$L_p$ Minkowski problem for electrostatic $p$-capacity

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Abstract Existence and uniqueness of the solution to the discrete $L_p$ Minkowski problem for $p$-capacity are proved when $p \geq 1$ and $1 < p < n$. For general $L_p$ Minkowski problem for $p$-capacity, existence and uniqueness of the solution are given when $p \geq 1$ and $1 < p \leq 2$. These results are non-linear extensions of the very recent solution to the $L_p$ Minkowski problem for $p$-capacity when $p = 1$ and $1 < p < n$ by CNSXYZ, and the classical solution to the Minkowski problem for electrostatic capacity when $p = 1$ and $p = 2$ by Jerison.

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1. Introduction

The setting for this paper is Euclidean $n$-space, $\mathbb{R}^n$. A convex body in $\mathbb{R}^n$ is a compact convex set that has a non-empty interior. A polytope in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided it has positive volume (i.e., $n$-dimensional volume).

The Brunn-Minkowski theory (or the theory of mixed volumes) of convex bodies, developed by Minkowski, Aleksandrov, Fenchel, et al., centers around the study of geometric functionals of convex bodies as well as the differentials of these functionals. Usually, the differentials of these functionals produce new geometric measures. The theory depends heavily on analytic tools such as the cosine transform on the unit sphere $S^{n-1}$ and Monge-Ampère type equations.

A Minkowski problem is a characterization problem for a geometric measure generated by convex bodies: It asks for necessary and sufficient conditions in order that a given measure arises as the measure generated by a convex body. The solution of a Minkowski problem, in general, amounts to solving a degenerate fully non-linear partial differential equation. The study of Minkowski problems has a long history and strong influence on both the Brunn-Minkowski theory and fully non-linear partial differential equations, see [66].

The classical Brunn-Minkowski theory begins with the variation of volume functional.

1.1. Volume, surface area measure and the classical Minkowski problem. Without doubt, the most fundamental geometric functional in the Brunn-Minkowski theory is volume
It is to see that via the variation of volume functional, it produces the most important geometric measure: surface area measure.

Specifically, if $K$ and $L$ are convex bodies in $\mathbb{R}^n$, then there exists a finite Borel measure $S(K, \cdot)$ on the unit sphere $\mathbb{S}^{n-1}$ known as the surface area measure of $K$, so that

$$
(1.1) \quad \frac{dV(K + tL)}{dt}
\bigg|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(\xi)dS(K, \xi),
$$

where $V$ is the $n$-dimensional volume (i.e., Lebesgue measure in $\mathbb{R}^n$); the convex body $K + tL = \{x + ty : x \in K, y \in L\}$ is the Minkowski sum of $K$ and $tL$; $h_L : \mathbb{S}^{n-1} \to \mathbb{R}$ is the support function of $L$, defined by $h_L(\xi) = \max\{\xi \cdot x : x \in L\}$, with $\xi \cdot x$ denoting the inner product of $\xi$ and $x$ in $\mathbb{R}^n$. Formula (1.1), also called the Aleksandrov variational formula, suggests that the surface area measure can be viewed as the differential of volume functional.

The surface area measure $S(K, \cdot)$ of a convex body $K$ can be defined directly, for each Borel set $\omega \subset \mathbb{S}^{n-1}$, by

$$
(1.2) \quad S(K, \omega) = \mathcal{H}^{n-1}(g_K^{-1}(\omega)),
$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Here the Gauss map $g_K : \partial'K \to \mathbb{S}^{n-1}$ is defined on $\partial'K$ of those points of $\partial K$ that have a unique outer normal and is hence defined $\mathcal{H}^{n-1}$-a.e. on $\partial K$. The integral in (1.1), divided by the ambient dimension $n$, is called the first mixed volume $V_1(K, L)$ of $K$ and $L$, i.e.,

$$
V_1(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(\xi)dS(K, \xi).
$$

It is a generalization of the well-known volume formula

$$
(1.3) \quad V(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(\xi)dS(K, \xi).
$$

The classical Minkowski problem, which characterizes the surface area measure, is one of the cornerstones of the Brunn-Minkowski theory of convex bodies. It reads: **Given a finite Borel measure $\mu$ on $\mathbb{S}^{n-1}$, what are the necessary and sufficient conditions on $\mu$ so that $\mu$ is the surface area measure $S(K, \cdot)$ of a convex body $K$ in $\mathbb{R}^n$?** More than a century ago, Minkowski himself [61] solved this problem for the case when the given measure is either discrete or has a continuous density. Aleksandrov [1, 2] and Fenchel-Jessen [21] independently solved the problem in 1938 for arbitrary measures: If $\mu$ is not concentrated on any great subsphere of $\mathbb{S}^{n-1}$, then $\mu$ is the surface area measure of a convex body if and only if $\int_{\mathbb{S}^{n-1}} \xi d\mu(\xi) = 0$.

Since for strictly convex bodies with smooth boundaries, the reciprocal of the Gauss curvature is the density of the surface area measure with respect to the spherical Lebesgue measure, the Minkowski problem in differential geometry is to characterize the Gauss curvature of closed convex hypersurfaces. Analytically, the Minkowski problem is equivalent to solving a degenerate Monge-Ampère equation. Establishing the regularity of the solution to the Minkowski problem is difficult and has led to a long series of highly influential works, see, e.g., Lewy [42], Nirenberg [63], Cheng and Yau [14], Pogorelov [64], Caffarelli [8, 9].
1.2. $L_p$ surface area measure and $L_p$ Minkowski problem for volume. The $L_p$ Brunn-Minkowski theory is an extension of the classical Brunn-Minkowski theory; see [22, 44, 45, 47, 49, 51, 57, 72]. In 1962, Firey [22] introduced $L_p$ sums for convex bodies. Let $1 \leq p < \infty$. If $K$ and $L$ are convex bodies with the origin in their interiors, then their $L_p$ sum $K +_p L$ is the convex body defined by

$$h_{K+_p L}(\xi)^p = h_K(\xi)^p + h_L(\xi)^p, \quad \xi \in \mathbb{S}^{n-1}.$$ 

See also, [22, 27, 47, 60]. Clearly, $K + _1 L = K + L$.

For $t > 0$, the $L_p$ scalar multiplication $t^p K$ is the convex body $t^p K$.

The $L_p$ surface area measure, introduced by Lutwak [47], is a fundamental notion in the $L_p$ theory. For fixed $p \in \mathbb{R}$, and a convex body $K$ in $\mathbb{R}^n$ with the origin in its interior, the $L_p$ surface area measure $S_p(K, \cdot)$ of $K$ is a Borel measure on $\mathbb{S}^{n-1}$ defined, for Borel $\omega \subset \mathbb{S}^{n-1}$, by

$$S_p(K, \omega) = \int_{x \in g_{K}^{-1}(\omega)} (x \cdot g_K(x))^{1-p} d\mathcal{H}^{n-1}(x).$$

The $L_p$ surface area measure $S_p(K, \cdot)$ can also be explicitly defined, for Borel $\omega \subset \mathbb{S}^{n-1}$, by

$$S_p(K, \omega) = \int_{\omega} h_K(\xi)^{1-p} dS(K, \xi).$$

Note that $S_1(K, \cdot)$ is just the surface area measure $S(K, \cdot)$. $\frac{1}{p} S_0(K, \cdot)$ is the cone-volume measure of convex body $K$, which is the only SL($n$) invariant measure among all the $L_p$ surface area measures. In recent years, cone-volume measures have been greatly investigated, e.g., [4, 29, 45, 61, 65, 68, 73]. $S_2(K, \cdot)$ is called the quadratic surface area measure of convex body $K$, which was studied in [44] and [52, 53, 59]. Applications of the $L_p$ surface area measure to affine isoperimetric inequalities were given in, e.g., [12, 50, 51, 56].

In [47], Lutwak established the following $L_p$ variational formula for volume

$$\left. \frac{dV(K+_{p} t \cdot p L)}{dt} \right|_{t=0^+} = \frac{1}{p} \int_{\mathbb{S}^{n-1}} h_L(\xi)^p dS_p(K, \xi),$$

which suggests that the $L_p$ surface area measure can be viewed as the differential of volume functional of $L_p$ combination of convex bodies. When $p = 1$, (1.5) is precisely (1.1).

Lutwak [47] initiated the following $L_p$ Minkowski problem.

$L_p$ Minkowski problem for volume. Suppose $\mu$ is a finite Borel measure on $\mathbb{S}^{n-1}$ and $p \in \mathbb{R}$. What are the necessary and sufficient conditions on $\mu$ so that $\mu$ is the $L_p$ surface area measure $S_p(K, \cdot)$ of a convex body $K$ in $\mathbb{R}^n$?

$L_1$ Minkowski problem is precisely the classical Minkowski problem. The $L_0$ Minkowski problem, which characterizes the cone-volume measure, is called the logarithmic Minkowski problem. In light of its strong geometric intuition and fundamental significance, the logarithmic Minkowski problem is regarded as the most important case. In 1999, Andrews [3] proved Firey’s conjecture [23] that convex surfaces moving by their Gauss curvature become spherical as they
contract to points. A major breakthrough was made by Böörcky and LYZ in 2013, who establish the sufficient and necessary conditions for the existence of a solution to the even logarithmic Minkowski problem. The $L_{-n}$ Minkowski problem is the centro-affine Minkowski problem. See Chou and Wang, and Zhu.

In the recent ground-breaking paper, Huang, Lutwak, Yang and Zhang introduced the dual curvature measures $\tilde{C}_i(K, \cdot)$, $i = 0, 1, \ldots, n$, of a convex body $K$ and solved their associated Minkowski problems. These new geometric measures are precisely the counterparts to the curvature measures in the dual Brunn-Minkowski theory and open up a new passage to the $L_p$ surface area measures, since $\tilde{C}_n(K, \cdot)$ is just the cone-volume measure of $K$.

By now, the $L_p$ Minkowski problem for volume has been intensively investigated and achieved great developments. See, e.g., [13, 15, 33, 38, 40, 43, 47, 55, 67, 72]. As applications, the solutions to $L_p$ Minkowski problem for volume have been used to establish sharp affine isoperimetric inequalities, such as the affine Moser-Trudinger and the affine Morrey-Sobolev inequalities, the affine $L_p$ Sobolev-Zhang inequality, etc. See, e.g., [6, 16, 34, 35, 54, 58, 69], for more details.

1.3. $p$-capacitary measure and Minkowski problem for $p$-capacity. It is worth mentioning that the Minkowski problem for electrostatic $p$-capacity is doubtless an extremely important variant among Minkowski problems. Recall that for $1 < p < n$, the electrostatic $p$-capacity of a compact set $K$ in $\mathbb{R}^n$ is defined by

$$C_p(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^p dx : u \in C_c^\infty(\mathbb{R}^n) \text{ and } u \geq \chi_K \right\},$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the set of functions from $C^\infty(\mathbb{R}^n)$ with compact supports, and $\chi_K$ is the characteristic function of $K$. $C_2(K)$ is the classical electrostatic (or Newtonian) capacity of $K$. Let $L$ be an arbitrary convex body. Via the variation of capacity functional $C_2(K)$, the classical Hadamard variational formula

$$\left. \frac{dC_2(K + tL)}{dt} \right|_{t=0^+} = \int_{S^{n-1}} h_L(\xi) d\mu_2(K, \xi)$$

and its special case, the Poincaré capacity formula

$$C_2(K) = \frac{1}{n-2} \int_{S^{n-1}} h_K(\xi) d\mu_2(K, \xi)$$

appear. Here, the new measure $\mu_2(K, \cdot)$ is a finite Borel measure on $S^{n-1}$, called the electrostatic capacitary measure of $K$. Formula (1.6) suggests that the electrostatic capacitary measure can be viewed as the differential of capacity functional.

In his celebrated article, Jerison pointed out the resemblance between the Poincaré capacity formula (1.7) and the volume formula (1.3) and also a resemblance between their variational formulas (1.6) and (1.1). Thus, he initiated to consider the Minkowski problem for electrostatic capacity: Given a finite Borel measure $\mu$ on $S^{n-1}$, what are the necessary and sufficient conditions on $\mu$ so that $\mu$ is the electrostatic capacitary measure $\mu_2(K, \cdot)$ of a convex body $K$ in $\mathbb{R}^n$?
Jerison [39] solved, in full generality, the Minkowski problem for electrostatic capacity. He proved the necessary and sufficient conditions for existence of a solution, which are unexpected identical to the corresponding conditions in the classical Minkowski problem. Uniqueness was settled by Caffarelli, Jerison and Lieb [11]. The regularity part of the proof depends on the ideas of Caffarelli [10] for regularity of solutions to Monge-Ampère equation.

Jerison’s work inspired much subsequent research on this topic. In the very recent article [19], the authors (CNSXYZ) extended Jerison’s work to electrostatic $p$-capacity. Let $K, L$ be convex bodies in $\mathbb{R}^n$ and $1 < p < n$. CNSXYZ established the Hadamard variational formula for $p$-capacity

$$
\frac{dC_p(K + tL)}{dt} \bigg|_{t=0^+} = (p - 1) \int_{\mathbb{S}^{n-1}} h_L(\xi)d\mu_p(K, \xi)
$$

and therefore the Poincaré $p$-capacity formula

$$
C_p(K) = \frac{p - 1}{n - p} \int_{\mathbb{S}^{n-1}} h_K(\xi)d\mu_p(K, \xi).
$$

Here, the new measure $\mu_p(K, \cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, called the electrostatic $p$-capacitary measure of $K$. Formula (1.8) suggests that $\mu_p(K, \cdot)$ can be viewed as the differential of $p$-capacity functional.

Consequently, the Minkowski problem for $p$-capacity was posed [19]: Given a finite Borel measure $\mu$ on $\mathbb{S}^{n-1}$, what are the necessary and sufficient conditions on $\mu$ so that $\mu$ is the $p$-capacitary measure $\mu_p(K, \cdot)$ of a convex body $K$ in $\mathbb{R}^n$? CNSXYZ proved the uniqueness of the solution when $1 < p < n$, and existence and regularity when $1 < p < 2$.

1.4. $L_p$ $p$-capacitary measure and $L_p$ Minkowski problem for $p$-capacity. By reviewing the Minkowski problems for volume and capacity respectively, we find that they have been intensively investigated along two parallel tracks, and their similarities are more highlighted therein. However, compared with a series of remarkable results on $L_p$ Minkowski problem for volume, the general $L_p$ Minkowski problem for capacity is hardly ever proposed yet. The time is ripe to initiate the research on general $L_p$ Minkowski problem for capacity.

In this paper, we generalize the Minkowski problem for $p$-capacity to general $L_p$ Minkowski problem for $p$-capacity. In this sense, this is the first paper to push the Minkowski problem for $p$-capacity to $L_p$ stage. Here, it is worth mentioning that to comply with the habits, we stick to using the terminology “$L_p$” Minkowski problem in our paper. But to avoid the confusion, we use “$p$-capacity”, instead of “$p$-capacity”, to distinguish the “$p$” in “$L_p$”.

In light of the fundamental significance of $L_p$ surface area measures $S_p(K, \cdot)$ in $L_p$ theory for convex bodies, we introduce the important geometric measure: $L_p$ $p$-capacitary measure.

