EXISTENCE AND NONEXISTENCE OF SOLUTIONS TO
THE HARDY PARABOLIC EQUATION

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Abstract. In this paper, we obtain necessary conditions and sufficient conditions on the initial data for the local-in-time solvability of the Cauchy problem
\[
\frac{\partial}{\partial t} u + (-\Delta)^{\frac{\theta}{2}} u = |x|^{-\gamma} u^p, \quad x \in \mathbb{R}^N, t > 0, \quad u(0) = \mu \quad \text{in} \quad \mathbb{R}^N,
\]
where \( N \geq 1, \ 0 < \theta \leq 2, \ p > 1, \ \gamma > 0 \) and \( \mu \) is a nonnegative Radon measure on \( \mathbb{R}^N \). Using these conditions, we attempt to identify the optimal strength of the singularity of \( \mu \) for the existence of solutions to this problem.

1. Introduction

Consider nonnegative solutions to the Cauchy problem for the Hardy parabolic equation
\[
\begin{aligned}
\frac{\partial}{\partial t} u + (-\Delta)^{\frac{\theta}{2}} u &= |x|^{-\gamma} u^p, \quad x \in \mathbb{R}^N, \ t > 0, \\
u(0) &= \mu \quad \text{in} \quad \mathbb{R}^N,
\end{aligned}
\]
where \( N \geq 1, \ 0 < \theta \leq 2, \ p > 1, \ \gamma > 0 \) and \( \mu \) is a nonnegative Radon measure in \( \mathbb{R}^N \). Here \((-\Delta)^{\theta/2}\) denotes the fractional power of the Laplace operator \(-\Delta\) in \( \mathbb{R}^N \). If \( \gamma = 0 \), this equation is the Fujita-type equation. Throughout this paper we assume
\[
0 < \gamma < \min\{\theta, N\}.
\]
In the case of \( 0 < \theta < 2 \), we assume the additional condition
\[
\gamma < \theta(p - 1).
\]
In the case of \( \theta = 2 \), \((1.2)\) is not necessary. In this paper, we attempt to give necessary conditions and sufficient conditions for the local-in-time solvability of the Cauchy problem \((1.1)\) and to identify the optimal strength of the singularity of \( \mu \) for the existence of local-in-time solutions to \((1.1)\). The potential term \( |x|^{-\gamma} \) promotes blow-up of solutions to \((1.1)\) at the origin, on the other hand, its effect is hardly noticeable far from the origin. For this reason, far from the origin, the optimal singularity of \( \mu \) is expected to be the same as that of the Fujita-type equation, while at the origin, it is expected to be weaker than that of the Fujita-type equation. However, to our knowledge, there seem to be no results describing this prediction.

We recall the local-in-time solvability of the Fujita-type equation and the optimal singularity of its initial data. Let us consider nonnegative solutions to the semilinear parabolic equation
\[
\frac{\partial}{\partial t} v + (-\Delta)^{\frac{\theta}{2}} v = v^p, \quad x \in \mathbb{R}^N, \ t > 0, \quad v(0) = \nu \quad \text{in} \quad \mathbb{R}^N,
\]
where \( N \geq 1, \ 0 < \theta \leq 2, \ p > 1 \) and \( \nu \) is a nonnegative Radon measure in \( \mathbb{R}^N \). The local-in-time solvability of Cauchy problem \((1.3)\) has been studied in many papers (see e.g. [13, 3]).
Let $G (1.4) \Psi (\theta)$ be the volume of $B (x, r)$ be the fundamental solution to the Cauchy problem (1.1). For $x \in \mathbb{R}^N$ and $r > 0$, let $B (x, r) := \{ y \in \mathbb{R}^N : |x - y| < r \}$ and $|B(x, r)|$ be the volume of $B(x, r)$. Furthermore, for $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, we set

$$\int_{B(x,r)} f(y) \, dy := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy.$$ 

Let $G = G(x, t)$ be the fundamental solution to

$$\partial_t v + (-\Delta)^{\theta \over 2} v = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty),$$

where $0 < \theta \leq 2$. 

(a) Let $0 < T < \infty$. Assume that problem (1.3) possesses a nonnegative solution. Then $\nu$ must satisfy the following:

- If $1 < p < p_0$, then $\sup_{x \in \mathbb{R}^N} \nu(B(x, 1)) < \infty$;

- If $p = p_0$, then $\sup_{x \in \mathbb{R}^N} \nu(B(x, \sigma)) < c_0 \left[ \log \left( e + \frac{T^{\frac{1}{p}}}{\sigma} \right) \right]^{-2 N \over p} \sigma^{N - \frac{p}{2 p - 1}}$ for all $0 < \sigma < T^{1 \over 2}$;

- If $p > p_0$, then $\sup_{x \in \mathbb{R}^N} \nu(B(x, \sigma)) < c_0 \sigma^{N - \frac{p}{2 p - 1}}$ for all $0 < \sigma < T^{1 \over p}$.

Here $p_0 := 1 + \theta / N$ and $c_0$ is a constant depending only on $N$, $\theta$ and $p$.

Then, we can find a large constant $C_* > 0$ with the following property: Set

$$\Psi(x) := \begin{cases} |x|^{-N} \left[ \log \left( e + \frac{1}{|x|} \right) \right]^{-N \over \theta} & \text{if} \quad p = p_0, \\ |x|^{-\theta \over p - 1} & \text{if} \quad p > p_0. \end{cases}$$

(b) Problem (1.3) possesses no local-in-time solutions if $\nu$ is a nonnegative measurable function in $\mathbb{R}^N$ satisfying $\nu(x) \geq c_* \Psi(x)$ in a neighborhood of the origin.

On the other hand, Kozono–Yamazaki [20], Robinson and the second author of this paper [24] and the first author of this paper and Ishige [13] obtained sufficient conditions for the local-in-time solvability. These results proved that there exists a small constant $c_* > 0$ such that if $\nu$ satisfies $0 \leq \nu(x) \leq c_* \Psi(x)$ in $\mathbb{R}^N$, then problem (1.3) possesses a local-in-time solution. By combining these conditions, we see that $\Psi(x)$ is the optimal singularity of $\nu$ for the existence of local-in-time solutions to problem (1.3). We are interested in finding similar necessary conditions and sufficient conditions for the local-in-time solvability of the Cauchy problem (1.1) and identifying the optimal singularity of the initial data.

Now let us return to the Cauchy problem (1.1). This problem has been studied in [2, 4, 5, 7–9, 12, 21, 23, 29, 30]. Among them, in the case of $\theta = 2$, Ben Slimene–Tayachi–Weissler [5] proved that the Cauchy problem (1.1) is locally well-posed in $L^q(\mathbb{R}^N)$ with a suitable $q \geq 1$. Moreover, in the case of $p > 1 + (2 - \gamma) / N$, they proved that if $\mu$ satisfies

$$0 \leq \mu(x) \leq c|x|^{-\frac{2 - \gamma}{N}} \quad \text{in} \quad \mathbb{R}^N,$$

for sufficiently small $c > 0$, then (1.1) possesses a global-in-time solution. However, there seem to be no results on necessary conditions such as assertion (a). Further still, it seems that in the case of $0 < \theta < 2$ no results covering the sufficient conditions are available.

