The Legendre-Hadamard condition in Cosserat elasticity theory

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Summary: The Legendre-Hadamard necessary condition for energy minimizers is derived in the framework of Cosserat elasticity theory.

1 Introduction

Cosserat elasticity \cite{1} is enjoying a resurgence as a framework for the modeling and analysis of scale effects in solids associated with the presence of microstructure. Definitive modern treatments of the subject may be found in \cite{2,11}. Here we supplement this literature with the relevant version of the Legendre-Hadamard necessary condition for energy minimizers. Thus we effectively extend the Legendre-Hadamard inequality of conventional elasticity theory \cite{5} to the Cosserat theory.

We work in the setting of classical nonlinear Cosserat theory, according to which the material comprising the considered body is endowed with independent deformation and rotation fields, the former describing the transplacements of material points as in conventional elasticity theory and the latter the change in microstructural orientation as the configurations of the body evolve.

Section 2 is devoted to a brief resumé of the basic theory for equilibria. We regard the latter as those states that satisfy an appropriate virtual work statement. Conditions under which this may be interpreted as a stationarity condition for a potential energy functional are identified in Section 3, and expressions for the first and second variations of this energy are obtained. In Section 4 we present a
detailed derivation, modelled after that given in [12], of the relevant Legendre-Hadamard inequality. This proceeds from the notion that the second variation is necessarily non-negative if an equilibrium state furnishes a minimum of the potential energy. We conclude in Section 5 with an application of the inequality to the particular strain-energy function proposed in [8].

Concerning notation, bold face is used for vectors and tensors and a dot interposed between bold symbols is used to denote the standard Euclidean inner product. For example, if $A$ and $B$ are second-order tensors, then their inner product is $A \cdot B = tr(AB^t)$, where $tr(\cdot)$ is the trace and the superscript $^t$ is used to denote the transpose. The induced norm is $\|A\| = \sqrt{A \cdot A}$. We make frequent use of the fact that $A \cdot BC = AC^t \cdot B$. The symbol $\otimes$ identifies the standard tensor product of vectors. We use $symA$ and $skewA$ respectively to denote the symmetric and skew parts of a tensor $A$, and $dev(symA)$ to denote the deviatoric part of $symA$. The axial vector of a skew tensor $W$ is denoted by $axlW$ and defined by $axlW \wedge v = Wv$ for any vector $v$. The symbols $\nabla$ and $Div$ respectively stand for the three-dimensional referential gradient and divergence operators. For a fourth-order tensor $A$, the notation $A[B]$ stands for the second-order tensor resulting from the linear action of $A$ on $B$ (see [5], eq. (7.10)). Its transpose $A^t$ is defined by $B \cdot A^t[A] = A \cdot A[B]$, and $A$ is said to possess major symmetry if $A^t = A$. The notation $G_S$ stands for the second-order-tensor-valued derivative of the scalar-valued function $G(S)$ with respect to the second-order tensor variable $S$. The second derivative is the fourth-order tensor $G_{SS}$; this possesses major symmetry if $G$ is twice differentiable. The second derivatives $G_{ST}$ and $G_{TS}$ of a twice differentiable scalar-valued function $G(S,T)$ satisfy $A \cdot G_{ST}[B] = B \cdot G_{TS}[A]$; accordingly, $G_{TS} = (G_{ST})^t$. Finally, we use superposed dots to denote variational derivatives. These are ordinary derivatives of one-parameter families of the varied functions with respect to the parameter, evaluated at parameter value zero, say, which we identify with an equilibrium state.

2 Cosserat elasticity

We present a brief outline of the equilibrium theory for the sake of completeness.

2.1 Kinematics and elasticity

The relevant kinematical variables of a Cosserat continuum are a deformation field $\chi(X)$ and a rotation field $R(X)$, where $X$ is the position of a material point in a reference configuration $\kappa$, say. Of course these may depend on time, but such dependence is not important for our purposes and is thus not made explicit. The deformation and rotation fields are regarded as being independent in the spirit of the conventional Cosserat theory (see [5], § 98).

