GEOMETRIC CAPACITY POTENTIALS ON CONVEX PLANE RINGS

JIE XIAO

Abstract. Under $1 < p \leq 2$, this paper presents some old and new convexity/isoperimetry based inequalities for the variational $p$-capacity potentials on convex plane rings.

1. Introduction

A pair $(K, \Omega)$ of sets in the plane $\mathbb{R}^2$ will be called a condenser if $\Omega$ is an open subset of $\mathbb{R}^2$ and $K$ is a compact subset of $\Omega$. And, a condenser $(K, \Omega)$ will be called a ring whenever $\Omega \setminus K$ is connected and $((\infty) \cup \mathbb{R}^2) \setminus (\Omega \setminus K)$ comprises only two components (cf. [19]). Under $1 < p \leq 2$ the variational $p$-capacity of a given condenser $(K, \Omega)$ is defined as

$$pcap(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla f|^p \, dA : f \in \dot{W}^{1,p}(\Omega), \ f \geq 1_K \right\}$$

where $\dot{W}^{1,p}(\Omega)$ is the completion of all infinitely differentiable functions $g$ on $\mathbb{R}^2$ with compact support in $\Omega$ under the Sobolev semi-norm $(\int_{\Omega} |\nabla g|^p \, dA)^{\frac{1}{p}} < \infty$, $dA$ is the differential element of area on $\mathbb{R}^2$, and $1_K$ represents the indicator function of $K$. A function that realizes the above infimum is called a $p$-capacity potential. When the infimum is finite, there exists a unique $p$-capacity potential $u$ which solves the following boundary value problem:

$$\begin{cases}
\text{div}(|u|^{p-2} \nabla u)|_{\Omega \setminus K} = 0; \\
u|_{\partial \Omega} = 0; \\
u|_{K} = 1.
\end{cases}$$

As an interesting topic in calculus of variations and mathematical physics (see e.g. [5] [16]), the variational $p$-capacity problem is to study the essential properties of a solution $u$ to (1.1) and hence of the capacity $pcap(K, \Omega)$. If a given condenser $(K, \Omega)$ is good enough - for example - $(K, \Omega)$ is a ring with $K$ and $\Omega$ being convex, then an integration-by-parts can be performed to establish the level curve representation (cf. [18] [15]) of the $p$-capacity of $(K, \Omega)$:

$$pcap(K, \Omega) = \int_{\{z \in \Omega \setminus K^\circ : u(z) = t\}} |\nabla u|^{p-1} \, dL \quad \forall \ t \in [0, 1],$$

where $\overline{\Omega}$, $K^\circ$ and $dL$ stand for the closure of $\Omega$, the interior of $K$ and the differential element of length on $\mathbb{R}^2$ respectively, and

$$|\nabla u(x)| \equiv \lim \inf_{\Omega \setminus K \ni y \rightarrow x} |\nabla u(y)| = \lim \sup_{\Omega \setminus K \ni y \rightarrow x} |\nabla u(y)| \quad \forall \ x \in \partial(\overline{\Omega} \setminus K^\circ).$$

2010 Mathematics Subject Classification. 31A15, 35J05, 35J92, 53A04, 53A30.

This project was supported by NSERC of Canada and Memorial’s University Research Professorship.
In accordance with [5, page 13], the $p$-modulus of a condenser $(K, \Omega)$ is decided by

\begin{equation}
\text{pmod}(K, \Omega) = (\text{pcap}(K, \Omega))^\frac{1}{p}.
\end{equation}

Of course, it is well-known that if $p = 2$ then (1.1), (1.2) and (1.3) are the conformal capacity problem, the conformal capacity and the conformal modulus of $(K, \Omega)$ respectively.

An important limiting case of (1.1) should get special treatment. More precisely, if $K$ reduces to a point $o \in \Omega$, then we are led to consider the singular $p$-capacity potential – the $p$-Green function $g_o(\cdot, \cdot)$ of $\Omega$ with singularity at $o$:

\begin{equation}
\begin{cases}
div((\nabla g_o(\cdot, \cdot))^{p-2}) = -\delta(o, \cdot); \\
g_o(\cdot, \cdot)|_{\partial \Omega} = 0,
\end{cases}
\end{equation}

where $\delta(o, \cdot)$ is the Dirac distribution at $o$. According to [5, pages 8, 13 and 63], if

\begin{equation}
k_p(r) = \begin{cases}
\frac{p-1}{2-p}(2\pi)^{\frac{1+p}{p}} r^{\frac{1}{p}} & \text{for} \quad 1 < p < 2; \\
(2\pi)^{-1} \ln r^{-1} & \text{for} \quad p = 2,
\end{cases}
\end{equation}

then

\begin{equation}
\tau_p(o, \Omega) = \lim_{\varepsilon \to 0} (k_p(|z - o|) - g_o(o, z))
\end{equation}

