Nonadiabatic Geometric Phase and Hannay Angle: A Squeezed State Approach

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The geometric phases of the cyclic states of a generalized harmonic oscillator with nonadiabatic time-periodic parameters are discussed in the framework of squeezed state. A class of cyclic states are expressed as a superposition of an infinite number of squeezed states. Then, their geometric phases are obtained explicitly and found to be $-(n+1/2)$ times the classical nonadiabatic Hannay angle. It is shown that the analysis based on squeezed state approach provide a clear picture of the geometric meaning of the quantal phase.

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Berry phase [1], which reveals the gauge structure associated with a phase shift in adiabatic processes in quantum mechanics, has attracted great theoretical interests and has been repeatedly corroborated by experiments (see e.g. [2]). This quantum adiabatic phase has a classical analog -Hannay angle [3]. The relaxation of the adiabatic approximation is an important step [4,5]. Aharanov and Anandan [6] studied the phase associated with a cyclic evolution in quantum mechanics (which occurs when a state returns to its initial condition), and shown that the phase is a geometric property of the curve in the projective Hilbert space which is naturally associated with the motion.

The significance of Aharanov and Anandan’s generalization are twofold. On the one hand, the cyclic evolution of a physical system is of most interest in physics both experimentally and theoretically. On the other hand, the universal existence of the cyclic evolution is guaranteed for any quantum system. This can be easily recognized by considering the eigenvectors of the unitary evolution operator for a quantum system. An explicit example is a time-periodic Hamiltonian system where the Floquet theorem applies. The eigenfunctions of the Floquet operator, which are so-called the Bloch wave functions in the condensed matter physics, are obviously cyclic solutions and of great interest in physics. Unlike the adiabatic case, however, in the nonadiabatic case, calculating the eigenvectors and extracting the nonadiabatic geometric phase from the quasi-energy term for a time-dependent Hamiltonian is far from trivial, except for such a special example as the spin particle in a magnetic field. Recent works of Ge and Child [7] made a step further in this direction. They found a special cyclic state of Gaussian wave packet’s form for a generalized harmonic oscillator. The nonadiabatic geometric phase is explicitly calculated and found to be one half of the classical nonadiabatic Hannay angle.

In this letter, we would like to study the nonadiabatic geometric phase of the general cyclic evolutions of the generalized harmonic oscillator. To this end, an alternative way - squeezed state approach will be used. In particular, we shall construct a class of quantum states based on a superposition of an infinite number of squeezed states. We find that the condition for them to be cyclic evolutions is nothing but a quantization rule without Maslov-Morse correction. The nonadiabatic geometric phases are obtained explicitly, and found to be related to the classical Hannay angle by a factor $n+1/2$. Furthermore, the quantum phase can be interpreted as a sum of the area difference on the expectation value plane through a canonical transformation and the area on the quantum fluctuation plane swept out by a periodic orbit. This interpretation gives a unified picture of the geometric meaning of the quantal phase for the adiabatic and nonadiabatic case.

Squeezed state approach has been successfully applied in many branches of physics such as quantum optics, high energy physics and condensed matter physics. Recent years have witnessed a growing application of squeezed state to study the chaotic dynamical systems [7,9]. In this letter, we shall employ this approach to discuss geometric phase and Hannay angle for a generalized harmonic oscillator. An apparent reason for this choice is that this system admits the squeezed state as an exact solution. The squeezed state approach [11,13] starts from the time-dependent variational principle (TDVP) formulation,

$$\delta \int dt \langle \Phi, t | i \hbar \frac{\partial}{\partial t} - \hat{H} | \Phi, t \rangle = 0. \quad (1)$$

Variation w.r.t $\langle \Phi, t |$ and $| \Phi, t \rangle$ gives rise to the Schrödinger equation and its complex conjugate, respectively. The squeezed state is chosen as the trial wave function, which is defined by the ordinary harmonic oscillator displacement operator acting on a squeezed vacuum state $|0\rangle$:

$$| \Psi \rangle = \exp \left( \alpha \hat{a}^+ - \alpha^* \hat{a} \right) | \phi \rangle,$$
\[ |\phi\rangle = \exp \left( \frac{1}{2} (\beta \hat{a}^+ - \beta^* \hat{a}) \right) |0\rangle. \quad (2) \]

\( \hat{a}^+ \) and \( \hat{a} \) are boson creation and annihilation operators which satisfy the canonical commutation relation: \([\hat{a}, \hat{a}^+] = 1\).

