A Ranking Approach to Global Optimization

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Abstract
In this paper, we consider the problem of maximizing an unknown and potentially nonconvex function \( f \) over a compact and convex set \( \mathcal{X} \subset \mathbb{R}^d \) using as few observations \( f(x) \) as possible. We observe that the optimization of the function \( f \) essentially relies on learning the induced bipartite ranking rule of \( f \). Based on this idea, we relate global optimization to bipartite ranking which allows to address problems with high dimensional input space, as well as cases of functions with weak regularity properties. The paper introduces novel meta-algorithms for global optimization which rely on the choice of any bipartite ranking method. Theoretical properties are provided as well as convergence guarantees and equivalences between various optimization methods are obtained as a byproduct. Eventually, numerical evidence is provided to show that the main algorithm of the paper which adapts empirically to the underlying ranking structure is efficient in practice and displays competitive results with regards to the existing state-of-the-art global optimization methods over a wide range of usual benchmarks.

Keywords: global optimization, ranking, statistical analysis, convergence rate bounds

1. Introduction
In many applications such as complex system design or hyperparameter calibration for learning systems, the goal is to optimize some output value of a non-explicit function with as few evaluations as possible. Indeed, in such contexts, one has access to the function values only through numerical evaluations by simulation or cross-validation with significant computational cost. Moreover, the operational constraints generally impose a sequential exploration of the solution space with small samples. This generic problem of sequentially optimizing the output of an unknown and potentially non-convex function is often referred to as global optimization (Pintér (1991)), black-box optimization (Jones et al. (1998)) or derivative-free optimization (Rios and Sahinidis (2013)). In particular, there are several algorithms based on various heuristics which have been introduced in order to address complicated optimization problems with limited regularity assumptions, such as genetic algorithms, Bayesian methods, multi-start algorithms, etc.

This paper follows the line of the approaches recently considered in the machine learning literature (Bull (2011); Munos (2014); Sergeyev et al. (2013)). These approaches extend the seminal work on Lipschitz optimization of Hansen et al. (1992) and Jones et al. (1993) and they led to significant relaxations of the conditions required for convergence, e.g., only the existence of a local smoothness around the optimum is required (Munos (2014); Grill et al. 2016 Cédric Malherbe and Nicolas Vayatis.
More precisely, in the work of Bull (2011) and Munos (2014), specific conditions have been identified to derive a finite-time analysis of the algorithms. However, these guarantees do not hold whenever the unknown function is not assumed to be locally smooth around (one of) its optimum. In the present work, we propose to explore concepts from ranking theory based on overlaying estimated level sets (Clémençon et al. (2010)) in order to develop global optimization algorithms that do not rely on the smoothness of the function. The idea behind this approach is simple: even if the unknown function presents arbitrary large variations, most of the information required to identify its optimum may be contained in its induced ranking rule, i.e. how the level sets of the function are included one in another. To exploit this idea, we introduce a novel optimization scheme where the complexity of the function is characterized by the underlying pairwise ranking which it defines. Our contribution is twofold: first, we introduce two novel global optimization algorithms that learn the ranking rule induced by the unknown function with a sequential scheme, and second, we provide mathematical results in terms of statistical consistency and convergence to the optimum. Moreover, the algorithms proposed lead to efficient implementation and display good performance on the classical benchmarks for global optimization as shown at the end of the paper.

This paper is structured as follows. In Section 2, we introduce the framework and the main definitions. In Section 3, we introduce and analyze the RankOpt algorithm which requires the knowledge of a ranking structure underlying the unknown function. In Section 4, an adaptive version of the algorithm is presented. Companion results which establish the equivalence between learning algorithms and optimization procedures are discussed in Section 5 as they support implementation choices. Finally, the adaptive version of the algorithm is compared to other global optimization algorithms in Section 6. All proofs are postponed to the Appendix section.

2. Global optimization and ranking structure

2.1 Setup and notations

Setup. Let $\mathcal{X} \subset \mathbb{R}^d$ be a compact and convex set and let $f : \mathcal{X} \to \mathbb{R}$ be an unknown function which is only supposed to admit a global maximum over its domain $\mathcal{X}$. The goal in global optimization consists in finding some point $x^* \in \arg\max_{x \in \mathcal{X}} f(x)$ with a minimal amount of function evaluations. More precisely, we wish to set up a sequential procedure which starts by evaluating the function at an initial point $X_1 \in \mathcal{X}$ and then selects at each step $t \geq 1$ an evaluation point $X_{t+1} \in \mathcal{X}$ which depends on the previous evaluations $\{(X_i, f(X_i))\}_{i=1}^t$ and receives the evaluation of the unknown function $f(X_{t+1})$ at this point. After $n$ iterations, we consider that the algorithm returns the argument of the highest evaluation observed so far:

$$X_{\hat{i}_n} \text{ where } \hat{i}_n \in \arg\max_{i=1...n} f(X_i).$$

The analysis provided in the paper considers that the number $n$ of evaluation points is not fixed and it is assumed that function evaluations are noiseless.
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Notations. For all \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), we define the standard \( \ell_2 \)-norm as \( \|x\|_2^2 = \sum_{i=1}^{d} x_i^2 \); we denote by \( \langle \cdot, \cdot \rangle \) the corresponding inner product and we denote by \( B(x, r) = \{ x' \in \mathbb{R}^d : \| x - x' \|_2 \leq r \} \) the \( \ell_2 \)-ball centered in \( x \) of radius \( r \geq 0 \). For any bounded set \( \mathcal{X} \subset \mathbb{R}^d \), we define its inner-radius as \( \text{rad}(\mathcal{X}) = \max \{ r > 0 : \exists x \in \mathcal{X} \text{ such that } B(x, r) \subseteq \mathcal{X} \} \), its diameter as \( \text{diam}(\mathcal{X}) = \max_{(x, x') \in \mathcal{X}^2} \| x - x' \|_2 \) and we denote by \( \mu(\mathcal{X}) \) its volume where \( \mu \) stands for the Lebesgue measure. We denote by \( C^0(\mathcal{X}, \mathbb{R}) \) the set of continuous functions defined on \( \mathcal{X} \) taking values in \( \mathbb{R} \), we denote by \( P_k(\mathcal{X}, \mathbb{R}) \) the set of (multivariate) polynomial functions of degree \( k \geq 1 \) defined on \( \mathcal{X} \), and for any function \( f : \mathcal{X} \rightarrow \mathbb{R} \), we denote by \( \text{Im}(f) = \{ f(x) : x \in \mathcal{X} \} \) its image. Finally, we denote by \( U(A) \) the uniform distribution over a bounded measurable domain \( A \), we denote by \( I\{\cdot\} \) the indicator function taking values in \( \{0, 1\} \) and we denote by \( \text{sgn}(\cdot) \) the standard sign function defined on \( \mathbb{R} \) and taking values in \( \{-1, 0, 1\} \).

2.2 The ranking structure of a real-valued function

In this section, we introduce the ranking structure as a characterization of the complexity for a general real-valued function to be optimized. First, we observe that real-valued functions induce an order relation over the input space \( \mathcal{X} \), and the underlying ordering induces a ranking rule which records pairwise comparisons between evaluation points.

**Definition 1 (Induced ranking rule)** The ranking rule \( r_f : \mathcal{X} \times \mathcal{X} \rightarrow \{-1, 0, 1\} \) induced by a function \( f : \mathcal{X} \rightarrow \mathbb{R} \) is defined by:

\[
r_f(x, x') = \begin{cases} 
+1 & \text{if } f(x) > f(x') \\
0 & \text{if } f(x) = f(x') \\
-1 & \text{if } f(x) < f(x') 
\end{cases}
\]

for all \( (x, x') \in \mathcal{X}^2 \).

The key argument of the paper is that the optimization of any weakly regular real-valued function only depends on the nested structure of its level sets. Hence there is an equivalence class of real-valued functions that share the same induced ranking rule as shown by the following proposition.

**Proposition 2 (Ranking rule equivalence)** Let \( h \in C^0(\mathcal{X}, \mathbb{R}) \) be any continuous function. Then, a function \( f : \mathcal{X} \rightarrow \mathbb{R} \) shares the same induced ranking rule with \( h \) (i.e. \( \forall (x, x') \in \mathcal{X}^2, r_f(x, x') = r_h(x, x') \)) if and only if there exists a strictly increasing, but not necessarily continuous function \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) such that \( h = \psi \circ f \).

![Figure 1: Three functions that share the same ranking rule](image-url)
The previous proposition states that even if the unknown function $f$ admits noncontinuous or large variations, up to a transformation $\psi$, there might exist a simpler function $h = \psi \circ f$ that shares the same induced ranking rule. Figure 2.2 gives an example of three functions that share the same ranking rule while they display highly different regularity properties. As a second example, we may consider the problem of maximizing the function $f(x) = 1 - 1/|\ln(x)|$ if $x \neq 0$ and 1 otherwise over $\mathcal{X} = [0, 1/2]$. In this case, the unknown function $f$ is not ‘smooth’ around its unique global maximizer $x^* = 0$ but shares the same induced ranking rule with $h(x) = -x$ over $\mathcal{X}$.

A ranking structure is a collection of ranking rules. The approach developed in this paper consists of seeing the ranking structure as a characterization of the complexity of the target function $f$ through the complexity of its induced ranking rule. We first introduce a very large class of ranking rules.

**Definition 3 (Continuous Ranking Structure and Continuous Ranking Rules)**

We say that a real-valued function $f : \mathcal{X} \rightarrow \mathbb{R}$ has a continuous ranking rule if $r_f \in \mathcal{R}_\infty$ where $\mathcal{R}_\infty := \{r_h \mid h \in C^0(\mathcal{X}, \mathbb{R})\}$ denotes the set of continuous ranking rules (i.e. the set of ranking rules induced by continuous functions).

In the continuation of this definition, we further introduce three examples of more stringent ranking structures.

**Definition 4 (Polynomial Ranking Rules)** The set of polynomial ranking rules of degree $k \geq 1$ is defined as

$$\mathcal{R}_{P_k} := \{r_h : (x, x') \mapsto \text{sgn}(h(x) - h(x')) \mid h \in P_k(\mathcal{X}, \mathbb{R})\}.$$ 

We point out that even a polynomial function of degree $k > 1$ may admit a lower degree polynomial ranking rule. For example, consider the polynomial function $f(x) = (x^2 - 3x + 1)^9$. Since $f(x) = \psi(x^2 - 3x)$ where $\psi : x \mapsto (x + 1)^9$ is a strictly increasing function, the ranking rule induced by $f$ is a polynomial ranking rule of (at most) degree 2. We may now introduce our second class of ranking structures which is an extension of the set of polynomial ranking rules.

**Definition 5 (Sinusoidal Ranking Rules)** The set of sinusoidal ranking rules of degree $k \geq 1$ is defined as

$$\mathcal{R}_{S_k} := \{r_h : (x, x') \mapsto \text{sgn}((h(\cos(2\pi x)) - h(\cos(2\pi x')))) \mid h \in P_k(\mathcal{X}, \mathbb{R})\}$$

where the cosine function is vectorized, i.e. $\forall x \in \mathbb{R}^d, \cos(x) = \{\cos(x_1), \ldots, \cos(x_d)\}$.

The last class of ranking structures we introduce is a class of non-parametric ranking rules.

**Definition 6 (Convex Ranking Rules)** The set of convex ranking rules of degree $k \geq 1$ is defined as

$$\mathcal{R}_{C_k} := \{r \in \mathcal{R}_\infty \text{ such that } \forall x' \in \mathcal{X}, \text{ the set } \{x \in \mathcal{X} : r(x, x') \geq 0\} \text{ is a union of } k \text{ convex sets}\}.$$ 

It is easy to see that the ranking rule of a function $f$ is a convex ranking rule of degree $k$ if and only all the level sets of the function $f$ are unions of at most $k$ convex sets.
2.3 Identifiability and regularity

We now state two conditions that will be used in the theoretical analysis: the first condition is about the identifiability of the maximum of the function and the second is about the regularity of the function around its maximum.

**Condition 1** (Identifiability) The maximum of a function \( f : \mathcal{X} \rightarrow \mathbb{R} \) is said to be identifiable if for any \( \varepsilon > 0 \) arbitrarily small,

\[
\mu(\{x \in \mathcal{X} : f(x) \geq \max_{x \in \mathcal{X}} f(x) - \varepsilon\}) > 0.
\]

This condition prevents the function from having a spike on its maximum. It will be useful to state asymptotic results of the type \( f(X_{i_n}) \rightarrow \max_{x \in \mathcal{X}} f(x) \) when \( n \rightarrow +\infty \).

**Condition 2** (Regularity of the level sets) A function \( f : \mathcal{X} \rightarrow \mathbb{R} \) has \((c_\alpha, \alpha)\)-regular level sets for some \( c_\alpha > 0 \), \( \alpha \geq 0 \) if:

1. The global optimizer \( x^* \in \mathcal{X} \) is unique;
2. For any \( y \in \text{Im}(f) \), the iso-level set \( f^{-1}(y) = \{x \in \mathcal{X} : f(x) = y\} \) satisfies

\[
\max_{x \in f^{-1}(y)} \|x^* - x\|_2 \leq c_\alpha \cdot \min_{x \in f^{-1}(y)} \|x^* - x\|_2^{1/(1+\alpha)}.
\]

Condition 2 guarantees that the points associated with high evaluations are close to the unique optimizer with respect to the Euclidean distance. Note however that for any iso-level set \( f^{-1}(y) \) with finite distance to the optimum, the condition is satisfied with \( \alpha = 0 \) and \( c_\alpha = \text{diam}(\mathcal{X}) / \min_{x \in f^{-1}(y)} \|x^* - x\|_2 \). Thus, this condition concerns the local behavior of the level sets when \( \min_{x \in f^{-1}(y)} \|x^* - x\|_2 \rightarrow 0 \). As an example, the iso-level sets of three simple functions satisfying the condition with different values of \( \alpha \) are shown in Figure 2.

![Figure 2: Illustration of the regularity of the level sets on three simple functions. Left: \( f(x_1, x_2) = -x_1^2 - 1.4x_2^2 \) where \( \alpha = 0 \). Middle: \( f(x_1, x_2) = \exp(-|x_1|^3 - 1.4x_2^2) \) where \( \alpha = 1/2 \). Right: \( f(x_1, x_2) = -x_1^4 - 1.4x_2^2 \) where \( \alpha = 1 \).](image)

3. Optimization with fixed ranking structure

In this section, we consider the problem of optimizing an unknown function \( f \) given the knowledge that its induced ranking rule \( r_f \) belongs to a given ranking structure \( \mathcal{R} \subseteq \mathcal{R}_\infty \).
3.1 The RankOpt algorithm

Definitions. In order to properly set up the algorithm, we first introduce some key concepts that will be at the core of its strategy and used throughout the paper. We start with the definition of the empirical ranking loss.

Definition 7 (Empirical ranking loss) The empirical ranking loss computed over a sample \((X_1, f(X_1)), \ldots, (X_t, f(X_t))\) of \(t \geq 2\) function evaluations is defined for all \(r : X \times X \to \{-1, 0, 1\}\) by

\[
L_t(r) := \frac{2}{t(t-1)} \sum_{1 \leq i < j \leq t} \mathbb{I}\{r(X_i, X_j) \neq r_f(X_i, X_j)\}
\]

where \(r_f(X_i, X_j) = \text{sgn}(f(X_i) - f(X_j))\) for all \((i, j) \in \{1, \ldots, t\}^2\).

Based on this definition, one might then recover among a ranking structure \(\mathcal{R}\) the subset of ranking rules \(r\) which are consistent with the ranking rule \(r_f\) induced by the unknown function over a sample of function evaluations.

Definition 8 (Active subset of consistent ranking rules) The active subset of a ranking structure \(\mathcal{R}\) which contains the ranking rules consistent with \(r_f\) over a sample \((X_1, f(X_1)), \ldots, (X_t, f(X_t))\) of \(t \geq 2\) function evaluations is defined by

\[
\mathcal{R}_t := \{r \in \mathcal{R} : L_t(r) = 0\}
\]

where \(L_t(\cdot)\) denotes the empirical ranking loss defined above.

We may now introduce the optimization algorithm.

Algorithm description. The input of the RankOpt algorithm (displayed in Figure 3) are a number \(n\) of iterations, the unknown function \(f\), a compact and convex set \(X \subset \mathbb{R}^d\) and a ranking structure \(\mathcal{R} \subseteq \mathcal{R}_\infty\). At each iteration \(t < n\), a point \(X_{t+1}\) is sampled uniformly

1. Initialization: Let \(X_1 \sim \mathcal{U}(X)\)
   \(\text{Evaluate } f(X_1), t \leftarrow 1\)
   \(\mathcal{R}_1 \leftarrow \mathcal{R}, \hat{i}_1 \leftarrow 1\)

2. Iterations: Repeat while \(t < n\):
   Let \(X_{t+1} \sim \mathcal{U}(X)\)
   If there exists \(r \in \mathcal{R}_t\) such that \(r(X_{t+1}, X_{\hat{i}_t}) \geq 0\) \{Decision rule\}
   \(\text{Evaluate } f(X_{t+1}), t \leftarrow t + 1\)
   \(\mathcal{R}_t \leftarrow \{r \in \mathcal{R} : L_t(r) = 0\}\)
   \(\hat{i}_t \in \text{arg max}_{i=1 \ldots t} f(X_i)\)

3. Output: Return \(X_{\hat{i}_n}\)

Figure 3: The RankOpt\((n, f, X, \mathcal{R})\) algorithm
over $\mathcal{X}$ and the algorithm decides, whether or not, to evaluate the function at this point. The decision rule involves the active subset $\mathcal{R}_t$ which contains the ranking rules that are consistent with the ranking rule induced by $f$ over the points sampled so far. More precisely the decision rule operates as follows: if there does not exist any ranking rule $r \in \mathcal{R}_t$ which satisfies $r(X_{t+1}, \hat{X}_i) \geq 0$, then we know from the definition of $\mathcal{R}_t$ that $f_r(X_{t+1}, \hat{X}_i) = -1$ which necessarily means that $f(X_{t+1}) < f(\hat{X}_i)$. Thus, the algorithm never evaluates the function at a point that will not return certainly an evaluation at least equal to the highest evaluation $f(\hat{X}_i)$ observed so far.

**Remark 9 (Connection with active learning)** Although the problem considered in this paper is very different, the RankOpt algorithm might be seen as an extension to ranking of the baseline active learning algorithm introduced in Cohn et al. (1994) and further analyzed by Hanneke (2011). However, the main difference with this algorithm lies in the fact that in active learning, one estimates a binary classifier $h : \mathcal{X} \to \{0, 1\}$ where the goal in global optimization is to estimate the winner of a tournament deriving from the ranking rule $r : \mathcal{X} \times \mathcal{X} \to \{-1, 0, 1\}$ and not the ranking rule itself.

**Remark 10 (Adaptation to noisy evaluations)** It is noteworthy that the proposed optimization scheme could be extended to settings with noisy evaluations by slightly adapting the ideas developed in Dasgupta (2011) and Hanneke (2011). More precisely, a straightforward strategy would consist in using a relaxed version of the active subset $\mathcal{R}_{\delta,t} := \{r \in \mathcal{R} : L_t(r) \leq \min_{r \in \mathcal{R}} L_t(r) + UB_{\delta,t}\}$ where the term $UB_{\delta,t}$ comes out of some standard generalization bound on $|L_t(r_f) - \min_{r \in \mathcal{R}} L_t(r)|$ (see, e.g., Clémencen et al. (2010)).

