Noise-induced amplification: Parametric amplifiers cannot simulate all phase-preserving linear amplifiers

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Introduction.— Linear amplification has long been an integral part of quantum measurements whereby a weak signal is amplified to a detectable level [1][2]. In more recent times, due to advances in quantum optics and quantum information, linear amplifiers are also seen as a facilitating component of many useful tasks such as state discrimination [3], quantum feedback [4], metrology [5], and entanglement distillation [6, 7].

Quantum mechanics commands any linear amplifier to add noise to its input [8]. Such noisy amplification of a single bosonic input can be modelled by a linear differential equation for its amplitude (annihilation operator) of the form

$$\frac{d}{dt} \hat{a}(t) = \kappa \hat{a}(t) + \hat{f}(t),$$

where $\kappa$ is the amplification rate (a positive real number), and $\hat{f}(t)$ represents noise added by the amplifier. Generally $\hat{f}(t)$ is assumed to be a zero-mean Markov process. It is due to amplifier degrees of freedom. The time-dependent solution $\hat{a}(t)$ to (1)—often at a designated time instant—is defined as the output of the amplifier corresponding to the input $\hat{a}(0)$. The ultimate performance of a linear amplifier is thus determined by the least amount of noise that it must add consistent with quantum theory. Such quantum-limited performance of linear amplifiers had been studied as early as the 1960s for phase-preserving amplifiers [9][10]. These results were later unified, and further generalised to phase-sensitive amplifiers by Caves [11].

It is a long held belief that amplifier-added noise is nothing but mere nuisance. This view has motivated noise-reduction methods in linear amplifiers [12][14], or methods which evade it altogether by trading the amplifier’s deterministic operation for noiseless gain [15][17]. It should not come as a surprise that amplifier-added noise is viewed as something purely negative since it has not been shown to behave otherwise in all known examples of linear amplifiers to date. In fact, the ubiquity of such added noise in linear amplifiers has led Caves et al. to argue that any phase-preserving linear amplifier, no matter how it is realised, is equivalent to a parametric amplifier (paramp) [18]. An implicit caveat for this to be true, as we will show, is that the added noise be independent of the signal (though it may not be obvious how other kinds of noise would arise). In other words, conventional wisdom regards the amplifier noise to be only additive, neglecting the possibility that it may also be multiplicative or otherwise [19].

In this Letter, we show how multiplicative noise may arise in phase-preserving linear amplifiers and that this has two very important implications for the theory of linear amplifiers: (i) Such noise serves to impose quantum mechanics on the amplifier and simultaneously amplify an input signal. This is in stark contrast to the conventional view that the sole function of added noise is to enforce quantum theory. (ii) Such multiplicative noise violates the equivalence of phase-preserving linear amplifiers to the paramp model claimed in Ref. [18].

The paper is organised as follows. We first define the model for noise-induced amplification and then describe its realisation, leaving the details to Ref. [20]. Its phase properties are then analysed and shown to satisfy the cri-
That is, it is possible to choose a noise process \( \hat{a} \) real if it is to be phase preserving \([11, 18]\)). We show that approximation does not treat \( \hat{a} \) certainly allowed within the Born–Markov framework. The fier and the signal that is being amplified. This is the to Ref. \([20]\).

Pauli lowering operators for each atom (the weights be-

\[ \frac{d}{dt} \hat{a}(t) = \hat{a}^\dagger(t) \hat{w}(t). \] (2)

That is, it is possible to choose a noise process \( \hat{w}(t) \) that ensures \( \langle \hat{a}(t), \hat{a}^\dagger(t) \rangle = 1 \) for all \( t \), and simultaneously satisfies the requirement for amplification

\[ \langle \hat{a}^\dagger(t) \hat{w}(t) \rangle = \gamma \langle \hat{a}(t) \rangle, \] (3)

where \( \gamma \geq 0 \). Unlike the ‘signal-plus-noise’ model of \([1]\), the noise in \([2]\) is multiplicative but otherwise \( \hat{w}(t) \) itself may be treated as zero-mean and Markovian, just like \( \hat{f}(t) \) in \([1]\). We refer to \([2]\) and \([3]\) as noise-induced amplification (noisi amplification), and to the amplifier as a noisiamp. The amplitude gain of the noisiamp is \( g(t) = \langle \hat{a}(t) \rangle / \langle \hat{a}(0) \rangle = \exp(\gamma t) \).

The model of \([2]\) and \([3]\) can be realised by using a gain medium that mediates two-photon interactions. An effective model for this is an interaction Hamiltonian that couples a single bosonic mode (representing the signal) to a collection of two-level atoms via two-photon exchanges in the rotating-wave approximation. One can show, that such an interaction Hamiltonian within the Born–Markov approximation leads to \([2]\) on setting the excited-state and ground-state atomic populations equal. The operator \( \hat{w}(t) \) is then realised by a weighted sum over the Pauli lowering operators for each atom (the weights being coupling constants). The detailed derivation is left to Ref. \([20]\).

Noisi amplification can be understood as classical correlation between the internal noise source of the ampli-

and the signal that is being amplified. This is the essential content of \([3]\). Though such equations are not typically encountered in the amplifier literature, it is certainly allowed within the Born–Markov framework. The important point to note here is that the Markov approximation does not treat \( \hat{w}(t) \) as truly white. All that is required is for \( \hat{w}(t) \) to have a small but nonzero correlation time, otherwise \( \hat{w}(t) \) would be zero and there would be no amplification. In other words, if there is no correlation between the noise and signal, there is no amplification.

