Quantum Spectral Curve for AdS$_5$/CFT$_4$

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We present a new formalism, alternative to the old TBA-like approach, for solution of the spectral problem of planar $\mathcal{N} = 4$ SYM. It takes a concise form of a non-linear matrix Riemann-Hilbert problem in terms of a few Q-functions. We demonstrate the formalism for two types of observables – local operators at weak coupling and cusped Wilson lines in a near BPS limit.

**INTRODUCTION**

The spectrum of anomalous dimensions in the planar $\mathcal{N} = 4$ SYM theory was successfully studied in the last decade, to great extent due to the ideas of AdS/CFT correspondence and integrability [1]. A conventional form of solution to the spectral problem is given by an infinite set of nonlinear integral TBA equations [2–4] for the functions of the spectral parameter $Y_{a,s}(u)$

$$\log Y_{a,s}(u) = \delta_a^s iP_a(u) + \int dv K^{a's'}_{as}(u,v) \log(1 + Y_{a',s'}(v))$$

where the sum over $a', s'$ in the r.h.s. goes along the internal nodes of the lattice (T-hook) in Fig. 1. The momentum $P_a$ and the kernels $K^{a's'}_{as}$ are explicit but rather complicated functions of the spectral parameters $u, v$ [3]. Their important analytic feature is the presence of cuts, parallel to $\mathbb{R}$, with fixed branch-points at $u, v \in \pm 2g + i\mathbb{Z}$ or $u, v \in \pm 2g + i(\mathbb{Z} + \frac{1}{2})$ where $g \equiv \sqrt{\lambda}/(4\pi)$ and $\lambda$ is the 't Hooft coupling. This TBA system fixes completely the Y-functions and hence the dimension of a particular operator specified by certain poles and zeros incorporated into the driving terms [3]. It was successfully used for the weak and strong coupling analysis [5–7] as well as for the first successful numerical computations of dimensions of Konishi [8, 9] and similar operators [10, 11]. However, this TBA system has very complex analyticity properties, which limits in practice its applications and obscures the long anticipated beauty of the whole problem.

An obvious sign of this hidden beauty is the direct equivalence of the TBA system to the AdS/CFT Y-system, originally proposed as a solution of the AdS/CFT spectral problem in [12], with additional analyticity conditions [13]. It is a universal set of equations equivalent, by the substitution $Y_{a,s} = T_{a+s}T_{-a-s}$, to the Hirota discrete bilinear equation (T-system) [32, 33]

$$T_{a,s}^+ T_{a,s}^- = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1},$$

which is integrable in its turn. Using this integrability the general solution of T-system can be explicitly parameterized in terms of Wronskians built from only 8 independent Q-functions [14, 15]. The Q-functions are the most elementary constituents of the whole construction with the analyticity properties much simpler than those of Y- or T-functions [16]. With a savvy choice of the basic Q-functions we managed in [16] to close a finite system of non linear integral equations (FiNLIE). It appeared to be an very efficient tool in multi-loop weak coupling computations [17, 18]. But it was clear that the somewhat bulky form of that FiNLIE [16] hides a much more beautiful and simple formulation, with a clear insight into the full analytic structure of the underlying functions.

We formulate in this note a new, much more transparent and concise system of the planar AdS$_5$/CFT$_4$ spectral equations of the Riemann-Hilbert type. It might represent the ultimate simplification of this spectral problem.

**P$_H$ SYSTEM FOR THE SPECTRUM**

We will demonstrate our new approach on the most important example of the left-right symmetric states for which $T_{a,s} = T_{a,-s}$ (in the appropriate gauge described in [16]). To start with, all T- and Y-functions can be expressed in terms of 4 + 4 Q-functions [15]. Let us exemplify this relation for T-functions of the right band (see Fig 1), where we have for $s > 0$ [16]

$$T_{1,s}(u) = P_1(u + \frac{i\pi}{2})P_2(u - \frac{i\pi}{2}) - P_2(u + \frac{i\pi}{2})P_1(u - \frac{i\pi}{2}),$$

where the symbol $P$ is used to denote the Q-functions in the right band, in order to avoid a clash with other
notations existing in the literature. An important feature of this parameterization is that $\mathbf{P}$’s have only one single cut between $-2g$ and $2g$, otherwise being analytic in the whole complex plane [34] [16]. This property is tightly related to what we refer to as $\mathbb{Z}_4$-symmetry [16, 19, 20].

