Bianchi type-I Universe with Cosmological constant and periodic varying deceleration parameter.

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This manuscript investigate the dark energy Bianchi type-I cosmological models in presence of generalized Chaplygin gas, variable gravitational and cosmological constants. In this manuscript, exact solutions of Einstein field equations are obtained under the assumption of time periodic varying deceleration parameter. The physical and dynamical behaviors of the models have been discussed with the help of graphical representations. Also we have discussed the stability and physical acceptability of the obtained solutions.

Keywords: \(f(R,T)\) gravity; Bianchi type-I space-time; periodic varying deceleration parameter

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1. Introduction

Cosmological and astronomical data such as the type Ia supernovae (SNe-Ia)\(^1\)\(^2\) cosmic microwave background (CMB)\(^3\), large scale structure (LSS)\(^4\), Wilkinson Microwave Anisotropy Probe (WMAP)\(^5\), Sloan Digital Sky Survey (SDSS)\(^6\) and the Plank 2015\(^7\) reveal that we live in some 13.8 billion years old Universe which experiences a speedy expansion stage that have been originated with a bang from a phase of very high density and temperature. For a long time, it was assumed that either the Universe expands eternally or the inward pull of gravity gradually slows down the expansion of the Universe and would ultimately came to a halt after which Universe start to contract into a big crunch. However, it was at the end of twentieth century, the unexpected discovery that the Universe might be expanding with an acceleration surprised the cosmologist as the idea of cosmic acceleration was against

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the standard predictions of decelerating expansion caused by gravity. Latter on it was observed that the expansion of the Universe is accelerated which is dominated by some exotic stuff called as “Dark energy” (DE) with highly negative pressure which is the cause of acceleration of the Universe. The successive disclosure in this direction gave more and more documentation for a flat, dark energy dominated accelerating Universe. However, by modifying the theory of gravity one can find the alternative way to explain the acceleration of the expanding Universe.

The Quantum field theory (QFT) and general theory of relativity (GTR) pointed out cosmological constant $\Lambda_{10^{-12}}$ introduced by Einstein as the most feasible candidate of dark energy. The cosmological constant is the simplest form of dark energy and though the model with cosmological constant is known as $\Lambda\text{CDM}$ model. It faces some serious problematic issues such as the fine tuning (a typical small value), cosmic coincidence problems (although the Universe is in accelerated phase of expansion, why the dark matter and the dark energy are of the same order?). Chaplygin gas (CG) with EoS $p = -\frac{B}{\rho}$ where $\rho$ and $p$ are energy density and pressure respectively, is one of the most expected candidate of dark energy (DE). Among the various class of dark energy models Chaplygin gas is a simple characterisation in order to understand the cosmic acceleration and has attracted large group of researchers in the field of cosmology. Chaplygin gas is also a fascinating subject of holography and string theory. Chaplygin gas depicts a transition to the present cosmic acceleration from a decelerated cosmic expansion and conceivably submit a deformation of $\Lambda\text{CDM}$ model. Chaplygin gas model fails the tests connected with structure formation and observed strong oscillations of matter power spectrum. To overcome this failure of Chaplygin gas model the generalised Chaplygin gas with EoS $p = -\frac{B}{\rho^\alpha}$ with $0 \leq \alpha \leq 1$ has been proposed. From the holographic point of view also the generalised Chaplygin gas is interesting. Further the generalised Chaplygin gas (GCG) model is modified as Modified Chaplygin gas (MGCG) model with EoS $p = \mu \rho - \frac{B}{\rho^\alpha}$, where $\mu$ is a positive constant. This modification of GCG to MGCG has been done because the inferences from GCG models are almost similar to the CDM models.

Several authors have studied the evolution of universe with time varying deceleration parameter in various therious of gravity. Shen and Zhao have given the model of oscillating universe with quintom matter in the framework of FRW line element. Shen has investigated the Barber’s second self-creation theory of gravitation with varying cosmological constant $\Lambda$ and field equations were solved using time periodic varying deceleration parameter. Actas and Aygun Sahoo et.al. Ahmed and Alamr and Bharadwaj and Ram have used the time varying deceleration parameter to study the evolution of universe in $f(R, T)$ gravity. In view of the above research, we have analysed the Bianchi type-I Universe with periodic varying deceleration parameter in presence of generalized Chaplygin gas.
2. Field equations

Let us consider the gravitational field of matter distribution represented by a Bianchi type I space-time as

\[ ds^2 = dt^2 - \left( R_2^2(t) dx^2 + R_2^2(t) dy^2 + R_3^2(t) dz^2 \right). \]

(1)

Einstein’s field equations with gravitational and cosmological constant for perfect fluid distribution are given as

\[ R_{ij} - \frac{1}{2} R g_{ij} = -8 \pi G T_{ij} + \Lambda g_{ij}, \]

(2)

where \( T_{ij} \) is the energy momentum tensor, which is given as

\[ T_{ij} = (\rho + p) u_i u_j - p g_{ij}. \]

(3)

Here \( \rho \) is the energy density, \( p \) represents perfect fluid pressure and \( u^i \) is the four velocity vector.

The Einstein’s field equation (2) for the space-time metric (1) yields following equations

\[ \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_2 \dot{R}_3}{R_2 R_3} = -8\pi G p + \Lambda. \]

(4)

\[ \frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_1 \dot{R}_3}{R_1 R_3} = -8\pi G p + \Lambda. \]

(5)

\[ \frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_1 \dot{R}_2}{R_1 R_2} = -8\pi G p + \Lambda. \]

(6)

\[ \frac{\ddot{R}_1}{R_1} \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_1 \dot{R}_3}{R_3 R_1} = 8\pi G \rho + \Lambda. \]

(7)

By a combination of equations (4)-(7) one can easily obtain

\[ \dot{\rho} + (\rho + p) \left( \frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} \right) + \rho \frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0. \]

(8)

The energy momentum conservation equation \( (T^i_{\ j}) \) suggests

\[ \dot{\rho} + (\rho + p) \left( \frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} \right) = 0. \]

(9)

From equations (8) and (9) we have

\[ \rho \frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0. \]

(10)

Further, equations (4)-(7) yield the solutions

\[ R_i = c_i Re^{k_i \int \frac{1}{R_i} dt}, \]

(11)

