Abstract
For compact manifolds with infinite fundamental group we present sufficient topological or metric conditions ensuring the existence of two geometrically distinct closed geodesics. We also extend results about generic Riemannian metrics to Finsler metrics. We show a bumpy metrics theorem for Finsler metrics and prove that a $C^4$-generic Finsler metric on a compact and simply-connected manifold carries infinitely many closed geodesics.

Keywords Closed geodesic · Fundamental group · Generic metric · Finsler metric

Mathematics Subject Classification 53C22 · 58E10

1 Introduction
In the first part of this paper we are interested in existence results for closed geodesics of a Riemannian or Finsler metric on a compact manifold with infinite fundamental group. The shortest closed curve in a non-trivial free homotopy class is a closed geodesic. Hence on a manifold with non-trivial fundamental group there always exists a non-contractible closed geodesic. It is not difficult to see that there are infinitely many geometrically distinct and non-contractible closed geodesics if the first Betti number satisfies $b_1(M) = \text{rk} H_1(M; \mathbb{Z}) \geq 2$.

The existence of infinitely many geometrically distinct closed geodesics in case of manifolds with fundamental group $\pi_1(M) \cong \mathbb{Z}$ was shown by Bangert and Hingston, cf. [7], and for an infinite solvable fundamental group in [31]. In [24] the authors present results about the

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existence of infinitely many non-contractible and geometrically distinct closed geodesics on connected sums $M = M_1 \# M_2$.

We investigate which topological assumptions imply the existence of at least two geometrically distinct and non-contractible closed geodesics for compact manifolds with infinite fundamental group.

**Theorem 1** Let $M$ be a compact manifold with infinite fundamental group $\pi_1(M)$ without torsion elements whose first homology group $H_1(M; \mathbb{Z})$ is neither trivial (then the fundamental group is perfect) nor cyclic of finite order (i.e. $H_1(M; \mathbb{Z}) \cong \mathbb{Z}/(m\mathbb{Z})$ for some $m \geq 2$). Then any Riemannian or reversible Finsler metric carries at least two geometrically distinct and non-contractible closed geodesics.

If the fundamental group does not satisfy the assumptions of Lemma 1, i.e. if the group is not isomorphic to $\mathbb{Z}$ and if there is no element $a$ of infinite order such that any other element is conjugate to a power of $a$ then it is clear that there are two geometrically distinct closed geodesics, cf. [6, (3.15)]. It is not known whether a finitely presented group with this property exists, cf. Remark 1. In Theorem 1 we give sufficient conditions for the first homology group of the manifold ensuring the existence of a second non-contractible closed geodesic.

We also present an existence result for non-reversible Finsler metrics.

**Theorem 2** Let $M$ be a compact manifold with non-trivial fundamental group $\pi_1(M)$ endowed with a non-reversible Finsler metric. If at least one of two following conditions:

(a) the first Betti number of $M$ is positive: $b_1(M) = \dim H_1(M; \mathbb{R}) > 0$,
(b) all closed geodesics of this metric are hyperbolic,

holds then there are two geometrically distinct and non-contractible closed geodesics.

The set of metrics all of whose closed geodesics are hyperbolic is important for statements about generic metrics. Therefore we study genericity statements about Finsler metrics in the second part of the paper.

For Riemannian metrics the *bumpy metrics theorem* due to Abraham [1] resp. Anosov [3] and perturbation arguments due to Klingenberg and Takens [16] can be used to prove that for a $C^4$-generic metric either all closed geodesics are *hyperbolic* or there is a non-hyperbolic closed geodesic of *twist type*. In the second case the Birkhoff-Lewis fixed point theorem implies the existence of infinitely many geometrically distinct closed geodesics, cf. [15, Thm.3.3.10] and [18].

For Finsler metrics corresponding genericity statements hold as we show in Sect. 5:

**Theorem 3** [Bumpy metrics theorem for Finsler metrics] For a compact differentiable manifold a $C^r$-generic Finsler for $r \geq 4$ metric is bumpy.

And for a $C^r$-generic Finsler metric for $r \geq 6$ on a compact manifold either all closed geodesics are hyperbolic, or there exists a non-hyperbolic closed geodesic of twist type, cf. Theorem 6. Hence for a statement about the existence of a second closed geodesic at least for a generic non-reversible Finsler metric it is in light of Theorem 2 sufficient to consider metrics all of whose closed geodesics are hyperbolic.

