SOME NUMERICAL RADIUS INEQUALITIES

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Abstract. In this work, a pre-Grüss inequality for positive Hilbert space operators is proved. So that, some numerical radius inequalities are proved. On the other hand, based on a non-commutative Binomial formula, a non-commutative upper bound for the numerical radius of the summand of two bounded linear Hilbert space operators is proved. A commutative version is also obtained as well.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathcal{B}(\mathcal{H})$. A bounded linear operator $A$ defined on $\mathcal{H}$ is selfadjoint if and only if $\langle Ax, x \rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$. The spectrum of an operator $A$ is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda 1_{\mathcal{H}} - A$ does not have a bounded linear operator inverse, and is denoted by $\text{sp}(A)$. Consider the real vector space $B_{sa}(\mathcal{H})$ of self-adjoint operators on $\mathcal{H}$ and its positive cone $B_{sa}(\mathcal{H})^{+}$ of positive operators on $\mathcal{H}$. Also, $B_{sa}(I)$ denotes the convex set of bounded self-adjoint operators on the Hilbert space $\mathcal{H}$ with spectra in a real interval $I$. A partial order is naturally equipped on $B_{sa}(\mathcal{H})$ by defining $A \leq B$ if and only if $B - A \in B_{sa}(\mathcal{H})^{+}$. We write $A > 0$ to mean that $A$ is a strictly positive operator, or equivalently, $A \geq 0$ and $A$ is invertible. When $\mathcal{H} = \mathbb{C}^n$, we identify $B_{sa}(\mathcal{H})$ with the algebra $\mathcal{M}_{n \times n}$ of $n$-by-$n$ complex matrices. Then, $\mathcal{M}_{n \times n}^{+}$ is just the cone of $n$-by-$n$ positive semidefinite matrices.

For a bounded linear operator $T$ on a Hilbert space $\mathcal{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \rightarrow \langle Tx, x \rangle$ associated with the operator. More precisely,

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$$

Also, the (maximum) numerical radius is defined by

$$w_{\text{max}}(T) = \sup_{\|x\| = 1} |\langle Tx, x \rangle| =: w(T)$$

and the (minimum) numerical radius is defined to be

$$w_{\text{min}}(T) = \inf_{\|x\| = 1} |\langle Tx, x \rangle| .$$

The spectral radius of an operator $T$ is defined to be

$$r(T) = \sup \{ |\lambda| : \lambda \in \text{sp}(T) \}$$

We recall that, the usual operator norm of an operator $T$ is defined to be

$$\|T\| = \sup \{ \|Tx\| : x \in H, \|x\| = 1 \} .$$

and

$$\ell(T) = \inf \{ \|Tx\| : x \in \mathcal{H}, \|x\| = 1 \}$$

$$= \inf \{ |\langle Tx, y \rangle| : x, y \in \mathcal{H}, \|x\| = \|y\| = 1 \} .$$
it is well known that \( w(\cdot) \) defines an operator norm on \( \mathcal{B}(\mathcal{H}) \) which is equivalent to operator norm \( \| \cdot \| \). Moreover, we have

\[
1 \frac{1}{2} \| T \| \leq w(T) \leq \| T \|
\]

for any \( T \in \mathcal{B}(\mathcal{H}) \). The inequality is sharp.

In 2003, Kittaneh [11] refined the right-hand side of (1.1), where he proved that

\[
w(T) \leq \frac{1}{2} \left( \| T \| + \| T^2 \|^{1/2} \right)
\]

for any \( T \in \mathcal{B}(\mathcal{H}) \).

After that in 2005, the same author in [10] proved that

\[
\frac{1}{4} \| A^* A + A A^* \| \leq w^2(A) \leq \frac{1}{2} \| A^* A + A A^* \|.
\]

The inequality is sharp. This inequality was also reformulated and generalized in [7] but in terms of Cartesian decomposition.

In 2007, Yamazaki [19] improved (1.1) by proving that

\[
w(T) \leq \frac{1}{2} \left( \| T \| + w(T) \right) \leq \frac{1}{2} \left( \| T \| + \| T^2 \|^{1/2} \right),
\]

where \( \bar{T} = |T|^{1/2}U|T|^{1/2} \) with unitary \( U \).

In 2008, Dragomir [5] (see also [4]) used Buzano inequality to improve (1.1), where he proved that

\[
w^2(T) \leq \frac{1}{2} \left( \| T \| + w(T^2) \right).
\]

This result was also recently generalized by Sattari et al. in [17].

