A global version of Günther’s polysymplectic formalism using vertical projections

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I construct a global version of the local polysymplectic approach to covariant Hamiltonian field theory pioneered by C. Günther. Beginning with the geometric framework of the theory, I specialize to vertical vector fields to construct the (poly)symplectic structures, derive Hamilton’s field equations, and construct a more-or-less natural Poisson bracket. I then examine a few key examples to determine the nature of the necessary vertical projections and find that the theory provides the geometric analog of the canonical transformation approach to covariant Hamiltonian field theory advanced by Struckmeier and Redelbach. I conclude with a few remarks about possible applications of this framework to the geometric quantization of classical field theories.

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I. INTRODUCTION

The study of covariant Hamiltonian field theory goes back at least to the pioneering works of Dedonder [6] and Weyl [27] in the 1930s. Geometric approaches to covariant Hamiltonian field theory really began to be pursued by Dedeker [7] and Goldschmidt and Sternberg [10] in the 1970s. This work was then taken up by Günther in the 1980s, and given a set of clear axiomatic foundations [15]. Though quite rigorous, Günther’s original paper only dealt with the case in which the fields are sections of a trivial vector bundle (that is, the local version of the theory). One transition to a global theory for Günther’s essential method was later provided by Carinena et al. in [5], and the method was more recently revived and re-formulated in more modern mathematical language by Munteanu et al. in [23]. In addition to these more-or-less direct extensions, Günther’s work has been quite influential in the move toward a differential geometric foundation for covariant Hamiltonian field theory generally, as noted in many papers detailing different perspectives (see [25], [12], and [16], to name just a few key examples).

Though Günther’s work was a major advance in the field, it was indeed only local in character, leaving an important gap that needed to be filled to provide a fully general treatment. In the extension of Carinena et al., the authors deviate substantially from the main path of Günther’s paper, deeming the approach that more closely follows Günther’s geometric foundation – the vector-valued polysymplectic structure – to be “artificial and unsatisfactory” [5]. Later expositions and continuations of Günther’s work like that of [23] also forgo its original geometric foundation in favor of other, more modern approaches (like k-symplectic structures). Though [3] provides a much needed update to the original at a much greater level of mathematical sophistication, to this author’s knowledge nowhere is Günther’s original polysymplectic structure in its original form used to produce the full range of phenomena of covariant Hamiltonian field theory in the general (global) case. The first goal of this paper is fill in this missing link.

In doing so, we will find the main critique of Carinena et al. in [5] to be correct: the

1 A few comments made by those authors just before and after the quoted text shows that they had understood several of the main points of this work. However, they seem never to have presented the details of that work.
path from the starting fiber bundle through the polysymplectic structure to Hamilton’s field equations, though feasible and well-defined, does indeed require extra structure beyond the original polysymplectic or standard multisymplectic approaches that make it seem artificial. Against this is (somewhat subjective) demerit, we must weigh the payoff of this approach: a directly and naturally defined Poisson bracket on generic functions on the phase space of the theory. Some of the common difficulties that surround the construction of natural, well-defined Poisson brackets in covariant Hamiltonian field theory (see, for example, [8]) are bypassed in this approach. Indeed, the outcome of the polysymplectic structure path seems to be the almost exact duplication of the natural results of the canonical transformation approach to covariant Hamiltonian field theory presented by Struckmeier and Redelbach in [26]. A second perspective on this paper, then, is that it presents a natural geometric setting for [26] that directly reproduces the most of the main results of that paper.

One of the longstanding hopes for covariant Hamiltonian field theory is that it will provide a path for the geometric quantization of classical fields [1], [17], [8] (but note also [11]) and/or especially challenging classical particle systems [14], [9]. From this point of view a third, more aspirational perspective on the paper is that it presents another geometric framework for covariant Hamiltonian field theory, one which is at least superficially rather different from many of the other modern approaches and which may therefore prove itself to be more applicable to the as-yet-unsolved problem of the geometric quantization of classical fields. I will make some preliminary remarks on this possibility in the conclusion. A more detailed exposition is planned in a forthcoming paper.

Given these three possible motivations for readers, I have tried to make my treatment as accessible as possible to as wide an audience as possible. I have avoided specialized or esoteric language and operations unless absolutely necessary, and I have explained such non-standard operations or language as I find it essential to employ. Though I give coordinate independent definitions of all structures and operations, I also provide local coordinate descriptions of all results, and the paper can be followed by looking only at those local coordinate results (though in this case much of the motivation will be lost). It is my hope that a reader with a solid grasp of the fundamentals of differential geometry and elementary variational calculus will be able to follow the entire paper.
In a field with so many different conventions and notations, a few remarks are in order about which ones I will adopt throughout the paper. Given a fiber bundle $\epsilon : E \rightarrow M$, I will denote the jet bundle simply as $JE$ instead of $J^1E$ (I will not use higher order jet bundles in this paper), and the vector bundle upon which $JE$ is modeled will be called $JE$. Given a map $f : M \rightarrow N$, I will denote the tangent map $Tf : TM \rightarrow TN$ as $Tf$ rather than $f_*$. I will use the notation $\Gamma(M, E)$ to denote sections of a fiber bundle $E$ with projection map $\epsilon : E \rightarrow M$, and will often use notation like $E_x$ to denote the part of one geometric structure (in this case the fiber of $E$) that lies over a specific point of another (in this case the point $x \in M$). I will use the symbol $X \lrcorner \omega$ to denote the interior product of the vector-field $X$ and the differential form $\omega$. I make use of fibered local coordinate systems throughout the paper, but nothing of the geometry is changed by using non-fibered coordinate systems (only the local coordinate descriptions). The Einstein summation convention is employed throughout.

II. A TENSOR ALGEBRA APPROACH TO GÜNTHER’S FORMULATION

Before beginning to embellish Günther’s original, flat-space approach, it makes sense to first restate it in the language of tensorial structures similar to those I will eventually use in the non-flat case. This will serve both as a reminder of the main line of reasoning in Günther’s paper and its results, as well as an introduction to the tensorial approach I will use. None of the material in this section is at all new; it is only a restatement of some of the main symplectic structures of [15] in slightly different terms.

As with the geometric approach to Hamiltonian particle theory, in Günther’s approach the parameter space of the theory is suppressed at the beginning. The approach begins directly with the space $Q$ in which the physical fields under consideration take their values. This space is taken to be a differentiable manifold with local coordinates $\{q^I\}$.

The next step is to form the phase space $P$. In geometric Hamiltonian particle theory, this would simply be the cotangent bundle $T^*Q$. However, in field theory the fact that the parameter space – suppressed though it may be at this stage – is no longer one dimensional means that care must be taken to produce enough momenta to encompass the covariant
dynamics of the theory. To do this, the space is instead taken to be the bundle $P$ over $Q$ with standard fiber

$$P_q = T_q^* \otimes \mathbb{R}^n$$

This bundle comes with a standard projection map $\pi : P \to Q$ and coordinates $\{q^I, p^i_I\}$, where the capital letter index $I$ runs over field degrees of the freedom and the lower case index $i$ runs over parameter space dimensions.