In order to state our main results, we introduce some notation and formulate the definition of solutions to (1.1). For $x \in \mathbb{R}^N$ and $r > 0$, let $B(x, r) := \{ y \in \mathbb{R}^N : |x - y| < r \}$ and $|B(x, r)|$ be the volume of $B(x, r)$. Furthermore, for $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, we set

$$\int_{B(x,r)} f(y) \, dy := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy.$$
**Definition 1.1.** Let \( u \) be a nonnegative measurable function in \( \mathbb{R}^N \times (0, T) \), where \( 0 < T \leq \infty \). We say that \( u \) is a solution to (1.1) in \( \mathbb{R}^N \times [0, T) \) if it satisfies

\[
\infty > u(x, t) = \int_{\mathbb{R}^N} G(x - y, t) \, d\mu(y) + \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s) |y|^{-\gamma} u(y, s)^p \, dy \, ds
\]

for almost all \( x \in \mathbb{R}^N \) and \( 0 < t < T \).

In what follows, define

\[ p_\gamma := 1 + \frac{\theta - \gamma}{N} \]

for \( N \geq 1, 0 < \theta \leq 2 \) and \( \gamma \geq 0 \).

Now we are ready to state the main results of this paper. In Theorem 1.1 and 1.2 we obtain necessary conditions for the local-in-time solvability of the Cauchy problem (1.1).

**Theorem 1.1.** Let \( u \) be a solution to (1.1) in \( \mathbb{R}^N \times (0, T) \), where \( 0 < T < \infty \). Then, there exists a constant \( C_1 > 0 \) depending only on \( N, \theta, p \) and \( \gamma \), such that

\[
\mu(B(0, \sigma)) \leq C_1 \sigma^{\frac{\theta - \gamma}{p - 1}} \left[ \log \left( e + \frac{\sigma^\frac{1}{p}}{T} \right) \right]^{-\frac{N}{p - 1}}
\]

for all \( 0 < \sigma < T^{\frac{1}{\theta}} \). In particular, in the case of \( p = p_\gamma \), there exists a constant \( C'_1 > 0 \) depending only on \( N, \theta \) and \( \gamma \), such that

\[
\mu(B(0, \sigma)) \leq C'_1 \sigma^{\frac{\theta - \gamma}{p - 1}} \left[ \log \left( e + \frac{\sigma^\frac{1}{p}}{T} \right) \right]^{-\frac{N}{p - 1}}
\]

for all \( 0 < \sigma < T^{\frac{1}{\theta}} \).

Since the effect of the singularity \( |x|^{-\gamma} \) is hardly noticeable away from the origin, the solution to (1.1) is expected to behave like that to (1.3). Theorem 1.2 shows a necessary condition for the local-in-time solvability of the Cauchy problem (1.1) far from the origin in the case of \( p = p_0 \) (compare with (a) as above).

**Theorem 1.2.** Let \( z \in \mathbb{R}^N \) and \( p = p_0 \). Let \( u \) be a solution to (1.1) in \( \mathbb{R}^N \times [0, T) \), where \( 0 < T < \infty \). Assume that \( |z| > T^{\frac{1}{\theta}} \). Then, there exists a constant \( C_2 > 0 \) depending only on \( N, \theta \) and \( \gamma \), such that

\[
\mu(B(z, \sigma)) \leq C_2 |z|^{\frac{\theta - \gamma}{p - 1}} \left[ \log \left( e + \frac{T^{\frac{1}{p}}}{\sigma} \right) \right]^{-\frac{N}{p - 1}}
\]

for all \( 0 < \sigma < T^{\frac{1}{\theta}} \).

From Theorems 1.1 and 1.2 we have the following remarks.

**Remark 1.1.** There exists a constant \( C_* > 0 \) with the following property:

(i) Problem (1.1) possesses no local-in-time solution if \( \mu \) is a nonnegative measurable function in \( \mathbb{R}^N \) satisfying

\[
\mu(x) \geq \begin{cases} 
C_* |x|^{-N} \left[ \log \left( e + \frac{1}{|x|} \right) \right]^{-\frac{N}{p-1}} & \text{if } p = p_\gamma, \\
C_* |x|^{\frac{\theta - \gamma}{p - 1}} & \text{if } p > p_\gamma,
\end{cases}
\]

in the neighborhood of the origin.
(ii) Let \( z \in \mathbb{R}^N \setminus \{0\} \). Problem \((1.1)\) possesses no local-in-time solution if \( \mu \) is a nonnegative measurable function in \( \mathbb{R}^N \) satisfying
\[
\mu(x) \geq C_* |z|^\frac{\gamma}{p-1} \Psi(x)
\]
in the neighborhood of \( z \).

**Remark 1.2.** Let \( \mu = \delta_z \) in \( \mathbb{R}^N \), where \( \delta_z \) is the Dirac measure concentrated at \( z \in \mathbb{R}^N \). Then, the following holds:
- If \( p_\gamma \leq p \) and \( z = 0 \), problem \((1.1)\) possesses no local-in-time solution;
- If \( p_0 \leq p \) and \( z \in \mathbb{R}^N \), problem \((1.1)\) possesses no local-in-time solution.

The proof of \((1.7)\) is based on [14, Theorem 1.1] and [19, Proposition 1]. In this proof, we consider the weak solution to \((1.1)\) and give an upper bound for \( \mu \) by substituting a suitable test function. On the other hand, the proofs of \((1.8)\) and Theorem 1.2 are based on [13, Lemma 3.2], which proved a necessary condition in the case of \( \gamma = 0 \). Let \( u \) be a solution (in the sense of Definition 1.1) to \((1.1)\) in \( \mathbb{R}^N \times [0,T) \) and \( z \in \mathbb{R}^N \). Following [13], we employ an iteration argument to get a lower estimate related to
\[
\int_{\mathbb{R}^N} G(x,t) u(x+z,t) \, dx,
\]
and we prove \((1.8)\) and Theorem 1.2. In particular, in the proof of Theorem 1.2, in order to apply an argument used previously for the Fujita-type equation [13], we have to estimate the potential term \( |x|^{-\gamma} \) properly, and assumption \( T^{1/\theta} < |z| \) plays an important role in it. Moreover, it is important to estimate the integral
\[
\int_{\mathbb{R}^N} G(y,t) |y+z|^\frac{\gamma}{p-1} \, dy
\]
from above for \( t > 0 \), and the assumption \((1.2)\) guarantees that this value is finite (see \((2.2)\) below).

We give a sufficient condition for the local-in-time solvability of problem \((1.1)\).