Ideally, we would like to reduce the whole problem to a single spectral curve, or a Riemann surface on which all $Q$-functions are defined. For that we need to know in particular the analytic continuations of $\mathbf{P}_1$ and $\mathbf{P}_2$ through the cut which we denote as $\tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2$. Quite expectedly, $\tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2$ have an infinite “ladder” of cuts, with branch points at $\pm 2g + in$ for any integer $n$. To describe completely the Riemann surface, one should know the analytic continuation through any of those new cuts, and so on. One of the main results of this note is that this complicated cut structure has a stunningly simple algebraic description!

Namely, inspecting the properties of $Q$-functions of [16] we managed to construct [21] two additional functions $\mathbf{P}_3$ and $\mathbf{P}_4$, again with only one single cut, such that after the analytic continuation the four functions $\tilde{\mathbf{P}}_a, a = 1, 2, 3, 4$, can be expressed as linear combinations of the initial $\mathbf{P}$’s

$$\tilde{\mathbf{P}}_a = -\mu_{ab}\chi^{bc}\mathbf{P}_c,$$

where $\mu_{ab}$ is a $4 \times 4$ antisymmetric matrix constrained by

$$\mu_{12}\mu_{34} - \mu_{13}\mu_{24} + \mu_{14}^2 = 1, \quad \mu_{23} = \mu_{14},$$

and $\chi$ is an antisymmetric constant $4 \times 4$ matrix with the only nonzero entries $\chi^{23} = \chi^{41} = -\chi^{14} = -\chi^{32} = 1$.

Furthermore, the analytic continuation around the branch point $2g$ of $\mu_{ab}$ itself, i.e. $\tilde{\mu}_{ab}$, has a very peculiar pseudo-periodicity condition (see Fig 2)

$$\tilde{\mu}_{ab}(u) = \mu_{ab}(u + i).$$

In other words, if we define a function $\tilde{\mu}$ such that it coincides with $\mu$ in the strip $0 < \mathrm{Im} \, u < 1$ but has all its cuts going to infinity then (5) simply tells us that $\tilde{\mu}$ is a truly $i$-periodic function: $\tilde{\mu}_{ab}(u + i) = \tilde{\mu}_{ab}$.

To close the system of equations on $\mathbf{P}, \mu$ we have to find a condition on $\mu_{ab}$ similar to (3). An important part of it is already dictated by (3): since the branch points are quadratic, we have $\tilde{\mathbf{P}}_a = \mathbf{P}_a$ which leads to $\mathbf{P} = -\tilde{\mu}\chi\tilde{\mathbf{P}}$. This, together with (3), gives a set of linear equations fixing the discontinuity of the matrix $\mu$ up to a single unknown factor $e(u)$: $\tilde{\mu}_{ab} - \mu_{ab} = e(u)(\mathbf{P}_a\mathbf{P}_b - \mathbf{P}_b\mathbf{P}_a)$. We argue below that $e(u) = 1$ and hence

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\mathbf{P}_b - \mathbf{P}_b\mathbf{P}_a.$$

Eqs. (3), (5), (6) represent our main result — a complete non-linear system of Riemann-Hilbert equations for the AdS/CFT spectral problem. They allow us to walk through the cuts to any sheet (out of infinite number) of the Riemann surface of the functions $\mathbf{P}$ and $\mu$. In this sense, they give the full description of the spectral curve of the problem. Indeed, by means of (3) and (5) it is easy to walk through the central cut in Fig 2. The other cuts are present only in $\mu$. To define the analytic continuation through them, we use a combination of (3), (5) and (6),

$$\mu_{ab}(u + i) = \mu_{ab} - \mathbf{P}_a\mathbf{P}_e\chi^{ec}\mu_{bc} - \mathbf{P}_b\mathbf{P}_e\chi^{ec}\mu_{ac},$$

which allows to recursively express $\mu_{ab}(u + in)$ through $\mu_{ab}(u)$ and shifted $\mathbf{P}$’s – the quantities with known monodromies. We refer to this new formulation of the spectral problem, given by eqs.(3)-(6), as to the $\mathbf{P} \mu$ system.

