SYMMETRIZATION PROCEDURES
FOR THE ISOPERIMETRIC PROBLEM
IN SYMMETRIC SPACES OF NONCOMPACT TYPE

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Abstract. We establish a new symmetrization procedure for the isoperimetric problem in symmetric spaces of noncompact type. This symmetrization generalizes the well known Steiner symmetrization in euclidean space. In contrast to the classical construction the symmetrized domain is obtained by solving a nonlinear elliptic equation of mean curvature type. We conclude the paper discussing possible applications to the isoperimetric problem in symmetric spaces of noncompact type.

Introduction

In this article we consider the isoperimetric problem in symmetric spaces of noncompact type, i.e., the problem of determining the domains minimizing surface area among all regions with a given volume. As existence and partial regularity of isoperimetric solutions in these spaces are given by geometric measure theory [Mo, pp. 129], the goal here is to get some information about the shape of isoperimetric solutions in these spaces.

In the history of the isoperimetric problem symmetrization procedures have been a very important tool. J. Steiner (1838), H. A. Schwarz (1884), and E. Schmidt (1943) used symmetrization arguments to get insight into the behavior of isoperimetric solutions in $\mathbb{R}^n$, $\mathbb{H}^n$, and $\mathbb{S}^n$, finally proving the isoperimetric property of metric balls in constant curvature spaces [BZ].

Beginning in 1989 with the work of W.-T. Hsiang and W.-Y. Hsiang [HS] the isoperimetric problem has been investigated in spaces like $\mathbb{H}^n \times \mathbb{R}^m$, $\mathbb{H}^n \times \mathbb{S}^1$, $\mathbb{R}^n \times \mathbb{S}^1$, $\mathbb{H}^n \times \mathbb{S}^1$, or $\mathbb{S}^n \times \mathbb{R}$ by R. Pedrosa, M. Ritoré, and D. John [P, PRi, J]. In these manifolds the initial technical tool always is a symmetrization argument reducing the problem to the 2–dimensional quotient of the product space by the isotropy group.

In some 3–dimensional space forms, for example $\mathbb{R}P^3$, stability arguments have been applied successfully by M. Ritoré and A. Ros [RiRo, Ro].

Up to now the isoperimetric problem has been investigated only in such special manifolds. Techniques suitable for more general symmetric spaces are largely unknown.

Date: February 1, 2008.

2000 Mathematics Subject Classification. Primary 49Q10; Secondary 53C42, 35J60.

Key words and phrases. Isoperimetric problem, symmetrization procedures, symmetric spaces of noncompact type.

The author was supported by the DFG Schwerpunkt 1154 “Globale Differentialgeometrie”. Special thanks are due to Prof. Dr. Uwe Abresch for his support during the work on this thesis.
The main goal of this paper is to establish a symmetrization procedure for domains in symmetric spaces of noncompact type. So far, it is not possible to conclude uniqueness or convexity of isoperimetric solutions by applying this symmetrization procedure. Nevertheless, it provides some interesting insights into the qualitative behavior of isoperimetric solutions.

1. Main Results

One of the fundamental features of symmetric spaces is the existence of special 1–parameter groups $\tau_t$ of isometries called transvections. Our main idea is to use these 1–parameter groups in order to establish a symmetrization procedure.

**Definition 1 (Symmetrization).** Let $\hat{M}^n$ be a symmetric space of noncompact type, $\hat{\Omega} \subset \hat{M}^n$ a given domain, and $\tau$ a transvection. Symmetrization of $\hat{\Omega}$ with respect to $\tau$ is defined to be the following: Determine a domain $S(\hat{\Omega})$ minimizing surface area among all volume preserving deformations of $\hat{\Omega}$ obtained by moving the line segments $\tau_R(x) \cap \hat{\Omega}$, $x \in \hat{M}^n$, along the orbits of $\tau$, compare Figure 1.

This obviously is a generalization of the well known Steiner symmetrization, since the orbits of transvections in $\mathbb{R}^n$ are just parallel lines. In euclidean space existence, uniqueness, and regularity properties of the symmetrized domain $S(\hat{\Omega})$ are immediate consequences of Minkowski’s inequality. Establishing these properties for $S(\hat{\Omega})$ in general symmetric spaces of noncompact type is much more involved and one of the main issues of this paper.

We will mainly consider domains $\hat{\Omega} \subset \subset \hat{M}^n$ for which $\tau_R(x) \cap \hat{\Omega}$ consists of a connected line segment for any $x \in \hat{M}^n$. These domains admit a simple representation in terms of the orbit space $M^{n-1} = \hat{M}^n/\tau$, a section $\sigma : M^{n-1} \to \hat{M}^n$, and appropriate functions $u, h : \Omega \to \mathbb{R}$, $h \geq 0$, where $\Omega := \hat{\Omega}/\tau$ denotes the quotient domain:

$$\hat{\Omega} = \{ \tau_t(\sigma(x)) \mid u(x) - h(x) \leq t \leq u(x) + h(x), x \in \Omega \}.$$

With these notations the main theorem of the present article can be stated as follows.

**Theorem 1 (Symmetrization).** Let $\hat{M}^n = G/K$ be a symmetric space of noncompact type. Consider a regular domain $\hat{\Omega} \subset \subset \hat{M}^n$ and a transvection $\tau$ such that the following holds:

1. $\tau_R(x) \cap \hat{\Omega}$ is connected for every $x \in \hat{M}^n$.
2. $h : \Omega \to \mathbb{R}$, $h(x) := \mathcal{H}^1(\tau_R(\sigma(x)) \cap \hat{\Omega})$ is smooth on the subset $\Omega$ of the orbit space. $\mathcal{H}^1$ denotes 1–dimensional Hausdorff measure.

Then the symmetrization procedure of Definition 1 assigns to $\hat{\Omega}$ a unique symmetrized domain $S(\hat{\Omega}) \subset \hat{M}^n$ of equal volume but smaller (or equal) surface area. The boundary of $S(\hat{\Omega})$ is smooth in those points $x \in \partial S(\hat{\Omega})$ where $\tau_R(x) \cap \partial \hat{\Omega}$ consists of precisely two different points.
This theorem can easily be extended to a more general class of domains \( \widehat{\Omega} \subset \widehat{M}^n \) for which assumption (1) does not hold for every \( x \in \widehat{M}^n \). Compare Section 2.3 for further details.

Regularity of \( \partial S(\widehat{\Omega}) \) is only investigated for those points described in the theorem but may also hold for other points.

Remark 1.1. The word symmetrization is used since the construction resembles Steiner symmetrization. However, in contrast to the case of constant curvature spaces, \( S(\widehat{\Omega}) \) is not necessarily invariant under a larger class of isometries than \( \widehat{\Omega} \). Moreover, in the general case the hypersurface defined by the midpoints of the segments \( \tau_R(x) \cap S(\widehat{\Omega}) \) does depend on the height function \( h \). It is not a priori given, whereas for domains in constant curvature spaces it is just a hyperplane.

Once we have established this symmetrization procedure we can apply it to the isoperimetric problem in \( \widehat{M}^n \). In analogy to Steiner symmetrization in \( \mathbb{R}^n \) we could try to show convexity of isoperimetric solutions in \( \widehat{M}^n \). Unfortunately this does not work (up to now). The main difficulty is illustrated in Section 5.1 where, by direct construction, we show the following.

**Theorem 2.** Consider a transvection \( \tau \) in a symmetric space of noncompact type \( \widehat{M}^n \) which is not a space of constant sectional curvature. Then there exists a domain which is not convex but invariant under symmetrization with respect to \( \tau \). This domain looks like a helix winding up in the direction of \( \tau \).

This example provides a good starting point for further investigations. If one assumes that there exists a nonconvex isoperimetric solution in some symmetric space of noncompact type, then our example gives some indications how such a solution could be constructed.

However, the helix is obviously far from being an isoperimetric solution, just think about symmetrization with respect to other transvections. Therefore, convexity of isoperimetric solutions in symmetric spaces of noncompact type remains a natural conjecture. Our example shows the main difficulties we will have to deal with proving convexity of isoperimetric domains.
The present paper is organized as follows: in Section 2 we shortly review the fundamental properties of symmetric spaces and transvections and use them to describe the symmetrization procedure in more detail. It turns out that, given \( \hat{\Omega} \) as above, the area of \( \partial \hat{\Omega} \) is computed by the functional

\[
F(u) = \int_{\Omega} \sqrt{1 + \|w + dh + du\|^2} + \sqrt{1 + \|w - dh + du\|^2} \, d\text{vol},
\]

where \( w \) is a 1–form on the orbit space \( M^{n-1} \).

Symmetrizing \( \hat{\Omega} \) with respect to \( \tau \) can be reduced to finding a function \( u_0 \) minimizing \( F \). Using \( u_0 \) the symmetrized set \( S(\hat{\Omega}) \) is given by

\[
S(\hat{\Omega}) = \{ \tau_t(\sigma(x)) \mid u_0(x) - h(x) \leq t \leq u_0(x) + h(x), x \in \Omega \}.
\]

The main work now consists in establishing existence and regularity of a minimizer \( u_0 \). For this purpose, in Section 3, we investigate the analytic properties of the area functional in more detail. It turns out that for the corresponding Euler-Lagrange equation we only have an estimate (3.1) providing nonuniform ellipticity. Our existence and regularity results are based on this estimate and involve ideas due to Ladyzhenskaya and Ural’tseva [LU1] as well as Giaquinta, Modica, and Soucek [GMS].

The main issue of Section 4 is to establish an a priori gradient estimate for minimizers of \( F \). Finally, the work of Sections 2, 3, and 4 is summarized in Theorem 4.21 which immediately implies Theorem 1.

In Section 5 we prove Theorem 2. Furthermore, the 1–form \( w \) involved in the definition of \( F \) is calculated explicitly for the case of complex hyperbolic space. Resorting to the properties of \( w \) in more detail should be important for future investigations of the isoperimetric problem in these spaces.

2. Symmetrization in symmetric spaces

2.1. Symmetric spaces and transvections. Let \( (\hat{M}^n, \hat{g}) \) be a Riemannian manifold of dimension \( n \). \( (\hat{M}^n, \hat{g}) \) is called locally symmetric, if for every point \( x \in \hat{M}^n \) there exists a neighbourhood \( U_x \) of \( x \) and a geodesic isometry \( I_x : U_x \to U_x \) such that

\[
I_x(x) = x \quad \text{and} \quad dI_x|_x = -\text{id} : T_x U_x \to T_x U_x.
\]

(\( \hat{M}^n, \hat{g} \)) is called (globally) symmetric if \( U_x = \hat{M}^n \) for every \( x \in \hat{M}^n \).

It is well known that for each locally symmetric space \( \hat{N}^n \) there exists a simply connected symmetric space \( \hat{M}^n \) and a group \( \Gamma \) operating on \( \hat{M}^n \) discretely, without fixed points, and isometrically, such that \( \hat{N}^n = \hat{M}^n / \Gamma \).

Isoperimetric domains in locally symmetric spaces \( \hat{M}^n / \Gamma \) depend strongly on the group \( \Gamma \). Therefore the isoperimetric problem in these spaces is mainly unsolved even in the easiest case of \( \hat{M} = \mathbb{R}^3 \) and \( \Gamma \) one of the cristallographic groups [HPRRo]. In our discussion of the isoperimetric problem we restrict our attention to the following class of symmetric spaces.

**Definition 2.1.** Let \( \hat{M}^n \) be a simply connected symmetric space of nonpositive curvature. Then \( \hat{M}^n \) is said to be a symmetric space of noncompact type, if it is not the Riemannian product of a euclidean space and another manifold.
From now on in this work, $\hat{M}^n$ always denotes a symmetric space of noncompact type. The irreducible symmetric spaces of noncompact type have been classified by Cartan [He, Chapter X].

The symmetrization procedure we will develop in this paper is based on special 1–parameter groups of isometries called transvections.

**Definition 2.2.** Consider a normal geodesic $\gamma : \mathbb{R} \to \hat{M}^n$. Then the 1–parameter group of isometries $\tau_\gamma : \mathbb{R} \times \hat{M}^n \to \hat{M}^n$ defined by

$$
\tau_\gamma(t,x) := I_{\gamma(t)} \circ I_{\gamma(0)}(x)
$$

is called a transvection along $\gamma$. We also write $\tau(t,x) = \tau_t(x)$ for short. The Killing field corresponding to $\tau$ is denoted by $K$.

We collect some basic properties of transvections. First of all, for all $s, t$ we have $\tau_t(\gamma(s)) = \gamma(s + t)$ and $\tau_{t+s} = \tau_t \circ \tau_s$. Furthermore, $\tau$ is a differentiable map and $d\tau_t$ is parallel translation along $\gamma$. For every $x \in \hat{M}^n$ the map $t \mapsto \tau(x,t)$ is injective because $\hat{M}^n$ is a Hadamard manifold. This property will be important for our surface area calculations. It does not hold in symmetric spaces of compact type and is one of the main reasons for restricting our attention to symmetric spaces of noncompact type.

