Proof of some conjectured formulas for $\frac{1}{\pi}$ by Z.W.Sun.

Gert Almkvist and Alexander Aycock

Recently Z.W.Sun found over hundred conjectured formulas for $\frac{1}{\pi}$. Many of them were proved by H.H.Chan, J.Wan and W.Zudilin (see [3], [9]). Here we show that several other formulas in [6] are simple transformations of known formulas for $\frac{1}{\pi}$, most of them due to Ramanujan. E.g. the following monstrous formula (not in [6])

$$\sum_{n=0}^{\infty} A_n P(n) \frac{1}{262537412640769728^n} = \frac{13803981511092062440689}{\pi^{163}}$$

where

$$P(n) = 4129922862271324476805 + 16564777691765267456000n$$

and

$$A_n = 1728^n \sum_{k=0}^{n} \binom{-1/12}{k} \binom{-7/12}{k} \binom{-5/12}{n-k} \binom{-11/12}{n-k}$$

is a transformation of Chudnovsky’s formula

$$\sum_{n=0}^{\infty} (-1)^n a_n (13591409 + 545140134n) \frac{1}{640320^{3n}} = \frac{53360\sqrt{640320}}{12\pi}$$

where

$$a_n = \binom{2n}{n} \binom{3n}{n} \binom{6n}{3n}$$

The transformation is

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{\sqrt{1-1728x}} \sum_{n=0}^{\infty} a_n \left(-\frac{x}{1-1728x}\right)^n$$

**General Transformation.**

Assume that we have a Ramanujan-like formula

$$\sum_{n=0}^{\infty} a_n (a + bn)x^n = \frac{1}{\pi}$$

We make the substitution

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{\sqrt{1-Mx}} \sum_{n=0}^{\infty} a_n \left(-\frac{x}{1-Mx}\right)^n$$

**Proposition 1.** Let

$$w_0 = \frac{x_0}{1-Mx_0}$$
Then we have the formula
\[
\sum_{n=0}^{\infty} A_n (A + Bn) w_n^0 = \frac{1}{\pi}
\]
where
\[
A = \left\{ \frac{1}{2} bMx_0 + a(1 - Mx_0) \right\} (1 - Mx_0)^{-3/2}
\]
and
\[
B = b(1 - Mx_0)^{-3/2}
\]

**Proof:** The transformation above is an involution, e.g. we also have
\[
\sum_{n=0}^{\infty} a_n x^n = \frac{1}{\sqrt{1 - Mx}} \sum_{n=0}^{\infty} A_n \left( -\frac{x}{1 - Mx} \right)^n
\]
Let \( \theta = x \frac{d}{dx} \). Then
\[
\sum_{n=0}^{\infty} a_n (a + bn) x^n = (a + b\theta) \sum_{n=0}^{\infty} a_n x^n
\]
\[
= (a + b\theta) \left\{ \frac{1}{\sqrt{1 - Mx}} \sum_{n=0}^{\infty} A_n \left( -\frac{x}{1 - Mx} \right)^n \right\}
\]
\[
= \sum_{n=0}^{\infty} (-1)^n A_n (a + b\theta) \left\{ \frac{x^n}{(1 - Mx)^{n+1/2}} \right\}
\]
\[
= \sum_{n=0}^{\infty} A_n \left\{ \frac{a + bn}{(1 - Mx)^{1/2}} + \frac{(n + \frac{1}{2})bx}{(1 - Mx)^{3/2}} \right\} \left( -\frac{x}{1 - Mx} \right)^n
\]
Substituting \( x = x_0 \) we are done.

The example in the introduction is the case \( s = \frac{1}{6} \) and \( M = 1728 \) of the hypergeometric case
\[
a_n = M^n (1/2)_n (s)_n (1 - s)_n \frac{1}{n!^3}
\]
Proving the transformation is a Maple exercise in each of the cases \( s = \frac{1}{3}, \frac{1}{4}, \frac{1}{6} \). E.g. in the case \( s = \frac{1}{6} \) one shows that both sides satisfy the differential equation
\[
y'''' + \frac{3(1 - 3456x)}{x(1 - 1728x)} y''' + \frac{1 - 11856x + 20155392x^2}{x^2(1 - 1728x)^2} y'' - \frac{24(31 - 93312x)}{x^2(1 - 1728x)^2} y' = 0
\]
and checks that the first four coefficients agree.
The case $s = \frac{1}{3}$.
Here we have $M = 108$ and
\[
a_n = 108^n \frac{(1/2)_n (1/3)_n (2/3)_n}{n!^3} = \left(\frac{2n}{n}\right)^2 \left(\frac{3n}{n}\right)
\]
with
\[
A_n = 108^n \sum_{k=0}^{n} \left(\frac{-2/3}{k}\right) \left(\frac{-1/6}{k}\right) \left(\frac{-1/3}{n-k}\right) \left(\frac{-5/6}{n-k}\right)
\]
In the table below the formula
\[
\sum_{n=0}^{\infty} a_n (a + bn) x_0^n = \frac{1}{\pi}
\]
is transformed to
\[
\sum_{n=0}^{\infty} A_n (A + Bn) w_0^n = \frac{1}{\pi}
\]

| # in [6] | $x_0$ | $a$ | $b$ | $w_0$ | $A$ | $B$ |
|---|---|---|---|---|---|---|
| 4.16 | $\frac{1}{192}$ | $\frac{\sqrt{3}}{4}$ | $\frac{5\sqrt{3}}{4}$ | 1 | $\frac{\sqrt{3}}{300}$ | $\frac{16\sqrt{3}}{25}$ |
| 4.18 | $\frac{1}{1728}$ | $\frac{7\sqrt{3}}{36}$ | $\frac{17\sqrt{3}}{12}$ | 1 | $\frac{11\sqrt{3}}{306}$ | $\frac{48\sqrt{3}}{153}$ |
| 4.20 | $\frac{1}{8640}$ | $\frac{\sqrt{15}}{12}$ | $\frac{3\sqrt{15}}{4}$ | 1 | $\frac{85\sqrt{3}}{8748}$ | 400$\sqrt{3}$ |
| 4.21 | $\frac{1}{108 \cdot 2^{10}}$ | $\frac{53\sqrt{3}}{288}$ | $\frac{205\sqrt{3}}{96}$ | $\frac{1}{110700}$ | $\frac{527\sqrt{123}}{18450}$ | $\frac{3072\sqrt{123}}{9225}$ |
| - | $\frac{1}{108 \cdot 3024}$ | $\frac{13\sqrt{7}}{108}$ | $\frac{55\sqrt{7}}{36}$ | $\frac{1}{326700}$ | $\frac{9989\sqrt{3}}{54450}$ | $\frac{127008\sqrt{3}}{54450}$ |
| 4.22 | $\frac{1}{108 \cdot 500^2}$ | $\frac{827\sqrt{3}}{4500}$ | $\frac{4717\sqrt{3}}{1500}$ | $\frac{1}{27000108}$ | $\frac{97659\sqrt{267}}{4500018}$ | $\frac{1500000\sqrt{267}}{4500018}$ |
| 4.17 | $\frac{1}{1458}$ | $\frac{8}{27}$ | $\frac{20}{9}$ | $\frac{1}{1350}$ | $\frac{52\sqrt{3}}{225}$ | $\frac{36\sqrt{3}}{25}$ |
| 4.19 | $\frac{1}{27 \cdot 125}$ | $\frac{8\sqrt{3}}{45}$ | $\frac{22\sqrt{3}}{45}$ | $\frac{1}{3267}$ | $\frac{100\sqrt{15}}{1089}$ | $\frac{250\sqrt{15}}{363}$ |
| 4.14 | $\frac{1}{27}$ | $\frac{4\sqrt{3}}{9}$ | $\frac{5\sqrt{3}}{3}$ | $\frac{1}{135}$ | $\frac{-2\sqrt{15}}{45}$ | $\frac{\sqrt{15}}{15}$ |

The last formula from columns 2-4 is divergent but results in the following supercongruence
\[
\sum_{n=0}^{p-1} a_n (4 + 15n) \frac{1}{(-27)^n} \equiv 4p \left(\frac{-3}{p}\right) \mod p^3
\]
conjectured by Z.W. Sun in [8].

