HYPERBOLIC HUBBARD-STRATONOVICH
TRANSFORMATION MADE RIGOROUS

by

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Abstract. — We revisit a long standing issue in the theory of disordered electron systems and their effective description by a non-linear sigma model: the hyperbolic Hubbard-Stratonovich (HS) transformation in the bosonic sector. For time-reversal invariant systems without spin this sector is known to have a non-compact orthogonal symmetry $O_{p,q}$. There exists an old proposal by Pruisken and Schäfer how to perform the HS transformation in an $O_{p,q}$-invariant way. Giving a precise formulation of this proposal we show that the HS integral is a sign-alternating sum of integrals over disjoint domains.

1. Introduction

Initiated by work of Wegner [1], Schäfer & Wegner [2], and Pruisken & Schäfer [3], non-compact non-linear sigma models as well as their supersymmetric generalizations due to Efetov [4] have long been a standard tool in the field of disordered electron physics. Areas of application include single electron motion in disordered and chaotic mesoscopic systems [5], chaotic scattering [6, 7], localization and delocalization in systems exhibiting the Integer Quantum Hall Effect [8], statistical properties of the Dirac spectrum in non-abelian gauge field backgrounds [9], to mention only a few.

The utility of the non-linear sigma model derives from the fact that it exposes the long-wavelength degrees of freedom of the disordered system, which are hidden in the original microscopic formulation by a Hamiltonian with quenched, i.e., time-independent, random parameters. These degrees of freedom correspond to interacting diffusion modes, known to be responsible for the universal behavior of spectral and eigenfunction statistics in a broad class of disordered systems [4, 5].

While the final results are expected to exhibit a high degree of universality, the mathematical tools employed to derive the non-linear sigma model may vary depending on the type of microscopic model under consideration. A concise introduction to this issue can be found in [10]; see also the recent papers [11, 12]. For some types of microscopic model the following mathematical identity plays a key role in the derivation:

$$C_0 \ e^{-\text{Tr} A^2} = \int_D e^{-\text{Tr} R^2 - 2i\text{Tr} AR} |dR|, \quad (1.1)$$
where \( C_0 \) is a constant and \(|dR|\) denotes the Lebesgue measure of a normed vector space of matrices \( R \). In a situation with compact orthogonal symmetry, which arises when the fermionic sector of the theory is considered, one has license to take the integration domain \( D \) to be the real symmetric matrices \( R \), and it is then trivial to do the Gaussian integral by completing the square and shifting variables. However, for reasons reviewed in Sect. 2 the bosonic sector (say, of time-reversal invariant systems without spin) calls for \( A \) to be a non-symmetric matrix composed of elements

\[
A_{ij} = \sum_{a=1}^{N} \varphi_{i,a} \varphi_{j,a} s_j ,
\]

with \((p+q) \times N\) real numbers \( \varphi_{i,a} \) and \( s_1 = \ldots = s_p = -s_{p+1} = \ldots = -s_{p+q} = 1 \).

The diagonal matrix \( s \) with entries \( s_{ij} = s_i \delta_{ij} \) determines a non-compact variant \( O_{p,q} \) of the real orthogonal group by the condition \( g^t s g = s \), where \( g^t \) means the transpose of the matrix \( g \). First discovered in the present context by Wegner [1], the group \( O_{p,q} \) is sometimes referred to as the ‘hyperbolic symmetry’ of the problem at hand.

In the situation with hyperbolic symmetry, choosing a good domain of integration for \( R \) and making rigorous sense of the integral (1.1) are non-trivial tasks. The difficulty arises from the general context of the so-called Hubbard-Stratonovich method: the exponential \( e^{-2i \text{Tr} A R} \) must be kept bounded – or else the next step of the method, which is to integrate over the microscopic fields \( \varphi_{i,a} \), would be invalid. Since the real matrix \( A \) satisfies the symmetry relation \( A = s A^t s \) one might think that one should try to integrate over the domain of all real matrices \( R \) subject to the symmetry \( R = s R^t s \). Unfortunately, the resulting quadratic form \( \text{Tr} R^2 = \text{Tr} R s R^t s \) is of indefinite sign and therefore such a choice of integration domain for \( R \) makes the \( R \)-integral divergent.

Nonetheless, a valid solution of the problem posed by (1.1), i.e., a choice of integration domain for \( R \) making the \( R \)-integral converge while keeping the integrand bounded as a function of \( A \), was offered in the paper by Schäfer and Wegner [2]; by deforming some of the real freedoms in \( R \) into the complex numbers they parameterize \( R \) as

\[
R = P - i \lambda T s T^{-1} ,
\]

where \( P \) runs through the real symmetric matrices that commute with \( s \), the constant \( \lambda \) is any positive real number, and \( T \in \text{SO}_{p,q} \). The reader is referred to the review [10] for a detailed discussion. Now the Schäfer-Wegner choice of domain has, besides its many merits, a certain drawback: it lacks invariance under the action of the symmetry group \( O_{p,q} \). This may be the reason why the Schäfer-Wegner choice was never much in use by the disordered electron physics community.

1.1. Statement of result. — Another choice of integration domain for (1.1), which meets the afore-mentioned requirements and also has the desirable property of \( O_{p,q} \)-invariance, first appeared in a paper by Pruisken and Schäfer [3]. Beginning with [6] this choice came to be used in numerous later works. In the present paper we are going to establish this so-called Pruisken-Schäfer domain as a rigorous alternative to that of Schäfer-Wegner. There exists, however, a mathematical subtlety: although our choice of domain agrees with Pruisken-Schäfer as a union \( D = \bigcup_\sigma D_\sigma \) of sets \( D_\sigma \), it differs as
an element $D = \sum_{\sigma} \text{sgn}(\sigma) D_\sigma$ of the vector space of integration chains (for the details, see below). Some low-dimensional special cases of it were worked out in [13, 14].

The Pruisken-Schäfer domain, $D$, is an open subspace of the $\frac{1}{2}(p+q)(p+q+1)$-dimensional space of real matrices $R$ obeying the mixed symmetry relation $R = sR^s$. Because such matrices may in general have complex eigenvalues and eigenvectors, one defines $D$ as the subspace of matrices $R = sR^s$ that can be diagonalized by conjugation $R \mapsto g^{-1} R g$ with an element $g \in O_{p,q}$ of the real (non-compact) orthogonal group. This set $D$ turns out to be the union $D = \bigcup \sigma D_\sigma$ of $\binom{p+q}{p}$ sub-domains $D_\sigma$.

The domains $D_\sigma$ are enumerated as follows. With the exception of cases forming a set of measure zero, an $O_{p,q}$-diagonalizable matrix $R = sR^s$ has $p$ ‘space-like’ and $q$ ‘time-like’ eigenvalues $\lambda \in \mathbb{R}$ — here we use the language of relativity theory to communicate that the corresponding eigenvector $v$ has the property $v^i s v > 0$ or $v^i s v < 0$, respectively. In the generic situation without degeneracies, the eigenvalues of $R$ can be arranged in decreasing order and this ordered sequence translates into a binary sequence $\sigma$ by writing, say, the symbols $\bullet$ for space-like and $\circ$ for time-like eigenvalues.

The binary sequence $\sigma$ encoding the relative order of space-like and time-like eigenvalues is an $O_{p,q}$-invariant. Moreover, collisions between eigenvalues of space-like and time-like type generically lead to the birth of complex eigenvalues, thereby taking us outside the integration domain for $R$. This means that the $O_{p,q}$-diagonalizable matrices $R$ of binary sequence $\sigma$ form an $O_{p,q}$-invariant domain, which we denote by $D_\sigma$, and any two such domains $D_\sigma$ and $D_\sigma'$ with $\sigma \neq \sigma'$ can only touch each other in subspaces of lower dimension (more precisely, of codimension two).

The integration measure $|dR|$ is defined to be the flat one for all domains $D_\sigma$:

$$|dR| = \prod_{i \leq j} dR_{ij}. \quad (1.3)$$

Now the startling feature of the following statement is that the integral on the right-hand side of (1.1) is proposed to be a sum of integrals $\int_{D_\sigma}$ with alternating sign!

**Theorem 1.** — There exists some choice of cutoff function $R \mapsto \chi_\varepsilon(R)$ (converging pointwise to unity as $\varepsilon \to 0$), and a unique choice of sign function $\sigma \mapsto \text{sgn}(\sigma) \in \{\pm 1\}$ and a constant $C_{p,q}$ such that

$$C_{p,q} \lim_{\varepsilon \to 0} \sum_{\sigma} \text{sgn}(\sigma) \int_{D_\sigma} e^{-\text{Tr} R^2 - 2i \text{Tr} A R} \chi_\varepsilon(R)|dR| = e^{-\text{Tr} A^2} \quad (1.4)$$

holds true for all matrices $A = sA^1$s with the positivity property $As > 0$.

**Remark.** — It will be shown that $\text{sgn}(\sigma)$ is the parity of the number of transpositions $\bullet \leftrightarrow \circ$ needed to reduce the binary sequence $\sigma$ to the extremal form $\sigma_0 = \bullet \cdots \circ \cdots \circ$. It should be emphasized that (1.4) is not a standard Gaussian integral; in fact, we have not succeeded in finding a proof of this formula by completing the square and shifting.

