Three-Way Serpentine Slow Wave Structures With Stationary Inflection Point and Enhanced Interaction Impedance

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Abstract—We introduce two novel variants of the serpentine waveguide (SWG) slow wave structure (SWS), often utilized in millimeter-wave traveling-wave tubes (TWTs), with enhanced interaction impedance. Using dispersion engineering in conjunction with transfer matrix methods, we tune the guiding wavenumber dispersion relation to exhibit stationary inflection points (SIPs), and also nonstationary, or “tilted” inflection points (TIPs), within the dominant TE10 mode of a rectangular waveguide. The degeneracy is found below the first upper band edge associated with the bandgap where neighboring spatial harmonics meet in the dispersion of the SWG which is threaded by a beam tunnel. The structure geometries are optimized to be able to achieve a SIP which allows for three-mode synchronism with an electron beam over a specified wavenumber interval in the desired Brillouin zone. Full-wave simulations are used to obtain and verify the existence of the SIP in the three-coupled waveguide and fine-tune the geometry such that a beam would be in synchronism at or near the SIP. The three-way waveguide SWS exhibits a moderately high Pierce impedance in the vicinity of a nearly stationary S waves, and therefore, the SWS geometry potentially useful for improving the power gain and basic extraction efficiency of millimeter-wave TWTs. Additionally, the introduced SWS geometries have directional coupler-like behavior, which enables distributed power extraction at frequencies near the SIP frequency.

Index Terms—Dispersion, distributed power extraction (DPE), interaction impedance, millimeter wave, serpentine ladder waveguide (SLWG), serpentine waveguide (SWG), stationary inflection point (SIP), three-coupled SWG (TCSWG), traveling-wave tube (TWT).

I. INTRODUCTION

TRAVELING-WAVE tubes (TWTs) are able to perform high-power amplification, often over a broad frequency range, due to the distributed transfer of energy from a beam of electrons to guided electromagnetic fields. There are two primary factors that control the strength of the interaction between the beam and the electromagnetic field: velocity synchronization between the beamline and guided modes and beam-wave interaction impedance, also called Pierce impedance, in the TWT slow wave structure (SWS) [1]. In 3-band helix TWTs, Pierce impedance is typically on the order of 100 $\Omega$, allowing for efficient basic beam-wave power conversion. However, at millimeter-wave frequencies, microfabricated slow wave structures, such as the serpentine waveguide (SWG) typically exhibit Pierce (or interaction) impedance on the order of 10 $\Omega$ or smaller, which drastically reduces their basic conversion efficiency. This is partly due to the fact that serpentine-type TWTs are typically synchronized to an electron beam in the first or second Brillouin zone, rather than the fundamental Brillouin zone to avoid the use of relativistic electron beam velocities, which require much larger voltages to accelerate electrons close to the speed of light [2]. Due to structure geometry that scales inversely with operating frequency, helix-type TWTs become extremely difficult to fabricate at millimeter-wave frequencies, making microfabricated structures like the SWG much more attractive. In this article, we propose two new types of dispersion-engineered SWSs based on the SWG geometry that are capable of exhibiting moderately high Pierce impedance over narrow bandwidths and can be microfabricated. These two structure variants, the serpentine ladder waveguide (SLWG) and the three-coupled SWG (TCSWG), with their longitudinal cross sections illustrated in Fig. 1(a) and (b), respectively, can have their geometries easily designed to exhibit stationary inflection points (SIPs) or nearly stationary tilted inflection points (TIPs), sometimes referred to as tilted SIPs, at specific frequencies, similar to the kind shown in Fig. 1(c). Such structures with SIPs are capable of exhibiting moderately high to very high Pierce (interaction) impedance comparable to the impedance observed near the band edge of SWG SWS, where the group velocity and power flow also vanish. Therefore, at the frequencies that the electron beam is in synchronism with the SIP, we expect the proposed SIP SWS to also have a high Pierce gain parameter, as $C^3 = Z_{\text{Pierce}} I_0/(4V_0)$, where $I_0$ and $V_0$ are the average current and equivalent
The SIP is a class of modal degeneracy, whereby three eigenmodes coalesce in both their wavenumbers and eigenvectors (polarization states). Such modal degeneracies of orders 2, 3, and 4 were originally investigated by Figotin and Vitebskiy [3], [4], [5], [6], [7], [8]. The SIP is a particular type of exceptional point of degeneracy (EPD) and is sometimes called a “frozen mode” in the literature. Exceptional points of the degeneracy of various orders have been previously investigated theoretically in gainless and lossless structures operating at both radio and optical frequencies in [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], and [23], and have also been experimentally demonstrated at radio frequency (RF) in [24], [25], [26], [27], [28], [29], and [30]. In particular, the first experimental demonstration of SIPs at radio frequencies in a reciprocal three-way waveguide SWS has been performed in [28]. The slow wave structures we introduce here are designed to operate with an electron beam synchronized to three degenerate “cold” eigenmodes. That is, the SIP is made to exist in the “cold” dispersion relation, i.e., before the introduction of an electron beam, which will perturb the dispersion relation. Note that this regime of operation is different from the regime of “exceptional synchronization” or “degenerate synchronization” studied in [31], [32], and [33], where an EPD is designed to occur in the “hot” system, i.e., it is visible in the modal dispersion relation only when an electron beam is present. These exceptionally synchronized modes of the structure become degenerate when a synchronized electron beam is coupled to the electromagnetic modes.

We define a “waveguide way” as an individual waveguide component (rectangular waveguide, SWG, and so on), which is not in cutoff over the designed operating frequency and can support two electromagnetic modes (one forward and one backward). A three-way waveguide supports six modes when considering propagation in both the $+z$- and $-z$-directions, i.e., three modes in each direction. The three-way microstrip structure with SIP in [28] is what inspired the design of the structures in this article.

The concept of a “three-mode synchronization regime” using SIPs in the cold SWS dispersion of linear-beam tubes was initially proposed in [22] and was based on ideal transmission lines, following a multitransmission line generalization of the Pierce model. In [22], it was shown analytically that a TWT with an SIP would have a narrower bandwidth than a conventional single-mode TWT, but could lead to a larger gain-bandwidth product. As the next step in this enhancement investigation, we first must demonstrate that it is possible to achieve an SIP in a realistic slow wave structure. Therefore, here, we show for the first time how one can design a realistic serpentine-like SWS to exhibit a cold SIP at millimeter-wave frequencies, leading to an enhancement of the interaction impedance, which is one parameter that controls the gain of a TWT, over a narrow frequency interval near the SIP.

