Rademacher Complexity of Stationary Sequences

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Abstract

We show how to control the generalization error of time series models wherein past values of the outcome are used to predict future values. The results are based on a generalization of standard i.i.d. concentration inequalities to dependent data without the mixing assumptions common in the time series setting. Our proof and the result are simpler than previous analyses with dependent data or stochastic adversaries which use sequential Rademacher complexities rather than the expected Rademacher complexity for i.i.d. processes. We also derive empirical Rademacher results without mixing assumptions resulting in fully calculable upper bounds.

1 Introduction

Statistical learning theory aims to bound the out-of-sample performance of prediction rules induced from finite data sets. The classical situation is where one wishes to predict one variable \( Y \in \mathcal{Y} \) from another \( X \in \mathcal{X} \), and has a training set of \( n \) pairs \((X_1, Y_1), \ldots, (X_n, Y_n)\), assumed to be drawn i.i.d. from a distribution \( \nu \) that will also generate future instances. Provided with a loss function \( \ell: \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+ \) and a class of prediction functions \( \mathcal{G} \), where each \( g \in \mathcal{G} \) is a map from \( \mathcal{X} \) to \( \mathcal{Y} \), the usual goal is to bound the supremum of the empirical process of the losses, \( \sup_{g \in \mathcal{G}} E_\nu [\ell(Y, g(X))] - \frac{1}{n} \sum_{i=1}^n \ell(Y_i, g(X_i)) \). Such bounds involve some notion of the flexibility or complexity of the model space \( \mathcal{G} \), and a particularly important one is the Rademacher complexity,

\[
\mathcal{R}_n(\mathcal{G}) = 2 \mathbb{E}_{\xi, \nu} \left[ \sup_{g \in \mathcal{G}} \frac{1}{n} \sum_{i=1}^n \xi_i \ell(Y_i, g(X_i)) \right]
\]

where \( \xi_i \) are a sequence of i.i.d. variables taking the values +1 and −1 with equal probability (see §2.3 for a fuller statement). While the Rademacher complexity was first used to bound generalization error for i.i.d. processes (e.g. Bartlett and Mendelson, 2002), it has been extended to situations where the \((X_i, Y_i)\) pairs are dependent but \((X_i, Y_i)\) becomes independent of \((X_j, Y_j)\) as \( |i - j| \to \infty \) (Mohri and Rostamizadeh, 2009), and even to adversarial settings, where the data source actively tries to fool the learner Rakhlin et al. (2010).

We build on the latter work to extend Rademacher complexity to the rather different problem of time-series forecasting. In that setting, we observe a single sequence of random variables \( Y_1, Y_2, \ldots, Y_n \) (for short, \( Y^n \)), taking values in \( \mathcal{Y} \), and wish to learn a function which extrapolates the sequence into the future, to forecast (say) the next\(^1\) value \( Y_{n+1} \). Given a predictor \( g: Y^n \to \mathcal{Y} \), a natural notion of generalization error for time series, the forecasting risk, is \( R(g) \equiv E [\ell(Y_{n+1}, g(Y^n))] \mid Y^n \).

While a precise statement needs some care (§2.4), we will show that forecasting risk, like the generalization error of classification and regression problems, can be bounded via the Rademacher complexity, despite the rather different nature of the problem. In particular, we are able to use the standard Rademacher complexity. Our result is comparable to that in Rakhlin et al. (2015, §9), which gives a bound for time series prediction in Banach spaces. Their result however is a consequence of results for the more difficult problem of prediction

\(^1\)Going beyond “one-step-ahead” forecasting, to longer horizons or whole blocks, involves mostly notational changes, which we will not note explicitly.
under stochastic adversaries. As such, our bound and the proof are simpler and tighter, though they apply to an easier (but still highly relevant) prediction task.

§2 gives background material essential for stating our results, on time series, model complexity, and forecasting risk. §3 derives risk bounds for time series, giving a novel proof that the standard Rademacher complexity characterizes the flexibility of \( \mathcal{G} \), even under stationarity, with concentration inequalities for non-mixing dependent variables. §4 carefully compares our results to others in the literature, sketches applications and algorithms, and concludes.

## 2 Time Series, Complexity, and Concentration of Measure

We introduce some of the concepts needed for our results: stationarity and ergodicity are required to control generalization error (unless we aim to predict only a single new observation); Rademacher complexity measures the flexibility of the model space \( \mathcal{G} \); forecasting risk measures the quality of a time-series prediction rule.

**Notation** \( Y = \{Y_t\}_{t=-\infty}^{\infty} \) is a sequence of random variables, i.e., each \( Y_t \) is a measurable mapping from some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) into a measurable space \( \mathcal{Y} \). We write \( Y_t^j \) for the block \( \{Y_t\}_{t=i,j+1,...,j} \) from the random sequence; either limit may be infinity. The \( \sigma \)-field generated by the block \( Y_t^j \) is \( \mathcal{F}_t^j \). \( \mathcal{L}(W) \) denotes the probability law of the random object \( W \), and \( \mathcal{L}(W|Y) \) the conditional law of \( W \) given \( Y \). Finally, if \( W \) has distribution \( \nu \) and \( f \) is a measurable function, we define \( \mathbb{E}_\nu[f(W)] = \mathbb{E}_W[f(W)] = \int d\nu f(W) \). We will try to use whichever notation is clearest in context, sticking to \( \mathbb{E}[f(W)] \) when that is unambiguous.

### 2.1 Stationarity and Ergodicity

We assume \( Y \) is (strictly or strongly) stationary.

**Definition 1** (Stationarity). A random sequence \( Y \) is stationary when all its finite-dimensional distributions are time invariant: for all \( t \) and all \( i \geq 0 \), \( \mathcal{L}(Y_t^{i+1}) = \mathcal{L}(Y_0^i) \).

Stationarity does not require the random variables \( Y_t \) to be independent across time, but does imply they all have the same distribution.

The infinite-dimensional distribution of \( Y \), \( \mathcal{L}(Y) \), is a probability measure on \( \mathcal{Y}^\infty \). In this space, the time-evolution of the process is just the shift map \( \tau \), which “moves the sequence a step to the right”: \((\tau Y)_t = Y_{t+1}\).

**Definition 2** (Ergodicity). A set \( A \subset \mathcal{Y}^\infty \) is shift-invariant, \( A \in \mathcal{I} \), when \( \tau^{-1}A = A \). A probability measure \( \mu \) on \( \mathcal{Y}^\infty \) is ergodic when shift-invariant sets have either probability 0 or probability 1, i.e., \( A \in \mathcal{I} \) only if \( \mu(A) = 0 \) or \( \mu(A) = 1 \).

Ergodicity is important for two reasons. The first is that it implies a law of large numbers for time series.

**Proposition 3** (Individual Ergodic Theorem; Gray 2009). If \( \mu \) is stationary and ergodic, and \( f \in L_1 \), then the time-average of \( f \) converges to its expectation \( \mu \)-almost-surely. That is, the set of \( y \in \mathcal{Y}^\infty \) such that \( \frac{1}{n} \sum_{i=0}^{n-1} f(\tau^i y) \to \mathbb{E}_\mu[f(Y)] \) has \( \mu \)-measure 1.

