JACOBI AND KUMMER’S IDEAL NUMBERS

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ABSTRACT. In this article we give a modern interpretation of Kummer’s ideal numbers and show how they developed from Jacobi’s work on cyclotomic, in particular the methods for studying “Jacobi sums” which he presented in his lectures on number theory and cyclotomy in the winter semester 1836/37.

Dem Andenken an Herbert Pieper (1943 – 2008) gewidmet.

Every mathematician nowadays is familiar with the notion of an ideal in a ring. Ideals were introduced by Dedekind when he generalized Kummer’s ideal numbers to general number fields. Kummer had invented ideal numbers in order to restore some kind of unique factorization in cyclotomic rings, and in the literature one usually finds the following characterizations of Kummer’s invention:

(1) Kummer’s idea was brilliant and new; there were no traces of it in the number theoretical work of his predecessors: it appeared out of the blue and solved the “problem” of nonunique factorization in a way reminiscent of Alexander the Great’s solution of the Gordian knot.

(2) Kummer’s definition of an ideal prime is difficult to understand and not easy to use in practice.

These opinions seem to be generally accepted: Dickson, to quote a typical example, wrote in connection with his review of Reid’s textbook on algebraic number theory:

He [Reid] wisely did not attempt to give any idea of Kummer’s ideal numbers, the operations on which are so delicate that one must use the utmost circumspection (as remarked by Dedekind in his important historical papers in Darboux’s Bulletin).

This explains why there are hardly any expositions of Kummer’s theory of ideal numbers; among the few exceptions are Edwards’ book and Soublin’s article. Nevertheless, Kummer’s articles on the arithmetic of cyclotomic number fields are fairly easy to read by simply replacing the expression “ideal number” by the word “ideal”. This works very well except for his work on the foundations of his theory of ideal numbers: the problems Kummer had to overcome when he introduced ideal numbers seem to differ fundamentally from the obstacles Dedekind had to deal with when he introduced ideals.

Let us now summarize the content of this article:

1Eine Würdigung von E. Knobloch zu Piepers 65. Geburtstag findet sich auf http://www.uni-potsdam.de/u/romanistik/humboldt/hin/hin16/knobloch.htm
2The majority of Kummer’s articles is written in German, only a few in Latin and French.
• We start by giving a brief summary of Jacobi’s lectures on number theory and cyclotomy in the winter semester 1836/37; as we will show, they played a major role in the development of Kummer’s notion of ideal numbers.
• Next we address the role of Fermat’s Last Theorem in Kummer’s early work.
• Afterwards we explain the “Jacobi maps”, certain substitutions used by Jacobi that turned out to be the key idea later used by Kummer when he invented ideal numbers.
• In our discussion of Kummer’s ideal numbers we offer a translation of ideal numbers into the modern mathematical language which is more faithful than the simple substitution of “ideal” for “ideal number”. We try to correct the historical picture of the development of Kummer’s ideal numbers by showing that the notion of ideal numbers used by Kummer is perfectly natural, and that it is based to a large degree on ideas put forth by Jacobi in his investigations in cyclotomy. Moreover, a theory of divisibility built on these ideas is hardly more complicated than Dedekind’s approach; Jung, in his introductory lectures on the arithmetic of quadratic number fields, uses an approach that is very close to Kummer’s first attempt at defining ideal numbers, and so do Stevenhagen & Lenstra in.
• Finally we discuss the relevance of the notion of integral closure for Kummer’s work by looking carefully at the concept of singularity in number theory and algebraic geometry.

It seems that the importance of Jacobi’s Königsberg lectures on number theory for the development of ideal numbers has not been recognized before. Apparently Kummer had carefully studied a copy of Jacobi’s lectures. We know from Jacobi himself that Kummer had access to these lecture notes: In 1846, Jacobi had a note from 1837 reprinted in Crelle’s Journal and added a footnote in which he said:

Diesen aus vielfach verbreiteten Nachschriften der oben erwähnten Vorlesung (an der Königsberger Universität) auch den Herren Dirichlet und Kummer seit mehreren Jahren bekannten Beweise sind neuerdings von Hrn. Dr. Eisenstein im 27ten Bande des Crelleschen Journals auf S. 53 publicirt worden.

Whether Eisenstein really had seen Jacobi’s lecture notes prior to 1846 is open to debate; most (if not all) historians of mathematics seem to agree that Eisenstein developed his proofs of quadratic, cubic and quartic reciprocity laws independently from Jacobi. Later, both Kummer and Eisenstein employed (without attribution) some form of p-adic development of the logarithm, which first appeared in Jacobi’s lectures.

1. Jacobi’s Königsberg Lectures

In his lectures on number theory during the winter semester 1836/37, Jacobi introduced his audience to the basic theory of Gauss and Jacobi sums (at that time, this was called the theory of cyclotomy (Kreisteilung)), and applied these techniques to derive the quadratic, cubic and quartic reciprocity laws, as well as results on the

3See Bölling, Edwards, Neumann, and Soublin.
4These proofs, which were known for several years to Mr. Dirichlet and Mr. Kummer, among others, through widely circulated notes of the lectures (at the University of Königsberg) mentioned above, have lately been published by Dr. Eisenstein on p. 53 of the 27th volume of Crelle’s Journal.
representation of primes by quadratic forms that led him to conjecture Dirichlet’s class number formula for binary quadratic forms of negative prime discriminants.

Today, these results are proved using the ideal decomposition of Gauss and Jacobi sums. The fact that Jacobi’s proofs are essentially equivalent to the modern proofs implies that he must have possessed a technique that allowed him to express the essential content of the prime ideal factorizations of Gauss and Jacobi sums. We will say more on this in Sections 4 and 5, and now turn to the content of Jacobi’s lectures.

In the first 5 lectures, Jacobi presents elementary number theory: congruences, primes, Euler’s phi function, the theorem of Euler-Fermat, quadratic residues, and the Legendre symbol. The second part deals with cyclotomy: he introduces Gauss and Jacobi sums (without their modern names, of course), develops their basic properties, and explains the connections between Gauss sums and certain binomial coefficients; in particular, he proves Gauss’s famous congruence $2a \equiv \pm \left(\frac{2n}{p}\right) \mod p$ for primes $p = a^2 + 4b^2 = 4n + 1$. The third part of the lectures is dedicated to applications of cyclotomy to number theory. Jacobi begins with his proof of the quadratic reciprocity law (the one Legendre included in his Théorie des Nombres, and which led to the priority dispute between Jacobi and Eisenstein that began with Jacobi’s footnote quoted above), then derives results by Dirichlet on quartic residues as well as the full quartic reciprocity law. After briefly discussing cubic residues and proving the cubic reciprocity law, Jacobi then shows that if $\lambda \equiv 3 \mod 4$ and $p = \lambda n + 1$ are primes (this implies that $\left(\frac{-\lambda}{p}\right) = +1$), then $4p^h = x^2 + \lambda y^2$ for integers $x, y$ and some positive integer $h$ that can be expressed as a sum of Legendre symbols, and which Jacobi conjectured to be equal to the number of classes of binary quadratic forms with discriminant $-\lambda$. In the last few lectures, he deals with similar problems for primes $\lambda \equiv 1 \mod 4$.

It is remarkable how quickly Jacobi led his audience from the basic facts of elementary number theory right into the middle of research problems he was working on.

Jacobi’s principal technique for studying Gauss and Jacobi sums were certain substitutions, whose key role in the era before Kummer’s ideal numbers was emphasized by Frobenius [35, p. 117–118]:

> Als Cauchy, Jacobi und Kummer angefangen hatten, die Untersuchungen von Gauss über complexe Zahlen auf allgemeinere aus Einheitswurzeln gebildete algebraische Zahlen auszudehnen, ergab sich das unerwünschte Resultat, daß in diesem Gebiete zwei Zahlen nicht immer einen größten gemeinsamen Divisor besitzen, und daß Produkte unzerlegbarer Factoren einander gleich sein können, ohne daß die Factoren einzeln übereinstimmen. Die Gleichheit solcher Produkte konnte man daher immer nur durch besondere Kunstgriffe beweisen, zu denen namentlich der gehörte, durch Substitution gewisser rationalen Zahlen für die algebraischen die untersuchten Gleichungen in Congruenzen zu verwandeln. Mit den Methoden,

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5 Many of the results that Jacobi presented in his lectures had also been obtained around 1830 by Cauchy, who published his theory in a long series of articles [5] in 1840. Lebesgue [31] later gave simplified proofs for the main results of Jacobi and Cauchy “on just a few pages”, as he proudly remarked.
We remark that the first part of Kronecker’s thesis (written under the supervision of Kummer) deals with the basic arithmetic of cyclotomic fields. The following result is related to the finiteness of the class number of cyclotomic number fields once Kummer had introduced ideal numbers and the class group: Let $M_\lambda$ denote the maximum of the norm of \( x + x_1 \varepsilon + \ldots + x_{\lambda-1} \varepsilon^{\lambda-1} \) as \(-1 \leq x_j \leq 1\) (here $\varepsilon$ is a primitive $\lambda$th root of unity). Then for any prime $p$ there is an integer $n < M_\lambda$ such that $np$ is a norm from $\mathbb{Q}(\varepsilon)$. Kronecker observes that this is analogous to the finiteness of the number of reduced forms. In the second part of his thesis, Kronecker proved “Dirichlet’s” unit theorem for cyclotomic extensions.

2. Kummer and Fermat’s Last Theorem

The story according to which Kummer, at the beginning of his career, gave a proof of Fermat’s Last Theorem in which he erroneously assumed unique factorization in the number rings $\mathbb{Z}[\alpha]$ of $\lambda$-th roots of unity, probably first appeared in the “Gedächtnisrede” on Kummer by Hensel [35], and is now believed to be false. Indeed, Edwards [12, 13, 14], Neumann [34] and Bölling [4] have shown that Kummer’s first article on cyclotomy dealt with the factorization of primes $\lambda m + 1$ in the rings $\mathbb{Z}[\alpha]$ of $\lambda$-th roots of unity, and that his (false) result implied unique factorization in $\mathbb{Z}[\alpha]$.

Hensel [35, p. 93] even claimed that Kummer developed his theory of ideal numbers only because of Fermat’s Last Theorem, and this is definitely not true (see also [6]). The weaker claim that Kummer invented his ideal numbers in connection with his work on Fermat’s Last Theorem is a story that perhaps originated in a short note by Kronecker [26], who claimed

So führte das Reciprocitätsgesetz für quadratische Reste schon zur weiteren Ausbildung der Theorie der Kreisteilung, und der berühmte Fermat’sche Satz gab Hrn. Kummer vor etwa dreissig Jahren die hauptsächlichste Anregung zu jenen von so glücklichem Erfolg gekrönten Untersuchungen, auf denen das Reuschle’sche Werk basirt und deren Weiterförderung es zugleich gewidmet ist.