In Section II, we explain the concept of the SIP, smooth-TIP, and alternating-TIP. In Section III, we introduce the two proposed waveguide structures and we describe our design methodology to introduce SIPs in the three-way coupled waveguides. In Section IV, we show the dispersion, scattering parameters, and Pierce (interaction) impedance enhancement
for our three-way coupled waveguides. All dimensions used in this article are in SI units unless otherwise stated.

II. COLD STATIONARY INFLECTION POINTS

SIPs are a special case of eigenmode degeneracy, whereby three eigenmodes coalesce at a single-frequency point in the dispersion diagram for the modes of a periodic structure, as has been explored in [10], [12], [22], [23], [30], [34], [35], [36], [37], [38], [39], and [40]. On the other hand, a nonstationary, or “tilted” inflection point (TIP) is a single-frequency point in the structure’s modal dispersion diagram where three eigenmodes are nearly coalescing, but not perfectly so. In the structure’s modal dispersion diagram, the dispersion relation local to an SIP or TIP will be cubic in shape. In general, these cubic-shaped dispersion relations will have an inflection point that occurs at a frequency-wavenumber combination, where the second derivative of the $\omega - k$ dispersion relation vanishes (i.e., $d^2\omega/dk^2 = 0$). These inflection points are classified here into two kinds: the SIP, and the nonstationary inflection point, or TIP. The SIP occurs where both the first and second derivative of the $\omega - k$ dispersion relation vanish at the same wavenumber, i.e., $d\omega/dk = 0$ and $d^2\omega/dk^2 = 0$. For the case of the TIP, only the second derivative of the $\omega - k$ dispersion relation vanishes at the inflection point. In other words, the SIP is a special case of TIP. Because the modal dispersion diagrams for our lossless, reciprocal structures are symmetric in each Brillouin zone, the usual classification of TIPs as rising or falling is ambiguous. Every reciprocal waveguide with a TIP in its dispersion relation will always exhibit two kinds of TIPs, rising and falling, for opposite signs of $k_{\text{TIP}}$. To remedy this, we further classify TIPs into two subcategories: smooth-TIPs, which have a nonvanishing group velocity that does not change the sign for wavenumbers slightly above and below the inflection point, and alternating-TIPs, which have a group velocity that alternates in sign as the wavenumber is swept near the inflection point, as illustrated in Fig. 1(c).

We were able to design both the SLWG and TCSWG structures to exhibit SIPs in their cold dispersion, that is, without an electron beam present. We stress that this form of modal degeneracy is different from the hot eigenmode degeneracy, or “exceptional synchronization” studied in [31] and [32]. The coalescence of the cold eigenmodes at an SIP results in a perfect cubic dispersion relation local to the SIP in the dispersion diagram

$$(f - f_{\text{SIP}}) \simeq h(k - k_{\text{SIP}})^3. \quad (1)$$

When the cold eigenmodes of a structure are nearly coalescing at a single wavenumber and the dispersion relation exhibits a TIP, the dispersion relation local to the inflection point may be represented by a depressed cubic function (the quadratic term is suppressed due to the shift in $k$ by the cubic function’s inflection point)

$$(f - f_{\text{TIP}}) \simeq h(k - k_{\text{TIP}})^3 + s(k - k_{\text{TIP}}). \quad (2)$$

The parameters $f_{\text{SIP}} \simeq f_{\text{TIP}}$ and $k_{\text{SIP}} \simeq k_{\text{TIP}}$ are the frequency and the Floquet–Bloch wavenumber, respectively, at which the three eigenmodes coalesce or are nearly coalescing to form an SIP or a TIP, respectively. We also note that, due to Floquet–Bloch spatial harmonics, the wavenumbers $k$, $k_{\text{TIP}}$, and $k_{\text{SIP}}$ in (1) and (2) have a periodicity of $2\pi/d$, where the pitch of the unit cell is $d$. That is, $k$, $k_{\text{TIP}}$, and $k_{\text{SIP}}$ in these formulae do not necessarily need to be within the fundamental Brillouin zone. The parameter $h$ is a scalar flatness coefficient that depends on the strength of eigenmode coupling and $s$ is a scalar coefficient that affects the “tilt” of the TIP, as demonstrated in Fig. 1(c). TIPs and their properties, such as improved gain-bandwidth products and power efficiency, have also been explored in [22], using a multitransmission line generalization of the Pierce model [41], [42].

At an SIP, the dispersion relation local to the inflection point is a cubic function similar to (2) with $s = 0$, as shown in solid black in Fig. 1(c). At, and very close to, an SIP or smooth-TIP, the eigenwaves all propagate in the same direction. That is, the group velocities of the eigenwaves do not change sign at frequencies slightly lower and higher than $f_{\text{TIP}}$. At the inflection point of an SIP, the group velocity becomes zero. In a smooth-TIP with $s > 0$, the group velocity $(d\omega/dk)$ at the inflection point is no longer zero [blue dash-dotted line in Fig. 1(c)]. Having the electron beam interact with a TIP with a near-zero positive group velocity (nearly stationary TIP) instead of a perfectly zero group velocity SIP may be preferable for TWT designs since the Pierce (interaction) impedance at the frequency of the inflection point will be sufficiently large but the device will not become absolutely unstable when the electron beam is introduced. On the other hand, having an electron beam interact with a TIP that has negative group velocity is useful for the design of BWOs. We call this interaction between an electron beam and a cold SIP, a “three-mode synchronization” (see [22] for more details). Beamline interactions at points of zero group velocity, like the band edge [43], [44], [45], [46] or the degenerate band edge (DBE) [17], [47], are to be avoided in the design of TWT amplifiers, as they are considered a source of instability. The alternating-TIP [magenta dotted line in Fig. 1(c)] is the second kind of TIP studied with $s < 0$, in which the group velocities of the eigenwaves will change sign at frequencies slightly lower and higher than $f_{\text{TIP}}$. If the geometry of a structure can be tuned to exhibit smooth-TIPs for one set of dimensions and alternating-TIPs for another set of dimensions, it is expected that such a structure can be made to exhibit an SIP.