The second reason is that every stationary process \( Y \) decomposes into a mixture of stationary and ergodic processes, and each realization of \( Y \) comes from just one of these ergodic components.

**Proposition 4** (Ergodic Decomposition; Dynkin 1978; Gray 2009). If \( \rho \) is a stationary but not ergodic distribution on \( \mathcal{Y}^\infty \), then \( \rho = \int \mu d\pi(\mu) \), where \( \pi \) is a measure on the space of stationary and ergodic processes. Moreover, for any \( f \in L_1 \), \( \frac{1}{n} \sum_{i=0}^{n-1} f(\tau^i y) \to \mathbb{E}_\rho[f(Y)|\mathcal{I}] \) for \( \rho \)-almost-all trajectories \( y \).

In words, to generate a trajectory from a stationary, non-ergodic process, first pick a stationary ergodic process (according to the distribution \( \pi \)), and then generate \( Y \) from that process.

To sum up, then, if we assume that the data source is stationary, and that we only get to see a single trajectory from it, there is no loss of generality in also assuming that the source is ergodic, and so the strong
law of large numbers, in the form of Prop. 3, applies. Non-ergodicity would only be relevant if we were to consider multiple independent trajectories from the same stationary process, which might sample different ergodic components Wiener (1956).

2.2 Empirical Processes

The standard device in learning theory for bounding the generalization error of a prediction function is to control the empirical process over a function space, i.e., the deviations of empirical means from their expectation values. We thus consider some convenient, if abstract, notation here.

Let \( Z_1, \ldots, Z_n \) be a sequence of \( Z \)-valued random variables (generally dependent), and \( \mathcal{H} \) a class of real-valued functions on \( Z \). We define the empirical mean or sample mean as \( \hat{h}_n \equiv \frac{1}{n} \sum_{i=1}^n h(Z_i) \) and the expectation value as

\[
E[h] \equiv E_{Z^n}[\hat{h}_n] = \frac{1}{n} \sum_{i=1}^n E_{Z_i}[h(Z_i)] = \frac{1}{n} \sum_{i=1}^n \gamma_i \equiv \frac{1}{n} \sum_{i=1}^n \gamma_i \hat{h}_n
\]

If the \( Z_i \) are i.i.d., then \( E[h] = E_{Z_1}[h(Z_1)] \). The empirical process\(^2\) at \( h \) is \( \gamma_n(h) = E[h] - \hat{h}_n \). We care particularly about the supremum of the empirical process: \( \Gamma_n(\mathcal{H}) \equiv \sup_{h \in \mathcal{H}} \gamma_n(h) \)

2.3 Rademacher Complexity

The Rademacher complexity of a function class is, in essence, how well it can (seem to) match pure noise. The formal definition is (after Bartlett and Mendelson 2002):

**Definition 5** (i.i.d. Rademacher Complexity). Let \( Z_1^n \) be a \( \mathcal{Y} \)-valued i.i.d. sequence, and \( \mathcal{H} \) a real-valued class of functions on \( \mathcal{Y} \). The empirical Rademacher complexity of \( \mathcal{H} \) on \( Z_1^n \) is\(^3\)

\[
\hat{\mathcal{R}}_n(\mathcal{H}) \equiv E_{\xi^n} \left[ \sup_{h \in \mathcal{H}} \frac{2}{n} \sum_{i=1}^n \xi_i h(Z_i) \right]
\]

The Rademacher complexity of \( \mathcal{H} \) is the expectation of the empirical Rademacher complexity over \( Z \):

\[
\mathcal{R}_n(\mathcal{H}) \equiv E_Z \left[ \hat{\mathcal{R}}_n(\mathcal{H}) \right]
\]

Rademacher complexity matters because it is closely related to the supremum of the empirical process over \( \mathcal{H} \). Specifically, \( E_Z [\Gamma_n(\mathcal{H})] \leq \mathcal{R}_n(\mathcal{H}) \). Its utility is that \( E_Z [\Gamma_n(\mathcal{H})] \) is almost never expressible, but one of \( \mathcal{R}_n(\mathcal{H}) \) of \( \mathcal{R}_n(\mathcal{H}) \) may be, thus allowing control of the generalization error with meaningful quantities.

The main burden of our paper is to show that, if \( \mathcal{H} \) is stationary and ergodic rather than i.i.d., we have the same result, though with a more involved proof. We rehearse the (now standard) i.i.d. Rademacher generalization error bound and its proof using our notation in the Supplement because of its importance for our own development.

This definition of i.i.d. Rademacher complexity will, it turns out, work for stationary processes almost unchanged.

**Definition 6** (Rademacher Complexity). Let \( Y_1^n \) be a time series generated from \( \mathbb{P} \). The empirical Rademacher complexity of the real-valued function class \( \mathcal{H} \) on \( Y_1^n \) is

\[
\hat{\mathcal{R}}_n(\mathcal{H}) \equiv E_{\xi^n} \left[ \sup_{h \in \mathcal{H}} \frac{2}{n} \sum_{i=1}^n \xi_i h(Y_i) \right]
\]

The Rademacher complexity is the expectation of the empirical Rademacher complexity: \( \mathcal{R}_n(\mathcal{H}) \equiv E_{Y_1^n} \left[ \hat{\mathcal{R}}_n(\mathcal{H}) \right] \).

\(^2\)Some authors would include an over-all scaling factor of \( \sqrt{n} \).

\(^3\)Some definitions have an absolute value inside the supremum after (Bartlett and Mendelson, 2002), but others avoid it (even the same authors in later work, e.g. Bartlett et al. 2005). As the eventual proof demonstrates, it isn’t required, so we drop it.
The term inside the supremum, \( \frac{1}{n} \sum_{i=1}^{n} \xi_i h_t(Y_t^i) \), is the sample covariance between the noise \( \xi \) and the values of a particular function sequence \( h \). The Rademacher complexity takes the largest value of this sample covariance over all models in the class (mimicking empirical risk minimization), then averages over realizations of the noise.

Relative to the i.i.d. Rademacher complexity, we have indexed the predictor \( h \) with a time dependent subscript. For time series, the goal is to forecast \( Y_{t+1} \) from the history \( Y_t^i \). Since a function \( Y^t \mapsto Y \) is not, technically, the same as a function \( Y^{t+1} \mapsto Y \), one must, strictly speaking, use a different prediction function at each time step. A single predictive model \( h \) thus is implemented as a whole series of functions \( h_t : Y^t \mapsto Y \). At some abuse of notation, we will write \( h \) for the name of this whole sequence of functions.\(^4\) We emphasize that the sequence \( h_1, h_2, \ldots \) does not represent infinitely many individually-learnable functions but rather stages of a single function sequence \( h \).