These properties imply that on the orbit space

$$
M^{n-1} := \hat{M}^n / \tau = \{ \tau_\mathbb{R}(x) \mid x \in \hat{M}^n \}
$$

there exists a unique structure of a Riemannian manifold such that the projection $\pi : \hat{M}^n \to M^{n-1}$, $\pi(x) = \tau_\mathbb{R}(x)$ is a Riemannian submersion.

Constructing our symmetrization procedure we will need the orbit space $M^{n-1}$ as well as a section $\sigma : M^{n-1} \to \hat{M}^n$. Such a section can be obtained, for example, as follows:

Corresponding to our geodesic $\gamma : \mathbb{R} \to \hat{M}^n$ consider the Busemann functions $\beta^+_\gamma : \hat{M}^n \to \mathbb{R}$ and $\beta^-_\gamma : \hat{M}^n \to \mathbb{R}$ defined by

$$
\beta^\pm_\gamma(x) := \lim_{t \to \infty} (t - \text{dist}(x, \gamma(\pm t))).
$$

Busemann functions in a symmetric space of noncompact type are known to be $C^\infty$–differentiable. The level sets of the Busemann functions are called horospheres. We now consider the function $\eta : \hat{M}^n \to \mathbb{R}$ defined by

$$
\eta = \eta_\gamma := \frac{1}{2}(\beta^+_\gamma - \beta^-_\gamma).
$$

As the isometry $\tau_\gamma$ transfers horospheres $(\beta^\pm_\gamma)^{-1}(\{b\})$ into corresponding horospheres $(\beta^\pm_\gamma)^{-1}(\{b \pm t\})$, the function $\eta$ obviously has the following properties:

1. $\eta$ is $C^\infty$–differentiable,
2. $\eta(\tau(t,x)) = \eta(x) + t$, and
3. $\langle \text{grad} \eta, K \rangle_{x} = d\eta_{x} : \frac{\partial}{\partial t} \tau(t,x)_{|t=0} = 1$.

Consequently 0 (as well as any other element of $\mathbb{R}$) is a regular value of $\eta$ and the orbits of $\tau$ intersect the level sets of $\eta$ transversally. Summarizing we can construct the desired section.
Lemma 2.3. Let $\pi : \tilde{M}^n \to M^{n-1} = \tilde{M}^n / \tau$ be the usual projection onto the orbit space. Then

$$\sigma : M^{n-1} \to \tilde{M}^n, \quad \sigma(\pi(x)) := \eta^{-1}(\{0\}) \cap \tau(\mathbb{R}, x)$$

is a section and

$$\Phi : \mathbb{R} \times M^{n-1} \to \tilde{M}^n, \quad \Phi(t, x) := \tau(t, \sigma(x))$$

is a diffeomorphism.

2.2. Surface area calculations. We consider a geodesic $\gamma : \mathbb{R} \to \tilde{M}^n$ and the corresponding 1–parameter group $\tau$ of transvections. Using the section $\sigma : M^{n-1} \to \tilde{M}^n$ constructed above, we can describe any hypersurface in $\tilde{M}^n$ intersecting the orbits of $\tau$ transversally by maps

$$\psi_u : \Omega \to \tilde{M}^n, \quad \psi_u(x) := \tau(u(x), \sigma(x)),$$

where $\Omega \subset M^{n-1}$ and $u : \Omega \to \mathbb{R}$ is an appropriate function. Using $u$ we can derive an easy formula computing the surface area of $\psi_u(\Omega) \subset (\tilde{M}^n, \tilde{g})$. For this issue we need the following notation.

For a vector field $X$ on the orbit space $M^{n-1}$ we denote by $\text{Hor}(X)$ the unique horizontal vector field on $\tilde{M}^n$ (with respect to the submersion $\pi$) such that $d\pi(\text{Hor} X) = X|_{\tau \gamma}$. $K$ is the Killing field corresponding to $\tau$.

Definition 2.4. The 1–forms $w_u$ and $w$ on the subset $\Omega \subset M^{n-1}$ of the orbit space are defined by requiring

$$d\psi_u : X = \text{Hor}(X)|_{\psi_u} + w_u(X) \cdot K|_{\psi_u},$$
$$d\sigma : X = \text{Hor}(X)|_{\sigma} + w(X) \cdot K|_{\sigma},$$

which is the splitting of $d\psi_u : X$ and $d\sigma : X$ into horizontal and vertical part. These 1–forms are obviously related by $w_u = w + du$.

We denote the volume form of the orbit space $(M^{n-1}, g)$ by $d\text{vol}_g$. The volume form of the hypersurface $\psi_u(\Omega) \subset \tilde{M}^n$ with respect to the metric induced by $\tilde{g}$ is denoted by $d\text{vol}_{\tilde{g}}$. Using this notation, the volume form of $(\Omega, \psi_u^* \tilde{g})$ is $\psi_u^* d\text{vol}_{\tilde{g}}$.

Lemma 2.5. Define $k : \Omega \to \mathbb{R}$, $k(x) := \|K|_{\psi_u(x)}\|_{\tilde{g}} = \|K|_{\sigma(x)}\|_{\tilde{g}}$. Then

$$\psi_u^* d\text{vol}_{\tilde{g}} = \sqrt{1 + k^2 \cdot \|w + du\|^2_{\tilde{g}}} d\text{vol}_g$$
$$= \sqrt{1 + k^2 \cdot \|W + \text{grad } u\|^2_{\tilde{g}}} d\text{vol}_g,$$

where $W$ is the vector field related to the 1–form $w$ by $w(X) = g(W, X)$ for all vector fields $X$ on $M^{n-1}$. For $\Omega \subset M^{n-1}$ and $u \in C^\infty(\Omega)$ the surface area of the set $\psi_u(\Omega) = \{\tau(u(x), \sigma(x)) \mid x \in \Omega\}$ is computed by

$$\text{area}(u) := \text{area}(\psi_u(\Omega)) = \int_\Omega \sqrt{1 + k^2 \cdot \|w + du\|^2_{\tilde{g}}} d\text{vol}_g$$
$$= \int_\Omega \sqrt{1 + k^2 \cdot \|W + \text{grad } u\|^2_{\tilde{g}}} d\text{vol}_g.$$
Proof. Writing \((\psi^* u \hat{g})(X, Y) = g(X, G \cdot Y)\) for an appropriate field of endomorphisms \(G\) we have \((\psi^* u \text{dvol}_{\hat{g}}) = \sqrt{\det G} \cdot \text{dvol}_g\). A short calculation shows that
\[
g(X, G \cdot Y) = g(X, Y) + w_u(X) \cdot w_u(Y) \cdot \hat{g}(K_{|\psi u}, K_{|\psi u}) \quad \text{and} \quad \det G = 1 + \|K_{|\psi u}\|_{\hat{g}}^2 \cdot \|w_u\|^2_{\hat{g}} + \|K_{|\psi u}\|_{\hat{g}}^2 \cdot \|w + du\|^2_{\hat{g}}.
\]

We establish some basic properties of the function \(k\) and the 1–form \(w\).

**Lemma 2.6.** For \(k : M^{n-1} \to \mathbb{R}, k(x) := \|K_{|\sigma(x)}\|_{\hat{g}}\) we have

1. \(k \in C^\infty(M^{n-1})\).
2. \(k\) is a convex function on \(M^{n-1}\) with \(k \geq 1\) and \(k(\pi(\gamma(t))) = 1\), where \(\gamma\) is the geodesic incorporated in the definition of \(\tau = \tau_\gamma\).
3. \(k\) is invariant under the isometries on the orbit space which are induced by those isometries of \(\hat{M}^n\) that transfer \(K_{|\gamma(0)}\) into \(\pm K_{|\gamma(0)}\).

Proof. (1) and (3) are clear. For (2) consider a geodesic \(c : \mathbb{R} \to M^{n-1}\) in the orbit space and a horizontal geodesic \(\hat{c}\) in \(\hat{M}^n\) with \(\pi \circ \hat{c} = c\). Then \(k(c(t)) = \|K_{|\sigma(c(t))}\| = \|K_{|\hat{c}(t)}\|\). Since \(\hat{K}\) is a Killing field, \(t \mapsto K_{|\hat{c}(t)}\) is a Jacobi field. Now \(\hat{M}\) has nonpositive curvature and therefore \(t \mapsto \|K_{|\hat{c}(t)}\|\) is a smooth, convex function. Hence \(k(x) \geq k(\pi(\gamma(t))) = 1\) for all \(x \in M^{n-1}\). \(\square\)

**Lemma 2.7.** \(W\) is a smooth vector field on \(M^{n-1}\). If \(\hat{\varphi} \in \text{Isom}(\hat{M}^n)\) is an isometry with \(d\hat{\varphi}(K_{|\gamma(0)}) = \pm K_{|\gamma(0)}\) and \(\varphi \in \text{Isom}(M^{n-1})\) the isometry induced by \(\hat{\varphi}\) on the orbit space \(M^{n-1}\), then \(d\varphi(W) = \pm W_{\varphi}\).

Proof. \(\hat{\varphi}\) maps \(\gamma\) to \(\gamma\). Therefore the Killing field \(\hat{K}\) is mapped to \(\pm K\). Furthermore, \(\hat{\varphi}\) leaves invariant the foliation of \(\hat{M}^n\) by the level sets of the function \(\beta_{\gamma+} - \beta_{\gamma-}\). Hence the unit normal field \(\nu\) on \(\sigma(M^{n-1})\) also remains invariant under \(\hat{\varphi}\) up to sign. \(\square\)

2.3. **Construction of the symmetrization procedure.** To introduce our generalized symmetrization procedure we concentrate on the following situation. Let \(\hat{\Omega} \subset \subset \hat{M}^n\) be a subset of the symmetric space of noncompact type and \(\tau = \tau_\gamma\) a transvection such that

1. \(\tau_{\hat{\Omega}}(x) \cap \hat{\Omega}\) is connected for every \(x \in \hat{M}^n\).
2. If \(\Omega := \pi(\hat{\Omega})\) is the projection of \(\hat{\Omega}\) to the orbit space then the function \(h : \Omega \to \mathbb{R}, h(x) := H^1(\tau(\mathbb{R}, \sigma(x)) \cap \hat{\Omega})\) is smooth. Here \(H^1\) denotes the 1–dimensional Hausdorff measure.
3. \(\partial \Omega\) is smooth.

These assumptions guarantee that \(\hat{\Omega}\) can be written as
\[
\hat{\Omega} = \{\tau(t, \sigma(x)) \mid u(x) + h(x) \leq t \leq u(x) + h(x), x \in \Omega\}, \tag{2.1}
\]
where \(u : \Omega \to \mathbb{R}\) is an appropriate smooth function.

According to Definition \(\square\) symmetrization of such an \(\hat{\Omega}\) with respect to \(\tau\) amounts to finding a symmetrized set \(S(\hat{\Omega})\) having least surface area among all deformations of \(\hat{\Omega}\) of the form
\[
\hat{\Omega}_v := \{\tau(t, \sigma(x)) \mid v(x) + h(x) \leq t \leq v(x) + h(x), x \in \Omega\},
\]
with \( v : \Omega \to \mathbb{R} \) any function (in a reasonable function space). Obviously all these domains \( \hat{\Omega}_v \) have the same volume since \( \tau \) is a 1–parameter group of isometries.

Therefore our surface area calculations imply that \( S(\hat{\Omega}) = \hat{\Omega}_{u_0} \) where \( u_0 : \Omega \to \mathbb{R} \) minimizes the surface area functional

\[
\mathcal{F}(v) := \text{area}(\partial \hat{\Omega}_v) = \int_{\hat{\Omega}} f(\text{grad } v) \, \text{dvol}_g
\]

with \( f : T \Omega \to \mathbb{R} \) defined by

\[
f(X) := \sqrt{1 + k^2 \|W_{|x} + \text{grad } h_{|x} + X\|_g^2} + \sqrt{1 + k^2 \|W_{|x} - \text{grad } h_{|x} + X\|_g^2}
\quad \text{for } X \in T_x \Omega.
\]

In other words, in order to symmetrize \( \hat{\Omega} \) we have to solve the variational problem \( \mathcal{F}(u) = \min \). We will start investigating the corresponding existence, uniqueness, and regularity questions in the next section. But before, we want to make some remarks concerning the assumptions (1) – (3).

**Remark 2.8.** Assumption (3), that is smoothness of \( \partial \hat{\Omega} \), can be made without loss of generality. This can be justified as follows: In geometric measure theory the isoperimetric problem is considered in the class of sets of finite perimeter \([\mathcal{H}]\). These sets can be seen as a special case of the more general notion of currents. The perimeter of a measurable set \( \hat{\Omega} \subset \hat{M} \) is defined by

\[
\text{Per}(\hat{\Omega}) := \inf(\lim \inf \mathcal{H}^{n-1}(\partial \hat{\Omega}_i)),
\]

where \( \mathcal{H}^{n-1} \) denotes the \((n - 1)\)–dimensional Hausdorff measure and the infimum is taken over all sequences of embedded \(n\)–dimensional manifolds \( \hat{\Omega}_i \) with smooth boundary \( \partial \hat{\Omega}_i \) such that the characteristic functions \( \chi_{\hat{\Omega}_i} \to \chi_{\hat{\Omega}} \) converge in \( L^1 \). \( \hat{\Omega} \subset \hat{M} \) is called a set of finite perimeter, if \( \text{Per}(\hat{\Omega}) < \infty \).

