear algebra that we may write $\psi(g_i) = \bar{t}_i + \bar{u} + \bar{v}$, where $\bar{m}_i \bar{u} = 0$, the coordinates of $\bar{v}$ lie in $\frac{Q}{r} \mathbb{Z}$ for some $Q = O_A(1)$ and each coordinate of $\bar{t}$ is $O_A(1/N^i)$. Now choose $\beta_i \in G(i)$ in such a way that $\psi(\beta_i) = \bar{t}$ and $dG(\beta_i, \text{id}) = O_A,M(1/N_i)$, choose an $O_A,M(1)$-rational element $\gamma_i \in G(i)$ with $\psi(\gamma_i) = \bar{v}$, and finally choose $g'_i$ so that $g_i = \beta_i g'_i \gamma_i$. Then one automatically has $\psi(g'_i) = \bar{u}$, which means that $g'_i$ lies in the kernel of the $i$-homomorphism $\xi_i$.

Finally, we record a convenient scaling lemma.

**Lemma A.8 (Scaling lemma).** Let $(G/\Gamma, G_\bullet)$ be a filtered nilmanifold of complexity $\leq M$. If $g \in \text{poly}(\mathbb{Z}, G_\bullet)$ is $(A,N)$-irrational, $r \in [-N,N]$, and $1 \leq q \leq M$, then the sequence $n \mapsto g(nq+r)$ is $(\gg_{M,\varepsilon} A, \varepsilon N)$-irrational for any $\varepsilon > 0$.

**Proof.** We need to show that the $i$th Taylor coefficient of $n \mapsto g(nq+r)$ is $(\gg_{M,\varepsilon} A, \varepsilon N)$-irrational for each $i \geq 0$. Note that we may assume $i \leq M$ since the filtered manifold has degree $\leq M$.

Fix $i$. We may quotient out the nilmanifold by the normal subgroups $G(i+1)$ and $[G(j), G(i+1)]$ for $0 \leq j \leq i$, since these do not affect the irrationality of the $i$th coefficient. We may then expand $g$ as a Taylor series

$$g(n) = \prod_{j=0}^{i} g_j^{(n)}$$

and thus

$$g(nq+r) = \prod_{j=0}^{i} g_j^{(q^n+r)}$$

Expanding out the binomial coefficient and using many applications of the Baker-Campbell-Hausdorff formula, we obtain

$$g(nq+r) = \left( \prod_{j=0}^{i-1} g_j^{(\gamma_j)} \right) g_i^{(n)}$$

for some $g'_j \in G(j)$; the point being that the Baker-Campbell-Hausdorff term cannot generate any terms involving polynomials in $n$ of degree $i$ or higher due to the fact that the groups $G(i+1)$ and $[G(j), G(i+1)]$ have been quotiented out. As a consequence, we see that the $i$th Taylor coefficient of $n \mapsto g(nq+r)$ is $q^i g_i$, and the claim is easily verified. \[\square\]
Appendix B. A multiparameter equidistribution result

The purpose of this appendix is to prove Theorem 3.6, which we recall here again.

Theorem 3.6. Suppose that \((G/\Gamma, G)\) is a filtered nilmanifold of complexity \(\leq M\) and that \(g \in \text{poly}(\mathbb{Z}^D, G)\) is a polynomial sequence for some \(D \leq M\). Suppose that \(\Lambda \subseteq \mathbb{Z}^D\) is a lattice of index \(\leq M\), that \(n_0 \in \mathbb{Z}^D\) has magnitude \(\leq M\), and that \(P \subseteq [-N,N]^D\) is a convex body. Suppose that \(\delta > 0\), and that \(\left| \sum_{n \in (n_0 + \Lambda) \cap P} F(g(n)\Gamma) - \frac{\text{vol}(P)}{[\mathbb{Z}^D : \Lambda]} \int_{G/\Gamma} F \right| > \delta N^D \|F\|_{\text{Lip}}\)

for some Lipschitz function \(F : G/\Gamma \to \mathbb{C}\). Then there is a nontrivial homomorphism \(\eta : G \to \mathbb{R}\) which vanishes on \(\Gamma\), has complexity \(O_M(1)\) and such that

\[ \|\eta \circ g\|_{C^\infty([-N]^D)} = O_{\delta,M}(1). \]

Recall from [28, Definition 8.2] that the norm \(\|g\|_{C^\infty([-N]^D)}\) of a polynomial sequence \(g : [-N]^D \to \mathbb{R}\) is given by the formula

\[ \|g\|_{C^\infty([-N]^D)} = \sup_{i \in [N]^D} N^{-|i|} \|g_i\|_{\mathbb{R}/\mathbb{Z}} \]

where \(g_i\) are the Taylor coefficients of \(g\), thus

\[ g(n) = \sum_{i \in [N]^D} \binom{n}{i} g_i. \]

We now prove the theorem, allowing all implied constants to depend on \(\delta\) and \(M\). We may assume that \(N\) is sufficiently large depending on \(\delta, M\), since the claim is trivial otherwise. A simple volume packing argument (using [29, Corollary A.2], for example, to control the boundary terms) shows that

\[ |(n_0 + \Lambda) \cap P| = \frac{\text{vol}(P)}{[\mathbb{Z}^D : \Lambda]} + o_{N \to \infty}(N^D). \]

As a consequence, for \(N\) large enough we may subtract off the mean of \(F\) and normalise \(F\) to have Lipschitz norm 1 and mean zero, thus

\[ \left| \sum_{n \in (n_0 + \Lambda) \cap P} F(g(n)\Gamma) \right| \gg N^D. \]

As \(\Lambda\) has index \(\leq M\) in \(\mathbb{Z}^D\), it contains the sublattice \(q\mathbb{Z}^D\) for some positive integer \(q = O(1)\). By the pigeonhole principle, we may thus find \(n_1 \in \mathbb{Z}^D\) of magnitude \(O(1)\) such that

\[ \left| \sum_{n \in (n_1 + q\mathbb{Z}^D) \cap P} F(g(n)\Gamma) \right| \gg N^D, \]
and thus
\[ \left| \sum_{n \in \mathbb{Z}^D \cap P'} F(g(qn + n_1)\Gamma) \right| \gg N^D. \]
for some convex body \( P' \) contains in a ball of radius \( O(N) \) centered at the origin.

By subdividing \( P' \) into cubes of sidelength \( \varepsilon N \) for some sufficiently small \( \varepsilon > 0 \) (and again using [29, Corollary A.2] to control the boundary terms), and then applying the pigeonhole principle, we see that
\[ \left| \sum_{n \in \mathbb{Z}^D \cap [\varepsilon N]^D} F(g(qn + n_2)\Gamma) \right| \gg N^D \]
for some \( \varepsilon \gg 1 \) and \( n_2 = O(N) \). We can rearrange this as
\[ \left| \sum_{n \in \mathbb{Z}^D \cap [\varepsilon N]^D} F(g(qn + n_3)\Gamma) \right| \gg N^D \]
for some \( n_3 = O(N) \).

We may now invoke [28, Theorem 8.6] to conclude that there exists a nontrivial homomorphism \( \eta : G \rightarrow \mathbb{R} \) which vanishes on \( \Gamma \), has complexity \( O(1) \) and such that
\[ \| \eta \circ g(q\cdot + n_3) \|_{C^\infty([N]^D)} \ll 1. \]
Applying [28, Lemma 8.4] we conclude that
\[ \| Q \eta \circ g(\cdot + n_3) \|_{C^\infty([N]^D)} \ll 1 \]
for some non-negative integer \( Q = O(1) \). Shifting the Taylor expansion by \( n_3 \), we conclude that
\[ \| Q \eta \circ g \|_{C^\infty([N]^D)} \ll 1. \]
The claim follows (with \( \eta \) replaced by \( Q \eta \)).

**Appendix C. The Baker-Campbell-Hausdorff Formula**

Let \( G \) be a connected, simply connected nilpotent Lie group, and let \( \exp : \mathfrak{g} \rightarrow G \) and \( \log : G \rightarrow \mathfrak{g} \) be the associated exponential and logarithm maps between \( G \) and its Lie algebra \( \mathfrak{g} \). The **Baker-Campbell-Hausdorff formula** asserts that
\[ \exp(X_1)\exp(X_2) = \exp(X_1 + X_2 + \frac{1}{2}[X_1, X_2] + \prod_{\alpha} c_\alpha X_\alpha) \]
for any \( X_1, X_2 \), where \( \alpha \) is a finite set of labels, \( c_\alpha \) are real constants, and \( X_\alpha \) are an iterated Lie bracket of \( k_1 = k_{1,\alpha} \) copies of \( X_1 \) and \( k_2 = k_{2,\alpha} \) copies of \( X_2 \) where \( k_1, k_2 \geq 1 \) and \( k_1 + k_2 \geq 2 \).

Using this formula, it is a routine matter to see that for any \( g_1, g_2 \in G \) and \( x \in \mathbb{R} \), we have
\[ (g_1 g_2)^x = g_1^x g_2^x \prod_{\alpha} g_\alpha^{Q_\alpha(x)} \]  \hfill (C.1)
where $\alpha$ is a finite set of labels, each $g_\alpha$ is an iterated of $k_1 = k_{1,\alpha}$ copies of $g_1$ and $k_2 = k_{2,\alpha}$ copies of $g_2$ where $k_1, k_2 \geq 1$ and $k_1 + k_2 \geq 2$, and the $Q_\alpha : \mathbb{R} \to \mathbb{R}$ are polynomials of degree at most $k_1 + k_2$ with no constant term.

In a similar vein, for any $g_1, g_2 \in G$ and $x_1, x_2 \in \mathbb{R}$, we have the formula

$$[g_1^{x_1}, g_2^{x_2}] = [g_1, g_2]^{x_1 x_2} \prod_\alpha g_\alpha^{P_\alpha(x_1, x_2)} \tag{C.2}$$

where $\alpha$ is a finite set of labels, each $g_\alpha$ is an iterated commutator of $k_1 = k_{1,\alpha}$ copies of $g_1$ and $k_2 = k_{2,\alpha}$ copies of $g_2$ where $k_1, k_2 \geq 1$ and $k_1 + k_2 \geq 3$, and the $P_\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are polynomials of degree at most $k_1$ in $x_1$ and at most $k_2$ in $x_2$ which vanish when $x_1 = 0$ or $x_2 = 0$.

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AN ARITHMETIC REGULARITY LEMMA, AN ASSOCIATED COUNTING LEMMA, AND APPLICATIONS

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Abstract. Szemerédi’s regularity lemma can be viewed as a rough structure theorem for arbitrary dense graphs, decomposing such graphs into a structured piece, a small error, and a uniform piece. We establish an arithmetic regularity lemma that similarly decomposes bounded functions \( f : [N] \to \mathbb{C} \), into a (well-equidistributed, virtual) \( s \)-step nilsequence, an error which is small in \( L^2 \) and a further error which is minuscule in the Gowers \( U^{s+1} \)-norm, where \( s \geq 1 \) is a parameter. We then establish a complementary arithmetic counting lemma that counts arithmetic patterns in the nilsequence component of \( f \).

We provide a number of applications of these lemmas: a proof of Szemerédi’s theorem on arithmetic progressions, a proof of a conjecture of Bergelson, Host and Kra, and a generalisation of certain results of Gowers and Wolf.

Our result is dependent on the inverse conjecture for the Gowers \( U^{s+1} \) norm, recently established for general \( s \) by the authors and T. Ziegler.

To Endre Szemerédi on the occasion of his 70th birthday.

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1. Introduction

Szemerédi’s celebrated regularity lemma [18, 19] is a fundamental tool in graph theory; see for instance [36] for a survey of some of its many applications. It is often described as a structure theorem for graphs \( G = (V, E) \), but
one may also view it as a decomposition for arbitrary functions $f : V \times V \to [0, 1]$. For instance, one can recast the regularity lemma in the following “analytic” form. Define a growth function to be any monotone increasing function $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+$ with $\mathcal{F}(M) \geq M$ for all $M$.

**Lemma 1.1** (Szemerédi regularity lemma, analytic form). Let $V$ be a finite vertex set, let $f : V \times V \to [0, 1]$ be a function, let $\varepsilon > 0$, and let $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+$ be a growth function. Then there exists a positive integer $M = O_{\varepsilon, \mathcal{F}}(1)$ and a decomposition

$$f = f_{\text{str}} + f_{\text{sml}} + f_{\text{unf}}$$

of $f$ into functions $f_{\text{str}}, f_{\text{sml}}, f_{\text{unf}} : V \times V \to [-1, 1]$ such that:

(i) ($f_{\text{str}}$ structured) $V$ can be partitioned into $M$ cells $V_1, \ldots, V_M$, such that $f_{\text{str}}$ is constant on $V_i \times V_j$ for all $i, j$ with $1 \leq i, j \leq M$;

(ii) ($f_{\text{sml}}$ small) The quantity $\|f_{\text{sml}}\|_{L^2(V \times V)} := (\mathbb{E}_{v, w \in V} |f_{\text{sml}}(v, w)|^2)^{1/2}$ is at most $\varepsilon$.

(iii) ($f_{\text{unf}}$ very uniform) The box norm $\|f_{\text{unf}}\|_{\boxplus(V \times V)}$, defined to be the quantity

$$(\mathbb{E}_{v_1, v_2, w_1, w_2 \in V} f_{\text{unf}}(v_1, w_1) f_{\text{unf}}(v_1, w_2) f_{\text{unf}}(v_2, w_1) f_{\text{unf}}(v_2, w_2))^{1/4},$$

is at most $1/\mathcal{F}(M)$.

(iv) (Nonnegativity) $f_{\text{str}}$ and $f_{\text{str}} + f_{\text{sml}}$ take values in $[0, 1]$.

Informally, this regularity lemma decomposes any bounded function into a structured part, a small error, and an extremely uniform error. While this formulation does not, at first sight, look much like the usual regularity lemma, it easily implies that result: see [53]. The idea of formulating the regularity lemma with an arbitrary growth function $\mathcal{F}$ first appears in [1], and is also very useful for generalisations of the regularity lemma to hypergraphs. See, for example, [52]. The bound on $M$ turns out to essentially be an iterated version of the growth function $\mathcal{F}$, with the number of iterations being polynomial in $1/\varepsilon$. In applications, one usually selects the growth function to be exponential in nature, which then makes $M$ essentially tower-exponential in $1/\varepsilon$. See [51] for a general discussion of these sorts of structure theorems and their applications in combinatorics. See also [52] for a related analytical perspective on the regularity lemma.

In applications the regularity lemma is often paired with a counting lemma that allows one to control various expressions involving the function $f$. For example, one might consider the expression

$$\mathbb{E}_{u, v, w \in V} f(u, v) f(v, w) f(w, u),$$

(1.2)
which counts triangles in $V$ weighted by $f$. Applying the decomposition (1.1) splits expressions such as (1.2) into multiple terms (in this instance, 27 of them). The key fact, which is a slightly non-trivial application of the Cauchy-Schwarz inequality, is that the terms involving the box-norm-uniform error $f_{\text{unf}}$ are negligible if the growth function $F$ is chosen rapidly enough. The terms involving the small error $f_{\text{sml}}$ are somewhat small, but one often has to carefully compare those errors against the main term (which only involves $f_{\text{str}}$) in order to get a non-trivial bound on the final expression (1.2). In particular, one often needs to exploit the positivity of $f_{\text{str}}$ and $f_{\text{str}} + f_{\text{sml}}$ to first localise expressions such as (1.2) to a small region (such as the portion of a graph between a “good” triple $V_i, V_j, V_k$ of cells in the partition of $V$ associated to $f_{\text{str}}$) before one can obtain a useful estimate.

The graph regularity and counting lemmas can be viewed as the first non-trivial member of a hierarchy of hypergraph regularity and counting lemmas, see e.g. [9, 19, 20, 43, 44, 52]. The formulation in [52] is particularly close to the formulation given in Theorem 1.1. These lemmas are suitable for controlling higher order expressions such as

$$E_{u,v,w,x \in V} f(u,v,w) f(v,w,x) f(w,x,u) f(x,u,v).$$

Our objective in this paper is to introduce an analogous hierarchy of such regularity and counting lemmas (one for each integer $s \geq 1$), in arithmetic situations. Here, the aim is to decompose a function $f : [N] \rightarrow [0,1]$ defined on an arithmetic progression $[N] := \{1, \ldots, N\}$ instead of a graph. One is interested in counting averages such as

$$E_{n,r \in [N]} f(n) f(n+r) f(n+2r),$$

which counts 3-term arithmetic progressions weighted by $f$, as well as higher order expressions such as

$$E_{n,r \in [N]} f(n) f(n+r) f(n+2r) f(n+3r).$$

As it turns out, the former average will be best controlled using the $s=1$ regularity and counting lemmas, while the latter requires the $s=2$ versions of these lemmas. In this paper we shall see several examples of these types of applications of the two lemmas.

**The arithmetic regularity lemma.** We begin with by formulating our regularity lemma. Following the statement we explain the terms used here.

**Theorem 1.2** (Arithmetic regularity lemma). Let $f : [N] \rightarrow [0,1]$ be a function, let $s \geq 1$ be an integer, let $\varepsilon > 0$, and let $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a growth function. Then there exists a quantity $M = O_{s,\varepsilon,F}(1)$ and a decomposition

$$f = f_{\text{nil}} + f_{\text{sml}} + f_{\text{unf}}$$

of $f$ into functions $f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}} : [N] \rightarrow [-1,1]$ of the following form:
(i) \((f_{\text{nil structured}})\, f_{\text{nil}}\) is a \((\mathcal{F}(M), N)\)-irrational virtual nilsequence of degree \(\leq s\), complexity \(\leq M\), and scale \(N\);
(ii) \((f_{\text{small}})\, f_{\text{small}}\) has an \(L^2[N]\) norm of at most \(\varepsilon\);
(iii) \((f_{\text{unif very uniform}})\, f_{\text{unif}}\) has a \(U^{s+1}[N]\) norm of at most \(1/\mathcal{F}(M)\);
(iv) \((\text{Nonnegativity})\) \(f_{\text{nil}}\) and \(f_{\text{nil}} + f_{\text{small}}\) take values in \([0, 1]\).

Remark. This result easily implies the recently proven inverse conjecture for the Gowers norms (Theorem 2.2). Conversely, this inverse conjecture, together with the equidistribution theory of nilsequences, will be the main ingredient used to prove Theorem 1.2.

We prove this theorem in §5. We turn now to a discussion of the various concepts used in the above statement. Readers who are interested in applications may skip ahead to the end of the section.

The \(L^2[N]\) norm, used to control \(f_{\text{small}}\), is simply
\[
\|f\|_{L^2[N]} := (\mathbb{E}_{n \in [N]} |f(n)|^2)^{1/2}.
\]

We turn next to the Gowers uniformity norm \(U^{s+1}[N]\), used to control \(f_{\text{unif}}\). If \(f : G \to \mathbb{C}\) is a function on a finite additive group \(G\), and \(k \geq 1\) is an integer, then the Gowers uniformity norm \(\|f\|_{U^k(G)}\) is defined by the formula
\[
\|f\|_{U^k(G)} := (\mathbb{E}_{x, h_1, \ldots, h_k \in G} \Delta_{h_1} \cdots \Delta_{h_k} f(x))^{1/2^k},
\]
where \(\Delta_h f : G \to \mathbb{C}\) is the multiplicative derivative of \(f\) in the direction \(h\), defined by the formula
\[
\Delta_h f(x) := f(x + h)\overline{f(x)} .
\]

In this paper we will be concerned with functions on \([N]\), which is not quite a group. To define the Gowers norms of a function \(f : [N] \to \mathbb{C}\), set \(G := \mathbb{Z}/\mathbb{N}\mathbb{Z}\) for some integer \(\bar{N} \geq 2^k\), define a function \(\tilde{f} : G \to \mathbb{C}\) by \(\tilde{f}(x) = f(x)\) for \(x = 1, \ldots, \mathbb{N}\) and \(\tilde{f}(x) = 0\) otherwise, and set \(\|f\|_{U^k[N]} := \|\tilde{f}\|_{U^k(G)}\|1_{[N]}\|_{U^k(G)}\), where \(1_{[N]}\) is the indicator function of \([N]\). It is easy to see that this definition is independent of the choice of \(\bar{N}\), and so for definiteness one could take \(\bar{N} := 2^k\). Henceforth we shall write simply \(\|f\|_{U^k}\), rather than \(\|f\|_{U^k[N]}\), since all Gowers norms will be on \([N]\). One can show that \(\|\cdot\|_{U^k}\) is indeed a norm for any \(k \geq 2\), though we shall not need this here; see [18]. For further discussion of the Gowers norms and their relevance to counting additive patterns see [18, 27, §5] or [55, §11].

Finally, we turn to the notion of an irrational virtual nilsequence, which is the concept that defines the structural component \(f_{\text{nil}}\). This is the most complicated concept, and requires a certain number of preliminary definitions. We first need the notion of a filtered nilmanifold. The first two sections of §30 may be consulted for a more detailed discussion.

**Definition 1.3** (Filtered nilmanifold). Let \(s \geq 1\) be an integer. A filtered nilmanifold \(G/T = (G/T,G_*)\) of degree \(\leq s\) consists of the following data:

(i) A connected, simply-connected nilpotent Lie group \(G\);
(ii) A discrete, cocompact subgroup $\Gamma$ of $G$ (thus the quotient space $G/\Gamma$ is a compact manifold, known as a nilmanifold);

(iii) A filtration $G_\bullet = (G(i))_{i=0}^\infty$ of closed connected subgroups $G = G(0) \supseteq G(1) \supseteq G(2) \supseteq \ldots$ of $G$, which are rational in the sense that the subgroups $\Gamma(i) := \Gamma \cap G(i)$ are cocompact in $G(i)$, such that $[G(i), G(j)] \subseteq G(i+j)$ for all $i,j \geq 0$, and such that $G(i) = \{\text{id}\}$ whenever $i > s$;

(iv) A Mal’cev basis $X = (X_1, \ldots, X_{\dim(G)})$ adapted to $G_\bullet$, that is to say a basis $X_1, \ldots, X_{\dim(G)}$ of the Lie algebra of $G$ that exponentiates to elements of $\Gamma$, such that $X_j, \ldots, X_{\dim(G)}$ span a Lie algebra ideal for all $j \leq i \leq \dim(G)$, and $X_{\dim(G)-\dim(G(i))}, \ldots, X_{\dim(G)}$ spans the Lie algebra of $G(i)$ for all $1 \leq i \leq s$. (For a detailed discussion of this concept, see [30, §2].)

Once a Mal’cev basis has been specified, notions such as the rationality of subgroups may be quantified in terms of it. Furthermore one may use a Mal’cev basis to define a metric $d_{G/\Gamma}$ on the nilmanifold $G/\Gamma$. The results of this paper are rather insensitive to the precise metric that one takes, but one may proceed for example as in [30, Definition 2.2]. We encourage the reader not to think too carefully about the precise definition (or about Mal’cev bases in general), but it is certainly important to have some definite metric in mind so that one can make sense of notions such as that of a Lipschitz function on $G/\Gamma$.

Observe that every filtered nilmanifold $G/\Gamma$ comes with a canonical probability Haar measure $\mu_{G/\Gamma}$, defined as the unique Borel probability measure on $G/\Gamma$ that is invariant under the left action of $G$. We abbreviate $\int_{G/\Gamma} F(x) \ d\mu_{G/\Gamma}(x)$ as $\int_{G/\Gamma} F$.

We will need a quantitative notion of complexity for filtered nilmanifolds, though once again, the precise definition is somewhat unimportant.

**Definition 1.4** (Complexity). Let $M \geq 1$. We say that a filtered nilmanifold $G/\Gamma = (G/\Gamma, G_\bullet)$ has complexity $\leq M$ if the dimension of $G$, the degree of $G_\bullet$, and the rationality of the Mal’cev basis $X$ (cf. [30, Definition 2.4]) are bounded by $M$.

**Heisenberg example.** The model example of a degree $\leq 2$ filtered nilmanifold is the Heisenberg nilmanifold

$$G/\Gamma := \left( \begin{pmatrix} 1 & R & R \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) / \left( \begin{pmatrix} 1 & Z & Z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

with the lower central series $G(0) = G(1) = G$ and $G(2) = [G, G] = \left( \begin{pmatrix} 1 & 0 & R \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$

with Mal’cev basis $X = \{X_1, X_2, X_3\}$ consisting of the matrices

$$X_1 = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad X_2 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \quad X_3 = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).$$
With the definition of filtered nilmanifold in place, the next thing we need is the idea of a *polynomial sequence*. The basic theory of such sequences was laid out in Leibman [37], and was extended slightly to general filtrations in [30]. An extensive discussion may be found in Section 6 of that paper.

**Definition 1.5** (Polynomial sequence). Let \((G/\Gamma, G_\bullet)\) be a filtered nilmanifold, with filtration \(G_\bullet = (G_\bullet(i))_{i=0}^\infty\). A *(multidimensional)* polynomial sequence adapted to this filtered nilmanifold is a sequence \(g : \mathbb{Z}^D \to G\) for some \(D \geq 1\) with the property that

\[ \partial_{h_1} \cdots \partial_{h_i} g(n) \in G_\bullet(i) \]

for all \(i \geq 0\) and \(h_1, \ldots, h_i, n \in \mathbb{Z}^D\), where \(\partial_h g(n) := g(n + h)g(n)^{-1}\) is the derivative of \(g\) with respect to the shift \(h\). The space of all such polynomial sequences will be denoted \(\text{poly}(\mathbb{Z}^D, G_\bullet)\). The space of polynomial sequences taking values in \(\Gamma\) will be denoted \(\text{poly}(\mathbb{Z}^D, \Gamma_\bullet)\). When \(D = 1\), we refer to multidimensional polynomial sequences simply as polynomial sequences.

