EVENT-BASED FAULT DETECTION FOR INTERVAL TYPE-2 FUZZY SYSTEMS WITH MEASUREMENT OUTLIERS

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Abstract. This paper investigates the event-based fault detection (FD) problem for a category of discrete-time interval type-2 fuzzy systems with measurement outliers. For the sake of decreasing the utilization of limited communication bandwidth, an event-based mechanism is introduced. Based on the saturation function technique, a novel event-based FD observer is first designed to reduce the influence of outliers in the dynamic systems. Then, on the basis of Lyapunov stability theory, sufficient conditions are provided to ensure that the error system satisfies the $\mathcal{H}_\infty$ performance and the $\mathcal{H}_\infty$ fault performance in different cases, respectively. In contrast to the existing event-based FD results, the false alarm, which is induced by measurement outliers, can be effectively avoided by the designed FD observer with saturation function. Lastly, some simulation results are given to verify the effectiveness of the method presented in this paper.

1. Introduction. Due to the evolution of industrial engineering, many systems become the nonlinear versions, and the existing linear control approaches cannot directly cope with the nonlinear systems [13, 14, 25, 32, 37, 39, 45]. Takagi-Sugeno (T-S) fuzzy model [15, 22], which can approximate the nonlinear systems by means of its precise approximation ability, has been widely applied in many significant results [5, 31]. In [31], a new fuzzy tracking control design method was presented on the basis of T-S fuzzy model. Considering the intermittent measurements, an $\mathcal{H}_\infty$ fuzzy filtering problem for a class of nonlinear systems was studied in [5]. However, the uncertainties, which usually exist in most physical systems, cannot be captured by the T-S fuzzy model with determinate membership functions [5, 31]. To this end, an interval type-2 fuzzy logic system was proposed in [21], and then, an interval type-2 (IT2) T-S fuzzy model was constructed in [12, 44] to cope with the uncertainties by using the lower and upper membership functions and relevant tradeoff coefficients. Inspired by the important technique in [12], significant strategy has been proposed on the basis of IT2 fuzzy dynamic systems [11]. Although the results in [11, 12, 44]...
have the ability to capture uncertainties existing in the considered systems, the sensor/actuator faults, which may influence the system stability, have not been considered.

Owing to the increasing demands of safety and higher performance in engineering, the FD problem has attracted a growing number of attentions in some automatic control fields, and fruitful significant results about the FD issue by the IT2 T-S fuzzy approach have been published. In [40], an FD filter was constructed as a residual generator to guarantee prescribed $\mathcal{H}_\infty$ performance. The FD problem was handled by a distributed filtering scheme proposed in [6]. However, in the aforementioned work [6], the FD approach is based on time-triggered condition, even the systems achieve the desired performance without the maintaining of control signals, abundant communication data are still transmitted into systems, which causes the waste of communication resources. Recently, an event-based scheme, which can determine the signal transmission by setting a triggered criterion, is developed to economize limited communication resources in [29]. Up to now, the event-based control condition has received remarkable attentions [3,18,28,29,41,42,46,47]. The event-based result in [29] was first extended to discrete-time T-S fuzzy dynamic models [28]. The authors in [41] developed a novel adaptive event-based scheme to decrease unnecessary signal communications. In [42], an $\mathcal{H}_\infty$ event-based filter was designed for a class of T-S fuzzy systems. Nevertheless, the problem of FD was neglected under the situation of introducing event-based mechanism in [28,29,41,42]. More recently, the authors in [24] designed an adaptive event-based FD filter for fuzzy stochastic models with missing measurements. Considering the uncertainties existing in the systems, in [26], an FD filter was designed by using the IT2 T-S fuzzy model, and event-based communication mechanism was introduced to save the utilization of limited communication bandwidth. In addition, the event-driven FD problem for discrete-time IT2 fuzzy networked control systems with nonlinear perturbations was studied in [27]. It is noteworthy that the above FD results cannot be employed to deal with the problem of false alarms which caused by measurement outliers.

Outliers usually appear in considerable systems and may produce unpredictable oscillatory of the dynamic systems. Therefore, measurement outliers issue has attracted much attentions and many results on outliers have been published. In [1], the authors designed a stubborn observer to estimate the states of linear time-variant systems subject to measurement outliers, and the influences of abnormal and isolated measurement noise were reduced. A novel robust Kalman filter, which can detect and restrict the effect of outliers in discrete linear systems, was constructed in [8]. The authors in [2] provided a method to handle state estimation issue for a class of discrete-time linear time-invariant systems in the case of outliers existing in considered systems. However, these important results only consider the problem of estimating the linear systems state subject to outliers. How to design a novel fuzzy FD strategy in the framework of IT2 T-S fuzzy model with outliers to avoid false alarms and decrease unnecessary utilization of transmission data? There are few attempts have been made to solve such a challenging problem, which motivates the current work.

This paper develops an event-based FD problem in the framework of IT2 T-S fuzzy model. In the design procedure, outliers are taken into account. The major contributions of this paper are summarized as follows:
1. Considering measurement outliers existing in the error systems, a new FD observer with event-based mechanism and saturation function is designed in this paper for the first time. The designed observer can reduce the effect of measurement outliers on the residual signals while reduce unnecessary signal transmissions.

2. This paper extends the method in [33] to a class of discrete-time fuzzy systems with uncertain parameters. Moreover, in the procedure of designing the FD observer, less network resources are utilized than [33] by introducing an event-based mechanism. Meanwhile, the mismatched membership functions can be handled by introducing a slack matrix technique.

3. In contrast to the existing event-based FD results of IT2 dynamic models [26, 27], this paper designs a novel FD observer with adaptive saturation on the output injection. On the basis of the designed method, the fault can be detected effectively and the false alarms can be also avoided. Furthermore, sufficient conditions are presented to sustain both the $H_\infty$ performance and $H_\infty$ fault performance for the error systems.

Lastly, some simulation results are given to identify the usefulness of the advanced approach.

2. Problem formulation.

2.1. IT2 T-S fuzzy model. In this subsection, the IT2 T-S fuzzy model is utilized to model the nonlinear discrete-time system.

**Plant Rule i:** IF $\ell_1(\varphi(k))$ is $N_{i1}$, and $\ell_2(\varphi(k))$ is $N_{i2}$ and ... and $\ell_ı(\varphi(k))$ is $N_{iı}$, THEN

$$x(k+1) = A_i x(k) + B_i w(k) + B_{i1} f(k),$$
$$y(k) = C x(k),$$  

(1)

where $\ell(\varphi(k)) = [\ell_1(\varphi(k)) \ \ell_2(\varphi(k)) \ \ldots \ \ell_ı(\varphi(k))]^T$ is the premise variable, $N_{iı}$ denotes the fuzzy set, $ı$ represents the amount of the fuzzy sets. $x(k) \in \mathbb{R}^n$ stands for the state vector, $y(k) \in \mathbb{R}^m$ refers to the measured output, $w(k) \in \mathbb{R}^w$ denotes the external interference which belongs to $\ell_2 [0, \infty)$, $f(k) \in \mathbb{R}^p$ stands for the fault signal. $A_i$, $B_i$, $C$ and $B_{i1}$ are system matrices with appropriate dimensions, $ı = 1, \ldots, r$, and $r$ refers to the amount of fuzzy rules.

Define

$$m_ı(\varphi(k)) = \prod_{p=1}^i u_{N_{ıp}}(\ell_p(\varphi(k))) \geq 0,$$
$$\bar{m}_ı(\varphi(k)) = \prod_{p=1}^i \bar{u}_{N_{ıp}}(\ell_p(\varphi(k))) \geq 0,$$
$$\bar{u}_{N_{ıp}}(\ell_p(\varphi(k))) \geq u_{N_{ıp}}(\ell_p(\varphi(k))) \geq 0,$$
$$\bar{m}_ı(\varphi(k)) \geq m_ı(\varphi(k)) \geq 0,$$

in which $u_{N_{ıp}}(\ell_p(\varphi(k)))$, $\bar{u}_{N_{ıp}}(\ell_p(\varphi(k)))$, $m_ı(\varphi(k))$ and $\bar{m}_ı(\varphi(k))$ stand for lower membership function, upper membership function, lower grade of membership and upper grade of membership, respectively.
Whereafter, the overall dynamic of (1) is described as the following form:

\[ x(k + 1) = \sum_{i=1}^{r} m_i(\phi(k))[A_i x(k) + B_i w(k) + B_{1i} f(k)], \]

\[ y(k) = C x(k), \]

where

\[ m_i(\phi(k)) = \frac{\alpha_i(\phi(k)) m_i(\phi(k)) + \bar{\alpha}_i(\phi(k)) \bar{m}_i(\phi(k))}{\sum_{d=1}^{r} [\alpha_d(\phi(k)) m_d(\phi(k)) + \bar{\alpha}_d(\phi(k)) \bar{m}_d(\phi(k))]}, \]

\[ 0 \leq \alpha_i(\phi(k)) \leq 1, 0 \leq \bar{\alpha}_i(\phi(k)) \leq 1, \]

\[ m_i(\phi(k)) \geq 0, \]

\[ \sum_{i=1}^{r} m_i(\phi(k)) = 1, \alpha_i(\phi(k)) + \bar{\alpha}_i(\phi(k)) = 1, \]

in which \( m_i(\phi(k)) \) is the grade of membership, \( \alpha_i(\phi(k)) \) and \( \bar{\alpha}_i(\phi(k)) \) denote weighting coefficients.

2.2. Event-based transmission mechanism. An event-based mechanism is introduced in this subsection to decrease unnecessary signal communications. The current measurement signal is represented by \( y(k) \), and \( y(i_k) \) denotes the last transmitted signal at the moment \( i_k \). The next triggered instant is provided as follows:

\[ i_{k+1} = \inf_{i_k} \{ k > i_k | [y(k) - y(i_k)]^T Q [y(k) - y(i_k)] \geq \varepsilon y^T(i_k) Q y(i_k) \}, \]

in which \( Q > 0 \) is a weighting matrix which should be determined, \( \varepsilon \in [0, 1) \) refers to the event-based threshold. The current data \( y(k) \) will be transmitted into the observer if it satisfies condition (3). If \( y(i_{k+1}) = y(i_k) \), it proves that the current signal will not be transmitted and the last released signal will not change.

