Connes’ metric for states in group algebras

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Abstract

In this article we follow the main idea of A. Connes for the construction of a metric in the state space of a C*-algebra. We focus in the reduced algebra of a discrete group Γ, and prove some equivalences and relations between two central objects of this category: the word-length growth (connected with the degree of the extension of Γ when the group is an extension of ℤ by a finite group), and the topological equivalence between the ω* topology and the one introduced with this metric in the state space of C*_r(Γ).

Keywords: group C*-algebra, state space, non commutative metric space.

1 Introduction

In [Connes1] and [Connes2], A. Connes introduced what he called non commutative metric spaces, which consist of triples (A, D, H) where A is a C*-algebra, acting on the Hilbert space H, and D is an unbounded operator in H, called the Dirac operator, satisfying

1. (D^2 + 1)^{-1} is compact
2. the set \{a \in A : [D, a] \text{ is bounded}\} is norm-dense in A

We are interested in the case when Γ is a discrete group with identity element e and the algebra A is the reduced C*-algebra C*_r(Γ). The Hilbert space is ℓ^2(Γ), with C*_r(Γ) acting as left convoluters (i.e. the left regular representation). The Dirac operator is defined in terms of a length function on Γ. A length function is a map L : Γ → ℝ_+ satisfying

1. L(gh) ≤ L(g) + L(h) for all g, h ∈ Γ.
2. L(g^{-1}) = L(g) for all g ∈ Γ.
3. L(e) = 0.

If Γ is given by generators and relations, the prototypical length function is the map which assigns to each word its (minimal) length. We shall fix this data L, and we will make the further assumption that the sets

\{g ∈ Γ : L(g) ≤ c\}

are finite for any c > 0. The Dirac operator [Connes2] is then defined as follows:

D(δ_g) = L(g)δ_g

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where \( \{ \delta_g : g \in \Gamma \} \) is the canonical orthonormal basis of \( \ell^2(\Gamma) \). As is custom, we shall denote by \( \lambda_e \) the element \( \delta_e \) regarded as an operator in \( \ell^2(\Gamma) \). The metric (of the non commutative metric space) is defined in the state space \( \mathcal{S}(C^*_r(\Gamma)) \) of \( C^*_r(\Gamma) \) by means of the formula

\[
d(\psi, \varphi) = \sup\{|\psi(a) - \varphi(a)| : a \in C^*_r(\Gamma) \text{ with } ||a, D|| \leq 1\}.
\]

Here \( [ , ] \) denotes the usual commutator of operators. This \( d \) is not necessarily finite. In this note we study situations in which it is finite, and consider a problem posed by M. Rieffel, asking under which assumptions the metric thus defined induces on the state space a topology which is equivalent to the \( w^* \) topology.

The basic example of this situation, which even justifies the name ”non commutative metric space”, occurs when \( \mathcal{A} = C(M) \), the algebra of continuous functions on a spin manifold \( M \) [Connes2], [GL]. M Rieffel found [Rieffel] a natural triple associated to the noncommutative tori. Also he pointed out that one can find a positive answer for matrix algebras.

In this note we consider this problem for group algebras arising from discrete groups and triples arising from length functions. Instead of dealing with the \( d \) metric directly, we refer it to two metrics, \( d_\infty \) and \( d_2 \), related with the asymptotic behaviour of the family \( \{ \frac{1}{L(g)} : e \neq g \in \Gamma \} \):

\[
d_\infty(\varphi, \psi) = \sup_{g \neq e} \frac{|\varphi(\lambda_g) - \psi(\lambda_g)|}{L(g)},
\]

and

\[
d_2(\varphi, \psi) = \left( \sum_{g \neq e} \frac{|\varphi(\lambda_g) - \psi(\lambda_g)|^2}{L(g)^2} \right)^{1/2}.
\]

First note that \( d_\infty \) is a well defined metric and that \( d_\infty(\varphi, \psi) \leq d(\varphi, \psi) \). The first fact is apparent. To prove the second, note that \( |||D, \lambda_g||| = L(g) \), and therefore

\[
d_\infty(\varphi, \psi) = \sup_{a=\sum_{\alpha \in F} \alpha e^{\alpha}} |\varphi(a) - \psi(a)| \leq d(\varphi, \psi).
\]