Kummer himself left no doubt as to which problems motivated his work; in [28], he writes

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6When Cauchy, Jacobi and Kummer started to extend Gauss’s investigations on complex numbers to general algebraic numbers formed with roots of unity, they came across the unpleasant fact that in this domain two numbers do not always have a greatest common divisor, and that products of irreducible factors can be equal without the factors being the same. The equality of such products could be proved only by resorting to certain tricks, notably the one that turns the equations under investigation into congruences by substitutions of certain rational numbers for the algebraic numbers. A large part of Kronecker’s dissertation deals with the methods for mastering such problems.

7In Jacobi’s and Kummer’s notation, $\lambda$ is an odd prime number and $\alpha$ a primitive root of the equation $\alpha^\lambda = 1$. The ring $\mathbb{Z}[\alpha]$ consists of all $\mathbb{Z}$-linear combinations of powers of $\alpha$.

8Thus the reciprocity law for quadratic residues led to the further development of the theory of cyclotomy, and Fermat’s famous theorem was thirty years ago Kummer’s main motivation for his successful investigations on which the work of Reuschle is built and to whose further advancement it is dedicated.
Thus in 1845, right after he had worked out the basic theory of ideal numbers, Kummer mentions the theory of cyclotomy and power residues as the driving force behind his work. But when Kronecker, who had turned from Kummer’s pupil to his closest friend, says that applications to Fermat’s Last Theorem had been on Kummer’s mind during his work on ideal numbers, we cannot simply dismiss this as nonsense. In fact, Kummer told Kronecker, in his letter from April 2, 1847, that he has found a proof of Fermat’s Last Theorem for a certain class of exponents, and writes:

> Der obige Beweis ist erst drei Tage alt, denn erst nach Beendigung der Recension fiel es mir ein wieder einmal diese alte Gleichung vorzunehmen, und ich kam diesmal bald auf den richtigen Weg.

We know from Kummer’s letters that he had discussed his ideal numbers with both Dirichlet and Jacobi, and although there cannot be any doubt that Kummer drew his motivation for introducing ideal numbers from Jacobi’s work on Jacobi sums, it is hard to believe that Dirichlet, who had proved Fermat’s Last Theorem for the exponents $n = 5$ and $n = 14$, failed to point out possible applications of unique factorization into ideal primes to the solution of Fermat’s Last Theorem. The fact that Kummer never mentioned Fermat’s Last Theorem as a possible motivation for his theory of ideal numbers before 1847 is also not surprising; if he was convinced that such an application was possible, it was natural for him to keep it to himself until he had worked out a proof.

Kummer found such a proof (for primes satisfying certain conditions, which, as he later showed, hold for regular primes) at the end of March 1847. At the beginning of March 1847, Dirichlet received a letter from his friend Liouville concerning Lamié’s attempted proof of Fermat’s Last Theorem; Liouville inquired what Dirichlet knew about unique factorization in cyclotomic rings of integers. It is not known whether Kummer had seen this letter before he took up the Fermat equation at the end of March (I think he did); what we do know is that Kummer sent his proof to Dirichlet on April 11, and that he sent a letter to Liouville on April 28. Kummer had already written a short note [27] on the Fermat equation $x^{2p} + y^{2p} = z^{2p}$ in 1837, but he had only used elementary means there. Kronecker’s statements make it plausible that Kummer was well aware of possible applications of the theory of cyclotomy to the proof of Fermat’s Last Theorem, and that he had looked at this problem occasionally while he developed his theory of ideal numbers. Nevertheless,
Fermat’s Last Theorem did apparently not play a decisive role in Kummer’s work before Liouville’s letter from March 1847. 

Kronecker also knew about Fermat’s Last Theorem very early on: in fact he had used the following claim as the last thesis for his disputation at his Ph.D. defense:

\[
\text{Fermatius theorema suum inclytum non demonstravit.}\]

3. Quadratic Forms vs. Quadratic Number Fields

In this section we will address two puzzling questions that do not seem to be related at first:

(1) Both Kummer and Kronecker knew that Kummer’s theory of ideal numbers, when applied to numbers of the form \(a + b\sqrt{m}\), is essentially equivalent to Gauss’s theory of quadratic forms of discriminant \(4m\). Yet it was Dedekind who worked out this theory, and he did so as late as 1871! Why wasn’t this done sooner?

(2) It is clear from the reactions of Jacobi, Dirichlet and Eisenstein to Kummer’s retraction of his manuscript that all three were fully aware of the failure of unique factorization in cyclotomic (and probably also in quadratic) rings of integers. Why were they all silent on this topic?

It was suggested that the reason why e.g. Kummer did not develop a theory of ideal numbers in quadratic fields was the problem coming from Kummer’s choice of the ring \(\mathbb{Z}[\sqrt{m}]\): this is not the maximal order in \(\mathbb{Q}(\sqrt{m})\), and we will see below that this makes his theory of ideal primes break down. This argument, however, is not fully convincing: in his proof of the \(p\)-th power reciprocity law, Kummer had to study ideal classes in certain orders of Kummer extensions \(\mathbb{Q}(\zeta_p, \sqrt{\mu})\), and these orders were also not maximal. Kummer avoided the problems caused by the primes dividing \(p\mu\) by simply excluding them and restricting his attention to elements coprime to \(p\mu\).

Dirichlet, in his proofs of Fermat’s Last Theorem for the exponents \(n = 5\) in 1828 [9] and \(n = 14\) in 1832 [10], did use algebraic numbers of the form \(a + b\sqrt{5}\) and \(a + b\sqrt{-7}\) for integers \(a, b\), but for deriving their basic properties he employed the theory of quadratic forms. Also in 1832, Gauss published his second memoir [19] on the theory of biquadratic residues. In this article, Gauss proved that \(\mathbb{Z}[i]\) is factorial and that \(\mathbb{Z}[i]\) is Euclidean, but the proof of unique factorization is not based on the Euclidean algorithm but rather on the fact that the binary quadratic forms with discriminant \(-4\) have class number 1. Dirichlet, in his article [11] on the quadratic reciprocity law in \(\mathbb{Z}[i]\), essentially copies Gauss’s proof. Only ten years later Dirichlet remarked that domains with a Euclidean algorithm have unique factorization.

The reason why Gauss, Dirichlet, Jacobi and Eisenstein preferred the theory of forms was that this was a perfectly general theory, whereas arguments based on unique factorization only worked in very special cases. This also seems to be the primary reason why the theory of ideal numbers in quadratic number fields was not seen as an important contribution to mathematics; the raison d’être of ideal numbers was their role in proofs of reciprocity laws and Fermat’s Last Theorem, which were based on cyclotomic number fields, and quadratic number fields were not yet studied for their own sake. Dirichlet’s class number formula was stated in

\[\text{Fermat did not prove his famous theorem.}\]
terms of binary quadratic forms, and even the class number formula for quadratic extensions of $\mathbb{Q}(i)$ was proved using the language of quadratic forms with coefficients from $\mathbb{Z}[i]$. But when Dirichlet learned from Kummer that he had found a substitute for unique factorization in general cyclotomic fields, he must have realized the potential of this theory right away.

4. Jacobi Maps

After Gauss had given two proofs (his fourth and sixth) of the quadratic reciprocity law using Gauss sums, it became clear that their generalization was the key to higher reciprocity laws. These Gauss and Jacobi sums for characters of higher order were studied by Jacobi, who first collected their basic properties.

In order to describe Jacobi’s results on Jacobi sums, let us first explain his notation. Let $p = m\lambda + 1$ denote a prime number, $g$ a primitive root modulo $p$, $\alpha$ a root of the equation $\alpha^\lambda = 1$, and $x$ a root of the equation $x^p = 1$. Then

$$(\alpha, x) = x + \alpha x g + \alpha^2 x g^2 + \ldots + \alpha^{p-2} x g^{p-2}$$

is a “Gauss sum”, whose $\lambda$-th power does not depend on $x$:

$$(\alpha, x)^\lambda \in \mathbb{Z}[\alpha].$$

When $x$ is fixed, or when he is studying expressions like $(\alpha, x)^\lambda$ that do not depend on $x$, Jacobi simply writes $(\alpha) = (\alpha, x)$.

Now assume that $r$ is a primitive root of the equation $r^{p-1} = 1$ (put $\alpha = r$ and $m = 1$ above). Then

$$\psi_r = (r^i r^k) (r^{i+k})$$

is a “Jacobi sum”, which also is independent of $x$. Its main properties are

$$\psi_r \cdot \psi(r^{-1}) = p,$$

as Jacobi proved in the XIIIth lecture, and the fact that $\psi_r$ is an element of $\mathbb{Z}[r]$ (see [24, XXVI. Vorl.]):

Die Funktion $\psi_r$ besteht aus ganzen positiven Zahlen, welche in die verschiedenen Potenzen von $r$ multiplicirt sind.

Jacobi then continues:

Wir wollen hier für $r$ eine ganze Zahl $g$ setzen, welche primitive Wurzel der Kongruenz $g^{p-1} \equiv 1 \pmod{p}$ ist. Dadurch ändern sich unsere Gleichungen nur so, dass sie Kongruenzen in bezug auf den Modul $p$ werden.

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13Eventually, however, it turned out that the decomposition of Gauss sums only gives a piece of the reciprocity law for $\ell$-th powers, namely Eisenstein’s reciprocity law. This is enough to derive the full version for cubic and quartic residues, but for higher powers, Kummer had to generalize Gauss’s genus theory from quadratic forms to class groups in Kummer extensions of cyclotomic number fields.

14Cauchy also studied these sums, but his lack of understanding higher reciprocity kept him from going as far as Jacobi did.

15The function $\psi_r$ consists of positive integers, multiplied by the different powers of $r$.

16We now want to substitute an integer $g$ for $r$, which is a primitive root of the congruence $g^{p-1} \equiv 1 \pmod{p}$. This will change our equations only in so far as they now become congruences with respect to the modulus $p$. 
That “equations become congruences” is Jacobi’s way of expressing the fact that this substitution commutes with addition and multiplication, so we can translate it into modern language as

\[ \text{“the substitution } r \mapsto g \mod p \text{ induces a ring homomorphism } \phi_g : \mathbb{Z}[r] \to \mathbb{Z}/p\mathbb{Z}. \]

Jacobi now writes \( \psi_g \) for the residue class mod \( p \) he gets by substituting \( g \mod p \) for \( r \), and proves the fundamental congruence

\[
(1) \quad \psi_g \equiv \begin{cases} 0 & \text{if } i + k < p - 1, \\ \frac{(2(p-1)-1-k)!}{(p-1-i)(p-1-k)!} \pmod p & \text{if } i + k > p - 1, \end{cases}
\]

where \( i, k \) denote integers \( 0 < i, k < p - 1 \).