It is informative to give an alternative expression for the integral (1.4) in terms of the eigenvalues of $R$. Let $R = g \lambda g^{-1}$ with $\lambda = \text{diag}(\lambda_1, \ldots, \lambda_{p+q}) \in \mathbb{R}^{p+q}$ and $g \in SO_{p,q}$. The volume element transforms as $|dR| = J(\lambda)|d\lambda| dg$ where $|d\lambda| = \prod_{i=1}^{p+q} d\lambda_i$ and $dg$
is a positive Haar measure for the connected group $SO_{p,q}$. The Jacobian is

$$J(\lambda) = \prod_{i<j} |\lambda_i - \lambda_j|.$$ 

Corollary 1. — Let a function $J'(\lambda)$ with alternating sign on $\mathbb{R}^{p+q}$ be defined by

$$J'(\lambda) = J(\lambda) \prod_{i=1}^{p} \prod_{j=p+1}^{p+q} \text{sign}(\lambda_i - \lambda_j).$$

Then, assuming the positivity $A \succ 0$ of the matrix $A = sA's$, we have

$$C_{p,q} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{p+q}} \left( \int_{SO_{p,q}} e^{-2i\text{Tr}A g \lambda g^{-1}} \chi(\lambda g) g^{-1} dg \right) e^{-\text{Tr}A^2} J'(\lambda) |d\lambda| = e^{-\text{Tr}A^2}. $$

Remark. — The first $p$ eigenvalues $\lambda_j$ are space-like, the last $q$ time-like. To deduce Cor. 1 from Thm. 1, we need only observe that the function $J'(\lambda)$ differs from the Jacobian $J(\lambda)$ precisely by our sign function $\lambda \mapsto \text{sgn}(\sigma(\lambda))$:

$$J'(\lambda) = J(\lambda) \text{sgn}(\sigma(\lambda)).$$

The presence of this alternating sign factor was conjectured in [14]; see also [13].

In previous work, beginning with [6], the identity of Cor. 1 was assumed to hold without the alternating sign. It became apparent in 1995 that this assumption was unfounded [15] though it remained unclear at that time how to correct the mistake. Problems with the Pruiksen-Schäfer domain were mentioned in [16], but for various reasons the issue was never much emphasized in the published literature. (For one thing, all results derived using $J(\lambda)$ instead of $J'(\lambda)$ stand correct after saddle-point approximation for large $N$, as the integral in the large-$N$ limit is dominated by contributions from a single domain $D_\sigma$. For another, the Schäfer-Wegner domain was available [16] as a rigorous alternative in order to do all intermediate steps correctly.) The present paper solves this long standing problem in a satisfactory if perhaps surprising manner.

Let it be pointed out that the matrix $A$s appearing in applications is not strictly positive but positive semi-definite. This is, however, a minor issue as an easy variant of the formula (1.4) takes care of the semi-definite case; see [13] for the details.

For the related case of non-compact unitary symmetry $U_{p,q}$ a formula analogous to that of Cor. 1 had been established in [13], using the special feature of semiclassical exactness by the Duistermaat-Heckman theorem [17]. Although it will not be shown here, the methods of the present paper are robust and can be adapted to handle the case of $U_{p,q}$ as well, without taking recourse to semiclassical exactness.

The paper is organized as follows. In Sect. 2 we present some background material concerning the Hubbard-Stratonovich method. Basic results needed from integral calculus are collected in Sect. 3. Then, in Sect. 4 we work out the simple but instructive case of $p = q = 1$ in detail. The case of general $p$ and $q$ is handled in Sect. 5 by reduction to $p = q = 1$. 

2. Background

To make the present paper self-contained, we now describe the steps prior to (1.4), called the Hubbard-Stratonovich transformation in the present context. Although the true power of this transformation is to allow the treatment of non-trivial models with local $O_N$ gauge symmetry [2] or even without such symmetry [3], we will restrict ourselves here to reviewing the basic steps at the example of the simplest random matrix model of this universality class: the Gaussian Orthogonal Ensemble (GOE).

In the present work we are concerned with time-reversal invariant systems without spin. For such systems, the quantum mechanics of stationary states can be done over the field $\mathbb{R}$ of real numbers. Assuming the real Hilbert space to be of finite dimension, $N$, we may express the Hamiltonian operators $H$ by real symmetric $N \times N$ matrices. The Gaussian Orthogonal Ensemble by definition is an $O_N$-invariant probability measure $d\mu(H)$ on the space of such matrices. For our purposes, the GOE measure $d\mu(H)$ is best characterized by its Fourier transform:

$$ \langle e^{i\text{Tr}HK} \rangle_{\text{GOE}} \equiv \int e^{i\text{Tr}HK} d\mu(H) = e^{-\frac{b^2}{2}\text{Tr}K^2}, \quad (2.1) $$

where the Fourier variable $K$ is any symmetric matrix, and $b \in \mathbb{R}$ is a parameter.

An important object of the theory (see, e.g., [10]) is the expectation value of the reciprocal of a product of (square roots of) characteristic polynomials of $H$:

$$ F(z_1, \ldots, z_{p+q}) := \langle \prod_{j=1}^{p+q} \text{Det}^{-1/2}(z_j - H) \rangle_{\text{GOE}} \quad (z_j \in \mathbb{C} \setminus \mathbb{R}). \quad (2.2) $$

We assume that $\Im z_j > 0$ for $j = 1, \ldots, p$ and $\Im z_j < 0$ for $j = p+1, \ldots, p+q$. To compute such an expectation value, we use the trick of representing the reciprocal square root of each determinant as a Gaussian integral over vector variables $\varphi$:

$$ \text{Det}^{-1/2}(z_j - H) = (i\pi s_j)^{-N/2} \int_{\mathbb{R}^N} e^{is_j(\varphi, \varphi z_j - H\varphi)} |d\varphi|, $$

where $|d\varphi| = \prod_{a=1}^N d\varphi_a$ and the factor $s_j := \text{sgn}(\Im z_j)$ ensures the convergence of the integral. $(\varphi, \psi) := \sum_{a=1}^N \varphi_a \psi_a$ means the Euclidean scalar product of $\mathbb{R}^N$.

Substituting such integrals into (2.2), one immediately performs the GOE average by employing the characteristic property (2.1). The result can then be expressed in terms of a matrix $A(\varphi)$ of size $(p+q) \times (p+q)$ with entries $A(\varphi)_{ij} = \sum_{a=1}^N \varphi_{i,a} \varphi_{j,a} s_j$:

$$ F(z_1, \ldots, z_{p+q}) = \int e^{i\sum_{j=1}^{p+q} s_j z_j (\varphi, \varphi)} e^{-\frac{b^2}{2}\text{Tr}A(\varphi)^2} \prod_j |d\varphi_j| (i\pi s_j)^{N/2}. \quad (2.3) $$

Here is where the identity (1.1) comes in: it enables us to linearize the quadratic term $\text{Tr}A(\varphi)^2$ in the exponent, thereby reducing the integral over the $N$-component vectors $\varphi_1, \ldots, \varphi_{p+q}$ to $N$ decoupled Gaussian integrals, one for each component $a = 1, \ldots, N$. When the latter integrals are carried out, (2.3) becomes an integral over the collective variables $R$ only. Using the rigorous form (1.4) of the relation (1.1) we obtain

$$ F(z) = C_{p,q} \sum_{\sigma} \text{sgn}(\sigma) \int_{D_\sigma} e^{-\frac{N}{2\sigma^2}\text{Tr}R^2} \text{Det}^{-N/2}(z - R) |dR|, \quad (2.4) $$
where \( z := \text{diag}(z_1, \ldots, z_{p+q}) \) and, with the assumption that \( N \) is large enough, the cutoff function \( \chi_\varepsilon \) has been removed by sending the regularization parameter \( \varepsilon \) to zero.

The integral representation (2.4) for \( F(z) \) is well suited for saddle-point analysis in the limit of large \( N \). We can now appreciate why the identity (1.1) is useful here: it serves the purpose of exposing the good variables in which to perform the large-\( N \) limit. Let this fact suffice as a motivation for our labors in the body of this paper.

### 3. Basics from calculus

Recalling from Sect.1.1 the definition of the connected domains \( D_\sigma \) (cf. also below) we consider the following alternating sum of integrals:

\[
I(A) = \lim_{\varepsilon \to 0^+} \sum_\sigma \text{sgn}(\sigma) \int_{D_\sigma} e^{-\text{Tr}(R+iA)^2} \chi_\varepsilon(R+iA) \, |dR| ,
\]

(3.1)

where \( \chi_\varepsilon \) is some smooth cutoff function which regularizes the integral and converges pointwise to unity in the limit \( \varepsilon \to 0 \). The real matrix \( A \) is subject to the conditions \( A = sA \) and \( sA > 0 \). Choosing the Lebesgue measure \( |dR| \) as in (1.3) we shall prove that the integral \( I(A) \) does not depend on the matrix \( A \), or equivalently, that all of its directional derivatives vanish:

\[
\frac{d}{dt} I(A + t\dot{A}) \bigg|_{t=0} = 0 .
\]

(3.2)

Once (3.2) has been established, the Hubbard-Stratonovich transformation (1.4) follows by multiplying both sides of (3.1) with a constant \( e^{-\text{Tr}A^2} \).

