An elliptic partial differential equations system and its application

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Abstract
This paper deals with the following elliptic system of equations

\[
\begin{aligned}
  -\frac{k_1}{2} \Delta z_1(x) + \frac{|\nabla z_1(x)|^2}{2} &= f_1(x) - (\lambda_1 + a_1)z_1(x) + a_1z_2(x), \\
  -\frac{k_2}{2} \Delta z_2(x) + \frac{|\nabla z_2(x)|^2}{2} &= f_2(x) - (\lambda_2 + a_2)z_2(x) + a_2z_1(x),
\end{aligned}
\]

\[x \in \mathbb{R}^N,\]

(1)

where \(\lambda_i > 0\) (\(i = 1, 2\)) are some real constants suitable chosen and \(k_i > 0\), \(a_i > 0\) (\(i = 1, 2\)) are some real arbitrary constants, and \(f_i\) are some continuous functions. The solution method is based on the sub- and super-solutions approach. An application to a stochastic control problem is presented. This system seemed not considered before.

1 Introduction

In this work we study the existence of positive solutions for the following partial differential equations (PDE) system

\[
\begin{aligned}
  -\frac{k_1}{2} \Delta z_1(x) + \frac{|\nabla z_1(x)|^2}{2} &= f_1(x) - (\lambda_1 + a_1)z_1(x) + a_1z_2(x), \\
  -\frac{k_2}{2} \Delta z_2(x) + \frac{|\nabla z_2(x)|^2}{2} &= f_2(x) - (\lambda_2 + a_2)z_2(x) + a_2z_1(x),
\end{aligned}
\]

\[x \in \mathbb{R}^N.\]

(2)

Here \(N \geq 1\) is the space dimension, \(|\cdot|\) is the Euclidean norm of \(\mathbb{R}^N\), \(f_i : \mathbb{R}^N \to [0, \infty)\) (\(i = 1, 2\)) are locally Hölder continuous of exponent \(\alpha \in (0, 1)\) convex functions satisfying

\[
\text{there exists } M_i > 0 \text{ such that } f_i \left( \right) \leq M_i \left( |x|^2 + 1 \right),
\]

(3)

\(\lambda_i > 0\) (\(i = 1, 2\)) are some real constants suitable chosen and \(k_i > 0\), \(a_i > 0\) (\(i = 1, 2\)) are some real arbitrary constants.
This system has received much attention in the last decades since it is related with several models that arise in different mathematical models of natural phenomena; for more on this see the papers of Akella and Kumar [1], Alvarez [2], Bensoussan, Sethi, Vickson and Derzko [5], the authors [9], Ghosh, Arapostathis and Marcus [15] and Lasry and Lions [17].

The system studied in this paper appears naturally in characterizing value function of a stochastic control problem with regime switching. Regime switching is a phenomena present in many real world problems. Let us point to some relevant works where regime switching is present. In finance [20] studies the effect of regime switching (high versus low discount rates) to a consumption and investment problem. In production management [6] studies the cost minimization problem of a company within an economy characterized by two regimes. In civil engineering [10] studies the optimal stochastic control problem for home energy systems with regime switching; the two regimes are the peak and off peak energy demand.

The principal device in studying this system comes from the recent work of [8], where the author obtained non-positive radial solutions for the system (2) and where we postulate an open problem regarding the existence of positive solution for this system. Another goal of this paper is to improve the model given in [5], [8], [15] and to give a verification result, i.e., show that the solution of the system yields the optimal control.

Furthermore, there seems to be no previous mathematical results about the existence of positive solutions for the semilinear system (2). This should not surprise us since there are some difficulties in analyzing this class of systems in $\mathbb{R}^N$ ($N \geq 1$), which will be revealed in the following sections organized as follows. In Section 2 we give our main theorem regarding the existence of positive solution for the problem (2) and its proof. Section 3 contains the context and the diffusion model from where such system appear. Section 4 presents a verification result. In Section 5 we obtain a closed form solution for our system in a special case.

2 Main Result

The main result of our paper, a basic existence theorem for (2), is presented.

