AUTOMORPHISMS OF GENERALIZED FERMAT MANIFOLDS

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ABSTRACT. Let \( d \geq 1, k \geq 2 \) and \( n \geq d + 1 \) be integers. A \( d \)-dimensional smooth complex algebraic variety \( M \) is called a generalized Fermat variety of type \((d; k, n)\) if there is a Galois holomorphic branched covering \( \pi : M \to \mathbb{P}^d \), with deck group \( H \cong \mathbb{Z}_k^n \), whose branch divisor consists of \( n + 1 \) hyperplanes in general position, each one of branch order \( k \). In this case, \( H \) is called a generalized Fermat group of type \((d; k, n)\). In previous work, we proved that the generalized Fermat group \( H \) is unique in the following cases: (i) \( d = 1 \) and \((k - 1)(n - 1) > 2\), or (ii) \( d \geq 2 \) and \((d; k, n) \notin \{(2; 2, 5), (2; 4, 3)\}\). To obtain this uniqueness fact, we used a differential method due to Kontogeorgis. This paper provides a different and shorter proof of the uniqueness of \( H \). We also study the locus of fixed points of subgroups of \( H \).

1. INTRODUCTION

Let \( M \subset \mathbb{P}^n \) be a \( d \)-dimensional smooth complex algebraic variety. In particular, as \( M \) carries a natural structure of complex manifold, we may consider the group \( \text{Aut}(M) \) of its holomorphic automorphisms. We denote by \( \text{Lin}(M) \) its subgroup consisting of those automorphisms that are restrictions of elements of \( \text{PGL}_{n+1}(\mathbb{C}) \).

If \( k, n \geq 2 \) are integers, then we say that a subgroup \( H \leq \text{Aut}(M) \) is a generalized Fermat group of type \((d; k, n)\) if (i) \( H \cong \mathbb{Z}_k^n \) and (ii) \( M/H \) is the orbifold whose underlying space is the \( d \)-dimensional projective space \( \mathbb{P}^d \) and whose branch locus is the complete intersection of \( n + 1 \) hyperplanes in general position, and each one with branch order \( k \).

In this case, we also say that \( M \) (respectively, \((M, H)\)) is a generalized Fermat manifold \( (\text{respectively, a generalized Fermat pair}) \) of type \((d; k, n)\). Necessarily, \( n \geq d \) (this follows from the fact that \( M \) is smooth), and if \( n = d \), then \( M \cong \mathbb{P}^d \) (see Theorem 2).

Let \((M, H)\) be a generalized Fermat pair of type \((d; k, n)\), where \( n \geq d + 1 \). Let us consider a Galois (branched) cover map \( \pi : M \to \mathbb{P}^d \), with deck group \( H \), and whose branch locus (the image of those points \( x \in M \) with a non-trivial \( H \)-stabilizer; in which case is a cyclic group of order \( k \)) consists of the \( n + 1 \) hyperplanes \( \Sigma_1, \ldots, \Sigma_{n+1} \), which are in general position. As a consequence of the classification of abelian coverings, due to Pardini in [14], the triple \((M, H, \pi)\) is completely determined by the above hyperplanes.

If \( n = d + 1 \), then (up to isomorphisms) \( M = M_{k+1}^d \) is the Fermat hypersurface of degree \( k \), for which \( \text{Lin}(M) = H \rtimes S_{n+1} \), where the permutation part acts as a permutation of coordinates (in particular, \( H \) is the unique generalized Fermat group of type \((d; k, d + 1)\)). If \( n \geq d + 2 \), then an explicit algebraic model of \( M \), for which \( H \) is given as a very simple linear group of automorphisms, is given as a complete intersection of \((n - d)\) Fermat hypersurfaces of dimension \( d \) and of degree \( k \) in \( \mathbb{P}^n \) (see Section 2 and also [4]). For \( d = 2 \), such an algebraic model was already known to Hirzebruch in [10] (in [11], Hirzebruch studied the arrangement of \( n + 1 \) lines in \( \mathbb{P}^2 \) which are not necessarily

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in general position). In [1], Bin studied some examples (case \(k = 2\) and \(n = 4\)) and their corresponding Pardini’s building data.

By results due to Kontogeorgis [13], if \((d; k, n) \notin \{(2; 2, 5), (2; 4, 3)\}\), then \(\text{Aut}(M) = \text{Lin}(M)\). The two exceptional tuples correspond to the unique generalized Fermat surfaces being K3 surfaces. If \((d; k, n) = (2; 4, 3)\), then \(M\) corresponds to the classical Fermat hypersurface of degree 4 in \(\mathbb{P}^3\) for which \(\text{Lin}(M) \cong \mathbb{Z}_4^3 \times S_4\) and \(\text{Aut}(M)\) infinite. If \((d; k, n) = (2; 2, 5)\), then \(\text{Lin}(M)\) is a finite extension of \(\mathbb{Z}_2^2\) (generically a trivial extension) and \(\text{Aut}(M)\) is infinite by results due to Shioda and Inose in [16, Thm 5] (in [18] Vinberg computed it for a particular case). In these exceptional cases, the generalized Fermat group \(H\) (which is unique in \(\text{Lin}(M)\)) cannot be a normal subgroup of \(\text{Aut}(M)\). Otherwise, every element of \(\text{Aut}(M)\) will induce an automorphism of \(\mathbb{P}^2\) (permuting the \(n + 1\) branched lines), so a finite group of linear automorphism. This will ensure that \(\text{Aut}(M)\) is a finite extension of \(H\), so a finite group, a contradiction.

In [8], it was proved that, for \(d = 1\) and \((n - 1)(k - 1) > 2\), \(M\) has a unique generalized Fermat group of type \((1; k, n)\). In [9], by applying a differential method due to Kontogeorgis [13], we proved that, for \(d \geq 2\), the group \(\text{Lin}(M)\) admits a unique generalized Fermat group of type \((d; k, n)\) (so, in the non-exceptional cases, the uniqueness in \(\text{Aut}(M)\)). In this paper, we provide a different argument by an inductive process to obtain the uniqueness. One of the interests of this uniqueness result is that it provides a natural short exact sequence \(1 \to \text{H} \to \text{Aut}(M) = \text{Lin}(M) \to \text{Aut}_{\text{orb}}(M/H) \to 1\), where \(\text{Aut}_{\text{orb}}(M/H)\) is the subgroup of the group \(\text{PGL}_{d+1}\) of conformal automorphisms of the \(\mathbb{P}^d\) that keeps invariant the \((n + 1)\) branch hyperplanes. This could be used to compute \(\text{Aut}(M)\) (see [5] for the case \(d = 1\)).

We also describe the locus of fixed points of subgroups of \(H\) (Proposition 3), and we compute the plurigenera and the arithmetic genus of \(M\) (Proposition 2). Moreover, we provide the arithmetic conditions of the tuple \((d; k, n)\) for \(M\) to be a Calabi-Yau variety (in particular, if \(d = 2\), to be K3).

In Section 4, we obtain information of \(\text{Aut}(M)\). For instance, by the Schur-Zassenhaus theorem, we obtain that if \(k\) is relatively prime to the \(\text{PGL}_{d+1}\) stabilizer \(G_0\) of the branch divisor of \(M/H\), then \(\text{Aut}(M) \cong H \rtimes G_0\).

In the last section, we provide some examples.