Definition. Let $p \in \mathbb{R}$ and $1 < p < n$. Suppose $K$ is a convex body in $\mathbb{R}^n$ with the origin in its interior. The $L_p$ $p$-capacitary measure $\mu_{p,p}(K, \cdot)$ of $K$ is a finite Borel measure on $\mathbb{S}^{n-1}$.
defined, for Borel $\omega \subseteq S^{n-1}$, by

$$\mu_{p,p}(K, \omega) = \int_{\omega} h_K(\xi)^{1-p} d\mu_p(K, \xi).$$

Soon Later, it will see that like the $L_p$ surface area measures $S_p(K, \cdot)$, the $L_p$ $p$-capacitary measure $\mu_{p,p}(K, \cdot)$ is resulted from the variation of $p$-capacity functional of $L_p$ sum of convex bodies. Specifically, if $K, L$ are convex bodies in $\mathbb{R}^n$ with origin in their interiors, then

$$\frac{dC_p(K+t \cdot L)}{dt} \bigg|_{t=0^+} = \frac{(p-1)}{p} \int_{S^{n-1}} h_{L}(\xi)^p d\mu_{p,p}(K, \xi),$$

where $1 \leq p < \infty$. See Corollary 3.3 for details.

Naturally, we pose the $L_p$ Minkowski problem for $p$-capacity.

**$L_p$ Minkowski problem for $p$-capacity.** Suppose $\mu$ is a finite Borel measure on $S^{n-1}$, $1 < p < n$ and $p \in \mathbb{R}$. What are the necessary and sufficient conditions on $\mu$ so that $\mu$ is the $L_p$ $p$-capacitary measure $\mu_{p,p}(K, \cdot)$ of a convex body $K$ in $\mathbb{R}^n$?

Jerison [39] solved the classical case when $p = 1$ and $p = 2$. CNSXYZ [19] studied the case when $p = 1$ and $1 < p < n$. For the general case when $p \neq 1$, the corresponding problem is completely new.

1.5. Main results. To state our main results, we need to explain something first. When $p + p = n$, the $L_{n-p}$ Minkowski problem for $p$-capacity is a bit troubling, since two convex bodies with the same $L_{n-p}$ $p$-capacitary measure are dilates each other, but not necessarily identical. For simplicity, we technically normalize the $L_p$ Minkowski problem for $p$-capacity as folows: Under what necessary and sufficient conditions on $\mu$ does there exist a convex body $K^*$ so that $C_p(K^*)^{-1} \mu_{p,p}(K^*, \cdot) = \mu$? Note that when $p + p \neq n$, two problems are essentially equivalent, in the sense that $K = C_p(K^*)^{1/(p+p-n)}K^*$.

In this article, we solve the discrete $L_p$ Minkowski problem for $p$-capacity when $1 < p < n$, and the general $L_p$ Minkowski problem for $p$-capacity when $1 < p \leq 2$.

**Theorem 1.1.** Suppose $1 < p < \infty$ and $1 < p < n$. If $\mu$ is a discrete measure on $S^{n-1}$ which is not concentrated on any closed hemisphere, then there exists a unique polytope $P$ with the origin in its interior, such that

$$\mu_{p,p}(P, \cdot) = c\mu,$$

where $c = 1$ if $p + p \neq n$, or $C_p(P)$ if $p + p = n$. Furthermore, $P$ is origin-symmetric if $\mu$ is even.

**Theorem 1.2.** Suppose $1 < p < \infty$ and $1 < p \leq 2$. If $\mu$ is a finite Borel measure on $S^{n-1}$ which is not concentrated on any closed hemisphere, then there exists a unique convex body $K$ containing the origin, such that

$$d\mu_p(K, \cdot) = ch_{K}^{p-1} d\mu,$$

where $c = 1$ if $p + p \neq n$, or $C_p(K)$ if $p + p = n$. Furthermore, $K$ contains the origin in its interior if $p \geq n$. Therefore, $\mu_{p,p}(K, \cdot) = \mu$. 

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Theorem 1.3. Suppose \( 1 < p < \infty \) and \( 1 < p \leq 2 \). If \( \mu \) is a finite even Borel measure on \( S^{n-1} \) which is not concentrated on any great subsphere, there exists a unique origin-symmetric convex body \( K \) such that \( \mu_{p,p}(K,\cdot) = c\mu \), where \( c = 1 \) if \( p + p \neq n \), or \( C_p(K) \) if \( p + p = n \).

Continuity of the solution to the \( L_p \) Minkowski problem for \( p \)-capacity is shown.

Theorem 1.4. Suppose that \( 1 < p < \infty \) and \( 1 < p \leq 2 \). Let \( \mu \) and \( \mu_j, j \in \mathbb{N} \), be finite Borel measures on \( S^{n-1} \) which are not concentrated on any closed hemisphere, and \( K \) and \( K_j \) be convex bodies containing the origin such that \( C_p(K)^{-1}\mu_{p,p}(K,\cdot) = \mu \) and \( C_p(K_j)^{-1}\mu_{p,p}(K_j,\cdot) = \mu_j \), respectively. If \( \mu_j \to \mu \) weakly, then \( K_j \to K \), as \( j \to \infty \).

CNSXYZ [19] demonstrated the weak convergence of \( L_p \) \( p \)-capacitary measure: If \( K_j \to K \), then \( \mu_{p,p}(K_j,\cdot) \to \mu_{p,p}(K,\cdot) \) weakly. Theorem 1.4 shows that the converse still holds: If \( \mu_{p,p}(K_j,\cdot) \to \mu_{p,p}(K,\cdot) \) weakly, then \( K_j \to K \).

We emphasize that, for \( p > 1 \), the \( L_p \) Minkowski problem for \( p \)-capacity is considerably more complicated than the \( p = 1 \) case, requiring both new ideas and techniques. Our approach to this problem is rooted in the ideas and techniques from convex geometry. So its proof exhibits rich geometric flavour. Specifically, to prove Theorem 1.1, techniques developed by Hug and LYZ [38], Klain [41] and Lutwak [47,55] are comprehensively employed. In addition, techniques developed by the authors themselves in [73–76] are also crucial to the proof. To prove Theorem 1.2, we turn the Minkowski problem into solving two dual optimization problems. This strategy was fist used by LYZ [57] to establish the \( L_p \) John ellipsoids, and then developed by Zou and Xiong to establish the Orlicz-John ellipsoids [73] and the Orlicz-Legendre ellipsoids [75].

This paper is organized as follows. In Section 2, we introduce necessary notations and collect some basic facts concerning the convex bodies, the \( p \)-capacity and the Aleksandrov bodies. Some basic facts of the \( L_p \) \( p \)-capacitary measures \( \mu_{p,p}(K,\cdot) \) are provided in Section 3. For example, to study the uniqueness of the \( L_p \) Minkowski problem for \( p \)-capacity, we prove the \( L_p \) Minkowski inequality for \( p \)-capacity and then characterize the uniqueness of \( \mu_{p,p}(K,\cdot) \). The proof of Theorem 1.1 is provided in Section 4. Along with the arguments in Section 4, we show that Theorem 1.1 still holds when \( p = 1 \) in Section 5, which solves CNSXYZ’s [19, p. 1517] open problem for discrete measures. Theorem 1.2 and Theorem 1.4 are provided in Section 8 and Section 9, respectively. To prove these theorems, we reformulate the \( L_p \) Minkowski problem for \( p \)-capacity into a pair of dual optimization problems. See Section 6 for details. More preliminaries about these optimization problems are provided in Section 7.

2. Preliminaries

2.1. Basics of convex bodies. For quick reference, we collect some basic facts on convex bodies. Excellent references are the books by Gardner [25], Gruber [30] and Schneider [66].

As usual, write \( x \cdot y \) for the standard inner product of \( x, y \in \mathbb{R}^n \). Each compact convex set \( K \) in \( \mathbb{R}^n \) is uniquely determined by its support function \( h_K : \mathbb{R}^n \to \mathbb{R} \), which is defined by
\[ h_K(x) = \max \{ x \cdot y : y \in K \}, \text{ for } x \in \mathbb{R}^n. \] It is easily seen that the support function is positively homogeneous of order 1.

The class of compact convex sets in \( \mathbb{R}^n \) is often equipped with the Hausdorff metric \( \delta_H \), which is defined for compact convex sets \( K \) and \( L \) by

\[ \delta_H(K, L) = \max \{ | h_K(\xi) - h_L(\xi) | : \xi \in \mathbb{S}^{n-1} \}. \]

Write \( K^n \) for the set of convex bodies in \( \mathbb{R}^n \), and write \( K^n_0 \) for the set of convex bodies with the origin \( o \) in their interiors. Let \( K \) and \( L \) be compact convex sets. For \( s > 0 \), the set \( sK = \{ sx : x \in K \} \) is called a dilate of \( K \). \( K \) and \( L \) are said to be homothetic, provided \( K = sL + x \), for some \( s > 0 \) and \( x \in \mathbb{R}^n \). The reflection of \( K \) is the set \( -K = \{ -x : x \in K \} \). The Minkowski sum of \( K \) and \( L \) is the set \( K + L = \{ x + y : x \in K, y \in L \} \).

For brevity, write \( h \) by \( h^t \). Let \( 1 \leq p < \infty \), \( K \in K^n_0 \) and \( t > 0 \), the \( L_p \) scalar multiplication \( t \cdot K \) is the convex body \( t^\frac{1}{p}K \). For \( K, L \in K^n_0 \), their \( L_p \) sum (See, e.g., [22, 27, 47, 60]) is the convex body \( K + pL \) defined by

\[ h_{K+pL}(\xi)^p = h_K(\xi)^p + h_L(\xi)^p, \quad \xi \in \mathbb{S}^{n-1}. \]

Clearly, \( K + pL = K + L \).

Let \( C(\mathbb{S}^{n-1}) \) be the set of continuous real functions on \( \mathbb{S}^{n-1} \), equipped with the metric induced by the maximal norm. Let \( C_+(\mathbb{S}^{n-1}) \) be the subset of \( C(\mathbb{S}^{n-1}) \), consisting of strictly positive functions.

For \( f, g \in C_+(\mathbb{S}^{n-1}) \) and \( t > 0 \), define

\[ h + p \cdot f = (h^p + tf^p)^\frac{1}{p}. \]

For brevity, write \( h + p \cdot f \) for \( h + p \cdot 1 \cdot f \).

For \( f \in C_+(\mathbb{S}^{n-1}) \), define

\[ [f] = \bigcap \{ x \in \mathbb{R}^n : x \cdot \xi \leq f(\xi) \}. \]

The set \([f]\) is called the Aleksandrov body (also known as Wulff shape) associated with \( f \). Obviously, \([f]\) is a convex body with the origin in its interior.

2.2. Basics of \( p \)-capacity. In this part, some basics of \( p \)-capacity are listed. For more details on \( p \)-capacity, see, e.g., [19, 20, 28, 39].

Let \( 1 < p < n \). The \( p \)-capacity \( C_p \) is increasing with respect to the inclusion of sets. That is, if \( E \subseteq F \), then \( C_p(E) \leq C_p(F) \). The \( p \)-capacity \( C_p \) is positively homogeneous of order \((n-p)\), i.e., \( C_p(sE) = s^{n-p}C_p(E) \), for \( s > 0 \). Also, it is rigid invariant, i.e., \( C_p(OE + x) = C_p(E) \), for \( x \in \mathbb{R}^n \) and \( O \in \mathbb{O}(n) \).

For \( K \in K^n \), the \( p \)-capacitary measure \( \mu_p(K, \cdot) \) is positively homogeneous of order \((n-p-1)\), i.e., \( \mu_p(sK, \cdot) = s^{n-p-1}\mu_p(K, \cdot) \), for \( s > 0 \). For \( x \in \mathbb{R}^n \), \( \mu_p(K + x, \cdot) = \mu_p(K, \cdot) \), i.e., it is translation invariant. The centroid of \( \mu_p(K, \cdot) \) is at the origin, i.e., \( \int_{\mathbb{S}^{n-1}} \xi d\mu_p(K, \xi) = 0 \).

The weak convergence of \( p \)-capacitary measures was proved by CNSXYZ [19] p. 1550: If \( \{K_j\}_{j \in \mathbb{N}} \subset K^n \) converges to \( K \in K^n \), then \( \{\mu_p(K_j, \cdot)\}_{j} \) converges weakly to \( \mu_p(K, \cdot) \).
Let $K \in \mathcal{K}_o^n$ and $f \in C(\mathbb{S}^{n-1})$. There exists $t_0 > 0$ such that $h_K + tf \in C_+(\mathbb{S}^{n-1})$, for $|t| < t_0$. So, there is a continuous family of Aleksandrov bodies $[h_K + tf]$ with $|t| < t_0$. The Hadamard variational formula for $p$-capacity (see [19] p. 1547) states that

$$
\left. \frac{dC_p(h_K + tf)}{dt} \right|_{t=0} = (p - 1) \int_{\mathbb{S}^{n-1}} f(\xi) d\mu_p(K, \xi).
$$

(2.1)

For $K, L \in \mathcal{K}_o^n$, the mixed $p$-capacity $C_p(K, L)$ (see [19] p. 1549) is defined by

$$
C_p(K, L) = \frac{1}{n - p} \left. \frac{dC_p(K + tL)}{dt} \right|_{t=0} = \frac{p - 1}{n - p} \int_{\mathbb{S}^{n-1}} h_L(\xi) d\mu_p(K, \xi).
$$

(2.2)

When $L = K$, it reduces to the Poincaré $p$-capacity formula (1.4). From the weak convergence of $p$-capacitary measures, it follows that $C_p(K, L)$ is continuous in $(K, L)$.

The $p$-capacitary Brunn-Minkowski inequality, proved by Colesanti and Salani [18], reads: If $K, L \in \mathcal{K}_o^n$, then

$$
C_p(K + L)^{1/p} \geq C_p(K)^{1/p} + C_p(L)^{1/p},
$$

(2.3)

with equality if and only if $K$ and $L$ are homothetic. When $p = 2$, the inequality was first established by Borell [5], and the equality condition was shown by Caffarelli, Jerison and Lieb [11]. For more details, see e.g., Colesanti [17], Gardner [24], and Gardner and Hartenstine [26].

The $p$-capacitary Brunn-Minkowski inequality is equivalent to the $p$-capacitary Minkowski inequality,

$$
C_p(K, L) \geq C_p(K)^{n-p} C_p(L),
$$

(2.4)

with equality if and only if $K$ and $L$ are homothetic. See [19] p. 1549 for its proof.

2.3. Basics of Aleksandrov bodies. For $f \in C_+(\mathbb{S}^{n-1})$, define

$$
C_p(f) = C_p(|f|).
$$

(2.5)

Obviously, $C_p(h_K) = C_p(K)$, for $K \in \mathcal{K}_o^n$.

The Aleksandrov convergence lemma reads: If the sequence $\{f_j\}_j \subset C_+(\mathbb{S}^{n-1})$ converges uniformly to $f \in C_+(\mathbb{S}^{n-1})$, then $\lim_{j \to \infty} [f_j] = [f]$. From this lemma and the continuity of $C_p$ on $\mathcal{K}_o^n$, we see that $C_p : C_+(\mathbb{S}^{n-1}) \to (0, \infty)$ is continuous.

Let $1 \leq p < \infty$ and $1 < p < n$. For $K \in \mathcal{K}_o^n$ and nonnegative $f \in C(\mathbb{S}^{n-1})$, define

$$
C_{p,p}(K, f) = \frac{p - 1}{n - p} \int_{\mathbb{S}^{n-1}} f(\xi)^p h_K(\xi)^{1-p} d\mu_p(K, \xi).
$$

(2.6)

For brevity, write $C_p(K, f)$ for $C_{1,p}(K, f)$. Obviously, $C_{p,p}(K, h_K) = C_p(K)$.

