SOME SCHEMES RELATED TO THE COMMUTING VARIETY

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ABSTRACT. The commuting variety is the pairs of \( n \times n \) matrices \((X, Y)\) such that \( XY = YX \). We introduce the diagonal commutator scheme, \( \{(X, Y) : XY - YX \text{ is diagonal}\} \), which we prove to be a reduced complete intersection, one component of which is the commuting variety. (We conjecture there to be only one other component.)

The diagonal commutator scheme has a flat degeneration to the scheme \( \{(X, Y) : XY \text{ lower triangular, } YX \text{ upper triangular}\} \), which is again a reduced complete intersection, this time with \( n! \) components (one for each permutation). The degrees of these components give interesting invariants of permutations.

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1. STATEMENTS OF RESULTS

The commuting variety of a reductive Lie algebra \( g \) is defined as the reduced sub-scheme of \( g \oplus g \) cut out by the equations \([X, Y] = 0\). It is known to be irreducible [Ri], but even for \( g = gl_n \) it is not presently known whether these equations serve to define it as a scheme (i.e. whether the ideal generated by these \( \dim g \) equations is radical).

Let \( h \) be a chosen Cartan subalgebra of \( g \) (we will work over \( \mathbb{C} \) in this paper, so any two Cartan subalgebras are conjugate). We introduce the diagonal commutator scheme

\[ D = \{(X, Y) \in g \oplus g : [X, Y] \in h\}. \]

This is now defined by only \( \dim g/h \) equations, rather than the \( \dim g \) (and who knows how many more) equations needed to define the commuting variety.

While our first theorem could be stated for general \( g \), we only prove it in the case \( g = gl_n \) with \( h \) the diagonal matrices (hence the name).

**Theorem 1.** The diagonal commutator scheme \( D \) is a reduced complete intersection in \( gl_n \oplus gl_n \), one component of which is the commuting variety.

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It is a well-known open problem, due to Artin and Hochster, to show that the commuting variety of $\mathfrak{gl}_n$ is Cohen-Macaulay. (Our reference is chapter 9 of [V].) The theorem above doesn’t address that directly, but implies something related:

**Corollary.** The commuting variety is Cohen-Macaulay if and only if the union of the other components is Cohen-Macaulay.

In fact we conjecture that for any reductive $\mathfrak{g}$, the diagonal commutator scheme has exactly two components – i.e. only one component other than the commuting one.

**Proof.** These two schemes are “directly linked,” and then the theory of linkage ([E], theorem 21.23) relates the Cohen-Macaulayness of one component to the union of the rest. □

We will prove this by studying a certain flat degeneration of the $\mathfrak{gl}_n$ diagonal commutator scheme. Let $\hat{\rho} : \mathbb{C}^x \rightarrow \text{GL}_n(\mathbb{C})$ denote the one-parameter subgroup

$$\hat{\rho}(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^{n-1} \\ & 1 & t & \cdots & t^{n-2} \\ & & 1 & \cdots & t^{n-3} \\ & & & \ddots & \cdots & \cdots \\ & & & & 1 & t^{n-1} \end{bmatrix}$$

of diagonal matrices. Define $D^z$ to be the subscheme of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$

$$D^z = \{(X', Y') : X'Y' = \hat{\rho}(z)Y'X'\hat{\rho}(z^{-1}) \text{ off the diagonal}\}$$

which is plainly isomorphic to $D^1 = D$ under the map $(X', Y') \mapsto (\hat{\rho}(z)X, Y\hat{\rho}(z^{-1}))$.

