The universal property of derived geometry

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September 11, 2017

Abstract

Derived geometry can be defined as the universal way to adjoin finite homotopical limits to a given category of manifolds compatibly with products and glueing. The point of this paper is to show that a construction closely resembling existing approaches to derived geometry in fact produces a geometry with this universal property.

I also investigate consequences of this definition in particular in the differentiable setting, and compare the theory so obtained to D. Spivak’s axioms for derived $C^\infty$ geometry.

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1 Introduction

Derived manifolds are what you get when you attach higher categorical fibre products to the category of manifolds. The purpose of this paper is to cast this idea in the form of a universal property, and hence give a self-contained definition of derived geometry in the syntax of higher category theory.

With this formulation, one can define derived geometry quite generally, and extract many elementary consequences for derived differentiable manifolds:
1.1 Definition. The ∞-category of derived manifolds is the universal finitely complete higher
categorical extension of the category of manifolds compatible with finite products and glueing.

With the notion of ‘glueing’ suitably interpreted (cf. §4), this gives a definition of the ∞-
category of derived $C^k$, real analytic, and complex manifolds, as well as numerous other vari-
ants - including, with a little more care, the algebraic category. What’s more, the universal
property here is nothing more than a local version of that enjoyed by:

- (strictly) commutative ring spaces in the algebraic category;
- $C^\infty$-ring spaces in the differentiable category.

Hence, it is comparable with typical approaches to derived algebraic [HAGII, §2.2] and syn-
thetic differential [MR91; Spi10] geometry. For example:

1.2 Theorem. Let $k = 0, \ldots, \infty, \omega$. The category of paracompact, second countable Hausdorff
spaces with a sheaf of $C^k$-algebras, locally isomorphic to a finite limit of $C^k$ manifolds, is a
category of derived $C^k$ manifolds.

The main result of this paper, theorem 9.2, constructs a category that satisfies definition
1.1 beginning with a rather general notion of ‘category of manifolds’. The glueing construction
uses topos theory in the style of [TV09] in place of the structured spaces favoured in [Spi10],
and a comparison leading to the preceding statement is sketched in 9.14.

Applying the universal property immediately yields several constructions that in other
works are carried out ‘by hand’, including:

- forgetful functors from derived $C^k$ to $C^{k'}$ manifolds for $k > k'$; from derived complex
  manifolds to derived real analytic manifolds; from derived C-schemes to derived complex
  manifolds...
- a version of the tangent bundle as a complex of topological vector bundles, and hence a
  virtual dimension invariant.
- a functor from so-called quasi-smooth derived manifolds - those representable as a fibre
  product of manifolds - to D. Joyce’s category of d-manifolds [Joy12].

In fact, we will argue in §2 that our definition implies:

1.3 Theorem (2.9). For $k > 0$, the category of quasi-smooth objects in $D\text{Man}_{C^k}$ satisfies the
first six of D. Spivak’s axioms for being ‘good for intersection theory on manifolds’.\footnote{[Spi10, def. 2.1]} Moreover,
the virtual dimension defined in loc. cit. is isomorphism-invariant.

1.4 Aside. In a sequel to this paper we will show that in fact, Spivak’s simplicially enriched
category is a model for the higher category of quasi-smooth derived manifolds defined here.
However, it is actually much easier to prove the axioms directly from the universal property
than to prove this equivalence and then appeal to Spivak’s arguments.

Finally, we extract from the construction a formula for the space of maps from a derived
manifold to a classical manifold via a simplicial resolution. A particularly appealing version
of this formula is available in the quasi-smooth case:

1.5 Theorem (11.7). Let $X \to Z \to Y$ be a diagram of manifolds. The space of functions on the
quasi-smooth derived manifold $X \times Z Y$ is represented as an $\mathbb{R}$-cdga by its Hochschild complex

$$
C^k(X \times Z Y, \mathbb{R})_{[n]} := C^k_{|X \times Z Y|}(X \times Z^n \times Y, \mathbb{R})
$$

$$
\quad df(x, \vec{z}, y) := f(x, x, \vec{z}, y) + \sum_{i=1}^n (-1)^i f(x, z_1, \ldots, z_i, z_i, \ldots, y) + (-1)^{n+1} f(x, \vec{z}, y, y),
$$

where the subscript $|X \times Z Y|$ means to take germs of functions along this subspace.