**Theorem 1.** There exist $\lambda_1^*, \lambda_2^* \in (0, \infty)$ such that for all $\lambda_1 \geq \lambda_1^*$ and $\lambda_2 \geq \lambda_2^*$ the system of equations (2) has a positive classical convex solution with quadratic growth, i.e.,

$$z_i(x) \leq K_i(1 + |x|^2), \text{ for some } K_i > 0, \quad i = 1, 2,$$

and, such that

$$|\nabla z_i(x)| \leq C_i(1 + |x|), \text{ for } x \in \mathbb{R}^N \text{ and for some positive constant } C_i,$$

(4)

We give a detailed proof of Theorem 1 which is based on the following two results.
Lemma 2. The system of partial differential equations with gradient term \( \Delta u = u(x)(f_1(x) + (\lambda_1 + a_1)k_1 \ln u - a_1 k_2 \ln v), \) \( \Delta v = v(x)(f_2(x) + (\lambda_2 + a_2)k_2 \ln v - a_2 k_1 \ln u), \) \( x \in \mathbb{R}^N. \) (6)

**Proof.** The change of variable

\[ z_1(x) = k_1 w_1(x) \quad \text{and} \quad z_2(x) = k_2 w_2(x), \]

transform the system (2) into

\[
\begin{align*}
-\frac{k_1^2}{2} \Delta w_1 + \frac{k_1^2}{2} |\nabla w_1|^2 &= f_1(x) - (\lambda_1 + a_1) k_1 w_1 + a_1 k_2 w_2, \\
-\frac{k_2^2}{2} \Delta w_2 + \frac{k_2^2}{2} |\nabla w_2|^2 &= f_2(x) - (\lambda_2 + a_2) k_2 w_2 + a_2 k_1 w_1,
\end{align*}
\]

or, equivalently

\[
\begin{align*}
-\Delta w_1 + |\nabla w_1|^2 &= \frac{2}{k_1^2} [f_1(x) - (\lambda_1 + a_1) k_1 w_1 + a_1 k_2 w_2], \\
-\Delta w_2 + |\nabla w_2|^2 &= \frac{2}{k_2^2} [f_2(x) - (\lambda_2 + a_2) k_2 w_2 + a_2 k_1 w_1].
\end{align*}
\]

The change of variable

\[ u(x) = e^{-w_1(x)} \quad \text{and} \quad v(x) = e^{-w_2(x)}, \]

transform the system (3) into

\[
\begin{align*}
\Delta u &= u(x) \left( f_1(x) + (\lambda_1 + a_1) k_1 \ln u - a_1 k_2 \ln v \right), \\
\Delta v &= v(x) \left( f_2(x) + (\lambda_2 + a_2) k_2 \ln v - a_2 k_1 \ln u \right),
\end{align*}
\]

since

\[
\begin{align*}
\Delta u(x) &= e^{-w_1(x)}(-\Delta w_1(x) + |\nabla w_1(x)|^2), \\
\Delta v(x) &= e^{-w_2(x)}(-\Delta w_2(x) + |\nabla w_2(x)|^2).
\end{align*}
\]

The existence of a solution \((u(x), v(x)) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)\) for the problem (3), such that \(0 < u(x) \leq 1\) and \(0 < v(x) \leq 1\), for all \(x \in \mathbb{R}^N\), is proved in the following:

**Theorem 3.** If there exist functions \(u, v, \overline{u}, \overline{v} : \mathbb{R}^N \rightarrow (0, 1)\) of class \(C^2(\mathbb{R}^N)\) such that

\[
\begin{align*}
-\Delta u(x) + u(x) \left[ \frac{k_1^2}{2} (f_1(x) + (\lambda_1 + a_1) k_1 \ln u(x)) \right] &\leq 2a_1 \frac{k_1^2}{2} u(x) \ln u(x), \\
-\Delta v(x) + v(x) \left[ \frac{k_2^2}{2} (f_2(x) + (\lambda_2 + a_2) k_2 \ln v(x)) \right] &\leq 2a_2 \frac{k_2^2}{2} v(x) \ln v(x), \\
-\Delta \overline{u}(x) + \overline{u}(x) \left[ \frac{k_1^2}{2} (f_1(x) + (\lambda_1 + a_1) k_1 \ln \overline{u}(x)) \right] &\geq 2a_1 \frac{k_1^2}{2} \overline{u}(x) \ln \overline{u}(x), \\
-\Delta \overline{v}(x) + \overline{v}(x) \left[ \frac{k_2^2}{2} (f_2(x) + (\lambda_2 + a_2) k_2 \ln \overline{v}(x)) \right] &\geq 2a_2 \frac{k_2^2}{2} \overline{v}(x) \ln \overline{v}(x),
\end{align*}
\]

