We discuss the problem of the information transfer (exchange of states configuration) between two interacting quantum systems along their evolution in time. We consider the specific case of two modes of the electromagnetic field with rotating wave coupling interaction (up-conversion in a nonlinear crystal). We verify that for certain initial states of the fields the swapping of state configuration occurs with conservation of the mean energy of each mode (without energy transfer), characterising thus pure information flow.

I. INTRODUCTION

The quantum two-mode harmonic oscillator (HO) allies simplicity together with richness of details and more importantly it is associated with a large variety of physical phenomena. Many properties of this system were studied in detail in the classical papers on quantum amplifiers and converters [1–7]. Among later publications we may refer to the considerations of various time-dependent couplings in [8–17]. The interaction between two electromagnetic modes, each one in a cavity, was considered in the analysis of reversible decoherence of superposition of field states in [18]. Quantum-statistical properties of two coupled modes of electromagnetic field were studied recently in [19–21]. A simple model of two coupled oscillators was used in [22] to illustrate Feynman’s concept of the “rest of the universe”. In quantum optics, one encounters the phenomena of two electromagnetic (EM) fields interacting in a nonlinear crystal [23–25], or through a beam splitter, or yet a coupler [26]. The two-mode HO also appears in problems involving an ion trapped in a two-dimensional well, with the ion center of mass oscillating harmonically and interacting with two electronic levels [27].

In another scenario, the problem of two-level atoms (the “reservoir”) interacting dispersively with one EM mode in a cavity was discussed in [28]. In their approach the field state is projected on photon number eigenstate without energy exchange between atom and field. The change in the field photon probability distribution is achieved through a dissipation free “information-gathering” process.

However, for our purpose of approaching pure information transfer between quantum subsystems, we are adopting here quantum optics (two EM fields at different frequencies and a classical driving field interact) as the scenario where the physical phenomena take place. We analyse the dynamical evolution of a two-mode harmonic system (interacting resonantly with the driving field) for some particular initial joint-state, one mode being a HO coherent state and the other is a special superposition of two coherent states, known as ‘cat’ state [29]. We are interested on two aspects of the quantum evolution, i) the recurrence to the initial state and ii) how and when information is transfered from one mode to the other, such that the states of the two modes exchange identity (or states configuration), meaning that during their evolution, at some times \( \tau_n \), they assume each other initial state, thus transferring completely their characteristics to one another. Although the interacting two-mode problem was already approached in the literature, we did not see it discussed as in the present physical context; in a more restricted sense a similar problem was considered earlier in [1–7], however, the authors of those papers were interested only in the energy exchange, and did not discuss the exchange of the quantum states configuration of the modes, as a pure transference of information. We will show that when both modes have the same number of quanta (not necessarily with the same energy) the information flows in two directions, without expense of energy for the modes. However, the variances of the number of photons oscillate in time, being a consequence of the exchange of the characteristics of the modes. Moreover, if the two fields have the same phase, looking at the reduced density matrix of one mode (call it \( A \)), we verify that the linear entropy functional of \( A \) is independent of mode \( B \), this tells that when the mean energy of each oscillator
remains constant during the evolution, the functional describes exclusively the flow of information to and from $A$. We also introduce a functional that measures the ‘degree of exchangeness’ between the modes.

The paper is planned as follows. In section II we consider the rotating wave coupling between two modes of the EM field in a nonlinear crystal and solve the dynamical equations for the mode operators. In section III we derive the joint-state vector and write it down at some characteristic times for the sake of comparison. In section IV we discuss the decoherence and entropy for one mode, the other mode assuming the role of ‘environment’. In section V we discuss the intermode exchange of information with and without energy transfer and section VI contains a summary and our conclusions.

II. DYNAMICS OF THE TWO-MODE SYSTEM AND CHARACTERISTIC TIMES

In a good approximation, the hamiltonian describing the interaction between two modes of EM field in the presence of a strong classical pumping in a nonlinear crystal can be written as

$$H = \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b + 2\hbar \lambda \cos(\nu t - \phi) \left( a^\dagger + a \right) \left( b^\dagger + b \right)$$

$$= \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b + \hbar \lambda \left[ e^{i(\nu t - \phi)} \left( a^\dagger b^\dagger + ab + a^\dagger b + ab^\dagger \right) + \text{h.c.} \right]$$

$$= H_a + H_b + V_I,$$  \hspace{1cm} (1)

where h.c. means hermitean conjugate, the coupling parameter $\lambda$ is the product of the pump amplitude and the coupling constant between the EM-field and the crystal, and $\nu$ is the pump field frequency. The operators $a, b$ and $a^\dagger, b^\dagger$ are the destruction and creation of the field quanta.

Since the Hamiltonian (1) is quadratic, the solutions of the Heisenberg equations of motion, $a(t)$ and $b(t)$, are linear combinations of all four initial operators, $a(0), b(0), a^\dagger(0), b^\dagger(0)$, with time-dependent coefficients $\tilde{a}$ and $\tilde{b}$. There are, however, two important special cases, when these linear combinations have only two terms.