**Remark 11 (Computational aspects)** Due to the theoretical nature and the genericity of the algorithm, several questions remain to be addressed in order to derive a practical implementation. In particular, the crucial steps of (i) identifying the set of ranking rules which minimize the empirical ranking loss and (ii) simulating the next evaluation points $X_{t+1}$ with the rejection method might not be trivial in practice. Nevertheless, we point out that, under specific conditions, a complete implementation of the algorithm can be proposed (see Section 5 for further discussions on these aspects).

### 3.2 Convergence analysis

We state here some convergence properties of the RankOpt algorithm. The results are stated in a probabilistic framework. Recall however that the source of randomness comes from the random variables generated by the algorithm and not from the evaluations which are assumed noiseless. We start by casting an intermediate result that will be important in order to formulate the consistency property of the algorithm and the upper bound on the convergence rate.

**Proposition 12** Let $\mathcal{X} \subset \mathbb{R}^d$ be any compact and convex set with non-empty interior, let $\mathcal{R}$ be any continuous ranking structure and let $f : \mathcal{X} \to \mathbb{R}$ be any function such that $r_f \in \mathcal{R}$. Then, for any $n \in \mathbb{N}^*$ and all $y \in \mathbb{R}$, we have that

$$
\mathbb{P}(f(X_{1:n}) \geq y) \geq \mathbb{P}(\max_{i=1 \ldots n} f(X_i') \geq y)
$$
where $X_{i_n}$ denotes the output of the RANKOPT$(n, f, \mathcal{X}, \mathcal{R})$ algorithm and $\{X'_i\}_{i=1}^n$ is a sequence of $n$ independent random variables uniformly distributed over $\mathcal{X}$.

One can then easily derive the next asymptotic result by combining Proposition 12 with the identifiability condition.

**Corollary 13 (Consistency)** Consider the same assumptions as in Proposition 12. Then, under Condition 1, we have that
\[ f(X_{i_n}) \xrightarrow{P} \max_{x \in \mathcal{X}} f(x). \]

Now we focus on the nonasymptotic performance of the algorithm. The next result provides our first finite-sample bound on the distance between the exact solution and the estimate provided by RANKOPT.

**Theorem 14 (Upper bound)** Suppose that the assumptions of Proposition 12 hold true. Then, under Condition 2, for any $n \in \mathbb{N}^*$ and $\delta \in (0, 1)$, we have with probability at least $1 - \delta$,
\[ \|x^* - X_{i_n}\|_2 \leq C_1 \cdot \left( \frac{\ln(1/\delta)}{n} \right)^{1/(1+\alpha)^2} \]
where $C_1 = c_\alpha^{(2+\alpha)/(1+\alpha)} \text{diam}(\mathcal{X})^{1/(1+\alpha)^2}$.

More surprisingly, a lower bound can also be derived by connecting the RANKOPT algorithm to a theoretical algorithm defined below which uses the knowledge of the level sets of the unknown function.

**Definition 15 (Pure Adaptive Search, from Zabinsky and Smith (1992))** We say that a sequence $\{X^*_{i}\}_{i=1}^n$ is distributed as a Pure Adaptive Search indexed by $f$ over $\mathcal{X}$ if it follows the Markov process defined by:
\[
\begin{align*}
X^*_1 &\sim U(\mathcal{X}) \\
X^*_{t+1} | X^*_t &\sim U(X^*_t) \quad \forall t \in \{1 \ldots n-1\}
\end{align*}
\]
where at each step $t \geq 1$ the next evaluation point $X^*_{t+1}$ is sampled uniformly over the level set of the previous evaluation $X^*_t := \{x \in \mathcal{X} : f(x) \geq f(X^*_t)\}$.

Precisely, the next result shows that the value of the highest evaluation observed by a Pure Adaptive Search is superior or equal, in the usual stochastic ordering sense, to the one observed by the RANKOPT algorithm tuned with the same number of function evaluations.

**Proposition 16** Consider the same assumptions as in Proposition 12. Then, for any $n \in \mathbb{N}^*$ and all $y \in \mathbb{R}$, we have that
\[ \mathbb{P}(f(X_{i_n}) \geq y) \leq \mathbb{P}(f(X^*_{i_n}) \geq y) \]
where $X_{i_n}$ denotes the output of the RANKOPT algorithm after $n$ iterations and $\{X^*_{i}\}_{i=1}^n$ is a sequence of $n$ evaluation points distributed as a Pure Adaptive Search indexed by $f$ over $\mathcal{X}$. 

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As the performance of the algorithm can now be controlled by Proposition 16, it is then possible to establish a second finite-time bound on the distance between the exact solution and its approximation.

**Theorem 17 (Lower bound)** Suppose that the assumptions of Proposition 12 hold true. Then, under Condition 2, for any \( n \in \mathbb{N}^* \) and \( \delta \in (0,1) \), we have with probability at least \( 1 - \delta \),

\[
C_2 \cdot e^{-\frac{(1+\alpha)^2}{4}(n+\sqrt{2n\ln(1/\delta)+\ln(1/\delta)})} \leq \|x^* - X_n\|_2
\]

where \( C_2 = c_\alpha^{-(1+\alpha)(2+\alpha)\text{rad}(\mathcal{X})^{(1+\alpha)^2}} \).

Note that, in addition to the following remarks, a complete discussion on the theoretical results obtained in this paper can be found in the next section where an adaptive version of the algorithm is presented.

**Remark 18 (Tightness of the bounds)** We stress that the RankOpt algorithm does achieve, for specific choices of ranking structures \( \mathcal{R} \) and functions \( f \), the polynomial and exponential rates exhibited in Theorems 14 and 17. Indeed, noticing that the algorithm is equivalent to a pure random search when \( \mathcal{R} \) is set to \( \mathcal{R}_\infty \), it can be easily shown by means of covering arguments that \( \|X_n - x^*\|_2 = \Omega_{\mathcal{P}}(n^{-1/d}) \) as soon as \( f \) admits a unique maximum and \( \mathcal{R} = \mathcal{R}_\infty \). Similarly, observing that the algorithm is equivalent to a pure adaptive search when \( \mathcal{R} \) is set to \( \{r_f\} \), one can also show by reproducing the same steps as in the proof of the lower bound that \( \|X_n - x^*\|_2 = \Omega_{\mathcal{P}}(e^{-n/d(1+\alpha+\epsilon)}) \) for any \( \epsilon > 0 \) when \( \mathcal{R} = \{r_f\} \) and \( f \) has regular level sets but no flat parts (i.e. \( \mu(\{x : f(x) = y\}) = 0 \) for all \( y \in \text{Im}(f) \)). As these bounds actually match the one reported above when \( \alpha = 0 \), we then deduce that the RankOpt algorithm does indeed achieve the near-optimal exponential and polynomial rates of \( \Theta_{\mathcal{P}}(e^{-n/d}) \) and \( \Theta_{\mathcal{P}}(n^{-1/d}) \) on any function \( f \) with \((1,0)\)-regular level sets and no flat parts when the ranking structure \( \mathcal{R} \) is respectively set to \( \{r_f\} \) and \( \mathcal{R}_\infty \).

**Remark 19 (Gap between the bounds)** As a direct consequence of the previous remark, we underline that whereas the upper and lower bounds reported in Theorems 14 and 17 display very different convergence rates, this gap can not be significantly reduced without imposing further conditions on both the function \( f \) and the ranking structure \( \mathcal{R} \) set as input. Indeed, observe that since the algorithm can achieve both the rates of \( \Theta_{\mathcal{P}}(e^{-n/d}) \) and \( \Theta_{\mathcal{P}}(n^{-1/d}) \) on the same function \( f \) depending on choice of the ranking structure \( \mathcal{R} \), then the gap between any generic lower and upper bounds on the convergence rate will necessarily be at least of the order of \([e^{-n/d},n^{-1/d}]\) as long as it is only assumed that \( \mathcal{R} \) is a continuous ranking structure and that \( f \) has regular level sets.

**Remark 20 (Choice of the ranking structure)** Finally, we point out that some simple indications on how to choose in practice the ranking structure \( \mathcal{R} \) set as input can be deduced from the previous remarks. Recall indeed that since the algorithm achieves its best performance when \( \mathcal{R} \) is set to \( \{r_f\} \), then an ideal but realistic ranking structure \( \mathcal{R} \) should be (i) large enough so that it actually has a chance to contain \( r_f \) and (ii) as small as possible in order to obtain similar performances as when \( \mathcal{R} = \{r_f\} \). Even though these indications seem to serve opposite goals, we will however see how to carefully combine them in the next section in order to derive an adaptive version of the algorithm which automatically selects the ranking structure \( \mathcal{R} \) among a series of ranking structures of different complexities.
1. Initialization: Let \( X_1 \sim \mathcal{U}(\mathcal{X}) \)
   Evaluate \( f(X_1), t \leftarrow 1 \)
   \( \mathcal{R} \leftarrow \mathcal{R}_1, \hat{i}_1 \leftarrow 1 \)

2. Iterations: Repeat while \( t < n \)
   Let \( B_{t+1} \sim \mathcal{B}(p) \)
   If \( B_{t+1} = 1 \) \{Exploration\}
     Let \( X_{t+1} \sim \mathcal{U}(\mathcal{X}) \)
   If \( B_{t+1} = 0 \) \{Exploitation\}
     Let \( X_{t+1} \sim \mathcal{U}(\{x \in \mathcal{X} : \exists r \in \mathcal{R} \text{ s.t. } r(x, X_i) \geq 0\}) \)
   Evaluate \( f(X_{t+1}), t \leftarrow t + 1 \)
   \( \hat{i}_t \in \arg\max_{i=1 \ldots t} f(X_i) \)
   \( \hat{k}_t \leftarrow \min\{k \in \mathbb{N}^*: \min_{r \in \mathcal{R}_k} L_t(r) = 0\} \) \{Model Selection\}
   \( \mathcal{R} \leftarrow \{r \in \mathcal{R}_{\hat{k}_t} : L_t(r) = 0\} \)

3. Output: Return \( X_{\hat{i}_n} \)

Figure 4: The AdaRankOpt\((n, f, \mathcal{X}, p, \{\mathcal{R}_k\}_{k \in \mathbb{N}^*})\) algorithm

4. Adaptive algorithm and stopping time analysis

We consider here the problem of optimizing an unknown function \( f \) when no information is available on its induced ranking rule \( r_f \).

4.1 The AdaRankOpt algorithm

The AdaRankOpt algorithm (shown in Figure 4) is an extension of the RankOpt algorithm which involves model selection. We consider a parameter \( p \in (0, 1) \) and a nested sequence of ranking structures \( \{\mathcal{R}_k\}_{k \in \mathbb{N}^*} \) satisfying

\[
\mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots \subset \mathcal{R}_\infty.
\] (1)

The algorithm is initialized by evaluating the function at a point \( X_1 \) uniformly distributed over \( \mathcal{X} \) and by considering the smallest ranking structure \( \mathcal{R}_1 \) of the sequence. At each iteration \( t < n \), a Bernoulli random variable \( B_{t+1} \) of parameter \( p \) is sampled. If \( B_{t+1} = 1 \), the algorithm explores the space by evaluating the function at a point uniformly sampled over \( \mathcal{X} \). If \( B_{t+1} = 0 \), the algorithm exploits the previous evaluations by making an iteration of the RankOpt algorithm with the smallest ranking structure \( \mathcal{R}_{\hat{k}_t} \) of the sequence that probably contains the true ranking \( r_f \). Once a new evaluation \( f(X_{t+1}) \) has been made, the index \( \hat{k}_t := \min\{k \in \mathbb{N}^*: \min_{r \in \mathcal{R}_k} L_t(r) = 0\} \) of the smallest ranking structure of the sequence \( \{\mathcal{R}_k\}_{k \in \mathbb{N}^*} \) which contains a ranking rule consistent with the sample is updated. Hence the parameter \( p \) drives the trade-off between the exploitation phase and the exploration phase which prevents the algorithm from getting stuck in a local maximum.

4.2 Theoretical properties of AdaRankOpt

We start by casting the consistency result for the algorithm.
Proposition 21 (Consistency) Fix any $p \in (0,1)$ and let $\{\mathcal{R}_k\}_{k \in \mathbb{N}^*}$ be any nested sequence of ranking structures. Then, under Condition 1, we have that

$$f(X_{i_n}) \xrightarrow{p} \max_{x \in \mathcal{X}} f(x)$$

where $X_{i_n}$ denotes the output of $\text{AdaRankOpt}(n, f, \mathcal{X}, p, \{\mathcal{R}_k\}_{k \in \mathbb{N}^*})$.

The previous result reveals however that the adaptive version of the algorithm remains consistent over the same set of identifiable functions regardless of its tuning (e.g. the choice of the sequence of ranking structures and the value of $p$). We thus have to examine its nonasymptotic performance in order to fully understand the impact of these parameters on its behavior.

We begin the finite-time analysis by investigating the number of iterations required to identify a ranking structure which contains the ranking rule $r_f$ induced by the unknown function.

Definition 22 (Stopping Time) Let $k^* = \min\{k \in \mathbb{N}^* : r_f \in \mathcal{R}_k\}$ be the index of the smallest ranking structure of the sequence which contains $r_f$ and let $\{\hat{k}_t\}_{t \in \mathbb{N}^*}$ be the sequence of random variables driving the model selection defined in the algorithm. We define the stopping time which corresponds to the number of iterations required to identify the index $k^*$ as

$$\tau_{k^*} := \min\{t \in \mathbb{N}^* : \hat{k}_t = k^*\}.$$

In order to bound $\tau_{k^*}$, we need to control the complexity of the sequence of ranking structures $\{\mathcal{R}_k\}_{k \in \mathbb{N}^*}$. Let us denote by $L(r) = \mathbb{P}(r(X, X') \neq r_f(X, X'))$ the true ranking loss where $(X, X')$ is a couple of independent random variables uniformly distributed over $\mathcal{X}$ and define the Rademacher average of a ranking structure $\mathcal{R}$ given $r_f$ as

$$R_n(\mathcal{R}) := \sup_{r \in \mathcal{R}} \frac{1}{\lfloor n/2 \rfloor} \left| \sum_{i=1}^{\lfloor n/2 \rfloor} \epsilon_i \cdot \mathbb{I}\{r(X_i, X_{\lfloor n/2 \rfloor + i}) \neq r_f(X_i, X_{\lfloor n/2 \rfloor + i})\} \right|$$

where $\{X_i\}_{i=1}^n$ are $n$ independent copies of $X \sim \mathcal{U}(\mathcal{X})$ and $\{\epsilon_i\}_{i=1}^{\lfloor n/2 \rfloor}$ are $\lfloor n/2 \rfloor$ independent Rademacher random variables (i.e., random symmetric sign variables), also independent of $\{X_i\}_{i=1}^n$.

Proposition 23 (Stopping Time Upper Bound) Assume that the index $k^* > 1$ is finite, assume that $\inf_{r \in \mathcal{R}_{k^*-1}} L(r) > 0$ and assume that there exists a constant $K > 0$ such that $\forall n \in \mathbb{N}^*$, the Rademacher complexity of $\mathcal{R}_{k^*-1}$ satisfies $\mathbb{E}[R_n(\mathcal{R}_{k^*-1})] \leq \sqrt{K/n}$. Then, for any $\delta \in (0,1)$, we have with probability at least $1 - \delta$,

$$\tau_{k^*} \leq \frac{10}{p} \cdot \left( \frac{K + \ln(2/\delta)}{\inf_{r \in \mathcal{R}_{k^*-1}} L(r)^2} \right).$$

In the situation described above, the smallest ranking structure of sequence which contains the true ranking rule can be identified in a finite number of iterations. One can then recover an upper bound similar to the one of Theorem 14 where a ranking structure containing $r_f$ is assumed to be known.
Theorem 24 (Upper Bound) Suppose that the assumptions of Proposition 23 hold true. Then, under Condition 2, for any \( \delta \in (0,1) \) and \( n \in \mathbb{N}^* \), we have with probability at least \( 1 - \delta \),
\[
\|X_{i_n} - x^*\|_2 \leq C_1 \cdot \left( \frac{11(K + \ln(4/\delta))}{p \inf_{r \in \mathcal{R}_{k^*}} L(r)} \right)^{\frac{1}{d(1+\alpha)^2}} \left( \frac{\ln(2/\delta)}{n} \right)
\]
where \( C_1 \) is the same constant as in Theorem 14 and \( X_{i_n} \) denotes the output of \( \text{AdaRankOpt}(n, X, \rho, \{\mathcal{R}_k\}_{k \in \mathbb{N}^*}) \).

The following remarks provide some insights on the different conditions and quantities involved in the theorem.

Remark 25 (On the Complexity Assumption) As pointed out in Clémençon (2011) (see Remark 2 therein), standard VC-type arguments can be used in order to bound \( \mathbb{E}[R_n(\mathcal{R}_{k^* - 1})] \). More specifically, if the set of functions \( \mathcal{R}_{k^* - 1} = \{(x, x') \in \mathcal{X}^2 \mapsto \mathbb{I}\{r(x, x') \neq r_f(x, x')\} \mid r \in \mathcal{R}_{k^* - 1}\} \) is a VC major class with finite VC dimension \( V \), then \( \mathbb{E}[R_n(\mathcal{R}_{k^* - 1})] \leq c \cdot V/n \) for a universal constant \( c > 0 \). This covers, in particular, the classes of polynomial and sinusoidal ranking rules of any degree \( k^* > 1 \).

Remark 26 (On the Infimum Ranking Loss) In order to grasp the meaning of the term \( \inf_{r \in \mathcal{R}_{k^* - 1}} L(r) \), observe first that since the function \( \rho : (r, r') \mapsto \mathbb{P}_{X, X' \sim \mu} (r(X, X') \neq r'(X, X')) \) defines a metric over the product space \( \mathcal{R}_\infty \times \mathcal{R}_\infty \), then the infimum ranking loss \( \inf_{r \in \mathcal{R}_{k^* - 1}} L(r) = \inf_{r \in \mathcal{R}_{k^* - 1}} \rho(r, r_f) \) can be interpreted as a measure of the distance between the ranking rule \( r_f \) and the ranking structure \( \mathcal{R}_{k^* - 1} \). As a consequence of this observation, we point out that the condition \( \inf_{r \in \mathcal{R}_{k^* - 1}} L(r) > 0 \) can then be easily checked to be fulfilled for the sequences of polynomial and sinusoidal ranking rules whenever \( r_f \in \mathcal{R}_{k^*} \) for some \( k^* > 1 \) by combining their parametric representation with the definition of the metric \( \rho(\cdot, \cdot) \).

4.3 Comparison with previous works

Our interest here is to compare the theoretical results obtained in this paper to existing results of the global optimization literature. We consider three different types of algorithms.

