Mathematical modeling of formation, evolution and interaction of galaxies in cosmological context

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Abstract. The results of mathematical modeling of formation of galaxies in cosmological context with using of multiphase hydrodynamical model are presented in the paper. Mathematical model of the problem of cosmological modeling, numerical methods for solving the hyperbolical equations and brief description of parallel implementation of the software complex CosmoPhi are described in details. The results of numerical experiments of large-scale cosmological simulations are presented.

1. Introduction

The subject of study of modern astrophysics is the study of physical processes in the Universe, how had they influenced on self-organization and evolution of astrophysical objects, their further dynamics and interaction. Despite the great progress in observation we usually can see only static image, and then the model (or theory) of behavior of astrophysical objects before or after the observational moment is built based on the observational data and known physical processes. 

Essentiality of accounting of gravitational and magnetic fields, the complexity of reproducing of cosmic conditions in vitro impose significant restrictions on experimental studying of astrophysical objects. In fact, mathematical modeling – is the main, if not the only one, way for theoretical research of astrophysical processes and objects.

The evolution of cold dark matter described with collisionless N-body model in the context of cosmological simulation of clusters of galaxies (see for example [1]) within the standard cosmological model $\Lambda$CDM was computed. In the recent years series of global cosmological simulations were performed [2, 3, 4]. But to get on the next level, individual galaxies, such approach has significant restrictions due to inability to obtain some of elliptical galaxies. So it is necessary to simulate also baryonic matter in hydrodynamic model and carry out a mutual cosmological modeling like in the projects ”Bolshoi” [5], ”Illustris” [6, 7, 8] and ”Eagle” [9, 10].

Refinement of cosmological model is done by not only using of multiphase model, but and sophistication of so-called subgrid physical model. Structural element of clusters of galaxies is actually galaxies themselves. The problems of simulation of dynamics of galaxies could be divided by time of their dynamics. Since, the time of evolution of particular galaxy is up to several billion of years, while the interaction between individual galaxies needs hundreds of millions of years. In the field of research of collisions between galaxies the GALMER project of the Paris observatory – the database of numerical experiments [11] on collision between different
types of galaxies – should be singled out. Subgrid processes (starburst [13], AGN [14], formation of supermassive double black holes [15, 16], chemokinetics [17]) are accelerated significantly in the problems of collision of galaxies [12], hence the explicit consideration of these processes in the mathematical model is required. The same consideration should be done in the problems of evolution of galaxies, for example, the separate proceedings [18] was devoted to the problem of consideration of different processes in the context of the problem of "galactic archaeology" of Milky Way.

The mathematical model and its implementation as software complex CosmoPhi for cosmological simulation of formation and collision of galaxies is described in the paper. The mathematical model is based on a joint solving of equations of multicomponent one-speed hydrodynamics, that describes gas component, and equations of the first momenta of Boltzmann collisionless equation, that describes collisionless components (stars and dark matter). The equations are written with consideration of cosmological expansion. The using of mathematical model written in form of hyperbolic equations allow us to use unified numerical method. This numerical method consist of special combination of Fluids-in-Cells method [19, 20], Godunov method [21] and piece-wise parabolic method on a local template [22, 23].

2. Mathematical model
The main ingredients of the cosmological simulation are collisionless dark matter and baryonic matter consisting of gas and stellar components, the dynamics of which is affected by dark energy. The dynamics of baryonic component is affected by the number of subgrid processes, such as chemokinetic and related processes of radiation and heating, and also processes of formation and destruction of stars.

2.1. Description of components of cosmological model
Redefined system of equations of gravitational gas dynamics describing baryonic gas component in comoving coordinate system with consideration of cosmological expansion is as follows:

\[
\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho_i \\ \rho i \\ \rho u \\ \rho E \\ pE \end{pmatrix} + \frac{1}{a} \nabla \cdot \begin{pmatrix} \rho \vec{u} \\ \rho_i \vec{u} \\ \rho u \vec{u} \\ \rho E \vec{u} \end{pmatrix} = \begin{pmatrix} S - D \\ s_i + S_{\rho u} - D_{\rho u} \\ -\frac{1}{a} \nabla \cdot p - H \rho \vec{u} - \frac{\rho u}{a^2} \nabla \cdot \Phi + S \vec{u} - D \vec{u} \\ s_i + S_{\rho E} - D_{\rho E} \\ -\frac{1}{a} \nabla \cdot (p \vec{u}) - 2H \rho E - \frac{1}{a^2} (\rho u, \nabla \Phi) + \Gamma - \Lambda + \varepsilon - \frac{\varepsilon}{p} - \frac{\varepsilon}{\rho} \end{pmatrix}
\]