Thinking about isoperimetric solutions as approximated by smooth domains, it is (almost) sufficient to develop a symmetrization procedure for sets \( \hat{\Omega} \subset \hat{M} \) with smooth boundary.

Another justification for assumption (3) can be given by the regularity part of geometric measure theory: Isoperimetric solutions are smooth up to a singular set of codimension \( \geq 7 \).

**Remark 2.9.** The symmetrization procedure can be easily extended to a much larger class of subsets of \( \hat{M} \) than those described above: Just think about \( \Omega \) not as a subset of the orbit space but as an open set such that \( \Omega \) is a compact \((n - 1)\)–dimensional Riemannian manifold with smooth boundary, isometrically immersed into the orbit space \( M^{n-1} \). Then consider again domains \( \Omega \subset \hat{M} \) whose boundary is described by functions \( u - h, u + h \) on \( \Omega \) the same way as above. Taking this point of view we can also apply our symmetrization procedure to domains such as a thickened helix winding up in the direction of the transvection \( \tau_\gamma \). However, one should be aware that it is not possible to describe every smooth domain \( \Omega \subset \hat{M} \) like this, i.e., using an \( \Omega \) isometrically immersed into the orbit space \( M^{n-1} \). An immediate counterexample is a torus \( T \) in \( \hat{M} \) where \( T \cap \gamma(\mathbb{R}) \) has two components.
Remark 2.10. Another strategy to deal with the fact that an arbitrary domain $\hat{\Omega} \subset \hat{M}^n$ can intersect the orbits $\tau_\gamma(\mathbb{R}, \sigma(x))$ in more than one component is to introduce for every component functions $u_i, h_i : \Omega_i \rightarrow \mathbb{R}$ such that $u_i - h_i, u_i + h_i$ describe the corresponding part of the boundary $\partial \hat{\Omega}$. Considering this account it seems appropriate to minimize the functional $F$ on subsets $U \subset \Omega_i$ with respect to Dirichlet boundary conditions on $\partial U$. For the (only minor) differences compared to the case of free boundary values treated in this paper we refer the reader to [GMS].

Remark 2.11. It should also be possible to generalize the symmetrization construction to domains $\hat{\Omega} = \hat{\Omega}_u$ where $u$ is not smooth but in a more general class of functions admitting jumps, such as $\text{BV}(\Omega)$, the class of functions of bounded variation. This would be an interesting issue. In this paper our focus is on the smooth case because we are interested in uniqueness and regularity properties of the symmetrized set $S(\hat{\Omega})$.

3. Analytic properties of the variational problem

3.1. The area functional. Resuming the above observations we now investigate existence, regularity, and uniqueness of solutions for the minimizing problem $F(u) = \min$.

First of all, we have to think about the appropriate function spaces concerning this problem. For this purpose we shortly review the basic properties and notions of Sobolev spaces on Riemannian manifolds:

Given the open subset $\Omega \subset \subset M^n$ we denote by $L^p(\Omega)$ the space of measurable functions $f$ on $\Omega$ for which $\int_\Omega |f|^p \, d\text{vol}_g < \infty$, where $1 \leq p < \infty$. For a vector field $X$ on $\Omega$ we define the norms $\|X\|_{L^p} := (\int_\Omega \|X\|^p \, d\text{vol}_g)^{1/p}$, $1 \leq p < \infty$. As usual, $L^p(\Omega)$ is the space of measurable vector fields $X$ with $\|X\|_{L^p} < \infty$.

Definition 3.1. Given a function $f \in L^1_{\text{loc}}(\Omega)$ we say that $Y \in L^1_{\text{loc}}(\Omega)$ is a weak derivative of $f$ if

$$\int_\Omega \langle X, Y \rangle \, d\text{vol}_g = - \int_\Omega f \cdot \text{div} X \, d\text{vol}_g$$

for all $C^1$–vector fields $X$ with compact support in $\Omega$. If such an $Y$ exists, it is unique and we write $\text{grad} f := Y$.

Definition 3.2. The Sobolev space $W^{1,p}(\Omega)$ consists of all those functions $f \in L^p(\Omega)$ for which the weak derivative exists and $\text{grad} f \in L^p(\Omega)$. For $f \in W^{1,p}(\Omega)$ we define its norm to be $\|f\|_{W^{1,p}} := (\|f\|_{L^p}^p + \|\text{grad} f\|_{L^p}^p)^{1/p}$.

As in the euclidean case, the usual Sobolev inequalities and embedding theorems also hold on manifolds [A]. Isoperimetric inequalities are closely related to Sobolev inequalities, more precisely to the optimal constant in the Sobolev inequality, compare [A, p. 39]. This has been of fundamental importance in the history of the Yamabe problem [A, p. 153].

Concerning the appropriate function spaces for our area functional $F$ we consider the following lemma.
Lemma 3.3. Let $\langle \cdot, \cdot \rangle$ be an arbitrary scalar product on $\mathbb{R}^m$, $\| \cdot \|$ the corresponding norm. Then for every $a, b \in \mathbb{R}^m$:

$$
\|a\| \leq \sqrt{1 + \|a\|^2} \leq \sqrt{2} \cdot \sqrt{1 + \|a\|^2 + \|b\|^2} \\
\leq \sqrt{1 + \|a + b\|^2 + \sqrt{1 + \|a - b\|^2}} \\
\leq 2\sqrt{1 + \|a\|^2 + \|b\|^2} \leq 2(1 + \|a\| + \|b\|).
$$

Using these inequalities and the Poincaré inequality we get the following.

Lemma 3.4. The area functional $F$ is a priori defined on the function space $W^{1,1}(\Omega)$. We can furthermore restrict our attention to those functions $u \in W^{1,1}(\Omega)$ with $\int_\Omega u \, d\operatorname{vol}_g = 0$, because $F(u) = F(u + \text{const})$.

The Banach space $W^{1,1}(\Omega)$ is not weakly sequentially compact. But $F$ can be naturally extended to the space of functions of bounded variation $\operatorname{BV}(\Omega)$, the bidual of $W^{1,1}(\Omega)$. Sets of functions uniformly bounded in the $\operatorname{BV}$–norm are relatively compact in $L^1(\Omega)$. It is this property that makes $\operatorname{BV}(\Omega)$ the suitable space for investigating variational problems corresponding to (area-like) functionals with linear growth. For more information on $\operatorname{BV}$–functions see [Gi], for example.

Extending $F$ to $\operatorname{BV}(\Omega)$, minimizing sequences have weak limits in $\operatorname{BV}(\Omega)$ as usual, compare [GMS] for this approach. The drawback is that $\operatorname{BV}$–functions can and do in general have jumps. We have to use other arguments to exclude this behaviour. The trick is to prove convergence in $W^{1,\infty}$ of a limiting sequence. For this we do not necessarily need the space $\operatorname{BV}(\Omega)$.

We now collect some of the main features of the area functional $F$ and its integrand $f$:

Lemma 3.5. 

1. $f : T_x \Omega \to \mathbb{R}$ is Lipschitz continuous for every $x \in \Omega$ with Lipschitz constant $2k(x)$.
2. $f : T_x \Omega \to \mathbb{R}$ is strictly convex for every $x \in \Omega$, i.e., for all $t \in (0, 1)$ and $X, Y \in T_x \Omega$, $X \neq Y$: $f(tX + (1 - t)Y) < tf(X) + (1 - t)f(Y)$.
3. A minimum of $F$ is unique (up to constants).
4. The functional $F$ is sequentially lower semicontinuous with respect to weak convergence in $W^{1,p}_{\text{loc}}(\Omega)$, $1 \leq p < \infty$.

Proof. (1) and (2) are easy computations. (3) is a direct consequence of the convexity property (2). For (4) see [Gi] p. 22, Theorem 2.5. \qed

3.2. The Euler-Lagrange equation. The next step is to introduce the Euler-Lagrange equation corresponding to our variational problem $F(u) = \min$ and to study its main properties.

Definition 3.6. Let $u : \Omega \to \mathbb{R}$ be fixed. Corresponding to the integrand $f : T\Omega \to \mathbb{R}$ of the area functional $F$ we define the 1–form $\tilde{a}$ and the symmetric 2–form $\tilde{\tilde{a}}$ on $\Omega$ by

$$
\tilde{a}(X) := \left. \frac{d}{dt} f(\text{grad } u + tX) \right|_{t=0} \\
\tilde{\tilde{a}}(X, Y) := \left. \frac{\partial^2}{\partial t \partial s} f(\text{grad } u + sX + tY) \right|_{s=t=0}.
$$

We also write $\tilde{a}(u; X)$ and $\tilde{\tilde{a}}(u; X, Y)$ to emphasize the dependence on $u$. 
Lemma 3.7. The Euler-Lagrange equation corresponding to the variational problem \( F(u) = \min \) is given by
\[
\text{div} \, \tilde{a} = \text{div} \, \tilde{a}(u, \cdot) = 0.
\]

Proof. Suppose \( u \) is a smooth minimum of \( F \). For \( \varphi \in C_0^\infty(\Omega) \) we get
\[
0 = \frac{d}{dt} F(u + t\varphi)|_{t=0} = \int_{\Omega} \tilde{a}(u; \text{grad} \, \varphi) \, d\text{vol}_g = -\int_{\Omega} \varphi \, \text{div}(\tilde{a}) \, d\text{vol}_g.
\]
As this holds for arbitrary \( \varphi \in C_0^\infty(\Omega) \) the claim follows. □

Unfortunately, this Euler-Lagrange equation is not uniformly elliptic. Nevertheless, we have estimates for \( \tilde{a} \) which allow existence and regularity theorems for our variational problem.

Lemma 3.8. For simplyfing notation from now on
\[
V := k \cdot (W + \text{grad} \, u),
\]
\[
V^\pm := k \cdot (W \pm \text{grad} \, h + \text{grad} \, u).
\]
Furthermore we will write \( \langle \cdot, \cdot \rangle := g(\cdot, \cdot) \) for the Riemannian metric on the orbit space \( M^{n-1} \) and \( ||\cdot|| = ||\cdot||_g \) for the corresponding norm. Using these abbreviations, the following holds:
\[
\tilde{a}(X) = \frac{k\langle V^+, X \rangle}{\sqrt{1 + ||V^+||^2}} + \frac{k\langle V^-, X \rangle}{\sqrt{1 + ||V^-||^2}}
\]
and
\[
\tilde{a}(X, Y) = \frac{k^2\langle X, Y \rangle}{\sqrt{1 + ||V^+||^2}} - \frac{k^2\langle V^+, X \rangle \langle V^+, Y \rangle}{(1 + ||V^+||^2)^{\frac{3}{2}}}
\]
\[
+ \frac{k^2\langle X, Y \rangle}{\sqrt{1 + ||V^-||^2}} - \frac{k^2\langle V^-, X \rangle \langle V^-, Y \rangle}{(1 + ||V^-||^2)^{\frac{3}{2}}}.
\]

Proof. Straightforward computation. □

A unit normal field on the surface \( \psi_u(\Omega) = \{ \tau(u(x), \sigma(x)) \mid x \in \Omega \} \subset \widehat{M}^n \) is given by
\[
\nu := \frac{-k \text{Hor}(V)|_{\psi_u} + K|_{\psi_u}}{k \sqrt{1 + ||V||^2}}.
\]
Analogously
\[
\nu^\pm := \frac{-k \text{Hor}(V^\pm)|_{\psi_u} + K|_{\psi_u}}{k \sqrt{1 + ||V^\pm||^2}}
\]
defines normal fields on \( \psi_{u+h}(\Omega) \) and \( \psi_{u-h}(\Omega) \), the boundary of the set \( \widehat{\Omega} \subset \widehat{M}^n \) we want to symmetrize, compare [21].