Defining the difference between \( y(k) \) and \( y(i_k) \) as follows:

\[ \delta(k) = y(k) - y(i_k). \]

For sake of economizing limited resources in information communication, the signals will be released only when the current measurement \( y(k) \) satisfies the following condition:

\[ \delta^T(k) Q \delta(k) \geq \varepsilon y^T(i_k) Q y(i_k). \]

2.3. Fault detection observer. For the sake of improving the flexibility of observer design, the premise variables of observer model are not the same as the system model. Meanwhile, we construct the following fuzzy FD observer with saturation function to avoid false alarms which brought by outliers.

\[ \textbf{Observer Rule j:} \text{ IF } \eta_1(\hat{x}(k)) \text{ is } \mathcal{O}_{j1}, \text{ and } \eta_2(\hat{x}(k)) \text{ is } \mathcal{O}_{j2}, \text{ and } \ldots, \text{ and } \eta_{\nu}(\hat{x}(k)) \text{ is } \mathcal{O}_{j\nu}, \text{ THEN} \]

\[ \begin{cases} \dot{x}(k + 1) = A_j \hat{x}(k) + L_j \text{sat}_\sigma(y(i_k) -  \hat{y}(k)), \\ \hat{y}(k) = C \hat{x}(k), \\ r(k) = \text{sat}_\sigma(y(i_k) - \hat{y}(k)), \end{cases} \]

where \( \mathcal{O}_{j\nu} \) means the fuzzy set of the designed observer, \( \hat{x}(k) \in \mathbb{R}^n \) is the estimation of the \( x(k) \), \( \hat{y}(k) \in \mathbb{R}^m \) represents the estimation of the output, \( r(k) \in \mathbb{R}^m \) stands
for the residual signal, $\mathcal{L}_j$ ($j = 1, \ldots, r$) refer to the gains of FD observer to be determined.

Define

$$
\begin{align*}
& w_j(\hat{x}(k)) = \prod_{q=1}^{\nu} \tilde{u}_{\sigma_q}(\eta_q(\hat{x}(k))) \geq 0, \\
& \bar{w}_j(\hat{x}(k)) = \prod_{q=1}^{\nu} \tilde{u}_{\sigma_q}(\eta_q(\hat{x}(k))) \geq 0, \\
& \tilde{u}_{\sigma_q}(\eta_q(\hat{x}(k))) \geq \bar{u}_{\sigma_q}(\eta_q(\hat{x}(k))) \geq 0,
\end{align*}
$$

in which $w_j(\hat{x}(k))$ and $\bar{w}_j(\hat{x}(k))$ stand for lower grade of membership and upper grade of membership, respectively. The overall FD observer is depicted as follows:

$$
\begin{align*}
\dot{x}(k + 1) &= \sum_{j=1}^{r} w_j(\hat{x}(k))[A_j\hat{x}(k) + \mathcal{L}_j s_{\sigma}(y(i_k) - \hat{y}(k))], \\
\dot{\hat{y}}(k) &= C\hat{x}(k), \\
r(k) &= s_{\sigma}(y(i_k) - \hat{y}(k)), \\
\end{align*}
$$

where

$$
\begin{align*}
& w_j(\hat{x}(k)) = \frac{b_j(\hat{x}(k))w_j(\hat{x}(k)) + \bar{b}_j(\hat{x}(k))\bar{w}_j(\hat{x}(k))}{\sum_{i=1}^{r} [b_i(\hat{x}(k))w_i(\hat{x}(k)) + \bar{b}_i(\hat{x}(k))\bar{w}_i(\hat{x}(k))]}, \\
& 0 \leq b_j(\hat{x}(k)) \leq 1, \quad 0 \leq \bar{b}_j(\hat{x}(k)) \leq 1, \\
& w_j(\hat{x}(k)) \geq 0, \\
& \sum_{i=1}^{r} w_j(\hat{x}(k)) = 1, \quad b_j(\hat{x}(k)) + \bar{b}_j(\hat{x}(k)) = 1,
\end{align*}
$$

in which $w_j(\hat{x}(k))$ denotes the grade of membership, $b_j(\hat{x}(k))$ and $\bar{b}_j(\hat{x}(k))$ represent weighting coefficients. For conciseness, define $m_i = m_i(\eta(k))$, $w_j = w_j(\hat{x}(k))$.

Let $\delta(k) = [\delta_1(k) \ldots \delta_m(k)]^T$. In the observer (5), $s_{\sigma}(v)$ is a symmetric vector saturation function, for each $v = [v_1 \ v_2 \ \ldots \ v_m]^T \in \mathbb{R}^m \geq 0$, the saturation restrictions $\sigma = [\sigma_1 \ \sigma_2 \ \ldots \ \sigma_m]^T \in \mathbb{R}^m \geq 0$ are defined as follows:

$$
\begin{align*}
\sigma = \left[ \begin{array}{c}
\sigma_{\sigma_1}(v_1) \\
\sigma_{\sigma_2}(v_2) \\
\vdots \\
\sigma_{\sigma_m}(v_m)
\end{array} \right],
\end{align*}
$$

where $\sigma_{\sigma_l}(v_l) = \max\{-\sigma_l(k), \min\{\sigma_l(k), v_l + \delta_l(k)\}\}$ ($l = 1, 2, \ldots, m$) denotes the standard scalar symmetric saturation function. The saturation limitation dynamic of $\sigma(k)$ is shown as

$$
\bar{\sigma}(k + 1) = \lambda \bar{\sigma}(k) + (y(i_k) - \hat{y}(k))^T \mathcal{R}(y(i_k) - \hat{y}(k)), \\
\sigma_l(k) = \sqrt{\bar{\sigma}(k)/\omega_l}, \quad l = 1, 2, \ldots, m,
$$

where $\lambda \in [0, 1]$, $\mathcal{R} = \mathcal{R}^T > 0$, $\bar{\sigma}(k) \in \mathbb{R} \geq 0$ and $\omega_l > 0$ ($l = 1, 2, \ldots, m$) are the parameters with suitable dimensions, and $\omega_l > 0$ is the $l$-th diagonal element of $\mathcal{W}$, where $\mathcal{R}$ and $\mathcal{W}$ are matrices to be determined. Furthermore, the expression (7) is always established based on $\bar{\sigma}(k) \in \mathbb{R} \geq 0$. 
Remark 1. For sake of detecting the system faults, a fuzzy FD observer with saturation function (5) is constructed. In the procedure of signal transmission, it is inevitable that the existence of outliers may effect the error dynamic and residual output directly. Accordingly, the adaptive saturation which exists in (5) can avoid the occurrence of this phenomenon. From these, the FD observer with this form can reduce the influence of outliers on the system performance, while false alarms which exist in the FD procedure can be avoided effectively.

Let $\tilde{x}(k) = x(k) - \hat{x}(k), e(k) = [x^T(k) \hat{x}^T(k)]^T$, and the fault estimation error $\bar{r}(k) = r(k) - Mf(k)$, in which $M = [I \ldots I]_{m \times p}$. Then, the error system is inferred below

$$e(k + 1) = \sum_{i=1}^{r} \sum_{j=1}^{r} m_{ij} w_{ij} [\bar{A}_{ij} e(k) + \bar{B}_{ij} w(k) + \bar{B}_{1ij} f(k) + \bar{D}_{ij} q(k)]$$

$$\bar{r}(k) = \bar{C} e(k) - q(k) - Mf(k),$$

$$\bar{\sigma}(k + 1) = \lambda \bar{\sigma}(k) - [\bar{D} \bar{\sigma}(k) [\bar{C} e(k) - \delta(k)],$$

where

$$\bar{A}_{ij} = \begin{bmatrix} A_i & 0 \\ A_i - A_j & A_j - L_j C \end{bmatrix}, \bar{B}_{ij} = \begin{bmatrix} B_{ij} \\ B_{ij} \end{bmatrix}, \bar{C} = \begin{bmatrix} 0 & C \end{bmatrix}.$$ 

The residual evaluation function $J(k)$ and the threshold $J_{th}$ are evaluated by the following equations:

$$J(k) = \left( \sum_{k=t_0}^{t_0+T} \bar{r}^T(k) \bar{r}(k) \right)^{\frac{1}{2}},$$

$$J_{th} = \sup_{0 \neq w(k) \in L_2, f(k) = 0} J(k).$$

The following strategy can detect the occurrence of faults:

$$J(k) > J_{th} \Rightarrow \text{with faults} \Rightarrow \text{alarm},$$

$$J(k) \leq J_{th} \Rightarrow \text{no faults}.$$

In addition, the FD observer which is designed in this paper satisfies the $H_\infty$ performance and the $H_\infty$ fault performance in different conditions, respectively.

- $H_\infty$ performance: the system (8) satisfies the $H_\infty$ performance if the condition (11) holds.

$$\|\bar{r}(k)\|_2 < \gamma \|w(k)\|_2.$$  

- $H_\infty$ fault performance: if the system (8) satisfies the condition (12), the $H_\infty$ fault performance is guaranteed.

$$\|\bar{r}(k)\|_2 < \gamma \|f(k)\|_2.$$
3. Main results.