To prepare our treatment, we recall a few basic facts from calculus. Given a vector field \( v \) on an \( n \)-dimensional differentiable manifold \( M \), let \( [-\delta, \delta] \ni t \mapsto \phi_t \) be the flow of \( v \), i.e., the one-parameter family of mappings \( \phi_t : M \to M \) determined by \( \phi_0(p) = p \) and \( \frac{d}{dt} \phi_t(p) = v(\phi_t(p)) \) for all \( p \in M \). In local coordinates \( x^1, \ldots, x^n \), we have

\[
\frac{d}{dt} x^i \circ \phi_t = v^i \circ \phi_t \quad (v = v^j \partial / \partial x^j) .
\]

We are using the summation convention: an index appearing twice (once as a covariant and once as a contravariant index) is understood to be summed over.

For a differential form \( \alpha = \alpha_{i_1 \ldots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \) on \( M \) let

\[
\phi_t^* \alpha = (\alpha_{i_1 \ldots i_k} \circ \phi_t) \, d(x^{i_1} \circ \phi_t) \wedge \cdots \wedge d(x^{i_k} \circ \phi_t)
\]

denote the pullback of \( \alpha \) by \( \phi_t \). The Lie derivative of \( \alpha \) w.r.t. the vector field \( v \) is then defined by differentiation at \( t = 0 \):

\[
\mathcal{L}_v \alpha = \frac{d}{dt} \phi_t^* \alpha \bigg|_{t=0} .
\]

By the relation \( \phi_t^* (\alpha \wedge \beta) = (\phi_t^* \alpha) \wedge (\phi_t^* \beta) \), the Lie derivative satisfies the Leibniz rule \( \mathcal{L}_v (\alpha \wedge \beta) = (\mathcal{L}_v \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_v \beta) \). Cartan’s formula expresses it as

\[
\mathcal{L}_v = d \circ t_v + t_v \circ d ,
\]
where $d$ is the exterior derivative,

$$d \left( \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) = d \alpha_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

and $t_v$ denotes the operation of contraction with the vector field $v$:

$$(t_v \alpha)_{i_1 \ldots i_k} = v^j \left( \alpha_{i_1 \ldots i_{k-1} - i_j i_{k-1} + \ldots + (-1)^{k-1} i_1 \ldots i_{k-1}} \right).$$

After this summary of basic facts, let $\omega$ be an $n$-form on $M$ and consider the integral $\int_M (\mathcal{L}_v f) \omega$ where $\mathcal{L}_v f = t_v df$ is the derivative of some function $f : M \to \mathbb{C}$ in the direction of the vector field $v$. Assuming that $\omega$ is invariant by the flow of $v$, the Leibniz rule gives

$$\int_M (\mathcal{L}_v f) \omega = \int_M \mathcal{L}_v (f \omega) = \int_M (d \circ t_v)(f \omega).$$

By Stokes’ theorem for the integral of the exact form $d(f \circ t_v \omega)$ it follows that

$$\int_M (\mathcal{L}_v f) \omega = \int_{\partial M} f \circ t_v \omega, \quad (3.3)$$

where $\partial M$ is the $(n-1)$-dimensional boundary of $M$.

In what follows, we will employ (3.3) in the closely related case where one is integrating against a density $|\omega|$ instead of an $n$-form $\omega$. Formula (3.3) still holds true in that case – assuming of course that $\mathcal{L}_v |\omega| = 0$. Note also that in order to integrate the twisted $(n-1)$-form $f \circ t_v |\omega|$, the boundary $\partial M$ must be equipped with an outer orientation, i.e., a transverse vector field pointing, say, from the inside to the outside of $\partial M$. (Technically speaking, a density $|\omega|$ on $M$ is a top-degree differential form $\omega$ tensored by a section $s$ of the orientation line bundle of $M$, and a choice of splitting $|\omega| = \omega \otimes s$ determines an isomorphism between outer and inner orientations for $\partial M$.)

To apply the formula (3.3) to the situation at hand, let $\tau(\hat{R}) = R_{ij} \partial / \partial R_{ij}$ denote the vector field generating translations $\phi_t(R) = R + t \hat{R}$. Then

$$\frac{d}{dt} I(A + t \hat{A}) \bigg|_{t=0} = \lim_{\epsilon \to 0+} i \sum_{\sigma} \text{sgn}(\sigma) \int_{D_{\sigma}} \mathcal{L}_{\tau(\hat{A})} \left( e^{-T_{(R + iA)^2}} \chi_{\epsilon}(R + iA) \right) |dR|,$$

where $\tau(\hat{A}) = \hat{A}_{ij} \partial / \partial R_{ij}$. Here, recognizing the fact that our integrand depends on $R$ and $A$ only through the combination $R + iA$, we have used the identity

$$\frac{\partial}{\partial A_{ij}} f(R + iA) = i \frac{\partial}{\partial R_{ij}} f(R + iA).$$

Now the Lebesgue measure (or positive density) $|dR|$ is invariant under translations $R \mapsto R + t \hat{A}$. Therefore $\mathcal{L}_{\tau(\hat{A})} |dR| = 0$, and we may apply formula (3.3) to obtain

$$\frac{d}{dt} I(A + t \hat{A}) \bigg|_{t=0} = \lim_{\epsilon \to 0+} i \sum_{\sigma} \text{sgn}(\sigma) \int_{\partial D_{\sigma}} e^{-T_{(R + iA)^2}} \chi_{\epsilon}(R + iA) \sigma_{\tau(\hat{A})} |dR|. \quad (3.4)$$

The main achievement of this paper is a proof that the alternating sum of integrals on the right-hand side vanishes in the limit $\epsilon \to 0$. For that purpose we need to develop a good understanding of the boundary components $\partial D_{\sigma}$ and the twisted form $t_{\tau(\hat{A})} |dR|$ restricted to them. As a first step, we illustrate the essential features of the argument at
the example of the symmetry group being $O_{1,1}$. This example plays a very important role as the general case will be handled by reduction to it.

4. The case of $O_{1,1}$-symmetry

Here we are going to deal with the special case of real $2 \times 2$ matrices $R$ subject to the linear symmetry relation

$$R = sR^ts, \quad s = \text{diag}(1, -1),$$

where $R \mapsto R^t$ means the operation of taking the matrix transpose. Such matrices can be parameterized by three real variables $r_{11}, r_{12}$ and $r_{22}$ as

$$R = \begin{pmatrix} r_{11} & r_{12} \\ -r_{12} & r_{22} \end{pmatrix}.$$

Thus our space of matrices $R$ is isomorphic as a vector space to $\mathbb{R}^3$. The measure of integration is the positive density $|dR| = dr_{11}dr_{12}dr_{22}$. The matrix $A$ is of the same form as $R$ but its matrix elements are constrained by

$$a_{11} > 0 > a_{22}, \quad |a_{12}| < \sqrt{-a_{11}a_{22}}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ -a_{12} & a_{22} \end{pmatrix}.$$

Another way of describing the setting is to equip the real vector space $\mathbb{R}^2$ with the sign-indefinite bilinear form $B$ determined by

$$B(u, v) \equiv u'sv := u_1v_1 - u_2v_2.$$

The matrices $R$ are then symmetric with respect to $B$ in the sense that

$$B(u, Rv) = B(Ru, v)$$

for all $u, v \in \mathbb{R}^2$. We denote the linear space of real matrices $R$ with this symmetry property by $\text{Sym}_B(\mathbb{R}^2)$. Since $B$ has signature $(1, 1)$, the symmetry group of $B$ is the non-compact real orthogonal group $O_{1,1}$. Elements $g \in O_{1,1}$ satisfy the equation

$$s = g'sg,$$

which is equivalent to saying that $B(u, v) = B(gu, gv)$ for all $u, v \in \mathbb{R}^2$.

For reasons indicated in the Introduction (see Sect. 1) – let us recall that in order for the integral (1.1) to exist one needs the inequality $\text{Tr}R^2 > 0$, at least asymptotically – we want our matrices $R \in \text{Sym}_B(\mathbb{R}^2)$ to be diagonalizable by the action $R \mapsto gRg^{-1}$ of the real group $O_{1,1}$ by conjugation. The condition of diagonalizability is formulated most clearly by expressing $R$ in a basis of light-like vectors of $\mathbb{R}^2$:

$$e_+ := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e_- := \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which obey $B(e_+, e_+) = B(e_-, e_-) = 0$ and $B(e_+, e_-) = 2$. Applying $R$ to this basis one has

$$Re_+ = \lambda e_+ + \eta e_-, \quad Re_- = \lambda e_- + \xi e_+,$$
where \( \lambda = \frac{1}{2}(r_{11} + r_{22}) \), \( \eta = \frac{1}{2}(r_{11} - r_{22}) + r_{12} \), and \( \xi = \frac{1}{2}(r_{11} - r_{22}) - r_{12} \). Since the diagonal piece of \( R \) in this basis is a multiple of unity, the condition for diagonalizability is that the product of off-diagonal matrix elements be positive: \( \xi \eta > 0 \).

The domain \( \xi \eta > 0 \) of \( O_{1,1} \)-diagonalizability of our matrices \( R \in \text{Sym}_2(\mathbb{R}^2) \) consists of two connected components:

\[
D_{\bullet} : \xi, \eta > 0, \quad D_{\circ} : \xi, \eta < 0.
\]

The motivation for the notation \((\bullet, \circ)\) will become clear shortly. Here we simply remark that if \( v_{\bullet}, v_{\circ} \) denotes a pair of \( R \)-eigenvectors with \( B(v_{\bullet}, v_{\bullet}) > B(v_{\circ}, v_{\circ}) \) and \( \lambda_{\bullet}, \lambda_{\circ} \) are the associated eigenvalues, then \( \lambda_{\bullet} > \lambda_{\circ} \) for \( R \in D_{\bullet} \) and \( \lambda_{\circ} > \lambda_{\bullet} \) for \( R \in D_{\circ} \).

