The critical point and the $p$-norm of $A_s$ and $C$-matrices

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Abstract

The $L$-matrix $A_s = [1/(n+s)]$ was introduced in [1]. As a surprising property, we showed that its 2-norm is constant for $s \geq s_0$, where the critical point $s_0$ is unknown but relies in the interval $(1/4, 1/2)$. In this note, using some delicate calculations we sharpen this result by improving the upper and lower bounds of the interval surrounding $s_0$. Moreover, we show that the same property persists for the $p$-norm of $A_s$ matrices. We also obtain the 2-norm of a family of $C$-matrices with lacunary sequences.

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1. Introduction

We encountered the $L$-matrices in studying the Hadamard multipliers in function spaces [2]. Characterizing $\mathcal{M}(X)$, the multiplier algebra of a Banach space $X$ of analytic functions on the open unit disc $\mathbb{D}$, is a very important subject in various studies of function spaces, e.g., zero sets, invariant subspaces, and cyclic elements [3]. In [2], it is shown that $h(z) = \sum_{n=0}^{\infty} c_n z^n$ is a Hadamard multiplier for every superharmonically weighted Dirichlet Space

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if and only if the infinite matrix

\[ T_h = \begin{pmatrix}
  c_1 - c_0 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \cdots \\
  0 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \cdots \\
  0 & 0 & c_3 - c_2 & c_4 - c_3 & \cdots \\
  0 & 0 & 0 & c_4 - c_3 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \]

acts as a bounded operator on \( \ell^2 \). This observation led to deeper study of \( C \)-matrices and \( L \)-matrices in [1] and [4], which are interesting subjects by themselves. More generally, it is often important to determine if an infinite matrix act as a bounded operator on some sequence space. Famous examples include the infinite Hilbert matrix [5] or the Cesàro matrix [6] which are important tools in approximation theory and in the study of divergent sequence, respectively.

Let \((a_n)_{n \geq 0}\) be a sequence of complex numbers. Then the infinite matrix

\[ A = \begin{pmatrix}
  a_0 & a_1 & a_2 & a_3 & \cdots \\
  a_1 & a_2 & a_3 & \cdots \\
  a_2 & a_2 & a_3 & \cdots \\
  a_3 & a_3 & a_3 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \]

is called an \( L \)-matrix. Abusing the notation, we will write \( A = [a_n] \) and, despite being slightly confusing, a general element of \( A \) will be denoted by \( a_{ij} \), where \( i \) and \( j \) run through \( \{0, 1, 2, \ldots\} \). Note that due to connections to function space theory and Taylor series of analytic functions on \( D \), the indices starts from zero. See also [4, Page 42] for another class of \( L \)-matrices, used in the theory of large linear systems. The infinite matrix

\[ C = \begin{pmatrix}
  a_0 & 0 & 0 & \cdots \\
  a_1 & a_1 & 0 & \cdots \\
  a_2 & a_2 & a_2 & \cdots \\
  a_3 & a_3 & a_3 & a_3 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \]

is also called a \( C \)-matrix, which as in the previous case will be denoted by \( C[a_n] \).
To the best of our knowledge, the first instance of an $L$-matrix being used in the literature is in 1983 in an article from Choi [8]. In this note, the author used the special $L$-matrix $A_1$ to show that the infinite Hilbert matrix act as a bounded operator on $\ell^2$. He even suggest that this is perhaps the quickest way to show the boundedness of the infinite Hilbert matrix as an operator from $\ell^2$ to $\ell^2$.

The $C$-matrices have received considerable attention over the past 30 years. Because of the close relation with the $L$-matrices, we will keep the notation $C$-matrices in this paper; however in most of the literature they are referred to by the name terraced matrices. This name seems to have been introduced by Rhaly in his note from 1989 [9]. The same author then published six more articles on this subject, with his latest being in 2013 [10]. Other examples of studies of the $C$-matrices include Roades [11], who provide lower bounds of $p$-norms of these matrices under certain restrictions on $p$, Almasri [12] show that the $C$-matrix defined by the sequence $1/n^\alpha$ is $p$-summing if and only if $\alpha > 1$, and Durna & Yildirim [13] introduced and studied the generalized terraced matrices.

