Geometric Bremsstrahlung in the Early Universe

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Abstract

We discuss photon emission from particles decelerated by the cosmic expansion. This can be interpreted as a kind of bremsstrahlung induced by the Universe geometry. In the high momentum limit its transition probability does not depend on detailed behavior of the expansion. This may play an important role when massless particle emission is discussed in the early Universe.
1 Introduction

Particles in the expanding Robertson-Walker Universe are decelerated in the comoving frame. Ignoring backreaction, they obey the geodesic equation and lose their physical momentum $P_{\text{phys}}$ as

$$P_{\text{phys}} = \frac{P_{\text{conf}}}{a} \to 0,$$

where $P_{\text{conf}}$ is a conserved conformal momentum and $a$ is a scale factor growing in time. In general particles in deceleration can emanate radiation, or some massless particles. We call this process geometric bremsstrahlung\(^1\) due to the cosmic expansion.

Many analyses on phenomena in the early Universe have been performed so far using results of high energy particle physics and proposed a lot of interesting features of the Universe\(^1\). However they are based on calculations of transition matrices in the flat spacetime, emphasizing the fact that rates of interactions are much larger than expansion rate given by Hubble parameter, and no careful attention seems to be paid to the geometric bremsstrahlung process, which actually gives no contribution in the flat spacetime.

We shall argue in this paper that geometric bremsstrahlung may play important roles in the early Universe. In the section 2, emission of electromagnetic wave from a classical charged particle in the expanding Universe is discussed, taking account of backreaction. Our argument in the classical level suggests that the emission rate may not be simply ignored and the damping time may nearly equal to expansion time. Thus in the section 3, we treat photon emission from charged a particle quantum mechanically. High momentum limit of the transition probability can be obtained analytically. We stress that massless limit should be treated carefully and is nontrivial.

In this paper, we adopt the natural units, the light velocity $c = 1$ and the Planck constant $\hbar = 1$. Signature of metric is taken as $(+,−,−,−)$.\(^1\)

\(^1\) This process is also called by DeWitt and Brehme electro-gravitic bremsstrahlung in ref.\(^3\).
2 Classical Geometric Bremsstrahlung in the Early Universe

The radiation reaction has been neglected in the study of the early Universe. However particles are deaccelerated due to the cosmic expansion and thus they will emit the radiation. If the damping time due to the radiation reaction is comparable to the expansion time, the effect of the radiation reaction may not be simply neglected. We shall study in detail this phenomenon in the case of classical charged particles.

The study of the radiation reaction for a charged particle has a long history. The first relativistic calculation was performed by Dirac[2]. His calculation has been generalized by DeWitt-Brehme[3] for the motion in gravitational field. They have shown that bremsstrahlung induced by the spacetime curvature which we call geometric bremsstrahlung occurred in addition to the usual radiation damping. The effect is nonlocal in general which is caused by the so-called tail term in the Green function. It was Hobbs[4] who corrected the result of De Witt-Brehme and pointed out that the tail term vanishes identically in the case of the conformally flat spacetimes. His equation of motion for a particle with 4 velocity $u^\mu$, mass $m$, charge $e$ without external electromagnetic field may be written in the following form in conformally flat spacetime.

$$m \frac{Du^\mu}{D\tau} = \frac{2e^2}{3} \left( \frac{D^2 u^\mu}{D\tau^2} + u^\mu \left( \frac{Du}{D\tau} \right)^2 \right) + \frac{2e^2}{3} (\Omega_{,\alpha\beta} - \Omega_{,\alpha}\Omega_{,\beta}) \left( g^{\mu\alpha} u^\beta - u^\mu u^\alpha u^\beta \right)$$  \hspace{1cm} (2)

where $D/D\tau$ is the absolute derivative along the worldline of the particle with $\tau$ the proper time and the $\exp(2\Omega)$ is the conformal factor, $g_{\mu\nu} = e^{2\Omega} \eta_{\mu\nu}$ with $\eta_{\mu\nu}$ the flat Minkowski metric.

Here we are interested in the radiation reaction induced by the cosmic expansion in the early Universe and thus we will restrict ourselves to the case where the conformal factor depends only on the time variable. We shall take a different approach from Hobbs and evaluate explicitly the damping time scale due to the geometric bremsstrahlung in the early Universe.

