Training highly effective connectivities within neural networks with randomly initialized, fixed weights

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Abstract

We present some novel, straightforward methods for training the connection graph of a randomly initialized neural network without training the weights. These methods do not use hyperparameters defining cutoff thresholds and therefore remove the need for iteratively searching optimal values of such hyperparameters. We can achieve similar or higher performances than in the case of training all weights, with a similar computational cost as for standard training techniques. Besides switching connections on and off, we introduce a novel way of training a network by flipping the signs of the weights. If we try to minimize the number of changed connections, by changing less than 10% of the total it is already possible to reach more than 90% of the accuracy achieved by standard training. We obtain good results even with weights of constant magnitude or even when weights are drawn from highly asymmetric distributions. These results shed light on the over-parameterization of neural networks and on how they may be reduced to their effective size.

1 Introduction

The use of deep neural networks in many challenging areas of computer science proved to be an indisputable success in recent years. The increase in computing power enabled researchers to build ever growing model architectures with millions and even billions of parameters for both supervised and unsupervised learning. Many successful applications of deep learning seem to favor large neural networks with intricate architectures. Despite their effectiveness, many aspects of deep neural networks are not well understood. One such aspect is why over-parameterized models are able to generalize so well.

A promising avenue of research towards a better understanding of deep learning architectures is neural network pruning. Recent work in this direction showed that large networks can be reduced to much smaller sub-networks while maintaining their accuracy. It has been found that even very aggressively pruned networks, with more than 95% of the weights removed, performed almost as well as the original [3]. This used a surprisingly simple heuristic: at the end of training, the weights with a magnitude below a certain threshold are set to zero, after which the network is reset to its original state and retrained. Setting weights to zero is functionally equivalent with their removal. This training and pruning procedure is repeated as long as the model maintains an accuracy as large as the full network. Although capable of finding very sparse networks, this mechanism requires an iterative procedure as well as a hyperparameter — a prior cutoff value for the threshold of the weight magnitudes. This makes it computationally expensive as well as prone to be sub-optimal due to the prior thresholds imposed on the weights. This pruning mechanism can be classified as a pruning after training approach.

Other works such as [15] [10] use a pruning before training approach, in order to save resources at training time. The end goal is to remove connections such that the resulting network is sparse and the weights are efficiently trainable after the pruning procedure. The third kind of approach is to prune
during training [2][12], where dynamical pruning strategies are used in order to both prune and train weights at the same time.

The main goal behind these pruning strategies is to find sparse neural networks that can be trained to large degrees of accuracy by changing the weights. However, it has been shown by [17] that there exist pruning masks which can be applied to an untrained network such that its performance is far better than chance. Furthermore, [13] developed an algorithm for finding good pruning masks for networks with fixed, random weights. They found that neural networks can be trained to performances close to state-of-the-art without changing the weights but training just a pruning mask. A downside of their algorithm is that, again, it requires an iterative procedure to find the optimal threshold value for the criterion upon which the weight removal is based.

In this work we further reap the seemingly unreasonable effectiveness of neural networks with randomly initialized, fixed weights, with an approach of **pruning without training** the weights. Our method is adjusting the connectivity graph of a randomly initialized neural network directly through back-propagation, without ever training the weights. As a result, our approach has several advantages:

- **No additional hyperparameters**: other than the network architecture (number of layers, nodes/filters per layer), learning rates and optimizer type, we do not use other predefined parameters.
- **Optimal pruning rates per layer**: as in [11], but in contrast to other approaches where the pruning percentages are not adapted to each layer, our algorithm finds the optimal pruning rates for each layer through back-propagation alone.
- **No additional computational cost**: since we do not use an iterative training and pruning procedure, our algorithm requires about as many computations as the standard way of training the weights of a neural network.

Besides the pruning paradigm where the connectivity graph is determined by switching connections on and off, we also introduce a novel, alternative algorithm that trains the connectivity graph by just flipping the signs of the connections.

The code used for the experiments presented here is available at https://github.com/rist-ro/training-neural-connectivities.