**Phase properties.**—The noisiamp is a phase-covariant amplifier. We define an arbitrary linear amplifier to be phase covariant if and only if the corresponding completely-positive trace-preserving map (hereafter simply map) \( \mathcal{A}(t) \) is such that

\[ \mathcal{A}(t) \mathcal{P}_\varphi = \mathcal{P}_\varphi \mathcal{A}(t), \] (4)

where \( \mathcal{P}_\varphi = \exp(-i\varphi \hat{a}^\dagger \hat{a}) \) is the phase-shift map. To prove phase covariance of the noisiamp it is useful to think of \( \mathcal{A}(t) \) as any compositions of \( \mathcal{A}(\delta t) \) in the limit that \( \delta t \to 0 \). That is, if we define \( \delta t = t/n \) then \( \mathcal{A}(t) = [\mathcal{A}(\delta t)]^n \) as \( n \to \infty \). Hence to show the noisiamp is phase covariant, all we have to do is show that its map satisfies \([1]\) for an infinitesimal time interval \( dt \). The map for the noisiamp can be shown to have the form

\[ \mathcal{L} = \frac{\gamma}{2} (\mathcal{D}[\hat{a}]^2 + \mathcal{D}[\hat{a}^\dagger]^2), \] (5)

where \( \mathcal{D}[\hat{A}] = \hat{A}\hat{D} - (\hat{A}^\dagger \hat{A} + \hat{A} \hat{A}^\dagger)/2 \) for any \( \hat{A} \). For an infinitesimal interval \( dt \) we may write \( \mathcal{N}(dt) = 1 + \mathcal{L} dt \) in which case the condition \( \mathcal{N}(dt) \mathcal{P}_\varphi = \mathcal{P}_\varphi \mathcal{N}(dt) \) becomes

\[ \mathcal{L} \mathcal{P}_\varphi = \mathcal{P}_\varphi \mathcal{L}. \] (6)

It is then possible to show that \([6]\) is true \([20]\).

One can also show, by an explicit calculation, that the noisiamp is phase insensitive in the conventional sense \([11, 21]\). That is, \( \hat{x}_\varphi = [\hat{a} \exp(-i\varphi) + \hat{a}^\dagger \exp(i\varphi)]/\sqrt{2} \) is such that it satisfies \( \langle \hat{x}_\varphi(t) \rangle = g \langle \hat{x}_\varphi(0) \rangle \) and \( \langle [\Delta \hat{x}_\varphi(t)]^2 \rangle = g^2 \langle [\hat{x}_\varphi(0)]^2 \rangle + N \) where \( g \) and \( N \) are independent of \( \varphi \), and we have defined \( \Delta \hat{x}_\varphi = \hat{x}_\varphi - \langle \hat{x}_\varphi \rangle \). For the noisiamp \( g \) is as defined already, and \( N = g^2 (g^2 - 1)(\langle n(0) \rangle + 1)/2 \) \( (\hat{n} = \hat{a}^\dagger \hat{a}) \). Both are indeed independent of \( \varphi \).

**Inequivalence to the paramp.**—A paramp has two inputs, \( \hat{a} \) and \( \hat{b} \). The input mode \( \hat{a} \) represents the signal amplitude to be amplified and acts on Hilbert space \( \mathbb{H}_A \). Mode \( \hat{b} \) is an ancillary system acting on \( \mathbb{H}_B \) and whose initial state is \( \sigma \). If we assume the signal mode to be prepared in an initial state \( \rho(0) \), then the paramp output in the Schrödinger picture is defined by

\[ \rho(t) = \mathcal{E}(t) \rho(0) = \text{Tr}_B [\hat{S}(t) \rho(0) \otimes \sigma \hat{S}^\dagger(t)]. \] (7)

Here \( \text{Tr}_B \) denotes a partial trace over \( \mathbb{H}_B \) and \( \hat{S}(t) = \exp[\kappa (\hat{a} \hat{b} - \hat{a}^\dagger \hat{b}^\dagger) t] \) when \( \kappa \geq 0 \) (see also Fig. \([1]\)). In Ref. \([18]\), it was asserted that any phase-preserving linear amplifier, no matter how it is physically realised, is always equivalent to the paramp for some \( (\kappa, \tau) \) (which determines its amplitude gain), and a physical choice of \( \tau \), thus leading to a complete classification of such amplifiers. The foregoing analysis of the noisiamp phase properties show that it satisfies all the assumptions required in Ref. \([18]\) to fall under the ambit of the paramp model. We now show that, despite this, the noisiamp is irreproducible by a paramp as illustrated in Fig. \([1]\).

For the paramp map \( \mathcal{E}(t) \) to be equivalent to \( \mathcal{N}(t) \), it is necessary that moments of the output \( \hat{a}(t) \) from the
requirement leads to Equation (12) clearly cannot be satisfied unless \( g \neq 1 \). This means no amplification. Thus, the paramp cannot be a universal model for all phase-preserving linear amplifiers. Note that it is the difference in how \( \langle \hat{n}(t) \rangle \) scales with \( g \) in the two types of amplifiers that makes \( \mathcal{E}(t) \neq \mathcal{N}(t) \). To the best of our knowledge, this is the first time that a phase-preserving linear amplifier has been shown to fall outside the reach of the paramp.