For studying e.g. Jacobi sums of order \( \lambda \) for some \( \lambda \mid (p - 1) \), Jacobi writes \( p - 1 = m \lambda \) and replaces \( r \) by \( \alpha = r^m \) and \( g \) by \( g^m \). Jacobi’s fundamental congruence \[ \text{(1)} \]
then shows that \( \psi g^m \) is divisible by \( p \) if and only if \( im + km < \lambda \).

5. A Modern Interpretation of Jacobi’s Results

Let us now see why Jacobi’s fundamental congruence \[ \text{(1)} \]
implies the prime ideal factorization of Jacobi sums (in the following, we assume familiarity with basic properties of character sums; see e.g. \[ \text{(2)} \].

Consider a prime \( p = m \lambda + 1 \), a primitive root \( g \mod p \), and a primitive \( \lambda \)-th root of unity \( \zeta \). Then there is a unique character \( \chi \) of order \( \lambda \) (a surjective homomorphism of \( (\mathbb{Z}/p\mathbb{Z})^\times \to \langle \zeta \rangle \)) such that \( \chi(g) = \zeta \). Jacobi’s functions

\[
\psi \alpha = \psi_{i,k} \alpha = \frac{(\alpha^i)(\alpha^k)}{(\alpha^{i+k})}
\]

for \( \lambda \)-th roots of unity \( \alpha = r^m \) then correspond to our Jacobi sums

\[
J(\chi^i, \chi^k) = -\sum_{t=1}^{p-1} \chi^i(t) \chi^k(1-t) = -\psi_{i,k} \alpha
\]

(this is the sign convention used in \[ \text{(2)} \]; in the notation used by \[ \text{(2)} \], we have \( J(\chi^i, \chi^k) = \psi_{i,k}(\alpha) \) with \( J(\chi^i, \chi^k) \in \mathbb{Z}[\zeta] \).

As a special case consider the character \( \chi \) of order \( \lambda = 4 \) defined by the quartic residue symbol \( \chi = \left[ \frac{\pi}{a} \right] \mod p \). We have \( \chi(g) \equiv g^{(p-1)/4} \mod \pi \), and we can choose the primitive root \( g \mod p \) in such a way that \( \chi(g) = i \). With these normalizations, we find \( \psi \alpha = -J(\chi, \chi) \) for \( i = k = \frac{p-1}{4} \), and \( \psi \alpha = -J(\chi^3, \chi^3) \) for \( i = k = \frac{3(p-1)}{4} \). Jacobi’s congruence implies that \( J(\chi, \chi) \equiv 0 \mod \pi \) and \( \pi \nmid J(\chi^3, \chi^3) \) (the actual congruence gives, as Jacobi observes, a proof of Gauss’s result that, for primes \( p = 4m + 1 \), we have \( p = a^2 + b^2 \), where the odd integer \( a \) is determined (up to sign) by the congruence \( a \equiv \frac{1}{2}(2m) \mod p \). Since Jacobi sums have absolute value \( \sqrt{p} \), this implies the “prime ideal factorization” \( J(\chi, \chi) = (\pi) \) of the quartic Jacobi sum.

In general, write \( p = m \lambda + 1 \), let \( \zeta = r^m \) be a primitive \( \lambda \)-th root of unity, and consider a character of order \( \lambda \) on \( \mathbb{Z}[\zeta] \) with \( \chi(g) = \zeta \). Letting \( p \) denote the prime ideal \( p = (p, \zeta - g^m) \) we find \( (\frac{\pi}{p}) \equiv g^m \equiv \zeta \mod \pi \) and thus \( (\frac{\pi}{p}) = \zeta \) for the \( \lambda \)-th power residue symbol \( (\cdot)^{\lambda} \). With this choice of \( p \), we therefore have \( \chi = (\frac{\pi}{p}) \).

Jacobi’s congruence \[ \text{(1)} \] then shows \( p \mid J(\chi^i, \chi^k) \) if and only if \( 0 < 2t < \lambda \). Let \( \sigma_j \) denote the automorphism \( \zeta \mapsto \zeta^j \) of \( \mathbb{Q}(\zeta)/\mathbb{Q} \); then \( p \mid J(\chi^i, \chi^k) \) if and only if
This implies that $J(\chi, \chi)$ is divisible by all the prime ideals $\sigma_t^{-1}p$ with $0 < 2t < \lambda$. Since these prime ideals are pairwise disjoint, and since $J(\chi, \chi)$ is an algebraic integer with absolute value $\sqrt{p}$, we conclude that

$$(J(\chi, \chi)) = p^s, \quad s = \sum_{t=1}^{\lambda-1} \left\lfloor \frac{\lambda - 2t}{\lambda} \right\rfloor \sigma_t^{-1}$$

is the complete prime ideal factorization of the Jacobi sum $J(\chi, \chi)$. Replacing $t$ by $-t$ in this summation gives [32, Cor. 11.5].

Jacobi maps (replacing roots of unity in $\mathbb{C}$ with roots of unity in $\mathbb{F}_p$) were a tool that allowed Jacobi to state and prove results that we would describe using prime ideals. As we will see in the next section, Jacobi maps were indeed used by Kummer for his first tentative definition of ideal numbers. Not only that, after Kummer had worked out the factorization of these Jacobi sums into ideal prime factors, he even remarked [29, p. 362]

Dieses Jacobische Resultat gibt unmittelbar die idealen Primfaktoren der complexen Zahl $\psi_k(\alpha)$.

Here Kummer’s $\psi_k(\alpha)$ is our $-J(\chi, \chi^k)$, and the result by Jacobi alluded to is his fundamental congruence (1); for its statement, Kummer refers to the publications [23] from 1837 and 1846, and then claims (see [29, p. 361]) that Jacobi proved this “at the place cited”. There are, however, no proofs in [23], just the statement

Die Beweise dieser Sätze konnten in den vergangenen Wintervorlesungen ohne Schwierigkeit meinen Zuhörern mitgetheilt werden.

followed by the footnote directed at Eisenstein that we quoted in the Introduction. Since Kummer apparently knew Jacobi’s proofs, he must have read them in a copy of Jacobi’s lectures in number theory.

6. Kummer’s Ideal Numbers

Let $\lambda$ be a prime and $\alpha$ a primitive root of the equation $\alpha^\lambda = 1$. The elements of the ring $\mathbb{Z}[\alpha]$ can be written as polynomials

$$f(\alpha) = a_0 + a_1 \alpha + \ldots + a_{\lambda-1} \alpha^{\lambda-1}$$

with coefficients $a_j \in \mathbb{Z}$. For an integer $k$ coprime to $\lambda$, the polynomial $f(\alpha^k)$ is then the conjugate of $f(\alpha)$ with respect to the automorphism $\sigma_k: \alpha \mapsto \alpha^k$. The norm of $f(\alpha)$ is the product $f(\alpha)f(\alpha^2) \cdots f(\alpha^{\lambda-1})$.

In [28], Kummer then explains that there are several possible ways of introducing ideal complex numbers; the simplest idea, and apparently the one that Kummer came up with first, is based on the following observation, which Kummer communicated to Kronecker in a letter from April 10, 1844:

Wenn $f(\alpha)$ die Norm $p$ hat ($p$ Primzahl $\lambda n + 1$), so ist jede complexe Zahl einer reellen congruent für den Modul $f(\alpha)$. Hierbei ist nur zu zeigen, daß $\alpha \equiv \xi \mod f(\alpha)$, wo $\xi$ reell. Dieß scheint sich von selbst zu verstehen, weil $\xi - \alpha$ wenn

$$1 + \xi + \xi^2 + \ldots + \xi^{\lambda-1} \equiv 0 \mod p$$

This result by Jacobi immediately gives the ideal prime factors of the complex number $\psi_k(\alpha)$.

The proofs of these theorems could be communicated without problems to my students in the lectures in the last winter semester.

Kummer’s letters to Kronecker can be found in Kummer’s Collected Papers [30].
stets einen Factor mit \( p \) gemein hat, wie in dem Beweise, daß jede Primzahl \( p \) sich in \( \lambda - 1 \) complexe Factoren zerlegen läßt gezeigt wird.

At this point, Kummer apparently still believed that he had proved that every prime \( p \equiv 1 \mod \lambda \), where \( \lambda \) is an odd prime, splits into \( \lambda - 1 \) factors in \( \mathbb{Z}[\alpha] \), where \( \alpha^\lambda = 1 \). A little later, Jacobi, upon his return from his journey to Italy on June 17, would point out this mistake to Kummer. Kummer then sat down to see what parts of his work survived, and was led to his first attempt at defining ideal numbers using Jacobi’s maps; in [28], he motivates his definition by assuming that the prime \( p = \lambda n + 1 \) splits into prime factors in \( \mathbb{Z}[\alpha] \):

\[
p = f(\alpha)f(\alpha^2)f(\alpha^3)\cdots f(\alpha^{\lambda-1}).
\]

Kummer then writes

\[
\text{Ist } f(\alpha) \text{ eine wirkliche complexe Zahl und ein Primfactor von } p,
\]

so hat sie die Eigenschaft, daß, wenn statt der Wurzel der Gleichung \( \alpha^\lambda = 1 \) eine bestimmte Congruenzwurzel von \( \xi^\lambda \equiv 1 \mod p \) substituiert wird, \( f(\xi) \equiv 0 \mod p \) wird. \[23\]

The main difference to what he wrote in his letter to Kronecker in April 1844 is that Kummer now explicitly assumes the existence of a decomposition of \( p \). He then continues:

Also auch, wenn in einer complexen Zahl \( \Phi(\alpha) \) der Primfactor \( f(\alpha) \) enthalten ist, wird \( \Phi(\xi) \equiv 0 \mod p \); und umgekehrt: wenn \( \Phi(\xi) \equiv 0 \mod p \), und \( p \) in \( \lambda - 1 \) complexe Primfactoren zerlegbar ist, enthält \( \Phi(\alpha) \) den Primfactor \( f(\alpha) \). Die Eigenschaft \( \Phi(\xi) \equiv 0 \mod p \) ist nun eine solche, welche für sich selbst von der Zerlegbarkeit der Zahl \( p \) in \( \lambda - 1 \) Primfactoren gar nicht abhängt; sie kann demnach als Definition benutzt werden, indem bestimmt wird, daß die complexe Zahl \( \Phi(\alpha) \) den idealen Primfactor von \( p \) enthält, welcher zu \( \alpha = \xi \) gehört, wenn \( \Phi(\xi) \equiv 0 \mod p \) ist. Jeder der \( \lambda - 1 \) complexen Primfactoren von \( p \) wird so durch eine Congruenzbedingung ersetzte. \[24\]

The Jacobi map \( \phi : \alpha \mapsto \xi \mod p \) has the property that \( \phi(f(\alpha)) = f(\xi) \equiv 0 \mod p \). Thus if \( f(\alpha) | \Phi(\alpha) \) in \( \mathbb{Z}[\alpha] \), then \( \Phi(\alpha) = f(\alpha)g(\alpha) \), and applying \( \phi \) shows that \( \Phi(\xi) = \phi(\Phi(\alpha)) = \phi(f(\alpha))\phi(g(\alpha)) = f(\xi)g(\xi) \equiv 0 \mod p \). Kummer then makes the crucial observation that the congruence \( \Phi(\xi) \equiv 0 \mod p \) makes sense whether \( f(\alpha) \) exists or not: it is a consequence of the existence of the Jacobi map!