From the TDVP, we can obtain the dynamical equations for the expectation values \((q, p)\) and the quantum fluctuations \(\Delta q^2 \equiv \langle \Psi, t |(\hat{p} - p)^2 |\Psi, t \rangle = \hbar \left( \frac{1}{4} + 4\Pi^2 G \right)\), \(\Delta p^2 \equiv \langle \Psi, t |(\hat{q} - q)^2 |\Psi, t \rangle = \hbar G\).

\[ \dot{q} = \frac{\partial H_{\text{eff}}}{\partial p}, \quad \dot{p} = -\frac{\partial H_{\text{eff}}}{\partial q}, \]

\[ \hbar \dot{\hat{G}} = \frac{\partial H_{\text{eff}}}{\partial \Pi}, \quad \hbar \dot{\Pi} = -\frac{\partial H_{\text{eff}}}{\partial G}, \quad (3) \]

where the dot denotes the time derivative. The effective Hamiltonian \(H_{\text{eff}}\) is defined on the extended space \((q, p, G, \Pi)\), taking the form \(H_{\text{eff}} = \langle \Psi, t |\hat{H} |\Psi, t \rangle\).

The time-dependent variational principle leaves an ambiguity of a time-dependent phase \(\lambda(t)\), which can be fixed with the aid of the Schrödinger equation,

\[ \dot{\lambda}(t) = \langle \Psi, t |i \frac{\partial}{\partial t} |\Psi, t \rangle - \frac{1}{\hbar} \langle \Psi, t |\dot{\hat{H}} |\Psi, t \rangle. \quad (4) \]

This phase is well defined for general non-adiabatic and non-cyclic evolution of a squeezed state. It represents a phase change of the squeezed state during time-evolution. Obviously, the phase consists of two parts. The meaning of the second part is obvious: a measure of the time of evolution. It is the dynamical phase and can be rewritten as,

\[ \lambda_{D}(t) = -\frac{1}{\hbar} \int_{0}^{t} H_{\text{eff}} dt. \quad (5) \]

The first part can be viewed as a difference of the total phase and the dynamical phase. We call it geometric phase since it just is the Aharonov-Anandan’s phase for the case of cyclic evolution. From the expression of the squeezed state, the geometric phase is equal to

\[ \lambda_{G}(t) = \int_{0}^{t} \left( \frac{1}{2\hbar} (p\dot{q} - \dot{p}q) - \hat{\Pi}G \right) dt. \quad (6) \]

It is clear that the evolution of expectation values \((q, p)\) as well as the evolution of the quantum fluctuations \((G, \Pi)\) contribute to the geometric phase. The contribution from the former one is explicitly \(\hbar\) dependent, while the contribution from quantum fluctuation is \(\hbar\) independent. For the case of cyclic evolution of squeezed state the quantal phase is equal to a sum of the projective areas on the coordinates plane \((q, p)\) and fluctuation plane \((G, \Pi)\) swept out by a periodic orbit of the effective Hamiltonian.

The Hamiltonian of the generalized harmonic oscillator takes the form,

\[ \hat{H}(q, p, t) = \frac{1}{2} \left( a(t)\dot{q}^2 + b(t)p^2 + c(t)(\dot{q}\dot{p} + \ddot{p}q) \right), \quad (7) \]

where real parameter \(a(t), b(t), c(t)\) are time-periodic functions with common period \(T\). Our discussions are restricted to the elliptic case, namely, \(a(t)b(t) > c^2(t)\).

Applying the squeezed state to this system, from Eq. (3) one obtains an effective Hamiltonian in the extended phase space \((q, p, G, \Pi)\),

\[ H_{\text{eff}}(q, p; G, \Pi; t) = H_{cl}(q, p, t) + \hbar H_{fl}(G, \Pi, t), \quad (8) \]

where

\[ H_{cl} = \frac{1}{2} \left( a(t)q^2 + b(t)p^2 + 2c(t)qp \right), \quad (9) \]

describes the motion of the expectation values:

\[ H_{fl} = \frac{1}{2} \left( a(t)G + b(t)\left( \frac{1}{4G} + 4\Pi^2 G \right) + 4c(t)G\Pi \right), \quad (10) \]

depicts the evolution of the quantum fluctuations.

