Leader Selection and Weight Redesign Problems for Multi-agent Systems

Bin Zhao, Yongqiang Guan, and Long Wang *

March 20, 2015

Abstract

For an uncontrollable system, adding leaders and adjusting edge weights are two methods to improve controllability. In this paper, controllability of multi-agent systems under directed topologies is studied, especially on leader selection problem and weight redesign problem. For the former one, necessary and sufficient algebraic conditions of single leader controllability are given. When a system cannot be controlled by only one leader, necessary and sufficient conditions for controllability with fewest leaders are proposed. To improve controllability by redesigning weights, the system is supposed to be structurally controllable, which holds if and only if the communication topology contains a spanning tree. It is proved that the number of fewest edges needed to adjust weights equals the rank deficiency of controllability matrix. An algorithm on how to perform weight redesign is presented. Simulation examples are provided to illustrate the theoretical results.

keywords: Multi-agent systems; Controllability; Leader selection; Weight redesign

1 Introduction

In the past few decades, due to the rapid development of computer science and communication technology, distributed cooperative control of multi-agent systems has become a hot topic in multidisciplinary research area. Many results have been obtained and applied in science and engineering areas, such as flocking in biology, formation of unmanned air vehicles, attitude alignment of satellite clusters and data fusion of sensors. Researches on multi-agent systems consist of several fundamental problems, including consensus [1], formation [2], flocking and swarming [3, 4], stabilizability [5, 6] and controllability [7], etc.

*Bin Zhao and Long Wang are with Center for Systems and Control, College of Engineering, Peking University, Beijing, 100871, China e-mail: bigbin@pku.edu.cn, longwang@pku.edu.cn.
†Yongqiang Guan is with College of Automation Science and Electrical Engineering, Beihang University, Beijing, 100191, China e-mail: guan-jq@163.com.
Controllability is a significant issue on multi-agent systems and attracts increasing attentions. A multi-agent systems is said to be controllable if appropriate external controls are put on the leaders such that all agents will achieve any designed configuration from any given initial states within a finite time. The controllability problem of multi-agent systems was put forward for the first time by Tanner [7], where an algebraic necessary and sufficient condition was presented under undirected communication topologies. Based on this, Ji et al. proposed a leader-follower connected structure and proved it to be a necessary condition to control a multi-agent system with multiple leaders [8, 9]. The models of agents used in the above are all with single-integrator dynamics. In [12], Wang et al. studied systems whose agents are with high-order dynamics and generic linear dynamics, and proved that controllability is congruously determined by the communication topology, regardless of agents’ dynamics. Further researches presented necessary and sufficient conditions for controllability on some special graphs, such as cycles and paths [13], stars and trees [14], grid graphs [15] and regular graphs [16], to name a few. Controllability problem from the perspective of graph theory was also investigated [10, 11]. Necessary conditions for controllability via equitable partitions and almost equitable partitions were discussed in [17, 18]. With respect to switching topologies, Liu et al. achieved several results on controllability [19, 20]. A parallel research in this field is structural controllability, which was proposed by Lin in [21] for linear time-invariant systems, and was brought into multi-agent systems in [22].

However, almost all the results are concerned with undirected graphs, few works consider directed topologies. The existing conclusions are only confined to strongly regular graphs and distance regular graphs [16], graph partitions [23] and some specific graphs [24]. In addition, the aforementioned results are practically focused on controllability conditions, rather than methods to turn an uncontrollable system into a controllable one. In fact, controllability is closely related to the communication topology. There are two methods to improve controllability, one is adding leaders and the other is adjusting edge weights. The former method derives the leader selection problem and the latter one derives the weight redesign problem. Leader selection requires the leaders to be as few as possible in order to minimize the quantity demand of external controllers. In this paper, we will first find out the necessary and sufficient condition for a multi-agent system to be controllable with only one leader. Then, a further topic on how to search for the fewest leaders will be investigated for a generic system. When the communication topology is weighted and the leaders are fixed, weight redesign is an effective method to improve controllability. The relevant results simply concern the algebraic or graphic conditions for structural controllability, which are merely qualitative researches [22]. In practice, it should be known exactly on which edges the weights are needed to be adjusted, as well as the new weights. Based on this, the problem of how to
redesign weights on fewest edges will be considered in this paper. Besides, there is no common algorithm on selecting leaders or adjusting weights for all systems. Fortunately, despite searching for the fewest leaders may be an NP-hard problem \[25\], we could reduce time complexity draw support from a greedy algorithm combined with an ergodic one. The algorithm on getting the fewest edges can be obtained via elementary row transformations on the controllability matrix.

Inspired by previous results, this paper studies leader selection problem and weight redesign problem for multi-agent systems under directed communication topologies. The contributions in our research are threefold: (i) Starting from single leader controllability, necessary and sufficient algebraic conditions of fewest leaders to ensure controllability are provided, via the Jordan form of Laplacian matrix and corresponding similarity transformation matrix; (ii) Necessary and sufficient graphic conditions are given for structural controllability, the exact fewest edges to be assigned new weights is given; (iii) Algorithms for leader selection and weight redesign are proposed.

This paper is organized as follows: In Section 2, basic concepts and preliminaries are given. In Section 3, problem of fewest leaders to control a system is proposed and solved; an algorithm to search for the leaders is also given here. In Section 4, weight redesign problem is investigated. Some typical examples are shown in Section 5 to illustrate the theoretical results. Finally, we draw the conclusions in Section 6.

**Notations**: Throughout this paper, the following notations are used. \(1_n\) is a vector with dimension \(n\) whose entries are all 1, and sometimes footprint \(n\) is omitted for convenience. If \(A\) is a square matrix, \(\det(A)\) denotes the determinant of \(A\). \(\text{diag}(a_1, a_2, \ldots, a_n)\) and \(\max\{b_1, b_2, \ldots, b_m\}\) represent the diagonal matrix with principal diagonals \(a_1, a_2, \ldots, a_n\) and the maximum value in \(b_1, b_2, \ldots, b_m\), respectively. The set of all real numbers is denoted by \(\mathbb{R}\). \(\binom{n}{k}\) is the number of combinations selecting \(k\) elements from \(n\) elements, \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\) where \(n!\) means the factorial from 1 to \(n\). A vector \(\alpha\) is called “all-0” if all the entries in \(\alpha\) are 0. A matrix is called “all-0” if all of its columns are all-0. \(|S|\) represents the cardinality of a set \(S\).

## 2 Preliminaries and problem formulation

### 2.1 Graph theory

A directed graph \(G = (V, E)\) consists of two parts, \(V = \{v_1, v_2, \ldots, v_n\}\) is the set of nodes in the graph, and \(E \subseteq V \times V\) represents the edge set. An edge in \(E\) is denoted by \((v_i, v_j)\) if the edge points at \(v_j\) from \(v_i\). \(v_i\) is called the parent node while \(v_j\) is called the child node and we say \(v_i\) is a neighbor of
v_j. The neighbor set of v_j is denoted by N_j = \{v_i \in V|(v_i, v_j) \in E\}. The in-degree of a node is the total number of its neighbors. Assume that there is no self-loop at any node, i.e., (v_i, v_i) \notin E, hence not any node is a neighbor of itself. A directed path \(P_n\) is a graph with \(n\) nodes and the edges are only \((v_1, v_2), (v_2, v_3), \cdots, (v_{n-1}, v_n)\). A directed cycle \(C_n\) is a directed path with an additional edge \((v_n, v_1)\). A complete graph \(K_n\) is the graph that each node is a neighbor of every other node. A tree graph \(T_v\) with root \(v\) is a graph that for each node other than \(v\), there exists one and only one path from \(v\) to this node. In a tree graph, a node is called a leaf if it has no child, and two nodes are said to be in different branches when there is no path from any one of them to the other. A graph \(G\) is said to contain a spanning tree if there exists a tree whose nodes are all those in \(V\) and edges in the tree are also in \(E\). A spanning forest of \(G\) is a set of trees covering \(V\) with no common nodes, and edges are all in \(E\). The minimal spanning forest is a spanning forest with fewest trees. A star graph is a kind of special tree graph whose root is a neighbor of all nodes rest. A directed graph is strongly connected if for any two nodes \(v_i\) and \(v_j\), there always exists a path from \(v_i\) to \(v_j\). A strong component in \(G\) is a maximal induced subgraph of \(G\), which is strongly connected. Since every single node is strongly connected, each node in \(G\) lies in a strong component. Length of the shortest path from \(v_i\) to \(v_j\) is called the distance from \(v_i\) to \(v_j\), denoted by \(d(v_i \rightarrow v_j)\). Especially, if \(v_i = v_j\), \(d(v_i \rightarrow v_j) = 0\). \(d(v_i \rightarrow v_j) = \infty\) when there is no path from \(v_i\) to \(v_j\).