By writing this object as a simplicial commutative algebra instead of a dga, this formula
extends to a description of $C^k(X \times Z Y, -)$ as a simplicial $C^k$-algebra - that is, it encodes the left
action by composition of $C^k$ functions.
1.6 Aside (Lurie’s DAG). Our definition of the universal property and construction has a substantial overlap with J. Lurie’s notion of a geometric envelope of a pregeometry \([\text{DAGV}, \S 3]\), which I learned of while putting the finishing touches to this paper.

The main contributions of this paper relative to [\text{DAGV}], apart from its brevity, are its more detailed discussions of the construction of what Lurie calls the geometric envelope, of the consequences of the universal property in the differentiable context, and the comparison with the work of Spivak.

Structure

Except for \(\S 2\), which collates references to results from later on, this paper is written in logical order. I define exactly what I mean by ‘glueing’ in sections 3-5 and by ‘derived’ in 7. The latter sections concern certain of Spivak’s axioms, those pertaining to transversality \(\S 8\) and embeddings \(\S 10\). The final section is a calculation of mapping spaces.

The actual constructions of categories satisfying the universal properties appearing in the definitions are found in \(\S 6\), which handles glueing, and \(\S 9\), which handles the attachment of fibre products. Hence, these sections are more technical than the others and may be skipped by readers who are prepared to take it on trust that categories with specified properties can typically be constructed using general nonsense.

In the appendices, I have collated some information about general category theory that is either inadequately covered in the literature or, in the case of appendix C, too fiddly to countenance putting in the main body of the text.

A note on philosophy

This paper is formulated entirely within the syntax of higher category theory; hence the word category actually means \(\infty\)-category throughout. Classical categories in which collections of maps between objects form a set are called 1-categories.

Syntax is model-independent: it can be successfully interpreted in any model - quasi-categories, simplically enriched categories, Segal spaces - sufficiently robust to capture the concepts of limits and colimits, functor categories, and the category of higher categories. The most important technical tool of this paper is the theory of the non-Abelian derived category as developed in \([\text{HTT}, \S 5.5.8]\) (in the quasi-category model).

As in type theory, I use a colon : to declare membership of a set, while the usual notation \(\in\) is reserved for statements. That is, \(x : S\) is a definition, while \(x \in S\) is a proposition.

Acknowledgements

I would like to thank D. Joyce for invigorating discussions on the subject of this paper, and for alerting me to the fact that a definition of transversality that I used in an earlier version of this paper was bogus. I also thank M. Porta and D. Carchedi for lending me expertise with derived geometry and higher topos theory.

Finally, I’d like to thank T. Coates, A. Corti, and R. P. Thomas for welcoming me back to Imperial during the months in which I completed a first draft of this work, and the people at MPIM in Bonn for hosting during work on the second.

2 Spivak’s axioms

In \([\text{Spi10}, \text{def. 2.1}]\), D. Spivak has gathered a number of axioms that could be considered prerequisites for a good derived differential geometry — in his language, to be good ‘for intersection theory on manifolds.’ This section is a guide to the present work from the perspective of these axioms.

2.1. Spivak’s first axiom, ‘geometric’ (definition 1.3 of \text{op. cit.}), is actually four axioms:

i) Spivak asks for derived manifolds to form a fibrant simplicially enriched category. Since this paper is model-independent, we will have nothing to say about this axiom.
Actually, Spivak’s work is entirely couched in the language of simplicially enriched categories, but all the subsequent axioms may equally be interpreted in higher category theory, so that is what we will do. In particular, our category DMan of derived manifolds is an ∞-category.

ii) The inclusion functor i : Man → DMan is fully faithful. This is not hypothesised in the definition 1.1, but it is a consequence of the general construction of completions of categories (cor. 9.3).

iii) The inclusion functor preserves transverse fibre products. By the implicit function theorem, transverse fibre products are preserved as soon as products and passing to covers is preserved, which is exactly our definition - see §8, especially prop. 8.2.

iv) There is a left exact underlying space functor that extends the usual one |−| : Man → Top for manifolds. This follows by applying the universal property to |−|.