In [2], Dragomir studied the Čebyšev functional

\[
C(f, g; A; x) = \langle f(A) g(A)x, x \rangle - \langle f(A)x, g(A)x \rangle
\]

for any selfadjoint operator \( A \in \mathcal{B}(H) \) and \( x \in H \) with \( \| x \| = 1 \). In particular, we have

\[
C(f, f; A; x) = \langle f^2(A)x, x \rangle - \langle f(A)x, f(A)x \rangle^2.
\]

In the several works, Dragomir proved various bounds for the Čebyšev functional. The most popular result concerning continuous synchronous (asynchronous) functions of selfadjoint linear operators in Hilbert spaces, which reads

**Theorem 1.** Let \( A \in \mathcal{B}(\mathcal{H})_{sa} \) with \( \sigma(A) \subset [\gamma, \Gamma] \) for some real numbers \( \gamma, \Gamma \) with \( \gamma < \Gamma \). If \( f, g : [\gamma, \Gamma] \to \mathbb{R} \) are continuous and synchronous (asynchronous) on \( [\gamma, \Gamma] \), then

\[
\langle f(A) g(A)x, x \rangle \geq (\leq) \langle g(A)x, x \rangle \langle f(A)x, x \rangle
\]

for any \( x \in H \) with \( \| x \| = 1 \).

This result was generalized recently by the author of this paper in [1]. For more related results concerning Čebyšev–Grüss type inequalities we refer the reader to [3], [14] and [15].

### 2. The Results

The following pre-Grüss inequality for linear bounded operators in inner product Hilbert spaces is valid.

**Theorem 2.** Let \( A \in \mathcal{B}(\mathcal{H})^+ \). If \( f, g \) are both measurable functions on \( [0, \infty) \), then we have the inequality

\[
|C(f, g; A; x)| \leq C^{1/2}(f; f; A; x)C^{1/2}(g, g; A; x)
\]
for any $x \in H$. In other words, we may write

$$|\langle f(A)g(A)x,x \rangle - \langle f(A)x,x \rangle \langle g(A)x,x \rangle| \leq \left( \langle f^2(A)x,x \rangle - \langle f(A)x,x \rangle^2 \right)^{1/2} \left( \langle g^2(A)x,x \rangle - \langle g(A)x,x \rangle^2 \right)^{1/2}$$

Proof. It’s not hard to show that

$$C(f,g;A;x) = \frac{1}{2} \int_0^\infty \int_0^\infty (f(t) - f(s))(g(t) - g(s)) \, d\langle E_t x,x \rangle \, d\langle E_s x,x \rangle$$

Utilizing the triangle inequality in (2.2) and then the Cauchy–Schwarz inequality, we get

$$|C(f,g;A;x)| = \frac{1}{2} \int_0^\infty \int_0^\infty |f(t) - f(s)||g(t) - g(s)| \, d\langle E_t x,x \rangle \, d\langle E_s x,x \rangle$$

$$\leq \frac{1}{2} \left( \int_0^\infty \int_0^\infty |f(t) - f(s)|^2 \, d\langle E_t x,x \rangle \, d\langle E_s x,x \rangle \right)^{1/2} \times \left( \int_0^\infty \int_0^\infty |g(t) - g(s)|^2 \, d\langle E_t x,x \rangle \, d\langle E_s x,x \rangle \right)^{1/2}$$

$$= \frac{1}{2} \left( \int_0^\infty d\langle E_s x,x \rangle \int_0^\infty f^2(t) \, d\langle E_t x,x \rangle - 2 \int_0^\infty f(t) \, d\langle E_t x,x \rangle \int_0^\infty f(s) \, d\langle E_s x,x \rangle \right)^{1/2}$$

$$+ \int_0^\infty d\langle E_t x,x \rangle \int_0^\infty f^2(s) \, d\langle E_s x,x \rangle \right)^{1/2} \times \left( \int_0^\infty d\langle E_s x,x \rangle \int_0^\infty g^2(t) \, d\langle E_t x,x \rangle - 2 \int_0^\infty g(t) \, d\langle E_t x,x \rangle \int_0^\infty g(s) \, d\langle E_s x,x \rangle \right)^{1/2}$$

$$+ \int_0^\infty d\langle E_t x,x \rangle \int_0^\infty g^2(s) \, d\langle E_s x,x \rangle \right)^{1/2}$$

$$= \left( 1_{\mathcal{H}} \cdot \int_0^\infty f^2(t) \, d\langle E_t x,x \rangle - \left( \int_0^\infty f(t) \, d\langle E_t x,x \rangle \right)^2 \right)^{1/2}$$