**Remark.** We will be primarily interested in the one-dimensional case \(D = 1\), but will need the higher \(D\) case in order to establish the counting lemma, Theorem 1.11.

One of the main reasons why we work with polynomial sequences, instead of just linear sequences such as \(n \mapsto g_0 g_n\), is that objects of the former type constitute a group.

**Theorem 1.6** (Lazard-Leibman). If \((G/\Gamma, G_\bullet)\) is a filtered nilmanifold and \(D \geq 1\) is an integer, then \(\text{poly}(\mathbb{Z}^D, G_\bullet)\) is a group (and \(\text{poly}(\mathbb{Z}^D, \Gamma_\bullet)\) is a subgroup).

**Proof.** See [38] or [30, Proposition 6.2].

With the concept of a polynomial sequence in hand, it is easy to define a polynomial orbit.

**Definition 1.7** (Orbits). Let \(D, s \geq 1\) be integers, and \(M, A > 0\) be parameters. A *(multidimensional)* polynomial orbit of degree \(\leq s\) and complexity \(\leq M\) is any function \(n \mapsto g(n)\Gamma\) from \(\mathbb{Z}^D \to G/\Gamma\), where \((G/\Gamma, G_\bullet)\) is a filtered nilmanifold of complexity \(\leq M\), and \(g \in \text{poly}(\mathbb{Z}^D, G_\bullet)\) is a (multidimensional) polynomial sequence.

Using the concept of polynomial orbit, we can define the notion of a (polynomial) nilsequence, as well as a generalisation which we call a virtual nilsequence, in analogy with virtually nilpotent groups (groups with a finite index nilpotent subgroup).

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3Strictly speaking, the orbit is the tuple of data \((G, \Gamma, G_\bullet, n \mapsto g(n)\Gamma)\), rather than just the sequence \(n \mapsto g(n)\Gamma\), but we shall abuse notation and use the sequence as a metonym for the whole orbit.
Definition 1.8 (Nilsequences). A \textit{nilsequence} of degree \( \leq s \) and complexity \( \leq M \) is any function \( f : \mathbb{Z}^D \to \mathbb{C} \) of the form \( f(n) = F(g(n)\Gamma) \), where \( n \mapsto g(n)\Gamma \) is a polynomial orbit of degree \( \leq s \) and complexity \( \leq M \), and \( F : G/\Gamma \to \mathbb{C} \) is a function of Lipschitz norm at most \( M \).

Definition 1.9 (Virtual nilsequences). Let \( N \geq 1 \). A \textit{virtual nilsequence} of degree \( \leq s \) and complexity \( \leq M \) at scale \( N \) is any function \( f : [N] \to \mathbb{C} \) of the form \( f(n) = F(g(n)\Gamma, n_1(n) \mod q, n/N) \), where \( 1 \leq q \leq M \) is an integer, \( n \mapsto g(n)\Gamma \) is a polynomial orbit of degree \( \leq s \) and complexity \( \leq M \), and \( F : G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R} \to \mathbb{C} \) is a function of Lipschitz norm at most \( M \). (Here we place a metric on \( G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R} \) in some arbitrary fashion, e.g. by embedding \( \mathbb{Z}/q\mathbb{Z} \) in \( \mathbb{R}/\mathbb{Z} \) and taking the direct sum of the metrics on the three factors.)

One concept that featured in Theorem 1.2 remains to be defined: that of an \textit{irrational} orbit. The definition is a little technical (see Definition A.6) and takes some setting up, and so we defer it and the discussion of some motivating examples to Appendix A. Very roughly speaking, an irrational orbit is one whose coefficients are not close to rationals (of bounded height) and for which the filtration \( G_i \) is as small as possible. For instance, with a polynomial sequence \( P : [N] \to \mathbb{R}/\mathbb{Z} \) of the form \( P(n) = \alpha s n^s + \ldots + \alpha_0 \), then (roughly speaking) this sequence would be considered irrational if one takes \( G(i) = \mathbb{R} \) for \( i \leq s \) and \( G(i) = \{0\} \) for \( i > s \), and if there was no positive integer \( q = O(1) \) for which \( \|q\alpha_s\|_{\mathbb{R}/\mathbb{Z}} \ll N^{-s} \). Again, we refer the reader to Appendix A for further examples and discussion.

This concludes our attempt to discuss all the concepts involved in the arithmetic regularity lemma, Theorem 1.2; we turn now to a statement and discussion of the counting lemma.

**Counting lemma.** In applications of the arithmetic regularity lemma, we will be interested in counting additive patterns such as arithmetic progressions or parallelepipeds. To understand the phenomena properly it is advantageous to work in a somewhat general setting similar to that taken in [22, 23, 24, 31]. In the latter paper one works with a family \( \Psi = (\psi_1, \ldots, \psi_t) \) of integer-coefficient linear forms (or equivalently, group homomorphisms) \( \psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z} \), and consider expressions such as
\[
\mathbb{E}_{n \in \mathbb{Z}^D} \psi_1(n) \ldots \psi_t(n) \tag{1.3}
\]

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4The (inhomogeneous) Lipschitz norm \( \|F\|_{\text{Lip}} \) of a function \( F : X \to \mathbb{C} \) on a metric space \( X = (X, d) \) is defined as
\[
\|F\|_{\text{Lip}} := \sup_{x \in X} |F(x)| + \sup_{x, y \in X : x \neq y} \frac{|F(x) - F(y)|}{|x - y|}.
\]
where $P$ is a convex subset of $\mathbb{R}^D$. Thus, for instance, if counting arithmetic progressions, one might use the linear forms
\begin{equation}
\psi_i(n_1, n_2) := n_1 + (i - 1)n_2; \quad i = 1, \ldots, k
\end{equation}
whilst for counting parallelepipeds one might instead use the linear forms
\begin{equation}
\psi_{\omega_1, \ldots, \omega_k}(n_0, n_1, \ldots, n_k) := n_0 + \omega_1 n_1 + \ldots + \omega_k n_k; \quad \omega_1, \ldots, \omega_k \in \{0, 1\}.
\end{equation}

In order to understand the contribution to (1.3) coming from the structured part $f_\text{nil}$ of $f$, one is soon faced with the question of understanding the equidistribution of the orbit
\begin{equation}
(g(\psi_1(n))\Gamma, \ldots, g(\psi_t(n))\Gamma)
\end{equation}
inside $(G/\Gamma)^t$, where $n = (n_1, \ldots, n_D)$ ranges over $\mathbb{Z}^D \cap P$. We abbreviate this orbit as $g^\Psi(n)\Gamma^t$, where $g^\Psi : \mathbb{Z}^D \to G^t$ is the polynomial sequence
\begin{equation}
g^\Psi(n) := (g(\psi_1(n)), \ldots, g(\psi_t(n))).
\end{equation}

A very useful model for this question, in which infinite orbits were considered in the “linear” case $g(n) = g^\Psi x$, was studied by Leibman [41]. His work leads one to the following definition.

**Definition 1.10** (The Leibman group). Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$. For any $i \geq 1$, define $\Psi[i]$ to be the linear subspace of $\mathbb{R}^k$ spanned by the vectors $(\psi_i^1(n), \ldots, \psi_i^j(n))$ for $1 \leq j \leq i$ and $n \in \mathbb{Z}^D$. Given a filtered nilmanifold $(G/\Gamma, G\star)$, we define the Leibman group $G^\Psi \triangleleft G^t$ to be the Lie subgroup of $G^t$ generated by the elements $g_i^{\vec{v}_i}$ for $i \geq 1$, $g_i \in G(\iota)$, and $\vec{v}_i \in \Psi[i]$, with the convention that for each $g \in G$. Note that $G^\Psi$ is normal in $G^t$ because $G(\iota)$ is normal in $G$. We will show in [33] that $G^\Psi$ is also a rational subgroup of $G^t$, thus $\Gamma^\Psi := \Gamma^t \cap G^\Psi$ is a discrete cocompact subgroup of $G^\Psi$.

**Examples.** Two particular instances of this construction correspond to the two lattices (1.4) and (1.3) above. In the case of arithmetic progressions, where $\Psi$ is as in (1.4), the Leibman group $G^\Psi$ is sometimes (see, for example, [12]) referred to as the *Hall-Petresco group* $\text{HP}^k(G\star)$ and has the particularly simple alternative description
\begin{equation}
\text{HP}^k(G\star) = G^\Psi = \{(g(0), \ldots, g(k - 1)) : g \in \text{poly}(G\star)\}.
\end{equation}

We will prove this fact in [33]. In the case of parallelepipeds, where $\Psi$ is as in (1.3), the Leibman group $G^\Psi$ has been referred to as the *Host-Kra cube group* [31] and it too has an alternative description. See [31] Appendix E.

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5We define $g^v$ for real $v$ by the formula $g^v := \exp(v \log(g))$, where $\exp : g \to G$ is the usual exponential map from the Lie algebra $\mathfrak{g}$ to $G$ (this is a homeomorphism since $G$ is nilpotent, connected, and simply connected).
for more information: we will not be making use of this particular group here.

Let \( g \in \text{poly}(\mathbb{Z}, G_\bullet) \) be a polynomial sequence, and let \( \Psi = (\psi_1, \ldots, \psi_t) \) be a collection of linear forms \( \psi_1, \ldots, \psi_t : \mathbb{Z}^d \to \mathbb{Z} \). It turns out (see Lemma 3.2) that the sequence \( g^\Psi \) takes values in \( G^\Psi \). More remarkably, the orbit (1.6) is in fact totally equidistributed on \( G^\Psi / \Gamma^\Psi \) if \( g \) is sufficiently irrational. It is this result that we refer to as our counting lemma.

**Theorem 1.11** (Counting lemma). Let \( M, D, t, s \) be integers with \( 1 \leq D, t, s \leq M \), let \( (G/\Gamma, G_\bullet) \) be a degree \( \leq s \) filtered nilmanifold of complexity \( \leq M \), let \( g : \mathbb{Z} \to G \) be an \((A, N)\)-irrational polynomial sequence adapted to \( G_\bullet \), let \( \Psi = (\psi_1, \ldots, \psi_t) \) be a collection of linear forms \( \psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z} \) with coefficients of magnitude at most \( M \), and let \( P \) be a convex subset of \([-N, N]^D\). Then for any Lipschitz function \( F : (G/\Gamma)^t \to \mathbb{C} \) of Lipschitz norm at most \( M \), one has

\[
\sum_{n \in \mathbb{Z}^D \cap P} F(g^\Psi(n)\Gamma^t) = \text{vol}(P) \int_{g(0)^\Delta G^\Psi / \Gamma^\Psi} F
+ o_{A \to \infty; M}(N^D) + o_{N \to \infty; M}(N^D),
\]

where \( g(0)^\Delta := (g(0), \ldots, g(0)) \in G^t \) and the integral is with respect to the probability Haar measure \( \mu_{g(0)^\Delta G^\Psi / \Gamma^\Psi} \) on the coset \( g(0)^\Delta G^\Psi / \Gamma^\Psi \), viewed as a subnilmanifold of \((G/\Gamma)^t\), and \( \text{vol}(P) \) is the Lebesgue measure of \( P \) in \( \mathbb{R}^D \).

More generally, whenever \( \Lambda \subseteq \mathbb{Z}^D \) is a sublattice of index \([\mathbb{Z}^D : \Lambda] \leq M\), and \( n_0 \in \mathbb{Z}^D \), one has

\[
\sum_{n \in (n_0 + \Lambda) \cap P} F(g^\Psi(n)\Gamma^t) = \frac{\text{vol}(P)}{[\mathbb{Z}^D : \Lambda]} \int_{g(0)^\Delta G^\Psi / \Gamma^\Psi} F
+ o_{A \to \infty; M}(N^D) + o_{N \to \infty; M}(N^D).
\]

The counting lemma is, of course, best understood by seeing it in action as we shall do several times later on. The errors \( o_{A \to \infty; M}(N^D) \) and \( o_{N \to \infty; M}(N^D) \) are negligible in most applications, as \( A \) will typically be a huge function \( F(M) \) of \( M \), and \( N \) can also be taken to be arbitrarily large compared to \( M \).

We remark that one could easily extend the above lemma to control averages of virtual irrational nilsequences, rather than just irrational sequences, by introducing some additional integrations over the local factors \( \mathbb{Z}/q\mathbb{Z} \) and \( \mathbb{R} \), but this would require even more notation than is currently being used and so we do not describe such an extension here.

\[^{6}\text{We use } o_{A \to \infty; M}(X) \text{ to denote a quantity bounded in magnitude by } c_M(A)X, \text{ where } c_M(A) \to 0 \text{ as } A \to \infty \text{ for fixed } M. \text{ Similarly for other choices of subscripts.}\]
APPLICATIONS. The proofs of the regularity and counting lemmas occupy about half the paper. In the remaining half, we give a number of applications of these results to problems in additive combinatorics. The scheme of the arguments in all of these cases is similar. First, one applies the arithmetic regularity lemma to decompose the relevant function \( f \) into structured, small, and (very) uniform components \( f = f_{\text{nil}} + f_{\text{sml}} + f_{\text{unf}} \). Very roughly speaking, these are analysed as follows:

(i) \( f_{\text{nil}} \) is studied using algebraic properties of nilsequences, particularly the counting lemma;
(ii) \( f_{\text{sml}} \) is shown to be negligible, though often (unfortunately) some additional algebraic input is required to ensure that this error does not conspire to destroy the contribution from \( f_{\text{nil}} \);
(iii) \( f_{\text{unf}} \) is easily shown to be negligible using results of “generalised von Neumann” type as discussed in \( \S \).

As we shall see, dealing with the error \( f_{\text{sml}} \) can cause a certain amount of pain. To show that this error is truly negligible, one often has to prove that patterns guaranteed by \( f_{\text{nil}} \) (such as arithmetic progressions) do not concentrate on some small set which might be contained in the support of \( f_{\text{sml}} \).

We now give specific examples of this paradigm. In \( \S 6 \) we give a “new” proof of Szemerédi’s famous theorem on arithmetic progressions. This is hardly exciting nowadays, with at least 16 proofs already in the literature \([2, 3, 6, 10, 14, 18, 44, 45, 48, 50, 52, 56]\) as well as (slightly implicitly) in \([4, 35, 58]\). However this proof makes the point that for a certain class of problems it suffices to “check the result for nilsequences”, and in so doing one really sees the structure of the problem. Just as random and structured graphs are two obvious classes to test conjectures against in graph theory, we would like to raise awareness of nilsequences as potential (and, in certain cases such as this one, the only) sources of counterexamples.

The second application, proven in \( \S 5 \), is to establish a conjecture of Bergelson, Host and Kra \([4]\). Here and in the sequel we use the notation \( X \ll_{\alpha, \varepsilon} Y \) or \( Y \gg_{\alpha, \varepsilon} X \) synonymously with \( X = O_{\alpha, \varepsilon}(Y) \), and similarly for other choice of subscripts.

Theorem 1.12 (Bergelson-Host-Kra conjecture). Let \( k = 1, 2, 3 \) or 4, and suppose that \( 0 < \alpha < 1 \) and \( \varepsilon > 0 \). Then for any \( N \geq 1 \) and any subset \( A \subseteq [N] \) of density \( |A| \geq \alpha N \), one can find \( \gg_{\alpha, \varepsilon} N \) values of \( d \in [-N, N] \) such that there are at least \( (\alpha^k - \varepsilon)N \) \( k \)-term arithmetic progressions in \( A \) with common difference \( d \).

Remarks. The claim is trivial for \( k = 1 \), and follows from an easy averaging argument when \( k = 2 \). This theorem was established in the case \( k = 3 \) by the first author in \([25]\): we give a new proof of this result which may be of independent interest. The case \( k = 4 \) is new, although a finite field analogue of this result previously appeared in lecture notes of the first author \([26]\) (reporting on joint work). Our proof of the \( k = 4 \) argument relies on
the inverse conjecture for the $U^3$ norm, proven in \cite{28}. A counterexample example of Ruzsa in the appendix to \cite{4} shows that Theorem 1.12 fails when $k \geq 5$. An ergodic counterpart to Theorem 1.12 (which, roughly speaking, replaces a single scale $N$ with a sequence of scales going to infinity and takes a limit), using a related but slightly different argument, was established in \cite{4}.

Finally, in \cite{7} we establish a generalisation of a recent result of Gowers and Wolf \cite{22, 23, 24} regarding the “true” complexity of a system of linear forms.

**Theorem 1.13.** Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms from $\mathbb{Z}^D \to \mathbb{Z}$, and let $s \geq 1$ be an integer such that the polynomials $\psi_1^{s+1}, \ldots, \psi_t^{s+1}$ are linearly independent. Then for any function $f : [N] \to \hat{\mathbb{C}}$ bounded in magnitude by 1 (and defined to be zero outside of $[N]$) obeying the bound $\|f\|_{U^{s+1}[N]} \leq \delta$ for some $\delta > 0$, one has

$$\mathbb{E}_{n \in [N]^D} \prod_{i=1}^t f(\psi_i(n)) = o_{\delta \to 0,s,D,t,\Psi}(1).$$

**Remarks.** This result was conjectured in \cite{22}, where it was shown that the linear independence hypothesis was necessary. The programme in \cite{22, 23, 24} gives an alternate approach to this result that avoids explicit mention of nilsequences, and in particular establishes the counterpart to Theorem 1.13 in finite characteristic; their work also gives a proof of this theorem in the case when the Cauchy-Schwarz complexity of the system (see Theorem 1.1) is at most two, and with better bounds than our result, which is all but ineffective. It is worth mentioning that the arguments in \cite{22, 23, 24} also develop several structural decomposition theorems along the lines of Theorem 1.2, but using the language of (high-rank) locally polynomial phases rather than (irrational) nilsequences.

**Relation to previous work.** A result closely related to Theorem 1.2 in the case $s = 1$ was proved by Bourgain as long ago as 1989 \cite{7}. In that paper, the decomposition was applied to give a different proof of *Roth’s theorem*, that is to say Szemerédi’s theorem for 3-term progressions. A different take on this result was supplied by the first author in \cite{25}, where the application to the case $k = 3$ of the Bergelson-Host-Kra conjecture was noted. In that same paper a construction of Gowers \cite{16} was modified to show that any application of the arithmetic regularity lemma must lead to awful (tower-type) bounds; the same kind of construction would show that the cases $s \geq 2$ of Theorem 1.2 lead to tower-type bounds as well. In \cite{26} the analogue of the case $s = 2$ of Theorem 1.2 in a finite field setting was stated, proved, and used to deduce the finite field analogue of the Bergelson-Host-Kra conjecture in the case $k = 4$. In that same paper the present work

\[\text{7The relevant part of these lecture notes by the first author reported on joint work of the two of us.}\]
was promised (as reference [22]) at “some future juncture”. Four years later we have reached that juncture and we apologise for the delay. We note, however, that until the very recent resolution of the inverse conjectures for the Gowers norms [33, 34] many of our results would have been conditional; furthermore, we are heavily dependent on our work [30], which had not been envisaged when the earlier promise was made.

In the meantime a greater general understanding of decomposition theorems of this type has developed through the work of Gowers [21], Reingold-Trevisan-Tulsiani-Vadhan [46], and Gowers-Wolf [22, 23, 24]; see also the survey [54] of the second author. While Theorem 1.2 is related to several of these general decomposition theorems, it also relies upon specific structure of nilmanifolds. In any case it seems appropriate, in this volume, to give a proof using the “energy increment argument” pioneered by Szemerédi.

The ergodic theory analogue of Theorem 1.2 is the classification of characteristic factors for the Gowers-Host-Kra seminorms $\| \cdot \|_{U^{s+1}(X)}$ (the ergodic theory counterpart of the Gowers norms) as inverse limits of nilsystems, which was first established by Host and Kra [35]. Roughly speaking, this classification allows one to decompose any bounded non-negative function $f \in L^\infty(X)$ in an (ergodic) measure-preserving system as a sum $f = f_{\text{str}} + f_{\text{sml}} + f_{\text{unf}}$, where $\|f_{\text{unf}}\|_{U^{s+1}(X)} = 0$, $f_{\text{sml}}$ is as small as one wishes in the $L^2(X)$ norm, and $f_{\text{str}}$ arises from an $s$-step nilsystem factor of $X$. This fundamental decomposition has many applications; for instance, in [4] it was used (together with the Furstenberg correspondence principle) to establish an ergodic analogue of Theorem 1.12 in which $A$ is a set of integers rather than a finite subset of $[N]$, with the notion of upper density replacing the notion of cardinality. It appears however that this correspondence principle does not directly yield “single-scale” results such as Theorem 1.12 from the ergodic theory results.

Acknowledgments. BG was, while this work was being carried out, a fellow at the Radcliffe Institute at Harvard. He is very happy to thank the Institute for proving excellent working conditions. TT is supported by a grant from the MacArthur Foundation, by NSF grant DMS-0649473, and by the NSF Waterman award. The authors also thank Tim Gowers, Julia Wolf, and Tamar Ziegler for helpful comments and corrections.

2. Proof of the arithmetic regularity lemma

We now prove Theorem 1.2. The proof proceeds in two main stages. Firstly, we establish a “non-irrational regularity lemma”, which establishes a weaker version of Theorem 1.2 in which the structured component $f_{\text{nil}}$ is a polynomial nilsequence, but one which is not assumed to be irrational. The main tool here is the inverse conjecture GI(s) for the Gowers norms [34], combined with the energy incrementation argument that appears in proofs of the graph regularity lemma. In the second stage, we upgrade this weaker regularity lemma to the full regularity lemma by converting the nilsequence
to a irrational nilsequence. The main tool here is a dimension reduction argument and a factorisation of nilsequences similar to that appearing in [30].

The non-irrational regularity lemma. We begin the first stage of the argument. As mentioned above, the key ingredient is the following result.

**Theorem 2.1 (GI(s)).** Let \( s \geq 1 \), and suppose that \( f : [N] \to \mathbb{C} \) is a function bounded in magnitude by 1 such that \( \|f\|_{L^{s+1}[N]} \geq \delta \) for some \( \delta > 0 \). Then there is a degree \( \leq s \) polynomial nilsequence \( \psi : \mathbb{Z} \to \mathbb{C} \) of complexity \( O_{s,\delta}(1) \) such that

\[
\langle f, \psi \rangle_{L^2[N]} := \mathbb{E}_{n \in [N]} f(n) \overline{\psi(n)}
\]

is the usual inner product.

*Remark.* The difficulty of this conjecture increases with \( s \). The case \( s = 1 \) easily follows from classical harmonic analysis. The case \( s = 2 \) was established by the authors in [28], building upon the breakthrough paper of Gowers [17]. The case \( s = 3 \) was recently established by the authors and Ziegler in [33], and the general case will appear in the forthcoming paper [34] by the authors and Ziegler.

For technical reasons, it is convenient to replace the notion of a degree \( \leq s \) polynomial nilsequence by a slightly different concept. The following definition is not required beyond the end of the proof of Proposition 2.7.

**Definition 2.2 (s-measurability).** Let \( \Phi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a growth function and \( s \geq 1 \). A subset \( E \subseteq [N] \) is said to be \( s \)-measurable with growth function \( \Phi \) if for every \( M \geq 1 \), there exists a degree \( \leq s \) polynomial nilsequence \( \psi : \mathbb{Z} \to [0,1] \) of complexity \( \leq \Phi(M) \) such that

\[
\| \psi - 1_E \|_{L^2[N]} \leq 1/M.
\]

An example of a 1-measurable function would be a regular Bohr set, as introduced in [8] and discussed further in [28, §2]. We will not need Bohr sets elsewhere in this paper, so we shall not dwell any longer on this example. However the reader will see ideas related to the basic theory of those sets in the proof of Corollary 2.3 below.

We make the simple but crucial observation that if \( E, F \) are \( s \)-measurable with some growth functions \( \Phi, \Phi' \) respectively, then boolean combinations of \( E, F \) such as \( E \cap F, E \cup F, \) or \( [N] \setminus E \) are also \( s \)-measurable with some growth function depending on \( \Phi, \Phi' \). Underlying this, of course, is that fact that the product and sum of two nilsequences is also a nilsequence, and hence the set of nilsequences form a kind of algebra (graded by complexity). The role of algebraic structure of this kind was brought to the fore in the work of Gowers [21] cited above.

Theorem 2.1 then implies
Corollary 2.3 (Alternate formulation of GI(s)). Let $s \geq 1$, and suppose that $f : [N] \to [-1, 1]$ is such that $\|f\|_{U^{s+1}[N]} \geq \delta$ for some $\delta > 0$. Then there exists a growth function $\Phi_{s, \delta}$ depending only on $s, \delta$, and an $s$-measurable set $E \subset N$ with growth function $\Phi_{s, \delta}$, such that

$$|\mathbb{E}_{n \in [N]} f(n) 1_E(n)| \gg_{s, \delta} 1.$$ 

Proof. We allow implied constants to depend on $s, \delta$. By Theorem 2.1, there exists a degree $\leq s$ polynomial nilsequence $\psi$ of complexity $O(1)$ such that $|\mathbb{E}_{n \in [N]} f(n) \psi(n)| \gg 1$.

