Gravitational radiation reaction

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We give a short personally-biased review on the recent progress in our understanding of gravitational radiation reaction acting on a point particle orbiting a black hole. The main motivation of this study is to obtain sufficiently precise gravitational waveforms from inspiraling binary compact stars with a large mass ratio. For this purpose, various new concepts and techniques have been developed to compute the orbital evolution taking into account the gravitational self-force. Combining these ideas with a few supplementary new ideas, we try to outline a path to our goal here.

§1. Introduction

Ground-based interferometric gravitational wave detectors have started operation and are also in the phase of rapid improvement.\textsuperscript{1)–4)} R&D studies of a space-based gravitational wave observatory project, the Laser Interferometer Space Antenna (LISA),\textsuperscript{5)} which observes in the mHz-band, are in rapid progress. There is also proposal for a DECi hertz Interferometer Gravitational wave Observatory (DECIGO/BBO),\textsuperscript{6), 7)} which will be a laser interferometer gravitational wave antenna in space sensitive at $f \sim 0.1$Hz.

Among the promising targets for space interferometers, the most precise theoretical prediction and observational measurement of gravitational waveform are expected from the inspiral stage of binary systems comprising of a supermassive black hole ($M \sim 10^5$–$8 \times 10^8 M_\odot$) and a compact object of solar mass ($\mu \sim 1$–$10 M_\odot$).\textsuperscript{8)} Here the meaning of “precise” is two-fold. One is that the number of cycles is large and the other is that higher order post-Newtonian corrections are necessary. These gravitational wave sources can provide the first high-precision test of general relativity in very strong gravitating regimes.\textsuperscript{9)}

On the theoretical side, to obtain sufficiently precise templates, the standard post-Newtonian approximation\textsuperscript{10)} seems to be insufficient since the accessible highest order of expansion is practically limited. However, there is a natural expansion parameter in this system, that is the mass ratio $\mu/M$. Therefore, in place of the standard post-Newtonian expansion, there is a possibility to develop perturbation theory based on the well-understood black hole linear perturbation theory\textsuperscript{11)–16)} for binaries with extreme mass ratios. To develop a method to compute waveforms on this line is our purpose of studying gravitational radiation reaction.

This paper is organized as follows. In Sec. 2 we briefly discuss the very basics of black hole perturbation approach, followed by a discussion on balance argument about radiation reaction, given in Sec. 3. In Secs. 4 and 5, we consider radiation reaction in adiabatic approximation. The basic idea is explained in Sec. 4, while the recent developments, which have led to a very concise formula for the change rate
of Carter constant, are presented in Sec. 5. A method for long time integration is also discussed. In Secs. 6 and 7, we discuss the instantaneous self-force. The main stream of the procedure about how to evaluate the self-force, although biased by my personal view, is explained in Sec. 6, supplemented with comments on some topics given in Sec. 7. The discussions presented in Sec. 5 might be beyond the level of an introductory review paper. Section 8 is devoted to Summary.

In this paper we use the units in which $8\pi G_N = 1$.

§2. Before radiation reaction

We model a binary system by a point particle of mass $\mu$ orbiting a black hole of mass $M$ assuming $\mu \ll M$. In the lowest order in the mass ratio, $O((\mu/M)^0)$, the particle moves along a geodesic on the background geometry. Using the black hole perturbation, we can compute the energy and angular momentum flux carried by gravitation waves emitted by the particle to the infinity or into the horizon. Already in this lowest order approximation, this black hole perturbation approach has proven to be very powerful for evaluating general relativistic gravitational waveforms.

The perturbed metric is expanded as

$$g_{\mu\nu} = g_{\mu\nu}^{BH} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + \cdots.$$  

Then the linearized perturbed Einstein equations

$$\delta G_{\mu\nu}[h^{(1)}] = T_{\mu\nu}^{(1)},$$

become coupled equations for metric perturbations $h^{(1)}$ in general, and are difficult to solve. However, when the background spacetime is given by Schwarzschild or Kerr black hole, perturbation equations can be written in terms of a single master equation due to symmetry. Here we schematically write the master equation as

$$L\zeta^{(1)} = \sqrt{-g} T^{(1)},$$

where $L$ is a second order differential operator and $\zeta^{(1)}$ is a master variable in the linear order. The source $T^{(1)}$ is a function obtained from the energy momentum tensor.

This reduction is called Regge-Wheeler-Zerilli formalism in the Schwarzschild case\cite{11,12} and Teukolsky-Sasaki-Nakamura formalism in the Kerr case\cite{13,14}. In the latter case, starting with metric perturbations is not very successful. Instead, we use Newman-Penrose quantities defined by, say,

$$-2\psi \approx -C_{\mu\nu\rho\sigma} n^{\mu} \bar{m}^{\rho} m^{\sigma},$$

as a master variable. This variable is a contraction of Weyl tensor $C_{\mu\nu\rho\sigma}$ with two null tetrad bases, $n^{\mu}$ (in-going) and $\bar{m}^{\mu}$ (angular). Here note that $m^{\mu}$ is complex valued and "$\bar{\phantom{a}}$" represents complex conjugation. We can similarly define $2\bar{\psi}$ by considering the contraction with $\ell^{\mu}$ (out-going) and $m^{\mu}$. As a result, we have two equations

$$sL_s\psi = \sqrt{-g} sT, \quad (s = \pm 2)$$

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$$sL_s\psi = \sqrt{-g} sT, \quad (s = \pm 2)$$
but they are not mutually independent. The source $sT$ is obtained by applying a corresponding second order differential operator $s\tau_{\mu\nu}$ to the energy momentum tensor $T_{\mu\nu}$.

Thus obtained master equation is separable. We introduce a harmonic function which describes angular dependence $sS_{\Lambda}(\theta, \varphi)$, where $\Lambda = \{\ell, m, \omega\}$ is a set of two angular eigenvalues and one frequency eigenvalue. Homogeneous solution can be found in the form of a linear combination of mode functions $s\Omega_{\Lambda} = sR_{\Lambda}(r) sS_{\Lambda}(\theta, \varphi) e^{-i\omega t}$. Substituting this form, the problem to find a homogeneous solution reduces to solving a second order ordinary differential equation

$$\left[\partial_r^2 + \cdots \right] sR_{\Lambda}(r) = 0. \quad (2.3)$$

By using homogeneous solutions which satisfy appropriate boundary conditions, we can construct Green function. Then, the solution with source is given by

$$s\psi \approx \sum_{\Lambda} \frac{1}{sW_{\Lambda}} s\Omega_{\Lambda}^{up}(x) \int \sqrt{-g} dt' x' s\Omega_{\Lambda}^{out}(x') T(x') \theta(r - r') + \cdots, \quad (2.4)$$

where $up$ and $out$, respectively, mean that there is no in-coming wave from the past infinity and that there is no wave absorbed into the future horizon. Here, “$\cdots$” stands for the part which contains $\theta(r' - r)$. We have introduced spin-flipped mode functions, $s\tilde{\Omega}_{\Lambda}(x') = -sR_{\Lambda}(r) sS_{\Lambda}(\theta, \varphi) e^{-i\omega t}$. Notice that $s\tilde{\Omega}_{\Lambda}^{out}(x') = sR_{\Lambda}^{in}(r) sS_{\Lambda}(\theta, \varphi) e^{-i\omega t}$. (Since the differential operator $L$ for the Teukolsky equation is not real, a simple complex conjugation does not give a solution of the same equation but it becomes a solution of the equation with its spin flipped from $s$ to $-s$. Therefore we cannot choose $sR_{\Lambda}^{in} = sR_{\Lambda}^{out}$ as usual, but we can choose $sR_{\Lambda}^{in} = -sR_{\Lambda}^{out}$. ) $sW_{\Lambda}$ is the Wronskian between these two radial functions, i.e.,

$$sW_{\Lambda} = W(sR_{\Lambda}^{up}, sR_{\Lambda}^{in}(r)) \approx sR_{\Lambda}^{up}(r) \frac{\partial sR_{\Lambda}^{in}(r)}{\partial r} - sR_{\Lambda}^{in}(r) \frac{\partial sR_{\Lambda}^{up}(r)}{\partial r}. \quad (2.5)$$

There is a systematic method to solve homogeneous Teukolsky equation as a series expansion. Using this method, one can rather easily obtain the solution, and we can write down the expression analytically in explicit form once we invoke the post-Newtonian expansion.

To estimate the fluxes at $r \to \infty$, one can use a simple relation, $-2\psi \approx \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_x)$. Using this relation, one can estimate, say, the energy loss rate due to gravitational wave emission per unit steradian as

$$\mu \frac{d^2 E_{GW}}{dt d\Omega} = \frac{r^2}{4\pi\omega^2} |-2\psi|^2.$$

In this paper, we use $E$ ($L$) as the specific energy (angular momentum) per unit mass of the particle. The contribution from the waves absorbed by the black hole can be evaluated in a similar manner.

§3. Balance argument and its limitation

When radiation reaction is not very significant, the trajectory of a particle is almost a geodesic. In such a case one can estimate the evolution of orbital frequency
by assuming that the orbit evolves losing its energy and angular momentum as much as those emitted as gravitational waves.\(^{17}\)

In the case of circular orbits, the evolution of the orbital frequency almost determines the gravitational waveform (except for the change of amplitude). Equating the energy lost through the gravitational wave emission with the minus of the binding energy, the change rate of the orbital frequency \(f\) is evaluated as

\[
\frac{df}{dt} = -\frac{dE_{GW}}{dt} \left( \frac{dE_{\text{orbit}}}{df} \right)^{-1}.
\] (3.1)

Then the leading order of \(dE_{GW}/dt\) starts with \(O(\mu)\). (Recall that \(E\) is the specific energy. In counting of the order of magnitude, we are assuming that \(M\) is \(O(1)\).) This is because the energy was exactly conserved, if the radiation reaction were turned off. On the other hand, the leading order of the relation between the binding energy and the orbital frequency, \(dE_{\text{orbit}}/df\), is determined by the geodesic motion on a given background black hole spacetime. The effect of the self-force, which is the force acting on a particle due to the metric perturbation caused by the particle itself, is higher order correction of \(O(\mu)\) in \(dE_{\text{orbit}}/df\). Roughly speaking, to obtain the leading order correction to the waveform, we therefore have only to know the effect of the self-force on \(dE_{GW}/dt\). (We will return to this point in Sec. 5.3.)

