Conditional sign flip via teleportation

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We present a model to realize a probabilistic conditional sign flip gate using only linear optics. The gate operates in the space of number state qubits and is obtained by a nonconventional use of the teleportation protocol. Both a destructive and a nondestructive version of the gate are presented. In the former case an Hadamard gate on the control qubit is combined with a projective teleportation scheme mixing control and target. The success probability is 1/2. In the latter case we need a quantum encoder realized via the interaction of the control qubit with an ancillary state composed of two maximally entangled photons. The success probability is 1/4.

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I. INTRODUCTION

Single photon qubits are a promising tool for quantum computation\textsuperscript{1}. The great advantage with respect to the others physical implementation\textsuperscript{2,3,4} is represented by the fact that photonic systems can be easily transferred from one place to another in the space and moreover the weak interaction with the environment makes the decoherence not so dangerous. These features permit secure transmission of information over long distances\textsuperscript{2,5,6}. On the other hand, the robustness of photons with respect to interactions creates a serious obstacle to the realization of conditional gates essential for quantum computation\textsuperscript{5} due to the large amount of resources required to create nonlinear coupling between qubits. Despite these considerations, Knill, Laflamme and Milburn (KLM) showed that quantum computation can be realized using only linear optics\textsuperscript{3}. This is done exploiting the nonlinearity induced by a measurement process. Probabilistic conditional gates are obtained using single photon sources, single photon detectors, ancilla photons and postselection measurements. More recently, Nielsen showed that any probabilistic gate based on linear optics is sufficient for the implementation of a quantum computer\textsuperscript{5}.

There is also some experimental realization of gates using only linear optics: Controlled-Not gate\textsuperscript{10,11,12} and Nonlinear sign shift\textsuperscript{13} have been recently reported.

On the other hand, it’s generally accepted that the teleportation protocol\textsuperscript{14} represents a fundamental resource for quantum computation, as already shown by Gottesman and Chuang\textsuperscript{15}.

Here we propose a model for a conditional sign flip gate based on photon number qubits, in agreement of most of features of KLM protocol, based on a nonconventional use of teleportation process.

In section\textsuperscript{11} we show how a destructive C-sign gate\textsuperscript{10} can be implemented starting from an Hadamard gate on the control qubit and a projective teleportation mixing control and target. How to obtain a non destructive gate is the subject of section\textsuperscript{11} while section\textsuperscript{15} will be devoted to conclusions.

II. DESTRUCTIVE C-SIGN FLIP GATE

A conditional sign flip gate is a two-qubit gate: the target qubit experiences a sign change between its components $|0\rangle$ and $|1\rangle$ if and only if the control qubit is in the logic state $|1\rangle$. In the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ the unitary operator representing the gate is $U = |0\rangle \langle 0|^{(1)} \otimes I^{(2)} + |1\rangle \langle 1|^{(1)} \otimes \sigma_z^{(2)}$ (I and $\sigma_z$ are respectively the identity operator and one of Pauli matrices) and has the following matrix representation:

$$U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$$

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On the other hand, the teleportation can be briefly described as follows. A quantum state $|\alpha_i\rangle = a|0_i\rangle + b|1_i\rangle$ is combined with a two qubit maximally entangled Bell state $|\Psi_{23}\rangle$. A Bell measurement, performed on the qubits 1 and 2, causes the transfer on the third qubit of the superposition initially encoded on the first one, except for a unitary transformation determined by the result of the Bell measurement. From a formal point of view, the teleportation is represented by a base change in the combined Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, plus a measurement. Usually the state $|\Psi_{23}\rangle$ is considered as fixed, but this is not a necessary prescription. In a more complete description, the global input state is written in terms of all possible Bell states, each of them with a probability amplitude $u_i$ where $i = 0, z, x, y$ (the choice of symbols will appear clear in what follows), that we can use to perform the process: recalling that the Bell states are $|\Phi^\pm\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$ and $|\Psi^\pm\rangle = 1/\sqrt{2}(|10\rangle \pm |01\rangle)$ we have

$$|\Phi\rangle = (\alpha_1|0_1\rangle + u_z|\Psi_{23}\rangle + u_x|\Phi_{23}\rangle + u_y|\Psi_{23}\rangle)$$

(2)

