SYMMETRIC QUASI-HEREDITARY ENVELOPES

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Abstract. We show how any finite-dimensional algebra can be realized as an idempotent subquotient of some symmetric quasi-hereditary algebra. In the special case of rigid symmetric algebras we show that they can be realized as centralizer subalgebras of symmetric quasi-hereditary algebras. We also show that the infinite-dimensional symmetric quasi-hereditary algebras we construct admit quasi-hereditary structure with respect to two opposite orders, that they have strong exact Borel and ∆-subalgebras and the corresponding triangular decompositions.

1. Introduction

The classical result of Dlab and Ringel (see [DR]) says that every finite-dimensional algebra can be realized as a centralizer subalgebra of some quasi-hereditary algebra. Motivated by the discovery of (infinite-dimensional) symmetric quasi-hereditary algebras in [Pe] (see also [CT, MT1, MT2, BS]) we address the question whether every symmetric finite-dimensional algebra can be realized as a centralizer subalgebra of some symmetric quasi-hereditary algebra (we will loosely call the latter algebra a symmetric quasi-hereditary envelope of the original algebra, although it is not uniquely defined in any reasonable sense). Unless the original algebra is semisimple, any symmetric quasi-hereditary envelope must be infinite-dimensional.

In the present paper we generalize the construction from [DR] and produce quasi-hereditary envelopes of finite-dimensional algebras. Under some mild natural restrictions on the original algebra, quasi-hereditary envelopes of symmetric algebras turn out to be symmetric. In particular, we show that every symmetric and rigid algebra can be realized as a centralizer subalgebra of some symmetric quasi-hereditary algebra. Furthermore, we show that every finite-dimensional algebra can be realized as an idempotent subquotient of some symmetric quasi-hereditary algebra. In particular, this gives many new examples of symmetric quasi-hereditary algebras.

The infinite-dimensional (symmetric) quasi-hereditary algebras, which we construct, have many interesting properties. To start with, all these algebras are quasi-hereditary with respect to two natural orders (one of them being the opposite of the other one). The standard
and costandard modules for these structures have a natural description in terms of the original algebra. We also show that all these algebras have $\Delta$-subalgebras in the sense of König ([Ko1, Ko2]). Assuming that the original algebra is graded, we show that our algebras have a strong exact Borel subalgebra in the sense of König ([Ko1, Ko2]), as well as the corresponding triangular decompositions.

The paper is organized as follows: In Section 2 we extend the construction from [DR] to produce quasi-hereditary envelopes of finite-dimensional algebra and show that these envelops are quasi-hereditary with respect to two natural opposite orders. Finite-dimensional algebras are realized as centralizer subalgebras of their quasi-hereditary envelopes. In Section 3 we prove that for symmetric rigid finite-dimensional algebras the quasi-hereditary envelopes, constructed in Section 2, are symmetric as well. For arbitrary algebras we show how the construction can be generalized to realize every finite-dimensional algebra as an idempotent subquotient of some symmetric quasi-hereditary algebra. In Section 4 we describe strong exact Borel and $\Delta$-subalgebras and the corresponding triangular decompositions for our infinite-dimensional quasi-hereditary algebras. Finally, in Section 5 we discuss some examples, in particular, those coming from Schur algebras and the BGG category $\mathcal{O}$.

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2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{k}$ be an algebraically closed field and $\mathcal{A}$ be a basic $\mathbb{k}$-linear category with at most countably many objects and finite-dimensional projectives and injectives (see [MOS] for details). We will often loosely call such categories “algebras” (as they can be realized using infinite-dimensional associative quiver algebras which do not have a unit element in the general case) and use for them standard matrix notation with infinite matrices. For $x \in \mathcal{A}$ we denote by $e_x$ the identity element in $\mathcal{A}(x, x)$.

Assume that for some $N \in \mathbb{N}$ we have a (fixed) finite filtration of $\mathcal{A}$ by two-sided ideals as follows:

\begin{equation}
\mathcal{A} = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_N = 0.
\end{equation}

Assume further that $I_iI_j \subseteq I_{i+j}$ and that $I_i/I_{i+1}$ are semi-simple as $\mathcal{A}$-bimodules.

Consider the new category $\mathfrak{A}$, whose objects are $x[i], x \in \mathcal{A}, i \in \mathbb{Z}$. For $x, y \in \mathcal{A}$ and $i, j \in \mathbb{Z}$ set $\mathfrak{A}(x[i], y[j]) = \mathcal{A}(x, y)$. Then the multiplication in $\mathcal{A}$ induces a multiplication in $\mathfrak{A}$, which makes $\mathfrak{A}$ into a category. The category $\mathfrak{A}$ comes together with the natural action of
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\( \mathbb{Z} \) by autoequivalences via shifts \([i], i \in \mathbb{Z} \) (here \([1]\) means “shift by one to the right”). The category \( \mathfrak{A} \) can be seen as a \( \mathbb{Z} \)-Morita-equivalent extension of \( \mathcal{A} \) (every object in \( \mathcal{A} \) is repeated \(|\mathbb{Z}|\) times). We shall think of \( \mathfrak{A} \) also as of infinite matrices of the form

\[
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \\
\vdots & \mathcal{A} & \mathcal{A} & \mathcal{A} & \\
\vdots & \mathcal{A} & \mathcal{A} & \mathcal{A} & \\
\vdots & \mathcal{A} & \mathcal{A} & \mathcal{A} & \\
\ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

Denote by \( \mathfrak{B} \) the subcategory of \( \mathfrak{A} \), which contains all objects but only the following morphisms: For \( x, y \in \mathcal{A} \) and \( i, j \in \mathbb{Z} \) set

\[
\mathfrak{B}(x[i], y[j]) = \begin{cases} 
\mathfrak{A}(x[i], y[j]), & i \geq j; \\
\mathcal{I}_{j-i}(x, y), & \text{otherwise.}
\end{cases}
\]

One can think of \( \mathfrak{B} \) also as of infinite matrices of the form

\[
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \\
\vdots & \mathcal{I}_1 & \mathcal{I}_1 & \mathcal{I}_1 & \\
\vdots & \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{I}_1 & \\
\vdots & \mathcal{I}_3 & \mathcal{I}_2 & \mathcal{I}_1 & \\
\ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

