A NEW PROOF OF THE NONSOLVABLE SIGNALIZER FUNCTOR THEOREM

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ABSTRACT. The Signalizer Functor Method as developed by Gorenstein and Walter played a fundamental role in the first proof of the Classification of the Finite Simple Groups. It plays a similar role in the new proof of the Classification in the Gorenstein-Lyons-Solomon book series. The key results are Glauberman’s Solvable Signalizer Functor Theorem and McBride’s Nonsolvable Signalizer Functor Theorem. Given their fundamental role, it is desirable to have new and different proofs of them. This is accomplished in A new proof of the Solvable Signalizer Functor Theorem, P. Flavell, J. Algebra, 398 (2014) 350–363 for Glauberman’s Theorem. The purpose of this paper is to give a new proof of McBride’s Theorem.

1. INTRODUCTION

The Signalizer Functor Method as developed by Gorenstein and Walter played a fundamental role in the first proof of the Classification of the Finite Simple Groups. It plays a similar role in the new proof of the Classification in the Gorenstein-Lyons-Solomon book series [11]. A discussion of the method may be found in [11][12][13]. The key results being Glauberman’s Solvable Signalizer Functor Theorem [10] and McBride’s Nonsolvable Signalizer Functor Theorem [15][16]. They are taken as background results in the Gorenstein-Lyons-Solomon project and not reproved there. Given their fundamental role, it is desirable to have new and different proofs. This is accomplished in [3] for Glauberman’s Theorem. The purpose of this paper is to give a new proof of McBride’s Theorem.

We have taken the liberty of combining the theorems of Glauberman and McBride into a single result. We shall prove:

The Signalizer Functor Theorem. Let $A$ be a finite abelian group of rank at least 3 that acts on the group $G$. Let $\theta$ be an $A$-signalizer functor on $G$ and assume that $\theta(a)$ is a $K$-group for all $a \in A^\#$. Then $\theta$ is complete.

Moreover, the composition factors of the completion of $\theta$ are to be found amongst the composition factors of the subgroups $\theta(a); a \in A^\#$.

Recall that by definition, $\theta$ ia a mapping that assigns to each $a \in A^\#$ a finite $A$-invariant subgroup $\theta(a)$ of $C_G(a)$ with order coprime to $|A|$ that satisfies

$$\theta(a) \cap C_G(b) \leq \theta(b)$$
for all \( a, b \in A^\# \). Note that \( G \) is not assumed to be finite. To say that \( \theta \) is complete means there exists a finite \( A \)-invariant subgroup \( K \), of order coprime to \( |A| \), such that

\[
\theta(a) = C_K(a)
\]

for all \( a \in A^\# \). In particular, the subgroup generated by the subgroups \( \theta(a) \) is finite with order coprime to \( |A| \). An exposition of elementary signalizer functor theory may be found in [3].

Recall also that a \( K \)-group is a finite group all of whose simple sections are known simple groups. The \( K \)-group assumption indicates that some portions of the argument rely on properties of simple groups that are established by taxonomy. The main application of the Signalizer Functor Theorem is to construct large subgroups in a minimal counterexample to the Classification Theorem. Thus, whilst not ideal, the \( K \)-group assumption causes no difficulty.

The proof of McBride’s Theorem presented here is very different from the original. It is based on the author’s proof of Glauberman’s Theorem and a general theory of automorphisms of finite groups as developed in [6]. We prefer the view that the Signalizer Functor Theorem is not a single isolated result but rather one of the high points of a well developed theory of automorphisms of finite groups. Indeed, although much of the material in [6] was motivated by the present work, it has been developed in much greater depth and generality than is required for the proof of the Signalizer Functor Theorem.

Sections §2,...,§7 consist mainly of statements of the general theory required and in §8 the proof begins.

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## 2. Preliminaries

The reader is assumed to be familiar with elementary signalizer functor theory, see for example [3] or [14]. An understanding of the author’s proof of the Solvable Signalizer Functor Theorem [3] would be advantageous.

Unless stated otherwise, the word group will mean finite group. The reader is assumed to be familiar with the notions of the Fitting subgroup, the set of components, the layer and the generalized Fitting subgroup of a group \( G \) denoted by \( F(G) \), \( \text{comp}(G) \), \( E(G) \) and \( F^*(G) \) respectively. See for example [14]. The notation \( \text{sol}(G) \) is used to denote the largest normal solvable subgroup of \( G \). We will need a number of variations of the notion of component as developed in [6].

**Definition 2.1.** A sol-component of \( G \) is a perfect subnormal subgroup of \( G \) that maps onto a component of \( G/\text{sol}(G) \). The set of sol-components of \( G \) is denoted by \( \text{comp}_{\text{sol}}(G) \) and we define

\[
E_{\text{sol}}(G) = \langle \text{comp}_{\text{sol}}(G) \rangle \quad \text{and} \quad O_4(G) = \text{sol}(G)E_{\text{sol}}(G).
\]

**Lemma 2.2.** Let \( G \) be a group.

(a) The sol-components of \( G \) are the minimal nonsolvable subnormal subgroups of \( G \).
(b) Set $\overline{G} = G/\text{sol}(G)$. The map $K \mapsto \overline{K}$ is a bijection $\text{comp}_{\text{sol}}(G) \rightarrow \text{comp}(G)$.
(c) If $K \in \text{comp}_{\text{sol}}(G)$ and $N \leq G$ then $K \leq N$ or $N \leq N_G(K)$.
(d) Distinct sol-components of $G$ normalize each other and commute modulo sol($G$).
(e) If $K \in \text{comp}_{\text{sol}}(G)$ then $K \leq O_s(G)$.
(f) (McBride) If $H$ satisfies $O_s(G) \leq H \leq G$ then $O_s(G) = O_s(H)$.

Proof. (a), . . . , (e) are well known and elementary, see for example [6, Lemma 3.2]. For (f) see [15, Lemma 2.15] or [6, Lemma 8.2].

Next we bring in a group of automorphisms.

**Definition 2.3.** Let the group $A$ act on the group $G$.
(a) $G$ is $A$-simple if $G$ is nonabelian and the only $A$-invariant normal subgroups of $G$ are 1 and $G$.
(b) $G$ is $A$-quasisimple of $G$ is perfect and $G/Z(G)$ is $A$-simple.
(c) An $A$-component of $G$ is the subgroup generated by an orbit of $A$ on $\text{comp}(G)$. The set of $A$-components of $G$ is denoted by $\text{comp}_A(G)$.
(d) An $(A, \text{sol})$-component of $G$ is the subgroup generated by an orbit of $A$ on $\text{comp}_{\text{sol}}(G)$. The set of $(A, \text{sol})$-components of $G$ is denoted by $\text{comp}_{A, \text{sol}}(G)$.

The $A$-components of $G$ are the subnormal $A$-quasisimple subgroups of $G$. The $(A, \text{sol})$-components of $G$ are the minimal nonsolvable $A$-invariant subnormal subgroups of $G$. A result exactly analogous to Lemma 2.2 holds for $(A, \text{sol})$-components.

Recall that a group $X$ is semisimple if $X = E(X)$ and constrained if $C_X(F(X)) \leq F(X)$. Then any $(A, \text{sol})$-component of $G$ is either semisimple or constrained.

The group $A$ acts coprimely on the group $G$ if $A$ acts on $G$; the orders of $A$ and $G$ are coprime; and $A$ or $G$ is solvable. If $p$ is a prime then we denote by $\text{Syl}_p(G; A)$ the set of maximal $A$-invariant $p$-subgroups of $G$ with respect to inclusion.

**Theorem 2.4 (Coprime Action).** Suppose the group $A$ acts coprimely on the group $G$.

(a) Let $p$ be a prime. Then $\text{Syl}_p(G; A) \subseteq \text{Syl}_p(G)$ and $C_G(A)$ acts transitively by conjugation on $\text{Syl}_p(G; A)$.
(b) Let $N$ be an $A$-invariant normal subgroup of $G$ and set $\overline{G} = G/N$. Then $C_{\overline{G}}(A) = C_G(A)$.
(c) $G = [G, A]C_G(A)$ and $[G, A] = [G, A, A]$.
(d) Suppose $A$ is elementary and noncyclic. Then $G = \langle C_G(B) \mid B \in \text{Hyp}(A) \rangle = \langle C_G(a) \mid a \in A^\# \rangle$.

Moreover if $T \leq A$ then $[G, T] = \langle [C_G(B), T] \mid B \in \text{Hyp}(A) \rangle = \langle [C_G(a), T] \mid a \in A^\# \rangle$.
(e) If $G = XY$ where $X$ and $Y$ are $A$-invariant subgroups of $G$ then $C_G(A) = C_X(A)C_Y(A)$. 

(f) Suppose $K \trianglelefteq G$ and $[K, A] = [C_G(K), A] = 1$. Then $[G, A] = 1$.

(g) If $[F^*(G), [A] = 1$ then $[G, A] = 1$.

(h) Suppose that $G$ is $p$-solvable for some prime $p$ and that $A$ centralizes a Sylow $p$-subgroup of $G$. Then $[G, A] \leq O_{p'}(G)$.

(i) Suppose that $G$ is a $p$-group for some prime $p$; that $A$ centralizes every characteristic abelian subgroup of $G$ and that $G = [G, A]$. Then

$$G' = \Phi(G) = Z(G) = C_G(A).$$

Proof. For (a),(b),(c),(e) see [14, p.184–187]. (d) is [17, p.484].

(f). By induction, we may suppose $K \trianglelefteq G$. Then $[G, A, A] \leq [C_G(K), A] = 1$.

Apply (c).

(g). Since $C_G(F^*(G)) = Z(F(G))$, this follows from (f).

(h). Set $\overline{G} = G/O_{p'}(G)$ so $F^*(\overline{G}) = O_{p'}(\overline{G})$. Then $[F^*(\overline{G}), A] = 1$. Apply (g).

(i). This is well known, see [5, Corollary 3.3] for example. □

Note that (a) implies that for each prime $p$, $G$ possesses a unique maximal $AC_G(A)$-invariant $p$-subgroup, namely the intersection of the members of $\operatorname{Syl}_p(G; A)$.

Definition 2.5. Suppose that group $A$ acts coprimely on the group $G$. Let $p$ be a prime. Then

$$O_p(G; A)$$

is the intersection of all the $A$-invariant $p$-subgroups of $G$.

Finally, we collect together some more specialized results.

Lemma 2.6. Let $R$ be an elementary abelian $r$-group that acts coprimely on the $K$-group $X$.

(a) $O_p(X; R)' \leq \operatorname{sol}(X)$ for all primes $p$.

(b) Suppose $R$ is noncyclic. Then

$$\bigcap_{b \in R^*} \operatorname{sol}(C_X(b)) \leq \operatorname{sol}(X).$$

(c) Suppose $R$ is cyclic, $X = [X, R]$, $t$ is a prime and $RX$ acts on the $t$-group $T$ with $C_T(R) = 1$. Set $\overline{X} = X/C_X(T)$. Then $\pi(\overline{X}) \subseteq \{2, t\}$ and $\overline{X}/O_t(\overline{X})$ is either trivial or a nonabelian $2$-group.

Proof. (a). This is [8, Theorem 3.1(c)].

(b). Because if $H$ is a simple $K$-group with order coprime to $r$ then the Sylow $r$-subgroups of $\operatorname{Aut}(K)$ are cyclic.

(c). This reduces to the case where $T$ is elementary abelian and $RX$ acts non-trivially and irreducibly on $T$. [6] Theorem 7.1 implies $\overline{X}$ is a special $2$-group. □

3. A-SIMPLE GROUPS

In the proof of the Signalizer Functor Theorem presented here, much of the argument concerns $A$-components. Consequently it is necessary to have an understanding of $A$-simple groups. Throughout this section,

$r$ is a prime and $A \neq 1$ is an elementary abelian $r$-group.

Theorem 3.1. Suppose that $A$ acts faithfully and coprimely on the $K$-group $K$ and that $K$ is $A$-simple.
(a) $K = K_1 \times \cdots \times K_n$ where $\{K_1, \ldots, K_n\}$ is a collection of simple subgroups of $K$ that is permuted transitively by $A$.

Define

\[ A_\infty = \ker A \rightarrow \text{Sym}(\{K_1, \ldots, K_n\}). \]

(b) $|A_\infty| = 1$ or $r$.

(c) Let $a \in A \setminus A_\infty$. Then $C_K(a)$ is a maximal $AC_K(A)$-invariant proper subgroup of $K$. It is $A$-simple and has $|A|/|A_\infty|$ components, each of which is normalized by $A_\infty$. Moreover $C_A(C_K(a)) = \langle a \rangle$.

(d) Let $a \in A_{\#}^\infty$. Then either $C_K(a)$ is solvable or $F^*(C_K(a))$ is $A$-simple. In the latter case, $C_K(a)/F^*(C_K(a))$ is abelian.

(e) Assume that $C_K(A)$ is solvable.

