FINITE DIMENSIONAL SMOOTH ATTRACTOR FOR THE BERGER PLATE WITH DISSIPATION ACTING ON A PORTION OF THE BOUNDARY

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Abstract. We consider a (nonlinear) Berger plate in the absence of rotational inertia acted upon by nonlinear boundary dissipation. We take the boundary to have two disjoint components: a clamped (inactive) portion and a controlled portion where the feedback is active via a hinged-type condition. We emphasize the damping acts only in one boundary condition on a portion of the boundary. In [24] this type of boundary damping was considered for a Berger plate on the whole boundary and shown to yield the existence of a compact global attractor. In this work we address the issues arising from damping active only on a portion of the boundary, including deriving a necessary trace estimate for $(\Delta u)|_{\Gamma_0}$ and eliminating a geometric condition in [24] which was utilized on the damped portion of the boundary.

Additionally, we use recent techniques in the asymptotic behavior of hyperbolic-like dynamical systems [11, 18] involving a “stabilizability” estimate to show that the compact global attractor has finite fractal dimension and exhibits additional regularity beyond that of the state space (for finite energy solutions).

1. Introduction. In this treatment, we study a partial differential equation (PDE) model that accounts for the nonlinear vibrations of a thin, elastic plate—in the sense of large deflections [6, 7, 21]. The physical derivation of this plate model, as well as a detailed discussion of the validity of the model, can be found in [24]. The particular interest of this treatment is to analyze Berger’s plate model from the point of view of long-time behavior. We are interested in global attractors and their structural properties (such as finite dimensionality and regularity) in the presence of nonlinear dissipation acting via moments on a portion of the boundary (the remaining, disjoint portion of the boundary will be taken to be clamped). To obtain asymptotic smoothness of the dynamics (see the Appendix for a definition)

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a standard geometric condition on the undamped portion of the boundary will be in force.

1.1. **Model.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma$. We assume the plate has negligible thickness $[15, 17, 18, 21, 30]$. We take $\Gamma = \Gamma_0 \sqcup \Gamma_1$, where $\sqcup$ denotes disjoint union and $\Gamma_0 \neq \emptyset$; $\Gamma_0$ will be considered the undamped portion of the boundary, while $\Gamma_1$ is the active portion of the boundary. Thus, invoking the hypothesis of (i) finite elasticity, (ii) assuming filaments of the plate remain perpendicular to the central plane of the plate throughout deflection, and (iii) neglecting in-plane accelerations $[30]$, we obtain:

\[
\begin{aligned}
&\left\{\begin{array}{l}
\ddot{u} + \Delta^2 u + f(u) = p(x) \quad \text{in } (0, T) \times \Omega \\
u(0) = u_0; \quad u_t(0) = u_1
\end{array}\right. \\
\end{aligned}
\]

with clamped-hinged boundary conditions and dissipation acting via moments $[15, 17, 18, 30]$, denoted (CHD)—clamped-hinged dissipation:

\[
\begin{aligned}
&\left\{\begin{array}{l}
u = \partial_x u = 0 \quad \text{on } \Gamma_0, \\
u = 0; \quad \Delta u = -g(\partial_x u_t) \quad \text{on } \Gamma_1.
\end{array}\right.
\]

The damping function $g(\cdot) \in C^1(\mathbb{R})$ is taken to be monotone increasing, and $g(0) = 0$. The Berger nonlinearity is given by

\[
f(u) = \left[\gamma - \Upsilon \cdot \left(\int_{\Omega} \nabla u \cdot \nabla u \, d\Omega\right)\right] \Delta u.
\]

This nonlinearity is a simplification of the scalar von Karman nonlinearity $[18, 21, 30]$ which is obtained by assuming the second strain-invariant to be negligible $[6, 24]$. Doing so is generally considered valid for clamped and hinged type plate boundary conditions $[43]$ (and references therein).

The source function $p \in L^2(\Omega)$ represents static pressure differences across the surface of the thin plate. The parameter $\Upsilon > 0$ is a physical parameter measuring the strength of the effect of stretching on bending, and $\gamma$ corresponds to in-plane tension ($\gamma < 0$) or compression ($\gamma > 0$) $[6]$. In this treatment (i) we consider the non-dissipative case $\gamma \geq 0$, as is by now standard in treatments of the Berger plate $[13, 16]$, and (ii) we normalize $\Upsilon = 1$.

1.2. **General discussion of previous work.** The asymptotic-in-time behavior of the hyperbolic-like dynamics has been a topic of immense study over the past 30 years $[4, 5, 12, 14, 22, 23, 33, 38, 41]$. We recommend the resource $[18]$—focused on von Karman plate dynamics—for a modern and comprehensive study of the long-time behavior (attractors and related topics) of second order (in time) abstract equations. In addition to its expansive coverage of von Karman plates, the monograph has a detailed exposition of the abstract theory of nonlinear dynamical systems, compact global attractors, and further asymptotic properties of solutions to hyperbolic-like PDEs, as well as key references for each of the principal topics previously mentioned. We also mention $[15, 17]$, which are precursors to much of the work in $[18]$ for boundary dissipation acting on a nonlinear plate.

In this treatment, the focus is the existence and structural properties of a compact global attractor for the non-rotational Berger plate (1) taken with (CHD) conditions. We base our approach on the modern analysis of dissipative dynamical systems $[11, 18]$. In such an approach, we firstly show that the dynamical system
generated by generalized (semigroup) solutions to (1), taken with (CHD) conditions, is (ultimately) dissipative; i.e., it has a bounded absorbing ball in the state space. (This dissipative property of the dynamical system does not require any geometric assumptions on \( \Gamma \).) To accomplish this, we use a multiplier method with the nonlinear energy functional, as developed in [14] (for the semilinear wave equation), and utilized later in [24] for the problem at hand taken with the entire boundary being damped. We then prove that the given dynamical system is asymptotically smooth, by using the well-known iterated compensated compactness criterion from [18] (first developed in [29]); this requires a standard geometric assumption on the uncontrolled portion of the boundary. After showing the existence of the compact attractor for the dynamics, we show that the attractor has finite fractal dimension in the state space, and that elements taken from this attractor have additional smoothness. This relies on modern techniques which utilize a “stabilizability” estimate (see [18]).

2. Notation and preliminaries. Throughout the paper, we denote the Sobolev space of order \( s \in \mathbb{R} \) on the domain \( \Omega \) by \( H^s(\Omega) \) with norms \( \| \cdot \|_s \), and \( \| \cdot \|_0 = \| \cdot \|_{L^2(\Omega)} \). We will use the notation \( \langle \cdot, \cdot \rangle_\Omega \) for inner-products in \( L^2(\Omega) \) and \( \langle \cdot, \cdot \rangle_\Gamma \) for those in \( L^2(\Gamma) \) (or a subset of \( \Gamma \), as indicated by the subscript where necessary).

For simplicity, norms and inner products written without subscript are taken to be \( L^2 \) of the appropriate domain (e.g., \( \langle \cdot, \cdot \rangle \) on \( \Omega \) and \( \langle \cdot, \cdot \rangle \) on \( \Gamma \)).

The natural energy for linear plate dynamics is given by the sum of the potential and kinetic energies

\[
E(t) = E(u, u_t) = \frac{1}{2} \left( \| \Delta u(t) \|^2 + \| u_t(t) \|^2 \right).
\]

The dynamics evolve in the state space

\[
\mathcal{H} \equiv (H^2_{0,0} \cap H^1_0)(\Omega) \times L^2(\Omega),
\]

where \( H^2_{0,0}(\Omega) \equiv \{ u \in H^2(\Omega) : u = \partial_\nu u = 0 \text{ on } \Gamma_0 \} \). We will also critically use the following nonlinear energies associated to equation in (1):

\[
\mathcal{E}(t) = \mathcal{E}(u, u_t) = E(t) + \Pi(u(t)), \quad \mathcal{E}(t) = E(t) + \frac{1}{4} \| \nabla u(t) \|^4,
\]

where the \( \Pi \) term represents the non-dissipative and nonlinear portion of the energy:

\[
\Pi(u) = \frac{1}{4} \left( \| \nabla u \|^4 - 2\gamma \| \nabla u \|^2 - 4 \int_\Omega pu \right).
\]

As in [18, Lemma 1.5.4], for any \( u \in (H^2_{0,0} \cap H^1_0)(\Omega) \), \( 0 < \eta \leq 2 \) and \( \epsilon > 0 \),

\[
\| u \|^2_2 - \eta \leq \epsilon \left( \| \Delta u(t) \|^2 + \frac{1}{2} \| \nabla u(t) \|^4 \right) + M(\epsilon).
\]

This yields the following crucial fact:

\[
\left[ \frac{\gamma}{2} \| \nabla u(t) \|^2 + (p, u) \right] \leq \epsilon \left[ \| \Delta u(t) \|^2 + \frac{1}{2} \| \nabla u(t) \|^4 \right] + M(\epsilon, \gamma, p)
\]

for all \( u \in (H^2_{0,0} \cap H^1_0)(\Omega) \). From (5), we have the bounds

\[
c_0 \mathcal{E}(u, u_t) - C \leq \mathcal{E}(u, u_t) \leq c_1 \mathcal{E}(u, u_t) + C,
\]

for some \( c_0, c_1, C > 0 \) depending on \( p \) and \( \gamma \). Accordingly, we introduce more notation for the study of long-time behavior (following [14]):

\[
0 \leq \mathcal{E}_M(t) \equiv \mathcal{E}(t) + M,
\]

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\]
where $M = M(\epsilon, \gamma, p)$ is the constant given in (5).

In the context of this treatment, we are interested in non-homogeneous boundary conditions where the dissipation mechanism is active only on a portion of the boundary. We consider generalized (nonlinear semigroup) solutions; these are strong limits of strong solutions in the energy topology which satisfy an integral formulation of (1). For more details on the notion of solution, and for precise definitions of strong and weak solutions, see [18] and [24]. Although the well-posedness of Berger plate equation has been known for some time in the case of homogeneous boundary conditions, it is more recent [24] under non-homogeneous boundary conditions when the damping is active via moments on the whole boundary. The approach relies on adaptations of the abstract results in [18] for second order (in time) evolutions with nonlinear damping. This result can easily be adapted to our case after some minor modifications of the abstract functions and operators defined in [24, Section 4.1]. But in order to formulate this result, we need the following assumption on the damping function (linear-like away from the origin):

**Assumption 1.** There exists positive constants $0 < m_1 < M_2 < \infty$ such that

\[ m_1 \leq g'(s) \leq M_2, \quad |s| \geq 1 \]

Here we note that—for well-posedness—one can make weaker assumptions on the damping mechanism $g(s)$; e.g., $g(s)$ could exhibit arbitrary polynomial growth. However, for our subsequent results for long time behavior it is necessary that $g(s)$ satisfy the stronger assumption above which is also utilized in [14, 18].

We now give the well-posedness result:

**Theorem 2.1.** Let Assumption 1 hold. With reference to (1) taken with (CHD) boundary conditions and with initial data $(u_0, u_1) \in \mathcal{H}$, for all $T > 0$ there exists a unique generalized solution $u \in C(0, T; \mathcal{H})$ depending continuously on the initial data. This implies that the map $((u(0), u_t(0)) \mapsto (u(t), u_t(t))$ defines a continuous (nonlinear) semigroup $S(t)$ on $\mathcal{H}$. Additionally, the following energy equality holds:

\[ E(t) + \int_0^t \int_{\Gamma} g(\partial_\nu u) (\partial_\nu u) = E(s). \tag{8} \]

**Remark 1.** Assumption 1 provides linear bounds for the damping function from below and above which ensures the validity of the energy equality (for more details, see [18, Section 4.2]).

3. Main results and discussion.

3.1. **Statement of main results.** Our first main result establishes the existence of the compact global attractor for the dynamical system $(S(t), \mathcal{H})$. As a first step, we show that the semigroup generated by this system is *dissipative*. This is to say that there exists a bounded absorbing set which captures (in finite time) all trajectories emanating from a given, bounded set of initial data.

**Theorem 3.1.** Let Assumption 1 hold. Then there exists an absorbing set $\mathcal{B} \subset \mathcal{H}$ for generalized solutions to (1) taken with (CHD) boundary conditions. This is to say: for all $R_0 > 0$ and initial data $(u_0, u_1) \in \mathcal{H}$ with $\| (u_0, u_1) \|_{\mathcal{H}} \leq R_0$, there exists a $t_0 = t(R_0)$ such that $(u(t), u_t(t)) \in \mathcal{B}$ for $t \geq t_0$. 
The asymptotic smoothness property of \((S(t), \mathcal{H})\) is the next supporting result. To achieve this, as in control problems when only a portion of the boundary is subjected to a feedback, we need an additional (standard) star-complemented geometric condition on the undamped portion of the boundary [18, 30]:

**Assumption 2.** There exists a point \(x_0 \in \mathbb{R}^2\) such that \(h(x) = x - x_0\) has the property that \(h \cdot \nu \leq 0\) for all \(x \in \Gamma_0\), where \(\nu\) is the outward normal vector to \(\Gamma\).