Recently, the study of the $p$-norm of different infinite matrices has been an active research area in computational mathematics. For example, Ilkhan [14] study the $p$-norms of some matrix operators on Fibonacci weighted difference sequence space. Jevtić and Karapetrović [15] obtained some result on the infinite Hilbert matrix on spaces of Bergman-type. A related matrix is the so-called multiplicative Hilbert matrix $M$, the infinite matrix with entries $(\sqrt{mn \log(mn)})^{-1}$, where $m, n \geq 2$. In their note, Brevig and al. [16] obtained some results on the $p$-norm of this matrix and even more. As a final example, we mention Chalendar and Partington [17] who showed in their article that if $T$ is a bounded operator on $H^2$, then under certain natural conditions it will act as a bounded operator on $H^2(\beta)$ and it will satisfy the inequality $\|T\|_{H^2} \leq \|T\|_{H^2(\beta)}$. 

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2. Main results

In [1], we studied the $L$-matrix

$$A_s = \begin{bmatrix} \frac{1}{n+s} \\ \frac{1}{n+s} & \frac{1}{1+s} & \frac{1}{2+s} & \frac{1}{3+s} & \cdots \\ \frac{1}{n+s} & \frac{1}{1+s} & \frac{1}{2+s} & \frac{1}{3+s} & \cdots \\ \frac{1}{n+s} & \frac{1}{1+s} & \frac{1}{2+s} & \frac{1}{3+s} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and, surprisingly enough, we could precisely determine the norm for some values of $s$. As a matter of fact, we showed that

$$\|A_s\|_{\ell^2 \to \ell^2} = 4, \quad (s \geq \frac{1}{2}). \quad (2.1)$$

Since $a_0 = 1/s$, we certainly have

$$\|A_s\|_{\ell^2 \to \ell^2} \geq \frac{1}{s} > 4, \quad (0 < s < \frac{1}{4}).$$

Therefore, in the light of (2.1), an interesting question is to determine the critical point $s_0$, where

$$s_0 := \inf \{ s : \|A_s\|_{\ell^2 \to \ell^2} = 4 \}.$$ 

From the previous observations, we know that

$$\frac{1}{4} \leq s_0 \leq \frac{1}{2}.$$

We sharpen these estimations as follows.

**Theorem 2.2.** We have

$$\frac{\sqrt{6(8 + 3\sqrt{3})} - \sqrt{3} - 3}{12} \leq s_0 \leq \frac{1}{2\sqrt{2}}.$$ 

The above upper and lower estimations are highly non-trivial and involve delicate calculations.

We also study the $p$-norm of $A_s$ defined as

$$\|A_s\|_{\ell^p \to \ell^p} := \sup_{x \neq 0} \frac{\|A_s x\|_p}{\|x\|_p}.$$ 

In this case, the same interesting phenomenon persist, albeit for $s \geq 1$. 

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Theorem 2.3. Let \( s \geq 1 \). We have
\[
\|A_s\|_{\ell^p \to \ell^p} = \frac{p^2}{p-1}, \quad (1 < p < \infty).
\]
\[
(2.4)
\]
Lastly, we study the 2-norm of a special family of \( C \)-matrices with lacunary coefficient. To show that the necessary condition \( a_n = O(1/\sqrt{n}) \) for the \( L \)-matrix \( A = [a_n] \) to be a bounded operator on \( \ell^2 \) is sharp, we introduced in [4] the \( L \)-matrices with lacunary coefficients. We say that the sequence \( (a_n) \) is lacunary if there is a constant \( \rho > 1 \) and a subsequence \( (n_j)_{j \geq 1} \) of positive integers such that
\[
\frac{n_{j+1}}{n_j} \geq \rho
\]
and \( a_n = 0 \) except possibly for indices \( n \in \{n_j : j \geq 1 \} \). In particular, we introduced the special Cesàro type matrix \( C = C[a_n] \) defined by
\[
a_{N^j} = \frac{1}{N^{j/2}}, \quad (j \geq 1),
\]
\[
(2.5)
\]
and \( a_n = 0 \) for other values of \( n \), where \( N \geq 2 \) in a fixed integer. Then we showed that
\[
\frac{\sqrt{N}}{\sqrt{N} - 1} \leq \|C\|_{\ell^2 \to \ell^2} \leq \frac{\sqrt{N}}{\sqrt{N} - 1}.
\]
This estimation is not far from being optimal. However, with more elaborate calculation, it is possible to precisely determine \( \|C\|_{\ell^2 \to \ell^2} \).