The concept of a *strongly bumpy metric* is introduced in [22, Def.4.1], cf. Sect. 5. We finally obtain the following result generalizing the statements of [22, Thm.4.3] and [22, Thm.5.7] to the Finsler case:

**Theorem 4** Let $M$ be a compact manifold.

(a) A $C^4$-generic Finsler metric is strongly bumpy.
(b) A strongly bumpy metric on a manifold with finite fundamental group carries infinitely many geometrically distinct and contractible closed geodesics.
For simply-connected and compact manifolds with a bumpy and non-reversible Finsler metric there are at least two geometrically distinct closed geodesics. This was shown by Duan, Long and Wang in [10, Cor.1.2]. It also follows from existence results for closed orbits of non-degenerate Reeb flows obtained by Abreu, Gutt, Kang & Macarini [2, Cor.1.14].

The structure of the paper is as follows: After explaining the setting in Sect. 2 we prove Theorem 1 in Sect. 3 and Theorem 2 in Sect. 4. In the last Sect. 5 we prove generic properties of Finsler metrics.

2 Preliminary facts

2.1 Functional spaces

Let \( M^n \) be a compact Riemannian or a Finsler manifold.

We denote by \( \Lambda(M^n) = H^1(S^1, M) \) the space of \( H^1 \)-maps

\[ \gamma : [0, 1] \rightarrow M^n, \quad \gamma(0) = \gamma(1), \]

of a circle \( S^1 = \mathbb{R}/\mathbb{Z} \) into \( M^n \) and by \( \Omega_1(M^n) \) the subspace of \( \Lambda(M^n) \) formed by loops starting and ending at \( \gamma(0) = \gamma(1) = x \in M^n \).

Let \( \Pi^+(M^n) \) and \( \Pi(M^n) \) be the quotients of \( \Lambda(M^n) \) with respect to the \( SO(2)(= S^1) \)-action:

\[ \varphi \cdot \gamma(t) = \gamma(t + \varphi), \quad \varphi \in S^1 = \mathbb{R}/\mathbb{Z}, \]

and the \( O(2) \) action generated by \( SO(2) \) and the inversion of the parameter:

\[ \gamma^{-1}(t) = \gamma(-t). \]

Let \( g : \Lambda M \rightarrow \Lambda M \) map a curve to be the map which corresponds to a curve the same line parametrized proportionally to the arc-length, such that \( (g \cdot \gamma)(0) = \gamma(0) \). We put

\[ L(M) = g(\Lambda(M)), \]

and denote by \( P^+(M) \) and \( P(M) \) the following quotient-spaces:

\[ P^+(M) = L(M)/SO(2), \quad P(M) = L(M)/O(2). \]

It is known that these spaces are deformation retracts of \( \Lambda(M), \Pi^+(M) \) and \( \Pi(M) \), respectively. We recall that the action of \( SO(2) \) is not free due to the iterates of prime curves.

There is the procedure which replaces \( L(M)/SO(2) \) by the quotient of a free action of \( SO(2) \). We recall it in the full generality. Let \( X \) be a topological space on which the group \( G \) acts continuously. The homotopy quotient \( X_G \) of the \( G \)-space \( X \) is the quotient of the product \( X \times EG \) with respect to the diagonal action of \( G \). Here

\[ EG \xrightarrow{G} BG \]

is the universal \( G \)-bundle. By definition, \( EG \) is contractible. The \( G \)-equivariant cohomology is defined as

\[ H^*_G(X) = H^*(X_G). \]

For \( G = SO(2) \) we have \( BG = \mathbb{C}P^\infty \), and, therefore, the \( SO(2) \)-equivariant cohomology of the point with the trivial \( SO(2) \) action on it are isomorphic to the cohomology of \( \mathbb{C}P^\infty \):

\[ H^*_{SO(2)}(pt; \mathbb{Z}) = H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[u], \quad \deg u = 2. \]
For $G = SO(2)$ with the standard action by left multiplications we have
\[ H^*_r(SO(2); \mathbb{Z}) = H^*(pt; \mathbb{Z}) = H^0(pt; \mathbb{Z}) = \mathbb{Z}, \]
compare for example [14, 21].