If $\nu \approx \omega_a + \omega_b$, then only the terms proportional to the products $e^{i\nu t}ab$ and $e^{-i\nu t}a^\dagger b^\dagger$ in the interaction part of the Hamiltonian (1) contribute effectively to the process (the free operators evolve in the Heisenberg picture as $a_0(t) = e^{-i\omega_a t}a(0)$ and $b_0(t) = e^{-i\omega_b t}b(0)$). Thus omitting all non-essential terms one arrives at the effective counterrotating wave coupling (CWC),

$$V_{CWC} = \hbar \lambda \left[ e^{i(\phi - \nu t)}a^\dagger b^\dagger + e^{i(\nu t - \phi)}ab \right],$$  \hspace{1cm} (2)

which describes the parametric amplification or parametric down conversion. In this case, $a(t)$ is a linear superposition of $a(0)$ and $b^\dagger(0)$ only, the coefficients being hyperbolic functions of time (if $\nu = \omega_a + \omega_b$), so no exchange between operators $a$ and $b$ is possible.

We concentrate on another special case, $\nu \approx \omega_a - \omega_b$, which corresponds to the up-conversion process. In this case, the most significant part of the interaction Hamiltonian is given by the rotating wave coupling (RWC)

$$V_{RWC} = \hbar \lambda \left[ e^{i(\nu t - \phi)} ab^\dagger + e^{i(\phi - \nu t)} a^\dagger b \right],$$  \hspace{1cm} (3)

so the equations of motion for the operators $\tilde{a}(t) = a(t)e^{i\omega_a t}$ and $\tilde{b}(t) = b(t)e^{i\omega_b t}$ (which correspond to the interaction picture) read

$$\frac{d\tilde{a}}{dt} = -i\lambda e^{i(\Omega t + \phi)}\tilde{b},$$  \hspace{1cm} (4)

$$\frac{d\tilde{b}}{dt} = -i\lambda e^{-i(\Omega t + \phi)}\tilde{a},$$  \hspace{1cm} (5)

where $\Omega \equiv \omega_a - \omega_b - \nu \ll \omega_a, \omega_b$. Recently, Senitzky analysed the physical nuances of the zero-point energy, considering four different kinds of two-mode interaction, identifying the presence of the van der Waals attraction in all but the rotating wave-coupling interaction. Nevertheless it is exactly this coupling that will play an essential role here, becoming the focus of our attention.

Deriving equation (4) with respect to time and substituting then equation (5), one obtains a second order differential equation to $\tilde{a}$ only,

$$\frac{d^2\tilde{a}}{dt^2} - i\Omega \frac{d\tilde{a}}{dt} + \lambda^2 \tilde{a} = 0,$$  \hspace{1cm} (6)
Solving (11) with the account of the initial conditions and returning back to the Heisenberg operators $a(t)$ and $b(t)$, we obtain

\begin{align}
a(t; \tau) &= e^{-i\omega t} \left[ u_1(\tau)a(0) + v_1(\tau)b(0) \right] = e^{-i\omega t} \tilde{a}(\tau), \quad (7) \\
b(t; \tau) &= e^{-i\omega t} \left[ u_2(\tau)b(0) + v_2(\tau)a(0) \right] = e^{-i\omega t} \tilde{b}(\tau), \quad (8)
\end{align}

where the time-dependent coefficients are given by

\begin{align}
u_1(\tau) &= e^{i\chi \tau} \left[ \cos \tau - i \chi \sin \tau \right], \quad u_2(\tau) = u_1^*(\tau); \\
v_1(\tau) &= -i \sqrt{1 - \chi^2} e^{i(\chi + \phi) \tau}, \quad v_2(\tau) = -v_1^*(\tau), \quad (10)
\end{align}

and

\begin{align}
\tau &= \omega \tau, \quad \chi &= \frac{\Omega}{2\omega}, \quad \omega = \sqrt{\Omega^2 + 4\lambda^2/2}. \quad (11)
\end{align}

Now we notice that the RWC approximation is justified provided the coupling and detuning are much smaller than the fields frequencies, i.e., $\lambda, \Omega \ll \omega_a, \omega_b$. Under these conditions, the true time $t$ and the dimensionless ‘slow time’ $\tau$ can be considered actually as independent variables, since even a shift of $t$ by $\pi/\omega_{a,b}$ practically does not affects the value of $\tau$ and, consequently, the values of coefficients $u_j(\tau)$ and $v_j(\tau)$, $j = 1, 2$.

Looking at equation (10) we notice that for the values of ‘slow time’ $\tau = n\pi, \ n = 1, 2, 3, \ldots$ the coefficients $v_{1,2}(\tau)$ turn into zero, so the operators $a(t)$ and $b(t)$ assume their initial values, up to unitary phase factors:

\begin{align}
\hat{a}(\tau_n) &= a(0) \exp \left[ i\pi(1 + \chi) \right], \\
\hat{b}(\tau_n) &= b(0) \exp \left[ i\pi(1 - \chi) \right]. \quad (12, 13)
\end{align}

These results are in agreement with the quantum recurrence theorem as proposed by Bocchieri and Loinger [31] and $T = \pi/\omega$ is the recurrence time.