Given an element $v \in TP$, the tautological vector-valued one-form $\theta$ is defined point-wise by:

$$\theta_p := p \circ T_p \pi (u)$$

where $T_p \pi : TP \to TQ$ is the differential of the projection map $\pi$. In fibered coordinates $\{q^I, p^i_I, v^I, v^i_I\}$ on $TP$, this map is given by

$$\theta_p = p^i_I dq^I \otimes e_i$$

where the $\{e_i\}$ are basis vectors for $\mathbb{R}^n$. The polysymplectic form $\omega$ is then given by

$$\omega := -d\theta$$

(The fact that this approach uses $\mathbb{R}^n$ – a single, fixed vector space – as the space in which the vector parts of $\theta$ take their values means that the exterior derivative here gives no trouble. See [15] for details.) In fibered coordinates on $TP$, this tensor is given by

$$\omega = dq^I \wedge dp^i_I \otimes e_i$$

Given a Hamiltonian $H : P \to \mathbb{R}$ that encodes the dynamics of the theory, a map $\gamma : \mathbb{R}^n \to P$ represents a physically realized field configuration if and only if it satisfies the condition

$$\omega(\iota \circ T \gamma) = dH$$

over every point $p \in \text{Im} \gamma$. Here $T_r \gamma : T_r \mathbb{R}^n \to T_{\gamma(r)} P$ is the differential of the map $\gamma$, and the natural identification $\iota_r : T_r \mathbb{R}^n \to \mathbb{R}^n$ of each tangent space $T_r \mathbb{R}^n$ with the vector space $\mathbb{R}^n$ is used when contracting the differential of $\gamma$ with the polysymplectic form. In the fibered coordinates used above, this condition amounts to the two equations

$$\frac{\partial \gamma^I}{\partial x^i} = \frac{\partial H}{\partial p^I_i}, \quad \frac{\partial \gamma^i_I}{\partial x^i} = -\frac{\partial H}{\partial q^I}$$
which are Hamilton’s field equations.

Formulated this way, it is not hard to see that one way to extend Günther’s approach to the case in which the parameter space \( \Lambda \) is not simply \( \mathbb{R}^n \) is to switch to a setup in which the map \( \iota : T\Lambda \to \mathbb{R}^n \) is some other isomorphism, and the fixed vector space \( \mathbb{R}^n \) matches the dimension of the tangent spaces \( T\Lambda \). Giving the most natural possible structures to make this generalization possible is the goal of the next two sections.

III. AN EXTENSION OF GÜNTHER’S APPROACH TO THE NON-FLAT CASE

My extension of Günther’s theory begins with a fiber bundle \( \epsilon : E \to M \) with total space \( E \), base manifold \( M \), projection map \( \epsilon : E \to M \), and standard fiber \( Q \). The base manifold \( M \) represents the space-time of our theory (typically represented by \( \mathbb{R}^4 \), but which we will assume is merely locally diffeomorphic to \( \mathbb{R}^n \) for generality), while the space \( Q \simeq E_x := \epsilon^{-1}(x) \) represents the space in which the physical field takes its values (typically \( \mathbb{R}^N \)). The total space \( E \), therefore, represents possible pairings between space-time values and field values; in other words, all possible field configurations over space-time. \( E \) is the extended configuration space of our theory. Local fibered coordinates on \( E \) are given by

\[
\{x^\alpha, \phi^I\} : E \to \mathbb{R}^{n+N} \ | \ e \mapsto x^\alpha e_\alpha + \phi^I e_I
\]

The next step is to construct the vector bundle

\[
V := V_E \otimes_E T^*M \simeq JE
\]

This vector bundle is isomorphic to the linearized first jet bundle over \( E \); that is, the vector bundle upon which the ordinary jet bundle \( JE \) is modeled. The first jet bundle \( JE \) (an affine bundle over \( E \)) is the foundation for most discussions of covariant Hamiltonian field theory (see \[8\] for notation and details). More specifically, this bundle is defined to be the bundle over \( E \) with fibers \( V_\epsilon E \otimes_E T^*_\epsilon(e)M \), where \( V_E \) is the vertical bundle over \( E \) (see, for example, \[8\] and \[12\]) with its fibers defined by \( V_\epsilon E := \{u \in T_e E \ | \ T_e \epsilon(u) = 0\} \), and \( \epsilon(e) \) is the base space point over which \( e \in E \) lies. We note, since it will be relevant later, that this bundle can be re-interpreted as a vector bundle over \( M \) rather than \( E \).
The bundle that forms the foundation for the geometric structures of this paper is really the dual of this bundle:

\[ P := V^*E \otimes_E TM \]

This bundle has standard fiber \( P_e := V^*_eE \otimes E T_{(e)}M \). Like the previous vector bundle, it can be interpreted either as a bundle over \( E \) with projection map \( \pi : P \to E \) or as a bundle over \( M \) with projection map \( \epsilon \circ \pi : P \to M \). We will make extensive use of both of these projections in defining the structures of our theory and their action on physical fields.

\( P \) represents possible combinations of three things: 1) a point in space-time, 2) the values of all components of a physical field, and 3) a conjugate momentum value to each field value for each dimension of space-time. \( P \) is the extended phase space of our theory. Local coordinates on \( P \) that are compatible with a coordinate system \( \{ x^\alpha, \phi^I \} \) on \( E \) are given by

\[ \{ x^\alpha, \phi^I, \pi^\alpha_I \} : P \to \mathbb{R}^{n+N+nN} \mid p \mapsto x^\alpha e_a + \phi^I e_I + \pi^\alpha_I d\phi^I \otimes \frac{\partial}{\partial x^\alpha} \]

Note that this space is almost identical to the one considered by Günther in [15], except that 1) the base space \( M \) (which I called \( \Lambda \) in the section [II]) is not suppressed and 2) the poly-momenta take their vector-values in each \( T_{(e)}E \), rather than a single copy of \( \mathbb{R}^n \). These two seemingly innocuous changes will turn out to make a good deal of difference.

To define the tautological tensor \( \theta : TM \to T^*P \) (a vector-valued one-form, as in section [II]) it is necessary to first consider an element of the space \( TP \otimes_p T^*M \). Given such an element \( u = v \otimes \alpha \), with \( v \in T_pP \) and \( \alpha \in T^*_{\epsilon \circ \pi(p)(e)}M \), the goal is to act upon it with the point \( p \) – considered as an element of the space \( P_e \); that is, as a map \( p : VP \otimes T^*M \to \mathbb{R} \) – as in section [III]. Certainly map \( p \) cannot be applied to \( u \) without modification. The first step, as before, is to apply \( T_p\pi : T_pP \to T_{\pi(p)}E \) by extending it to the tensor product \( TP \otimes T^*M \). (Since push-forwards by projection maps and identity maps on vector spaces are linear, the universal property of the tensor product [2] guarantees that the map \( T\pi : TP \to TE \) can be uniquely extended to \( TP \otimes T^*M \).) But the result is an element of \( TE \), not \( V E \), a natural consequence of the fact that I have not suppressed the parameter space \( M \) as in section [III]. To map this element \( u \) appropriately requires additional structure, namely a vertical projection (more abstractly called an Ehresmann connection) on \( TE \). This is a map \( V_E : TE \to VE \)
such that 1) $VE = \text{Im } V_E$ and 2) $V^2_E = V_E$. Given such a map, $\theta$ can be defined by

$$\theta_p(u) := p \circ V_E \circ T\pi(u)$$  \hspace{1cm} (1)

