A Framework for Decentralised Resolvent Splitting

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Abstract

Decentralised optimisation is typically concerned with problems having objective functions with finite-sum structure that are distributed over a network. Although there are several decentralised algorithms in the literature for solving minimisation problems with the aforementioned form, relatively few of these generalise to the abstraction of monotone inclusions. In this work, we address this by developing a new framework for decentralised resolvent splitting for finding a zero in the sum of finitely many set-valued monotone operators over regular networks. Our framework also simplifies and extends non-decentralised splitting algorithms in the literature.

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1 Introduction and Preliminaries

Throughout this work, \( \mathcal{H} \) denotes a real Hilbert space with inner-product \( \langle \cdot, \cdot \rangle \) and induced norm \( \| \cdot \| \). We are interested in solving the monotone inclusion given by

\[
\text{find } x \in \left( \sum_{i=1}^{n} F_i \right)^{-1}(0) \subseteq \mathcal{H},
\]

where \( F_1, \ldots, F_n : \mathcal{H} \rightharpoonup \mathcal{H} \) are (set-valued) maximally monotone operators \([3, 16]\). Recall that \( F_i \) is said to be maximally monotone if it is monotone in the sense

\[
\langle x - \bar{x}, y - \bar{y} \rangle \geq 0 \quad \forall y \in F_i(x), \forall \bar{y} \in F_i(\bar{x})
\]

and its graph contains no proper monotone extensions. The abstraction given by (1) covers many fundamental problems arising in mathematical optimisation involving objectives with finite-sum structure. Most notable are the following two types of examples:

**Example 1.1** (nonsmooth minimisation). Let \( f_1, \ldots, f_n : \mathcal{H} \to (-\infty, +\infty] \) be proper, lsc, convex functions. Consider the finite-sum nonsmooth minimisation problem given by

\[
\min_{x \in \mathcal{H}} \sum_{i=1}^{n} f_i(x).
\]

The first-order optimality condition for (2) is given by the inclusion (1) with \( F_i = \partial f_i \). Note that here “\( \partial \)” denotes the subdifferential in the sense of convex analysis.

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Example 1.2 (nonsmooth min-max). Let \( \phi_1, \ldots, \phi_n : \mathcal{H}_1 \times \mathcal{H}_2 \to [-\infty, +\infty] \) be proper, saddle-functions \cite{14}. That is, \( \phi_i \) is convex-concave and there exists a point \( (u, v) \in \mathcal{H}_1 \times \mathcal{H}_2 \) such that \( \phi_i(\cdot, v) < +\infty \) and \( \phi_i(u, \cdot) > -\infty \). Consider the finite-sum nonsmooth minmax problem given by

\[
\min_{u \in \mathcal{H}_1} \max_{v \in \mathcal{H}_2} \sum_{i=1}^n \phi_i(u, v)
\]

The first-order optimality condition for (3) is given by the inclusion (1) with \( \mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2 \), \( x = (u, v) \) and \( F_i := (\partial_u \phi_i, -\partial_v \phi_i) \). Here “\( \partial_u \)” (resp. \( \partial_v \)) denotes the partial subdifferential with respect to the variable \( u \) (resp. \( v \)).

Let us now return our attention back to the general problem (1). The particular focus of this work is on distributed algorithms \cite{5} for solving (1) under the following conditions:

(a) Each operator \( F_i \) in (1) is known by single node in a graph \( G = (V, E) \) with \(|V| = n\). More precisely, \( F_i \) is known by the \( i \)th node.

(b) Each node has its own local variables, and can compute standard arithmetic operations as well as the resolvent of its monotone operator \( F_i \) given by \( J_{F_i} := (\text{Id} + F_i)^{-1} \) (see \cite{3, Section 23}).

(c) Two nodes in \( G \) can communicate the results of their computations directly with one another only if they are connected by an edge in \( G \).

(d) The algorithm is “decentralised” in the sense that its does not necessarily require the computation of a global sum across all local variables of the nodes.

Stated at the abstraction of monotone inclusions, the classical way to solve (1) (without distributed considerations) involves using the Douglas–Rachford algorithm applied to a reformulation in the product-space \( \mathcal{H}^n \) (see, for instance, \cite{13, Example 2.7} or \cite{3, Proposition 26.12}). Given an initial point \( z^0 = (z^0_1, \ldots, z^0_n) \in \mathcal{H}^n \) and a constant \( \gamma \in (0, 2) \), this approach gives rise to the iteration given by

\[
\begin{cases}
  x^k = \frac{1}{n} \sum_{j=1}^n z^k_j \\
  z^k_{i+1} = z^k_i + \gamma (J_{F_i}(2x^k - z^k_i) - x^k) \quad \forall i \in \{1, \ldots, n\}.
\end{cases}
\]

In terms of distributed computing, this iteration requires the global sum across the local variables \( z^k_1, \ldots, z^k_n \) to compute the aggregate \( x^k \) and so it does not satisfy requirements (c) or (d) above. On the other hand, the \( z_i \)-update in (4) does satisfy properties (a) and (b), assuming it is computed by node \( i \). For recent variations on (4), the reader is referred to \cite{7, 11}.

