Complex Numbers in 6 Dimensions

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Abstract

Two distinct systems of commutative complex numbers in 6 dimensions of the polar and planar types having the form \( u = x_0 + h_1 x_1 + h_2 x_2 + h_3 x_3 + h_4 x_4 + h_5 x_5 \), are described in this work, where the variables \( x_0, x_1, x_2, x_3, x_4, x_5 \) are real numbers. The polar 6-complex numbers introduced in this paper can be specified by the modulus \( d \), the amplitude \( \rho \), and the polar angles \( \theta_+, \theta_- \), the planar angle \( \psi_1 \), and the azimuthal angles \( \phi_1, \phi_2 \). The planar 6-complex numbers introduced in this paper can be specified by the modulus \( d \), the amplitude \( \rho \), the planar angles \( \psi_1, \psi_2 \), and the azimuthal angles \( \phi_1, \phi_2, \phi_3 \). Exponential and trigonometric forms are given for the 6-complex numbers. The 6-complex functions defined by series of powers are analytic, and the partial derivatives of the components of the 6-complex functions are closely related. The integrals of polar 6-complex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the 6-complex numbers depends on cyclic variables leads to the concept of pole and residue for integrals on closed paths. The polynomials of polar 6-complex variables can be written as products of linear or quadratic factors, the polynomials of planar 6-complex variables can always be written as products of linear factors, although the factorization is not unique.

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1 Introduction

A regular, two-dimensional complex number $x + iy$ can be represented geometrically by the modulus $\rho = (x^2 + y^2)^{1/2}$ and by the polar angle $\theta = \arctan(y/x)$. The modulus $\rho$ is multiplicative and the polar angle $\theta$ is additive upon the multiplication of ordinary complex numbers.

The quaternions of Hamilton are a system of hypercomplex numbers defined in four dimensions, the multiplication being a noncommutative operation, and many other hypercomplex systems are possible, but these hypercomplex systems do not have all the required properties of regular, two-dimensional complex numbers which rendered possible the development of the theory of functions of a complex variable.

Two distinct systems of commutative complex numbers in 6 dimensions having the form

$$u = x_0 + h_1 x_1 + h_2 x_2 + h_3 x_3 + h_4 x_4 + h_5 x_5$$

are described in this work, for which the multiplication is associative and commutative, where the variables $x_0, x_1, x_2, x_3, x_4, x_5$ are real numbers. The first type of 6-complex numbers described in this article is characterized by the presence of two polar axes, so that these numbers will be called polar 6-complex numbers. The other type of 6-complex numbers described in this paper will be called planar 6-complex numbers.

The polar 6-complex numbers introduced in this paper can be specified by the modulus $d$, the amplitude $\rho$, and the polar angles $\theta_+, \theta_-$, the planar angle $\psi_1$, and the azimuthal angles $\phi_1, \phi_2$. The planar 6-complex numbers introduced in this paper can be specified by the modulus $d$, the amplitude $\rho$, the planar angles $\psi_1, \psi_2$, and the azimuthal angles $\phi_1, \phi_2, \phi_3$. Exponential and trigonometric forms are given for the 6-complex numbers. The 6-complex functions defined by series of powers are analytic, and the partial derivatives of the components of the 6-complex functions are closely related. The integrals of polar 6-complex functions are independent of path in regions where the functions are regular. The fact that the exponential form of their 6-complex numbers depends on cyclic variables leads to the concept of pole and residue for integrals on closed paths. The polynomials of polar 6-complex variables can be written as products of linear or quadratic factors, the polynomials of planar 6-complex variables can always be written as products of linear factors, although the factorization is not unique.
This paper belongs to a series of studies on commutative complex numbers in \( n \) dimensions. The polar 6-complex numbers described in this paper are a particular case for \( n = 6 \) of the polar hypercomplex numbers in \( n \) dimensions, and the planar 6-complex numbers described in this section are a particular case for \( n = 6 \) of the planar hypercomplex numbers in \( n \) dimensions[3],[3]

### 2 Polar complex numbers in 6 dimensions

#### 2.1 Operations with polar complex numbers in 6 dimensions

The polar hypercomplex number \( u \) in 6 dimensions is represented as

\[
u = x_0 + h_1x_1 + h_2x_2 + h_3x_3 + h_4x_4 + h_5x_5.\tag{1}\]

The multiplication rules for the bases \( h_1, h_2, h_3, h_4, h_5 \) are

\[
h_1^2 = h_2, \quad h_2^2 = h_4, \quad h_3^2 = 1, \quad h_4^2 = h_2, \quad h_5^2 = h_4, \quad h_1h_2 = h_3, \quad h_1h_3 = h_4, \quad h_1h_4 = h_5, \quad h_1h_5 = 1, \quad h_2h_3 = h_5, \quad h_2h_4 = 1, \quad h_2h_5 = h_1, \quad h_3h_4 = h_1, \quad h_3h_5 = h_2, \quad h_4h_5 = h_3.\tag{2}\]

The significance of the composition laws in Eq. (2) can be understood by representing the bases \( h_j, h_k \) by points on a circle at the angles \( \alpha_j = \pi j/3, \alpha_k = \pi k/3 \), as shown in Fig. 1, and the product \( h_jh_k \) by the point of the circle at the angle \( \pi(j + k)/3 \). If \( 2\pi \leq \pi(j + k)/3 < 4\pi \), the point represents the basis \( h_l \) of angle \( \alpha_l = \pi(j + k)/3 - 2\pi \).

The sum of the 6-complex numbers \( u \) and \( u' \) is

\[
u + u' = x_0 + x'_0 + h_1(x_1 + x'_1) + h_1(x_2 + x'_2) + h_3(x_3 + x'_3) + h_4(x_4 + x'_4) + h_5(x_5 + x'_5).\tag{3}\]

The product of the numbers \( u, u' \) is

\[
nu' = x_0x'_0 + x_1x'_1 + x_2x'_2 + x_3x'_3 + x_4x'_4 + x_5x'_5 + h_1(x_0x'_1 + x_1x'_0 + x_2x'_3 + x_3x'_4 + x_4x'_5 + x_5x'_2) + h_2(x_0x'_2 + x_1x'_1 + x_2x'_0 + x_3x'_5 + x_4x'_4 + x_5x'_3) + h_3(x_0x'_3 + x_1x'_2 + x_2x'_1 + x_3x'_0 + x_4x'_5 + x_5x'_4) + h_4(x_0x'_4 + x_1x'_3 + x_2x'_2 + x_3x'_1 + x_4x'_0 + x_5x'_5) + h_5(x_0x'_5 + x_1x'_4 + x_2x'_3 + x_3x'_2 + x_4x'_1 + x_5x'_0).\tag{4}\]
The relation between the variables \( v_+, v_, v_1, v_2, \tilde{v}_1, \tilde{v}_2 \) and \( x_0, x_1, x_2, x_3, x_4, x_5 \) are

\[
\begin{pmatrix}
v_+\\
v_-\\
v_1\\\tilde{v}_1\\v_2\\\tilde{v}_2
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 \\
1 & \frac{1}{2} & -\frac{1}{2} & -1 & -\frac{1}{2} & \frac{1}{2} \\
0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\
1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}.
\]  

(5)

The other variables are \( v_4 = v_2, \tilde{v}_4 = -\tilde{v}_2, v_5 = v_1, \tilde{v}_5 = -\tilde{v}_1 \). The variables \( v_+, v_, v_1, v_2, \tilde{v}_1, \tilde{v}_2 \) will be called canonical polar 6-complex variables.

### 2.2 Geometric representation of polar complex numbers in 6 dimensions

The 6-complex number \( u = x_0 + h_1 x_1 + h_2 x_2 + h_3 x_3 + h_4 x_4 + h_5 x_5 \) is represented by the point \( A \) of coordinates \( (x_0, x_1, x_2, x_3, x_4, x_5) \). The distance from the origin \( O \) of the 6-dimensional space to the point \( A \) has the expression

\[
d^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2.
\]  

(6)

The distance \( d \) is called modulus of the 6-complex number \( u \), and is designated by \( d = |u| \).