The flat family $\{D^z\}, z \in \mathbb{C}^x$ has a unique extension to a flat family over $\mathbb{C}$, with special fiber $D^0$. This sort of flat limit – by rescaling some of the coordinates of the ambient vector space – is called a Gröbner degeneration. We know from [KS] that such a limit must be again equidimensional (up to embedded components). But in our instance much more is true:

**Theorem 2.** The flat limit $D^0 := \lim_{z \to 0} D^z$ of this family is again a reduced complete intersection.

Moreover, it is defined by the limiting equations on $D^z$,

$$\{(X, Y) : XY \text{ lower triangular, } YX \text{ upper triangular}\}.$$ 

Ordinarily more equations are needed in such a limit; we give a typical example. Consider the $z \to 0$ limit of the equations $X = 0, X = zY$ in the $X, Y$ plane. For each nonzero $z$, these two equations describe two lines intersecting at the origin. But in the limit $z = 0$ the two lines are equal and the condition $Y = 0$ is lost; the correct limit is only obtained if that equation $Y = 0$ is added to the list. The second conclusion of the above theorem says that this unfortunate phenomenon doesn’t occur in our case: our list of $\dim \mathfrak{g}/\mathfrak{h}$ equations is already enough for this limit.

(For readers familiar with Gröbner bases: this list of equations is *not* a Gröbner basis, but is “Gröbner enough” for this limit defined by a partial term order.)

We have a better handle on the components of this scheme $D^0$, because of the large group

$$B_- \times B_+ = \{(L, U) \in \text{GL}_n(\mathbb{C})^2 : L \text{ lower triangular, } U \text{ upper triangular}\}$$
acting on it by the rule
\[(L, U) \cdot (X', Y') = (LX'U^{-1}, UY'L^{-1}).\]

This group is of dimension \(n^2 + n\), slightly larger than the \(GL_n(\mathbb{C})\) acting on the commuting variety. Since this group is connected, it preserves (and acts on) each component of \(D^0\).

For \(\pi\) an \(n \times n\) permutation matrix, define \(D^0_\pi\) as the closure
\[D^0_\pi := (B_- \times B_+) \cdot \bigcup_{t, s \in H} (\pi t, s\pi^{-1}).\]

**Theorem 3.** The components of \(D^0\) are exactly the \(\{D^0_\pi\}, \pi \in S_n\).

Inside the flat family \(\{D^z\}_{z \in \mathbb{C}}\), consider the component whose generic fiber is the commuting variety as a subfamily. The special fiber of this subfamily is \(D^z_1\), plus possibly some nonreduced structure.

In the rest of the paper we prove these statements, in reverse order. Theorem 2 builds on the first half of theorem 3 whose proof uses simple facts about matrix Schubert varieties (our reference is [MS]), and we prove them together in subsection 2.1. Theorem 1 is then a consequence of theorem 2.

We close with a number of conjectures, generalizing the standard ones about the commuting variety.

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2. A Gröbner Degeneration Using \(\check{\rho}\)

Consider the scheme of pairs of matrices
\[E := \{(X, Y) : XY, YX \text{ both upper triangular}\}\]
which we’ll call the upper-upper scheme.

**Proposition 1.** Recall the scheme
\[D^0 := \lim_{z \to 0} \{ (X', Y') : X'Y' = \check{\rho}(z)Y'X'\check{\rho}(z^{-1}) \text{ off the diagonal} \}\]
defined as the flat limit of this one-parameter family.

This scheme embeds in \(E\), via the map
\[\tau : (X, Y) \in D^0 \mapsto (w_0X, Yw_0) \in E\]
where \(w_0\) is the permutation matrix with 1s along the antidiagonal.

**Proof.** Consider the equations defining \(D^z\). When we conjugate \(Y'X'\) by \(\check{\rho}(z)\), it multiplies the upper triangle by positive powers of \(z\), the lower triangle by negative powers (and leaves the diagonal alone, though we don’t care). So the upper triangle of \(X'Y'\) is equal to the upper triangle of \(Y'X'\) times positive powers of \(t\), and after rescaling, the lower triangle of \(Y'X'\) is equal to the lower triangle of \(X'Y'\) times positive powers of \(z\). In the limit, we get the equations \(X'Y'\) lower triangular, \(Y'X'\) upper triangular on \(D^0\). Therefore \((w_0X', Y'w_0) \in E\).

\[\square\]
We will eventually prove in theorem 2 that this map is an isomorphism. We’ve twisted this scheme by \( w_0 \) only because it’s less confusing to deal always with upper triangular matrices, rather than to mix upper and lower.