We shall take the standard form for the action of a charged particle with mass $m$ and charge $e$ in the gravitational field,
\[ S = -m \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, d\tau - e \int A_{\mu} \dot{x}^\mu \, d\tau - \frac{1}{4} \int d^4 x \sqrt{-g} g^{\alpha\beta} g_{\mu\nu} F_{\alpha\mu} F_{\beta\nu}. \]

The dot denotes the derivative with respect to the proper time \( \tau \). The equation of motion derived from the above action may be written as follows.

\[ m \frac{D u^\mu}{D \tau} = m \left( \frac{d u^\mu}{d \tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \right) = e F^{\mu\nu} u_\nu. \]

Since we are interested in the early Universe, we may neglect the spatial curvature and thus take the spatially flat Robertson-Walker model as our background geometry,

\[ ds^2 = a^2(\eta)(d\eta^2 - d\vec{x}^2) = dt^2 - a^2 d\vec{x}^2. \]

Since the metric is conformally flat, it is convenient to work in the conformally related flat spacetime. Defining the conformally related proper time \( d\tau_f = a^{-1} d\tau \), we shall define the conformally related 4 velocity

\[ \tilde{u}^\mu = \frac{dx^\mu}{d\tau_f}. \]

Then the equation of motion may be written as follows.

\[ m \frac{D u^\mu}{D \tau} = ma^{-2} \left( \frac{d \tilde{u}^\mu}{d \tau_f} + \frac{\dot{a}}{a} \frac{d \eta}{d \tau_f} \tilde{u}^\mu - \frac{a'}{a} \delta^\mu_0 \right) = e a^{-3} \eta^\mu\alpha F_{\alpha\beta} \tilde{u}^\beta. \]

where the prime denotes the derivative with respect to the conformal time \( \eta \). By using the self field of the particle in the right hand side of the above equation, we shall obtain the radiation reaction force.

Before calculating the reaction force explicitly, let us compare the timescale due to the radiation damping with that due to the cosmic expansion to see the importance of the radiation reaction in the early Universe. The damping time may be roughly evaluated as follows.

\[
\frac{1}{t_r} \sim \left| \frac{1}{E_{\text{conf}}} \frac{d E_{\text{conf}}}{d \eta} \right| = \frac{1}{p_{\text{conf}}} \frac{2}{3a} \frac{e^2}{3} \left( \frac{d \tilde{u}}{d \eta} \right)^2 = \frac{2e^2}{3p_{\text{conf}}a} \left( \frac{p_{\text{conf}} da}{m^2 a^2 d \eta} \right)^2 = \frac{2}{3} e^2 p_{\text{phys}} H^2 \frac{m^2}{H^2}.
\]
where
\[ H = \frac{1}{a} \frac{da}{dt} \]
is the Hubble parameter, \( p_{\text{conf}} \) is the conformal momentum and we have used the fact that the physical momentum \( p_{\text{phys}} = m \ddot{u} = p_{\text{conf}}/a \) decays as \( a^{-1} \) if the radiation reaction is neglected. Thus the ratio between the Hubble time \( t_{\text{exp}} = H^{-1} \) and the damping time is
\[
\frac{t_{\text{exp}}}{t_r} \sim 2 e^2 \frac{m^2}{3 m^2 p_{\text{phys}} H}.
\]
This ratio is much larger than the unity for a relativistic particle at sufficiently early times in the Universe. Thus the radiation reaction may not be simply ignored and might play an important role in the early Universe.

For the calculation of the reaction force, we shall need the field equation derived from the above action,
\[
\eta^{\alpha\beta} F_{\alpha \nu,\beta} = e \int d\tau f \delta^4(x - x(\tau_f)) \tilde{u}^\mu.
\]
Taking the following non-covariant gauge
\[
\eta^{\mu\nu} A_{\mu,\nu} = 0,
\]
we arrive at the field equation which has the same form with that in the flat spacetime,
\[
\eta^{\alpha\beta} A_{\alpha,\beta} = e \int d\tau f \tilde{u}^\mu \delta^4(x - x(\tau_f)).
\]
Then the calculation by Dirac \[2\] applies here and we obtain the standard expression for the reaction force in the flat spacetime,
\[
F_{\mu,\text{react}} = e \eta^{\mu\alpha} F_{\alpha \beta} \ddot{u}^\beta = 2 e^2 \left[ \frac{d^2 \ddot{u}^\mu}{d\tau^2} + \left( \frac{d\ddot{u}}{d\tau} \right) \ddot{u}^\mu \right].
\]
It can be shown by a direct calculation that our expression of the equation of motion with radiation reaction in the conformally related flat spacetime coincides with eqn(4) when transformed back to the original physical frame.
In order to see the effect of the radiation reaction explicitly, we shall focus our attention to an 1-dimensional motion. Then the above equation is simplified as

