Simulation of interaction Hamiltonians
by quantum feedback:
a comment on the dynamics of information
exchange between coupled systems

Holger F. Hofmann
Graduate School of Advanced Sciences of Matter, Hiroshima University, Kagamiyama
1-3-1, Higashi Hiroshima 739-8530, Japan

Abstract. Since quantum feedback is based on classically accessible measurement
results, it can provide fundamental insights into the dynamics of quantum systems
by making available classical information on the evolution of system properties and
on the conditional forces acting on the system. In this paper, the feedback-induced
interaction dynamics between a pair of quantum systems is analyzed. It is pointed
out that any interaction Hamiltonian can be simulated by local feedback if the levels
of decoherence are sufficiently high. The boundary between genuine entanglement
generating quantum interactions and non-entangling classical interactions is identified
and the nature of the information exchange between two quantum systems during an
interaction is discussed.

PACS numbers: 03.65.Yz, 03.67.-a, 42.50.Dv, 03.65.Ta

E-mail: h.hofmann@osa.org
1. Introduction

In recent years, there has been rapid progress in the analysis of quantum systems at a fundamental level based on precise measurements and optimized control. Experimentally, optical systems have provided a fertile testing ground for this new understanding of quantum effects due to the availability of highly coherent lasers and sensitive detectors. In the light of these new technologies, the discussion about quantum measurement has obtained a practical relevance that sometimes challenges the seemingly well-defined notions of quantum states and Hamiltonians conveyed by typical introductions to quantum mechanics.

A particularly instructive example is the development of quantum feedback theory, which originated from a formal analysis of open system dynamics that was largely motivated by the intention to identify the proper pure state description in the presence of noise, but resulted instead in a new formulation of conditional quantum dynamics that actually highlights the inadequacy of a measurement independent definition of pure states in the dynamics of open systems [1, 2, 3, 4, 5]. By emphasizing the importance of classically available measurement information, the theory of quantum feedback then permitted the identification of interesting parallels between the classical notion of control and its quantum mechanical equivalent [6], a result that may be of significant practical use.

My own interest in quantum feedback originated from studies of quantum noise in lasers, where the classical description of light is surprisingly successful even when the intensities studied are so small that individual photons could be resolved. This observation can be explained in detail by quantum measurement theory: in the presence of a strong laser field, the main effect of the spontaneous emission from a single atom is its interference with the laser light, not the energy contributed by the single atom. It is thus reasonable to analyze the dynamics of single atom emission in terms of homodyne detection of the electromagnetic field. Interestingly, quantum dynamics then corresponds closely to classical electrodynamics, and the effects of quantum noise and of classical dipole radiation can be identified in the measurement statistics. By using feedback, it is therefore possible to eliminate the quantum noise effects and to reduce the effect of spontaneous emission to a quantum nondemolition measurement of the atomic dipole [7, 8]. This method may have interesting applications to the stabilization of atomic quantum states, and it has been pointed out that, in principle, this kind of quantum feedback can indeed stabilize almost any quantum state of a two level atom [9].

Quantum feedback can thus restore some of the classical concepts of dynamics that seem to be lost in the transition from classical systems to quantum systems. In particular, the measurement information used in feedback can always be interpreted as a minimal back-action measurement of a system variable and a conditional unitary evolution of the system [10]. A feedback setup therefore keeps track of some of the observable properties of the system, in addition to the information available on the
conditional forces responsible for the deterministic evolution of the system \[11\]. It may be worthwhile to pursue this line of thought a bit further in order to gain a better understanding of the relationship between the unobservable closed system dynamics and the observable (and therefore controllable) dynamics of quantum feedback. Specifically, this approach could shed some light on the quantum mechanical interaction dynamics between two coupled systems by providing a description of interaction that combines the classical notion of deterministic conditional forces with the quantum notion of the interaction Hamiltonian.

In the following, I will therefore describe systems where the interaction is realized entirely by quantum feedback. According to standard quantum feedback theory, the dynamics is then described by an effective interaction Hamiltonian and a (seemingly separate) superoperator describing the decoherence associated with the measurement interaction. The feedback system can thus simulate a genuine quantum interaction. By itself, the effective Hamiltonian would entangle the interacting system, but in the context of the feedback setup, the quantum state must remain separable because the quantum operations on the individual systems are in fact local. In the terminology of quantum information theory \[12\], the interaction has been realized by local operations and classical communication between the systems, whereas the Hamiltonian interaction itself corresponds to quantum communication between the systems.