In order to describe other algorithms in the literature, it is convenient to introduce some notation. Given a matrix \( W = (w_{ij}) \in \mathbb{R}^{p \times l} \), we denote \( W := W \otimes \text{Id} \) were \( \text{Id} \) denotes the identity operator on \( \mathcal{H} \) and \( \otimes \) denotes the Kronecker product. In other words, \( W \) can be identified with the bounded linear operator from \( \mathcal{H}^l \) to \( \mathcal{H}^p \) given by

\[
W = W \otimes \text{Id} = \begin{bmatrix}
w_{11} \text{Id} & w_{12} \text{Id} & \cdots & w_{1l} \text{Id} \\
w_{21} \text{Id} & w_{22} \text{Id} & \cdots & w_{2l} \text{Id} \\
\vdots & \vdots & \ddots & \vdots \\
w_{p1} \text{Id} & w_{p2} \text{Id} & \cdots & w_{pl} \text{Id}
\end{bmatrix}.
\]

Given maximally monotone operators \( F_1, \ldots, f_n : \mathcal{H}^n \rightrightarrows \mathcal{H}^n \), we denote the maximally monotone operator \( F : \mathcal{H}^n \rightrightarrows \mathcal{H}^n \) by \( F(\mathbf{x}) := (F_1(x_1), \ldots, F_n(x_n)) \) for all \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{H}^n \). Consequently, its resolvent \( J_F : \mathcal{H}^n \rightrightarrows \mathcal{H}^n \) is given by \( J_F = (J_{F_1}, \ldots, J_{F_n}) \). Recall also that, when \( F_i = \partial f_i \) where of \( f_i \) is proper, lsc and convex, then the resolvent coincides with proximity operator \cite{3, Section 24} given

\[
J_{F_i} = \text{prox}_{f_i} = \arg \min_{y \in \mathcal{H}} \left\{ f_i(y) + \frac{1}{2} \| y \|_2^2 \right\}.
\]
Now, in the special case of Example 1.1 where $J_{\partial f_i} = \text{prox}_{f_i}$, the proximal exact first-order algorithm (P-EXTRA) [18] can be used to solve (1). Given $x^0 \in \mathcal{H}^n$ and initialising with $y^0 = Wx^0$ and $x^1 = \text{prox}_{\alpha f}(y^0)$, this method iterates according to

$$
\begin{align*}
    y^k &= Wx^k + y^{k-1} - \tilde{W}x^{k-1} \\
    x^{k+1} &= \text{prox}_{\alpha f}(y^k).
\end{align*}
$$

(5)

where $W, \tilde{W} \in \mathbb{R}^{n \times n}$ are “mixing matrices” satisfying (c) that describe the communication between nodes. For specific examples, the reader is referred to [18, Assumption 1]. This method satisfies conditions (a)-(d) in the minimisation setting with $x, y, z \in \mathcal{H}^n$. For specific examples, the reader is referred to [18, Assumption 1]. This method satisfies conditions (a)-(d) with $x^i, y^i, z^i$ being the local variables of the $i$th node.

Let $L \in \mathbb{R}^{n \times n}$ denote the graph Laplacian of $G = (V, E)$ [9] where $|V| = n$. For the general monotone inclusion (1), the primal dual hybrid gradient (PDHG) algorithm [8, 10] can be used. After a change of variables (see [13, Section 5.1]), the PDHG can be expressed as

$$
\begin{align*}
    x^{k+1} &= J_{\gamma F}(x^k - \tau v^k) \\
    v^{k+1} &= v^k + \sigma L(2x^{k+1} - x^k)
\end{align*}
$$

(6)

where the stepsizes are required to satisfy $\tau \sigma \|L\| \leq 1$. This method satisfies conditions (a)-(d) with $x_i, v_i$ being the variables local to the $i$th node.