in the entire Euclidean space \(\mathbb{R}^N\), then system (4) possesses an entire solution \((u, v) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)\) with \(u(x) \leq u(x) \leq \overline{u}(x)\) in \(\mathbb{R}^N\) and \(v(x) \leq v(x) \leq \overline{v}(x)\) in \(\mathbb{R}^N\).

Let us point out that the functions \((u, v)\) (resp. \((\overline{u}, \overline{v})\)) are called sub-solution (resp. super-solution) for the system (4).
Proof. In the following we construct the functions \((u, v)\), \((\overline{u}, \overline{v})\) which satisfies the inequalities in \(\mathbb{R}^N\). We proceed as in Bensoussan, Sethi, Vickson and Derzko, for the scalar case. More exactly, we observe that there exist
\[
(u(x), v(x)) = \left(e^{m_1(|x|^2+1)}, e^{m_2(|x|^2+1)}\right), \quad \text{with } m_1, m_2 \in (-\infty, 0),
\]
and \(\lambda_1^*, \lambda_2^* \in (0, \infty)\) such that for all \(\lambda_1 \geq \lambda_1^*\) and \(\lambda_2 \geq \lambda_2^*\) the following hold
\[
\begin{align*}
-\Delta u(x) + u(x) \left[\frac{2}{k_1} \left(f_1(x) + (\lambda_1 + a_1) k_1 \ln u(x)\right)\right] &\leq 2a_1 \frac{k_1}{k_2} u(x) \ln u(x), \\
-\Delta v(x) + v(x) \left[\frac{2}{k_2} \left(f_2(x) + (\lambda_2 + a_2) k_2 \ln v(x)\right)\right] &\leq 2a_2 \frac{k_2}{k_1} v(x) \ln v(x),
\end{align*}
\]
and
\[
\begin{align*}
|\nabla u(x)|^2 &\leq 4m_1^2 + \frac{4M_1}{k_1} + (\lambda_1 + a_1) k_1 m_1 - 2a_1 \frac{mk_{1k_2}}{k_1} + 4m_1^2 - 2m_1 N \leq 0, \\
|\nabla v(x)|^2 &\leq 4m_2^2 + \frac{4M_2}{k_2} + (\lambda_2 + a_2) k_2 m_2 - 2a_2 \frac{mk_{2k_1}}{k_2} + 4m_2^2 - 2m_2 N \leq 0.
\end{align*}
\]

Then, we can choose for example \(m_1, m_2 \in (-\infty, 0)\) such that
\[
\begin{align*}
\lambda_1^* &\geq \max \left\{2a_1 \frac{mk_{1k_2}}{m_1k_1} + 4m_1^2 - \frac{2M_1}{m_1k_1} - a_1, 2a_1 \frac{mk_{1k_2}}{m_1k_1} - 4m_1^2 + \frac{4M_1}{k_1} - \frac{2M_1}{m_1k_1} - a_1\right\}, \\
\lambda_2^* &\geq \max \left\{2a_2 \frac{mk_{2k_1}}{m_2k_2} + 4m_2^2 - \frac{2M_2}{m_2k_2} - a_2, 2a_2 \frac{mk_{2k_1}}{m_2k_2} - 4m_2^2 + \frac{4M_2}{k_2} - \frac{2M_2}{m_2k_2} - a_2\right\},
\end{align*}
\]
are positive and the inequalities in (12) hold.

To construct a super-solution it is useful to remember that \(\ln 1 = 0\) and then a simple calculation shows that
\[
(\overline{u}(x), \overline{v}(x)) = (1, 1),
\]
is a super-solution of the problem (11).