3.1. $\mathcal{H}_\infty$ performance analysis. In this subsection, sufficient criteria are given for the system (8) with $f(k) = 0$ to guarantee the $\mathcal{H}_\infty$ performance. Rewriting the system (8) as follows:

$$e(k + 1) = \sum_{i=1}^{r} \sum_{j=1}^{r} m_{ij} w_{ij}[\bar{A}_{ij} e(k) + \bar{B}_{ij} w(k) + \bar{D}_{ij} q(k)],$$

$$\bar{r}(k) = \bar{C} e(k) - q(k),$$

$$\bar{\sigma}(k + 1) = \lambda \bar{\sigma}(k) + [\bar{C} e(k) - \delta(k)]^T R[\bar{C} e(k) - \delta(k)],$$

$$q(k) = d z_{\sigma(k)}(\bar{C} e(k))$$

$$= \bar{C} e(k) - \text{sat}_{\sigma(k)}(\bar{C} e(k)) - \delta(k).$$

(13)

Theorem 3.1. Given FD observer gains $\mathcal{L}_j$ ($j = 1, \ldots, r$), the constants $\lambda \in [0, 1)$, $\gamma > 0$, $\varepsilon \in [0, 1)$, and considering $w_{ij} - \mu_j m_{ij} \geq 0$, the $\mathcal{H}_\infty$ performance can be guaranteed for the system (13) if there exist matrices $P > 0$, $W > 0$, $U > 0$, $R > 0$, $Q > 0$ make the following conditions hold:

$$\begin{bmatrix} -P & \mathcal{J}_i^T \\ \star & -\omega_i \end{bmatrix} < 0,$$

(14)

$$F_{ij} - \Gamma < 0,$$

(15)

$$\Phi_{ij} - \Lambda < 0,$$

(16)

$$\mu_i \bar{\Phi}_{ii} - \mu_i \Gamma + \Gamma < 0,$$

(17)

$$\rho_i \bar{\Phi}_{ii} - \rho_i \Lambda + \Lambda < 0,$$

(18)

$$\mu_j \bar{\Phi}_{ij} - \mu_j \Gamma + \mu_j \Gamma + 2\Gamma < 0,$$

(19)

$$\rho_j \bar{\Phi}_{ij} - \rho_j \Lambda + \rho_j \Phi_{ji} - \rho_i \Lambda + 2\Lambda < 0,$$

(20)

where

$$\bar{F}_{ij} = F_{ij} + F'_{ij}, \bar{\Phi}_{ij} = \Phi_{ij} + \Phi''_{ij},$$

$$F_{ij} = \begin{bmatrix} F_{1ij} & F_{2ij} \\ \star & F_{3ij} \end{bmatrix},$$

$$F_{1ij} = A_{ij}^T P A_{ij} + \bar{C}^T \bar{R} + \varepsilon \bar{C}^T Q A_{ij}^T P \bar{B}_{ij},$$

$$F_{2ij} = A_{ij}^T P D_{ij} - \bar{C}^T F_{3ij} = \bar{D}_{ij}^T P D_{ij} + I,$$

$$F_{3ij} = \bar{B}_{ij}^T P \bar{B}_{ij} - \gamma^2 I, \bar{C} = [ C \ 0 ],$$

$$\bar{F}''_{ij} = \begin{bmatrix} 0 & \bar{C}^T W^T + H^T W^T \\ \star & -W - W^T \end{bmatrix},$$

$$\bar{\Phi}''_{ij} = \begin{bmatrix} 0 & \bar{C}^T U^T \\ \star & -U - U^T \end{bmatrix}. $$
Proof. Consider the Lyapunov function as the following form:

\[ V(e(k), \dot{\sigma}(k)) = V_1(e(k), \dot{\sigma}(k)) + \eta V_2(e(k), \dot{\sigma}(k)) \tag{21} \]

where

\[ V_1(e(k), \dot{\sigma}(k)) = e^T(k) P e(k) + \dot{\sigma}(k), \]

\[ V_2(e(k), \dot{\sigma}(k)) = \max \{ e^T(k) P e(k) - \dot{\sigma}(k), 0 \}. \]

Due to that the value of \( V_2(e(k), \dot{\sigma}(k)) \) is uncertain, two cases are taken into account below.

**Case 1:** \( e^T(k) P e(k) < \dot{\sigma}(k) \)

Firstly, define \( \mathcal{H} = W^{-1} \mathcal{V} \). Then, utilizing the Schur complement and multiplying \( e^T(k) \) and its transposition in (14), then, we can obtain that \( \varpi e^T(k) \mathcal{H} e(k) - e^T(k) P e(k) \leq 0 \). Under the condition of \( e^T(k) P e(k) < \dot{\sigma}(k) \), we can get

\[ \varpi \| \mathcal{H} e(k) \|^2 \leq e^T(k) P e(k) < \dot{\sigma}(k) = \varpi \dot{\sigma}_1^2(k). \]

By the saturation function \( sat_{\sigma_i}(\mathcal{H} e(k) - \delta_i(k)) = \mathcal{H} e(k) \), we can known that \( dz_{\sigma_i}(\mathcal{H} e(k)) = 0 \) for all \( i \in \{1, 2, \ldots, m\} \), which implies that \( dz_{\sigma}(\mathcal{H} e(k)) = 0 \). Based on [1], the following sector condition can be inferred:

\[ q^T(k) \mathcal{V}(\dot{C} e(k) + \mathcal{H} e(k) - q(k)) \geq 0. \tag{22} \]

Define

\[ J_1 = \Delta \mathcal{V}(e(k), \dot{\sigma}(k)) + \dot{r}(k) \dot{r}(k) - \gamma^2 w^T(k) w(k), \]

it follows that

\[
J_1 = \Delta V_1(e(k), \dot{\sigma}(k)) + \dot{r}(k) \dot{r}(k) - \gamma^2 w^T(k) w(k) \\
\leq \sum_{i=1}^r \sum_{j=1}^r m_i w_j [\bar{A}_{ij} e(k) + \bar{B}_{ij} w(k) + \bar{D}_{ij} q(k)]^T P \\
\times [\bar{A}_{ij} e(k) + \bar{B}_{ij} w(k) + \bar{D}_{ij} q(k)] - e^T(k) P e(k) \\
\times P e(k) + [\bar{C} e(k) - \delta(k)]^T R [\bar{C} e(k) - \delta(k)] \\
+ (\lambda - 1) \dot{\sigma}(k) + [\bar{C} e(k) - q(k)]^T [\bar{C} e(k) - q(k)] \\
- \gamma^2 w^T(k) w(k) \\
\leq \sum_{i=1}^r \sum_{j=1}^r m_i w_j \xi^T(k) \xi(k) + (\lambda - 1) \dot{\sigma}(k),
\]

\[ (\lambda - 1) \dot{\sigma}(k). \]
where $\xi^T(k) = [e^T(k) \; q^T(k) \; \delta^T(k) \; w^T(k)]$. Owing to $2q^T(k)\mathcal{W}(\hat{c}e(k) + \mathcal{H}e(k) - q(k)) \geq 0$, we have

$$J_1 = \Delta V_1(e(k), \bar{\sigma}(k)) + \dot{\xi}^T(k)\dot{\xi}(k) - \gamma^2 w^T(k)w(k)$$

$$\leq \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \xi^T(k)\mathcal{F}_{ij} \xi(k) + (\lambda - 1)\bar{\sigma}(k).$$

For reducing the conservatism, the following slack matrix is introduced:

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i (m_j - w_j)\Gamma = 0.$$

Thereby, the following expression is obtained:

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \mathcal{F}_{ij}$$

$$= \sum_{i=1}^{r} m_i^2 (\mu_i \mathcal{F}_{ii} - \mu_i \Gamma + \Gamma) + \sum_{i=1}^{r-1} \sum_{j=i}^{r} m_i m_j (\mu_j \mathcal{F}_{ij})$$

$$- \mu_j \Gamma + \mu_j \mathcal{F}_{ji} - \mu_i \Gamma + 2\Gamma) + \sum_{i=1}^{r} \sum_{j=1}^{r} m_i (w_j)$$

$$- \mu_j m_j) (\mathcal{F}_{ij} - \Gamma).$$

Based on (15), (17) and (19), we can obtain that $\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \mathcal{F}_{ij} < 0$. As stated in [1], there must be a small enough scalar $\epsilon > 0$ such that the following inequality holds:

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \mathcal{F}_{ij} < -2\epsilon \mathcal{I}. \quad (23)$$

Multiplying $\xi^T(k)$ and its transposition in left and right sides to (23), one gets

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \xi^T(k)\mathcal{F}_{ij} \xi(k)$$

$$< -2\epsilon |e(k)|^2 - 2\epsilon |q(k)|^2 - 2\epsilon |\delta(k)|^2 - 2\epsilon |w(k)|^2.$$

As a result, the following inequality can be derived:

$$J_1 \leq -2\epsilon |e(k)|^2 - 2\epsilon |q(k)|^2 - 2\epsilon |\delta(k)|^2$$

$$- 2\epsilon |w(k)|^2 + (\lambda - 1)\bar{\sigma}(k) < 0.$$

Thereby, one can get $\|\dot{\xi}(k)\|_2 < \gamma^2 \|w(k)\|_2$, it means that the $\mathcal{H}_\infty$ performance is guaranteed for the system (13).

**Case 2:** $e^T(k)\mathcal{P}e(k) > \bar{\sigma}(k)$

In this case, the Lyapunov function becomes the following form:

$$\mathcal{V}(e(k), \bar{\sigma}(k)) = e^T(k)\mathcal{P}e(k) + \bar{\sigma}(k)$$

$$+ \eta(e^T(k)\mathcal{P}e(k) - \bar{\sigma}(k)).$$
Define
\[ J_2 = \Delta V_2(e(k), \bar{\sigma}(k)) + \bar{r}^T(k) \bar{r}(k) - \gamma^2 w^T(k)w(k) \]
\[
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \left[ \bar{A}_{ij} e(k) + \bar{B}_{ij} w(k) + \bar{D}_{ij} q(k) \right]^T \mathcal{P} \\
\times \left[ \bar{A}_{ij} e(k) + \bar{B}_{ij} w(k) + \bar{D}_{ij} q(k) \right] - e^T(k) \mathcal{P} e(k) \\
- \gamma^2 w^T(k)w(k) \\
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \xi^T(k) \bar{\Phi}_{ij} \xi(k).
\]

Since \( dz_\sigma(0) = 0 \), we reuse (22) with \( \mathcal{W} = \mathcal{U} \) and \( \mathcal{H} = 0 \), then, the following condition can be derived:
\[ J_2 \leq \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \xi^T(k) \bar{\Phi}_{ij} \xi(k). \]

In the same way, a slack matrix is also utilized in this case
\[ \sum_{i=1}^{r} \sum_{j=1}^{r} m_i (m_j - w_j) \Lambda = 0. \]

Then, one gets
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \bar{\Phi}_{ij} \\
= \sum_{i=1}^{r} m_i^2 (\rho_i \bar{\Phi}_{ij} - \rho_i \Lambda + \Lambda) + \sum_{i=1}^{r} \sum_{i<j} m_i w_j (\rho_j \bar{\Phi}_{ij} - \rho_j \Lambda + 2 \Lambda) + \sum_{i=1}^{r} \sum_{j=1}^{r} m_i (w_j - \mu_j m_j) (\bar{\Phi}_{ij} - \Lambda).
\]