Intuitively, Rademacher complexity shows how well our models could seem to fit outcomes which were really just noise, giving a baseline against which to assess over-fitting or failing to generalize. Since \( Y \) is stationary and ergodic, and \( \xi \) is i.i.d. and independent of \( Y \), the joint process \((Y, \xi)\) is also stationary and ergodic. Thus by the ergodic tower property (van Handel, 2014), for a fixed function sequence \( h \), the sample covariance tends to zero almost surely:

\[
\frac{1}{n} \sum_{t=1}^{n} \xi_i h_t(Y_t^i) \rightarrow E_{Y, \xi} [h(Y)] = E_\xi [\xi] E_Y [h(Y)] = 0
\]

The overall Rademacher complexity should also shrink, though more slowly, unless the model class is so flexible that it can fit absolutely anything, in which case we can infer nothing about how well it will predict in the future from the fact that it performed well in the past.

Showing that this heuristic reasoning is valid, and that the Rademacher complexity of Definition 6 continues to control the empirical process when forecasting stationary time series, is the main aim of our paper. We note that Kuznetsov and Mohri (2014, 2015) prove generalization error bounds for the forecasting risk under non-stationarity with and without mixing assumptions, but these results rely on the intricate sequential complexities introduced by Rakhlin et al. (2010), which replace the outer expectation over the observations \( Y_t^i \) with a supremum over such observations. (§4.1 carefully compares these results and ours.)

### 2.4 Forecast Risk

In classification or regression problems, we obtain data points \( Z_t = (X_t, Y_t) \), and the goal is to predict one part of the data, \( Y_t \), from the other, \( X_t \). The risk of a prediction function \( g : X \mapsto Y \), can be sensibly defined as an expectation over data points: \( R(g) = E_{X,Y} [\ell(Y, g(X))] \). This risk is well-defined so long as the marginal distribution of the data is shift-invariant (\( L(Z_i) = L(Z_1) \) for all \( i \)). For an i.i.d. data source, it is of course true that

\[
E_{X_{n+1}, Y_{n+1}} [\ell(Y_{n+1}, g(X_{n+1})) \mid X_n^n, Y_n^n] = E_{X_{n+1}, Y_{n+1}} [\ell(Y_{n+1}, g(X_{n+1}))] = R(g)
\]

so that averaging over the marginal distribution of the next data point indicates the expected loss of continuing to use the predictor \( g \) on new data. This is no longer true for dependent data. However, for a stationary ergodic source, one has that (Shalizi and Kontorovich, 2013)

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} E_{X_{n+1}, Y_{n+1}} [\ell(Y_{n+1}, g(X_{n+1})) \mid X_1^n, Y_1^n] = E_{X,Y} [\ell(Y, g(X))] = R(g)
\]

so the expectation over new data would still be a good indicator of long-run performance.

All of this is subtly changed for time series, where the goal is to forecast \( Y_{t+1} \) from the history \( Y_t^i \). As discussed above, we abuse notation and denote a single predictive model \( g \) even though it really represents a of functions \( g_t : Y^t \mapsto Y \).

\(^4\)If we wanted to be purists, we could introduce a parameter space \( \Theta \) (not necessarily finite-dimensional), and consider the collection of prediction functions \( g_t(Y_t^i; \theta) \) for all \( t \).
**Definition 7** (Forecast risk). Given a stationary and ergodic stochastic process $Y$, and a loss function $\ell$, the **finite-history risk** of the predictive model $g$ is

$$R_n(g) \equiv \mathbb{E}_{Y^n} \left[ \frac{1}{n} \sum_{t=1}^{n} \ell(Y_t, g(Y_{t-1}^{t-1})) \right]$$

and the **forecast risk** is $R(g) = \lim_{n \to \infty} R_n(g)$ when the limit exists.

For brevity, we introduce the notation $Z_t \equiv (Y_t, Y_{t-1}^{t-1})$, and $h_t(Z_t) \equiv \ell(Y_t, g(Y_{t-1}^{t-1}))$, defining $Y_t^0 \equiv \emptyset$. The forecast risk can thus be also written as $\lim_{n \to \infty} \sum_{t=1}^{n} h_t(Z_t)$. So $R(g)$, again, captures the long-run average cost of using the predictive model $g$. By contrast, $R_n(g)$ is the average risk of $g$ if used on an independent realization of $Y_t^n$.

Having an infinite-time limit in the definition of forecast risk is irksome. It can be evaded if the predictive model has only a finite memory length $d \geq 0$, so nothing more than $d$ time steps old matters for predictions (formally, $g_t(Y_t^1) \in \sigma(Y_t^{t-d})$ for all $t > d$). Then, by stationarity, we may simplify $R(g) = \mathbb{E}_{Y^{d+1}} \left[ \ell(Y_{t+1}, g(Y_{t-d}^t)) \right]$. In fact, in the finite-memory-length case, as soon as $n > d$,

$$R(g) = \frac{1}{n-d} \sum_{t=d+1}^{n} \mathbb{E}_{Y^n} \left[ \ell(Y_{t+1}, g(Y_{t-d}^t)) \right] = R_n(g)$$

and it follows from the ergodic theorem that $\frac{1}{n-d} \sum_{t=d+1}^{n} \ell(Y_{t+1}, g(Y_{t-d}^t)) \to R(g)$ almost surely.

Predictive models with infinite-range memories are however actually fairly common in forecasting practice, including not just hidden Markov models but also things as basic as moving-average models. We therefore *posit* that $R(g)$ exists for such models, writing the gap between the forecast risk and the finite-history risk by $\Delta_n(g) \equiv R(g) - R_n(g)$. We also posit\(^5\) that the time-averaged loss converges to the forecast risk: $\frac{1}{n} \sum_{t=1}^{n} h_t(Z_t) \to R(g)$. With finite amounts of data, we thus focus on control of $R_n(g)$. Whether $\Delta_n(g) \to 0$ is a property of both the function class and the dependence structure, and is outside our scope, though see [van Handel (2014)](https://example.com) for related discussion.

### 3 Risk Bounds

Generalization error bounds follow from deriving high probability upper bounds on the quantity

$$\Gamma_n(\mathcal{H}) := \sup_{h \in \mathcal{H}} R_n(h) - \hat{R}_n(h),$$

which is the worst case difference between the true risk $R_n(h)$ and the empirical risk $\hat{R}_n(h)$ over all functions in the class of losses $\mathcal{H} = \{h = \ell(\cdot, g(\cdot)) : g \in \mathcal{G}\}$ defined over a particular class of prediction functions $\mathcal{G}$. We first present our main result, which bounds $\mathbb{E}_Z[\Gamma_n(\mathcal{H})]$ with the Rademacher complexity and discuss its proof. We then use our Rademacher bound to derive risk bounds for time-series forecasters which are fully calculable from data.

