DEGREE FOUR PLANE SPANNERS: SIMPLER AND BETTER

Iyad Kanj,† Ljubomir Perković,† and Duru Türkoğlu†

ABSTRACT. Let $P$ be a set of $n$ points embedded in the plane, and let $C$ be the complete Euclidean graph whose point-set is $P$. Each edge in $C$ between two points $p, q$ is realized as the line segment $[pq]$ and is assigned a weight equal to the Euclidean distance $|pq|$. In this paper, we show how to construct in $O(n \log n)$ time a plane spanner of $C$ of maximum degree at most 4 and of stretch factor at most 20. This improves a long sequence of results on the construction of bounded degree plane spanners of $C$. Our result matches the smallest known upper bound of 4 by Bonichon et al. on the maximum degree while significantly improving their stretch factor upper bound from 156.82 to 20. The construction of our spanner is based on Delaunay triangulations defined with respect to the equilateral-triangle distance, and uses a different approach than that used by Bonichon et al. Our approach leads to a simple and intuitive construction of a well-structured spanner and reveals useful structural properties of Delaunay triangulations defined with respect to the equilateral-triangle distance.

The structure of the constructed spanner implies that when $P$ is in convex position, the maximum degree of the spanner is at most 3. Combining the above degree upper bound with the fact that 3 is a lower bound on the maximum degree of any plane spanner of $C$ when the point-set $P$ is in convex position, the results in this paper give a tight bound of 3 on the maximum degree of plane spanners of $C$ for point-sets in convex position.

1 Introduction

Let $P$ be a set of $n$ points embedded in the plane and let $C$ be the complete Euclidean graph whose point-set is $P$. Each edge in $C$ between two points $p, q$ is realized as the line segment $[pq]$ and is assigned a weight equal to the Euclidean distance $|pq|$. In this paper, we consider the problem of constructing a plane spanner of $C$ of small degree and small stretch factor. This problem has received considerable attention, and there is a long list of results on the construction of plane spanners of $C$ that achieve various trade-offs between the degree and the stretch factor of the spanner.

The problem of constructing a plane spanner of $C$ was considered as early as the 1980s, if not earlier. Chew [14] proved that the $L_1$-Delaunay triangulation of $P$, which is the Delaunay triangulation of $P$ defined with respect to the $L_1$-distance, is a spanner of $C$ of stretch factor at most $\sqrt{10}$. Chew’s result was followed by a series of papers showing that other Delaunay triangulations are plane spanners (of $C$) as well. In 1987, Dobkin et al. [18]

*A preliminary version of this paper appears in proceedings of the 32nd International Symposium on Computational Geometry [20].
†School of Computing, DePaul University, Chicago, USA. {ikanj,lperkovic,dturkogl}@cs.depaul.edu
showed that the classical $L_2$-Delaunay triangulation of $P$ is a plane spanner of stretch factor at most $\frac{\pi(1+\sqrt{5})}{2}$. This bound was subsequently improved by Keil and Gutwin [21] to $\frac{4\pi}{3\sqrt{3}}$.

In the meantime, Chew [15] showed that the $TD$-Delaunay triangulation defined using a distance function based on an equilateral triangle — rather than a square ($L_1$-distance) or a circle ($L_2$-distance) — is a spanner of stretch factor 2. This result was generalized by Bose et al. [8], who showed that the Delaunay triangulation defined with respect to any convex distance function (i.e., based on a convex shape) is a plane spanner. The bound on the stretch factor of the $L_2$-Delaunay triangulation by Keil and Gutwin stood unchallenged for many years until Xia recently improved the bound to below 2 [26]. Recently, Bonichon et al. [5] improved Chew’s original bound on the stretch factor of the $L_1$-Delaunay triangulation to $\sqrt{4 + 2\sqrt{2}}$ and showed this bound to be tight.

All the Delaunay triangulations mentioned above can have unbounded degree. In recent years, bounded degree plane spanners have been used as the building block of wireless network topologies. Wireless distributed system technologies, such as wireless ad-hoc and sensor networks, are often modeled as proximity graphs in the Euclidean plane. Spanners of proximity graphs represent topologies that can be used for efficient communication. For these applications, in addition to having low stretch factor, spanners are typically required to be plane and have bounded degree, where both requirements are useful for efficient routing [11, 25].

The wireless network applications motivated researchers to turn their attention to minimizing the maximum degree of the plane spanner as well as its stretch factor. It can be readily seen that 3 is a lower bound on the maximum degree of a spanner of $C$ because a Hamiltonian path/cycle through a set of points arranged in a grid has unbounded stretch factor. Work on bounded degree but not necessarily plane spanners of $C$ closely followed the above-mentioned work on plane spanners. In a 1992 breakthrough, Salowe [24] proved the existence of spanners of maximum degree 4. The question was then resolved by Das and Heffernan [16] who showed that spanners of maximum degree 3 always exist. The focus in this line of research was to prove the existence of low degree spanners and the techniques developed to do so were not tuned towards constructing spanners that had both low degree and low stretch factor. For example, the bound on the stretch factor of the degree-4 spanner by Salowe [24] is greater than $10^9$, which is far from practical. Furthermore, these bounded-degree spanners are not guaranteed to be plane.

Bose et al. [10] were the first to show how to extract a subgraph of the $L_2$-Delaunay triangulation that is a bounded-degree, plane spanner of $C$. The maximum degree and stretch factor they obtained were subsequently improved by Li and Wang [22], by Bose et al. [12], and by Kanj and Perković [19] (see Table 1). The approach used in all these results was to extract a bounded degree spanning subgraph of the $L_2$-Delaunay triangulation and the main goal was to obtain a bounded-degree plane spanner of $C$ with the smallest possible stretch factor. In a breakthrough result, Bonichon et al. [4] lowered the bound on the maximum degree of a plane spanner from 14 to 6. Instead of using the $L_2$-Delaunay triangulation as the starting point of the spanner construction, they used the $TD$-Delaunay triangulation and exploited a connection between these Delaunay triangulations and $1/2-\theta$ graphs. The plane spanner they constructed also has a small stretch factor of 6. Independently, Bose et
Table 1: Results on plane spanners with maximum degree bound $\Delta$. The constant $C_0 = 1.998$ is the best known upper bound on the stretch factor of the $L_2$-Delaunay triangulation [26].

| Paper                | $\Delta$ | Stretch factor bound                  |
|----------------------|----------|----------------------------------------|
| Bose et al. [10]     | 27       | $(\pi + 1)C_0 \approx 8.27$           |
| Li and Wang [22]     | 23       | $(1 + \pi \sin \frac{\pi}{3})C_0 \approx 6.43$ |
| Bose et al. [12]     | 17       | $(2 + 2\sqrt{3} + \frac{3\pi}{2} + 2\pi \sin(\frac{\pi}{12}))C_0 \approx 23.56$ |
| Kanj and Perković [19]| 14       | $(1 + \frac{2\pi}{14 \cos(\frac{\pi}{14})})C_0 \approx 2.91$ |
| Bonichon et al. [4]  | 6        | 6                                      |
| Bose et al. [7]      | 6        | $1/(1 - \tan(\pi/7)(1 + 1/\cos(\pi/14)))C_0 \approx 81.66$ |
| Bonichon et al. [6]  | 4        | $(19 + 29\sqrt{2})\sqrt{4 + 2\sqrt{2}} \approx 156.82$ |
| This paper           | 4        | 20                                     |

In this paper, we present a construction of a plane spanner $S$ of $C$ of degree at most 4 and of stretch factor at most 20. This result matches the smallest known upper bound of 4 on the maximum degree of the spanner by Bonichon et al. [6], while significantly improving their stretch factor bound from 156.82 to 20. Our construction is also simpler and more intuitive. It is based on Delaunay triangulations defined with respect to the equilateral-triangle distance, similar to the degree 6 spanner construction used by Bonichon et al. [4], which could be viewed as the starting point of our construction. To get down to maximum degree 4, our approach introduces fresh techniques in both the construction and the analysis of the spanner. Unlike the approach in [4], our approach has a bias — from the beginning — towards certain edges of the Delaunay triangulation; this bias breaks symmetry and ensures that the constructed spanner is well structured. To make up for edges not in the spanner, we make use of recursion which, unlike the construction in [6], may have depth not bounded by a constant. To ensure that the recursion is contained and yields short paths, we carefully add shortcut edges to the spanner to ensure the existence of paths with specific properties, which we refer to as monotone paths. Finally, in our analysis we use a new type of distance metric and we also take the extra step of analyzing the stretch factor of our spanner with respect to $C$ directly, rather than with respect to the underlying Delaunay triangulation.

The structure of our spanner guarantees that if the given point-set is in convex position then the constructed spanner has maximum degree at most 3. Therefore, for any point-set in convex position, there exists a plane spanner of $C$ of maximum degree at most 3. We mention that, very recently and independently, Biniaz et al. [1] proved the existence of degree-3 plane spanners of stretch factor $3 + 4\pi/3$ for points in convex position. We also show that 3 is a lower bound on the maximum degree of plane spanners of $C$ for point-sets in convex position. This completely resolves the question about minimizing the maximum degree of plane spanners of $C$ for point-sets in convex position.
2 Preliminaries

Given a set of points $P$ embedded in the Euclidean plane, we consider the complete weighted graph $C(P)$, or simply $C$, where each edge between any two points $p,q \in P$ is associated with the line segment $[pq]$ and is assigned a weight equal to the Euclidean distance $|pq|$. Given a subgraph $G$ of $C$, $G$ is said to be plane if the edges of $G$ do not cross each other, i.e., the line segments associated with the edges of $G$ intersect only at their endpoints. The maximum degree of $G$ is the maximum degree (in $G$) over all points in $P$; we say that a family of graphs has bounded degree if there is an integer constant $c \geq 0$ such that every graph in the family has a maximum degree at most $c$. If graph $G$ is connected, we define the distance between any two points $p,q \in P$, denoted $d_G(p,q)$, to be the weight of a minimum-weight path between $p$ and $q$ in $G$, where the weight of a path is the sum of the weights of its edges.

Given a constant $\rho \geq 1$, we say that $G$ is a $\rho$-spanner of $C$ if for any two points $p,q \in P$, $d_G(p,q) \leq \rho \cdot |pq|$; we refer to the minimum such constant $\rho$ as the stretch factor of $G$. We also say that a family of geometric graphs, one for every finite set of points $P$ in the plane, is a spanner if there is a constant $\rho \geq 1$ such that every graph $G(P)$ in the family is a $\rho$-spanner of $C(P)$; we refer to the minimum such constant $\rho$ as the stretch factor of the family. In this paper, the family we construct consists of the set of spanners $G(P)$, where $G(P)$ is the spanner obtained by applying our algorithm to a point-set $P$.

In this paper, we rely on a metric that is different from the Euclidean metric. In order to define this metric, we fix an equilateral triangle with two of its points lying on the $x$-axis at coordinates $(0,0)$ and $(1,0)$, and the third point lying below the $x$-axis; we use the symbol $\triangledown$ to refer to this equilateral triangle. We define a triangle to be a $\triangledown$-homothet if it can be obtained through a translation of $\triangledown$ followed by a scaling. We define the triangular metric, $d_{\triangledown}$, as follows:

**Definition 1.** For any two points $p,q \in P$, define $d_{\triangledown}(p,q)$ to be the side-length of the smallest $\triangledown$-homothet that contains $p$ and $q$ on its boundary; we denote this triangle $\triangledown(p,q)$.

It is easy to verify that $d_{\triangledown}$ is indeed a metric. In particular, for any two points $p,q$, we have $d_{\triangledown}(p,q) = 0 \iff p = q$, we have symmetry as in $d_{\triangledown}(p,q) = d_{\triangledown}(q,p)$, and for any third point $r$, we have the triangle inequality $d_{\triangledown}(p,q) \leq d_{\triangledown}(p,r) + d_{\triangledown}(r,q)$. It is also easy to see that $p$ or $q$ must be a vertex of the triangle $\triangledown(p,q)$ and that $|pq| \leq d_{\triangledown}(p,q)$.

Using the triangular metric $d_{\triangledown}$, we define a subgraph $D$ of $C$ as follows. For every point $w \in P$, we partition the space around $w$ into six equiangular cones whose common apex is $w$, three above and three below the horizontal line passing through $w$, as illustrated in Figure 1-(a). We denote the middle cone above the horizontal line and the two outer cones below the horizontal line as the positive cones of $w$, and the remaining three cones as the negative cones of $w$. Each point $w$ chooses an edge in each of its three positive cones by selecting the point $v \neq w$ in the cone such that $d_{\triangledown}(w,v)$ is minimum. Assuming that $P$ is in general position for any two distinct points $v,v'$ in a positive cone of $w$, we obtain
Figure 1: (a) To construct graph $D$, every point $w$ chooses the shortest edge, according to the $d_\vartriangle$ distance, in every positive cone. (b) Edge $(v_2, w)$ is the anchor of $w$ in the negative cone shown, because it is the shortest edge according to the $d_\vartriangle$ distance, among edges incoming to $w$ in the cone; the path $v_1, \ldots, v_k$ is the canonical path of anchor $(v_2, w)$.

