Junction Conditions in $f(R)$ Theories of Gravity

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Abstract

Taking advantage of the conformal equivalence of $f(R)$ theories of gravity with General Relativity coupled to a scalar field we generalize the Israel junction conditions for this class of theories by direct integration of the field equations. We suggest a specific non-minimal coupling of matter to gravity which opens the possibility of a new class of braneworld scenarios.
§1. Introduction

In 1918 Weyl\(^1\) was the first to consider Lagrangians for gravity which are a linear combination of the three quadratic scalars \(R^2\), \(R_{AB}R^{AB}\) and \(R_{ABCD}R^{ABCD}\) formed out of the scalar curvature, the Ricci and Riemann tensors of a metric \(g_{AB}\)\(^\)(a). Pauli\(^5\) and Eddington\(^6\) then noticed that the Schwarzschild metric was a solution of the corresponding vacuum field equations. Since at that time all precision tests of General Relativity theories relied on the Schwarzschild metric, these authors concluded that quadratic theories were a priori just as viable as ordinary General Relativity based on the Einstein–Hilbert Lagrangian \(R\). (As noted by Chiba et al.,\(^7\) this seems to be still a fairly widespread belief.)

The question of the unicity of the Schwarzschild metric in such theories in vacuo was first investigated by Buchdahl\(^8\) who showed that the Schwarzschild metric was the only spherically symmetric, asymptotically flat, solution of the vacuum pure \(R^2\) field equations. The subject was considered anew much later by Whitt\(^9\) (in the case of the \(R + a_2 R^2\) Lagrangian in 4 dimensions), and Mignemi and Wiltshire\(^10\) who showed that the Schwarzschild metric was the only \(D\)-dimensional, spherically symmetric, asymptotically flat, vacuum solution with a regular horizon, for all polynomial \(f(R) = R + \sum a_n R^n\) with \(a_2 > 0\)\(^\ast\).

Now, since the \(f(R)\) field equations are fourth order differential equations for the metric, they possess “runaway” solutions on top of solutions which smoothly tend to solutions of the Einstein equations in the limit \(f(R) \to R\)\(^\ast\ast\). The question of whether the Schwarzschild metric is the \(f(R)\) solution outside a distribution of ordinary matter (rather than a black hole), either point-like or extended, must therefore be raised. Pechlaner and Sexl\(^12\) showed that, in fact, in pure \(R^2\) theory the metric cannot be asymptotically flat as soon as the field equations have a right-hand side describing matter with positive energy density. They also showed that in \(R + a_2 R^2\) theory the metric can be asymptotically flat but that, at linear order around Minkowski spacetime, it is not the linearized Schwarzschild metric. The origin of such results is clearly explained by Havas\(^13\)): The Green function for the (second order) Einstein equations, which at lowest order reduce to \(\nabla^2 G_N = -\delta(r)\), takes the familiar form \(G_N = 1/r\), which yields the linearized Schwarzschild solution. On the other hand, in the case of the (fourth order) pure \(R^2\) theory the Green function solving the linearized field equations

\(^{\ast}\) The fact that the \(R_{ABCD}R^{ABCD}\) can be traded for \(-(R^2 - 4 R_{AB}R^{AB})\) (in 4 dimensions) was apparently first shown by Bach, and then by Lanczos\(^2\) (the Gauss–Bonnet theorem\(^3\)). For a historical review, see 4.

\(^{\ast\ast}\) As emphasized in 10), other solutions exist if one relaxes the condition of asymptotic flatness and allows for asymptotically de Sitter or anti-de Sitter spacetimes.

\(^{\ast\ast\ast}\) This fact was used by Starobinsky\(^11\) to build the first inflationary cosmological model.
\( \nabla^4 G = -\delta(r) \) is \( G = r/2 \), which yields a divergent metric. As for \( G_N = 1/r \) it satisfies \( \nabla^4 G_N = -\nabla^2 \delta(r) \); this means that in pure \( R^2 \) theory the source for the Schwarzschild metric is not a delta function but its second derivative which does not represent a point-like distribution of matter with positive energy density. Finally the Green function for \( R + a^2 R^2 / 6 \) theory is \( G = (1 - e^{-r/a})/r \), yielding the Pechlaner–Sexl metric (after correcting a couple of typos in their equation (23)). (See also 6, 14.)

This question of unicity or non-unicity of the Schwarzschild (or Schwarzschild–(A)dS) solution was revived recently when \( f(R) \) theories were invoked in an attempt to explain the observed present acceleration of the universe by means other than a cosmological constant\(^{15}\) (see also 16) and references therein). In that context the question became whether or not the Schwarzschild (or Schwarzschild–de Sitter) metric is, at least approximately, a solution of the \( f(R) \) field equations in the presence of localized sources such as the Sun. In 17), static spherically symmetric solutions for the special case \( f(R) = R - \mu^4/R \) were built numerically with matter represented by a perfect fluid (see also references in 17)). It was found that, if the metric tends to an appropriate de Sitter limit to explain the acceleration of the universe then the PPN parameter \( \gamma \) measuring, e.g., the light bending by the Sun is of order 1/2 instead of 1,\(^{18}\) which rules out these “dark energy” models. In 7) and 19) (see also references therein), the field equations were solved at the linear approximation around Minkowski spacetime and similar results were found for a wide class of \( f(R) \) theories.