Until now, we constructed the corresponding sub- and super-solutions employed in the scalar case by [5]. Clearly, (11) holds and then in Theorem 3 it remains to prove that there exists \(u(x), v(x) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)\) with \(u(x) \leq u(x) \leq \overline{u}(x) \in \mathbb{R}^N\) and \(u(x) \leq v(x) \leq \overline{v}(x) \in \mathbb{R}^N\) satisfying (11).

To do this, let \(B_k\) be the ball whose center is the origin of \(\mathbb{R}^N\) and which has radius \(k = 1, 2, \ldots\). We consider the boundary value problem
\[
\begin{align*}
\Delta u &= u \left[\frac{2}{k_1^2} \left(f_1(x) + (\lambda_1 + a_1) k_1 \ln u - a_1 k_2 \ln v\right)\right], \quad x \in B_k, \\
\Delta v &= v \left[\frac{2}{k_2^2} \left(f_2(x) + (\lambda_2 + a_2) k_2 \ln v - a_2 k_1 \ln u\right)\right], \quad x \in B_k, \\
u(x) &= \underline{u}_k(x), \quad v(x) = \underline{v}_k(x), \quad x \in \partial B_k,
\end{align*}
\]
where \(\underline{u}_k = \underline{u}|_{\partial B_k}\) and \(\underline{v}_k = \underline{v}|_{\partial B_k}\). In a similar way, we define \(\overline{u}_k = \overline{u}|_{\partial B_k}\) and \(\overline{v}_k = \overline{v}|_{\partial B_k}\) then \(\underline{u}_k, \overline{u}_k, \underline{v}_k, \overline{v}_k \in C^2(\overline{B_k})\).
Observing that
\[
\inf_{x \in \mathbb{R}^N} u(x) \leq \min_{x \in \mathbb{R}^N} u_k(x) \quad \text{and} \quad \sup_{x \in \mathbb{R}^N} u(x) \geq \max_{x \in \mathbb{R}^N} \bar{u}_k(x),
\]
\[
\inf_{x \in \mathbb{R}^N} v(x) \leq \min_{x \in \mathbb{R}^N} v_k(x) \quad \text{and} \quad \sup_{x \in \mathbb{R}^N} v(x) \geq \max_{x \in \mathbb{R}^N} \bar{v}_k(x),
\]
a result of Reis Gaete [14] (see also the pioneering papers of Kawano [13] and Lee, Shivaji and Ye [15]), proves the existence of a solution \((u_k, v_k) \in \left[C^2(B_n) \cap C(B_n)\right]^2\) satisfying the system (14). The functions \((u_k, v_k)\) also satisfy
\[
\begin{align*}
&u_k(x) \leq u_k(x) \leq \bar{u}_k(x), \quad x \in \overline{B}_n, \\
v_k(x) \leq v_k(x) \leq \bar{v}_k(x), \quad x \in \overline{B}_n.
\end{align*}
\]

By a standard regularity argument based on Schauder estimates, see Tolksdorf [22, 17, proposition 3.7, p. 806] and Reis Gaete [14] for details, we can see that for all integers \(k \geq n + 1\) there are \(\alpha_1, \alpha_2 \in (0, 1)\) and positive constants \(C_1, C_2, C_3\) independent of \(k\) such that
\[
\begin{align*}
&u_k \in C^{2, \alpha_1}(\overline{B}_n) \quad \text{and} \quad |u_k|_{C^{2, \alpha_1}(\overline{B}_n)} < C_1, \\
v_k \in C^{2, \alpha_2}(\overline{B}_n) \quad \text{and} \quad |v_k|_{C^{2, \alpha_2}(\overline{B}_n)} < C_2,
\end{align*}
\]
where \(|\cdot|_{C^{2, \alpha}}\) is the usual norm of the space \(C^{2, \alpha}(\overline{B}_n)\). Moreover, there exist constants: \(C_3\) independent of \(u_k\), \(C_4\) independent of \(v_k\) and such that
\[
\begin{align*}
&\max_{x \in \overline{B}_n} |\nabla u_k(x)| \leq C_3 \max_{x \in \overline{B}_n} |u_k(x)|, \\
&\max_{x \in \overline{B}_n} |\nabla v_k(x)| \leq C_4 \max_{x \in \overline{B}_n} |v_k(x)|,
\end{align*}
\]
The information from (15) and (16) implies that \(((\nabla u_k, \nabla v_k))_k\) as well as \(((u_k, v_k))_k\) are uniformly bounded on \(\overline{B}_n\). We wish to show that this sequence \(((u_k, v_k))_k\) contains a subsequence converging to a desired entire solution of (8). Next, we concentrate our attention to the sequence \((u_k)_k\).