**Remark 2** (Geometric Condition and Trace Estimates on \(\Gamma_0\)). To provide some context, we note that early control and stabilization studies for hyperbolic-like problems required geometric conditions on the entire boundary (the active and inactive portions) [30] (and references therein). The reference [32] concerning the linear Euler-Bernoulli plate equation with given RHS (which we critically use in [24] and here) allowed the removal of “unnatural” geometric restrictions on the controlled portion of the boundary. (Note that in [24], the main result appeals to the star-shaped assumption on \(\Gamma_1 = \Gamma\) due to the contribution of the nonlinear term on the boundary.) Most control and stabilization considerations (for instance, those in [15, 17, 18]) retain a geometric assumption on the uncontrolled portion of the boundary \(\Gamma_0\)—namely the star-complemented condition—for their principal results; this is to provide control of \(\Delta u\) on \(\Gamma_0\), which is not directly controlled by the estimates in [32]. In our analysis, to obtain attractors with a “split” boundary, we require control of the term \(\Delta u\big|_{\Gamma_0}\) in the estimates. In utilizing the multiplier approach in [2, 3], we will obtain control of this term using the nonlinear structure of the problem in showing dissipativity of the dynamics. This does not eliminate the need for a geometric (sign control) assumption on \(\Gamma_0\) in showing the asymptotic smoothness property of the dynamics. We will elaborate on this below (see Remark 5).

**Remark 3.** When dealing with a solution to (1) we use the nonlinear energy to our advantage. By utilizing the so called trace-moment inequality (Theorem 5.1), and exploiting the super-quadratic nature of the nonlinear energy, we dispense with the star-shaped geometric condition on \(\Gamma_1\), which would be analogous to what is assumed in [24].

**Theorem 3.2.** Let Assumptions 1 and 2 be in force. The dynamical system \((S(t), \mathcal{H})\) generated by generalized solutions to (1) is asymptotically smooth.

Consequently, by Theorem 3.1 and Theorem 3.2, the application of the abstract Theorem 5.3 gives the following result:

**Theorem 3.3.** Under Assumptions 1 and 2, the dynamical system \((S(t), \mathcal{H})\) possesses a compact global attractor \(A\).

Let \(A\) be the global attractor corresponding to the flow \(S(t)\), as established above. The second focus of this treatment is devoted to the analysis of structural properties of this attractor. In this context, we show finite dimensionality and additional regularity of \(A\). At this point we recall that the fractal dimension \(\dim_f A\) of a compact set \(A\) is defined by

\[
\dim_f A = \limsup_{\epsilon \to 0} \frac{\ln N(A, \epsilon)}{\ln(1/\epsilon)}
\]

where \(N(A, \epsilon)\) is the minimal number of closed sets of the diameter \(2\epsilon\) which cover the set \(A\).
Now, finite dimensionality and the smoothness of the global attractor $A$ is seen in the following theorem:

**Theorem 3.4.** Suppose that Assumption 1 holds for all $s \in \mathbb{R}$ and Assumption 2 is satisfied. Then

(a) The global attractor $A$ has finite fractal dimension in $H$,

(b) This attractor is smooth in the sense that $A$ is a bounded set in the space $H^3(\Omega) \times H^2(\Omega)$.

### 3.2. Main results in relation to previous literature.

The treatment [24] is the precursor to the work at hand. In [24], the focus is on the general qualitative effects of boundary damping in Berger’s (non-rotational) plate model. (In [24], as part of the discussion of earlier results, long-time behavior of Berger plates and beams with interior dissipation is also addressed.) Thus, the work [24] discusses plate modeling considerations, as well as provides a comparative study between the effect of boundary damping on Berger’s versus von Karman’s plate for both hinged and free boundary conditions with dissipation. In many configurations (since Berger’s plate model is a simplification of von Karman’s) the analysis of Berger’s plate is subsumed by the analysis of von Karman’s plate. In the case of boundary damping, however, this is not so; indeed, for hinged dissipation type boundary conditions and free-clamped dissipation we notice distinct differences in the qualitative behavior of solutions, as well as distinctions in the applicability of certain analytical techniques.

One of the principal results in [24] concerns the existence of a compact global attractor when hinged boundary damping is active in Berger’s plate model on the entire boundary. This result utilizes a different technique than what has been used in the analogous configuration for von Karman’s plate; specifically, in [24] the absorbing ball is constructed directly—rather than by appealing to a gradient structure for the dynamics (which requires some additional interior damping, as in [17, 18]). In [24], a geometric assumption is made on the (controlled) boundary; this assumption was made to accommodate a boundary contribution from Berger’s nonlinearity which does not occur in the presence of the von Karman nonlinearity.

In [18], when the (von Karman) dynamics are under hinged boundary dissipation, the damping need not be active on the entire boundary in order to obtain a compact global attractor. Indeed, with a geometric assumption on the uncontrolled, clamped boundary portion (which is also assumed here), active damping via a hinged type dissipation on a strict subset of the boundary will yield the existence of a smooth global attractor of finite dimension. Subsequently, it is a natural question to investigate whether active damping on a portion of the boundary will lead to a compact global attractor for Berger’s dynamics. The answer to this question is non-trivial, and the main subject at hand.

In the course of establishing dissipativity—Theorem 3.1—we eliminate the need for a geometric condition (as was appealed to in [24]) on the controlled portion of the boundary. We achieve a smooth global attractor of finite fractal dimension with damping active only on this portion of the boundary. For the existence of the global attractor, our technique is a refinement of the approach in [24], as we must utilize

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1 This is to say that the linear-like assumption holds for all $s \in \mathbb{R}$, rather than just for $|s| > 1$.

2 Though not mentioned in [24], the references [9] and [10] contain results on long-time behavior and qualitative properties of solutions for certain composite wave-plate systems (with Berger’s plate nonlinearity and/or thermal effects); some of these results also apply to a stand-alone Berger plate or beam, and include discussions of finite dimensionality and regularity of the global attractor, as well as the existence of exponential attractors in the presence of thermal effects [10].
technical trace estimates on the term $\Delta u|_{\Gamma_0}$ (a high order trace on the uncontrolled portion of the boundary) in constructing the absorbing ball. Doing so requires a particular multiplier analysis, and a few critical amendments to the approach in [24] to obtain the compact global attractor. Additionally, we show that dissipativity of the dynamical system can be obtained without any geometric assumptions on $\Gamma$.

After showing the existence of a compact global attractor for these (CHD) boundary conditions, we proceed to show via a stabilizability approach the finite dimensionality and additional smoothness of the global attractor constructed at the previous step. This represents a substantial improvement of the results in [24], and brings the analysis of Berger’s non-rotational plate with (CHD) boundary conditions inline with the results in [18] for von Karman plate dynamics. (Of course the results on smoothness and finite dimensionality here also apply to the case of an entirely damped boundary—the model in [24].)

Lastly, we mention the new treatise [11] on quasi-stable dynamical systems. The abstract analysis in this reference on certain dynamical systems generated by second order evolutions is pertinent to our long-time behavior studies here. See Remark 7 for further discussion.

Owing to the approach described above (due to the special structure of Berger’s nonlinearity) our results here represent an improvement over those presented in [18] for von Karman’s dynamics with (CHD) in the following senses:

1. We obtain a bound on the size of the global attractor, via a direct dissipativity analysis on the dynamics with no geometric assumptions on the domain. In the most up-to-date analysis of von Karman’s non-rotational plate with boundary damping, an indirect approach is used (appealing to a gradient structure for the dynamics); we opt for a direct approach which applies Theorem 5.3.

2. We do not require interior damping in addition to the boundary damping to obtain smooth, finite dimensional global attractors. In [18], this additional damping is necessary and yields the gradient structure for the dynamics, as mentioned in point (1).

4. Proof of main results.

4.1. Existence of the compact global attractor.

4.1.1. Proof of Theorem 3.1—Dissipativity. The proof relies on the technique applied in [14] for the semilinear wave equation. This approach was adapted to our current situation in the proof of [24, Theorem 4.1]. However, at this point we note that unlike [24]—since the boundary has two disjoint portions—the proof has critical differences. It is divided into several steps, which are presented below.

Step 1: Observability estimate

We begin with the following estimate:
Lemma 4.1. Any generalized solution $u$ to (1) taken with (CHD) boundary conditions satisfies the following inequality

$$\int_0^T \mathcal{E}(\tau) \leq C \mathcal{E}(0) + \epsilon \left[ \mathcal{E}(T) + \int_0^T \mathcal{E}(\tau) \right] + C(p, \epsilon, \gamma) \cdot T$$

where we have noted that $C$ does not depend on $T$.

**Proof.** The proof uses the equipartition and flux multipliers $u$ and $h \cdot \nabla u$ (for $h(x) = x - x_0$ with $x_0 \in \mathbb{R}^2$). Then, using (5) and standard quadratic estimation techniques, the bound follows. (See the proof of [24, Lemma 4.2].)

Now, in order to estimate the RHS of the above inequality—which involves higher order trace terms—the steps mirror those in the stability analysis of the Kirchoff plate in [28] (which itself critically relies on Theorem 4.3 below—from [32]). However, we approach higher order trace term involving $\partial_n(\Delta u)$ in a different way which, ultimately, benefits our analysis and fundamentally exploits the structure of the Berger nonlinearity.

**Step 2: Estimating nonlinear terms via higher order traces**

The steps below are reminiscent of [28], adapted in a few critical places. Let the operator $A = \Delta$, acting on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H^1_0(\Omega)$, and let $D$ be the associated Dirichlet “lift” map defined by

$$Dg = f \in L^2(\Omega) \iff \Delta f = 0 \text{ in } \Omega \text{ and } f = g \text{ on } \Gamma.$$

Let $F \equiv \{-f(u) + p\}$. Accounting for boundary conditions, the operator representation of (1) is then

$$u_{tt} + A \left[ Au - D (\Delta u|_\Gamma) \right] = F. \quad (10)$$

Applying $A^{-1}$ to (10) (justified on strong solutions, and a posteriori on generalized solutions via the estimate produced) we obtain

$$\Delta u = Au = -A^{-1}u_{tt} + D (\Delta u|_\Gamma) + A^{-1}F \text{ in } \mathcal{D}'(\Omega).$$

Now, taking the normal derivative of both sides of above equality, multiplying by $h \cdot \nabla u$, integrating over the space-time cylinder $[0, T] \times \Gamma$, and reading off from the equation we have the relation

$$\int_0^T \int_\Gamma \partial_n(\Delta u)(h \cdot \nabla u) = -\int_0^T \int_\Gamma \partial_n(A^{-1}u_{tt})(h \cdot \nabla u)$$

$$+ \int_0^T \int_{\Gamma} \partial_n D (\Delta u|_{\Gamma}) (h \cdot \nabla u) + \int_0^T \int_\Gamma (\partial_n A^{-1}F)(h \cdot \nabla u), \quad (11)$$

where we have noted that

$$\int_0^T \int_{\Gamma} \partial_n D (\Delta u|_{\Gamma}) (h \cdot \nabla u) = 0,$$

since $h \cdot \nabla u = (h \cdot \nu)\partial_n u$ (owing to $u|_{\Gamma} = 0$ and $\partial_n u = 0$ on $\Gamma_0$).
The following is the critical step which allows our approach to the higher order trace terms to obtain (as our tack is distinct from [28]). This relies specifically on the structure of the Berger nonlinearity. If we note that
\[
A^{-1}F = A^{-1}[[|\nabla u|^2 - \gamma] \Delta u + A^{-1}p
\]
= \[|\nabla u|^2 - \gamma] u + A^{-1}p, \tag{12}
\]
and again that \(h \cdot \nabla = (h \cdot \nu)\partial_{\nu}u\) on \(\Gamma\), we have:
\[
-2 \int_0^T \int_{\Gamma} \partial_{\nu}(\Delta u)(h \cdot \nabla u) - \int_0^T (\gamma - |\nabla u|^2) \int_{\Gamma_1} (h \cdot \nu)|\partial_{\nu}u|^2
\]
\[
= 2 \int_0^T \int_{\Gamma} \partial_{\nu}(A^{-1}u_t)(h \cdot \nabla u) - 2 \int_0^T \int_{\Gamma_1} \partial_{\nu}D(\Delta u_{\Gamma})(h \cdot \nabla u)
\]
\[
+ 2 \int_0^T (\gamma - |\nabla u|^2) \int_{\Gamma_1} (h \cdot \nu)(\partial_{\nu}u)^2 - 2 \int_0^T \int_{\Gamma} \partial_{\nu}(A^{-1}p(x))(h \cdot \nabla u)
\]
\[
- \int_0^T (\gamma - |\nabla u|^2) \int_{\Gamma_1} (h \cdot \nu)|\partial_{\nu}u|^2. \tag{13}
\]

**Remark 4.** In [24] the final term in the above identity is handled by way of a geometric assumption on \(\Gamma_1\); namely by assuming that we have chosen \(h = x - x_0\) so that \(h \cdot \nu \geq 0\) for \(x \in \Gamma_1\). This allows us to discard the nonlinear boundary contribution. Here, we opt to estimate it directly using the trace-moment inequality (Appendix, Theorem 5.1).