In many of the above mentioned analyses advantage is taken from the fact that \( f(R) \) field equations are conformally equivalent to Einstein gravity with a minimally coupled scalar field (this was first shown by Higgs\(^{20}\) for \( f(R) = R^2 \) and by Teyssandier and Tourrenc\(^{21}\) in the general case). This property puts the theories on a more familiar footing but, in itself, does not modify either the mathematics or the physics of the problem. Nevertheless conclusions concerning what are the “correct” solutions of the field equations are still controversial. In particular the validity of the linear approximation has been challenged (see e.g. 22) and references therein) and the question of how one recovers the (de Sitter–) Schwarzschild solution in the Einstein limit does not seem to be settled yet (see e.g. 23) and references therein).

In this paper, taking advantage of this conformal equivalence, we investigate junction conditions for a brane (i.e. an infinitesimally thin domain wall) in \( f(R) \) theories. The topic has already been investigated in 24), where, however, the questions of the Einstein limit and the coupling to matter on the brane were not addressed. Here, as shown below, we are able to see clearly the possible irregularity which may appear in the Einstein limit \( f(R) \rightarrow R \). Then we suggest a specific non-minimal coupling of matter on the brane to gravity which opens the possibility of a new class of braneworld scenarios.
The paper is organized as follows. In Sec. 2 we consider general quadratic theories of gravity and discuss possible freedom in the choice of the junction conditions depending on how singular one allows the metric to be. Then, as an example, we formulate the junction conditions by requiring the metric to be least singular, namely its first and second derivatives are continuous across the brane. Then, in Sec. 3 focusing of \( f(R) \) theories, we formulate the junction conditions which allow for discontinuities in the first derivatives of the metric. In Sec. 4 we consider \( f(R) \) theories in the Einstein frame, i.e., Einstein–scalar theories conformally equivalent to \( f(R) \) theories, and formulate the junction conditions in the Einstein frame. In doing so, we suggest a new type of gravitational coupling for the matter on the brane. Finally, in Sec. 5 we translate these generalized junction conditions back to the original, Jordan, frame. We work in \( D \) spacetime dimensions. The gravitational constant \( 8\pi G \) in \( D \) dimensions is set to unity; the signature is \(-++\cdots+\).

\textbf{§2. “Weak” junction conditions in quadratic theories of gravity}

Consider the quadratic Lagrangian,

\[
L = -2\Lambda + R + \gamma R^{ABCD}R_{ABCD} - 4\beta R_{AB}R^{AB} + \alpha R^2. \tag{2.1}
\]

The variational derivative of \( \sqrt{-g}L \) with respect to the metric \( g_{AB} \) yields, up to a divergence, \( \delta(\sqrt{-g}L) = -\sqrt{-g}\sigma^{AB}\delta g_{AB} \) with (see e.g. 25))

\[
\sigma_{AB} = -\frac{1}{2}Lg_{AB} + R_{AB} + 2\gamma R_{ALNP}R_{B}^{LNP} + 4(2\beta - \gamma)R_{C}^{D}R_{BAC}^{D} - 4\gamma R_{C}^{A}R_{CB} + 2\alpha R_{AB} + 4(\gamma - \beta)R_{AB} + 2(\alpha - \beta)g_{AB}R - 2(\alpha + \gamma - 2\beta)D_{A}D_{B}R \tag{2.2}
\]

where \( D \) is the covariant derivative associated with \( g_{AB} \). The field equations are, in the bulk,

\[
\sigma_{AB} = 0. \tag{2.3}
\]

These equations are fourth order in the derivatives of the metric, except for the Gauss–Bonnet combination \( \alpha = \beta = \gamma \) that we shall exclude.

We use Gaussian-normal coordinates in which the metric is

\[
ds^2 = dy^2 + \gamma_{\mu\nu}dx^\mu dx^\nu, \tag{2.4}
\]

where the brane is assumed to be located at \( y = 0 \). In terms of the extrinsic curvature,

\[
K_{\mu\nu} = -\frac{1}{2}\frac{\partial \gamma_{\mu\nu}}{\partial y}, \tag{2.5}
\]

\textsuperscript{*)} The junction conditions in Einstein–Gauss–Bonnet theory are well-known. See 26), and, e.g. 27).
the Riemann tensor is
\[ R_{\mu\nu} = \partial K_{\mu\nu} + K_{\rho\nu} K_{\mu}^\rho, \quad R_{\mu\nu\rho} = \bar{D}_{\nu} K_{\mu\rho} - \bar{D}_{\rho} K_{\mu\nu}, \tag{2.6} \]

where \( \bar{D}_\rho \) and \( \bar{R}_{\nu\rho} \) are the covariant derivative and the Riemann tensor associated with the metric \( \gamma_{\mu\nu} \), with the Greek indices being raised or lowered by the metric \( \gamma_{\mu\nu} \).

For convenience, we introduce the tensor
\[ H_{\mu\nu} \equiv \frac{\partial^2 K_{\mu\nu}}{\partial y^2} = -\frac{1}{2} \frac{\partial^3 \gamma_{\mu\nu}}{\partial y^3}. \tag{2.7} \]

Keeping in \( \sigma_{AB} \) only the terms with highest derivatives in \( y \), i.e., the terms proportional to \( \partial H_{\mu\nu}/\partial y \), one finds (see e.g. 25)) that there are no such terms in \( \sigma_{yy} \) and \( \sigma_{y\mu} \), and that they appear in \( \sigma_{\mu\nu} \) under the following combination:
\[ \sigma_{\mu\nu} = 4 \frac{\partial}{\partial y} [(\gamma - \beta) H_{\mu\nu} + (\alpha - \beta) \gamma_{\mu\nu} H] + \cdots, \tag{2.8} \]
with \( H = \gamma^{\rho\sigma} H_{\rho\sigma} \).