#### 3.1 Stationary Rademacher Bounds

The symmetrization arguments used to prove Rademacher bounds for the i.i.d. case fail for time series prediction. However, as we now show, for stationary time series, bounds of the same form are still valid, albeit with a somewhat more involved proof. This is in contrast to the far more intricate constructions needed to establish bounds using generalized Rademacher complexities for online learning ([Rakhlin et al., 2010, 2011](https://example.com)) or for non-stationary processes ([Kuznetsov and Mohri, 2015](https://example.com)). (We give more detailed contrasts in §4.1.) Our first principle result is simply:

**Theorem 8.** For a time series prediction problem based on a sequence $Y_t^n$,

$$\mathbb{E}[\Gamma_n(\mathcal{H})] \leq \mathcal{R}_n(\mathcal{H}).$$

\(^5\)If the loss function is the negative log-likelihood, this posit is the generalized asymptotic equipartition property, or Shannon-McMillan-Breiman theorem [Algoet and Cover (1988); Gray (1990)].
We note here that unless \( \sup_{h \in \mathcal{H}} \|h\|_\infty < \infty \), \( \mathcal{R}_n(\mathcal{H}) = \infty \) by its definition. Thus, this result, like all Rademacher results, is only useful with bounded predictors or losses. Of course if \( \sup_{h \in \mathcal{H}} \|h\|_\infty = \infty \), the theorem holds trivially.

The standard proof for i.i.d. classification or regression introduces a “ghost sample”, an independent sample of size \( n \) from the same distribution that produced the original data, before using a symmetrization argument. For forecasting, however, introducing an independent copy of the original time series will not produce the necessary symmetry. Rather, following an idea introduced by Rakhlin et al. (2010, 2011) for dealing with adversarial data, we work with a tangent sequence, where the surrogate value introduced at each time point is conditioned on the actual time series up to that point. That is, the tangent sequence \( \mathbf{Y} \) is defined recursively: \( \mathcal{L} \left( \tilde{Y}_1 \right) = \mathcal{L} \left( Y_1 \right) \), and \( \mathcal{L} \left( \tilde{Y}_t | Y_{1:t-1} \right) = \mathcal{L} \left( Y_t | Y_{1:t-1} \right) \). Furthermore, \( \tilde{Y}_t \) is independent of all other \( \tilde{Y}_s \) and of all \( Y_s \), conditional on \( Y_1^{t-1} \). (In directed graphical models terms, \( Y_1^{t-1} \) are the parents of \( \tilde{Y}_t \), which has no children. See Figure 1.) The time series \( \mathbf{Y} \) and the tangent sequence do not have the same joint distributions.\(^6\)

**Proof of Thm. 8.** With both the original time series \( \mathbf{Y} \) and the tangent sequence \( \mathbf{Y} \) in hand, we construct \( Z_t \) and \( \tilde{Z}_t \) variables as follows: \( Z_t \equiv (Y_t, Y_1^{t-1}) \) and \( \tilde{Z}_t \equiv (\tilde{Y}_t, Y_1^{t-1}) \) (with the convention that \( Z_1 = Y_1 \), \( \tilde{Z}_1 = \tilde{Y}_1 \)). Notice that \( \tilde{Z} \) combines the original time series and its tangent sequence, but in such a way that \( \mathcal{L} \left( Z_t \right) = \mathcal{L} \left( \tilde{Z}_t \right) \). Furthermore, since \( \tilde{Y}_t \perp Y_t^{t-1} | Y_1^{t-1} \), it follows that \( \tilde{Z}_t \perp Z_t | Z_1^{t-1} \). As \( R_n(h) = \mathbb{E}_Z \left[ \frac{1}{n} \sum_{t=1}^{n} h_t(Z_t) \right] \) for some \( h \in \mathcal{H} = \ell \circ \mathcal{G} \), we may equally well write the risk in terms of the tangent sequence: \( R_n(h) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}_{\tilde{Z}_t} \left[ h_t(\tilde{Z}_t) \right] \). Therefore,

\[
\mathbb{E} \left[ \Gamma_n(\mathcal{H}) \right] = \mathbb{E}_Z \left[ \sup_{h \in \mathcal{H}} \left( \mathbb{E}_Z \left[ \frac{1}{n} \sum_{i=1}^{n} h(Z_i) \right] - \frac{1}{n} \sum_{i=1}^{n} h(Z_i) \right) \right]
\]

\[
= \mathbb{E}_Z \left[ \sup_{h \in \mathcal{H}} \left( \mathbb{E}_Z \left[ \frac{1}{n} \sum_{i=1}^{n} h(\tilde{Z}_i) \right] - \frac{1}{n} \sum_{i=1}^{n} h(Z_i) \right) \right]
\]

\[
\leq \mathbb{E}_Z \mathbb{E}_{\tilde{Z}^\bullet} \left[ \frac{1}{n} \sum_{i=1}^{n} h(\tilde{Z}_i) - h(Z_i) \right] \quad (\text{Jensen’s inequality})
\]

\[
= \mathbb{E}_{Z_1} \mathbb{E}_{Z_2 | Z_1} \cdots \mathbb{E}_{Z_n | Z_{n-1}, \ldots, Z_1} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} h(\tilde{Z}_i) - h(Z_i) \right] \quad (\text{Iterated expectation}). \quad (1)
\]

Now, due to dependence, Rademacher variables must be introduced carefully as in the adversarial case. Rademacher variables create two tree structures, one associated to the \( Z \) sequence, and one associated to the \( \tilde{Z} \) sequence (see Rakhlin et al., 2010, 2011 for a thorough treatment). We write these trees as \( Z(\xi) \) and

\(^6\)Let \( Y_t \) be 0 or 1 with equal probability, and \( Y_{t+1} = Y_t \) with probability 0.9 and \( = 1 - Y_t \) otherwise. (\( \mathbf{Y} \) is a stationary and ergodic Markov chain.) Because \( \tilde{Y}_t \perp Y_t^{t-1} | Y_1^{t-1} \), the probability that \( \tilde{Y}_2 = \tilde{Y}_1 \) is not 0.9 but 0.5.
\( \tilde{Z}(\xi) \), where \( \xi \) is a particular sequence of Rademacher variables, e.g. \((1,-1,-1,1,\ldots,1)\), which creates a path along each tree. For example, consider \( \xi = 1 \). Then, \( Z(\xi) = (Z_1, \ldots, Z_n) \) and \( \tilde{Z}(\xi) = (\tilde{Z}_1, \ldots, \tilde{Z}_n) \), the “right” path of both tree structures. For \( \xi = -1 \), then \( Z(\xi) = (\tilde{Z}_1, \ldots, \tilde{Z}_n) \) and \( \tilde{Z}(\xi) = (Z_1, \ldots, Z_n) \), the “left” path of both tree structures. Changing \( \xi \), from +1 to −1 exchanges \( Z_i \) for \( \tilde{Z}_i \) in both trees and chooses the left child of \( Z_{i-1} \) and \( \tilde{Z}_{i-1} \) rather than the right child. Figure 1 displays both trees. In order to talk about the probability of \( Z_i \) conditional on the “past” in the tree, we need to know the path taken so far. For this, we define a selector function \( \chi(\xi) := \chi(\xi, \rho, \varrho) = \rho I(\xi = 1) + \varrho I(\xi = -1) \). Distributions over trees then become the objects of interest.