Here \( k_i \) and \( c_i \) \( (i = 1, 2, 3) \) are constants obeying the relation \( \sum_{i=1}^{3} k_i = 0 \) and \( \prod_{i=1}^{3} c_i = 1 \). The above results suggest that the metric potential can be explicitly expresses in terms of scale factor represented by Bianchi type I space-time.
3. Cosmological Solutions

It can be easily seen that we have four equations (4)-(7) with six unknowns $R_1$, $R_2$, $R_3$, $\rho$, $G$ and $\Lambda$. Hence to solve the system of equations completely we need two additional physically plausible relations among these variables. To obtain the cosmological solution we considered the deceleration parameter of the form

$$q = m \cos(nt) - 1$$

(12)

where $m$ and $n$ are positive constant quantity. This type of deceleration parameter is known as time periodic varying deceleration parameter (TPVDP). The deceleration parameter play a crucial role in determining the nature of the constructed models of our universe i.e. decelerating or accelerating in nature. According to the value/ranges of $q$ the universe exhibits the expansion in the following ways:

- $q > 0$: Decelerating expansion
- $q = 0$: Expansion with constant rate
- $-1 < q < 0$: Accelerating power law expansion
- $q = -1$: Exponential expansion/de Sitter expansion
- $q < -1$: Super exponential expansion

From the considered form of $q$ in equation (12), the deceleration parameter shows periodic nature due to the presence of $\cos(nt)$. The deceleration parameter lies in the interval $-(m+1) \leq q \leq m-1$. Here we observed that (i) for $m = 0$, the deceleration parameter $q$ is equal to $-1$ and the universe exhibits exponential expansion/de Sitter expansion. (ii) for $m \in (0, 1)$, the deceleration parameter $q$ becomes negative and leads to accelerated expansion in a periodic way. (iii) for $m = 1$, $q$ lies in the interval $[-2, 0]$. This shows that, the universe evolves from expansion with constant rate to super exponential expansion in a periodic way followed by accelerating power law expansion to de Sitter expansion. (iv) $m > 1$, phase transition takes place from decelerating phase to accelerating phase in a periodic way and the universe start with a decelerating expansion and evolves to super exponential expansion. The present observational limit of the considered deceleration parameter $q_0 = -0.53^{+0.17}_{-0.13}$ suggests the following values of $m$ and $n$, which is given in Table I. In order to obtain the Hubble parameter from equation (12), we used the relation among Hubble parameter and $q$ as $q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1$, which leads to

$$H = \frac{n}{m \sin(nt) + nc_4}, \quad c_4 \text{ is the constant of integration}$$

(13)

The value of the integration constant $c_4$ does not affect the qualitative behaviour of the Hubble parameter but it effect the scale factor. Thus we have classified our cosmological models in three different cases as per $c_4 = 0$, $c_4 > 0$ and $c_4 < 0$, i.e. $c_4 = -c_5$, $c_5 > 0$. 
Table 1. Computational range of the parameter \( m \) for fixed values of \( n \), obtained from the present value of the deceleration parameter.

| \( n \) | Interval of \( m \) |
|-------|---------------------|
| 0.01  | 0.34 \( \leq m \leq 0.64 \) |
| 0.02  | 0.35 \( \leq m \leq 0.66 \) |
| 0.03  | 0.37 \( \leq m \leq 0.69 \) |
| 0.04  | 0.39 \( \leq m \leq 0.74 \) |
| 0.05  | 0.43 \( \leq m \leq 0.82 \) |
| 0.06  | 0.49 \( \leq m \leq 0.94 \) |
| 0.07  | 0.59 \( \leq m \leq 1.11 \) |
| 0.08  | 0.74 \( \leq m \leq 1.39 \) |
| 0.09  | 1.02 \( \leq m \leq 1.93 \) |
| 0.10  | 1.70 \( \leq m \leq 3.20 \) |

Fig. 1. Profile of deceleration parameter \( q \) against cosmic time for \( 0.34 \leq m \leq 0.64 \) and \( n = 0.01 \).

Fig. 2. Profile of deceleration parameter \( q \) against cosmic time for \( 1.70 \leq m \leq 3.20 \) and \( n = 0.10 \).

In order to analyse the behaviour of the deceleration parameter \( q \), we have represented graphically in Figures 1 and 2 for different values of \( m \) and \( n \). From the figures, it is noticed that (i) for \( n = 0.01 \) and \( 0.34 \leq m \leq 0.64 \) our models are accelerating in nature i.e \( q < 0 \) and (ii) for \( n = 0.10 \) and \( 1.70 \leq m \leq 3.20 \) our models show phase transition i.e decelerating phase to accelerating phase. In this situation \( q \) take values from positive to negative. We are mainly focus to investigate the phase transition scenario so in all our discussed models as a representative case we have considered the value of \( m \) in the range \( 1.70 \leq m \leq 3.20 \) and \( n = 0.10 \).

3.1. **Case-I: \( c_4 = 0 \)**

In this case the Hubble parameter in equation (13) leads to

\[
H = \frac{n}{m \sin(nt)}
\]
We know the relationship among Hubble parameter and scale factor as \( H = \frac{\dot{R}}{R} \), which along with (14) leads to the scale factor of the form
\[
R = c_6 \left[ \frac{1 - \cos(nt)}{\sin(nt)} \right]^{\frac{1}{m}} = c_6 \left[ \tan \left( \frac{nt}{2} \right) \right]^{\frac{1}{m}}, \ c_6 \text{ is the constant of integration.}
\]

Further, equations (11) and (15) yields the following metric potentials as
\[
R_i = c_{i1} \left[ \tan \left( \frac{nt}{2} \right) \right]^{\frac{1}{m}} \exp \left[ c_{i2} \int \left[ \tan \left( \frac{nt}{2} \right) \right]^{-\frac{1}{m}} \ dt \right],
\]
where \( c_{i1} = c_i c_6 \) and \( c_{i2} = \frac{k_i}{3c_6^3} \) for \( i = 1, 2, 3 \). It is known that ordinary matter fields available from standard model of particle physics in general relativity, fails to account the present observations. Therefore modifications of the matter sector of the Einstein-Hilbert action with exotic matter are considered in the literature. Chaplygin gas (CG) is considered to be one such candidate for dark energy so in this case we have considered the equation of state for an exotic background fluid, the generalized Chaplygin gas, described by equation of state
\[
p = -\frac{A}{\rho^\alpha}, \ A > 0 \text{ and } 0 \leq \alpha \leq 1
\]