Theorem 2.6. Let \( N \geq 2 \) be a fixed positive integer, and let \( C \) be defined by (2.5). Then
\[
\|C\|_{\ell^2 \to \ell^2} = \frac{\sqrt{N} - 1}{\sqrt{N} - 1}.
\]

3. Approximating \( s_0 \)

In this section, we present the proof of Theorem 2.2. Theorem 2 from [4] tells us that if \( A = [a_n] \) is an \( L \)-matrix and that there is a sequence of strictly decreasing positive numbers \( \delta_n, n \geq 0 \), such that
\[
\Delta := \sup_{n \geq 1} \frac{(|a_n| + \delta_{n-1})(|a_n| + \delta_n)}{\delta_{n-1} - \delta_n} < \infty.
\]
Then \( A \in \mathcal{L}(\ell^2) \) and, moreover,

\[
\|A\|_{\ell^2 \rightarrow \ell^2} \leq \max\{\delta_0 + |a_0|, \Delta\}.
\]

Consider the sequence \((\delta_n)\) defined by \( \delta_n = (n+s+\frac{1}{2})^{-1} \).

Then we know that

\[
\|A_s\|_{\ell^2 \rightarrow \ell^2} \leq \max\left\{ \delta_0 + |a_0|, \frac{|a_n| + \delta_{n-1}|a_n| + \delta_n}{\delta_{n-1} - \delta_n} (n \geq 1) \right\} = \max\left\{ \frac{1}{s + \frac{1}{2}} + \frac{1}{s}, 4 - \frac{1}{4(n+s)^2} (n \geq 1) \right\} = \max\left\{ \frac{1}{s + \frac{1}{2}} + \frac{1}{s}, 4 \right\}.
\]

Observe that \( f(s) := \frac{1}{s + \frac{1}{2}} + \frac{1}{s} \) is a strictly decreasing function. Thus, if \( s \geq \frac{1}{2\sqrt{2}} \), we have

\[
f(s) \leq f\left(\frac{1}{2\sqrt{2}}\right) = 4.
\]

Hence, we conclude that \( \|A_s\|_{\ell^2 \rightarrow \ell^2} \leq 4 \) whenever \( s \geq \frac{1}{2\sqrt{2}} \). Moreover, from the previous observations, we have \( \|A_s\|_{\ell^2 \rightarrow \ell^2} \geq 4 \) whenever \( s \leq 1/2 \) so we conclude that \( s_0 \leq \frac{1}{2\sqrt{2}} \). It is easy to show that the method outlined above is optimal for sequences \((\delta_n)\) of the form \( \delta_n = \alpha/(n + \beta) \).

Finding a good lower bound for the number \( s_0 \) turned out to be harder than the upper bound, mainly due to the fact that we do not have a similar result to Theorem 2 from [1] for lower bounds. However, we can show that

\[
s_0 \geq \frac{\sqrt{6(8 + 3\sqrt{3})} - \sqrt{3} - 3}{12} =: s^* \approx 0.347
\]

by using the fact that \( \|A_s\|_{\ell^2 \rightarrow \ell^2} \geq \|A_s x\|_2 / \|x\|_2 \) for every \( x \in \ell^2 \) and by carefully choosing the entries of the sequence \( x \). Note that the upper bound is \( s_0 \leq 1/(2\sqrt{2}) \approx 0.354 \).