The generalized distance partition is defined as follow. 

**Definition 1** The generalized distance partition of graph \(G\) relative to node \(v\) consists of a series of sets \(D_0, D_1, D_2, \cdots, D_l\) and \(D_{\infty}\), where \(D_0 = \{v\}\), \(D_l = \{w \in V|d(v \rightarrow w) = i\}\) and \(D_{\infty} = \{w \in V|d(v \rightarrow w) = \infty\}\). \(\bigcup_i D_i = V, i = 0, 1, 2, \cdots, l, \infty\).

In this paper, \(G\) is fixed. The adjacency matrix of \(G\) is \(A(G) = [a_{ij}] \in \mathbb{R}^{n \times n}\), where \(a_{ij}\) is the weight of edge \(e_{ij}\), and \(a_{ij} = 0\) if \((v_j, v_i) \notin E\). The Laplacian matrix of \(G\) is \(L = D - A\), \(D = \text{diag}(d_1, d_2, \cdots, d_n)\) where \(d_k = \text{deg}(k)\) is the in-degree of node \(k, k = 1, 2, \cdots, n\). A matrix \(M\) is said to be cyclic if its eigenpolynomial equals the minimal polynomial.

Since the mapping between the communication topology of a system and the corresponding graph is a bijection, “node” and “agent” are not distinguished in this paper for convenience.

### 2.2 Problem formulation

Consider a multi-agent system with \(n\) single-integrator dynamic agents:

\[
\dot{x}_i = u_i, \quad i = 1, 2, \cdots, n.
\]
Agents that can be driven by external inputs are called leaders. The set of leaders are denoted by $\mathcal{V}_l = \{i_1, i_2, \cdots, i_m\}$. The rest agents are called followers, denoted by $\mathcal{V}_f = \mathcal{V}/\mathcal{V}_l$. The control inputs on the agents obey a distributed consensus-based protocol:

$$u_i = \begin{cases} 
\sum_{j \in N_i} (x_j - x_i) + u_{o,i}, & i \in \mathcal{V}_l, \\
\sum_{j \in N_i} (x_j - x_i), & i \in \mathcal{V}_f,
\end{cases} \quad (2)$$

where $u_{o,i}$ is the external control on agent $i$.

The compact form of system (1) with protocol (2) is summarized as follows.

$$\dot{x} = -Lx + Bu, \quad (3)$$

where $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ and $u = (u_1, u_2, \cdots, u_n)^T \in \mathbb{R}^n$ represent the states and control inputs, respectively. $L$ is the Laplacian matrix and $B = (e_{i_1}, e_{i_2}, \cdots, e_{i_m}) \in \mathbb{R}^{n \times m}$. $e_i \in \mathbb{R}^n$ is a vector with the $i$-th entry 1 and the rest 0. For simplicity, only one dimensional states are considered in the following, i.e. $x_i \in \mathbb{R}$. However, the results obtained from this paper can be extended to arbitrary dimensional systems via Kronecker products.

**Definition 2** Multi-agent system (3) (or the corresponding communication graph) is said to be controllable if for any initial state $x(t_0)$ and target state $x^*$, $x(t_0)$ can be transferred to $x(t_1) = x^*$ in finite time $t_1 > t_0$ by putting external controls on leaders.

Especially, if all 0 entries in the adjacency matrix of the communication graph remain to be 0, and all other entries can be weighted positive numbers freely, the concept of structural controllability is proposed.

**Definition 3** Multi-agent system (3) (or the corresponding communication graph) is said to be structurally controllable if there exists a set of weights to make the system controllable.

As introduced, if multi-agent system (3) is not controllable, there are two methods to improve controllability. The two problems derived from these methods are defined as follow.

**Problem 1** Leader selection problem: For multi-agent system (3), find a set of nodes $\mathcal{V}_l \subseteq \mathcal{V}$ with minimum $|\mathcal{V}_l|$, such that when all nodes in $\mathcal{V}_l$ are chosen as leaders, the system is controllable.
Problem 2 Weight redesign problem: For multi-agent system (3), find a set of edges $E_m \subseteq E$ with minimum $|E_m|$, such that when the weights are properly adjusted on edges in $E_m$, the system is controllable without changing any leader.

3 Leader selection problem

Leader selection problem will be studied in this section. The investigation begins with a basic concept.

Definition 4 $r$ Leaders Controllable System: Multi-agent system (3) is said to be $r$ leaders controllable if the minimum $|\mathcal{V}_l| = r$. Especially, if $r = 1$, system (3) is called single leader controllable (SLC).

3.1 Single leader controllability

The leader selection problem is started by single leader controllability. As a matter of fact, controllability of system (3) is invariant under any labeling of the nodes in communication graph $G$. Suppose $B = e \triangleq e_i \in \mathbb{R}^n$. Since the controllability of system (3) is same to that of system $\dot{x} = Lx + Bu$, the latter system is studied for simplicity. Consider the controllability matrix $C = (e, Le, L^2e, \cdots, L^{n-1}e)$, system (3) is controllable if and only if $\text{rank}(C) = n$.

Denote the Jordan form of $L$ as $J = \text{diag}(J_0, J_1, J_2, \cdots, J_s)$, and the corresponding similarity transformation matrix is $P = (\xi_1, \xi_2, \xi_3, \cdots, \xi_n)$, $P^{-1}LP = J$. Here

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & & & \\ & & \ddots & & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{pmatrix}_{n_i \times n_i} , n_0 + n_1 + n_2 + \cdots + n_s = n. $$

Denote $m_i = n_0 + n_1 + n_2 + \cdots + n_i$, obviously $m_s = n$. $\xi_1, \xi_{m_0+1}, \xi_{m_1+1}, \cdots, \xi_{m_s+1}$ are the linearly independent eigenvectors of $L$. Correspondingly, $\xi_{m_0+2}, \xi_{m_0+3}, \cdots, \xi_{m+i}$ are the linearly independent generalized eigenvectors of $\lambda_i$. $C$ can be rewritten as:

$$C = P(P^{-1}e, JP^{-1}e, J^2P^{-1}e, \cdots, J^{n-1}P^{-1}e).$$

Let $\eta = P^{-1}e$, then $\text{rank}(C) = n$ if and only if $\det(\bar{C}) \neq 0$, where $\bar{C} = (\eta, J\eta, J^2\eta, \cdots, J^{n-1}\eta)$. 

6
Lemma 1  The determinant
\[
\det(\tilde{C}) = (-1)^{\frac{1}{2}(\sum_{i=1}^{s} n_i^2 + 1 - n)} \prod_{k=0}^{s} \eta_{m_k} \prod_{0 \leq i < j \leq s} (\lambda_j - \lambda_i)^{m_i, m_j},
\]
where \( m_0 = n_0 = 1, \lambda_0 = 0 \).

Proof: See Appendix in 7.1. □

If \( s = n - 1 \), namely \( C \) is diagonalizable, let \( \eta = 1_n \), conclusion of Lemma 1 degrades into the determinant of a Vandermonde matrix:
\[
\det \begin{pmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\
1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda_n & \cdots & \lambda_n^{n-1}
\end{pmatrix} = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i).
\]

Theorem 1  Multi-agent system (3) is SLC if and only if the following two conditions are satisfied simultaneously:
1. The Laplacian matrix \( L \) is cyclic;
2. There exists a column \( \eta^i \) in \( P^{-1} = (\eta^1, \eta^2, \cdots, \eta^n) \) such that \( \eta_{m_j}^i \neq 0 \) for all \( j = 1, 2, \cdots, s \).

In this circumstance, agent \( i \) is able to control the system, where \( i \) is also the column index of \( \eta^i \) in \( P^{-1} \).

Proof: Let \( \eta = P^{-1} e_i \) if agent \( i \) is selected as the leader.

(Sufficiency) Apparently \( \eta_{m_j}^i \) is the \( m_j \)-th entry in \( \eta^i \), and \( \eta_{m_j}^i \neq 0 \) for all \( j = 1, 2, \cdots, s \). Since \( L \) is cyclic, eigenvalues in different Jordan blocks are different. According to Lemma 1, \( \det(\tilde{C}) \neq 0 \), i.e. \( C \) is of full rank. Therefore, system (3) is controllable with the single leader agent \( i \).

(Necessity) Assume system (3) is SLC, there exists an \( i \) such that \( \det(\tilde{C}) \neq 0 \). This means agent \( i \) is the leader. According to Lemma 1, \( \det(\tilde{C}) \neq 0 \) leads to \( \prod_{0 \leq i < j \leq s} (\lambda_j - \lambda_i)^{m_i, m_j} \neq 0 \) and \( \prod_{k=0}^{s} \eta_{m_k}^i \neq 0 \), which ensure condition 1 and condition 2, respectively. □

Corollary 1  The following three assertions hold:
1. For multi-agent system (3), suppose that eigenvalues of the Laplacian matrix satisfy condition 1 of Theorem 7. Agent \( i \) can be selected as the single leader to control the system if and only if the \( i \)-th column in \( P^{-1} \) satisfies condition 2 in Theorem 7.
2. Suppose that the communication graph is undirected. Systems (3) is SLC if and only if all the eigenvalues of \( L \) are distinct and there exists a column in \( P^{-1} \) that none of the entries in this column is 0.
3. If multi-agent system (3) is SLC, there must be a spanning tree in the communication graph with the root being the leader.