Kummer then attaches an ideal prime to every Jacobi map \( \alpha \mapsto \xi \mod p \). Then an integer \( \Phi(\alpha) \in \mathbb{Z}[\alpha] \) will be divisible by the ideal prime attached to \( \phi \) if and only if \( \Phi(\xi) \equiv 0 \mod p \).

---

If \( f(\alpha) \) is an actual complex number and a prime factor of \( p \), then it has the property that when a root of the congruence \( \xi^\lambda \equiv 1 \mod p \) is substituted for the root of the equation \( \alpha^\lambda = 1 \), we get \( f(\xi) \equiv 0 \mod p \).

Thus if the prime \( f(\alpha) \) divides a complex number \( \Phi(\alpha) \), then we will have \( \Phi(\xi) \equiv 0 \mod p \); and conversely: if \( \Phi(\xi) \equiv 0 \mod p \), and \( p \) can be decomposed into \( \lambda - 1 \) complex prime factors, then \( \Phi(\alpha) \) contains the prime factor \( f(\alpha) \). The property \( \Phi(\xi) \equiv 0 \mod p \) is such that it does not depend on the possibility of decomposing the number \( p \) into \( \lambda - 1 \) prime factors; thus we can use it as a definition, by demanding that the complex number \( \Phi(\alpha) \) contain the ideal prime factor of \( p \) belonging to \( \alpha = \xi \) if \( \Phi(\xi) \equiv 0 \mod p \). Each of the \( \lambda - 1 \) complex prime factors of \( p \) is replaced by a congruence condition in this way.
Nowadays we would not hesitate defining an ideal prime to be the Jacobi map, but such an idea would probably have been too revolutionary even for Dedekind, and certainly must have been out of reach for Kummer, who talked about the ideal prime belonging to $\alpha = \xi$ instead. As Kummer explains, however, there are problems connected with this approach:

In der hier gegebenen Weise aber gebrauchen wir die Congruenzbedingungen nicht als Definitionen der idealen Primfaktoren, weil diese nicht hinreichend sein würden, mehrere gleiche, in einer komplexen Zahl vorkommende ideale Primfaktoren vorzustellen, und weil sie, zu beschränkt, nur ideale Primfaktoren der realen Primzahlen von der Form $m\lambda + 1$ geben würden.

Thus the problems Kummer was facing were

(A) Inertia: the Jacobi maps $\mathbb{Z}[\alpha] \rightarrow \mathbb{F}_p$ only provide ideal numbers dividing primes $p \equiv 1 \mod \lambda$.

(B) Multiplicity: there is no obvious way of defining the exact power of an ideal prime dividing a given element in $\mathbb{Z}[\alpha]$.

(C) Completeness: how can we be sure that we have found “all” ideal primes?

Kummer’s solution of these problems will be discussed in the next few sections. Afterwards we will explain the close connection between Kummer’s ideas and modern valuation theory.

It follows immediately from Kummer’s definition that ideal primes behave like primes: the ideal prime attached to the Jacobi map $\phi$ divides $f(\alpha) \in \mathbb{Z}[\alpha]$ if and only if $\phi(f(\alpha)) = 0$; if it divides a product $f(\alpha)g(\alpha)$, then $0 = \phi(fg) = \phi(f)\phi(g)$, hence it divides a factor.

Before we start addressing the problems listed above, we remark that the ideal prime dividing $p = \lambda$ is easy to deal with: there is only one, it is “real” (namely $\pi = 1 - \alpha$), and the corresponding Jacobi map is defined by $\phi(\alpha) = 1 + \lambda\mathbb{Z}$.

7. Solving Problem (A): Decomposition Fields

The first problem is easy to solve for us: instead of looking at homomorphisms $\phi : \mathbb{Z}[\alpha] \rightarrow \mathbb{F}_p$, we consider surjective homomorphisms $\phi : \mathbb{Z}[\alpha] \rightarrow \mathbb{F}_q$ for finite fields with $q = p^f$ elements. Let $\Phi(X) = 1 + X + X^2 + \ldots + X^{\lambda - 1}$ denote the $\lambda$-th cyclotomic polynomial; if $\Phi(X) \equiv P_1(X) \cdots P_f(X) \mod p$ splits into $f$ irreducible factors $P_j(X)$ over $\mathbb{F}_p$, then reduction modulo $p$ gives us surjective ring homomorphisms $\phi_j : \mathbb{Z}[\alpha] \simeq \mathbb{Z}[X]/(\Phi) \rightarrow \mathbb{F}_p[X]/(P_j) \simeq \mathbb{F}_q$. These Jacobi maps can then be used to define ideal numbers for general primes $p$.

This was, however, not an option for Kummer: although Gauss had already introduced residue class fields in the ring of Gaussian integers, some of which do have $p^2$ elements, the general theory of finite fields originated in the work of Galois, which was completely unknown in Germany at the time Kummer started working on these problems. In a development independent of the work of Galois, Schönemann started studying “higher congruences”, as the theory of polynomial rings over $\mathbb{F}_p$ was called at the time, at about the same time Kummer worked on ideal numbers.

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22We do not use the congruence conditions in the way given here as definitions of ideal prime factors since these would not suffice to detect several equal ideal prime factors occurring in a complex number, and since they, being too narrow, would only yield ideal prime factors of the real prime numbers of the form $m\lambda + 1$.

23Galois’ work was published in 1846 by Liouville.
So how did Kummer proceed then? With hindsight, Kummer’s solution is simple and ingenious, and in order to explain why it works, we will use the language of Dedekind’s ideal theory. Let \( p \neq \lambda \) be a prime with order \( f \) in \( (\mathbb{Z}/\mathbb{Z})^x \); then \( p \) splits into \( e = \frac{\lambda - 1}{f} \) distinct primes of degree 1 in the decomposition field \( F \) of \( p \) (since \( K/\mathbb{Q} \) is cyclic, this is the unique subfield of degree \( e \) over \( \mathbb{Q} \)), say \( p\mathcal{O}_F = p_1 \cdots p_e \).

These prime ideals \( p_j \) remain inert in \( K/F \). Since \( \mathcal{O}_F/p_j \cong \mathbb{Z}/p\mathbb{Z} \), every element of \( \mathcal{O}_F \) is congruent to an integer modulo \( p_j \), hence reduction modulo \( p_j \) defines a Jacobi map \( \phi_j : \mathcal{O}_F \to \mathbb{Z}/p\mathbb{Z} \). The Gaussian periods \( \eta_1, \ldots, \eta_e \) form an integral basis of \( \mathcal{O}_F \), so we have \( \mathcal{O}_F = \mathbb{Z}[\eta_1, \ldots, \eta_e] \), and in particular there are integers \( u_1, \ldots, u_e \) such that \( \eta_1 \equiv u_1, \ldots, \eta_e \equiv u_e \) mod \( p_j \). The integers \( u_1, \ldots, u_e \) completely characterize the Jacobi map \( \phi_j \), and, therefore, the prime ideal \( p_j \). Thus we can avoid the introduction of finite fields at the cost of replacing \( \mathcal{O}_K = \mathbb{Z}[\alpha] \) by \( \mathcal{O}_F \).

None of the facts used above were known to Kummer, who proved the existence of these integers \( u_j \) ab ovo (they have the property that the substitution \( \eta_i \to u_i \) turns equations into congruences modulo \( p \)), and then could describe the associated Jacobi maps using systems of congruences in \( \mathbb{Z} \).

For each prime \( p \), Kummer attaches an ideal prime to each Jacobi map \( \mathcal{O}_F \to \mathbb{Z}/p\mathbb{Z} \). This does not really solve Kummer’s first problem: for deciding whether an element \( f(\alpha) \in \mathbb{Z}[\alpha] \) is divisible by an ideal prime attached to \( \phi \), we would like to evaluate \( \phi(f) \) and thus face the problem that \( \phi \) is defined on the subring \( \mathcal{O}_F \) of \( \mathbb{Z}[\alpha] \), but not on \( \mathbb{Z}[\alpha] \). Kummer’s solution of problem (B), namely defining a factorization of cyclotomic integers into powers of ideal prime numbers, also provided him with a clever way around having to extend \( \phi \) from \( \mathcal{O}_F \) to \( \mathbb{Z}[\alpha] \).

Before we turn to Kummer’s solution of problem (B), let us address question (C): Suppose we have what we believe to be a complete set of Jacobi maps from our domains \( \mathbb{Z}[\alpha] \) to certain finite fields; how can we be sure to have found “all” of them? Kummer’s answer was as follows: he proved the “fundamental theorem” that \( f(\alpha) \mid g(\alpha) \) if and only if each ideal prime divides \( g(\alpha) \) with at least the same multiplicity with which it divides \( f(\alpha) \). This property can be formulated in a slightly different way: Each Jacobi map \( \phi \) is defined at \( f(\alpha)/g(\alpha) \) if and only if \( f(\alpha) \mid g(\alpha) \). Clearly \( \phi \) is defined at \( f(\alpha)/g(\alpha) \) if \( f(\alpha) \mid g(\alpha) \), since then the quotient is an element of \( \mathbb{Z}[\alpha] \). The essential criterion for completeness therefore is the following:

If \( h \in \mathbb{Q}(\alpha) \) is an element at which every Jacobi map \( \phi : \mathbb{Z}[\alpha] \to \mathbb{F}_q \) is defined, then \( h \in \mathbb{Z}[\alpha] \).

As we will see, this is not just a statement on the completeness of the Jacobi maps, but also on the correct choice of the ring of integers we are working with — in our case \( \mathbb{Z}[\alpha] \) and not some smaller ring.