Another important characteristic time of the two-mode system is the period when an exchange (but for a phase factor) of the two modes operators takes place, namely,

\begin{align}
\hat{a}(\tau'_n) &= e^{i\delta_n} b(0), \quad \hat{b}(\tau'_n) = e^{i\delta_n} a(0). \quad (14)
\end{align}

These conditions imply $u_1(\tau'_n) = u_2(\tau'_n) = 0$, which can happen only provided $\Omega = \chi = 0$ and $\tau'_n = (n - 1/2)\pi, \ n = 1, 2, 3, \ldots$. The choice $\phi = \pi/2$ for the pump field then gives

\begin{align}
\hat{a}(\tau'_n) &= b(0) e^{i(n+1)\pi} = e^{i\delta_n} b(0), \\
\hat{b}(\tau'_n) &= a(0) e^{i\pi n} = e^{i\delta_n} a(0), \quad (15)
\end{align}

thus identifying $\theta_n$ and $\delta_n$. The exchange time, $T' = \pi/(2\lambda)$, is the fundamental period when operators are exchanged, it is half the recurrence time when one sets $\Omega = 0$, $T = 2T'$. Summarizing, when this resonance condition is introduced into the problem the time-evolving operators $\hat{a}(\tau)$ and $\hat{b}(\tau)$, will recur at times multiples of $\pi/\lambda$ and exchange identity at odd multiples of $\pi/(2\lambda)$ (except for a phase factor).

### III. Joint State Vector in the RWC: Recurrence and Exchange of Identity

Now we are proceeding to calculate the time evolution of the two-mode state vector, with the goal to determine the times at which the system recurs and the times when the modes exchange their states, i.e., each one assuming the state of the other at $\tau = 0$. We confine ourselves to a special class of initial states, which can be described as a finite superposition of the coherent states. In this case one can easily transform the expressions for the time-dependent Heisenberg operators found above to the expressions for the state vectors, using the method of characteristic functions [2, 32, 33].

The symmetric form of the characteristic function is

\begin{align}
\chi_S(\eta, \zeta, t) &= \text{Tr}_{AB} \left[ \rho_{AB}(t) e^{\eta a^\dagger + \zeta b^\dagger - \eta^* a - \zeta^* b} \right] \\
&= \text{Tr}_{AB} \left[ \rho_{AB}(0) e^{\eta a^\dagger(t) + \zeta b^\dagger(t) - \eta^* a(t) - \zeta^* b(t)} \right], \quad (17)
\end{align}
where the RHS of the first (second) line stands for the Schrödinger (Heisenberg) picture, $\rho_{AB}$ being the density operator for the two-mode system, $A + B$. Whenever the field operators $a(t)$ and $b(t)$ (Heisenberg picture) depend linearly on $a(0)$ and $b(0)$ (Schrödinger picture) it becomes possible to define new time-dependent functions

\[
\tilde{\eta} \equiv \tilde{\eta}(\eta, \zeta; t), \quad \tilde{\zeta} \equiv \tilde{\zeta}(\eta, \zeta; t),
\]

and rewrite (17) as

\[
\chi_S(\eta, \zeta, t) = \text{Tr}_{AB} \left[ \rho_{AB}(0) e^{\tilde{\eta}a^\dagger + \tilde{\zeta}b^\dagger - \tilde{\eta}^* a - \tilde{\zeta}^* b} \right] \equiv \chi_S(\tilde{\eta}, \tilde{\zeta}, 0). \tag{19}
\]

The normal form of the characteristic function can be found through the relation

\[
\chi_N(\eta, \zeta, t) = e^{\frac{\chi_S(\eta, \zeta, t)}{2} - \frac{\chi_S(0)}{2}} \chi_S(\tilde{\eta}, \tilde{\zeta}, 0),
\]

which can be written in terms of (18) as

\[
\chi_N(\eta, \zeta, t) = e^{\frac{1}{4} \left[ \left| \eta \right|^2 + \left| \zeta \right|^2 - \left| \eta \right|^2 - \left| \zeta \right|^2 \right]} \chi_N(\tilde{\eta}, \tilde{\zeta}, 0), \tag{21}
\]

with

\[
\chi_N(\tilde{\eta}, \tilde{\zeta}, 0) = e^{\frac{\chi_S(0)}{2} + \frac{\chi_S(\eta)}{2}} \chi_S(\tilde{\eta}, \tilde{\zeta}, 0). \tag{22}
\]

Substituting Eqs. (18) in the second line of Eq. (20) and after rearranging the terms we obtain

\[
\chi_S(\eta, \zeta, t) = \text{Tr}_{AB} \left[ \rho_{AB}(0) e^{\tilde{\eta}a^\dagger - \tilde{\zeta}b^\dagger} \right] = \chi_S(\tilde{\eta}, \tilde{\zeta}, 0), \tag{23}
\]

where the dynamical evolution is present only in the parameters

\[
\tilde{\eta} = \eta u_1^*, \quad \tilde{\zeta} = \eta^* u_2^*, \tag{24}
\]

\[
\tilde{u}_1 \equiv e^{-i\omega_0 t}u_1(t), \quad \tilde{v}_1 \equiv e^{-i\omega_0 t}v_1(t), \quad \tilde{u}_2 \equiv e^{-i\omega_0 t}u_2(t), \quad \tilde{v}_2 \equiv e^{-i\omega_0 t}v_2(t). \tag{25}
\]

From the definitions (24) and (25) one verifies that

\[
\left| \eta \right|^2 + \left| \zeta \right|^2 = \left| \eta \right|^2 + \left| \zeta \right|^2,
\]

then equation (21) becomes

\[
\chi_N(\eta, \zeta, t) = \chi_N(\tilde{\eta}, \tilde{\zeta}, 0). \tag{26}
\]

If the initial joint density operator is factorised (absence of initial correlations),

\[
\rho_{AB}(0) = \rho_A(0) \otimes \rho_B(0), \tag{27}
\]

then the normal joint characteristic function (Fourier space) factorises as

\[
\chi_N(\eta, \zeta, t) = \text{Tr}_{A} \left[ \rho_A(0) e^{\tilde{\eta}a^\dagger} \right] \text{Tr}_{B} \left[ \rho_B(0) e^{\tilde{\zeta}b^\dagger} \right] = \chi_A^A(\tilde{\eta}, 0) \chi_B^B(\tilde{\zeta}, 0). \tag{28}
\]

Thus, contrarily to the density operator which correlates in the course of its evolution, in the Fourier space the modes evolve in an apparent uncorrelated fashion, however there is a hidden correlation which is present in the parameters (24) and (25), since they depend on the functions (4) and (10).