It is natural to wonder whether the noisiamp is something of a special case. One can show that \( \mathcal{M}(t) = \exp(\mathcal{E}t) \) where now

\[
\mathcal{L} = \frac{\gamma}{9} \left( D[\hat{a}^3] + D[\hat{a}^{13}] \right) + \gamma D[\hat{a}^2] ,
\]

defines a phase-covariant linear amplifier with the same gain as (2) but may have a different Heisenberg equation. In this case a simple analytic expression like (8) cannot be found for the average output photon number. It is nevertheless possible to show that \( \mathcal{M}(t) \) leads to average photon numbers which are irreproducible by the paramp. See the detailed proof in Ref. [20].

**Atom-photon interactions.**—We stated underneath how the noisiamp may be realised using a gain medium with two-photon transitions. We also discussed how it can be understood in terms of correlations from the perspective of stochastic processes. Here we delve into these issues a little bit more. Namely, we show how the noisiamp operates at the level of elementary atom-photon interactions by taking the white-noise limit of (2). It will be instructive to first see how the standard linear amplifier of (1) can be understood in terms of elementary atom-photon interactions when it is realised using a gain medium with one-photon transitions. Such a realisation of (1) would result in \( \kappa = \kappa_\uparrow - \kappa_\downarrow \) where \( \kappa_\uparrow \) and \( \kappa_\downarrow \) are effective populations of the excited and ground states of the gain-medium atoms [24]. In this case, the average photon number of the signal evolves according to

\[
\frac{d}{dt} \langle \hat{n}(t) \rangle = \frac{1}{2} (\kappa_\uparrow - \kappa_\downarrow) \langle \hat{n}(t) \rangle + \kappa_\uparrow .
\]

Each term on the right-hand side can be linked to an elementary atom-photon interaction, namely \( \kappa_\uparrow \langle \hat{n}(t) \rangle \) originates from stimulated emission while \(-\kappa_\downarrow \langle \hat{n}(t) \rangle \) is due to absorption since these are the only two atom-photon interactions that can depend on the signal strength \( \langle \hat{n}(t) \rangle \).

The one process which depends solely on the excited-state population is spontaneous emission which is associated with \( \kappa_\uparrow \). These processes are illustrated in Fig. 2(a) for later comparison with the noisiamp. Equation (14) suggests that if one wants to understand the noisiamp in terms of elementary atom-photon interactions then one should derive its corresponding equation of motion for the average photon number. Furthermore, if we want to see clearly the effect of the multiplicative noise on the evolution of \( \langle \hat{n}(t) \rangle \) then this calculation is best done in the Heisenberg picture. However, the correlation between \( \hat{\psi}(t) \) and \( \hat{a}_2(t) \) makes such an analysis rather cumbersome so to simplify the calculation we take the white-noise limit of (2). By the Wong–Zakai theorem, (2) must then be interpreted as a Stratonovich quantum stochastic differential equation [24][25]. The Itô equivalent of (2)

![Diagram](image-url)
would then have decorrelated noise, and becomes easier to analyse [27–29]. Employing Itô calculus then simplifies the operator algebra that one has to do for higher-order moments of $\hat{a}(t)$. It is this simplicity of the Itô formalism that we will now exploit to understand the noisiank defined by (2) and (3) in more detail.

Within Itô quantum stochastic calculus, the noisiank may be defined by a two-photon Hudson–Parthasarathy equation for the Stinespring dilation $\hat{U}(t,0)$ of the atom-field system [30, 31]:

$$d\hat{U}(t,0) = \left\{ -\frac{\gamma}{4} \left[ \hat{a}^2(t) \hat{a}^\dagger(t) + \hat{a}^\dagger(t) \hat{a}^2(t) \right] dt + \frac{1}{2} \left[ \hat{a}^2(t) d\hat{W}^\dagger(t) - \hat{a}^\dagger(t) d\hat{W}(t) \right] \right\} \hat{U}(t,0) ,$$

where $d\hat{A}(t) = \hat{A}(t+dt) - \hat{A}(t)$ for any operator $\hat{A}(t)$ and $\hat{W}(t)$ is a quantum Wiener process [integral of $\hat{w}(t)$]. Its increment has zero mean, and the second-order moments (quantum Itô rules) $d\hat{W}^\dagger(t) d\hat{W}(t) = d\hat{W}(t) d\hat{W}^\dagger(t) = 2\gamma dt$ [30, 31]. The $2\gamma$ here may be taken as an effective excited-state population of the gain medium which is set equal to the ground-state population. The quantum Itô rules essentially assume the range of transition frequencies in the atoms of the gain medium to be infinite. It is now simple to show from (15) that the Itô equivalent of (2) is

$$d\hat{a}(t) = \gamma \hat{a}(t) dt + \hat{a}(t) d\hat{W}(t) .$$

Using (16) the average photon number evolves as

$$d\langle \hat{n}(t) \rangle = 2\gamma \langle \hat{n}(t) \rangle dt + \langle \hat{a}(t) \hat{a}^\dagger(t) d\hat{W}^\dagger(t) d\hat{W}(t) \rangle = 2\gamma \langle \hat{n}(t) \rangle dt + \left[ \langle \hat{n}(t) \rangle + 1 \right] 2\gamma dt .$$