Definition 3.9. We define the “projections” \( P : T\Omega \to T(\psi_u(\Omega)) \subset \widehat{T}M^n \) and \( P^\pm : T\Omega \to T(\psi_{u\pm h}(\Omega)) \subset TM^n \) by
\[
P(X) := \text{Hor}(X)|_{\psi_u} - \langle \text{Hor}(X)|_{\psi_u}, \nu|_{\psi_u} \rangle \hat{g} \cdot \nu|_{\psi_u},
\]
\[
P^\pm(X) := \text{Hor}(X)|_{\psi_{u\pm h}} - \langle \text{Hor}(X)|_{\psi_{u\pm h}}, \nu^\pm|_{\psi_{u\pm h}} \rangle \hat{g} \cdot \nu^\pm|_{\psi_{u\pm h}}.
\]
Using Pythagoras theorem we get
\[ \|P(X)\|_g^2 = \|X\|^2 - \frac{(V \cdot X)^2}{1 + \|V\|^2} \quad \text{and} \quad \|P^\pm(X)\|^2 = \|X\|^2 - \frac{(\pm V \cdot X)^2}{1 + \|V\|^2}. \]
Consequently
\[
\tilde{a}(u; X, X) = k^2 \frac{\|P^+(X)\|^2}{\sqrt{1 + \|V^+\|^2}} + k^2 \frac{\|P^-(X)\|^2}{\sqrt{1 + \|V^-\|^2}}.
\]

**Lemma 3.10.** For \( \tilde{a} \) we have the following estimate:
\[
\mu_1 \cdot \frac{\|P(X)\|^2_g}{\sqrt{1 + \|\text{grad} \, u\|^2}} \leq \tilde{a}(u; X, X) \leq \mu_2 \cdot \frac{\|P(X)\|^2_g}{\sqrt{1 + \|\text{grad} \, u\|^2}} \tag{3.1}
\]
The constants \( \mu_1, \mu_2 \) depend on the functions \( k \) and \( h \). More precisely:
\[
\mu_1 = \frac{k}{2 \sqrt{1 + k^2 \|W\|^2 (1 + k^2 \|\text{grad} \, h\|^2)}},
\]
\[
\mu_2 = 16k^2 \sqrt{1 + k^2 \|W\|^2 (1 + k^2 \|\text{grad} \, h\|^2)}.
\]

**Proof.** This is a straightforward computation: It is easy to see
\[
1 + \|V^\pm\|^2 \leq 4k^2 (1 + k^2 \|W\|^2) (1 + k^2 \|\text{grad} \, h\|^2) (1 + \|\text{grad} \, u\|^2), \quad \text{and} \quad 1 + \|\text{grad} \, u\|^2 \leq 4 (1 + k^2 \|W\|^2) (1 + k^2 \|\text{grad} \, h\|^2) (1 + \|V^\pm\|^2).
\]
A short calculation then yields
\[
\tilde{a}(X, X) \geq \frac{k^2 \left( \|P^+(X)\|^2_g + \|P^-(X)\|^2_g \right)}{2k \sqrt{(1 + k^2 \|W\|^2)(1 + k^2 \|\text{grad} \, h\|^2)(1 + \|\text{grad} \, u\|^2)}} \geq \mu_1 \cdot \frac{\|P(X)\|^2_g}{\sqrt{1 + \|\text{grad} \, u\|^2}}.
\]
For the second inequality one computes
\[
\tilde{a}(X, X) \leq \frac{2k^2 \sqrt{(1 + k^2 \|W\|^2)(1 + k^2 \|\text{grad} \, h\|^2)}}{\sqrt{1 + \|\text{grad} \, u\|^2}} \left( \|P^+(X)\|^2_g + \|P^-(X)\|^2_g \right) \leq \mu_2 \cdot \frac{\|P(X)\|^2_g}{\sqrt{1 + \|\text{grad} \, u\|^2}}.
\]

Summarizing, instead of uniform ellipticity we only have inequality (3.1). A priori gradient estimates for \( C^2 \)-solutions of partial differential equations which fulfill such an inequality have already been obtained by Ladyzhenskaya and Ural’tseva [LU1]. Giaquinta, Modica, and Soucek then used these a priori estimates in order to obtain existence and regularity results for Dirichlet boundary value problems corresponding to functionals with linear growth [GMS]. We will apply these ideas to our minimizing problem \( \mathcal{F}(u) = \min. \)
3.3. **Approximating minimizing problems.** One of the main properties of BV–functions is that they may have jumps. In our situation we would not like a minimum of the area functional $F$ to have jumps, because that would mean our symmetrization procedure could tear sets $\hat{\Omega} \subset \hat{M}^n$ apart. In order to remedy this problem we need gradient estimates for minimizers of $F$. It turns out that this can be done without extending $F$ to BV($\Omega$), we can simply use Sobolev spaces. For this purpose we will consider the approximating functionals $F_\varepsilon$ defined by

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(\text{grad } u) \, d\text{vol}_g := \int_\Omega \left( f(\text{grad } u) + \varepsilon \|\text{grad } u\|^2 \right) \, d\text{vol}_g$$

with $\varepsilon > 0$ and

1. $u \in W^{1,2}(\Omega) \subset W^{1,1}(\Omega)$, because $\text{vol}(\Omega) < \infty$,
2. $\int_\Omega u \, d\text{vol}_g = 0$ as a normalization, because $F_\varepsilon(u) = F_\varepsilon(u + \text{const})$.

**Definition 3.11.** Let $u : \Omega \to \mathbb{R}$ be fixed. Corresponding to the integrand $f_\varepsilon : T\Omega \to \mathbb{R}$ of the functional $F_\varepsilon$ we define the 1–form $\tilde{a}_\varepsilon$ and the symmetric 2–form $\tilde{\tilde{a}}_\varepsilon$ on $\Omega$ by

$$\tilde{a}_\varepsilon(X) := \left. \frac{d}{dt} f_\varepsilon(\text{grad } u + tX) \right|_{t=0}$$

$$\tilde{\tilde{a}}_\varepsilon(X, Y) := \left. \frac{\partial^2}{\partial t \partial s} f_\varepsilon(\text{grad } u + sX + tY) \right|_{s=0, t=0}.$$ 

We also write $\tilde{a}_\varepsilon(u; X)$ and $\tilde{\tilde{a}}_\varepsilon(u; X, Y)$ to emphasize the dependence on $u$.

$\tilde{a}_\varepsilon$ and $\tilde{\tilde{a}}_\varepsilon$ can easily be computed as

$$\tilde{a}_\varepsilon(u; X) = a(u; X) + 2\varepsilon \langle \text{grad } u, X \rangle$$

$$\tilde{\tilde{a}}_\varepsilon(u; X, Y) = a(u; X, Y) + 2\varepsilon \langle X, Y \rangle.$$ 

Furthermore, the functional $F_\varepsilon$ corresponds to the uniformly elliptic Euler-Lagrange equation

$$\text{div} \tilde{a}_\varepsilon = \text{div} \tilde{\tilde{a}}_\varepsilon(u; \cdot) = 0. \quad (3.2)$$

Therefore the standard theory of elliptic partial differential equations yields

**Lemma 3.12.** For every $\varepsilon > 0$ the variational problem $F_\varepsilon(u) = \min$ has a unique solution $u_\varepsilon$ such that

$$u_\varepsilon \in W^{1,2}(\Omega) \cap C^\infty(\Omega) \text{ and } \int_\Omega u_\varepsilon \, d\text{vol}_g = 0.$$ 

For any $\varphi \in W^{1,2}(\Omega)$, with $\int_\Omega \varphi \, d\text{vol}_g = 0$ we have

$$F(u_\varepsilon) + \varepsilon \int_\Omega \|\text{grad } u_\varepsilon\|^2 \, d\text{vol}_g \leq F_\varepsilon(\varphi) \leq F_1(\varphi) = \text{const < } \infty.$$ 

Therefore we have $\varepsilon \int_\Omega \|\text{grad } u_\varepsilon\|^2 \, d\text{vol}_g \leq \text{const < } \infty$ for $0 < \varepsilon \leq 1$ and by Lemma 3.13 also $\int_\Omega \|\text{grad } u_\varepsilon\| \, d\text{vol}_g \leq F(u_\varepsilon) \leq \text{const < } \infty$ for $0 < \varepsilon \leq 1$. Consequently, applying Poincaré’s inequality yields for all $0 < \varepsilon \leq 1$

$$\varepsilon \int_\Omega |u_\varepsilon|^2 \, d\text{vol}_g \leq \text{const < } \infty \quad \text{and} \quad \int_\Omega |u_\varepsilon| \, d\text{vol}_g \leq \text{const < } \infty. \quad (3.3)$$
Suppose now that for every \( U \subset\subset \Omega \) and \( 0 < \varepsilon \leq 1 \) we have estimates
\[
\sup_U |u_\varepsilon| \leq C_U < \infty, \quad \text{and} \quad \sup_U \|\text{grad } u_\varepsilon\| \leq C_U < \infty,
\] (3.4)
where \( C_U \) is a constant depending on \( U \) but independent of \( \varepsilon \). Then the existence of a locally uniformly convergent subsequence \( u_{\varepsilon_i} \to u_0 \) follows from the Arzela-Ascoli theorem. \( u_0 \) is a locally Lipschitz continuous function.

Since \( W^{1,2}(U) \) is weakly sequentially compact for \( U \subset\subset \Omega \), we can assume \( u_{\varepsilon_i} \to u_0 \) weakly in \( W^{1,2}(\Omega) \). Weakly lower semicontinuity of \( \mathcal{F} \) in \( W^{1,2}_{\text{loc}}(\Omega) \) implies \( \mathcal{F}(u_0) \leq \liminf_{i \to \infty} \mathcal{F}(u_{\varepsilon_i}) \). Hence \( \mathcal{F}_{\varepsilon_i}(u_{\varepsilon_i}) \leq \mathcal{F}_{\varepsilon_i}(\varphi) \) implies
\[
\mathcal{F}(u_0) \leq \mathcal{F}(\varphi) \quad \text{for all } \varphi \in W^{1,2}(\Omega).
\] (3.5)

We consider now the minimizing problem
\[
\mathcal{F}(u) \to \min, \quad u \in W^{1,1}(\Omega), \quad \int_\Omega u \, d\text{vol}_g = 0.
\] (3.6)

\( \partial \Omega \) is smooth and therefore any \( u \in W^{1,1}(\Omega) \) can be approximated in the \( W^{1,1}(\Omega) \)-norm by a sequence \( \varphi_j \in C^{\infty}(\overline{\Omega}) \subset W^{1,2}(\Omega) \). Since by Lemma 3.13 \( f \) is Lipschitz continuous we have
\[
|\mathcal{F}(u) - \mathcal{F}(\varphi_j)| \leq \int_\Omega 2k \|\text{grad } u - \text{grad } \varphi_j\| \, d\text{vol}_g \to 0
\]
for \( j \to \infty \). In other words \( \mathcal{F}(u) = \lim_{j \to \infty} \mathcal{F}(\varphi_j) \) for \( \varphi_j \to u \) in \( W^{1,1}(\Omega) \). Henceforth we know that for the minimizing problem (3.6) there exists a minimizing sequence \( \varphi_j \in C^{\infty}(\overline{\Omega}) \subset W^{1,2}(\Omega) \). Inserting this sequence into inequality (3.5) yields \( \mathcal{F}(u_0) \leq \mathcal{F}(u) \) for all \( u \in W^{1,1}(\Omega) \).

Now the locally uniform convergence of \( u_{\varepsilon_i} \to u_0 \) and \( \int_\Omega u_{\varepsilon_i} \, d\text{vol}_g = 0 \) imply \( \int_\Omega u_0 \, d\text{vol}_g = 0 \). Furthermore Lemma 3.13 yields
\[
\int_\Omega \|\text{grad } u_0\| \, d\text{vol}_g \leq \mathcal{F}(u_0) \leq \text{const} < \infty.
\]
By Poincaré’s inequality \( \int_\Omega |u_0| \leq \text{const} < \infty \). Summarizing we have

**Proposition 3.13.** Suppose estimates (3.4) hold for any \( U \subset\subset \Omega \) and \( 0 < \varepsilon \leq 1 \). Then the minimizing problem \( \mathcal{F}(u) \to \min, u \in W^{1,1}(\Omega), \int_\Omega u \, d\text{vol}_g = 0 \) has a locally Lipschitz continuous solution
\[
u_0 \in W^{1,1}(\Omega) \cap C^{0,1}_{\text{loc}}(\Omega).
\]
As the integrand of \( \mathcal{F} \) is strictly convex and independent of the value of \( u \), this is even the unique solution of the minimizing problem.

**Remark 3.14.** Applying standard regularity theory for elliptic partial differential equations of second order we get
\[
\nu_0 \in C^{\infty}(\Omega)
\]
For a short overview of regularity theory see [Gi] Appendix C.

**Remark 3.15.** What remains to be done is to show that the estimates (3.5) hold for any \( U \subset\subset \Omega \) and \( 0 < \varepsilon \leq 1 \). Proving \( \sup_U |u_\varepsilon| \leq C_U < \infty \) can be accomplished using a hair cutting argument, see [Gi] Thm. 14.10, p. 167] or [U] Section 3.5. As this is quite standard, we will omit the proof. It turns
out that the constant $C_U$ in this inequality only depends on $\text{dist}(\partial U, \Omega)$, $\int_{\Omega}|u_\varepsilon|\,d\text{vol}_g$ and $\varepsilon \int_{\Omega}|u_\varepsilon|^2\,d\text{vol}_g$. By (3.3) we already know that $\int_{\Omega}|u_\varepsilon|\,d\text{vol}_g$ and $\varepsilon \int_{\Omega}|u_\varepsilon|^2\,d\text{vol}_g$ can easily be estimated by constants independent of $\varepsilon$.