(i) $|A_\infty| = r$ and $K_1$ is isomorphic to $L_2(2^r)$, $L_2(3^r)$, $U_3(2^r)$ or $Sz(2^r)$.

(ii) $K$ possesses a unique maximal $AC_K(A)$-invariant solvable subgroup $S$.

(iii) $C_K(A_\infty) \leq S$ and $S$ is maximal subject to being an $AC_K(A)$-invariant proper subgroup of $K$.

(f) Assume that $C_K(A)$ is nonsolvable.

(i) $F^*(C_K(A))$ is simple and $C_K(A)/F^*(C_K(A))$ is cyclic.

(ii) $K$ does not possess a nontrivial $AC_K(A)$-invariant solvable subgroup.

Proof. See [3] §6.

We note in particular that if $a \in A^\#$ then either

\[ F^*(C_K(a)) \] is $A$-simple or $C_K(a)$ is solvable.

By (d), the following balance property holds, for all $a, b \in A^\#

\[ E(E(C_K(a)) \cap C_K(b)) \leq E(C_K(b)). \]

These properties characterize $K$ and the collection $\{C_K(a) \mid a \in A^\#\}$ of fixed point subgroups. It is convenient to state this characterization in the language of signalizer functor theory.

**Theorem 3.2** (Characterization of $A$-Simple Groups [3]). Suppose that $\text{rank}(A) \geq 3$ and that $A$ acts on the (possibly infinite) group $G$. Assume the following:

(i) $\theta$ is an $A$-signalizer functor on $G$.

(ii) $\theta(a)$ is a $K$-group for all $a \in A^\#$.

(iii) If $a \in A^\#$ with $E(\theta(a)) \neq 1$ then $E(\theta(a))$ is $A$-simple, $F(\theta(a)) = 1$ and $C_A(E(\theta(a))) = \langle a \rangle$.

(iv) For all $a, b \in A^\#$,

\[ E(E(\theta(a)) \cap C_G(b)) \leq E(\theta(b)). \]

(v) $G = \langle E(\theta(a)) \mid a \in A^\# \rangle \neq 1$.

Then $G$ is a finite $r'$-group, it is $A$-simple, a $K$-group and

\[ \theta(a) = C_G(a) \]

for all $a \in A^\#$. In particular $\theta$ is complete and $G$ is its completion.

We close this section with three results on $A$-quasisimple groups.

**Definition 3.3.** Whenever $K$ is an $A$-quasisimple group define

\[ C_K^A(\alpha) = \begin{cases} C_K(A) & \text{if } C_K(A) \text{ is solvable} \\ E(C_K(A)) & \text{if } C_K(A) \text{ is nonsolvable}. \end{cases} \]
Lemma 3.4. Suppose that $K$ is an $A$-quasisimple $K$-group on which $A$ acts coprimely.

(a) $C_K^*(A)$ is nonabelian.

(b) Suppose that $H$ is an $AC_K(A)$-invariant nonsolvable subgroup of $K$. Then $H^{(\infty)} = E(H)$ is $A$-quasisimple and $C_{E(H)}^*(A) = C_K^*(A)$.

Proof. Set $\overline{K} = K/Z(K)$, so $\overline{K}$ is $A$-simple. Coprime Action implies $C_K(A) = \overline{C_K(A)}$. Write $\overline{K} = \overline{K}_1 \times \cdots \times \overline{K}_n$ where $\{\overline{K}_1, \ldots, \overline{K}_n\}$ is a collection of simple subgroups of $\overline{K}$ that is permuted transitively by $A$. Set $A_\infty = \ker A \rightarrow \text{Sym}(\{\overline{K}_1, \ldots, \overline{K}_n\})$. Recall that if $X$ is a group and $Z \leq Z(X)$ then $E(X)$ maps onto $E(X/Z)$.

(a). If $A_\infty = C_A(\overline{K})$ then [6] Lemma 6.5 implies $C_K(A) \cong \overline{K}_1$, so $C_K^*(A)$ is simple. If $A_\infty \not\cong C_A(\overline{K})$ then [6] Lemma 6.5 implies $C_K(A) \cong C_K(\{\overline{A}_\infty\})$ and then [6] Theorem 4.1 implies $F^*(C_K(\{\overline{A}_\infty\}))$ is simple or $C_K^*(A_\infty)$ is solvable and nonabelian. Since $E(C_K(A))$ maps onto $E(C_K(A))$ it follows that $C_K^*(A)$ is nonabelian.

(b). Recall from [6] that $H$ is overdiagonal if $H$ projects onto each $\overline{K}_i$. In the contrary case, $H$ is underdiagonal. Suppose that $H$ is underdiagonal. [6] Lemma 6.6 implies $H = C_K(B)(H \cap Z(K))$ for some $B \leq A$ with $B \cap A_\infty = C_A(K)$. Then [6] Lemma 6.5 implies $C_K(B)$ is $A$-quasisimple. Consequently $H^{(\infty)} = E(H) = C_K(\{\overline{A}_\infty\})$ and as $B \leq A$ we have $C_K(E(H)) = C_K(A)$ and the conclusion holds in this case. Hence we may assume that $H$ is underdiagonal.

If $\overline{K}$ possesses a nontrivial $AC_K(A)$-invariant solvable subgroup then all $AC_K(A)$-invariant underdiagonal subgroups are solvable by [6] Lemma 6.7. Thus $\overline{K}$ possesses no such subgroup. In particular, $C_K^*(A)$ is nonsolvable, whence $F^*(C_K^*(A))$ is simple. Also, $F(H) = 1$ so we may choose $H_0 \in \text{comp}_A(H)$.

Since $H_0$ is nonsolvable, [6] Theorem 4.4 implies $C_{H_0}(A) \not\leq C_K^*(A)$ whence $F^*(C_{H_0}(A))$ and $F^*(C_K^*(A))$ are uniquely determined. Then $E(H_0) = H_0$ and $E(H)$ is $A$-simple. Now $F^*(C_K^*(A)) \leq E(H)$. Recall that $C_K^*(A)/F^*(C_K^*(A))$ is cyclic. Consequently $F^*(C_K^*(A))$ is cyclic so [6] Theorem 4.4 implies $H/E(H)$ is solvable. We have shown that $\overline{H}^{(\infty)} = E(H)$ is $A$-simple and $E(C_K(E(H))) = E(C_K^*(A)) \neq 1$.

Then $H^{(\infty)}Z(K) = E(H)Z(K)$ so taking the derived group yields $H^{(\infty)} = E(H)$. Similarly $E(C_K(E(H))) = E(C_K(A))$, completing the proof. □

Lemma 3.5 ([6] Lemma 6.12]). Suppose $A$ acts coprimely on the $A$-quasisimple group $K$.

(a) If $A$ is noncyclic then $K = \{C_K(D) \mid D \in \text{Hyp}(A) \text{ and } C_K(d) \text{ is } A\text{-quasisimple for all } d \in D^\#\}$.

(b) If $D \in \text{Hyp}(A)$ and $D$ is noncyclic then $K = \{C_K(d) \mid d \in D^\# \text{ and } C_K(d) \text{ is } A\text{-quasisimple}\}$.

Lemma 3.6 ([6] Theorem 4.4(c)]). Suppose $A$ acts coprimely on the $K$-group $G$ and that $K \in \text{comp}_A(G)$. Then $C_G(C_K(A)) = C_G(K)$. 
4. Automorphisms

Throughout this section we assume:

**Hypothesis 4.1.**
- $r$ is a prime and $A \neq 1$ is an elementary abelian $r$-group.
- $A$ acts coprimely on the $K$-group $G$.
- $a \in A^\#$.
- $H$ is an $AC_G(a)$-invariant subgroup of $G$.

The following result relates the structure of $H$ to the structure of $G$ in the case that $G$ is solvable.

**Theorem 4.2.** Assume that $G$ is solvable and $H = [H,a]$.

(a) Let $p$ be a prime. Then
$$O_p(H) \leq O_p(G)$$

or all of the following hold: $p = 2$, $r$ is Fermat and the Sylow 2-subgroups of $H$ are nonabelian.

(b) $O_2(O_2(H)) \leq O_2(G)$.

This result is fundamental to the author’s proof of the Solvable Signaler Functor Theorem. It is a consequence of well known results on the representation theory of solvable groups. See [5, Corollary 5.2] for example. To deal with nonsolvable signalizer functors, it is necessary to have analogous results for nonsolvable groups.

**Theorem 4.3.**

(a) Suppose $K \in \text{comp}_{A,\text{sol}}(H)$ and $K = [K,a]$. Then $K \in \text{comp}_{A,\text{sol}}(G)$.

(b) $[O_*(H),a]^{(\infty)} \leq O_*(G)$.

**Proof.** (a). This is [7, Theorem 7.5(a)].

(b). Note that if $K \in \text{comp}_{A,\text{sol}}(H)$ then either $[K,a] \leq \text{sol}(K) \leq \text{sol}(H)$ or $K = [K,a]$. Then $[O_*(H),a]^{(\infty)} = \langle K \in \text{comp}_{A,\text{sol}}(H) \mid K = [K,a] \rangle$. Apply (a). \qed

**Theorem 4.4.** Suppose $K \in \text{comp}_{A}(H)$. Then there exists $\bar{K}$ with
$$K \leq \bar{K} \in \text{comp}_{A,\text{sol}}(G).$$

Moreover:

(a) If $[K,a] \neq 1$ then $K = [K,a] = \bar{K}$.

(b) If $[K,a] = 1$ then $K = E(C_{\bar{K}}(a))$.

(c) Suppose $\bar{K}$ is constrained. Then $[K,a] = 1$ and
$$\bar{K} = K \text{sol}(\bar{K}).$$

Moreover if $b \in A \setminus C_A(K)$ then $\bar{K} = (K, C_{\text{sol}(\bar{K})}(b))$.

(d) Let $L \in \text{comp}_{A,\text{sol}}(G)$. Assume
$$L \neq \bar{K} \text{ and } L = [L,a].$$

Then $[\bar{K},L] = 1$.

**Proof.** This is [6, Theorem 9.8] except for the final assertion in (c) which is [6, Lemma 8.2]. \qed
Remark. In the constrained case, it is in fact possible to show that $\tilde{K} = KF(\tilde{K})$, but we do not need this stronger result. Recall that distinct $A$-components of $G$ commute. This fact is very useful. However, the same is not necessarily true of $(A,\text{sol})$-components. (d) circumvents this difficulty.

5. $\mathcal{P}$-subgroups

Throughout this section, we assume:

Hypothesis 5.1.

- $r$ is a prime and $A \neq 1$ is an elementary abelian $r$-group.
- $\mathcal{P}$ is a group theoretic property that is closed under subgroups, quotients and extensions.

Definition 5.2. Suppose $A$ acts on the group $G$.

$O_P(G) = \langle X \mid X$ is an $A$-invariant normal $\mathcal{P}$-subgroup of $G\rangle$.

$O_P(G; A) = \langle X \mid X$ is an $AC_G(A)$-invariant $\mathcal{P}$-subgroup of $G\rangle$.

It is clear that $O_P(G)$ is itself a $\mathcal{P}$-group and is thus the unique maximal normal $\mathcal{P}$-subgroup of $G$. The following is less clear:

Theorem 5.3 ([7, Theorem 5.2]). Suppose $A$ acts coprimely on the $K$-group $G$. Then $O_P(G; A)$ is a $\mathcal{P}$-group. In other words, $G$ possesses a unique maximal $AC_G(A)$-invariant $\mathcal{P}$-subgroup.

A useful corollary is the following:

Corollary 5.4. Assume the hypotheses of Theorem 5.3. Suppose that $N$ is an $A$-invariant subnormal subgroup of $G$. Then

$O_P(N; A) = O_P(G; A) \cap N$.

Proof. Consider first the case that $N \triangleleft G$. Then $O_P(G; A) \cap N$ is an $AC_N(A)$-invariant $\mathcal{P}$-subgroup of $N$ so $O_P(G; A) \cap N \leq O_P(N; A)$. Now $C_N(A) \leq C_G(A)$ so it follows that $C_G(A)$ permutes the $AC_N(A)$-invariant $\mathcal{P}$-subgroups of $N$. Then $C_G(A)$ normalizes $O_P(N; A)$. Theorem 5.3 implies $O_P(N; A)$ is a $\mathcal{P}$-subgroup so $O_P(N; A) \leq O_P(G; A)$ and the result follows in this case.

Suppose that $N$ is not normal in $G$. Set $G_0 = \langle N^G \rangle$. Since $N$ is a proper subnormal subgroup of $G$ it follows that $G_0$ is a proper $A$-invariant normal subgroup of $G$. Apply the previous case and induction. 

The main result of this section is the following:

Theorem 5.5. Suppose $A$ acts on the (possibly infinite) group $G$ and that $\theta$ is an $A$-signalizer functor on $G$. Assume that $\theta(a)$ is a $K$-group for all $a \in A^\#$. Define $\theta_\mathcal{P}$ by

$\theta_\mathcal{P}(a) = O_P(\theta(a); A)$.

