Topological Conditions for Identifiability of Dynamical Networks with Partial Node Measurements

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Abstract: This paper deals with dynamical networks where the relations between node signals are described by proper transfer functions and external and disturbance signals can influence each of the node signals. In particular, we are interested in graph-theoretic conditions for identifiability of such dynamical networks, where we assume that only a subset of nodes is measured but the underlying graph structure of the network is known. This problem has recently been investigated in the case of generic identifiability. In this paper, we investigate a stronger notion of identifiability for all network matrices associated with a given graph. For this, we introduce a new graph-theoretic concept called constrained vertex-disjoint paths. As our main results, we state conditions for identifiability based on these constrained vertex-disjoint paths.

Keywords: System identification, identifiability, dynamical networks, graph theory

1. INTRODUCTION

Networks of dynamical systems appear in a variety of contexts, including power systems, robotic networks and water distribution networks. In this paper, we consider dynamical networks where the relations between node signals are modelled by proper transfer functions and external and disturbance signals can influence each of the node signals. Such network models have received much attention (see, e.g., [van den Hof et al. (2013)], [Dankers (2014)], [Hendrickx et al. (2018)], [Weerts et al. (2018)]).

The interconnection structure of a dynamical network can be represented by a directed graph, where vertices (or nodes) represent scalar signals, and edges correspond to transfer functions connecting different node signals. We will assume that the underlying graph (i.e., the topology) of the dynamical network is known. Note that methods for identifying the topology of dynamical networks are available (see, e.g., [Shahrampour and Preciado (2015)], [van Waarde et al. (2017)]).

We are interested in conditions for identifiability of dynamical networks. Loosely speaking, identifiability comprises the ability to distinguish between certain network models on the basis of measured (input/output) data. In this work, we assume that each node of the network is externally excited by a known signal. However, the node signals of only a subset of nodes is measured. Within this setup, we are interested in two identifiability problems. Firstly, we want to find conditions under which the transfer functions from a given node to its out-neighbours can be identified. Secondly, we wonder under which conditions all transfer functions in the network can be identified. In particular, our aim is to find graph-theoretic conditions for the above problems. Conditions based on the topology of the network are desirable, since they give insight in the types of network structures that allow identification, and in addition may aid in the selection of measured nodes.

In previous work [Hendrickx et al. (2018)], graph-theoretic conditions have been established for generic identifiability. That is, conditions were given under which a subset of transfer functions in the network can be identified for “almost all” network matrices associated with a given graph. In contrast to [Hendrickx et al. (2018)], we are interested in graph-theoretic conditions for a stronger notion, namely identifiability for all network matrices associated with the graph. The difference between generic identifiability and identifiability for all network matrices might seem subtle at first, however, similar differences in the controllability literature have led to different graph-theoretic characterizations (cf. [Liu et al. (2011)], [Chapman and Mesbahi (2013)])

In this paper, we introduce a new graph-theoretic concept called constrained vertex-disjoint paths. Such paths are a generalization of constrained matchings in bipartite graphs [Hershkowitz and Schneider (1993)]. As our main result, we state conditions for identifiability in terms of constrained vertex-disjoint paths.

This paper is organized as follows. In Section 2 we introduce notation and preliminaries. Subsequently, in Section 3 we state the problem. Next, Section 4 contains the main results, and in Section 5 we give our conclusions.
2. PRELIMINARIES

We denote the set of natural numbers by \( \mathbb{N} \), real numbers by \( \mathbb{R} \), and real \( m \times n \) matrices by \( \mathbb{R}^{m \times n} \). The \( n \times n \) identity matrix is denoted by \( I_n \). When its dimension is clear from the context, we simply write \( I \). Consider a rational function \( f(z) = \frac{p(z)}{q(z)} \), where \( p(z) \) and \( q(z) \) are polynomials with real coefficients. Then, \( f \) is called proper if the degree of \( p(z) \) is less than or equal to the degree of \( q(z) \). We say \( f \) is strictly proper if the degree of \( p(z) \) is less than the degree of \( q(z) \). An \( m \times n \) matrix \( A(z) \) is called rational if its entries are rational functions in the indeterminate \( z \). In addition, \( A(z) \) is proper if its entries are proper rational functions in \( z \). We omit the argument \( (z) \) whenever the dependency of \( A \) on \( z \) is clear from the context. The normal rank of \( A \) is denoted by rank \( A \). We denote the \((i, j)\)-th entry of \( A \) by \( A_{ij} \). Moreover, the \( j \)-th column of \( A \) is given by \( A_{*j} \). More generally, let \( M \subseteq \{1, 2, \ldots, m\} \) and \( N \subseteq \{1, 2, \ldots, n\} \). Then, \( A_{M,N} \) denotes the submatrix of \( A \) containing the rows of \( A \) indexed by \( M \) and the columns of \( A \) indexed by \( N \). Next, consider the case that \( A \) is square, i.e., \( m = n \). The determinant of \( A \) is denoted by det \( A \), while the adjugate of \( A \) is denoted by adj \( A \). A principal submatrix of \( A \) is a submatrix \( A_{M,M} \), where \( M \subseteq \{1, 2, \ldots, m\} \). The determinant of \( A_{M,M} \) is called a principal minor of \( A \).