2.2. Axioms (2-6) capture fairly general features of derived geometry:

• Axioms (2-3) concern glueing, and are subsumed in the general notion of a spatial geometry discussed in this paper, discussed in §4, especially 5.11.

• Axioms (4-5) say, respectively, that fibre products of manifolds exist in DMan, and that it is moreover locally generated this way; cf. part iv) of 7.5.

In fact, our universal property says that it is generated freely, subject to preserving products and glueing, by these fibre products.

These axioms are suitable for quasi-smooth derived manifolds (def. 7.3). A fibre product of quasi-smooth derived manifolds over a manifold is again quasi-smooth. Meanwhile, the full category of derived manifolds defined in 1.1 contains all finite limits, that is, iterated fibre products of manifolds.

• Axiom (6) asks that each ‘model imbedding’ X → Y - principal closed immersions in the sense of [DAGIX] - induce a surjection π0C∞(Y, R) → C∞(X, R). This axiom is a consequence of the description of regular epimorphisms in the category of C∞-rings; see §10, especially cor. 10.5.

2.3. Spivak’s last axiom asks for a well-defined normal bundle to an embedding X ⊆ M, and that up to replacing M with a (‘virtual’) neighbourhood of X, that it be cut out by a section of this normal bundle. The local existence version of this axiom is vacuous: by definition, X is locally the fibre over 0 of some map f : M → Rk, and so the outer square in the diagram

\[ \begin{array}{ccc}
X & \rightarrow & M \\
\downarrow & & \downarrow \\
M & \rightarrow & Rk \\
M \Gamma(f) & \rightarrow & M × Rk & \rightarrow & Rk
\end{array} \]

is Cartesian. By transversality — see part iii) of 2.1 — the right-hand square is also Cartesian. Hence, the left-hand square exhibits X as the zero set of the f, regarded as a section of the trivial vector bundle.\(^2\)

The uniqueness clause, meanwhile, is material: it implies the isomorphism-invariance of the rank of this normal bundle, and hence of the virtual dimension introduced in axiom (5). (Note that as phrased in loc. cit. the axiom does not ask that the normal bundle be natural, but only unique up to unspecified isomorphism.) In this paper, we regard the existence of such an invariant to be the essential feature.

2.4 Definition. A virtual dimension invariant on DMan is a section

v.dim : DMan → ZDMan

of the constant integer sheaf on DMan, satisfying the following identities:

i) v.dim(X) = dim(X) when X is a manifold;

\(^2\)Global existence of this presentation is special to the C∞ setting, having something to do with partitions of unity; since it is somewhat tangential to the theme of this paper, we won’t discuss it.
ii) \( \dim(X \times Y) = \dim(X) + \dim(Y) \);

iii) if \( X \to Y \twoheadrightarrow Z \) is an equaliser, then \( \dim(X) = \dim(Y) - \dim(Z) \).

Evidently, if each object of \( \text{DMan} \) is locally a finite limit of manifolds — as in Spivak’s axiom (5) — then a virtual dimension invariant, if it exists, is uniquely determined by these properties.

It is well-documented — see, for example, the introduction to [Spi10] — that a virtual dimension invariant cannot be defined on any 1-categorical extension of \( \text{Man} \): if \( f : X \to Y \) is any map of manifolds, then the additivity formula for the equaliser

\[
\begin{array}{c}
X \\
\downarrow \\
\downarrow \\
\downarrow \\
X \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
Y
\end{array}
\]

of \( f \) with itself forces \( \dim Y = 0 \), which of course is not the case for all \( Y \). In higher category theory, on the other hand, this fork needn’t be an equaliser.

**2.5 (Virtual tangent bundle).** The existence of a virtual dimension invariant for manifolds of class \( C^k \) with \( k > 0 \) follows from a pared-back version of the theory of the cotangent complex as a complex of \( C^0 \) vector bundles. Again, this may be deduced immediately from the universal property: the functor

\[
X \mapsto (|X|, T_X)
\]

from \( \text{Man} \) to the category \( \text{Top} \ltimes \text{Vect}_\mathbb{R} \) of topological spaces equipped with a complex of real vector bundles gets a unique left exact extension

\[
T : \text{DMan} \to \text{Top} \ltimes \text{Vect}_\mathbb{R}
\]

that puts a (coconnective) tangent bundle (complex) \( T \) on the underlying space of a derived manifold.