\[
\frac{d}{d\tau_f} (am\ddot{u}) = \frac{2}{3} e^2 \left[ \frac{d^2 \ddot{u}}{d\tau_f^2} - \frac{\dddot{u}}{1 + \dddot{u}^2} \left( \frac{d\dddot{u}}{d\tau_f} \right)^2 \right].
\]

Without the radiation reaction, the conformal momentum \( p_{\text{conf}} = am\ddot{u} \) is conserved as expected.

Now we shall rewrite the above equation using the conformal momentum \( p_{\text{conf}} \) and the background time \( dt = ad\eta \),

\[
\frac{d^2 p_{\text{conf}}}{dt^2} = \left( H + \frac{3m}{2e^2 \sqrt{1 + (p_{\text{phys}}/m)^2}} \right) \frac{dp_{\text{conf}}}{dt} + \frac{dH}{dt} p_{\text{conf}}. \tag{3}
\]

Notice that there will be no classical geometric bremsstrahlung in the case of de Sitter expansion, namely \( H = \text{const} \). We shall be interested in the relativistic case in the early universe, namely

\[ p_{\text{phys}} \gg \frac{3m^2}{2e^2 H}, \quad m. \]

Then the second term in the coefficient of \( dp_{\text{conf}}/dt \) in eqn (3) is negligible. Thus when the particle is relativistic, its evolution is governed by the reaction force only and the Hubble time will be the only available time scale in this situation. In fact, the solution in this case may be written as follows.

\[
p_{\text{conf}}(t) = p_0 \left( 1 - H(t_0) \int_{t_0}^{t} dt' \exp \left( - \int_{t_0}^{t'} H(x) dx \right) \right) \exp \left( \int_{t_0}^{t} H(t') dt' \right)
\]

where we have taken the following initial conditions;

\[
p_{\text{conf}}(t = t_0) = p_0, \quad \frac{dp_{\text{conf}}}{dt}(t = t_0) = 0
\]

The second condition expresses the fact that the reaction force is absent at the initial time. The solution shows that the momentum decays in the Hubble time. Thus this process should not be simply neglected. However the above conclusion is obtained as a classical effect and it is not clear if the
geometric bremsstrahlung is still effective when the quantum effect is taken into account. We shall discuss quantum geometrobremsstrahlung in the next section.

3 Quantum Geometric Bremsstrahlung

As argued in the section 2, geometric bremsstrahlung may work efficiently classically in the early Universe. Whether this notable process survives or not after taking quantum effects into account is rather nontrivial and this question will be addressed next.

To define well-behaved quantum amplitudes in the expanding Universe, we consider spacetimes with Minkowskian in- and out- regions. The way of expansion is chosen arbitrary. The scale factor is described as

$$a(\eta) = C(\lambda \eta)$$

where $\eta$ is the conformal time, $\lambda^{-1}$ is a constant exhibiting typical time scale of the expansion. The function $C(x)$ in eqn(4) is arbitrary except the following constraints,

$$C(x \sim -\infty) = b,$$  
$$C(x \sim \infty) = 1,$$  
$$C(x) > 0,$$

where $b$ is some positive constant describing ratio of initial scale factor to final one.