2.1 Graph theory

Let \( G = (V, E) \) be a directed graph, with vertex set \( V = \{1, 2, \ldots, n\} \) and edge set \( E \subseteq V \times V \). Unless explicitly mentioned, the graphs considered in this paper are simple, i.e., without self-loops (edges of the form \((i, i) \in E \)) and with at most one edge from one node to another. A node \( j \in V \) is said to be an out-neighbour of \( i \in V \) if \((i, j) \in E \). The set of out-neighbours of node \( i \in V \) is denoted by \( N_i \). For any subset \( S = \{v_1, v_2, \ldots, v_k\} \subseteq V \) we define the \( s \times n \) matrix \( P(V; S) \) as \( P_{ij} := 1 \) if \( j = v_i \), and \( P_{ij} := 0 \) otherwise. The complement of \( S \) in \( V \) is defined as \( S^c := V \setminus S \). Moreover, the cardinality of \( S \) is denoted by \( |S| \). A path \( P \) is a set of edges in \( G \) of the form \( P = \{(v_i, v_{i+1}) \mid i = 1, 2, \ldots, k-1\} \subseteq E \), where the vertices \( v_1, v_2, \ldots, v_k \) are distinct. The vertex \( v_1 \) is called a starting node of \( P \), while \( v_k \) is the end node. The cardinality of \( P \) is called the length of the path. By convention, each node \( i \in V \) has a path of “length zero” to itself. A collection of paths \( P_1, P_2, \ldots, P_l \) is called vertex-disjoint if the paths have no vertex in common, that is, if for all distinct \( i, j \in \{1, 2, \ldots, l\} \), we have that \((u_i, w_i) \in P_i, (u_j, w_j) \in P_j \implies u_i = u_j, w_i, w_j \) are distinct. Consider two subsets \( V_1, V_2 \subseteq V \). We say there exist \( m \) vertex-disjoint paths from \( V_1 \) to \( V_2 \) if there exist \( m \) vertex-disjoint paths in \( G \) with starting nodes in \( V_1 \) and end nodes in \( V_2 \). A cycle \( K \) is a set of edges in \( G \) of the form \( K = \{(v_i, v_{i+1}) \mid i = 1, 2, \ldots, k-1\} \subseteq E \), where \( v_1, v_2, \ldots, v_{k-1} \) are distinct, and \( v_k = v_1 \). The cardinality of \( K \) is called the length of the cycle. A collection of cycles is called vertex-disjoint if the cycles have no vertex in common. Moreover, a spanning cycle family in \( G \) is a collection of vertex-disjoint cycles such that each vertex in \( V \) is contained in exactly one cycle.

Next, consider a weighted directed graph \( G = (V, E) \), that is, a directed graph where each edge \((i, j) \in E \) has an associated rational function \( f_{ij}(z) \) called the weight of the edge \((i, j) \). The weight of a path \( P \) in \( G \) is defined as the product of the weights of all edges in \( P \). Moreover, the weight of a set of vertex-disjoint paths is defined as the product of the weights of all paths in the set. Similarly, the weight of a cycle \( K \) is defined as the product of the weights of all edges in \( K \). Finally, the weight of a set of vertex-disjoint cycles is defined as the product of weights of all cycles in the set.

3. PROBLEM STATEMENT AND MOTIVATION

Let \( G = (V, E) \) be a simple directed graph, with vertex set \( V = \{1, 2, \ldots, n\} \) and edge set \( E \subseteq V \times V \). Following the setup of [Hendrickx et al. (2018)] (see also [van den Hof et al. (2013)], [Weerts et al. (2018)])), we associate the following dynamical system with the graph \( G \):

\[
\begin{align*}
  w(t) &= G(q)w(t) + r(t) + v(t) \\
  y(t) &= Cw(t).
\end{align*}
\]

Here \( w, r, v \) are \( n \)-dimensional vectors of node signals, known external signals, and unknown disturbances, respectively. The (measured) output vector \( y \) is \( p \)-dimensional, and consists of the node signals of a subset \( C \subseteq V \) of nodes, with \(|C| = p \). Consequently, the matrix \( C \) is defined as \( C := P(V, C) \). Moreover, \( q^{-1} \) denotes the unit delay operator, i.e., \( q^{-1}w_i(t) = w_i(t-1) \), and \( G(z) \) is an \( n \times n \) rational matrix, called the network matrix, satisfying the following properties [van den Hof et al. (2013)].