$$\times \left( 1_{\mathcal{H}} \cdot \int_0^\infty g^2(t) \, d\langle E_t x,x \rangle - \left( \int_0^\infty g(t) \, d\langle E_t x,x \rangle \right)^2 \right)^{1/2}$$

$$= \left( \langle f^2(A)x,x \rangle - \langle f(A)x,x \rangle^2 \right)^{1/2} \left( \langle g^2(A)x,x \rangle - \langle g(A)x,x \rangle^2 \right)^{1/2}$$

for any $x \in \mathcal{H}$, which gives the desired result (2.1). \qed

Corollary 1. Let $A \in \mathcal{B}(\mathcal{H})^+$. Then

$$|\langle Ax,x \rangle - \langle A^\alpha x,x \rangle \langle A^{1-\alpha} x,x \rangle|$$

$$\leq \left( \langle A^{2\alpha}x,x \rangle - \langle A^\alpha x,x \rangle^2 \right)^{1/2} \left( \langle A^{2(1-\alpha)}x,x \rangle - \langle A^{1-\alpha} x,x \rangle^2 \right)^{1/2}$$

for any $x \in \mathcal{H}$ and all $\alpha \in \left[ 0, \frac{1}{2} \right]$.

Theorem 3. Let $A \in \mathcal{B}(\mathcal{H})^+$. If $f,g$ are both measurable functions on $[0,\infty)$, then we have the inequality

$$w_{\max}(f(A)g(A)) - w_{\min}(f(A)) \cdot w_{\min}(g(A))$$

$$\leq \left[ \|f(A)\|^2 - \ell^2 \left( f^{1/2}(A) \right) \right]^{1/2} \cdot \left[ \|g(A)\|^2 - \ell^2 \left( g^{1/2}(A) \right) \right]^{1/2}$$

(2.3)
Proof. Using the basic triangle inequality \(|a| - |b| \leq |a - b|\), we have from (2.1) that
\[
|\langle (f (A) g (A) x, x) \rangle - |\langle (f (A) x, x) \rangle g (A) x, x)| |\langle g (A) x, x) | |g (A) x, x)| |
\leq |\langle f (A) g (A) x, x) - f (A) x, x) g (A) x, x)|
\leq \left( \langle f^2 (A) x, x) - f (A) x, x) \right)^{1/2} \left( g^2 (A) x, x) - g (A) x, x) \right)^{1/2}
\]
Taking the supremum over \( x \in \mathcal{H} \), we obtain
\[
\sup_{\|x\|=1} |\langle f (A) g (A) x, x) - |\langle f (A) x, x) g (A) x, x)| |
\leq \sup_{\|x\|=1} |\langle f (A) g (A) x, x) - f (A) x, x) g (A) x, x)|
\leq \sup_{\|x\|=1} |\langle f (A) g (A) x, x) - \inf_{\|x\|=1} |\langle f (A) x, x) g (A) x, x)| |
\leq \sup_{\|x\|=1} |\langle f (A) g (A) x, x) - \inf_{\|x\|=1} |\langle f (A) x, x) g (A) x, x)| |
\leq \sup_{\|x\|=1} \left[ |\langle f (A) x, x) - f (A) x, x) g (A) x, x)| \right]^{1/2} \cdot \sup_{\|x\|=1} \left[ |g (A) x, x) - g (A) x, x)| \right]^{1/2}
\leq \left[ \sup_{\|x\|=1} |f (A) x, x) | ^2 - \inf_{\|x\|=1} |f (A) x, x) | ^2 \right]^{1/2} \cdot \left[ \sup_{\|x\|=1} |g (A) x, x) | ^2 - \inf_{\|x\|=1} |g (A) x, x) | ^2 \right]^{1/2}
= \left[ \|f (A) \|^2 - \ell^2 \left( f^{1/2} (A) \right) \right]^{1/2} \cdot \left[ \|g (A) \|^2 - \ell^2 \left( g^{1/2} (A) \right) \right]^{1/2}.
\]
It follows that
\[
w_{\text{max}} (f (A) g (A) - w_{\text{min}} (f (A)) w_{\text{min}} (g (A))
\leq \left[ \|f (A) \|^2 - \ell^2 \left( f^{1/2} (A) \right) \right]^{1/2} \cdot \left[ \|g (A) \|^2 - \ell^2 \left( g^{1/2} (A) \right) \right]^{1/2},
\]
or equivalently we have
\[
w_{\text{max}} (f (A) g (A) - w_{\text{min}} (f (A)) \cdot w_{\text{min}} (g (A))
\leq \left[ \|f (A) \|^2 - \ell^2 \left( f^{1/2} (A) \right) \right]^{1/2} \cdot \left[ \|g (A) \|^2 - \ell^2 \left( g^{1/2} (A) \right) \right]^{1/2},
\]
which proves the desired result. \(\square\)

