MORE SUMS OF HILBERT SPACE FRAMES

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Abstract. In this paper we have some new results on sums of Hilbert space frames and Riesz bases. We also have a correction for some results in "S. Obeidat et al., Sums of Hilbert space frames, J. Math. Anal. Appl. 351 (2009) 579–585."

1. Introduction

Throughout this paper, $\mathcal{H}$ denotes a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. Recall that a sequence $\{f_i\}_{i \in I} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ if there exist $0 < A \leq B < \infty$ such that

\begin{equation}
A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2
\end{equation}

for all $f \in \mathcal{H}$. The constants $A$ and $B$ are called a lower and upper frame bound.

If $\{f_i\}_{i \in I} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$, the frame operator for $\{f_i\}_{i \in I}$ is the bounded linear operator $S : \mathcal{H} \to \mathcal{H}$ given by $Sf = \sum_{i \in I} \langle f, f_i \rangle f_i$. Therefore $\langle Sf, f \rangle = \sum_{i \in I} |\langle f, f_i \rangle|^2$ for all $f \in \mathcal{H}$. It follows that $S$ is positive and invertible. This provides the frame decomposition

$$f = \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle S^{-1}f_i$$

for all $f \in \mathcal{H}$.

2. Main results

The following is proved in [3, Proposition 2.1].

Proposition 2.1. [3] Let $\{f_i\}_{i \in I}$ be a frame for $\mathcal{H}$ with the frame operator $S$, frame bounds $A \leq B$ and let $L : \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then $\{Lf_i\}_{i \in I}$ is a frame for $\mathcal{H}$ if and only if $L$ is invertible.

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on $\mathcal{H}$. Moreover, in this case the frame operator for $\{L f_i\}_{i \in I}$ is $L S L^*$ and the new frame bounds are $A \|L^{-1}\|^{-2}$, $B \|L\|^2$.

In this note, we show that Proposition 2.1 is not true in general. Indeed, if $\{f_i\}_{i \in I}$ is a frame for Hilbert space $\mathcal{H}$ and $L : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded invertible operator, then $\{L f_i\}_{i \in I}$ is a frame for $\mathcal{H}$ but the inverse is not true in general. In the proof of Proposition 2.1, the authors proved that $LSL^*$ is invertible. It does not imply that $L$ is invertible on $\mathcal{H}$. It should be noted that Proposition 2.1 has been used in Corollaries 2.2, 2.3 and in the proof of Proposition 4.1 of [3].

**Example 2.2.** Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Define a shift operator $L$ on $\mathcal{H}$ by $L(e_n) = e_{n-1}$ if $n > 1$ and $L(e_1) = 0$. It is clear that $\{L(e_n)\}_{n=1}^\infty$ is a frame for $\mathcal{H}$, but $L$ is not invertible although $LL^* = I$. Moreover, $\{L^*(e_n)\}_{n=1}^\infty$ is not a frame for $\mathcal{H}$.

We can improve Proposition 2.1 as follows:

**Proposition 2.3.** Let $\{f_i\}_{i \in I}$ be a frame for $\mathcal{H}$ with the frame operator $S$, frame bounds $A \leq B$ and let $L : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. Then $\{L f_i\}_{i \in I}$ is a frame for $\mathcal{H}$ if and only if $L$ is surjective. Moreover, in this case the frame operator for $\{L f_i\}_{i \in I}$ is $LSL^*$ and the new frame bounds are $A \|L^\dagger\|^{-2}$ and $B \|L\|^2$, where $L^\dagger$ is the pseudo-inverse of $L$.

**Proof.** If $\{L f_i\}_{i \in I}$ is a frame for $\mathcal{H}$, then its frame operator $LSL^*$ is invertible. So $L$ is surjective. The converse follows from Corollary 5.3.2 of [2].

We also have

**Proposition 2.4.** Let $\{f_i\}_{i \in I}$ be a frame for $\mathcal{H}$ with the frame operator $S$ and let $L : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. Then $\{L f_i\}_{i \in I}$ and $\{L^* f_i\}_{i \in I}$ are frame for $\mathcal{H}$ if and only if $L$ is invertible. Moreover, in this case the frame operators for $\{L f_i\}_{i \in I}$ and $\{L^* f_i\}_{i \in I}$ are $LSL^*$ and $L^* SL$, respectively.

**Proof.** If $\{L f_i\}_{i \in I}$ and $\{L^* f_i\}_{i \in I}$ are frames for $\mathcal{H}$, then their frame operators $LSL^*$ and $L^* SL$ are invertible. So $L$ is invertible. The converse is clear.