There seems to be relatively few fixed point algorithms for the general problem that are suitable for decentralised implementation, as outlined in conditions (a)-(d). In this work, we aim to partially rectify this by proposing a new framework that is naturally suited to decentralised resolvent splitting. Given $\gamma \in (0, 1)$ and $\tau := \frac{\gamma}{\|F\|}$, the most similar realisation of our approach to (6) (see Example 3.5) is given by

$$
\begin{align*}
    x^k_i &= J_{f_i}(v^k_i + \tau \sum_{j=1}^{i-1} A_{ij} x^k_j) \quad \forall i \in \{1, \ldots, n\} \\
    v^{k+1}_i &= v^k_i - \gamma \tau \sum_{j=1}^{n} L_{ij} x^k_j
\end{align*}
$$

(7)

where $A \in \mathbb{R}^{n \times n}$ denotes the adjacency matrix of the graph $G$. Note that this method satisfies conditions (a)-(d) with $x_i, v_i$ be the variable local to the $i$th node. By letting $N \in \mathbb{R}^{n \times n}$ denote the lower triangular part of $A$, the iteration (7) can be compactly expressed as

$$
\begin{align*}
    x^k_i &= J_{F_i}(v^k_i + \tau N x^k) \\
    v^{k+1} &= v^k - \gamma \tau L x^k.
\end{align*}
$$

(8)

Here we note that $x^k_i$ is well-defined due to the fact that $N$ is lower triangular. A feature of (8), which is not present in (5) or (6), is that its $x_i$-updates in (8) are performed sequentially within each iteration. Furthermore, the update (8) require no knowledge of past iterates.

The remainder of the work is structured as follows. In Section 2, we develop an abstract framework for resolving splitting algorithms as fixed point iterations, which extend ideas form [13], and analyse its convergence properties. We believe this framework is of interest in its own right. In Section 3, we provide various concrete realisation of the framework. In particular, this includes a previously unknown extension of Ryu’s method [15] to finitely many operators, the method proposed in [13], as well as the new scheme (8) when the communication graph $G$ is regular. A numerical example is reported in Section 4.

2 A Framework for Resolvent Splitting

In this section, we consider an operator $T: \mathcal{H}^m \to \mathcal{H}^m$ of the form

$$
T(z) := z + \gamma Mx, \quad y = Sx + Nx, \quad x = J_F(y),
$$

(9)
where $M \in \mathbb{R}^{m \times n}, S \in \mathbb{R}^{n \times m}$ and $N \in \mathbb{R}^{n \times n}$ are coefficient matrices. Moreover, we assume that the resolvents are evaluated in the order $J_{A_1}, \ldots, J_{A_n}$, which means that the matrix $N$ is necessarily lower triangular with zeros on the diagonal. Altogether, we require the following assumptions on the coefficient matrices.

**Assumption 2.1.** The coefficient matrices $M \in \mathbb{R}^{m \times n}, S \in \mathbb{R}^{n \times m}$ and $L \in \mathbb{R}^{n \times n}$ satisfy the following properties.

(a) ker $M \subseteq \mathbb{R}^n$ where $e = (1, \ldots, 1)^T \in \mathbb{R}^n$.

(b) $\sum_{i,j=1}^n N_{ij} = n$ and $N$ is lower triangular with zeros on the diagonal.

(c) $S^* = -M$.

(d) $M^* M + N + N^* - 2 \text{Id} \preceq 0$.

**Remark 2.2.** When Assumption 2.1 holds, properties of the Kronecker product imply that

$$\ker M = \Delta := \{x = (x_1, \ldots, x_n) \in \mathcal{H} : x_1 = \cdots = x_n\},$$

$S^* = -M$ and $M^* M + N + N^* - 2 \text{Id} \preceq 0$.

**Lemma 2.3** (Fixed points and zeros). Suppose Assumptions 2.1(a)-(c) holds, and denote

$$\Omega := \{(z, x) \in \mathcal{H}^m \times \mathcal{H} : x = J_F(Sz + Nx) \text{ where } x = (x, \ldots, x) \in \Delta\}.$$  

Then the following assertions hold.

(a) If $z \in \text{Fix } T$, then there exists $x \in \mathcal{H}$ such that $(z, x) \in \Omega$.

(b) If $x \in \text{zer} (\sum_{i=1}^n F_i)$, then there exists $z \in \mathcal{H}^m$ such that $(z, x) \in \Omega$.

(c) If $(z, x) \in \Omega$, then $z \in \text{Fix } T$ and $x \in \text{zer}(\sum_{i=1}^n F_i)$.

Consequently,

$$\text{Fix } T \neq \emptyset \iff \Omega \neq \emptyset \iff \text{zer} \left( \sum_{i=1}^n F_i \right) \neq \emptyset.$$  

**Proof.** (a): Let $z \in \text{Fix } T$. Set $y := Sz + Nx$ and $x := J_F(y)$. Since $z \in \text{Fix } T$, we have $x \in \ker M$ and so Assumption 2.1(a) implies $x \in \Delta$.