As a linear map from $TP \otimes T^*M \to \mathbb{R}$, $\theta$ is naturally a section of $T^*P \otimes TM$. The exact coordinate representation of $\theta$ depends upon the vertical projection $V_E$. Choosing an arbitrary set of local fibered coordinates for $E$ gives $V_E : TE \to VE \mid u^\alpha \frac{\partial}{\partial x^\alpha} + u^I \frac{\partial}{\partial x^I} \mapsto (V^I_\alpha u^\alpha + u^I) \frac{\partial}{\partial x^I}$, so the result for $\theta$ is

$$\theta_p = \pi^\alpha_I d\phi^I \otimes \frac{\partial}{\partial x^\alpha} + \pi^\alpha_I V^I_\beta dx^\beta \otimes \frac{\partial}{\partial x^\alpha}$$

where the $V^I_\alpha$ are the (arbitrary) coefficients of the vertical projection $V_E$. In the case that these coefficients are 0 (as is natural in the case where $E = \mathbb{R}^n \times Q$) the result is exactly the same as in section II

$$\theta_p = \pi^\alpha_I d\phi^I \otimes \frac{\partial}{\partial x^\alpha}$$

I will call local coordinates in which $\theta = \pi^\alpha_I d\phi^I \otimes \frac{\partial}{\partial x^\alpha}$ canonical coordinates 2.

Since $\theta$ no longer takes its values in a single vector space but instead in the many tangent spaces $T_mM$, it does not have a canonical exterior derivative. Indeed, given its rather strange pedigree, defining any exterior derivative on $\theta$ is somewhat challenging. (It is not even a vector-valued exterior form in the usual sense, since our vector fields are sections of $TM$, rather than $P$ or $TP$). Based on the fact that the parameter space was suppressed throughout section II, we expect that the polysymplectic structure should be vertical in some sense. Indeed, something similar to Günther’s basic approach can be made to work with some modification.

Given any two vector fields $u, v \in \Gamma(P, TP)$ and any one-form $\beta \in \Gamma(M, T^*M)$, define the (non-tensorial) map $\omega$ via

$$\omega(u, v, \beta) := -d(\theta \lrcorner \beta)(u, v)$$

where $\lrcorner$ denotes the contraction of the contravariant part of the tensor on the left with the covariant components of the form on the right. Though this is a well-defined map, it

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2 Though this terminology matches that of many works on symplectic geometry, it does not match [15]. I use it because it seems to be the most sensible definition in this particular extension.
is not a tensor: the result of \( d(\theta \lrcorner \beta) \) depends upon the particular one-form \( \beta \), whereas a polysymplectic structure should be multi-linear. But looking at things in local coordinates makes it clear that defining \( \omega \) only on vertical vector fields makes this structure unique:

\[
-d(\theta \lrcorner \beta) = -d(\beta_\alpha \pi_I^\alpha d\phi^I + \beta_\alpha \pi_I^\alpha \beta V_I^\beta dx^\beta) = \\
\beta_\alpha d\phi^I \wedge d\pi_I^\alpha + \beta_\alpha V_I^\beta dx^\beta \wedge d\pi_I^\alpha + (\pi_I^\beta \frac{\partial \beta_\alpha}{\partial x^\gamma} - \pi_I^\alpha \beta_\alpha \frac{\partial V_I^\beta}{\partial \phi^J}) d\phi^I \wedge dx^\gamma + (\pi_I^\alpha \beta_\alpha \frac{\partial V_I^\beta}{\partial x^\gamma}) dx^\beta \wedge dx^\gamma
\]

so inputting two vertical vector fields \( u, v \in \Gamma(P, VP) \) gives

\[
-d(\theta \lrcorner \beta)(u,v) = \beta_\alpha (u_\ell^\alpha v_I^I - u_I^I v_\ell^\alpha)
\]

Therefore, in any local canonical fibered coordinates where \( \theta \) can be represented as \( \theta = \pi_I^\alpha d\phi^I \otimes \frac{\partial}{\partial x^\alpha} + \pi_I^\alpha V_I^\beta dx^\beta \otimes \frac{\partial}{\partial x^\alpha} \), the foregoing means that we can write the resulting tensor \( \omega_p : V_pP \times V_pP \times T^{\ast}_{\pi_\circ(p)} M \to \mathbb{R} \) as

\[
\omega = d\phi^I \wedge d\pi_I^\alpha \otimes \frac{\partial}{\partial x^\alpha}
\]

which is directly analogous to what was found in section III. Note that this (local) result is independent of the particular vertical projection \( V_E \) used to define the tautological tensor of eq. (I). In the end, the apparent complication of this additional geometric structure does not affect the local physical result.

To make contact with physics, the last step is to introduce a Hamiltonian function \( H : P \to \mathbb{R} \) that encodes the dynamics of the theory. (Note that \( H \) can now depend explicitly on the spacetime coordinates \( x^\alpha \), since the base space \( M \) is incorporated into \( P \), unlike in section III.) It would be nice to proceed exactly as before: given a section \( \gamma : M \to P \) of \( P \), take the differential \( T_m \gamma : T_mM \to T_{\gamma(m)}P \), contract it with the polysymplectic structure \( \omega \), and demand that it match the exterior derivative of \( H \). However, this does not work because the domain of \( \omega \) is restricted to vertical vectors. Once again, the straightforward solution is to introduce a vertical projection \( V_P \) – this time on \( TP \) instead of \( TE \) – and to extend it to the tensor product \( TP \otimes T^\ast M \) as above. Given the polysymplectic structure defined above, physical solution sections \( \gamma \) can be identified as those that satisfy

\[
\omega(V_P \circ T\gamma, v) = dH(v) \tag{2}
\]
at every point \( p \in \text{Im} \gamma \), for all vertical vectors fields \( v \in \Gamma(P, VP) \). Here, \( H : P \to \mathbb{R} \) is the covariant Hamiltonian function. Just as in section III this function encodes the dynamical properties of the physical system under consideration. The exact coordinate representation of these sections now depends upon the vertical projection \( V_P \).

With all this structure now in place, it is worth noting that what looks like two new structures – the vertical projections \( V_E \) and \( V_P \) – need really only be one new structure. For every vertical projection \( V_P \) on \( TP \) determines a unique compatible vertical projection \( V_E \) on \( TE \) through the requirement that the following diagram commute:

\[
\begin{array}{ccc}
TP & \xrightarrow{V_P} & VP \\
\downarrow{T \pi} & & \downarrow{T \pi} \\
TE & \xrightarrow{V_E} & VE
\end{array}
\]

Again, what naively looks like two independent new structures is actually just one. In compatible local fibered coordinates on \( P \) and \( E \) in which \( V_P : TP \to TE \mid \ u^\alpha \frac{\partial}{\partial x^\alpha} + u^I \frac{\partial}{\partial \phi^I} + u_I^\alpha \frac{\partial}{\partial \pi^\alpha} \mapsto (V^I_\alpha u^\alpha + u^I) \frac{\partial}{\partial \phi^I} + (V^\alpha_\beta u^\beta + u^\alpha) \frac{\partial}{\partial \pi^\alpha} \), this just amounts to the requirement that \( V_E^I_\alpha = V_P^I_\alpha \).