Consider the subset \( \mathcal{I} \) of \( \mathfrak{B} \) with the same set of objects and morphisms given by

\[
\mathcal{I}(x[i], y[j]) = \begin{cases} 
\mathcal{I}_{N-(i-j)}(x, y), & i - N < j < i; \\
\mathfrak{B}(x, y), & j \leq i - N; \\
0, & \text{otherwise.}
\end{cases}
\]

The set \( \mathcal{I} \) is not a subcategory as it does not contain identity morphisms on objects. One can think of \( \mathcal{I} \) also as of infinite matrices of the form

\[
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \\
\vdots & 0 & \mathcal{I}_{N-1} & \mathcal{I}_{N-2} & \mathcal{I}_{N-3} & \\
\vdots & 0 & 0 & \mathcal{I}_{N-1} & \mathcal{I}_{N-2} & \\
\vdots & 0 & 0 & 0 & \mathcal{I}_{N-1} & \\
\ddots & 0 & 0 & 0 & 0 & \ddots
\end{pmatrix}
\]
It is easy to see that $\mathcal{I}$ is an ideal of $\mathfrak{B}$. Define the category $\mathcal{C} = \mathcal{C}(\mathcal{A}) = \mathfrak{B}/\mathcal{I}$. One can think of $\mathcal{C}$ as of infinite matrices of the form

$$
\begin{pmatrix}
\vdots & \vdots & \vdots & \ddots & \vdots \\
\cdots & \mathcal{A} & \mathcal{A}/\mathcal{I}_{N-1} & \mathcal{A}/\mathcal{I}_{N-2} & \mathcal{A}/\mathcal{I}_{N-3} & \cdots \\
\cdots & \mathcal{I}_1 & \mathcal{A} & \mathcal{A}/\mathcal{I}_{N-1} & \mathcal{A}/\mathcal{I}_{N-2} & \cdots \\
\cdots & \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{A} & \mathcal{A}/\mathcal{I}_{N-1} & \cdots \\
\cdots & \mathcal{I}_3 & \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{A} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

Observe that, given $x \in \mathcal{I}_i$ and some class $a + \mathcal{I}_j \in \mathcal{A}/\mathcal{I}_j$ we have $x(a + \mathcal{I}_j) \subset xa + \mathcal{I}_{i+j}$ due to our assumption that $\mathcal{I}_i\mathcal{I}_j \subseteq \mathcal{I}_{i+j}$, so multiplication of these matrices is well-defined. Note that, using the matrix notation, left modules are columns, while right modules are rows.

We consider two natural linear orders on $\mathbb{Z}$, we call the order where $i < i + 1$ the first order, and the one where $i > i + 1$ the second order. These orders induce partial orders on the equivalence classes of primitive idempotents in $\mathcal{C}(\mathcal{A})$, which we will also call the first and the second orders, respectively. The following statement is a generalization of the main construction from [DR]:

**Proposition 1.** (i) Left standard modules in the first order are given by direct summands of the following module:

$$
\Delta_{\mathcal{C}}^{1,l} = \begin{pmatrix}
\vdots \\
\mathcal{A}/\mathcal{I}_i \\
\mathcal{A}/\mathcal{I}_1 \\
\mathcal{A}/\mathcal{I}_1 \\
0 \\
0 \\
\vdots
\end{pmatrix}.
$$

(ii) Left standard modules in the second order are given by direct summands of the following module:

$$
\Delta_{\mathcal{C}}^{2,l} = \begin{pmatrix}
\vdots \\
0 \\
0 \\
\mathcal{A}/\mathcal{I}_1 \\
\mathcal{I}_1/\mathcal{I}_2 \\
\mathcal{I}_2/\mathcal{I}_3 \\
\vdots
\end{pmatrix}.
$$

(iii) Right standard modules for the first order are given by direct summands of the following module:

$$
\Delta_{\mathcal{C}}^{1,r} = ( \cdots \mathcal{I}_3/\mathcal{I}_4 \mathcal{I}_2/\mathcal{I}_3 \mathcal{I}_1/\mathcal{I}_2 \mathcal{A}/\mathcal{I}_1 0 0 \cdots ) \quad .
$$
(iv) Right standard modules for the second order are given by direct summands of the following module:

\[ \Delta_{e^r}^2 = (\ldots 0 0 A/I_1 A/I_1 A/I_1 \ldots ) \]

(v) The category \( \mathcal{C} \) is quasi-hereditary with respect to both orders.

Proof. Let \( i \in \mathbb{Z} \). For the first order, the quotient of \( \mathcal{C} \) modulo the two-sided ideal, generated by all idempotents \( e_x[j], x \in A, j \in \mathbb{Z}, j > i \), looks as follows:

\[
\begin{pmatrix}
\ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\
\ldots & * & * & A/I_1 & 0 & \ldots \\
\ldots & * & * & A/I_1 & 0 & \ldots \\
\ldots & I_2/I_3 & I_1/I_2 & A/I_1 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \ldots \\
\end{pmatrix}
\]

(here we do not care about the asterisks).

Similarly, for the second order, the quotient of \( \mathcal{C} \) modulo the two-sided ideal, generated by all idempotents \( e_x[j], x \in A, j \in \mathbb{Z}, j < i \), looks as follows:

\[
\begin{pmatrix}
\ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & A/I_1 & A/I_1 & A/I_1 & \ldots \\
\ldots & 0 & I_1/I_2 & * & * & \ldots \\
\ldots & 0 & I_2/I_3 & * & * & \ldots \\
\ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\
\end{pmatrix}
\]

As left modules are columns and right modules are rows, the claims (i)–(iv) follow.