Proof: Assertion 1 and assertion 2 are direct conclusions of Theorem 1 and the proof is omitted. For assertion 3, if the graph doesn’t contain a spanning tree, then rank(L) < n – 1, which means rank(Le, L²e, ⋯, Ln⁻¹e) < n – 1 and rank(C) < n. □

Remark 1 Although it is intuitive to judge controllability of multi-agent system (3) from the perspective of graph theory, to find a graphic necessary and sufficient condition for controllability is rather difficult. Ji et al. achieved a necessary and sufficient condition via an algebraic property of eigenvalues of the Laplacian matrix [9], but the conclusion is only applicable to judge controllability of some specific systems with given leaders. However, Theorem 1 and Corollary 1 showed necessary and sufficient conditions based on the Jordan form of L, which could not only judge controllability, but also solved SLC problem and contribute to searching for the fewest leaders. Jordan form of Laplacian matrix becomes a new channel to investigate controllability of multi-agent systems.

In this paper, checking single leader controllability is a fundamental step of searching for the fewest leaders. The whole algorithm will be introduced in the next subsection.

3.2 r leaders controllability

For a generic directed topology, the next theorem shows how to check controllability of a system with multiple leaders, as well as whether |V_i| is minimum. A proposition of C is needed to prove it.

Proposition 1 If C is rank deficient, there must be at least one of the two situations appears:

1. There exists an i, 1 ≤ i ≤ s such that the last row of Ĉ_i is all-0;
2. There exist i and j, 1 ≤ i < j ≤ s and θ ≠ 0 such that Ĉ_{i, i,j} = θĈ_{j, i,j} for k = 1, 2, ⋯, n.

Moreover, if η_{m_i} = η_{m_{i-1}} = ⋯ = η_{m_{s-r+1}} = 0, and η_{m_{s-r}} ≠ 0, 1 ≤ r ≤ n_i - 1, then exactly the last r rows of Ĉ_i are all-0; if Ĉ_{i, i-r} = θĈ_{j, j-r} for some 0 ≤ r ≤ min{r, n_j} , then Ĉ_{i, i-r} = θĈ_{j, j-r} for all 0 ≤ s ≤ min{r, n_j}.

Proof: According to Lemma 1, if C is rank deficient, there must be 1 ≤ i < j ≤ s such that λ_i = λ_j or η_{m_i} = 0. If η_{m_i} = 0, the last row of Ĉ_i is all-0, which results in situation 1. If η_{m_i} ≠ 0, λ_i = λ_j must holds, and corresponding rows are proportional between Ĉ_i and Ĉ_j with a same θ = η_{m_i}/η_{m_j}, which becomes situation 2. It can be verified, if the last r rows of Ĉ_i are all-0, since η_{m_{s-r}} ≠ 0, the reciprocal (r + 1)-th row of Ĉ_i is (η_{m_{s-r}}/η_{m_i}, η_{m_{s-r}}/η_{m_i}, λ_{1}j^2 η_{m_{s-r}}, ⋯ , λ_{n-1}j η_{m_{s-r}}). Proceed the elementary row transformations on
As seen from Proposition \[\text{III}\], if \( C \) is rank deficient, there must be an all-0 row or proportional rows in \( \tilde{C} \), and rows corresponding to different eigenvalue Jordan blocks will always be linearly independent.

For multi-agent system (3), \( J = \text{diag}(J_0, J_1, \cdots, J_s) \) is the Jordan form of Laplacian matrix \( L \). Distinct eigenvalues of \( L \) are denoted as \( \lambda_0, \lambda_1, \cdots, \lambda_t, t \leq s \). \( P^{-1}LP = J = P^{-1} = (\eta^1, \eta^2, \cdots, \eta^n) \).

**Theorem 2** System (3) is \( r \) leaders controllable if and only if there exist \( r \) columns in \( P^{-1} \), denoted as \( \bar{P} = (\eta_{c_1}, \eta_{c_2}, \cdots, \eta_{c_r}) \), satisfying the following two conditions simultaneously:

1. Assume the geometric multiplicity of eigenvalue \( \lambda_i \) is \( k_i \), with the corresponding Jordan blocks \( J_{i1}, J_{i2}, \cdots, J_{iki} \), then \( \text{rank}(\Omega_{\lambda_i}) = k_i \), where

\[
\Omega_{\lambda_i} = \begin{pmatrix}
\eta_{m_{i1}}^{c_1} & \eta_{m_{i1}}^{c_2} & \cdots & \eta_{m_{i1}}^{c_r} \\
\eta_{m_{i2}}^{c_1} & \eta_{m_{i2}}^{c_2} & \cdots & \eta_{m_{i2}}^{c_r} \\
\vdots & \vdots & \ddots & \vdots \\
\eta_{m_{iki}}^{c_1} & \eta_{m_{iki}}^{c_2} & \cdots & \eta_{m_{iki}}^{c_r}
\end{pmatrix},
\]

\( i = 0, 1, 2, \cdots, t; \ k_0 + k_1 + \cdots + k_t = s \);

2. Any combination of less than \( r \) columns in \( P^{-1} \) couldn’t satisfy condition 1.

In this circumstance, agents \( c_1, c_2, \cdots, c_r \) are able to control the system together, where \( c_1, c_2, \cdots, c_r \) are also the column indices of \( \bar{P} \) in \( P^{-1} \).

Proof: (Sufficiency) The proof is divided into three parts. Firstly, prove that when agents \( c_1, c_2, \cdots, c_r \) are selected as leaders, there is no all-0 row in \( \tilde{C} = (\tilde{C}_{c_1}, \cdots, \tilde{C}_{c_r}) \), where \( \tilde{C}_{c_i} \) is the transformed controllability matrix when taking agent \( c_i \) as the single leader. Next, prove that there will never be an all-0 row no matter how we take elementary row transformations on \( \tilde{C} \). Finally, condition 2 guarantees the minimality of \(|V_L|\).

Part 1: If there is an all-0 row in \( \tilde{C} \), according to Proposition \[\text{III}\] there exists an \( i, 1 \leq i \leq s \) such that the \( m_i \)-th row of \( \tilde{C} \) is all-0, which means \( \eta_{m_i}^{c_1} = \eta_{m_i}^{c_2} = \cdots = \eta_{m_i}^{c_r} = 0 \). This will lead to \( \text{rank}(\Omega_{\lambda_i}) < k_i \), which is in contradiction of condition 1.

Part 2: Since \( \Omega_{\lambda_i} \) is of full row rank, according to Proposition \[\text{III}\] all rows that correspond to \( \lambda_i \) in \( \tilde{C} \) are linearly independent. Considering that there is no all-0 row in \( \tilde{C} \), rows that correspond to different eigenvalues in \( \tilde{C} \) are linearly independent. This means \( \tilde{C} \) is of full row rank. With proper elementary row
transformations,
\[
\tilde{C} \rightarrow \begin{pmatrix}
I_0 & * & \cdots & * \\
I_1 & \cdots & * \\
\vdots & & \ddots & \vdots \\
I_t & & & I_t
\end{pmatrix}
\]
where \( I_s \) is the identity matrix with proper dimension, and \(*\) are blocks that may not be zeros but do not influence the rank of \( \tilde{C} \). Combine part 1 and part 2, once condition 1 is satisfied, the controllability matrix is of full row rank and thus the system is controllable.

Part 3: Condition 2 implies that any combination of less than \( r \) leaders will lead to an all-0 row or linearly dependent rows in \( \tilde{C} \). Therefore, \( C \) is not of full row rank.

(Necessity) Assume system (3) is controllable, \( C \) must be of full row rank. According to the proof of sufficiency and Proposition 1 if condition 1 is not satisfied, either an all-0 row or linearly dependent rows will appear in \( C \), which will lead to rank deficiency. For condition 2, if there exists an \( r' < r \) such that \( r' \) columns of \( P^{-1} \) satisfy condition 1, the proof of sufficiency demonstrates that the system can be controlled by \( r' \) leaders, which leads to a contradiction. This completes the proof. □

If only condition 1 of Theorem 2 is satisfied, the system is also controllable, whereas \( |V_l| \) may not be minimum.