Keeping in mind the fact that there is only a single vertical projection \( V_P = V \) to be specified and choosing local fibered coordinates in which \( V : TP \to VP \mid u^\alpha \frac{\partial}{\partial x^\alpha} + u^I \frac{\partial}{\partial \phi^I} + u_I^\alpha \frac{\partial}{\partial \pi^\alpha} \mapsto (V^I_\alpha u^\alpha + u^I) \frac{\partial}{\partial \phi^I} + (V^\alpha_\beta u^\beta + u^\alpha) \frac{\partial}{\partial \pi^\alpha} \), (2) amounts to the local fibered coordinate equations

\[
\frac{\partial \gamma^I_\alpha}{\partial x^\alpha} + V^I_\alpha = \frac{\partial H}{\partial \pi^\alpha_I} \quad (3)
\]

\[
\frac{\partial \gamma^\alpha_i}{\partial x^\alpha} + V^\alpha_i = -\frac{\partial H}{\partial \phi^i} \quad (4)
\]

so in the case that \( E = \mathbb{R}^n \times Q \) and the vertical projection is the natural flat one the result is exactly the same as in section III

\[
\frac{\partial \gamma^I_\alpha}{\partial x^\alpha} = \frac{\partial H}{\partial \pi^\alpha_I}
\]

\[
\frac{\partial \gamma^\alpha_i}{\partial x^\alpha} = -\frac{\partial H}{\partial \phi^i}
\]

which are Hamilton’s field equations.
However, if the vertical projection is not naturally flat – and generically it cannot be – then we do not recover Hamilton’s field equations unless we use a modified Hamiltonian function that now depends upon the vertical projection $V$ as well as the system under consideration:

$$H' := H - \pi^\alpha_\beta v_\alpha^I + \phi^I V^\alpha_{I\alpha}$$

This seems a very artificial way to produce the physically correct field equations. We shall find in section $\textbf{V}$ that there are other, more elegant ways to circumvent this problem in the cases of greatest physical interest.

### IV. POISSON BRACKETS

Poisson brackets are not strictly necessary for doing covariant Hamiltonian field theory, as evidenced by their absence from most versions [8]. However, as Hamiltonian vector fields and Poisson brackets play an essential role in the most straightforward geometric quantization methods for Hamiltonian particle theory, they may be essential to formulating a quantum counterpart to covariant Hamiltonian field theory. Therefore, I take the attitude that the formulation of a field theoretic counterpart to the Poisson bracket of Hamiltonian particle theory is an important component of any covariant Hamiltonian field theory.

To define a second symplectic tensor that is the counterpart to the Poisson structure in geometric Hamiltonian particle theory, I begin by noting that it is possible to associate with any function $f$ on $P$ a family of tensors $\mathcal{J}_f$, where each $s_f \in \mathcal{J}_f : TM \to VP$ must satisfy:

$$\omega(s_f, v) = v^\alpha_\beta s^I_\alpha - v^I s^\alpha_{I\alpha} = df(v)$$

and the relation is expected to hold for all vertical vector fields $v : P \to VP$.

In coordinates, these sections have components obeying the relations

$$s^I_\alpha = -\frac{\partial f}{\partial \pi^\alpha_\beta}, \quad s^\alpha_{I\alpha} = \frac{\partial f}{\partial \phi^I}$$

There is a family of sections rather than a single section associated with each function $f$ because the second relation specifies only the trace of the second set of coordinate functions $\sigma^\alpha_{I\beta}$, rather than specifying every coordinate function uniquely. In local fibered coordinates,
the sections \( s_f \) look like:
\[
s_f = -\frac{\partial f}{\partial \pi_I^\alpha} \frac{\partial}{\partial \phi^I} \otimes dx^\alpha + \frac{\partial f}{\partial \phi^I} \frac{\partial}{\partial \pi_I^\alpha} \otimes dx^\alpha + s_{TF} I_\alpha^\alpha \frac{\partial}{\partial \pi_I^\alpha} \otimes dx^\beta
\]
where the components \( s_{TF} \) (TF stands for “trace-free”) are arbitrary other than that they must obey the condition \( s_{TF} I_\alpha^\alpha = 0 \).

These families of sections allow us to start defining a new tensor \( \Sigma \in \Gamma(TP \otimes TP \otimes T^*M) \). This can be done by requiring that
\[
\Sigma(df) \in \mathcal{F}_f
\]
for all functions \( f \in C^\infty(P) \). In local fibered coordinates, this definition implies that
\[
\Sigma = -\delta_\beta^\alpha \frac{\partial}{\partial \pi^\alpha_I} \otimes \frac{\partial}{\partial \phi^I} \otimes dx^\beta + (\Sigma_{TF} J_\alpha^\beta + \delta_\beta^\alpha \delta_\alpha^\beta) \frac{\partial}{\partial \phi^J} \otimes \frac{\partial}{\partial \pi^\alpha_I} \otimes dx^\beta
\]
where once again the \( \Sigma_{TF} \) components are arbitrary except that they must obey \( \frac{\partial f}{\partial \phi^J} \Sigma_{TF} J_\alpha^\alpha = 0 \). If this tensor is required to be anti-symmetric in its first two arguments so that \( \Sigma(\alpha, \beta, v) = -\Sigma(\beta, \alpha, v) \forall \alpha, \beta \in TP, v \in TM \) (as is natural if it is to be in any sense an inverse of \( \omega \)), then in fact the components \( \Sigma_{TF} J_\alpha^\beta \) are all forced to zero. The tensor \( \Sigma \) is then given in local fibered coordinates as:
\[
\Sigma = \frac{\partial}{\partial \phi^I} \wedge \frac{\partial}{\partial \pi^\alpha_I} \otimes dx^\alpha
\]

The primary significance of \( \Sigma \) is that it allows us to define Poisson brackets in a natural and invariant manner: for any two functions \( f, g \in C^\infty(P) \), we define their Poisson bracket \( \{f, g\} \) by
\[
\{f, g\} := \Sigma(df, dg)
\]
In coordinates, this reads
\[
\{f, g\} = \left( \frac{\partial f}{\partial \phi^I} \frac{\partial g}{\partial \pi^\alpha_I} - \frac{\partial f}{\partial \pi^\alpha_I} \frac{\partial g}{\partial \phi^I} \right) dx^\alpha
\]
This means that the Poisson bracket of two functions is represented in this theory by a one-form over \( T^*M \). It arises more-or-less naturally in this geometric setting, with all the natural transformation properties of the Poisson bracket of [26].