From the Pierce theory, the Pierce (interaction) impedance for a specific Floquet–Bloch spatial harmonic $p$ and specific wavenumber corresponding to the frequency of interest is defined as

$$Z_{\text{Pierce}}(k_p) = \frac{|E_{z,p}(k)|^2}{2[\text{Re}(k_p)]^2 P(k)} \quad (3)$$

where $k_p = k + 2\pi p/d$ is the wavenumber corresponding to the appropriate $p$th Floquet–Bloch spatial harmonic, $p = 0, \pm 1, \pm 2, \ldots$, and the wavenumber $k$ is in the fundamental Brillouin zone defined here as $kd/\pi \in [-1, 1]$, i.e., with $p = 0$. Furthermore, $|E_{z,p}(k)|$ is the magnitude of the phasor of the electric field component along the center of the beam tunnel in the z-direction for a given wavenumber and $p$th Floquet–Bloch spatial harmonic, and $P(k)$ is the time-average power flux at the fundamental wavenumber corresponding to the frequency of interest (the time average power flux is the sum of power contributions from all spatial harmonics) [48]. To obtain the magnitude of the
axial electric field phasor, corresponding to the appropriate spatial harmonic, the complex axial electric field along the beam tunnel axis is decomposed into Floquet–Bloch spatial harmonics as \( E_z(z, k) = \sum_{p=-\infty}^{\infty} E_{z,p}(k)e^{jkp}, \) where the harmonic weights are computed as \( E_{z,p}(k) = (1/d) \int_0^d E_z(z, k)e^{jkz}dz \) [49]. Both the complex axial electric field \( E_z(z, k) \) and the time-average power flux \( P(k) \) through the cross section of the unit cell are calculated for the cold structure (i.e., without the electron beam) using the eigenmode solver in CST Studio Suite. However, one must pay careful attention to how the phase across the periodic boundaries is defined in the CST model to correctly compute the interaction impedance. Since the \( \exp(j\omega t) \) time convention is used by CST, the formula for calculating \( E_{z,p}(k) \) requires a delaying phase from the lower periodic boundary to the upper periodic boundary of the simulated unit cell, i.e., phase of \( E_z(z, k) \) must decrease from \( z_{\text{min}} \) to \( z_{\text{max}} = z_{\text{min}} + d \) for a positive value of \( k \). Conveniently, the electromagnetic energy simulated within the enclosed vacuum space of the unit cell between periodic boundaries in the eigenmode solver, which is based on the finite element method (FEM) implemented in the software CST Studio Suite, is always assumed to be 1 J for each eigenmode solution. Therefore, the power flux is calculated using the formula \( P = (1 \text{ Joule})v_g/d \), where \( d \) is the unit cell pitch, and the group velocity \( v_g = \frac{d\omega}{dk} \) is determined directly from the dispersion diagram via numerical differentiation (the group velocity is the same at every spatial harmonic).

In order for interaction impedance to be large, the ratio in (3), \( |E_{z,p}(k)|^2/P(k) \), must become large in magnitude or the wavenumber in the denominator must become very small (i.e., operating closer to the fundamental spatial harmonic). At a nearly stationary TIP, which is close to becoming an SIP, the power flow at the inflection point is indeed smaller than the power flow of conventional SWG at the same wavenumber of the inflection point, this is because the power flow is proportional to the group velocity. Assuming that the magnitude of the axial electric field component is comparable for both cases, one concludes that the Pierce impedance will be larger for the structure with an inflection point than in a conventional SWG, at the wavenumber corresponding to the inflection point.

Due to this phenomenon, it is possible to obtain a moderately high, narrowband Pierce impedance at an SIP or nearly stationary TIP, which is several times larger than the Pierce impedance observed in a conventional SWG, as demonstrated in Section IV.

Conventional TWT SWS exhibit higher symmetries, such as glide symmetry in the serpentine-type TWT or screw symmetry in the helix-type TWT [50], [51]. Nearly parallel dispersion curves are formed for our structures by breaking glide symmetry. Glide symmetry can be broken in our structures by introducing minor dimensional differences between two similar waveguide sections. This allows us to readily tune the tilt of TIPs in our dispersion relation by simply varying one or more of our structure’s dimensions.

Next, we show the design methodology for two kinds of SWS geometries, whose unit cells are shown in Fig. 2, that can be dispersion engineered to exhibit SIPS or nearly stationary TIPs.

### III. Proposed Waveguides Exhibiting SIP

#### A. Serpentine Ladder Waveguide

The SLWG was our first attempt at obtaining an SWG-like structure that is capable of exhibiting an SIP and has its geometry shown in Fig. 2(a). As we will show in our dispersion diagrams in Section IV, the SLWG structure can be potentially designed to operate as a BWO due to the backward waves that are exhibited, where the beamline interacts with an SIP or smooth-TIP. This behavior is also illustrated by the intersection of the inflection point (solid brown curve) and the beamline (red dashed line) in the dispersion diagram of Fig. 3(a). Furthermore, since the guided electromagnetic modes of the SLWG structure are distributed over a larger cross section than a conventional SWG due to the two lateral waveguides that couple to the middle SWG, the power handling capability of this structure may be enhanced. The SLWG structure is an SWG SWS, which is sandwiched between two straight, parallel rectangular waveguides with similar broad wall dimensions. A single-beam tunnel is through the center of the SWG structure. The structure resembles a ladder due to the rung-like appearance of transverse serpentine sections running between the parallel straight waveguides. The parallel waveguides are
metallic obstacles of different dimensions in the top and bottom parallel waveguides break glide symmetry. Breaking glide-symmetry provides a simple route to achieve a cold SIP in the structure’s dispersion relation once the periodic coupling is introduced between individual-waveguide modes. These differences between the top and bottom waveguides directly control the tilt of the TIP. In general, breaking glide symmetry is not a necessary condition to have an SIP, as three-way waveguide structures with unbroken glide-symmetry have been previously shown to exhibit SIPs [28]. However, here it is has been found to be convenient to achieve and manipulate SIPs in structures with broken glide-symmetry, as the SIP can occur within the fundamental TE_{10} mode of the SWG below the upper band edge associated with the bandgap of the SWG which occurs at $k = 2\pi p/d$.

B. Three-Coupled Serpentine Waveguide

The TCSWG structure is our second example of a serpentine-type SWS that is capable of exhibiting SIPs in its dispersion relation and has its geometry shown in Fig. 2(b). The TCSWG seems to be better suited for use in a TWT than the SLWG structure since the example we show tends to exhibit forward waves in proximity to the synchronization point where the beamline interacts with an SIP or a smooth-TIP, as we will show in our dispersion diagrams in Section IV. The TCSWG structure is constructed similarly to the SLWG structure. However, the top and bottom rectangular waveguides, which sandwich the center SWG are also made to be serpentine in shape, giving them a longer path length and similar dispersion shape to that of the center SWG, as shown in the longitudinal cross section of Fig. 1(b) and in the dimensional markup of the unit cell in Fig. 2(b). No periodic obstacles or broad wall dimension variations are required, as the mean path lengths of each individual waveguide way may be altered to break glide-symmetry and obtain an SIP. In our TCSWG structure, the top, bottom, and middle SWGs each have the same pitch and broad wall dimension. The straight section height of each SWG way ($H_t$, $H_m$, $H_b$) is varied to alter the shape of its respective dispersion curve, allowing the prementioned SIP conditions to occur. The tilt of the TIP is controlled by the size of the coupling slots placed between bends of adjacent SWSs, as well as minor path length differences (between the serpentine height dimensions $H_t$ and $H_b$) introduced to break glide symmetry. We have also found that if the path length of the middle SWG, mainly controlled by $H_m$, is made to be a scalar multiple of the top or bottom path length that is greater than two, it is possible to have multiple SIPs or TIPs in synchronism with the beamline at different frequencies, though we do not show it in this article.