**Notations:**

1. If \(n \geq 1\) and \(k \geq 2\) are integers, then \(\mathbb{Z}_k^n := \mathbb{Z}_k \times \cdots \times \mathbb{Z}_k\), where \(\mathbb{Z}_k\) denotes the cyclic group of order \(k\).
2. We denote by \(M_{d+1}^k\) the degree \(k\) Fermat hypersurface in \(\mathbb{P}^d\). For \(d = 1\) (resp., \(d = 2\)) we also use the notation \(C_2^k\) (resp., \(S_3^k\)).
3. The divisor \(D\) given by the union of \(n + 1\) hyperplanes \(L_1, \ldots, L_{n+1} \subset \mathbb{P}^d\), where \(n \geq 2\), which are in general position, is called a strict normal crossing divisor.
4. For \(n \geq d + 1\) and a strict normal crossing divisor \(D = L_1 + \cdots + L_{n+1}\) (where, for \(d = 1\), \(D\) consists of \(n + 1\) different points of \(\mathbb{P}^1\)), we will use the notation \(M_{n+1}^k(D)\) for a generalized Fermat manifold of type \((d; k, n)\) associated to \(D\). In the special case \(d = 1\) (resp., \(d = 2\)), we will also use the notation \(C_{n+1}^k(D)\) (resp., \(S_{n+1}^k(D)\)) for a generalized Fermat manifold of type \((1; k, n)\) (resp., \((2; k, n))\). In [5], it is proven that \(C_{n+1}^k(D)\) is uniquely determined by the isomorphism class of \(D\) (class defined by the automorphisms of \(\mathbb{P}^1\)).
2. Generalized Fermat Manifolds

In this section, we provide suitable algebraic models for generalized Fermat manifolds of type \((d; k, n)\), obtained as a fiber product of \((n - d)\) classical Fermat hypersurfaces of degree \(k\).

2.1. Hyperplanes in general position. Let’s start by recalling some basic facts on hyperplanes in general position as it is important in defining generalized Fermat pairs. First, note that each hyperplane in \(L \subset \mathbb{P}^d\) has the form \(L_q := \{\rho_1t_1 + \cdots + \rho_{d+1}t_{d+1} = 0\}\), where \(q := [\rho_1 : \cdots : \rho_{d+1}] \in \mathbb{P}^d\).

A collection \(L_{q_1}, \ldots, L_{q_{n+1}} \subset \mathbb{P}^d\), where \(n \geq d + 1\), of hyperplanes are in general position if the corresponding (pairwise different) points \(q_1, \ldots, q_{n+1} \in \mathbb{P}^d\) are in general position. This means that, for \(d \geq 2\), any subset of \(3 \leq s \leq d + 1\) of these points spans a \((s - 1)\)-plane \(\Sigma \subset \mathbb{P}^d\). In this above situation, the divisor \(D\), formed by the hyperplanes \(L_{q_1}, \ldots, L_{q_{n+1}}\), is a strict normal crossing divisor. If \(q_i = [\rho_{1,i} : \cdots : \rho_{d+1,i}] \in \mathbb{P}^d\), for \(j = 1, \ldots, n + 1\), then the hyperplanes \(L_{q_1}, \ldots, L_{q_{n+1}}\) are in general position if and only if, for every all \((d + 1 \times d + 1)\)-minors of the matrix

\[
M(q_1, \ldots, q_{n+1}) := \begin{pmatrix}
\rho_{1,1} & \cdots & \rho_{1,n+1} \\
\vdots & \ddots & \vdots \\
\rho_{d+1,1} & \cdots & \rho_{d+1,n+1}
\end{pmatrix} \in M_{(d+1) \times (n+1)}(\mathbb{C})
\]

are nonzero. For example, if \(e_1 = [1 : 0 : \cdots : 0], e_2 = [0 : 1 : 0 : \cdots : 0], \ldots, e_{d+1} = [0 : \cdots : 0 : 1]\) and \(e_{d+2} = [1 : \cdots : 1]\), then the hyperplanes \(L_{e_1}, \ldots, L_{e_{d+2}}\) are in general position.

Let \(n \geq d + 1\). Two ordered tuples \((L_{q_1}, \ldots, L_{q_{n+1}})\) and \((L_{p_1}, \ldots, L_{p_{n+1}})\), of hyperplanes in \(\mathbb{P}^d\) in general position, are equivalent if there is a \(T \in \text{PGL}_{d+1}(\mathbb{C})\) such that \(L_{p_j} = T(L_{q_j})\), for \(j = 1, \ldots, n + 1\). We denote by \(X_{n,d}\) the set of such equivalence classes.

If \((L_{q_1}, \ldots, L_{q_{n+1}})\) is one of such tuples, then there is a unique \(T \in \text{PGL}_{d+1}(\mathbb{C})\) such that \(T(L_{q_j}) = L_{e_j}\), for \(j = 1, \ldots, d + 2\). Then, for each \(j = d + 3, \ldots, n + 1\), there is some unique \(\Lambda_i := [\lambda_{i,1} : \cdots : \lambda_{i,d+2} : 1] \in \mathbb{P}^d\), \(i = 1, \ldots, n - d - 1\), such that \(T(L_{q_j}) = L_{\Lambda_j}\). In this case, we set \(\Lambda := (\lambda_1, \ldots, \lambda_{d}) \in \mathbb{C}^{d(n-d-1)}\), where \(\lambda_j := (\lambda_{1,j}, \ldots, \lambda_{n-d-1,j}) \in \mathbb{C}^{n-d-1}\), for \(j = 1, \ldots, d\). So, if we set

\[
L_j(\Lambda) := L_{e_j} \subset \mathbb{P}^d, \quad j = 1, \ldots, d + 2, \\
L_j(\Lambda) := L_{\Lambda_{j-d-2}} \subset \mathbb{P}^d, \quad j = d + 3, \ldots, n + 1,
\]

then \((L_{1}(\Lambda), \ldots, L_{n+1}(\Lambda))\) is equivalent to \((L_{q_1}, \ldots, L_{q_{n+1}})\). We call the tuple \(\Lambda\) a standard parameter.

This observation permits us to identify (i) \(X_{d+1,d}\) with the one set-point \(\{(1, \cdots, 1)\}\) and (ii) for \(n \geq d + 2\), \(X_{n,d}\) with the set of tuples \(\Lambda \in \mathbb{C}^{d(n-d-1)}\) such that the \(n + 1\) hyperplanes \(L_1(\Lambda), \ldots, L_{n+1}(\Lambda)\) are in a general position. We observe that in case (ii), \(X_{n,d}\) is a (non-empty) open set.

Remark 1 (Moduli spaces of generalized Fermat manifolds). The symmetric group \(\mathfrak{S}_{n+1}\) acts by permuting the \(n + 1\) coordinates of an ordered tuple of \(n + 1\) hyperplanes in general position. Such an action induces an action of a group \(\mathfrak{S}_{n,d}\) of holomorphic automorphisms on \(X_{n,d}\). The quotient complex orbifold \(X_{n,d} = X_{n,d}/\mathfrak{S}_{n,d}\) provides a parameter space of (unordered) collection of \(n + 1\) hyperplanes in general position. As a consequence of the uniqueness of the generalized Fermat groups of type \((d; k, n)\) \(\not\in \{(2; 2, 5), (2; 4, 3)\},\)
the orbifold $\mathcal{X}_{n,d}$ can be identified with the moduli space of generalized Fermat manifolds of type $(d; k, n)$.

2.2. Algebraic models of generalized Fermat pairs for $n \geq d + 1$. Let us consider integers $n \geq d + 1$, $k \geq 2$ and $d \geq 1$. Let $(M, H)$ be a generalized Fermat pair of type $(d; k, n)$ and $\pi : M \to \mathbb{P}^d$ be an abelian branched covering with deck group $H$. Up to post-composition by some linear automorphism of $\mathbb{P}^d$, we may assume that its branch locus consists of the hyperplanes $L_1(\Lambda), \ldots, L_{n+1}(\Lambda)$, $\Lambda : (\lambda_1, \ldots, \lambda_d) \in X_{n,d}$, where $
abla_{\Lambda}$ consists of the hyperplanes $L_1(\Lambda), \ldots, L_{n+1}(\Lambda)$. In the orbifold $\mathcal{X}_{n,d}$, we see that a point which is an irreducible nonsingular complete intersection.

**Proposition 1.** If $\Lambda \in X_{n,d}$, then $M^n(\Lambda)$ is an irreducible nonsingular complete intersection.

**Proof.** Let us observe that the matrix of coefficients of (1) has rank $(m + 1) \times (m + 1)$ and all of its $(m + 1) \times (m + 1)$-minors are different from zero (this is the general position condition of the $n + 1$ hypersurfaces). The result follows from [17, Proposition 3.1.2]. For explicitness, let us work out the case $d = 2$ (the case $d = 1$ was noted in [5]).