Based on (16), (18) and (20), one has
\[ \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \bar{\Phi}_{ij} < -2 \epsilon \mathcal{I}. \tag{24} \]

Multiplying \( \xi^T(k) \) and its transposition in left and right sides to (24), one gets
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \xi^T(k) \bar{\Phi}_{ij} \xi(k) \\
< -2 \epsilon |e(k)|^2 - 2 \epsilon |q(k)|^2 - 2 \epsilon |\delta(k)|^2 - 2 \epsilon |w(k)|^2.
\]

Hence, the following inequality is derived:
\[ J_2 \leq -2 \epsilon |e(k)|^2 - 2 \epsilon |q(k)|^2 - 2 \epsilon |\delta(k)|^2 - 2 \epsilon |w(k)|^2. \]
Defining $\eta = \frac{1}{2\pi}$, one can have $J = J_1 + \frac{1}{2\pi} J_2$, and

$$J = \Delta V(k) + 2\gamma^T(k)\tilde{r}(k) - 2\gamma^Tw^T(k)w(k) \leq -\varepsilon |e(k)|^2 - \varepsilon |q(k)|^2 - \varepsilon |\delta(k)|^2 - \varepsilon |w(k)|^2 + (\lambda - 1)\sigma(k).$$

It can be obtained that $J < 0$, which refers to that the system (13) satisfies the $\mathcal{H}_\infty$ performance. In addition, according to Theorem 1, we can get that $\Delta V(k) < 0$ with $w(k) = 0$, which implies that the system (13) is exponentially stable. \hfill $\square$

**Remark 2.** The conditions of observer design cannot be derived directly by the existing result [33] when the mismatched membership functions of the error system are introduced. A slack matrix, which has the membership functions information of both the system model and observer, is introduced in the design procedure to reduce the conservativeness.

### 3.2. $\mathcal{H}_\infty$ fault performance.

In this subsection, sufficient conditions are provided to ensure the $\mathcal{H}_\infty$ fault performance for the system (8). With the influence of the fault, the system is rewritten as follows:

$$e(k + 1) = \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j [\bar{A}_{ij} e(k) + \bar{B}_{1ij} f(k) + \bar{D}_{ij} q(k)],$$

$$\bar{r}(k) = \bar{C}_{e}(k) - q(k) - M f(k),$$

$$\bar{\sigma}(k + 1) = \lambda \bar{\sigma}(k) + [\bar{C}_{e}(k) - \delta(k)]^T \bar{R} [\bar{C}_{e}(k) - \delta(k)],$$

$$q(k) = dz_{\bar{\sigma}}(k) (\bar{C}_{e}(k))$$

$$= \bar{C}_{e}(k) - \text{sat}_{\bar{\sigma}(k)}(\bar{C}_{e}(k) - \delta(k)).$$

**Theorem 3.2.** Given the constants $\lambda \in [0,1]$, $\varepsilon \in [0,1]$, $\gamma > 0$, and observer gains $L_j$, under $w_j - \bar{\mu}_j m_j \geq 0$ if there exist matrices $P > 0$, $W > 0$, $U > 0$, $\mathcal{R} > 0$, $Q > 0$, $M > 0$ such that the following conditions are satisfied:

$$\begin{bmatrix} -P & V_i^T \\ * & -\varepsilon \omega_i \end{bmatrix} < 0, \quad (26)$$

$$\bar{\Psi}_{ij} - \bar{\Gamma} < 0, \quad (27)$$

$$\bar{\Omega}_{ij} - \bar{\Lambda} < 0, \quad (28)$$

$$\bar{\mu}_i \bar{\Psi}_{ii} - \bar{\mu}_i \bar{\Gamma} + \bar{\Gamma} < 0, \quad (29)$$

$$\bar{\rho}_j \bar{\Omega}_{ii} - \bar{\rho}_j \bar{\Lambda} + \bar{\Lambda} < 0, \quad (30)$$

$$\bar{\mu}_j \bar{\Psi}_{jj} - \bar{\mu}_j \bar{\Gamma} + \bar{\mu}_i \bar{\Psi}_{ji} - \bar{\mu}_i \bar{\Gamma} + 2 \bar{\Gamma} < 0, \quad (31)$$

$$\bar{\rho}_j \bar{\Omega}_{ij} - \bar{\rho}_j \bar{\Lambda} + \bar{\rho}_i \bar{\Omega}_{ji} - \bar{\rho}_i \bar{\Lambda} + 2 \bar{\Lambda} < 0, \quad (32)$$

where

$$\bar{\Psi}_{ij} = \Psi_{ij} + \Theta, \quad \bar{\Omega}_{ij} = \Omega_{ij} + \Xi,$$

$$\Psi_{ij} = \begin{bmatrix} \Psi_{11ij} & \Psi_{21ij} & -\bar{C}^T \mathcal{R} - \varepsilon \bar{C}^T Q & \Psi_{41ij} \\ * & \Psi_{31ij} & 0 & \Psi_{51ij} \\ * & * & \mathcal{R} + (\varepsilon - 1) Q & 0 \\ * & * & * & \Psi_{61ij} \end{bmatrix},$$

$$\Psi_{11ij} = A_{11}^T P \bar{A}_{ij} + \bar{C}^T \mathcal{R} \bar{C} + \bar{C}^T \bar{C} + \varepsilon \bar{C}^T Q \bar{C} - P,$$
Consider the following Lyapunov function:

\[ V_k = \Psi_{2ij} \bar{P} \bar{D}_{ij} - \bar{C}^T \bar{V}_k, \]  
\[ \forall k \in \mathbb{N}, \]  
\[ \text{s.t.} \quad \Psi_{3ij} = \bar{D}_{ij} \bar{P} \bar{D}_{ij} + I, \]

where \( \Psi_{6ij} = \bar{B}_{1ij} \bar{P} \bar{B}_{1ij} - \gamma^2 I + M^T M, \)

\[
\begin{bmatrix}
0 & \bar{C}^T \bar{W}^T + H^T \bar{V}_k & 0 & 0 \\
* & -W - W^T & 0 & 0 \\
* & * & 0 & 0 \\
* & * & * & 0 
\end{bmatrix},
\]

\[
\Theta = \begin{bmatrix}
\Omega_{1ij} & \Omega_{2ij} & -\bar{C}^T \bar{R} & \bar{A}_j^T \bar{P} \bar{B}_{1ij} - \bar{C}^T M \\
* & \Omega_{3ij} & 0 & \bar{D}^T \bar{P} \bar{B}_{1ij} + M \\
* & * & -R & 0 \\
* & * & * & \Omega_{4ij}
\end{bmatrix},
\]

\[
\Omega_{1ij} = \bar{A}_{ij}^T \bar{P} \bar{A}_{ij} - \bar{C}^T \bar{R} \bar{C} + \bar{C}^T \bar{C} - \lambda \bar{P},
\]

\[
\Omega_{2ij} = \bar{A}_{ij}^T \bar{P} \bar{D}_{ij} - \bar{C}^T \frac{1}{e} \Omega_{3ij} = \bar{D}_{ij} \bar{P} \bar{D}_{ij} + I,
\]

\[
\Omega_{4ij} = \bar{B}_{1ij} \bar{P} \bar{B}_{1ij} - \gamma^2 I + M^T M,
\]

\[
\Xi = \begin{bmatrix}
0 & \bar{C}^T \bar{U}^T & 0 & 0 \\
* & -U - U^T & 0 & 0 \\
* & * & 0 & 0 \\
* & * & * & 0
\end{bmatrix},
\]

then, the \( H_\infty \) fault performance can be ensured for the system (25).

**Proof.** Consider the following Lyapunov function:

\[
V(e(k), \bar{\sigma}(k)) = V_1(e(k), \bar{\sigma}(k)) + \eta V_2(e(k), \bar{\sigma}(k))
\]

where

\[
V_1(e(k), \bar{\sigma}(k)) = e^T(k) \bar{P} e(k) + \bar{\sigma}(k),
\]

\[
V_2(e(k), \bar{\sigma}(k)) = \max\{e^T(k) \bar{P} e(k) - \bar{\sigma}(k), 0\}.
\]

Similarity, two cases are also considered.

**Case 1:** \( e^T(k) \bar{P} e(k) < \bar{\sigma}(k) \)

Define

\[
J^f_k = \Delta V_1(e(k), \bar{\sigma}(k)) + \bar{\tau}^T(k) \bar{\tau}(k) - \gamma^2 f^T(k) f(k)
\]

\[
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j [\bar{A}_{ij} e(k) + \bar{B}_{1ij} f(k) + \bar{D}_{ij} q(k)] T \bar{P}
\]

\[
\times [\bar{A}_{ij} e(k) + \bar{B}_{1ij} f(k) + \bar{D}_{ij} q(k)] - e^T(k) \bar{P} e(k) + [\bar{C} e(k) - \delta(k)] T \bar{R} [\bar{C} e(k) - \delta(k)]
\]

\[
+ (\lambda - 1) \bar{\sigma}(k) + [\bar{C} e(k) - q(k) - M f(k)] T + [\bar{C} e(k) - q(k) - M f(k)] - \gamma^2 f^T(k) f(k)
\]

\[
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \xi^T(k) \bar{\xi}(k) + (\lambda - 1) \bar{\sigma}(k),
\]  

(34)
where $\xi^T(k) = [e^T(k) \ q^T(k) \ \delta^T(k) \ f^T(k)]$. Based on $2q^T(k)\mathcal{W}(\bar{c}e(k) + \mathcal{H}e(k) - q(k)) \geq 0$, which has been validated in Theorem 1, we know that

$$J_1^f = \Delta \mathcal{V}_1(e(k), \bar{\sigma}(k)) + \bar{r}^T(k) \bar{r}(k) - \gamma^2 f^T(k)f(k)$$

$$\leq \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \xi^T(k) \tilde{\Psi}_{ij} \xi(k) + (\lambda - 1) \bar{\sigma}(k).$$