is called the $p$-Robin function of $\Omega$ at $o$, and hence the function $\rho_p(o, \Omega)$ determined by

\begin{equation}
k_p(\rho_p(o, \Omega)) = \tau_p(o, \Omega)
\end{equation}

is said to be the $p$-harmonic (conformal as $p = 2$) radius. Interestingly, such a radius can be utilized to estimate $\text{pcap}(\overline{D(o, r)}, \Omega)$ for $D(o, r)$ being the closed disk centered at $o$ with radius $r \to 0$ (cf. [5, pages 81-82]):

\begin{equation}
\tau_p(o, \Omega) = \begin{cases}
\lim_{r \to 0} \frac{\text{pcap}(\overline{D(o, r)}, \Omega) - \text{pcap}(\overline{D(o, r)}, \mathbb{R}^2)}{(p-1)(\text{pcap}(\overline{D(o, r)}, \mathbb{R}^2))^{\frac{1}{p-1}}} & \text{for} \quad 1 < p < 2; \\
\lim_{r \to 0} \left((\text{pcap}(\overline{D(o, r)}, \Omega))^{-1} - (2\pi)^{-1} \ln r^{-1}\right) & \text{for} \quad p = 2,
\end{cases}
\end{equation}

where

\begin{equation}
\text{pcap}(\overline{D(o, r)}, \mathbb{R}^2) = \begin{cases}
2\pi \left(\frac{p-1}{2-p}\right) r^{2-p} & \text{for} \quad 1 < p < 2; \\
r & \text{for} \quad p = 2.
\end{cases}
\end{equation}

A careful look at (1.2) reveals that the level curve of the $p$-capacity potential $u$ of (1.1) plays a decisive role. Moreover, for $t \in [0, 1]$, let $A(t)$ be the area of the set bounded by the closed level curve $\{z \in \Omega \setminus K^\circ : u(z) = t\}$ whose length is denoted by $L(t)$. Then we have the isoperimetric inequality below:

\begin{equation}
A(t) \leq (4\pi)^{-1}(L(t))^2.
\end{equation}

This (1.10), together with the closely related works [13, 1, 10], suggests us to further find geometric properties induced by (1.1)-(1.9) such as optimal estimates for the area and perimeter of a level set of either $p$-capacity potential of a convex ring and or $p$-Green function of a bounded convex domain, as well as sharp isoperimetric inequalities involving $p$-capacity - the details will be respectively presented in the forthcoming three sections: [2, 3, 4]
2. Longinetti’s convexity for $p$-capacity potentials of convex rings

Referring to [13 Section 2], in what follows we always suppose that $\Omega$ is a planar convex domain containing the origin, $\nu = (\cos \theta, \sin \theta)$ is the exterior unit normal vector to the boundary $\partial \Omega$ at the point $z = (z_1, z_2) \in \partial \Omega$, and

$$h(\theta) = z \cdot \nu = z_1 \cos \theta + z_2 \sin \theta$$

is the support function $h_K$ of $K = \overline{\Omega}$ which measures the Euclidean distance from the origin to the support line $\ell$ supporting $\partial \Omega$ at $z$ orthogonal to $\nu$. Here it is worth recalling the differentiability and integrability of the support function below:

- if $\partial \Omega$ is strictly convex and $\ell$ supports $\partial \Omega$ at $z$ only then $h$ is of class $C^1$ and

$$h'(\theta) = \frac{d}{d\theta} h(\theta) = -z_1 \sin \theta + z_2 \cos \theta;$$

- if $\partial \Omega$ is of class $C^2$ and its curvature $\kappa(\theta)$ is positive then $h$ is of class $C^2$ and

$$h''(\theta) = \frac{d^2}{d\theta^2} h(\theta) = (\kappa(\theta))^{-1} - h(\theta).$$

- the area $A = A(\Omega)$ and the length $L = L(\Omega)$ of $\partial \Omega$ are determined by

$$A = 2^{-1} \int_{S^1} h(\theta)(\kappa(\theta))^{-1} \, d\theta,$$

$$L = \int_{S^1} h(\theta) \, d\theta = \int_{S^1} (\kappa(\theta))^{-1} \, d\theta,$$

where $S^1$ is the unit circle and may be identified with the interval $[0, 2\pi)$.

With the help of (2.1)-(2.4), Longinetti obtained the following assertion - see also [13 Theorems 3.1-3.2]:

**Longinetti’s convexity.** Given $p \in (1, 2]$ and two convex domains $\Omega_0$ and $\Omega_1$ with $0 \in \overline{\Omega}_1 \subset \overline{\Omega}_0 \subset \mathbb{R}^2$. For a solution $u$ to (1.1) with $K = \overline{\Omega}_1$ and $\Omega = \Omega_0$ and $t \in [0, 1]$ let $\Omega_t$ be the convex domain bounded by the level curve $\Gamma_t = \{ z \in \overline{\Omega_0} \setminus \Omega : u(z) = t \}$ as well as $A(t)$ and $L(t)$ be the area of $\Omega_t$ and the length of $\partial \Omega_t = \Gamma_t$, respectively. Then

$$A'(t)A'''(t) - 2p^{-1} (A''(t))^2 \geq 0;$$

$$L(t)L'''(t) - (p - 1)^{-1} (L'(t))^2 \geq 0,$$

where each equality in (2.3) holds when and only when all level curves $\{ \Gamma_t \}_{t \in [0, 1]}$ are concentric circles.