To model elasticity, we introduce an energy density $U(F, R, \nabla R; X)$, per unit volume of $\kappa$, where $F = \nabla \chi$ is the deformation gradient and $\nabla R$ is the rotation gradient. In Cartesian index notation, these are

$$F = F_{iA} e_i \otimes E_A, \quad R = R_{iA} e_i \otimes E_A \quad \text{and} \quad \nabla R = R_{iA,B} e_i \otimes E_A \otimes E_B \quad (2.1)$$

with

$$F_{iA} = \chi_{i,A}, \quad (2.2)$$
where \((\cdot)_{,A} = \partial(\cdot)/\partial X_A\). Here \(\{e_i\}\) and \(\{E_A\}\) are fixed orthonormal bases associated with Cartesian coordinates \(x_i\) and \(X_A\), where \(x_i = \chi_i(X_A)\).

We assume the strain energy to be Galilean-invariant and thus impose

\[ U(F, R, \nabla R, X) = U(QF, QR, Q\nabla R, X), \quad (2.3) \]

where \(Q\) is an arbitrary spatially uniform rotation and \((Q\nabla R)_{AB} = (Q_{ij}R_{jA})_{,B} = Q_{ij}R_{jA,B}\). The restriction

\[ U(F, R, \nabla R, X) = W(E, \Gamma; X), \quad (2.4) \]

where \[E = R^iF = EAEBE_A \otimes E_B; \quad EAEB = R_{iA}F_{iB},\]

and

\[ \Gamma = \Gamma_{DC}E_D \otimes E_C; \quad \Gamma_{DC} = \frac{1}{2}e_{BAD}R_{iA}R_{iB,C}, \quad (2.5) \]

with \(W\) the reduced energy and \(e_{ABC}\) the permutation symbol \((e_{123} = 1, \text{ etc.})\), furnishes the necessary and sufficient condition for Galilean invariance. Sufficiency is obvious, whereas necessity follows by choosing \(Q = R\big|_{x}\), where \(x\) is the material point in question, and making use of the fact that for each fixed \(C \in \{1, 2, 3\}\), the matrix \(R_{iA}R_{iB,C}\) is skew. This follows by differentiating \(R_{iA}R_{iB} = \delta_{AB}\) (the Kronecker delta). The associated axial vectors \(\gamma_C\) have components

\[ \gamma_{D(C)} = \frac{1}{2}e_{BAD}R_{iA}R_{iB,C}, \quad (2.6) \]

yielding \[\Gamma = \gamma_C \otimes E_C, \quad (2.7)\]

and so \(\Gamma\) - the \textit{wryness tensor} - is isomorphic to the Cosserat strain measure \(R^i\nabla R\). The strain measures \(E\) and \(\Gamma\) are generally non-symmetric.

We note that the considerations of \[8\] and \[9\] are based on strain measures that differ from those adopted here. However, in these works it is demonstrated that the various sets of measures adopted therein are equivalent to those used in the present work.

Henceforth we assume \(W\) to be a continuous function of \(X\) and twice continuously differentiable with respect to \(E\) and \(\Gamma\).

2.2 Virtual power and equilibrium

We define equilibria to be states that satisfy the virtual-power statement

\[ \dot{S} = P, \quad (2.9) \]

where \(P\) is the virtual power of the loads acting on the body, the explicit form of which is deduced below,

\[ S = \int_S U\, dv \]

is the total strain energy, and, here and henceforth, superposed dots identify variational derivatives. Thus, by the chain rule,

\[ \dot{U} = W = \sigma \cdot \dot{E} + \mu \cdot \dot{\Gamma}, \quad (2.10) \]
where
\[ \sigma = W_E \quad \text{and} \quad \mu = W_\Gamma \]  \hspace{1cm} (2.12)
are evaluated at equilibrium, i.e., at states satisfying (2.9).

It follows from (2.5) that
\[ \dot{E} = R^t(\nabla u - \Omega F) \], where \( u = \dot{\chi} \) and \( \Omega = \dot{R} R^t \), \hspace{1cm} (2.13)
where \( \Omega \) is an arbitrary skew tensor (see the Appendix).

Then,
\[ \sigma \cdot \dot{E} = R \sigma \cdot \nabla u - \Omega \cdot \text{skew}(R \sigma F^t) \]. \hspace{1cm} (2.14)

Let \( \omega = axl \Omega \). If \( \alpha \) is a skew tensor and \( a = axl \alpha \), then it is easy to show that \( \Omega \cdot \alpha = 2 \omega \cdot a \). Further, \( R \sigma F^t = R \sigma E^t R^t \) and \( \text{skew}(R \sigma E^t R^t) = R \text{skew}(\sigma E^t) R^t \), yielding
\[ \sigma \cdot \dot{E} = R \sigma \cdot \nabla u - 2 axl[R \text{skew}(\sigma E^t) R^t] \cdot \omega \]. \hspace{1cm} (2.15)

The reduction
\[ \dot{\Gamma} = R^t \nabla \omega \] \hspace{1cm} (2.16)
is somewhat more involved. Reference may be made to [13] for a detailed derivation.