The case $s = \frac{1}{4}$

Here we have $M = 256$ and

$$a_n = 256^n \frac{(1/2)_n (1/4)_n (3/4)_n}{n!^3} = \left(\frac{2n}{4n}\right)^2$$

with

$$A_n = 256^n \sum_{k=0}^n \left(\frac{-1/8}{k}\right) \left(\frac{-5/8}{k}\right) \left(\frac{-3/8}{n-k}\right) \left(\frac{-7/8}{n-k}\right)$$

| # in [6] | $x_0$ | $a$ | $b$ | $w_0$ | $A$ | $B$ |
|----------|-------|-----|-----|-------|-----|-----|
| 4.23     | $-\frac{1}{1024}$ | 3 | $\frac{5}{8}$ | $\frac{1}{1280}$ | $\sqrt{3}$ | $\frac{4\sqrt{3}}{5}$ |
| -        | $-\frac{1}{63^2}$ | $8\sqrt{2}$ | $\frac{25}{63}$ | $\frac{1}{4225}$ | $392\sqrt{7}$ | $3969\sqrt{7}$ |
| 4.27     | $-\frac{1}{3 \cdot 2^{12}}$ | $3\sqrt{3}$ | $\frac{7\sqrt{3}}{16}$ | $\frac{1}{12544}$ | 57 | 144 |
| 4.29     | $-\frac{1}{288^2}$ | 23 | $\frac{260}{72}$ | $\frac{1}{83200}$ | $113\sqrt{13}$ | $324\sqrt{13}$ |
| -        | $-\frac{1}{1280 \cdot 72^2}$ | $41\sqrt{5}$ | $\frac{256161\sqrt{5}}{288}$ | $\frac{1}{6635776}$ | $32995$ | $518400$ |
| -        | $-\frac{1}{14112^2}$ | 1123 | $\frac{5365}{882}$ | $\frac{1}{199148800}$ | $162833\sqrt{37}$ | $3111696\sqrt{37}$ |
| 4.26     | $\frac{1}{648}$ | 2 | $\frac{14}{9}$ | $\frac{1}{392}$ | 46 | 162 |
| 4.25     | $\frac{1}{48^2}$ | $\sqrt{3}$ | $\frac{4\sqrt{3}}{6}$ | $\frac{1}{2048}$ | $3\sqrt{3}$ | $9\sqrt{3}$ |
| 4.28     | $\frac{1}{144^2}$ | $2\sqrt{2}$ | $\frac{20\sqrt{2}}{9}$ | $\frac{1}{20480}$ | $17\sqrt{10}$ | $81\sqrt{10}$ |
| 4.30     | $\frac{1}{784^2}$ | $9\sqrt{3}$ | $\frac{120\sqrt{3}}{49}$ | $\frac{1}{614400}$ | $361\sqrt{2}$ | $2401\sqrt{2}$ |
| 4.31     | $\frac{1}{16^2 \cdot 99^2}$ | $19\sqrt{11}$ | $\frac{140\sqrt{11}}{99}$ | $\frac{1}{2508800}$ | $1331\sqrt{22}$ | $9801\sqrt{22}$ |
| -        | $\frac{1}{16^2 \cdot 99^4}$ | $2206\sqrt{2}$ | $\frac{52780\sqrt{2}}{9801}$ | $\frac{1}{24591257600}$ | $8029841\sqrt{58}$ | $192119202\sqrt{58}$ |
| 4.24     | $-\frac{1}{144}$ | $\sqrt{3}$ | $\frac{5\sqrt{3}}{3}$ | $\frac{1}{400}$ | $3\sqrt{3}$ | $9\sqrt{3}$ |

The last Ramanujan-like formula is divergent (for a proof see Guillera [2]) but leads
to the conjectured supercongruence (already in [8])

\[ \sum_{n=0}^{p^{-1}} a_n (1 + 5n) \frac{1}{(-144)^n} \equiv p \left( \frac{-3}{p} \right) \mod p^3 \]

The case \( s = \frac{1}{6} \).

Here \( M = 1728 \) and

\[ a_n = 1728^n (1/2) n (1/6) n (5/6) n \]

with

\[ a_n = 1728^n \sum_{k=0}^{n} \left( \frac{-1}{12} \right) \left( \frac{-7}{12} \right) \left( \frac{-5}{12} \right) \left( \frac{-11}{12} \right) \]

We have deleted the formula obtained from Chudnovsky’s formula since it is in the Introduction.

So far we have only considered Ramanujan series with rational \( x_0 \), found in [1]. We give one example in case \( s = \frac{1}{3} \) with \( x_0 = \frac{1}{72} (7\sqrt{3} - 12) \).
\[ \sum_{n=0}^{\infty} a_n (1 + (5 + \sqrt{3})n) \left( \frac{7\sqrt{3} - 12}{72} \right)^n = \frac{2 + \sqrt{3}}{\pi} \]

giving
\[ \sum_{n=0}^{\infty} A_n \left\{ 2(27\sqrt{3} - 41) + 8(5 + \sqrt{3})n \right\} \left( \frac{15 - 14\sqrt{3}}{66^2} \right)^n = \frac{254 - 134\sqrt{3}}{\pi} \]

In the paper [7] by Z.W. Sun there are some formulas for \( \frac{1}{\pi} \) which are special cases of identities for the hypergeometric function
\[ F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n \]

Thus Theorem 1.1 (i) in [7] is the special cases \( s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6} \) of

**Proposition 2:**

We have
\[ \sum_{n=0}^{\infty} n \left( \frac{1}{2} \right)^n \sum_{k=0}^{n} \frac{(s)_k(1-s)_k (s)_{n-k}(1-s)_{n-k}}{k!^2 (n-k)!^2} = \frac{2}{\pi} \sin(\pi s) \]

**Proof:** Let
\[ f(x) = F(s, 1-s; 1; x)^2 \]
\[ = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{(s)_k(1-s)_k (s)_{n-k}(1-s)_{n-k}}{k!^2 (n-k)!^2} \]

Then
\[ \theta f(x) = \sum_{n=0}^{\infty} n x^n \sum_{k=0}^{n} \frac{(s)_k(1-s)_k (s)_{n-k}(1-s)_{n-k}}{k!^2 (n-k)!^2} \]
\[ = 2s(1-s)x \cdot F(s, 1-s; 1; x) \cdot F(1+s, 2-s; 2; x) \]