**Theorem 1.3.** Let \( p > p_\gamma \), \( \alpha > 1 \), \( r > 1 \) and \( T > 0 \). Assume that \( r > 1 \) satisfies
\[
(1.9) \quad r \in \left(\frac{N(p-1)}{\theta - \gamma} - \epsilon, \frac{N(p-1)}{\theta - \gamma}\right)
\]
for sufficiently small \( \epsilon > 0 \). Then there exists a constant \( C_3 > 0 \), depending only on \( N, \theta, p, \gamma \) and \( r \), such that, if \( \mu \) is a nonnegative measurable function in \( \mathbb{R}^N \) satisfying
\[
(1.10) \quad \sup_{z \in \mathbb{R}^N} \left( \int_{B(z,\sigma)} \mu(y)^r \, dy \right)^{\frac{1}{r}} \leq C_3 \sigma^{-\frac{\theta - \gamma}{p-1}}
\]
for all \( 0 < \sigma < T^{1/\theta} \), then problem \((1.1)\) possesses a solution in \( \mathbb{R}^N \times [0,T) \).

As a corollary of Theorem 1.3, we have

**Corollary 1.1.** Let \( p > p_\gamma \). There exists a constant \( c_* > 0 \) such that if \( \mu \) satisfies
\[
(1.11) \quad 0 \leq \mu(x) \leq c_* |x|^{-\frac{\theta - \gamma}{p-1}} \quad \text{in} \quad \mathbb{R}^N,
\]
then problem \((1.1)\) possesses a local-in-time solution.
By Corollary [1.11] we see that the singularity of \( \mu \) as in (1.11) is the optimal one at the origin in the case of \( p > p_* \). However, in other cases, the optimal singularity has not been obtained.

The rest of this paper is organized as follows. In Section 2 we collect some properties of the fundamental solution \( G \) and prepare some preliminary lemmas. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2. In Section 5 we prove Theorem 1.3.

2. Preliminaries

In this section, we collect some properties of the fundamental solution \( G \) to (1.5) and prepare preliminary lemmas. In what follows the letter \( C \) denotes a generic positive constant depending only on \( N, \theta, p \) and \( \gamma \).

Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). The fundamental solution \( G \) to (1.5) is a positive and smooth function in \( \mathbb{R}^N \times (0, \infty) \) and has the following properties:

\[
G(x,t) = t^{-\frac{\theta}{p}} G\left(t^{-\frac{1}{p}} x, 1\right),
\]

\[
C^{-1}(1 + |x|)^{-N-\theta} \leq G(x,1) \leq C(1 + |x|)^{-N-\theta} \quad \text{if } 0 < \theta < 2,
\]

\[
G(\cdot,1) \text{ is radially symmetric and } G(x,1) \leq G(y,1) \text{ if } |x| \geq |y|,
\]

\[
G(x,t) = \int_{\mathbb{R}^N} G(x-y,t-s)G(y,s)dy,
\]

\[
\int_{\mathbb{R}^N} G(x,t) \, dx = 1,
\]

for all \( x, y \in \mathbb{R}^N \) and \( 0 < s < t \). For any \( \phi \in L^1_{\text{loc}}(\mathbb{R}^N) \), we identify \( \phi \) with the Radon measure \( \phi \, dx \). For any Radon measure \( \mu \) in \( \mathbb{R}^N \), we define

\[
[S(t)\mu](x) := \int_{\mathbb{R}^N} G(x-y,t) \, d\mu(y), \quad x \in \mathbb{R}^N, t > 0.
\]

Furthermore, we have the following lemmas.

**Lemma 2.1.** There exists a constant \( C > 0 \) such that

\[
\|S(t)\mu\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{\theta}{p}} \sup_{y \in \mathbb{R}^N} \mu(B(y,t^{\frac{1}{p}}))
\]

for any Radon measure \( \mu \) in \( \mathbb{R}^N \) and \( t > 0 \).

**Proof.** This lemma was proved in [13, Lemma 2.1].

**Lemma 2.2.** Let \( \mu \) be a nonnegative Radon measure in \( \mathbb{R}^N \) and \( 0 < T \leq \infty \). Assume that there exists a supersolution \( v \) to (1.1) in \( \mathbb{R}^N \times (0,T) \). Then, there exists a solution to (1.1) in \( \mathbb{R}^N \times [0,T) \).

**Proof.** Set \( u_1 := S(t)\mu \). Define \( u_k (k = 2, 3, \cdots) \) inductively by

\[
u_k(t) := S(t)\mu + \int_0^t S(t-s)|\cdot|^{-\gamma} u_{k-1}(s)^p.
\]

Let \( v \) be a supersolution to (1.1) in \( \mathbb{R}^N \times [0,T) \), where \( 0 < T \leq \infty \). Then, it follows inductively that

\[
0 \leq u_1(x,t) \leq u_2(x,t) \leq \cdots \leq u_k(x,t) \leq \cdots \leq v(x,t) < \infty
\]

for almost all \( x \in \mathbb{R}^N \) and \( t \in (0,T) \). This implies that

\[
u(x,t) := \lim_{k \to \infty} u_k(x,t)
\]
is well-defined for almost all \( x \in \mathbb{R}^N \) and \( t \in (0, T) \). Furthermore, by (2.6), we see that \( u \) satisfies (1.6) for almost all \( x \in \mathbb{R}^N \) and \( t \in (0, T) \). Thus, Lemma 2.2 follows.

\[ \square \]

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. First, we prove (1.7). The proof is based on the argument in [14, Theorem 1.1] and [19, Proposition 1]. Therefore, we note the following remark.

**Remark 3.1.** Let \( 0 < T < \infty \). If \( u \) satisfies (1.6), then \( u \) also satisfies

\[
\int_0^T \int_{\mathbb{R}^N} (u(-\varphi_t + (-\Delta)^{\frac{\theta}{2}} \varphi) - |x|^{-\gamma} u^p \varphi) \, dx \, dt = \int_{\mathbb{R}^N} \varphi(0) \, d\mu
\]

for all \( \varphi \in C_0^\infty(\mathbb{R}^N \times [0, T]) \) with \( \varphi(T) = 0 \).

The proof of (1.7) relies on substituting a suitable test function into (3.1).

**Proof of (1.7).** Let \( u \) be a solution to (1.1) in \( \mathbb{R}^N \times [0, T) \), where \( 0 < T < \infty \). For \( z \in \mathbb{R}^N \) and \( \sigma \in (0, T^{1/\theta}) \), let \( \zeta : \mathbb{R}^N \times [0, \infty) \to [0, 1] \) be a \( C_0^\infty \)-function which satisfies

\[
\zeta(x, t) = \begin{cases} 
1 & \text{in } B(z, 2^{-1/\theta} \sigma) \times [0, 2^{-1} \sigma^\theta], \\
0 & \text{outside } B(z, \sigma) \times [0, \sigma^\theta]. 
\end{cases}
\]

By substituting \( \varphi(x, t) = \zeta(x, t)^s \) as a test function in (3.1), where \( s \) is an integer and satisfies \( s > p/(p-1) \), we get

\[
-s \int_0^\sigma \int_{B(z, \sigma)} u \zeta_t \zeta^{s-1} \, dx \, dt + \int_0^\sigma \int_{\mathbb{R}^N} u(-\Delta)^{\frac{\theta}{2}} \zeta^s \, dx \, dt = \int_0^\sigma \int_{B(z, \sigma)} |x|^{-\gamma} u^p \zeta^s \, dx \, dt + \int_{B(z, \sigma)} \zeta(0)^s \, d\mu.
\]

