Probing the Almeida-Thouless line away from the mean-field model

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Results of Monte Carlo simulations of the one-dimensional long-range Ising spin glass with power-law interactions in the presence of a (random) field are presented. By tuning the exponent of the power-law interactions, we are able to scan the full range of possible behaviors from the infinite-range (Sherrington-Kirkpatrick) model to the short-range model. A finite-size scaling analysis of the correlation length indicates that the Almeida-Thouless line does not occur in the region with non-mean-field critical behavior in zero field. However, there is evidence that an Almeida-Thouless line does occur in the mean-field region.

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I. INTRODUCTION

The behavior of spin glasses in a magnetic field is still controversial. While the infinite-range (mean-field) Sherrington-Kirkpatrick (SK) model\textsuperscript{2} has a line of transitions at finite field known as the Almeida-Thouless (AT) line\textsuperscript{3,4,5,6,7,8,9,10,11,12} it has not been definitely established whether an AT line occurs in more realistic models with short range interactions. Previous numerical studies\textsuperscript{2,7,8,9,10,11,12,13,14,15} have yielded conflicting results: some data support the existence of an AT line in short-range spin glasses while others claim its absence. Recently,\textsuperscript{11} a new approach using the correlation length\textsuperscript{12,13,14,15} at finite fields has been applied to the three-dimensional Edwards-Anderson Ising spin-glass model. The data of Ref.\textsuperscript{11} indicate that, even for small fields, there is no AT line in three-dimensional spin glasses.

II. MODEL, OBSERVABLES, AND NUMERICAL DETAILS

The Hamiltonian of the one-dimensional long-range Ising spin glass with random power-law interactions\textsuperscript{16,17} is given by

\[ \mathcal{H} = -\sum_{(i,j)} J_{ij} S_i S_j - \sum_i h_i S_i , \tag{1} \]

where \( S_i = \pm 1 \) represents Ising spins evenly distributed on a ring of length \( L \) in order to ensure periodic boundary conditions. The sum is over all spins on the chain and the couplings \( J_{ij} \) are given by

\[ J_{ij} = c(\sigma) \frac{\epsilon_{ij}}{r_{ij}^\sigma} , \tag{2} \]

where the \( \epsilon_{ij} \) are chosen according to a Gaussian distribution with zero mean and standard deviation unity, and \( r_{ij} = (L/\pi) \sin(|i-j|/L) \) represents the geometric distance between the spins on the ring.\textsuperscript{10,18,19} The power-law exponent \( \sigma \) characterizes the interactions and, hence, determines the universality class of the model. The constant \( c(\sigma) \) in Eq. (2) is chosen to give a mean-field transition temperature \( T_{MF}^c = 1 \), where

\[ (T_{MF}^c)^2 = \sum_{j \neq i} [J_{ij}^2]_{av} = c(\sigma)^2 \sum_{j \neq i} \frac{1}{r_{ij}^{2\sigma}} . \tag{3} \]

Here \( [\cdots]_{av} \) denotes an average over disorder.

In Eq. (1), the spins couple to site-dependent random fields \( h_i \) chosen from a Gaussian distribution with zero mean \( [h_i]_{av} = 0 \) and standard deviation \( [h_i^2]_{av}^{1/2} = H_R \). For a symmetric distribution of bonds, the sign of \( h_i \) can be “gauged away” so a uniform field is completely equivalent to a bimodal distribution of fields with \( h_i = \pm H_R \). While the AT line is usually studied for the case of a uniform field, the SK model with Gaussian random fields (as considered here) also shows an AT line. For short-range three-dimensional spin glasses it has been shown in Ref.\textsuperscript{11} that results for Gaussian-distributed random

The paper is organized as follows. In Sec. I\textsuperscript{11} we introduce in detail the model, observables, and numerical method used. In Sec. II\textsuperscript{11} we present our results, and in Sec. IV\textsuperscript{11} we summarize our findings.
The behavior is mean-field-like, while for $2 < \sigma \leq 3$ the critical behavior is non-mean field like. This behavior is summarized in Table I. Critical exponents depend continuously on $\sigma$ in the LR regime, but are independent of $\sigma$ in the SR regime.