The extension to more general orbits is straightforward in the case of Schwarzschild background. In this case one can assume that the orbit is on the equatorial plane without loss of generality. Then, the geodesics are specified by the energy \(E\) and the \(z\)-component of the angular momentum \(L\). Both “constants of motion” have the associated timelike and rotational Killing vectors, \(\eta^\mu_{(t)} \equiv (\partial_t)^\mu\) and \(\eta^\mu_{(\varphi)} \equiv (\partial_\varphi)^\mu\). Therefore we can define conserved current from the effective energy momentum tensor \(t_{\mu\nu}\), which satisfies the conservation law \(t^{\mu\nu};_\nu = 0\) with respect to the background covariant derivative, as

\[
j^{(E)}_{\mu} = \mu^{-1}t_{\mu\nu}\eta_{(t)}^\nu.
\] (3.2)

Then \(j^{(E)}_{\mu}\) satisfies \(j^{(E)}_{\mu};^\mu = 0\), and hence one can define the conserved (specific) energy by the integral over spatial three surface \(\Sigma\) as

\[
E = -\int_\Sigma d\Sigma^\mu j^{(E)}_{\mu}.
\] (3.3)

Let’s choose the boundary of \(\Sigma\) to be a sphere \(S\) at a fixed radius, which is supposed to be sufficiently large. In this case the change rate of \(E\) is given by

\[
\frac{dE}{dt} = -\int_S dS j^{(E)}_{r},
\] (3.4)

where \(dS\) is an infinitesimal element of area on \(S\). In the above discussion, we assumed the existence of the effective energy momentum tensor \(t_{\mu\nu}\) that satisfies the conservation law. Now we directly derive it from the Einstein equation. The Einstein tensor can be expanded as

\[
G_{\mu\nu}[g + h] = G^{[0]}_{\mu\nu} + G^{[1]}_{\mu\nu}[h] + G^{[2]}_{\mu\nu}[h, h] + \cdots.
\] (3.5)
Since the background metric $g$ satisfies the vacuum Einstein equations in the present case, $G^{[0]}_{\mu\nu} = 0$. Using this relation, the contracted Bianchi identity at the linear order in $h$ becomes

$$G^{[1]}_{\mu\nu}[h] = 0$$  \hspace{1cm} (3.6)

Here a semicolon denotes a covariant differentiation with respect to the background metric. Substituting $h = h^{(1)} + h^{(2)} + \ldots$ into the Einstein equations and keeping the terms up to second order, we obtain

$$G^{[1]}_{\mu\nu}[h^{(1)}] + G^{[1]}_{\mu\nu}[h^{(2)}] + G^{[2]}_{\mu\nu}[h^{(1)}, h^{(1)}] = T_{\mu\nu}. \hspace{1cm} (3.7)$$

From the background covariant derivative of this equation, we have

$$G^{[2]}_{\mu\nu}[h^{(1)}] + G^{[2]}_{\mu\nu}[h^{(2)}] + G^{[2]}_{\mu\nu}[h^{(1)}, h^{(1)}] = T_{\mu\nu}. \hspace{1cm} (3.8)$$

Hence we find that

$$t_{\mu\nu} \equiv T_{\mu\nu} - G^{[2]}_{\mu\nu}[h^{(1)}, h^{(1)}] \hspace{1cm} (3.9)$$

satisfies the conservation law with respect to the background covariant derivative. One may identify the second term as the effective energy momentum tensor of gravitational waves $t^{(GW)}_{\mu\nu}$. Accordingly, one can divide the energy $E$ into two parts: $E = E_{\text{orbit}} + E_{GW}$. $E_{\text{orbit}}$ is the contribution from $T_{\mu\nu}$ while $E_{GW}$ is that from $t^{(GW)}_{\mu\nu}$.

Now we should note that $E_{\text{orbit}}$ is $O(1)$ while $E_{GW}$ is $O(\mu)$. Keeping this fact in mind, we look at Eqs. (3.3) and (3.4) again. Then $E$ is $O(1)$ and the leading term comes from $E_{\text{orbit}}$. On the other hand, $dE/dt$ does not have contribution from $T_{\mu\nu}$, and hence it is $O(\mu)$. However, we do not conclude that the radiation reaction is unimportant. After integration over a long period of $O(\mu^{-1})$ the change in $E$ becomes $O(1)$. This change in $E$ must be attributed to the change of $T_{\mu\nu}$ because $E_{GW}$ should stay $O(\mu)$. This consideration establishes the balance argument to the motion of a particle:

$$\frac{dE_{\text{orbit}}}{dt} \approx - \int_S dS j^r(\Sigma). \hspace{1cm} (3.10)$$

In the case of Kerr background, we do not have spherical symmetry. Hence, one cannot say that the orbits are on the equatorial plane in general. To specify geodesics off the equatorial plane, we need to consider another “constant of motion”. It is well known that the geodesics on the Kerr background possess the third “constant of motion” called Carter constant. However, Carter constant is not associated with any Killing vector field. Instead, it is related to a rank two Killing tensor.

Let us examine the “constants of motion” in Kerr in more detail. The background Kerr spacetime in the Boyer-Lindquist coordinates is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\varphi + \sum_{\Delta} dr^2 + \Sigma d\theta^2 + \left[r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right]\sin^2\theta d\varphi^2. \hspace{1cm} (3.11)$$
where Kerr parameter \( a \) is the angular momentum of the black hole divided by its mass, \( \Sigma := r^2 + a^2 \cos^2 \theta \), and \( \Delta := r^2 - 2Mr + a^2 \). Killing tensor in Kerr spacetime is given by

\[
K_{\mu\nu} := 2\Sigma l(\mu) n(\nu) + r^2 g_{\mu\nu},
\]

where the parentheses denote symmetrization on the indices enclosed, and \( l(\mu) := (r^2 + a^2, \Delta, 0, a) / \Delta \) and \( n(\mu) := (r^2 + a^2, -\Delta, 0, a) / 2\Sigma \) are out-going and in-going radial null vectors, respectively. Killing tensor satisfies the equation

\[
K(\mu\nu; \rho) = 0.
\]

Using this Killing tensor, the Carter constant is defined as

\[
Q \equiv K_{\alpha\beta} u^\alpha u^\beta,
\]

where \( u^\alpha := dz^\alpha / d\tau \) is the four velocity of an orbiting particle. We often use another notation for the Carter constant, \( C := Q - (a\mathcal{E} - \mathcal{L})^2 \), defined in such a way that it vanishes for orbits on the equatorial plane. By using the symmetry of the Killing tensor, it is easy to check that thus defined Carter constant does not vary along geodesic. For Carter constant, however, we cannot define a quantity corresponding to \( E_{GW} \). Therefore there is no counterpart of Eq. (3.10) for \( dQ / dt \).

\section{Adiabatic approximation for \( dQ / dt \)}

As we have seen in the previous section, one cannot use the balance argument to evaluate the change rate of \( Q \). Then, to evaluate \( dQ / dt \), we need to compute the self-force acting on the particle directly.\(^{20} \) Though the prescription to calculate the self-force is formally established,\(^{21)-23} \) performing explicit calculation is not so straightforward. However, it was found to be easier to compute the averaged value of \( dQ / dt \). As an approximation, we may use the averaged values of the change rates for the “constants of motion” instead of those evaluated by using the instantaneous self-force. We call it adiabatic approximation. This approximation will be as good as an estimate using the balance argument, and moreover is also applicable to \( dQ / dt \).

\subsection{Use of radiative field}

Gal’tsov\(^{24} \) advocated to use the radiative part of the metric perturbation, which was introduced earlier by Dirac,\(^{25} \) to calculate \( d\mathcal{E} / dt \) and \( d\mathcal{L} / dt \). The radiative field is defined by half retarded field minus half advanced one. The divergent part contained in the retarded field is common to that contained in the advanced field. Therefore the combination of the radiative field is free from divergence. Hence as far as we discuss the self-force composed of the radiative field, we do not have to worry about how to regularize the force. The question however remains whether the radiative self-force correctly reproduces the result which is obtained by using the retarded field with the aid of appropriate regularization procedure.

Gal’tsov has shown that the radiative field correctly reproduces the results obtained by using the balance argument for \( d\mathcal{E} / dt \) and \( d\mathcal{L} / dt \) when they are averaged over an infinitely long time interval assuming a geodesic motion as a source of metric
perturbation. When $\mu$ is thought to be an infinitesimal expansion parameter, the trajectory of the particle follows exactly a geodesic at the lowest order. In this sense, in the lowest order, this averaging over a long period of time assuming a geodesic motion is justified when we evaluate the self-force to the lowest order in $\mu$.

However, there had been no justification for applying the same scheme to $dQ/dt$ until very recently. The breakthrough was brought by Mino, who gave a justification for applying the same scheme to $dQ/dt$.\(^{26}\) (See also Ref. 27). Namely, he has proven that

$$\left\langle \frac{dQ}{d\tau} \right\rangle = \frac{1}{\mu} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\tau \frac{\partial Q}{\partial u^\alpha} F^\alpha [h^{(rad)}], \quad (4.1)$$

where $F^\alpha [h^{(rad)}]$ is the self-force evaluated by using the radiative field $h^{(rad)} \equiv (h^{(ret)} - h^{(adv)})/2$.\(^{26}\)

The key observation in his proof is the invariance of the geodesics under the transformation:

$$a \to -a, \quad (t, r, \theta, \varphi) \to (-t, r, \theta, -\varphi).$$

All geodesics, unless the values of $(E, L, Q)$ are fine tuned, transform into themselves under the above transformation if we choose the origin of $t$ and $\varphi$ appropriately. Once we notice this symmetry, it is easy to show that

$$\left\langle \left( \frac{dQ}{d\tau} \right)_{\text{ret}} \right\rangle = - \left\langle \left( \frac{dQ}{d\tau} \right)_{\text{adv}} \right\rangle, \quad (4.2)$$

where the superscripts $(\text{ret})$ and $(\text{adv})$ indicate that the retarded field $h^{(ret)}$ and the advanced field $h^{(adv)}$ are used instead of $h^{(rad)}$ in Eq. (4.1). The relation (4.2) justifies the use of the formula (4.1).