DIRECT and SOO (Jones et al. (1993) and Munos (2014)). These algorithms use a splitting technique of the search space and sequentially evaluate the function on subdivisions of the space that have recorded the highest evaluation among all the subdivisions of similar size. To the best of our knowledge, there is no finite-time analysis of the DIRECT algorithm (only the consistency was proven by Finkel and Kelley (2004)). However, Munos (2014) identified some local smoothness conditions allowing to derive a finite-time analysis of the algorithms. Precisely, assuming there exists \( x^* \in X \), \( \eta, c_1, c_2, \nu > 0 \) and \( \alpha \geq 0 \) such that \( \forall x \in B(x^*, \eta), c_1 \|x^* - x\|_\nu \leq f(x^*) - f(x) \leq c_2 \|x^* - x\|^{\nu/(1+\alpha)} \) for some norm \( \|\cdot\| \) (e.g., \( l_2 \), \( l_\infty \)), the author reports for the SOO algorithm a polynomial upper bound on the difference between the maximum and its estimation of \( \max_{x \in X} f(x) - f(X_{i_n}) = O(n^{-\nu/(d+\alpha)}) \) when \( \alpha > 0 \) and an exponential decay of \( O(e^{-c(\sqrt{n})}) \) for some \( c > 0 \) when \( \alpha = 0 \). As a comparison, we obtain for AdaRankOpt a polynomial bound of \( \max_{x \in X} f(x) - f(X_{i_n}) = O_{\mathbb{P}}(n^{-\nu/d}) \) for all \( \alpha \geq 0 \) by assuming that both the conditions of Proposition 23 and the local smoothness
condition are fulfilled. Hence the bound we obtain turns out to be better when $\alpha > 1$ and worse for $\alpha < 1$, which is consistent with the fact that while the asymmetry in the smoothness of the function around its maximum (captured here by $\alpha$) strongly impacts the performance of SOO in both ways, it does not impact AdaRankOpt which remains invariant to the variations of the local smoothness of the unknown function around its maximum.

**Evolution Strategies** (Eigen (1973)). We now consider the class of $(\mu, \lambda)$-Evolution Strategies which use mutation, recombination, and selection in order to iteratively evolve the set of evaluation points. As far as we know, no consistency results or generic upper bounds have been proven for these algorithms. However, Teytaud and Fournier (2008) were able to derive exponential lower bounds for several extensions of the $(\mu, \lambda)$-ES using the VC-dimension $V$ of the level sets of the unknown function. Precisely, they showed that if $V$ is finite, then $\|X_{\hat{t}} - x^*\|_2 = \Omega_p(e^{-c(V)n/d})$ where $c(V)$ is a constant that depends on both the extension under consideration and $V$. Moreover Auger (2005) also analyzed the convergence of the $(1, \lambda)$-SA-ES algorithm on the simple sphere function $f(x) = -\|x\|_2^2$ and proved specific conditions on the parameters of the algorithm in order to ensure that $\ln(\|X_{\hat{t}}\|_2)/n \xrightarrow{a.s.} c$ for some constant $c \in \mathbb{R}$. However, since the sign of the limit of $\ln(\|X_{\hat{t}}\|_2)/n$ remains unknown, this result only proves the exponential convergence or divergence of the algorithm and can therefore not be cast into our framework. More specifically, we point out all the results reported in those works can not be directly compared to the one obtained in this paper, as they are opposed by nature. Indeed, recall that while we were able to derive (i) a generic upper bound for AdaRankOpt and (ii) a lower bound for its nonadaptive version, they obtained on the contrary (i) generic lower bounds for various extensions of the $(\mu, \lambda)$-ES and (ii) an asymptotic upper bound which might be only valid for a specific version of the algorithm in the case where $f$ is the sphere function.

**Expected Improvement Strategy** (Moćkus (1975)). The last algorithm we consider is a Bayesian optimization strategy which selects at each step $t \geq 2$ an evaluation point $x_{t+1} = \arg \max_{x \in X} E_{f \sim \pi}[\max( f(x) - \max_{i=1, \ldots, t} f(x_i), 0)]$ where $f$ is assumed to be drawn from a law $\pi$ set as input. Vazquez and Bect (2010) showed that when $\pi$ is a fixed Gaussian process prior with a finite smoothness, the EI strategy converges on the maximum of any function $f$ of the reproducing kernel Hilbert space $\mathcal{H}$ canonically attached to $\pi$. Moreover Bull (2011) went on to prove that an adaptive version of the EI algorithm they define could achieve a near-optimal polynomial bound of $\max_{x \in \mathcal{X}} f(x) - f(X_{\hat{t}}) = O_P(n^{-\nu/d})$ for all $f \in \mathcal{H}$ when $\pi$ is a prior of smoothness $\nu$. As a comparison, considering that both the conditions of Proposition 23 are fulfilled and that $f \in \mathcal{H}$, we obtain for AdaRankOpt the exact same polynomial bound of $\max_{x \in \mathcal{X}} f(x) - f(X_{\hat{t}}) = O_P(n^{-\nu/d})$. But we point out that this similarity simply comes from the fact that the author also used a very similar—and potentially suboptimal—covering argument of the search space in order to prove their result.

These comparisons suggest that although the upper bounds provided in this paper are generic, they could certainly be improved in order to obtain the exponentially decreasing loss exhibited in Theorem 17 and observed in Munos (2014). Nonetheless, as detailed in Remark 19, such an analysis would require a refinement of the characterization of a real-valued function with regards to a ranking structure and is therefore left as future work.
5. Implementation and computational aspects

In this section, we discuss some technical aspects involved in the practical implementation of AdaRankOpt. In particular, we provide some equivalences that can be used in order to implement the algorithm for the classes of ranking structures introduced in Section 2 without explicitly maintaining the active subset of consistent ranking rules.

5.1 Notations

We collect here the specific notations used in this section. For any sample \(\{(X_i, f(X_i))\}_{i=1}^{t+1}\) of \(t+1\) function evaluations with distinct values (i.e. any sample such that \(f(X_i) \neq f(X_j)\) for all \(i \neq j\)), we denote by \(\{1, 2, \ldots, (t+1)\}\) the indexes corresponding to the strictly increasing reordering: \(f(X_{(1)}) < f(X_{(2)}) < \cdots < f(X_{(t+1)})\). For any dimension \(d \geq 1\), we respectively denote by \(0 = (0, \ldots, 0) \in \mathbb{R}^d\) and by \(\hat{1} = (1, \ldots, 1) \in \mathbb{R}^d\) the zero and the unit vector of \(\mathbb{R}^d\). The notation \(x \succeq x'\) corresponds to the component-wise inequality (i.e. \(x \succeq x' \iff \forall i \in \{1 \ldots d\}, \ x_i \geq x'_i\)) and we denote by \(\text{ConvHull}\{x_i\}_{i=1}^{t}\) the convex hull of any set \(\{x_i\}_{i=1}^{t}\) of \(t \geq 1\) points in \(\mathbb{R}^d\). For any degree \(k \geq 1\), the function that maps \(\mathbb{R}^d\) into the corresponding polynomial feature space of degree \(k\) is denoted by \(\Phi_k: \mathbb{R}^d \rightarrow \mathbb{R}^{\dim(\Phi_k)}\) where \(\dim(\Phi_k) = \binom{k+d}{d}-1\). For instance, in the case where \(k = d = 2\), we have that \(\Phi_2(x) = (x_1, x_2, x_1x_2, x_1^2, x_2^2) \in \mathbb{R}^5\) for all \(x = (x_1, x_2) \in \mathbb{R}^2\). Finally, we denote by \(M^k_{Ck} = [C_1 | \cdots | C_t]\) the \((\dim(\Phi_k), t)\)-matrix with its \(i\)-th column \(C_i\) equal to \((\Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)}))^\top\) and we denote for all \(i \leq t+1\) by \(M_i = [C_1 | \cdots | C_i]\) the \((d, i)\)-matrix where its \(j\)-th column \(C_j\) is equal to \(X_{(t+2-j)}^\top\).

5.2 General ranking structures

Suppose now that we have collected a sample \(\{(X_i, f(X_i))\}_{i=1}^{t}\) of \(t \geq 2\) observations generated by AdaRankOpt tuned with any nested sequence of ranking structures \(\mathcal{R}_k\). We address here the questions of (i) sampling the next evaluation point \(X_{t+1}\) and (ii) updating the index \(k_{t+1}\) of the model selection once \(f(X_{t+1})\) has been evaluated.

(i) We first consider the problem of sampling the next evaluation point \(X_{t+1} \sim U(\mathcal{X})\) over the non-trivial subset \(\mathcal{X}_t := \{x \in \mathcal{X} : \exists r \in \mathcal{R}_{k_t} \text{ such that } L_t(r) = 0 \text{ and } r(x, X_{i_t}) \geq 0\}\). To do so, we propose to use the rejection which consists in sampling \(X' \sim U(\mathcal{X})\) until \(X' \in \mathcal{X}_t\).

We thus need to set up a procedure that tests if any point \(X' \in \mathcal{X}\) belongs to \(\mathcal{X}_t\). By definition of \(\mathcal{X}_t\), we know that \(X' \in \mathcal{X}_t\) if and only if there exists a ranking rule \(r \in \mathcal{R}_{k_t}\) which satisfies \(L_t(r) = 0\) and \(r(X', X_{i_t}) = 0\) or 1. Therefore, we obtain by rewriting the previous statement in terms of minimal error that \(X' \in \mathcal{X}_t\) if and only if:

- either \(\min_{r \in \mathcal{R}_{k_t}} L_{t+1}(r) = 0\) where the empirical ranking loss is taken over the sample \(\{(X_i, f(X_i))\}_{i=1}^{t} \cup (X', f(X_{i_t}))\) (case \(r(X', X_{i_t}) = 0\));

- or \(\min_{r \in \mathcal{R}_{k_t}} L_{t+1}(r) = 0\) where \(L_{t+1}(\cdot)\) is taken over the sample \(\{(X_i, f(X_i))\}_{i=1}^{t} \cup (X', f(X_{i_t}) + c)\) where \(c > 0\) is any positive constant (case \(r(X', X_{i_t}) = 1\)).

Hence \(X_{t+1}\) can be generated by sequentially sampling \(X' \sim U(\mathcal{X})\) until there exists a ranking rule \(r \in \mathcal{R}_{k_t}\) that perfectly ranks the initial set of \(t\) observations where we
added a supplementary ghost evaluation \( \{(X_i, f(X_i))\}_{i=1}^{t+1} \) for some \( c \geq 0 \).

(ii) We now consider the problem of updating the index \( \hat{k}_{t+1} \) of the model selection once \( f(X_{t+1}) \) has been evaluated. Since \( \{R_k\}_{k \in \mathbb{N}^*} \) forms, by assumption, a nested sequence, it necessarily follows that the sequence of indexes \( \{\hat{k}_t\}_{t \in \mathbb{N}^*} \) is also increasing. One can thus write that \( \hat{k}_{t+1} = \hat{k}_t + \min\{i \in \mathbb{N}^* : \min_{r \in R_{\hat{k}_t+i}} L_{t+1}(r) = 0 \} \) where the empirical ranking loss \( L_{t+1}(\cdot) \) is computed over the sample \( \{(X_i, f(X_i))\}_{i=1}^{t+1} \). Hence, the index \( \hat{k}_{t+1} \) can be updated by sequentially testing if \( \min_{r \in R_{\hat{k}_t+i}} L_{t+1}(r) = 0 \) for \( i = 0, 1, 2, \ldots \)

As shown above, both the steps (i) and (ii) can be done using a single generic procedure that determines if \( \min_{r \in R_k} L_{t+1}(r) = 0 \) holds true for any ranking structure \( R_k \) of the sequence with \( k \geq 1 \) and where the empirical ranking loss \( L_{t+1}(\cdot) \) is computed over any sample of \( t + 1 \) function evaluations. In the next subsections, we provide some equivalences that can be used in order to design such a procedure for the classes of ranking structures introduced in Section 2. For simplicity, we will consider in the sequel that all the function evaluations of the sample have distinct values.

5.3 Polynomial and sinusoidal ranking rules

We consider here the sequence of polynomial ranking rules \( \{R_{P_k}\}_{k \in \mathbb{N}^*} \) and we recall that \( \Phi_k(\cdot) \) denotes the function that maps \( \mathbb{R}^d \) into the corresponding polynomial feature space of degree \( k \). However, we point out that the results stated below can easily be adapted for the sequence of sinusoidal ranking rules by considering the adequate feature space. The first result we establish relates the existence of a consistent polynomial ranking rule to the linear separability of a sample-dependent set of points which belong to the corresponding feature space.

**Proposition 27** (Separability) Let \( \{(X_i, f(X_i))\}_{i=1}^{t+1} \) be any sample of \( t + 1 \) function evaluations with distinct values. Then, there exists a polynomial ranking rule of degree \( k \geq 1 \) that perfectly ranks the sample (i.e. \( \min_{r \in R_{P_k}} L_{t+1}(r) = 0 \)) if and only if there exists an axis \( \omega \in \mathbb{R}^{\dim(\Phi_k)} \) satisfying:

\[
\langle \omega, \Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)}) \rangle > 0, \quad \forall i \in \{1 \ldots t\}.
\]

where \( (1), (2), \ldots (t + 1) \) denote the indexes of the strictly increasing reordering of the sample.

Unfortunately, the equivalence exhibited in Proposition 27 might not be always convenient in practice since the computational cost of estimating such an axis \( \omega \in \mathbb{R}^{\dim(\Phi_k)} \) can be prohibitive when the dimensionality of the feature space \( \dim(\Phi_k) \) is large. Nonetheless, as the previous result only makes the link with the existence of a separating axis, one can then use the following lemma presented in the generic framework of binary classification and illustrated in Figure 5 in order get an equivalence generally easier to check in practice.

**Lemma 28** Let \( \{(x_i, y_i)\}_{i=1}^{t} \) be any set of binary classification samples where \( (x_i, y_i) \in \mathbb{R}^d \times \{-1, +1\} \). Then, there exists a separating axis \( \omega \in \mathbb{R}^d \) satisfying

\[y_i \cdot \langle \omega, x_i \rangle > 0, \quad \forall i \in \{1 \ldots t\}\]
Figure 5: Illustration of Lemma 28. \textit{Left:} A separable sample \(\{(x_i, y_i)\}_{i=1}^n\). \textit{Middle:} The sample \(\{y_i \cdot x_i\}_{i=1}^n\). \textit{Right:} The convex hull of \(\{y_i \cdot x_i\}_{i=1}^n\) next to the zero vector.

if and only if

\[ \vec{0} \notin \text{ConvHull}\{y_i \cdot x_i\}_{i=1}^t. \]

One can then deduce from the combination of Proposition 27 and Lemma 28 that testing the existence of a consistent polynomial ranking rule can simply be performed by checking the emptiness of a specific polyhedron built from the sample as detailed in the next corollary.

\textbf{Corollary 29} Consider the same assumptions as in Proposition 27. Then, there exists a polynomial ranking rule of degree \(k \geq 1\) that perfectly ranks the sample if and only if the polyhedral set \(\Omega_t^{\Phi_k}\) defined by

\[ \Omega_t^{\Phi_k} := \{ \lambda \in \mathbb{R}^t : M_t^{\Phi_k} \lambda^T = \vec{0}, \langle \vec{1}, \lambda \rangle = 1, \lambda \succeq \vec{0} \} \]

is empty where \(M_t^{\Phi_k} = [C_1 | \cdots | C_t]\) is the \((\dim(\Phi_k), t)\)-matrix with its \(i\)-th column \(C_i\) equal to \((\Phi_k(X_{i+1}) - \Phi_k(X_i))^T\).

As a full implementation of the algorithm can be derived at this point (see Figure 6 in Section 6 for more details), a few comments are in order.

\textbf{Remark 30} (Algorithmic aspects) Notice that, in practice, the problem of testing the emptiness of a polyhedron admits a tractable solution. Indeed, it can be seen as the problem of determining if a particular linear program admits a feasible point and can therefore be solved with the simplex algorithm. For further details on this topic, we refer to Chapter 11.4 in Boyd and Vandenberghe (2004) where practical examples as well as algorithmic solutions are discussed.

\textbf{Remark 31} (Numerical complexity) In contrast, the numerical complexity of the proposed implementation can not be tracked precisely due to the stochastic nature of the rejection method. Nonetheless, we point out that a simple union bound indicates that the complexity of generating the next evaluation point \(X_{t+1} | \{X_i\}_{i=1}^t \sim \mathcal{U}(X_t)\) given a sample \(\{(X_i, f(X_i))\}_{i=1}^t\) is upper bounded, with probability at least \(1 - \delta\), by the complexity of testing the emptiness of a polyhedron multiplied by \([\ln(\delta)/\ln(1 - \mu(X_t)/\mu(X))]\). But, we stress that the value of the ratio \(\mu(X_t)/\mu(X)\) which controls the upper bound depends on both the random evaluations previously made and the nested structure of the level sets of the unknown function and can therefore not be developed further.
5.4 Convex ranking rules

We now consider the nonparametric sequence of convex ranking rules \( \{R_C_k\}_{k \in \mathbb{N}^*} \). The equivalences provided below essentially rely on the fact that any bipartite ranking rule can be approximated by overlaying a finite sequence binary classifiers as previously shown in Clémencéon and Vayatis (2010). We start with the one-dimensional case.

**Proposition 32** (Overlaying classifiers) Set \( d = 1 \) and assume that we have collected a sample \( \{(X_i, f(X_i))\}_{i=1}^{t+1} \) of \( t + 1 \) function evaluations with distinct values. Then, there exists a convex ranking rule of degree \( k \geq 1 \) that perfectly ranks the sample if and only if there exists a sequence of classifiers \( \{h_i\}_{i=1}^{t+1} \) of the form \( h_i(x) = \sum_{m=1}^k I\{l_{i,m} \leq x \leq u_{i,m}\} \) satisfying:

1. \( h_i(X(j)) = I\{j \geq i\}, \forall (i, j) \in \{1 \ldots t + 1\}^2 \);
2. \( h_1 \geq h_2 \geq \cdots \geq h_{t+1} \).

where \( (1), (2) \ldots (t+1) \) denote the indexes of the strictly increasing reordering of the sample.

In the specific case where \( d > 1 \) and \( k = 1 \), we further argue that checking the existence of a consistent and finite collection of nested convex classifiers can be performed by determining the emptiness of a cascade of polyhedral sets.

**Proposition 33** Set any \( d \in \mathbb{N}^* \) and assume that we have collected a sample \( \{(X_i, f(X_i))\}_{i=1}^{t+1} \) of \( t + 1 \) function evaluations with distinct values. Then, there exists a convex ranking rule of degree \( k = 1 \) that perfectly ranks the sample if and only if for each \( i = 1, \ldots, t \), the polyhedral set \( \Omega_i \) defined by

\[
\Omega_i := \left\{ \lambda \in \mathbb{R}^d : M_i \lambda = X^T_{(t+1-i)}, \langle \vec{1}, \lambda \rangle = 1, \lambda \succeq 0 \right\}
\]

is empty where \( M_i = [C_1 \mid \cdots \mid C_i] \) is the \((d, i)\)-matrix with its \( j \)-th column \( C_j \) is equal to \( X^T_{(t+2-j)} \).

6. Numerical experiments

In this section, we compare the empirical performance of the main algorithm of the paper to the existing state-of-the-art global optimization methods on real and synthetic problems.

**Algorithms.** We compared AdaRankOpt with five different types of algorithms, developed from various approaches of global optimization:

- **BayesOpt** (Martinez-Cantin (2014)) is a Bayesian optimization algorithm. It uses a distribution over functions to build a surrogate model of the unknown function. The parameters controlling the distribution are estimated during the optimization process.