Suppose now that there exists a sub-class of metrics \( \gamma_{\mu\nu}(y, x^\rho) \) solving the bulk equations \( \sigma_{AB} = 0 \), whose third order derivatives \( H_{\mu\nu} \) jump across \( y = 0 \); that is, such that \( H_{\mu\nu} \) can be written as, e.g., \( H_{\mu\nu} = h_{\mu\nu}(x^\rho) \tanh(y/\ell) \) with \( \ell \to 0 \), in a region of order \( \ell \) around \( y = 0 \) ("Z_2-symmetric" case). For this sub-class of metrics \( \sigma_{\mu\nu} \) exhibits a Dirac distribution-like behavior at \( y = 0 \):
\[ 4 \frac{\partial}{\partial y} [(\gamma - \beta) H_{\mu\nu} + (\alpha - \beta) \gamma_{\mu\nu} H] \equiv \delta(y) D_{\mu\nu}, \tag{2.9} \]
where \( D_{\mu\nu} \) is the "strength" of the singularity. Integration across the brane then yields
\[ 4 [(\gamma - \beta) H_{\mu\nu} + (\alpha - \beta) \gamma_{\mu\nu} H]_+^- = D_{\mu\nu}, \tag{2.10} \]
where \( [F]_+^- \equiv \lim_{y \to 0^+} F(y) - \lim_{y \to 0^-} F(y) \). If we require this class of metrics to be solutions of the field equations, then we must have
\[ D_{\mu\nu} = S_{\mu\nu}, \tag{2.11} \]
where we may naturally interpret \( S_{\mu\nu} \) as the total energy–momentum tensor of the brane. These equations, together with Eq. (2.10), give the junction conditions.

These junction conditions were first given in 28) (in the particular case \( D = 4 \) and hence \( \gamma = 0 \), because of the Gauss–Bonnet theorem) and generalized recently in 29) to Lagrangians of the type \( L = f(R, R_{AB} R^{AB}, R_{ABCD} R^{ABCD}) \). It is clear that such discontinuities are
specific to higher derivative theories and smoothly disappear in the Einstein limit when the parameters $\alpha$, $\beta$ and $\gamma$ are “switched off.”

An important point (apparently overlooked in 29)) is that the contracted Bianchi identities, $D_B \sigma^B_A \equiv 0$, imply that the brane energy–momentum tensor is conserved:

$$\bar{D}_\nu S^\nu_\mu = 0. \quad (2.12)$$

We leave to further work the question of finding a bulk metric, a solution to the bulk equations (e.g., anti-de Sitter), which can be written in Gaussian coordinates in such a way as to exhibit a discontinuity in its third derivative across $y = 0$. (The forms of the metric given in, e.g., 30) do not belong to this desired sub-class as they exhibit a jump in their first derivatives.)

A few remarks to conclude this section are in order:

(a) For $\gamma = 2\beta - \alpha$ (generalization of Eddington’s choice\(^{31}\)), the junction conditions (2.11) take the form (in the $\mathbb{Z}_2$-symmetric case)

$$H_{\mu\nu} - \gamma_{\mu\nu} H = \frac{S_{\mu\nu}}{8(\beta - \alpha)}. \quad (2.13)$$

(b) For $\alpha - \beta = -\frac{\gamma - \beta}{D-1}$ (that is Weyl’s choice in $D=4$ and with $\gamma = 0^{11}$), they become

$$H_{\mu\nu} - \frac{H}{D-1} \gamma_{\mu\nu} = \frac{S_{\mu\nu}}{8(\gamma - \beta)}. \quad (2.14)$$

Therefore the total energy–momentum tensor on the brane must be traceless.

(c) Finally in the pure $f(R)$ case ($\beta = \gamma = 0$, $L = -2\Lambda + R + \alpha R^2$), the total energy–momentum tensor on a $\mathbb{Z}_2$-symmetric brane is constrained to be of the form

$$S_{\mu\nu} = 8\alpha H \gamma_{\mu\nu}. \quad (2.15)$$

The fact that $S_{\mu\nu}$ is conserved implies $H$ is a constant. Thus, the matter on the brane must be vacuum energy.

This final example indicates that imposing the metric to be of class $C^2$ (continuity of $\gamma_{\mu\nu}$ and its first and second derivatives) is probably too restrictive to allow for physically interesting braneworld scenarios in higher derivative theories of gravity. We shall therefore seek junction conditions that allow for more singular metrics. In the rest of the paper, we concentrate on $f(R)$ theories.
§3. Junction conditions in the Jordan frame: the standard approach

The field equations derived from the variation of $\sqrt{-gf(R)}$ with respect to the metric $g_{AB}$ are $\sigma_{AB} = 0$ in the bulk, with

$$\sigma_{AB} = f'(R)G_{AB} + \frac{1}{2} g_{AB}(R f'(R) - f(R)) + g_{AB}\square f'(R) - D_A D_B f'(R), \quad (3.1)$$

where $f'(R) \equiv df/dR$ and $G_{AB} = R_{AB} - \frac{1}{2} g_{AB} R$ is the Einstein tensor\(^*)\). In a Gaussian normal coordinate system, $ds^2 = dy^2 + \gamma_{\mu\nu} dx^\mu dx^\nu$, the Einstein tensor is decomposed as

