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Dynamical systems generated by quadratic homeomorphisms of 2D simplex

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Abstract. In this paper we analyze a general form of homeomorphisms of the two-dimensional simplex and study the asymptotical behavior of their trajectories.

1. Introduction

The notion of quadratic stochastic operators (QSO) was introduced by Bernstein [2], and the theory of QSO has been developed for more than 90 years. In recent years, there has been a considerable growth of interest in this theory due to its numerous applications to problems in mathematics, biology, and physics ([4],[6],[11]).

The quadratic stochastic operator (QSO) is a mapping of the simplex

\[ S^{m-1} = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \right\} \]

into itself, of the form

\[ V : x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = 1, \ldots, m, \]

where \( P_{ij,k} \) are coefficients of heredity and

\[ P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1, (i,j,k = 1, \ldots, m). \]

Thus, each QSO \( V \) can be uniquely defined by a cubic matrix \( P = (P_{ij,k})_{i,j,k=1}^{m} \) with conditions (1.2). Note that each element \( x \in S^{m-1} \) is a probability distribution on \( E = \{1, \ldots, m\} \).

Let \( x^0 = (x_1^0, \ldots, x_m^0) \) be a probability distribution (where \( x_i^0 = P(i) \) is the probability of \( i \), \( i = 1,2,\ldots,m \) of species in the initial generation, and \( P_{ij,k} \) is the probability that individuals in the \( i \)-th and \( j \)-th species interbred to produce an individual \( k \).

In this paper, we consider a model of free population, i.e. there is no difference of sex and in any generation, the parents \( ij \) are independent, i.e. \( P(i,j) = P(i)P(j) = x_i x_j \). Then the

\(^1\) To the memory of our teacher T. A. Sarymsakov on the occasion of his 100th birthday.
probability distribution $x' = (x'_1, \ldots, x'_m)$ (the state) of the species in the first generation can be found by the total probability

$$x'_k = \sum_{i,j=1}^m P(k|i,j)P(i)P(j) = \sum_{i,j=1}^m P_{ij,k}x_0^i x_0^j, \quad k = 1, \ldots, m. \quad (1.3)$$

This means that the correspondence $x_0 \rightarrow x'$ defines a map $V$ called the evolution operator. The population evolves by starting from an arbitrary state $x_0$, then passing to the state $x' = V(x_0)$ (in the next generation), after to the state $x'' = V(V(x_0))$, and so on. Thus, the states of the population are described by the following discrete-time dynamical system

$$x^{(0)}, x^{(1)} = V(x^{(0)}), x^{(2)} = V^2(x^{(0)}), \ldots \quad (1.4)$$

where $V^n(x) = V(V^{n-1}(x))$ denotes the $n$ times iteration of $V$ to $x$.

For a given $x^{(0)} \in S^{m-1}$ the trajectory (orbit) $x^{(n)}, n = 0, 1, 2, \ldots$ of $x^{(0)}$ under action of QSO (1.1) is defined by $x^{(n+1)} = V(x^{(n)})$, where $n = 0, 1, 2, \ldots$.

One of the main problems in mathematical biology consists in the study of the asymptotical behavior of the trajectories. The difficulty of the problem depends on the given matrix $P$. An asymptotic behavior of the QSO even on the small dimensional simplex is complicated. In order to solve this problem, many researchers always introduced a certain class of QSO and studied their behavior (see for example, [8, 12, 13, 16, 17]). In this paper we shall consider several particular cases of $P$ for which the above-mentioned problem is (particularly) solved.