P1. For all \( i, j \in V \), the entry \( G_{ji}(z) \) is a proper rational (transfer) function.

P2. The function \( G_{ji}(z) \) is nonzero if and only if \((i, j) \in E \).

A matrix \( G(z) \) that satisfies this property is said to be consistent with the graph \( G \).

P3. Every principal minor of \( \lim_{z \to \infty}(I - G(z)) \) is nonzero. This implies that the network model (1) is well-posed in the sense of Definition 2.11 of [Dankers (2014)].

Remark 1. Note that in this paper, we restrict \( G_{ji}(z) \) to be nonzero if \((i, j) \in E \), while in [Hendrickx et al. (2018)] the transfer function \( G_{ji}(z) \) can be zero even though there is an edge \((i, j) \) in the graph. The reason for the choice of nonzero transfer functions is the fact that we want to find identifiability conditions for all network matrices consistent with the graph. Within this framework, allowing zero transfer functions would result in very restrictive conditions for identifiability.

A network matrix \( G \) satisfying Properties P1, P2, and P3 is called admissible. In what follows, we use the shorthand notation \( T(z) := (I - G(z))^{-1} \), where \( G \) is assumed to be admissible. Note that (1) implies that \( y(t) = CT(q)r(t) + CT(q)v(t) \), which shows that the transfer matrix from \( r \) to \( y \) is given by \( CT(z) \). In this paper, we are interested in the question of which transfer functions in \( G(z) \) can be uniquely identified from input/output data, that is, from the external signals \( r(t) \) and output signals \( y(t) \). To this end, we assume that the graph \( G = (V, E) \) is known. Moreover, we assume that the excitation signal \( r(t) \) is sufficiently rich such that, under suitable assumptions on the disturbance \( v(t) \), the transfer matrix \( CT(z) \) can be identified from \((r, y)\)-data [Jung (1999)]. Since this paper...
focuses on identifiability and not on identification, we assume that the transfer matrix $CT(z)$ is known. Under the latter assumption, the question is which transfer functions in $G(z)$ can be uniquely reconstructed from the transfer matrix $CT(z)$. In recent work [Hendrickx et al. (2018)], this question has been considered for generic identifiability. Graph-theoretic conditions were given under which a set of transfer functions can be uniquely identified from $CT(z)$ for almost all network matrices $G(z)$ consistent with the graph $G$. For a formal definition of generic identifiability we refer to Definition 1 of [Hendrickx et al. (2018)]. Here, we will informally illustrate the approach of [Hendrickx et al. (2018)].

**Example 2.** Consider the graph $G = (V, E)$ depicted in Figure 1. We assume that the node signals of nodes 4 and 5 can be measured, and therefore we have $C = \{4, 5\}$. Suppose that we want to identify the transfer functions from node 1 to its out-neighbours, i.e., the transfer functions $G_{21}(z)$ and $G_{31}(z)$. According to Corollary 5.1 of [Hendrickx et al. (2018)], this is possible if and only if there exist two vertex-disjoint paths from $N_1^i$ to $C$. Note that this is the case in this example, since the edges (2, 4) and (3, 5) are two vertex-disjoint paths. To see why we can generically identify the transfer functions $G_{21}$ and $G_{31}$, we compute the transfer matrix $CT$ as follows:

$$CT = \begin{bmatrix} G_{42}G_{21} + G_{43}G_{31} & G_{42}G_{43} & 0 \\ G_{52}G_{21} + G_{53}G_{31} & G_{52}G_{42} & G_{53}G_{43} \end{bmatrix}.$$  

Clearly, we can uniquely identify the transfer functions $G_{42}, G_{43}, G_{52},$ and $G_{53}$ from $CT$. Moreover, the transfer matrices $G_{21}$ and $G_{31}$ satisfy

$$\begin{pmatrix} G_{42} & G_{43} \\ G_{52} & G_{53} \end{pmatrix} \begin{pmatrix} G_{21} \\ G_{31} \end{pmatrix} = \begin{pmatrix} T_{41} \\ T_{51} \end{pmatrix}.$$  

Equation (2) has a unique solution in the unknowns $G_{21}$ and $G_{31}$ if $G_{42}G_{53} - G_{43}G_{52} \neq 0$, which means that we can identify $G_{21}$ and $G_{31}$ for almost all $G$ consistent with the graph $G$ (specifically, for all $G$ except those for which $G_{42}G_{53} - G_{43}G_{52} = 0$).  