## 2 Pruning algorithm

For each weight of the network we assign a trainable variable, $t$, which is passed to a masking function defined in the following way:

$$m(t) = \text{sign}(\max(0, t)) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases}$$

Applying this function on the tensor $T$ with elements $t \in \mathbb{R}$, we obtain a binary mask $M$ with elements $m$ from $\{0, 1\}$. This mask is applied to the network connections: for a feed-forward network with $L$ hidden layers where each layer $l$ has $n_l$ units, the expression for the output of node $j$ in layer $l$ is then given by:

$$h^l_j = \sigma \left( \sum_{i=0}^{n_{l-1}} h^{l-1}_i w^l_{ij} m(t^l_{ij}) \right)$$

where $\sigma$ is the non-linear activation function, $h^{l-1}_i$ the output of a node from the previous layer, $w^l_{ij}$ are the incoming weights for the current node, and $m(t^l_{ij})$ is the masking function applied on the trainable value $t^l_{ij}$ associated with each weight $w^l_{ij}$. Note that we do not use biases in this approach.

For each forward pass a connection between two nodes can be enabled or disabled based on the output of $m(t)$ and is calculated automatically by the network, without the need to use ad-hoc heuristics. For the backward pass the gradient of the masking function is not defined and to overcome this issue we use the straight through estimator [1]. This (biased) estimator was first proposed by Hinton [7] and treats the gradient of a hard threshold function as if it was the identity function. Therefore the gradient is always 1. Unlike the approach of magnitude based pruning where a connection once
removed it is unable to “grow back”, with this method connections are dynamically added or removed depending on how well the network performs.

When we initialize the network, each layer’s weights, $W$, are drawn from a distribution $D$ and the associated values, $T$, from a uniform distribution in the interval $(0, 0.1]$. During training, the network weights are kept fixed and only $T$ is updated via back-propagation. Given that $T$ is initially strictly positive, the masks associated with the weights are initially 1 everywhere, effectively creating a network which uses all weights in the first forward pass.

### 2.1 Loss function

We used both a straightforward pruning method, which we call free pruning, as well as a method where we try to minimize the number of pruned weights while training the network, which we call minimal pruning.

For free pruning, we minimize the loss

$$L_{\text{free}} = \frac{1}{N} \sum_{i=0}^{N} L((x_i, y_i); W)$$

(3)

where $N$ is the number of samples and $L$ is the categorical cross-entropy loss. In this case, the network reduces the amount of connections between layers as much as needed in order to minimize the loss and therefore finds the optimal pruning factor for each layer. This eliminates any biased priors for the amount of pruning, which were present in other works, e.g. [3] and [13], where the pruning factor is specified explicitly for each layer.

For minimal pruning, we added a regularization term such that we can also minimize the amount of weights that the network prunes:

$$L_{\text{minimal}} = L_{\text{free}} - \frac{1}{M} \sum_{i=0}^{M} m(t_i)$$

(4)

where $M$ is the number of weights in the network. In this case the regularization term keeps the number of non-zero components in the mask as high as possible since it essentially counts the average number of 1’s in the masks. Therefore the network is constrained to prune as few weights as possible while concomitantly minimizing $L_{\text{free}}$. This allows us to investigate what would be a minimum amount of weights to be removed from a randomly initialized network such that it is still able to achieve a good performance.

### 2.2 Experiments

We have run experiments on MNIST [9] using a LeNet-300 [8] architecture and CIFAR-10 using three variations of a VGG-like [14] network. These networks have exactly the same architectures used in [3] [17] [13]. They are listed in appendix A (see supplementary material) and we refer to them in the same manner: LeNet, Conv2, Conv4 and Conv6. For the weight initialization we have used the two popular distributions Glorot Normal [4] and He Normal [6] as well as the Signed He Constant distribution, as used in [13]. For this latter distribution, each weight of a layer is set to a constant value, $\sqrt{2/n_{l-1}}$ (where $n_{l-1}$ is the number of nodes in the previous layer), and its sign is chosen randomly.