$$G_{yy} = -\frac{1}{2} (K_{\mu\nu} K^{\mu\nu} - K^2 + R),$$

$$G_{y\mu} = -\bar{D}_\nu (K^\nu_\mu - \delta^\nu_\mu K), \quad (3.2)$$

$$G_{\mu\nu} = \partial_y (K_{\mu\nu} - K \gamma_{\mu\nu}) + 2 K^\rho_\mu K_{\rho\nu} - 3 K K_{\mu\nu} + \frac{1}{2} \gamma_{\mu\nu} (K_{\alpha\beta} K^{\alpha\beta} + K^2) + G_{\mu\nu},$$

where the extrinsic curvature is $K_{\mu\nu} \equiv -\frac{1}{2} \partial_y \gamma_{\mu\nu}$. We note that

$$R = 2 \partial_y K - K_{\mu\nu} K^{\mu\nu} - K^2 + \bar{R}$$

$$= 2 \partial_y K - K^*_{\mu\nu} K^{*\mu\nu} - \frac{D}{D-1} K^2 + \bar{R}, \quad (3.3)$$

where $K^*_{\mu\nu}$ is the traceless part of the extrinsic curvature: $K^*_{\mu\nu} \equiv K_{\mu\nu} - \frac{K}{D-1} \gamma_{\mu\nu}$.

For convenience, we decompose $\sigma_{AB}$ into

$$\sigma_{AB} = Q_{AB} + L_{AB}, \quad (3.4)$$

with

$$Q_{AB} = f'(R) G_{AB} + \frac{1}{2} (R f'(R) - f(R)) g_{AB},$$

$$L_{AB} = -D_A D_B f'(R) + g_{AB}\square f'(R). \quad (3.5)$$

Their components are given by

$$Q_{yy} = \frac{1}{2} (R f'(R) - f(R)) + f'(R) G_{yy},$$

$$Q_{y\mu} = f'(R) G_{y\mu},$$

$$Q_{\mu\nu} = \frac{1}{2} \gamma_{\mu\nu} (R f'(R) - f(R)) + f'(R) G_{\mu\nu}, \quad (3.6)$$

and

$$L_{yy} = -K \partial_y f'(R) + \square f'(R),$$

$$L_{y\mu} = -\partial_\nu \partial_y f'(R) - K^\nu_\mu \partial_\nu f'(R),$$

$$L_{\mu\nu} = -\bar{D}_\mu f'(R) + K_{\mu\nu} \partial_y f'(R) + \gamma_{\mu\nu} (\partial_{yy} f'(R) + \square f'(R) - K \partial_y f'(R)). \quad (3.7)$$

\(^*)\ We consider here metric $f(R)$ theories. For variation à la Palatini, see 32) and references therein.
In Sec. 2 we considered the class of metrics $\gamma_{\mu \nu}$ which were continuous across $y = 0$ with continuous first and second derivatives. In that case all the components of the Einstein tensor and of $Q_{AB}$ are well behaved and a Dirac distribution appears in $L_{\mu \nu}$ in the term $\gamma_{\mu \nu} \partial_{y y} f'(R)$. Now, since, for that sub-class, $R$ is continuous and hence $\partial_y R$ is at most discontinuous, the delta distribution behavior of $\partial_{y y} f'(R)$ comes from $\partial_{y y} R$:

$$\partial_{y y} f'(R) = f''(R) \partial_{y y} R + f'''(R)(\partial_y R)^2 = 2 f''(R) \partial_{y y} K + \cdots = 2 f''(R) \partial_y H + \cdots,$$

with $H_{\mu \nu} = \partial_{y y} K_{\mu \nu}$. In the particular case $f(R) = -2\Lambda + R + \alpha R^2$ we thus recover the result (2.15) of Sec. 2.

Here, on the other hand, we shall consider the class of metrics which are continuous across $y = 0$ but which allow for (a certain type of) discontinuity in their first derivatives.

The scalar curvature $R$ which could be now, a priori, a delta function, must be at most discontinuous; otherwise unacceptable $\delta(y)^2$ terms would appear in $Q_{AB}$ (unless of course $f(R) = -2\Lambda + R$). Furthermore, an inspection of $L_{\mu \nu}$ tells us that $R$ must be in fact continuous across $y = 0$; otherwise a $\delta(y)^2$ term would appear in $L_{\mu \nu}$ (unless $f(R)$ is quadratic, because it precisely arises from the term $f'''(R)(\partial_y R)^2$, see (3.8)). We shall therefore restrict ourselves to the sub-class of metrics with continuous scalar curvature $R$.

Again we leave to further work the question of finding a bulk metric, solution to the bulk equations, which can be written in Gaussian coordinates in such a way as to exhibit discontinuities in its first order derivative across $y = 0$, but not in $R$. (It seems that the fact that the scalar curvature must be continuous has been overlooked in the recent paper.)

We then see by inspection that the $y y$ and $y \mu$ components of $\sigma_{AB}$ at most jump across $y = 0$ and that the delta-like part of $\sigma_{\mu \nu}$ is

$$\partial_y [f'(R)(K_{\mu \nu} - K_{\gamma \mu \nu}) + \gamma_{\mu \nu} f''(R) \partial_y R] \equiv \delta(y) D_{\mu \nu}.$$  

Integration across the brane then yields the junction conditions

$$D_{\mu \nu} = [f'(R)(K_{\mu \nu} - K_{\gamma \mu \nu}) + \gamma_{\mu \nu} f''(R) \partial_y R]_{+} = S_{\mu \nu},$$

where $S_{\mu \nu}$ is the total energy momentum tensor on the brane. From the contracted Bianchi identities, we have that it must be conserved: $\bar{D}_\nu S^\nu_{\mu} = 0$.