The modulus has the property that

\[
|u'\overline{u''}| \leq \sqrt{6}|u'||u''|.
\]  

(7)

The exponential and trigonometric forms of the 6-complex number \( u \) can be obtained conveniently in a rotated system of axes defined by a transformation which has the form

\[
\begin{pmatrix}
\xi_+ \\
\xi_- \\
\xi_1 \\
\xi_1 \\
\xi_2 \\
\xi_2
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\
\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}.
\]  

(8)
The lines of the matrices in Eq. (8) gives the components of the 6 basis vectors of the new system of axes. These vectors have unit length and are orthogonal to each other. The relations between the two sets of variables are

\[ v_+ = \sqrt{6} \xi_+, v_- = \sqrt{6} \xi_-, v_k = \sqrt{3} \xi_k, \tilde{v}_k = \sqrt{3} \eta_k, k = 1, 2. \] (9)

The radius \( \rho_k \) and the azimuthal angle \( \phi_k \) in the plane of the axes \( v_k, \tilde{v}_k \) are

\[ \rho_k^2 = v_k^2 + \tilde{v}_k^2, \cos \phi_k = v_k/\rho_k, \sin \phi_k = \tilde{v}_k/\rho_k, 0 \leq \phi_k < 2\pi, k = 1, 2, \] (10)

so that there are 2 azimuthal angles. The planar angle \( \psi_1 \) is

\[ \tan \psi_1 = \rho_1/\rho_2, 0 \leq \psi_1 \leq \pi/2. \] (11)

There is a polar angle \( \theta_+ \),

\[ \tan \theta_+ = \frac{\sqrt{2} \rho_1}{v_+}, 0 \leq \theta_+ \leq \pi, \] (12)

and there is also a polar angle \( \theta_- \),

\[ \tan \theta_- = \frac{\sqrt{2} \rho_1}{v_-}, 0 \leq \theta_- \leq \pi. \] (13)

The amplitude of a 6-complex number \( u \) is

\[ \rho = \left( v_+ v_- - \rho_1^2 \rho_2^2 \right)^{1/6}. \] (14)

It can be checked that

\[ d^2 = \frac{1}{6} v_+^2 + \frac{1}{6} v_-^2 + \frac{1}{3} (\rho_1^2 + \rho_2^2). \] (15)

If \( u = u'u'' \), the parameters of the hypercomplex numbers are related by

\[ v_+ = v_+ v'_+, \] (16)

\[ v_- = v_- v'_-, \] (18)

\[ v_+ = v_+ v'_+, \] (16)

\[ v_- = v_- v'_-, \] (18)

\[ \tan \theta_+ = \frac{1}{\sqrt{2}} \tan \theta'_+ \tan \theta''_+, \] (17)

\[ \tan \theta_- = \frac{1}{\sqrt{2}} \tan \theta'_- \tan \theta''_-, \] (19)

\[ \tan \psi_1 = \tan \psi'_1 \tan \psi''_1, \] (20)
\[ \rho_k = \rho_k' \rho_k'', \quad (21) \]

\[ \phi_k = \phi_k' + \phi_k'', \quad (22) \]

\[ v_k = v_k' v_k'' - \tilde{v}_k' \tilde{v}_k'', \quad (23) \]

\[ \rho = \rho' \rho'', \quad (24) \]

where \( k = 1, 2 \).

The 6-complex number \( u = x_0 + h_1 x_1 + h_2 x_2 + h_3 x_3 + h_4 x_4 + h_5 x_5 \) can be represented by the matrix

\[
U = \begin{pmatrix}
    x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\
    x_5 & x_0 & x_1 & x_2 & x_3 & x_4 \\
    x_4 & x_5 & x_0 & x_1 & x_2 & x_3 \\
    x_3 & x_4 & x_5 & x_0 & x_1 & x_2 \\
    x_2 & x_3 & x_4 & x_5 & x_0 & x_1 \\
    x_1 & x_2 & x_3 & x_4 & x_5 & x_0
\end{pmatrix}. \quad (25)
\]

The product \( u = u' u'' \) is represented by the matrix multiplication \( U = U' U'' \).

### 2.3 The polar 6-dimensional cosexponential functions

The polar cosexponential functions in 6 dimensions are

\[ g_{6k}(y) = \sum_{p=0}^{\infty} y^{k+6p}/(k+6p)!, \quad (26) \]

for \( k = 0, ..., 5 \). The polar cosexponential functions \( g_{6k} \) of even index \( k \) are even functions, \( g_{6,2p}(-y) = g_{6,2p}(y) \), and the polar cosexponential functions of odd index \( k \) are odd functions, \( g_{6,2p+1}(-y) = -g_{6,2p+1}(y) \), \( p = 0, 1, 2 \).

It can be checked that

\[ \sum_{k=0}^{5} g_{6k}(y) = e^y, \quad (27) \]

\[ \sum_{k=0}^{5} (-1)^k g_{6k}(y) = e^{-y}. \quad (28) \]
The exponential function of the quantity $h_y$ is

\[ e^{h_y} = g_0(y) + h_1 g_1(y) + h_2 g_2(y) + h_3 g_3(y) + h_4 g_4(y) + h_5 g_5(y), \]

\[ e^{h_2 y} = g_0(y) + g_3(y) + h_2 \{ g_1(y) + g_4(y) \} + h_4 \{ g_2(y) + g_5(y) \}, \]

\[ e^{h_3 y} = g_0(y) + g_2(y) + g_4(y) + h_3 \{ g_1(y) + g_3(y) + g_5(y) \}, \]

\[ e^{h_4 y} = g_0(y) + g_3(y) + h_2 \{ g_2(y) + g_5(y) \} + h_4 \{ g_1(y) + g_4(y) \}, \]

\[ e^{h_5 y} = g_0(y) + h_1 g_5(y) + h_2 g_4(y) + h_3 g_3(y) + h_4 g_2(y) + h_5 g_1(y). \]

The relations for $h_2$ and $h_4$ can be written equivalently as $e^{h_2 y} = g_0 + h_2 g_{31} + h_4 g_{32}$, $e^{h_4 y} = g_0 + h_2 g_{32} + h_4 g_{31}$, and the relation for $h_3$ can be written as $e^{h_3 y} = g_20 + h_3 g_{21}$, which is the same as $e^{h_3 y} = \cosh y + h_3 \sinh y$.

The expressions of the polar 6-dimensional cosexponential functions are

\[ g_0(y) = \frac{1}{6} \cosh y + \frac{2}{3} \cosh \frac{\sqrt{3}}{2} y, \]

\[ g_1(y) = \frac{1}{6} \sinh y + \frac{1}{2} \sinh \frac{\pi}{6} \cosh \frac{\sqrt{3}}{2} y, \]

\[ g_2(y) = \frac{1}{6} \cosh y - \frac{1}{3} \cosh \frac{\pi}{6} \cosh \frac{\sqrt{3}}{2} y, \]

\[ g_3(y) = \frac{1}{6} \sinh y - \frac{2}{3} \sinh \frac{\pi}{6} \cosh \frac{\sqrt{3}}{2} y, \]

\[ g_4(y) = \frac{1}{3} \cosh y - \frac{1}{2} \cosh \frac{\pi}{6} \sinh \frac{\sqrt{3}}{2} y, \]

\[ g_5(y) = \frac{1}{3} \sinh y + \frac{1}{2} \sinh \frac{\pi}{6} \sinh \frac{\sqrt{3}}{2} y. \]

The cosexponential functions (30) can be written as

\[ g_{6k}(y) = \frac{1}{6} \sum_{l=0}^{5} \exp \left[ y \cos \left( \frac{2\pi l}{6} \right) \right] \cos \left[ y \sin \left( \frac{2\pi l}{6} \right) - \frac{2\pi kl}{6} \right], \]

for $k = 0, \ldots, 5$. The graphs of the polar 6-dimensional cosexponential functions are shown in Fig. 2.

It can be checked that

\[ \sum_{k=0}^{5} g_{6k}^2(y) = \frac{1}{3} \cosh 2y + \frac{2}{3} \cosh y. \]

The addition theorems for the polar 6-dimensional cosexponential functions are

\[ g_0(y + z) = g_0(y) g_0(z) + g_0(y) g_{65}(z) + g_0(y) g_{64}(z) + g_0(y) g_{63}(z) + g_0(y) g_{62}(z) + g_0(y) g_{61}(z), \]

\[ g_1(y + z) = g_0(y) g_0(z) + g_0(y) g_{60}(z) + g_0(y) g_{65}(z) + g_0(y) g_{64}(z) + g_0(y) g_{63}(z) + g_0(y) g_{62}(z), \]

\[ g_2(y + z) = g_0(y) g_0(z) + g_0(y) g_{61}(z) + g_0(y) g_{60}(z) + g_0(y) g_{65}(z) + g_0(y) g_{64}(z) + g_0(y) g_{63}(z) + g_0(y) g_{62}(z), \]

\[ g_3(y + z) = g_0(y) g_0(z) + g_0(y) g_{61}(z) + g_0(y) g_{60}(z) + g_0(y) g_{65}(z) + g_0(y) g_{64}(z) + g_0(y) g_{63}(z) + g_0(y) g_{62}(z), \]

\[ g_4(y + z) = g_0(y) g_0(z) + g_0(y) g_{61}(z) + g_0(y) g_{60}(z) + g_0(y) g_{65}(z) + g_0(y) g_{64}(z) + g_0(y) g_{63}(z) + g_0(y) g_{62}(z), \]

\[ g_5(y + z) = g_0(y) g_0(z) + g_0(y) g_{61}(z) + g_0(y) g_{60}(z) + g_0(y) g_{65}(z) + g_0(y) g_{64}(z) + g_0(y) g_{63}(z) + g_0(y) g_{62}(z). \]
It can be shown that
\[ \{g_60(y) + h_1 g_61(y) + h_2 g_62(y) + h_3 g_63(y) + h_4 g_64(y) + h_5 g_65(y)\}^\top = g_60(l) + h_1 g_61(l) + h_2 g_62(l) + h_3 g_63(l) + h_4 g_64(l) + h_5 g_65(l), \]
\[ \{g_60(y) + g_63(y) + h_2 \{g_61(y) + g_64(y)\} + h_4 \{g_62(y) + g_65(y)\}\}^\top = g_60(l) + g_63(l) + h_2 \{g_61(l) + g_64(l)\} + h_4 \{g_62(l) + g_65(l)\}, \]
\[ \{g_60(y) + g_62(y) + g_64(y) + h_3 \{g_61(y) + g_63(y) + g_65(y)\}\}^\top = g_60(l) + g_62(l) + g_64(l) + h_3 \{g_61(l) + g_63(l) + g_65(l)\}, \]
\[ \{g_60(y) + g_63(y) + h_2 \{g_62(y) + g_65(y)\} + h_4 \{g_61(y) + g_64(y)\}\}^\top = g_60(l) + g_63(l) + h_2 \{g_62(l) + g_65(l)\} + h_4 \{g_61(l) + g_64(l)\}, \]
\[ \{g_60(y) + h_1 g_65(y) + h_2 g_64(y) + h_3 g_63(y) + h_4 g_62(y) + h_5 g_61(y)\}^\top = g_60(l) + h_1 g_65(l) + h_2 g_64(l) + h_3 g_63(l) + h_4 g_62(l) + h_5 g_61(l). \]