The indecomposable right projective \( \mathcal{C} \)-module, generated by \( e_x[i], x \in A \), is a direct summands of the following module \( P \):

\[
(\ldots 0 I_{N-1} I_{N-2} \ldots I_1 A A/I_{N-1} \ldots A/I_1 0 \ldots ).
\]

The filtration (1) induces a filtration on every component of \( P \), whose subquotients could be organized into the following rhombal picture:

\[
\begin{array}{cccccccc}
I_{N-1} & \ldots & I_2 & I_1 & A & A/I_{N-1} & A/I_{N-2} & \ldots & A/I_1 \\
I_{N-1} & \ldots & I_2 & I_1 & A & A/I_{N-1} & A/I_{N-2} & \ldots & A/I_1 \\
\end{array}
\]

\[
(2)
\begin{array}{cccccccc}
I_{N-1} & I_{N-2} & I_{N-3} & I_{N-4} & \ldots & I_0/I_1 \\
I_{N-1} & I_{N-2} & I_{N-3} & I_{N-4} & \ldots & I_0/I_1 \\
\end{array}
\]

\[
(3)
\begin{array}{cccccccc}
I_{N-1} & I_{N-2}/I_{N-1} & I_{N-3}/I_{N-2} & I_{N-4}/I_{N-3} & \ldots & I_0/I_1 \\
I_{N-1} & I_{N-2}/I_{N-1} & I_{N-3}/I_{N-2} & I_{N-4}/I_{N-3} & \ldots & I_0/I_1 \\
\end{array}
\]

\[
(4)
\begin{array}{cccccccc}
I_{N-1} & I_{N-2}/I_{N-1} & I_{N-3}/I_{N-2} & I_{N-4}/I_{N-3} & \ldots & I_0/I_1 \\
I_{N-1} & I_{N-2}/I_{N-1} & I_{N-3}/I_{N-2} & I_{N-4}/I_{N-3} & \ldots & I_0/I_1 \\
\end{array}
\]
Organizing these subquotients into a filtration of $P$ as shown on the following pictures:

(3)

we obtain a filtration of $P$ by direct summands of the module $\Delta^{1,r}_e$ and $\Delta^{2,r}_e$, respectively. This means that right $\mathcal{C}$-projectives are filtered by standard modules for both orders. The claim (v) follows and the proof is complete.

Corollary 2. (i) Left costandard modules for the first order are given by direct summands of the following module:

$$\nabla^{1,l}_e = \begin{pmatrix} 
\vdots \\
(\mathcal{I}_2/\mathcal{I}_3)^* \\
(\mathcal{I}_1/\mathcal{I}_2)^* \\
(\mathcal{A}/\mathcal{I}_1)^* \\
0 \\
0 \\
\vdots 
\end{pmatrix}$$

(ii) Left costandard modules for the second order are given by direct summands of the following module:

$$\nabla^{2,l}_e = \begin{pmatrix} 
\vdots \\
0 \\
0 \\
(\mathcal{A}/\mathcal{I}_1)^* \\
(\mathcal{A}/\mathcal{I}_1)^* \\
(\mathcal{A}/\mathcal{I}_1)^* \\
\vdots 
\end{pmatrix}$$

(iii) Right costandard modules for the first order are given by direct summands of the following module:

$$\nabla^{1,r}_e = (\ldots (\mathcal{A}/\mathcal{I}_1)^* (\mathcal{A}/\mathcal{I}_1)^* (\mathcal{A}/\mathcal{I}_1)^* 0 0 \ldots )$$

(iv) Right costandard modules for the second order are given by direct summands of the following module:

$$\nabla^{2,r}_e = (\ldots 0 0 (\mathcal{A}/\mathcal{I}_1)^* (\mathcal{I}_1/\mathcal{I}_2)^* (\mathcal{I}_2/\mathcal{I}_3)^* \ldots )$$

Proof. This follows from Proposition 1 applying duality.

Corollary 3. For every $x \in \mathcal{A}$ and every $i \in \mathbb{Z}$ there is an isomorphism $\nabla^{2,l}_e(x,i) \cong \Delta^{1,l}_e(x,i + N)$. 
Proof. Since the $A$-module $A/I_1$ is semi-simple by our assumptions, the claim follows directly from Proposition 1(i) and Corollary 2(ii). □

Note that, by construction, the original category $A$ is a centralizer subcategory of the category $C$.

3. Symmetric quasi-hereditary envelopes of algebras

From now on we assume that $A$ has finitely many objects. Let $A$ be the path algebra of $A$. Then $A$ is a finite-dimensional algebra and we may assume that it is given by a quiver $Q$ with set of vertices $\{1, \ldots, n\}$ and relations $R$. As in the previous section, we fix a filtration of $A$ by two-sided ideals

\[ A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_N = 0 \]

with semisimple subquotients and such that $I_i I_j \subset I_{i+j}$. For example, we can take (4) to be the radical filtration of $A$. For $k \in \{1, \ldots, n\}$ we denote by $e_k$ the idempotent corresponding to the vertex $k$ in $A$, and we denote the corresponding idempotent of $A[i]$ (that is in the $(i, i)$th matrix position) by $e_{k,i}$. Set $C := C(A)$.

**Theorem 4.** Assume that $A$ is symmetric with the symmetric trace form $(\cdot, \cdot)$ and that $(\cdot, \cdot)$ induces a non-degenerate pairing between $A/I_j$ and $I_{N-j}$ for every $j$. Then the algebra $C$ is symmetric.

**Proof.** Define a bilinear form $(\cdot, \cdot)_{C}$ on $C$, by setting

\[ (a_{i,j}, b_{k,l})_C := \delta_{j,k} \delta_{i,l} (a, b), \]

where $a, b \in A$ (in a suitable ideal if $i > j$ resp. $k > l$), $i, j, k, l \in \mathbb{Z}$, and $a_{i,j}$ means the element $a$ in matrix position $(i, j)$.

The form $(\cdot, \cdot)_C$ is bilinear, symmetric and associative by construction. Again, by construction, the form $(\cdot, \cdot)_C$ pairs matrix positions $(i, j)$ and $(j, i)$. By the definition of $C$, the corresponding components in these positions are $A/I_s$ and $I_{N-s}$ for some $s$. By our assumption, the form $(\cdot, \cdot)$ induces a nondegenerate pairing of $A/I_s$ and $I_{N-s}$. This yields that $(\cdot, \cdot)_C$ is nondegenerate as well, completing the proof. □

**Corollary 5.** Assume that $A$ is symmetric and that (4) is both the radical and the socle filtration of $AA$ (i.e. $AA$ is rigid). Then $C$ is symmetric.