8. Solving Problem (B): Valuations

Now let us look at problem (B): defining multiplicity. We will immediately discuss the general case of primes \( q \) with \( q^f \equiv 1 \mod \lambda \). If we think of ideal numbers not, as Kummer did, as systems of congruences but as being attached to Jacobi maps, then it is not difficult to define when two numbers \( f(\alpha), g(\alpha) \in \mathbb{Z}[\alpha] \) are divisible by the same power of an ideal prime attached to \( \phi \): in such a case we would expect that this ideal prime can be cancelled in the fraction \( \frac{f(\alpha)}{g(\alpha)} \), i.e., that there exist elements \( f'(\alpha), g'(\alpha) \in \mathbb{Z}[\alpha] \) such that \( \frac{f'}{g'} = \frac{f}{g} \) with \( \phi(f')\phi(g') \neq 0 \).
This idea can be extended immediately: if we can write \( \frac{f}{g} = \frac{f'}{g'} \) with \( \phi(g') \neq 0 \), we say that the ideal number attached to \( \phi \) divides \( f \) at least as often as \( g \). If we let \( v_\phi(f) \) denote the hypothetical exponent with which the ideal prime attached to \( \phi \) divides a number \( f(\alpha) \), then the above definitions will tell us when \( v_\phi(f) < v_\phi(g) \), \( v_\phi(f) = v_\phi(g) \), or \( v_\phi(f) > v_\phi(g) \). The fundamental problem now is to show that, given elements \( f, g \in \mathbb{Z}[\alpha] \), we always are in exactly one of these three situations:

(B') Given nonzero elements \( f(\alpha), g(\alpha) \in \mathbb{Z}[\alpha] \), there exist \( f'(\alpha), g'(\alpha) \in \mathbb{Z}[\alpha] \) with \( \frac{f}{g} = \frac{f'}{g'} \) and \( \phi(f') \neq 0 \) or \( \phi(g') \neq 0 \).

Let us say that the Jacobi map \( \phi \) is defined at a nonzero element \( h(\alpha) \in \mathbb{Q}(\alpha) \) if we can write \( h = \frac{f}{g} \) with \( \phi(g) \neq 0 \). Then (B') can be formulated in the following way:

(B') Given an element \( h(\alpha) \in \mathbb{Q}(\alpha) \), a Jacobi map \( \phi \) is defined at \( h \) or at \( \frac{1}{h} \).

Subrings \( R \) of a field \( K \) with the property that for every \( h \in K \) we have \( h \in R \) or \( \frac{1}{h} \in R \) are called valuation rings; each Jacobi map satisfying property (B") defines a valuation ring in \( \mathbb{Q}(\alpha) \).

For stating (B') and (B") we have assumed the existence of the exponent \( v_\phi(f) \). In order to guarantee its existence we have to assume that the valuation ring defined by \( \phi \) has additional properties. Everything we need will follow from

(B") For each Jacobi map \( \phi \) there exists a \( \psi(\alpha) \in \mathbb{Z}[\alpha] \) with the following properties:

(a) \( \phi(\psi(\alpha)) = 0 \);

(b) \( \phi \) is defined at \( \frac{f(\alpha)}{\psi(\alpha)} \) for all \( f(\alpha) \in \mathbb{Z}[\alpha] \) with \( \phi(f(\alpha)) = 0 \);

(c) if \( \phi \) is defined at \( f(\alpha)/\psi(\alpha)^n \) for all \( n \in \mathbb{N} \), then \( f(\alpha) = 0 \).

Such an element \( \psi \) is called a uniformizer for (the valuation defined by) \( \phi \). Let us first show that for any \( f \in K^\times \) there is an integer \( n \in \mathbb{Z} \) such that \( \phi \) is defined at \( f/\psi^n \): write \( f = \frac{f}{g} \) for \( g, h \in R \); if \( \phi(h) \neq 0 \), then we can choose \( n = 0 \). If \( \phi(h) = 0 \), let \( n \geq 0 \) be the maximal integer such that \( \phi \) is defined at \( h/\psi^n \). Then \( \phi(h/\psi^n) \neq 0 \) by the maximality of \( n \) and (b), hence \( \phi \) is also defined at \( \psi^n/h \). But then \( \phi \) is defined at \( \psi^n = f/\psi^n \). Using this argument it is easy to show the existence of \( v_\phi(f) \) as well as property (B").

The properties (B'), (B") and (B"') will be discussed, within the context of the theory of valuations, in the sections below; in the rest of this section, let us see how Kummer solved problem (B):

Die von mir gewählte Definition der idealen complexen Primfaktoren, welche im Wesentlichen zwar mit den hier angedeuteten übereinstimmt, aber einfach er und allgemeiner ist, beruht darauf, daß sich, wie ich besonders beweise, immer eine aus Perioden gebildete Zahl \( \psi(\eta) \) finden läßt von der Art, daß

\[
\psi(\eta) \psi(\eta_1) \psi(\eta_2) \cdots \psi(\eta_{\ell-1})
\]

(welches eine ganze Zahl ist) durch \( q \) theilbar sei, aber nicht durch \( q^2 \). Diese complexe Zahl \( \psi(\eta) \) hat alsdann immer die obige Eigenschaft, daß sie congruent Null wird, modulo \( q \), wenn statt der Perioden die entsprechenden Congruenzwurzeln gesetzt werden, also \( \psi(\eta) \equiv 0 \mod q \), für \( \eta = u_1, \eta_1 = u_1, \eta_2 = u_2 \) etc. Ich setze nun \( \psi(\eta_1) \psi(\eta_2) \cdots \psi(\eta_{\ell-1}) = \Psi(\eta) \) und definire die idealen Primzahlen folgendermaßen:
"Wenn $f(\alpha)$ die Eigenschaft hat, daß das Produkt $f(\alpha) \cdot \Psi(\eta_r)$ durch $q$ teilbar ist, so soll dies so ausgedrückt werden: Es enthält $f(\alpha)$ den idealen Primfaktor von $q$, welcher zu $u = \eta_r$ gehört. Ferner, wenn $f(\alpha)$ die Eigenschaft hat, daß $f(\alpha)(\Psi(\eta_r))^\mu$ durch $q^\mu$ teilbar ist, aber $f(\alpha)(\Psi(\eta_r))^{\mu+1}$ nicht teilbar durch $q^{\mu+1}$, so soll dies heißen: Es enthält $f(\alpha)$ den zu $u = \eta_r$ gehörigen idealen Primfaktor von $q$ genau $\mu$ mal.\[24\]

Kummer’s element $\psi(\eta)$ is a uniformizer for the ideal prime attached to the generalized “Jacobi map” $\phi : \eta \mapsto u, \eta_1 \mapsto u_1, \ldots, \eta_{e-1} \mapsto u_{e-1}$; its norm $N_{F/Q}(\psi(\eta)) = \psi(\eta)\Psi(\eta)$ is divisible by the prime $q$, but not by $q^2$. The multiplicity $\mu$ with which the ideal prime attached to $\phi$ divides some $f(\alpha) \in \mathbb{Z}[\alpha]$ is the maximal natural number $\mu$ with the property that $f(\alpha)(\Psi(\eta))^\mu$ is divisible by $q^\mu$. Rephrasing this condition slightly, we see that it is equivalent to the condition that $\phi$ is defined at $\frac{f(\alpha)(\Psi(\eta))^\mu}{\psi(\eta)^\mu}$, or, equivalently (observe that $\phi$ is defined at both $\frac{\psi(\eta)}{\psi(\eta)}$ and its inverse), at $\frac{f(\alpha)}{\psi(\eta)^\mu}$.

9. Jacobi Maps and Valuations

Kummer developed a “complete” theory of ideal numbers only for cyclotomic fields and their subfields. Although he ran into insurmountable problems when trying to extend his construction to general number fields, he did not investigate exactly which properties of the rings he was working in were responsible for his success (it is no exaggeration to claim that no one before Dedekind understood such questions properly).

In order to display the gaps in Kummer’s theory as clearly as possible we now try to transfer his construction to more general situations. Let $R$ be a domain with quotient field $K$. A Jacobi map is a ring homomorphism $\phi : R \rightarrow F$ onto a field $F$. The kernel $p = \ker \phi$ satisfies $R/p \simeq F$, hence is a maximal ideal (and fortiior a prime ideal) in $R$; Jacobi maps with the same kernel will be identified.

A Jacobi map $\phi$ is said to be defined at $t \in K$ if there exist $r, s \in R$ with $t = \frac{r}{s}$ and $\phi(s) \neq 0$. The set of all $t \in K$ at which $\phi$ is defined is a subring of $K$ denoted by $R_\phi$.

A subring $R$ of a field $K$ is called a valuation ring if for any $t \in K$ we have $t \in R$ or $\frac{1}{t} \in R$. The unit group of a valuation ring consists of all elements $a \in K^\times$ for

\[24\] The definition of the ideal complex prime factors I have chosen, which coincides essentially with the ones sketched above, but is simpler and more general, is based on the fact that, as I will prove, there always exists a number $\psi(\eta)$, formed out of periods, with the property that

$$
\psi(\eta_1)\psi(\eta_2)\cdots\psi(\eta_{e-1}),
$$

which is an integer, is divisible by $q$, but not by $q^2$. This complex number $\psi(\eta)$ has the property above of becoming congruent to 0 mod $q$ if we replace the periods by the corresponding roots of the congruence, i.e., $\psi(\eta) \equiv 0 \pmod{q}$ for $\eta = u, \eta_1 = u_1, \eta_2 = u_2$ etc. Now I set

$$
\psi(\eta_1)\psi(\eta_2)\cdots\psi(\eta_{e-1}) = \Psi(\eta)
$$

and define the ideal prime factors as follows: If $f(\alpha)$ has the property that the product $f(\alpha) \cdot \Psi(\eta_r)$ is divisible by $q$, then we shall express this by saying that $f(\alpha)$ contains the ideal prime factor of $q$ belonging to $u = \eta_r$. Moreover, if $f(\alpha)$ has the property that $f(\alpha)(\Psi(\eta))^\mu$ is divisible by $q^\mu$ without $f(\alpha)(\Psi(\eta))^\mu+1$ being divisible by $q^{\mu+1}$, then this shall mean: $f(\alpha)$ contains the prime ideal factor of $q$ belonging to $u = \eta_r$ exactly $\mu$ times.
which \( a \in R \) and \( \frac{1}{a} \in R \). The set \( m = R \setminus R^\times \) of nonunits is an ideal, and in fact the unique maximal ideal of \( R \).

If \( \phi \) is a Jacobi map defined on \( R \), then \( R_\phi \) is a valuation ring if and only if the analog of property (B”) holds, that is: given any \( t \in K \), a Jacobi map \( \phi \) is defined at \( t \) or at \( \frac{1}{t} \).