Contrary to the online-learning scenario, the dependence between future and past paths makes the adversary is not free to change predictors and responses separately. Once a branch of the tree is chosen, the distribution of future data points is fixed, and depends only on the preceding sequence. Because of this, the joint distribution of any path along the tree is the same as any other path, i.e. for any two paths \( \xi, \xi' \), \( \mathcal{L}(Z(\xi)) = \mathcal{L}(Z(\xi')) \) and \( \mathcal{L}(\tilde{Z}(\xi)) = \mathcal{L}(\tilde{Z}(\xi')) \). Similarly, due to the construction of the tangent sequence, we have that \( \mathcal{L}(Z(\xi)) = \mathcal{L}(\tilde{Z}(\xi)) \). This equivalence between paths allows us to introduce Rademacher variables swapping \( Z_i \) for \( \tilde{Z}_i \) as well as the ability to combine terms below:

\[
\begin{align*}
(1) &= \mathbb{E}_{Z_i} \mathbb{E}_{\xi_1} \mathbb{E}_{Z_i|\chi(\xi_i, Z_i, Z_{i-1})} \mathbb{E}_{\xi_2} \cdots \mathbb{E}_{Z_i|\chi(\xi_{i-1}, \ldots, \chi(\xi_1) \mathbb{E}_{\xi_n} \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \xi_i(h(\tilde{Z}_i) - h(Z_i)) \\
&= \mathbb{E}_{Z, \xi} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \xi_i(h(\tilde{Z}_i) - h(Z_i)) \right] \\
&\leq \mathbb{E}_{Z, \xi} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \xi_i h(Z_i) \right] + \mathbb{E}_{Z, \xi} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \xi_i h(\tilde{Z}_i) \right] \\
&= 2 \mathbb{E}_{Z, \xi} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \xi_i h(\tilde{Z}_i) \right] = \mathcal{R}_n(\mathcal{H}).
\end{align*}
\]

Rademacher complexity gets its utility from bounding the prediction risk of forecasters. For i.i.d. data, the main tools for proving risk bounds are the inequalities of Hoeffding (1963) and McDiarmid (1989). Extensions of learning theory to dependent data have relied on strong mixing properties to approximate weakly-dependent processes by i.i.d. ones, recovering i.i.d. results with a reduced effective sample size. We instead use generalizations of Hoeffding and McDiarmid to dependent sequences based on results of van de Geer (2002), which do not need mixing at all. Rather than deriving bounds under the condition \( \sup_{h \in \mathcal{H}} \|h\|_{\infty} < \infty \), we use a weaker hypothesis on the tails of conditional distributions. We first state this more general result, giving the bounded case as a corollary. We discuss the concentration bound and its derivation in §3.3.

**Theorem 9.** Suppose that there exist constants \( \tau \) and \( c \) such that

\[
\mathbb{E} \left[ \psi \left( \frac{\|\Gamma_n(\mathcal{H})\|}{c} \right) \bigg| \mathcal{F}_0 \right] \leq \tau \quad \forall t,
\]

where \( \psi(x) = \exp(x^2) - 1 \). Then, for \( \epsilon > 0 \) and \( n \) large enough, for all \( h \in \mathcal{H} \), with probability at least \( 1 - \delta \),

\[
R_n(h) \leq \tilde{R}_n(h) + \mathcal{R}_n(\mathcal{H}) + 4c(\tau + 1) \sqrt{\frac{2 \log 1/\delta}{n}}.
\]

The following corollary is immediate by noting that \( \sup_{h \in \mathcal{H}} \|h\|_{\infty} \leq M < \infty \) implies that \( \mathbb{E} \left[ \exp \left( \left( \frac{\|\Gamma_n(\mathcal{H})\|}{M} \right)^2 \right) \right] \leq e < 3 \). We made no effort to optimize the constant before the confidence penalty.

**Corollary 10.** If \( \sup_{h \in \mathcal{H}} \|h\|_{\infty} \leq M \), then for all \( h \in \mathcal{H} \), with probability at least \( 1 - \delta \),

\[
R_n(h) \leq \tilde{R}_n(h) + \mathcal{R}_n(\mathcal{H}) + 12M \sqrt{\frac{2 \log 1/\delta}{n}}.
\]
3.2 Empirical Rademacher Bounds

Unfortunately $\mathcal{R}_n(H)$ may itself be hard or impossible to calculate for some classes $H$. However, under our assumptions, we show that $\mathcal{R}_n(H)$ is closely approximated by the empirical Rademacher complexity. That is, the same data can estimate both $\tilde{R}_n$ and $\mathcal{R}_n(H)$.

**Theorem 11 (Empirical Rademacher Complexity Bound).** Assume eq. (2) holds. Then, for all $h \in H$, with probability at least $1 - \delta$,

$$R_n(h) \leq \tilde{R}_n(h) + \mathcal{R}_n(H) + 12c(\tau + 1)\sqrt{\frac{2\log 2/\delta}{n}}.$$

To apply Thm. 11, we can estimate $\mathcal{R}_n(H)$ by drawing $m$ independent Rademacher samples of size $n$, and use

$$\frac{1}{mn} \sum_{i=1}^{m} \sup_{h \in H} \sum_{t=1}^{n} \xi_i h_t \approx \mathcal{R}_n(H). \quad (3)$$

The approximation is $O(1/m)$-accurate. Thus, given one sample of data, the entire risk bound is fully calculable. If $\mathcal{R}_n(H)$ is known (the case for many common classes $H$, see Section 4.2) we may apply Thm. 9. For any other class of predictors, we can estimate the complexity with (3) and apply Thm. 11. Finally, we present a corollary for the case that $\sup_{h \in H} \|h\|_{\infty} < \infty$.

**Corollary 12.** If $\sup_{h \in H} \|h\|_{\infty} \leq M < \infty$, then for all $h \in H$, with probability at least $1 - \delta$,

$$R_n(h) \leq \tilde{R}_n(h) + \mathcal{R}_n(H) + 36M\sqrt{\frac{2\log 2/\delta}{n}}.$$

Both of these results can be seen as penalizing the empirical risk with a term that accounts for the complexity of $H$ along with a second penalty for the amount of confidence we require.

3.3 Necessary Concentration Inequalities

For i.i.d. data, the main tools for developing risk bounds are the inequalities of Hoeffding (1963) and McDiarmid (1989). As discussed above, extensions of learning theory to dependent data have relied on strong mixing properties to approximate weakly-dependent processes by i.i.d. ones, and so recover the i.i.d. results with a reduced effective sample size. We will instead use a generalization applying to dependent sequences based on results due to van de Geer (2002), which do not require mixing at all.

We need some conditions on the tails of the random variables. Suppose that $X_t$ is a martingale, e.g. a real-valued $\mathcal{F}_t$-measurable random variable satisfying $\mathbb{E}[X_t \mid \mathcal{F}_{t-1}^t] = 0$ with the convention $\mathcal{F}_0 = \emptyset$. For a constant $c$, define

$$B_n^2 = \sum_{t=1}^{n} c^2 \left(1 + \mathbb{E} \left[ \psi \left( \frac{|X_t|}{c} \right) \mid \mathcal{F}_{t-1}^t \right] \right),$$

where $\psi(x) = \exp(x^2) - 1$.