It follows from (3.2) and the Young inequality that

\[
\begin{align*}
&\int_0^\sigma \int_{B(z, \sigma)} |x|^{-\gamma} u^p \zeta^s \, dx \, dt + \int_{B(z, \sigma)} \zeta(0)^s \, d\mu \\
= &-s \int_0^\sigma \int_{B(z, \sigma)} u \zeta_t \zeta^{s-1} \, dx \, dt + \int_0^\sigma \int_{\mathbb{R}^N} u(-\Delta)^{\frac{\theta}{2}} \zeta^s \, dx \, dt \\
\leq &-s \int_0^\sigma \int_{B(z, \sigma)} u \zeta_t \zeta^{s-1} \, dx \, dt + s \int_0^\sigma \int_{B(z, \sigma)} u \zeta^{s-1}(-\Delta)^{\frac{\theta}{2}} \zeta \, dx \, dt \\
\leq &C \int_0^\sigma \int_{B(z, \sigma)} u |\zeta_t| \zeta^{s-1} \, dx \, dt + C \int_0^\sigma \int_{B(z, \sigma)} u \zeta^{s-1}(-\Delta)^{\frac{\theta}{2}} |\zeta| \, dx \, dt \\
\leq &\int_0^\sigma \int_{B(z, \sigma)} |x|^{-\gamma} u^p \zeta^s \, dx \, dt + C \int_0^\sigma \int_{B(z, \sigma)} |x|^{\frac{\theta}{p-\theta}} |\zeta_t|^{\frac{\theta}{p-\theta}} \zeta^{s-\frac{\theta}{p-\theta}} \, dx \, dt \\
&+ C \int_0^\sigma \int_{B(z, \sigma)} |x|^{\frac{\theta}{p-\theta}} |(-\Delta)^{\frac{\theta}{2}} \zeta|^{\frac{\theta}{p-\theta}} \zeta^{s-\frac{\theta}{p-\theta}} \, dx \, dt.
\end{align*}
\]
Here, we also used the inequality \((-\Delta)^{\theta/2} \zeta \leq s\zeta^{s-1}(-\Delta)^{\theta/2} \zeta\) (see \[18\] for details). Since \(s > p/(p-1)\), we have
\[
\int_{B(z, \sigma)} \zeta(0)^s \, d\mu \leq C \int_0^{\sigma^\theta} \int_{B(z, \sigma)} |x|^{\frac{\gamma}{\sigma^\gamma}} |\zeta|^{p^\gamma} \, dx \, dt + C \int_0^{\sigma^\theta} \int_{B(z, \sigma)} |x|^{\frac{\gamma}{\sigma^\gamma}} \left|(-\Delta)^{\frac{\theta}{p}} \zeta\right|^{p^\gamma} \, dx \, dt.
\]
(3.3)

Now, we choose in \((3.3)\) the function \(\zeta(x, t) = \psi(\sigma^{-\theta} t) \xi(\sigma^{-1} x)\), where \(\psi : [0, \infty) \to [0, 1]\) is a smooth function which satisfies
\[
\psi(t) = 1 \quad \text{on} \quad [0, 2^{-1}], \quad \psi(t) = 0 \quad \text{outside} \quad [0, 1)
\]
and \(\xi : \mathbb{R}^N \to [0, 1]\) is a smooth function which satisfies
\[
\xi(x) = 1 \quad \text{in} \quad B(z, 2^{-\frac{1}{\theta}}), \quad \xi(x) = 0 \quad \text{in} \quad B(z, 1).
\]
Since the functions \(\psi\) and \(\xi\) can be chosen such that
\[
|\partial_t \psi(\sigma^{-\theta} t)| \leq C \sigma^{-\theta} \quad \text{and} \quad |(-\Delta)^{\frac{\theta}{p}} \xi(\sigma^{-1} x)| \leq C \sigma^{-\theta},
\]
(3.4) with \((3.3)\) yields
\[
\mu(B(z, 2^{-\frac{1}{\theta}} \sigma)) \leq C \sigma^{-\frac{\theta}{p-1}} \int_{B(z, \sigma)} |x|^{\frac{\gamma}{p-1}} \, dx
\]
for all \(z \in \mathbb{R}^N\) and \(0 < \sigma < T^{\frac{1}{\theta}}\). Then, we can find a positive constant \(m\) depending only \(N, \theta, p\) and \(\gamma\), such that
\[
\sup_{z \in \mathbb{R}^N} \left( \int_{B(z, \sigma)} |x|^{\frac{\gamma}{p-1}} \, dx \right)^{-1} \mu(B(z, \sigma)) \leq m \sup_{z \in \mathbb{R}^N} \left( \int_{B(z, \sigma)} |x|^{\frac{\gamma}{p-1}} \, dx \right)^{-1} \mu(B(z, 2^{-\frac{1}{\theta}} \sigma)) \leq C m \sigma^{N-\frac{\theta}{p-1}}
\]
for all \(0 < \sigma < T^{\frac{1}{\theta}}\). Therefore, we obtain the desired estimate and the proof of \((1.7)\) is complete. \(\square\)

In order to prove \((1.8)\) and complete the proof of Theorem \((1.1)\) we prepare the following lemma, which has been obtained in \[13, \text{Lemma 3.2}\].

**Lemma 3.1.** Let \(u\) be a solution to \((1.1)\) in \(\mathbb{R}^N \times [0, T)\), where \(0 < T < \infty\). Let \(z \in \mathbb{R}^N\) and \(\rho > 0\) with \((2\rho)^{\theta} < T\). Then, there exists a constant \(c_* > 0\) depending only on \(N\) such that
\[
u(x + z, (2\rho)^{\theta}) \geq c_* G(x, \rho^{\theta}) \mu(B(z, \rho))
\]
for almost all \(x \in \mathbb{R}^N\).

Now we are ready to prove \((1.8)\). The proof is based on the argument in \[13, \text{Lemma 3.3}\].

**Proof of \((1.8)\).** We assume \(p = p_{\gamma}\). Let \(\nu > 0\) be a sufficiently small constant and \(\rho\) be such that
\[
0 < \rho < (\nu T)^{\frac{1}{\theta}}.
\]
Set \(v(x, t) := u(x, t + (2\rho)^{\theta})\) for almost all \(x \in \mathbb{R}^N\) and \(t \in (0, T - (2\rho)^{\theta})\). Since \(u\) is a solution to \((1.1)\) in \(\mathbb{R}^N \times [0, T)\), it follows from \((1.6)\) that for \(0 < \tau < T - (2\rho)^{\theta}\),
\[
v(x, t) = \int_{\mathbb{R}^N} G(x - y, t - \tau) v(y, \tau) \, dy + \int_{\tau}^t \int_{\mathbb{R}^N} G(x - y, t - s) |y|^{-\gamma} v(y, s) \, dy \, ds
\]
(3.6)
holds for almost all $x \in \mathbb{R}^N$, $t \in (\tau, T - (2\rho)^\theta)$. In the case of $0 < \theta < 2$, by (2.1) and (2.2) we have

$$G(x - t, \tau) \geq C\tau^{-\frac{N}{\theta}} \left( 1 + \frac{|x| + |y|}{\tau^{\frac{\theta}{2}}} \right)^{-N-\theta} \geq C\tau^{-\frac{N}{\theta}} \left( 2 + \frac{|y|}{\tau^{\frac{\theta}{2}}} \right)^{-N-\theta} \geq CG(y, \tau)$$

for all $x \in \mathbb{R}^N$ with $|x| < \tau^\frac{\theta}{2}$, $y \in \mathbb{R}^N$ and $\tau > 0$. This time (3.6) with $t = 2\tau$ gives