(a) $\theta_\mathcal{P}$ is an $A$-signalizer functor on $G$.

(b) Assume that $A$ is noncyclic; that $\theta_\mathcal{P}$ is complete; and that $\theta_\mathcal{P}(G)$ is a $K$-group. Then $\theta_\mathcal{P}(G)$ is the unique maximal $\theta(A)$-invariant $(\mathcal{P}, \theta)$-subgroup of $G$. (A $(\mathcal{P}, \theta)$-subgroup is a $\theta$-subgroup that is a $\mathcal{P}$-group.)

We also need the following:
Lemma 5.6. Suppose that $A$ acts coprimely on the $K$-group $G$. Assume that $A$ is noncyclic and that $C_G(a)$ is a $P$-group for all $a \in A^\#$. Then $G$ is a $P$-group.

Proof. Using Coprime Action(b) we may suppose that 1 and $G$ are the only $A$-invariant normal subgroups of $G$. Then $G$ is characteristically simple. Suppose $G$ is abelian. Then $G$ is an elementary abelian $p$-group for some prime $p$. Coprime Action(d) implies $C_G(a) \neq 1$ for some $a \in A^\#$. Since $C_G(a)$ is $A$-invariant and normal we have $G = C_G(a)$ so $G$ is a $P$-group. Hence we may suppose that $G$ is nonabelian. Then $G = G_1 \times \cdots \times G_n$ where $\{G_1, \ldots, G_n\}$ is a collection of simple subgroups of $G$ that is permuted transitively by $A$. Suppose that $n > 1$. Choose $a \in A$ such that $G_1^a \neq G_1$. Then $\{gg^a \cdots g^{n-1} \mid g \in G_1\}$ is a normal subgroup of $C_G(a)$ that is isomorphic to $G_1$. Then $G_1$ is a $P$-group, whence $G$ is also. Suppose that $n = 1$. Then $G$ is a simple $K$-group. Consequently the Sylow $r$-subgroups of $\mathrm{Aut}(G)$ are cyclic so $G = C_G(a)$ for some $a \in A^\#$ and $G$ is a $P$-group. □

Proof of Theorem 5.5 (a). Let $a, b \in A^\#$. Note that $C_{\theta(a)}(A) = \theta(A) = C_{\theta(b)}(A)$. Now

$$\theta_P(a) \cap C_G(b) \leq \theta_P(a) \cap \theta(b) \leq \theta_P(b),$$

the first inclusion because $\theta$ is an $A$-signalizer functor and the second because $\theta_P(a) \cap \theta(b)$ is an $A\theta(A)$-invariant $P$-subgroup of $\theta(b)$. Hence $\theta_P$ is an $A$-signalizer functor.

(b). Set $K = \theta_P(G)$. Since $\theta_P$ is complete, $\theta_P(a) = C_K(a)$ for all $a \in A^\#$. Lemma 5.6 implies $K$ is a $P$-group. Suppose $L$ is a $\theta(A)$-invariant $(P, \theta)$-subgroup of $G$. If $a \in A^\#$ then $C_L(a)$ is a $\theta(A)$-invariant $(P, \theta)$-subgroup of $\theta(a)$, whence $C_L(a) \leq \theta_P(a) \leq K$. Coprime Action(d) implies $L \leq K$. □

6. Bender’s Maximal Subgroup Theorem

The aim of this section is to prove slight extension of a result of Bender [2, 1.7]. Bender’s result gave a criterion for two maximal subgroups $M$ and $N$ of a simple group to be equal. First we need some definitions.

Definition 6.1. Suppose that $M$ and $N$ are finite subgroup of a (possibly infinite) group.

- $M$ is maximal with respect to $N$ if

$$N_N(T) \leq M$$

whenever $1 \neq T \operatorname{char} M$ with $T \leq M \cap N$.

- $M$ and $N$ are comaximal if $M$ is maximal with respect to $N$ and $N$ is maximal with respect to $M$.

- $M \leadsto N$ means

$$X C_{F^*(M)}(X) \leq N \quad \text{for some} \quad X \leq F^*(M).$$

- If $p$ is a prime then $M$ has characteristic $p$ if $F^*(M) = O_p(M)$.

Theorem 6.2 (Bender’s Maximal Subgroup Theorem). Suppose that $M$ and $N$ are finite subgroups of a (possibly infinite) group, that $M$ is maximal with respect to $N$ and that $M \leadsto N$.

(a) $E(M) \leq N$ and $M \cap O_p(N) = 1$ for all $p \notin \pi(F(M))$.

(b) Assume that $E(M) \neq 1$ or $|\pi(F(M))| \geq 2$. Then $O_p(N) \leq M$ for all $p \in \pi(F(M))$. 
(c) Assume in addition that $N$ is maximal with respect to $M$ and that
(i) $N \hookrightarrow M$ or
(ii) $E(N) \leq M$ and $\pi(F(N)) \subseteq \pi(F(M))$.

Then $M = N$ or $M$ and $N$ have characteristic $p$ for some prime $p$.

Proof. This is proved in [2, 1.7] under the assumption that $M$ and $N$ are maximal subgroups of a simple group. However, only the stated hypotheses are required. □

The result stated below is used to handle the characteristic $p$ case. Under the given hypotheses, it leads to the same conclusion.

**Theorem 6.3** ([8, Theorem A]). Let $p$ be a prime and suppose $M_1$ and $M_2$ are finite subgroups of a (possibly infinite) group with the following properties:
- $M_1$ and $M_2$ are comaximal.
- $M_1$ and $M_2$ are $K$-groups with characteristic $p$.
- For each $i$ there is an elementary abelian group $A_i$ that acts coprimely on $M_i$ and $O_p(M_1; A_1) = O_p(M_2; A_2)$.

Then $M_1 = M_2$.

Unfortunately, at one point in the argument this result is not strong enough. However, the following result, provides the necessary extra leverage. Note that Theorem 6.3 is a trivial corollary.

**Theorem 6.4** ([8, Theorem 4.3]). Let $p$ be a prime and suppose that $M$ and $S$ are subgroups of a group. Assume that:
- $M$ and $S$ are finite $K$-groups with characteristic $p$.
- $M$ is maximal with respect to $S$.
- There exist elementary abelian groups $A_m$ and $A_s$ that act coprimely on $M$ and $S$ respectively and $O_p(M; A_m) = O_p(M; A_s)$.

Set
$$P = O_p(M)O_p(S).$$

Then the following hold:
(a) If $O_p(M)$ is abelian then $J(P) = J(O_p(M))$.
(b) $J(P) = J(O_p(S))$.

Note that $O_p(M) \leq O_p(M; A_m) = O_p(S; A_s) \leq S$ whence $P$ is a $p$-group.

7. Elementary results

A number of elementary results are presented. In particular, to any signalizer functor $\theta$ we associate a positive integer $|\theta|$. Note that the group $G$ in the statement of the Signalizer Functor Theorem is not assumed to be finite. Hence this device is needed to enable inductive arguments.

Throughout this section we assume the following:

**Hypothesis 7.1.**
- $A$ is an noncyclic abelian group that acts on the (possibly infinite) group $G$.
- $\theta$ is an $A$-signalizer functor on $G$.

**Lemma 7.2.** Let $B \leq A$ be noncyclic and define a $B$-signalizer functor $\theta_0$ by $\theta_0(b) = \theta(b)$ for all $b \in B^\theta$. If $\theta_0$ is complete then so is $\theta$ and $\theta(G) = \theta_0(G)$. 
Proof. Let $K = \theta_0(G)$, so $K$ is $A$-invariant and $\theta(b) = \theta_0(b) = C_K(b)$ for all $b \in B^\#$. Let $a \in A^\#$. Note that $\theta(a) \cap C_G(b) = C_G(a) \cap \theta(b)$ for all $b \in B^\#$. Using Coprime Action we have

$$\theta(a) = \langle \theta(a) \cap C_G(b) \mid b \in B^\# \rangle = \langle C_G(a) \cap \theta(b) \mid b \in B^\# \rangle = \langle C_G(a) \cap C_K(b) \mid b \in B^\# \rangle = C_K(a).$$

The conclusion follows. □

Henceforth we assume in addition to Hypothesis 7.1 that $A$ is an elementary abelian $r$-group for some prime $r$. Recall (see [3] for example) that if $1 \neq B \leq A$ then $\theta(B)$ is defined by

$$\theta(B) = \bigcap_{b \in B^\#} \theta(b).$$

Moreover if $H$ is a $\theta$-subgroup then $C_H(B) = H \cap \theta(B)$.

**Definition 7.3.**

$$||\theta|| = |\theta(A)| \prod_{B \in \text{Hyp}(A)} |\theta(B) : \theta(A)|.$$

Note that $||\theta|| < \infty$ since by the definition of signalizer functor, the subgroups $\theta(a)$ are finite. The definition is motivated by the following:

**Theorem 7.4 (The Wielandt Order Formula).** Suppose that $A$ acts coprimely on the group $H$. Then

$$|H| = |C_H(A)| \prod_{B \in \text{Hyp}(A)} |C_H(B) : C_H(A)|.$$ 

**Lemma 7.5.**

(a) Let $\psi$ be a subfunctor of $\theta$. Then $||\psi|| \leq ||\theta||$ with equality if and only if $\psi = \theta$.

(b) Let $H$ be a $\theta$-subgroup of $G$. Then $|H| \leq ||\theta||$ with equality if and only if $\theta(a) \leq H$ for all $a \in A^\#$.

(c) If $\theta$ is complete then $|\theta(G)| = ||\theta||$.

(d) Suppose that $N$ is a normal $\theta$-subgroup of $G$. Set $\overline{G} = G/N$ and define $\overline{\theta}$ by

$$\overline{\theta}(a) = \overline{\theta(a)}$$

for all $a \in A^\#$. Then:

(i) $\overline{\theta}$ is an $A$-signalizer functor on $\overline{G}$.

(ii) $\overline{\theta}(B) = \overline{\theta(B)}$ for all $1 \neq B \leq A$.

(iii) $\theta$ is complete if and only if $\overline{\theta}$ is complete.

(iv) $||\overline{\theta}|| \leq ||\theta||$ with equality if and only if $N = 1$.

(e) Let $a \in A^\#$. Then $\theta(a) = \langle \theta(B) \mid a \in B \in \text{Hyp}(A) \rangle$.

**Proof.** This follows from Coprime Action and the Wielandt Order Formula. □

Finally we develop an idea of McBride that results in a fundamental dichotomy in the proof of the Signalizer Functor Theorem.

**Definition 7.6.**
• $\theta$ is semisimple if 1 is the only $\theta(A)$-invariant solvable $\theta$-subgroup.

• $\theta$ is nearsolvable if $\theta(A)$ is solvable and every composition factor of every proper $\theta$-subgroup is isomorphic to $L_2(2^r)$, $L_2(3^r)$, $U_3(2^r)$ or $Sz(2^r)$.

McBride’s idea was to separate out the nonsolvable pieces of $\theta$ from the solvable pieces. This is not possible – but it nearly is. The difficulty arises because the groups listed possess an automorphism of order $r$ whose fixed point subgroup is solvable. The following result is [15, Theorem 6.6], a presentation of which may also be found in [7, Theorem 8.8].

**Theorem 7.7** (McBride’s Dichotomy). Suppose that $\theta$ is a minimal counterexample to the Signalizer Functor Theorem. Then $\theta$ is either semisimple or nearsolvable.

8. The minimal counterexample

Henceforth we assume the Signalizer Functor Theorem to be false and let $(A, G, \theta)$ be a counterexample. By Lemma 7.2 we may suppose that $A$ is an elementary abelian $r$-group with rank 3 for some prime $r$. Then we may assume that $|\theta|$ has been minimized. Without loss

$$G = \langle \theta(a) \mid a \in A^\# \rangle.$$  (1)

In broad outline, the proof proceeds as follows: show that the family of subgroups \{ $\theta(a) \mid a \in A^\#$ \} resembles the family of centralizers \{ $C_G(a) \mid a \in A^\#$ \} of some $A$-simple group $G^*$. Then invoke a suitable characterization theorem, namely Theorem 3.2.

Most of the difficulty lies in establishing

$$E(\theta(a)) \neq 1$$

for some $a \in A^\#$ and then that

$$E(E(\theta(a)) \cap C_G(b)) \leq E(\theta(b))$$

for all $a, b \in A^\#$. We define some notation:

• $\Theta$ is the set of proper $\theta$-subgroups of $G$.

• $\mathcal{L}$ is the set of $\theta(A)$-invariant members of $\Theta$.

• $\Theta^*$ and $\mathcal{L}^*$ denote the sets of maximal members of $\Theta$ and $\mathcal{L}$ respectively.

Note that it could be the case that $G$ is itself a $\theta$-subgroup, but in that case, $G$ is not a $K$-group.