In order to work out the area-analogue of the second inequality in (2.5), we have the following assertion whose (i) and (ii) with $p = 2$ are due to Longinetti - see also [13 (3.28), (3.29) and (5.13)].

**Theorem 2.1.** Under the same assumptions as in Longinetti’s convexity, one has:

- if $\partial \Omega_1$ is a circle then

$$2(p - 1)A(t)A''(t) \geq p(A'(t))^2.$$

- if $|\nabla u|$ equals a constant on $\partial \Omega_0$ then

$$2(p - 1)A(t)A''(t) \leq p(A'(t))^2.$$
Proof. (i) This follows from [13] (3.29]).
(ii) To verify (2.7), let
\[
M_p(t) = \frac{A(t)A''(t) - p(2p - 2)^{-1}(A'(t))^2}{(A(t))^{\frac{2p}{p - 1}}} \quad \forall \quad t \in [0, 1].
\]
Then a straightforward computation, along with the first inequality of (2.5) and the fact
\( A'(t) \leq 0 \), gives
\[
M'_p(t) = (A(t))^{\frac{1}{p - 1}} \left( A''(t)A(t) + A'(t)A'(t) - p(2p - 2)^{-1}A'(t)A'(t) \right)
\]
\[
+ (A'(t))A'(t) - p(2p - 2)^{-1}(A'(t))^3(A(t))^{-1}(1 - p)^{-1}
\]
\[
\leq 2p^{-1}(A'(t)(A(t))^{\frac{2p}{p - 1}}(M_p(t))^2 \leq 0.
\]
As a consequence, one has
\[
M_p(t) \leq M_p(0) \quad \forall \quad t \in [0, 1].
\]
If \( M_p(0) \leq 0 \) then the argument is complete. To see this, we are required to calculate
\( A'(0) \) and \( A''(0) \). Note that \( |\nabla u| \) is a constant, say, \( c \) on \( \partial \Omega_0 \). So, an application of (1.2) yields
\[
pcap(K_1, \Omega_0) = c^{p - 1}L(0).
\]
For each \( t \in [0, 1] \) suppose that \( h(\theta, t) \) and \( \kappa(\theta, t) \) are the support function and the curvature function of the level curve \( \Gamma_t = \{ z \in \overline{\Omega_0} \setminus \Omega_1 : u(z) = t \} \). Then, according to the Lewis convexity in [11, Theorem 1], one has that for each \( t \in (0, 1) \) the level curve \( \Gamma_t \) is strictly convex and \(|\nabla u| \neq 0 \) in \( \overline{\Omega_0} \setminus \Omega_1 \). Moreover, one has the following formulas for the first-order and second-order derivatives of
(2.8)
\[
\begin{align*}
A(t) &= 2^{-1} \int_{\gamma^1} h(\theta, t)(\kappa(\theta, t))^{-1} d\theta; \\
L(t) &= \int_{\gamma^1} h(\theta, t) d\theta,
\end{align*}
\]
(cf. [13] 3.11-3.12 and 3.5-3.6]):
(2.9)
\[
\begin{align*}
A'(t) &= \int_{\gamma^1} \left( \frac{\partial}{\partial t} h(\theta, t) \right)(\kappa(\theta, t))^{-1} d\theta; \\
A''(t) &= \int_{\gamma^1} \left( \left( \frac{\partial}{\partial t} h(\theta, t) \right)^2 + \left( \frac{\partial^2}{\partial t^2} h(\theta, t) \right)(\kappa(\theta, t))^{-1} \right) d\theta; \\
L'(t) &= \int_{\gamma^1} \frac{\partial}{\partial t} h(\theta, t) d\theta; \\
L''(t) &= \int_{\gamma^1} \frac{\partial^2}{\partial t^2} h(\theta, t) d\theta.
\end{align*}
\]
The preceding formulas (2.8)-(2.9), along with [13] (2.13) which particularly ensures
(2.10)
\[
|\nabla u|_{l_0} \frac{\partial}{\partial t} h(\theta, t) \bigg|_{l=0} = -1
\]
and so (cf. [13] (3.20))
\[
\frac{\partial^2}{\partial t^2} h(\theta, t) \bigg|_{l=0} = (p - 1)^{-1} \left( \frac{\partial}{\partial t} h(\theta, t) \bigg|_{l=0} \right)^2 \kappa(\theta, t) \bigg|_{l=0},
\]
give
\[
A'(0) = -L(0)c^{-1} = -(L(0))^{\frac{p-1}{p}}(pcap(\overline{\Omega_1}, \Omega_0))^{\frac{1}{p-1}}.
\]
and
\[ A''(0) = -L'(0)c^{-1} + 2\pi(p - 1)^{-1}c^{-2} = 2\pi p(p - 1)^{-1}\left(\frac{L(0)}{pcap(\Omega_1, \Omega_0)}\right)^{\frac{1}{p}}. \]