It is worth noting that the existence of this structure is by no means guaranteed. Indeed, as mentioned at the beginning of this section, these physically desirable Poisson brackets are difficult or impossible to generate in many covariant Hamiltonian field theories [8]. The existence of such a bracket structure is one of the major successes of the theory.
V. APPLICATIONS AND REFINEMENTS

Note that equations (3) and (4) do not constrain all the derivatives \( \frac{\partial \gamma}{\partial x} \), only a particular sum of them (as is necessary to avoid over-determining the solution sections). Taken together, they serve to identify physically realizable field configurations (and their conjugate momenta) from un-physical configurations. The first equation defines the relationship between the conjugate momentum coordinates and derivatives of the field-value part of the solution section, while the second equation mimics the Euler-Lagrange equation of motion of the system when the constraints imposed by the first equation are satisfied. The specific field configuration taken by a physical system is then determined by imposing appropriate initial conditions and solving the system of partial differential equations. \cite{26} provides the details of how these field equations apply a wide range of examples of physical interest. As an indication of how this abstract geometric framework pertains to specific physical fields – and in particular how the unusual ramifications of eqs. (3) and (4) are to be understood – I consider a few simple examples here. Perhaps more importantly, I will show that an alternative approach allows us to side-step the complications of adding a vertical projection to the ordinary covariant Hamiltonian approach.

A. Scalar Field Theory

The first and simplest example of a classical field is a single, real-valued (i.e., uncharged) scalar field. I will now examine how this simplest case fits into the framework of the previous sections.

In the most basic case, the extended configuration space for the scalar field is \( E = \mathbb{R}^4 \times \mathbb{R} \). It carries global fibered coordinates \( \{ x^\alpha, \phi \} \) such that a point \( e \in E \) is given by \( e = x^\alpha e_\alpha + \phi e_\phi \). The appropriate extended phases space is then \( P := V^* E \otimes_E TM = \mathbb{R}^4 \times \mathbb{R} \times (\mathbb{R} \otimes \mathbb{R}^4) \) with global fibered coordinates \( \{ x^\alpha, \phi, \pi^\alpha \} \) such that a point \( p \in P \) is given by \( e = x^\alpha e_\alpha + \phi e_\phi + \pi^\alpha d\phi \otimes \frac{\partial}{\partial x^\alpha} \). The key symplectic structures are given in these global fibered coordinates by \( \theta = \pi^\alpha d\phi \otimes \frac{\partial}{\partial x^\alpha}, \omega = d\pi^\alpha \wedge d\phi \otimes \frac{\partial}{\partial x^\alpha} \), and (though not necessary for the present analysis) \( \Pi = \frac{\partial}{\partial \phi} \wedge \frac{\partial}{\partial \pi^\alpha} \otimes dx^\alpha \). Note that I have made use of the natural (flat) vertical projection in defining \( \theta \), and will make use of it again below to derive Hamilton’s field equations. Given the standard Klein-Gordon Hamiltonian \( H = \frac{1}{2} \eta_{\alpha\beta} \pi^\alpha \pi^\beta + \frac{1}{2} m^2 \phi^2 \)
(where $\eta$ is the Minkowski metric in the “mostly-minus” or “West coast” metric signature convention) and the fact that there is a natural (flat) vertical projection on $P$, eq. (2) implies that physically realizable sections $\gamma$ must obey

$$\frac{\partial \gamma}{\partial x^\alpha} = \frac{\partial H}{\partial \pi^\alpha} = g_{\alpha\beta} \pi^\beta$$

and

$$\frac{\partial \gamma^\alpha}{\partial x^\alpha} = -\frac{\partial H}{\partial \phi} = -m^2 \phi$$

Solving the first equation for $\pi^\alpha$, noting that $\frac{\partial \eta_{\alpha\beta}}{\partial x^\gamma} = 0$, and substituting into the second gives

$$\eta^{\alpha\beta} \frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} + m^2 \phi = 0 \quad (7)$$

which is the standard, flat space Klein-Gordon equation.

As a step toward more interesting results, consider a single, real-valued scalar field over a flat space-time (so that is still $E = \mathbb{R}^4 \times \mathbb{R}$), but with a non-flat metric; this is the situation usually encountered in general relativity. In this case, the flat vertical projection is still available, but it is important to be careful about deriving the correct Hamiltonian from the Lagrangian formulation. If we think of the classical action as $S = \int \mathcal{L} \, d^4x$, picking off

$$\mathcal{L} = \frac{1}{2\sqrt{-\det g}} (g^{\alpha\beta} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} - m^2 \phi^2)$$

as everything in the integrand except the top form $d^4x$, then the appropriate covariant conjugate momenta are

$$p^\alpha := \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial x^\alpha}} = \frac{g^{\alpha\beta} \frac{\partial \phi}{\partial x^\beta}}{\sqrt{-\det g}}$$

so that

$$H = p^\alpha \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} = \frac{1}{2} \sqrt{-\det g} \, g_{\alpha\beta} p^\alpha p^\beta + \frac{m^2 \phi^2}{2\sqrt{-\det g}}$$

and Hamilton’s field equations read

$$\frac{\partial \gamma}{\partial x^\alpha} = \frac{\partial H}{\partial \pi^\alpha} = \sqrt{-\det g} \, g_{\alpha\beta} \pi^\beta$$

and

$$\frac{\partial \gamma^\alpha}{\partial x^\alpha} = -\frac{\partial H}{\partial \phi} = -\frac{m^2 \phi}{\sqrt{-\det g}}$$

Solving the first equation for $p^\alpha$ and substituting it into the second give

$$0 = m^2 \phi + \sqrt{-\det g} \frac{\partial}{\partial x^\alpha} \left( \frac{1}{\sqrt{-\det g}} g^{\alpha\beta} \frac{\partial \phi}{\partial x^\beta} \right) = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + m^2 \phi \quad (8)$$
which is indeed the appropriate equation of motion for a single scalar field in a curved space-time.

But what happens when the underlying space-time is not \( \mathbb{R}^4 \) and one cannot access the flat vertical projection? In this formalism there are two options. The first is to begin with the same Lagrangian density as in the previous case, perform the appropriate Legendre transformation, choose an arbitrary vertical projection on the underlying space to find Hamilton’s equations, then finally alter the Hamiltonian function as noted at the end of section III. This has the advantage of being guaranteed to work not just in this special case but in any case, and the two disadvantages of 1) undoing the naturality of the Legendre transformation by amending its result and 2) being dependent – in construction, not in result – upon the specific vertical projection chosen. A third, more philosophical disadvantage is that the process seems quite artificial: the natural, necessary modification for the non-flat case is removed by hand because it is physically incorrect! (This third disadvantage is particularly damning here because there are so many other, seemingly more natural, alternatives.) However, there is an another approach that solves all three of these problems.

Let us begin again with the action, but in this case let us write it as:

\[
S = \int \text{vol } \mathcal{L} = \int \frac{1}{\sqrt{-\text{det } g}} d^4x \left( \frac{\hbar^2}{m} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \frac{\partial \phi}{\partial x^\mu} - m c^2 \phi^2 \right)
\]

Rather than use the ordinary separation of \( \mathcal{L} \) from \( d^4x \) used successfully above, let us consider what happens when we single out the Lagrangian differently, being careful to isolate \( \mathcal{L} \) not just from the top form \( d^4x \) but from the volume form \( \text{vol} = \frac{1}{\sqrt{-\text{det } g}} d^4x \). This will have important ramifications later.