As mentioned before, the approach based on vertex-disjoint paths [Hendrickx et al. (2018)] gives necessary and sufficient conditions for generic identifiability. This implies that for some network matrices $G$, it might be impossible to identify the transfer functions, even though the path-based conditions are satisfied. For instance, in Example 2 we cannot identify the transfer functions $G_{21}$ and $G_{31}$ if the network matrix $G$ is such that $G_{42} = G_{43} = G_{52} = G_{53}$. Nonetheless, a scenario in which some of the transfer matrices in the network are equal may occur in practice. Instead of generic identifiability, in this paper we are interested in graph-theoretic conditions that guarantee identifiability for all network matrices consistent with a given graph. Such a problem might seem like a simple extension of the work on generic identifiability [Hendrickx et al. (2018)]. However, to analyze strong structural network properties (for all network matrices), we typically need different graph-theoretic tools than the ones used in the analysis of generic network properties. For instance, in the literature on controllability of dynamical networks, generic controllability (often called (weak) structural controllability) is related to maximal matchings [Liu et al. (2011)], while strong structural controllability is related to constrained matchings [Chapman and Mesbahi (2013)]. To make the problem of this paper more precise, we state a few definitions. Firstly, we are interested in conditions under which all transfer functions from a node $i$ to its out-neighbours $N_i^i$ are identifiable (for any admissible network matrix $G$, i.e., any $G$ that satisfies properties P1, P2, and P3). If this is the case, we say $(i, N_i^i)$ is identifiable. More precisely, we have the following definition.

**Definition 3.** Consider a directed graph $G = (V, E)$, let $C \subseteq V$, and define $C := P(V, C)$. Moreover, let $i \in V$ and consider its out-neighbours $N_i^i$. We say $(i, N_i^i)$ is identifiable from $C$ if for all admissible network matrices $G(z)$ and $\bar{G}(z)$, we have that

$$C(I - G(z))^{-1} = C(I - \bar{G}(z))^{-1} \implies \bar{G}_i(z) = \bar{G}_i(z).$$

In addition to identifiability of $(i, N_i^i)$, we are interested in conditions under which the entire network matrix $G$ can be identified from the transfer matrix $CT$. If this is the case, we say the graph $G$ is identifiable.

**Definition 4.** Consider a directed graph $G = (V, E)$, let $C \subseteq V$, and define $C := P(V, C)$. We say $G$ is identifiable from $C$ if for all admissible network matrices $G(z)$ and $\bar{G}(z)$, we have that

$$C(I - G(z))^{-1} = C(I - \bar{G}(z))^{-1} \implies G(z) = \bar{G}(z).$$

The main goal of this paper is to find graph-theoretic conditions under which $(i, N_i^i)$ is identifiable. Furthermore, based on such conditions, we want to establish graph-theoretic conditions under which $G$ is identifiable.

4. MAIN RESULTS

In this section we will present our main results. In Section 4.1 we give conditions for necessary and sufficient rank conditions for identifiability. Subsequently, in Section 4.2 we use these rank conditions to derive graph-theoretic tests for identifiability.

4.1 Rank conditions for identifiability

First, we give a condition for identifiability of $(i, N_i^i)$ in terms of the normal rank of $T_{C, N_i^i}(z)$ in Lemma 5. Note that this condition is similar to the one in Lemma 5.1 of [Hendrickx et al. (2018)], however, since we have additional assumptions (P2 and P3) on the network matrix, the result of [Hendrickx et al. (2018)] is not directly applicable to our setup. Therefore, we provide a proof of Lemma 5.

**Lemma 5.** Consider a directed graph $G = (V, E)$, and let $C \subseteq V$. Moreover, let $i \in V$ and consider its out-neighbours $N_i^i$. Then $(i, N_i^i)$ is identifiable from $C$ if and only if for any admissible network matrix $G(z)$ we have

$$\text{rank} T_{C, N_i^i}(z) = |N_i^i|,$$

where $T(z) := (I - G(z))^{-1}$.
Proof. For the ‘if’ part, suppose that rank $T_{C, N_i}(z) = |N_i|$ for any admissible network matrix $G(z)$. Let $G(z)$ be an admissible network matrix satisfying $C(I - G(z))^{-1} = C(I - G(z)) - 1$. We define $D(z) := G(z) - G(z)$, and note that the following four statements are equivalent:

$$
\begin{align*}
C &= C(I - G(z))^{-1}(I - G(z)) \\
C &= C(I - G(z))^{-1}(I - G(z) + D(z)) \\
C &= C + C(I - G(z))^{-1}D(z) \\
0 &= CT(z)D(z).
\end{align*}
$$

In particular, we obtain $CT(z)D_{\bullet,i}(z) = 0$. Since both $G$ and $\hat{G}$ are consistent with the graph, we have that $D_{\bullet,j}(z) = 0$ if $j \notin N_i$. Consequently, $T_{C, N_i}(z)D_{\bullet,j}(z) = 0$. By hypothesis, rank $T_{C, N_i}(z) = |N_i|$, and therefore $D_{\bullet,j}(z) = 0$. As a consequence, also $D_{\bullet,i}(z) = 0$. We conclude that $G_{\bullet,i}(z) = \hat{G}_{\bullet,i}(z)$, which shows that $(i, N_i)$ is identifiable.