We assume that the mode $A$ was prepared ‘initially’ ($\tau = 0$) in a superposition of two coherent states,

\[
|\Psi_A(t; \tau = 0)\rangle = \frac{1}{N} \left( |\alpha e^{-i\omega_0 t} \rangle + e^{i\Phi} |\alpha e^{-i\omega_0 t} \rangle \right), \quad N = \sqrt{2 \left( 1 + \cos \Phi e^{-2|\alpha|^2} \right)}. \tag{29}
\]
The special cases of the state \( \omega \), for \( \Phi = 0, \pi/2, \pi \) are known as even cat state \([34]\), Yurke-Stoler state \([35]\) and odd cat state \([34]\), respectively. The ‘initial’ density operator is therefore \( \rho_A(t; \tau = 0) = \left| \psi_A(t; \tau = 0) \right\rangle \left\langle \psi_A(t; \tau = 0) \right| \). The mode \( B \) is assumed to be ‘initially’ in the coherent state \( |\beta e^{-i\omega_B t}\rangle \) with \( \rho_B(t; \tau = 0) = |\beta(t; \tau = 0)\rangle \langle \beta(t; \tau = 0)| \). By ‘initial’ we do not mean \( t = 0 \), but when the interaction is yet not turned on, the slow motion beginning at \( \tau = 0 \) when \( \lambda = 0 \) (not \( t = 0 \)).

Substituting \( \rho_A(t; 0) \) and \( \rho_B(t; 0) \) into the characteristic function \([30]\) one obtains
\[
\chi_N(\eta, \zeta, t) = \frac{1}{N^2} \left[ e^{i\eta \alpha^* + i\eta^* \alpha} + e^{-i\eta \alpha + i\eta^* \alpha} + e^{-2|\alpha|^2} \left( e^{i\Phi} e^{i\eta \alpha + i\eta^* \alpha} + e^{-i\Phi} e^{-i\eta \alpha + i\eta^* \alpha} \right) \right] e^{i\bar{\beta} \alpha^* - i\bar{\zeta} \alpha} \chi_N(\eta, \zeta, t) \] (32)

Using equations \([24]\) and \([25]\) and after some algebraic manipulation one gets
\[
\chi_N(\eta, \zeta, t) = \frac{1}{N^2} \left[ \exp \left[ \eta z_1^* - \eta^* z_1 + \chi z_3^* - \chi^* z_3 \right] + \exp \left[ -\eta z_2^* + \eta^* z_2 - \chi z_4^* + \chi^* z_4 \right] + e^{-2|\alpha|^2} \left( e^{i\Phi} \exp \left[ \eta z_1^* + \eta^* z_2 + \chi z_3^* + \chi^* z_4 \right] + e^{-i\Phi} \exp \left[ -\eta z_2^* - \eta^* z_1 - \chi z_4^* - \chi^* z_3 \right] \right) \right] \] (33)

where
\[
\begin{align*}
z_1 &= u_1 \alpha + \bar{v}_1 \beta, \quad z_2 = u_1 \alpha - \bar{v}_1 \beta, \quad z_3 = \bar{v}_2 \alpha + \bar{v}_2 \beta, \quad z_4 = \bar{v}_2 \alpha - \bar{v}_2 \beta,
\end{align*}
\] (34)

thus the time-dependent new labels \( z_i \) are the linear superpositions of the initial ones, \( \alpha, \beta, \) for the joint-state written in the Schrödinger picture, where the multiplicative time-dependent phases \( e^{-i\omega_B t} \) of the functions \( u_i, \bar{v}_i (i = a, b) \) in the definition \([26]\) stand for the ‘fast’ oscillation of the field whereas the time dependence in \( u_i(\tau) \) and \( \bar{v}_i(\tau) \) correspond to the ‘slow’ motion of the field configuration. We can write the two-mode normal characteristic function in the Schrödinger picture also as
\[
\chi_N(\eta, \zeta, t) = \text{Tr}_{AB} \left[ \rho_{AB}(t) e^{i\eta \alpha^* + i\eta^* \alpha} e^{i\bar{\beta} \alpha^* - i\bar{\zeta} \alpha} \right] = \langle \Psi_{AB}(t) | e^{i\eta \alpha^* + i\eta^* \alpha} e^{i\bar{\beta} \alpha^* - i\bar{\zeta} \alpha} | \Psi_{AB}(t) \rangle \] (35)

For a superposition \( |\Psi_{AB}(t)\rangle = e_1|\Psi_1(t)\rangle + e_2|\Psi_2(t)\rangle \), a comparison between Eqs. \([35]\) and \([33]\) permits to identify the joint statevector, at any time \( t \), showing an entanglement of the modes,
\[
|\Psi_{AB}(t)\rangle = \frac{1}{N} \left( |z_1, z_3\rangle + e^{i\Phi} | -z_2, -z_4\rangle \right) = \frac{1}{N} \left( |z_1\rangle_A \otimes |z_3\rangle_B + e^{i\Phi} | -z_2\rangle_A \otimes |-z_4\rangle_B \right),
\] (36)

where the \( |\pm z_i\rangle \) are coherent states.