### 2.2 The exact homotopy sequence for the spaces of non-contractible closed curves

Given $h \in \pi_1(M, x_0)$, we denote by $[h] \in [S^1, M]$ the corresponding free homotopy class of closed curves and by
\[ \Lambda M[h] \subset \Lambda M \quad \text{and} \quad LM[h] \subset LM \]
the connected components of $\Lambda M$ and $LM$ consisting of curves from $[h]$.

Let $h$ be realized by a map $\omega : [0, 1] \to M$ with $\omega(0) = x_0$, and let $h_i$ be the automorphism
\[ h_i : \pi_i(M, x_0) \to \pi_i(M, x_0) \]
corresponding to the standard action of $h \in \pi_1$ on $\pi_i$.

We have

**Theorem 5** [5, 29] The mapping
\[ \pi : \Lambda M \to M, \quad \pi(\gamma) = \gamma(0), \]
which maps a closed curve $\gamma$ onto the marked point $\gamma(0)$, is a Serre fibration with the fibre $\Omega M$:
\[ \Lambda M \xrightarrow{\Omega M} M. \]

The exact homotopy sequence for this fibration restricted onto $\Lambda M[h]$ takes the form
\[ \cdots \to \pi_i(\Lambda M[h], \omega) \xrightarrow{\pi_*} \pi_i(M, x_0) \xrightarrow{f_i} \pi_{i-1}(\Omega x_0(M), \omega) = \pi_{i-1}(\Lambda M[h], \omega) \to \cdots \]
where
(a) $\pi_* (\pi_1(\Lambda M[h], \omega)) = St(h_i)$, where $St(h_i)$ is the subgroup of $\pi_i(M, x_0)$ consisting of all elements fixed under $h_i$;
(b) $f_i = h_i - \text{id}$ for $i \geq 2$.

Moreover, the homotopy sequence finishes as
\[ \to \pi_2(M) = \pi_1(\Omega M[h]) \to \pi_1(\Lambda M[h]) \to St(h) \to 1, \]
where $St(h)$ is the subgroup of $\pi_1(M)$ formed by all elements which commute with $h$.

The maps $f_i$ are written uniformly in the simple form
\[ f_i(g) = [h, g], \quad g \in \pi_i(M, x_0), \quad i \geq 1, \]
where $[h, g]$ is the Whitehead product of $h \in \pi_1$ and $g \in \pi_i$.

By the Serre theorem,
\[ \pi_{i-1}(\Omega M) = \pi_i(M), \]
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and for the simplest case $h = 1$ (the unit of $\pi_1$) we have $h_*=\text{id}$ and the exact sequence splits into short exact subsequence of the form

$$0 \to \pi_{i+1}(M) \to \pi_{i}(\Lambda M) \overset{\pi_*}{\longrightarrow} \pi_{i}(M) \to 0.$$  

In the sequel we take tensor products of all entries with $\mathbb{Q}$ over $\mathbb{Z}$. Therewith all higher homotopy groups become $\mathbb{Q}[\pi_1]$-modules (a priori such modules can be even infinitely generated) which we denote by $\pi_i \mathbb{Q}$.

### 2.3 The type numbers of closed geodesics

A point $\gamma \in \Lambda M$ is a closed geodesic if it is a critical point of the length functional

$$S(\gamma) = \int_{\gamma} F(x, \dot{x}) dt,$$

where $F(x, v), x \in M, v \in T_x M$, is positive for $v \neq 0$ and homogeneous of first order in $v$:

$$F(x, \lambda v) = \lambda F(x, v) \quad \text{for} \quad \lambda \geq 0,$$

and the set $S_x M = \{v \in T_x M : F(x, v) = 1\} \subset T_x M$ of unit vectors is a strictly convex subset. This action functional is called the Finsler length. If $F(x, v) = \sqrt{g_{ik}(x)v^iv^k}$, where $g_{ik}$ is the Riemannian metric tensor, we get the usual length with respect to the Riemannian metric. The Finsler metric is called reversible if $F(x, v) = F(x, -v)$ for all $x$ and $v \in T_x M$.

The classical Morse theory is applicable to the study of critical points of $S$ on the spaces $\Lambda M$. We omit the exposition of the standard things and only discuss an important statement due to Schwarz.

It is clear that every iterate of a closed geodesic is a closed geodesic itself. For a reversible Finsler metric the inverse $\gamma^{-1}$ of the geodesic $\gamma$ is also geodesic. For a non-reversible Finsler metric we say that two closed geodesics are (geometrically) distinct if they are not iterates of the same closed curve with the same orientation. Geodesics of a Riemannian metric or, more general, reversible Finsler metric are (geometrically) distinct if they are not iterates of a certain geodesic and its inverse.