Starting from this effective Hamiltonian, it is easy to analyse the dynamical properties. The motions of both degree of freedom are decoupled. In the fluctuation plane \((G, \Pi)\), whole motions are restricted on the invariant tori except for a unique T-periodic solution denoted by \((G_{p}(t), \Pi_{p}(t))\). The Hamiltonian \(H_{cl}\) which describes the motion of the expectation values \((q, p)\) is identical to the classical version of the system \((\bar{q}, \bar{p})\). The \((q = 0, p = 0)\) is obviously a fixed point. Other motions are quasi-periodic trajectories confined on the tori. Through a canonical transformation, \(q = q(\bar{I}, \bar{\phi}, t), p = p(\bar{I}, \bar{\phi}, t)\), the Hamiltonian \(H_{cl}(q, p, t)\) can be transformed to a new Hamiltonian \(\bar{H}(\bar{I}, t)\) which does not contain angle variable \(\bar{\phi}\). Its solution is described by \(\bar{I} = I_{0}; \bar{\phi}(t) = \phi_{0} + \int_{0}^{t} \frac{\partial \bar{H}}{\partial I_{0}} dt\).

For this canonical transformation is explicitly time dependent, the new Hamiltonian \(\bar{H}\) differs from the old one \(H_{cl}\) both in value and in functional form. Thus, we introduce a function \(A\) to measure the difference,

\[ A(\bar{\phi}, \bar{I}, t) = \bar{H}(\bar{I}, t) - H_{cl} ((\phi(\bar{\phi}, \bar{I}, t), I(\bar{\phi}, \bar{I}, t), t). \quad (11) \]

Therefore the classical non-adiabatic Hannay angle is

\[ \Theta_{H} = \langle I_{0}^{T} \frac{\partial A}{\partial I_{0}} dt \rangle_{\phi_{0}}, \quad (12) \]

where the bracket denotes averaging around the invariant torus, \(\langle \cdots \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} \cdots d\phi_{0}\).

Now we turn to the quantum system \((\bar{q}, \bar{p})\). Since it is a time-periodic Hamiltonian system, the Floquet theory applies. A unitary time evolution operator referring to one period \(T\), the so-called Floquet operator \(U(T)\) is worthy of consideration. We can construct a state as a superposition of infinite number of squeezed states
where $|\bar{I}_0, \bar{\phi}_0; G_0, \Pi_0\rangle$ represents a squeezed state centered at $q(\bar{I}_0, \bar{\phi}_0, t = 0), p(\bar{I}_0, \bar{\phi}_0, t = 0)$ with fluctuations $G_0, \Pi_0$. The $G_0$ and $\Pi_0$ are chosen on the unique periodic orbit $(G_0 = G_p(t = 0), \Pi_0 = \Pi_p(t = 0))$; $c$ is a normalization constant.

Consider the situation that $\dot{U}(mT)$ (or $\dot{U}^m(T)$) acts on the state $|S_1\rangle$,

$$\dot{U}(mT)|S_1\rangle = c \int_0^{2\pi} e^{i\lambda_0 \bar{\phi}_0} e^{i\lambda}\bar{I}_0, \bar{\phi}_0 + \bar{\phi}_m; G_0, \Pi_0\rangle d\bar{\phi}_0,$$

(14)

where $\bar{\phi}_m = \int_0^m \frac{dH}{d\bar{I}_0} dt$, and $\lambda = \lambda_D(mT) + \lambda_G(mT)$. The dynamical part is $\lambda_D(mT) = -\frac{1}{h} \int_0^m H_{eff} dt$, and the geometric part $\lambda_G(mT) = \frac{1}{2} \int_0^m (\bar{p}\dot{q} - \bar{q}\dot{p}) dt - \int_0^m \Pi_p G_p dt$. They can be expressed as,

$$\lambda_D(mT) = \langle \lambda_D(mT) \rangle_{\bar{\phi}_0} + \{\lambda_D(mT)\}(\bar{\phi}_0),$$

(15)

$$\lambda_G(mT) = \langle \lambda_G(mT) \rangle_{\bar{\phi}_0} + \{\lambda_G(mT)\}(\bar{\phi}_0),$$