Since for \( s < \frac{1}{4}, \) we have \( \|A_s\|_{\ell^2 \rightarrow \ell^2} \geq a_0 > 4, \) we assume without loss of generality that \( s \geq \frac{1}{4} \). Let \( x = (x_n) \) be a sequence of real numbers defined by

\[
x_n = \begin{cases} 
1 & \text{if } n = 0, \\
s(n+s)K_n & \text{if } n \geq 1,
\end{cases}
\]
where
\[ K_n = \frac{\Gamma(\beta)\Gamma(n + \beta - \alpha)}{\Gamma(n + \beta + 1)\Gamma(\beta - \alpha + 1)} \]

and
\[ \alpha = \frac{2}{4 + \varepsilon - \sqrt{(4 + \varepsilon)\varepsilon}}, \quad \beta = \frac{s^2}{\alpha((4 + \varepsilon)s - 1)}. \]

for \( \varepsilon > 0. \) Using Stirling’s formula, we see that
\[ K_n \approx n^{\alpha - 1} \]
and thus 
\[ x_n \approx n^{\alpha}, \]
since \( x_n \approx nK_n. \) Therefore, since
\[ \alpha = \frac{2}{4 + \varepsilon - \sqrt{(4 + \varepsilon)\varepsilon}} > \frac{1}{2} \quad \iff \quad \varepsilon > 0, \]
we have \( x \in \ell^2. \) Write \( y = A_s x. \) Hence,
\[
y_n = a_n \sum_{j=0}^{n} x_j + \sum_{j=n+1}^{\infty} a_j x_j
= a_n + s \left( a_n \sum_{j=1}^{n} (j + s)K_j + \sum_{j=n+1}^{\infty} K_j \right)
= a_n + s \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} \left( a_n \sum_{j=1}^{n} (j + s) \frac{\Gamma(j + \beta - \alpha)}{\Gamma(j + \beta + 1)} \right)
+ \sum_{j=n+1}^{\infty} \frac{\Gamma(j + \beta - \alpha)}{\Gamma(j + \beta + 1)}. \]

We know from a combinatorial identity that
\[
\sum_{j=1}^{n} \frac{\Gamma(j + b)}{\Gamma(j + c)} = \frac{\Gamma(n + b + 1)}{(1 + b - c)\Gamma(n + c)} - \frac{\Gamma(b + 1)}{(1 + b - c)\Gamma(c)}. \]

Thus, we can use this fact to find that
\[
y_n = (4 + \varepsilon)s \frac{\Gamma(n + b - \alpha + 1)\Gamma(\beta)}{(n + s)\Gamma(n + \beta)\Gamma(\beta - \alpha + 1)}; \quad (n \geq 0). \]

Observe that for \( n = 0, \) we have \( y_0 = 4 + \varepsilon \) so \( y_0 \geq (4 + \varepsilon)x_0. \) We will now show that this inequality is true for all \( n \geq 1, \) as long that \( \frac{1}{4} \leq s < s^*. \) We can write
\[
y_n = (4 + \varepsilon)\frac{(n + \beta - \alpha)(n + \beta)}{(n + s)^2} x_n, \quad (n \geq 1). \]
Thus we need to show that 
\[(n + \beta - \alpha)(n + \beta) \geq (n + s)^2, \text{ i.e., that}
\]
\[h_{\varepsilon,s}(n) := (2(\beta - s) - \alpha)n + \beta(\beta - \alpha) - s^2 \geq 0, \quad (n \geq 1). \tag{3.1}\]
Observe that \(\alpha, \beta\) are right-continuous function relative to \(\varepsilon\) at \(\varepsilon = 0\). Hence, if we define \(g(\varepsilon) := 2(\beta - s) - \alpha\), we have
\[|g(\varepsilon) - g(0)| < \eta \]
provided that \(0 < \varepsilon < \mu\), where \(\mu = \mu(\eta) > 0\). However, \(g(0) = \frac{1 - 8s^2}{2(4s - 1)} > 0\) if \(\frac{1}{4} < s < s^*\); thus we can set \(\eta = g(0)\) to be assured that there exist an \(\varepsilon\) small enough so that \(g(\varepsilon) > 0\).

We have just shown that the leading coefficient of (3.1) is non-negative. Therefore, we just have to make sure that \(h_{\varepsilon,s}(1) \geq 0\) to make sure that (3.1) hold. Similarly to what we just did, we can write \(f(\varepsilon) := h_{\varepsilon,s}(1)\) and observe that \(f\) is a right continuous function relative to \(\varepsilon\) at \(\varepsilon = 0\). So if \(f(0) > 0\), we are assured that there exist a small enough \(\varepsilon\) such that \(f(\varepsilon) = h_{\varepsilon,s}(1) \geq 0\).