where \( \rho = \sum_i \rho_i \) – density, \( \rho_i \) – density of each component of gaseous mixture, \( s_i \) – the speed of formation of \( i \)-th component of mixture, \( S \) – the speed of formation of supernovae, \( D \) – the speed of starburst, \( \vec{u} \) – velocity, \( p = \rho \varepsilon (\gamma - 1) \) – pressure, \( \varepsilon \) – internal energy, \( \gamma \) – adiabatic index, \( \rho E = \rho \varepsilon + \rho u^2/2 \) – full mechanical energy, \( \Phi \) – gravitational potential, \( \Gamma \) – heating function, \( \Lambda \) – cooling function, \( H \) – Hubble constant, \( a = 1/(1 + z) \) – the expansion parameter, computed from the Einstein equation:

\[
\frac{da}{dt} = H \sqrt{\Omega_M (a^{-1} - 1) + \Omega_\Lambda (a^2 - 1)} + 1
\]

where \( \Omega_M = \Omega_B + \Omega_D \) – the part of matter in the total mass, \( \Omega_B \) – the part of visible baryonic matter (gas and stars), \( \Omega_D \) – the part of dark matter, \( \Omega_\Lambda \) – the part of dark energy. To describe the baryonic stellar component and dark matter we use collisionless hydrodynamical approach based on the equations for the first momenta of collisionless Boltzmann equation. It was successfully used for the problems of evolution and collision of galaxies [19, 24, 25]. In this approach we will use only diagonal tensor of dispersion of velocities because collisions almost
play no role, hence, the off-diagonal elements of tensor could be neglected. The system of first moments of collisionless Boltzmann equation in comoving coordinate system with consideration of cosmological expansion is as follows:

\[
\frac{\partial}{\partial t} \begin{pmatrix} n \\ n\vec{v} \\ \Pi_{xx} \\ \Pi_{yy} \\ \Pi_{zz} \\ nW \end{pmatrix} + \frac{1}{a} \nabla \cdot \begin{pmatrix} n \\ n\vec{v} \\ \Pi_{xx} \\ \Pi_{yy} \\ \Pi_{zz} \\ nW \end{pmatrix} = \begin{pmatrix} \mathcal{D} - \mathcal{S} \\ -\frac{1}{a} \nabla \cdot (\Pi) - Hn\vec{v} - \frac{n}{a^2} \nabla \Phi + \mathcal{D}\vec{u} - S\vec{v} \\ -\frac{2}{a} \Pi_{xx} \frac{\partial n}{\partial x} - 2H\Pi_{xx} + \frac{\mathcal{D}}{3p} - \frac{\mathcal{S}}{3p} \\ -\frac{2}{a} \Pi_{yy} \frac{\partial n}{\partial y} - 2H\Pi_{yy} + \frac{\mathcal{D}}{3p} - \frac{\mathcal{S}}{3p} \\ -\frac{2}{a} \Pi_{zz} \frac{\partial n}{\partial z} - 2H\Pi_{zz} + \frac{\mathcal{D}}{3p} - \frac{\mathcal{S}}{3p} \\ -\frac{1}{a} \nabla \cdot (\Pi\vec{v}) - 2HnW - \frac{1}{a^2} (n\vec{v}, \nabla \Phi) + \frac{\mathcal{D}}{p} - \frac{\mathcal{S}}{p} \end{pmatrix}
\]

(3)

where \( n \) – the density of stellar component, \( \vec{u} \) – velocity, \( \Pi_{\xi\xi} \) – diagonal tensor of dispersion of velocities, \( nW = \frac{1}{2} (\Pi_{xx} + \Pi_{yy} + \Pi_{zz}) + \frac{\Pi}{2} \) – full mechanical energy of the stellar component.

The system of equations is supplemented by the equation for the gravitational potential:

\[
\Delta \Phi = 4\pi G (\rho + n - \rho_0)
\]

where \( \rho_0 \) – average density of baryonic component of the dark matter.