The last equality gives
\[
-2 \int_0^T \int_{\Gamma} \partial_{\nu}(\Delta u)(h \cdot \nabla u) - \int_0^T (\gamma - |\nabla u|^2) \int_{\Gamma_1} (h \cdot \nu)|\partial_{\nu}u|^2
\]
\[
= 2 \int_0^T \int_{\Gamma} \partial_{\nu}(A^{-1}u_t)(h \cdot \nabla u) - 2 \int_0^T \int_{\Gamma_1} \partial_{\nu}D(\Delta u_{\Gamma})(h \cdot \nabla u)
\]
\[
- 2 \int_0^T \int_{\Gamma} \partial_{\nu}(A^{-1}p(x))(h \cdot \nabla u) + \int_0^T (\gamma - |\nabla u|^2) \int_{\Gamma_1} (h \cdot \nu)(\partial_{\nu}u)^2. \tag{14}
\]

Now, we estimate the integrals on the RHS of the above inequality, term by term. Using integration by parts in space that we note that
\[
\int_0^T \int_{\Gamma} \partial_{\nu}(A^{-1}u_t)(h \cdot \nabla u) = (\partial_{\nu}(A^{-1}u_t), \partial_{\nu}u)_{\Gamma} \int_0^T - \int_0^T \int_{\Gamma} (h \cdot \nu)(\partial_{\nu}(A^{-1}u_t))(\partial_{\nu}u_t).
\]
From the elliptic regularity theorem (\(\Gamma_0\) smooth, with \(\Gamma_0\) and \(\Gamma_1\) disjoint), for any \(h \in L^2(\Omega): |A^{-1}h|_2 \leq C||h||_{0,\Omega};\) thus we have:
\[
||\partial_{\nu}(A^{-1}h)||_{0,\Gamma} \leq C||A^{-1}h||_{3/2+\epsilon,\Omega} \leq C||A^{-1}h||_{2,\Omega} \leq C||h||_{0,\Omega}.
\]
Corollary 1, the Hölder–Young inequalities, and (5) yield that
\[
\int_0^T \int_{\Gamma} (\partial_{\nu}(A^{-1}u_t)(h \cdot \nabla u) \leq \epsilon \left( \tilde{E}(T) + \tilde{E}(0) + \int_0^T \tilde{E}(\tau) \right) + C(\epsilon) \int_0^T ||\partial_{\nu}u_t||_{0,\Gamma_1}^2. \tag{15}
\]
Again, since from standard elliptic theory, \( \partial_\nu D \in L^2(\Gamma_1) \), we have
\[
\left| \int_0^T \int_{\Gamma_1} \partial_\nu D (\Delta u|_T) (h \cdot \nabla u) \right| \leq \int_0^T \left| \partial_\nu D (\Delta u|_T) \right| ||h \cdot \nabla u||_{H^{-1}(\Gamma_1)} \\
\leq C_h \left[ ||\Delta u||_{L^2(0,T;L^2(\Gamma_1))} + ||g(\partial_\nu u_t)||_{L^2(0,T;L^2(\Gamma_1))} \right] \\
\times \left\{ \int_0^T \left| \frac{\partial^2 u}{\partial \nu^2} \right|^2_{L^2(\Gamma_1)} + \left| \frac{\partial^2 u}{\partial \tau \partial \nu} \right|^2_{L^2(\Gamma_1)} \right\}^{1/2}.
\] (16)

We note that although the first derivatives of \( \nabla u \) on \( \Gamma_1 \) produces double tangential derivatives \( ||u_\tau||^2_{L^2(\Gamma_1)} \), we omit this term since \( u = 0 \) on \( \Gamma_1 \) and so \( \partial_\tau u = 0 \) on \( \Gamma_1 \).

Similarly, by elliptic theory, we obtain
\[
\int_0^T \int_{\Gamma_1} \partial_\nu (A^{-1} p(x))(h \cdot \nabla u) \leq \epsilon \int_0^T \hat{E}(\tau) + C(p, \epsilon) \cdot T.
\] (17)

Finally, we estimate the nonlinear boundary term \( \int_0^T (\gamma - ||\nabla u||^2) \int_{\Gamma_1} (h \cdot \nu) ||\partial_\nu u||^2 \).

Invoking Corollary 1 we get
\[
\left| \int_0^T (\gamma - ||\nabla u||^2) \int_{\Gamma_1} (h \cdot \nu) ||\partial_\nu u||^2 \right| \leq C(h, \nu) \int_0^T (\gamma + ||\nabla u||^2) ||u||_{1} ||u||_{2} \\
\leq C(h, \nu) \int_0^T ||\nabla u||^3 ||\Delta u|| \\
+ C(\gamma, h, \nu) \int_0^T ||\nabla u|| ||\Delta u||
\]

From Young’s inequality and (4) it follows that
\[
\left| \int_0^T (\gamma - ||\nabla u||^2) \int_{\Gamma_1} (h \cdot \nu) ||\partial_\nu u||^2 \right| \leq \epsilon \int_0^T \hat{E}(\tau) + C(\epsilon) \int_0^T (\hat{E}(\tau))^2 + C(\epsilon, T).
\] (18)

Taking into account (15)–(18) in (14), using (5), and applying the Hölder-Young inequalities we arrive at the next preliminary estimate which can be implemented in (9):
\[
-2 \int_0^T \int_{\Gamma_1} \partial_\nu (\Delta u)(h \cdot \nabla u) - \int_0^T (\gamma - ||\nabla u||^2) \int_{\Gamma_1} (h \cdot \nu)(\partial_\nu u)^2 \\
\leq C(h) \left( ||\Delta u||_{L^2(0,T;L^2(\Gamma_1))} + ||g(\partial_\nu u_t)||_{L^2(0,T;L^2(\Gamma_1))} \right) \\
\times \left\{ \int_0^T \left| \frac{\partial^2 u}{\partial \nu^2} \right|^2_{L^2(\Gamma_1)} + \left| \frac{\partial^2 u}{\partial \tau \partial \nu} \right|^2_{L^2(\Gamma_1)} \right\}^{1/2} \\
+ C(\epsilon) \int_0^T ||\partial_\nu u||_{L^2(\Gamma_1)} + \epsilon \left\{ \hat{E}(T) \\
+ \int_0^T \hat{E}(\tau) \right\} + C(\epsilon) \int_0^T (\hat{E}(\tau))^2 + C(\epsilon, 0) + C(p, \epsilon, \gamma, h) \cdot T.
\] (19)
For the terms in the last line of (9) involving the higher order trace term \( \partial_v(h \cdot \nabla u) \), we note the boundary condition \( \Delta u = -g(\partial_v u_t) \) on \( \Gamma_1 \), and we have
\[
\int_0^T \int_{\Gamma_1} g(\partial_v u_t)(\partial_v u) + \int_0^T \int_{\Gamma_1} 2(\Delta u)\partial_v(h \cdot \nabla u) \\
\leq \|g(\partial_v u_t)\|_{L^2(0,T;L^2(\Gamma_1))}\|\partial_v u\|_{L^2(0,T;L^2(\Gamma_1))} \\
+ \|g(\partial_v u_t)\|_{L^2(0,T;L^2(\Gamma_1))}\|\partial_v(h \cdot \nabla u)\|_{L^2(0,T;L^2(\Gamma_1))}.
\]
(20)
We now combine (19) and (20) in (9) and absorb terms to obtain:
\[
\int_0^T \bar{E}(\tau) \leq C(h) \left( \|\Delta u\|_{L^2(0,T;L^2(\Gamma_0))} + \|g(\partial_v u_t)\|_{L^2(0,T;L^2(\Gamma_1))} \right) \\
\times \left\{ \int_0^T \left[ \frac{\partial^2 u}{\partial \nu^2} \right]_{L^2(\Gamma_1)}^2 + \left\| \frac{\partial^2 u}{\partial r \partial \nu} \right\|_{L^2(\Gamma_1)}^2 \right\}^{1/2} \\
+ C \int_0^T \|\partial_v u_t\|_{L^2(\Gamma_1)}^2 + C \bar{E}(T) + C(p, \gamma, h)T \\
+ \|g(\partial_v u_t)\|_{L^2(0,T;L^2(\Gamma_1))}\|\partial_v u\|_{L^2(0,T;L^2(\Gamma_1))} \\
+ \|g(\partial_v u_t)\|_{L^2(0,T;L^2(\Gamma_1))}\|\partial_v(h \cdot \nabla u)\|_{L^2(0,T;L^2(\Gamma_1))} \\
+ \int_0^T \int_{\Gamma_0} (h \cdot \nu)\|\Delta u\|^2 + C \int_0^T (\bar{E}(\tau))^2.
\]
(21)

**Step 3: Trace estimate for \( (\Delta u)|_{\Gamma_0} \)**

By way of estimating (21), we will require the following estimate for \( (\Delta u)|_{\Gamma_0} \).

**Lemma 4.2.** Let \( u \) be the solution to (1) under (CHD) boundary conditions. Then \( (\Delta u)|_{\Gamma_0} \in L^2(0,T;L^2(\Gamma_0)) \) with the following estimate:
\[
\int_0^T \|\Delta u(t)\|^2_{L^2(\Gamma_0)}dt \leq C\left[ \int_0^T \bar{E}(t)dt + \bar{E}(T) + \bar{E}(0) \right] \\
+ \|g(\partial_v u_t)\|_{L^2(0,T;L^2(\Gamma_1))}\|\partial_v(m \cdot \nabla u)\|_{L^2(0,T;L^2(\Gamma_1))},
\]
(22)
where
\[
m(x) = \begin{cases} 
\nu(x) & \text{on } \Gamma_0 \\
0 & \text{on } \Gamma_1.
\end{cases}
\]
(23)

**Proof of Lemma 4.2.** For this estimate, we will use a multiplier approach which was also used critically in [2, 3] (though it was used earlier in other contexts [30, 35]). A vector field as in (23) exists when \( \Gamma_0 \) and \( \Gamma_1 \) are disjoint [2, 3]. We then consider the multiplier \( m \cdot \nabla u \) for (1). Though formally this multiplier has the same structure as the \( h \cdot \nabla u \) multiplier (with \( h(x) = x - x_0 \) for some \( x_0 \)) used in obtaining (9), the specification of \( m(x) \) in (23) will aid in the derivation of the estimate (22).