(b): Let $x \in \text{zer}(\sum_{i=1}^n F_i)$ and set $x := (x, \ldots, x) \in \Delta$. Then there exists $v = (v_1, \ldots, v_n) \in F(x)$ such that $\sum_{i=1}^n v_i = 0$. Define $y = (y_1, \ldots, y_n) \in \mathcal{H}^n$ according to $y := v + x$, so that $x = J_F(y)$. To complete the proof, we must show there exists $z \in \mathcal{H}^m$ such that $y = Sz + Nx$, which is equivalent to the $y - Nx \in \text{range } S$. To this end, first note that Assumption 2.1(a) & (c) imply

$$\text{range } S = (\ker S^*)^\perp = (\ker M)^\perp \supseteq \Delta^\perp = \{(x_1, \ldots, x_n) \in \mathcal{H}^n : \sum_{i=1}^n x_i = 0\}.$$  

With the help of Assumption 2.1(b), we see that

$$\sum_{i=1}^n (y - Nx)_i = \sum_{i=1}^n y_i - \sum_{i,j=1}^n N_{ij} x = \left(\sum_{i=1}^n v_i + nx\right) - nx = 0,$$

Hence $y - Nx \in \text{range } S$, as required.

(c): Let $(z, x) \in \Omega$ and set $y := Sz + Nx$ where $x = (x, \ldots, x) \in \Delta$. Since $x = J_F(y) \in \ker M$, we have $z \in \text{Fix } T$ by Assumption 2.1(a). Denote $e = (1, \ldots, 1)^T \in \mathbb{R}^n$. Then $Me = 0$ by Assumption 2.1(a). Since $F(x) \ni y - x = -M^* z + Nx - x$, Assumption 2.1(b) gives

$$\sum_{i=1}^n F_i(x_1) \ni -((e^T M^* \otimes \text{Id}) z) + \sum_{i,j=1}^n N_{ij} x - nx = -((Me)^T \otimes \text{Id}) z + nx - nx = 0,$$

This shows that $x \in \text{zer}(\sum_{i=1}^n F_i)$ and so completes the proof. 

\[\blacksquare\]
Lemma 2.4 (Nonexpansivity). Let $F_1, \ldots, F_n: \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone. Suppose Assumptions 2.1(c) & (d) hold, and let $\gamma > 0$. Then, for all $z, \tilde{z} \in \mathcal{H}^n$, we have

$$
\|T(z) - T(\tilde{z})\|^2 + \frac{1-\gamma}{\gamma}(\|\text{Id} - T\|) \leq \|z - \tilde{z}\|^2.
$$

where $x = J_F(Sz + Nx)$ and $\tilde{x} = J_F(S\tilde{z} + N\tilde{x})$. In particular, $T$ is $\gamma$-averaged nonexpansive whenever $\gamma \in (0, 1)$.

Proof. Let $z, \tilde{z} \in \mathcal{H}^n$, set $x := J_F(y)$ where $y := Sz + Nx$ and $\tilde{x} := J_F(\tilde{y})$ where $\tilde{y} := S\tilde{z} + N\tilde{x}$. Since $y - x \in \lambda F(x)$ and $\tilde{y} - \tilde{x} \in \lambda F(\tilde{x})$, monotonicity of $F = (F_1, \ldots, F_n)$ gives

$$
0 \leq (x - \tilde{x}, (y - x) - (\tilde{y} - \tilde{x})) = (x - \tilde{x}, (Sz + Nx - x) - (S\tilde{z} + N\tilde{x} - \tilde{x}))
$$

By Assumption 2.1(c), $S^* = -M$ and so the first term in (11) can be expressed as

$$
\langle S^*x - S^*\tilde{x}, z - \tilde{z} \rangle = \langle (-Mx) - (-M\tilde{x}), z - \tilde{z} \rangle = \frac{1}{\gamma}((\text{Id} - T)(z) - (\text{Id} - T)(\tilde{z}), z - \tilde{z})
$$

The second term in (11) can be expressed as

$$
\langle x - \tilde{x}, (N - \text{Id})x - (N - \text{Id})\tilde{x} \rangle = \frac{1}{2}\langle x - \tilde{x}, [M^*M + 2N - 2\text{Id}](x - \tilde{x}) \rangle = \frac{1}{2\gamma}(\|x - \tilde{x}\|^2 - \|\text{Id} - T\|)^2.
$$

Substituting (12) and (13) into (11), followed by multiplying by $2\gamma$, gives (10). In particular, if Assumption 2.1(d) holds, then the inner-product on the LHS of (10) is non-negative and hence $T$ is $\gamma$-averaged nonexpansive whenever $\gamma \in (0, 1)$.