We observe that for any point $w \in \mathcal{P}$ there is at most one edge outgoing from $w$ in a positive cone of $w$, but there can be an unbounded number of edges incoming to $w$ in a positive cone of $w$, and we use the term path to refer to weak paths in $D$; we always use the term directed path when edge orientations are relevant.

---

$2$A TD-Delaunay triangulation of $\mathcal{P}$ is not necessarily a triangulation of $\mathcal{P}$ as defined traditionally (a triangulation of the convex hull of the set of points). Just as Chew [15] did, we abuse the term triangulation because TD-Delaunay triangulations are closely related to classical $L_2$-Delaunay triangulations.
negative cone of \( w \), and that in such cases all these edges have the same color as the cone itself (e.g., see Figure 1-(b)). We follow the same approach as Bonichon et al. [4], and identify in each negative cone of point \( w \) an edge that plays a key role in the spanner construction:

**Definition 2.** For any point \( w \in \mathcal{P} \), and for each negative cone of \( w \) that contains at least one edge incoming to \( w \), let (directed) edge \((v, w) \in \mathcal{D}\) be the edge in the cone such that \( d_{\mathcal{G}}(v, w) \) is minimum. We define \((v, w)\) to be the anchor of \( w \) in the cone.

We say that anchors incident to the same point \( w \) are *adjacent* if their cones are adjacent. Note that for any two adjacent anchors incident to \( w \), one of the two adjacent anchors must lie in a positive cone of \( w \) and must be an anchor of a point other than \( w \).

Consider a negative cone of a point \( w \in \mathcal{P} \) containing at least one incoming edge to \( w \) in \( \mathcal{D} \). Let \((v_1, w), \ldots, (v_k, w) \in \mathcal{D}\), where \( k \geq 1 \), be all the incoming edges to \( w \) that lie in the cone, listed in counterclockwise order, as illustrated in Figure 1-(b), and let \((v_j, w)\), for some \( j \) such that \( 1 \leq j \leq k \), be the anchor of \( w \) in the cone. We call \((v_1, w), \ldots, (v_k, w)\) the *fan* of the anchor \((v_j, w)\). We identify \((v_j, w)\) as the anchor of each edge in the fan. Note that every edge in \( \mathcal{D} \) has an anchor which could be itself. We call the first edge \((v_1, w)\) and the last edge \((v_k, w)\) of the fan the *boundary edges* of the anchor \((v_j, w)\). Note that either one (possibly both) of the boundary edges of an anchor could be the anchor itself. If \( k \geq 2 \), since \( \mathcal{D} \) is a triangulation, it follows that \((v_i, v_{i+1}, w)\) is a triangle in \( \mathcal{D} \), for \( i = 1, \ldots, k-1 \). Hence, \( v_1, \ldots, v_k \) is a (weak) path in \( \mathcal{D} \) between the endpoints \( v_1 \) and \( v_k \). We call this path the *canonical path* of \( w \) in the designated cone; we also call each edge on this path a *canonical edge* of \( w \). Finally, we refer to the (weak) subpath \( v_r, \ldots, v_s \) of the canonical path \( v_1, \ldots, v_k \) of \( w \) as the canonical path between \( v_r \) and \( v_s \) of \( w \). The two *sides* of an edge are the two half-planes defined by the line obtained by extending the edge. Given a canonical edge \( e \), we distinguish \( e \) being canonical on a particular side (or on both sides). If \( e \) is a canonical edge of a point that lies on a given side of \( e \), we say that \( e \) is canonical on that given side. Note that a canonical edge can be canonical on both sides. Using these definitions, we state the following useful properties related to canonical edges:

**Lemma 1.** Let \((s, t)\) be a canonical edge of a point \( w \), and let \((s', t)\) be the anchor of \((s, t)\).

a. The edges \((s, w)\) and \((t, w)\) are in \( \mathcal{D} \).

b. The edge \((s, w)\) cannot be a canonical edge on the side containing \( t \).

c. The edge \((t, w)\) is not an anchor.

d. The edge \((s, t)\) is a boundary edge of its anchor \((s', t)\).

**Proof.** By definition of canonical edges, \((s, t, w)\) forms a triangle in \( \mathcal{D} \) and the two edges of this triangle incident to \( w \) are directed towards \( w \), hence part (a) follows. Since \( \mathcal{D} \) is a triangulation, there can be only two triangles in \( \mathcal{D} \) that contain \((s, w)\) as an edge, one on each side of \((s, w)\). On the side of \((s, w)\) that contains \( t \), that triangle is \((s, t, w)\), so \((s, w)\) can only be a canonical edge of point \( t \) (on the side that contains \( t \)). But since the edge \((t, w)\) is directed towards \( w \), that is not possible either, and part (b) follows. Next, since the edge \((s, t) \in \mathcal{D} \) is directed towards \( t \), it is easy to verify that \( d_{\mathcal{G}}(s, w) < d_{\mathcal{G}}(t, w) \). Since
both \((s, w)\) and \((t, w)\) are in the same negative cone of \(w\), it follows that \((t, w)\) cannot be an anchor proving part (c). Finally, consider the fan of the anchor \((s', t)\). By definition, \((s, t)\) is an edge of that fan and the edge \((t, w)\) is not. Since \((s, t, w)\) is a triangle in \(\mathcal{D}\), the edges \((t, w)\) and \((s, t)\) must be consecutive around \(t\), and thus \((s, t)\) must be the first or the last edge of the fan. This proves part (d).

\[\square\]

3 Monotone paths

In this section, we define a type of path in \(\mathcal{C}\) that generalizes canonical paths and that will be a key tool in our analysis.

**Definition 3.** A path in \(\mathcal{C}\) is **monotone** if the edges of the path are colored with at most two colors, and any two consecutive edges of the path lie in different non-adjacent cones of the shared endpoint.

The key property of a monotone path between \(u\) and \(v\) is that its length can be bounded by twice the side-length of \(\nabla(u, v)\), i.e., by \(2d_{\nabla}(u, v)\). This follows from a stronger insight which we develop next. To facilitate our discussion, we label the vertices of a \(\nabla\)-homothet green, blue, and red, in clockwise order starting from the upper left vertex.

**Definition 4.** Let \(v\) be a point lying in a positive cone of \(u\) of color \(c_1\). With \(u\) being the vertex of \(\nabla(u, v)\) of color \(c_1\), let \(y\) and \(z\) be the vertices of \(\nabla(u, v)\) of colors \(c_2\) and \(c_3\) respectively (refer to Figure 2 where \(c_1 = \text{green}\), \(c_2 = \text{red}\), and \(c_3 = \text{blue}\)). We define the following distance functions \(\delta_{\nabla}^{c_2}, \delta_{\nabla}^{c_3}\), and \(\delta_{\nabla}^{\min}\):

1. \(\delta_{\nabla}^{c_2}(u, v) = \delta_{\nabla}^{c_2}(v, u) = |yv|\).
2. \(\delta_{\nabla}^{c_3}(u, v) = \delta_{\nabla}^{c_3}(v, u) = |zv|\).
3. \(\delta_{\nabla}^{\min}(u, v) = \delta_{\nabla}^{\min}(v, u) = \min\{\delta_{\nabla}^{c_2}(u, v), \delta_{\nabla}^{c_3}(u, v)\}\).

![Figure 2](image_url)

**Figure 2:** (a) If point \(v\) lies in the positive green cone of point \(u\), and the vertices of \(\nabla(u, v)\), \(u, y, z, v\), are colored green, red, and blue, respectively, then \(d_{\nabla}(u, v) = |zu|, \delta_{\nabla}^{\text{blue}}(u, v) = |zv|,\) and \(\delta_{\nabla}^{\text{red}}(u, v) = |yv|\). (b) and (c) \(P\) is a monotone path between \(u\) and \(v\) with edges colored green or red. The projection onto \(zu\) (resp. \(zy\)) of \(vp, pq,\) and \(qu\) do not overlap and are contained within \([zu]\) (resp. \([zv]\)).
Given the assumptions of Definition 4, let $P$ be a monotone path in $D$ between $u$ and $v$ whose edges are colored $c_1$ or $c_2$. We define, using the lines $zu$ and $zu$ as axes of a coordinate system of the Euclidean plane, the projection onto $zu$ and onto $zu$. In the following lemma, we use this projection to map the edges of $P$ and derive an upper bound on the length of $P$:

**Lemma 2.** Let $v$ be a point lying in a positive cone of $u$ of color $c_1$. Let $P$ be a monotone path between $u$ and $v$ whose edges are colored $c_1$ or $c_2$ (refer to Figure 2 where $c_1 = \text{green}$, $c_2 = \text{red}$, and $c_3 = \text{blue}$). With $u$ being the vertex of $\nabla(u,v)$ of color $c_1$, let $z$ be the vertex of $\nabla(u,v)$ of color $c_3$. Then, the monotone path $P$ satisfies the following:

a. The projections of all edges of $P$ onto $zu$ (resp., $zv$) do not overlap and are contained within the segment $[zu]$ (resp., $[zv]$); see Figures 2-(b) and 2-(c).

b. For any $c_1$ (resp., $c_2$) colored edge of $P$ between points $p$ and $q$, the projection of this edge onto $zu$ (resp., $zv$) has length $\delta_{zu}(p,q) \geq |pq|$.

c. The sum of the lengths with respect to the triangular metric of the $c_1$-colored edges of $P$ is at most $\delta^2_{zu}(u,v) = |zu|$, i.e., $\sum_{c_1\text{-colored } p \in P} \delta_{zu}(p,q) \leq \delta^2_{zu}(u,v) = |zu|$.

d. The sum of the lengths with respect to the triangular metric of the $c_2$-colored edges of $P$ is at most $|zv|$, i.e., $\sum_{c_2\text{-colored } p \in P} \delta_{zv}(p,q) \leq \delta^2_{zv}(u,v) = |zv|$.

e. The length of $P$ is at most $\delta^2_{zu}(u,v) + \delta^2_{zv}(u,v) \leq 2\delta^2_{zu}(u,v)$.

**Proof.** For part (a), we consider the coordinates of the points of $P$ in the coordinate system of the Euclidean plane defined by using the lines $zu$ and $zu$ as axes. Since $P$ is monotone, any two consecutive edges of $P$ lie in different non-adjacent cones of their shared endpoint. Thus, when visiting the points of $P$ in the order in which they appear on $P$, the coordinates of the points along the $zu$ (resp., $zv$) axis form a monotonic sequence (decreasing or increasing) between the coordinates of $z$ and $u$ (resp., $z$ and $v$), and part (a) follows. For part (b), we consider a $c_1$ (resp., $c_2$) colored edge of $P$ between points $p$ and $q$, and without loss of generality, we assume that $p$ lies on the $c_1$ (resp., $c_2$) colored corner of $\nabla(p,q)$. Since $zu$ (resp., $zv$) is parallel to a side of $\nabla(p,q)$ with $p$ as an endpoint, the projection of the edge between $p$ and $q$ onto $zu$ (resp., $zv$) has length $\delta_{zu}(p,q)$, proving part (b). Parts (c) and (d) follow from parts (a) and (b), and part (e) follows from parts (c) and (d). 

Implied by Lemma 2, the following lemma makes explicit an insight implicit in Lemma 2 of [4] on canonical paths (see Figure 3-(a)).

**Lemma 3.** For any two edges $(v,w)$ and $(u,w)$ that lie in the same fan:

a. The canonical path $P$ between $v$ and $u$ is monotone.

b. The total length of the edges on $P$ of the same color is at most $\delta^2_{zu}(v,u)$.

c. The length of the canonical path $P$ between $v$ to $u$ is at most $2\delta^2_{zu}(v,u)$. 

Figure 3: (a) Illustration of Lemma 3. (b) \( w \) is incident to an anchor in every cone. In step 1, both blue anchors are added. In step 2, no more than one white anchor is added below the horizontal line through \( w \) and no more than one white anchor is added above.

**Proof.** For part (a), we assume, without loss of generality, that \( w \) lies in the red positive cones of \( v \) and of \( u \). We then observe that for every point \( p \) on \( P \), \((p, w)\) is an edge in \( D \). Therefore, every edge of \( P \) must lie in the blue or green positive cones of its tail, and thus the edges of \( P \) are colored with at most two colors. Furthermore, since \( D \) is planar, at every intermediate point \( p \) of \( P \), the two edges of \( P \) incident to \( p \) must lie in different non-adjacent cones. The canonical path \( P \) between \( v \) and \( u \) is thus monotone. Hence, parts (b) and (c) follow by Lemma 2.