An additive valuation \(^{25}\) is a map from the nonzero elements of a field \( K \) to an ordered group \( G \) with the properties

1. \( v(ab) = v(a) + v(b) \),
2. \( v(a + b) \geq \min\{v(a), v(b)\} \)

for all \( a, b \in K \). We also set \( v(0) = \infty \) and \( \infty \geq g \) for all \( g \in G \); then \( v \) is a map \( K \to G \cup \{\infty\} \). The set of all \( a \in K \) with \( v(a) \geq 0 \) forms a valuation ring. For example, let \( p \) be a prime and let \( v_p(a) \) denote the exponent of \( p \) in the prime factorization of the nonzero rational number \( a \in \mathbb{Q} \); then \( v_p(a) \) is a valuation.

Thus every valuation determines a valuation ring (valuations giving rise to the same valuation ring are called equivalent). Conversely, to every valuation ring we can find a corresponding valuation: to this end, define an order on the “divisibility group” \( G_K = K^\times / R^\times \) (in \(^{33}\), this group shows up in a different connection) via \( yR^\times \leq xR^\times \) if and only if \( \frac{y}{x} \in R \). It is a simple exercise to show that the map \( v: K^\times \to G_K \) sending \( x \in K^\times \) to its coset \( xR^\times \) is indeed a valuation.

For solving Kummer’s problem (B) of defining multiplicity, a valuation is not good enough: we want a valuation with values not in some ordered group but in \( \mathbb{Z} \)! Such valuations are called discrete valuations; they can be characterized by the fact that the corresponding valuation rings are discrete valuation rings, i.e., their maximal ideal \( m \) must be principal. In fact, if \( m = (t) \) is generated by an element \( t \in K^\times \) (such elements are called uniformizers for the corresponding valuation), then any \( a \in K^\times \) can be written uniquely in the form \( a = ut^m \) for some unit \( u \in R^\times \) and some \( m \in \mathbb{Z} \), and setting \( v(a) = m \) gives us a discrete valuation.

In Kummer’s case of cyclotomic rings of integers, the existence of a uniformizing element \( h(\alpha) \) is exactly what we asked for in property (B”). Thus in terms of modern algebra, Kummer used Jacobi maps to construct discrete valuation rings.

10. Examples of Jacobi Maps

The major sources of Jacobi maps and their associated valuations are number theory and algebraic geometry. In addition to the two most important examples coming from number fields and algebraic curves, we will also discuss certain monoids that behave a lot like them.

**Hilbert Monoids.** In his lectures on number theory in the winter semester 1897/98, Hilbert \(^{22}\) used the monoid \( \{1, 6, 11, 16, 21, \ldots\} \) of natural numbers \( \equiv 1 \mod 5 \) for motivating the introduction of ideals. We would like to show now that these monoids can also be used for explaining Kummer’s solution of problem (B), namely defining multiplicity. A general theory of divisibility in monoids was given by Rychlik \(^{37}\).

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\(^{25}\) For the history of valuation theory, beginning in 1912 with Kürschak’s work, see Roquette’s article \(^{36}\).

\(^{26}\) If \( v \) is an additive valuation and \( c \) any real number \( > 1 \), then \( |a| = e^{-v(a)} \) defines a “multiplicative valuation”, or an “absolute value” on \( K \) (we put \( |0| = 0 \)).
Consider the monoid $M = \{1, 5, 9, 13, \ldots \}$ of natural numbers $\equiv 1 \mod 4$. In this monoid, factorization into irreducibles is not unique, as the example $21 \cdot 21 = 9 \cdot 49$ shows. One can restore unique factorization by adjoining “ideal numbers” such as $\gcd(9, 21)$ representing a common divisor of 9 and 21.

Kummer’s approach using Jacobi maps also works here: For each odd prime $p \equiv 1 \mod 4$, there is a surjective homomorphism of monoids $\phi_p : M \to \mathbb{F}_p$. For primes $p \equiv 1 \mod 4$, the kernels of the $\phi_p$ are “principal” in the sense that $\ker \phi_p = pM$. For primes $p \equiv 3 \mod 4$, this is not the case: $\ker \phi_3 = \{9, 21, 33, 45, \ldots \}$ cannot be written in the form $aM$ for some $a \in M$, but we can think of the Jacobi map $\phi_3$ as representing an “ideal prime” 3 in $M$.

In order to define multiplicity we extend the Jacobi maps $\phi_p$ to the quotient monoid $Q(M)$ of all fractions $\frac{a}{b}$ with $a, b \in M$ in the following way: if we can write $\frac{a}{b} = \frac{c}{d}$ with $c, d \in M$ and $\phi_p(d) \neq 0$, then we set $\phi_p(\frac{a}{b}) = \frac{\phi_p(c)}{\phi_p(d)}$ and say that $\phi_p$ is defined at $\frac{a}{b}$; we also set $\phi_p(\frac{a}{b}) = \infty$ if $\phi_p(\frac{a}{b}) = 0$. Since $\frac{2}{21} = \frac{9}{49} = \frac{21}{21} = \frac{21}{49}$, we see that $\phi_3$ is defined at $\frac{9}{21}$, and that $\phi_3(\frac{9}{21}) = 0$. Similarly, $\phi_3$ is defined at $\frac{9}{49} = \frac{1}{49}$.

The following fact is a fundamental property of $M$: if $\phi_p$ is not defined at $\frac{a}{b}$, then it is defined at $\frac{b}{a}$. In fact, if both $c$ and $d$ are divisible in $\mathbb{N}$ by $p$, then we can cancel $p$ immediately if $p \equiv 1 \mod 4$, or we can find a natural number $r \equiv 3 \mod 4$ coprime to $p$ and write $\frac{a}{b} = \frac{rc}{rd}$ with $rc, rd \in M$. Continuing in this way we see that $\frac{a}{b} = \frac{c}{d}$ with $p \nmid c$ or $p \nmid d$, and then $\phi_p$ is defined at $\frac{b}{a}$ and $\frac{a}{b}$, respectively.

An element $q \in M$ is called a uniformizer for $p$ if $\phi_p$ is defined at $\frac{1}{q}$ for all $c \in M$ with $\phi_p(c) = 0$. It is easy to see that the elements $21, 33, \ldots$ are uniformizers for $p = 3$, and that 9 is not. More generally, given a prime $p \equiv 3 \mod 4$, any element $q = pr$, where $r \equiv 3 \mod 4$ is coprime to $p$, is a uniformizer for $p$.

Now we say that the ideal prime $p$ divides some $a \in M$ with exponent $m$ if and only if $\phi_p$ is defined at $\frac{a}{q^m}$, but not at $\frac{a}{q^{m+1}}$. Such an integer $m$ exists, and it does not depend on the choice of the uniformizer $q$. Since e.g. $\phi_3$ is defined at $\frac{9}{21}$, but not at $\frac{9}{49}$, we find that 9 is exactly divisible by the square of the ideal prime 3.

It can be shown that every element in $M$ can be written uniquely as a product of powers of ideal primes. We leave it as an exercise for the reader to show that unique factorization into ideal primes in $M$ can be used to show that $a \in M$ is a square in $Q(M)$ if and only if it is a square in $M$.

We have seen above that the “Hilbert monoid” $M$ is quite well behaved in that it satisfies the analog of property (B’): for odd primes $p$ and all $a, b \in M$, the “Jacobi maps” $\phi_p$ are defined either at $\frac{a}{b} \in Q(M)$ or at $\frac{b}{a}$ (or at both).

In fact, the same thing holds in general monoids of Hilbert type: for any natural number $m > 1$, let $H$ be a subgroup of $G = (\mathbb{Z}/m\mathbb{Z})^\times$, and define the monoid $M^G_H$ as the set of all natural numbers $a \in \mathbb{N}$ whose residue classes mod $m$ lie in $H$. The monoid $M = \{1, 5, 9, \ldots \}$ considered above, for example, is $M^G_H$ for the trivial subgroup $H$ of $G = (\mathbb{Z}/4\mathbb{Z})^\times$. It is easy to show that all such monoids $M^G_H$ are nonsingular in the sense that Jacobi maps $\phi_p$, for all primes $p$ coprime to $m$, are defined at $\frac{b}{a} \in Q(M^G_H)$ or at $\frac{a}{b}$. 
It is actually not difficult to prove that all these monoids have unique factorization into ideal primes, to define an analog $\text{Cl}(M_H^G)$ of the ideal class group$^{27}$, and to show that$^{28} \text{Cl}(M_H^G) \simeq G/H$.

**Number Fields.** In a Dedekind domain $R$, every prime ideal $p \neq (0)$ is maximal, hence induces a Jacobi map $\phi_p : R \longrightarrow F = R/p$. These maximal ideals also give rise to valuations $v_p$: given any $a \in K^\times$, write $(a) = p^m a$ for some ideal $a$ in whose prime ideal factorization $p$ does not occur, and set $v_p(a) = m$. The valuations $v_p$ attached to nonzero prime ideals are non-archimedean in the sense that the corresponding absolute values $| \cdot |_p$ defined by $|\alpha|_p = (Np)^{-v_p(\alpha)}$ satisfy the strong triangle inequality $|\alpha + \beta|_p \leq \max\{|\alpha|_p, |\beta|_p\}$. It can be shown that every archimedean valuation in a number field $K$ is equivalent to a valuation $v_p$ for a suitable prime ideal $p$.

In addition, every number field has some archimedean valuations: if $K = \mathbb{Q}(\alpha)$ and $\alpha$ is the root of the monic polynomial $f \in \mathbb{Z}[X]$ of degree $n$, let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ denote the roots of $f$ in $\mathbb{C}$. Every element of $K$ can be written as a polynomial in $\alpha$, say as $g(\alpha)$ with $g \in \mathbb{Q}[X]$; therefore we can define an absolute value $| \cdot |_j$ on $K$ by setting $|g(\alpha)|_j = |g(\alpha_j)|$. It turns out that complex conjugate roots give rise to the same absolute value, and that there are $r + s$ independent absolute values if the number of real and complex roots of $f$ is $r$ and $2s$, respectively (complex roots come in pairs since $f$ has real coefficients).

The main difference between archimedean and non-archimedean valuations from our point of view is that non-archimedean valuations come from (resp. give rise to) an additive valuation and thus to a valuation ring, whereas archimedean valuations do not.

**Algebraic Curves.** Let $K$ be an algebraically closed field, and $f \in K[X,Y]$ an irreducible polynomial. The zero set $\mathcal{C}(K) = \{(x,y) \in K \times K : f(x,y) = 0\}$ of $f$ is called a plane algebraic curve. Its coordinate ring is defined to be the ring $\mathcal{O} = K[\mathcal{C}] = K[X,Y]/(f)$, whose elements are polynomials in $x = X + (f)$ and $y = Y + (f)$: since $f$ was assumed to be irreducible in the factorial domain $K[X,Y]$, the coordinate ring of $\mathcal{C}$ is actually a domain, and its quotient field $K(\mathcal{C})$ is called the function field of $\mathcal{C}$.