Our goal is to understand to what extent random weight initialization is sufficient for constructing highly accurate sub-networks within larger ones. In order to isolate as much as possible the effect of pruning, we train all our networks with a minimal setup. We do not use data augmentation, batch normalization nor any regularization techniques which may interfere with the randomness of the initialization.

Figure 1 (top row panels) shows the accuracy of LetNet trained on MNIST with weights initialized from the three different distributions. In blue we show the baseline performance, training all weights of the network. In orange/green are the curves for the free/minimal pruning algorithms. In every experiment throughout this paper each data-point is the average of 5 runs and the shaded area is the minimum and maximum of the 5 runs. The bottom row panels (corresponding colors) indicate the fraction of pruned weights as a function of the training epoch. For the baseline network, all
weights are changed at each iteration, therefore we omitted the curve in the bottom panels. One can observe that the randomly initialized network trained through free pruning reaches almost the same accuracy as the fully trained network. This is true for all types of weight initializations. The fraction of removed connections is about 45%.

Among the used distributions for the initial weights, the He Constant distribution yields the best results. With this distribution, minimal pruning achieves an accuracy of 96.8% compared to 97.9% for the baseline, even though the amount of pruned weights is less than 8%, while the rest of 92% are randomly generated and remain untrained.

Figure 2 shows a similar behaviour also when training a convolutional network on CIFAR-10. The randomly initialized network trained through free pruning reaches almost the same accuracy as the fully trained network. Minimal pruning in this case achieves an accuracy of about 72.3% compared to 79.6% for the fully trained network with less than 10% of the weights removed. A similar conclusion can be drawn as in the case of LeNet trained on MNIST – random weights are well enough suited for large performance.

2.3 Weight removal

In general the forward propagation through a neural network’s layer is a non-linear activation function of the weighted sum of the nodes from the previous layer. The formal expression for this is given by Eq. 2. Consider the special case when we draw weights from the He Constant distribution. In each layer the weights are set to have the same magnitude and a randomly chosen sign: \( w'_{l_{ij}} = |w^l| s_{l_{ij}} \) where \( s_{l_{ij}} \) is the sign of the weight. It follows that the weights from Eq. 2 can be factored out of the sum. Furthermore, if we choose ReLU as the activation function then we have the convenient
property that for any $a > 0$ the function $\max(0, ax) = a \max(0, x)$. Eq. (2) can then be rewritten as:

$$h^l_j = |w^l_i| \cdot \sigma \left( \sum_{i=0}^{n_{l-1}} h^l_{i-1} s_{ij} m(t^l_{ij}) \right)$$  (5)

The above equation is recursively applied and as such the weights can be factored out of each layer. Therefore we can initialize a neural network by setting all weights to 1, choosing a sign randomly and scaling the training data by the product of each layer’s weight magnitude, i.e. $\sqrt{2/n_{l-1}}$. More precisely, for a network with $L$ hidden layers where each layer has $n_l$ nodes, the training data $X$ becomes:

$$\hat{X} = X \cdot \prod_{l=1}^{L} |w^l_i| = X \cdot \prod_{l=1}^{L} \sqrt{2/n_{l-1}}$$  (6)

with $n_0$ being the number of incoming connections from layer zero (the input data). This expression is general and can be applied for fully connected layers as well as convolutional layers. Training the mask of a network with constant weights per layer is completely equivalent with training the mask of a network with unitary weights and input data scaled appropriately. A result of this training technique is that at inference time the magnitude of the weights becomes irrelevant to the classification accuracy because the output nodes are scaled by the same $W$ and we are only interested in the node with the highest value: $\arg \max(ax) = \arg \max(x)$ for any $a > 0$.

In order to verify that this procedure is numerically stable we have repeated the same experiments described in the previous section with the new initialization scheme. The results for the Conv2, Conv4 and Conv6 networks are shown in Figure 3. We compare the accuracy of the standard procedure, where we train all weights, to the free pruning algorithm, with and without weight removal. The figure shows that the accuracy curves of the free pruning algorithm, with and without weight removal, are almost identical in each case, and also almost identical to the ones of the baseline, for Conv2 and Conv4.

