Universal moduli spaces of surfaces with flat connections and cobordism theory

Ralph L. Cohen ∗ Soren Galatius † Nitu Kitchloo ‡
Dept. of Mathematics Dept. of Mathematics Dept. of Mathematics
Stanford University Stanford University UC San Diego
Stanford, CA 94305 Stanford, CA 94305 La Jolla, CA 92093
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Abstract

Given a semisimple, compact, connected Lie group $G$ with complexification $G^c$, we show there is a stable range in the homotopy type of the universal moduli space of flat connections on a principal $G$-bundle on a closed Riemann surface, and equivalently, the universal moduli space of semistable holomorphic $G^c$-bundles. The stable range depends on the genus of the surface. We then identify the homology of this moduli space in the stable range in terms of the homology of an explicit infinite loop space. Rationally this says that the stable cohomology of this moduli space is generated by the Mumford-Morita-Miller $\kappa$-classes, and the ring of characteristic classes of principal $G$-bundles, $H^*(BG)$. We then identify the homotopy type of the category of one-manifolds and surface cobordisms, each equipped with a flat $G$-bundle. We also explain how these results may be generalized to arbitrary compact connected Lie groups. Our methods combine the classical techniques of Atiyah and Bott, with the new techniques coming out of Madsen and Weiss’s proof of Mumford’s conjecture on the stable cohomology of the moduli space of Riemann surfaces.

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1 Introduction and statement of results

Let $G$ be a fixed, connected, semisimple, compact Lie group $G$. Our goal is to study the moduli spaces of flat $G$-connections on principal bundles over Riemann surfaces. By allowing the complex structures of the surfaces to vary, we are able to prove a stability theorem in homology, and to study a cobordism category built out of such moduli spaces.

The study of moduli spaces of flat connections, and its connection with holomorphic bundles on Riemann surfaces goes back to the seminal work of Atiyah-Bott [1]. Given a Riemann surface $\Sigma$ without boundary, and a principal $G$-bundle $E$, Atiyah and Bott studied the space of holomorphic structures on the complexification $E^c$. This space admits a canonical (complex-gauge) equivariant stratification that was first described in the work of Harder-Narasimhan [7]. The open-dense stratum for this stratification is also known as the space of semistable complex structures on $E^c$. The codimension of the remaining strata grows linearly in the genus, and hence the semistable stratum approximates the whole space increasingly with genus of the curve.

The space of holomorphic structures may be identified with the space of principal connections on $E$. Atiyah and Bott also study the Yang-Mills functional on this space of all connections. They show that the Yang-Mills functional behaves like a perfect, (gauge) equivariant Morse-Bott function, with critical subspace given by the Yang-Mills connections. The analytical aspects of the Yang-Mills flow were not studied by Atiyah and Bott in [1]. However, the authors do motivate the reason why the Harder-Narasimhan stratification represents the descending strata for the critical level sets of the Yang-Mills functional. In particular, this suggests that the open stratum (identified with semistable complex structures on $E^c$) must equivariantly deform onto the space of minimal Yang-Mills connections on $E$. This minima can be described in terms of central connections (see the final section). In the setting where $G$ is semisimple, these Yang-Mills minima are simply the flat connections.

The Morse theoretic program suggested above was completed by G. Daskalopoulos [3] and J. Råde [16]. In [3] and [16] the authors succeeded in proving the long time convergence of the Yang-Mills flow, thereby rigorously establishing the correspondance between the Yang-Mills moduli spaces, and the moduli spaces of semistable complex structures.

In this paper we consider the moduli space of flat connections on bundles, parametrized over the moduli space of Riemann surfaces of a fixed genus. We call this the universal moduli space of flat
connections. More specifically, we let

\[ \mathcal{M}_g^G = \{ (\Sigma, E, \omega) : \Sigma \text{ is a closed Riemann surface of genus } g, \ E \to \Sigma \text{ is a principal } G \text{-bundle, and } \omega \text{ is a flat connection on } E \}/ \sim. \]  

The relation denoted by \( \sim \) is induced by diagrams of the form,

\[
\begin{array}{cc}
E_1 & \xrightarrow{\tilde{\phi}} & E_2 \\
\downarrow & & \downarrow \\
\Sigma_1 & \xrightarrow{\phi} & \Sigma_2
\end{array}
\]

where \( \tilde{\phi} \) is an isomorphism of \( G \)-bundles living over an orientation preserving diffeomorphism, \( \phi \). Here \( \phi \) takes the complex structure of \( \Sigma_1 \) to the complex structure of \( \Sigma_2 \), and \( \tilde{\phi} \) pulls back the flat connection \( \omega_2 \) to the flat connection \( \omega_1 \).

Strictly speaking, the moduli space \( \mathcal{M}_G^g \) should be viewed as a topological stack. Alternatively, one takes the homotopy orbit space of appropriate group actions defining this equivalence relation. The details of this topology will be described in section 2.1.

Our main theorem about this moduli space is theorem 1 below, which calculates the homology \( H_q(\mathcal{M}_G^g) \) for \( 2q + 4 \leq g \). Before stating it, let us introduce some notation. Let \( L \) denote the canonical line bundle over \( \mathbb{CP}^\infty \), and let \( \mathbb{CP}^\infty_1 = (\mathbb{CP}^\infty)^{-L} \) be the Thom spectrum of the virtual inverse \( -L \) (graded so that the Thom class is in \( H^{-2} \)). Let \( BG_+ \) denote the classifying space \( BG \), with a disjoint basepoint added. Roughly, theorem 1 says that \( \mathcal{M}_G^g \) and the infinite loop space \( \Omega^\infty(\mathbb{CP}^\infty_1 \wedge BG_+) \) have isomorphic homology up to degree \((g - 4)/2\). We need to be precise about connected components, however (the two spaces have non-isomorphic sets of path components).

The topological type of a principal \( G \)-bundle \( E \to \Sigma \) is determined by the homotopy class of a map \( \Sigma \to BG \). Since \( BG \) is simply connected, this in turn is determined by the element \( f_*[\Sigma] \in H_2(BG) = \pi_1(G) \). The correspondence \((\Sigma, E, \omega) \mapsto f_*[\Sigma] \) defines an isomorphism \( \pi_0 \mathcal{M}_G^g \to \pi_1(G) \).

For completeness, we describe \( \pi_0 \Omega^\infty(\mathbb{CP}^\infty_1 \wedge BG_+) \) are of course homotopy equivalent, and we let

\[ \mathcal{M}^G_{g, \gamma} \subseteq \mathcal{M}_G^g \]

denote the corresponding connected component. All components of \( \Omega^\infty(\mathbb{CP}^\infty_1 \wedge BG_+) \) are of course homotopy equivalent, and we let

\[ \Omega^\infty(\mathbb{CP}^\infty_1 \wedge BG_+) \subseteq \Omega^\infty(\mathbb{CP}^\infty_1 \wedge BG_+) \]

denote the component containing the basepoint.

For completeness, we describe \( \pi_0 \Omega^\infty(\mathbb{CP}^\infty_1 \wedge BG_+) \). The collapse maps \( BG_+ \to S^0 \) and \( BG_+ \to BG \) define a homotopy equivalence

\[ \Omega^\infty(\mathbb{CP}^\infty_1 \wedge BG_+) \xrightarrow{\sim} \Omega^\infty(\mathbb{CP}^\infty_1 \wedge BG_+). \]
It is well known (see e.g. [10]) that \(\pi_0 \mathbb{C}P^{-1}_\infty \cong \mathbb{Z}\). The Hurewicz homomorphism defines an isomorphism

\[ \pi_0(\mathbb{C}P^{-1}_\infty \wedge BG) \cong H_2(BG) = \pi_1(G), \]

and we have described an isomorphism \(\pi_0 \Omega^\infty(\mathbb{C}P^{-1}_\infty \wedge BG_+) \cong \mathbb{Z} \times \pi_1(G)\).

**Theorem 1.** 1. Let \(G\) be a connected, compact, semisimple Lie group. Then the homology group \(H_q(M_{g,\gamma}^G)\) is independent of \(g\) and \(\gamma \in \pi_1(G)\), so long as \(2q + 4 \leq g\).

2. For \(q\) in this range,

\[ H_q(M_{g,\gamma}^G) \cong H_q(\Omega^\infty(\mathbb{C}P^{-1}_\infty \wedge BG_+)). \]

**Remarks.** 1. The isomorphism in the theorem is induced by an explicit map \(M_{g,\gamma}^G \to \Omega^\infty(\mathbb{C}P^{-1}_\infty \wedge BG_+)\), defined by a Pontryagin-Thom construction, cf. [11], [6], [2]. The direct definition gives a map into the component labelled \((g - 1, \gamma) \in \mathbb{Z} \times \pi_1(G)\), rather than the base point component.