Lemma 2.1. Suppose $1 \leq p < \infty$ and $1 < p < n$. If $f \in C_+(\mathbb{S}^{n-1})$, then

$$
C_{p,p}(tf, f) = C_p(|f|) = C_p(f).
$$

\[9\]
Corollary 3.3. Let \( h_{[\Omega]} \leq f \). A basic fact established by Aleksandrov is that \( h_{[\Omega]} = f \), a.e. with respect to \( S_{[\Omega]} \). That is, \( S_{[\Omega]}(\{h_{[\Omega]} < f\}) = 0 \). Since \( \mu_p([f], \cdot) \) is absolutely continuous with respect to \( S_{[\Omega]} \), it follows that \( \mu_p([f], \{h_{[\Omega]} < f\}) = 0 \). Combining this fact and the inequality \( h_{[\Omega]} \leq f \), it follows that

\[
C_{p,p}([f], f) - C_p(f) = \frac{p-1}{n-p} \int_{\{f > h_{[\Omega]}\}} (f^p - h_{[\Omega]}^p) h_{[\Omega]}^{1-p} d\mu_p([f], \xi) = 0,
\]
as desired. \( \Box \)

Note that for \( K \in \mathcal{K}^n_0 \) and \( f \in C_+(S^{n-1}) \), we have \( C_p(K, h_{[\Omega]}) \leq C_p(K, f) \).

3. The \( L_p \) \( p \)-capacitary measure \( \mu_{p,p}(K, \cdot) \)

3.1. The first \( L_p \) variational of \( p \)-capacity.

**Lemma 3.1.** Let \( I \subset \mathbb{R} \) be an interval containing both \( 0 \) and some positive number, and let \( h_t(\xi) = h(t, \xi) : I \times S^{n-1} \to (0, \infty) \) be continuous, such that the convergence in

\[
h'(0, \xi) = \lim_{t \to 0} \frac{h(t, \xi) - h(0, \xi)}{t}
\]
is uniform on \( S^{n-1} \). Then

\[
\lim_{t \to 0^+} \frac{C_p(h_t) - C_p(h_0)}{t} = (p-1) \int_{S^{n-1}} h'(0, \xi) d\mu_p([h_0], \xi).
\]

**Lemma 3.2.** Suppose \( 1 \leq p < \infty \) and \( 1 < p < n \). If \( K \in \mathcal{K}^n_0 \) and \( f \in C(S^{n-1}) \) is nonnegative, then

\[
\left. \frac{dC_p(h_K + p t \cdot f)}{dt} \right|_{t=0^+} = \frac{n-p}{p} C_{p,p}(K, f).
\]

**Proof.** Take an interval \( I = [0, t_0] \) for \( 0 < t_0 < \infty \). Since \( h_t(\xi) = h(t, \xi) = (h_K + p t \cdot f)(\xi) : I \times S^{n-1} \to (0, \infty) \) is continuous, and

\[
\lim_{t \to 0^+} \frac{(h_K + p t \cdot f) - h_K}{t} = \frac{f p h_K^{1-p}}{p}
\]
uniformly on \( S^{n-1} \), the desired lemma is a consequence of Lemma 3.1 and (2.6). \( \Box \)

Note that when \( p = 1 \), Lemma 3.2 reduces to the Hadamard variational formula (2.1).

**Corollary 3.3.** Suppose \( 1 \leq p < \infty \) and \( 1 < p < n \). If \( K \in \mathcal{K}^n_0 \) and \( L \) is a compact convex set containing the origin, then

\[
\left. \frac{dC_p(K + p t \cdot L)}{dt} \right|_{t=0^+} = \frac{p-1}{p} \int_{S^{n-1}} h_L(\xi)^p h_K(\xi)^{1-p} d\mu_p(K, \xi).
\]

Let \( 1 < p < n \). Now, we can introduce the following definitions.
Definition 3.4. If $1 \leq p < \infty$, $K \in \mathcal{K}_n^o$ and $L$ is a compact convex set containing the origin, then the quantity $C_{p,p}(K, L)$ defined by

$$C_{p,p}(K, L) = \frac{\frac{p}{n-p}}{\int_{\mathbb{S}^{n-1}} h_L(\xi)^p h_K(\xi)^{1-p} d\mu_p(K, \xi)}$$

is called the \textit{Lp mixed p-capacity} of $K$ and $L$.

Definition 3.5. If $p \in \mathbb{R}$ and $K \in \mathcal{K}_n^o$, then the Borel measure $\mu_{p,p}(K, \cdot)$ on $\mathbb{S}^{n-1}$, defined by

$$\mu_{p,p}(K, \omega) = \int_{\omega} h_K^{1-p} d\mu_p(K, \cdot),$$

for Borel $\omega \subseteq \mathbb{S}^{n-1}$, is called the \textit{Lp p-capacitary measure} of $K$.

Obviously, $C_{1,p}(K, L) = C_p(K, L)$, $C_{p,p}(K, K) = C_p(K)$ and $C_{p,p}(K, h_L) = C_{p,p}(K, L)$. Also, $\mu_{1,p}(K, \cdot) = \mu_p(K, \cdot)$, $\frac{p-1}{n-p} \mu_{0,p}(K, \mathbb{S}^{n-1}) = C_p(K)$. In addition, $C_{p,p}(OK, OL) = C_{p,p}(K, L)$, for $O \in O(n)$.

As the $L_p$ mixed volume $V_p(K, L)$ and the $L_p$ surface area measure $S_p(K, \cdot)$ greatly extend the first mixed volume $V_1(K, L)$ and the classical surface area measure $S(K, \cdot)$ in convex geometry, respectively, $C_{p,p}(K, L)$ and $\mu_{p,p}(K, \cdot)$ are precisely the $L_p$ extensions of the mixed $p$-capacity $C_p(K, L)$ and the $p$-capacitary measure $\mu_p(K, \cdot)$, respectively.

The next lemma shows that $C_{p,p}(K, L)$ is continuous in $(K, L, p)$.

Lemma 3.6. Suppose that $K_i, L_i, K, L \in \mathcal{K}_n^o$, $p_i, p \in [1, \infty)$, $i \in \mathbb{N}$ and $1 < p < n$. If $(K_i, L_i) \to (K, L)$ and $p_i \to p$, as $i \to \infty$, then $C_{p_i,p}(K_i, L_i) \to C_{p,p}(K, L)$.

Proof. Since $h_{K_i}, h_{L_i} > 0$ and $h_{K_i} \to h_K, h_{L_i} \to h_L$ uniformly on $\mathbb{S}^{n-1}$, it follows that $h_{L_i}/h_{K_i} \to h_L/h_K$ uniformly on $\mathbb{S}^{n-1}$. Clearly, there exists a compact interval $I \subset (0, \infty)$, such that $h_{L_i}/h_{K_i} \in I$ for all $i$. Since the sequence $t^{p_i}$ converges uniformly to $t^p$ on $I$, it follows that $(h_{L_i}/h_{K_i})^{p_i} \to (h_L/h_K)^p$, uniformly on $\mathbb{S}^{n-1}$. Meanwhile, the convergence $K_i \to K$ implies that $\mu_p(K_i, \cdot) \to \mu_p(K, \cdot)$ weakly. By Definition 3.4, the desired limit is obtained. \hfill \Box

The weak convergence of $p$-capacitary measures implies the weak convergence of $\mu_{p,p}$.

Lemma 3.7. Suppose that $K_i, K \in \mathcal{K}_n^o$, $i \in \mathbb{N}$, $1 \leq p < \infty$ and $1 < p < n$. If $K_i \to K$, as $i \to \infty$, then $\mu_{p,p}(K_i, \cdot) \to \mu_{p,p}(K, \cdot)$ weakly.

From the $(n - p - 1)$-order positive homogeneity of $p$-capacitary measures, the positive homogeneity of support functions and Definition 3.5 we obtain the following result.

Lemma 3.8. Suppose that $K \in \mathcal{K}_n^o$, $1 \leq p < \infty$ and $1 < p < n$. Then for $s > 0$,

$$\mu_{p,p}(sK, \cdot) = s^{n-p-p} \mu_{p,p}(K, \cdot).$$
3.2. $L_p$ Minkowski inequality for $p$-capacity. In this part, we will show that associated with $C_{p,p}(K, L)$, there is a natural $L_p$ extension of the $p$-capacitary Minkowski inequality. Then we will use it to extend the $p$-capacitary Brunn-Minkowski inequality to the $L_p$ stage. It is interesting that the $L_p$ Brunn-Minkowski type inequality for $p$-capacity was previously established in [77] by the authors’ $L_p$ transference principle.

Theorem 3.9. Suppose $1 \leq p < \infty$ and $1 < p < n$. If $K \in K_o^n$ and $f \in C_+(S^{n-1})$, then

$$C_{p,p}(K, f)^{n-p} \geq C_p(K)^{n-p-p}C_p(f)^p,$$

with equality if and only if $K$ and $|f|$ are dilates.

Proof. From (2.6), (2.5) and the Hölder inequality, it follows that

$$C_p(K, f) = \frac{p-1}{n-p} \int_{S^{n-1}} f(\xi)h_K(\xi)^{\frac{p-1}{p}}d\mu_p(K, \xi)$$

$$\leq \left( \frac{p-1}{n-p} \int_{S^{n-1}} f(\xi)^p h_K(\xi)^{1-p}d\mu_p(K, \xi) \right)^{\frac{1}{p}} \left( \frac{p-1}{n-p} \int_{S^{n-1}} h_K(\xi)d\mu_p(K, \xi) \right)^{\frac{p-1}{p}}$$

$$= C_{p,p}(K, f)^{\frac{1}{p}} C_p(K)^{\frac{n-1}{p}}.$$

Thus,

$$C_{p,p}(K, f) \geq C_p(K, f)^p C_p(K)^{1-p}.$$

From this inequality, the fact that $C_p(K, f) \geq C_p(K, |f|)$ and the $p$-capacitary Minkowski inequality, it follows that

$$C_{p,p}(K, f) \geq C_p(K, |f|)^p C_p(K)^{1-p}$$

$$\geq \left( C_p(K)^{\frac{n-p-1}{n-p}} C_p(|f|)^{\frac{1}{n-p}} \right)^p C_p(K)^{1-p}$$

$$= C_p(K)^{\frac{n-p-2}{n-p}} C_p(|f|)^{\frac{p}{n-p}}.$$

In the next, we prove the equality condition.

Assume that equality holds in (3.1). By the equality condition of $p$-capacitary Minkowski inequality, there exist $x \in \mathbb{R}^n$ and $s > 0$, such that $|f| = sK + x$. Meanwhile, by the equality condition of the Hölder inequality, $C_p(K, |f|)h_K(\xi) = C_p(K)h_{|f|}(\xi)$, for $\mu_p(K, \cdot)$-almost all $\xi \in S^{n-1}$. Hence, for $\mu_p(K, \cdot)$-almost all $\xi \in S^{n-1},$

$$\left( sC_p(K) + \frac{p-1}{n-p} x \cdot \int_{S^{n-1}} \xi d\mu_p(K, \xi) \right) h_K(\xi) = C_p(K)(sh_K(\xi) + x \cdot \xi).$$

Since the centroid of $\mu_p(K, \cdot)$ is at the origin, this implies that $x \cdot \xi = 0$, for $\mu_p(K, \cdot)$-almost all $\xi \in S^{n-1}$. Note that the $p$-capacitary measure $\mu_p(K, \cdot)$ is not concentrated on any great subsphere of $S^{n-1}$. Hence, $x = 0$, which in turn implies that $K$ and $|f|$ are dilates.

Conversely, assume that $K$ and $|f|$ are dilates, say, $K = s|f|$ for some $s > 0$. From our assumption, (2.6) combined with the fact that $\mu_p(s|f|, \cdot) = s^{n-p-1}\mu_p(|f|, \cdot)$, Lemma 2.1, the
Theorem 3.11. extension of the Colesanti-Salani Brunn-Minkowski inequality. Again, it follows that

\[ C_{p,p}(K, f) = C_p(s[f], f) \]
\[ = s^{n-p}C_p(f, f) \]
\[ = s^{n-p}C_p(f) \]
\[ = s^{n-p}C_p(f) \frac{n-p}{n-p} C_p(n-p) \]
\[ = C_p(K) \frac{n-p}{n-p} C_p(n-p) \]

This completes the proof. \( \square \)

From Theorem 3.8, we have that for any \( L \in \mathcal{K}_o^n \),

\[ C_{p,p}(K, L)^{n-p} \geq C_p(K)^{n-p}C_p(L)^p, \]
with equality if and only if \( K \) and \( L \) are dilates.

The next result is an \( L_p \) extension of the \( \mathfrak{p} \)-capacitary isoperimetric inequality on the total mass of the measure \( \mu_{p,p}(K, \cdot) \).

**Corollary 3.10.** Suppose \( 1 \leq p < \infty \) and \( 1 < \mathfrak{p} \leq n \). If \( K \in \mathcal{K}_o^n \), then

\[ \mu_{p,p}(K, S^{n-1})^{n-p} \geq n^{p-p}n \left( \frac{n-p}{p-1} \right)^{(p-1)p} C_p(K)^{n-p}, \]

with equality if and only if \( K \) is an origin-symmetric ball.

**Proof.** Let \( L \) be the unit ball \( B \) in \( \mathbb{R}^n \). Since \( C_p(B) = n \omega_n \left( \frac{n-p}{p-1} \right)^{p-1} \), from the \( L_p \) capacitary Minkowski inequality, the desired inequality with its equality condition is obtained. \( \square \)

Let \( f_1, f_2, g \in C_+(S^{n-1}) \). From the definition of \( f_1 +_p f_2 \) and (2.6), it follows that

\[ C_{p,p}(g, f_1 +_p f_2) = C_{p,p}(g, f_1) + C_{p,p}(g, f_2). \]

This, combined with Theorem 3.3, yields the inequality

\[ C_{p,p}(g, f_1 +_p f_2) \geq C_p(g)^{n-p} \left( C_p(f_1)^{p-p} + C_p(f_2)^{p-p} \right), \]

with equality if and only if \( f_1 \) and \( f_2 \) are dilates of \( g \). Hence, let \( g = f_1 +_p f_2 \), it yields an \( L_p \) extension of the Colesanti-Salani Brunn-Minkowski inequality.

**Theorem 3.11.** Suppose \( 1 \leq p < \infty \) and \( 1 < \mathfrak{p} \leq n \). If \( f_1, f_2 \in C_+(S^{n-1}) \), then

\[ C_p(f_1 +_p f_2)^{n-p} \geq C_p(f_1)^{n-p} + C_p(f_2)^{n-p}, \]

with equality if and only if \( f_1 \) and \( f_2 \) are dilates.