2.2. Model of subgrid physics

The model of subgrid physics play a major role in cosmological problems. In our approach we will focus on the following, in our opinion, the most important physical processes.

2.2.1. Process of star formation and effect of the supernovae explosion

Let us formulate the necessary conditions for star formation process as follows [26]:

\[
T < 10^4 K \quad \nabla \cdot \vec{u} < 0 \quad \rho > 1.64 \frac{M_\odot}{pc^3}
\]

(5)

Then the star formation rate will be used in the following form [1]:

\[
\Delta^{SFR} = \frac{dn}{dt} = C \rho \tau_{dyn} = C \rho^{3/2} \sqrt{\frac{32G}{3\pi}}
\]

(6)

where \( C = 0.03 \) – the coefficient of efficiency of star formation. We use the collapse rate of stellar component in a result of supernovae explosion as follows [27]:

\[
\Delta^{SN} = \frac{d\rho}{dt} = \beta C \frac{n}{\tau_{dyn}} = \beta C n^{3/2} \sqrt{\frac{32G}{3\pi}}
\]

(7)

where the coefficient of explosion of young star is \( \beta = 0.1 \). Also with each explosion of supernova with mass equal to \( M^{SN} = M_\odot \) the energy in an amount of \( 10^{51} \text{erg} \) is released, which will play the role of one of members of the function of gas heating

\[
\Gamma_{SN} = 10^{51} \frac{M^{SN}}{M_\odot} \text{erg}
\]

(8)

2.2.2. Cooling function

Intergalactic gas, heated to a temperature \( \sim 10^4 - 10^8 K \) while collision, will be cooled with time. The cooling rate of gas with a temperature more than \( \sim 10^4 \) is estimated in paper [28] and is used as follows:

\[
\Lambda_{cooling} = 10^{-22} n^2 \frac{\text{erg}}{cm^3}
\]

(9)

where \( n \) – the concentration of atomic hydrogen.
2.2.3. **Primordial model of chemical kinetics** Let us specially focus on the model of chemokinetics. We will treat chemical evolution of the basic forms of hydrogen and helium [29, 30, 31]. For this we will use the following 20 chemical reactions.

1. Collisional ionization $H + e \rightarrow H^+ + 2e$, the reaction rate (near and further in cm$^3$/s) $k_1$ is given in table (1).

2. Collisional recombination with radiation $H^+ + e \rightarrow H + \gamma$, reaction rate $k_2$ in case with $T > 5500K$ is given in table (1), with lower temperature $k_2 = 3.92 \times 10^{-13} \times T^{-0.6353}$.

3. Collisional ionization with radiation $H + e \rightarrow H^+ + \gamma$, reaction rate $k_3 = 6.77 \times 10^{-15} \times T^{0.8779}$.

4. Dissociative electron attachment $H^- + H \rightarrow H_2 + e$, reaction rate $k_4$ in case with $T > 1160K$ is given in table (1), with a lower temperature $k_4 = 1.43 \times 10^{-9}$.

5. Association with radiation $H + H^+ \rightarrow H_2^+ + \gamma$, reaction rate in case with $T > 6700K$ $k_5 = 5.81 \times 10^{-16} \left( \frac{T}{56200} \right)^{-0.6657 \times \log_{10}(T/56200)}$, with a lower temperature $k_5 = 1.85 \times 10^{-23} \times T^{1.8}$.

6. Recombination of hydrogen $H_2^+ + H \rightarrow H_2 + H^+$, reaction rate $k_6 = 6 \times 10^{-10}$.

7. Recombination of molecular hydrogen $H_2 + H^+ \rightarrow H_2^+ + H$, reaction rate $k_7$ in case with $T > 350K$ is given in table (1), with no reaction with a lower temperature.

8. Collisional dissociation $H_2 + e \rightarrow 2H + e$, reaction rate in case with $T > 350K$ $k_8 = 5.6 \times 10^{-11} \times \sqrt{T} \times \exp^{-102124/T}$, there is no reaction with a lower temperature.

9. The first reaction of collisional detachment of electrons $H^- + e \rightarrow H + 2e$, reaction rate $k_9$ in case with $T > 45K$ is given in table (1), with no reaction with a lower temperature.