The time-dependent functions \([30]\) and \([32]\) calculated at ‘slow’ recurrence times \( t_n = n \pi \), with \( n = 1, 2, 3, ..., \) are
\[
\begin{align*}
u_1(\tau_n) &= e^{i\eta \alpha^* + i\eta^* \alpha} e^{i\Phi} | -\alpha(t)\rangle \rangle_A \otimes | \beta(t)\rangle_B, \end{align*}
\] (37)

So the joint statevector \([32]\) is
\[
|\Psi_{AB}(t; \tau_n)\rangle = \frac{1}{N} \left( |\alpha_n(t)\rangle + e^{i\Phi} | -\alpha_n(t)\rangle \right) \rangle_A \otimes | \beta_n(t)\rangle_B,
\] (38)

with
\[
\alpha_n(t) \equiv \alpha \exp \left[ -i\omega_B t + i\pi (1 + \chi) \right], \quad \beta_n(t) \equiv \beta \exp \left[ -i\omega_B t + i\pi (1 - \chi) \right].
\] (39)

Thus \( |\Psi_{AB}(t; \tau_n)\rangle \) has the same functional form as the initial state, although the positions of the centers of the peaks are rotated in the complex planes \( \alpha \) and \( \beta \). However, exact recurrence of the fields can happen for the choice \( \chi = m/n (m < n), \ m = 1, 2, 3, ... \) and \( n = m + 1, m + 2, ..., m + n \) is even, thus one gets \( \alpha_n(t) = \alpha e^{-i\omega_B t}, \) and \( \beta_n(t) = \beta e^{-i\omega_B t}. \)

For \( \Omega = \chi = 0, \phi = \pi/2 \) and at ‘slow’ times \( \tau_n = (n - 1/2)\pi \) with \( n = 1, 2, 3, ... \), the functions \([30]\) and \([32]\) take the values
\[
\begin{align*}
u_1(\tau_n) &= u_2(\tau_n) = 0, \quad v_1(\tau_n') = (1)^{n+1}, \quad v_2(\tau_n') = (1)^n,
\end{align*}
\] (40)

so, the joint statevector writes
\[
|\Psi_{AB}(t; \tau_n')\rangle = |(1)^{n+1} \beta e^{-i\omega_B t}\rangle_A \otimes \frac{1}{N} \left( |(1)^n \alpha e^{-i\omega_B t}\rangle + e^{i\Phi} | - (1)^n \alpha e^{-i\omega_B t}\rangle \right). \] (41)
Now the initial superposition state (mode A) and coherent state (mode B) do an exchange of configuration at ‘slow’ times $\tau_n' = (n - 1/2)\pi$, although the phases of the field variables are not exactly the same as in the original state (at $\tau = 0$). For $n = 2, 4, 6, ...$ the state of mode $A$ is $| - \beta e^{-i\omega_i t} \rangle$, thus the field acquires an extra phase (it suffers a rotation by an angle $\pi$ in phase space) but the mode $B$ becomes exactly the original superposition state of mode $A$. For $n = 1, 3, 5, ...$ the state of mode $A$ is $| \beta e^{i\omega_i t} \rangle$ and of mode $B$ is the superposition $| - \alpha e^{-i\omega_i t} + e^{i\Phi}|\alpha e^{-i\omega_i t} \rangle$, where $\alpha$ changed sign, thus not repeating the original superposition of mode $A$. However, for $\Phi = 0$ the state of mode $B$ becomes identical to the state of mode $A$ at $\tau = 0$ and vice versa; for $\Phi = \pi$ the only difference is that state $B$ becomes equal to the initial superposition of mode $A$ times a global phase (a minus sign). So, for $\Phi = 0, \pi$ and at times $t_n, n = odd$ an exchange of state configuration between the two modes takes place, the coherent state goes to the superposition state and vice-versa, this shows that (at $\tau_n$'s there is a full transfer of information among the modes, allowing an exchange of identity. Moreover, it is worth observing that in (41) factors $\nu$ frequency $\tau$ to the superposition state and vice-versa, this shows that (at times $t$ $A$ becomes identical to the state of mode $B$ becomes a pure state when again $\tau = 0$), when the states of modes are uncorrelated, then the initial pure state decoheres and becomes a mixed state (the ascending curve), the nondiagonal terms of the density operator will be the decoherence, attaining its climax at $T = 1/\lambda$. For a statistical mixture $A$ recoheres, returning to the initial state at $T = \pi/\lambda$. The periodicity of decoherence and recoherence occurs because the ‘environment’ of mode $A$ is constituted by a single mode $B$ [6]; for a multimode environment the recoherence time would be quite larger and for an infinite number of modes the decoherence becomes irreversible [10,17].