Accordingly,
\[ \mu \cdot \dot{\Gamma} = R \mu \cdot \nabla \omega \] \hspace{1cm} (2.17)
and on substituting (2.11), (2.15) and (2.16) into (2.9) we obtain
\[ P = \int_{\partial \kappa} \left[ (R \sigma) \nu \cdot u + (R \mu) \nu \cdot \omega \right] da + \int_{\kappa} \left[ u \cdot \text{Div}(R \sigma) + \omega \cdot [\text{Div}(R \mu) + 2 axl(R \text{skew}(\sigma E^t) R^t)] \right] dv, \] \hspace{1cm} (2.18)
where \( \nu \) is the exterior unit normal to the (piecewise smooth) surface \( \partial \kappa \). The virtual power is thus of the form
\[ P = \int_{\partial \kappa} (t \cdot u + c \cdot \omega) da + \int_{\kappa} (g \cdot u + \pi \cdot \omega) dv, \] \hspace{1cm} (2.19)
where \( t \) and \( c \) are densities of force and couple acting on \( \partial \kappa \), and \( g \) and \( \pi \) are densities of force and couple acting in \( \kappa \).

If there are no kinematical constraints; that is, if \( u \) and \( \omega \) can be chosen independently and arbitrarily, then, by the Fundamental Lemma,
\[ g = -\text{Div}(R \sigma) \quad \text{and} \quad \pi = -\text{Div}(R \mu) - 2 axl[R \text{skew}(\sigma E^t) R^t] \] \hspace{1cm} in \( \kappa \), \hspace{1cm} (2.20)
whereas
\[ t = (R \sigma) \nu \quad \text{on} \ \partial \kappa_t \quad \text{and} \quad c = (R \mu) \nu \quad \text{on} \ \partial \kappa_c, \] \hspace{1cm} (2.21)
where \( \partial \kappa_t \) is a part of \( \partial \kappa \) where position is not assigned and \( \partial \kappa_c \) is a part where rotation is not assigned.

We assume position to be assigned on \( \partial \kappa \setminus \partial \kappa_t \), so that \( u = 0 \) there, and rotation to be assigned on \( \partial \kappa \setminus \partial \kappa_c \), where \( \omega = 0 \). These, in addition to the degree of smoothness implied by the foregoing reduction, are the admissibility conditions on \( u \) and \( \omega \).

Equations (2.20) and (2.21) are the equilibrium conditions for an elastic Cosserat continuum.
3 Conservative problems and potential energy

We are concerned in this work with conservative problems for which a potential energy is available. These are such that there exists a load potential $L$, say, whose variational derivative is identical to the virtual power. Thus,

$$\dot{L} = P$$

and the potential energy is

$$E = S - L,$$

apart from an unimportant constant. Equilibria are thus seen to be those states that render the potential energy stationary, i.e.,

$$\dot{E} = 0,$$

for all admissible $u$ and $\omega$.

3.1 Dead-load problems

For the sake of simplicity and definiteness we confine attention to dead-load problems with vanishing volumetric densities of force $g$ and couple $\pi$. These are characterized by load potentials of the form

$$L = \int_{\partial\Omega} t \cdot \chi da + \int_{\partial\Omega} M \cdot R da$$

in which $t$ and $M$ respectively are assigned configuration-independent vector and tensor fields. Here $t$ is as in (2.21), and the (configuration dependent) couple traction in (2.21) is

$$c = 2ax[S\text{ skew}(MR^t)].$$

The first variation of the energy is

$$\dot{E} = \int_{K} (RW_T \cdot \nabla \omega + RW_E \cdot \nabla u - RW_E F^t \cdot \Omega)dv - \int_{\partial\Omega} t \cdot u da - \int_{\partial\Omega} c \cdot \omega da,$$

and vanishes if and only if the state $\{\chi, R\}$ is equilibrated.