If $H$ is an $A$-invariant subgroup of $G$ and of the $\theta$-subgroups of $G$ contained in $H$ there is a unique maximal one, then we denote that $\theta$-subgroup by

$$\theta(H)$$

and say that $\theta(H)$ is defined. If $X \leq G$ then we abbreviate $N_G(X)$ and $C_G(X)$ to $N(X)$ and $C(X)$ respectively.

**Lemma 8.1.**

(a) The members of $\Theta$ are $K$-groups.

(b) Every member of $\Theta$, resp. $\mathcal{L}$, is contained in a member of $\Theta^*$, resp. $\mathcal{L}^*$.

(c) If $H$ is a proper $A$-invariant subgroup of $G$ then $\theta(H)$ is defined and $\theta(H) \in \Theta$.

(d) If $1 \neq H \in \Theta$ then $N(H) \neq G$.

(e) If $M \in \Theta^*$ then $M = \theta(N(X))$ for all $1 \neq X \text{ char } M$.

(f) If $M, L \in \Theta^*$ then $N_L(X) \leq M$ for all $1 \neq X \text{ char } M$. 


that the composition factors of $\theta$ to the Signalizer Functor Theorem. □

Proof. This is a consequence of Lemma 7.5 (1) and the minimality of $|\theta|$.

**Corollary 8.2.** Let $M, N \in \Theta^*$. Suppose that $O_*(M) \leq N$ and $O_*(N) \leq M$. Then $M = N$.

Proof. We have $O_*(M) \leq M \cap N \leq N$ so Lemma 7.2(f) implies $O_*(M) = O_*(M \cap N)$. Similarly $O_*(N) = O_*(M \cap N)$. Then $O_*(M) = O_*(N)$. If $O_*(M) \neq 1$ then the conclusion follows from Lemma 8.1(e). If $O_*(M) = 1$ then $M = N = 1$ and again the conclusion holds. □

If $M, N \in L^*$ then $M$ and $N$ are comaximal by Lemma 8.1(f). However, a little more can be said.

**Lemma 8.3.** Let $M, N \in L^*$, $B \in \text{Hyp}(A)$ and $x \in \theta(B)$. Then $M$ and $N^x$ are comaximal.

Proof. Suppose $1 \neq T \text{char } M$ with $T \leq M \cap N^x$. Set $L = N_{N^x}(T)$. Let $b \in B^\#$. Then

$$C_L(b) \leq C_{M^x}(b) = (C_M(b))^x \leq \theta(b)^x = \theta(b)$$

so $C_L(b) \leq \theta(b) \cap N(T) \leq \theta(N(T)) = M$. Since $B$ is noncyclic, Coprime Action(d) implies $L \leq M$. Hence $M$ is maximal with respect to $N^x$. Similarly, $N^x$ is maximal with respect to $M$. □

Recall from 36 that $\theta_{sol}$ is defined by

$$\theta_{sol}(a) = O_{sol}(\theta(a); A)$$

for each $a \in A^\#$, where $O_{sol}(\theta(a); A)$ is the largest $AC_{\theta(a)}(A)$-invariant solvable subgroup of $\theta(a)$. Note that $C_{\theta(a)}(A) = \theta(A)$. Theorem 6.7 implies that $\theta_{sol}(a)$ is itself solvable and that $\theta_{sol}$ is an $A$-signalizer functor on $G$. The Solvable Signalizer Functor Theorem implies that $\theta_{sol}$ is complete. Let

$$S = \theta_{sol}(G).$$

**Lemma 8.4.** $S$ is the unique maximal $\theta(A)$-invariant solvable $\theta$-subgroup of $G$.

Proof. Apply Theorem 5.5 □

McBride’s Dichotomy implies that if $S = 1$ then $\theta$ is semisimple and if $S \neq 1$ then $\theta$ is nearsolvable.

This section concludes by eliminating a certain configuration.

**Lemma 8.5.** The following is impossible: $e \in A^\#$, $M \in \Theta$ and

$$[\theta(a), e] \leq M$$

for all $a \in A^\#$.

Proof. Assume that it does hold. An argument of Bender, see 34 Theorem 4.2], implies that $\theta(e)$ normalizes $[M, e]$ and that $\theta$ is complete with $\theta(G) = \theta(e)|[M, e]$. Then $\theta(G)$ is a $K$-group because $\theta(e)$ and $[M, e]$ are $K$-groups. Lemma 5.6 implies that the composition factors of $\theta(G)$ are to be found amongst the composition factors of the subgroups $\theta(a); a \in A^\#$, contrary to $(A, G, \theta)$ being a counterexample to the Signalizer Functor Theorem. □
Corollary 8.6. Suppose that $\psi$ is a subfunctor of $\theta$, $e \in A^#$ and

$$[\theta(a), e] \leq \psi(a)$$

for all $a \in A^#$. Then $\psi = \theta$.

Proof. Suppose that $\psi \neq \theta$. Lemma 7.5 implies that $||\psi|| < ||\theta||$ so the minimality of $||\theta||$ implies that $\psi$ is complete and that $\psi(G)$ is a $K$-group. Note that $C_{\psi(G)}(a) = \psi(a) \leq \theta(a)$ for all $a \in A^#$ so $\psi(G)$ is a $\theta$-subgroup. In particular, $\psi(G) \neq G$ as $(A, G, \theta)$ is a counterexample to the Signalizer Functor Theorem. Lemma 8.5 with $\psi(G)$ in the role of $M$, supplies a contradiction. \hfill \Box

9. SUBFUNCTORS

Recall from [5] that if $p$ is a prime then a $(p, \theta)$-subgroup is a $\theta$-subgroup that is also a $p$-group. The collection of $(p, \theta)$-subgroups is partially ordered by inclusion and its set of maximal elements is denoted by

$$\text{Syl}_p(G; \theta).$$

The Transitivity Theorem asserts that $\theta(A)$ acts transitively on $\text{Syl}_p(G; \theta)$. In the proof of the Solvable Signalizer Functor Theorem it was necessary to show that $C_A(P) = 1$ whenever $1 \neq P \in \text{Syl}_p(G; \theta)$. This was accomplished using the subfunctor $\theta'$. We shall extend those ideas to obtain information in the case $C_A(P) \neq 1$ and $\theta$ is nearsolvable. First, a simple criterion for $C_A(P)$ to be nontrivial.

Lemma 9.1. Let $e \in A^#$ and suppose $\theta(e) \leq M \in \mathcal{L}^*$. Assume $p \in \pi(F(M))$ and $[M, e]$ is a $p'$-group. Then $e$ centralizes every $(p, \theta)$-subgroup of $G$.

Proof. Choose $P \in \text{Syl}_p(M; A)$. Then $[P, e] = 1$. Also, $1 \neq O_p(M) \leq P$ so as $M \in \mathcal{L}^*$ we have $\theta(C(P)) \leq \theta(N(O_p(M))) = M$. Choose $Q$ with $P \leq Q \in \text{Syl}_p(G; \theta)$. By Coprime Action(e), $N_Q(P) = [N_Q(P), e](N_Q(P) \cap C(e))$. Now $[P, e] = 1$ so $[N_Q(P), e] \leq C_Q(P) \leq \theta(C(P)) \leq M$. Also $N_Q(P) \cap C(e) \leq \theta(e) \leq M$ whence $N_Q(P) \leq M$. Since $P \in \text{Syl}_p(M)$ this forces $N_Q(P) = P$ and then $P = Q \in \text{Syl}_p(G; \theta)$. As $[P, e] = 1$ and $\theta(A) \leq C(e)$, the Transitivity Theorem implies that $e$ centralizes every member of $\text{Syl}_p(G; \theta)$. The conclusion follows. \hfill \Box

Recall that if $p$ is a prime and $X$ is a group then $O_{p-}\text{sol}(X)$ is the largest normal $p$-solvable subgroup of $X$. Theorem 5.3 asserts that the map $\theta_{p-}\text{sol}$ defined by

$$\theta_{p-}\text{sol}(a) = O_{p-}\text{sol}(\theta(a); A)$$

is the unique maximal $A\theta(A)$-invariant $p$-solvable subgroup of $\theta(a)$

We state the main result of this section.

Theorem 9.2. Assume the following:

- $p \in \pi(\theta)$.
- $e \in A^#$ and $e$ centralizes every $(p, \theta)$-subgroup of $G$.
- $\theta$ is nearsolvable.

Then the following hold:

(a) $G$ possesses a unique maximal $\theta(A)$-invariant $p$-solvable $\theta$-subgroup.
We claim that ψ recall the following: let case arising in conclusion (c) of Bender’s Maximal Subgroup Theorem . First we

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Proof of Theorem 9.2(a). Assume that θp-sol = θ. Let a ∈ A#. By hypothesis, e centralizes every A-invariant Sylow p-subgroup of θ(a) so using Coprime Action(h) we have

[θ(a), e] ≤ O_p(θ(a)) ≤ θ_p(a).

Corollary 8.6 implies that θ = θ_p. But then θ(a) is a p'-group for all a ∈ A#, contrary to p ∈ π(θ). We deduce that θp-sol ̸= θ. The minimality of ||θ|| implies that θp-sol is complete and that θp-sol(G) is p-solvable. Theorem 9.2 implies that θp-sol(G) is the unique maximal θ(A)-invariant p-solvable θ-subgroup.

Lemma 9.3. Assume the hypotheses of Theorem 9.2. Let X ∈ L and suppose that X = [X, e]. Set X = X/O_p-sol(X).

(a) F^*(X) = E(X), Z(E(X)) = 1 and each component of X is normalized but not centralized by e.

(b) C_X(e) is p-solvable.

Proof. Since O_p-sol(X) = 1 we have Z(E(X)) = F(X) = O_p(X) = 1. In particular, each component of X has order divisible by p. By hypothesis, e centralizes a Sylow p-subgroup of E(X). Then e acts trivially on comp(X). Since X = [X, e] it follows that each component of X is normal in X. If K ∈ comp(X) and [X, e] = 1 then X = [X, e] centralizes K, a contradiction. Thus (a) holds.

Let K ∈ comp(X). Since θ is nonsolvable, X is nonsolvable and K ≃ L_2(2^r), L_2(3^r), U_3(2^r) or Sz(2^r). Now e induces a nontrivial automorphism of order r on K and K is an r^r-group. It follows that C_K(e) is solvable. Then C_{E[X]}(e) is solvable. By (a) and the Schreier Property, X/E(X) is solvable. Now X = X/O_p-sol(X), whence C_X(e) is p-solvable and (b) holds.

Proof of Theorem 9.2(b). For each a ∈ A# define ψ(a) by

ψ(a) = [θ(a), e]θ_p-sol(a).

We claim that ψ is an A-signalizer functor. Indeed, let a, b ∈ A#. Coprime Action(e) implies

ψ(a) ∩ C(b) = ([θ(a), e] ∩ C(b))(θ_p-sol(a) ∩ C(b)).

Let Y = [θ(a), e] ∩ C(b). Then Y ≤ θ(a) ∩ C(b) ≤ θ(b). Now Y = [Y, e]C_Y(e) by Coprime Action(e) and [Y, e] ≤ ψ(b). Lemma 9.3(b) implies that [θ(a), e] ∩ C(e) is p-solvable. Then C_Y(e) is an A0(A)-invariant p-solvable subgroup of θ(b), so C_Y(e) ≤ θp-sol(b) ≤ ψ(b). Thus Y ≤ ψ(b). As θp-sol is an A-signalizer functor we have θp-sol(a) ∩ C(b) ≤ θp-sol(b) ≤ ψ(b). Hence ψ(a) ∩ C(b) ≤ ψ(b) and the claim is established. Corollary 8.6 implies ψ = θ. Also, θ(e) = ψ(e) = θ_p-sol(e) so θ(e) is p-solvable.

10. The First Uniqueness Theorem

The aim of this section is to prove a result that deals with the characteristic p case arising in conclusion (c) of Bender’s Maximal Subgroup Theorem. First we recall the following: let p be a prime.

- A group M has characteristic p if F^*(M) = O_p(M).
Theorem 10.1 (The First Uniqueness Theorem). Let \( p \) be a prime and suppose \( M \in \mathcal{L}^* \) has characteristic \( p \).

(a) \( O_p(G; \theta) \leq M \).
(b) \( M \) is the only member of \( \mathcal{L}^* \) with characteristic \( p \).

Proof of Theorem 10.1. Let \( N = \theta(N(O_p(M; A))) \). Note that \( O_p(M) \leq O_p(M; A) \) so \( O_p(M; A) \neq 1 \) since \( M \) has characteristic \( p \). Let \( B \in \text{Hyp}(A) \) and \( x \in C_N(B) \). Then

\[
O_p(M; A) = O_p(M; A)^x = O_p(M^x; A^x).
\]

Using Lemma 6.3 we see that the hypotheses of Theorem 10.1 are satisfied with \( M_1 = M, A_1 = A, M_2 = M^x \) and \( A_2 = A^x \). Consequently \( M = M^x \). Then \( x \in \theta(B) \cap N(M) \leq \theta(N(M)) = M \). We deduce that \( C_N(B) \leq M \) for all \( B \in \text{Hyp}(A) \).