By taking real and imaginary parts of $\psi$, and then positive and negative parts, and rescaling, we may assume without loss of generality that $\psi$ takes values in $[0, 1]$. By Fubini’s theorem, we then have

$$|\int_{0}^{1} \mathbb{E}_{n \in [N]} f(n) 1_{E_t}(n) \, dt| \gg 1$$

where $E_t := \{n \in [N] : \psi(n) \geq t\}$. We thus see that there is a subset $\Omega \subset [0, 1]$ of Lebesgue measure $|\Omega| \gg 1$ such that $|\mathbb{E}_{n \in [N]} f(n) 1_{E_t}(n)| \gg 1$ uniformly for all $t \in \Omega$.

It remains to show that at least one of the $E_t$ is $s$-measurable with respect to a suitable growth function. For any $t \in \mathbb{R}$, we consider the maximal function

$$M(t) := \sup_{r > 0} \frac{1}{2r \cdot N} |\{n \in [N] : |\psi(n) - t| \leq r\}|.$$ 

From the Hardy-Littlewood maximal inequality or the Besicovitch covering lemma we have that the set $\{t \in \mathbb{R} : M(t) \geq \lambda\}$ has Lebesgue measure $O(1/\lambda)$ for any $\lambda > 0$. Thus, we can find $t \in \Omega$ such that $M(t) = O(1)$. Fixing such a $t$, we then see that

$$|\{n \in [N] : |\psi(n) - t| \leq r\}| \ll rN$$

for all $r > 0$. As a consequence, for any $r > 0$, one can then approximate $1_{E_t}$ to within $O(\sqrt{r})$ in $L^2[N]$ norm by a Lipschitz function of $\psi$ with Lipschitz norm $O(1/r)$. This implies that $1_{E_t}$ is $s$-measurable with some growth function $\Phi$ depending only on $s, \delta$, and the claim follows. \[\square\]

We rephrase this fact in terms of conditional expectations. The following definition, like Definition 2.2, will only be needed until the end of the proof of Proposition 2.7.

---

8Here we are, in some sense, finding a “regular” nil-Bohr set $\{n \in [N] : \psi(n) \geq t\}$, that is to say one rather insensitive to small changes in the value of $t$. A similar idea also appears in [46] Claim 2.2.
**Definition 2.4** (s-factors). An s-factor \( \mathcal{B} \) of complexity \( \leq M \) and growth function \( \Phi \) is a partition of \( [N] \) into at most \( M \) sets (or cells) \( E_1, \ldots, E_m \) which are \( s \)-measurable of growth function \( \Phi \). Given an s-factor \( \mathcal{B} \) and a function \( f : [N] \to \mathbb{C} \), we define the conditional expectation \( \mathbb{E}(f|\mathcal{B}) : [N] \to \mathbb{C} \) of \( f \) with respect to the s-factor to be the function which equals \( \mathbb{E}_{n \in E_j}f(n) \) on each cell of the partition. We define the index or energy \( \mathcal{E}(\mathcal{B}) \) of the s-factor \( \mathcal{B} \) relative to \( f \) to be the quantity \( \|\mathbb{E}(f|\mathcal{B})\|_{L^2[N]}^2 \).

An s-factor \( \mathcal{B}' \) is said to refine another \( \mathcal{B} \) if every cell of \( \mathcal{B}' \) is contained in a cell of \( \mathcal{B} \).

**Corollary 2.5** (Lack of uniformity implies energy increment). Let \( s \geq 1 \), let \( \mathcal{B} \) be an s-factor of complexity \( \leq M \) and some growth function \( \Phi \), and suppose that \( f : [N] \to [0,1] \) is such that \( \|f - \mathbb{E}(f|\mathcal{B})\|_{U^{s+1}[N]} \geq \delta \) for some \( \delta > 0 \). Then there exists a refinement \( \mathcal{B}' \) of \( \mathcal{B} \) of complexity \( \leq 2M \) and some growth function depending on \( s, \delta, M, \Phi \), such that
\[
\mathcal{E}(\mathcal{B}') - \mathcal{E}(\mathcal{B}) \geq_{s, \delta} 1.
\]

**Proof.** By Corollary 2.3, we can find an \( s \)-measurable set \( E \) with a growth function depending on \( s, \delta \) such that
\[
|\langle f - \mathbb{E}(f|\mathcal{B}), 1_E \rangle_{L^2[N]}| \geq_{s, \delta} 1 \tag{2.1}
\]
Now let \( \mathcal{B}' \) be the partition generated by \( \mathcal{B} \) and \( E \); then \( \mathcal{B}' \) clearly has complexity \( \leq 2M \) and a growth function depending on \( s, \delta, M, \Phi \). Since \( 1_E \) is \( s \)-measurable with respect to the partition \( \mathcal{B}' \) (that is to say it is constant on each cell of this partition), we can rewrite the left-hand side of (2.1) as
\[
|\langle \mathbb{E}(f|\mathcal{B}'), \mathbb{E}(f|\mathcal{B}), 1_E \rangle_{L^2[N]}|
\]
and hence by the Cauchy-Schwarz inequality
\[
\|\mathbb{E}(f|\mathcal{B}') - \mathbb{E}(f|\mathcal{B})\|_{L^2[N]} \geq_{s, \delta} 1.
\]
The claim then follows from Pythagoras’ theorem.

We can iterate this to obtain a weak regularity lemma, analogous to the weak graph regularity lemma of Frieze and Kannan [15].

**Corollary 2.6.** Let \( s \geq 1 \), let \( \mathcal{B} \) be an s-factor of complexity \( \leq M \) and some growth function \( \Phi \), let \( f : [N] \to [0,1] \), and let \( \varepsilon > 0 \). Then there exists a refinement \( \mathcal{B}' \) of \( \mathcal{B} \) of complexity \( O(s, M, \varepsilon)(1) \) and some growth function depending on \( s, \varepsilon, M, \Phi \), such that
\[
\|f - \mathbb{E}(f|\mathcal{B}')\|_{U^{s+1}[N]} \leq \varepsilon. \tag{2.2}
\]

**Proof.** We define a sequence of successively more refined factors \( \mathcal{B}' \), starting with \( \mathcal{B}' := \mathcal{B} \). If (2.2) already holds then we are done, so suppose that this is not the case. Then by Corollary 2.5 we can find a refinement \( \mathcal{B}'' \) of complexity \( O(s, M, \varepsilon)(1) \) and some growth function depending on \( s, \varepsilon, M, \Phi \) whose energy is larger than that of \( \mathcal{B}' \) by a factor \( \gg_{s, \varepsilon} 1 \). On the other hand, the energy clearly ranges between 0 and 1. Thus after replacing \( \mathcal{B}' \)
with $B''$ and iterating this algorithm at most $O_{s,\varepsilon}(1)$ times we obtain the claim. 

One final iteration then gives the full non-irrational regularity lemma.

**Proposition 2.7.** Let $f : [N] \to [0, 1]$, let $s \geq 1$, let $\varepsilon > 0$, and let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a growth function. Then there exists a quantity $M = O_{s,\varepsilon,F}(1)$ and a decomposition

$$f = f_{\text{nil}} + f_{\text{sml}} + f_{\text{unf}}$$

of $f$ into functions $f_{\text{nil}}, f_{\text{unf}} : [N] \to [-1,1]$ such that:

(i) ($f_{\text{nil}}$ structured) $f_{\text{nil}}$ equals a degree $\leq s$ polynomial nilsequence of complexity $\leq M$.

(ii) ($f_{\text{sml}}$ small) $\|f_{\text{sml}}\|_{L^2[N]} \leq \varepsilon$.

(iii) ($f_{\text{unf}}$ very uniform) $\|f_{\text{nil}}\|_{U^{s+1}[N]} \leq 1/F(M)$.

(iv) (Nonnegativity) $f_{\text{nil}}$ and $f_{\text{nil}} + f_{\text{sml}}$ take values in $[0,1]$.

**Proof.** We need a growth function $\tilde{F} : \mathbb{R}^+ \to \mathbb{R}^+$, somewhat more rapidly growing than $F$ in manner that depends on $F, s, \varepsilon$. We will specify the exact requirements we have of it later. We then define a sequence $1 = M_0 \leq M_1 \leq \ldots$ by setting $M_0 := 1$ and $M_{i+1} := \tilde{F}(M_i)$.

Applying Corollary 2.6 repeatedly, we may find for each $i \geq 0$ an $s$-factor $B_i$ of complexity $O_{s,M_i}(1)$ and a growth function depending on $s, M_i$ such that each $B_i$ refines $B_{i-1}$, and such that

$$\|f - \mathbb{E}(f|B_i)\|_{U^{s+1}[N]} \leq 1/M_i$$

for all $i \geq 0$.

By Pythagoras’ theorem, the energies $\mathcal{E}(B_i)$ are non-decreasing, and also range between 0 and 1. Thus by the pigeonhole principle, one can find $i = O_{\varepsilon}(1)$ such that

$$\mathcal{E}(B_{i+1}) - \mathcal{E}(B_i) \leq \varepsilon^2/4,$$

which by Pythagoras’ theorem again is equivalent to

$$\|\mathbb{E}(f|B_{i+1}) - \mathbb{E}(f|B_i)\|_{L^2[N]} \leq \varepsilon/2.$$ 

Meanwhile, as $B_i$ is an $s$-factor and $f$ is bounded, we can find a degree $\leq s$ polynomial nilsequence $f_{\text{nil}} : [N] \to \mathbb{R}$ of complexity $O_{s,M_i}(1)$ such that

$$\|\mathbb{E}(f|B_i) - f_{\text{nil}}\|_{L^2[N]} \leq \varepsilon/2.$$ 

Since $\mathbb{E}(f|B_i)$ ranges in $[0,1]$, we may retract $f_{\text{nil}}$ to $[0,1]$ also (note that this does not increase the complexity of $f_{\text{nil}}$). If we then set $f_{\text{unf}} := f - \mathbb{E}(f|B_{i+1})$ and $f_{\text{sml}} := \mathbb{E}(f|B_{i+1}) - f_{\text{nil}}$, we obtain the claim. 

**Remark.** The application of the Hardy-Littlewood maximal inequality in the proof of Corollary 2.6 makes for a reasonably tidy argument. A more direct approach would be to carve up $[N]$ into approximate level sets of nilsequences, and then to approximate the projections onto the factors thus defined by nilsequences using the Weierstrass approximation theorem. There are a number of technicalities involved in this approach, chiefly involving the
need to choose the approximate level sets randomly. This kind of argument was employed, in a closely related context, in [27, Chapter 7]. One can also use utilise arguments based on the Hahn-Banach theorem instead; see [21], [46], and [22, 23, 24].

Obtaining irrationality. Our task now is to replace the nilsequence $f_{nil}$ appearing in Proposition 2.7 with a highly “irrational” nil sequence as advertised in the statement of our main theorem, Theorem 1.2. It turns out to be sufficient to establish the following claim.

**Proposition 2.8.** Let $s, M_0 \geq 1$, let $F$ be a growth function, and let $f : \mathbb{Z} \rightarrow [0, 1]$ be a degree $\leq s$ nilsequence of complexity $\leq M_0$. Then there exists an $M = O_{s, M_0, F}(1)$, such that $f$ (when restricted to $[N]$) is also a $(F(M), N)$-irrational degree $\leq s$ virtual nilsequence of complexity $\leq M$ at scale $N$.

To establish Theorem 1.2 from this and Proposition 2.7 one first applies the latter result with $F$ replaced by a much more rapid growth function $F'$, and then one applies Proposition 2.8 to the structured component $f_{nil}$ obtained in Theorem 2.6.

It remains to prove Proposition 2.8. Let $s, M_0, F, \psi$ be as in that proposition. By definition, we have $\psi = F_0(g_0(n)\Gamma)$ for some degree $\leq s$ filtered nilmanifold $(G/\Gamma, G_*)$ of complexity $\leq M_0$, a polynomial sequence $g_0 \in \text{poly}(\mathbb{Z}, G_*)$, and a function $F_0 : G/\Gamma \rightarrow \mathbb{C}$ which has a Lipschitz norm of at most $M_0$. Since $\psi$ takes values in $[0, 1]$, we may assume without loss of generality that $F_0$ is real, and by replacing $F_0$ with the retraction $\max(\min(F_0, 1), 0)$ to $[0, 1]$ if necessary, we may assume that $F_0$ also takes values in $[0, 1]$. Henceforth $(G/\Gamma, G_*)$, $g_0$, and $F_0$ are fixed.

Factorisation results. One of the main results of our paper [30] was a decomposition of an arbitrary polynomial nilsequence $g$ on $G/\Gamma$ into a product $\beta g' \gamma$, where $\beta$ is “smooth”, $\gamma$ is “rational”, and $g'(n)\Gamma$ is equidistributed inside some possibly smaller nilmanifold $G'/\Gamma'$. We need a similar result here, but with $g'$ having the somewhat stronger property of being irrational that we mentioned in the introduction. The notion of irrationality is discussed in more detail in Appendix A.

We will be also using the notions of smooth and rational polynomial sequences from [30]. Again, the basic definitions and properties of these concepts are recalled in Appendix A.

Define a complexity $\leq M$ sub-nilmanifold of $(G/\Gamma, G_*)$ to be a degree $\leq s$ filtered nilmanifold $(G'/\Gamma', G'_*)$ of complexity $\leq M$, where each subgroup $G'_{(i)}$ in the filtration $G'_*$ is a rational subgroup of the associated subgroup $G_{(i)}$ of complexity $\leq M$, $\Gamma = G' \cap \Gamma$, and each element of the Mal’cev basis of $(G'/\Gamma', G'_*)$ is a rational linear combination of the Mal’cev basis of

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9In our paper [30] the letter $\varepsilon$ was used for a smooth nilsequence, but we use $\beta$ here to avoid conflict with various uses of $\varepsilon$ to denote a small positive real number.
By commutating all the $\beta$ notation, I suppose that $(\Gamma, G_\bullet)$ rewrite this as where

$$\sum_{i=0}^s \dim(G'_i);$$

this is also the dimension of poly($\mathbb{Z}, G_\bullet$) (thanks to the Taylor series expansion, Lemma A.4).

We make the easy remark that if $(G'/\Gamma', G'_\bullet)$ is a subnilmanifold of $(G/\Gamma, G_\bullet)$ for some $M \geq M_0$, and $(G''/\Gamma'', G''_\bullet)$ is a complexity $\leq M$ subnilmanifold of $(G'/\Gamma', G'_\bullet)$, then $(G''/\Gamma'', G''_\bullet)$ is a complexity $O_M(1)$ subnilmanifold of $(G/\Gamma, G_\bullet)$.

Our first lemma is very similar in form to [30, Lemma 7.9].

Lemma 2.9 (Initial factorisation). Let $(G'/\Gamma', G'_\bullet)$ be a complexity $\leq M$ subnilmanifold of $(G/\Gamma, G_\bullet)$ for some $M \geq M_0$, let $g' \in \text{poly}(\mathbb{Z}, G'_\bullet)$, and let $A > 0$ and $N \geq 1$. Then at least one of the following statements hold:

(Irrationality) $g'$ is $(A, N)$-irrational in $(G'/\Gamma', G'_\bullet)$.

(Dimension reduction) There exists a factorisation

$$g' = \beta g'' \gamma$$

where $\beta \in \text{poly}(\mathbb{Z}, G'_\bullet)$ is $(O_{M,A}(1), N)$-smooth, $g'' \in \text{poly}(\mathbb{Z}, G''_\bullet)$ takes values in a subnilmanifold $(G''/\Gamma'', G''_\bullet)$ of $(G'/\Gamma', G'_\bullet)$ of strictly smaller total dimension and of complexity $O_{M,A}(1)$, and $\gamma \in \text{poly}(\mathbb{Z}, G'_\bullet)$ is $O_{M,A}(1)$-rational.

Proof. To make this proof a little more readable, we drop one dash from every expression. Thus $g'$ becomes $g$, $G''$ becomes $G'$, and so on. Suppose that $g$ is not $(A, N)$-irrational. Recall (see Lemma A.4) that $g$ has a Taylor expansion that we may write in the form

$$g(n) = g_0 g_1^{(a)}(n) g_2^{(a)}(n) \ldots g_s^{(a)}(n),$$

where $g_i \in G(i)$ for each $i$. It follows from Lemma A.7 that for some $i$, $1 \leq i \leq s$, we can factorise

$$g_i = \beta_i g_i' \gamma_i,$$

where $g_i' \in G(i)$ lies in the kernel of some horizontal character $\xi_i : G(i) \rightarrow \mathbb{R}$ of complexity $O_{A,M}(1)$, $\gamma_i \in G(i)$ is $O_{A,M}(1)$-rational in the sense that $\gamma_i^m \in \Gamma(i)$ for some $m = O_{A,M}(1)$, and $\beta_i \in G(i)$ has distance $O_{A,M}(1/N^i)$ from the origin.

We now divide into two cases, depending on whether $i > 1$ or $i = 1$. First suppose that $i > 1$. Then the Taylor expansion of $g$ reads, with an obvious notation,

$$g(n) = g_{<i}(n) (\beta_i g_i' \gamma_i)(n) g_{>i}(n).$$

By commutating all the $\beta_i$s to the left and all the $\gamma_i$s to the right, and using the group properties of polynomial sequences (Theorem 1.6), one can rewrite this as

$$g(n) = \beta_i(g'(n)) \gamma_i,$$

where

$$g'(n) := g_{<i}(n) g_i' \gamma_i g_{>i}(n)$$
and \( g_{j}(n) \) is another polynomial sequence taking values in \( G_{(i+1)} \). Observe that \( g' \) is then a polynomial sequence adapted to the subnilmanifold \( (G'/\Gamma', G'_*) \), where \( G'/\Gamma' = G/\Gamma \) and \( G'_{(j)} = G_{(j)} \) for \( j \neq i \), but \( G'_{(i)} = \ker(\xi_{i}) \). This is indeed a subnilmanifold, with complexity \( O_{A,M}(1) \); note that \( (G'_{(i)})_{i \geq 0} \) is a filtration, thanks to our insistence in the definition of \( i \)-horizontal character (cf. Definition 2.6) that \([G_{(j)}, G_{(i-j)}] \subseteq \ker(\xi_{j})\) for all \( 0 \leq j \leq i \). Meanwhile, \( \beta_{i}(\gamma) \) is a \((O_{A,M}(1), N)\)-smooth sequence and \( \gamma_{i}(\gamma) \) is a \( O_{A,M}(1) \)-rational sequence, so we have the desired factorisation in the \( i > 1 \) case.

When \( i = 1 \), the above argument does not quite work, because \( G'_{(1)} \) would be distinct from \( G_{(0)} \) and would thus not qualify as a filtration. But this can be easily remedied by performing an additional factorisation

\[
g_{0} = \beta_{0}g'_{0}
\]

where \( \beta_{0} \in G' \) is a distance \( O_{A,M}(1) \) from the identity, and \( g'_{0} \) lies in the kernel of \( \xi'_{1} \). This leads to a factorisation of the form

\[
g(n) = \beta_{0}\beta_{1}^{n}g'(n)\gamma_{1}^{n}
\]

where

\[
g'(n) = g_{0}g_{1}g_{2} \cdots (n)
\]

and \( g_{j} \) is a polynomial sequence taking values in \( G_{(2)} \). One then argues as before, but now one sets both \( G'_{(0)} \) and \( G'_{(1)} \) equal to the kernel of \( \xi'_{1} \).

We can iterate the above lemma to obtain the following result, which is analogous to [30, Theorem 1.19]. Apart from dealing with irrationality rather than equidistribution, the following result is somewhat different to that just cited in that one requires an arbitrary (rather than polynomial) growth function, but one does not (of course) need polynomial complexity bounds. A variant of [30, Theorem 1.19] was also given in [33, Theorem 4.2].

**Lemma 2.10** (Complete factorisation). Let \((G/\Gamma, G_{*})\) be a degree \( \leq s \) filtered nilmanifold of complexity \( \leq M_{0} \), and let \( g \in \text{poly}(\mathbb{Z}, G_{*}) \). For any growth function \( F' \), we can find a quantity \( M_{0} \leq M \leq O_{M,F'}(1) \) and a factorisation \( g = \beta g' \gamma \) where:

(i) \( \beta \in \text{poly}(\mathbb{Z}, G_{*}) \) is \((O_{M}(1), N)\)-smooth;

(ii) \( g' \in \text{poly}(\mathbb{Z}, G_{*}) \) is \((F'(M), N)\)-irrational in a subnilmanifold \((G'/\Gamma', G'_*)\) of \((G/\Gamma, G_{*})\) of complexity \( O_{M}(1) \), and

(iii) \( \gamma \in \text{poly}(\mathbb{Z}, G_{*}) \) is \( O_{M}(1) \)-periodic.

**Proof.** We use an iterative argument, setting \( \beta = \gamma = \text{id} \), \( g' = g \), \( M = M_{0} \), and \((G'/\Gamma', G'_*) = (G/\Gamma, G_{*})\) to begin with. In particular, \((G', \Gamma', G'_*)\) is initially a subnilmanifold of \((G/\Gamma, G_{*})\) of complexity \( O_{M}(1) \). If \( g' \) is \( F'(M) \)-equidistributed in \((G'/\Gamma', G'_*)\) then we are done; otherwise, by Lemma 2.9 we may factorise \( g' = \beta g'' \gamma \) where \( \beta \) is \((O_{F'(M)}(1), N)\)-smooth, \( \gamma \) is \( O_{F'(M)}(1) \)-periodic, and \( g'' \) now takes values in a subnilmanifold \((G''/\Gamma'', G''_{*})\) of \((G'/\Gamma', G'*_{*)}\).
We then replace $\beta$ by $\beta \beta'$, $\gamma$ by $\gamma' \gamma$, $g'$ by $g''$, $(G'/\Gamma', G_{\bullet}')$ by $(G''/\Gamma'', G''_{\bullet})$, and increase $M$ to a quantity of the form $O_{\mathcal{F}'(M)}(1)$, using Lemma A.4 to conclude that the new $\beta$ is smooth and the new $\gamma$ is rational. We then iterate this process. Since the total dimension of $(G/\Gamma, G_{\bullet})$ is initially $O_{M_0}(1)$, this process can iterate at most $O_{M_0}(1)$ times, and the claim follows. \qed

With this lemma we can now establish Proposition 2.8 and hence Theorem 1.2. Let $\mathcal{F}'$ be a rapid growth function (depending on $\varepsilon, M_0, \mathcal{F}$) to be chosen later. We apply Lemma 2.10, obtaining some $M$ with $M_0 \leq M \leq O_{M_0, \mathcal{F}'}(1)$ and a factorisation

$$
\psi(n) = F(\beta(n)g'(n)\gamma(n)\Gamma)
$$

with $\beta, g'$ and $\gamma$ having the properties described in that lemma.

The sequence $\gamma$ is $O_M(1)$-rational and so, by Lemma A.4, the orbit $n \mapsto \gamma(n)\Gamma$ is periodic with some period $q = O_M(1)$, and thus $\gamma(n)\Gamma$ depends only on $n \mod q$.

For each $n$, the rationality of $\gamma(n)$ ensures that $\gamma(n)\Gamma$ intersects $\Gamma$ in a subgroup of $\Gamma$ of index $O_M(1)$. Since there are only $O_M(1)$ different possible values of $\gamma(n)\Gamma$, we may thus find a subgroup $\Gamma'$ of $\Gamma$ of index $O_M(1)$ such that $\Gamma' \subseteq \gamma(n)\Gamma$ for all $n$.

We can thus express $\psi$ as a virtual nilsequence

$$
\psi(n) = \tilde{F}(g'(n)\Gamma', n \mod q, n/N)
$$

where $\tilde{F} : G/\Gamma' \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R}$ is defined by the formula

$$
\tilde{F}(x, a, y) := F(\beta(Ny)\tilde{x}\gamma(\tilde{a})\Gamma)
$$

whenever $y \in \frac{1}{N}\mathbb{Z}$ and by Lipschitz extension to all $y \in \mathbb{R}$, where $\tilde{a}$ is any integer with $\tilde{a} = a \mod q$, and $\tilde{x}$ is any element of $G$ such that $\tilde{x}\Gamma' = x$.

One easily verifies that $\tilde{F}$ is well-defined and has a Lipschitz norm of $O_M(1)$.

Also, since $g'$ was already $(\mathcal{F}(M), N)$-irrational in $G/\Gamma$, and $\Gamma'$ has index $O_M(1)$ in $\Gamma$, we see that $g'$ is $(\mathcal{F}(M), N)$-irrational in $G/\Gamma'$. Proposition 2.8 now follows by replacing $M$ by a suitable quantity of the form $O_M(1)$, and choosing $\mathcal{F}'$ sufficiently rapidly growing depending on $\mathcal{F}$.

3. Proof of the counting lemma

The purpose of this section is to prove the counting lemma, Theorem 1.11. We begin by recalling from the introduction the definition of the Leibman group $G^\Psi$.

**Definition 3.1 (The Leibman group).** Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$. For any $i \geq 1$, define $\Psi[i]$ to be the linear subspace of $\mathbb{R}^t$ spanned by the vectors $(\psi_1^j(n), \ldots, \psi_t^j(n))$ for $1 \leq j \leq i$ and $n \in \mathbb{Z}^D$. Given a filtered nilmanifold $(G/\Gamma, G_{\bullet})$, we define the Leibman group $G^\Psi \triangleleft G^t$ to be the Lie subgroup of $G^t$ generated by the elements $g_{\psi_n}^{\lambda}$.
for $i \geq 1$, $g_i \in G_{(i)}$, and $\vec{v}_i \in \Psi^{[i]}$, with the convention that if $\vec{v} = (v_1, \ldots, v_t)$ then

$$g^{\vec{v}} := (g^{v_1}, \ldots, g^{v_t}).$$

Now might be a good time to remark explicitly that we have introduced a slightly vulgar convention that we hope will help the reader follow this section and other parts of the paper. Bold font letters such as $\mathbf{n} \in \mathbb{R}^D$ denote $D$-dimensional vectors, whilst arrows such as $\vec{v} \in \mathbb{R}^t$ denote $t$-vectors. Occasionally we shall write $m_i := \dim(\Psi^{[i]})$.