**Corollary 2** The following two assertions hold:

1. Assume the geometric multiplicity of eigenvalues \( \lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_t \) are \( k_0, k_1, k_2, \cdots, k_t \) respectively, denote \( k = \max \{k_0, k_1, k_2, \cdots, k_t\} \), then the minimum \( |V_l| = r \) satisfies \( k \leq r \leq \sum_{l=0}^{t} k_l \).

2. \( K_n \) is controllable with at least \( n - 1 \) leaders, \( S_n \) is controllable with at least \( n - 1 \) leaders and the central node must be a leader, \( T \) with \( m \) leaves needs at least \( m \) leaders to be controllable.

Proof: For assertion 1, without loss of generality, suppose \( k = k_1 \). On the one hand, if \( |V_l| < k \), \( \Omega_{\lambda_l} \) is not of full row rank, i.e. \( \text{rank}(\Omega_{\lambda_l}) < k_1 \), which contradicts condition 1 in Theorem 2. On the other hand, there always exist \( k_i \) linearly independent columns in \( P^{-1} \) to ensure \( \text{rank}(\Omega_{\lambda_i}) = k_i \), therefore \( |V_l| \leq \sum_{l=0}^{t} k_l \).

For assertion 2, since \( n \) is an eigenvalue of the Laplacian matrix of \( K_n \) with geometric multiplicity \( n - 1 \), according to assertion 1, \( n - 1 \) leaders are needed to ensure controllability. Conclusions on trees and stars can be achieved by calculating the Jordan blocks and eigenvectors. □

Be worth mentioning, in claim 1, \( k \) leaders are not always enough to control the whole system, see Example 2. Theorem 2 is just a theoretical result, not a direct method to search for the fewest leaders. For a generic directed graph, this problem is extremely similar to the minimal controllability problem.
Algorithm 1

For multi-agent system (3), get the Laplacian matrix \( L \); Find a node \( n^* \) with minimum in-degree;
Calculate the controllability matrix \( C; m = 1; \)
if \( \text{rank}(C) = n \)
   Output “The system is SLC. Agent \( n^* \) could be the leader.”
else
   while \( \text{rank}(C) \neq n \)
      Find the node adding whom to the leader set could maximize the increase of the rank of \( C \),
      select it as a new leader; Update \( C; m = m + 1; \)
   end while
Get all identifiers of the leaders.
end if
while \( \text{rank}(C) = n \)
   \( m = m - 1; \)
   Try all combinations of \( m \) agents to be leaders, get a \( C \) with the maximum rank and the
   corresponding leader identifiers \( i_1, i_2, \ldots, i_m); \)
end while
Output “The system is \( m + 1 \) leaders controllable. Agents \( i_1, i_2, \ldots, i_m, i_{m+1} \) could be the leaders”.

Proposed in [25], which appears to be NP-hard. However, a greedy-based algorithm could reduce time complexity.

Remark 2 Algorithm 1 consists of a greedy part and an ergodic part. The greedy part is to lessen the upper bound of \( |V| \) rapidly, and the ergodic part is to get this number precisely. Although usually the greedy algorithm is enough to find the least leaders, it’s not suitable for all situations, see Example (2).

3.3 Leader selection for in-degree graphs

In this subsection, we show a kind of special graphs named in-degree regular graphs. The controllability of in-degree regular graphs can be validated more intuitively.

Definition 5 In-degree Regular Graph: A directed graph is called in-degree regular if the in-degrees of each node are equal, i.e. \( \deg_{\text{in}}(i) = \deg_{\text{in}}(j) \) for all \( 1 \leq i, j \leq n \).

Theorem 3 An in-degree regular graph can be controlled by agent 1 if and only if matrix \( M = [m_{ij}] \in \mathbb{R}^{(n-1) \times (n-1)} \) is invertible, where \( m_{ij} \) is the number of different paths from agent 1 to agent \( i+1 \) with length \( j \).

Proof: According to Definition 5, the Laplacian matrix \( L = D - A = dI - A, L^k = \sum_{i=0}^{k} C_i d^i (-A)^{k-i} \). The controllability matrix \( C = (e, Le, L^2e, \cdots, L^{n-1}e) \), and

\[
\text{rank}(C) = \text{rank}(e, (dI - A)e, \cdots, \sum_{i=0}^{n-1} C_i d^i (-A)^{n-i} e) = \text{rank}(e, Ae, \cdots, A^{n-1}e).
\]
As assumed, \( e = (1, 0, \cdots, 0)^T \) and the \( i \)-th entry of \( A^j e \) is the number of paths from agent 1 to agent \( i \) with length \( j \). \( M \) is the submatrix of \( C \) by deleting the first row and the first column. When \( M \) is invertible, \( C \) is of full rank and thus the system is controllable, vice versa. \( \square \)

**Corollary 3** The next two assertions on in-degree regular graphs hold.

1. For an in-degree regular graph with \( n \) nodes, whose adjacency matrix is \( A \). If each column in \( \sum_{i=1}^{n-1} A^i \) contains at least one 0, the system is not SLC.

2. For an in-degree regular graph, denote \( S = \sum_{k=1}^{n-1} A^k \), if at least \( m \) columns of \( S \) are needed to ensure the sum of them contains no 0 entry, then it should be \(|V| \geq m \) to make system (3) controllable.

Proof: For assertion 1, if each column of \( \sum_{i=1}^{n-1} A^i \) contains at least one 0, no matter which agent is selected as the leader, there will be at least one agent that couldn’t get information from leader, and this makes an in-degree regular graph uncontrollable.

For assertion 2, choosing \( m \) columns of \( S \) whose sum contains no 0 entry is to ensure a leader-follower connected structure. Therefore, at least \( m \) leaders are needed for controllability. \( \square \)

**Remark 3** According to Theorem 1, Corollary 1 and Theorem 3, \( K_n \) (\( n \geq 3 \)) is not SLC because its eigenpolynomial has a repeated root \( n \) with multiplicity \( n - 1 \), as well as all the rows in \( M \) are exactly the same. If any node in \( C_n \) is selected as the leader, the corresponding \( M \) is upper triangular with no 0 in the principal diagonal, and the graph is controllable. A comparison of different existing results on a path graph is shown in Example 1.

4 **Weight redesign problem**

When the communication graph is weighted and the leaders are fixed, controllability could be achieved by assigning new weights to different edges. In this section, weight redesign problem will be discussed.

4.1 **Structural controllability**

In order to control a system by weight redesign, the system should be structurally controllable. We start investigating structural controllability from a property of tree graphs.

**Proposition 2** \( T_v \) with root \( v \) as the single leader is controllable if and only if edges in different branches in \( T_v \) have no equal weights.

Proof: This conclusion is first achieved in [24]. Here we prove it in a pure algebraic method, see Appendix in 7.2. \( \square \)
Theorem 4 System (3) is structurally controllable with one leader if and only if the communication graph contains a spanning tree with the root being the leader.

Proof: (Necessity) If the communication graph doesn’t contain a spanning tree, there must be at least two agents that couldn’t get information from each other, and thus the minimum spanning forest contains more than one trees, denoted as $T_1, T_2, \cdots, T_r$. Once the leader is selected in some $T_i$, there always be agents in other trees that couldn’t get information from the leader and apparently the system is not controllable.

(Sufficiency) Suppose the Laplacian matrix of the spanning tree $T$ is $L_T$, and the corresponding similarity transformation matrix is $P_T$. Consider the communication topology $G$ with the Laplacian matrix $L = L_T + \varepsilon L_R$, where $L_R$ is the Laplacian matrix of the subgraph of $G$ by deleting the edges in $T$. Suppose the similarity transformation matrix of $L$ is $P, \Delta P \triangleq P^{-1} - P_T^{-1}$. Apparently, when $\varepsilon \to 0, \Delta P \to 0$. There exists a combination of weights that makes all eigenvalues of $L_T$ distinct and $T$ is controllable by Proposition 2. Therefore, all entries in the first column of $P_T^{-1}$ are not 0. When $\varepsilon$ is small enough, all eigenvalues of $L$ remain distinct and all entries in $\Delta P$ will be small enough such that the first column of $P^{-1}$ contains no 0. Therefore, system (3) is structurally controllable with the root of $T$ being the single leader. □

Corollary 4 System (3) is structurally controllable with at least $r$ leaders if and only if the minimum spanning forest of communication graph contains $r$ trees with the roots being the leaders.

Proof: (Necessity) Refer to the necessity proof of Theorem 4.