Let us argue now that $\mathbf{P}$ and $\mu$ contain the complete information about the initial Y-system. Indeed, from (2) we restore $T_{1,s}$ for $s > 0$. Furthermore, $T_{2,s} = T_{1,-s}T_{1,1}^{-1}, T_{0,s} = 1$ and with a help of one extra relation $T_{3,2} = T_{2,3}\mu_{12}$ (see [16], where $T = \mu_{12}$) we have just enough of information to recover any $T_{a,s}$ using solely the Hirota equation (1), for any left-right symmetric state. It is just a matter of elementary algebra to write any Y-function explicitly through $\mathbf{P}, \mu$. In particular, we find

$$Y_{11}Y_{22} = 1 + \frac{\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_2\mathbf{P}_1}{\mu_{12}} = \frac{\mu_{12}(u + i)}{\mu_{12}(u)}.$$

We note that the first equality holds for any $e(u)$, but imposing [16] $\frac{Y_{11}Y_{22}}{Y_{11}} = \frac{1}{1 + \frac{1}{\mu_{12}}}$ we fix $e(u) = 1$.

**Asymptotics and charges.** The quantity (8) is known to contain the energy/dimension $\Delta$ of the state in its large $u$ asymptotics [16]: $\log Y_{11}Y_{22} \simeq i\frac{\Delta - \Delta_0}{2\mu_0}$. Similarly, the large $u$ asymptotics of $\mathbf{P}$ and $\mu$ contains the information about other conserved charges of the state. In fact, $\mathbf{P}_a^\pm/\mathbf{P}_a^\pm$ is the exact quantum analogue of the $S^5$ eigenvalues of the monodromy matrix [20] of classical strings moving in $AdS_5 \times S^5$ and thus $\mathbf{P}_a^\pm/\mathbf{P}_a^\pm \simeq 1 + M_a/(2iu)$, where $M_a$ are integer charges of the global SO(6) symmetry. For instance, in the $\mathfrak{sl}_2$ sector, i.e. for spin $S$ twist $T$ operators of the type $T_{k}Z^{-S_k}Z^{-S-L-1}$ dual to the string which is point-like in $S^5$ and moves there with the angular momentum $L$, one has the following asymptotics

$$\mathbf{P}_a \simeq (A_1u^\frac{i}{2}, A_2u^\frac{i}{2}, A_3u^\frac{i}{2}, A_4u^\frac{i}{2})a.$$

Note that at odd $L$’s $\mathbf{P}_a$ have a sign ambiguity (see [34]).

Next, we also have to specify the asymptotics of $\mu$. Assuming its power-like behavior we immediately get from
\( \mu_{12} \simeq u^{\Delta - L} \). To deduce the asymptotics of the remaining \( \mu \)'s we consider \( \mathbf{P}_1 = -\mu_{12} \mathbf{P}_1 + \mu_{11} \mathbf{P}_2 - \mu_{12} \mathbf{P}_3 \) and assume that all the terms in the r.h.s. scale in the same way. This gives e.g. \( \mu_{13} \sim u^{\Delta+1} \) and, similarly, \( (\mu_{14}, \mu_{24}, \mu_{34}) \sim (u^{\Delta}, u^{\Delta-1}, u^{\Delta+1}) \). This strategy allows one to easily determine the asymptotics for any state even outside of the \( \mathfrak{s}(2) \) sector.

Finally, let us fix the coefficients \( A_i \) in (9). Note that (7) becomes at large \( u \) a homogeneous differential equation on the 5 independent components of \( \mu_{ab} \). By plugging into this equation the asymptotics for \( \mu_{ab} \) and \( \mathbf{P}_a \) we get a 5'th order algebraic equation on \( \Delta \). Its roots are of the form \( (\pm \alpha, \pm \beta, 0) \) where \( \alpha, \beta \) are functions of \( A_i \). The root \( \alpha = \Delta \) reproduces the correct asymptotics of \( \mu_{ab} \), whereas one can show (see a motivation in discussion) that \( \beta + 1 = S \) is the Lorentz spin of the state. By inverting these relations one gets

\[
A_2A_3 = \frac{[(L - S + 2)^2 - \Delta^2][(L + S)^2 - \Delta^2]}{16\Gamma(L + 1)}, \\
A_4A_1 = \frac{[(L + S - 2)^2 - \Delta^2][(L - S)^2 - \Delta^2]}{16\Gamma(L - 1)},
\]

(10) Note that \( \Delta \) enters (10) only as \( \Delta^2 \), which suggests that the function \( S(\Delta) \) is even, as claimed in [22]. Interestingly, \( A_i \) enter only through the products (10), due to a rescaling symmetry of the \( \mathbf{P}_\mu \) system [21].

**Regularity condition.** To single out physical solutions of the \( \mathbf{P}_\mu \) system we impose the regularity condition: \( \mathbf{P}_a \) and \( \mu \)'s do not have poles on their defining sheet, and hence, due to (3),(5),(7), on the whole Riemann surface.