10. The second reaction of collisional detachment of electrons $H^- + H \rightarrow 2H + e$, reaction rate $k_{10}$ in case with $T > 45K$ is given in table (1), with a lower temperature $k_{10} = 2.56 \times 10^{-9} \times T^{1.78186}$.

11. Neutralization $H^- + H^+ \rightarrow 2H + \gamma$, reaction rate $k_{11} = 6.5 \times 10^{-9} \times T^{-1/2}$.

12. Collisional association $H^- + H^+ \rightarrow H^- + e$, reaction rate $k_{12} = 10^{-8} \times T^{-0.4}$.

13. Dissociative recombination $H_2^+ + e \rightarrow 2H + \gamma$, reaction rate with a temperature $T > 617K$ $k_{13} = 1.32 \times 10^{-6} \times T^{-0.76}$, with a lower temperature $k_{13} = 10^{-8}$.

14. Molecular neutralization $H_2^+ + H^- \rightarrow H + H_2$, reaction rate $k_{14} = 5 \times 10^{-7} (100 \times T)^{-0.5}$.

15. Collisional molecular association $3H \rightarrow H + H_2 + H$, reaction rate with a temperature $T > 300K$ $k_{15} = 1.3 \times 10^{-32} (T/300)^{-1}$, with a lower temperature $k_{15} = 1.3 \times 10^{-32} (T/300)^{-0.38}$.

16. Collisional molecular dissociation $H_2 + H \rightarrow 3H$, reaction rate $k_{16} = 5 \times 10^{-7} (100T)^{-0.5} \exp^{4.667/T} (1+0.2472/T)^{1.517}$.

17. Collisional ionization $He + e \rightarrow He^+ + 2e$, reaction rate $k_{17}$ in case with $T > 900K$ is given in table (1), there is no reaction with a lower temperature.

18. Collisional recombination with radiation $He^+ + e \rightarrow He + \gamma$, reaction rate with a temperature $T > 900K$ $k_{18} = 3.92 \times 10^{-12} \times T^{-0.6353} + 1.54 \times 10^{-9} \times T^{-1.5} \times \exp^{-0.8925/T}$, with a lower temperature $k_{18} = 3.92 \times 10^{-13} T^{-0.6353}$.

19. Collisional ionization $He^+ + e \rightarrow He^{++} + 2e$, reaction rate $k_{19}$ in case with a temperature $T > 900K$ is given in table (1), there is no reaction with a lower temperature.

20. Collisional molecular dissociation $He^{++} + e \rightarrow He^{++} + \gamma$, reaction rate $k_{20} = 3.36 \times 10^{-10} \times T^{-0.5} (T/1000)^{-0.2} \exp^{0.496/T} (1+(T/1000)^{0.7}$.
Cooling functions related to reactions are given in appendix. We will calculate effective adiabatic index from the equation:

\[ \gamma = \frac{5n_H + 5n_{He} + 5n_e + 7n_{H_2}}{3n_H + 3n_{He} + 3n_e + 5n_{H_2}} \]  

(10)

where \( n_H \) – concentration of atomic hydrogen, \( n_{He} \) – atomic helium, \( n_e \) – electrons, \( n_{H_2} \) – molecular hydrogen. We will use the classical initial distribution of elements in the Universe [31] in the numerical experiments.

3. Numerical method

The original numerical method based on the combination of operator splitting approach, Godunov method, Roe scheme and piece-wise parabolic reconstruction on a local template [22, 23] is used to solve the hydrodynamical equations. The method based on Fast Fourier Transform is used to solve the Poisson equation. At the eulerian stage of the numerical method the equations without consideration of advective members and function, that simulate “subgrid” processes, are solved. To approximate spatial derivatives the solution of the linearized Riemann problem is used. For this purpose the averaging of density and pressure (in the case of collisionless processes, are solved. To approximate spatial derivatives the solution of the linearized Riemann problem is used. For this purpose the averaging of density and pressure (in the case of collisionless component we use diagonal member of the tensor of dispersion of velocities) is made on all of the interfaces between cells (\( L \) – is the designation of the left cell, \( R \) – is the designation of the right cell) with using of the following formulae:

\[ \rho = \frac{\rho_L^{3/2} + \rho_R^{3/2}}{\sqrt{\rho_L} + \sqrt{\rho_R}} \quad \rho = \frac{p_L\sqrt{\rho_L} + p_R\sqrt{\rho_R}}{\sqrt{\rho_L} + \sqrt{\rho_R}} \]