The derivatives of the polar cosexponential functions are related by
\[ \frac{d g_{60}}{d u} = g_{65}, \quad \frac{d g_{61}}{d u} = g_{60}, \quad \frac{d g_{62}}{d u} = g_{61}, \quad \frac{d g_{63}}{d u} = g_{62}, \quad \frac{d g_{64}}{d u} = g_{63}, \quad \frac{d g_{65}}{d u} = g_{64}. \] (35)

2.4 Exponential and trigonometric forms of polar 6-complex numbers

The exponential and trigonometric forms of polar 6-complex numbers can be expressed with the aid of the hypercomplex bases
\[
\begin{pmatrix}
  e_+ \\
  e_- \\
  e_1 \\
  \tilde{e}_1 \\
  e_2 \\
  \tilde{e}_2
\end{pmatrix}
= \begin{pmatrix}
  \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
  -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\
  \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\
  0 & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & 0 & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} \\
  \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\
  0 & -\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & 0 & \frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6}
\end{pmatrix}
\begin{pmatrix}
  1 \\
  h_1 \\
  h_2 \\
  h_3 \\
  h_4 \\
  h_5
\end{pmatrix}. \] (36)

The multiplication relations for these bases are
\[ e_+^2 = e_+, \quad e_-^2 = e_-, \quad e_+ e_- = 0, \quad e_+ e_k = 0, \quad e_+ \tilde{e}_k = 0, \quad e_- e_k = 0, \quad e_- \tilde{e}_k = 0, \]
\[ e_k^2 = e_k, \quad \tilde{e}_k^2 = -e_k, \quad e_k \tilde{e}_k = \tilde{e}_k e_k = 0, \quad e_k \tilde{e}_l = 0, \quad \tilde{e}_k \tilde{e}_l = 0, \quad k, l = 1, 2, k \neq l. \] (37)
The bases have the property that
\[ e_+ + e_- + e_1 + e_2 = 1. \] (38)

The moduli of the new bases are
\[ |e_+| = \frac{1}{\sqrt{6}}, \quad |e_-| = \frac{1}{\sqrt{6}}, \quad |e_k| = \frac{1}{\sqrt{3}}, \quad k = 1, 2. \] (39)

It can be shown that
\[ x_0 + h_1 x_1 + h_2 x_2 + h_3 x_3 + h_4 x_4 + h_5 x_5 = e_+ v_+ + e_- v_- + e_1 v_1 + e_2 v_2 + \tilde{e}_1 \tilde{v}_1 + \tilde{e}_2 \tilde{v}_2. \] (40)

The ensemble \( e_+, e_-, e_1, \tilde{e}_1, e_2, \tilde{e}_2 \) will be called the canonical polar 6-complex base, and Eq. (40) gives the canonical form of the polar 6-complex number.

The exponential form of the 6-complex number \( u \) is
\[ u = \rho \exp \left\{ \frac{1}{6} (h_1 + h_2 + h_3 + h_4 + h_5) \ln \frac{\sqrt{2}}{\tan \theta_+} - \frac{1}{6} (h_1 - h_2 + h_3 - h_4 + h_5) \ln \frac{\sqrt{2}}{\tan \theta_-} + \frac{1}{6} (h_1 + h_2 - 2h_3 + h_4 + h_5) \ln \tan \psi_1 + \tilde{e}_1 \phi_1 + \tilde{e}_2 \phi_2 \right\}, \] (41)

for \( 0 < \theta_+ < \pi/2, 0 < \theta_- < \pi/2 \).

The trigonometric form of the 6-complex number \( u \) is
\[ u = d \sqrt{3} \left( \tan^2 \theta_+ + \tan^2 \theta_- + 1 + \frac{1}{\tan^2 \psi_1} \right)^{-1/2} \left( \frac{e_+ \sqrt{2}}{\tan \theta_+} + \frac{e_- \sqrt{2}}{\tan \theta_-} + e_1 + \frac{e_2}{\tan \psi_1} \right) \exp (\tilde{e}_1 \phi_1 + \tilde{e}_2 \phi_2). \] (42)

The modulus \( d \) and the amplitude \( \rho \) are related by
\[ d = \rho \frac{2^{1/3}}{\sqrt{6}} \left( \tan \theta_+ \tan \theta_- \tan^2 \psi_1 \right)^{1/6} \left( \frac{1}{\tan^2 \theta_+} + \frac{1}{\tan^2 \theta_-} + 1 + \frac{1}{\tan^2 \psi_1} \right)^{1/2}. \] (43)

### 2.5 Elementary functions of a polar 6-complex variable

The logarithm and power functions of the 6-complex number \( u \) exist for \( v_+ > 0, v_- > 0 \), which means that \( 0 < \theta_+ < \pi/2, 0 < \theta_- < \pi/2 \), and are given by
\[ \ln u = \ln \rho + \frac{1}{6} (h_1 + h_2 + h_3 + h_4 + h_5) \ln \frac{\sqrt{2}}{\tan \theta_+} - \frac{1}{6} (h_1 - h_2 + h_3 - h_4 + h_5) \ln \frac{\sqrt{2}}{\tan \theta_-} + \frac{1}{6} (h_1 + h_2 - 2h_3 + h_4 + h_5) \ln \tan \psi_1 + \tilde{e}_1 \phi_1 + \tilde{e}_2 \phi_2, \] (44)
\[ u^m = e_+ v_+^m + e_- v_-^m + \rho_1^m (e_1 \cos m \phi_1 + \tilde{e}_1 \sin m \phi_1) + \rho_2^m (e_2 \cos m \phi_2 + \tilde{e}_2 \sin m \phi_2). \] (45)

The exponential of the 6-complex variable \( u \) is
\[ e^u = e_+ e^{v_+} + e_- e^{v_-} + e^{v_1} (e_1 \cos \tilde{v}_1 + \tilde{e}_1 \sin \tilde{v}_1) + e^{v_2} (e_2 \cos \tilde{v}_2 + \tilde{e}_2 \sin \tilde{v}_2). \] (46)

The trigonometric functions of the 6-complex variable \( u \) are
\[ \cos u = e_+ \cos v_+ + e_- \cos v_- + \sum_{k=1}^{2} (e_k \cos v_k \cosh \tilde{v}_k - \tilde{e}_k \sin v_k \sinh \tilde{v}_k), \] (47)
\[ \sin u = e_+ \sin v_+ + e_- \sin v_- + \sum_{k=1}^{2} (e_k \sin v_k \cosh \tilde{v}_k + \tilde{e}_k \cos v_k \sinh \tilde{v}_k). \] (48)

The hyperbolic functions of the 6-complex variable \( u \) are
\[ \cosh u = e_+ \cosh v_+ + e_- \cosh v_- + \sum_{k=1}^{2} (e_k \cosh v_k \cos \tilde{v}_k + \tilde{e}_k \sin v_k \sin \tilde{v}_k), \] (49)
\[ \sinh u = e_+ \sinh v_+ + e_- \sinh v_- + \sum_{k=1}^{2} (e_k \sinh v_k \cos \tilde{v}_k + \tilde{e}_k \cosh v_k \sin \tilde{v}_k). \] (50)

### 2.6 Power series of 6-complex numbers

A power series of the 6-complex variable \( u \) is a series of the form
\[ a_0 + a_1 u + a_2 u^2 + \cdots + a_l u^l + \cdots. \] (51)

Since
\[ |a^l| \leq 6^{l/2} |a||u|^l, \] (52)
the series is absolutely convergent for
\[ |u| < c, \] (53)
where
\[ c = \lim_{l \to \infty} \frac{|a_l|}{\sqrt{6}|a_{l+1}|}. \] (54)