Note that when $f(R) \to R$ the junction conditions (3.10) do not reduce to the familiar Israel conditions as they have to be supplemented by the condition of continuity of $R$. What happens when $f(R) = -2\Lambda + R + \ell^2 R^2 + \cdots$ when $\ell \to 0$ is that the bulk geometry may approach a solution of the Einstein bulk equations (e.g., AdS) everywhere, to the exception
of a region of size $\ell$ in the vicinity of the brane, so that when $\ell$ becomes very small the thin shell limit is no longer valid and the thickness of the brane must be taken into account. To render this irreducible difference between Einstein and $f(R)$ theories manifest, let us split the junction conditions into their trace and traceless parts, recalling that $\gamma_{\mu\nu}$ as well as $R$ are continuous (see (3.3)):

\[ \gamma_{\mu\nu}\big|_+ = 0, \quad [R]_+ = 0 \quad \implies \quad [K]_+ = 0, \quad [2\partial_\gamma K - K^*_\mu K^*_{\mu\nu}]_+ = 0, \quad (3.11) \]

\[ f'(R)[K^*_\mu]_+ = S^*_\mu, \]

\[ (D - 1)f''(R)[\partial_\gamma R]_+ = S, \]

where $K^*_\mu$ and $S^*_\mu$ are the traceless parts of the extrinsic curvature and brane energy–momentum tensor, respectively. The “weak” junction conditions considered in Sec. 2 are just a particular sub-class of the above such that $S^*_\mu = 0$.

For further reference, we shall also generalize them to a non-Gaussian coordinate system, $ds^2 = N^2dy^2 + \gamma_{\mu\nu}dx^\mu dx^\nu$, where $N$ is a continuous lapse function:

\[ \gamma_{\mu\nu}\big|_+ = 0, \quad [K]_+ = 0, \quad [R]_+ = 0, \quad f'(R)[K^*_\mu]_+ = S^*_\mu, \quad (3.12) \]

\[ (D - 1)f''(R)\frac{1}{N}[\partial_\gamma R]_+ = S, \]

where, now, the extrinsic curvature is defined as $K^\mu_\nu = -\frac{1}{2N}\partial_\gamma \gamma^\mu_\nu$.

In this and the preceding section we just stated that the jumps in some derivatives of the bulk metric coefficients must be equal to the total energy–momentum tensor of matter on the brane. We have yet to decide on the type of matter we want to have on the brane. An even more crucial issue is to decide how matter on the brane couples to gravity, that is, how it couples to the metric $\gamma_{\mu\nu}$ (which is the only dynamical variable at our disposal).

It is natural (and this is the hypothesis which is “universally” made in the literature) to assume that matter is minimally coupled to the metric. For example, if matter on the brane is taken to be a scalar field $\psi$ with potential $V(\psi)$, we have

\[ S_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \gamma_{\mu\nu} \left( \frac{1}{2} \gamma^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi + V(\psi) \right). \quad (3.13) \]

As we shall see in an accompanying paper,\textsuperscript{35} the junction conditions (3.11) and (3.12) thus obtained are the standard ones, that is, those one derives from the action $\frac{1}{2}\int d^Dx \sqrt{-g}f(R)$ supplemented by the Hawking–Luttrell boundary term $\int d^{D-1}x \sqrt{-\gamma}f'(R)K$.\textsuperscript{36}

In the next section, we shall question this conventional wisdom, taking advantage of the conformal equivalence of $f(R)$ theories of gravity with General Relativity coupled to a scalar
field. This will lead us to treat the scalar curvature as an independent field and allow us to propose more general junction conditions than (3.11) and (3.12).

§4. Junction conditions in the Einstein frame: extra degree of freedom and its coupling to matter on the brane

In order to simplify the notation, we denote quantities in the Jordan frame with tildes, e.g., \( \tilde{R} \) for the scalar curvature, and those in the Einstein frame without tildes. Thus all the quantities that appeared in the previous section should be tilded.

As is well-known,\(^{20,21}\) the bulk \( f(\tilde{R}) \) field equations for the Jordan frame metric \( \tilde{g}_{AB} \), i.e., \( \tilde{\sigma}_{AB} = 0 \) with \( \tilde{\sigma}_{AB} \) given by (3.4), are equivalent to bulk Einstein equations for the “Einstein frame” metric,

\[
\tilde{g}_{AB} = \tilde{g}_{AB} \exp \left( \frac{2\phi}{\sqrt{(D - 1)(D - 2)}} \right),
\]

with a scalar field minimally coupled to gravity:

\[
G_{AB} = \partial_A \phi \partial_B \phi - g_{AB} \left( \frac{1}{2} \partial_C \phi \partial^C \phi + W(\phi) \right),
\]

where the potential \( W(\phi) \) is implicitly defined as a function of \( \phi \) via

\[
W(\tilde{R}) = \frac{1}{2} \left( \tilde{R} f'(\tilde{R}) - f(\tilde{R}) f'(\tilde{R}) \right) - \frac{\phi}{D - 2}, \quad \phi = \sqrt{\frac{D - 1}{D - 2}} \ln f'(\tilde{R}).
\]

Because of the Bianchi identities, the Einstein equations (4.2) are consistent only if \( \phi \) satisfies the Klein–Gordon equation,

\[
\Box \phi - \frac{dW}{d\phi} = 0.
\]

Mathematically, this conformal transformation transforms the original, fourth-order differential equation (3.11) into two second-order differential equations, one for the Einstein frame metric \( g_{AB} \), the other for \( \phi \). Physically, it shows that \( f(\tilde{R}) \) is a “scalar–tensor” theory of gravity where \( \phi \), that is, the bulk Jordan frame scalar curvature \( \tilde{R} \), is an extra, independent, degree of freedom.\(^{37}\) What we shall dwell upon in the following is the coupling of this extra degree of freedom with matter on the brane.