Essentially, controlling $B_n^2$ by bounding the expectation of $\psi(|X_t|/c)$ controls the tails of $X_t$. The function $\psi$ can be any non-decreasing, convex function satisfying $\psi(0) = 0$, but the use of $\psi(x) = \exp(x^2) - 1$ is most common. In general, $\inf_{c>0} \mathbb{E} \psi(|X_t|/c) \leq 1$ is referred to as the Orlicz norm of $X_t$ denoted as $\|X_t\|_{\psi}$. In the simplest case, if $c < \infty$ and $\mathbb{E}[|X_t|] = 0$ it holds that $\mathbb{P}(|X_t| > x) \leq 2 \exp(-x^2/c^2)$. This is the definition of sub-Gaussian tails: $X_t$ has tails which decrease at least as quickly as those of a standard Gaussian random variable. In particular, bounded random variables satisfy this condition. As our data come from a time-dependent process $Y$, we require the conditional version of this idea.

**Lemma 13 (van de Geer 2002; Theorem 2.2).** Suppose $X_t$ is a martingale. Then, for all $\epsilon > 0$, $b > 0$, for $n$ large enough,

$$\mathbb{P} \left( \sum_{t=1}^{n} X_t \geq \epsilon \text{ and } B_n^2 \leq b^2 \right) \leq \exp\{-\epsilon^2/8b^2\}.$$
Table 1: Comparison of existing risk bounds. We use the notation \( \text{polylog}(n) \) to mean \( \log^k(n) \) for some \( k > 0 \).

| Assumptions                  | Reference                  | Complexity                  | Calculable | Best-case convergence rate |
|------------------------------|----------------------------|-----------------------------|------------|----------------------------|
| I.i.d.                       | (Many, e.g. Bartlett and Mendelson, 2002) | Rademacher                  | Yes        | \( O(\sqrt{\frac{1}{n}}) \) |
| Stationary & mixing          | (Mohri and Rostamizadeh, 2009) | Blocked Rademacher           | If \( \beta \)-mixing coef are known | \( O(\sqrt{\frac{1}{n}}) \) |
| Non-stationary & mixing      | (Kuznetsov and Mohri, 2014, 2017) | Blocked or Sequential Rademacher | If \( \beta \)-mixing coef are known | \( O(\sqrt{\frac{1}{n}}) \) |
| Non-stationary, non-mixing   | (Kuznetsov and Mohri, 2015) | Expected covering number     | Depends on \( H \) | \( O(\sqrt{\frac{1}{n}}) \) |
| Adversarial                  | (Rakhlin et al., 2010, 2011, 2015) | Sequential Rademacher       | Depends on \( H \) | \( O(\sqrt{\frac{1}{n}}) \) |

This result generalizes Hoeffding’s inequality to the case of conditionally sub-Gaussian random variables from a dependent sequence. As long as the tails of the next observation are well controlled conditional on the past, we can still control the size of deviations from the mean with high probability.

We now present the following extension, analogous to McDiarmid’s inequality, but for dependent sequences with sub-Gaussian tails (rather than bounded differences).

**Theorem 14.** Let \( X_t \) be \( \mathcal{F}_0 \)-measurable with

\[
E \left[ \psi \left( \frac{|X_t|}{c} \right) \right] \leq \tau, \tag{4}
\]

for some \( \tau > 0 \) and all \( t > 0 \). Then for all \( \epsilon > 0 \) and \( n \) large enough,

\[
P \left( X_n - E[X_n] > \epsilon \right) \leq \exp \left\{ -\frac{\epsilon^2}{32nc^2(\tau + 1)^2} \right\}.
\]

Thm. 14 can be generalized to allow both \( c \) and \( \tau \) to depend on \( t \) with appropriate modifications as in Lem. 13. Typically, we would expect better control over the tails as we condition on more data, resulting in a decreasing sequence of \( \tau \), though we will not pursue this generality further here. Because we were unable to find a comparable result in the literature, and this one may be useful in its own right, we have chosen to include it here. The proof is given in the Supplement.

## 4 Discussion

In this section, we give a careful explanation, situating our results in the context of existing bounds. We then provide a few simple (standard) examples of cases in which our bounds are calculable, as well as a generalized algorithm for classes which don’t admit calculable expected Rademacher complexities. Finally, we conclude.

### 4.1 Relationship with Existing Work

As discussed in the introduction, existing work has developed risk bounds for dependent data under a number of assumptions which are more or less general than ours. In order to give context for our results, we compare the assumptions and benefits of each of these here. This comparison is summarized in Table 4.1.

The first risk bounds for time series are, like our result, based on standard Rademacher complexities. Mohri and Rostamizadeh (2009) assume that \( Y \) is a stationary \( \beta \)-mixing process. Like our results (Cor. 10 and Cor. 12), they are able to prove bounds based on both the expected and empirical Rademacher complexities. Their results however, do not apply to the full time-series forecasting setting we present here—predictions in their setting may depend only on a fixed lag \( d \) of previous observations. Furthermore, both the Rademacher complexity and the confidence penalty depend on blocks of data rather than individual data points. The number of blocks, \( \mu \), then replaces \( n \) in both terms, where \( \mu \) depends on the unknown mixing coefficients. Thus, convergence rates are slightly slower—because the size of the blocks should increase with \( n \), \( \mu \) must be sublinear in \( n \)—and cannot be directly calculated without knowledge of the mixing coefficients. McDonald et al. (2011, 2015) give an estimator for the mixing coefficients with nearly parametric rates, though bounds which replace known coefficients with estimates have not been derived. Our results
subsume the stationary and mixing results because our convergence rate is faster without assuming any type of asymptotic decay of dependence.

Alternatively, Rakhlin et al. (2010, 2011, 2015) develop truly ingenious techniques for an adversarial data generating process, a much more general condition wherein not only is the process potentially non-stationary and non-mixing, but subsequent data points may be chosen based on previous predictions to make the learner perform as poorly as possible. These results rely instead on the sequential Rademacher complexity defined in our notation as

\[
\mathfrak{R}_n^{seq}(\mathcal{H}) = \sup_{Z} \mathbb{E}_\xi \left[ \sup_{h \in \mathcal{H}} \frac{2}{n} \sum_{i=1}^{n} \xi_i h(Z_i(\xi)) \right],
\]

where the outer supremum is taken over all \( \mathbb{Y} \)-valued trees of depth \( n \). Because their results are more general, one could simply apply them to our setting. However, \( \mathfrak{R}_n^{seq}(\mathcal{H}) \) is more difficult to calculate than \( \mathfrak{R}_n(\mathcal{H}) \), is looser, and does not admit an empirical version (analogous to our Cor. 12) because it replaces the outer expectation over \( Z \) with a supremum.