Put \( x = \frac{1}{2} \) and use the evaluation
\[ F(a, b; \frac{a+b+1}{2}; 1) = \sqrt{\pi} \frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})} \]

We obtain
\[ 2^{\frac{1}{2}} s(1-s) \pi \frac{\Gamma(1)}{\Gamma(\frac{1}{2} + \frac{s}{2})\Gamma(1 - \frac{s}{2})} \frac{\Gamma(2)}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2} - \frac{s}{2})} \]
\[ = s(1-s) \pi \frac{1}{\Gamma(\frac{1}{2} - \frac{s}{2})\Gamma(\frac{s}{2})} \frac{1}{\Gamma(\frac{1}{2} + \frac{s}{2})(\frac{1}{2} - \frac{s}{2})\Gamma(\frac{1}{2} - \frac{s}{2})} \]

6
\[
\sin\left(\frac{\pi s}{2}\right) \sin\left(\frac{\pi \left(\frac{1}{2} - \frac{s}{2}\right)}{\pi}\right) = \frac{4}{\pi} \sin\left(\frac{\pi s}{2}\right) \cos\left(\frac{\pi s}{2}\right) = \frac{2}{\pi} \sin(\pi s)
\]

Some other transformations.
We start with proving Conjecture 4 in [6]. We have

**Proposition 3.** Let

\[
A_n = \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} (-s)^k \binom{-1-s}{n-k}
\]

Then the following formula is valid

\[
\sum_{n=0}^{\infty} A_n x^n = \frac{1}{\sqrt{1 + 4x}} 3F_2(1/2, s, 1-s; 1, 1; -\frac{4x^2}{1+4x})
\]

**Classical Proof:**

We first note the identities

\[
\binom{2k}{k} = 4^k \frac{(1/2)_k}{k!}
\]

\[
\binom{-s}{k} = (-1)^k \frac{(s)_k}{k!}
\]

where \((a)_0 = 1\) and \((a)_k = a(a+1)...(a+k-1)\) for \(k > 0\)

We get

\[
L = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} (-s)^k \binom{-1-s}{n-k} x^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1/2)_k (s)_k (1/2)_{n-k} (1-s)_{n-k}}{k!^2 (n-k)!^2} (-4x)^n
\]

\[
= F(1/2, s; 1; -4x) F(1/2, 1-s; 1; -4x)
\]

Now Euler’s identity

\[
F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x)
\]

leads to

\[
F(1/2, 1-s; 1; -4x) = (1+4x)^{-1/2+s} F(1/2, s; 1; -4x)
\]

Hence

\[
L = \frac{1}{\sqrt{1+4x}} \left\{(1+4x)^{s/2} F(1/2, s; 1; -4x)\right\}^2
\]
Now we use the following identity (see [2], p.176, Exercise 1b)

\[ F(2a, b; 2b; x) = (1 - x)^{-a} F(a, b - a; b + 1/2; \frac{x^2}{4(x - 1)} ) \]

with \( a = s/2, b = 1/2 \) to get

\[ F(s, 1/2; 1; -4x) = (1 + 4x)^{-s/2} F(s/2, (1 - s)/2; 1; -\frac{4x^2}{1 + 4x}) \]

Finally Clausen’s identity

\[ F(a, b; a + b + 1/2; x)^2 = F(2a, 2b, a + b; a + b + 1/2, 2a + 2b; x) \]

gives

\[ L = \frac{1}{\sqrt{1 + 4x}} F(1/2, s, 1 - s; 1, 1; -\frac{4x^2}{1 + 4x}) \]

and the proof is finished.

**Maple Proof:**

Using Maple one verifies that both sides satisfy the differential equation

\[ y''' + \frac{3(1 + 8x)}{x(1 + 4x)} y'' + \frac{1 + 28x + (108 + 16s - 16s^2)x^2}{x^2(1 + 4x)^2} y' + \frac{2(1 + (6 + 8s - 8s^2)x) - 1}{x^2(1 + 4x)^2} y = 0 \]

Then we check that the first terms in the power series solutions agree.

**Proposition 4.** Let

\[ a_n = \frac{(1/2)_n(s)_n(1 - s)_n}{n!^3} \]

Given a formula for \( \frac{1}{\pi} \) of Ramanujan type

\[ \sum_{n=0}^{\infty} a_n (a + bn)x_0^n = \frac{1}{\pi} \]

Let

\[ w_0 = \frac{1}{2} (-x_0 \pm \sqrt{x_0^2 - x_0}) \]

Then the transformation above gives the formulas

\[ \sum_{n=0}^{\infty} A_n (A + Bn)w_0^n = \frac{1}{\pi} \]

where

\[ A = \sqrt{1 + 4w_0} \left\{ a + \frac{bw_0}{1 + 2w_0} \right\} \]

\[ B = \frac{b(1 + 4w_0)^{3/2}}{2(1 + 2w_0)} \]
Proof: We have
\[ \sum_{n=0}^{\infty} A_n w^n = \frac{1}{\sqrt{1 + 4w}} \sum_{n=0}^{\infty} a_n \left( -\frac{w^2}{1 + 4w} \right)^n \]

Take \( A + B\theta \) on both sides (\( \theta = w \frac{d}{dw} \))

\[ \sum_{n=0}^{\infty} A_n(A + Bn)w^n = \sum_{n=0}^{\infty} a_n \left\{ \frac{A}{\sqrt{1 + 4w}} + \frac{2B}{(1 + 4w)^{3/2}} (-w + (2w + 1)n) \right\} \left( -\frac{w^2}{1 + 4w} \right)^n \]

Now put \( w = w_0 \) so \( -\frac{w_0^2}{1 + 4w_0} = x_0 \) and the right hand is \( \sum_{n=0}^{\infty} a_n(a + bn)x_0^n \). We get

\[ a = \frac{A}{\sqrt{1 + 4w_0}} - \frac{2Bw_0}{(1 + 4w_0)^{3/2}} \]

\[ b = \frac{2B(2w_0 + 1)}{(1 + 4w_0)^{3/2}} \]

and solving for \( A \) and \( B \) we are done.