Corollary 2. Let \( A \in S (\mathcal{H})^+ \). Then,
\[
\begin{aligned}
(2.4) \quad & w_{\text{max}} (A) - w_{\text{min}} (A^\alpha) \cdot w_{\text{min}} (A^{1-\alpha}) \leq \left[ \|A^\alpha \|^2 - \ell^2 \left( A^{1/2} \right) \right]^{1/2} \cdot \left[ \|A^{1-\alpha} \|^2 - \ell^2 \left( A^{1/2} \right) \right]^{1/2} \\
\end{aligned}
\]
for each \( x \in \mathcal{H} \). In particular, we have
\[
\begin{aligned}
(2.5) \quad & w_{\text{max}} (A) - w_{\text{min}}^2 \left( A^{1/2} \right) \leq \left[ \|A^{1/2} \|^2 - \ell^2 \left( A^{1/4} \right) \right]
\end{aligned}
\]
for each \( x \in \mathcal{H} \).

Corollary 3. Let \( A \in S (\mathcal{H})^+ \). If \( f \) is measurable functions on \([0, \infty)\), then we have the inequality
\[
\begin{aligned}
(2.6) \quad & w_{\text{max}} \left( f^2 (A) \right) - w_{\text{min}}^2 (f (A)) \leq \|f (A) \|^2 - \ell^2 \left( f^{1/2} (A) \right)
\end{aligned}
\]
for each \( x \in \mathcal{H} \).

A generalization of (2.5) can be deduced from (2.6) as follows:
Corollary 4. Let $A \in \mathcal{B}(\mathcal{H})^+$. Then, for any $p > 0$ the inequality
\begin{equation}
\begin{split}
w_{\text{max}}(A^{2p}) - w_{\text{min}}^2(A^p) & \leq \|A^p\|^2 - \ell^2 \left( A^{p/2} \right) \\
\end{split}
\end{equation}
holds for each $x \in \mathcal{H}$.

The Schwarz inequality for positive operators reads that if $A$ is a positive operator in $\mathcal{B}(\mathcal{H})$, then
\begin{equation}
\begin{split}
|\langle Ax, y \rangle|^2 & \leq \langle Ax, x \rangle \langle Ay, y \rangle, \quad 0 \leq \alpha \leq 1.
\end{split}
\end{equation}
for any vectors $x, y \in \mathcal{H}$.

In 1951, Reid [16] proved an inequality which in some senses considered a variant of Schwarz inequality. In fact, he proved that for all operators $A \in \mathcal{B}(\mathcal{H})$ such that $A$ is positive and $AB$ is selfadjoint then
\begin{equation}
\begin{split}
|\langle ABx, y \rangle| & \leq \|B\| \langle Ax, x \rangle,
\end{split}
\end{equation}
for all $x \in \mathcal{H}$. In [8], Halmos presented his stronger version of Reid inequality (2.9) by replacing $r(B)$ instead of $\|B\|$.

In 1952, Kato [9] introduced a companion inequality of (2.8), called the mixed Schwarz inequality, which asserts
\begin{equation}
\begin{split}
|\langle Ax, y \rangle|^2 & \leq \left| |A|^{2\alpha} x, x \right| \left| |A^*|^{2(1-\alpha)} y, y \right|, \quad 0 \leq \alpha \leq 1.
\end{split}
\end{equation}
for all positive operators $A \in \mathcal{B}(\mathcal{H})$ and any vectors $x, y \in \mathcal{H}$, where $|A| = (A^* A)^{1/2}$.

In 1988, Kittaneh [13] proved a very interesting extension combining both the Halmos–Reid inequality (2.9) and the mixed Schwarz inequality (2.10). His result reads that
\begin{equation}
\begin{split}
|\langle ABx, y \rangle| & \leq r(B) \|f(|A|) x\| \|g(|A^*|) y\|
\end{split}
\end{equation}
for any vectors $x, y \in \mathcal{H}$, where $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^* |A|$ and $f, g$ are nonnegative continuous functions defined on $[0, \infty)$ satisfying that $f(t)g(t) = t \ (t \geq 0)$. Clearly, choose $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ with $B = 1_{\mathcal{H}}$ we refer to (2.10). Moreover, choosing $\alpha = \frac{1}{2}$ some manipulations refer to Halmos version of Reid inequality.