Also note that \( d_2 \) may fail to be finite. Indeed, consider \( \Gamma = \mathbb{Z} \times \mathbb{Z} \). Then the family \( \{ \frac{1}{L(g)} : g \neq e \} \) does not belong to \( \ell^2(\mathbb{Z} \times \mathbb{Z}) \). Consider the positive definite functions \( f(g) = 1 \) for all \( g \) and \( h = \delta_e \). These functions induce states \( \varphi_f \) and \( \varphi_h \) on \( C^*_r(\mathbb{Z} \times \mathbb{Z}) \) satisfying \( \varphi_f(\lambda_g) = f(g) \) and \( \varphi_h(\lambda_g) = h(g) \). It follows that

\[
d_2(\varphi_f, \varphi_h) = \sum_{e \neq g \in \mathbb{Z} \times \mathbb{Z}} \frac{1}{L(g)^2} = \infty.
\]

Denote by \( K(\Gamma) \) the group algebra of \( \Gamma \), i.e. the set of elements of the form \( \sum_{g \in F} \alpha_g \lambda_g \), where \( \alpha_g \in \mathbb{C} \) and \( F \subset \Gamma \) is a finite set.

**Lemma 1.1** \( d_\infty \) is a metric in \( \mathcal{S}(C^*_r(\Gamma)) \) which induces a topology equivalent to the \( w^* \)-topology.

**Proof.** If \( d_\infty(\varphi_n, \varphi) \to 0 \), then clearly \( \varphi_n(\lambda_g) \to \varphi(\lambda_g) \) for all \( g \neq e \). Since \( \varphi_n, \varphi \) are states, \( \varphi_n(1) = 1 = \varphi(1) \). It follows that \( \varphi_n(a) \to \varphi(a) \) for all \( a \in K(\Gamma) \). Since \( \varphi_n, \varphi \) have their norms bounded (by 1), and since \( K(\Gamma) \) is dense in \( C^*_r(\Gamma) \), it follows that \( \varphi_n \to \varphi \) in the \( w^* \) topology. Conversely, suppose that \( \varphi_n(a) \to \varphi(a) \) for all \( a \in C^*_r(\Gamma) \) and fix \( \epsilon > 0 \). Let \( F = \{ g \in \Gamma : L(g) < 4/\epsilon \} \), which is a finite set, say \( F = \{ g_1, ..., g_k \} \). If \( g \in F \), one has

\[
|\varphi_n(\lambda_g)| - |\varphi(\lambda_g)| \leq \frac{|\varphi_n(\lambda_g)| + |\varphi(\lambda_g)|}{4/\epsilon} \leq \epsilon/2.
\]
On the other hand, there exists \( n_0 \) such that for all \( n \geq n_0 \),
\[
|\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})| < \frac{\epsilon}{2} \min\{L(g_1), ..., L(g_k)\}, \text{ for } i = 1, ..., k.
\]
It follows that
\[
\frac{|\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})|}{L(g_i)} < \epsilon/2.
\]
Therefore, if \( n \geq n_0 \), then
\[
\sup_{g \neq e} \frac{|\varphi_n(\lambda_g) - \varphi(\lambda_g)|}{L(g)} = d_\infty(\varphi_n, \varphi) \to 0.
\]
\[\Box\]

\section{Comparison between \( d, d_\infty \) and \( d_2 \)}

Here we establish the basic inequality for these metrics, namely \( d_\infty \leq d \leq d_2 \).

\textbf{Lemma 2.1} Let \( a = \sum_{g \in F} \alpha_g \lambda_g \in K(\Gamma) \), then
\[
\|[D, a]\| \geq (\sum_{g \in F} |\alpha_g|^2L(g)^2)^{1/2}.
\]

\textbf{Proof.} Note that
\[
[D, a]\delta_e = \sum_{g \in F} \alpha_g [D, \lambda_g] \delta_e = - \sum_{g \in F} \alpha_g L(g) \delta_g,
\]
because \( D \lambda_g \delta_e = D \delta_g = L(g) \delta_g \), and in particular \( D \delta_e = 0 \). Therefore
\[
\|[D, a]\delta_e\|_2^2 = (\sum_{g \in F} |\alpha_g|^2L(g)^2)^{1/2}.
\]
It follows that \( \|[D, a]\| \geq \|[D, a]\delta_e\|_2 = (\sum_{g \in F} |\alpha_g|^2L(g)^2)^{1/2} \). \[\Box\]