\[ s = \frac{1}{2} \]

| \( x_0 \) | \( a \) | \( b \) | \( w_0 \) | \( A \) | \( B \) |
|-----|-----|-----|-----|-----|-----|
| -1/2 | 1/2 | 2 | 1/2 \( (1 - \sqrt{2}) \) | -3 + 2\( \sqrt{2} \) | -4 + 3\( \sqrt{2} \) |
| -1/2 | \( \sqrt{2} / 4 \) | 3\( \sqrt{2} / 2 \) | 1/16 \( (1 \pm 3) \) | 1/2 \( (1 \pm 1) \) | 1/4 \( (5 \pm 3) \) |
| -1/8 | \( \sqrt{3} / 4 \) | 3\( \sqrt{3} / 2 \) | 1/160 \( (1 \pm 9) \) | 3\( \sqrt{3} / 96 \) | 3\( \sqrt{3} / 48 \) |
| -1/16 | \( \sqrt{3} / 12 \) | 5\( \sqrt{3} / 12 \) | 3\( \sqrt{3} / 48 \) | 1/160 \( (1 \pm 9) \) | 1/4 \( (5 \pm 3) \) |
| -1/1024 | \( 53\sqrt{3} / 288 \) | 205\( \sqrt{3} / 96 \) | 1/2048 \( (1 \pm 5\sqrt{11}) \) | 533\( \sqrt{123} / 18432 \) | 513\( \sqrt{123} / 3072 \) |
| -1/3024 | \( 13\sqrt{7} / 108 \) | 55\( \sqrt{7} / 36 \) | 1/2048 \( (1 \pm 5\sqrt{11}) \) | \( \sqrt{3} / 7776 \) | \( \sqrt{3} / 1296 \) |
| -1/5002 | \( 827\sqrt{3} / 4500 \) | 4717\( \sqrt{3} / 1500 \) | 1/500000 \( (1 \pm 53\sqrt{89}) \) | 17533\( \sqrt{267} / 900000 \) | 1250001\( \sqrt{267} / 750000 \) |

\( s = \frac{1}{3} \)

| \( x_0 \) | \( a \) | \( b \) | \( w_0 \) | \( A \) | \( B \) |
|-----|-----|-----|-----|-----|-----|
| -9/16 | \( \sqrt{3} / 4 \) | 5\( \sqrt{3} / 4 \) | 3/32 \( (3 \pm 5) \) | \( \sqrt{3} / 32 \) \( (19 \pm 21) \) | \( \sqrt{3} / 16 \) \( (17 \pm 15) \) |
| -1/16 | \( \sqrt{3} / 36 \) | 17\( \sqrt{3} / 12 \) | 1/32 \( (1 \pm \sqrt{17}) \) | 17\( \sqrt{51} \pm 65\sqrt{3} \) / 288 | 9\( \sqrt{51} \pm 17\sqrt{3} \) / 48 |
| -1/80 | \( \sqrt{15} / 12 \) | 3\( \sqrt{15} / 4 \) | 1/160 \( (1 \pm 9) \) | \( \sqrt{3} / 96 \) \( (19 \pm 11) \) | \( \sqrt{3} / 48 \) \( (41 \pm 9) \) |
| -1/1024 | \( 53\sqrt{3} / 288 \) | 205\( \sqrt{3} / 96 \) | 1/2048 \( (1 \pm 5\sqrt{11}) \) | 533\( \sqrt{123} \pm 721\sqrt{3} \) / 18432 | 513\( \sqrt{123} \pm 205\sqrt{3} \) / 3072 |
| -1/3024 | \( 13\sqrt{7} / 108 \) | 55\( \sqrt{7} / 36 \) | 1/2048 \( (1 \pm 5\sqrt{11}) \) | \( \sqrt{3} / 7776 \) | \( \sqrt{3} / 1296 \) |
| -1/5002 | \( 827\sqrt{3} / 4500 \) | 4717\( \sqrt{3} / 1500 \) | 1/500000 \( (1 \pm 53\sqrt{89}) \) | 17533\( \sqrt{267} / 900000 \) | 1250001\( \sqrt{267} / 750000 \) |
\[ s = \frac{1}{4} \]

| \( x_0 \) | \( a \) | \( b \) | \( w_0 \) | \( A \) | \( B \) |
|---|---|---|---|---|---|
| \(-\frac{1}{4}\) | \frac{3}{8} | \frac{5}{7} | \frac{1}{8}(1 + \sqrt{5}) | \( \pm 13 + 5\sqrt{5} \) | \( \pm 5 + 3\sqrt{5} \) |
| \(-\frac{16}{63}\) | \frac{8\sqrt{7}}{63} | \frac{65\sqrt{7}}{63} | \frac{63}{8}(16 + 65) | \( \frac{8\sqrt{7} (1 \pm 1)}{69} \) | \( \frac{\sqrt{7} (4481 \pm 2080)}{7038} \) |
| \(-\frac{1}{48}\) | \frac{1}{12} | \frac{1}{96}(1 \pm 7) | \frac{1}{192}(23 \pm 17) | \( \frac{1}{48}(25 \pm 7) \) |
| \(-\frac{1}{324}\) | \frac{23}{72} | \frac{65}{18} | \frac{1}{648}(1 \pm 5\sqrt{13}) | \( \frac{1}{144}(\pm 17 + 13\sqrt{13}) \) |
| \(-\frac{1}{5 \cdot 72^2}\) | \frac{41\sqrt{5}}{288} | \frac{161\sqrt{5}}{72} | \frac{1}{51840}(1 \pm 161) | \( \frac{1}{324}(2201 \pm 121) \) |
| \(-\frac{1}{882^2}\) | \frac{1123}{3528} | \frac{5365}{882} | \frac{1}{1553848}(1 \pm 145\sqrt{37}) | \( \frac{1}{77924}(\pm 5365 + 388963\sqrt{37}) \) |

\[ s = \frac{1}{6} \]

| \( x_0 \) | \( a \) | \( b \) | \( w_0 \) | \( A \) | \( B \) |
|---|---|---|---|---|---|
| \(-\left(\frac{4}{5}\right)^3\) | \frac{8\sqrt{15}}{75} | \frac{21\sqrt{15}}{25} | \frac{32}{125}(12 + 21\sqrt{7}) | \( \frac{168\sqrt{7} \pm 316\sqrt{3}}{375} \) | \( \frac{25\sqrt{7} \pm 336\sqrt{3}}{250} \) |
| \(-\left(\frac{3}{5}\right)^3\) | \frac{15\sqrt{2}}{64} | \frac{77\sqrt{2}}{32} | \frac{27}{1024}(21 \pm 3\sqrt{11}) | \( \frac{33\sqrt{11} \pm 69\sqrt{3}}{512} \) |
| \(-\frac{1}{512}\) | \frac{25\sqrt{6}}{192} | \frac{57\sqrt{6}}{32} | \frac{1}{1024}(3 \pm 3\sqrt{57}) | \( \frac{57\sqrt{19} \pm 49\sqrt{3}}{768} \) | \( \frac{257\sqrt{19} \pm 57\sqrt{3}}{512} \) |
| \(-\frac{9}{40}\) | \frac{93\sqrt{30}}{1600} | \frac{759\sqrt{30}}{800} | \frac{9}{12800}(9 \pm 759\sqrt{3}) | \( \frac{\sqrt{3}(96027 \pm 2277)}{64000} \) |
| \(-\frac{1}{80}\) | \frac{263\sqrt{15}}{3200} | \frac{2709\sqrt{15}}{1600} | \frac{1}{1250}(63 \pm 129) | \( \frac{12427 \pm 743\sqrt{3}}{32000} \) | \( \frac{256001 \sqrt{15} \pm 2709\sqrt{3}}{512000} \) |
| \(-\frac{1}{440^2}\) | \frac{10177\sqrt{339}}{580800} | \frac{43617\sqrt{339}}{96800} | \frac{1}{17036800}(1 \pm 651\sqrt{201}) | \( \frac{4968921 \sqrt{67} \pm 35257\sqrt{3}}{8518400} \) |
| \(-\frac{1}{53360^2}\) | \frac{13591409\sqrt{1009}}{227897009584000} | \frac{90856689\sqrt{1009}}{37982843264000} | \( \frac{1 \pm 557403\sqrt{1009}}{122051200} \) | \( \frac{75965686528001 \sqrt{15} \pm 90856689\sqrt{7}}{810705866268160000} \) |

This takes care of formulas 4.2-4.13 except 4.7 which comes from a divergent series with \( x_0 = -\frac{16}{9} \). Note that we find a new formula with rational \( w_0 \) for \( s = \frac{1}{6} \).

