On Hofmann-Streicher universes

Steve Awodey

December 16, 2022

to the memory of Erik Palmgren

Abstract

We have another look at the construction by Hofmann and Streicher of a universe \((U, \mathbb{E})\) for the interpretation of Martin-Löf type theory in a presheaf category \([\mathbb{C}^{\text{op}}, \text{Set}]\). It turns out that \((U, \mathbb{E})\) can be described as the categorical nerve of the classifier \(\mathbb{S} \to \text{Set}^{\text{op}}\) for discrete fibrations in \(\text{Cat}\), where the nerve functor is right adjoint to the so-called “Grothendieck construction” taking a presheaf \(P : \mathbb{C}^{\text{op}} \to \text{Set}\) to its category of elements \(\int_{\mathbb{C}} P\).

Let \(\hat{\mathbb{C}} = [\mathbb{C}^{\text{op}}, \text{Set}]\) be the category of presheaves on the small category \(\mathbb{C}\).

1. The Hofmann-Streicher universe

In [HS97] the authors define a (type-theoretic) universe \((U, \mathbb{E})\) with \(U \in \hat{\mathbb{C}}\) and \(\mathbb{E} \in \int_{\mathbb{C}} \hat{U}\) as follows. For \(I \in \mathbb{C}\), set

\[
U(I) = \text{Cat}(\mathbb{C}/I^{\text{op}}, \text{Set}) \quad (1)
\]
\[
\mathbb{E}(I, A) = A(id_I) \quad (2)
\]

with an evident associated action on morphisms. A few comments are required:

1. In (1), we have taken the underlying set of objects of the category \(\mathbb{C}/I = [\mathbb{C}/I^{\text{op}}, \text{Set}]\) (in contrast to the specification in [HS97]).

2. In (2), and throughout, the authors steadfastly adopt a “categories with families” point of view in describing a morphism

\[
E \rightarrow U \quad (3)
\]
in $\hat{C}$ equivalently as an object in

$$\hat{C}/U \simeq \int_{\hat{C}} U,$$

that is, as a presheaf on the category of elements $\int_{\hat{C}} U$, where:

$$E(I) = \coprod_{A \in U(I)} El(I, A).$$

Thus the argument $(I, A) \in \int_{\hat{C}} U$ in (2) consists of an object $I \in C$ and an element $A \in U(I)$.

3. In order to account for size issues, the authors assume a Grothendieck universe $U$ in $\text{Set}$, the elements of which are called small. The category $C$ is then assumed to be small, as are the values of the presheaves, unless otherwise stated.

The presheaf $U$, which is not small, is regarded as the Grothendieck universe $U$ “lifted” from $\text{Set}$ to $[\hat{C}, \text{Set}]$. We will analyse the construction of $(U, El)$ from a slightly different perspective in order to arrive at its basic property as a classifier for small families in $\hat{C}$.

2. An unused adjunction

For any presheaf $X$ on $C$, recall that the category of elements is the comma category,

$$\int_{\hat{C}} X = \gamma_{C}/X,$$

where $\gamma_C : C \to [\hat{C}, \text{Set}]$ is the Yoneda embedding, which we may suppress and write simply $C/X$. While the category of elements $\int_{\hat{C}} X$ is used in the specification of the Hofmann-Streicher universe $(U, El)$ at the point (4), the authors seem to have missed a trick which would have simplified things:

**Proposition 1** ([Gro83],§28). The category of elements functor $\int_{\hat{C}} : \hat{C} \longrightarrow \text{Cat}$ has a right adjoint, which we denote

$$\nu_C : \text{Cat} \longrightarrow \hat{C}.$$

For a small category $\mathbb{A}$, we call the presheaf $\nu_C(\mathbb{A})$ the $C$-nerve of $\mathbb{A}$.

