TWO MORE CHARACTERISATIONS OF NEARLY PSEUDOCOMPACT SPACE

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Abstract. In this paper we have obtained two more characterizations of nearly pseudocompact spaces.

1. INTRODUCTION

In this paper by a space we always mean Tychonoff unless otherwise mentioned. As usual βX and υX denote respectively the Stone-Čech compactification and Hewitt realcompactification of X. We know that the property of being pseudocompact can be described in terms of βX and υX; that is X is pseudocompact if and only if βX\X = υX\X. In the year 1980 Henriksen and Rayburn in [3] studied those spaces X where υX\X is dense in βX\X referred as nearly pseudocompact, obviously a generalization of pseudocompactness. In that paper they have extensively studied different properties of nearly pseudocompact space, furnishing two characterization of these spaces. Infact they have shown that [3], Theorem 3.10 X is nearly pseudocompact if and only if X can be expressed as X₁ \cup X₂, where X₁ is a regular closed almost locally compact pseudocompact subset and X₂ is regular closed anti-locally realcompact and int_X(X₁ ∩ X₂) = ∅ if and only if [3], Theorem 3.18 every countable family of pairwise disjoint non-empty regular hard subset of X has a limit point in X. Further John J. Schommer furnished few characterizations of nearly pseudocompact spaces in his paper [6], Theorem 1.3, Theorem 3.1, Corollary 3.2. In the year 2005 B. Mitra and S.K. Acharyya in their paper [5], Theorem 2.5, Theorem 3.4, Theorem 3.6, Theorem 3.7 produced few more characterizations of nearly pseudocompact spaces.

In this paper we have further shown two more characterizations of nearly pseudocompact space. In fact we have proved the following statements. X is nearly pseudocompact if and only if X does not contain any C-embedded copy of N which is hard in X if and only if any family of hard sets with finite intersection property has non-void intersection.

2. PRELIMINARIES

All the basic symbols, preliminary ideas, and terminologies are taken from the book of L. Gillman and M. Jerison, Rings of Continuous Functions [2]. Now we recall few notations which are used several times over here. If f ∈ C(X) or C*(X), we call the set Z(f) = \{x ∈ X : f(x) = 0\}, the zero set of f and its complement is called cozero set or cozero part of f which is denoted as cozf. Support of a continuous function is the closure of a set in X where the function does not vanish; that is, if f ∈ C(X), then cl_x(X\Z(f)) is the support of f. Let Y ⊆ X. If for every f ∈ C(Y)(or C*(Y)) there exists g ∈ C(X) such that g(y) = f(y) for all y ∈ Y, then Y is called C-embedded (or respectively C*-embedded) in X. When an f ∈ C(X) is unbounded on E, then E has a C-embedded copy of N along which f goes to ∞ [2], Theorem 1.21. Rayburn [1] introduced the notion of hard set in 1976.
**Definition 1.** A subspace $H$ of $X$ is called hard in $X$ if $H$ is closed in $X \cup K$, $K = \text{cl}_{\beta X}(vX \setminus X)$ where $\beta X$ and $vX$ are the Stone-Čech compactification and Hewitt realcompactification of $X$ respectively.

It immediately follows that every hard set is closed in $X$, but the converse is obviously not true. Again every compact subset of $X$ is hard, but the converse may not be true. In the year 1980, Henriksen and Rayburn in [3], Theorem 3.8 proved the following theorem in general set up.

**Theorem 2.** The followings are equivalent.
1. $X$ is nearly pseudocompact
2. Every hard set is compact
3. Every regular hard set is compact

In the year 2005, Mitra and Acharyya in [5], Theorem 2.5 furnished some characterizations of nearly pseudocompact space using the notion of $C_H(X)$ and $H_\infty(X)$ among them the following characterization is relevant to this paper.

**Theorem 3.** $X$ is nearly pseudocompact if and only if $C_H(X) \subseteq C^*(X)$, where $C_H(X) = \{ f \in C(X) | cl_X(X \setminus Z(f))$ is hard in $X$ $\}$.

In the same paper [5] under section three, the authors introduced an intermediate subring $\chi(X)$ of $C(X)$ to characterise nearly pseudocompact space. The ring $\chi(X) = C^*(X) + C_H(X)$, is the smallest subring of $C(X)$ containing $C^*(X)$ and $C_H(X)$, which is isomorphic with $C(Z)$ for some space $Z$. In the same paper [5], Theorem 2.8 it was proved that $v_{C_H}X = X \cup K$ and from the definition of $\chi(X)$ it follows that $v_{\chi}X = v_{C_H}X$ where for any subset $A$ of $C(X)$, $v_{\chi}X = \{ p \in \beta X : f^*(p) \in R, \forall f \in A \}$. So finally we have $v_{\chi}X = X \cup K$, where $K = cl_{\beta X}(vX \setminus X)$.

