SYMMETRY AND NON-EXISTENCE OF SOLUTIONS TO AN INTEGRAL SYSTEM

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Abstract. In this paper, we consider the nonnegative solutions of the following system of integral form:

$$\begin{cases}
    u_i(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f_i(u(y)) \, dy, & x \in \mathbb{R}^n, \ i = 1, \ldots, m, \\
    0 < \alpha < n, \text{ and } u(x) = (u_1(x), u_2(x), \ldots, u_m(x)).
\end{cases}$$

(1)

Here $f_i(u) \in C^1(\mathbb{R}_+^m) \cap C^0(\mathbb{R}_+^m)$ are real-valued functions, nonnegative, homogeneous of degree $\beta_i$, where $0 < \beta_i \leq \frac{n+\alpha}{n-\alpha}$, and monotone nondecreasing with respect to the variables $u_1, u_2, \ldots, u_m$. We show that the nonnegative solution $u = (u_1, u_2, \ldots, u_m)$ is radially symmetric in the critical and subcritical case by method of moving planes in an integral form and $u$ must be zero in the subcritical case.

Furthermore, we consider the form of $f_i(u) = \sum_{r=1}^k f_{ir}(u)$, where $f_{ir}(u)$ are real-valued homogeneous functions of various degrees $\beta_{ir}$, $r = 1, 2, \ldots, k$ and $0 < \beta_{ir} \leq \frac{n+\alpha}{n-\alpha}$. We also show that the radial symmetry property of the nonnegative solution. Due to the homogeneous of degree can be different, the more intricate method is needed to deal with this difficulty.

1. Introduction. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $\alpha$ be a real number satisfying $0 < \alpha < n$. In [5, 7], W.Chen, C.Li and B.Ou considered the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^p \, dy.$$  

(2)

They showed that the nonnegative solution of the above integral equation is symmetric in the so-called critical case $p = \alpha^* = \frac{n+\alpha}{n-\alpha}$, and the non-trivial solution doesn’t exist in the so-called subcritical case $1 < p < \alpha^*$.

Latter, the system for integral form have been studied by many authors. In particular, in [6], W.Chen, C.Li, and B.Ou considered the integral system

$$\begin{cases}
    u(x) = \int_{\mathbb{R}^n} \frac{v^q(y)}{|x-y|^\lambda} \, dy \\
    v(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^\lambda} \, dy
\end{cases}$$

(3)

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with $\frac{1}{q+1} + \frac{1}{p+1} = \frac{1}{n}$. This system can be regarded as the Euler-Lagrange equations of the well-known Hardy-Littlewood-Sobolev inequalities.

To establish the symmetry of the solutions to (2) and (3), W. Chen, C. Li, and B. Ou [5, 6] introduced the method of moving planes in an integral form which is different from those for differential equations. Further, Chen, Li, and Ou [7], Jin and Li [11] and many others have used this method to treat the symmetry of the solutions for a large class of general integral system. It is well known that the method of moving planes for differential equations was invented by the Soviet mathematician Alexanderoff in the early 1950s. Decades later, it was further developed by Serrin [12], Gidas, Ni, and Nirenberg [1]. It turns out that this method is powerful. It makes us to handle the integral equation in its global form, though it is lack of knowledge of the local properties. It has been applied to free boundary problems, semilinear differential equations, and other problems. In particular, the method of moving planes in an integral form is also powerful. It makes us to handle the integral equation in its global form, though it is lack of knowledge of the local properties.

In this paper, we consider the nonnegative solutions of the following system of integral form:

$$
\begin{align*}
&\begin{cases}
  u_i(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f_i(u(y)) \, dy, \quad x \in \mathbb{R}^n, \quad i = 1, \cdots, m, \\
  0 < \alpha < n, \quad \text{and} \quad u(x) = (u_1(x), u_2(x), \cdots, u_m(x)).
\end{cases}
\end{align*}
$$

Here $f_i(u)$, $i = 1, \cdots, m$ satisfy the conditions (C), i.e. $f_i(u) \in C^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ are real-valued functions, non-negative, homogeneous of degree $\beta_i$, where $0 < \beta_i \leq \frac{n+\alpha}{n-2}$, and monotone nondecreasing with respect to the variables $u_1, u_2, \cdots, u_m$. Here $\mathbb{R}^n_{++}$ consists of the points whose components are all positive real numbers. According to the definition of critical and subcritical case, we consider the properties of the solutions in both cases.

For $f_i$ are positive linear combinations of terms like $\prod_{i=1}^m u_i^{\gamma_i}$ with $\gamma_i \geq 0$ and $\sum_{i=1}^m \gamma_i = \frac{n+2}{n-2}$.

C. Li and L. Ma in [14] studied the following system

$$
\begin{align*}
&\begin{cases}
  u(x) = \int_{\mathbb{R}^n} \frac{u^{\alpha} v^\beta}{|x-y|^{n-2}} \, dy \\
  v(x) = \int_{\mathbb{R}^n} \frac{v^{\alpha} u^\beta}{|x-y|^{n-2}} \, dy
\end{cases}
\end{align*}
$$

with $\alpha + \beta = \frac{n+2}{n-2}$, which is the critical exponents. In the special case, when $n = 3$ and $\alpha = 2$, $\beta = 3$, the system is closely related to the ones from the stationary Schrödinger system with critical exponents for the Bose-Einstein condensate. In [14], they proved the radial symmetry of the positive solution to the elliptic system (5) and then obtained the uniqueness of the solution.

In general case, a system of integral form like (4) was studied in a paper [4] by W. Chen and C. Li. They considered that $f_1(u), f_2(u), \cdots, f_m(u)$ are homogeneous of the same degree $\gamma$, here $\gamma = \frac{n+\alpha}{n-\alpha}$ is critical. They obtained that the solution is radially symmetric with the same center, also gave the form of the positive solution.

If $f_1(u), f_2(u), \cdots, f_m(u)$ are homogeneous of the same subcritical degree $\gamma$ with $1 < \gamma < \frac{n+\alpha}{n-\alpha}$, R. Zhuo, W. Chen, X. Cui and Z. Yuan [15] proved the nonnegative
solution of the system of integral form is also radially symmetric with the same center, but must be zero.

According to the results in [4] and [15], a natural question is that whether we have the same results if the homogeneous degrees for $f_1(u), f_2(u), \ldots, f_m(u)$ are different?

