TIME-DEPENDENT SOBOLEV INEQUALITY ALONG THE RICCI FLOW

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Abstract. In this article, we get a time-dependent Sobolev inequality along the Ricci flow in a more general situation than those in Zhang [7], Ye [6] and Hsu [2] which also generalizes the results of them. As an application of the time-dependent Sobolev inequality, we get a growth of the ratio of non-collapsing along immortal solutions of Ricci flow.

1. Introduction

Consider the Ricci flow

\begin{equation}
\left\{\begin{array}{l}
\frac{4}{n} g = -2 Rc \\
g(0) = g_0
\end{array}\right.
\end{equation}

on a closed manifold $M^n$. An important ingredient of Perelman’s proof of geometrization conjecture is the non-collapsing theorem of Ricci flow which makes sure that we can get a singularity model of the flow when a singularity exists. In [7], Zhang gave an easier way to prove the non-collapsing theorem of Ricci flow via a uniform Sobolev inequality along the flow. Unfortunately, there is a mistake in the proof of Zhang [7]. Later, Ye [6] corrected the error and Zhang [8] also corrected the error by himself.

In [6] and [7], Ye and Zhang only considered Sobolev inequalities with $L^2$ right hand side so that the surface case was excluded. In Hsu [2], she got uniform Sobolev inequalities with general right hand side so that the surface case was also included.

In [6] and [2], Ye and Hsu got uniform Sobolev inequalities along the Ricci flow with the assumption that we are only considering Ricci flow in a finite time interval or that $\lambda_0(g_0) > 0$ where $g_0$ is the initial metric and $\lambda_0(g)$ is the first eigenvalue of $-\Delta_g + \frac{R(g)}{4}$. Note that, we have the following evolution inequality of $\lambda_0$ along the Ricci

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Therefore, when $\lambda_0(g_0) > 0$, the Ricci flow exists only for finite time. So, the second case is included in the first case that assumes the time interval is finite.

In this article, we remove the assumption of finite time interval and get a time-dependent Sobolev inequality along the Ricci flow with Sobolev constants varying when time is varying which generalizes the results of Zhang [7], Ye [6] and Hsu [2]. The main result of this article is as follows.

**Theorem 1.1.** Let $g(t)$ with $t \in [0, T)$ be a solution to the Ricci flow on a closed manifold $M^n$ where $T$ can be $\infty$. Then, there are two positive constants $A$ and $B$ depending only on the initial metric $g(0)$ and $n$, such that

$$\left( \int_M |u|^\frac{np}{n-p} \, dV_t \right)^\frac{n-p}{np} \leq \frac{Ae^{Bt}}{n-p} \left( \int_M \left( \|\nabla u\|^2 + \frac{R + \max_M R_-(0) + 4}{4} u^2 \right) dV_t \right)^\frac{1}{p}$$

for any $p \in [1, n)$, $t \in [0, T)$ and $u \in C^\infty(M)$.

Our proof of the main result is mainly the same as in Zhang [7], Ye [6] and Hsu [2]. Arguments are mainly contained in Davies [1]. First, by the monotonicity of Perelman’s W-entropy, we can get a uniform logarithmic Sobolev inequality. Then, the remaining arguments are somehow standard. We get the result just by calculating the constants of estimations more carefully via a trick in Saloff-Coste [5].

2. **Time-dependent Sobolev inequality and the growth of the ratio of non-collapsing**

Let $M^n$ be a closed manifold with dimension $n$ ($n \geq 2$). $g(t)$ with $t \in [0, T)$ be a solution to the Ricci flow where $T$ may be $\infty$. By Ye [6] and Hsu [2], we have the following uniform logarithmic Sobolev inequality.

**Theorem 2.1.** There are two positive constants $A, B$ depending only on the initial metric $g(0)$ and $n$, such that for any $\sigma > 0$ and $t \in [0, T)$

$$\int_M v^2 \log v^2 \, dv_t \leq \sigma \int_M \left( \|\nabla v\|^4 + \frac{R}{4} v^2 \right) \, dv_t - \frac{n}{2} \log \sigma + A(t + \sigma/4) + B$$
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for any \( v \in C^\infty(M) \) with \( \int_M v^2 dV = 1 \).