We remind that the exchange of identity between the modes is possible only for $\Omega = 0$, which means that the frequency $\nu$ of the pump field must be ‘resonant’ with the difference of frequencies $\nu_a - \nu_b$ (if $\nu_a > \nu_b$, otherwise one should choose $\nu = \nu_b - \nu_a$). At ‘slow’ times $\tau_n = \pi(n - 1/2)$ the states are maximally correlated, attaining the maximal entanglement.

IV. DECOHERENCE AND ENTROPY

In order to analyse the decoherence of a single mode, for instance mode $A$, we have to follow the time evolution of its state independently of that for mode $B$. The joint density operator is

$$\rho_{AB}(t) = \frac{1}{N^2} \left( |z_1, z_3 \rangle \langle z_1, z_3| + | - z_2, -z_4 \rangle \langle -z_2, -z_4| + e^{i\Phi} |z_1, z_3 \rangle \langle -z_2, -z_4| + e^{-i\Phi} | - z_2, -z_4 \rangle \langle z_1, z_3| \right),$$

and the calculation of $\rho_A = \text{Tr}_B [\rho_{AB}]$ gives the reduced density operator (hereafter we set $\Omega = 0$)

$$\rho_A(t) = \frac{1}{N^2} \left( |z_1 \rangle \langle z_1| + | - z_2 \rangle \langle -z_2| \right) + \exp \left[ - |\alpha|^2 (1 - \cos(2\lambda t)) \right] \left[ e^{-i\Phi} c^{2\text{Im}(\omega_2 r^2 \alpha^* \beta)} |z_1 \rangle \langle -z_2| + H.c. \right).$$

The first exponential factor is responsible for the decoherence (reduction of the nondiagonal terms in (43)). The decoherence process begins altogether with the evolution of the two modes, and the larger is $|\alpha|$ the more pronounced will be the decoherence, attaining its climax at $T_d = \pi/(2\lambda)$ (the exponential factor becomes $\exp(-2|\alpha|^2)$), then it begins its way back, the state of mode $A$ recoheres, returning to the initial state at $T_r = \pi/\lambda$. The periodicity of decoherence and recoherence occurs because the ‘environment’ of mode $A$ is constituted by a single mode $B$ [6]; for a multimode environment the recoherence time would be quite larger and for an infinite number of modes the decoherence becomes irreversible [10,17].

The linear entropy

$$S_A(t) = \text{Tr}_A [\rho_A(t) - \rho^2_A(t)],$$

is well suited for the analysis of the decoherence; from the norm conservation, $\text{Tr}_A [\rho_A(t)] = 1$, (43) becomes $S_A(t) = 1 - \text{Tr}_A \rho^2_A(t)$. For a pure state $\rho^2_A(t) = \rho_A(t)$, $\text{Tr}_A \rho^2_A(t) = 1$, thus $S_A(t) = 0$. For a statistical mixture $0 \leq \text{Tr}_A [\rho^2_A(t)] \leq 1$, then $0 \leq S_A(t) \leq 1$. For (43) the entropy writes as

$$S_A(t) = 1 - \frac{2}{N^4} \left[ 1 + e^{-2|\alpha|^2(1 + \cos(2\lambda t))} + e^{-2|\alpha|^2(1 - \cos(2\lambda t))} + 4 \cos \Phi e^{-2|\alpha|^2} 
+ e^{-4|\alpha|^2} \cos [2\Phi + 2 \sin(2\lambda t) \text{Im}(\alpha \beta^*)] \right],$$

and it is plotted in Fig. 1 for $\text{Im}(\alpha \beta^*) = 0$, $\Phi = 0, \pi/2, \pi$ and $|\alpha|^2 = 1$, where one can see the decoherence-recoherence cycles. Initially the entropy is minimum ($S_A = 0$), when the states of modes are uncorrelated, then the initial pure state decoheres and becomes a mixed state (the ascending curve), the nondiagonal terms of the density operator attain their lowest value (the maxima of the curve), then they increase back, recohering (the descending curve) and the reduced density operator for $A$ becomes a pure state when again $S_A = 0$ at $\lambda t = \pi/2$, and the joint state
disentangles (in particular, when recurrence or exchange of identity of the states takes place). This cycle repeats with period $\pi/2\lambda$. Since the two modes are symmetrical the entropy of mode $B$ is the same as $\mathcal{E}_B(t) = \mathcal{E}_A(t)$, thus during the interaction the information flows at the same rate in the two ‘directions’, $A \leftrightarrow B$. In Fig. 2 we show the entropy with the same settings as in Fig. 1, except that here $|\alpha|^2 = 5$; the three ‘cat’ states have nearly the same entropy and the duration of the entangled state is larger (the plateau at the maxima) compared to curves in Fig. 1. The mean energy of two modes (for the state (42)) are given by

$$
\langle E_A(t) \rangle = \text{Tr}_{AB} \left[ \rho_{AB}(t) \hbar \omega_a \alpha^\dagger \alpha \right] = \frac{2\hbar\omega_a}{N^2} \left[ |\alpha|^2 \cos^2(\lambda t) \left( 1 - \cos \Phi \ e^{-2|\alpha|^2} \right) + |\beta|^2 \sin^2(\lambda t) \left( 1 + \cos \Phi \ e^{-2|\alpha|^2} \right) + \text{Im}(\alpha\beta^*) \sin(2\lambda t) \sin \Phi \ e^{-2|\alpha|^2} \right],
$$