Instead of inserting a beam tunnel only in the center SWG, it is also possible to add beam tunnels to the center of the top and bottom SWGs. This makes the structure operate with two beams propagating parallel to each other, provided that the two beams are not too close together and an external magnetic field can be used to confine both beams. This dual-beam structure can potentially benefit from increased power output at the SIP/TIP due to beamline synchronism in both the top and bottom SWG sections.
C. Design Methodology

A minimum of three ways is required to obtain an SIP in a reciprocal, lossless, cold structure. This is because the SIP is a synchronous coalescence of three eigenmodes. In order to design three-way SWG SWSs that are both consistently synchronized to a beamline and exhibit SIPs, we utilize a design methodology based on the work of [52]. We use a dispersion approximation for the initial design of both the individual straight rectangular waveguide and SWG ways. The design process begins by selecting a fixed center operating frequency, the spatial harmonic number \( n \), cell pitch \( d \), and average electron beam velocity \( u_0 \) determined from the cathode–anode voltage of an electron gun. The full-cell pitch \( d \) in our work is chosen to be equal to \( \lambda_g/4 \), where \( \lambda_g = 2\pi/\left(k_0(1 - c^2/(2af_{\text{center}})^2)^{1/2}\right) \) is the guided wavelength at the center operating frequency within the SWG containing the beam tunnel, where \( k_0 \) is the free space wavenumber, \( c \) is the velocity of light in free space, \( a \) is the broad-wall dimension of the individual waveguide cross section, as shown in Fig. 2, and \( f_{\text{center}} \) is the center operating frequency.

Starting with the average beam velocity, \( u_0 \), the beamline’s linear relation (neglecting space charge effects) between the frequency and average electronic phase constant, \( \beta_0 \), is

\[
\beta_0 = \frac{2\pi f}{u_0}.
\]  

Then, the dispersion for the \( p \)th spatial harmonics of the individual serpentine and/or straight waveguide sections is calculated from the relation found in [52]

\[
f = f_c \sqrt{1 + \left(\frac{\alpha \sqrt{2}}{L} \left(\frac{k d}{\pi} - 2 p\right)^2\right)}
\]

where \( f_c = c/(2a) \) is the cutoff frequency of the rectangular waveguide cross section and \( L = 2H + \pi d/2 \) is the mean path length of the individual uncoupled waveguide section within the unit cell, as can be observed in the SWG sections of Fig. 2.

To model the dispersion of straight rectangular waveguide sections of Fig. 2(a), the path length simply becomes equal to the pitch of the unit cell, \( L = d \). We describe our structure using a full unit cell notation (of period \( d \)), rather than the half unit cell notation (of period \( d/2 \)) commonly used in literature. The half-cell notation is often used because the beam “sees” two beam tunnel intersections per geometric period of the full unit cell, which only differ by a sense-inversion of \( E_z \) fields at each consecutive beam tunnel intersection. The primary difference is that the path lengths of each individual waveguide of our full unit cell are twice as long as they would be in the half-cell notation. Much of the fundamental spatial harmonic of the full-cell dispersion diagram lies above the light-line of \( k_0 = \pm \omega_0 (\mu_0 \epsilon_0)^{1/2} \) and cannot be utilized for amplification without the use of a relativistic beam velocity [2]. Because the full-cell notation is being used in this article, rather than the half-cell notation, the additional \( \pi \) phase shift considered in some other papers, such as [52], [53], and [54], is no longer needed, so the term \( 2p + 1 \) in [52] has been replaced with \( 2p \) in (5). As long as the coupling between waveguide sections in each unit cell is weak, this dispersion relation will serve as a reasonably accurate approximation of actual dispersion below the frequency of the first \( k = 2\pi p/d \) bandgap, which occurs at the intersection of two neighboring spatial harmonic curves corresponding to the same individual waveguide way which contains the beam tunnel.

The dimensions \( a \) and \( H \) of the SWG sections containing beam tunnels, as shown in Fig. 2, are selected using an optimization algorithm, which minimizes the integrated frequency error between the beamline and SWG dispersion curves over the wavenumber interval of \( kd/\pi \in [2p, 2(p + 1)] \). This wavenumber interval corresponds to the frequency range over which we wish to obtain an inflection point, as shown in Fig. 3(a) and (b). Once suitable \( a \) and \( H \) dimensions are determined for the SWG with a beam tunnel, the narrow wall \( b \) dimension of our SWG way was chosen to be \( b = a/6 \) to provide adequate spacing between the SWG beam tunnel intersections. Of course, in most SWG structures, the \( a \) and \( b \) dimensions are rarely close to standard waveguide sizes due to the need for synchronization with a specific beamline, so waveguide transitions are needed at the input and output ports to allow connections for standard waveguide sizes, in addition to RF windows to maintain the vacuum within the tube. However, for simplicity, we do not consider such waveguide transitions or windows in our study, and we only focus on the beam-wave interaction region. The beam tunnel diameter \( d_t \) is selected based on the empirical formula from [52]

\[
d_t = L a \left(1 + \left(\frac{L}{2a}\right)^2\right)^{-1/2}
\]

which minimizes the width of the bandgap caused by the beam tunnel. The ratio of the beam tunnel radius to the free space wavelength at the \( 2\pi \) frequency, \( a \approx 0.115 \), was used in the design of our structures, as well. The bandgap normally caused by the beam tunnel is significantly widened due to the additional periodic reactive loading introduced by coupling slots. Increasing the size of the coupling slots introduces stronger coupling between the waveguide sections, but enlarges the bandgap. Therefore, simply having a beam tunnel, which is sufficiently in cutoff appears to be adequate for these kinds of structures. A square-shaped beam tunnel with side length \( d_t \) may also be used in place of a conventional cylindrical beam tunnel, as shown in Fig. 2, to make the structures more compatible with multistep LIGA (lithographie, galvanoformung, abformung; German for lithography, electroplating, and molding) processes [55]. TWT amplifiers with square beam tunnels may be fabricated using LIGA processes like in [56], [57], [58], [59], [60], [61], and [62]. In the two-step LIGA process, the SWSs are electroformed out of two symmetric halves, which are later bonded together. However, more than two steps will likely be necessary for our structures due to the coupling slot lengths differing from the beam tunnel width or differing broad wall dimensions in each waveguide way. Additionally, the use of a square beam tunnel may slightly degrade the hot operation of TWTs, as mentioned in [60] and [63]. While it is potentially challenging to fabricate such structures using LIGA fabrication, it should not be significantly more challenging than it already is for conventional SWGs fabricated by two-step LIGA. For example, each additional LIGA step required for the SLWG and TCSWG structures corresponds to a repetition of procedures 7–13 (lapping/polishing, photoresist attachment, mask, \( n \)th layer alignment,
exposure, development, and electroplating) after procedure 13 shown in [60].