The first term in (17) is inherited from the $\gamma \hat{a}(t) dt$ term in (16) and contributes $2\gamma \langle \hat{n}(t) \rangle dt$ number of photons to the signal on average over an infinitesimal interval $dt$. By analogy to (14) it corresponds to one-photon stimulated emission even though only two-photon processes are covered in our model! The effect of the multiplicative processes enters through the second term. As can be seen from (17), it contributes $\{ \langle \hat{n}(t) \rangle + 1 \} 2\gamma dt$ noise photons (on average) to the output in a time of $dt$. In contrast to (14), there are now two types of noise photons. The first is linear in $\langle \hat{n}(t) \rangle$, so it again corresponds to a one-photon process but note that it also depends on the signal strength reaching the atom. The fact that it is a noise photon means that it must have come from spontaneous emission. The seemingly strange possibility of getting one-photon processes in a two-photon model can now be resolved when we combine this noise photon with the previous stimulated photon to arrive at the two-photon process shown on the left of Fig. 2(b). This is the underlying mechanism responsible for linear (i.e. one-photon) amplification in a two-photon process. It also gives us a physical picture of why it must be correlated with the amplifier’s internal noise. The second type of noise photon is the constant $2\gamma dt$ in (17) which corresponds to two-photon spontaneous emission. This is shown on the right in Fig. 2(b).

Discussion.—Interestingly the claimed universality of the paramp model can be understood in terms of the concept of programmability defined for a set of maps [32]. A family of completely-positive maps $\{ \Phi_k \}$ acting on $\rho \in \mathbb{H}_S$ is defined to be programmable if and only if there exist a set of states $\{ \sigma_k \}$ ($\sigma_k \in \mathbb{H}_S$) and a fixed unitary $U$ (independent of $k$) acting on $\mathbb{H}_S \otimes \mathbb{H}_B$ such that $\Phi_k \rho = \text{Tr}_B[U(\rho \otimes \sigma_k)U^\dagger]$. If we let $\{ \Phi_k \}$ describe the set of phase-preserving linear amplifiers with a common gain, then Ref. [18] may be understood as claiming $\{ \Phi_k \}$ to be programmable by a paramp i.e. with $\hat{U} = \hat{S}$ [two-mode squeezing with a fixed $\kappa t$, defined underneath (1)]. We have thus shown that there are in fact some phase-preserving linear amplifiers (namely ones with multiplicative noise) that a paramp cannot be programmed to simulate.

The use of the Heisenberg picture should also be highlighted. A Schrödinger-picture treatment where the input state evolves under a master equation (an equation of motion for the signal state) has the noise averaged over. The Heisenberg picture has the advantage that it does not do this averaging and this is what allowed us to see explicitly its effect on the input signal [as demonstrated in (2), (3), and (17)]. The possibility of linear amplification and the atom-photon interaction necessary for this to occur in a two-photon device had been noted earlier [33]. But no explicit connection to noise was made (such as addressing the type of noise the amplifier was adding). The Heisenberg-picture analysis used here is thus critical for the insight on amplifier-added noise. It is what allowed us to cleanly pick out the noise contribution to the output to arrive at Fig. 2(b).
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[34] This same property has been referred to as phase-preserving in the strict sense in Ref. [18].
Supplementary Material for “Noise-induced amplification: Parametric amplifiers cannot simulate all phase-preserving linear amplifiers”

I. REALISATION OF THE NOISE-INDUCED AMPLIFIER

A. Amplitude equation of motion

The noisiamp may be realised as an open system modelled by a single bosonic oscillator (with Hilbert space $H_S$) coupled to a bath of two-level atoms (with Hilbert space $H_B$) that mediate two-photon transitions. The atoms model the gain medium that is used for amplification. The full Hamiltonian $\hat{H}$ on $H_S \otimes H_B$ is

$$\hat{H} = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \sum_n \frac{\hbar \omega_n}{2} \hat{\pi}_n^+ \hat{\pi}_n^- + \hbar^2 \hat{\Pi}^\dagger \hat{\Pi},$$

where $\omega_0$ is the oscillator’s natural frequency and $\hat{\pi}_n^+, \hat{\pi}_n^-$, and $\hat{\pi}_n^-$ are atomic operators for the $n$th atom is defined by

$$\hat{\pi}_n^+ = \hat{\pi}_n^- \hat{\pi}_n^+ - \hat{\pi}_n^- \hat{\pi}_n^+, \quad \hat{\pi}_n^+ = |\uparrow\rangle_n \langle \downarrow|, \quad \hat{\pi}_n^- = |\downarrow\rangle_n \langle \uparrow|.$$  

We have also defined the bath operators

$$\hat{\Pi} = \hbar \sum_n \xi_n \hat{\pi}_n^-, \quad \hat{\Pi}^\dagger = \hbar \sum_n \xi_n^* \hat{\pi}_n^+.$$  