Consequently, for any \( K, L \in \mathcal{K}_o^n \),

\[ C_p(K +_p L)^{n-p} \geq C_p(K)^{n-p} + C_p(L)^{n-p}, \]

with equality if and only if \( K \) and \( L \) are dilates.
Remark 3.12. The p-capacitary Brunn-Minkowski inequality also yields the p-capacitary Minkowski inequality. Indeed, consider the nonnegative concave function

\[ f(t) = C_p(K + pt \cdot L)^{\frac{\mu}{p\nu}} - C_p(K)^{\frac{\mu}{p\nu}} - tC_p(L)^{\frac{\mu}{p\nu}}. \]

The p-capacitary Brunn-Minkowski inequality and Corollary 3.3 yield

\[ \lim_{t \to 0^+} \frac{f(t) - f(0)}{t} = C_p(K)^{\frac{\mu}{p\nu}} - 1C_p(K, L) - C_p(L)^{\frac{\mu}{p\nu}} \geq 0. \]

By the equality condition of p-capacitary Brunn-Minkowski, if equality holds on the right, the function \( f \) must be linear and thus \( K, L \) must be dilates.

Remark 3.13. Suppose that \( K, L \in K^n_o, 1 \leq p < \infty \) and \( 1 < p < n \). Let \( 0 < s < 1 \). From the \((n-p)\)-ordered positive homogeneity of \( C_p \) and the definition of \( L_p \) scalar multiplication, the inequality (3.2) has the following equivalent forms:

1. \( C_p((1-s) \cdot K + p s \cdot L)^{\frac{\mu}{p\nu}} \geq (1-s)(C_p(K)^{\frac{\mu}{p\nu}} + sC_p(L)^{\frac{\mu}{p\nu}}). \)
2. \( C_p((1-s) \cdot K + p s \cdot L) \geq C_p(K)^{1-s}C_p(L)^s. \)
3. \( C_p((1-s) \cdot K + p s \cdot L) \geq \min \{C_p(K), C_p(L)\}. \)
4. If \( C_p(K) = C_p(L) = 1, \) then \( C_p((1-s) \cdot K + p s \cdot L) \geq 1. \)

Recall that \( K + \infty L = \text{conv}(K \cup L) \). From the monotonicity of \( C_p \), it yields that

\[ C_p(K + \infty L) \geq \max \{C_p(K), C_p(L)\}. \]

In fact, from the continuity of \( K + \infty L \) in \( p \) and the continuity of \( C_p \) on \( K^n_o \), the inequality (3.2) will become the above, as \( p \to \infty \).

3.3. Uniqueness of the \( L_p \) p-capacitary measures. In this part, we show an immediate application of the \( L_p \) Minkowski inequality for p-capacity to the uniqueness of the \( L_p \) Minkowski problem for p-capacity, which is closely related with the following question:

If \( K, L \in K^n_o \) are such that \( \mu_{p,p}(K, \cdot) = \mu_{p,p}(L, \cdot) \), then is this the case that \( K = L ? \)

Theorems 3.14 (2) and 3.16 (2) affirm this question. In fact, we show a series of characterizations for identity of convex bodies.

Theorem 3.14. Suppose that \( K, L \in K^n_o \) and \( C \) is a subset of \( K^n_o \) such that \( K, L \in C \). Let \( 1 < p < \infty, 1 < q < n \) and \( n - p \neq p \). Then the following assertions hold.

1. If \( C_{p,p}(K, Q) = C_{p,p}(L, Q) \) for all \( Q \in C \), then \( K = L \).
2. If \( \mu_{p,p}(K, \cdot) = \mu_{p,p}(L, \cdot) \), then \( K = L \).
3. If \( C_{p,p}(Q, K) = C_{p,p}(Q, L) \) for all \( Q \in C \), then \( K = L \).

Proof. Since \( C_{p,p}(K, K) = C_p(K) \), it follows that \( C_{p,p}(L, K) = C_p(K) \) by the assumption. By the p-capacitary Minkowski inequality \( C_{p,p}(K, L) \geq C_p(L)^{(n-p-\nu)/(n-p)}C_p(K)^{\nu/(n-p)} \), we have

\[ C_p(K)^{\frac{n-p-\nu}{n-p}} \geq C_p(L)^{\frac{n-p-\nu}{n-p}}, \]

with equality if and only if \( K \) and \( L \) are dilates. This inequality is reversed if interchanging \( K \) and \( L \). So, \( C_p(K) = C_p(L) \), and \( K \) and \( L \) are dilates. Assume that \( K = sL \), for some \( s > 0 \). Since \( C_p(sL) = s^{n-p}C_p(L) \), it follows that \( s = 1 \). Thus, \( K = L \).
If $\mu_{p,p}(K,\cdot) = \mu_{p,p}(L,\cdot)$, then $C_{p,p}(K,Q) = C_{p,p}(L,Q)$ for any $Q \in K^n_o$. Thus, $K = L$ by (1). (3) can be proved by the similar arguments in (1). □

If $p = 1$ in Theorem 3.14, then $K$ and $L$ are translates each other.

**Theorem 3.15.** Suppose that $K, L \in K^n_o$ are such that $\mu_{p,p}(K,\cdot) \leq \mu_{p,p}(L,\cdot)$. Let $1 < p < \infty$, $1 < p < n$ and $n - p \neq p$. Then the following assertions hold.

1. If $C_p(K) \geq C_p(L)$ and $p < n - p$, then $K = L$.
2. If $C_p(K) \leq C_p(L)$ and $p > n - p$, then $K = L$.

**Proof.** From $C_{p,p}(L,L) = C_p(L)$, together with the assumption $\mu_{p,p}(K,\cdot) \leq \mu_{p,p}(L,\cdot)$ and Definition 3.4 the $p$-capacitary Minkowski inequality, and the assumptions in (1) or (2), we have

$$C_p(L) \geq C_{p,p}(K,L) \geq C_p(K)^{\frac{n-p}{n-p}}C_p(L)^{\frac{p}{n-p}} \geq C_p(L)^{\frac{n-p-p}{n-p}}C_p(L)^{\frac{p}{n-p}} = C_p(L).$$

Thus, $C_p(K) = C_p(L)$, and $K$ and $L$ are dilates. Hence, $K = L$. □

When $n - p = p$, we have the following result.

**Theorem 3.16.** Suppose that $K, L \in K^n_o$ and $C$ is a subset of $K^n_o$ such that $K, L \in C$. Let $1 < p < \infty$ and $1 < p < n$. Then the following assertions hold.

1. If $C_{n-p,p}(K,Q) \geq C_{n-p,p}(L,Q)$ for all $Q \in C$, then $K$ and $L$ are dilates.
2. If $\mu_{n-p,p}(K,\cdot) \geq \mu_{n-p,p}(L,\cdot)$, then $K$ and $L$ are dilates. Therefore, $\mu_{n-p,p}(K,\cdot) = \mu_{n-p,p}(L,\cdot)$.

**Proof.** Take $Q = K$. From the fact $C_{n-p,p}(K,K) = C_p(K)$, the assumption in (1) and the $p$-capacitary Minkowski inequality, we have

$$C_p(K) \geq C_{n-p,p}(L,K) \geq C_p(K).$$

Thus, all the equalities in the above hold and $K$ and $L$ are dilates by the equality condition of the $p$-capacitary Minkowski inequality. Incidentally, we obtain $\mu_{n-p,p}(K,\cdot) = \mu_{n-p,p}(L,\cdot)$ by Lemma 3.8 With (1) in hand, (2) can be derived directly. □

### 4. The discrete $L_p$ Minkowski problem for $p$-capacity

Throughout this section, let $1 < p < \infty$ and $1 < p < n$. Suppose that $\xi_1, \ldots, \xi_m \in \mathbb{S}^{n-1}$ are pairwise distinct and not contained in a closed hemisphere, and $c_1, \ldots, c_m$ are positive numbers. Denote by $\delta_{\xi_i}$ the probability measure with unit point mass at $\xi_i$. We focus on the following.

**Problem 1.** Among all polytopes in $\mathbb{R}^n$ with the origin in their interiors, find a polytope $P$ such that $\frac{\mu_{p,p}(P,\cdot)}{C_p(P)} = \sum_{i=1}^{m} c_i \delta_{\xi_i}$.

We present a solution to Problem 1.
Lemma 4.3. Let $P$ be a polytope and $y$ a vector. Then the $p$-capacity of $P$ with respect to $y$ is given by

$$
\frac{\mu_{p,p}(P, \cdot)}{C_p(P)} = \sum_{i=1}^{m} c_i \delta_{\xi_i}.
$$

Proof. By Aleksandrov’s convergence theorem, $\mu_{p,p}(P, \cdot) / C_p(P)$ is continuous with respect to $\delta_{\xi_i}$. Thus for each $i$, there exists a unique convex polytope $P_i \in \mathcal{K}_0^n$ such that

$$
\mu_{p,p}(P, \cdot) = \sum_{i=1}^{m} c_i \delta_{\xi_i}.
$$

To prove this theorem, we need to make some preparations. Let $\mathbb{R}_+^m = [0, \infty)^m$. For each nonzero $y = (y_1, \ldots, y_m) \in \mathbb{R}_+^m$, define

$$
P(y) = \bigcap_{i=1}^{m} \{ x \in \mathbb{R}^n, x \cdot \xi_i \leq y_i \}.
$$

Then the unit outer normals to facets of $P(y)$ belong to $\{\xi_1, \ldots, \xi_m\}$, and $P(y)$ is a polytope containing $o$. Since $\mu_p(P(y), \cdot)$ is absolutely continuous with respect to $S_{P(y)}$, we have

$$
C_p(P(y), P(z)) = \frac{p-1}{n-p} \sum_{i=1}^{m} h_{P(z)}(\xi_i) \mu_p(P(y), \{\xi_i\}).
$$

Since $h_{P(y)}(\xi_i) \leq y_i$, with equality if $S_{P(y)}(\{\xi_i\}) > 0$, for $i = 1, \ldots, m$, we have

$$
C_p(P(y)) = \frac{p-1}{n-p} \sum_{i=1}^{m} y_i \mu_p(P(y), \{\xi_i\}).
$$

To solve Problem 1, our strategy is to attack the following Problem 2. In the proof of Theorem 5.1, we can see that Problem 1 is essentially solved once we solve Problem 2. Precisely, we show that Problem 1 and Problem 2 have the identical solution.

Problem 2. Among all elements $y$ in $\mathbb{R}_+^m$, find an element which solves the following constrained maximization problem

$$
\max_{y} C_p(P(y)) \quad \text{subject to} \quad \frac{p-1}{n-p} \sum_{i=1}^{m} c_i y_i^n = 1.
$$

Lemma 4.2. $C_p(P(y))$ is continuous with respect to $y \in \mathbb{R}_+^m \setminus \{o\}$.

Proof. By Aleksandrov’s convergence theorem, $P(y)$ is continuous with respect to $y \in \mathbb{R}_+^m \setminus \{o\}$. So, by the continuity of $p$-capacity with respect to the Hausdorff metric, $C_p(P(y))$ is continuous with respect to $y \in \mathbb{R}_+^m \setminus \{o\}$.

Lemma 4.3. $P \left( \frac{y' + y''}{2} \right) \supseteq \frac{1}{2} P(y') + \frac{1}{2} P(y'')$, for any nonzero $y', y'' \in \mathbb{R}_+^m$.

Proof. Let $x \in \frac{1}{2} P(y') + \frac{1}{2} P(y'')$. Then there exist $x' \in P(y')$ and $x'' \in P(y'')$, such that $x = \frac{x' + x''}{2}$ and for each $i$,

$$
x' \cdot \xi_i \leq y'_i \quad \text{and} \quad x'' \cdot \xi_i \leq y''_i.
$$

Thus for each $i$, we have

$$
x \cdot \xi_i = \frac{x' + x''}{2} \cdot \xi_i \leq \frac{y'_i + y''_i}{2},
$$

which implies that $x \in P \left( \frac{y' + y''}{2} \right)$. \hfill \Box
To prove Lemma 4.4, we adopt the elegant deformation technique, which was previously employed by Hug and LYZ [38].

Lemma 4.4. If \( y \in \mathbb{R}^m \) solves Problem 2, then \( o \in \text{int} P(y) \).

Proof. We argue by contradiction and assume that \( o \in \partial P(y) \). Let \( y = (y_1, \ldots, y_m) \) and \( h_i = h_{P(y)}(\xi_i) \), for \( i = 1, \ldots, m \). Since \( o \in \partial P(y) \), w.l.o.g., assume that \( h_1 = \cdots = h_k = 0 \) and \( h_{k+1}, \ldots, h_m > 0 \), for some \( 1 \leq k < m \). In the next, we will construct a new polytope \( P(z) \) with \( o \) in its interior, such that \( z \) satisfies the constraint in Problem 2 but \( C_p(P(z)) > C_p(P(y)) \).

Let \( c = \sum_{i=1}^k c_i / \sum_{i=k+1}^m c_i \) and \( t_0 = \min \{h_i^p / c : 1 \leq i \leq k\}^{1/p} \). For \( 0 \leq t < t_0 \), let

\[
y_t = \left(t, \ldots, t, (h_{k+1}^p - ct^p)^{1/p}, \ldots, (h_m^p - ct^p)^{1/p}\right).
\]

Then, \( y_t \in (0, \infty)^m \) for \( 0 < t < t_0 \) and \( P(y_0) = P(y) \). From (4.1) combined with (4.2), and then the fact \( \lim_{t \to 0^+} P(y_t) = P(y) \) combined with the weak convergence of \( p \)-capacitary measures, we have

\[
\lim_{t \to 0^+} \frac{C_p(P(y_t)) - C_p(P(y_t), P(y))}{t} = \frac{p - 1}{n - p} \left( \sum_{i=1}^k \lim_{t \to 0^+} \frac{t - 0}{t} \mu_p(P(y_t), \{\xi_i\}) + \sum_{i=k+1}^m \lim_{t \to 0^+} \frac{(h_i^p - ct^p)^{1/p} - h_i}{t} \mu_p(P(y_t), \{\xi_i\}) \right)
\]

\[
= \frac{p - 1}{n - p} \sum_{i=1}^k \mu_p(P(y), \{\xi_i\}).
\]

Since there is at least one facet of \( P(y) \) containing \( o \), it follows that \( \sum_{i=1}^k S_{P(y)}(\{\xi_i\}) > 0 \). Also, by CNSXYZ [19], Lemma 2.18, there exists a positive constant \( c \) depending on \( n, p \) and the radius of a ball containing \( P(y) \), such that \( \mu_p(P(y), \cdot) \geq c^{-p} S_{P(y)} \). So, \( \sum_{i=1}^k \mu_p(P(y), \{\xi_i\}) > 0 \). This in turn implies that

\[
\lim_{t \to 0^+} \frac{C_p(P(y_t)) - C_p(P(y_t), P(y))}{t} > 0.
\]

Hence, by the \( p \)-capacitary Minkowski inequality and continuity of \( C_p(P(y_t)) \) in \( t \), we have

\[
C_p(P(y))^{n-p-1} \liminf_{t \to 0^+} \frac{C_p(P(y_t))^{1/n-p} - C_p(P(y))^{1/n-p}}{t}
\]

\[
= \liminf_{t \to 0^+} \frac{C_p(P(y_t)) - C_p(P(y))}{t} \frac{C_p(P(y))^{n-p-1}}{n-p} C_p(P(y))^{1/n-p}
\]

\[
\geq \liminf_{t \to 0^+} \frac{C_p(P(y_t)) - C_p(P(y), P(y))}{t}
\]

\[
> 0.
\]

Consequently, for sufficiently small \( t \), we have \( C_p(P(y_t)) > C_p(P(y)) \).