### 4 The Spanner

In this section, we describe the construction of a plane spanner of \( C \) of maximum degree at most 4 and stretch factor at most 20. In our construction, we will bias blue – positive and negative – cones and edges. This bias breaks symmetry and results in a spanner satisfying structural properties that allow us to prove the desired upper bounds on the spanner degree and stretch factor. These structural properties also ensure that the spanner has maximum degree at most 3 when the point-set \( P \) is in convex position. In the algorithm description and the remainder of the paper, we find it convenient to refer to the four non-blue cones, as well as all the red and green edges, as *white* (see Figure 3-(b)). We also use some new terminology which we define next.

If \( e \) is a canonical edge of a point \( w \) that lies in a white (resp., blue) cone of \( w \), we say that \( e \) is a canonical edge in a *white* (resp., *blue*) cone. We note that a canonical edge could be in a white cone of one point and in a blue cone of another. Given a white anchor \((v, w)\), the ray starting from \( w \) extending \((v, w)\) partitions the (white) negative cone of \( w \) containing \( v \) into two *sides*: we refer to the side of the cone that is adjacent to a blue cone as the *blue side*, and we refer to the other side that is adjacent to a white cone as the *white side*. We say that an edge \((u, w)\) in the fan of \((v, w)\) is on the *white side* (resp. *blue side*) if it is on the white side (resp. blue side) of \((v, w)\).

The following describes the construction of the spanner \( S \) of \( C \). The construction is based on the underlying triangulation \( D \) of \( C \). We start by constructing a degree-4 anchor.
subgraph $A$ of $S$ that includes all blue anchors (steps 1 and 2). We then augment $S$ by adding some white canonical edges (steps 3 and 5) and shortcut edges (steps 4 and 6).

1. We add to $A$ (which is initially empty) every blue anchor (see Figure 3-(b)).

2. In increasing order of length with respect to the metric $d_\varphi$, for every white anchor $a$, we add $a$ to $A$ if no white anchor adjacent to $a$ is already in $A$ (see Figure 3-(b)).

3. We set $S$ to $A$ and then add to $S$ every (white) canonical edge in a blue cone if the edge is not in $A$ (see Figure 4-(a)).

4. For every pair of canonical edges $(p, q), (r, q)$ in a blue cone such that $(p, q), (r, q) \in S \setminus A$, we remove $(p, q)$ and $(r, q)$ from $S$ and add a (white) shortcut edge between $p$ and $r$ (see Figure 4-(a)).

5. We add to $S$ every white canonical edge that is on the white side of its (white) anchor, but only if its anchor is not in $A$ (see Figure 4-(b)).

6. For every white anchor $(v, w)$ and its boundary edge $(u, w) \neq (v, w)$ on the white side, let $(u = p_0, p_1, \ldots, p_k = v)$ be the canonical path between $u$ and $v$ (see Figure 4-(b)). We apply the following procedure at a current point $p_i$ starting with $i = 0$ and stopping when $i = k$:

   (a) If the canonical edge $(p_{i+1}, p_i)$ is white, we skip this edge and set $i$ to $i + 1$;

   (b) otherwise, $(p_i, p_{i+1})$ must be blue. Let $j$ be the smallest index of a point in $P_i = \{p_{i+1}, \ldots, p_k\}$ such that $p_j$ lies in a (negative) white cone of $p_i$ and all points in $P_i$ are on the same side of the straight line $p_ip_j$ (possibly on $p_ip_j$ itself).\footnote{By Lemma 4, point $p_j$ is well defined, and the shortcut $(p_j, p_i)$ selected in this step is an edge of the convex hull of $\{p_i, \ldots, p_k\}$.}

   We add the (white) shortcut $(p_j, p_i)$ to $S$; we remove the (white) canonical edge $(p_j, p_{j-1})$ from $S$ if $(p_j, p_{j-1}) \in S \setminus A$; and we set $i$ to $j$.

In the following section we prove that this algorithm yields a plane spanner of maximum degree at most 4 and stretch factor at most 20. We provide here a high-level overview of our arguments.

To show planarity, we note that the underlying graph $D$ is planar and that the only edges of $S$ not in $D$ are the shortcut edges added in steps 4 and 6-(b). We prove in Lemmas 5 and 6 that each such edge does not intersect any other edge of $S$. For the degree upper bound, we note that the first two steps of the algorithm yield the subgraph $A$ of maximum degree at most 4. In the remaining steps, we carefully add additional edges, whether canonical edges or shortcuts of canonical paths. To prove the degree bound, we develop a charging argument that assigns each edge of $S$ to a cone at each endpoint and show, in Lemma 7, that no more than 4 cones are charged at every point and each cone is charged at most once.

To prove that $S$ is a spanner, we show that every edge $(u, w)$ in $D$ but not in $S$ can be reconstructed, by which we mean that there is a short path between $u$ and $w$ in $S$. To do
Figure 4: (a) In step 3, white canonical edges of \( w \) in the negative blue cone of \( w \) are added to \( S \) if not in \( A \) already; in step 4, any pair of canonical edges of \( w \) added in step 3 that are incoming at the same point are replaced by a shortcut between the outgoing endpoints ((\( p_6, p_5 \) and \( p_4, p_5 \)) replaced by shortcut \( (p_6, p_4) \) and \( (p_3, p_1) \) ) and \( (p_1, p_2) \) replaced by shortcut \( (p_3, p_1) \) ). (b) Shortcut edges \( (p_3, p_0) \) and \( (p_7, p_4) \) are added to \( S \) in step 6; edges \( (p_7, p_6) \) and \( (p_3, p_2) \) are not in \( S \) unless they are anchors in \( A \).

In this section, we consider the path between \( u \) and \( w \) in \( D \) consisting of the anchor \( (v, w) \) of \( (u, w) \) and the canonical path from \( u \) to \( v \) of anchor \( (v, w) \), and we argue that every edge on that path can be reconstructed. Because canonical edges are boundary edges, it is sufficient to show that all anchors and boundary edges can be reconstructed. We offer some further intuition on how we are going to do this.

In step 1, we add all blue anchors to \( S \) and in steps 3 and 4 we add to \( S \) all white canonical edges in blue cones, except for some consecutive pairs of canonical edges that are replaced with shortcut edges. Together, these steps ensure that (almost) all blue edges are reconstructible as we show in Lemma 8; in particular, all blue boundary edges are reconstructible.

If \( (u, w) \) is a white boundary edge, the edges on the canonical path between \( u \) and \( v \) are either blue boundary edges (which are reconstructible, as discussed above) or white boundary edges on the white side of their anchor. Steps 5 and 6 ensure that white boundary edges on the white side of their anchor are reconstructible with a monotone path that is constructed recursively using shortcuts added in step 6. Therefore, if white anchor \( (v, w) \) is in \( S \), edge \( (u, w) \) is reconstructible as we show in Lemma 9. If white anchor \( (v, w) \) is not in \( S \) then there must exist a shorter anchor adjacent to \( (v, w) \) in \( S \) (by step 2). In Lemma 12, we show that this shorter anchor can be used to reconstruct anchor \( (v, w) \) implying that \( (u, w) \) is reconstructible as well.

5 Properties of the Spanner

In this section, we prove the three properties of the spanner \( S \) obtained using our algorithm: planarity, the maximum degree upper bound of 4, and the stretch factor bound of 20. We start with the following lemma validating the construction of the shortcuts in step 6 of the spanner construction:
Lemma 4. Let \( p_i \) be a point considered in step 6-(b) of the construction, and let \( P_i \) be the canonical path \( \{p_{i+1}, \ldots, p_k\} \). Then there exists a point \( p_j \in P_i \) such that \( p_j \) lies in a negative white cone of \( p_i \) and all points in \( P_i \) are on the same side of the straight line \( p_ip_j \). Moreover, the shortcut \((p_j, p_i)\) selected in step 6-(b) is an edge of the convex hull of \( \{p_i, \ldots, p_k\} \).

Proof. Let \((v, w)\) be the white anchor of the canonical path \( P_i \). Assume, without loss of generality, that \((v, w)\) is red; the arguments in the other case are similar. We start by showing that \( p_k \) lies in the negative green cone of \( p_i \). Since \( d_{\bigtriangleup}(p_k, w) < d_{\bigtriangleup}(p_i, w) \), \( (p_k, w), (p_i, w) \in D \) lie in the same fan, and \((p_k, w)\) is clockwise from \((p_i, w)\), \( p_k \) must lie in the negative green cone of \( p_i \). Consider the set \( Q \) of points on the convex hull of \( \{p_i, \ldots, p_k\} \) that lie on the same side of the line \( p_ip_k \) as point \( w \) (including \( p_i \) and \( p_k \)). Observe that every point \( q \in Q \setminus \{p_i\} \) falls in the negative green cone of \( p_i \); \( q \) cannot be in a blue cone of \( p_i \) by definition of \( Q \) and the fact that \( p_k \) lies in the negative green cone of \( p_i \); also, \( q \) cannot be in the positive red cone of \( p_i \) by the fact that \((p_i, w) \in D \). Let \( p_j \in Q \) be the point such that \( p_ip_j \) is an edge of the above convex hull. Clearly, \((p_j, p_i)\) satisfies the statement of the lemma and coincides with the choice of the point in step 6-(b) of the construction. □

Next, we show that \( S \) is plane. We first need the following definition and lemma.

Definition 5. An edge \((u, w)\) \( \in D \) is uncrossed if no shortcut in \( S \) crosses \((u, w)\).

Lemma 5. All anchors, all canonical edges, and all boundary edges are uncrossed.

Proof. Let \((p, r)\) be a shortcut. Then either \((p, r)\) was added in step 4 or in step 6 of the spanner construction. If \((p, r)\) was added in step 4, let \((p, q)\) and \((r, q)\) be the pair of canonical edges in the blue cone as described in step 4, and let \( w \) be the apex of this blue cone. Since both \((p, q)\) and \((r, q)\) are incoming to \( q \), it is easy to verify that the two adjacent triangles \((p, q, w)\) and \((q, w, r)\) in \( D \) form a convex quadrilateral. Therefore, the shortcut \((p, r)\) crosses only the diagonal \((q, w)\) \( \in D \) of this quadrilateral. Because \((p, q)\) (or \((r, q)\)) is in \( D \), by part (c) of Lemma 1 \((q, w) \in D \) is not an anchor. Since \((q, w)\) falls between \((p, w)\) and \((r, w)\), it cannot be a boundary edge, and by part (b) of Lemma 1 it cannot be a canonical edge either.

Next, consider a shortcut \((p_j, p_i)\) that was added in step 6 of the spanner construction, and let \((v, w)\) be the white anchor and \( p_{i+1}, p_{i+2}, \ldots, p_{j-1} \) be the points on the canonical path between \( p_i \) and \( p_j \) as described in step 6. Observe that, by the choice of \( p_j \), \((p_{i+1}, w), (p_{i+2}, w), \ldots, (p_{j-1}, w) \) are the only edges in \( D \) that the shortcut \((p_j, p_i)\) crosses. None of these edges is an anchor, and because they fall between \((p_i, w)\) and \((p_k, w)\), by the same token as above, it follows (using Lemma 1) that none of these edges is a boundary edge or a canonical edge. □

Lemma 6. The subgraph \( S \) is a plane subgraph of \( C \).

Proof. Let \( S_1 = D \cap S \) and \( S_2 = S \setminus S_1 \). Note that \( S_1 \) consists of \( A \) plus those canonical edges that are added in steps 3 or 5 and kept after steps 4 and 6. Note also that \( S_2 \) consists only of the shortcuts which are added in steps 4 and 6. Since \( S_1 \) is a subgraph of \( D \), \( S_1 \) is plane. By Lemma 5, all the edges in \( S_1 \) are uncrossed, i.e., no shortcut (edge in \( S_2 \)) crosses an edge in \( S_1 \). To conclude the proof, we show that no two edges in \( S_2 \) cross either. Observe
that any two shortcuts connect pairs of endpoints of canonical paths that either belong to different fans or that belong to the same fan. In the former case, the shortcuts do not cross because they are contained in different fans. In the latter case, the shortcuts do not cross because they connect the endpoints of non-overlapping canonical subpaths of the same canonical paths.