Set $\lambda_{1,1} = \lambda_{0,2} = 1$ and consider the following degree $k$ homogeneous polynomials $f_i := \lambda_{1,1} x_1^k + \lambda_{2,1} x_2^k + x_3^k + x_4^k \in \mathbb{C}[x_1, \ldots, x_{n+1}]$, where $i \in \{0, 1, \ldots, n - 3\}$. Let $V_i \subset \mathbb{P}^n$ be the hypersurface given as the zero locus of $f_i$.

The algebraic set $S^n_{n}(\lambda_1, \lambda_2)$ is the intersection of the $(n-2)$ hypersurfaces $V_0, \ldots, V_{n-3}$. We consider the matrix of $\nabla f_i$ written as rows.

$$
\begin{pmatrix}
  k x_1^{k-1} & k x_2^{k-1} & k x_3^{k-1} & k x_4^{k-1} & 0 & \ldots & 0 \\
  \lambda_{1,1} x_1^{k-1} & \lambda_{2,1} x_2^{k-1} & x_3^{k-1} & x_4^{k-1} & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \lambda_{n-3,1} x_1^{k-1} & \lambda_{n-3,2} x_2^{k-1} & k x_3^{k-1} & 0 & \ldots & 0 & k x_4^{k-1}
\end{pmatrix}
$$

By the defining equations of the curve, and as $(\lambda_1, \lambda_2) \in X_n$, we see that a point which has three variables $x_i = x_j = x_l = 0$ for $i \neq j \neq l$ and $1 \leq i, j, l \leq n + 1$ has also $x_t = 0$ for $t = 1, \ldots, n + 1$. Therefore, the above matrix has the maximal rank $n - 2$ at all points of the surface. So, the defining hypersurfaces are intersecting transversally, and the corresponding algebraic surface they define is a nonsingular complete intersection.

Next, we proceed to see that the ideal $I_{k,n}$, defined by the $n - 2$ equations defining $S^n_{n}(\lambda_1, \lambda_2) \subset \mathbb{P}^n$ is prime. This follows similarly as in [13, sec. 3.2.1]. Observe first that the defining equations $f_0, \ldots, f_{n-3}$ form a regular sequence, that $\mathbb{C}[x_1, \ldots, x_{n+1}]$ is a Cohen-Macaulay ring and that the ideal $I_{k,n}$ defined is of codimension $n - 2$. The ideal $I_{k,n}$ is prime as a consequence of the Jacobian Criterion [3, Th. 18.15], [13, Th. 3.1] and that the fact that the matrix (2) is of maximal rank $n - 2$ on $S^n_{n}(\lambda_1, \lambda_2)$. In remark [13, 3.4], it is pointed out that an ideal $I$ is prime if the singular locus of the algebraic set defined by $I$ has big enough codimension (in our case, the singular set is the empty set).
2.2.2. The group $H_0$. Let us now consider the group $H_0 = \langle \varphi_1, \ldots, \varphi_{n+1} \rangle$, where
$$\varphi_j([x_1 : \cdots : x_j : \cdots : x_{n+1}]) := [x_1 : \cdots : w_k x_j : \cdots : x_{n+1}],$$
and $w_k$ is a primitive $k$-th root of unity. Let $\text{Fix}(\varphi_j) \subset M_n^k(\Lambda)$ be the set of fixed points of $\varphi_j$ and $F(H_0) := \cup_{j=1}^{n+1} \text{Fix}(\varphi_j)$. The following facts can be deduced from the above.

I) $H_0 \cong \mathbb{Z}_k^n$.
II) $\varphi_1 \varphi_2 \cdots \varphi_{n+1} = 1$.
III) $H_0 < \text{Aut}(M_n^k(\Lambda)) < \text{PGL}_{n+1}(\mathbb{C})$.
IV) The only non-trivial elements of $H_0$ with fixed set points being of maximal dimension $d - 1$ are the non-trivial powers of the generators $\varphi_1, \ldots, \varphi_{n+1}$ (see Proposition 3). Moreover, $\text{Fix}(\varphi_j) := \{x_j = 0\} \cap M_n^k(\Lambda)$, which is isomorphic to a generalized Fermat manifold of type $(d-1; k, n-1)$ for $d \geq 2$ and a collection of $k^{n-1}$ points for $d = 1$. We call the set $\varphi_1, \ldots, \varphi_{n+1}$ a canonical generators of $H_0$ (canonical generators are not unique, but any other set of them is of the form $\varphi_1^r, \ldots, \varphi_{n+1}^r$, where $r \in \{1, \ldots, k-1\}$ is relatively prime to $k$).

2.2.3. An algebraic models of $(M, H)$.  

Theorem 1. $(M_n^k(\Lambda), H_0)$ is a generalized Fermat pair of type $(d; k, n)$ and there is a biholomorphism between $M$ and $M_n^k(\Lambda)$ conjugating $H$ to $H_0$. In particular, every generalized Fermat manifold of type $(d; k, n)$ whose associated branching divisor is $\text{PGL}_{d+1}(\mathbb{C})$ equivalent to the divisor associated to $\Lambda$ is isomorphic to $M_n^k(\Lambda)$.

Proof. The map
$$\pi_0 : M_n^k(\Lambda) \to \mathbb{P}^d : [x_1 : \cdots : x_{n+1}] \mapsto [x_1^k : \cdots : x_{d+1}^k]$$
is a regular branched cover with deck group $H_0$ and whose branch set is the union of the previous hyperplanes $L_1(\Lambda), \ldots, L_{n+1}(\Lambda)$, each one of order $k$. In other words, the pair $(M_n^k(\Lambda), H_0)$ is a generalized Fermat pair of type $(d; k, n)$. The last part is a consequence of Pardini’s building data.

Remark 4. (1) If $\Lambda = (\lambda_1, \ldots, \lambda_d) \in X_{n,d}$, where $n \geq d + 1$, and setting $\lambda_{0,i} = 1$, for $i = 1, \ldots, d$ and, for each $j \in \{0, 1, \ldots, n-d-1\}$, we consider the classical degree $k$ Fermat hypersurface $F_j = \{\lambda_j x_1^k + \cdots + \lambda_d x_d^k + x_{d+1}^k + x_{d+2+j}^k = 0\} \subset \mathbb{P}^{d+1}$ and the rational map $\pi_j : F_j \to \mathbb{P}^d$ defined by $\pi_j([x_1 : \cdots : x_{d+1} : x_{d+2+j}]) = [x_1^k : \cdots : x_{d+1}^k]$. The branch values of $\pi_j$ are given by the hyperplanes $L_i(\Lambda)$, for $i = 1, \ldots, d+1$, and $L_{d+2+j}(\Lambda)$. If we consider the fiber product of all these pairs $(F_0, \pi_0), \ldots, (F_{n-d-1}, \pi_{n-d-1})$, then we obtain a reducible projective algebraic variety with the action of the group $\mathbb{Z}_k^{n(n-d)}$ and quotient $\mathbb{P}^d$ with branching divisor given by the previous $n + 1$ hyperplanes. This fiber product has $k^{(n-d)(d+1)-n}$ irreducible components, each of them isomorphic to $M_n^k(\Lambda)$.

(2) (An inductive process) If we set $H_j := \langle \varphi_1, \ldots, \varphi_{j-1}, \varphi_{j+1}, \ldots, \varphi_n \rangle$, where $1 \leq j \leq n$, and $H_{n+1} := \langle \varphi_1, \ldots, \varphi_{n-2}, \varphi_{n+1} \rangle$, then $H_j \cong \mathbb{Z}_k^{n-1}$. For $d \geq 2$, $H_j < \text{Aut}(F_j)$ and there is a holomorphic map $\pi_j : F_j \to L_j(\Lambda) = \mathbb{P}^{d-1}$ (given by the restriction of $\pi_0$), which is a Galois holomorphic branched cover with deck group $H_j$, such that its branch locus is given by the collection of the $n$ intersections $\pi(F_j) \cap \pi(F_i)$, $i \neq j$, which are copies of $\mathbb{P}^{d-2}$ in general position. In particular, $(F_j, H_j)$ is a generalized...
2.3. Algebraic model of a generalized Fermat pair for \( n = d \). In the above, we have assumed the condition \( n \geq d + 1 \). When \( 2 \leq n = d \), then we have the following fact.