A QSO given by (1.1), (1.2) is called Volterra if one has

$$P_{ij,k} = 0, \quad \text{if} \quad k \in \{i, j\}, \forall i, j, k \in E. \quad (1.5)$$

The biological treatment of condition (1.5) is clear: the offspring repeats the genotype of one of its parents. In [5], the general form of Volterra QSO is given by

$$x'_k = x_k \left(1 + \sum_{i=1}^m a_{ki}x_i\right), \quad k \in E, \quad (1.6)$$

where

$$a_{ki} = 2P_{ik,k} - 1, \quad \text{for} \quad i \neq k \text{ and } a_{ii} = 0, i \in E. \quad (1.7)$$

Moreover, $a_{ki} = -a_{ik}$ and $|a_{ki}| \leq 1$.

Denote by $A = (a_{ij})_{i,j=1}^m$ the skew-symmetric matrix with entries (1.7). Note that the operator (1.6) is a discretization of the Volterra model which describes interacting, competing species in population. Such a model has attracted considerable attention in the fields of biology, ecology, mathematics. It is known from [5], that Volterra operators are quadratic homeomorphism of the simplex.

**Theorem 1.1.** [5] Let $V$ be a quadratic homeomorphism of the simplex. Then there is a Volterra operator $V_0$ and a permutation $\pi$ of $1, 2, \ldots, n$ such that $V = V_0 \circ \pi$.

**Corollary 1.2.** [5] The set of all quadratic homeomorphisms of the simplex $S^{m-1}$ can be geometrically presented as the union of $m!$ disjoint cubes of dimension $\frac{m(m-1)}{2}$.

**Remark 1.3.** In [1, 15] quadratic homeomorphisms of the simplex are characterized as orthogonal preserving QSO.
2. Main result
In this section, we consider the case \( m = 3 \) and study the dynamical systems of quadratic homeomorphisms of the two-dimensional simplex. In [15] these quadratic homeomorphisms classified into three disjoint classes. Namely, let \( \pi \) be a permutation of the set \( \{1, 2, 3\} \). One has the following three possibilities \( \pi = \text{id}, \pi^2 = \text{id} \) and \( \pi^3 = \text{id} \). The cases \( \pi = \text{id} \) and \( \pi^3 = \text{id} \), respectively, define two disjoint classes, which were studied in [8], [7], [17]. The remaining class corresponds to \( \pi^2 = \text{id} \).

In this paper, we study the mentioned class. It is easy to see that there are three possibilities
\[
\begin{align*}
\pi_3 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.
\end{align*}
\]
Let us consider the case \( \pi_3 \) (the cases \( \pi_4 \) and \( \pi_5 \) can be considered in a similar way). By \( T\pi_3 : S^2 \to S^2 \) we denote the corresponding linear transformation defined by \( T\pi_3(x_1, x_2, x_3) = (x_2, x_1, x_3) \).

In [6] the dynamics of the following operator
\[
V : \begin{cases} 
    x'_1 = x_1(1 + ax_2 - bx_3) \\
    x'_2 = x_2(1 + cx_3 - ax_1) \\
    x'_3 = x_3(1 + bx_1 - cx_2)
\end{cases}
\]
has been studied.

Let us define
\[
V_{\pi_3} = V \circ T\pi_3 : \begin{cases} 
    x'_1 = x_2(1 + ax_1 - bx_3) \\
    x'_2 = x_1(1 + cx_3 - ax_2) \\
    x'_3 = x_3(1 + bx_2 - cx_1)
\end{cases},
\]
where \(-1 \leq a, b, c \leq 1, abc \neq 0.\)