Theorem 4. Let $A \in \mathcal{B}(\mathcal{H})$. If $f, g$ are both positive continuous and $f(t)g(t) = t$ for all $t \in [0, \infty)$, then we have the inequality
\begin{equation}
\begin{split}
w_{\text{max}}(A) - w_{\text{min}}(f(A)) \cdot w_{\text{min}}(g(A)) & \leq \frac{1}{2} \left[ f^2(|A|) + g^2(|A^*|) \right] - \ell^2 \left( f^{1/2}(A) \right) : \ell^2 \left( g^{1/2}(A) \right).
\end{split}
\end{equation}
Proof. Since $f(t)g(t) = t$ for all $t \in [0, \infty)$, then from the proof of Theorem 3 we have
\begin{equation}
\begin{split}
& \sup \limits_{\|x\|=1} \left\{ |\langle f(A)g(A) x, x \rangle| - |\langle f(A) x, x \rangle| \right\} |\langle g(A) x, x \rangle| \\
& \leq \sup \limits_{\|x\|=1} |\langle f(A)g(A) x, x \rangle| - \inf \limits_{\|x\|=1} \left\{ |\langle f(A) x, x \rangle| \right\} |\langle g(A) x, x \rangle| \\
& = \sup \limits_{\|x\|=1} |\langle Ax, x \rangle| - \inf \limits_{\|x\|=1} |\langle f(A) x, x \rangle| \cdot \inf \limits_{\|x\|=1} |\langle g(A) x, x \rangle| \quad \text{(by (2.11)) with } B = 1_{\mathcal{H}} \} \\
& \leq \sup \limits_{\|x\|=1} \left\{ f^2(|A|) x, x \right\}^{1/2} \left\{ g^2(|A^*|) x, x \right\}^{1/2} - \inf \limits_{\|x\|=1} |\langle f(A) x, x \rangle| \cdot \inf \limits_{\|x\|=1} |\langle g(A) x, x \rangle| \\
& \leq \sup \limits_{\|x\|=1} \left\{ f^2(|A|) x, x \right\}^{1/2} \left\{ g^2(|A^*|) x, x \right\}^{1/2} - \inf \limits_{\|x\|=1} |\langle f(A) x, x \rangle| \cdot \inf \limits_{\|x\|=1} |\langle g(A) x, x \rangle| \\
& \leq \frac{1}{2} \sup \limits_{\|x\|=1} \left\{ \left[ f^2(|A|) + g^2(|A^*|) \right] x, x \right\} - \inf \limits_{\|x\|=1} |\langle f(A) x, x \rangle| \cdot \inf \limits_{\|x\|=1} |\langle g(A) x, x \rangle|
\end{equation}
which proves the required result.\qed
Corollary 5. Let \( A \in \mathcal{B}(\mathcal{H})^+ \). If \( f, g \) are both positive continuous and \( f(t)g(t) = t \) for all \( t \in [0, \infty) \). Then

\[
(2.13) \quad w_{\max}(A) - w_{\min}(A^\alpha) \cdot w_{\min}(A^{1-\alpha}) \leq \frac{1}{2} \left\| \left| A^{\alpha} + |A^\star|^2(1-\alpha) \right| \right\| - \ell^2 \left( A^{\frac{\alpha}{2}} \right) \cdot \ell^2 \left( A^{\frac{1-\alpha}{2}} \right)
\]

In particular, we have

\[
(2.14) \quad w_{\max}(A) - w_{\min}(A^{1/2}) \leq \frac{1}{2} \left\| |A| + |A^\star| \right\| - \ell^4 \left( A^{1/4} \right)
\]

Theorem 5. Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then,

\[
(2.15) \quad w \left( (A + B)^2 \right) \leq w \left( A^2 \right) + w \left( B^2 \right) + \frac{1}{4} \min \left\{ w \left( BA^2B \right) + \| AB \|^2, w \left( AB^2A \right) + \| BA \|^2 \right\}
\]

Proof. Let us first note that the Dragomir refinement of Cauchy-Schwarz inequality reads that

\[
\langle \langle x, y \rangle \rangle \leq |\langle x, e \rangle \langle e, y \rangle | + |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle | \leq \| x \| \| y \|
\]

for all \( x, y, e \in \mathcal{H} \) with \( \| e \| = 1 \).