Similar to Theorem 1, the slack matrix is introduced

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i \bar{w}_j = 0.$$

Then, we can obtain

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \tilde{\Psi}_{ij} = \sum_{i=1}^{r} m_i^2 (\bar{\mu}_i \bar{\Psi}_{ii} - \bar{\mu}_i \bar{\Gamma}) + \sum_{i=1}^{r} \sum_{i<j} m_i m_j (\bar{\mu}_j \bar{\Psi}_{ij}$$

$$- \bar{\mu}_j \bar{\Gamma} + \bar{\mu}_i \bar{\Psi}_{ji} - \bar{\mu}_i \bar{\Gamma} + 2\bar{\Gamma}) + \sum_{i=1}^{r} \sum_{j=1}^{r} m_i (w_j$$

$$- \mu_j m_j)(\bar{\Psi}_{ij} - \bar{\Gamma}).$$

Thus, we can get that $\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \tilde{\Psi}_{ij} < 0$ under the conditions of (27), (29) and (31), thus the following inequality holds:

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \tilde{\Psi}_{ij} < -2\epsilon \mathcal{I}. \quad (35)$$

Multiplying $\xi^T(k)$ and its transposition in left and right sides to (35), one has

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \xi^T(k) \tilde{\Psi}_{ij} \xi(k)$$

$$< -2\epsilon |e(k)|^2 - 2\epsilon |q(k)|^2 - 2\epsilon |\delta(k)|^2 - 2\epsilon |f(k)|^2.$$

Whereafter, by calculating (34), one can obtain

$$J_1^f \leq -2\epsilon |e(k)|^2 - 2\epsilon |q(k)|^2 - 2\epsilon |\delta(k)|^2$$

$$- 2\epsilon |f(k)|^2 + (\lambda - 1) \bar{\sigma}(k) < 0.$$

Therefore, condition $\|\bar{r}(k)\|_2 < \gamma^2 \|f(k)\|_2$ is guaranteed for the system (25).

**Case 2:** $e^T(k)\mathcal{P}e(k) > \bar{\sigma}(k)$

In this case, the Lyapunov function becomes the following form:

$$\mathcal{V}(e(k), \bar{\sigma}(k)) = e^T(k)\mathcal{P}e(k) + \bar{\sigma}(k)$$

$$+ \eta(e^T(k)\mathcal{P}e(k) - \bar{\sigma}(k)).$$

Define

$$J_2^f = \Delta \mathcal{V}_2(e(k), \bar{\sigma}(k)) + \bar{r}^T(k) \bar{r}(k) - \gamma^2 f^T(k)f(k)$$

$$\leq \sum_{i=1}^{r} \sum_{j=1}^{r} m_i \bar{w}_j \xi^T(k) \Omega_{ij} \xi(k). \quad (36)$$
Substituting condition (22) with $W = U$ and $H = 0$ into (36) and introducing a slack matrix, one has

$$J_f^2 \leq \sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \Omega_{ij}$$

$$= \sum_{i=1}^{r} m_i^2 (\overline{\rho_i} \overline{\Omega_{ii}} - \overline{\rho_i} \overline{\Lambda} + \overline{\Lambda}) + \sum_{i=1}^{r-1} \sum_{j<i}^{r} m_i m_j (\overline{\rho_j} \overline{\Omega_{ij}} - \overline{\rho_j} \overline{\Lambda} + \overline{\rho_i} \overline{\Lambda} - \overline{\Lambda}) + \sum_{i=1}^{r} m_i (w_j - \mu_j m_j)(\overline{\Omega_{ij}} - \overline{\Lambda}).$$

On the basis of (28), (30) and (32), \(\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \Omega_{ij} < 0\) is derived. The following inequality holds if there exists a small enough \(\epsilon > 0\):

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \Omega_{ij} < -2\epsilon I. \quad (37)$$

Multiplying \(\xi^T(k)\) and its transposition in left and right sides to (37), one can get

$$\sum_{i=1}^{r} \sum_{j=1}^{r} m_i w_j \xi^T(k) \overline{\Omega_{ij}} \xi(k) < -2\epsilon |e(k)|^2 - 2\epsilon |q(k)|^2 - 2\epsilon |\delta(k)|^2 - 2\epsilon |f(k)|^2.$$

Accordingly, \(J_f^2\) is calculated as follows:

$$J_f^2 \leq -2\epsilon |e(k)|^2 - 2\epsilon |q(k)|^2 - 2\epsilon |\delta(k)|^2 - 2\epsilon |f(k)|^2.$$

Selecting \(\eta = \frac{1}{2\epsilon}\), one has \(J_f = J_1^f + \frac{1}{2\epsilon} J_2^f\), and

$$J_f \leq -\epsilon |e(k)|^2 - \epsilon |q(k)|^2 - \epsilon |\delta(k)|^2 - \epsilon |f(k)|^2 + (\lambda - 1)\overline{\delta}(k).$$

Hence, the system (25) satisfies the $H_\infty$ fault performance. Moreover, when \(f(k) = 0\), it can be derived that \(\Delta V(k) < 0\). \(\Box\)

**Remark 3.** In contrast with the existing results [1] and [33], an event-based mechanism is introduced in this paper to decrease unnecessary signal communications. Besides, the adaptive saturation function which exists in above results cannot effectively deal with triggered signals. In this paper, we design a new adaptive saturation function to cope with this problem.

### 3.3. FD observer design

The method of the FD observer design is given in this subsection.

**Theorem 3.3.** For known scalars \(\lambda \in [0, 1), \epsilon \in [0, 1)\) and \(\gamma > 0\), the $H_\infty$ performance and $H_\infty$ fault performance are guaranteed if there exist matrices \(P, Q, W, U, R, X, S > 0\), symmetric matrices \(\Gamma, \Lambda, \overline{\Gamma}, \overline{\Lambda}\) and FD observer gains \(L_j\),
j = 1, \ldots, r \text{ such that the following conditions are satisfied:}

$$
\begin{align*}
\begin{bmatrix}
\Delta_{ii} & \Upsilon_{ii} \\
\ast & f_{ii}
\end{bmatrix} & < 0, \\
\begin{bmatrix}
\Delta_{ij} & \Upsilon_{ij} \\
\ast & f_{ij}
\end{bmatrix} & < 0,
\end{align*}
$$

(38)

$$
\begin{align*}
\begin{bmatrix}
\tilde{\Delta}_{ii} & \tilde{\Upsilon}_{ii} \\
\ast & \tilde{f}_{ii}
\end{bmatrix} & < 0, \\
\begin{bmatrix}
\tilde{\Delta}_{ij} & \tilde{\Upsilon}_{ij} \\
\ast & \tilde{f}_{ij}
\end{bmatrix} & < 0,
\end{align*}
$$

(41)

$$
\begin{align*}
\begin{bmatrix}
\tilde{\Delta}_{ij} & \tilde{\Upsilon}_{ij} \\
\ast & \tilde{f}_{ij}
\end{bmatrix} +
\begin{bmatrix}
\tilde{\Delta}_{ji} & \tilde{\Upsilon}_{ji} \\
\ast & \tilde{f}_{ji}
\end{bmatrix} & < 0,
\end{align*}
$$

(42)

where

$$
\begin{align*}
\Delta_{ij} & = 
\begin{bmatrix}
\Delta_{11} & \Delta_{12} \\
\ast & \Delta_{13}
\end{bmatrix},
\Delta_{11} & = 
\begin{bmatrix}
\Delta_{111} & \Delta_{112} \\
\ast & \Delta_{113}
\end{bmatrix},
\end{align*}
$$