A metric is called bumpy if all its closed geodesics are non-degenerate in the Morse theoretic sense except the trivial one-point extremals.

We assign to every nontrivial closed geodesic its type numbers which are

$$t_i = \text{rank } H_i \left( \{ S < S_0 \} \cup \gamma, \{ S < S_0 \}; \mathbb{Q} \right) \quad \text{where } S_0 = S(\gamma)$$

and $S$ is considered as a functional on $\Lambda / SO(2)$ or on $\Lambda SO(2)$ (on the quotient or on the homotopy quotient of $\Lambda = \Lambda M$). It is evident that for nondegenerate closed geodesics $t_i \in \{0, 1\}$.

It was established by Schwarz that, if $\gamma$ is a prime closed geodesic, then

$$t_i(\gamma^k) = \delta_{i, \text{ind } \gamma^k}$$

if the Morse index of $\gamma^k$, i.e., $\text{ind } \gamma^k$, has the same parity as $\text{ind } \gamma$ and

$$t_i(\gamma^k) = 0 \quad \text{otherwise.}$$
Therefore, we have

**Proposition 1** [27] If for two nondegenerate closed geodesics $\gamma_1$ and $\gamma_2$, we have

$$t_i(\gamma_1) = 1, \quad t_j(\gamma_2) = 1, \quad i - j \equiv 1 \mod 2,$$

then these geodesics are distinct.

This proposition is very useful in combination with another statement derived from the Bott index formula for iterates of geodesics:

**Proposition 2** [8] If $\gamma$ is a nondegenerate prime closed geodesic, then the indices of $\gamma$ and its iterates of odd order have the same parity:

$$\text{ind} \gamma^{2k+1} - \text{ind} \gamma \equiv 0 \mod 2.$$

2.4 $\pi_1(\Lambda M[h])$ and geometrically distinct closed geodesics in $\Lambda M[h]$

**Proposition 3** [31] If $\text{St}(h)$ is non-cyclic, then there are at least two geometrically distinct closed geodesics in $\Lambda M[h]$.

**Proof** First, let us recall the original argument from [31] (see Lemma 3). Let two loops $\gamma$ and $\eta$ realize elements $h$ and $g$ from $\text{St}$ which do not lie in a cyclic subgroup of $\text{St}(h)$. Moreover we assume that $\gamma$ realizes a closed geodesic of minimal length in this homotopy class. Take a mapping of a two-torus $T^2 \to M$ such that one of the parallels is mapped into $\gamma$ and another parallel is mapped into $\eta$. This torus is swept out by closed curves homotopic to $\gamma$ and deforming this loop in $\Lambda M[h]$ we have either to stop at some nontrivial closed geodesic $\gamma_1$ or to deform the whole loop into $\gamma$. In the latter case the second parallel is deformed to some iterate of a curve $\gamma_0$ such that $\gamma$ is also an iterate of $\gamma_0$. However that contradicts the choice of $\eta$. Hence we have a closed geodesic in $\Lambda [h]$ which is distinct from $\gamma$.

Algebraically this argument is reformulated as follows. Let us consider the homotopy quotient $\Lambda[h]_SO(2)$. We have the fibration $\Lambda[h] \times ESO(2) \to S^1 \to \Lambda[h]_SO(2)$ and from the exact homotopy sequence for this fibration we have

$$\cdots \to \pi_1(S^1) \to \pi_1(\Lambda[h]) \to \pi_1(\Lambda[h]_SO(2)) \to 1.$$  \hspace{1cm} (3)

By construction, the image of $\pi_1(S^1)$ is just the cyclic subgroup generated by $h$. Hence

$$\pi_1(\Lambda[h]_SO(2)) \neq 1$$  \hspace{1cm} (4)

and therefore $\Lambda[h]_SO(2)$ is not contractible and contains critical points different from the minimal closed geodesic.

**Corollary 1** If the metric is bumpy and the conditions of Proposition 3 hold then there are two geometrically distinct closed geodesics in $\Lambda M[h]$.