Let \( P = O_p(G; \theta) \). Then \( P \) contains every \( \theta(A) \)-invariant \( (p, \theta) \)-subgroup. Also \( M \in \mathcal{L}^* \) so \( \theta(A) \leq M \), in fact \( \theta(A) = C_M(A) \). It follows that

\[
P \cap M = O_p(M; A).
\]

Then \( N_p(P \cap M) = \theta(N(O_p(M; A))) = N \leq M \) so \( N_p(P \cap M) \leq P \cap M \). This forces \( P = P \cap M \leq M \) so \( P = O_p(M; A) \) and (a) holds.

To prove (b), suppose \( N \in \mathcal{L}^* \) also has characteristic \( p \). Then \( O_p(M; A) = P = O_p(N; A) \). Another application of Theorem 6.3 forces \( M = N \). 

11. The subgroups \( M_a \)

The main result of this section is the following:

Theorem 11.1. Let \( a \in A^\# \). There exists \( M_a \) such that the following hold:

(a) \( \theta(a) \leq M_a \in \mathcal{L}^* \).
(b) If \( N \in \mathcal{L}^* \) satisfies

\[
M_a \hookrightarrow N \quad \text{and} \quad \theta(a) \leq N
\]

then \( M_a = N \).
(c) If \( X \neq 1 \) is an \( A\theta(a) \)-invariant subnormal subgroup of \( M_a \) then \( \theta(N(X)) \leq M_a \).

Throughout the remainder of this paper, we let \{ \( M_a \mid a \in A^\# \) \} be the family of subgroups constructed in Theorem 11.1.

It is a trivial consequence of Coprime Action and the fact that \( G = \langle \theta(a) \mid a \in A^\# \rangle \) that if \( B \in \text{Hyp}(A) \) then there exists \( b, b' \in B^\# \) with \( M_b \neq M_{b'} \). In fact, we can go a little further.

Lemma 11.2. Let \( B \in \text{Hyp}(A) \).

(a) \( B_0 \in \text{Hyp}(B) \). Then \( M_b \) takes at least two values as \( b \) ranges over \( B \setminus B_0 \).
(b) $M_a$ takes at least three values as a ranges over $A \setminus B$.

Proof of Theorem [11.1]. Choose $W$ maximal subject to
\[ W \in L, \text{ } W \text{ is } O(a)\text{-invariant and } W = E(W) = [W, a]. \]

If $W \neq 1$ choose $M$ with $\theta(N(W)) \leq M \in L^*$. If $W = 1$ choose $M$ with $\theta(a) \leq M \in L^*$ and if possible with $C_{O_q}(M)(a) = 1$ for some $p \in \pi(F(M))$. In both cases, $\theta(a) \leq M$ and $\theta(a) = C_M(a)$. Moreover, $W$ is $AC_M(a)$-invariant so as $W = E(W) = [W, a]$, Theorem 4.3(a) implies $W \leq E(M)$.

Suppose $N$ satisfies
\[ \theta(a) \leq N \in L^* \text{ and } M \leadsto N. \]

We will prove that
\[ (*) \quad M = N. \]

Since $M \leadsto N$ we have $W \leq E(M) \leq N$ by Theorem 6.2(a) so another application of Theorem 4.3(a) implies $W \leq E(N)$. Then $W = [W, a] \leq [E(N), a]$. Now $\theta(a) \leq N$ so $[E(N), a]$ is $\theta(a)$-invariant. It is also normal in $E(N)$ so it is the central product of its components. The maximal choice of $W$ forces
\[ W = [E(N), a] \leq F^*(N). \]

Suppose $W \neq 1$. Then $F^*(N) \leq \theta(N(W)) \leq M$ so $N \leadsto M$. Since $E(N) \neq 1$, Theorem 6.2(c) forces $M = N$. Hence we may assume that $W = 1$. In particular
\[ E(N) \leq \theta(a) \leq M. \]

We claim that
\[ \pi(F(N)) \leq \pi(F(M)). \]

Assume false and choose $q \in \pi(F(N)) \setminus \pi(F(M))$. Theorem 6.2(a) implies $M \cap O_q(N) = 1$. Now $C_{O_q(N)}(a) \leq \theta(a) \cap O_q(N) \leq M \cap O_q(N)$ so $C_{O_q(N)}(a) = 1$. Recall that $\theta(a) \leq N$. The choice of $M$ implies that there exists $p \in \pi(F(M))$ with $C_{O_p(M)}(a) = 1$. As $M \leadsto N$ we have $Z(O_p(M)) \leq N$. Set $X = Z(O_p(M))O_q(N)$. Then $Z(O_p(M))$ is an $a$-invariant subgroup of $X$ and Coprime Action(c) implies $Z(O_p(M)) = [Z(O_p(M)), a]$. Theorem 12.9(a) implies $Z(O_p(M)) \leq O_{X}(X)$ whence $O_q(N) \leq \theta(N(Z(O_p(M)))) = M$, contrary to $M \cap O_q(N) = 1$. The claim is established.

Theorem 6.2(c) and the First Uniqueness Theorem imply $M = N$, which proves $(*)$.

Suppose $1 \neq X \leq M$ is $A\theta(a)$-invariant. Choose $N$ with $\theta(N(F^*(X))) \leq N \in L^*$. Now $F^*(X) \leq F^*(M) \leq N$ so $M \leadsto N$. Then $M = N$ by $(*)$ and so $\theta(N(X)) \leq \theta(N(F^*(X))) \leq M$.

Set $M_a = M$ to complete the proof. \hfill $\square$

Proof of Lemma 11.2. (a). Recall that rank($A$) = 3 so $B_0$ is cyclic. Let $e$ be a generator for $B_0$. Assume the result is false and let $M$ denote the common value of $M_b$ as $b$ ranges over $B \setminus B_0$. By Coprime Action(d), for each $a \in A^\#$,
\[
[\theta(a), e] = \{[\theta(a) \cap C(b), e] \mid b \in B \setminus B_0\} \\
\leq \{\theta(b) \mid b \in B \setminus B_0\} \leq M.
\]

Lemma 8.3 supplies a contradiction.
(b). Assume the result is false. Then there exist \( M, L \) with \( M_a \in \{ M, L \} \) for all \( a \in A \setminus B \). Let \( a \in A \setminus B \). Then
\[
\theta(a) = \langle \theta(D) \mid a \in D \in \text{Hyp}(A) \rangle.
\]
If \( a \in D \in \text{Hyp}(A) \) then \( D \neq B \) so \( D \cap B \in \text{Hyp}(D) \). By (a), with \( D \) in the role of \( B \), \( M_d \) takes at least two values as \( d \) ranges over \( D \setminus D \cap B \). Hence \( M_d = M \) for some \( d \in D \setminus D \cap B \). Consequently \( \theta(D) \leq \theta(d) \leq M \) and we deduce that
\[
\theta(a) \leq M
\]
for all \( a \in A \setminus B \).

Choose \( D \in \text{Hyp}(A) \) with \( D \neq B \). Set \( D_0 = D \cap B \in \text{Hyp}(D) \) and let \( e \) be a generator for \( D_0 \). Let \( T \) be any \( \theta \)-subgroup. By Coprime Action(d),
\[
[T, e] = \langle [C_T(d), e] \mid d \in D \setminus D_0 \rangle \leq M.
\]
In particular, \( [\theta(a), e] \leq M \) for all \( a \in A^\# \). Again, Lemma 8.5 supplies a contradiction. \( \square \)

12. The Fermat case

Since \( S \) is the unique maximal \( \theta(A) \)-invariant solvable \( \theta \)-subgroup it follows that \( F(M_a) \leq S \) for all \( a \in A^\# \). The goal of this section is to prove:

**Theorem 12.1.** Let \( a \in A^\# \) and suppose \( E(M_a) = 1 \). Then \( [F(M_a), a]F(S) \) is nilpotent.

In the case that \( r \) is not a Fermat prime, this follows readily from Theorem 4.2(a), with \( [F(M_a), a] \) in the role of \( H \). Just as in the author’s proof of the Solvable Signalizer Functor Theorem, the Fermat case requires special treatment.

Throughout the remainder of this section we assume Theorem 12.1 to be false. Theorem 4.2(a) implies \( r \) is Fermat and that \( [O_2(M_a), a]F(S) \) is not nilpotent. Set
\[
Q = [O_2(M_a), a]
\]
and choose an odd prime \( p \) such that
\[
[O_p(S), Q] \neq 1.
\]

**Lemma 12.2.**
(a) \( O_p(M_a) = 1, M_a \cap O_p(S) = 1 \) and \( C_{O_p(S)}(a) = 1 \).
(b) \( Q = [Q, a] \) and \( Q' = \Phi(Q) = Z(Q) = C_Q(a) \).

**Proof.** (a). Since \( F(M_a) \leq S \) and \( E(M_a) = 1 \) we have \( M_a \sim S \). Suppose \( p \in \pi(F(M_a)) \). Then \( \{ 2, p \} \subseteq \pi(F(M_a)) \) so Theorem 6.2(b) implies \( O_p(S) \leq M_a \). Then \( [O_p(S), Q] \leq O_p(S) \cap O_2(M_a) = 1 \), a contradiction. Thus \( p \notin \pi(F(M_a)) \). Theorem 6.2(a) implies \( M_a \cap O_p(S) = 1 \). Finally \( C_{O_p(S)}(a) \leq \theta(a) \cap O_p(S) \leq M_a \cap O_p(S) = 1 \).

(b). The first assertion is Coprime Action(c) and the second is Coprime Action(i). Provided we can show \( [U, a] = 1 \) whenever \( U \) is a characteristic abelian subgroup of \( Q \). Assume this to be false. Now \( [U, a] \leq O_2(M_a) \) so \( U \) is an \( A \theta(a) \)-invariant subgroup of \( M_a \) and Theorem 11.1 implies \( \theta(N([U, a])) \leq M_a \). On the other hand, Lemma 2.3(c) implies \( [U, a] \) acts trivially on \( O_p(S) \). Hence \( O_p(S) \leq \theta(N([U, a])) \) contrary to \( M_a \cap O_p(S) = 1 \) which completes the proof. \( \square \)

Set
\[
W = \langle [C_Q(B), a]^\prime \mid B \in \text{Hyp}(A) \rangle.
\]
Lemma 12.3. \( (a) \) \( W \leq Z(Q) \) and \( 1 \neq W \leq \theta(A) \).
\( (b) \) If \( H \in \mathcal{L}^* \) then \( W \leq \text{sol}(H) \).
\( (c) \) \( C_{O_p(S)}(W) = 1 \).

Proof. Lemma \([12.2]_{(b)}\) implies \( W \leq Q' = Z(Q) = C_Q(a) \). Let \( B \in \text{Hyp}(A) \), set \( Q_0 = [C_Q(B), a] \) and suppose \( Q_0 \neq 1 \). Then \( a \notin B \) and \( A = (B, a) \). As \( Q_0' \leq W \leq C_Q(a) \) we obtain \( [Q_0', A] = 1 \). Moreover \( \theta(A) \) normalizes \( Q, B \) and \( a \) so \( Q_0 \) is \( \theta(A) \)-invariant and \( Q_0' \leq \theta(A) \). Consequently \( W \leq \theta(A) \).

For each \( b \in B^* \) we have \( Q_0 \leq O_2(\theta(b) A) \) and then Lemma \([2.6]_{(a)}\) implies \( Q_0' \leq \text{sol}(\theta(b)) \). Let \( H \in \mathcal{L}^* \). Then \( Q_0' \leq \theta(A) \leq H \). Using Lemma \([2.6]_{(b)}\) for the last containment, we have

\[
Q_0' \leq \bigcap_{b \in B^*} \text{sol}(\theta(b)) \cap H \leq \bigcap_{b \in B^*} \text{sol}(C_H(b)) \leq \text{sol}(H).
\]

This proves \( (b) \).

By Coprime Action\((d)\),

\[
(\ast) \quad Q = ([C_Q(B), a] \mid B \in \text{Hyp}(A)).
\]

As \([O_p(S), Q] \neq 1\) we may choose \( B \) with \([O_p(S), Q_0] \neq 1\). Lemma \([2.6]_{(c)}\), with \( Q_0 \) in the role of \( X \), implies that \( Q_0 \) is nonabelian. Then \( 1 \neq Q_0' \leq W \), which completes the proof of \( (a) \).