**Proof.** By our assumptions, the filtration (4) is the unique Loewy filtration of $AA$. The form $(\cdot, \cdot)$ pairs it with another Loewy filtration, and hence with itself. This yields that $(\cdot, \cdot)$ induces a non-degenerate pairing between $A/I_j$ and $I_{N-j}$ for every $j$ and the claim follows from Theorem 4. □

Some other examples to which Theorem 4 can be applied come from the category $O$ and will be discussed later on (see Example 24). If $A$ is not symmetric (or if it is symmetric but does not satisfy the
assumptions of Theorem 4) it is more reasonable to try to embed $A$ into its quasi-hereditary “envelope” not as a centralizer subalgebra, but as an idempotent subquotient. This goes as follows:

Assume that (4) is the radical filtration of $A$. We form a new algebra $	ilde{A}$ by attaching, for every vertex $k$, a vertex $\tilde{k}$ and an arrow $k \to \tilde{k}$, keeping the original relations $R$, defining the algebra $A$. Then $A$ is a centralizer subalgebra of $\tilde{A}$ (corresponding to nontilded vertices) in the natural way, and $\text{Rad} \tilde{A}$ has nilpotency degree $N + 1$. Moreover, the algebra $A$ is also an idempotent quotient of $\tilde{A}$, obtained by factoring out the two-sided ideal, generated by idempotents, associated with the new (tilded) vertices. Set $N = \{1, \ldots, n\}$, $\tilde{N} = \{\tilde{1}, \ldots, \tilde{n}\}$, and $N = N \cup \tilde{N}$.

Now $\text{soc} \tilde{A} \tilde{A}$ consists of simple modules with indices $\tilde{k}$. The right projective $e_k \tilde{A}$ for $\tilde{A}$, corresponding to a vertex $k \in N$, is the same as the right projective for $A$ at the same vertex. The right projective $e_k \tilde{A}$ at vertex $\tilde{k} \in \tilde{N}$ is an extension of the simple at $\tilde{k}$ with the right projective at $k$ (the simple extending the top of $e_k A$), hence has a longer Loewy length. Therefore $e_k \text{Rad}^N \tilde{A} = 0$ or, equivalently, the Loewy length $N_k'$ of $e_k \tilde{A}$ is strictly less than the nilpotency degree of $\text{Rad} \tilde{A}$ (which is $N + 1$). Let $\tilde{A} e_k$ be the left projective at vertex $k$, $N_k'$ its Loewy length.

We now take $\mathcal{C} = \mathcal{C}(\tilde{A})$ (with respect to the radical filtration) and form the trivial extension $\mathcal{D} = \mathcal{D}(\tilde{A})$ of $\mathcal{C}$ with its “restricted dual” $\mathcal{C}$-bimodule

$$
\mathcal{C}^* := \bigoplus_{i,j \in \mathbb{Z}; x,y \in N} \text{Hom}_k(e_{y,j} \mathcal{C} e_{x,i}, k).
$$

(see [Ha, 3.1]). Being a trivial extension of $\mathcal{C}$, the algebra $\mathcal{D}$ is automatically symmetric. To make the notation consistent with the previous section, from now on we assume that the nilpotency degree of $\text{Rad} \tilde{A}$ is $N$.

We now extend our first order in the following way: for $(k, i), (l, j) \in N \times \mathbb{Z}$ we set $(k, i) > (l, j)$ if $i > j$ or if $i = j, k \in N$ and $l \in \tilde{N}$. We will again call this order the first order.

**Proposition 6.** The algebra $\mathcal{D}(\tilde{A})$ is quasi-hereditary with respect to the first order and for left standard $\mathcal{D}$-modules we have $\Delta^1_{\mathcal{D}}(k, i) = \Delta^1_{\mathcal{C}}(k, i), k \in N, i \in \mathbb{Z}$.

**Proof.** We first consider $\mathcal{C}$. Let $e_{k,i}$ denote the idempotent in $\tilde{A}$ at the vertex $k \in N$, in matrix position $i, i$. With respect to our first order left standard modules $\Delta^1_{\mathcal{C}}(k, i)$ are uniserial with a filtration with composition factors

$$L^1(k, i), L^1(k, i - 1), \ldots, L^1(k, i - N + 1)$$
read from top to bottom (see Proposition 1(i)). Then, by (2), the left projective $C e_{k,i}$ for $C$ has a filtration with subquotients
\[ \Delta_{e}^{1,l}(k, i), \bigoplus_{j \in J_{i}} \Delta_{e}^{1,l}(j, i + 1), \ldots, \bigoplus_{j \in J_{N_{k}^{l}}} \Delta_{e}^{1,l}(j, i + N_{k}^{l}), \]
where $\text{Rad}^{m} \tilde{A} e_{k}/\text{Rad}^{m+1} \tilde{A} e_{k} \cong \bigoplus_{j \in J_{m}} L^{l}(j)$.

Similarly, the right projective $e_{k,i}C$ has a filtration with subquotients
\[ \Delta_{e}^{2,r}(k, i), \bigoplus_{j \in J_{i}} \Delta_{e}^{2,r}(j, i - 1), \ldots, \bigoplus_{j \in J_{N_{k}^{r}}} \Delta_{e}^{2,r}(j, i - N_{k}^{r}), \]
where $e_{k} \text{Rad}^{m} \tilde{A}/e_{k} \text{Rad}^{m+1} \tilde{A} \cong \bigoplus_{j \in J_{m}} L^{r}(j)$. Hence the left injective $(e_{k,i}C)^{*}$ has a filtration with subquotients
\[ \bigoplus_{j \in J_{N_{k}^{l}}} \nabla_{e}^{2,l}(j, i - N_{k}^{r}), \ldots, \bigoplus_{j \in J_{i}} \nabla_{e}^{2,l}(j, i - 1), \nabla_{e}^{2,l}(k, i) \]
and thus, by the isomorphism $\nabla_{e}^{2,l}(k, i) \cong \Delta_{e}^{1,l}(k, i + N)$ (Corollary 3), a filtration with subquotients
\[ \bigoplus_{j \in J_{N_{k}^{l}}} \Delta_{e}^{1,l}(j, i + N - N_{k}^{r}), \ldots, \bigoplus_{j \in J_{i}} \Delta_{e}^{1,l}(j, i + N - 1), \Delta_{e}^{1,l}(k, i + N). \]

We now claim that $D = D(\tilde{A})$ is quasi-hereditary with $\Delta_{D}^{1,l}(k, i) = \Delta_{e}^{1,l}(k, i)$. As the projective module $D e_{k,i}$ has a filtration with subquotients $C e_{k,i}$ and $(e_{k,i}C)^{*}$, which both have $\Delta_{D}^{1,l}$-filtrations by above, $D e_{k,i}$ also has a $\Delta_{D}^{1,l}$-filtration. So it suffices to check that all standard modules appearing in $(e_{k,i}C)^{*}$ have larger index than $(k, i)$. To see this, we need to distinguish two cases.