- **CMA-ES** (Hansen (2006)) is an evolutionary algorithm. At each iteration, the new evaluation points are sampled according to a multivariate normal distribution with a mean vector and a covariance matrix computed from the previous evaluations.
• **CRS** (Kaelo and Ali (2006)) is a variant of the Controlled Random Search of Price (1983) which includes local mutations. It starts with a random population and randomly evolves these points by an heuristic rule.

• **DIRECT** (Jones et al. (1993)) is a Lipschitz optimization algorithm where the Lipschitz constant is unknown. It uses a deterministic splitting technique of the search space and it is therefore the only purely deterministic algorithm of the benchmark.

• **MLSL** (Kan and Timmer (1987)) is a multistart algorithm. It performs a series of local optimizations starting from points randomly chosen by a clustering heuristic that helps to avoid repeated searches of the same local optima.

For a fair comparison, the tuning parameters of the algorithms were all set to default and the AdaRankOpt algorithm was used in all the experiments with the sequence of polynomial ranking rules and with a parameter \( p \) fixed to \( 1/10 \). The detailed implementation of the AdaRankOpt algorithm used in the experiments can be found in Figure 6. The source of the implementations of the remaining algorithms are also reported in Table 1.

**Data sets.** We considered a series of nonconvex optimization problems which involve real data sets and naturally arise in the tuning of machine learning algorithms, and two series of artificial problems that are commonly met in standard global optimization benchmarks:

1. We first studied the task of estimating the regularization parameter \( \lambda \) and the bandwidth \( \sigma \) of a gaussian kernel ridge regression that minimize the empirical mean squared error of the predictions over a 10-fold cross validation. We employed five data sets from the UCI Machine Learning Repository (Lichman (2013)): Auto-MPG, Breast Cancer Wisconsin (Prognostic), Concrete slump test, Housing and Yacht Hydrodynamics. For each dataset, we only considered the real-valued attributes which were centered and normalized so that \( \sum_{i=1}^{n} X_i = 0 \) and \( \frac{1}{n} \sum_{i=1}^{n} X_i^2 = 1 \) for all the attributes.

2. We then compared the algorithms on a series of five bidimensional problems taken from Jamil and Yang (2013) and Surjanovic and Bingham (2013). The dimensionality of these problems allows an easy visualization of the test functions and it can be seen that this series covers a wide variety of situations, including multimodal and non-linear functions as well as ill-conditioned and well-shaped functions.

3. The last benchmark we used to assess the performance of the algorithms consists of a set of five synthetic functions with a dimensionality varying from three to seven taken from Finck et al. (2010) and Jamil and Yang (2013). Remark that, due to the high dimensionality of the input spaces, only few information is available on the structure of the test functions of this series.

A complete description of the test functions of the benchmark can be found in Table 2.
1. **Initialization:** Let $X_1 \sim \mathcal{U}(X)$
   Evaluate $f(X_1)$, $t \leftarrow 1$, $\hat{k}_1 \leftarrow 1$

2. **Iterations:** Repeat while $t < n$:
   Let $B_{t+1} \sim \mathcal{B}(1/10)$
   If $B_{t+1} = 1$
     Let $X_{t+1} \sim \mathcal{U}(X)$
   If $B_{t+1} = 0$
     Bool $\leftarrow$ True
     While Bool {Rejection Method}
       Let $X_{t+1} \sim \mathcal{U}(X)$
       Let $\Omega_{t}^{\Phi_{k_t}}$ be the polyhedron of Corollary 29 computed over the sample
       $\{(X_i, f(X_i))\}_{i=1}^{t} \cup (X_{t+1}, \max_{i=1..t} f(X_i) + c)$ where $c > 0$ is any strictly positive constant and the degree $k$ set to $\hat{k}_t$
       Test if $\Omega_{t}^{\Phi_{k_t}}$ is empty with the simplex algorithm
       If $\Omega_{t}^{\Phi_{k_t}}$ is empty
         Bool $\leftarrow$ False
     Else
       $\hat{k}_t \leftarrow \hat{k}_t + 1$

3. **Output:** Return $X_{\hat{i}_n}$

Figure 6: Implementation of the ADARankOpt algorithm with the sequence of polynomial ranking structures and with a parameter $p$ set to 1/10.

| Library                      | Algorithm(s)       |
|------------------------------|--------------------|
| The CMA 1.1.06 package       | CMA-ES             |
| NLOpt Library                | CRS, DIRECT, MLSL  |
| BayesOpt Library             | BayesOpt           |

Table 1: Source of the implementations of the algorithms used for comparison.
### Problem Objective function

| Problem  | Objective function                                                                 | Domain       | Local max. |
|----------|-----------------------------------------------------------------------------------|--------------|------------|
| Auto MPG | \(- \frac{1}{10} \sum_{k=1}^{10} \sum_{i \in D_k} (\hat{f}_k(x_i) - Y_i)^2\)          | \([-2,4] \times [-5,5]\) | -          |
| Breast Cancer | where: \(\hat{f}_k \in \arg\min_{f \in H} \frac{1}{n_k} \sum_{i \in D_k} (f(x_i) - Y_i)^2 + \lambda \|f\|_{H,\sigma}\) |
| Concrete | - the data set \((X_i, Y_i)\) is split into 10 folds \(D_1, \ldots, D_{10}\)       | \([-2,4] \times [-5,5]\) | -          |
| Yacht    | - \(\|f\|_{H,\sigma}\) is the corresponding norm                                    | \([-2,4] \times [-5,5]\) | -          |
| Housing  | - \(\sigma = 10^{\xi}\)                                                          | \([-2,4] \times [-5,5]\) | -          |
| Branim-Hoo | \(10(1 - 1/(8\pi))\cos(x_1) + 10 + (x_2 - 5.1x_1^2/(4\pi^2)) + 5x_1/\pi - 6)^2\) | \([-5,10] \times [0,15]\) | 3          |
| Himmelblau | \(-(x_1^2 + x_2 - 11)^2\)                                                    | \([-5,5] \times [-5,5]\) | 4          |
| Styblinski | \(8x_1^2 - 0.5x_2^2 - 2.5x_2 + 8x_1^2 - 0.5x_1^2 - 2.5x_1\)                    | \([-5,5] \times [-5,5]\) | 4          |
| Holder Table | \(|\sin(x_1)| \times |\cos(x_2)||\times \exp((1-(x_1^2 + x_2^2)^{1/2}/\pi))\)                      | \([-10,10]^2\) | 36         |
| Levy N.13 | \(- (x_1 - 1)^2(1+\sin^2(3\pi x_2))\)                                        | \([-10,10]^2\) | >100       |
| Rosenbrock | \(-\sum_{i=1}^{3} (x_i - 1)^2 - \sum_{i=1}^{3} 100(x_{i+1} + x_i)^2\)          | \([-2.048,2.048]^3\) | -          |
| Mishra N.2 | \(- (6 - \sum_{i=1}^{6} 0.5(x_i + x_{i+1}))^{5-\sum_{i=1}^{3} 0.5(x_{i+1})}\) | \([0,1]^6\) | -          |
| Linear Slope | \(\sum_{i=1}^{7} 10^{(i-1)/6} (x_1 - 5)\)                                        | \([-5,5]^7\) | 1          |
| Deb N.1          | \(\frac{1}{5} \sum_{i=1}^{5} \sin(6\pi x_i)\)                                      | \([-5,5]^5\) | 36         |
| Griewank N.4     | \(-1 - \sum_{i=1}^{4} x_i^2/4000 + \prod_{i=1}^{4} \cos(x_i / \sqrt{1})\)    | \([-300,600]^4\) | >100       |

**Table 2**: Description of the test functions of the benchmark. Dash symbols are used when a value can not be calculated.
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Protocol and performances. For each problem and each algorithm, we performed $K = 100$ distinct runs with a budget of $n = 1000$ function evaluations. For each target parameter $t = 90\%$, $95\%$ and $99\%$, we have collected the stopping times corresponding to the number of evaluations required by each method to reach the specified target

$$\tau_k := \min \{ i = 1, \ldots, n : f(X^{(k)}_i) \geq f_{\text{target}}(t) \}$$

where $\min \{ \emptyset \} = 1000$ by convention, $\{ f(X^{(k)}_i) \}_{i=1}^n$ denotes the evaluations made by a given method on the $k$-th run, $k \leq 100$ and the target value is set to

$$f_{\text{target}}(t) := \max_{x \in X} f(x) - \int_{x \in X} f(x) \, dx / \mu(X) - (\max_{x \in X} f(x) - \int_{x \in X} f(x) \, dx / \mu(X)) \times (1 - t).$$

Note that the target is normalized to the average value of the function over the domain to prevent the performance measures from being dependent of any constant term in the unknown function. In practice, the average value was estimated from a Monte Carlo sampling of $10^6$ evaluations and the maximum of the function was estimated, for the real task problems, by taking the best value observed over all the sets of experiments. Based on these stopping times, we then measured performance through a collection of indicators:

I) Average and standard deviation of the number of evaluations required to reach the specified target: $\bar{\tau}_K = \frac{1}{K} \sum_{k=1}^K \tau_k$ and $\hat{\sigma}_\tau = \left( \frac{1}{K} \sum_{k=1}^K (\tau_k - \bar{\tau}_K)^2 \right)^{1/2}$.

II) Proportion of runs that reached the specified target in terms of function evaluations: $\forall i \leq n, \hat{P}_K(\tau \leq i) = \frac{1}{K} \sum_{k=1}^K \mathbb{I}\{ \tau_k \leq i \}$.

III) Number of runs for which a method has executed less (or more) evaluations to reach the target than ADA_RANKOPT. Precisely, we have collected the following win/tie/loss indicators: $W = \sum_{k=1}^K \mathbb{I}\{ \tau_k < (1 - 0.1) \tau_k^{\text{ada}} \}$, $L = \sum_{k=1}^K \mathbb{I}\{ \tau_k > (1 + 0.1) \tau_k^{\text{ada}} \}$ and $T = K - (W + L)$ where $\tau_k^{\text{ada}}$ denotes the hitting times of the ADA_RANKOPT algorithm.

These indicators capture the most important properties of global optimization algorithms, such as accuracy, stability and velocity of convergence.

Results and comments. Results are collected in a series of Tables 3, 4, 5. We also report the proportion of runs that reached the different targets in terms of function evaluations in Figures 7, 8, 9. Our main observations are the following:

- The proposed method displays—as one should expect—very competitive results on test problems with estimated ranking rules of moderate complexity with regards to the sequence of ranking structures set as input (see, e.g., Breast Cancer, Concrete, Housing, Himmelblau or Styblinski). Moreover, experiments Linear Slope and Mishra N.2 also confirm that the algorithm can be robust against the dimensionality of the input space in the case of test functions with estimated ranking rule of low complexity.

- In contrast the method stalls on test problems which do not admit an estimated ranking rule of moderate complexity (see, e.g., Deb N.1 and Holder Table). Indeed, the algorithm can not estimate efficiently the ranking rules induced by some classes of
functions with a single sequence of ranking structures set as input. Considering at the same time multiple sequences of ranking structures might be a promising approach to address this issue, allowing the algorithm to adapt to wider varieties of shapes.

- Finally, it can be observed that in the case of test functions with strong global structure but many local optima, the algorithm reaches the 95% target with few function evaluations but fails at getting to the 99% target (see, e.g., Griewank N.4 or Levy N.13). Indeed, the algorithm starts moving toward the global optima by learning the global structure of the function but then considers ranking rules of a level of complexity higher than required when many local variations are met. As detailed in Remark 10, adding a noise parameter would allow the algorithm to be more robust against this type of local perturbations.

These empirical results aim at (i) providing numerical evidence that the main algorithm of the paper is competitive with the state-of-the-art methods and effective on a wide range of optimization problems and (ii) identifying some limits of the ranking-based approach we developed that could be solved with further extensions. However, a complete and detailed empirical analysis of the merits and limitations of the algorithm with these extensions is beyond the scope of this paper and will be carried out in future work.

7. Conclusion and future work

The major contribution of the paper was to show how to apply the principles of bipartite ranking to the global optimization problem. We introduced two novel global optimization strategies based on a sequential estimation of the ranking rule induced by the unknown function: RANKOPT which requires the knowledge of a ranking structure containing the induced ranking rule of the unknown function and its adaptive version ADA RANKOPT which performs model selection. A theoretical analysis of the algorithms is provided and empirical results based on synthetic and real problems have also been obtained, demonstrating the competitiveness of the adaptive version of the algorithm with regards to the existing state-of-the-art global optimization methods. Several questions are left open for future work. First, our theoretical analysis suggest that the characterization of real-valued functions given a particular ranking structure could be refined in order to identify the classes of functions providing the exponentially decreasing loss obtained in the lower bound. Second, our numerical studies also suggest that the empirical performance of the algorithm could be improved by (i) investigating the choice of the sequence of ranking structures set as input and (ii) allowing the algorithm do deal with noisy evaluations. Last, determining whether using a more aggressive sampling strategy would lead to better empirical results without deteriorating the convergence properties of the algorithm is left as a completely open question.
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| Problem     | Auto MPG | Breast Cancer | Concrete | Housing | Yacht |
|-------------|----------|---------------|----------|---------|-------|
| **Target 90%** |          |               |          |         |       |
| AdaRank     | 14.77 (+007) | 6.14 (+003)  | 5.82 (+003) | **6.64** (+003) | 17.33 (+008) |
| BayesOpt    | **10.84** (+003) | 6.83 (+003)  | 6.40 (+004)  | 7.67 (+003)  | 13.81 (+002) |
| CMA-ES      | 29.27 (+024)  | 11.10 (+009) | 10.41 (+008) | 12.84 (+012) | 29.61 (+025) |
| CRS         | 28.73 (+014)  | 8.87 (+008)  | 10.03 (+009) | 14.15 (+011) | 32.63 (+015) |
| DIRECT      | **11.00** (+000) | **6.00** (+000) | **6.00** (+000) | **11.00** (+000) | **11.00** (+000) |
| MLSL        | 13.06 (+015)  | 6.59 (+004)  | **3.85** (+004) | 7.19 (+003)  | 14.36 (+013) |

| **Target 95%** |          |               |          |         |       |
| AdaRank     | 17.14 (+008)  | **6.89** (+004) | **6.69** (+003) | 12.25 (+004) | 23.45 (+012) |
| BayesOpt    | 12.20 (+006)  | 8.35 (+004)  | 7.94 (+004)  | 14.10 (+022) | **15.91** (+021) |
| CMA-ES      | 42.90 (+031)  | 13.71 (+010) | 13.45 (+011) | 23.53 (+016) | 40.49 (+016) |
| CRS         | 35.82 (+012)  | 13.58 (+010) | 14.60 (+011) | 23.00 (+013) | 38.28 (+014) |
| DIRECT      | **11.00** (+000) | **11.00** (+000) | **11.00** (+000) | **11.00** (+000) | **11.00** (+000) |
| MLSL        | 14.97 (+015)  | 7.64 (+003)  | 7.31 (+004)  | **11.82** (+007) | 16.25 (+013) |

| **Target 99%** |          |               |          |         |       |
| AdaRank     | 41.75 (+033)  | 22.09 (+011) | 24.51 (+016) | 448.7 (+438) |
| BayesOpt    | **13.97** (+007) | 28.15 (+033) | 18.54 (+022) | **18.84** (+022) |
| CMA-ES      | 73.74 (+049)  | 46.31 (+029) | 62.14 (+085) | 70.87 (+049) |
| CRS         | 48.48 (+016)  | 36.55 (+014) | 44.09 (+014) | 52.89 (+018) |
| DIRECT      | **47.00** (+000) | **37.00** (+000) | **14.00** (+000) | **49.00** (+000) |
| MLSL        | **20.60** (+017) | 7.31 (+004)  | **14.73** (+010) | **16.38** (+010) | **21.43** (+014) |

Table 3: Results achieved on the cross-validation problems. The top of the table displays the number of evaluations to reach the specified targets (mean ± standard deviation). In bold, the best result obtained for each target in terms of average of evaluations. The bottom of the table displays the number of win/tie/loss to AdaRankOpt.
| Problem     | Branin-Hoo (mean ± std) | Himmelblau (mean ± std) | Styblinski (mean ± std) | Holder Table (mean ± std) | Levy N.13 (mean ± std) |
|-------------|-------------------------|-------------------------|-------------------------|---------------------------|------------------------|
| **Target 90%** |                         |                         |                         |                           |                        |
| AdaRank     | 7.23 (±0.004)           | 12.24 (±0.009)          | 27.5 (±0.010)           | 170.8 (±185)              | 13.10 (±0.012)         |
| BayesOpt    | 6.46 (±0.004)           | 12.69 (±0.013)          | 79.9 (±0.079)           | 410.0 (±417)              | 10.37 (±0.006)         |
| CMA-ES      | 20.61 (±0.017)          | 18.04 (±0.014)          | 259.6 (±385)            | **79.9** (±115)           | 16.98 (±0.014)         |
| CRS         | 8.77 (±0.007)           | 13.41 (±0.013)          | 160.3 (±297)            | 307.9 (±422)              | 17.89 (±0.016)         |
| DIRECT      | **4.00** (±0.000)      | **2.00** (±0.000)       | **20.0** (±0.000)       | **8.00** (±0.000)         | **1.00** (±0.000)      |
| MLSL        | 8.91 (±0.005)           | 7.60 (±0.005)           | 116.4 (±0.090)          | 305.0 (±379)              | 35.57 (±0.035)         |
| **Target 95%** |                         |                         |                         |                           |                        |
| AdaRank     | **8.79** (±0.005)      | 18.86 (±0.011)          | 34.5 (±0.011)           | 285.4 (±276)              | 19.67 (±0.022)         |
| BayesOpt    | 10.40 (±0.004)          | 20.77 (±0.020)          | 150.3 (±146)            | 417.8 (±410)              | 14.64 (±0.006)         |
| CMA-ES      | 29.28 (±0.021)          | 38.17 (±0.027)          | 339.5 (±406)            | 557.9 (±447)              | 26.99 (±0.023)         |
| CRS         | 18.89 (±0.017)          | 31.31 (±0.029)          | 170.6 (±294)            | 580.1 (±444)              | 27.56 (±0.020)         |
| DIRECT      | 11.00 (±0.000)          | 26.00 (±0.000)          | **34.0** (±0.000)       | **8.00** (±0.000)         | **1.00** (±0.000)      |
| MLSL        | 14.53 (±0.017)          | **10.07** (±0.005)      | 118.0 (±0.090)          | 315.7 (±384)              | 43.10 (±0.160)         |
| **Target 99%** |                         |                         |                         |                           |                        |
| AdaRank     | 16.08 (±0.006)          | 35.80 (±0.013)          | 58.3 (±0.023)           | 808.6 (±301)              | 184.2 (±0.230)         |
| BayesOpt    | 14.99 (±0.005)          | 32.19 (±0.023)          | 602.5 (±376)            | 422.0 (±407)              | 37.17 (±0.028)         |
| CMA-ES      | 55.83 (±0.041)          | 96.71 (±0.083)          | 426.5 (±399)            | 214.6 (±198)              | 105.7 (±125)           |
| CRS         | 57.06 (±0.057)          | 88.97 (±0.045)          | 212.9 (±280)            | 599.1 (±427)              | 90.87 (±0.039)         |
| DIRECT      | 11.00 (±0.000)          | 55.00 (±0.000)          | **34.0** (±0.000)       | **8.00** (±0.000)         | **1.00** (±0.000)      |
| MLSL        | 61.79 (±177)            | **15.17** (±0.005)      | 121.2 (±0.090)          | 321.7 (±382)              | 67.41 (±0.201)         |