3.2 The second variation at equilibrium

To secure an expression for the second variation, we define $v = \dot{\chi}$, and note, from (2.13), that

$$\ddot{R} = R + \Omega^2 R,$$

where $\Omega$ is defined in (2.13) and $\Phi$ is an arbitrary skew tensor (see the Appendix).

On taking a further variation of (3.6), after some effort we obtain

$$\ddot{E} = \int_{K} (RW_T \cdot \nabla \varphi + RW_E \cdot \nabla v - RW_E F^t \cdot \Phi)dv - \int_{\partial\Omega} t \cdot v da - \int_{\partial\Omega} c \cdot \varphi da$$

$$+ \int_{K} [\Omega RW_T \cdot \nabla \omega + RW_E \cdot \nabla u - \Omega RW_E F^t \cdot \Omega - RW_E (\nabla u)^t \cdot \Omega] dv$$

$$+ \int_{K} R(W_E)^t \cdot \nabla u + R(W_T)^t \cdot \nabla \omega - R(W_E)^t \cdot \Omega \right] dv - \int_{\partial\Omega} MR^t \cdot \Omega^2 da,$$
where \( \varphi = axl\Phi \) and, by the chain rule,

\[
(W_E)' = W_{EE}[\dot{E}] + W_{ET}[\dot{\Gamma}] \quad \text{and} \quad (W_\Gamma)' = W_{\Gamma E}[\dot{E}] + W_{\Gamma T}[\dot{\Gamma}]
\]

(3.9)

with \( \dot{E} \) and \( \dot{\Gamma} \) given by (2.13) and (2.14), respectively. Here \( \upsilon \) vanishes on \( \partial \chi \setminus \partial \kappa \) and \( \varphi \) vanishes on \( \partial \kappa \setminus \partial \kappa_c \).

If the state \( \left\{ \chi, R \right\} \) is equilibrated then the first line of (3.8) vanishes by (3.3) and (3.6). The second variation at equilibrium becomes

\[
\dot{E} = \int_\kappa \{ R' \nabla u \cdot W_{EE}[R' \nabla u] + R' \nabla \omega \cdot W_{ET}[R' \nabla \omega] + R' \nabla \omega \cdot W_{\Gamma E}[R' \nabla u] + R' \nabla \omega \cdot W_{\Gamma T}[R' \nabla \omega] \} dv
+ \int_\kappa F(\nabla u, \nabla \omega, \Omega)dv - \int_{\kappa_c} MR' \cdot \Omega^2 da,
\]

(3.10)

where

\[
F(\nabla u, \nabla \omega, \Omega) = 2\Omega R(W_E) \cdot \nabla u + \Omega R(W_\Gamma) \cdot \nabla \omega - \Omega R(W_E) F' \cdot \Omega + R' \Omega F \cdot W_{EE}[R' \Omega F] - 2R' \nabla u \cdot W_{EE}[R' \Omega F] - R' \nabla \omega \cdot W_{ET}[R' \Omega F] - R' \Omega F \cdot W_{ET}[R' \nabla \omega].
\]

(3.11)

If the equilibrium state is an energy minimizer, it is necessary that

\[
\dot{E} \geq 0
\]

(3.12)

for all \( u \) and \( \omega \) such that \( u \) vanishes on \( \partial \kappa \setminus \partial \kappa_t \) and \( \omega \) vanishes on \( \partial \kappa \setminus \partial \kappa_c \).

### 4 The Legendre-Hadamard inequality

**Theorem:** If (3.12) is satisfied then it is necessary that the Legendre-Hadamard inequality

\[
a \otimes n \cdot W_{EE}[a \otimes n] + a \otimes n \cdot W_{ET}[b \otimes n] + b \otimes n \cdot W_{\Gamma E}[a \otimes n] + b \otimes n \cdot W_{\Gamma T}[b \otimes n] \geq 0
\]

(4.1)

be satisfied at every \( X \in \kappa \) and for all vectors \( a, b \) and \( n \).

