Stochastic Patching Process

Xuhui Fan, Bin Li, Yi Wang, Yang Wang, Fang Chen

Data61, CSIRO
firstname.lastname@data61.csiro.au

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Abstract

Stochastic partition models tailor a product space into a number of rectangular regions such that the data within each region exhibit certain types of homogeneity. Due to constraints of partition strategy, existing models may cause unnecessary dissections in sparse regions when fitting data in dense regions. To alleviate this limitation, we propose a parsimonious partition model, named Stochastic Patching Process (SPP), to deal with multi-dimensional arrays. SPP adopts an “enclosing” strategy to attach rectangular patches to dense regions. SPP is self-consistent such that it can be extended to infinite arrays. We apply SPP to relational modeling and the experimental results validate its merit compared to the state-of-the-arts.

1 Introduction

Stochastic partition processes on a product space have found many real-world applications, such as relational modeling [9, 2], community detection [17, 8], collaborative filtering [19], and random forests [12]. By tailoring the product space into rectangular regions, the partition model aims to fit data using these “blocks” such that the data within each block exhibit certain types of homogeneity. As one can choose an arbitrarily fine resolution of partition, the data can be fitted reasonably well.

The cost of data fitness is that the partition model may cause unnecessary dissections in sparse regions. Compared to regular-grid partitions, the Mondrian process (MP) [22] is a hierarchical partition process which has been more parsimonious for data fitting. However, the strategy of recursively cutting the space still cannot avoid unnecessary dissections in sparse regions. Take community detection for example, where a “block” corresponds to a community. When tailoring a block out of the relational matrix, cutting-based models will unavoidably separate some uninvolved users. As a result, some meaningless communities are generated as an undesired by-product (see Figure 1).

Instead of “cutting”, we propose an enclosing-based partition process, named Stochastic Patching Process (SPP), to alleviate the above limitation. SPP attaches patches on a multi-dimensional array to enclose dense regions. In this way, “significant” regions of the space can be comprehensively modeled. Each patch can be generated by an outer product of multiple binary vectors, with a segment of consecutive “1” entries to indicate the initial and terminal positions of the patch. As patches are independently generated, the layout of patches can be quite flexible. This improves its expressiveness to describe those regions with complicated patterns. Thus, SPP is able to use fewer patches (blocks) than those cutting-based models to achieve similar modeling capability.

An important property of SPP is self-consistency. This means that, by restricting the patches generated from an SPP on a multi-dimensional array \( Y \) to its sub-array \( X \), the resulting patches are distributed as if they are directly generated on \( X \) through another SPP (given the same budget). The property will be verified in three steps: (1) the number distribution of nonempty patches is self-consistent; (2) the position distribution of a nonempty patch is self-consistent; (3) based on the

*corresponding author: xuhui.fan@data61.csiro.au
We consider a projective system of stochastic partitions: Let \( \{ (\Omega_X, \mathcal{B}_X) \} \) be a family of measurable spaces, where \( \Omega_X \) is the partition space, \( \mathcal{B}_X \) is a \( \sigma \)-algebra on \( \Omega_X \), and \( \mathcal{F}(\mathbb{N}^D) \) denotes the collection of all finite subsets of the infinite \( D \)-dimensional array \( \mathbb{N}^D \). For each \( X \in \mathcal{F}(\mathbb{N}^D) \), \( \mathbb{P}_X \) is a probability measure on \( \mathcal{B}_X \). \( \mathcal{F}(\mathbb{N}^D) \) is a partially ordered set; while \( X \preceq Y \in \mathcal{F}(\mathbb{N}^D) \) the projection \( \pi_{Y,X} \) restricts the partition \( \mathbb{P}_Y \) on \( Y \) into \( X \), by keeping \( \mathbb{P}_Y \)'s entries within \( X \) unchanged and removing the remaining entries. For \( B_X \in \mathcal{B}_X \), the pre-image under projection is defined as \( \pi_{Y,X}^{-1} B_X = \{ Y \subseteq \Omega_Y | \pi_{Y,X}^{-1} B_Y = B_X \} \) and the projection also satisfies \( \pi_{Y,X} \circ \pi_{X,W} = \pi_{Y,W} \), \( W \preceq X \preceq Y \). This family defines the projective limit measurable space \( (\Omega_{\mathbb{N}^D}, \mathcal{B}_{\mathbb{N}^D}) \).

**Theorem 1 (\[11\] Theorem 3.3.6).** For a set of probability spaces \( \{ (\Omega_X, \mathcal{B}_X, \mathbb{P}_X) \} \) such that projection \( \pi_{Y,X} : \Omega_Y \to \Omega_X \), \( X \preceq Y \in \mathcal{F}(\mathbb{N}^D) \) and \( \mathbb{P}_X(\pi_{Y,X}^{-1} B_X) = \mathbb{P}_Y(B_X) \) holds for all \( B_X \in \mathcal{B}_X \). Then \( \mathbb{P}_X \) can be uniquely extended to measure \( \mathbb{P}_{\mathbb{N}^D} \) on \( (\Omega_{\mathbb{N}^D}, \mathcal{B}_{\mathbb{N}^D}) \) as the projective limit measurable space.

The Kolmogorov extension theorem provides us a constructive way to extend SPP to the infinite \( D \)-dimensional array \( \mathbb{N}^D \), which will be discussed in Section 4.

The merit of SPP is investigated in the application of relational modeling, where patches can be viewed as communities while the “price” of a patch (cost per unit area) can be viewed as the interacting intensity within the community. A sampling based approximate inference solution is also proposed for SPP relational modeling. The experimental results on a number of real-world relational data sets demonstrate that SPP can outperform the compared state-of-the-arts.

### 2 Preliminaries

#### 2.1 Stochastic Partition Processes

Stochastic partition processes partition a product space into blocks. A popular application of such processes is modeling relational data such that the intensity of interactions is homogeneous within each block. In terms of partitioning strategy, state-of-the-art stochastic partition processes can be roughly categorized into regular-grid partitions and flexible axis-aligned partitions.