A computation gives us
\[f(0) = \frac{-24s^4 - 24s^3 + 8s^2 + 4s - 1}{2(4s - 1)^2}.\]

Now, a simple analysis of this equation shows that \(f(0) > 0\) for \(\frac{1}{4} \leq s \leq s^*\), with equality if and only if \(s = s^*\). Hence, \(y_n \geq (4 + \varepsilon)x_n\) for every \(n \geq 0\), provided that \(s < s^*\) and \(\varepsilon\) is small enough.

Since \(y = A_s x\) is a positive sequence and \(x \in \ell^2\),
\[\|A_s x\|_2^2 = \sum_{n=0}^{\infty} y_n^2 \geq (4 + \varepsilon)^2 \sum_{n=0}^{\infty} x_n^2 = (4 + \varepsilon)^2 \|x\|_2^2,\]
if \(\varepsilon\) is small enough and \(s < s^*\). It follows that \(\|A\|_{\ell^2 \to \ell^2} > 4\) for all \(s \in (0, s^*)\) and thus, \(s_0 \geq s^*\).

4. The \(p\)-norm of \(A_s\)

In this section, we present the proof of Theorem \[\text{[2.3]}\]. Parallel to the definition of \(A_s\), consider the generalized Cesàro matrix
\[C_s = \begin{pmatrix}
\frac{1}{s} & 0 & 0 & 0 & \cdots \\
\frac{1}{1+s} & \frac{1}{1+s} & 0 & 0 & \cdots \\
\frac{1}{2+s} & \frac{1}{2+s} & \frac{1}{2+s} & 0 & \cdots \\
\frac{1}{3+s} & \frac{1}{3+s} & \frac{1}{3+s} & \frac{1}{3+s} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},\]
and the generalized Copson matrix $C_{s}^{tr}$. We know from [18] that, for $s \geq 1$,

$$
\|C_{s}\|_{\ell^{p} \to \ell^{p}} = q.
$$

It also follows that

$$
\|C_{s}^{tr}\|_{\ell^{p} \to \ell^{p}} = \|C_{s}\|_{\ell^{q} \to \ell^{q}} = p
$$

where $q$ is the Hölder conjugate of $p$, i.e., $1/p + 1/q = 1$.

As the first step, note that each of the entries of $A_{s}$ are positive and less than or equal to those of $C_{s} + C_{s}^{tr}$. Hence, we have

$$
\|A_{s}\|_{\ell^{p} \to \ell^{p}} \leq \|C_{s} + C_{s}^{tr}\|_{\ell^{p} \to \ell^{p}} \leq \|C_{s}\|_{\ell^{p} \to \ell^{p}} + \|C_{s}^{tr}\|_{\ell^{p} \to \ell^{p}} = q + p = pq = \frac{p^{2}}{p-1}.
$$

We then proceed to show that $\|A_{s}\|_{\ell^{p} \to \ell^{p}} \geq \frac{p^{2}}{p-1}$. Let

$$
x_{m} := \left( s^{-\frac{1}{p}}, (1 + s)^{-\frac{1}{p}}, (2 + s)^{-\frac{1}{p}}, \ldots, (m + s)^{-\frac{1}{p}}, 0, 0, \ldots \right)^{tr},
$$

We then have

$$
\|A_{s}x_{m}\|_{\ell^{p} \to \ell^{p}}^{p} \geq \sum_{n=0}^{m} \frac{1}{n + s} \left( \frac{1}{n + s} \sum_{k=0}^{n-1} (k + s)^{-\frac{1}{p}} + \sum_{k=n}^{m} (k + s)^{-\frac{1}{p}} \right)^{p} \geq \sum_{n=0}^{m} \frac{1}{n + s} \left( \frac{1}{n + s} \int_{0}^{n} (x + s)^{-\frac{1}{p}} dx + \int_{n}^{m+1} (x + s)^{-\frac{1}{p}} dx \right)^{p} = (pq)p \sum_{n=0}^{m} \frac{1}{n + s} \left( 1 - \frac{1}{p} \left( \frac{s}{n + s} \right)^{\frac{1}{q}} - \frac{1}{q} \left( \frac{m + s + 1}{n + s} \right)^{-\frac{1}{p}} \right)^{p}.
$$