It’s easy to deduce the inequality

\[
(2.16) \quad |\langle x, e \rangle \langle e, y \rangle | \leq \frac{1}{2} \left( |\langle x, y \rangle | + \| x \| \| y \| \right)
\]

Utilizing the triangle inequality we have

\[
(2.17) \quad \left| \left\langle (A + B)^2, x, x \right\rangle \right| \leq |\langle A^2x, x \rangle | + |\langle ABx, x \rangle | \| x, A^*B^*x \rangle | + |\langle B^2x, x \rangle |
\]

so that by setting \( e = u, x = ABu, y = A^*B^*u \) in (2.16) we get

\[
|\langle ABu, u \rangle \langle u, A^*B^*u \rangle | \leq \frac{1}{2} \left( |\langle ABu, A^*B^*y \rangle | + \| ABu \| \| A^*B^*u \| \right)
\]

Substituting in (2.17) and taking the supremum over all unit vector \( x \in \mathcal{H} \) we get

\[
w \left( (A + B)^2 \right) \leq w \left( A^2 \right) + w \left( B^2 \right) + \frac{1}{2} \left( w \left( BA^2B \right) + \| AB \|^2 \right)
\]

Replacing \( B \) by \( A \) and \( A \) by \( B \) in the previous inequality we get that

\[
w \left( (B + A)^2 \right) \leq w \left( B^2 \right) + w \left( A^2 \right) + \frac{1}{2} \left( w \left( AB^2A \right) + \| BA \|^2 \right)
\]

Adding the above two inequalities we get the desired result. \( \square \)

Corollary 6. Let \( A \in \mathcal{B}(\mathcal{H}) \). Then,

\[
(2.18) \quad w \left( A^2 \right) \leq \frac{1}{8} \left( w \left( A^4 \right) + \| A^2 \|^2 \right)
\]

Proof. Setting \( A = B \) in (2.15) we get the desired result. \( \square \)

Let \( \mathcal{A} \) be an associative algebra, not necessarily commutative, with identity \( 1_{\mathcal{A}} \). For two elements \( A \) and \( B \) in \( \mathcal{A} \), that commute; i.e., \( AB = BA \). It’s well known the Binomial Theorem reads that

\[
(2.19) \quad (A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}
\]

In [18], Wyss derived an interesting non-commutative Binomial formula for commutative algebra \( \mathcal{A} \) with identity \( 1_{\mathcal{A}} \). Denotes \( \mathcal{L}(\mathcal{A}) \) the algebra of linear transformations from \( \mathcal{A} \) to \( \mathcal{A} \). Let \( A, X \in \mathcal{A} \), the element (commutator) \( d_A \) in \( \mathcal{L}(\mathcal{A}) \) is defined by

\[
d_A \left( X \right) = [A, X] = AX - XA.
\]

It follows that, \( A \) and \( d_A \) are element of \( \mathcal{L}(\mathcal{A}) \). Moreover, \( A \) can be looked upon as an element in \( \mathcal{L}(\mathcal{A}) \) by \( A(X) = AX \), which is the left multiplication.

The following properties are hold [18]:

1. \( A \) and \( d_A \) commute; i.e., \( A d_A(X) = d_A A(X) \).
(2) $d_A$ is a derivation on $\mathcal{A}$; i.e., $d_A(XY) = (d_A X) Y + X (d_A Y)$.
(3) $(A - d_A) X = X A$.
(4) The Jacobi identity $d_A d_B (C) + d_B d_C (A) + d_C d_A (B) = 0$ holds.

Using these properties Wyss proved the following non-commutative version of Binomial theorem [18]:

\[(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\} B^{n-k} \]

for all elements $A, B$ in the associative algebra $\mathcal{A}$ with identity $1_{\mathcal{A}}$.

We write

\[(A + d_B)^n 1_{\mathcal{A}} = A^n + D_n (B, A) \]

For a commutative algebra, $D_n (B, A)$ is identically zero. We thus call $D_n (B, A)$ the essential non-commutative part. Moreover, $D_n (B, A)$ satisfies the following recurrence relation

\[D_{n+1} (B, A) = d_B A^n + (A + d_B) D_n (B, A), \quad n \geq 0\]

with $D_0 (B, A) = 0$.

A non-commutative upper bound for the summand of two bounded linear Hilbert space operators is proved in the following result.