To determine the existence of an AT line, we compute the two-point correlation length $\xi_L$ of the finite system is then given by

$$\xi_L = \frac{1}{2 \sin(k_{\text{min}}/2)} \left[ \frac{\chi_{\text{SG}}(0)}{\chi_{\text{SG}}(k_{\text{min}})} - 1 \right]^{1/(2\sigma - 1)},$$

where $k_{\text{min}} = (2\pi/L, 0, 0)$ is the smallest nonzero wave vector. The reason for the power $1/(2\sigma - 1)$ is that long wavelengths, we expect a modified Ornstein-Zernicke form

$$\chi_{\text{SG}}(k) \propto (v + k^{2\sigma - 1})^{-1}$$

for the long-range case, where $v$ is a measure of the deviation from criticality. It follows that the bulk correlation length $\xi$ diverges for $v \to 0$ like $v^{-1/(2\sigma - 1)}$.

The correlation length divided by the system size $\xi_L/L$ has the following scaling property:

$$\frac{\xi_L}{L} = \bar{X} \left( \frac{1}{\nu} (T - T_c(H_R)) \right),$$

where $\nu$ is the correlation length exponent and $T_c(H_R)$ is the transition temperature for a field of strength $H_R$. This behavior is similar to that of the Binder ratio but it shows a clearer signature of the transition as the data are not restricted to a finite interval.

In order to test equilibration of the Monte Carlo method, we also compute the link overlap $q_l$ given by

$$q_l = \frac{2}{N} \sum_{\langle i,j \rangle} \text{[Eq. (3)]}^{1/2}$$

where $T_c^{\text{MF}}$ is given by Eq. (3) and $\alpha$ and $\beta$ refer to two replicas of the system with the same disorder. In

| $\sigma$ | behavior |
|---------|----------|
| $\sigma = 0$ | SK model |
| $0 < \sigma \leq 1/2$ | IR |
| $1/2 < \sigma < 2/3$ | LR (mean field with $T_c > 0$) |
| $2/3 < \sigma \leq 1$ | LR (non-mean field with $T_c > 0$) |
| $1 < \sigma \leq 2$ | LR ($T_c = 0$) |
| $\sigma \geq 2$ | SR ($T_c = 0$) |

TABLE I: A summary of the behavior for different ranges of $\sigma$ in one space dimension and at zero field. IR means infinite range, i.e., $\sum_{i \neq j} J_{ij}$ diverges unless the bonds $J_{ij}$ are scaled by an inverse power of the system size. LR means that the behavior is dominated by the long-range tail of the interactions, and SR means that the behavior is that of a short-range system.
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in a field,
Monte Carlo method.

As shown in Fig. 1, when starting from a random spin
configuration, \( U(q, q) \) approaches its equilibrium value from
above while \( U(q, q) \) approaches its equilibrium value from below. Once \( U = U(q, q) \), the data do not change
by further increasing the number of Monte Carlo steps, which
shows that the system is in equilibrium. It is also
important to ensure that other observables are also in
equilibrium once \( U = U(q, q) \), and this is shown in the
inset to Fig. 1 for the case of the correlation length.

The simulations are done using the parallel tempering
Monte Carlo method. The method is not as efficient
in a field but nevertheless it performs considerably
better than simple Monte Carlo. In order to compute
the products of up to four thermal averages in Eq. (4)
without bias, we simulate four copies (replicas) of the
system with the same bonds and fields at each temper-
ate. Simulations are performed at zero field, as well
as at \( H_R = 0.1 \), a field that is considerably smaller than
the \( \sigma \)-dependent transition temperature \( T_c \). Parameters of the
simulations at zero and finite fields are presented in
Tables II and III, respectively.