If \( a_t = \sum_{p=0}^{5} h_p a_{tp} \), where \( h_0 = 1 \), and
\[ A_{t+} = \sum_{p=0}^{5} a_{tp}, \] (55)
\[ A_l = \sum_{p=0}^{5} (-1)^p a_{lp}, \quad (56) \]

\[ A_{lk} = \sum_{p=0}^{5} a_{lp} \cos \frac{\pi kp}{3}, \quad (57) \]

\[ \tilde{A}_{lk} = \sum_{p=0}^{5} a_{lp} \sin \frac{\pi kp}{3}, \quad (58) \]

for \( k = 1, 2 \), the series (51) can be written as

\[ \sum_{l=0}^{\infty} \left[ e_{+} A_{l+} v_{+}^{l} + e_{-} A_{l-} v_{-}^{l} + 2 \sum_{k=1}^{2} (e_{k} A_{lk} + \tilde{e}_{k} \tilde{A}_{lk}) (e_{k} v_{k} + \tilde{e}_{k} \tilde{v}_{k})^{l} \right]. \quad (59) \]

The series in Eq. (51) is absolutely convergent for

\[ |v_{+}| < c_{+}, \quad |v_{-}| < c_{-}, \quad \rho_{k} < c_{k}, k = 1, 2, \quad (60) \]

where

\[ c_{+} = \lim_{l \to \infty} \frac{|A_{l+}|}{|A_{l+1,+}|}, \quad c_{-} = \lim_{l \to \infty} \frac{|A_{l-}|}{|A_{l+1,-}|}, \quad c_{k} = \lim_{l \to \infty} \frac{\left( A_{lk}^{2} + \tilde{A}_{lk}^{2} \right)^{1/2}}{\left( A_{l+1,k}^{2} + \tilde{A}_{l+1,k}^{2} \right)^{1/2}}, \quad k = 1, 2. \quad (61) \]

### 2.7 Analytic functions of a polar 6-complex variable

The expansion of an analytic function \( f(u) \) around \( u = u_{0} \) is

\[ f(u) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(u_{0})(u - u_{0})^{k}. \quad (62) \]

Since the limit \( f'(u_{0}) = \lim_{u \to u_{0}} \{ f(u) - f(u_{0}) \} / (u - u_{0}) \) is independent of the direction in space along which \( u \) is approaching \( u_{0} \), the function \( f(u) \) is said to be analytic, analogously to the case of functions of regular complex variables. 

If \( f(u) = \sum_{k=0}^{5} h_{k} P_{k}(x_{0}, x_{1}, x_{2}, x_{3}, x_{4}, x_{5}) \), then

\[ \frac{\partial P_{0}}{\partial x_{0}} = \frac{\partial P_{1}}{\partial x_{1}} = \frac{\partial P_{2}}{\partial x_{2}} = \frac{\partial P_{3}}{\partial x_{3}} = \frac{\partial P_{4}}{\partial x_{4}} = \frac{\partial P_{5}}{\partial x_{5}}, \quad (63) \]

\[ \frac{\partial P_{1}}{\partial x_{0}} = \frac{\partial P_{2}}{\partial x_{1}} = \frac{\partial P_{3}}{\partial x_{2}} = \frac{\partial P_{4}}{\partial x_{3}} = \frac{\partial P_{5}}{\partial x_{4}} = \frac{\partial P_{0}}{\partial x_{5}}, \quad (64) \]

\[ \frac{\partial P_{2}}{\partial x_{0}} = \frac{\partial P_{3}}{\partial x_{1}} = \frac{\partial P_{4}}{\partial x_{2}} = \frac{\partial P_{5}}{\partial x_{3}} = \frac{\partial P_{0}}{\partial x_{4}} = \frac{\partial P_{1}}{\partial x_{5}}, \quad (65) \]

\[ \frac{\partial P_{3}}{\partial x_{0}} = \frac{\partial P_{4}}{\partial x_{1}} = \frac{\partial P_{5}}{\partial x_{2}} = \frac{\partial P_{0}}{\partial x_{3}} = \frac{\partial P_{1}}{\partial x_{4}} = \frac{\partial P_{2}}{\partial x_{5}}, \quad (66) \]
\[
\frac{\partial P_4}{\partial x_0} = \frac{\partial P_3}{\partial x_1} = \frac{\partial P_0}{\partial x_2} = \frac{\partial P_1}{\partial x_3} = \frac{\partial P_2}{\partial x_4} = \frac{\partial P_3}{\partial x_5}, \\
(67) \\
\frac{\partial P_5}{\partial x_0} = \frac{\partial P_0}{\partial x_1} = \frac{\partial P_1}{\partial x_2} = \frac{\partial P_2}{\partial x_3} = \frac{\partial P_3}{\partial x_4} = \frac{\partial P_4}{\partial x_5}, \\
(68)
\]
and
\[
\frac{\partial^2 P_k}{\partial x_0 \partial x_l} = \frac{\partial^2 P_k}{\partial x_1 \partial x_{l-1}} = \cdots = \frac{\partial^2 P_k}{\partial x_{[l/2]} \partial x_{l-[l/2]}} = \frac{\partial^2 P_k}{\partial x_{l+1} \partial x_5} = \frac{\partial^2 P_k}{\partial x_{l+2} \partial x_4} = \cdots = \frac{\partial^2 P_k}{\partial x_{l+[(4-l)/2]} \partial x_{5-[(4-l)/2]}}, \\
(69)
\]
for \(k, l = 0, \ldots, 5\). In Eq. (69), \([a]\) denotes the integer part of \(a\), defined as \([a] \leq a < [a] + 1\).

In this work, brackets larger than the regular brackets \([\ ]\) do not have the meaning of integer part.

### 2.8 Integrals of polar 6-complex functions

If \(f(u)\) is an analytic 6-complex function, then
\[
\oint_{\Gamma} f(u) \frac{du}{u-u_0} = 2\pi f(u_0) \left[ \hat{e}_1 \text{ int}(u_0 \xi_1 \eta_1, \Gamma \xi_1 \eta_1) + \hat{e}_2 \text{ int}(u_0 \xi_2 \eta_2, \Gamma \xi_2 \eta_2) \right], \\
(70)
\]
where
\[
\text{int}(M, C) = \begin{cases} 
1 & \text{if } M \text{ is an interior point of } C, \\
0 & \text{if } M \text{ is exterior to } C,
\end{cases} \\
(71)
\]
and \(u_0 \xi_k \eta_k\) and \(\Gamma \xi_k \eta_k\) are respectively the projections of the pole \(u_0\) and of the loop \(\Gamma\) on the plane defined by the axes \(\xi_k\) and \(\eta_k\), \(k = 1, 2\).

### 2.9 Factorization of 6-complex polynomials

A polynomial of degree \(m\) of the 6-complex variable \(u\) has the form
\[
P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1} u + a_m, \\
(72)
\]
where \(a_l\), for \(l = 1, \ldots, m\), are 6-complex constants. If \(a_l = \sum_{p=0}^{5} h_p a_{lp}\), and with the notations of Eqs. (53)-(58) applied for \(l = 1, \cdots, m\), the polynomial \(P_m(u)\) can be written as
\[
P_m = e_+ \left( v_+^m + \sum_{l=1}^{m} A_{l+} v_+^{m-l} \right) + e_- \left( v_-^m + \sum_{l=1}^{m} A_{l-} v_-^{m-l} \right) \\
+ \sum_{k=1}^{2} \left[ (e_k v_k + \tilde{e}_k \tilde{v}_k)^m + \sum_{l=1}^{m} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk})(e_k v_k + \tilde{e}_k \tilde{v}_k)^{m-l} \right], \\
(73)
\]
where the constants $A_{t+}, A_{t-}, A_{l+k}, \tilde{A}_{l+k}$ are real numbers.

The polynomial $P_m(u)$ can be written, as

$$P_m(u) = \prod_{p=1}^{m} (u - u_p),$$

(74)

where

$$u_p = e_+ v_{p+} + e_- v_{p-} + (e_1 v_{1p} + \tilde{e}_1 \tilde{v}_{1p}) + (e_2 v_{2p} + \tilde{e}_2 \tilde{v}_{2p}), \quad p = 1, \ldots, m.$$  

(75)

The quantities $v_{p+}, v_{p-}, e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}, p = 1, \ldots, m, k = 1, 2$, are the roots of the corresponding polynomial in Eq. (74). The roots $v_{p+}, v_{p-}$ appear in complex-conjugate pairs, and $v_{kp}, \tilde{v}_{kp}$ are real numbers. Since all these roots may be ordered arbitrarily, the polynomial $P_m(u)$ can be written in many different ways as a product of linear factors.