**Remark:** Choosing \( a \) or \( b \) to vanish in this inequality yields the further necessary conditions

\[
a \otimes n \cdot W_{EE}[a \otimes n] \geq 0 \quad \text{and} \quad b \otimes n \cdot W_{\Gamma T}[b \otimes n] \geq 0,
\]

(4.2)

again for every \( X \in \kappa \) and arbitrary \( a, b \) and \( n \). Clearly these are also sufficient for (4.1) in the case of a decoupled energy with \( W_{ET} = 0 \) and \( W_{\Gamma E} = 0 \). Further, (4.1) follows if \( W \) is convex in the strain measures \( E \) and \( \Gamma \) jointly. Indeed this hypothesis underpins existence theorems for equilibria proved in [6,8,9,11] and guarantees that (4.1) is automatically satisfied at any equilibrium state. However, this does not imply convexity of the overall minimization problem due to the nonlinear nature of the strain measures.

**Proof of the Theorem:** Following [12] we consider variations

\[
u(X) = \epsilon \Xi(Y) \quad \text{and} \quad \omega(X) = \epsilon \eta(Y) \quad \text{with} \quad Y = \epsilon^{-1}(X - X_0),
\]

(4.3)
where $X_0$ is an interior point of $\kappa$, $\epsilon$ is a positive constant, and $\xi, \eta$ are compactly supported in a region $D$, the image of a strictly interior neighborhood $\kappa' \subset \kappa$ of $X_0$ under the map $Y(\cdot)$. Accordingly $u$ and $\omega$ (hence $\Omega$) vanish on $\partial \kappa$ and are therefore admissible. For these variations reduces, after dividing by $\epsilon^3$, passing to the limit $\epsilon \to 0$ and invoking the Dominated Convergence Theorem, to
\[
\int_D \{ R_0' \nabla \xi \cdot A[R_0' \nabla \xi] + R_0' \nabla \xi \cdot B[R_0' \nabla \eta] + R_0' \nabla \eta \cdot B'[R_0' \nabla \xi] + R_0' \nabla \eta \cdot C[R_0' \nabla \eta] \} \, dv \geq 0, \tag{4.4}
\]
where $R_0 = R(X_0)$, and with $A = W_{EE}X_0 = A^t$, $B = W_{ET}X_0$, $B' = W_{TE}X_0$ and $C = W_{TT}X_0 = C^t$. Here and henceforth $\nabla$ is the gradient with respect to $Y$ and we have used the fact that $F$, defined by \([3.11]\), vanishes in the limit.

We extend $\xi$ and $\eta$ to complex-valued vector fields as
\[
\xi = \xi_1 + i \xi_2 \quad \text{and} \quad \eta = \eta_1 + i \eta_2, \tag{4.5}
\]
where $\xi_{1,2}$ and $\eta_{1,2}$ are real-valued, and use these to derive
\[
R_0' \nabla \xi \cdot B[R_0' \nabla \eta] = R_0' \nabla \eta \cdot B'_0 \nabla \xi.
\]
in which an overbar is used to denote the complex conjugate. The imaginary part of this expression vanishes by virtue of the fact that $A \cdot B^t[B] = B \cdot B^t[A]$ for arbitrary $A, B$. In the same way, we obtain
\[
R_0' \nabla \xi \cdot A[R_0' \nabla \xi] + R_0' \nabla \xi \cdot C[R_0' \nabla \eta]
\]
so that if \([4.3]\) holds for real-valued $\xi$ and $\eta$, then it follows that
\[
\int_D \{ R_0' \nabla \xi \cdot A[R_0' \nabla \xi] + R_0' \nabla \xi \cdot B[R_0' \nabla \eta] + R_0' \nabla \eta \cdot B'[R_0' \nabla \xi] + R_0' \nabla \eta \cdot C[R_0' \nabla \eta] \} \, dv \geq 0 \tag{4.8}
\]
for complex-valued $\xi$ and $\eta$.