Using the compactness of the embedding \(C^{2, \alpha_1}(\overline{B}_n) \hookrightarrow C^2(\overline{B}_n)\), enables us to define the subsequence
\[
u_n^k := u_k|_{\overline{B}_n}, \quad \text{for all } k \geq n + 1.
\]

Then for \(n = 1, 2, 3, ...\) there exist a subsequence \((u_n^{k_n})_{k \geq n + 1, j \geq 1}\) of \((u_n^k)_{k \geq n + 1}\) and a function \(u_n\) such that
\[
u_n^{k_n} \rightarrow u_n,
\]
uniformly in the \(C^2(\overline{B}_n)\) norm. More exactly, we get through a well-known diagonal process that
\[
\begin{align*}
u_1^{k_1}, \nu_1^{k_2}, \nu_1^{k_3}, \ldots & \rightarrow u_1 \text{ in } C^2(\overline{B}_1), \\
u_2^{k_1}, \nu_2^{k_2}, \nu_2^{k_3}, \ldots & \rightarrow u_2 \text{ in } C^2(\overline{B}_2), \\
u_3^{k_1}, \nu_3^{k_2}, \nu_3^{k_3}, \ldots & \rightarrow u_3 \text{ in } C^2(\overline{B}_3), \\
\ldots
\end{align*}
\]
Since $\mathbb{R}^N = \bigcup_{n=1}^{\infty} B_n$, we can define the function $u : \mathbb{R}^N \to [0, \infty)$ such that

$$u(x) = \lim_{n \to \infty} u_n(x).$$

Let us give the construction of the function $u$ for the problem (10). This is obtained by considering the sequence $(u_d^{k, d})_{d \geq 1}$ and the sequence $(u_n^{k, d})_{k \geq n+1}$, restricted to the ball $B_n$, which are such that

$$u_n^{k, d} \xrightarrow{d \to \infty} u_n := u(x) \text{ for all } x \in B_n,$$

and then, for $d \to \infty$ we obtain

$$u_d^{k, d} \xrightarrow{d \to \infty} u(x) \text{ in } C^2(\mathbb{R}^N),$$

according with the diagonal process. Furthermore, since

$$u(x) \leq u_d^{k, d} \leq \varpi(x), \text{ for } x \in \mathbb{R}^N,$$

and for each $d = 1, 2, 3, \ldots$ the following relation is valid

$$u(x) \leq u(x) \leq \varpi(x), \text{ for } x \in \mathbb{R}^N.$$

We employ the same iteration scheme to construct the function $v : \mathbb{R}^N \to [0, \infty)$ such that

$$v(x) = \lim_{n \to \infty} v_n(x).$$

From the regularity theory the solution $(u, v)$ belongs to $C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ and satisfies (10). This completes the proof of Theorem 3.

**Proof of Theorem 1** As easily verified, the existence of solutions is proved by Lemma 2 and Theorem 3. Then it remains to prove (4).

A recapitulation of the changes of variables say that

$$z_1(x) = -k_1 \ln u(x) \text{ and } z_2(x) = -k_2 \ln v(x), \quad (18)$$

is a solution for (2). Observing that

$$u(x) = e^{m_1(|x|^2 + 1)} \leq \ln u(x) \leq \ln 1, \quad x \in \mathbb{R}^N,$$

it follows that

$$m_1(|x|^2 + 1) \leq \ln u(x) \leq \ln 1,$$

and, then

$$0 \leq -k_1 \ln u(x) \leq -k_1 m_1(|x|^2 + 1),$$

or equivalently

$$0 \leq z_1(x) \leq K_1(|x|^2 + 1), \text{ for } x \in \mathbb{R}^N \text{ and } K_1 = -k_1 m_1.$$

In the same way

$$0 \leq z_2(x) \leq K_2(|x|^2 + 1), \text{ for } x \in \mathbb{R}^N \text{ and } K_2 = -k_2 m_2,$$

and the proof is completed.