If $P(u) = u^2 - 1$, the degree is $m = 2$, the coefficients of the polynomial are $a_1 = 0, a_2 = -1$, the coefficients defined in Eqs. (54)-(58) are $A_{2+} = -1, A_{2-} = -1, A_{21} = -1, \tilde{A}_{21} = 0, A_{22} = -1, \tilde{A}_{22} = 0$. The expression of $P(u)$, Eq. (73), is $v_+^2 + v_-^2 - e_- + (e_1 v_1 + \tilde{e}_1 \tilde{v}_1)^2 - e_1 + (e_2 v_2 + \tilde{e}_2 \tilde{v}_2)^2 - e_2$. The factorization of $P(u)$, Eq. (74), is $P(u) = (u - u_1)(u - u_2)$, where the roots are $u_1 = \pm e_+ \pm e_- \pm e_1 \pm e_2, u_2 = -u_1$. If $e_+, e_-, e_1, e_2$ are expressed with the aid of Eq. (36) in terms of $h_1, h_2, h_3, h_4, h_5$, the factorizations of $P(u)$ are obtained as

$$u^2 - 1 = (u + 1)(u - 1),$$

$$u^2 - 1 = \left[u + \frac{1}{3}(1 + h_1 + h_2 - 2h_3 + h_4 + h_5)\right] \left[u - \frac{1}{3}(1 + h_1 + h_2 - 2h_3 + h_4 + h_5)\right],$$

$$u^2 - 1 = \left[u + \frac{1}{3}(1 - h_1 + h_2 + 2h_3 + h_4 - h_5)\right] \left[u - \frac{1}{3}(1 - h_1 + h_2 + 2h_3 + h_4 - h_5)\right],$$

$$u^2 - 1 = \left[u + \frac{1}{3}(2 + h_1 - h_2 + h_3 - h_4 + h_5)\right] \left[u - \frac{1}{3}(2 + h_1 - h_2 + h_3 - h_4 + h_5)\right],$$

$$u^2 - 1 = \left[u + \frac{1}{3}(-1 + 2h_2 + 2h_4)\right] \left[u - \frac{1}{3}(-1 + 2h_2 + 2h_4)\right],$$

$$u^2 - 1 = \left[u + \frac{1}{3}(2h_1 - h_3 + 2h_5)\right] \left[u - \frac{1}{3}(2h_1 - h_3 + 2h_5)\right],$$

$$u^2 - 1 = (u + h_3)(u - h_3),$$

$$u^2 - 1 = \left[u + \frac{1}{3}(-2 + h_1 + h_2 + h_3 + h_4 + h_5)\right] \left[u - \frac{1}{3}(-2 + h_1 + h_2 + h_3 + h_4 + h_5)\right].$$

(76)

It can be checked that $(\pm e_+ \pm e_- \pm e_1 \pm e_2)^2 = e_+ + e_- + e_1 + e_2 = 1$. 

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2.10 Representation of polar 6-complex numbers by irreducible matrices

If the unitary matrix which appears in the expression, Eq. (8), of the variables $\xi_+, \xi_-, \xi_1, \eta_1, \xi_k, \eta_k$ in terms of $x_0, x_1, x_2, x_3, x_4, x_5$ is called $T$, the irreducible representation of the hypercomplex number $u$ is

$$TUT^{-1} = \begin{pmatrix} v_+ & 0 & 0 & 0 \\ 0 & v_- & 0 & 0 \\ 0 & 0 & V_1 & 0 \\ 0 & 0 & 0 & V_2 \end{pmatrix},$$

(77)

where $U$ is the matrix in Eq. (25), and $V_k$ are the matrices

$$V_k = \begin{pmatrix} v_k & \tilde{v}_k \\ -\tilde{v}_k & v_k \end{pmatrix}, \quad k = 1, 2.$$  

(78)

3 Planar complex numbers in 6 dimensions

3.1 Operations with planar complex numbers in 6 dimensions

The planar hypercomplex number $u$ in 6 dimensions is represented as

$$u = x_0 + h_1 x_1 + h_2 x_2 + h_3 x_3 + h_4 x_4 + h_5 x_5.$$  

(79)

The multiplication rules for the bases $h_1, h_2, h_3, h_4, h_5$ are

$$h_1^2 = h_2, \quad h_2^2 = h_4, \quad h_3^2 = 1, \quad h_4^2 = -h_2, \quad h_5^2 = -h_4, \quad h_1 h_2 = h_3, \quad h_1 h_3 = h_4, \quad h_1 h_4 = h_5, \quad h_1 h_5 = -1,$$

$$h_2 h_3 = h_5, \quad h_2 h_4 = -1, \quad h_2 h_5 = -h_1, \quad h_3 h_4 = -h_1, \quad h_3 h_5 = -h_2, \quad h_4 h_5 = -h_3.$$  

(80)

The significance of the composition laws in Eq. (80) can be understood by representing the bases $1, h_1, h_2, h_3, h_4, h_5$ by points on a circle at the angles $\alpha_k = \pi k/6$. The product $h_j h_k$ will be represented by the point of the circle at the angle $\pi(j+k)/12$, $j, k = 0, 1, ..., 5$. If $\pi \leq \pi(j+k)/12 \leq 2\pi$, the point is opposite to the basis $h_l$ of angle $\alpha_l = \pi(j+k)/6 - \pi$.

The sum of the 6-complex numbers $u$ and $u'$ is

$$u + u' = x_0 + x'_0 + h_1(x_1 + x'_1) + h_1(x_2 + x'_2) + h_3(x_3 + x'_3) + h_4(x_4 + x'_4) + h_5(x_5 + x'_5).$$  

(81)
The product of the numbers \( u, u' \) is

\[
    uu' = x_0 x'_0 - x_1 x'_1 - x_2 x'_2 + x_3 x'_3 - x_4 x'_4 - x_5 x'_5
\]
\[
    + h_1(x_0 x'_1 + x_1 x'_0 - x_2 x'_2 - x_3 x'_3 - x_4 x'_4 - x_5 x'_5)
\]
\[
    + h_2(x_0 x'_2 + x_1 x'_1 + x_2 x'_0 - x_3 x'_3 - x_4 x'_4 - x_5 x'_5)
\]
\[
    + h_3(x_0 x'_3 + x_1 x'_2 + x_2 x'_1 + x_3 x'_0 - x_4 x'_4 - x_5 x'_5)
\]
\[
    + h_4(x_0 x'_4 + x_1 x'_3 + x_2 x'_2 + x_3 x'_1 + x_4 x'_0 - x_5 x'_5)
\]
\[
    + h_5(x_0 x'_5 + x_1 x'_4 + x_2 x'_3 + x_3 x'_2 + x_4 x'_1 + x_5 x'_0).
\]

The relation between the variables \( v_1, \tilde{v}_1, v_2, \tilde{v}_2, v_3, \tilde{v}_3 \) and \( x_0, x_1, x_2, x_3, x_4, x_5 \) are

\[
    \begin{pmatrix}
        v_1 \\
        \tilde{v}_1 \\
        v_2 \\
        \tilde{v}_2 \\
        v_3 \\
        \tilde{v}_3
    \end{pmatrix}
    =
    \begin{pmatrix}
        1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
        0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\
        1 & 0 & -1 & 0 & 1 & 0 \\
        0 & 1 & 0 & -1 & 0 & 1 \\
        1 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
        0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & -\frac{\sqrt{3}}{2} & \frac{1}{2}
    \end{pmatrix}
    \begin{pmatrix}
        x_0 \\
        x_1 \\
        x_2 \\
        x_3 \\
        x_4 \\
        x_5
    \end{pmatrix}.
\]

The other variables are \( v_4 = v_3, \tilde{v}_4 = -\tilde{v}_3, v_5 = v_2, \tilde{v}_5 = -\tilde{v}_2, v_6 = v_1, \tilde{v}_6 = -\tilde{v}_1 \). The variables \( v_1, \tilde{v}_1, v_2, \tilde{v}_2, v_3, \tilde{v}_3 \) will be called canonical planar 6-complex variables.

### 3.2 Geometric representation of planar complex numbers in 6 dimensions

The 6-complex number \( u = x_0 + h_1 x_1 + h_2 x_2 + h_3 x_3 + h_4 x_4 + h_5 x_5 \) is represented by the point \( A \) of coordinates \((x_0, x_1, x_2, x_3, x_4, x_5)\). The distance from the origin \( O \) of the 6-dimensional space to the point \( A \) has the expression

\[
    d^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2,
\]

is called modulus of the 6-complex number \( u \), and is designated by \( d = |u| \). The modulus has the property that

\[
    |u'u''| \leq \sqrt{3}|u'||u''|.
\]
The exponential and trigonometric forms of the 6-complex number $u$ can be obtained conveniently in a rotated system of axes defined by a transformation which has the form

$$
\begin{pmatrix}
\xi_1 \\
\eta_1 \\
\xi_2 \\
\eta_2 \\
\xi_3 \\
\eta_3
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2} \\
0 & \frac{1}{2\sqrt{3}} & \frac{1}{2} & \frac{1}{\sqrt{3}} & \frac{1}{2} & \frac{1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & -\frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2} \\
0 & \frac{1}{2\sqrt{3}} & -\frac{1}{2} & \frac{1}{\sqrt{3}} & -\frac{1}{2} & \frac{1}{2\sqrt{3}}
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}.
$$

The lines of the matrices in Eq. (86) give the components of the 6 vectors of the new basis system of axes. These vectors have unit length and are orthogonal to each other. The relations between the two sets of variables are

$$v_k = \sqrt{3} \xi_k, \tilde{v}_k = \sqrt{3} \eta_k,$$

for $k = 1, 2, 3$.