When reading this section, it might be found helpful to have a running example in mind. We will take as an illustrative example the case $D = 2$, $t = 4$ and $\Psi = (\psi_1, \ldots, \psi_4)$, where $\psi_i(\mathbf{n}) = n_1 + in_2$ for $i = 0, 1, 2, 3$. The system $\Psi$, of course, defines a 4-term arithmetic progression. As we remarked in the introduction the corresponding Leibman group $G^\Psi$ is also known as the Hall-Petresco group $\text{HP}^4(G)$. The reader will easily confirm that in this case we have

$$\Psi^{[1]} = \mathbb{R}(1, 1, 1) \oplus \mathbb{R}(0, 1, 2, 3)$$

and

$$\Psi^{[2]} = \mathbb{R}(1, 1, 1) \oplus \mathbb{R}(0, 1, 2, 3) \oplus \mathbb{R}(0, 0, 1, 3)$$

and

$$\Psi^{[3]} = \mathbb{R}(1, 1, 1) \oplus \mathbb{R}(0, 1, 2, 3) \oplus \mathbb{R}(0, 0, 1, 3) + \mathbb{R}(0, 0, 0, 1) = \mathbb{R}^4.$$ 

Some work must be done before we can describe $G^\Psi = \text{HP}^4(G)$ in a pleasant way. However we can already establish the following lemma, whose statement and proof go some way towards explaining the introduction of the Leibman group.

**Lemma 3.2.** Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$. Suppose that $(G/\Gamma, G_\bullet)$ is a filtered nilmanifold and that $g \in \text{poly}(\mathbb{Z}, G_\bullet)$ is a polynomial sequence. Then the sequence $g^\Psi : \mathbb{Z}^D \to G^t$ defined by $g^\Psi(\mathbf{n}) := (g(\psi_1(\mathbf{n})), \ldots, g(\psi_t(\mathbf{n})))$ takes values in $G^\Psi$.

**Proof.** The sequence $g(\mathbf{n})$ has a (unique) Taylor expansion

$$g(\mathbf{n}) = g_0 g_1^{(n_1)} \cdots g_s^{(n_s)},$$

with $g_i \in G_{(i)}$ for all $i$ (see Lemma [A]). Substituting in, it follows that

$$g^\Psi(\mathbf{n}) = \prod_{i=0}^s g_i^{(\psi_1^{(n)}) \cdots (\psi_t^{(n)})},$$

and it is immediate from the definition that each element in this product lies in $G^\Psi$. □

The counting lemma, whose proof is the main objective of this section, was stated as Theorem [1.11]. Essentially, it states that $g^\Psi(\mathbf{n})\Gamma^\Psi$ is equidistributed in $G^\Psi/\Gamma^\Psi$ as $\mathbf{n}$ ranges over “nice” subsets of “big” lattices, provided that the original sequence $g$ is suitably irrational. We will recall what that
means in due course, but our first task is to develop the basic theory of the Leibman group \( G^\Psi \). At the moment, for example, we have not established that \( G^\Psi \) is a connected Lie subgroup of \( G^t \) or that \( G^\Psi / \Gamma^\Psi \) has the structure of a filtered nilmanifold. Nor have we developed tools for calculating inside this group.

**Basic facts about the Leibman group and nilmanifold.** We can endow \( \mathbb{R}^t \) with the structure of a commutative algebra over \( \mathbb{R} \) by using the pointwise product

\[
\vec{x} \cdot \vec{y} = (x_1 y_1, \ldots, x_t y_t)
\]

and setting \( \vec{1} = (1, \ldots, 1) \) to be the multiplicative identity. With this algebra structure, one can view the spaces \( \Psi[i] \) defined in Definition 1.10 as the span of the powers \( \Psi(n)^j \) for \( n \in \mathbb{Z}^D \) and \( 1 \leq j \leq i \), where we view \( \Psi \) as a homomorphism from \( \mathbb{Z}^D \) to \( \mathbb{Z}^t \). We have the following alternate definition of the \( \Psi[i] \).

**Lemma 3.3 (Depolarisation).** \( \Psi[i] \) is the span of the products

\[
\Psi(n_1) \ldots \Psi(n_j),
\]

where \( 1 \leq j \leq i \) and \( n_1, \ldots, n_j \in \mathbb{Z}^D \).

**Proof.** Clearly \( \Psi[i] \) is contained in this span. To establish the reverse containment, we observe the elementary depolarisation identity

\[
\Psi(n_1) \ldots \Psi(n_j) = \frac{(-1)^j}{j!} \sum_{\omega \in \{0,1\}^j} (-1)^{\omega} \Psi(\omega_1 n_1 + \ldots + \omega_j n_j)^j
\]

where \( \omega = (\omega_1, \ldots, \omega_j) \) and \( |\omega| := \omega_1 + \ldots + \omega_j \), and the claim follows. \( \Box \)

As an immediate consequence we have

**Corollary 3.4 (Filtration property).** For any \( i, j \geq 0 \), we have \( \Psi[i] \cdot \Psi[j] \subseteq \Psi[i+j] \).

Let \((G/\Gamma, G_\bullet)\) be a degree \( \leq s \) filtered nilmanifold. From Definition 1.10, the Leibman group \( G^\Psi \) is the subgroup of \( G^t \) generated by the group elements \( g_i^{v_i} \) for \( i \geq 1 \), \( v_i \in \Psi[i] \), and \( g_i \in G(i) \). For any \( i_0 \geq 1 \), let \( G^\Psi(i_0) \) be the subgroup of \( G^\Psi \) generated by those \( g_i^{v_i} \) with \( i \geq i_0 \), \( \vec{v}_i \in \Psi[i] \), \( g_i \in G(i) \), with the convention that \( G^\Psi(i_0) := G^\Psi \).

**Lemma 3.5 (Filtration property for \( G^\Psi_\bullet \)).** \( G^\Psi_\bullet := (G^\Psi(i_0))_{i_0=0}^\infty \) is a filtration on \( G^\Psi \). In other words, the \( G^\Psi(i) \) are nested with \( [G^\Psi(i), G^\Psi(j)] \subset G^\Psi(i+j) \) for all \( i, j \geq 0 \).

**Proof.** It suffices to check that if \( g_i \in G(i), g_j \in G(j), \vec{v}_i = (v_{i1}, \ldots, v_{it}) \in \Psi[i] \) and \( \vec{v}_j = (v_{j1}, \ldots, v_{jt}) \in \Psi[j] \) then \( [g_i^{\vec{v}_i}, g_j^{\vec{v}_j}] \in G^\Psi(i+j) \). But this follows from the Baker-Campbell-Hausdorff formula (see (C.2)), the filtration property of \( G(i) \) and Corollary 3.4. \( \Box \)
The spaces $\Psi_i[i]$ form a flag

$$0 \leq \Psi^{[1]} \leq \ldots \leq \Psi^{[s]} \leq \mathbb{R}^t$$

of subspaces which are rational (i.e. they can be defined over $\mathbb{Q}$). From a greedy algorithm (and clearing denominators) we may thus find a basis $\vec{v}_1, \ldots, \vec{v}_{m_s} \in \Psi^{[s]}$ with the following properties:

(i) (Integrality) $\vec{v}_1, \ldots, \vec{v}_{m_s}$ all lie in $\mathbb{Z}^t$;
(ii) (Partial span) For every $1 \leq i \leq s$, $\vec{v}_1, \ldots, \vec{v}_{m_i}$ span $\Psi^{[i]}$;
(iii) (Row echelon form) For each $1 \leq j \leq m_s$, there exists $l_j, 1 \leq l_j \leq t$, such that $\vec{v}_j$ has a non-zero $l_j$ coordinate, but such that $\vec{v}_{j'}$ has a zero $l_j$ coordinate for all $j < j' \leq m_s$.

For instance, the basis

$\vec{v}_1 := (1, 1, 1, 1); \quad \vec{v}_2 := (0, 1, 2, 3); \quad \vec{v}_3 := (0, 0, 1, 3); \quad \vec{v}_4 := (0, 0, 0, 1)$

we implicitly gave above for our running example is already in this form.

For each basis element $\vec{v}_j$, we can define the degree $\deg(\vec{v}_j)$ of that element to be the first $i$ for which $j \leq m_i$, thus $\deg(\vec{v}_j)$ is an integer between 1 and $s$, and $\vec{v}_j \in \Psi^{[\deg(\vec{v}_j)]}$.

Fix such a basis. For each basis element $\vec{v}_j$, we can define the degree $\deg(\vec{v}_j)$ of that element to be the first $i$ for which $j \leq m_i$, thus $\deg(\vec{v}_j)$ is an integer between 1 and $s$, and $\vec{v}_j \in \Psi^{[\deg(\vec{v}_j)]}$.

Observe that an arbitrary element of $G^\Psi$ can be expressed as a product of finitely many elements of the form $g_j^{\vec{v}_j}$ for $0 \leq j \leq m_s$ and $g_j \in G^{(\deg(\vec{v}_j))}$. By many applications of the Baker-Campbell-Hausdorff formula (see (C.1)) and Lemma 3.5, we can now express any element of $G^\Psi$ in the form

$$\prod_{j=1}^{m_s} g_j^{\vec{v}_j}$$

where $g_j \in G^{(\deg(\vec{v}_j))}$ for all $1 \leq j \leq m_s$.

Thus, in our running example, we have the explicit description of $G^\Psi = \text{HP}^4(G)$ as

$$\{(g_0, g_0 g_1, g_0 g_1^2, g_2, g_0 g_1^3 g_2^3 g_3) : g_0 \in G(0), g_1 \in G(1), g_2 \in G(2), g_3 \in G(3)\}.$$ 

Note that from results on the Taylor expansion (see Lemma A.1) this group may also be identified as

$$\{(g(0), g(1), g(2), g(3)) : g \in \text{poly}(\mathbb{Z}, G_{\bullet})\}.$$ 

The group nature of $\text{HP}^4(G)$ is then easily deduced from Theorem 1.6 but this presentation is somewhat specific to the Hall-Petresco case and we shall not require it further.

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10 Indeed, one uses (C.1) and Lemma 3.5 to extract out and collects all terms with degree $\deg(\vec{v}_j) = 1$, leaving only terms with base $g_j$ in $G(2)$. Then one extracts out those terms with degree 2 (merging them with the $i = 1$ terms as necessary), leaving only terms with base in $G(3)$. Continuing this process gives the desired factorisation.
From the row-echelon form one can verify inductively that the representation (3.1) is unique (this can be seen clearly by working with the Hall-Petresco example presented above). This gives $G^{\Psi}$ the structure of a connected, simply connected Lie group, with dimension
\[
\dim(G^{\Psi}) = \sum_{i=1}^{s} \dim(G^{(i)})(\dim(\Psi[i]) - \dim(\Psi[i-1]))
\] (with the convention that $\Psi[0]$ is trivial). A similar argument also shows that every element of $G^{\Psi}(i_0)$ can be expressed uniquely in the form (3.1), where now $g_j$ is constrained to lie in $G(\max(\deg(\nu_j),i_0))$ rather than $G(\deg(\nu_j))$.

In particular, by reading off the coefficients $g_j$ one at a time, this implies the pleasant identity
\[
G^{\Psi}(i) = G^{\Psi} \cap (G(i))^k.
\] (3.3)

**Remark.** From Taylor expansion (see Lemma A.1) we see that the sequence $g^{\Psi}_n$ in (3.7) lies in poly($\mathbb{Z},G^{\Psi}$). While we do not directly use this fact here, it may help explain why the filtration $G^{\Psi}_{(i)}$ will plays a prominent role in the proof of the counting lemma that we will shortly come.

Recall that we normalised the basis vectors $\vec{v}_j \in \mathbb{Z}^t$ to have integer coefficients. As a consequence, we see that if the $g_j$ are in $\Gamma$, then the expression (3.1) lies in $\Gamma^k$. From this (and many applications of Lemma 3.5) we see that $G^{\Psi}(i) := \Gamma^i \cap G^{\Psi}_{(i)}$ is cocompact in $G^{\Psi}_{(i)}$ for each $i$, and so $(G^{\Psi}/\Gamma^{\Psi},G^{\Psi}_{(i)})$ is a filtered nilmanifold. Furthermore, the same argument shows that the $G^{\Psi}_{(i)}$ are rational subgroups of $G^k$ and so $(G^{\Psi}/\Gamma^{\Psi},G^{\Psi}_{(i)})$ is a subnilmanifold of $(G^k/\Gamma^k,G^k_{(i)})$.

**The Counting Lemma: Preliminary Maneuvers.** Now that we have verified that $G^{\Psi}/\Gamma^{\Psi}$ is indeed a nilmanifold, we can begin the proof of Theorem 1.11.

We begin with some easy reductions. First, observe that for fixed $M$, there are only finitely many possibilities for $s,D,t,\Psi$, and (up to isomorphism) there are only finitely many possibilities for $(G/\Gamma,G_{(i)})$ and $\Gamma$. Thus it will suffice to establish the result for a single choice of $s,D,t,\Psi,(G/\Gamma,G_{(i)})$, with the bounds depending on these quantities. Hence, we fix these quantities and allow all implicit constants to depend on these quantities (thus, in this section, we will not explicitly subscript out $O(1)$ quantities).

Similarly, because the space of Lipschitz functions with Lipschitz norm $O(1)$ is precompact in the uniform topology (by the Arzelà-Ascoli theorem), it suffices to prove the desired bound for each fixed $F$, as the uniformity in $F$ then follows from an easy approximation argument. Thus we fix $F$ and allow all quantities to depend on $F$.

Next, we observe that we may normalise $g(0) = \text{id}$. Indeed, we may factorise $g(0) = c_0 \gamma_0$ where $d_G(c_0,\text{id}) = O(1)$ and $\gamma_0 \in \Gamma$. Factorising, we obtain
\[
g(n) = c_0 g'(n) \gamma_0
\]
where \( g'(n) := c_0 g_0(\gamma_0^{-1} g(n) \gamma_0) \). Note that \( g'(0) = \text{id} \) and that Taylor coefficients of \( g' \) are given by \( g'_i = \gamma_0^{-1} g_i \gamma_0 \), and so \( g' \) is also \((A,N)\)-irrational.

It is then an easy matter to see that Theorem 1.11 for \( g \) and \( F \) follows from Theorem 1.11 for \( g' \) and for the shifted function \( F'(x) := F(c_0 x) \), which is still Lipschitz with norm \( O(1) \).

Note that we may assume that \( A \) and \( N \) are large, as the claim is trivial otherwise.

Equidistribution in the Leibman group. Let us recall what we are trying to prove. In the counting lemma, Theorem 1.11, our aim is to show that if \( g(n) \) is suitably irrational then the orbit \( (g_\psi(n))_{n \in (n_0 + \Lambda) \cap P} \) is equidistributed on the Leibman nilmanifold \( G^\Psi/\Gamma^\Psi \). We shall proceed by contradiction, supposing this orbit is not equidistributed and deducing that \( g(n) \) could not have been irrational. The reader should recall the definition of \textit{irrational} in this context: it is given in Definition A.6.

Our main tool will be a mild generalisation of the “multiparameter Leibman criterion”, which is [30, Theorem 8.6]. Here is the statement we shall use.

**Theorem 3.6.** Suppose that \((G/\Gamma, G_\bullet)\) is a filtered nilmanifold of complexity \( \leq M \) and that \( g \in \text{poly}(\mathbb{Z}^D, G_\bullet) \) is a polynomial sequence for some \( D \leq M \). Suppose that \( \Lambda \subseteq \mathbb{Z}^D \) is a lattice of index \( \leq M \), that \( n_0 \in \mathbb{Z}^D \) has magnitude \( \leq M \), and that \( P \subseteq [-N,N]^D \) is a convex body. Suppose that \( \delta > 0 \), and that

\[
| \sum_{n \in (n_0 + \Lambda) \cap P} F(g(n) \Gamma) - \frac{\text{vol}(P)}{|\mathbb{Z}^D : \Lambda|} \int_{G/\Gamma} F | > \delta N^D \|F\|_{\text{Lip}}
\]

for some Lipschitz function \( F : G/\Gamma \to \mathbb{C} \). Then there is a nontrivial homomorphism \( \eta : G \to \mathbb{R} \) which vanishes on \( \Gamma \), has complexity \( O_M(1) \) and such that

\[
\|\eta \circ g\|_{C^\infty([-N]^D)} = O_{\delta,M}(1).
\]

**Remarks.** This differs from [30, Theorem 8.6] in several insubstantial ways. On the one hand we have no concern here with the polynomial bounds that were important in that setting. However, we are dealing here with a sublattice \( \Lambda \subseteq \mathbb{Z}^D \) rather than \( \mathbb{Z}^D \) itself, and with an arbitrary convex body \( P \) rather than the box \([-N]^D\). This more general result can be deduced from [30, Theorem 8.6] in a somewhat routine, though slightly tedious, manner.

We sketch the details in Appendix 13. The notation \( C^\infty([-N]^D) \) is recalled both in the appendix and later in this section.

Later on, the notation will get a little complicated. Let us, then, first apply Theorem 3.6 to establish the following very simple special case of the counting lemma (it is, of course, the special case in which \( \Psi \) consists of the single form \( \psi_1(n) = n_1 \)).

**Lemma 3.7** (Irrational implies equidistributed). Suppose that \((G/\Gamma, G_\bullet)\) is a filtered nilmanifold of complexity at most \( M \) and that \( g : \mathbb{Z} \to G \) is an
(A, N)-irrational polynomial sequence. Then we have the equidistribution property
\[ E_{n \in [N]} F(g(n)\Gamma) = \int_{G/\Gamma} F + O_M(A^{-c_M}\|F\|_{\text{Lip}}) \]
for all Lipschitz \( F : G/\Gamma \to \mathbb{C} \) and some \( c_M > 0 \).

Proof. Suppose the conclusion is false. Then by Theorem 3.6 there is some continuous homomorphism \( \eta : G \to \mathbb{R} \) which vanishes on \([G, G]\) and \( \Gamma \), has complexity \( O_\delta(1) \), and for which \( \|\eta \circ g\|_{C^\infty([N])} \leq \delta^{-O(1)} \). Recall (cf. Definition 2.7) what this means: in the Taylor expansion
\[ \eta \circ g(n) = \alpha_0 + \alpha_1(n) + \cdots + \alpha_s(n), \]
the \( j \)th coefficient \( \alpha_j \) satisfies \( \|\alpha_j\|_{\mathbb{R}/\mathbb{Z}} \leq \delta^{-O(1)}/N^j \) for \( j = 1, \ldots, s \). If the sequence \( g \) is developed as a Taylor expansion
\[ g(n) = g_0 g_1^{(n)} \cdots g_s^{(n)} \]
then of course we have \( \alpha_j = \eta(g_j) \). Choose \( i \) maximal so that the restriction \( \eta|_{G(i)} \) is nontrivial. Then certainly \( \|\eta(g_i)\|_{\mathbb{R}/\mathbb{Z}} \leq \delta^{-O(1)}/N^i \). We claim that \( \eta \) is an \( i \)-horizontal character in the sense of Definition A.5, a statement which will clearly contradict the supposed \((A, N)\)-irrationality of \( g \) if \( \delta \) is a sufficiently small power of \( 1/A \). To this end we need only confirm that \( \eta \) vanishes on \( G(i+1), \Gamma(i) \) and on \([G(j), G(i-j)]\) for \( 0 \leq j \leq i \). The first of these follows from the maximality of \( i \), whilst the second and third follow immediately from the properties of \( \eta \) stated at the beginning of the proof. \[ \square \]

Let us turn now to the more notationally intensive general case. Now, we apply Theorem 3.6 to \( G^\Psi/\Gamma^\Psi \) to conclude that there is a non-trivial continuous homomorphism \( \eta : G^\Psi \to \mathbb{R} \) which maps \( \Gamma^\Psi \) to \( \mathbb{Z} \), has complexity \( O_\delta(1) \), and satisfies
\[ \|\eta \circ g^\Psi\|_{C^\infty([N]^D)} = O_\delta(1). \] (3.4)

Much as in the proof of Lemma 3.7 what this means is that if \( \eta \circ g^\Psi(n) \) is developed as a Taylor series in multi-binomial coefficients \( ^n_p = ^{n_1}_{p_1} \cdots ^{n_D}_{p_D} \) (see Lemma A.1), the coefficient \( \alpha_j \) satisfies \( \|\alpha_j\|_{\mathbb{R}/\mathbb{Z}} \ll \delta N^{1-j} \). Our aim is to use this information to contradict the assumption that \( g(n) \) is \((A, N)\)-irrational.

Let us once again take \( i \) maximal such that \( \eta|_{G^\Psi(i)} \) is nontrivial. Considering again the Taylor expansion of \( g(n) \), we have
\[ (\eta \circ g^\Psi)(n) = \sum_{j=1}^i \eta(g_j^{(\psi_1(n))}, \ldots, g_j^{(\psi_t(n))}). \] (3.5)

\[ \text{In fact here we only need the rather simpler 1-parameter version, which is [30 Theorem 1.16].} \]
Take the basis $\vec{v}_1, \vec{v}_2, \ldots$ for $\Psi^{[i]}$ described earlier. Then, since the vector
$$(\psi_j(n), \ldots, \psi_j(n))$$
lies in $\Psi^{[j]}$, there is an expansion
$$(\psi_j(n), \ldots, \psi_j(n)) = P_{j,1}(n)\vec{v}_1 + \cdots + P_{j,m_j}(n)\vec{v}_{m_j} \tag{3.6}$$
for $j = 1, \ldots, i$, where the $P_{j,k}: \mathbb{Z}^D \to \mathbb{R}$ are polynomials of degree at most $j$, recalling that $m_j := \dim(\Psi^{[j]})$. Comparing with $[3.5]$, we obtain
$$(\eta \circ g^\Psi)(n) = \sum_{j=1}^{i} \sum_{k=1}^{m_j} P_{j,k}(n)\eta(g^{\vec{v}_k}). \tag{3.7}$$

We are going to look at the coefficients $\alpha_i$ of $[3.7]$ for the monomial $n^i := n_1^{i_1} \cdots n_D^{i_D}$, where $i = (i_1, \ldots, i_D)$ and $|i| := |i_1| + \cdots + |i_D| = i$. We are assuming that every such coefficient satisfies $\|\alpha_i\|_{\mathbb{R}/\mathbb{Z}} \ll \delta N^{-i}$. Note also that
$$\alpha_i = \sum_{k=1}^{m_i} (P_{i,k})_i \eta(g^{\vec{v}_k}), \tag{3.8}$$
where $(P_{i,k})_i$ is the $n^i$ coefficient of $P_{i,k}(n)$; this is because terms of total degree $i$ cannot arise from the terms $j = 1, \ldots, i - 1$ in the sum on the right hand side of $[3.7]$.

On the other hand by taking $j = i$ in $[3.6]$ we have
$$(P_{i,1}(n))_i \vec{v}_1 + \cdots + (P_{i,m_i}(n))_i \vec{v}_{m_i} = \frac{1}{i_1! \cdots i_D!}(\psi_1(e_1)^{i_1} \cdots \psi_1(e_D)^{i_D}, \ldots, \psi_l(e_1)^{i_1} \cdots \psi_l(e_D)^{i_D})$$
$$= \frac{1}{i_1! \cdots i_D!}\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D}, \tag{3.9}$$
where $e_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^D$, the 1 being in the $j$th position, and $\Psi(e_j) := (\psi_1(e_j), \ldots, \psi_l(e_j)) \in \mathbb{R}^l$.

Comparing $[3.8]$ and $[3.9]$ and using the fact that $\eta$ is a homomorphism on $G^\Psi$, we obtain
$$\alpha_i = \frac{1}{i_1! \cdots i_D!}\eta(g^{\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D}}).$$
Thus, for each $i$ with $|i| = |i_1| + \cdots + |i_D| = i$, we have
$$\|\eta(g^{\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D}})\|_{\mathbb{R}/\mathbb{Z}} \ll \delta N^{-i} \tag{3.10}$$
To obtain the desired contradiction with the $(A,N)$-irrationality hypothesis and thus complete the proof, it suffices (after taking $A$ sufficiently large depending on $\delta$) to establish that for at least one choice of $i$, the map $\xi_i : G_{(i)} \to \mathbb{R}$ defined by
$$\xi_i(g) := \eta(g^{\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D}})$$
is a nontrivial horizontal $i$-character of complexity $O_\delta(1)$.
The complexity bound follows from the fact that the coefficients of the forms $\psi_i$ are integers of size $O(1)$ and the Baker-Campbell-Hausdorff formula (Appendix C). That at least one of these maps is nontrivial follows from that fact that $\eta$ is nontrivial on $G^\Psi_i$ and the fact that the vectors $\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D}$, $i_1 + \cdots + i_D = i$, span $\Psi[i]$ (a consequence of Lemma 3.3).