We assume that solution in treated cells is piece-wise parabolic function. The procedure of construction of local parabolas is described in classic work [32] in details. Omitting the detailed calculations of the development of scheme with using of the Godunov method (the numerical method and its verification are described in [33] in details) the solution of the Riemann problem at the eulerian stage with consideration of cosmological expansion is:

\[ U = \frac{u_L(-\frac{\lambda \tau}{a}) + u_R(\frac{\lambda \tau}{a})}{2} + \frac{p_L(-\frac{\lambda \tau}{a}) - p_R(\frac{\lambda \tau}{a})}{2} \sqrt{\frac{2^{\frac{\gamma + 1}{2}}(\rho_L^{3/2} + \rho_R^{3/2})(p_L\sqrt{\rho_L} + p_R\sqrt{\rho_R})}{\lambda^{3/2}(\rho_L^{3/2} + \rho_R^{3/2})(p_L\sqrt{\rho_L} + p_R\sqrt{\rho_R})^2}} \]

\[ P = \frac{p_L(\frac{-\lambda \tau}{a}) + p_R(\frac{\lambda \tau}{a})}{2} + \frac{u_L(-\frac{\lambda \tau}{a}) - u_R(\frac{\lambda \tau}{a})}{2} \sqrt{\frac{2^{\frac{\gamma + 1}{2}}(\rho_L^{3/2} + \rho_R^{3/2})(p_L\sqrt{\rho_L} + p_R\sqrt{\rho_R})^2}{(\sqrt{\rho_L} + \sqrt{\rho_R})^2}} \]

where

\[ \lambda = \sqrt{\frac{2^{\frac{\gamma + 1}{2}}(p_L\sqrt{\rho_L} + p_R\sqrt{\rho_R})}{\rho_L^{3/2} + \rho_R^{3/2}}} \]

\[ q_L(-\nu t) = q_i^R - \frac{\nu t}{2h} \left( \triangle q_i - q_i^6 \left( 1 - \frac{2\nu t}{3h} \right) \right) \quad q_R(\nu t) = q_i^L + \frac{\nu t}{2h} \left( \triangle q_i + q_i^6 \left( 1 - \frac{2\nu t}{3h} \right) \right) \]

At the lagrangian stage the advection of hydrodynamical quantities takes place and the equations has a following form:

\[ \frac{\partial f}{\partial t} + \frac{1}{a} \nabla \cdot (f\vec{v}) = 0 \]

where \( f \) is symbolic notation of the density \( \rho \), \( n \), momentum \( \rho \vec{u}, n \vec{v} \), density of the full mechanical \( \rho E, nW \) or internal \( \rho_\epsilon, \Pi_{\xi\xi} \) energy of the gas. To solve the equation we use the same approach,
that is used at the eulerian stage. To compute the flow $F = f \vec{v}$ with $\lambda = |\vec{v}|$ the following formula is used:

$$F = v \times \begin{cases} f_L(-\frac{\lambda \tau}{\alpha}), v \geq 0 \\ f_R(\frac{\lambda \tau}{\alpha}), v < 0 \end{cases}$$

where $f_L(-\frac{\lambda \tau}{\alpha})$, $f_R(\frac{\lambda \tau}{\alpha})$ – piece-wise parabolic function for the quantity $f$. To construct the piece-wise parabolic solution the similar procedure is used. At the final stage of solving of the
Figure 3. The density of baryon matter in $z = 2$ (top left), $z = 1$ (top right), $z = 0.4$ (bottom left), $z = 0$ (bottom right).

hydrodynamical equations the correction of solution is done. At the gas-vacuum interface [34] the correction of velocity takes place, in the remaining region the internal energy is corrected, that guarantees nondecreasing entropy [35]. Such modification provides detailed balance of energies and guarantees nondecreasing entropy. Method of solving of the Poisson equation is based on Fast Fourier Transform. Geometrical decomposition of computational domain is in a basis of parallel implementation. Unified numerical method allow us to use the same scheme of geometrical decomposition, that leads us to more simple and effective implementation. The numerical experiments on the study of performance of the CosmoPhi code on the cluster RSC PetaStream were carried out. As a result, 134-fold acceleration (strong scalability) on one accelerator Intel Xeon Phi, 77% of efficiency (weak scalability) with using of 224 accelerators Intel Xeon Phi (more than 50 000 logical cores), and 40% of peak performance with using of scalar computations on the accelerator Intel Xeon Phi, were achieved. Using of simulation of behaviour of programm implementation [36] on such architecture had shown 80% of efficiency on up to 983040 cores, that corresponds to exaflop computational level.