Next, let \( G_{A,\epsilon} \) denote the regularized Gaussian integrand

\[
G_{A,\epsilon}(R) := e^{-\text{Tr}(R+iA)^2} \chi_\epsilon(R+iA),
\]

where we choose the cutoff function to be

\[
\chi_\epsilon(R) := e^{-\frac{\epsilon}{2} \text{Tr}(R-R_s)^2} = e^{-4\epsilon r_{12}^2}.
\]

Using the coordinates \( \lambda, \xi, \eta \) we have the expression \( G_{A,\epsilon} = e^{-f_2-f_1-f_0} \) with

\[
f_0 = -(a_{11}^2 + a_{22}^2) + 2a_{12}^2(1 - 2\epsilon), \quad f_2 = 2(\lambda^2 + \xi \eta) + \epsilon(\eta - \xi)^2,
\]

\[
f_1 = 2i(a_{11} + a_{22})\lambda + i(a_{11} - a_{22})(\xi + \eta) + 2ia_{12}(1 - 2\epsilon)(\xi - \eta).
\]

Let us observe that the integral \( I(A) \) defined in (3.1) can now be written as

\[
I(A) = \lim_{\epsilon \to 0^+} \left( \int_{D_{\bullet}} - \int_{D_{\circ}} \right) G_{A,\epsilon} |dR|.
\]

**4.1. Description of boundaries.** — By their definition via inequalities, the domains \( D_\sigma \) (for \( \sigma = \bullet, \circ \)) are open in the three-dimensional space \( \text{Sym}_2(\mathbb{R}^2) \) and have boundaries \( \partial D_\sigma \) which are two-dimensional. Each of the boundaries \( \partial D_\sigma \) is a union of 2 half-planes. To describe this in detail, let 4 half-planes \( E_\sigma \) and \( F_\sigma \) be defined by

\[
E_{\bullet} : \xi = 0, \quad \eta > 0, \quad F_{\bullet} : \eta = 0, \quad \xi > 0,
\]

\[
E_{\circ} : \xi = 0, \quad \eta < 0, \quad F_{\circ} : \eta = 0, \quad \xi < 0.
\]

\( E_\sigma \) and \( F_\sigma \) make up the boundary of \( D_\sigma \) (for both \( \sigma = \bullet, \circ \)). Being part of \( \partial D_{\bullet} \), the half-plane \( E_{\bullet} \) inherits from \( D_{\bullet} \) an outer orientation by the transverse vector field \( \partial_\xi \equiv \partial/\partial \xi \). (Indeed, starting from any point \( R \in D_{\bullet} \), very close to the boundary \( E_{\bullet} \) and making a small step in the direction of \( \partial_\xi \), one crosses \( E_{\bullet} \).) In the same sense, \( F_{\bullet} \subset \partial D_{\bullet} \) is oriented by \( \partial_\eta \) while \( E_\sigma \subset \partial D_\sigma \) and \( F_\sigma \subset \partial D_\sigma \) for \( \sigma = \bullet \) are oriented by the opposite vector fields \( -\partial_\xi \) and \( -\partial_\eta \) respectively; see Fig. 1.

With these orientations being understood, we have the relations

\[
\partial D_\sigma = E_\sigma + F_\sigma \quad (\sigma = \bullet, \circ),
\]

which are equalities in the sense of chains, or distributions. (To achieve equality in the sense of sets, we would have to include the real line \( \xi = \eta = 0 \), which is part of both \( \partial D_{\bullet} \) and \( \partial D_{\circ} \). However, our aim is to integrate bounded differential forms, and for that purpose the lower-dimensional parts of the boundary are of no concern.)
The outer orientation of these boundaries is determined by the rule of crossing the boundary “from the inside to the outside”.

With $\partial D_\sigma$ as described above, formula (3.4) now reads

$$\frac{d}{dt} I(A + t\dot{A})\bigg|_{t = 0} = \lim_{\varepsilon \to 0^+} \left( \int_{\partial D_\circ} - \int_{\partial D_*} \right) iG_{A,\varepsilon} l_{\tau(A)} |dR| \quad (4.1)\) $$

**4.2. Reorganization of boundary pieces.** — The key idea is to reorganize the boundary half-planes into consistently oriented closed manifolds. Define the full planes

- $E : \xi = 0$ (oriented by $-\partial_\xi$),
- $F : \eta = 0$ (oriented by $-\partial_\eta$),

so that we have

$$E = E_\circ - E_\bullet, \quad F = F_\circ - F_\bullet, $$

in the sense of chains. Since

$$\partial D_\circ - \partial D_\bullet = E_\circ + F_\circ - E_\bullet - F_\bullet = E + F$$

(still as chains), we obtain the following relation between integrals:

$$\int_{\partial D_\circ} \omega - \int_{\partial D_\bullet} \omega = \int_E \omega + \int_F \omega$$

for any twisted 2-form $\omega$ which is bounded (if not continuous or smooth). In particular, this relation holds for $\omega = iG_{A,\varepsilon} l_{\tau(A)} |dR|$. Thus (4.1) turns into

$$\frac{d}{dt} I(A + t\dot{A})\bigg|_{t = 0} = \lim_{\varepsilon \to 0^+} \left( \int_E + \int_F \right) iG_{A,\varepsilon} l_{\tau(A)} |dR| \quad (4.2)$$
A graphical sketch of the situation is shown in Fig. 2.

4.3. Computation of boundary integral. — We are now going to show that the integral \( \int_E \omega \) vanishes for \( \omega = iG_{A,\varepsilon} t_{\tau(\dot{A})} |dR| \) in the limit \( \varepsilon \to 0 \). Essentially the same calculation shows that \( \int_F \omega \) vanishes in this limit.

To handle the case of \( E \), let \( \psi : E \to \text{Sym}_B(\mathbb{R}^2) \) be the identical embedding and introduce coordinates \( \lambda, \eta \) on \( E \) so that 

\[
\psi^\ast \lambda = \lambda \quad \psi^\ast \xi = 0 \quad \psi^\ast \eta = \eta .
\]

Recalling \( G_{A,\varepsilon} = e^{-f_2-f_1-f_0} \) we pull back the functions \( f_2 \) and \( f_1 \) to \( E \):

\[
\psi^\ast f_2 = 2\lambda^2 + \varepsilon\eta^2 \quad \psi^\ast f_1 = 2i(a_{11} + a_{22})\lambda + i(a_{11} - a_{22} - 2a_{12}(1 - 2\varepsilon))\eta .
\]

Next we compute the pullback of the twisted 2-form \( t_{\tau(\dot{A})}|dR| \). For this, we write

\[
|dR| = dr_{11} dr_{12} dr_{22} = d\lambda d\xi d\eta := (d\lambda \wedge d\xi \wedge d\eta) \otimes \text{Or}
\]

where \( \text{Or} = +1 \) when \( \text{Sym}_B(\mathbb{R}^2) \) is oriented by the ordered set of linear coordinates \( \lambda, \xi, \eta \) (or any even permutation thereof), and \( \text{Or} = -1 \) when the ordering \( \lambda, \eta, \xi \) (or any other odd permutation of \( \lambda, \xi, \eta \)) is chosen. Using the coordinate expression of the vector field \( \tau(\dot{A}) \),

\[
\tau(\dot{A}) = \dot{a}_{11} \frac{\partial}{\partial r_{11}} + \dot{a}_{12} \frac{\partial}{\partial r_{12}} + \dot{a}_{22} \frac{\partial}{\partial r_{22}},
\]

\[
= \frac{1}{2}(\dot{a}_{11} + \dot{a}_{22}) \partial_\lambda + \frac{1}{2}(\dot{a}_{11} - \dot{a}_{22} - 2\dot{a}_{12}) \partial_\xi + \frac{1}{2}(\dot{a}_{11} - \dot{a}_{22} + 2\dot{a}_{12}) \partial_\eta ,
\]
we then obtain
\[ \psi^* \tau(\Delta) |dR| = \frac{1}{2} (\hat{a}_{11} - \hat{a}_{22} - 2\hat{a}_{12}) (d\eta \wedge d\lambda) \otimes \text{Or}. \]

Now the presence of the orientation factor Or tells us that an outer orientation by \( \pm \partial_x \) translates into an inner orientation by \( \pm d\eta \wedge d\lambda \). Since our \( \lambda, \eta \)-plane \( E \) is oriented by \( -\partial_x \), we must assign to it the orientation given by \(-d\eta \wedge d\lambda\). In this way, assembling everything we arrive at the expression
\[
\int_E \psi^* G_{A,\epsilon} \tau(\Delta) |dR| = -\frac{1}{2} (\hat{a}_{11} - \hat{a}_{22} - 2\hat{a}_{12}) e^{a_{11}^2 + a_{22}^2 - 2a_{12}(1-2\epsilon)} \times \]
\[
\times \int_{\mathbb{R}} e^{-2\lambda^2 - 2i(a_{11} + a_{22})\lambda} d\lambda \int_{\mathbb{R}} e^{-\eta^2 - i(a_{11} - a_{22} - 2a_{12}(1-2\epsilon))\eta} d\eta,
\]
where the right-hand side has been reduced to a product of Riemann integrals by Fubini’s theorem and the definition of what it means to integrate a differential form.