### III. RESULTS

We first consider the case of zero field and take \( \sigma = 0.55, 0.65, 0.75, \) and 0.85. The values \( \sigma = 0.75 \) and
0.85 are in the non-MF region (see Table II) while \( \sigma = 0.55 \) is in the MF region and, furthermore, is close to the
value \( \sigma = 1/2 \) where the system becomes infinite
range. The value \( \sigma = 0.65 \) is close to the point \( \sigma = 2/3 \)
where the critical behavior changes from MF to non-MF.
The data are shown in Fig. 2. In all cases, the data
cross at a transition temperature which we determine as
\( T_c = 1.03(3) \) for \( \sigma = 0.55, 0.86(2) \) for \( \sigma = 0.65, 0.69(1) \)
for \( \sigma = 0.75, \) and 0.49(1) for \( \sigma = 0.85 \). Note that \( T_c \)
decreases continuously with increasing \( \sigma \) and is expected
to drop to zero at \( \sigma = 1 \). For the SK model \( \sigma = 0 \),
one has \( T_c = 1 \), essentially the result we find for \( \sigma = 0.55, \) so it is possible that \( T_c \) has little variation with \( \sigma \) for

### Table II: Parameters of the simulations for \( H_R = 0.0 \).

| \( \sigma \) | \( L \) | \( N_{sa} \) | \( N_{sw} \) | \( T_{min} \) | \( N_T \) |
|---|---|---|---|---|---|
| 0.55 | 32 | 5000 | 10240 | 0.405 | 15 |
| 0.55 | 64 | 5000 | 10240 | 0.405 | 15 |
| 0.55 | 128 | 5000 | 20480 | 0.405 | 15 |
| 0.55 | 256 | 5000 | 102400 | 0.405 | 15 |
| 0.55 | 512 | 5000 | 32768 | 0.630 | 11 |
| 0.65 | 32 | 5000 | 10240 | 0.405 | 15 |
| 0.65 | 64 | 5000 | 10240 | 0.405 | 15 |
| 0.65 | 128 | 5000 | 20480 | 0.405 | 15 |
| 0.65 | 256 | 5000 | 102400 | 0.405 | 15 |
| 0.65 | 512 | 5000 | 32768 | 0.630 | 11 |
| 0.75 | 32 | 5000 | 10240 | 0.405 | 15 |
| 0.75 | 64 | 5000 | 10240 | 0.405 | 15 |
| 0.75 | 128 | 5000 | 20480 | 0.405 | 15 |
| 0.75 | 256 | 5000 | 102400 | 0.405 | 15 |
| 0.75 | 512 | 5000 | 32768 | 0.630 | 11 |
| 0.85 | 32 | 5000 | 10240 | 0.405 | 15 |
| 0.85 | 64 | 5000 | 10240 | 0.405 | 15 |
| 0.85 | 128 | 5000 | 20480 | 0.405 | 15 |
| 0.85 | 256 | 5000 | 102400 | 0.405 | 15 |
| 0.85 | 512 | 5000 | 32768 | 0.630 | 11 |

### Table III: Parameters of the simulations for \( H_R = 0.1 \).

| \( \sigma \) | \( L \) | \( N_{sa} \) | \( N_{sw} \) | \( T_{min} \) | \( N_T \) |
|---|---|---|---|---|---|
| 0.55 | 32 | 5000 | 81920 | 0.100 | 26 |
| 0.55 | 64 | 5000 | 327680 | 0.100 | 26 |
| 0.55 | 128 | 5000 | 1310720 | 0.100 | 26 |
| 0.55 | 256 | 2000 | 1048576 | 0.405 | 15 |
| 0.55 | 512 | 2000 | 65536 | 0.760 | 9 |
| 0.65 | 32 | 5000 | 81920 | 0.100 | 26 |
| 0.65 | 64 | 5000 | 327680 | 0.100 | 26 |
| 0.65 | 128 | 5000 | 1310720 | 0.100 | 26 |
| 0.65 | 256 | 2000 | 1048576 | 0.195 | 20 |
| 0.65 | 512 | 2000 | 65536 | 0.500 | 13 |
| 0.75 | 32 | 5000 | 81920 | 0.100 | 26 |
| 0.75 | 64 | 5000 | 327680 | 0.100 | 26 |
| 0.75 | 128 | 5000 | 1310720 | 0.100 | 26 |
| 0.75 | 256 | 2000 | 8388608 | 0.100 | 26 |
| 0.75 | 512 | 2000 | 524288 | 0.100 | 13 |
| 0.85 | 32 | 5000 | 81920 | 0.100 | 26 |
| 0.85 | 64 | 5000 | 327680 | 0.100 | 26 |
| 0.85 | 128 | 5000 | 1310720 | 0.100 | 26 |
| 0.85 | 256 | 2000 | 16777216 | 0.100 | 26 |
FIG. 2: (Color online) Each figure shows data for $\xi_L/L$ vs $T$ for $H_R = 0$ for different system sizes, for a particular value of $\sigma$. For all values of $\sigma$, the data cross indicating that there is a spin-glass transition at finite temperature.