With the vector field \( m \) as in (23), multiplying (1) by \( m \cdot \nabla u \) and integrating on the space-time cylinder \( \Omega \times [0, T] \), we have:
\[
\int_0^T (u_t, m \cdot \nabla u)_\Omega + \int_0^T (\Delta^2 u, m \cdot \nabla u)_\Omega = \int_0^T (p, m \cdot \nabla u)_\Omega \\
- \int_0^T (\gamma - \|\nabla u\|^2)\Delta u, m \cdot \nabla u)_\Omega.
\]
(24)
For the first term, we have, exactly as derived in [2]—see p. 18 (and as \( u_t \big|_{\Gamma} = 0 \)),
\[
\int_0^T (u_{tt}, m \cdot \nabla u)_{\Omega} = \frac{1}{2} \int_0^T \int_{\Omega} \text{div}(m)u_t^2\,d\Omega dt + (u_t, m \cdot \nabla u)_{\Gamma_0}^T. \tag{25}
\]
From here it is clear that
\[
\left| \int_0^T (u_{tt}, m \cdot \nabla u)_{\Omega} \right| \leq C(m) \left[ \int_0^T \tilde{E}(t)\,dt + \tilde{E}(T) \right]. \tag{26}
\]
The two critical terms in the estimate are the biharmonic term and the nonlinear term. We begin with the biharmonic term. Noting that we have clamped boundary conditions on \( \Gamma_0 \) and that \( m \) vanishes on \( \Gamma_1 \), we have the following equality:
\[
\int_0^T (\Delta^2 u, m \cdot \nabla u)_{\Omega} \,d\tau = \int_0^T (\Delta u, \Delta(m \cdot \nabla u))_{\Omega} \,d\tau - \int_0^T \langle \Delta u, \partial_\nu(m \cdot \nabla u) \rangle_{\Gamma_1} \,d\tau \tag{27}
\]
We utilize the following facts: (i) the relation (due to \( u = 0 \)) [28, p. 463]:
\[
\Delta u \big|_{\Gamma_1} = \partial_\nu u + k(x)\partial_n u,
\]
where \( k(x) \) is the mean curvature of \( \Gamma \) at \( x \); (ii) the decomposition [30, p. 82] on \( \Gamma \)
\[
\partial_\nu (m \cdot \nabla u) = \partial_\nu u + (m \cdot \nu)\partial_\nu u + (m \cdot \tau)\partial_\tau u. \tag{28}
\]
With (23), these yield:
\[
\partial_\nu (m \cdot \nabla u) = \partial_\nu u = \Delta u \text{ on } \Gamma_0.
\]
Thus with the above, and the hinged boundary conditions on \( \Gamma_1 \), (27) becomes:
\[
\int_0^T (\Delta^2 u, m \cdot \nabla u)_{\Omega} \,d\tau = \int_0^T (\Delta u, \Delta(m \cdot \nabla u))_{\Omega} \,d\tau - \int_0^T ||\Delta u||_{\Gamma_1}^2 \,d\tau \\
+ \int_0^T \langle g(\partial_\nu u_t), \partial_\nu (m \cdot \nabla u) \rangle_{\Gamma_1} \,d\tau. \tag{29}
\]
Moreover, for the first term on RHS, we have initially,
\[
\int_0^T (\Delta u, \Delta(m \cdot \nabla u))_{\Omega} \,dt = \frac{1}{2} \int_0^T \int_{\Omega} m \cdot \nabla [u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}] \,d\Omega dt + \Theta_1, \tag{30}
\]
where
\[
\Theta_1(u) = \int_0^T \int_{\Omega} (\Delta u, m_{1xx}u_{xx} + 2m_{1x}u_{xx} + m_{2xx}u_y + 2m_{2x}u_{xy})_{\Omega} \,dt \\
+ \int_0^T \int_{\Omega} (\Delta u, m_{1yy}u_{xx} + 2m_{1y}u_{x}u_{yy} + m_{2yy}u_y + 2m_{2y}u_{yy})_{\Omega} \,dt. \tag{31}
\]
Proceeding, we then rewrite first term on right hand side of (30) as
\[
\frac{1}{2} \int_0^T \int_{\Omega} m \cdot \nabla [u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}] \,d\Omega dt = \frac{1}{2} \int_0^T \int_{\Omega} \text{div} \{m(\Delta u)^2\} \,d\Omega dt + \Theta_2, \tag{32}
\]
where
\[
\Theta_2(u) = -\frac{1}{2} \int_0^T \int_{\Omega} (\Delta u)^2 \text{div} \{m\} \,d\Omega dt. \tag{33}
\]
Combining (30) and (32), and subsequently applying the Divergence Theorem of Gauss (and also recalling the definition of $m$ in (4.16)), we have now
\[
\int_0^T (\Delta u, \Delta (m \cdot \nabla u))_\Omega dt = \frac{1}{2} \int_0^T \int_{\Gamma_0} (\Delta u)^2 d\Gamma_0 dt + \Theta_1 + \Theta_2.
\] (34)

Using (34) in (29), we have now the key identity:
\[
\int_0^T (\Delta^2 u, m \cdot \nabla u)_\Omega d\tau = -\frac{1}{2} \int_0^T \|\nabla u\|^2 d\tau + \int_0^T \langle g(\partial_\nu u_t), \partial_\nu (m \cdot \nabla u) \rangle_{\Gamma_1} d\tau + \Theta_1 + \Theta_2
\] (35)

We note that clearly
\[
|\Theta_1| \leq C(m)\|u\|_{L^2(0,T;H^2(\Omega))}^2.
\] (36)

Next, we consider the nonlinear term:
\[
\int_0^T [\gamma - \|\nabla u\|^2] (\Delta u, m \cdot \nabla u)_\Omega d\tau = \int_0^T \|\nabla u\|^2 - \gamma \|\nabla u, \nabla (m \cdot \nabla u)\|_\Omega d\tau,
\]
where the boundary terms are null, owing to the clamped boundary conditions on $\Gamma_0$ and $m \equiv 0$ on $\Gamma_1$. Explicitly computing $\nabla (m \cdot \nabla u)$, we have the identity (where $H(u)$ is the Hessian of $u$, and $D(m)$ the Jacobian of $m$):
\[
\nabla u \cdot \nabla (m \cdot \nabla u) = m \cdot H(u) \nabla u + [D(m)^T \nabla u] \cdot \nabla u,
\]
as well as
\[
m \cdot H u \nabla u = \frac{1}{2} \text{div}(m|\nabla u|^2) - \frac{1}{2} \text{div}(m)|\nabla u|^2.
\]

These calculations, along with the Divergence Theorem, result in the identity:
\[
\int_0^T [\gamma - \|\nabla u\|^2] (\Delta u, m \cdot \nabla u)_\Omega d\tau = \int_0^T \|\nabla u\|^2 - \gamma \|\nabla u, \nabla (m \cdot \nabla u)\|_\Omega d\tau - \int_\Omega \left\{ \frac{1}{2} \int_\Gamma m|\nabla u|^2 d\Gamma - \int_\Omega \frac{\text{div}(m)}{2}|\nabla u|^2 - D(m)^T \nabla u \cdot \nabla u \right\} d\Omega d\tau.
\]

We note that the boundary term is again null, since on $\Gamma_0$ we have that $\nabla u = \partial_\nu u = 0$, and on $\Gamma_1 m \equiv 0$. This results in our key identity:
\[
\int_0^T [\gamma - \|\nabla u\|^2] (\Delta u, m \cdot \nabla u)_\Omega d\tau
\]
\[
= \int_0^T \|\nabla u\|^2 - \gamma \int_\Omega \left[ Dm^T \nabla u \cdot \nabla u - \frac{\text{div}(m)}{2}|\nabla u|^2 \right] d\Omega d\tau
\] (37)

From the above, we now get that
\[
\int_0^T (f(u), m \cdot \nabla u)_\Omega d\tau \leq C(m) \int_0^T \|\nabla u\|^4 + \gamma \|\nabla u\|^2] d\tau
\]
\[
\leq C(m, \gamma) \int_0^T \hat{E}(t) dt
\] (38)

From (24), considering (26), (35), (36) and (38), we arrive at (22).

\textbf{Step 4: Estimating higher order traces via [32]}

Now, considering (21) over the interval $(\alpha, T - \alpha)$ instead of $(0, T)$ (hence, performing the calculations above on $(\alpha, T - \alpha)$), we can use the decreasing nature (modulo
a constant) of the energy functional ((6) and (8)), and then extend some benign integrals back on (0, T). We obtain:

\[
\int_0^T -E(\tau) \leq C(h) \left[ \|\Delta u\|_{L^2(\alpha, T-\alpha; L^2(\Gamma_0))} + \|g(\partial_\nu u_t)\|_{L^2(0, T; L^2(\Gamma_1))} \right] \\
\times \left[ \int_0^T \|\partial_{\nu\nu} u\|^2_{L^2(\Gamma_1)} + \|\partial_{\nu\gamma} u\|^2_{L^2(\Gamma_1)} \right]^{1/2} + C(p, \gamma, h)[T - 2\alpha] \\
+ \int_\alpha^T \int_\Gamma (h \cdot \nu)|\Delta u|^2 + C \int_0^T \|\partial_\nu u_t\|^2_{L^2(\Gamma_1)} + C \hat{E}(T - \alpha) + C \hat{E}(\alpha) \\
+ \|g(\partial_\nu u_t)\|_{L^2(0, T; \Gamma_1)} \|\partial_\nu u\|_{L^2(\alpha, T-\alpha; \Gamma_1)} + C \int_\alpha^T (\hat{E}(\tau))^2 \\
+ \|g(\partial_\nu u_t)\|_{L^2(0, T; L^2(\Gamma_1))} \left[ \|\partial_\nu(h \cdot \nabla u)\|_{L^2(\alpha, T-\alpha; L^2(\Gamma_1))} \\
+ \|\partial_\nu(m \cdot \nabla u)\|_{L^2(\alpha, T-\alpha; L^2(\Gamma_1))} \right]. \tag{39}
\]

Now, for the second order trace terms \(\partial_{\nu\nu} u|_{\Gamma_1}\) and \(\partial_{\nu\gamma} u|_{\Gamma_1}\) above, we will use the following sharp regularity result for the boundary traces of solutions to the Euler-Bernoulli equation (linear, with given RHS):

**Theorem 4.3.** ([32, p.281, Theorem 2.4]) Let \(0 < \alpha < T, 0 < \delta < 1/2, \) and \(s_0 < 1/2\) be arbitrary. Then the generalized solutions of (1) with (CHD) boundary conditions enjoy the bound

\[
\int_\alpha^T \int_\Gamma |\partial_{\gamma\gamma} u|^2 + |\partial_{\gamma\delta} u|^2 + |\partial_{\gamma\gamma} u^2| d\Gamma dt \leq C(T, \alpha, \delta) \left\{ ||f(u)||_{L^2(0, T; H^{-\delta^2}(\Omega))} \\
+ ||u||_{L^2(0, T; H^{2-\delta}(\Gamma))} + ||g(\partial_\nu u_t)||_{L^2(0, T; L^2(\Gamma))} + ||\partial_\nu u_t||_{L^2(0, T; L^2(\Gamma_1))} \right\}. \tag{40}
\]

Hereafter, the explicit dependence of the above constants on \(\alpha, \delta, \gamma, h\) will be suppressed. We now proceed to estimate the key terms in the first line of (39).

(I) From Theorem 4.3 we have:

\[
\|g(\partial_\nu u_t)\|_{L^2(0, T; L^2(\Gamma_1))} \left\{ \int_\alpha^T \left[ \left| \frac{\partial^2 u}{\partial \nu^2} \right|_{L^2(\Gamma_1)}^2 + \left| \frac{\partial^2 u}{\partial \tau \partial \nu} \right|_{L^2(\Gamma_1)}^2 \right]^{1/2} \right\} \\
\leq C(T) \left[ ||g(\partial_\nu u_t)||_{L^2(0, T; L^2(\Gamma_1))} ||f(u)||_{L^2(0, T; H^{-\delta^2}(\Omega))} \\
+ ||u||_{L^2(0, T; H^{2-\delta}(\Gamma))} + ||g(\partial_\nu u_t)||_{L^2(0, T; L^2(\Gamma))} + ||\partial_\nu u_t||_{L^2(0, T; L^2(\Gamma_1))} \right] \right\} \\
\leq C + C(T) ||g(\partial_\nu u_t)||_{L^2(0, T; L^2(\Gamma_1))} ||f(u)||_{L^2(0, T; H^{-\delta^2}(\Omega))} \\
+ ||u||_{L^2(0, T; H^{2-\delta}(\Gamma))} + ||g(\partial_\nu u_t)||_{L^2(0, T; L^2(\Gamma_1))} + ||\partial_\nu u_t||_{L^2(0, T; L^2(\Gamma_1))} \right] \\
+ C \|u\|_{L^2(0, T; H^{2-\delta}(\Omega))}. \tag{41}
\]
Then, via (4)
\[
\|g(\partial_{\nu} u_i)\|_{L^2(0,T;L^2(\Gamma_1))} \left\{ \int_{\alpha}^{T-\alpha} \left| \frac{\partial^2 u}{\partial \nu^2} \right|_{L^2(\Gamma_1)}^2 + \left| \frac{\partial^2 u}{\partial \tau \partial \nu} \right|_{L^2(\Gamma_1)}^2 \right\}^{1/2}
\]
\[
\leq C + C(T)\|g(\partial_{\nu} u_i)\|_{L^2(0,T;L^2(\Gamma_1))}\left[ \int_0^T \left| \left( \gamma - |\nabla u|^2 \right) \Delta u \right|_{L^2\gamma}^2 \, d\tau \right]
\]
\[
+ C(T) \|g(\partial_{\nu} u_i)\|_{L^2(0,T;L^2(\Gamma_1))} + \|\partial_{\nu} u_i\|_{L^2(0,T;L^2(\Gamma_1))}^2 + C\|u\|_{H^2(\Omega)}^2
\]
\[
\leq C + C(T)\|g(\partial_{\nu} u_i)\|_{L^2(0,T;L^2(\Gamma_1))}\left[ \int_0^T (\hat{E}(\tau))^2 \, d\tau \right]
\]
\[
+ C(T) \|g(\partial_{\nu} u_i)\|_{L^2(0,T;L^2(\Gamma_1))} + \|\partial_{\nu} u_i\|_{L^2(0,T;L^2(\Gamma_1))}^2 + C\|u\|_{H^2(\Omega)}^2
\]
(42)