In this paper, we answer this question. We consider $f_i(u)$ is homogeneous of degree $\beta_i$, where $0 < \beta_i \leq \frac{n+\alpha}{n-\alpha}$ and $i = 1, 2, \ldots, m$. We show that the symmetry property of the solutions in both critical and subcritical cases, and also obtain that the solution must be zero in subcritical case.

When we study the homogeneous of degree $\beta_i$, where $0 < \beta_i \leq \frac{n+\alpha}{n-\alpha}$, it is easy to see that the homogeneous of degree $\beta_i$ can be different. And we consider the more general case such as the form of $f_i(u) = \sum_{r=1}^k f_{ir}(u)$, where $f_{ir}(u)$ are real-valued homogeneous functions of degree $\beta_{ir}$, and $0 < \beta_{ir} \leq \frac{n+\alpha}{n-\alpha}$. Here the homogeneous of degree $\beta_{ir}$ can also be different. Using the same technique, we also show that the radial symmetry property of the nonnegative solutions. Our conditions here are more general than the above papers, but can also establish similar results for system (4).

For the above two cases, we give several examples, the following system

$$\begin{align*}
\begin{cases}
  u_1(x) = \int_{\mathbb{R}^n} \frac{u_2^2 u_3^{p-\frac{1}{2}}}{|x-y|^{n-\alpha}}, \\
  u_2(x) = \int_{\mathbb{R}^n} \frac{u_1^2 u_3^{p-\frac{1}{2}}}{|x-y|^{n-\alpha}}, \\
  u_3(x) = \int_{\mathbb{R}^n} \frac{u_1^2 u_2^{p-\frac{1}{2}}}{|x-y|^{n-\alpha}}, \\
  x \in \mathbb{R}^n, 0 < \alpha < n.
\end{cases}
\end{align*}$$

(6)

with $p = \frac{n+\alpha}{n-\alpha}$. Set $f_1(u) = u_2^2 u_3^{p-\frac{1}{2}}, f_2(u) = u_1^2 u_3^{p-\frac{1}{2}}$ and $f_3(u) = u_1^2 u_2^{p-\frac{1}{2}}$. We can see that the homogeneous degree for $f_1(u)$ is $p$, but the homogeneous degrees for $f_2(u)$ and $f_3(u)$ are less than $p$, it means that the homogeneous degrees for $f_1(u)$, $f_2(u)$ and $f_3(u)$ are different.

When we focus on the form of $f_i(u) = \sum_{r=1}^k f_{ir}(u)$, for example, the following system

$$\begin{align*}
\begin{cases}
  u(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} (u_1^2 v_1^{p-\frac{1}{2}} + u_2^2 v_2^{p-\frac{1}{2}}), \\
  v(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} (u_1^2 v_1^{p-\frac{1}{2}} + u_2^2 v_2^{p-\frac{1}{2}}),
\end{cases}
\end{align*}$$

(7)

with $p = \frac{n+\alpha}{n-\alpha}$. Set $f_1(u) = f_{11}(u) + f_{12}(u)$, where $f_{11}(u) = u_1^2 v_1^{p-\frac{1}{2}}, f_{12}(u) = u_1^2 v_2^{p-\frac{1}{2}}$, and $f_2(u) = f_{21}(u) + f_{22}(u)$, where $f_{21}(u) = u_2^2 v_1^{p-\frac{1}{2}}, f_{22}(u) = u_2^2 v_2^{p-\frac{1}{2}}$. We can see that the the homogeneous degree for $f_{11}(u)$ is $p$, but the homogeneous degrees for $f_{12}(u)$, $f_{21}(u)$ and $f_{22}(u)$ are less than $p$, it means that the homogeneous degrees for $f_{11}(u), f_{12}(u), f_{21}(u)$ and $f_{22}(u)$ are different.

From the above examples, we also can see that the homogeneous degree can be less than $1$. It is difficult to deal since in the proof of the theorems, we need to linearize the system by using the mean value theorem, then the power of functions would be negative. To overcome this difficulty, we adapt the method in the paper
the method of moving planes in an integral form. Theorem 1.2 where $f_2$. Proof of Theorem 1. of positive linear combinations is derived in Section 3. a key role to get the only zero solution.

The Kelvin transform is defined by

$$f(x) = \frac{1}{|x - x_0|^{n-\alpha}} u(\frac{x - x_0}{|x - x_0|^2 + x_1})$$

In this subcritical case, it will turn out that a singular term $\frac{1}{|x|^\alpha}$ plays a key role to get the only zero solution.

Our paper is structured as follows: In Section 2, we prove Theorem 1.1 by using the method of moving planes in an integral form. Theorem 1.2 where $f_i$ are terms of positive linear combinations is derived in Section 3.

2. Proof of Theorem 1. In this part, by using the method of moving planes in an integral form introduced by Chen-Li-Ou [5], we prove symmetry property of the solution in critical and subcritical cases. We also obtain that the solution must be zero in subcritical case.

First, we recall the Kelvin transform and derive some simple formulas and corresponding estimates. The Kelvin transform is defined by

$$\tilde{u}(x) = K(u)(x) = K(n, \alpha, x_0, x_1)(u)(x) = \frac{1}{|x - x_0|^{n-\alpha}} u(\frac{x - x_0}{|x - x_0|^2 + x_1})$$
for any \( x_0, x_1 \in \mathbb{R}^n \) and \( 0 < \alpha < n \). For simplicity, we derive it for the case \( x_0 = x_1 = 0 \):

\[
\tilde{u}_i(x) = \frac{1}{|x|^{\alpha-n}} u_i \left( \frac{x}{|x|^2} \right)
\]

\[
= \frac{1}{|x|^{\alpha-n}} \int_{\mathbb{R}^n} \frac{1}{|x|^{\alpha-n}} f_i(u(y)) dy
\]

\[
= \frac{1}{|x|^{\alpha-n}} \int_{\mathbb{R}^n} \frac{1}{|x|^{\alpha-n}} f_i \left( \frac{y}{|y|^2} \right) \frac{1}{|y|^{\alpha-n}} dy
\]

\[
= \frac{1}{|x|^{\alpha-n}} \int_{\mathbb{R}^n} \frac{(|y|y)^{\alpha-n}}{|x-y|^{\alpha-n}} f_i(\frac{|y|^{\alpha-n}}{y\tilde{u}(y)}) \frac{1}{|y|^{\alpha-n}} dy
\]

\[
= \int_{\mathbb{R}^n} \frac{1}{|y|^{\alpha-n}} f_i \left( \frac{y}{|y|^{\alpha-n}} \tilde{u}(y) \right) \frac{1}{|y|^{\alpha-n}} dy
\]

\[
= \int_{\mathbb{R}^n} \frac{1}{|y|^{\alpha-n}} f_i(\tilde{u}(y)) \frac{1}{|y|^{\alpha-n}} dy, \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}.
\]

Here, we have used the assumption that \( f_i \) is homogeneous of degree \( \beta_i \).