4. Results of cosmological simulation

From the moment of time, corresponding to $z = 99$, we will deal with expanding cubic region with the length of cube $L = 100/h$ Mpc $= 3 \times 10^{23}/h$ m and periodic border conditions on each dimension. As typical value of the density we will get $\rho = 1.88 \times 10^{-28}h^2$ kg/m$^3$. Portion of the dark energy $\Omega_\Lambda = 0.73$, dark matter $\Omega_D = 0.226$, visible baryonic matter $\Omega_B = 0.044$ (in the beginning a lack of stars is presumed). The temperature of the gas component is $T = 10^4$ K. The Hubble constant is $H = 67.8$ km/s/Mpc. To set the initial conditions the small fluctuations with uniform density distribution are used. To set the random fluctuations the normal distribution
Figure 4. The temperature of gas in $z = 2$ (top left), $z = 1$ (top right), $z = 0.4$ (bottom left), $z = 0$ (bottom right).

with the range, corresponding to energetic spectrum (equation (25) in the paper [37]), is formed. Then the inverse Fourier transform is performed.

To study the need of using of equilibrium chemokinetics in problems of cosmological simulation the typical concentrations of different forms of hydrogen and helium [29, 30, 31] with a temperature $T = 10^4$ K were taken. Then, with using of software package ChemPAK [38] developed earlier, the numerical experiment, which had shown that all of the chemical reactions happens for about a ten of thousands of years, was carried out. The model of equilibrium chemokinetics, with consideration that typical timestep in problems of cosmological simulation is equal to one million of years, was chosen and implemented.

Formation of large-scale cosmological structures – filaments, pancakes, clusters of galaxies, voids – within two-phase multicomponent hydrodynamical model with consideration of cosmological expansion and subgrid processes was simulated. These large-scale objects are formed as a result of evolution of instability. Then individual cluster of galaxies was selected with using of zoom-in approach, approximate exponential profile of the density of gas and collisionless components was build and numerical experiment in the region containing only this cluster was carried out. As a result of the numerical experiment, qualitative compliance of structure of simulated and observed clusters, quantitative compliance of masses of simulated and observed galaxies and distances between them, were shown.

Also the behavior of starburst process as a result of cosmological simulation was studied. Simulated value of fraction of the stars in the total mass corresponds to the observed value. Also the process of active starburst begins with $z = 2$, that corresponds to the observed data.
5. Conclusion
The new hydrodynamical model of formation of galaxies in cosmological context within multiphase hydrodynamical model, based on the joint solving of equations of multicomponent gas dynamics, first momenta of collisionless Boltzmann equation and Poisson equation for the potential, with consideration of cosmological expansion and subgrid processes, is shown in paper. The mathematical model of problem of cosmological simulation and numerical methods for solving of hyperbolical equations are described in details, also parallel implementation as software complex CosmoPhi is given briefly in the paper. The using of such model allowed us to formulate the unified numerical method for solving of hyperbolical equations. This numerical method is implemented effectively on a heterogeneous supercomputer equipped with accelerators Intel Xeon Phi. As a result, 134-fold acceleration on one accelerator Intel Xeon Phi, 77% of efficiency with using of 224 accelerators Intel Xeon Phi, and 40% of peak performance with using of scalar computations on the accelerator Intel Xeon Phi, were achieved. The results of numerical experiments on large-scale cosmological simulation are presented.