By the same arguments as in [3] Theorem 3, p. 278 the solution $(z_1(x), z_2(x))$ is convex. Since $(z_1(x), z_2(x))$ verifies (11) the inequality (10) follows from [11] Lemma 1, p. 24 (see also the arguments in [12] Theorem 1, p. 236). The proof is completed.
3 Context and the Diffusion Model

Let us present the setting. Consider $W$ a $N$-dimensional Brownian motion on a filtered probability space

$$(\Omega, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathcal{F}, P),$$

(19)

where $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is a completed filtration, and $T = \infty$ (we deal with the infinite horizon case). We allow for regime switching in our model; regime switching refers to the situation when the characteristics of the state process are affected by several regimes (e.g. in finance bull and bear market with higher volatility in the bear market). The regime switching is captured by a continuous time homogeneous Markov chain $\epsilon(t)$ adapted to $\mathcal{F}_t$ with two regimes good and bad, i.e., for every

$$t \in [0, \infty) \text{ and } \epsilon(t) \in \{1, 2\}.$$ 

In a specific application, $\epsilon(t) = 1$ could represent a regime of economic growth while $\epsilon(t) = 2$ could represent a regime of economic recession. In another application, $\epsilon(t) = 1$ could represent a regime in which consumer demand is high while $\epsilon(t) = 2$ could represent a regime in which consumer demand is low.

The Markov chain’s rate matrix is

$$A = \begin{pmatrix} -a_1 & a_1 \\ a_2 & -a_2 \end{pmatrix},$$

(20)

for some $a_1 > 0, a_2 > 0$. Diagonal elements $A_{ii}$ are defined such that

$$A_{ii} = -\sum_{j \neq i} A_{ij},$$

(21)

where

$$A_{11} = -a_1, A_{12} = a_1, A_{21} = a_2, A_{22} = -a_2.$$ 

In this case, if $p_t = \mathbb{E}[\epsilon(t)] \in \mathbb{R}^2$, then

$$\frac{dp(t)}{dt} = A \epsilon(t).$$

(22)

Moreover

$$\epsilon(t) = \epsilon(0) + \int_0^t A \epsilon(u) \, du + M_t,$$

(23)

where $M(t)$ is a martingale with respect to $\mathcal{F}_t$.

Let us consider a Markov modulated controlled diffusion with controls in feedback form

$$dX^i(t) = c^i_{\epsilon(t)}(X(t)) \, dt + k_{\epsilon(t)} \, dW^i(t), i = 1, \ldots, N,$$

(24)

for some constants $k_1 > 0, k_2 > 0$, and $X(0) = x \in \mathbb{R}^N$. Here, at every time $t$, the control $c_{\epsilon(t)}$ (for instance the demand of certain items), and the volatility $k_{\epsilon(t)}$ depend on the regime $\epsilon(t)$. We allow the demand to take on negative values, which represent items return (due to spoilage). We consider the class of admissible controls, $\mathcal{A}$, which are the feedback controls for which the SDE (24) has a unique strong solution.
The infinitesimal generator \( L \) of diffusion \( X \) is second order differential operator defined by

\[
L^c v(x, 1) = \frac{1}{2} k_1 \Delta v(x, 1) + c_1 \nabla v(x, 1) + A_{11} v(x, 1) + A_{12} v(x, 2),
\]

(25)

\[
L^c v(x, 2) = \frac{1}{2} k_2 \Delta v(x, 2) + c_2 \nabla v(x, 2) + A_{22} v(x, 2) + A_{21} v(x, 1),
\]

(26)

(see [18] for more on this). Following this we can state Itô’s formula for Markov modulated diffusion

\[
dv(X(t), \epsilon(t)) = L^c v(X(t), \epsilon(t)) dt + k_{\epsilon(t)} \nabla v(X(t), \epsilon(t)) dW(t).
\]

(27)

Next, for each \( c \in A \) the cost functional is defined by

\[
J(x, c, i) = E\left[ \int_0^\infty e^{-\lambda_i t} [f_{\epsilon(t)}(X(t)) + \frac{1}{2} |c|^2 \epsilon(t)](X(t)) dt \mid \epsilon(0) = i \right].
\]

(28)

Our objective is to minimize the functional \( J \), i.e. determine the value function

\[
z_i(x) = \inf_{c \in A} J(x, c, i),
\]

(29)

and to find the optimal control. The infimum is taken over all admissible controls \( c \in A \). Notice that the discount rate depends on the regime; for more on this modelling approach see [20].

