Rotating relativistic stars: two problems

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Abstract. In this article we discuss two problems related to rotating stars. The first part of
the article deals a recent suggestion [11] that w-modes of supercompact stars might become CFS
unstable, while in the second part we present the evolution equations for fluid perturbations of
fast rotating stars and we present some preliminary numerical results.

1. Introduction
The oscillations and instabilities of relativistic stars gained a lot of interest in the last decades
due to the possible detection of their associated gravitational waves. It is not impossible
that every compact star in a specific period of its life will undergo an oscillatory phase, in which
it might become rotationally unstable. Only the dynamical instabilities were thought to be a
relevant source of gravitational waves, whereas instabilities due to dissipation mechanisms were
believed to be only of academic interest. This belief has been dramatically altered during the
last seven years after it was discovered that for a specific class of rotational perturbations, the
so-called r-modes [1, 2], the instability due to gravitational radiation [3, 4, 5, 6] (CFS instability)
has the potential of being a prime source for gravitational waves. It was subsequently shown that
this instability has many interesting astrophysical implications, which attracted the attention
of both relativists and astrophysicists. For an extensive review refer to [7]. Recent numerical
studies of the CFS instability of the f-mode suggested that the instability might be considerably
more important than what was thought before [8, 9, 10].

In a recent article it has been shown [11] that the mainly spacetime, w-modes may become
unstable in a rotating ultra-compact (R < 3M) relativistic star. The study of the axial modes of
a rotating star in the slow rotation approximation suggested also that the CFS instability may
become active in stars with an ergosphere. These results suggest that the so-called “ergoregion
instability” discussed by Friedman [12], in fact, is associated with the w-modes of the star. The
unstable modes are found to grow on a timescale of seconds to minutes.

In most of the studies related to the rotational instabilities the slow rotation approximation
has been used either in Newtonian theory or in General Relativity. In the second part of
this article we derive the perturbation equations for the fluid (Cowling approximation) for fast
rotating relativistic stars and we show some initial numerical results.

2. The CFS instability for the w-modes
Just like a rotating black hole, a spinning neutron star can develop an ergosphere. The ergosphere
is defined as a region where all observers must be dragged along with the rotation, and no
observer can remain at rest. It is also associated with the Penrose process and superradiance
and the instability that we discuss sets in due to the existence of the ergosphere. All three phenomena are emerging from the the basic property of the ergosphere that “...no observer can remain at rest”. Mathematically, the ergosphere is distinguished as the part of a stationary spacetime where a spacelike Killing vector becomes timelike. Hence, the boundary of the ergosphere is defined by the change in the sign of the $[tt]$ component of the metric. In the case of relativistic stars, an ergosphere may develop inside the stellar fluid if the star is sufficiently compact ($R < 3M$).

It is by now well known that a neutron star spacetime with an ergosphere may become unstable. This was first demonstrated by Friedman [12] for scalar and electromagnetic perturbations. The problem was studied in detail by Comins and Schutz [13] who considered scalar fields in the background of a slowly rotating star. Their approach had the advantage that the problem could be reduced to one dimension, which facilitated the use of the WKB approximation to calculate the unstable modes of oscillation (for large $m$, where the perturbations are assumed to be proportional to $e^{im\phi}$). Comins and Schutz found that the growth time of the instability would be very large (of the order of the age of the universe). This would mean that, even if sufficiently compact and rapidly rotating stars would form, the instability would not be astrophysically important. More recently, the problem was revisited by Yoshida and Eriguchi [14] who developed a new numerical scheme which allowed them to extended the results of Comins and Schutz to smaller values of $m$. Their results showed that the low-order (in $m$) modes would grow much faster than indicated by the asymptotic results of Comins and Schutz.