Finally, work on non-stationary, mixing processes (Kuznetsov and Mohri, 2014, 2017) and non-stationary, non-mixing processes (Kuznetsov and Mohri, 2015) has also appeared. In the mixing case, the complexity is either the blocked version as in (Mohri and Rostamizadeh, 2009) adjusted to handle non-stationarity, or the sequential complexity above with an additional discrepancy penalty which “measures” non-stationarity in view of \( \mathcal{H} \). The discrepancy measure can be calculated from data as can the blocked Rademacher complexity, though again, the mixing coefficients cannot. The non-stationary, non-mixing setting replaces Rademacher complexities with an expected sequential covering number. This results in bounds which are looser than ours by poly-logarithmic factors in \( n \). If the covering number can be computed for the function class \( \mathcal{H} \) of interest, than these results are wholly calculable, but if the class does not have known covering number, there is no analogue to Cor. 12 which can be estimated from the given data.

Thus, the benefits of our work are that, if we are willing to assume stationarity, our results are tighter than previous results, easier to calculate based on known expected Rademacher formulas, and admit empirical Rademacher complexities which can always be calculated given sufficient computational resources. None of these benefits require untestable mixing assumptions or knowledge of the associated coefficients.

4.2 Examples and Algorithms

In some cases, the expected (or empirical) Rademacher complexity is easily calculated from data. In these cases, one can derive simple algorithms for time-series prediction. Our first two examples, give complete risk bounds for algorithms which predict future observations based on previous observations for clarity. These follow from results of Bartlett and Mendelson (2002).

Consider first the case of a 2-layer Neural Network which makes predictions based on \( d \) previous values and let \( \mathcal{Y} = \mathbb{R}^p \). Suppose that the activation function \( \sigma : \mathbb{R} \to [-1,1] \) is 1-Lipschitz with \( \sigma(0) = 0 \). For \( v_i \in \mathbb{R}^{pd} \) define

\[
\mathcal{G}_N = \left\{ y \mapsto \sum_i w_i \sigma(v_i \cdot y) : \| w \|_1 \leq 1, \| v_i \|_1 \leq 1 \right\}.
\]

Suppose further that \( \ell \) is 1-Lipschitz. Then,

\[
\hat{\mathfrak{R}}_n(\ell \circ \mathcal{G}_N) \leq \frac{2c \log^{1/2}(pd)}{n} \max_{1 \leq i,j \leq p} \sqrt{\sum_{i=1}^{n-d} (y_{ij} - y_{ij'})^2}
\]

for some \( c > 0 \). Thus, \( \hat{\mathfrak{R}}_n(\ell \circ \mathcal{G}_N) = O_P(n^{-1/2}) \) as usual. The Lipschitz conditions and norm constraints can easily be exchanged for other constants without altering the rate, and the number of layers is easily altered.

Consider now regularized Kernel methods. Suppose \( \ell \) is \( M \)-Lipschitz and consider the class \( \mathcal{G}_K = \left\{ y \mapsto w \cdot \Phi(y) : \| w \|_\Psi \leq B^2 \right\} \), where \( \Phi(y) : \mathcal{Y} \to \Psi \) is the feature map associated with the Hilbert space \( \Psi \).
Algorithm 1: Generic ERM Algorithm

Input: data \( Y^n \), models \( H_1, \ldots, H_k \), integer \( m \)

for \( i = 1 \) to \( k \) do
    Estimate a predictor \( h_i \in H_i \) as usual
    Compute the training error \( \hat{R}_n(h_i) \).
    Compute \( \hat{R}_n(H_i) \) using (3)
end for

Choose \( i^* = \text{argmin}_i \hat{R}_n(h_i) + \hat{R}_n(H_i) \).

Return \( h_{i^*}, \hat{R}_n(h_{i^*}) + \hat{R}_n(H_{i^*}) \) and calculate the complexity penalty to form the bound in Cor. 12.

\( k \) is the corresponding kernel function, and \( \| \cdot \|_\Psi \) denotes the norm in \( \Psi \). Then, we have that

\[
\hat{R}_n(\ell \circ G_K) \leq \frac{4MB}{n} \sqrt{\sum_{i=1}^{n-d} k(y_i, y_i)} = O_P(n^{-1/2}).
\]

Finally, using Cor. 12, we can derive a generic empirical risk minimization-type (ERM) algorithm for learning without any knowledge of complexity measurements. Algorithm 1 shows how to choose a predictor from among a collection of bounded function classes \( H_1, \ldots, H_k \).

4.3 Conclusion

In this paper, we have demonstrated how to control the generalization of time series prediction algorithms. These methods use some or all of the observed past to predict future values of the same series. In order to handle the complicated Rademacher complexity bound for the expectation, we have followed the approach used in the online learning case pioneered by Rakhlin et al. (2010, 2011), but we show that in our particular case, much of the structure needed to deal with the adversary is unnecessary. This results in clean risk bounds which have a form similar to the i.i.d. case. As these results take expectations over \( Y^n \) rather than a supremum, empirical counterparts which are estimable can also be derived. Extending our results to local Rademacher complexities with faster convergence rates is left for future work.

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A Additional Proofs

Proposition (Standard i.i.d. Rademacher bound). If \( Z_1, \ldots, Z_n \) is an i.i.d. sample from some probability distribution \( P \), then \( E_Z [\Gamma_n(H)] \leq \mathcal{R}_n(H) \).

Proof. The usual proof introduces a “ghost sample” \( \tilde{Z}_1^n \), where the \( \tilde{Z}_t \) have the same distribution as the \( Z_t \), but are independent of the latter and of each other. Then expectations may as well be taken over the ghost sample as the real one: \( E[h] = E_{\tilde{Z}_1^n} [h(\tilde{Z}_1)] = \frac{1}{n} \sum_{i=1}^{n} E_{\tilde{Z}_i} [h(\tilde{Z}_t)] \). Hence (using the notation from §2.2)

\[
\gamma_n(h) = \frac{1}{n} \sum_{i=1}^{n} E_{\tilde{Z}_i} [h(\tilde{Z}_t)] - \frac{1}{n} \sum_{i=1}^{n} h(Z_t) = \frac{1}{n} \sum_{i=1}^{n} E_{\tilde{Z}_i} [h(\tilde{Z}_t) - h(Z_t)],
\]

\[
\Gamma_n(H) \leq E_{\tilde{Z}_1^n} \left[ \sup_{h \in H} \frac{1}{n} \sum_{i=1}^{n} h(\tilde{Z}_t) - h(Z_t) \right],
\]

and

\[
E_Z [\Gamma_n(H)] \leq E_{\tilde{Z}_1^n} \left[ \sup_{h \in H} \frac{1}{n} \sum_{i=1}^{n} h(\tilde{Z}_t) - h(Z_t) \right].
\]

Eq. 5 holds because the supremum of expectations is less than or equal to the expected supremum, and Eq. 6 just takes the expectation of both sides with respect to \( Z \). Since \( Z_t \) and \( \tilde{Z}_t \) have the same marginal distribution and are independent, \( \mathcal{L}(h(\tilde{Z}_t) - h(Z_t)) = \mathcal{L}(h(Z_t) - h(\tilde{Z}_t)) \), and the signs of summands in Eq. 6 can be flipped arbitrarily, according to the Rademacher variables, without effect:

\[
E_Z [\Gamma_n(H)] \leq E_{\tilde{Z}_1^n} \left[ \sup_{h \in H} \frac{1}{n} \sum_{i=1}^{n} \xi_i (h(\tilde{Z}_t) - h(Z_t)) \right] \\
\leq E_{\tilde{Z}_1^n} \left[ \sup_{h \in H} \frac{1}{n} \sum_{i=1}^{n} \xi_i h(Z_t) \right] + E_{\tilde{Z}_1^n} \left[ \sup_{h \in H} \frac{1}{n} \sum_{i=1}^{n} \xi_i h(\tilde{Z}_t) \right] \\
= 2E_{\tilde{Z}_1^n} \left[ \sup_{h \in H} \frac{1}{n} \sum_{i=1}^{n} \xi_i h(Z_t) \right] = \mathcal{R}_n(H).
\]

Proof of Thm. 9. This result follows immediately from Thm. 14 upon setting the right hand side equal to \( \delta \) and solving for \( \epsilon \).