The following characterizations of nearly pseudocompact space in [5], Theorem 3.4 were proved using the notion of $\chi(X)$ and $v_{\chi}X$

**Theorem 4.** For a space $X$, the followings are equivalent.
1. $X$ is nearly pseudocompact.
2. $\chi(X) = C^*(X)$
3. $|\beta X \setminus v_{\chi}(X)| < 2^\omega$

Mitra and Das in their communicated paper [7] constructed nearly pseudocompact extension of a space, referred as nearly pseudocompactification of the space, with the help of special class of $\mathbb{z}$-ultrafilters, called $\mathbb{h}z$-ultrafilter.

**Definition 5.** An $\mathbb{h}z$-filter is a $\mathbb{z}$-filter, having a base consisting of hard-zero sets in $X$. Maximal $\mathbb{h}z$-filter is called $\mathbb{h}z$-ultrafilter.

Using the notion of $\mathbb{h}z$-filters and $\mathbb{h}z$-ultrafilters, Mitra and Das obtained the following another characterization of nearly pseudocompact space[7], Theorem 2.14 which has been used in the present paper.

**Theorem 6.** The followings are equivalent.
1. $X$ is nearly pseudocompact
2. Every $\mathbb{h}z$-filter is fixed
3. Every $\mathbb{h}z$-ultrafilter is fixed

3. **TWO MORE CHARACTERIZATIONS**

In this section we shall prove two more characterizations of nearly pseudocompact space. We know from [[2], Theorem 1.21] that a space $X$ is pseudocompact if and only if $X$ does not contain any $C$-embedded copy of $\mathbb{N}$. We have obtained the following theorem for nearly pseudocompact space in the similar fashion to the above statement for pseudocompact space.

**Theorem 7.** A space $X$ is nearly pseudocompact if and only if $X$ does not contain any $C$-embedded copy of $\mathbb{N}$ which is hard in $X$. 
Proof. Suppose X contains a C-embedded copy of N which is hard in X. So cl$_{\beta \chi} N \subseteq \beta X \setminus v_\chi X$. We know N is C-embedded in X ⇒ N is C*-embedded in X. Therefore cl$_{\beta \chi} N \subseteq \beta X \setminus v_\chi X \Rightarrow \beta X \setminus v_\chi X \setminus \beta N \subseteq \beta X \setminus v_\chi X$ i.e. $|\beta X \setminus v_\chi X| \geq 2^\omega \Rightarrow X$ is not nearly pseudocompact. Contrapositively, if X is nearly pseudocompact then X does not contain a C-embedded copy of N which is hard in X.

Conversely, suppose X does not contain a C-embedded copy of N which is hard in X. We have to show that X is nearly pseudocompact. Let X be not nearly pseudocompact. Then $C_H(X) \not\subseteq C^*(X)$; i.e. there exists a function $f \in C_H(X)$ but $f \notin C^*(X)$. So f is unbounded on $cl_X(X \setminus Z(f))$. So there exists a C-embedded copy D of N in $cl_X(X \setminus Z(f))$. As D is closed in X, it is closed in $cl_X(X \setminus Z(f))$. Since $f \in C_H(X)$, $cl_X(X \setminus Z(f))$ is hard in X and hence D is hard in X, which contradicts our first assumption. Hence X is nearly pseudocompact. □

It is well known that a space is compact if and only if any family of non-empty closed sets with finite intersection property has non-empty intersection. We have proved the following similar type of result for nearly pseudocompact space.

Theorem 8. X is nearly pseudocompact if and only if any family of non-empty hard sets with finite intersection property has non-void intersection.

Proof. Let X be nearly pseudocompact. Then every hard set is compact and any family of compact sets having finite intersection property has non-empty intersection. Hence the result follows.

Conversely, suppose any family of hard sets with finite intersection property has a non-void intersection. Let $\mathcal{F}$ be a $hz$-filter in X. So there exists a base $\mathcal{H}$ for $\mathcal{F}$ consisting of hard sets. Since $\mathcal{F}$ is a filter , it has finite intersection property. Then $\mathcal{H}$ has also finite intersection property and by our assumption $\cap \mathcal{H} \neq \phi$. Again $\cap \mathcal{H} \subseteq \cap \mathcal{F} \Rightarrow \cap \mathcal{F} \neq \phi$. Hence $\mathcal{F}$ being arbitrary, every $hz$-filter in X is fixed. Therefore X is nearly pseudocompact. □

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