By the same arguments as in Zhang \cite{Zhang} and Ye \cite{Ye} which is mainly contained in Davies \cite{Davies}, we have the following ultracontractivity of heat kernel assuming a logarithmic Sobolev inequality. Because our statement is a little different with that in Ye \cite{Ye}, we also give the proof here.

**Theorem 2.2.** Let \((M^n, g)\) be a closed Riemannian manifold,

\[
H = -\Delta + \Psi \quad \text{and} \quad Q(u) = \int_M uH u dV = \int_M (\|\nabla u\|^2 + \Psi u^2) dV
\]

with \(\Psi \in C^\infty(M)\) and \(\Psi \geq 0\). Let \(\sigma_0\) be a positive constant such that for any \(\sigma \in (0, \sigma_0]\), the following logarithmic Sobolev inequality

\[
\int_M u^2 \log u^2 dV \leq \sigma Q(u) + \beta(\sigma)
\]

holds for any smooth function \(u\) with \(\int_M u^2 dV = 1\) where \(\beta(\sigma)\) is an integrable function on \((0, \sigma_0]\). Then,

\[
\|e^{-tH} f\|_\infty \leq \exp \left( \frac{1}{pt} \int_0^t \beta(4\sigma) d\sigma \right) \|f\|_p
\]

for any \(p \geq 1\), \(t \in (0, \sigma_0/4]\) and \(f \in C^\infty(M)\).

**Proof.** The logarithmic Sobolev inequality reads

\[
\int_M u^p \log u^p \leq \sigma Q(u^\frac{2}{p}) + \beta(\sigma)\|u\|_p^p + \|u\|_p^p \log \|u\|_p^p
\]

for any nonnegative function \(u \in C^\infty(M)\), \(p \geq 1\) and \(\sigma \in (0, \sigma_0]\).

Let \(u(t)\) be a solution to the Schrodinger equation

\[
\frac{\partial u}{\partial t} = -Hu.
\]

Let \(p(t)\) be an increasing function to be determined \((p(t) \geq 1)\). Compute as follows (assuming that \(u\) is nonnegative),

\[
\frac{d}{dt} \log \|u\|_{p(t)} = \frac{1}{p(t)} \int_M u^p
\]

\[
= - \frac{p'(t)}{p(t)^2} \log \left( \int_M u^p \right) + \frac{1}{p(t) \int_M u^p} \left( \int_M pu^{p-1} u' + \int_M u^p p' \log u \right)
\]

\[
= - \frac{p'(t)}{p(t)^2} \log \int_M u^p - \frac{1}{\int_M u^p} \int_M u^{p-1} H u + \frac{p'}{p^2 \int_M u^p} \int_M u^p \log u^p
\]
Note that

\[
\int_M u^{p-1} Hu = \int_M u^{p-1}(-\Delta u + \Psi u)
= \int_M \langle \nabla u^{p-1}, \nabla u \rangle + \Psi u^p
= \frac{4(p-1)}{p^2} \int_M \|\nabla u^{p/2}\|^2 + \int_M \Psi u^p
= \frac{4(p-1)}{p^2} Q(u^{p/2}) + \frac{(p-2)^2}{p^2} \int_M \Psi u^p
\geq \frac{4(p-1)}{p^2} Q(u^{p/2})
\]

Therefore,

\[
\frac{d}{dt} \log \|u\|_{p(t)}\]
\[
\leq - \frac{p'(t)}{p(t)^2} \log \int_M u^p - \frac{4(p-1)}{p^2} \int_M Q(u^{p/2}) + \frac{p'}{p^2} \int_M u^p \log u^p
= \frac{p'(t)}{p(t)^2} \int_M u^p \left( \int_M u^p \log u^p - \frac{4(p-1)}{p'} Q(u^{p/2}) - \int_M u^p \log \int_M u^p \right).
\]

In order to use the logarithmic Sobolev inequality, we need

\[
\frac{4(p(t)-1)}{p'(t)} \leq \sigma_0.
\]