Proof of Thm. 11. Write \( h \in \mathbb{R}^n \) for the vector \( h_1(Z_1), \ldots, h_n(Z_n) \). Note that as the range of \( \ell \) is \( \mathbb{R}^+ \), \( h \) lies
in the non-negative orthant of \( \mathbb{R}^n \) \(( h \geq 0 \)). Now,

\[
n\Gamma_n = \sup_{h \in \mathcal{H}} \left( \sum_{t=1}^{n} h_t - \mathbb{E}_Z \left[ \sum_{t=1}^{n} h_t \right] \right)
\]

\[
\geq \sup_{h \in \mathcal{H}} \sum_{t=1}^{n} h_t - \sup_{h \in \mathcal{H}} \mathbb{E}_Z \left[ \sum_{t=1}^{n} h_t \right] \quad \text{(property of sup)}
\]

\[
\geq \sup_{h \in \mathcal{H}} \sum_{t=1}^{n} h_t - \mathbb{E}_Z \left[ \sup_{h \in \mathcal{H}} \sum_{t=1}^{n} h_t \right] \quad \text{(Jensen’s ineq.)}
\]

\[
= \sup_{h} 1^T h - \mathbb{E}_Z \left[ \sup_{h} 1^T h \right]
\]

\[
\geq \mathbb{E}_\xi \left[ \sup_{h} \xi^T h \right] - \mathbb{E}_Z \left[ \sup_{h} 1^T h \right]
\]

\[
= \frac{n}{2} \tilde{\mathbb{R}}_n(h) - K
\]

where \( K \) is a constant. Therefore,

\[
\mathbb{E}_Z \left[ \frac{n}{2} \tilde{\mathbb{R}}_n(h) - K \right] = \frac{n}{2} \tilde{\mathbb{R}}_n(h) - K
\]

Since \( \psi \) is increasing in its argument and we assumed that \( n\Gamma_n \) satisfied eq. (2) for constants \( c \), and \( \tau \), we can apply Thm. 14 with \( Z_n = \tilde{\mathbb{R}}_n(h) - K \) with constants \( c \to 2c/n \) and \( \tau \) as before. Thus,

\[
\mathbb{P}(\tilde{\mathbb{R}}_n(h) - \mathbb{R}_n(h) > \epsilon) \leq \exp \left\{ \frac{-nc^2}{128c^2(\tau + 1)^2} \right\}.
\]

Setting the right hand side equal to \( \delta/2 \) and combining with Thm. 9 applied with \( \delta \to \delta/2 \) via the union bound gives the result. \( \square \)

**Proof of Thm. 14.** Write \( X_n - \mathbb{E}[X_n] = \sum_{i=1}^{n} W_i \), where \( W_t = \mathbb{E}[X_n | \mathcal{F}_t^0] - \mathbb{E}[X_n | \mathcal{F}_t^{t-1}] \), for all \( t = 1, \ldots, n \). Then \( W_t \) is \( \mathcal{F}_0^t \)-measurable, and \( \mathbb{E}[W_t | \mathcal{F}_0^t] = 0 \) for all \( t \). Now, let \( K > 0 \) to be chosen. Then

\[
\mathbb{E} \left[ \psi \left( |W_t|/K \right) \mid \mathcal{F}_0^{t-1} \right] = \mathbb{E} \left[ \psi \left( \frac{\left| \mathbb{E}[X_n | \mathcal{F}_0^t] - \mathbb{E}[X_n | \mathcal{F}_0^{t-1}] \right|}{K} \right) \mid \mathcal{F}_0^{t-1} \right]
\]

\[
= \mathbb{E} \left[ \exp \left( \frac{1}{K^2} \left( \mathbb{E}[X_n | \mathcal{F}_0^t] - \mathbb{E}[X_n | \mathcal{F}_0^{t-1}] \right)^2 \right) \right] - 1 \mid \mathcal{F}_0^{t-1}
\]

\[
\leq \mathbb{E} \left[ \exp \left( \frac{2}{K^2} \left( \mathbb{E}[X_n | \mathcal{F}_0^t]^2 + \mathbb{E}[X_n | \mathcal{F}_0^{t-1}]^2 \right) \right) \right] - 1 \mid \mathcal{F}_0^{t-1}
\]

\[
= \mathbb{E} \left[ \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^t]^2}{K^2} \right) \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^{t-1}]^2}{K^2} \right) \right] - 1 \mid \mathcal{F}_0^{t-1}
\]

\[
= \mathbb{E} \left[ \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^t]^2}{K^2} \right) \right] \mathbb{E} \left[ \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^{t-1}]^2}{K^2} \right) \right] - 1 \mid \mathcal{F}_0^{t-1}
\]

\[
\leq \mathbb{E} \left[ \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^t]^2}{K^2} \right) \right] \mathbb{E} \left[ \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^{t-1}]^2}{K^2} \right) \right] - 1
\]

\[
= \mathbb{E} \left[ \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^t]^2}{K^2} \right) \right] \mathbb{E} \left[ \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^{t-1}]^2}{K^2} \right) \right] - 1
\]

\[
= \mathbb{E} \left[ \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^t]^2}{K^2} \right) \right] \mathbb{E} \left[ \exp \left( \frac{\mathbb{E}[X_n | \mathcal{F}_0^{t-1}]^2}{K^2} \right) \right] - 1
\]

\[
\leq \mathbb{E} \left( \frac{|X_n|^2}{K^2} \right)^{t-1} - 1
\]

\[
\leq (\tau + 1)^2 - 1
\]
for $K = c\sqrt{2}$. Therefore, we have

$$B_n^2 = \sum_{i=1}^{n} 2c^2 \left( 1 + E \left[ \psi(|W_i|/\sqrt{2}c) \mid \mathcal{F}_0^{i-1} \right] \right) \leq 2nc^2(\tau + 1)^2,$$

and so,

$$P(X_n - E[X_n] > \epsilon) = P\left( \sum_{i=1}^{n} W_i > \epsilon \right) \leq \exp\left\{ -\frac{\epsilon^2}{32nc^2(\tau + 1)^2} \right\},$$

by Lem. 13. \qed