To prove \( (c) \) consider the action of \( Q_0 \) on \( C_{O_p(S)}(W) \). Note that \( C_{O_p(S)}(W) \) is \( Q \)-invariant because \( W \leq Z(Q) \). Now \( Q_0' \leq W \) so \( Q_0 \) induces an abelian group on \( C_{O_p(S)}(W) \). Lemma \([2.6]_{(c)}\) implies \([C_{O_p(S)}(W), Q_0] = 1 \). Then \( (\ast) \) implies \([C_{O_p(S)}(W), Q] = 1 \). Now \( Q \) is an \( A\theta(a) \)-invariant subnormal subgroup of \( M_a \) so Theorem \([11.1]\) implies \( \theta(N(Q)) \leq M_a \). Then \( C_{O_p(S)}(W) \leq M_a \cap O_p(S) = 1 \). \( \square \)

Lemma 12.4. Let \( H \in \mathcal{L}^* \) and suppose \( O_p(S) \cap H \neq 1 \). Then \( O_p(S) \leq O_p(H) \).

Proof. Set \( P = O_p(S) \cap H \). Lemma \([12.3]_{(b, c)}\) and Coprime Action\((c)\) imply

\[
P = [P, W] \leq \text{sol}(H).
\]

Now \( H \in \mathcal{L}^* \) so \( \text{sol}(H) \leq S \) and then \( 1 \neq P \leq O_p(\text{sol}(H)) \leq O_p(H) \). Also \( O_p(H) \leq S \) so \( O_p(H)O_p(S) \) is a \( p \)-group. Since \( N_{O_p(S)}(O_p(H)) \leq O_p(S) \cap H = P \leq O_p(H) \) it follows that \( O_p(H)O_p(S) = O_p(H) \). As \( O_p(H) \leq S \), the conclusion follows. \( \square \)

Choose \( N \) with \( \theta(N(O_p(S))) \leq N \in \mathcal{L}^* \).

Lemma 12.5. \( E(N) = 1 \).

Proof. We have \( Q \leq N \) whence \( Q \leq O_2(N; A) \). Lemma \([2.6]_{(a)}\) implies \( Q' \leq \text{sol}(N) \), so \([Q', E(N)] = 1 \). Now \( Q' \neq 1 \) so Theorem \([11.1]\) implies \( \theta(N(Q')) \leq M_a \), whence \( E(N) \leq M_a \). Now \( F(M_a) \leq S \leq N \) and \( E(N) \) and \( F(M_a) \) normalize each other. Then \([E(N), F(M_a)] = 1 \). By hypothesis, \( E(M_a) = 1 \) so it follows that \( E(N) = 1 \). \( \square \)

We can now complete the proof of Theorem \([12.1]\) By Coprime Action\((d)\) there exists \( B \in \text{Hyp}(A) \) with \( C_{O_p(S)}(B) \neq 1 \). By Lemma \([11.2]\) there exists \( b \in B^* \) with \( M_b \neq N \). Now \( 1 \neq C_{O_p(S)}(B) \leq \theta(b) \cap O_p(S) \leq M_b \cap O_p(S) \) so Lemma \([12.3]\) implies
$O_p(S) \leq O_p(M_b)$. Then $F^*(M_b) \leq N$ and $M_b \sim N$. Since $E(N) = 1$, Theorem 6.2 and the First Uniqueness Theorem imply there exists a prime $t$ with

$$O_t(N) \neq 1, O_t(M_b) = 1 \text{ and } M_b \cap O_t(N) = 1.$$ 

As $\theta(C_{O_t(N)}(b)) \leq \theta(b) \leq M_b$ we also have

$$C_{O_t(N)}(b) = 1.$$ 

Since $p \neq 2$, Lemma 2.1(c) implies $[O_p(M_b), b]$ centralizes $O_t(N)$. If $[O_p(M_b), b] \neq 1$ then $O_t(N) \leq \theta(N([O_p(M_b), b])) \leq M_b$ by Theorem 11.1, a contradiction. Thus $[O_p(M_b), b] = 1$. Then as $O_p(S) \leq O_p(M_b)$, we have

$$[M_b, b] \leq C_{M_b}(O_p(M_b)) \leq \theta(N(O_p(S))) \leq N.$$ 

Now, $U = [M_b, b]$. Lemma 2.1(c) implies that $U/C_U(O_t(N))$ is a solvable $\{2, t\}$-group. Let $V$ be the subgroup of $U$ generated by $U^{(\infty)}$ and the $\{2, t\}'$-elements of $U$. Then $[O_t(N), V] = 1$. Also $V \dechar U \leq M_b$ so $V \leq M_b$. If $V \neq 1$ then $O_t(N) \leq M_b$, a contradiction. Thus $V = 1$ and $U$ is a solvable $\{2, t\}$-group.

Note that $p \in \pi(F(M_b))$ since $O_p(S) \leq O_p(M_b)$ and so $p \neq t$ as $O_t(M_b) = 1$. Also, $p \neq 2$. Thus $[M_b, b]$ is a $p'$-group. McBride’s Dichotomy, Lemma 9.1 and Theorem 9.2 imply that there exists a unique maximal $\theta(A)$-invariant $p$-solvable $\theta$-subgroup $K$ and that $\theta(b)$ is $p$-solvable. Now $M_b = C_{M_b}(b)U = \theta(b)U$. Since $U$ is a normal solvable subgroup of $M_b$ we deduce that $M_b$ is $p$-solvable and then that $M_b = K$. But $O_t(N)$ is $p$-solvable and $\theta(A)$-invariant, whence $O_t(N) \leq M_b$. This contradiction completes the proof of Theorem 12.1.

13. THE SECOND UNIQUENESS THEOREM

The goal of this section is the prove the following:

**Theorem 13.1** (The Second Uniqueness Theorem). Let $a \in A^\#$ and suppose that $E(M_a) = 1$. Then:

(a) $S \leq M_a$.

(b) If $b \in A^\#$ and $E(M_b) = 1$ then $M_b = M_a$.

**Lemma 13.2.** Suppose $I$ and $J$ are subgroups of the group $X$. Suppose also that $E(X) = 1$ and that $IF(X)$ and $JF(X)$ are nilpotent. Let $p$ and $q$ be distinct primes. Then $[O_p(I), O_q(J)] = 1$.

**Proof.** Set $Z = Z(F(X))$. Since $E(X) = 1$ we have $C_X(F(X)) = Z$. Now

$$[O_p(I), O_q(J)] \leq [C_X(O_p(F(X))), C_X(O_q(F(X)))] \leq C_{F(X)}(=)Z.$$ 

Hence $O_p(I)$ normalizes the nilpotent group $O_q(X)Z$. Then $[O_p(I), O_q(J)] \leq O_q(O_q(J)Z)$ and the commutator is a $q$-group. Similarly, it is a $p$-group and hence is trivial. \hfill \Box

**Proof of Theorem 13.1**. Set $M = M_a$. Since $E(M) = 1$ we have $F(M) \neq 1$ so McBride’s Dichotomy implies that $\theta$ is nearsolvable.

(a) Suppose first that $[F(M), a] = 1$. Coprime Action(g) implies $[M, a] = 1$. Thus $M = \theta(a)$. Choose $p \in \pi(F(M))$. Theorem 9.1 implies that there exists a unique maximal $\theta(A)$-invariant $p$-solvable $\theta$-subgroup and that $\theta(a)$, and hence $M$ is $p$-solvable. Since $M \in L^*$ it follows that $M$ is the said subgroup. Now $S$ is $\theta(A)$-invariant and nearsolvable so $S \leq M$. Hence we may assume that $[F(M), a] \neq 1$.
Note that
\[ F(M) \leq S \]
because \( F(M) \) is \( \theta(A) \)-invariant and solvable. In particular, \( M \leadsto S \).

We claim that \( \pi(F(S)) \leq \pi(F(M)) \). Indeed, suppose \( q \) is a prime with \( q \notin \pi(F(M)) \). Using Theorems 12.1 and 11.1 we have \( O_p(S) \leq \theta(N(F(M),a)) \leq M \). On the other hand, Theorem 6.2(a) implies \( M \cap O_q(S) = 1 \). Hence \( O_q(S) = 1 \) and the claim is established.

Consider the case that \( |\pi(F(M))| \geq 2 \). Theorem 6.2(b) implies
\[ F(S) \leq M. \]

Since \( F(M) \leq S \) we deduce that \( F(M)F(S) \) is nilpotent. Let \( x \in S \) and let \( p, q \in \pi(F(M)) \) be distinct. Now \( F(M)x F(S) \) is nilpotent so Lemma 13.2 implies \( [O_p(M), O_q(M)]^x = 1 \). Then \( O_q(M)^x \leq N_S(O_p(M)) \leq M \). We deduce that \( F(M)^x \leq M \) and so \( M^x \leadsto M \). If \( x \in C_S(B) \) for some \( B \in \text{Hyp}(A) \) then Lemma 6.3 implies that \( M \) and \( M^x \) are comaximal and so \( M = M^x \) by Theorem 6.2(c)(ii). Then \( x \in N_S(M) \leq M \) and we deduce that \( C_S(B) \leq M \) for all \( B \in \text{Hyp}(A) \).

Coprime Action(d) forces \( S \leq M \). Hence we may assume that
\[ F(M) \text{ is a } p\text{-group} \]
for some prime \( p \).

The First Uniqueness Theorem implies \( O_p(G; \theta) \leq M \) so \( O_p(G; \theta) = O_p(M; A) \). As \( O_p(G; \theta) \) is \( \theta(A) \)-invariant and solvable we have \( O_p(G; \theta) \leq S \) and so \( O_p(G; \theta) = O_p(S; A) \). Consequently \( O_p(M; A) = O_p(S; A) \). Note that \( O_p(M) \neq 1 \) since \( F(M) \) is a \( p \)-group. If \( O_p(M) \) is abelian then Theorem 6.4 implies \( J(O_p(M)) = J(O_p(S)) \). Then \( S \leq \theta(N(J(O_p(M)))) = M \). Hence we may assume that \( O_p(M) \) is nonabelian.

Choose \( N \) with \( S \leq N \in \mathcal{L} \). As above, \( O_p(G; \theta) = O_p(N; A) \) whence \( O_p(M) \leq O_p(N; A) \). Lemma 12.2(a) implies \( O_p(M)^f \leq \text{sol}(N) \). Consequently \( E(N) \leq \theta(N(O_p(M)^f)) \leq M \). As \( S \leq N \) and \( F(N) \) is \( \theta(A) \)-invariant and solvable we have \( F(N) \leq F(S) \). Then \( \pi(F(N)) \leq \pi(F(S)) \leq \pi(F(M)) = \{ p \} \). Theorem 6.2(c) and the First Uniqueness Theorem imply \( M_0 = N \), so \( S \leq M \).

(b). Recall that \( S \) contains every \( \theta(A) \)-invariant solvable \( \theta \)-subgroup. Using (a) we have \( F^*(M_a) = F(M_a) \) \( \leq S \leq M_b \) so \( M_a \leadsto M_b \). Similarly \( M_b \leadsto M_a \). Theorem 6.2(c) and the First Uniqueness Theorem force \( M_b = M_a \).

\textit{Remark.} In [5] it is conjectured that if \( p \) is a prime then to each nontrivial \( p \)-group \( P \) there exists a nontrivial characteristic subgroup \( W(P) \) such that whenever \( A \) acts coprimely on the group \( M \) and \( M \) has characteristic \( p \) then \( W(O_p(M; A)) \leq M \). A proof of this conjecture would lead to a much cleaner proof of the Second Uniqueness Theorem. The conjecture is known to be true if \( p > 3 \), see [4].

14. \( A \)-COMPONENTS

For each \( a \in A^\# \) let
\[ \Omega_a = \{ K \mid K \in \text{comp}_A(M) \text{ for some } M \text{ with } \theta(a) \leq M \in \mathcal{L} \}. \]
In particular, \( \text{comp}_A(\theta(a)) \cup \text{comp}_A(M_a) \subseteq \Omega_a \). Let
\[ \Omega = \bigcup_{a \in A^\#} \Omega_a. \]
The Second Uniqueness Theorem and Lemma [112] imply
\[ \Omega \neq \emptyset. \]

The subsequent analysis is dominated by the elements of \( \Omega \). Recall the definition of \( C_K^*(A) \) given in (3).

**Theorem 14.1.** Let \( K, L \in \Omega \). The following are equivalent:

(a) \([K, L] \neq 1\).
(b) \( C_K^*(A) = C_L^*(A) \).
(c) \([C_K^*(A), C_L^*(A)] \neq 1\).

In particular 'does not commute' is an equivalence relation on \( \Omega \).

**Lemma 14.2.** Suppose \( a, b \in A^\#, K \in \Omega_a, L \in \Omega_b \) and \( N \in L \). Set \( K_0 = E(K \cap N) \) and \( L_0 = E(L \cap N) \). Assume that \([K_0, L_0] \neq 1\). Then there exists \( X \) such that:

(a) \([K_0, L_0] \leq X \in \text{comp}_{A, \text{sol}}(N)\).
(b) If \([K_0, a] \neq 1 \) then \( X = K_0 \) and if \([L_0, b] \neq 1 \) then \( X = L_0 \).
(c) \( C_K^*(A) = C_L^*(A) \).
(d) If \( X \) is constrained then \( K_0 = L_0 \) and \( X = K_0 \text{sol}(X) \).