The expression for the energy density can be obtained from equations (9), (14) and (17)-(18) as
\[
\rho = \left[ A + \frac{c_7}{R^{3(1+\alpha)}} \right]^{\frac{1}{m}} = \left[ A + \rho_0 \left( \sin(nt) \right)^{-\frac{3(1+\alpha)}{m}} \left( 1 + \cos(nt) \right)^{\frac{3(1+\alpha)}{m}} \right]^{\frac{1}{m}}
\]
where \( \rho_0 = c_7 c_6^{3(1+\alpha)} \) and \( c_7 \) is the constant of integration. The directional Hubble parameter \( H_i \) are given by
\[
H_i = \frac{\dot{R}_i}{R_i} = \frac{3c_i^2 n + \left( \sin(nt) \right)^{\frac{m-3}{m}} \left( 1 + \cos(nt) \right)^{\frac{2}{m}} \ k_i m}{3c_6^3 m \sin(nt)}, \ i = 1, 2, 3
\]
Eqs. (4), (7), (16) and (17)-(18), we can obtain the expression for Gravitational constant as
\[
G = \frac{1}{8\pi (\rho + p)} \left[ k_4 \left( \sin(nt) \right)^{\frac{2}{m}} \left( \cos(nt) - 1 \right) \left( \cos(nt) + 1 \right)^{\frac{m+6}{m}} - 18c_6^2 n^2 \cos(nt) \right] \frac{9c_6^3 m \sin(nt)}{\cos^2(nt) - 1}
\]
where \( k_4 = m(k_1 k_2 + k_1 k_3 - k_2^2 - k_3^2) \). With the help of equations (4) and (7), one can obtain the expression for cosmological constant as
\[
\Lambda = \frac{1}{\rho + p} \left[ \frac{m^2 \left( \sin(nt) \right)^{\frac{2}{m}} \left( \cos(nt) - 1 \right) \left( \cos(nt) + 1 \right)^{\frac{m+6}{m}} \left( k_3 \rho + k_6 p \right)}{9c_6^3 m^2 \left( \cos^2(nt) - 1 \right) - 27c_6^2 n^2 \left( \rho + p \right) + 18c_6^3 n^2 m \cos(nt) \rho} \right]
\]
where \( k_5 = k_2 k_3 + k_2^2 + k_3^2 \) and \( k_6 = k_1 k_2 + k_1 k_3 + k_2 k_3 \).
The physical quantities of the observational interest are expansion scalar (\( \Theta \)), shear...
scalar ($\sigma^2$) and the anisotropic parameter ($A_m$), which are defined as follows:

$$\Theta = 3H$$

$$\sigma^2 = \frac{1}{2} [H_x^2 + H_y^2 + H_z^2] - \frac{\Theta^2}{6}$$

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2$$

In this case, the physical quantities are obtained as

$$\Theta = \frac{3n}{m \sin(nt)}$$

$$\sigma^2 = \left( k_1^2 + k_2^2 + k_3^2 \right) \left[ 1 + \cos(nt) \right]^{\frac{m}{3}} \left[ \sin(nt) \right]^{-\frac{6+2m}{m}} = \frac{k_1^2 + k_2^2 + k_3^2}{18c_0^6} \left[ \frac{1 + \cos(nt)}{\sin(nt)} \right]^{\frac{m}{3}}$$

$$A_m = \left( k_1^2 + k_2^2 + k_3^2 \right) m^2 \left[ 1 + \cos(nt) \right]^{\frac{m}{3}} \left[ \sin(nt) \right]^{-\frac{6+2m}{m}}$$

In this case the state finder parameters are defined and expressed as

$$r = \frac{R \dot{R}}{R^2} = m^2 \cos^2(nt) - 3m \cos(nt) + 1 + m^2$$

$$s = \frac{r - 1}{3(q - 0.5)} = \frac{2m(m \cos^2(nt) - 3 \cos(nt) + m)}{3(2m \cos(nt) - 3)}$$

The relationship among $r$ and $s$ is given by

$$r = 1 + \frac{9}{2} s^2 \pm \frac{3}{2} \sqrt{9 + 9s^2 - 4m^2}$$

In terms of the deceleration parameter the state finder parameters are given as

$$r = -1 - q + q^2 + m^2$$

$$s = \frac{2(-2 - q + q^2 + m^2)}{3(-1 + 2q)}$$

The profile of scale factor $R$, energy density $\rho$, pressure $p$, cosmological constant $\Lambda$ and gravitational constant $G$ against time is presented in the Figure 3 to Figure 7 respectively for fix $n$ and different values of $m$ with suitable choice of arbitrary constant involve in the expressions of the physical quantities. The scale factor is increasing with the evolvement of time in the provided range of time, which can be seen from the Figure 3. However, the qualitative behaviour is similar to tan function due to the presence of $\tan \left( \frac{n_1 \pi}{2} \right)$ in equation (15) and $R \to \infty$ at $t = \frac{(2n_1+1)n}{n}$, $n_1 \in \mathbb{Z}$. 
Table 2. \( \{r,s\} = (1,0) \) corresponding to the different values of \( m \) and \( q \).

| \( q \)  | \( m \)  | \( \{r,s\} \) pair          |
|--------|--------|-----------------------------|
| -1.00  | 0.000  | \( (1.000,0) \)            |
| -0.80  | 0.748  | \( (1.0,2.846726e-017) \approx (1,0) \) |
| -0.60  | 1.020  | \( (1.0,0) \)              |
| -0.40  | 1.200  | \( (1.0,-8.223874e-017) \approx (1,0) \) |
| -0.20  | 1.327  | \( (1.0,1.057355e-016) \approx (1,0) \) |
| 0.00   | 1.411  | \( (1.0,-2.960595e-016) \approx (1,0) \) |
| 0.20   | 1.470  | \( (1.0,0) \)              |
| 0.40   | 1.497  | \( (1.0,0) \)              |
| 0.60   | 1.497  | \( (1.0,0) \)              |
| 0.80   | 1.470  | \( (1.0,0) \)              |
| 1.00   | 1.411  | \( (1.0,2.960595e-016) \approx (1,0) \) |

Fig. 3. Profile of Scale factor parameter \( R \) against cosmic time for different \( m \) and \( n \).