Now, choose a sufficiently small \( t > 0 \) and let

\[
z = \left((y_1^p + t^p)^{1/p}, \ldots, (y_k^p + t^p)^{1/p}, (y_{k+1}^p - ct^p)^{1/p}, \ldots, (y_m^p - ct^p)^{1/p}\right).
\]
Then $z$ satisfies the constraint in Problem 2. Since $0 < h_i \leq y_i$, $k + 1 \leq i \leq m$, it follows that $P(y_t) \subseteq P(z)$. So, $C_p(P(z)) > C_p(P(y))$. In light of $o \in \text{int}P(y_t)$, it yields that $o \in \text{int}P(z)$. □

Let $y = (y_1, \ldots, y_m) \in \mathbb{R}^m_+(0, +\infty)^m$. For $z \in \mathbb{R}^m$, applying the Hadamard variational formula to $P(y + tz)$, it yields that

$$
\frac{dC_p(P(y + tz))}{dt} \bigg|_{t=0} = (p - 1) \sum_{i=1}^{m} z_i \mu_p(P(y), \{\xi_i\}).
$$

Thus, we obtain the following useful formula.

**Lemma 4.5.** $\frac{\partial C_p(P(y))}{\partial y_i} = (p - 1) \mu_p(P(y), \{\xi_i\})$, for $y \in \mathbb{R}^m_+$ and $i = 1, \cdots, m$.

**Lemma 4.6.** Suppose $1 < p < \infty$ and $1 < p < n$. If $K, L \in \mathcal{K}_n$ are such that $C_p(K)^{-1} \mu_p(K, \cdot) = C_p(L)^{-1} \mu_p(L, \cdot)$, then $K = L$.

**Proof.** From the Poincaré $p$-capacity formula together with Definition 3.4, the supposition that $C_p(K)^{-1} \mu_p(K, \cdot) = C_p(L)^{-1} \mu_p(L, \cdot)$, Definitions 3.4 and 3.5, and Theorem 3.9, it follows that

$$
1 = \frac{p - 1}{(n - p)C_p(L)} \int_{S^{n-1}} h_L^p d\mu_p(L, \cdot)
= \frac{p - 1}{(n - p)C_p(K)} \int_{S^{n-1}} h_K^p d\mu_p(K, \cdot)
= \frac{C_{p,p}(K, L)}{C_p(K)}
\geq \left( \frac{C_p(L)}{C_p(K)} \right)^{\frac{p}{n-p}}.
$$

Thus, $C_p(K) \geq C_p(L)$. Interchanging $K$ and $L$, we have $C_p(L) \geq C_p(K)$. So, by Theorem 3.9 the convex bodies $K$ and $L$ are dilates, so that $C_p(K) = C_p(L)$. In other words, $K = L$. □

What follows provides the proof of Theorem 4.1

**Proof of Theorem 4.1** Let

$$
\mathcal{B} = \left\{ y \in \mathbb{R}^m_+ : \frac{p - 1}{n - p} \sum_{i=1}^{m} c_i y_i^p \leq 1 \right\}
$$

and

$$
\mathcal{E}_t = \left\{ y \in \mathbb{R}^m_+ : C_p(P(y)) \geq t \right\}, \text{ for } t > 0.
$$

Then $\mathcal{B}$ is a convex body in $\mathbb{R}^m$. By Lemma 4.2 $\mathcal{E}_t$ is a closed set.
Pick up $y', y'' \in \mathcal{E}_t$. From Lemma 4.3, the monotonicity of $p$-capacity and the $p$-capacitary Brunn-Minkowski inequality, it follows that

$$C_p \left( P \left( \frac{y' + y''}{2} \right) \right) \geq C_p \left( \frac{1}{2} P(y') + \frac{1}{2} P(y'') \right) \geq \left( \frac{1}{2} C_p(P(y'))^{\frac{1}{n-p}} + \frac{1}{2} C_p(P(y''))^{\frac{1}{n-p}} \right)^{n-p} = t,$$

which implies that $\frac{y' + y''}{2} \in \mathcal{E}_t$. Hence, $\mathcal{E}_t$ is convex. Since $C_p(P(sy)) = s^{n-p}C_p(P(y))$, for nonzero $y \in \mathbb{R}^n$ and $s > 0$, it follows that $\mathcal{E}_t$ is unbounded and strictly decreasing (with respect to set inclusion) when $t$ is increasing, and its interior is nonempty. So, when $t$ is sufficiently big, $\mathcal{E}_t \cap B = \emptyset$; when $t$ is sufficiently small, $\text{int}(\mathcal{E}_t) \cap \text{int}(B) \neq \emptyset$.

Consequently, there exists a unique $t_0 > 0$ such that $\mathcal{E}_t \cap B = \partial \mathcal{E}_t \cap \partial B$. Since the set $\{y \in \mathbb{R}^m : \frac{p-1}{n-p} \sum_{i=1}^m c_i |y_i|^p \leq 1\}$ is a strictly convex body in $\mathbb{R}^m$ with smooth boundary, the sets $\mathcal{E}_{t_0}$ and $B$ necessarily share a unique common boundary point, say $\tilde{y}$. In other words, for any $y \in \partial B$, we have

$$C_p(P(\tilde{y})) \geq C_p(P(y)),$$

with equality if and only if $y = \tilde{y}$. This proves the unique existence of solution to Problem 2.

We proceed to prove that $P(\tilde{y})$ uniquely solves Problem 1.

By Lemma 4.4, the polytope $P(\tilde{y})$ contains the origin in its interior. Therefore, $\tilde{y} \in \mathbb{R}^m$. Since

$$\nabla \left( \sum_{i=1}^m c_i y_i^p \right) |_{\tilde{y}}$$

is a normal of $B$ at $\tilde{y}$ with components $pc_i \tilde{y}_i^{p-1}$, and $\nabla C_p(P(y)) |_{\tilde{y}}$ is a normal of $\mathcal{E}_{t_0}$ at $\tilde{y}$ with components $(p-1)\mu_p(P(\tilde{y}), \{\xi_i\})$ by Lemma 4.5 so there exists a unique $s_0 > 0$ such that for each $i$, $c_i \tilde{y}_i^p = s_0 \tilde{y}_i \mu_p(P(\tilde{y}), \{\xi_i\})$. Since for each $i$, $c_i > 0$ and $\tilde{y}_i > 0$, this in turn implies that $\mu_p(P(\tilde{y}), \{\xi_i\}) > 0$. In light of $\mu_p(P(\tilde{y}), \cdot)$ is absolutely continuous with respect to $S_{P(\tilde{y})}$, so each $\xi_i$ is a unit normal of $P(\tilde{y})$. Hence, $h_{P(\tilde{y})}(\xi_i) = y_i$, for each $i$. Consequently,

$$s_0 C_p(P(\tilde{y})) = s_0 \cdot \frac{p-1}{n-p} \sum_{i=1}^m \tilde{y}_i \mu_p(P(\tilde{y}), \{\xi_i\})$$

$$= \frac{p-1}{n-p} \sum_{i=1}^m s_0 \tilde{y}_i \mu_p(P(\tilde{y}), \{\xi_i\})$$

$$= \frac{p-1}{n-p} \sum_{i=1}^m c_i \tilde{y}_i^p$$

$$= 1,$$

which yields that

$$s_0 = \frac{1}{C_p(P(\tilde{y}))},$$
Furthermore,

\[
\sum_{i=1}^{m} c_i \delta_{\xi_i} = \frac{\sum_{i=1}^{m} \frac{1}{p} \mu_p(P(\bar{y}), \{\xi_i\}) \delta_{\xi_i}}{C_p(P(\bar{y}))}
\]

\[
= \frac{\sum_{i=1}^{m} h_{P(\bar{y})}(\xi_i)^{1-p} \mu_p(P(\bar{y}), \{\xi_i\}) \delta_{\xi_i}}{C_p(P(\bar{y}))}
\]

\[
= \frac{\mu_{p,p}(P(\bar{y}), \cdot)}{C_p(P(\bar{y}))}.
\]

Put it in other words, \(P(\bar{y})\) is a solution to Problem 1, and is unique by Lemma 4.6.

From Theorem 4.1 we immediately obtain the following results.

**Corollary 4.7.** Suppose \(1 < p < \infty\), \(1 < p < n\) and \(n - p \neq 0\). If \(\mu\) is a finite discrete measure on \(S^{n-1}\) which is not concentrated on a closed hemisphere, then there exists a unique convex polytope \(P \in \mathcal{K}^n_o\) such that \(\mu_{p,p}(P, \cdot) = \mu\).

**Proof.** By Theorem 4.1 there exists a unique convex polytope \(P^* \in \mathcal{K}^n_o\), such that \(\frac{\mu_{p,p}(P^*, \cdot)}{C_p(P^*)} = \mu\). Let \(P = C_p(P^*)^{-\frac{1}{1-p}} P^*\). Then,

\[
\mu = \frac{\mu_{p,p} \left( C_p(P^*)^{-\frac{1}{1-p}} P, \cdot \right)}{C_p(P^*)} = \frac{C_p(P^*) \mu_{p,p}(P, \cdot)}{C_p(P^*)} = \mu_{p,p}(P, \cdot),
\]

as desired.

The following lemma shows the solution to the even \(L_p\) Minkowski problem for \(p\)-capacity is symmetric.

**Lemma 4.8.** Suppose \(1 \leq p < \infty\) and \(1 < p < n\). If \(K \in \mathcal{K}^n_o\), then the following statements are equivalent.

1. \(K\) is origin-symmetric when \(p > 1\), or centrally symmetric when \(p = 1\).
2. \(\mu_{p,p}(K, \cdot)\) is even.
3. \(C_{p,p}(K, -Q) = C_{p,p}(K, Q)\), for all \(Q \in \mathcal{K}^n_o\).
4. \(C_{p,p}(K, -K) = C_p(K)\).

**Proof.** When \(p = 1\), the implication “(1) \(\Rightarrow\) (2)” is obvious. When \(p > 1\), the implication “(1) \(\Rightarrow\) (2)” follows from the facts that \(\mu_p(K, \cdot)\) is even, \(h_K = h_{-K}\) and Definition 3.5.

The implication “(2) \(\Rightarrow\) (3)” follows from Definition 3.4 and the fact that \(h_Q(-\xi) = h_{-Q}(\xi)\) for all \(\xi \in S^{n-1}\).

The implication “(3) \(\Rightarrow\) (4)” is obvious, since \(C_{p,p}(K, -K) = C_{p,p}(K, K) = C_p(K)\).

Assume that \(C_{p,p}(K, -K) = C_p(K)\). From the \(p\)-capacitary Minkowski inequality and the fact \(C_p(K) = C_p(-K)\), it follows that

\[
C_p(K) = C_{p,p}(K, -K) \geq C_p(K)^{\frac{n-p}{1-p}} C_p(-K)^{\frac{p}{1-p}} = C_p(-K).
\]

So, \(K\) and \(-K\) are dilates when \(p > 1\), or homothetic when \(p = 1\). \(\square\)
Corollary 4.9. Suppose \( 1 < p < \infty \), \( 1 < p < n \) and \( n - p \neq p \). If \( \mu \) is a finite even discrete measure on \( S^{n-1} \) which is not concentrated on any great subsphere, then there exists a unique origin-symmetric convex polytope \( P \in K_n^o \) such that \( \mu_{p,p}(P, \cdot) = \mu \).

Proof. Since \( \mu \) is even and not concentrated on any great subsphere, it is not concentrated on any closed hemisphere. By Corollary 4.7, there exists a unique polytope \( P \in K_n^o \) such that \( \mu_{p,p}(P, \cdot) = \mu \). Since \( \mu_{p,p}(P, \cdot) \) is even, it implies that \( P \) is origin-symmetric by Lemma 4.8. \( \square \)

5. Revisiting the discrete Minkowski problem for \( p \)-capacity: CNSXYZ’s problem

Let \( \mu \) be a finite Borel measure on the unit sphere \( S^{n-1} \). Consider the following conditions.

\((A_1)\) The measure \( \mu \) is not concentrated on any great subsphere.

\((A_2)\) The centroid of \( \mu \) is at the origin.

\((A_3)\) The measure \( \mu \) does not have a pair of antipodal point masses; that is, if \( \mu(\{\xi\}) > 0 \), then \( \mu(\{-\xi\}) = 0 \), for \( \xi \in S^{n-1} \).

Under these conditions, CNSXYZ [19, pp. 1570-1572] proved the following important result.

**Theorem A.** Suppose \( 1 < p < 2 \leq n \). If \( \mu \) is a finite Borel measure on \( S^{n-1} \) satisfying conditions \((A_1)-(A_3)\), then there exists a convex body \( K \in \mathbb{R}^n \) such that \( \mu_{p,p}(K, \cdot) = \mu \).

Conditions \((A_1)\) and \((A_2)\) are both necessary. They are exactly the same sufficient and necessary conditions as in Jerison’s solution to the Minkowski problem for electrostatic capacity [39], as well as in the Aleksandrov [1] and Fenchel and Jessen’s [21] solution to the classical Minkowski problem for the surface area measure.

CNSXYZ [19] emphasized that \((A_3)\) is instead not a necessary condition. They pointed out that: It would be interesting if the assumption \((A_3)\) could be removed, and it is a very interesting open problem to naturally extend their result to the range \( 2 < p < n \).

In this part, we solve CNSXYZ’s problem for discrete measures.

**Theorem 5.1.** Suppose \( 1 < p < n \). If \( \mu \) is a discrete measure on \( S^{n-1} \) satisfying conditions \((A_1)\) and \((A_2)\), then there exists a unique (up to a translation) polytope \( P \) such that

\[ \frac{\mu_{p}(P, \cdot)}{C_{p}(P)} = \mu. \]

If in addition \( \mu \) is even, then \( P \) is centrally symmetric.

Proof. The argument is similar to the proof of Theorem 4.1 so we have to use the notations and lemmas provided in Section 5. Represent \( \mu \) as the form \( \sum_{i=1}^{m} c_i \delta_{\xi_i} \), where \( c_1, \ldots, c_m > 0 \), and \( \xi_1, \ldots, \xi_m \) are unit vectors which are not contained on any great subsphere.

We start with considering the simplex

\[ S = \left\{ y \in \mathbb{R}^m : \frac{p}{n-p} \sum_{i=1}^{m} c_i y_i = 1 \right\}. \]

By Lemma 4.2 and the compactness of \( S \), the functional \( C_{p}(P(y)) \) can attain its maximum on \( S \) at a point \( z \), say \( z = (z_1, \ldots, z_m) \).
If $z \notin \text{relint} S$ (i.e., $z$ is not a relative interior point of $S$), then at least one $z_i$ is 0, and therefore $o \in \partial P(z)$. Choose a nonzero $\Delta z \in \mathbb{R}^m$, such that $o \in \text{int}(P(z) + \Delta z)$. Let

$$
\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_m) = z + (\xi_1 \cdot \Delta z, \ldots, \xi_m \cdot \Delta z).
$$

Then,

$$
P(\tilde{y}) = \{x \in \mathbb{R}^n : \xi_i \cdot x \leq \tilde{y}_i, \text{ for } i = 1, \ldots, m\}
= \{x \in \mathbb{R}^n : \xi_i \cdot x \leq z_i + \xi_i \cdot \Delta z, \text{ for } i = 1, \ldots, m\}
= \{x \in \mathbb{R}^n : \xi_i \cdot x \leq z_i, \text{ for } i = 1, \ldots, m\} + \Delta z
= P(z) + \Delta z.
$$

Since $o \in \text{int}(P(z) + \Delta z)$, it follows that

$$
\tilde{y}_1 > 0, \ldots, \tilde{y}_m > 0.
$$

From $z \in S$ and the centroid of $\sum_{i=1}^m c_i \delta_{\xi_i}$ is at the origin, it follows that

$$
\sum_{i=1}^m c_i \tilde{y}_i = \sum_{i=1}^m c_i (z_i + \xi_i \cdot \Delta z)
= \sum_{i=1}^m c_i z_i + \left( \sum_{i=1}^m c_i \xi_i \right) \cdot \Delta z
= \frac{n-p}{p-1} + o \cdot \Delta z
= \frac{n-p}{p-1},
$$

i.e., $\frac{n-p}{p-1} \sum_{i=1}^m c_i \tilde{y}_i = 1$, which implies that $\tilde{y} \in S$. Hence, $C_p/(P(\tilde{y}))$ attains its maximum on $S$ at a relative interior point $\tilde{y}$.

