Riemannian isometries of twisted magnetic flux tube metric and stable current-carrying solar loops

by

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Abstract

Two examples of the use of differential geometry in plasma physics are given: The first is the computation and solution of the constraint equations obtained from the Riemann metric isometry of the twisted flux tube. In this case a constraint between the Frenet torsion and curvature is obtained for inhomogeneous helical magnetic flux tube axis. In the second one, geometrical and topological constraints on the current-carrying solar loops are obtained by assuming that the plasma filament is stable. This is analogous to early computations by Liley [(Plasma Physics (1964)] in the case of hydromagnetic equilibria of magnetic surfaces. It is shown that exists a relationship between the ratio of the current components along and cross the plasma filament and the Frenet torsion and curvature. The computations are performed for the helical plasma filaments where torsion and curvature are proportional. The constraints imposed on the electric currents by the energy stability condition are used to solve the remaining magnetohydrodynamical (MHD) equations which in turn allows us to compute magnetic helicity and from them the twist and writhe topological numbers. Magnetic energy is also computed from the solutions of MHD equations. PACS numbers: 02.40.Hw-Riemannian geometries
I Introduction

The tools of differential geometry and, in particular, Riemannian geometry [1] have been proved very useful in handling problems in areas of physics ranging from plasma astrophysics [2] to Einstein’s general relativity [3]. This common feature can be better appreciated when one notices that in one of the problems we shall address in this paper, the isometry which is the vanishing of the variation of the metric $g_{ik}$, or $\delta g_{ik} = 0$ ($i, j = 1, 2, 3$) either in gravitational spacetime, or as we shall see here, in the magnetic flux tube axis geometry. Another application of differential geometry and topology to plasma physics is the investigation of topology and geometrical constraints in fast kinematic dynamos [4, 5] where writhe number and twist were computed from magnetic helicity, or as shall address in this paper, to the investigation of the stability of solar loops in plasma astrophysics. Riemannian geometry has also been used with great success to investigate the tokamaks and stellarators metric oscillations and the instabilities of confined plasmas [6]. In this paper we show that when one imposes the constraint of twisted magnetic flux tube [7] metric isometry a relation between Frenet curvature [8] and inhomogeneous helical solar loops are obtained. The helicoidal form of the device implies that Frenet torsion of magnetic field lines is compulsory. Earlier Lileys [9] has computed the current conditions for static hydromagnetic equilibria. In plasmas the perturbation of energy $\delta W$ of magnetic surfaces is computed, and one is able to say if the magnetic surface is stable or unstable according the $\delta W \geq 0$ or $\delta W < 0$ respectively. The extreme condition $\delta W = 0$ is considered to be the case of stable equilibria in the case understand the mechanism which allow the highly twisted coronal magnetic flux tube to emerge from the solar surface and produce those beautiful solar flares and loops. Recently Berger and Prior [10] have presented a detailed discussion of twist and writhe of open and closed curves presenting a formula for the twist of magnetic filaments in terms of parallel electric currents. s paper we also compute the writhe and twist numbers investigated previously by Moffat and Ricca [11] and Berger and Field [12] to obtain an expression for the writhe number for the magnetically perturbed vortex filamentary twisted structure. To obtain the writhe number we make use of the helicity local expression for the magnetic field proportional to the magnetic field itself and decompose the magnetic vector field along the Serret-Frenet frame in 3D dimensions. This
allows us to solve scalar MHD equations to obtain constraints on the global helicity expression which will allow us to compute the writhe number. The knowledge of tilt, twist and writhe of solar filaments for example has recently helped solar physicists [13] to work out data obtained from the vector magnetograms placed in solar satellites. This is already an strong motivation to go on investigating topological properties of these filamentary twisted magnetic structures. Earlier the Yokkoh solar mission has shown that the sigmoids which are nonplanar solar filaments are obtained due to the action of electric currents along these filaments which was used recently [14] as motivation to investigate current-carrying torsioned twisted magnetic curves. The paper is organized as follows: In section 2 we compute the conditions imposed on the current-carrying solar loops by their stability. Section 3 addresses the computation of writhe and twist numbers from the magnetic helicity while in section 4 we investigate the Riemann isometry constraint on the twisted thin filament and solve the scalar equations obtained for the curvature and torsion of the magnetic axis of the flux tube. In section 5 we present the conclusions.