Since the first studies of the ergoregion instability were carried out the existence of pulsation modes that are essentially due to the spacetime itself has been established [15]. These are known as the w-modes [15], and they only rely on the background curvature generated by the fluid in order to exist. The actual coupling to the stellar fluid is weak, and the mode mainly excites oscillations of the spacetime. An extreme case concerns axial perturbations of a non-rotating perfect fluid star. In this case no non-trivial fluid modes of oscillations exist. Yet they have an infinite set of w-modes. A similar set of spacetime modes exists for polar perturbations, although in that case there are also non-trivial fluid modes (for example, the f-mode and the acoustic p-modes), see [16] for an exhaustive discussion. In general, the w-modes have high frequencies, with the fundamental mode in the range 5-12kHz, and very short damping times, of the order of a tenth of a millisecond. Before the w-modes were discovered Chandrashekhar and Ferrari [17] suggested that ultra-compact stars (with radius smaller than $3M$) can support oscillation modes due to the trapping of gravitational waves by the curvature potential (familiar from studies of perturbed black holes). Detailed studies [16, 18] have shown that these long-lived modes are, in fact, the first few w-modes of an ultra-compact star. The key difference is that, because these modes are trapped inside the peak of the curvature potential their damping can be very slow (the damping time is roughly proportional to the width of the potential barrier).

As we will demonstrate below, these trapped w-modes may become unstable if the star is set into rotation. We will show that this instability only becomes active after the star has developed an ergosphere. This means that the gravitational-wave version of the ergoregion instability [12, 13, 14] sets in through the spacetime w-modes.

Assuming that the star is slowly rotating with a uniform angular velocity $\Omega$, we neglect all terms of order higher than $\Omega$. In this approximation, the star remains spherical, because the deformation due to centrifugal forces is of order $\Omega^2$. Thus, the metric can be written in the form

$$ds_0^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) - 2\omega r^2 \sin^2 \theta dt d\phi ,$$

(1)

where $\nu$, $\lambda$ and the “frame dragging” $\omega$ are functions of the radial coordinate $r$ only. With the neutron star matter described by a perfect fluid with pressure $p$, energy density $\epsilon$, and
four-velocity
\[ (u^t, u^r, u^\theta, u^\phi) = (e^{-\nu}, 0, 0, \Omega e^{-\nu}) \]
the Einstein equations, together with a one-parameter equation of state \( p = p(\epsilon) \), yield the well-known TOV equations plus an extra equation for the function \( \varpi \), defined as \( \varpi = \Omega - \omega \). To linear order, this equation is
\[ \varpi'' - 4[\pi r e^{2\lambda}(p + \epsilon) - r^{-1}]\varpi' - 16\pi e^{2\lambda}(p + \epsilon) \varpi = 0. \]
In the exterior, it reduces to \( \varpi'' + 4r^{-1}\varpi' = 0 \), for which we have the solution \([19]\]
\[ \varpi = \Omega - 2Jr^{-3}, \]
with \( J \) being the total angular momentum of the neutron star.

In order to establish the existence of the w-mode instability we study the effect of rotation on the axial w-modes within the slow rotation formalism. In writing down the perturbation equations we will keep only terms of first order in the angular frequency of the star (\( \Omega \)). Hence, we account for the relativistic frame dragging, but do not consider the rotational deformation of the star (which is an order \( \Omega^2 \) effect). In addition, we neglect the coupling between axial and polar perturbations.

If we focus on purely axial perturbations, the perturbed metric can be written as
\[ ds^2 = ds_0^2 + 2 \sum_{l,m} \left( h_0^{lm}(t,r) dt + h_1^{lm}(t,r) dr \right) \left( -\sin^{-1} \theta \partial_\phi Y_{lm} d\theta + \sin \theta \partial_\theta Y_{lm} d\phi \right), \]
where \( Y_{lm} = Y_{lm}(\theta, \phi) \) denote the scalar spherical harmonics. The fluid is assumed to rotate with uniform angular velocity \( \Omega \), and the axial component of the contravariant fluid velocity perturbation can be expanded as
\[ 4\pi r^2(p + \epsilon) \left( \delta u^\theta, \delta u^\phi \right) = e^\nu \sum_{l,m} U^{lm}(t,r) \left( -\sin^{-1} \theta \partial_\phi Y_{lm} \sin \theta \partial_\theta Y_{lm} \right) \]
where \( p \) is the pressure, \( \epsilon \) the energy density.