The dispersion condition of the waveguides (when uncoupled to each other) utilized by our group to consistently obtain SIPs is to have two nearly parallel uncoupled (individual waveguide) modes crossover a third (individual waveguide) mode, which is nearly perpendicular to the other two modes on the dispersion diagram, as shown in Fig. 3. If a periodic coupling is introduced between all three of the individual-waveguide modes and the nearly parallel individual-waveguide modes are in close proximity in the dispersion diagram, then two phase- and frequency-shifted bandgaps will form at the intersection points. If the top band edge of one bandgap tangentially touches the bottom band edge of another bandgap, an inflection point is able to form between these band edges, as illustrated in the inset of Fig. 3(a) and (b). Varying the proximity of near-parallel individual-waveguide modes for a given coupling strength directly controls the tilt of the TIP. Near-parallel individual-waveguide modes which are close to each other tend to form smooth-TIPs, whereas near-parallel individual-waveguide modes which are further from each other will typically form alternating-TIPs. Between smooth-TIP and alternating-TIP conditions, an SIP condition is expected to exist.

Once the basic dimensions of the SWGs with beam tunnels are established, coupling slots are positioned between the bends of adjacent waveguide ways to periodically couple the individual-waveguide modes. The coupling slots have a vertical thickness $t$ in the $y$-direction, width $w$ in the $z$-direction, and length $l$ in the $x$-direction, as shown in Fig. 2. The length of the coupling slot, which is also along the same axis as the broad wall dimension of the waveguide ways, is the dimension that strongly controls the evanescent coupling of modes between waveguide ways. The width of the coupling slot controls the wave impedance within the slot, and the slot thickness primarily controls the extent of evanescent decay and phase delay for waves, which are below the slot cutoff frequency. In this article, we use a coupling slot length equal to half the $a$ dimension of the SWG section containing the beam tunnel. The slot width and thickness are arbitrarily chosen in this article to demonstrate that SIPs can be attained in our structures. Larger slot lengths will strengthen mode coupling, but the dispersion relation of the actual structure will be strongly dissimilar to the dispersion relation of the individual uncoupled waveguide modes from before. While large slot lengths and widths enhance mode coupling, the reflections introduced by the periodic slot reactance in a finite-length structure may also make the hot structure more susceptible to regenerative oscillations.

Finally, a small geometric difference is introduced between the waveguide sections corresponding to near-parallel dispersion curves to control the frequency and wavenumber spacing between near-parallel dispersion curves. This directly controls the tilt of the TIP. Large geometric differences typically result in an alternating-TIP, whereas small geometric differences make the TIP smooth. For the TCSWG structure, geometric differences may be introduced as a height difference $\Delta H = |H_t - H_b|$ between the top and bottom serpentine sections $[H_t$ and $H_b$ in Fig. 2(b), respectively]. For the SLWG structure, a broad wall dimension difference $\Delta a = |a_t - a_b|$ may be introduced between the top and bottom straight waveguide sections $[a_t$ and $a_b$ in Fig. 2(a), respectively]. Alternatively, geometric differences may be introduced in periodic capacitive obstacles loading the top and bottom straight waveguides of the SLWG structure to achieve the same effect, reducing the number of steps required for LIGA fabrication. However, it may not be desirable to use periodic obstacles due to the large reflections they introduce in the top and bottom waveguide sections of the SLWG structure. Conversely, the use of unloaded parallel waveguides also reduces the complexity and reflection coefficients of SWS at the cost of more steps with LIGA fabrication.

Once all initial structure dimensions are determined, the full unit cell geometry may be simulated in an eigenmode solver to obtain the Pierce (interaction) impedance and $a - \kappa$ dispersion relation with real-valued wavenumber, $k$. In this article, we use the software CST Studio Suite to obtain the modal dispersion for each periodic structure. Using simulated dispersion data, the $a$ and $H$ dimensions of SWG ways containing beam tunnels are then further tuned to recover beamline synchronism. Adjusting the $a$ dimensions of each SWG shown in Fig. 2(a) ($a_m$, for the middle SWG of the SLWG structure; $a_t$ and $a_b$ for the top and bottom SWGs of the TCSWG structure) primarily serves to shift each respective SWG dispersion curve up or down. Adjusting the $H$ dimension primarily controls the slope of the SWG dispersion curve, which may have changed due to periodic loading from the slots. When the modal dispersion curves and TIP are satisfactorily synchronized to the beamline, the geometric difference between waveguide sections corresponding to parallel dispersion curves (for example, $\Delta a$ in the SLWG structure or $\Delta H$ in the TCSWG structure) may then be tuned to adjust the tilt of the TIP to make it close to an SIP.

IV. Results

Following the aforementioned design procedures of the previous section, we obtain the real part of the modal dispersion relation (for the lossless and cold structures shown, imaginary parts of the dispersion relation correspond to evanescent modes, e.g., below the cutoff frequency of the waveguide or in bandgaps where neighboring spatial harmonics meet on the dispersion diagram) for both SLWG and two-beam TCSWG structures, as shown in Fig. 4(a) and (b), obtained using the methods shown in Appendix B and verified using the eigenmode solver in CST Studio Suite. In the insets of Fig. 4, we also demonstrate how it is possible to vary the tilt of the inflection point for three different cases simply by altering the difference in straight waveguide widths ($a_t$ and $a_b$ for top and bottom rectangular waveguides, respectively) for the SLWG structure, or the SWG heights ($H_t$ and $H_b$) of the top and bottom waveguides for the TCSWG structure. The dimensions of each case are provided in Appendix A. From the dispersion relation shown for the SLWG in Fig. 4(a), the point where the beamline intersects an SIP or smooth-TIP (solid black and dashed blue curves, respectively) is a backward-wave interaction, making the SLWG design better suited for use in a BWO, rather than a TWT. While an alternating TIP [magenta dotted curve in the inset of Fig. 4(a)] might enable forward-wave interactions in the SLWG structure, the upper and lower band edges on either side of the inflection point still
pose a significant risk for oscillations. Backward wave oscillators constructed with the SLWG structure may also benefit from improved power handling capability compared with a conventional SWG BWO due to the guided electromagnetic mode being distributed over a larger cross section in the two lateral waveguides.