A regular-grid stochastic partition process is constituted by two separate partition processes on each dimension of the multi-dimensional array. The resulting orthogonal interactions between two dimensions will exhibit regular grids, which can represent interacting intensities. Typical regular-grid partition models include the infinite relational model (IRM) \[9\] and the infinite extension of mixed-membership stochastic blockmodels \[2\]. Regular-grid partition models are widely used in real-world applications for modeling graph data \[7, 23\].

#### 2.2 Kolmogorov Extension Theorem

We consider a projective system of stochastic partitions: Let \( \{ (\Omega_X, \mathcal{B}_X) \} \) be a family of measurable spaces, where \( \Omega_X \) is the partition space, \( \mathcal{B}_X \) is a \( \sigma \)-algebra on \( \Omega_X \), and \( \mathcal{F}(\mathbb{N}^D) \) denotes the collection of all finite subsets of the infinite \( D \)-dimensional array \( \mathbb{N}^D \). For each \( X \in \mathcal{F}(\mathbb{N}^D) \), \( \mathbb{P}_X \) is a probability measure on \( \mathcal{B}_X \). \( \mathcal{F}(\mathbb{N}^D) \) is a partially ordered set; while \( X \preceq Y \in \mathcal{F}(\mathbb{N}^D) \) the projection \( \pi_{Y,X} \) restricts the partition \( \mathbb{P}_Y \) on \( Y \) into \( X \), by keeping \( \mathbb{P}_Y \)'s entries within \( X \) unchanged and removing the remaining entries. For \( B_X \in \mathcal{B}_X \), the pre-image under projection is defined as \( \pi_{Y,X}^{-1} B_X = \{ Y \subseteq \Omega_Y | \pi_{Y,X}^{-1} B_Y = B_X \} \) and the projection also satisfies \( \pi_{Y,X} \circ \pi_{X,W} = \pi_{Y,W} \), \( W \preceq X \preceq Y \). This family defines the projective limit measurable space \( (\Omega_{\mathbb{N}^D}, \mathcal{B}_{\mathbb{N}^D}) \).

**Theorem 1 (\[11\] Theorem 3.3.6).** For a set of probability spaces \( \{ (\Omega_X, \mathcal{B}_X, \mathbb{P}_X) \} \) such that projection \( \pi_{Y,X} : \Omega_Y \to \Omega_X \), \( X \preceq Y \in \mathcal{F}(\mathbb{N}^D) \) and \( \mathbb{P}_X(\pi_{Y,X}^{-1} B_X) = \mathbb{P}_Y(B_X) \) holds for all \( B_X \in \mathcal{B}_X \). Then \( \mathbb{P}_X \) can be uniquely extended to measure \( \mathbb{P}_{\mathbb{N}^D} \) on \( (\Omega_{\mathbb{N}^D}, \mathcal{B}_{\mathbb{N}^D}) \) as the projective limit measurable space.

The Kolmogorov extension theorem provides us a constructive way to extend SPP to the infinite \( D \)-dimensional array \( \mathbb{N}^D \), which will be discussed in Section 4.
2.3 Exchangeable Arrays

The Aldous–Hoover theorem \[18\] provides the theoretical foundation to model exchangeable multidimensional arrays conditioned on a stochastic partition model. A random 2-dimensional array is called separately exchangeable if its distribution is invariant under separate permutations of rows and columns.

**Theorem 2.** \[18\] A random array \((R_{ij})\) is separately exchangeable if and only if it can be represented as follows: There exists a random measurable function \(F : [0, 1] \times [0, 1] \rightarrow X\) such that \((R_{ij}) \sim \frac{d}{m} F(\xi_{ij}, \eta_{ij})\), where \(\{\xi_{ij}\}_{i,j}, \{\eta_{ij}\}_{i,j}\) and \(\nu_{ij}\) are, respectively, two sequences and an array of i.i.d. uniform random variables in \([0, 1]\).

Relational modeling based on the stochastic partition models is a typical application of the Aldous–Hoover theorem \[18\]. By defining a graph function \(W(\xi, \eta, \nu, \nu_{ij}) := P(F(\xi, \eta, \nu_{ij}) = 1|F)\), every exchangeable array can be represented by a random graph function. The SPP relational model introduced in Section 5 is implicitly built on this theorem.

3 Stochastic Patching Process

SPP is defined on a measurable space \((\Omega_X, B_X), X \in \mathcal{F}(\mathbb{N}^D)\). Each element in \(\Omega_X\) denotes a partition \(\Xi_X\), constituted by a collection of rectangular nonempty patches \(\{\square_k\}_{k}\) with corresponding costs \(\{m_k\}_k\) where \(k \in \mathbb{N}\) indexes the patch number in \(\Xi_X\). In particular, a patch is defined by an outer product \(\square_k := \bigotimes_{d=1}^D u_k^{(d)}\), where \(u_k^{(d)} \in \{0, 1\}^{N_X(d)}\) is a position indicator vector for the \(d\)th dimension of \(\square_k\), with the constraint that \(u_k\) only comprises a segment of \(l_k^{(d)} \in \{1, \ldots, N_X(d)\}\) consecutive “1” entries which starts at an initial position \(s_k^{(d)} \in \{1, \ldots, N_X(d)\}\).

Given an array \(X\) and a budget \(\tau\), we can sample a random partition from an SPP: \(\Xi_X \sim \text{SPP}(X, \tau)\). We assume that the costs of patches are i.i.d. sampled from the same exponential distribution, which implies there exists a homogeneous Poisson process on the time (cost) line. The generating time of each patch is uniform in \((0, \tau]\) and the number of patches has a Poisson distribution. We represent a random partition as \(\Xi_X := \{m_k, \square_k\}_{k=1}^{K_\tau} \in \Omega_X\), which is generated as follows:

1. Sample the number of candidate patches \(K_\tau \sim \text{Poisson}(\tau S_X)\);
2. Sample \(K_\tau\) i.i.d. candidate patches. For \(k' = 1, \ldots, K_\tau, d = 1, \ldots, D\)
   (a) Sample the initial position \(s_{k'}^{(d)}\) of the \(k'\)th candidate uniformly from \(\{1, \ldots, N_X(d)\}\);
   (b) If \(s_{k'}^{(d)} = 1\), the side-length \(l_{k'}^{(d)}\) increments from 0 to 1 with probability \(\theta_0\); otherwise \(l_{k'}^{(d)}\) increments from 0 to 1 with probability \(\theta(1 - \theta)\), where \(\theta_0, \theta \in [0, 1]\);
   (c) If \(l_{k'}^{(d)} = 1\), generate the remaining side-length from a Geometric distribution by \((l_{k'}^{(d)} - 1) \sim \text{Geometric}(1 - \theta)\);
3. Remove all empty patches and retain \(K_\tau\) nonempty patches \(\{\square_k|S_{\square_k} = \prod_{d=1}^D l_{k_d}^{(d)} > 0\}_{K_\tau}\)
   Sample \(K_\tau\) i.i.d. time points uniformly in \((0, \tau]\) and index them to satisfy \(t_1 < \ldots < t_{K_\tau}\). Set the cost of \(\square_k\) as \(m_k = t_k - t_{k-1}\) when \(t_0 = 0\) and the rate of \(\omega_k\) of \(\square_k\) as \(\omega_k = m_k/S_{\square_k}\).

We use the initial position \(s_{k'}^{(d)}\) and the side-length \(l_{k'}^{(d)}\) of \(\square_k\) in the \(d\)th dimension to determine the position indicator vector \(u_k^{(d)} \in \{0, 1\}^{N_X(d)}\), which further constitutes \(\square_k\). Thus, given \(K_\tau\), all patches are i.i.d. generated and the layout of patches can be quite flexible. Patches can be overlapped or even contained by others. In the two-dimensional case, a partition sampled from an SPP has the following interpretation: \(X\) can be viewed as a piece of cloth while \(\square_k\) can be viewed as a patch; the

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1 An equivalent construction is to directly generate nonempty patches through thinning the Poisson process used for generating candidate patches (see “Alternative Construction of SPP” in Supplementary Material).

2 In Section 5 we will show that such rate is useful when we use \(\square_k\) and \(\omega_k\) as the priors of a community and the intensity of interactions within the community (large communities have relatively weak interactions).
material of \( \Box_k \) has rate (price) \( \omega_k \) and the number of “patches” on the “cloth” is determined by \( S_X \) and the budget \( \tau \) – This is where the name of “Stochastic Patching Process” come from.

Due to such a flexible layout of patches, SPP is parsimonious to model multi-dimensional arrays, especially in sparse scenarios – SPP is able to describe “significant” parts of the array (e.g. active communities in a social network) through small patches; after patching these “significant” parts, the rest are usually large and irregular sparse areas which may be neglected.

4 Self-Consistency

Section 3 has defined SPP on a finite array given a budget. To further extend SPP to the infinite array \( \mathbb{N}^D \), an essential property of SPP is self-consistency. That is to say, while restricting an SPP on a finite \( D \)-dimensional array \( Y \), say \( SPP(Y, \tau) \), to its sub-array \( X, X \subset Y \in \mathcal{F}(\mathbb{N}^D) \), the resulting patches restricted to \( X \) are distributed as if they are directly generated on \( X \) through \( SPP(X, \tau) \). The property is verified in three steps: (1) the number distribution of nonempty patches is self-consistent; (2) the position distribution of a nonempty patch is self-consistent; (3) SPP is self-consistent.

Following the notations used in Sections 2.2 and 3, we use \( \pi_{Y,X} \) to denote the projection that restricts \( \Omega_Y \in \Omega_Y \) to \( X \) by keeping \( \Omega_Y \)'s entries in \( X \) unchanged and removing the rest. An “empty patch” is referred to the case \( S_{\Box} = 0 (\exists d, l^{(d)} = 0) \), where \( S_{\Box} \) denotes the volume of the candidate patch.

4.1 Number of Nonempty Patches

**Proposition 1.** While restricting \( SPP(Y, \tau) \) to \( X, X \subset Y \in \mathcal{F}(\mathbb{N}^D) \), the time points of nonempty patches crossing into \( X \) from \( Y \) follows the same Poisson process for generating the time points of nonempty patches in \( SPP(X, \tau) \).

**Proof.** According to the definition, the candidate patches sampled from \( SPP(Y, \tau) \) (or \( SPP(X, \tau) \)) follows a homogeneous Poisson process with intensity \( S_Y \) (or \( S_X \)). Since there exists possibility to generate empty patches, we use intensity \( S_X \cdot \Pr(S_{\Box} > 0) \) for thinning the Poisson process to generate nonempty patches. Given the same budget \( \tau \), Proposition 1 holds if we can prove the following equality of two Poisson process intensities

\[
S_Y \cdot \Pr(S_{\pi_{Y,X}(\Box^Y)} > 0) = S_X \cdot \Pr(S_{\Box} > 0)
\]  

(1)

Due to the independence of dimensions, we have

\[
\Pr(S_{\Box} > 0) = \prod_d \Pr(l^{(d)}_{X} > 0)
\]

(2)

Assuming \( N_{X}^{(d)} \geq 2 \), we have

\[
\Pr(l^{(d)}_{X} > 0) = \frac{1}{N_{X}^{(d)}} \cdot \theta_0 + \frac{N_{X}^{(d)} - 1}{N_{X}^{(d)}} \cdot \theta_0(1 - \theta) = \frac{\theta_0}{N_{X}^{(d)}} \cdot \left[ \theta + N_{X}^{(d)}(1 - \theta) \right]
\]

(3)

W.l.o.g, we assume that \( X \) and \( Y \) have the same shape apart from \( d \)'th dimension where \( Y \) has one additional column (the general case follows by induction), then \( S_X / S_Y = N_X^{(d)} / N_Y^{(d)} \). There are two cases to consider: (1) \( X \) and \( Y \) do not share the initial boundary in the \( d \)'th dimension; (2) \( X \) and \( Y \) share the initial boundary. In either case, we have, by independence of dimensions

\[
\Pr(S_{\pi_{Y,X}(\Box^Y)} > 0) = \Pr(\pi_{Y,X}(l^{(d)}_{Y}) > 0) \cdot \prod_{d \neq d'} \Pr(l^{(d')}_{Y} > 0)
\]