The Euler characteristic of this complex is locally constant and additive on finite limits, and hence provides a virtual dimension invariant in the sense of definition 2.4.

**2.6 Example (Pathology of derived \( C^0 \)-manifolds).** Conversely, the category of \( C^0 \) manifolds does not admit a virtual dimension invariant. Let \( X \to Y \) be an embedding of topological manifolds. I claim that the square

\[
\begin{array}{c}
X \\
\downarrow \\
\downarrow \\
\downarrow \\
X \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
Y
\end{array}
\]

is Cartesian in \( \text{DMan}_{C^0} \). This easiest to explain in the case \( X \) is a point, \( Y = \mathbb{R} \). We will show that the fibre product \( \Omega_0 Y \) is a retract of the terminal object, hence isomorphic to it.

Because transverse fibre products are preserved in \( \text{DMan} \), the square

\[
\begin{array}{c}
\text{pt} \\
\downarrow \\
\mathbb{R}^1 \\
\downarrow \\
(1,0) \\
\downarrow \\
\mathbb{R}^2 \\
\downarrow \\
(0,1) \\
\downarrow \\
\text{pt}
\end{array}
\]

exhibiting the intersection of the axes in the plane is Cartesian. The unique map from \( \Omega_0 Y \) to the point can hence be written as a map of cospans

\[
\begin{array}{c}
\text{pt} \\
\downarrow \\
\mathbb{R}^1 \\
\downarrow \\
0 \\
\downarrow \\
\mathbb{R}^1 \\
\downarrow \\
\Delta \\
\downarrow \\
\mathbb{R}^2 \\
\downarrow \\
\text{pt}
\end{array}
\]

where the arrows in the lower line are the inclusions of the axes.
If we are in the category of $C^0$ or PL manifolds, then this map has a retraction given by the unique map $\mathbb{R}^1 \to \mathbb{R}^0$ and on $\mathbb{R}^2$ by the PL function

$$
\pi(x, y) = \begin{cases} 
\min\{x, y\} & x, y \geq 0 \\
\max\{x, y\} & x, y \leq 0 \\
0 & \text{otherwise}
\end{cases}
$$

represented by the picture

\[
\begin{array}{c|c|c}
& x & y \\
\hline
0 & 0 \\
\hline
x & y \\
\hline
y & x \\
\end{array}
\]

Note that the existence of such a $\pi$ for manifolds of class $C^1$ or better is obstructed by the first derivative.

In generalisation of the above example, I conjecture:

**2.7 Conjecture.** The underlying space functor of $\text{DMan}_{C^0}$ is fully faithful. In other words, the closure in $\text{Top}$ under finite limits of the category of topological manifolds is a category of derived topological manifolds.

2.8 Aside. A true theory of the cotangent complex of a derived $C^k$-manifold $X$ ought to yield an object $L$ in some category of perfect modules over the $C^{k-1}$-algebra space $C^{k-1}(X)$. A candidate for this theory could be extracted from the general framework of [HA, §7.3], or from the theory of $C^\infty$-derivations of [CR12, §2]. Since the algebra $C^k(X)$ itself can fail to be homologically bounded, the Euler characteristic of this object isn’t straightforward to define.

Our tangent complex 2.5 is something more basic — it arranges the Zariski tangent spaces, considered simply as real complexes, into a topological bundle.

If conjecture 2.7 is true, then a perfect module on a derived topological manifold is nothing more than a bounded complex of finite rank real vector bundles on the underlying space. We therefore might speculate that our $T$ is related to the true cotangent complex by the formula $L \otimes_{C^k} C^0 = T^v$.

**2.9 Theorem.** The category of quasi-smooth derived $C^k$ manifolds satisfies the first six of Spivak’s axioms of being good for doing intersection theory. It admits a virtual dimension invariant if and only if $k > 0$.