The bath will be assumed to be at temperature $T$ so that its state $\rho_B$ is given by

$$\rho_B = \bigotimes_n \rho_n, \quad \rho_n = \frac{\exp\left(-\beta \hbar \omega_n \hat{\pi}_n^+ \hat{\pi}_n^- / 2\right)}{Z_n},$$

where we have defined $\beta = 1/k_B T$, and $k_B$ is the Boltzmann constant. The normalisation of $\rho_n$ (also the partition function) is

$$Z_n = \text{Tr}\left[\exp\left(-\beta \hbar \omega_n \hat{\pi}_n^+ \hat{\pi}_n^- / 2\right)\right] = 2 \cosh\left(\frac{\beta \hbar \omega_n}{2}\right).$$

The Heisenberg equation of motion for the oscillator’s amplitude $\hat{a}$ is defined by

$$\frac{d}{dt}\hat{a}(t) = -\frac{i}{\hbar} e^{i \hat{H} t / \hbar} \left[\hat{a}(0), \hat{H}(0)\right] e^{-i \hat{H} t / \hbar},$$

where $\hat{H}(0) = \hat{H}$. Noting that at the initial time the system and bath operators commute, we find

$$\frac{d}{dt}\hat{a}(t) = -i \omega_0 \hat{a}(t) - \frac{i}{\hbar} 2 \hat{a}^\dagger(t) \hat{\Pi}(t),$$

where $\hat{\Pi}(t) = \exp(i \hat{H} t / \hbar) \hat{\Pi} \exp(-i \hat{H} t / \hbar)$. It helps to move into a rotating frame at the oscillator frequency by defining

$$\hat{a}(t) = \hat{a}(t) e^{i \omega_0 t}.$$  

Differentiating $\hat{a}(t)$ and using we get

$$\frac{d}{dt}\hat{a}(t) = -\frac{i}{\hbar} 2 \hat{a}^\dagger(t) \hat{\Pi}(t) e^{i 2 \omega_0 t}.$$  

We see that $\hat{a}(t)$ is coupled to $\hat{\Pi}(t)$. To deal with this one may substitute the formal solution for $\hat{\Pi}(t)$ back into it iteratively. However, if the system and bath are only weakly coupled then we can approximate the system evolution
up to second order in the interaction strength. This step constitutes the so-called Born approximation, after which we arrive at
\[
\frac{d}{dt} \bar{a}(t) = -\frac{i}{\hbar} 2 \bar{a}^\dagger(t) \bar{\Pi}(t) e^{i 2\omega_0 t} - \frac{2}{\hbar^2} \bar{a}^\dagger(t) \int_0^t dt' \bar{a}^2(t') \left[ \bar{\Pi}(t), \bar{\Pi}^\dagger(t') \right] e^{i 2\omega_0(t-t')},
\]
where we have defined
\[
\bar{\Pi}(t) = \hbar \sum_n \xi_n \bar{\pi}_n(0) e^{-i \omega_n t}.
\]
The system’s evolution is now affected by the history of \(\bar{\Pi}(t)\) in the commutator inside the integrand. We can simplify this by first replacing the bath commutator by its average, which is justified if we are going to use the Heisenberg equation of motion for calculating expectation values. The dependence on the history of \(\bar{\Pi}(t)\) can then be simplified by making the Markov approximation: This relies on the characteristic timescale over which \(\bar{a}(t)\) evolves to be much longer than the timescale over which bath correlations decay. In this regime we can then replace the system operators at time \(t'< t\), by the present time \(t\) and extend the top limit of the time integrals to infinity. Doing so allows us to compute the time integral in (28) by assuming the distribution of transition frequencies of the atoms to be sufficiently dense. We may then convert the sum over atomic degrees of freedom in \(\bar{\Pi}(t)\) into an integral by introducing a function \(D(\omega)\) which counts how many atoms there are per transition frequency in the bath. That is, \(D(\omega) d\omega\) is the number of atoms in the bath with a transition frequency in the range from \(\omega\) to \(\omega + d\omega\). We then have
\[
\frac{d}{dt} \bar{a}(t) = (\kappa_{\bar{h}} - \kappa_{\bar{g}}) \bar{a}^\dagger(t) \bar{a}^2(t) - \frac{i}{\hbar} 2 \bar{a}^\dagger(t) \bar{\Pi}(t) e^{i 2\omega_0 t},
\]
where we have further defined
\[
\kappa_{\bar{h}} = \gamma N_{\bar{h}}(2\omega_0, T), \quad \kappa_{\bar{g}} = \gamma N_{\bar{g}}(2\omega_0, T), \quad \gamma \equiv 2\pi D(2\omega_0) |\xi(2\omega_0)|^2.
\]
In (30) we have neglected shifts in the oscillator’s frequency due to the bath correlation functions on the grounds that they are typically very small [1, 2]. Physically this can be expected since atoms that are detuned from \(2\omega_0\) would not be expected to have a strong two-photon coupling. The dominant coupling occurs for the on-resonance case and they give rise to \(\kappa_{\bar{h}}\) and \(\kappa_{\bar{g}}\).