**Proof** Indeed, from (4) and the bumpyness of the metric it follows that there is closed geodesic of index one. By Propositions 1 and 2, it has to be geometrically distinct from the minimal closed geodesic in this homotopy class.
3 The second closed geodesic of a Riemannian or reversible Finsler metric

By using Propositions 1 and 2 and computations of certain homologies in [27], Fet proved that a bumpy Riemannian metric on a simply-connected closed manifold has at least two distinct closed geodesics [11]. For compact manifolds with infinite fundamental group the analogue of this statement is not yet established.

In this and the next sections we present topological and generic conditions under which the existence of second closed geodesic on a non-simply-connected manifold can be proved. We want to give a sufficient condition for the fundamental group of a compact manifold to ensure the existence of a second closed geodesic. On a compact manifold with a non-trivial fundamental group \(\pi_1(M)\) and with a torsion element \(a\) such that \(a^m = 1\) for some \(m \geq 2\) there are infinitely many closed geodesics for a Riemannian metric all of whose closed geodesics are hyperbolic. This is shown in [4, Thm.A], cf. Proposition 4. If we assume that a compact manifold with non-trivial fundamental group and without torsion elements is endowed with an arbitrary Riemannian metric which carries only one closed geodesic then its fundamental group satisfies the assumption of the following algebraic lemma, cf. [6, (3.15)]:

**Lemma 1** Let \(G\) be a group with the following property: There is an element \(a \in G\) such that the subgroup \(\langle a \rangle = \{a^m; m \in \mathbb{Z}\} \subset G\) generated by \(a\) is isomorphic to the group of integers, i.e. \(\langle a \rangle \cong \mathbb{Z}\) and such that for all \(g \in G\) there is an element \(h \in G\) and \(m \in \mathbb{Z}\) with \(g = ha^m h^{-1}\).

Then the abelianization \(G^{ab} = H_1(G; \mathbb{Z}) = G/[G, G]\) is finite and cyclic unless \(G \cong \mathbb{Z}\). Hence the commutator subgroup \([G, G]\) is infinite and has finite index.

**Proof** If \(G \not\cong \mathbb{Z}\) then there is \(\tilde{b} \in G\) and \(h \in G, m \in \mathbb{Z}\) with \(\tilde{b} = ha^m h^{-1} \notin \langle a \rangle\). It follows that \(b := hah^{-1} \notin \langle a \rangle\). Then

\[
1 \neq ba^{-1} = hah^{-1} a^{-1} = [h, a].
\]

On the other hand there is an element \(f \in G, k \in \mathbb{Z}, k \neq 0\) with

\[
1 \neq ba^{-1} = [h, a] = fa^k f^{-1}.
\]

Then

\[
[h, a]a^{-k} = [f, a^k]
\]

which implies

\[
a^k = [a^k, f][h, a] \in [G, G].
\]

Hence the normal subgroup generated by \(a^k\) is a subgroup of the commutator subgroup \([G, G]\) and the quotient group \(H_1(G; \mathbb{Z}) = G/[G, G]\) is a finite cyclic group whose order divides \(k\). \(\square\)

**Proof of Theorem 1 in the Introduction** Assume there is a prime closed geodesic \(c\) such that all closed geodesics are geometrically equivalent to \(c\). Then there is a nontrivial homotopy class \(a \in \pi_1(M)\) such that \(c\) lies in the free homotopy class generated by \(a\) and such that the following holds: For every \(g \in \pi_1(M)\) there exists \(h \in \pi_1(M), m \in \mathbb{Z}\) with \(g = ha^m h^{-1}\). By assumption the subgroup \(\langle a \rangle\) is isomorphic to \(\mathbb{Z}\). If \(\pi_1(M) \cong \mathbb{Z}\) then there are infinitely many geometrically distinct closed geodesics by a result due to Bangert-Hingston, cf. [7]. Hence
the fundamental group satisfies the assumptions of Lemma 1. Therefore the abelianization 
$H_1(M; \mathbb{Z})$ of the fundamental group is either trivial (then the fundamental group is perfect) 
or it is isomorphic to $\mathbb{Z}_m, m \geq 2$.

\textit{Remark 1} Problem 8.8 in the Kourovka notebook [17] posed by Anosov asks whether a 
finitely generated group satisfying the assumption of Lemma 1 besides the integers exists. Guba [13] constructed such a group, but this group is not finitely presented and hence it is not the 
fundamental group of a compact manifold, cf. [30, p. 206] and [4, Rem.(3) following Thm.B].