2.1. Dimensions of the components. The upper-upper scheme \( E \) carries an obvious action of pairs of upper triangular matrices, \( B_+ \times B_+ \):

\[
(U_1, U_2) \cdot (X, Y) := (U_1 X U_2^{-1}, U_2 Y U_1^{-1})
\]

The projection \( p : (X, Y) \mapsto X \) is \( B_+ \times B_+ \)-equivariant with respect to the action \( (U_1, U_2) : X := U_1 X U_2^{-1} \) on the space of single matrices.

We know the orbits of \( B_+ \times B_+ \) on the space of matrices: each orbit contains a unique partial permutation matrix, a \( 0,1 \)-matrix with at most one \( 1 \) in any row and column.

We will need also a slightly more specific fact, that each orbit of \( N_+ \times N_+ \) contains a unique monomial matrix, which has at most one nonzero entry in each row and column.

Our reference for facts about these orbits, in particular their dimensions, is [MS]:

**Proposition 2 ([MS], theorem 15.28).** Let \( \pi \) be a partial permutation matrix. The dimension of \( B_+ \pi B_+ \) inside \( M_n(\mathbb{C}) \) is the number of matrix entries such that a \( 1 \) entry in \( \pi \) is either on it, directly below, or directly to the left.

Given a partial permutation matrix \( \pi \), let

\[
E_\pi := p^{-1} ((B_+ \times B_+) \cdot \pi)
\]

so the \( \{E_\pi\} \) give a finite decomposition of \( E = \bigsqcup E_\pi \) into \( B_+ \times B_+ \)-invariant locally closed subsets.

**Lemma 1.** The stratum \( E_\pi \) is smooth and irreducible, of dimension \( n^2 + (\text{rank of } \pi) \).

If \( \pi \) is a permutation matrix (not just partial), then the set

\[
(N_+ \times N_+) \cdot \{(\pi t, s\pi^{-1}), \ s, t \text{ invertible diagonal}\}
\]

is an open dense subset of \( E_\pi \).

**Proof.** The fiber over the “central” point \( \pi \in E_\pi \) is

\[
\{Y : \pi Y, Y\pi \text{ both upper triangular}\}
\]

which is a vector space. Since \( B_+ \times B_+ \) acts transitively on \( p(E_\pi) \), \( E_\pi \) is a vector bundle over the smooth irreducible \( p(E_\pi) \), and therefore smooth and irreducible. Moreover, the dimension of \( E_\pi \) is the dimension of the orbit in the base (given by proposition 2), plus the dimension of the fiber.

So let’s compute the fiber dimension. The conditions \( \pi Y, Y\pi \) upper triangular become, on matrix entries, that \( Y_{ij} \) must be zero if there is a \( 1 \) in \( \pi \) directly (and strictly) to the left, or directly (and strictly) below, entry \( ij \). Otherwise \( Y_{ij} \) is free, which includes the case when there is a \( 1 \) in \( \pi \) actually in entry \( ij \).

Every matrix entry therefore “counts” for the dimension of the base (by proposition 2) if it has a \( 1 \) in \( \pi \) below or to the left, counts for the fiber if it doesn’t, and counts for both if it is actually placed at a \( 1 \) in \( \pi \). So the total count is \( n^2 \) plus the number of \( 1 \)s, as was to be shown.
Consider now the \( N^+ \times N^+ \) orbit of the point \((\pi t, s\pi^{-1})\), and assume \( s \) invertible, and that \( s^{-1}t \) has no repeated entries. (Even excluding those \((s, t)\) it turns out that we’ll still get a dense open set.) Plainly, this orbit is contained in \( E_{\pi} \) by its definition. The infinitesimal stabilizer of \((\pi t, s\pi^{-1})\) consists of those pairs \((A, B)\) of strictly upper-triangular matrices such that
\[
A\pi t - \pi tB = 0, \quad Bs\pi^{-1} - s\pi^{-1}A = 0.
\]
So
\[
\pi^{-1}A\pi = tBt^{-1} = s^{-1}Bs
\]
making \( B \) commute with the generic diagonal matrix \( s^{-1}t \). Therefore \( B \) is diagonal, hence zero, and so too is \( A \).