Table 4: Results achieved on the first series of synthetic problems. The top of the table displays the number of evaluations to reach the specified targets (mean ± standard deviation). In bold, the best result obtained for each target in terms of average of evaluations. The bottom of the table displays the number of win/tie/loss to AdaRankOpt.
### Table 5: Results achieved on the second series of synthetic problems. The top of the table displays the number of evaluations to reach the specified targets (mean ± standard deviation). In bold, the best result obtained for each target in terms of average of evaluations. The bottom of the table displays the number of win/tie/loss to AdaRankOpt.

| Problem      | Rosenbrock | Mishra N.2 | Linear Slope | Deb N.1 | Griewank N.4 |
|--------------|------------|------------|--------------|---------|--------------|
| **Target 90%** |            |            |              |         |              |
| AdaRank      | 10.53      | 4.84       | 54.60        | 950.0   | 35.87        |
|              | (±009)     | (±003)     | (±180)       | (±016)  |              |
| BayesOpt     | 11.97      | 5.56       | 319.2        | 814.7   | 27.67        |
|              | (±008)     | (±003)     | (±406)       | (±276)  |              |
| CMA-ES       | 16.30      | 5.00       | 213.1        | 930.1   | 66.79        |
|              | (±012)     | (±004)     | (±105)       | (±166)  |              |
| CRS          | 15.08      | 5.10       | 368.1        | 980.7   | 76.70        |
|              | (±014)     | (±005)     | (±239)       | (±108)  |              |
| DIRECT       | 10.00      | 16.00      | 390.0        | 1000    | 103.0        |
|              | (±000)     | (±000)     | (±000)       | (±000)  |              |
| MLSL         | 8.82       | 5.51       | 27.48        | 198.0   | 218.1        |
|              | (±005)     | (±003)     | (±036)       | (±326)  |              |
| **Target 95%** |            |            |              |         |              |
| AdaRank      | 14.92      | 7.89       | 76.15        | 991.8   | 185.0        |
|              | (±014)     | (±004)     | (±015)       | (±091)  |              |
| BayesOpt     | 17.39      | 7.76       | 467.0        | 949.1   | 46.43        |
|              | (±014)     | (±003)     | (±455)       | (±153)  |              |
| CMA-ES       | 22.09      | 10.12      | 279.9        | 952.0   | 138.3        |
|              | (±015)     | (±008)     | (±100)       | (±127)  |              |
| CRS          | 21.95      | 10.38      | 553.5        | 997.1   | 136.3        |
|              | (±018)     | (±009)     | (±319)       | (±038)  |              |
| DIRECT       | 10.00      | 36.00      | 512.0        | 1000    | 130.0        |
|              | (±000)     | (±000)     | (±000)       | (±000)  |              |
| MLSL         | 10.01      | 7.40       | 37.74        | 215.8   | 282.2        |
|              | (±006)     | (±004)     | (±057)       | (±328)  |              |
| **Target 99%** |            |            |              |         |              |
| AdaRank      | 33.62      | 19.33      | 127.5        | 1000    | 1000         |
|              | (±029)     | (±005)     | (±032)       | (±000)  |              |
| BayesOpt     | 27.71      | 22.63      | 468.3        | 1000    | 422.7        |
|              | (±027)     | (±019)     | (±468)       | (±000)  |              |
| CMA-ES       | 43.59      | 35.75      | 380.0        | 962.1   | 267.5        |
|              | (±043)     | (±021)     | (±106)       | (±106)  |              |
| CRS          | 43.58      | 93.78      | 612.9        | 1000    | 424.8        |
|              | (±043)     | (±037)     | (±322)       | (±000)  |              |
| DIRECT       | 24.00      | 98.00      | 910.0        | 1000    | 908.0        |
|              | (±000)     | (±000)     | (±000)       | (±000)  |              |
| MLSL         | 19.72      | 13.66      | 50.5         | 256.7   | 451.1        |
|              | (±051)     | (±007)     | (±080)       | (±334)  |              |

Table 5: Results achieved on the second series of synthetic problems. The top of the table displays the number of evaluations to reach the specified targets (mean ± standard deviation). In bold, the best result obtained for each target in terms of average of evaluations. The bottom of the table displays the number of win/tie/loss to AdaRankOpt.
Figure 7: Proportion of runs that reached the targets 90%, 95% and 99% in terms of function evaluations on each of the cross-validation problems.
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Figure 8: Proportion of runs that reached the targets 90%, 95% and 99% in terms of function evaluations on each problem of the first series of synthetic functions.
Figure 9: Proportion of runs that reached the targets 90%, 95% and 99% in terms of function evaluations on each problem of the second series of synthetic functions.
Appendix A. Proof of Proposition 2

We develop here the proof for the equivalence class of real-valued functions sharing the same induced ranking stated in Proposition 2.

Proof of proposition 2 \((\Leftarrow)\) Assume that there exists a strictly increasing function \(\psi : \mathbb{R} \to \mathbb{R}\) such that \(h = \psi \circ f\). Since \(\psi\) is strictly increasing, it directly follows that \(\forall (x, x') \in \mathcal{X}^2, r_h(x, x') = \text{sgn}(\psi \circ f(x) - \psi \circ f(x')) = \text{sgn}(f(x) - f(x')) = r_f(x, x')\).

\((\Rightarrow)\) Assume now that \(\forall (x, x') \in \mathcal{X}^2, r_f(x, x') = r_h(x, x')\). First, note that if \(\forall (x, x') \in \mathcal{X}^2 r_f(x, x') = r_h(x, x') = 0\), both \(f = c_1\) and \(h = c_2\) are constant over \(\mathcal{X}\) and then \(h = \psi \circ f\) where \(\psi : x \mapsto x + (c_2 - c_1)\) is a strictly increasing function. We now consider the case where \(f\) is not constant over \(\mathcal{X}\) and we start to show that there exists a strictly increasing function \(\psi : \mathbb{R} \to \mathbb{R}\) such that \(f = \psi \circ M\) where \(M : \mathcal{X} \to [0, 1]\) is defined for all \(x \in \mathcal{X}\) by

\[
M : x \mapsto \int_{x' \in \mathcal{X}} \mathbb{I}\{r_f(x, x') < 0\} \, dx' = \mu(\{x' \in \mathcal{X} : f(x') < f(x)\}).
\]

To properly define \(\psi\), we first need to ensure that the function \(f\) is constant over the iso-level set \(M^{-1}(y) = \{x \in \mathcal{X} : M(x) = y\}\) for all \(y \in \text{Im}(M)\). To do so, fix any \(y \in \text{Im}(M)\), pick any \((x_1, x_2) \in M^{-1}(y) \times M^{-1}(y)\) and assume by contradiction and without loss of generality that \(f(x_1) < f(x_2)\). As the ranking rules \(r_f\) and \(r_h\) are assumed to be equal over \(\mathcal{X} \times \mathcal{X}\), we have that (i) \(h(x_1) < h(x_2)\) and (ii) \(M(x_i) = \mu(\{x' : h(x') < h(x_i)\})\), \(i \in \{1, 2\}\). Hence putting (i) and (ii) altogether with the continuity of \(h\) leads us to the contradiction

\[
M(x_1) = \mu(\{x' \in \mathcal{X} : h(x') < h(x_1)\}) < \mu(\{x' \in \mathcal{X} : h(x') < h(x_2)\}) = M(x_2)
\]

and we deduce that \(f\) is constant over any iso-level set of \(M\). Now, denoting by \(f(M^{-1}(y))\) the unique value of \(f\) over \(M^{-1}(y)\), we are ready to introduce the restriction of the function \(\psi\) over \(\text{Im}(M)\) defined by

\[
\psi_{\text{Im}(M)} : y \in \text{Im}(M) \mapsto f(M^{-1}(y)) \in \mathbb{R}.
\]

As \(\forall x \in \mathcal{X}, \psi_{\text{Im}(M)}(M(x)) = f(x)\), it follows from the continuity of \(h\) that \(\forall y_1 < y_2 \in \text{Im}(M) \times \text{Im}(M), \psi_{\text{Im}(M)}(y_1) < \psi_{\text{Im}(M)}(y_2)\). Hence \(\psi_{\text{Im}(M)}\) is strictly increasing over \(\text{Im}(f)\) and one can then write that \(f = \psi \circ M\) where \(\psi : \mathbb{R} \to \mathbb{R}\) is any strictly increasing extension of the function \(\psi_{\text{Im}(M)}\) over \(\mathbb{R}\). In addition, it can easily be shown by reproducing the same steps as previously with the function \(h\) that there also exists a strictly increasing function \(\psi' : \mathbb{R} \to \mathbb{R}\) such that \(h = \psi' \circ M\). Hence, the desired result follows by writing that \(h = \psi' \circ M = (\psi' \circ \psi^{-1}) \circ f\) where \(\psi' \circ \psi^{-1} : \mathbb{R} \to \mathbb{R}\) is a strictly increasing function. \(\square\)

Appendix B. Analysis of the RankOpt algorithm

In this section, we develop the full proofs of Proposition 12, Corollary 13, Theorem 14, Proposition 16 and Theorem 17.
B.1 Generic results and technical lemmas

We start by casting a simple property (Proposition 34) and two technical lemmas (Lemma 35 and Lemma 36) that will be used throughout the proofs.

**Proposition 34 (RankOpt process)** Consider that the assumptions of Proposition 16 are fulfilled. Then, the sequence of evaluation points \( \{X_t\}_{t=1}^n \) generated by the RankOpt algorithm after \( n \) iterations, that will be denoted in the sequel by \( \{X_t\}_{t=1}^n \sim \text{RankOpt}(n, f, X, R) \), is distributed as follows:

\[
\begin{cases}
X_1 \sim U(\mathcal{X}) \\
X_{t+1} | \{X_i\}_{i=1}^t \sim U(\mathcal{X}) & \forall t \in \{1 \ldots n-1\}
\end{cases}
\]

where at each step \( t \geq 1 \) the sampling area \( \mathcal{X}_t := \{x \in \mathcal{X} : \exists r \in R_t \text{ such that } r(x, X_{i_t}) \geq 0\} \) satisfies

\[
\{x \in \mathcal{X} : f(x) \geq f(X_{i_t})\} \subseteq \mathcal{X}_t \subseteq \mathcal{X}
\]

where \( i_t = \arg \max_{i=1 \ldots t} f(X_i) \).

**Proof** The first part of the proposition is a direct consequence of the definition of the algorithm. Noticing that \( \mathcal{X}_t \) is a subset of \( \mathcal{X} \) gives the second inclusion of the second part of the proposition. To prove the first inclusion fix any \( t \geq 1 \), pick any \( x \in \mathcal{X} \) satisfying \( f(x) \geq f(X_{i_t}) \) and observe that \( r_f(x, X_{i_t}) = \text{sgn}(f(x) - f(X_{i_t})) \geq 0 \). As \( L_t(r_f) = 0 \) by definition, it necessarily follows that \( r_f \in R_t \). Hence there exists \( r = r_f \in R_t \) such that \( r(x, X_{i_t}) \geq 0 \) and we deduce that \( \{x \in \mathcal{X} : f(x) \geq f(X_{i_t})\} \subseteq \mathcal{X}_t \) which concludes the proof. \( \square \)

The next lemmas (Lemma 35 and Lemma 36) will be important in order to state the nonasymptotic performance of the algorithm (i.e. Theorems 14 and 17).

**Lemma 35 (From Zabinsky and Smith (1992), see Appendix Section)** Let \( \mathcal{X} \subset \mathbb{R}^d \) be any compact and convex set with non-empty interior. Then, for any \( x^* \in \mathcal{X} \) and any \( r \in (0, \text{diam}(\mathcal{X})) \), we have that

\[
\frac{\mu(B(x^*, r) \cap \mathcal{X})}{\mu(\mathcal{X})} \geq \left( \frac{r}{\text{diam}(\mathcal{X})} \right)^d.
\]

**Proof** Introduce the similarity transformation \( S : \mathbb{R}^d \to \mathbb{R}^d \) defined by

\[
S : x \mapsto x^* + \frac{r}{\text{diam}(\mathcal{X})} (x - x^*)
\]

and let \( S(\mathcal{X}) = \{S(x) : x \in \mathcal{X}\} \) be the image of \( \mathcal{X} \) by \( S \). Since \( x^* \in \mathcal{X} \) and \( \max_{x \in \mathcal{X}} \|x - x^*\|_2 \leq \text{diam}(\mathcal{X}) \) by definition, it follows from the convexity of \( \mathcal{X} \) that \( S(\mathcal{X}) \subseteq B(x^*, r) \cap \mathcal{X} \) which directly implies that \( \mu(B(x^*, r) \cap \mathcal{X}) \geq \mu(S(\mathcal{X})) \). Moreover, as \( S \) is a similarity transformation conserves the ratios of the volumes before/after transformation, we have that

\[
\frac{\mu(B(x^*, r) \cap \mathcal{X})}{\mu(\mathcal{X})} \geq \frac{\mu(S(\mathcal{X}))}{\mu(\mathcal{X})} = \frac{\mu(S(B(x^*, \text{diam}(\mathcal{X}))))}{\mu(B(x^*, \text{diam}(\mathcal{X})))} = \frac{\mu(B(x^*, r))}{\mu(B(x^*, \text{diam}(\mathcal{X})))}
\]

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which combined with the fact that $\forall r \geq 0$, $\mu(B(x^*, r)) = \pi^{d/2} r^d / \Gamma(d/2 + 1)$ where $\Gamma(\cdot)$ stands for the standard gamma function gives the result. $\square$

The next lemma will be useful in order to control the volume of level sets of a function with $(c_\alpha, \alpha)$-regular level sets.

**Lemma 36** Let $\mathcal{X} \subset \mathbb{R}^d$ be any compact and convex set and let $f \in C^0(\mathcal{X}, \mathbb{R})$ be any continuous function with $(c_\alpha, \alpha)$-regular level sets (Condition 2). Then, for any radius $r \in (0, \max_{x \in \mathcal{X}} \|x^* - x\|_2)$, we have that

$$
\mathcal{X} \cap B(x^*, (r/c_\alpha)^{1+\alpha}) \subseteq \{ x \in \mathcal{X} : f(x) \geq \min_{x_r \in S_r} f(x_r) \} \subseteq B(x^*, c_\alpha \cdot r^{1/(1+\alpha)})
$$

where $S_r = \{ x \in \mathcal{X} : \|x^* - x\|_2 = r \}$.

**Proof** We start with the second inclusion. First, we show that for all $y \in [\min_{x \in S_r} f(x), f(x^*)]$, there exists $x_y \in f^{-1}(y) := \{ x \in \mathcal{X} : f(x) = y \}$ such that $\|x^* - x_y\| \leq r$. Consider any $y \in [\min_{x \in S_r} f(x), f(x^*)]$, pick any $x_r \in \text{arg min}_{x \in S_r} f(x)$ and introduce the function $F : [0, 1] \to \mathbb{R}$ defined by

$$
F : \lambda \mapsto f((1-\lambda)x^* + \lambda x_r),
$$

which returns the value of the function $f$ over the segment $[x^*, x_r]$. As (i) $F$ is continuous, (ii) $F(0) = f(x^*)$, (iii) $F(1) = \min_{x \in S_r} f(x)$ and (iv) $y \in [F(1), F(0)]$, it follows from the intermediate value theorem that there exists $\lambda_y \in [0, 1]$ such that $F_{x_r}(\lambda_y) = y$. Hence there exists $x_y = \lambda_y x^* + (1-\lambda_y)x_r \in f^{-1}(y)$ such that $\|x^* - x_y\| \leq \|x^* - x_r\|_2 = r$. Keeping in mind the previous statement, we may now prove the second inclusion. Assume by contradiction that there exists $x'_y \in f^{-1}(y)$ such that $\|x^* - x'_y\|_2 > c_\alpha r^{1/(1+\alpha)}$. Then, it directly from the definition of the maximum that $\max_{x \in f^{-1}(y)} \|x^* - x\|_2 \geq \|x^* - x'_y\|_2 > c_\alpha r^{1/(1+\alpha)}$. However, since $c_\alpha \cdot \min_{x \in f^{-1}(y)} \|x^* - x\|_2^{1/(1+\alpha)} \leq c_\alpha \cdot \|x^* - x'_y\|_2^{1/(1+\alpha)} \leq c_\alpha \cdot r^{1/(1+\alpha)}$ by definition of the minimum, we get the following contradiction by combining the previous statements with the regularity of the level set of the function:

$$
\max_{x \in f^{-1}(y)} \|x^* - x\|_2 \leq c_\alpha \cdot \min_{x \in f^{-1}(y)} \|x^* - x\|_2^{1/(1+\alpha)} < \max_{x \in f^{-1}(y)} \|x^* - x\|_2.
$$

As the previous contradiction holds for any $y \in [\min_{x \in S_r} f(x_r), f(x^*)]$, we deduce that $\{ x \in \mathcal{X} : f(x) \geq \min_{x \in S_r} f(x_r) \} \subseteq B(x^*, c_\alpha \cdot r^{1/(1+\alpha)})$ which proves the second inclusion.

We use similar arguments to prove the first inclusion. Suppose by contradiction that there exists $x' \in \mathcal{X} \cap B(x^*, (r/c_\alpha)^{1+\alpha})$ such that $f(x) < f(x_r)$ and introduce the function $F : \lambda \in [0, 1] \mapsto f((1-\lambda)x^* + \lambda x')$. First, we know from the intermediate value theorem that there exists $x'_r \in f^{-1}(f(x_r))$ such that $\|x^* - x'_r\|_2 < \|x^* - x'_r\|_2^{1/(1+\alpha)}$. Hence we have that $c_\alpha \cdot \min_{x \in f^{-1}(f(x_r))} \|x^* - x\|_2^{1/(1+\alpha)} \leq c_\alpha \|x^* - x'_r\|_2^{1/(1+\alpha)} < r$. However, as $\max_{x \in f^{-1}(f(x_r))} \|x^* - x\|_2 \geq \|x^* - x_r\|_2 = r$, we get a similar contradiction as the one obtained previously which proves that the first inclusion. $\square$
B.2 Consistency and upper bound

In this subsection, we provide the proofs of Proposition 12, Corollary 13 and Theorem 14.