The crucial point now is that the integral over \( \eta \) vanishes in the limit \( \epsilon \to 0 \). Indeed, writing \( b_\epsilon := \frac{1}{2} (a_{11} - a_{22}) - a_{12} (1 - 2\epsilon) \) for short, the \( \eta \)-integral gives
\[
\int_{\mathbb{R}} e^{-\eta^2 - 2ib_\epsilon \eta} d\eta = e^{-b_\epsilon^2/\epsilon} \sqrt{\pi/\epsilon},
\]
and this does go to zero for \( \epsilon \to 0 \) as long as the real number \( b_0 \) is non-zero. The latter condition is always satisfied, as the constraints on the matrix elements of \( A \) ensure that
\[
b_0 = \frac{1}{2} (a_{11} - a_{22}) - a_{12} \geq \sqrt{-a_{11}a_{22} - |a_{12}|} > 0.
\]
Thus we have shown that \( \lim_{\epsilon \to 0^+} \int_E \psi^* i G_{A,\epsilon} \tau(\Delta) |dR| = 0 \).

The situation is no different for the integral over the 2-plane \( F \). We therefore conclude that \( \frac{d}{dt} I(A + tA) |_{t=0} \). Hence \( I(A) = c_{1,1} \) is a constant independent of \( A \) for \( A > 0 \). This proves the validity of the Hubbard-Stratonovich transformation in the following precise form with cutoff function \( \chi_\epsilon(R) = e^{-\frac{\epsilon}{2} \text{Tr} (sR - R^2)} \):
\[
e^{-\text{Tr} A^2} = c_{1,1} \lim_{\epsilon \to 0^+} \sum_{\epsilon \in \{\bullet, \times\}} \text{sgn}(\epsilon) \int_{D_\epsilon} e^{-\text{Tr} R^2 - 2i\text{Tr} AR} \chi_\epsilon(R) |dR|,
\]
where \( \text{sgn}(\bullet) = 1 \) and \( \text{sgn}(\times) = -1 \), and we have replaced \( \chi_\epsilon(R + iA) \) by \( \chi_\epsilon(R) \).

Let us emphasize once again that the right-hand side is not a standard Gaussian integral, and that our proof of this formula was not by completing the square and shifting. We will show in the next subsection that \( C_{1,1} = i2\pi^{-\frac{3}{2}} \).

4.4. Short proof. — We now spell out another line of approach for the case of \( p = q = 1 \), which is a variant of that given in [13]. This calculation will be much quicker, but does not extend (at least not in any way known to us) to higher values of \( p, q \).

Recall that in light-cone coordinates \( \xi, \eta, \) and \( \lambda \), we have the expression
\[
e^{-\frac{1}{2} \text{Tr} R^2 - i\text{Tr} AR} = e^{-\lambda^2 - i(a_{11} + a_{22})\lambda} e^{-\xi \eta - i b_0 \eta - i b_1 \xi},
\]
where \( b_0 = \frac{1}{2} (a_{11} - a_{22}) - a_{12} > 0 \) and \( b_1 = \frac{1}{2} (a_{11} - a_{22}) + a_{12} > 0 \). It is clear that this can be integrated against \( \int_{\mathbb{R}} d\lambda \) resulting in a factor \( \sqrt{\pi} e^{-\frac{1}{2} (a_{11} + a_{22})^2} \). It then remains to integrate the other factor \( e^{-\xi \eta - i b_0 \eta - i b_1 \xi} \) against \( d\xi d\eta \) over the two quadrants \( \xi, \eta > 0 \).
and $\xi, \eta < 0$. For this we make the substitution of variables $\eta = \rho e^\tau$ and $\xi = \rho e^{-\tau}$ which transforms the measure $d\xi d\eta$ to $2|\rho| d\rho d\tau$. Noticing that $\rho > 0$ on $D_{\bullet \bullet}$, $\rho < 0$ on $D_{\bullet \circ}$, and our chain of integration is $D = D_{\bullet \circ} - D_{\bullet \bullet}$, we drop the absolute value on $|\rho|$ and integrate against the sign-alternating distribution $\int_{\mathbb{R}^2} \rho d\rho d\tau$. More precisely, we take the $\rho$ integration to be the inner one (this obviates the need for regularization albeit at the expense of Fubini’s theorem becoming inapplicable, so that the order of integrations is now fixed and can no longer be changed). The resulting $\rho$ integral is

$$\int_{\mathbb{R}} e^{-\rho^2 - 2i\rho \beta} \rho \, d\rho = -i\sqrt{\pi} \beta e^{-\beta^2}, \quad \beta = \frac{1}{2}(b_0 e^\tau + b_1 e^{-\tau}).$$

We now observe the relation

$$\beta e^{-\beta^2} d\tau = e^{-b_0 b_1} e^{-\frac{i}{2}(b_0 e^\tau - b_1 e^{-\tau})^2} \frac{1}{2} d\left(b_0 e^\tau - b_1 e^{-\tau}\right),$$

which shows that the remaining $\tau$ integral is another standard Gaussian integral in the variable $\frac{1}{2}(b_0 e^\tau - b_1 e^{-\tau})$, yielding the value $\sqrt{\pi} e^{-b_0 b_1}$. Altogether we then obtain

$$C_{1,1} \int_D e^{-\frac{1}{4} \text{Tr}R^2 - i\text{Tr}AR} |dR| = 2\frac{i}{2} e^{-\frac{1}{4} \text{Tr}A^2}, \quad C_{1,1} = i2\frac{1}{2} \pi^{-\frac{3}{2}},$$

for $D = D_{\bullet \circ} - D_{\bullet \bullet}$, which is the desired result but for scaling $A \to \sqrt{2}A$ and $R \to \sqrt{2}R$. (Of course it must be understood here that the $\rho$ integration has to be done first.)

This computation, while direct and short, does not seem to carry over to the case of higher values of $p$ and $q$, whereas our conceptual proof of Sects. 4.1, 4.3 does.

5. The general case of $O_{p,q}$-symmetry

We now turn to the general case of real matrices $R$ of size $(p + q) \times (p + q)$,

$$R = \begin{pmatrix} R_{p,p} & R_{p,q} \\ -(R_{p,q})^t & R_{q,q} \end{pmatrix},$$

where the blocks $R_{p,p}$ and $R_{q,q}$ are symmetric. Such matrices $R$ as a whole satisfy the linear symmetry relation

$$R = sR^t s, \quad s = \text{diag}(1_{p}, -1_{q}).$$

Equivalently, if we equip the real vector space $\mathbb{R}^{p+q}$ with the indefinite bilinear form

$$B(u, v) \equiv u^t s v := \sum_{i=1}^{p} u_i v_i - \sum_{j=p+1}^{p+q} u_j v_j,$$

then our matrices $R$ form the linear space $\text{Sym}_B(\mathbb{R}^{p+q})$ of $B$-symmetric matrices, i.e.,

$$\forall u, v \in \mathbb{R}^{p+q} : \quad B(u, R v) = B(v, R u).$$

The symmetry group of the bilinear form $B$ with signature $(p, q)$ is the non-compact real orthogonal group $O_{p,q}$ of matrices $g$ satisfying the equation $s = g^t s g$ or equivalently, $B(u, v) = B(g u, g v)$ for all $u, v \in \mathbb{R}^{p+q}$.

We still want our matrices $R \in \text{Sym}_B(\mathbb{R}^{p+q})$ to be diagonalizable by the $O_{p,q}$-action $R \mapsto g R g^{-1}$. In the simple case $p = q = 1$ we saw that diagonalizability fails when
the off-diagonal block $R_{p,q}$ is larger than the difference of the diagonal blocks $R_{p,p}$ and $R_{q,q}$. A similar phenomenon is expected to occur for general $p,q$. Roughly speaking, $R$ will fail to have real eigenvectors and eigenvalues when the degree of non-Hermiticity coming from the off-diagonal blocks becomes too large.

5.1. Description of components $D_\sigma$. — In the Hubbard-Stratonovich integral (3.1) we integrate over the set $D$, of $R \in \text{Sym}_B(\mathbb{R}^{p+q})$ that can be brought to diagonal form by the $O_{p,q}$-action, i.e., matrices of the form $R = g\lambda g^{-1}$ with $\lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{p+q})$ and $g \in O_{p,q}$. We know from our considerations for $p = q = 1$ that $D$ decomposes into connected sets $D_\sigma$ whose boundaries intersect in lower-dimensional sets (of codimension 2). Here we describe these connected components $D_\sigma$.

Let $R = g\lambda g^{-1}$ be $O_{p,q}$-diagonalizable in the sense just described, so that the $p+q$ column vectors which make up $g$ form a basis of $\mathbb{R}^{p+q}$ consisting of eigenvectors of the matrix $R$. Let $v^{(k)}$ denote the $k$th column of $g$. Then by the equation $g^t s g = s$ the vector $v^{(k)}$ has positive norm square $B(v^{(k)}, v^{(k)}) = 1 > 0$ for $k = 1, \ldots, p$ and negative norm square $B(v^{(k)}, v^{(k)}) = -1 < 0$ for $k = p + 1, \ldots, p + q$. To distinguish between these two cases, we use the language of relativity theory and call the corresponding eigenvalue $\lambda_k$ of ‘space-like’ type in the former case and ‘time-like’ in the latter.