With these definitions Einstein’s field equations reduce to three equations for the three variables \( h_0^{lm}, h_1^{lm} \) and \( U^{lm} \) \([20]\). Assuming a harmonic time dependence \( e^{i\sigma t} \), the perturbation equations can be written as two ODEs for \( h_0 \) and \( h_1 \) and an algebraic relation for \( U \);
\[ h_1' = \left[ \lambda' - \nu' + \frac{m\omega'\Lambda - 2}{\Lambda(\sigma + m\omega)} \right] h_1 + \left[ \frac{2m^2 r^2}{\Lambda} \left( \frac{\omega'^2 e^{-2\lambda}}{\Lambda(\sigma + m\omega)} - \frac{16\pi m^2(p + \epsilon)}{2m\varpi - \Lambda(\sigma + m\Omega)} \right) + (\sigma + m\omega) \right] e^{2\lambda - 2\nu} h_0 \]
\[ h_0' = 2 \left[ \frac{1}{r} + \frac{m\omega'}{\Lambda(\sigma + m\omega)} \right] h_0 + \left[ (\sigma + m\omega) - \frac{e^{2\nu}(\Lambda - 2)}{r^2(\sigma + m\omega)} \right] ih_1 \]
\[ U = 4\pi e^{-2\nu}(p + \epsilon) \left[ 1 - \frac{2m\varpi}{2m\varpi - \Lambda(\sigma + m\Omega)} \right] h_0 \]
where \( \Lambda = l(l + 1) \).

Our numerical results for uniform density ultra-compact stars are illustrated in Fig. 1. The two frames show the oscillation frequency and damping/growth time of the first w-mode for a star with mass \( 1.4M_\odot \), the canonical value for a neutron star and four different compactness ratios. The illustrated mode is such that its pattern moves backwards with respect to the
Figure 1. Oscillation frequency (left frame) and damping/growth time (right frame) of the first \(w\)-mode of ultra-compact stars. The rotation rate of the star is represented by \(\varepsilon = \Omega / \Omega_K\). For positive real frequencies, the modes are stable and the right frame shows the damping times. The \(w\)-mode instability is active beyond the point where a neutral mode exists, i.e. whenever the mode frequency is negative. In this case the right frame shows the growth times. As is clear from the data, the instability can grow on a timescale of a few tenths of a second to minutes in very rapidly rotating stars.

rotation as \(\Omega \to 0\). As the rotation rate, represented by \(\varepsilon = \Omega / \Omega_K\) (where \(\Omega_K\) is the mass shedding limit), increases the frequency of the mode decreases and passes through zero at a critical value. Beyond this point the emerging gravitational waves drive the mode unstable. The onset of instability is also signaled by a singularity in the growth time since a marginally unstable mode would not radiate at all. These singularities are obvious in the right frame of Fig. 1. For the calculation of the complex quasi-normal frequencies we have assumed an outgoing boundary condition at infinity. The numerical techniques used are described in [21].

It is relevant to compare the critical rotation rate at which the first \(w\)-mode becomes unstable to the Kepler limit. For the cases shown in Fig. 1 we find that the instability becomes active at 0.19\(\Omega_K\) for \(R/M = 2.26\), at 0.33\(\Omega_K\) for \(R/M = 2.28\), at 0.44\(\Omega_K\) for \(R/M = 2.30\) and at 0.83\(\Omega_K\) for \(R/M = 2.40\). In other words, for stars near the absolute limit of compactness allowed by General Relativity the \(w\)-modes become unstable already at quite low rates of rotation (below 20\% of the mass-shedding limit). As is clear for the right frame in Fig. 1 the associated growth times can be as short as a few tenths of seconds to minutes. Thus the growth time of the \(w\)-mode instability can be very short (Our results indicate that the unstable gravitational perturbations grow at least four orders of magnitude faster than the scalar field perturbations considered by e.g. Yoshida and Eriguchi [14]. Nevertheless, we find that the unstable \(w\)-modes grow slower than the \(l = m = 2\) \(r\)-mode in all cases we have considered. We have also verified that the growth times scale with the rotation rate as \(\sim \Omega^{2l+2}\) as one would expect. One word of caution should be placed. As are working in the slow rotation limit and, in addition, neglect the coupling of our axial equations to the polar ones, it is clear that we cannot expect our results to be valid in the limit \(\varepsilon \to 1\). Nevertheless we do not expect these approximation to alter the qualitative features of our results. Moreover, we also expect the polar \(w\)-modes to become unstable, hence the inclusion of the polar equations should not be able to weaken the instability.