**Proof.** References to the results implying (1-6) are cited in 2.1 and 2.2. For the last statement, the case $k > 0$ follows from the existence of the tangent complex 2.5, while a counterexample 2.6 is provided for the case $k = 0$.

For $k > 0$, this also appears as a result in Spivak’s paper - though the arguments therein are occasionally incomplete.

**2.10 Corollary.** Let $W \subset \text{DMan}_{C^k}$ be a class of morphisms such that:

- terms of $W$ induce homeomorphisms of underlying spaces;
- a pullback of a morphism in $W$ is in $W$;
- if the domain or codomain of $f : W$ is a manifold, then $f$ is a $C^k$ diffeomorphism.

Then there is a category $\text{DMan}_{C^k}[W^{-1}]$ equipped with a left exact functor from $\text{DMan}_{C^k}$ that satisfies Spivak’s axioms (1-5).

If terms of $W$ induce bijections on $\pi_0 \text{Map}(-, \mathbb{R})$, then this category satisfies (6) as well.
Proof. Because the terms of $W$ are stable for pullback, the localisation $\text{DMan} \to \text{DMan}[W^{-1}]$ is left exact and has a natural calculus of right fractions. This proves axioms (4-5) as well as part iii) of (1). Moreover,

$$\text{Map}_{\text{DMan}[W^{-1}]}(X,Y) = \colim_{\overline{W}|X} \text{Map}_{\text{DMan}}(-,Y).$$

The hypothesis that $W$ does not touch Man implies that the inclusion functor of Man into $\text{DMan}[W^{-1}]$ remains fully faithful — part ii) of (1). Since the terms of $W$ are homeomorphisms, the underlying space functor descends to the localisation. This completes the proof of Spivak’s ‘geometricity’.

If all maps in $W$ induce bijections on $\pi_0$, then $\text{Map}_C(-,R) \to \text{Map}_{\text{C}[W^{-1}]}(-,R)$ is a bijection on $\pi_0$ as well, by the preceding formula for the mapping space and because taking $\pi_0$ commutes with colimits. Let $X \to Y$ be a model embedding in $\text{DMan}[W^{-1}]$, so that $X \cong Y \times \mathbb{R}^k$ pt for some $f : Y \to \mathbb{R}^k$. By the calculus of right fractions, there exists a lift $\tilde{X} : Y \to \mathbb{R}^k$ of $f$ to $\text{DMan}$, whence $\tilde{X} := \tilde{Y} \times \mathbb{R}^k$ pt is a lift of $X$. Hence by consideration of the square

$$
\begin{array}{ccc}
\pi_0(Y,R) & \to & \pi_0(X,R) \\
\downarrow & & \downarrow \\
\pi_0(Y,R) & \to & \pi_0(X,R)
\end{array}
$$

axiom (6) is preserved in this case as well.

For axioms (2-3), we will need to show that $\text{DMan}[W^{-1}]$ retains the structure of a spatial geometry (def. 4.1). It remains to show that open immersions (retain the Cartesian property in $\text{DMan}[W^{-1}]$, and hence that this latter is a spatial geometry (not necessarily subcanonical!).

Lemma. Let $C \to \text{C}[W^{-1}]$ be a left exact localisation, and let $|-| : \text{C}[W^{-1}] \to T$ be any functor. If a map $U \to X$ in $C$ is Cartesian over its image in $T$, then so is its image in $\text{C}[W^{-1}]$.

Proof. By left exactness, for any $V : C$ the mapping space is computed

$$\text{Map}_{\text{C}[W^{-1}]}(V,-) = \colim_{V \subseteq W} \text{Map}_C(\tilde{V},-)$$

by a filtered colimit. The Cartesian property for $U \to X$ states that

$$
\begin{array}{ccc}
\text{Map}_C(-,U) & \to & \text{Map}_C(-,X) \\
\downarrow & & \downarrow \\
\text{Map}_T(|-|,|U|) & \to & \text{Map}_T(|-|,|X|)
\end{array}
$$

is a pullback square. Passing to the colimit over $W \mid V$, the corresponding square in $\text{C}[W^{-1}]$ remains Cartesian by the exactness of filtered colimits of spaces.