**Proof.** As suggested by the name, the adjunction $\int_{\hat{C}} \dashv \nu_C$ can be seen as the familiar “realization $\dashv$ nerve” construction with respect to the covariant
functor \( C/- : C \to \text{Cat} \), as indicated below.

\[
\begin{array}{c}
\tilde{C} \\
\nu_C \\
\downarrow \\
\int_C \\
\downarrow \\
\nu_C(C/-) \\
\end{array}
\]

In detail, for \( A \in \text{Cat} \) and \( c \in C \), let \( \nu_C(A)(c) \) be the Hom-set of functors,

\[
\nu_C(A)(c) = \text{Cat}(C/_{c} , A) ,
\]

with contravariant action on \( h : d \to c \) given by pre-composing a functor \( P : C/_{c} \to A \) with the post-composition functor

\[
C/h : C/d \to C/c .
\]

For the adjunction, observe that the slice category \( C/_{c} \) is the category of elements of the representable functor \( yc \),

\[
\int_C yc \cong C/_{c}.
\]

Thus for representables \( yc \), we have the required natural isomorphism

\[
\tilde{C}(yc, \nu_C(A)) \cong \nu_C(A)(c) = \text{Cat}(C/_{c} , A) \cong \text{Cat}(\int_C yc, A).
\]

For arbitrary presheaves \( X \), one uses the presentation of \( X \) as a colimit of representables over the index category \( \int_C X \), and the easy to prove fact that \( \int_C \) itself preserves colimits. Indeed, for any category \( D \), we have an isomorphism in \( \text{Cat} \),

\[
\lim_{d \in D} D/d \cong D .
\]

When \( C \) is fixed, as here, we may omit the subscript from the notation \( yc \) and \( \int_C \) and \( \nu_C \). The unit and counit maps of the adjunction \( \int - \nu \),

\[
\eta : X \longrightarrow \nu \int X ,
\]

\[
\epsilon : \int \nu A \longrightarrow A ,
\]

are as follows. At \( c \in C \), for \( x : yc \to X \), the functor \( (\eta_X)_c(x) : C/_{c} \to C/X \) is just composition with \( x \),

\[
(\eta_X)_c(x) = C/_{x} : C/_{c} \longrightarrow C/_{x} .
\]
For $A \in \text{Cat}$, the functor $\epsilon : \int \nu A \to A$ takes a pair $(c \in C, f : C/c \to A)$ to the object $f(1_c) \in A$,

$$\epsilon(c, f) = f(1_c).$$

**Lemma 2.** For any $f : Y \to X$, the naturality square below is a pullback.

$$
\begin{array}{ccc}
Y & \xrightarrow{\eta Y} & \nu\int Y \\
\downarrow f & & \downarrow \nu ff \\
X & \xrightarrow{\eta X} & \nu\int X.
\end{array}
$$

**Proof.** It suffices to prove this for the case $f : X \to 1$. Thus consider the square

$$
\begin{array}{ccc}
X & \xrightarrow{\eta X} & \nu\int X \\
\downarrow & & \downarrow \\
1 & \xrightarrow{m} & \nu\int 1.
\end{array}
$$

Evaluating at $c \in C$ and applying (6) then gives the following square in $\text{Set}$.

$$
\begin{array}{ccc}
Xc & \xrightarrow{C/c} & \text{Cat}(C/c, C/X) \\
\downarrow & & \downarrow \\
1c & \xrightarrow{\mathbb{C}/-} & \text{Cat}(C/c, C/1)
\end{array}
$$

The image of $* \in 1c$ along the bottom is the forgetful functor $U_c : C/c \to C$, and its fiber under the map on the right is therefore the set of functors $F : C/c \to C/X$ such that $U_X \circ F = U_c$, where $U_X : C/X \to C$ is also a forgetful functor. But any such $F$ is easily seen to be uniquely of the form $C/x$ for $x = F(1_c) : yc \to X$.

\[\square\]

**3. Classifying families**

For the terminal presheaf $1 \in \mathbb{C}$, we have an iso $\int 1 \cong C$. So for every $X \in \mathbb{C}$ there is a canonical projection $\int X \to C$, which is easily seen to be a discrete fibration. It follows that for any map $Y \to X$ of presheaves, the associated map $\int Y \to \int X$ is also a discrete fibration. Ignoring size issues for the moment, recall that discrete fibrations in $\text{Cat}$ are classified by the forgetful functor $\text{Set}^{\text{op}} \to \text{Set}^{\text{op}}$ from (the opposites of) the category of pointed sets.
to that of sets (cf. [Web07]). For every presheaf $X \in \tilde{C}$, we therefore have a pullback diagram in $\mathbf{Cat}$,

$$
\begin{array}{ccc}
\int X & \longrightarrow & \mathbf{Set}^{\mathbf{op}} \\
\downarrow & & \downarrow \\
\mathbb{C} & \longrightarrow & \mathbf{Set}^{\mathbf{op}}.
\end{array}
$$