II Stability of current-carrying solar loops

Let us now start by considering the MHD field equations

\[ \nabla \cdot \vec{B} = 0 \]  
\[ \nabla \cdot \vec{j} = 0 \]

where \( \alpha \) is the magnetic twist and the magnetic field \( \vec{B} \) along the filament is defined by the expression \( \vec{B} = B_s \vec{t} \) and \( \vec{j} = j_s \vec{t} + j_n \vec{n} + j_b \vec{b} \) is the electric current density where \( j_s \) and \( j_b \) are respectively the components along and cross to the solar filament all along its extension, and \( B_s \) is the component along the arc length s of the filament. The vectors \( \vec{t} \) and \( \vec{n} \) along with binormal vector \( \vec{b} \) together form the Frenet frame which obeys the Frenet-Serret equations

\[ \vec{t}' = \kappa \vec{n} \]  
\[ \vec{n}' = -\kappa \vec{t} + \tau \vec{b} \]  
\[ \vec{b}' = -\tau \vec{n} \]
the dash represents the ordinary derivation with respect to coordinate \( s \), and \( \kappa(s, t) \) is the curvature of the curve where \( \kappa = R^{-1} \). Here \( \tau \) represents the Frenet torsion. We follow the assumption that the Frenet frame may depend on other degrees of freedom such as that the gradient operator becomes

\[
\nabla = \hat{t} \frac{\partial}{\partial s} + \hat{n} \frac{\partial}{\partial n} + \hat{b} \frac{\partial}{\partial b} \tag{II.6}
\]

The other equations for the other legs of the Frenet frame are

\[
\frac{\partial}{\partial n} \hat{t} = \theta_{ns} \hat{n} + [\Omega_{b} + \tau] \hat{b} \tag{II.7}
\]

\[
\frac{\partial}{\partial n} \hat{n} = -\theta_{ns} \hat{t} - (\text{div} \hat{b}) \hat{b} \tag{II.8}
\]

\[
\frac{\partial}{\partial n} \hat{b} = -[\Omega_{n} + \tau] \hat{t} - (\text{div} \hat{n}) \hat{n} \tag{II.9}
\]

\[
\frac{\partial}{\partial b} \hat{t} = \theta_{bs} \hat{b} - [\Omega_{n} + \tau] \hat{n} \tag{II.10}
\]

\[
\frac{\partial}{\partial b} \hat{n} = [\Omega_{n} + \tau] \hat{t} - \kappa - (\text{div} \hat{n}) \hat{b} \tag{II.11}
\]

\[
\frac{\partial}{\partial b} \hat{b} = -\theta_{bs} \hat{n} - [\kappa + (\text{div} \hat{n})] \hat{b} \tag{II.12}
\]

Now that we have this mathematical machinery at our disposal, let us consider the Bernstein et al [15] plasma energy stability relation

\[
\delta W = \frac{1}{2} \int ds[(\vec{Q} + \vec{n}.\vec{\varepsilon}\times\vec{n})^2 + \gamma p(\text{div} \vec{\varepsilon})^2 - 2(\vec{n}.\vec{\varepsilon})^2\vec{j} \times \vec{n}.\vec{B}.\nabla \vec{n})] \tag{II.13}
\]

where

\[
\vec{Q} = \text{curl}(\hat{\varepsilon} \times \vec{B}) \tag{II.14}
\]

and \( \gamma \) is the ratio of the specific heats. According to Lileys [14] if one puts

\[
S = \vec{j} \times \vec{n}.(\vec{B}.\nabla \vec{n}) \tag{II.15}
\]

and if there is no discontinuities in the equilibrium values, then a sufficient condition for stability, i.e, \( \delta W \geq 0 \), is

\[
S \leq 0 \tag{II.16}
\]

substitution of the expressions for \( \vec{j} \) and for \( \vec{B} \)

\[
\vec{j} \times \vec{n} = j_b \hat{t} - j_s \hat{b} \tag{II.17}
\]
into this expression yields the following geometrical constraint in the solar loop

\[
[j_b \kappa(s) + \tau(s)j_s]B \geq 0 \quad (\text{II.18})
\]

By assuming that \( B \geq 0 \) one obtains in the stable equilibrium case \( \delta W = 0 \) or \( S = 0 \) which reads

\[
\frac{j_b}{j_s} = -\frac{\tau}{\kappa} \quad (\text{II.19})
\]

Note that in planar (torsionless) solar loops the normal current \( j_n \) vanishes. In helical solar loops where the ratio between torsion and curvature is constant or \( \beta = -\frac{\tau}{\kappa} \) one has the following relation

\[
\frac{dj_s}{ds} = -\beta \frac{dj_b}{ds} \quad (\text{II.20})
\]

Physically this means that as observe the solar loop along its extension the toroidal current increases while the normal current decreases as going to a focal point. A long and straightforward computation allows us to write down the following expression

\[
\nabla \cdot \vec{j} = 0 \quad (\text{II.21})
\]

implies

\[
\partial_s j_s + [\theta_{bs} + \text{div}\vec{b}]j_s = -j_b (\text{div}\vec{b} + \theta_{bs}) \quad (\text{II.22})
\]

substitution of expression (II.19) into (II.22) allows us to obtain a differential equation for the current \( j_s \) as

\[
\partial_s j_s + \theta_{bs} [1 - \frac{\tau_0}{\kappa_0}]j_s = 0 \quad (\text{II.23})
\]

which yields the solution

\[
j_s = j_s^0 \exp\left[-\int \theta_{bs} (1 - \frac{\tau_0}{\kappa_0}) ds\right] \quad (\text{II.24})
\]