**Theorem 2.** If \((M, H)\) is a generalized Fermat pair of type \((d; k, d)\), where \( d, k \geq 2 \), then (up to isomorphisms) \( M = \mathbb{P}^d \) and \( H = \langle \varrho_1, \ldots, \varrho_d \rangle \), where \( \varrho_j([x_1 : \cdots : x_{d+1}]) = [x_1 : \cdots : x_{j-1} : \omega_k x_j : x_{j+1} : \cdots : x_{d+1}] \).

**Proof.** Let us consider a Galois branched covering \( \pi : M \to \mathbb{P}^d \) with deck group \( H \), and whose branch divisor locus is given by the hyperplanes \( L_1, \ldots, L_{n+1} \). Up to postcomposition of \( \pi \) by a suitable projective linear automorphisms, we may assume \( L_j = \{t_j = 0\} \). Now, we add the hyperplane \( L_{d+2} \), where \( L_{d+2} = \{t_1 + \cdots + t_{d+1} = 0\} \). Next, consider the orbifold \( M_{\text{orb}} \), whose underlying complex manifold is \( M \) and whose orbifold divisor is obtained by lifting the hyperplane \( L_{d+2} \) and giving to each component of its lifting the branching order equal to \( k \). By construction, the group \( H \) preserves that branch divisor of \( M_{\text{orb}} \). Let \( M_{d+1}^k \subset \mathbb{P}^{d+1} \) be the generalized Fermat manifold of type \((d; k, d+1)\) (the Fermat hypersurface of degree \( k \)) together its generalized Fermat group \( H_0 = \langle \varphi_1, \ldots, \varphi_{d+1} \rangle \) and the corresponding Galois branched covering \( \pi_0 : M_{d+1}^k \to \mathbb{P}^d \) with deck group \( H_0 \). Note that \( M_{\text{orb}}/H = M_{d+1}^k/H_0 \) (which is \( \mathbb{P}^d \) with branch divisor \( L_1 + \cdots + L_{d+2} \), each one with order \( k \)). As the universal covering of this last orbifold is \( M_{d+1}^k \subset \mathbb{P}^{d+1} \), the branched covering \( \pi_0 \) it must factors through \( \pi \), that is, there is some \( \mathbb{Z}_k \cong K < H_0 \) such that \( M = M_{d+1}^k/K \). Because of the choices of the hyperplanes \( L_j \), we must have \( K = \langle \varphi_{d+2} \rangle \), where \( \varphi_1, \ldots, \varphi_{d+2} \in H_0 \) are the corresponding standard set of generators. We may consider the (branched) Galois cover, with deck group \( K \),

\[
\pi_K : M_{d+1}^k \to \mathbb{P}^d : [x_1 : \cdots : x_{d+2}] \mapsto [x_1 : \cdots : x_{d+1}] = [y_1 : \cdots : y_{d+1}] .
\]

In this way, \( M \cong \mathbb{P}^d \) and \( H = \langle \varrho_1, \ldots, \varrho_d \rangle \), where \( \varrho_j \) is as described in the theorem. The corresponding Galois branched cover, with deck group \( H \), is \( \pi([y_1 : \cdots : y_{d+1}]) = [y_1^k : \cdots : y_{d+1}^k] \) whose branch divisor is \( L_1 + \cdots + L_{d+1} \). \( \square \)

2.4. A remark on the cohomological information of generalized Fermat manifolds. The fact that \( M := M_n^k(\Lambda) \) is a complete intersection variety allows us to compute the cohomology groups of the twisting sheaf \( \mathcal{O}_M(r) \) in a relatively direct way, and in particular, to obtain the following.

**Proposition 2.** Let \( \Lambda \in X_{n,d} \), \( n \geq d + 1 \), and \( M := M_n^k(\Lambda) \). Then

1. The plurigenera \( P_m(M) \) of \( M \) satisfies

\[
P_m(M) = \frac{k^{n-d}((n-d)k-n-1)^d}{d!} m^d + O(m^{d-1}).
\]
(2) The arithmetic genus \( p_a(M) \) and the geometric genus \( p_g(M) \) are given by

\[
p_a(M) = p_g(M) = \begin{cases} 
0 & \text{if } r_1 < 0 \\
\left( \frac{-1}{n} \right) & \text{if } 0 \leq r_1 < k \\
\sum_{j \in \Delta_1} \left( \frac{r_1 - j + 1}{d} \right) & \text{if } r_1 \geq k 
\end{cases}
\]

(3) If \((n - d)k - n - 1 = 0\), then \(M\) is a Calabi-Yau variety.

(4) If \(d = 2\), then \(M\) is a general type surface except for the rational varieties cases \((k, n) \in \{(2, 3), (3, 3), (2, 4)\}\) and the \(K3\) varieties \((k, n) \in \{(4, 3), (2, 5)\}\).

Proof:

Let \(\mathbb{C}[x_1, \ldots, x_m]_l\) be the homogeneous polynomials of degree \(l\).

(a) We first proceed to describe the cohomology groups of the twisting sheaf \(\mathcal{O}_M(r)\), \(r \in \mathbb{Z}\).

(a1) Let \(\Delta_r := \{(j_1, \ldots, j_{n-d}) \in \mathbb{Z}^{n-d} : 0 \leq j_i \leq k - 1, 0 \leq i \leq n-d, \text{ and } \bar{r} := j_1 + j_2 + \cdots j_{n-d} \leq r\}\). Then

\[
H^0(M, \mathcal{O}_M(r)) := \begin{cases} 
\mathbb{C}[x_1, \ldots, x_{n+1}]_r & \text{if } r < 0 \\
\bigoplus_{j \in \Delta_r} Q_j & \text{if } r \geq k
\end{cases}
\]

where \(Q_j := \mathbb{C}[x_1, \ldots, x_{d+1}]_{(r-\bar{r})+j_1} \cdot x_{d+2}^{j_2} \cdots x_{n+1}^{j_{n-d}}\), \(j := (j_1, \ldots, j_{n-d})\).

(a2) By Grothendieck’s vanishing theorem, we have that:

\[
H^i(M, \mathcal{O}_M(r)) = 0 \quad \text{for } i > d \text{ and } r \in \mathbb{Z}.
\]

(a3) and as \(M\) is a complete intersection variety we obtain that:

\[
H^i(M, \mathcal{O}_M(r)) = 0 \quad \text{for } 0 < i < d \text{ and } r \in \mathbb{Z}.
\]

(see page 231 of [7]).

(a4) Finally, using the Serre duality, we obtain that:

\[
H^d(M, \mathcal{O}_M(r)) \cong H^0(M, \mathcal{O}_M(r_1 - r))
\]

Remember that \(\omega_M \cong \mathcal{O}_M(r_1), r_1 = (n - d)k - n - 1\), (see page 188 of [7]).

(b) With the former, we can calculate the plurigenus of \(M\)

\[
P_m(M) = \dim_{\mathbb{C}} H^0(M, \omega_M^\otimes m) = \dim_{\mathbb{C}} H^0(M, \mathcal{O}_M(r_m))
\]

where \(r_m := mr_1 = m((n - d)k - n - 1)\).

(b1) If \((n - d)k - n - 1 < 0\), we obtain that \(P_m(M) = 0\). This implies that the Kodaira dimension of \(M\) is \(\kappa(M) = -\infty\).

(b2) If \((n - d)k - n - 1 = 0\), we obtain that \(P_m(M) = 1\). This implies that the Kodaira dimension of \(M\) is \(\kappa(M) = 0\).