(4)

In case (1) where \( X \) and \( Y \) do not share the initial boundary in the \( d \)'th dimension, we have

\[
\Pr(\pi_{Y,X}(l^{(d')}_{Y}) > 0) = \frac{\theta_0 \theta}{N_{Y}^{(d')}} + \frac{N_{X}^{(d')}}{N_{Y}^{(d')}} \cdot \theta_0(1 - \theta) = \frac{\theta_0}{N_{Y}^{(d')}} \cdot \left[ \theta + N_{Y}^{(d')}(1 - \theta) \right]
\]

(5)
By combining Eqs. (2), (3), (4) and (5), we have

\[
\frac{\Pr(S_{Y,X \mid X'} > 0)}{\Pr(S_{\Box X} > 0)} = \frac{\Pr(\pi_{Y,X}(l_Y^{(d')}) > 0)}{\Pr(l_Y^{(d')}) > 0)} = \frac{N_X^{(d')}}{N_Y^{(d')}} = S_X \quad \frac{S_Y}{S_Y}
\]

(6)

In case (2) where \(X \text{ and } Y\) share the initial boundary in the \(d'\)th dimension, we have

\[
\Pr(\pi_{Y,X}(l_Y^{(d')}) > 0) = \frac{\theta_0}{N_Y^{(d')}} + \frac{N_X^{(d')}-1}{N_Y^{(d')}} \cdot \theta_0(1-\theta) = \frac{\theta_0}{N_Y^{(d')}} \cdot \left[\theta + N_X^{(d')}(1-\theta)\right]
\]

(7)

The conclusion can be similarly derived.

Because of the same Poisson process intensity Eq. (1), the following equality also holds

\[
\mathbb{P}_{K_r, \{m_k\}_{k}}^X \left(\pi_{Y,X}^{-1} \left(\mathbb{K}_r, \{m_k X\}_{k=1}^{K_X} \right) \right) = \mathbb{P}_{K_r, \{m_k\}_{k}}^X \left(\mathbb{K}_r', \{m_k X\}_{k=1}^{K_X} \right)
\]

(8)

### 4.2 Position of Nonempty Patches

**Proposition 2.** While restricting \(SPP(Y, \tau)\) to \(X, X \subset Y \in \mathcal{F}(\mathbb{N}^D), \) the marginal probability of the pre-images of a nonempty patch \(\Box^X\) in \(Y\) equals to the probability of \(\Box^X\) directly sampled from \(SPP(X, \tau)\), that is \(\mathbb{P}_{\Box^X}(\pi_{Y,X}^{-1}(\Box^X)) = \mathbb{P}_{\Box^X}(\Box^X).\)

**Proof.** W.l.o.g, we assume that \(X \text{ and } Y\) have the same shape apart from \(d'\)th dimension where \(Y\) has one additional column (the general case follows by induction). For dimensions \(d \neq d'\), it is obvious that the law of patches are consistent under projection because the projection is the identity. Given the same budget \(\tau\), Proposition 2 holds if we can prove the following equality

\[
\mathbb{P}_{l(d')}^Y \left(\pi_{Y,X}^{-1}(l_X^{(d')}) \right) = \mathbb{P}_{l(d')}^X \left(l_X^{(d')} \right)
\]

(9)

where \(l_X^{(d')}\) denotes the \(d'\)th side-length of \(\Box^X\). Consider the \(d'\)th dimension, there are two cases: (1) \(X \text{ and } Y\) share the initial boundary; (2) \(X \text{ and } Y\) do not share the initial boundary.

In case (1) where \(X \text{ and } Y\) share the initial boundary. For \(0 < s_X^{(d')} + l_X^{(d')} - 1 < N_X^{(d')} < N_Y^{(d')}\),

\[
\mathbb{P}_{l(d')} \left(\pi_{Y,X}^{-1}(l_X^{(d')}) \right) = \theta_0 \theta_X^{(d')}-1 \cdot (1-\theta) = \mathbb{P}_{l(d')} \left(l_X^{(d')} \right)
\]

(10)

where \(\theta_0\) if \(s_X^{(d')} = 1; \theta_1 = \theta_0(1-\theta)\) if \(s_X^{(d')} > 1.\) For \(0 < s_X^{(d')} + l_X^{(d')} - 1 = N_X^{(d')} < N_Y^{(d')}\),

\[
\mathbb{P}_{l(d')} \left(\pi_{Y,X}^{-1}(l_X^{(d')}) \right) = \mathbb{P}_{l(d')} \left(l_X^{(d')} \right) = \theta_0 \theta_X^{(d')}-1 \cdot (1-\theta) = \mathbb{P}_{l(d')} \left(l_X^{(d')} \right)
\]

(11)

In case (2) where \(X \text{ and } Y\) do not share the initial boundary. For \(\pi_{Y,X}(s_Y^{(d')}) = 1\), we have

\[
\mathbb{P}_{l(d')} \left(\pi_{Y,X}^{-1}(l_X^{(d')}) \right) = \mathbb{P}_{l(d')} \left(l_Y^{(d')} = l_X^{(d')}, s_Y^{(d')} = 2\right) + \mathbb{P}_{l(d')} \left(l_Y^{(d')} = l_X^{(d')} + 1, s_Y^{(d')} = 1\right)
\]

(12)

where \(\theta_1 = 1\) if \(0 < \pi_{Y,X}(s_Y^{(d')}) + l_X^{(d')} - 1 = N_X^{(d')}\); and \(\theta_1 = 1-\theta\) if \(0 < \pi_{Y,X}(s_Y^{(d')}) + l_X^{(d')} - 1 < N_X^{(d')}\). For \(\pi_{Y,X}(s_Y^{(d')}) > 1\), we have

\[
\mathbb{P}_{l(d')} \left(\pi_{Y,X}^{-1}(l_X^{(d')}) \right) = \theta_0(1-\theta) \theta_X^{(d')}-1 \cdot (1-\theta) = \mathbb{P}_{l(d')} \left(l_X^{(d')} \right)
\]