Remark 2.5. As we will explain in Section 3, the setting considered here includes Ryu’s method [15] and the resolvent splitting with minimal lifting from [13] as special cases. Consequently, Lemma 2.4 provides a very concise proof covering [15, Section 4.2] and [13, Lemma 4.3] (each of which took several pages of algebraic manipulation).

The following theorem is our main result regarding convergence of the fixed point iteration corresponding to $T$ in (9).

Theorem 2.6. Let $F_1, \ldots, F_n: \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone operators with $\text{zer}(\sum_{i=1}^n F_i) \neq \emptyset$. Suppose Assumption 2.1 holds and let $\gamma \in (0, 1)$. Let $z^0 \in \mathcal{H}$ and define the sequences $(z^k)$ and $(x^k)$ according to

$$
z^{k+1} = T(z^k) = z^k + \gamma Mz^k, \quad x^k = J_F(Sz^k + Nx^k).
$$

Then the following assertions hold.

(a) As $k \rightarrow \infty$, we have $z^k - z^{k+1} \rightarrow 0$ and $\sum_{i=1}^n s_i x_i^k \rightarrow 0$ for all $s \in \mathbb{R}^n$ with $\sum_{i=1}^n s_i = 0$.

(b) The sequence $(z^k)$ converges weakly to a point $\tilde{z} \in \text{Fix}T$. 
(c) The sequence \((x^k)\) converges weakly to a point \((\bar{x}, \ldots, \bar{x}) \in \mathcal{H}^n\) such that \(\bar{x} \in \text{zer}(\sum_{i=1}^n F_i)\).

**Proof.** (a) & (b): Since \(\text{zer}(\sum_{i=1}^n F_i) \neq \emptyset\), Lemma 2.3 implies \(\text{Fix} T \neq \emptyset\). By Lemma 2.4, the operator \(T\) is averaged nonexpansive and so it follows that \(z^k - z^{k+1} \to 0\) and \((z^k)\) converges weakly to a point \(\bar{z} \in \text{Fix} T\). Next, let \(s \in \{s \in \mathbb{R}^n : \sum_{i=1}^n s_i = 0\} = \Delta_\bot \subseteq (\ker M)_\bot = \text{range } M^* = \text{range } S\). Then, there exists \(v \in \mathbb{R}^m\) such that \(s = Sv = -M^*v\) and hence

\[-\sum_{i=1}^n s_i x_i^k = -(s^\top \otimes \text{Id}) x^k = ((v^\top M) \otimes \text{Id}) x^k = \frac{1}{\gamma} (v^\top \otimes \text{Id}) (z^{k+1} - z^k) \to 0.\]

(c): Denote \(y^k = Sz^k + Nz^k\). We claim that the sequence \((x^k)\) is bounded. To see this, let \(S_1 \in \mathbb{R}^{1 \times n}\) denote the first row of the matrix \(S\) and let \(N_1 = (0, \ldots, 0) \in \mathbb{R}^{1 \times n}\) denote the first row of the matrix \(N\) (which is zero by Assumption 2.1(b)). Thus, by definition, we have

\[x_1^k = J_{F_1}(y_1^k) = J_{F_1}(S_1 z^k + N_1 x^k) = J_{F_1}(S_1 z^k).\]

By Lemma 2.3, \((\bar{z}, \bar{x}) \in \Omega\) where \(\bar{x} = J_{F_1}(S_1 \bar{z})\). Thus, by nonexpansivity of resolvents [3, Proposition 23.8] and boundedness of \((z^k)\), it then follows that \((x_1^k)\) is also bounded. By (a), it follows that \((x^k)\) is bounded, as claimed.

Next, using the identity \(x^k = J_F(y^k)\), followed by the identity \(y^k = Sz^k + Nz^k\) and the fact that range \(S \subseteq \Delta_\bot\), we deduce that

\[S \begin{pmatrix} y_1^k - x_1^k \\ \vdots \\ y_{n-1}^k - x_{n-1}^k \\ x_n^k \\ \sum_{i=1}^n y_i^k - \sum_{i=1}^n x_i^k \end{pmatrix} \ni \begin{pmatrix} x_1^k - x_n^k \\ \vdots \\ x_{n-1}^k - x_{n-1}^k \\ \sum_{i=1}^n x_i^k \\ \sum_{i=1}^n (\sum_{j=1}^n N_{ij} - 1) x_i^k \end{pmatrix} = \begin{pmatrix} x_1^k - x_n^k \\ \vdots \\ x_{n-1}^k - x_{n-1}^k \\ \sum_{i=1}^n (\sum_{j=1}^n N_{ij} - 1) x_i^k \end{pmatrix} \quad (15)\]

where \(S : \mathcal{H}^n \Rightarrow \mathcal{H}^n\) denotes operator given by

\[S := \begin{pmatrix} (F_1)^{-1} \\ \vdots \\ (F_{n-1})^{-1} \end{pmatrix} + \begin{bmatrix} 0 & \ldots & 0 & -\text{Id} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & -\text{Id} \\ \text{Id} & \ldots & \text{Id} & 0 \end{bmatrix}.\]

Note that \(S\) is maximally monotone operator as the sum of the two maximally monotone operators, the latter having full domain [3, Corollary 24.4(i)]. Consequently, its graph is sequentially closed in the weak-strong topology [3, Proposition 20.32]. Note also that the RHS of (15) converges strongly to zero as a consequence of (a) and Assumption 2.1(b).