(16)

respectively. Where the symbols $\langle \cdots \rangle_{\bar{\phi}_0}$ denotes the average over the $\bar{\phi}_0$ as in Eq. (2) and $\{\cdots\}(\bar{\phi}_0)$ represent the terms relating to $\bar{\phi}_0$. Then,

$$\langle \lambda_G(mT) \rangle_{\bar{\phi}_0} = \frac{m}{h} \int_0^m \left( \frac{1}{2} (\bar{p}\dot{q} - \bar{q}\dot{p}) \right) dt_{\bar{\phi}_0} - m \int_0^m G_p d\Pi_p.$$

(17)

Making variables transformation $\bar{\phi}_0' = \bar{\phi}_0 + \bar{\phi}_m$, we have

\[
\dot{U}(mT)|S_1\rangle = c e^{i\lambda_0 \bar{\phi}_0} \int_0^{2\pi + \bar{\phi}_m} e^{i\lambda_0 \bar{\phi}_0} \left( e^{i\lambda_D(mT)}(\bar{\phi}_0') + i\{\lambda_D(mT)\}(\bar{\phi}_0') \right) |\bar{I}_0, \bar{\phi}_0', G_0, \Pi_0\rangle d\bar{\phi}_0',
\]

(18)

where $\lambda_0^1 = m(\lambda_G^R + \lambda_D^R)$. The geometrical part and the dynamical part take the forms as follows,

$$\lambda_G^R = \frac{1}{h} \left( \int_0^T \frac{1}{2} (\bar{p}\dot{q} - \bar{q}\dot{p}) dt_{\bar{\phi}_0} - \bar{I}_0 \int_0^T \frac{\partial H}{\partial \bar{I}_0} dt \right) - \int_0^T G_p d\Pi_p.$$  

(19)

$$\lambda_D^R = -\frac{1}{h} \int_0^T H_{eff} dt_{\bar{\phi}_0}.$$  

(20)

The integral in Eq. (18) can be written as $\int_0^{2\pi} + \int_0^{2\pi + \bar{\phi}_m} \cdots - \int_0^{2\pi} \cdots$. The last two terms will cancel each other if and only if $e^{i\lambda_0 \bar{\phi}} = 1$, which gives rise to

$$\bar{I}_0 = n \hbar.$$  

(21)

This is nothing but the quantization rule without Maslov-Morse correction.

The motion of the expectation values $(q, p)$ confined on the invariant torus $\bar{I}_0$ is quasi-periodic. The ergodicity of the motion guarantees that temporal average is equivalent to the spatial average suppressing that the time is long enough. Then, from the ergodicity principle, we can choose an integer $r$, which is large enough so that the phase change (see [12] and [10]) during the time interval $rT$ does not relate to $\bar{\phi}_0$. Then, we construct a state $|S_r\rangle$ like [11],

$$|S_r\rangle = |S_1\rangle + \cdots + e^{-i\lambda_0^1 \bar{I}_0} \dot{U}(mT)|S_1\rangle + \cdots$$

(22)

and under the condition [21] we can prove that [14],

$$\dot{U}(T)|S_r\rangle = e^{i(\lambda_G^R + \lambda_D^R)}|S_r\rangle.$$  

(23)

In fact, the above relation indicates that the state $|S_r\rangle$ is an eigenstate of the Floquet operator, $n$ is the state number. $\lambda_G^R$ and $\lambda_D^R$ is the dynamical and geometric phase relating to the cyclic states, respectively.

To see the meaning of the geometric phase $\lambda_G^R$ expressed by Eq. (19), let us consider following differential 2-form which is preserved under the canonical transformation, i.e. $dp \wedge dq - dH \wedge dt = dI \wedge d\phi - dH \wedge dt$. We rewrite this into another form,

$$dp \wedge dq - d\bar{I} \wedge d\bar{\phi} = -d(\bar{H} - H_{eff}) \wedge dt.$$  

(24)

Let us first make an integration of the above equation for one period $(T)$ and then average over the variable $\phi_0$. Keeping in mind that the area meaning of the differential 2-form, One will find immediately that the term bracketed in the expression of the geometric phase Eq. (19) corresponds to the left hand side of the above equation, whereas the right hand side will equal to $n \hbar$ times the classical Hanny’s angle (see Eq. [12]). The relation between the last term in Eq. (19) and the classical angle is given by Ge and Child [3] and verified by our explicit perturbative results in follows. Then, we can reach a simple relation between the geometric phase and non-adiabatic Hannay angle,

$$\lambda_G^R = -(n + \frac{1}{2}) \Theta_H.$$  

(25)