Consider first the photon emission process in massive scalar QED with conformal coupling to the background curvature. The action reads

$$S = \int d^4x \sqrt{-g} \left( (\nabla_\mu + ieA_\mu)\Phi^*(\nabla^\mu - ieA^\mu)\Phi 
+ \left( \frac{1}{6}R - m^2 \right)\Phi^*\Phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \right).$$

The photon emission process is prohibited in the flat spacetime by energy-momentum conservation. However in the expanding spacetimes the energy conservation law gets broken and the transition can take place. The transition amplitude is given in the lowest order of perturbation such that

$$Amp = -ie \int d^4x \sqrt{-g} i \left( \Phi_j^* \nabla_\mu \Phi_j - \nabla_\mu \Phi_j^* \Phi_j \right) A^*_\mu$$  

(8)
where $\Phi_i (\Phi_f)$ is initial(final) mode function of massive charged scalar field and $A^*_\mu$ is final mode function of electromagnetic field. The scalar mode functions satisfy
\[
\left( \nabla^2 + m^2 - \frac{1}{6} R \right) \Phi = 0. \tag{9}
\]
Redefining the field $\tilde{\Phi} = \Phi \cdot a(\eta)$, the wave equation becomes the Klein-Gordon equation with a time-dependent mass,
\[
(\partial^2 + m^2 a(\eta))^2 \tilde{\Phi} = 0. \tag{10}
\]
Here we introduce $g^{\text{in}}_{\vec{p}_i}(\eta)$ and $g^{\text{out}}_{\vec{p}_f}(\eta)$ satisfying a Schrödinger-type equation,
\[
\left[ -\frac{d^2}{d\eta^2} - m^2 a(\eta)^2 \right] g_{\vec{p}} = \vec{p}^2 g_{\vec{p}}, \tag{11}
\]
with the boundary conditions in the asymptotic in and out regions as
\[
g^{\text{in}}_{\vec{p}_i}(\eta) \to \exp \left( -i\eta \sqrt{p_i^2 + m^2 b^2} \right) \frac{1}{\sqrt{(2\pi)^3 2 \sqrt{p_i^2 + m^2}}} \quad (\eta \sim -\infty),
\]
\[
g^{\text{out}}_{\vec{p}_f}(\eta) \to \exp \left( -i\eta \sqrt{p_f^2 + m^2} \right) \frac{1}{\sqrt{(2\pi)^3 2 \sqrt{p_f^2 + m^2}}} \quad (\eta \sim \infty).
\]
They also satisfy the following normalization condition.
\[
i \left( g^{*}_{\vec{p}} g'_{\vec{p}} - g^{*'}_{\vec{p}} g_{\vec{p}} \right) = \frac{1}{(2\pi)^3}, \tag{12}
\]
where the prime denotes the derivative with respect to $\eta$. Then the mode functions can be expressed as
\[
\Phi_{f}^* = \frac{1}{a} \tilde{\Phi}_f^* = \frac{1}{a} e^{-i\vec{p}_f \cdot \vec{x}} g^{\text{out}}_{\vec{p}_f},
\]
\[
\Phi_{i} = \frac{1}{a} \tilde{\Phi}_i = \frac{1}{a} e^{i\vec{p}_i \cdot \vec{x}} g^{\text{in}}_{\vec{p}_i}.
\]
The electromagnetic final mode function satisfies the Maxwell equation in curved spacetime,
\[
\nabla^\mu (\nabla_\mu A^*_\nu - \nabla_\nu A^*_\mu) = 0. \tag{13}
\]

Notice that in 4-dimensional conformally flat spacetimes this eqn[13] can be reduced into the same form in the flat spacetime,

\[ \partial^\mu (\partial_\mu A^*_\nu - \partial_\nu A^*_\mu) = 0. \]  

(14)

Therefore we get easily the final mode function

\[ A^*_\mu = \epsilon^*_\mu(k) \exp \left( i|k|\eta - ik \cdot \vec{x} \right) \sqrt{\frac{(2\pi)^3}{2|k|}}, \]

(15)

where \( \epsilon^*_\mu \) is a helicity factor.

Using the rescaled field, the amplitude, eqn[8], is rewritten as

\[ Amp = -ie \int d^4x \left( \tilde{\Phi}^*_f \partial^\mu \tilde{\Phi}_i - \partial^\mu \tilde{\Phi}^*_f \tilde{\Phi}_i \right) A^*_\mu. \]  

(16)

Because the photon emission lasts only during the epoch of expansion, the concept of the probability per unit time is ambiguous. So we shall use the transition probability itself. The transition probability \( W \) can be obtained from the amplitude such that

\[ W = \sum_{h=L,R} \frac{(2\pi)^3}{V} \int d^3p d^3k |Amp|^2, \]