Let \(x \in \mathcal{H}^n\) be an arbitrary weak cluster point of the sequence \((x^k)\). As a consequence of (a), it follows that \(x = (x, \ldots, x)\) for some \(x \in \mathcal{H}\). Denote \(y = Sz + Nx\). Taking the limit in (15) along a subsequence of \((x^k)\) which converges weakly to \(x\) gives

\[S \begin{pmatrix} y_1 - x \\ \vdots \\ y_{n-1} - x \\ x \end{pmatrix} \ni \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} F_1(x) \ni y_i - x & \forall i \in \{1, \ldots , n-1\} \\ F_n(x) \ni (n-1)x - \sum_{i=1}^{n-1} y_i \end{cases}.\]

Altogether, this shows that \(x = J_{F_1}(y_1) = J_{F_1}(S_1 \bar{z}) = \bar{x}\). In other words, \(x = (\bar{x}, \ldots, \bar{x})\) is the unique weak sequential cluster point of the bounded sequence \((x^k)\). We therefore deduce that \((x^k)\) converges weakly to \(x\), which completes the proof. \(\square\)

**Remark 2.7.** A closer look at the iteration (14) shows that, although convergence is analysed with respect to the sequence \((z^k)\), this sequence does not need to be directly computed. Indeed, using the fact that \(S = -M^*\) and making the change of variables \(v^k = Sz^k\), we may write (14) as

\[v^{k+1} = v^k - \gamma M^* M x^k, \quad x^k = J_F(v^k + N x^k).\]

In the case where \(m > n\), this transformation also has the advantage of replacing a variable \(z^k \in \mathcal{H}^m\) in the larger space with a variable \(z^k \in \mathcal{H}^n\) in the smaller space.
3 Realisations of the Framework

In what follows, we provide several realisations of Theorem 2.6 for solving the monotone inclusion (1). Examples 3.1, 3.2 and 3.3 are already known in the literature. However, Examples 3.4 and 3.5 are new.

Example 3.1 (Douglas–Rachford splitting). Douglas–Rachford splitting [12, 19, 4, 6] for (1) with $n = 2$ is defined by the operator $T: \mathcal{H} \to \mathcal{H}$ given by

$$T(z) = z + \gamma (x_2 - x_1) \quad \text{where} \quad \begin{cases} x_1 = J_{F_1}(z) \\ x_2 = J_{F_2}(2x_1 - z) \end{cases}.$$ 

Thus, the corresponding coefficient matrices $M \in \mathbb{R}^{1 \times 2}, S \in \mathbb{R}^{2 \times 1}$ and $N \in \mathbb{R}^{2 \times 2}$ are given by

$$M = -S^* = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$ 

Here, we have

$$M^* M + N + N^* - 2 \text{Id} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \succeq 0.$$

Example 3.2 (Ryu splitting for $n = 3$). Ryu’s splitting algorithm [15, 1] for (1) with $n = 3$ is defined by the operator $T: \mathcal{H}^2 \to \mathcal{H}^2$ given by

$$T(z) = z + \gamma \begin{pmatrix} x_3 - x_1 \\ x_3 - x_2 \end{pmatrix} \quad \text{where} \quad \begin{cases} x_1 = J_{F_1}(z_1) \\ x_2 = J_{F_2}(z_2 + x_1) \\ x_3 = J_{F_3}(x_1 - z_1 + x_2 - z_2) \end{cases}.$$ 

Thus, the corresponding coefficient matrices $M \in \mathbb{R}^{2 \times 3}, S \in \mathbb{R}^{3 \times 2}$ and $N \in \mathbb{R}^{3 \times 3}$ are given by

$$M = -S^* = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

(16)

It is straightforward to verify that Assumption 2.1(a)-(c) holds. To verify Assumption 2.1(d), observe that

$$M^* M + N + N^* - 2 \text{Id} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq 0.$$