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Appendix
Cooling function (here and further in ergs cm$^{-3}$ s$^{-1}$ [29, 30, 31]) for collisional reactions:

$$Q_{\text{col}}^{\text{ol}} = 7.50 \times 10^{-19} \left(1 + \sqrt{T/10^5}\right)^{-1} \exp^{-118349/T} \times n_e n_H$$
Table 1. Coefficients of rates of reactions in form $A(N) = A \times 10^{-N}$

| N  | $k_1$  | $k_2$   | $k_3$   | $k_4$   | $k_5$   | $k_6$   | $k_7$   | $k_8$   | $k_9$   |
|----|--------|---------|---------|---------|---------|---------|---------|---------|---------|
| $A_0$ | -32.71 | -28.61  | -20.07  | -24.24  | -18.01  | -20.37  | -40.37  | -44.09  | -68.71  |
| $A_1$ | 13.53  | -7.24(1)| 2.28(1) | 3.40    | 2.36    | 1.13    | 23.91   | 43.93   |         |
| $A_2$ | -5.73  | -2.02(2)| 3.59(2) | -3.89   | -2.82(1)| -1.42(1)| -10.75  | -18.48  |         |
| $A_3$ | 1.56   | -2.38(3)| -4.55(3)| 2.04    | 1.62(2) | 8.46(3) | 3.05    | 4.71    |         |
| $A_4$ | -0.28  | -3.21(4)| -3.10(4)| -5.41(1)| -3.36(2)| -1.43(3)| -5.68(1)| -7.69(1)|         |
| $A_5$ | 3.48(2)| 1.42(5) | 1.07(5) | 8.41(2) | 1.17(2) | 2.01(4) | 6.79(2) | 8.11(2) |         |
| $A_6$ | -2.63(3)| 4.98(6)| -8.36(6)| -7.87(3)| -1.65(3)| 8.66(5) | -5.01(3)| -5.32(3)|         |
| $A_7$ | 1.12(4)| 5.75(7) | 2.23(7) | 4.13(4) | 1.06(4) | -2.58(5)| 2.06(4) | 1.97(4) |         |
| $A_8$ | -2.04(6)| -1.85(8)| 0       | -9.36(6)| -2.63(8)| 2.45(8) | -3.64(6)| -3.16(6)|         |

$$Q_2^{\text{col}} = 9.10 \times 10^{-27} \left(1 + \sqrt{T/10^5}\right)^{-1} T^{-0.1687} \exp^{-13179/T} \times n_e^2 n_{He}$$

$$Q_3^{\text{col}} = 5.54 \times 10^{-17} \left(1 + \sqrt{T/10^5}\right)^{-1} T^{-0.397} \exp^{-473638/T} \times n_e n_{He+}$$

for reactions of ionizations:

$Q_1^{\text{ion}} = 2 \times 10^{-11} \times k_1 n_e n_H$ \quad $Q_2^{\text{ion}} = 4 \times 10^{-11} \times k_1 n_e n_{He}$ \quad $Q_3^{\text{ion}} = 9 \times 10^{-11} \times k_1 n_e n_{He+}$

$Q_4^{\text{ion}} = 5 \times 10^{-27} \left(1 + \sqrt{T/10^5}\right)^{-1} T^{-0.1687} \exp^{-55338/T} \times n_e^2 n_{He+}$

for reaction of recombination:

$Q_1^{\text{rec}} = 8.71 \times 10^{-27} (T/1000)^{-0.2} \left(1 + (T/10^6)^{0.7}\right)^{-1} \times n_e n_{H+}$

$Q_2^{\text{rec}} = 1.55 \times 10^{-26} T^{0.3647} \times n_e n_{He+}$

$Q_3^{\text{rec}} = 1.24 \times 10^{-13} T^{-1.5} \left(1 + 0.3 \exp^{-94000/T}\right) \exp^{-94000/T} \times n_e n_{He+}$

$Q_4^{\text{rec}} = 3.48 \times 10^{-26} \sqrt{T} (T/1000)^{-0.2} \left(1 + (T/10^6)^{0.7}\right)^{-1} \times n_e n_{He+}$

where $n_H$ – concentration of atomic hydrogen, $n_{He}$ – atomic helium, $n_e$ – electrons, $n_{H2}$ – molecular hydrogen, $n_{H+}$ – ionized hydrogen, $n_{He+}$ – ionized helium, $n_{He+}$ – twice ionized helium. We will write the speeds of chemical reaction in the following form $k = \exp \sum_{i=0}^{N} A_i \times \log^2(T)$. Coefficients $A_i$ for reactions will be written in the table (1).

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