The radius $\rho_k$ and the azimuthal angle $\phi_k$ in the plane of the axes $v_k, \tilde{v}_k$ are

$$\rho_k^2 = v_k^2 + \tilde{v}_k^2, \quad \cos \phi_k = v_k / \rho_k, \quad \sin \phi_k = \tilde{v}_k / \rho_k,$$

where $0 \leq \phi_k < 2\pi$, $k = 1, 2, 3$, so that there are 3 azimuthal angles. The planar angles $\psi_{k-1}$ are

$$\tan \psi_1 = \rho_1 / \rho_2, \quad \tan \psi_2 = \rho_1 / \rho_3,$$

where $0 \leq \psi_1 \leq \pi/2$, $0 \leq \psi_2 \leq \pi/2$, so that there are 2 planar angles. The amplitude of an 6-complex number $u$ is

$$\rho = (\rho_1 \rho_2 \rho_3)^{1/3}.$$  

It can be checked that

$$d^2 = \frac{1}{3} (\rho_1^2 + \rho_2^2 + \rho_3^2).$$

If $u = u'u''$, the parameters of the hypercomplex numbers are related by

$$\rho_k = \rho_k^i \rho_k^u,$$
\[
\tan \psi_k = \tan \psi'_k \tan \psi''_k, \quad (93)
\]
\[
\phi_k = \phi'_k + \phi''_k, \quad (94)
\]
\[
v_k = v'_k v''_k - \tilde{v}'_k \tilde{v}''_k, \quad \tilde{v}_k = v'_k \tilde{v}''_k + \tilde{v}'_k v''_k, \quad (95)
\]
\[
\rho = \rho' \rho'', \quad (96)
\]

where \( k = 1, 2, 3 \).

The 6-complex planar number \( u = x_0 + h_1 x_1 + h_2 x_2 + h_3 x_3 + h_4 x_4 + h_5 x_5 \) can be represented by the matrix

\[
U = \begin{pmatrix}
x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\
-x_5 & x_0 & x_1 & x_2 & x_3 & x_4 \\
-x_4 & -x_5 & x_0 & x_1 & x_2 & x_3 \\
-x_3 & -x_4 & -x_5 & x_0 & x_1 & x_2 \\
-x_2 & -x_3 & -x_4 & -x_5 & x_0 & x_1 \\
-x_1 & -x_2 & -x_3 & -x_4 & -x_5 & x_0
\end{pmatrix}. \quad (97)
\]

The product \( u = u' u'' \) is represented by the matrix multiplication \( U = U' U'' \).

### 3.3 The planar 6-dimensional cosexponential functions

The planar cosexponential functions in 6 dimensions are

\[
f_{6k}(y) = \sum_{p=0}^{\infty} (-1)^p \frac{y^{k+6p}}{(k+6p)!}, \quad (98)
\]

for \( k = 0, \ldots, 5 \). The planar cosexponential functions of even index \( k \) are even functions, \( f_{6,2l}(-y) = f_{6,2l}(y) \), and the planar cosexponential functions of odd index are odd functions, \( f_{6,2l+1}(-y) = -f_{6,2l+1}(y) \), \( l = 0, 1, 2 \). The exponential function of the quantity \( h_k y \) is

\[
e^{h_1 y} = f_{60}(y) + h_1 f_{61}(y) + h_2 f_{62}(y) + h_3 f_{63}(y) + h_4 f_{64}(y) + h_5 f_{65}(y),
\]
\[
e^{h_2 y} = g_{60}(y) - g_{63}(y) + h_2 \{g_{61}(y) - g_{64}(y)\} + h_4 \{g_{62}(y) - g_{65}(y)\},
\]
\[
e^{h_3 y} = f_{60}(y) + f_{62}(y) + f_{64}(y) + h_3 \{f_{61}(y) - f_{63}(y) + f_{65}(y)\},
\]
\[
e^{h_4 y} = g_{60}(y) + g_{63}(y) - h_2 \{g_{62}(y) + g_{65}(y)\} + h_4 \{g_{61}(y) + g_{64}(y)\},
\]
\[
e^{h_5 y} = f_{60}(y) + h_1 f_{65}(y) - h_2 f_{64}(y) + h_3 f_{63}(y) - h_4 f_{62}(y) + h_5 f_{61}(y).
\]
The relations for $h_2$ and $h_4$ can be written equivalently as $e^{h_2 y} = f_{30} + h_2 f_{31} + h_4 f_{32}, e^{h_4 y} = g_{30} - h_2 f_{32} + h_4 g_{31},$ and the relation for $h_3$ can be written as $e^{h_3 y} = f_{20} + h_3 f_{21},$ which is the same as $e^{h_3 y} = \cos y + h_3 \sin y.$

The planar 6-dimensional cosexponential functions $f_{6k}(y)$ are related to the polar 6-dimensional cosexponential function $g_{6k}(y)$ by the relations

$$f_{6k}(y) = e^{-i\pi k/6} g_{6k} \left( e^{i\pi/6} y \right),$$

for $k = 0, ..., 5$. The planar 6-dimensional cosexponential functions $f_{6k}(y)$ are related to the polar 6-dimensional cosexponential function $g_{6k}(y)$ also by the relations

$$f_{6k}(y) = e^{-i\pi k/2} g_{6k}(iy),$$

for $k = 0, ..., 5$. The expressions of the planar 6-dimensional cosexponential functions are

$$f_{60}(y) = \frac{1}{3} \cos y + \frac{2}{3} \cosh \frac{\sqrt{3}}{2} y \cos \frac{y}{2},$$

$$f_{61}(y) = \frac{1}{3} \sin y + \frac{\sqrt{3}}{3} \sinh \frac{\sqrt{3}}{2} y \cos \frac{y}{2} + \frac{1}{3} \cosh \frac{\sqrt{3}}{2} y \sin \frac{y}{2},$$

$$f_{62}(y) = -\frac{1}{3} \cos y + \frac{1}{3} \cosh \frac{\sqrt{3}}{2} y \cos \frac{y}{2} + \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} y \sin \frac{y}{2},$$

$$f_{63}(y) = -\frac{1}{3} \sin y + \frac{2}{3} \cosh \frac{\sqrt{3}}{2} y \sin \frac{y}{2},$$

$$f_{64}(y) = \frac{1}{3} \cos y - \frac{1}{3} \cosh \frac{\sqrt{3}}{2} y \cos \frac{y}{2} + \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} y \sin \frac{y}{2},$$

$$f_{65}(y) = \frac{1}{3} \sin y - \frac{\sqrt{3}}{3} \sinh \frac{\sqrt{3}}{2} y \cos \frac{y}{2} + \frac{1}{3} \cosh \frac{\sqrt{3}}{2} y \sin \frac{y}{2}.$$

The planar 6-dimensional cosexponential functions can be written as

$$f_{6k}(y) = \frac{1}{6} \sum_{l=1}^{6} \exp \left[ y \cos \left( \frac{\pi (2l - 1)}{6} \right) \right] \cos \left[ y \sin \left( \frac{\pi (2l - 1)}{6} \right) \right],$$

for $k = 0, ..., 5$. The graphs of the planar 6-dimensional cosexponential functions are shown in Fig. 4.

It can be checked that

$$\sum_{k=0}^{5} f_{6k}^2(y) = \frac{1}{3} + \frac{2}{3} \cosh \sqrt{3} y.$$
It can be shown that

\[
\{f_60(y) + h_1f_61(y) + h_2f_62(y) + h_3f_63(y) + h_4f_64(y) + h_5f_65(y)\}^l = f_60(ly) + h_1f_61(ly) + h_2f_62(ly) + h_3f_63(ly) + h_4f_64(ly) + h_5f_65(ly),
\]
\[
\{g_60(y) - g_63(y) + h_2\{g_61(y) - g_64(y)\} + h_4\{g_62(y) - g_65(y)\}\}^l = g_60(ly) - g_63(ly) + h_2\{g_61(ly) - g_64(ly)\} + h_4\{g_62(ly) - g_65(ly)\},
\]
\[
\{f_60(y) - f_62(y) + f_64(y) + h_3\{f_61(y) - f_63(y) + f_65(y)\}\}^l = f_60(ly) - f_62(ly) + f_64(ly) + h_3\{f_61(ly) - f_63(ly) + f_65(ly)\},
\]
\[
\{g_60(y) + g_63(y) - h_2\{g_62(y) + g_64(y)\} + h_4\{g_61(y) + g_64(y)\}\}^l = g_60(ly) + g_63(ly) - h_2\{g_62(ly) + g_64(ly)\} + h_4\{g_61(ly) + g_64(ly)\},
\]
\[
\{f_60(y) + h_1f_65(y) - h_2f_64(y) + h_3f_63(y) - h_4f_62(y) + h_5f_61(y)\}^l = f_60(ly) + h_1f_65(ly) - h_2f_64(ly) + h_3f_63(ly) - h_4f_62(ly) + h_5f_61(ly).
\]

The derivatives of the planar cossexponential functions are related by

\[
\frac{df_{60}}{du} = -f_{65}, \quad \frac{df_{61}}{du} = f_{60}, \quad \frac{df_{62}}{du} = f_{61}, \quad \frac{df_{63}}{du} = f_{62}, \quad \frac{df_{64}}{du} = f_{63}, \quad \frac{df_{65}}{du} = f_{64}. \tag{107}
\]