Example 3.3 (Resolvent splitting with minimal lifting). The resolvent splitting algorithm considered in [13, 2] for (1) with $n \geq 2$ is defined by the operator $T: \mathcal{H}^{n-1} \to \mathcal{H}^{n-1}$ given by

$$T(z) = z + \gamma \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{pmatrix} \quad \text{where} \quad \begin{cases} x_1 = J_{F_1}(z_1) \\ x_i = J_{F_i}(z_i + x_{i-1} - z_{i-1}) \quad \forall i \in \{2, \ldots, n-1\} \\ x_n = J_{F_n}(x_1 + x_{n-1} - z_{n-1}) \end{cases}.$$ 

Thus, the coefficient matrices $M \in \mathbb{R}^{(n-1) \times n}, S \in \mathbb{R}^{n \times (n-1)}$ and $N \in \mathbb{R}^{n \times n}$ are given by

$$M = -S^* = \begin{bmatrix} -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$
It is straightforward to verify that Assumption 2.1(a)-(c) holds. To verify Assumption 2.1(d), observe that

\[ M^*M + N + N^* - 2 \text{Id} = \begin{bmatrix} -1 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & -1 \end{bmatrix} \preceq 0. \quad (17) \]

**Example 3.4 (An extension of Ryu splitting for \( n \geq 3 \) operators).** As explained in [13, Remark 4.7], the naïve extension of Ryu splitting to \( n > 3 \) operators fails to converge. However, the more general perspective offered by our framework suggests an alternative.

Indeed, for (1) with \( n \geq 2 \), consider the operator \( T : \mathcal{H}^{n-1} \rightarrow \mathcal{H}^{n-1} \) given by

\[ T(z) = z + \gamma \sqrt{\frac{2}{n-1}} \begin{pmatrix} x_n - x_1 \\ x_n - x_2 \\ \vdots \\ x_n - x_{n-1} \end{pmatrix} \]

where \( x_1, \ldots, x_n \in \mathcal{H} \) are given by

\[
\begin{aligned}
x_i &= J_{Fi}(\sqrt{\frac{2}{n-1}} z_i + \frac{2}{n-1} \sum_{j=1}^{i-1} x_i) & \forall i \in \{1, \ldots, n-1\} \\
x_n &= J_{Fn}(\frac{2}{n-1} \sum_{j=1}^{n-1} x_i - \sqrt{\frac{2}{n-1} \sum_{i=1}^{n-1} z_i}).
\end{aligned}
\]

This has coefficient matrices \( M \in \mathbb{R}^{(n-1)\times n}, S \in \mathbb{R}^{n\times(n-1)} \) and \( L \in \mathbb{R}^{n\times n} \) are given by

\[
M = -S^* = \sqrt{\frac{2}{n-1}} \begin{bmatrix} -1 & 0 & \ldots & 0 & 1 \\ 0 & -1 & \ldots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & 1 \end{bmatrix}, \quad N = \frac{2}{n-1} \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 0 \end{bmatrix}.
\]

Note that, when \( n = 3 \), these matrices are precisely those given in (16) of Example 3.2. Consequently, the setting considered in this example provides a one parameter family that coincides with Ryu splitting in the special case when \( n = 3 \).

It is straightforward to verify that Assumption 2.1(a)-(c) holds. To verify Assumption 2.1(d), observe that

\[ M^*M + N + N^* - 2 \text{Id} = \frac{2}{n-1} \begin{bmatrix} 2-n & 1 & \ldots & 1 & 0 \\ 1 & 2-n & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 2-n & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \]

This matrix is diagonally dominate with non-positive diagonal entries, hence it is negative definite.

Before our next example, we first recall some notational from graph theory (see, for instance, [9]). Consider an (undirected) graph \( G = (V, E) \) with vertex set \( V = \{v_1, \ldots, v_n\} \) and edge set \( E = \{e_1, \ldots, e_m\} \). The degree matrix of \( G \) is the diagonal matrix \( D \in \mathbb{R}^{n\times n} \) such that \( D_{ii} \) is equal to the degree of \( v_i \). The adjacency matrix of \( G \) is the matrix \( A \in \{0, 1\}^{n\times n} \) such that \( A_{ij} = 1 \) if only if \( v_i \) and \( v_j \) are adjacent. The graph Laplacian is the matrix \( L \in \mathbb{R}^{V\times|V|} \) given by \( L = D - A \). An orientation of an undirected graph is a directed graph obtained by assigning a direction to each edge in the original graph. The incidence matrix of a directed graph \( G = (V, E) \) is the matrix
$B \in \{0, \pm 1\}^{|V| \times |E|}$ given by

$$B_{ij} = \begin{cases} -1 & \text{if } e_j \text{ leaves vertex } v_i, \\ +1 & \text{if } e_j \text{ enters vertex } v_i, \\ 0 & \text{otherwise.} \end{cases}$$  \hspace{1cm} (18)$$

Let $B_\sigma$ denote an oriented incidence matrix of an undirected graph $G$, that is, an incidence matrix for any orientation of $G$. Then its graph Laplacian can also satisfies $L = B_\sigma B_\sigma^T$.