B. Effects of noise—linear amplification

The model of (30) now becomes a linear system on setting \(N_{\bar{h}} = N_{\bar{g}} = 1/2\) so that
\[
\kappa_{\bar{h}} = \kappa_{\bar{g}} = \frac{\gamma}{2}.
\]
Thus to obtain at least a linear system we now restrict our attention on the limit of (32). A linear amplifier should then have a linear equation of motion in the amplitude when averaged. The dynamics of \(\langle \bar{a}(t) \rangle\) can be found directly by taking the average of the nonlinear Heisenberg equation (30):
\[
\frac{d}{dt} \langle \bar{a}(t) \rangle = -\frac{i}{\hbar} 2 \langle \bar{a}^\dagger(t) \bar{\Pi}(t) \rangle e^{i 2\omega_0 t}.
\]
The difficulty in calculating \(\langle \bar{a}^\dagger(t) \bar{\Pi}(t) \rangle\) is because \(\bar{\Pi}(t)\) is correlated with \(\bar{a}(t)\). To deal with this we integrate (30) to get
\[
\langle \bar{a}^\dagger(t) \bar{\Pi}(t) \rangle = \langle \bar{a}^\dagger(0) \bar{\Pi}(t) \rangle + (\kappa_{\bar{h}} - \kappa_{\bar{g}}) \int_0^t dt' \langle \bar{a}^\dagger(t') \bar{a}(t') \bar{\Pi}(t) \rangle + 2 \frac{i}{\hbar} \int_0^t dt' \langle \bar{\Pi}^\dagger(t') \bar{a}(t') \bar{\Pi}(t) \rangle e^{-i 2\omega_0 t'}.
\]
Because we are assuming \(\bar{\Pi}(t)\) to have very short correlation times (the Markov approximation) we can approximate factorise multitime averages between noise operators at time \(t\) and system operators at time \(t'\) provided that \(t > t'\). For example, for any system operator \(\hat{s}\),
\[
\langle \hat{s}(t') \bar{\Pi}(t) \rangle = \langle \hat{s}(t') \rangle \langle \bar{\Pi}(t) \rangle = 0, \quad t > t',
\]
where we have noted that $\tilde{\Pi}(t)$ has zero mean. When $t > t'$ $\tilde{\Pi}(t)$ and $\tilde{s}(t')$ are in fact independent so we have
\[
[\tilde{s}(t'), \tilde{\Pi}(t)] = 0 , \quad t > t'.
\]
Hence by Markovianity we have,
\[
\langle \hat{a}^\dagger(t) \tilde{\Pi}(t) \rangle = 2 \frac{i}{\hbar} \langle \tilde{a}(t) \rangle e^{-i 2 \omega_0 t} \int_0^\infty d\tau \langle \tilde{\Pi}(t-\tau) \tilde{\Pi}(t) \rangle e^{i 2 \omega_0 \tau} = i \hbar \kappa_\tilde{\Pi} \langle \tilde{a}(t) \rangle e^{-i 2 \omega_0 t}.
\]
Substituting this back into (33) thus gives
\[
\frac{d}{dt} \langle \tilde{a}(t) \rangle = -\frac{i}{\hbar} 2 \langle \hat{a}^\dagger(t) \tilde{\Pi}(t) \rangle e^{i 2 \omega_0 t} = 2 \kappa_\tilde{\Pi} \langle \tilde{a}(t) \rangle.
\]
Equations (30) and (38) (for $\kappa_\tilde{\Pi} = \kappa_\tilde{\omega}$) can now be seen to reproduce the noisiamp model of (2) and (3) in the main text by defining
\[
\hat{w}(t) = -\frac{i}{\hbar} 2 \tilde{\Pi}(t) e^{i 2 \omega_0 t},
\]
and relabelling $\tilde{a}(t)$ as $\hat{a}(t)$ (keeping in mind that it is an equation of motion in the rotating frame).

C. Consistency with quantum mechanics and photon-number evolution

We can show that (30) preserves the canonical commutation relation for $\hat{a}$ and $\hat{a}^\dagger$. It is sufficient to show that if at time $t$ the commutator is true, then
\[
\frac{d}{dt} [\hat{a}(t), \hat{a}^\dagger(t)] = 0 ,
\]
where $\hat{d}s(t) = \hat{s}(t+dt) - \hat{s}(t)$ for any operator $\hat{s}$. From now on we shall simply write $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ for the annihilation and creation operators in the rotating frame. A calculation akin to Sec. I A is rather cumbersome so we use the Itô equivalent of (30) [see the discussion around (16) in the main text]:
\[
d\hat{a}(t) = 2 \kappa_\tilde{\Pi} \hat{a}(t) dt + (\kappa_\tilde{\Pi} - \kappa_\tilde{\omega}) \hat{a}^\dagger(t) \hat{a}^2(t) + \hat{a}^\dagger(t) d\hat{W}(t)
\]
where $d\hat{W}(t) = \hat{w}(t) dt$ satisfies the Itô rules
\[
d\hat{W}(t) d\hat{W}(t) = 4 \kappa_\tilde{\Pi} dt , \quad d\hat{W}(t) d\hat{W}^\dagger(t) = 4 \kappa_\tilde{\omega} dt.
\]
Omitting the time argument for ease of writing we have
\[
\frac{d}{dt} [\hat{a}, \hat{a}^\dagger] = (\hat{d}\hat{a}) \hat{a}^\dagger + \hat{a} (\hat{d}\hat{a}^\dagger) + (\hat{d}\hat{a}^\dagger) \hat{a} - (\hat{d} \hat{a}) \hat{a}^\dagger - (\hat{d} \hat{a}^\dagger) \hat{a} + (\hat{d}\hat{a}^\dagger) \hat{a}^2
\]
\[
= (\kappa_\tilde{\Pi} - \kappa_\tilde{\omega}) (\hat{a}^\dagger \hat{a}^2 \hat{a}^\dagger + \hat{a} \hat{a}^2 \hat{a}^\dagger - 2 \hat{a} \hat{a}^2 \hat{a}^\dagger) dt - 4 (\kappa_\tilde{\Pi} - \kappa_\tilde{\omega}) \hat{a}^\dagger dt.
\]
On normal ordering the first two terms in the parentheses on the right-hand side we arrive at (40). As part of the proof of (40) we have also worked out the photon-number evolution in the general case when $\kappa_\tilde{\Pi} \neq \kappa_\tilde{\omega}$. Its average gives
\[
\frac{d}{dt} \langle \hat{a}^\dagger \hat{a} \rangle = 2 (\kappa_\tilde{\Pi} - \kappa_\tilde{\omega}) \langle \hat{a}^\dagger \hat{a}^2 \rangle + 8 \kappa_\tilde{\Pi} \langle \hat{a}^\dagger \hat{a} \rangle + 4 \kappa_\tilde{\omega}.
\]
In the main text we worked out the corresponding atom-photon interactions taking place when $\kappa_\tilde{\Pi} = \kappa_\tilde{\omega}$ so that the nonlinear term in (15) does not contribute to the noisiamp [see Fig. 2(b) of the main text]. The nonlinear term here represents a two-photon generalisation of the linear (i.e. one-photon) amplifier. We depict the necessary atom-photon interactions associated with the general two-photon amplifier in Fig. 3.