Now we take a specific choice of the periodic parameters as an example to demonstrate the above approach and verify our findings. Set that $a(t) = 1 +
\[ \epsilon \cos(\omega t), b(t) = 1 - \epsilon \cos(\omega t), c(t) = \epsilon \sin(\omega t). \]

Our discussions are restricted to the elliptic case, namely, \( a(t) \) is of the form \( c^2(t) \), i.e. \( \epsilon < 1 \). The perturbation method will be employed in the following discussions. Our solutions of power series are accurate to second order.

Now, we rewrite the classical Hamiltonian in terms of the action-angle variables, i.e. \( q = \sqrt{2I} \cos \phi \), \( p = \sqrt{2I} \sin \phi \),

\[ H_{cl} = H_0(I) + \epsilon H_1(I, \phi), \tag{26} \]

where \( H_0 = I, H_1 = -I \cos(\omega t + 2\phi) \). It is convenient to employ the Lie transformation \([13]\) method to make a canonical transformation, so that the new Hamiltonian \( \bar{H}(\bar{I}) \) contains the action variable only,

\[ \bar{H}(\bar{I}) = \bar{I} - \frac{\bar{I}}{\omega + 2} \epsilon^2. \tag{27} \]

The generating functions are \( w_1 = I \sin(\omega t + 2\phi)/(\omega + 2) \) and \( w_2 = 0 \), respectively. The relation between the old variables and the new variables is given by \( (\phi, I) = T^{-1}(\bar{I}, \bar{\phi}) \), where the transformation operator \( T^{-1} = 1 + \epsilon L_1 + \epsilon^2 (L_2/2 + L_1^2/2), L_n \) is Lie operator defined by \( L_n = \{w_n, \}, \{, \} \) represents a Poisson Bracket.

With the help of Eqs. (12) and (26, 27), we arrive at the expression of the classical angle analytically,

\[ \Theta_H = \frac{2\pi \epsilon^2}{(\omega + 2)^2}. \tag{28} \]

Obviously, this classical non-adiabatic Hannay’s angle is independent of the action. T-periodic solution \((G_p(t), \Pi_p(t))\) of the Hamiltonian \( H_{cl} \), can be derived by the power-series expansion,

\[ G_p(t) = \frac{1}{2} - \frac{\cos(\omega t)}{\omega + 2} \epsilon, \quad \Pi_p(t) = -\frac{\sin(\omega t)}{\omega + 2} \epsilon. \tag{29} \]

Notice the fact that an arbitrary \( \omega \) can be approached by a series of rational number like \( q/p \), we can repeat the above process by constructing a state as in Eq. (22), where the \( r = q \). Finally, we obtain the analytic expression of the geometric phase,

\[ \lambda_{q/p} = -\frac{I_0}{\hbar} + \frac{1}{2} \frac{2\pi \epsilon^2}{(\omega + 2)^2}. \tag{30} \]

Considering the quantization rule \((I_0 = n\hbar)\) and the explicit expression of Hannay’s angle \([24]\), the above equation coincide with the relation \([24]\).

An interesting example is given by the case \( n = 0 \), i.e. the ground state of the Floquet states. It corresponds to a cyclic squeezed state with period \( T \), whose expectation values keep fixed at the zero point, while its fluctuations change periodically (see Eq. (22)). The geometric phase of this cyclic state resulting only from the periodic evolution of the fluctuations’ part, is equal to one half of the classical Hannay angle. This is just what obtained by Ge and Child \([3]\).

In summary, the squeezed state approach is used to study the nonadiabatic geometric phase relating to the cyclic evolutions of a generalized harmonic oscillator. The quantum phases are obtained explicitly and found to be \(-n + 1/2\) times the Hannay angle. The quantum phase can be interpreted as a sum of the area difference on the expectation value plane through the canonical transformation and the area on the quantum fluctuation plane swept out by a periodic orbit. The explanation given here provides a unified picture of the geometric meaning of the quantal phase for the adiabatic case as well as the nonadiabatic case. In the adiabatic limit, our \( n + 1/2 \) relation is identical to the elegant formula of Berry \([1]\). However, the semiclassical approximation has not been envoked.

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