\begin{align*}
\Delta_{111} & = -\mu_i \mathcal{P}_1 - \mu_i \Gamma_{111} + \Gamma_{111}, \\
\Delta_{112} & = -\mu_i \mathcal{P}_2 - \mu_i \Gamma_{112} + \Gamma_{112}, \\
\Delta_{113} & = -\mu_i \mathcal{P}_3 - \mu_i \Gamma_{122} + \Gamma_{122} + \mu_i \mathcal{C}^T \mathcal{C} + \mu_i \mathcal{C}^T \mathcal{R} \mathcal{C}, \\
\Delta_{12} & = 
\begin{bmatrix}
\Delta_{121} & \Delta_{122} & \Delta_{123} \\
\Delta_{124} & \Delta_{125} & \Delta_{126}
\end{bmatrix},
\Delta_{121} & = \mu_i \mathcal{J}_1^T - \mu_i \Gamma_{211} + \Gamma_{211}, \Delta_{123} = -\mu_i \Gamma_{411} + \Gamma_{411}, \\
\Delta_{122} & = -\mu_i \mathcal{C}^T \mathcal{Q} - \mu_i \Gamma_{311} + \Gamma_{311}, \\
\Delta_{124} & = -\mu_i \mathcal{C}^T + \mu_i \mathcal{C}^T \mathcal{W}^T + \mu_i \mathcal{J}_2^T - \mu_i \Gamma_{221} + \Gamma_{221}, \\
\Delta_{125} & = -\mu_i \mathcal{C}^T \mathcal{R} - \mu_i \Gamma_{321} + \Gamma_{321}, \Delta_{126} = -\mu_i \Gamma_{421} + \Gamma_{421},
\end{align*}
$$
\[
\Delta_{131} = -\mu_iI - \mu_iW - \mu_iW^T - \mu_i\Gamma_5 + \Gamma_5,
\]
\[
\Delta_{132} = -\mu_i\Gamma_6 + \Gamma_6, \Delta_{133} = -\mu_i\Gamma_7 + \Gamma_7,
\]
\[
\Delta_{134} = \mu_iR + \mu_i(\varepsilon - 1)Q - \mu_i\Gamma_8 + \Gamma_8,
\]
\[
\Delta_{135} = -\mu_i\Gamma_9 + \Gamma_9, \Delta_{136} = -\mu_i\gamma^2I - \mu_i\Gamma_{10} + \Gamma_{10},
\]
\[
\eta_{ij} = \begin{bmatrix}
\frac{A_i^T}{D_{ij}^T} & \frac{A_i^T}{D_{ij}^T} & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \Delta_{13} = \begin{bmatrix}
\Delta_{131} & \Delta_{132} & \Delta_{133} \\
* & \Delta_{134} & \Delta_{135} \\
* & * & \Delta_{136}
\end{bmatrix},
\]
\[
F_{ij} = \begin{bmatrix}
-\mu_i^{-1}\chi & 0 \\
* & -\mu_i^{-1}\varepsilon^{-1}S
\end{bmatrix},
\]
\[
\hat{\Delta}_{ij} = \begin{bmatrix}
\hat{\Delta}_{11} & \hat{\Delta}_{12} \\
* & \hat{\Delta}_{13}
\end{bmatrix}, \hat{\Delta}_{12} = \begin{bmatrix}
\hat{\Delta}_{121} & \hat{\Delta}_{122} & -\Gamma_{411} \\
* & \hat{\Delta}_{123} & \hat{\Delta}_{124} & -\Gamma_{421}
\end{bmatrix},
\]
\[
\hat{\Delta}_{11} = \begin{bmatrix}
-\rho_1 - \Gamma_{111} & -\rho_2 - \Gamma_{112} \\
* & -\rho_3 + C^T C + C^T R C - \Gamma_{122}
\end{bmatrix},
\]
\[
\hat{\Delta}_{121} = \gamma_i^T - \Gamma_{211}, \hat{\Delta}_{122} = -\varepsilon C^T Q - \Gamma_{311},
\]
\[
\hat{\Delta}_{123} = -C^T + C^T W + \gamma_i^2 - \Gamma_{221}, \hat{\Delta}_{124} = -C^T R - \Gamma_{321},
\]
\[
\hat{\Delta}_{13} = \begin{bmatrix}
\hat{\Delta}_{131} & -\Gamma_6 & -\Gamma_7 \\
* & \hat{\Delta}_{132} & -\Gamma_9 \\
* & * & \hat{\Delta}_{136}
\end{bmatrix},
\]
\[
\hat{\Delta}_{131} = -I - W - W^T - \Gamma_5, \hat{\Delta}_{132} = \mathcal{R} + (\varepsilon - 1)Q - \Gamma_8,
\]
\[
\hat{f}_{ij} = \begin{bmatrix}
-\chi & 0 \\
* & -\varepsilon^{-1}S
\end{bmatrix}, \hat{\eta}_{ij} = \begin{bmatrix}
\hat{\Delta}_{11} & \hat{\Delta}_{12} \\
* & \hat{\Delta}_{13}
\end{bmatrix},
\]
\[
\hat{\Delta}_{11} = \begin{bmatrix}
\hat{\Delta}_{111} & \hat{\Delta}_{112} \\
* & \hat{\Delta}_{113}
\end{bmatrix}, \hat{\Delta}_{12} = \begin{bmatrix}
\hat{\Delta}_{121} & \hat{\Delta}_{122} & \hat{\Delta}_{123} \\
* & \hat{\Delta}_{124} & \hat{\Delta}_{125} & \hat{\Delta}_{126}
\end{bmatrix},
\]
\[
\hat{\Delta}_{111} = -\lambda_i\rho_1 P_1 - \rho_i\Lambda_{111} + \Lambda_{111}, \hat{\Delta}_{112} = -\lambda_i\rho_2 P_2 - \rho_i\Lambda_{112} + \Lambda_{112},
\]
\[
\hat{\Delta}_{113} = -\lambda_i\rho_3 + \rho_iC^T C - \rho_iC^T R C - \rho_i\Lambda_{112} + \Lambda_{112},
\]
\[
\hat{\Delta}_{121} = -\rho_i\Lambda_{211} + \Lambda_{211}, \hat{\Delta}_{122} = -\rho_i\Lambda_{311} + \Lambda_{311},
\]
\[
\hat{\Delta}_{123} = -\rho_i\Lambda_{411} + \Lambda_{411}, \hat{\Delta}_{124} = -\rho_i\Lambda_{421} + \Lambda_{421},
\]
\[
\hat{\Delta}_{124} = -\rho_iC^T + \rho_iC^T U^T - \rho_i\Lambda_{221} + \Lambda_{221}, \hat{\Delta}_{125} = -\rho_iC^T R - \rho_i\Lambda_{321} + \Lambda_{321},
\]
\[
\hat{\Delta}_{13} = \begin{bmatrix}
\hat{\Delta}_{131} & \hat{\Delta}_{132} & -\rho_i\Lambda_7 + \Lambda_7 \\
* & \hat{\Delta}_{133} & -\rho_i\Lambda_9 + \Lambda_9 \\
* & * & -\rho_i\gamma^2 I - \rho_i\Lambda_{10} + \Lambda_{10}
\end{bmatrix},
\]
\[
\hat{\Delta}_{131} = \rho_iI - \rho_iU - \rho_iU^T - \rho_i\Lambda_5 + \Lambda_5, \hat{\Delta}_{132} = -\rho_i\Lambda_6 + \Lambda_6,
\]
\[
\tilde{\eta}_{ij} = \begin{bmatrix}
\tilde{A}_{ij}^T \\
\tilde{D}_{ij}^T \\
0
\end{bmatrix}, \tilde{f}_{ij} = -\rho_i^{-1}\chi, \tilde{\eta}_{133} = -\rho_i\mathcal{R} - \rho_i\Lambda_8 + \Lambda_8,
\]
\[
\tilde{\eta}_{ij} = \begin{bmatrix}
\tilde{\Delta}_{11} & \tilde{\Delta}_{12} \\
* & \tilde{\Delta}_{13}
\end{bmatrix}, \tilde{f}_{ij} = -\chi, \tilde{\Delta}_{124} = -C^T + C^T U^T - \Lambda_{221},
\]
\[
\tilde{\Delta}_{11} = \begin{bmatrix}
-\lambda_iP_1 - \Lambda_{111} & -\lambda_iP_2 - \Lambda_{112} \\
* & \tilde{\Delta}_{111}
\end{bmatrix},
\]
\[
\tilde{\Delta}_{111} = -\lambda_i\rho_3 + C^T C - C^T R C - \Lambda_{112}, \tilde{\Delta}_{131} = I - U - U^T - \Lambda_5,
\]
\[
\begin{align*}
\hat{\Delta}_{12} &= \begin{bmatrix}
-\Lambda_{211} & -\Lambda_{311} & -\Lambda_{411} \\
-\Lambda_{124} & C^T R - \Lambda_{321} & -\Lambda_{421}
\end{bmatrix}, \\
\hat{\Delta}_{13} &= \begin{bmatrix}
\hat{\Delta}_{131} & -\Lambda_6 & -\Lambda_7 \\
* & -R - \Lambda_8 & -\Lambda_9 \\
* & * & -\gamma^2 I - \Lambda_{10}
\end{bmatrix}, \\
\mathcal{G}_{ij} &= \begin{bmatrix}
G_{11} & G_{12} \\
* & G_{13}
\end{bmatrix}, \quad \mathcal{G}_{11} = \begin{bmatrix}
G_{111} & G_{112} \\
* & G_{113}
\end{bmatrix}, \\
G_{111} &= -\hat{\mu}_i R_1 - \hat{\mu}_i \hat{\Gamma}_{111} + \hat{\Gamma}_{111}, \quad G_{112} = -\hat{\mu}_i R_2 - \hat{\mu}_i \hat{\Gamma}_{112} + \hat{\Gamma}_{112}, \\
G_{113} &= -\hat{\mu}_i R_3 + \hat{\mu}_i C^T R C + \hat{\mu}_i C^T C - \hat{\mu}_i \hat{\Gamma}_{112} + \hat{\Gamma}_{112}, \\
G_{12} &= \begin{bmatrix}
G_{121} & G_{122} \\
G_{124} & G_{125} \\
G_{126}
\end{bmatrix}, \quad G_{123} = -\hat{\mu}_i \hat{\Gamma}_{411} + \hat{\Gamma}_{411}, \\
G_{121} &= \hat{\mu}_i Y^T - \hat{\mu}_i \hat{\Gamma}_{211} + \hat{\Gamma}_{211}, \quad G_{122} = -\hat{\mu}_i C^T Q - \hat{\mu}_i \hat{\Gamma}_{311} + \hat{\Gamma}_{311}, \\
G_{124} &= -\hat{\mu}_i C^T + \hat{\mu}_i C^T W^T + \hat{\mu}_i Y^T - \hat{\mu}_i \hat{\Gamma}_{221} + \hat{\Gamma}_{221}, \\
G_{125} &= -\hat{\mu}_i C^T R - \hat{\mu}_i \hat{\Gamma}_{321} + \hat{\Gamma}_{321}, \quad G_{126} = \hat{\mu}_i C^T M - \hat{\mu}_i \hat{\Gamma}_{421} + \hat{\Gamma}_{421}, \\
G_{13} &= \begin{bmatrix}
G_{131} & G_{132} & G_{133} \\
* & G_{134} & G_{135} \\
* & * & G_{136}
\end{bmatrix}, \quad G_{132} = -\hat{\mu}_i \hat{\Gamma}_6 + \hat{\Gamma}_6, \\
G_{131} &= \hat{\mu}_i I - \hat{\mu}_i W - \hat{\mu}_i W^T - \hat{\mu}_i \hat{\Gamma}_5 + \hat{\Gamma}_5, \\
G_{133} &= \hat{\mu}_i M - \hat{\mu}_i \hat{\Gamma}_7 + \hat{\Gamma}_7, \quad G_{135} = -\hat{\mu}_i \hat{\Gamma}_9 + \hat{\Gamma}_9, \\
G_{134} &= \hat{\mu}_i R + \hat{\mu}_i (\varepsilon - 1) Q - \hat{\mu}_i \hat{\Gamma}_8 + \hat{\Gamma}_8, \\
G_{136} &= -\hat{\mu}_i \gamma^2 I + \hat{\mu}_i M^T M - \hat{\mu}_i \hat{\Gamma}_{10} + \hat{\Gamma}_{10}, \\
\dot{\mathcal{G}}_{ij} &= \begin{bmatrix}
-\hat{\mu}_i^{-1} \mathcal{X} & 0 \\
* & -\hat{\mu}_i^{-1} \varepsilon^{-1} Q^{-1}
\end{bmatrix}, \quad \dot{\mathcal{G}}_{13} = \begin{bmatrix}
\dot{\mathcal{G}}_{131} & -\dot{\Gamma}_6 & M - \dot{\Gamma}_7 \\
* & \dot{\mathcal{G}}_{132} & -\dot{\Gamma}_9 \end{bmatrix}, \\
\dot{\mathcal{G}}_{ij} &= \Bigg[ \begin{bmatrix}
\dot{G}_{11} & \dot{G}_{12} \\
* & \dot{G}_{13}
\end{bmatrix}, \quad \dot{\mathcal{G}}_{12} = \begin{bmatrix}
\dot{G}_{121} & \dot{G}_{122} & -\dot{\Gamma}_{411} \\
\dot{G}_{123} & \dot{G}_{124} & \dot{G}_{125}
\end{bmatrix}, \\
\dot{\mathcal{G}}_{11} &= \begin{bmatrix}
\dot{G}_{111} & \dot{G}_{112} \\
* & \dot{G}_{113}
\end{bmatrix}, \quad \dot{\mathcal{G}}_{12} = \begin{bmatrix}
\dot{G}_{121} & \dot{G}_{122} & \dot{G}_{123} \\
\dot{G}_{124} & \dot{G}_{125} & \dot{G}_{126}
\end{bmatrix}, \\
\dot{G}_{11} &= \begin{bmatrix}
\dot{G}_{111} & \dot{G}_{112} \\
* & \dot{G}_{113}
\end{bmatrix}, \quad \dot{G}_{12} = \begin{bmatrix}
\dot{G}_{121} & \dot{G}_{122} & \dot{G}_{123} \\
\dot{G}_{124} & \dot{G}_{125} & \dot{G}_{126}
\end{bmatrix}, \\
\dot{G}_{111} &= -\hat{\rho}_i \lambda P_1 - \hat{\rho}_i \hat{\Lambda}_{111} + \hat{\Lambda}_{111}, \quad \dot{G}_{112} = -\hat{\rho}_i \lambda P_2 - \hat{\rho}_i \hat{\Lambda}_{112} + \hat{\Lambda}_{112}, \\
\dot{G}_{113} &= -\hat{\rho}_i \lambda P_3 + \hat{\rho}_i C^T C - \hat{\rho}_i C^T R C - \hat{\rho}_i \hat{\Lambda}_{112} + \hat{\Lambda}_{112}, \\
\dot{G}_{121} &= -\hat{\rho}_i \hat{\Lambda}_{211} + \hat{\Lambda}_{211}, \quad \dot{G}_{122} = -\hat{\rho}_i \hat{\Lambda}_{311} + \hat{\Lambda}_{311}, \\
\dot{G}_{123} &= -\hat{\rho}_i \hat{\Lambda}_{411} + \hat{\Lambda}_{411}, \quad \dot{G}_{125} = \hat{\rho}_i C^T R - \hat{\rho}_i \hat{\Lambda}_{321} + \hat{\Lambda}_{321}, \\
\dot{G}_{124} &= -\hat{\rho}_i C^T + \hat{\rho}_i C^T U^T - \hat{\rho}_i \hat{\Lambda}_{221} + \hat{\Lambda}_{221}, \\
\dot{G}_{126} &= -\hat{\rho}_i C^T M - \hat{\rho}_i \hat{\Lambda}_{421} + \hat{\Lambda}_{421}, \quad \dot{G}_{134} = -\hat{\rho}_i R - \hat{\rho}_i \hat{\Lambda}_8 + \hat{\Lambda}_8,
\end{align*}
\]
\[
\begin{align*}
\hat{G}_{13} &= \begin{bmatrix} \hat{G}_{131} & \hat{G}_{132} & \hat{G}_{133} \\ \ast & \hat{G}_{134} & \hat{G}_{135} \\ \ast & \ast & \hat{G}_{136} \end{bmatrix}, \quad \hat{v}_{ij} = -\hat{p}_i \lambda', \quad \hat{G}_{132} = -\hat{p}_i \hat{\lambda}_6 + \hat{\lambda}_6, \\
\hat{G}_{131} &= \hat{\rho}_i \mathcal{I} - \hat{\rho}_i \mathcal{U} - \hat{\rho}_i \mathcal{U}^T - \hat{\rho}_i \hat{\lambda}_5 + \hat{\lambda}_5, \quad \hat{G}_{133} = \hat{\rho}_i \mathcal{M} - \hat{\rho}_i \hat{\lambda}_7 + \hat{\lambda}_7, \\
\hat{G}_{135} &= -\hat{\rho}_i \hat{\lambda}_9 + \hat{\lambda}_9, \quad \hat{G}_{136} = -\hat{\rho}_i \gamma^2 \mathcal{I} + \hat{\rho}_i \mathcal{M}^T \mathcal{M} - \hat{\rho}_i \hat{\lambda}_{10} + \hat{\lambda}_{10}, \\
\hat{G}_{ij} &= \begin{bmatrix} \hat{G}_{111} & \hat{G}_{112} & \hat{G}_{113} \\ \ast & \hat{G}_{12} & \hat{G}_{13} \\ \ast & \ast & \hat{G}_{131} \end{bmatrix}, \\
\hat{g}_{111} &= -\lambda \hat{\rho}_1 - \hat{\lambda}_{111}, \quad \hat{g}_{112} = -\lambda \hat{\rho}_2 - \hat{\lambda}_{112}, \\
\hat{g}_{113} &= -\lambda \hat{\rho}_3 + \hat{\lambda}_{111} \mathcal{C} - \hat{\lambda}_{111} \mathcal{C}^T \mathcal{R} - \hat{\lambda}_{111}, \quad \hat{g}_{121} = -\hat{\lambda}^T + \hat{\lambda} \mathcal{U}^T - \hat{\lambda}_{221}, \\
\hat{g}_{12} &= \begin{bmatrix} -\hat{\lambda}_{211} & -\hat{\lambda}_{311} \\ -\hat{\lambda}_{121} & \hat{\lambda}_{321} \end{bmatrix}, \\
\hat{g}_{13} &= \begin{bmatrix} \hat{\lambda}_6 & \mathcal{M} - \hat{\lambda}_{311} \\ \ast & \hat{\lambda}_8 - \hat{\lambda}_{131} \\ \ast & \ast & \hat{\lambda}_{132} \end{bmatrix}, \\
\hat{g}_{131} &= \mathcal{I} - \mathcal{U} - \mathcal{U}^T - \hat{\lambda}_5, \quad \hat{g}_{132} = \mathcal{M}^T \mathcal{M} - \gamma^2 \mathcal{I} - \hat{\lambda}_{10}.
\end{align*}
\]