(10)

Using $\int 1 \cong \mathbb{C}$ and transposing by the adjunction $\int \dashv \nu$ then gives a commutative square in $\tilde{\mathbb{C}}$,

$$
\begin{array}{ccc}
X & \longrightarrow & \nu\mathbf{Set}^{\mathbf{op}} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \nu\mathbf{Set}^{\mathbf{op}}.
\end{array}
$$

(11)

Lemma 3. The square (11) is a pullback in $\tilde{\mathbb{C}}$. More generally, for any map $Y \to X$ in $\tilde{\mathbb{C}}$, there is a pullback square

$$
\begin{array}{ccc}
Y & \longrightarrow & \nu\mathbf{Set}^{\mathbf{op}} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \nu\mathbf{Set}^{\mathbf{op}}.
\end{array}
$$

(12)

Proof. Apply the right adjoint $\nu$ to the pullback square (10) and paste the naturality square (7) from Lemma 2 on the left, to obtain the transposed square (12) as a pasting of two pullbacks. □

Let us write $\dot{V} \to V$ for the vertical map on the right in (12); that is, let

$$
\begin{align*}
\dot{V} &= \nu\mathbf{Set}^{\mathbf{op}} \\
V &= \nu\mathbf{Set}^{\mathbf{op}}.
\end{align*}
$$

(13)

We can then summarize our results so far as follows.

Proposition 4. The nerve $\dot{V} \to V$ of the classifier for discrete fibrations $\mathbf{Set}^{\mathbf{op}} \to \mathbf{Set}^{\mathbf{op}}$, as defined in (13), classifies natural transformations $Y \to X$ in $\tilde{\mathbb{C}}$, in the sense that there is always a pullback square,

$$
\begin{array}{ccc}
Y & \longrightarrow & \dot{V} \\
\downarrow & & \downarrow \\
X & \longrightarrow & V.
\end{array}
$$

(14)
The classifying map \( \tilde{Y} : X \to V \) is determined by the adjunction \( f \dashv \nu \) as the transpose of the classifying map of the discrete fibration \( \int Y \to \int X \).

The classifying map \( \tilde{Y} : X \to V \) of a given a natural transformation \( Y \to X \) is, of course, not in general unique. Nonetheless, we can make use of the construction of \( \tilde{V} \to V \) as the nerve of the discrete fibration classifier \( \hat{\text{Set}}^{\text{op}} \to \text{Set}^{\text{op}} \), for which classifying functors \( C \to \text{Set}^{\text{op}} \) are unique up to natural isomorphism, to infer the following proposition, which plays a role in [Shu15, GSS22] and elsewhere.

**Proposition 5** (Realignment). Given a monomorphism \( c : C \to X \) and a family \( Y \to X \), let \( y_c : C \to V \) classify the pullback \( c^*Y \to C \). Then there is a classifying map \( y : X \to V \) for \( Y \to X \) with \( y \circ c = y_c \).

\[
\begin{array}{ccc}
c^*Y & \to & \tilde{V} \\
\downarrow & & \downarrow \\
C & \xrightarrow{y_c} & V \\
\downarrow & \searrow & \downarrow \\
X & \xrightarrow{y} & V \\
\end{array}
\]

**Proof.** Transposing the realignment problem (15) for presheaves across the adjunction \( f \dashv \nu \) results in the following realignment problem for discrete fibrations.

\[
\begin{array}{ccc}
\int c^*Y & \to & \int \text{Set}^{\text{op}} \\
\downarrow & & \downarrow \\
\int C & \xrightarrow{\tilde{y}_c} & \int \text{Set}^{\text{op}} \\
\downarrow & \searrow & \downarrow \\
\int c & \xrightarrow{\tilde{y}} & \int \text{Set}^{\text{op}} \\
\end{array}
\]