Consider
\[
\xi(Y) = \alpha \exp(ikn \cdot Y)f(Y) \quad \text{and} \quad \eta(Y) = \beta \exp(ikn \cdot Y)f(Y), \tag{4.9}
\]
where $\alpha, \beta$ and $n$ are real fixed vectors, $k$ is a non-zero real number and $f$ is a real-valued differentiable function compactly supported in $D$. These yield
\[
R_0' \nabla \xi = \exp(ikn \cdot Y)(ik f a \otimes n + a \otimes \nabla f) \quad \text{and} \quad R_0' \nabla \eta = \exp(ikn \cdot Y)(ik f b \otimes n + b \otimes \nabla f), \tag{4.10}
\]
with $a = R_0' \alpha$ and $b = R_0' \beta$. Substitution into \([4.8]\) and division by $k^2$ results in
\[
0 \leq \{ a \otimes n \cdot A[a \otimes n] + 2a \otimes n \cdot B[b \otimes n] + b \otimes n \cdot C[b \otimes n] \} \int_D f^2 \, dv
+ k^{-2} \int_D \{ a \otimes \nabla f \cdot A[a \otimes \nabla f] + 2a \otimes \nabla f \cdot B[b \otimes \nabla f] + b \otimes \nabla f \cdot C[b \otimes \nabla f] \} \, dv. \tag{4.11}
\]
Finally, as $k \to \infty$ we recover
\[
a \otimes n \cdot A[a \otimes n] + 2a \otimes n \cdot B[b \otimes n] + b \otimes n \cdot C[b \otimes n] \geq 0, \tag{4.12}
\]
which is just \([1.1]\) on account of the arbitrariness of $X_0$. 

7
5 Example

By way of illustration we apply inequalities (4.2) to the quadratic, decoupled energy

\[ W = \mu \| \text{sym}(E - I) \|^2 + \mu_c \| \text{skew}(E - I) \|^2 + \frac{1}{2} \lambda \| \text{tr}(E - I) \|^2 + a_1 \| \text{dev}(\text{sym} \Gamma) \|^2 + a_2 \| \text{skew} \Gamma \|^2 + \frac{1}{3} a_3 (\text{tr} \Gamma)^2, \]  

(5.1)

proposed in [10] to model isotropic materials, where \( \mu, \mu_c, \lambda \) and \( a_{1-3} \) are material constants and \( I \) is the identity. Using the variational formulas (\( \| A \|^2 = 2 A \cdot \hat{A} \) and \( \| \text{tr} A \|^2 = 2 (\text{tr} A) I \cdot \hat{A} \)), together with the orthogonality of symmetric and skew tensors, and also that of deviatoric and spherical tensors, we obtain

\[ \dot{W} = [2 \mu \text{sym}(E - I) + 2 \mu_c \text{skew}(E - I) + \lambda \text{tr}(E - I) I] \cdot \dot{E} \]

\[ + [2 a_1 \text{dev}(\text{sym} \Gamma) + 2 a_2 \text{skew} \Gamma + \frac{2}{3} a_3 (\text{tr} \Gamma) I] \cdot \dot{\Gamma}, \]  

(5.2)

from which it follows that

\[ W_E = 2 \mu \text{sym}(E - I) + 2 \mu_c \text{skew}(E - I) + \lambda \text{tr}(E - I) I \]  

(5.3)

and

\[ W_\Gamma = 2 a_1 \text{dev}(\text{sym} \Gamma) + 2 a_2 \text{skew} \Gamma + \frac{2}{3} a_3 (\text{tr} \Gamma) I. \]  

(5.4)

A further variation yields

\[ W_{EE} [\dot{E}] = 2 \mu \text{sym} \dot{E} + 2 \mu_c \text{skew} \dot{E} + \lambda (\text{tr} \dot{E}) I \]  

(5.5)

and

\[ W_{TT} [\dot{\Gamma}] = 2 a_1 \text{dev}(\text{sym} \dot{\Gamma}) + 2 a_2 \text{skew} \dot{\Gamma} + \frac{2}{3} a_3 (\text{tr} \dot{\Gamma}) I. \]  

(5.6)

Accordingly,

\[ W_{EE} [a \otimes n] = \mu (a \otimes n + n \otimes a) + \mu_c (a \otimes n - n \otimes a) + \lambda (a \cdot n) I \]  

(5.7)

and

\[ W_{TT} [b \otimes n] = a_1 [b \otimes n + n \otimes b - \frac{2}{3} (b \cdot n) I] + a_2 [b \otimes n - n \otimes b] + \frac{2}{3} a_3 (b \cdot n) I. \]  

(5.8)

These in turn yield

\[ a \otimes n \cdot W_{EE} [a \otimes n] = \mu (\| a \| \| n \|^2 + (a \cdot n)^2) + \mu_c (\| a \| \| n \|^2 - (a \cdot n)^2) + \lambda (a \cdot n)^2 \]

(5.9)

and

\[ b \otimes n \cdot W_{TT} [b \otimes n] = a_1 (\| b \| \| n \|^2 + (b \cdot n)^2) + a_2 (\| b \| \| n \|^2 - (b \cdot n)^2) + \frac{2}{3} (a_3 - a_1) (b \cdot n)^2. \]  