Assuming that we are in the generic situation, where all eigenvalues $\lambda_k$ of $R$ differ from each other, we can arrange them as a descending sequence. Given any such sequence, we assign to it a motif, $\sigma$: if an eigenvalue of the ordered sequence is space-like, then we represent it as a full dot, otherwise as an empty dot. Thus our motifs $\sigma$ are sequences of $p$ full and $q$ empty dots; e.g., $\sigma = \bullet \circ \circ \bullet \bullet$ is a motif for $p = q = 3$.

With each motif $\sigma$ we now associate a domain $D_\sigma$ as follows. While insisting that eigenvalues of opposite type (space-like vs. time-like) must never be equal, we lift the condition of non-degeneracy of eigenvalues partially: eigenvalues of the same type are now allowed to collide and exchange positions, but eigenvalues of opposite type still are not. This rule of eigenvalue motion defines a connected domain $D_\sigma$ for each $\sigma$.

We define a sign function $\sigma \mapsto \text{sgn}(\sigma) \in \{\pm 1\}$ by setting $\text{sgn}(\sigma) := (-1)^n$ where $n$ is the number of transpositions $\bullet \circ \leftrightarrow \circ \bullet$ which are needed to reduce $\sigma$ to the reference motif $\sigma_0 = \bullet \bullet \cdots \circ \circ \bullet \circ \cdots \circ$ made of $q$ empty dots following $p$ full dots. The parity of this number $n$ is well-defined although $n$ itself is not. Indeed, another expression for $\text{sgn}(\sigma) = (-1)^n$ is

$$\text{sgn}(\sigma) = \prod_{i=1}^{p} \prod_{j=p+1}^{p+1} \text{sign}(\lambda_i - \lambda_j),$$

where the $\lambda$‘s are the eigenvalues of any $R \in D_\sigma$.

5.2. Description of boundary. — As was explained above, a point in the interior of $D_\sigma$ is a matrix $R \in \text{Sym}_B(\mathbb{R}^{p+q})$ which is diagonalizable by the $O_{p,q}$-action and has eigenvalues that order according to the motif $\sigma$ – in particular, no two eigenvalues of opposite type are equal. This means that if $t \mapsto R(t)$ is a continuous curve which starts in $D_\sigma$ but leads to a collision of two or more eigenvalues of opposite type at $t_c$, then $R(t_c)$ is a point in the boundary of $D_\sigma$. The generic boundary situation is
that (i) exactly two eigenvalues of opposite type collide, (ii) the norms of the two corresponding eigenvectors go to zero, and (iii) the boundary point \( R(t_\epsilon) \) is a matrix which fails to be diagonalizable by any real transformation \( g \in O_{p,q} \).

To carry out the computations leading to the desired result (5.2), we need a good description of the interior of \( D = \bigcup_q D_q \) as well as the boundary \( \partial D \). This is achieved as follows. Let us first make a slight change of perspective: abandoning the viewpoint of \( R \) as a matrix, we switch to regarding \( R \) as an invariantly defined linear operator (giving rise to a matrix when expressed w.r.t. some fixed basis); i.e., taking the matrix elements \( R^{ij} \) we let \( R := R^{ij}E_{ij} \) (summation convention!) with \( E_{ij} \) the standard basis of \( \text{End}(\mathbb{R}^{p+q}) \) defined in terms of the standard basis \( e_i \) of \( \mathbb{R}^{p+q} \) by \( E_{ij}e_k = e_i\delta_{jk} \).

Consider now the Grassmannian \( \text{Gr}_{1,1}(\mathbb{R}^{p+q}) \) of \( B \)-orthogonal decompositions \( \mathbb{R}^{p+q} = V \oplus V^\perp \), where \( V \subset \mathbb{R}^{p+q} \) is any Lorentzian plane, i.e., a 2-dimensional subspace \( V = L^+ \oplus L^- \) spanned by a pair of null lines or light-like directions \( L^\pm \) such that \( s(L^\pm) = L^\mp \). The restriction of \( B \) to \( V \) is then non-degenerate, and the \( B \)-orthogonal complement \( V^\perp \) of \( V \) is spanned by \( p-1 \) space-like and \( q-1 \) time-like vectors. Note that \( \text{Gr}_{1,1}(\mathbb{R}^{p+q}) \) is acted upon transitively by the group \( SO_{p,q} \), which is to say that every Lorentzian plane \( V \subset \mathbb{R}^{p+q} \) can be regarded as the image \( g(\mathbb{R}^{1+1}) \) of the standard Lorentzian plane \( \mathbb{R}^{1+1} \) by some transformation \( g \in SO_{p,q} \). Since \( \mathbb{R}^{1+1} \) has isotropy \( S(O_{1,1} \times O_{p-1,q-1}) \), our manifold \( \text{Gr}_{1,1}(\mathbb{R}^{p+q}) \) is the base of a principal fibre bundle

\[
SO_{p,q} \to SO_{p,q}/S(O_{1,1} \times O_{p-1,q-1}) \simeq \text{Gr}_{1,1}(\mathbb{R}^{p+q}) \, .
\]

Moreover, since the Lorentzian planes \( V \subset \mathbb{R}^{p+q} \) are in one-to-one correspondence with pairs of real lines, one in \( \mathbb{R}^p \) and \( \mathbb{R}^q \) each, \( \text{Gr}_{1,1}(\mathbb{R}^{p+q}) \) is a product

\[
\text{Gr}_{1,1}(\mathbb{R}^{p+q}) \simeq \mathbb{R}P_{p-1} \times \mathbb{R}P_{q-1}
\]

of real projective spaces \( \mathbb{R}P_{n-1} \equiv \mathbb{R}^n/(\mathbb{R} \setminus \{0\}) \) for \( n = p \) and \( n = q \). From this identification it follows immediately that the manifold \( \text{Gr}_{1,1}(\mathbb{R}^{p+q}) \) is compact, non-simply connected, and non-orientable unless both \( p \) and \( q \) are even. It is also seen that (5.1) decomposes as a direct product of two principal fibre bundles

\[
SO_n \to SO_n/S(O_1 \times O_{n-1}) \simeq \mathbb{R}P_{n-1}
\]

for \( n = p, q \). We note that these principal fibre bundles are non-trivial.

Choosing a point \( x \in \text{Gr}_{1,1}(\mathbb{R}^{p+q}) \) is equivalent to choosing an orthogonal projector

\[
\Pi_x : \mathbb{R}^{p+q} \to V_x \equiv V \, .
\]

By orthogonality, \( B \) restricts to a bilinear form of signature \((1,1)\) on \( V_x \) and a bilinear form of signature \((p-1,q-1)\) on the orthogonal complement \( V_x^\perp \equiv (\text{Id} - \Pi_x)\mathbb{R}^{p+q} \). We denote by \( O(V_x) \) resp. \( O(V_x^\perp) \) the symmetry groups of these symmetric bilinear forms. Let \( \text{Sym}_B(V_x) \) denote the subspace of linear operators \( r : V_x \to V_x \) which are symmetric with respect to \( B \) restricted to \( V_x \), and let \( D(V_x^\perp) \subset \text{Sym}_B(V_x^\perp) \) denote the subspace of \( O(V_x^\perp) \)-diagonalizable linear transformations \( t : V_x^\perp \to V_x^\perp \).
Now consider the set, $\mathcal{X}$, of triples $(x; r, t)$ consisting of any point $x \in \text{Gr}_{1,1}(\mathbb{R}^{p+q})$ and two linear operators $r \in \text{Sym}_B(V_x)$ and $t \in D(V_x^\perp)$. This set $\mathcal{X}$ has the structure of a fibre bundle

$$
\pi : \mathcal{X} \to \text{Gr}_{1,1}(\mathbb{R}^{p+q}), \quad (x; r, t) \mapsto x,
$$

where the fibre over $x$ is a direct product

$$
\pi^{-1}(x) = \text{Sym}_B(V_x) \times D(V_x^\perp).
$$

One may also view $\mathcal{X}$ as an associated bundle $\mathcal{X} = G \times_K W$ with $G = \text{SO}_p \times \text{SO}_q$,

$$
K = S(O_1 \times O_{p-1}) \times S(O_1 \times O_{q-1}), \quad W = \text{Sym}_B(\mathbb{R}^{1+1}) \times D(\mathbb{R}^{(p-1)+(q-1)}),
$$

where the action of $K$ on $W$ is by conjugation. Since the principal bundle $G \to G/K = \text{Gr}_{1,1}(\mathbb{R}^{p+q})$ is non-trivial, so is the associated bundle $\mathcal{X} \to \text{Gr}_{1,1}(\mathbb{R}^{p+q})$.