The covariant momenta are then defined as the appropriate partial derivatives of the Lagrangian density \( \mathcal{L} \):

\[
p^\mu := \frac{\partial \mathcal{L}}{\partial \dot{\phi} \frac{\partial \phi}{\partial x^\mu}} = \frac{2\hbar^2}{m} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu}
\]

so that the standard construction for the Hamiltonian density gives

\[
\mathcal{H} := p^\mu \frac{\partial \phi}{\partial x^\mu} - \mathcal{L} = \frac{m}{4\hbar^2} g_{\mu\nu} p^\mu p^\nu + m c^2 \phi^2
\]
Given local coordinates in which we have

\[ p \in P = x^\mu e_\mu + \phi e_\phi + p^\mu d\phi \otimes \frac{\partial}{\partial x^\mu} \]

so that

\[ \omega = d\phi \wedge dp^\mu \otimes \frac{\partial}{\partial x^\mu} \]

we turn to the critical question of how to define the vertical projection. Let us take the very unusual approach of allowing each (prospective) solution section \( \gamma : M \rightarrow P \) to define its own vertical projection operator by

\[ V_\gamma = \Gamma^\mu_{\sigma \nu} x^\mu \otimes \frac{\partial}{\partial \phi} + (\partial p^\mu_{\sigma} + \Gamma^\mu_{\sigma \nu} p^\nu) \cdot d\phi \otimes \frac{\partial}{\partial x^\nu} \]

where the second equality comes from realizing that there is a canonical flat connection (just the exterior derivative \( d \)) on the scalar fields themselves, even though there is not on the conjugate momenta. Then the fact that

\[ T_\gamma = dx^\mu \otimes \frac{\partial}{\partial x^\mu} + \partial \phi \otimes \frac{\partial}{\partial \phi} + \partial p^\mu \otimes \frac{\partial}{\partial p^\mu} \]

gives the composition

\[ V_\gamma \circ T_\gamma = \frac{\partial \phi}{\partial x^\mu} dx^\mu \otimes \frac{\partial}{\partial \phi} + (\partial p^\mu_{\sigma} + \Gamma^\mu_{\sigma \nu} p^\nu) \cdot dx^\nu \otimes \frac{\partial}{\partial p^\mu} \]

so that Hamilton’s equations in the form of \( (\ref{eq:2}) \) read

\[ \omega(V_\gamma \circ T_\gamma, -) = \frac{\partial \phi}{\partial x^\mu} dp^\mu - (\partial p^\mu_{\sigma} + \Gamma^\mu_{\sigma \nu} p^\nu) \cdot d\phi \]

Given that

\[ d\mathcal{H} = \frac{m}{2\hbar^2} g_{\mu \nu} p^\nu dp^\mu + 2mc^2 \phi \cdot d\phi \]

the geometric form of Hamilton’s equations then implies that

\[ p^\mu = \frac{2\hbar^2}{m} g^{\mu \nu} \frac{\partial \phi}{\partial x^\nu} \]

and, simultaneously,

\[ \frac{\partial p^\mu}{\partial x^\mu} + \Gamma^\mu_{\sigma \nu} p^\nu + 2mc^2 = 0 \]

Putting the first of these into the second then gives

\[ \frac{\hbar^2}{m} g^{\mu \nu} \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu} - \frac{\hbar^2}{m} g^{\mu \nu} \Gamma^\mu_{\nu \sigma} \frac{\partial \phi}{\partial x^\sigma} + mc^2 \phi = 0 = \frac{\hbar^2}{m} g^{\mu \nu} \nabla_\mu \nabla_\nu \phi + mc^2 \phi \]
which are precisely the curved space-time analog of the Klein-Gordon equation \([7\).\]

This is certainly the right result, but it is important to understand where it has come from. By isolating the Lagrangian from the entire volume form, we have caused the Lagrangian (and therefore the Hamiltonian also) to “forget” about the (potentially) non-flat nature of the underlying space-time. The correction from a non-flat space-time then comes from the vertical projection operator used to give us our Hamiltonian equations of motion. If we had singled out the Lagrangian as everything except the top form \(d^4x\) instead, we would have over-corrected for the underlying space-time. We will see that a similar procedure works equally well for electrodynamics, too, but we will have to be equally careful to define the correct vertical projection operator.

### B. Electromagnetism

In the case of electromagnetism, the most basic extended configuration space is \(T^*\mathbb{R}^4 \simeq \mathbb{R}^4 \times \mathbb{R}^4\), so that \(P\) has global fibered coordinates \(\{x^\alpha, A_\alpha, p^{\alpha\beta}\}\) and a point \(p \in P\) is given by \(p = x^\alpha e_\alpha + A_\alpha dx^\alpha + p^{\alpha\beta} dA_\alpha \otimes \frac{\partial}{\partial x^\beta}\).

Given the Lagrangian density \(L = -\frac{1}{4} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} - g^{\alpha\beta} A_\alpha J_\beta\) (where \(F_{\alpha\beta} := \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\)) care must be taken in how one proceeds in order to successfully carry out the covariant Legendre transformation and recover the correct equations of motion; see [26]. One finds that

\[
p^{\alpha\beta} := \frac{\partial L}{\partial \frac{\partial A_\beta}{\partial x^\alpha}} = g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta}
\]

and so

\[
H = -\frac{1}{4} g_{\alpha\gamma} g_{\beta\delta} p^{\alpha\beta} p^{\gamma\delta} + g^{\alpha\beta} A_\alpha J_\beta
\]

Assuming a flat vertical projector, eq. (3) gives:

\[
\frac{\partial A_\alpha}{\partial x^\beta} = \frac{\partial H}{\partial p^{\alpha\beta}} = -\frac{1}{2} g_{\alpha\gamma} \beta \delta p^{\gamma\delta} = -\frac{1}{2} F_{\alpha\beta} = +\frac{1}{2} F_{\beta\alpha}
\]

so that we have

\[
F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}
\]

while eq. (4) gives:

\[
\frac{\partial p^{\alpha\beta}}{\partial x^\alpha} = \frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = g^{\alpha\beta} J_\beta
\]
which is the usual inhomogeneous Maxwell equation.

However, the alternative approach of section [VA] can be applied here, too, in order to extend this result to the non-flat case. This time, we must note that the use of the metric \( g_{\alpha \beta} \) in the definition of the Maxwell Lagrangian density gives us access to the unique Levi-Civita connection \( \nabla \) compatible with \( g \); I shall use this connection to define the appropriate vertical projection to use.