(13)

Consider all \(D\) dimensions we have \(\mathbb{P}_{l(d')}^Y \left(\pi_{Y,X}^{-1}(\Box^X) \right) = \mathbb{P}_{l(d')}^X (\Box^X)\).
4.3 Self-Consistency of SPP

Now we are ready to prove \( P_Y^X(\pi_X^{-1}(\mathcal{X})) = P_X(\mathcal{X}) \).

\[
P_Y^X(\pi_X^{-1}(\mathcal{X})) = P_Y^X_K,\{\omega_k\}_h \left( \pi_X^{-1} K^X, \{m^X_{k} \}_{k=1}^{K} \right) \prod_{k=1}^{K} P^X_\varnothing \left( \pi_X^{-1} \left( \varnothing^X \right) \right) \tag{14}
\]

\[
= P_X^X_K,\{\omega_k\}_h \left( K^X, \{m^X_{k} \}_{k=1}^{K} \right) \prod_{k=1}^{K} P^X_\varnothing \left( \varnothing^X \right) = P^X_\mathcal{H}(\mathcal{X}) \tag{15}
\]

where \( P^X_K,\{\omega_k\}_h (\cdot) \) and \( \prod_{k=1}^{K} P^X_\varnothing (\cdot) \) in Eq. (15) are obtained by applying Propositions 1 and 2 respectively. According to the Kolmogorov extension theorem (see Section 2.3), we can conclude

**Theorem 3.** The probability measure \( P^X_\mathcal{H} \) on a measurable space \((\Omega_X, \mathcal{B}_X)\) of SPP, \( X \in \mathcal{F}(N^D) \), can be uniquely extended to \( P^D_\mathcal{H} \) on \((\Omega^{\mathcal{H}}, \mathcal{B}^{\mathcal{H}})\) as the projective limit measurable space.

5 Application to Relational Modeling

5.1 SPP Relational Model

A typical application of SPP is relational modeling. Given the relational data as an asymmetric matrix \( R \in \{0, 1\}^{N \times N} \), with \( R_{ij} \) indicating the relation from node \( i \) to node \( j \), the patches \( \{\varnothing_k\}_h \) with different rates \( \{\omega_k\}_h \) of a partition \( \mathcal{H} \) are used for modeling communities with different intensities of relations. Because \( \{\varnothing_k\}_h \) can be overlapped, the intensity of relations in an overlapped part on \( R \) is synthesized by the rates of the involved patches (see Figure 2).

The generative process of an SPP relational model is as follows: (1) Generate a partition \( \mathcal{H} \); (2) For \( i = 1, \ldots, N \), generate row index \( r_i \) of \( R \); (3) for \( j = 1, \ldots, N \), generate column index \( c_j \) of \( R \); (4) for \( i, j = 1, \ldots, N \), generate relational data \( R_{r_i,c_j} \sim \text{Bernoulli}(\sigma(\sum_{k=1}^{K} \omega_k \cdot u_{k,i}^{(1)} u_{k,j}^{(2)})) \), where \( \sigma(x) = \frac{\exp(x + e^{-6}) - 1}{\exp(x + e^{-6}) + 1} \) is a selected function for mapping the aggregated rate from \([0, \infty)\) to \((0, 1)\) as intensity of relations and \( \gamma \) is a scaling parameter. While here we instantiate an SPP relational model with binary interactions (Bernoulli likelihood), other types of relations (e.g., Categorical likelihood) can also be plugged in.

Actually, SPP and the mapping function \( \sigma(\cdot) \) play together as the role of random function \( W(\cdot) \) defined in Section 2.3. The uniformly exchanged row and column indices \( r_i \) and \( c_j \) resemble the row and column indices \((\xi_i^{\text{row}} \text{ and } \eta_j^{\text{col}})\) which are uniformly sampled in \([0, 1] \). By re-arranging the rows and columns of \( R \) according to the inferred indices, the SPP relational model is expected to uncover homogeneous interactions in \( R \) as compact patches.

The joint probability of the data \( \{R_{ij}\}_{i,j} \), the number of nonempty patches \( K \), the variables of the nonempty patches \( \{m_k, u_k^{(1)}, u_k^{(2)}\}_{k=1}^{K} \), and the indices \( \{r_i\}_{i}, \{c_j\}_{j} \) gives

\[
p(\{R_{ij}\}_{i,j}, K, \{m_k, u_k^{(1)}, u_k^{(2)}\}_{k=1}^{K}, \{r_i\}_{i}, \{c_j\}_{j} | \theta, \tau, \gamma, N) = \prod_{i,j} \Pr(R_{r_i,c_j} | K, \{m_k, u_k^{(1)}, u_k^{(2)}\}_{k=1}^{K}, \gamma) \cdot \Pr(\{r_i\}_{i} | N) \cdot \Pr(\{c_j\}_{j} | N) \cdot \Pr(K, \{m_k\} | \tau, \theta, N) \cdot \prod_k \Pr(u_k^{(1)} | \theta, \tau, N) \Pr(u_k^{(2)} | \theta, \tau, N)
\]

Figure 2: SPP relational model: \( \{\omega_k\}_k \) denote intensities of relations within communities \( \{\varnothing_k\}_h \).
We empirically test the SPP relational model (SPP-RM) for link prediction. We compare SPP-RM with four state-of-the-arts: (1) Infinite Relational Model (IRM) [9] (regular grids); (2) Latent Feature Relational Model (LFRM) [15] (plaid grids); (3) MP Relational Model (MP-RM) [22] (hierarchical $kd$-tree); (4) Matrix Tile Analysis Relational Model (MTA-RM) [5] (noncontiguous tiles). All these models except MP-RM can be represented as a (weighted) sum of outer products of binary latent feature (community) vectors (see Figure 3). For IRM and LFRM, we adopt the collapsed Gibbs sampling algorithms for inference; for MP-RM, we adopt the reversible-jump MCMC algorithm for inference [24]; for MTA-RM, we adopt the Iterative Conditional Modes algorithm used in [5].