Again we shall use Gaussian coordinates, that is

\[
ds^2 = dz^2 + \gamma_{\mu\nu} dx^\mu dx^\nu,
\]

and consider a class of metrics which are continuous across \( z = 0 \), but which allow for discontinuities in their first derivatives. We shall also impose the scalar field to be continuous,
and allow for a discontinuity in its first derivative. Now, \( \phi \) is directly related to the scalar curvature of the bulk Jordan frame metric; one must not however deduce hastily that the condition of continuity of \( \phi \) is equivalent to imposing the continuity of the scalar curvature of the Jordan frame metric; indeed the relation between \( \phi \) and \( \tilde{R} \) holds in the bulk only and they may differ, as we shall see, by a term confined on the brane.

Thus, allowing for discontinuities of the z-derivatives of the metric and of \( \phi \), the right-hand side of the field equations jumps at most. The \( G_{zz} \) and \( G_{z\mu} \) components of \( G_{\alpha\beta} \) also jump at most. As for the delta-like part of the Einstein tensor \( G_{\mu\nu} \), it is, see (3.2):

\[
\partial_z(K_{\mu\nu} - K\gamma_{\mu\nu}) \equiv \delta(z)D_{\mu\nu}.
\]

Integration across the brane then yields the Israel junction conditions:

\[
D_{\mu\nu} = [K_{\mu\nu} - K\gamma_{\mu\nu}]^+ = T_{\mu\nu},
\]

where \( T_{\mu\nu} \) is the total energy–momentum tensor of the brane in the Einstein frame. Later we shall relate it to the energy–momentum tensor of the brane in the Jordan frame \( \tilde{S}_{\mu\nu} \).

Since the first derivative of \( \phi \) is allowed to be discontinuous, \( \Box \phi \) also exhibits a delta function-like behavior. From the Bianchi identities,

\[
0 = \partial_B \phi \left( \Box \phi - \frac{dW}{d\phi} \right) + D_A(T^A B \delta(z)),
\]

we have

\[
\partial_\mu \varphi [\partial_z \phi]^+ = -\bar{D}_\nu T^\nu_\mu,
\]

where \( \varphi(x^\mu) = \phi(z = 0, x^\mu) \).

Just as in the case of working in the Jordan frame, the last task is to express the total stress–energy tensor of the brane matter in terms of the matter variables. Since the gravitational variables are \( \gamma_{\mu\nu} \) and \( \varphi \) we have to decide how matter couples to those gravitational fields. For example, if matter on the brane is taken to be a scalar field \( \psi \) with potential \( V(\psi) \), we may consider as the matter action,

\[
S_m = -\int d^{D-1}x \sqrt{-\gamma} \left( F_1(\varphi) \frac{1}{2} \gamma^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + F_2(\varphi)V(\psi) \right),
\]

where \( F_1 \) and \( F_2 \) are two a priori arbitrary functions of \( \varphi \). The associated energy–momentum tensor is

\[
T_{\mu\nu} = F_1(\varphi) \partial_\mu \psi \partial_\nu \psi - \gamma_{\mu\nu} \left( \frac{1}{2} F_1(\varphi) \gamma^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi + F_2(\varphi)V(\psi) \right).
\]

The field equation for \( \psi \) is

\[
\bar{D}^\nu (F_1(\varphi) \partial_\nu \psi) - F_2(\varphi)V'(\psi) = 0.
\]
Thus the divergence of the energy–momentum tensor gives

\[ \bar{D}_\nu T^\nu_\mu = -\partial_\mu \varphi \left( \frac{dF_1}{d\varphi} \gamma^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi + \frac{dF_2}{d\varphi} V(\psi) \right). \] (4.13)

Following Einstein’s suggestion to Nordström,\(^3^8\) we require that the source for \( \phi \) in (4.9) be related to the trace of the matter energy–momentum tensor,

\[ \bar{D}_\nu T^\nu_\mu \propto \partial_\mu \varphi T. \] (4.14)

This imposes

\[ F_1(\varphi) = \exp[(D - 3)k(\varphi)], \quad F_2(\varphi) = \exp[(D - 1)k(\varphi)], \] (4.15)

where \( k(\varphi) \) still has to be determined. Plugging these back into (4.10), we see the meaning of the condition (4.14). Namely, the matter should be minimally coupled to a metric \( \bar{\gamma}_{\mu\nu} \),

\[ S_m = S_m[\bar{\gamma}_{\mu\nu}; \psi], \quad \bar{\gamma}_{\mu\nu} = e^{2k(\varphi)}\gamma_{\mu\nu}, \] (4.16)

where \( \psi \) now represents general matter variables not restricted to a scalar field.