The difficult part is to show the estimate $\sup_U \|\text{grad} u_\varepsilon\| \leq C_U < \infty$. This is the objective of Section 4.

### 3.4. Alternative symmetrization procedures.

The initial idea for our symmetrization procedure was to deform a given set $\hat{\Omega} \subset \hat{M}^n$ along the integral lines of the Killing field $K$ corresponding to a transvection $\tau$, compare Section 2.3. A question that naturally arises in this context is the following:

Besides Killing fields, which other vector fields $X$ could be used to establish symmetrization procedures?

As we want the symmetrization procedure to be volume preserving, an immediate consequence is to restrict attention to vector fields $X$ that are divergence free. Considering an arbitrary vector field of this kind, the resulting symmetrization procedure will typically have some major (analytical) disadvantages. To illustrate this just take the vector field $\text{grad} \beta_\gamma$, where $\beta_\gamma$ is a Busemann function. Rescaling, we easily obtain a divergence free vector field $X = (\varphi \circ \beta_\gamma) \cdot \text{grad} \beta_\gamma$, where $\varphi \in C^\infty(\mathbb{R})$ is an appropriate function. In this setting the area functional that corresponds to the area functional in Lemma 2.5 takes the form

$$\text{area}(u) = \int_{\Omega} \sqrt{u^2 + \frac{1}{m^2} \sum_{i=1}^{n-1} \frac{d(uX_i)^2}{2^\lambda_i m}} \,d\text{vol},$$

where $\Omega$ is an open subset of a horosphere and the $X_i$ are an orthonormal frame of eigenvectors corresponding to the eigenvalues $\lambda_i \geq 0$ of the second fundamental form for the horosphere.

The first problem here is that the area functional depends on the values of $u$ directly. This causes severe difficulties already for the existence problem for weak minimizers. Another problem emerges from the fact that the integrand is not convex in $(u, du)$, making this area functional more or less useless for our purposes.

### 4. The gradient estimate

The core of this section is to establish the estimate

$$\sup_U \|\text{grad} u_\varepsilon\| \leq C_U < \infty, \quad U \subset \subset \Omega, \quad 0 < \varepsilon \leq 1,$$

for the gradient of the minimizer $u_\varepsilon$ of $\mathcal{F}_\varepsilon$. For this purpose we will introduce a differential equation for $\text{grad} u_\varepsilon$, which will subsequently be used to estimate this gradient.

#### 4.1. Differential equation for the gradient.

A differential equation for $\text{grad} u_\varepsilon$ is obtained by differentiating the Euler-Lagrange equation for the minimizer $u_\varepsilon$ of $\mathcal{F}_\varepsilon$ and exchanging the order of differentiation.

**Definition 4.1.** From now on, we will use the abbreviation

$$p_\varepsilon := \|\text{grad} u_\varepsilon\|^2.$$
Observe that we have the following identity
\[
\text{grad } p_\varepsilon = 2 \nabla_{\text{grad } u_\varepsilon} \text{grad } u_\varepsilon = 2 \text{Hess } u_\varepsilon (\text{grad } u_\varepsilon).
\] (4.1)

**Lemma 4.2.** Let \( X_1, \ldots, X_{n−1} \) be an arbitrary orthonormal frame on \( \Omega \). For \( X \in T\Omega \) and the 1-form \( \tilde{a}_\varepsilon \) (or any other 1-form instead) we have
\[
d_X(\text{div } \tilde{a}_\varepsilon) = \text{div}(\nabla_X \tilde{a}_\varepsilon) − (\nabla \tilde{a}_\varepsilon)(\nabla_X X, X_j) − \tilde{a}_\varepsilon \cdot \text{Ric}(X).
\]

**Proof.** Here and in the following computations we use the convention to sum over repeated indices.
\[
d_X(\text{div } \tilde{a}_\varepsilon) = \nabla_X ((\nabla \tilde{a}_\varepsilon)(X_j, X_j)) = (\nabla^2_{X,X_j} \tilde{a}_\varepsilon)(X_j)
= (\nabla^2_{X_j,X} \tilde{a}_\varepsilon)(X_j) − \tilde{a}_\varepsilon(R(X, X_j)X_j)
= \text{div}(\nabla_X \tilde{a}_\varepsilon) − (\nabla \tilde{a}_\varepsilon)(\nabla_X X, X_j) − \tilde{a}_\varepsilon \cdot \text{Ric}(X).
\]

\( \square \)

From now on in this chapter, we will always consider \( \tilde{a}_\varepsilon \) and \( \tilde{a}_\varepsilon \) corresponding to the unique minimizer \( u_\varepsilon \) of \( \mathcal{F}_\varepsilon \), that is \( \tilde{a}_\varepsilon = \tilde{a}_\varepsilon(u_\varepsilon; \cdot, \cdot) \), and \( \tilde{a}_\varepsilon = \tilde{a}_\varepsilon(u_\varepsilon; \cdot) \).

**Definition 4.3.** For \( \varepsilon > 0 \) the 2–form \( \overline{b}_\varepsilon \) on \( \Omega \) is defined by
\[
\overline{b}_\varepsilon(X,Y) := \nabla_X \tilde{a}_\varepsilon \cdot Y − \tilde{a}_\varepsilon(\nabla_X \text{grad } u_\varepsilon, Y).
\]

**Lemma 4.4.** Let \( X_1, \ldots, X_{n−1} \) be an orthonormal frame on \( \Omega \). Then the following differential equation holds for \( \text{grad } u_\varepsilon \):
\[
0 = \frac{1}{2} \text{div } (\tilde{a}_\varepsilon(\text{grad } p_\varepsilon, \cdot)) − \tilde{a}_\varepsilon(\nabla_X \text{grad } u_\varepsilon, \nabla_{X_j} \text{grad } u_\varepsilon) + B
\] (4.2)

where
\[
B := \text{div } (\overline{b}_\varepsilon(\text{grad } u_\varepsilon, \cdot)) − \overline{b}_\varepsilon(\nabla_X \text{grad } u_\varepsilon, X_j) − \tilde{a}_\varepsilon \cdot \text{Ric}(\text{grad } u_\varepsilon).
\] (4.3)

**Proof.** Differentiating the Euler-Lagrange equation \( \text{div } \tilde{a}_\varepsilon = 0 \) and applying Lemma 4.2 with \( X = \text{grad } u_\varepsilon \) yields
\[
0 = d_{\text{grad } u_\varepsilon}(\text{div } \tilde{a}_\varepsilon)
= \text{div } (\nabla \tilde{a}_\varepsilon(\nabla_{\text{grad } u_\varepsilon} \text{grad } u_\varepsilon, \cdot)) + \overline{b}_\varepsilon(\text{grad } u_\varepsilon, \cdot)
= (\nabla \tilde{a}_\varepsilon)(\nabla_{X_j} \text{grad } u_\varepsilon, X_j) − \tilde{a}_\varepsilon \cdot \text{Ric}(\text{grad } u_\varepsilon).
\]

Now
\[
(\nabla \tilde{a}_\varepsilon)(\nabla_{X_j} \text{grad } u_\varepsilon, X_j)
= \tilde{a}_\varepsilon(\nabla_{X_j} \text{grad } u_\varepsilon, \nabla_{X_j} \text{grad } u_\varepsilon) + \overline{b}_\varepsilon(\nabla_{X_j} \text{grad } u_\varepsilon, X_j).
\]

Combining these identities with 4.2 gives the differential equation. \( \square \)

**Lemma 4.5.** For the quantity \( B \) defined in (4.3) we have the estimate
\[
B \leq |B| \leq \mu_3 \left( \sum_{i=1}^{n−1} \|P(\nabla_{X_i} \text{grad } u_\varepsilon)\| \right) + \mu_4 \sqrt{1 + p_\varepsilon}
\] (4.4)

with \( \mu_3 = \mu_3(x) \), \( \mu_4 = \mu_4(x) \). \( X_1, \ldots, X_{n−1} \) denotes an arbitrary orthonormal frame on \( \Omega \).

**Proof.** This is a long but straightforward calculation. \( \square \)
4.2. Preliminary estimates. For $U \subset \subset \Omega$ we want to estimate $\sup_{U} \| \text{grad } u_{\varepsilon} \|$. Therefore we define

**Definition 4.6.** For $p_{\varepsilon} = \| \text{grad } u_{\varepsilon} \|^2$

$$ q_{\varepsilon} := \log(1 + p_{\varepsilon}). $$

Observe that we have the identities

$$ \text{grad } q_{\varepsilon} = \frac{\text{grad } p_{\varepsilon}}{1 + p_{\varepsilon}} \quad \text{and} \quad P(\text{grad } q_{\varepsilon}) = \frac{P(\text{grad } p_{\varepsilon})}{1 + p_{\varepsilon}}. $$

**Definition 4.7.** We define the sets

$$ \Omega_{\lambda} := \{ x \in \Omega \mid q_{\varepsilon}(x) > \lambda \} $$

$$ \Omega_{\lambda, \rho} := \Omega_{\lambda} \cap B_{\rho}(x_0), $$

where $x_0 \in \Omega$ is an arbitrary but fixed point and $\rho \leq R_0$ with $R_0$ such that $B_{R_0}(x_0) \subset \subset \Omega$. Furthermore

$$ S_{\lambda} := \{(x, u_{\varepsilon}(x)) \in \Omega \times \mathbb{R} \mid x \in \Omega_{\lambda}\}, $$

$$ S_{\lambda, \rho} := (\Omega_{\lambda, \rho} \times \mathbb{R}) \cap S_{\lambda}. $$

The idea for estimating $\sup \| \text{grad } u_{\varepsilon} \|$ works as follows: Consider

$$ \beta(\lambda, \rho) := \int_{S_{\lambda, \rho}} (q_{\varepsilon} - \lambda)^2 d\mathcal{H}_{n-1} + \varepsilon \int_{\Omega_{\lambda, \rho}} (1 + p_{\varepsilon})(q_{\varepsilon} - \lambda)^2 d\text{vol}_g $$

$$ = \int_{\Omega_{\lambda, \rho}} (q_{\varepsilon} - \lambda)^2 \cdot \sqrt{1 + p_{\varepsilon}} d\text{vol}_g + \varepsilon \int_{\Omega_{\lambda, \rho}} (1 + p_{\varepsilon})(q_{\varepsilon} - \lambda)^2 d\text{vol}_g $$

where $\mathcal{H}_{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure corresponding naturally to $S_{\lambda, \rho} \subset \Omega \times \mathbb{R}$. Using the differential equation (4.2) we will derive some estimates for the terms involved in the definition of $\beta$. These will be applied to show that there exist $0 < \rho_0, \lambda_0 < \infty$ such that $\beta(\lambda_0, \rho_0) = 0$. This is just equivalent to

$$ \text{ess sup}_{x \in B_{\rho_0}(x_0)} q_{\varepsilon}(x) \leq \lambda_0 < \infty, $$

which is nothing but the desired gradient estimate.

We will now establish the basic estimates involved into these computations.

**Lemma 4.8.** For $p_{\varepsilon} = \| \text{grad } u_{\varepsilon} \|^2$ we have

$$ \| \text{grad } p_{\varepsilon} \|^2 \leq 4 \| \text{grad } u_{\varepsilon} \|^2 \cdot \sum_{i=1}^{n-1} \| \nabla X_i \text{ grad } u_{\varepsilon} \|^2, $$

where $X_1, \ldots, X_{n-1}$ is an arbitrary orthonormal frame on $\Omega$.

**Proof.** This is just an application of the Cauchy-Schwarz inequality. \( \square \)

**Lemma 4.9.** Let $X_1, \ldots, X_{n-1}$ be an orthonormal frame of $\Omega$, $P$ the projection as in Definition 3.9. Then

$$ \| P(\text{grad } p_{\varepsilon}) \|^2 \leq 4 \| \text{grad } u_{\varepsilon} \|^2 \sum_{i=1}^{n-1} \| P(\nabla X_i \text{ grad } u_{\varepsilon}) \|^2. \quad (4.5) $$
Proof. It is easy to show that the expressions in (4.5) are independent of the choice of the orthonormal frame. Therefore we can choose an orthonormal frame \( \{X_i\} \) such that \( X_1 = \frac{\operatorname{grad} u_\varepsilon}{\|\operatorname{grad} u_\varepsilon\|} \). We abbreviate \( V_\varepsilon := k \cdot (W + \operatorname{grad} u_\varepsilon) \). The Cauchy-Schwarz inequality now implies
\[
\left( 4p_\varepsilon \langle \nabla X_1, \operatorname{grad} u_\varepsilon, V_\varepsilon \rangle^2 - \langle \operatorname{grad} p_\varepsilon, V_\varepsilon \rangle^2 \right) + 4p_\varepsilon \sum_{i=2}^{n-1} \langle \nabla X_i, \operatorname{grad} u_\varepsilon, V_\varepsilon \rangle^2 \leq (1 + \|V_\varepsilon\|^2) \left( 4p_\varepsilon \|\nabla X_1, \operatorname{grad} u_\varepsilon\|^2 - \|\operatorname{grad} p_\varepsilon\|^2 + 4p_\varepsilon \sum_{i=2}^{n-1} \|\nabla X_i, \operatorname{grad} u_\varepsilon\|^2 \right).
\]
An easy computation yields the claim. \( \square \)

Lemma 4.10. Let \( \zeta \in C_0^\infty(\Omega), \lambda \geq 0 \). Then
\[
\int_{S_\lambda} \|P(\operatorname{grad} q_\varepsilon)\|^2 \zeta^2 \, d\mathcal{H}_{n-1} + \varepsilon \int_{\Omega_\lambda} (1 + p_\varepsilon) \|\operatorname{grad} q_\varepsilon\|^2 \zeta^2 \, d\operatorname{vol}_g \leq C \cdot \left\{ \int_{S_\lambda} (q_\varepsilon - \lambda)^2 \|P(\operatorname{grad} \zeta)\|^2 + (q_\varepsilon - \lambda)^2 \zeta^2 \, d\mathcal{H}_{n-1} \right. \\
+ \left. \varepsilon \int_{\Omega_\lambda} (1 + p_\varepsilon)(q_\varepsilon - \lambda)^2 \|\operatorname{grad} \zeta\|^2 \, d\operatorname{vol}_g \right\}.
\]
\( C \) depends on \( \inf_{\operatorname{supp} \zeta} \mu_1, \sup_{\operatorname{supp} \zeta} \mu_2, \sup_{\operatorname{supp} \zeta} \mu_3, \) and \( \sup_{\operatorname{supp} \zeta} \mu_4 \).