The radiative Green function has a simple structure which does not contain any step function $\theta(r - r')$:

$$G^{(rad)}(x, x') = \sum_A \frac{1}{W(sR^{up}_A, sR^{in}_A)W(sR^{down}_A, sR^{out}_A)} \times \left( W(sR^{in}_A, sR^{out}_A)S_A^{down}(x)\tilde{G}^{down}_A(x') + W(sR^{down}_A, sR^{up}_A)S_A^{out}(x)\tilde{G}^{out}_A(x') \right), \quad (4.3)$$

To show this, let us start with the following defining expression of the radiative Green function for $r > r'$:

$$\sum_A e^{-i\omega(t-t')} \left[ \frac{sR^{up}_A(r)sR^{in}_A(r')} {W(sR^{up}_A, sR^{in}_A)} - \frac{sR^{down}_A(r)sR^{out}_A(r')} {W(sR^{down}_A, sR^{out}_A)} \right] sS_A(\theta, \varphi)\tilde{S}_A(\theta', \varphi'). \quad (4.4)$$

Since our goal is to obtain an expression in terms of $\text{down}$-field and $\text{out}$-field, we want to eliminate $sR^{up}_A(r)$ and $sR^{out}_A(r') (= -sR^{in}_A(r'))$ in Eq. (4.4). Hence we expand $sR^{up}$ and $sR^{out}$ as

$$sR^{up} = \alpha sR^{out} + \beta sR^{down}$$
Substituting these relations, the expression (4.4) reduces to (4.3). We can do an analogous reduction for one can easily obtain taking the Wronskian of both sides of Eqs. (4.5) with appropriate radial functions, one can easily obtain

\[ W(sR^{up}, sR^{down}) = \alpha W(sR^{out}, sR^{down}), \quad W(sR^{up}, sR^{out}) = \beta W(sR^{down}, sR^{out}), \quad W(sR^{out}, sR^{in}) = \gamma W(sR^{up}, sR^{in}), \quad W(sR^{out}, sR^{up}) = \delta W(sR^{in}, sR^{up}). \]

Substituting these relations, the expression (4.4) reduces to (4.3). We can do an analogous reduction for the factorized form of the tensor Green function as analogous reduction for the factorized form of the tensor Green function. This assumption is not necessary if we follow the derivation given by Wald, \(^{29}\) and actually this assumption itself is not correct. \(^{30}\)

4.2. metric reconstruction \(^{28}\)

In order to use the formula (4.3), we need to know how to reproduce the metric perturbations from the master variable \(s\psi\). In the present case, what we have to deal with is a source-free homogeneous solution. This fact simplifies the reconstruction significantly \(^{s}\). The method was originally given by Chrzanowski. \(^{29, 31}\) Here, we present the basic idea of the derivation of reconstruction formula, neglecting details.

We formally write the metric perturbation induced by a source \(T^{\alpha \beta}\) as

\[ h_{\mu \nu}(x) = \int \sqrt{-g} d^4 x' G^{(ret)}_{\mu \nu \alpha \beta}(x, x') T^{\alpha \beta}(x') \] (4.6)

We assume that \(T^{\alpha \beta}(x')\) is localized within \(r' < r_0\). And we assume a factorized form of the tensor Green function as

\[ G^{(ret)}_{\mu \nu \alpha \beta}(x, x') = \sum_A \frac{1}{sN_A sW_A} \left( \Pi^{up}_{\alpha \mu}(x) \Pi^{out}_{\alpha \beta}(x') \theta(r - r') + \cdots \right), \] (4.7)

where \(\Pi_{\alpha \mu}\) is the mode function for the metric perturbation, whose explicit form is unknown at this point. Since we do not know how to normalize the mode function for metric perturbation, we have introduced a constant \(sN\) to take care of this normalization. \(sW_A\) is the Wronskian for the corresponding mode function for the master variable defined by (2.5).

The master variable can be computed from the metric perturbation by applying a second order differential operator as \(s\Omega_A = sD^{\mu \nu} \Pi_{\alpha \mu\nu}\). Hence, we have

\[ s\psi = sD^{\mu \nu} h_{\mu \nu} = \sum_A \frac{1}{sN_A sW_A} s\Omega^{up}_A(x) \int \sqrt{-g} d^4 x' \bar{\Pi}^{out}_{\alpha \beta}(x') T^{\alpha \beta}(x'), \] (4.8)

for \(r > r_0\), outside the source distribution. On the other hand, we can evaluate the same quantity by using the Green function for the master variable, which leads to

\[ s\psi = sD^{\mu \nu} h_{\mu \nu}(x) = \sum_A \frac{1}{sW_A} s\Omega^{up}_A(x) \int \sqrt{-g} d^4 x' \bar{\Omega}^{out}_A(x') sT(x'). \] (4.9)

\(^{s}\) When we reconstruct the metric perturbation from a solution of the master variable within the source distribution, more delicate treatment is necessary. \(^{30}\) Below we use the assumption of the factorized form of the tensor Green function. This assumption is not necessary if we follow the derivation given by Wald, \(^{29}\) and actually this assumption itself is not correct. \(^{30}\)
Comparing these two expressions, we find

\[ \int \sqrt{-g} \, d^4x' \, \Pi_{A\mu}^{\text{out}}(x') T^{\mu\nu}(x') = sN \int \sqrt{-g} \, d^4x' \, s\bar{\Omega}_A^{\text{out}}(x') s\bar{\tau}_{\mu\nu} T^{\mu\nu}(x'). \]  

(4.10)

The same relation holds for the modes with other boundary conditions. Since this relation holds for arbitrary \( T^{\mu\nu} \) as long as it satisfies the conservation law, \( T^{\mu\nu}_{;\nu} = 0 \), one can establish the relation

\[ \Pi_{A\mu}(x) = sN \, s\bar{\tau}_{\mu}^* \, s\bar{\Omega}_A(x'), \]  

(4.11)

where \( s\bar{\tau}_{\mu}^* \) is the second order differential operator that is obtained by integration by parts from \( s\bar{\tau}_{\mu\nu} \). This equation still has an ambiguity of adding pure gauge term \( \xi_{(\mu\nu)} \) since its contraction with \( T^{\mu\nu} \) vanishes after integration over four volume.

The above is a very crude explanation about the reason why we can reconstruct the metric perturbation form the master variable just by operating a second order differential operator. More rigorous derivation was given by Wald.\(^{20} \) With the aid of Starobinsky-Teukolsky identity\(^{8} \), we can explicitly show that

\[ s\bar{\Omega}_A = sD^{\mu\nu} s\Pi_{A\mu} = sD^{\mu\nu} sN \, s\bar{\tau}_{\mu}^* \, \bar{\Omega}_A, \]  

(4.12)

simultaneously fixing the normalization constant \( sN \).

When the source is composed of a point particle, i.e., when the energy-momentum tensor takes the form \( T^{\mu\nu} = \int d\tau (-g)^{-1/2} \, u^\mu u^\nu \delta^4(x - z(\tau)) \), the radiative metric perturbation is given by

\[ h_{\mu\nu}^{(\text{rad})} = \sum_A sN \, W(sR_A^{\text{up}}, sR_A^{\text{in}}) W(sR_A^{\text{down}}, sR_A^{\text{out}}) \times \left( W(sR_A^{\text{in}}, sR_A^{\text{out}}) s\Pi_{A\mu}(x) \int \frac{d\tau}{\Sigma} \bar{\phi}_A^{\text{up}}(z(\tau)) \right) + (\text{c.c.}), \]  

(4.13)

where

\[ \phi_A^{\text{up}} = \sum \bar{u}^\mu \bar{u}^\nu \Pi_{A\mu}^{\text{down}} = sN \Sigma \bar{u}^\mu \bar{u}^\nu s\bar{\tau}_{\mu\nu}^* \bar{\Omega}_A^{\text{up}}, \]  

(4.14)

and \( \phi^{\text{up}} \) is defined in a similar manner. For future convenience, instead of the four velocity \( u^\mu \), we used \( \bar{u}^\mu \), an extension of \( u^\mu \) to a vector field, whose definition is given below Eq. (5.4). The factor \( \Sigma(= r^2 + a^2 \cos^2 \theta) \) is also introduced for future convenience.

We can easily confirm that \( \phi_A^{\text{up}} h_{\mu\nu}^{(\text{rad})} \) with the substitution of the above expression reproduces the same \( \psi^{(\text{rad})} \) that is obtained by using Eq. (4.14) neglecting the last complex conjugate term. This term is necessary to make the expression real. To show that adding this term does not disturb the property of reproducing \( s\psi^{(\text{rad})} \), a more detailed discussion is necessary.\(^{24} \) However, we will not go into such a technical detail here.

\(^{8} \) “\( sD^{\mu\nu} s\tau_{\mu\nu}^* \)” reduces to a forth order differential operator which transforms the radial function \( -sR_A \) to the spin-flipped one \( sR_A \). This identity is called Starobinsky-Teukolsky identity.\(^{15} \)
§5. Simplified $dQ/dt$ formula

Now we know how to compute $dQ/dt$ in principle. However, actual implementation of calculation is not so straightforward. In this section we introduce a simpler expression for the adiabatic evolution of Carter constant.\(^{32}\)

5.1. property of geodesics in Kerr

We first discuss in a little more detail about geodesics in Kerr spacetime: \(z^\alpha(\tau) = (t_z(\tau), r_z(\tau), \theta_z(\tau), \varphi_z(\tau))\). Here \(\tau\) is the proper time along the orbit. Using a new parameter \(\lambda\) defined by \(d\lambda := d\tau/\Sigma\), which was recently reintroduced by Mino in the context of radiation reaction,\(^{26}\) the geodesic equations are written as

\[
\frac{\left(\frac{dr_z}{d\lambda}\right)^2}{\Sigma} = R(r_z), \quad \frac{\left(\frac{d\cos \theta_z}{d\lambda}\right)^2}{\Sigma} = \Theta(\cos \theta_z), \quad \tag{5.1}
\]

\[
\frac{dt_z}{d\lambda} = -a(E \sin^2 \theta_z - \mathcal{L}) + \frac{r_z^2 + a^2}{\Delta} P(r_z), \quad \frac{d\varphi_z}{d\lambda} = -aE + \frac{\mathcal{L}}{\sin^2 \theta_z} + \frac{a}{\Delta} P(r_z), \quad \tag{5.2}
\]

where \(P(r) = E(r^2 + a^2) - a \mathcal{L}, \quad R(r) = [P(r)]^2 - \Delta[r^2 + \mathcal{Q}]\) and \(\Theta(\cos \theta) = \mathcal{C} - (\mathcal{C} + a^2(1 - \mathcal{E}^2) + \mathcal{L}^2) \cos^2 \theta + a^2(1 - \mathcal{E}^2) \cos^4 \theta\). It should be noted that the equation for the \(r\)-component and the one for the \(\theta\)-component are decoupled when we use \(\lambda\). The solutions of the first two equations (5.1) are periodic. We denote the periods by \(2\pi/\Omega_r\) and \(2\pi/\Omega_\theta\), respectively. The other two equations (5.2) are integrated as

\[
t_z(\lambda) = \langle t^{(r)}(\lambda) \rangle + \left\langle \frac{dt_z}{d\lambda} \right\rangle \lambda, \quad \tag{5.3}
\]

\[
\varphi_z(\lambda) = \langle \varphi^{(r)}(\lambda) \rangle + \left\langle \frac{d\varphi_z}{d\lambda} \right\rangle \lambda, \quad \tag{5.3}
\]

where \(\langle \cdots \rangle\) means time average along the geodesic.