The junction condition (4.9) then becomes

\[ [\partial_z \phi]^+ = -\frac{dC}{d\varphi} T, \] (4.17)

with

\[ T = -2 \frac{\gamma_{\mu\nu}}{\sqrt{-\bar{\gamma}}} \frac{\delta S_m[e^{2k(\varphi)}\gamma_{\rho\sigma}; \psi]}{\delta \gamma_{\mu\nu}} \]
\[ = -e^{(D-1)k(\varphi)} \left( \frac{D-3}{2} e^{-2k(\varphi)} \bar{\gamma}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + (D-1)V(\psi) \right), \] (4.18)

where the second line is the case when the matter \( \psi \) is a scalar field. The junction condition (4.17) is nothing but the one used when studying the brane cosmology with a bulk scalar field.\(^3^9\)

We also note that the matter action can then be rewritten in terms of the Jordan metric \( \bar{\gamma}_{\mu\nu} \) as

\[ S_m = S_m[\bar{\gamma}_{\mu\nu}; \psi] = S_m[e^{2C(\varphi)}\bar{\gamma}_{\mu\nu}; \psi] \]
\[ = - \int d^{D-1}x \sqrt{-\bar{\gamma}} e^{(D-1)C(\varphi)} \left( \frac{1}{2} e^{-2C(\varphi)} \bar{\gamma}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + V(\psi) \right), \] (4.19)

with

\[ C(\varphi) = k(\varphi) + \frac{\varphi}{\sqrt{(D-1)(D-2)}}. \] (4.20)

This is where the fact that we are treating a \( f(\bar{R}) \) theory in the Einstein frame comes into play. Indeed, if we impose the matter on the brane to be minimally coupled to the Jordan
metric, then we must choose (a point already known to Einstein\textsuperscript{40}) (see also 41))) $C = 0$, that is,

$$k(\varphi) = -\frac{\varphi}{\sqrt{(D - 1)(D - 2)}}. \quad (4.21)$$

This reduces the junction conditions to

$$[K_{\mu\nu} - K\gamma_{\mu\nu}]^+ = T_{\mu\nu},$$
$$[\partial_\varphi^+ = \frac{T}{\sqrt{(D - 1)(D - 2)}}. \quad (4.22)$$

We shall now translate back to the Jordan frame the generalized junction conditions (4.17) and (4.17), and show that they reduce to those obtained in (3.12) with (3.13) only when $k(\varphi)$ is imposed to be given by (4.21).

§5. Back to the Jordan frame: generalized junction conditions

In a Gaussian normal coordinate system in which the line element reads $ds^2 = dz^2 + \gamma_{\mu\nu} dx^\mu dx^\nu$, the junction conditions we have obtained in the Einstein frame are

$$[\phi]^+ = 0, \quad [\gamma_{\mu\nu}]^+ = 0,$$
$$[K_{\mu\nu} - \gamma_{\mu\nu} K]^+ = T_{\mu\nu}, \quad [\partial_\nu \phi]^+ = -\frac{dk}{d\varphi} T, \quad (5.1)$$

where $K_{\mu\nu} = -\frac{1}{2} \partial_\nu \gamma_{\mu\nu}$ and the second line recalls that the induced metric as well as $\phi$ have to be continuous across the brane. We assume that the matter on the brane couples minimally to the metric $\tilde{\gamma}_{\mu\nu} = e^{2k(\varphi)} \gamma_{\mu\nu}$ as given by (4.16).

Let us perform the following transformations

$$\phi \to \phi = \sqrt{\frac{D - 1}{D - 2}} \ln f'(\rho),$$
$$W(\phi) \to W(\rho) = \frac{1}{2} (\rho f'(\rho) - f(\rho)) f'(\rho) - \frac{\rho^2}{2},$$
$$g_{AB} \to g_{AB} = \tilde{g}_{AB} f'(\rho) \frac{\rho^2}{2}. \quad (5.2)$$

It is a side exercise to show that if $g_{AB}(x^C)$ and $\phi(x^C)$ are solution of the bulk field equations (4.2) then $\rho(x^C)$ is the scalar curvature $\tilde{R}$ of the bulk Jordan metric $\tilde{g}_{AB}$. However if one includes the presence of matter on the brane, there appears a delta function-like singularity in $\tilde{R}$ while $\rho$ is continuous unless the matter is minimally coupled on the brane, i.e., unless $C(\varphi) = 0$ (modulo a constant), as we shall see below.

Note that the coordinates are no longer Gaussian for the Jordan line element

$$ds^2 = \tilde{g}_{AB} dx^A dx^B = f'(\rho)^{-\frac{2D^2 - 2}{D - 2}} g_{AB} dx^A dx^B = f'(\rho)^{-\frac{2D^2}{D - 2}} dz^2 + \tilde{\gamma}_{\mu\nu} dx^\nu dx^\nu, \quad (5.3)$$
with \( \tilde{\gamma}_{\mu\nu} = f'(\rho)^{-\frac{1}{D-2}} \gamma_{\mu\nu} \) the induced metric of the Jordan brane. We introduce

\[
\tilde{K}_{\mu\nu} = -\frac{1}{2} f'^{-\frac{1}{D-2}} \partial_z \tilde{\gamma}_{\mu\nu}, \quad \tilde{K} = \tilde{\gamma}^{\mu\nu} \tilde{K}_{\mu\nu}.
\] (5.4)

The junction conditions (5.1) then translate as follows. As already mentioned, the continuity of \( \phi = \sqrt{\frac{D-1}{D-2}} \ln f'(\rho) \) translates into the continuity of \( \rho \) and the continuity of the metric \( g_{AB} \) translates into the continuity of the Jordan induced metric \( \tilde{\gamma}_{\mu\nu} \):

\[
[\rho]^+ = 0, \quad [\tilde{\gamma}_{\mu\nu}]^+ = 0.
\] (5.5)