After the base change we obtain a new expression in terms of Bell states on 1 and 2:

$$|\Phi\rangle = \sum_i \left( |\Psi_{12}^+\rangle u_i a_{0i} \sigma_i |\alpha_3\rangle + |\Psi_{12}^-\rangle u_i a_{z1} \sigma_z \sigma_i |\alpha_3\rangle + |\Phi_{12}^+\rangle u_i a_{x2} \sigma_x \sigma_i |\alpha_3\rangle + |\Phi_{12}^-\rangle u_i a_{y2} \sigma_y \sigma_i |\alpha_3\rangle \right)$$

(3)

having introduced the Pauli matrices acting on the third qubit and

$$a_{ij} = \begin{pmatrix} 1 & -1 & 1 & i \\ 1 & -1 & -i & -1 \\ 1 & -i & 1 & 1 \\ -i & 1 & 1 & 1 \end{pmatrix}$$

If a measurement is done by projection, e.g. on the the singlet state $|\Phi_{12}^-\rangle$, we obtain a different state of the third qubit according to the $u_i$ selected. This result shows that teleportation acts as a controlled gate: the teleported state experiences a unitary transformation determined by the Bell state used as an input. In the most general case both C-NOT and C-sign are contemplated respectively when $u_0$ and $u_x$ or $u_0$ and $u_z$ are nonvanishing and by the establishment of a connection between the logic value of a qubit used as control and the suitable pair of Bell states $|\Psi_{23}\rangle$ selected. In particular, we found a simple model where this behavior emerges giving rise to a C-sign flip gate.

Let us explain our proposal. As requested in \S each qubit is realized on two spatial modes $|1\rangle$, $|2\rangle$: the presence of the photon in the first (second) rail corresponds to the logic state $|1\rangle$ ($|0\rangle$). For the sake of clarity we shall utilize the second quantization language, using occupation numbers instead of logic values, writing $|01\rangle$ for $|0\rangle$ and $|10\rangle$ for $|1\rangle$.

The rails of the control qubit are the input arms of a 50% beam splitter ($BS_1$) that acts as an Hadamard gate:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(4)

Then, if the input photon is in the state $|01\rangle$ the output state is an entangled singlet state, while if it is in the state $|10\rangle$ we deal with a triplet one on the output arms.

The entangled states created are used to perform teleportation. We refer to the experimental realization of vacuum-one photon qubit teleportation [20]. One of spatial modes outgoing from $BS_1$ is mixed on a second 50% beam splitter ($BS_2$) with one of spatial modes of the target qubit. With reference to figure 1 we denote with 1 and 2 the modes associated to the control qubit, with 1' and 2' the output modes of $BS_1$ and with 3 and 4 the modes corresponding to the target qubit, while the output modes of $BS_2$ will be labelled with 5 and 6. Let us consider first the case that the control qubit is in the state $|1_02\rangle$. Due to the action of the Hadamard gate the state after the photon has impinged $BS_1$ is $1/\sqrt{2}(|0_1, 1_2\rangle + |1_1, 0_2\rangle)$. This is a triplet entangled state realized over the output spatial modes of $BS_1$.

Being the target qubit in an arbitrary superposition $\alpha|0_3, 1_4\rangle + \beta|1_3, 0_4\rangle$ the whole state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( \alpha|0_1, 1_2, 0_3, 1_4\rangle + \beta|0_1, 1_2, 1_3, 0_4\rangle + \alpha|1_1, 0_2, 0_3, 1_4\rangle + \beta|1_1, 0_2, 1_3, 0_4\rangle \right)$$

(5)

The portion of this state corresponding to the spatial modes $2'$ and 3 is conveniently rewritten in terms of Bell states $|\Phi^\pm\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$ and $|\Psi^\pm\rangle = 1/\sqrt{2}(|10\rangle \pm |01\rangle)$. After this substitution we have

$$|\Psi\rangle = \frac{1}{2} \left[ |\Psi_{23}^+\rangle (\alpha|0_1, 1_4\rangle + \beta|1_1, 0_4\rangle) + |\Psi_{23}^-\rangle (\alpha|0_1, 1_4\rangle - \beta|1_1, 0_4\rangle) + |\Phi_{23}^+\rangle (\alpha|1_1, 1_4\rangle + \beta|0_1, 0_4\rangle) + |\Phi_{23}^-\rangle (\alpha|1_1, 1_4\rangle - \beta|0_1, 0_4\rangle) \right]$$

(6)
Our idea is to perform a projective measurement over the modes $2'$ and 3 by selecting only those events corresponding to the state $|\Psi_{23}^-\rangle$ as result. The measurement is performed using the modes $2'$ and 3 as the input arms of $BS_2$. The state $|\Psi_{23}^-\rangle$ corresponds to the detection of one and only one photon on the detector $D_1$ (see figure 1) and to the absence of counts on the second detector $D_2$.