(b3) If \((n - d)k - n - 1 > 0\), the canonical sheaf in very ample and

\[
P_m(M) = \begin{cases} 
\left( \frac{-1}{n} \right) & \text{if } 0 \leq r_m < k \\
\sum_{j \in \Delta_m} \left( \frac{r_m - j + 1}{d} \right) & \text{if } r_m \geq k
\end{cases}
\]

In particular, if \(r_m \geq \max\{k, (n - d)(k - 1)\}\), we obtain that

\[
P_m(M) = \frac{(n - d)(k - n - 1)^d}{kd} m^d + O(m^{d-1})
\]

This implies that the Kodaira dimension of \(M\) is \(\kappa(M) = d\).

(c) The former also permits us to determine the arithmetic genus and geometric genus of \(M\). As seen from the above,

\[
p_a(M) = p_g(M) = \dim_{\mathbb{C}} H^0(M, \mathcal{O}_M) = \dim_{\mathbb{C}} H^0(M, \mathcal{O}_M(r_1))
\]
so
\[ p_\alpha(M) = p_\beta(M) = \begin{cases} 
0 & \text{if } r_1 < 0 \\
\left(\frac{r_1 + n}{n}\right) & \text{if } 0 \leq r_1 < k \\
\sum_{j \in \Delta r_1} \left(\frac{r_1 - j + d}{d}\right) & \text{if } r_1 \geq k 
\end{cases} \]

\[ \square \]

3. Uniqueness of the Generalized Fermat Group

In this section, we provide an alternative (and short) proof of the uniqueness of generalized Fermat groups to the one given in [9].

**Theorem 3.** Every generalized Fermat manifold \( M \) of type \( (d; k, n) \), where \( n \geq d + 2 \), has a unique generalized Fermat group of the same type in \( \operatorname{Lin}(M) \). In particular, if \( (d; k, n) \notin \{(2; 2, 5), (2; 4, 3)\} \), then the uniqueness holds in \( \operatorname{Aut}(M) \).

**Proof.** Let us assume \( n \geq d + 2 \) and that \( H < \operatorname{Lin}(M) \) is another generalized Fermat group of \( M = M^k_{\Lambda} \) of type \( (d; k, n) \). Let \( \varphi_1, \ldots, \varphi_{n+1} \) be an standard set of generators of \( H_0 \) and \( \varphi_1^*, \ldots, \varphi_{n+1}^* \) be an standard set of generators for \( H \). Following in a similar way as done in the proof of Claim 1 of Section 5 in [8], we may assume that there are \( i, j \in \{1, \ldots, n\} \) such that \( \langle \varphi_i \rangle \sim \langle \varphi_j \rangle \). Up to a permutation of indices, we may assume \( i = j = 1 \). So, let \( \langle \varphi_1 \rangle \sim \langle \varphi_1^* \rangle \) and set \( F_1 \subset M \) the locus of fixed points of \( \varphi_1 \), which is a generalized Fermat manifold of type \( (d-1; k, n-1) \), there are two generalized Fermat groups of that type, these being \( H_0/\langle \varphi_1 \rangle \) and \( H/\langle \varphi_1 \rangle \). So, if we have the uniqueness for dimension \( d-1 \), then we will be done by an induction process. The starting case is \( d = 2 \), for which \( F_1 \) is a generalized Fermat curve of type \( (k, n-1) \). The uniqueness, in this case, works if \( (k-1)(n-2) > 2 \) [8]. As we are assuming \( n \geq d + 2 = 4 \), the uniqueness will hold if either (i) \( n = 4 \) and \( k \geq 3 \) or (ii) \( n \geq 5 \). So, in these cases, \( H_0 = H \).

Let us now consider the left case \( n = 4, k = 2 \), and, by contradiction, assume \( H \neq H_0 \). In this case, \( F_1 \) is a genus one Riemann surface. On it we have the group \( J = H_0/\langle \varphi_1 \rangle \cong \mathbb{Z}_3^2 \) of conformal automorphisms. If \( j \neq 1 \), then \( F_1 \) is tangent to each \( F_j \) (the locus of fixed points of \( \varphi_j \)) at four points (these are the fixed points of the action of \( \varphi_j \) on \( F_1 \)). The torus \( F_1 \) corresponds to the plane generalized Fermat curve of genus one

\[ F_1 = \left\{ \begin{array}{l}
y_1^2 + y_2^2 + y_3^2 = 0 \\
y_1^2 + y_2^2 + y_4^2 = 0
\end{array} \right\} \subset \mathbb{P}^4.\]

In the above model, the group \( J \) is generated by \( a_1([y_1 : y_2 : y_3 : y_4]) = [-y_1 : y_2 : y_3 : y_4], a_2([y_1 : y_2 : y_3 : y_4]) = [y_1 : -y_2 : y_3 : y_4] \) and \( a_3([y_1 : y_2 : y_3 : y_4]) = [y_1 : y_2 : y_3 : y_4] = -(y_2/y_1)^2 = z \) whose branched values are \( \infty, 0, 1 \) and \( \mu \). Set \( a_4 = a_1 a_2 a_3 \). As we have assumed above that \( \varphi_1 = \varphi_1^* \), then \( F_1 \) also admits the group \( J' = H/\langle \varphi_1^* \rangle \cong \mathbb{Z}_2^2 \) of conformal automorphisms; and it normalizes \( J \). So, \( J' \) induces (under \( \pi \)) a group of conformal automorphisms of the Riemann sphere that permutes the branch values of \( \pi \) and isomorphic to \( \mathbb{Z}_2^t \), for some \( t > 0 \) (as we are assuming \( H \neq H' \)). There are only two possibilities: \( t \in \{1, 2\} \) (as the only finite abelian subgroups of Möbius transformations are cyclic or \( \mathbb{Z}_2^2 \)). It follows that \( J \cap J' \) can be either \( \mathbb{Z}_2 \) or \( \mathbb{Z}_2^3 \). Since \( H \neq H_0 \), one of the standard generators of \( H \) is different from the standard generators of \( H_0 \), say \( \varphi_2^* \). This generator induces an involution \( \alpha \) with four fixed points on \( F_1 \) different from those in \( J \). We may assume that it induces the involution \( a(z) = z/\mu \), in which case \( \alpha([y_1 : y_2 : y_3 : y_4]) = [y_2 : \epsilon_1 \sqrt{\mu} y_1 : \epsilon_2 y_4 : \epsilon_3 \sqrt{\mu} y_3] \), where \( \sqrt{\mu} \) is a fixed square root of \( \mu \), \( \epsilon_j \in \{\pm 1\} \) and \( \epsilon_1 = \epsilon_2 \epsilon_3 \). As \( \alpha a_1 \alpha = a_2 \) and \( \alpha a_3 \alpha = a_4 \), there is no subgroup
isomorphic to \( \mathbb{Z}_2^d \) inside \( J \) such that the group generated by it and \( \alpha \) is \( \mathbb{Z}_2^d \). It follows that \( J \cap J' \neq \mathbb{Z}_2^d \). If \( J \cap J' = \mathbb{Z}_2^d \), then \( J' \) must induces \( \mathbb{Z}_2^d \), one generator being \( a \) and the other being \( b(z) = (z - \mu)/(z - 1) \), this last one induced by some involution \( \beta \in J' \backslash J \). It can be seen that \( \beta([y_1 : y_2 : y_3 : y_4]) = [y_3 : \epsilon_1 iy_4 : \epsilon_2 \sqrt{1 - \mu} y_1 : \epsilon_3 i \sqrt{1 - \mu} y_2] \), where \( \sqrt{1 - \mu} \) is a fixed square root of \( 1 - \mu \), \( \epsilon_j \in \{ \pm 1 \} \) and \( \epsilon_2 = -\epsilon_1 \epsilon_3 \). Similarly as for \( \alpha \), one may see that \( \beta a \beta = a_3 \) and \( \beta a_2 \beta = a_4 \). It can be checked that there is no involution of \( J \) such that together \( \alpha \) and \( \beta \) generate \( \mathbb{Z}_2^d \).