Fig. 4. Profile of energy density \( \rho \) against cosmic time for different \( m, n = 0.1, \alpha = 0.5, \rho_0 = 1 \) and \( A = 1 \).

Fig. 5. Profile of pressure \( p \) against cosmic time for different \( m, n = 0.1, \alpha = 0.5, \rho_0 = 1 \) and \( A = 1 \).
The energy density, pressure, cosmological constant and gravitational constant are periodic in nature, which is noticed from the Figure 6 to Figure 7 respectively due to the presence of cos(nt) and sin(nt) terms in the expressions of these physical quantities. Here also we noted that $\rho, p, \Lambda, G \to \infty$ at $t = \frac{n\pi}{2}, \forall n_2 \in \mathbb{Z}$. The energy density is positive whereas pressure is negative for different values of $m$ with evolution of time. In this case, the qualitative behavior of energy density follow the pattern of higher energy density value to lower energy density value (approaching to zero) to higher energy density value. This process will continue due to the periodic nature of the terms cos(nt) and sin(nt) involve in the expression of energy density \[(18)\]. Again it is also pointed out that, cosmological constant is positive for some values of $m$ and positive to negative values for some $m$ (See Figure 6). Gravitational constant takes values from negative to positive values and positive to negative values in a periodic way for different values of $m$ (See Figure 7). Further, it is noticed that the expansion scalar, shear scalar and anisotropy parameter have also singularity at $t = \frac{n\pi}{n}, \forall n_2 \in \mathbb{Z}$ and they behaves periodically.

### 3.2. Case-II: $c_4 > 0$

In this case the Hubble parameter in equation \[(13)\] takes the form

$$H = \frac{n}{m \sin(nt) + nc_4} \quad (33)$$

We know the relationship among Hubble parameter and scale factor as $H = \frac{\dot{R}}{R}$, which along with \[(33)\] leads to the scale factor of the form

$$R = c_8 e^{\frac{-2}{\sqrt{n^2 - m^2}} \arctan \left( \frac{nc_4 \tan \left( \frac{nt}{2} \right) + m}{\sqrt{n^2 - m^2}} \right)}, \quad c_8 \text{ is the constant of integration.} \quad (34)$$
Further, equations (11) and (34) yields the following metric potentials as

\[
R_i = c_{i3} e \sqrt{n^2 c_i^4 - m^2} \exp \left[ c_{i4} \int e^{-6 \arctan \left( \frac{n c_i^4 \tan(n t)^2 + m}{\sqrt{n^2 c_i^4 - m^2}} \right)} dt \right], \quad i = 1, 2, 3
\]

where \( c_{i3} = c_i c_8 \) and \( c_{i4} = k_i \frac{i}{c_8} (i = 1, 2, 3) \). The directional Hubble parameters are expressed as

\[
H_i = \frac{3 c_8^3 e}{3 c_8^3 (nc_4 + m \sin(nt))}
\]

The expression for the energy density can be obtained from equations (9) and (34) as

\[
\rho = \left[ A + \frac{c_7}{R^{3(1+\alpha)}} \right]^{\frac{1}{1+\alpha}} = A + \rho_1 e^{-6(1+\alpha) \arctan \left( \frac{n c_4 \tan(n t) + m}{\sqrt{n^2 c_4^4 - m^2}} \right)}
\]

where \( \rho_1 = c_7 c_8^{-3(1+\alpha)} \). The expression for gravitational and cosmological constant are obtained as

\[
G = \frac{1}{12 \arctan \left( \frac{nc_4 \tan \left( \frac{n t}{2} \right) + m}{\sqrt{n^2 c_4^4 - m^2}} \right)} - m^2 \cos^2 \left( \frac{nt}{2} \right) - nmc_4 \sin \left( \frac{nt}{2} \right) \cos \left( \frac{nt}{2} \right) - \frac{n^2 c_4^4}{4} k_i \frac{i}{c_8}
\]

\[
72 \pi c_8^6 (\rho + p) \left( m^2 \cos^4 \left( \frac{nt}{2} \right) - m^2 \cos^2 \left( \frac{nt}{2} \right) \right) \left( 1 - nmc_4 \sin \left( \frac{nt}{2} \right) \cos \left( \frac{nt}{2} \right) - \frac{n^2 c_4^4}{4} \right) e^{12 \arctan \left( \frac{nc_4 \tan \left( \frac{n t}{2} \right) + m}{\sqrt{n^2 c_4^4 - m^2}} \right)}
\]
In this case, the physical quantities are obtained as

\[
\Lambda = \frac{9n^2c_8^6 (\rho m \cos^2 (\frac{nt}{2}) - \frac{1}{2}) - \frac{4}{9}(\rho + p) e}{1 + (m^2 \cos^4 (\frac{nt}{2}) - m^2 \cos^2 (\frac{nt}{2}) - nmc_4 \sin (\frac{nt}{2}) \cos (\frac{nt}{2}) - \frac{n^2c_4^2}{4})} \left(k_5 \rho + k_6 p\right)
\]

In this case the state finder parameters are defined and expressed as

\[
\Theta = \frac{3n}{m \sin(nt) + nc_4}
\]

\[
\sigma^2 = \left(\frac{k_1^2 + k_2^2 + k_3^2}{18c_8^6}\right) e - \frac{12 \arctan \left(\frac{nc_4 \tan (\frac{nt}{2}) + m}{\sqrt{n^2c_4^2 - m^2}}\right)}{\sqrt{n^2c_4^2 - m^2}}
\]

\[
A_m = -\left(\frac{k_1^2 + k_2^2 + k_3^2}{27n^2c_8^6}\right) (m^2 \cos^2(nt) - 2nmc_4 \sin(nt) - m^2 - n^2c_4^2) e - \frac{12 \arctan \left(\frac{nc_4 \tan (\frac{nt}{2}) + m}{\sqrt{n^2c_4^2 - m^2}}\right)}{\sqrt{n^2c_4^2 - m^2}}
\]