As a result, the state emerging on the spatial modes $1'$ and 4 is $\alpha |0_1'1_4\rangle - \beta |1_1'0_4\rangle$. We observe that an entanglement swapping has been realized together with a sign flip with respect to the incoming target state.

Next, we study the situation corresponding to a control qubit in the $|0_11_2\rangle$. In such a situation the Hadamard gate creates a singlet entangled state on the output modes of $BS_1$: $1/\sqrt{2} (|0_1'1_2\rangle - |1_1'0_2\rangle)$. Then Eq. 6 has to be opportune modified. Limiting our interest to the term associated with the singlet as output result, now we have $|\Psi_{23}^-\rangle (\alpha |0_1'1_4\rangle + \beta |1_1'0_4\rangle)$. Thus, we observe again an entanglement swapping, but the difference with the former situation is that no sign flip arises from the process.

The previous results can be synthesized stating that the target qubit, initially encoded using the modes 3 and 4, is transferred on $1'$ and 4 with a sign change conditional to the logic state of the control qubit, as required from the definition of the C-sign gate. The gate is deterministic: it does not work with a success probability equal to 1, but we know whether it works correctly. In our case the probability is $1/4$, determined by the postselection procedure selecting one of four Bell states, and it can increased up to $1/2$ accepting single counts on $D_2$, with an adjunctive single qubit rotation.

Unfortunately, the control qubit is destroyed by the projection and the gate above illustrated is not complete. To make the scheme useful for quantum computation a method to restore the control state has to be introduced.

### III. NONDESTRUCTIVE GATE

To overcome the previous obstacle we use the technique of quantum encoding. From the “no cloning theorem” [21] we learn that a physical machine able to copy an arbitrary quantum state in a blank state cannot be realized. However, the theorem does not exclude the possibility of copying two selected orthogonal states and this is the working principle of a quantum encoder. Roughly speaking, the conversion $(\alpha |0\rangle + \beta |1\rangle) \rightarrow (\alpha |0\rangle + \beta |1\rangle) \otimes (\alpha |0\rangle + \beta |1\rangle)$ is forbidden while $(\alpha |0\rangle + \beta |1\rangle) \rightarrow (\alpha |0\rangle \otimes |0\rangle + \beta |1\rangle \otimes |1\rangle)$ is (at least in a probabilistic way) allowed leaving $\alpha$ and $\beta$ out of consideration.

A quantum encoder operating on polarization qubits is described in [14, 17]. It applies also in our case due to the existence of converters from polarization to dual rail and vice versa that are easily realizable using a polarizing beam splitter and a $\lambda/2$ waveplate.

On the other hand, we will show that a quantum encoder working only with photon number qubits is feasible using non polarizing beam splitters. The scheme is depicted in figure 2. The control qubit $(\alpha_1 |01\rangle + \alpha_2 |10\rangle)$ we want to copy is defined on the modes 1 and 2, while modes $a_1, a_2, b_1, b_2$ correspond to two ancilla qubits previously prepared in an maximally entangled state $1/\sqrt{2} (|0_{a_1}a_20_{b_1}b_2\rangle - |1_{a_1}a_21_{b_1}b_2\rangle)$. The modes $b_2$ and 1 are mixed on a beam splitter ($BS_a$) and a projective measurement analogous to that one described in section II takes place selecting only the singlet state $|\Psi_{a1}^-\rangle = 1/\sqrt{2} (|0_{a_1}1_{b_1}\rangle - |1_{a_1}0_{b_1}\rangle)$. The projection is performed measuring one and only one photon on $D_{a_1}$ and zero photons on $D_{a_2}$. As a result of the projection, it remains $1/\sqrt{2} (\alpha_1 |0_{a_1}1_{a_2}0_{b_1}1_{b_2}\rangle + \alpha_2 |1_{a_1}0_{a_2}1_{b_1}0_{b_2}\rangle)$. Thus, we have realized the quantum encoding operation, apart from a swapping from mode 1 to $b_1$. This gate is
The target qubit are yet implemented respectively on the modes 1 and 2 and 3 and 4. The auxiliary beam splitter $BS_2$, and the auxiliary detectors $D_a$ and $D_b$, are used to “double” the control qubit in an entangled state on $a_1$, $a_2$, $b_1$ and 2. $BS_1$ and $BS_2$ perform the conditional gate, as illustrated in Fig. 2 and the output is represented by the control qubit on the modes $a_1$ and $a_2$ and the (modified by the gate) target qubit on the modes $1'$ and 4.