Define \( \tau_i = 2n - (\beta_i + 1)(n - \alpha) \), then the above formula becomes

\[
\tilde{u}_i(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{\alpha-n}} f_i \left( \frac{y}{|y|^{\alpha-n}} \tilde{u}(y) \right) \frac{1}{|y|^{\alpha-n}} dy
\]

\[
= \int_{\mathbb{R}^n} \frac{1}{|y|^{\alpha-n}} f_i(\tilde{u}(y)) \frac{1}{|y|^{\alpha-n}} dy, \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}.
\]

Notice that \( \tau_i \) satisfies \( 0 \leq \tau_i < n + \alpha \), only in critical case, we have \( \tau_i = 0 \). Thus, we consider two cases together without loss of generality.

Second, we recall the Hardy-Littlewood-Sobolev inequality:

\[
\| Tf \|_p \leq C(n, p) \| f \|_{\frac{n}{n-\alpha}}^{\frac{n}{n+\alpha p}}, \quad (8)
\]

where \( \| \cdot \|_p \) is the \( L^p \) norm and \( \frac{n}{n-\alpha} < p < \infty \), \( C(n, p) \) is a uniform positive constant, and

\[
Tf(x) = \int_{\mathbb{R}^n} |x-y|^{n-n} f(y) dy.
\]

From the assumptions of Theorem 1, it’s easy to see that

\[
\frac{c}{|x|^{\alpha-n}} \leq \tilde{u}_i(x) \leq \frac{C}{|x|^{\alpha-n}} \quad (9)
\]

for \( |x| \) large, where \( c, C \) are positive constants. Another important observation is that if \( u_i \in L_{loc}^\infty(\mathbb{R}^n) \), then \( \tilde{u}_i \in L_{loc}^\infty(\mathbb{R}^n \setminus \{0\}) \). In fact, \( \tilde{u}_i \) is also bounded near infinity.

To prove that \( u \) is radially symmetric about a point \( x_0 \in \mathbb{R}^n \), we only need to derive these properties for the Kelvin transform of \( u \). For simplicity, we still denote it by \( u \).

Now, we begin the proof. For each \( \lambda \in \mathbb{R} \), define the half-space

\[
\Sigma_\lambda = \{ x \in \mathbb{R}^n; x_1 < \lambda \},
\]
and plane
\[ T_\lambda = \{ x \in \mathbb{R}^n; x_1 = \lambda \}, \]
for each \( x = (x_1, x') \in \mathbb{R}^n \), let
\[ x^\lambda = (2\lambda - x_1, x') \]
be the reflection point of \( x \) with respect to the hyperplane \( T_\lambda \).

In the following part, we mainly prove the results in subcritical case. The proof of the critical case is similar by taking \( \tau_i = 0 \).

Define
\[ u^\lambda(x) = u(x^\lambda), \quad \Omega^\lambda_i = \{ x \in \Sigma_\lambda \setminus \{0^\lambda\}; u^\lambda_i(x) = u_i(x^\lambda) < u_i(x) \} \text{ for } i = 1, 2, \ldots, m. \]

For any solution \( u_i(x) \) of (4), since \( u_i(x) \) and \( u^\lambda_i(x) \) can be rewritten as
\[ u_i(x) = \int_{\Sigma_\lambda} \frac{f_i(u(y))}{|y|^\tau_i} \frac{1}{|x - y|^{n-\alpha}} dy + \int_{\Sigma_\lambda} \frac{f_i(u^\lambda(y))}{|y^\lambda|^\tau_i} \frac{1}{|x^\lambda - y|^{n-\alpha}} dy \]
and
\[ u^\lambda_i(x) = \int_{\Sigma_\lambda} \frac{f_i(u^\lambda(y))}{|y^\lambda|^\tau_i} \frac{1}{|x - y|^{n-\alpha}} dy + \int_{\Sigma_\lambda} \frac{f_i(u(y))}{|y|^\tau_i} \frac{1}{|x^\lambda - y|^{n-\alpha}} dy, \]
we have
\[ u_i(x) - u^\lambda_i(x) = \int_{\Sigma_\lambda} \left( \frac{f_i(u(y))}{|y|^\tau_i} - \frac{f_i(u^\lambda(y))}{|y^\lambda|^\tau_i} \right) \left( \frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) dy. \]

The proof consists of three main steps. In Step 1, we show that there exists an \( N > 0 \) such that for \( \lambda \leq -N \), we have \( \Omega^\lambda_i = \bigcup_{i=1}^m \Omega^\lambda_i \) is empty. In Step 2, we increase \( \lambda \) continuously to a critical level \( \lambda_0 \) for some \( \lambda_0 < 0 \), then \( u_i \) must be symmetric in the \( x_1 \) direction about that critical plane \( x_1 = \lambda_0 \). Otherwise, we can move the plane all the way to \( x_1 = 0 \). In Step 3, we prove that the solution must be zero in subcritical case. Notice that in Step 1 and 2, we consider both cases. However, in Step 3, we only focus on the subcritical case.

In Step 1. When we consider the subcritical case, since the degree \( \beta_i \) can be different, according to the definition of \( \tau_i \), then \( \tau_i \) can be different. Without loss of generality, we suppose \( \tau_1 \) satisfies the subcritical condition.