\textbf{Proposition 2.2}
\[
d_\infty(\varphi, \psi) \leq d(\varphi, \psi) \leq d_2(\varphi, \psi).
\]

\textbf{Proof.} Pick \( a = \sum_{g \in F} \alpha_g \lambda_g \in K(\Gamma) \), with \( \|[D, a]\| \leq 1 \) (note that for any \( a \in K(\Gamma) \), \( [D, a] \) is a bounded operator). Then
\[
|\varphi(a) - \psi(a)| = |\sum_{g \in F} \alpha_g (\varphi(\lambda_g) - \psi(\lambda_g))| = \sum_{\epsilon \neq g \in F} \alpha_g L(g) \left( \frac{(\varphi(\lambda_g) - \psi(\lambda_g))}{L(g)} \right),
\]
which by the Cauchy-Schwartz inequality is less than or equal to
\[
\left( \sum_{\epsilon \neq g \in F} |\alpha_g|^2L(g)^2 \right)^{1/2} \left( \sum_{\epsilon \neq g \in F} \left| \frac{(\varphi(\lambda_g) - \psi(\lambda_g))}{L(g)} \right|^2 \right)^{1/2} \leq \|[D, a]\|d_2(\varphi, \psi) \leq d_2(\varphi, \psi).
\]
The proof finishes by observing that the set of elements \( a \in K(\Gamma) \) with \( \|[D, a]\| \leq 1 \) is dense among elements \( b \in C_\ast^r(\Gamma) \) with \( \|[D, b]\| \leq 1 \). Indeed, let \( b = \sum_{g \in F} \beta_g \lambda_g \in C_\ast^r(\Gamma) \) with \( \|[D, b]\| \leq 1 \). For finite sets \( F \subset \Gamma \), the truncated elements \( b_F = \sum_{g \in F} \beta_g \lambda_g \in K(\Gamma) \) converge in norm to \( b \). Clearly also the (bounded) commutants \( [D, b_F] \) converge in norm to \( [D, b] \).

Denote by \( N_F = \|[D, b]\|\|[D, b_F]\|^{-1} \) (after dropping the elements \( b_F \) with \( [D, b_F] = 0 \)). Then \( N_Fb_F \) lies in \( K(\Gamma) \), the commutants \( [D, N_Fb_F] \) have norm less than or equal to 1, and converge to \( b \). \[\Box\]
We emphasize that \( d_2 \) might be infinite. It would be finite if for example the family \( \{ \frac{1}{L(g)} : e \neq g \in \Gamma \} \) would lie in \( \ell^2(\Gamma) \). This imposes a strong condition on \( \Gamma \), namely, that the group \( \Gamma \) has linear growth (polynomial growth with degree 1), see [Gromov] and [Connes2]. This means, that there exists constants \( k, l \) such that \( \# \{ g \in \Gamma : L(g) \leq c \} \sim kc + l \).

**Example 2.3** Let us consider the following examples, of groups \( \Gamma \) which satisfy that the family \( \{ \frac{1}{L(g)} : e \neq g \in \Gamma \} \) lies in \( \ell^2 \).

1. Let \( \Gamma = \mathbb{Z} \). Here the length function is \( L(m) = |m|, m \in \mathbb{Z} \). The group \( C^* \)-algebra equals in this case \( C(S^1) \).

2. Let \( \Gamma \) be a finite extension of \( \mathbb{Z} \), i.e. a group \( \Gamma \) which has a copy of \( \mathbb{Z} \) inside, as a normal subgroup, and the quotient \( \mathcal{F} = \Gamma/\mathbb{Z} \) is finite. Then, as a set, \( \Gamma \) is \( \mathbb{Z} \times \mathcal{F} \). Let \( \mathcal{F} = \{ f_1, ..., f_n \} \). Then the classes of \( (1, f_1), ..., (1, f_n) \) (i.e. these elements regarded as elements of \( \Gamma \)) are generators for \( \Gamma \). Let us consider the length function \( L \) given by word length with respect to this set of generators. Note that for this \( L \), there are at most \( 2n \) elements of \( \Gamma \) with any given length. It follows that \( \{ \frac{1}{L(g)} : e \neq g \in \Gamma \} \) lies in \( \ell^2 \). The (reduced) \( C^* \)-algebra of such \( \Gamma \) can be computed. They consist of algebras of \( n \times n \) matrices with entries in \( C(S^1) \), see chapter VIII of [Davidson] for a complete description of this computation. Let us point out two special cases of this type