### 2.4 Weight sign imbalance

Standard weight initialization procedures use symmetric distributions: standard normal, truncated normal (Glorot, He) etc. The special He Constant distribution is symmetric but bimodal, having only two values, $\pm \sqrt{2/n_{l-1}}$, with the sign of the weight chosen randomly from a uniform distribution. Drawing weights from this distribution is essentially a Bernoulli trial where a weight has $p = 0.5$ probability of being positive and $q = 1 - p = 0.5$ of being negative. The amount of negative/positive weights follows a binomial distribution and due to the large amount of parameters in standard neural networks it is extremely unlikely that there is a significant imbalance between the two.

We have varied the probability $p$ of obtaining a positive weight and experimented with different values. We obtain therefore networks with a significant imbalance between the number of negative and positive weights. Figure 4 (left panels) shows the dependence on $p$ of the accuracy of LeNet trained on MNIST and of the fraction of pruned weights. The network maintains its performance when $0.3 \leq p \leq 0.7$. The accuracy drops by 1% only when reaching $p \leq 0.1$ and by 10% when $p \leq 0.01$. A notable result is the extreme case when $p = 1$, which corresponds to a network where there are only positive weights between neurons. The network is still trainable and reaches 95% accuracy with about 65% of the weights pruned.
In the case of Conv2, Conv4 and Conv6, right panels in Figure 4, we observe a similar behaviour as for LeNet: high accuracies are reached even when initializing weights from asymmetric distributions. However, convolutional networks are more sensitive to large asymmetries in the sign distributions.

3 Sign flipping

Consider a small standard neural network with no biases as shown in Figure 5. For this simple setup one can easily work out the full propagation through the network. For a given node, e.g. $a_0$, the forward propagation is written as $a_0 = \sigma \left( x_0 W_{x_0} + x_1 W_{x_1} \right)$ where $\sigma$ is the activation function. If we choose ReLU as an activation function then we have the advantage that it can also be written as $\sigma(x) = x H(x)$ where $H$ is the unit step function. With this in mind we can rewrite $a_0 = \left( x_0 W_{a_0} + x_1 W_{a_1} \right) H_{a_0}$ where we used $H_{a_0}$ as a short notation for the step function applied at $a_0$. It can be shown that an output node, e.g. $c_0$, can be expressed as:

$$c_0 = x_0 \left[ W_{x_0 a_0} W_{a_0 b_0} W_{b_0 c_0} H_{a_0} H_{b_0} + W_{x_0 a_1} W_{a_1 b_0} W_{b_0 c_0} H_{a_1} H_{b_0} + \right.$$  (*)

$$\left. W_{x_1 a_0} W_{a_0 b_1} W_{b_1 c_0} H_{a_0} H_{b_1} + W_{x_1 a_1} W_{a_1 b_1} W_{b_1 c_0} H_{a_1} H_{b_1} + \right.$$  (**)

$$x_1 \left[ W_{x_1 a_0} W_{a_0 b_0} W_{b_0 c_0} H_{a_0} H_{b_0} + W_{x_1 a_1} W_{a_1 b_0} W_{b_0 c_0} H_{a_1} H_{b_0} + \right.$$  (***)

$$\left. W_{x_1 a_0} W_{a_0 b_1} W_{b_1 c_0} H_{a_0} H_{b_1} + W_{x_1 a_1} W_{a_1 b_1} W_{b_1 c_0} H_{a_1} H_{b_1} \right]$$

We can observe the following pattern: the value of the output node depends on the input values $x_0$ and $x_1$, each multiplied by the sum of terms corresponding to all paths which connect these nodes. Each path has an associated coefficient $H$, which is either 0 or 1, depending on the weighted sum at each intermediate node along the path. A more detailed derivation of the equations in this section is given in the supplementary material.