To facilitate the discussion in the proof of the degree upper bound, we refer to the two adjacent white cones above (resp., below) the horizontal line through a point \( p \in \mathcal{P} \) as the upper (resp., lower) white sector of \( p \); we also refer to the two blue cones at \( p \) as the left and right blue sectors of \( p \). We develop a charging scheme to show that, for each point \( p \), each edge incident to \( p \) in \( S \) can be mapped in a one-to-one fashion to one of the four sectors at \( p \). To describe the charging scheme for every edge \( e \in S \) and for every endpoint \( p \) of \( e \), we define \( \sigma(e,p) \) to be the sector of \( p \) that contains \( e \). Also for a point \( p \), we denote by \( LB_p \), \( RB_p \), \( UW_p \), and \( LW_p \), the left blue, the right blue, the upper white, and the lower white sectors of \( p \) respectively. We describe in Table 2 below the charging scheme for every edge \( e = (x,y) \in S \) based on which step of the construction \( e \) is added to \( S \).

| Step | Classification of \( e = (x,y) \) | Charge at \( x \) | Charge at \( y \) |
|------|---------------------------------|------------------|------------------|
| 1    | Blue anchor in \( \mathcal{A} \) | \( \sigma(e,x) = LB_x \) | \( \sigma(e,y) = RB_y \) |
| 2    | White anchor in \( \mathcal{A} \) | \( \sigma(e,x) = UW_x \) or \( LW_x \) | \( \sigma(e,y) = LW_y \) or \( UW_y \) |
| 3    | White canonical edge in a blue cone | \( \sigma(e,x) = UW_x \) or \( LW_x \) | \( LB_y \) |
| 4    | (White) shortcut in a blue cone | \( \sigma(e,x) = LW_x \) or \( LW_x \) | \( \sigma(e,y) = LW_y \) or \( UW_y \) |
| 5    | White canonical edge in a white cone | \( RB_x \) | \( \sigma(e,y) = UW_y \) or \( LW_y \) |
| 6    | (White) shortcut in a white cone | \( RB_x \) | \( \sigma(e,y) = UW_y \) or \( LW_y \) |

Table 2: The charging scheme.

**Lemma 7.** For any point \( p \in \mathcal{P} \), the charging scheme in Table 2 charges each sector of \( p \) with at most one edge in \( S \). Therefore, the maximum degree of \( S \) is at most 4.

**Proof.** Let \( p \in \mathcal{P} \). We will show that the charging scheme in Table 2 charges every edge incident to \( p \) in \( S \) to one of the four sectors at \( p \) in a one-to-one fashion. Clearly, this will imply that the maximum degree of \( S \) is at most 4.

First, consider the left blue sector of \( p \). Observe that \( LB_p \) could potentially be charged in each of the following two cases:

i. In step 1 by a blue anchor \( (x = p, y) \in \mathcal{A} \); and

ii. in step 3 by a white canonical edge \( (x, y = p) \) in a blue cone.

Observe that (i) and (ii) cannot apply simultaneously by part (c) of Lemma 1. Observe also that \( LB_p \) cannot be charged twice according to (ii) because this would mean that there are two white canonical edges incoming to \( p \) in a blue cone, which would imply that step 4 of the spanner construction applies, and both incoming edges to \( p \) would be removed from \( S \). It follows that \( LB_p \) is charged at most once.
Next, we consider the right blue sector of $p$. Observe that $RB_p$ could potentially be charged in each of the following three cases:

i. In step 1 by a blue anchor $(x, y = p) \in A$;  
ii. in step 5 by a white canonical edge $(x = p, y)$ in a white cone of apex $w$; and  
iii. in step 6 by a shortcut $(x = p, y)$ in a white cone of apex $w$.

Observe that in both (ii) and (iii), there is a white canonical edge $(p, q)$ in the white cone with apex $w$, and hence $(p, q, w)$ is a triangle in $D$: in (ii) $(p, q)$ is $(x, y)$, and in (iii) $(p, q)$ is the (last) edge of the canonical path between $x = p$ and $y$, in step 6 of the spanner construction, by the choice of point $x$ in this step. The existence of this triangle $(p, q, w)$ implies that $RB_p$ must be empty of edges of $D$. Thus, (i) cannot apply if either (ii) or (iii) applies. It follows that if $RB_p$ is charged according to (i) then it is charged with at most one edge. We may assume now that $RB_p$ is not charged according to (i). By part (b) of Lemma 1, $(p, w)$ is not a canonical edge on the side that contains $q$, i.e., $(p, w)$ cannot be a canonical edge in a white cone. Therefore, $(p, q)$ is the only white canonical edge in a white cone that is outgoing from $p$, and hence, $RB_p$ cannot be charged twice according to (ii), or twice according to (iii). Observe that (ii) and (iii) cannot apply simultaneously because step 6 of the construction removes this white canonical edge $(p, q)$ when it adds the shortcut $(x, y)$. It follows from the above that $RB_p$ is charged at most once.

Finally, we consider the upper sector of $p$, and note that the arguments for the lower sector of $p$ follow similarly. Sector $UW_p$ could potentially be charged in each of the following five cases (see Figure 5):

i. In step 2 by a white anchor $(x, y) \in A$, $x = p$ or $y = p$;  
ii. in step 3 by a white canonical edge $(x = p, y)$ in a blue cone;  
iii. in step 4 by a shortcut $(x, y)$ in a blue cone, $x = p$ or $y = p$;  
iv. in step 5 by a white canonical edge $(x, y = p)$ in a white cone; and  
v. in step 6 by a shortcut $(x, y = p)$ in a white cone.

Since step 2 of the construction does not include two adjacent anchors in $A$, $UW_p$ cannot be charged twice according to (i). We show next that if either case (ii) or (iii) applies, then none of the other cases (i), (iv), or (v) applies. Suppose that either (ii) or (iii) applies. Then, in either case, there must exist a white canonical edge $(p, q)$ in a blue cone with apex $w$, and hence, $(p, w)$ and $(q, w)$ are blue edges in $D$; in (ii) $(p, q)$ is $(x, y)$ and in (iii) $(p, q)$ is the canonical edge incident to $p$ that step 4 of the construction removes after adding the shortcut $(x, y)$. Note that $(p, q) \notin A$ because (ii) charges $(p, q)$ to $UW_p$, if $(p, q)$ was added in step 3 of the construction, and step 3 adds an edge $(p, q)$ to $S$ only if $(p, q) \notin A$; similarly, (iii) charges $(x, y)$ to $UW_p$, if $(p, q)$ was added in step 3 of the construction, and step 3 adds an edge $(p, q)$ to $S$ only if $(p, q) \notin A$. The existence of this white canonical edge
Figure 5: Illustration of all the possible charges at $UW_p$: (i) white anchor \( (x = p, y) \in \mathcal{A} \) (left) or \( (x, y = p) \in \mathcal{A} \) (right); (ii) white canonical edge \( (x = p, y = q) \) in a blue cone with apex \( w \); (iii) shortcut \( (x, y = p) \) in a blue cone with apex \( w \) (the other very similar case of shortcut \( (x = p, y) \) is omitted); (iv) white canonical edge \( (x, y = p) \) in a white cone with apex \( w \); and (v) shortcut \( (x, y = p) \) in a white cone with apex \( w \).

\((p,q)\) implies that the negative white cone of \( UW_p \) is empty of edges of \( D \) and of shortcuts incident to \( p \); hence, neither (iv) nor (v) applies. Since \((p,q) \notin \mathcal{A}\) is the only outgoing edge in \( UW_p \), (i) does not apply either.

Observe that \( UW_p \) cannot be charged twice according to (ii) or according to (iii) simply because \((p,q)\) is the only canonical edge in a blue cone that lies in \( UW_p \). Cases (ii) and (iii) cannot apply simultaneously because — as described above — step 4 of the construction removes this canonical edge \((p,q)\) if a shortcut is added.

Assuming that (ii) and (iii) do not apply, we proceed to analyze case (iv). In this case, we know that the white canonical edge \((x,p)\) is in a white cone with apex \( w \); that is \((x,w)\) and \((p,w)\) are white edges in \( D \). Since step 5 of the construction ensures that the anchor of the canonical edge \((x,p)\) is not in \( \mathcal{A} \), and since by part (c) of Lemma 1 edge \((p,w)\) is not an anchor — and hence not in \( \mathcal{A} \), we conclude that (i) does not apply. Step 6 of the construction ensures that (v) does not apply either in this case. Observe that \( UW_p \) cannot be charged twice according to (iv) because \((x,p)\) is the only canonical edge in a white cone that lies in \( UW_p \).

Finally, we analyze case (v). From the way shortcuts are added in step 6 of the construction, there is a blue canonical edge \((p,q)\) in the white cone, say with apex \( w \), under consideration that contains the shortcut \( (x,y = p) \), and hence \((p,w)\) and \((q,w)\) are white edges in \( D \); assume that this white cone is a red cone, and the arguments are similar in the case when the cone is green. The existence of the blue canonical edge \((p,q)\) implies that the green cone contained in \( UW_p \) is empty of edges of \( D \). From the way shortcuts are added in step 6 of the construction, we know that the only outgoing white edge \((p,w)\) in \( D \) in the red cone contained in \( UW_p \) is not an anchor. Since the green cone contained in \( UW_p \) is empty, it follows from the preceding that case (i) does not apply when (v) does. Finally, the remaining case is covered after observing that \( UW_p \) cannot be charged twice according to (v) simply because step 6 of the spanner construction adds at most one shortcut for any blue canonical edge, and because \((p,q)\) is the only blue (canonical) edge outgoing from \( p \).

It follows from the above that each of the four sectors at \( p \) is charged with at most one edge incident to \( p \) in \( S \). This completes the proof.
The remainder of this section is devoted to proving the upper bound of 20 on the stretch factor of $\mathcal{S}$. We do so by first proving a sequence of lemmas that derive upper bounds on the distance in $\mathcal{S}$ between the endpoints of different types of edges in $\mathcal{D}$; we then use these lemmas to derive the upper bound of 20 on the stretch factor of $\mathcal{S}$.

**Lemma 8.** For any uncrossed blue edge $(u, w) \in \mathcal{D}$, $d_\mathcal{S}(u, w) \leq 3d_\mathcal{V}(u, w)$.

*Proof.* Let $(v, w)$ be the blue anchor of the blue edge $(u, w)$. In step 1 of the algorithm, we add all the blue anchors in $\mathcal{S}$, and thus $(v, w) \in \mathcal{S}$. Also, in step 3 of the algorithm, we add in $\mathcal{S}$ all the canonical edges in blue cones except that, in step 4, we substitute some pairs of these canonical edges with shortcuts. Since $(u, w)$ is uncrossed, these canonical edges and shortcuts provide a path $P$ for connecting $v$ and $u$. Note that $P$ is a white monotone path: it consists of only white edges and any two consecutive edges in $P$ lie in different non-adjacent cones of their shared endpoint. Lemma 2 bounds the length of $P$ by $2d_\mathcal{V}(v, u) \leq 2d_\mathcal{V}(u, w)$. Also, the path $P$ plus the anchor $(v, w)$ constitutes a path between $u$ and $w$. Since $|vw| \leq d_\mathcal{V}(v, w) \leq d_\mathcal{V}(u, w)$, $d_\mathcal{S}(u, w)$ is therefore bounded by $d_\mathcal{V}(u, w)$ (anchor) plus $2d_\mathcal{V}(u, w)$ (path $P$), proving the lemma. \hfill $\square$

The following is a key lemma of this paper that takes advantage of monotone paths:

**Lemma 9.** For any white anchor $(v, w)$ and any uncrossed white edge $(u, w) \in \mathcal{D}$ that lies on the white side of $(v, w)$, $d_\mathcal{S}(v, u) \leq d_\mathcal{V}(v, u) + \delta_\mathcal{V}^{\text{blue}}(v, u) \leq 2d_\mathcal{V}(v, u)$. Furthermore, if $(v, w) \in \mathcal{S}$, then $d_\mathcal{S}(u, w) \leq d_\mathcal{V}(u, w) + \delta_\mathcal{V}^{\text{blue}}(u, w) \leq 2d_\mathcal{V}(u, w)$.

*Proof.* To prove this lemma, we consider step 6 of the algorithm for the anchor $(v, w)$ and its boundary edge $(y, w) \neq (v, w)$ on the white side; in the case when $(y, w) = (v, w)$, we note that the lemma statement becomes trivial. Without loss of generality, assuming that $(v, w)$ is red, we note that the path between $y$ and $v$ consisting of the canonical edges described in step 6-(a) and the shortcuts added in step 6-(b) is green and monotone. Considering the only edge incident to $v$ on this path, say $(v, v')$, we note that the red anchor $(v, w)$ and the green edge $(v, v')$ lie in non-adjacent cones of $v$. Hence, this green path plus $(v, v')$ forms a white monotone path, which we will refer to as the white monotone connection of the boundary edge $(y, w)$. Furthermore, this white monotone connection starts at a cone adjacent to the cone of $y$ that contains $(y, w)$ and ends at the same cone of $w$ that contains $(y, w)$. More precisely, the only edge incident to $y$ on the white monotone connection, say $(y', y)$, lies in a cone adjacent to the green cone of $y$ that contains $(y, w)$; and the red edge $(v, w)$ of the white monotone connection lies in the same red cone of $w$ that contains $(y, w)$. This shows that for any white monotone path that includes the boundary edge $(y, w)$ as an edge, the path obtained by replacing $(y, w)$ with its white monotone connection is still a white monotone path. More generally, for any white boundary edge $e$ that lies on the white side of its anchor, and any white monotone path that includes $e$, the path obtained by replacing $e$ with its white monotone connection is another white monotone path.