In order, to obtain the HJB equation, we apply the martingale/supermartingale principle; search for a function \( u(x, i) \) such that the stochastic process \( M^c(t) \) defined below

\[
M^c(t) = e^{-\lambda_i t} u(X(t), \epsilon(t)) - \int_0^t e^{-\lambda_i u} [f_{\epsilon(t)}(X(u)) + \frac{1}{2} |c|^2 \epsilon(u)](X(u)) du.
\]

(30)

is supermartingale and martingale for the optimal control. If this is achieved together with the following transversality condition

\[
\lim_{t \to \infty} E[e^{-\lambda_i t} u(X(t), \epsilon(t))] = 0,
\]

(31)

and some estimates on the value function yield that

\[
z_i(x) = -u(x, i) = \inf_{c \in A} J(x, c, i).
\]

(32)

The proof of this statement is done in the Verification subsection.

The supermartingale/martingale requirement leads to the following HJB equation

\[
\frac{k_j}{2} \Delta u(x, i) + \sup_{c \in A} (\nabla u(x, i) c - \frac{|c|^2}{2}) = f_i(x) + (\lambda_i + a_i) u(x, i) - a_j u(x, j),
\]

(33)

for \( i, j \in \{1, 2\} \). First order condition yields the candidate optimal control

\[
\hat{c}_i(x) = \nabla u(x, i) = -\nabla z_i(x),
\]

(34)
and this leads to the system

\[ \frac{k_i}{2} \Delta u(x, i) + \frac{\| \nabla u(x, i) \|^2}{2} = f_i(x) + (\lambda_i + a_i) u(x, i) - a_i u(x, j), \]

for \( i, j \in \{1, 2\} \). Alternatively this system can be written in terms of \( z_i(x), (i = 1, 2) \) to get \( \text{(2)} \).

### 4 Verification

In this section we establish the optimality of control

\[ \hat{c}_i(x) = \nabla u(x, i) = -\nabla z_i(x). \]

Its associated Markov modulated diffusion is

\[ dX^i(t) = \hat{c}_i^\prime(t)(X(t))dt + k_i(t)dW_i(t), i = 1, \ldots, N. \]

The verification theorem proceeds with the following steps:

**First Step:** Girsanov theorem for Markov-modulated processes (Lemma 1 page 286 in [21]) together with (5) yield a weak solution for SDE (37).

**Second Step:** Let \( X(t) \) be the solution of (37). In light of (5) one can get using exercise 7.5 of [19] that

\[ E|X(t)|^2 \leq C_1 e^{C_2 t}, \]

for some positive constants \( C_1, C_2 \).

**Third Step:** The set of acceptable controls that we consider is encompassing of controls \( c \) for which

\[ J(x, c, i) = E[\int_0^\infty e^{-\lambda_i t} |f_i(t)(X(t)) + \frac{1}{2} \| \hat{c}_i(t)(X(t)) \|^2 dt|e(0) = i ] < \infty, \]

and the following transversality condition

\[ \lim_{t \to \infty} E e^{-\lambda_i t} |X(t)|^2 = 0, \]

is met. Because of (39), estimates (41), (38), the candidate optimal control \( \hat{c} \) of (36) verifies that \( J(x, c, i) < \infty \), for \( \lambda_1, \lambda_2 \) large enough. Moreover, there exist \( \lambda_1^* > 0 \) and \( \lambda_2^* > 0 \) large enough such that for all \( \lambda_1 \geq \lambda_1^*, \lambda_2 \geq \lambda_2^* \) the transversality condition (31) is met because of (4) and (38). Also the control \( c = 0 \), is an acceptable control.