We notice if such a group exists and appears as the fundamental group of a closed $K(\pi, 1)$ 
manifold, then we do not see how to prove the existence of at least two closed geodesics on 
such a manifold with the methods used in this text.

4 The second closed geodesic of a non-reversible Finsler metrics

In this section we consider non-reversible Finsler metrics. In particular two closed geodesics 
c_1, c_2 : S^1 \longrightarrow M are geometrically equivalent if their images coincide, i.e. $c_1(S^1) = c_2(S^1)$ 
and if their orientations coincide.

In case of non-reversible Finsler metrics a result by Ballmann, Thorbergsson and Ziller 
gives the following

\textbf{Proposition 4} [4, Thm.A] Let $M$ be a compact manifold with non-trivial fundamental group 
$\pi_1(M)$. Let $a \in \pi_1(M), a \neq 1$. We assume that for some positive integers $0 < n < m$ the 
conjugacy classes $[a^n] = [a^m]$ of the iterates $a^n, a^m$ coincide. If the manifold carries a 
non-reversible Finsler metric all of whose closed geodesics are hyperbolic then 
\[
\liminf_{t \to \infty} \frac{N^a(t)}{t} > 0. 
\]

Here $N^a(t)$ is the number of geometrically distinct closed geodesics of length $\leq t$ which 
are freely homotopic to a power $a^k, k \geq 1$. The proof of the Birkhoff-Lewis fixed point 
thorem [18] also implies that for a closed geodesic $c$ of twist type freely homotopic to 
a $\in \pi_1(M)$ there are infinitely many closed geodesics freely homotopic to a power $a^k$ and 
the function $N^a(t)$ satisfies Equation (5). Using this Proposition we give the following

\textbf{Proof of Theorem 2 in the Introduction} Since the metric is non-reversible, we say that two 
closed geodesics are geometrically distinct if they are not positive iterates $c^k$ and $c^l, k, l > 0$, 
of a closed curve $c$.

If condition a) holds then the minimal closed curves in the homology classes $[a]$ and 
$[a^{-1}]$, where $[a]$ is of infinite order in $H_1(M)$, are definitely geometrically distinct.

We are left to consider the case when condition b) holds. We assume that there is only one 
prime closed geodesic $c : S^1 \longrightarrow M$. Choose $h \in \pi_1(M)$ such that $c$ is freely homotopic to 
h : $c \in \Lambda M[h]$. If $h$ is a torsion element in $\pi_1$, then Proposition 4 implies that there are 
ininitely many geometrically distinct closed geodesics. Hence we assume that 
\[
h^m \neq 1 \quad \text{for all } m = 1, 2, \ldots
\]

Since there is only one closed geodesic there exists a $k \geq 1$ such that the closed geodesic $c^k$ 
is up to parametrization the unique shortest closed curve in the free homotopy class $[h^{-1}]$. 
Hence there exists $a \in \pi_1(M)$ such that for some $k \geq 1$:
\[
h^{-1} = ah^k a^{-1}.
\]
Then we conclude
\[ h = (h^{-1})^{-1} = ah^{-k}a^{-1} = a^2h(k^2)a^{-2}. \]

If \( k \geq 2 \) then we obtain from Proposition 4 that there are infinitely many closed geodesics.

Hence we can assume \( k = 1 \). There are two commuting elements \( h, a^2 \in \pi_1(M) \) and we can assume that both are of infinite order. If they do not lie in some cyclic subgroup, then by Proposition 3, there exist two geometrically distinct closed geodesics. Therefore we are left to assume that \( h \) and \( a^2 \) belong to some cyclic subgroup. We notice that \( h \) is a prime element, otherwise \( h = g^m \) for some \( g \in \pi_1 \) and \( m > 1 \) and the homotopy class \([h]\) contains an \( m\)-th iterate of the minimal closed geodesic \( c' \) in \([g]\). In this case either there are at least two closed geodesics \( c \) and \( c' \), or \( c' = c, l > 0 \), and \( c \) is conjugate to \( c^l \) which implies, by Proposition 4, the existence of infinitely many closed geodesics.