The result of this analysis shows that the decoherence term is not really separable from the Hamiltonian dynamics, but describes the quantum noise required to reduce the exchange of quantum information between the systems to zero. The transition from a purely classical exchange of traceable information to the more intimate entanglement generating quantum interaction is therefore a quantitative one based on the precise relation between the noise levels and the coupling constants representing the forces acting between the systems. This observation is consistent with recent results on the robustness of quantum gate operations against noise \[15\], \[16\], \[17\], \[18\], \[19\] and may therefore have interesting implications for the evaluation of experimental quantum devices. The quantum feedback analysis of interactions between quantum systems thus reveals an interesting connection between basic concepts of quantum information and physical interactions. Moreover, the qualitative correspondence between the feedback dynamics and the quantum interaction described by the Hamiltonian indicates that the interpretation of interactions in terms of conditional forces acting between systems might have its applications even in the context of genuine quantum interactions.

The rest of the paper is organized as follows. In section \[2\] the possibility of modifying the Hamiltonian dynamics by quantum feedback is reviewed. In section \[3\] these results are applied to a pair of non-interacting systems to generate an effective interaction Hamiltonian. In section \[4\] the results are applied to the analysis of noisy interactions to derive a criterion for the separability of the interaction dynamics. In section \[5\] the results are illustrated for the case of a feedback induced interaction between a pair of optical cavity modes. It is shown that the feedback analysis provides exact uncertainty limits for the separability of the two mode squeezing interaction. In section
Simulation of interaction Hamiltonians by quantum feedback

The implications of separability for the information exchange between interacting quantum systems is discussed. The results are summarized in Section 7.

2. Quantum feedback and effective Hamiltonians

First, it may be useful to review some of the central results of continuous quantum feedback theory \cite{5} in terms of the relation between measurement information and conditional dynamics. For this purpose, let us consider the dynamics of a quantum system coupled to the environment in such a way that some observable property $\hat{X}$ causes the emission of a corresponding signal in a quantum fluctuating field propagating away from the system to a detector setup. The effects of the environment on the system state then causes dephasing between eigenstates of $\hat{X}$. If this is the only relevant dynamics of the system, the evolution of the density matrix $\hat{\rho}$ can be written as

$$\frac{d}{dt} \hat{\rho} = \gamma \left( \hat{X} \hat{\rho} \hat{X} - \frac{1}{2} \hat{X}^2 \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{X}^2 \right),$$

(1)

where $\gamma$ is the coupling rate determining the strength of the interaction between the system variable $\hat{X}$ and the signal field. At the detector, the emitted signal can then be measured with a resolution limited by the quantum fluctuations of the field \cite{3}. If the signal is integrated over a finite time interval $\Delta t$, this resolution is given by

$$\frac{1}{\delta X^2} = 4 \gamma \Delta t,$$

(2)

where $\delta X$ is the expected error in the measurement result $X_m$ of the observable $\hat{X}$ obtained during the time interval $\Delta t$.

The measurement result $X_m$ is now a classical record of the quantum property $\hat{X}$. In quantum feedback, this record is used to condition the quantum dynamics of the system \cite{5}. In the case of linear feedback, the feedback can be described by a Hamiltonian of the form

$$\hat{H}_{\text{feedback}}(X_m) = -2\hbar \gamma X_m \hat{Y}. $$

(3)

If the time delay between the emission of the signal and the application of the feedback can be neglected, the effective dynamics of the system can be obtained by applying the operators representing the measurement and the feedback for the time interval $\Delta t$ to both sides of the density matrix. The feedback dynamics of the density matrix can then be determined by averaging over all possible measurement results. Due to this averaging procedure, only quadratic terms in $\hat{X}$ and $\hat{Y}$ contribute, and the result reads

$$\frac{d}{dt} \hat{\rho} = \gamma \left( \hat{X} \hat{\rho} \hat{X} - \frac{1}{2} \hat{X}^2 \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{X}^2 \right) + \gamma \left( \hat{Y} \hat{\rho} \hat{Y} - \frac{1}{2} \hat{Y}^2 \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{Y}^2 \right) + \gamma \left( i \hat{Y} \hat{\rho} \hat{X} - i \hat{X} \hat{\rho} \hat{Y} + i \hat{Y} \hat{\rho} \hat{X} - i \hat{X} \hat{\rho} \hat{Y} \right).$$

(4)

In this representation, the first term represents the original dephasing in $\hat{X}$ caused by the emission of the signal, the second term represents the dephasing in $\hat{Y}$ caused...
by the quantum noise in the feedback, and the third term represents the effects of the correlation between the measurement results obtained for $\hat{X}$ and the feedback defined by $\hat{Y}$. Note that the third term is not symmetric in $\hat{X}$ and $\hat{Y}$ due to the temporal sequence of the feedback. Where $\hat{X}$ and $\hat{Y}$ are applied to the same side of $\hat{\rho}$, the measurement term $\hat{X}$ is always applied before the feedback term $\hat{Y}$. As will be shown below, this physically motivated sequence establishes a necessary correlation between the unitary evolution and the decoherence of the system.