**Remark 2.1.** From the followings
\[
V_{\pi_3}^2 : \begin{cases} 
    x'_1 = x_1(1 + cx_3 - ax_2)(1 + ax_2 - bx_3 - a^2x_1x_2 - b^2x_1x_3 + (ac - bc)x_2x_3) \\
    x'_2 = x_2(1 + ax_1 - bx_3)(1 + cx_3 - ax_1 - a^2x_1x_2 - (bc - ac)x_1x_3 - c^2x_2x_3) \\
    x'_3 = (x_3(1 + bx_2 - cx_1)(1 + bx_1 - cx_2 - (ab + ac)x_1x_2 + bxx_1x_3 + bcxx_2x_3)
\end{cases}
\]
and
\[
V^2 : \begin{cases} 
    x'_1 = x_1(1 + ax_2 - bx_3)(1 + ax_2 - bx_3 - a^2x_1x_2 - b^2x_1x_3 + (ac + bc)x_2x_3) \\
    x'_2 = x_2(1 + cx_3 - ax_1)(1 + cx_3 - ax_1 - a^2x_1x_2 - (bc + ab)x_1x_3 - c^2x_2x_3) \\
    x'_3 = x_3(1 + bx_1 - cx_2)(1 + bx_1 - cx_2 - c^2x_2x_3 - b^2x_1x_3 + (ab + ac)x_1x_2)
\end{cases}
\]
we obtain that the operator \( V_{\pi_3}^2 \) is not equal to operator \( V^2.\)

Recall that a point \( x \in S^{m-1} \) is called a fixed point of \( V \) if \( V(x) = x \). The set of all fixed points of \( V \) is denoted by \( \text{Fix}(V) \).

Let \( D_xV(a) = (\frac{\partial V}{\partial x_j})(a) \) be the Jacobian of \( V \) at the point \( a \in S^2 \). A fixed point \( a \) is called hyperbolic if the Jacobian \( D_xV(a) \) at \( a \) has no eigenvalues on the unit circle. A hyperbolic
fixed point \( a \) is called: *attracting* if all the eigenvalues of the Jacobian \( D_x V(a) \) are in the unit ball; *repelling* if all the eigenvalues of the Jacobian \( D_x V(a) \) are outside the unit ball; a *saddle* otherwise.

We denote one dimensional face \( \Gamma_{i,j} = \{ x = (x_1, \ldots, x_m) \in S^{m-1} : x_i + x_j = 1 \} \) and \( \text{int} \Gamma_{i,j} = \{ x = (x_1, \ldots, x_m) \in S^{m-1} : x_i + x_j = 1, x_ix_j \neq 0 \} \).

We have

\[
D_x V_{\pi_3} = \begin{pmatrix}
ax_2 & 1 + ax_1 - bx_3 & -bx_2 \\
1 + cx_3 - ax_2 & -ax_1 & cx_1 \\
-ax_3 & bx_3 & 1 + bx_2 - cx_1
\end{pmatrix}.
\]

**Definition 2.2.** [3] If \( f : I \to I \) is a \( C^3 \) map and \( Df(x) \neq 0 \), the Schwarzian derivative of \( f \) at \( x \) is

\[
(Sf)(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2} \left( \frac{D^2 f(x)}{Df(x)} \right)^2,
\]

where \( Df(x) \) is derivation of \( f(x) \).

Denote \( e_1 = (1,0,0), e_2 = (0,1,0) \) and \( e_3 = (0,0,1) \).

**Definition 2.3.** A curve \( \gamma \) is called an invariant curve, if \( x \in \gamma \) implies that \( V^i(x) \in \gamma \), for all \( i = 1,2,\ldots \).

It is evident that the face \( \Gamma_{12} \) is invariant set with respect to \( V_{\pi_3} \).

For the sake of simplicity, we denote \( W = V^2_{\pi_3} \). First we study dynamics of \( W \) on the invariant face \( \Gamma_{12} \).

**Theorem 2.4.** The operator \( W \) has a unique fixed point \( x^3 = (x^3_1, x^3_2, 0) \) belonging to \( \text{int} \Gamma_{12} \). If \( x \in \{ x = (x_1, x_2, 0) : x_1 \in [0, x^3_1) \} \) then \( \lim_{n \to \infty} W^{(n)}(x) = e_1 \) and if \( x \in \{ x = (x_1, x_2, 0) : x_1 \in (x^3_1, 1] \} \) then \( \lim_{n \to \infty} W^{(n)}(x) = e_2 \).