Since the \( N^+ \times N^+ \)-stabilizer of \((\pi t, s\pi^{-1})\) is trivial, its orbit is \(2(n^2)\) dimensional. No two of these orbits intersect (see the comment before the lemma about monomial matrices), so we have a \(2n\)-dimensional family of them, in all comprising \(n^2 + n\) dimensions. This is the same dimension as \( E_{\pi} \), so this subset is open (hence dense, since \( E_{\pi} \) is irreducible). □

The following lemma tells us some (but not all) of the equations separating the components of \( E \).

**Lemma 2.** Let \( \pi \) be a permutation matrix. Then
\[
(X, Y) \in E_{\pi} \quad \implies \quad \text{diag}(XY) = \pi \cdot \text{diag}(YX),
\]
i.e. \((XY)_{ii} = (YX)_{\pi(i),\pi(i)}, i = 1 \ldots n\).

**Proof.** It’s enough to test this equality on the dense subset given us by lemma\[\text{II}\] consisting of elements of the form \((X, Y) = (U_1\pi tU_2^{-1}, U_2s\pi^{-1}U_1^{-1})\), where \(U_1, U_2 \in N^+\) and \(s, t\) are diagonal.
\[
XY = U_1\pi t s\pi^{-1}U_1^{-1}
\]
\[
YX = U_2 stU_2^{-1}
\]
So their diagonals are the same as those of \(\pi t s\pi^{-1}\) and \(st\). □

We give a precise conjecture of the equations defining the closure of \( E_{\pi} \) in section \[\text{III}\]. Note that the obvious component of \( E \), in which both \(X\) and \(Y\) are themselves upper triangular, is \(E_1\), whereas the component that interests us most is \(E_{\pi_0}\).

**Proofs of theorems\[\text{II}\text{and III}\]** The scheme \( E \) is defined by \(n^2 - n\) equations, and by lemma\[\text{II}\] is a finite union of pieces \(\{E_{\pi}\}\) each of codimension \(\geq n^2 - n\). So it is a complete intersection.

Therefore it is pure, and only those pieces \(E_{\pi}\) of codimension exactly \(n^2 - n\) are components (the others lie in the closure). These are the \(E_{\pi}\) for which \(\pi\) has rank \(n\), i.e. is a permutation matrix and not just a partial permutation matrix. In particular this proves the first statement in theorem\[\text{III}\].

It remains to show that \( E \) is reduced. Since it is a complete intersection and therefore Cohen-Macaulay, being generically reduced implies that it is reduced (see \[\text{II}\], exercise 18.9). We will now find a smooth, reduced point \((\pi t, s\pi^{-1})\) on each \(E_{\pi}\).

Let \(t, s\) be generic diagonal matrices (the genericity condition will be specified in due course). The scheme \(E\) is the zero set of the map
\[
(X, Y) \mapsto \text{strict lower triangles of } XY, YX,
\]
whose differential at the point \((\tau t, s\tau^{-1})\) is

\[(A, B) \mapsto \text{strict lower triangles of } AY + XB, YA + BX = \text{strict lower triangles of } As\tau^{-1} + \tau tB, s\tau^{-1}A + B\tau t.
\]

We want to show this differential is onto. Consider \(A = \lambda e_{\pi(i),j}\), \(B = \mu e_{k,\pi(l)}\). Then

\[(A, B) \mapsto \text{strict lower triangles of } \lambda e_{\pi(i),j}s\tau^{-1} + \mu\tau t e_{k,\pi(l)}, s\tau^{-1}\lambda e_{\pi(i),j} + \mu e_{k,\pi(l)}\tau t = \text{strict lower triangles of } \lambda s_j e_{\pi(i),\pi(j)} + \mu t_k e_{\pi(k),\pi(l)}, \lambda s_i e_{ij} + \mu t_i e_{kl}
\]

In particular \(\lambda = 1, \mu = 0\) gives us

\[(A, B) \mapsto \text{strict lower triangles of } s_j e_{\pi(i),\pi(j)}, s_i e_{ij}.
\]

If \(i > j\) but \(\pi(i) < \pi(j)\), we can use this to produce pairs \((0, e_{ij})\). If \(i < j\) but \(\pi(i) > \pi(j)\), we can use this to produce pairs \((e_{\pi(i),\pi(j)}, 0)\).