2. While this description of this stable homology might seem quite complicated, the infinite loop space appearing in this theorem has quite computable cohomology (see [5]). In particular the stable rational cohomology is essentially the free, graded commutative algebra generated by the Miller-Morita-Mumford \(\kappa\)-classes, and the rational cohomology of \(BG\). See Corollary 7 below.

3. Notice that when \(G = \{id\}\), \(M_g^G = M_g\) is the moduli space of Riemann surfaces. In this case, part one of this theorem is the Harer-Ivanov stability theorem [8], [9]. Part 2 of this theorem is the Madsen-Weiss theorem [11], proving the Mumford conjecture.

**Note.** The homology groups in this theorem can be taken with any coefficients. Indeed the theorem is true for any connective generalized homology theory.

We then go on to interpret this theorem in terms of a stability result for the homology of the universal moduli space of semistable holomorphic bundles, and also in terms of the \(\text{Out}(\pi_1(\Sigma_g))\)-equivariant homology of the representation variety, \(\text{Rep}(\pi_1(\Sigma_g), G)\). Here \(\pi_1(\Sigma_g)\) is the fundamental group of a closed, connected, oriented surface \(\Sigma_g\) of genus \(g\), and \(\text{Out}(\pi_1(\Sigma_g))\) is the outer automorphism group. See Theorems [10] and [11] below.

The second main result of the paper regards a cobordism category of surfaces with flat connections. We call this category \(C^F_G\) whose objects are closed, oriented one-manifolds \(S\) equipped with connections on the trivial principal bundle \(S \times G\), and whose morphisms are surface cobordisms \(\Sigma\) between the one-manifold boundary components, equipped with flat \(G\) bundles \(E \to \Sigma\) that restrict on the boundaries in the obvious way. (See section two for a careful definition.) Our result is the identification of the homotopy type of the geometric realization of this category.

**Theorem 2.** There is a homotopy equivalence,

\[ |C^F_G| \simeq \Omega^\infty(\Sigma(\mathbb{C}P^{-1}_\infty \wedge BG_+)). \]
In order to prove this theorem, we will compare the category $C^F_G$ of surfaces with flat connections to the category of surfaces with any connection, $C_G$. That is, this category is defined exactly as was the category $C^F_G$, except that we omit the requirement that the connection $\omega$ on the principal $G$-bundle $E \to \Sigma$ be flat. We will use results of [4] and [13] to prove that the inclusion of cobordism categories $C^F_G \hookrightarrow C_G$ induces a homotopy equivalence on their geometric realizations, and then use the results of [6] to identify the resulting homotopy type.

We have chosen to work with semisimple, compact, connected Lie groups so as to ensure that the minimal Yang-Mills connections are flat. All our theorems and techniques have obvious extensions to arbitrary connected, compact Lie groups without much more effort, (see the final section of this paper).

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2 A stability theorem for the universal moduli space of flat connections

2.1 The main theorem

The goal of this section to prove Theorem 1 as stated in the introduction. We begin by describing the topology of the universal moduli space $M^G_g$ more carefully.

Recall that if $H$ is a group acting on a space $X$, the homotopy orbit space $X//H$ is defined to be the orbit space, $(EH \times X)/H$, where $EH$ is a contractible space with an action of $H$, such that the projection $EH \to E/H$ is a principal $H$-bundle (equivalently: it has local sections). Note that if the action of $H$ on $X$ is free (and $X \to X/H$ is a principal $H$-bundle), then $X//H$ is homotopy equivalent to the geometric orbit space, $X/H$.

Definition 1. a. Let $E$ be a principal $G$-bundle over a fixed closed, smooth, oriented surface $\Sigma_g$ of genus $g$. Let $J(\Sigma_g)$ be the space of (almost) complex structures on $\Sigma_g$, let $A^F(E)$ be the space of flat connections on $E$, and define $N(E) = J(\Sigma_g) \times A^F(E)$.

b. Let $\text{Aut}(E)$ be the group of bundle automorphisms of $E$, living over diffeomorphisms of $\Sigma_g$. That is, an element of $\text{Aut}(E)$ is an isomorphism of principal $G$-bundles,

$$
\begin{array}{ccc}
E & \xrightarrow{\phi} & E \\
\downarrow & & \downarrow \\
\Sigma_g & \xrightarrow{\phi} & \Sigma_g
\end{array}
$$
where $\phi \in \text{Diff}(\Sigma_g)$, the group of orientation preserving diffeomorphisms of $\Sigma_g$.

c. $\text{Aut}(E)$ has a natural action on $\mathcal{N}(E)$. A diffeomorphism transforms one complex structure into another, and a bundle automorphism pulls back a flat connection to a new flat connection. We define $\mathcal{M}_g^G(E)$ to be the homotopy orbit space of this action:

$$\mathcal{M}_g^G(E) = \mathcal{N}_g^G(E)/\text{Aut}_g^G(E).$$

d. We define the universal moduli space of flat connections, $\mathcal{M}_g^G$ to be the disjoint union,

$$\mathcal{M}_g^G = \bigsqcup_{\{E\}} \mathcal{M}_g^G(E)$$

where the union is taken over isomorphism classes of principal $G$-bundles $E \to \Sigma_g$.

**Remarks.**

a. It is not hard to see that $\text{Aut}(E)$ fits into a short exact sequence:

$$1 \to G(E) \to \text{Aut}(E) \to \text{Diff}(\Sigma_g) \to 1$$

where $G(E)$ is the gauge group of smooth bundle automorphisms $\tilde{\phi} : E \to E$ living over the identity of $\Sigma_g$.

b. The space $\mathcal{M}_g^G(E)$ was denoted $\mathcal{M}_g^G,\gamma$ in the introduction. Here $\gamma = f_*[\Sigma] \in H_2(BG) = \pi_1(G)$ for a classifying map $f : \Sigma \to BG$ of the bundle $E \to G$.

c. Instead of defining $\mathcal{M}_g^G(E)$ as the homotopy orbit space of $\text{Aut}_g^G(E)$ acting on $\mathcal{N}_g^G(E)$, we could consider the quotient as a topological stack.

Let $\mathcal{A}(E)$ be the affine space of all connections on the bundle $E$ (no flatness required). Define the configuration space, $\mathcal{B}_g^G(E)$ to be the homotopy orbit space,

$$\mathcal{B}_g^G(E) = (J(\Sigma_g) \times \mathcal{A}(E))/\text{Aut}_g^G(E).$$

(2)

Including $\mathcal{A}_F(E) \hookrightarrow \mathcal{A}(E)$ defines a natural map $j_g : \mathcal{M}_g^G(E) \hookrightarrow \mathcal{B}_g^G(E)$. The following is a straightforward consequence of the works [1, 3, 16] (see also [17], section 3.1).

**Theorem 3.** The map $j_g : \mathcal{M}_g^G(E) \hookrightarrow \mathcal{B}_g^G(E)$ is $2(g-1)r$-connected. Here $g$ is the genus of $\Sigma_g$, and $r$ denotes the smallest number of the form $\frac{1}{2} \dim(G/Q)$, where $Q \subset G$ is any proper compact subgroup of maximal rank. In particular $j_g$ induces an isomorphism in homotopy groups and in homology groups in dimensions less than $2(g-1)r$.

**Proof.** As done by Atiyah-Bott [1], the space of $G$-connections on $E$ can be identified with the space of holomorphic structures on the induced complexified bundle, $E^c = E \times_G G^c$. Moreover in the Atiyah-Bott stratification of the space of holomorphic bundles, one has that the space of flat connections, $\mathcal{A}_F(E)$, is homotopy equivalent to the stratum of semistable holomorphic bundles.
By considering the codimension of the next smallest stratum (in the partial order described in [1]) one knows that the inclusion of the semistable stratum into the entire space of holomorphic bundles is $2(g-1)r$-connected. Translating to the setting of connections, this says that the the inclusion,

$$j : \mathcal{A}_F(E) \hookrightarrow \mathcal{A}(E)$$

is $2(g-1)r$-connected. Since $\mathcal{A}(E)$ is affine, and thus contractible, we can conclude that the space of flat connections, $\mathcal{A}_F(E)$ is $2(g-1)r$-connected (i.e. its homotopy groups vanish through this dimension). Therefore the product $\mathcal{N}(E) = J(\Sigma_g) \times \mathcal{A}_F(E)$ is $2(g-1)r$-connected.