The class of “trapped” \(w\)-modes which becomes unstable consists of a finite number of modes. The number of the “trapped” modes increases with the compactness of the star and several of these modes become unstable for higher rotational rates. For example for a star with compactness \(R/M = 2.26\) near the mass shedding limit more than 4 modes become unstable. We now want to establish that the \(w\)-modes will only be unstable in stars that have developed
an ergosphere. To test the existence of an ergosphere we follow Schutz and Comins [22] and approximate $g_{tt}$ as

$$g_{tt} = -e^{2\nu} + \omega^2 r^2 \sin^2 \theta \quad (10)$$

Given this, an ergosphere is present whenever $g_{tt} > 0$, i.e. restricting ourselves to the equatorial plane $\theta = \pi/2$ we require $e^{2\nu} < r \omega$. Figure 2 illustrates the ergosphere (in the equatorial plane) for our ultra-compact stellar models and varying rates of rotation. Also shown in the figure are the critical values of the rotation parameter $\varepsilon$ at which the first $w$-mode becomes unstable. As is clear from the figure, the $w$-mode only becomes unstable once the star has developed an ergosphere.

Our results show that the instability come into operation after the star has developed an ergosphere, and that the unstable modes can grow quite fast—on a timescale of seconds to minutes. The $w$-mode instability is a variation of the gravitational-wave driven (CFS) instability, which was discovered by Chandrasekhar [3, 4] for the Maclaurin spheroids and subsequently investigated in detail by Friedman and Schutz [5, 6]. The key difference between the $w$-mode instability and the standard CFS instability of fluid oscillation modes (like the axial $r$-mode [7]) is that the $w$-modes are mainly due to spacetime oscillations. They are gravitational waves trapped (in the present case) inside the peak of the curvature potential. Their instability can be understood from the fact that an oscillation mode can be associated with a negative energy provided that the star has an ergosphere.

As an interesting aside it is worth commenting on the fact that despite them being endowed with an ergosphere, rotating black holes do not suffer an analogous instability. The reason for this is simple: In the black hole case, the leakage of energy through the horizon proceeds fast enough to stabilize the system.

The demonstration that the $w$-modes may become unstable is interesting from a conceptual point of view. However, the astrophysical relevance of this instability is questionable. The main reason for this is that it is very difficult to construct realistic stable stellar models which would exhibit an ergosphere [23, 24]. Indeed, there are very few “realistic” supra-nuclear equations of state that permit stable stellar models significantly more compact than $R \approx 3M$. A possibility that cannot yet be completely ruled out is that of ultracompact strange stars. One can, in principle, generate very compact quark stars but we have not yet any observational evidence for...
the existence of such objects.

Finally, it is worth discussing two features that make the w-mode instability differ somewhat from the more familiar CFS instability of fluid oscillations modes. The first concerns dissipation. Since the w-modes are spacetime modes that hardly excite any fluid motion there are no apparent dissipation mechanisms. The viscous damping of the instability will certainly be significantly weaker than that of (say) the r-modes since the viscosity will only act on a relatively small part of the w-mode energy. The second point concerns the saturation of the unstable mode. Recent work suggests that the unstable r-modes saturate at a relatively low amplitude because of strong coupling to short wavelength inertial modes. The weak coupling to the stellar fluid in the present case may mean that this mechanism will be significantly less efficient, if of any relevance, in the case of unstable w-modes. In fact, one could argue that these modes ought to saturate because of nonlinear spacetime effects. More detailed investigations into these issues would be of interest since the may help improve our understanding of nonlinear dynamical spacetimes.