The hard case is when \(i > j\) and \(\pi(i) > \pi(j)\), then \((i, j) = (k, l)\) gives us

\[(A, B) \mapsto \text{strict lower triangles of } (\lambda s_j + \mu t_e) e_{\pi(i),\pi(j)}, (\lambda s_i + \mu t_i) e_{ij}
\]

and as long as \(s_j/s_i \neq t_j/t_i\) for any \(i, j\), we can adjust \(\lambda, \mu\) to obtain \((0, e_{ij})\) and \((e_{\pi(i),\pi(j)}, 0)\) in the image. So we’ve gotten every pair of matrices where one has zero strict lower triangle and the other has exactly one entry in the strict lower triangle. These generate the target so the differential is indeed onto.

We’ve found a reduced point in each component \(E_\pi\) of \(E\). Therefore \(E\) is generically reduced, hence by its Cohen-Macaulayness it’s reduced.

We’re now ready to knock off theorem 2 and the remainder of theorem 3. Theorem 2 amounts to the statement that the map in proposition 1 is an isomorphism, which we’ll now prove.

Since \(D^1\) and \(E\) are complete intersections defined by \(n^2 - n\) quadratics, they have the same degree, \(2n^2 - n\). This map \(\tau: (X,Y) \mapsto (w_0X, Yw_0)\) from proposition 1 is linear, so preserves degree. Taking flat limits also preserves degree. So the image \(\tau(D^0)\) inside \(E\) has the same degree as \(E\). Since \(E\) is equidimensional of the same dimension as \(\tau(D^0)\), we find that \(\tau(D^0)\) must include all of \(E\)’s components. But this lower bound on \(\tau(D^0)\) is already \(E\), since \(E\) is reduced.

By [KS], the (reduction of the) \(z \to 0\) limit of any component of \(D^z\) is again equidimensional, hence a union of some components of \(E\). We want to see that the commuting component of \(D\) limits only to \(E_1\), and that the non-commuting components of \(D\) accounts for all the other components of \(E\). A priori one might expect some components \(\{E_\pi\}\) to arise as components of both limits, but as \(E\) is generically reduced this does not happen.

Let \(t, s\) be generic and \(\pi \neq 1\). Now note that each point \((\tau t, s\tau^{-1})\) is in every \(D^z\). For \(z = 1\), this point is in a non-commuting component of \(D = D^1\). For \(z = 0\), this point is in the \(E_\pi\) component (and no other, by the genericity). So the limit of the non-commuting components of \(D\) is all the non-identity components of \(E\).

Unfortunately, the result of [KS] doesn’t let us rule out the possibility that the \(z \to 0\) limit of the commuting variety has embedded components. To show this doesn’t happen, it would be enough to know that the variety \(E_{w_0}\) is defined by the additional equations
\[
\text{diag}(XY) = w_0 \cdot \text{diag}(YX)
\] with no more needed. That would also imply that the commuting scheme is reduced, which remains unknown at the time of this writing.

2.2. Some results about (multi)degrees of components. Let \(d_\pi\) denote the degree of the homogeneous affine variety \(E_\pi\).

**Proposition 3.**

- The sum \(\sum_{\pi \in S_n} d_\pi\) is \(2^{n^2-n}\).
- Denote by \(\star : S_k \times S_{n-k} \to S_n\) the standard concatenation of permutations. Then \(d_{\pi \star \rho} = d_\pi d_\rho\).
- If \(w_0\) is the permutation of length \(n\) of maximum length, then
  \[
  d_\pi = d_{\pi^{-1}} = d_{w_0 \pi w_0} = d_{w_0 \pi^{-1} w_0}.
  \]

**Proof.** For the first statement, the right-hand side is the degree of the quadratic complete intersection \(E_\pi\), which is the sum of the degrees of its components.