We will apply this formalism to our constrained model where weights of the neural network are drawn from the Signed He Constant distribution, i.e. each layer has its own $\pm u_i$ value for the weights. This means that each term in Eq. (*) corresponding to a path has essentially the same magnitude but with a different sign. We can replace the product of all three weights along a path with $W$ and pull out the sign of the product in a separate factor $S_{i,j,k}$ which represents the sign of the path.
from input node $x_i$ to output node $c_0$ through the intermediate hidden nodes $a_j$ and $b_k$. As such $W^{x_i}_a W^a_{j/k} W^b_{b_0} = |W| S_{i,j,k}$. The value of $W$ is determined by the network architecture as shown in Eq. (6). Rewriting all terms in Eq. (7) in the previous manner we obtain:

$$c_0 = W x_0 [S_{000} H_{a_0} H_{b_0} + S_{010} H_{a_1} H_{b_0} + S_{001} H_{a_0} H_{b_1} + S_{011} H_{a_1} H_{b_1}] + W x_1 [S_{100} H_{a_0} H_{b_0} + S_{110} H_{a_1} H_{b_0} + S_{101} H_{a_0} H_{b_1} + S_{111} H_{a_1} H_{b_1}]$$

The above equation can be extended for networks with arbitrary depth and width where the number of terms scales with the product of the number of nodes in each layer, $\prod_{i=1}^L n_i$.

Empirically we know that, in general, many weights can be set to zero while keeping the network performance at the same level. If we consider a network with a fixed set of removed connections we have essentially a network with many paths of zero contribution to the sum in Eq. (8). One can observe that the contribution of sets of paths can also become zero if the signs of $S_{i,j,k}$ are carefully chosen such that the sum of terms in these sets is effectively zero.

Therefore, we hypothesise that, analogously to how pruning can be used to train networks by setting to zero the values of some paths in Eq. (8), networks can also be trained by changing the signs of paths, since such changes may lead to some paths cancelling the contributions of others, which is equivalent to pruning weights associated to all these sets of paths.

We have verified this hypothesis by replacing the masking function $m(t)$ with a simpler function, $f(t) = \text{sign}(t)$, such that instead of a binary mask $M$ with elements from \{0, 1\} we obtain a filter $F$ with elements from \{-1, 1\} which flips the sign of a weight. The training procedure is the same as before: the weights are kept at their fixed, randomly initialized values and we train only a variable $t$ which is passed to the masking function.

By analogy to the pruning technique, we perform either free flipping where the network is flipping as many signs as needed in order to minimize the cross-entropy loss, or minimal flipping where a regularization term is added in order to constrain the network to flip as few weight signs as possible.

Figure 6 shows the performance achieved for LeNet-300 on MNIST. We found that flipping the signs of the weights drawn from all three distributions works at least as good as when training all weights of the network in a classical manner. It is also superior to the pruning mechanism in both variants, free pruning as well as minimal pruning.

For Conv6 (see Figure 7) the accuracy is equal to or higher than the baseline for the He and He Constant initializations.

\section{Conclusion and discussion}

We have presented a simple and straightforward method for training the connectivity graph of a randomly initialized neural network directly through back-propagation, without ever training the weights, even if the weights have constant magnitude. With our algorithms we can either switch
on and off the connections between neurons or flip their signs. Our novel methods do not need hyperparameters defining cutoff thresholds, which removes the need of searching optimal values for such hyperparameters. Both methods yield very good results compared to training all weights of a network, in some cases even outperforming it, with free flipping of connection signs. We have also shown that it is possible to train a neural network connectivity graph essentially with a single weight which can be absorbed in the input data at training time, while at inference time the weight becomes irrelevant to the classification accuracy. We achieved good results even when the weights are drawn from very skewed distributions.