In [3], corollary 2.2 can be improved as below.
Corollary 2.5. Let \( \{f_i\}_{i \in I} \) be a frame for \( \mathcal{H} \) with the frame operator \( S \), frame bounds \( A \leq B \) and let \( L : \mathcal{H} \rightarrow \mathcal{H} \) be a bounded operator, then \( \{f_i + Lf_i\}_{i \in I} \) is a frame for \( \mathcal{H} \) if and only if \( I + L \) is surjective. Moreover, in this case the frame operator for the new frame is \( (I + L)S(I + L^*) \) with the frame bounds \( A\|(I + L)^\dagger\|^{-2} \) and \( B\|I + L\|^2 \), where \( (I + L)^\dagger \) is the pseudo-inverse of \( I + L \). In particular, if \( L \) is a positive operator (or just \( L > -I \)), then \( \{f_i + Lf_i\}_{i \in I} \) is a frame for \( \mathcal{H} \) with the frame operator \( S + SL + SL^* + LSL^* \).

Corollary 2.6. Let \( \{f_i\}_{i \in I} \) be a frame for \( \mathcal{H} \) and \( P : \mathcal{H} \rightarrow \mathcal{H} \) be a bounded operator. If \( P^2 = P \), then for all \( a \neq -1 \), \( \{f_i + aPf_i\}_{i \in I} \) is a frame for \( \mathcal{H} \).

Proof. If \( a \neq -1 \), then we have \( (I + aP)(I - \frac{a}{a+1}P) = I \). This implies that \( I + aP \) is invertible and so \( \{f_i + aPf_i\}_{i \in I} \) is a frame for \( \mathcal{H} \). \( \square \)

Proposition 2.7. Let \( \{f_i\}_{i \in I} \) be a sequence in \( \mathcal{H} \) such that \( \sum_{i \in I} \langle f, f_i \rangle f_i \) converges for all \( f \in \mathcal{H} \). If \( L : \mathcal{H} \rightarrow \mathcal{H} \) is a bounded operator such that \( \{Lf_i\}_{i \in I} \) and \( \{L^*f_i\}_{i \in I} \) are frames for \( \mathcal{H} \), then \( \{f_i\}_{i \in I} \) is a frame for \( \mathcal{H} \).

Proof. Let us define

\[
U : \mathcal{H} \rightarrow \mathcal{H}, \quad U(f) := \sum_{i \in I} \langle f, f_i \rangle f_i.
\]

Let \( S_L \) be the frame operator for \( \{Lf_i\}_{i \in I} \). Then \( S_L = LUL^* \) is invertible. So \( L \) is surjective. Similarly, we infer that \( L^* \) is surjective. Therefore \( L \) is invertible and so \( \{f_i\}_{i \in I} \) is a frame for \( \mathcal{H} \) with the frame operator \( L^{-1}S_L(L^*)^{-1} \). \( \square \)

Proposition 2.8. Let \( \{f_i\}_{i \in I} \) be a Riesz basis for \( \mathcal{H} \) with analysis opeartor \( T \), Riesz basis bounds \( A \leq B \), and let \( L : \mathcal{H} \rightarrow \mathcal{H} \) be a bounded opeartor. Then \( \{Lf_i\}_{i \in I} \) is a Riesz basis for \( \mathcal{H} \) if and only if \( L \) is invertible on \( \mathcal{H} \). Moreover in this case the analysis opeartor for \( \{Lf_i\}_{i \in I} \) is \( T_L = TL^* \) and the new Riesz basis bounds are \( \| L^{-1} \|^{-2} A, \| L \|^{-2} B \).

Proof. Since the analysis opeartor for \( \{Lf_i\}_{i \in I} \) is \( T_L = TL^* \), \( L \) is invertible if and only if \( \{Lf_i\}_{i \in I} \) is a Riesz basis for \( \mathcal{H} \). \( \square \)

Corollary 2.9. If \( \{f_i\}_{i \in I} \) is a Riesz basis for \( \mathcal{H} \) and \( L : \mathcal{H} \rightarrow \mathcal{H} \) is a bounded operator, then \( \{f_i + Lf_i\}_{i \in I} \) is a Riesz basis for \( \mathcal{H} \) if and
only if $I + L$ is invertible on $H$. In this case the synthesis operator for new frame is $T_{I+L} = T(I + L^*)$ and the new Riesz basis bounds are $\| (I + L)^{-1} \|^{-2} A, \| I + L \|^{2} B$.

**Corollary 2.10.** Let $\{f_i\}_{i \in I}$ be a Riesz basis for $H$ with frame operator $S$ and $\{g_i\}_{i \in I}$ be its alternative dual frame. Suppose that $-1 \notin \sigma(S^{-a+b^{-1}})$. Then $\{S^af_i + S^bg_i\}_{i \in I}$ is a Riesz basis for $H$ for all real numbers $a, b$.

Here, we also show that the equivalence of part (1) and (2) in Proposition 3.1 of [3], is not true in general. Indeed, if $T_1L_1^* + T_2L_2^*$ is an invertible operator, then $\{L_1f_i + L_2g_i\}_{i \in I}$ is a frame for $H$ but the inverse is not true.