**Remark:** Formula (4.11) in [6] is false. The right hand side should be \( \frac{162\sqrt{7}}{343\pi} \).

Formula 4.1 is of different kind. It is a special case of...
Proposition 5.

We have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{-s}{k} \right)^2 \frac{(-1-s)^2}{(n-k)!} x^n = \frac{1}{1-x} F(1/2, s, 1-s; 1, 1; -\frac{4x}{(1-x)^2})
\]

Classical Proof:

The left hand side is

\[
L = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(s)^2}{k!^2} \frac{(-1)^2}{(n-k)!} x^n = F(s, s, 1; x) F(1-s, 1-s; 1; x)
\]

after using Pfaff’s identity twice

\[
F(a, b; c; x) = (1-x)^{-a} F(a, c-b; c; x)
\]

Now we use

\[
F(2a, 2b; a+b+1/2; x) = F(a, b; a+b+1/2; 4x(1-x))
\]

again to get

\[
L = \frac{1}{1-x} F(s/2, (1-s)/2; 1; -\frac{4x}{(1-x)^2})^2
\]

by Clausen’s identity.

Maple Proof:

Both sides satisfy

\[
y''' + \frac{3(2-5x)}{x(1-x)} y'' + \frac{1-(10+s-s^2)x+(12+s-s^2)x^2}{x^2(1-x)^2} y' + \frac{1}{2} \frac{12+s-s^2-(6+3s-3s^2)x}{x^2(1-x)^2} y = 0
\]

Then we check the first terms in the power series.

Let

\[
a_n = \frac{(1/2)_n (s)_n (1-s)_n}{n!^3}
\]

and

\[
A_n = \sum_{k=0}^{n} \left( \frac{-s}{k} \right)^2 \frac{(-1-s)^2}{(n-k)!}
\]

Then copying the proof of Proposition 4 we get for every formula

\[
\sum_{n=0}^{\infty} a_n (a+bn)x_0^n = \frac{1}{\pi}
\]

a new formula

\[
\sum_{n=0}^{\infty} A_n (A+bn)w_0^n = \frac{1}{\pi}
\]

where

\[
w_0 = 1 - \frac{2}{x_0} (1 - \sqrt{1-x_0})
\]
$$A = (1 - w_0)(a - \frac{bw_0}{1 + w_0})$$

$$B = \frac{b(1 - w_0)^2}{1 + w_0}$$

We give only the rational $w_0$

| $s$ | $x_0$ | $a$   | $b$   | $w_0$ | $A$   | $B$   |
|-----|-------|-------|-------|-------|-------|-------|
| $\frac{1}{3}$ | $\frac{9}{16}$ | $\frac{\sqrt{3}}{4}$ | $\frac{5\sqrt{3}}{4}$ | $\frac{1}{9}$ | $\frac{\sqrt{3}}{9}$ | $\frac{8\sqrt{3}}{9}$ |
| $\frac{1}{4}$ | $\frac{1}{63}$ | $\frac{8\sqrt{7}}{63}$ | $\frac{65\sqrt{7}}{63}$ | $\frac{1}{64}$ | $\frac{7\sqrt{7}}{64}$ | $\frac{63\sqrt{7}}{64}$ |
| $\frac{1}{4}$ | $\frac{32}{81}$ | $\frac{2}{9}$ | $\frac{14}{9}$ | $-\frac{1}{8}$ | $\frac{1}{2}$ | $\frac{9}{4}$ |

The formulas (2.2)-(2.4) in [6] due to the twin brother Z.H. Sun are special cases of the following

**Proposition 6:** We have

$$F(s, 1-s; 1; \frac{1}{2}(1 - \sqrt{1-x}))^2 = F(\frac{1}{2}, s, 1-s; 1, 1; x)$$

**Classical Proof:**

Solving for $x$ we have the equivalent statement

$$F(s, 1-s; 1; w)^2 = F(\frac{1}{2}, s, 1-s; 1, 1; 4w(1-w))$$

Using (formula 3.1.3, p.125 in [2])

$$F(2a, 2b; a + b + 1/2; w) = F(a, b; a + b + 1/2; 4w(1-w))$$

we get

$$F(s, 1-s; 1; w)^2 = F(s/2, (1-s)/2; 1; 4w(1-w))^2$$

and finish by Clausen’s identity.

**Maple Proof:**

One verifies that both sides satisfy the differential equation

$$y'' + \frac{3}{2} \frac{2 - 3x}{x(1-x)} y'' + \frac{1 - (3 + s - s^2)x}{x^2(1-x)} y' - \frac{1}{2} \frac{s(1-s)}{x^2(1-x)} y = 0$$

One expands both sides in power series and checks the first few coefficients.

Theorem 1.3 in [7] is a special case of the following transformation.

Let

$$A_n = \sum_{k=0}^{n} \frac{(s)_k(1-s)_k(s)_{n-k}(1-s)_{n-k}}{k!(n-k)!^2}$$

so

$$F(s, 1-s; 1; x)^2 = \sum_{n=0}^{\infty} A_n x^n$$

so we have the following result.
Proposition 7.
Assume we have a formula
\[
\sum_{n=0}^{\infty} \frac{(1/2)_n (s)_n (1 - s)_n}{n!^3} (a + bn)x_0^n = \frac{1}{\pi}
\]
Then we have
\[
\sum_{n=0}^{\infty} A_n(A + Bn)w_0^n = \frac{1}{\pi}
\]
where
\[
w_0 = \frac{1}{2} (1 - \sqrt{1 - x_0})
\]
and
\[
A = a \\
B = b \frac{1 - w_0}{1 - 2w_0}
\]
**Proof:** Let \( \theta = x \frac{d}{dx} \) and \( a_n = \frac{(1/2)_n (s)_n (1 - s)_n}{n!^3} \). Then
\[
\sum_{n=0}^{\infty} a_n(a + bn)x^n = (a + b\theta) \sum_{n=0}^{\infty} a_n x^n
\]
\[
= (a + b\theta) \sum_{n=0}^{\infty} A_n \left( \frac{1}{2} (1 - \sqrt{1 - x}) \right)^n = \sum_{n=0}^{\infty} A_n(a + \frac{bn}{2\sqrt{1 - x} (1 - \sqrt{1 - x})}) \left( \frac{1}{2} (1 - \sqrt{1 - x}) \right)^n
\]
Putting \( x = x_0 \) we get
\[
A = a
\]
and
\[
B = \frac{b}{2\sqrt{1 - x_0} (1 - \sqrt{1 - x_0})} = \frac{b}{1 - 2w_0}
\]
\[
s = \frac{1}{2}
\]