**Theorem 6.** Let $A, B \in \mathcal{B} (H)$. If $f, g$ are both positive continuous and $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

\[(2.22) \quad w ((A + B)^n) \leq \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\} B^{n-k} \]

where $d_B (A) = [B, A] = BA - AB$ and $d_B^* (A) = [B, A]^* = A^* B^* - B^* A^*$.

**Proof.** By Utilizing the triangle inequality in (2.20) and by employing (2.11) we have

\[|\langle (A + B)^n x, y \rangle| \]

\[= \left| \left\langle \left( \sum_{k=0}^{n} \binom{n}{k} \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\} B^{n-k} \right) x, y \right\rangle \right| \]

\[\leq \sum_{k=0}^{n} \binom{n}{k} \left| \left\langle \left( (A + d_B)^{k} 1_{\mathcal{A}} \right) B^{n-k} x, y \right\rangle \right| \]

\[\leq \sum_{k=0}^{n} \binom{n}{k} \left\| f \left( \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\} B^{n-k} \right) \right\| \cdot \left\| g \left( \left\{ (B^{n-k})^{*} \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\}^{*} \right\} \right) \right\| \]

\[\leq \sum_{k=0}^{n} \binom{n}{k} \left\langle f \left( \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\} B^{n-k} \right) x, x \right\rangle^{1/2} \left\langle g \left( \left\{ (B^{n-k})^{*} \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\}^{*} \right\} \right) y, y \right\rangle^{1/2} \]

\[\leq \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left[ \left\langle f \left( \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\} B^{n-k} \right) x, x \right\rangle + \left\langle g \left( \left\{ (B^{n-k})^{*} \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\}^{*} \right\} \right) y, y \right\rangle \right], \]

where the last inequality follows by applying AM-GM inequality. Hence, by letting $y = x$, we get

\[|\langle (A + B)^n x, x \rangle| \]

\[\leq \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left[ \left\langle f \left( \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\} B^{n-k} \right) x, x \right\rangle + \left\langle g \left( \left\{ (B^{n-k})^{*} \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\}^{*} \right\} \right) x, x \right\rangle \right] \]

\[\leq \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left\langle f \left( \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\} B^{n-k} \right) + g \left( \left\{ (B^{n-k})^{*} \left\{ (A + d_B)^{k} 1_{\mathcal{A}} \right\}^{*} \right\} \right) x, x \right\rangle. \]
Taking the supremum over all unit vector $x \in \mathcal{H}$ we get the required result. \hfill \Box

**Remark 1.** Taking the supremum over all unit vectors $x, y \in \mathcal{H}$ in the proof of Theorem 6 we get the following power norm inequality:

$$\| (A + B)^n \| \leq \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left\| f \left( \{ (A + d_B)^k \} B^{n-k} \right) + g \left( \{ (B^{n-k})^* (A + d_B)^k \}^* \right) \right\|$$

for all $A, B \in \mathcal{B} (\mathcal{H})$.

**Corollary 7.** Let $A, B \in \mathcal{B} (\mathcal{H})$. If $f, g$ are both positive continuous and $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$w (A + B) \leq \frac{1}{2} \left\| f (|B|) + g (|B^*|) + f (|A + d_B A|) + g (|(A^* + A^* d_B^*)|) \right\|$$

where $d_B (A) = [B, A] = BA - AB$ and $d_B^* (A) = [B, A]^* = A^* B^* - B^* A^*$.

**Proof.** Setting $n = 1$ in (2.22) we get that

$$w (A + B) \leq \frac{1}{2} \left\| f (|B|) + f (|A + d_B A|) + f (|A + d_B A|) + f (|A + d_B A|) + g (|(A^* + A^* d_B^*)|) \right\|$$

Making use of (2.21), we have

$$(A + d_B) 1_{\mathcal{H}} = A + D_1 (B, A) = A + d_B A,$$

and

$$(A + d_B)^* 1_{\mathcal{H}} = (A^* + d_B^*) 1_{\mathcal{H}} = A^* + D_1 (B^*, A^*) = A^* + A^* d_B^*.$$

Hence,

$$w (A + B) \leq \frac{1}{2} \left\| f (|B|) + f (|A + d_B A|) + g (|(A^* + A^* d_B^*)|) \right\|$$

which gives the required result. \hfill \Box

**Remark 2.** As noted in Remark 1 and deduced in Corollary 7, we may observe that

$$\| A + B \| \leq \frac{1}{2} \left\| f (|B|) + f (|A + d_B A|) + g (|(A^* + A^* d_B^*)|) \right\|$$

$A, B \in \mathcal{B} (\mathcal{H})$.