(5.10)

Introducing angles \( \alpha \) and \( \beta \) defined by \( a \cdot n = \| a \| \| n \| \cos \alpha \) and \( b \cdot n = \| b \| \| n \| \cos \beta \) we find that inequalities (4.2) are satisfied if and only if

\[ 0 \leq \mu + \mu_c + (\mu - \mu_c + \lambda) \cos^2 \alpha \]

\[ = (\mu + \mu_c) (\cos^2 \alpha + \sin^2 \alpha) + (\mu - \mu_c + \lambda) \cos^2 \alpha \]

\[ = (2 \mu + \lambda) \cos^2 \alpha + (\mu + \mu_c) \sin^2 \alpha \]  

(5.11)
and

\[ 0 \leq a_1 + a_2 + \left[ a_1 - a_2 + \frac{2}{3}(a_3 - a_1) \right] \cos^2 \beta \]
\[ = (a_1 + a_2)(\cos^2 \beta + \sin^2 \beta) + [a_1 - a_2 + \frac{2}{3}(a_3 - a_1)] \cos^2 \beta \]
\[ = \frac{2}{3}(2a_1 + a_3) \cos^2 \beta + (a_1 + a_2) \sin^2 \beta, \]  
(5.12)

for all \( \alpha \) and \( \beta \). The necessary and sufficient conditions

\[ 2\mu + \lambda \geq 0, \quad \mu + \mu_c \geq 0, \quad 2a_1 + a_3 \geq 0 \quad \text{and} \quad a_1 + a_2 \geq 0 \]  
(5.13)

follow immediately, and coincide with the Legendre-Hadamard conditions derived in \([10]\) for linearized, isotropic Cosserat elasticity.

We observe that in general (4.1) and (4.2) do not impose restrictions on the constitutive function \( W \), but rather on the configuration fields \( \{\chi(X), R(X)\} \). In the present example, however, these emerge as constitutive inequalities due to the quadratic nature of the energy (5.1).

A Appendix

To confirm the kinematic admissibility of the first and second variations \( \dot{R} \) and \( \ddot{R} \) defined by (2.13) and (3.7), consider a tensor-valued function \( Q(X; \epsilon) \) satisfying the differential equation

\[ Q' = WQ \quad \text{with} \quad Q(X; 0) = R(X), \]  
(A.1)

where \((\cdot)' = \partial(\cdot)/\partial \epsilon\), \( R \) is a rotation, and \( W(X; \epsilon) \) is an arbitrary differentiable skew tensor function. Let \( Z(X; \epsilon) = QQ' \). Then,

\[ Z' = WZ - ZW \quad \text{with} \quad Z(X; 0) = I. \]  
(A.2)

This has the unique solution \( Z(X; \epsilon) = I \), implying that \( Q(X; \epsilon) \) is orthogonal with \( \det Q = \pm 1 \).

Further,

\[ (\det Q)' / \det Q = tr(Q'Q^{-1}) = trW = 0, \]  
(A.3)

implying that \( \det Q(X; \epsilon) = \det R = 1 \) and hence that \( Q(X; \epsilon) \) is an admissible Cosserat rotation field. The notation \( \dot{R} = Q'_{|\epsilon=0} \) then yields (2.13) with \( \Omega(X) = W(X; 0) \).

From (A.1) we have

\[ Q'' = (WQ)' = W'Q + W^2Q \]  
(A.4)

in which \( W' \) is skew. This integrates to

\[ Q' = WQ + C \]  
(A.5)

in which \( C \) independent of \( \epsilon \). Evaluating at \( \epsilon = 0 \) yields \( C = \dot{R} - \Omega R \), which vanishes by (2.13)3.

Accordingly \( Q' = WQ \), which, as we have seen, ensures that \( Q(X; \epsilon) \) is a rotation provided that \( R = Q(X; 0) \) is a rotation. On setting \( \epsilon = 0 \) in (A.4) we recover (3.7) in which \( \dot{R} = Q''_{|\epsilon=0} \) and \( \Phi = W'_{|\epsilon=0}. \) The arbitrariness of the skew function \( W(X; \epsilon) \) implies that the skew tensor fields \( \Omega(X) \) and \( \Phi(X) \) in (2.13)3 and (3.7) can be chosen independently and arbitrarily.
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