| \( x_0 \) | \( w_0 \) | \( a = A \) | \( b \) | \( B \) |
|------|------|------|------|------|
| \( 1/4 \) | \( 1/2 - \sqrt{3}/4 \) | \( 1/4 \) | \( 3/2 \) | \( 3 + 2\sqrt{3}/4 \) |
| \( 1/64 \) | \( 1/2 - 3\sqrt{7}/16 \) | \( 5/16 \) | \( 21/8 \) | \( 21 + 8\sqrt{7}/16 \) |
| \( -1 \) | \( 1/2 - \sqrt{2}/2 \) | \( 1/2 \) | \( 2 \) | \( 2 + \sqrt{2}/2 \) |
| \( -1/8 \) | \( 1/2 - 3\sqrt{2}/8 \) | \( \sqrt{2}/4 \) | \( 3\sqrt{7}/2 \) | \( 4 + 3\sqrt{2}/4 \) |
\( s = \frac{1}{3} \)

| \( x_0 \) | \( w_0 \) | \( a = A \) | \( b \) | \( B \) |
|---|---|---|---|---|
| \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{\sqrt{2}}{4} \) | \( \frac{\sqrt{3}}{9} \) | \( \frac{2\sqrt{3}}{3} \) | \( \sqrt{3} + \sqrt{6} \) |
| \( \frac{2}{27} \) | \( \frac{1}{2} \) | \( \frac{5\sqrt{3}}{18} \) | \( \frac{8}{27} \) | \( \frac{20}{9} \) | \( 10 + 6\sqrt{3} \) |
| \( \frac{4}{125} \) | \( \frac{1}{2} \) | \( \frac{11\sqrt{5}}{50} \) | \( \frac{8\sqrt{5}}{45} \) | \( \frac{22\sqrt{5}}{45} \) | \( 11\sqrt{3} + 5\sqrt{15} \) |
| \( -\frac{9}{16} \) | \( \frac{1}{8} \) | \( \frac{\sqrt{3}}{4} \) | \( \frac{5\sqrt{3}}{4} \) | \( 9\sqrt{3} \) |
| \( -\frac{1}{16} \) | \( \frac{1}{8} \) | \( \frac{\sqrt{17}}{8} \) | \( \frac{7\sqrt{17}}{36} \) | \( \frac{17\sqrt{17}}{12} \) | \( 17\sqrt{3} + 4\sqrt{51} \) |
| \( -\frac{1}{80} \) | \( \frac{1}{40} \) | \( \frac{9\sqrt{5}}{12} \) | \( \frac{\sqrt{15}}{4} \) | \( \frac{3\sqrt{15}}{24} \) | \( 20\sqrt{3} + 9\sqrt{15} \) |
| \( -\frac{1}{1024} \) | \( \frac{1}{64} \) | \( \frac{5\sqrt{11}}{288} \) | \( \frac{53\sqrt{3}}{96} \) | \( \frac{205\sqrt{3}}{96} \) | \( 205\sqrt{3} + 32\sqrt{123} \) |
| \( -\frac{1}{3024} \) | \( \frac{1}{504} \) | \( \frac{55\sqrt{21}}{108} \) | \( \frac{13\sqrt{7}}{36} \) | \( \frac{55\sqrt{7}}{36} \) | \( 84\sqrt{3} + 55\sqrt{7} \) |
| \( -\frac{1}{500^2} \) | \( \frac{1}{1000} \) | \( \frac{53\sqrt{89}}{4500} \) | \( \frac{827\sqrt{3}}{1500} \) | \( \frac{4717\sqrt{3}}{3000} \) | \( 4717\sqrt{3} + 500\sqrt{267} \) |
$$s = \frac{1}{4}$$

| $x_0$ | $w_0$ | $a = A$ | $b$ | $B$ |
|------|-------|---------|-----|-----|
| $\frac{32}{81}$ | $\frac{1}{9}$ | $\frac{2}{9}$ | $14$ | $\frac{16}{9}$ |
| $\frac{1}{9}$ | $\frac{1}{2} - \frac{\sqrt{2}}{3}$ | $\frac{\sqrt{3}}{6}$ | $\frac{4\sqrt{3}}{3}$ | $\frac{4\sqrt{3} + 3\sqrt{6}}{6}$ |
| $\frac{1}{81}$ | $\frac{1}{2} - \frac{2\sqrt{5}}{9}$ | $\frac{2\sqrt{2}}{9}$ | $\frac{20\sqrt{2}}{9}$ | $\frac{20\sqrt{2} + 9\sqrt{10}}{18}$ |
| $\frac{1}{40^2}$ | $\frac{1}{2} - \frac{10\sqrt{2}}{49}$ | $\frac{9\sqrt{3}}{49}$ | $\frac{120\sqrt{3}}{49}$ | $\frac{147\sqrt{2} + 120\sqrt{3}}{98}$ |
| $\frac{1}{99^2}$ | $\frac{1}{2} - \frac{35\sqrt{2}}{99}$ | $\frac{19\sqrt{11}}{198}$ | $\frac{140\sqrt{11}}{99}$ | $\frac{140\sqrt{11} + 99\sqrt{22}}{198}$ |
| $\frac{1}{99^2}$ | $\frac{1}{2} - \frac{910\sqrt{29}}{9801}$ | $\frac{2206\sqrt{2}}{9801}$ | $\frac{52780\sqrt{2}}{9801}$ | $\frac{52780\sqrt{2} + 9801\sqrt{58}}{19602}$ |
| $-\frac{1}{4}$ | $\frac{1}{2} - \frac{\sqrt{5}}{4}$ | $\frac{3}{8}$ | $\frac{5}{2}$ | $\frac{5 + 2\sqrt{5}}{4}$ |
| $-(\frac{16}{63})^2$ | $-\frac{1}{63}$ | $\frac{8\sqrt{7}}{63}$ | $\frac{65\sqrt{7}}{63}$ | $\frac{64\sqrt{7}}{63}$ |
| $-\frac{1}{48}$ | $\frac{1}{2} - \frac{7\sqrt{3}}{24}$ | $\frac{3\sqrt{3}}{16}$ | $\frac{7\sqrt{3}}{4}$ | $\frac{12 + 7\sqrt{3}}{8}$ |
| $-\frac{1}{324}$ | $\frac{1}{2} - \frac{5\sqrt{13}}{36}$ | $\frac{23}{72}$ | $\frac{65}{18}$ | $\frac{65 + 18\sqrt{13}}{36}$ |
| $-\frac{1}{72^2}$ | $\frac{1}{2} - \frac{161\sqrt{5}}{720}$ | $\frac{41\sqrt{5}}{288}$ | $\frac{161\sqrt{5}}{72}$ | $\frac{360 + 161\sqrt{5}}{144}$ |
| $-\frac{1}{882^2}$ | $\frac{1}{2} - \frac{145\sqrt{37}}{1764}$ | $\frac{1123}{3528}$ | $\frac{5365}{882}$ | $\frac{5365 + 882\sqrt{37}}{1764}$ |
\[ s = \frac{1}{6} \]