Now consider the following diagram of principal $\text{Aut}^G_{\text{G}}(E)$- fibrations,

$$
\begin{array}{ccc}
J(\Sigma_g) \times \mathcal{A}_F(E) & \longrightarrow & J(\Sigma_g) \times \mathcal{A}(E) \\
\downarrow & & \downarrow \\
\mathcal{M}_g^G(E) & \longrightarrow & \mathcal{B}_g^G(E) \\
\downarrow & & \downarrow \\
\mathcal{B}\text{Aut}(E) & \longrightarrow & \mathcal{B}\text{Aut}(E)
\end{array}
$$

The above discussion implies that the top horizontal arrow induces an isomorphism of homotopy groups through dimension $2(g-1)r$. Applying the five-lemma to the long exact sequences in homotopy groups induced by the two bundles, we get that the middle horizontal arrow, $j_g : \mathcal{M}_g^G(E) \rightarrow \mathcal{B}_g^G(E)$ also induces an isomorphism of homotopy groups in this range.

**Remark 4.** The formula for $r$ may be derived easily from [1, equation 10.7]. By the formula given there, the connectivity of the map $j_g$ is at least $2(g-1)r$, where $r$ denotes the minimum number (over the set of all proper parabolic subgroups of $G^r$) of positive roots of $G$, which are not roots of the parabolic subgroup. This number may be rewritten as we have stated above. We note that if $G$ is the special unitary group $SU(n)$, then the largest parabolic in $SL_n(\mathbb{C})$ is $GL_{n-1}(\mathbb{C})$. Hence the number $r$ is given by $n-1$ in this case.

This theorem states that through a range of dimensions, the universal moduli space $\mathcal{M}_g^G(E)$ has the homotopy type of the classifying space of the automorphism group, $\mathcal{B}\text{Aut}_g^G(E)$. Now observe that this classifying space has the following description.

Let $EG \rightarrow BG$ be a smooth, universal principal $G$-bundle, so that $EG$ is contractible with a free $G$-action. Notice that the mapping space of smooth equivariant maps, $C^\infty_\mathcal{G}(E, EG)$ is also contractible, and has a free action of the group $\mathcal{G}(E)$. The action is pointwise, and clearly has slices. Thus one has a model for the classifying space of this gauge group,

$$B(\mathcal{G}(E)) \simeq C^\infty_\mathcal{G}(E, EG)/\mathcal{G}(E) \cong C^\infty(\Sigma_g, BG)_E,$$

where $C^\infty(\Sigma_g, BG)_E$ denotes the component of the mapping space classifying the isomorphism class of the bundle $E$. 

7
Let $E(\text{Diff}(\Sigma_g)) \to B\text{Diff}(\Sigma_g)$ be a smooth, universal principal $\text{Diff}(\Sigma_g)$-bundle. A nice model for $E(\text{Diff}(\Sigma_g))$ is the space of smooth embeddings, $E(\text{Diff}(\Sigma_g)) = \text{Emb}(\Sigma_g, \mathbb{R}^\infty)$. The product of the action of $\text{Aut}(E)$ on $C_\infty^\infty(E, E \text{G})$ and the action (through $\text{Diff}(\Sigma_g)$) on $E(\text{Diff}(\Sigma_g))$ gives a free action on the product $E(\text{Diff}(\Sigma_g)) \times C_\infty^\infty(E, E \text{G})$. The quotient of this action is a model of the classifying space $B\text{Aut}(E)$. But notice that this quotient is given by the homotopy orbit space,

$$B\text{Aut}(E) \simeq E(\text{Diff}(\Sigma_g)) \times \text{Diff}(\Sigma_g) C_\infty^\infty(\Sigma_g, B\text{G})_E \tag{3}$$

where $\text{Diff}(\Sigma_g)$ acts on $C_\infty^\infty(\Sigma_g, B\text{G})_E$ by precomposition. We therefore have the following corollary to Theorem 3.

**Corollary 5.** There is a natural map

$$\tilde{j}_g : M_g^G(E) \to E(\text{Diff}(\Sigma_g)) \times \text{Diff}(\Sigma_g) C_\infty^\infty(\Sigma_g, B\text{G})_E$$

which is $2(g - 1)r$-connected. By taking the disjoint union over isomorphism classes of $G$-bundles $E$, we then have a map

$$\tilde{j}_g : M_g^G \to E(\text{Diff}(\Sigma_g)) \times \text{Diff}(\Sigma_g) C_\infty^\infty(\Sigma_g, B\text{G})$$

which is $2(g - 1)r$-connected.

We recall from [2] that the space $E(\text{Diff}(\Sigma_g)) \times \text{Diff}(\Sigma_g) C_\infty^\infty(\Sigma_g, X)$ can be viewed as the space of smooth surfaces in the background space $X$ in the following sense. As in [2], define

$$S_g(X) = \{(S_g, f) : \text{where } S_g \subset \mathbb{R}^\infty \text{ is a smooth oriented surface of genus } g \text{ and } f : S_g \to X \text{ is a smooth map.}\}$$

The topology was described carefully in [2], which used the embedding space $\text{Emb}(\Sigma_g, \mathbb{R}^\infty)$ for $E(\text{Diff}(\Sigma_g))$. In particular, $S_g(B\text{G})$ is a model for $E(\text{Diff}(\Sigma_g)) \times \text{Diff}(\Sigma_g) C_\infty^\infty(\Sigma_g, B\text{G})$, and therefore corollary 5 defines a $2(g - 1)r$-connected map $\tilde{j}_g : \bigsqcup_{[E]} M_g^G(E) \xrightarrow{\simeq} S_g(B\text{G})$. Again, $S_g(X)$ need not be connected. The correspondence $(S, f) \mapsto f_*[S]$ defines an isomorphism $\pi_0 S_g(X) \cong H_2(X) = \pi_2(X)$. For $\gamma \in H_2(X)$, we let

$$S_{g,\gamma}(X) \subseteq S_g(X)$$

be the corresponding connected component.

Now in [2], the stable topology of $S_{g,\gamma}(X)$ was studied, for a simply connected space $X$. The following is the main result of [2].

**Theorem 6.** For $X$ simply connected, the homology group $H_q(S_g(X))$ is independent of $g$ and $\gamma$, so long as $2q + 4 \leq g$. For $q$ in this range,

$$H_q(S_{g,\gamma}(X)) \cong H_q(\Omega^\infty_\bullet(\mathbb{CP}^\infty_1 \wedge X_+)).$$
Notice that since $G$ is assumed to be a compact, connected Lie group, $BG$ is simply connected, so we can apply theorem 6.

**Note.** The homology groups in this theorem can be taken with any coefficients. Indeed the theorem is true for any connective generalized homology theory.

We now observe that if let $X = BG$, and put Corollary 5 and Theorem 6 together, Theorem 1 follows.

In [2], the stable rational cohomology of the spaces $S_g(X)$ was described. This then gives the stable rational cohomology of the universal moduli space, $M^G_g$. This stable cohomology is generated by the Miller-Morita-Mumford $\kappa$-classes, and the rational cohomology of $BG$. For the sake of completeness, we state this result more carefully, and give a geometric description of how these generating classes arise.

For a graded vector space $V$ over the rationals, let $V_+$ be positive part of $V$, i.e.

$$V_+ = \bigoplus_{n=1}^{\infty} V_n.$$

Let $A(V_+)$ be the free graded-commutative $\mathbb{Q}$-algebra generated by $V_+$. Given a basis of $V_+$, $A(V_+)$ is the polynomial algebra generated by the even dimensional basis elements, tensor the exterior algebra generated by the odd dimensional basis elements. Let $K$ be the graded vector space $H^*(\mathbb{CP}^{\infty}_1; \mathbb{Q})$. It is generated by one basis element, $\kappa_i$, of dimension $2i$ for each $i \geq -1$. Explicitly, $\kappa_{-1}$ is the Thom class, and $\kappa_i = c_i^{i+1} \kappa_{-1}$, for $c_1 = c_1(L) \in H^2(\mathbb{CP}^{\infty})$. Consider the graded vector space

$$V = H^*(\mathbb{CP}^{\infty}_1 \wedge BG_+; \mathbb{Q}) = K \otimes H^*(BG; \mathbb{Q}).$$

Then $H^*(\Omega^\infty_+ (\mathbb{CP}^{\infty}_1 \wedge BG_+); \mathbb{Q})$ is canonically isomorphic to $A(V_+)$, and we get the following corollary of the stable rational cohomology $H^*(S_{g,\gamma}(X))$ given in [2] and Corollary 5 above.

**Corollary 7.** There is a homomorphism of algebras,

$$\Theta : A((K \otimes H^*(BG; \mathbb{Q}))_+) \rightarrow H^*(M^G_g; \mathbb{Q})$$

which is an isomorphism in dimensions less than or equal to $(g - 4)/2$.