In this case the state finder parameters are defined and expressed as

\[
r = \frac{\ddot{R}}{R H^2} = -\frac{(m \sin(nt) + nc_4)^3 \left(1 + 3m + 2m^2 + nmc_4 \sin(nt)\right)}{1 + \frac{12n^2c_4^2 + 4m^3 \sin(nt) \cos^4 (\frac{nt}{2}) - (12n^2c_4^2 + 4m^3 \sin(nt) \cos^2 (\frac{nt}{2}) - 3n^2mc_4^2 \sin(nt) - n^3c_4^3}{2} (43)}
\]

\[
s = \frac{r - 1}{3(q - 0.5)} = \frac{2}{3(2m \cos(nt) - 3) - \frac{12n^2c_4^2 + 4m^3 \sin(nt) \cos^4 (\frac{nt}{2}) - (12n^2c_4^2 + 4m^3 \sin(nt) \cos^2 (\frac{nt}{2}) - 3n^2mc_4^2 \sin(nt) - n^3c_4^3}{2} (44)}
\]

\[
\left[\frac{1}{12n^2c_4^2 + 4m^3 \sin(nt) \cos^4 (\frac{nt}{2}) - (12n^2c_4^2 + 4m^3 \sin(nt) \cos^2 (\frac{nt}{2}) - 3n^2mc_4^2 \sin(nt) - n^3c_4^3}{2} (44)}
\]

\[
\left[\left(m \sin(nt) + nc_4\right)^3 \left(1 + 3m + 2m^2 + nmc_4 \sin(nt)\right)_{1} + 4m^2 \cos^4 (\frac{nt}{2}) - (4m^2 + 6m) \cos^2 (\frac{nt}{2}) - \frac{12n^2c_4^2 + 4m^3 \sin(nt) \cos^2 (\frac{nt}{2}) - 3n^2mc_4^2 \sin(nt) - n^3c_4^3}{2} (44)}
\]

\[
\left[\left(m \sin(nt) + nc_4\right)^3 \left(1 + 3m + 2m^2 + nmc_4 \sin(nt)\right)_{1} + 4m^2 \cos^4 (\frac{nt}{2}) - (4m^2 + 6m) \cos^2 (\frac{nt}{2}) - \frac{12n^2c_4^2 + 4m^3 \sin(nt) \cos^2 (\frac{nt}{2}) - 3n^2mc_4^2 \sin(nt) - n^3c_4^3}{2} (44)}
\]
In terms of the deceleration parameter the state finder parameters are given as

\[
r = \frac{\left(\sqrt{m^2 - 1 - 2q - q^2 + nc_4}\right)^3 \left(m^2 - 1 - q + q^2 + nc_4 \sqrt{m^2 - 1 - 2q - q^2}\right)}{(m^2 - 1 - 2q - q^2 + 3n^2c_4^2) \sqrt{m^2 - 1 - 2q - q^2} + 3nc_4 \left(m^2 - 1 - 2q - q^2 + \frac{n^2c_4^2}{3}\right)}
\]

\[
s = \frac{2 \left(\sqrt{m^2 - 1 - 2q - q^2 + nc_4}\right)^3 \left(m^2 - 1 - q + q^2 + nc_4 \sqrt{m^2 - 1 - 2q - q^2}\right)}{3(2q - 1) \left(m^2 - 1 - 2q - q^2 + 3n^2c_4^2\right) \sqrt{m^2 - 1 - 2q - q^2} + 3nc_4 \left(m^2 - 1 - 2q - q^2 + \frac{n^2c_4^2}{3}\right)} - \frac{2}{3(2q - 1)}
\]

Figure 8 indicates the profile of scale factor against the cosmic time. Here we noticed that, the scale factor is increasing with respect to cosmic time in a periodic way.

Figure 9 and Figure 10 are representing the profile of pressure and energy density. As per the evolution of time, energy density and pressure are positive and negative respectively in a periodic way for different values of m.

Figure 11 and Figure 12 shows the qualitative behaviour of gravitational constant and cosmological constant with respect to cosmic time. We noticed that, both behaves in a periodic way with time and also \(\Lambda > 0\) and \(G < 0\) with the evolution of cosmic time. \(\Lambda > 0\) and \(G < 0\) are free from the initial singularity.

Figure 13 and Figure 14 shows the qualitative behaviour of shear scalar and anisotropic parameter respect to cosmic time. Both the parameters behave in a periodic way with time and positive with the evolution of cosmic time.
3.3. Case 3: \( c_4 < 0 \) i.e. \( c_4 = -c_5, \ c_5 > 0 \)

In this case the Hubble parameter in (13) takes the form

\[
H = \frac{n}{m \sin(nt) - nc_5}
\]  

(47)
We know the relationship among Hubble parameter and scale factor as $H = \frac{\dot{R}}{R}$, which along with (47) leads to the scale factor of the form

$$R = c_9 e^{-\frac{2}{\sqrt{n^2 c_5^2 - m^2}}} \arctan \left( \frac{nc_5 \tan \left( \frac{nt}{2} \right) - m}{\sqrt{n^2 c_5^2 - m^2}} \right),$$

$c_9$ is the constant of integration. (48)

Further, equations (11) and (48) yields the following metric potentials as

$$R_i = c_{i5} e^{-\frac{2}{\sqrt{n^2 c_5^2 - m^2}}} \arctan \left( \frac{nc_5 \tan \left( \frac{nt}{2} \right) - m}{\sqrt{n^2 c_5^2 - m^2}} \right) \exp \left[ c_{i6} \int e^{-\frac{6}{\sqrt{n^2 c_5^2 - m^2}}} dt \right]$$

where $c_{i5} = c_i c_9$ and $c_{i6} = \frac{k_i}{3c_9}$ ($i = 1, 2, 3$). The directional Hubble parameters are expressed as

$$H_i = \left[ \frac{6 \arctan \left( \frac{nc_5 \tan \left( \frac{nt}{2} \right) - m}{\sqrt{n^2 c_5^2 - m^2}} \right)}{\sqrt{n^2 c_5^2 - m^2}} - nc_5 k_i + mk_i \sin(nt) \right] e^{-\frac{6}{\sqrt{n^2 c_5^2 - m^2}}}$$

The expression for the energy density can be obtained from equations (9) and (21) as

$$\rho = \left[ A + \frac{c_7}{R^{(1+\alpha)}} \right]^{\frac{1}{1+\alpha}} = A + \rho_2 e^{-\frac{6 \arctan \left( \frac{nc_5 \tan \left( \frac{nt}{2} \right) - m}{\sqrt{n^2 c_5^2 - m^2}} \right)}{\sqrt{n^2 c_5^2 - m^2}}}$$