(17)

where the summation is performed over the photon helicity and \( V \) is the conformal volume of the space, which is cancelled by the factor \( (2\pi)^3\delta(\vec{0}) \) coming from the conformal momentum conservation in \(|Amp|^2\). After the helicity summation, the explicit form of \( W \) is obtained as follows.

\[ W = (2\pi)^6 e^2 \int \frac{d^3k}{(2\pi)^3|k|} \int d^3p d^3k \delta(\vec{k} + \vec{p}_f - \vec{p}_i) \]

\[ \times \left[ (\vec{p}_f + \vec{p}_i)^2 \left| \int d\eta e^{i|k|\eta} g^i_{\vec{p}_f} g^i_{\vec{p}_i} \right|^2 - \left| \int d\eta e^{i|k|\eta} (g^i_{\vec{p}_f} g^{i*}_{\vec{p}_i} - g^{i*}_{\vec{p}_f} g^i_{\vec{p}_i}) \right|^2 \right]. \]

By virtue of the Wronskian relation,

\[ \frac{d}{d\eta} \left( g^i_{\vec{p}_f} g^{i*}_{\vec{p}_i} - g^{i*}_{\vec{p}_f} g^i_{\vec{p}_i} \right) = (\vec{p}_f^2 - \vec{p}_i^2) g^i_{\vec{p}_f} g^{i*}_{\vec{p}_i}, \]

(18)
the form of $W$ is more simplified such that

$$W = (2\pi)^3 e^2 \int \frac{d^3k}{2|k|} 4 \left( \vec{p}_i^2 - \frac{(\vec{k} \cdot \vec{p}_i)^2}{k^2} \right) \left| \int_{-\infty}^{\infty} d\eta \ e^{i|k|\eta} g^{out*}_{\vec{p}_i - \vec{k}} g^{in}_{\vec{p}_i} \right|^2. \quad (19)$$

To grasp the behavior of $W$ in the $|\vec{p}_i| \rightarrow \infty$ limit, we first argue a case with scale factor

$$a(\eta) = \Theta(\eta) + b\Theta(-\eta). \quad (20)$$

Then the exact mode functions are derived as

$$g^{in}_{\vec{p}_i} = \Theta(-\eta) \exp\left(-i\eta\sqrt{\vec{p}_i^2 + m^2 b^2}\right) \sqrt{(2\pi)^3 2\sqrt{\vec{p}_i^2 + m^2 b^2}} + \Theta(\eta) \frac{A(\vec{p}_i) \exp\left(-i\eta\sqrt{\vec{p}_i^2 + m^2}\right) + B(\vec{p}_i) \exp\left(i\eta\sqrt{\vec{p}_i^2 + m^2}\right)}{\sqrt{(2\pi)^3 2\sqrt{\vec{p}_i^2 + m^2 b^2}}}, \quad (21)$$

$$g^{out}_{\vec{p}_f} = \Theta(\eta) \exp\left(-i\eta\sqrt{\vec{p}_f^2 + m^2}\right) \sqrt{(2\pi)^3 2\sqrt{\vec{p}_f^2 + m^2}} + \Theta(-\eta) \frac{A(\vec{p}_f) \exp\left(-i\eta\sqrt{\vec{p}_f^2 + m^2 b^2}\right) + B(\vec{p}_f) \exp\left(i\eta\sqrt{\vec{p}_f^2 + m^2 b^2}\right)}{\sqrt{(2\pi)^3 2\sqrt{\vec{p}_f^2 + m^2}}}, \quad (22)$$

where

$$A(\vec{p}) = \frac{1}{2} \left( 1 + \left| \frac{\vec{p}^2 + m^2 b^2}{\vec{p}^2 + m^2} \right| \right),$$

$$B(\vec{p}) = \frac{1}{2} \left( 1 - \left| \frac{\vec{p}^2 + m^2 b^2}{\vec{p}^2 + m^2} \right| \right).$$

Substituting eqn\((21)\) and eqn\((22)\) into eqn\((15)\) and taking the high momentum limit, $|\vec{p}_i| \rightarrow \infty$, it is shown that the terms proportional to $B$ do not contribute to $W$ because of the damping behavior of $B$. Notice that taking
the high momentum limit, the energy conservation law almost restores in the following sense.

\[ |\vec{p}_i| \sim |\vec{k}| + |\vec{p}_i - \vec{k}|. \]  

(23)