Furthermore $\xi_i$ always annihilates $\Gamma^\Psi_i$ and $G^\Psi_{i+1}$ (by the asserted maximality of $i$). To qualify as an $i$-horizontal character we must also show that it vanishes on $[G^\Psi_i, G^\Psi_{i+1}]$ for each $0 \leq j \leq i$. To this end, note that we may factor $\Psi(e_1)^{i_1} \cdots \Psi(e_D)^{i_D} = w w'$, where $w \in \Psi[j]$ and $w' \in \Psi[i-j]$. Indeed, we may take $w = \Psi(e_1)^{j_1} \cdots \Psi(e_D)^{j_D}$, $w' = \Psi(e_1)^{i_1-j_1} \cdots \Psi(e_D)^{i_D-j_D}$ for any indices $j_1, \ldots, j_D$ with $j_i \leq i_i$ and $j_1 + \cdots + j_D = j$, whereupon the relevant containments follow from Lemma 3.3. Now if $g \in G^\Psi_{i+1}$ and $g' \in G^\Psi_{i-j}$ are arbitrary then we have

$$[g^w, g'^{w'}] \equiv [g, g']^{ww'} \pmod{G^\Psi_{i+1}}$$

by the Baker-Campbell-Hausdorff formula [C.2]. Applying $\eta$, which is trivial on $G^\Psi_{i+1}$ by assumption, we obtain

$$\xi_i([g, g']) = \eta([g, g']^{ww'}) = \eta([g^w, g'^{w'}]) = 0,$$

the last step being a consequence of the fact that $\eta$ has abelian image and hence vanishes on $[G^\Psi_i, G^\Psi_{i+1}]$. This concludes the proof of the counting lemma, Theorem 1.11.

4. Generalised von Neumann type theorems

In this section we recall a number of results asserting the connection between Gowers norms and various types of linear configuration. These results are collectively known in the literature as “generalised von Neumann theorems”. The connection between Gowers norms (not called by that name, of course) and linear configurations was first made in [17]. A fairly general result of this type, which appears in [31], is the following.

**Theorem 4.1** (Generalised von Neumann Theorem). Let $\Psi = (\psi_1, \ldots, \psi_t)$ be a collection of linear forms $\psi_1, \ldots, \psi_t : \mathbb{Z}^D \to \mathbb{Z}$ for some $t, D \geq 1$, any two of which are linearly independent. Then there exists an integer $s = s(\Psi)$ with the property that one has the inequality

$$|\mathbb{E}_{n \in [N]} e^{\sum_{i=1}^t f_i(\psi_i(n))}| \ll_{t, D, \Psi} \inf_{1 \leq i \leq m} \|f_i\|_{U^{s+1}[N]} \quad (4.1)$$

for all $N \geq 1$ and all $f_1, \ldots, f_m : [N] \to \mathbb{C}$ bounded in magnitude by 1.
Remarks. A natural value of $s(\Psi)$ comes from the proof in [31], which proceeds via $s$ applications of the Cauchy-Schwarz inequality. For this reason Gowers and Wolf [22] call $s(\Psi)$ the Cauchy-Schwarz complexity of the system $\Psi$. There is a linear-algebraic recipe for computing $s(\Psi)$ which is not especially enlightening but sufficiently simple that we can give it here (see the introduction to [31] for more details). If $1 \leq i \leq t$ and $s \geq 0$ then we say that $\Psi$ has $i$-complexity at most $s$ if one can cover the $t-1$ forms $\{\psi_j : j \in [t] \setminus \{i\}\}$ by $s+1$ classes, such that $\psi_i$ does not lie in the linear span of the forms in any one of these classes. Then $s(\Psi)$ is the smallest $s$ for which the system has $i$-complexity at most $s$ for all $1 \leq i \leq t$. Note, then, that the Cauchy-Schwarz complexity of the system $\Psi = \{n_1, n_1 + n_2, \ldots, n_1 + (k-1)n_2\}$ corresponding to a $k$-term arithmetic progression is $k-2$. As a final remark, let us note that Theorem 4.1, as proved in [31, Appendix C], is regrettable somewhat difficult to understand as we had to establish a more general result in which the functions $f_i$ were bounded by an arbitrary pseudorandom measure, and this is notationally heavy. For a gentle explanation of the special case we may rewrite the estimate to be proven as

$$\|E_{n \in [N]} F(n \Gamma) \prod_{i=0}^{k-1} f_i(n + c_id)\|_{U^{k-1}[N]} \leq k, M, c_0, \ldots, c_{k-1}, \inf_{0 \leq i \leq k-1} \|f_i\|_{U^k[N]}.$$

Proof. We induct on $k$, starting with the case $k = 3$. The underlying nilmanifold $G/\Gamma$ is then a torus $\mathbb{R}/\mathbb{Z}^m$ with $m = O_M(1)$, and $g(n) = \theta n + \theta_0$ may be taken to be linear. By a standard Fourier decomposition we may assume that $F(x) = e(\xi \cdot x)$ for some $\xi \in \mathbb{Z}^m$ with $|\xi| = O_M(1)$, in which case we may rewrite the estimate to be proven as

$$\|E_{n \in [N]} F_{x \in [-N,N]} f_0(n + c_0d) f_1'(n + c_1d) f_2'(n + c_2d)\|_{U^{2}[N]} \leq k, M, \inf_{i=0,1,2} \|f_i\|_{U^2[N]}.$$

where $f_1'(n) = f_1(n)e(-(c_2-c_1)^{-1} \xi \cdot \theta n)$ and $f_2'(n) = f_2(n)e((c_2-c_1)^{-1} \xi \cdot \theta n)$. However it is easy to establish the invariance properties $\|f_1\|_{U^2} = \|f_1'\|_{U^2}$ and $\|f_2\|_{U^2} = \|f_2'\|_{U^2}$, and so the result follows immediately from Theorem 4.1.
Now suppose that \( k \geq 4 \) and the claim has already been proven for smaller \( k \). By permuting indices and then translating \( n \), it suffices to show that

\[
|E_{n \in [N]; d \in [-N,N]} F(g(d)\Gamma) \prod_{i=0}^{k-1} f_i(n+c_id)| \ll_{k,M,\epsilon_0,\ldots,\epsilon_{k-1}} \|f_{k-1}\|_{U^{k-1}[N]} \quad (4.2)
\]

under the assumption that \( c_0 = 0 \).

Recall from [30] that we define a *vertical character* to be a continuous homomorphism \( \xi : G_{(k-2)}/(G_{(k-2)} \cap \Gamma) \to \mathbb{R}/\mathbb{Z} \). We say that \( F \) has *vertical frequency* \( \xi \) if one has \( F(g_{k-2}x) = e(\xi(g_{k-2}))F(x) \) for all \( x \in G/\Gamma \) and \( g_{k-2} \in G_{(k-2)} \). By a standard Fourier decomposition in the vertical direction (e.g. by arguing exactly as in [30 Lemma 3.7]) we may assume without loss of generality that \( F \) has a vertical frequency \( \xi \).

Applying the Cauchy-Schwarz inequality, we can bound the left-hand side of (4.2) by

\[
\ll |E_{n \in [N]; h, d \in [-N,N]} F(g(d+h)\Gamma) \overline{F(g(d)\Gamma)} \prod_{i=0}^{k-1} f_i(n+c_id+c_id) f_i(n+c_id)|^{1/2}.
\]

Because \( F \) has a vertical frequency, \( F(g(d+h))\overline{F(g(d)\Gamma)} \) is a degree \( \leq (k-3) \) nilsequence of complexity \( O_{M,k}(1) \) (see [30 Proposition 7.2]). Applying the induction hypothesis, we may thus bound the above expression by

\[
\ll_{M,k,c_0,\ldots,c_{k-1}} (E_{h \in [-N,N]} \|\Delta_{c_i} f_i\|_{U^{k-2}[N]}^2)^{1/2}
\]

which by Hölder’s inequality can be bounded by

\[
\ll_{M,k,c_0,\ldots,c_{k-1}} (E_{h \in [-N,N], c_i \in [N]} \|\Delta_{c_i} f_i\|_{U^{k-2}[N]}^{2k-2})^{1/2k-2}
\]

and the claim follows from the recursive definition of the Gowers norms. \( \square \)

**Remark.** The above argument is very similar to the short proof presented in [33] Appendix G that \( s \)-step nilsequences obstruct uniformity in the \( U^{s+1} \)-norm (that is, the inverse conjecture GI(\( s \)) is an if-and-only if statement).

### 5. On a Conjecture of Bergelson, Host, and Kra

We now apply the arithmetic regularity and counting lemmas to establish Theorem 1.12, the proof of the conjecture of Bergelson, Host and Kra. Our strategy here can be viewed as a finitary analogue of the ergodic theory arguments in [31], however there are some slight differences in our approach which we comment on at the end of this section.

It will suffice to prove the following claim.

**Theorem 5.1.** Let \( k = 1, 2, 3 \) or 4, and suppose that \( 0 < \alpha < 1 \) and \( \epsilon > 0 \).

Then for any \( N \geq 1 \) and any subset \( A \subseteq [N] \) of density \( |A| \geq \alpha N \), one can find a function \( \mu : \mathbb{Z} \to \mathbb{R}^+ \) such that

\[
E_{d \in [-N,N]} \mu(d) = 1 + O(\epsilon) \quad (5.1)
\]
and
\[
\sup_{d \in [-N,N]} \mu(d) \ll_{\alpha, \varepsilon} 1
\]

such that
\[
\mathbb{E}_{n \in [N], d \in [-N,N]} 1_A(n) 1_A(n + d) \ldots 1_A(n + (k - 1)d) \mu(d) \geq \alpha^k - O(\varepsilon). \tag{5.3}
\]

Indeed, from (5.1), (5.3), we see that we have
\[
\mathbb{E}_{n \in [N]} 1_A(n) 1_A(n + d) \ldots 1_A(n + (k - 1)d) \geq \alpha^k - O(\varepsilon)
\]
for all \(d\) in a subset \(E\) of \([-N,N]\) with \(\mathbb{E}_{d \in [-N,N]} 1_E(d) \mu(d) \gg_{\alpha, \varepsilon} 1\). (Here we crucially use the trivial but fundamental fact that \(1_A\) is nonnegative.) From (5.2) we conclude that \(|E| \gg_{\alpha, \varepsilon} N\), and Theorem 1.12 follows (after shrinking \(\varepsilon\) by an absolute constant). Conversely, it is not difficult to deduce Theorem 1.12 from Theorem 5.1.

It remains to establish Theorem 5.1. We may assume that \(N\) is large depending on \(\alpha, \varepsilon\) as the claim is trivial otherwise (just take \(\mu\) to be the Kronecker delta function at 0).

For \(k = 1\) one can simply take \(\mu \equiv 1\). For \(k = 2\), we first observe that
\[
\mathbb{E}_{n \in [N]} \mathbb{E}_{h \in [-\varepsilon N, \varepsilon N]} 1_A(n + h) = \alpha + O(\varepsilon);
\]
applying Cauchy-Schwarz we conclude that
\[
\mathbb{E}_{h, h' \in [-\varepsilon N, \varepsilon N]} \mathbb{E}_{n \in [N]} 1_A(n + h) 1_A(n + h') \geq \alpha^2 - O(\varepsilon).
\]
The claim then follows, with \(\mu\) being the probability density function of \(h - h'\) as \(h, h'\) range uniformly in \([-\varepsilon N, \varepsilon N]\).

Now we turn to the cases \(k = 3, 4\). Here, one has to be more sophisticated about how one chooses \(\mu\) (for instance, by using a Behrend set construction it is not hard to see that the previous choices of \(\mu\) do not always work). Let \(F : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be a sufficiently rapidly growing function depending on \(\alpha, \varepsilon\) in a manner to be specified later. We apply Theorem 1.12 with \(s := k - 2\) to obtain a quantity \(M = O_{\varepsilon,F}(1)\) and a decomposition
\[
1_A(n) = f_{\text{nil}}(n) + f_{\text{sml}}(n) + f_{\text{unf}}(n) \tag{5.4}
\]
such that
(i) \(f_{\text{nil}}(n)\) is a \((F(M), N)\)-irrational degree \(\leq k - 2\) virtual nilsequence of complexity at most \(M\) and scale \(N\);
(ii) \(f_{\text{sml}}\) has an \(L^2[N]\) norm of at most \(\varepsilon/100\);
(iii) \(f_{\text{unf}}\) has an \(U^{k-1}[N]\) norm of at most \(1/F(M)\);
(iv) \(f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}}\) are all bounded in magnitude by 1; and
(v) \(f_{\text{nil}}\) and \(f_{\text{unf}} + f_{\text{sml}}\) are non-negative.

It is clear that \(|\mathbb{E}_{n \in [N]} f_{\text{sml}}(n)| = O(\varepsilon)\), and furthermore, by Theorem 1.1 (setting all but one of the functions equal to 1) we also have \(|\mathbb{E}_{n \in [N]} f_{\text{unf}}(n)| = O(\varepsilon)\) if \(F\) grows rapidly enough. Therefore
\[
\mathbb{E}_{n \in [N]} f_{\text{nil}}(n) \geq \alpha - O(\varepsilon). \tag{5.5}
\]
The heart of the matter is the following proposition.
Proposition 5.2 (Bergelson-Host-Kra for \( f_{\text{nil}} \)). Let \( k = 3, 4 \). Then there exists a non-negative \((k - 2)\)-step nilsequence \( \mu : \mathbb{Z} \to \mathbb{R}^+ \) of complexity \( O_{\alpha, \varepsilon, M}(1) \) obeying the normalisation
\[
\mathbb{E}_{d \in [N]} \mu(d) = 1 + O(\varepsilon)
\]
and such that
\[
\mathbb{E}_{n, d \in [N]} f_{\text{nil}}(n) f_{\text{nil}}(n + d) \ldots f_{\text{nil}}(n + (k - 1)d) \mu(d) \geq \alpha^k - O(\varepsilon). \tag{5.7}
\]

Deduction of Theorem 5.1 from Proposition 5.2. Using (5.4), one can expand the left-hand side of (5.3) into \( 3^k \) terms, one of which is (5.7). As for the other terms, any term involving at least one copy of \( f_{\text{nil}} \) is of size \( O_{\alpha, \varepsilon, M}(1/F(M)) \) by Lemma 4.2 and the \( U^{k-1} \) norm bound on \( f_{\text{nil}} \). Finally, consider a term that involves at least one copy of \( f_{\text{sm}} \). Suppose first that we have a term that involves \( f_{\text{sm}}(n) \). Then after performing the average in \( d \) using (5.6), we see that this term is \( O(\mathbb{E}_{n \in [N]} |f_{\text{sm}}(n)|) \), which is \( O(\varepsilon) \) by the \( L^2[N] \) bound on \( f_{\text{sm}} \) and the Cauchy-Schwarz inequality. Similarly for any term that involves \( f_{\text{sm}}(n + id) \), after making a change of variables \((n', d) := (n + id, d)\). Putting all this together we obtain the result. \( \square \)

It remains, of course, to establish Proposition 5.2. We may assume that \( N \) is sufficiently large depending on \( \alpha, \varepsilon, M \), as the claim is trivial otherwise by taking \( \mu \) to be a delta function.

We first establish the proposition in the easier of the two cases, namely the case \( k = 3 \). This was previously considered in [25]. In this case it is actually easier to work with the (easier) weak regularity lemma, Proposition 2.7, in which the degree 1 polynomial sequence \( g(n) \) is not required to be irrational. Note that we have not made any use of irrationality so far, though we shall do so later when discussing the case \( k = 4 \). We may identify \( G/\Gamma \) with \((\mathbb{R}/\mathbb{Z})^m\) for some \( m = O_M(1) \) and, by modulating \( F \) if necessary, we may suppose that \( g(n) = \theta n \) is linear with no constant term, where \( \theta \in \mathbb{R}^m \).

Then
\[
f_{\text{nil}}(n) = F(n\theta),
\]
where \( F : (\mathbb{R}/\mathbb{Z})^m \to \mathbb{C} \) has Lipschitz norm \( O_M(1) \).

Let \( \varepsilon' > 0 \) be a small number depending on \( \varepsilon \) and \( M \) to be chosen later, and let \( B_1, B_2 \subseteq [-N, N] \) denote be the two Bohr sets
\[
B_1 := \{ d \in [-\varepsilon' N, \varepsilon' N] : \text{dist}(\mathbb{R}/\mathbb{Z})^m(\theta d, 0) \leq \varepsilon' \}
\]
and
\[
B_2 := \{ d \in [-\varepsilon' N, \varepsilon' N] : \text{dist}(\mathbb{R}/\mathbb{Z})^m(\theta d, 0) \leq \varepsilon'/2 \}.
\]
By the usual Dirichlet pigeonhole argument we see that \( |B_2| \gg \varepsilon', M N \). Also, from the Lipschitz nature of \( F \), we see that
\[
f_{\text{nil}}(n + d) = f_{\text{nil}}(n) + O_M(\varepsilon')
\]
whenever \( d \in B_1 \) and \( n \in [-1 - \varepsilon')N, (1 - \varepsilon')N] \). As a consequence, it follows that
\[
\mathbb{E}_{n \in [N]} f_{\text{nil}}(n) f_{\text{nil}}(n + d) f_{\text{nil}}(n + 2d) = \mathbb{E}_{n \in [N]} f_{\text{nil}}(n)^3 + O_M(\varepsilon')
\]
for such $d$. However from (5.5) and Hölder’s inequality one has
\[ \mathbb{E}_{n \in \mathbb{N}} f(n)^3 \geq C \mathcal{O}(\varepsilon). \]  
(5.8)

Proposition 5.2 (in the case $k = 3$) now follows by taking $\mu(d) = c N d^3$, where $\psi : (\mathbb{R}/\mathbb{Z})^m \to [0, 1]$ is an $O(1)$-Lipschitz function which is 1 on $B_2$ and 0 outside $B_1$, $c = O(1)$ is a suitable normalisation constant, and by taking $\varepsilon'$ to be suitably small.

It is important to note here that the error term $O(\varepsilon)$ in (5.8) is uniform in $M$, as otherwise the argument would not work (recall that $M$ will depend on $\varepsilon$). The dependence on $M$ is instead manifested where it does not do significant damage to the argument, namely in the complexity of the weight $\mu$.

We now turn to the $k = 4$ case of Proposition 5.2. For simplicity, let us first consider the model case when $f$ is a genuine nilsequence and not just a virtual nilsequence, that is to say
\[ f(n) = F(g(n) \Gamma) \]  
(5.9)

where $(G/\Gamma, G \cdot \Gamma)$ is a degree $\leq 2$ filtered nilmanifold of complexity $O(1)$, and $g \in \text{poly}(\mathbb{Z}, G \cdot \Gamma)$ is $\mathcal{F}(M, N)$-irrational. By Taylor expansion (see Appendix A), we have
\[ g(n) = g_0(n) + g_1(n) + g_2(n) \]  
for some $g_0, g_1 \in G$ and $g_2 \in G_{(2)}$. The $\mathcal{F}(M, N)$-irrationality of $g$ ensures certain irrationality properties on $g_1$ and $g_2$, though we will not need these properties explicitly here, as we will only be using them through the counting lemma (Theorem 1.11), which we shall be using as a black box.

Let $\pi : G \to T_1$ be the projection homomorphism to the torus $T := G/(G_{(2)} \Gamma)$. Then
\[ \pi(g(n)) = \pi(g_0) \pi(g_1)^n. \]

Let $\varepsilon' > 0$ be a small quantity depending on $\varepsilon, M$ to be chosen later. We set
\[ \mu(d) := c \int_{[\varepsilon', N]} \phi(d) \phi(\pi(g_1)^d), \]
where, much as in the analysis of the case $k = 3$, $\phi : T_1 \to \mathbb{R}^+$ is a smooth non-negative cutoff to the ball of radius $\varepsilon'$ centered at the origin that is not identically zero, and $c$ is a normalisation constant to be chosen shortly. From Theorem 1.11 one has
\[ \mathbb{E}_{d \in [\varepsilon', N]} \phi(d) \phi(\pi(g_1)^d) = \int_{T_1} \phi + o_{\mathcal{F}(M) \to \infty, \varepsilon', M}(1) + o_{N \to \infty, \varepsilon'}(1). \]

Thus if we set
\[ c := \frac{1}{\int_{T_1} \phi} = O_{\varepsilon', M}(1) \]  
(5.10)

then we have the normalisation (5.6), if $\mathcal{F}$ is sufficiently rapid, depending on the way in which $\varepsilon'$ depends on $\varepsilon, M$, and $N$ is sufficiently large depending

\[ \text{Note this is not quite the same thing as the horizontal torus, which is so important in [30], which is } (G/\Gamma)_{ab} := G/[G, G] \Gamma. \]
on $\varepsilon, \varepsilon', M$. From the bound on $c$ we see that $\mu$ is a degree $\leq 1$ (and hence also degree $\leq 2$) nilsequence of complexity $O_{\varepsilon', M}(1)$.

We now apply the counting lemma, Theorem 1.11, to conclude that
\[
\mathbb{E}_{n, d \in [N]} f_{\text{nil}}(n) f_{\text{nil}}(n + d) f_{\text{nil}}(n + 2d) f_{\text{nil}}(n + 3d) \mu(d)
= \int_{G^\psi/\Gamma^\psi} \tilde{F} + o_{\mathcal{F}(M) \to \infty; \varepsilon', M}(1) + o_{N \to \infty; \varepsilon', M}(1)
\]
(5.11)
where $G^\psi \subseteq G^4$ is the Leibman group associated to the collection $\Psi = (\psi_0, \psi_1, \psi_2, \psi_3) : \mathbb{Z}^2 \to \mathbb{Z}^4$ of linear forms $\psi_i(n) := n_1 + in_2$, $i = 0, 1, 2, 3$, that is to say the Hall-Petresco group $\text{HP}^4(G)$, and $\tilde{F} : G^\psi \to \mathbb{C}$ is the function
\[
\tilde{F}(x_0, x_1, x_2, x_3) := c\phi(\pi(x_1)\pi(x_0)^{-1})F(x_0)F(x_1)F(x_2)F(x_3)
\]
(here we use the identity $\pi(g(n+d))^{-1}\pi(g(n)) = \pi(g_1)^d$, immediately verified from the Taylor expansion).

We now do some calculations in the Hall-Petresco group very similar to those in [3]. We saw in [3] that
\[
G^\psi = \{(g_0, g_0g_2, g_0g_1^2g_2; g_0g_1^3g_2^3) : g_0, g_1 \in G, g_2 \in G(2)\}
\]
(note, of course, that $G(3) = \text{id}$ in the case we are considering). For our calculations it is convenient to use the following obviously equivalent representation:
\[
G^\psi = \{(g_0g_2, g_0g_1g_2, g_0g_1^2g_22; g_0g_1^3g_2^3) : g_0, g_1 \in G; \ g_2, \ldots, g_23 \in G(2); g_20g_21g_22g_23 = \text{id}\}.
\]
Here we have taken note of the fact that $\Psi^2 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 - 3x_1 + 3x_3 - x_3 = 0\}$. This last equation is quite special in that it exhibits a certain “positivity”, as we shall see later; this is key to our argument. The lattice $\Gamma^\psi$ can be similarly described by requiring $g_0, g_1, g_2, \ldots, g_23$ to also lie in $\Gamma$. As a consequence of this, an arbitrary point of the nilmanifold $G^\psi/\Gamma^\psi$ can be parameterised uniquely as
\[
(g_0g_2, g_0g_1g_2, g_0g_1^2g_22; g_0g_1^3g_2^3)\Gamma^\psi
\]
(5.12)
where $g_0, g_1$ lie in a fundamental domain $\Sigma_1 \subset G$ of the horizontal torus $T_1$ (i.e. a smooth manifold with boundary on which $\pi$ is a bijection from $\Sigma_1$ to $T_1$), and $g_2, g_22, \ldots, g_23$ lie in a fundamental domain $\Sigma_2 \subset G(2)$ of the vertical torus $T_2 := G(2)/\Gamma_{(2)}$ subject to the constraint $g_20g_21g_22g_23^{-1} \in \Gamma_{(2)}$. For such a point (5.12), the function $\tilde{F}$ takes the value
\[
c\phi(\pi(g_1)) \prod_{j=0}^3 F(g_0g_1^jg_2j\Gamma).
\]
On the support of $\phi$, $g_1$ is a distance $O_M(\varepsilon')$ from the identity (if the fundamental domain $\Sigma_1$ was chosen in a suitably smooth fashion), and so by the Lipschitz nature of $F$ and the boundedness of $g_0$ we have

$$F(g_0 g_1 g_2 g_3) = F(g_0 g_2_3 \Gamma) + O_M(\varepsilon').$$

As a consequence, the integral $\int_{G^{\Psi \Psi}} \hat{F}$ can be expressed as

$$c \int_{g_0 \in \Sigma_1} \int_{g_1 \in \Sigma_1} \phi(\pi(g_1)) \left( \int_{g_2,0,\ldots,2,3 \in \mathbb{T}_2} \prod_{j=0}^3 F(g_0 g_2_3 \Gamma) + O_M(\varepsilon') \right) \quad (5.13)$$

where all integrals are with respect to Haar measure.