The category of elements functor \( \int \) is easily seen to preserve pullbacks, hence monos; thus let us consider the general case of a functor \( C : C \to D \) which is monic in \( \text{Cat} \), a pullback of discrete fibrations as on the left below, and a
presheaf $E : \mathcal{C} \to \text{Set}^{\text{op}}$ with $\int E \cong E$ over $\mathcal{C}$.

\[
\begin{array}{c}
\text{E} \\
\downarrow \\
\text{Set}^{\text{op}} \\
\downarrow \\
\text{F} \\
\downarrow \\
\text{D} \\
\end{array}
\]

We seek $F : \mathcal{D} \to \text{Set}^{\text{op}}$ with $\int F \cong F$ over $\mathcal{D}$ and $F \circ \mathcal{C} = E$. Let $F_0 : \mathcal{D} \to \text{Set}^{\text{op}}$ with $\int F_0 \cong F$ over $\mathcal{D}$. Since $F_0 \circ \mathcal{C}$ and $E$ both classify $\mathcal{E}$, there is a natural iso $e : F_0 \circ \mathcal{C} \cong E$. Consider the following diagram

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow \\
\text{Set}^{\cong} \text{op} \\
\downarrow \\
\mathcal{D} \\
\end{array}
\]

where $\text{Set}^{\cong}$ is the category of isos in $\text{Set}$, with $p_1, p_2$ the (opposites of the) domain and codomain projections. There is a well-known weak factorization system on $\text{Cat}$ (part of the “canonical model structure”) with injective-on-objects functors on the left and isofibrations on the right. Thus there is a diagonal filler $f$ as indicated. The functor $F := p_2 f : \mathcal{D} \to \text{Set}^{\text{op}}$ is then the one we seek.

\[\square\]

Of course, as defined in (13), the classifier $\hat{\mathcal{V}} \to \mathcal{V}$ cannot be a map in $\hat{\mathcal{C}}$, for reasons of size; we now address this.

4. Small maps

Let $\alpha$ be a cardinal number, and call the sets that are strictly smaller than it $\alpha$-small. Let $\text{Set}_\alpha \hookrightarrow \text{Set}$ be the full subcategory of $\alpha$-small sets. Call a presheaf $X : \mathcal{C}^{\text{op}} \to \text{Set}$ $\alpha$-small if all of its values are $\alpha$-small sets, and thus if, and only if, it factors through $\text{Set}_\alpha \hookrightarrow \text{Set}$. Call a map $f : Y \to X$ of presheaves $\alpha$-small if all of the fibers $f_c^{-1}\{x\} \subseteq Yc$ are $\alpha$-small sets (for all
$c \in \mathcal{C}$ and $x \in Xc$). The latter condition is of course equivalent to saying that, in the pullback square over the element $x : yc \to X$,

\[
\begin{array}{ccc}
Y_z & \longrightarrow & Y \\
\downarrow^d & & \downarrow^f \\
yc & \longrightarrow & X,
\end{array}
\]

(19)

the presheaf $Y_z$ is $\alpha$-small.

Now let us restrict the specification (13) of $\hat{V} \to V$ to the $\alpha$-small sets:

\[
\begin{aligned}
\hat{V}_\alpha &= \nu \text{Set}_\alpha \\
V_\alpha &= \nu \text{Set}_\alpha.
\end{aligned}
\]

(20)

Then the evident forgetful map $\hat{V}_\alpha \to V_\alpha$ is a map in the category $\hat{\mathcal{C}}$ of presheaves, and it is in fact $\alpha$-small. Moreover, it has the following basic property, which is just a restriction of the basic property of $\hat{V} \to V$ stated in Proposition 4.

**Proposition 6.** The map $\hat{V}_\alpha \to V_\alpha$ classifies $\alpha$-small maps $f : Y \to X$ in $\hat{\mathcal{C}}$, in the sense that there is always a pullback square,

\[
\begin{array}{ccc}
Y & \longrightarrow & \hat{V}_\alpha \\
\downarrow^d & & \downarrow \\
X & \longrightarrow & V_\alpha.
\end{array}
\]

(21)

The classifying map $\check{Y} : X \to V_\alpha$ is determined by the adjunction $\int \dashv \nu$ as (the factorization of) the transpose of the classifying map of the discrete fibration $\int X \to \int Y$.