(51)
In this case, the physical quantities are obtained as

$$
\Lambda = \frac{-9n^2mc^6\left(\cos^2\left(\frac{n t}{T}\right) - \frac{1}{2}\right) e}{\sqrt{n^2c^2 - m^2}} + \left(m^2 \cos^4\left(\frac{n t}{T}\right) - \frac{n^2c^2}{4}\right) k_7
$$

where

$$
G = \frac{1}{\sqrt{n^2c^2 - m^2}} \left(-m^2 \cos^2\left(\frac{n t}{T}\right) + nmc_5 \sin\left(\frac{n t}{T}\right) \cos\left(\frac{n t}{T}\right) - \frac{n^2c^2}{4}\right) e
$$

and

$$
\Theta = \frac{3n}{m \sin(nt) - nc_5}
$$

In this case, the physical quantities are obtained as

$$
\Theta = \frac{3n}{m \sin(nt) - nc_5}
$$

$$
\sigma^2 = \frac{(k_1^2 + k_2^2 + k_3^2)}{18c_3^6} e - \frac{1}{\sqrt{n^2c^2 - m^2}} \left(-m^2 \cos^2\left(\frac{n t}{T}\right) + nmc_5 \sin\left(\frac{n t}{T}\right) \cos\left(\frac{n t}{T}\right) - \frac{n^2c^2}{4}\right) k_7
$$

$$
A_m = \frac{27n^2m^2c^6\left(-2 \tan\left(\frac{n t}{T}\right) + \tan^2\left(\frac{n t}{T}\right) \sin(nt) + \sin(nt)\right)^2 e}{\sqrt{n^2c^2 - m^2}} - \frac{1}{\sqrt{n^2c^2 - m^2}} \left(-m^2 \cos^2\left(\frac{n t}{T}\right) + nmc_5 \sin\left(\frac{n t}{T}\right) \cos\left(\frac{n t}{T}\right) - \frac{n^2c^2}{4}\right) k_7
$$

$$
A_m = \frac{27n^2c^6\left(-nc_5 - 2m \tan\left(\frac{n t}{T}\right) + 2m \tan\left(\frac{n t}{T}\right)^2\right)^2 e}{\sqrt{n^2c^2 - m^2}}
$$
In this case the state finder parameters are defined and expressed as

\[
\frac{\dot{R}}{RH^3} = \left( m \sin(nt) - nc_5 \right)^3 \left( \frac{-4m^2 \cos^4 \left( \frac{nt}{2} \right) + \left( 6m + 4m^2 \right) \cos^2 \left( \frac{nt}{2} \right)}{4m^2(m \sin(nt) - 3nc_5) \cos^2 \left( \frac{nt}{2} \right) \left( \cos^2 \left( \frac{nt}{2} \right) - 1 \right) + n^3c_3^2 - 3mn^2c_2^2 \sin(nt)} + \frac{nm \sin(nt) - 3m - 2m^2 - 1}{3} \right)
\]

\[
s = \frac{r - 1}{3(q - 0.5)} = \frac{2}{3(2m \cos(nt) - 3)} \times \left( \frac{m \sin(nt) - nc_5 \left( \frac{-4m^2 \cos^4 \left( \frac{nt}{2} \right) + \left( 6m + 4m^2 \right) \cos^2 \left( \frac{nt}{2} \right)}{4m^2(m \sin(nt) - 3nc_5) \cos^2 \left( \frac{nt}{2} \right) \left( \cos^2 \left( \frac{nt}{2} \right) - 1 \right) + n^3c_3^2 - 3mn^2c_2^2 \sin(nt)} + \frac{nm \sin(nt) - 3m - 2m^2 - 1}{3} \right)
\]

In terms of the deceleration parameter the state finder parameters are given as

\[
r = \left( \frac{\sqrt{m^2 - 1 - 2q - q^2} - nc_5}{m^2 - 1 - 2q - q^2 + 3n^2c_2^2} \right) \left( \frac{m^2 - 1 - q + q^2 - nc_5 \sqrt{m^2 - 1 - 2q - q^2}}{m^2 - 1 - 2q - q^2 - 3nc_5 \left( m^2 - 1 - 2q - q^2 + \frac{n^2c_3^2}{3} \right)} \right)
\]

\[
s = \frac{2}{3(2q - 1)} \times \left( \frac{\sqrt{m^2 - 1 - 2q - q^2} - nc_5}{m^2 - 1 - 2q - q^2 + 3n^2c_2^2} \right) \left( \frac{m^2 - 1 - q + q^2 - nc_5 \sqrt{m^2 - 1 - 2q - q^2}}{m^2 - 1 - 2q - q^2 - 3nc_5 \left( m^2 - 1 - 2q - q^2 + \frac{n^2c_3^2}{3} \right)} \right) - 1
\]

Figure 15 and Figure 16 indicates the profile of Hubble parameter against cosmic time for different values of \( c_4 \). Similar qualitative behaviour as that of case-II is observed for physical parameters like energy density [Figure 17], pressure [Figure 18], gravitational constant [Figure 19], cosmological constant [Figure 20], shear scalar [Figure 21] and anisotropic parameter [Figure 22].

Further, we have analyzed whether these derived models are approaching ΛCDM model or not for the computational range of \( n \) and \( m \) (Table 1). From equations (31), (32), (45), (46), (59) and (60), we have evaluated the \( \{r, s\} \) pair for different model parameters, which are presented in Table 2 to Table 4 respectively for three cases. Here we have noticed that model discussed in case-I approaches to ΛCDM model for different value of \( q \) and \( m \). At current value of \( q = \frac{3}{2} \), the model in case-I approaches to ΛCDM model for \( m = \frac{\sqrt{5}}{2} \). The models in case-II and Case-III, the \( \{r, s\} \) pair depends on the model parameters \( n, m, c_4, c_5 \) and \( q \). At current value of \( q = \frac{1}{2} \), the model in case-II approaches to ΛCDM model whereas model in Case-III does not approaches to ΛCDM model in view of the values of \( n \) and \( m \) presented in Table 1.
Bianchi type-I Universe with Cosmological constant and periodic varying deceleration parameter

Fig. 15. Profile of Hubble parameter $H$ against cosmic time for $m = 2.2$, $n = 0.1$ and different $c_4$.