5.2. simplified formula

We start to simplify the expression of the formula for \(dQ/dt\). The self-force \(f^\alpha\), which is defined by \(u^\nu u^\mu_{\nu} = f^\mu\), is derived from the geodesic equation on a perturbed spacetime. Then \(f^\mu\) is basically given by \(-\delta \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma\), where \(\delta \Gamma^\mu_{\rho\sigma}\) is the contribution to the Christoffel symbol from the metric perturbation \(\mathbf{h}\). Taking into account the redefinition of the proper time so that \(u^\mu u_\mu = -1\) is maintained, we obtain

\[
f^\mu[\mathbf{h}] := -\frac{1}{2}(g^\mu_{\nu} + u^\mu u^\nu)(h_{\nu\rho;\sigma} + h_{\nu\sigma;\rho} - h_{\rho\sigma;\nu})u^\rho u^\sigma. \quad \tag{5.4}
\]

Using this force, the evolution of Carter constant is given by

\[
\frac{dQ}{d\tau} = 2K^\nu_{\mu} u^\mu f^\nu. \quad \tag{5.5}
\]
where \( \Psi(x) = \Sigma \bar{u}^\mu h_{\mu\nu}/2 \) and \((\bar{u}_t, \bar{u}_r, \bar{u}_\theta, \bar{u}_\phi) := (-\mathcal{E}, \pm \sqrt{R(r)/\Delta}, \pm \sqrt{\Theta(\cos \theta)}/\sin \theta, \mathcal{L})\). This vector field \( \bar{u}_\mu \) is an extension of the four velocity of a particle in the sense that it satisfies \( \bar{u}_\mu(z(\lambda)) = u_\mu(\lambda) \), but it differs from the parallel transport of the four velocity. Using the fact that \( \bar{u}_r \) and \( \bar{u}_\theta \), respectively, depend only on \( r \) and \( \theta \), we can easily verify the relation, \( \bar{u}_{\alpha; \beta} = \bar{u}_{\beta; \alpha} \).

If we take the long-time average of Eq. (5.4), the second term in the last line vanishes because it is a total derivative. Furthermore we can show that the long-time average of the third term also becomes higher order in \( \mu \) by using the relations \( \bar{u}_{\alpha; \beta} = \bar{u}_{\beta; \alpha} \) and \( K_{(\mu; \rho)} \). Finally, we obtain

\[
\left\langle \frac{dQ}{dt} \right\rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} d\lambda \Sigma K_{\mu}^\sigma \bar{u}_\sigma \frac{\Psi(x)}{\Sigma}. \tag{5.5}
\]

We can derive an analogous expression more easily for the energy loss rate as \(^{24})\)

\[
\left\langle \frac{dE}{d\lambda} \right\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\lambda \Sigma (-\eta_{(t)}^\alpha) f_\alpha [h_{\mu \nu}]
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\lambda \left[ (-\eta_{(t)}^\alpha) \partial_\alpha \Psi(x) \right]_{x \to z(\lambda)}, \tag{5.6}
\]

where \( \eta_{(t)}^\alpha \) is the timelike Killing vector.

In any cases, all we need to know is \( \Psi(x) \) for the radiative field. Using the formulas (4.13) and (4.14), we find

\[
\Psi^{(\text{rad})}(x) = i \int \frac{d\omega}{2\pi \omega} \sum_{\ell, m, n_r, n_\theta} \mathcal{N}^{\text{in}} \phi^{(\text{in})}_{\ell, m}(x) \int d\lambda' \phi^{(\text{in})}_{\ell', m}(z(\lambda')) + \cdots, \tag{5.7}
\]

where the normalization constant \( \mathcal{N}^{\text{in}} \) is defined by \( \mathcal{N}^{\text{in}} := -2\pi \omega W(sR_{A}^{\text{down}}, sR_{A}^{\text{up}}) s\mathcal{N}^{-1} W(sR_{A}^{\text{up}}, sR_{A}^{\text{down}})^{-1} W(sR_{A}^{\text{down}}, sR_{A}^{\text{down}})^{-1} \). Hereafter we neglect the contribution from waves absorbed by black hole, since the extension is trivial.

We will not repeat here the technical issues for further reduction of the formula discussed in Ref. 32). Instead, we just briefly mention essential points. First point is that the frequency \( \omega \) is discretized as

\[
\int d\lambda' \phi^{(\text{in})}_{\ell, m}(z(\lambda')) = \sum_{n_r, n_\theta} 2\pi \delta (\omega - \omega^{n_r, n_\theta}_m) \tilde{Z}_\lambda, \tag{5.8}
\]

with

\[
\omega^{n_r, n_\theta}_m := (dt_z/d\lambda)^{-1} (m \langle d\varphi_z / d\lambda \rangle + n_r \Omega_r + n_\theta \Omega_\theta).
\]

Here \( \tilde{Z}_\lambda \) represents a set of eigenvalues \( \{\ell, m, n_r, n_\theta\} \). For \( \langle \frac{dE}{dt} \rangle \), we obtain

\[
\left\langle \frac{dE}{dt} \right\rangle = -\sum_{\lambda} |Z_\lambda|^2, \tag{5.10}
\]
and similarly,

$$\langle \frac{dL}{dt} \rangle = - \sum_{\Lambda} \frac{m}{\omega_{mn}^2} |Z_{\Lambda}|^2. \quad (5\cdot11)$$

In the above, we have set $2N^m = 1$ by rescaling the amplitude of radial functions $sR_A$ appropriately. Although it is not manifest from our definition of $N^m$, we can show that it is real. For $\langle dQ/dt \rangle$, writing down the expression in Eq. (5.5) explicitly, we arrive at

$$\int d\lambda \left[ \sum K^r_\mu \bar{u}^\mu_\nu \partial_\nu \phi^{(rad)}(x) \right]_{x=z(\lambda)} = \int d\lambda \left[ \left( \frac{P(r)}{\Delta} (r^2 + a^2) \partial_t + a \partial_\varphi - \frac{dr_c}{d\lambda} \partial_r \right) \phi^{(rad)}(x) \right]_{x=z(\lambda)}. \quad (5\cdot12)$$

Using Eqs. (5.7) and (5.8), the above expression reduces to

$$\Re \left\{ \sum_{\Lambda} iZ_{\Lambda} \int d\lambda \left[ \left( \frac{P(r)}{\Delta} (r^2 + a^2) \partial_t + a \partial_\varphi - \frac{dr_c}{d\lambda} \partial_r \right) \phi^{(rad)}(x) \right]_{x=z(\lambda)} \right\} \quad (5\cdot13)$$

The second point is to notice that the integrand is now a double periodic function with periods $2\pi/\Omega_r$ and $2\pi/\Omega_\theta$. Suppose $f$ is a function of $g^{(r)}$ and $g^{(\theta)}$, and $g^{(\theta)}(\lambda)$ are periodic with periods $2\pi/\Omega_r$ and $2\pi/\Omega_\theta$, respectively. Then, in general,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\lambda f(g^{(r)}(\lambda), g^{(\theta)}(\lambda)) = \frac{\Omega_r \Omega_\theta}{(2\pi)^2} \int_{0}^{2\pi/\Omega_r^{-1}} d\lambda_r \int_{0}^{2\pi/\Omega_\theta^{-1}} d\lambda_\theta f(g^{(r)}(\lambda_r), g^{(\theta)}(\lambda_\theta)). \quad (5\cdot14)$$

holds. Using this formula, the integral (5.13) can be integrated by parts. Then finally we arrive at the formula

$$\langle \frac{dQ}{dt} \rangle = 2 \left\langle \frac{r^2 + a^2}{\Delta} \right\rangle \langle \frac{dE}{dt} \rangle - 2 \left\langle \frac{aP}{\Delta} \right\rangle \langle \frac{dL}{dt} \rangle + 2 \sum_{\Lambda} \frac{n_r \Omega_r}{\omega_{mn}^2} |Z_{\Lambda}|^2. \quad (5\cdot15)$$

This expression is as easy to evaluate as $\langle dE/dt \rangle$ and $\langle dL/dt \rangle$. To evaluate the last term, we have only to replace $m$ with $n_r \Omega_r$ in the expression for $\langle dL/dt \rangle$, (5.11).

Although the final expressions for $\langle dL/dt \rangle$ and $\langle dQ/dt \rangle$ are quite similar, there is a big difference between them. The squared amplitude of each partial wave, $|Z_{\Lambda}|^2$, is a measurable quantity in the asymptotic regions, i.e., near the future null infinity or the future event horizon. The eigenvalues $\omega$ and $m$ also can be read from the waveform in the asymptotic region. Hence, the expressions for $\langle dE/dt \rangle$ and $\langle dL/dt \rangle$ are solely written in terms of the asymptotic waveform. This is consistent with the fact that the balance argument applies for these quantities. On the other hand, the eigenvalue $n_r$, which appears in the expression for $\langle dQ/dt \rangle$, is not a quantity which
can be read from the asymptotic waveform without knowing the particle orbit. We can read the frequencies from the asymptotic waveform, but the frequency itself does not tell the numbers \( n_r \) and \( n_\theta \), without an input of additional information about the orbit. This is consistent with our understanding that the balance argument cannot be used for evaluating \( \langle dQ/dt \rangle \).

5.3. long period orbital evolution

One may ask how we can use the knowledge about the adiabatic evolution of \( Q \) to evaluate the long time orbital evolution. Just to be consistent with a given set of constants of motion \( I^i \), we can choose \( r, \theta \) and \( \varphi \) arbitrarily at each time \( t \) as far as \( r \) and \( \theta \) are within the allowed range. Hence, the evolution of the constants of motion is not complete at all as a description of the orbital evolution. Nevertheless, combined with the normalization of four velocity,

\[
g_{\mu\nu}^{BH} \frac{dz^\mu}{\Sigma d\lambda} \frac{dz^\nu}{\Sigma d\lambda} = -1, \quad (5.16)
\]

three constants of motion are sufficient to specify the four velocity as a function of \( r \) and \( \theta \). Even if we take into account the instantaneous self-force, the equations of motion integrated once \((5.11)\) and \((5.2)\) are kept unchanged since they are merely algebraic relations between \( u^\mu \) and \( I^i \). Hence, if the evolution of the “constants of motion” \( I^i \) is given, we do not need further information about the self-force in order to evolve \((r(\lambda), \theta(\lambda), t(\lambda), \varphi(\lambda))\). Here we give a prescription how to integrate the orbit for a long period of time. The properties of orbits in Kerr spacetime which we have already mentioned largely simplify the problem of solving the orbital evolution. We will find that the leading order approximation can be obtained by using the adiabatic approximation as expected. We will also find that the leading order corrections to the adiabatic approximation coming from the instantaneous self-force in the linear order are given by a few time-averaged quantities constructed from the self-force. The basic idea of the discussion in this subsection is found in Ref. 33).