**Proof.** If $Y \to X$ is $\alpha$-small, its classifying map $\check{Y} : X \to V$ factors through $V_\alpha \hookrightarrow V$, as indicated below,

\[
\begin{array}{c}
\begin{array}{ccc}
Y & \longrightarrow & \nu \text{Set}_\alpha \hookrightarrow \nu \text{Set}_\alpha \\
\downarrow & & \downarrow \\
X & \longrightarrow & \nu \text{Set}_\alpha \hookrightarrow \nu \text{Set}_\alpha
\end{array}
\end{array}
\]

(22)
in virtue of the following adjoint transposition,

\[ \int Y \xrightarrow{\text{Set}_\alpha^{\text{op}}} \text{Set}^{\text{op}} \xleftarrow{\text{Set}^{\text{op}}} \int X \]

Note that the square on the right is evidently a pullback, and the one on the left therefore is, too, because the outer rectangle is the classifying pullback of the discrete fibration \( \int Y \to \int X \), as stated. Thus the left square in (22) is a pullback. \( \square \)

5. Examples

1. Let \( \alpha = \kappa \) a strongly inaccessible cardinal, so that \( \text{ob}(\text{Set}_\kappa) \) is a Grothendieck universe. Then the Hofmann-Streicher universe of (3) is recovered in the present setting as the \( \kappa \)-small map classifier

\[ E \cong \hat{V}_\kappa \to V_\kappa \cong U \]

in the sense of Proposition 6. Indeed, for \( c \in C \), we have

\[ V_\kappa c = \nu(\text{Set}_\kappa^{\text{op}})(c) = \text{Cat}(C/c, \text{Set}^{\text{op}}_\kappa) = \text{ob}(\vdash C/c) = Uc. \quad (24) \]

For \( \hat{V}_\kappa \) we then have,

\[ \hat{V}_\kappa c = \nu(\text{Set}_\kappa^{\text{op}})(c) = \text{Cat}(C/c, \text{Set}^{\text{op}}_\kappa) \cong \coprod_{A \in \hat{V}_\kappa c} \text{Cat}_{C/c}(C/c, A^* \text{Set}^{\text{op}}_\kappa) \quad (25) \]

where the \( A \)-summand in (25) is defined by taking sections of the pullback indicated below.

\[ A^* \text{Set}^{\text{op}}_\kappa \to \text{Set}^{\text{op}}_\kappa \]

But \( A^* \text{Set}^{\text{op}}_\kappa \cong \int_{C/c} A \) over \( C/c \), and sections of this discrete fibration in \( \text{Cat} \) correspond uniquely to natural maps \( 1 \to A \) in \( \vdash C/c \). Since 1 is
representable in \( \hat{\mathcal{C}}/c \) we can continue (25) by

\[
\mathcal{V}_c \cong \prod_{A \in \mathcal{V}_c} \mathbf{Cat}_{\mathcal{C}/c}(\mathcal{C}/c, A^{\mathbf{Set}^{\mathsf{op}}}) \\
\cong \prod_{A \in \mathcal{V}_c} \hat{\mathcal{C}}/c(1, A) \\
\cong \prod_{A \in \mathcal{V}_c} A(1_c) \\
= \prod_{A \in \mathcal{V}_c} \mathbf{El}(\langle c, A \rangle) \\
= Ec.
\]

2. By functoriality of the nerve \( \nu : \mathbf{Cat} \to \hat{\mathbf{C}} \), a sequence of Grothendieck universes

\[ \mathcal{U} \subseteq \mathcal{U}' \subseteq \ldots \]

in \( \mathbf{Set} \) gives rise to a (cumulative) sequence of type-theoretic universes

\[ \mathcal{V} \vdash \mathcal{V}' \vdash \ldots \]

in \( \hat{\mathbf{C}} \). More precisely, there is a sequence of cartesian squares,

\[
\begin{array}{c}
\hat{\mathbf{V}} & \longrightarrow & \mathcal{V}' & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{V} & \hookrightarrow & \mathcal{V}' & \hookrightarrow & \ldots
\end{array}
\]

in the image of \( \nu : \mathbf{Cat} \to \hat{\mathbf{C}} \), classifying small maps in \( \hat{\mathbf{C}} \) of increasing size, in the sense of Proposition 6.