First, we need estimate \( |u(x) - u^\lambda(x)| \). Let
\[ D^\lambda_i = \{ x \in \Sigma_\lambda \setminus \{0^\lambda\}; f_i(u^\lambda(x)) < f_i(u(x)) \} \text{ for } i = 1, 2, \ldots, m. \]

We estimate (10) for \( i = 1 \) when \( x \in B^1_\lambda \). For \( x, y \in \Sigma_\lambda \setminus \{0^\lambda\} \), due to
\[ \frac{1}{|x - y|^{n-\alpha}} > \frac{1}{|x^\lambda - y|^{n-\alpha}}, \]

...
we have
\[ u_1(x) - u_1^\lambda(x) \]
\[ \leq \int_{D_1^\lambda} \left( f_1(u(y)) - f_1(u^\lambda(y)) \right) \left( \frac{1}{|y|^{\tau_1}} - \frac{1}{|x-y|^{n-\alpha} - |x^\lambda - y|^{n-\alpha}} \right) dy \]
\[ = \int_{D_1^\lambda} \left( \frac{f_1(u(y)) - f_1(u^\lambda(y))}{|y|^{\tau_1}} + f_1^\lambda(u^\lambda(y)) - f_1^\lambda(u^\lambda(y)) \right) \left( \frac{1}{|x-y|^{n-\alpha} - |x^\lambda - y|^{n-\alpha}} \right) dy \]
\[ = \int_{D_1^\lambda} \left( f_1(u(y)) - f_1(u^\lambda(y)) \right) \left( \frac{1}{|y|^{\tau_1}} - \frac{1}{|x-y|^{n-\alpha} - |x^\lambda - y|^{n-\alpha}} \right) dy \]
\[ \leq \int_{D_1^\lambda} f_1(u(y)) - f_1(u^\lambda(y)) \left( \frac{1}{|y|^{\tau_1}} - \frac{1}{|x-y|^{n-\alpha}} \right) dy. \]

For \( y \in D_1^\lambda \), since \( f_1 \) is monotone nondecreasing with respect to its components, we must have \( u_i(y) > u_i^\lambda(y) \) for some \( i \). For simplicity, we assume that:
\[ u_k(y) > u_k^\lambda(y) \quad \text{for} \quad k = 1, \ldots, l \]

and
\[ u_k(y) \leq u_k^\lambda(y) \quad \text{for} \quad k = l + 1, \ldots, m. \]

Let
\[ u_k^{\lambda+}(y) = \begin{cases} 0, & \text{when } u_k(y) \leq u_k^\lambda(y), \\ u_k(y) - u_k^\lambda(y), & \text{when } u_k(y) > u_k^\lambda(y), \end{cases} \]

and write
\[ w^{\lambda+}(y) = (w_1^{\lambda+}(y), \ldots, w_m^{\lambda+}(y)). \]

It is clear that \( w_k^{\lambda+}(y) \equiv 0 \) for \( k = l + 1, \ldots, m \). When \( k = 1, \ldots, l \), we will develop the argument with respect to the following two possibilities:

(i) \( u_k^\lambda(y) \leq \frac{1}{2} u_k(y) \) for some \( k \), or
(ii) \( u_k(y) > u_k^\lambda(y) > \frac{1}{2} u_k(y) \) for \( k = 1, \ldots, l \).

In case (i), we have
\[ |w_k^{\lambda+}(y)| \geq \frac{1}{2} u_k(y) \geq \frac{c}{|y|^{n-\alpha}} \cong |u(y)|, \]

and thus
\[ f_1(u(y)) - f_1(u^\lambda(y)) \leq f_1(u(y)) \leq C|u(y)|^{\beta_1} \leq C|u(y)|^{\beta_1-1} |w_k^{\lambda+}(y)|. \]

\[ u_1(x) - u_1^\lambda(x) \leq \int_{D_1^\lambda} \left( f_1(u(y)) - f_1(u^\lambda(y)) \right) \left( \frac{1}{|y|^{\tau_1}} - \frac{1}{|x-y|^{n-\alpha} - |x^\lambda - y|^{n-\alpha}} \right) dy \]
\[ \leq \int_{D_1^\lambda} \left( C|u(y)|^{\beta_1-1} |w_k^{\lambda+}(y)| \right) \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) dy \]
\[ \leq C \int_{D_1^\lambda} \frac{|u(y)|^{\beta_1-1} |w^{\lambda+}(y)|}{|y|^{\tau_1} |x-y|^{n-\alpha}} dy. \]
While in case (ii), we obtain
\begin{align*}
& f_1(u(y)) - f_1(u^\lambda(y)) = f_1(u_1, u_2, \ldots, u_m) - f_1(u_1^\lambda, u_2, \ldots, u_m) \\
& + f_1(u_1^\lambda, u_2, \ldots, u_m) - f_1(u_1^\lambda, u_2^\lambda, \ldots, u_m) \\
& + \cdots + f_1(u_1^\lambda, \ldots, u_{l-1}^\lambda, u_l, \ldots, u_m) - f_1(u_1^\lambda, \ldots, u_{l-1}^\lambda, u_l^\lambda, u_{l+1}, \ldots, u_m) \\
& + f_1(u_1^\lambda, \ldots, u_{l-1}^\lambda, u_l^\lambda, u_{l+1}, \ldots, u_m) - f_1(u^\lambda) \\
& \leq f_1(u_1, u_2, \ldots, u_m) - f_1(u_1^\lambda, u_2, \ldots, u_m) \\
& + f_1(u_1^\lambda, u_2, \ldots, u_m) - f_1(u_1^\lambda, u_2^\lambda, \ldots, u_m) \\
& + \cdots + f_1(u_1^\lambda, \ldots, u_{l-1}^\lambda, u_l, \ldots, u_m) - f_1(u_1^\lambda, \ldots, u_{l-1}^\lambda, u_l^\lambda, u_{l+1}, \ldots, u_m) \\
& = \frac{\partial f_1(\xi_1, u_2, \ldots, u_m)}{\partial u_1} u_1^{\lambda+}(y) + \frac{\partial f_1(\xi_2, u_3, \ldots, u_m)}{\partial u_2} u_2^{\lambda+}(y) \\
& + \cdots + \frac{\partial f_1(u_1^\lambda, \ldots, u_{l-1}^\lambda, u_l, u_{l+1}, \ldots, u_m)}{\partial u_l} u_l^{\lambda+}(y). \quad (11)
\end{align*}

Notice that for \( k = 1, \ldots, l, \)
\[ u_k(y) > u_k^\lambda(y) > \frac{1}{2} u_k(y), \quad \frac{\partial f_1}{\partial u_k} \leq C \frac{f_1}{u_k}, \]
and the fact
\[ \frac{c}{|y|^{n-\alpha}} \leq u_k(y) \leq \frac{C}{|y|^{n-\alpha}}, \]
we deduce that
\[ u_k(y) > \xi_k(y) > \frac{1}{2} u_k(y), \]
and thus
\begin{align*}
f_1(u(y)) - f_1(u^\lambda(y)) & \leq C \left[ \frac{f_1(u(y))}{\xi_1(y)} + \cdots + \frac{f_1(u(y))}{\xi_l(y)} \right] |u^{\lambda+}(y)| \\
& \leq C \left[ \frac{f_1(u(y))}{u_1(y)} + \cdots + \frac{f_1(u(y))}{u_l(y)} \right] |u^{\lambda+}(y)| \\
& \leq C |u(y)|^{\beta_l-1} |u^{\lambda+}(y)|. \quad (12)
\end{align*}
\begin{align*}
u_1(x) - u_1^\lambda(x) & \leq \int_{D_1^\lambda} \left( \frac{f_1(u(y)) - f_1(u^\lambda(y))}{|y|^r_1} \right) \left( \frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) dy \\
& \leq \int_{D_1^\lambda} \left( C |u(y)|^{\beta_l-1} |u^{\lambda+}(y)| \right) \left( \frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) dy \\
& \leq C \int_{D_1^\lambda} \frac{|u(y)|^{\beta_1-1} |u^{\lambda+}(y)|}{|y|^r} \frac{dy}{|x - y|^{n-\alpha}} dy.
\end{align*}