We begin with discussing orbits for fixed constants of motion \( I^i \). It is convenient to introduce phase functions \( \chi^r \) and \( \chi^\theta \) by \( \chi^a = \Omega_a(I^i)\lambda \), where \( \Omega_r(I^i) \) and \( \Omega_\theta(I^i) \) are angular frequencies of oscillations in \( r \) and \( \theta \) directions respectively. Note that we can choose the initial values of the phases \( \chi^a(0) \) arbitrarily. We define functions \( \hat{r}(I^i, \chi^r) \) and \( \hat{\theta}(I^i, \chi^\theta) \) as solutions of the \( r \)-and \( \theta \)-components of the geodesic equations. Here we fix the ambiguity in the choice of phases of these functions so as to satisfy \( \chi^r = 0 \mod 2\pi \) for \( \hat{r}(\chi^r) = r_- \) and \( \chi^\theta = 0 \mod 2\pi \) for \( \hat{\theta}(\chi^\theta) = \theta_- \). Here \( r_- \) and \( \theta_- \) are minima of \( r \) and \( \theta \) for given \( I^i \). Since the evolutions of \( r \) and \( \theta \) for \( 0 < \chi^a < \pi \) and for \( \pi < \chi^a < 2\pi \) are symmetric, we automatically have \( \chi^r = \pi \mod 2\pi \) for \( \hat{r}(\chi^r) = r_+ \) and \( \chi^\theta = \pi \mod 2\pi \) for \( \hat{\theta}(\chi^\theta) = \theta_+ \) with this choice of phases. Further we redefine the functions \( t^{(r)} \) and \( \varphi^{(r)} \) here as functions of \( I^i \) and \( \chi^r \), and similarly \( t^{(\theta)} \) and \( \varphi^{(\theta)} \) as functions of \( I^i \) and \( \chi^\theta \). Again, for definiteness we fix these functions so that \( t^{(a)}(I^i, \chi^a) = 0 \) and \( \varphi^{(a)}(I^i, \chi^a) = 0 \) for \( \chi^a = 0 \mod 2\pi \).

Now we are ready to introduce our parametrization to describe orbits when we take into account the self-force. Our proposal is to promote \( I^i \) and \( \chi^a \) to functions
of $\lambda$ as
\[ r(\lambda) = \hat{r}(I^i(\lambda), \chi^r(\lambda)), \]
\[ \theta(\lambda) = \hat{\theta}(I^i(\lambda), \chi^\theta(\lambda)), \]
\[ t(\lambda) = t^i(\lambda), \chi^\theta(\lambda)) + t^0(\lambda, \chi^\theta(\lambda)) + \hat{\chi}^i(\lambda), \]
\[ \varphi(\lambda) = \varphi^r(\lambda) + \varphi^0(\lambda, \chi^\theta(\lambda)) + \hat{\chi}^\varphi(\lambda). \] (5.17)

Here we have also introduced $\hat{\chi}^A(\lambda)$ instead of the integrals such as $\int (dt/d\lambda) (I^i(\lambda))d\lambda$. Below we derive equations for $I^i(\lambda)$, $\chi^\alpha(\lambda)$ and $\hat{\chi}^A(\lambda)$.

First we examine the evolution equation for $r$. As we have mentioned earlier, Eqs. (5.14) and (5.22) are kept unchanged even if the self-force is taken into account. Therefore we have
\[ \frac{dr}{d\lambda} = \frac{\partial \hat{r}(I^i(\lambda), \chi^r)}{\partial \chi^r} \Omega_r(I^i(\lambda)), \] (5.18)
since $\hat{r}$ is the solution of the geodesic equation for fixed $I^i$. On the other hand, taking the $\lambda$-derivative of $r(\lambda)$ in the form given in Eq. (5.14), we obtain
\[ \frac{d}{d\lambda} \hat{r}(I^i(\lambda), \chi^r(\lambda)) = \frac{\partial \hat{r}}{\partial I^i} \frac{dI^i}{d\lambda} + \frac{\partial \hat{r}}{\partial \chi^r} \frac{d\chi^r}{d\lambda}. \] (5.19)

Comparing these two expressions, we obtain an equation for $d\chi^r/d\lambda$ as
\[ \frac{d\chi^r}{d\lambda} = \Omega_r + \delta \left( \frac{d\chi^r}{d\lambda} \right), \] (5.20)
where
\[ \delta \left( \frac{d\chi^r}{d\lambda} \right) = - \left( \frac{\partial \hat{r}}{\partial \chi^r} \right)^{-1} \frac{\partial \hat{r}}{\partial I^i} \frac{dI^i}{d\lambda}. \] (5.21)

Near the turning points $r = r_\pm$, $\hat{r}$ can be expanded as $\hat{r} = r_\pm(I^i) + O((\Delta \chi^r)^2)$, where $\Delta \chi^r$ is the difference of $\chi$ from its value at $r = r_\pm$. Then $\delta (d\chi^r/d\lambda)$ looks singular since $\partial \hat{r}/\partial \chi^r$ behaves like $\approx \Delta \chi$ near the turning points, but in fact this term is not singular. As we shall see immediately below, the other factor $(\partial \hat{r}/\partial I^i)(dI^i/d\lambda)$ simultaneously goes to 0 at the turning points as long as the self-force stays finite.

We shall show $(\partial \hat{r}/\partial I^i)(dI^i/d\lambda) = O((\Delta \chi^r)^2)$. Differentiating the equation $(dr/d\lambda)^2 = R(I^i, r)$ with respect to $\lambda$, we have
\[ 2 \left( \frac{dr}{d\lambda} \right) \frac{d^2 r}{d\lambda^2} = \frac{\partial R}{\partial I^i} \frac{dI^i}{d\lambda} + \frac{\partial R}{\partial r} \frac{dr}{d\lambda}. \] (5.22)

Hence, we conclude that $(\partial R(I^i, r)/\partial I^i)(dI^i/d\lambda) = 0$ for $r = r_\pm$. On the other hand, by the definition of $r_\pm$, we have $R(I^i, r_\pm(I^i)) \equiv 0$. Differentiating this identity with respect to $I^i$, and contracting it with $dI^i/d\lambda$, we have
\[ \left( \frac{\partial R}{\partial r} \right)_{r=r_\pm} \frac{dr_\pm}{dI^i} \frac{dI^i}{d\lambda} = - \left( \frac{\partial R}{\partial I^i} \right)_{r=r_\pm} \frac{dI^i}{d\lambda} = 0. \] (5.23)

Hence, except for circular orbits, in which $(\partial R/\partial r)_{r=r_\pm} = 0$, we establish $(\partial r_\pm/\partial I^i)(dI^i/d\lambda) = 0$. When the orbit is circular, $r$ is given as a function of $I^i$. Hence, we do not have to care about the evolution of $\chi^r$. 

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Now we return to Eq. (5.20). To evaluate $dI^i/d\lambda$, here we accept the use of the geodesic momentarily tangential to the orbit in evaluating the self-force. The errors caused by this approximation need future investigation. Under this approximation, the change of the “constants of motion” becomes a double periodic function in $\lambda$, and is expanded as

$$
\frac{dI^i}{d\lambda} = \left< \frac{dI^i}{d\lambda} \right> + \sum_{(n_r, n_\theta) \neq (0,0)} J^i_{n_r, n_\theta}(I^j(\lambda)) e^{i(n_r \chi^r(\lambda) + n_\theta \chi^\theta(\lambda))} + O(\mu^2),
$$

(5.24)

where $J^i_{n_r, n_\theta}(I^j(\lambda))$ are coefficients to be computed from the instantaneous self-force. The last term of $O(\mu^2)$ is the correction due to the second order self-force, which currently we do not know how to evaluate. In what follows we will give a rough estimate of the error due to this second order contribution, but its derivation is not rigorous at all. The second term in (5.20) is also a regular double periodic function. Hence it accept a similar expansion. We can write Eq. (5.20) as

$$
\frac{d\chi^{r}}{d\lambda} = \Omega_r + \left< \delta \left( \frac{d\chi^{r}}{d\lambda} \right) \right> (I^i(\lambda)) 
+ \sum_{(n_r, n_\theta) \neq (0,0)} \delta \Omega^{n_r, n_\theta}_{r, I^j}(\lambda) e^{i(n_r \chi^r(\lambda) + n_\theta \chi^\theta(\lambda))} + O(\mu^2).
$$

(5.25)

The effect of the third term looks as large as the second one at first sight. However, if we integrate the above equation once, one finds that the second time grows as $O(\mu \Delta \lambda)$, where $\Delta \lambda$ is the length of integration time. Since we are interested in long term orbital evolution, $\Delta \lambda$ is assumed to be large. On the other hand, the third term can be integrated by parts as

$$
\int d\lambda \delta \Omega^{n_r, n_\theta}_{r, I^j}(\lambda) e^{i(n_r \chi^r(\lambda) + n_\theta \chi^\theta(\lambda))}
= \frac{\delta \Omega^{n_r, n_\theta}_{r, I^j}(\lambda)}{i(n_r (d\chi^r/d\lambda) + n_\theta (d\chi^\theta/d\lambda))} e^{i(n_r \chi^r(\lambda) + n_\theta \chi^\theta(\lambda))}
- \int d\lambda e^{i(n_r \chi^r(\lambda) + n_\theta \chi^\theta(\lambda))} \frac{d}{d\lambda} \frac{\delta \Omega^{n_r, n_\theta}_{r, I^j}(\lambda)}{i(n_r (d\chi^r/d\lambda) + n_\theta (d\chi^\theta/d\lambda)).}
$$

(5.26)

The first term in the last expression does not grow even for a large value of $\Delta \lambda$, while the second term becomes higher order in $\mu$ due to the appearance of the factor $dI^i/d\lambda$. We express the result of integration of (5.25) as

$$
\chi^r(\lambda) = \int \left\{ \Omega_r + \left< \delta \left( \frac{d\chi^r}{d\lambda} \right) \right> \right\} d\lambda + O(\mu(\Delta \lambda)^0) + O(\mu^2 \Delta \lambda).
$$

(5.27)

The last term is the correction coming from the second order self-force. Discussion about the $\theta$-component is completely parallel to the $r$-component.