**Proof.** According to the conditions (50) and (51), we can obtain that \( \lambda' = \mathcal{P}^{-1}, \ S = \mathcal{Q}^{-1} \). By utilizing Schur complement, (38), (39) and (40) can be converted into (17), (15) and (19), (18), (16) and (20) can be derived from (41), (42) and (43), (44), (45) and (46) can be changed into (29), (27) and (31), (47), (48) and (49) can be transformed into (30), (28) and (32), respectively.

Because of the existence of bilinear terms, the conditions in Theorem 3 cannot be handled by MATLAB LMI Toolbox directly. A cone complementarity linearization algorithm [7] is an effective approach to solve such a problem. The detailed operations are shown below

**Algorithm 1:**

**Step 1:** Provide a group of initial solution

\[
\left( \mathcal{P}^{(0)}, \lambda^{(0)}, \mathcal{Q}^{(0)}, \mathcal{S}^{(0)} \right),
\]

so that (38)-(49) and the following conditions are satisfied:

\[
\begin{bmatrix} \mathcal{P} & \mathcal{I} & \lambda' \\ \mathcal{I} & \mathcal{I} & \mathcal{S} \end{bmatrix} \geq 0, \begin{bmatrix} \mathcal{Q} & \mathcal{I} & \mathcal{S} \end{bmatrix} \geq 0.
\]

Set \( t = 0 \).

**Step 2:** Define the optimal issue as follows:

\[
\begin{aligned}
\min \operatorname{tr} \left( \mathcal{P} \lambda^{(t)} + \mathcal{P}^{(t)} \lambda' + \mathcal{Q} \mathcal{S}^{(t)} + \mathcal{Q}^{(t)} \mathcal{S} \right),
\end{aligned}
\]

subject to (38)-(49) and

\[
\begin{bmatrix} \mathcal{P} & \mathcal{I} & \lambda' \\ \mathcal{I} & \mathcal{I} & \mathcal{S} \end{bmatrix} \geq 0, \begin{bmatrix} \mathcal{Q} & \mathcal{I} & \mathcal{S} \end{bmatrix} \geq 0.
\]

**Step 3:** Substituting the derived matrix variables (\( \mathcal{P}, \lambda' \)) into (15)-(20) and (27)-(32). If (15)-(20) and (27)-(32) satisfy the following form:

\[
|\operatorname{tr}(\mathcal{P} \lambda'+ \mathcal{Q} \mathcal{S}) - r\kappa| < \kappa,
\]

for a sufficient small scalar \( \kappa > 0 \), and (\( \mathcal{P}, \lambda', \mathcal{Q}, \mathcal{S} \)) are practicable, and then, output the practicable solutions. EXIT.
Step 4: If $K > E$, in which $K$ and $E$, respectively, stand for the number of iterations and the maximum number iterations. EXIT.

Step 5: Let $t = t + 1$, $(P^{(t)}, X^{(t)}, Q^{(t)}, S^{(t)}) = (P, X, Q, S)$. Then, return to Step 2.

4. Demonstrative examples. Two examples are given to confirm the usefulness of the method put forward in this paper.

Example 1. A numerical example is provided with two fuzzy rules in this subsection. The corresponding parameters are given as follows:

$$
A_1 = \begin{bmatrix} -0.6889 & -0.3035 \\ 1.1002 & -0.5989 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2001 \\ 0.1012 \end{bmatrix},
$$
$$
B_{11} = \begin{bmatrix} 0.1038 \\ 0.4852 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.4064 \\ -0.0933 \end{bmatrix},
$$
$$
C = \begin{bmatrix} 0.1019 \\ 0.0986 \end{bmatrix},
$$
$$
A_2 = \begin{bmatrix} 0.2012 \\ -0.3015 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.3991 \\ 0.8124 \end{bmatrix},
$$
$$
B_{12} = \begin{bmatrix} 0.0102 \\ -0.0826 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.1110 \\ -2016 \end{bmatrix}.
$$

Tables 1 and 2 present the lower and upper membership functions of the system and FD observer, respectively.