Note that the current along the solar loop decays as the coordinate \( s \to \infty \). From this last expression one notes that

\[
\frac{j_s}{j_s^0} = \frac{\exp[-\int \theta_{bs} ds]}{\exp[-\int \frac{\tau_0}{\kappa_0} ds]} \leq 1 \quad (\text{II.25})
\]

where the RHS inequality is valid since the initial current is greater than the resultant current. This inequality is equivalent to

\[
\exp[-\int \theta_{bs} ds] \leq \exp[-\int \frac{\tau_0}{\kappa_0} ds] \quad (\text{II.26})
\]
which yields the following constraint between the ratio between torsion and curvature and the factor $\theta_{bs}$ as

$$\theta_{bs} \geq \frac{\tau_0}{\kappa_0} \quad (\text{II.27})$$

in all points of the solar loops which are possibly broken in the inflexionary points of the loops [7]. Our computations we neglect $\text{div}\, \vec{b}$ and also consider that $\theta_{bs}$ varies very slowly along the solar loop. Now let us compute the magnetic field from the equation $\nabla.\vec{B} = 0$ which becomes

$$\partial_s B_s + [\theta_{bs} + \theta_{ns}] B_s = 0 \quad (\text{II.28})$$

A simple solution of this equation can be obtained as

$$B_s = B_0 \exp\left(\int (\theta_{bs} + \theta_{ns}) ds\right) \quad (\text{II.29})$$

where $B_0$ is an integration constant. Substitution into equation (II.17) assuming that the flow is geodesic along the filament and abnormality relation $\Omega_s = 0$ along with $\theta_{bs} = \theta_{ns} = 0$ one obtains

$$B_n = B_0 (\text{div}\, \vec{b})^{-1} - \int \kappa ds \quad (\text{II.30})$$

In the next section we shall compute the magnetic energy in terms of the magnetic vector potential $\vec{A}$, which allows us to compute the helicity integral and the writhe number and twist of the solar loops.

### III Magnetic energy, Twist and Writhe of Solar Loops

In this section we compute the magnetic energy, twist and writhe of the solar filament based on the vector potential, assuming that it obeys the Coulomb gauge $\nabla.\vec{A} = 0$ and the definition $\vec{B} = \nabla \times \vec{A}$. The Coulomb gauge becomes

$$\partial_s A_s + [\theta_{bs} + \theta_{ns}] A_s = 0 \quad (\text{III.31})$$

assuming that $A = A(s, n)$ together with the equations for the definition of $\vec{B}$, taking $\vec{A} = A_{s,n} \vec{t}$ yields

$$\partial_n A + \kappa A = 0 \quad (\text{III.32})$$

$$[2\tau + \Omega_n + \Omega_s] A = 0 \quad (\text{III.33})$$
which upon separation of variables $A = \Phi(s)\Theta(n)$ one is able to find

$$A(s, n) = A_0 e^{\kappa_0 n + \int (\theta_{bs} + \theta_{ns}) ds} \quad (III.34)$$

Let us now compute the magnetic energy

$$E_B = \frac{1}{8\pi} \int B^2 dV \quad (III.35)$$

which yields

$$E_B = \frac{(B_0 a)^2}{8} e^{\kappa_0 n + \int (\theta_{bs} + \theta_{ns}) ds} \quad (III.36)$$

Note that from above constraints $\theta_{bs} = a_{\kappa_0}$ and allowing $\theta_{ns}$ to vanish one obtains the energy

$$E_B = \frac{(B_0 a)^2}{8} e^{\kappa_0 n + \int (\theta_{bs} + \theta_{ns}) ds} \quad (III.37)$$

which is the energy of solar loops based on the assumption of its stability. Now let us compute the global helicity

$$H = \int \vec{A} \cdot \vec{B} dV \quad (III.38)$$

which is equivalent to

$$H = \int [A(s, n)B_s] dV \quad (III.39)$$

substitution of the relations above yields

$$H = -\pi a^2 \int [2\tau + \Omega_s + \Omega_n] A^2 ds \quad (III.40)$$

Besides if one considers that the plasma flow in solar loops is geodesic the abnormality factor $\Omega_s$ vanishes and the magnetic helicity reduces to

$$H = -\pi a^2 \int [2\tau + \Omega_n] A^2 ds \quad (III.41)$$

If one assumes that the magnetic vector potential varies also quite slow along the solar filament one has

$$H = -\pi a^2 A^2 \int [2\tau + \Omega_n] ds \quad (III.42)$$

The first integral on the RHS in (III.42) is the total torsion and considering that the helicity is proportional to the sum of twist and writhe and yet that the twist is proportional to the total torsion, we conclude that the writhe number Wr is proportional to the integral of the abnormality $\Omega_n$. 