By the same arguments as in Sec. I A one can derive the equation of motion for the oscillator’s state in the rotating frame. This is a master equation of the form
\[
\frac{d}{dt} \rho(t) = \kappa_\tilde{\omega} D[\hat{a}^2] \rho(t) + \kappa_\tilde{\Pi} D[\hat{a}^\dagger \hat{a}] \rho(t) \equiv \mathcal{L} \rho(t).
\]
This provides an alternative check for the consistency of the moment equations obtained using the Heisenberg equations of motion. The master-equation calculation however would not reveal which terms originate from noise.
II. PHASE COVARIANCE

We have already argued in the main text that phase covariance follows from (6) (recalled here for convenience):

$$\mathcal{L} \mathcal{P}_\varphi = \mathcal{P}_\varphi \mathcal{L} ,$$

with $\mathcal{L}$ given by (46) and (32) for the noisamp and

$$\mathcal{P}_\varphi \rho = e^{-i\varphi \hat{n}} \rho e^{i\varphi \hat{n}} \equiv \rho_\varphi ,$$

where $\hat{n} = \hat{a}^\dagger \hat{a}$.

Here we will in fact prove that any $m$-photon dissipator leads to a phase-covariant channel, namely,

$$\mathcal{D}[\hat{a}^m] \mathcal{P}_\varphi = \mathcal{P}_\varphi \mathcal{D}[\hat{a}^m] , \quad \mathcal{D}[\hat{a}^\dagger m] \mathcal{P}_\varphi = \mathcal{P}_\varphi \mathcal{D}[\hat{a}^\dagger] .$$

This is useful for showing that all other linear amplifier constructed using (49) are all phase covariant as in our second counterexample in the next section. For $\mathcal{D}[\hat{a}^m]$ we have,

$$\mathcal{D}[\hat{a}^m] \mathcal{P}_\varphi \rho = \hat{a}^m \rho_\varphi \hat{a}^\dagger m - \frac{1}{2} \hat{a}^\dagger m \hat{a}^m \rho_\varphi - \frac{1}{2} \rho_\varphi \hat{a}^\dagger m \hat{a}^m$$

$$= \left( e^{-i\varphi \hat{n}} e^{i\varphi \hat{n}} \right) \hat{a}^m \rho_\varphi \hat{a}^\dagger m \left( e^{-i\varphi \hat{n}} e^{i\varphi \hat{n}} \right) - \frac{1}{2} \hat{a}^\dagger m \left( e^{-i\varphi \hat{n}} e^{i\varphi \hat{n}} \right) \hat{a}^m \rho_\varphi$$

$$- \frac{1}{2} \rho_\varphi \hat{a}^\dagger m \left( e^{-i\varphi \hat{n}} e^{i\varphi \hat{n}} \right) \hat{a}^m \left( e^{-i\varphi \hat{n}} e^{i\varphi \hat{n}} \right)$$

$$= e^{-i\varphi \hat{n}} \left( e^{i\varphi \hat{n}} \hat{a}^m \rho_\varphi \hat{a}^\dagger m e^{-i\varphi \hat{n}} \right) e^{i\varphi \hat{n}} - \frac{1}{2} \left( e^{-i\varphi \hat{n}} \left( e^{i\varphi \hat{n}} \hat{a}^\dagger m e^{-i\varphi \hat{n}} \right) \left( e^{i\varphi \hat{n}} \hat{a}^m e^{-i\varphi \hat{n}} \right) \rho_\varphi \right)$$

$$= e^{-i\varphi \hat{n}} \left( e^{i\varphi \hat{n}} \hat{a}^m \rho_\varphi \hat{a}^\dagger m e^{-i\varphi \hat{n}} \right) e^{i\varphi \hat{n}} - \frac{1}{2} \left( e^{i\varphi \hat{n}} \left( e^{-i\varphi \hat{n}} \hat{a}^\dagger m \hat{a}^m e^{-i\varphi \hat{n}} \right) \left( e^{-i\varphi \hat{n}} \hat{a}^\dagger m \hat{a}^m e^{-i\varphi \hat{n}} \right) \rho_\varphi \right) .$$