According to the cases (i),(ii), for \( x \in D_1^\lambda, \) we obtain the estimates which we need to use in the moving planes method,
\[ u_1(x) - u_1^\lambda(x) \leq C \int_{D_1^\lambda} \frac{|u(y)|^{\beta_1-1} |u^{\lambda+}(y)|}{|y|^r} \frac{dy}{|x - y|^{n-\alpha}} dy. \]

Arguing by the same way, we can get the similar estimate that
\[ u_i(x) - u_i^\lambda(x) \leq C \int_{D_1^\lambda} \frac{|u(y)|^{\beta_i-1} |u^{\lambda+}(y)|}{|y|^r} \frac{dy}{|x - y|^{n-\alpha}} dy. \]
for each $i = 1, \cdots, m$. Then we have

$$|u(x) - u^\lambda(x)| \leq C \sum_{i=1}^{m} \int_{D^\lambda_i} \frac{|u(y)|^{\beta_i - 1}}{|y|^r} \frac{|w^{\lambda,+}(y)|}{|x - y|^{n-\alpha}} dy.$$  

Noticing that $D^\lambda_i \subseteq \Omega^\lambda$, and applying the Hardy-Littlewood-Sobolev and Hölder inequalities, we obtain:

$$\|w^{\lambda,+}\|_{p+1, \Sigma_\lambda \setminus \{0^\lambda\}} \leq C \sum_{i=1}^{m} \left\{ \int_{\Omega^\lambda} \left( \frac{|u(y)|^{\beta_i - 1}}{|y|^r} \right)^{\frac{\alpha}{\beta_i - 1}} dy \right\}^{\frac{1}{\alpha}} \|w^{\lambda,+}\|_{p+1, \Sigma_\lambda \setminus \{0^\lambda\}}, \quad (13)$$

here $p = \frac{n+\alpha}{n-\alpha}$, and $\| \cdot \|_{q, \Gamma}$ represents the $L^q$ norm on the set $\Gamma$.

From $u_i(x) \in L^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$ and estimates (9), we have

$$\int_{\Omega^\lambda} \left( \frac{|u(y)|^{\beta_i - 1}}{|y|^r} \right) dy \leq C \int_{\Omega^\lambda} \frac{1}{|y|^{2n}} dy < \infty. \quad (14)$$

Hence, for $\lambda$ sufficiently negative, the quantity $C(\sum_{i=1}^{m} \{ \int_{\Omega^\lambda} \left( \frac{|u(y)|^{\beta_i - 1}}{|y|^r} \right)^{\frac{\alpha}{\beta_i - 1}} dy \right\}^{\frac{1}{\alpha}} \|w^{\lambda,+}\|_{p+1, \Sigma_\lambda \setminus \{0^\lambda\}})$ can be very small, so we have $\|w^{\lambda,+}\|_{p+1, \Sigma_\lambda \setminus \{0^\lambda\}} = 0$. That means the Lebesque measure of $\Omega^\lambda$ is zero. This implies that

$$u_i(y) \leq u_i^\lambda(y) \quad \text{and} \quad f_i(u(y)) \leq f_i(u^\lambda(y)) \quad \text{for almost all } y \in \Sigma_\lambda \setminus \{0^\lambda\}.$$

Thus, we can move the plane to the right continuously as long as the above inequality holds.

By the way, in critical case, if all $\beta_i$ equal to $\frac{n+\alpha}{n-\alpha}$, we have the estimate

$$\|w^{\lambda,+}\|_{p+1, \Sigma_\lambda \setminus \{0^\lambda\}} \leq C \|u\|_{p+1, \Omega^\lambda}^{-1} \|w^{\lambda,+}\|_{p+1, \Sigma_\lambda \setminus \{0^\lambda\}},$$

here $p = \frac{n+\alpha}{n-\alpha}$. Similarly, for $\lambda$ sufficiently negative, the quantity $C \|u\|_{p+1, \Omega^\lambda}^{-1}$ can be very small, which gives us that the Lebesque measure of $\Omega^\lambda$ is zero.

In Step 2, we define

$$\lambda_0 = \sup\{ \lambda \in \mathbb{R}; \mu(\Omega^\lambda) = 0 \quad \text{for all } \lambda' \leq \lambda \}.$$

It is clear $\lambda_0 \leq 0$, and $u^{\lambda_0}(x) \geq u(x)$.

Next we show that $u_i$ is symmetric in the $x_1$ direction about that critical plane $x_1 = \lambda_0$. If $\lambda_0 = 0$, since we can move the plane from $x_1 = +\infty$ to the left, then it is obviously to see that $u_i$ is symmetric in the $x_1 = 0$.

Now we suppose that $\lambda_0 < 0$, $u_i$ is not symmetric in the $x_1$ direction at $\lambda_0$, we show that the plane can be moved further to the right to get the contradiction.