(Sufficiency) When $r = 2$, i.e. the spanning forest $F = \{T_{v_1}, T_{v_2}\}$. Since the graph is not structurally controllable, at least two leaders are needed. Select $v_1$ and $v_2$ as leaders. With proper weights, $T_{v_1}$ and $T_{v_2}$ could be controlled by $v_1$ and $v_2$ respectively. For each edge whose parent node lies in $T_{v_1}$ (or $T_{v_2}$) and the child node lies in $T_{v_2}$ (or $T_{v_1}$), assign a weight small enough to neglect the effect of it, then the whole graph remains controllable. Hence the conclusion holds for $r = 2$. Suppose the conclusion holds for $r = n$. When $r = n+1$, i.e. $F = \{T_{v_1}, T_{v_2}, \cdots, T_{v_{n+1}}\}$, by the induction hypothesis, any $n$ trees in $F$ are structurally controllable with their roots being the leaders. Without loss of generality, suppose $T_{v_1}, T_{v_2}, \cdots, T_{v_n}$ are controllable. Select $v_{n+1}$ as a new leader, assign proper weights to $T_{v_{n+1}}$. $F$ is controllable. Assign small weights to the connections among the trees could make the whole graph controllable. According to mathematical induction, the conclusion holds for any positive integer $r$. □

Proposition 3 An in-degree regular graph is structurally controllable if and only if there exists one
column of $\sum_{k=1}^{n-1} A^k$ that contains no 0 entry except for the principal diagonal elements, where A is the adjacency matrix of the graph.

Proof: Without loss of generality, consider the first column of $\sum_{k=1}^{n-1} A^k$, delete the first entry and denote the remained vector as $\eta$. $\eta_i \neq 0$ if and only if there exists a path from agent 1 to agent $i+1$. Therefore, $\eta$ contains no 0 entry if and only if the graph contains a spanning tree, which is a necessary and sufficient condition of structural controllability. □

4.2 Fewest edges to be assigned new weights

The next theorem puts forward the number of fewest edges needed to adjust weights.

**Theorem 5** Suppose that the communication graph of multi-agent system (3) contains a directed spanning tree, and the root is the single leader. If the rank of controllability matrix is $n - r$, then there exist $r$ edges such that the system could be controllable by adjusting weights on them, and any adjustment on less than $r$ edges cannot make system (3) controllable.

Proof: When $r = 0$, the conclusion is obvious. Without loss of generality, assume the agents are labeled as follow: Label the root as 1, get the generalized distance partition $\{D_0, D_1, \cdots, D_p\}$ where $D_0$ is the root, and there is no $D_\infty$ due to the existence of spanning tree in the graph. Label the nodes from those in $D_1$ to those in $D_p$ successively. With this method, for agent $i$ in $D_q, q = 2, 3, \cdots, p$, the first $q-1$ entries of the $i$-th row in $L$ are 0.

For $1 \leq r < n$, we prove that there exists one edge whose weight if be adjusted properly, could increase the rank of controllability matrix $C$ by 1. Mathematically, this equals to prove that if $k_i \neq 0, i = 1, 2, \cdots, s,$

$$(k_1e_{i_1} + k_2e_{i_2} + \cdots + k_se_{i_s})^T L^m e_1 = 0, \quad (4)$$

for $m = 0, 1, 2, \cdots, n-1$, then, there exist $\Delta L$ and $m_0$ such that $(k_1e_{i_1} + k_2e_{i_2} + \cdots + k_se_{i_s})^T (L + \Delta L)^{m_0} e_1 \neq 0$ where $\Delta L$ contains only two opposite nonzero elements who lie in a same row. The positive one is in the principal diagonal and the other is in front of it, $m_0 \leq n - 1$.

Suppose the two nonzero elements are in the $j$-th row, $j \neq i_t, t = 1, 2, \cdots, s$, it can be verified,

$$(k_1e_{i_1} + k_2e_{i_2} + \cdots + k_se_{i_s})^T (L + \Delta L)^m e_1 = 0 \quad (5)$$

for all $m = 0, 1, 2, \cdots, n-1$. If we intend to increase rank of $C$, the revised edge must be selected from one of the $i_1, i_2, \cdots, i_s$-th rows in $L$. Take the $i_1$-th row as an example.
Next we show the existence of $\Delta L$ and $m_0$. Suppose that there is an edge from agent $j$ to agent $i_1$, i.e. $L_{j,i_1} \neq 0$, and the equation (5) holds for $m = 0, 1, 2, \cdots, n - 1, \varepsilon > 0$. Here $\Delta L_{j,i_1} = -\varepsilon, \Delta L_{i_1,i_1} = \varepsilon$ and the other entries in $\Delta L$ are all 0. Under the assumption (4), combined with (5), we get

$$(k_1e_{i_1} + k_2e_{i_2} + \cdots + k_se_{i_s})^T((L + \Delta L)^m - L^m)e_1 = 0$$

for $m = 0, 1, 2, \cdots, n - 1$. When $m = 1$, $(k_1e_{i_1} + k_2e_{i_2} + \cdots + k_se_{i_s})^T\Delta Le_1 = \varepsilon L_{i_1,1} = 0$ yields $L_{i_1,1} = 0$, which means agent $i_1$ couldn’t get information from the root of the spanning tree. Denote $D_m = (L + \Delta L)^m - L^m$, therefore $D_{m+1} = D_mL + D_m\Delta L + L^m\Delta L$. Since the $(i_1,1)$ entry in $D_m\Delta L$ and $L^m\Delta L$ are both 0, the $(i_1,1)$ entry in $D_mL$ must also be 0 to satisfy (6). Considering that the $(i_1,1)$ entry in $D_mL$ is $\varepsilon(L_{i_1i_1} - L_{i_1,j})$, and this will lead to $L_{i_1i_1} = L_{i_1,j}$, which contradicts the fact that $L_{i_1i_1} > 0$ and $L_{i_1,j} < 0$. Here we get that there exist $\Delta L$ and $m_0$ such that $(k_1e_{i_1} + k_2e_{i_2} + \cdots + k_se_{i_s})^T((L + \Delta L)^m_0 - L^m_0)e_1 \neq 0$.

Apparently, $\Delta L$ only changes the $i_1$-th row of $C$. Without loss of generality, suppose no $s - 1$ vectors of $C_{r_1}, C_{r_2}, \cdots, C_{r_s}$ are linearly dependent. Next we prove that there exists a proper $\varepsilon > 0$ such that $C_{r_1} + \delta C_{r_1} + C_{r_2} + \cdots + C_{r_s}$ are linearly independent. Consider the equation $k_1(C_{r_1} + \delta C_{r_1}) + k_2C_{r_2} + \cdots + k_sC_{r_s} = 0$, it is equal to $(k_1 - k_1)C_{r_1} + (k_2 - k_2)C_{r_2} + \cdots + (k_s - k_s)C_{r_s} + k_1\delta C_{r_1} = 0$. With the discussion above, we can get that if $\delta C_{r_1} \neq 0$, $\delta C_{r_1}$ will change with $\varepsilon$ nonlinearly and thus a proper $\varepsilon$ will ensure that $\delta C_{r_1}$ is linearly independent to $C_{r_1} + \delta C_{r_1}, C_{r_2}, \cdots, C_{r_s}$. If $k_1' \neq 0$, $(k_1' - k_1)C_{r_1} + (k_2' - k_2)C_{r_2} + \cdots + (k_s' - k_s)C_{r_s} + k_1'\delta C_{r_1} = 0$ will never be 0, therefore $k_1' = 0$. Since $C_{r_1} + \delta C_{r_1}, C_{r_2}, \cdots, C_{r_s}$ are linearly independent, once $k_1' = 0$, $k_j' - k_i = -k_i$ for all $i = 2, 3, \cdots, s$. Finally, $k_1' = k_2' = \cdots = k_s' = 0$. This means that, there exist proper $\Delta L$ and $m_0$ to eliminate one of the linearly dependent rows in the controllability matrix. Based on this, assigning a proper weight to one proper edge, the rank of $C$ will increase 1 and only 1. This implies that exactly $r$ different edges from different rows are needed to be adjusted to fulfill the decreased rank of $C$. □

**Remark 4** Refer to the proof of Theorem 5, the next two conclusions can be achieved.

1. Suppose that controllability of system (3) can be improved by adjusting the weight on edge $e^*$ and the weight increment is $\varepsilon$, i.e. $\Delta L_{e^*} = -\varepsilon$, then there exists an $n - 1$ order polynomial of $\varepsilon$, say $f(\varepsilon)$, such that $\Delta L_{e^*}$ fails to increase rank of $L$ if and only if $f(\varepsilon) = 0$. Hence, there are no more than $n - 1$ values of $\varepsilon$ that would fail to improve controllability. Therefore, if we randomly endue a new weight to $e^*$, the probability of successfully increase the rank of $C$ is 1.