**Proof.** Since \( \Gamma_{12} \) is invariant subset of the operator \( (2.1) \), then the restriction of \( (2.1) \) on \( \Gamma_{12} \) has the following form

\[
V_{\pi_3}(x_0) : \begin{cases}
x'_1 = x_2(1 + ax_1), \\
x'_2 = x_1(1 - ax_2).
\end{cases}
\]

Due to \( x_1 + x_2 = 1 \) one has \( x'_1 = x_2(1 + ax_1) = (1 - x_1)(1 + ax_1) \).

Using notation \( x = x_1, \) we have \( x' = g(x) \), where \( g(x) = (1 - x)(1 + ax), x \in [0,1]. \)

Simple algebra gives that

\[
x^3_i = \begin{cases}
a - \sqrt{a^2 + 4} & \text{if } a > 0, \\
\frac{a - \sqrt{a^2 + 4}}{2a} & \text{if } a < 0,
\end{cases}
\]

is a fixed point of \( g(x) \) and \( |Dg(x^3_i)| > 1 \), i.e. \( x^3 \) is a repelling fixed point of \( g(x) \). One can see that \( D(g(x)) < 0 \) for any \( x \in (0,1) \) and \( g(x) \) decreasing function in \([0,1] \).

It is easy to check that \( g^2(0) = 0 \) and \( g^2(1) = 1 \), besides \( x = 0 \) and \( x = 1 \) are attracting fixed points of \( g^2(x) \). From \( (2.3) \) one has that \( (Sg)(x) < 0 \) and \( (Sg^2)(x) < 0 \).

From Singer’s Theorem [18] we infer that if \( x \in \{ x = (x_1, x_2, 0) : x_1 \in [0, x^3_1) \} \) then \( \lim_{n \to \infty} W^{(n)}(x) = e_1 \) and if \( x \in \{ x = (x_1, x_2, 0) : x_1 \in (x^3_1, 1] \} \) then \( \lim_{n \to \infty} W^{(n)}(x) = e_2 \).
Note that the operator $W$ has the following form

$$x'_i = x_i(1 + f_i(x)), \quad i = 1, 2, 3$$

where $f_1(x)$, $f_2(x)$, $f_3(x)$ are some polynomials.

Note that this kind of operators are called Lotka-Volterra operators. Some properties of such kind operators have been studied in [14].

Denote

$$P = \{x \in S^2 : f_i(x) \geq 0, \quad i = 1, 2, 3\}, \quad Q = \{x \in S^2 : f_i(x) \leq 0, \quad i = 1, 2, 3\}. $$

**Theorem 2.5.** [9] For any $x \in P$ and $y \in Q$ there exists an invariant curve $\gamma$ connecting these two points.

Next theorem gives full dynamics of the operator $W$ in $S^2 \setminus \Gamma_{12}$.

**Theorem 2.6.** The following statements hold:

(i) If $(1-b)(1+c) < 1$ and $(1+b)(1-c) > 1$, then for every $x \in S^2 \setminus \Gamma_{12}$ one has

$$\lim_{n \to \infty} W^{(n)}(x) = e_3;$$

(ii) If $(1-b)(1+c) > 1$ and $(1+b)(1-c) < 1$, then there exist an invariant curve $\gamma$ which passes from $e_3$ and the fixed point $x^3$. This curve divides the simplex into two open parts such that $e_1$ and $e_2$ vertices are situated in different parts, respectively. If $x$ belongs to the part of simplex where the vertex $e_1$ is located, then $\lim_{n \to \infty} W^{(n)}(x) = e_1$, if $x \in \gamma$ then $\lim_{n \to \infty} W^{(n)}(x) = x^3$, otherwise one has $\lim_{n \to \infty} W^{(n)}(x) = e_2$.