The dynamics given by equation (4) above can be transformed to a more conventional form by ordering the operator products according to the position of $\hat{\rho}$,

$$\frac{d}{dt} \hat{\rho} = \gamma \left( (\hat{X} + i\hat{Y}) \hat{\rho} (\hat{X} - i\hat{Y}) - \frac{1}{2}(\hat{X}^2 - i2\hat{Y}\hat{X} + \hat{Y}^2)\hat{\rho} - \frac{1}{2}\hat{\rho}(\hat{X}^2 + i2\hat{X}\hat{Y} + \hat{Y}^2) \right).$$

(5)

Here, the effects of measurement and feedback have been combined in such a way that their different physical origin is not recognizable anymore. In fact, the dynamics can now be summarized using only two operators, one to describe decoherence, and one to describe the unitary evolution of the quantum state. The dynamics then read

$$\frac{d}{dt} \hat{\rho} = \gamma \left( \hat{c}\hat{c}^\dagger - \frac{1}{2}\hat{c}\hat{c}^\dagger - \frac{1}{2}\hat{c}\hat{c}^\dagger \right) - \frac{i}{\hbar} [\hat{H}_{\text{eff.}}, \hat{\rho}],$$

(6)

where the decoherence operator is $\hat{c} = \hat{X} + i\hat{Y}$ and the effective Hamiltonian $\hat{H}_{\text{eff.}}$ is given by

$$\hat{H}_{\text{eff.}} = -\frac{\hbar\gamma}{2} \left( \hat{X}\hat{Y} + \hat{Y}\hat{X} \right).$$

(7)

It is interesting to compare this effective Hamiltonian with the actual feedback operator of equation (3). The essential difference is that the measurement value $X_m$ has now been replaced with the operator $\hat{X}$. While this replacement appears to be a rather intuitive result since $X_m$ is the measurement result of the observable property represented by $\hat{X}$, it is important to recognize that the replacement of a (generally continuous) real number with a hermitian operator completely changes the dynamics described by the Hamiltonian. In particular, the effective Hamiltonian is symmetric in $\hat{X}$ and $\hat{Y}$, indicating that the same Hamiltonian could be obtained from a measurement of $\hat{Y}$ followed by a feedback in $\hat{X}$. The difference between the two scenarios only appears in the exchanged roles of $\hat{c}$ and $\hat{c}^\dagger$.

Equation (6) now gives the feedback dynamics of a system emitting a signal dependent on only a single observable of the system. The dynamics are therefore based on the case of an ideal quantum non-demolition measurement, as described by equation (1). Nevertheless the resulting decoherence operators $\hat{c}$ in equation (6) have the form of annihilation operators, and the decoherence dynamics after feedback appears similar to typical photon emission dynamics. Interestingly, this observation has a simple classical explanation. In classical electrodynamics, radiation losses can be described by the back-action of the emitted electric field on the oscillating dipole itself. If the emitted dipole field is known, this kind of back-action can be represented by a feedback Hamiltonian.
As reported in [7,8], homodyne detection of the emitted radiation can be used to identify and to compensate this back-action by applying a feedback Hamiltonian that cancels the effects of this deterministic back-action. It is thus possible to interpret the natural relaxation processes of a system interacting with the environment in terms of a hypothetical combination of minimal back-action measurement and quantum feedback, where the feedback represents the quantum version of classical back-action [10].

The analysis given above shows how the classical information flow in a quantum feedback system can be identified with elements of the quantum dynamics of the density matrix. In the next section, we will consider the consequences of these results for interactions between two separate systems by analyzing the simulation of interaction Hamiltonians by quantum feedback.

3. Information dynamics in basic interactions

In classical physics, interactions can always be described in terms of local forces acting on local systems. The fact that the forces originate from other systems can be taken into account by simply correlating the specific value of the force with the value of the corresponding system property. A classical interaction between a system $A$ and a system $B$ can thus be described as shown in figure 1: A force $F_{A \rightarrow B}$ depending on the value of the property $X_A$ of system $A$ acts locally on system $B$, while a force $F_{B \rightarrow A}$ depending on the value of the property $X_B$ of system $B$ acts locally on system $A$. The dynamics of each system can therefore be treated locally, while the connection between the systems is established by the transfer of the values of $X_A$ and of $X_B$ from one system to the other. This transfer of classical variables corresponds to a classical communication channel and could be realized by a classical feedback line using precise measurement data from the remote system.

In the quantum case, things are a bit more complicated. As illustrated by figure 2, the Hamiltonian $\hat{H}_{AB}$ describing the interaction dynamics necessarily acts on both systems at once. In general, it is not possible to define the local dynamics, as evidenced by the possibility of generating non-separable entangled states in the interaction. Effectively, the systems exchange genuine quantum information, and their interaction cannot be described in terms of local operations based on classical communication between the systems. As a result of this quantum communication, the density matrix $\hat{\rho}_{AB}$ usually becomes entangled and cannot be represented by products of local density matrices. However, it is a well-known fact that entangled states become separable when a given amount of noise is added. Likewise, quantum interactions can be analyzed in terms of a feedback model closely corresponding to the classical case shown in figure 1 if the decoherence rate of the interacting systems is sufficiently high.