The first case is when $k \in \mathbb{N}$. In this case, the smallest second index of the standard modules appearing in $(e_{k,i}C)^{*}$ is $i + N - N_{k}^{r}$. But, as seen above, for $k \in \{1, \ldots, n\}$, $N_{k}^{r} < N$, so $i + N - N_{k}^{r} > i$, which is what we need.

The second case is when $k \in \tilde{\mathbb{N}}$. In this case the smallest second index of the standard modules appearing in $(e_{k,i}C)^{*}$ can well be $i$, however, in this case $P^{r}(k)$ has simple top $L^{r}(k)$ and all other composition factors are of the form $L^{r}(j)$, with $j \in \{1, \ldots, n\}$. Therefore, the standard modules appearing in $(e_{k,i}C)^{*}$ with smallest second index, namely $\Delta_{e}^{1,l}(j, i)$, have first index $j$ where $L^{r}(j)$ occurs in $e_{k} \text{Rad}^{N_{k}^{r}} \tilde{A}$, so $j \in \mathbb{N}$, and $(k, i) < (j, i)$. This completes the proof that $D$ is quasi-hereditary.

From Proposition 6 and [MT1, Corollary 5] it follows that, with respect to the first order, right $D$-projectives also have standard filtrations. The corresponding standard modules are described as follows:
Lemma 7. The right standard module $\Delta_{D}^{1r}(i,k)$ for $D$ is an extension of the $C$-modules $\Delta_{C}^{1r}(i,k)$ and $\nabla_{C}^{2r}(i - N + 1,k)$.

Proof. The right projective module $e_{k,i}D$ has a filtration with subquotients $e_{k,i}C$ and $(Ce_{k,i})^*$. The module $Ce_{k,i}$ is filtered by $\Delta_{C}^{1r}(k,i)$, $\Delta_{C}^{1r}(k,i + 1)$, $\ldots$, $\Delta_{C}^{1r}(k,i + N - 1)$ and the module $Ce_{k,i}$ is filtered by $\Delta_{C}^{2l}(k,i)$, $\Delta_{C}^{2l}(k,i - 1)$, $\ldots$, $\Delta_{C}^{2l}(k,i - N + 1)$. Therefore the module $(Ce_{k,i})^*$ is filtered by $\nabla_{C}^{2r}(k,i - N + 1)$, $\nabla_{C}^{2r}(k,i - N + 2)$, $\ldots$, $\nabla_{C}^{2r}(k,i)$.

Let $X$ denote the quotient of $e_{k,i}D$ modulo the trace of all $e_{k,j}D$, $j > i$. Obviously $\Delta_{C}^{1r}(k,i)$ is a quotient of $X$. Since none of modules $e_{k,j}D$, $j > i$, contains $L^{r}(k,i - N + 1)$, $\nabla_{C}^{2r}(k,i - N + 1)$ is a subquotient of $X$ as well. By definition, none of other $\Delta_{C}^{1r}(k,j)$ contributes to $X$, which yields that $X$ has a quotient $\tilde{X}$, which is an extension of $\Delta_{C}^{1r}(k,i)$ by $\nabla_{C}^{2r}(k,i - N + 1)$.

As $C$ is quasi-hereditary with respect to the second order, we also have a quotient $\Delta_{C}^{2l}(k,i)$ which is uniserial with a filtration $\nabla^{r}(k,i)$, $\nabla^{r}(k,i + 1)$, $\ldots$, $\nabla^{r}(k,i + N - 1)$. Since $\nabla^{r}(k,i + 1)$ is in the top of the kernel of $e_{k,i}D \to X$, we know that $\Delta_{D}^{1r}(k,i + 1)$ appears as a subquotient of a standard filtration of $e_{k,i}D$. Inductively, we obtain that the modules $\Delta_{D}^{1r}(k,i)$, $\Delta_{D}^{1r}(k,i + 1)$, $\ldots$, $\Delta_{D}^{1r}(k,i + N - 1)$ appear as subquotients of a standard filtrations of $e_{k,i}D$. Each of those $\Delta_{D}^{1r}(k,j)$ has a quotient which is an extension of $\Delta_{C}^{1r}(k,j)$ by $\nabla_{C}^{2r}(k,j - N + 1)$ and we see that this exhausts the whole module. Hence the surjection of $X$ onto $\tilde{X}$ must be an isomorphism and right standard modules in the first order for $D$ are of the desired form. \hfill $\Box$

Corollary 8. The algebra $D$ is quasi-hereditary with respect to the second order as well.

Proof. Since $D$ is symmetric, projective and injective $D$-modules coincide. By Proposition 6, left standard $D$-modules with respect to the first order are uniserial and coincide with the corresponding $C$-modules. Take a standard filtration of a left projective $D$-module. Applying duality we get a costandard filtration of a right injective $D$-module, which is also a right projective.

Taking into account that these right costandard modules coincide, up to shift, with right standard modules with respect to the second order, we obtain that right projective $D$-modules have a filtration by right standard modules with respect to the second order. Since all
shifts are the same (by $N$), it follows that this filtration satisfies the necessary ordering condition. The claim follows. □

**Remark 9.** Assume that the right projective $\tilde{A}$-module at vertex $k$ has radical filtration with subquotients $P^1, \ldots, P^s$. Then the indecomposable right projective $\mathcal{C}$-module at $(k, i)$ looks as follows:

\begin{equation}
\begin{array}{cccccccc}
P^1_i \\
P^1_{i+1} & P^2_{i-1} \\
P^1_{i+2} & P^2_i & \ddots \\
\vdots & \vdots & \ddots & P^s_{i-s+1} \\
P^1_{i+N-1} \\
P^2_{i+N-2} \\
P^s_{i+N-s}
\end{array}
\end{equation}

If the left projective $\tilde{A}$-module at vertex $k$ has a radical filtration with subquotients $Q^1, \ldots, Q^t$, then the indecomposable right injective $\mathcal{C}$-module at $(k, i)$ looks as follows:

\begin{equation}
\begin{array}{cccccccc}
Q^t_{i-N+t} \\
Q^t_{i-N+t+1} & Q^{t-1}_{i-N+t-1} \\
Q^t_{i-N+t+2} & Q^{t-1}_{i-N+t} & \ddots \\
\vdots & \vdots & \ddots & Q^1_{i-t+1} \\
Q^t_{i+t-1} \\
Q^{t-1}_{i+i-2} \\
Q^1_i
\end{array}
\end{equation}
For $k \in \tilde{N}$, $s$ can reach $N$, but $t = 1$. For $k \in N$, $t$ can reach $N$, but $s$ is always less than $N$ and $Q^t$ only has composition factors indexed by $k \in N$.