Let $\xi \in \mathcal{T}_2$ be a vertical character, i.e. a continuous homomorphism from $\mathbb{T}_2$ to $\mathbb{R}/\mathbb{Z}$. For any $x \in G/\Gamma$, we can define the vertical Fourier transform $\hat{F}(x, \xi)$ to be the quantity

$$\hat{F}(x, \xi) := \int_{g_2 \in \mathcal{T}_2} e(-\xi(g_2))F(g_2 x).$$

From the Fourier inversion formula we have

$$\int_{g_2,0,\ldots,2,3 \in \mathbb{T}_2} \prod_{j=0}^3 F(g_0 g_2_3 \Gamma) = \sum_{\xi \in \mathcal{T}_2} |\hat{F}(g_0, \xi)|^2 |\hat{F}(g_0, 3\xi)|^2.$$  

In particular, we have\footnote{This is the “positivity” alluded to earlier. The argument is essentially that used in \cite{4} and it is special to the $k = 4$ case, which is of course consistent with the failure of Theorem (5.1) to extend to $k \geq 5$.}

$$\int_{g_2,0,\ldots,2,3 \in \mathbb{T}_2} \prod_{j=0}^3 F(g_0 g_2_3 \Gamma) \geq |\hat{F}(g_0, 0)|^4.$$  

Inserting this bound and (5.10) into (5.13), we conclude that

$$\int_{G^{\Psi \Psi}} \hat{F} \geq \int_{g_0 \in \Sigma_1} |\hat{F}(g_0, 0)|^4 - O_M(\varepsilon') - o_{F(M) \to \infty; \varepsilon', M}(1).$$

From Fubini’s theorem we have

$$\int_{g_0 \in \Sigma_1} \hat{F}(g_0, 0) = \int_{G/\Gamma} F$$

and from Theorems 1.11, (5.9) and (5.13), we have

$$\int_{G/\Gamma} F = \alpha + O(\varepsilon) + o_{F(M) \to \infty; \varepsilon', M}(1) + o_{N \to \infty; \varepsilon', M}(1).$$

Applying Hölder’s inequality, we conclude that

$$\int_{G^{\Psi \Psi}} \hat{F} \geq \alpha^4 - O(\varepsilon') - O_M(\varepsilon') - o_{F(M) \to \infty; \varepsilon', M}(1) - o_{N \to \infty; \varepsilon', M}(1),$$
and so (5.7) follows from (5.11), if \( \varepsilon' \) is sufficiently small depending on \( \varepsilon, M \), \( F \) is sufficiently rapid depending on \( \varepsilon \), and \( N \) is sufficiently large depending on \( \varepsilon', M \).

This concludes the proof of the \( k = 4 \) case of Proposition 5.2 in the special case when \( f_{\text{nil}}(n) = F(g(n)\Gamma) \) with \( g \) irrational. Unfortunately Theorem 1.2 requires us to deal with the somewhat more general setting of virtual nilsequences, in which there is dependence on \( n \mod q \) or \( n/N \). The extra details required are fairly routine but notationally irritating. Let us now suppose, then, that

\[
f_{\text{nil}}(n) = F(g(n)\Gamma, n \mod q, n/N).
\]  

We let \( \varepsilon' \) be as before, but modify \( \mu \) to now be given by

\[
\mu(d) := q1_{[d \in [-\varepsilon'N, \varepsilon'N]]}(d)\phi(\pi(g_1)^d),
\]

with \( c \) still chosen by (5.10). As before, one can use Theorem 1.11 to establish (5.6).

Now consider the left-hand side of the expression (5.7) we are to bound in Proposition 5.2, that is to say

\[
E_{n,d \in [N]} f_{\text{nil}}(n)f_{\text{nil}}(n + d)f_{\text{nil}}(n + 2d)f_{\text{nil}}(n + 3d)\mu(d).
\]

Splitting into residue classes modulo \( q \), we can express this as

\[
cE_{r \in [q]}E_{n \in [N/q]}E_{d \in [-\varepsilon'N/q, \varepsilon'N/q]} \prod_{i=0}^{3} F(g(qn + qid + r)\Gamma, r, q(n + ir)/N)\phi(\pi(g_1)^{qd}) + O_{N \to \infty; \varepsilon', M}(1).
\]

We partition \([N/q]\) into intervals \( P \) of length \( \lfloor \varepsilon'N \rfloor \) (plus a remainder of cardinality \( O(\varepsilon'N) \)). We can then rewrite the above expression as

\[
cE_{P}E_{r \in [q]}E_{n \in P}E_{d \in [-\varepsilon'N/q, \varepsilon'N/q]} \prod_{i=0}^{3} F(g(qn + qid + r)\Gamma, r, q(n + ir)/N)\phi(\pi(g_1)^{qd}) + O(\varepsilon') + O_{N \to \infty; \varepsilon', M}(1).
\]

For each such expression, we can use the Lipschitz nature of \( F \) to replace \( q(n + ir)/N \) by \( qn_P/N \), where \( n_P \) is an arbitrary element of \( P \), losing only an error of \( O_M(\varepsilon') \). The above expression thus becomes

\[
cE_{P}E_{r \in [q]}E_{n \in P}E_{d \in [-\varepsilon'N/q, \varepsilon'N/q]} \prod_{i=0}^{3} F(g(qn + qid + r)\Gamma, r, qn_P/N)\phi(\pi(g_1)^{qd}) + O_{M}(\varepsilon') + O_{N \to \infty; \varepsilon', M}(1).
\]

Because the orbit \( n \mapsto g(n)\Gamma \) is \((F(M), N)\)-irrational, we see from Lemma A.8 that shifted translate \( n \mapsto g(q(n + n_P) + r)\Gamma \) is \((\gg_M F(M), N)\)-irrational. We may then argue as in the previous case and bound the above...
average below by
\[ \geq E_P E_{r \in \{q\}} \int_{G/\Gamma} F(\cdot, r, qn_P / N)|^4 - O(\varepsilon) - O_M(\varepsilon') \\
\quad - o_F(M) \rightarrow \infty; \varepsilon', M(1) - o_N \rightarrow \infty; \varepsilon', M(1). \]

Using Theorem 1.11 again, we have
\[ \mathbb{E}_{n \in P} f_{\text{unif}}(qn + r) = \int_{G/\Gamma} F(\cdot, r, qn_P / N) + o_F(M) \rightarrow \infty; \varepsilon', M(1) + o_N \rightarrow \infty; \varepsilon', M(1) \]
and so \((5.15)\) is at least
\[ \geq E_P E_{r \in \{q\}} |\mathbb{E}_{n \in P} f_{\text{unif}}(qn + r)|^4 - O(\varepsilon) - O_M(\varepsilon') \\
\quad - o_F(M) \rightarrow \infty; \varepsilon', M(1) - o_N \rightarrow \infty; \varepsilon', M(1). \]

Now from \((5.5)\) and double-counting one has
\[ E_P E_{r \in \{q\}} \mathbb{E}_{n \in P} f_{\text{unif}}(qn + r) = \alpha + O(\varepsilon) \]
and so, from Hölder’s inequality, we deduce that \((5.15)\) is
\[ \geq \alpha^4 - O(\varepsilon) - O_M(\varepsilon') - o_F(M) \rightarrow \infty; \varepsilon', M(1) - o_N \rightarrow \infty; \varepsilon', M(1). \]

Proposition 5.2 now follows by once again choosing \(\varepsilon'\) small enough depending on \(\varepsilon, M\), and choosing \(F\) rapid enough depending on \(\varepsilon\), and \(N\) sufficiently large depending on \(\varepsilon, \varepsilon', M\).

Remark. Our arguments are similar to, but slightly different from, the ergodic theory arguments in [4]. However it is likely that the argument in [4] can be translated to a finitary setting; we sketch how this would proceed as follows, restricting attention to the \(k = 4\) case for concreteness. The goal is to obtain a lower bound \(\mathbb{E}_{n \in \{N\}} f(n) f(n + d) f(n + 2d) f(n + 3d) \geq \alpha^4 - O(\varepsilon)\) for some positive density set of values of \(d\). The analogue to the argument in [4] would proceed by performing the regularity lemma decomposition at step \(s = 3\) rather than \(s = 2\), so that the error \(f_{\text{unif}}\) is tiny in the \(U^4\) norm and not just the \(U^3\) norm. From this and Theorem 4.1, one can show that
\[ E_{d \in \{N\}} |\mathbb{E}_{n \in \{N\}} f_1(n) f_2(n + d) f_3(n + 2d) f_4(n + 3d)|^2 \]
is tiny whenever at least one of \(f_1, f_2, f_3, f_4\) is equal to \(f_{\text{unif}}\). As a consequence, \(\mathbb{E}_{n \in \{N\}} f_1(n) f_2(n + d) f_3(n + 2d) f_4(n + 3d)\) is negligible for almost all \(d\). We can thus ignore the contribution of \(f_{\text{unif}}\). The remainder of the argument proceeds along similar lines as above, but at one higher step (though the 3-step nilsequences involved can quickly be reduced to 2-step nilsequences, cf. [4] Section 8.1) or Section 7 below).

One of the innovations in this paper is to introduce weights such as \(\mu(d)\), controlling the double average \(\mathbb{E}_{n, d \in \{N\}} f(n) f(n + d) f(n + 2d) f(n + 3d) \mu(d)\) rather than controlling the single average \(\mathbb{E}_{n \in \{N\}} f(n) f(n + d) f(n + 2d) f(n + 3d)\) for many values \(d\). Thanks to the twisted generalised von Neumann theorem (Lemma 1.12), the “complexity” of such double averages is slightly less than that of the single averages, and in particular our proof of Theorem
1.12 requires only the inverse $U^3$ theorem from [28] rather than the more difficult inverse $U^4$ theorem from [33].

6. Proof of Szemerédi’s theorem

We turn now to the proof of Szemerédi’s theorem. We deemed this result too famous to state in the introduction but, for the sake of fixing notation, we recall it here now. It is most natural to establish what might be called the “functional” form of the theorem which is \textit{a priori} a stronger statement (though quite easily shown to be equivalent to the standard formulation by an argument of Varnavides [57]).

\begin{theorem}[Szemerédi’s theorem] Let $0 < \alpha \leq 1$, let $k \geq 3$, and let $N \geq 1$. If $f : [N] \to [0,1]$ is a function with $E_{n \in [N]} f(n) \geq \alpha$ then
\[
\Lambda_k(f, f, \ldots, f) \gg_{k,\alpha} 1,
\]
where
\[
\Lambda_k(f_1, \ldots, f_k) := E_{n \in [N]; d \in [-N,N]} f_1(n)f_2(n + d)\ldots f_k(n + (k - 1)d)
\]
is the multilinear operator counting arithmetic progressions.
\end{theorem}

We now prove this theorem. We fix $k, \alpha$, and allow implied constants to depend on these quantities. Our argument has some similarities with the ergodic theory proof of (a polynomial generalisation of) Szemerédi’s theorem in [6], in particular in first reducing the problem to a problem concerning nilsystems, which one then solves by the equidistribution theory of such systems. However, one of the key steps in [6], in which one shows that multiple recurrence is preserved under inverse limits, is more difficult to replicate in the finitary setting than in the ergodic one (see [50]). Our argument thus differs somewhat from [6], most notably by inserting a carefully chosen weight $\mu(n,d)$ before proceeding.

As usual, we begin by applying the regularity lemma, Theorem 1.2. In view of the generalised von Neumann theorem, Theorem 4.1, it is natural to apply this theorem with $s = k - 2$ (which, as remarked in [14], is the Cauchy-Schwarz complexity $s = s(\Psi)$ of the system $\Psi$ of linear forms $n_1, n_1 + n_2, \ldots, n_1 + (k - 1)n_2$). If we do so, with a small parameter $\varepsilon > 0$ depending on $\alpha, k$ to be chosen later, and a growth function $F$ depending on $\alpha, k, \varepsilon$ to be specified later, we obtain a decomposition
\[
f(n) = f_{\text{nil}}(n) + f_{\text{sml}}(n) + f_{\text{unf}}(n) \tag{6.1}
\]
where
\begin{enumerate}
\item $f_{\text{nil}}$ is a $(\mathcal{F}(M), N)$-irrational degree $\leq k - 2$ virtual nilsequence of complexity $\leq M$ and scale $N$;
\item $f_{\text{sml}}$ has an $L^2[N]$ norm of at most $\varepsilon$;
\item $f_{\text{unf}}$ has an $U^{k-1}[N]$ norm of at most $1/\mathcal{F}(M)$;
\item $f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}}$ are all bounded in magnitude by 1; and
\item $f_{\text{nil}}$ and $f_{\text{nil}} + f_{\text{sml}}$ are non-negative.
\end{enumerate}
As we shall soon see, the contribution of $f_{unf}$ can be quickly discarded using the generalised von Neumann theorem. If one could also easily discard the contribution of the small term $f_{sml}$, then matters would simply reduce to verifying that the contribution of $f_{nil}$ is bounded away from zero, which would be an easy consequence of the counting lemma. Unfortunately the small term $f_{sml}$ is only moderately small (of size $O(\varepsilon)$) rather than incredibly small (e.g. of size $O(1/F(M))$), and so one has to take a certain amount of care in dealing with this term, which makes the analysis significantly more delicate.

We turn to the details. Much as the key to proving Theorem 1.12 was to establish Proposition 5.2, the key to establishing Szemerédi’s theorem is the following proposition.

**Proposition 6.2 (Szemerédi for $f_{nil}$).** Let $f_{nil}$ be as above, and let $\varepsilon > 0$. Then there exists a function $\mu : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^+$ supported on the set

$$\{(n, d) \in \mathbb{Z} \times \mathbb{Z} : d \in [-\varepsilon N, \varepsilon N]; n + id \in [N] \text{ for all } i = 0, \ldots, k - 1\}$$

with

$$\mathbb{E}_{n \in [N]} |d \in [-\varepsilon N, \varepsilon N]| \mu(n, d) = 1 + O(\varepsilon)$$

and with $\mu$ bounded in magnitude by $O_{M, \varepsilon}(1)$, such that

$$f_{nil}(n + id) = f_{nil}(n) + O(\varepsilon)$$

whenever $0 \leq i \leq k - 1$ and $\mu(n, d) \neq 0$, and such that one has the equidistribution property

$$\mathbb{E}_{n \in [N]} \mathbb{E}_{d \in [-\varepsilon N, \varepsilon N]} | \mu(n - id, d) |^2 = 1 + O(\varepsilon)$$

for all $0 \leq i \leq k - 1$.

The crucial feature of Proposition 6.2 for us is that, with the exception of the uniform bound on $\mu$, the error terms here decay as $\varepsilon \to 0$, even if the complexity bound $M$ on $f_{nil}$ is extremely large compared to $1/\varepsilon$.

The reader may benefit from a few words about the role of the function $\mu$. Supposing that $f_{nil}(n) = F(g(n)\Gamma)$ is a genuine nilsequence, this function acts like a kind of “weight” on progressions $(n, n + d, \ldots, n + (k-1)d)$ which are “almost diagonal” in the sense that $g(n)\Gamma \approx \cdots \approx g(n + (k-1)d)\Gamma$ in $G/\Gamma$. The condition (6.5) reflects the fact that the weighted number of almost diagonal progressions whose $i$th point is $n$ is roughly independent of $n$. This “non-concentration” of almost diagonal progressions ultimately means that the error $f_{sml}$ cannot destroy too many of these progressions, a fact that is crucial for our argument.

\[14\] In the language of ergodic theory, the problem here is that the characteristic factor is not necessarily a nilsystem, but may merely be a pro-nilsystem - an inverse limit of nilsystems. A short, but not entirely trivial, argument of Furstenberg [11] shows that multiple recurrence is preserved under inverse limits. This argument was adapted with some difficulty to the finitary setting in [50]; our approach here is different and exploits some additional equidistribution properties of nilsystems, as well as using a carefully chosen weight $\mu(n, d)$.
Let us assume Proposition 6.2 for now and see how it implies Theorem 4.1. We use (6.1) to expand out the form \( \Lambda_k(f, \ldots, f) \) into \( 3^k \) terms. By Theorem 4.1 any term that involves \( f_{\text{unf}} \) will be of size \( O(1/\mathcal{F}(M)) \), thus
\[
\Lambda_k(f, \ldots, f) = \Lambda_k(f_{\text{nil}} + f_{\text{sml}}, \ldots, f_{\text{nil}} + f_{\text{sml}}) + O(1/\mathcal{F}(M)).
\]
Next, we use the weight \( \mu \) arising from Proposition 6.2 and the non-negativity of \( f_{\text{nil}} + f_{\text{sml}} \) guaranteed by Theorem 1.2 to write
\[
\Lambda_k(f_{\text{nil}} + f_{\text{sml}}, \ldots, f_{\text{nil}} + f_{\text{sml}}) \gg M, \varepsilon \mathbb{E}_{n \in [N]: d \in [-\varepsilon N, \varepsilon N]} (f_{\text{nil}} + f_{\text{sml}})(n) \ldots (f_{\text{nil}} + f_{\text{sml}})(n + (k - 1)d) \mu(n, d).
\]
We then expand this latter average into the sum of \( 2^k \) terms. The main term is
\[
\mathbb{E}_{n \in [N]: d \in [-\varepsilon N, \varepsilon N]} f_{\text{nil}}(n) \ldots f_{\text{nil}}(n + (k - 1)d) \mu(n, d),
\]
and the other terms are error terms, involving at least one factor of \( f_{\text{sml}} \).

Consider one of the error terms, involving the factor \( f_{\text{sml}} \) (say) for some \( 0 \leq i \leq k - 1 \). We can bound the contribution of this term by
\[
\mathbb{E}_{n \in [N]: d \in [-\varepsilon N, \varepsilon N]} f_{\text{sml}}(n + id) \mu(n, d),
\]
which by a change of variables \( n \mapsto n - id \) we can write as
\[
\mathbb{E}_{n \in [N]} |f_{\text{sml}}(n)| \mathbb{E}_{d \in [-\varepsilon N, \varepsilon N]} \mu(n - id, d).
\]
By Cauchy-Schwarz, (6.5), and the \( L^2[N] \) bound on \( f_{\text{sml}} \), this is \( O(\varepsilon) \).

Finally, we look at the main term (6.7). Using (6.4) we can approximate
\[
f_{\text{nil}}(n) \ldots f_{\text{nil}}(n + (k - 1)d) = f_{\text{nil}}(n)^k + O(\varepsilon)
\]
and so (using (6.3)) we can write (6.7) as
\[
\mathbb{E}_{n \in [N]} f_{\text{nil}}(n)^k \mathbb{E}_{d \in [-\varepsilon N, \varepsilon N]} \mu(n, d) + O(\varepsilon).
\]
Now, from (6.3) one has
\[
\mathbb{E}_{n \in [N]} \mathbb{E}_{d \in [-\varepsilon N, \varepsilon N]} \mu(n, d) = 1 + O(\varepsilon)
\]
and hence by (6.5)
\[
\mathbb{E}_{n \in [N]} \mathbb{E}_{d \in [-\varepsilon N, \varepsilon N]} |\mu(n, d) - 1|^2 = O(\varepsilon).
\]
In particular, by Chebyshev’s inequality, we have
\[
\mathbb{E}_{d \in [-\varepsilon N, \varepsilon N]} \mu(n, d) = 1 + O(\varepsilon^{1/3})
\]
for all \( n \in E \), where \( E \subseteq [N] \) has cardinality \(|E| \geq (1 - O(\varepsilon^{1/3})) N \). Thus, for \( \varepsilon \) small enough, we can bound (6.7) from below by
\[
\gg \mathbb{E}_{n \in [N]} \frac{1}{|E(n)|} f_{\text{nil}}(n)^k - O(\varepsilon^{1/3}).
\]
Now from hypothesis we have \( \mathbb{E}_{n \in [N]} f(n) \gg 1 \). From Cauchy-Schwarz we have
\[
\mathbb{E}_{n \in [N]} f_{\text{sml}}(n) = O(\varepsilon),
\]
and from Theorem 4.1 we also have
\[
\mathbb{E}_{n \in [N]} f_{\text{unf}}(n) = O(\varepsilon)
if $F$ is rapid enough. Thus if $\varepsilon$ is small enough we have $E_{n \in [N]} f_{\text{nil}}(n) \gg 1$, which implies that $E_{n \in [N]} 1_E(n) f_{\text{nil}}(n) \gg 1$, and hence by Hölder’s inequality that $E_{n \in [N]} 1_E(n) f_{\text{nil}}^k(n) \gg 1$. Putting all this together, we conclude that (6.7) is $\gg 1$ if $\varepsilon$ is small enough, and thus

$$\Lambda_k(f_{\text{nil}} + f_{\text{sml}}, \ldots, f_{\text{nil}} + f_{\text{sml}}) \gg M, \varepsilon 1.$$ 

Inserting this bound into (6.6) we obtain the claim, completing the proof of Szemerédi’s theorem, if $F$ is chosen sufficiently rapid.

**Proof of Proposition 6.2.** Let us first establish this in the easy case $k = 3$. In this case, $f_{\text{nil}}$ is essentially quasiperiodic, which will allow us to take $\mu(n, d)$ to be of the form

$$\mu(n, d) = 1_{[2\varepsilon N, (1-2\varepsilon)N]}(n) \mu(d)$$

with $\mu(d)$ normalised by requiring

$$E_{d \in [-\varepsilon N, \varepsilon N]} \mu(d) = 1 + O(\varepsilon).$$

It is then easy to verify that both (6.3) and (6.5) follow from this normalisation. To establish the remaining claims in Proposition 6.2 we use the degree $\leq 1$ nature of the orbit $n \mapsto g(n)\Gamma$ as in Section 5 to write $f_{\text{nil}}$ as

$$f_{\text{nil}}(n) = F(n\theta)$$

for some $\theta \in (\mathbb{R}/\mathbb{Z})^D$ with $D = O_M(1)$ and some $F : (\mathbb{R}/\mathbb{Z})^D \to \mathbb{C}$ of Lipschitz constant $O_M(1)$. If one then sets $\mu$ to equal

$$\mu(d) := \frac{|[\varepsilon N, \varepsilon N]|}{|B|} 1_B(d)$$

where $B$ is the Bohr set

$$\{d \in [-\varepsilon N, \varepsilon N] : d_{(\mathbb{R}/\mathbb{Z})^D}(d\theta, 0) \leq \delta\}$$

and $\delta > 0$ is sufficiently small depending on $\varepsilon, M$, one easily verifies all the required claims.

We now turn to the case $k > 3$, which is harder because $f_{\text{nil}}$ is no longer quasiperiodic, and so $\mu(n, d)$ will have to depend more heavily on $n$ and not just on $d$. By arguing as in the previous section we can normalise $g(0)$ to equal id. We may also assume $N$ is sufficiently large depending on $\varepsilon, M$, since otherwise we may simply take $\mu(n, d) = 1_{[N]}(n) \delta_0(d)$ where $\delta_0$ is the Kronecker delta function at 0. We may of course also assume that $\varepsilon < 1$.

We take an $O_M(1)$-rational Mal’cev basis $X_1, \ldots, X_{\dim(G)}$ for the Lie algebra $\mathfrak{g} = \log G$ adapted to the filtration $G_\bullet$ as described in [30, Appendix A]. For any radius $r > 0$, we define the “ball” $B_r$ in $G$ to be the set of all group elements of the form

$$\exp(\sum_{j=1}^{\dim(G)} t_j X_j) \quad (6.8)$$
where the $t_j$ are real numbers with $t_j \leq r^{s+1-i}$ whenever $1 \leq i \leq s$ and $j \leq \dim(G) - \dim(G_{(i)})$. Thus, when $r$ is small, $B_r$ is quite “narrow” (of diameter comparable to $r^s$) when projected down to $G/G(2)$, but is relatively large when restricted to the top order component $G_{(s)}$ (of diameter comparable to $r$). This type of eccentricity is necessary in order to make $B_r$ approximately “normal” with respect to conjugations. Indeed, we have

**Lemma 6.3** (Approximate normality). Let $A, \delta > 0$, and let $g \in G$ be such that $d_G(g, \text{id}) \leq A$. Then we have the containments

$$B_{(1-\delta)r} \subseteq gB_rg^{-1} \subseteq B_{(1+\delta)r}.$$  \hspace{1cm} (6.9)

whenever $r > 0$ is sufficiently small depending on $A, \delta, M$.

**Proof.** We prove the second inclusion only, as the first is similar (and can also be deduced from the second). The conjugation action $h \mapsto ghg^{-1}$ on $G$ induces a Lie algebra automorphism $\exp(\text{ad}(\log g)) : g \to g$. If we conjugate the group element (6.8) by $g$, we thus obtain

$$\exp(\sum_{j=1}^{\dim(G)} t_j \exp(\text{ad}(\log g))(X_j)).$$

But if $1 \leq i \leq s$ and $j \leq \dim(G) - \dim(G_{(i)})$, we see from the Baker-Campbell-Hausdorff formula (C.2) that

$$\exp(\text{ad}(\log g))(X_j) = X_j + \sum_{j'=\dim(G)-\dim(G_{(i)})+1}^{\dim(G)} c_{j,j'} X_{j'}$$

for some coefficients $c_{j,j'}$ of size $O_{A,M}(r^{s+1-i})$. Collecting all the coefficients together, we obtain the claim for $r$ small enough. \hfill \Box

Let $0 < \delta < 1/10$ be a small quantity (depending on $\varepsilon, M$), let $R$ be a large quantity depending on the same parameters, and let $r_0 > 0$ be an even smaller quantity than $\delta$ (depending on $\varepsilon, M, \delta, R$) to be chosen later. For each $r$ with $0 < r < r_0$ take a Lipschitz function $\phi_r : G \to \mathbb{R}^+$ of Lipschitz norm $O_{M,r,\delta}(1)$ which is supported on $B_r$ and equals one on $B_{(1-\delta)r}$, and choose these functions so that $\phi_r \leq \phi_{r'}'$ pointwise whenever $0 < r < r' < r_0$. For each such $r$, let $\Phi_r : G/\Gamma \times G/\Gamma \to \mathbb{R}^+$ be the induced function

$$\Phi_r(x, x') := \sum_{g \in G: gx = x'} \phi_r(g).$$

This function $\Phi_r$ is supported near the diagonal of $G/\Gamma \times G/\Gamma$; indeed, $\Phi_r(x, x')$ is only non-zero when $x' \in B_r x$, and furthermore if $x' \in B_{(1-\delta)r} x$

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15Readers may find it helpful to keep the hierarchy of scales

$$1 \sim 1/k, \alpha \gg \varepsilon \gg 1/M \gg \delta \gg 1/R \gg r_0 \gg r \gg 1/F(M) \gg 1/N > 0$$

in mind.
then $\Phi_r(x, x') = 1$. If $r_0$ is chosen sufficiently small depending on $M, \delta$, we conclude from Lemma 6.3 that we have the approximate shift-invariance

$$
\Phi_r(1-3\delta)(x, x') \leq \Phi_r(gx, gx') \leq \Phi_r(1+3\delta)(x, x')
$$

(6.10)

whenever $x, x' \in G/\Gamma$ and $g \in G$ is such that $d_G(g, \text{id}) \leq R$ (say).