Figure 3 shows the quantum feedback setup realizing an effective Hamiltonian of $2\hbar \gamma \hat{X}_A \hat{X}_B$ by correlated local operations on systems $A$ and $B$. In this setup, the systems are now connected by classical communication lines carrying information on the measurement outcomes $X_{m,A}$ and $\hat{X}_{m,B}$ for the system properties $\hat{X}_A$ and $\hat{X}_B$ from
Figure 1. Illustration of the information exchange in the classical interaction between a system $A$ and a system $B$. The property $X_A$ of system $A$ causes the action of a force $F_{A\rightarrow B}$ in system $B$, and vice versa.

Figure 2. Illustration of the closed-system interaction between two quantum systems $A$ and $B$. The interaction Hamiltonian $\hat{H}_{AB}$ acts on the joint state $\hat{\rho}_{AB}$ of both systems. As a result of the interaction, $\hat{\rho}_{AB}$ is usually not separable into a product state of $A$ and $B$. 
Simulation of interaction Hamiltonians by quantum feedback

Figure 3. Illustration of a quantum feedback setup realizing an effective Hamiltonian of $\hat{H}_{\text{eff.}} = 2\hbar\gamma \hat{X}_A \hat{X}_B$. The property $\hat{X}_k$ of each system emits an observable signal $X_{m,k}$ measured at the detectors. This measurement value is then used to define a feedback Hamiltonian $\hat{H}_{k\rightarrow l}$ acting on the opposite system $l$ ($k, l = A, B$).

Using the quantum feedback theory introduced in section 2 above, it is then possible to determine the effective interaction dynamics between the systems. Specifically, the initial decoherence dynamics without feedback can be written as

$$\frac{d}{dt} \hat{\rho}_{AB} = D(\hat{\rho}_{AB}), \quad \text{with}$$

$$D(\hat{\rho}_{AB}) = \gamma \left( \hat{X}_A \hat{\rho}_{AB} \hat{X}_A - \frac{1}{2} \hat{X}_A^2 \hat{\rho}_{AB} - \frac{1}{2} \hat{\rho}_{AB} \hat{X}_A^2 \right) + \gamma \left( \hat{X}_B \hat{\rho}_{AB} \hat{X}_B - \frac{1}{2} \hat{X}_B^2 \hat{\rho}_{AB} - \frac{1}{2} \hat{\rho}_{AB} \hat{X}_B^2 \right).$$

(8)

where the indices of the operators indicate which system they act on. This decoherence permits a continuous measurement of $\hat{X}_A$ and $\hat{X}_B$ with a resolution of $4\gamma \Delta t$ per time interval $\Delta t$ as given by equation (2). The measurement results $X_{m,A}$ and $X_{m,B}$ obtained during each time interval can then be used to generate a linear feedback acting on the opposite system with feedback Hamiltonians given by

$$\hat{H}_{A\rightarrow B}(X_{m,A}) = -2\hbar \gamma X_{m,A} \hat{X}_B \quad \text{and}$$

$$\hat{H}_{B\rightarrow A}(X_{m,B}) = -2\hbar \gamma X_{m,B} \hat{X}_A.$$  

(9)

The feedback dynamics can then be determined most effectively by using equation (5), since a number of imaginary terms generated by the two feedback lines cancel. As a result, the decoherence term retains its original form, with the decoherence rate being doubled by the noise in the feedback. The joint dynamics of the feedback-coupled systems can then be written as

$$\frac{d}{dt} \hat{\rho}_{AB} = -\frac{i}{\hbar} [\hat{H}_{\text{eff.}}, \hat{\rho}_{AB}] + 2D(\hat{\rho}_{AB}),$$

(10)
where the effective interaction Hamiltonian is given by
\[ \hat{H}_{\text{eff.}} = 2\hbar \gamma \hat{X}_A \hat{X}_B. \] (11)

The non-local quantum interaction represented by the Hamiltonian \( \hat{H}_{\text{eff.}} \) above can thus be simulated by a feedback setup in which only classical information is exchanged between the systems. The price to be paid for the replacement of quantum interactions with a classical signal transfer is given by the decoherence operator \( 2D \). The qualitative effects of the interaction Hamiltonian \( \hat{H}_{\text{eff.}} \) can then be analyzed in terms of an \( \hat{X}_A \)-dependent unitary transform acting on system \( B \) and a \( \hat{X}_B \)-dependent unitary transform acting on \( A \).