To this purpose, by using the expression (10), we first claim that

$$u_i(x^{\lambda_0}) > u_i(x) \quad \text{for} \quad x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\} \quad \text{and} \quad i = 1, 2, \cdots, m. \quad (15)$$

Actually, the assumption that $u_i$ is not symmetric in the $x_1$ direction at $\lambda_0$ implies that there exists $y_i^0 \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}$ such that

$$u_i(y_i^0) < u_i^{\lambda_0}(y_i^0). \quad (16)$$

Consequently, we have

$$u_i(y) < u_i^{\lambda_0}(y) \quad \text{on a set of positive measure in } \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\} \quad \text{for some } i. \quad (17)$$
For simplicity, we assume that 
\[ u_i(y) < u_i^{\lambda_0}(y) \] on a set of positive measure for \( i = 1, 2, \ldots, l \)
and 
\[ u_i(y) = u_i^{\lambda_0}(y), \quad y \in \Sigma_{\lambda_0} \backslash \{0^{\lambda_0}\}, \quad \text{and} \quad i = l + 1, \ldots, m. \]
If \( l = m \), then our claim is proved. If on the other hand \( 1 < l < m \), we derive a contradiction via the assumption that system (4) is non-degenerate. This assumption implies that on a set of positive measure of \( \Sigma_{\lambda_0} \backslash \{0^{\lambda_0}\} \) we have 
\[ (f_{i+1}(u(y)), \ldots, f_m(u(y))) \neq (f_{i+1}(u(y^{\lambda_0})), \ldots, f_m(u(y^{\lambda_0}))). \]
Hence there exists \( i_0 \), which is between \( l + 1 \) and \( m \), such that 
\[ f_{i_0}(u(y)) \neq f_{i_0}(u(y^{\lambda_0})) \] on a set of positive measure.
Since \( f_{i_0} \) is monotone nondecreasing with respect to its components, then \( f_{i_0}(u(y)) < f_{i_0}(u(y^{\lambda_0})) \) on a set of positive measure. Since from (10)
\[ u_{i_0}(x) - u_{i_0}^{\lambda_0}(x) = \int_{\Sigma_{\lambda_0}} \left( \frac{f_{i_0}(u(y)) - f_{i_0}(u^{\lambda_0}(y))}{|y|^{\tau_0}} \right) \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{n-\alpha}} \right) dy \]
then we get \( u_{i_0}(x) < u_{i_0}^{\lambda_0}(x) \) for almost all \( x \in \Sigma_{\lambda_0} \backslash \{0^{\lambda_0}\} \). Thus we get a contradiction. So the claim (15) holds.

The claim (15) implies that the measure of \( \Omega_{\lambda_0} \) is zero.

Similarly we can use the method in Step 1 to yield that
\[ \|w^{\lambda_0}+\|_{p+1,\Sigma_{\lambda_0}\{0^\lambda\}} \leq C(\sum_{i=1}^m \int_{\Omega_{\lambda}} \left( \frac{|u(y)|^{\beta_i-1}}{|y|^{\tau_i}} \right) dy \|^2) \|w^{\lambda_0}_{\cdot+}\|_{p+1,\Sigma_{\lambda_0}\{0^\lambda\}}, \] (18)
for \( p = \frac{n+\alpha}{n-\alpha} \).

From \( u_i(x) \in L_{loc}^\infty(\mathbb{R}^n \backslash \{0\}) \), for any small \( \eta \), we can choose sufficiently large \( R \) such that
\[ \left( \int_{\mathbb{R}^n \backslash B_R(0)} \left( \frac{|u(y)|^{\beta_i-1}}{|y|^{\tau_i}} \right) dy \right)^{\frac{2}{n}} < \eta. \] (19)
Then fix \( R \), we have \( \lim_{\lambda \to \lambda_0} \mu(\Omega_{\lambda} \cap B_R(0)) = 0 \), you can see the details in [10], and thus
\[ \left( \int_{\Omega_{\lambda}} \left( \frac{|u(y)|^{\beta_i-1}}{|y|^{\tau_i}} \right) dy \right)^{\frac{2}{n}} < \eta. \] (20)
Then we have \( \|w^{\lambda_0}_{\cdot+}\|_{p+1,\Sigma_{\lambda_0}\{0^\lambda\}} = 0 \). That means \( \mu(\Omega_{\lambda_0}) = 0 \) for \( \lambda \) close to \( \lambda_0 \). This contradicts with the definition of \( \lambda_0 \). Similarly, we also can obtain this conclusion in the critical case.

Now, we obtain the symmetry property of the solution in critical case and subcritical case. This means \( u_i(x) = u_i^{\lambda_0}(x) \), and consequently \( f_i(u(y)) = f_i(u^{\lambda_0}(y)) \) for all \( i \), and for \( x, y \in \Sigma_{\lambda_0} \backslash \{0^{\lambda_0}\} \). In subcritical case, since one of \( \tau_0 \neq 0 \), and by (10) we can get that 
\[ u_{i_0}(y) < u_{i_0}^{\lambda_0}(y), \]
if \( \lambda_0 \neq 0 \). This is a contradiction. Hence, we conclude that we must have \( \lambda_0 = 0 \) in subcritical case.
In all, we have \( u_i \) is symmetric in the \( x_1 \) direction about the plane \( x_1 = \lambda_0 \). Since the direction of \( x_1 \)-direction is arbitrary, we conclude that the solution \( u \) is radially symmetric and \( u \) is radially symmetric about the origin in the subcritical case.

In the above part, we consider \( u \) which is the Kelvin transform of \( u \) and obtain radial symmetry property of \( u \). If the symmetric point is singular point, it is easy to see that \( u \) is radially symmetric. If not, we can carry on the moving planes on \( u \) directly to obtain the radial symmetry.

In Step 3, we prove that the solution must be zero in subcritical case. Considering the Kelvin type transform centered at 0, 

\[
\bar{u}(x) = \frac{1}{|x|^{n-\alpha}} u \left( \frac{x}{|x|^2} \right).
\]

Let \( x^1 \) and \( x^2 \) be any two points in \( \mathbb{R}^n \setminus \{0\} \), we choose the coordinate system so that the midpoint \( \frac{x^1 + x^2}{2} \) is at the origin. According to the arguments as in Step 1 and 2 in subcritical case, we can see that \( \bar{u}(x) \) must be radially symmetric about 0. Let \( x^i_\ast = \frac{x^i}{|x^i|^2}, i = 1, 2 \) be the inversions of \( x^i \), then we have \( \bar{u}(x^1_\ast) = \bar{u}(x^2_\ast) \), and therefore, \( u(x^1) = u(x^2) \). Since \( x^1 \) and \( x^2 \) are any two points in \( \mathbb{R}^n \setminus \{0\} \), we conclude that \( u \) must be constant.

Now, we claim \( u \) must be zero. According to the system (4), since \( u \) is constant, we can see that the right hand side of the system is divergent. This is possible. we conclude that \( u \equiv 0 \).