**Proof.** Choose \( M \in L \) with \( \theta(a) \leq M \) and \( K \in \text{comp}_A(M) \). Now \( C_N(a) \leq \theta(a) \cap N \leq M \cap N \) so Hypothesis 4.4 is satisfied with \( N \) and \( M \cap N \) in the roles of \( G \) and \( H \) respectively. Also \( K \leq \leq M \) so \( K_0 \leq \leq M \cap N \). As \([K_0, L_0] \neq 1\) we have \( K_0 \neq 1 \) and then Lemma 3.4 implies \( K_0 \) is \( A \)-quasisimple, so \( K_0 \in \text{comp}_A(M \cap N) \). Theorem 4.3 implies there exists \( \tilde{K} \) with
\[ K_0 \leq \tilde{K} \in \text{comp}_{A, \text{sol}}(N). \]

Similarly there exists \( \tilde{L} \) with
\[ L_0 \leq \tilde{L} \in \text{comp}_{A, \text{sol}}(N). \]

Since \([K_0, L_0] \neq 1\) we have \([\tilde{K}, \tilde{L}] \neq 1\). Consequently either \( \tilde{K} \) and \( \tilde{L} \) are both semisimple or both constrained.

Suppose that \( \tilde{K} \) and \( \tilde{L} \) are both semisimple. Since \([\tilde{K}, \tilde{L}] \neq 1\) this forces \( \tilde{K} = \tilde{L} \). Put \( X = \tilde{K} \). Then (a) holds and (b) follows from Theorem 4.4(a). We claim that \( C_{\tilde{K}}^*(A) = C_{K_0}^*(A) \). If \([K_0, a] \neq 1\) then (b) implies \( \tilde{K} = K_0 \) and the claim is clear. Suppose \([K_0, a] = 1\). Theorem 4.3(b) implies \( K_0 = E(C_{\tilde{K}}^*(a)) \). In particular, \( K_0 \) is \( C_{\tilde{K}}^*(A) \)-invariant. The claim follows from Lemma 3.4. Similarly \( C_{\tilde{L}}^*(A) = C_{L_0}^*(A) \). Also by Lemma 3.4 \( C_{\tilde{K}}^*(A) = C_{K_0}^*(A) \) and \( C_{\tilde{L}}^*(A) = C_{L_0}^*(A) \). Since \( \tilde{K} = \tilde{L} \), (c) follows. Note that (d) is not applicable in this case.

Suppose \( \tilde{K} \) and \( \tilde{L} \) are both constrained. Theorem 4.3 implies
\[ [K_0, a] = 1 \quad \text{and} \quad \tilde{K} = K_0 \text{sol}(\tilde{K}). \]

In particular \([\tilde{K}, a] \leq \text{sol}(\tilde{K})\). Recall that \( K_0 \in \text{comp}_A(M \cap N) \). As \( \theta(a) \leq M \) and \([K_0, a] = 1\) it follows that \( K_0 \in \text{comp}_A(\theta(a) \cap N) \). Similarly
\[ L_0 \in \text{comp}_A(\theta(b) \cap N). \]

Since \( L_0 \) is \( A \)-quasisimple, we have \([L_0, a] = 1 \) or \( L_0 \). Suppose \([L_0, a] = L_0 \). Then \([\tilde{L}, a] \leq \text{sol}(\tilde{L})\) and since every proper \( A \)-invariant normal subgroup of \( \tilde{L} \) is solvable
it follows that $\tilde{L} = [\tilde{L}, a]$. As $[\tilde{K}, a] \leq \text{sol}(\tilde{K})$, we have $\tilde{K} \neq \tilde{L}$. Theorem 4.3(d) implies $[\tilde{K}, \tilde{L}] = 1$, a contradiction. We deduce that $[L_0, a] = 1$. Then (*) implies $L_0 \in \text{comp}_A(\theta(a) \cap \theta(b) \cap N)$.

Similarly $K_0 \in \text{comp}_A(\theta(a) \cap \theta(b) \cap N)$ so as $[K_0, L_0] \neq 1$ we must have $K_0 = L_0$. In particular, $\tilde{K} \cap \tilde{L}$ is nonsolvable so $\tilde{K} = \tilde{L}$. Put $X = \tilde{K}$. Then (a) holds. (b) is not applicable in this case. Lemma 3.4 implies that $C^*_K(A) = C^*_K(\theta(A)) = C^*_L(A)$. Then (c) holds. (d) has also been proved. □

**Proof of Theorem 14.7** Suppose (a) holds, so $[K, L] \neq 1$. Lemma 3.5(a) implies there exists $D \in \text{Hyp}(A)$ such that $C^*_L(d)$ is $A$-quasisimple for all $d \in D^\#$ and $[K, C^*_L(D)] \neq 1$. Lemma 3.5(b) implies there exists $d \in D^\#$ such that $C^*_K(d)$ is $A$-quasisimple and $[C^*_K(d), C^*_L(D)] \neq 1$. Then $[C^*_K(d), C^*_L(d)] \neq 1$.

Put $N = \theta(d)$. Now $C^*_K(d) = K \cap N$ so $C^*_K(d) = E(K \cap N)$. Similarly $C^*_L(d) = E(L \cap N)$. Lemma 4.2 implies $C^*_K(A) = C^*_L(A)$ so (b) holds.

Lemma 3.4 implies that $C^*_K(A)$ is nonabelian. The remaining implications follow trivially. □

### 15. The Balance Theorem

The aim of this section is to prove the following.

**Theorem 15.1** (The Balance Theorem). Suppose $a, b \in A^\#, K \in \Omega_a$ and $E(K \cap M_b) \neq 1$. Then $E(K \cap M_b)$ is $A$-quasisimple and is contained in an $A$-component of $M_b$. In particular

$$E(K \cap M_b) \leq E(M_b).$$

A number of lemmas are required. Recall that $S$ is the unique maximal $\theta(A)$-invariant solvable $\theta$-subgroup.

**Lemma 15.2.** Suppose $a \in A^\#, K \in \Omega_a$ and $\theta$ is nearsolvable. Let $Y$ be a non-solvable $AC_K(A)$-invariant subgroup of $K$. Then

$$K = \langle Y, K \cap S \rangle.$$  

**Proof.** Choose $M$ with $\theta(a) \leq M \in \mathcal{L}$ and $K \in \text{comp}_A(M)$. Since $\theta(A) \leq M$ we have $S \cap M = \text{sol}_A(M; A)$. Since $K$ is an $A$-invariant subnormal subgroup of $M$, Corollary 5.3 implies $\text{sol}_A(K) = K \cap \text{sol}_A(M; A)$. Then $\text{sol}_A(K; A) = K \cap S$. Now $\theta$ is nearsolvable so $\theta(A)$ and hence $C_K(A)$ is solvable. Apply Theorem 3.1(c) to $K/Z(K)$. □

**Lemma 15.3.** Suppose $a, b \in A^\#, K \in \Omega_a$ and $E(K \cap M_b) \neq 1$. Then $E(K \cap M_b)$ is $A$-quasisimple. Suppose also that $E(K \cap M_b)$ is not contained in an $A$-component of $M_b$. Then:

(a) $K \leq M_b$ and there exists $\tilde{K}$ with

$$K \leq \tilde{K} \in \text{comp}_{A, \text{sol}}(M_b).$$

(b) $\tilde{K}$ is constrained, $\tilde{K} = K \text{sol}(\tilde{K})$ and $K \cap E(M_b) \leq Z(K)$.

(c) If $c \in A \setminus C_A(K)$ then $\tilde{K} = [\tilde{K}, c] = \langle K, C_{\text{sol}(\tilde{K})}(c) \rangle$.

(d) $\theta$ is nearsolvable.
Proof. Choose \( M \) with \( \theta(a) \leq M \in \mathcal{L} \) and \( K \in \text{comp}_A(M) \). Set \( K_0 = E(K \cap M_b) \). Lemma 5.4 implies that \( K_0 \) is \( A \)-quasisimple. Since \( C_{M_b}(a) \leq \theta(a) \cap M_b \leq M \cap M_b \), Hypothesis 14.1 is satisfied with \( M_b \) and \( M \cap M_b \) in the roles of \( G \) and \( H \) respectively. As \( K \cap M_b \leq M \cap M_b \) we have \( K_0 \in \text{comp}_A(M \cap M_b) \). Theorem 4.4 implies that there exists \( \tilde{K} \) with

\[
K_0 \leq \tilde{K} \in \text{comp}_{A,\text{sol}}(M_b).
\]

By assumption, \( K_0 \) is not contained in an \( A \)-component of \( M_b \) so \( \tilde{K} \) is constrained. Then \( 1 \neq \text{sol}(\tilde{K}) \leq \text{sol}(M_b) \in \mathcal{L} \) and McBride’s Dichotomy implies that \( \theta \) is nearsolvable.

Lemma 3.3 implies that \( C_{K_0}(A) = C_{\tilde{K}}(A) \). Since \( \tilde{K} \) is constrained we have

\[
[K, E(M_b)] = 1, \quad \text{whence} \quad [C_{\tilde{K}}(A), E(M_b)] = 1.
\]

Recall that \( E(M_b) \) is generated by the \( A \)-components of \( M_b \). Theorem 14.1 implies \( [K, E(M_b)] = 1 \). If \( E(M_b) \neq 1 \) then \( K \leq M_b \). If \( E(M_b) = 1 \) then the Second Uniqueness Theorem implies \( S \leq M_b \) and then Lemma 15.2 yields \( K = (K_0, K \cap S) \leq M_b \). In both cases, \( K \leq M_b \) so \( K_0 = K \).

Also \( K \cap E(M_b) \leq \tilde{K} \cap E(M_b) \leq \text{sol}(\tilde{K}) \) so \( K \cap E(M_b) \) is solvable normal subgroup of the \( A \)-quasisimple group \( \tilde{K} \). Hence it is contained in \( Z(K) \).

To prove (c), suppose \( c \in A \setminus C_A(K) \). Then \( K = [K, c] \leq [\tilde{K}, c] \) so as \( \tilde{K} \in \text{comp}_{A,\text{sol}}(M_b) \) it follows that \( \tilde{K} = [\tilde{K}, c] \). The remaining assertion follows from Theorem 14.1(c).

\[\square\]

Proof of the Balance Theorem. Assume false. Lemma 15.3 implies there exists a constrained \( \tilde{K} \) with

\[
K \leq \tilde{K} \in \text{comp}_{A,\text{sol}}(M_b).
\]

We may suppose that \( (a, b, k) \) has been chosen to maximize \( \tilde{K} \).

Claim 1. Let \( L \in \Omega \) and suppose \( [K, L] \neq 1 \). Then \( K = L \).

Proof. Lemma 5.5 implies there exists \( D \in \text{Hyp}(A) \) and \( d \in D^\# \) such that \( C_K(d) \) and \( C_L(d) \) are \( A \)-quasisimple. Let \( K_0 = E(K \cap M_d) \) and \( L_0 = E(L \cap M_d) \). Since \( C_K(d) \leq K \cap \theta(d) \leq K \cap M_d \), Lemma 3.3 implies \( C_K(d) \leq K_0 \). Similarly \( C_L(d) \leq L_0 \). Now \( [K, L] \neq 1 \) so Theorem 14.1 implies \( [C_K(A), C_L(A)] \neq 1 \). Then \( [K_0, L_0] \neq 1 \).

Lemma 14.2 with \( M_d \) in the role of \( N \), implies there exists \( X \) with

\[
\langle K_0, L_0 \rangle \leq X \in \text{comp}_{A,\text{sol}}(M_d).
\]

Suppose \( X \) is constrained. Then \( K_0 \) is not contained in a component of \( M_d \). As \( K_0 = E(K \cap M_d) \), Lemma 15.3(a) implies \( K \leq M_d \), so \( K_0 = K \). Similarly \( L_0 = L \). Lemma 14.2(d) implies \( K_0 = L_0 \) so we are done in this case.

Suppose \( X \) is semisimple. Then \( X \) is \( A \)-quasisimple. Now \( C_K(d) \leq X \cap M_b \) so as \( C_K(d) \) is \( A \)-quasisimple, Lemma 15.4(b), with \( X \) and \( X \cap M_b \) in the roles of \( K \) and \( H \) respectively, implies \( C_K(d) \leq E(X \cap M_b) \). By Lemma 15.3 \( K \cap E(M_b) \leq Z(K) \) so \( E(X \cap M_b) \leq E(M_b) \). In particular, \( E(X \cap M_b) \) is not contained in an \( A \)-component of \( M_b \). Now \( X \in \text{comp}_A(M_d) \subseteq \Omega_d \) so Lemma 15.3(a) forces \( X \leq M_b \) and \( X \cap E(M_b) \leq Z(X) \).