3. Perturbations of fast rotating relativistic stars

The slow rotation approximation that has been used in the previous section is a very good approximation for cases related even to millisecond pulsars. Actually, for the fast known rotating millisecond pulsar, $P = 1.56\text{ms}$ the expansion parameter $\varepsilon$ is about $\sim 0.25$. Still, the nascent neutron stars can rotate even faster and in this case the slow rotation approximation breaks down. The study of the eigenfrequencies of fast rotating neutron stars has begun only recently e.g. for the r-modes in the Cowling approximation (the spacetime perturbations are considered as frozen) using a boundary value problem approach [25] or via studies of nonlinear oscillations [26, 27]. Here we present an attempt to derive the eigenfrequencies of fast rotating compact stars via the time dependent perturbation equations in the Cowling approximation.

The equilibrium configuration of a fast rotating relativistic star can be described by standard tools: a metric, an energy-momentum tensor and an equation of state. For a detailed review of the properties of fast rotating relativistic stars, see [28].

A fast rotating relativistic star in equilibrium is described by a stationary, axisymmetric metric of the form:

$$ds_0^2 = -e^{2\nu}dt^2 + e^{2\psi}r^2\sin^2\theta(d\phi - \omega dt)^2 + e^{2\mu}(dr^2 + r^2d\theta^2)$$  \hspace{1cm} (11)

where $\nu$, $\mu$, $\omega$ and $\psi$ are four functions, known as “metric potentials”, which depend on the coordinates $r$ and $\theta$ only.

Like in Eq. (2), $u^r = u^\theta = 0$, and the non-vanishing components of the fluid’s four-velocity, $u^t$ and $u^\phi$, are related by the angular velocity $\Omega$:

$$\Omega = \frac{u^\phi}{u^t}$$  \hspace{1cm} (12)

Additionally, the four-velocity components should satisfy the condition $u^au_a = -1$ which is the so-called normalization condition. This condition determines the exact form of the four-velocity component $u^t$ and, in favor of Eq.(12), of $u^\phi$. So, we have:

$$u^au_a = -1 \rightarrow u^t = \frac{e^{-\nu}}{\sqrt{1 - e^{-2\nu+2\psi}r^2\sin^2\theta(\Omega - \omega)^2}}$$  \hspace{1cm} (13)

Both $u^t$ and $u^\phi$ are functions of $r$ and $\theta$. Furthermore, we should make a special comment for the angular velocity $\Omega$. In general, this quantity is also a function of $r$ and $\theta$ allowing the star to rotate differentially. However, shortly after the birth of the neutron stars, differential rotation is expected to become uniform due to viscosity. So it is enough to limit ourselves in the case of $\Omega$ being just a constant.
We assume a perfect fluid in the well-known form for the energy-momentum tensor:

$$T^{\alpha \beta} = (\epsilon + p) u^\alpha u^\beta + pg^{\alpha \beta}$$  \hspace{1cm} (14)$$

where \( \epsilon = \epsilon(r, \theta) \) is the energy density and \( p = p(r, \theta) \) the pressure. Energy density and pressure are not independent. They are related by a one-to-one relation, a barotropic equation of state. A commonly used equation of state is the polytropic one which relates energy density \( \epsilon \) and pressure \( p \) via the notion of rest-mass density \( \rho \). The defining relations are:

$$p = K \rho^\gamma = K \rho^{1+\frac{1}{n}} \quad \text{with} \quad \epsilon = \rho \frac{p}{\gamma - 1} = \rho + np$$  \hspace{1cm} (15)$$

where \( K \) is the polytropic constant, \( \gamma \) the polytropic exponent and \( n \) the polytropic index. The sound speed \( c_s^2 \) for a polytropic equation of state is becoming:

$$c_s^2 = \frac{dp}{d\epsilon} = \ldots = \frac{p\gamma}{\epsilon + p}$$  \hspace{1cm} (16)$$