For the ‘only if’ part, suppose that rank $T_{C, N_i}(z) < |N_i|$ for some admissible network matrix $G$. We want to prove that $(i, N_i)$ is not identifiable, that is, we want to prove the existence of an admissible network matrix $G$ such that $C(I - G(z))^{-1} = C(I - G(z))^{-1}$, but $G_{\bullet,i}(z) \neq \hat{G}_{\bullet,i}(z)$. Note that by our hypothesis, there exists a nonzero rational vector $\hat{w}(z)$ such that $T_{C, N_i}(z)\hat{w}(z) = 0$. Obviously, this means that $T_{C, N_i}(z)\hat{v}(z) = 0$, where $\hat{v}(z) := \alpha^{-1}\hat{w}(z)$ for any $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. We choose $k \in \mathbb{N}$ such that the nonzero entries of $\hat{v}(z)$ are strictly proper. Moreover, we choose $\alpha \in \mathbb{R} \setminus \{0\}$ such that all entries of $G_{\bullet,i}(z)\hat{v}(z)$ are nonzero, where $G_{\bullet,i}(z)$ is the vector obtained from $G_{\bullet,i}(z)$ by removing the entries corresponding to $V \setminus N_i$. Note that this is always possible, since $G$ is consistent with the graph, and therefore all entries of $G_{\bullet,i}(z)$ are nonzero. Let $v(z)$ denote the $n$-dimensional rational vector with the following two properties. Firstly, $v(z)$ has zero entries in positions corresponding to nodes in $V \setminus N_i$. Secondly, the vector obtained from $v(z)$ by removing all entries corresponding to $V \setminus N_i$ equals $\hat{v}(z)$. In addition, let the versor $u \in \mathbb{R}^n$ be such that $u_1 = 1$ and $u_j = 0$ for all $j \in V \setminus \{i\}$. We define the matrix $G(z) := G(z) - v(z)u^T$. Moreover, define $D(z) := v(z)u^T$, and note that (3) yields $C(I - G(z))^{-1} = C(I - G(z))^{-1}$. Furthermore, since $v(z)$ is nonzero, we immediately obtain $G_{\bullet,i}(z) \neq \hat{G}_{\bullet,i}(z)$. It remains to be shown that the matrix $G$ is admissible, i.e., that $G$ satisfies properties P1, P2, and P3.

Firstly, note that $G(z) = G(z) - v(z)u^T$ is the difference of a proper transfer matrix $G(z)$ and a matrix $v(z)u^T$ with entries that are either zero or strictly proper transfer functions. Consequently, each entry of $G(z)$ is a proper rational function, and $G$ satisfies P1. Secondly, to prove that $G$ satisfies P2, note that for all $k \in V \setminus \{i\}$ and all $j \in V$ we have $G_{jk}(z) = G_{jk}(z)$, and consequently, $G_{jk}(z) \neq 0$ if and only if $(k, j) \in E$. Moreover, by construction of $v(z)$, we have that $G_{ji}(z) \neq 0$ if and only if $j \in N_i$. We conclude that $G$ is consistent with the graph. Finally, to prove that $G$ satisfies property P3, note that

$$
\lim_{z \to \infty} (I - G(z)) = \lim_{z \to \infty} (I - G(z)) + \lim_{z \to \infty} v(z)u^T. \tag{4}
$$

since both limits on the right-hand side of (4) exist. In fact, since an entry of $v(z)$ is either zero or a strictly proper rational function, we have $\lim_{z \to \infty} v(z)u^T = 0$, and consequently

$$
\lim_{z \to \infty} (I - G(z)) = \lim_{z \to \infty} (I - G(z)).
$$

This shows that $G$ satisfies property P3 (as $G$ satisfies P3). To conclude, we have shown the existence of an admissible $G$ such that $C(I - G(z))^{-1} = C(I - G(z))^{-1}$, but $G_{\bullet,i}(z) \neq \hat{G}_{\bullet,i}(z)$. That is, $(i, N_i)$ is not identifiable. This proves the lemma.

As an immediate consequence of Lemma 5, we find conditions for the identifiability of $G$ based on the normal rank of transfer matrices.

Corollary 6. Consider a directed graph $G = (V, E)$, and let $C \subseteq V$ be a subset. Then $G$ is identifiable from $C$ if and only if for any admissible network matrix $G(z)$, the condition rank $T_{C, N_i}(z) = |N_i|$ holds for all $i \in V$, where $T(z) := (I - G(z))^{-1}$.

In the case that $T_{C, N_i}(z)$ is square, we can relate the rank condition with $T_{C, N_i}(z) = |N_i|$ to a rank condition of a certain submatrix of $I - G(z)$. Specifically, the following lemma states the equivalence of det $T_{C, N_i}(z) = 0$ and det $((I - G(z))N_i, C_i) = 0$, where the nonzero condition should be understood as nonzero as a rational function.