Now, using the case \( t = 0 \) of (1.10) one gets
\[ M_p(0) = \left(\frac{p(2p - 2)^{-1}}{(pcap(\Omega_1, \Omega_0))^{\frac{1}{p}}}\right)\left(\frac{(L(0))^2}{(A(0))^{2-p}}\right)^{\frac{1}{p^*}} \left(4\pi - \frac{(L(0))^2}{A(0)}\right) \leq 0, \]
as desired. \( \square \)

3. ISOPERIMETRY FOR P-CAPACITIES OF CONVEX RINGS AND BOUNDED CONDENSERS

In [9, Theorem 1.1] (cf. [12, Theorem 4.2]) Henrot-Shahgholian showed that for \( 1 < p \leq 2 \), a bounded convex domain \( \Omega_1 \subset \mathbb{R}^2 \) and a constant \( c > 0 \), there is a unique convex domain \( \Omega_0 \supset K_1 = \overline{\Omega_1} \) in \( \mathbb{R}^2 \) such that

\[
\left\{
\begin{array}{l}
\text{div}(\nabla u)^{p-2}\nabla u)|_{\Omega_0\setminus K_1} = 0; \\
u|_{\partial \Omega_0} = 0; \\
u|_{\partial \Omega_1} = 1; \\
|\nabla u||_{\partial \Omega_0} = c.
\end{array}
\right.
\]

(3.1)

This fact leads to an isoperimetry for \( pcap(\Omega_1, \Omega_0) \) that extends Carleman’s inequality [13 (5.10)] (cf. [20, Proposition A.1]), Longinetti’s inequality [13 (5.4)] (cf. [6, Theorem] for another lower bound estimate for the case \( p = 2 \)) and Longinetti’s isoperimetric deficit monotonicity [13 (5.12)].

**Theorem 3.1.** For \( 1 < p \leq 2, c > 0 \) and \( K_1 = \overline{\Omega_1} \) let \( c_p = ((2\pi)^{-1}pcap(K_1, \Omega_0))^{\frac{1}{p-1}} \), \( \Omega_0 \supset K_1 \) and \( (u, \Omega_0 \setminus K_1, c) \) satisfy (3.1). Then one has:

(i) an isoperimetry for the variational capacity

\[
\left\{
\begin{array}{l}
\left(\frac{L(1)}{2\pi}\right)^{\frac{2}{p}} - \left(\frac{L(0)}{2\pi}\right)^{\frac{2}{p^*}} \leq \frac{2-p}{p-1}c_p^{-1} \leq \left(\frac{A(1)}{\pi}\right)^{\frac{2}{p^*}} - \left(\frac{A(0)}{\pi}\right)^{\frac{2}{p^*}} \text{ for } 1 < p < 2; \\
\ln\left(\frac{L(0)}{L(1)}\right) \leq c_p^{-1} \leq \ln\left(\frac{A(0)}{A(1)}\right) \text{ for } p = 2,
\end{array}
\right.
\]

where (3.2) holds with the sign of equality if \( \Omega_0 \setminus K_1 \) is a circular annulus.

(ii) a monotonicity for the isoperimetric deficit

\[
\frac{d}{dt}\left((L(t))^2 - 4\pi A(t)\right) \geq 0 \quad \forall \quad t \in [0, 1],
\]

and consequently

\[
\left(\frac{L(0)}{2\pi}\right)^2 - \left(\frac{L(1)}{2\pi}\right)^2 \leq \left(\frac{A(0)}{\pi}\right) - \left(\frac{A(1)}{\pi}\right),
\]

(3.4)

with equality if \( \Omega_0 \setminus K_1 \) is a circular annulus.