(3.7) $\int_{\mathbb{R}^N} G(y, \tau)v(y, \tau) \, dy < \infty$

for almost all $\tau \in (0, [T - (2\rho)^\theta]/2)$. On the other hand, in the case of $\theta = 2$, we have

$$G(x - y, 2\tau) \geq (8\pi t)^{-\frac{N}{2}} \exp \left( -\frac{2|x|^2 + 2|y|^2}{8\pi t} \right) \geq C(4\pi t)^{-\frac{N}{2}} \exp \left( -\frac{|y|}{4\tau} \right) = CG(y, \tau)$$

for all $x \in \mathbb{R}^N$ with $|x| < \tau^\frac{1}{2}$, $y \in \mathbb{R}^N$ and $\tau > 0$. Then (3.6) with $t = 3\tau$ yields

(3.8) $\int_{\mathbb{R}^N} G(y, \tau)v(y, \tau) \, dy < \infty$

for almost all $\tau \in (0, [T - (2\rho)^\theta]/3)$. Furthermore, by (2.4), (3.5) and (3.6) with $\tau = 0$ we have

$$v(x, t) - \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s)|y|^{-\gamma}v(y, s)^p \, dyds \geq c_*\mu(B(0, \rho)) \int_{\mathbb{R}^N} G(x - y, t)G(y, \rho^\theta) \, dy$$

(3.9) $= c_*\mu(B(0, \rho))G(x, t + \rho^\theta)$

for almost all $x \in \mathbb{R}^N$ and $0 < t < T - (2\rho)^\theta$, where $c_*$ is the constant in Lemma 3.1. Set

$$w(t) := \int_{\mathbb{R}^N} G(x, t)v(x, t) \, dx.$$

By (3.7) and (3.8), we see that $w(t) < \infty$ for almost all $t \in (0, [T - (2\rho)^\theta]/3)$. Then, it follows from (2.4) and (3.9) that

(3.10) $\infty > w(t) \geq c_*\mu(B(0, \rho)) \int_{\mathbb{R}^N} G(x, t + \rho^\theta)G(x, t) \, dx$

$$+ \int_{\mathbb{R}^N} \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s)G(x, t)|y|^{-\gamma}v(y, s)^p \, dydsdx \geq c_*\mu(B(0, \rho))G(0, 2t + \rho^\theta) + \int_{\rho^\theta}^{2t} \int_{\mathbb{R}^N} G(y, 2t - s)|y|^{-\gamma}v(y, s)^p \, dyds$$

for almost all $\rho^\theta < t < [T - (2\rho)^\theta]/3$. Now it follows from (2.1) and (2.3) that

$$G(y, 2t - s) = (2t - s)^{-\frac{N}{\theta}} G \left( \frac{y}{(2t - s)^{\frac{\theta}{2}}}, 1 \right)$$

(3.11) $\geq \left( \frac{s}{2t} \right)^{-\frac{N}{\theta}} s^{-\frac{N}{\theta}} G \left( \frac{y}{s^{\frac{\theta}{2}}}, 1 \right) = \left( \frac{s}{2t} \right)^{-\frac{N}{\theta}} G(y,s)$
for $y \in \mathbb{R}^N$ and $0 < s < t$. By (2.1), (3.10) and (3.11), we obtain

$$\infty > w(t) \geq c_1 \mu(B(0, \rho)) G(0, 2t + \rho^\theta) + \int_{\rho^\theta}^t \left( \frac{s}{2t} \right)^{\frac{N}{p}} \int_{\mathbb{R}^N} G(y, s) |y|^{-\gamma} v(y, s)^p \, dy \, ds$$

(3.12)

$$\geq C \mu(B(0, \rho)) t^{-\frac{N}{p}} + \int_{\rho^\theta}^t \left( \frac{s}{2t} \right)^{\frac{N}{p}} \int_{\mathbb{R}^N} G(y, s) |y|^{-\gamma} v(y, s)^p \, dy \, ds$$

for almost all $\rho^\theta < t < \lfloor T - (2\rho)^\theta \rfloor / 3$. By the Hölder inequality, we have

$$\int_{\mathbb{R}^N} G(y, s) |y|^{-\gamma} v(y, s)^p \, dy \geq \left( \int_{\mathbb{R}^N} G(y, s) |y|^ {-\frac{\gamma}{p-1}} \, dy \right)^{-(p-1)} w(s)^p.$$  

(3.13)

In the case of $\theta = 2$, by direct computations we obtain

$$\int_{\mathbb{R}^N} G(y, s) |y|^ {-\frac{\gamma}{p-1}} \, dy \leq C s^{\frac{\gamma}{p(p-1)}}.$$

(3.14)

On the other hand, in the case of $0 < \theta < 2$, by virtue of assumption (1.2) we see that

$$\int_{\mathbb{R}^N} G(y, s) |y|^ {-\frac{\gamma}{p-1}} \, dy \leq C s^{\frac{\gamma}{p(p-1)}} \int_{\mathbb{R}^N} (1 + |y|)^{-N-\theta} \, dy$$

(3.15)

$$= C s^{\frac{\gamma}{p(p-1)}} \int_{\mathbb{R}^N} (1 + |y|)^{-N-\theta} |y|^{\frac{\gamma}{p-1}} \, dy$$

$$\leq C s^{\frac{\gamma}{p(p-1)}}.$$

By (3.12), (3.13), (3.14) and (3.15), we get

$$\infty > w(t) \geq c_1 \mu(B(0, \rho)) t^{-\frac{N}{p}} + c_2 t^{-\frac{N}{p}} \int_{\rho^\theta}^t \left( \frac{s}{2t} \right)^{\frac{N}{p}} w(s)^p \, ds$$

(3.16)

for almost all $\rho^\theta < t < \lfloor T - (2\rho)^\theta \rfloor / 3$, where $c_1 > 0$ and $c_2 > 0$ are constants depending only on $N$, $\theta$, $p$ and $\gamma$.