$$
\langle E_B(t) \rangle = \text{Tr}_{AB} \left[ \rho_{AB}(t) \hbar \omega_b \beta^\dagger \beta \right] = \frac{2\hbar\omega_b}{N^2} \left[ |\alpha|^2 \sin^2(\lambda t) \left( 1 - \cos \Phi \ e^{-2|\alpha|^2} \right) + |\beta|^2 \cos^2(\lambda t) \left( 1 + \cos \Phi \ e^{-2|\alpha|^2} \right) - \text{Im}(\alpha\beta^*) \sin(2\lambda t) \sin \Phi \ e^{-2|\alpha|^2} \right],
$$

both mean energies oscillate in time but out of phase by $\pi$, their sum oscillates in time too since $\omega_a \neq \omega_b$. For the interaction (3) the total number of photons of the two modes, $n_A + n_B = a^\dagger a + b^\dagger b$, is a conserved quantity, however, the number of photons of each mode is not,

$$
[H, n_A + n_B] = 0, \quad [H, n_A] \neq 0, \quad [H, n_B] \neq 0.
$$

The expectation value of $n_A + n_B$ in (3) is given by

$$
\bar{n}_A + \bar{n}_B = \frac{\langle E_A(t) \rangle}{\hbar \omega_a} + \frac{\langle E_B(t) \rangle}{\hbar \omega_b} = \frac{2}{N^2} \left[ |\alpha|^2 \left( 1 - \cos \Phi \ e^{-2|\alpha|^2} \right) + |\beta|^2 \left( 1 + \cos \Phi \ e^{-2|\alpha|^2} \right) \right],
$$

and is a constant of motion.

V. INFORMATION EXCHANGE WITHOUT INTERMODE ENERGY TRANSFER

In general, decoherence of a subsystem is presented altogether with energy flow to the “other” degrees of freedom (the environment) [39], thus transference of information between two interacting subsystems goes along with a transference of energy. For instance, the processes of decoherence and damping of a system coupled to a thermal reservoir is interpreted as a leak of information from the system to the environment altogether with a loss of energy, the dynamical evolution terminating with the thermalization of the system [57]. Now we shall verify that for some particular relation between $\alpha$ and $\beta$, the transference of information between the modes may happen without energy transfer.

Looking at the RHS of Eqs.(46) and (47) one notes that if the modes are prepared such that $\text{Im}(\alpha\beta^*) = 0$, i.e. $\alpha$ and $\beta$ have the same phase, and the intensities are related by the relation

$$
|\beta|^2 = |\alpha|^2 \frac{1 - \cos \Phi \ e^{-2|\alpha|^2}}{1 + \cos \Phi \ e^{-2|\alpha|^2}}, \quad (50)
$$

each mean energy becomes constant in time. In particular, for $\Phi = 0$, $|\beta|^2 = |\alpha|^2 \text{tanh} |\alpha|^2$, when

$$
\langle E_A(t) \rangle = \hbar \omega_a |\alpha|^2 \text{tanh} |\alpha|^2, \quad \langle E_B(t) \rangle = \hbar \omega_b |\alpha|^2 \text{tanh} |\alpha|^2, \quad (51)
$$

and for $\Phi = \pi$, $|\beta|^2 = |\alpha|^2 \text{coth} |\alpha|^2$, when

$$
\langle E_A(t) \rangle = \hbar \omega_a |\alpha|^2 \text{coth} |\alpha|^2, \quad \langle E_B(t) \rangle = \hbar \omega_b |\alpha|^2 \text{coth} |\alpha|^2. \quad (52)
$$

So, optimal transference of information between modes (in the sense of exchange of states configuration) without energy transfer occurs for a particular relation between the field amplitudes, namely, for the modes possessing the same mean number of photons, with conservation of mean energy for each mode and the ratio between mean energies,

$$
\frac{\langle E_B(t) \rangle}{\langle E_A(t) \rangle} = \frac{\omega_b}{\omega_a}. \quad (53)
$$
depending only on the frequencies ratio.

However, the variance of the mean number of photons of each mode, \( n_A \) and \( n_B \) is not null neither constant, but it oscillates in time, for instance,

\[
V_{\Phi=0,\pi}(n_A) \equiv \langle n_A^2 \rangle - \langle n_A \rangle^2 = |\alpha|^4 \left[ \cos^4(\lambda t) - (1 - \sin^4(\lambda t)) \right] \left( \frac{1 + e^{-2|\alpha|^2}}{1 + e^{-2|\alpha|^2}} \right)^2 + |\alpha|^2 \cos^2(\lambda t) \left( \frac{1 + e^{-2|\alpha|^2}}{1 + e^{-2|\alpha|^2}} \right) \left[ 1 + 4|\alpha|^2 \left( \frac{1 + e^{-2|\alpha|^2}}{1 + e^{-2|\alpha|^2}} \right) \sin^2(\lambda t) \right],
\]

(54)

where upper (lower) sign stands for \( \Phi = 0(\pi) \), and for mode \( B \), one obtains an expression similar to (54), however oscillating out of phase. So, when there is no transference of energy between the modes the transference of information induces fluctuations in the number of photons. Therefore we can assert that if the modes have an equal mean number of photons the information transfer occurs without energy transfer, transference of energy taking place only when the number of photons is unequal, however a fluctuation in their number is verified. Since we assumed that the fields have the same phase, \( \text{Im}(\alpha^*\beta) = 0 \), the expression for the entropy (15) is the same whether the mean energies of the modes are conserved or not, so, the linear entropy (14) measures the flow of information from and to mode \( A \), independently of its mean energy being conserved or not. So, in this enlightening example we have seen that decoherence is a process of pure information transfer without any need of transference of energy.