**Proof of Proposition 12.** The statement is proved by induction. Since $X_1 \sim \mathcal{U}(\mathcal{X})$, the result directly holds for $n = 1$. Assume now that the statement holds for a given $n \in \mathbb{N}^*$ and let $\{X_i\}_{i=1}^{n+1} \sim \text{RANKOPT}(n + 1, f, \mathcal{X}, \mathcal{R})$. As the result also trivially holds whenever $y \notin \text{Im}(f)$, consider any $y \in \text{Im}(f)$ and let $\mathcal{X}_y = \{x \in \mathcal{X} : f(x) \geq y\}$ be the corresponding level set. We start with the following decomposition:

$$
P \left( \max_{i=1, \ldots, n+1} f(X_i) \geq y \right) = P \left( \max_{i=1, \ldots, n} f(X_i) \geq y \right) + P \left( \left\{ \max_{i=1, \ldots, n} f(X_i) < y \right\} \cap \{X_{n+1} \in \mathcal{X}_y\}\right). \tag{2}$$

From Proposition 34, we know that $X_{n+1}\mid \{X_i\}_{i=1}^n \sim \mathcal{U}(\mathcal{X}_n)$ where the sampling area $\mathcal{X}_n$ has a strictly positive Lebesgue measure whenever $\{\max_{i=1, \ldots, n} f(X_i) < y\}$. Hence conditioning upon $\{X_i\}_{i=1}^n$ gives that

$$
P \left( \left\{ \max_{i=1, \ldots, n} f(X_i) < y \right\} \cap \{X_{n+1} \in \mathcal{X}_y\} \right) = E \left[ \left\{ \max_{i=1, \ldots, n} f(X_i) < y \right\} \cdot P(X_{n+1} \in \mathcal{X}_y\mid \{X_i\}_{i=1}^n) \right] = E \left[ \left\{ \max_{i=1, \ldots, n} f(X_i) < y \right\} \cdot \frac{\mu(\mathcal{X}_n \cap \mathcal{X}_y)}{\mu(\mathcal{X}_n)} \right].$$

From Proposition 34 again, we also know that the sampling area $\mathcal{X}_n$ satisfies $\mathcal{X}_y \subseteq \mathcal{X}_n$ and $\mathcal{X}_n \subseteq \mathcal{X}$ whenever $\{\max_{i=1, \ldots, n} f(X_i) < y\}$. Therefore

$$
P \left( \left\{ \max_{i=1, \ldots, n} f(X_i) < y \right\} \cap \{X_{n+1} \in \mathcal{X}_y\} \right) \geq \frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X})} \cdot E \left[ \left\{ \max_{i=1, \ldots, n} f(X_i) < y \right\} \right] \geq \frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X})} \cdot \left( 1 - P \left( \max_{i=1, \ldots, n} f(X_i) \geq y \right) \right).$$

Finally, successively plugging the previous inequality into (2) and applying the induction assumption gives us that

$$
P \left( \max_{i=1, \ldots, n+1} f(X_i) \geq y \right) \geq P \left( \max_{i=1, \ldots, n} f(X_i) \geq y \right) + \frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X})} \cdot \left( 1 - P \left( \max_{i=1, \ldots, n} f(X_i) \geq y \right) \right)$$

$$
\geq P \left( \max_{i=1, \ldots, n} f(X'_i) \geq y \right) + \frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X})} \cdot \left( 1 - P \left( \max_{i=1, \ldots, n} f(X'_i) \geq y \right) \right)
$$

where $\{X'_i\}_{i=1}^{n+1} \sim \mathcal{U}(\mathcal{X})$ and the desired result follows by noticing that the right hand term of the previous inequality is equal to $P(\max_{i=1, \ldots, n+1} f(X'_i) \geq y)$. \hfill\Box

Equipped with Proposition 12, we may now easily prove the consistency property of the algorithm.

**Proof of Corollary 13.** Pick any $\varepsilon > 0$ and let $\mathcal{X}_{f^* - \varepsilon} = \{x \in \mathcal{X} : f(x) \geq \max_{x \in \mathcal{X}} f(x) - \varepsilon\}$ be the corresponding level set. By Proposition 12, we have that $\forall n \in \mathbb{N}^*$,

$$
P \left( f(X_{in}) < \max_{x \in \mathcal{X}} f(x) - \varepsilon \right) \leq P \left( \max_{i=1, \ldots, n} f(X'_i) < \max_{x \in \mathcal{X}} f(x) - \varepsilon \right)$$

where $\{X'_i\}_{i=1}^n \sim \mathcal{U}(\mathcal{X})$. 

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Therefore, using the fact that $0 < \mu(X_{f^* - \epsilon})/\mu(X) \leq 1$ by Condition 1, we directly get that

$$
\mathbb{P}\left(f(X_{in}) < \max_{x \in X} f(x) - \epsilon\right) \leq \mathbb{P}\left(X_1' \notin X_{f^* - \epsilon}\right)^n = \left(1 - \frac{\mu(X_{f^* - \epsilon})}{\mu(X)}\right)^n \xrightarrow{n \to \infty} 0
$$

which proves the result.

We now turn to the proof of the upper bound.

**Proof of Theorem 14.** Note first that since $r_f \in R \subseteq R_\infty$ is a continuous ranking rule, we know from Proposition 2 that there exists a continuous function $h \in C^0(\mathcal{X}, \mathbb{R})$ which shares the same ranking rule with $f$. One can then consider, without loss of generality, that $f \in C^0(\mathcal{X}, \mathbb{R})$ as all the arguments used in the proof only use function comparisons. Additionally, since the result trivially holds whenever the upper bound of the theorem, denoted here by $r_{\delta,n}$, satisfies $r_{\delta,n} \geq \max_{x \in \mathcal{X}} \|x - x^*\|_2$, we consider that $r_{\delta,n} < \max_{x \in \mathcal{X}} \|x - x^*\|_2$ which also implies by the level set assumption that $\ln(1/\delta) < n$. Last, we also set some notations, set $S_{\delta,n} = \{x \in \mathcal{X} : \|x^* - x\|_2 = (r_{\delta,n}/c_0)^{1+\alpha}\}$ and let $R_{\delta,n} = ((r_{\delta,n}/c_0)^{1+\alpha}/c_0)^{1+\alpha}$. Equipped with these notations, we may now prove the result. By Lemma 36, we have that

$$
\mathbb{P}(\|X_{in} - x^*\|_2 \leq r_{\delta,n}) = \mathbb{P}(X_{in} \in B(x^*, r_{\delta,n})) \geq \mathbb{P}\left(f(X_{in}) \geq \min_{x \in S_{\delta,n}} f(x)\right)
$$

which together with Proposition 12 gives that

$$
\mathbb{P}(\|X_{in} - x^*\|_2 \leq r_{\delta,n}) \geq \mathbb{P}\left(\max_{i=1...n} f(X_i') \geq \min_{x \in S_{\delta,n}} f(x)\right)
$$

where $\{X_i'\}_{i=1}^n \sim \mathcal{U}(\mathcal{X})$. Therefore, successively using independence and the second inclusion of Lemma 36 gives that

$$
\mathbb{P}(\|X_{in} - x^*\|_2 \leq r_{\delta,n}) \geq \mathbb{P}\left(\bigcup_{i=1}^n \{X_i' \in \mathcal{X} \cap B(x^*, R_{\delta,n})\}\right) = 1 - \left(1 - \frac{\mu(\mathcal{X} \cap B(x^*, R_{\delta,n}))}{\mu(\mathcal{X})}\right)^n.
$$

Finally, as $R_{\delta,n}$ was defined so that Lemma 35 ensures that

$$
\frac{\mu(\mathcal{X} \cap B(x^*, R_{\delta,n}))}{\mu(\mathcal{X})} \geq \left(\frac{R_{\delta,n}}{\text{diam}(\mathcal{X})}\right)^d = \frac{\ln(1/\delta)}{n},
$$

it follows that

$$
\mathbb{P}(\|X_{in} - x^*\|_2 \leq r_{\delta,n}) \geq 1 - \left(1 - \frac{\ln(1/\delta)}{n}\right)^n
$$

which combined with the elementary inequality $1 - x \leq e^{-x}$ gives the result.

**B.3 Lower bound**

In order to prove Theorem 17, we start by developing the full proof for Proposition 16 and we provide two technical lemmas (Lemma 37 and Lemma 38) that will used in the proof of the lower bound.

**Proof of Proposition 16.** Again, the result is proved by induction. Since $X_1$ and $X_1'$ are
both uniformly distributed over $\mathcal{X}$, the result directly holds for $n = 1$. Assume now that the statement holds for a given $n \in \mathbb{N}^*$ and let $\{X_i\}_{i=1}^{n+1} \sim \text{RANKOPT}(n + 1, f, \mathcal{X}, \mathcal{R})$. As the result also trivially holds whenever $y \notin \text{Im}(f)$, consider any $y \in \text{Im}(f)$ and let $\mathcal{X}_y = \{x \in \mathcal{X} : f(x) \geq y\}$ be the corresponding level set. We start with a similar decomposition as the one used in the proof of Proposition 12:

$$
\mathbb{P} \left( \max_{i=1 \ldots n+1} f(X_i) \geq y \right) = \mathbb{E} \left[ \mathbb{I} \{ \max_{i=1 \ldots n} f(X_i) \geq y \} \right] + \frac{\mu(\mathcal{X}_y \cap \mathcal{X}_n)}{\mu(\mathcal{X}_n)} \cdot \mathbb{I} \{ \max_{i=1 \ldots n} f(X_i) < y \}.
$$

Observe now that if $\mu(\mathcal{X}_y) = 0$, then $\mathbb{P} \left( \max_{i=1 \ldots n+1} f(X_i) \geq y \right) = \mathbb{P}(X_1 \in \mathcal{X}_y) = 0$ and the result directly holds. We thus consider the case where $\mu(\mathcal{X}_y) > 0$ and we set some additional notations to clarify the proof: let $f(X_{i_n}) = \max_{i=1 \ldots n} f(X_i)$ and let $\mathcal{X}_f(X_{i_n}) = \{x \in \mathcal{X} : f(x) \geq f(X_{i_n})\}$. From Proposition 34, we know that on the event $\{f(X_{i_n}) < y\}$ the sampling area $\mathcal{X}_n$ satisfies both $\mathcal{X}_f(X_{i_n}) \subseteq \mathcal{X}_n$ and $\mathcal{X}_y \subseteq \mathcal{X}_f(X_{i_n})$. Therefore we have that

$$
\mathbb{P} \left( \max_{i=1 \ldots n+1} f(X_i) \geq y \right) \leq \mathbb{E} \left[ \mathbb{I} \{ f(X_{i_n}) \geq y \} + \frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X}_f(X_{i_n}))} \cdot \mathbb{I} \{ f(X_{i_n}) < y \} \right].
$$

which combined with the fact that for any random variable $X \in [0,1]$, $\mathbb{E}[X] = \int_0^1 \mathbb{P}(X \geq t) \, dt$ gives that

$$
\mathbb{P} \left( \max_{i=1 \ldots n+1} f(X_i) \geq y \right) \leq \int_0^1 \mathbb{P} \left( \mathbb{I} \{ f(X_{i_n}) \geq y \} + \frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X}_f(X_{i_n}))} \cdot \mathbb{I} \{ f(X_{i_n}) < y \} \geq t \right) \, dt.
$$

(3)

Now, observe that since the volume of the sampling area always satisfies $\mu(\mathcal{X}_f(X_{i_n})) \leq \mu(\mathcal{X}_n) \leq \mu(\mathcal{X})$ by Proposition 34, then (i) the probability under the integral in (3) is equal to 1 whenever $t \leq \mu(\mathcal{X}_y)/\mu(\mathcal{X})$ and (ii) for any $t > \mu(\mathcal{X}_y)/\mu(\mathcal{X})$, the following events are equivalent:

$$
\left\{ \mathbb{I} \{ f(X_{i_n}) \geq y \} + \frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X}_f(X_{i_n}))} \cdot \mathbb{I} \{ f(X_{i_n}) < y \} \geq t \right\} = \left\{ \mu(\mathcal{X}_f(X_{i_n})) \leq \frac{\mu(\mathcal{X}_y)}{t} \right\}.
$$

Therefore plugging the inequalities obtained in (i) and (ii) into (3) gives us that

$$
\mathbb{P} \left( \max_{i=1 \ldots n+1} f(X_i) \geq y \right) \leq \frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X})} + \int_{\frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X})}}^{1} \mathbb{P} \left( \mu(\mathcal{X}_f(X_{i_n})) \leq \frac{\mu(\mathcal{X}_y)}{t} \right) \, dt.
$$

(4)

We now turn to the analysis of the probability under the integral in (4). By successively using the continuity of the ranking rule induced by the unknown function and applying the induction assumption, we obtain for any $t \in (\mu(\mathcal{X}_y)/\mu(\mathcal{X}), 1)$ the following bound:

$$
\mathbb{P} \left( \mu(\mathcal{X}_f(X_{i_n})) \leq \frac{\mu(\mathcal{X}_y)}{t} \right) = \mathbb{P} \left( f(X_{i_n}) \geq \min \left\{ y' \in \text{Im}(f) : \mu(\mathcal{X}_{y'}) \leq \frac{\mu(\mathcal{X}_y)}{t} \right\} \right)
\leq \mathbb{P} \left( f(X_n^*) \geq \min \left\{ y' \in \text{Im}(f) : \mu(\mathcal{X}_{y'}) \leq \frac{\mu(\mathcal{X}_y)}{t} \right\} \right)
= \mathbb{P} \left( \mu(\mathcal{X}_f(X_n^*)) \leq \frac{\mu(\mathcal{X}_y)}{t} \right)
$$

(5)
where \( \{X^*_i\}_{i=1}^n \) is a sequence of \( n \) random variables distributed as Pure Adaptive Search indexed by \( f \) over \( \mathcal{X} \) and \( \mathcal{X}_{f(X^*_n)} = \{ x \in \mathcal{X} : f(x) \geq f(X^*_n) \} \). Therefore, plugging (5) into (4) gives that

\[
\Pr \left( \max_{i=1\ldots n+1} f(X_i) \geq y \right) \leq \frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X})} + \int_{\frac{\mu(\mathcal{X}_y)}{\mu(\mathcal{X})}}^{1} \Pr \left( \mu(\mathcal{X}_{f(X^*_i)}) \leq \frac{\mu(\mathcal{X}_y)}{t} \right) dt
\]

and the desired result follows by noticing that the right hand term of the previous equation is equal to \( \Pr(f(X^*_{n+1}) \geq y) \) (which can be easily shown by reproducing the same steps as previously with a sequence of \( n + 1 \) random variables distributed as a Pure Adaptive Search).

The next lemma will be used in the proof of Theorem 17 to control the volume of the level set of the highest value observed by a Pure Adaptive Search.

**Lemma 37** Let \( \mathcal{X} \subset \mathbb{R}^d \) be any compact and convex set with non-empty interior, let \( f : \mathcal{X} \rightarrow \mathbb{R} \) be any function such that \( r_f \in \mathcal{R}_{\infty} \) and let \( \{X^*_i\}_{i=1}^n \) be a sequence of \( n \) random variables distributed as a Pure Adaptive Search indexed by \( f \) over \( \mathcal{X} \). Then, for any \( u \in [0, 1] \), we have that

\[
\Pr \left( \frac{\mu(\mathcal{X}^*_n)}{\mu(\mathcal{X})} \leq u \right) \leq \Pr \left( \prod_{i=1}^n U_i \leq u \right)
\]

where \( \mathcal{X}^*_n := \{ x \in \mathcal{X} : f(x) \geq f(X^*_n) \} \) and \( \{U_i\}_{i=1}^n \sim \mathcal{U}([0, 1]) \).

**Proof** Observe first that if \( u^* = \mu(\{ x \in \mathcal{X} : f(x) \geq \max_{x \in \mathcal{X}} f(x) \}) / \mu(\mathcal{X}) > 0 \), then the result trivially holds for all \( u < u^* \) and \( n \geq 1 \). For simplicity, we thus consider that \( u^* = 0 \) and we set some notations: \( \forall u \in [0, 1] \), let \( y_u := \min \{ y \in \text{Im}(f) : \mu(\{ x \in \mathcal{X} : f(x) \geq y \}) \leq u \cdot \mu(\mathcal{X}) \} \) and let \( \mathcal{X}_{y_u} = \{ x \in \mathcal{X} : f(x) \geq y_u \} \) be the corresponding level set. Keeping in mind that \( \mu(\mathcal{X}_{y_u}) \leq u \cdot \mu(\mathcal{X}) \) for all \( u \in [0, 1] \), we may now prove the result by induction.

Set \( n = 1 \), pick any \( u \in [0, 1] \) and let \( U_1 \sim \mathcal{U}([0, 1]) \). Since \( \mathcal{X}^*_1 \sim \mathcal{U}(\mathcal{X}) \) and \( \Pr(U_1 \leq u) = u \), it directly follows that

\[
\Pr \left( \frac{\mu(\mathcal{X}^*_1)}{\mu(\mathcal{X})} \leq u \right) = \Pr(\mathcal{X}^*_1 \in \mathcal{X}_{y_u}) = \frac{\mu(\mathcal{X}_{y_u})}{\mu(\mathcal{X})} \leq u = \Pr(U_1 \leq u)
\]

which proves the result for \( n = 1 \). Assume now that the statement holds for a given \( n \in \mathbb{N}^* \). Fix any \( u \in [0, 1] \) and let \( \{X^*_i\}_{i=1}^{n+1} \) be a sequence of \( n + 1 \) random variables distributed as Pure Adaptive Search indexed by \( f \) over \( \mathcal{X} \). From definition 15, we know that \( \mathcal{X}^*_{n+1} \sim \mathcal{U}(\mathcal{X}^*_n) \) where \( \mathcal{X}^*_n = \{ x \in \mathcal{X} : f(x) \geq f(X^*_n) \} \). Therefore, conditioning upon \( \mathcal{X}^*_n \) gives that

\[
\Pr \left( \frac{\mu(\mathcal{X}^*_{n+1})}{\mu(\mathcal{X})} \leq u \right) = \mathbb{E} \left[ \frac{\mu(\mathcal{X}_{y_u} \cap \mathcal{X}^*_{n+1})}{\mu(\mathcal{X}^*_{n+1})} \cdot \mathbb{I} \{ \mu(\mathcal{X}^*_n) > \mu(\mathcal{X}_{y_u}) \} + \mathbb{I} \{ \mu(\mathcal{X}^*_{n+1}) \leq \mu(\mathcal{X}_{y_u}) \} \right].
\]

Since the level sets of the unknown function form a nested sequence, we know that the following events are equivalent \( \{ \mu(\mathcal{X}^*_n) > \mu(\mathcal{X}_{y_u}) \} = \{ \mathcal{X}^*_n \subset \mathcal{X}_{y_u} \} \). Hence, using the convention
\(1/0 = +\infty\), we obtain that
\[
\Pr \left( \frac{\mu(X_{n+1}^*)}{\mu(X)} \leq u \right) = \mathbb{E} \left[ \min \left( 1, \frac{\mu(X_{n+1})}{\mu(X_n^*)} \right) \right].
\]

Now, since for any random variable \(U_{n+1} \sim \mathcal{U}([0, 1])\) independent of \(Y \in [0, 1]\), \(\Pr(U_{n+1} \leq Y \mid Y) = Y\), we have that
\[
\Pr \left( \frac{\mu(X_{n+1}^*)}{\mu(X)} \leq u \right) = \mathbb{E} \left[ \Pr \left( U_{n+1} \leq \min \left( 1, \frac{\mu(X_{n+1})}{\mu(X_n^*)} \right) \mid \mu(X_n^*) \right) \right].
\]

Therefore using the independence of \(U_{n+1}\) and \(\{X_i^*\}_{i=1}^n\) gives that
\[
\Pr \left( \frac{\mu(X_{n+1}^*)}{\mu(X)} \leq u \right) = \Pr \left( U_{n+1} : \frac{\mu(X_{n+1}^*)}{\mu(X)} \leq \frac{\mu(X_{n+1})}{\mu(X)} \right). \tag{6}
\]

Finally, successively using the fact that \(\mu(X_{n+1}^*) \leq u \cdot \mu(X)\) and plugging the induction assumption into (6) gives that
\[
\Pr \left( \frac{\mu(X_{n+1}^*)}{\mu(X)} \leq u \right) \leq \Pr \left( U_{n+1} : \frac{\mu(X_{n+1}^*)}{\mu(X)} \leq u \right) \leq \Pr \left( \prod_{i=1}^{n+1} U_i \leq u \right)
\]
where \(\{U_i\}_{i=1}^{n+1} \sim \mathcal{U}([0, 1])\) and the proof is complete. \(\square\)

The concentration inequality provided in the next lemma will be important in order to control the volume of the level set of the highest value observed by a Pure Adaptive Search.