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IV  Riemann isometry of magnetic flux tube metric

Since in the limit of thin magnetic flux tube one could represent it as a mathematical model for solar loop, it is interesting to argue if there is a similar way to constraint the magnetic flux tube as was done for the current-carrying solar loops in previous sections. A simple way to address and respond to this question is to resource to a simple tool of Riemannian geometry [7, 16] used in general relativity, which is called metric isometry and is represented by the expression

$$\delta g_{ik} = 0$$ (IV.43)

If one applies the isometry hypothesis to the flux tube metric

$$dl^2 = dr^2 + r^2d\theta + K^2(s)ds^2$$ (IV.44)

one obtains that $\delta g_{rr} = 0$ and that

$$\delta g_{\theta\theta} = [\frac{\partial}{\partial r} g_{rr}] \delta r = 2r \delta r = 0$$ (IV.45)

which yields $r = 0$ and means we are close to the magnetic axis of the tube, while

$$\delta g_{ss} = [\frac{\partial}{\partial r} g_{ss}] \delta r + [\frac{\partial}{\partial s} g_{ss}] \delta s$$ (IV.46)

which yields

$$\delta g_{ss} = [r(\kappa(s) \sin \theta \frac{\partial}{\partial s} \theta - \frac{\partial}{\partial s} \kappa(\cos \theta))] \delta s$$ (IV.47)

where we assume that $\delta r = 0$ or that we are over the magnetic axis of the flux tube. Here we also consider as in Riccas paper [8] that $\theta = \theta_R - \int \tau ds$. Applying the isometry condition above for this component one finally obtains the constraint equation for the Frenet curvature as

$$\frac{\partial}{\partial s} \kappa = -\tau_0^2 s + c_0$$ (IV.48)

which after a simple integration yields

$$\frac{\partial}{\partial s} \kappa = -\frac{1}{2} \tau_0^2 s^2 + c_0s + c_1$$ (IV.49)

This equation was obtained under the assumption that the magnetic axis of the twisted magnetic flux tube torsion is constant $\tau = \tau_0$ and that $c_0$ and $c_1$ are integration constants.
V Conclusions

In conclusion, topological and geometrical constraints on solar loops were obtained through the use of the tools of differential geometry. Other kinds of isometries such as the Riemann curvature tensor $R_{ijkl}$ isometry $\delta R_{ijkl} = 0$ could be calculated for tokamaks and stellarators as well as for solar loops. This may appear elsewhere. We also hope that the analytical models discussed here may be useful to solar physicists in the checking their observational results in magnetic twist and writhe in solar activity regions.

Acknowledgements

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References

[1] E. Cartan, Riemannian Geometry in Orthonormal Frames (2001) world scientific.

[2] C.T. Russell, E.R. Priest, L.C. Lee (editors) Physics of Magnetic Flux Ropes (1990) Geophysical Monograph 58.

[3] MacCallum et al. Exact Solutions of Einstein Field Equations (2001) Cambridge University Press.

[4] V. Arnold and B. Khesin, Topological Methods in Hydrodynamics, Applied Mathematics Sciences 125 (1991).

[5] S. Childress and A. Gilbert, Twist, Stretch and Fold: The fast dynamo.

[6] A.B. Mikhailovskii, Instabilities in a Confined Plasma (1998) IOP.

[7] R.L. Ricca (Editor) An Introduction to the Geometry and Topology of Fluid Flows (2001) II/47, NATO Science series. R. Ricca, Fluid Dynamics Research 36 (2005) 319. R. Ricca, Solar Physics 172, 241 (1997).

[8] L.C. Garcia de Andrade, Phys Scripta 73 (2006).

[9] B. S. Liley, Plasma Physics (J. Nuclear Energy Part C) vol. 4, (1962), 325.

[10] M. Berger and C. Prior, J. Phys. A: Mathematical and General (2006) 8321.

[11] K. Moffatt and R.L. Ricca, Proc. Roy. Society London A (1992) 439, 411.

[12] M. A. Berger and G. B. Field, J. Fluid Mech. 147 (1984), 133.

[13] D.W. Longscope and I. Klapper, Astrophysical J., 488:433 (1997).

[14] L. C. Garcia de Andrade, Curvature and Torsion effects on carrying currents twisted solar loops, (2006) Phys of Plasmas nov issue.

[15] I. Bernstein et al. Proc. Roy Soc A244 (1958), 17.

[16] L.C. Garcia de Andrade, Physics of Plasmas 13, 022309 (2006).