**Proof.** (i) The case \( p = 2 \) of (3.2) can be seen from [13 (5.4) and (5.10)]. So it remains to check the case \( 1 < p < 2 \) of (3.2).
Let us begin with proving the left-hand-inequality of (3.2) in the case $1 < p < 2$. Note that the second inequality of (2.3) is equivalent to

$$\left( \frac{L''(t)}{-L'(t)} \right) \geq (p - 1)^{-1} \left( \frac{-L'(t)}{L(t)} \right).$$

So, integrating this inequality over $[0, t] \subset [0, 1]$ derives

$$L'(0) \leq \left( \frac{L(t)}{L(0)} \right)^{(p-1)^{-1}},$$

namely,

$$(L(t))^{-(p-1)^{-1}} L'(t) \geq L'(0)(L(0))^{-(p-1)^{-1}}.$$

An integration of the last inequality over $[0, 1]$ yields that if $1 < p < 2$ then

$$\left( \frac{p-1}{p-2} \right) \left( (L(1))^{\frac{p-2}{p}} - (L(0))^{\frac{p-2}{p}} \right) \geq L'(0)(L(0))^{-(p-1)^{-1}}.$$

Since $|\nabla u|$ is just the constant $c$ on $\partial \Omega_0$, an application of (1.2) and (2.9), and (2.10) gives

$$\frac{L'(0)}{(L(0))^{(p-1)^{-1}}} = -\frac{(2\pi)^{\frac{p-2}{p-1}}}{c_p}.$$  

Upon putting this formula into the right-hand-side of the last inequality, we obtain the required estimate.

Next, let us verify the right-hand-inequality of (3.2) in the case $1 < p < 2$. In doing so, let $(D(0, r_0), D(0, r_1))$ be the origin-centered disk pair with $A(j) = \pi r_j^2$ for $j = 0, 1$. Then an application of [19, 7.5 The main theorem] and [8, (2.13)] gives

$$c_p \geq ((2\pi)^{-1} p \text{cap}(\overline{D(0, r_1)}, D(0, r_0)))^{\frac{1}{p-1}}$$

$$= \left( \frac{2 - p}{p - 1} \right) \left( r_1^{\frac{p-2}{p}} - r_0^{\frac{p-2}{p}} \right)^{-1},$$

$$= \left( \frac{2 - p}{p - 1} \right) \left( \frac{A(1)}{\pi} \right)^{\frac{p-2}{p-1}} - \left( \frac{A(0)}{\pi} \right)^{\frac{p-2}{p-1}},$$

as desired.

(ii) To establish (3.3), we just observe (1.2) and the following formula

$$\begin{cases} -A'(t) = \int_{\Gamma_t} |\nabla u|^{-1} dL; \\ 2\pi c_p^{p-1} = \int_{\Gamma_t} |\nabla u|^{p-1} dL, \end{cases}$$

and then utilize the Hölder inequality to achieve

$$L(t) \leq \left( \int_{\Gamma_t} |\nabla u|^{p-1} dL \right)^{1-p} \left( \int_{\Gamma_t} |\nabla u|^{-1} dL \right)^{p-1} = (2\pi c_p^{p-1})^{\frac{1}{p-1}} (A'(t))^{1-p^{-1}}.$$  

This, along with (3.5) and (3.6), derives

$$\frac{d}{dt} \left( (L(t))^2 - 4\pi A(t) \right) \geq -4\pi \left( 2\pi c_p^{p-1} \right)^{\frac{1}{p-1}} (L(t))^{p(p-1)^{-1}} + A'(t) \geq 0,$$

as desired. Of course, this gives (3.4) right away.  \( \square \)
Needless to say, the right-hand-inequalities of (3.2) are still valid for more general condensers. In addition, we can find out their pure capacity versions. Given a compact subset $K$ of $\mathbb{R}^2$. If $r > 0$ is so large that $K$ is contained in the origin-centered open disk $D(0, r)$, then it is not hard to see that

$$r \mapsto F_p(K, r) \equiv \left\{ \begin{array}{ll} \left( \left( \text{pcap}(K, D(0, r)) \right)^{1/p} + (2\pi)^{1/p} \left( \frac{p}{2\pi} \right)^{2/p-1} r^{p-2} \right)^{1-1/p} & \text{for } 1 < p < 2; \\
\exp \left( -2\pi \left( \left( \text{pcap}(K, D(0, r)) \right)^{1/p} + (2\pi)^{-1} \ln r^{-1} \right) \right) & \text{for } p = 2,
\end{array} \right.$$ 

is a decreasing function on $[0, \infty)$ (cf. [5, Lemma 2.1] and [4, Section 3.1]). So, it is reasonable to define

$$(3.7) \quad \text{pcap}(K) = \lim_{r \to \infty} F_p(K, r).$$

Below is a known chain of the isocapacitary/isoperimetric inequalities (cf. [14], [17] pages 140-141 and [2, 7, 21, 3]) for the closure $\overline{\Omega}$ of a bounded domain $\Omega \subset \mathbb{R}^2$ with its area $A(\overline{\Omega})$ and diameter $\text{diam}(\overline{\Omega})$ as well as length $L(\overline{\Omega})$ of the boundary $\partial \overline{\Omega}$:

$$\left( \frac{A(\overline{\Omega})}{\pi} \right)^{1/p} \leq \left\{ \begin{array}{ll} \left( \left( \frac{p-1}{2-p} \right)^{1/p-1} \text{pcap}(\overline{\Omega}) \right)^{1/p} & \text{for } 1 < p < 2; \\
\text{pcap}(\overline{\Omega}) & \text{for } p = 2 
\end{array} \right.$$ 

which holds with the sign of equality in the first two estimates if $\overline{\Omega}$ is a closed disk $\overline{D}(0, r)$ with (cf. (1.9))

$$\overline{\text{pcap}(D(0, r))} = \text{pcap}(D(0, r), \mathbb{R}^2).$$

As an extension of [4, Lemma 1], the following isocapacitary deficit result gives a sharp lower bound of $\text{pcap}(K, \Omega)$ in terms of $\text{pcap}(K)$ and $\text{pcap}(\overline{\Omega})$.