$\mathcal{X}$ can be viewed as a subset of our space $\text{Sym}_B(\mathbb{R}^{p+q})$ of operators $R$. Indeed, for $R \equiv (x; r, t)$ we may use the projector $\Pi_x$ to define $Rv$ for any vector $v \in \mathbb{R}^{p+q}$ by

$$
Rv := r\Pi_x v + t(\text{Id} - \Pi_x)v.
$$

In other words, $R$ acts on $V_x$ as $r$ and on $V_x^\perp$ as $t$. Since both $r$ and $t$ are symmetric with respect to $B$, so is $R = r\Pi_x + t(\text{Id} - \Pi_x)$. More formally, this correspondence $(x; r, t) \mapsto r\Pi_x + t(\text{Id} - \Pi_x)$ defines a mapping $\phi : \mathcal{X} \to \text{Sym}_B(\mathbb{R}^{p+q})$. Note that the mapping $\phi$ is not surjective, as the operators $R$ with more than two non-real eigenvalues cannot be presented in this form. However, the image $\phi(\mathcal{X})$ is large enough to contain our integration domain $D$ and also (in the measure-theoretic sense) the boundary $\partial D$.

Now for any point $x$ of the Grassmann manifold $\text{Gr}_{1,1}(\mathbb{R}^{p+q})$ let $L_x^\pm$ denote the two null lines of the Lorentzian vector space $V_x$. In other words $B(u, u) = 0$ for $u \in L_x^+$ or $u \in L_x^-$. The three-dimensional space $\text{Sym}_B(V_x)$ then decomposes as

$$
\text{Sym}_B(V_x) = \mathbb{R}\text{Id}_{V_x} \oplus \text{Hom}(L_x^+, L_x^-) \oplus \text{Hom}(L_x^-, L_x^+) \quad (5.2)
$$

Let us choose a basis $e_x^\pm \in L_x^\pm$ of null vectors of $V_x$ so that $se_x^\pm = e_x^\mp$ and $B(e_x^+, e_x^-) = 1$. In such a basis $\Pi_x$ is expressed as $\Pi_x = e_x^+ B(e_x^+, \cdot) + e_x^- B(e_x^-, \cdot)$. Linear coordinates $\lambda$, $\xi$, $\eta$ for $\text{Sym}_B(V_x)$ are then introduced by decomposing $r$ according to (5.2):

$$
r = \lambda \Pi_x + \xi e_x^+ B(e_x^+, \cdot) + \eta e_x^- B(e_x^-, \cdot).
$$

Extending the notation used before, let $E_x \in \text{Sym}_B(V_x)$ and $F_x \subset \text{Sym}_B(V_x)$ denote the linear subspaces which are defined by the equations $\xi = 0$ and $\eta = 0$ respectively. These 2-planes $E_x$ and $F_x$ will assume the roles played by $E$ and $F$ in Sect. 4.2. It should be emphasized, however, that $\xi$ and $\eta$ do not extend to globally defined coordinate functions for $\mathcal{X}$. The point here is that transporting the pair of null lines $L_x^\pm$ around a closed path in $\text{Gr}_{1,1}(\mathbb{R}^{p+q})$ one may find that $L_x^+$ and $L_x^-$ get interchanged. In other words, although $L_x^\pm$ are invariantly defined as a pair of lines, there is no invariant way of telling which line is $L_x^+$ and which is $L_x^-$. As a result, the linear coordinate functions $\xi$ and $\eta$ projecting from $\text{Sym}_B(V_x)$ to $\text{Hom}(L_x^-, L_x^+)$ and $\text{Hom}(L_x^+, L_x^-)$ are only defined locally in the variable $x \in \text{Gr}_{1,1}(\mathbb{R}^{p+q})$. By the same token, the union of planes $E_x \cup F_x$ is invariantly defined for all $x$, whereas the planes $E_x$ and $F_x$ individually are not.
The smooth assignment \( \text{Sym}_B(V_\perp) \supset E_\perp \sqcup F_\perp \mapsto x \) gives a bundle \( \mathcal{E} \subset \mathcal{X} \) with fibre \((E_\perp \sqcup F_\perp) \times D(V_\perp^+)\) over \( x \in \text{Gr}_{1,1}(\mathbb{R}^{p+q}) \). By applying the map \( \phi : \mathcal{X} \to \text{Sym}_B(\mathbb{R}^{p+q}) \) we then obtain a submanifold \( \phi(\mathcal{E}) \subset \partial D \). As a matter of fact, since the generic situation at the boundary of \( D = D(\mathbb{R}^{p+q}) \) is that exactly two eigenvalues of opposite type collide, the set \( \phi(\mathcal{E}) \) agrees with the set \( \partial D \) up to lower-dimensional pieces.

5.3. Orientation of boundary. — We now introduce an outer orientation on \( \mathcal{E} \) to achieve the stronger property that \( \phi(\mathcal{E}) \) agrees with the boundary \( \partial D \) as a chain (or distribution). For that, fix any \( x \in \text{Gr}_{1,1}(\mathbb{R}^{p+q}) \) and consider some \((r,t) \in E_\perp \times D(V_\perp^+)\). Arrange the single eigenvalue \( \lambda(r) \) of \( r \) and the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_{p+q-2} \) of \( t \) as a decreasing sequence. (This ordering is uniquely defined except for the set of measure zero where some eigenvalues are degenerate.) Following the prescription of Sect. 5.1 draw the space-like eigenvalues of \( t \) as full dots and the time-like eigenvalues as empty dots. In view of Fig. 2 if \( \eta(r) > 0 \) then represent \( \lambda(r) \) by \( \bullet \circ \), else by \( \circ \bullet \). This defines a motif \( \sigma(r,t) \) for almost every pair \((r,t)\). For \( p = q = 2 \) for example, if \( \lambda_1 > \lambda_2 > \lambda(r) \), \( \lambda_1 \) is space-like, \( \lambda_2 \) is time-like, and \( \eta(r) > 0 \), then \( \sigma(r,t) = \bullet \circ \circ \).

The point \( \phi(x;r,t) \) is a point in the boundary of \( D_{\sigma(r,t)} \subset \partial D \). According to the alternating sign of the sum \( \partial D = \sum \text{sgn}(\sigma) \partial D_\sigma \) the outer orientation of \( \partial D \) at \( \phi(x;r,t) \) is directed from the inside to the outside if \( \text{sgn}(\sigma(r,t)) = +1 \) and from the outside to the inside if \( \text{sgn}(\sigma(r,t)) = -1 \). Consulting again Fig. 2 we assign to the boundary point \( \phi(x;r,t) \) the transverse vector

\[- \text{sign}(\eta(r)) \text{sgn}(\sigma(r,t)) (\phi_\ast)_{(x;r,t)}(\partial_\xi) \, ,\]

where \( (\phi_\ast)_{x;r,t} \) means the differential \( d\phi \equiv \phi_\ast \) of the map \( \phi \) at the point \((x;r,t)\). The motif \( \sigma(r,t) \) switches sign as \( \eta(r) \) passes through zero; therefore our assignment is smooth along the \( \eta \)-axis of \( E_\perp \), although discontinuities may occur when the eigenvalue \( \lambda(r) \) hits eigenvalues of \( t \) or when eigenvalues of \( t \) of opposite type collide.

In the case of the other component, \( F_\perp \times D(V_\perp^+) \), we proceed in the same way, albeit with the two boundary coordinate functions \( \xi \) and \( \eta \) exchanging roles. Thus we assign to a point \( \phi(x;r,t) \) for \((r,t) \in F_\perp \times D(V_\perp^+) \) the transverse vector

\[- \text{sign}(\xi(r)) \text{sgn}(\sigma(r,t)) (\phi_\ast)_{(x;r,t)}(\partial_\eta) \, ,\]

which is again smooth as a function of \( \xi \).

Now while the distinction between \( \xi \) and \( \eta \) (or \( E_\perp \) and \( F_\perp \)) is only defined locally in \( x \), the outcome of our discussion is free of the ambiguity of exchanging \( \xi \leftrightarrow \eta \). Indeed, a glance at Fig. 2 shows that the picture is invariant under the reflection \((\xi \leftrightarrow \eta)\) at the axis dividing the first quadrant. Hence our procedure gives a globally defined piecewise smooth transverse vector field and hence a piecewise smooth outer orientation of the boundary \( \phi(\mathcal{E}) \). Adopting this outer orientation we have

\[ \phi(\mathcal{E}) = \sum_{\sigma} \text{sgn}(\sigma) \partial D_\sigma \]

as an equality between chains, i.e., in the weak sense:

\[ \int_{\phi(\mathcal{E})} \omega = \sum_{\sigma} \text{sgn}(\sigma) \int_{D_\sigma} d\omega \, , \quad (5.3) \]

\[ \text{sgn}(\sigma) \]

\[ \int_{\phi(\mathcal{E})} \omega = \sum_{\sigma} \text{sgn}(\sigma) \int_{D_\sigma} d\omega \, , \quad (5.3) \]
for any bounded twisted differential form \(d\omega\) of top degree. This result holds true, in particular, for the integrand \(\omega = iG_{A,\varepsilon} t_{\tau(A)}|dR|\) of equation (3.4) with the Gaussian
\[
G_{A,\varepsilon}(R) = e^{-\text{Tr}(R+iA)^2} \chi_{\varepsilon}(R+iA)
\]
regularized by the cutoff function \(\chi_{\varepsilon}(R) = e^{-\frac{\varepsilon}{2}\text{Tr}(\varepsilon R-R)^2}\).