Data Sets: Five social network data sets are used: Digg, Flickr [25], Enron [10], Gplus [14], and Wiki-Vote [13]. We extract a subset of nodes (top 500 active nodes based on their interactions with others) from each data set for constructing the relational data matrix.

Experimental Setting: For a fair comparison that all the compared methods result in a similar number of blocks/patches (or latent features), we set the hyper-parameters for each method as follows: In IRM $\alpha = 1$ such that an expectation of $4 \times 4$ blocks would be generated. In LFRM $\alpha = 2.5$ such that an expectation of 17 latent features would be generated. The budget parameter in MP-RM is set to 3, which suggests that around $(3 + 1) \times (3 + 1)$ blocks would be generated. For parametric model MTA-RM, we simply set the number of tiles to 16. In SPP-RM, we set $\theta_0 = 0.9, \theta = 0.95$ and $\tau = 3, \gamma = 10^{-2}$, which leads to an expectation of 16.5 patches. The reported performance is averaged over 10 randomly selected hold-out test sets (Train : Test = 9 : 1).

Results: Table 1 reports the performance comparison results on the five data sets. We can see that SPP-RM consistently outperforms the other four methods in all cases, with around 0.02 improvement compared to the runner-up in prediction AUC. The overall results validate that SPP-RM is effective in relational modeling due to its flexibility via attaching patches to dense regions. MTA-RM also performs well – One reason may be that MTA-RM and SPP-RM have similar modeling philosophy.
(see Figure 3); another reason is that, being a parametric model, MTA-RM converges faster and can be fitted better than MP-RM and SPP-RM using same wall-clock time.

Figure 4 (rows 2–6) illustrates the visual patterns of the partition results. As expected, our enclosing-based method SPP-RM indeed focuses on describing dense regions of relational data matrices with fewer patches, while the two representative cutting-based methods, IRM and MP-RM, may cut sparse regions into more blocks. An interesting observation of SPP-RM is that overlapped patches are very useful in describing inter-community interactions (e.g., lower-right corners in Digg and Flickr) and community-in-community interactions (e.g., upper-right corner in Gplus).

Figure 5 (rows 7–8) plot the average performance versus the wall-clock time for investigating the convergence behavior of the compared methods. IRM and LFRM converge fastest because of efficient collapsed Gibbs sampling. MTA-RM also converges fast because it is trained using a simple iterative algorithm. MP-RM and SPP-RM have similar convergence rate after first 200 seconds. SPP-RM has an inferior initial performance since it starts with no patches; however it overtakes MP-RM very soon because SPP-RM updates all the patches simultaneously in each iteration while MP-RM only updates one leaf block of the \( kd \)-tree partition structure.

6 Conclusion

A parsimonious partition process, named Stochastic Patching Process (SPP), is proposed. Instead of the cutting-based strategy, we adopt an enclosing-based strategy to attach \( i.i.d. \) rectangular patches to model dense data regions in the space such that it can avoid unnecessary dissections in sparse regions. We apply SPP to relational modeling and find that SPP can achieve clear performance gain with fewer patches (blocks) compared to the state-of-the-art relational modeling methods.

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Figure 4: Partition structure visualization and performance comparison on the five data sets: (from left to right) Digg, Flickr, Enron, Gplus and Wiki-Vote. The rows correspond to (from top to bottom) (1) the original data, (2) IRM, (3) LFRM, (4) MP-RM, (5) MTA-RM, (6) SPP-RM, (7) training log-likelihood vs. wall-clock time (s) and (8) testing AUC vs. wall-clock time (s).
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Alternative Construction of SPP

An alternative construction of SPP which is equivalent to the one introduced in Section 3 is as follows:

1. Sample the number of nonempty patches $K_{\tau} \sim \text{Poisson}\left(\tau S_X \cdot \Pr(S_{\Box} > 0)\right)$, where

   \[
   \Pr(S_{\Box} > 0) = \prod_{d=1}^{D} \frac{\theta_0}{N_X^{(d)}} \cdot \left[ \theta + N_X^{(d)}(1 - \theta) \right];
   \]

2. Given $K_{\tau}$, sample i.i.d. nonempty patches $\{\Box_k\}_{k=1}^{K_{\tau}}$. For $k = 1, \ldots, K_{\tau}, d = 1, \ldots, D$

   (a) Sample the initial position $\tau^{(d)}_k$ of $\Box_k$ from $\{1, 2, \ldots, N_X^{(d)}\}$ in proportion to $\{\theta_0, \theta_0(1 - \theta), \cdots, \theta_0(1 - \theta)\}$ and set $l^{(d)}_k = 1$;

   (b) Sample the remaining side-length from a Geometric distribution by $(l^{(d)}_k - 1) \sim \text{Geometric}(1 - \theta)$;

3. Sample $K_{\tau}$ i.i.d. time points uniformly in $(0, \tau]$ and index them to satisfy $t_1 < \ldots < t_{K_{\tau}}$.

   Set the cost of $\Box_k$ as $m_k = t_k - t_{k-1}$ ($t_0 = 0$) and the rate of $\Box_k$ as $\omega_k = m_k/S_{\Box_k}$, where $S_{\Box_k} = \prod_{d=1}^{D} l^{(d)}_k$.

In this way, one can directly sample nonempty patches through thinning the Poisson process which is used for generating candidate patches.