Since the summand in the second line of the chain of equations is positive,

$$
\left( 1 - \frac{1}{p} \left( \frac{s}{n + s} \right)^{1/q} - \frac{1}{q} \left( \frac{m + s + 1}{n + s} \right)^{-1/p} \right)
$$

must also be positive. Hence, we can use Bernoulli inequality to deduce that $\|A_{s}x_{m}\|_{\ell^{p} \to \ell^{p}}^{p}$ is

$$
\geq (pq)^{p} \sum_{n=0}^{m} \frac{1}{n + s} \left( 1 - \frac{1}{p} \left( \frac{s}{n + s} \right)^{\frac{1}{q}} + \frac{1}{q} \left( \frac{m + s + 1}{n + s} \right)^{-\frac{1}{p}} \right) = (pq)^{p} \|A_{s}x_{m}\|_{\ell^{p} \to \ell^{p}}^{p} - \gamma_{m},
$$

where

$$
\gamma_{m} := (pq)^{p} \sum_{n=0}^{m} \frac{1}{n + s} \left( \left( \frac{s}{n + s} \right)^{\frac{1}{q}} + \frac{p}{q} \left( \frac{m + s + 1}{n + s} \right)^{-\frac{1}{p}} \right).
$$
This implies that
\[ \| A_s \|^p_{\ell^p \to \ell^p} \geq \frac{\| A_s x_m \|^p_{\ell^p \to \ell^p}}{\| x_m \|^p_{\ell^p \to \ell^p}} \geq (pq)^p - \frac{\gamma_m}{\| x_m \|^p_{\ell^p \to \ell^p}}. \]

It is enough now to show that \( \gamma_m / \| x_m \|^p_{\ell^p \to \ell^p} \to 0 \) whenever \( m \to \infty \). First, note that
\[ \lim_{m \to \infty} \| x_m \|^p_{\ell^p \to \ell^p} = \lim_{m \to \infty} \sum_{n=0}^m \frac{1}{n+s} = \infty. \quad (4.1) \]

Moreover, we have
\[ \gamma_m = (pq)^p \sum_{n=0}^m \left( \frac{1}{n+s} \left( \frac{s}{n+s} \right)^{\frac{1}{q}} + p \frac{1}{q} \frac{1}{n+s} \left( \frac{m+s+1}{n+s} \right)^{-\frac{1}{p}} \right) \]
\[ \leq c_1 \sum_{n=0}^m (n+s)^{-1/q-1} + c_2 (m+s+1)^{-1/p} \sum_{n=0}^m (n+s)^{1/p-1} \]
\[ \leq c'_1 (m+s+1)^{-1/q} + c'_2 (m+s+1)^{-1/p} (m+s+1)^{1/p} \]
\[ \leq c'_1 (s+1)^{-1/q} + c'_2. \]

Thus, the sequence \( \gamma_m \) is bounded and there exist a constant \( c \) such that \( \gamma_m \leq c \) for every \( m \geq 0 \). Hence,
\[ 0 \leq \frac{\gamma_m}{\| x_m \|^p_{\ell^p \to \ell^p}} \leq \frac{c}{\| x_m \|^p_{\ell^p \to \ell^p}}. \]

From (4.1), it follows that \( \gamma_m / \| x_m \|^p_{\ell^p \to \ell^p} \to 0 \) whenever \( m \to \infty \). Therefore, \( \| A_s \|_{\ell^p \to \ell^p} \geq pq = \frac{p^2}{p-1} \) and we are done.