We now define our cutoff function $\mu = \mu_r$ by

$$
\mu_r(n, d) := c_r \frac{1}{q} |d| |k\varepsilon N, (1-k\varepsilon)N| (n) \prod_{i=1}^{k-1} \Phi_r(g(n)\Gamma, g(n+id)\Gamma),
$$

(6.11)

where $c_r > 0$ is a normalisation constant to be chosen later. This function, as discussed immediately following the statement of Proposition 6.2, is a smooth cutoff to the set of “almost-diagonal” progressions in $G/\Gamma$. Specifically, $\mu_r$ is supported in (6.2), and also in the region where $g(n+id)\Gamma \in B_r g(n)\Gamma$, $|d| \leq \delta N$, and $q|d$ for $i = 0, \ldots, k - 1$. From the Lipschitz nature of $F$ we thus have

$$
F(g(n+id)\Gamma, (n + id)(\text{mod } q), (n + id)/N) = F(g(n)\Gamma, n(\text{mod } q), n/N) + O_M(r_0)
$$

for $(n, d)$ in the support of $\mu_r$, which gives (6.4) for $\mu_r$ if $r_0$ is sufficiently small depending on $\varepsilon, M$.

Next, we compute the expectation of $\mu_r(n, d)$, in order to work out what the normalisation constant $c_r$ should be. Observe that

$$
\mathbb{E}_{n \in [N], d \in [-\varepsilon N, \varepsilon N]} \mu_r(n, d)
$$

$$
= \frac{\delta}{q\varepsilon} \frac{1}{1+O(\varepsilon)c_r} \times
$$

$$
= \frac{\delta}{q\varepsilon} \frac{1}{1+O(\varepsilon)c_r} \times \Phi_r(g(n)\Gamma, \ldots, g(n + (k - 1)d)\Gamma),
$$

where $\Phi_r : (G/\Gamma)^k \to \mathbb{R}^+$ is the function

$$
\tilde{\Phi}_r(x_0, \ldots, x_{d-1}) := \prod_{i=1}^{k-1} \Phi_r(x_0, x_i).
$$

(6.13)

Observe that $\tilde{\Phi}$ has a Lipschitz norm of $O_M, r, \delta(1)$. Applying Theorem 1.11, we can express (6.12) as

$$
\frac{\delta}{q\varepsilon} (1 + O(\varepsilon))c_r \frac{1}{1+O(\varepsilon)c_r} \int_{G^\Psi/\Gamma^\Psi} \tilde{\Phi}_r + o_{F(M) \to \infty; M, r, \delta}(1) + o_{N \to \infty; M, r, \delta}(1),
$$

where $G^\Psi \subseteq G^k$ is the $k$th Hall-Petresco group, that is to say the Leibman group associated to the collection $\Psi = (\psi_0, \ldots, \psi_{k-1})$ of linear forms $\Psi_i := (n, d) \mapsto n + id$ for $i = 0, \ldots, k - 1$.

The group $G^\Psi$ is a $O_M(1)$-rational subgroup of $G^k$, which itself has complexity $O_M(1)$. Meanwhile, the function $\tilde{\Phi}_r$ equals 1 on a ball of radius
\( r^{O_M(1)} \) centred at the identity, and is bounded by 1 throughout. We conclude that the quantity

\[
v_r := \int_{G^\psi / T^\psi} \tilde{\Phi}_r
gives the bounds
\]

\[
r^{O_M(1)} \ll_M v_r \leq 1.
\]

Furthermore, from the properties of the functions \( \phi_r \), we have the monotonicity property

\[
v_{(1-\delta)r} \leq v_r
\]

for any \( 0 < r < r_0 \). Applying the pigeonhole principle (using the fact that polynomial growth is always slower than exponential growth), and choosing \( \delta \gg \epsilon, M \) sufficiently small depending on \( \epsilon, M \), one can thus find a radius

\[
r_0 > r > r_0, \epsilon, \delta, M
\]

such that we have the regularity property

\[
(1 - O(\epsilon))v_r \leq v_{(1-3\delta)r} \leq v_{(1+3\delta)r} \leq (1 + O(\epsilon))v_r.
\]

(6.14)

Note that this idea of picking a “regular” radius originates, in additive combinatorics, in Bourgain’s paper [8]. Fix from now on a value of \( r \) with this property. If we then set

\[
c_r := \frac{q\epsilon}{\delta v_r}
\]

we conclude that

\[
c_r \ll_M, r_0, \epsilon
\]

and

\[
\mathbb{E}_{n \in [N], d \in [-\epsilon N, \epsilon N]} \mu_r(n, d) = 1 + O(\epsilon) + o_{F(M) \to \infty; M, \epsilon, r_0}(1) + o_{N \to \infty; M, \epsilon, r_0}(1).
\]

This will give (6.3) provided that \( r_0 \) is chosen to depend on \( M, \epsilon, \delta \), that \( F \) is sufficiently rapid depending on \( \epsilon \), and \( N \) is sufficiently large depending on \( M, \epsilon \).

Our remaining task, and the most difficult one, is to study the expression in (6.5). That is to say, we fix \( 0 \leq i \leq k-1 \) and consider

\[
\mathbb{E}_{n \in [N]} \mathbb{E}_{d \in [-\epsilon N, \epsilon N]} \mu_r(n - id, d)^2.
\]

(6.17)

Using (6.11), we can write this expression as

\[
(1 + O(\epsilon))(\frac{\epsilon}{q\theta})^2 \mathbb{E}_{n \in [k \epsilon N, (1-k)\epsilon N]} \mathbb{E}_{d, d' \in [-\delta N, \delta N]} \Phi_{r, \theta}^2(g(n - id)\Gamma, \ldots, g(n + (k-1 - i)d)\Gamma, g(n - id')\Gamma, \ldots, g(n + (k-1 - i)d')\Gamma)
\]

where \( \Phi_{r, \theta}^2 : (G/\Gamma)^k \times (G/\Gamma)^k \to \mathbb{R}^+ \) is the tensor square

\[
\Phi_{r, \theta}^2(x, x') := \Phi_r(x) \Phi_r(x').
\]
Applying Theorem 1.1, we can thus express (6.17) as

\( (1 + O(\varepsilon))(\frac{\varepsilon}{q^d} c_T)^2 \left( \int_{G^{\Psi(i)} / T^{\Psi(i)}} \tilde{\Phi}_r \otimes \tilde{\Phi}_r + o_{F(M) \to \infty; \varepsilon, M, r_0}(1) + o_{N \to \infty; \varepsilon, M, r_0}(1) \right) \)  

(6.18)

where \( G^{\Psi(i)} \subset G^{2k} \) is the Leibman group associated to the collection

\( \Psi(i) := (\psi_{0,i}, \ldots, \psi_{k-1,i}, \psi'_{0,i}, \ldots, \psi'_{k-1,i}) \)

of linear forms

\( \psi_{j,i}: (n, d, d') \mapsto n + (j - i)d \)

and

\( \psi'_{j,i}: (n, d, d') \mapsto n + (j - i)d' \)

for \( j = 0, \ldots, k - 1 \).

We will be establishing the following claim.

Claim 6.4 (Approximate factorisation). We have

\( \int_{G^{\Psi(i)} / T^{\Psi(i)}} \tilde{\Phi}_r \otimes \tilde{\Phi}_r = (1 + O(\varepsilon)) v_r^2. \)  

(6.19)

Proof of Proposition 6.2 assuming Claim 6.4. Substitute back into (6.18) and use (6.15), (6.16) to conclude that

\( (6.17) = 1 + O(\varepsilon) + o_{F(M) \to \infty; \varepsilon, M, r_0}(1) + o_{N \to \infty; \varepsilon, M, r_0}(1). \)

This gives the result upon choosing \( r_0 \) sufficiently small depending on \( \varepsilon, M, \delta \), \( F \) sufficiently rapid depending on \( \varepsilon \), and \( N \) sufficiently large depending on \( \varepsilon, M \).

It remains to establish Claim 6.4. For notational simplicity we establish only the claim \( i = 0 \) (the others being very similar). The intuition behind this claim (and behind the key assertion that the number of almost-diagonal progressions whose \( i \)th term is \( n \) does not depend on \( n \)) is that the linear forms \( (\psi_{0,0}, \ldots, \psi_{k-1,0}) \) and \( (\psi'_{0,0}, \ldots, \psi'_{k-1,0}) \) are almost independent of each other, except for the fact that they are coupled via the obvious identity \( \psi_{0,0} = \psi'_{0,0} \).

One way to encode this formally is to note that the Leibman group \( G^{\Psi(0)} \) is given by

\( H := \{(x, x') \in G^{\Psi} \times G^{\Psi} : x_0 = x'_0\} \),

a product of two copies of the Hall-Petresco group \( G^{\Psi} = HP^k(G) \) fibred over the zeroth coordinate. To prove this, one may note that the containment \( G^{\Psi(0)} \subseteq H \) is obvious. On the other hand, one may compute directly using the dimension formula (3.1) that

\( \dim(G^{\Psi}) = \dim(G) + \sum_{i=1}^{k-2} \dim(G^{(i)}) \)
and
\[ \dim(G^{\Psi(0)}) = \dim(G) + 2 \sum_{i=1}^{k-2} \dim(G^{(i)}) \]
and thus
\[ \dim(G^{\Psi(0)}) = 2 \dim(G^{\Psi}) - \dim(G) = \dim(H), \]
and so since both sides are connected, simply-connected nilpotent Lie groups (and so both are homeomorphic to their Lie algebras) we have \( G^{\Psi(0)} = H \).

Write \( J_r \) for the integral appearing in (6.19), that is to say
\[ J_r := \int_{(x,x') \in G^{\Psi}/\Gamma^{\Psi} \times G^{\Psi}/\Gamma^{\Psi}} \tilde{\Phi}_r^{\otimes 2}(x,x'). \]

Let \( R \) be some quantity, and suppose that \( \text{dist}_G(g,\text{id}) \leq R \). Then by the almost-invariance property (6.10) we have
\[ \int_{(x,x') \in (G^{\Psi}/\Gamma^{\Psi})^2} \lambda(x,x') \tilde{\Phi}_r^{\otimes 2}(x,x') \geq \text{vol}(B_R) J_r, \]
where \( \lambda(x,x') \) is the number of \( g \in B_R \) for which \( x_0 = gx'_0 \pmod{\Gamma} \), or equivalently
\[ \lambda(x,x') := |\Gamma \cap x_0^{-1} B_R x'_0|. \]
Choose representatives \( x_0, x'_0 \) in some fundamental domain with \( x_0, x'_0 = O_M(1) \). By a volume-packing argument and simple geometry we then have
\[ \lambda(x,x') = \text{vol}(B_R)(1 + o_{R \to \infty; M}(1)). \]
Comparing with the above we have
\[ v_r^{2(1-3\delta)} = \int_{(x,x') \in (G^{\Psi}/\Gamma^{\Psi})^2} \tilde{\Phi}_r^{\otimes 2}(x,x') \geq J_r(1 + o_{R \to \infty; M}(1)), \]
and so by (6.19) we have
\[ J_r \leq (1 + O(\varepsilon) + o_{R \to \infty; M}(1)) v_r^2. \]

This gives the upper bound for Claim 6.4. The lower bound is proven similarly. This concludes the proof of Proposition 6.2 and thus Theorem 6.1.
7. On a theorem of Gowers and Wolf

Our aim in this section is to prove Theorem 1.13 whose statement we recall now.

**Theorem 7.1** (Theorem 1.13). Let \( \Psi = (\psi_1, \ldots, \psi_t) \) be a collection of linear forms \( \psi_1, \ldots, \psi_t : \mathbb{Z}^D \rightarrow \mathbb{Z} \), and let \( s \geq 1 \) be an integer such that the polynomials \( \psi_{s+1}, \ldots, \psi_{t+1} \) are linearly independent. Then for any function \( f : [N] \rightarrow \mathbb{C} \) bounded in magnitude by 1 (and defined to be zero outside of \([N]\)) obeying the bound \( \|f\|_{U^{s+1}[N]} \leq \delta \) for some \( \delta > 0 \), one has

\[
\mathbb{E}_{n \in [N]^D} \prod_{i=1}^t f(\psi_i(n)) = o_{\delta \rightarrow 0,s,D,t,\Psi}(1).
\]

Henceforth we allow all implied constants to depend on \( d, t, s, \Psi \) without indicating this explicitly. Let \( s' = s'(\Psi) \) be the Cauchy-Schwarz complexity of the linear forms \( \Psi \), as defined in Theorem 4.1. We may of course assume that \( s' > s \), as Theorem 1.13 is immediate otherwise. We may also assume that \( N \) is large depending on \( \delta \), since otherwise the claim is trivial from a compactness argument.

Let \( \varepsilon > 0 \) be a small number depending on \( \delta \) to be chosen later, and let \( \mathcal{F} \) be a growth function depending on \( \varepsilon \) to be chosen later. Applying Theorem 1.2 at degree \( s' \) (after first decomposing \( f \) as a linear combination of \( O(1) \) functions taking values in \([0,1]\) ), we can find a positive quantity \( M = O_{\varepsilon, \mathcal{F}}(1) \) and a decomposition

\[
f = f_{\text{nil}} + f_{\text{sml}} + f_{\text{unf}} \tag{7.1}
\]

where:

(i) \( f_{\text{nil}} \) is a \( (\mathcal{F}(M), N) \)-irrational virtual nilsequence of degree \( \leq s' \), complexity \( \leq M \), and scale \( N \);

(ii) \( f_{\text{sml}} \) has \( L^2[N] \) norm at most \( \varepsilon \);

(iii) \( f_{\text{unf}} \) has \( U^{s'+1}[N] \) at most \( 1/\mathcal{F}(M) \);

(iv) All functions \( f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}} \) are bounded in magnitude by \( O(1) \).

We apply this decomposition to split the expression

\[
\mathbb{E}_{n \in [N]^D} \prod_{i=1}^t f(\psi_i(n)) \tag{7.2}
\]

as the sum of \( 3^t \) terms, in which each copy of \( f \) has been replaced with either \( f_{\text{nil}}, f_{\text{sml}}, \) or \( f_{\text{unf}} \).

Any term involving at least one factor of \( f_{\text{sml}} \) can be easily seen to be of size \( O(\varepsilon) \) by crudely estimating all other factors by 1. By (4.1), any term involving at least one factor of \( f_{\text{unf}} \) is of size \( O(1/\mathcal{F}(M)) \), which is also of size \( O(\varepsilon) \) if \( \mathcal{F} \) is chosen to be sufficiently rapidly growing depending on \( \varepsilon \).
We can therefore express (7.2) as

$$E_{n \in [N]^D} \prod_{i=1}^t f_{\text{nil}}(\psi_i(n)) + O(\varepsilon).$$

By hypothesis, we can write

$$f_{\text{nil}}(n) = F(g(n)\Gamma, n(\text{mod } q), n/N)$$

for some $q$ with $1 \leq q \leq M$, some degree $\leq s$, $(\mathcal{F}(M), N)$-irrational, orbit $n \mapsto g(n)\Gamma$ of complexity $\leq M$ and some Lipschitz function $F : G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R}$ of norm at most $M$. The mod $q$ and Archimedean behaviour in $f_{\text{nil}}$ are nothing more than technical annoyances, and we set about eliminating them now. We encourage the reader to work through the heart of the argument, starting at (7.3) below, in the model case $f_{\text{nil}} = F(g(n)\Gamma)$. Let $\varepsilon'$ be a small quantity depending on $\varepsilon, M$ to be chosen later\(^1\). We partition $[N]$ into progressions $P$ of spacing $q$ and length $\varepsilon' N$, plus a remainder set of size at most $O_M(1)$. We can then rewrite the above expression as

$$E_{P_1, \ldots, P_D} \prod_{n \in P_1 \times \cdots \times P_D} \prod_{i=1}^t f_{\text{nil}}(\psi_i(n)) + O(\varepsilon).$$

We abbreviate $P_1 \times \ldots \times P_D$ as $P$. For a given $P$, observe that as $n$ ranges in $P$, the residue class of $\psi_i(n)$ modulo $q$ is equal to a fixed class $a_{P,i}$, and the value of $\psi_i(n)/N$ differs by at most $O_M(\varepsilon')$ from a fixed number $x_{P,i}$. We may assume that $x_{P,i} \in [0, 1]$ for each $i$, otherwise the inner expectation is zero (except for a few “boundary” values of $P$ which give a net contribution of $O_M(\varepsilon')$).

If $\varepsilon'$ is small enough depending on $\varepsilon, M$, the $O_M(\varepsilon')$ error in the above discussion can be absorbed in the $O(\varepsilon)$ error, and so we have

$$E_{n \in [N]^D} \prod_{i=1}^t f(\psi_i(n)) = E_{P} E_{n \in P} \prod_{i=1}^t F(g(\psi_i(n))\Gamma, a_{P,i}, x_{P,i}) + O(\varepsilon).$$

We now apply Theorem 1.11, which tells us the right-hand side here is

$$E_{P} \int_{G^\Psi / \Gamma^\Psi} \tilde{F}_P + O(\varepsilon) + o_{\mathcal{F}(M) \to \infty; \varepsilon, \varepsilon'}(1) + o_{N \to \infty; \varepsilon, \varepsilon'}(1), \quad (7.3)$$

where as usual $G^\Psi \leq G'$ is the Leibman group associated to the system of forms $\Psi = \{\psi_1, \ldots, \psi_t\}$, and here $\tilde{F}_P : G^\Psi / \Gamma^\Psi \to \mathbb{C}$ is the function

$$\tilde{F}_P((g_1, \ldots, g_t)\Gamma^\Psi) := \prod_{i=1}^t F(g_i\Gamma, a_{P,i}, x_{P,i}).$$

Readers may find it helpful to keep the hierarchy of scales

$$1 \gg \varepsilon \gg 1/M, 1/q \gg \varepsilon' \gg 1/\mathcal{F}(M) \gg \delta \gg 1/N > 0$$

in mind.
The heart of the matter is to obtain an upper bound on the quantity $\mathbb{E}_{\mathbf{P}} \int_{G^\psi \cap \Gamma^\psi} \tilde{F}_{\mathbf{P}}$ appearing in (7.3). To do this, of course, we need to make use of the assumption on the forms $\psi_1, \ldots, \psi_t$, as well as the fact that $\|f\|_{U^{s+1}} \leq \delta$.

The aforementioned assumption, namely that $\psi_{s+1}^1, \ldots, \psi_{s+1}^t$ are linearly independent, implies that $\Psi[s+1]$ is the whole of $\mathbb{R}^t$ which, in view of the definition of the Leibman group $G^\Psi$, implies that $G_{s+1}^t \leq G^\Psi$. By Fubini’s theorem, we thus have

$$\int_{G^\psi \cap \Gamma^\psi} \tilde{F}_{\mathbf{P}} = \int_{G^\psi \cap \Gamma^\psi} \tilde{F}_{\mathbf{P}, s}$$

where

$$\tilde{F}_{\mathbf{P}, s}(g_1, \ldots, g_t) := \prod_{i=1}^t F_{s}(g_i \Gamma, a_{\mathbf{p}, i}, x_{\mathbf{p}, i})$$

(7.4)

and $F_{s}$ is defined by averaging over cosets of the normal subgroup $G_{s+1}$, specifically

$$F_{s}(g \Gamma, a, x) := \int_{G_{s+1} \cap \Gamma_{s+1}} F(g g_{s+1} \Gamma, a, x) \, dg_{s+1}.$$ 

Since $F$ was Lipschitz with norm $O_M(1)$, we see that $F_{s}$ is Lipschitz with norm $O_M(1)$ also. Also, since $F$ is bounded in magnitude by $O(1)$, so is $F_{s}$.

As the forms $\psi_{s+1}^1, \ldots, \psi_{s+1}^t$ are independent, we see in particular that $\psi_1$ is non-zero. This implies that the projection of $G^\Psi$ to the first coordinate $G$ is surjective. Meanwhile, from (7.4) and the boundedness of $F_{s}$ we have the crude upper bound

$$|\tilde{F}_{\mathbf{P}, s}(g_1, \ldots, g_t)| \ll |F_{s}(g_1 \Gamma, a_{\mathbf{p}, 1}, x_{\mathbf{p}, 1})|.$$ 

From Fubini’s theorem, we obtain the bound

$$\left| \int_{G^\psi \cap \Gamma^\psi} \tilde{F}_{\mathbf{P}} \right| \ll \int_{G^\Gamma} |F_{s}(\cdot, a_{\mathbf{p}, 1}, x_{\mathbf{p}, 1})|.$$ 

(7.5)

To proceed further, we need a crucial smallness estimate on $F_{s}$:

**Proposition 7.2** ($F_{s}$ small in $L^2$). For any $a \in \mathbb{Z}/q\mathbb{Z}$ and $x \in [0, 1]$, one has

$$\int_{G^\Gamma} |F_{s}(\cdot, a, x)|^2 \ll O(\varepsilon) + O_M(\varepsilon') +$$

$$o_{\delta \to \infty; M, \varepsilon, \varepsilon'(1)} + o_{F(M) \to \infty; M, \varepsilon, \varepsilon'(1)} + o_{N \to \infty; M, \varepsilon, \varepsilon'(1)}.$$ 

**Proof.** By reflection symmetry we may assume that $x \leq 1/2$. We may also round $x$ so that $x = qn_0/N$ for some $n_0 \in [N/2q]$, as the error in doing so can be easily absorbed by the Lipschitz properties of $F_{s}$.
By construction, $F_{\leq s}$ is invariant on $G_{(s+1)}$-cosets, while $F - F_{\leq s}$ integrates to zero on any such coset. In particular, $F_{\leq s}(\cdot, a, x)$ and $F - F_{\leq s}(\cdot, a, x)$ are orthogonal, and thus

$$\int_{G/\Gamma} |F_{\leq s}(\cdot, a, x)|^2 = \int_{G/\Gamma} F \overline{F_{\leq s}}(\cdot, a, x).$$

Applying Theorem 1.11 (really just the special case of this result asserting that $(g(n)\Gamma)$ is equidistributed, cf. Lemma 3.7) and the Lipschitz nature of $\overline{F_{\leq s}}$, the right-hand side can be written as

$$\mathbb{E}_{n \in [\varepsilon N]} F \overline{F_{\leq s}}(g(qn + qn_0 + a)\Gamma, a, x) + o_{\mathcal{F}(M) \to \infty; M, \varepsilon, \varepsilon'}(1) + o_{N \to \infty; M, \varepsilon, \varepsilon'}(1).$$

Let $P$ be the progression $\{qn + qn_0 + a : n \in [\varepsilon'N]\}$. Then by a further use of the Lipschitz properties of $F$, we can rewrite the above expression as

$$\mathbb{E}_{n \in P} F(g(n)\Gamma, n \mod q, n/N) \psi(n) + O_M(\varepsilon')$$

$$+ o_{\mathcal{F}(M) \to \infty; M, \varepsilon, \varepsilon'}(1) + o_{N \to \infty; M, \varepsilon, \varepsilon'}(1) \quad (7.6)$$

where

$$\psi(n) := \overline{F_{\leq s}}(g(n)\Gamma, a, x).$$

Note that, as a consequence of the $G_{(s+1)}$-invariance of $F_{\leq s}$, $\psi(n)$ is a degree $\leq s$ nilsequence of complexity $O_M(1)$. Now by (7.1) we have

$$F(g(n)\Gamma, n \mod q, n/N) = f(n) - f_{\text{unif}}(n) - f_{\text{smil}}(n).$$

The contribution of $f_{\text{smil}}(n)$ to (7.6) is $O(\varepsilon)$ by the Cauchy-Schwarz inequality. Now consider the contribution of $f$. Observe that because $F_{\leq s}$ is $G_{(s+1)}$-invariant, $\psi$ is a degree $\leq s$ nilsequence of complexity $O_M(1)$. Meanwhile, $\|f\|_{U^{s+1}[N]} \leq \delta$ by hypothesis. Applying the converse to the inverse conjecture for the Gowers norms (first established in [28], though for a simple proof see [33, Appendix G]), we see that

$$\mathbb{E}_{n \in P} f(n) \psi(n) = o_{\delta \to 0; M, \varepsilon, \varepsilon'}(1).$$

Similarly, since $\|f_{\text{unif}}\|_{U^{s'+1}[N]} \leq 1/\mathcal{F}(M)$ and $s' \geq s$, we have

$$\mathbb{E}_{n \in P} f(n) \psi(n) = o_{\mathcal{F}(M) \to 0; M, \varepsilon, \varepsilon'}(1).$$

Putting all of these estimates together, we obtain the claim. \qed

Applying this bound and (7.5), we can thus bound (7.3) in magnitude by

$$O(\varepsilon) + O_{\mathcal{F}(M)}(\varepsilon') + o_{\delta \to \infty; M, \varepsilon, \varepsilon'}(1) + o_{\mathcal{F}(M) \to \infty; M, \varepsilon, \varepsilon'}(1) + o_{N \to \infty; M, \varepsilon, \varepsilon'}(1).$$

Choosing $\varepsilon'$ sufficiently small depending on $M$ and $\varepsilon$, and choosing $\mathcal{F}$ sufficiently rapidly growing depending on $\varepsilon$, and then using the bound $M = O_{\varepsilon, \mathcal{F}}(1)$ (and recalling that $N$ can be chosen large depending on $\delta$), we conclude that

$$|\mathbb{E}_{n \in [N]^D} \prod_{i=1}^t f(\psi_i(n))| \ll \varepsilon$$

whenever $\delta$ is sufficiently small depending on $\varepsilon$. Theorem 1.13 follows.
Remark. It seems certain that one can extend this result to the case when one has \( t \) distinct functions \( f_1, \ldots, f_t : [N] \to \mathbb{C} \) rather than a single function \( f : [N] \to \mathbb{C} \). The main change in the argument would be to use a version of the regularity lemma (Theorem 1.2) valid for several functions simultaneously, in which one regularises the \( f_1, \ldots, f_t \) using the same data \( M, q, (G/\Gamma, G_\bullet), g() \) (but allows each function \( f_i \) to be given a separate Lipschitz function \( F_i : G/\Gamma \times \mathbb{Z}/q \mathbb{Z} \times \mathbb{R} \to \mathbb{C} \)). Such a result could be obtained by straightforward modifications to the proof of Theorem 1.2, but we do not pursue this matter here.