(a) \( \Gamma = \mathbb{Z} \times \mathcal{F} \) with the usual product for pairs. In this case the \( C^* \)-algebra is \( C^*_r(\mathbb{Z} \times \mathcal{F}) \simeq C(S^1) \otimes C^*_r(\mathcal{F}) \). The algebra \( C^*_r(\mathcal{F}) \) is finite dimensional, therefore in this case \( C^*_r(\Gamma) \) consists of a direct sum of full matrix algebras with entries in \( C(S^1) \). In particular, if \( \mathcal{F} = S_k \), the group of permutations of order \( k \), then \( C^*_r(\Gamma) = M_k(C(S^1)) \).

(b) Consider the (unique) nontrivial automorphism of \( \mathbb{Z} \), \( \theta(m) = -m \). Then one has a \( \mathbb{Z}_2 \) extension of \( \mathbb{Z} \), \( \Gamma = \mathbb{Z} \times \theta \mathbb{Z}_2 \), and the corresponding \( C^* \)-algebra \( C^*_r(\Gamma) \) is the cross product \( C(S^1) \rtimes \theta \mathbb{Z}_2 \), which identifies with the algebra of \( 2 \times 2 \) matrices with entries in \( C(S^1) \) of the form

\[
\begin{pmatrix}
f(z) & g(z) \\
\overline{f(\overline{z})} & \overline{g(\overline{z})}
\end{pmatrix},
\]

where \( f \) and \( g \) are continuous functions in \( S^1 \).

**Proposition 2.4** If \( \Gamma \) has a length function \( L \) which satisfies that \( \{ \frac{1}{L(g)} : e \neq g \in \Gamma \} \) is square summable, then the metric \( d \) is well defined (is finite) and induces on the state space of \( C^*_r(\Gamma) \) the \( w^* \) topology.

**Proof.** Since \( \{ \frac{1}{L(g)} \} \in \ell^2, d \leq d_2 < \infty \). By the above results it suffices to prove that if a sequence \( \varphi_n \) converges to \( \varphi \) in the \( w^* \) topology, then it converges in the \( d \) metric. We claim that it converges in the \( d_2 \) metric. Fix \( \epsilon > 0 \). There exists a finite set \( F = \{ g_1, ..., g_k \} \) such that

\[
\left( \sum_{g \in \Gamma - F} \frac{|\varphi_n(\lambda_g) - \varphi(\lambda_g)|^2}{L(g)^2} \right)^{1/2} \leq 2 \left( \sum_{g \in \Gamma - F} \frac{1}{L(g)^2} \right)^{1/2} < \epsilon/2.
\]

Put \( c = (\sum_{i=1}^k \frac{1}{L(g_i)^2})^{1/2} \). There exists \( n_0 \) such that for all \( n \geq n_0 \), one has \( |\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})| < \epsilon/2c \). Therefore

\[
\left( \sum_{i=1}^k \frac{|\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})|^2}{L(g_i)^2} \right)^{1/2} < \epsilon/2.
\]

Then \( d_2(\varphi_n, \varphi) \rightarrow 0 \). \( \square \)
Corollary 2.5 If $\Gamma$ is a finite extension of $\mathbb{Z}$, then in $S(C_0^*(\Gamma))$ the $d$ metric is well defined and induces the $\omega^*$-topology.

3 Normal states which are bounded with respect to the trace

We shall prove that the metric $d$ is finite on the set of normal states of $C_0^*(\Gamma)$ which are bounded with respect to the trace of $C_\infty(\Gamma)$, i.e., the states $\varphi$ which extend to normal states of the Von Neumann algebra $\mathcal{L}_\Gamma$ of $\Gamma$, and verify that there exists a constant $\kappa > 0$ such that

$$\varphi(a^*a) \leq \kappa \tau(a^*a),$$

or shortly, $\varphi \leq \kappa \tau$. Recall that the trace $\tau$ is given by $\tau(a) = \langle a \delta_e, \delta_e \rangle$. There is a Radon-Nykodim derivative for all such $\varphi$ [Araki]. Namely, there exists an element $\rho_\varphi \geq 0$ in $\mathcal{L}_\Gamma$ such that

$$\varphi(a) = \tau(\rho_\varphi a), \quad \text{with } \|\rho_\varphi\| \leq \kappa^{1/2}.$$  