Therefore \( a^2 = h^q \) for some \( q > 0 \). By (7), for \( k = 1 \) we have
\[ h^{-q} = ah^q a^{-1} = a a^2 a^{-1} = a^2 = h^q, \]
which implies that
\[ h^{2q} = 1, \]
which contradicts Eq. (6).

\[ \square \]

5 Generic conditions for Riemannian and Finsler metrics and the existence of closed geodesics

In this section we discuss how known generic properties of Riemannian metrics are also generic for Finsler metrics. A Finsler metric \( F : TM \rightarrow \mathbb{R} \) defines for a point \( x \in M \) a family of Riemannian metrics \( g^V \) on the tangent space \( T_xM \) parametrized by unit vectors \( y \in S_xM := \{ w \in T_xM ; F(w) = 1 \} \). For a coordinate system \( x = (x_1, \ldots, x_n) \) for a chart \( U \subset M \) of the manifold \( M \) with corresponding chart \( (x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \) for the tangent bundle \( TU \) the metric coefficients \( g_{ij}(x, y) = g^V_{ij}(x) \) are given by:
\[ g_{ij}(x, y) = g^V_{ij}(x) = \frac{1}{2} \frac{\partial F^2}{\partial y_i \partial y_j}(x, y). \]

For a closed geodesic \( c : \mathbb{R} \rightarrow M \) of the Finsler metric parametrized by arc length we can extend the velocity vector field \( c' \) along \( c \) to a non-zero vector field \( V \) in a tubular neighborhood \( U \subset M \) of the closed geodesic. Then the Riemannian metric \( g^V \) is called osculating Riemannian metric. Now we follow the notation of Anosov [3]. We choose Fermi coordinates \( (u_0, u) = (u_0, u_1, \ldots, u_n) \) (here \( n + 1 = \dim M \)) in a tubular neighborhood of the closed geodesic with respect to the osculating Riemannian metric \( g^V \). Hence \( u_0(t) = t, u(t) = 0 \) parametrizes the closed geodesic and for any \( t_0 \) and any \( v \in \mathbb{R}^n \) the curve \( u_0(t) = t_0, u(t) = tv \) is a geodesic, too. Then for the corresponding metric coefficients we obtain
\[ g^V_{ij}(t, 0) = \delta_{ij}, 0 \leq i, j \leq n; \frac{\partial}{\partial u_k}g^V_{ij}(t, 0) = 0. \]

Then \( c \) is also a closed geodesic of \( g^V \) with the same length. The flag curvature \( K(V(x); \sigma) \) of the Finsler metric of a flag \( (V(x), \sigma) \) with a plane \( \sigma \in T_xM, V(x) \in \sigma \) agrees with the sectional curvature \( K(\sigma) \) of the osculating Riemannian metric \( g^V \), cf. [23]. Parallel transport along \( c \) as well as the index and nullity of the closed geodesic of the Finsler metrics agree
with the parallel transport and the index and nullity of the osculating Riemannian metric. This allows to extend the proof of the bumpy metric theorem for Riemannian metrics presented by Anosov [3, Thm.1] to the Finsler case. This seems to be obvious to experts, for example Ginzburg and Gürel mention that the statement follows from Anosov’s proof without giving details, cf. [12, Example 1.15]. In the sequel we indicate how this extension works.

For a compact manifold $M$ denote by $\mathcal{F}'(M)$ the set of $C^r$-Finsler metrics $F : TM \to \mathbb{R}$ endowed with the strong $C^r$-topology. Similar to the bumpy metrics theorem for Riemannian manifolds due to Abraham [1] resp. Anosov [3] the corresponding statement holds for Finsler metrics, cf. Theorem 3. In contrast to the Riemannian case the order $r$ is at least 4 instead of 2. This is due to the fact that Equation (8) shows that we obtain the osculating metric by differentiating the Finsler metric twice fibrewise.

**Proof of Theorem 3** We can extend the proof given by Anosov using the following argument: Let $c$ be a closed geodesic $c$ of the Finsler metric $F$ with an osculating Riemannian metric $g^V$ in a tubular neighborhood of $c$ defined for the vector field $V$ as above and let $x \in U \mapsto \delta g(x)$ be a field of symmetric bilinear forms on the tangent space $T_x M$ with $\delta g_00(u_0, 0) = 2 \sum_{i=1}^n f_i(u_0, u)u_i$ and $\delta g_{ij}(u_0, u) = 0$ for all $i, j; i + j \geq 1$. Here $f_i, i = 1, \ldots, n + 1$ are smooth functions with support in a neighborhood of a point $(t_0, 0)$, cf. [3, §5]. We define a one-parameter family $s \in (-\epsilon, \epsilon) \mapsto F_s$ of Finsler metrics with $F = F_0$ by the Equation:

$$F_s^2(u_0, u, y_0, y) = F^2(u_0, u, y_0, y) + s\phi(y)\sum_{k=1}^n f_k(u_0, u)u_k y_0^2. \quad (10)$$