Remark 2. We can see that in Step 1, we consider two possibilities instead of considering \( u_k^\lambda(y) < u_k(y) \) directly. The reason is that when we calculate

\[
f_1(u(y)) - f_1(u^\lambda(y)) \leq C \left[ \frac{f_1(u(y))}{\xi_1(y)} + \cdots + \frac{f_1(u(y))}{\xi_k(y)} \right] |w^{\lambda, +}(y)|
\]

\[
\leq C |u(y)|^{\beta_1 - 1} |w^{\lambda, +}(y)|,
\]

(21)

where

\[
u_k(y) > \xi_k(y) > u_k^\lambda(y) > \frac{1}{2} u_k(y).
\]

If \( \xi_k(y) \) is sufficiently small, then the second inequality would be failed in (21). Hence, we need consider two possibilities, that means giving the lower bound of \( \xi_i(y) \) to overcome this difficulty.

3. Proof of Theorem 2. In this part, we prove Theorem 2. Since the proof of Theorem 2 is similar to the idea of Theorem 1, then we only state the main steps in this section.

Again using the Kelvin transform which is defined by

\[
\bar{u}(x) \equiv K(u)(x) \equiv K(n, \alpha, x_0, x_1)(u)(x) \equiv \frac{1}{|x - x_0|^{n-\alpha}} u \left( \frac{x - x_0}{|x - x_0|^2 + x_1} \right)
\]
for any \(x_0, x_1 \in \mathbb{R}^n\) and \(0 < \alpha < n\). Similar to the proof of Theorem 1, we derive
\[
\bar{u}_i(x) = \frac{1}{|x|^{n-\alpha}} u_i\left(\frac{x}{|x|^2}\right)
\]
\[
= \frac{1}{|x|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} f_i(u(y)) dy
\]
\[
= \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f_i(|y|^{n-\alpha} \bar{u}(y)) \frac{1}{|y|^{n+\alpha}} dy
\]
\[
= \sum_{r=1}^k \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} \left(|y|^{n-\alpha} f_r(\bar{u}(y))\right) \frac{1}{|y|^{n+\alpha}} dy
\]
\[
= \sum_{r=1}^k \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f_r(\bar{u}(y)) |y|^{2n-(\beta_r+1)(n-\alpha)} dy, \quad \text{for} \ x \in \mathbb{R}^n \backslash \{0\}.
\]

Here, we have used the assumption that \(f_r\) is homogeneous of degree \(\beta_r\).

Define \(\tau_r = 2n - (\beta_r + 1)(n-\alpha)\), then the above formula becomes
\[
\bar{u}_i(x) = \sum_{r=1}^k \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} \frac{f_r(\bar{u}(y))}{|y|^{2n-(\beta_r+1)(n-\alpha)}} dy
\]
\[
= \sum_{r=1}^k \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f_r(\bar{u}(y)) |y|^{\tau_r} dy.
\]

Notice that \(\tau_r\) satisfies \(0 \leq \tau_r < n + \alpha\), and only when all \(\beta_r = \frac{n+\alpha}{n-\alpha}\) we have \(\tau_r = 0\).

To prove that \(u\) is radially symmetric about a point \(x_0 \in \mathbb{R}^n\), we only need to derive these properties for the Kelvin transform of \(u\). For simplicity, we still denote it by \(u\).

Now, we begin the proof. Using the notation as in Theorem 1, for any solution \(u_i(x)\) of (4), we have
\[
u_i(x) = \sum_{r=1}^k \int_{\Sigma_\lambda} \left( \frac{f_r(u(y))}{|y|^\tau_r} - \frac{f_r(u^\lambda(y))}{|y^\lambda|^\tau_r} \right) \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x-y^\lambda|^{n-\alpha}} \right) dy,
\]
(22)
since
\[
u_i(x) = \sum_{r=1}^k \int_{\Sigma_\lambda} \frac{f_r(u(y))}{|y|^\tau_r} \frac{1}{|x-y|^{n-\alpha}} dy + \int_{\Sigma_\lambda} \frac{f_r(u^\lambda(y))}{|y^\lambda|^\tau_r} \frac{1}{|x-y^\lambda|^{n-\alpha}} dy
\]
and
\[
u^\lambda_i(x) = \sum_{r=1}^k \int_{\Sigma_\lambda} \frac{f_r(u^\lambda(y))}{|y^\lambda|^\tau_r} \frac{1}{|x-y|^{n-\alpha}} dy + \int_{\Sigma_\lambda} \frac{f_r(u(y))}{|y|^\tau_r} \frac{1}{|x-y^\lambda|^{n-\alpha}} dy.
\]

In Step 1, when we consider \(\beta_r\) satisfies \(0 < \beta_r \leq \frac{n+\alpha}{n-\alpha}\), since the degree \(\beta_r\) can be different, according to the definition of \(\tau_r\), then \(\tau_r\) can be different.

First, we need estimate \(|u(x) - u^\lambda(x)|\). Let
\[
D_r^1 = \{x \in \Sigma_\lambda \backslash \{0^\lambda\}; f_r(u^\lambda(x)) < f_r(u(x))\} \quad \text{for} \ i = 1, 2, \cdots, m, r = 1, 2, \cdots, k.
\]
We estimate (22) for \(i = 1\) when \(x \in B^1_\lambda\).
For \( x, y \in \Sigma_\lambda \setminus \{0^\lambda\} \), due to
\[
\frac{1}{|x - y|^{n - \alpha}} > \frac{1}{|x^\lambda - y|^{n - \alpha}} ,
\]
we have
\[
|u(x) - u^\lambda(x)| \leq \sum_{r=1}^{k} \int_{D^{r}_{1\lambda}} \left( \frac{f_{1r}(u(y)) - f_{1r}(u^\lambda(y))}{|y|^{\tau_{rr}}} \right) \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) dy .
\]

On the right hand side of the above inequality, each term
\[
\int_{D^{r}_{1\lambda}} \left( f_{1r}(u(y)) - f_{1r}(u^\lambda(y)) \right) \left( \frac{1}{|x - y|^{n - \alpha}} - \frac{1}{|x^\lambda - y|^{n - \alpha}} \right) dy
\]
can be dealt with as the proof of Theorem 1. So we can get that
\[
|u(x) - u^\lambda(x)| \leq C \sum_{r=1}^{k} \sum_{i=1}^{m} \int_{D^{r}_{1\lambda}} \frac{|u(y)|^{\beta_{rr} - 1}}{|y|^{\tau_{rr}}} \frac{|u^\lambda + (y)|}{|x - y|^{n - \alpha}} dy .
\]
Furthermore, we obtain
\[
|u(x) - u^\lambda(x)| \leq C \sum_{r=1}^{k} \int_{D^{r}_{1\lambda}} \frac{|u(y)|^{\beta_{rr} - 1}}{|y|^{\tau_{rr}}} \frac{|u^\lambda + (y)|}{|x - y|^{n - \alpha}} dy.
\]

Notice that \( D^{r}_{1\lambda} \subseteq \Omega_\lambda \), and applying the Hardy-Littlewood-Sobolev and Hölder inequalities, we obtain:
\[
\|u^\lambda + \|_{p+1, \Sigma_\lambda \setminus \{0\}^\lambda} \leq C \left( \sum_{r=1}^{k} \sum_{i=1}^{m} \int_{\Omega_\lambda} \left( \frac{|u(y)|^{\beta_{rr} - 1}}{|y|^{\tau_{rr}}} \right)^{\frac{p}{\alpha}} dy \right)^{\frac{\alpha}{p}} \|u^\lambda + \|_{p+1, \Sigma_\lambda \setminus \{0\}^\lambda} ,
\]
here \( p = \frac{n + \alpha}{n - \alpha} \).