5. The norm of a special lacunary C-matrix

In this section, we present the proof of Theorem 2.6. Suppose that \( y = C x \). Then, we have
\[ N^{n/2} y = \sum_{j=0}^N x_j + \sum_{j=N+1}^{N^2} x_j + \cdots + \sum_{j=N^{n-1}+1}^N x_j \]
and $y_k = 0$ for the other indices $k$. By Cauchy–Schwartz,

$$N^{n/2}y_{N^n} \leq (N + 1)^{\frac{1}{2}} \left( \sum_{j=0}^{N} x_j^2 \right)^{\frac{1}{2}} + (N^2 - N)^{\frac{1}{2}} \left( \sum_{j=N+1}^{N^2} x_j^2 \right)^{\frac{1}{2}} + \cdots + (N^n - N^{n-1})^{\frac{1}{2}} \left( \sum_{j=N^{n-1}+1}^{N^n} x_j^2 \right)^{\frac{1}{2}}.$$

Once more, use the Cauchy–Schwartz inequality to get

$$N^{n/2}y_{N^n} \leq \left[ (N + 1)^{t} + (N^2 - N)^{\frac{1}{2}} + \cdots + (N^n - N^{n-1})^{\frac{1}{2}} \right]^2.$$

Write

$$B_n := (N + 1)^{t} + (N^2 - N)^{\frac{1}{2}} + \cdots + (N^n - N^{n-1})^{\frac{1}{2}}$$

$$= (N + 1)^{t} + \sqrt{N - 1} \frac{\sqrt{N^n} - \sqrt{N}}{\sqrt{N} - 1}.$$

Then, for each $0 \leq t \leq 1$, we have

$$y^2_{N^n} \leq \frac{B_n(N + 1)^{1-t}}{N^n} \sum_{j=0}^{N} x_j^2 + \frac{B_n \sqrt{N^2 - N}}{N^n} \sum_{j=N+1}^{N^2} x_j^2 + \cdots + \frac{B_n \sqrt{N^n - N^{n-1}}}{N^n} \sum_{j=N^{n-1}+1}^{N^n} x_j^2.$$

Therefore,

$$\|Cx\|_2^2 = \sum_{n=0}^{\infty} y_n^2 = \sum_{n=1}^{\infty} y_{N^n}^2$$

$$\leq \eta_0 \sum_{j=0}^{N} x_j^2 + \eta_1 \sum_{j=N+1}^{N^2} x_j^2 + \eta_2 \sum_{j=N^2+1}^{N^3} x_j^2 + \cdots, \quad (5.1)$$
where

\[
\eta_0 = \sum_{n=1}^{\infty} \frac{B_n(N+1)^{1-t}}{N^n} = \sum_{n=1}^{\infty} \frac{1}{N^n} \left( (N+1)^t + \sqrt{N} - 1 \frac{\sqrt{N} - \sqrt{N}^n}{1 - \sqrt{N}} \right)
\]

\[
= (N+1)^{1-t} \frac{(N+1)^t(\sqrt{N} - 1) + \sqrt{N} - 1}{(\sqrt{N} - 1)(N-1)},
\]

and

\[
\eta_k = \sum_{n=k+1}^{\infty} \frac{B_n \sqrt{N^{k+1}} - N^k}{N^n} = \sqrt{N} - 1 \sqrt{N}^k \sum_{n=k+1}^{\infty} \frac{B_n}{N^n}
\]

\[
= \frac{N-1}{(\sqrt{N} - 1)^2} + \left( \frac{(N+1)^t}{\sqrt{N+1}} - \frac{\sqrt{N}}{\sqrt{N} - 1} \right) \frac{1}{\sqrt{N}^k}, \quad (k \geq 1).
\]

The upper bound (5.1) is valid for any \( t \in [0,1] \). However, the optimal \( t \) is

\[
t = 1 - \log_{N+1} \sqrt{N - 1}
\]

for which \( \eta_k \) reduces to

\[
\eta_k = \frac{N-1}{(\sqrt{N} - 1)^2} - \frac{1}{(\sqrt{N} + 1)\sqrt{N}^k}, \quad (k \geq 0).
\]

Thus, we have

\[
\eta_k \leq \frac{N-1}{(\sqrt{N} - 1)^2}, \quad (k \geq 0).
\]