This can be read easily from the \( \eta \) integration part in eqn (19). Contribution from the region where eqn (23) does not hold is severely suppressed by the energy conservation factor. Using the polar coordinate decomposition \( \vec{p}_i \cdot \vec{k} = |\vec{p}_i|k \cos \theta \) with \( k = |\vec{k}| \) and taking \( |\vec{p}_i| \) much larger than \( m \), it is easily derived that no contribution comes from the \( k \) integral region lying between \( |\vec{p}_i| \) and \( \infty \). Hence we get

\[
W(|\vec{p}_i| \sim \infty) = \frac{e^2|\vec{p}_i|}{2(2\pi)^2} \int_{|\vec{p}_i|\delta}^{|\vec{p}_i|} dk \frac{k}{|\vec{p}_i| - k} \int_0^{\pi} d\theta \sin^3 \theta 
\times \left[ k - \sqrt{\vec{p}_i^2 + m^2} + \sqrt{(|\vec{p}_i| - k)^2 + 2|\vec{p}_i|k(1 - \cos \theta) + m^2} \right]^{-1} 
- \left[ k - \sqrt{\vec{p}_i^2 + m^2b^2} + \sqrt{(|\vec{p}_i| - k)^2 + 2|\vec{p}_i|k(1 - \cos \theta) + m^2b^2} \right]^{-1}.
\]  

(24)

We need infra-red cutoff \( |\vec{p}_i|\delta \) in eqn(24) due to the existence of massless photon. This infra-red divergence is well known one in flat spacetime quantum field theories with massless particles and it should be cancelled by an infra-red divergence of the self energy term [5]. The cutoff \( \delta \) is physically determined by resolving power of soft photon observation. The \( \theta \) integration in eqn(24) can be straightforwardly calculated. After performing this integration and taking the high momentum limit \( |\vec{p}_i| \to \infty \), the \( k \) integration is simplified and we finally obtain

\[
W(|\vec{p}_i| \to \infty) = \frac{e^2}{4\pi^2} \left( \ln \frac{1}{\delta} + \delta - 1 \right) \left( \frac{1 + b^2}{1 - b^2} \ln \frac{1}{b^2} - 2 \right).
\]  

(25)

Note that we have taken the helicity sum in eqn(24). Instead, it is also possible to evaluate \( W(b) \) independently with a fixed photon helicity. For left and right handed helicity, each probability is the same, a half of \( W \) in eqn(24).

Furthermore we can also obtain the analytic forms of \( W(b) \) in the spinor QED for the case of eqn(21). Because both of the charged fermion and photon
have degree of helicity freedom, 4 helicity contributions must be considered separately. The probability in the high momentum limit for 1/2 helicity fermion decaying into fermion with helicity $h_{\text{fermion}}$ and photon with helicity $h_{\text{photon}}$ is denoted by $W(1/2; h_{\text{fermion}}, h_{\text{photon}})$ and is given for each case as follows.

\[
W(1/2; 1/2, 1) = \frac{e^2}{8\pi^2} \left( \ln \frac{1}{\delta} \right) \left( \frac{1 + b^2}{1 - b^2} \ln \frac{1}{b^2} - 2 \right). \tag{26}
\]

\[
W(1/2; 1/2, -1) = \frac{e^2}{8\pi^2} \left( \ln \frac{1}{\delta} - \frac{\delta^2}{2} + 2\delta - \frac{3}{2} \right) \left( \frac{1 + b^2}{1 - b^2} \ln \frac{1}{b^2} - 2 \right). \tag{27}
\]

\[
W(1/2; -1/2, 1) = \frac{e^2}{8\pi^2} \left( 1 - \frac{b}{1 - b^2} \ln \frac{1}{b^2} \right). \tag{28}
\]

\[
W(1/2; -1/2, -1) = 0. \tag{29}
\]

No infra-red cutoff $\delta$ appears in eqn(28) and eqn(29) because spinflip of the fermion enables observers to distinguish the bremsstrahlung from the self energy process.