The equilibrium configuration will be constructed by using the local law of energy-momentum conservation:

$$\nabla_\beta T^{\alpha \beta} = 0$$  \hspace{1cm} (17)$$

When expanded, this law leads to the following set of elliptic equations:

$$\left[ e^{2\psi} r^2 \sin^2 \theta (\Omega - \omega)^2 - e^{2\nu} \right] \frac{\partial p}{\epsilon + p} - e^{2\nu} \partial_r \nu + e^{2\psi} r^2 \sin^2 \theta (\Omega - \omega)^2 \partial_\psi$$

$$- e^{2\psi} r^2 \sin^2 \theta (\Omega - \omega) \partial_r \omega + e^{2\psi} r^2 \sin^2 \theta (\Omega - \omega)^2 \frac{1}{r} = 0$$  \hspace{1cm} (18)$$

$$\left[ e^{2\psi} r^2 \sin^2 \theta (\Omega - \omega)^2 - e^{2\nu} \right] \frac{\partial p}{\epsilon + p} - e^{2\nu} \partial_\theta \nu + e^{2\psi} r^2 \sin^2 \theta (\Omega - \omega)^2 \partial_\psi$$

$$- e^{2\psi} r^2 \sin^2 \theta (\Omega - \omega) \partial_\theta \omega + e^{2\psi} r^2 \sin^2 \theta (\Omega - \omega)^2 \frac{\cos \theta}{\sin \theta} = 0$$  \hspace{1cm} (19)$$

In the non-rotating limit Eq.(18) reduces to the well-known TOV equation \( \partial_r p + (\epsilon + p) \partial_r \nu = 0 \). One can easily verify this by setting \( \Omega = \omega = 0 \) in Eq.(18).

In what follows, we assume that all equilibrium quantities are known at every point of the two-dimensional subspace \((r, \theta)\). Actually, this information is provided by Stergioulas’ numerical code rns which is available as a public domain code and can be downloaded from [29].

Here we use the so-called Cowling approximation in which all metric perturbations are set to zero, i.e. \( \delta g_{\alpha \beta} = 0 \). The components of the fluid’s four-velocity are \( (\delta u^t, \delta u^r, \delta u^\theta, \delta u^\phi) \). In fact, two of these perturbations are related i.e. the normalization of the four-velocity \((u^\alpha + \delta u^\alpha)(u^\alpha + \delta u^\alpha) = -1\) leads to the relation:

$$\delta u^t = -\frac{u_\phi}{u_t} \delta u^\phi$$  \hspace{1cm} (20)$$

In this way the only independent components of the 4-velocity are the spatial ones i.e. \( \delta u^r, \delta u^\theta \) and \( \delta u^\phi \) while the perturbation of the temporal component \( \delta u^t \) is given by Eq.(20).

The energy-momentum tensor, Eq.(14), is also perturbed linearly and it has six independent components \( \delta T^{tt} = Q_1, \delta T^{t\phi} = Q_2, \delta T^{rr} = Q_3, \delta T^{r\phi} = Q_4, \delta T^{\phi\phi} = Q_5 \) and \( \delta T^{\theta\theta} = Q_6 \). The perturbations of energy density \( \epsilon \) and pressure \( p \) are \( \delta \epsilon \) and \( \delta p \) respectively and they are related as \( \delta p = p(\epsilon + p)^{-1} \delta \epsilon \). This is the so-called adiabatic condition. By making use of this condition and Eq.(20) the number of independent components are reduced to four. Furthermore, the
adiabatic constant $\Gamma$ is taken to be equal to the polytropic constant $\gamma$ of Eq.(16), ($\Gamma = \gamma$), i.e. we are dealing with barotropic stars. In this case, the sound speed $c_s^2$ relates the perturbations of energy density and pressure: $\delta p = c_s^2 \delta \epsilon$.