Assuming this, we get

\[
\frac{d}{dt} \log \|u\|_{p(t)} \leq \frac{p'(t)}{p(t)^2} \beta(4(p-1)/p').
\]

For each \(s > 0\), let \(p(t) = \frac{ps}{s-t}\), then

\[
\frac{4(p(t)-1)}{p'} = \frac{4(ps/(s-t)-1)}{ps/(s-t)^2} = \frac{4(ps-(s-t))(s-t)}{ps} \leq 4(s-t).
\]

So, when \(s \in (0, \sigma_0/4]\), we have

\[
\frac{d}{dt} \log \|u\|_{p(t)} \leq \frac{1}{ps} \beta(4(s-t))
\]

for any \(t \in (0, s]\). Integrating against \(t\) on \([0, s]\), we get

\[
\|u(s)\|_\infty \leq e^{\frac{1}{ps} \int_0^s \beta(4\sigma) d\sigma} \|u(0)\|_p.
\]

\[
\square
\]
From the ultracontractivity of heat kernel, we can get a Sobolev inequality with effective constants by a combination of a trick in Davies [1] and a trick in Saloff-Coste [5].

**Theorem 2.3.** Let \((M^n, g)\) be a closed Riemannian manifold, 
\[
H = -\Delta + \Psi \text{ and } q(u) = \|\nabla u\|^2 + \Psi u^2
\]
with \(\Psi \in C^\infty(M)\) and \(\Psi \geq 1\). Let \(p \in [1, n)\), \(A \geq 1\) be such that
\[
\|e^{-tH} f\|_\infty \leq At^{-\frac{n}{2p}} \|f\|_p
\]
for any \(t \in (0, 1]\) and \(f \in C^\infty(M)\). Then,
\[
\|u\|_\frac{np}{n-p} \leq \frac{32An^np}{n-p} \left( \int_M q(u)^{\frac{p}{2}} dV \right)^{\frac{1}{p}}
\]
for any \(u \in C^\infty(M)\).

**Proof.** When \(t > 1\), by maximum principle, we have
\[
\|e^{-tH} f\|_\infty \leq e^{-(t-1)} \|e^{-H} f\|_\infty \leq Ae^{-(t-1)} \|f\|_p \leq n^p At^{-\frac{n}{2p}} \|f\|_p
\]
for any \(f \in C^\infty(M)\). Obviously, the inequality is also true for \(t \in (0, 1]\).

Note that,
\[
H^{-1/2} f = \Gamma \left( \frac{1}{2} \right)^{-1} \int_0^\infty t^{-1/2} e^{-tH} f dt = \Gamma \left( \frac{1}{2} \right)^{-1} \int_0^T t^{-1/2} e^{-tH} f dt + \Gamma \left( \frac{1}{2} \right)^{-1} \int_T^\infty t^{-1/2} e^{-tH} f dt := g + h
\]
where \(T\) is a positive constant to be determined. Moreover, note that
\[
\|h\|_\infty \leq n^p A \Gamma(1/2)^{-1} \int_T^\infty t^{-1/2-n/(2p)} dt \|f\|_p = \frac{2n^p A p T^{-\frac{n-p}{2p}}}{(n-p) \Gamma(1/2)} \|f\|_p.
\]
For any \(\lambda > 0\), choose \(T > 0\) such that
\[
(2.1) \quad \frac{2n^p A p T^{-\frac{n-p}{2p}}}{(n-p) \Gamma(1/2)} \|f\|_p = \lambda/2.
\]
Then,
\[
m(\{|H^{-1/2} f| \geq \lambda\}) \leq m(\{|g| \geq \lambda/2\}) \leq (\lambda/2)^{-p} \|g\|_p^p \\
\leq (\lambda/2)^{-p} \Gamma(1/2)^{-p} \left( \int_0^T t^{-1/2} e^{-tH} f \|_p dt \right)^p \\
\leq \Gamma(1/2)^{-p} A^p \lambda^{-p} T^{-\frac{np}{2p}} \|f\|_p^p.
\]
where we have used the Minkowski inequality and the fact
\[ \|e^{-tH}f\|_p \leq \|f\|_p. \]