It is not clear why deep neural networks generalize so well despite having far more trainable model parameters than the number of samples they are trained on [16]. According to the Minimum Description Length principle [5], generalization capacity is correlated with a short size of the computer program performing the classification. In the case of a neural network, the program includes the actual computational steps needed for running the inference as well as any needed parameters or subprograms for generating these parameters. Our results contribute to finding ways of reducing the program length of deep neural networks. As also mentioned in [17], if a network uses randomly initialized, untrained weights, then the weights do not need to be stored and may be represented as just a random number generator and its seed. In such a case, just a representation of the connectivity mask has to be integrated in the program. With our sign flipping training, using constant weights, the weights do not have to be represented at all, and just the filter $F$ of one bit signs has to be represented in the program. With our minimal pruning or minimal flipping training procedures, we can generate sparse masks or filters, which also have compressed representations. Jointly with previous results that show that, through pruning, the number of weights in a classical neural network can be reduced significantly without loss of performance [3], our results suggest that classical deep neural networks are indeed over-parameterized, and that, through pruning or through alternative training methods like the sign flipping introduced here, their program size can be reduced to an effective, shorter one, which may explain their generalization power.
5 Acknowledgements

This work was supported by the European Regional Development Fund and the Romanian Government through the Competitiveness Operational Programme 2014–2020, project ID P_37_679, MySMIS code 103319, contract no. 157/16.12.2016.

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Appendices

A  Experimental setup

Table A.2 lists the architectures for all networks as well as the optimizer and learning rates we used for each experiment and training/pruning methods.

| Model  | Dataset | LeNet | Conv2 | Conv4 | Conv6 |
|--------|---------|-------|-------|-------|-------|
|        | MNIST   | CIFAR10 | CIFAR10 | CIFAR10 | CIFAR10 |
| Conv Layers | None | 2x64, pool | 2x64, pool | 2x64, pool | 2x64, pool |
| FC Layers  | 300, 100, 10 | 256, 256, 10 | 256, 256, 10 | 256, 256, 10 | 256, 256, 10 |
| Batch size | 25 | | | | |
| Optimizer  | Adam | | | | |
| **Learning rates** | | | | | |
| Baseline training | $10^{-3}$ | $2 \cdot 10^{-4}$ | $3 \cdot 10^{-4}$ | $3 \cdot 10^{-4}$ | |
| Free pruning | $10^{-3}$ | $3 \cdot 10^{-3}$ | $3 \cdot 10^{-3}$ | $3 \cdot 10^{-3}$ | |
| Minimal pruning | $10^{-3}$ | $3 \cdot 10^{-3}$ | $3 \cdot 10^{-3}$ | $3 \cdot 10^{-3}$ | |
| Free flipping | $10^{-3}$ | $5 \cdot 10^{-4}$ | $5 \cdot 10^{-4}$ | $5 \cdot 10^{-4}$ | |
| Minimal flipping | $10^{-3}$ | $5 \cdot 10^{-4}$ | $5 \cdot 10^{-4}$ | $5 \cdot 10^{-4}$ | |
\[ a_0 = \sigma \left( x_0 W_{x_0}^{x_0} + x_1 W_{x_1}^{x_1} \right) \quad b_0 = \sigma \left( a_0 W_{a_0}^{a_0} + a_1 W_{a_1}^{a_1} \right) \quad c_0 = b_0 W_{b_0}^{b_0} + b_1 W_{b_1}^{b_1} \]  
(9)

\[ a_1 = \sigma \left( x_0 W_{a_0}^{x_0} + x_1 W_{a_1}^{x_1} \right) \quad b_1 = \sigma \left( a_0 W_{b_0}^{a_0} + a_1 W_{b_1}^{a_1} \right) \quad c_1 = b_0 W_{c_0}^{b_0} + b_1 W_{c_1}^{b_1} \]  
(10)

where \( \sigma(x) \) is a non-linear activation function. Note that for the output nodes \( c_i \) we do not apply an activation function, we are only interested in their values. Choosing \( \sigma(x) = \max(0, x) \) we have the convenient property that \( \sigma(x) = xH(x) \), where \( H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases} \) is the step function.