| \( x_0 \) | \( w_0 \) | \( a = A \) | \( b \) | \( B \) |
|-------|-------|-------|------|------|
| 27/125 | 1/2 | 7\sqrt{10}/50 | 3\sqrt{5}/25 | 28\sqrt{5}/25 | 25\sqrt{2} + 14\sqrt{5} |
| 4/125 | 1/2 | 11\sqrt{5}/50 | 2\sqrt{15}/25 | 22\sqrt{15}/25 | 25\sqrt{3} + 11\sqrt{15} |
| 8/113 | 1/2 | 21\sqrt{33}/363 | 20\sqrt{33}/121 | 84\sqrt{33}/121 | 2 + 42\sqrt{33} |
| 64/853 | 1/2 | 171\sqrt{7785}/14450 | 144\sqrt{255}/7225 | 2394\sqrt{255}/7225 | \sqrt{7} + 1197\sqrt{255} |
| -(4/5)^3 | 1/2 | 3\sqrt{105}/50 | 8\sqrt{15}/25 | 21\sqrt{15}/25 | 25\sqrt{7} + 21\sqrt{15} |
| -(3/8)^3 | 1/2 | 7\sqrt{22}/64 | 15\sqrt{2}/32 | 77\sqrt{2}/32 | 77\sqrt{2} + 32\sqrt{11} |
| -1/8^3 | 1/2 | 3\sqrt{114}/64 | 25\sqrt{6}/32 | 57\sqrt{6}/32 | 57\sqrt{6} + 32\sqrt{19} |
| -9/40^2 | 1/2 | 253\sqrt{10}/1600 | 93\sqrt{30}/800 | 759\sqrt{30}/800 | 2400\sqrt{3} + 759\sqrt{30} |
| 1/80^2 | 1/2 | 63\sqrt{645}/3200 | 263\sqrt{15}/1600 | 2709\sqrt{15}/1600 | 2709\sqrt{15} + 1600\sqrt{43} |
| -1/440^3 | 1/2 | 651\sqrt{22110}/193600 | 10177\sqrt{330}/580800 | 43617\sqrt{330}/968000 | 96800\sqrt{67} + 43617\sqrt{330} |
| -1/53360^3 | 1/2 | 651\sqrt{22110}/193600 | 13591409\sqrt{10005}/37982843264000 | 90856689\sqrt{10005} | see below |

In the last row
\[ B = \frac{711822400\sqrt{163} + 90856689\sqrt{10005}}{75965686528000} \]

**Remark:** When \( x_0 \) is positive then we get a (slowly) convergent series with \( \frac{1}{2}(1 + \sqrt{1-x_0}) \) but the sum is not \( \frac{1}{\pi} \) (rather a negative multiple of it). Why?

**References.**
1. G. Almkvist, *Strängar i människan*, Normat, 51 (2003), 22-33.
2. G.E. Andrews, R.Askey, R.Roy, *Special Functions*, Cambridge University Press, 1999.
3. H.H. Chan, J. Wan, W. Zudilin, Legendre polynomials and Ramanujan-type series for \( \frac{1}{\pi} \),
4. J. Guillera, Tables of Ramanujan series with rational values of \( z \), Guillera’s home page
5. J. Guillera, WZ-proofs of ”divergent” Ramanujan-type series, NT/1012.2681.
6. Z. W. Sun, List of conjectural series for powers of \( \pi \) and other constants, CA/1102.5649
7. Z. W. Sun, Some new series for \( \frac{1}{\pi} \) and related congruences, NT/1104.3856.
8. Z. W. Sun, Supercongruences and Euler numbers, Sci. China Math. 54 (2011), 2509-2535.
9. J. Wan, W. Zudilin, Generating functions of Legendre polynomials: A tribute to Fred Brafman,
Appendix: A class of slowly converging series for \(1/\pi\).

Arne Meurman

In the final remark Almkvist and Aycock ask why, when one considers the power series at \(w_1 = \frac{1}{2}(1 + \sqrt{1 - x_0})\), instead of at \(w_0 = \frac{1}{2}(1 - \sqrt{1 - x_0})\), one gets formulas for negative multiples of \(1/\pi\). Here we shall prove such formulas in the cases \(s = 1/2, 1/3, 1/4, 1/6\) in Proposition 7.

Following [2] we set

\[F(t) = F(s, t) = 2F_1(s, 1-s; 1; t),\]

\[G(t) = t \frac{dF}{dt},\]

and let for \(s = 1/2, 1/3, 1/4, 1/6\) \(t(\tau) = t_N(\tau)\) be given by

\[t_4(\tau) = \left(1 + \frac{1}{16} \left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8\right)^{-1}, \quad t_3(\tau) = \left(1 + \frac{1}{27} \left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12}\right)^{-1},\]

\[t_2(\tau) = \left(1 + \frac{1}{64} \left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24}\right)^{-1}, \quad t_1(\tau) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{1728}{\eta(\tau)}}.\]

Let \(U\) be the connected component of \(\{\tau \in \mathbb{C} | \Im(\tau) > 0, |t(\tau)| < 1\}\) which contains all \(\tau\) with sufficiently large imaginary part, a "neighborhood of \(i\infty\)". Let \(\tau_0 \in U\) such that

\[w_0 = t(\tau_0), \quad w_1 = 1 - t(\tau_0)\]

satisfy

\[|w_0| < 1, \quad |w_1| < 1.\]

In the Ramanujan-type formulas, \(\tau_0\) is usually a quadratic irrationality. Let \(A_n\) be defined by the power series expansions

\[F^2(t) = \sum_{n=0}^{\infty} A_n t^n,\]

as in Proposition 7. Set

\[C_s = \frac{1}{2\sin(\pi s)}.\]

**Theorem 1** Assume that there is an identity

\[\sum_{n=0}^{\infty} (A + Bn) A_n w_0^n = \frac{C}{\pi},\]

equivalently

\[A F^2(w_0) + 2BF(w_0)G(w_0) = \frac{C}{\pi}.\]
Then
\[ \sum_{n=0}^{\infty} (\hat{A} + Bn) A_n w_1^n = \frac{\hat{C}}{\pi}, \]  
where
\[ \hat{A} = A, \]
\[ \hat{B} = -B \frac{w_0}{w_1}, \]
\[ \hat{C} = \frac{C \left( \frac{\tau_0}{i} \right)^2 - B \left( \frac{\tau_0}{i} \right)}{C^2 w_1}. \]  