Substituting (2.1) into the last equation, we get
\[ \lambda \frac{n_p}{n-p} m(\{|H^{-1/2}f| \geq \lambda\}) \leq 4 \frac{n_p}{n-p} \left( \frac{n^n A p}{n - p} \right)^{\frac{n_p}{n-p}} \|f\|_p^{\frac{n_p}{n-p}}. \]

In another word, we get
\[ \lambda \frac{n_p}{n-p} m(\{|u| \geq \lambda\}) \leq 4 \frac{n_p}{n-p} \left( \frac{n^n A p}{n - p} \right)^{\frac{n_p}{n-p}} \left( \int_M \omega(u) \frac{q}{2} dV \right)^{\frac{n_p}{n-p}} \]
for any \( \lambda > 0 \) and \( u \in C^\infty(M) \).

Let \( u \) be a nonnegative function. Let \( \rho > 1 \) be a constant to be determined and Let
\[ u_{\rho,k} = (u - \rho^k)_+ \land (\rho^{k+1} - \rho^k) = \begin{cases} 
0 & \text{if } u \leq \rho^k \\
u - \rho^k & \text{if } \rho^k < u \leq \rho^{k+1} \\
rho^{k+1} - \rho^k & \text{if } u > \rho^{k+1}
\end{cases}, \]
where \( k \) is any integer. Then,
\[ (\rho^{k+1} - \rho^k) \frac{n_p}{n-p} m(\{u \geq \rho^{k+1}\}) \]
\[ = (\rho^{k+1} - \rho^k) \frac{n_p}{n-p} m(\{u_{\rho,k} \geq \rho^{k+1} - \rho^k\}) \]
\[ \leq 4 \frac{n_p}{n-p} \left( \frac{n^n A p}{n - p} \right)^{\frac{n_p}{n-p}} \left( \int_M \omega(u_{\rho,k}) \frac{q}{2} dV \right)^{\frac{n_p}{n-p}} \]
So,

\[
\sum_{k=-\infty}^{\infty} (\rho^{k+1} - \rho^k)^{\frac{np}{n-p}} m(\{u \geq \rho^{k+1}\})
\leq 4\frac{np}{n-p} \left( \frac{n^Ap}{n-p}\right)^{\frac{p^2}{n-p}} \sum_{k=-\infty}^{\infty} \left( \int_M q(u_{\rho,k})^\frac{p}{2} dV \right)^{\frac{n}{n-p}}
\leq 4\frac{np}{n-p} \left( \frac{n^Ap}{n-p}\right)^{\frac{p^2}{n-p}} \left( \sum_{k=-\infty}^{\infty} \int_M q(u_{\rho,k})^\frac{p}{2} dV \right)^{\frac{n}{n-p}}
\leq 4\frac{np}{n-p} \left( \frac{n^Ap}{n-p}\right)^{\frac{p^2}{n-p}} \left( \sqrt{2} \sum_{k=-\infty}^{\infty} \int_M (\|\nabla u_{\rho,k}\|^p + \Psi u_{\rho,k}^p) dV \right)^{\frac{n}{n-p}}
\leq 4\frac{np}{n-p} \left( \frac{n^Ap}{n-p}\right)^{\frac{p^2}{n-p}} \left( \sqrt{2} \int_M (\|\nabla u\|^p + \Psi^p u^p) dV \right)^{\frac{n}{n-p}}
\leq 4\frac{np}{n-p} \left( \frac{n^Ap}{n-p}\right)^{\frac{p^2}{n-p}} \left( \sqrt{2} \cdot \sqrt{2} \int_M q(u)^\frac{p}{2} dV \right)^{\frac{n}{n-p}}
\leq 8\frac{np}{n-p} \left( \frac{n^Ap}{n-p}\right)^{\frac{p^2}{n-p}} \left( \int_M q(u)^\frac{p}{2} dV \right)^{\frac{n}{n-p}}
\]

where we have used the inequalities

\[
\frac{1}{\sqrt{2}}(a^\frac{p}{2} + b^\frac{p}{2}) \leq (a + b)^\frac{p}{2} \leq \sqrt{2}(a^\frac{p}{2} + b^\frac{p}{2})
\]