Given an element $\alpha \in H^*(BG; \mathbb{Q})$, we describe the image

$$\Theta(\kappa_i \otimes \alpha) \in H^*(M^G_g; \mathbb{Q}).$$

Consider the universal surface bundle over $M^G_g$:

$$\Sigma_g \rightarrow M^G_g \rightarrow M^G_g.$$
Here $\mathcal{M}_{g,1}^G = \{(\Sigma, E, \omega, x), \text{where } x \in \Sigma\}/\sim$. In other words, a point in $\mathcal{M}_{g,1}^G$ is a point in the universal moduli space $\mathcal{M}_g^G$, together with a marked point in $\Sigma$. The topology of $\mathcal{M}_{g,1}^G$ is defined in the obvious way, so that the projection map $p : \mathcal{M}_{g,1}^G \to \mathcal{M}_g^G$ is a fiber bundle.

Notice that the space $\mathcal{M}_{g,1}^G$ has two canonical bundles over it. The first is the “vertical tangent bundle”, $T_{\text{vert}} \mathcal{M}_{g,1}^G$. This is an oriented, two dimensional vector bundle, whose fiber over $(\Sigma, E, \omega, x)$ is the tangent space $T_x \Sigma$. The second canonical bundle is a principal $G$-bundle, $E_{g,1}^G \to \mathcal{M}_{g,1}^G$, whose fiber over $(\Sigma, E, \omega, x)$ is $E_x$.

View a class $\alpha \in H^*(BG; \mathbb{Q})$ as a characteristic class for $G$-bundles. Then $\alpha(E_{g,1}^G) \in H^*(\mathcal{M}_{g,1}^G; \mathbb{Q})$ is a well defined cohomology class. Similarly, since $T_{\text{vert}} \mathcal{M}_{g,1}^G$ is an oriented, two dimensional bundle, it has a well defined Chern class $c_1 \in H^2(\mathcal{M}_{g,1}^G; \mathbb{Q})$. One then defines $\Theta(\kappa_i \otimes \alpha) \in H^*(\mathcal{M}_{g,1}^G; \mathbb{Q})$ to be the image under integrating along the fiber,

$$\Theta(\kappa_i \otimes \alpha) = \int_{\text{fiber}} c_1^{i+1} \cup \alpha(E_{g,1}^G).$$

**Remarks.**

1. The smoothness of the moduli spaces, $\mathcal{M}_g^G$ and $\mathcal{M}_{g,1}^G$ have not been discussed, so that fiberwise integration has not been justified. However, as described in [2] and [10], the Pontrjagin-Thom construction, which realizes fiberwise integration in the smooth setting, is well defined, and gives the definition of the map $\Theta$ we are using.

2. When $\alpha = 1$, $\Theta(\kappa_i)$ is exactly the Miller-Morita-Mumford class coming from $H^*(\text{BDiff}(\Sigma_g); \mathbb{Q}) = H^*(\mathcal{M}_g; \mathbb{Q})$, the cohomology of the moduli space of Riemann surfaces.

3. Notice that the above formula makes good sense, even when $i = -1$, in that $\Theta(\kappa_{-1} \otimes \alpha) = \int_{\text{fiber}} \alpha(E_{g,1}^G)$.

2.2 Applications to semistable bundles and surface group representations

We now deduce two more direct corollaries of Theorem 1 that stem from the close relationship between the space of flat connections, the space of semistable holomorphic bundles on a Riemann surface, and the space of representations of the fundamental group of the surface.

As above, let $\Sigma_g$ be a fixed closed, oriented, smooth surface of genus $g$, and let $J(\Sigma_g)$ be the space of (almost) complex structures on $\Sigma_g$. Let $E$ be a principal $G$ bundle over $\Sigma_g$, where as before, $G$ is a compact, connected, semisimple Lie group. For a fixed $J \in J(\Sigma_g)$, let $C_{\text{ss}}^J(E) \subset C^J(E)$ be the space of semistable $G^\circ$- holomorphic bundles inside the full affine space of all holomorphic structures on the bundle, $(E, J) \times_G G^\circ$. Define the space

$$C_{\text{ss}}^J(E) = \{(J, B) : J \in J(\Sigma_g), \text{ and } B \in C_{\text{ss}}^J(E)\}. \quad (4)$$

Let $C^g(E)$ be the full space of holomorphic bundles, $C^g(E) = \{(J, B) : J \in J(\Sigma_g), \text{ and } B \in C^J(E)\}$. As before $\text{Aut}_g^G(E)$ acts on $C^g(E)$ with the semistable bundles $C_{\text{ss}}^g(E)$ as an invariant subspace.
Namely, an automorphism $(\tilde{\phi}, \phi)$ of the principal bundle, 

$$
\begin{array}{c}
E \\
\downarrow \\
\Sigma_g \\
\downarrow \\
\Sigma_g
\end{array}
\quad
\xrightarrow{\tilde{\phi}}
\quad
\begin{array}{c}
E \\
\downarrow \\
\Sigma_g \\
\downarrow \\
\Sigma_g
\end{array}
$$

pulls back a holomorphic structure on $(E, J) \times_G G^e$ to a holomorphic structure on the bundle $(E, g^*(J)) \times_G G^e$. We define the universal moduli space of semistable holomorphic structures, and the universal moduli space of all holomorphic structures:

$$
\mathcal{M}^g_{ss}(E) = \mathcal{C}^g_{ss}(E) / \text{Aut}^G(E) \quad \mathcal{D}^g(E) = \mathcal{C}^g(E) / \text{Aut}^G(E)
$$

to be the homotopy orbit spaces. Notice that the space of all holomorphic structures $\mathcal{C}^g_{ss}(E)$ is contractible, so the homotopy orbit space $\mathcal{D}^g(E)$ is a model of the classifying space, $B\text{Aut}^G_g(E) \simeq E(\text{Diff}(\Sigma_g)) \times \text{Diff}(\Sigma_g) C^\infty(\Sigma_g, BG)_E$.

**Remark 8.** To be faithful to the literature, we must define the universal moduli space of semistable holomorphic structures as the homotopy orbit space with respect to the automorphism group that extends $\text{Diff}(\Sigma_g)$ by the complexified Gauge group. However, this will not change the homotopy type of the space.

We now have the following:

**Theorem 9.** The inclusion of the universal moduli space of semistable holomorphic bundles into all holomorphic bundles,

$$
\mathcal{M}^g_{ss}(E) \hookrightarrow \mathcal{D}^g(E) \simeq E(\text{Diff}(\Sigma_g)) \times \text{Diff}(\Sigma_g) C^\infty(\Sigma_g, BG)_E
$$

is a $2(g-1)r$-connected map.

**Proof.** This follows from Theorem 3 together with the equivariant homotopy equivalence between the space of flat connections and Atiyah-Bott’s semistable stratum of the space of holomorphic bundles [1, 3, 10].

We therefore have the following stability theorem for the universal moduli space of semistable holomorphic bundles as a corollary to Theorem 6.

**Theorem 10.** Let $G$ be a connected, compact Lie group. Then the homology group of the universal moduli space of semistable holomorphic bundles, $H_q(\mathcal{M}^g_{ss}(E))$ is independent of $g$ and $E$, so long as $2q + 4 \leq g$. For $q$ in this range,

$$
H_q(\mathcal{M}^g_{ss}(E)) \cong H_q(\Omega^\infty(\mathbb{CP}^{\infty}_+ \wedge BG_+)).
$$
We conclude with an application to the space of representations of the fundamental group. Choose a fixed basepoint \( x_0 \in \Sigma_g \), and let \( \pi = \pi_1(\Sigma_g, x_0) \) be the fundamental group based at that point. Let \( \text{Hom}(\pi, G) \) denote the space of homomorphisms from \( \pi \) to \( G \). We topologize this space as a subspace of the mapping space, \( \text{Map}(\pi, G) \).

Let \( \text{Aut}(\pi) \) denote the the group of homotopy classes of basepoint preserving, orientation preserving, homotopy equivalences of \( \Sigma_g \). As suggested by the notation, we may identify \( \text{Aut}(\pi) \) with the subgroup of automorphisms of the fundamental group that acts by the identity on \( H_2(\pi, \mathbb{Z}) = \mathbb{Z} \).