More complicated interactions can be simulated if measurement information on other system variables is available in the emitted fields. In principle, any interaction Hamiltonian can be simulated by decomposing it into a sum of bilinear terms of the form \( 2\hbar \gamma \hat{X}_A \hat{X}_B \), implementing each term by a separate feedback. Quantum feedback interactions can therefore be used to implement a wide range of interaction Hamiltonians between systems that are only connected by classical communication lines.

4. Interactions in a noisy environment: quantum limits of decoherence rates

The special feature of an interaction realized entirely by quantum feedback is that the conditional evolution of the two systems is fully defined by the available classical information. In a direct quantum interaction between two systems, this kind of information is not necessarily available. However, it is possible that the interaction of the systems with the environment makes such information available even when the interaction is not implemented by feedback. In this case, the interaction is too noisy to entangle the system, and the classical information necessary to identify the local system dynamics is in principle available in the local environments of the interacting systems. Figure 4 illustrates this case: the measurement of \( X_A \) in the emitted fields allows an identification of the Hamiltonian \( H_{A \rightarrow B} \) determining the evolution of system \( B \), and the measurement of \( X_B \) in the emitted fields allows an identification of the Hamiltonian \( H_{B \rightarrow A} \) determining the evolution of system \( A \). The measurements in the environment can thus resolve the entanglement between the two systems and the environment by projecting the systems into a product state.

As figure 4 suggests, the noise levels required to achieve this identification of local dynamics are equal to the noise levels generated by the corresponding quantum feedback based interaction. It is therefore possible to give some quantitative limits beyond which the information in the environment is definitely sufficient for an identification of the local dynamics. This feedback-based analysis can then be used to obtain quantitative results on the robustness of the entangling capabilities of interaction Hamiltonians against quantum noise [15, 16, 17, 18, 19].

If the system dynamics is described by
\[ \frac{d}{dt} \hat{\rho}_{AB} = -\frac{i}{\hbar} [\hat{H}_0, \hat{\rho}_{AB}] + L(\hat{\rho}_{AB}), \] (12)
Simulation of interaction Hamiltonians by quantum feedback

Figure 4. Separation of quantum interaction dynamics by measurements of the information emitted into the environment. $\hat{X}_A$ and $\hat{X}_B$ can be measured with sufficient precision so that the interaction dynamics can be identified with the local Hamiltonians $\hat{H}_{A\rightarrow B}(X_{m,A})$ and $\hat{H}_{B\rightarrow A}(X_{m,B})$ conditioned by the measurement outcomes $X_{m,A}$ and $X_{m,B}$.

where $\hat{H}_0$ is the Hamiltonian describing the unitary part of the dynamics and $L$ is the superoperator describing the non-unitary part of the dynamics, the condition for separability of the dynamics is given by a relation between the frequencies defining $\hat{H}_0/h$ and the rates defining $L$. Using the results of section 3 above, it is possible to define the separability limit for the case of a bilinear interaction in the presence of dephasing between the eigenstates of the interaction Hamiltonian,

$$\hat{H}_0 = \hbar g_{AB} \hat{X}_A \hat{X}_B \quad \text{and}$$

$$L(\hat{\rho}_{AB}) = \gamma_A \left( \hat{X}_A \hat{\rho}_{AB} \hat{X}_A - \frac{1}{2} \hat{X}_A^2 \hat{\rho}_{AB} - \frac{1}{2} \hat{\rho}_{AB} \hat{X}_A^2 \right)$$

$$+ \gamma_B \left( \hat{X}_B \hat{\rho}_{AB} \hat{X}_B - \frac{1}{2} \hat{X}_B^2 \hat{\rho}_{AB} - \frac{1}{2} \hat{\rho}_{AB} \hat{X}_B^2 \right),$$

(13)

where the strength of the interaction is defined by the coupling frequency $g_{AB}$ and the local decoherence rates of systems $A$ and $B$ are given by $\gamma_A$ and by $\gamma_B$, respectively. The separate variation of $\gamma_A$ and of $\gamma_B$ can be obtained by rescaling the operators $\hat{X}_A$ and $\hat{X}_B$ in equation (6) while leaving the product $\hat{X}_A \hat{X}_B$ unchanged. This equation of motion for the density matrix can then be separated into an effective local feedback scenario if (and only if)

$$g_{AB}^2 \leq \gamma_A \gamma_B.$$  

(14)

That this condition is in fact both necessary and sufficient is a consequence of the conservation of eigenstates of $\hat{X}_A$ and $\hat{X}_B$ in the noisy dynamics given by equations (13). The only possible local feedback model consistent with this conservation of $\hat{X}_A$ and $\hat{X}_B$ is the model based on quantum non-demolition measurements of $\hat{X}_A$ and $\hat{X}_B$,
as given by equations (8) to (11). Consequently, it is not possible to construct any local feedback scenario for equations (13) if $g_{AB}^2 > \gamma_A \gamma_B$.