**Example 2.11.** Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $H$ and $T$ be the analysis operator of $\{e_n\}_{n=1}^{\infty}$. Define a shift operator $L$ on $H$ as in Example 2.2. Letting $L_1 = L_2 = L$ and $f_n = g_n = e_n$ for each $n \in \mathbb{N}$, in Proposition 3.1 of [3], we see that $\{2L(e_n)\}_{n=1}^{\infty}$ is a frame for $H$ but $2TL^*$ is not a surjective operator. If $TL^*$ is a surjective operator, then for $\delta_1 \in \ell^2(\mathbb{N})$, there exists $h \in H$ such that $TL^*(h) = \delta_1$ and so $\langle L(e_1), h \rangle = 1$, which is a contradiction.

**Proposition 2.12.** Let $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ be Bessel sequences in $H$ with analysis operators $T_1, T_2$ and frame operators $S_1, S_2$, respectively. Also let $L_1, L_2 : H \rightarrow H$. Then the following are equivalent:

1. $\{L_1f_i + L_2g_i\}_{i \in I}$ is a Riesz basis for $H$.
2. $T_1L_1^* + T_2L_2^*$ is an invertible operator on $H$.

**Proof.** (1) $\Leftrightarrow$ (2) $\{L_1f_i + L_2g_i\}_{i \in I}$ is a Riesz basis for $H$ if and only if its analysis operator $T$ is invertible on $H$ where

$$Tf = \{\langle f, L_1f_i + L_2g_i \rangle \}_{i \in I} = \{\langle L_1^*f, f_i \rangle + \langle L_2^*f, g_i \rangle \}_{i \in I} = T_1L_1^*f + T_2L_2^*f.$$ 

\[\square\]

### 3. Applications to Gabor frames

For $x, y \in \mathbb{R}$ we consider the operators $E_x$ and $T_y$ on $L^2(\mathbb{R})$ defined by $(E_xf)(t) = e^{2\pi int}f(t)$ and $(T_yf)(t) = f(t-y)$. It is easy to prove that $E_x$ and $T_y$ are unitary with $E_x^* = E_{-x}$ and $T_y^*y = T_{-y}$. A Gabor frame
is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$, where $a, b > 0$ and $g \in L^2(\mathbb{R})$ is a fixed function. We use $(g, a, b)$ to denote $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$.

**Lemma 3.1.** Let $x, y \in \mathbb{R}$ and $c \in \mathbb{C}$ with $|c| = 1$. Then the following are equivalent:

(i) $I + cT_x E_y$ is a surjective operator on $L^2(\mathbb{R})$.

(ii) $I + cE_y T_x$ is a surjective operator on $L^2(\mathbb{R})$.

**Proof.** Using Proposition 2 of [1], we infer that $I + cT_x E_y$ is surjective if and only if $I + cT_x E_y$ is invertible. So $I + cT_x E_y$ is invertible if and only if $I + c T_x E_y$ is invertible, and $I + c T_x E_y$ is invertible if and only if $I + cE_y T_x$ is invertible. \[\square\]

**Corollary 3.2.** Let $x, y \in \mathbb{R}$ and $c \in \mathbb{C}$. If $I + cT_x E_y$ is a surjective operator on $L^2(\mathbb{R})$, then there exists $\delta > 0$ such that $\|(I+cT_x E_y)(g)\| \geq \delta\|g\|$ for all $g \in L^2(\mathbb{R})$.

In the following, we intend to improve Proposition 4.1 of [3].

**Theorem 3.3.** Let $x, y \in \mathbb{R}$ such that $x \neq 0$, $xy \in \mathbb{Z}$ and let $c \in \mathbb{C}$ with $|c| = 1$. Then $I + cE_y T_x : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is not surjective.

**Proof.** It is enough we take $x > 0$. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function defined by

$$f(t) := \sum_{k=1}^{n} (-1)^k e^{2\pi i k y^t} \chi_{[kx,(k+1)x]}(t).$$

By a simple computation, we get

$$\|f\|^2 = \int_{\mathbb{R}} |f(t)|^2 dt = nx, \quad \|(I + cE_y T_x)f\|^2 = 2x.$$ 

Therefore $f \in L^2(\mathbb{R})$ and Corollary 3.2 implies that $I + cE_y T_x$ is not surjective. \[\square\]

**Corollary 3.4.** Let $x, y \in \mathbb{R}$ such that $x \neq 0$, $xy \in \mathbb{Z}$ and let $c \in \mathbb{C}$ with $|c| = 1$. If $(g, a, b)$ is a Gabor frame, then $(g + cE_y T_x g, a, b)$ is not a Gabor frame.

**Proof.** There exists $d \in \mathbb{C}$ with $|d| = 1$ such that $E_{mb} T_n (g + cE_y T_x g) = (I + dT_x E_y)(E_{mb} T_n g)$. If $(g + cE_y T_x g, a, b)$ is a Gabor frame, then $I + dT_x E_y$ is surjective (invertible) on $L^2(\mathbb{R})$ by Proposition 2.3. So $I + dE_y T_x$ is surjective by Lemma 3.1. Using Theorem 3.3, we get a contradiction. \[\square\]


**References**

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