2. If $e^*$ should be selected from the $i$-th row of $L$, denoted as $L_{i,*}$, then the first nonzero entry in $L_{i,*}$ could be the edge to adjust weight. This means we could redesign the weight of the edge that connects to agent
Algorithm 2

Get all the nodes that could be a root of a spanning tree, identify them as \(v_1, v_2, \cdots, v_m\), \(C = 0_{n \times n}\);

For \(k = 1 : m\)

Label the root \(v_k\) as 1, get the generalized distance partition \(\{D_0 = v_k, D_1, \cdots, D_p\}\), label the whole system from nodes in \(D_1\) to nodes in \(D_p\) successively, get the Laplacian matrix \(L\) and the controllability matrix \(C\);

if \(\text{rank}(C) = n\)

Output “The system is controllable with leader agent \(k\)”, exit;

else if \(\text{rank}(C) < r\)

\(r = \text{rank}(C), s = k\);

end if

end for

Use row elimination to get the all-0 rows, get their row identifiers \(i_1, i_2, \cdots, i_s\);

while \(\text{rank}(C) < n\)

for \(1 \leq j \leq s\)

Add weight \(j\theta\) to the edge corresponding to the first nonzero element in the \(i_j\)-th row of \(L\);

end for

Get \(\tilde{L}, L = \tilde{L}\); Calculate \(C; \theta = 1.1\theta\);

end while

Output “The number of fewest edges to be assigned new weights is \(s\), and an available graph Laplacian is \(L\)”.

\(i\) from the agent with the minimum identifier in \(N_i\).

Next we show an algorithm on how to perform weight redesign on proper edges. To express more explicitly, the algorithm here is designed for the graphs that contain spanning trees. However, it can be improved to fit for generic directed graphs.

5 Simulation

Examples are presented in this section to illustrate the effectiveness of theoretical results.

Example 1 Figure 1 shows a path graph with \(n = 5\) nodes. The eigenvalues of its Laplacian matrix are \(0.382, 2.618, 0, 3.618, 1.382,\) and the corresponding \(P^{-1}\) is

\[
P^{-1} = \begin{pmatrix}
-0.3618 & -0.2236 & 0 & 0.2236 & 0.3618 \\
-0.1382 & 0.2236 & 0 & -0.2236 & 0.1382 \\
0.2 & 0.2 & 0.2 & 0 & 0.2 \\
0.0382 & -0.1 & 0.1236 & -0.1 & 0.0382 \\
0.2618 & -0.1 & -0.3236 & -0.1 & 0.2618
\end{pmatrix}.
\]

Since 0 only exists in the third column of \(P^{-1}\), and all the eigenvalues of \(L\) are distinct, Figure 1 is controllable with any single agent except agent 3. A same conclusion can be also obtained from [13].
and [17], in the aspect of number theory and symmetry of the graph, respectively. This suggests that our result is an algebraic extension from the former results.

**Example 2** Figure 2 shows a system with four agents. The Laplacian matrix and the corresponding $P^{-1}$ are shown as follow:

\[
L = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
-1 & -1 & 2 & 0 \\
-1 & -1 & -1 & 3
\end{pmatrix},
\]

\[
P^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{pmatrix}.
\]

There are at least two 0 in each column of $P^{-1}$, therefore Figure 2 is not SLC. If agent 1 and agent 3 are selected as leaders, the graph is controllable, but once agent 2 is selected as a leader, two more leaders are needed. This suggests that the greedy algorithm couldn’t always ensure fewest leaders. Moreover, it also demonstrates that even if the eigenvalues of the Laplacian matrix are distinct, Figure 2 may also not be SLC.

**Example 3** Figure 3 shows a directed communication graph of system (3). The Laplacian matrix is
\begin{equation*}
L = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & -1 & -1 & 2
\end{pmatrix}
\end{equation*}

with eigenvalues 0, 2, 0.2451, 1.8774 ± 0.7449i, and

\begin{equation*}
P^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 \\
-1.2672 & 0.3106 & 0.5451 & 0.2345 & 0.1770 \\
0.1336 + 0.1283i & -0.1553 - 0.3404i & -0.2726 + 0.0740i & -0.1172 + 0.4143i & 0.4115 - 0.2762i \\
0.1336 - 0.1283i & -0.1553 + 0.3404i & -0.2726 - 0.0740i & -0.1172 - 0.4143i & 0.4115 + 0.2762i
\end{pmatrix}
\end{equation*}

Since there is at least one 0 in each column, the system is not SLC. Actually it can be controlled by two leaders, one of which must be agent 1 and the other could be agent 4 or 5. Meanwhile, the graph contains a spanning tree, hence system (3) is structurally controllable with one leader. Select agent 1 as the leader, we can get that rank of the controllability matrix is 4, thus the system can be controlled only adjusting the weight on one edge. Here revise \(w_{35}\) from the original 1 to 1.1, and the Laplacian matrix turns to be

\begin{equation*}
\bar{L} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & -1.1 & -1 & 2.1
\end{pmatrix}
\end{equation*}

with the eigenvalues 0, 0.2493, 1.8930, 1.9788 ± 0.7305i, and the first column of the corresponding \(\bar{P}^{-1}\) is \([1, -1.2639, 0.0105, 0.1267 + 0.1412i, 0.1267 - 0.1412i]^T\), which mean the system becomes controllable. This illustrates the conclusions in Section 4.

\section{Conclusion}

This paper has studied the controllability of multi-agent systems with directed communication topologies. The concept of single leader controllability was proposed and the problem of fewest leaders to control a system was investigated. Algebraic necessary and sufficient conditions on how to select the fewest leaders were presented based on the Jordan blocks of \(L\) and the corresponding transformation matrix \(P^{-1}\). For in-degree regular graphs, simplified conditions were also given to check controllability.
Denote $\tilde{\eta}_i = (\eta_{m_i+1}, \eta_{m_i+2}, \ldots, \eta_{m_i-1}, \eta_{m_i})^T$, $i = 1, 2, \ldots, s$, especially $\tilde{\eta}_0 = \eta_1$, and $\tilde{C}$ can be turned into blocks:

$$
\tilde{C} = \begin{pmatrix}
\tilde{\eta}_0 & J_0 \tilde{\eta}_0 & J_0^2 \tilde{\eta}_0 & \cdots & J_0^{n-1} \tilde{\eta}_0 \\
\tilde{\eta}_1 & J_1 \tilde{\eta}_1 & J_1^2 \tilde{\eta}_1 & \cdots & J_1^{n-1} \tilde{\eta}_1 \\
\tilde{\eta}_2 & J_2 \tilde{\eta}_2 & J_2^2 \tilde{\eta}_2 & \cdots & J_2^{n-1} \tilde{\eta}_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\tilde{\eta}_s & J_s \tilde{\eta}_s & J_s^2 \tilde{\eta}_s & \cdots & J_s^{n-1} \tilde{\eta}_s
\end{pmatrix} = 
\begin{pmatrix}
\tilde{C}_0 \\
\tilde{C}_1 \\
\tilde{C}_2 \\
\vdots \\
\tilde{C}_s
\end{pmatrix},
$$

where $\tilde{C}_i = (\tilde{\eta}_i, J_i \tilde{\eta}_i, J_i^2 \tilde{\eta}_i, \ldots, J_i^{n-1} \tilde{\eta}_i)$, $i = 0, 1, 2, \ldots, s$.

Expand each element in $\tilde{C}_i$ yields

$$
\tilde{C}_i = \begin{pmatrix}
\eta_{i,1} & \lambda_i \eta_{i,2} + \eta_{i,3} & \lambda_i^2 \eta_{i,3} + 2\lambda_i \eta_{i,4} + \eta_{i,5} & \lambda_i^3 \eta_{i,4} + 3\lambda_i^2 \eta_{i,5} + 3\lambda_i \eta_{i,6} + \eta_{i,7} & \cdots & \lambda_i^{n-1} \eta_{i,n+1} \\
\eta_{i,2} & \lambda_i \eta_{i,3} + \eta_{i,4} & \lambda_i^2 \eta_{i,4} + 2\lambda_i \eta_{i,5} + \eta_{i,6} & \lambda_i^3 \eta_{i,5} + 3\lambda_i^2 \eta_{i,6} + 3\lambda_i \eta_{i,7} + \eta_{i,8} & \cdots & \lambda_i^{n-2} \eta_{i,n-1} + (n-1)\lambda_i^{n-3} \eta_{i,n-2} \\
\eta_{i,3} & \lambda_i \eta_{i,4} + \eta_{i,5} & \lambda_i^2 \eta_{i,5} + 2\lambda_i \eta_{i,6} + \eta_{i,7} & \lambda_i^3 \eta_{i,6} + 3\lambda_i^2 \eta_{i,7} + 3\lambda_i \eta_{i,8} + \eta_{i,9} & \cdots & \lambda_i^{n-3} \eta_{i,n-2} + (n-2)\lambda_i^{n-4} \eta_{i,n-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\eta_{i,n-1} & \lambda_i \eta_{i,n} + \eta_{i,n+1} & \lambda_i^2 \eta_{i,n+1} + 2\lambda_i \eta_{i,n+2} + \eta_{i,n+3} & \lambda_i^3 \eta_{i,n+2} + 3\lambda_i^2 \eta_{i,n+3} + 3\lambda_i \eta_{i,n+4} + \eta_{i,n+5} & \cdots & \lambda_i^{n-2} \eta_{i,n-1} + (n-1)\lambda_i^{n-3} \eta_{i,n-2} + (n-2)\lambda_i^{n-4} \eta_{i,n-3} \\
\tilde{\eta}_{i,n} & \lambda_i \tilde{\eta}_{i,n} + \lambda_i^2 \tilde{\eta}_{i,n} + \lambda_i^3 \tilde{\eta}_{i,n} + \cdots + \lambda_i^{n-1} \eta_{i,n} + \lambda_i^{n} \eta_{i,n}
\end{pmatrix}
$$

If there exists a $j$ such that $\eta_{m_j} = 0$, i.e. $\tilde{\eta}_{j,n_j} = 0$, then $\det(\tilde{C}) = 0$ holds. When $\eta_{m_j} \neq 0$ for all $j$, this paper also studied the weight redesign problem, which aims to adjust weights on fewest edges to ensure controllability. The result showed that a multi-agent system is structurally controllable if and only if the communication graph contains a spanning tree. The number of fewest edges equals the rank deficiency of controllability matrix. Algorithms to search for the fewest leaders and edges were also provided.