As before, we must isolate \( \mathcal{L} \) from the entire volume form, writing

\[
S = \int \text{vol } \mathcal{L} = \int \frac{1}{\sqrt{-\det g}} \, d^4x \left( \frac{1}{4} g^{\mu \rho} g^{\nu \sigma} F_{\mu \nu} F_{\rho \sigma} - \frac{4\pi}{c} g^{\mu \nu} A_\mu J_\nu \right)
\]

where we are using as before that the definition

\[
F_{\mu \nu} := \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}
\]

Then we have

\[
p^{\mu \nu} := \frac{\partial \mathcal{L}}{\partial g^{\mu \nu}} = g^{\mu \sigma} g^{\nu \rho} F_{\mu \nu}
\]

so that

\[
\mathcal{H} := p^{\mu \nu} \frac{\partial A_\nu}{\partial x^\mu} - \mathcal{L} = \frac{1}{2} p^{\mu \nu} \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) - \frac{1}{4} g^{\mu \rho} g^{\nu \sigma} F_{\mu \nu} F_{\rho \sigma} + \frac{4\pi}{c} g^{\mu \nu} A_\mu J_\nu
\]

\[
= -\frac{1}{4} p^{\mu \nu} F_{\mu \nu} + \frac{4\pi}{c} g^{\mu \nu} A_\mu J_\nu = -\frac{1}{4} g_{\mu \sigma} g_{\nu \rho} p^{\mu \rho} p^{\sigma \nu} + \frac{4\pi}{c} g^{\mu \nu} A_\mu J_\nu
\]

Given local fibered coordinates in which

\[
p \in P = x^\mu e_\mu + A_\mu dx^\mu + P^{\mu \nu} dA_\mu \otimes \frac{\partial}{\partial x^\nu}
\]

we have

\[
\omega = dA_\mu \wedge dP^{\mu \nu} \otimes \frac{\partial}{\partial x^\nu}
\]

Now we must define our vertical projection operator. As before, we will make use of the most natural possible connection on our space, inherited from the metric \( g_{\alpha \beta} \) and its associated Levi-Civita connection and the fact that the underlying fields \( A(x) = A_\mu dx^\mu \) are one-forms and therefore carry an exterior derivative that remains viable despite the non-flat underlying space-time. Letting each prospective solution section define its own vertical projection
operator as in $V\Lambda$ gives

$$V_\gamma = (\gamma_\sigma \Gamma^\sigma_{\mu \nu} + \gamma_\nu \Gamma^\nu_{\mu \rho}) \, dx^\nu \otimes \frac{\partial}{\partial A_\mu} + \left( \gamma^\rho \Gamma^\mu_{\rho \sigma} + \gamma^\mu \Gamma^\nu_{\rho \sigma} \right) \, dx^\sigma \otimes \frac{\partial}{\partial P^{\mu \nu}}$$

$$= 0 \, dx^\nu \otimes \frac{\partial}{\partial A_\mu} + \left( \gamma^\rho \Gamma^\mu_{\rho \sigma} + \gamma^\mu \Gamma^\nu_{\rho \sigma} \right) \, dx^\sigma \otimes \frac{\partial}{\partial P^{\mu \nu}}$$

Then the fact that

$$T_\gamma = \frac{\partial}{\partial x^\mu} \otimes dx^\mu + \frac{\partial A_\mu}{\partial x^\nu} \otimes dx^\nu + \frac{\partial P^{\mu \nu}}{\partial x^\sigma} \otimes \frac{\partial}{\partial P^{\mu \nu}}$$

gives us

$$\omega(V_\gamma \circ T_\gamma, -) = \frac{\partial A_\mu}{\partial x^\nu} P^{\mu \nu} - \left( \frac{\partial P^{\mu \nu}}{\partial x^\nu} + \rho \Gamma^\mu_{\rho \sigma} + \rho \Gamma^\nu_{\rho \sigma} \right) \, dA_\mu$$

Taking the exterior derivative of $\mathcal{H}$ gives

$$d \mathcal{H} = -\frac{1}{2} g^{\mu \sigma} g^{\nu \rho} F_{\mu \nu} \, dp^\rho + \frac{4\pi}{c} g^{\mu \nu} J_\nu \, dA_\mu$$

Then equation (2) gives us

$$-\frac{1}{2} g^{\mu \sigma} g^{\nu \rho} F_{\mu \nu} = \frac{\partial A_\mu}{\partial x^\nu}$$

and

$$- \left( \frac{\partial P^{\mu \nu}}{\partial x^\nu} + \rho \Gamma^\mu_{\rho \sigma} + \rho \Gamma^\nu_{\rho \sigma} \right) = \frac{4\pi}{c} g^{\mu \nu} J_\nu$$

The first equation is equivalent to

$$-\frac{1}{2} F_{\mu \nu} = \frac{\partial A_\mu}{\partial x^\nu} = +\frac{1}{2} F_{\nu \mu}$$

which is in turn equivalent to

$$\frac{1}{2} F_{\mu \nu} - \frac{1}{2} F_{\nu \mu} = F_{\mu \nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

which is equivalent to the definition of $p^{\mu \nu}$ from the Legendre transformation. (See [26] for the subtleties of this manipulation.) Meanwhile, the second of Hamilton’s field equations is equivalent to

$$\frac{\partial F_{\mu \nu}}{\partial x^\nu} + F_{\nu \rho} \gamma^\rho_{\mu \nu} = \nabla_\nu F^\rho_{\mu \nu} = \nabla_\nu F_{\mu \nu} = \frac{4\pi}{c} g^{\mu \nu} J_\nu = \frac{4\pi}{c} J^\mu$$

which is the curved space-time inhomogeneous Maxwell equation. Once again, the procedure of $V\Lambda$ works.
C. General Relativity?

Every covariant Hamiltonian formulation seems to struggle with the case of general relativity \[12\]. Fundamentally, this comes from the fact that the Legendre transformation of the Einstein-Hilbert Lagrangian density (considered as a function of the metric \( g \)) does not access all the information encoded in that function, as the Lagrangian density contains second derivatives of the metric which the Hamiltonian theory never becomes “aware” of. Many attempts have been made to solve this problem, for instance by introducing higher order analogs of the Legendre transformation \[20\] or by using a different mechanism to produce a covariant Hamiltonian \[24\] \[22\]. To my knowledge, none of these attempts has been completely successful. Though it is disappointing, it should therefore be no great surprise that the Legendre transformation fails in this formalism, too.