Next we consider $H_R = 0.10$ and show the data in Fig. 3. The results for $\sigma = 0.75$ and 0.85, which are in the non-mean-field regime, show no sign of a transition. However, the data for $\sigma = 0.55$ do show a signature of a transition at $T_c = 0.96(2) < T_c(H_R = 0)$. Whether this would persist up to infinite system sizes is not clear, but it certainly cannot be ruled out. The results for $\sigma = 0.55$ show that the method used here is capable of detecting an AT line in the presence of a field. For $\sigma = 0.65$, the data shows a marginal behavior. Since $\sigma = 0.65$ is close to the value of $2/3$ which separates MF and non-MF behavior in zero field, this marginal behavior may indicate that $2/3$ is also the borderline value below which an AT line occurs. An alternative possibility, which we cannot rule out, is that an AT line only occurs in the infinite-range region ($\sigma < 1/2$) but that as $\sigma$ is decreased toward $1/2$, one needs to study larger system sizes to see the absence of a transition.

IV. CONCLUSIONS

We have considered a one-dimensional spin-glass model with long-range interactions that allows the universality class to be changed from the infinite-range limit to the short-range case by tuning the power-law exponent $\sigma$ of the interactions. We find that there does not appear to be an AT line in a field for models with $\sigma$ in the range where there is non-mean-field critical behavior at zero field. However, in the region of $\sigma$ that is not infinite-range but has mean-field critical behavior ($1/2 < \sigma < 2/3$),
FIG. 3: (Color online) Each figure shows results for $\xi_L/L$ vs $T$ for $H_R = 0.10$ for different system sizes, for a particular value of $\sigma$. For $\sigma = 0.55$, the data cross at $T_c = 0.96(2)$ showing that an AT line seems to be present. For $\sigma = 0.65$, the data show close to marginal behavior. This may indicate that $\sigma = 0.65$ is close to the borderline value for having an AT line. For $\sigma = 0.75$ and $\sigma = 0.85$, the data do not cross for any temperature down to $T = 0.10$, which is considerably smaller than the zero field transition temperatures. This indicates that there is no AT line. Overall the results indicate that, on increasing $\sigma$, the AT line disappears.

There does appear to be an AT line.

These conclusions rely on extrapolating from finite sizes to the thermodynamic limit. It would be particularly interesting to know if the conclusion that there is an AT line for $\sigma = 0.55$ persists in the thermodynamic limit, or whether the AT line really only occurs in the infinite-range case ($\sigma < 1/2$). It is possible that as $\sigma$ is decreased, larger sizes or larger values of $H_R$ are needed to probe the asymptotic behavior. Therefore, it would be desirable to simulate a range of values of $H_R$, especially for $\sigma = 0.55$. We have some results for $H_R = 0.2$ for a relatively small number of samples and for sizes only up to $L = 96$, which indicate a crossing at a lower temperature than for $H_R = 0.1$. However, we are unable to carry out a systematic study of the dependence on $H_R$ because the results presented above already required considerable computer time, and the parallel tempering algorithm becomes less efficient at larger fields.

Making an analogy between the one-dimensional long-range model for different values of $\sigma$ and short-range models for different values of space dimension $d$, we infer that there is no AT line for short-range spin glasses in the non-mean-field regime, i.e., below the upper critical dimension $d_u = 6$. However, there may be an AT line
above the upper critical dimension. Speculations along these lines have also been made very recently by Moore.\textsuperscript{25}

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Note that the distance $r_{ij}$ between two spins $i$ and $j$ on the chain is determined by $r_{ij} = 2R\sin(\alpha/2)$, where $R$ is the radius of the chain and $\alpha$ is the angle between the two sites on the circle. The previous expression can be rewritten in terms of the system size $L$ and the positions of the spins to obtain $r_{ij} = (L/\pi)\sin(\pi |i-j|/L)$.

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