### Weak Coupling

Let us demonstrate the weak coupling limit for the \( \mathfrak{s}(2) \) sector. First, (10) gives an idea about the scaling of \( \mathbf{P}_a \)'s at weak coupling: since \( \Delta = L + S + \mathcal{O}(g^2) \), we see that \( A_2A_3 = \mathcal{O}(g^2) \rightarrow 0 \) which suggests also that \( \mathbf{P}_2 \mathbf{P}_3 = \mathcal{O}(g^2) \). Hence at the leading order \( \mathbf{P}_2 \simeq 0 \) and (7) simplifies considerably: Equations for \( \mu_{12} \) and \( \mu_{24} \) decouple from the rest. Excluding \( \mu_{24} \) we get a 2-n order difference equation for \( \mu_{12}^+ = Q + \mathcal{O}(g^2) \)

\[
T \mathbf{Q} + \frac{1}{(P_1)^2} Q^{[-2]} + \frac{1}{(P_1)^2} Q^{[+2]} = 0,
\]

(11)

\[
T = \frac{P_+}{P_-} - \frac{P_-}{P_+} - \frac{1}{(P_1)^2} - \frac{1}{(P_1)^2},
\]

which is strikingly similar to the Baxter equation for the Heisenberg spin chain; this analogy goes even further as the zeros of \( \mu_{12} \) are indeed exact Bethe roots [16]. To demonstrate the actual equivalence with Baxter equation one should show that the coefficients in (11) do have the desired analytic properties. Omitting details in this short letter, we only mention that from explicit expression [16] it follows that \( \mathbf{P}_1 = A_1 u^{-L/2} + \mathcal{O}(g^2) \), i.e. the leading order of \( \mathbf{P}_1 \) coincides with its large \( u \) asymptotics. Furthermore, the ratio \( \mathbf{P}_4/\mathbf{P}_1 \) behaves asymptotically as \( u^{L-1} \) and by construction it has no poles when \( u \neq 0 \). \( u = 0 \) is the place where the branch points merge, hence this point is potentially singular. However, one can advocate that at the leading order \( \mathbf{P}_4/\mathbf{P}_1 \) is also regular at \( u = 0 \) and hence this ratio is simply a polynomial. For the same reason of regularity, \( \mathbf{Q} \) should be also a polynomial, of degree \( S \) as it follows from the asymptotics of \( \mu_{12} \sim u^{\Delta - L} \simeq u^S \).

By standard arguments, zeros of \( \mathbf{Q} \) should satisfy Bethe equations, which singles out a discrete set of possible \( Q \)'s and hence of solutions of the \( \mathbf{P}_\mu \) system corresponding to the states from the \( \mathfrak{s}(2) \) sector. For AdS/CFT, we have an additional zero-momentum constraint \( Q(+i/2)/Q(-i/2) = 1 \) which is due to the cyclicality of trace. The \( \mathbf{P}_\mu \) system also encodes this constraint! Indeed, in the limit \( g \to 0 \), it is nothing but (5) evaluated at the branch point \( u = 2g \) where we used the analyticity condition \( \mu_{ab}(2g) = \mu_{ab}(2g) \).

To compute the one-loop energy we have to compute the large \( u \) asymptotics of \( \mu_{12} \) to the next order. From \( \mu_{12}/Q \sim u^{\Delta - S - L} \simeq 1 + (\Delta - S - L) \log u + \mathcal{O}(g^2) \) we see that we have to find the pre-factor of log \( u \) term. Such large \( u \) behavior clearly shows that at the next order \( \mu_{12} \) can no longer be a polynomial. Instead, \( \mu_{12} \) develops singularities at the collapsing branch cuts \( u = i\pi \), \( n \in \mathbb{Z} \) in addition to a modified polynomial part. We denote the singular part of \( \mu_{12} \) by \( R \). To separate the regular and singular parts we write \( \mu_{12} \) in the following way

\[
\mu_{12} = \left( \frac{\mu_{12}^{++}}{2} \right) + \sqrt{u^2 - 4g^2} \left[ \frac{\mu_{12}^{++}}{2\sqrt{u^2 - 4g^2}} \right],
\]

(12)

where, due to (5), both expressions inside the brackets have a trivial monodromy on the cut \([-2g, 2g]\), thus being very smooth near the origin. The singularity comes solely from the square root factor whose small \( g \) expansion reads: \( \sqrt{u^2 - 4g^2} = u - \frac{4g^2}{2} + \cdots \), which allows us to fix \( R'^- \equiv g^2 r' \) with \( r' \equiv Q(\frac{\pi}{2}) - Q(-\frac{\pi}{2}) \) in the vicinity of \( u = 0 \). From (5) and (12) we also get \( R'^+ \equiv -g^2 r' /u \).