**Theorem 3.2.** Let $(K, \Omega)$ be a condenser in $\mathbb{R}^2$ with $\Omega$ being bounded.

$$(3.8) \quad \left( \frac{\text{pcap}(K, \Omega)}{2\pi} \right)^{1/p} \leq \left\{ \begin{array}{ll} \left( \frac{\text{pcap}(K)}{2\pi} \right)^{1/p} - \left( \frac{\text{pcap}(\overline{\Omega})}{2\pi} \right)^{1/p} & \text{for } 1 < p < 2; \\
\ln \left( \frac{\text{pcap}(\overline{\Omega})}{\text{pcap}(K)} \right) & \text{for } p = 2,
\end{array} \right.$$ 

with equality if $\Omega \setminus K$ is a circular annulus.

**Proof.** For such a large $R > 0$ that $\Omega \subset D(0, R)$, let

$$M(p, R) \equiv \left\{ \begin{array}{ll} (2\pi)^{1/p} \left( \frac{p-1}{2-p} \right)^{2/p-1} R^{2/p-2} & \text{for } 1 < p < 2; \\
(2\pi)^{-1} \ln R & \text{for } p = 2.
\end{array} \right.$$ 

Then, an application of [5, Lemma 2.1] - the subadditivity of $p$-modulus and (3.7) yields

$$\left( \frac{\text{pcap}(K, \Omega)}{2\pi} \right)^{1/p} = \text{pmod}(K, \Omega)$$

$$\leq \text{pmod}(K, D(0, R)) + M(p, R) - \text{pmod}(\overline{\Omega}, D(0, R)) - M(p, R)$$

$$= \left( \frac{\text{pcap}(K, D(0, R))}{2\pi} + M(p, R) - \left( \frac{\text{pcap}(\overline{\Omega}, D(0, R))}{2\pi} - M(p, R) \right) \right)^{1/p}$$

$$\to \left\{ \begin{array}{ll} \left( \left( \frac{\text{pcap}(K)}{2\pi} - \left( \frac{\text{pcap}(\overline{\Omega})}{2\pi} \right) \right)^{1/p} \right) & \text{for } 1 < p < 2; \\
(2\pi)^{-1} \ln \left( \frac{\text{pcap}(\overline{\Omega})}{\text{pcap}(K)} \right) & \text{for } p = 2,
\end{array} \right.$$ 

as $R \to \infty,$
whence reaching (3.8) whose equality follows from the following formula for $0 < r < R < \infty$:

$$\left(\frac{pcap(D(0, r), D(0, R))}{2\pi}\right)^{\frac{1}{p-2}} = \begin{cases} (\frac{p-2}{2p})^\frac{p}{p-2} \left( (\frac{p-2}{2p}) \left( \ln \frac{R}{r} \right) \right)^{\frac{2}{p-2}} \text{ for } 1 < p < 2; \\
\ln \frac{R}{r} \text{ for } p = 2,
\end{cases}$$

see also [8, p.35].

\[\square\]

4. Convexity for $p$-Green functions of convex domains

The following is a generalization of [13] Theorems 4.1-4.2] from $p = 2$ to $p \in (1, 2]$.

**Theorem 4.1.** Let $1 < p \leq 2$ and $\Omega \subset \mathbb{R}^2$ be a convex domain containing a given point $o$. For $t \geq 0$ set

$$\begin{align*}
A_g(t) &= \int_{\{z \in \Omega; g(z) \geq t\}} dA; \\
L_g(t) &= \int_{\{z \in \Omega; g(z) = t\}} dL.
\end{align*}$$

Then

(i)

$$\begin{align*}
A_g'(t)A_g''(t) - 2p^{-1}(A_g''(t))^2 &\geq 0; \\
L_g(t)L_g''(t) - (p - 1)^{-1}(L_g''(t))^2 &\geq 0,
\end{align*}$$

with equality if $\Omega$ is a disk centered at $o$.

(ii)

$$\begin{align*}
A_g(t)A_g''(t) &\geq 2^{-1}p(p - 1)^{-1}(A_g'(t))^2; \\
A_g''(t) &\geq 2\pi p(p - 1)^{-1}(A_g'(t))^{2p-1}; \\
(A_g'(t))^{2-\frac{2}{p}} &\geq 4\pi A_g(t),
\end{align*}$$

with equality if $\Omega$ is a disk centered at $o$.