The group \( \text{Aut}(\pi) \) acts on the space of homomorphisms \( \text{Hom}(\pi, G) \), by precomposition. This action descends to an action of the outer automorphism group, \( \text{Out}(\pi) = \text{Aut}(\pi)/\text{Inn}(\pi) \) on the strict quotient variety, \( \text{Hom}(\pi, G)/G \), where \( G \) acts by conjugation. Here \( \text{Inn}(\pi) \) is the normal subgroup of inner automorphisms. We now study how this action of \( \text{Out}(\pi) \) lifts to an action of the orientation preserving diffeomorphism group, \( \text{Diff}(\Sigma_g) \) on the homotopy quotient space, \( \text{Rep}(\pi, G) = \text{Hom}(\pi, G)/G. \)

Recall that holonomy defines a homeomorphism from the space of based gauge equivalence classes of flat connections to the corresponding component of the space of homomorphisms:

\[
h : \mathcal{A}_F(E)/\mathcal{G}_0(E) \cong \text{Hom}(\pi, G)_E
\]

where \( \mathcal{G}_0(E) \) is the based gauge group (which fixes a fiber pointwise), and acts freely on \( \mathcal{A}_F(E) \) (see [1]). This holonomy map is \( G \)-equivariant, where \( G \) acts as usual on \( \text{Hom}(\pi, G) \) by conjugation, and on the space of flat connections, \( \mathcal{A}_F(E)/\mathcal{G}_0(E) \), it acts by identifying \( G \) as the quotient group \( G = \mathcal{G}(E)/\mathcal{G}_0(E) \), and by using the action of the full gauge group \( \mathcal{G}(E) \) on \( \mathcal{A}_F(E) \). We therefore have a homeomorphism,

\[
\mathcal{A}_F(E)/\mathcal{G}(E) = EG \times_G (\mathcal{A}_F(E)/\mathcal{G}_0(E)) \cong EG \times_G \text{Hom}(\pi, G)_E = \text{Rep}(\pi, G)_E,
\]

where the subscript denotes the representations that correspond to the holonomy map for \( E \). Notice that we can take an alternative model of \( \mathcal{A}_F(E)/\mathcal{G}(E) \) as

\[
\mathcal{A}_F(E)/\mathcal{G}(E) \cong (\mathcal{A}_F(E) \times E(\text{Aut}(E)))/\mathcal{G}(E)
\]

which has a residual action of \( \text{Aut}(E)/\mathcal{G}(E) \cong \text{Diff}(\Sigma_g) \). Thus the representation space, \( \text{Rep}(\pi, G)_E \), is homotopy equivalent to a space with a \( \text{Diff}(\Sigma_g) \) action, and this action clearly lifts the action of \( \text{Out}(\pi) \) on the honest quotient space, \( \mathcal{A}_F(E)/\mathcal{G}(E) \cong \text{Hom}(\pi, G)/G \). Furthermore, for genus \( g \geq 2 \), this diffeomorphism group has contractible path components, and so is homotopy equivalent to its discrete group of path components, the mapping class group, \( \text{Diff}(\Sigma_g) \cong \Gamma_g = \text{Out}(\pi) \). We therefore define the \( \text{Out}(\pi) \)-equivariant homology

\[
H^q_{\text{Out}(\pi)}(\text{Rep}(\pi, G)_E) = H^q_{\text{Diff}(\Sigma_g)}(\mathcal{A}_F(E) \times E(\text{Aut}(E)))/\mathcal{G}(E))
\]
Theorem 11. Let $g \geq 2$. Then the $\text{Out}(\pi)$-equivariant homology of the representation variety is independent of the genus $g$, so long as $2q + 4 \leq g$. For $q$ in this range,

$$H^q_{\text{Out}(\pi)}(\text{Rep}(\pi, G)_E) \cong H_q(\Omega^\infty_*(\mathbb{CP}^{\infty}_1 \wedge BG_+)).$$

Proof. $H^q_{\text{Out}(\pi)}(\text{Rep}(\pi, G)_E) = H^q_{\text{Diff}(\Sigma_g)}(A_F(E)\!/\!/G(E))$, but the latter group is equal to

$$H^q(E\text{Diff}(\Sigma_g) \times \text{Diff}(\Sigma_g) A_F(E)\!/\!/G(E)) = H^q(A_F(E)\!/\!/\text{Aut}_g^G(E)) = H^q(\mathcal{M}_g^G(E)).$$

The result follows by Theorem 6.

3 The cobordism category of surfaces with flat connections

In this section we study the cobordism category $\mathcal{C}_G^F$ of surfaces equipped with flat $G$-bundles. In definition 1 above, we defined moduli spaces $\mathcal{M}_g^G$ of pairs $(\Sigma, E)$ consisting of a closed Riemann surface $\Sigma$ and a flat $G$-bundle $E$ over $\Sigma$. In this section we generalize to Riemann surfaces with boundary. These moduli spaces form morphisms in $\mathcal{C}_G^F$, and gluing along common boundaries define composition of morphisms. Some care is needed to make this well defined (composition must be associative). We then identify the the homotopy type of its classifying space.

We first define the relevant moduli spaces. Let $\Sigma$ be a compact oriented 2-manifold (not necessarily connected, possibly with boundary). Let $J(\Sigma)$ be the space of (almost) complex structures on $\Sigma$. A principal $G$-bundle $E \to \Sigma$ restricts to a principal $G$-bundle $\partial E \to \partial \Sigma$. For a flat connection $\omega$ on $\partial E$, let $A_F(E, \omega)$ denote the space of flat connections on $E$ which restrict to $\omega$ on $\partial E$. Let $\text{Aut}(E; \partial)$ denote the group of automorphisms of $E$, which restrict to the identity on a neighborhood of $\partial E$. Thus, $\text{Aut}(E)$ fits into an exact sequence

$$1 \to G(E; \partial) \to \text{Aut}(E, \partial) \to \text{Diff}(\Sigma) \to 1,$$

where $G(E; \partial)$ is the group of gauge transformations of $E$ which restrict to the identity near the boundary. Let $\mathcal{M}(E, \omega)$ be the homotopy orbit space

$$\mathcal{M}(E, \omega) = (J(\Sigma) \times A_F(E, \omega))\!/\!\text{Aut}(E; \partial)$$

(5)

An imprecise definition of $\mathcal{C}_G^F$ goes as follows.

Definition 12. An object is a triple $x = (S, E, \omega)$, where $S$ is a closed 1-manifold, $E \to S$ is a principal $G$-bundle, and $\omega$ is a connection on $E$. The space of morphisms from $x_0 = (S_0, E_0, \omega_0)$ to $x_1 = (S_1, E_1, \omega_1)$ is the disjoint union

$$\mathcal{C}_G^F(x_0, x_1) = \coprod_E \mathcal{M}(E, \omega),$$

where the disjoint union is over all $E \to \Sigma$ with $\partial E = E_0 \cup E_1$, one $E$ in each diffeomorphism class, and $\omega = \omega_0 \cup \omega_1$. 

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This definition is imprecise because it only defines the homotopy type of the space of morphisms, not the underlying set (the homotopy quotient involved in defining $\mathcal{M}(E)$ involves a choice). We must give a precise, set-level description of the homotopy quotient, and define an associative composition on the point set level. We present a way of doing this. Recall that the definition of homotopy quotient involves the choice of a free, contractible $\text{Aut}(E; \partial)$-space $E(\text{Aut}(E; \partial))$. As constructed in equation \ref{equation:3}, a convenient choice of this space is given by:

$$E \text{Aut}(E; \partial) = \mathbb{R}_+ \times \text{Emb}(\Sigma_g, [0, 1] \times \mathbb{R}^\infty) \times C^\infty_C(E, EG),$$

where $\mathbb{R}_+$ denote the positive real numbers, $\text{Emb}(\Sigma_g, [0, 1] \times \mathbb{R}^\infty)$ denote the space of embeddings, which restricts to embeddings of incoming and outgoing boundaries $S_\nu \to \{\nu\} \times \mathbb{R}^\infty$, $\nu = 0, 1$. $C^\infty_C(E, EG)$ denotes the space of $G$-equivariant smooth maps. Using this space in the definition of the homotopy quotient, we get the following definition of the set of objects and the set of morphisms.

**Definition 13.** A point in the space of objects $\text{Ob}(C^C_G)$, is given by a triple $(S, c, \omega)$, where $S \subset \mathbb{R}^\infty$ is an embedded, closed, oriented one-manifold, $c : S \to BG$ is a smooth map, and $\omega$ is a principal connection on the pullback along $c$ of $EG \to BG$.

A point in the space of morphisms $\text{Mor}(C^C_G)$, is given by the data: $(t, M, i, c, \sigma)$, where $t$ is a positive real number, $M \subset [0, t] \times \mathbb{R}^\infty$ is a 2-dimensional cobordism, $i \in J(\Sigma)$ is a complex structure, and $c : \Sigma \to BG$ is a smooth map. Let $E \to \Sigma$ be the pullback along $c$ of the universal smooth $G$-bundle $EG \to BG$. Finally, $\sigma$ is a flat connection on $E$.