From \( u_i(x) \in L^{\infty}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) and estimates (9), we have
\[
\int_{\Omega_\lambda} \left( \frac{|u(y)|^{\beta_{rr} - 1}}{|y|^{\tau_{rr}}} \right)^{\frac{p}{\alpha}} dy \leq C \int_{\Omega_\lambda} \frac{1}{|y|^{2\alpha}} dy < \infty.
\]
Hence, for \( \lambda \) sufficiently negative, the quantity
\[
C \left( \sum_{r=1}^{k} \sum_{i=1}^{m} \left( \int_{\Omega_\lambda} \left( \frac{|u(y)|^{\beta_{rr} - 1}}{|y|^{\tau_{rr}}} \right)^{\frac{p}{\alpha}} dy \right)^{\frac{\alpha}{p}} \right)
\]
can be very small, so we have
\[
\|u^\lambda + \|_{p+1, \Sigma_\lambda \setminus \{0\}^\lambda} = 0.
\]
That means the Lebesgue measure of \( \Omega_\lambda \) is zero. This implies that
\[
u_i(y) \leq u^\lambda_i(y) \quad \text{and} \quad f_{ir}(u(y)) \leq f_{ir}(u^\lambda(y)) \quad \text{for almost all} \quad y \in \Sigma_\lambda \setminus \{0^\lambda\}.
\]
Thus, we can move the plane to the right continuously as long as the above inequality holds.

By the way, if all \( \beta_{ir} \) equal to \( \frac{n + \alpha}{n - \alpha} \), we also can prove that the Lebesgue measure of \( \Omega_\lambda \) is zero as the proof of Theorem 1.

In Step 2, we define
\[
\lambda_0 = \sup \left\{ \lambda \in \mathbb{R} : \mu(\Omega^\lambda) = 0 \right\} \quad \text{for all} \quad \lambda^\prime \leq \lambda.
\]
It is clear \( \lambda_0 \leq 0 \), and \( u^{\lambda_0}(x) \geq u(x) \).

If \( \lambda_0 = 0 \), it is obviously to see that \( u_1 \) is symmetric in the \( x_1 \) direction at \( \lambda_0 \).

Now we suppose that \( \lambda_0 < 0 \), \( u_1 \) is not symmetric in the \( x_1 \) direction at \( \lambda_0 \), we show that the plane can be moved further to the right to get the contradiction.
To this purpose, it is suffice to prove the following claim by using the expression (22) and non-degeneracy of the system

\[ u_i(x^{\lambda_0}) > u_i(x) \text{ for } x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\} \text{ and } i = 1, 2, \cdots, m. \]  

(25)

Actually, the assumption that \( u_i \) is not symmetric in the \( x_1 \) direction at \( \lambda_0 \) implies that

\[ u_i(y) < u_i^{\lambda_0}(y) \text{ on a set of positive measure in } \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\} \text{ for some } i. \]  

(26)

For simplicity, we assume that

\[ u_i(y) < u_i^{\lambda_0}(y) \text{ on a set of positive measure for } i = 1, 2, \cdots, l \]

and

\[ u_i(y) = u_i^{\lambda_0}(y), \text{ for } y \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}, \text{ and } i = l + 1, \cdots, m. \]

If \( l = m \), then our claim is proved. If on the other hand \( 1 < l < m \), we derive a contradiction via the assumption that system (4) is non-degenerate. This assumption implies that on a set of positive measure in \( \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\} \) we have

\[ (f_{l+1}(u(y)), \cdots, f_m(u(y))) \neq (f_{l+1}(u(y^{\lambda_0})), \cdots, f_m(u(y^{\lambda_0}))). \]

Hence there exists \( i_0 \), which is between \( l + 1 \) and \( m \), such that

\[ f_{i_0}(u(y)) \neq f_{i_0}(u(y^{\lambda_0})) \]

on a set of positive measure. Since \( f_{i_0} \) is monotone nondecreasing with respect to its components, then \( f_{i_0}(u(y)) < f_{i_0}(u(y^{\lambda_0})) \) on a set of positive measure. That is

\[ \sum_{r=1}^{k} f_{i_0r}(u(y)) < \sum_{r=1}^{k} f_{i_0r}(u(y^{\lambda_0})) \]

on a set of positive measure. By using \( f_{i_0r} \) is monotone nondecreasing with respect to its components again there exists \( r_0 \) between 1 and \( k \) such that

\[ f_{i_0r_0}(u(y)) < f_{i_0r_0}(u(y^{\lambda_0})) \]

on a set of positive measure. Since by (22)

\[ u_{i_0}(x) - u_{i_0}^{\lambda_0}(x) \]

\[ = \sum_{r=1}^{k} \int_{\Sigma_{\lambda_0}} \left( f_{i_0r}(u(y)) - f_{i_0r}(u^{\lambda_0}(y)) \right) \left( \frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x^{\lambda_0} - y|^{n-\alpha}} \right) dy \]

\[ \leq \sum_{r=1}^{k} \int_{\Sigma_{\lambda_0}} \left( f_{i_0r}(u(y)) - f_{i_0r}(u^{\lambda_0}(y)) \right) \left( \frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x^{\lambda_0} - y|^{n-\alpha}} \right) dy, \]

then we have \( u_{i_0}(x) < u_{i_0}^{\lambda_0}(x) \) for almost all \( x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\} \). Thus we get a contradiction. So the claim contradicts to our assumption here. That means the claim (25) holds.

As the proof of Theorem 1, we can use the similar arguments to prove the following part. Hence, we finish the proof.

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