Next, we recursively describe a white monotone path in $\mathcal{S}$ between $v$ and $u$. We start by considering the subpath between $v$ and $u$ of the white monotone connection of the boundary edge $(y, w)$; it is a subpath because $(u, w)$ is uncrossed, i.e., $u$ is a point on this...
white monotone connection. We note that all the shortcuts in this path are in \( \mathcal{S} \), but some of the canonical edges may not be in \( \mathcal{S} \). For those edges, step 5 of the algorithm ensures that their anchors are in \( \mathcal{S} \). Then, we replace all of those missing canonical edges with their white monotone connections; each of them is a white boundary edge on the white side of its anchor. We proceed recursively until all of the edges in this path are in \( \mathcal{S} \). Using the above property of white monotone connections, we deduce that the final path is a white monotone path in \( \mathcal{S} \). Therefore, using Lemma 2, we bound the length of this path, and hence \( d_\mathcal{S}(v, u) \), by \( d_\nabla(v, u) + \delta_\nabla(v, u) \leq 2d_\nabla(v, u) \) as desired. Furthermore, if \((v, w) \in \mathcal{S}\), we expand this path to include the white anchor \((v, w)\) while preserving its monotonicity. Thus, we obtain the desired bound \( d_\mathcal{S}(u, w) \leq d_\nabla(u, w) + \delta_\nabla(u, w) \leq 2d_\nabla(u, w) \) by using Lemma 2.

**Lemma 10.** For any white anchor \((v, w)\) and any white edge \((u, w) \in \mathcal{D}\) that lies on the blue side of \((v, w)\), \( d_\mathcal{S}(v, u) \leq 5d_\nabla(v, u) \). Furthermore, if \((v, w) \in \mathcal{S}\), then \( d_\mathcal{S}(u, w) \leq 6d_\nabla(u, w) \).

**Proof.** The canonical path from \( v \) to \( u \) consists of blue and white canonical edges. The total length of the blue canonical edges does not exceed \( d_\nabla(v, u) \), and the total length of the white canonical edges does not exceed \( d_\nabla(v, u) \) by Lemma 3. By Lemma 5, we know that all of these canonical edges are uncrossed. By Lemma 8, the total length of the paths needed to reconstruct these blue canonical edges can be bounded by \( 3d_\nabla(v, u) \). Also, since either the white canonical edges themselves or their anchors are in \( \mathcal{S} \), the total length of the white canonical edges can be bounded by \( 2d_\nabla(v, u) \) by Lemma 9. Therefore, \( d_\mathcal{S}(v, u) \) can be bounded by \( 5d_\nabla(v, u) \) for the edge \((v, u)\) as stated. Furthermore, if \((v, w) \in \mathcal{S}\), \( d_\mathcal{S}(u, w) \) can be bounded by \( 5d_\nabla(v, u) + d_\nabla(v, w) \), which in turn is bounded by \( 6d_\nabla(u, w) \).

**Definition 6.** For any two points \( p, q \in \mathcal{P} \) such that \( p \) lies in a white cone of \( q \), we define \( \delta_\nabla(v, p) = \delta_\nabla(v, p) = d_\nabla(p, q) - \delta_\nabla(p, q) \) (see Figure 6-(a)).

**Lemma 11.** For any two white edges \((v, w), (u, w) \in \mathcal{D}\) that lie in the same negative cone of \( w \) such that \( u \) lies in a positive white cone of \( v \), we have: \( d_\nabla(v, u) = \delta_\nabla(u, w) - \delta_\nabla(v, w) \), \( \delta_\nabla(v, u) = d_\nabla(u, w) - d_\nabla(v, w) \), and \( \delta_\nabla(v, u) = \delta_\nabla(v, w) - \delta_\nabla(u, w) \).

![Figure 6: (a) Illustration of \( \delta_\nabla(v, p) \). (b) Illustration of the proof of Lemma 11 depicting \( \nabla(v, w) \) (vertices \( a, c, v \)), \( \nabla(u, w) \) (vertices \( b, d, u \)), and \( \nabla(v, u) \) (vertices \( v, y, z \)).](image-url)
Figure 7: Illustrations of the proof of Lemma 12. (a) The case when \((v, w) \notin \mathcal{A}\) because a shorter adjacent anchor \((w, w')\) was added first. Edge \((w', u')\) is a blue boundary edge and there is a white monotone path from \(u'\) to \(v\) in \(\mathcal{S}\). (b) The case when \((v, w) \notin \mathcal{A}\) because a shorter adjacent anchor \((v', v)\) was added first. Edge \((w, u')\) is a blue boundary edge and \((u', v)\) is a white boundary edge on the white side of its cone and there is a white monotone path between \(u'\) and \(v'\) in \(\mathcal{S}\).

Proof. Since \((v, w) \in \mathcal{D}\), \(u\) must lie in the positive white cone of \(v\) that does not contain \(w\). The rest of the proof follows from the definitions of \(\delta_{\mathcal{V}}^{\text{blue}}\) and \(\delta_{\mathcal{V}}^{\text{white}}\); more specifically, as illustrated in Figure 6-(b), the proof follows from the equalities \(d_{\mathcal{V}}(v, u) = |yz| = |cd| = |wd| - |wc|\), \(\delta_{\mathcal{V}}^{\text{blue}}(v, u) = |yu| = |ud| - |vc|\), and \(\delta_{\mathcal{V}}^{\text{white}}(v, u) = |uz| = |ab| = |wa| - |wb|\).

Lemma 12. For any white anchor \((v, w)\), \(d_{\mathcal{S}}(v, w) \leq 9d_{\mathcal{V}}(v, w)\). Furthermore, for any uncrossed white edge \((u, w)\) in the fan of \((v, w)\), we have \(d_{\mathcal{S}}(u, w) \leq 9d_{\mathcal{V}}(u, w) + \delta_{\mathcal{V}}^{\text{blue}}(u, w)\) if \((u, w)\) lies on the white side of \((v, w)\), and \(d_{\mathcal{S}}(u, w) \leq 9d_{\mathcal{V}}(u, w)\) otherwise.

Proof. If \((v, w) \in \mathcal{S}\), then clearly \(d_{\mathcal{S}}(v, w) \leq d_{\mathcal{V}}(v, w)\). As for any uncrossed edge \((u, w)\) in the fan, by Lemma 9 we get a bound of \(2d_{\mathcal{V}}(u, w)\) on \(d_{\mathcal{S}}(u, w)\) if \((u, w)\) lies on the white side of \((v, w)\), and by Lemma 10 we get a bound of \(6d_{\mathcal{V}}(u, w)\) on \(d_{\mathcal{S}}(u, w)\) if \((u, w)\) lies on the blue side of \((v, w)\), proving the lemma in these cases. We assume next that \((v, w) \notin \mathcal{S}\) and analyze two cases: \((v, w)\) was not added in \(\mathcal{A}\) because of an adjacent anchor at \(w\), and \((v, w)\) was not added in \(\mathcal{A}\) because of an adjacent anchor at \(v\).

If \((v, w)\) was not added in \(\mathcal{A}\) because of an adjacent (white) anchor at \(v\), let \((w, w')\) be that anchor (see Figure 7-(a)). We first prove that \(d_{\mathcal{S}}(v, w) \leq 9d_{\mathcal{V}}(v, w)\) and use it to prove that \(d_{\mathcal{S}}(u, w) \leq 9d_{\mathcal{V}}(u, w)\) for any uncrossed white edge \((u, w)\). By the construction of \(\mathcal{A}\) in step 2 of the algorithm, \((w, w')\) must be shorter than \((v, w)\), i.e., \(d_{\mathcal{V}}(w, w') < d_{\mathcal{V}}(v, w)\). Therefore, as \((v, w)\) is a white anchor, we conclude that \(u\) lies in the positive blue cone of \(w'\). Hence, there must be an outgoing blue edge from \(w'\); let \((w', u)\) be that blue edge. The edge \((u', v)\) must be a white boundary edge of \((v, w)\), with the possibility of \(u' = v\). To prove that \(d_{\mathcal{S}}(v, w) \leq 9d_{\mathcal{V}}(v, w)\), we first upper bound \(d_{\mathcal{S}}(v, w)\) as follows: \(d_{\mathcal{S}}(v, w) \leq d_{\mathcal{S}}(w, w') + d_{\mathcal{S}}(w', u') + d_{\mathcal{S}}(v, u')\). Using the fact that \((w, w') \in \mathcal{S}\) and using Lemmas 5, 8, and 9, we have the following inequalities bounding \(d_{\mathcal{S}}(w, w')\), \(d_{\mathcal{S}}(w', u')\), and \(d_{\mathcal{S}}(v, u')\):

\[
d_{\mathcal{S}}(w, w') \leq d_{\mathcal{V}}(w, w') < d_{\mathcal{V}}(v, w), \quad d_{\mathcal{S}}(w', u') \leq 3d_{\mathcal{V}}(w', u'), \quad d_{\mathcal{S}}(v, u') \leq 2d_{\mathcal{V}}(v, u').
\]
From the above inequalities, to prove \(d_\mathcal{S}(v, w) \leq 9d_\mathcal{V}(v, w)\), it suffices to show that 
\[ d_\mathcal{V}(v, u') \leq d_\mathcal{V}(v, w) \] 
and 
\[ d_\mathcal{V}(w', u') \leq 2d_\mathcal{V}(v, w). \] 
We will actually prove stronger upper bounds as we need them in the upper bound proof for the uncrossed white edges. In particular, we show the following upper bounds on \(d_\mathcal{V}(v, u')\) and \(d_\mathcal{V}(w', u')\):

\[
\begin{align*}
d_\mathcal{V}(w', u') &\leq d_\mathcal{V}(v, w) + \delta_\mathcal{V}\text{white}(v, w). \tag{1} \\
d_\mathcal{V}(v, u') &\leq \delta_\mathcal{V}\text{white}(v, w). \tag{2}
\end{align*}
\]

To prove both bounds, we use the fact that \(u'\) lies in a positive white cone of \(v\) and that \(u'\) lies in the positive blue cone of \(w'\). More specifically, the edge \((w', u')\) lies inside \(\nabla(w', v)\) which has side length no more than \(d_\mathcal{V}(w', w') + \delta_\mathcal{V}\text{white}(v, w)\). Therefore, the inequality (1) follows from the fact that \(d_\mathcal{V}(w, w') < d_\mathcal{V}(v, w)\). Furthermore, the inequality \(d_\mathcal{V}(w, w') < d_\mathcal{V}(v, w)\) implies that \(\nabla(v, u')\) lies fully inside the negative white cone of \(w\) that contains \(v\), or in other words, that the side length of \(\nabla(v, u')\) is no longer than \(\delta_\mathcal{V}\text{white}(v, w)\). This proves the inequality (2).

Next, we consider an uncrossed white edge \((u, w)\) in the fan of \((v, w)\). Since \((v, w)\) is the anchor, we have \(d_\mathcal{V}(v, w) \leq d_\mathcal{V}(u, w)\). If \((u, w)\) is on the white side of \((v, w)\), the bound on \(d_\mathcal{S}(v, w)\) applies directly to \(d_\mathcal{S}(u, w)\) because the path between \(v\) and \(w\) already connects \(u\) and \(w\). We immediately get \(d_\mathcal{S}(u, w) \leq 9d_\mathcal{V}(v, w) \leq 9d_\mathcal{V}(u, w)\). If \((u, w)\) is on the blue side of \((v, w)\), by Lemma 10, we get \(d_\mathcal{S}(v, u) \leq 5d_\mathcal{V}(v, u)\). Also, in this case, it is easy to verify that \(\delta_\mathcal{V}\text{white}(v, w) \leq d_\mathcal{V}(u, w) - d_\mathcal{V}(v, u)\). Combining this inequality with (1) and (2), we obtain the inequalities \(d_\mathcal{V}(v, u') \leq d_\mathcal{V}(u, w) - d_\mathcal{V}(v, u)\) and \(d_\mathcal{V}(w', u') \leq 2d_\mathcal{V}(u, w) - d_\mathcal{V}(v, u)\). Therefore:

\[
\begin{align*}
d_\mathcal{S}(u, w) &\leq d_\mathcal{S}(w, w') + d_\mathcal{S}(w', u') + d_\mathcal{S}(v, u') + d_\mathcal{S}(v, u) \\
&\leq d_\mathcal{V}(w, w') + 3d_\mathcal{V}(w', u') + 2d_\mathcal{V}(v, u') + 5d_\mathcal{V}(v, u) \\
&\leq d_\mathcal{V}(u, w) + 6d_\mathcal{V}(u, w) - 3d_\mathcal{V}(v, u) + 2d_\mathcal{V}(u, w) - 2d_\mathcal{V}(v, u) + 5d_\mathcal{V}(v, u) \\
&= 9d_\mathcal{V}(u, w).
\end{align*}
\]

In the case when \((v, w)\) was not added in \(\mathcal{A}\) because of an adjacent (white) anchor at \(v\) (see Figure 7-(b)), we know that \(v'\) lies in the positive blue cone of \(w\) and that \((v', v)\) is shorter than \((v, w)\), i.e., \(d_\mathcal{V}(v', v) < d_\mathcal{V}(v, w)\). Therefore, there must be an outgoing blue edge from \(v\); let \((w, u')\) be that blue edge. The edge \((u', v)\) must be a white boundary edge of \((v', v)\), with the possibility of \(u' = v'\). Using the fact that \((v', v) \in \mathcal{S}\) and using Lemmas 5, 8, and 9, we obtain the following inequalities to bound \(d_\mathcal{S}(v, w)\): 
\[
\begin{align*}
d_\mathcal{S}(v', v) &\leq d_\mathcal{V}(v', v) < d_\mathcal{V}(v, w), \\
d_\mathcal{S}(v', u') &\leq 2d_\mathcal{V}(v', u'), \\
d_\mathcal{S}(w, u') &\leq 3d_\mathcal{V}(w, u').
\end{align*}
\]

Using arguments similar to those used in the previous case, we get \(d_\mathcal{V}(v, u') \leq d_\mathcal{V}(v, w)\) and \(d_\mathcal{V}(w, u') \leq 2d_\mathcal{V}(v, w)\). Consequently, we get the desired upper bound \(d_\mathcal{S}(v, w) \leq 9d_\mathcal{V}(v, w)\).