Lemma 7. Consider a directed graph $G = (V, E)$, let $i \in V$ and consider its out-neighbours $N_i$. Suppose that $C \subseteq V$ satisfies $|C| = |N_i|$. Let $G(z)$ be an admissible network matrix, and define $T(z) := (I - G(z))^{-1}$. We have det $T_{C, N_i}(z) = 0$ if and only if det $((I - G(z))N_i, C_i) = 0$.

Proof. We define $A(z) := \text{adj}(I - G(z))$. Note that det $T_{C, N_i}(z) = 0 \iff$ det $A_{C, N_i}(z) = 0 \iff$ det $((A^T(z))_{N_i, C_i}) = 0$.

Next, we apply Jacobi's identity for the determinant of a submatrix of the adjugate (cf. Theorem 2.5.2 of [Prasolov (1994)]), which shows that det $((A^T(z))_{N_i, C_i})$ is equal to $\pm$ det $((I - G(z))N_i, C_i)$ det $(I - G(z)|C_i|^{-1})$.

Since det $(I - G(z)) \neq 0$, this proves the lemma.

4.2 Graph-theoretic conditions for identifiability

In this section, we provide graph-theoretic conditions for identifiability. Before we continue with the technical details, we start with a simple example to give some intuition for the approach.

Example 8. Consider the graph in Figure 1. We saw in Example 2 that $(1, N_1)$ is not identifiable from $C = \{4, 5\}$, since (2) has multiple solutions in $G_{21}$ and $G_{31}$ in the case that $G_{42}G_{33} \sim G_{43}G_{52} = 0$. Suppose that we consider a slightly different graph, namely the one in Figure 2. We are

Fig. 2. Graph used in Example 8.
Consequently, we can identify the transfer functions identified from CT \((1)\) where \(G_{V1}\) and \(G_{V2}\) still interested in identifiability of \((1,N_1)\) from \(C = \{4, 5\}\). A simple calculation shows that in this case
\[
\begin{pmatrix}
G_{42} & G_{43} \\
0 & G_{33}
\end{pmatrix}
\begin{pmatrix}
G_{21} \\
G_{31}
\end{pmatrix} =
\begin{pmatrix}
T_{41} \\
T_{31}
\end{pmatrix},
\]
where \(G_{42}, G_{43}, G_{33}\) are transfer functions that can be identified from CT. Note that the matrix on the left-hand side of (5) has full rank for all nonzero \(G_{42}, G_{43}, \) and \(G_{33}\). Consequently, we can identify the transfer functions \(G_{21}\) and \(G_{31}\) (for all admissible network matrices \(G\)). That is, \((1,N_1)\) is identifiable from \(C\). We observe the following difference between the graphs in Figures 1 and 2: In Figure 1, there are two different sets of two vertex-disjoint paths from \(\{2, 3\}\) to \(\{4, 5\}\), while in the graph of Figure 2, there exists exactly one set of two vertex-disjoint paths between these vertices. Therefore, it seems that identifiability of \((i,N_i)\) does not only depend on the existence of \(\{N_i\}\) vertex-disjoint paths from \(N_i\) to \(C\) (as is the case for generic identifiability [Hendrickx et al. (2018)]) , but also depends on the number of such sets of vertex-disjoint paths.

To make the idea of Example 8 more precise, we need the notion of constrained vertex-disjoint paths.

**Definition 9.** Consider a directed graph \(G = (V,E)\), and let \(V_1, V_2 \subseteq V\) be two subsets. Consider a set of \(m\) vertex-disjoint paths from \(V_1\) to \(V_2\), and let \(V_1 \subseteq V_1 \) and \(V_2 \subseteq V_2\) be the sets of starting nodes and end nodes, respectively. We say that the set of vertex-disjoint paths from \(V_1\) to \(V_2\) is constrained if it is the only set of \(m\) vertex-disjoint paths from \(V_1\) to \(V_2\).

**Remark 10.** Note that for a set of \(m\) vertex-disjoint paths from \(V_1\) to \(V_2\) to be constrained, we do not require to be a unique set of \(m\) vertex-disjoint paths from \(V_1\) to \(V_2\). In fact, we only require there to be a unique set of vertex-disjoint paths between the starting nodes \(V_1\) of the paths and the end nodes \(V_2\). We will illustrate the definition of constrained vertex-disjoint paths in Example 12.

**Remark 11.** The notion of constrained vertex-disjoint paths is strongly related to the notion of constrained matchings in bipartite graphs [Hershkowitz and Schneider (1993)]. In fact, a constrained matching can be seen as a special case of a constrained set of vertex-disjoint paths, where all paths are of length one.