The perturbation equations are derived by the linear variation of the local law of energy-momentum conservation, Eq.(17):

$$\delta \left( \nabla_\beta T^{\alpha \beta} \right) = 0 \rightarrow \partial_\beta \delta T^{\alpha \beta} + \delta \Gamma^\alpha_{\gamma \beta} T^{\gamma \beta} + \delta \Gamma^\gamma_{\gamma \beta} T^{\alpha \gamma} + \Gamma^\beta_{\gamma \beta} \delta T^{\alpha \gamma} = 0 \quad (21)$$

In the Cowling approximation $\delta \Gamma^\alpha_{\beta \gamma} = 0$ which leads to further simplification:

$$\partial_\beta \delta T^{\alpha \beta} + \delta \Gamma^\alpha_{\gamma \gamma} T^{\gamma \beta} + \Gamma^\beta_{\gamma \beta} \delta T^{\alpha \gamma} = 0 \rightarrow \nabla_\beta \delta T^{\alpha \beta} = 0 \quad (22)$$

This relation provides four evolution equations for the quantities $Q_1$, $Q_2$, $Q_3$ and $Q_4$ and one additional constraint equation for $Q_1$, $Q_2$ and $Q_6$ which is derived from the adiabatic condition. In particular, the expansion of Eq.(22) results in four evolution equations:

\[
\begin{align*}
\partial_t Q_1 &= -\partial_\phi Q_2 - \partial_r Q_3 - \partial_\theta Q_4 \\
&- \left[ e^{-2\nu+2\mu} r^2 \sin^2 \theta (\Omega - \omega) \partial_r \omega + \partial_r \psi + 2 \partial_r \mu + 3 \partial_\nu + \frac{2}{r} \right] Q_3 \\
&- \left[ e^{-2\nu+2\mu} r^2 \sin^2 \theta (\Omega - \omega) \partial_\theta \omega + \partial_\theta \psi + 2 \partial_\theta \mu + 3 \partial_\nu + \cot \theta \right] Q_4 \\
\partial_t Q_2 &= \Omega^2 \partial_\phi Q_1 - 2 \Omega \partial_\phi Q_2 - \Omega \partial_\phi Q_3 - \Omega \partial_\phi Q_4 + \left[ e^{-2\nu+2\mu} (\Omega - \omega)^2 - e^{-2\nu+2\mu} \sin^2 \theta \right] \partial_\phi Q_6 \\
&- \left[ e^{-2\nu+2\mu} r^2 \sin^2 \theta (\Omega - \omega) \omega \partial_r \omega + \partial_r (\Omega - \omega) + (3 \Omega - 2 \omega) \partial_r \psi \\
&+ 2 \Omega \partial_r \mu + (\Omega + 2 \omega) \partial_r \nu + (2 \Omega - \omega) \frac{2}{r} \right] Q_3 \\
&+ \left[ e^{-2\nu+2\mu} r^2 \sin^2 \theta (\Omega - \omega) \omega \partial_\theta \omega + \partial_\theta (\Omega - \omega) + (3 \Omega - 2 \omega) \partial_\theta \psi \\
+ 2 \Omega \partial_\theta \mu + (\Omega + 2 \omega) \partial_\theta \nu + (3 \Omega - 2 \omega) \cos \theta \frac{\cos \theta}{\sin \theta} \right] Q_4 \\
\partial_t Q_3 &= -\Omega \partial_\phi Q_3 - \partial_r Q_6 \\
&- \left\{ e^{-2\nu+2\mu} \partial_r \nu + e^{-2\nu+2\mu} r \sin^2 \theta \left[ (\Omega^2 - \omega^2) \cos \theta + \sin \theta \partial_\theta \nu \right] - \sin \theta \partial_\theta \omega \right\} Q_1 \\
&- e^{-2\nu+2\mu} r \sin^2 \theta \left[ \sin \theta \partial_\nu \omega - 2 (\Omega - \omega) (1 + \nu \partial_r \psi) \right] Q_2 \\
&- \left[ \partial_r \nu + 2 \partial_r \mu + e^{-2\nu+2\mu} r^2 \sin^2 \theta (\Omega - \omega)^2 (1 + \nu \partial_r \psi) \right] Q_6 \\
\partial_t Q_4 &= -\Omega \partial_\phi Q_4 - \frac{1}{r^2} \partial_\theta Q_6 \\
&- \frac{1}{r^2} \left\{ e^{-2\nu+2\mu} \partial_\theta \nu + e^{-2\mu+2\nu} r^2 \sin \theta \cos \theta \sin \theta + \sin \theta \omega \partial_\theta \omega \right\} Q_1 \\
&- \frac{1}{r^2} e^{-2\mu+2\nu} r^2 \sin \theta \left[ \sin \theta \partial_\theta \omega - 2 (\Omega - \omega) (\cos \theta + \sin \theta \partial_\theta \psi) \right] Q_2 \\
&- \frac{1}{r^2} \left[ \partial_\theta \nu + 2 \partial_\theta \mu + e^{-2\nu+2\mu} r^2 \sin \theta (\Omega - \omega)^2 (\cos \theta + \sin \theta \partial_\theta \psi) \right] Q_6 \\
\end{align*}
\]