### 3.4 Exponential and trigonometric forms of planar 6-complex numbers

The exponential and trigonometric forms of planar 6-complex numbers can be expressed with the aid of the hypercomplex bases

\[
\begin{pmatrix}
  e_1 \\
  \tilde{e}_1 \\
  e_2 \\
  \tilde{e}_2 \\
  e_3 \\
  \tilde{e}_3
\end{pmatrix} = \begin{pmatrix}
  \frac{\sqrt{3}}{6} & \frac{1}{3} & 0 & -\frac{1}{6} & -\frac{\sqrt{3}}{6} \\
  0 & \frac{1}{6} & \frac{\sqrt{3}}{6} & \frac{1}{3} & \frac{\sqrt{3}}{6} \\
  \frac{1}{3} & 0 & -\frac{1}{3} & 0 & \frac{1}{3} \\
  0 & \frac{1}{3} & 0 & -\frac{1}{3} & 0 \\
  \frac{1}{3} & -\frac{\sqrt{3}}{6} & \frac{1}{6} & 0 & -\frac{\sqrt{3}}{6} \\
  0 & \frac{1}{6} & -\frac{\sqrt{3}}{6} & \frac{1}{3} & -\frac{\sqrt{3}}{6}
\end{pmatrix} \begin{pmatrix}
  1 \\
  h_1 \\
  h_2 \\
  h_3 \\
  h_4 \\
  h_5
\end{pmatrix}. \tag{108}
\]

The multiplication relations for the bases \(e_k, \tilde{e}_k\) are

\[
e_k^2 = e_k, \tilde{e}_k^2 = -e_k, e_k\tilde{e}_k = \tilde{e}_k, e_k e_l = 0, e_k \tilde{e}_l = 0, \tilde{e}_k \tilde{e}_l = 0, k, l = 1, 2, 3, k \neq l. \tag{109}
\]

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The moduli of the bases $e_k, \tilde{e}_k$ are

$$|e_k| = \sqrt{\frac{1}{3}}, |\tilde{e}_k| = \sqrt{\frac{1}{3}},$$

(110)

for $k = 1, 2, 3$. It can be shown that

$$x_0 + h_1 x_1 + h_2 x_2 + h_3 x_3 + h_4 x_4 + h_5 x_5 = \sum_{k=1}^{3} (e_k v_k + \tilde{e}_k \tilde{v}_k).$$

(111)

The ensemble $e_1, \tilde{e}_1, e_2, \tilde{e}_2, e_3, \tilde{e}_3$ will be called the canonical planar 6-complex base, and Eq. (111) gives the canonical form of the planar 6-complex number.

The exponential form of the 6-complex number $u$ is

$$u = \rho \exp \left\{ \frac{1}{3} (h_2 - h_4) \ln \tan \psi_1 + \frac{1}{6} (\sqrt{3} h_1 - h_2 + h_4 - \sqrt{3} h_5) \ln \tan \psi_2 + \tilde{e}_1 \phi_1 + \tilde{e}_2 \phi_2 + \tilde{e}_3 \phi_3 \right\}.$$

(112)

The trigonometric form of the 6-complex number $u$ is

$$u = d \sqrt{3} \left( 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} \right)^{-1/2} \left( e_1 + \frac{e_2}{\tan \psi_1} + \frac{e_3}{\tan \psi_2} \right) \exp (\tilde{e}_1 \phi_1 + \tilde{e}_2 \phi_2 + \tilde{e}_3 \phi_3).$$

(113)

The modulus $d$ and the amplitude $\rho$ are related by

$$d = \rho \frac{2^{1/3}}{\sqrt{6}} (\tan \psi_1 \tan \psi_2)^{1/3} \left( 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} \right)^{1/2}.$$

(114)

### 3.5 Elementary functions of a planar 6-complex variable

The logarithm and power functions of the 6-complex number $u$ exist for all $x_0, ..., x_5$ and are

$$\ln u = \ln \rho + \frac{1}{3} (h_2 - h_4) \ln \tan \psi_1 + \frac{1}{6} (\sqrt{3} h_1 - h_2 + h_4 - \sqrt{3} h_5) \ln \tan \psi_2 + \tilde{e}_1 \phi_1 + \tilde{e}_2 \phi_2 + \tilde{e}_3 \phi_3,$$

(115)

$$u^m = \sum_{k=1}^{3} \rho_k^m (e_k \cos m \phi_k + \tilde{e}_k \sin m \phi_k).$$

(116)

The exponential of the 6-complex variable $u$ is

$$e^u = \sum_{k=1}^{3} e^{\nu_k} (e_k \cos \tilde{v}_k + \tilde{e}_k \sin \tilde{v}_k).$$

(117)

The trigonometric functions of the 6-complex variable $u$ are

$$\cos u = \sum_{k=1}^{3} (e_k \cos v_k \cosh \tilde{v}_k - \tilde{e}_k \sin v_k \sinh \tilde{v}_k),$$

(118)
\[
\sin u = \sum_{k=1}^{3} (e_k \sin v_k \cosh \tilde{v}_k + \tilde{e}_k \cos v_k \sinh \tilde{v}_k).
\]  \hfill (119)

The hyperbolic functions of the 6-complex variable \( u \) are
\[
cosh u = \sum_{k=1}^{3} (e_k \cosh v_k \cos \tilde{v}_k + \tilde{e}_k \sinh v_k \sin \tilde{v}_k),
\]  \hfill (120)
\[
\sinh u = \sum_{k=1}^{3} (e_k \sinh v_k \cos \tilde{v}_k + \tilde{e}_k \cosh v_k \sin \tilde{v}_k).
\]  \hfill (121)

### 3.6 Power series of 6-complex numbers

A power series of the 6-complex variable \( u \) is a series of the form
\[
a_0 + a_1 u + a_2 u^2 + \cdots + a_l u^l + \cdots.
\]  \hfill (122)

Since
\[
|au^l| \leq 3^{l/2}|a||u|^l,
\]  \hfill (123)

the series is absolutely convergent for
\[
|u| < c,
\]  \hfill (124)

where
\[
c = \lim_{l \to \infty} \frac{|a_l|}{\sqrt{3} |a_{l+1}|}.
\]  \hfill (125)

If \( a_l = \sum_{p=0}^{5} h_p a_{lp} \), and
\[
A_{lk} = \sum_{p=0}^{5} a_{lp} \cos \frac{p(2k-1)}{6},
\]  \hfill (126)
\[
\tilde{A}_{lk} = \sum_{p=0}^{5} a_{lp} \sin \frac{p(2k-1)}{6},
\]  \hfill (127)

where \( k = 1, 2, 3 \), the series (122) can be written as
\[
\sum_{l=0}^{\infty} \left[ \sum_{k=1}^{3} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk})(e_k v_k + \tilde{e}_k \tilde{v}_k)^l \right].
\]  \hfill (128)

The series is absolutely convergent for
\[
\rho_k < c_k, k = 1, 2, 3,
\]  \hfill (129)

where
\[
c_k = \lim_{l \to \infty} \frac{[A_{lk}^2 + \tilde{A}_{lk}^2]^{1/2}}{[A_{l+1,k}^2 + \tilde{A}_{l+1,k}^2]^{1/2}}.
\]  \hfill (130)
3.7 Analytic functions of a planar 6-complex variable

The expansion of an analytic function \( f(u) \) around \( u = u_0 \) is

\[
f(u) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(u_0)(u - u_0)^k.
\]

(131)

If \( f(u) = \sum_{k=0}^{5} h_k P_k(x_0, ..., x_5) \), then

\[
\frac{\partial P_0}{\partial x_0} = \frac{\partial P_1}{\partial x_1} = \frac{\partial P_2}{\partial x_2} = \frac{\partial P_3}{\partial x_3} = \frac{\partial P_4}{\partial x_4} = \frac{\partial P_5}{\partial x_5},
\]

(132)

\[
\frac{\partial P_1}{\partial x_0} = \frac{\partial P_2}{\partial x_1} = \frac{\partial P_3}{\partial x_2} = \frac{\partial P_4}{\partial x_3} = \frac{\partial P_5}{\partial x_4} = -\frac{\partial P_0}{\partial x_5},
\]

(133)

\[
\frac{\partial P_2}{\partial x_0} = \frac{\partial P_3}{\partial x_1} = \frac{\partial P_4}{\partial x_2} = \frac{\partial P_5}{\partial x_3} = -\frac{\partial P_0}{\partial x_4} = -\frac{\partial P_1}{\partial x_5},
\]

(134)

\[
\frac{\partial P_3}{\partial x_0} = \frac{\partial P_4}{\partial x_1} = \frac{\partial P_5}{\partial x_2} = -\frac{\partial P_0}{\partial x_3} = -\frac{\partial P_1}{\partial x_4} = -\frac{\partial P_2}{\partial x_5},
\]

(135)

\[
\frac{\partial P_4}{\partial x_0} = \frac{\partial P_5}{\partial x_1} = -\frac{\partial P_0}{\partial x_2} = -\frac{\partial P_1}{\partial x_3} = -\frac{\partial P_2}{\partial x_4} = -\frac{\partial P_3}{\partial x_5},
\]