Here $\phi : \mathbb{R}^n \to [0, 1]$ is 0-homogeneous smooth function which is locally constant with value 1 in a neighborhood of $(1, 0, \ldots, 0)$. Hence $F_s$ is a smooth perturbation of the Finsler metric $F$ with the property that $c$ is a closed geodesic for all Finsler metrics $F_s$. The osculating Riemannian metric $g^V_s$ of $F_s$ with respect to the vector field $V$ near $c$ is given by:

$$(g^V_s)_{ij}(u_0, u) = g^V_{ij}(u_0, u) + \delta_{i0}\delta_{j0}s\sum_{k=1}^n f_k(u_0, u)u_k. \quad (11)$$

Then for all $s$ the curve $c$ is still a geodesic for $F_s$ and the osculating Riemannian metric $g^V_s$ near the closed geodesic $c$ of is of the form: $g^V_s = g^V + s\delta g$. Therefore Lemma 1 and Lemma 2 in Anosov’s proof of the bumpy metrics theorem carry over to the Finsler setting which is sufficient for the proof of the Theorem. \hfill $\square$

In [28] it was already shown that one can perturb a closed geodesic of a Finsler metric to become non-degenerate using the osculating Riemannian metric. The proof of Theorem 3 could also follow the approach presented by Rifford and Ruggiero [26]. The linearized Poincaré map $P_c$ of a closed geodesic is a linear symplectic map. If $c$ is not hyperbolic denote by $\exp(2\pi i \lambda_j(c))$, $\lambda_j(c) \in [0, 1/2], 1 \leq j \leq l, l \leq n - 1$ the eigenvalues of $P_c$ of modulus 1. The Finsler metric is called strongly bumpy if all eigenvalues of the prime closed geodesics are simple and if any finite set of the disjoint union of the Poincaré exponents $\lambda_j(c)$ is algebraically independent, cf. [22, Def.4.1]. Then the proofs in [22] carry over to Finsler metrics and starting from Theorem 3 one obtains Theorem 4. Using the Birkhoff normal form of a symplectic mapping one defines the concept of a non-hyperbolic closed geodesic of (generalized) twist type. It is an open and dense condition for the 3-jet of the Poincaré mapping of a non-hyperbolic closed geodesic. For a $C^4$-Riemannian metric on a compact manifold either all closed geodesics are hyperbolic or there exists a closed geodesic of twist type. This can be shown starting from a bumpy metric by perturbing the 3-jet of the Poincaré...
mapping of single closed geodesics, cf. [16] or [15, Thm. 3.3.10]. Similar arguments can be used to show the following

**Theorem 6** Let $M$ be a compact manifold.

(a) For a $C^r$-generic Finsler metric with $r \geq 6$ either all closed geodesics are hyperbolic or there exists a non-hyperbolic closed geodesic of twist type.

(b) In a tubular neighborhood of a non-hyperbolic closed geodesic of twist type there are infinitely many geometrically distinct closed geodesics.

The necessary local perturbation argument in (a) in the larger class of Tonelli Hamiltonians was shown by Carballo and Miranda [9, Cor.5]. The order 6 instead of 4 as in the Riemannian case comes from taking the Hamiltonian $H : T^* M \to \mathbb{R}$ corresponding to the Lagrangian $L = F^2 : TM \to \mathbb{R}$ by applying the Legendre transformation: $y \in T_x M \to g^x(y, .) \in T^* M$, cf. Equation (8). Part (b) is a direct consequence of the Birkhoff-Lewis fixed point theorem, cf. [15, Thm. 3.3.10] and [18].

In [32, p. 141] Ziller sketches how Pugh’s closing lemma [19] implies that for a $C^2$-generic Finsler metric on a compact manifold the initial vectors to closed geodesics are dense in the unit tangent bundle, cf. also [20] for a Hamiltonian closing lemma.

Hence for these metrics there are in particular infinitely many geometrically distinct closed geodesics. The perturbations used in the closing lemma do not preserve particular flows as the geodesic flow of a Riemannian metric or a magnetic geodesic flow. A $C^1$-closing lemma for Riemannian metrics was obtained by Rifford [25].

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