From the dispersion relation for the TCSWG structure in Fig. 4(b), the point where the beamline intersects the inflection point is a forward wave for the SIP and smooth-TIP in Fig. 4(b), the point where the beamline intersects the inflection point is achieved by fine-tuning one or more of the structure dimensions. Dimensions for the smooth-TIP, alternating-TIP, and SIP are available in Appendix A.

One must also consider how the electron beam perturbs the inflection point in the hot dispersion relation. The amount that the inflection point is deformed in the hot dispersion relation depends on factors, such as velocity synchronism with the beamline, which is directly controlled by the accelerating voltage of the electron gun, to avoid striking dispersion branches that have zero group velocity, such as the band edge, as it can lead to instability [43], [44], [45], [46]. Because there are upper and lower band edges at frequencies close to the inflection point, the tunability of the beam voltage is limited. For instance, in Fig. 4(a), neglecting space charge (i.e., at low beam currents), the average beam velocity can only be varied by approximately \( u_0 = 0.200c \pm 0.001c \) to avoid striking the neighboring upper or lower band edges on other dispersion branches. This beam velocity range corresponds to an approximate kinetic equivalent voltage tunable range of \( V_0 = 10.54 \pm 0.11 \text{ kV} \) from the relativistic relation \( V_0 = (c^2/\eta)(1 - (u_0/c)^2)^{-1/2} - 1 \) in [64], where \( \eta \) is the charge-to-mass ratio of an electron at rest. For the case of the TCSWG structure we show, the tunable range of the beam velocity equivalent kinetic voltage is better due to the neighboring upper/lower band edges near the inflection point being separated at higher/lower frequencies, respectively, as can be observed in Fig. 4(b). However, there is still a risk of oscillations at the lower band edge corresponding to a frequency of approximately 133 GHz for the structure shown, so the tunable range of beam velocity for the TCSWG structure is \( u_0 = 0.300c \pm 0.002c \), which corresponds to an approximate beam voltage range of \( V_0 = 24.67 \pm 0.36 \text{ kV} \).

Fig. 4. Modal dispersion diagrams for (a) SLWG unit cell and (b) TCSWG unit cell, with beamline (red dashed). Insets: Smooth-TIP, alternating-TIP, and SIP in blue (dashed line), magenta (dotted line), and black (solid line), respectively, for each structure. Only wavenumbers that are purely real are shown. The dispersion diagrams in the figure are obtained using the methods shown in Appendix B, and the dispersion relations were verified using the full-wave eigenmode solver of CST Studio Suite. Tilting of the inflection point is achieved by fine-tuning one or more of the structure dimensions. Dimensions for the smooth-TIP, alternating-TIP, and SIP are available in Appendix A.
its SIP/TIP frequencies, as demonstrated with a finite-length structure of 32 unit cells, shown in Fig. 5(a) and (b). The \( S \)-parameters of the finite-length structure were calculated using the methods explained in Appendix B. The port numbering scheme for our structures is that the input ports at the electron-gun end of the structure (on the left) are odd-numbered from top to bottom, and output ports at the collector end of our structure (on the right) are even-numbered from top to bottom. While it is highly important to consider the effect of waveguide transitions and RF windows, our study focuses primarily on the interaction region of linear beam tubes, so for brevity, we do not consider the effect of input–output coupling structures; i.e., we only consider the \( S \)-parameters at reference planes between the SWS and where an RF window would be placed in a fabricated device. This directional coupler-like behavior enables distributed power extraction (DPE) which can be directed either backward toward the cathode-end of the structure or forward toward the collector-end of the structure. However, only forward-directive DPE may be desired for amplification, due to the potential risk of regenerative oscillations introduced by amplified waves returning to the electron gun-end of the structure, like in [68].

The introduction of DPE for linear beam tubes was necessary to conceive the degenerate (or exceptional) synchronization in the hot systems studied in [31], [32], and [33]. For the TCSWG structure, increasing the path length of the middle SWG can allow forward-directive DPE to occur at certain frequencies and for power to be extracted in the top, middle, and bottom SWG outputs. However, there may still be SIP/TIP frequencies, where backward-directive DPE continues to occur. Shifting the frequency above or below the SIP/TIP in the vicinity of the inflection point directly controls whether the top or bottom SWG section contributes more power to the output of the middle SWG, as can be seen from the scattering parameters. This dual-beam TCSWG structure may also be excited either from the middle SWG input or top and bottom SWG inputs to achieve amplification and DPE at the SIP/TIP frequency if the coupling is sufficient. Increasing the size of coupling slots enhances DPE; however, this also exacerbates reflections in the finite-length structure and increases the risk of BWO. Because these structures can still be designed to be well-matched with small coupling slots, longer finite-length structures, and higher beam currents may potentially be used before unstable BWO occurs in particle-in-cell (PIC) simulations and hot testing.

Finally, we compute the Pierce impedance as discussed in Section II for a fourth case of the SLWG structure in the vicinity of a nearly stationary smooth-TIP, which has dispersion relation similar to the black curve, but the inflection point is not as tilted as the blue curve, as shown in Fig. 4(a), with dimensions provided in Appendix A. We demonstrate the benefit of using nearly stationary TIPS to enhance the Pierce impedance of serpentine-like structures, as shown in Fig. 6. We also compare the Pierce impedance of the SLWG (blue solid line) to the Pierce impedance of a conventional individual SWG (red dotted line) (i.e., the serpentine of the SLWG structure, with removed coupling slots and straight waveguide ways). We find that the pierce impedance of the full SLWG structure is several times higher than a conventional simple SWG at the frequency corresponding to a nearly stationary TIP. We also note that, while the interaction impedance of the SLWG appears quite small relative to the simple SWG at frequencies beyond the inflection point, the interaction impedance is comparable to that of an SWG on other higher/lower
frequency branches of the SLWG's dispersion diagram, which are not shown.

Similarly, we compute the Pierce impedance for the fourth case of the dual-beam TCSWG structure in the vicinity of the nearly stationary smooth-TIP, which has a dispersion similar to the black curve, but not as tilted as the blue curve shown in the inset of Fig. 4, with dimensions provided in Appendix A. We demonstrate that the nearly stationary TIP can be used to enhance the Pierce impedance in both beam tunnels, as shown in Fig. 7. We compare the Pierce impedance of the TCSWG structure (solid blue line) to the Pierce impedance of a conventional individual SWG (i.e., with coupling slots and adjacent waveguide sections removed) for each respective beam tunnel (dashed red for the top SWG and dotted magenta for the bottom SWG). We find that, just like with the SLWG structure, the Pierce impedance is several times higher than a conventional SWG at the frequency corresponding to the nearly stationary TIP. Interestingly, below the frequency of the TIP, the interaction impedance in the lower beam tunnel is higher than in the upper beam tunnel, whereas the opposite occurs at frequencies above the TIP. Since glide symmetry is slightly broken due to the top and bottom SWGs having different $H_1$ and $H_2$ dimensions, respectively, the dispersion branches of the individual SWGs (red dashed line for the top SWG and magenta dotted line for the bottom SWG) are dissimilar. Due to broken glide symmetry, the peak values of interaction impedance in the top and bottom tunnels are also different at the inflection point. If an electron beam is introduced to the SLWG or TCSWG structures and the beam is velocity synchronized to the SIP/TIP, we say that the electron beam is synchronized to three degenerate modes, i.e., we have three-mode synchronization, as was described in [22].