Proof. Choose the test function
\[
\varphi(x) := \zeta^2(x) \cdot \max\{q_\varepsilon(x) - \lambda, 0\},
\]
with \( \zeta \in C_0^\infty(\Omega) \). Multiplying the differential equation (4.2) with \( \varphi \), integrating over \( \Omega \) and applying the divergence theorem yields
\[
\int_{\Omega_\lambda} \frac{1}{2} \tilde{a}_\varepsilon(\operatorname{grad} p_\varepsilon, \operatorname{grad} q_\varepsilon) \cdot \zeta^2 \\
+ \tilde{a}_\varepsilon(\nabla X_1, \operatorname{grad} u_\varepsilon, \nabla X_1, \operatorname{grad} u_\varepsilon) \cdot \zeta^2 \cdot (q_\varepsilon - \lambda) \, d\operatorname{vol}_g \\
- \int_{\Omega_\lambda} \frac{1}{2} \tilde{a}_\varepsilon(\operatorname{grad} p_\varepsilon, \operatorname{grad} \zeta) \cdot \zeta \cdot (q_\varepsilon - \lambda) + B \cdot \zeta^2 \cdot (q_\varepsilon - \lambda) \, d\operatorname{vol}_g
\]
We now estimate the expressions (1) – (4) separately. For (1) we apply (3.1) to obtain
\[
\frac{1}{2} \tilde{a}_\varepsilon(\operatorname{grad} p_\varepsilon, \operatorname{grad} q_\varepsilon) \cdot \zeta^2 \\
= \frac{1}{2} \tilde{a}(\operatorname{grad} q_\varepsilon, \operatorname{grad} q_\varepsilon) + 2\varepsilon \langle \operatorname{grad} q_\varepsilon, \operatorname{grad} q_\varepsilon \rangle (1 + p_\varepsilon) \zeta^2 \\
\geq \frac{1}{2} \mu_1 \|P(\operatorname{grad} q_\varepsilon)\|^2 \sqrt{1 + p_\varepsilon} \cdot \zeta^2 + \varepsilon \|\operatorname{grad} q_\varepsilon\|^2(1 + p_\varepsilon) \zeta^2.
\]
For (2) we use (3.1) again to compute
\[
\tilde{a}_\varepsilon(\nabla X_1, \operatorname{grad} u_\varepsilon, \nabla X_1, \operatorname{grad} u_\varepsilon) \geq \mu_1 \sum_i \frac{\|P(\nabla X_i, \operatorname{grad} u_\varepsilon)\|^2}{\sqrt{1 + p_\varepsilon}}.
\]
As $f|_{\Omega}$ is strictly convex for every $x \in \Omega$, we know that $\tilde{a}$ is positive definite. Therefore the Cauchy-Schwarz inequality and applied to (3.1) gives

$$-\tilde{a}_\varepsilon(\nabla p_\varepsilon, \nabla \zeta) \leq \mu_2 \|P(\nabla p_\varepsilon)\| \|P(\nabla \zeta)\| \sqrt{1 + p_\varepsilon} + 2\varepsilon \|\nabla p_\varepsilon\| \|\nabla \zeta\|.$$ 

(4.) can obviously be estimated by (4.4).

We define the constants

$$C_1 := \inf_{\text{supp } \zeta} \mu_1 \quad \text{and} \quad C_i := \sup_{\text{supp } \zeta} \mu_i, \quad i = 2, 3, 4.$$ 

Now we insert the estimates for (1.) – (4.) into (4.7) to calculate

$$\frac{C_1}{2} \int_{S_\lambda} \|P(\nabla q_\varepsilon)\|^2 \zeta^2 dH_{n-1}$$

$$+ \int_{S_\lambda} \sum_i \|P(\nabla X_i \nabla u_\varepsilon)\|^2 \zeta^2 (q_\varepsilon - \lambda) dH_{n-1}$$

$$+ \varepsilon \int_{\Omega} \|\nabla q_\varepsilon\| (1 + p_\varepsilon) \zeta^2 d\text{vol}_g$$

$$\leq \frac{C_2}{2} \int_{S_\lambda} \|P(\nabla p_\varepsilon)\| \|P(\nabla \zeta)\| \zeta (q_\varepsilon - \lambda) dH_{n-1}$$

$$+ \int_{S_\lambda} \sum_i \|P(\nabla X_i \nabla u_\varepsilon)\| (q_\varepsilon - \lambda) \zeta^2 dH_{n-1}$$

$$+ \int_{S_\lambda} (q_\varepsilon - \lambda) \zeta^2 dH_{n-1}$$

$$+ 2\varepsilon \int_{\Omega} \|\nabla p_\varepsilon\| \|\nabla \zeta\| \zeta (q_\varepsilon - \lambda) d\text{vol}_g.$$

Using Hölder’s inequality and the Cauchy inequality $ab \leq \varepsilon_l a^2 + \frac{1}{4\varepsilon_l} b^2$, $l = 1, 2, 3$, we continue

$$\leq C_2 \varepsilon_1 \int_{S_\lambda} \|P(\nabla q_\varepsilon)\|^2 \zeta^2 dH_{n-1}$$

$$+ \frac{C_2}{4\varepsilon_1} \int_{S_\lambda} \|P(\nabla \zeta)\|^2 (q_\varepsilon - \lambda)^2 dH_{n-1}$$

$$+ C_3 \varepsilon_2 \int_{S_\lambda} \sum_i \|P(\nabla X_i \nabla u_\varepsilon)\|^2 (q_\varepsilon - \lambda) \zeta^2 dH_{n-1}$$

$$+ \left( \frac{C_3}{4\varepsilon_2} + C_4 \right) \int_{S_\lambda} (q_\varepsilon - \lambda) \zeta^2 dH_{n-1}$$

$$+ 2\varepsilon_3 \int_{\Omega} \|\nabla q_\varepsilon\|^2 \zeta^2 (1 + p_\varepsilon) d\text{vol}_g$$

$$+ \frac{\varepsilon}{2\varepsilon_3} \int_{\Omega} \|\nabla \zeta\|^2 (q_\varepsilon - \lambda)^2 (1 + p_\varepsilon) d\text{vol}_g.$$ 

Set $C_2 \varepsilon_1 = \frac{C_4}{4}$, $C_3 \varepsilon_2 = C_1$ and $\varepsilon_3 = \frac{1}{4}$. Performing a short calculation and choosing $C$ appropriately, inequality follows. $\square$
Lemma 4.11. Let $\zeta \in C_0^\infty(\Omega)$, $\lambda \geq 0$. Then

$$
\varepsilon \int_{\Omega} \sum_{i=1}^{n-1} \|
abla X_i \text{grad} u_\varepsilon\|^2 (q_\varepsilon - \lambda)^2 \zeta^2 d\text{vol}_g \\
\leq C \cdot \left\{ \varepsilon \int_{\Omega} (1 + p_\varepsilon)(q_\varepsilon - \lambda)^2 \|	ext{grad} \zeta\|^2 d\text{vol}_g + \int_{S_\lambda} (q_\varepsilon - \lambda)^2 (\|	ext{P(grad} \zeta)\|^2 + \zeta^2) dH_{n-1} \right\}.
$$

(4.8)

$C$ depends on $\inf_{\text{supp} \zeta} \mu_1$, $\sup_{\text{supp} \zeta} \mu_2$, $\sup_{\text{supp} \zeta} \mu_3$, and $\sup_{\text{supp} \zeta} \mu_4$.

Proof. Similar to the proof of Lemma 4.10 we now consider the test function $\varphi(x) := \zeta^2 \cdot \max\{q_\varepsilon(x) - \lambda, 0\}^2$.

Multiplying the differential equation (4.2) with $\varphi$, integrating over $\Omega$ and applying the divergence theorem yields

$$
\int_{\Omega} \begin{aligned}
\bar{a}_\varepsilon(\text{grad} p_\varepsilon, \text{grad} q_\varepsilon) \cdot (q_\varepsilon - \lambda) \cdot \zeta^2 \\
+ \bar{a}_\varepsilon(\nabla X_i \text{grad} u_\varepsilon, \nabla X_i \text{grad} u_\varepsilon) \cdot \zeta^2 \cdot (q_\varepsilon - \lambda)^2 \\
= \int_{\Omega} -\bar{a}_\varepsilon(\text{grad} p_\varepsilon, \text{grad} \zeta) \cdot (q_\varepsilon - \lambda)^2 + B \cdot \zeta^2 \cdot (q_\varepsilon - \lambda)^2 d\text{vol}_g.
\end{aligned}
$$

(4.9)

Again we examine the terms (1.) – (4.) separately. The first term is simply estimated by

$$
\bar{a}_\varepsilon(\text{grad} p_\varepsilon, \text{grad} q_\varepsilon) \cdot (q_\varepsilon - \lambda) \cdot \zeta^2 \geq 0 \quad \text{on } \Omega_\lambda.
$$

For (2.) we get, using (3.1),

$$
\bar{a}_\varepsilon(\nabla X_i \text{grad} u_\varepsilon, \nabla X_i \text{grad} u_\varepsilon) \\
\geq 2\mu_1 \frac{1}{2} \sum_i \|P(\nabla X_i \text{grad} u_\varepsilon)\|^2 \sqrt{1 + p_\varepsilon} + 2\varepsilon \sum_i \|
abla X_i \text{grad} u_\varepsilon\|^2
$$

The last two terms are estimated independently: We use Lemma 4.8 to obtain

$$
\sum_i \|P(\nabla X_i \text{grad} u_\varepsilon)\|^2 \sqrt{1 + p_\varepsilon} \geq \frac{1}{4} \|P(\text{grad} q_\varepsilon)\|^2 \sqrt{1 + p_\varepsilon}
$$

and by Lemma 4.8

$$
\sum_i \|
abla X_i \text{grad} u_\varepsilon\|^2 \geq \frac{1}{4} \|\text{grad} q_\varepsilon\|^2 (1 + p_\varepsilon).
$$
Summarizing, (2.) can be estimated as
\[
\tilde{a}_\varepsilon(\nabla X_\varepsilon, \nabla X_\varepsilon) \geq \frac{\mu_1}{2} \cdot \sum_i \frac{\|P(\nabla X_\varepsilon, \nabla u_\varepsilon)||^2}{1 + p_\varepsilon} \sqrt{1 + p_\varepsilon} + \frac{\mu_1}{8} \|P(\nabla q_\varepsilon)\|^2 \sqrt{1 + p_\varepsilon} + \varepsilon \sum_i \|\nabla X_\varepsilon \cdot \nabla u_\varepsilon\|^2 + \frac{\varepsilon}{4} \|\nabla q_\varepsilon\|^2 (1 + p_\varepsilon).
\]

For (3.) inequality 3.11 yields
\[
-\tilde{a}_\varepsilon(\nabla p_\varepsilon, \nabla \zeta) \leq \frac{\mu_2}{4} \frac{\|P(\nabla p_\varepsilon)\| \|P(\nabla \zeta)\|}{\sqrt{1 + p_\varepsilon}} + 2\varepsilon \|\nabla p_\varepsilon\| \|\nabla \zeta\|.
\]

(4.) can again be estimated by 4.3.

These estimates can now be inserted into (4.9). Performing then the same steps as in the proof of Lemma 4.10, that is, applying Hölder’s inequality and Cauchy’s inequality with \(\varepsilon_1\), we finally get the desired inequality (4.8).