Proof. By formulas (8), (9) in [2] we have, for \( \tau \in U \),
\[ \tau = i C \frac{F(1-t)}{F(t)}, \]  
and
\[ \frac{1}{2\pi i} \frac{dt}{d\tau} = q \frac{dt}{dq} = t(1-t) F^2(t), \]  
where \( q = e^{2\pi i \tau} \). Take \( \frac{1}{2\pi i} \) times the logarithmic derivative with respect to \( \tau \) in (9):
\[ \frac{dF}{dt}(1-t) \cdot (-1) \cdot \frac{1}{2\pi i} \frac{dt}{d\tau} = \frac{1}{2\pi i}. \]  
Substitute (10) to obtain
\[ \frac{G(1-t) t F^2(t)}{F(1-t)} - G(1-t) F(t) = \frac{1}{2\pi i}. \]
Multiply by \( 2i\tau \) and substitute (9):
\[ 2 C_s t G(1-t) F(t) + 2 \left( \frac{\tau_0}{i} \right) (1-t) G(t) F(t) = \frac{1}{\pi}. \]  
Evaluate (11) at \( \tau = \tau_0 \):
\[ 2 C_s w_0 G(w_1) F(w_0) + 2 \left( \frac{\tau_0}{i} \right) w_1 G(w_0) F(w_0) = \frac{1}{\pi}. \]  
By assumption (6)
\[ 2 B F(w_0) G(w_0) = \frac{C}{\pi} - A F^2(w_0), \]  
and eliminating \( F(w_0) G(w_0) \) by (12), (13) we have
\[ 2 B C_s w_0 G(w_1) F(w_0) - A \left( \frac{\tau_0}{i} \right) w_1 F^2(w_0) = \frac{1}{\pi} \left( B - C \left( \frac{\tau_0}{i} \right) w_1 \right). \]
Substitute \( F(w_0) = \left( \frac{i}{\tau_0} \right) C_s F(w_1) \) from (9) to obtain
\[ -A \left( \frac{i}{\tau_0} \right) C_s^2 w_1 F^2(w_1) + 2 B \left( \frac{i}{\tau_0} \right) C_s^2 w_0 F(w_1) G(w_1) = \frac{1}{\pi} \left( B - C \left( \frac{\tau_0}{i} \right) w_1 \right). \]
Dividing by \(-\left(\frac{t}{\tau_0}\right) C_2^2 w_1\) we get

\[
AF^2(w_1) - 2B \frac{w_0}{w_1} F(w_1)G(w_1) = \frac{1}{\pi} \left( C \left( \frac{w_0}{w_1} \right)^2 - B \left( \frac{w_0}{w_1} \right) \right),
\]

which is equivalent to (7).

**Remark.** The arguments in the proof above are analogous to some arguments in the proof of [1] Theorem 2.1.

Theorem 1 applies to all of the identities in the Tables following Proposition 7 where \(x_0 > 0\). We present 4 examples of such.

**Example 1.** In case \(s = \frac{1}{4}, \tau_0 = i\) we have \(w_0 = t_2(i) = \frac{1}{\pi}\), and there is in [4], (1.12) the identity

\[
\sum_{n=0}^{\infty} (1 + 8n) A_n \left( \frac{8}{9} \right)^n = \frac{9}{2\pi},
\]

where

\[
A_n = \frac{1}{64} \sum_{k=0}^{n} \binom{2k}{k} \binom{4k}{2k} \binom{2(n-k)}{n-k} \binom{4(n-k)}{2(n-k)}.
\]

In this case \(A = 1, \ B = 8, \ C = \frac{9}{\sqrt{2}}, \ C_s = \frac{1}{\sqrt{2}},\) and we obtain

\[
w_1 = \frac{8}{9},
\]

\[
\hat{A} = 1, \ \hat{B} = -1, \ \hat{C} = -9,
\]

\[
\sum_{n=0}^{\infty} (1 - n) A_n \left( \frac{8}{9} \right)^n = -\frac{9}{\pi},
\]

which proves [3], (2.10).

**Example 2.** In case \(s = \frac{1}{4}, \tau_0 = \sqrt{\frac{55}{2}} i\) we have

\[
w_0 = t_2 \left( \frac{\sqrt{55}}{2} i \right) = \frac{1}{\pi} - \frac{910}{9801} \sqrt{29},
\]

and there is in Proposition 7, Table \(s = \frac{1}{4}\), the identity

\[
\sum_{n=0}^{\infty} \left( \frac{2206\sqrt{2}}{9801} + \frac{52780\sqrt{2} + 9801\sqrt{55}}{19602} n \right) A_n w_0^n = \frac{1}{\pi},
\]

where \(A_n\) is as in [15]. In this case

\[
C_s = \frac{1}{\sqrt{2}}, \ \ w_1 = \frac{1}{2} + \frac{910}{9801} \sqrt{29},
\]

and we obtain

\[
\sum_{n=0}^{\infty} \left( \frac{2206\sqrt{2}}{9801} + \frac{52780\sqrt{2} - 9801\sqrt{55}}{19602} n \right) A_n w_1^n = -\frac{29}{\pi}.
\]
Example 3. In case $s = \frac{1}{2}$, $\tau_0 = \frac{\sqrt{2}}{2}i$ we have

$$w_0 = t_4 \left( \frac{\sqrt{2}}{2}i \right) = \frac{1}{2} - \frac{\sqrt{3}}{4},$$

and there is in Proposition 7, Table $s = \frac{1}{2}$, the identity

$$\sum_{n=0}^{\infty} \left( \frac{1}{4} + \frac{3 + 2\sqrt{3}}{4} n \right) A_n w_0^n = \frac{1}{\pi},$$

(19)

where

$$A_n = \frac{1}{16^n} \sum_{k=0}^{n} \binom{2k}{k} \left( \frac{2(n-k)}{n-k} \right)^2.$$  (20)

In this case

$$C_s = \frac{1}{2}, \quad w_1 = \frac{1}{2} + \frac{\sqrt{3}}{4},$$

and we obtain

$$\sum_{n=0}^{\infty} \left( \frac{1}{4} + \frac{3 - 2\sqrt{3}}{4} n \right) A_n w_1^n = -\frac{3}{\pi}.$$  (21)

Example 4. In case $s = \frac{1}{6}$, $\tau_0 = \sqrt{7}i$ we have

$$w_0 = t_4 \left( \sqrt{7}i \right) = \frac{1}{2} - \frac{171}{14450} \sqrt{1785},$$

and there is in Proposition 7, Table $s = \frac{1}{6}$ the identity

$$\sum_{n=0}^{\infty} (A + Bn) A_n w_0^n = \frac{1}{\pi},$$

(22)

where

$$A = \frac{144\sqrt{255}}{7225}, \quad B = \sqrt{7} + \frac{1197\sqrt{255}}{7225},$$

$$A_n = \frac{1}{432^n} \sum_{k=0}^{n} \binom{3k}{k} \left( \frac{6k}{3k} \right) \left( \frac{3(n-k)}{n-k} \right) \left( \frac{6(n-k)}{3(n-k)} \right).$$

(23)

In this case

$$C_s = 1, \quad w_1 = \frac{1}{2} + \frac{171}{14450} \sqrt{1785},$$

and we obtain

$$\sum_{n=0}^{\infty} (\hat{A} + Bn) A_n w_1^n = -\frac{7}{\pi},$$

(24)

where

$$\hat{A} = A, \quad \hat{B} = -\sqrt{7} + \frac{1197\sqrt{255}}{7225}.$$
References

[1] H.H.Chan, S.H.Chan, Z.Liu, Domb's numbers and Ramanujan-Sato type series for $1/\pi$, Advances in Math. 186 (2004), 396–410.

[2] H.H.Chan, J.Wan, W.Zudilin, Legendre polynomials and Ramanujan-type series for $\frac{1}{\pi}$, Max-Planck Institute preprint MPIM 11–36.

[3] Z.-W.Sun, List of conjectural series for powers of $\pi$ and other constants, arXiv:1102.5649

[4] Z.-W.Sun, Some new series for $\frac{1}{\pi}$ and related congruences, arXiv:1104.3856

Centre for Mathematical Sciences
Mathematics
Lund University
Box 118
SE-22100 Lund
Sweden
arnem@maths.lth.se