For each point $P \in \mathcal{C}(K)$, we can define a “Jacobi map” $\phi_P : \mathcal{O} \longrightarrow K$ via $\phi_P(g) = g(P)$. Since $\mathcal{O}$ contains all the constant functions, the $\phi_P$ are surjective ring homomorphisms. An element $\frac{r}{s} \in K(\mathcal{C})$ is said to be defined at a point $P \in K \times K$ if $\frac{r}{s} = \frac{r}{s}$ for $r, s \in \mathcal{O}$ with $\phi_P(s) = s(P) \neq 0$.

The functions from $K(\mathcal{C})$ defined at a fixed point $P$ form a ring $\mathcal{O}_P$ called the local ring at $P$, and $\mathcal{O}_P$ clearly contains $K$ (here we are identifying $K$ with the constant functions, which are defined everywhere) and even $\mathcal{O}$.

---

$^{27}$Call two ideal numbers $p, q$ equivalent if there is an ideal number $r$ such that $pr, qr \in M_H^G$.

$^{28}$There is an exact sequence

\[ 1 \longrightarrow H \longrightarrow G \longrightarrow \text{Cl}(M_H^G) \longrightarrow 1, \]

where $\pi$ maps the residue class $a + m\mathbb{Z}$ to the ideal class generated by the ideal number $a$.

$^{29}$In order to keep things as simple as possible, we only consider the affine plane here, and remark that, in algebraic geometry, the projective point of view is usually to be preferred. We also have to admit that our definition of a plane algebraic curve is quite naive; but it suffices for our purposes.
As an example, consider the curve defined by \( Y^2 = X^3 + X^2 \). The function \( h(x, y) = \frac{x}{y+1} \in K(C) \), where \( x = X + (f) \) and \( y = Y + (f) \), is not defined at \( P = (-1, 0) \), but its inverse \( \frac{x+1}{y} \) is: we have \( \frac{x+1}{y} = \frac{(x+1)y}{y^2} = \frac{(x+1)y}{x^2+x} = \frac{y}{x} \), and this function \( \frac{1}{h} \) is defined at \( P \) and vanishes there.

11. Singularities

After having gone through several examples where Jacobi maps \( \phi : R \rightarrow F \) give rise to valuation rings and, therefore, to valuations, we will now present examples of Jacobi maps \( \phi \) for which \( R_\phi \) is not a valuation ring.

**Singular Monoids.** In Hilbert monoids \( M[H] \), every Jacobi map \( \phi : M[H] \rightarrow F \) gives rise to a “valuation”. The situation is completely different for the monoid \( N = \{1, 2, 4, 5, 6, 8, 9, \ldots \} \) of natural numbers congruent to 0, 1, 2 mod 4. Here, the Jacobi map \( \phi_2 : N \rightarrow F \) cannot be extended to the quotient module \( Q(N) \) by imitating the process that worked so well for \( M[H] \): the map \( \phi_2 \) is undefined both for \( \frac{5}{2} \) and \( \frac{6}{2} \). In fact, if we had \( \frac{a}{2} \), for some odd \( b \), then \( b \equiv 1 \) mod 4; on the other hand, \( 6b = 2a \) implies \( a = 3b \) in \( N \), and thus \( a \equiv 3 \) mod 4, contradicting the assumption that \( a \in N \).

The element \( \frac{9}{2} \in Q(N) \) displays a singular behavior in more than one way: the element \( 9 = (\frac{9}{2})^2 \) is a square in the quotient monoid \( Q(N) \), but not in \( N \). In the natural numbers \( N \), on the other hand, square roots are either integers or irrational. The following proof of the irrationality of \( \sqrt{m} \) for nonsquares \( m \) brings out a fundamental concept that we will need to explain Kummer’s success at introducing ideal numbers in cyclotomic rings of integers:

\[
\sqrt{m} \text{ is rational } \iff \text{ the roots of } X^2 - m \text{ are rational } \\
\iff X^2 - m \text{ factors over } \mathbb{Q} \\
\iff X^2 - m \text{ factors over } \mathbb{Z}
\]

Here, the only nontrivial step is the last equivalence, which is a special case of

**Gauss’s Lemma.** Let \( f \) be a monic polynomial with integral coefficients. Then \( f \) factors over \( \mathbb{Q}[X] \) if and only if it factors over \( \mathbb{Z}[X] \).

As a matter of fact, Gauss’s Lemma (which can be found in Gauss’s Disquisitiones \cite{18}) implies the following general result: algebraic integers (roots of monic polynomials \( \in \mathbb{Z}[X] \)) are either elements of \( \mathbb{Z} \) or irrational.

Although it means stretching the analogy beyond its limit, let us now talk about “polynomials” with coefficients in the monoid \( N \) (we can do so since \( N \) is a subset of \( \mathbb{N} \); note, however, that \( N \) is not closed with respect to addition). Then the natural analog of Gauss’s Lemma does not hold in \( N \): the polynomial \( f(X) = X^2 - 9 \) is irreducible over \( N \) in the sense that \( f \) cannot be written as a product of two linear factors with coefficients in \( N \), whereas \( X^2 - 9 = (X - \frac{3}{2})(X + \frac{3}{2}) \) factors over the quotient monoid \( Q(N) \).

**Singular Curves.** Consider the plane algebraic curve defined by the polynomial \( f(X, Y) = Y^2 - X^3 - X^2 \). Then both \( \frac{x}{y} \in K(C) \) and its inverse \( \frac{y}{x} \) are undefined at the singular point \( P = (0, 0) \). In fact, assume that \( \frac{x}{y} = \frac{y}{h} \) for \( g, h \in \mathcal{O} \) with \( h(P) \neq 0 \). Then \( xh(x, y) = yg(x, y) \), that is, \( Xh(X, Y) - Yg(X, Y) = r(X, Y)f(X, Y) \) for polynomials \( g, h, r \in K[X, Y] \). Plugging in \( Y = 0 \) yields \( Xh(X, 0) = -(X^3 + \)
$X^2)r(X,0)$, and cancelling $X$ gives $h(X,0) = -(X^2 + X)r(X,0)$. Plugging in $X = 0$ now shows that $h(P) = 0$. A similar calculation shows that $\frac{2}{x}$ is not defined at $P$.

As in the case of the monoid $N$, the singular behavior of $P = (0,0)$ is connected with the failure of Gauss’s Lemma: the polynomial $F(T) = T^2 - x - 1 \in R[T]$ is irreducible over $R$, but factors as $F(T) = (T - \frac{1}{2}) (T + \frac{1}{2})$ over the field of fractions $K = k(x,y)$ of $R$. Similarly, the element $x + 1 \in \O = K[C]$ is not a square in $\O$, but becomes a square in the quotient field $K(C)$ of $\O$.

The notion of singularity has its origin in the theory of algebraic curves, and is connected with the existence of tangents. In fact, let $P$ denote the partial derivatives of $f$ with respect to $X$ and $Y$. Then the tangent to $C$ at $P$ is defined to be the line $f_X(P)(X - a) + f_Y(P)(Y - b) = 0$; points with $f_X(P) = f_Y(P) = 0$ are called singular points.

In our example of the curve $C$ defined by $f(X,Y) = Y^2 - X^3 - X^2 = 0$, the only common solution of the two equations $f_X = -3X^2 - 2X = 0$ and $f_Y = 2Y = 0$ is $P = (0,0)$, which therefore is the only singular point on $C$ in the affine plane.

It can be shown that $\O_P$ is a valuation ring if and only if $P$ is a nonsingular point of $C$. The connection between the definition of singularity above and the failure of Gauss’s Lemma in the coordinate ring is provided by the crucial observation that Gauss’s Lemma for monic polynomials holds in a domain $R$ if and only if $R$ is integrally closed (see e.g. [33]).

Recall that a domain $R$ with quotient field $K$ is called integrally closed if every $a \in K$ that is a root of a monic polynomial in $R[T]$ actually belongs to $R$. The coordinate ring $R = K[C]$ in the example above is not integrally closed: the root $\frac{1}{2}$ of the monic polynomial $T^2 - x - 1$ is an element of $K$ that is integral over $R$, yet does not belong to $R$.

An irreducible algebraic curve is called normal if its coordinate ring $K[C]$ is integrally closed. Normality is a local property: $C$ is normal if and only if all its local rings $\O_P$ are integrally closed ([33] Chap. II, § 5). Finally, an algebraic curve is normal if and only if none of its points is singular.

**Singular Orders.** Consider the ring $\O = \Z[\sqrt{-3}]$; the Jacobi map $\phi_2 : \O \rightarrow F_2$ defined by $\phi_2(a + b\sqrt{-3}) \equiv a + b \pmod{2}$ is a surjective ring homomorphism, and thus should correspond to an ideal prime. Since $\phi_2(1 + \sqrt{-3}) = \phi_2(2) = 0$, we expect that $\phi_2$ is defined at $\frac{1 + \sqrt{-3}}{2}$ or $\frac{2}{1 + \sqrt{-3}}$, but a simple calculation shows that this is not the case. This shows that Kummer’s idea of attaching an ideal prime to each Jacobi map does not work here.

Dedekind’s ideal theory, by the way, also fails in $\O$: consider the ideal $\ker \phi_2 = \mathfrak{p} = (2,1 + \sqrt{-3})$ in $\O$; since $\mathfrak{O}/\mathfrak{p}$ has two elements, $\mathfrak{p}$ is prime (even maximal). Yet $\mathfrak{p}^2 = (4,2 + 2\sqrt{-3},-2 + 2\sqrt{-3}) = (2)(2, 1 + \sqrt{-3}) = (2)\mathfrak{p}$, and if we had unique factorization into prime ideals, this would imply $\mathfrak{p} = (2)$, which is not true, however.

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30We say that Gauss’s Lemma holds over a domain $R$ with quotient field $K$ if monic polynomials in $R[X]$ that factor in $K[X]$ also factor in $R[X]$.  
31This is a notion introduced by Emmy Noether, modeled on Dedekind’s definition of an algebraic integer.  
32This definition also applies to algebraic varieties of higher dimension.
Similar examples are provided by the subrings \( \mathcal{O} = \mathbb{Z}[p_i] = \{a + pbi : a, b \in \mathbb{Z}\} \) of \( \mathbb{Z}[i] \), where \( p \) is any rational prime. Here \( \phi(a + pbi) = a + p\mathbb{Z} \) defines a Jacobi map \( \mathcal{O} \to \mathbb{F}_p \) with kernel \( p = \ker \phi = p\mathbb{Z} \oplus p\mathbb{Z} \). Since \( \mathcal{O}/p \simeq \mathbb{F}_p \), \( p \) is a prime ideal. On the other hand, \( \phi \) is not defined at \( \frac{a}{p} \) and \( \frac{b}{p} \).