All the above produces a contradiction under the assumption that \( H \neq H_0 \). \( \square \)

4. AUTOMORPHISMS OF GENERALIZED FERMAT MANIFOLDS

4.1. Upper bounds. One direct consequence of the uniqueness of generalized Fermat groups is that one may obtain some information on the groups of automorphisms of generalized Fermat manifolds.

**Corollary 1.** Let \( d \geq 2, k \geq 2, n \geq d + 1 \) be integers and \( (d; k; n) \notin \{(2; 2, 5), (2; 4, 3)\} \). Let \( M \) be a generalized Fermat manifold of type \( (d; k; n) \) and let \( H \) be its unique generalized Fermat group of type \( (d; k; n) \). If \( G_0 \) is the \( \text{PGL}_{d+1}(\mathbb{C}) \)-stabilizer of the \( n + 1 \) branch hyperplanes of \( M/H = \mathbb{P}^d \), then \( |\text{Aut}(M)| = |G_0|k^n \) and, if the order of \( G_0 \) is relatively prime with \( k \), then \( \text{Aut}(M) \cong H \times G_0 \).

**Proof.** We know that \( M \) admits a unique generalized Fermat group \( H \) of type \( (d; k; n) \). Let \( \pi : M \to \mathbb{P}^d \) be a Galois branched covering, with \( H \) as its desk group, and let \( \{L_1, \ldots, L_{n+1}\} \) be its set of branch hyperplanes. Let \( G_0 \) be the \( \text{PGL}_{d+1}(\mathbb{C}) \)-stabilizer of these \( n + 1 \) branch hyperplanes. As \( H \) is a normal subgroup of \( \text{Aut}(M) \), it follows the existence of a homomorphism \( \theta : \text{Aut}(M) \to G_0 \), with kernel \( H \). As \( M \) is a universal branched cover, every element \( Q \) of \( G_0 \) lifts to a holomorphic automorphism \( \tilde{Q} \) of \( M \). Then there is a short exact sequence \( 1 \to H \to \text{Aut}(M) \to G_0 \to 1 \). In particular, \( |\text{Aut}(M)| = |G_0|k^n \). Also, by the Schur-Zassenhaus theorem [2], in the case that the order of \( G_0 \) is relatively prime with \( k \), then \( \text{Aut}(M) \cong H \times G_0 \). \( \square \)

**Corollary 2.** Let \( d \geq 2, k \geq 2 \). If \( G_0 \) be a finite subgroup of \( \text{PGL}_{d+1}(\mathbb{C}) \), then there exists a generalized Fermat pair \( (M, H) \) of type \( (d; k; n) \), for some \( n \geq d + 1 \), such that \( \text{Aut}(M/H) \cong G_0 \). In fact, for \( |G_0| \leq d + 1 \) we may assume \( n = d + 1 \) and, for \( |G_0| \geq d + 2 \), we may assume \( n = |G_0| - 1 \).

**Proof.** If \( |G_0| \leq d + 1 \), then take \( n = d + 1 \) and note that for the classical Fermat hypersurface \( M^d_n \subset \mathbb{P}^n \) of degree \( k \) one has that \( \text{Aut}(M^d_n)/H \) contains the permutation group of \( d + 1 \) letters. Let us assume \( |G_0| \geq d + 2 \). The linear group \( G_0 \) induces a linear action on the space \( \mathbb{P}^d_{\text{hyper}} \) of hyperplanes of \( \mathbb{P}^d \). As \( G_0 \) is finite, we may find (generically) a point \( q \in \mathbb{P}^d_{\text{hyper}} \) whose \( G_0 \)-orbit is a generic set of points. Such an orbit determines a collection of \( |G_0| \) lines in general position in \( \mathbb{P}^d \). Let us observe that, by the generic choice, we may even assume the above set of points to have \( \text{PGL}_{d+1}(\mathbb{C}) \)-stabilizer exactly \( G_0 \), so the same situation for our collection of hyperplanes. Now, the results follow from Corollary 1. \( \square \)

4.2. Fixed points of elements of \( H_0 \). Next, we consider the generalized Fermat pair \( (M^d_n(\Lambda), H_0) \) of type \( (d; k; n) \). This section describes the elements of \( H_0 \) having fixed points.

The diagonal presentation of the generators \( \varphi_j \in H_0 \), for \( j = 1, \ldots, n + 1 \), permits us to obtain the locus of fixed points of every element of \( H_0 \) (see Proposition 3 below).
First, let us recall that each element of $H_0$ has the form $\varphi := \varphi_1^{m_1} \cdots \varphi_{n+1}^{m_{n+1}} \in H_0$, where $m_1, \ldots, m_{n+1} \in \{0, 1, \ldots, k-1\}$.

Let us start with some definitions and notations. For each tuple $(m_1, \ldots, m_{n+1}) \in \{0, 1, \ldots, k-1\}^{n+1}$ and each $l \in \{0, 1, \ldots, k-1\}$, we set

$$L_l(m_1, \ldots, m_{n+1}) := \{j \in \{1, \ldots, n+1\} : m_j = l\},$$

and the (possibly empty) algebraic sets

$$\bar{F}_l(m_1, \ldots, m_{n+1}) = \{[x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n : x_i = 0, \forall i \notin L_l(m_1, \ldots, m_{n+1})\},$$

and

$$F_l(m_1, \ldots, m_{n+1}) := \bar{F}_l(m_1, \ldots, m_{n+1}) \cap M_n^k(\Lambda).$$

**Proposition 3.** Let $d \geq 1$, $n \geq d+1$, $k \geq 2$, $\lambda \in X_{n,d}$. $M_n^k(\Lambda)$, $H_0 \cong \mathbb{Z}_k^n$ and $\varphi_1, \ldots, \varphi_{n+1} \in H_0$ as before. Then:

1. If $F_l(m_1, \ldots, m_{n+1}) \neq \emptyset$ and only if $\#L_l(m_1, \ldots, m_{n+1}) = n+1 - d$.
2. If $F_l(m_1, \ldots, m_{n+1}) \neq \emptyset$, then it is a generalized Fermat manifold of dimension $\#L_l(m_1, \ldots, m_{n+1}) + d - n - 1$.
3. Let $\varphi := \varphi_1^{m_1} \cdots \varphi_{n+1}^{m_{n+1}} \in H_0$ different from the identity, where $m_1, \ldots, m_{n+1} \in \{0, 1, \ldots, k-1\}$. Then its locus of fixed points is the disjoint union of the sets $F_l(m_1, \ldots, m_{n+1})$, where $l \in \{0, 1, \ldots, k-1\}$. In particular, the number of (non-empty) connected components of its locus of fixed points (if non-empty) equals the number of exponents $l$ appearing in $\varphi$ at least $n+1 - d$ times.

**Proof.** The locus of fixed points, in $\mathbb{P}^n$, of $\varphi$ is the disjoint union of the algebraic sets

$$\bar{F}_l(m_1, \ldots, m_{n+1}) = \{[x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n : x_i = 0, \forall i \notin L_l(m_1, \ldots, m_{n+1})\},$$

where the union is taking on all those $l \in \{0, 1, \ldots, k-1\}$ such that $\#L_l(m_1, \ldots, m_{n+1}) \geq 1$. Note that each $\bar{F}_l(m_1, \ldots, m_{n+1})$ is:

1. (i) just a point if $\#L_l(m_1, \ldots, m_{n+1}) = 1$, and
2. (ii) a projective space of dimension $\#L_l(m_1, \ldots, m_{n+1}) - 1$ if $\#L_l(m_1, \ldots, m_{n+1}) > 1$.