Appendix A. Properties of polynomial sequences

In this appendix we collect a variety of facts and definitions concerning polynomial sequences in nilpotent groups, all of which were required at some point in the paper proper. We take for granted the definition of filtration \( G_\bullet \) and of the group \( \text{poly}(\mathbb{Z}^d, G_\bullet) \) of polynomial sequences \( g : \mathbb{Z}^d \to G \) adapted to \( G_\bullet \); these notions were recalled in the introduction.

Taylor expansions. Polynomial sequences may be described in terms of so-called Taylor expansions. In the lemma that follows we make use of the generalised binomial coefficients \( \binom{n}{i} \) are the generalised binomial coefficients

\[
\binom{n_1, \ldots, n_D}{i_1, \ldots, i_D} := \binom{n_1}{i_1} \cdots \binom{n_D}{i_D}
\]

where

\[
\binom{n}{i} := \frac{n(n - 1) \ldots (n - i + 1)}{i!}.
\]

If \( i = (i_1, \ldots, i_d) \in \mathbb{N}^D \) is a \( D \)-tuple of non-negative integers we define the degree \( |i| := i_1 + \ldots + i_D \). Choose an arbitrary ordering on \( \mathbb{N}^D \) with the property that \( |i| \geq |j| \) whenever \( i \geq j \).

Lemma A.1 (Taylor expansions). Suppose that \( g \in \text{poly}(\mathbb{Z}^D, G_\bullet) \). Then there are unique Taylor coefficients \( g_i \in G_{|i|} \) with the property that

\[
g(n) = \prod_{i \in \mathbb{N}^d} g_i^{\binom{n}{i}}
\]

for all \( n \in \mathbb{Z}^D \). Conversely, every Taylor expansion of this type gives rise to a polynomial sequence \( g \in \text{poly}(\mathbb{Z}^D, G_\bullet) \).

Remarks. This is proven in [30, Lemma 6.7]. Note that, since \( G \) is nilpotent, this is a finite expansion. In the case \( D = 1 \) (which will feature most prominently in the paper) the it takes the form

\[
g(n) = g_0 g_1^{(n_1)} \cdots g_s^{(n_s)}.
\]

Note how, from the presentation of polynomial sequences as Taylor expansions, it is by no means clear (and somewhat remarkable) that they form a group under pointwise multiplication (Theorem 1.6).
Polynomial sequences that vary slowly, in a certain sense, are called *smooth*. We employ the following definition, which is the same as the one given in the introduction to [30].

**Definition A.2** (Smooth sequences). Let \( A \) be a positive parameter and let \( N \geq 1 \) be an integer. Let \( \beta \in \text{poly}(\mathbb{Z}, G) \). We say that \( \beta \) is \((A, N)\)-smooth if we have \( d_G(\beta(n), \text{id}) \leq A \) and \( d_G(\beta(n), \beta(n + 1)) \leq A/N \) for all \( n \in [N] \).

Here \( d_G \) is a metric on the group \( G \) constructed using the Mal’cev basis, see [30, Definition 2.2]. The precise definition of this metric is not terribly important for our analysis.

In counterpoint to the notion of a smooth sequence is that of a *rational* sequence.

**Definition A.3** (Rational sequences). Let \( A \geq 1 \) be an integer, and let \((G/\Gamma, G)\) be a filtered nilmanifold. Then an element \( g \in G \) is \( A \)-rational if there is some \( q \), \( 1 \leq q \leq A \), such that \( g^q \in \Gamma \). If \( \gamma \in \text{poly}(\mathbb{Z}, G) \) is a polynomial sequence then we say that it is \( A \)-rational if \( \gamma(n) \) is \( A \)-rational for every integer \( n \).

We have the following basic facts about smooth and rational sequences:

**Lemma A.4** (Basic facts). Let \((G/\Gamma, G)\) be a filtered nilmanifold of complexity \( \leq M_0 \). By a “sequence”, we mean an element of \( \text{poly}(\mathbb{Z}, G) \). Then:

(i) The product of two \((A, N)\)-smooth sequences is \( O_{M_0, A}(1) \)-smooth;

(ii) The product of two \( A \)-rational sequences is \( O_{M_0, A}(1) \)-rational;

(iii) Any \( A \)-rational sequence is periodic with period \( O_{M_0, A}(1) \).

**Proof.** For (i), see [30, Lemma 10.1]; for (ii), see [30, Lemma A.11 (v)]; and for (iii), see [30, Lemma A.12 (ii)]. In fact these results hold in the multiparameter setting, with polynomially effective bounds, but we will not need these facts here. \( \square \)

We turn now to an important new definition for this paper, that of an *irrational* polynomial sequence. In [30], much emphasis was placed on the notion of an *equidistributed* polynomial sequence \( g : \mathbb{Z} \to G \) one for which the orbit \((g(n)\Gamma)_{n \in [N]}\) is close to equidistributed on \( G/\Gamma \). The notion of an irrational sequence implies equidistribution (see Lemma 3.7, which is also a special case of Theorem 1.11), but also encodes an assertion that the filtration \( G \) is in some sense “minimal” for the sequence. To illustrate the difference, let us think about a simple abelian case in which \( G/\Gamma \) is just the unit circle \( \mathbb{R}/\mathbb{Z} \) (written additively), and \( g : \mathbb{Z} \to \mathbb{R} \) is a polynomial

\[
g(n) = \alpha_0 + \alpha_1 \binom{n}{1} + \ldots + \alpha_s \binom{n}{s}.
\] (A.1)

\(\text{One could take an “adelic” perspective here and view smooth sequences as those that are local to the Archimedean place } \infty, \text{ while rational sequences are those that are local to finite places } p.\)
This sequence is adapted to the filtration in which $G_{(i)} = \mathbb{R}$ for $i \leq s$ and $G_{(i)} = \{0\}$ for $i > s$. Qualitatively speaking, $g$ is equidistributed if at least one of $\alpha_1, \ldots, \alpha_s$ is irrational; in contrast, $g$ is irrational with respect to this filtration if it is $\alpha_s$ which is irrational. Note that if $s > 1$ and $\alpha_s$ is rational, then (after removing the periodic component $\alpha_s n^s$ from $g$) $g$ is now adapted to the filtration $G'_s$ in which $G'_{(i)} = \mathbb{R}$ for $i \leq s - 1$ and $G'_{(i)} = \{0\}$ for $i > s - 1$, which has a strictly smaller total dimension. This basic example is the model for the more sophisticated result in Lemma 2.9.

Let us turn now to the precise definition in the more general setting of Lie group-valued polynomial sequences, in which the role of the $\alpha_i$ is played by the Taylor coefficients of $g$. We need a preliminary definition.

**Definition A.5** $(i$-horizontal characters). Let $(G/\Gamma, G_\bullet)$ be a filtered nilmanifold of degree $\leq s$ with filtration $G_\bullet = (G_{(i)})^\infty_{i=0}$. Then by an $i$-horizontal character we mean a continuous homomorphism from $\xi_i : G_{(i)} \to \mathbb{R}$ which vanishes on $G_{(i+1)}$ and all $[G_{(j)}, G_{(i-j)}]$ for any $0 \leq j \leq i$. We say that such a character is non-trivial if it is not constant. We can assign a notion of complexity by taking a Mal’cev basis adapted to $G_\bullet$, whereupon one has a natural isomorphism $G_{(i)}/G_{(i+1)} \cong \mathbb{R}^k$. Writing $\psi(g_i)$ for the coordinates of $g_i(\mod G_{(i+1)})$, any $i$-horizontal character has the form $\xi_i(g_i) = \vec{m} \cdot \psi(g_i)$, for some vector $\vec{m} = (m_1, \ldots, m_k)$ of integers. We may then define the complexity of $\xi_i$ to be $|m_1| + \cdots + |m_k|$.

The list of subgroups on which $\xi_i$ is required to vanish looks rather restrictive and slightly unnatural at first sight. Roughly speaking, this list is intended to isolate that behaviour which genuinely “belongs” to the degree $i$ portion of the filtered nilmanifold, as opposed to arising from those terms of higher or lower degree, or which disappear after quotienting out by the lattice $\Gamma$.

**Definition A.6** (Irrationality). Let $(G/\Gamma, G_\bullet)$ be a filtered nilmanifold of degree $\leq s$ with filtration $G_\bullet = (G_{(i)})^\infty_{i=0}$. Let $g_i \in G_{(i)}$. Let $A, N > 0$. Then we say that $g_i$ is $(A, N)$-irrational in $G_{(i)}$ if for every non-trivial $i$-horizontal character $\xi_i : G_{(i)} \to \mathbb{R}$ of complexity $\leq A$ one has $\|\xi_i(g_i)\|_{\mathbb{R}/\mathbb{Z}} \geq A/N^i$. We say that the sequence $g(n)$ is $(A, N)$-irrational if its $i$th Taylor coefficient $g_i$ is $(A, N)$-irrational in $G_{(i)}$ for each $i$, $1 \leq i \leq s$.

To understand this definition, it is helpful to consider examples. We leave it as an exercise to check that in the abelian case $A_1$, this amounts to stipulating that the top coefficient of $g$ is poorly approximated by rationals, thus $\|q \alpha_i\|_{\mathbb{R}/\mathbb{Z}} \geq A'/N^s$ whenever $1 \leq q \leq A'$.

A second interesting case to examine is that in which $g(n) = g^n$ is a linear polynomial sequence adapted to the lower central series filtration $(G_i)_{i=0}^\infty$. For the lower central series filtration there are nontrivial $i$-horizontal characters when $i \geq 2$, and $1$-horizontal characters are the same thing as horizontal characters in the sense of [30] Definition 1.5. It follows from this and [30] Theorem 1.16 that $g(n)$ is irrational if and only if $(g(n)\Gamma)_{n\in[N]}$ is
equidistributed. Now polynomial sequences that are not linear do not arise naturally in ergodic-theoretic settings such as those considered in [30][41], and thus the equivalence of the notions of “irrational” and “equidistributed” in this setting explains why the former concept has not appeared in the literature before. The need for it is a new feature of the quantitative world, as is the need for polynomial nilsequences themselves, for reasons explained on [30] §1.  

The following third example is also edifying. Take \( g(n) \) to be any polynomial sequence on the Heisenberg group, for example \( g(n) = \begin{pmatrix} 1 + \gamma_n & \alpha_n & \beta_n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). This sequence is a polynomial sequence adapted to the lower central series filtration \( G_0 = G_1 = G, G_2 = [G, G], G_3 = \{\text{id}\} \), and it will be equidistributed in that setting for generic \( \alpha, \beta, \gamma \). However \( g \) is also a polynomial sequence with respect to some much flabbier filtrations, for example the one in which \( G(i) = G(1) = G(2) = \cdots = G(10) = G, G(11) = \cdots = G(100) = [G, G] \) and \( G(i) = \{\text{id}\} \) for \( i \geq 101 \). It is easy to check that \( g \) is not irrational in this setting, and indeed irrationality is somehow detecting the fact that a given filtration \( G_* \) is minimal for \( g \). This point is quite clear in the proof of Lemma A.7 (which itself depends on Lemma A.7 below), where the failure of a sequence to be irrational is used to create a coarser filtration for a polynomial sequence related to \( g \).

**Lemma A.7.** Suppose that \( (G/\Gamma, G_*) \) is a filtered nilmanifold of degree \( \leq s \) with filtration \( G_* = (G(i))_{i=0}^{\infty} \). Suppose that \( g \) is not \((A, N)\)-irrational. Then there is an index \( i, 1 \leq i \leq s \), such that the \( i \)-th Taylor coefficient \( g_i \) factors as \( \beta_i g'_i \gamma_i \), where \( \beta_i, g'_i, \gamma_i \in G(i) \), \( g'_i \) lies in the kernel of some \( i \)-horizontal character \( \xi : G(i) \to \mathbb{R} \) of complexity at most \( A \), \( d_G(\beta_i, \text{id}) = O_{A, M}(N^{-1}) \) and \( \gamma_i \) is \( O_{A, M}(1) \)-rational.

**Proof.** The proof is (unsurprisingly) extremely similar to that of [30] Lemma 7.9. Reversing the definition of irrational polynomial sequence, we see that there is an index \( i \) together with an \( i \)-horizontal character \( \xi_i : G(i) \to \mathbb{R} \) such that \( \|\xi_i(g_i)\|_{\mathbb{R}/\mathbb{Z}} \leq A/N^i \). It is convenient at this point to work in a Mal’cev coordinate system adapted to \( G_* \), whereby \( G(i)/G(i+1) \) may be identified with \( \mathbb{R}^k \) and \( \Gamma(i)/G(i+1) \) with \( \mathbb{Z}^k \). If \( g_i \in G(i) \) then, as above, we write \( \psi(g) \in \mathbb{R}^k \) for the corresponding coordinates. Then \( \xi_i \) has the form \( \xi_i(g_i) = \vec{m}_i \psi(g) \) for some vector \( \vec{m} = (m_1, \ldots, m_k) \) of integers with \( |m_1| + \cdots + |m_k| \leq A \). Now by assumption we have \( \|\vec{m}_i \psi(g_i)\|_{\mathbb{R}/\mathbb{Z}} \leq A/N^i \), and therefore \( \vec{m}_i \psi(g_i) = r + O(A/N^i) \) for some integer \( r \). It follows from simple linear algebra that we may write \( \psi(g_i) = \vec{t} + \vec{u} + \vec{v} \), where \( \vec{m}_i \vec{u} = 0 \), the coordinates of \( \vec{v} \) lie in \( \mathbb{Z}^k \) for some \( Q = O_A(1) \) and each coordinate of \( \vec{t} \) is \( O_A(1/N^i) \). Now choose \( \beta_i \in G(i) \) in such a way that \( \psi(\beta_i) = \vec{t} \) and \( d_G(\beta_i, \text{id}) = O_{A, M}(1/N_i) \), choose an \( O_{A, M}(1) \)-rational element \( \gamma_i \in G(i) \) with \( \psi(\gamma_i) = \vec{v} \), and finally choose \( g'_i \) so that \( g_i = \beta_i g'_i \gamma_i \). Then one automatically
has \( \psi(g'_i) = \bar{u} \), which means that \( g'_i \) lies in the kernel of the \( i \)-homomorphism \( \xi_i \). 

Finally, we record a convenient scaling lemma.

**Lemma A.8** (Scaling lemma). Let \((G/\Gamma, G_\bullet)\) be a filtered nilmanifold of complexity \( \leq M \). If \( g \in \text{poly}(\mathbb{Z}, G_\bullet) \) is \((A, N)\)-irrational, \( r \in [-N, N] \), and \( 1 \leq q \leq M \), then the sequence \( n \mapsto g(nq + r) \) is \((\gg_{M, \varepsilon} A, \varepsilon N)\)-irrational for any \( \varepsilon > 0 \).

**Proof.** We need to show that the \( i \)-th Taylor coefficient of \( n \mapsto g(nq + r) \) is \((\gg_{M, \varepsilon} A, \varepsilon N)\)-irrational for each \( i \geq 0 \). Note that we may assume \( i \leq M \) since the filtered manifold has degree \( \leq M \).

Fix \( i \). We may quotient out the nilmanifold by the normal subgroups \( G_{(i+1)} \) and \([G_{(j)}, G_{(i-j)}]\) for \( 0 \leq j \leq i \), since these do not affect the irrationality of the \( i \)-th coefficient. We may then expand \( g \) as a Taylor series

\[
g(n) = \prod_{j=0}^{i} g_j^{(n)}_j,
\]

and thus

\[
g(nq + r) = \prod_{j=0}^{i} g_j^{(qn+r)}_j.
\]

Expanding out the binomial coefficient and using many applications of the Baker-Campbell-Hausdorff formula, we obtain

\[
g(nq + r) = \left( \prod_{j=0}^{i-1} (g_j^{(q)}) g_i^{(n)} \right)
\]

for some \( g'_j \in G_{(j)} \); the point being that the Baker-Campbell-Hausdorff term cannot generate any terms involving polynomials in \( n \) of degree \( i \) or higher due to the fact that the groups \( G_{(i+1)} \) and \([G_{(j)}, G_{(i-j)}]\) have been quotiented out. As a consequence, we see that the \( i \)-th Taylor coefficient of \( n \mapsto g(nq + r) \) is \( q^i g_i \), and the claim is easily verified. \( \square \)

**APPENDIX B. A MULTIPARAMETER EQUIDISTRIBUTION RESULT**

The purpose of this appendix is to prove Theorem 3.6 which we recall here again.

**Theorem 3.6.** Suppose that \((G/\Gamma, G_\bullet)\) is a filtered nilmanifold of complexity \( \leq M \) and that \( g \in \text{poly}(\mathbb{Z}^D, G_\bullet) \) is a polynomial sequence for some \( D \leq M \). Suppose that \( \Lambda \subseteq \mathbb{Z}^D \) is a lattice of index \( \leq M \), that \( n_0 \in \mathbb{Z}^D \) has magnitude \( \leq M \), and that \( P \subseteq [-N, N]^D \) is a convex body. Suppose that \( \delta > 0 \), and that

\[
\left| \sum_{n \in (n_0 + \Lambda) \cap P} F(g(n)) - \frac{\text{vol}(P)}{[\mathbb{Z}^D : \Lambda]} \int_{G/\Gamma} F \right| > \delta N^D \| F \|_{\text{Lip}}
\]
for some Lipschitz function $F : G/\Gamma \to \mathbb{C}$. Then there is a nontrivial homomorphism $\eta : G \to \mathbb{R}$ which vanishes on $\Gamma$, has complexity $O_M(1)$ and such that

$$\|\eta \circ g\|_{C^\infty([N]^D)} = O_{\delta,M}(1).$$

Recall from [31, Definition 8.2] that the norm $\|g\|_{C^\infty([N]^D)}$ of a polynomial sequence $g : [N]^D \to \mathbb{R}$ is given by the formula

$$\|g\|_{C^\infty([N]^D)} = \sup_{i \in [N]^D} N^{-|i|} \|g_i\|_{\mathbb{R}} / \mathbb{Z}$$

where $g_i$ are the Taylor coefficients of $g$, thus

$$g(n) = \sum_{i \in [N]^D} \binom{n}{i} g_i.$$

We now prove the theorem, allowing all implied constants to depend on $\delta$ and $M$. We may assume that $N$ is sufficiently large depending on $\delta, M$, since the claim is trivial otherwise. A simple volume packing argument (using [31, Corollary A.2], for example, to control the boundary terms) shows that

$$|(n_0 + \Lambda) \cap P| = \frac{\text{vol}(P)}{|\mathbb{Z}^D : \Lambda|} + o_{N \to \infty}(N^D).$$

As a consequence, for $N$ large enough we may subtract off the mean of $F$ and normalise $F$ to have Lipschitz norm 1 and mean zero, thus

$$\left| \sum_{n \in (n_0 + \Lambda) \cap P} F(g(n)\Gamma) \right| \gg N^D.$$

As $\Lambda$ has index $\leq M$ in $\mathbb{Z}^D$, it contains the sublattice $q\mathbb{Z}^D$ for some positive integer $q = O(1)$. By the pigeonhole principle, we may thus find $n_1 \in \mathbb{Z}^D$ of magnitude $O(1)$ such that

$$\left| \sum_{n \in (n_1 + q\mathbb{Z}^D) \cap P} F(g(n)\Gamma) \right| \gg N^D,$$

and thus

$$\left| \sum_{n \in \mathbb{Z}^D \cap P'} F(g(qn + n_1)\Gamma) \right| \gg N^D.$$

for some convex body $P'$ contains in a ball of radius $O(N)$ centered at the origin.

By subdividing $P'$ into cubes of sidelength $\varepsilon N$ for some sufficiently small $\varepsilon > 0$ (and again using [31, Corollary A.2] to control the boundary terms), and then applying the pigeonhole principle, we see that

$$\left| \sum_{n \in \mathbb{Z}^D \cap [n_2 + \varepsilon N]^D} F(g(qn + n_1)\Gamma) \right| \gg N^D$$

for some $\varepsilon \gg 1$ and $n_2 = O(N)$. We can rearrange this as

$$\left| \sum_{n \in \mathbb{Z}^D \cap [n_3^D]} F(g(qn + n_3)\Gamma) \right| \gg N^D.$$
for some $\mathbf{n}_3 = O(N)$.

We may now invoke [30, Theorem 8.6] to conclude that there exists a nontrivial homomorphism $\eta : G \to \mathbb{R}$ which vanishes on $\Gamma$, has complexity $O(1)$ and such that

$$\|\eta \circ g(q \cdot + \mathbf{n}_3)\|_{C^\infty([N]^D)} \ll 1.$$  

Applying [30, Lemma 8.4] we conclude that

$$\|Q \eta \circ g(\cdot + \mathbf{n}_3)\|_{C^\infty([N]^D)} \ll 1$$

for some non-negative integer $Q = O(1)$. Shifting the Taylor expansion by $\mathbf{n}_3$, we conclude that

$$\|Q \eta \circ g\|_{C^\infty([N]^D)} \ll 1.$$  

The claim follows (with $\eta$ replaced by $Q \eta$).

Appendix C. The Baker-Campbell-Hausdorff formula

Let $G$ be a connected, simply connected nilpotent Lie group, and let $\exp : \mathfrak{g} \to G$ and $\log : G \to \mathfrak{g}$ be the associated exponential and logarithm maps between $G$ and its Lie algebra $\mathfrak{g}$. The Baker-Campbell-Hausdorff formula asserts that

$$\exp(X_1) \exp(X_2) = \exp(X_1 + X_2 + \frac{1}{2}[X_1, X_2] + \prod_\alpha c_\alpha X_\alpha)$$

for any $X_1, X_2$, where $\alpha$ is a finite set of labels, $c_\alpha$ are real constants, and $X_\alpha$ are an iterated Lie bracket of $k_1 = k_{1,\alpha}$ copies of $X_1$ and $k_2 = k_{2,\alpha}$ copies of $X_2$ where $k_1, k_2 \geq 1$ and $k_1 + k_2 \geq 2$.

Using this formula, it is a routine matter to see that for any $g_1, g_2 \in G$ and $x \in \mathbb{R}$, we have

$$(g_1 g_2)^x = g_1^x g_2^x \prod_\alpha g_Q^{c_\alpha(x)} \prod_\alpha g_{Q_\alpha(x)}$$  

(C.1)

where $\alpha$ is a finite set of labels, each $g_\alpha$ is an iterated of $k_1 = k_{1,\alpha}$ copies of $g_1$ and $k_2 = k_{2,\alpha}$ copies of $g_2$ where $k_1, k_2 \geq 1$ and $k_1 + k_2 \geq 2$, and the $Q_\alpha : \mathbb{R} \to \mathbb{R}$ are polynomials of degree at most $k_1 + k_2$ with no constant term.

In a similar vein, for any $g_1, g_2 \in G$ and $x_1, x_2 \in \mathbb{R}$, we have the formula

$$[g_1^{x_1}, g_2^{x_2}] = [g_1, g_2]^{x_1 x_2} \prod_\alpha g_{P_\alpha(x_1, x_2)}$$  

(C.2)

where $\alpha$ is a finite set of labels, each $g_\alpha$ is an iterated commutator of $k_1 = k_{1,\alpha}$ copies of $g_1$ and $k_2 = k_{2,\alpha}$ copies of $g_2$ where $k_1, k_2 \geq 1$ and $k_1 + k_2 \geq 3$, and the $P_\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are polynomials of degree at most $k_1$ in $x_1$ and at most $k_2$ in $x_2$ which vanish when $x_1 = 0$ or $x_2 = 0$.  

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