Next we consider the $t$-component. By the definitions of $t^{(r)}$ and $t^{(\theta)}$, we have

$$
\frac{dt}{d\lambda} = \frac{\partial t^{(r)}}{\partial \chi^r} \Omega_r + \frac{\partial t^{(\theta)}}{\partial \chi^\theta} \Omega_\theta + \left< \frac{dt}{d\lambda} \right>.
$$

(5.28)
On the other hand, differentiating $t(\lambda)$ in the form of (5.17), we obtain

$$\frac{dt}{d\lambda} = \frac{\partial t^r}{\partial I^r} \frac{dI^r}{d\lambda} + \frac{\partial t^{\theta}}{\partial I^{\theta}} \frac{dI^{\theta}}{d\lambda} + \frac{\partial t^r}{\partial I^r} \frac{dI^r}{d\lambda} + \frac{\partial t^{\theta}}{\partial I^{\theta}} \frac{dI^{\theta}}{d\lambda} + \frac{d\chi^r}{d\lambda}.$$  (5.29)

From a comparison of these two equations, we find that the evolution of $\tilde{\chi}^r$ is determined by

$$\frac{d\tilde{\chi}^r}{d\lambda} = \left( \frac{dt}{d\lambda} \right) (I^j(\lambda)) + \left( \delta \left( \frac{d\tilde{\chi}^r}{d\lambda} \right) \right) (I^j(\lambda)) + \sum_{(n_r, n_\theta) \neq (0,0)} \left( \frac{d\chi^r}{d\lambda} \right) J_n^{(n_r, n_\theta)} (I^j(\lambda)) e^{i(n_r\chi^r(\lambda) + n_\theta \theta(\lambda))} + O(\mu^2),$$  (5.30)

with

$$\delta \left( \frac{d\tilde{\chi}^r}{d\lambda} \right) := -\frac{\partial t^r}{\partial I^r} \frac{dI^r}{d\lambda} - \frac{\partial t^{\theta}}{\partial I^{\theta}} \frac{dI^{\theta}}{d\lambda} - \frac{\partial t^r}{\partial I^r} \frac{dI^r}{d\lambda} - \frac{\partial t^{\theta}}{\partial I^{\theta}} \frac{dI^{\theta}}{d\lambda}.$$  (5.31)

Then, this equation can be integrated as in the case of $\chi^a$. A completely parallel discussion goes through for the $\varphi$-component, too.

Now we consider the evolution of $I^i(\lambda)$. The evolution equations for $I^i(\lambda)$ are already given by Eq. (5.24). We express the result of integration of (5.24) as

$$I^i(\lambda) = I^i_{\text{Ad}}(\lambda) + \delta I^i(\lambda).$$  (5.32)

The first term $I^i_{\text{Ad}}(\lambda)$ is the contribution only from the first term on the right hand side of Eq. (5.24). We find that the contribution to $\delta I^i(\lambda)$ from the terms including $J_n^{i, n_r, n_\theta}$ remains $O(\mu(\Delta \lambda)^0)$ or higher. On the other hand, the second order self-force contribute to $\delta I^i$ as terms of $O(\mu^2 \Delta \lambda)$.

The errors in $I^i(\lambda)$ propagate to $\chi^a(\lambda)$ and $\tilde{\chi}^A(\lambda)$. By using the same argument that we have already used many times, the propagated errors caused by the terms which include $J_n^{i, n_r, n_\theta}$ are at most $O(\mu(\Delta \lambda)^0)$, while the effect of second order self-force can be as large as $O(\mu^2 \Delta \lambda^2)$ for both $\chi^a(\lambda)$ and $\tilde{\chi}^A(\lambda)$. To conclude, we found

$$I^i(\lambda) = I^i_{\text{Ad}}(\lambda) + O(\mu(\Delta \lambda)^0) + O(\mu^2 \Delta \lambda),$$

$$\chi^a(\lambda) = \chi^a_{\text{Ad}}(\lambda) + \int \left\langle \delta \left( \frac{d\chi^r}{d\lambda} \right) \right\rangle d\lambda + O(\mu(\Delta \lambda)^0) + O(\mu^2 \Delta \lambda^2),$$

$$\tilde{\chi}^A(\lambda) = \tilde{\chi}^A_{\text{Ad}}(\lambda) + \int \left\langle \delta \left( \frac{d\tilde{\chi}^r}{d\lambda} \right) \right\rangle d\lambda + O(\mu(\Delta \lambda)^0) + O(\mu^2 \Delta \lambda^2),$$  (5.33)

with

$$I^i_{\text{Ad}}(\lambda) := \int \left\langle \frac{dI^i}{d\lambda} \right\rangle (I^i_{\text{Ad}}(\lambda)) d\lambda,$$

$$\chi^a_{\text{Ad}}(\lambda) := \int \Omega_a(I^i_{\text{Ad}}(\lambda)) d\lambda,$$

$$\tilde{\chi}^A_{\text{Ad}}(\lambda) := \int \left\langle \frac{dt}{d\lambda} \right\rangle (I^i_{\text{Ad}}(\lambda)) d\lambda,$$

$$\tilde{\chi}^A_{\text{Ad}}(\lambda) := \int \left\langle \frac{dt}{d\lambda} \right\rangle (I^i_{\text{Ad}}(\lambda)) d\lambda.$$  (5.34)
The last term for each $I^i(\lambda)$, $\chi^a(\lambda)$ and $\tilde{\chi}^A(\lambda)$ comes from the second order self-force. The second terms in the expressions for $\chi^a(\lambda)$ and $\tilde{\chi}^A(\lambda)$ are $O(\mu \Delta \lambda)$. When $\mu \Delta \lambda \ll 1$, the deviation from the adiabatic approximation is small. In this sense adiabatic approximation is a good approximation. On the other hand, once $\mu \Delta \lambda$ becomes $O(1)$, we cannot neglect the second order self-force, either. Hence, roughly speaking, the orbital evolution which takes into account the instantaneous self-force but only at the linear order will not be a better approximation than that obtained by using the adiabatic approximation. Even when $\mu \Delta \lambda$ is moderately small, the leading corrections are given by quantities averaged over a long period of time, $\langle \delta (d\chi^a/d\lambda) \rangle$ and $\langle \delta (d\tilde{\chi}^A/d\lambda) \rangle$. Again we can repeat the same argument that was used to justify replacing the retarded field with the radiative one when we evaluate $dQ/dt$. Therefore adiabatic approximation is sufficient to calculate these averaged quantities.

Nevertheless, we are not saying that the study of the instantaneous self-force is not important. When we consider the whole process of inspiral of binaries, the time scale is inversely proportional to $\mu$. Therefore, both the terms of $O(\mu \Delta \lambda)$ and of $O(\mu^2 \Delta \lambda^2)$ potentially cause measurable effects. To evaluate these effects, we need to know (maybe some time average of) the second order self-force, and for this purpose the study of the instantaneous self-force at linear order will be necessary. Furthermore, there might be situations in which the effects of the oscillating part of the self-force, i.e., the terms which contain $J^i_{n_r,n_\theta}$ are significantly enhanced. In the above discussion, when we perform integration by parts such as shown explicitly in (5.26), a factor $n_r(d\chi^r/d\lambda) + n_\theta (d\chi^\theta/d\lambda)$ appeared in the denominator. As the “constants of motion” $I^i$ evolve, this denominator eventually can cross 0 for a certain combination of $n_r$ and $n_\theta$. In such a case, the contribution from $J^i_{n_r,n_\theta}$ may leave some significant effect on the orbital evolution. Suppose that $n_r(d\chi^r/d\lambda) + n_\theta (d\chi^\theta/d\lambda)$ vanishes at $\lambda = \lambda_0$. Such a stationary phase point (SPP) will cause an additional shift in $I^i(\lambda)$ which will be estimated by the Gaussian integral

$$
\Delta I^i_{SPP} \approx \int d\lambda J^i_{n_r,n_\theta}(I^i(\lambda)) e^{i(n_r \chi^r(\lambda_0) + n_\theta \chi^\theta(\lambda_0)) + \frac{1}{2} n_r (d^2 \chi^r/d\lambda^2) + n_\theta (d^2 \chi^\theta/d\lambda^2))(\lambda - \lambda_0)^2
$$

$$
\approx \frac{2\pi}{\sqrt{i(n_r \frac{d^2}{d\lambda^2} + n_\theta \frac{d^2}{d\lambda^2})}} J^i_{n_r,n_\theta}(I^i(\lambda_0)) e^{i(n_r \chi^r(\lambda_0) + n_\theta \chi^\theta(\lambda_0))}. \quad (5.35)
$$

This correction is $O(\sqrt{\mu})$, and it will induce shifts in phases $\chi^a$ and $\tilde{\chi}^A$ of $O(\sqrt{\mu} \Delta \lambda)$. Hence, this naive order counting indicates that this correction is the leading order correction to the adiabatic approximation. However, when we consider relatively non-relativistic orbits with small eccentricity, such stationary points will appear only when the values of $n_r$ and/or $n_\theta$ are large. In such cases the coefficients $J^i_{n_r,n_\theta}$ will be suppressed by some large powers of the eccentricity or $(v/c)$. Hence, the effect will practically remain small. To the contrary, when we consider highly eccentric orbits, the corrections due to stationary points may become really $O(\sqrt{\mu} \Delta \lambda)$ without any significant additional suppression. Again, to evaluate these effect quantitatively, we need to compute the instantaneous self-force.

Before closing this subsection, we want to emphasize that the errors in phases $\chi^a$
and $\tilde{\chi}^A$ can grow in a dynamical time scale if we do not use an appropriate integration scheme as presented here. Namely, they can be as large as $O(\mu(e^{\alpha \Delta \lambda} - 1))$ with $\alpha$ being a constant of $O(1)$.

§6. Toward post-Teukolsky formalism

Now we consider to extend the Teukolsky formalism to the second order, which seems to be indispensable to improve predictions for waveforms beyond the level of adiabatic approximation. Perturbed Einstein equations up to the second order take the form

$$
\delta G_{\mu\nu} \left[ h^{(2)} \right] = T^{(2)}_{\mu\nu} - G_{\mu\nu}^{[2]} \left[ h^{(1)}, h^{(1)} \right],
\tag{6.1}
$$

where $\delta G_{\mu\nu}$ is nothing but $G_{\mu\nu}^{[1]}$ in Eq. (3.5). Projection of this equation can be done formally as in the case of linear perturbation as

$$
L \zeta^{(2)} = \sqrt{-g} T^{(2)}.
\tag{6.2}
$$

Here $\zeta^{(2)}$ is defined in the same way as $\zeta^{(1)}$ just substituting $h^{(1)}$ with $h^{(2)}$. The second order source term $T^{(2)}$ has a spatially extended distribution due to the non-linearity of gravity. The differential operator $L$ is the same as the one for the linear perturbation. Therefore all the technical difficulties which arise in the second order for the first time are in how to evaluate the source term. To obtain the corrections to the energy momentum tensor $T^{(2)}_{\mu\nu}$, we need to know the correction to the trajectory of the particle taking into account the self-force. Hence as a first step toward the post-Teukolsky formalism, we examine the self-force.