On the other hand, the adiabatic condition, expressed in terms of $Q_i$'s, gives an additional algebraic constraint equation:

\[
Q_6 = \frac{e^{-2\nu+2\mu} r^2}{1 - e^{-2\nu+2\mu} r^2 \sin^2 \theta (\Omega - \omega)^2 c_s^2} \times \left\{ \left[ 1 + e^{-2\nu+2\mu} r^2 \sin^2 \theta \left( \Omega^2 - \omega^2 \right) \right] Q_1 - 2 e^{-2\nu+2\mu} r^2 \sin^2 \theta (\Omega - \omega) Q_2 \right\} \quad (27)
\]
Table 1. The eigenfrequencies of the $\ell = 0$ radial modes ($F$, $H_1$, $H_2$ and $H_3$) and the $\ell = 1$ dipole modes ($f_{\ell=1}$, $p_{1}^{\ell=1}$, $p_{2}^{\ell=1}$ and $p_{3}^{\ell=1}$) are shown in comparison with the results by Font et al. [26].

| Mode   | Present code | Font et al. |
|--------|--------------|-------------|
| $F$    | 2.684        | 2.706       |
| $H_1$  | 4.550        | 4.547       |
| $H_2$  | 6.354        | 6.320       |
| $H_3$  | 8.140        | 8.153       |
| $f_{\ell=1}$ | 1.378     | 1.335       |
| $p_{1}^{\ell=1}$ | 3.475     | 3.473       |
| $p_{2}^{\ell=1}$ | 5.342     | 5.335       |
| $p_{3}^{\ell=1}$ | 7.161     | 7.136       |

If $Q = (Q_1, Q_2, Q_3, Q_4)^\top$, we can rewrite the system of equations (23)-(26) in a more compact form:

$$\partial_t Q = \alpha_1 \partial_\phi Q + \alpha_2 \partial_r Q + \alpha_3 \partial_\theta Q + \alpha_4 Q$$

(28)

together with the constraint equation:

$$Q_6 = \gamma_1 Q_1 + \gamma_2 Q_2$$

(29)

The coefficients $\alpha_i$ and $\gamma_i$ are functions of $r$ and $\theta$ their numerical values are calculated by the (rms code [29]).

Our main aim is to evolve numerically the system of equations (28). This numerical procedure in general suffers from a number of numerical instabilities.

3.1. Results

The previous analysis has an $\Omega = 0$ limit. This is the limit where the unperturbed star is non-rotating. This is natural limit to test a numerical code and for this case, the numerical time-evolution of our system of equations was stable. Stability was achieved not only in spherical but also in cylindrical coordinate systems. In both cases we were able to take the Fourier transform of the evolution and extract the stellar eigenfrequencies. These frequencies are presented in Table 1 in comparison to the ones derived by the full nonlinear evolution [26].

For a fast rotating stellar models, the numerical procedure did not have the same success. The numerical evolution was stable only for small rotation rates and the numerical instability is developing after a few rotations. Work is in progress using surface-fitted coordinates in order to overcome the stability problems described.

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