### 5.4. Pulling back to \(\mathscr{E}\). —

In the next step the left-hand side of the integral (5.3) for \(\omega = iG_{A,\varepsilon} t_{\tau(A)}|dR|\) is pulled back to \(\mathscr{E} \subset \mathcal{X}\). This results in an iterated integral where the outer integral is over the compact manifold \(\text{Gr}_{1,1}(\mathbb{R}^{p+q}) \supset x\) and the inner integral is over \((E_x \cup F_x) \times D(V_x^\perp)\). It will now be shown that the inner integral over \(E_x \cup F_x\) is of exactly the same form as in the case of \(p = q = 1\) and hence, for the reasons explained in Sect.[4.3] vanishes in the limit \(\varepsilon \to 0\). We will do this by inspection of \(\phi^* \omega\).

First we compute the pull back \(\phi^*(t_{\tau(A)}|dR|)\). Denoting by \((\phi^{-1})_* = (\phi_*)^{-1} \equiv \phi_*^{-1}\) the differential of the inverse map \(\phi^{-1}\), we have
\[
\phi^*(t_{\tau(A)}|dR|) = 1_{\phi_*^{-1}(\tau(A))} \phi^*|dR|.
\]
With this relation in mind we now compute \(\phi^*|dR|\). For this we notice that the map \(\phi : \mathcal{X} \to \text{Sym}_B(\mathbb{R}^{p+q}), (x; r, t) \mapsto r\Pi_x + t(\text{Id} - \Pi_x)\) schematically has the variation
\[
\delta\phi = (\delta r)\Pi_x + (\delta t)(\text{Id} - \Pi_x) + (r - t)\delta\Pi_x.
\]
The first two terms contribute factors of unity to the Jacobian when one pulls back the translation-invariant measure \(|dR|\) of \(\text{Sym}_B(\mathbb{R}^{p+q})\).

Now the third term is purely off-diagonal r.w.t. the decomposition \(\mathbb{R}^{p+q} = V_x \oplus V_x^\perp:\)
\[
\delta\Pi_x \in (\text{Hom}(V_x, V_x^\perp) \oplus \text{Hom}(V_x^\perp, V_x)) \cap \text{Sym}_B(\mathbb{R}^{p+q}),
\]
and may be expressed as a commutator \(\delta\Pi_x = [\alpha, \Pi_x]\) with a generator \(\alpha \in \text{Lie}(\text{O}(p,q))\).

The two components of \(\delta\Pi_x\) in \(\text{Hom}(V_x, V_x^\perp)\) and \(\text{Hom}(V_x^\perp, V_x)\) are related to each other by the condition of \(B\)-symmetry. Identifying \(V_x\) with its dual \(V_x^*\) by means of the non-degenerate form \(B|_{V_x \times V_x}\) we have an isomorphism \(\text{Hom}(V_x, V_x^\perp) \simeq V_x \otimes V_x^\perp\).

By this isomorphism we may view the multiplication operator \(\delta\Pi_x \mapsto (r - t)\delta\Pi_x\) as a linear transformation \(r \otimes \text{Id} - \text{Id} \otimes t\) on \(V_x \otimes V_x^\perp\). We thus arrive at the formula
\[
\phi^*\Pi_x|dR| = J_x(r_x, t_x) dx \, dr_x \, dt_x,
\]
with the Jacobian of the transformation being
\[
J_x(a, b) = \left|\text{Det}(a \otimes \text{Id}_{V_x^\perp} - \text{Id}_{V_x} \otimes b)\right|.
\]
Here \(dx\) denotes a suitably normalized \(\text{SO}(p,q)\)-invariant measure for the compact manifold \(\text{Gr}_{1,1}(\mathbb{R}^{p+q})\), while \(dr_x\) and \(dt_x\) are translation-invariant measures for \(\text{Sym}_B(V_x)\) and \(D(V_x^\perp)\). As an observation to be used presently, we note that the Jacobian
\[
J_x(a, b) = \prod_{k=1}^{p+q-2} |(\lambda(a) - \lambda_k(b))^2 - \xi(a)\eta(a)|
\]
becomes independent of \(\xi\) and \(\eta\) for \(a \in E_x \cup F_x\).
We now have to contract the density $\phi^*|dR|$ with the vector field $\phi^{-1}_x(\tau(\hat{A}))$ and make the restriction to $\mathcal{E}$. Since $d\xi = 0$ on $E_x$ and $d\eta = 0$ on $F_x$, the contraction after restriction depends only on the $\xi$- and $\eta$-components of $\phi^{-1}_x(\tau(\hat{A}))$:

$$\phi^{-1}_x(\tau(\hat{A})) = \hat{A}\xi + \hat{A}\eta + \ldots .$$

While these components vary with $x$, \[ \hat{A}\xi(x) = B(e^-_x, \hat{A}e^-_x), \quad \hat{A}\eta(x) = B(e^+_x, \hat{A}e^+_x), \] they are constant along $E_x$ and $F_x$. Hence, recalling the expression $dr_x = d\lambda d\xi d\eta$ and assembling factors, we see that $\phi^*(t(\tau(A))|dR|$) for fixed $x$ is proportional to the form $\pm d\lambda d\eta$ (with constant coefficient) along $E_x$ and $\pm d\lambda d\xi$ (still with constant coefficient) along $F_x$. In other words, the inner dependence (along $E_x$ and $F_x$) of the integration form $\phi^*(t(\tau(A))|dR|$) is the same as in Sect. 4.3.

### 5.5. Gaussian integrand on boundary.

Our final step is to compute the restriction to $\mathcal{E}$ of the pull back $\phi^*G_{A,\varepsilon}$ of the Gaussian $G_{A,\varepsilon}$. For this we decompose

$$\text{Tr}(R + iA)^2 = \text{Tr}R^2 + 2i\text{Tr}AR - \text{Tr}A^2 =: f_2(R) + f_1(R) + f_0 .$$

Fixing any $x \in \text{Gr}_{1,1}(\mathbb{R}^{p+q})$ and inserting $R = \phi(x; r, t) = r\Pi_x + t(\text{Id} - \Pi_x)$ we obtain

$$\phi^*f_2(x; r, t) = \text{Tr}_{V_x}(r^2) + \text{Tr}_{V^\perp_x}(t^2),$$
$$\phi^*f_1(x; r, t) = 2i\text{Tr}_{V_x}((A\Pi_x)r) + 2i\text{Tr}_{V^\perp_x}((A(\text{Id} - \Pi_x)t).$$

Next we fix $t \in D(V_x^\perp)$ and restrict the range of the variable $r \in \text{Sym}_B(V_x)$ to $E_x \cup F_x$. With this restriction we get

$$\phi^*f_2(x; \cdot, t)|_{E_x \cup F_x} = 2\lambda^2 + \text{const},$$
$$\phi^*f_1(x; \cdot, t)|_{E_x} = 2i\lambda\text{Tr}_{V_x}(A) + 2i\eta B(e^+_x, Ae^+_x) + \text{const},$$
$$\phi^*f_1(x; \cdot, t)|_{F_x} = 2i\lambda\text{Tr}_{V^\perp_x}(A) + 2i\xi B(e^-_x, Ae^+_x) + \text{const} .$$

The coefficients $B(e^+_x, Ae^+_x)$ never vanish. Indeed, from $A = \sum_{a=1}^N \varphi_a B(\varphi_a, \cdot)$ we have

$$B(e^+_x, Ae^+_x) = \sum_{a=1}^N B(\varphi_a, e^+_x)^2 > 0 ,$$

by the positivity assumption $A \geq 0$ and the non-degeneracy of $B$. As a consequence of the fact that $B(e^+_x, Ae^+_x) \neq 0$, it will suffice for our purpose of taking the limit $\varepsilon \rightarrow 0$ to examine the cutoff function $R \mapsto \chi_\varepsilon(R + iA)$ at $A = 0$.

By the relations $se^-_x = e^-_x$ and $se^+_x = e^+_x$ the operator $s$ commutes with the projector $\Pi_x = e^+_x B(e^-_x, \cdot) + e^-_x B(e^+_x, \cdot)$. Therefore the pull back $\phi^*\chi_\varepsilon$ of the cutoff function $R \mapsto e^{-\frac{1}{\varepsilon}2\text{Tr}(sR^- - Rs^-)^2}$ separates into two factors, one for $V_x$ and $V_x^\perp$ each. We again keep $t \in D(V_x^\perp)$ fixed and investigate the dependence on $r \in \text{Sym}_B(V_x)$. Using the easily verified relation $\Pi_x(sr - rs)\Pi_x = (\xi(r) - \eta(r))\Pi_x$, we obtain the expression

$$\phi^*\chi_\varepsilon(x; \cdot, t) = e^{-\varepsilon^2(\xi - \eta)^2 + \text{const}} .$$
We finally conclude that \( \int_{E} \phi \ast \omega \) vanishes for \( \omega = iG_{A, \epsilon} e_{\tau(tA)} dR \) in the limit \( \epsilon \to 0 \). Indeed, we now see from the above expressions that the inner integral over \( E_x \cup F_x \) is

\[
\int_{\mathbb{R}} e^{-2\lambda^2 - 2i\lambda \text{Tr} A} d\lambda \left( \int_{\mathbb{R}} e^{-\epsilon \eta^2 - 2i\eta B(e_x^+, Ae_x^+)} d\eta + \int_{\mathbb{R}} e^{-\epsilon \xi^2 - 2i\xi B(e_x^-, Ae_x^-)} d\xi \right),
\]

and this always goes to zero when the regularization \( \epsilon \) is sent to zero.

This shows that \( \frac{d}{dt} I(A + t\dot{A})|_{t=0} = 0 \) and completes the proof of Thm. 1.

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