(II) By (22) we have
\[
\int_{\Gamma_0}^{T-\alpha} \int_0^T (h \cdot \nu) |\Delta u|^2 \leq C \int_{\alpha}^{T} \hat{E}(T) + \hat{E}(T) + \int_{\alpha}^{T} (\hat{E}(t))^2 \, d\tau
\]
\[
+ g(\partial_{\nu} u_i)\|_{L^2(0,T;L^2(\Gamma_1))} \|\partial_{\nu} (m \cdot \nabla u)\|_{L^2(0,T;L^2(\Gamma_1))},
\]
and using (40) we arrive at
\[
\|\Delta u\|_{L^2(\alpha,T-\alpha;L^2(\Gamma_0))} \left\{ \int_{\alpha}^{T-\alpha} \left| \frac{\partial^2 u}{\partial \nu^2} \right|_{L^2(\Gamma_1)}^2 + \left| \frac{\partial^2 u}{\partial \tau \partial \nu} \right|_{L^2(\Gamma_1)}^2 \right\}^{1/2}
\]
\[
\leq \epsilon \|\Delta u\|_{L^2(\alpha,T-\alpha;L^2(\Gamma_0))} + \epsilon \int_{\alpha}^{T} \hat{E}(\tau) + C(T, \epsilon) \int_{\alpha}^{T} (\hat{E}(\tau))^2 \, d\tau + C(T, \epsilon)
\]
\[
+ C(T, \epsilon) \|g(\partial_{\nu} u_i)\|_{L^2(0,T;L^2(\Gamma_1))} + \|\partial_{\nu} u_i\|_{L^2(0,T;L^2(\Gamma_1))}^2
\]
\[
\leq \epsilon \left( \hat{E}(\alpha) + \hat{E}(T - \alpha) + \int_{\alpha}^{T-\alpha} \hat{E}(\tau) \right) + C(T, \epsilon) \int_{\alpha}^{T} (\hat{E}(\tau))^2 \, d\tau + C(T, \epsilon)
\]
\[
+ C(T, \epsilon) \|g(\partial_{\nu} u_i)\|_{L^2(0,T;L^2(\Gamma_1))} + \|\partial_{\nu} u_i\|_{L^2(0,T;L^2(\Gamma_1))}^2 \right].
\]
(43)

(III) To proceed with estimating the RHS of (39): Using Young’s inequality and (4) we get
\[
\left| g(\partial_{\nu} u_i) \right|_{L^2(0,T;L^2(\Gamma_1))} \leq \epsilon \int_{\alpha}^{T-\alpha} \hat{E}(\tau) + \int_{0}^{T} \left| g(\partial_{\nu} u_i) \right|_{L^2(\Gamma_1)} + C(\epsilon, T).
\]

Considering (28) and applying the same steps as in the estimate (42), we have:
\[
\left| g(\partial_{\nu} u_i) \right|_{L^2(0,T;L^2(\Gamma_1))} \leq \epsilon \int_{\alpha}^{T-\alpha} \hat{E}(\tau) + \int_{0}^{T} \left| g(\partial_{\nu} u_i) \right|_{L^2(\Gamma_1)}^2 + C(\epsilon, T).
\]

\[
\left| g(\partial_{\nu} u_i) \right|_{L^2(0,T;L^2(\Gamma_1))} \leq \epsilon \int_{\alpha}^{T-\alpha} \hat{E}(\tau) + \int_{0}^{T} \left| g(\partial_{\nu} u_i) \right|_{L^2(\Gamma_1)}^2 + C(T) \left[ \left| g(\partial_{\nu} u_i) \right|_{L^2(0,T;L^2(\Gamma_1))}^2 + \|\partial_{\nu} u_i\|_{L^2(0,T;L^2(\Gamma_1))}^2 \right].
\]
If we take into account the last two inequalities we have:
\[
\begin{align*}
| g(\partial_t u_t) |_{L^2(0,T;\Gamma_1)} &\leq C(T,\Gamma_1) + ||| g(\partial_t u_t) |_{L^2(0,T;\Gamma_1)}^2 + ||| g(\partial_t u_t) |_{L^2(0,T;\Gamma_1)}^2 + ||| g(\partial_t u_t) |_{L^2(0,T;\Gamma_1)}^2 + \cdots (44)
\end{align*}
\]

(IV) Returning to (39) and collecting (42)–(44), we have:
\[
\begin{align*}
\int_\alpha^{T-\alpha} \tilde{E}(t) dt &\leq C \tilde{E}(T-\alpha) + C(\alpha, T) \\
&\quad + C(\alpha, T) \left[1 + ||| g(\partial_t u_t) |_{L^2(0,T;\Gamma_1)}^2 + \cdots (45)\right]
\end{align*}
\]

Now, using the decreasing property of the energy and (6) we finally get
\[
\begin{align*}
(T-2\alpha) \tilde{E}(T) &\leq C \tilde{E}(0) + C(T) \left[1 + ||| g(\partial_t u_t) |_{L^2(0,T;\Gamma_1)}^2 + \cdots (46)\right]
\end{align*}
\]

Using the properties of the damping (Assumption 1), we note these facts from [18, 24]:
\[
\begin{align*}
g(s) &\geq \frac{s^2}{2} m \quad \text{for } |s| \geq 2 \quad \text{and} \quad g(s) s \geq m_* \frac{s^2}{2} \quad \text{for } |s| \geq \epsilon . \nn\end{align*}
\]

\[
\begin{align*}
g(s) &\leq s \cdot \max \left\{ \sup_{0 \leq \xi \leq 1} g'(\xi), M \right\}, \quad s \geq 0. \nn\end{align*}
\]

From which it is straight-forward to obtain (see [24]):
\[
\begin{align*}
||| g(\partial_t u_t) |_{L^2(0,T;\Gamma_1)}^2 + ||| g(\partial_t u_t) |_{L^2(0,T;\Gamma_1)}^2 &\leq C(T,\Gamma_1) + \max \left\{ \frac{2}{m_* \sup_{0 \leq \xi \leq 1} g'(\xi), M} \right\} D_0^T, \quad (47)\nn\end{align*}
\]

where
\[
D_0^T = \int_0^T \int_{\Gamma_1} g(\partial_t u_t)(\partial_t u_t). \nn\]

Now, if we take into account (47) in our observability estimate (46) we have
\[
\begin{align*}
(T-2\alpha) \tilde{E}(T) &\leq C \tilde{E}(0) + C(T) D_0^T + C(T) \left[1 + D_0^T \right] \int_0^T (\tilde{E}(\tau))^2 + K(T). \nn\end{align*}
\]

Again, via (6) and (8)
\[
\tilde{E}(0) \leq CD_0^T + C \tilde{E}(T) + C(p, \gamma), \nn\]
where $C$ here does not depend on $T$. Thus, scaling $T$ to be sufficiently large, we have
\[(T - 2\alpha - C)\hat{E}(T) \leq C(T)D_0^T + C(T)[1 + D_0^T]\int_0^T (\hat{E}(\tau))^2d\tau + K(T).
\]

**Step 5: Stabilization-type argument**

Now, if we rewrite the last inequality in terms of the full nonlinear energy $\mathcal{E}(T)$, use the relation (6) between $\mathcal{E}$ and $\hat{E}$, and employ the notation $\mathcal{E}_M(T) \equiv \mathcal{E}(T) + M$, where $M$ is the constant coming from (7), we obtain:
\[(T - 2\alpha - C)\mathcal{E}_M(T) \leq C(T)\int_0^T \mathcal{E}_M^2(\tau)d\tau + C(T)D_0^T\left[1 + \int_0^T \mathcal{E}_M^2(\tau)d\tau\right] + K(T),
\]
where $K(T)$ does not depend on $\mathcal{E}(0)$ and $T > \max\{1, 2\alpha + C\}$ is large enough but fixed. Using the fact that $\mathcal{E}_M(T) \leq \mathcal{E}_M(0)$, and $D_0^T = \mathcal{E}_M(0) - \mathcal{E}_M(T)$, we have
\[(T - 2\alpha - C)\mathcal{E}_M(T) \leq C(T)T\mathcal{E}_M^2(0) + C(T)[\mathcal{E}_M(0) - \mathcal{E}_M(T)][1 + T\mathcal{E}_M^2(0)] + K(T).
\]
Note that we can freely specify $C(T) > 1$ for $T$ chosen large enough. Rearranging, we have:
\[
\left[(T - 2\alpha - C) + C(T)[1 + T\mathcal{E}_M^2(0)]\right]\mathcal{E}_M(T)
\leq C(T)T\mathcal{E}_M^2(0) + C(T)[1 + T\mathcal{E}_M^2(0)]\mathcal{E}_M(0) + K(T).
\]

Then, denoting $\overline{K}(T) = \max\{C(T)T, K(T)\}$ and
\[
\eta = \frac{C(T)[1 + T\mathcal{E}_M^2(0)]}{(T - 2\alpha - C) + C(T)[1 + T\mathcal{E}_M^2(0)]},
\]
we have:
\[
\mathcal{E}_M(T) \leq \eta(\hat{E}(0), T)\mathcal{E}_M(0) + \frac{1 + \mathcal{E}_M^2(0)}{(T - 2\alpha - C) + C(T)[1 + T\mathcal{E}_M^2(0)]}\overline{K}(T).
\]

Since $C(T) > 1$ for the chosen $T$, we get
\[
\mathcal{E}_M(T) \leq \eta(\hat{E}(0), T)\mathcal{E}_M(0) + \overline{K}(T).
\]
Now, we can reiterate the same estimate on each subinterval $(mT, (m + 1)T)$ via the semigroup property. We note that the constants $\eta(\hat{E}(0), T)$ and $\overline{K}(T)$ will be same at each step. Then we obtain
\[
\mathcal{E}_M((m + 1)T) \leq \eta^m\mathcal{E}_M(mT) + \overline{K}(T)
\]
\[
\leq \eta^m\mathcal{E}_M(0) + \sum_{i=0}^m \eta^i[\overline{K}(T)]
\]
\[
\leq \eta^m\mathcal{E}_M(0) + \frac{1}{1 - \eta}\overline{K}(T)
\]
\[
\equiv \eta^m\mathcal{E}_M(0) + \hat{K}(T)
\]
As is standard, the monotonicity and continuity of the energy functional $\mathcal{E}$, and the fact that $\eta < 1$, yield
\[
\mathcal{E}_M(t) \leq \hat{K}(T) + 1, \text{ for all } t > t_0(\hat{E}(0)),
\]
which, taken together with (6), completes the proof of Theorem 3.1.
4.1.2. Proof of Theorem 3.2—Asymptotic Smoothness. Now, as the second step of the existence of the global attractor, we give the asymptotically smoothness criterion for the solutions to (1). For this, we are interested in the difference of two solutions \( z = u - w \), where \( U(t) = (u(t), u_t(t)) = S(t)y_1 \) and \( W(t) = (w(t), w_t(t)) = S(t)y_2 \) solve (1) corresponding to initial conditions \( y_1 = (u_0, u_1) \) and \( y_2 = (w_0, w_1) \) (respectively), taken from an invariant, bounded set. Then, \( z \) will solve the following problem:

\[
\begin{align*}
  z_{tt} + \Delta^2 z + F(z) &= 0 \text{ in } (0, T) \times \Omega; \quad z = 0, \quad \Delta z = -[g(\partial_{\nu}u_t) - g(\partial_{\nu}w_t)] \text{ on } \Gamma_1, \\
  z(0) &= u_0 - w_0, \quad z_t(0) = u_1 - w_1; \quad z = \partial_{\nu}z = 0 \text{ on } \Gamma_0
\end{align*}
\]

(49)

where \( F(z) \equiv f(u) - f(w) \). The energy of the system is given by

\[
E_z(t) = \frac{1}{2} \left\{ ||\Delta z||^2 + ||z_t(t)||^2 \right\}.
\]

The relation (6) between the energies gives that there exists an \( R_* \) such that the set

\[
\mathcal{W}_R \equiv \{(u_0, u_1) \in \mathcal{H} : \mathcal{B}(u_0, u_1) \leq R\}
\]

is a non-empty bounded set in \( \mathcal{H} \) for all \( R \geq R_* \). Moreover, by (6) and (8), any bounded set \( \mathcal{B} \subset \mathcal{H} \) is contained in \( \mathcal{W}_R \) for some \( R \), and the set \( \mathcal{W}_R \) is invariant with respect to \( S(t) \). Then, we consider the restriction of the dynamical system \( (S(\cdot), \mathcal{H}) \) to \( (S(\cdot), \mathcal{W}_R) \) in showing the asymptotic smoothness property, and thus we consider the solutions \( u, w \) satisfying

\[
||u(t)||_2 + ||u_t(t)||_0 + ||w(t)||_2 + ||w_t(t)||_0 \leq C(R), \quad t > 0.
\]

The proof of asymptotic smoothness of the dynamical system \( (S(t), \mathcal{H}) \) follows the same tack as in [24]. We provide some details here.