As for the uncrossed white edges in this case, we first note that the anchor \((v, w)\) itself is the boundary edge on the blue side and that there are no white edges on the blue side. Thus, there is nothing to prove on the blue side. As for any uncrossed white edge \((u, w)\) on the white side of \((v, w)\), using the triangle inequality \(d_\mathcal{S}(u, w) \leq d_\mathcal{S}(v, w) + d_\mathcal{S}(v, u)\), the bound \(d_\mathcal{S}(v, w) \leq 9d_\mathcal{V}(v, w)\) from above, and Lemmas 5 and 9 to bound \(d_\mathcal{S}(v, u) \leq d_\mathcal{V}(v, u) + \delta_\mathcal{V}\text{blue}(v, u)\), we get the upper bound \(d_\mathcal{S}(u, w) \leq 9d_\mathcal{V}(v, w) + d_\mathcal{V}(v, u) + \delta_\mathcal{V}\text{blue}(v, u)\).
Since the white edges \((v, w)\) and \((u, w)\) satisfy the premise of Lemma 11, we have \(\delta_{\text{blue}}(v, u) = d_{\text{\triangle}}(v, w) - d_{\text{\triangle}}(v, w)\), and \(d_{\text{\triangle}}(v, u) \leq \delta_{\text{blue}}(u, w)\). Also using the fact that \(d_{\text{\triangle}}(v, w) \leq d_{\text{\triangle}}(u, w)\), we get \(d_S(u, w) \leq 9d_{\text{\triangle}}(u, w) + \delta_{\text{blue}}(u, w)\) as desired.

\[\square\]

**Lemma 13.** For any crossed blue edge \((u, w) \in D\), \(d_S(u, w) \leq 3d_{\text{\triangle}}(u, w) + 9\delta_{\text{\triangle}}^\text{min}(u, w)\).

**Proof.** For the blue edge \((u, w)\), we have \(\delta_{\text{\triangle}}^\text{min}(u, w) = \min\{\delta_{\text{\triangle}}^\text{red}(u, w), \delta_{\text{\triangle}}^\text{green}(u, w)\}\) by definition. We prove the lemma by showing that \(d_S(u, w) \leq 3d_{\text{\triangle}}(u, w) + 9\delta_{\text{\triangle}}^\text{red}(u, w)\), and the proof that \(d_S(u, w) \leq 3d_{\text{\triangle}}(u, w) + 9\delta_{\text{\triangle}}^\text{green}(u, w)\) follows similarly. The combination of the two inequalities yields the statement of the lemma.

Let \((p, q)\) be a shortcut that crosses \((u, w)\). Since this shortcut is in a blue cone (see Figure 4-(a)), it must have been added in \(S\) to replace a green and a red canonical edges incoming at \(u\), namely \((p, u)\) and \((q, u)\); we assume, without loss of generality, that \((p, u)\) is green. Since \(p\) is an endpoint of the shortcut \((p, q)\), the blue edge \((p, w)\) is uncrossed, and we have \(d_S(p, w) \leq 3d_{\text{\triangle}}(p, w)\) by Lemma 8. Furthermore, by Lemmas 5 and 12, we have \(d_S(p, u) \leq 9d_{\text{\triangle}}(p, u)\) since the canonical edge \((p, u)\) is either an anchor or lies on the blue side of its anchor. Finally, by applying Lemma 2 to the monotone path consisting of the two edges \((p, u)\) and \((p, w)\), we obtain \(d_{\text{\triangle}}(p, w) \leq d_{\text{\triangle}}(u, w)\) and \(d_{\text{\triangle}}(p, u) \leq \delta_{\text{\triangle}}^\text{red}(u, w)\). Combining all the above bounds, we conclude that \(d_S(u, w) \leq 3d_{\text{\triangle}}(u, w) + 9\delta_{\text{\triangle}}^\text{red}(u, w)\). \[\square\]

**Lemma 14.** For any crossed white edge \((u, w) \in D\), \(d_S(u, w) \leq 10d_{\text{\triangle}}(u, w) + 10\delta_{\text{\triangle}}^\text{min}(u, w)\).

**Proof.** Since there are no shortcuts on the blue side of a white anchor, all white edges on the blue side are uncrossed. Therefore, we can assume that \((u, w)\) is a white edge on the white side of its anchor crossing a shortcut \((p, q)\) in the same cone. From the way shortcuts are taken in step 6 of the construction, \(p\) is in a white cone of \(q\), and \(u\) is below line \(pq\), which implies the following inequality:

\[d_{\text{\triangle}}(p, w) \leq d_{\text{\triangle}}(u, w).\] (3)

To prove the lemma, we consider two specific routes between \(u\) and \(w\): route \(A\) that consists of the uncrossed white edge \((p, w)\) and the canonical path from \(p\) to \(u\); and route \(B\) that consists of \((p, w)\), the shortcut \((p, q)\), and the canonical path from \(q\) to \(u\) (see Figure 8-(a)). Even though the edge \((p, w)\), or any of the canonical edges on the canonical path between \(p\) and \(q\) may not be included in \(S\), we show that one of these two routes bounds \(d_S(u, w)\) by at most \(10d_{\text{\triangle}}(u, w) + 10\delta_{\text{\triangle}}^\text{min}(u, w)\).

First, we bound the total length of route \(A\). Since \((p, w)\) is uncrossed, by Lemma 12, \(d_S(p, w)\) is bounded by \(9d_{\text{\triangle}}(p, w) + \delta_{\text{\triangle}}^\text{blue}(p, w)\). Also, by Lemmas 2 and 3, the total length (with respect to triangular metric) of the white and blue canonical edges on the canonical path from \(p\) to \(u\) are bounded by \(d_{\text{\triangle}}(p, u)\) and \(\delta_{\text{\triangle}}^\text{white}(p, u)\), respectively. Since by Lemma 5 any canonical edge \((x, y)\) on this canonical path is uncrossed, Lemmas 8 and 12 bound \(d_S(x, y)\) by \(3d_{\text{\triangle}}(x, y)\) if \((x, y)\) is blue, and by \(9d_{\text{\triangle}}(x, y) + \delta_{\text{\triangle}}^\text{blue}(x, y)\leq 10d_{\text{\triangle}}(x, y)\) if \((x, y)\) is
white. Thus, route $A$ allows us to conclude that:

$$d_S(u, w) \leq d_S(p, w) + d_S(p, u)$$

$$\leq 9d_{\nabla}(p, w) + 9\delta_{\nabla}^{blue}(p, w) + 10\delta_{\nabla}(p, u) + 10\delta_{\nabla}(p, w) + 3\delta_{\nabla}(u, w). \quad (4)$$

Since the edges $(p, w)$ and $(u, w)$ satisfy the premise of Lemma 11, we substitute in (4) $d_{\nabla}(p, u) = \delta_{\nabla}^{blue}(u, w) - \delta_{\nabla}^{blue}(p, w)$ and $\delta_{\nabla}(p, u) = \delta_{\nabla}^{white}(p, w) - \delta_{\nabla}^{white}(u, w)$ to get:

$$d_S(u, w) \leq 9d_{\nabla}(p, w) - 9\delta_{\nabla}^{blue}(p, w) + 10\delta_{\nabla}^{blue}(u, w) + 3\delta_{\nabla}^{white}(p, w) - 3\delta_{\nabla}^{white}(u, w)$$

$$= 9\delta_{\nabla}^{white}(p, w) + 10\delta_{\nabla}^{blue}(u, w) + 3\delta_{\nabla}^{white}(p, w) - 3\delta_{\nabla}^{white}(u, w)$$

$$= 12\delta_{\nabla}^{white}(p, w) + 10\delta_{\nabla}^{blue}(u, w) - 3\delta_{\nabla}^{white}(u, w)$$

$$\leq 12\delta_{\nabla}^{white}(p, w) + 10d_{\nabla}(u, w) - 13\delta_{\nabla}^{white}(u, w)$$

$$= 10d_{\nabla}(u, w) + 5\delta_{\nabla}^{white}(u, w) + 12\delta_{\nabla}^{white}(p, w) - 18\delta_{\nabla}^{white}(u, w). \quad (5)$$

Using (3) we obtain $\delta_{\nabla}^{white}(p, w) \leq d_{\nabla}(u, w)$ and using this inequality in (5) we get:

$$d_S(u, w) \leq 10d_{\nabla}(u, w) + 10\delta_{\nabla}^{blue}(u, w) + 2\delta_{\nabla}^{white}(p, w) - 3\delta_{\nabla}^{white}(u, w). \quad (6)$$

If $2\delta_{\nabla}^{white}(p, w) \leq 3\delta_{\nabla}^{white}(u, w)$, we get $d_S(u, w) \leq 10d_{\nabla}(u, w) + 10\delta_{\nabla}^{white}(u, w)$ by (6) and (7), and the proof is done. Otherwise, we have $3\delta_{\nabla}^{white}(u, w) < 2\delta_{\nabla}^{white}(p, w) \leq 2d_{\nabla}(p, w)$, and using (3) we get $3\delta_{\nabla}^{white}(u, w) < 2d_{\nabla}(p, w) = 2\delta_{\nabla}^{white}(u, w) + 2\delta_{\nabla}^{blue}(u, w)$, which implies $\delta_{\nabla}^{white}(u, w) < 2\delta_{\nabla}^{blue}(u, w)$. With this inequality, we analyze route $B$ below and prove that $d_S(u, w) \leq 10d_{\nabla}(u, w) + 5\delta_{\nabla}^{white}(u, w)$, which implies $d_S(u, w) \leq 10d_{\nabla}(u, w) + 10\delta_{\nabla}^{blue}(u, w)$ and concludes the proof.

We now analyze route $B$. By Lemma 12, $d_S(p, w)$ is bounded by $9d_{\nabla}(p, w) + \delta_{\nabla}^{blue}(p, w)$. The shortcut $(p, q) \in S$, so $d_S(p, q) = |pq| \leq d_{\nabla}(p, q)$. Moreover, $d_{\nabla}(p, q) \leq d_{\nabla}(p, u) + d_{\nabla}(q, u)$ by the triangle inequality, and $d_{\nabla}(p, u) = \delta_{\nabla}^{blue}(u, w) - \delta_{\nabla}^{blue}(p, w)$ by Lemma 11. Hence, route $B$ allows us to conclude that:

$$d_S(u, w) \leq d_S(p, w) + d_S(p, q) + d_S(q, u)$$

$$\leq 9d_{\nabla}(p, w) + \delta_{\nabla}^{blue}(u, w) + d_{\nabla}(q, u) + d_S(q, u). \quad (8)$$

If $d_{\nabla}(q, w) \leq d_{\nabla}(u, w)$, by Lemma 3, the total length (with respect to triangular metric) of the white edges and the total length of the blue edges on the canonical path from $q$ to $u$ are each bounded by $d_{\nabla}(q, u)$. Once again, all canonical edges are uncrossed by Lemma 5. Thus, for any blue canonical edge $(x, y)$ in this canonical path, $d_S(x, y) \leq 3d_{\nabla}(x, y)$ by Lemma 8. And, for any white canonical edge $(x, y)$ in this canonical path, either $(x, y)$ or its anchor is in $S$ by step 5 of our algorithm, and $(x, y)$ is not removed in step 6 of the construction. Therefore, $d_S(x, y) \leq 2d_{\nabla}(x, y)$ by Lemma 9. Thus:

$$d_S(u, w) \leq 9d_{\nabla}(p, w) + \delta_{\nabla}^{blue}(u, w) + 6d_{\nabla}(q, u).$$