This special choice of \( t \) implies

\[
\|Cx\|_2^2 \leq \frac{N-1}{(\sqrt{N} - 1)^2} \left( \sum_{j=0}^{N} x_j^2 + \sum_{j=N+1}^{N^2} x_j^2 + \cdots \right) = \frac{N-1}{(\sqrt{N} - 1)^2} \|x\|_2^2.
\]

Hence,

\[
\|C\|_{\ell^2 \to \ell^2} \leq \frac{\sqrt{N-1}}{\sqrt{N-1}}.
\]

We now proceed to show that this upper bound is attained.
Let \( x := (x_k) \) be the following sequence: \( x_k \) equals 1 for indices between 0 and \( N \), and equals \( 1/\sqrt{N} \) for every indices between \( N^n + 1 \) to \( N^{n+1} \), for \( n = 1, 2, \ldots, m \), and finally equals 0 everywhere else. Then

\[
\|x\|_2^2 = \sum_{j=0}^{N} x_j^2 + \sum_{j=N+1}^{N^2} x_j^2 + \cdots + \sum_{j=N^{m-1}+1}^{N^m} x_j^2 \\
= N + 1 + (N^2 - N) \frac{1}{\sqrt{N}} + \cdots + (N^m - N^{m-1}) \frac{1}{\sqrt{N^{m-1}}} \\
= 2 + (N - 1)m.
\]

Once more, write \( y = Cx \). Then

\[
N^{n/2} y_{N^n} = \sum_{j=0}^{N} x_j + \sum_{j=N+1}^{N^2} x_j + \cdots + \sum_{j=N^{n-1}+1}^{N^n} x_j \\
= N + 1 + (N^2 - N) \frac{1}{\sqrt{N}} + (N^3 - N^2) \frac{1}{\sqrt{N^2}} \\
+ \cdots + (N^n - N^{n-1}) \frac{1}{\sqrt{N^{n-1}}} \\
= (\sqrt{N} + 1)\sqrt{N^n} - (\sqrt{N} - 1),
\]

for every \( n \leq m \), and thus

\[
y_{N^n}^2 = \left( (\sqrt{N} + 1) - \frac{\sqrt{N} - 1}{\sqrt{N^n}} \right)^2.
\]

Hence,

\[
\|Cx\|_2^2 \geq \sum_{n=1}^{m} y_{N^n}^2 = \sum_{n=1}^{m} \left( (\sqrt{N} + 1) - \frac{\sqrt{N} - 1}{\sqrt{N^n}} \right)^2 \\
\geq (\sqrt{N} + 1)^2 m + c,
\]

for a certain constant \( c \) (for example, \( c = -2(\sqrt{N} + 1) \) works). Therefore,

\[
\|C\|_{\ell^2 \to \ell^2}^2 \geq \frac{\|Cx\|_2^2}{\|x\|_2^2} \geq \frac{(\sqrt{N} + 1)^2 m + c}{2 + (N - 1)m}
\]

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for every $m \geq 1$. By letting $m \to \infty$, we get

$$\|C\|_{\ell^2 \to \ell^2}^2 \geq \frac{(\sqrt{N} + 1)^2}{N - 1} = \frac{N - 1}{(\sqrt{N} - 1)^2}.$$

Therefore,

$$\|C\|_{\ell^2 \to \ell^2} \geq \sqrt{\frac{N - 1}{\sqrt{N} - 1}},$$

and thus the equality follows.

6. Concluding remarks

(i) The precise value of $s_0$ is still unknown. Find $s_0$?

(ii) In the light of Theorem 2.3, we define

$$s_0^{(p)} := \inf \{ s : \|A_s\|_{\ell^p \to \ell^p} = \frac{p^2}{p-1} \}.$$

That theorem ensures

$$s_0^{(p)} \leq 1.$$

Find $s_0^{(p)}$.

(iii) Find $\|A_s\|_{\ell^p \to \ell^q}$, where $p \neq q$.

(iv) As in the case $p = q$, does $\|A_s\|_{\ell^p \to \ell^q}$ remain constant for large values of $s$? If so, we define $s_0^{(pq)}$ by slightly modifying the definition of $s_0^p$ given in (ii). Then how does $s_0^{(pq)}$ depend on the parameters $p$ and $q$?

Declaration of competing interest

The authors declare that they have no competing interests.

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