Finally, to get (3) we replace the localisation with its category of manifold objects. Again because $W$ does not touch Man, the inclusion of Man remains fully faithful. Meanwhile, the Yoneda functor is left exact, so iii-iv) of (1) and (4) is true, while (5) remains true by definition. Finally, (6-loc) remains true because of the square above: the upper terms are calculated locally, hence so too are the lower ones.

2.11 Example. On the $\infty$-category of quasi-smooth derived manifolds, the tangent complex functor is 2-truncated. Hence, it and the virtual dimension invariant factor through its 2-truncation. This localisation satisfies all of the conditions of corollary 2.10; hence this 2-category satisfies all of Spivak’s axioms for being ’good for intersection theory’.

2.12 Example. Denote by $D$ the zero set of the function $x^2 : \mathbb{R} \to \mathbb{R}$. The projection $D \to \text{pt}$ admits a unique section and induces a homeomorphism on the underlying space. Moreover, the virtual dimensions on both sides are zero.

Let $W$ be the smallest set of morphisms of $\text{DMan}$ containing this map and stable under composition and base change. Then $W$ satisfies the first two hypotheses of corollary 2.10.
Moreover, because $D$ admits a unique point, $\text{pt}$ is coinitial in $W \downarrow \text{pt}$; hence although $W$ identifies a manifold with a non-manifold, the inclusion $\text{Man} \to \text{Man}[W^{-1}]$ is still fully faithful.

Thus $\text{Man}_{\text{v}}[W^{-1}]$ satisfies Spivak’s axioms (1-6) and admits a virtual dimension invariant. It seems likely that it also satisfies (7). However, the tangent complex and useful homological invariants like multiplicities do not descend to this category.

## 3 Atlases in topology

Our definition of scheme revolves around a certain notion of atlas, which is a diagram of open sets that contains enough information for descent of sheaves of types on a topological space.

### 3.1 (Sieves).

Let $I$ be a partially ordered set. Recall that a sieve of $I$ is a lower subset, that is, a subset $K \subseteq I$ such that $i \leq j$ and $j : K$ implies $i \in K$. The smallest sieve containing a given element $i : I$ is denoted $I \downarrow i$. If $K$ is a sieve and $j$ an element, we abbreviate $K \cap (I \downarrow j)$ to $K \neg j$.

We denote by $\mathcal{U}(I)$ the complete, distributive lattice of sieves of $I$. It is a frame in the sense of locale theory, or a 0-topos in the sense of [HTT, §6.4].

### 3.2 (Atlas).

Let $X$ be a topological space. We denote by $\mathcal{U}(X)$ the lattice of open subsets of $X$. A diagram of open subsets of $X$ is a monotone map of posets $U : I \to \mathcal{U}(X)$. We write $U_i \subseteq X$ for the open set associated to $i : I$ by $U$.

**Proposition.** Let $U : I \to \mathcal{U}(X)$ be a diagram of open subsets of $X$. The following conditions are equivalent:

1. $X = \bigcup_{i \in I} U_i$, and for any $i, j : I$, $U_i \cap U_j = \bigcup_{k \leq i, j} U_k$;
2. for any finite $J \subseteq I$,
   $$\prod_{i : J \cap (I \downarrow j)} U_i \to \bigcap_{j : J} U_j;$$
3. $U_I$ is left exact — that is, the extension $\mathcal{U}(I) \to \mathcal{U}(X)$ preserves meets ([HTT, §5.3.2]).

In this case, $X$ is a colimit in $\text{Top}$ of $U$.

**Proof.** The second condition is just an explicit restatement of the third. Meanwhile, by induction it is enough to check ii) on sets of cardinality 0 and 2, which is the content of i). □

**Definition.** An atlas $U \downarrow I$ for the topological space $X$ is a map $U : I \to \mathcal{U}(X)$ satisfying the equivalent conditions of the proposition.

### 3.3 Aside.

It is not always convenient to assume a priori that the map $U$ is an injective map of posets — for instance, the construction of 3.17 does not preserve this property. Note, however, that if $U$ factors through a surjective map $I \to J \subseteq \mathcal{U}(X)$, then $U \downarrow I$ an atlas entails that the descended map $U \downarrow J$ is also an atlas.