3. Let \( \alpha = 2 \) so that \( 1 \to 2 \) is the subobject classifier of \( \mathbf{Set} \), and

\[ 1 = \mathbf{Set}^{\mathsf{op}}_2 \to \mathbf{Set}^{\mathsf{op}} = 2 \]

is then a classifier in \( \mathbf{Cat} \) for sieves, i.e. full subcategories \( \mathcal{S} \hookrightarrow \mathcal{A} \) closed under the domains of arrows \( a \to s \) for \( s \in \mathcal{S} \). The nerve \( \hat{\mathcal{V}}_2 \to \mathcal{V}_2 \) is then exactly the subobject classifier \( 1 \to \Omega \) of \( \hat{\mathbf{C}} \),

\[ 1 = \nu 1 = \hat{\mathcal{V}}_2 \hookrightarrow \mathcal{V}_2 = \nu 2 = \Omega. \]

4. Let \( i : 2 \hookrightarrow \mathbf{Set}_k \) and \( p : \mathbf{Set}_k \to 2 \) be the embedding-retraction pair with \( i : 2 \hookrightarrow \mathbf{Set}_k \) the inclusion of the full subcategory on the sets \( \{0,1\} \) and \( p : \mathbf{Set}_k \to 2 \) the retraction that takes \( 0 = \emptyset \) to itself,
and everything else (i.e. the non-empty sets) to 1 = \{\emptyset\}. There is a retraction (of arrows) in \textit{Cat},

\[
\begin{array}{ccc}
1 & \xrightarrow{\cdot} & \text{Set}_\kappa \\
\downarrow & & \downarrow \\
2 & \xleftarrow{\cdot} & \text{Set}_\kappa \\
\end{array}
\]

(28)

where the left square is a pullback.

By the functoriality of \((\cdot \text{-} \text{op})\) and \(\nu: \text{Cat} \to \widehat{\mathcal{C}}\) we then have a retract diagram in \(\widehat{\mathcal{C}}\), again with a pullback on the left,

\[
\begin{array}{ccc}
1 & \xrightarrow{\cdot} & \text{\(\nu\kappa\)} \\
\downarrow & & \downarrow \\
\Omega & \xleftarrow{\{-\}} & \text{\(\nu\kappa\)} \\
\end{array}
\]

(29)

where for any \(\phi: X \to \Omega\) the subobject \(\{\phi\} \to X\) is classified as a small map by the composite \(\{\phi\}: X \to \nu\kappa\), and for any small map \(A \to X\), the image \([A] \to X\) is classified as a subobject by the composite \([\alpha]: X \to \nu\kappa \to \Omega\), where \(\alpha: X \to \nu\kappa\) classifies \(A \to X\).

The idempotent composite

\[
\|\| = \{[-]\}: \nu\kappa \to \nu\kappa
\]

is the \textit{propositional truncation modality} in the natural model of type theory given by \(\hat{\nu}\kappa \to \nu\kappa\) (see [AGH21]).

**Acknowledgement**

Thanks to Mathieu Anel and Emily Riehl for discussions, and to Evan Cavallo, Ivan Di Liberti, and Taichi Uemura for help with the references. This material is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-21-1-0009.

**References**

[AGH21] S. Awodey, N. Gambino, and S. Hazratpour. Kripke-Joyal forcing for type theory and uniform fibrations, October 2021. Preprint available as \(https://arxiv.org/abs/2110.14576\).
[Gro83] Alexander Grothendieck. Pursuing stacks. 1983. Unpublished.

[GSS22] Daniel Gratzer, Michael Shulman, and Jonathan Sterling. Strict universes for grothendieck topoi. arXiv preprint arXiv:2202.12012, 2022.

[HS97] Martin Hofmann and Thomas Streicher. Lifting Grothendieck universes. Spring 1997. Unpublished.

[Shu15] Michael Shulman. The univalence axiom for elegant reedy presheaves. Homology, Homotopy and Applications, 17(2):81–106, 2015.

[Web07] Mark Weber. Yoneda structures from 2-toposes. Applied Categorical Structures, 15(3):259–323, 2007.