It should not be surprising that Gauss’s Lemma also fails in \( R = \mathbb{Z}[\sqrt{-3}] \): the polynomial \( T^2 + T + 1 \) is irreducible over \( R \), but factors as \( T^2 + T + 1 = (T - \rho)(T - \rho^2) \) over its field of fractions \( K = \mathbb{Q}(\sqrt{-3}) \), where \( \rho = \frac{-1 + \sqrt{-3}}{2} \).

Orders \( \mathcal{O} \) are subrings of the rings of integers \( \mathcal{O}_K \) of a number field containing \( \mathbb{Z} \); a typical example is \( \mathcal{O} = \mathbb{Z}[\sqrt{-3}] \). The “bad” prime ideals (those attached to Jacobi maps not satisfying condition (B')) are those dividing the conductor of the order; the conductor is an ideal that measures how far the order is from being maximal: the maximal order \( \mathcal{O}_K \), for example, is nonsingular and has conductor (1). The order \( \mathbb{Z}[\sqrt{-3}] \), on the other hand, has conductor (2), and the prime ideal dividing (2) shows a singular behavior.

## 12. Integral Closure

We now would like to use a given Jacobi map \( \phi : R \to F \) for defining a valuation on \( R \). As we have seen, there are situations in which this does not work, for example if the domain \( R \) is not integrally closed. It is actually not hard to show that valuation rings \( R \) are integrally closed: if \( t \in K \) is integral over \( R \), then \( t^n = \sum_{j=0}^{n-1} a_j t^j \). If \( t \notin R \), then \( t^{-1} \in R \) since \( R \) is a valuation ring; this implies \( t = \sum_{j=0}^{n-1} a_j t^{j+1-n} \). But then \( t \in R \), which is a contradiction.

Assume now that \( K \) is a number field with ring of integers \( \mathcal{O}_K \), and that \( \mathcal{O} \subset \mathcal{O}_K \) is a proper subring containing \( \mathbb{Z} \) (such subrings are called orders). Then there is an integral element \( \alpha \in K \setminus \mathcal{O} \). Since \( \alpha \) is not in \( \mathcal{O} \), there must be a Jacobi map \( \phi : \mathcal{O} \to \mathbb{F}_q \) such that \( \phi \) is not defined at \( \alpha \). We claim that \( \phi \) is also not defined at \( \frac{\alpha}{1} \).

Suppose that we can write \( \alpha = \frac{\beta}{\gamma} \) with \( \phi(\beta) \not\equiv 0 \). From \( \alpha^n = \sum_{j=0}^{n-1} a_j \alpha^j \) for integral coefficients \( a_j \in \mathbb{Z} \) we deduce, after multiplying through by \( \gamma^n \), that \( \beta^n = \gamma (a_{n-1} \beta^{n-1} + \ldots + a_0 \beta^{n-1}) \). This implies that \( \phi(\gamma) \not\equiv 0 \). But then \( \phi \) is defined at \( \alpha = \frac{\beta}{\gamma} \), contrary to our assumption.

Thus if \( \mathcal{O} \) is not integrally closed, then there exist elements \( \alpha \in \mathcal{O}_K \setminus \mathcal{O} \) and Jacobi maps \( \phi : \mathcal{O} \to \mathbb{F}_q \) which are not defined at \( \alpha \) and \( \frac{\alpha}{1} \). In these cases, Kummer’s method of attaching an ideal prime to Jacobi maps fails.

## 13. Ideal Numbers and Integral Closure

Although Kummer did not isolate the property (B') (let alone (B'') or even (B''')) in his work, for verifying that his prime ideal exponents have the desired properties he implicitly had to prove some form of property (B'). In fact, we have seen above that Kummer’s method of attaching an ideal prime to each Jacobi map sometimes does not work. The reason why Kummer did not run into these problems, at least not until he had to study ideal numbers in Kummer extensions of cyclotomic fields (these are extensions of \( K = \mathbb{Q}(\zeta_p) \) of the form \( L = K(\sqrt[p]{\mu}) \); for the rings \( \mathcal{O} \subset L \) that Kummer considered, he had to exclude all ideal primes dividing \( (1 - \zeta_p)\mu \), was clarified much later by Dedekind. Dedekind was the first
to give the correct definition of algebraic integers and showed that the ring \( \mathcal{O}_K \) of all algebraic integers contained in a number field \( K \) is “nonsingular” in the sense above. When Emmy Noether later characterized “Dedekind domains” (these are domains in which every ideal can be written uniquely as a product of prime ideals) axiomatically, integral closure turned out to be one of the axioms.

Luckily for Kummer, the obvious choice of the ring \( \mathbb{Z}[\alpha] \) in a cyclotomic field \( K = \mathbb{Q}(\alpha) \) turns out to be the full ring \( \mathcal{O}_K \) of integers. But Kummer found the correct ring of integers even for the subfields of cyclotomic fields: the rings \( \mathbb{Z}[\eta_1, \ldots, \eta_e] \) generated by the Gaussian periods \( \eta_i \), which Gauss had introduced in the seventh section of his Disquisitiones Arithmeticae [18]. Note that for quadratic number fields \( \mathbb{Q}(\sqrt{p}) \) with \( p \equiv 1 \mod 4 \), the ring of periods is \( \mathbb{Z}[1 + \sqrt{-p}] \).

Kummer mentioned the ring of periods already in his letter to Kronecker from October 2, 1844; the main part of his letter was, however, devoted to a proof that \( \mathbb{Z}[\zeta_5] \) is a Euclidean ring. But although Kummer suspected early on that the solution to his problems could be found in this ring of periods, it took him a whole year to work out the details: in his letter from October 18, 1845, he finally could explain his new theory of ideal numbers to Kronecker.

After having discussed the relevance of integral closure for Kummer’s theory in the last few sections, we are left with the following question: where exactly did Kummer use the fact that the rings \( \mathbb{Z}[\alpha] \) and \( \mathbb{Z}[\eta_1, \ldots, \eta_e] \) are integrally closed? Kummer’s first construction of ideal numbers contained a serious gap (already noticed by Eisenstein, and later by Cauchy and Dedekind), which is why Edwards presents Kummer’s “second” proof in [12]; but even there it is not at all clear whether Kummer actually used the integral closure of the rings he considered. Simply going through his claims with the example \( \mathcal{O} = \mathbb{Z}[\sqrt{-3}] \) and the ideal prime attached to the Jacobi map \( \phi : \mathcal{O} \to \mathbb{Z}/2\mathbb{Z} \) sending 1 and \( \sqrt{-3} \) to the residue class \( 1 + 2\mathbb{Z} \) does not help very much: Kummer’s construction is so tied to special properties of the rings he is working in that one has to make various choices when attempting to transfer his theory to general number fields, and the places where integral closure is needed depends on the choices that are made.

Consider, for example, the question whether there exist elements \( \Psi \in \mathcal{O} \) such that \( \alpha \cdot \Psi \equiv 0 \mod 2 \) if and only if \( \phi(\alpha) = 0 \) for \( \alpha \in \mathcal{O} \). It is easy to verify that \( \Psi = 1 + \sqrt{-3} \) has this property, but this is not the element you get by faithfully imitating Kummer’s construction. In particular, our \( \Psi \) has the property \( \Psi^2 \equiv 0 \mod 2 \), whereas Kummer proves and uses the fact that \( \Psi^2 \not\equiv 0 \mod p \) for his elements.

In his dissertation [20] on the genesis of Dedekind’s ideal theory, Haubrich also discussed Kummer’s construction. A look into his very informative thesis quickly reveals that even Kummer’s second proof contains a gap! This gap was first noticed by Dedekind in his review [21, p. 418] of Bachmann’s book [11]: Kummer (as well

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33Dirichlet, for example, proved his unit theorem in orders \( \mathbb{Z}[\alpha] \), where \( \alpha \) is an algebraic integer, i.e., a root of a monic polynomial with integral coefficients.

On the other hand E. Heine defined algebraic integers in [21] as numbers that can be constructed from the rational integers by addition, multiplication, and raising to \( m/n \)-th powers with \( m, n \) positive integers. He then goes on to show that every such number is integral in the modern sense, i.e. it is a root of a monic polynomial with integral coefficients. The converse is, of course, false, as Heine’s construction gives only algebraic numbers that are solvable, i.e., that can be expressed in terms of radicals; Heine claimed that any root of a solvable monic polynomial with integral coefficients is integral in his sense, but the proof he gave is not valid.
as Bachmann) did not prove that if an ideal prime divides a number \( m \) times it also divides it \( m - 1 \) times; Dedekind remarked that, as long as this has not been accomplished, it is conceivable that an ideal prime divides a number exactly six times and exactly eight times. Haubrich also explains that Dedekind’s proof of the corresponding fact for ideals uses the integral closure of the domain he was working in.

14. Summary

We have seen that Jacobi introduced techniques which, in our language, give rise to ring homomorphisms \( \phi \) from the ring \( \mathbb{Z}[\alpha] \) of the ring of \( \lambda \)-th roots of unity to \( \mathbb{Z}/p\mathbb{Z} \) for primes \( p \equiv 1 \) mod \( \lambda \). These maps, as Kummer realized, could be used for defining “ideal numbers”. Kummer generalized the Jacobi maps to all primes \( p \) and characterized them by certain sets of congruences (this is completely in line with the spirit of Kronecker, whose ultimate goal was to reduce mathematics to working with natural numbers). While these congruences were easy to work with in practice, they also made the algebraic structure behind the construction of ideal numbers invisible, and Dedekind had to struggle for quite a while before he could uncover this structure again: he arrived at his ideals by looking at the set of all algebraic integers of a number field divisible by Kummer’s ideal numbers. Had he started with Jacobi maps instead of Kummer’s sets of congruences, he might have arrived earlier at the correct definition of ideals: these are exactly the kernels of Jacobi’s ring homomorphisms \( \phi \).

Expositions of Kummer’s theory of ideal numbers were given by Bachmann [1] and Edwards [12]. Both authors give Kummer’s description of ideal numbers in terms of sets of congruences; Edwards ([12 Sect. 4.9, Ex. 10]) gives the bijection between ideal numbers and ring homomorphisms \( \mathbb{Z}[\alpha] \rightarrow \mathbb{F}_q \) as an exercise, and mentioned the connection to valuation theory in [16]. Dieudonné, in his review (MR1160701) of [12], also recognized the algebraic structure behind Kummer’s construction when he wrote

\[
\text{what [...] Kummer did was the determination of discrete valuations on a cyclotomic field.}
\]

Nevertheless, the simple algebraic idea behind Kummer’s construction remained almost as unknown as Jacobi’s role in the creation of ideal numbers.

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