Under the three-mode synchronization regime, the Pierce gain parameter $C$ will also become larger than that of a conventional SWG TWT due to the enhanced interaction impedance.

Using the Pierce theory [1], the gain and 3-dB bandwidth of such a TWT can be predicted in terms of its interaction impedance, length, and beam parameters at synchronism as $G = A + BCN$, where $A$ is the launching loss, $B = 47.3$ dB, $C = Z_{\text{Pierce}}A_0/(4\pi\varepsilon_0)$ is the Pierce gain parameter, and $N = nd/\lambda_e$ is the length of the structure in terms of guided wavelengths. Furthermore, $n$ is the number of unit cells of period $d$ in a finite-length structure, and $\lambda_e = 2\pi/k$ is the guided wavelength.

As a numerical example, suppose we have a TCSWG structure which is $n = 32$ unit cells (43.62 mm) in length, as shown in Fig. 5(b). At the SIP-beamline synchronism point for this structure in Fig. 4(b), we have $kd/\pi = 3.55$, which corresponds to a guided wavelength of $\lambda_e = 0.768$ mm. Thus, the structure in this example will be $N = 56.8$ guided wavelengths in length. For demonstration purposes, we assume that there is a launching loss of approximately $A = -9.54$ dB, like in [1]. If a small-signal gain of 20 dB is desired for such a single-stage TWT constructed with the TCSWG structure, then the required Pierce gain parameter will be $C = 0.011$.

To find the 3-dB bandwidth in terms of the interaction impedance that varies with frequency (assuming that the beam remains velocity synchronized to the guided wave over such a frequency range), the required Pierce gain parameter is calculated at $-3$ dB below the frequency where maximum gain occurs, i.e., for a small-signal gain of 17 dB, which is $C_{\text{dB}} = 0.0099 = 0.9C$. Since the Pierce gain parameter is proportional to the cube root of the interaction impedance ($C \propto Z_{\text{Pierce}}^{1/3}$) [1], we find that a 3-dB reduction in gain will correspond to an interaction impedance that is reduced to 73% of its peak value for the numerical example given. Thus, the absolute bandwidth of such a TWT constructed using the TCSWG structure will be approximately 150 MHz, which corresponds to a relative bandwidth of approximately 0.13% at 116.7 GHz from the interaction impedance plotted in Fig. 7.

Depending on the application, efficient and high-power narrowband TWTs may still be useful. For example, narrowband millimeter-wave TWT designs have been reported by TWT manufacturers in [69] with a 1-dB bandwidth of 2.2% at 28 GHz, in [70] with a small-signal 3-dB bandwidth of approximately 0.6% at approximately 34.55 GHz, and in [71] with a 1-dB bandwidth of 1% at 42 GHz. In these examples, space communication and telemetry appear to be the primary application cited for such narrowband millimeter-wave TWTs. While the relative bandwidth of our structure is even smaller than the narrowband TWT papers cited, this proposed scheme can be used in amplifiers to achieve a large gain in exchange for narrow bandwidth, as was indicated theoretically in [22] using a generalized Pierce model. However, before further investigating the enhancement of the gain-bandwidth product in such a tube using PIC simulations, we first must demonstrate that it is possible to design structures, which exhibit SIPs at millimeter-waves, which is the purpose of this article.

V. CONCLUSION

We have showcased two novel dispersion-engineered three-way SWSs for use in linear electron beam devices: the SLWG and TCSWG geometries. Such geometries are capable of...
exhibiting SIPs or TIPs in their dispersion relations, and larger Pierce (interaction) impedance than that of a conventional SWG at the frequency corresponding to the inflection point. Using our design methodology, we were able to demonstrate simple conditions which enable one or more SIP/TIP to occur in a three-way waveguide periodic structure once the weak periodic coupling is introduced between individual waveguides. We have shown the first known example of a millimeter-wave SWS for linear beam tubes which exhibits stationary or nearly SIPs in its dispersion relation. A previous example of a waveguide, which exhibits an SIP at radio frequencies was demonstrated using microstrip technology in [28] and was the inspiration for this article.

What is of interest in both of our introduced structures is that the group velocity in the vicinity of the SIP/TIP may be easily controlled by slightly breaking glide symmetry in our geometries. With weak coupling between waveguides, the dispersion relation of the introduced structures will not be significantly different from the dispersion relations of the individual (uncoupled) waveguides. Due to the “three-mode synchronization” regime, the Pierce impedance, and consequently the Pierce gain parameter, can be drastically enhanced over narrow bandwidths (3 dB relative bandwidth of 0.13% in the example shown) near the SIP/TIP when compared with a conventional SWG commonly used for millimeter-wave TWTs and BWOs. We believe that the introduction of the three-mode synchronization regime in such structures may enable the design of more efficient, compact linear beam tubes, as it was speculated in [22]. There is a lot of room for improvement if one wishes to focus on improving the interaction impedance or bandwidth further using SIPs. However, we believe that the design methodology shown in this article is still useful for designing realistic millimeter-wave SWSs with inflection points in their dispersion relation for use in linear beam tubes. Additionally, we have shown how to obtain TIPs with either backward or forward mode interactions by simply varying how much glide symmetry is broken in our structures, potentially enabling the design of either BWOs or TWTs from the same initial geometry. We have also showcased the directional coupler-like properties of both the TCSWG and SLWG structures at frequencies near-to the SIP/TIP. This property may be exploited in the design of specialized TWTs or BWOs which can be simultaneously excited from multiple ports or simultaneously drive multiple loads.

### APPENDIX A

**Dimensions Used in Figures**

For the dispersion of the individual-waveguide ways of the serpentine ladder waveguide (SLWG) structure in Fig. 3(a) (i.e., without coupling), the dimensions are identical to those in Table I with \( \alpha = 0.1 \) mm, with the exception of \( H = 0.843 \) mm, which differs from the final \( H \) dimension used for shown in Fig. 4(a). Note that, unlike the three-coupled serpentine waveguide (TCSWG) structure, the SLWG structure only has one \( H \) dimension.