Explicitly, given elements $(j, \tau) \in J(\Sigma) \times \mathcal{A}_F(E; \omega)$, and $(t, \phi, b) \in E \text{Aut}(E; \partial)$, let $M \subseteq [0, t] \times \mathbb{R}^\infty$ be obtained by stretching the first coordinate of the image $\phi(\Sigma) \subseteq [0, 1] \times \mathbb{R}^\infty$, and letting $i, c$, and $\sigma$ be induced from $j, b$, and $\tau$ by the identification $M \cong \Sigma$. This defines an $E \text{Aut}(E; \partial)$-invariant map

$$E \text{Aut}(E; \partial) \times (J(\Sigma) \times \mathcal{A}_F(E; \omega)) \longrightarrow \text{Mor}(C^C_G),$$

(with $C^C_G$ defined as in definition \ref{definition:13} which descends to an injection of $E \text{Aut}(E; \partial)$-orbits

$$\mathcal{M}(E, \omega) \longrightarrow \text{Mor}(C^C_G). \quad (6)$$

Taking disjoint union over $\Sigma$’s and $E$’s, we get an identification of the morphism spaces in definition \ref{definition:12} and those in definition \ref{definition:13}. Moreover, it is now clear how to define an associative composition rule: take union of subsets $M_0 \subseteq [0, t_1] \times \mathbb{R}^\infty$ and $M_1 \subseteq [t_1, t_1 + t_2] \times \mathbb{R}^\infty$. (For this to be a smooth submanifold, we should insist that all cobordism $M \subseteq [0, t] \times \mathbb{R}^\infty$ are “collared” as in \ref{remark:1}. Similarly, $c$ and $\omega$ should be constant in the collar direction. We omit the details.)

**Remark 14.** 1. If all connected components of $\partial \Sigma$ have non-empty boundary, the action of $\text{Aut}(E; \partial)$ on $J(\Sigma) \times \mathcal{A}_F(E; \omega)$ is free, and we can replace the homotopy quotient in \ref{equation:4} with the strict quotient. For a fixed choice of complex structure on $\Sigma$, an explicit description of this strict moduli space, parametrized over all values of $\omega$ is given by \ref{equation:5}:

$$\mathcal{M}(E) = \{ (a, c, \omega) \in G^2 \times G^{n-1} \times \mathcal{A}(\partial E) \mid \prod [a_{2i}, a_{2i-1}] = \prod Ad_{c_j} \text{Hol}(\omega_j) \}$$

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where $c_1 = 1$, and $\text{Hol}(\omega_j)$ denotes the holonomy of the connection $\omega$ about the $j$-th boundary circle over a fixed basepoint. In addition, the composition map in our category may be interpreted as a suitable symplectic reduction:

Let $\Sigma$ be a connected Riemann surface, with non-empty boundary, obtained from a (possibly disconnected) Riemann surface $\hat{\Sigma}$ by gluing along two boundary components $B_\pm \subseteq \partial \hat{\Sigma}$. Let $E$ denote a bundle over $\Sigma$ obtained by identifying a bundle $\hat{E}$ on $\hat{\Sigma}$ along $\partial \hat{E}$, then one may identify the strict moduli space $\mathcal{M}(E)$ with the symplectic reduction of the gauge group $\mathcal{G}(G \times B)$ acting on $\mathcal{M}(\hat{E})$. Here $B$ is the one-manifold that both components $B_\pm$ are identified with, and $\mathcal{G}(G \times B)$ is identified with the gauge group of the trivial bundle over $B$. The moment map for the $\mathcal{G}(G \times B)$-action along which the symplectic reduction is carried out is given by $\omega \mapsto \omega_+ - \omega_-$, where $\omega_\pm$ denote the restrictions of the connection $\omega \in \mathcal{M}(\hat{E})$ to the boundary components $B_\pm$.

2. Forgetting the bundle $E$ gives a functor $\mathcal{C}_F^G \to \mathcal{C}_{SO(2)}$, where $\mathcal{C}_{SO(2)}$ is the cobordism category or oriented 2-manifolds from [6]. An oriented 2-dimensional cobordism $\Sigma$ defines a morphism in $\mathcal{C}_{SO(2)}$. The inverse image of $\Sigma$ in $\mathcal{C}_G^F$ is the moduli space

$$\mathcal{A}_F(E) / \mathcal{G}(E; \partial)$$

of equivalence classes of flat connections on $E$ under the action of the gauge group $\mathcal{G}(E; \partial)$ of gauge transformations of $E$ relative to the boundary. When $E$ is trivialized, this group is identified with the group of smooth maps $(\Sigma, \partial \Sigma) \to (G, e)$.

The goal of this section is to identify the homotopy type of the geometric realization of this category $|\mathcal{C}_G^F|$. (Strictly speaking this is the geometric realization of the simplicial nerve of the category, sometimes known as the classifying space of the category.) More specifically, our goal is to prove Theorem 2 as stated in the introduction.

In order to prove this theorem, we will compare the category $\mathcal{C}_G^F$ of surfaces with flat connections to the category of surfaces with any connection. Namely, let $\mathcal{C}_G$ be the category defined exactly as was the category $\mathcal{C}_G^F$, except that we omit the requirement that the principal connection $\omega$ be flat.

The inclusion of flat connections into all connections defines a functor

$$\iota : \mathcal{C}_G^F \hookrightarrow \mathcal{C}_G.$$

We will observe that this inclusion restricts to the “positive boundary subcategories” defined as follows. Let $\mathcal{C}$ represent either of the cobordism categories, $\mathcal{C}_G^F$ or $\mathcal{C}_G$. Let $\mathcal{C}_+$ denote the subcategory that has the same objects as $\mathcal{C}$, but the morphisms of $\mathcal{C}_+$ are those morphisms of $\mathcal{C}$ that involve surfaces, each path component of which has a non-empty “outgoing” boundary. (The “outgoing boundary” of a surface $\Sigma \subset \mathbb{R}^\infty \times [0, t]$ is $\Sigma \cap (\mathbb{R}^\infty \times \{t\})$.) An important step in proving Theorem 2 is the following.
Proposition 15. The inclusion functor \( \iota : (C^+_G) \hookrightarrow (C^+_F) \) induces a homotopy equivalence of geometric realizations,
\[
\iota : |(C^+_G)| \xrightarrow{\simeq} |(C^+_F)|.
\]

This proposition will allow us to identify the homotopy type of \(|C^+_G|\), because we will be able to identify \(|(C^+_G)|\) with the geometric realization of a cobordism category studied in [6]. In that paper, the authors identified the homotopy type of a broad range of topological cobordism categories. We will be interested in a particular such category we call \(C^+_2(BG)\), that is defined as follows.

Definition 16. The space of objects \(\text{Ob}(C^+_2(BG))\) is given by pairs \((S, c)\), where \(S \subset \mathbb{R}^\infty\) is an embedded, closed, oriented one-manifold, and \(c : S \to BG\) is a continuous map.

The space of morphisms \(\text{Mor}(C^+_2(BG))\), is given by triples, \((t, \Sigma, c)\), where \(t\) is a nonnegative real number, \(\Sigma \subset [0, t] \times \mathbb{R}^\infty\) is an oriented cobordism, and \(c : \Sigma \to BG\) is a continuous map. The embedded surface is collared at the boundaries as in [6]. In particular, \(\Sigma_0 = \Sigma \cap (\mathbb{R}^\infty \times \{0\})\) and \(\Sigma_t = \Sigma \cap (\mathbb{R}^\infty \times \{t\})\) are smoothly embedded, oriented one-manifolds. Again, morphisms are assumed to be collared, and composition is defined by gluing of cobordisms and maps. (See [6] for details.)

The homotopy type of the geometric realization \(|C^+_2(BG)|\) was determined in [6]. Namely, the following was proved there.

Theorem 17. ([6]) a. Let \((C^+_2(BG))_+\) denote the positive boundary subcategory. Then the inclusion functor, \((C^+_2(BG))_+ \hookrightarrow C^+_2(BG)\) induces a homotopy equivalence of geometric realizations,
\[
|(C^+_2(BG))_+| \xrightarrow{\simeq} |C^+_2(BG)|.
\]

b. There is a homotopy equivalence,
\[
|C^+_2(BG)| \simeq \Omega^\infty(\Sigma(\mathbb{C} \mathbb{P}^\infty_1 \wedge BG_+)).
\]

Because of this theorem, Theorem 2 will follow from Proposition 15 and the following two results.

Proposition 18. The functor \(C^+_G \to C^+_2(BG)\) which on the level of morphisms is given by \((t, \Sigma, j, c, \omega) \mapsto (t, \Sigma, c)\) induces homotopy equivalences of geometric realizations
\[
|C_G| \simeq |C^+_2(BG)|
\]
\[
|(C^+_G)_+| \simeq |(C^+_2(BG))_+|.
\]

Proposition 19. The inclusion of the positive boundary subcategory, \((C^+_F)_+ \hookrightarrow C^+_F(G)\) induces a homotopy equivalence of geometric realizations,
\[
|(C^+_F)_+| \simeq |C^+_F|.
\]
3.1 The positive boundary subcategories

In this subsection we prove Propositions 15 and 18. In view of Theorem 17, this will imply that there is an equivalence of the geometric realization of the positive boundary subcategory,

\[ |(\mathcal{C}_F^G)_+| \simeq \Omega^\infty(\Sigma(CP_{\infty}^\infty \wedge BG_+)). \]  

Thus the proof of Theorem 2 would then be completed once we prove Proposition 19 which we will do in the next subsection.