For $k = 1, 2, \cdots$, we define the sequence $\{a_k\}$ inductively as

$$a_1 := c_1, \quad a_{k+1} := c_2 a_k \frac{p-1}{p} \left( \log \frac{a_k}{\rho^\theta} \right)^{\frac{k-1}{p-1}}, \quad k = 1, 2, \cdots.$$  

(3.17)

Furthermore, set

$$f_k(t) := a_k \mu(B(0, \rho)) \left( \log \frac{t}{\rho^\theta} \right)^{\frac{k-1}{p-1}}, \quad k = 1, 2, \cdots.$$  

(3.18)

We claim that

$$w(t) \geq f_k(t), \quad k = 1, 2, \cdots,$$  

(3.19)
for almost all $\rho^0 < t < \lfloor T - (2\rho)^0 \rfloor / 3$. By (3.16), we see that (3.19) holds for $k = 1$. We assume that (3.19) holds with some $k \in \{1, 2, \cdots \}$. Then, due to (3.16), we infer that

\[
\begin{align*}
   w(t) & \geq c_2 t^{-\frac{N}{\sigma}} \int_{\rho^0}^{t} s^{-\frac{N}{\sigma} - \frac{\gamma}{\sigma}} f_k(s)^p \, ds \\
& = c_2 t^{-\frac{N}{\sigma}} \int_{\rho^0}^{t} s^{-\frac{N}{\sigma} - \frac{\gamma}{\sigma}} \left[ a_k \mu(B(0, \rho))^{\mu k-1} s^{-\frac{N}{\sigma}} \left( \log \frac{s}{\rho^0} \right)^{\frac{\mu k-1}{p-1}} \right]^p \, ds \\
& = c_2 t^{-\frac{N}{\sigma}} \int_{\rho^0}^{t} s^{-\frac{N}{\sigma} - \frac{\gamma}{\sigma}} \left[ \mu(B(0, \rho))^{\mu k-1} \left( \log \frac{s}{\rho^0} \right)^{\frac{\mu k-1}{p-1}} \right] \, ds \\
& = c_2 a_k^p \mu(B(0, \rho))^{\mu k} t^{-\frac{N}{\sigma}} \left( \log \frac{t}{\rho^0} \right)^{\frac{\mu k-1}{p-1}} = f_{k+1}(t)
\end{align*}
\]

for almost all $\rho^0 < t < \lfloor T - (2\rho)^0 \rfloor / 3$. Therefore, we conclude that (3.19) holds for all $k = 1, 2, \cdots$.

Next, we claim that there exists a constant $\beta > 0$ such that

\[
   a_k \geq \beta^p, \quad k = 1, 2, \cdots.
\]

Set $b_k := -p^{-k} \log a_k$. We prove that there exists a constant $C > 0$ such that $b_k \leq C$. By (3.17), we see that

\[
   -\log a_{k+1} = -p \log a_k + \log \left[ c_2 \frac{p - 1}{p^k - 1} \right].
\]

This implies that

\[
   b_{k+1} - b_k = p^{-k-1} \log \left[ c_2 \frac{p - 1}{p^k - 1} \right] \leq C p^{-k-1} (k + 1), \quad k = 1, 2, \cdots
\]

for some constant $C > 0$. By (3.21) we see that

\[
   b_{k+1} = b_1 + \sum_{j=1}^{k} (b_{j+1} - b_j) \leq b_1 + C \sum_{j=1}^{k} p^{-j-1} (j + 1) \leq C
\]

for $k = 1, 2, \cdots$. This implies (3.20). Taking a sufficiently small $\nu$ if necessary, by (3.18), (3.19) and (3.20) we see that

\[
   \infty > w(t) \geq f_{k+1}(t) \leq \left[ \beta^p \mu(B(0, \rho)) \left( \log \frac{t}{\rho^0} \right)^{\frac{1}{p-1}} \right]^{\frac{1}{p-1}} t^{-\frac{N}{\sigma}} \left( \log \frac{t}{\rho^0} \right)^{-\frac{1}{p-1}} \left[ \beta^p \mu(B(0, \rho)) \left( \log \frac{T}{\rho^0} \right)^{\frac{1}{p-1}} \right]^{\frac{1}{p-1}} t^{-\frac{N}{\sigma}} \left( \log \frac{t}{\rho^0} \right)^{-\frac{1}{p-1}}, \quad k = 1, 2, \cdots
\]

for almost all $T/5 < t < T/4$. Then it follows that

\[
   \beta^p \mu(B(0, \rho)) \left[ \log \frac{T}{\rho^0} \right]^{\frac{1}{p-1}} \leq 1,
\]
which in turn implies that
\[(3.22) \quad \mu(B(0, \rho)) \leq C \left[ \log \frac{T}{\rho^\theta} \right]^{\frac{1}{p-1}} \leq C \left[ \log \frac{T}{\rho^\theta} \right]^{\frac{1}{p-1}} \leq C \left[ \log \left( e + \frac{T}{\rho^\theta} \right) \right]^{\frac{1}{p-1}} \]
for \(0 < \rho < (\nu T)^{1/\theta}\). By (1.7) there exists a constant \(C_* > 0\) such that
\[
\mu(B(0, \rho)) \leq C_*
\]
for \((\nu T)^{1/\theta} \leq \rho < T^{1/\theta}\). By (1.7) there exists a constant \(C_* > 0\) such that
\[
\mu(B(0, \rho)) \leq C_*
\]
for \((\nu T)^{1/\theta} \leq \rho < T^{1/\theta}\). Since \(p = p_\gamma\), we see that
\[(3.23) \quad \left[ \log \left( e + \frac{T}{\rho^\theta} \right) \right]^{\frac{1}{p-1}} \geq \left[ \log \left( e + \nu^{-\frac{1}{\theta}} \right) \right]^{\frac{1}{p-1}} = C \frac{C^*}{C_*} \mu(B(0, \rho))
\]
for \((\nu T)^{1/\theta} \leq \rho < T^{1/\theta}\). Combining (3.22) and (3.23), we obtain
\[
\mu(B(0, \rho)) \leq C \left[ \log \left( e + \frac{T}{\sigma^\theta} \right) \right]^{\frac{1}{p-1}}
\]
for all \(0 < \sigma < T^{1/\theta}\). Therefore, we obtain the desired result and the proof of Theorem 1.1 is complete.

4. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. The proof is based on the argument found in [13, Lemma 3.3]. The following lemma is the key to the proof.

**Lemma 4.1.** Assume the same conditions as in Theorem 1.2. Then there exists a constant \(C > 0\) depending only on \(N, \theta\) and \(\gamma\) such that
\[(4.1) \quad \int_{R^N} G(y, s) |y + z|^{\frac{p-1}{p}} dy \leq C \rho^{-N} \int_{B(z, \rho)} |y|^{\frac{p-1}{p}} dy
\]
for almost all \(\rho^\theta < s < T/3\).

**Proof.** The proof is divided into two steps.

**1st step.** We prove that
\[
\int_{R^N} G(y, s) |y + z|^{\frac{p-1}{p}} dy \leq C \int_{B(0, \rho^\theta)} G(y, s) |y + z|^{\frac{p-1}{p}} dy
\]
for almost all \(\rho^\theta < s < T/3\). For this purpose, it is sufficient to show that
\[(4.2) \quad \int_{B(0, \rho^\theta)^c} G(y, s) |y + z|^{\frac{p-1}{p}} dy \leq C \int_{B(0, \rho^\theta)^c} G(y, s) |y + z|^{\frac{p-1}{p}} dy
\]
for almost all $\rho^\theta < s < T/3$. First, we give an upper estimate to the integral on $B(0, s^{1/\theta})^c$. By virtue of (1.2), we see that

$$\int_{B(0, s^{1/\theta})^c} G(y, s)|y + z|^{\frac{\gamma}{p-1}} \, dy = \int_{B(0, s^{1/\theta})^c} G(y, s)|y|^{\frac{\gamma}{p-1}} \left(\frac{|y + z|}{|y|}\right)^{\frac{\gamma}{p-1}} \, dy$$

$$\leq \int_{B(0, s^{1/\theta})^c} G(y, s)|y|^{\frac{\gamma}{p-1}} (1 + |z|/|y|)^{\frac{\gamma}{p-1}} \, dy$$

$$\leq C \left(1 + |z|/s^\theta\right)^{\frac{\gamma}{p-1}} \int_{B(0, s^{1/\theta})^c} G(y, s)|y|^{\frac{\gamma}{p-1}} \, dy$$

$$\leq C(|z| + s^\theta)^{\frac{\gamma}{p-1}}$$

for almost all $\rho^\theta < s < T/3$.