By Lemma 4.5 and the Lagrange multiplier theorem, there exists a suitable constant $s$, such that for each $i = 1, \ldots, m$,

$$
\frac{\partial}{\partial y_i} \left( \frac{C_p(P(\tilde{y}))}{p-1} - s \sum_{i=1}^m c_i \tilde{y}_i \right) = \mu_p(P(\tilde{y}), \{\xi_i\}) - sc_i = 0.
$$

Since $P(\tilde{y})$ is $n$-dimensional and all the $\tilde{y}_i$ are positive, there is at least one $i_0$ such that $S_{P(\tilde{y})}(\{\xi_{i_0}\}) > 0$. Meanwhile, by CNSXYZ [19] Lemma 2.18, there is a positive constant $c$ depending on $n$, $p$ and and the radius of a ball containing $P(\tilde{y})$, such that $\mu_p(P(\tilde{y}), \cdot) \geq c^{-p} S_{P(\tilde{y})}$. So, $\mu_p(P(\tilde{y}), \{\xi_{i_0}\}) > 0$, which implies that $s > 0$, and therefore $\mu_p(P(\tilde{y}), \{\xi_i\}) > 0$ for all $i$. In light of $\mu_p(P(\tilde{y}), \cdot)$ is absolutely continuous with respect to $S_{P(\tilde{y})}$, it follows that $S_{P(\tilde{y})}(\{\xi_i\}) > 0$ for all $i$. So, each $\xi_i$ is an outer unit normal to the facet of $P(\tilde{y})$, and $h_{P(\tilde{y})}(\xi_i) = \tilde{y}_i$. 


Hence,
\[
C_p(P(\tilde{y})) = \frac{p - 1}{n - p} \sum_{i=1}^{m} \tilde{y}_i \mu_p(P(\tilde{y}), \{\xi_i\}) \\
= s \cdot \frac{p - 1}{n - p} \sum_{i=1}^{m} \tilde{y}_i c_i \\
= s.
\]

Therefore,
\[
\mu = m \sum_{i=1}^{m} c_i \delta_{\xi_i} = s^{-1} \sum_{i=1}^{m} s c_i \delta_{\xi_i} = \frac{\sum_{i=1}^{m} \mu_p(P(\tilde{y}), \{\xi_i\})}{C_p(P(\tilde{y}))}.
\]

Take \(P = P(\tilde{y})\). Then \(P\) is a desired polytope of this theorem.

What follows shows the uniqueness. Assume the polytope \(P'\) satisfies \(C_p(P')^{-1} \mu_p(P', \cdot) = \mu\). We will show that \(P\) and \(P'\) differ only by a translation.

From the Poincaré \(p\)-capacity formula, the assumptions that \(\mu = C_p(P')^{-1} \mu_p(P', \cdot)\) and \(\mu = C_p(P)^{-1} \mu_p(P, \cdot)\), the definition of mixed \(p\)-capacity, and finally the \(p\)-capacitary Minkowski inequality, it follows that
\[
1 = \frac{\frac{p-1}{n-p} \int_{S^{n-1}} h_{P'} d\mu_p(P', \cdot)}{C_p(P')} \\
= \frac{p - 1}{n - p} \int_{S^{n-1}} h_{P'} d\mu \\
= \frac{\frac{p-1}{n-p} \int_{S^{n-1}} h_{P'} d\mu_p(P, \cdot)}{C_p(P)} \\
= \frac{C_p(P, P')}{C_p(P)} \\
\geq \left( \frac{C_p(P')}{C_p(P)} \right)^{\frac{1}{p-1}}.
\]

All the above still hold, if interchanging \(P\) and \(P'\). So, \(C_p(P') = C_p(P)\). By the equality condition of the \(p\)-capacitary Minkowski inequality, \(P\) and \(P'\) differ only by a translation.

Assume that \(\mu\) is even. Since \(\mu = C_p(P)^{-1} \mu_p(P, \cdot)\), it follows that the \(p\)-capacitary measure \(\mu_p(P, \cdot)\) is even. By Theorem 4.8 the polytope \(P\) is centrally symmetric. \(\square\)

6. Two dual extremum problems for \(p\)-capacity

Throughout this section, let \(1 < p < \infty\) and \(1 < p < n\). Suppose that \(\mu\) is a finite Borel measure on \(S^{n-1}\), which is not concentrated on any closed hemisphere. We focus on the general \(L_p\) Minkowski problem for \(p\)-capacity.
**Problem 3.** Among all convex bodies $Q$ in $\mathbb{R}^n$ containing the origin, find a body to solve the following constrained maximization problem

$$\sup_{Q} C_p(Q) \quad \text{subject to} \quad F_p(Q) = 1.$$  

Here,

$$F_p(Q) = \frac{p - 1}{n - p} \int_{S^{n-1}} h_Q^p d\mu.$$  

Naturally, we also consider the dual problem of Problem 3.

**Problem 4.** Among all convex bodies $Q$ in $\mathbb{R}^n$ containing the origin, find a body to solve the following constrained minimization problem

$$\inf_{Q} F_p(Q) \quad \text{subject to} \quad C_p(Q) = 1.$$  

When $p = 1$ and $p = 2$, Problem 4 is the Minkowski problem for classical Newtonian capacity, which was solved by Jerison [39], and Caffarelli, Jerison and Lieb [11]. When $p = 1$ and $1 < p < 2$, Problem 4 was solved by CNSXYZ [19]. For $p > 1$, Problem 4 is totally new.

In Section 8, we will solve the general $L_p$ ($p > 1$) Minkowski problem for $p$-capacity (i.e., Problem 5) with $1 < p \leq 2$, under the basis of Theorem [4,1]. To achieve this goal, our strategy is first to demonstrate the duality of Problem 3 and Problem 4, in the sense that their solutions only differ by a scale factor. Then we show that Problem 5 is equivalent to Problem 3, in the sense that their solutions are identical.

**Lemma 6.1.** (1) If convex body $K$ solves Problem 3, then convex body

$$\tilde{K} = \frac{K}{C_p(K)^{\frac{1}{n-p}}}$$

solves Problem 4.

(2) If convex body $\tilde{K}$ solves Problem 4, then convex body

$$K = \frac{\tilde{K}}{F_p(\tilde{K})^\frac{1}{p}}$$

solves Problem 3.

**Proof.** (1) Assume that $K$ solves Problem 3. Let $Q$ be a convex body containing the origin such that $C_p(Q) = 1$. Since $F_p(K) = 1$ and $F_p(\frac{Q}{F_p(Q)^\frac{1}{p}}) = 1$, we have

$$F_p(\tilde{K}) = F_p\left(\frac{K}{C_p(K)^{\frac{1}{n-p}}}\right) = F_p(\frac{K}{C_p(K)^{\frac{1}{n-p}}}) = \frac{1}{C_p(K)^{\frac{1}{n-p}}} \leq \frac{1}{C_p(\frac{Q}{F_p(Q)^\frac{1}{p}})^{\frac{1}{n-p}}} = \frac{F_p(Q)}{C_p(Q)^{\frac{1}{n-p}}} = F_p(Q),$$

which shows that $\tilde{K}$ solves Problem 4.
(2) Assume that $\bar{K}$ solves Problem 4. Let $Q$ be a convex body containing the origin such that $F_p(Q) = 1$. Since $C_p(\bar{K}) = 1$ and $C_p(\frac{Q}{C_p(Q)^{\frac{1}{p}}}) = 1$, we have

$$C_p(K)^{\frac{n}{n-p}} = \frac{C_p(\bar{K})^{\frac{n}{n-p}}}{F_p(\bar{K})} = \frac{1}{F_p(\bar{K})} \geq \frac{C_p(Q)^{\frac{n}{n-p}}}{F_p(Q)} = C_p(Q)^{\frac{n}{n-p}},$$

which shows that $K$ solves Problem 3.

\[\Box\]

**Lemma 6.2.** If $\mu$ is a discrete measure, then Problem 3 and Problem 2 are identical.

**Proof.** Assume that $\mu$ is a discrete measure, say, $\mu = \sum_i c_i \delta_{\xi_i}$. For any convex body $Q$ containing the origin, since $[h_Q|_{\text{supp}\ \mu}] \supseteq Q$, it follows that $C_p([h_Q|_{\text{supp}\ \mu}]) \geq C_p(Q)$. Since $F([h_Q|_{\text{supp}\ \mu}]) = F(Q) = 1$, it follows that the domain of Problem 3 can be restricted to the class of proper convex polytopes $P(y)$ generated by

$$P(y) = \bigcap_{i=1}^m \{x \in \mathbb{R}^n, x \cdot \xi_i \leq y_i\},$$

for $y = (y_1, \ldots, y_m) \in \mathbb{R}^m_+$. \[\Box\]

Therefore, for a discrete measure $\mu$, Problem 3 and Problem 2, even further as well as Problem 1 (i.e., the discrete $L_p$ Minkowski problem for $p$-capacity) have the same unique solution. A generalization of Problem 1 is as follows.

**Problem 5.** Among all convex bodies in $\mathbb{R}^n$ that contain the origin, find a body $K$ such that

$$d\mu_p(K, \cdot) = \frac{h_p^{p-1} d\mu}{C_p(K)}.$$

The equivalence between Problem 3 and Problem 5 is shown by the next lemma.

**Lemma 6.3.** Let $1 < p < \infty$ and $1 < p < n$. Suppose that $\mu$ is a finite Borel measure on $\mathbb{S}^{n-1}$ and is not concentrated on any closed hemisphere. Then a convex body $K$ solves Problem 3, if and only if $K$ solves Problem 5. Moreover, if Problem 5 (or equivalently, Problem 3) has a solution, then such solution is unique.

**Proof.** First, assume that $K$ solves Problem 3. We prove that $K$ also solves Problem 5.

Let $f \in C(\mathbb{S}^{n-1})$ be nonnegative. For $t \geq 0$, let

$$K_t = [h_K + tf]$$

and

$$F_p(h_K + tf) = \frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} (h_K + tf)^p d\mu.$$

Then, $F_p(h_K + tf) \geq F_p(K_t)$. Since $K$ solves Problem 3, and $F_p(K_t)^{\frac{1}{p}} K_t$ satisfies the constraint in Problem 3, it follows that for $t \geq 0$,

$$G(t) := C_p \left( \frac{K_t}{F_p(h_K + tf)^{\frac{1}{p}}} \right) \leq C_p(K).$$
Clearly, \(G(t)\) is continuous in \(t \geq 0\), and \(G(0) = C_p(K)\). Since

\[
\frac{dF_p(h_K + tf)}{dt} \bigg|_{t=0^+} = \frac{p(p - 1)}{n - p} \int_{\mathbb{S}^{n-1}} fh_{h_K}^{p-1} d\mu
\]

and

\[
\frac{dC_p(K)}{dt} \bigg|_{t=0^+} = (p - 1) \int_{\mathbb{S}^{n-1}} f d\mu_p(K, \cdot),
\]

it follows that

\[
0 = G'_+(0) = (p - 1) \int_{\mathbb{S}^{n-1}} f d\mu_p(K, \cdot) - (p - 1)C_p(K) \int_{\mathbb{S}^{n-1}} fh_{h_K}^{p-1} d\mu.
\]

Thus,

\[
\int_{\mathbb{S}^{n-1}} fh_{h_K}^{p-1} d\mu = \frac{1}{C_p(K)} \int_{\mathbb{S}^{n-1}} f d\mu_p(K, \cdot).
\]

That is, the above equality holds for any nonnegative \(f \in C(\mathbb{S}^{n-1})\). Therefore, it also holds for any \(f \in C(\mathbb{S}^{n-1})\), which concludes that \(C_p(K)^{-1}d\mu_p(K, \cdot) = h_{h_K}^{p-1}d\mu\).

Conversely, assume that \(K\) solves Problem 5. Let \(Q\) be a convex body containing the origin, such that \(1 = \frac{p - 1}{n - p} \int_{\mathbb{S}^{n-1}} h_Q^p d\mu\). Our aim is to prove that \(C_p(K) \geq C_p(Q)\). That is, \(K\) also solves Problem 3.

Using the condition that \(C_p(K)h_K^{p-1}d\mu = d\mu_p(K, \cdot)\), we have

\[
1 = \frac{p - 1}{n - p} \int_{\{h_K > 0\}} h_Q^p d\mu + \frac{p - 1}{n - p} \int_{\{h_K = 0\}} h_Q^p d\mu
\]

\[
\geq \frac{p - 1}{n - p} \int_{\{h_K > 0\}} h_Q^p d\mu
\]

\[
= \frac{p - 1}{n - p} \int_{\{h_K > 0\}} \left(\frac{h_Q}{h_K}\right)^p \frac{h_K}{C_p(K)} d\mu_p(K, \cdot).
\]

From the Poincaré \(p\)-capacity formula, it follows that

\[
C_p(K) = \frac{p - 1}{n - p} \int_{\{h_K > 0\}} h_K d\mu_p(K, \cdot).
\]
So, the measure $\frac{p-1}{(n-p)C_p(K)}h_Kd\mu_p(K, \cdot)$ is a Borel probability measure on the set $\{h_K \neq 0\}$. From the Jensen inequality, we have

$$1 \geq \left( \frac{p-1}{(n-p)C_p(K)} \int_{\{h_K>0\}} \left( \frac{h_Q}{h_K} \right)^p h_Kd\mu_p(K, \cdot) \right)^{\frac{1}{p}}$$

$$\geq \frac{p-1}{(n-p)C_p(K)} \int_{\{h_K>0\}} h_Qh_Kd\mu_p(K, \cdot)$$

$$= \frac{p-1}{(n-p)C_p(K)} \int_{\{h_K>0\}} h_Qd\mu_p(K, \cdot).$$

Furthermore, from the $p$-capacitary Minkowski inequality, we have

$$1 \geq \frac{C_p(K, Q)}{C_p(K)} - \frac{p-1}{(n-p)C_p(K)} \int_{\{h_K=0\}} h_Qd\mu_p(K, \cdot)$$

$$\geq \left( \frac{C_p(Q)}{C_p(K)} \right)^{\frac{1}{p-1}} - \frac{p-1}{(n-p)C_p(K)} \int_{\{h_K=0\}} h_Qd\mu_p(K, \cdot).$$

By the condition that $C_p(K)h_K^{p-1}d\mu = d\mu_p(K, \cdot)$, it follows that

$$\int_{\{h_K=0\}} h_Qd\mu_p(K, \cdot) = \int_{\{h_K=0\}} h_Qh_K^{p-1}d\mu = 0.$$

Thus,

$$1 \geq \left( \frac{C_p(Q)}{C_p(K)} \right)^{\frac{1}{n-p}},$$

as desired.