**Detail Proof of** $\Pr(Y \in \Pi^{-1}_Y (\Box_X)) = \Pr_X(\Box_X)$

\[
\Pr_Y(\pi^{-1}_Y (\Box_X)) = \Pr_Y(\pi^{-1}_Y (K_{\tau}, \{m_k, \Box_k\}_{k=1}^{K_{\tau}}))
\]

\[
= \Pr_{K_{\tau}, \{m_k\}_k}^Y (\pi^{-1}_Y (K_{\tau}, \{m_k^{K_{\tau}}\}_{k=1}^{K_{\tau}})) \cdot \Pr_{\Box}^Y (\pi^{-1}_Y (\{\Box_k\}_{k=1}^{K_{\tau}}, \{m_k^{K_{\tau}}\}_{k=1}^{K_{\tau}}))
\]

\[
= \Pr_{K_{\tau}, \{m_k\}_k}^Y (\pi^{-1}_Y (K_{\tau}, \{m_k^{K_{\tau}}\}_{k=1}^{K_{\tau}})) \cdot \Pr_{\Box}^Y (\pi^{-1}_Y (\{\Box_k\}_{k=1}^{K_{\tau}}(\Box_k)))
\]

\[
= \Pr_{K_{\tau}, \{m_k\}_k}^X (K_{\tau}, \{m_k^{K_{\tau}}\}_{k=1}^{K_{\tau}}) \cdot \prod_{k=1}^{K_{\tau}} \Pr_{\Box}^X (\pi^{-1}_X (\Box_k))
\]

\[
= \Pr_{K_{\tau}, \{m_k\}_k}^X (K_{\tau}, \{m_k^{K_{\tau}}\}_{k=1}^{K_{\tau}}) \cdot \prod_{k=1}^{K_{\tau}} \Pr_{\Box}^X (\Box_k)
\]

\[
= \Pr_{\Box}^X (K_{\tau}, \{m_k, \Box_k\}_{k=1}^{K_{\tau}}) = \Pr_X(\Box_X)
\]

We can obtain Eq. (17) from Eq. (16) because

\[
p \left( \{m_k, \Box_k\}_{k=1}^{K_{\tau}} | K_{\tau}\right) = p \left( \{m_k\}_{k=1}^{K_{\tau}} | K_{\tau}\right) p \left( \{\Box_k\}_{k=1}^{K_{\tau}} | K_{\tau}\right)
\]

which indicates

\[
p \left( \{\Box_k\}_{k=1}^{K_{\tau}} | K_{\tau}, \{m_k\}_{k=1}^{K_{\tau}}\right) = p \left( \{\Box_k\}_{k=1}^{K_{\tau}} | K_{\tau}\right)
\]

We can obtain Eq. (18) from Eq. (17) because of independence of patches. Eq. (19) is derived from Eq. (18) by applying Proposition 1 and Eq. (20) is derived from Eq. (19) by applying Proposition 2.
Algorithm 1 Sampling for SPP Relational Model

Input: Relational data R, budget τ, hyper-parameters θ₀, θ, γ, iteration time T

Output: K, \{m_k, u_k^{(1)}, u_k^{(2)}\}_k, \{r_i\}_i and \{c_j\}_j

for t = 1, \cdots, T do
    Sample K; 
    for k = 1, \cdots, K do
        Sample m_k; 
        end for 
    for k = 1, \cdots, K do
        Sample u_k^{(1)} and u_k^{(2)}; 
        end for 
    for i, j = 1, \cdots, N do
        Sample r_i and c_j; 
        end for 
end for

Sampling for SPP Relational Model

We adopt the Metropolis-Hastings algorithm for sampling the posteriors of K, \{m_k, u_k^{(1)}, u_k^{(2)}\}_k, \{r_i\}_i and \{c_j\}_j. Let \(\ell(r_i, c_j, \rho_{ij}) = \Pr(R_{r_i, c_j} | K, \{m_k, u_k^{(1)}, u_k^{(2)}\}_k, \gamma) = \rho_{ij}^{R_{r_i, c_j}} (1 - \rho_{ij})^{1 - R_{r_i, c_j}}\) be the likelihood function, where \(\rho_{ij} = \sigma(\sum_{k=1}^{K} \omega_{k} \cdot u_{k,i}^{(1)} u_{k,j}^{(2)})\) denotes the parameter of the Bernoulli likelihood; \(\Pr(\{r_i\}_i | N) = \Pr(\{c_j\}_j | N) = \frac{1}{N}\) denotes the probability of indexing orders; \(\Pr(K, \{m_k\}_k | \theta_0, \theta, \gamma, N) = (\gamma N^2 \theta_0)^K e^{-\gamma N^2 \theta_0}\) denotes the joint probability of K and \{m_k\}_k (where \(\theta_0 = \frac{\theta^2}{2N} + \frac{\theta N(1 - \theta)}{2}\)).

By an abuse of notation, in the following \(\rho_{ij}^{x \rightarrow x^*}\) is used to represent the case that the likelihood is updated by replacing \(x\) with \(x^*\), keeping the other variables unchanged. Also, we use \(\rho_{ij}^{k}\) to denote the likelihood computed excluding the \(k\)th patch. The sampling algorithm is outlined in Algorithm 1.

Sample K. We use a similar strategy of [1] for updating K. First, we use probability \(P_0 = \frac{1}{2}\) (or \(1 - P_0\)) to choose proposing adding (or removing) a nonempty patch. The proposal probability of adding a nonempty patch is

\[q_{\text{add}}(K \rightarrow K + 1) = P_0 \cdot \frac{1}{\tau} \cdot \Pr(\Box_\star)\]

(21)

where \(\Box_\star\) denotes a newly added patch; the proposal probability of deleting an existing patch is

\[q_{\text{del}}(K \rightarrow K - 1) = \frac{1 - P_0}{\tau}\]

(22)

We accept adding or removing a patch with a ratio of \(\min(1, \alpha_{\text{add}})\) or \(\min(1, \alpha_{\text{del}})\), where

\[\alpha_{\text{add}} = \frac{\prod_{i,j} \ell(r_i, c_j, \rho_{ij}^{K \rightarrow K + 1})}{\prod_{i,j} \ell(r_i, c_j, \rho_{ij})} \cdot \frac{(\gamma N^2 \theta_0)^K e^{-\gamma N^2 \theta_0}}{(\gamma N^2 \theta_0)^K e^{-\gamma N^2 \theta_0}} \cdot \frac{\Pr(\Box_\star)}{1} \cdot \frac{q_{\text{del}}(K + 1 \rightarrow K)}{q_{\text{add}}(K \rightarrow K + 1)}\]

(23)

\[\alpha_{\text{del}} = \frac{\prod_{i,j} \ell(r_i, c_j, \rho_{ij}^{K \rightarrow K - 1})}{\prod_{i,j} \ell(r_i, c_j, \rho_{ij})} \cdot \frac{(\gamma N^2 \theta_0)^{K - 1} e^{-\gamma N^2 \theta_0}}{(\gamma N^2 \theta_0)^K e^{-\gamma N^2 \theta_0}} \cdot \frac{1}{\Pr(\Box_\star)} \cdot \frac{q_{\text{add}}(K - 1 \rightarrow K)}{q_{\text{del}}(K \rightarrow K - 1)}\]

(24)
It is worth noting that in the proposal of adding a new nonempty patch, it is generated by following Step 2 (a) and (b) of “Alternative Construction of SPP” in Supplementary Material.