Beginning with the standard Einstein-Hilbert action but identifying \( S_{EH} = \int \text{vol} \ L = \frac{c^4}{16\pi G} \int \frac{d^4 x}{\sqrt{-\text{det} g}} R \)

To evaluate the covariant momenta, we need to recall that, in order of increasing complexity

\[
R = g^{\alpha\beta} R_{\alpha\beta}
\]

and

\[
R_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta}
\]

and

\[
R^\alpha_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\epsilon\gamma} \Gamma^\epsilon_{\beta\delta} - \Gamma^\epsilon_{\beta\gamma} \Gamma^\alpha_{\epsilon\delta}
\]

where

\[
\Gamma^\alpha_{\beta\gamma} = g^{\alpha\epsilon} \Gamma_{\epsilon\beta\gamma} = \frac{1}{2} g^{\alpha\epsilon} \left( -\frac{\partial g_{\beta\gamma}}{\partial x^\epsilon} + \frac{\partial g_{\epsilon\beta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\epsilon}}{\partial x^\beta} \right)
\]

Remembering that this is not a variational problem, we will need the identity

\[
\frac{\partial g^{\alpha\beta}}{\partial x^\gamma} = -g^{\alpha\delta} g^{\beta\epsilon} \frac{\partial g_{\delta\epsilon}}{\partial x^\gamma}
\]

Together will a great deal of index gymnastics, this chain of equalities gives

\[
\pi^{\zeta\eta} := \frac{\partial L}{\partial \dot{g}_{\alpha\beta}} = \frac{c^4}{16\pi G} \frac{\partial R}{\partial \dot{g}_{\alpha\gamma\beta}} = \frac{c^4}{16\pi G} g^{\beta\gamma} \frac{\partial R^\alpha_{\beta\gamma\delta}}{\partial \dot{g}_{\alpha\gamma\beta}}
\]

\[
= \frac{c^4}{32\pi G} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \times \left( -4 g^{\alpha\eta} g^{\beta\gamma} g^{\zeta\theta} + 2 g^{\alpha\zeta} g^{\beta\eta} g^{\gamma\theta} - 2 g^{\alpha\beta} g^{\gamma\theta} g^{\zeta\eta} + g^{\beta\gamma} g^{\gamma\eta} g^{\zeta\theta} + g^{\gamma\gamma} g^{\beta\theta} g^{\zeta\eta} \right)
\]
Unfortunately, this is not invertible to get $\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$ in terms of $\pi^{\zeta\eta}$. Even if it were, it would not give us access to the second derivatives $\frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}$ that appear in the $\frac{\partial \Gamma^\alpha}{\partial x^\gamma}$ terms in the Einstein-Hilbert action. The Legendre transformation fails.

In light of the aspirational perspective I mentioned in the introduction, this is a serious problem: one of the primary reasons to try to formulate a geometric approach to the quantization of classical fields is to feed in general relativity with the hope that the results will be better defined than those of the canonical approach. However, for this to be reasonable it is at minimum necessary that the classical theory be well defined within the framework used for quantization! The special case of general relativity is therefore an important area for future work within the field of polysymplectic Hamiltonian field theory.

VI. CONCLUSIONS

In this paper, I have presented a modified approach to polysymplectic, covariant Hamiltonian field theory that generates many of the key results of [26]—most importantly Hamilton’s field equations, the form of the Poisson bracket, and the covariant Hamiltonian—in a more-or-less natural manner. In addition, the path I have taken generalizes Günther’s approach in [15] to the global case in a manner that seems to me much more in keeping with Günther’s original approach and spirit than most recent work. This accounts for two of the three motivations I outlined in the introduction to justify this enterprise.

The third motivation I offered for this particular treatment of covariant Hamiltonian field theory was that it might make possible a novel approach to the geometric quantization of classical field theories. Though I have not yet successfully used this framework to reproduce all the results of even scalar quantum field theory, there are a few promising points worth mentioning. First, it is a relatively simple matter to reproduce the canonical commutation relations for standard particle theories using the geometric structures of this paper; indeed, the analysis can be successfully extended substantially farther, depending upon exactly which desirable properties one wishes the quantization procedure to reproduce [21]. (That it will never produce all the properties one might reasonably expect for all smooth functions on the phase space is a result of Groenewald’s theorem [13].) On the field theory side, this procedure does not produce the canonical commutation relations. However, it is
possible to reproduce the results of the canonical commutation relations _after integration_; see [22] for these preliminary results in the context of Minkowski space-time. This is a more reasonable expectation for a finite dimensional, differential geometric analysis of quantum field theory, as it is not clear how operator-valued distributions would ever arise from a well-defined differential geometric structure (see, for example, [18] and [19] for similar efforts, and [4] for a more closely related, though more mathematical, perspective). This seems a promising beginning, and a full analysis of the possibilities for geometric quantization from the perspective of this particular covariant Hamiltonian framework will be an important area for future work. It will be particularly interesting to see what role the importance of the vertical projection operator in defining Hamilton’s field equations plays in defining the quantum theory in curved space-times.

**Appendix A: Vertical projection operators**

Since vertical projection operators play such a major role in the construction of this version of polysymplectic covariant Hamiltonian field theory – see, for example, eqs. (1) and (2) – it is worth briefly reviewing their properties for those unfamiliar with them.

Formally, a vertical projection operator on a fiber bundle $E$ over a base manifold $M$ is defined as a map $V : TE \to TE$ such that the following two properties hold:

1. $\text{Im} V = VE$
2. $V^2 = V$

More intuitively, the first property means both that the vertical projection operator must project onto the vertical space at each point and that it hit every element in each vertical space, while the second property means that the vertical projection operator is indeed a projection operator: once it has been applied once, that’s all there is to it.

Since this paper deals exclusively with vector bundles, let us consider the specialized case of a vertical projection operator on a vector bundle more carefully. Let us choose coordinates $\{x^\alpha, \phi^I\}$ for $E$ such that an arbitrary tangent vector $u \in T_x E$ can be represented as $u = u^\alpha \frac{\partial}{\partial x^\alpha} + u^I \frac{\partial}{\partial \phi^I}$. Then any endomorphism $E : TE \to TE$ can be written locally as
\[ E = E_\beta^\gamma(e) \frac{\partial}{\partial x^\alpha} \otimes dx^\beta + E_I^\alpha(e) \frac{\partial}{\partial x^\alpha} \otimes d\phi^I + E_I^\alpha(e) \frac{\partial}{\partial \phi^I} \otimes dx^\alpha + E_J^I(e) \frac{\partial}{\partial \phi^I} \otimes d\phi^J \]

If these local coordinates are adapted so that \( v \in V \), \( v_I^J \frac{\partial}{\partial \phi^I} \), then the first criteria for vertical projection operators implies that we must have \( E_\beta^\gamma = E_I^\alpha = 0 \) if \( E \) is to represent a vertical projection operator. Calling our prospective vertical projection operator \( V \) instead of \( E \), this means that we have

\[ V = V^I_\alpha(e) \frac{\partial}{\partial \phi^I} \otimes dx^\alpha + V^I_J(e) \frac{\partial}{\partial \phi^I} \otimes d\phi^J \]

The second criteria now implies that we must have

\[ V^2 = \frac{\partial}{\partial \phi^I} \otimes (V^I_J V^J_i dx^i + V^I_J V^K_J d\phi^K) \]

This is satisfied whenever \( V^I_J = \delta^I_J \). In this case, we have

\[ V = \frac{\partial}{\partial \phi^I} \otimes d\phi^I + V^i_I \frac{\partial}{\partial \phi^I} \otimes dx^i \]

Specifying the arbitrary coefficients \( V^i_J \) therefore specifies a vertical projection operator in adapted local coordinates. The case dealt with in the main article is similar, just a larger vertical space \( VP \) in place of \( VE \). In the local canonical coordinates used throughout most of the main article, this means that a vertical projection operator \( V \) looks like

\[ V = \frac{\partial}{\partial \phi^I} \otimes (V^i_I dx^i + d\phi^I) + \frac{\partial}{\partial \pi^i_I} \otimes (V_{ij}^i dx^j + d\pi^i_I) \]

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