Fig. 16. Profile of Hubble parameter $H$ against cosmic time for $m = 0.5$, $n = 0.01$ and different $c_4$.

Fig. 17. Profile of energy density $\rho$ against cosmic time for $1.70 \leq m \leq 3.20$ and $n = 0.10$.

Fig. 18. Profile of pressure $p$ against cosmic time for $1.70 \leq m \leq 3.20$ and $n = 0.10$.

Fig. 19. Profile of Gravitational constant $G$ against cosmic time for $1.70 \leq m \leq 3.20$, $c_7 = c_8 = A = 1$, $\alpha = 0.5$, $c_4 = 55$ and $n = 0.10$.

Fig. 20. Profile of Cosmological constant $\Lambda$ against cosmic time for $1.70 \leq m \leq 3.20$, $c_7 = c_8 = A = 1$, $\alpha = 0.5$, $c_4 = 55$ and $n = 0.10$. 
Fig. 21. Profile of shear scalar \( \sigma^2 \) against cosmic time for \( 1.70 \leq m \leq 3.20, c_7 = c_8 = A = 1, \alpha = 0.5, c_4 = 55 \) and \( n = 0.10 \).

Fig. 22. Profile of anisotropic parameter \( A_m \) against cosmic time for \( 1.70 \leq m \leq 3.20, c_7 = c_8 = A = 1, \alpha = 0.5, c_4 = 55 \) and \( n = 0.10 \).

Table 4. \( \{r, s\} \) pair for different model parameters with fixed \( q \)

| \( q \) | \( n \) | \( m \) | \( c_5 \) | \( r \) | \( s \) |
|-------|-------|-------|-------|-------|-------|
| -0.5  | 0.02  | 1.127031913 | 1      | 1.000000001≈1 | -3.333333333×10^{-10}≈0 |
| -0.5  | 0.04  | 1.130137315 | 1      | 1.02043999     | -0.006801333000 |
| -0.5  | 0.06  | 1.144360162 | 1      | 1.041217996    | -0.01373937200  |
| -0.5  | 0.08  | 1.154670505 | 1      | 1.062447982    | -0.01373937200  |

4. Stability and physical acceptability of the solutions

4.1. The squared sound speed

Here we introduced the adiabatic squared sound speed for the stability of our system. It is one of the important quantity in cosmology. The squared of sound speed, for any fluid is defined as

\[
c_s^2 = \frac{\partial p}{\partial \rho},
\]

(61)

here \( c_s^2 \) has three possibilities i.e \( c_s^2 < 0 \) or \( c_s^2 = 0 \) or \( c_s^2 > 0 \). The sign of \( c_s^2 \) is very important to investigate as it leads to the instability of the cosmological models through which one can reject or accept the constructed cosmological models. The case when, \( c_s^2 < 0 \), leads to classical instability of the cosmological models due to the uncontrolled grow of the energy density perturbation. The case when \( c_s^2 > 0 \) may leads to the issue of occurrence of casuality. As a matter of fact, it is usually considered as \( c_s \leq 1 \) and the bound on \( c_s^2 \) is \( 0 \leq c_s^2 \leq 1 \). In addition to that, the complementary bound \( c_s > 1 \) is used as a condition for rejecting the theories. The details regarding \( c_s^2 \) can be found in [22].
In our present study, $c_s^2$ is obtained as

$$c_s^2 = \frac{\partial p}{\partial \rho} = \frac{A\alpha}{\rho^{1+\alpha}} = \begin{cases} \frac{A\alpha}{A+\rho_0(\sin(nt))^{-\frac{1+\alpha}{n}}(1+\cos(nt))^{-\frac{1+\alpha}{m}}}, & \text{for } c_4 = 0 \\ \frac{6(1+\alpha)\arctan\left(\frac{mc_4}{\sqrt{n^2c_5^2-m^2}}\right)}{A+\rho_{1e}}, & \text{for } c_4 > 0 \\ \frac{6(1+\alpha)\arctan\left(\frac{mc_4}{\sqrt{n^2c_5^2-m^2}}\right)}{A+\rho_{2e}}, & \text{for } c_4 < 0 \end{cases}$$

(62)

In the Figure 23, we have presented the profile of squared of sound speed against time for different values of $m$. It is observed from the Figure 23 that, $c_s^2$ is periodic in nature and $0 \leq c_s^2 < 0.6$ in all the cases. Thus in account of the prescribe range of $c_s^2$, the solution presented here are stable.

4.2. Energy conditions

The role of energy conditions (ECs) can not be ignored in defining the cosmological evolution. The energy conditions are contractions of time like or null vector field with respect to the Einstein’s tensor and energy momentum tensor coming from Einstein’s field equations. Energy conditions can be imposed in order to investigate the constraints to the free parameters involved in the cosmological models. For example the evolution of acceleration or deceleration of cosmic fluid and the emergence of Big Rip singularities, can be related to the constraints imposed by the energy conditions. In our study, we have investigated the four ECs namely

- Null energy condition (NEC): $\rho + p \geq 0$
- Weak energy condition (WEC): $\rho \geq 0$ and $\rho + p \geq 0$
- Strong energy condition (SEC): $\rho + p \geq 0$ and $\rho + 3p \geq 0$
- Dominant energy condition (DEC): $\rho \geq 0$ and $\rho \pm p \geq 0$

It is important to note that, the above ECs are formulated from the Raychaudhuri equation. We have used the above ECs to investigate the energy conditions in this
discussed gravity. Cosmological model should satisfy WEC and SEC and violets SEC for late time accelerated expansion of the universe. For each of the three cases, Figures 24, 25 and 26 represent the profile of WEC, DEC and SEC respectively versus cosmic time. From these figures it can been that profile of energy conditions follows periodic variation for each of the cases. Considered model satisfies WEC and DEC whereas SEC is satisfied and violated periodically throughout the evolution of the universe.