(iii)

$$\begin{align*}
A_g(t) &\leq \begin{cases} 
((A_g(0))^{\frac{p}{2p-2}} + (\frac{2p}{2p-2})^\frac{p}{2p-2} t^{\frac{2p-2}{2p}})^{\frac{2p-2}{p}} \text{ for } 1 < p < 2; \\
A_g(0) \exp(-4\pi t) \text{ for } p = 2,
\end{cases} \\
L_g(t) &\leq \begin{cases} 
((L_g(0))^{\frac{p}{2p-2}} + (\frac{2p}{2p-2})^\frac{p}{2p-2} t^{\frac{2p-2}{2p}})^{\frac{2p-2}{p}} \text{ for } 1 < p < 2; \\
L_g(0) \exp(-2\pi t) \text{ for } p = 2,
\end{cases}
\end{align*}$$

with equality if $\Omega$ is a disk centered at $o$.

**Proof.** (i) Since $\Omega$ is convex, each level curve of $g_\Omega(o, \cdot)$ is strictly convex (cf. [11, Theorem 1]). This fact, plus (2.5), implies (4.1).

(ii) Next, noticing the following fundamental formula for $g_\Omega(o, \cdot)$ (cf. [5, Lemma 9.1]):

$$\begin{align*}
-A_g'(t) &= \int_{\{z \in \Omega; g_\Omega(o, z) = t\}} \left| \nabla g_\Omega(o, z) \right|^{-1} dL(z); \\
1 &= \int_{\{z \in \Omega; g_\Omega(o, z) = t\}} \left| \nabla g_\Omega(o, z) \right|^{p-1} dL(z),
\end{align*}$$
we employ the Hölder inequality to derive

\[(4.4) \quad L_g(t) \leq ( - A_g'(t))^{1-p^{-1}} \left( \int_{\{z \in \Omega : g_{\Omega}(o, z) = t\}} |\nabla g_{\Omega}(o, z)|^{p-1} dL(z) \right)^{p^{-1}} = ( - A_g'(t))^{1-p^{-1}}. \]

Clearly, the following isoperimetric inequality

\[(4.5) \quad A_g(t) \leq (4\pi)^{-1}(L_g(t))^2 \]

holds. So, a combination of (4.4) and (4.5) gives the third inequality of (4.2). This, plus the first inequality of (4.2), implies the second inequality of (4.2). Thus, it remains to verify the first inequality of (4.2). In doing so, let us choose a sequence of open disks \(\{D_j\}_{j=1}^\infty\) centered at \(o\) with radius tending to 0. If

\[
\begin{align*}
  a_j &= \min_{z \in \partial D_j} g_{D_j}(o, z); \\
  b_j &= \max_{z \in \partial D_j} g_{D_j}(o, z),
\end{align*}
\]

then (1.6) can be used to deduce \(\lim_{j \to \infty}(b_j - a_j) = 0\). Also, if \(u_j\) and \(v_j\) are \(p\)-harmonic in \(\Omega \setminus D_j\), i.e.,

\[
div(|\nabla u_j|^{p-2} \nabla u_j) = 0 = div(|\nabla v_j|^{p-2} \nabla v_j) \quad \text{on} \quad \Omega \setminus D_j,
\]

subject to

\[
\begin{align*}
  u_j(z) &= v_j(z) = 0 \quad \forall \quad z \in \partial \Omega; \\
  u_j(z) &= b_j \quad \forall \quad z \in \partial D_j; \\
  v_j(z) &= a_j \quad \forall \quad z \in \partial D_j,
\end{align*}
\]

then an application of the comparison principle for \(p\)-harmonic functions (cf. [9]) derives

\[
v_j(z) \leq g_{\Omega}(o, z) \leq u_j(z) \quad \forall \quad z \in \Omega \setminus \overline{D_j}.
\]

This, together with \(\lim_{j \to \infty}(b_j - a_j) = 0\), implies that

\[
\lim_{j \to \infty} u_j(z) = \lim_{j \to \infty} v_j(z) = g_{\Omega}(o, z) \quad \forall \quad z \in \Omega \setminus \overline{D(o, r)}
\]

holds for any small \(r > 0\) such that the open disk \(D(o, r)\) is contained in \(\Omega\). Now, using (2.6) for \(u_j\) and \(v_j\) and letting \(j \to \infty\) we arrive at the first inequality of (4.2).