It remains to prove that if $K$ and $L$ are solutions to Problem 5, then $K = L$. From the above argument and the equality condition of the $p$-capacitary Minkowski inequality, we see that $K$ and $L$ are homothetic, so that $C_p(K) = C_p(L)$. In other words, $K = L + x$, for some $x \in \mathbb{R}^n$. From the translation invariance of $p$-capacitary measure and the assumptions, it follows that

$$(h_L(\xi) + x \cdot \xi)^{p-1}d\mu(\xi) = h_L(\xi)^{p-1}d\mu(\xi).$$

In other words,

$$(6.1) \quad (h_L(\xi) + x \cdot \xi)^{p-1} = h_L(\xi)^{p-1}, \quad \text{for } \mu-\text{almost all } \xi \in S^{n-1}.$$ 

Note that $\mu$ is not concentrated on any closed hemisphere. If $x$ is nonzero, then on the open hemisphere $U := \{\xi \in S^{n-1} : x \cdot \xi > 0\}$, we have $\mu(U) > 0$ and $(h_L(\xi) + x \cdot \xi)^{p-1} > h_L(\xi)^{p-1}$, for all $\xi \in U$, which contradicts (6.1). Hence, $K = L$.

The proof is complete. $\square$

By now, we propose 5 related problems in variant disguises. For convenience, it is necessary to summarize their relationship here.
(1). Problem 1 and Problem 2 are proposed exclusively for *discrete measures*. They have the identical unique solution;

(2). Problem 3 and Problem 4 are *dual* each other. Their solutions only differ by a scale factor. For discrete measures, Problem 3 and Problem 2 are identical.

(3). Problem 5 generalizes Problem 1 to *general measures*.

(4). Problem 5 and Problem 3 are equivalent. They have the identical unique solution.

7. Several useful lemmas for Section 8

In light of the equivalence of Problem 3 and Problem 5, we will solve Problem 5 in Section 8 via the passage by firstly solving Problem 3. For this aim, we have to make more preparatory works. Throughout this section, let $1 < p < \infty$ and $1 < p < n$.

Suppose that $\mu$ and $\mu_j$, $j \in \mathbb{N}$, are finite Borel measures on $\mathbb{S}^{n-1}$ and not concentrated on any closed hemisphere. For each $j$, assume that $K_j$ is the solution to Problem 5 for $\mu_j$.

Let

$$\bar{K}_j = \frac{K_j}{C_p(K_j)^{\frac{1}{p-n}}}.$$ 

From Lemma 6.3 and Lemma 6.1 (1), it implies that $\bar{K}_j$ is the solution to Problem 4 for $\mu_j$.

For a convex body $Q$ in $\mathbb{R}^n$ containing the origin, let

$$F_{p,j}(Q) = \frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} h_{\mathbb{Q}}^p d\mu_j \quad \text{and} \quad F_p(Q) = \frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} h_{\mathbb{Q}}^p d\mu.$$ 

**Lemma 7.1.** If $\{\mu_j\}_j$ converges weakly to $\mu$, then $\{K_j\}_j$ and $\{\bar{K}_j\}_j$ are bounded from above.

**Proof.** For each $j$, there is a $\xi_j \in \mathbb{S}^{n-1}$ such that $h_{K_j}(\xi_j) = \max_{\mathbb{S}^{n-1}} h_{K_j}$. Since the segment joining the origin and $(\max_{\mathbb{S}^{n-1}} h_{K_j}) \xi_j$ is contained in $K_j$, it follows that for all $\xi \in \mathbb{S}^{n-1},$

$$(\max_{\mathbb{S}^{n-1}} h_{K_j})(\xi_j \cdot \xi)_+ \leq h_{K_j}(\xi),$$

where $(\xi_j \cdot \xi)_+ = \max\{0, \xi_j \cdot \xi\}$. Thus,

$$1 = \frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} h_{K_j}^p d\mu_j$$

$$\geq (\max_{\mathbb{S}^{n-1}} h_{K_j})^p \frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} (\xi_j \cdot \xi)_+^p d\mu_j(\xi)$$

$$\geq (\max_{\mathbb{S}^{n-1}} h_{K_j})^p \frac{p-1}{n-p} \min_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} (\xi' \cdot \xi)_+^p d\mu_j(\xi).$$

Consider the functional $\mathbb{R}^n \to \mathbb{R},$

$$x \mapsto \left( \frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} (x \cdot \xi)_+^p d\mu_j(\xi) \right)^{\frac{1}{p}}.$$
Since \((x + x') \cdot \xi_+ \leq (x \cdot \xi)_+ + (x' \cdot \xi)_+\), by the Minkowski integral inequality, it implies that this functional is convex. Since \(\mu_j\) is not concentrated on any closed hemisphere, this functional is strictly positive for any nonzero \(x\). Thus, this functional is the support function of a unique convex body, say \(\Pi_{p,p\mu_j} \in \mathcal{K}_n^p\). So, \(\min_{\mathbb{S}^{n-1}} h_{\Pi_{p,p\mu_j}} > 0\) and

\[
\max_{\mathbb{S}^{n-1}} h_{K_j} \leq \frac{1}{\min_{\mathbb{S}^{n-1}} h_{\Pi_{p,p\mu_j}}} < \infty.
\]

Similarly, define the convex body \(\Pi_{p,p\mu} \in \mathcal{K}_n^p\) by

\[
h_{\Pi_{p,p\mu}}(x) = \left(\frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} |x \cdot \xi|^p d\mu(\xi)\right)^{\frac{1}{p}}.
\]

Since the weak convergence \(\mu_j \to \mu\) yields the pointwise convergence \(h_{\Pi_{p,p\mu_j}} \to h_{\Pi_{p,p\mu}}\) on \(\mathbb{S}^{n-1}\), and the pointwise convergence of support functions on \(\mathbb{S}^{n-1}\) is also a uniform convergence, it follows that the sequence \(\{h_{\Pi_{p,p\mu_j}}\}_j\) on \(\mathbb{S}^{n-1}\) is uniformly bounded from below by a constant \(m > 0\). So, we have

\[
\sup_j \left\{\max_{\mathbb{S}^{n-1}} h_{K_j}\right\} \leq \frac{1}{\inf_j \left\{\min_{\mathbb{S}^{n-1}} h_{\Pi_{p,p\mu_j}}\right\}} \leq \frac{1}{m} < \infty,
\]

which implies that \(\{K_j\}_j\) is bounded from above.

To prove that \(\{K_j\}_j = \left\{\frac{K_j}{C_p(K_j)^{\frac{1}{p}}}\right\}_j\) is also bounded from above, two observations are in order. First, by the fact that \(F_{p,j}\left(\left(\frac{p-1}{n-p}\right)^{-1/p} B\right) = 1\), where \(|\mu_j|\) denotes the total mass of \(\mu_j\), the ball \(\left(\frac{p-1}{n-p}\right)^{-1/p} B\) satisfies the constraint in Problem 3 for \(\mu_j\). Thus,

\[
C_p(K_j) \geq C_p\left(\left(\frac{p-1}{n-p}\right)^{-1/p} |\mu_j|\right)^{\frac{1}{p}} B.
\]

Second, the weak convergence \(\mu_j \to \mu\) yields the convergence \(|\mu_j| \to |\mu|\), which implies that

\[
\sup_j \{|\mu_j|\} < \infty.
\]

So,

\[
\max_{\mathbb{S}^{n-1}} h_{K_j} = \frac{\max_{\mathbb{S}^{n-1}} h_{K_j}}{C_p(K_j)^{\frac{1}{p}}}
\leq \frac{\max_{\mathbb{S}^{n-1}} h_{K_j}}{C_p\left(\left(\frac{p-1}{n-p}\right)^{-1/p} B\right)^{\frac{1}{p}}}
\leq \frac{(p-1)^{\frac{1}{p}} |\mu_j|^{\frac{1}{p}} \max_{\mathbb{S}^{n-1}} h_{K_j}}{(n-p)^{\frac{1}{p}} C_p(B)^{\frac{1}{p}}}
\leq M := \left(\frac{p-1}{n-p}\right)^{\frac{1}{p}} C_p(B)^{\frac{1}{n-p}} \sup_j \{|\mu_j|\}^{\frac{1}{p}} \sup_j \{\max_{\mathbb{S}^{n-1}} h_{K_j}\}
\leq \infty,
\]

which concludes that \(\{K_j\}_j\) is bounded from above.
Lemma 7.2. If \( \{\bar{K}_j\}_j \) converges to a compact convex set \( \bar{K} \), then \( \dim(\bar{K}) \neq n - 1 \).

Proof. Recall that \( \bar{K}_j \) is the solution to Problem 4 for \( \mu_j \). By Lemma 6.3 and Lemma 6.1 (2), \( K_j = F_{p,j}(\bar{K}_j)^{-1/p} \bar{K}_j \) is the solution to Problem 5 for \( \mu_j \). Since \( C_p(K_j)h_{K_j}^{p-1}d\mu_j = d\mu_p(K_j, \cdot) \), it follows that

\[
C_p \left( \frac{\bar{K}_j}{F_{p,j}(\bar{K}_j)^{1/p}} \right) h_{K_j}^{p-1}d\mu_j = d\mu_p \left( \frac{\bar{K}_j}{F_{p,j}(\bar{K}_j)^{1/p}} \right).
\]

From this, the fact that \( C_p(\bar{K}_j) = 1 \), together with the positive homogeneity of \( p \)-capacity, support functions and \( p \)-capacitary measure, it follows that

\[
h_{K_j}^{p-1}d\mu_j = F_{p,j}(\bar{K}_j)d\mu_p(\bar{K}_j, \cdot).
\]

By CNSXYZ [19] Lemma 2.18, there is a positive constant \( c \) depending on \( n, p \) and \( M \), such that \( \mu_p(\bar{K}_j, \cdot) \geq c^{-p}S_{K_j} \). Thus,

\[
h_{K_j}^{p-1}d\mu_j \geq c^{-p}F_{p,j}(\bar{K}_j)dS_{K_j}.
\]

Let \( f \in C(S^{n-1}) \) be non-negative. Then,

\[
\int_{S^{n-1}} fh_{K_j}^{p-1}d\mu_j \geq c^{-p}F_{p,j}(\bar{K}_j) \int f dS_{K_j}.
\] (7.1)

Here, several facts are in order. First, the convergence \( \bar{K}_j \to \bar{K} \) is equivalent to the uniform convergence \( h_{K_j} \to h_{\bar{K}} \) over the sphere \( S^{n-1} \). Second, the uniform convergence \( h_{K_j} \to h_{\bar{K}} \) together with the weak convergence \( \mu_j \to \mu \) yields the convergence \( F_{p,j}(\bar{K}_j) \to F_p(\bar{K}) \). Third, the convergence \( \bar{K}_j \to \bar{K} \) again yields the weak convergence \( S_{K_j} \to S_{\bar{K}} \). Hence, let \( j \to \infty \), (7.1) yields that

\[
\int_{S^{n-1}} fh_{K_j}^{p-1}d\mu \geq c^{-p}F_p(\bar{K}) \int f dS_{\bar{K}}.
\] (7.2)

With this inequality in hand, we devote to showing that \( \dim(\bar{K}) \neq n - 1 \).

Assume that \( \dim(\bar{K}) = n - 1 \) and \( \bar{K} \) is contained in an \((n-1)\)-dimensional linear subspace with normal \( \xi_0 \in S^{n-1} \). By the definition of surface area measure, \( S_{\bar{K}} = V_{n-1}(\bar{K})(\delta_{\xi_0} + \delta_{-\xi_0}) \), where \( V_{n-1}(\bar{K}) \) is the \((n-1)\)-dimensional volume of \( \bar{K} \). Now, (7.2) can be reformulated as

\[
\int_{S^{n-1}} f d\bar{\mu} \geq c' \cdot (f(\xi_0) + f(-\xi_0)),
\] (7.3)

where \( \bar{\mu} \) is the Borel measure on \( S^{n-1} \) defined by \( d\bar{\mu} = h_{\bar{K}}^{p-1}d\mu \), and \( c' = c^{-p}F(\bar{K})V_{n-1}(\bar{K}) \).

Recall that \( \bar{K} \) contains the origin. So, \( h_{\bar{K}} \geq 0 \), which in turn gives \( F_p(\bar{K}) \geq 0 \). Now, we prove that \( F_p(\bar{K}) > 0 \). Assume that \( F_p(\bar{K}) = 0 \). Since

\[
0 = \int_{S^{n-1}} h_{\bar{K}}^pd\mu = \int_{\{h_{\bar{K}} > 0\}} h_{\bar{K}}^pd\mu + \int_{\{h_{\bar{K}} = 0\}} h_{\bar{K}}^pd\mu = \int_{\{h_{\bar{K}} > 0\}} h_{\bar{K}}^pd\mu,
\]

it follows that \( \mu(\{h_{\bar{K}} > 0\}) = 0 \). Thus,

\[
\text{supp} \mu \subseteq S^{n-1} \setminus \{h_{\bar{K}} > 0\} = \{h_{\bar{K}} = 0\}.
\]
Since \( \{h_K = 0\} \) is contained in some closed hemisphere, it follows that \( \mu \) is concentrated on some closed hemisphere, which is a contradiction. Hence, \( F_p(K) > 0 \), and therefore, \( c' > 0 \).

With \( c' > 0 \) and \( f \in C(S^{n-1}) \) is non-negative, by Evans and Gariepy [20, Theorem 3, p. 42], (7.3) implies that the Borel measure \( \bar{\mu} \) satisfies
\[
\bar{\mu}(\{\xi_0\}) = \bar{\mu}(\{-\xi_0\}) > 0.
\]
However, from the assumption that \( h_K(\pm \xi_0) = 0 \) and the definition of \( \bar{\mu} \), it follows that
\[
\bar{\mu}(\{\xi_0\}) = \bar{\mu}(\{-\xi_0\}) = 0.
\]
A contradiction occurs. Hence, \( \dim(K) \neq n - 1 \).

**Lemma 7.3.** Suppose \( 1 < p \leq 2 \). If \( \{\bar{K}_j\}_j \) converges to a compact convex set \( \bar{K} \), then \( \dim(\bar{K}) \neq 0, 1, \ldots, n - 2 \).

**Proof.** The arguments here is similar to that from CNSXYZ [19, p. 1571]. If \( 1 < p \leq 2 \) and \( \dim(\bar{K}) \leq n - 2 \), then \( \dim(\bar{K}) \leq n - p \) and thus \( \mathcal{H}^{n-p}(\bar{K}) < \infty \). According to Evans and Gariepy [20, Theorem 3, p. 154]: if \( \mathcal{H}^{n-p}(\bar{K}) < \infty \), then \( C_p(\bar{K}) = 0 \), it follows that \( C_p(\bar{K}) = 0 \). This is impossible, because of the continuity of \( C_p \) and the fact that \( C_p(K_j) = 1 \) for each \( j \).

**Lemma 7.4.** Suppose \( 1 < p \leq 2 \). If \( \{K_j\}_j \) converges to a compact convex set \( K \), then the following assertions hold.
\begin{enumerate}
\item \( K \) is a convex body containing the origin.
\item \( 0 < \int_{S^{n-1}} h_K^p \, d\mu < \infty \).
\item The convex body
\[
K = \left( \frac{p-1}{n-p} \int_{S^{n-1}} h_{K_j}^p \, d\mu \right)^{-1/p} \bar{K}
\]
\end{enumerate}
is the unique solution to Problem 5 for \( \mu \).