There exists a very useful aspect of $W$ in the high momentum limit. It is supposed that the results eqn(23)~eqn(29) are exact not only for the special way of the expansion given by eqn(20) but also arbitrary way satisfying eqn(4)~eqn(7). This implies that $W(|\vec{p}_i| \to \infty)$ possesses a remarkable universality with respect to the ways of the cosmic expansion. This property may be explained as the Lorentz contraction effect from the view point of the high energy particle. Imagine a particle running in the comoving frame. Suppose that the Universe begins to expand when the particle passes through a point A and the Universe ceases to expand when the particle reaches point B. The particle catches energy from the expansion only while running from A to B. Taking the high momentum limit, the length between A and B contracts to zero in the rest frame of the particle. Therefore the particle cannot see the details of the way how the Universe expands and thus the universality of $W$ crops up.

To see the universality more quantitatively, we shall discuss the scalar QED with an adiabatically slow evolution of the scale factor $a(\eta)$ satisfying eqn(4)~eqn(7). In the zeroth order adiabatic approximation(WKB
approximation) the mode functions satisfying eqn(11) is written as

\[
g_{p}^{in} \sim g_{p}^{out} \sim \exp\left[i\vec{p} \cdot \vec{x} - i \int_{0}^{\eta} d\eta' \sqrt{\vec{p}^{2} + m^{2}a(\eta')^{2}}\right] \sqrt{(2\pi)^{3/2}\sqrt{\vec{p}^{2} + m^{2}a(\eta)^{2}}}.
\]

(30)

Substituting eqn(30) into eqn(19) and introducing the polar coordinate decomposition; \(\vec{p}_{\parallel} \cdot \vec{k} = |\vec{p}_{\parallel}|k \cos \theta\), we get

\[
W \sim \frac{e^{2}|\vec{p}_{\parallel}|^{2}}{2(2\pi)^{2}} \int_{|\vec{p}_{\parallel}|\delta}^{\infty} dk \int_{0}^{\theta} d\theta \sin^{3} \theta \times \left| \int_{-\infty}^{\infty} d\eta \exp\left[ik\eta - i \int_{0}^{\eta} d\eta' \sqrt{\vec{p}_{\parallel}^{2} + m^{2}a(\eta')^{2}} + i \int_{0}^{\eta} d\eta' \omega(\vec{p}_{\parallel} - \vec{k}, \eta')\right] \right|^{2}
\]

(31)

where \(\omega(\vec{p}_{\parallel} - \vec{k}, \eta) = \sqrt{(|\vec{p}_{\parallel}| - k)^{2} + 2|\vec{p}_{\parallel}|k(1 - \cos \theta) + m^{2}a^{2}}\) and \(|\vec{p}_{\parallel}|\delta\) is the inra-red cutoff. Consider the high momentum limit in eqn(31). As mentioned before nonvanishing contribution to \(W\) comes from the integral region where the momentum holds the relation eqn(23). Thus it is enough to restrict the momentum region between \(|\vec{p}_{\parallel}|\) and \(|\vec{p}_{\parallel}|\delta\). Here it should be searched which integral region of \(\theta\) contributes, accompanied with the influence of \(a(\eta)\), to the nonvanishing value of \(W\) in eqn(31). Due to eqn(23), only emittion to nearly forward direction \((\theta \sim 0)\) is permitted and especially the integral region of \(\theta\) satisfying

\[
0 \leq \theta \leq O\left(m/|\vec{p}_{\parallel}|\right)
\]

gives the scale factor dependence to the \(W\). Several expansions like

\[
\sqrt{\vec{p}_{\parallel}^{2} + m^{2}a(\eta')^{2}} \sim |\vec{p}_{\parallel}| + \frac{m^{2}a(\eta')^{2}}{2|\vec{p}_{\parallel}|}
\]

yield finally

\[
W \sim \frac{e^{2}|\vec{p}_{\parallel}|^{2}}{2(2\pi)^{2}} \int_{|\vec{p}_{\parallel}|\delta}^{|\vec{p}_{\parallel}|} dk \frac{k}{|\vec{p}_{\parallel}| - k} \int_{0}^{O\left(m/|\vec{p}_{\parallel}|\right)} d\theta \theta^{3} \times \left| \int_{-\infty}^{\infty} d\eta \exp\left[-i \int_{0}^{\eta} d\eta' \left(\frac{m^{2}a(\eta')^{2}}{2|\vec{p}_{\parallel}|} - \frac{m^{2}a(\eta')^{2}}{2(|\vec{p}_{\parallel}| - k)} - \frac{|\vec{p}_{\parallel}|k\theta^{2}}{2(|\vec{p}_{\parallel}| - k)}\right)\right]^{2}
\]