In light of the quadratic estimate on the value function (see 4 in theorem 2.1), the transversality condition implies that

\[ \lim_{t \to \infty} E e^{-\lambda_i t} u(X(t), e(t)) = 0. \]

**Fourth Step:** Recall that

\[ M^c(t) = e^{-\lambda_i t} u(X(t), e(t)) - \int_0^t e^{-\lambda_i (u)} u[f_i(t)(X(u)) + \frac{1}{2} \| \hat{c}_i(t)(X(u)) \|^2] du. \]
Therefore, the Itô’s Lemma yields for the optimal control candidate, \( \hat{c} \)
\[
dM^\epsilon(t) = e^{-\lambda(t)}k_{\epsilon(t)} \nabla u(X(t), \epsilon(t))dW(t).
\]

Consequently \( M^\epsilon(t) \) is a local martingale. Moreover, for \( \lambda_1, \lambda_2 \) large enough, in light of (33) and (38),
\[
E \int_0^1 e^{-2\lambda_1}k_{\epsilon(t)}^2 |\nabla u(X(s), \epsilon(s))|^2 ds \leq C,
\]
for some positive constants \( C \). This in turn makes \( M^\epsilon(t) \) a (true) martingale.

**Fifth Step:** This step establishes the optimality of \( \hat{c} \) of (39). The HJB equation (33) is equivalent to
\[
\sup_{c} L^\epsilon u(x, i) = 0, \quad L^\epsilon u(x, i) = 0, \quad i = 1, 2.
\]

The martingale/supermartingale principle yields
\[
E e^{-\lambda(t)} u(X(t), \epsilon(t)) \leq E \int_0^1 e^{-\lambda(s)} u[f_{\epsilon(t)}(X(u)) + \frac{1}{2}[\partial^2_{\epsilon}(X(u))] du = u(x, \epsilon(0)),
\]
and
\[
E e^{-\lambda(t)} u(X(t), \epsilon(t)) \leq E \int_0^1 e^{-\lambda(s)} u[f_{\epsilon(t)}(X(u)) + \frac{1}{2}[\partial^2_{\epsilon}(X(u))] du \leq u(x, \epsilon(0)).
\]

By passing \( t \to \infty \) and using transversality condition (40) we get the optimality of \( \hat{c} \).

5 Special Case

In the following we manage to obtain a closed form solution for our system given a special discount \( \lambda_1, \lambda_2 \) and the loss functions of the type \( f_1(x) = f_2(x) = |x|^2 \). That is, assume
\[
\lambda_1 = -a_1 + Nk_1 + \frac{1}{4}a_1 \left( \sqrt{N^2k_1^2 + 8} - Nk_1 \right) \left( \sqrt{N^2k_2^2 + 8} - Nk_2 \right) + \frac{1}{2}Na_1k_1 \left( \sqrt{N^2k_2^2 + 8} - Nk_2 \right),
\lambda_2 = -a_2 + Nk_2 + \frac{1}{4}a_2 \left( \sqrt{N^2k_2^2 + 8} - Nk_1 \right) \left( \sqrt{N^2k_1^2 + 8} - Nk_2 \right) + \frac{1}{2}Na_2k_2 \left( \sqrt{N^2k_1^2 + 8} - Nk_1 \right),
\]
are such that \( \lambda_1 > 0, \lambda_2 > 0 \). Then, one solution for the problem (6) is
\[
u(|x|) = e^{m_1(|x|^2 + 1)}, \quad m_1 = -\frac{1}{4k_1} \left( \sqrt{N^2k_1^2 + 8} - Nk_1 \right),
\]
\[
u(|x|) = e^{m_2(|x|^2 + 1)}, \quad m_2 = -\frac{1}{4k_2} \left( \sqrt{N^2k_2^2 + 8} - Nk_2 \right).
\]

Let us point out that (138) implies
\[
z_1(x) = -k_1m_1 (|x|^2 + 1) > 0 \text{ and } z_2(x) = -k_2m_2 (|x|^2 + 1) > 0 \text{ for all } x \in \mathbb{R}^N,
\]
(42)
i.e. \((z_1(x), z_2(x))\) is the positive solution obtained with the above procedure. For the stochastic control problem we choose the positive solution, i.e., the one given in \((12)\).

We also point that in the special case \(f_i(x) = a_i |x|^2 + b_i |x| + c_i\), with \(a_i, b_i, c_i \in \mathbb{R}\) suitable chosen, many others exact solutions can be constructed.

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