(iii) Finally, let us check (4.3). Thanks to the first inequality of (4.2) and the second inequality of (4.2), it is enough to verify the area part of (4.3). Note that the first inequality of (4.2) yields that

\[
t \mapsto A_g'(t)(A_g(t))^{p(2-2p)^{-1}}
\]

is an increasing function on \([0, \infty)\). So, it follows that

\[
A_g'(t)(A_g(t))^{p(2-2p)^{-1}} \leq \lim_{s \to \infty} A_g'(s)(A_g(s))^{p(2-2p)^{-1}} \equiv \gamma_p \quad \forall \quad t \in [0, \infty).
\]

This, along with an integration of the last inequality over \([0, t]\), gives

\[
A_g(t) \leq \begin{cases} 
  \left((A_g(0))^{p-2} + \frac{(p-2)^{\frac{p-2}{2p-2}}}{2p-2} \gamma_p t \right)^{\frac{2p}{p-2}} & \text{for } 1 < p < 2; \\
  A_g(0) \exp(\gamma_p t) & \text{for } p = 2.
\end{cases}
\]
Thus, it remains to show $\gamma_p = -(4\pi)^{p(2p-2)-1}$. But, this follows from the basic fact (cf. [5] Lemma 9.1) and (1.7) that when $t \to \infty$ the level set $\{z \in \Omega : g_\Omega(o,z) \geq t\}$ approaches a closed disk centered at $o$ with radius

$$r = \begin{cases} \left( \frac{(2-p)(p-1)^{-1}}{2} \right)^{p-1} \left( t + \tau_p(o,\Omega) \right)^{\frac{p-1}{2}} \text{ for } 1 < p < 2; \\ \exp \left( -2\pi(t + \tau_p(o,\Omega)) \right) \text{ for } p = 2. \end{cases}$$

$\square$

REFERENCES

[1] G. Alessandrini, *Isoperimetric inequalities for the length of level lines of solutions of quasilinear capacity problems in the plane*, J. Appl. Math. Phy. (ZAMP) 40(1989)920-924.

[2] V. Andrievskii, W. Hasen and N. Nadirashvilli, *Isoperimetric inequalities for capacities in the plane*, Math. Ann. 292(1992)191-195.

[3] R. W. Barnard, K. Pearce and A. Y. Solynin, *An isoperimetric inequality for logarithmic capacity*, Ann. Acad. Sci. Fenn. Math. 27(2002)419-436.

[4] D. Betsakos, *Geometric versions of Schwarz’s lemma for quasiregular mappings*, Proc. Amer. Math. Soc. 139(2010)1397-1407.

[5] M. Flucher, *Variational Problems with Concentration*, Birkhäuser, 1999.

[6] L. E. Fraenkel, *A lower bound for electrostatic capacity in the plane*, Proc. Royal Soc. Edin. 88A(1981)267-273.

[7] W. Hansen and N. Nadirashvili, *Isoperimetric inequalities in potential theory*, Potential Anal. 3(1994)1-14.

[8] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Dover Publications, Inc., Mineola, New York, 2006.

[9] A. Henrot and H. Shahgholian, *Existence of classical solutions to a free boundary problem for the p-Laplace operator: (I) the exterior convex case*, J. reine angew. Math. 521(2000)85-97.

[10] P. Laurence, *On the convexity of geometric functionals of level for solutions of certain elliptic partial differential equations*, J. Appl. Math. Phy. (ZAMP) 40(1989)258-284.

[11] J. Lewis, *Capacity functions in convex rings*, Arch. Rational Mech. Anal. 66(1977)201-224.

[12] J. Lewis, *Applications of Boundary Harnack Inequalities for p Harmonic Functions and Related Topics*, C.I.M.E Summer Course: Regularity Estimates for Nonlinear Elliptic and Parabolic Problems, Cetraro (Cosenza) Italy, June 21-27, 2009.

[13] M. Longinetti, *Some isoperimetric inequalities for the level curves of capacity and Green’s functions on convex plane domains*, SIAM J. Math. Anal. 19(1988)377-389.

[14] V. Maz’ya, *Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev-type imbeddings*, J. Funct. Anal. 224(2005)408-430.

[15] G. A. Philippin and L. E. Payne, *On the conformal capacity problem*, Symposia Mathematica, Vol. XXX (Cortona) 1988)119-136.

[16] G. Pólya, *Estimating electrostatic capacity*, Amer. Math. Monthly 54(1947)201-206.

[17] T. Ransford, *Potential Theory in the Complex Plane*, London Math. Soc. Student Texts 28, Cambridge University Press 1995.

[18] A. S. Romanov, *Capacity relations in a flat quadrilateral*, Siberian Math. J. 49(2008)709-717.

[19] J. Sarvas, *Symmetrization of condensers in n-space*, Ann. Acad. Sci. Fenn. Ser. AI 522, 1972.

[20] D. Smets and J. Schaftingen, *Desingularization of vortices for the Euler equation*, Arch. Rational Mech. Anal. 198(2010)869-925.

[21] A. Y. Solynin and V. A. Zalgaller, *An isoperimetric inequality for logarithmic capacity of polygons*, Ann. Math. 159(2004)277-303.

Department of Mathematics and Statistics, Memorial University, St. John’s, NL A1C 5S7, Canada

E-mail address: jxiao@mun.ca