To find all other possible singularities at \( u \sim i(n+1/2) \) we notice that in the vicinity of each regularity, \( \mathbf{R} \) must satisfy the same Baxter equation as \( \mathbf{Q} \), up to some regular terms. With the poles of \( \mathbf{R}(u) \) defined above at \( u = \pm i/2 \) the solution is unique and is given by \( \mathbf{R}(u) = ig^2r\frac{Q(u)}{Q(\frac{\pi}{2})} (\psi(\frac{\pi}{2} - iu) + \psi(\frac{\pi}{2} + iu)) \). Now we can expand \( \mathbf{R} \) at large \( u \) to get \( \mathbf{R}(u)/Q(u) \approx \frac{2i g^2 \log u}{Q'(\frac{\pi}{2})} \), thus reproducing the well known expression for the one-loop dimension \( \Delta = L + S + 2i g^2 \partial_\phi \log \frac{Q'}{Q} \bigg|_{u=0} \).

### Cusp Anomalous Dimension

It was shown in [23, 24] that the Wilson line with a cusp of an angle \( \phi \) can be described by essentially the
same system of TBA equations. As a consequence, it can be also studied via the $P_\mu$ system which turns out to be a very efficient approach, as we are going to demonstrate. We consider a particular limit of small $\phi$. For a more general case, with more details of the derivation, see [25].

Whereas $P_\mu$-equations remain unaltered, it is the large $u$ behaviour which distinguishes this case from the case of local operators. In particular, one finds that $P_\mu \simeq (A_1 u^{-L+1/2}, A_2 u^{-L-1/2}, A_3 u^{+L+3/2}, A_4 u^{L+1/2})$ instead of (9). Even though (10) is not fully applicable now, it appears to capture correctly the behaviour of $P_\mu$ at small $\phi$: For the case of the vacuum state $S = 0$ and $\Delta = L+O(\phi^2)$. We see that at $\phi = 0$ $A_2 - A_3 \simeq A_4 - A_1 \rightarrow 0$ suggesting that to the leading order $P_\mu = 0$. Hence one gets from (6) $\hat{\mu}_{ab} = \mu_{ab}$, i.e. $\mu_{ab}$ has no cuts; it is then just a periodic function as follows from (5).

Another specific feature of this case is that $Y$-functions have poles which originate from the boundary dressing phase. In particular, the product (8) has simple poles at $u = in/2$ for any integer $n \neq 0$ [24]. By requiring the regularity of the $P_\mu$-system, we see that in (8) the poles can only originate from zeros of $\mu_{12}$. Hence, $\mu_{12}$ is a periodic entire function with simple zeros at $in/2$. In addition, as $Y$-functions are even for the vacuum, each $\mu_{ab}$ has a certain parity w.r.t. $u$: for instance $\mu_{12}$ is odd and hence it has the form $\mu_{12} = C \sinh(2\pi u)$.

We have no physical reason to introduce infinite sets of zeros for other $\mu_{ab}$’s and we assume from their periodicity that they are just constants which are further constrained by the parity: $\mu_{13} = \mu_{24} = 0$ because they are odd. Then $\mu_{34} = 0$ and $\mu_{14} = \pm 1$, in order to satisfy (4). A consistent choice of the sign is $\mu_{14} = -1$. Then (3) gives

$$P_1 - P_2 = -C \sinh(2\pi u)P_3 \quad , \quad P_3 + P_4 = 0 \quad (13)$$

$$P_4 - P_2 = -C \sinh(2\pi u)P_4 \quad , \quad P_4 - P_4 = 0 \quad (14)$$

In what follows we consider for simplicity the case $L = 0$. The generalization to arbitrary $L$ can be done very similarly. First we notice that in order to cancel the pole in the denominator of (8) at $u = 0$ we have to assume $P_1 P_2 = 0$ at $u = 0$. If we “split” this zero between all $P$’s by introducing a $\sqrt{u}$ factor into each of them, we also ensure a half-integer asymptotics of $P$’s. From (14) we see that $P_4 / \sqrt{u}$ should have no cut and behaves as $u^0$ at infinity, so it is simply $P_4 = A_4 \sqrt{u}$. On the other hand, $P_3 / \sqrt{u}$ should flip its sign when crossing the cut $[-2g, 2g]$ and thus $P_3 = A_3 \sqrt{u} / u^2 - 4g^2$. $P_2$ is given from (14) by the Hilbert transform of $\sinh(2\pi u)$:

$$\frac{-P_2}{CA_4 \sqrt{u}} = \frac{1}{2\pi} \left[ \frac{u^2 - 4g^2}{\sqrt{u^2 - 4g^2}} \frac{\sinh(2\pi u)}{4\pi i(u - u)} \right] = \frac{\pi}{\sinh(2\pi u)} \sum_{n=\pm 1}^\infty I_{2n+1}(4\pi g u^{2n+1}),$$

where $x(u)$ is defined by $x + \frac{1}{x} = \frac{u}{g}$ so we have to set $A_2 = -gCA_4 (4\pi g)$. Finally, the solution for $P_1$ is

$$P_1 = -\frac{A_1}{A_4} \sqrt{u^2 - 4g^2} P_2 + (A_1 + A_4 A_2 / A_4) \sqrt{u}.$$ 

Now, we introduce $\phi$ by requiring that $1 + Y_{11} \simeq -\phi^2 / 2$ for $u \rightarrow \infty$ and find the energy from $Y_{11} Y_{22} - 1 \simeq -2\Delta / u$ (note an extra two in this equation which is due to the open boundary conditions). We notice that to match these expansions we should first assume $A_1 A_4 = A_2 A_3$ as otherwise $Y_{11} Y_{22} - 1$ would grow linearly. Then the first condition gives $-\phi^2 / 2 = \frac{1}{2} A_1 A_4$ and from the second

$$\Delta = -\phi^2 g^2 \left(1 - \frac{I_3(4\pi g)}{I_1(4\pi g)}\right)$$

the same result as found from localization in [26, 27] or using TBA/FiNLIE approach in [28].

**DISCUSSION**

In this letter we formulated the $P_\mu$ system – a new way to describe the AdS/CFT spectrum. This system seems to be well suited to address various long-standing open problems, including a systematic study of strong coupling of short operators and the BFKL regime. We can also benefit from it for a systematic weak coupling expansion and the study of Wilson loops.

At the same time, the $P_\mu$ system provides a new conceptual insight into the AdS/CFT integrability. In particular, the present $P_\mu$-system, with $P_\mu / P_\mu$ corresponding to the $S^5$ eigenvalues of the quasiclassical monodromy matrix, is the perfect counterpart of the $Q\omega$-system to be described in [21]; the four fundamental fermionic $Q$-functions $Q_a$ have only one long cut ($-\infty, -2g] \cup [2g, \infty$) and their monodromies are expressed through a $4 \times 4$ matrix $\omega$ (periodic on the sheet with short cuts). We believe that $Q_a / Q_b$ correspond to the $AdS_5$ eigenvalues of $\Omega$.

These two systems are related by linear relations of the type $\mu_{ab} = Q_{ab} 1^{\omega_{ab}}$, which allowed us to explicit the Lorenz spin $S$ dependence of the coefficients in (10) and thus to close the $P_\mu$-system on itself [21]. In addition, the symmetry between these two systems would a priori allow to interchange the role of $P_\mu$ and $Q_a$. One interesting application of this is the possibility to construct the “physical T-hook” – where the $Y$- and $T$-systems have the same algebraic formulation as in the original mirror T-hook, but all cuts are short instead. At weak coupling, short cuts collapse and we expect the 1-loop physical T-functions to be the eigenvalues of transfer matrices of the $\mathfrak{psu}(2, 2|4)$ XXX spin chain [29]. We describe this construction and the full derivation of the $P_\mu$ system in our future work [21]. The exact physical T-functions seem to represent the eigenvalues of, yet to be constructed, T-operators of all-loop $N=4$ SYM spin chain.

Finally, let us note that the monodromy around a branch point corresponds to the crossing transformation and one can speculate that (3) is a crossing QQ-relation, related to the bubble $Y$-system of [30, 31].
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[33] With the present definition of $T_{a,s}$, (1) holds when $|\text{Im}(u)| < s-a$, and an analytic continuation is necessary to make (1) hold everywhere by imposing long cuts for $T_{a,s}$.
[34] More precisely, it is in general the square $P_2$ of these Q-functions which has a single cut. For operators with half-integer asymptotic behaviour, $P_a$ has an additional quadratic branch point at infinity. This branch point is absent from the physical quantities, as they are expressed through products and ratios of two $P_a$ (cf. (8)).