Now \( C_L(d) \) is \( A \)-quasisimple and \( C_L(d) \leq X \leq M_b \). Then \( C_L(d) \leq E(L \cap M_b) \leq E(M_b) \) and Lemma 15.2 forces \( L \leq M_b \). We apply Lemma 14.2 with \( M_b, K \) and
L in the roles of \( N, K_0 \) and \( L_0 \) respectively. Since \( K \) is not contained in an \( A \)-component of \( M_b \), Lemma (14.2 d) forces \( K = L \). □

**Claim 2.** Suppose \( c \in A \setminus C_A(K) \) and \( E(M_c) \neq 1 \). Then \( \tilde{K} \in \text{comp}_{A,\text{sol}}(M_c) \).

**Proof.** Claim (1) implies \( K \) normalizes \( E(M_c) \) so \( K \leq M_c \). Using Lemma (16.3 c) we have

\[
\tilde{K} = \langle K, C_{\text{sol}}(\tilde{K})(c) \rangle \leq \langle K, \theta(c) \rangle \leq M_c.
\]

As previously, Theorem (4.4) implies that there exists \( K^* \) with

\[
K \leq K^* \in \text{comp}_{A,\text{sol}}(M_c).
\]

Then \( K \leq K^* \cap \tilde{K} \leq \tilde{K} \). But \( \tilde{K} \) contains no proper \( A \)-invariant nonsolvable subnormal subgroups, whence \( K^* \cap \tilde{K} = \tilde{K} \) and \( \tilde{K} \leq K^* \). As \( \tilde{K} \) is constrained we have \( K < \tilde{K} \leq K^* \) and then Claim (1) implies \( K^* \notin \Omega \), so \( K^* \) is not semisimple. The maximal choice of \( \tilde{K} \) forces \( \tilde{K} = K^* \), which proves the claim. □

Choose \( N \) with

\[
\theta(N(\tilde{K})) \subseteq N \in \Theta^*.
\]

**Claim 3.** Suppose \( c \in A \setminus C_A(K) \) and \( E(M_c) \neq 1 \). Then \( M_c = N \).

**Proof.** Claim (2) implies \( \tilde{K} \in \text{comp}_{A,\text{sol}}(M_c) \). In particular, \( \tilde{K} \leq O_s(M_c) \) and then \( O_s(M_c) \leq N \). Lemma (15.3) implies \( \tilde{K} = [\tilde{K}, c] \) so using Theorem (4.3 b) we have

\[
1 \neq \tilde{K} \leq [O_s(M_c), c]^{(\infty)} \leq O_s(N).
\]

Theorem (11.1) implies \( O_s(N) \leq M_c \) and then Corollary (8.2) forces \( M_c = N \). □

It is straightforward to complete the proof of the Balance Theorem. Now \( \theta \) is nearsolvable so \( \theta(A) \) is solvable and we may choose \( B \) with \( C_A(K) \leq B \in \text{Hyp}(A) \). The Second Uniqueness Theorem and Claim (4) implies that \( M_c \) takes at most two values as \( c \) ranges over \( A \setminus B \). Lemma (11.2 b) supplies a contradiction. □

16. THE STRUCTURE THEOREM

The following result will be proved. Once it has, Theorem (3.2) and Lemma (5.6) will supply a contradiction and complete the proof of the Signalizer Functor Theorem.

**Theorem 16.1** (The Structure Theorem).

(a) If \( a \in A^\# \) with \( E(\theta(a)) \neq 1 \) then \( E(\theta(a)) \) is \( A \)-simple, \( F(\theta(a)) = 1 \) and \( C_A(E(\theta(a))) = \langle a \rangle \).

(b) For all \( a, b \in A^\# \),

\[
E(E(\theta(a)) \cap C(b)) \leq E(\theta(b)).
\]

(c) \( G = \langle E(\theta(a)) \mid a \in A^\# \rangle \).

**Lemma 16.2.** Let \( B \in \text{Hyp}(A) \). Then \( G = \langle E(M_b) \mid b \in B^\# \rangle \).

**Proof.** For each \( D \leq A \) set \( G_D = \langle E(M_d) \mid d \in D^\# \rangle \). Let \( D \in \text{Hyp}(A) \) and \( a \in A^\# \). We claim that \( E(M_a) \leq G_D \). Indeed, let \( K \in \text{comp}_A(M_a) \). If \( d \in D^\# \) and \( C_K(d) \) is \( A \)-quasisimple then \( C_K(d) \leq E(K \cap M_d) \) by Lemma (3.4) and the Balance Theorem yield

\[
K = \langle C_K(d) \mid d \in D^\# \rangle \text{ and } C_K(d) \text{ is } A\text{-quasisimple}
\]

\[
\leq \langle E(M_d) \mid d \in D^\# \rangle \leq G_D.
\]
The claim is established. In particular, $G_A = G_D$. Note that $\theta(D)$ normalizes $G_D$ and hence $G_A$. Using Lemma 7.3 and the fact that $G = \langle \theta(a) \mid a \in A^\#$ we have $G = \langle \theta(D) \mid D \in \text{Hyp}(A) \rangle$ and it follows that $G_A \leq G$. We have previously observed that $E(M_a) \neq 1$ for some $a$ by the Second Uniqueness Theorem. Hence $G_A = G$. Then $G = G_B$. □

Lemma 16.3. (a) Let $K, L \in \Omega$. Then $[K, L] \neq 1$ and $C^*_K(A) = C^*_L(A)$.

(b) Let $a \in A^\#$ with $E(M_a) \neq 1$. Then $E(M_a)$ is $A$-quasisimple.

Proof. (a). Theorem 14.1 implies that ‘does not commute’ is an equivalence relation on $\Omega$. Let $K_1, \ldots, K_n$ be representatives for the equivalence classes and let $[K_i]$ denote the subgroup generated by the class of $K_i$. Suppose $n \geq 2$. Then for all $i \neq j$ we have $[K_i] \leq \theta(C(K_j))$ so $[K_i]$ is a $\theta$-subgroup. Moreover $[K_i]$ and $[K_j]$ commute. Then $[K_i] \leq \langle \Omega \rangle$. Lemma 16.2 implies $\langle \Omega \rangle = G$ and then Lemma 8.1(d) supplies a contradiction. Hence there is only one equivalence class so $[K, L] \neq 1$. Theorem 14.1 implies $C^*_K(A) = C^*_L(A)$.

(b). Distinct $A$-components of $M_a$ commute. Then (a) implies $E(M_a)$ is $A$-quasisimple. □

Lemma 16.4. Let $a \in A^\#$. Then $E(M_a) = 1$ or $F(M_a) = 1$.

Proof. Assume false. Then $F(M_a) \neq 1$ and McBride’s Dichotomy implies that $\theta$ is nearsolvable, whence $\theta(A)$ is solvable. By Coprime Action(d) there exists $B \in \text{Hyp}(A)$ with $C_{F(M_a)}(B) \neq 1$. Set $Z = C_{F(M_a)}(B)$. Let $b \in B^\#$ and suppose $E(M_b) \neq 1$. Lemma 16.3 implies $C^*_E(M_a)(A) = C^*_E(M_b)(A)$ so as $[Z, E(M_b)] = 1$ we have $[Z, C^*_E(M_a)(A)] = 1$. Now $\theta$ is nearsolvable so $C^*_E(M_a)(A) = C^*_E(M_b)(A)$ and $Z \leq \theta(b) \leq M_b$. Lemma 3.6 implies $[Z, E(M_b)] = 1$. But then Lemma 16.2 implies $Z \leq G$, contrary to Lemma 8.1(d). □

Lemma 16.5. Let $a \in A^\#$ with $E(M_a) \neq 1$. Then

$$M_a = \theta(a).$$

Proof. Assume false. Recall that $\theta(a) = C_{M_a}(a)$. Lemma 16.4 implies $F(M_a) = 1$ so Coprime Action(g) implies $E(M_a, a) \neq 1$. Set $K = E(M_a)$, so $K \in \Omega$ by Lemma 16.3(b). Lemma 8.5(a) implies there exists $D \in \text{Hyp}(A)$ such that $[C_K(D), a] \neq 1$ and $C_K(D)$ is $A$-quasisimple for all $d \in D^\#$.

Let $L \in \Omega$. Lemma 16.3(a) and Theorem 14.1 imply $[C^*_K(A), C^*_L(A)] \neq 1$. Let $d \in D^\#$ and suppose $C_L(d)$ is $A$-quasisimple. Note that $[C_K(d), a] \neq 1$ and that $1 \neq [C^*_K(A), C^*_L(A)] \leq [C_K(d), C_L(d)]$. Lemma 14.2(b), with $\theta(d), C_K(d)$ and $C_L(d)$ in the roles of $N, K_0$ and $L_0$ respectively, forces $C_L(d) \leq C_K(d) \leq K$. Then Lemma 8.5(b) implies $L \leq K$. But then, by Lemma 16.2 $G = \langle \Omega \rangle \leq K$, a contradiction. □

Proof of the Structure Theorem. (a). The Balance Theorem implies $E(\theta(a)) \leq E(M_a)$ so $E(M_a) \neq 1$. Lemma 16.5 implies $M_a = \theta(a)$. Lemmas 16.4 and 16.3(b) imply that $F(\theta(a)) = 1$ and then $E(\theta(a))$ is $A$-simple. Let $B = C_A(E(\theta(a)))$. Coprime Action(g) implies $B = C_A(\theta(a))$, so as $\theta(a) = M_a$ we have $M_a \leq \theta(b) \leq M_b$ and then $M_a = M_b$ for all $b \in B^\#$. Recall that rank($A$) = 3. Then Lemma 11.2 implies $B$ is cyclic. Consequently $B = \langle a \rangle$ as required.

(b). Set $J = E(E(\theta(a)) \cap C(b)) \leq \theta(b)$ and $H = E(\theta(a)) \cap M_b$. We may assume that $J \neq 1$. Then $E(\theta(a)) \neq 1$. (a) implies $E(\theta(a))$ is $A$-simple, so $E(\theta(a)) \in \Omega$. □
Since \( J \leq H \), Lemma 3.4 implies \( H^{(\infty)} \) is \( A \)-quasisimple. The Balance Theorem implies \( H^{(\infty)} \leq E(M_b) \), whence \( M_b = \theta(b) \) by Lemma 16.5. Since \( J = J^{(\infty)} \leq H^{(\infty)} \) we have \( J \leq E(\theta(b)) \).

Finally, (c) follows from Lemma 16.2 and Lemma 16.5. This completes the proof of the Structure Theorem and hence of the Signalizer Functor Theorem. \( \square \)

References

1. M. Aschbacher, R. Lyons, S.D. Smith, R.M. Solomon, The Classification of the Finite Simple Groups: Groups of Characteristic 2-type, Mathematical Surveys and Monographs, 172, American Math. Soc., Providence Rhode Island 2011
2. H. Bender, On groups with abelian Sylow 2-subgroups, Math. Z., 117 (1970) 164–176
3. P. Flavell, A new proof of the Solvable Signalizer Functor Theorem, J. Algebra 398 (2014) 350–363
4. P. Flavell, An equivariant analogue of Glauberman’s ZJ-Theorem, J. Algebra 257(2) (2002) 249–264
5. P. Flavell, Automorphisms of soluble groups, Proceedings of the London Mathematical Society 2016 112 (4): 623–650 doi: 10.1112/plms/pdw005
6. P. Flavell, Automorphisms of K-groups I, Preprint: http://arxiv.org/abs/1609.01969
7. P. Flavell, Automorphisms of K-groups II, Preprint: http://arxiv.org/abs/1609.02380
8. P. Flavell, Primitive pairs of K-groups, Preprint: http://arxiv.org/abs/1609.03026
9. P. Flavell, A characterization of A-simple groups, Preprint: http://arxiv.org/abs/1609.03028
10. G. Glauberman, On solvable signalizer functors in finite groups, Proc. Lond. Math. Soc. (3) 33 (1976) 1–27.
11. D. Gorenstein, R. Lyons, R.M. Solomon, The classification of the finite simple groups, Mathematical Surveys and Monographs, 40, American Math. Soc., Providence Rhode Island 1994
12. D. Gorenstein, R. Lyons, R.M. Solomon, The classification of the finite simple groups. Number 2, Mathematical Surveys and Monographs, 40, American Math. Soc., Providence Rhode Island 1996
13. D. Gorenstein, Finite Simple Groups: An Introduction to their Classification. Plenum Press, New York 1982.
14. H. Kurzweil and B. Stellmacher, The theory of finite groups. An introduction. Universitext, Springer-Verlag, New York 2004.
15. P.P. McBride, Near solvable signalizer functors on finite groups, J. Algebra 78(1) (1982) 181–214
16. P.P. McBride, Nonsolvable signalizer functors on finite groups, J. Algebra 78(1) (1982) 215–238
17. M. Suzuki, Group Theory II. Springer-Verlag, Berlin 1986

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