### 3.4 (Universality).

Let $U \downarrow I$ be an atlas for a space $Y$, $f : X \to Y$ a continuous map. Because the preimage map $f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$ commutes with finite intersections, the composite diagram $f^* U \downarrow I := f^{-1} \circ U \downarrow I$ is an atlas of $X$, the pullback of $U \downarrow I$ along $f$.

By way of converse, let $f : V \to U$ be a natural transformation of functors $I \to \text{Top}$ such that for each $i \leq j$ the diagram

$$
\begin{array}{ccc}
V_i & \rightarrow & U_i \\
\downarrow & & \downarrow \\
V_j & \rightarrow & U_j
\end{array}
$$

is a pullback; in other words, $i$ is a Cartesian natural transformation of diagrams [HTT, Def. 6.1.3.1]. In particular, for $i \leq j$ in $I$ the map $V_i \to V_j$ is an open immersion. Then setting $V_I = \text{colim}_I V$, for each $i : I$,

$$V_i \cong U_i \times_{U_j} V_j,$$

that is, colimits over atlases are universal in $\text{Top}$. It follows immediately that in this case, $V \downarrow I$ is the pullback atlas of $U \downarrow I$ along $V_I \to Y$. 

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Proof. The statement is true in the category of sets. It therefore suffices to observe that the induced maps \( V_i \to V_j \) are (open) immersions with respect to the strong topology on \( V_i \).

Alternatively, the claim may be deduced from the fact 3.10 that colimits over atlases remain colimits in the hypercomplete topos \( \text{ShTop} \), and that colimits are universal there.

3.5 (Descent for open immersions). In the notation of the previous paragraph, if for each \( i : I \) \( f^i : V_i \to U_i \) is an open immersion, then \( V_i \to Y \) is an open immersion: in fact, \( V_i = \bigcup_{j \in I} V_j \subseteq Y \).

To see this, it is enough to show that \( V | I \) is an atlas for this union. By construction, we only have to check the second part of condition i) for binary overlaps. Let \( i, j : I \). Because \( f \) is Cartesian, \( V_{i \cap j} = U_{i \cap j} \times_{U_i} V_i \), thus because \( U | I \) is an atlas, \( V_{i \cap j} = V_i \cap V_j \).

3.6 Examples. Let \( X \) be a topological space. Then the identity map \( \mathcal{V}(X) \to \mathcal{V}(X) \) is an atlas, the tautological atlas. Dually, the map \( 1 \to \mathcal{V}(X) \) from the one-element set that picks out the whole space \( X : \mathcal{V}(X) \) is called the trivial atlas. A pullback of a trivial atlas is trivial.

3.7 Example (Covers). Let \( \{ U_i \}_{i : I} \) be an open covering of \( X \) in the usual sense, i.e. \( \bigcup_{i : I} U_i = X \). Denote by \( I^\wedge \) the opposite to the set of finite, inhabited subsets \( \emptyset \neq J \subseteq I \). Then the set map \( I \to \mathcal{V}(X) \) extends uniquely to a binary-meet-preserving map on \( I^\wedge \) by the formula \( U_J := \bigcap_{j \in J} U_j \). Since by hypothesis \( X = \bigcup_{i : I} U_i \), this also preserve the empty meet, hence is an atlas in the sense of definition 3.2: the atlas completion of \( \{ U_i \}_{i : I} \).

3.8 Example (Atlases of a point). A partially ordered set \( I \) is filtered if and only if the constant diagram \( i \to 1 \) is an atlas of the one-point topological space 1.

3.9 Example (Hypercovers). Let \( X : \Delta^{op} \to \text{SR}(\mathcal{V}(X)) \) be a hypercover of \( X \) in the sense of [SGA4, Exp. V, §7.3]. Then \( X \) can be thought of as a diagram in \( \mathcal{V}(X) \) indexed by the totalisation of some simplicial set \( I \). This diagram is an atlas in the sense of definition 3.2. See the appendix C for a proof.

3.10 Aside (Higher atlases). Following the logic of [HTT, §6.4.5], we could also add to proposition 3.2 the alternatives

iii) the \( n \)-truncated extension \( \text{PSh}_{\leq n} I \to \text{Sh}_{\leq n} X \) is a left exact functor of \( n \)-categories;

iv) the hypercompleted extension \( \text{PSh}^\wedge I \to \text{Sh}^\wedge X \) is a left exact functor of \( \infty \)-categories.