(136)

\[
\frac{\partial P_5}{\partial x_0} = -\frac{\partial P_0}{\partial x_1} = -\frac{\partial P_1}{\partial x_2} = -\frac{\partial P_2}{\partial x_3} = -\frac{\partial P_3}{\partial x_4} = -\frac{\partial P_4}{\partial x_5},
\]

(137)

and

\[
\frac{\partial^2 P_k}{\partial x_0 \partial x_l} = \frac{\partial^2 P_k}{\partial x_1 \partial x_{l-1}} = \cdots = \frac{\partial^2 P_k}{\partial x_{[l/2]} \partial x_{l-[l/2]}} = -\frac{\partial^2 P_k}{\partial x_{l+1} \partial x_5} = -\frac{\partial^2 P_k}{\partial x_{l+2} \partial x_4} = \cdots = -\frac{\partial^2 P_k}{\partial x_{l+[4-l/2]} \partial x_{5-[4-l/2]}}.
\]

(138)

3.8 Integrals of planar 6-complex functions

If \( f(u) \) is an analytic 6-complex function, then

\[
\oint_{\Gamma} \frac{f(u)}{u - u_0} \, du = 2\pi f(u_0) \left\{ \tilde{e}_1 \int \text{int} (u_0, \xi_1, \eta_1) + \tilde{e}_2 \int \text{int} (u_0, \xi_2, \eta_2) + \tilde{e}_3 \int \text{int} (u_0, \xi_3, \eta_3) \right\},
\]

(139)

where \( u_0, \eta_k \) and \( \xi_k, \eta_k \) are respectively the projections of the point \( u_0 \) and of the loop \( \Gamma \) on the plane defined by the axes \( \xi_k \) and \( \eta_k, \) \( k = 1, 2, 3.\)
### 3.9 Factorization of 6-complex polynomials

A polynomial of degree $m$ of the 6-complex variable $u$ has the form

$$P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1} u + a_m,$$  \hspace{1cm} (140)

where $a_l$, for $l = 1, \ldots, m$, are 6-complex constants. If $a_l = \sum_{p=0}^{5} h_p a_{lp}$, and with the notations of Eqs. (126)-(127) applied for $l = 1, \cdots, m$, the polynomial $P_m(u)$ can be written as

$$P_m = \sum_{k=1}^{3} \left[ (e_k v_k + \tilde{e}_k \tilde{v}_k)^m + \sum_{l=1}^{m} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk})(e_k v_k + \tilde{e}_k \tilde{v}_k)^{m-l} \right],$$  \hspace{1cm} (141)

where the constants $A_{lk}, \tilde{A}_{lk}$ are real numbers.

The polynomial $P_m(u)$ can be written as a product of factors

$$P_m(u) = \prod_{p=1}^{m} (u - u_p),$$  \hspace{1cm} (142)

where

$$u_p = \sum_{k=1}^{3} (e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}),$$  \hspace{1cm} (143)

for $p = 1, \ldots, m$. The quantities $e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}$, $p = 1, \ldots, m, k = 1, 2, 3$, are the roots of the corresponding polynomial in Eq. (141) and are real numbers. Since these roots may be ordered arbitrarily, the polynomial $P_m(u)$ can be written in many different ways as a product of linear factors.

If $P(u) = u^2 + 1$, the degree is $m = 2$, the coefficients of the polynomial are $a_1 = 0, a_2 = 1$, the coefficients defined in Eqs. (126)-(127) are $A_{21} = 1, \tilde{A}_{21} = 0, A_{22} = 1, \tilde{A}_{22} = 0, A_{23} = 1, \tilde{A}_{23} = 0$. The expression, Eq. (141), is $P(u) = (e_1 v_1 + \tilde{e}_1 \tilde{v}_1)^2 + e_1 + (e_2 v_2 + \tilde{e}_2 \tilde{v}_2)^2 + e_2 + (e_3 v_3 + \tilde{e}_3 \tilde{v}_3)^2 + e_3$. The factorization of $P(u)$, Eq. (142), is $P(u) = (u - u_1)(u - u_2)$, where the roots are $u_1 = \pm \tilde{e}_1 \pm \tilde{e}_2 \pm \tilde{e}_3, u_2 = -u_1$. If $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ are expressed with the aid of Eq. (108) in terms of $h_1, h_2, h_3, h_4, h_5$, the factorizations of $P(u)$ are obtained as

\[
\begin{align*}
    u^2 + 1 &= \left[ u + \frac{1}{3}(2h_1 + h_3 + 2h_5) \right] \left[ u - \frac{1}{3}(2h_1 + h_3 + 2h_5) \right], \\
    u^2 + 1 &= \left[ u + \frac{1}{3}(h_1 + \sqrt{3}h_2 - h_3 + \sqrt{3}h_4 + h_5) \right] \left[ u - \frac{1}{3}(h_1 + \sqrt{3}h_2 - h_3 + \sqrt{3}h_4 + h_5) \right], \\
    u^2 + 1 &= (u + h_3)(u - h_3), \\
    u^2 + 1 &= \left[ u + \frac{1}{3}(-h_1 + \sqrt{3}h_2 + h_3 + \sqrt{3}h_4 - h_5) \right] \left[ u - \frac{1}{3}(-h_1 + \sqrt{3}h_2 + h_3 + \sqrt{3}h_4 - h_5) \right].
\end{align*}
\](144)

It can be checked that $(\pm \tilde{e}_1 \pm \tilde{e}_2 \pm \tilde{e}_3)^2 = -e_1 - e_2 - e_3 = -1$. 

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3.10 Representation of planar 6-complex numbers by irreducible matrices

If the unitary matrix written in Eq. (86) is called \(T\), the matrix \(TUT^{-1}\) provides an irreducible representation of the planar hypercomplex number \(u\),

\[
TUT^{-1} = \begin{pmatrix}
V_1 & 0 & 0 \\
0 & V_2 & 0 \\
0 & 0 & V_3 
\end{pmatrix},
\]

(145)

where \(U\) is the matrix in Eq. (74) used to represent the 6-complex number \(u\), and the matrices \(V_k\) are

\[
V_k = \begin{pmatrix}
v_k & \tilde{v}_k \\
-\tilde{v}_k & v_k 
\end{pmatrix},
\]

(146)

for \(k = 1, 2, 3\).

4 Conclusions

The operations of addition and multiplication of the polar 6-complex numbers introduced in this work have a geometric interpretation based on the amplitude \(\rho\), the modulus \(d\) and the polar, planar and azimuthal angles \(\theta_+, \theta_-, \psi_1, \phi_1, \phi_2\). If \(v_+ > 0\) and \(v_- > 0\), the polar 6-complex numbers can be written in exponential and trigonometric forms with the aid of the modulus, amplitude and the angular variables. The polar 6-complex functions defined by series of powers are analytic, and the partial derivatives of the components of the 6-complex functions are closely related. The integrals of polar 6-complex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the polar 6-complex numbers depends on the cyclic variables \(\phi_1, \phi_2\) leads to the concept of pole and residue for integrals on closed paths. The polynomials of polar 6-complex variables can be written as products of linear or quadratic factors.

The operations of addition and multiplication of the planar 6-complex numbers introduced in this work have a geometric interpretation based on the amplitude \(\rho\), the modulus \(d\), the planar angles \(\psi_1, \psi_2\) and the azimuthal angles \(\phi_1, \phi_2, \phi_3\). The planar 6-complex numbers can
be written in exponential and trigonometric forms with the aid of these variables. The planar 6-complex functions defined by series of powers are analytic, and the partial derivatives of the components of the 6-complex functions are closely related. The integrals of planar 6-complex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the 6-complex numbers depends on the cyclic variables $\phi_1, \phi_2, \phi_3$ leads to the concept of pole and residue for integrals on closed paths. The polynomials of planar 6-complex variables can always be written as products of linear factors, although the factorization is not unique.

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FIGURE CAPTIONS

Fig. 1. Representation of the polar hypercomplex bases $1, h_1, h_2, h_3, h_4, h_5$ by points on a circle at the angles $\alpha_k = 2\pi k/6$. The product $h_j h_k$ will be represented by the point of the circle at the angle $2\pi(j + k)/6$, $i, k = 0, 1, ..., 5$, where $h_0 = 1$. If $2\pi \leq 2\pi(j + k)/6 \leq 4\pi$, the point represents the basis $h_l$ of angle $\alpha_l = 2\pi(j + k)/6 - 2\pi$.

Fig. 2. Polar cosexponential functions $g_{60}, g_{61}, g_{62}, g_{63}, g_{64}, g_{65}$.

Fig. 3. Representation of the planar hypercomplex bases $1, h_1, h_2, h_3, h_4, h_5$ by points on a circle at the angles $\alpha_k = \pi k/6$. The product $h_j h_k$ will be represented by the point of the circle at the angle $\pi(j + k)/12$, $i, k = 0, 1, ..., 5$. If $\pi \leq \pi(j + k)/12 \leq 2\pi$, the point is opposite to the basis $h_l$ of angle $\alpha_l = \pi(j + k)/6 - \pi$.

Fig. 4. Planar cosexponential functions $f_{60}, f_{61}, f_{62}, f_{63}, f_{64}, f_{65}$.
Fig. 1
Fig. 3