For the second, note that \(\pi \mapsto \pi^{-1}\) symmetry comes from the map \((X, Y) \mapsto (Y, X)\). The map \((X, Y) \mapsto (w_0 X^T w_0, w_0 Y^T w_0)\) is also easily seen to give a linear isomorphism of \(E_\pi\) and \(E_{w_0 \pi^{-1} w_0}\).

With Macaulay 2 \([M2]\), we computed these degrees for small \(n\) and \(\star\)-irreducible \(\pi\):

\[
\begin{align*}
  d_1 &= 1 \\
  d_2 &= 3 \\
  d_{23} &= d_{31} = 13, \quad d_{32} = 31
\end{align*}
\]

So for example when \(n = 3\),

\[
\begin{align*}
  2^{3^2-3} &= d_{123} + d_{213} + d_{132} + d_{312} + d_{231} + d_{321} \\
          &= d_3^2 + d_2 + d_1 + d_1 d_2 + d_1 d_3 + d_2 + d_3 + d_3^2 \\
          &= 1 + 3 + 3 + 13 + 13 + 31.
\end{align*}
\]

In fact one can sharpen proposition \(\Box\) a great deal using the theory of multidegrees \([MS]\) (also known as equivariant multiplicities \([Ro]\)), using not only the rescaling action but the full torus action on \(D^0\). Since we don’t want to recapitulate this theory here – except to say that it assigns each cycle a homogeneous polynomial, rather than just a number – we give only a little taste, using the 2-torus action that scales \(X\) and \(Y\) individually.

**Proposition 4.** Let \(A, B\) be the usual generators of the weight lattice of the 2-torus scaling \(X, Y\) individually. Let \(d_\pi'\) denote the bidegree of \(E_\pi\), a homogeneous polynomial in \(\mathbb{N}[A, B]\).

- The sum \(\sum_{\pi \in S_n} d_\pi'\) is \((A + B)^{n^2-n}\).
- Denote by \(\star : S_k \times S_{n-k} \to S_n\) the standard concatenation of permutations. Then
  \[
  d_{\pi \star \rho}' = d_\pi' d_\rho' (AB)^k (n-k).
  \]
If \( w_0 \) is the permutation of length \( n \) of maximum length, then \( d'_\pi = d'_{w_0 \pi^{-1} w_0} \).

- \( d'_\pi(A, B) = d'_{\pi^{-1}}(B, A) \).

The proofs are exactly as in proposition 3. The \( n = 3 \) example now becomes

\[
\begin{align*}
d'_{1} &= 1, \\
d'_{21} &= A^2 + AB + B^2, \\
d'_{231} &= 2A^5B + 4A^4B^2 + 4A^3B^3 + 2A^2B^4 + AB^5, \\
d'_{312} &= A^5B + 2A^4B^2 + 4A^3B^3 + 4A^2B^4 + 2AB^5, \\
d'_{321} &= A^6 + 3A^5B + 7A^4B^2 + 9A^3B^3 + 7A^2B^4 + 3AB^5 + B^6,
\end{align*}
\]

\[
(A + B)^{3^2 - 3} = d'_{123} + d'_{213} + d'_{132} + d'_{312} + d'_{231} + d'_{321} = (AB)^3(d'_1)^3 + ABd'_2d'_1 + ABD'_1d'_2 + d'_{312} + d'_{231} + d'_{321}
\]

\[
= A^3B^3 + 2(A^3B + A^2B^2 + AB^3)
\]

\[
+ (2A^5B + 4A^4B^2 + 4A^3B^3 + 2A^2B^4 + AB^5)
\]

\[
+ (A^5B + 2A^4B^2 + 4A^3B^3 + 4A^2B^4 + 2AB^5)
\]

\[
+ (A^6 + 3A^5B + 7A^4B^2 + 9A^3B^3 + 7A^2B^4 + 3AB^5 + B^6)
\]

3. Conjectures

There are two main conjectures about the commuting scheme: that it is reduced, and

that it is Cohen-Macaulay. We state some conjectures sharpening these two.