**Proof.** By Lemma 7.2 and Lemma 7.3 it follows that \( \bar{K} \) is a convex body containing the origin.

From the facts that \( \max_{S^{n-1}} h_K^p < \infty \) and \( |\mu| < \infty \), it follows that \( \int_{S^{n-1}} h_K^p \, d\mu < \infty \). Now, we show \( \int_{S^{n-1}} h_K^p \, d\mu > 0 \) by contradiction. Assume that \( \int_{S^{n-1}} h_K^p \, d\mu = 0 \). Then, \( 0 = \int_{(h_K > 0)} h_K^p \, d\mu \), and therefore, \( \mu(\{h_K > 0\}) = 0 \). If \( K \) contains the origin in its interior, then \( \{h_K > 0\} = S^{n-1} \) and \( \mu(\{h_K > 0\}) = \mu(S^{n-1}) = |\mu| > 0 \). So, the origin is on the boundary of \( K \), and therefore \( \{h_K = 0\} \) is contained in some closed hemisphere. Note that \( \text{supp} \mu \subseteq \{h_K = 0\} \). So, \( \mu \) is concentrated on some closed hemisphere. It is a contradiction.

The assertions (1) and (2) imply that \( K \) is a convex body containing the origin. Since
\[
K_j = \left( \frac{p-1}{n-p} \int_{S^{n-1}} h_{K_j}^p \, d\mu_j \right)^{-1/p} \bar{K}_j \quad \text{and} \quad \lim_{j \to \infty} \int_{S^{n-1}} h_{K_j}^p \, d\mu_j = \int_{S^{n-1}} h_K^p \, d\mu,
\]
it follows that \( \{K_j\}_j \) converges to \( K \). From \( C_p(K_j) h_{K_j}^p \, d\mu_j = d\mu_p(K_j, \cdot) \), and the facts that the uniform convergence \( h_{K_j} \to h_K \) yields the convergence \( C_p(K_j) \to C_p(K) \) and the weak
convergence $\mu_p(K_j, \cdot) \to \mu_p(K, \cdot)$, it follows that $C_p(K)h_K^{p-1}d\mu = d\mu_p(K, \cdot)$. So, $K$ is a solution to Problem 5 for $\mu$. As far the uniqueness, it is guaranteed by Lemma 6.3.  

8. The $L_p$ Minkowski problem for $p$-capacity when $1 < p \leq 2$

With the preparatory works in Section 6 and Section 7, we set out to prove Theorem 1.2.

**Theorem 8.1.** Suppose $1 < p < \infty$ and $1 < p \leq 2$. If $\mu$ is a finite Borel measure on $S^{n-1}$ which is not concentrated on any closed hemisphere, then there exists a unique convex body $K$ in $\mathbb{R}^n$ containing the origin, such that $C_p(K)h_K^{p-1}d\mu = d\mu_p(K, \cdot)$. If in addition $p \geq n$, then $K$ contains the origin in its interior.

**Proof.** Take a sequence of discrete measures $\{\mu_j\}_j$ on $S^{n-1}$, such that each $\mu_j$ is not concentrated on any closed hemisphere and $\mu_j \to \mu$ weakly. By Theorem 4.1 and Lemma 6.2, for each $j$, Problem 5 for $\mu_j$ has a unique solution $P_j$, a convex polytope containing the origin in its interior.

Let $\bar{P}_j = \frac{P_j}{C_p(P_j)^{\frac{1}{p-1}}}$.

By Lemma 6.3 and Lemma 6.1, $P_j$ is the unique solution to Problem 4 for $\mu_j$. Since $\mu_j \to \mu$ weakly, the sequence $\{\bar{P}_j\}_j$ is bounded from above by Lemma 7.1. From the Blaschke selection theorem, $\{\bar{P}_j\}_j$ has a convergent subsequence $\{\bar{P}_{j_l}\}_l$, which converges to a compact convex set, say $\bar{K}$. By Lemma 7.4 (1), $\bar{K}$ is a convex body containing the origin. By Lemma 7.4 (2), $0 < \int_{S^{n-1}} h_K^p d\mu < \infty$. Thus, we get a convex body

$$K := \left(\frac{p-1}{n-p} \int_{S^{n-1}} h_K^p d\mu\right)^{-\frac{1}{p}}.$$  

By Lemma 7.4 (3), the convex body $K$ is the unique solution to Problem 5 for $\mu$.

It remains to prove that if in addition $p \geq n$, then $K$ contains the origin in its interior.

Several useful facts are listed. First, $\sup_l \{\|\mu_{jl}\|\} < \infty$. Second, $d\mu_{jl} = \frac{h_{P_{jl}}^{p}}{C_p(P_{jl})}d\mu_p(P_{jl}, \cdot)$, for each $l$. Third, from the convergence $P_{jl} \to K$ and CNSXYZ [19 Lemma 2.18], there is a positive constant $c_1$ depending on $n$, $p$ and $\max \{h_{P_{jl}}(\xi) : \xi \in S^{n-1}, l \in \mathbb{N}\}$, such that $\mu_p(P_{jl}, \cdot) \geq c_1^{-p}S_{P_{jl}}$. Finally, from the convergence $P_{jl} \to K$ again and the continuity of $p$-capacity, it follows that $0 < \sup_l \{C_p(P_{jl})\} < \infty$. Hence, $\infty > \sup_l \{\|\mu_{jl}\|\} \geq |\mu_{jl}| = \frac{1}{C_p(P_{jl})} \int_{S^{n-1}} h_{P_{jl}}^{1-p}d\mu_p(P_{jl}, \cdot) \geq c_2 \int_{S^{n-1}} h_{P_{jl}}^{1-p}dS_{P_{jl}}$, where $c_2 = \frac{c_1}{\sup_l \{C_p(P_{jl})\}}$.

Assume that the origin is on the boundary of $K$. We derive that $p < n$ by adapting an argument from Hug and LYZ [33, p.713]. Let $\xi_K \in S^{n-1}$ be such that $\partial K$ can be locally represented as the graph of a convex function over (a neighborhood of) $B_r := \xi_K^\perp \cap rB$, $r > 0$.  


and \( x \cdot \xi_K \geq 0 \) for any \( x \in K \). There exists a subsequence \( \{ j_k \} \) of \( \{ j_l \} \) tending to \( \infty \) and a constant \( c_3 > 0 \) independent of \( l \), such that

\[
\lim_{k \to \infty} \int_{S^{n-1}} h_{P, j_k}^{1-p} dS_{P, j_k} \geq c_3 \int_0^r t^{n-p-1} dt.
\]

Hence,

\[
\infty > \sup_l \{ |\mu_{j_l}| \} \geq c_2 c_3 \int_0^r t^{n-p-1} dt,
\]

which implies that \( p < n \). □

From Theorem \( \text{8.1} \), we immediately obtain the following results.

**Corollary 8.2.** Suppose \( 1 < p < \infty \), \( 1 < p \leq 2 \) and \( n - p \neq p \). If \( \mu \) is a finite Borel measure on \( S^{n-1} \) which is not concentrated on any closed hemisphere, then there exists a unique convex body \( K \) in \( \mathbb{R}^n \) containing the origin, such that

\[
h_K^{p-1} d\mu = d\mu_p(K, \cdot).
\]

If in addition \( p \geq n \), then \( K \in \mathcal{K}_o^n \).

*Proof.* By Theorem \( \text{8.1} \) there exists a unique convex body \( K^* \) containing the origin, such that \( C_p(K^*)h_{K^*}^{p-1} d\mu = d\mu_p(K^*, \cdot) \). Let \( K = C_p(K^*)^{1/(p+n)} K^* \). Then, \( h_K^{p-1} d\mu = d\mu_p(K, \cdot) \). □

**Corollary 8.3.** Suppose \( 1 < p < \infty \) and \( 1 < p \leq 2 \). If \( \mu \) is a finite even Borel measure on \( S^{n-1} \) which is not concentrated on any great subsphere, then there exists a unique origin-symmetric convex body \( K \) in \( \mathbb{R}^n \), such that \( C_p(K)^{-1} \mu_{p,p}(K, \cdot) = \mu \).

*Proof.* By Theorem \( \text{8.1} \) there exists a unique convex body containing the origin, such that \( h_K^{-p} d\mu = C_p(K)^{-1} d\mu_p(K, \cdot) \). Since \( \mu \) is even, it implies that \( h_K^{-p} d\mu = C_p(-K)^{-1} d\mu_p(-K, \cdot) \). So, the uniqueness of \( K \) in turn implies that \( -K = K \). □

Consequently, if \( n - p \neq p \), then there exists a unique origin-symmetric convex body \( K' \) in \( \mathbb{R}^n \), such that \( \mu = \mu_{p,p}(K', \cdot) \).

9. Continuity

Let \( 1 < p < \infty \), \( 1 < p \leq 2 \) and \( p < n \). Write \( \mathcal{M} \) for the set of finite Borel measures on \( S^{n-1} \) which are not concentrated on any closed hemisphere. For each \( \mu \in \mathcal{M} \), denote by \( C_p^\mu \) the unique solution to Problem 5 for \( (\mu, p, p) \), i.e., the unique convex body containing the origin such that

\[
d\mu_p(K, \cdot) = h_K^{p-1} d\mu.
\]

A natural question about the continuity of solution to \( L_p \) Minkowski problem for \( p \)-capacity asks the following: *If \( \{ \mu_j \} \subset \mathcal{M} \) converges to \( \mu \in \mathcal{M} \) weakly, is this the case that \( C_p^\mu \mu_j \to C_p^\mu \mu \)?*

We answer this question affirmatively.
Theorem 9.1. Suppose that $\mu_j, \mu \in \mathcal{M}$, $j \in \mathbb{N}$, $1 < p < \infty$ and $1 < p \leq 2$. If $\mu_j \rightarrow \mu$ weakly as $j \rightarrow \infty$, then $C_p^p\mu_j \rightarrow C_p^p\mu$.

Proof. For the sake of simplicity, write $K_j$ and $K$ for $C_p^p\mu_j$ and $C_p^p\mu$, respectively. By Lemma 6.3, $K_j$ and $K$ are also the unique solutions to Problem 3 for $\mu_j$ and $\mu$, respectively. From Lemma 7.1, it follows that the sequence $\{K_j\}_j$ is bounded from above. Hence, to prove that $K_j \rightarrow K$, it suffices to prove each convergent subsequence $\{K_{j_l}\}_l$ of $\{K_j\}_j$ converges to $K$.

Assume that $\{K_{j_l}\}_l$ is a convergent subsequence of $\{K_j\}_j$. Let $\bar{K}_j = C_p(K_j)^{-1/(n-p)}K_j$, $j \in \mathbb{N}$. By Lemma 6.1 (1), $\bar{K}_{j_l}$ is the unique solution to Problem 4 for $\mu_{j_l}$. From Lemma 7.1, the sequence $\{\bar{K}_{j_l}\}_l$ is bounded from above. Thus, by the Blaschke selection theorem, $\{\bar{K}_{j_l}\}_l$ has a subsequence $\{\bar{K}_{j_{l_l}}\}_l$, converging to a compact convex set $\bar{K}_0$. By Lemma 7.1 (1), $\bar{K}_0$ is a convex body containing the origin; by Lemma 7.1 (2), $0 < \int_{S^{n-1}} h^p_{K_0} d\mu < \infty$. Thus,

$$K_0 = \left( \frac{p-1}{n-p} \int_{S^{n-1}} h^p_{K_0} d\mu \right)^{-\frac{1}{p}} \bar{K}_0$$

is indeed a convex body. By Lemma 7.2 (3), $K_0$ is the unique solution to Problem 5 for $\mu$.

In light of $K$ is also the unique solution to Problem 5 for $\mu$, we have $K_0 = K$. Therefore, $\lim_{l \rightarrow \infty} K_{j_{l_l}} = K$. Since $\{K_{j_l}\}_l$ is a convergent sequence, it follows that $\lim_{l \rightarrow \infty} K_{j_l} = K$. □

For each $\mu \in \mathcal{M}$, if $n - p \neq p$, we can define $C_p^p\mu$. Then $C_p^p\mu$ is the unique convex body which contains the origin and is such that $h_{C_p^p\mu}^{-1}d\mu = d\mu_p(C_p^p\mu, \cdot)$.

Corollary 9.2. Suppose that $\mu_j, \mu \in \mathcal{M}$, $j \in \mathbb{N}$, $1 < p < \infty$ and $1 < p \leq 2$, $n - p \neq p$. If $\mu_j \rightarrow \mu$ weakly as $j \rightarrow \infty$, then $C_p^p\mu_j \rightarrow C_p^p\mu$.

Proof. Since $\mu_j \rightarrow \mu$ weakly, we have $C_p^p\mu_j \rightarrow C_p^p\mu$ by Theorem 9.1. So, $C_p(C_p^p\mu_j) \rightarrow C_p(C_p^p\mu)$, and therefore $C_p(C_p^p\mu_j)^{-1/(n-p-p)} \rightarrow C_p(C_p^p\mu)^{-1/(n-p-p)}$, as $j \rightarrow \infty$. Consequently,

$$\lim_{j \rightarrow \infty} \bar{C}_p^p\mu_j = \lim_{j \rightarrow \infty} C_p(C_p^p\mu_j)^{-1/(n-p-p)}C_p^p\mu_j = C_p(C_p^p\mu)^{-1/(n-p-p)}C_p^p\mu = \bar{C}_p^p\mu,$$

as desired. □

Corollary 9.3. Suppose that $K_j, K \in \mathcal{K}^n$, $j \in \mathbb{N}$, $1 < p < \infty$ and $1 < p \leq 2$, $n - p \neq p$. If $\mu_{p,\mu_j}(K_j, \cdot) \rightarrow \mu_{p,\mu}(K, \cdot)$ weakly as $j \rightarrow \infty$, then $K_j \rightarrow K$.

Proof. Let $\mu_j = \mu_{p,\mu_j}(K_j, \cdot)$ and $\mu = \mu_{p,\mu}(K, \cdot)$. Then, $h_{K_j}^{p-1}d\mu_j = d\mu_p(K_j, \cdot)$, and $h_{K}^{p-1}d\mu_j = d\mu_p(K, \cdot)$. From the uniqueness of $C_p^p\mu$ it follows that $K_j = C_p^p\mu_j$ and $K = C_p^p\mu$. Since $\mu_j \rightarrow \mu$ weakly, Corollary 9.2 implies that $C_p^p\mu_j \rightarrow C_p^p\mu$, as $j \rightarrow \infty$. That is, $K_j \rightarrow K$ as $j \rightarrow \infty$. □

Remark 9.4. After this work, we further study the $L_p$ Minkowski problem for $p$-capacity when the given measure is even, it will be dealt with in a separate paper as a sequel.
10. **Open problem**

Since the logarithmic Minkowski problem is the most important case, we pose the following

**Logarithmic Minkowski problem for capacity.** *Suppose that μ is a finite Borel measure on S^{n-1} and 1 < p < n. What are the necessary and sufficient conditions on μ so that μ is the L_0 p-capacitary measure μ_0,p(K, ·) of a convex body K in \( \mathbb{R}^n \)?*

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