The locus of fixed points of $\varphi$ on $M_n^k(\Lambda)$ is then given as the disjoint union of the sets

$$F_l(m_1, \ldots, m_{n+1}) = \bar{F}_l(m_1, \ldots, m_{n+1}) \cap M_n^k(\Lambda).$$

But on $M_n^k(\Lambda)$ we cannot have points $[x_1 : \cdots : x_{n+1}]$ with at least $d+1$ coordinates equal to zero. This fact asserts that for $\#L_l(m_1, \ldots, m_{n+1}) \leq n - d$ one has that $F_l(m_1, \ldots, m_{n+1}) = \emptyset$. Also, for $\#L_l(m_1, \ldots, m_{n+1}) \geq n+1 - d$, we obtain that $F_l(m_1, \ldots, m_{n+1}) \neq \emptyset$ is a generalized Fermat manifold of dimension $\#L_l(m_1, \ldots, m_{n+1}) + d - n - 1$. \hfill $\square$

**Remark 5.** (1) If $k = 2$, and $\varphi := \varphi_1^{m_1} \cdots \varphi_{n+1}^{m_{n+1}} \in H_0 \cong \mathbb{Z}_2^n$, then $m_1, \ldots, m_{n+1} \in \{0, 1\}$. Proposition 3 asserts that $\varphi$ has no fixed points on $M_n^2(\Lambda)$ if and only if $\#L_0(m_1, \ldots, m_{n+1}), \#L_1(m_1, \ldots, m_{n+1}) \leq n - d$. As these two cardinalities must add to $n+1$, this is only possible for $n \geq 1 + 2d$.

(2) Let $k = 2$, $d = 1$, $n \geq 4$, and $C := C_2^4(\Lambda)$, so $H_0 \cong \mathbb{Z}_2^n$. If $n \geq 5$ odd, then there is a (unique) subgroup $K \cong \mathbb{Z}_2^{n-1}$ of $H_0$ acting freely on $C$. For $n \geq 4$ even, the above is not true, but there are subgroups $K \cong \mathbb{Z}_2^{n-2}$ acting freely.

(3) Let $d \geq 2$, $k = p \geq 2$ be a prime integer, $n \geq d+1$, and $M = M_n^p(\Lambda)$. Assume $K \cong \mathbb{Z}_p^{d-1}$ is a subgroup of $H_0$ acting freely on $M$. Let $F_j \subset M$, $j = 1, \ldots, n+1$, be the locus of fixed points of the canonical generator $\varphi_j$. As $H_0$ is an abelian group, each $F_j$ is invariant under $K$ and acts freely on it. Let $N = M/K$ (which is a compact complex manifold of dimension $d$) and $X_j = F_j/K$ (a connected complex submanifold of $N$). The $(n+1)$ connected sets $X_j$ are the locus of fixed points of the induced holomorphic automorphism by $\varphi_j$. As each two different $F_j$ and $F_j$ always intersect transversely, it follows that the same happens for $X_i$ and $X_j$. As the locus of fixed points of (finite)
holomorphic automorphisms are smooth, it follows that different \( X_i \) and \( X_j \) are the fixed points of different cyclic groups of \( A = H_0/K \cong \mathbb{Z}_p^r \). This in particular asserts that \( n + 1 \leq (p^r - 1)/(p - 1) \). So, for instance, the cases (i) \( r = 1 \) and (ii) \( r = 2 \) and \( p = 2 \), are impossible (note that this is in contrast to the case \( p = 2 \) and \( d = 1 \), where these subgroups exist and are related to hyperelliptic Riemann surfaces).

Example 1. Let us take \( n = k = 3 \) and \( d = 2 \). In this case, \( S_3^n \) is just the Fermat hypersurface \( \{ x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \} \subset \mathbb{P}^3 \). If \( \varphi = \varphi_1 \varphi_2 \varphi_3^2 \), then \( \{ m_1, m_2, m_3, m_4 \} = \{ 1, 1, 2, 0 \} \) and \( L_0(1, 1, 2, 0) = \{ 4 \} \), \( L_1(1, 1, 2, 0) = \{ 1, 2 \} \), \( L_2(1, 1, 2, 0) = \{ 3 \} \). The locus of fixed points (in \( \mathbb{P}^3 \)) of \( \varphi \) is given by

\[
\tilde{F}_0(1, 1, 2, 0) \cup \tilde{F}_1(1, 1, 2, 0) \cup \tilde{F}_2(1, 1, 2, 0) = \\
\{ [0 : 0 : 0 : 1] \} \cup \{ [x_1 : x_2 : 0 : 0] \} \subset \{ [0 : 0 : 1 : 0] \}.
\]

As the cardinalities of \( L_0(1, 1, 2, 0) \) and \( L_2(1, 1, 2, 0) \) are equal to \( 1 \leq n - d \), these two do not introduce fixed points of \( \varphi \) on \( S_3^3 \) (this can be seen also directly). The set \( L_1(1, 1, 2, 0) \) has cardinality \( 2 \geq n - d + 1 \), so it produces a zero-dimensional set of fixed points consisting of the three points \( [1 : -1 : 0] \), \( [1 : \omega_6 : 0] \) and \( [1 : \omega_6^{-1} : 0] \), where \( \omega_6 = e^{\pi i/3} \).

5. Examples

5.1. Jacobians of genus two surfaces. In [12, 15], it was observed that the desingularized Kummer surface of the Jacobian variety \( JC \) of the genus two hyperelliptic curve \( C : y^2 = (x - \alpha_1) \cdots (x - \alpha_6) \), where \( \alpha_1, \ldots, \alpha_6 \in \mathbb{C} \) are different points, is isomorphic to the following complete intersection

\[
S = \left\{ \begin{array}{l}
x_1^2 + \cdots + x_6^2 = 0 \\
\alpha_1 x_1^2 + \cdots + \alpha_6 x_6^2 = 0 \\
\alpha_1^2 x_1^2 + \cdots + \alpha_6^2 x_6^2 = 0
\end{array} \right\} \subset \mathbb{P}^9.
\]

The surface \( S \) admits the group \( H \cong \mathbb{Z}_2^5 \) of automorphisms (generated by those elements that multiply each coordinate by \(-1\)), and the quotient \( S/H \) is \( \mathbb{P}^2 \) branched at six lines in general position. This, in particular, asserts that \( S \) is a generalized Fermat surface of type \((2; 2, 5)\). Below, we proceed to find an equation of it in the form \( S_5^2(\lambda_1, \lambda_2) \). If we set \( t_1 = (\alpha_1 - \alpha_5)(\alpha_1 - \alpha_6)x_1^2, t_2 = (\alpha_2 - \alpha_5)(\alpha_2 - \alpha_6)x_2^2, t_3 = (\alpha_3 - \alpha_5)(\alpha_3 - \alpha_6)x_3^2 \), then \( P : S \to \mathbb{P}^2 : [x_1 : \cdots : x_6] \mapsto [t_1 : t_2 : t_3] \) defines a Galois branched covering with deck group \( H \). In this case, the six branched lines are

\[
L_1 = \{ t_1 = 0 \}, L_2 = \{ t_2 = 0 \}, L_3 = \{ t_3 = 0 \}, L_4 = \{ t_1 + t_2 + t_3 = 0 \},
\]

\[
L_5 = \left\{ \frac{(\alpha_1 - \alpha_4)(\alpha_3 - \alpha_5)}{(\alpha_1 - \alpha_5)(\alpha_3 - \alpha_4)}t_1 + \frac{(\alpha_2 - \alpha_4)(\alpha_3 - \alpha_5)}{(\alpha_2 - \alpha_5)(\alpha_3 - \alpha_4)}t_2 + t_3 = 0 \right\},
\]

\[
L_6 = \left\{ \frac{(\alpha_1 - \alpha_4)(\alpha_3 - \alpha_6)}{(\alpha_1 - \alpha_6)(\alpha_3 - \alpha_4)}t_1 + \frac{(\alpha_2 - \alpha_4)(\alpha_3 - \alpha_6)}{(\alpha_2 - \alpha_6)(\alpha_3 - \alpha_4)}t_2 + t_3 = 0 \right\}.
\]