5. Outlook
In this work we have considered Bianchi type-I cosmological models with generalised Chaplygin gas EoS represented by equation (17) and cosmological solutions are obtained by using time periodic varying deceleration parameter represented by
equation (12). By fixing the value of constant parameter $n$, we have computed the range of parameter $m$ (see Table 1 for present observational value (range) of deceleration parameter. Figures 1 and 2 shows the variation of deceleration parameter with respect to cosmic time for $n = 0.01$ and $n = 0.10$ and corresponding range of $m$. For $n = 0.01$ deceleration parameter is negative throughout the evolution of the universe whereas for $n = 0.10$ it shows transition from decelerated phase to accelerated phase periodically throughout the evolution of the universe. Hubble parameter represented by (13) is computed by using its relation with deceleration parameter. The constant of integration $c_4$ affects the scale factor so we have discussed three different cases for the considered model depending on the positive negative and neutral nature of $c_4$. In each of the cases we have calculated metric potentials, energy density ($\rho$), pressure ($p$) directional Hubble parameters, Gravitational constant ($G$), cosmological constant ($\Lambda$), physical quantities of the observational interest (such as expansion scalar ($\Theta$), shear scalar ($\sigma^2$) and the anisotropy parameter ($A_m$) and state finder parameters ($r$ and $s$). In this work we have considered the time periodic deceleration parameter and EoS of generalised Chaplygin gas as exotic fluid background in the framework of Bianchi type-I space-time. The conclusion of our study are as follows

- Almost all the parameters which we have discussed in our study shows periodic behaviour which is because of the choice of deceleration parameter.
- For each of the cases energy $\rho$ is positive and $p$ is negative throughout the evolution of the universe and this negative pressure guarantees the late time expansion of the universe.
- Models discussed in Case-I and Case-II, tending to $\Lambda$CDM model whereas model in Case-III fails for the value of $n$ and $m$ provided in Table 1. The expressions obtained for expansion scalar, shear scalar, and anisotropic parameter bears singularities in Case-I whereas these are free from singularities in Case-II and Case-III.
- The square sound speed satisfies the bounds $0 \leq c_s^2 \leq 1$ for each of the cases. So the considered model is stable and physically acceptable.
- For each of the cases considered model satisfies WEC and DEC whereas it violate SEC periodically throughout the evolution of the universe. So from the results of energy conditions one can conclude that the violation of SEC may leads to the accelerating Universe.

References
1. Perlmutter, S. and Supernova Cosmology Project, Nature, 391 51 (1998).
2. Perlmutter, S., Aldering, G., Goldhaber, G., Knop, R.A., Nugent, P., Castro, P.G., Deustua, S., Fabbro, S., Goobar, A., Groom, D.E. and Hook, I.M., The Astrophysical Journal, 517(2) 565 (1999).
3. Torbet, E., Devlin, M.J., Dorwart, W.B., Herbig, T., Miller, A.D., Nolta, M.R., Page, L., Puchalla, J. and Tran, H.T., The Astrophysical Journal Letters, 521(2) L79 (1999).
4. Bahcall, N.A., Ostriker, J.P., Perlmutter, S. and Steinhardt, P.J., *Science*, 284(5419) 1481 (1999)
5. Bennett, C.L., 2003. *Astrophys. J.(Suppl.)* 148, 1 (2003) ; DN Spergel, et al. arXiv preprint astro-ph/0603449
6. Bridle, S.L., Lahav, O., Ostriker, J.P. and Steinhardt, P.J., 2003. *Science*, 299(5612), 1532 (2003)
7. Spergel, D.N., Verde, L., Peiris, H.V., Komatsu, E., Nolta, M.R., Bennett, C.L., Halpern, M., Hinshaw, G., Jarosik, N., Kogut, A. and Limon, M., 2003. *The Astrophysical Journal Supplement Series*, 148(1), p.175.
8. Tegmark, M., Strauss, M.A., Blanton, M.R., Abazajian, K., Dodelson, S., Sandvik, H., Wang, X., Weinberg, D.H., Zehavi, I., Bahcall, N.A. and Hoyle, F., 2004. *Physical review D*, 69(10), p.103501.
9. Ade, P.A., Aghanim, N., Arnaud, M., Ashdown, M., Aumont, J., Baccigalupi, C., Banday, A.J., Barreiro, R.B., Bartlett, J.G., Bartolo, N. and Battaner, E., 2016. *Astronomy & Astrophysics*, 594, p.A13.
10. Peebles, P.J.E. and Ratra, B., 2003. *Reviews of modern physics*, 75(2), p.559.
11. Sahni, V. and Starobinsky, A., 2000. *International Journal of Modern Physics D*, 9(04), pp.373-443.
12. Padmanabhan, T., 2003. Physics Reports, 380(5), pp.235-320.
13. Cline, D.B. ed., 2001. Sources and detection of dark matter and dark energy in the universe. Springer.
14. Sharif, M. and Jawad, A., 2012. *Astrophysics and Space Science*, 337(2), pp.789-794.
15. Chaplygin, S., 1944. On gas jets, Sci. Mem., Moscow Univ. Phys-Math. 21 (1904) pp. 1-127. Trans. by M. Shud, Brown University.
16. Kamenshchik, A., Moschella, U. and Pasquier, V., 2001. *Physics Letters B*, 511(2-4), pp.265-268.
17. Setare, M.R., 2007. *Physics Letters B*, 648(5), pp.329-332.
18. Bordemann, M. and Hoppe, J., 1993. *Physics Letters B*, 317(3), pp.315-320.
19. Sandvik, H.B., Tegmark, M., Zaldarriaga, M. and Waga, I., 2004. *Physical Review D*, 69(12), p.123524.
20. Zhang, X., Wu, F.Q. and Zhang, J., 2006. *Journal of Cosmology and Astroparticle Physics*, 2006(01), p.003.
21. Wu, Y., Deng, X., Lu, J., Li, S. and Yang, X., 2006. *Modern Physics Letters A*, 21(15), pp.1233-1239.
22. Ming, S. and Liang, Z., 2014. *Chinese Physics Letters*, 31(1), p.010401.
23. Shen, M., 2016. *Astrophysics and Space Science*, 361(9), p.319.
24. Akta, C. and Aygn, S., 2017. *Archives of Current Research International*, pp.1-6.
25. Sahoo, P.K., Tripathy, S.K. and Sahoo, P., 2018. *Modern Physics Letters A*, 33(33), p.1850193.
26. Ahmed, N. and Alamri, S.Z., 2019. *Canadian Journal of physics*, 97(10), pp.1075-1082.
27. Bhardwaj, V.K. and Rana, M.K., *International Journal of Geometric Methods in Modern Physics*, 16(12) 1950195 (2019).
28. Garcia-Salesco, R., Gonzalez, T. and Quiros, I., 2014. *Physical Review D*, 89(8) 084047 (2014).