for any \(a, b \geq 0\),

\[
\sum_{k=-\infty}^{\infty} \int_M \|\nabla u_{\rho,k}\|^p dV = \int_M \|\nabla u\|^p dV.
\]
and
\[
\sum_{k=-\infty}^{\infty} \int_{M} \Psi^p u_{\rho,k}^p \, dV = \sum_{k=-\infty}^{\infty} \int \Psi^p (u - \rho^k)^p \, dV + \sum_{k=-\infty}^{\infty} \int \Psi^p (\rho^{k+1} - \rho^k)^p \, dV
\]
\[
\leq \sum_{k=-\infty}^{\infty} \int \Psi^p (u - \rho^k)^p \, dV + \sum_{k=-\infty}^{\infty} \int \Psi^p (\rho^{k+1} - \rho^k)^p \, dV
\]
\[
= \sum_{k=-\infty}^{\infty} \int \Psi^p (u - \rho^k)^p \, dV + \sum_{k=-\infty}^{\infty} \int \Psi^p (\rho^{k+1} - \rho^k)^p \, dV
\]
\[
= \sum_{k=-\infty}^{\infty} \int \Psi^p ((u - \rho^k)^p + \rho^{kp}) \, dV
\]
\[
\leq \sum_{k=-\infty}^{\infty} \int \Psi^p u^p \, dV = \int_{M} \Psi^p u^p \, dV.
\]
On the other hand,
\[
\sum_{k=-\infty}^{\infty} (\rho^{k+1} - \rho^k)^{\frac{np}{n-p}} m(\{u \geq \rho^{k+1}\})
\]
\[
= \sum_{k=-\infty}^{\infty} (\rho - 1)^{\frac{np}{n-p}} \rho^{\frac{kn_p}{n-p}} m(\{u \geq \rho^{k+1}\})
\]
\[
\geq (\rho - 1)^{\frac{np}{n-p}} \sum_{k=-\infty}^{\infty} \rho^{\frac{kn_p}{n-p}} - \frac{(np+n-k)(k+2)}{n-p} (\rho^{k+2} - \rho^{k+1})^{-1} \int_{\rho^{k+1}}^{\rho^{k+2}} t^{\frac{n_p+n-k}{n-p}} m(\{u \geq t\}) \, dt
\]
\[
= (\rho - 1)^{\frac{np+n-k}{n-p}} - \frac{2np+n-k}{n-p} \int_{0}^{\infty} t^{\frac{n-p+n}{n-p}} m(\{u \geq t\}) \, dt
\]
\[
\leq (\rho - 1)^{\frac{np+n-k}{n-p}} - \frac{2np+n-k}{n-p} \int_{M} u^{\frac{np}{n-p}} \, dV
\]
Letting $\rho = 2$ and combining the last two inequalities, we get
\[
\int_{M} u^{\frac{np}{n-p}} \, dV \leq \frac{np}{n-p} \times 2^{\frac{2np+n-k}{n-p}} \times 8^{\frac{np}{n-p}} \left( \frac{n^A p}{n-p} \right) \frac{n^2}{n-p} \left( \int_{M} q(u)^2 \, dV \right)^{\frac{np}{n-p}}
\]
\[
\leq n^{\frac{np}{n-p}} \left( 32 \frac{Ap}{n-p} \right)^{\frac{np}{n-p}} \left( \int_{M} q(u)^2 \, dV \right)^{\frac{np}{n-p}}
\]
Hence,
\[ \| u \|_{\frac{n}{n-p}}^n \leq \frac{32A}{n-p} \left( \int_M q(u)^{\frac{p}{n-p}} dV \right)^{\frac{1}{p}}. \]

Combining the last three theorems, we get the following time-dependent Sobolev inequality along the Ricci flow.