The success probability is 1/4 and again it reaches 1/2 if also $|\Psi^+_{231}\rangle = 1/\sqrt{2}(|0_20_11_1\rangle + |1_20_10_1\rangle)$ is accepted via a classically feed-forwarded one qubit rotation. Notice that a qubit can be encoded also on a string of n qubits simply using a generalized maximally entangled state $1/\sqrt{2}(|0010.....01\rangle - |1010.....10\rangle)$ and performing the projection measurement mixing one of the 2n modes with one mode of the incoming qubit.

Let us return to our main problem. We want to build a gate that transforms a two qubit state, defined on four spatial modes, in accordance with the operator $U$ introduced in Eq. 1.

$$U(\alpha_1 |0_11_2\rangle + \alpha_2 |1_10_2\rangle)(\alpha_3 |0_31_4\rangle + \alpha_4 |1_30_4\rangle) = \alpha_1\alpha_3 |0_11_20_31_4\rangle + \alpha_1\alpha_4 |0_11_21_30_4\rangle + \alpha_2\alpha_3 |1_10_20_31_4\rangle - \alpha_2\alpha_4 |1_10_21_30_4\rangle$$

(7)

The control state is doubled via the quantum encoder above introduced and, under the probabilistic condition relied to the postselection process, we deal with the initialized three qubit state

$$|\Psi\rangle = (\alpha_1 |0_a1_20_a0_2\rangle + \alpha_2 |1_a0_a1_20_a\rangle)(\alpha_3 |0_31_4\rangle + \alpha_4 |1_30_4\rangle)$$

(8)

The procedure described in the previous section can now start: the modes $b_1$ and 2 are rearranged in $1'$ and $2'$ via the $BS_1$, the modes $2'$ and 3 are mixed on $BS_2$, the postselection measurement on $|\Psi^+_{231}\rangle$ is performed, and as a result of the complete set of operations we find that $U$ creates the state

$$\alpha_1\alpha_3 |0_a1_a0_11_4\rangle + \alpha_1\alpha_4 |1_a0_a1_10_4\rangle + \alpha_2\alpha_3 |1_a0_a0_11_4\rangle - \alpha_2\alpha_4 |0_a1_a0_11_4\rangle$$

in perfect agreement with the definition of the C-sign flip gate. Furthermore, the scheme realizes a teleported gate, as outlined in Fig. 2.

Due to the nondeterministic nature of the destructive gate and the quantum encoder, the nondestructive C-sign flip can reach 1/4 as overall efficiency.

IV. CONCLUSIONS

We have proposed a method to realize a probabilistic C-sign flip gate for number state qubits based only on few linear optics elements, specifically three balanced beam splitters, one source of entangled photons for auxiliary states, two single photon sources for target and control qubits, photodetectors and postselection measurements. All these elements are contained in the KLM scheme, for which our model seems to be tailored. In the original proposal contained in Fig. 8, the C-sign gate was achieved via two Nonlinear sign shift combined with two beam splitters. The network created in such a scheme was very intricate, and the simplification arising from the idea previously illustrated is remarkable. Furthermore, the maximum success probability of the gate is the same reported in the KLM work. To achieve the gate, a four fold coincidences measurement is required, fully available with the present technology. This scheme, being based on manipulations of number states, could be extended to solid state devices, where the
degenerate ground state is used both for transferring information and performing the unitary rotation associated to a beam splitter \[22\].

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