In the case of a sum of several bilinear interactions in the Hamiltonian, sufficient conditions for separability can be obtained by simply combining all individual separability criteria. Obtaining a necessary conditions for separability is usually more difficult, since a general superoperator $L$ can have infinitely many possible decompositions [15, 18]. It is therefore generally unclear what selection of quantum measurements are optimal as a starting point for the feedback model.

5. Application to the two mode squeezing Hamiltonian

It may now be instructive to consider a specific case of feedback induced interactions in optical systems. The most simple example is perhaps given by the dynamics of two resonant optical cavity modes described by the annihilation operators $\hat{a}_1 = \hat{x}_1 + i\hat{y}_1$ and $\hat{a}_2 = \hat{x}_2 + i\hat{y}_2$, respectively. If the attenuation rate of both cavities is $\kappa$, the initial dynamics of the two cavity modes is simply given by

$$\frac{d}{dt} \hat{\rho}_j = 2\kappa \left( \hat{a}_j \hat{\rho}_j \hat{a}_j^\dagger - \frac{1}{2} \hat{a}_j^\dagger \hat{a}_j \hat{\rho}_j - \frac{1}{2} \hat{\rho}_j \hat{a}_j^\dagger \hat{a}_j \right),$$

(15)

describing the emission of light from the cavities. If the emitted light is detected by heterodyne detection, both quadrature components can be measured with a resolution of

$$\frac{1}{\Delta x^2} = \frac{1}{\Delta y^2} = 4\kappa \Delta t.$$  

(16)

Therefore, both components can be used to generate feedback, where the cavity emission rate $\kappa$ corresponds to the decoherence rate $\gamma$ of sections 2 and 3.

It is then possible to implement a two mode squeezing Hamiltonian by using the following feedback Hamiltonians,

$$\hat{H}_{x_1 \rightarrow x_2}(x_{m,1}) = +2\hbar \kappa x_{m,1} \hat{x}_2, \quad \hat{H}_{x_2 \rightarrow x_1}(x_{m,2}) = +2\hbar \kappa x_{m,2} \hat{x}_1,$$

$$\hat{H}_{y_1 \rightarrow y_2}(y_{m,1}) = -2\hbar \kappa y_{m,1} \hat{y}_2, \quad \hat{H}_{y_2 \rightarrow y_1}(y_{m,2}) = -2\hbar \kappa y_{m,2} \hat{y}_1.$$  

(17)

If the total interaction dynamics of this feedback setup is written as

$$\frac{d}{dt} \hat{\rho}_{12} = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}_{12}] + 2D(\hat{\rho}_{12}),$$

(18)

the effective two mode squeezing Hamiltonian is given by

$$\hat{H}_{\text{eff}} = 2\hbar \kappa (\hat{x}_1 \hat{x}_2 - \hat{y}_1 \hat{y}_2) = \hbar \kappa \left( \hat{a}_1 \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2^\dagger \right).$$

(19)

In the absence of the decoherence represented by $D$, this two mode squeezing would squeeze the noise in the two mode quadratures $\hat{x}_1 - \hat{x}_2$ and $\hat{y}_1 + \hat{y}_2$ to zero at an exponential relaxation rate of $\kappa$, creating standard squeezed state entanglement in the process. However, the decoherence effects of the feedback add noise to this relaxation.
will reduce the noise level of both $\hat{a}$ and $\hat{b}$. However, it should be noted that even a slight increase in the squeezing interaction rates are expressed by $\gamma$. The dynamics is then described by the squeezing Hamiltonian $\hat{H}$. The two-mode squeezing operation can be given in the form defined by equation (12). The feedback induced interaction described above is therefore the strongest two-mode squeezing interaction that can be realized by local operations and classical communication only. This result can be used to analyze the entangling capability of a two-mode squeezing interaction in the presence of noise according to the procedure outlined in section 4 above. Specifically, a noisy two-mode squeezing operation can be given in the form defined by equation (12). The dynamics is then described by the squeezing Hamiltonian $\hat{H}_0$ and a decoherence operator $L$.

$$\dot{H}_0 = 2\hbar g_{12} (\hat{x}_1 \hat{x}_2 - \hat{y}_1 \hat{y}_2) \quad \text{and}$$

$$L(\hat{\rho}_{12}) = \gamma_- \left( \hat{a}_1 \hat{\rho}_{12} \hat{a}_1^\dagger - \frac{1}{2} \hat{a}_1^\dagger \hat{a}_1 \hat{\rho}_{12} - \frac{1}{2} \hat{\rho}_{12} \hat{a}_1^\dagger \hat{a}_1 \right)$$

$$+ \gamma_+ \left( \hat{\rho}_{12} \hat{a}_1^\dagger \hat{a}_1 - \frac{1}{2} \hat{a}_1^\dagger \hat{a}_1 \hat{\rho}_{12} - \frac{1}{2} \hat{\rho}_{12} \hat{a}_1^\dagger \hat{a}_1 \right)$$

$$+ \gamma_- \left( \hat{a}_2 \hat{\rho}_{12} \hat{a}_2^\dagger - \frac{1}{2} \hat{a}_2^\dagger \hat{a}_2 \hat{\rho}_{12} - \frac{1}{2} \hat{\rho}_{12} \hat{a}_2^\dagger \hat{a}_2 \right)$$

$$+ \gamma_+ \left( \hat{\rho}_{12} \hat{a}_2^\dagger \hat{a}_2 - \frac{1}{2} \hat{a}_2^\dagger \hat{a}_2 \hat{\rho}_{12} - \frac{1}{2} \hat{\rho}_{12} \hat{a}_2^\dagger \hat{a}_2 \right).$$  (22)