The corresponding indecomposable injective $\mathcal{D}$-module is obtained by gluing (5) and (6). The standard filtrations of this module with respect to the first and the second order can be organized using the left and the right diagrams from (3), respectively.

**Proposition 10.** $A$ is an idempotent subquotient of $\mathcal{D}$ as follows:

$$A \cong 1_{\tilde{A}_i} \mathcal{D} 1_{\tilde{A}_i}/1_{\tilde{A}_i} \mathcal{D} \tilde{e}_i \mathcal{D} 1_{\tilde{A}_i},$$

where $\tilde{e}_i = \sum_{k=1}^{n} e_{k,i} \in \mathcal{D}$.

**Proof.** It is obvious that $1_{\tilde{A}_i} \mathcal{D} 1_{\tilde{A}_i}$ is isomorphic to the trivial extension $S$ of $\tilde{A}$ by $\tilde{A}^*$. Now we claim that the ideal $\tilde{A}^*$ is contained in the ideal, generated by $\tilde{e} := \sum_{k=1}^{n} e_{k,i}$.

Consider the right projective $S$-module $e_k S$ at vertex $k \in N$. This module has a filtration by the right projective $\tilde{A}$-module $e_k \tilde{A}$ (which is the dual of the corresponding left injective $A$-module and only has composition factors $L^r(j)$ for $j \in N$), and the right injective $\tilde{A}$-module at the vertex $k$, which has a semisimple quotient consisting of simples $L^r(r)$ for $r \in \tilde{N}$ and sitting on a submodule isomorphic to the right injective $A$-module at the vertex $k$. Hence, right projectives for $S/S \tilde{e} S$ look like right projectives for $A$, so $S/S \tilde{e} S \cong A$. □

4. **Triangular decomposition**

Recall (see [Ko1, Ko2]) that a directed subalgebra $B$ of a basic quasi-hereditary algebra $A$ is called a (strong) exact Borel subalgebra provided that $A$ and $B$ have the same simple modules, the tensor induction functor $A \otimes_B -$ is exact and maps simple modules to standard modules. Dually one defines (strong) $\Delta$-subalgebras (again see [Ko1]). There is an obvious generalization of these notions to $k$-linear categories (our algebras). We keep the setup of the previous section and identify the algebra $A/I_1$ with some maximal semisimple subalgebra of $A$, say $S$. Then $S$ is a maximal semisimple subalgebra (in particular, a subspace) of all algebras $A/I_i$ for all $i > 0$. 
Proposition 11. The algebra

\[
\tilde{B} := \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & S & S & \ddots & S & 0 & 0 & \vdots \\
\vdots & 0 & S & S & \ddots & S & 0 & \vdots \\
\vdots & 0 & 0 & S & S & \ddots & S & \vdots \\
\vdots & 0 & 0 & 0 & S & S & \ddots & \vdots \\
\vdots & 0 & 0 & 0 & 0 & S & S & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\]

(here each row contains exactly \(N\) nonzero entries) is a strong exact \(\Delta\)-subalgebra of both \(\mathfrak{C}\) and \(\mathfrak{D}\) with respect to the first order.

Proof. The algebra \(\tilde{B}\) is obviously a subalgebra of both \(\mathfrak{C}\) and \(\mathfrak{D}\). It is directed by definition and thus quasi-hereditary with respect to the first order. Corresponding right standard modules are just simple modules, corresponding left standard modules are projectives and look as follows:

\[
\begin{pmatrix}
\vdots \\
A/I_1 \\
A/I_1 \\
0 \\
0 \\
\vdots
\end{pmatrix}
\]

These coincide with left standard modules for both \(\mathfrak{C}\) and \(\mathfrak{D}\) (by Proposition 1(i) and Proposition 6). Therefore, using [Ko1, Theorem A], we deduce that \(\tilde{B}^{\text{op}}\) is an exact Borel subalgebra for \(\mathfrak{C}^{\text{op}}\) and \(\mathfrak{D}^{\text{op}}\). Thus, by [Ko1, Theorem B], we have that \(\tilde{B}\) is a \(\Delta\)-subalgebra for \(\mathfrak{C}\) and \(\mathfrak{D}\). That \(\tilde{B}\) is strong follows from the definitions. This completes the proof. \(\square\)

Assume now that the algebra \(A\) is positively graded, \(A = \bigoplus_{i=0}^{\infty} A_i\) and that the filtration (4) coincides with the grading filtration, that is \(I_j = \bigoplus_{i=j}^{\infty} A_i\). In this case we have \(I_j/I_{j+1} \cong A_j\) for all \(i\), in particular, \(I_j/I_{j+1}\) can be realized as a canonical subspace of \(A\).
Proposition 12. Under the above assumptions, the algebra

\[
\mathcal{B} := \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \\
\vdots & A_0 & 0 & 0 & 0 \\
\vdots & A_1 & A_0 & 0 & 0 \\
\vdots & A_2 & A_1 & A_0 & 0 \\
\vdots & A_3 & A_2 & A_1 & A_0 \\
\ddots & \ddots & \ddots & \ddots & \\
\end{pmatrix}
\]

is a strong exact Borel subalgebra of \( \mathcal{C} \) with respect to the first order.

Proof. That \( \mathcal{B} \) is a subalgebra follows from the definitions and the fact that \( \mathcal{A} \) is graded (i.e. \( \mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{i+j} \)). Note that \( \mathcal{A}_0 \) is a maximal semi-simple subalgebra of \( \mathcal{A} \) and hence simple \( \mathcal{A} \)-modules can be identified with simple \( \mathcal{A}_0 \)-modules. Therefore simple \( \mathcal{C} \)-modules (shifted simple \( \mathcal{A} \)-modules) and \( \mathcal{B} \)-modules (shifted simple \( \mathcal{A}_0 \)-modules) can be identified as well.