**Conjecture.** The variety \( \overline{E_{\pi}} \) of \( E \) is defined as a scheme by three sets of equations:

1. those defining \( E \), which say \( XY, YX \) upper triangular
2. those given by lemma 2, that \( \text{diag}(XY) = \pi \cdot \text{diag}(YX) \)
3. those defining the \( \pi, \pi^{-1} \) matrix Schubert varieties: for each pair \( i, j \) the rank of the lower-left \( i \times j \) rectangle in \( X \) (resp. in \( Y \)) is bounded above by the number of \( 1 \)s in that rectangle in \( \pi \) (resp. in \( \pi^{-1} \)).

Note that for \( \pi = w_0 \), the third set is empty.

Moreover, if we impose just the first and third set of equations, we get the reduced scheme \( \bigcup_{\rho \leq \pi} \overline{E_{\rho}} \).

If this is proved for \( \pi = w_0 \), it implies that the commuting scheme (to which \( \overline{E_{w_0}} \) deforms) is reduced, i.e. is the commuting variety.

**Conjecture.** Each individual \( \{E_{\pi}\} \), and each union \( \bigcup_{\rho \leq \pi} \overline{E_{\rho}} \), is Cohen-Macaulay.

Note that these statements are trivial for the component \( \overline{E_1} \), being a linear subspace \( \{(X, Y) : \text{both upper triangular}\} \). Perhaps they can be proved by induction in the Bruhat order.

We repeat our earlier-stated conjecture (which doesn’t seem to imply anything directly about the commuting scheme):

**Conjecture.** For \( g \) a reductive Lie algebra, the diagonal commutator scheme of \( g \) is a reduced complete intersection with two components.
In the $\mathfrak{gl}_n$ case, Terry Tao conjectured in a discussion the equations defining the other component. First, we find some equations that do in fact hold.

**Proposition 5.** Consider the $n \times 2n$ matrix, whose first $n$ columns are the diagonals of $X^i$, $i = 0 \ldots n - 1$, and next $n$ are the diagonals of $Y^i$, $i = 0 \ldots n - 1$.

If $(X,Y) \in H$ but $[X,Y] \neq 0$, then the rank of this $n \times 2n$ matrix is at most $n - 1$. In particular every size $n$ minor vanishes.

**Proof.** Let $K = [X,Y]$. Then the nonzero diagonal matrix $K$ is trace-perpendicular to any element $Z_X$ in the centralizer $C_X$ of $X$:

$$\text{Tr} (KZ_X) = \text{Tr} ([X,Y]Z_X) = \text{Tr} ([Z_X,X]Y) = \text{Tr} 0 = 0$$

The same argument holds for any $Z_Y$ in the centralizer $C_Y$ of $Y$ (rotating the opposite direction), and any linear combination $Z_X + Z_Y$.

The functional $\text{Tr} (K \cdot)$ is only sensitive to the diagonal entries, and the trace form $\text{Tr} (\cdot \cdot)$ is nondegenerate. So the projection “take diagonals” from $C_X + C_Y$ to $t$ is not onto, since it only hits $K^\perp$. (This is where we use $K \neq 0$.)

The argument so far would work fine in any semisimple $\mathfrak{g}$, with the Killing form in place of the trace form. In the $\mathfrak{gl}_n$ case, we have a bunch of matrices we know to be in $C_X$ (resp. $C_Y$), namely the powers of $X$ (resp. $Y$). The non-ontoness of the projection then gives us the rank claim in the proposition. \qed

Note that this gives one equation each on $X$ and $Y$ individually – while every $X$ commutes with some $Y$ (e.g. $0$ or $X$ itself), not every $X$ has a nonzero diagonal commutator with some $Y$.

**Conjecture.** The equations in proposition 5 define the other component(s) of the diagonal commutator scheme.

With Macaulay 2, we verified this in the $\mathfrak{gl}_3$ case – first by finding the equations, then using them to suggest the conjecture. The other conjectures were also all verified in this case.

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