### 7 Appendix

#### 7.1 Proof of Lemma 1

Figure 3: Interaction topology of Example 3
through a series of elementary row transformations,

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 1 & C_{n_i}^{-1} \lambda_i & \cdots & C_{n_i}^{-1} \lambda_i^{n_i - n_i + 1} \\
0 & 0 & 0 & 0 & \cdots & (n_i - 1) \lambda_i & C_{n_i}^{-2} \lambda_i^2 & \cdots & C_{n_i}^{-2} \lambda_i^{n_i - n_i + 2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 2 \lambda_i & 3 \lambda_i^2 & \cdots & (n_i - 1) \lambda_i^{n_i - 2} & n_i \lambda_i^{n_i - 1} & \cdots & (n - 1) \lambda_i^{n_i - 2} \\
1 & \lambda_i & \lambda_i^2 & \lambda_i^3 & \cdots & \lambda_i^{n_i - 1} & \lambda_i^{n_i - 1} & \cdots & \lambda_i^{n_i - 1}
\end{pmatrix}
\]

(7)

Denote \( \tilde{C} = (\tilde{C}_1^T, \tilde{C}_2^T, \cdots, \tilde{C}_s^T)^T \), obviously, \( \det(\tilde{C}) = \prod_{k=0}^{s} n_i^{n_i} \det(\tilde{C}) \). Next, we show how to transform \( \tilde{C} \) into blocked upper triangular form, taking \( \tilde{C}_0 \) and \( \tilde{C}_i \) as examples. Here suppose \( \lambda_i \neq \lambda_0 \) and 0 is a simple eigenvalue of \( L \).

\[
\begin{pmatrix}
\tilde{C}_0 \\
\tilde{C}_i
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^{n_i - 1} & \cdots & \lambda_0^{n_i - 1} \\
0 & 0 & 0 & \cdots & 1 & \cdots & C_{n_i}^{-1} \lambda_i^{n_i - n_i + 1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 1 & 2 \lambda_i & 3 \lambda_i^2 & \cdots & (n_i - 1) \lambda_i^{n_i - 2} & (n - 1) \lambda_i^{n_i - 2} \\
0 & \lambda_i - \lambda_0 & \lambda_i^2 - \lambda_0^2 & \cdots & \lambda_i^{n_i - 1} - \lambda_0^{n_i - 1} & \cdots & \lambda_i^{n_i - 1} - \lambda_0^{n_i - 1}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^{n_i - 1} & \cdots & \lambda_0^{n_i - 1} \\
0 & 0 & 0 & \cdots & \lambda_i - \lambda_0 & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \lambda_i - \lambda_0 & \cdots & \alpha_1 & \cdots & \alpha_2 \\
0 & \lambda_i - \lambda_0 & \lambda_i^2 - \lambda_0^2 & \cdots & \lambda_i^{n_i - 1} - \lambda_0^{n_i - 1} & \cdots & \lambda_i^{n_i - 1} - \lambda_0^{n_i - 1}
\end{pmatrix}
\]

where \( \alpha_1 = (n_i - 2) \lambda_i^{n_i - 2} - \lambda_0 \lambda_i^{n_i - 3} - \lambda_0^\alpha_i - \lambda_0^{n_i - 2}, \) \( \alpha_2 = (n - 2) \lambda_i^{n_i - 2} - \lambda_0 \lambda_i^{n_i - 3} - \lambda_0^{n_i - 3} \lambda_i - \lambda_0^{n_i - 2} \), and * makes no effect on \( \det(\tilde{C}) \). Both \( \alpha_1 \) and \( \alpha_2 \) have a factor \( \lambda_i - \lambda_0 \).

Repeat the elementary row transformation as above yields
\[
\det(\mathcal{C}) = \prod_{0 \leq i < j \leq s} (\lambda_j - \lambda_i)^{p_{ij}} \det \left( \begin{array}{cccc}
B_0 & * & * & * \\
* & B_1 & * & * \\
* & * & B_2 & * \\
& \ddots & \ddots & * \\
& & & B_s
\end{array} \right),
\]

where \( B_0 = 1 \), \( B_i = \begin{pmatrix} 1 \\ \vdots \\ 1 & * \\ 1 & * & * \\ \end{pmatrix}, \) \( i = 1, 2, \ldots, s \) and * represents zero or nonzero block that do not influence the value of the determinant. Obviously,

\[
\det \left( \begin{array}{cccc}
B_0 & * & * & * \\
* & B_1 & * & * \\
* & * & B_2 & * \\
& \ddots & \ddots & * \\
& & & B_s
\end{array} \right) = \prod_{i=0}^{s} \det(B_i) = \prod_{i=0}^{s} (-1)^{n_i-1} \sum_{j=1}^{n_i-1} j = 0.
\]

especially, if \( n_i = 1 \), define \( \sum_{j=1}^{n_i-1} j = 0 \). This completes the proof. □

7.2 Proof of Proposition

(Necessity) If two edges of \( T_v \) in different branches share a common weight, there must be two equal elements \( L_{ii} = L_{jj} = \lambda \) in the principal diagonal of \( L \), and \( L_{ij} = L_{ji} = 0 \). Since \( \lambda \) is an eigenvalue of \( L \), there must be two linearly independent eigenvectors of \( \lambda \). Therefore, in the Jordan form of \( L \), two Jordan blocks share a common eigenvalue \( \lambda \). According to Theorem 1, the system is not SLC.

(Sufficiency) The proof is divided into four parts. First assume all the edges have different weights, hence \( P^{-1}LP = D \) where \( D \) is the diagonal form of \( L \), and we prove that \( p_{ij} = 0 \) for all \( i < j \). Then we show the expressions of \( p_{ij} \) for all \( 1 \leq i, j \leq n \). Next we prove that none of the elements in the first column of \( P^{-1} \) is 0. Finally we prove that controllability will not be broken when adding one leaf to a controllable tree with any weight different from weights in other branches.

Part 1: Denote \( R = \lambda_j I - L \), obviously when \( j \geq 1 \), the only nonzero element in the first row of \( R \) is \( r_{11} \), thus \( p_{1j} = 0 \). Suppose when \( i < i' \), where \( i' \leq j - 1 \), \( p_{ij} = 0 \), since \( 0 = \sum_{k=1}^{n} l_{r_{1k}p_{k}} - \lambda_j p_{jj} = \sum_{k=1}^{i'} l_{r_{1k}p_{k}} + \sum_{k=i'+1}^{i} l_{r_{1k}p_{k}} = l_{r_{1r}} p_{r_{r},j} \) and \( l_{r_{1r}} \neq 0 \), so \( p_{r_{r},j} = 0 \). According to mathematical induction, \( p_{ij} = 0 \) for all \( i < j \).
Part 2: Since \( p_i = (p_{i1}, p_{i2}, \ldots, p_{in})^T \) is an eigenvector of \( L, (L - \lambda_i I)p_i = 0 \), which means \( \sum_{k=1}^{n} l_{ik} p_{ki} - \lambda_i p_{ii} = 0 \). From part 1 we know \( p_{i1} = p_{i2} = \cdots = p_{i-1,j} = 0 \), and for \( T, l_{i,i+1} = l_{i,i+2} = \cdots = l_{i,n} = 0 \), this yields \( (l_{ii} - \lambda_i)p_{ii} = 0 \). Owing to \( l_{ii} - \lambda_i = 0 \), \( p_{ii} \) could be any number, and \( p_{ii} = 1 \) is chosen here without loss of generality. For each \( i > j \), we get \( 0 = \sum_{k=1}^{n} l_{ik} p_{ki} - \lambda_j p_{ij} = \sum_{k=1}^{n} l_{ik} p_{ki} - \lambda_j p_{ij} \). As mentioned before, there exists one and only one \( k_i < i \) for each \( i \) such that \( l_{ik} \neq 0 \). Correspondingly, \( p_{ij} = \frac{l_{ik} p_{ij}}{\lambda_j - \lambda_k} \). Combine the results afore yields

\[
p_{ij} = \begin{cases} 
0, & i < j \\
1, & i = j \\
\frac{l_{ik} p_{ij}}{\lambda_j - \lambda_k}, & i > j 
\end{cases}
\]