This super operator still describes a relaxation of the field components at a rate of $\kappa$, but the feedback has doubled the rate at which quantum noise enters the cavities. The relaxation dynamics of the squeezed field fluctuations is therefore given by

$$\frac{d}{dt} \langle (\hat{x}_1 - \hat{x}_2)^2 \rangle = -4\kappa \langle (\hat{x}_1 - \hat{x}_2)^2 \rangle + 2$$

$$\frac{d}{dt} \langle (\hat{y}_1 + \hat{y}_2)^2 \rangle = -4\kappa \langle (\hat{y}_1 + \hat{y}_2)^2 \rangle + 2,$$  (21)

and the stationary solutions are exactly equal to the vacuum noise level of 1/2.

As expected, the feedback interaction therefore cannot entangle the cavity fields. However, it should be noted that even a slight increase in the squeezing interaction will reduce the noise level of both $\hat{x}_1 - \hat{x}_2$ and $\hat{y}_1 + \hat{y}_2$ and lead to a violation of local uncertainties, indicating entanglement [13, 14]. The feedback induced interaction described above is therefore the strongest two-mode squeezing interaction that can be realized by local operations and classical communication only. This result can be used to analyze the entangling capability of a two-mode squeezing interaction in the presence of noise according to the procedure outlined in section 4 above. Specifically, a noisy two-mode squeezing operation can be given in the form defined by equation (12). The dynamics is then described by the squeezing Hamiltonian $\hat{H}_0$ and a decoherence operator $L$.

$$\hat{H}_0 = 2\hbar g_{12} (\hat{x}_1 \hat{x}_2 - \hat{y}_1 \hat{y}_2) \quad \text{and}$$

$$L(\hat{\rho}_{12}) = \gamma_- \left( \hat{a}_1 \hat{\rho}_{12} \hat{a}_1^\dagger - \frac{1}{2} \hat{a}_1^\dagger \hat{a}_1 \hat{\rho}_{12} - \frac{1}{2} \hat{\rho}_{12} \hat{a}_1^\dagger \hat{a}_1 \right)$$

$$+ \gamma_+ \left( \hat{\rho}_{12} \hat{a}_1^\dagger \hat{a}_1 - \frac{1}{2} \hat{a}_1^\dagger \hat{a}_1 \hat{\rho}_{12} - \frac{1}{2} \hat{\rho}_{12} \hat{a}_1^\dagger \hat{a}_1 \right)$$

$$+ \gamma_- \left( \hat{a}_2 \hat{\rho}_{12} \hat{a}_2^\dagger - \frac{1}{2} \hat{a}_2^\dagger \hat{a}_2 \hat{\rho}_{12} - \frac{1}{2} \hat{\rho}_{12} \hat{a}_2^\dagger \hat{a}_2 \right)$$

$$+ \gamma_+ \left( \hat{\rho}_{12} \hat{a}_2^\dagger \hat{a}_2 - \frac{1}{2} \hat{a}_2^\dagger \hat{a}_2 \hat{\rho}_{12} - \frac{1}{2} \hat{\rho}_{12} \hat{a}_2^\dagger \hat{a}_2 \right).$$  (22)

Here, the squeezing rate is expressed by the coupling frequency $g_{12}$ and the decoherence rates are expressed by $\gamma_-$ for photon loss and by $\gamma_+$ for photon gain ($\gamma_- > \gamma_+$ for stationary solutions). Comparison with equations (19) and (20) then shows that the two-mode squeezing Hamiltonian could only originate from local measurements and feedback if

$$g_{12} \leq \gamma_+. $$  (23)
This separability limit is consistent with the uncertainty limit obtained from the steady state of the squeezing dynamics,

\[
\frac{d}{dt}\langle(\hat{x}_1 - \hat{x}_2)^2\rangle = -(\gamma_- - \gamma_+ + 2g_{12})\langle(\hat{x}_1 - \hat{x}_2)^2\rangle + \frac{1}{2}(\gamma_- + \gamma_+)
\]

\[
\frac{d}{dt}\langle(\hat{y}_1 + \hat{y}_2)^2\rangle = -(\gamma_- - \gamma_+ + 2g_{12})\langle(\hat{y}_1 + \hat{y}_2)^2\rangle + \frac{1}{2}(\gamma_- + \gamma_+).
\]

Entanglement is obtained when the two quadrature components drop below the standard quantum limit, a result that is obtained in the steady state for \(g_{12} > \gamma_+\). The example of two mode squeezing thus illustrates how quantum feedback scenarios can be used to identify the separability limit of a specific entanglement generating Hamiltonian.