Using the current assumption $d_{\nabla}(q, w) \leq d_{\nabla}(u, w)$, we deduce that $d_{\nabla}(q, u) \leq \delta_{\nabla}^{white}(u, w)$. Using this inequality:

$$d_S(u, w) \leq 9d_{\nabla}(p, w) + \delta_{\nabla}^{blue}(u, w) + 6\delta_{\nabla}^{white}(u, w)$$

$$\leq 10d_{\nabla}(u, w) + 5\delta_{\nabla}^{white}(u, w).$$
Also in this case, by Lemma 11, we obtain the inequalities
\[ d_{\triangledown}(u, w) \leq d_{\triangledown}(q, w). \]
Therefore:
\[ d_{S}(u, w) \leq 9d_{\triangledown}(p, w) + 3d_{\triangledown}(q, u) + 3d_{\triangledown}(q, w) + 3\delta_{\triangledown}(q, u). \]

Also in this case, by Lemma 11, we obtain the inequalities $\delta_{\triangledown}(q, u) \leq \delta_{\triangledown}(u, w)$ and $d_{\triangledown}(q, u) \leq d_{\triangledown}(q, w) - \delta_{\triangledown}(u, w)$. Therefore:
\[ d_{S}(u, w) \leq 9d_{\triangledown}(p, w) + 3d_{\triangledown}(q, w) - 2\delta_{\triangledown}(u, w) + 3\delta_{\triangledown}(q, u). \]
Furthermore, using the assumption that $3\delta_{\triangledown}(u, w) < 2\delta_{\triangledown}(p, w)$, and the fact that $\delta_{\triangledown}(q, w) \geq 0$, we deduce that $3\delta_{\triangledown}(q, u) < 2\delta_{\triangledown}(q, p)$. Using Lemma 11 on both terms, we obtain $d_{\triangledown}(q, w) < 3d_{\triangledown}(p, w) - 2d_{\triangledown}(p, w)$. Finally we obtain:
\[ d_{S}(u, w) \leq 9d_{\triangledown}(p, w) + 9d_{\triangledown}(u, w) - 6d_{\triangledown}(p, w) - 2\delta_{\triangledown}(u, w) + 3\delta_{\triangledown}(u, w) \]
\[ = 3d_{\triangledown}(p, w) + 7d_{\triangledown}(u, w) + 5\delta_{\triangledown}(u, w) \]
\[ \leq 10d_{\triangledown}(u, w) + 5\delta_{\triangledown}(u, w). \]

Showing $d_{S}(u, w) \leq 10d_{\triangledown}(u, w) + 10\delta_{\triangledown}(u, w)$ in all cases, we conclude the proof. \[\Box\]
By Lemmas 8, 12, 13, and 14 we have that $S$ is a 20-spanner of $D$. Since $D$ is a 2-spanner of $C$ ([15]) it follows that $S$ is a 40-spanner of $C$. We prove, however, a much better stretch factor upper bound of 20.

**Lemma 15.** For any two points $p, q \in \mathcal{P}$, we have $d_S(p, q) \leq 20|pq|$.

*Proof.* We prove the lemma by first defining, in $D$, a monotone path $\pi$ between $p$ and $q$ that lies inside $\nabla(p, q)$. The definition of the path $\pi$ is due to Bonichon et al. and Bose et al. [2, 9], who use it to give an alternative proof that the stretch factor of $D$ is at most 2.

We define the path $\pi$ between $p$ and $q$ recursively. Starting with the pair of points $\{p, q\} = \{p_0, q_0\}$, either $p_0$ or $q_0$ lies in a positive cone of the other. If $q_0$ lies in the positive cone of $p_0$, we add the unique (shortest) edge $(p_0, r) \in D$ to $\pi$, and set $q_1 = q_0$ and $p_1 = r$. Otherwise, $p_0$ lies in a positive cone of $q_0$, we set $p_1 = p_0$ and $q_1 = r'$, where $(q_1, r') \in D$, and add $(q_1, r')$ to $\pi$. We then recurse on the pair $(p_1, q_1)$. The recursion stops when $p_k = q_k$. We point out that the recursion must stop, as the set of points $\mathcal{P}$ is finite and the length $d_{\pi}(p_k, q_k)$ strictly decreases, thereby forming a sequence of pairs of points $\{p, q\} = \{p_0, q_0\}, \{p_1, q_1\}, \ldots, \{p_k, q_k\}$. Furthermore, we note that the sequence of $\nabla$-homothets $\nabla(p_0, q_0), \nabla(p_1, q_1), \ldots, \nabla(p_k, q_k)$ form a nested sequence of triangles, which implies that the aforementioned path $\pi$ lies within $\nabla(p, q)$, and that the subpath of $\pi$ that is between $p_i$ and $q_i$ lies within $\nabla(p_i, q_i)$.

We then inductively prove that all the edges of the form $(p_i, p_{i+1})$ are of the same color (red, green, or blue): the very first edge determines the color of the rest of these edges. The base case follows trivially and for the inductive step we need to prove that the next edge is of the same color as the previous edge. To prove the inductive step, we consider the last edge $(p_\ell, p_{\ell+1})$ for some $\ell$, and the next edge $(p_\lambda = p_{\ell+1}, p_{\lambda+1})$ for some $\lambda > \ell$. By definition of $D$, we know that $\nabla(p_\ell, p_\lambda = p_{\ell+1})$ is empty of points of $\mathcal{P}$ in its interior. From the previous paragraph, we also know that both edges $(p_\ell, p_{\ell+1})$ and $(p_\lambda, p_{\lambda+1})$ lie within $\nabla(p_\ell, q_\ell)$. Therefore, two of the three positive cones of $p_\lambda$ restricted to $\nabla(p_\ell, q_\ell)$ do not contain any points of $\mathcal{P}$ in its interior. As a result, $p_{\lambda+1}$ must lie in the same corresponding cone of $p_\lambda$, i.e., the two edges $(p_\ell, p_{\ell+1})$ and $(p_\lambda, p_{\lambda+1})$ must be of the same color. This completes the inductive proof.

Having proven this critical color property about the path consisting of all the edges of the form $(p_i, p_{i+1})$, we denote this path by $\pi_p$ and refer to it as one of the two monochromatic monotone branches, where the other branch, which we denote by $\pi_q$, is defined analogously consisting of all the edges of the form $(q_j, q_{j+1})$. Since the last edge in both branches are incoming at the same point $p_k = q_k$, these two edges connecting $\pi_p$ and $\pi_q$ lie in different non-adjacent cones of their shared endpoint $p_k = q_k$. Hence, we conclude that the path $\pi$, which consists of these two branches $\pi_p$ and $\pi_q$ is monotone.

Finally, we prove the lemma statement using path $\pi$. First, we consider the case when $\pi_p$ and $\pi_q$ are both white. Letting $z$ be the blue vertex of $\nabla(p, q)$, by Lemma 2, the projections of all edges of $\pi$ onto $zp$ (resp. $zq$) do not overlap, are contained within $[zp]$ (resp. $[zq]$), $|zp| = d_\nabla(p, q)$, and $|zq| = d_\nabla^\text{blue}(p, q)$. Furthermore, we know that $|zp| + |zq| \leq 2|pq|$, which proves $d_D(p, q) \leq 2|pq|$. As for $d_S(p, q)$, in the worst case, each edge of $\pi$ is crossed and
Lemma 14 applies to each edge \((s, t)\) of \(\pi\) (Lemma 12 provides better bounds):
\[
d_\mathcal{S}(p, q) \leq \sum_{(s, t) \in \pi} 10d_{\mathcal{V}}(s, t) + 10s_{\min}(s, t) \tag{9}
\]
\[
\leq 10 \sum_{(s, t) \in \pi} d_{\mathcal{V}}(s, t) + \delta_{\text{blue}}(s, t)
\]
\[
= 10(|zp| + |zq|).
\]

Using the above inequality \(|zp| + |zq| \leq 2|pq|\), we conclude that \(d_\mathcal{S}(p, q) \leq 20|pq|\) as desired. For the remaining cases when either \(\pi_p\) or \(\pi_q\) is blue, similar arguments can be used since Lemmas 8 and 13 provide better bounds for \(9\). \(\square\)

**Theorem 1.** \(\mathcal{S}\) is a plane spanner of \(\mathcal{C}\) with maximum degree at most 4 and stretch factor at most 20 that can be constructed in \(O(n \log n)\) time.

**Proof.** The planarity, maximum degree, and stretch factor properties of \(\mathcal{S}\) were proven in Lemmas 6, 7, and 15, respectively. It remains to argue that the construction of \(\mathcal{S}\) can be performed in \(O(n \log n)\) time.

The TD-Delaunay triangulation \(\mathcal{D}\) can be constructed in \(O(n \log n)\) time [15], and we can compute a rotation system (i.e., a cyclic ordering of the edges incident to each vertex) of \(\mathcal{D}\) within the same time upper bound. Using the rotation system, for each point \(p \in \mathcal{P}\), we can identify the edges in \(\mathcal{D}\) in each of the six cones of apex \(p\). Since \(\mathcal{D}\) is plane, and hence has \(O(n)\) edges, we can identify all anchors in \(O(n)\) time overall since the shortest edge (with respect to \(d_{\mathcal{V}}\)) in each cone can be identified in time linear in the number of edges in the cone. Therefore, step 1 of the construction can be implemented in \(O(n)\) time. Step 2 can be implemented in \(O(n \log n)\) time, which is the time needed to sort all white anchors. Clearly, step 3 can be implemented in \(O(n)\) time. Using the rotation system of \(\mathcal{D}\), we can traverse the fan of each blue anchor to determine the shortcuts taken in step 4 in \(O(n)\) time overall. Similarly, by traversing the fan of each white anchor, we can implement step 5 in \(O(n)\) time. We show next how to implement step 6 in \(O(k \log k)\) for a canonical path consisting of \(k\) canonical edges; by additivity, and since the number of edges in \(\mathcal{D}\) is \(O(n)\), this will imply that step 6 can be implemented in \(O(n \log n)\) time overall.

Let \((v, w)\) be a white anchor and let \(P = (u_i = p_0, p_1, \ldots, p_k = v)\) be the canonical (sub)path described in step 6 of the spanner construction. Also, let \(C_i\) be the convex hull of \(\{p_i, \ldots, p_k\}\). We start by computing the convex hull \(C_0\) in \(O(k \log k)\) time [17]. For each point \(p_i\) in step 6, for substep (a) we update \(C_i\) to \(C_{i+1}\), and for substep (b) we update \(C_i\) to \(C_j\), where \(p_ip_j\) is the shortcut taken in this substep (Lemma 4). To update \(C_i\), we rely on the data structure described by Brodal and Jacob [13] for representing and maintaining a dynamic convex hull. This data structure can perform the operations of deleting a vertex and finding the neighbor of a vertex of the convex hull of a set of \(k\) points in \(O(\log k)\) amortized time per operation. Assuming that \(C_0\) is initially computed in \(O(k \log k)\) time, the implementation of step 6 described above performs only deletion operations, and operations that find the neighbor of a vertex of the convex hull. Each vertex that gets deleted is removed from further consideration, and hence at most \(k\) deletion operations are performed in total. Moreover,
there are at most \( k \) operations that find the neighbor of a vertex on the convex hull. It follows that the convex hull \( C_i \), can be initially computed and dynamically maintained over all points \( p_i \) in an \( \mathcal{O}(k \lg k) \) time in total, and hence, step 6 can be performed on \( P \) in time \( \mathcal{O}(k \lg k) \). This completes the proof.

6 Tight degree bound for points in convex position

In this section, we show that if the set \( P \) of points is in convex position, then the same spanner \( S \) constructed in the previous section has maximum degree at most 3. Therefore, for any set of points \( P \) in convex position, there exists a plane spanner of \( C \) of maximum degree at most 3. We also show in this section that 3 is a lower bound on the maximum degree of plane spanners of \( C \) for point-sets in convex position. The preceding implies that 3 is a tight bound on the maximum degree of plane geometric spanners of \( C \), in the case when \( P \) is in convex position.

Lemma 16. Let \( P \) be a set of points in convex position in the plane, and let \( S \) be the spanner of \( C \) constructed as described in Section 4. Then the maximum degree of \( S \) is at most 3.