(iii) If $(1-b)(1+c) < 1$ and $(1+b)(1-c) < 1$ then there are $x^1 \in \text{int}\Gamma_{23}$, $x^2 \in \text{int}\Gamma_{13}$ and $x^3 \in \text{int}\Gamma_{12}$ fixed points of $W$. There are three invariant curves which pass over the fixed points. These curves divide the simplex into three open parts and vertices of the simplex are situated in each parts of the simplex, respectively. If $x$ belongs to a containing the vertex part, then the trajectory of $x$ converges to the corresponding vertex.

**Remark 2.7.** $$(1-b)(1+c)(1+b)(1-c) = (1-b^2)(1-c^2) < 1, \quad \text{because} \quad (1-b)(1+c) \geq 1 \quad \text{and} \quad (1+b)(1-c) \geq 1 \quad \text{inequalities does not hold.}$$

**Proof.** From (2.2) we have

$$W = \begin{cases} x'_1 = x_1(1 + f_1(x)) \\ x'_2 = x_2(1 + f_2(x)) \\ x'_3 = x_3(1 + f_3(x)) \end{cases}$$

where $f_1, f_2, f_3$ are some polynomials.

One can see that $e_1$ and $e_2$ are fixed points of $W$. The eigenvalues of $D_xW$ at points $e_1$ and $e_2$ are equal to $|\lambda_1| = 1$, $|\lambda_2| = 1 - a^2 < 1$, $|\lambda_3| = (1-c)(1+b)$, and eigenvalues of $D_xW$ at $e_3$ are $|\lambda_1| = |\lambda_2| = \sqrt{(1-b)(1+c)}$ and $|\lambda_3| = 1$.

Let $x \in \Gamma_{13}$ then the restriction of $W$ to $\Gamma_{13}$ has the following form

$$W(x) : \begin{cases} x'_1 = f(k(x_1)), \\ x'_3 = 1 - f(k(x_1)) \end{cases}$$

where $f(x) = (1-x)(1-bx)$ and $k(x) = (1-x)(1-cx)$.

Using notion $x = x_1$ we have $x' = g_1(x)$, where $g_1(x) = f(k(x))$, $x \in [0, 1]$. From (2.3) one find $(Sg_1)(x) < 0$. 


It is easy to check that

\[
\begin{cases}
&Dg_1(0) < 1 \text{ and } |Dg_1(1)| > 1 \text{ if } (1 - b)(1 + c) < 1 \text{ and } (1 + b)(1 - c) > 1 \\
&Dg_1(0) < 1 \text{ and } |Dg_1(1)| > 1 \text{ if } (1 - b)(1 + c) > 1 \text{ and } (1 + b)(1 - c) < 1 \\
&Dg_1(0) < 1 \text{ and } |Dg_1(1)| > 1 \text{ if } (1 - b)(1 + c) < 1 \text{ and } (1 + b)(1 - c) < 1
\end{cases}
\]

(2.5)

(i) From \((1 - b)(1 + c) < 1\) we have \(|\lambda_1| = |\lambda_2| < 1\). Therefore \(e_3\) is an attracting fixed point of \(W\).

From Singer’s theorem for \(x \in (0, 1)\), it follows \(\lim g_1(x) = 0\). Therefore if \(x \in \text{int}\Gamma_{13}\) we have \(\lim_{n \to \infty} W^{(n)}(x) = e_3\).

Similarly, for \(\Gamma_{23}\) one can prove that \(\lim_{n \to \infty} W^{(n)}(x) = e_3\).

From Lemma 2.2 of [10] we obtain

\[\lim_{n \to \infty} W^{(n)}(x) = e_3\]

for all \(x \in S^2 \setminus \Gamma_{12}\).

(ii) Let \((1 - b)(1 + c) > 1\) and \((1 + b)(1 - c) < 1\). Then \(e_1\) and \(e_2\) are attracting fixed points, and \(e_3\) is repelling fixed point for \(W\).

Note that \((x^3, e_3) \in P \times Q\). There is an invariant curve \(\gamma\) which passes over two given fixed points.