For the dispersion of the individual waveguide ways of the TCSWG structure in Fig. 3(b), the dimensions are identical to those in Table I with the exception of the normalized beam velocity being \( u_0/c = 0.20 \) and the top, middle, and bottom SWG heights being \( H_t = 0.518 \) mm, \( H_m = 1.161 \) mm, and \( H_b = 0.418 \) mm, respectively.

Table I reports the dimensions for Fig. 4(a) and (b) as follows. In the inset of Fig. 4(a), we have the dimensions used in the SLWG column of Table I, with \( \Delta a = 0.21 \) mm for the stationary inflection point (SIP) (black solid lines), \( \Delta a = 0.24 \) mm for the alternating-tilted inflection point (TIP) (group velocity alternates in sign at frequencies slightly above or below TIP frequency) (magenta dotted line), and \( \Delta a = 0.18 \) mm for the smooth-TIP (group velocity does not change sign at frequencies slightly above or below TIP frequency) (blue dashed line).

In the inset of Fig. 4(b), we have the dimensions used in the TCSWG column of Table I, with \( \Delta H = 0.15 \) mm for the SIP (black solid line), \( \Delta H = 0.10 \) mm for the smooth-TIP (blue dashed line), and \( \Delta H = 0.20 \) mm for the alternating-TIP (magenta dotted line).

| Dimensions used for SLWG | Dimensions used for TCSWG | Notes |
|--------------------------|--------------------------|-------|
| \( \alpha_t \), \( \beta_t \) | 1.879 - \( \Delta \alpha/2 \), \( \alpha_t/2 \) | 1.726, 0.432 | Cross-section dimensions for top waveguide |
| \( \alpha_m \), \( \beta_m \) | 1.879, \( \alpha_m/6 \) | 1.726, 0.432 | Cross-section dimensions for middle waveguide |
| \( \alpha_b \), \( \beta_b \) | 1.879 + \( \Delta \alpha/2 \), \( \alpha_b/2 \) | 1.726, 0.432 | Cross-section dimensions for bottom waveguide |

| \( l \) | 0.940 | 0.863 | Rectangular slot length along x-dimension |
| \( W \) | 0.100 | 0.259 | Rectangular slot width along x-dimension |
| \( t \) | 0.100 | 0.050 | Rectangular slot thickness between waveguide bends |
| \( d_t \) | 0.282 | 0.259 | Width of square beam tunnel |
| \( d \) | 0.989 | 1.363 | Unit cell pitch |
| \( H_t \), \( H_m \), \( H_b \) | \( H = 0.784 \), 0.472 + \( \Delta H/2 \), 1.312, 0.472 - \( \Delta H/2 \) | Height of straight sections for top, middle, and bottom SWGs. SLWG only has one \( H \) dimension |
| \( u_0/c \) | 0.200 | 0.300 | Average electron beam velocity normalized to the speed of light |
greatly on the mesh density and step size of the boundary phase.

**APPENDIX B**

**T-Parameters**

Using frequency-domain simulations, it is possible to rapidly and accurately compute the complex dispersion and approximate finite-length \( S \)-parameters of a periodic structure without an eigenmode solver using the frequency-dependent scattering parameters (\( S \)-parameters) of a single unit cell having separable pairs of ports for each way of the multiway waveguide structure. The benefit of computing dispersion in this manner is that it is possible to obtain both real and imaginary solutions for the Bloch wavenumber, \( k \), whereas it is only possible to obtain the real part of the Bloch wavenumber using the eigenmode solver of CST Studio Suite. For lossless and cold structures, the imaginary part of the Bloch wavenumber corresponds to evanescent modes, e.g., below the cutoff frequency of the waveguide or in bandgaps that form where neighboring spatial harmonics meet. It is also notably faster to obtain the unit cell \( S \)-parameters using the frequency-domain solver than it is to directly obtain the modal dispersion from a phase sweep of the periodic boundary in the eigenmode solver. Using this method, it was possible for us to tune the geometry of our structures to obtain beamline synchronism and desired inflection point tilt in a reasonable time frame. The real dispersion obtained by this method was found to be in excellent agreement with CST Studio Suite’s eigenmode solutions.

The method we use to obtain the complex dispersion and approximate finite-length \( S \)-parameters of our periodic structures involves converting between \( S \)-parameters and scattering transmission matrices (\( T \)-parameters) in intermediate steps. \( T \)-parameters may be directly obtained through algebraic manipulation of each \( S \)-parameter, as both are defined in terms of the same \( a \)- and \( b \)-waves

\[
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
  b_5 \\
  b_6 \\
\end{bmatrix} = S
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
\end{bmatrix} \quad \iff \quad T = \begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
\end{bmatrix}
\]

By converting our frequency-dependent \( S \)-parameters to \( T \)-parameters at each frequency for the unit cell, it is possible to solve the following Floquet–Bloch eigenvalue problem at each frequency to obtain the complex dispersion diagram for the modes of the periodic structure [42], [72], [73]

\[
T(z + d, z)\Psi(z) = e^{-jkd}\Psi(z)
\]

where \( \Psi \) is the complex state vector for the unit cell composed of \( a \) and \( b \) waves at each port, as shown in (7) and Fig. 5(a) and (b), and \( d \) is the unit cell pitch. In other words, the wavenumbers of the fundamental spatial harmonic may be evaluated directly from the eigenvalues \( \lambda = \exp(-jkd) \) of the \( T \) matrix through the relation

\[
k = \frac{-\ln(\lambda)}{jd}
\]

The modal dispersion diagrams in Fig. 4(a) and (b) (only showing the branches with purely real \( k \)) are calculated from the eigenvalue problem shown earlier and are verified using the eigenmode solver of CST Studio Suite. It is also possible to estimate the \( S \)-parameters of our periodic structure with finite length by cascading \( T \) matrices and converting the resultant parameters back into \( S \)-parameters using the same algebraic manipulation as before [74]. This is how the finite-length \( S \)-parameters shown in Fig. 5(a) and (b) were computed. This method may be readily generalized for \( 2N \)-port periodic structures. It is also possible to solve the eigenvalue for spatial harmonics other than the fundamental, as the solutions \( k_n \) are periodic as \( k_n = k + 2\pi n/d, \) with \( n = 0, \pm 1, \pm 2, \ldots \).

Due to the presence of periodic coupling slots in both of our unit cell designs, our single-cell frequency-domain model for our structures must be slightly modified in order to avoid placing ports at coupling slots or in waveguide bends. Our unit cells were modified for the frequency-domain solver by horizontally shifting the reference planes of all ports by \( d/4 \) along each waveguide path and adding deembedding to the ports to account for small reflections caused by transitions between straight waveguide sections and waveguide bends and slots, as shown in Fig. 8(a) and (b). This modification does not appear to significantly affect the dispersion relation of the periodic structure under study.

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