6. Information inside and outside the system: separability and its interpretation

The quantum feedback analysis of interactions between quantum systems illustrates the correspondence between classical interactions and quantum interactions by restoring the concept of local forces to the quantum formalism. Instead of treating the interaction process as an inseparable whole, it is now possible to analyze the sequence of cause and effect in the Hamiltonian dynamics. The difference between entanglement generating dynamics and separable dynamics is then a quantitative one, depending only on the noise levels of the systems.

Nevertheless, there remains one significant difference that appears to introduce a qualitative element to the distinction between quantum systems and classical systems. In the completely classical interaction scenario shown in figure 1, we would naturally identify the properties of the systems directly with their measurement record in the environment. In the quantum case, however, this identification of measurement record and system property is not even legitimate in the separable case, since the resolution of quantum measurements is necessarily limited by quantum fluctuations. Consequently, it is not possible to access the “real” system properties at all, and a description of the systems in terms of the available information is all that we can achieve. In this sense, even local quantum systems do not have a classical description. What then is the significance of the distinction between separable and entangled systems?

Quantum feedback provides some insight into this fundamental question by achieving a partial separation of information and physical causality. In the case of the separable interaction dynamics induced by quantum feedback, the information exchanged between the systems is available as classical information outside the systems. The information exchange between the systems is therefore completely detached from the actual system properties once the measurement has been performed. This detachment of information and physical properties is the decisive difference between the separable feedback and the direct entanglement generating interaction. In the purely Hamiltonian interaction, the exchange of quantum information implies that the information about the exact forces acting on each local system remains attached to the
physical properties of the systems. No information about the interaction is available outside of the systems. Separability therefore indicates that the information about the interaction has detached itself from the quantum systems and is now available as classical information in the environment outside the systems, while entanglement generation indicates that the quantum information shared between the systems is exclusively confined inside the two interacting systems.

7. Conclusions

As the analysis above has shown, the simulation of interaction Hamiltonians by quantum feedback can provide fundamental insights into the dynamics of information exchange between interacting systems. In particular, the dynamics expressed by an interaction Hamiltonian can be interpreted in terms of a completely classical information exchange if the decoherence rates are comparable to the frequencies defining the strength of the interaction in the Hamiltonian. The difference between a separable interaction that can be represented by local operations and classical communication between the systems and a genuine entanglement generating quantum interaction can then be expressed quantitatively in terms of the decoherence rates associated with the availability of information in the environment outside the interacting systems. It may thus be possible to identify the robustness of quantum interactions against various noise effects by applying an appropriate quantum feedback model.

Acknowledgments

Part of this work has been supported by the JST-CREST project on quantum information processing.

[1] Dalibard J, Castin Y and Molmer K, 1992 Phys. Rev. Lett. 68 580
[2] Wiseman H M and Milburn G J 1993 Phys. Rev. A 47 1652
[3] Carmichael H 1993 An Open Systems Approach to Quantum Mechanics (Berlin: Springer).
[4] Wiseman H M and Milburn G J 1993 Phys. Rev. Lett. 70 548
[5] Wiseman H M 1994 Phys. Rev. A 49 2133
[6] Doherty A C, Habib S, Jacobs K, Mabuchi H and Tan S M 2000 Phys. Rev. A 62 012105
[7] Hofmann H F, Mahler G and Hess O 1998 Phys. Rev. A 57 4877
[8] Hofmann H F, Hess O, and Mahler G 1998 Opt. Express 2 339
[9] Wang J and Wiseman H M 2001 Phys. Rev. A 64 063810
[10] Wiseman H M 1995 Phys. Rev. A 51 2459
[11] Wiseman H M 1996 1996 Quantum Semiclass. Opt. 8 205 (1996).
[12] See for example Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[13] Duan L-M, Giedke G, Cirac J I and Zoller P 2000 Phys. Rev. Lett. 84, 2722
[14] Hofmann H F and Takeuchi S 2003 Phys. Rev. A 68, 032103
[15] Harrow A W and Nielsen M A 2003 Phys. Rev. A 68, 012308
[16] Montangero S, Benenti G and Fazio R 2003 Phys. Rev. Lett. 91, 187901
[17] Bandyopadhyay S and Lidar D A 2004 Phys. Rev. A 70, 010301(R)
[18] Hofmann H F 2005 Phys. Rev. Lett. 94, 160504
[19] Hofmann H F 2004 Preprint quant-ph/0407165