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where we change the integral variables in the following way,

\[ k = \frac{\vec{p}_i}{y}, \]
\[ \theta = \frac{m}{|\vec{p}_i|} z, \]
\[ \eta = \frac{2|\vec{p}_i|}{m^2} \tilde{\eta}. \]

Note that the function \( C\left(\frac{2\lambda|\vec{p}_i|}{m^2} \tilde{\eta}\right) \) in eqn(32) approaches in the high momentum limit to a step function,

\[ C\left(\frac{2\lambda|\vec{p}_i|}{m^2} \tilde{\eta}\right) \sim \Theta(\tilde{\eta}) + b \Theta(-\tilde{\eta}). \]  

Therefore the value of \( W \) for arbitrary adiabatical cosmic expansion satisfying eqn(4) \( \sim \) eqn(7) must equal to the specified value for eqn(20), and the universality is surely realized. If we dismiss the adiabatic approximation, the mode functions have reflection wave terms like in eqn(21) and eqn(22). However amplitude of the reflection waves vanishes in the high momentum limit and the universality are thought to survive.

Here we have a comment on the rate of the geometric bremsstrahlung. Comparing with the classical results in the section 2, it is noticed from eqn(25) \( \sim \) eqn(29) that interaction rate of quantum geometric bremsstrahlung does not so large compared with classical one. This is due to the fact that quantum effect smears position of the classical point particle and dilutes the charge density.

Now it is worth considering implication of the results; eqn(25) \( \sim \) eqn(29) in connection with the conformal symmetry. Since the massless limit \( m \to 0 \) forces the speed of the particle to reach the light velocity, the universality
with respect to the way of the cosmic expansion is maintained. In the lowest perturbation of the QED, the massless limit is shown to be equivalent with the $|\vec{p}_i| \to \infty$ limit. Therefore again the same results, eqn(25) $\sim$ eqn(29), come up in the $m \to 0$ limit and $W$ really possesses the non-vanishing value. One might naively expect for the massless case that the amplitude in the conformally flat spacetime vanishes as in the Minkowskian spacetime, by virtue of the conformal symmetry guaranteed at least classical mechanically. However this is not true unless $b = 1$ as argued above.

Next let us discuss the rate of the geometric bremsstrahlung. Taking large expansion limit $b \sim 0$, the transition probability can be typically expressed as

$$W(b \sim 0) \sim O(1) \frac{N^* e^2}{4\pi^2} \ln \frac{1}{b}, \quad (34)$$

where $N^*$ is the number of final modes. In our spinor QED model we take $N^* = 2$ due to contributions of eqns (26) and (27). However in the standard model we have more particles and $N^* \sim O(10)$. Moreover extended theories (like GUT and SUSY) can give us $N^* \sim O(100)$. The process is thought to occur when $W \sim 1$. Thus when the Universe expands enough to satisfy

$$\frac{1}{b} = \frac{a_f}{a_i} \sim \exp \left[ \frac{4\pi^2 O(1)}{N^* e^2} \right] \quad (35)$$

the event will take place. If we specify a model of the Universe evolution, we can get the rate itself. For example let us assume that the expansion is dominated by the radiation, $a(t) \propto t^{1/2}$. Then we get the following estimation for the transition rate.

$$\Gamma \sim \frac{1}{t_f} = b^2 \frac{1}{t_i} \sim \exp \left[ -\frac{8\pi^2 O(1)}{N^* e^2} \right] H_i \sim e^{-\frac{O(100)}{N^*}} H_i, \quad (36)$$

where $H_i = 1/t_i$ is the Hubble parameter at the initial time. Thus we cannot neglect the quantum bremsstrahlung naively when $N^* \sim O(100)$. It might be also worth reminding that rates of other gauge interaction processes different from the geometric bremsstrahlung is much smaller than the Hubble parameter when the temperature of the Universe is higher than $O(10^{15})$ GeV. Thus the bremsstrahlung may be the most dominant process in such an early era.
Acknowledgement

The authors wish to thank T. Goto, I. Joichi, T. Moroi, M. Tanaka and M. Yoshimura for fruitful discussions. We also thank to K. Hikasa and J. Arafune for their critical comments.

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