**Example 12.** Consider the graph \(G = (V,E)\) in Figure 3. Moreover, consider the subsets of vertices \(V_1 := \{2, 3\}\) and \(V_2 := \{6, 7, 8\}\). Clearly, the edges \((2, 4), (4, 6)\) together are also other sets of vertex-disjoint paths from \(V_1\) to \(V_2\). For example, also the edges \((2, 4), (4, 7)\) together with \((3, 5), (5, 8)\) form two vertex-disjoint paths. However, this set of vertex-disjoint paths is not constrained. To see this, note that we have another set of vertex-disjoint paths from \(V_1 = \{2, 3\}\) to \(V_2 = \{7, 8\}\), namely the paths formed by the edges \((2, 4), (4, 8)\) and \((3, 5), (5, 7)\).

The following theorem gives graph-theoretic conditions for identifiability of \((i,N_i)\) in terms of constrained vertex-disjoint paths.

**Theorem 13.** Consider a directed graph \(G = (V,E)\), let \(i \in V\) and consider its out-neighbours \(N_i\). Moreover, let \(C \subseteq V\) be a subset. Then, \((i,N_i)\) is identifiable from \(C\) if there exists a constrained set of \(\{N_i\}\) vertex-disjoint paths from \(N_i\) to \(C\).

**Remark 14.** We emphasize that, by convention, there is a path of length zero from each node in \(V\) to itself (see Section 2.1). Consequently, if \(N_i \cap C \neq \emptyset\), then the condition of Theorem 13 is equivalent with the existence of a constrained set of \(\{N_i\} \setminus C\) vertex-disjoint paths from \(N_i \setminus C\) to \(\emptyset\). In particular, \((i,N_i)\) is identifiable in the case that \(N_i \subseteq C\).

**Proof.** Suppose there exists a constrained set of \(\{N_i\}\) vertex-disjoint paths from \(N_i\) to \(C\). This means that there exists a subset \(C \subseteq C\) with \(\{N_i\} = \{|C|\}\) such that there is a constrained set of \(\{N_i\}\) vertex-disjoint paths from \(N_i\) to \(C\). We want to prove that \((i,N_i)\) is identifiable from \(C\). By Lemma’s 5 and 7, this is equivalent to proving that det \((I - G(z))_{N_i,C}^{-1} \geq 0\) for all admissible network matrices \(G(z)\).

We partition the vertex set \(V\) into four disjoint subsets, namely \(R, N_i \setminus C, N_i \cap C, C \setminus N_i\). Here the set \(R\) contains all the vertices that are not contained in any of the three other sets. Let \(G(z)\) be an admissible network matrix. In accordance with the above partition of the vertex set, we write
\[M := (I - G)_{N_i,C}^{-1} = \left( \begin{array}{cc}
(I - G_{R,R})^{-1} & -G_{R,N_i,C}^{-1} \\
-G_{C,R,N_i}^{-1} & (I - G_{C,N_i,C})^{-1}
\end{array} \right),\]
where we have omitted the dependence on \(z\) for the sake of brevity. Let \(p = n - |N_i|\) be the number of rows (and columns) of \(M\). With \(M\), we associate a weighted directed graph \(G_M = (V_M, E_M)\), where \(V_M = \{1, 2, \ldots, p\}\) and \(E_M := \{(k,l) \mid M_{kl} \neq 0\}\). Furthermore, each edge \((k,l) \in E_M\) is weighted by \(M_{kl}\). Such a graph is sometimes called a Coates graph (see, e.g., [Chen (1971)], [van der Woude (1991)]). Note that the graph \(G_M\) contains self-loops even though \(G\) was assumed to be simple. We observe that vertex-disjoint paths from \(N_i \setminus C\) to \(C \setminus N_i\) in \(G\) correspond with vertex-disjoint cycles in the graph \(G_M\) since columns in \(M\) corresponding to \(N_i \setminus C\) have the same indices as the rows of \(M\) corresponding to \(C \setminus N_i\).

It is known that the determinant of \(M\) can be expressed as a sum of the weights of spanning cycle families in \(G_M\). Recall from Section 2.1 that a spanning cycle family in \(G_M\) is a collection of vertex-disjoint cycles such that each vertex in \(V_M\) appears in one of the cycles. To be precise, we express det \(M\) as (cf. Theorem 3.1 of [Chen (1971)]
\[\det M = \pm \sum_{F} (-1)^{\nu_F} w(F),\]

![Fig. 3. Graph used in Example 12.](image-url)
where $\mathcal{F}$ is a spanning cycle family in $\mathcal{G}_M$. $w(\mathcal{F})$ denotes the weight of the spanning cycle family (i.e., the product of the weights of all cycles in $\mathcal{F}$), the integer $N_{\mathcal{F}} \in \mathbb{N}$ denotes the number of cycles in $\mathcal{F}$, and the sum is taken over all spanning cycle families in $\mathcal{G}_M$.