This can simplified by noting that $\exp(i\varphi \hat{n}) \hat{a} \exp(-i\varphi \hat{n}) = \hat{a} \exp(-i\varphi)$ from which we can also see that

$$\exp(i\varphi \hat{n}) \hat{a}^m \exp(-i\varphi \hat{n}) = \hat{a}^m e^{-i m \varphi} .$$

Equation (52) is thus

$$\mathcal{D}[\hat{a}^m] \mathcal{P}_\varphi \rho = e^{-i\varphi \hat{n}} \hat{a}^m \rho_\varphi \hat{a}^\dagger m e^{i\varphi \hat{n}} - \frac{1}{2} e^{-i\varphi \hat{n}} \hat{a}^\dagger m \hat{a}^m \rho_\varphi - \frac{1}{2} e^{i\varphi \hat{n}} \rho_\varphi \hat{a}^\dagger m \hat{a}^m$$

$$= e^{-i\varphi \hat{n}} \left( \hat{a}^m \rho_\varphi \hat{a}^\dagger m - \frac{1}{2} \hat{a}^\dagger m \hat{a}^m \rho_\varphi - \frac{1}{2} \rho_\varphi \hat{a}^\dagger m \hat{a}^m \right) e^{i\varphi \hat{n}}$$

$$= \mathcal{P}_\varphi \mathcal{D}[\hat{a}^m] \rho .$$

The proof for $\mathcal{D}[\hat{a}^\dagger m]$ follows similarly on replacing $\hat{a}$ with $\hat{a}^\dagger$ and using

$$\exp(i\varphi \hat{n}) \hat{a}^\dagger m \exp(-i\varphi \hat{n}) = \hat{a}^\dagger m e^{i m \varphi} .$$
FIG. 4: No matter how the paramp operating state $\sigma$ is chosen to change $\langle \hat{b}(0)\hat{b}^\dagger(0) \rangle$, it cannot produce photon-number outputs that are entirely within the allowed region above the red-dashed line.

III. A THREE-PHOTON COUNTEREXAMPLE

To demonstrate the non-uniqueness of the noisamp as an example which cannot be described by the paramp, we proposed in the main text the example defined by

$$\frac{d}{dt} \rho(t) = \frac{\gamma}{9} D[\hat{a}^3] \rho(t) + \frac{\gamma}{9} D[\hat{a}^4] \rho(t) + \gamma D[\hat{a}^2] \rho(t).$$

(57)

This is clearly a phase-covariant linear amplifier by the results of Sec. I. It is simple to show from this that

$$\frac{d}{dt} \langle \hat{a} \rangle = \gamma \langle \hat{a} \rangle,$$

(58)

$$\frac{d}{dt} \langle \hat{n} \rangle = 2\gamma \langle \hat{n}^2 \rangle + 6\gamma \langle \hat{n} \rangle + 2\gamma.$$

(59)

It is obvious that $\langle \hat{a}(t) \rangle = g \langle \hat{a}(0) \rangle$ where $g = \exp(\gamma t)$. However, the output average photon number is now coupled to its second moment. We can still show that it leads to unattainable values for the paramp by considering a lower bound of $\langle \hat{n}(t) \rangle$ by ignoring the first term of (59). Solving the resulting differential equation gives

$$\langle \hat{n}(t) \rangle_{\text{LB}} = g^6 \langle \hat{n}(0) \rangle + \frac{g^6 - 1}{3}.$$

(60)

The paramp with identical amplitude gain has

$$\langle \hat{n}(t) \rangle = g^2 \langle \hat{n}(0) \rangle + (g^2 - 1) \langle \hat{b}(0)\hat{b}^\dagger(0) \rangle.$$

(61)

When considered a function of $\langle \hat{n}(0) \rangle$, $\langle \hat{n}(0) \rangle_{\text{LB}}$ is a straight line with gradient $g^2$ and vertical intercept $(g^2 - 1) \langle \hat{b}(0)\hat{b}^\dagger(0) \rangle$. This is shown as the black line in Fig. 4. On the same axes, $\langle \hat{n}(0) \rangle_{\text{LB}}$ is shown as the red dashed line (not to scale). It is also a straight line but with a larger gradient and vertical intercept $(g^6 - 1)/3$. The actual solution to (59) must therefore
lie above the red-dashed line while the area below it (shaded region) is forbidden. Figure 4 clearly illustrates that no matter how $\langle \hat{b}(0)\hat{b}^\dagger(0) \rangle$ is chosen (by choosing $\sigma$ in the ancillary mode), the paramp $\langle \hat{n}(t) \rangle$ always has a segment in the forbidden region of the three-photon example.

[1] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, (Oxford University Press, 2002).
[2] H. J. Carmichael, *Statistical Methods in Quantum Optics 1* (Second-corrected-printing), (Springer, 2002).