**Lemma 38** Let \(\{U_i\}_{i=1}^n\) be a sequence of \(n\) independent copies of \(U \sim \mathcal{U}([0, 1])\). Then, for any \(\delta \in (0, 1)\), we have that \(\Pr \left( \prod_{i=1}^n U_i < \delta \cdot e^{-n - \sqrt{2n \ln(1/\delta)}} \right) < \delta\).

**Proof** Taking the logarithm on both sides gives that \(\prod_{i=1}^n U_i < \delta \cdot e^{-n - \sqrt{2n \ln(1/\delta)}}\) if and only if \(\sum_{i=1}^n -\ln(U_i) > n + \sqrt{2n \ln(1/\delta)} + \ln(1/\delta)\). As \(U_i \sim \mathcal{U}([0, 1])\) for \(i \leq n\), we have that \(-\ln(U_i) \sim \text{Exp}(1)\) which combined with independence gives that \(\sum_{i=1}^n -\ln(U_i) \sim \text{Gamma}(n, 1)\). Therefore, the desired result follows from the application of a standard concentration inequality for sub-gamma random variables (see Chapter 2.4 in Boucheron et al. (2013)). \(\square\)

Equipped with Proposition 16, Lemma 37 and Lemma 38, we may now prove the lower bound.

**Proof of Theorem 17.** As in the proof of Theorem 14, we may consider without loss of generality that \(f \in \mathcal{C}^6(X, \mathbb{R})\). Now, fix any \(\delta \in (0, 1)\), let \(r_{\delta,n}\) be the lower bound of the theorem, set \(S_{\delta,n} = \{x \in X : \|x^* - x\|_2 = c_n r_{\delta,n}^{1/(1+\alpha)}\}\) and let \(R_{\delta,n} = \text{rad}(X) \delta^{1/d} \exp(-n + \sqrt{2n \ln(1/\delta)})/d\). From the first inclusion of Lemma 36, we have that
\[
\Pr(\|X_{i_n} - x^*\|_2 \leq r_{\delta,n}) = \Pr(X_{i_n} \in B(x^*, r_{\delta,n}) \cap X) \leq \Pr \left( f(X_{i_n}) \geq \min_{x \in S_{\delta,n}} f(x) \right)
\]

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which together with Proposition 16 gives that
\[ \mathbb{P}(\|X_n - x^*\|_2 \leq r_{\delta,n}) \leq \mathbb{P}\left(f(X_n) \geq \min_{x \in S_{\delta,n}} f(x)\right) \]
where \( \{X_i^t\}_{i=1}^n \) is a sequence of \( n \) random variables distributed as Pure Adaptive Search indexed by \( f \) over \( \mathcal{X} \). Now, observing that if \( \mathcal{X}^* = \{x \in \mathcal{X} : f(x) \geq f(X_n^*)\} \) denotes the level set of \( f(X_n^*) \), then the following events are equivalent:
\[ \{f(X_n^*) \geq \min_{x \in S_{\delta,n}} f(x)\} = \{\mu(\mathcal{X}^*_n) \leq \mu(\{x \in \mathcal{X} : f(x) \geq \min_{x \in S_{\delta,n}} f(x)\})\}, \]
we obtain by applying the second inclusion of Lemma 36 that
\[ \mathbb{P}(\|X_n - x^*\|_2 \leq r_{\delta,n}) \leq \mathbb{P}\left(\frac{\mu(\mathcal{X}^*_n)}{\mu(\mathcal{X})} \leq \frac{\mu(B(x^*, R_{\delta,n}))}{\mu(B(x, \text{rad}(\mathcal{X}))}\right). \]
As \( \text{rad}(\mathcal{X}) > 0 \) is assumed to be finite, we know that there exists \( x \in \mathcal{X} \) such that \( B(x, \text{rad}(\mathcal{X})) \subseteq \mathcal{X} \) which implies that \( \mu(\mathcal{X}) \geq \mu(B(x, \text{rad}(\mathcal{X}))) = \pi^{d/2} \text{rad}(\mathcal{X})^d / \Gamma(d/2 + 1) \). Hence, we deduce that
\[ \mathbb{P}(\|X_n - x^*\|_2 \leq r_{\delta,n}) \leq \mathbb{P}\left(\frac{\mu(\mathcal{X}^*_n)}{\mu(\mathcal{X})} \leq \left(\frac{R_{\delta,n}}{\text{rad}(\mathcal{X})}\right)^d\right). \]
Finally, as \( R_{\delta,n} \) was defined so that \( (R_{\delta,n}/\text{rad}(\mathcal{X}))^d = \delta \cdot \exp (-n - \sqrt{2n \ln(1/\delta)}) \), we obtain that
\[ \mathbb{P}(\|X_n - x^*\|_2 \leq r_{\delta,n}) \leq \mathbb{P}\left(\prod_{i=1}^n U_i \leq \delta \cdot e^{-n-\sqrt{2n\ln(1/\delta)}}\right) \]
by using Lemma 37 where \( \{U_i\}_{i=1}^n \overset{i.d.}{\sim} \mathcal{U}([0, 1]) \) and the desired result naturally follows from the application of the concentration inequality of Lemma 38.

\[ \square \]

Appendix C. Analysis of the AdaRankOpt algorithm

We develop here the proofs of Proposition 21, Proposition 23 and Theorem 24. For convenience, we start recalling the definition of the sequence of evaluation points generated by the algorithm.

**Definition 39 (AdaRankOpt process)** Pick any \( p \in (0, 1) \), let \( \{R_k\}_{k \in \mathbb{N}^*} \) be any sequence of nested ranking structures, let \( \mathcal{X} \subseteq \mathbb{R}^d \) be any compact and convex set with non-empty interior and let \( f : \mathcal{X} \to \mathbb{R} \) be any function such that \( r_f \in \mathcal{R}_\infty \). We say that a sequence \( \{X_i\}_{i=1}^n \) is distributed as an AdaRankOpt\( (n, f, \mathcal{X}, p, \{R_k\}_{k \in \mathbb{N}^*}) \) process if it follows the process defined by:
\[ \begin{align*}
X_1 &\sim \mathcal{U}(\mathcal{X}) \\
X_{t+1} | B_{t+1}, \{X_i\}_{i=1}^t &\sim B_{t+1} \cdot \mathcal{U}(\mathcal{X}) + (1 - B_{t+1}) \cdot \mathcal{U}(\mathcal{X}) & \forall t \in \{1 \ldots n - 1\}
\end{align*} \]

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where at each step $t \geq 1$, $B_{t+1}$ is a Bernoulli random variable of parameter $p$ independent of $\{(X_i, B_i)\}_{i=1}^{n}$, $\mathcal{X}_t := \{ x \in \mathcal{X} : \exists r \in \mathcal{R}_{k_t}, s.t. r(x, X_i) \geq 0 \}$ and $\hat{t}$ and $\bar{k}_t$ are defined as in the algorithm.

The proof of the consistency property of the algorithm is straightforward.

**Proof of Proposition 21.** Pick any $\varepsilon > 0$ and let $\mathcal{X}_{f^*-\varepsilon} = \{ x \in \mathcal{X} : f(x) \geq \max_{x \in \mathcal{X}} f(x) - \varepsilon \}$ be the corresponding level set. Since $p \in (0, 1)$ and $0 < \mu(\mathcal{X}_{f^*-\varepsilon})/\mu(\mathcal{X}) \leq 1$ by Condition 1, it is sufficient to show that the following holds true to prove the result:

\[
\forall n \in \mathbb{N}^*, \quad P \left( f(X_n) < \max_{x \in \mathcal{X}} f(x) - \varepsilon \right) \leq \left( 1 - p \cdot \frac{\mu(\mathcal{X}_{f^*-\varepsilon})}{\mu(\mathcal{X})} \right)^n. \tag{7}
\]

We prove (7) by induction. As $\hat{t}_1 = 1$, $X_1 \sim \mathcal{U}(\mathcal{X})$ and $0 < p \leq 1$, we directly get that

\[
P \left( f(X_1) < \max_{x \in \mathcal{X}} f(x) - \varepsilon \right) = P(X_1 \notin \mathcal{X}_{f^*-\varepsilon}) = \left( 1 - \frac{\mu(\mathcal{X}_{f^*-\varepsilon})}{\mu(\mathcal{X})} \right) \leq \left( 1 - p \cdot \frac{\mu(\mathcal{X}_{f^*-\varepsilon})}{\mu(\mathcal{X})} \right)
\]

which proves the result for $n = 1$. Assume now that (7) holds for a given $n \in \mathbb{N}^*$, let $\{X_i\}_{i=1}^{n+1} \sim \text{ADARANKOPT}(n, f, \mathcal{X}, p, \{R_k\}_{k \in \mathbb{N}^*})$ and consider following decomposition:

\[
P \left( \max_{i=1}^{n+1} f(X_i) < \max_{x \in \mathcal{X}} f(x) - \varepsilon \right) = \mathbb{E} \left[ P(X_{n+1} \notin \mathcal{X}_{f^*-\varepsilon} \mid \{X_i\}_{i=1}^{n}) \cdot P \left( \bigcap_{i=1}^{n} \{X_i \notin \mathcal{X}_{f^*-\varepsilon} \} \right) \right].
\]

As Definition 39 guarantees that $\forall(x_1, \ldots, x_n) \in \mathcal{X}^n$

\[
P \left( X_{n+1} \notin \mathcal{X}_{f^*-\varepsilon} \bigcap \bigcup_{i=1}^{n} \{X_i = x_i\} \right) \leq 1 - \mathbb{P} \left( X_{n+1} \in \mathcal{X}_{f^*-\varepsilon} \bigcap B_{n+1} = 1, \bigcup_{i=1}^{n} \{X_i = x_i\} \right) \times \mathbb{P} \left( B_{n+1} = 1 \bigcap \bigcup_{i=1}^{n} \{X_i = x_i\} \right)
\]

it follows that

\[
P \left( \max_{i=1}^{n+1} f(X_i) < \max_{x \in \mathcal{X}} f(x) - \varepsilon \right) \leq \left( 1 - p \cdot \frac{\mu(\mathcal{X}_{f^*-\varepsilon})}{\mu(\mathcal{X})} \right) \times \mathbb{P} \left( \bigcap_{i=1}^{n} \{X_i \notin \mathcal{X}_{f^*-\varepsilon} \} \right)
\]

which combined with the induction assumption proves (7). \[\square\]

We now prove the stopping time upper bound.

**Proof of Proposition 23.** Let $\{(X_i, B_i)\}_{i \in \mathbb{N}^*}$ be the sequence of random variables defined in the ADARANKOPT algorithm. Fix any $\delta \in (0, 1)$ and set $n'_\delta = \lfloor p \cdot n_\delta - \sqrt{n_\delta \log(2/\delta)}/2 \rfloor$ where $n_\delta = \lceil 10 \cdot (K + \log(2/\delta))/p \cdot \inf_{r \in \mathcal{R}_{X^*}} L(r)^2 \rceil$ denotes the integer part of the upper bound. First, observe that since $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots \subset \mathcal{R}_\infty$ forms a nested sequence,

\[
\min_{r \in \mathcal{R}_1} L_{n_\delta}(r) \leq \min_{r \in \mathcal{R}_2} L_{n_\delta}(r) \leq \cdots \leq \min_{r \in \mathcal{R}_{n_\delta}} L_{n_\delta}(r)
\]

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where \( L_{n\delta} \) denotes the empirical ranking loss taken over the first \( n\delta \) samples \( \{X_i\}_{i=1}^{n\delta} \). One might then start with the following decomposition:

\[
\mathbb{P}(\tau_k^* \leq n\delta) = \mathbb{P}\left( \min_{r \in \mathcal{R}_{k^*-1}} L_{n\delta}(r) > 0 \right)
\geq \mathbb{P}\left( \min_{r \in \mathcal{R}_{k^*-1}} L_{n\delta}(r) > 0 \mid \sum_{i=1}^{n\delta} B_i \geq n'_{\delta} \right) \times \mathbb{P}\left( \sum_{i=1}^{n\delta} B_i \geq n'_{\delta} \right).
\]  

(8)

We now focus on the first term of the right hand side of (8) and we start to lower bound the empirical risk by only keeping the first \( n'_{\delta} \) i.i.d. exploratory samples:

\[
L_{n\delta}(r) \geq \frac{2}{n\delta(n\delta - 1)} \sum_{1 \leq i < j \leq n\delta} \mathbb{P}\{r(X_i, X_j) \neq r_f(X_i, X_j)\} \times \mathbb{P}\{r_f(X_i, X_j) \neq r_f(X_i', X_j')\} \times L_{n'_{\delta}}(r)
\]

where \( I = \{i \leq n\delta : B_i = 1\} \) and \( \sum_{j=1}^{n\delta} B_i \leq n'_{\delta} \). By definition 39, we know that conditioned upon \(|I|\), the sequence \( \{X_i\}_{i \in I}|I| \) is a sequence of \(|I|\) independent random variables uniformly distributed over \( \mathcal{X} \). Therefore, conditioned upon the event \( \{\sum_{i=1}^{n\delta} B_i \geq n'_{\delta}\} = \{|I| = n'_{\delta}\} \), the right hand term of (9) has the same distribution as

\[
\frac{2}{n\delta(n\delta - 1)} \sum_{1 \leq i < j \leq n\delta} \mathbb{P}\{r(X_i', X_j') \neq r_f(X_i', X_j')\} \sim \mathcal{U}(\mathcal{X}) \text{ is independent of } \{B_i\}_{i=1}^{n\delta}.
\]

Hence, we have that

\[
\mathbb{P}\left( \min_{r \in \mathcal{R}_{k^*-1}} L_{n\delta}(r) > 0 \mid \sum_{i=1}^{n\delta} B_i \geq n'_{\delta} \right) \geq \mathbb{P}\left( \min_{r \in \mathcal{R}_{k^*-1}} L_{n'_{\delta}}(r) > 0 \right)
\]

where \( L_{n'_{\delta}} \) denotes the empirical ranking loss is taken over \( \{X_i'\}_{i=1}^{n'_{\delta}} \). Now, by slightly adapting the generalization bound on bipartite ranking rules of Cléménçon et al. (2010) \((i.e.,\, see\, the\, proof\, of\, Corollary\, 3\, in\, Section\, 3\, therein)\) we obtain that with probability at least \( 1 - \delta/2 \),

\[
\sup_{r \in \mathcal{R}_{k^*-1}} \left| L_{n'_{\delta}}(r) - L(r) \right| \leq 2\mathbb{E}\left[ R_{n'_{\delta}}(\mathcal{R}_{k^*-1}) \right] + 2\sqrt{\frac{\log(2/\delta)}{n'_{\delta} - 1}},
\]

which combined with the fact that \( \mathbb{E}\left[ R_{n'_{\delta}}(\mathcal{R}_{k^*-1}) \right] \leq \sqrt{K/n} \) gives that

\[
\min_{r \in \mathcal{R}_{k^*-1}} L_{n'_{\delta}}(r) \geq \inf_{r \in \mathcal{R}_{k^*-1}} L(r) - 2\sqrt{\frac{K}{n'_{\delta}}} - 2\sqrt{\frac{\log(2/\delta)}{n'_{\delta} - 1}}.
\]

Finally, as \( n'_{\delta} \) and \( n\delta \) were defined (with express purpose) so that (i) the right hand term of the previous inequality is strictly positive and so that (ii) Hoeffding’s inequality ensures that

\[
\mathbb{P}\left( \sum_{i=1}^{n\delta} B_i \geq n\delta \right) \geq 1 - \delta/2,
\]

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we deduce from (8) that $\mathbb{P}(\tau_{k*} \leq n_\delta) \geq (1 - \delta/2)^2 \geq 1 - \delta$ and the proof is complete. □

Theorem 24 is obtained by combining the upper bounds of Proposition 23 and Theorem 14.

**Proof of Theorem 24.** Fix $\delta \in (0, 1)$, let $n_{\delta/2} = \lfloor 10(K + \ln(4/\delta))/(p \cdot \inf_{r \in \mathcal{R}_{k*}} L(r)^2) \rfloor$ be the integer part of the upper bound of Proposition 23 (set with probability $1 - \delta/2$) and let $r_{\delta/2,n}$ be the upper bound of the Theorem 14 (also set with probability $1 - \delta/2$). We use the following decomposition:

$$
\mathbb{P}(\|X_i - x^*\|_2 \leq r_{\delta/2,n}) \geq \mathbb{P}(\|X_i - x^*\|_2 \leq r_{\delta/2,n} | \tau_{k*} \leq n_{\delta/2}) \times \mathbb{P}(\tau_{k*} \leq n_{\delta/2}).
$$

(10)

First, as on the event $\{\tau_{k*} \leq n_{\delta/2}\} = \bigcap_{t \geq n_{\delta/2}} \{\hat{k}_t = k^*\}$ the smallest ranking structure $\mathcal{R}_{k*}$ containing the true ranking rule $r_f$ is identified for all $t \geq n_{\delta/2}$, one can easily check that

$$
\mathbb{P}(\|X_i - x^*\|_2 \leq r_{\delta/2,n} | \tau_{k*} \leq n_{\delta/2}) \geq 1 - \delta/2
$$

by reproducing the same steps as in Theorem 14’s proof with the last $n - n_{\delta/2}$ samples. Second, as Proposition 23 also guarantees that $\mathbb{P}(\tau_{k*} \leq n_{\delta/2}) \geq 1 - \delta/2$, we then obtain from (10) that $\mathbb{P}(\|X_i - x^*\|_2 \leq r_{\delta/2,n}) \geq (1 - \delta/2)^2 \geq 1 - \delta$. Hence, for all $n > n_{\delta/2}$, we have with probability at least $1 - \delta$,

$$
\|X_i - x^*\|_2 \leq C_1 \cdot \left( \frac{\ln(2/\delta)}{n - n_{\delta/2}} \right)^{1/(1 + \alpha)^2} 
= C_1 \cdot \left( 1 + \frac{n_{\delta/2}}{n - n_{\delta/2}} \right)^{1/(1 + \alpha)^2} \cdot \left( \frac{\ln(2/\delta)}{n} \right)^{1/(1 + \alpha)^2} 
\leq C_1 \cdot \left( \frac{11(K + \ln(4/\delta))}{p \cdot \inf_{r \in \mathcal{R}_{k*}} L(r)^2} \right)^{1/(1 + \alpha)^2} \cdot \left( \frac{\ln(2/\delta)}{n} \right)^{1/(1 + \alpha)^2}
$$

and the proof is complete by noticing that the right hand term of the previous inequality is superior or equal to $\text{diam}(X)$ whenever $n \leq n_{\delta/2}$.