The algebra \( \mathcal{B} \) is directed by definition hence quasi-hereditary with respect to the first order. Left standard \( \mathcal{B} \)-modules are simple. Right standard \( \mathcal{B} \)-modules are projective. Left costandard \( \mathcal{B} \)-modules are dual to right standard \( \mathcal{B} \)-modules and hence have the following form:

\[
\begin{pmatrix}
\vdots \\
A_2^* \\
A_1^* \\
A_0^* \\
0 \\
\vdots \\
\end{pmatrix}
\]

As \( A_j \cong I_j/I_{j+1} \) for all \( j \), from Corollary 2(i) we obtain that these costandard modules are restrictions of costandard \( \mathcal{C} \)-modules. Hence \( \mathcal{B} \) is an exact Borel subalgebra by [Ko1, Theorem A]. That \( \mathcal{B} \) is strong follows from the definitions. This completes the proof. \( \square \)

Remark 13. If we assume the existence of a Borel subalgebra, the condition of left costandard modules for this algebra being isomorphic to left costandard modules for \( \mathcal{C} \) forces the Borel subalgebra to have the following form:

\[
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \\
\vdots & \vdots & X_0,0 & 0 & 0 \\
\vdots & \vdots & X_1,0 & X_0,0 & 0 \\
\vdots & \vdots & X_2,0 & X_1,0 & X_0,0 \\
\ddots & \ddots & \ddots & \ddots & \\
\end{pmatrix}
\]
where $X_{j,i}$ are subspaces of $I_j$ providing a splitting of $I_j \rightarrow I_j/I_{j+1}$. Furthermore we must have $X_{j,i}X_{i-k,k} \subseteq X_{j+i-k,k}$ for this to be a subalgebra. If we assume that the Borel subalgebra is stable under the shift, i.e. that $X_{j,i} = X_{j,i+1}$ for all $i,j$, then the above is simply the condition that $A$ is graded. Hence the existence of a Borel subalgebra which is invariant under the shift is equivalent to $A$ being graded with respect to the filtration (4).

We further assume that $A$ is positively graded. Then the trivial extension $A = A \oplus A^*$ of $A$ inherits a natural $\mathbb{Z}$-grading by assigning degree $-i$ to the space $A_i^*$, $i \geq 0$. We would need to redefine this natural grading as follows: set $\deg A_i^* = N - 1 - i$. For $i \in \mathbb{Z}$ set $\overline{A}_i = A_i \oplus A_{N-1-i}$ and, because of $\text{Rad}^N(A) = 0$, we have $\overline{A}_i = 0$ for all $i < 0$.

**Proposition 14.** Under the assumptions of Proposition 12, the algebra

$$\overline{B} := \left( \begin{array}{ccccccccc}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \overline{A}_0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \overline{A}_1 & \overline{A}_0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \overline{A}_2 & \overline{A}_1 & \overline{A}_0 & 0 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \overline{A}_3 & \overline{A}_2 & \overline{A}_1 & \overline{A}_0 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right)$$

is a strong exact Borel subalgebra of $\mathcal{D}$ with respect to the first order.

**Proof.** That $\overline{B}$ is a directed subalgebra of $\mathcal{D}$ and that simple $\overline{B}$ and $\mathcal{D}$ modules can be identified follows from the construction. Using Lemma 7, the rest is proved just as in the proof of Proposition 12. □

Denote by $S_Z$ the subalgebra

$$\tilde{B} := \left( \begin{array}{ccccccccc}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & A_0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & 0 & A_0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & 0 & 0 & A_0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right)$$

of $\mathcal{C}$. Note that $S_Z$ is a semi-simple subalgebra of $\mathcal{D}$, $\tilde{B}$, $B$ and $\overline{B}$. Propositions 12 and 14 allow us to deduce the following *triangular decompositions* for the algebras $\mathcal{C}$ and $\mathcal{D}$:

**Theorem 15.** Under the assumptions of Proposition 12 we have:

1. Multiplication in $\mathcal{C}$ induce the following isomorphism of left $\tilde{B}$- and right $B$-modules: $\mathcal{C} \cong \tilde{B} \otimes_{S_Z} B$.

2. Multiplication in $\mathcal{D}$ induce the following isomorphism of left $\overline{B}$- and right $\overline{B}$-modules: $\mathcal{D} \cong \overline{B} \otimes_{S_Z} \overline{B}$.

**Proof.** This follows from Propositions 12 and 14 and [Ko2]. □
Similarly one obtains the following:

**Theorem 16.** With respect to the second order we have the following:

1. The algebra $\tilde{B}$ is a strong exact Borel subalgebra of both $\mathfrak{C}$ and $\mathfrak{D}$.
2. Under the assumptions of Proposition 12, the algebra $B$ is a strong exact $\Delta$-subalgebra of $\mathfrak{C}$.
3. Under the assumptions of Proposition 12, the algebra $\overline{B}$ is a strong exact $\Delta$-subalgebra of $\mathfrak{D}$.

**Proof.** Left to the reader. \qed

**Corollary 17.** Under the assumptions of Proposition 12, we have that $A_{\text{mod}}$ embeds into $F(\Delta_{\mathfrak{C}}^{1,r})$.

**Proof.** As $B$ is a Borel subalgebra of $\mathfrak{C}$, we have that $B_{\text{mod}}$ embeds into $F(\Delta_{\mathfrak{C}}^{1,r})$ via exact tensor induction. As $A$ is an idempotent subquotient of $B$ by construction, the claim follows. \qed

Similarly we have the following:

**Corollary 18.** Under the assumptions of Proposition 12, we have that $\text{mod} - A$ embeds into $F(\Delta_{\mathfrak{C}}^{2,r})$.

Let $B$ be the path algebra of the quiver

$$
\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots
$$

modulo the relations that any composition of $N$ arrows is zero.