6.1. gravitational self-force

When we consider the point particle limit, the full self-force diverges at the location of the particle, and hence needs to be regularized. It is known that the properly regularized self-force is given by the tail part of the self-field, which is obtained by subtracting the direct part from the full field. Here we do not give the precise definition of the direct part, but, roughly speaking, it is the part that propagates along the light cone unaffected by curvature scattering. Since direct part does not contain curvature scattering effect, the direct part of the local field near the particle is solely determined by the local geometry and the orbital elements of the particle. The justification of this prescription is given in\textsuperscript{21} for the scalar and electro-magnetic cases, and in\textsuperscript{22,23} for the gravitational case. An equivalent but more elegant decomposition of the Green function was proposed by Detweiler and Whiting\textsuperscript{34,35} in which the direct part is replaced by the $S$-part and the tail part by the $R$-part. The $S$-part is defined so as to vanish when two arguments $x$ and $x'$ are timelike. When $S$-part is subtracted from the full field, the remainder $R$-part gives the regularized self-force. The advantage of this new decomposition is that the $S$-part is symmetric with respect to $x$ and $x'$, and it satisfies the same equation as the retarded Green function does. This implies that the $R$-part now satisfies the source-free, homogeneous equation.
6.2. subtraction and regularization

Since we do not know a direct way to compute the $R$-part, we instead compute $F^\alpha \left[ \psi^R \right] (\tau) = F^\alpha \left[ \psi^{full} \right] (\tau) - F^\alpha \left[ \psi^S \right] (\tau)$. Since both the terms on the right hand side are divergent, this expression does not make sense unless we regularize the divergent quantities. A practical way of regularization is the so-called “point splitting” regularization. We evaluate the expression for the force not exactly at the location of the particle but slightly off the point. If we subtract the $S$-part appropriately, the coincidence limit must be well-defined.

For an actual computation, we decompose divergent full- and $S$-parts of the force into terms labelled by the total angular momentum $\ell, (37)-45)$

$$F^\alpha \left[ \psi^R \right] (\tau) = \lim_{x \to z(\tau)} \sum_{\ell=0}^{\infty} \left(F^\alpha_{\ell} \left[ \psi^{full} \right] (\tau) - F^\alpha_{\ell} \left[ \psi^S \right] (\tau)\right). \quad (6.3)$$

Then each $\ell$-order term stays finite even in the coincidence limit. Here the question is whether we can change the order of the two operations as

$$F^\alpha \left[ \psi^R \right] (\tau) = \sum_{\ell=0}^{\infty} \lim_{x \to z(\tau)} \left(F^\alpha_{\ell} \left[ \psi^{full} \right] (\tau) - F^\alpha_{\ell} \left[ \psi^S \right] (\tau)\right). \quad (6.4)$$

In general these two operations do not commute.

For example, let us define two functions $A(x)$ and $B(x)$ as $A(x) = \sum_{\ell=0}^{\infty} (1-x)^\ell$ and $B(x) = \sum_{\ell=1}^{\infty} (1-x)^{\ell-1}$ for $x \geq 0$. For $x \neq 0$, both $A(x)$ and $B(x)$ are convergent. Therefore we can compute the expression $A(x) - B(x)$ unambiguously, and it is 1. In this case it is trivial that the difference $A(x) - B(x) = 1 + \sum_{\ell=1}^{\infty} 0$ is uniformly convergent, and each $\ell$-th order term is continuous. Therefore, one can change the order of two operations. Irrespective of the order of operations we arrive at the same answer. The importance of the conditions of uniform convergence will become clearer if we consider the case with $B(x) = \sum_{\ell=1}^{\infty} (1-x)^{\ell-1}$. In this case $A(x) - B(x) = 0$. Nevertheless, if we take the limit $x \to 0$ first, we end up with $A(0) - B(0) = 1$. Now the term at the $\ell$-th order is $(1-x)^{\ell} - (1-x)^{\ell-1} = -x(1-x)^{\ell-1}$. As $x$ becomes closer to 0, the convergence becomes worse. This series is convergent at each point $x$ but not uniformly.

A practical way to guarantee the uniform convergence in the harmonic expansion of the force is to use the same prescription for both full- and $S$-part to extend the force off the trajectory and to use the same harmonics. The $S$-part of the force is determined by the local expansion near the particle. It is composed of terms like

$$\sum R^b \Theta^c \Phi^d f_{abcd}[z^\mu(\tau), u^\mu(\tau)],$$

Here, the separation from the trajectory of the particle $(x - z(\tau))^\mu$ was denoted by $(T, R, \Theta, \Phi)$, and $\epsilon$ is the spatial distance between $x^\mu$ and the trajectory. This type of function can be expanded in terms of spherical harmonics. On the other hand, the full-part composed of the master variable is also computed by using the
spherical harmonic decomposition\(^\ast\)). After averaging over the angular direction from the location of the particle, we will obtain the full- and the S-parts of the force in the form of

\[
F^\alpha(\tau, R, X) = \sum_{\ell=0}^{\infty} F^\alpha_\ell(\tau, R) P_{\ell}(\cos X),
\]

where \(X\) is the angle between \(x^\mu\) and \(z^\mu(\tau)\). These expansions have a little ambiguity since the extension of the self-force is given only locally, in the vicinity of the particle. Nevertheless, the ambiguity does not affect the force evaluated near the particle after summation over \(\ell\) since that is the basic requirement for the construction of harmonic expansion.\(^{37), 39), 41)}\)

Now we consider the difference between the full-part and the S-part of the self-force. If both parts are computed appropriately, the difference must become \(R\)-part, which is regular at least near the location of the particle. Therefore, the series

\[
\sum_{\ell=0}^{\infty} \left( F^\alpha_{\text{full},\ell}(\tau, R) - F^\alpha_{S,\ell}(\tau, R) \right) P_{\ell}(\cos X),
\]

must be uniformly convergent. Thus we can take the coincidence limit, and we arrive at

\[
F^\alpha_R(\tau) = \sum_{\ell=0}^{\infty} \left( F^\alpha_{\text{full},\ell}(\tau, R = 0) - F^\alpha_{S,\ell}(\tau, R = 0) \right).
\]

The \(S\)-part and hence also full-part of the self-force in the harmonic gauge takes the form of\(^{37), 39), 41), 45)}\)

\[
F^\alpha_{\ell}(\tau, R = 0) = A^\alpha \left( \ell + \frac{1}{2} \right) + B^\alpha \left( \ell + \frac{1}{2} \right)^{-1} + C^\alpha \ell + D^\alpha_{\ell},
\]

The constants \(A^\alpha, B^\alpha, C^\alpha\) and the residual part \(D^\alpha := \sum_{\ell=0}^{\infty} D^\alpha_{\ell}\) are called “regularization parameters”. The choice of the residual for each \(\ell\) mode, \(D^\alpha_{\ell}\), can be changed rather freely by changing the behavior at a large separation angle, but the value of regularization parameter \(D^\alpha\) is not affected. The first three regularization parameters \(A^\alpha, B^\alpha\) and \(C^\alpha\) must be common for both full- and S-parts since otherwise the \(R\)-part diverges. Hence, what we need to evaluate is the difference

\[
F^\alpha_R(\tau) = \sum_{\ell} \left( D^\alpha_{\text{full},\ell} - D^\alpha_{S,\ell} \right).
\]

The terminology “mode-sum regularization” or “mode decomposition regularization” is often used for the above prescription. However, we want to stress that the mode decomposition itself is just a useful tool to simplify the procedure of the point splitting regularization.

\(^\ast\) The way how to extend the force must be carefully chosen so that the harmonic decomposition of the full-part of the force is computable.\(^{37}\) Alternative way to avoid this problem is to subtract S-part at the level of the metric perturbations\(^{51}\) (or at the level of the master variable\(^{49}\)).
6.3. gauge problem and use of intermediate gauge.

The formal expression for the self-force is derived in harmonic gauge, and \( S \)-part is also given by using a local Hadamard expansion of the Green function in the harmonic gauge. However, when we calculate the full metric perturbations by using some metric reconstruction method from a master variable, the gauge that we can practically use is restricted to Regge-Wheeler-Zerilli gauge for Schwarzschild case or radiation gauge for Kerr case. Since the force is a gauge dependent quantity, regularization parameters can be different in different gauges. Hence, simple subtraction of \( S \)-part for harmonic gauge does not work.

Here we present the basic idea of the intermediate gauge method\(^{47}\) in a manner slightly modified from the original. We associate subscripts, \( \text{full}, R, S \) to denote full perturbations, \( R \)-part and \( S \)-part, respectively. In any gauge \( G \) we can define the \( R \)-part as

\[
    h_R^{(G)} := h_{\text{full}}^{(G)} - h_S^{(G)}.
\]

(6.11)

We consider a gauge transformation from the harmonic gauge to the gauge \( G \):

\[
    h^{(H)} = h^{(H)} + \nabla \xi^{(H \to G)} [h^{(H)}],
\]

(6.12)

where \( (H) \) represents harmonic gauge, and \( \nabla \xi \) denotes the change of metric generated by an infinitesimal coordinate transformation \( x^\mu \to x^\mu - \xi^\mu \). \( \xi^{(H \to G)} [h^{(H)}] \) is the generator that transforms the metric perturbation in harmonic gauge \( h^{(H)} \) into that in the gauge \( G \). Then \( R \)-part of metric perturbations in the harmonic gauge can be rewritten as

\[
    h_R^{(H)} = h_{\text{full}}^{(G)} - h_S^{(G')} - \nabla \xi^{(H \to G)}_R,
\]

(6.13)

where

\[
    h_S^{(G')} = h_S^{(H)} + \nabla \xi_S^{(H \to G)},
\]

(6.14)

and

\[
    \xi^{(H \to G)}_R = \xi^{(H \to G)} [h_{\text{full}}^{(H)}] - \xi_S^{(H \to G)}.
\]

(6.15)

Here \( \xi_S^{(H \to G)} \) is not specified yet, and it is not necessarily equal to \( \xi^{(H \to G)} [h_S^{(H)}] \).

The idea is to drop the last gauge term \( \nabla \xi^{(H \to G)} [h_{R}^{(H)}] \) in (6.13). Namely, the force is computed from the regularized metric

\[
    h_R^{(\text{int})} = h_{\text{full}}^{(G)} - h_S^{(G')}.
\]

(6.16)

A trajectory in this intermediate gauge is related to that in the harmonic gauge via the gauge transformation specified by \( \xi_R^{(H \to G)} \). For the intermediate gauge to be useful, it should satisfy the following two conditions.