**Example 3.5 (d-regular networks).** Consider a simple, connected $d$-regular graph $G = (V, E)$ with $|V| = n$ vertices and $|E| = \frac{nd}{2}$ edges. Let $B_\sigma$ be an oriented incidence matrix for $G$. Consider the coefficient matrices given by

$$-S^* = M = \sqrt{\frac{2}{d}} B_\sigma^T \in \mathbb{R}^{|E| \times |V|}$$

and $N \in \mathbb{R}^{|V| \times |V|}$ being the lower triangular matrix (with zero diagonal) satisfying $N + N^* = \frac{2d}{d} A$. We claim that these matrices satisfy Assumptions 2.1. Indeed, we have:

(a) By (18), we have $\ker M = \ker B_\sigma^T = \mathbb{R} e$ where $e = (1, \ldots, 1)^T \in \mathbb{R}^{|E|}$.

(b) Since $\sum_{i,j=1}^n A_{ij} = 2|E|$, it follows that $\sum_{i,j=1}^n N_{ij} = \frac{1}{d} \sum_{i,j=1}^n A_{ij} = \frac{2|E|}{d} = n$.

(c) $S^* = -M$ holds by definition.

(d) Since $G$ is $d$-regular, $L = B_\sigma B_\sigma^T = d \text{Id} - A$. Hence

$$M^* M + N + N^* - 2 \text{Id} = \frac{2}{d} (d \text{Id} - A) + \frac{2}{d} A - 2 \text{Id} = 0 \preceq 0.$$

Altogether, we have that the operator $T : \mathcal{H}^{|E|} \to \mathcal{H}^{|E|}$ given by

$$T(z) = z + \gamma M x, \quad x = J_F (S z + N x)$$  \hspace{1cm} (19)$$

satisfies the conditions of Section 2. It is interesting to interpret the variables in the context of the above example. Note that $z \in \mathcal{H}^{|E|}$ and $x \in \mathcal{H}^{|V|}$, so that $z$ represents edges of $G$ and $x$ represents vertices of $G$. The matrix $M$ describes information flow from vertices to their incident edges. Similarly, the matrix $S$ describes information flow from edges to their adjacent vertices. Finally, $N$ describes a direct information flow between adjacent vertices.

Since $|E|$ is potentially large, it is often better to avoid using the operator $T$ directly. Using the observation outlined in Remark 2.7 and the fact that $SM = -M^* M = -L$, (19) can be rewritten in terms of the operator $\tilde{T} : \mathcal{H}^{|V|} \to \mathcal{H}^{|V|}$ given by

$$\tilde{T}(v) = v - \gamma \frac{2}{d} L x, \quad x = J_F (v + N x),$$  \hspace{1cm} (20)$$

This is suited for a distributed implementation with node $i$ responsible for computing $v_i$ and $x_i$. The corresponding iteration for (20) is given explicitly in (7) (or (8)).

### 4 Computational Example

In this section, we demonstrate the algorithm presented in Example 3.5 numerically (more precisely iteration (8) or (20)) and compare it to the P-EXTRA and the PDHG algorithm on a toy problem.

Given a vector $c \in \mathbb{R}^n$, we consider the $\ell_1$-consensus problem given by

$$\min_{x \in \mathbb{R}} \sum_{i=1}^n |x - c_i|.  \hspace{1cm} (21)$$
To examine the role of connectivity, we consider solving this problem over randomly generated $d$-connected graphs with $n = 10$ nodes and $d \in \{4, 8, 10\}$. In this context, node $i$ is responsible for computing the resolvent of the operator $\partial \cdot - c_i$.

In our implementation of P-EXTRA, the mixing matrices are taken as $W = I - \frac{L}{d+1}$ and $\tilde{W} = \frac{4d+W}{2}$ as per the suggestion in [17, Section 2.4(ii)]. For PDHG, we take the stepsizes equal to $(\tau, \sigma) = \left(\frac{1}{10\|L\|}, \frac{10}{\sqrt{\|L\|}}\right)$ as this was observed to perform best for this problem in [13, Section 5.1].

All computations are performed in Python and random graphs were generated using the networkx package. All algorithms were initialised with the zero vector for the starting points.

To compare the different algorithms, each was run for 10,000 iterations at which point it was considered to have “converged”. This final point was used in the y-axis in Figure 1 which shows the distance to the (approximate) limit point as a function of iterations. Figure 1 suggests favourable performance of the proposed algorithm for lower values of $d$.

![Figure 1: Distance to solution as a function of iterations for the three methods for (21).](image)

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