□

**Appendix D. Proofs of the implementation details**

We state here the proofs of Proposition 27, Lemma 28, Corollary 29, Proposition 32 and Proposition 33.

**D.1 Polynomial and sinusoidal ranking rules**

We start with the proofs of Proposition 27, Lemma 28 and Proposition 33.

**Proof of Proposition 27 ($\Rightarrow$)** Assume that there exists $r \in \mathcal{R}_{P(k)}$ such that $L_{t+1}(r) = 0$. By definition of $\mathcal{R}_{P(k)}$, we know that there exists a polynomial function $f_r$ of degree $k$ such that $\forall (x, x') \in X^2$, $r(x, x') = \text{sgn}(f(x) - f(x'))$. Moreover, as $f_r \in P_k(X, \mathbb{R})$, we also know that there exists $(\omega_r, c_r) \in \mathbb{R}^{\dim(\Phi_k)} \times \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $f_r(x) = \langle \omega_r, \Phi_k(x) \rangle + c_r$. 


Therefore, putting the previous statements altogether with the fact that $L_{t+1}(r) = 0$ gives that $\forall i \leq t$,

$$1 = r(X_{(i+1)}, X_{(i)}) = \text{sgn}(f_r(X_{(i+1)}) - f_r(X_{(i)})) = \text{sgn}(\langle \omega_r, \Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)}) \rangle)$$

and we thus deduce that there exists $\omega = \omega_r \in \mathbb{R}^{\dim(\Phi_k)}$ such that $\forall i \leq t$, $\langle \omega, \Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)}) \rangle > 0$

($\Leftarrow$) Assume now that there exists $\omega \in \mathbb{R}^{\dim(\Phi_k)}$ such that $\forall i \in \{1 \ldots t\}$, $\langle \omega, \Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)}) \rangle > 0$ and introduce the polynomial function of degree $k$ defined by $f_\omega : x \mapsto \langle \omega, \Phi_k(x) \rangle + c$ where $c \geq 0$ is any arbitrary constant. Now, if $r_{f_\omega}$ denotes the polynomial ranking rule induced by $f_\omega$ we obtain from the first assumption that $\forall i \leq t$,

$$r_{f_\omega}(X_{(i+1)}, X_{(i)}) = \text{sgn}(f_\omega(X_{(i+1)}) - f_\omega(X_{(i)})) = \text{sgn}(\langle \omega, \Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)}) \rangle) = 1.$$

Hence $L_{t+1}(r) = 0$ and we deduce that there exists $r = r_{f_\omega} \in \mathcal{R}_{\mathcal{P}_k}$ such that $L_{t+1}(r) = 0.$

\[\square\]

**Proof of Lemma 28.** Observe first that for all $i \leq t$, $Y_i \cdot \langle \omega, X_i \rangle > 0 \iff \langle \omega, Y_i \cdot X_i \rangle > 0$. One can then consider without loss of generality that $Y_i = 1$ for all $i \leq t$, by replacing $X_i$ with $Y_i \cdot X_i$.

($\Rightarrow$) Assume that there exists $\omega \in \mathbb{R}^d$ such that $\forall i \in \{1 \ldots t\}$, $\langle \omega, X_i \rangle > 0$. If $\vec{0} \in \mathrm{CH}\{X_i\}_{i=1}^t$, this would mean that there exists $(\lambda_1, \ldots, \lambda_t) \in \mathbb{R}^t$ such that $(i)$ $\vec{0} = \sum_{i=1}^t \lambda_i \cdot X_i$, (ii) $\sum_{i=1}^n \lambda_i = 1$ and (iii) $\lambda_i \geq 0$, $i = 1 \ldots t$ and it would give us to the following contradiction:

$$0 = \langle \omega, \vec{0} \rangle = \sum_{i=1}^t \lambda_i \cdot \langle \omega, X_i \rangle > 0.$$ 

Hence $\vec{0} \notin \mathrm{CH}\{X_i\}_{i=1}^t$.

($\Leftarrow$) Assume now that $\vec{0} \notin \mathrm{CH}\{X_i\}_{i=1}^t$. Since $t$ and $d$ are finite, $\mathrm{CH}\{X_i\}_{i=1}^t$ is a closed, compact and convex set and thus $\min_{x \in \mathrm{CH}\{X_i\}_{i=1}^t} \|x\|_2 = d_{\min}$ exists and the condition $\vec{0} \notin \mathrm{CH}\{X_i\}_{i=1}^t$ implies that $d_{\min} > 0$. Now, let $x_d \in \mathrm{CH}\{X_i\}_{i=1}^t$ be the (unique) point of the convex hull which satisfies $\|x_d\|_2 = d_{\min}$. We now prove by contradiction that $\forall x \in \mathrm{CH}\{X_i\}_{i=1}^t$, $(x, x_d) \geq d_{\min}^2$. Suppose that there exists $x \in \mathrm{CH}\{X_i\}_{i=1}^t$ such that $(x, x_d) < d_{\min}^2$. First, we know from the convexity of the convex hull that the whole line $L = \langle x, x_d \rangle$ also belongs to $\mathrm{CH}\{X_i\}_{i=1}^t$. However, since $\|x_d\|_2 = d_{\min}$ and $(x, x_d) < \|x_d\|_2^2$, the line $L$ is not tangent to the ball $B(\vec{0}, d_{\min})$ and intersects it. Therefore, we deduce that there necessarily exists $x' \in L \cap B(\vec{0}, d_{\min})$ such that $\|x'\|_2 < d_{\min}$. Nonetheless, as $x' \in L \subseteq \mathrm{CH}\{X_i\}_{i=1}^t$ also belongs to the convex hull, we obtain the following contradiction:

$$\min_{x \in \mathrm{CH}\{X_i\}_{i=1}^t} \|x\|_2 \leq \|x'\|_2 < d_{\min} = \min_{x \in \mathrm{CH}\{X_i\}_{i=1}^t} \|x\|_2$$

and we deduce that $\forall x \in \mathrm{CH}\{X_i\}_{i=1}^t$, $(x_d, x) \geq d_{\min} > 0$. Finally, as $\{X_i\}_{i=1}^t \in \mathrm{CH}\{X_i\}_{i=1}^t$, it directly follows that there exists $\omega = x_d \in \mathbb{R}^d$ such that $\forall i \in \{1 \ldots t\}$, $\langle \omega, X_i \rangle > 0$ and the proof is complete. \[\square\]
Corollary 29 is obtained by combining Proposition 27 with Lemma 28.

**Proof of Corollary 29** From Proposition 27, we have the following equivalence:

\[
\min_{r \in \mathcal{R}_{P_k}} L_{t+1}(r) = 0 \iff \exists \omega \in \mathbb{R}^{\dim(\phi_k)} \text{ s.t. } \langle \omega, \Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)}) \rangle > 0, \forall i \in \{1 \ldots t\}
\]

which combined with Lemma 28 gives

\[
\min_{r \in \mathcal{R}_{P_k}} L_{t+1}(r) = 0 \iff \vec{0} \notin \text{CH}\{(\Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)}))\}_{i=1}^t.
\]

In addition, we know from the vertex representation of convex hulls that \(\vec{0} \notin \text{CH}\{(\Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)}))\}_{i=1}^t\) if and only if there does not exist any \(\lambda = (\lambda_1, \ldots, \lambda_t) \in \mathbb{R}^t\) such that (i) \(\sum_{i=1}^{t} \lambda_i (\Phi_k(X_{(i+1)}) - \Phi_k(X_{(i)})) = \vec{0}\), (ii) \(\sum_{i=1}^{t} \lambda_i = 1\) and (iii) \(\lambda_i \geq 0\), \(i = 1, \ldots, t\) and therefore putting those constraints (i), (ii) and (iii) into matrix form leads us to the desired equivalence:

\[
\min_{r \in \mathcal{R}_{P_k}} L_{t+1}(r) = 0 \iff \{ \lambda \in \mathbb{R}^t : M_{\Phi_k}^\top \lambda^\top = \vec{0}, \langle \vec{1}, \lambda \rangle = 1, \lambda \geq \vec{0} \} = \emptyset
\]

where \(M_{\Phi_k}\) is the matrix defined in the corollary. \(\square\)

**D.2 Convex ranking rules**

In this subsection, we provide the proofs for Proposition 32 and Proposition 33.

**Proof of Proposition 32** \((\Rightarrow)\) Assume that there exists \(r \in \mathcal{R}_{C_k}\) such that \(L_{t+1}(r) = 0\) and let \(\{h_i\}_{i=1}^{t+1}\) be the sequence of classifiers defined \(\forall i \leq t + 1\) by \(h_i(x) = \mathbb{I}\{r(x, X_{(i)}) \geq 0\}\). First, we know from the definition of \(\mathcal{R}_{C_k}\) that all the classifiers are of the form 

\(h_i(x) = \sum_{m=1}^{k} \mathbb{I}\{l_{i,m} \leq x \leq u_{i,m}\}\). Second, since \(L_{n+1}(r) = 0\), it directly follows that \(\forall (i,j) \in \{1,\ldots, t+1\}^2\), \(h_i(X_{(j)}) = \mathbb{I}\{j \geq i\}\). Finally, as \(r\) is transitive and \(\forall i \leq t\), \(r(X_{(i+1)}), X_{(i)}) = 1\), we have that \(h_1 \geq h_2 \geq \cdots \geq h_{t+1}\).

\((\Leftarrow)\) Assume now that there exists a sequence of classifiers \(\{h_i\}_{i=1}^{t+1}\) of the form \(h_i(x) = \sum_{m=1}^{k} \mathbb{I}\{l_{i,m} \leq x \leq u_{i,m}\}\) satisfying: (i) \(h_1 \geq h_2 \geq \cdots \geq h_{t+1}\) and (ii) \(\forall (i,j) \in \{1\ldots t+1\}^2\), \(h_i(X_{(j)}) = \mathbb{I}\{j \geq i\}\). Define the step function \(f_{\text{step}}(x) = \sum_{i=1}^{t+1} h_i(x)\) and observe that \(L_{t+1}(r_{\text{step}}) = 0\) since \(\forall (j,k) \in \{1 \ldots t+1\}^2\),

\[
r_{\text{step}}(X_{(j)}, X_{(k)}) = \text{sgn} \left( \sum_{i=1}^{t+1} \mathbb{I}\{i \geq j\} - \mathbb{I}\{k \geq i\} \right) = \text{sgn}(j - k) = r_f(X_{(j)}, X_{(k)}),
\]

To prove the result, we will simply construct a continuous approximation of the function \(f_{\text{step}}\) which (i) induces a ranking rule which perfectly ranks the sample and (ii) admits level sets which are unions of at most \(k\) convex set. Let \(f_k : X \rightarrow \mathbb{R}\) be the continuous function
defined by \( \hat{f}_\epsilon(x) = \sum_{i=1}^{t+1} \sum_{m=1}^{k} \hat{f}_{\epsilon,i,m,u,m}(x) \) where \( \forall l \leq u \),

\[
\hat{f}_{\epsilon,l,u}(x) = \begin{cases} 
1 & \text{if } x \in [l, u] \\
1 - \frac{l-x}{\epsilon} & \text{if } x \in [l, l] \\
1 - \frac{u-x}{\epsilon} & \text{if } x \in [u, u + \epsilon] \\
0 & \text{otherwise.}
\end{cases}
\]

Observe now that \( \forall \epsilon < \min\{|x_1 - x_2| : x_1 \neq x_2 \in \{X(i)_{i=1}^{t+1} \cup \{l_i,m\}_{i=1..t+1} \cup \{u_i,m\}_{i=1..t+1} \} \) and \( \forall i \leq t \), we have that \( \hat{f}_\epsilon(X_{(i)}) = f_{\text{step}}(X_{(i)}) \). Hence \( L_{t+1}(r_{f_\epsilon}) = L_{t+1}(r_{f_{\text{step}}}) = 0 \) which proves (i). Moreover, as for any \( \epsilon < \min\{|x_1 - x_2| : x_1 \neq x_2 \in \{l_i,m\}_{i=1..t+1} \cup \{u_i,m\}_{i=1..t+1} \} / 2 \), the level sets of \( \hat{f}_\epsilon \) are by construction a union of at most \( k \) segments (convex sets) and (ii) holds true. We then deduce from (i) and (ii) that for \( \epsilon \) small enough there exists \( r = r_{f_\epsilon} \in \mathcal{R}_C \) such that \( L_{t+1}(r) = 0. \) \( \square \)

The next lemma will be used in the proof of Proposition 33.

**Lemma 40** Let \( \mathcal{X} \subset \mathbb{R}^d \) be any compact and convex set and define for any \( \epsilon > 0 \) the \( \epsilon \)-ball of \( \mathcal{X} \) as \( B(\mathcal{X}, \epsilon) = \{x \in \mathbb{R}^d : \min_{x' \in \mathcal{X}} \|x - x'\|_2 \leq \epsilon \} \). Then, for any \( \epsilon > 0 \), the \( \epsilon \)-ball of \( \mathcal{X} \) is also a convex set.

**Proof** Pick any \( (b_1, b_2) \in B(\mathcal{X}, \epsilon)^2 \). By definition of \( B(\mathcal{X}, \epsilon) \), we know that there exists \( (x_1, \epsilon_1) \in \mathcal{X} \times \mathbb{R}^d \) such that \( b_1 = x_1 + \epsilon_1 \) and \( \|\epsilon_1\|_2 \leq \epsilon \) (resp. \( b_2 = x_2 + \epsilon_2 \) where \( x_2 \in \mathcal{X} \) and \( \|\epsilon_2\|_2 \leq \epsilon \)). Then, by convexity of \( \mathcal{X} \), we have that \( \forall \lambda \in [0,1], \)

\[
(1 - \lambda)b_1 + \lambda b_2 = \lambda x_1 + (1 - \lambda)x_2 + \lambda \epsilon_1 + (1 - \lambda)\epsilon_2. \\
\|\|_2 \leq \epsilon
\]

Hence \((1 - \lambda)b_1 + \lambda b_2 \in B(\mathcal{X}, \epsilon)\) and we deduce that \( B(\mathcal{X}, \epsilon) \) is a convex set. \( \square \)

Equipped with Lemma 40, we may now prove Proposition 33.

**Proof of Proposition 33** (\( \Rightarrow \)) Assume that there exists \( r \in \mathcal{R}_C \) such that \( L_{t+1}(r) = 0 \). Observe first that since \( \forall j \neq k \leq t + 1 \), \( r(X_{(j)}, X_{(k)}) = 2 \{ j > k \} - 1 \), we have that \( \forall k \leq t \),

i) \( \{X(i)_{i=k+1}^{t+1} \in \{x \in \mathcal{X} : r(x, X_{(k+1)}) \geq 0 \}; \)

ii) \( X(k) \notin \{x \in \mathcal{X} : r(x, X_{(k+1)}) \geq 0 \}. \)

Now, pick any \( k \in \{ 1, \ldots, t \} \) and notice that, by definition of \( \mathcal{R}_C \), the level set \( \{x \in \mathcal{X} : r(x, X_{(k+1)}) \geq 0 \} \) is also a convex set. However, since \( \text{CH}\{X(i)_{i=k+1}^{t+1} \} \) is the smallest convex set which contains \( \{X(i)_{i=k+1}^{t+1} \}, \) we deduce from (i) that \( \text{CH}\{X(i)_{i=k+1}^{t+1} \} \subseteq \{x \in \mathcal{X} : r(x, X_{(k+1)}) \geq 0 \}. \) Therefore, combining the previous statement with (ii) gives that \( \forall k \leq t, \)

\[
X(k) \notin \text{CH}\{X(i)_{i=k+1}^{t+1} \}.
\]

Finally, using the vertex representation of convex hulls we know that \( X(t+1-i) \notin \text{CH}\{X(i)_{i=t+2-i}^{t+1} \} \) if and only if there does not any \( \lambda = (\lambda_1, \ldots, \lambda_i) \in \mathbb{R}^i \) such that (i)
\[ \sum_{j=1}^{j} \lambda_j \cdot X_{(t+2-j)} = X_{(k)}, \quad (\text{ii}) \quad \sum_{j=1}^{j} \lambda_j = 1 \quad \text{and} \quad (\text{iii}) \quad \lambda_j \geq 0, \quad j = 1 \ldots i \quad \text{and putting those constraints (i), (ii) and (iii) into matrix form gives the result.} \]

\( \Leftarrow \) Assume now that the cascade of polyhedrons is empty. First, we point out that it can easily check by reproducing the same steps as in the first part of the proof that \( \forall k \leq t, \: X_{(k)} \notin \text{CH} \{X_{(i)}\}_{i=t}^{t+1}, \) which implies that

\[ \text{CH} \{X_{(t+1)}\} \subset \text{CH} \{X_{(i)}\}_{i=t}^{t+1} \subset \cdots \subset \text{CH} \{X_{(i)}\}_{i=1}. \quad (11) \]

Now define the step function \( f_{\text{step}} : x \in \mathcal{X} \mapsto \sum_{i=1}^{t+1} \| x \in \text{CH} \{X_{(j)}\}_{j=i}^{i+1} \) and observe that \( L_{t+1}(r_{f_{\text{step}}}) = 0 \) by (11). To prove the result, we will simply construct a continuous approximation of the function \( f_{\text{step}} \) which (i) induces a ranking rule that perfectly ranks the sample and (ii) has convex level sets. Let \( \hat{f}_\varepsilon \) be the continuous function defined by \( \hat{f}_\varepsilon(x) = \sum_{i=1}^{t+1} \phi_{i,\varepsilon}(x) \) where \( \forall i \leq t + 1, \)

\[ \phi_{i,\varepsilon}(x) = \begin{cases} 1 - \frac{d(x, B(\text{CH} \{X_{(j)}\}_{j=i}^{i+1}, 2(t+1-i)\varepsilon))}{\varepsilon} & \text{if } d(x, B(\text{CH} \{X_{(j)}\}_{j=i}^{i+1}, 2(t+1-i)\varepsilon)) \leq \varepsilon \\ 0 & \text{otherwise} \end{cases} \]

Observe now that for any \( \varepsilon < \min_{i=1 \ldots t} d(X_{(i)}, \text{CH} \{X_{(j)}\}_{j=i}^{i+1})/(2t + 2), \) we have that \( \forall i \leq t + 1, \: \hat{f}_\varepsilon(X_{(i)}) = f_{\text{step}}(X_{(i)}). \) Hence \( L_{t+1}(r_{\hat{f}_\varepsilon}) = L_{t+1}(r_f) = 0, \) which proves (i). Moreover, we know from Lemma 40 that for any \( \varepsilon < \min_{i=1 \ldots t} d(X_{(i)}, \text{CH} \{X_{(j)}\}_{j=i}^{i+1})/(2t + 2) \) and any \( x \in \mathcal{X}, \) the level set \( \{ x' \in \mathcal{X} : \hat{f}_\varepsilon(x') \geq \hat{f}_\varepsilon(x) \} \) is a convex set. Hence (ii) holds true and we then deduce from (i) and (ii) that for \( \varepsilon \) small enough there exists \( r = r_{\hat{f}_\varepsilon} \in \mathcal{R}_{C_1} \) such that \( L_{t+1}(r_{\hat{f}_\varepsilon}) = 0 \) and the proof is complete. \( \square \)

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