1. We need to choose \( \xi_S^{(H \to G)} \) so that \( h_S^{(G')} \) completely cancels the singular part in \( h_{\text{full}}^{(G)} \) in the coincidence limit.
2. There is no secular growth in the generator of the gauge transformation \( \xi_R^{(H \to G)} \).
The condition (1) is the requirement that $\xi^{(H \rightarrow G)}_{S}$ is a good approximation of the true $S$-part gauge transformation, $\xi^{(H \rightarrow G)}_{S}[h^{(H)}_{S}]$. If the difference between $\xi^{(H \rightarrow G)}_{S}$ and $\xi^{(H \rightarrow G)}_{S}[h^{(H)}_{S}]$ is regular and finite, we have $\xi^{(H \rightarrow G)}_{R} = \xi^{(H \rightarrow G)}_{S}[h^{(H)}_{R}]$ except for regular term. Hence, the coincidence limit of $\xi^{(H \rightarrow G)}_{R}$ is guaranteed to be finite. To obtain a sufficiently good approximation of $\xi^{(H \rightarrow G)}_{S}[h^{(H)}_{S}]$, we just need to know $h^{(H)}_{S}$ as a local expansion near the trajectory, and that is the best we can do.

Here it is very convenient if the gauge $G$ satisfies a property that the gauge parameters to transform metric perturbations in another gauge to the specified gauge $G$ are determined without temporal or radial integration. This means that the equations to determine the gauge parameters are solved locally on a sphere. Then transformation into $G$-gauge is fixed unambiguously just by looking at the perturbations in the vicinity of a sphere. We call such a gauge as a *Gauge Operationally Deterministic on a Sphere* (GODS). Regge-Wheeler gauge is GODS.\(^{46}\) Radiation gauge in Kerr case does not seem to be GODS in its original form, but there seems to be a modification which allows it to transform into GODS in the sense of expansion with respect to Kerr parameter $a^{*}$.

In understanding the true difficulty about the gravitational radiation reaction, it is very important to realize that the condition (2) is really necessary. First of all, if the condition (2) is not required, we can add any finite gauge transformation. As a result, the trajectory in the coordinate representation can be arbitrarily changed, which is quite unsatisfactory situation. If the amplitude of gauge parameters is large, the coordinate values of the particle in $H$ and $G$ gauges are quite different. Even in that case, although it is quite counter-intuitive, one can cancel the divergent pieces in metric perturbations as far as linear perturbation is concerned, in which the gauge transformation does not couple with metric perturbations. However, once we consider the second order perturbation, large gauge parameters will cause disaster to the perturbative expansion.

Now we come back to the issue how to guarantee the condition (2). When the gauge $G$ is GODS, we can define $\xi^{(H \rightarrow G)}_{S}$ in terms of local quantities without including any integration over $t$ or $r$. With such a choice of $\xi^{(H \rightarrow G)}_{S}$, we do not have to worry about the condition (2) for $S$-part since there is no secular growth in $\xi^{(H \rightarrow G)}_{S}$. What we need to guarantee to satisfy the condition (2) is the absence of secular growth in $\xi^{(H \rightarrow G)}_{S}[h^{(H)}_{full}]^{**}$. We first recall that $h^{(H)}_{full}$ must stay finite by assumption. If this assumption does not hold, the whole formulation of self-force based on harmonic gauge

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*\(^{46}\) We would like to come back to this point in future publication.

**\(^{**}\) There is another way given by Amos Ori during post-Capra discussion meeting (2003 Kyoto), which does not rely on the presence of GODS. We give a brief explanation of his argument here although it might be inaccurate. When the source can be decomposed into Fourier mode in time direction, both $h^{(H)}_{full}$ and $h^{(G)}_{full}$ corresponding to a partial wave will be periodic. Namely, we assume $h^{(H)}_{full}(t, x) = h^{(H)}_{full}(t + T, x)$. Then, the gauge transformation connecting between two such metrics
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breaks down, and hence we lose the whole foundation. For GODS, \( \xi^{(H \to G)}[h_{\text{full}}^{(H)}] \) is guaranteed to stay finite since so \( h_{\text{full}}^{(H)} \) is.

§7. Other topics

Here we briefly mention a few topics which we have not yet discussed at all.

7.1. analytic approach to the self-force

There are a few calculation of the instantaneous self-force using numerical approach for a limited class of orbits.\(^{48}\) If we do not use the fully numerical approach, the full-part of the self-force is given in the form of Fourier expansion in time. On the other hand, the \( S \)-part is given in the local expansion, and hence is expressed in the time domain. To perform the subtraction, we need to transform the full-part (or the \( S \)-part) into the expression into the time domain (or into the frequency domain). As was mentioned earlier, a systematic method to solve the linear perturbation equation analytically is already known.\(^{18},^{19}\) Taking advantage of it, we proposed a method to perform this transformation between different domains analytically.\(^{49}\)

7.2. \( \ell = 0 \) and \( \ell = 1 \) modes

There was a debate on how to treat \( \ell = 0 \) and \( \ell = 1 \) modes.\(^{50},^{52}\) Here we give a simple-minded understanding of this complicated issue from the view-point of the intermediate gauge approach, although deeper understanding might indeed become indispensable when we consider the second order perturbation.

The treatment of these lower lying modes in black hole perturbation theory is different from the other modes. The Regge-Wheeler-Zerilli formalism can handle these modes separately, but their equations are not hyperbolic. Therefore the retarded boundary conditions do not fix the boundary conditions for these modes. In the Teukolsky formalism, these modes are absent from the beginning.

Those \( \ell = 0 \) and \( 1 \) modes are composed of physical part and gauge part. In the intermediate gauge approach, no debate can arise in the gauge part. As long as it does not secularly increase, we do not care about it at all. The problem arises only in the point how to determine the physical part. There exists some information which is not encoded in the master variable, which is what we call \( \ell = 0 \) and \( 1 \) modes.

The particle motion defines a three dimensional tube in four dimensional space-time specified by the trajectory \((t, r) = (t_z(\tau), r_z(\tau))\). This tube divide the background spacetime into two pieces: one containing the infinity \( i^0 \) and the other containing black hole horizon.

In both inner and outer regions, the metric perturbation is a homogeneous solution of Einstein equations. It is given by the part reconstructed from the master

\[
\nabla_{(\mu} \xi_{\nu)} \text{ satisfies the same periodicity. Hence, we have } [\xi_{\mu,\nu} + \xi_{\nu,\mu}](t, x) = [\xi_{\mu,\nu} + \xi_{\nu,\mu}](t + T, x). \text{ This implies that } \xi_{\mu}(t + T, x) - \xi_{\mu}(t, x) \text{ satisfies the Killing equation. Therefore, } \xi_{\mu} = \text{(periodic piece)} + tK_{\mu}, \text{ where } K_{\mu} \text{ is a Killing vector. For black hole background the variation of } K_{\mu} \text{ is limited, and hence for any choice of non-vanishing } K_{\mu}, \text{ the gauge transformation } \nabla_{(\mu}(tK_{\nu)}) \text{ does not vanish at } r^* \to \pm \infty. \text{ As far as the } h_{\text{full}}^{(G)} \text{ is guaranteed to go to zero at infinity or on the horizon, we find } K_{\mu} = 0. \text{ Hence, the full-part of the gauge transformation does not have any secular growth.}
\]
variables with $\ell \geq 2$ with an additional piece which does not affect the master variables with $\ell \geq 2$. Possibly such additional perturbations are those which are given by a change of the parameters contained in the background metric:

$$\delta h_{\mu\nu} = \frac{\partial g_{\mu\nu}^{BH}}{\partial M} \delta M + \frac{\partial g_{\mu\nu}^{BH}}{\partial a} \delta a,$$

(7.1)

where $M$ and $a$ are the mass and Kerr parameter of the central black hole, respectively. For vanishing of the master variable, no other perturbations are possible. The Schwarzschild case should be considered as a special case of Kerr. If we consider an orbit with inclination, one may think that these two parameters are insufficient because we do not have parameters corresponding to the rotation in $x$ and $y$ axes. However, these rotations other than that in the $z$-direction can be absorbed by a gauge transformation, i.e., by global rotation of angular coordinates.

If we do not care about small error of $O(\mu)$ in the estimate of the mass and the angular momentum of the central black hole, $\delta M$ and $\delta a$ are unimportant. They are renormalized in the definition of the mass and the angular momentum of the central black hole. Hence, this issue about $\ell = 0$ and 1 is not an issue of debate at the lowest order in $\mu$. It becomes an issue only when we discuss it in connection with the second order perturbation.

§8. Summary

In this paper, we gave a brief review of the recent development in the study of gravitational radiation reaction problem in the context of generating gravitational wave templates, with some new insights.

First, we reported the adiabatic approximation to the radiation reaction, in which long time average is assumed in evaluating the change rates of the “constants of motion”. The radiation reaction to the Carter constant had been a long-standing issue, but now we are ready to compute it in the adiabatic approximation. We explained the formulation to evaluate the change rate of the Carter constant. We have presented a method to integrate the evolution of orbits for a long period taking into account the radiation reaction. We have shown that the errors in the phases of the orbits obtained by using the adiabatic approximation are $O(\mu \Delta \lambda, \mu^2 \Delta \lambda^2)$ or higher, where $\mu$ is the mass of the small compact star orbiting the central black hole and $\Delta \lambda$ is the time duration of integrating the orbit. In this order counting, $\Delta \lambda$ is supposed to be large, but $\mu \Delta \lambda$ is small. Typically, $\Delta \lambda \propto \mu^{-1}$ since the evolution due to radiation reaction is slower for a smaller mass. The phase errors caused by ignorance of the instantaneous leading order self-force is $O(\mu \Delta \lambda)$. But when those errors become large, the contribution from the second order self-force of $O(\mu^2 \Delta \lambda^2)$ becomes comparable. However, there is a possibility that further study on the leading order self-force alone, beyond the level of adiabatic approximation, can improve the gravitational wave templates dramatically in some cases. As the orbit evolves adiabatically, the frequencies of oscillating part of the self-force change, and eventually one of them may cross zero. A rough order of magnitude estimate suggests that the corrections in phases due to this accidental appearance of zero
frequency modes can be as large as $O(\sqrt{\mu \Delta \lambda})$. For more definite estimate of this effect we need to study the instantaneous self-force.

Furthermore, once we start to detect gravitational waves from binary inspiral with an extreme mass ratio, the detailed comparison of the signal with the theoretical prediction will become possible. In such a situation, we will wish to have a theoretical tool which can predict the phase evolution with an accuracy less than $O(1)$. For this purpose, we need to understand the self-force not only at the leading order but also up to the second order, although probably some kind of averaged values will be sufficient for the second order self-force.

Also as a step toward this goal, complete understanding about the first order self-force will be necessary. Recently, there have been a lot of developments also in this direction. We have presented here a biased summary about this issue, partly due to my understanding and also due to lack of space. So far, the lowest order self-force has not been discussed extensively as a step toward the second order self-force. In order to reflect the stored knowledge about the self-force to the improvement of the theoretical prediction of gravitational waveforms, we need to develop a formalism to evaluate the second order self-force.

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