**Theorem 2.4.** Let \( g(t) \) with \( t \in [0, T) \) be a solution to the Ricci flow on a closed manifold \( M^n \) where \( T \) can be \( \infty \). Then, there are two positive constants \( A \) and \( B \) depending only on the initial metric \( g(0) \) and \( n \), such that

\[ \left( \int_M |u|^{\frac{n_p}{n-p}} dV_t \right)^{\frac{n_p}{n-p}} \leq A e^{Bt} \left( \int_M \left( \| \nabla u \|^2 + \frac{R + \max_M R_-(0) + 4}{4} |u|^2 \right)^{\frac{p}{2}} dV_t \right)^{\frac{1}{p}} \]

for any \( p \in [1, n) \), \( t \in [0, T) \) and \( u \in C^\infty(M) \).

**Proof.** By Theorem 2.1, we have two positive constants \( C_1 \) and \( C_2 \) depending only on the initial metric \( g(0) \), such that for any \( \sigma \in (0, 4] \) and \( t \in [0, T) \)

\[ \int_M v^2 \log v^2 dV_t \leq \sigma \int_M \left( \| \nabla v \|^2 + \frac{R}{4} v^2 \right) dV_t - \frac{n}{2} \log \sigma + C_1 t + C_2 \]

for any \( v \in C^\infty(M) \) with \( \int_M v^2 dV_t = 1 \).

For each \( s \in [0, T) \), let

\[ H_s = -\Delta_s + \frac{R(s) + \max_M R_+(s)}{4} + 1, \]

\[ q_s(u) = \langle \nabla^s u, \nabla^s u \rangle_s + \left( \frac{R(s) + \max_M R_+(s)}{4} + 1 \right) u^2 \]

and

\[ Q(u) = \int_M q_s(u) dV_s. \]

Then,

\[ \int_M u^2 \log u^2 dV_s \leq \sigma Q_s(u) - \frac{n}{2} \log \sigma + C_1 s + C_2 \]

for any \( \sigma \in (0, 4] \) and for any \( u \in C^\infty(M) \) with \( \int_M u^2 dV_s = 1 \).

By Theorem 2.2, we have

\[ \| e^{-tH_s} f \|_{\infty} \leq e^{C_1 s + C_2 t^{-\frac{2}{n}}} \| f \|_p \]
for any $f \in C^\infty(M)$, $t \in (0,1]$ and $p \geq 1$, where $C_3 = C_2 + \frac{n}{2}$.

Finally, by Theorem 2.3 we have
\[ \|u\|_{\frac{np}{n-p}} \leq \frac{32n^p p \exp(C_1 s + C_3)}{n-p} \sqrt{q_s(u)} \| \nabla u \| p. \]
on $(M^n, g(s))$, for any $u \in C^\infty(M)$ and $p \in [1,n)$.

Noting that $\max R_-(t)$ decreases along the Ricci flow, we get the following time-dependent Sobolev inequality along the Ricci flow:
\[ \left( \int_M |u|^{\frac{np}{n-p}} \, dV_t \right)^{\frac{n-p}{np}} \leq \frac{16n^p p \exp(C_1 t + C_3)}{n-p} \left( \int_M \left( \| \nabla u \|^2 + \frac{R + \max R_- (0) + 4}{4} u^2 \right)^{\frac{p}{2}} \, dV_t \right)^{\frac{1}{p}} \]
for any $t \in [0,T)$ and $u \in C^\infty(M)$.

**Remark 2.1.** When $T$ is finite, we get the uniform Sobolev inequalities in Zhang [7], Ye [6] and Hsu [2].

By a standard arguments via iteration as in Zhang [7], Ye [6] and Hsu [2], we get the following growth of the ratio non-collapsing along Ricci flow.

**Theorem 2.5.** Let $g(t)$ with $t \in [0,T)$ be a solution to the Ricci flow on a closed manifold $M^n$ where $T$ can be $\infty$. Then, there are two positive constants $\kappa$ and $A$ depending only on the initial metric $g(0)$ and $n$, such that
\[ V_x(r, t) \geq \kappa e^{-At} r^n \]
whenever $r \leq 1$ and $r^2 R(t) \leq 1$ on $B_x(r, t)$, where $B_x(r, t)$ is the geodesic ball of radius $r$ centered $x$ with respect to $g(t)$ and $V_x(r, t)$ is the volume of $B_x(r, t)$ with respect to $g(t)$.

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