Sample \( \{m_k\}_k \) For the \( k \)th patch, \( k \in \{1, \cdots, K_r\} \), a new \( m^*_k \) is sampled from the proposal distribution, which is a truncated Exponential distribution \( f(m^*_k) \propto e^{-\gamma N^2 \theta, m^*_k} \mathbb{I}[m^*_k \in (0, \tau - \sum_{k' \neq k} m_{k'})] \). We then accept \( m^*_k \) with a ratio of \( \min(1, \alpha) \), where

\[
\alpha = \frac{\prod_{i,j} \ell(r_{i'}c_j, \rho_{ij}^{m^*_k-m_k^*})}{\prod_{i,j} \ell(r_{i'}c_j, \rho_{ij})} \frac{e^{-\gamma N^2 \theta, m_k}}{e^{-\gamma N^2 \theta, m_k^*}}
\]

Sample \( \{u_k^{(1)}, u_k^{(2)}\}_k \) Each pair \( \{u_k^{(1)}, u_k^{(2)}\} \) is determined by the initial positions \( \{s_k^{(1)}, s_k^{(2)}\} \) and the terminal positions \( \{e_k^{(1)}, e_k^{(2)}\} \) of the consecutive “1” entries. We take \( s_k^{(1)} \) for example, the other three position variables can be inferred in a similar manner. First, we use probability \( P_0 = \frac{1}{2} \) (or \( 1 - P_0 \)) to choose proposing moving the position \( s_k^{(1)} \) one entry forward (or backward) on \( u_k^{(1)} \). In the case of “forward” (\( s_k^{(1)} < N \) and \( s_k^{(1)} + 1 < e_k^{(1)} \)), we propose to move \( s_k^{(1)} \) to \( s_k^{(1)} + 1 \) and accept the proposed position with a ratio of \( \min(1, \alpha_{\text{fwd}}) \)

\[
\alpha_{\text{fwd}} = \frac{\prod_{i,j} \ell(r_{i'}c_j, \rho_{ij}^{s_k^{(1)}-s_k^{(1)}+1}) \Pr(s_k^{(1)} + 1, e_k^{(1)})}{\prod_{i,j} \ell(r_{i'}c_j, \rho_{ij}) \Pr(s_k^{(1)}, e_k^{(1)})} \frac{1 - P_0}{P_0}
\]

where

\[
\frac{\Pr(s_k^{(1)} + 1, e_k^{(1)})}{\Pr(s_k^{(1)}, e_k^{(1)})} = \begin{cases} 
(1 - \theta)/\theta, & s_k^{(1)} = 1 \\
1/\theta, & s_k^{(1)} > 1 
\end{cases}
\]

In the case of “backward” (\( s_k^{(1)} > 1 \)), we propose to move \( s_k^{(1)} \) to \( s_k^{(1)} - 1 \) and accept the proposed position with a ratio of \( \min(1, \alpha_{\text{bwd}}) \)

\[
\alpha_{\text{bwd}} = \frac{\prod_{i,j} \ell(r_{i'}c_j, \rho_{ij}^{s_k^{(1)}-s_k^{(1)}-1}) \Pr(s_k^{(1)} - 1, e_k^{(1)})}{\prod_{i,j} \ell(r_{i'}c_j, \rho_{ij}) \Pr(s_k^{(1)}, e_k^{(1)})} \frac{P_0}{1 - P_0}
\]

where

\[
\frac{\Pr(s_k^{(1)} - 1, e_k^{(1)})}{\Pr(s_k^{(1)}, e_k^{(1)})} = \begin{cases} 
\theta/(1 - \theta), & s_k^{(1)} = 2 \\
\theta, & s_k^{(1)} > 2 
\end{cases}
\]

Sample \( \{r_i\}_i, \{c_j\}_j \) The prior distributions of \( \{r_i\}_i \) and \( \{c_j\}_j \) are discrete uniform distributions over all the \( N! \) permutations. For each \( r_i \) (similar for \( c_j \)), we propose an exchange between \( r_i \) and \( r_{i'} \), where \( r_{i'} \) is randomly chosen. Since the prior distribution and the proposal distribution are identical, the acceptance ratio \( \min(1, \alpha) \) of the exchange is simply a likelihood ratio

\[
\alpha = \frac{\prod_j \ell(r_i^*, c_j, \rho_{ij}) \ell(r_{i'}^*, c_j, \rho_{ij})}{\prod_j \ell(r_i, c_j, \rho_{ij}) \ell(r_{i'}, c_j, \rho_{ij})}
\]

where \( r_i^* = r_{i'} \) and \( r_{i'}^* = r_i \) (row index exchange).

Complexity In general, the inference scheme involves two major parts: updating the SPP partition structure \( K_r, \{m_k, u_k^{(1)}, u_k^{(2)}\}_k \) and updating the indexing variables \( \{r_i\}_i, \{c_j\}_j \). The complexity of partition structure updating is \( O(K_r N^2) \) as there exist \( K_r \) patches and \( O(N^2) \) likelihood evaluations required for a structure change proposal to each patch. The complexity of row/column index updating is \( O(N^2) \) since each node can be exchanged with all the other nodes. Thus, the complexity for each iteration is \( O(K_r N^2) \). While the RJMCMC algorithm used in the MP relational model [24] has a lower theoretical complexity of \( O(N^2) \) in each iteration, however only one block can be updated each
time; while the SPP relational model updates all the patches simultaneously, which will significantly shorten the outer loop and the mixing rate can be comparable with the MP relational model in practice (see Figure 4 in the paper).