**Corollary 19.** The category $B_{\text{mod}}$ embeds into $F(\Delta_{\mathfrak{C}}^{2,l})$.

**Proof.** The algebra $\tilde{B}$ consists of direct summands, each of which is isomorphic to $B$. As $\tilde{B}$ is a $\Delta$-subalgebra of $\mathfrak{C}$, we have that $\tilde{B}_{\text{mod}}$, and hence $B_{\text{mod}}$, embeds into $F(\nabla_{\mathfrak{C}}^{1,l})$. However, up to a shift, costandard modules in the first order are the same as standard modules in the second order by Corollary 3, so $F(\nabla_{\mathfrak{C}}^{1,l}) = F(\Delta_{\mathfrak{C}}^{2,l})$. This completes the proof. \qed

Similarly we have:

**Corollary 20.** The category $\text{mod} - B$ embeds into $F(\Delta_{\mathfrak{C}}^{1,r})$.

**Corollary 21.**

1. The category $\text{mod} - B$ embeds into $F(\Delta_{\mathfrak{D}}^{1,r})$.
2. The category $B_{\text{mod}}$ embeds into $F(\Delta_{\mathfrak{D}}^{2,l})$.
3. Under the assumptions of Proposition 12, we have that $\overline{A}_{\text{mod}}$ embeds into $F(\Delta_{\mathfrak{D}}^{1,l})$.
4. Under the assumptions of Proposition 12, we have that $\text{mod} - \overline{A}$ embeds into $F(\Delta_{\mathfrak{D}}^{2,r})$.  

5. Examples

Example 22 (An easy quiver algebra). Let $A$ be the path algebra of the following quiver:

$$
\begin{array}{c}
1 \\
\downarrow \ a \\
\downarrow \\
2
\end{array}
$$

Assume that (4) is the radical filtration of $A$. Let $e_1$ and $e_2$ be the idempotents of $A$, corresponding to the vertices 1 and 2, respectively. In this case the algebra $\mathcal{C}(A)$ is the path algebra of the following quiver:

$$
\begin{array}{ccccccc}
\cdots & e_0^1 & 1_1 & e_1^1 & e_2^1 & 1_2 & e_3^1 & \cdots \\
\downarrow & a^1 & \downarrow & a^2 & \downarrow & a^3 & \downarrow & a^4 \\
\cdots & e_0^2 & 2_1 & e_1^2 & e_2^2 & 2_2 & e_3^2 & \cdots
\end{array}
$$

modulo the ideal, generated by the following relations:

$$
e_{i+1}e_i = e_{i+1}e_2 = 0, \quad e_{i-1}a_i = a^{i+1}e_i,
$$

where $i \in \mathbb{Z}$.

We also have $A \cong \mathbb{K}$ (where $2 = \mathbf{1}$). In this case the algebra $\mathcal{D}(\mathbb{K})$ is the path algebra of the following quiver (the dual part $\mathcal{C}^*$ is depicted using the dotted arrows):

$$
\begin{array}{ccccccc}
\cdots & (e_0^1)^* & e_1^1 & (e_2^1)^* & e_3^1 & \cdots \\
\downarrow & (a^1)^* & \downarrow & (a^2)^* & \downarrow & (a^3)^* \\
\cdots & (e_0^2)^* & e_1^2 & (e_2^2)^* & e_3^2 & \cdots
\end{array}
$$

(here $x^i = (e_{i-1}a_i)^*$) modulo the ideal, generated by the relations (7), the relations saying that the product of any two dotted arrows is zero, and the relations defining the natural $\mathcal{C}$-bimodule structure on $\mathcal{C}^*$.

Example 23 (Schur algebras for $GL_2$). Let $A$ be a block of a Schur algebras for $GL_2$, say with $ap^k + r$ simple modules ($1 \leq a \leq p - 1, k \geq 0, 1 \leq r \leq p^k$). These have been extensively studied in [MT1] and [MT2] and in particular have been shown to be a hereditary idempotent subquotients of an infinite-dimensional symmetric quasi-hereditary algebra. Instead of taking an idempotent subquotient, one might also take a centralizer subalgebra $B$ which is again symmetric, such that it corresponds to the endomorphism ring of the first $ap^k$ projectives for the Schur algebra. From the explicit description in terms of quivers and relations in [MT2], it is easily seen that this has a $\mathbb{Z}$-grading, which coincides with the radical filtration, hence has semisimple subquotients. By [MV, Theorem 3.3], any connected finite-dimensional
self-injective positively graded algebra is rigid. Therefore we can apply Corollary 5 to obtain a symmetric quasi-hereditary algebra. This will however give an algebra that is significantly larger than the symmetric quasihereditary envelope given in [MT1, MT2].

Example 24 (Category $\mathcal{O}$). Let $\mathfrak{g}$ be a semi-simple finite-dimensional complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ be a parabolic subalgebra of $\mathfrak{g}$. Let $\mathcal{O}_\mathfrak{p}^0$ denote the principal block of the $\mathfrak{p}$-parabolic category $\mathcal{O}$ for $\mathfrak{g}$, and $A^\mathfrak{p}$ denote the endomorphism algebra of the multiplicity-free direct sum of all indecomposable projective-injective modules in $\mathcal{O}_\mathfrak{p}^0$.

The algebra $A^\mathfrak{p}$ is positively graded and symmetric (see [MS]) and simple $A^\mathfrak{p}$-modules are naturally indexed by the elements of some right cell for the Weyl group $W$ of $\mathfrak{g}$. In the special case $\mathfrak{g} = \mathfrak{sl}_n$, the parabolic subalgebra $\mathfrak{p}$ is given by some composition of $\mathfrak{n}$ and the algebra $A^\mathfrak{p}$ can be used to model the corresponding Specht module (for the symmetric group or Hecke algebra) via the action of some exact functors on $A^\mathfrak{p}-\text{mod}$, see [KMS]. The algebra $A^\mathfrak{p}$ has a simple preserving duality, which yields that all indecomposable projective $A^\mathfrak{p}$-modules are self-dual. Since the trace form on $A^\mathfrak{p}$ respects grading, it follows that this form induces a nondegenerate pairing between the components of the grading filtration of $A^\mathfrak{p}$ as required in the formulation of Theorem 4. Thus, from Theorem 4 it follows that the quasi-hereditary envelope $\mathfrak{E}(A^\mathfrak{p})$ of $A^\mathfrak{p}$ is symmetric and thus $A^\mathfrak{p}$ is a centralizer subalgebra of a symmetric quasi-hereditary algebra. It would be interesting to understand the algebra $\mathfrak{E}(A^\mathfrak{p})$. Note that the natural grading filtration on $A^\mathfrak{p}$ does not have to coincide with the radical filtration.

In the special case $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_+$, the algebra $\mathfrak{E}(A^\mathfrak{p})$ is closely related to the algebras from [MT1].

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