Thus we can rewrite the output for the nodes \( a_i \) and \( b_i \) as:

\[ a_0 = (x_0 W_{a_0}^{x_0} + x_1 W_{a_1}^{x_1}) H_{a_0} \quad a_1 = (x_0 W_{a_1}^{x_0} + x_1 W_{a_1}^{x_1}) H_{a_1} \]  
(11)

\[ b_0 = (a_0 W_{b_0}^{a_0} + a_1 W_{b_1}^{a_1}) H_{b_0} \quad b_1 = (a_0 W_{b_1}^{a_0} + a_1 W_{b_1}^{a_1}) H_{b_1} \]  
(12)

where \( H_{a_i} \) and \( H_{b_i} \) are short notations for the step function applied at \( a_i \) and \( b_i \) in order to avoid rewriting each time the long expression for the argument of \( H \). Replacing \( a_i \) in \( b_i \) we obtain:

\[ b_0 = (x_0 W_{a_0}^{x_0} + x_1 W_{a_1}^{x_1}) H_{a_0} W_{b_0}^{a_0} + (x_0 W_{a_1}^{x_0} + x_1 W_{a_1}^{x_1}) H_{a_1} W_{b_0}^{a_1} \]  
(13)

\[ b_1 = (x_0 W_{a_0}^{x_0} + x_1 W_{a_1}^{x_1}) H_{a_0} W_{b_1}^{a_0} + (x_0 W_{a_1}^{x_0} + x_1 W_{a_1}^{x_1}) H_{a_1} W_{b_1}^{a_1} \]  
(14)

We can regroup terms and give \( x_0 \) and \( x_1 \) common factors:

\[ b_0 = \left[ x_0 (W_{a_0}^{x_0} W_{b_0}^{a_0} H_{a_0} + W_{a_1}^{x_0} W_{b_1}^{a_1} H_{a_1}) + x_1 (W_{a_0}^{x_1} W_{b_0}^{a_0} H_{a_0} + W_{a_1}^{x_1} W_{b_1}^{a_1} H_{a_1}) \right] H_{b_0} \]  
(15)

\[ b_1 = \left[ x_0 (W_{a_0}^{x_0} W_{b_1}^{a_0} H_{a_0} + W_{a_1}^{x_0} W_{b_1}^{a_1} H_{a_1}) + x_1 (W_{a_0}^{x_1} W_{b_0}^{a_0} H_{a_0} + W_{a_1}^{x_1} W_{b_1}^{a_1} H_{a_1}) \right] H_{b_1} \]  
(16)

Replacing \( b_i \) in \( c_i \), multiplying and giving \( x_0 \) and \( x_1 \) common factors results in:

\[ c_0 = x_0 \left[ W_{a_0}^{x_0} W_{b_0}^{a_0} W_{c_0}^{b_0} H_{a_0} H_{b_0} + W_{a_1}^{x_0} W_{b_1}^{a_1} W_{c_0}^{b_0} H_{a_1} H_{b_0} + W_{a_0}^{x_1} W_{b_0}^{a_0} W_{c_0}^{b_1} H_{a_0} H_{b_1} + W_{a_1}^{x_1} W_{b_1}^{a_1} W_{c_0}^{b_1} H_{a_1} H_{b_1} + x_1 \left[ W_{a_0}^{x_1} W_{b_0}^{a_0} W_{c_0}^{b_0} H_{a_0} H_{b_0} + W_{a_1}^{x_1} W_{b_0}^{a_1} W_{c_0}^{b_0} H_{a_1} H_{b_0} + W_{a_0}^{x_1} W_{b_1}^{a_0} W_{c_0}^{b_1} H_{a_0} H_{b_1} + W_{a_1}^{x_1} W_{b_1}^{a_1} W_{c_0}^{b_1} H_{a_1} H_{b_1} \right] \right] \]  
(17)
C Full comparison between algorithms

The figures in this section indicate the performance of the free/minimal pruning and flipping algorithms compared to the baseline for all networks. The learning rates for each algorithm are listed in table A.2.

Figure C.1: LeNet on MNIST with different initializations.

Figure C.2: Conv6 on CIFAR10 with different weight initializations.
Figure C.3: Conv4 on CIFAR10 with different weight initializations.

Figure C.4: Conv2 on CIFAR10 with different weight initializations.