Proof of Proposition 15. Morphisms in both categories are given by tuples \((t, \Sigma, j, c, \omega)\), where \(\omega\) is a connection on a principal \(G\) bundle \(E \to \Sigma\). The only difference between the two categories is that in one of them, \(\omega\) is required to be flat. Let \((t, \Sigma, j, c)\) be fixed. We prove that under the “positive boundary” assumption on \(\Sigma\), the inclusion of flat connections into all connections, \(\mathcal{A}_F(E) \to \mathcal{A}(E)\), is a homotopy equivalence.

The “positive boundary” assumption implies that no connected component of \(\Sigma\) is a closed 2-manifold. Hence \(\Sigma\) deformation retracts onto its 1-skeleton \(X \subseteq \Sigma\). Choose a 1-parameter family \(\phi_t : \Sigma \to \Sigma\) of smooth maps which start at the identity, and such that \(\phi_1\) retracts \(\Sigma\) onto its 1-skeleton. We can lift this family to a 1-parameter family \(\Phi_t : E \to E\) of maps of principal \(G\)-bundles with \(\Phi_0\) the identity. Then we can let \(\omega_t\) be the connection on \(E\) obtained by pullback along \(\Phi_t\). The curvature of \(\omega_t\) can be computed by naturality:

\[ F_{\omega_t} = (\phi_t)^*(F_{\omega}), \]

and therefore \(\omega_1\) is flat, because \(\phi_1\) has one-dimensional image and the curvature is a two-form. Thus the identity map of \(\mathcal{A}(E)\) is homotopic to a map into \(\mathcal{A}_F(E)\), and since \(\mathcal{A}(E)\) is contractible, \(\mathcal{A}_F(E)\) is contractible too.

We have proved that the functor \((\mathcal{C}_G^F)_+ \to (\mathcal{C}_G)_+\) induces a homotopy equivalence on morphisms or, in other words,

\[ N_1(\mathcal{C}_G^F)_+ \to N_1(\mathcal{C}_G)_+. \]

For \(k \geq 2\), the argument is similar: An element in \(N_k(\mathcal{C}_G)_+\) is given by \((\Sigma, j, c, \omega)\) as before, together with a \(k\)-tuple \((t_1, \ldots, t_k)\) of positive real numbers. Here \(\Sigma \subseteq [0, t] \times \mathbb{R}^\infty\) with \(t = t_1 + \cdots + t_k\). The same procedure as for \(k = 1\) gives a path from \(\omega\) to a flat connection. Therefore the functor induces homotopy equivalences on \(k\)-nerves for all \(k\), and hence on the geometric realization.

**Remark 20.** Notice that the above proof holds for any Lie group. In particular, it shows that the space of flat connections is gauge equivariantly contractible, for any principal bundle over a connected Riemann surface with non-empty boundary.
We now go about proving Proposition 18.

**Proof of Proposition 18.** The map in the proposition is induced by the functor $(t, \Sigma, j, c, \omega) \mapsto (t, \Sigma, c)$ which forgets the complex structure $j$ on the oriented surface $\Sigma$, and forgets the connection $\omega$ on the principal $G$-bundle $E \to \Sigma$. Thus the functor gives a fibration

$$N_1C_G \to N_1C_2(BG)$$

whose fiber over $(t, \Sigma, c)$ is the space

$$N(\Sigma, c) = J(\Sigma) \times A(E),$$

where $J(\Sigma)$ is the space of (almost) complex structures on the oriented surface $\Sigma$ and $A(E)$ is the space of connections on $E$. But both spaces are contractible, so (8) is a homotopy equivalence. The higher levels of the simplicial nerve are completely similar, and we get a homotopy equivalence of geometric realizations.

**Proof of Proposition 19.**

In the case where $G$ is the trivial group (in other words, omit the flat $G$-bundle $E \to X$ from definition 12), we recover the cobordism category $C_d$ from [6], when $d = 2$. The positive boundary subcategory is denoted $C^0_d$. In [6] Section 6], it is proved that the inclusion $|C^0_d| \to |C_d|$ is a weak homotopy equivalence when $d \geq 2$. (In [6] the notation $BC$ was used to denote the geometric realization of the nerve of a category $C$, rather than $|C|$.) The proof of proposition 19 will follow [6, Section 6] very closely. We first recall an outline of that argument. Let $C = C_{\{e\}}$, the cobordism category in the case $G$ is the trivial group.

In [6] a functor $D$ from smooth manifolds to sets was defined, where $D(X)$ is the set of smooth manifolds $W \subseteq X \times \mathbb{R} \times \mathbb{R}^\infty$ such that the projection $(\pi, f) : W \to X \times \mathbb{R}$ is proper, and the projection $\pi : W \to X$ is a submersion with 2-dimensional fibers. A concordance is an element $W \in D(X \times \mathbb{R})$; in that case the restrictions to $X \times \{0\}$ and $X \times \{1\}$ are called concordant. This is an equivalence relation on $D(X)$, and the set of equivalence classes is denoted $D[X]$. The equivalence $|C| \to |C^0|$ was proved in [6] by proving two natural isomorphisms:

$$D[X] \cong [X, |C|]$$
$$D[X] \cong [X, |C^0|]$$

The first is proved as follows (again, in outline). Given an element $W \in D(X)$ and a point $x \in X$, let $W_x \subseteq \mathbb{R} \times \mathbb{R}^\infty$ be the $d$-manifold $W_x = \pi^{-1}(x)$ and let $f_x : W_x \to \mathbb{R}$ denote the projection to the first factor. A choice of regular value $a \in \mathbb{R}$ for $f_x$ defines an object $(f_x)^{-1}(a)$ of $C$, and if $a_0 < a_1$ are both regular values, then $(f_x)^{-1}([a_0, a_1])$ is a morphism in $C$ between the two corresponding objects. This is used to define a map from left to right in (9) which is an isomorphism.

To construct (10), we need to ensure that only morphisms satisfying the positive boundary condition arise as $f^{-1}([a_0, a_1])$. At the heart of this is [6] lemma 6.2], which constructs a continuous
family of pairs $(K_t, f_t)$, $t \in \mathbb{R}$ consisting of a $d$-manifold $K_t$ containing the open subset $U = \mathbb{R}^d - D^d \subseteq K_t$, and a smooth function $f_t : K_t \to \mathbb{R}$ which is constant on $U$ and proper when restricted to $K_t - U$. Furthermore $K_0 = \mathbb{R}^d$ and $f_0$ is constant, and $K_1 = \mathbb{R}^d - \{0\}$ and $f_1(x)$ goes to infinity as $x \to 0$.

Now let $W \in D(X)$ and let $a_0 < a_1$ be regular values of $f_x : W_x \to \mathbb{R}$. If we are lucky, $f_x^{-1}([a_0, a_1])$ already satisfies the positive boundary condition. If not, let $Q \subseteq f_x^{-1}([a_0, a_1])$ be a connected component not touching $f_x^{-1}(a_1)$, and let $e : \mathbb{R}^d \to Q$ be an embedding. Gluing in the family $(K_t, f_t)$, we get a one-parameter family of pairs $(W_t, f_t)$. Repeat this procedure for each such component $Q$, and we get a one-parameter family $(W_t, f_t)$ starting at $(W_x, f_x)$ at time 0, and ending at some other pair $(W_t', f_t')$ at time 1, for which $(f_t')^{-1}([a_0, a_1])$ satisfies the positive boundary condition. The rest of the proof in [6, section 6] describes how this construction and a (somewhat complicated) gluing procedure can be used to construct the map in (10).

For the purposes of proving Proposition [19] we need to construct a version of the “standard” one-parameter family $(K_t, f_t)$ where $K_t$ is equipped with a flat $G$-bundle $E_t \to K_t$, which is specified over $K_0$. Fortunately, this is not hard. Namely, the proof of [6, lemma 6.2] also constructs an immersion $j_t : K_t \to K_0$ which is the identity for $t = 0$. Then any given any flat $G$-bundle $E_0 \to K_0$, it has a canonical extension to a flat $G$-bundle $E_t = (j_t)^*(E_0)$ over $K_t$. With this extension of [6, lemma 6.2] in place, the rest of the proof in [6, section 6] applies verbatim if we add flat $G$-bundles to every surface in sight.

This completes the proof of Theorem [17].