Proof. Let \( p \in P \). It suffices to show that the degree of \( p \) in \( S \) is at most 3. To this effect, we show that at most 3 edges incident to \( p \) could be added in steps 1 – 6 of the construction of \( S \). Since \( P \) is in convex position, there exists a support line, \( D_p \), passing through \( p \) such that all the points of \( P \) lie in one closed half plane, \( H \), of the two half planes delimited by \( D_p \) [23]. Observe that — by the construction of \( S \) — each of the two blue sectors of \( p \) contains at most one edge incident to \( p \) in \( S \). In Lemma 7, we showed that each of the at most 4 edges incident to a point \( p \in S \) is charged by the charging scheme (see Section 5) to one of the 4 sectors \( LB_p, RB_p, UW_p, \) and \( LW_p \), such that each of the 4 sectors is charged with at most one edge. In this charging scheme, a blue anchor at \( p \) is charged to the blue sector containing it, and a white edge incident to \( p \) is either charged to a blue sector at \( p \) (and in such case the blue sector does not contain a blue anchor), or to the white sector containing the edge. We distinguish the following cases, based on the angle, \( \alpha \), that \( D_p \) makes with the positive \( x \)-axis:

Case 1: \( 0 \leq \alpha \leq \pi/3 \) (\( D_p \) passes through the two blue sectors at \( p \)). In this case the half plane \( H \) entirely contains one of the two white sectors at \( p \). Suppose that \( H \) contains \( UW_p \); the case is symmetric if \( H \) contains \( LW_p \). Since (1) each of the three sectors \( LB_p, UW_p, \) and \( RB_p \) is charged with at most one edge incident to \( p \), (2) each of \( LB_p \) and \( RB_p \) contains at most one blue anchor incident to \( p \), and (3) each edge incident to \( p \) in \( UW_p \) is either charged to \( UW_p, LB_p, \) or \( RB_p \), it follows that the number of edges incident to \( p \) in \( H \), and hence in \( S \), is at most 3.

Case 2: \( \pi/3 < \alpha \leq 2\pi/3 \) (\( D_p \) passes through the two red cones at \( p \)). Assume, to get a contradiction, that there are 4 edges incident to \( p \) in \( H \). Suppose first that \( H \) entirely contains \( RB_p \), and hence is disjoint from \( LB_p \). In this case the part of \( UW_p \) in \( H \) is contained in a positive cone at \( p \), and hence, there is at most one edge incident to \( p \) in \( S \) that lies in \( H \cap UW_p \). Since each of the 4 sectors at \( p \) is charged with at most one edge, it follows
from the preceding that there is a white edge $e$ incident to $p$ in $H \cap LW_p$ that is charged to $LB_p$. This could only happen if $e$, a white edge, is a canonical edge in a blue cone, added according to step 3 in the spanner construction, and is charged to $LB_p$ according to step 3 of the charging scheme. This, however, implies that $LB_p$ is not empty (a white canonical edge $e \in LW_p$ in a blue cone could exist only if $RB_p$ contains points of $\mathcal{P}$), contradicting that all points of $\mathcal{P}$ lie in $H$. Suppose now that $H$ entirely contains $LB_p$, and hence is disjoint from $RB_p$. By the same argument as above, there must exist a white canonical edge, $e$, incident to $p$ and lying in a white sector at $p$, that is charged to $LB_p$. By the charging scheme, the edge $e$ cannot lie in $UW_p$ because in such case $e$ would not be charged to $UW_p$. Therefore, $e$ must lie in $LW_p$. But then $e$ must lie in the negative cone adjacent to $p$ in $LW_p$, which is (the cone) disjoint from $H$, again contradicting that $H$ contains all points of $\mathcal{P}$.

Case 3: $2\pi/3 < \alpha \leq \pi$ ($D_p$ passes through the two green cones at $p$). The proof is analogous to that of Case 2 above.

**Lemma 17.** For any constant $\rho \geq 1$, there exists a point-set $\mathcal{P}$ in convex position such that any plane spanner of $\mathcal{C}$ of maximum degree at most 2 has stretch factor $> \rho$.

**Proof.** Let $\rho \geq 1$ be a given constant. Choose an integer $b > \rho$, and an integer $N > 3(\rho \cdot b + 1)$. Consider an orthogonal rectangle of vertical dimension $a = N - 1$ and horizontal dimension $b$. Let $n = 2N$, and let $\mathcal{P} = \{p_1, \ldots, p_N, q_1, \ldots, q_N\}$ be a set of $n$ points placed on the rectangle as follows. Points $p_1, \ldots, p_N$ are placed on one vertical side of the rectangle such that $|p_ip_{i+1}| = 1$, for $i = 1, \ldots, N - 1$, so that $p_1$ and $p_N$ end up on the two vertices of the vertical side of the rectangle. Points $q_1, \ldots, q_N$ are placed symmetrically on the other vertical side of the rectangle so that the $p_iq_i$’s are all parallel and $|p_iq_i| = b$, for all $i = 1, \ldots, N$. We note that one can choose $\mathcal{P}$ to be in convex position while respecting the standard general position assumptions (no 3 points on a line, etc.) by slightly modifying the set of points chosen above (e.g., rotating each of $p_ip_{i+1}$ and $q_iq_{i+1}$, $i = 1, \ldots, N$, slightly but increasingly inwards towards the interior of the rectangle, and slightly modifying the choice of the parameters $a, b, N$). However, we decided to go with the above configuration for clarity and ease of presentation. See Figure 9 for an illustration of the point set $\mathcal{P}$.

Suppose, to get a contradiction, that there is a plane spanner $S$ of $\mathcal{C}$ of maximum degree at most 2 and stretch factor $\rho$. Since $S$ is connected, $S$ is either a Hamiltonian path or a Hamiltonian cycle. Without loss of generality, we will assume in what follows that $S$ is a Hamiltonian cycle, as the proof in the other case is simpler. Let $L = \{p_1, \ldots, p_N\}$ and $R = \{q_1, \ldots, q_N\}$. We distinguish the following two cases:

Case 1: The subgraph of $S$ induced by either $L$ or $R$ is disconnected. Suppose that the subgraph of $S$ induced by $L$, $S_L$, is disconnected; the proof follows by symmetry in the other case. Then there must exist $i \in \{1, \ldots, N - 1\}$ such that the two consecutive points
\(p_i, p_{i+1}\) belong to two different connected components of \(S_L\). Therefore, any path between \(p_i\) and \(p_{i+1}\) in \(S\) must contain an edge \(p_rq_s\), \(r, s \in \{1, \ldots, N\}\). Since the distance between any point from \(L\) and any point from \(R\) is at least \(b\), we have \(|p_rq_s| \geq b\). It follows that the length of any path in \(S\) between \(p_i\) and \(p_{i+1}\) is at least \(b > \rho = \rho \cdot |p_ip_{i+1}|\), which contradicts that \(S\) is a spanner of stretch factor \(\rho\).

Case 2: Each of the two subgraphs \(S_L\) and \(S_R\) of \(S\), induced by \(L\) and \(R\), respectively, is connected. Then each of \(S_L\) and \(S_R\) must be a Hamiltonian path on \(L\) and \(R\), respectively. Let \(p_r\) and \(p_s\), \(r, s \in \{1, \ldots, N\}\), be the points of degree 1 in \(S_L\), and \(q_r', q_s', r', s' \in \{1, \ldots, N\}\), be those of degree 1 in \(S_R\). Since \(S\) is a Hamiltonian cycle, \(S\) consists of the edges in the Hamiltonian path \(S_L\), the edges in the Hamiltonian path \(S_R\), plus a matching between \(\{p_r, p_s\}\) and \(\{q_r', q_s'\}\), say \(p_rq_{s'}\) and \(p_sq_{r'}\). Since there are \(N\) points in \(L\), there must exist a point \(p_i \in L\), \(i \in \{1, \ldots, N\}\), such that the number of edges on each of the two subpaths between \(p_i\) and \(p_r\) and between \(p_i\) and \(p_s\) in \(S_L\), is at least \(N/3 - 1 > \rho \cdot b\) by the choice of \(N\). Consider point \(q_i \in R\). Any path between \(p_i\) and \(q_i\) in \(S\) must contain either the subpath of \(S_L\) between \(p_i\) and \(p_r\) or the subpath of \(S_L\) between \(p_i\) and \(p_s\), and hence must contain more than \(\rho \cdot b\) edges of \(S_L\). Since each edge of \(S_L\) has length at least 1, the length of any path between \(p_i\) and \(q_i\) in \(S\) is more than \(\rho \cdot b = \rho \cdot |p_rq_s|\). This again contradicts the assumption that \(S\) has stretch factor \(\rho\), and completes the proof.

Combining Lemmas 16 and 17, we conclude with the following theorem:

**Theorem 2.** The constant 3 is a tight bound on the maximum degree of geometric plane spanners of \(C\) for point-sets in convex position.

### References

[1] A. Biniaz, P. Bose, J.-L. De Carufel, C. Gavoille, A. Maheshwari, and M. Smid. Towards Plane Spanners of Degree 3. In *27th International Symposium on Algorithms and Computation (ISAAC)*, volume 64, pages 19:1–19:14, 2016.

[2] N. Bonichon, C. Gavoille, N. Hanusse, and D. Ilcinkas. Personal communication, July 2009.

[3] N. Bonichon, C. Gavoille, N. Hanusse, and D. Ilcinkas. Connections between theta-graphs, Delaunay triangulations, and orthogonal surfaces. In *Proceedings of the 36th International Workshop on Graph Theoretic Concepts in Computer Science (WG)*, volume 6410 of *Lecture Notes in Computer Science*, pages 266–278. Springer, 2010.

[4] N. Bonichon, C. Gavoille, N. Hanusse, and L. Perković. Plane spanners of maximum degree six. In *Proceedings of the 37th International Colloquium on Automata, Languages and Programming (ICALP)*, volume 6198 of *Lecture Notes in Computer Science*, pages 19–30. Springer, 2010.

[5] N. Bonichon, C. Gavoille, N. Hanusse, and L. Perković. The stretch factor of \(L_1\)- and \(L_\infty\)-Delaunay triangulations. In *Proceedings of the 20th Annual European Symposium on Algorithms (ESA)*, volume 7501 of *Lecture Notes in Computer Science*, pages 205–216. Springer, 2012.
[6] N. Bonichon, I. Kanj, L. Perkovic, and G. Xia. There are plane spanners of degree 4 and moderate stretch factor. *Discrete & Computational Geometry*, 53(3):514–546, 2015.

[7] P. Bose, P. Carmi, and L. Chaitman-Yerushalmi. On bounded degree plane strong geometric spanners. *Journal of Discrete Algorithms*, 15:16–31, 2012.

[8] P. Bose, P. Carmi, S. Collette, and M. Smid. On the stretch factor of convex Delaunay graphs. *Journal of Computational Geometry*, 1(1):41–56, 2010.

[9] P. Bose, R. Fagerberg, A. van Renssen, and S. Verdonschot. Competitive routing in the half-$\theta_6$-graph. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1319–1328, 2012.

[10] P. Bose, J. Gudmundsson, and M. Smid. Constructing plane spanners of bounded degree and low weight. *Algorithmica*, 42(3-4):249–264, 2005.

[11] P. Bose, P. Morin, I. Stojmenović, and J. Urrutia. Routing with guaranteed delivery in ad hoc wireless networks. *Wireless Networks*, 7(6):609–616, 2001.

[12] P. Bose, M. Smid, and D. Xu. Delaunay and diamond triangulations contain spanners of bounded degree. *International Journal of Computational Geometry and Applications*, 19(2):119–140, 2009.

[13] G. S. Brodal and R. Jacob. Dynamic planar convex hull. In *Proceedings of the 43rd Symposium on Foundations of Computer Science (FOCS)*, pages 617–626. IEEE Computer Society, 2002.

[14] L. P. Chew. There is a planar graph almost as good as the complete graph. In *Proceedings of the 2nd Annual Symposium on Computational Geometry (SoCG)*, pages 169–177, 1986.

[15] L. P. Chew. There are planar graphs almost as good as the complete graph. *Journal of Computer and System Sciences*, 39(2):205–219, 1989.

[16] G. Das and P. J. Heffernan. Constructing degree-3 spanners with other sparseness properties. *International Journal of Foundations of Computer Science*, 7(2):121–136, 1996.

[17] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf. *Computational Geometry: Algorithms and Applications*. Springer-Verlag, 2008.

[18] D. Dobkin, S. Friedman, and K. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete & Computational Geometry*, 5(4):399–407, December 1990.

[19] I. Kanj and L. Perković. On geometric spanners of Euclidean and unit disk graphs. In *Proceedings of the 25th Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 409–420. HAL, 2008.

[20] I. Kanj, L. Perković, and D. Türkoğlu. Degree four plane spanners: Simpler and better. In *32nd International Symposium on Computational Geometry (SoCG)*, volume 51, pages 45:1–45:15, 2016.
[21] J. M. Keil and C. A. Gutwin. Classes of graphs which approximate the complete Euclidean graph. *Discrete & Computational Geometry*, 7(1):13–28, 1992.

[22] X.-Y. Li and Y. Wang. Efficient construction of low weight bounded degree planar spanner. *International Journal of Computational Geometry and Applications*, 14(1-2):69–84, 2004.

[23] F. Preparata and M. Shamos. *Computational Geometry - An Introduction*. Springer, 1985.

[24] J. Salowe. Euclidean spanner graphs with degree four. *Discrete Applied Mathematics*, 54(1):55–66, 1994.

[25] Y. Wang and X.-Y. Li. Localized construction of bounded degree and planar spanner for wireless ad hoc networks. *Mobile Networks and Applications*, 11(2):161–175, 2006.

[26] G. Xia. The stretch factor of the Delaunay triangulation is less than 1.998. *SIAM Journal on Computing*, 42(4):1620–1659, 2013.