**Lemma 4.12.** Let \(u_\varepsilon \in C^2(\overline{\Omega})\) be a solution of the differential equation (3.2) and \(\Omega' \subset \Omega\) such that even the convex hull of \(\Omega'\) is contained in \(\Omega\). Then for any \(\varphi \in C^\infty(\overline{\Omega'})\) with \(\varphi|_{\partial \Omega'} = 0\)
\[
\int_{\partial \Omega'} \varphi^2 \, dH_{n-1} \leq C \cdot \int_{\partial \Omega'} \|P(\nabla \varphi)\|^2 \, dH_{n-1}.
\]

Here \(\partial \Omega' = \{(x, u_\varepsilon(x)) \in \Omega' \times \mathbb{R}\}\) and \(C\) depends on \(\text{osc}_{\Omega'} u_\varepsilon\), as well as on \(\text{osc}_{\Omega'} u_\varepsilon\), as well as on constants involving the values of \(k\), \(\|W\|\), \(\|\nabla h\|\) on \(\Omega\).

**Proof.** The proof is based on isoperimetric inequalities and can be accomplished in analogy to [LU1 Lemma 2, p. 697]. See also [GMS Lemma 3.8]. □

### 4.3. The final estimates.
We have now established all the inequalities necessary to derive the final estimate for the function
\[
\beta(\lambda, \rho) := \int_{S_{\lambda, \rho}} (q_\varepsilon - \lambda)^2 \, dH_{n-1} + \varepsilon \int_{\Omega_{\lambda, \rho}} (1 + p_\varepsilon)(q_\varepsilon - \lambda)^2 \, dvol_g,
\]
which can then be used to show that there exist \(0 < \lambda_0, \rho_0 < \infty\) such that \(\beta(\lambda_0, \rho_0) = 0\). We already mentioned that this is equivalent to the desired bound for \(\sup_{\Omega'} \|\nabla u_\varepsilon\|\).

**Definition 4.13.** Let \(x_0 \in \Omega\) be fixed, \(R_0 \in \mathbb{R}, B_{R_0}(x_0) \subset \Omega\). We define \(\zeta_{\rho, R} : \Omega \to \mathbb{R}\), \(0 < \rho < R \leq R_0\), to be a \(C^\infty(\Omega)\)-test function with \(\zeta_{\rho, R}(x) = 1\) for \(x \in B_{\rho}(x_0)\), \(\zeta_{\rho, R}(x) = 0\) for \(x \in \Omega \setminus B_{R}(x_0)\), \(\|\nabla \zeta_{\rho, R}\| \leq \frac{\text{const}}{R - \rho}\).

**Lemma 4.14.** For all \(0 < \rho < R \leq R_0\) we have
\[
\beta(\lambda, \rho) \leq C \cdot \frac{2}{\mathcal{H}_{n-1}(S_{\lambda, \rho})} \left( \frac{1}{(R - \rho)^2} \beta(\lambda, R) + \mathcal{H}_{n-1}(S_{\lambda, R}) \right).
\]

The constant \(C\) depends on the constants appearing in the inequalities of the previous section.
Proof. Applying Lemma \(4.12\) we estimate
\[
\beta(\lambda, \rho) \leq \int_{S_{\lambda, R}} (q_{\varepsilon} - \lambda)^2 \zeta^2_{\rho, R} \, d\mathcal{H}_{n-1} + \varepsilon \int_{\Omega_{\lambda, R}} (1 + p_{\varepsilon})(q_{\varepsilon} - \lambda)^2 \zeta^2_{\rho, R} \, d\text{vol}_g
\]
\[
\leq C \mathcal{H}_{n-1}^{\frac{2}{n-1}}(S_{\lambda, R}) \left( \int_{S_{\lambda, R}} (q_{\varepsilon} - \lambda)^2 \|P(\text{grad} \, \zeta_{\rho, R})\|^2 \, d\mathcal{H}_{n-1} + \varepsilon \int_{\Omega_{\lambda, R}} (1 + p_{\varepsilon})(q_{\varepsilon} - \lambda)^2 \|\text{grad} \, \zeta_{\rho, R}\|^2 \, d\text{vol}_g + \varepsilon \int_{\Omega_{\lambda, R}} \|\text{grad} \, q_{\varepsilon}\|^2 \zeta^2_{\rho, R} \, d\text{vol}_g \right)
\]
\[
\leq C \mathcal{H}_{n-1}^{\frac{2}{n-1}}(S_{\lambda, R}) \left( \int_{S_{\lambda, R}} (q_{\varepsilon} - \lambda)^2 \|P(\text{grad} \, \zeta_{\rho, R})\|^2 \, d\mathcal{H}_{n-1} + \varepsilon \int_{\Omega_{\lambda, R}} (1 + p_{\varepsilon})(q_{\varepsilon} - \lambda)^2 \|\text{grad} \, \zeta_{\rho, R}\|^2 \, d\text{vol}_g + \varepsilon \int_{\Omega_{\lambda, R}} \|\text{grad} \, q_{\varepsilon}\|^2 \zeta^2_{\rho, R} \, d\text{vol}_g \right).
\]
We now insert \(\|P(\text{grad} \, \zeta_{\rho, R})\|^2 \leq \|\text{grad} \, \zeta_{\rho, R}\|^2 \leq \text{const} \, (R - \rho)^2\), Lemma \(4.10\) and Lemma \(4.8\) to obtain
\[
\beta(\lambda, \rho) \leq C \mathcal{H}_{n-1}^{\frac{2}{n-1}}(S_{\lambda, R}) \cdot \left( \beta(\lambda, R) + \int_{S_{\lambda, R}} (q_{\varepsilon} - \lambda)^2 \zeta^2_{\rho, R} \, d\mathcal{H}_{n-1} \right)
\]
\[
\leq C \mathcal{H}_{n-1}^{\frac{2}{n-1}}(S_{\lambda, R}) \cdot \left( \beta(\lambda, R) + \int_{S_{\lambda, R}} (q_{\varepsilon} - \lambda)^2 \zeta^2_{\rho, R} \left( \sum_{i=1}^{n-1} \|\nabla X_i \, \text{grad} \, u_{\varepsilon}\|^2 \right) \, d\text{vol}_g \right)
\]
Using the Hölder inequality (1.) can be estimated as
\[
\text{(1.)} \leq \frac{1}{2} \int_{S_{\lambda, R}} (q_{\varepsilon} - \lambda)^2 \zeta^2_{\rho, R} \, d\mathcal{H}_{n-1} + \frac{1}{2} \mathcal{H}_{n-1}(S_{\lambda, R}).
\]
(2.) can be estimated using Lemma \(4.11\). Summarizing
\[
\beta(\lambda, \rho) \leq C \mathcal{H}_{n-1}^{\frac{2}{n-1}}(S_{\lambda, R}) \cdot \left( \beta(\lambda, R) \frac{(R - \rho)^2}{(R - \rho)^2} + \mathcal{H}_{n-1}(S_{\lambda, R}) + \varepsilon \int_{\Omega_{\lambda, R}} (1 + p_{\varepsilon})(q_{\varepsilon} - \lambda)^2 \|\text{grad} \, \zeta_{\rho, R}\|^2 \, d\text{vol}_g \right)
\]
\[
+ \int_{S_{\lambda, R}} (q_{\varepsilon} - \lambda)^2 \|P(\text{grad} \, \zeta_{\rho, R})\|^2 \zeta^2_{\rho, R} \, d\mathcal{H}_{n-1} \right).
\]
Obviously \(\zeta^2_{\rho, R} \leq \text{const} \, (R - \rho)^2\) and consequently (4.11) follows.

Remark 4.15. Obviously \((\lambda, \rho) \mapsto \beta(\lambda, \rho)\) and \((\lambda, \rho) \mapsto \mathcal{H}_{n-1}(S_{\lambda, \rho})\) are non-negative functions which are monotone increasing in \(\rho\) and monotone decreasing in \(\lambda\).
Lemma 4.16. For \( \lambda < \Lambda \) and \( R \leq R_0 \)
\[
\mathcal{H}_{n-1}(S_{\lambda,R}) \leq \frac{\beta(\lambda, R)}{(\Lambda - \lambda)^2}. \tag{4.12}
\]

Proof. \( q_\varepsilon(x) - \lambda \geq \Lambda - \lambda > 0 \) for all \( x \in \Omega_{\lambda,R} \). Therefore
\[
\beta(\lambda, R) \geq \int_{S_{\lambda,R}} (q_\varepsilon - \lambda)^2 \, d\mathcal{H}_{n-1} \geq (\Lambda - \lambda)^2 \int_{S_{\lambda,R}} 1 \, d\mathcal{H}_{n-1}.
\]

Corollary 4.17. For \( 0 \leq \lambda < \Lambda \) and \( 0 \leq \rho < R \leq R_0 \)
\[
\beta(\lambda, \rho) \leq C \left( \frac{1}{(\Lambda - \lambda)^{n-1}} \right) \beta(\lambda, R) \left( \frac{R - \rho}{R - \lambda} \right)^2. \tag{4.14}
\]

Proof. This follows directly by combining (4.11) and (4.12). \( \square \)

Corollary 4.18. For \( \lambda_0 \geq 0 \), \( k = 0, 1, 2, \ldots \) we define
\[
\rho_k := \frac{R_0}{2} + \frac{R_0}{2^k}, \quad \lambda_k := 2\lambda_0 - \frac{\lambda_0}{2^k}, \quad J_k := \beta(\lambda_k, \rho_k).
\]
Then
\[
J_{k+1} \leq C(\lambda_0) \cdot \left(2^{2+\frac{4}{n-1}}\right)^k \cdot J_k^{1+\frac{2}{n-1}}, \tag{4.13}
\]
where
\[
C(\lambda_0) := C \cdot \left( \frac{1}{\lambda_0^{n-1}} \cdot \frac{R_0^2}{\lambda_0^{2+\frac{4}{n-1}}} \right) \cdot 2^{2+\frac{4}{n-1}}.
\]
Here \( C \) is the constant from the previous corollary.

Proof. This follows immediately from the previous corollary. Just choose \( \lambda := \lambda_k \), \( \Lambda := \lambda_{k+1} \), \( \rho := \rho_{k+1} \), \( R := \rho_k \). Then we have \( \Lambda - \lambda = \frac{\lambda_0}{2^k} \) and \( R - \rho = \frac{R_0}{2^{k+1}} \). \( \square \)

Remark 4.19. Reviewing the involved inequalities, we see that the constant \( C(\lambda_0) \) in the previous corollary depends on \( R_0 \), the functions \( k, h \in C^\infty(\Omega) \), as well as the smooth vector field \( W \). What is more important is the fact that \( C(\lambda_0) \) is independent of \( \varepsilon \).

Proposition 4.20. Consider \( U \subset \subset \Omega \). Then there exists a constant \( C_U \) depending on \( U \) such that for all \( 0 < \varepsilon \leq 1 \) the unique minimizer \( u_\varepsilon \) of \( \mathcal{F}_\varepsilon \) satisfies
\[
\sup_U \|\nabla u_\varepsilon\| \leq C_U < \infty.
\]

Proof. Without loss of generality we can restrict our attention to subsets \( U \) of the form \( U := B_{2R_0}(x_0) \) with \( R_0 \) such that \( B_{2R_0}(x_0) \subset \Omega \). Choosing now \( \lambda_0 \) large, the constant \( C(\lambda_0) > 0 \) in the previous corollary can be made arbitrarily small. Therefore, taking \( \lambda_0 \) large enough, we can achieve
\[
J_1 \leq a < 1 \quad \text{and} \quad \left(2^{2+\frac{4}{n-1}} \cdot a^{\frac{2}{n-1}}\right)^k \leq \frac{a}{C(\lambda_0)} \quad \text{for all} \quad k = 0, 1, 2, \ldots
\]
for an appropriate constant \( 0 < a < 1 \). Hence
\[
C(\lambda_0) \left(2^{2+\frac{4}{n-1}}\right)^k \left(a^k\right)^{1+\frac{2}{n-1}} \leq a^{k+1}.
\]
Applying (4.13) and proceeding by induction we obtain $J_k \leq a^k$ for the quantity $J_k$ of the previous corollary. Now $\lim_{k \to \infty} J_k = 0$ because $a < 1$ and consequently $\beta \left( 2\lambda_0, \frac{\rho_0}{2} \right) = 0$. □

Summarizing our work of Sections 2, 3, and 4, Proposition 3.1 implies

**Theorem 4.21.** The minimizing problem $\mathcal{F}(u) \to \min$, $u \in W^{1,1}(\Omega)$, $\int_{\Omega} u \, d\text{vol}_g = 0$ has a unique solution

$$u_0 \in W^{1,1}(\Omega) \cap C^\infty(\Omega).$$

In other words: Our symmetrization procedure has the desired properties. That is, given a transvection $\tau$, and a bounded subset $\hat{\Omega} \subset \hat{M}^n$ as described in Section 2.3, the symmetrization procedure yields a unique set $S_\gamma(\hat{\Omega})$ which has the same volume as $\hat{\Omega}$ but minimal surface area among all sets obtained as variations of $\hat{\Omega}$ along the orbits of $\tau$. $S_\gamma(\hat{\Omega})$ is given by

$$S_\gamma(\hat{\Omega}) = \{ \tau(x, \sigma(x)) \mid u_0(x) - h(x) \leq t \leq u_0(x) + h(x), \ x \in \Omega \} \subset \hat{M}^n.$$

Using this description, we see that the boundary of $S_\gamma(\hat{\Omega})$ is smooth in points corresponding to the interior of $\Omega$. Boundary regularity of $u_0$ on $\partial \Omega$ has not been investigated here.