Part 3: Now consider the first column of \( P^{-1} \), denoted as \( q = (q_1, q_2, \ldots, q_n)^T \). Obviously, \( q_1 \neq 0 \), otherwise, \( Pq \neq e \) where \( e = (1, 0, 0, \ldots, 0)^T \). Suppose \( q_1 \neq 0 \) with \( i < i^* \), and \( \sum_{k=1}^{i} p_{ik} q_k = e_i \). Since \( e_i = \sum_{k=1}^{i} p_{ik} q_k, q_{i^*} = -\sum_{k=1}^{i-1} p_{ik} q_k = -l_{i^*k_1}(\frac{p_{i^*k_1}}{\lambda_{i^*-i_1}}, \frac{p_{i^*k_2}}{\lambda_{i^*-i_2}}, \ldots, \frac{p_{i^*k_{i^*-1}}}{\lambda_{i^*-i_{i^*-1}}})(q_1, q_2, \ldots, q_{i^*-1})^T \). It follows from part 2 that the only nonzero element in the \( i^* \) row of \( P \) except for \( p_{i^*,1} = p_{i^*,2} = 1 \) is \( p_{i^*,k_{i^*}} \).

Denote \( \xi_{i^*} = -l_{i^*k_1}(\frac{p_{i^*k_1}}{\lambda_{i^*-i_1}}, \frac{p_{i^*k_2}}{\lambda_{i^*-i_2}}, \ldots, \frac{p_{i^*k_{i^*-1}}}{\lambda_{i^*-i_{i^*-1}}})^T, \) and \( \eta_j = (p_{1j}, p_{2j}, \ldots, p_{j,i^*-1})^T \). Obviously, \( \eta_j \) are linear independent, \( i = 1, 2, \ldots, i^*-1 \). Thus, there exist \( c_1, c_2, \ldots, c_{i^*-1} \) such that \( \sum_{k=1}^{i^*-1} c_k \eta_k^T = \xi_{i^*}^T \).

Consider \( \eta_2, \ldots, \eta_{i^*-1}, \xi \), which are also linearly independent, so that \( c_1 \neq 0 \). According to the induction hypothesis, \( q_{i^*} = (q_1, q_2, \ldots, q_{i^*-1}) \sum_{k=1}^{i^*-1} c_k \eta_k = \sum_{k=1}^{i^*-1} c_k e_k = c_1 \neq 0 \). Thus, each element in \( q \) is not 0, and in this case, condition 2 in Theorem 1 is satisfied.

Part 4: Suppose the diagonal form of \( L \) is \( J, P^{-1}LP = J \). If weight \( \mu \) of the new edge is different from every other weight, the Jordan form of the new graph is

\[
\hat{J} = \begin{pmatrix} J & \alpha^T \mu \\
\mu & \beta^T \end{pmatrix} = \begin{pmatrix} P^{-1} & 0 \\
\alpha^T & 1 \end{pmatrix} \begin{pmatrix} L & 0 \\
\gamma^T & \mu \end{pmatrix} \begin{pmatrix} P & 0 \\
\beta^T & 1 \end{pmatrix}.
\]

This means \( \alpha^T P + \beta^T = 0 \) and

\[
\alpha^T LP + \gamma^T P + \mu \beta^T = \beta^T (\mu I - J) + \gamma^T P = 0.
\]

The first entry of \( \beta \) is not 0, otherwise, the first element of \( \beta^T (\mu I - J) \) is 0. Yet the first column of \( P \) is \( 1_n \). It follows from Theorem 1 that the system is controllable. On the other hand, if \( \mu \) equals to the weight of an edge in the path from the root to the new edge, \( \mu \) is different from any weights in other branches due to the necessity conclusion. This guarantees that there is only one eigenvector.
correspond to eigenvalue $\mu$, where $c$ is a nonzero coefficient, and thus condition 1 in Theorem 1 is satisfied. Condition 2 can be proved similarly. □

References

[1] Xiao F, Wang L. Consensus problems for high-dimensional multi-agent systems. *IET Control Theory and Applications* 2007; 1(3):830–837.

[2] Xiao F, Wang L, Chen J, Gao Y. Finite-time formation control for multi-agent systems. *Automatica* 2009; 45(11):2605–2611.

[3] Olfati-Saber R. Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Transactions on Automatic Control* 2006; 51(3): 401–420.

[4] Shi H, Wang L, Chu T. Swarming behavior of multi-agent systems. *Journal of Control Theory and Applications* 2004; 2(4): 313–318.

[5] Guan Y, Ji Z, Zhang Z, Wang L. Decentralized stabilizability of multi-agent systems under fixed and switching topologies. *System & Control Letters* 2013; 62(2): 438–446.

[6] Guan Y, Ji Z, Zhang L, Wang L. Quadratic stabilisability of multi-agent systems under switching topologies. *International Journal of Control* 2014; 87(12): 2657-2668.

[7] Tanner H G. On the controllability of nearest neighbor interconnections. *Proceedings of the 43rd IEEE Conference on Decision and Control*, Paradise Island, 2004; 2467–2472.

[8] Ji Z, Lin H, Lee T H. A graph theory based characterization of controllability for multi-agent systems with fixed topology. *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, 2008; 5262–5267.

[9] Ji Z, Wang Z D, Lin H, Wang Z. Interconnection topologies for multi-agent coordination under leader-follower framework. *Automatica* 2009; 45(12): 2857–2863.

[10] Ji M, Egerstedt M. A graph-theoretic characterization of controllability for multi-agent systems. *Proceedings of the 2007 American Control Conference*. New York, 2007; 4588–4593.

[11] Najafi M, Sheikholeslam F. Graph theoretical methods to study controllability and leader selection for dead-time systems. *Transactions on Combinatorics* 2013; 2(4): 25–36.
[12] Wang L, Jiang F, Xie G, Ji Z. Controllability of multi-agent systems based on agreement protocols. *Science in China Series F: Information Science* 2009; 52(11): 2074–2088.

[13] Parlangeli G, Notarstefano G. On the reachability and observability of path and cycle graphs. *IEEE Transaction on Automatic Control* 2012; 57(3): 743–748.

[14] Ji Z, Lin H, Yu H. Leaders in multi-agent controllability under consensus algorithm and tree topology. *System & Control Letters* 2012; 61(9): 918–925.

[15] Notarstefano G, Parlangeli G. Controllability and observability of grid graphs via reduction and symmetries. *IEEE Transaction on Automatic Control* 2013; 58(7): 1719–1731.

[16] Kibangou A Y, Commault C. Observability in connected strongly regular graphs and distance regular graphs. *IEEE Transactions on Control of Networks Systems*; DOI: 10.1109/TCNS.2013.2357532.

[17] Rahmani A, Ji M, Mesbahi M, Egerstedt M. Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization* 2009; 48(1): 162–186.

[18] Martini S, Egerstedt M, Bicchi A. Controllability analysis of multi-agent systems using relaxed equitable partitions. *International Journal of Systems, Control and Communications* 2010; 2(1,2,3): 100–121.

[19] Liu B, Chu T, Wang L, Xie G. Controllability of a leader-follower dynamic network with switching topology. *IEEE Transactions on Automatic Control* 2008; 53(4): 1009–1013.

[20] Liu B, Chu T, Wang L, Zuo Zh, Chen G, Su H. Controllability of switching networks of multi-agent systems. *International Journal of Robust and Nonlinear Control* 2012; 22(6): 630–644.

[21] Lin C T. Structural controllability. *IEEE Transactions on Automatic Control* 1974; 19(3): 201–208.

[22] Zamani M, Lin H. Structural controllability of multi-agent systems. *Proceedings of the 2009 American Control Conference*, St. Louis, 2009; 5743–5748.

[23] Lou Y, Hong Y. Controllability analysis of multi-agent systems with directed and weighted interconnection. *International Journal of Control* 2012; 85(10): 1486–1496.

[24] Guan Y, Ji Z, Zhang L, Wang L. Controllability of multi-agent systems under directed topology. *Submitted to System & Control Letters*.

[25] Olshevsky A. Minimal controllability problems. *IEEE Transactions on Control of Network Systems* 2014; 1(4): 249–258.