By our hypothesis, there exists a constrained set of $|N_i|$ vertex-disjoint paths from $N_i$ to $C$. This implies that there exists a constrained set of $|N_i| \setminus C$ vertex-disjoint paths from $N_i \setminus C$ to $C \setminus N_i$. In terms of $\mathcal{G}_M$, this means that there is exactly one set of $|N_i| \setminus C$ vertex-disjoint cycles in $\mathcal{G}_M$ that contain the nodes $\{1, \ldots, p\} \subseteq V_M$. We will denote this set of vertex-disjoint cycles by $\mathcal{F}_1$. The previous discussion implies that each spanning cycle family in $\mathcal{G}_M$ contains the cycles in $\mathcal{F}_1$. In addition, the weight of $\mathcal{F}_1$, denoted by $w(\mathcal{F}_1)$, equals the weight of the set of vertex-disjoint paths from $N_i \setminus C$ to $C \setminus N_i$ in $\mathcal{G}$. Since each spanning cycle family in $\mathcal{G}_M$ contains the cycles in $\mathcal{F}_1$, we can rewrite the formula for $\det M$ as

$$\det M = \pm w(\mathcal{F}_1) \sum_{\mathcal{F} \in \mathcal{I}} (-1)^{|N_{\mathcal{F}}|} w(\mathcal{F}_2).$$

Here $\mathcal{F}_2$ is a spanning cycle family of the subgraph of $\mathcal{G}_M$ obtained by removing all nodes (and incident edges) from $\mathcal{G}_M$ that appear in a cycle in $\mathcal{F}_1$. Moreover, $N_{\mathcal{F}_2}$ denotes the number of cycles in $\mathcal{F}_2$, and the sum is taken over all spanning cycle families of the subgraph of $\mathcal{G}_M$. Again, using Theorem 3.1 of [Chen (1971)], we obtain

$$\det M = \pm w(\mathcal{F}_1) \det(I - G_{\mathcal{R}, \mathcal{R}}),$$

where $\mathcal{R} \subseteq \mathcal{R}$ is the set of nodes in $\mathcal{R}$ that do not appear in one of the vertex-disjoint paths from $N_i \setminus C$ to $C \setminus N_i$ in $\mathcal{G}$ and $G_{\mathcal{R}, \mathcal{R}}$ is the principal submatrix of $G$ corresponding to the rows and columns indexed by $\mathcal{R}$. Recall that $w(\mathcal{F}_1)$ is equal to the weight of the set of vertex-disjoint paths from $N_i \setminus C$ to $C \setminus N_i$. Hence, $w(\mathcal{F}_1) \neq 0$. Furthermore, since $G$ is admissible, it satisfies property P3, and therefore $\det(I - G_{\mathcal{R}, \mathcal{R}}) \neq 0$. We conclude that $\det M \neq 0$. By combining Lemma 5 and Lemma 7, we conclude that $(i, N_i)$ is identifiable from $C$ (and hence, from $\mathcal{C}$).

Example 15. Consider the graph depicted in Figure 3. For this example, let $\mathcal{C} := \{6, 7, 8\}$. Suppose we are interested in identifying the transfer functions associated with the edges from node 1 to its out-neighbours $N_1 = \{2, 3\}$. To check that $(1, N_1)$ is identifiable from $C$, we use Theorem 13. In Example 12, we already saw that there exists a constrained set of two vertex-disjoint paths from $N_1$ to $C$, formed by the edges $(2, 4), (4, 6)$ and $(3, 5), (5, 7)$. Therefore, we conclude by Theorem 13 that $(1, N_1)$ is identifiable. That is, for any admissible network matrix $G(z)$ associated with $\mathcal{G}$ in Figure 3, we can uniquely identify the transfer functions $G_{21}(z)$ and $G_{31}(z)$.

The following result gives graph-theoretic conditions under which all transfer functions in $G$ are identifiable. The result follows from Theorem 13.

Theorem 16. Consider a directed graph $G = (V, E)$, and let $C \subseteq V$. Then, $G$ is identifiable from $C$ if for each $i \in V$ there exists a constrained set of $|N_i|_v$ vertex-disjoint paths from $N_i \setminus C$ to $C$.

Using Theorem 16 we can show that all transfer functions appearing in the network of Figure 3 are identifiable, by measuring the node signals of just nodes 6, 7, and 8.

5. CONCLUSIONS

In this paper we have considered the problem of identifiability of dynamical networks with partial node measurements. Unlike previous work [Hendrickx et al. (2018)] that considers generic identifiability, we consider identifiability for all network matrices (satisfying certain properties). We have introduced the new notion of constrained vertex-disjoint paths, and as our main result we have given a sufficient graph-theoretic condition for identifiability in terms of such paths. The concept of constrained vertex-disjoint paths seems to be an interesting generalization of constrained matchings, and the authors are currently investigating whether this concept can be used to provide necessary and sufficient conditions for identifiability.

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