Second, we give a lower estimate to the integral on $B(0, s^{1/\theta})$. Since $|z| > s^{1/\theta} > |y|$ for $y \in B(0, s^{1/\theta})$, we see that

$$\int_{B(0, s^{1/\theta})} G(y, s)|y + z|^{\frac{\gamma}{p-1}} \, dy \geq \int_{B(0, s^{1/\theta})} G(y, s)(|z| - |y|)^{\frac{\gamma}{p-1}} \, dy$$

$$\geq (|z| - s^\theta)^{\frac{\gamma}{p-1}} \int_{B(0, s^{1/\theta})} G(y, s) \, dy$$

$$\geq C(|z| - s^\theta)^{\frac{\gamma}{p-1}}$$

for almost all $0 < s < T/3$.

Combining (4.3) and (4.4), we obtain

$$\int_{B(0, s^{1/\theta})} G(y, s)|y + z|^{\frac{\gamma}{p-1}} \, dy \leq C \left(\frac{|z| + s^\theta}{|z| - s^\theta}\right)^{\frac{\gamma}{p-1}} \int_{B(0, s^{1/\theta})} G(y, s)|y + z|^{\frac{\gamma}{p-1}} \, dy$$

(4.5)

for almost all $\rho^\theta < s < T/3$. Since $\rho^\theta < s < T/3$ and $|z| > T^{1/\theta}$, we have

$$\frac{|z| + s^\theta}{|z| - s^\theta} \leq \frac{|z| + (T/3)^{\frac{1}{\theta}}}{|z| - (T/3)^{\frac{1}{\theta}}} \leq \frac{|z| + |z|/3^{\frac{1}{\theta}}}{|z| - |z|/3^{\frac{1}{\theta}}} \leq C.$$

This together with (4.5) yields (4.2).

2nd step. We give an upper estimate to the integral on $B(0, s^{1/\theta})$. By (2.1) and (2.3), we see
we obtain

for almost all

By combining (4.2) and (4.6), we obtain (4.1).

We assume

Proof of Theorem 1.2. We assume \( p = p_0 \). Let \( \nu > 0 \) be a sufficiently small constant. Let \( \rho \) be such that

\[
0 < \rho < (\nu T)^\frac{1}{\gamma}.
\]

Set \( v(x, t) := u(x + z, t + (2\rho)^{\theta}) \) for almost all \( x \in \mathbb{R}^N \) and \( t \in (0, T - (2\rho)^{\theta}). \) Since \( u \) is a solution to (1.1) in \( \mathbb{R}^N \times [0, T) \), it follows from (1.6) that

\[
v(x, t) = \int_{\mathbb{R}^N} G(x - y, t)v(y, 0) \, dy + \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s)|y + z|^{-\gamma}v(y, s)^p \, dyds
\]

holds for almost all \( x \in \mathbb{R}^N \) and \( t \in (0, T - (2\rho)^{\theta}). \). Then, we see that

\[
\int_{\mathbb{R}^N} G(x, t)v(x, t) \, dy < \infty
\]

holds for almost all \( 0 < t < [T - (2\rho)^{\theta}]/3 \). Set

\[
\overline{w}(t) := \int_{\mathbb{R}^N} G(x, t)v(x, t) \, dx
\]

for almost all \( 0 < t < [T - (2\rho)^{\theta}]/3 \). By an argument similar to that used in the proof of (1.8), we obtain

\[
\infty > \overline{w}(t) \geq c_1 \mu(B(z, \rho))t^{-\frac{N}{\theta}}
\]

\[
+ c_2 t^{-\frac{N}{\theta}} \int_{\rho^\theta}^t s^{\frac{N}{\theta}} \left( \int_{\mathbb{R}^N} G(y, s)|y + z|^{\frac{N}{\theta - 1}} \, dy \right)^{-(p - 1)} \overline{w}(s) \, ds
\]

for almost all \( 0 < t < [T - (2\rho)^{\theta}]/3 \).
for almost all $\rho^0 < t < [T - (2\rho)^0]/3$. By applying Lemma 4.1 to (1.7), we have
\[
\infty > \overline{w}(t) \geq c_1 \mu(B(z, \rho)) t^{-N}\frac{w}{\alpha}
+ c_2 t^{-\frac{N}{\alpha}} \left( \rho^{-N} \int_{B(z, \rho)} |y|^{\frac{\alpha}{p-1}} dy \right)^{-(p-1)} \int_{\rho^0}^{t} s^{\gamma} \overline{w}(s)^p ds
\]
for almost all $\rho^0 < t < [T - (2\rho)^0]/3$. Again, by an iteration scheme similar to the one used in the proof of (1.8), we have
\[
\infty > \overline{w}(t) \geq \left[ \beta^p \left( \rho^{-N} \int_{B(z, \rho)} |y|^{\frac{\alpha}{p-1}} dy \right)^{-1} \mu(B(z, \rho)) \left( \log \frac{t}{\rho^0} \right)^{\frac{1}{p-1}} \right]^{p^k}
\times t^{-\frac{N}{\alpha}} \rho^{-N} \int_{B(z, \rho)} |y|^{\frac{\alpha}{p-1}} dy \left( \log \frac{t}{\rho^0} \right)^{-\frac{1}{p-1}}
\]
for almost all $T/5 < t < T/4$, where $\beta > 0$ is a constant. Then it follows that
\[
\beta^p \left( \rho^{-N} \int_{B(z, \rho)} |y|^{\frac{\alpha}{p-1}} dy \right)^{-1} \mu(B(z, \rho)) \left( \log \left( e + \frac{T}{5\rho^0} \right) \right)^{\frac{1}{p-1}} \leq 1
\]
for all $0 < \rho < (\nu T)^{1/\alpha}$. Similarly to the proof of (1.8), we obtain
\[
\left( \int_{B(z, \sigma)} |y|^{\frac{\alpha}{p-1}} dy \right)^{-1} \mu(B(z, \sigma)) \leq C \left[ \log \left( e + \frac{T}{\sigma^\theta} \right) \right]^{\frac{1}{p-1}}
\]
for all $z \in \mathbb{R}^N$ with $|z| > T^{1/\theta}$ and $0 < \sigma < T^{1/\theta}$. This is the desired estimate and the proof of Theorem 1.2 is complete. \qed

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. To simplify the notation, we set $V(x) := |x|^{-\gamma}$.