**Proof of Theorem (3.2).** As in the proof of [24, Theorem 4.4], the key point is the observability inequality given in [18, (10.5.15), p. 618]:

**Lemma 4.4.** Let \( T > 0 \) and \( \beta \in C^2(\mathbb{R}) \) be a given function satisfying (i) \( \text{supp}(\beta) \subset [\alpha, T - \alpha] \) (with \( \alpha < T/2 \)); (ii) \( 0 \leq \beta \leq 1 \) and \( \beta \equiv 1 \) on \( [\alpha, T - \alpha] \). Assume there exists an \( x_0 \) such that for \( h(x) = x - x_0 \) we have \( h \cdot \nu \leq 0 \) on \( \Gamma_0 \). Then, any generalized solution \( z \) to equation (49) satisfies

\[
\int_0^T E_z(t) \beta(t) dt \\
\leq C_1 \int_0^T E_z(t) |\beta'(t)| dt + C_2(T) \left\{ \int_0^T \int_{\Gamma_1} \left[ g(\partial_{\nu}u_t) - g(\partial_{\nu}w_t) \right] \partial_{\nu}z_t + |\partial_{\nu}z_t|^2 d\Gamma \right\} \\
+ \text{l.o.t.}(z),
\]

(50)

where

\[
\text{l.o.t.}(z) \equiv C(T, R) \left[ \sup_{[0, T]} ||z(t)||_{2-\eta}^2 + \int_0^T ||z_t(t)||_{2-\eta}^2 d\tau \right], \quad 0 < \eta < 1/2.
\]

The above inequality is proved in [18, Section 10.3] and is valid for both von Karman and Berger dynamics in the configuration of interest: (1) with (CHD) conditions. It relies on the equipartition and flux multiplier analysis on (49), and critically uses: (i) the sharp trace estimates from (4.3) for the linear plate equation, as well as (ii) the geometric condition on \( \Gamma_0 \) there.
Remark 5. In the proof of the above inequality we discarded the term
\[ \int_0^T \int_{\Gamma_1} (h \cdot \nu)|\Delta z|^2 d\Gamma d\tau \]
via the geometric condition that \( h \cdot \nu \leq 0 \) on \( \Gamma_0 \). We note that in the construction of the absorbing ball the analogous term (coming from the biharmonic term) was accommodated by utilizing \( \int_0^T \frac{d}{dt} E_{z(t)}^2d\tau \), and we dispensed there with the need for the geometric condition on \( \Gamma_0 \).

Now, by the energy relation
\[ E_z(T) + \int_t^T \int_{\Gamma_1} [g(\partial_\nu u_t) - g(\partial_\nu w_t)] (\partial_\nu z_t) + \int_t^T \mathcal{F}(z), z_t) = E_z(t), \quad T \geq t, \quad (51) \]
we have:

(i) \[ \int_0^T E_z(t)(1 - \beta(t) + |\beta'(t)|)dt \leq \int_0^T [1 - \beta(t) + |\beta'(t)|] dt \left[ E_z(T) + \int_0^T \int_{\Gamma_1} [g(\partial_\nu u_t) - g(\partial_\nu w_t)] (\partial_\nu z_t) \right] + \mathcal{F}_*(z), \]

where
\[ \mathcal{F}_*(z) \equiv \int_0^T (1 - \beta(t) - |\beta'(t)|) \int_t^T \mathcal{F}(z), z_t) d\tau dt, \]

(ii) \[ T E_z(T) \leq \int_0^T E_z(t)dt + \mathcal{F}_{**}(z), \]

where
\[ \mathcal{F}_{**}(z) \equiv - \int_0^T \int_t^T \mathcal{F}(z), z_t) d\tau dt. \]

Combining (i) and (ii), we obtain that there exists \( T_0 > 0 \), and constants \( C_1(T) \) and \( C_2(R, T) \), such that
\[ T E_z(T) + \int_0^T E_z(t)dt \leq C_1(T) \left[ \int_0^T \int_{\Gamma_1} [g(\partial_\nu u_t) - g(\partial_\nu w_t)] (\partial_\nu z_t) d\Gamma dt \right. \]
\[ + \int_0^T \int_{\Gamma_1} |\partial_\nu z_t|^2 d\Gamma dt \]
\[ + \mathcal{F}_*(z) + \mathcal{F}_{**}(z) + C_2(R, T)l.o.t.(z), \forall \quad T \geq T_0. \quad (52) \]

For detailed calculations see [18, p. 619]. In particular, we are using the fact that the cutoff function \( \beta \) may be chosen so that \( \int_0^T (1 - \beta(t) + |\beta'(t)|) dt \) is independent of \( T > 0 \). Using the assumption on the structure of the damping (Assumption 1), we have
\[ \int_0^T \int_{\Gamma_1} |\partial_\nu z_t|^2 \leq \epsilon + C(\epsilon) \int_0^T \int_{\Gamma_1} [g(\partial_\nu u_t) - g(\partial_\nu w_t)] (\partial_\nu z_t), \forall \quad \epsilon > 0. \]

On the other hand, it follows immediately from the energy relation (51) that
\[ \int_0^T \int_{\Gamma_1} [g(\partial_\nu u_t) - g(\partial_\nu w_t)] (\partial_\nu z_t) \leq E_z(0) - E_z(T) + \int_0^T \mathcal{F}(z), z_t). \]
Now, invoking the decomposition of \((\mathcal{F}(z), z_t)\) [24, Theorem 4.6] we have
\[
\left| \int_0^T (\mathcal{F}(z), z_t) dt \right| \leq C(R, T) \sup_{[0,T]} \|z\|_{\mathcal{X}}^2 + \epsilon \int_0^T E_z(s) ds, \; \eta > 0. \tag{53}
\]
As the analogous bounds on \(\mathcal{F}_*\) and \(\mathcal{F}_{**}\) follow immediately—since \(f(\cdot)\) is locally Lipschitz—we have now for \(T \geq T_0 > 1\),
\[
E_z(T) \leq \epsilon + C_1(\epsilon, T)[E_z(0) - E_z(T)] + l.o.t.(z). \tag{54}
\]
Applying the same procedure followed in the proof of [24, Theorem 4.4] (and Step 5 [18, pp. 619–620]) the desired result follows.

4.2. Further properties of the global attractor.

4.2.1. Proof of (a) of Theorem 3.4—Finite dimensionality. In the proof, a critical role is played by the following stabilizability estimate valid for the system generated by the solutions to (1).

Lemma 4.5. Let Assumptions 1 and 2 (for all \(s \in \mathbb{R}\), as discussed above) be in force. In addition, assume that \(S(t)y_1 = (u(t), u_t(t))\) and \(S(t)y_2 = (w(t), w_t(t))\) solve (1) corresponding to initial conditions \(y_1 = (u_0, 1)\) and \(y_2 = (w_0, 1)\) (respectively), taken from the set \(\mathcal{W}_R\). Then there exists positive constants \(\omega, C_1\), and \(C_2, C_3\) (both depending on \(R\)), such that
\[
\|S(t)y_1 - S(t)y_2\|_{\mathcal{X}}^2 \leq C_1 e^{-\omega t} \|y_1 - y_2\|_{\mathcal{X}}^2 + C_2 \sup_{[0,t]} \|z(t)\|_{\mathcal{X}}^2 \omega + C_3 \int_0^t e^{-\omega(t-\tau)} \|z(t)\|_{\mathcal{X}}^2 d\tau, \tag{55}
\]
and
\[
\int_0^t \int_{\Gamma_1} |\partial_\nu z_t|^2 d\Gamma d\tau + \|S(t)y_1 - S(t)y_2\|_{\mathcal{X}}^2 \leq C(T) \|y_1 - y_2\|_{\mathcal{X}}^2, \tag{56}
\]
where \(z = u - w\).

Proof of Lemma 4.5. Our beginning point will be the observability estimate (52). Since by the linear growth condition \(g'(s) \geq m_1\) for every \(s \in \mathbb{R}\), we have
\[
\int_0^T \int_{\Gamma_1} |\partial_\nu z_t|^2 d\Gamma d\tau \leq CD_0^T(z)
\]
where
\[
D_0^T(z) \equiv \int_0^T \int_{\Gamma_1} [g(\partial_\nu u_t) - g(\partial_\nu w_t)] (\partial_\nu z_t).
\]
Considering the last inequality in (52) we have
\[
TE_z(T) + (1 - \epsilon) \int_0^T E_z(t) dt \leq C_1(T) D_0^T(z) + C_2(R, T)l.o.t.(z), \; \forall \; T \geq T_0, \tag{57}
\]
which together with (51) and (53) yields
\[
E_z(T) \leq \sigma E_z(0) + C_2(T, R) \left( \sup_{[0,T]} \|z(t)\|_{\mathcal{X}}^2 \right) + \int_0^T \|z(t)\|_{\mathcal{X}}^2 d\tau, \; 0 < \eta < 1/2, \tag{58}
\]
where \(\sigma = \sigma(T) = \frac{C_1(T)}{1 + C_1(T)} < 1\). Then, the last inequality gives
\[ E_z(mT) \leq \sigma E_z((m-1)T) + l.o.t.^m(z), \quad m = 1, 2, \ldots \]

with
\[ l.o.t.^m(z) = C(\epsilon, R, T) \left[ \sup_{[(m-1)T, mT]} \|z(\tau)\|_{L^2-\eta}^2 + \int_{(m-1)T}^{mT} \|z(\tau)\|_{L^2-\eta}^2 d\tau \right] \]

After iteration, we observe that
\[ E_z(mT) \leq \sigma^m E_z(0) + \sum_{j=0}^{m-1} \sigma^j l.o.t.^m-j(z) \]

Since \( \sigma < 1 \), we obtain that there exists \( \omega > 0 \) such that for all \( t \geq 0 \),
\[ E_z(t) \leq C_1 e^{-\omega t} E_z(0) + C_2 \sup_{[0,T]} \|z(\tau)\|_{L^2-\eta}^2 + C_3 \int_0^t e^{-\omega(t-\tau)} \|z(\tau)\|_{L^2-\eta}^2 d\tau \]

which gives (55).
\[ \square \]

Recalling \( \mathcal{H} = (H_0^1 \cap H_0^{2,0})(\Omega) \times L^2(\Omega) \) we have by (55) that
\[ E_z(t) \leq C_1 e^{-\omega t} \|y_1 - y_2\|^2_{\mathcal{H}} + C_2 \sup_{[0,t]} \|z(\tau)\|^2_{(H_0^1 \cap H_0^{2,0})(\Omega)} \]
\[ + C_3 \int_0^t e^{-\omega(t-\tau)} \|z(\tau)\|^2 d\tau. \]  

Then the Intermediate Derivatives Theorem—see, e.g., Theorem 1.1 of [34]—implies that
\[ E_z(t) \leq C_1 e^{-\omega t} \|y_1 - y_2\|^2_{\mathcal{H}} + C(T) \int_0^t E_z(\tau) d\tau \]

which together with the application of Gronwall’s inequality gives the following a priori bound:
\[ E_z(t) \leq C(T) \|y_1 - y_2\|^2_{\mathcal{H}}. \]

Finally, the relation (51) and additional assumption on \( g \) yield (56). Now, to finish the proof of (a) in Theorem 3.4 we also need the next assertion:

**Lemma 4.6.** The difference of generalized solutions \( u - w = z \) (satisfying (49)) has the property that:
\[ \int_0^T \|z_{tt}(t)\|^2_{(H_0^1 \cap H_0^{2,0})(\Omega))} dt \leq C(R, T) \sup_{[0,T]} E_z(t) \leq C(R, T) \|y_1 - y_2\|^2_{\mathcal{H}} \]

where \( R \) denotes dependence of the estimate on the ball from which \( u, w \) are chosen.