Denote by $\mathcal{S}_\kappa$ the set

$$\mathcal{S}_\kappa = \{ \varphi \in \mathcal{S}(\mathcal{L}_\Gamma) : \varphi \leq \kappa \tau \}.$$

First note that a state which lies in $\mathcal{S}_\kappa$ is necessarily normal. Indeed, let $\{p_i : i \in I\}$ be an arbitrary family of pairwise orthogonal projections in $\mathcal{L}_\Gamma$. Fix $\epsilon > 0$ and let $J \subset I$ be a finite set such that $\tau(\sum_{j \in J} p_i) \leq \epsilon/\kappa$. Then $\varphi(\sum_{j \in J} p_i) \leq \epsilon$. Therefore $0 \leq \varphi(\sum_{i \in I} p_i) = \sum_{j \in J} \varphi(p_j) + \varphi(\sum_{i \in I \setminus J} p_i) \leq \sum_{j \in J} \varphi(p_j) + \epsilon$. That is, $\sum_{i \in I} \varphi(p_i) = \varphi(\sum_{i \in I} p_i)$, and $\varphi$ is normal. Also it is apparent that $\mathcal{S}_\kappa$ is $w^*$ compact and convex.

Proposition 3.1 The metrics $d$ and $d_2$ are well defined on $\mathcal{S}_\kappa$ and induce the $w^*$ topology.

Proof. Note that if $\varphi \in \mathcal{S}_\kappa$ then

$$\varphi(\lambda_g) = \tau(\rho_\varphi \lambda_g) = \langle \rho_\varphi \delta_g, \delta_e \rangle = \rho_\varphi(g^{-1}),$$

where $\rho_\varphi(g^{-1})$ denotes the $g^{-1}$-coordinate of $\rho_\varphi$ regarded as an element of $\ell^2(\Gamma)$. In particular, it follows that the family $\{\varphi(\lambda_g) : g \in \Gamma\}$ is square summable. Moreover,

$$\left( \sum_{g \in \Gamma} |\varphi(\lambda_g)|^2 \right)^{1/2} = \|\rho_\varphi\|_2 \leq \|\rho_\varphi\| \leq \kappa^{1/2}.$$

It follows that if $\varphi, \psi \in \mathcal{S}_\kappa$, then

$$d(\varphi, \psi) \leq d_2(\varphi, \psi) = \left( \sum_{e \neq g \in \Gamma} \frac{|\varphi(\lambda_g) - \psi(\lambda_g)|^2}{L(g)^2} \right)^{1/2} \leq 2\kappa.$$

Suppose now that $\varphi_n \to \varphi$ in the $w^*$ topology. Fix $\epsilon > 0$. Then the set $F = \{ g \in \Gamma : L(g) \leq 2\kappa/\epsilon\}$ is finite. Say $F = \{g_1, \ldots, g_n\}$. If $g$ lies outside $F$ one has

$$\left( \sum_{e \neq g \in \Gamma \setminus F} \frac{|\varphi_n(\lambda_g) - \varphi(\lambda_g)|^2}{L(g)^2} \right)^{1/2} \leq \epsilon/2 \kappa \left( \sum_{g \in \Gamma \setminus F} |\varphi_n(\lambda_g) - \varphi(\lambda_g)|^2 \right)^{1/2} \leq \epsilon.$$

Let $C = \sum_{i=1}^n \frac{1}{L(g_i)^2}$. There exists $n_0$ such that if $n \geq n_0$ then

$$|\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})| < \epsilon/C \quad \text{for } i = 1, \ldots, n.$$

It follows that $d(\varphi_n, \varphi) \leq d_2(\varphi_n, \varphi) < \epsilon$ if $n \geq n_0$. \qed
Remark 3.2 1. The first part of the proof in fact shows that if $\varphi$ and $\psi$ are normal states of $L_\Gamma$ whose Radon-Nykodim derivatives with respect to the trace $\tau$ lie in $\ell^2(\Gamma)$, then $d(\varphi,\psi) \leq d_2(\varphi,\psi) < \infty$.

2. It is apparent that the metrics $d$ and $d_2$ are also finite on the set $\bigcup_{\kappa > 0} S_\kappa$.

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