**Proof of Theorem 1.3** By Lemma 2.2, it is sufficient to construct a supersolution to (1.1). Furthermore, it is sufficient to consider the case of $T = 1$. Indeed, for any solution $u$ to (1.1) in $\mathbb{R}^N \times [0, T)$, where $0 < T < \infty$, we see that $u_\lambda(x, t) := \lambda^{(\theta-\gamma)/(p-1)} u(\lambda^x, \lambda^\theta t)$ with $\lambda = T^{1/\theta}$ is also a solution to (1.1) in $\mathbb{R}^N \times [0, 1)$. Let $\alpha > 1$ be sufficiently close to $N/\gamma$. Set
\[
\rho(t) := t^{1-\frac{\theta-\gamma}{p-1}} \left( \frac{1}{\alpha'} \right)^{\frac{1}{p-1}}
\]
and
\[
W(t) := S(t) \mu + \rho(t)(S(t) \mu^\gamma)^{\frac{1}{\alpha'}}
\]
where $\alpha' = \alpha/(\alpha - 1)$. We will show that $W(t)$ is a supersolution to (1.1) in $\mathbb{R}^N \times [0, 1)$. Since $p > p_\gamma$, $\alpha > 1$ is sufficiently close to $N/\gamma$ and (1.9) holds, we see that
\[
1 - \frac{\theta - \gamma}{\theta} \frac{1}{p-1} \left( p - 1 \right) \left( \frac{1}{\alpha'} \right) > 0
\]
(5.1)
and
\[
1 + \frac{\theta - \gamma}{\theta} p \left( 1 - \frac{1}{p - 1} \left( p - \frac{1}{\alpha'} \right) \right) - \frac{\theta - \gamma}{\theta} \frac{1}{\alpha'} > 0.
\]

Since \( G \) satisfies \((2.4)\) and \((2.5)\), by the Hölder inequality and the Jensen inequality, we have
\[
S(t - s)V(S(s)\mu)^p \leq (S(t - s)V^\alpha)^{\frac{1}{\alpha}} (S(t - s)(S(s)\mu)^{\rho s})^{\frac{1}{p'}}
\]
\[
\leq \|S(t - s)V^\alpha\|_{L^\infty(SN)} \|\|S(s)\mu\|_{L^\infty(SN)}^{\rho s} (S(t)\mu)^{\frac{1}{p'}}
\]
\[
\leq \|S(t - s)V^\alpha\|_{L^\infty(SN)} \|S(s)\mu\|_{L^\infty(SN)}^{\rho s} \|S(t)\mu\|_{L^\infty(SN)}^{\frac{1}{p'}} \|S(t)\mu\|_{L^\infty(SN)}^{\frac{1}{p'}}.
\]

Since \( 1 < \alpha < N/\gamma \), by \((1.10)\) and Lemma \(2.1\) we have
\[
\|S(t - s)V^\alpha\|_{L^\infty(SN)} \leq C(t - s)^{-\frac{\gamma}{s}}
\]
and
\[
\|S(s)\mu\|_{L^\infty(SN)}^{\frac{1}{p'}} \leq CC_3 \frac{\gamma - \alpha}{\alpha} \frac{1}{p - 1}
\]
for almost all \( 0 < s < t \). Then by \((5.3)\), \((5.4)\) and \((5.5)\) we have
\[
S(t - s)V(S(s)\mu)^p \leq CC_3^{\frac{p - 1}{p'}} (t - s)^{-\frac{\gamma}{s} - \frac{\gamma - \alpha}{\alpha} \frac{1}{p - 1}} \|S(t)\mu\|_{L^\infty(SN)}^{\frac{1}{p'}}
\]
for almost all \( 0 < s < t \). By \((5.6)\), the right hand side is integrable with respect to \( s \). Then we obtain
\[
\int_0^t S(t - s)V(S(s)\mu)^p \, ds \leq CC_3^{\frac{p - 1}{p'}} t^{1 - \frac{\gamma}{s} - \frac{\gamma - \alpha}{\alpha} \frac{1}{p - 1}} \|S(t)\mu\|_{L^\infty(SN)}^{\frac{1}{p'}}
\]
\[
= CC_3^{\frac{p - 1}{p'}} \rho(t)(S(t)\mu)^{\frac{1}{p'}}
\]
for almost all \( 0 < t < 1 \). Similarly to \((5.3)\), we see that
\[
S(t - s)V(p(s)(S(s)\mu)^{p})^{\frac{1}{p'}} \leq \rho(s)^p (S(t - s)V^\alpha)^{\frac{1}{\alpha}} (S(t - s)(S(s)\mu)^{\rho s})^{\frac{1}{p'}}
\]
\[
\leq C \rho(s)^p (t - s)^{-\frac{\gamma}{s} - \frac{\gamma - \alpha}{\alpha} \frac{1}{p - 1}} \|S(s)\mu\|_{L^\infty(SN)}^{\frac{1}{p'}} \|S(t)\mu\|_{L^\infty(SN)}^{\frac{1}{p'}}
\]
\[
\leq C \rho(s)^p (t - s)^{-\frac{\gamma}{s} - \frac{\gamma - \alpha}{\alpha} \frac{1}{p - 1}} \rho \frac{1}{p'} \|S(t)\mu\|_{L^\infty(SN)}^{\frac{1}{p'}}
\]
\[
\leq CC_3^{\frac{p - 1}{p'}} (t - s)^{-\frac{\gamma}{s} - \frac{\gamma - \alpha}{\alpha} \frac{1}{p - 1}} \rho(t)(S(t)\mu)^{\frac{1}{p'}}
\]
for almost all \( 0 < s < t \). By \((5.2)\) we have
\[
\int_0^t S(t - s)V(\rho(s)(S(s)\mu)^{p})^{\frac{1}{p'}} \, ds \leq CC_3^{\frac{p - 1}{p'}} t^{1 - \frac{\gamma}{s} - \frac{\gamma - \alpha}{\alpha} \frac{1}{p - 1}} \rho(t)(S(t)\mu)^{\frac{1}{p'}}
\]
\[
= CC_3^{\frac{p - 1}{p'}} \rho(t)(S(t)\mu)^{\frac{1}{p'}}
\]
for almost all $0 < t < 1$. Combining (5.6) and (5.7), we see that
\[
S(t)\mu + \int_0^t S(t-s)V(W(s))^p \, ds \\
\leq S(t)\mu + 2^{p-1} \int_0^t S(t-s)V(S(s)\mu)^p \, ds + 2^{p-1} \int_0^t S(t-s)V(\rho(s)(S(s)\mu)^{\frac{1}{\alpha^*}})^p \, ds \\
\leq S(t)\mu + C(C_3^{p-1} + C_3^{-1})\rho(t)(S(t)\mu)^{\frac{1}{\alpha^*}}
\]
for almost all $0 < t < 1$. Taking a sufficiently small constant $C_3 > 0$ if necessary, $W(t)$ is a supersolution to (1.1) in $\mathbb{R}^N \times [0,1)$. Thus, the proof of Theorem 1.3 is complete. \hfill \Box

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