**Table 1.** Lower and upper membership functions of the system

| $u_{L1}(x_1)$ | $1 - e^{-\frac{x_1}{15}}$ | $\bar{u}_{L1}(x_1)$ | $1 - e^{-\frac{x_1}{12}}$ |
|---------------|--------------------------|----------------------|--------------------------|
| $u_{L21}(x_1)$ | $1 - \bar{u}_{L1}(x_1)$ | $\bar{u}_{L21}(x_1)$ | $1 - \bar{u}_{L1}(x_1)$ |

**Table 2.** Lower and upper membership functions of the observer

| $u_{V11}(x_1)$ | $e^{-\frac{x_1}{14}}$ | $\bar{u}_{V11}(x_1)$ | $e^{-\frac{x_1}{12}}$ |
|---------------|--------------------------|----------------------|--------------------------|
| $u_{V21}(x_1)$ | $1 - u_{V11}(x_1)$ | $\bar{u}_{V21}(x_1)$ | $1 - \bar{u}_{V11}(x_1)$ |

Define the weighting coefficients as follows:

$$
\alpha_i(x_1(k)) = \sin^2(x_1(k)), \quad \bar{\alpha}_i(x_1(k)) = 1 - \alpha_i(x_1(k)),
$$
$$
\beta_i(x_1(k)) = \cos^2(x_1(k)), \quad \bar{\beta}_i(x_1(k)) = 1 - \beta_i(x_1(k)).
$$

By addressing conditions (38)-(51), the weighting matrix is obtained as $Q = 1.4913$, and the observer gains are shown as follows:

$$
L_1 = \begin{bmatrix} 2.8825 \\ 3.1942 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 7.0949 \\ 3.0144 \end{bmatrix},
$$
$$
W = 18.1808, \quad \mathcal{R} = 0.4159.
$$

For the sake of verifying the advantage of the presented approach in this paper, three cases are given as follows:

**Case 1:** In this case, the $H_\infty$ performance of the error system is testified with $f(t) = 0$ and $w(k) = 0.4 \sin(0.5k)(1 < k < 20)$. Fig. 1 is the state estimation errors of the system, it means that interference signal can be suppressed effectively. Fig.
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Figure 1. Estimation errors of the system.

Figure 2. Trajectory of $\bar{\sigma}(k)$.

Figure 3. Event-based release instants and release interval.

2 stands for the trajectory of $\bar{\sigma}(k)$. From Fig. 2, we can see that $\bar{\sigma}(k)$ can grow fast enough when the errors change quickly. At the same time, it can be seen that $\bar{\sigma}(k)$ will converge to zero under the situations of $w(k) = 0$ and the errors converge
to zero. Fig. 3 displays the event-based release instants and release interval. From Fig. 3, we can see that only 63% communication resource is used.
Case 2: Assuming \( f(k) = 0.1(14 < k < 100) \) and \( w(k) = 0.4\sin(0.5k)(0 < k < 100) \), and in this case, the threshold \( J_{th} \) and the evaluation function \( J(k) \) are given in Fig. 4. From Fig. 4, we can know that the fault is detected.

Case 3: Suppose that outliers exist in the measurement and happen at \( 22 < k < 25 \), fault signal satisfies \( f(k) = 0 \) and interference signal satisfies \( w(k) = 0.4\sin(0.5k)(0 < k < 25) \). Then, the threshold \( J_{th} \) and the evaluation function \( J(k) \) are shown in Fig. 5. From Fig. 5, it can be obtained that when outliers exist in the error system, false alarms are not generated by the proposed method in this paper. Fig. 6 displays the threshold \( J_{th} \) and the evaluation function \( J(k) \) by using the general observer [27] without adaptive saturation. We can see that in Fig. 6, the system generates the false alarms when outliers exist in the measurement. According to this phenomenon, the usefulness of the proposed approach can be identified.

Example 2. A tunnel diode circuit is introduced for sake of further explaining the feasibility of the method which developed in this paper. According to the circuit knowledge, the following equation can be derived:

\[
\begin{align*}
i_D(t) &= 0.002v_D(t) + \partial v_D^3(t),
\end{align*}
\]

where \( \partial \) is a parameter which is uncertain and belongs to \([0.01, 0.03]\).

Letting \( x_1(t) = v_C(t) \), \( x_2(t) = i_L(t) \) and \( \bar{v} = 0.002 + \partial v_D^3(t) \), one gets

\[
\begin{align*}
C \dot{x}_1(t) &= -\bar{e}x_1(t) + x_2(t), \\
L \dot{x}_2(t) &= -x_1(t) - R x_2(t) + w(t),
\end{align*}
\]

in which \( C = 20mF \), \( L = 1000mH \), \( R = 10\Omega \). Assume that \( x_1(t) \in [-2, 2] \), the following IT2 T-S fuzzy model is get:

\[
x(k + 1) = \sum_{i=1}^{2} m_i(\varrho(k))[A_i x(k) + B_i w(k)],
\]

where

\[
A_1 = \begin{bmatrix} -\bar{e}_{\min} & 50 \\ -1 & -10 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\bar{e}_{\max} \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Figure 7. Tunnel diode circuit.
The weighting coefficients are the same as that of Example 1. Tables 3 and 4, respectively, refer to the lower and upper membership functions of the system and FD observer.

**Table 3.** Lower and upper membership functions of the system

| Function | Lower Bound | Upper Bound |
|----------|-------------|-------------|
| $u_{c11}(x_1)$ | $e_{\max} - e_{\min}$, $\partial = 0.03$ | $e_{\max} - e_{\min}$, $\partial = 0.01$ |
| $u_{c21}(x_1)$ | $e_{\min}$, $\partial = 0.01$ | $e_{\min}$, $\partial = 0.03$ |

**Table 4.** Lower and upper membership functions of the observer

| Function | Lower Bound | Upper Bound |
|----------|-------------|-------------|
| $u_{v11}(x_1)$ | $0.4e^{-\frac{x_1}{T}}$ | $0.4e^{-\frac{x_1}{T}}$ |
| $u_{v21}(x_1)$ | $1 - u_{v11}(x_1)$ | $1 - u_{v11}(x_1)$ |

Given the sampling period $T = 0.06 s$, matrices can be derived as follows:

$$A_1 = \begin{bmatrix} 0.9210 & 2.1821 \\ -0.0436 & 0.4889 \end{bmatrix}, B_1 = \begin{bmatrix} 0.0732 \\ 0.0438 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.4000 & 1.4368 \\ -0.0287 & 0.5035 \end{bmatrix}, B_2 = \begin{bmatrix} 0.0562 \\ 0.0440 \end{bmatrix}.$$

Define remaining matrices as follows:

$$B_{11} = \begin{bmatrix} 0.2019 \\ 0.3987 \end{bmatrix}, C_1 = \begin{bmatrix} 0.1934 & 0.1012 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.8019 & 0.9986 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} 0.4988 \\ 0.3987 \end{bmatrix}, C_2 = \begin{bmatrix} 0.3058 & 0.2315 \end{bmatrix}.$$

**Figure 8.** Estimation errors of the system.

Select $\lambda = 0.8$, $\gamma = 1.7$, $\mu_1 = 0.2$, $\mu_2 = 0.4$, $\rho_1 = 0.3$, $\rho_2 = 0.6$, $\bar{\mu}_1 = 0.3$, $\bar{\mu}_2 = 0.2$, $\bar{\rho}_1 = 0.8$, $\bar{\rho}_2 = 0.2$, $\varepsilon = 0.34$. According to the conditions (38)-(49), the
weighting matrix can be got as $Q = 0.8729$, and then, we can get that the following
observer gains:

\[
\mathcal{L}_1 = \begin{bmatrix} 1.9069 \\ 0.3511 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 0.8610 \\ 0.3538 \end{bmatrix}, \quad \mathcal{W} = 25.3221, \quad R = 0.2647.
\]

Similar to Example 1, three cases are given to identify the effectiveness of the presented method in this paper.

**Case 1:** Assuming \( f(t) = 0 \) and the disturbance signal as \( w(k) = 0.5 \sin(0.48k) \) \((0 < k < 14)\). Fig. 8 refers to the state estimation error of the fuzzy system, and it can be seen that the interference signal is effective suppressed by the presented approach. The trajectory of \( \hat{\sigma}(k) \) is shown in Fig. 9, it can be derived that \( \hat{\sigma}(k) \) can react fast enough when the errors alter quickly. With Fig. 10, we can see that only 43% communication resource is used, therefore, the effectiveness of the method which presented in this paper is verified.

**Case 2:** In this case, we suppose that the fault signal satisfies \( f(k) = 0.1(14 < k < 100) \) and disturbance signal satisfies \( w(k) = 0.5 \sin(0.48k) (0 < k < 100) \). Fig. 11 displays the FD threshold \( J_{th} \) and evaluation function \( J_k \), and by Fig. 11, the fault is detected.
**Case 3:** For the sake of illustrating the effectiveness of the presented approach, it is supposed that the outliers exist in the measurement and occur at $25 < k < 27$. Defining $f(k) = 0$, $w(k) = 0.5 \sin(0.48k)(0 < k < 26)$, then the evaluation function $J_k$ and FD threshold $J_{th}$ can be depicted in Fig. 12. As can be seen from Fig. 12, the false alarms are not produced when outliers exist in the measurement by applying the proposed approach in this paper. Fig. 13 shows the evaluation function $J_k$ and FD threshold $J_{th}$, and it can be found that false alarm is presented by utilizing the observer [27] when outliers exist in the measurement. Based on above analysis, the effectiveness of the method which presented in this paper is identified.

5. **Conclusion.** This paper has focused on the FD observer design problem for a category of nonlinear systems subject to outliers. For the sake of avoiding false alarms which caused by outliers, an FD observer that has the saturated limit scheme has been constructed in this paper. In addition, an event-based mechanism has been introduced to decrease unnecessary signal communications. Sufficient criteria have been given so that both the $H_\infty$ performance and the $H_\infty$ fault performance can be guaranteed, respectively. At last, according to simulation results, the effectiveness of the advanced method has been verified. In future work, we will utilize the FD method to impulsive control [16], sliding mode control [23,36], tracking control [30], a class of Convolutional Broad Network [43] and neural networks [20, 34, 35], and meanwhile, we will apply the FD method for switched systems [38] and multi-agent systems [19,48] under the condition of considering the impulsive effects [17]. In addition, optimal control technique [4,9,10] will be attempted to solve the parameters.

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