It follows that \( S \cong S_5^2(\lambda_1, \lambda_2) \), where

\[
\lambda_1 = \left( \frac{(\alpha_1 - \alpha_4)(\alpha_3 - \alpha_5)}{(\alpha_1 - \alpha_5)(\alpha_3 - \alpha_4)}, \frac{(\alpha_1 - \alpha_4)(\alpha_3 - \alpha_6)}{(\alpha_1 - \alpha_6)(\alpha_3 - \alpha_4)} \right),
\]

\[
\lambda_2 = \left( \frac{(\alpha_2 - \alpha_4)(\alpha_3 - \alpha_5)}{(\alpha_2 - \alpha_5)(\alpha_3 - \alpha_4)}, \frac{(\alpha_2 - \alpha_4)(\alpha_3 - \alpha_6)}{(\alpha_2 - \alpha_6)(\alpha_3 - \alpha_4)} \right).
\]
5.2. A connection between generalized Fermat surfaces and curves of the same type.
Let \((\lambda, \mu) \in X_{n,2}\), where \(n \geq 4\), \(\lambda = (\lambda_1, \ldots, \lambda_{n-3})\), \(\mu = (\mu_1, \ldots, \mu_{n-3}) \in \mathbb{C}^{n-3}\).
Associated with this pair is the generalized Fermat surface \(S^k_{n}(\lambda, \mu)\), \(k \geq 2\), together its (unique) generalized Fermat group \(H = H_0 \cong \mathbb{Z}^k_p\) and the Galois branched holomorphic map \(\pi : S^k_{n}(\lambda, \mu) \rightarrow \mathbb{P}^2\) where \(\pi[x_1 : \cdots : x_{n+1}] = [t_1 = x_1^k : t_2 = x_2^k : t_3 = x_3^k]\) (whose deck group is \(H\)). Its branch locus set is given by the union of the lines \(L_j(\lambda, \mu)\), \(j = 1, \ldots, n+1\), as previously defined.
Let us consider a line \(L = \{(\rho_1 t_1 + \rho_2 t_2 + \rho_3 t_3 = 0) \subset \mathbb{P}^2, [\rho_1 : \rho_2 : \rho_3] \in \mathbb{P}^2\}, \) such that the collection of \(n+2\) lines \(L_1(\lambda, \mu), \ldots, L_{n+1}(\lambda, \mu), L\) are in general position. On \(L\) we have exactly \(n+1\) intersection points with the branched lines \(L_1(\lambda, \mu), \ldots, L_{n+1}(\lambda, \mu)\). These points are given by

\[
p_j = \left[ \frac{\mu_{j-4} \rho_3 - \rho_2}{\rho_1 - \lambda_{j-4} \rho_3} : 1 \right], j = 5, \ldots, n + 1.
\]

The preimage \(\pi^{-1}(L) \subset S^k_{n}(\lambda, \mu)\) provides the following curve

\[
\pi^{-1}(L) := \left\{ \begin{array}{l}
\rho_1 x_1^k + \rho_2 x_2^k + \rho_3 x_3^k = 0 \\
x_1^k + x_2^k + x_3^k = 0 \\
\lambda_1 x_1^k + \mu_1 x_2^k + x_3^k = 0 \\
\vdots \\
\lambda_{n-3} x_1^k + \mu_{n-3} x_2^k + x_{n+1}^k = 0 \\
\end{array} \right\} \subset \mathbb{P}^n.
\]

By taking \(y_1^k = \rho_1 x_1^k, y_2^k = \rho_2 x_2^k, y_3^k = \rho_3 x_3^k, y_4^k = \frac{\rho_2 \rho_3 - \rho_2}{\rho_3 - \rho_2} x_4^k, y_j^k = \frac{\rho_j \rho_3 - \rho_2}{\rho_3 - \rho_2} x_j^k,\) for \(j = 5, \ldots, n + 1\), we observe that the above is isomorphic to the following generalized Fermat curve of type \((k, n)\)

\[
C^k_n(\eta_{\lambda;\mu}, \mu) = \left\{ \begin{array}{l}
\eta_1 y_1^k + y_2^k + y_3^k = 0 \\
\eta_2 y_1^k + y_2^k + y_4^k = 0 \\
\vdots \\
\eta_{n-2} y_1^k + y_2^k + y_{n+1}^k = 0 \\
\end{array} \right\} \subset \mathbb{P}^n,
\]

where \(\eta_{\lambda;\mu} = (\eta_1, \ldots, \eta_{n-2})\) and

\[
\eta_1 := \frac{\rho_2 (\rho_3 - \rho_1)}{\rho_1 (\rho_3 - \rho_2)}, \quad \eta_2 := \frac{\rho_2 (\lambda_1 \rho_3 - \rho_1)}{\rho_1 (\mu_1 \rho_3 - \rho_2)}, \ldots, \eta_{n-2} := \frac{\rho_2 (\lambda_{n-3} \rho_3 - \rho_1)}{\rho_1 (\mu_{n-3} \rho_3 - \rho_2)}.
\]

By varying \(L\) (and including the non-general ones as above), we obtain a two-dimensional family of such generalized Fermat curves for the fixed parameter \((\lambda, \mu)\). This process provides the 3-fold \(X(\lambda, \mu) \subset \mathbb{P}^2 \times \mathbb{P}^n\), defined by those tuples \([(\rho_1 : \rho_2 : \rho_3), [x_1 : \cdots : x_{n+1}]]) \in \mathbb{P}^2 \times \mathbb{P}^n\) satisfying the following equations

\[
\left\{ \begin{array}{l}
\rho_2 (\rho_3 - \rho_1) y_1^k + \rho_1 (\rho_3 - \rho_2) y_2^k + \rho_1 (\rho_3 - \rho_1) y_3^k = 0 \\
\rho_2 (\lambda_1 \rho_3 - \rho_1) y_1^k + \rho_1 (\mu_1 \rho_3 - \rho_2) y_2^k + \rho_1 (\mu_1 \rho_3 - \rho_1) y_3^k = 0 \\
\vdots \\
\rho_2 (\lambda_{n-3} \rho_3 - \rho_1) y_1^k + \rho_1 (\mu_{n-3} \rho_3 - \rho_2) y_2^k + \rho_1 (\mu_{n-3} \rho_3 - \rho_1) y_3^k = 0 \\
\end{array} \right\},
\]

together the morphism \(\pi_{\lambda,\mu} : X(\lambda, \mu) \rightarrow \mathbb{P}^2 : ([\rho_1 : \rho_2 : \rho_3], [y_1 : \cdots : y_{n+1}]) \mapsto [\rho_1 : \rho_2 : \rho_3],\) such that \(C^k_n(\eta_{\lambda;\mu}, \mu) \cong \pi_{\lambda,\mu}^{-1}([\rho_1 : \rho_2 : \rho_3]).\)
The (possible singular) generalized Fermat curve \( \pi_{\lambda,\mu}^{-1}(\rho) \) is invariant under the group \( H \cong \mathbb{Z}^n_k = \{ b_1, \ldots, b_n \} \) of automorphisms of \( X(\lambda,\mu) \), where \( b_j \) is amplification of \( y_j \) by \( e^{2\pi i/k} \), an such a restriction to \( \pi_{\lambda,\mu}^{-1}(\rho) \) is its generalized Fermat group of type \((k,n)\).

**Remark 6.** (1) If \( \rho := [\rho_1 : \rho_2 : \rho_3] \in \mathbb{P}^2 \) corresponds to a line \( L \) such that the collection of lines \( L, L_1(\lambda,\mu), \ldots, L_{n+1}(\lambda,\mu) \) is not in general position, then \( C_n^k(\eta_{L,\lambda,\mu}) \cong \pi_{\lambda,\mu}^{-1}(\rho) \subset X(\lambda,\mu) \) is a singular curve (a singular generalized Fermat curve of type \((k,n)\)).

(2) The above provides a nice relation between the surface \( S_n^k(\lambda,\mu) \) and the 3-fold \( X(\lambda,\mu) \).

The last one parametrizes all the (singular) generalized Fermat curves of type \((k,n)\) inside the surface, each one being invariant under the generalized Fermat group \( H \) of \( S_n^k(\lambda,\mu) \).

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