It’s easy to see that \(x^3\) is a saddle point for operator \(W\). From Grobman-Hartman’s theorem it follows that from \(x^3\) one passes only two invariant curves: the fist curve is \(\Gamma_{12}\) and the second one is \(\gamma\). So \(\gamma\) is a unique invariant curve.

From (2.5) we find that \(x = 1\) is attracting and \(x = 0\) is repelling points of \(g_1(x)\).

From Singer’s Theorem we infer that if \(x \in (0, 1)\), then \(\lim g_1(x) = 1\). Therefore, if \(x \in \Gamma_{13}\), then \(\lim W^{(n)}(x) = e_1\). Similarly, for \(\Gamma_{23}\) one can prove that \(\lim_{n \to \infty} W^{(n)}(x) = e_2\).

If follows from Lemma 2.2 of [10] that if \(x\) belongs to the part of simplex where the vertex \(e_1\) is located, then \(\lim W^{(n)}(x) = e_1\), if \(x \in \gamma\) then \(\lim W^{(n)}(x) = x^3\). Otherwise, we obtain \(\lim_{n \to \infty} W^{(n)}(x) = e_2\).

(iii) From \((1 - b)(1 + c) < 1\) and \((1 + b)(1 - c) < 1\), one has that \(e_1\), \(e_2\) and \(e_3\) are attracting fixed points of \(W\).

It follows from (2.5) that \(x = 1\) and \(x = 0\) are attracting points of \(g_1(x)\).

The Singer’s Theorem yields that there exists another fixed point \(x_1^2 \in (0, 1)\) of \(g_1(x)\).

Moreover, if \(x \in [0, x_1^2]\), then \(\lim g_1(x) = 0\), and if \(x \in (x_1^2, 1]\), then \(\lim g_1(x) = 1\).

Therefore, if \(x \in \{x = (x_1, 0, x_3) : x_1 \in [0, x_1^2]\}\) then \(\lim_{n \to \infty} W^{(n)}(x) = e_1\). If \(x \in \{x = (x_1, 0, x_3) : x_1 \in (x_1^2, 1]\} \) then \(\lim_{n \to \infty} W^{(n)}(x) = e_3\).

Similarly, for \(\Gamma_{23}\) one can prove that there exists a fixed point \(x^1 \in (0, x^1_2, x^1_3)\). Moreover, if \(x \in \{x = (x_1, 0, x_3) : x_1 \in [0, x_1^2]\}\) then \(\lim_{n \to \infty} W^{(n)}(x) = e_2\), and if \(x \in \{x = (x_1, 0, x_3) : x_1 \in (x^1_2, 1]\} \) then \(\lim_{n \to \infty} W^{(n)}(x) = e_3\).

For \(\Gamma_{12}\), the claim of theorem was proved in the proof of Theorem 2.5.

In the case (iii) there is a fixed point

\[x_0 = \left(\frac{b - c + bc}{c(a + b + c)}, \frac{b - c + bc}{b(a + b + c)}\right) = \frac{b - c + bc}{c(a + b + c) b(a + b + c)} \in \text{int}S^2\]
It is easy to check that $x_0$ is repelling fixed point.

Note that $(x^1, x_0), (x^2, x_0), (x^3, x_0)$ are in $P \times Q$. Consequently there are invariant curves each of which passes over two given fixed points. Since $x_0$ is a repelling point, the points $x^1, x^2$ and $x^3$ are saddle points, these invariant curves divide the simplex into three parts and the boundaries of the parts are invariant in the corresponding part.

Let us consider the part containing the vertex $e_1$. Since $e_1$ is an attracting point by Lemma 2.2 of [10] one has that for an arbitrary initial point takes from the part where $e_1$ is located, then the trajectory of $W$ converges to $e_1$. This completes the proof.

**Remark 2.8.** If we know trajectory of operator $W$ than we can know the trajectory of the operator $V_{x_3}$.

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