**Proof.** Given \( \phi \in L^2(0, T; (H_0^1 \cap H_0^2)(\Omega)) \), we have from (49) that
\[ \int_0^t (z_{tt}(t), \phi) = - \int_0^t (\Delta^2 z, \phi) + (F(z), \phi) \]
\[ = \int_0^t (\Delta z, \Delta \phi) - \int_0^t (g(\partial_\nu u_t) - g(\partial_\nu w_t) + (\Delta w, \phi))_\Gamma + \int_0^t (F(z), \phi) \]

Estimating RHS via (56) and the locally Lipschitz property of the Berger nonlinearity gives the desired result. \( \square \)
Completion of the Proof of (a) of Theorem 3.4

In order to prove the finiteness of fractal dimension of the attractor we utilize Theorem 5.4. For this, we apply the method of “short trajectories”, inspired by \[36, 40\] (and used often in [11, 18]) and make use of Lemma 4.5 and 4.6. Let us introduce the extended space

\[ X = \mathcal{H} \times W_2(0, T) = (H^2_{T_0} \cap H^1_{T})(\Omega) \times L^2(\Omega) \times W_2(0, T) \]

where

\[ W_2(0, T) = \{ z \in L^2(0, T; (H^2_{T_0} \cap H^1_{T})(\Omega)) : z_t \in L^2(0, T; L^2(\Omega)) \} \]

endowed with the norm

\[ ||z||_{W_2(0, T)} = \int_0^T (||z||^2 + ||z_t||^2) \, dt. \]

Then the norm in \( X \) is given by

\[ ||U||^2_X = ||u_0||^2 + ||u_1||^2 + ||z||^2_{W_2(0, T)}, \quad U = (u_0, u_1, z). \]

Let \( \mathbf{A} \) be the global attractor for the semiflow \( S(t) \). Consider the set \( \mathbf{A}_T \subset X \) which is a suitable extension of \( \mathbf{A} \),

\[ \mathbf{A}_T = \{ U = (u(0), u_t(0), u(t)) : t \in [0, T] : (u(0), u_t(0)) \in \mathbf{A} \}. \]

Our aim is to show that \( \mathbf{A}_T \) has finite fractal dimension in the extended space \( X \). Because, in this case, since the operator

\[ \mathcal{P} : X \to \mathcal{H}; \quad (u_0, u_1, z(t)) \to (u_0, u_1) \]

is Lipschitz continuous with \( \mathcal{P} \mathbf{A}_T = \mathbf{A} \) this yields that

\[ \dim \mathcal{H} \mathbf{A} \leq \dim X \mathbf{A}_T < \infty, \]

which then concludes the proof.

To this end, on the set \( \mathbf{A}_T \) we define the operator

\[ V : \mathbf{A}_T \to X; (u(0), u_t(0), u(t)) \to (u(T), u_t(T), u(t + T)) \]

which is a translation by \( T \) of the solution. Now, to apply the abstract Theorem 5.4, we need to show that the map \( V \) satisfies the conditions of this theorem.

The conditions (i) and (ii) in Theorem 5.4 follow easily from the definition of the map \( V \) with the invariance property of the attractor \( \mathbf{A} \), (56) and Lemma 4.6. For the proof of (iii), we rely on the stabilizability estimate given in Lemma 4.5 and Lemma 4.6.

Invoking (55) twice–and in particular, integrating (59) from \( T \) to \( 2T \) with respect to \( t \) we obtain

\[ E_z(T) + \int_T^{2T} E_z(t) \, dt \leq C_1(T + 1)e^{-\omega T} E_z(0) + C_2(T + 1) \sup_{[T, 2T]} ||z(\tau)||^2_{(H^2_{T_0} \cap H^1_{T})(\Omega)} \]

\[ + C_3(T + 1) \int_T^{2T} e^{-\omega (T - \tau)} ||z(\tau)||^2_{H^2_{T_0} \cap H^1_{T}} \, d\tau. \]  

(61)

Now, if we choose

\[ \eta = \eta_T = C_1(T + 1)e^{-\omega T}, \quad n_1 = \sup_{[T, 2T]} ||z(\tau)||^2_{(H^2_{T_0} \cap H^1_{T})(\Omega)}, \quad n_2 = \int_T^{2T} ||z(\tau)||^2_{H^2_{T_0} \cap H^1_{T}} \, d\tau. \]
then we have that \( \eta_T < 1 \). Moreover, note that by (56), Lemma 4.6 and [42, Corollary 4, p. 85], chosen seminorm \( n_1 = \sup_{[T,T]} \|z(\tau)\|_{(H^1_0 \cap H^2_0 \cap \eta)}(\Omega) \) is compact [42], as is \( n_2 = \left( \int_T^{2T} |z(\tau)|^2 d\tau \right)^{1/2} \) by Aubin’s Compactness Lemma [1]. This finishes the proof of (a) of Theorem 3.4.

4.2.2. Proof of (b) of Theorem 3.4—Smoothness of the attractor. To prove the smoothness of the global attractor we firstly choose a full trajectory \( \gamma = \{ y(t) \equiv (u(t), u_t(t)) : t \in \mathbb{R} \} \) from this attractor. We work with the estimate (58) on the interval \([s, s + T]\), for some fixed \( T \). Let \( |\alpha| < 1 \). If we apply (58) with \( y_1(s) = y(s + T + \alpha) \) and \( y_2 = y(s + T) \) we have

\[
\|y(T + s + \alpha) - y(T + s)\|_{\mathcal{H}}^2 \leq \sigma \|y(s + \alpha) - y(s)\|_{\mathcal{H}}^2 + C_2(T, R) \left[ \sup_{\tau \in [0,T]} \|u(s + \alpha) - u(s + \alpha)\|_{\mathcal{H}}^2 - \eta \right.
\]

\[
+ \int_0^T \|u_t(s + \alpha) - u_t(s + \alpha)\|_{\mathcal{H}}^2 dt \right].
\]

Taking \( 0 < \alpha < 1 \) and denoting \( y^\alpha(s + T) = \frac{y(s + T + \alpha) - y(s + T)}{\alpha} \) we have

\[
\|y^\alpha(s + T)\|_{\mathcal{H}}^2 \leq \|y(s)\|_{\mathcal{H}}^2 + C(T, R) \left[ \sup_{\tau \in [0,T]} \|u^\alpha(s + \alpha)\|_{\mathcal{H}}^2 - \eta \right] + \int_0^T \|u^\alpha_t(s + \alpha)\|_{\mathcal{H}}^2 dt \right].
\]

For any \( \delta > 0 \), interpolation and the Hölder Inequality lead to the following relations:

\[
\|u^\alpha(\tau)\|_{\mathcal{H}}^2 \leq \delta \|u^\alpha(\tau)\|_{\mathcal{H}}^2 + C(\delta, \eta) \|u^\alpha(\tau)\|_{\mathcal{H}}^2,
\]

\[
\int_0^T \|u^\alpha_t(\tau)\|_{\mathcal{H}}^2 dt \leq \delta \sup_{\tau \in [0,T]} \|u^\alpha_t(\tau)\|_{\mathcal{H}}^2 + C(\delta, T, \eta) \int_0^T \|u^\alpha_t(\tau)\|_{(H^1_0 \cap H^2_0)(\Omega)} dt.
\]

Applying these estimates to (62), we then have

\[
\|y^\alpha(T + s)\|_{\mathcal{H}}^2 \leq \sigma \|y^\alpha(s)\|_{\mathcal{H}}^2 + \delta \sup_{\tau \in [0,T]} \|y^\alpha(s + \alpha)\|_{\mathcal{H}}^2
\]

\[
+ C(T, R, \delta, \eta) \left[ \sup_{\tau \in [0,T]} \|u^\alpha(s + \alpha)\|_{\mathcal{H}}^2 \right. + \delta \sup_{\tau \in [0,T]} \|u^\alpha_t(\tau)\|_{\mathcal{H}}^2
\]

\[
\left. + \int_0^T \|u^\alpha_t(s + \alpha)\|_{(H^1_0 \cap H^2_0)(\Omega)} dt \right].
\]

To handle the last two terms on RHS: since we are working on the attractor, an invariant set with associated bounds, we have

\[
\|u^\alpha(\tau)\| \leq \frac{1}{\alpha} \int_0^\alpha \|u_t(\tau + t)\| dt \leq \frac{1}{\alpha} \sup_{\tau \in [0,T]} \|u_t(\tau)\| \leq C(R).
\]

Moreover, as in the proof of Lemma 4.6, we have for any \( \phi \in (H^1_0 \cap H^2_0)(\Omega) \):

\[
\|(u_{tt}, \phi)\| = \|\Delta u, \Delta \phi\| + \|u_t, \phi\| + \|f(u), \phi\| \leq C(\|\Delta u\| + \|u_t\| + \|\nabla u\|^2) \|\phi\|_{\mathcal{H}}^2,
\]
and thus

$$\int_0^T \left\| u_t^2(s + \tau) \right\|_{([H_0^2 \cap H_0^{2n})(\Omega)])}^2 d\tau$$

$$\leq \frac{1}{\alpha^2} \int_0^T \left\| \int_0^\alpha u_t(s + \tau + \xi) d\xi \right\|_{([H_0^2 \cap H_0^{2n})(\Omega)])}^2 d\tau$$

$$\leq \frac{1}{\alpha^2} \int_0^T \left( \int_0^\alpha \left\| u_t(s + \tau + \xi) \right\|_{([H_0^2 \cap H_0^{2n})(\Omega)])} d\xi \right)^2 d\tau$$

$$\leq \frac{C}{\alpha^2} \int_0^T \left[ \alpha \| p \| + \alpha \sup_{\xi \in \mathbb{R}} \left\{ \| \Delta u(\xi) \| + \alpha \| \nabla u(\xi) \|^2 \right\} \right]^2 d\tau$$

$$\leq C(R, T, p). \tag{66}$$

Applying (64) and (56) to the RHS of (63), choosing $\delta(\sigma)$ sufficiently small, and taking the supremum for $s \in \mathbb{R}$, we have:

$$\sup_{s \in \mathbb{R}} \| y^\alpha(s) \|^2_{\mathcal{B}} \leq C(T, R, p).$$

We may repeat the same argument for $-1 < \alpha < 0$, and finally obtain (passing with the limit as $\alpha \to 0$):

$$\| u_t(t) \|^2_2 + \| u_{tt}(t) \|^2 \leq C(R, p),$$

for any full trajectory $\gamma = \{ y(t) \equiv (u(t), u_t(t)) : t \in \mathbb{R} \}$ from the attractor. (We have dropped the dependence of $T$, as it is arbitrary here.)

Furthermore, taking into account the boundary conditions (CHD), combining the assumptions on $g(\cdot)$, and Assumption 1 together with Mean Value Theorem, we can read off from (1) taken with (CHD) that $g(\partial_\nu u_t) \in H^{1/2}(\Gamma_1)$ and $\Delta^2 u \in L^2(\Omega)$. This translates into the following elliptic regularity problem:

$$\Delta^2 u \in L^2(\Omega), \ u = 0 \text{ on } \Gamma; \ \Delta u \in H^{1/2}(\Gamma_1), \ \partial_\nu u = 0 \text{ on } \Gamma_0.$$ 

Noting that the boundary components $\Gamma_0$ and $\Gamma_1$ are disjoint, standard elliptic regularity (e.g., [18] and [31]) yields that $u \in H^3(\Omega)$ with a bound depending only on the size of the attractor in the finite energy topology. This finishes the proof of (b) of Theorem 3.4.

**Remark 6.** We remark that the proof strategy employed above is streamlined in comparison to what is done in [18, Chapter 10.5] (which we now outline) for the non-rotational von Karman plate in the (CHD) configuration. This is due to the fact that obtaining the stabilizability estimate (55) (or its supporting estimate (58)) is more subtle there; indeed, obtaining (53) in [18] can only be done on the attractor. In [18, Chapter 10.5] this is done by appealing to the gradient nature of the dynamics (and corresponding characterization of the global attractor as the unstable manifold of the stationary points). By considering large negative times, the dynamics on the attractor are near an equilibrium point (and hence the velocity is small). This smallness is used to obtain (58), and additional smoothness of the dynamics is obtained and propagated forward in time via forward well-posedness of strong solutions (with a uniform bound for the dynamics in the higher topology obtained in a separate step). Following this argument, the additional smoothness of the attractor is used to obtain the stabilizability (55) on the attractor, and hence to obtain finite dimensionality.
Remark 7. We mention that the new monograph [11] contains a general definition of quasi-stable dynamical systems (which is broader than the definition in [18]). In essence, a quasi-stable dynamical system is one where the difference of two trajectories can be decomposed into a uniformly stable part and compact part. The theory of quasi-stable dynamical systems has been developed rather thoroughly in recent years [11, 18]. In the arguments preceding this section, we opted for a direct proof of finite dimensionality and smoothness via the so-called stabilizability estimate. However, with notions of quasi-stability in [11], it seems possible to show that the dynamical system \( (S(t), \mathcal{H}) \) above is quasi-stable on the attractor \( \mathcal{A} \). At which point, abstract theorems following the quasi-stability property lead to finite fractal dimension and smoothness of the attractor (as well as existence of an exponential attractor)\(^3\).