The time-dependent entropy (15) becomes equal to zero at times when the joint state (42) disentangles, however it is immediate to verify that at times \( \tau_n = n\pi/2 \), \( n = 1, 5, 9, \ldots \), \( \mathcal{E}(\tau_n) \) assumes the same value,

\[
\mathcal{E} \left( \frac{\pi}{2} \right) = \left( \frac{\cos \Phi + e^{-2|\alpha|^2}}{1 + \cos \Phi \ e^{-2|\alpha|^2}} \right)^2,
\]

(57)

the fact that it is independent of \( \beta \) means that mode \( B \) transferred exactly its original configuration (coherent state) to mode \( A \), but simultaneously mode \( A \) cannot transfer its original ‘cat’ state form to mode \( B \) for any value of \( \Phi \), but only for the special cases of even and odd ‘cat’ states, \( \Phi = 0, \pi \), when \( \mathcal{E}(\pi/2) = 1 \). For \( |\alpha|^2 \gg 1 \), \( \mathcal{E}(\pi/2) = \cos^2 \Phi \), becoming independent of \( |\alpha| \). The Yurke-Stoler state (\( \Phi = \pi/2 \)) could not participate in the state-exchange scheme.

At times \( \tau_n = n\pi/2 \), \( n = 3, 7, 11, \ldots \), \( \mathcal{E}(\tau_n) \) assumes always the same value,

\[
\mathcal{E} \left( \frac{3\pi}{2} \right) = e^{-4|\beta|^2},
\]

(58)

being independent of \( \alpha \) and \( \Phi \), thus the mode \( A \) transferred information to mode \( B \) in order to reproduce the original state of \( A \), but the inverse did not occured, the mode \( A \) was unable to reproduce exactly the state of mode \( B \); an exact exchange of identity only happens if the original state of mode \( B \) is the vacuum state, \( \beta = 0 \), independently of the value the phase \( \Phi \) can take. In Fig. 3 we plotted (55) versus time for \( \Phi = 0 \) and \( |\alpha|^2 = 5 \), \( |\beta|^2 = |\alpha|^2 \tanh(|\alpha|^2) \), that illustrates our comments.
VI. SUMMARY AND CONCLUSIONS

We considered the problem of the coupling between two monochromatic modes (treated quantically) pumped by another (classical) monochromatic field, into a nonlinear crystal. For the RWC we determined the periods of recurrence (when modes $A$ and $B$ return to their initial joint state) and of state exchange, when information between the modes is exchanged, becoming possible that mode $A$ assumes the initial state of mode $B$ and vice-versa. We attributed to modes $A$ and $B$ the generic ‘cat state’ and the coherent state, respectively, as initial joint state; we analyzed its evolution during the interaction, verifying that an exact interchange between the states at certain periodic times occurs only when the ‘cat state’ is even or odd, or, the interchange occurs for any kind of ‘cat state’ only if mode $B$ is initially in the vacuum state. We introduced a time-dependent state-exchange functional, which showed to be quite helpful for measuring the degree of exchangeness of the state as times go on. We calculated the reduced entropy of mode $A$ and the linear entropy associated to it and analyzed the periodic decoherence and recoherence processes, and identified the maximum entanglement between the modes for the maximum of the entropy and disentanglement (at recurrence or state-exchange times) when the entropy becomes zero.

As the mean energy of each of the modes oscillates in time, their sum remaining constant for arbitrary number of photons in each mode, we verified that when the modes have the same mean number of photons the mean energies become constant in time, although the joint state goes on in its evolution, therefore intermode state-exchange (exchange of information) is possible without transference of energy, although the variance of the number of photons oscillates in time. We verified that the expression for the entropy related to the reduced density operator of mode $A$ does not change whether the mean energies are constant or not, thus it becomes clear the entropy is a measure of the flux of information, loss (decoherence) or gain (recoherence), without discerning on the direction of the flux of energy. In sum, by going back to an ‘old problem’ we analysed it from a different point of view from the other papers in the literature, unveiling interesting new features in the realm of pure information transfer between systems. As a last remark, we are considering the dynamics of the single mode phase operator $[38]$ which may lead to a deeper insight on the interaction process between the quantum fields; the calculations are in progress and results will be presented in a forthcoming publication.

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FIG. 1. The linear entropy of mode $A$ versus time in units of the parameter $\lambda$ and $|\alpha|^2 = 1$. The solid, dashed and dot-dashed lines correspond to the even ($\Phi = 0$), Yurke-Stoler ($\Phi = \pi/2$) and odd ($\Phi = \pi$) ‘cat’ states.

FIG. 2. The same settings as Fig. 1, except that $|\alpha|^2 = 5$. The three curves of Fig. 1 are practically coincident.

FIG. 3. The state-exchange functional versus time in units of $\lambda$, showing a coincident curve for even and odd ‘cat’ states ($\Phi = 0, \pi$) and attaining value 1 for $\lambda t = \pi/2, 5\pi/2, 9\pi/2, \ldots$.

FIG. 4. The same as Fig. 3 but for Yurke-Stoler state $\Phi = \pi/2$, the state-exchange functional never attains value 1.