As for the jumps in \( \partial_z \phi \) and the extrinsic curvature they translate into

\[
f'^{-\frac{1}{D-2}} f''[\partial_z \rho]^+] = \frac{1}{D-1} \left( 1 - \sqrt{(D-1)(D-2)} \frac{dC}{d\phi} \right) \tilde{T},
\]

\[
f'[\tilde{K}]^+ = -\sqrt{\frac{D-1}{D-2}} \frac{dC}{d\phi} \tilde{T}, \quad f'[\tilde{K}_{\mu\nu}]^+ = \tilde{T}^*_{\mu\nu},
\] (5.6)

where

\[
\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{\gamma}}} \frac{\delta S_m(e^{2C(\phi)} \tilde{\gamma}_{\rho\sigma}; \psi)}{\delta \tilde{\gamma}^{\rho\sigma}},
\] (5.7)

and a star means taking the traceless part. We note that because of the assumption that the matter is minimally coupled to the metric \( \tilde{\gamma}_{\mu\nu} \), we have

\[
\sqrt{-\tilde{\gamma}} \tilde{T} = -2 \tilde{\gamma}^{\mu\nu} \frac{\delta S_m(e^{2C(\phi)} \tilde{\gamma}_{\rho\sigma}; \psi)}{\delta \tilde{\gamma}^{\rho\sigma}} = \sqrt{-\gamma} T.
\] (5.8)

If matter on the brane is a scalar field \( \psi \), then \( S_m \) is given by (4.17) so that

\[
\tilde{T}_{\mu\nu} = e^{(D-1)C(\phi)} \left( e^{-2C(\phi)} \partial_\mu \psi \partial_\nu \psi - \tilde{\gamma}_{\mu\nu} \left( \frac{1}{2} e^{-2C(\phi)} \tilde{\gamma}_{\rho\sigma} \partial_\rho \psi \partial_\sigma \psi + V(\psi) \right) \right).
\] (5.9)

Now, generalizing (3.3) to a non-Gaussian coordinate, the scalar curvature \( \tilde{R} \) may be expressed as

\[
\tilde{R} = 2N^{-1} \partial_z \tilde{K} - \tilde{K}_{\mu\nu} \tilde{K}^{\mu\nu} - \tilde{K}^2 + \tilde{R}; \quad N = f'(\rho)^{-\frac{1}{D-2}}.
\] (5.10)

Integrating this across \( z = 0 \) and using the junction conditions (5.6), one finds that

\[
\int_{-\epsilon}^{\epsilon} \tilde{R} N dz = 2[\tilde{K}]^+ = -2 \sqrt{\frac{D-1}{D-2}} \frac{dC}{d\phi} \frac{\tilde{T}}{f'}. \]

(5.11)

From this, we deduce that

\[
\tilde{R} = \rho - 2 \sqrt{\frac{D-1}{D-2}} \frac{dC}{d\phi} f'^{-\frac{1}{D-2}} \frac{\tilde{T}}{f'} \delta(z).
\] (5.12)
Therefore, as anticipated, the continuity of $\phi$ translates in the continuity of $\rho$ but not of $\tilde{R}$.

The junction conditions (5.5) and (5.6) are the central result of this paper. When the arbitrary function $C$ vanishes, we have from (5.12) that $\tilde{R} = \rho$ everywhere including on the brane and we have that $\tilde{T}_{\mu \nu}$ coalesces with $\tilde{S}_{\mu \nu}$, the stress–energy tensor of matter minimally coupled to the brane metric introduced in (3.10). The junction conditions thus reduce to those obtained in Sec. 3.

When $C$ is a non-trivial function, they generalize them to the case when matter on the brane is coupled not only to the brane metric $\tilde{\gamma}_{\mu \nu}$ but to the extra degree of freedom of $f(\tilde{R})$ gravity which, in the Jordan frame, is the quantity $\rho$, equal everywhere but on the brane to the scalar curvature $\tilde{R}$.

We just note here, to conclude, that, if we choose $C(\phi) = \phi/\sqrt{(D-1)(D-2)}$, that is $k(\phi) = 0$, then the junction conditions in the Jordan frame closely resemble the standard Israel junction conditions

$$f'[\tilde{K}_{\mu \nu} - \tilde{K}\tilde{\gamma}_{\mu \nu}] = \tilde{T}_{\mu \nu}. \quad (5.13)$$

However the coupling of matter to the brane metric is not minimal, as given by (5.7).

§6. Conclusion

We thoroughly investigated the junction conditions in $f(R)$ theories of gravity. We found that in a pure $f(R)$ theory in which matter on the brane couples minimally to the metric, the bulk scalar curvature $R$ must be continuous across the brane, which is in marked contrast with the case of Einstein gravity. Then taking advantage of the conformal equivalence of $f(R)$ theories with Einstein gravity with a scalar field, we clarified the importance of identifying the scalar curvature $R$ as an extra degree of freedom and the specific form of the coupling of matter to this extra gravitational degree of freedom on the brane.

Then as a bonus of working in the Einstein frame, we presented a natural generalization of the coupling of matter to gravity. In the original frame, this leads to a non-trivial coupling of the matter on the brane to the extra degree of freedom, which allows a delta function-like behavior of the scalar curvature. This suggests a new class of braneworld models whose solutions may have a smooth limit in the Einstein limit $f(R) \rightarrow R$.

It is known that, in the bulk, an $f(R)$ theory may be rewritten as a Brans–Dicke theory with $\omega = 0$ but with a potential. If we use this equivalence, the generalization mentioned above may be regarded as a non-trivial coupling of the Brans–Dicke scalar to the matter on the brane.

We leave to an accompanying paper the derivation of the junction conditions (5.6) via a first order description of the $f(R)$ action as well as an analysis of the braneworld models.
that they may lead to.

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