5. Appendix.

5.1. Trace estimate. The following theorem we use is referred to as the Trace-Moment Inequality [8].

**Theorem 5.1.** Suppose \( \Omega \) has a Lipschitz boundary and \( 1 < p < \infty \). Then there is a constant, \( C \), such that

\[
\|v\|_{L^p(\partial \Omega)} \leq C\|v\|_{L^p(\Omega)}^{1-1/p} \times \|v\|_{W^{1,p}(\Omega)}^{1/p}.
\]

(67)

In practice, we utilize it as follows:

**Corollary 1.** Suppose \( u \in H^2(\Omega) \). Then for \( \partial_{\nu} u = (\nabla u) \cdot \nu \) (valid in a collar of the boundary) we have

\[
\|\partial_{\nu} u\|_{L^2(\partial \Omega)} \leq C(\partial \Omega)\|u\|_{H^1(\Omega)}^{1/2}\|u\|_{H^2(\Omega)}^{1/2}.
\]

(68)

5.2. Long-time behavior of dynamical systems. In the context of this paper we use a few keys theorems (which we now formally state) to prove the existence of the attractor and determine its properties, as well as provide some context for other results mentioned in the discussions above. For general dynamical systems references, see [4, 18, 37, 41] (and references therein). For proofs pertinent to what is presented here, and more references, see [18].

Let \( (S_t, H) \) be a dynamical system on a complete metric space \( H \).

1. \( (S_t, H) \) is said to be dissipative iff it possesses a bounded absorbing set \( B^4 \).
   This is to say that for any bounded set \( B \), there is a time \( t_B \) so that \( S_{t_B}(B) \subset B \).

2. A dynamical system is asymptotically compact if there exists a compact set \( K \) which is uniformly attracting: for any bounded set \( D \subset H \) we have that
   \[
   \lim_{t \to +\infty} d_H(S_t D, K) = 0 \text{ in the sense of the Hausdorff semidistance.}
   \]

3. \( (S_t, H) \) is said to be asymptotically smooth if for any bounded, forward invariant \( (t > 0) \) set \( D \) there exists a compact set \( K \subset \bar{D} \) which is uniformly attracting with respect to \( D \) (as in the preceding definition) (see [18, p. 338]).

4. A global attractor \( \mathcal{A} \) is a closed, bounded set in \( H \) which is invariant (i.e. \( S_t \mathcal{A} = \mathcal{A} \) for all \( t > 0 \)) and uniformly attracting (see [18, p. 344]).

\(^3\)The slightly less recent quasi-stability analysis in [18] does not have a broad enough definition of quasi-stability to encompass situations like the one considered here.

\(^4\)In other references, to avoid ambiguity with a dissipative operator, authors use the phrase ultimately dissipative.
We recall the following useful criterion (first appearing [29] and stated in the present version in [18]) for the asymptotic smoothness:

**Theorem 5.2** ([16]—Proposition 2.10). Let \((S(t), H)\) be a dynamical system, \(H\) a Banach space with norm \(\| \cdot \|\). Assume that for any bounded positively invariant set \(B \subset H\) and for all \(\epsilon > 0\) there exists a \(T = T_{\epsilon,B}\) such that
\[
\|S_T x_1 - S_T x_2\|_H \leq \epsilon + \Psi_{\epsilon,T}(x_1, x_2), \quad x_i \in B
\]
with \(\Psi\) a functional defined on \(B \times B\) depending on \(\epsilon, T,\) and \(B\) such that
\[
\liminf_m \liminf_n \Psi_{\epsilon,T,B}(x_m, x_n) = 0
\]
for every sequence \(\{x_n\} \subset B\). Then \((S, H)\) is an asymptotically smooth dynamical system.

Now, we give the well known criterion for the existence of the global attractors:

**Theorem 5.3.** Let \((S_t, H)\) be a dissipative dynamical system in a complete metric space \(H\). Then \((S_t, H)\) possesses a compact global attractor \(A\) if and only if \((S_t, H)\) is asymptotically smooth.

In the theory of infinite-dimensional dynamical systems, finite fractal dimensionality is proved by an approach which is related to the squeezing property. This useful tool is given by the following theorem:

**Theorem 5.4.** Let \(H\) be a separable Hilbert space and \(M\) be a bounded closed set in \(H\). Assume that there exists a mapping \(V : M \rightarrow H\) such that
i) \(M \subseteq VM\),
ii) \(V\) is Lipschitz on \(M\); that is, there exists \(L > 0\) such that
\[
\|V v_1 - V v_2\| \leq L \|v_1 - v_2\|, \quad v_1, v_2 \in M,
\]
iii) There exists compact seminorms \(n_1(x)\) and \(n_2(x)\) on \(H\) such that
\[
\|V v_1 - V v_2\| \leq \eta \|v_1 - v_2\| + K [n_1(v_1 - v_2) + n_2(V v_1 - V v_2)]
\]
for any \(v_1, v_2 \in M\), where \(0 < \eta < 1\) and \(K > 0\) are constants. Here, a seminorm \(n(x)\) on \(H\) is said to be compact iff \(n(x_n) \rightarrow 0\) for any sequence \(x_n \subset H\) such that \(x_n \rightarrow 0\) weakly in \(H\). Then \(M\) is a compact set in \(H\) of finite fractal dimension.

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**REFERENCES**

[1] J. P. Aubin, Une théorème de compacité, C.R. Acad. Sci. Paris, **256** (1963), 5042–5044.
[2] G. Avalos and I. Lasiecka, Exponential stability of a thermoelastic system without mechanical dissipation, Rend. Ist. Mat. Univ. Trieste, **28** (1997), 1–28.
[3] G. Avalos and I. Lasiecka, Boundary controllability of thermoelastic plates via the free boundary conditions, SIAM J. Control. Optim., **38** (2000), 337–383.
[4] A. Babin and M. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.
[5] J. M. Ball, Global attractors for damped semilinear wave equations, Discrete Cont. Dyn. Sys, **10** (2004), 31–52.
[6] H. M. Berger, A new approach to the analysis of large deflections of plates, J. Appl. Mech., **22** (1955), 465–472.
[7] V. V. Bolotin, *Nonconservative Problems of Elastic Stability*, Pergamon Press, Oxford, 1963.
S. C. Brenner and R. Scott, The Mathematical Theory of Finite Element Methods, 15, Springer Science & Business Media, 2008.

F. Bucci and I. Chueshov, Global attractor for a composite system of nonlinear wave and plate equations, Comm. Pure and Appl. Anal., 6 (2007), 113–140.

F. Bucci and I. Chueshov, Long-time dynamics of a coupled system of nonlinear wave and thermoelastic plate equations, Dynam. Sys., 22 (2008), 557–586.

I. Chueshov, Dynamics of Quasi-Stable Dissipative Systems, Springer, 2015.

I. Chueshov, Long-time dynamics of Kirchhoff wave models with strong nonlinear damping, J. Diff. Equ., 252 (2012), 1229–1262.

I. Chueshov, Introduction to the Theory of Infinite Dimensional Dissipative Systems, Acta, Kharkov, 1999, in Russian; English translation: Acta, Kharkov, 2002; http://www.emis.de/monographs/Chueshov/

I. Chueshov, M. Eller and I. Lasiecka, Finite dimensionality of the attractor for a semilinear wave equation with nonlinear boundary dissipation, Comm. PDE, 29 (2004), 1847–1976.

I. Chueshov and I. Lasiecka, Global attractors for von Karman evolutions with a nonlinear boundary dissipation, J. Differ. Equ., 198 (2004), 196–231.

I. Chueshov and I. Lasiecka, Long-time behavior of second-order evolutions with nonlinear damping, Memoires of AMS, 195, 2008.

I. Chueshov and I. Lasiecka, Long-time dynamics of von Karman semi-flows with non-linear boundary/interior damping, J. Differ. Equ., 233 (2008), 42–86.

I. Chueshov and I. Lasiecka, Von Karman Evolution Equations, Springer-Verlag, 2010.

I. Chueshov, I. Lasiecka and D. Toundykov, Global attractor for a wave equation with nonlinear localized boundary damping and a source term of critical exponent, J. Dyn. Diff. Equ., 21 (2009), 269–314.

I. Chueshov, I. Lasiecka and J. T. Webster, Attractors for delayed, non-rotational von Karman plates with applications to flow-structure interactions without any damping, Comm. in PDE, 39 (2014), 1965–1997.

P. Ciarlet and P. Rabier, Les Equations de Von Karman, Springer, 1980.

A. Eden and A. J. Milani, Exponential attractors for extensible beam equations, Nonlinearity, 6 (1993), 457–479.

P. Fabrie, C. Galusinski, A. Miranville and S. Zelik, Uniform exponential attractors for a singularly perturbed damped wave equation, Discrete Cont. Dyn. Sys, 10 (2004), 211–238.

P. G. Geredeli and J. T. Webster, Qualitative results on the dynamics of a Berger plate with nonlinear boundary damping, Nonlin. Anal: Real World Applications, 31 (2016), 227–256.

P. G. Geredeli, I. Lasiecka and J. T. Webster, Smooth attractors of finite dimension for von Karman evolutions with nonlinear frictional damping localized in a boundary layer, J. Diff. Eqs., 254 (2013), 1193–1229.

P. G. Geredeli and J. T. Webster, Decay rates to equilibrium for nonlinear plate equations with geometrically constrained, degenerate dissipation, Appl. Math. and Optim., 68 (2013), 361–390. Erratum, Appl. Math. and Optim., 70 (2014), 565–566.

J. K. Hale and G. Raugel, Attractors for dissipative evolutionary equations, In International Conference on Differential Equations (Vol. 1, p. 2), 1993, World Scientific River Edge, NJ.

G. Ji and I. Lasiecka, Nonlinear boundary feedback stabilization for a semilinear Kirchhoff plate with dissipation acting only via moments-limiting behavior, JMAA, 229 (1999), 452–470.

A. Kh. Khanmamedov, Global attractors for von Karman equations with non-linear dissipation, J. Math. Anal. Appl, 318 (2006), 92–101.

J. Lagnese, Boundary Stabilization of Thin Plates, SIAM, 1989.

I. Lasiecka and R. Triggiani, Control Theory for Partial Differential Equations, Cambridge University Press, Cambridge, 2000.

I. Lasiecka and R. Triggiani, Sharp trace estimates of solutions to Kirchhoff and Euler-Bernoulli equations, Appl. Math Optim, 28 (1993), 277–306.

V. Kalantarov and S. Zelik, Finite-dimensional attractors for the quasi-linear strongly-damped wave equation, J. Diff. Equ., 247 (2009), 1120–1155.

J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer, 1971.

J. L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, Vol. I, Masson, Paris, 1989.
[36] J. Málek and D. Pražák, Large time behavior via the method of I-trajectories, *J. Diff. Eqs.*, 181 (2002), 243–279.

[37] A. Miranville and S. Zelik, Attractors for dissipative partial differential equations in bounded and unbounded domains, in *Handbook of Differential Equations: Evolutionary Equations* (M. C. Dafermos and M. Pokorny eds.) v.4, Elsevier, Amsterdam, 2008.

[38] V. Pata and S. Zelik, Smooth attractors for strongly damped wave equations, *Nonlinearity*, 19 (2006), 1495–1506.

[39] J.-P. Puel and M. Tucsnak, Boundary stabilization for the von Karman equations, *SIAM J. Control and Optim.*, 33 (1995), 255–273.

[40] D. Pražák, On finite fractal dimension of the global attractor for the wave equation with nonlinear damping, *J. Dyn. Diff. Eqs.*, 14 (2002), 764–776.

[41] G. Raugel, Global attractors in partial differential equations, in *Handbook of Dynamical Systems* (B. Fiedler ed.), v. 2, Elsevier Sciences, Amsterdam, 2002.

[42] J. Simon, Compact sets in the space $L^p(0, T; B)$, *Annali di Matematica pura ed applicata IV*, CXLVI (1987), 65–86.

[43] C. P. Vendhan, A study of Berger equations applied to nonlinear vibrations of elastic plates, *Int. J. Mech. Sci.*, 17 (1975), 461–468.

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