**Remark 4.22.** Our symmetrization construction can be carried out in any simply connected symmetric space of nonpositive curvature. All its properties discussed above remain valid without modifications of the proofs. In other words: An euclidean factor does not disturb our symmetrization constructions. Nevertheless, the case of symmetric spaces of noncompact type is the most interesting for investigating the isoperimetric problem.

### 5. Application to the isoperimetric problem

#### 5.1. Convexity of isoperimetric solutions.

Our symmetrization argument based on transvections coincides with Steiner symmetrization in the case $\hat{M}^n = \mathbb{R}^n$. In this case the symmetrization procedure shows that isoperimetric solutions are convex: Suppose a geodesic $\gamma : \mathbb{R} \to \mathbb{R}^n$ intersects a (smooth) domain $\hat{\Omega} \subset \mathbb{R}^n$ at least twice, then the symmetrized set $S_\gamma(\hat{\Omega})$ will intersect $\gamma(\mathbb{R})$ only once. This is just a consequence of the fact $w \equiv 0$ if $\hat{M}^n = \mathbb{R}^n$. Here $w$ is the 1–form introduced in Section 2 and incorporated in the area functional

$$\mathcal{F}(u) = \int_{\Omega} \sqrt{1 + k^2\|w + dh + du\|^2 + \sqrt{1 + k^2\|w - dh + du\|^2}} \, d\text{vol}_g.$$

In a general symmetric space $\hat{M}^n$ of noncompact type we would immediately get convexity of isoperimetric solutions if (for every geodesic $\gamma : \mathbb{R} \to \hat{M}^n$) we had $w = \text{grad} \ v$ for an appropriate function $v$ on the orbit space $M^{n-1} = \hat{M}^n/\tau$. In this case we could just set $u = -v$ to achieve convexity.

The first de Rham cohomology group of the orbit space $M^{n-1}$ is trivial, because $M^{n-1}$ is diffeomorphic to $\mathbb{R}^{n-1}$. Therefore $w = \text{grad} \ v$ for an appropriate function $v$ if and only if $dw = 0$. Investigating for which class of symmetric spaces of noncompact type we have $dw = 0$ (for all directions of
symmetrization given by geodesics $\gamma$), it turns out that this only holds for spaces with constant sectional curvature.

Suppose we are given a geodesic $\gamma: \mathbb{R} \to \hat{M}^n$ with $dw \neq 0$ on the corresponding orbit space $M^{n-1} = \hat{M}^n/\tau_\gamma$. Then there exists a set $\hat{\Omega} \subset \hat{M}^n$ which is invariant under symmetrization with respect to $\tau_\gamma$ but not convex. Such an example can be constructed as follows:

As $dw \neq 0$, there exists a 2-dimensional submanifold $B^2 \subset M^{n-1}$ homeomorphic to a disc such that $\int_{B^2} dw \neq 0$. Using Stoke's theorem

$$\int_{\partial B^2} w = \int_{B^2} dw \neq 0$$

for the closed loop $\partial B^2$, parametrized by $s: (0,1) \to \partial B^2$. Now we choose a function $u: M^{n-1} \to \mathbb{R}$ such that $u$ is smooth almost everywhere, $u \circ s$ is strictly monotone increasing and $\text{grad} u_{|s}$ is tangential to $\partial B^2$ with $\|\text{grad} u_{|s}\| \equiv 1$. Then obviously $\int_{\partial B^2} w(\text{grad} u) = \int_{\partial B^2} \langle w, du \rangle \neq 0$. Choosing now a neighborhood $\Omega$ of $\partial B^2$ and a constant $t \in \mathbb{R}$ in an appropriate way, we can therefore achieve

$$\int_{\Omega} \sqrt{1 + k^2 \|w + d(t \cdot u)\|^2} \text{dvol}_g$$

$$= \int_{\Omega} \sqrt{1 + k^2 (\|w\|^2 + 2t \langle w, du \rangle + t^2 \|du\|^2)} \text{dvol}_g$$

$$< \int_{\Omega} \sqrt{1 + k^2 \|w\|^2} \text{dvol}_g.$$

Using this it is now clear how to construct a set $\hat{\Omega} \subset \hat{M}^n$ which is not convex but invariant under symmetrization with respect to the translation $\tau$. Just think about $\hat{\Omega}$ as a neighborhood of the "lifted loop" $s$, that is of the curve

$$\tau(t \cdot u \circ s, \sigma \circ s): (0,1) \to \hat{M}^n.$$

In other words, our standard counterexample against an immediate convexity proof by symmetrization looks like a helix winding up and overlapping only over a very small part of the projection $\pi(\hat{\Omega}) \subset M^{n-1}$ in the orbit space. Of course, we expect that such a helix will not survive as a candidate for an isoperimetric solution if we consider symmetrization with respect to another direction, but unfortunately this is hard to control.

5.2. **Complex hyperbolic space.** It is well known that the boundaries of metric balls in complex hyperbolic spaces provide surfaces of constant mean curvature. That is, they are critical points of the area functional with respect to volume preserving deformations. However, for large volumes it is not known that they are isoperimetric solutions. Symmetrization with respect to transvections corresponds to a special class of volume preserving deformations. Consequently metric balls in complex hyperbolic spaces remain invariant under our symmetrization procedure. The fact that the area functional is convex with respect to our restricted class of deformations provides some evidence that isoperimetric solutions in complex hyperbolic spaces are balls and hence unique.
For studying our symmetrization procedure, up to now we have only used very basic properties of the function $k$ and the 1–form $w$ involved in the area functional $\mathcal{F}$. In fact, we did not need much more than smoothness and $k \geq 1$. But since we are in a symmetric space of noncompact type, these quantities should have a lot of nice properties, remember for example Lemma 2.6 and 2.7. This provides an interesting starting point for future research. For intuition, we will finish this paper by explicitly computing the 1–form $w$ for the case of the complex hyperbolic space.

The 1–form $w$ on the orbit space $M^{n-1} = \hat{M}^n / \gamma$ is defined by

$$d\sigma(X) = \text{Hor}(X)_{|\sigma} + w(X) \cdot K_{|\sigma},$$

where we choose $\sigma : M^{n-1} \to \hat{M}^n$ to be the section defined in 2.1 such that $\sigma(M^{n-1}) = \left(\frac{1}{2}(\beta^+ - \beta^-)\right)^{-1} \{0\}$. Taking the scalar product with the unit normal field $\nu$ on $\sigma(M^{n-1})$ (where $\nu_{|\gamma(0)} = K_{|\gamma(0)}$) we obtain

$$\langle \text{Hor}(W), \text{Hor}(X) \rangle = \langle W, X \rangle = w(X) = \frac{-\langle \nu, \text{Hor}(X) \rangle}{\langle K, \nu \rangle} = \frac{-1}{\langle K, \nu \rangle} \left( \nu - \frac{\nu \cdot K}{\|K\|} \right) \frac{K}{\|K\|} \cdot \text{Hor}(X).$$

Therefore

$$\text{Hor}(W)_{|\sigma} = \frac{K_{|\sigma}}{k^2} - \frac{\nu_{|\sigma}}{\langle K, \nu \rangle_{|\sigma}}.$$

As an easy application we can compute

$$\|W\|^2 = \frac{1}{k^2} + \frac{1}{\langle K, \nu \rangle^2}$$

and

$$\sqrt{1 + k^2 \|W\|^2} = \frac{k}{\langle K, \nu \rangle}.$$

For the rest of this section we specialize to the case of complex hyperbolic space $\hat{M}^{2n} = CH^n$. For every $X \perp K_{|\gamma(0)}$ we then have an isometry $\varphi$ with $d\varphi(X) = X$ and $d\varphi(K_{|\gamma(0)}) = -K_{|\gamma(0)}$. These isometries can be easily obtained by direct construction. In particular the existence of such isometries implies $\sigma(M^{2n-1}) = \exp(\hat{\gamma}(0) \text{^1}).$

Now we will compute the vector field $W$ explicitly. For this we only have to determine $\nu$ and $K$ along the geodesics $c : \mathbb{R} \to \hat{M}^{2n}$ with $c(0) \perp K_{|\gamma(0)}$. As $\hat{M}^{2n}$ is the complex hyperbolic space, the operator $R(\cdot, c(t))c(t)$ has eigenvalues $0$, $-1$ and $-4$. More precisely, along $c$ we may choose $2n$ orthonormal parallel vector fields $X_1, JX_1 = c, \ldots, X_n, JX_n$ , where $J$ denotes the almost complex structure, such that

$$R(X_1(t), c(t))c(t) = -4X_1(t),$$

$$R(JX_1(t), c(t))c(t) = 0,$$

$$R(X_i(t), c(t))c(t) = -X_i(t), \quad \text{for } i \geq 2$$

$$R(JX_i(t), c(t))c(t) = -JX_i(t), \quad \text{for } i \geq 2.$$

Furthermore, we can assume $\hat{\gamma}(0) = \cos(\vartheta) \cdot X_1(0) + \sin(\vartheta) \cdot X_2(0)$ for an appropriate $\vartheta \in [0, 2\pi]$ without restriction. Since $K$ is a Killing field corresponding to a transvection, $K(t) := K_{|c(t)}$ is a Jacobi field with initial data $K(0) = \hat{\gamma}(0)$ and $K'(0) = 0$. This implies

$$K(t) = \cos(\vartheta) \cosh(2t)X_1(t) + \sin(\vartheta) \cosh(t)X_2(t).$$
For calculating the normal field $\nu$ along $c$, we consider curves of the form $Y: (-\varepsilon, \varepsilon) \to T_{c(0)}\sigma(M^{2n-1})$ with $\|Y(s)\| = 1$. Applying the usual Jacobi field techniques to the “radial” geodesic variations $V(t, s) := \exp_{c(0)}(t \cdot Y(s))$, it turns out that $T_{c(t)}\sigma(M^{2n-1})$ is spanned by the Jacobi fields

$$JX_1(t) = \dot{c}(t),$$
$$\sinh(t) \cdot X_i(t) \quad \text{for } i = 3, \ldots, n,$$
$$\sinh(t) \cdot JX_1(t) \quad \text{for } i = 2, \ldots, n,$$
$$\sin(\vartheta) \cdot \frac{1}{2} \sinh(2t) \cdot X_1(t) - \cos(\vartheta) \sinh(t) \cdot X_2(t).$$

As $\nu(t) := \nu_{c(t)}$ has to be perpendicular to these vector fields

$$\nu(t) = \frac{\cos(\vartheta) \cdot X_1(t) + \sin(\vartheta) \cosh(t) \cdot X_2(t)}{\sqrt{\sin^2(\vartheta) \cdot \cosh^2(t) + \cos^2(\vartheta)}}. $$

Combining these results we obtain

$$\text{Hor}(W)|_{c(t)} = \frac{\cos(\vartheta) \cosh(2t) \cdot X_1(t) + \sin(\vartheta) \cosh(t) \cdot X_2(t)}{\cos^2(\vartheta) \cosh^2(2t) + \sin^2(\vartheta) \cosh^2(t)}$$

$$- \frac{\cos(\vartheta) \cdot X_1(t) + \sin(\vartheta) \cosh(t) \cdot X_2(t)}{\cos^2(\vartheta) \cosh(2t) + \sin^2(\vartheta) \cosh^2(t)}.$$  \hfill (5.1)

Observe that $\cos(\vartheta) = \langle J\dot{\gamma}(0), X_1(0) \rangle = \langle J\dot{\gamma}(0), JX_1(0) \rangle = \langle J\dot{\gamma}(0), \dot{c}(0) \rangle$. In other words, $\vartheta$ is the angle between $J\dot{\gamma}(0)$ and $\dot{c}(0)$. Using\footnote{5.1} we can immediately derive the following qualitative properties of $\text{Hor}(W)|_{c(t)}$:

- $\text{Hor}(W)|_{c(t)} \perp \dot{c}(t)$.
- $\text{Hor}(W)|_{c(t)} \to 0$ for $t \to \infty$.
- $\text{Hor}(W)|_{c(t)} = 0$ for $\vartheta = 0$ and $\vartheta = \pi$.

\textbf{Remark 5.1.} The 1–form $\omega$ that appears in our symmetrization construction clearly has some interesting special properties with respect to the geometry of the symmetric space. This information might provide further insight into the shape of isoperimetric domains. There also might be interesting connections to stability of isoperimetric domains $\hat{\Omega}$. After all, Killing fields on $\hat{M}^n$ induce elements in the kernel of the Jacobi operator of $\partial\hat{\Omega}$.

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