**Corollary 8.** For $A, B \in \mathcal{B} (\mathcal{H})$ that commute. If $f, g$ are both positive continuous and $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$w ((A + B)^n) \leq \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left\| f (|A^k B^{n-k}|) + g \left( (B^{n-k})^* (A^k)^* \right) \right\|.$$

In particular, we have

$$w (A + B) \leq \frac{1}{2} \left\| f (|B|) + f (|A|) + g (|A^*|) \right\|.$$

**Proof.** Since $AB = BA$, then $d_B = 0$ in (2.23). Alternatively, we may use (2.19) and proceed as in the proof of Theorem 6. \hfill \Box

**Remark 3.** As in the same way we previously remarked, for $A, B \in \mathcal{B} (\mathcal{H})$ that commute, we can have

$$\| (A + B)^n \| \leq \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left\| f (|A^k B^{n-k}|) + g \left( (B^{n-k})^* (A^k)^* \right) \right\|.$$ 

In particular,

$$\| A + B \| \leq \frac{1}{2} \left\| f (|B|) + f (|A|) + g (|A^*|) \right\|.$$
Setting \( f(t) = t^\alpha \) and \( g(t) = t^{1-\alpha} \) for all \( \alpha \in [0,1] \), in the last inequality above we get
\[
\|A + B\| \leq \frac{1}{2} \left\| |B|^\alpha + |B^*|^{1-\alpha} + |A|^\alpha + |A^*|^{1-\alpha} \right\|.
\]

In special case for \( \alpha = \frac{1}{2} \) we have,
\[
\|A + B\| \leq \frac{1}{2} \left\| |B|^{1/2} + |B^*|^{1/2} + |A|^{1/2} + |A^*|^{1/2} \right\|.
\]

**Corollary 9.** For \( A \in B(\mathcal{H}) \). If \( f, g \) are both positive continuous and \( f(t)g(t) = t \) for all \( t \in [0,\infty) \).

Then
\[
(2.25) \quad w(A^n) \leq \frac{1}{2} \left( \|f(|A^n|) + g(|(A^n)^*|)\right)
\]

**Proof.** Setting \( B = 0 \) in (2.22) we get the desired result. In another way, one may set \( B = A \) in Corollary 8, so that we get
\[
w(A^n) \leq \frac{1}{2^n} \left( \|f(|A^n|) + g(|(A^n)^*|)\right) \cdot \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right),
\]
but since \( \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) = 2^n \), then we get the required result. \( \square \)

**Corollary 10.** Let \( A \in B(\mathcal{H}) \). Then,
\[
(2.26) \quad w(A^n) \leq \frac{1}{2} \left( \left\| A^n \right\|^\alpha + \left\| (A^n)^* \right\|^{1-\alpha} \right).
\]

In particular, we have
\[
(2.27) \quad w(A) \leq \frac{1}{2} \left( \left\| A \right\|^\alpha + \left\| A^* \right\|^{1-\alpha} \right).
\]

**Proof.** Setting \( f(t) = t^\alpha \) and \( g(t) = t^{1-\alpha} \) in (2.25). \( \square \)

**Corollary 11.** Let \( A \in B(\mathcal{H}) \). Then,
\[
(2.28) \quad w(A) \leq \frac{1}{2} (\|A\| + \|A\|) \leq \frac{1}{4} \left( 1 + \|A\| + \sqrt{\|A\|^2 - 4 \|A\|} \right)
\]

**Proof.** Letting \( \alpha = 1 \) in (2.27), we get the first inequality. The second inequality follows by employing the norm estimates [12]:
\[
\|A + B\| \leq \frac{1}{2} \left( \|A\| + \|B\| + \sqrt{\|A\|^2 - \|B\|^2 + 4 \|A^{1/2}B^{1/2}\|^2} \right),
\]
and then
\[
\left\| A^{1/2}B^{1/2} \right\| \leq \|AB\|^{1/2}.
\]

in the first inequality and use the fact that \( \|A\| = \|A\| \). In other words, we have
\[
\|A\| + \|A\| \leq \frac{1}{2} \left( \|A\| + \|A\| + \sqrt{\left(\|A^{1/2}\|\right)^2 - 1)^2 + 4 \|A^{1/2}1\|}\right) = \frac{1}{2} \left( 1 + \|A\| + \sqrt{\|A\|^2 - 1)^2 + 4 \|A\|}\right)
\]
which proves the required result. \( \square \)
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