### 3.2 The universal moduli space and the loop space of the cobordism category

We connect our two main theorems (Theorem [1] and Theorem [2]). First note that the empty set $\emptyset$ is one-manifold, and thereby an object in $C^F_G$. In the case $\partial \Sigma = \emptyset$, the definition of $\mathcal{M}(E; \omega)$ in [5] agrees with our previous definition of $\mathcal{M}^G_D(E)$, and as in [10] above, we get a map

$$\mathcal{M}^G_D(E) \to C^F_G(\emptyset, \emptyset),$$

which identifies $\mathcal{M}^G_D(E)$ as the (open and closed) subset of $C^F_G(\emptyset, \emptyset)$ where the topological type of $M$ is fixed to be that of a connected genus $g$ surface, and the homotopy class of $e : M \to BG$ is fixed to be that of a classifying map for $E$. In particular, an element of $\mathcal{M}^G_D$ defines a loop in the classifying space $|C^F_G|$ that start and end at the vertex $\emptyset \in |C^F_G|$.

**Corollary 21.** The induced map to the loop space of the cobordism category,

$$i_g : \mathcal{M}^G_D(E) \to \Omega |C^F_G|$$

induces an isomorphism in homology, $H_q(\mathcal{M}^G_D(E)) \xrightarrow{\sim} H_q(\Omega \cdot |C^F_G|)$ for $2q \leq 2g - 4$. 

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Proof. Consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{M}_g^G & \xrightarrow{\iota_g} & \Omega|\mathcal{C}_G^F| \\
\tilde{j}_g & \Downarrow & j \\
S_g(BG) & \xrightarrow{\iota_g} & \Omega|\mathcal{C}_2(BG)|
\end{array}
\]

where \(\mathcal{C}_2(BG)\) is the cobordism category of oriented surfaces in the background space \(BG\), as described in section one. By construction, this diagram commutes.

We proved above that the map \(j : |\mathcal{C}_G^F| \to |\mathcal{C}_2(BG)|\) is an equivalence. So the right hand vertical map in this diagram is an equivalence. By the work of [2] one knows that the bottom horizontal map \(\iota_g : S_g(BG) \to \Omega|\mathcal{C}_2(BG)|\) induces an isomorphism in homology through dimension \(2g - 2\). By Corollary \(5\), the left hand vertical map \(\tilde{j}_g : \mathcal{M}_g^G \to S_g(BG)\) is \(2(g - 1)r\)-connected. The result now follows. \(\square\)

4 The case of a general compact connected Lie group

We may extend the above constructions to an arbitrary compact connected Lie group \(G\). This involves little more than a cosmetic change in definitions. Let \(\mathfrak{g}\) be the Lie algebra of \(G\), with center \(\mathfrak{z} \subseteq \mathfrak{g}\). Given a Riemann surface \(\Sigma\) with a fixed metric, we define say that a connection \(\omega\) on a bundle \(E\) over \(\Sigma\) is central if the curvature of \(\omega\) is a constant multiple of the volume form, with values in \(\mathfrak{z}\). Notice that this makes sense since the Lie algebra \(\mathfrak{z}\) generates a trivial summand in the adjoint bundle \(E \times_G \mathfrak{g}\). As mentioned in the introduction, the Yang-Mills functional on a closed Riemann surface achieves a minimum on the space of central connections. The curvature for any connection in this space is independent of the connection, and is given by a topological invariant of the bundle.

Given a bundle \(E\) with a fixed identification of \(\partial(E)\), notice that the relative gauge group \(\mathcal{G}(E; \partial)\) acts on the space of central connections on \(E\). Let \(\mathcal{M}_G^C(E, \omega)\) denote the universal moduli space of central connections on the bundle \(E\) which restrict to \(\omega\) on the boundary. We may now define the category of central connections \(\mathcal{C}_G^C\):

**Definition 22.** An object of \(\mathcal{C}_G^C\) is a quadruple \(x = (S, \sigma, E, \omega)\), where \(S\) is a closed 1-manifold, endowed with a metric \(\lambda\), \(E \to S\) is a principal \(G\)-bundle, and \(\omega\) is a connection on \(E\). The space of morphisms from \(x_0 = (S_0, \lambda_0, E_0, \omega_0)\) to \(x_1 = (S_1, \lambda_1, E_1, \omega_1)\) is the disjoint union

\[
\mathcal{C}_G^C(x_0, x_1) = \coprod_E \mathcal{M}_G^C(E, \omega),
\]

where the disjoint union is over all \(E \to \Sigma\) with \(\partial E = E_0 \coprod E_1\), one \(E\) in each diffeomorphism class, and \(\omega = \omega_0 \coprod \omega_1\). This morphism space is parametrized over the moduli space of Riemann surfaces with metric (which is equivalent to the one without metrics).

As before, this definition needs to be replaced with a working definition involving the classifying space \(BG\) to make it well defined.
For a closed surface $S$, the results in [1] can be invoked again to show that the space $\mathcal{M}^C(E)$ approximates the classifying space of the group of total automorphisms of $E$ with connectivity given by Theorem [3]. On the other hand if $S$ is a connected surface with non-empty boundary, then it is easy to show as before that the space $\mathcal{M}^C(E, \bullet)$ (where $\omega$ is free) is homotopy equivalent to the classifying space of the total automorphism group of $E$ relative to the boundary.

Using these facts, an easy extension of previous arguments shows that our main theorems [1] [2] and [10] remain valid for a general compact connected Lie group $G$.

Theorem [11] about the equivariant cohomology of the representation variety may also be interpreted in this context. Recall [1], that any central connection on a bundle $E$ over a closed Riemann surface $S$ yields a $G$-representation of the fundamental central extension $\hat{\pi}$ of $\pi_1(S)$ by $\mathbb{R}$. Furthermore, this representation preserves the respective centers.

Let $\text{Rep}^C(\hat{\pi}, G)_E = \text{Hom}^C(\hat{\pi}, G)_E/\!\!/G$ denote the component of the variety of such central representations. It is not hard to see that $\text{Rep}^C(\hat{\pi}, G)_E$ admits an induced action of the group of Symplectomorphism of $S$. For genus $g \geq 2$, this group is homotopy equivalent to $\text{Out}(\pi)$. Theorem [11] is now true for all compact connected Lie groups once we replace $\text{Rep}(\pi, G)_E$ by $\text{Rep}^C(\hat{\pi}, G)_E$. The equivariant cohomology is to be understood with respect to the group $\text{Symp}(S) \simeq \Gamma_g = \text{Out}(\pi)$.

References

[1] M. Atiyah, and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. Lond. A 308, 523-615 (1982)

[2] R.L. Cohen, and I. Madsen, *Surfaces in a background space and the homology of mapping class groups*. preprint, arXiv:math.GT/0601750 (2006)

[3] G. Daskalopoulos, *The topology of the space of stable bundles on a compact Riemann surface*, Journal of Differential Geometry 36, (1992), No. 3, 699-746.

[4] S.K. Donaldson, *Boundary value problems for Yang-Mills fields*, Journal of Geometry and Physics 8, (1992), 89-122.

[5] S. Galatius, *Mod p homology of the stable mapping class group*, Topology 43, (2004), 1105-1132.

[6] S. Galatius, I. Madsen, U. Tillmann, and M. Weiss, *The homotopy type of the cobordism category*, Acta Math., to appear. Available as arXiv:math/0605249

[7] G. Harder, M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles over curves*, Math. Ann. 212, (1975), 215-248.

[8] J.L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces* Ann. Math. 121, (1985), 215-249.
[9] N.V. Ivanov, *On stabilization of the homology of Teichmüller modular groups*, Algebra i Analyz, V. 1 No. 3, (1989), 110-126; **English translation**: Leningrad J. of Math., V. 1, No. 3, (1990), 675-691.

[10] I. Madsen and U. Tillmann, *The stable mapping class group and Q(\mathbb{C}P^\infty)*, Invent. Math., 145 (3), (2001), 409–544.

[11] I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford’s conjecture*, Ann. Math. 165 (2007) 843–941.

[12] E. Meinrenken, C. Woodward *Hamiltonian loop group actions and Verlinde factorization*, J. Diff. Geom. 50, no. 3, 417-469. (1998) [arXiv:dg-ga/9612018]

[13] E. Meinrenken, C. Woodward *Cobordism for Hamiltonian loop group actions and flat connections on the punctured two-sphere*, Math. Zeit. 231, 133-168. (1999) [arXiv:dg-ga/9707018]

[14] J.W. Milnor and J.C. Moore, *On the structure of Hopf algebras*, Annals of Math. 81 (1965), 211-264.

[15] A. Pressley and G. Segal, *Loop Groups*, Oxford Math. Monographs, Clarendon Press (1986).

[16] J. Råde, *On the Yang-Mills heat equation in two and three dimensions*, J. Reine. angew. Math. 431 (1992), 123-163.

[17] C. Teleman and C. Woodward, *Parabolic bundles, products of conjugacy classes, and quantum cohomology*, Ann. Inst. Fourier (Grenoble) (53) (2003), 713-748.