More precisely, proposition 7 of loc. cit. gives the unique existence of the extension of \( f^* \) to a geometric morphism between \( n \)-topoi. (Actually, the statement of loc. cit. is couched incorrectly in terms of the pushforward functor \( f_* \); nonetheless, the proof of the key lemma 5 uses pullbacks, and hence the actual conclusion is as I claimed.)

Condition iv) follows from iii) because \( \text{Sh}^\wedge X = \lim_{\to \infty} \text{Sh}_{\leq n} X \). As usual, if \( X \) is paracompact, the word ‘hypercompleted’ may be suppressed.

3.11 Aside (General index categories). It is also possible to define atlases more generally indexed by an arbitrary \( n \)-category \( I \). At least the case \( n = 1 \) would be necessary for generalisations in which \( X \) is some kind of stack rather than an honest topological space. In this case, the atlas condition should be replaced with one of the conditions of the preceding remark 3.10 (depending on \( n \)).

The explicit condition ii) of proposition 3.2 must be strengthened in this setting as follows: if \( K \) is an arbitrary finite, directed 1-category and \( p : K \to I \) a functor, then

\[
\prod_{e : I_p} U_e \rightrightarrows \lim_{k : K} U_k
\]

is surjective, where \( I_p \) denotes the category of cones over the diagram \( p \) (called ‘overcategory’ in [HTT, §1.2.9]). By iterating this condition, it follows that in fact

\[
\colim_{e : I_p} U_e \rightrightarrows \lim_{k : K} U_k
\]

in \( \text{Sh}^\wedge(X) \), whence this functor is left exact. A formal proof of this could follow the lines of that of theorem C.6.

The condition of definition 3.2 is just the special case where \( K \) is discrete. If \( I \) is a 1-category, it is enough to consider the case where \( K \) has length at most one.
3.12 (Hereditary). If $K \subseteq I$ is a sieve and $I$ is an atlas for $X$, then the pair $U \mid K$ obtained by restricting $U$ to $K$ is an atlas for the open subset $U_K \subseteq X$ covered by members of $K$, as can easily be seen by the first criterion of the definition.

In particular, if $K$ is a covering sieve, then $U \mid K$ is an atlas for $X$.

3.13 Definition (Site). An atlas $U \mid I$ for a space $X$ is said to be a site for $X$ if its Yoneda extension $\mathcal{U}(I) \to \mathcal{U}(X)$ is surjective.

Let $V \subseteq X$ be an open subset. Write $I \mid V$ for the sieve generated by $V$. It is just $I \times_{\mathcal{U}(X)} \mathcal{U}(V)$. We may therefore write $U \mid I \mid V$ for the restricted diagram.

Then an atlas $U \mid I$ is a site if it satisfies the following hereditary property: for every open $V \subseteq X$ the restricted diagram $U \mid I \mid V$ is an atlas for $V$.

3.14 Example (Basis). If $I \subseteq \mathcal{U}(X)$ is a subset, then the inclusion map makes $I$ into a site if and only if every open subset of $X$ is a union of elements of $I$.

3.15 (Composition of atlases). Let $U : I \to \mathcal{U}(X)$ be a diagram of open subsets of a space $X$, and let $\eta : J \to \mathcal{U}(I)$ be a left exact map of posets. Write $U \mid J$ for the $J$-indexed diagram obtained by restriction of the Yoneda extension of $U$ along $\eta$.

If $U \mid I$ is an atlas, then so is its restriction $U \mid J$ of $U$ along $J\eta$. Moreover, for each $j : J$, $U \mid I \mid j$ is an atlas for $U_j$. We will show that a converse is true:

**Lemma.** Suppose that $U \mid J$ is an atlas for $X$ and $U \mid I \mid j$ is an atlas of $U_j$ for each $j : J$. Then $U \mid I \mid J$ is an atlas for $X$.

In this statement, $I \mid J$ denotes the partially ordered set of pairs $(i, j) : I \times J$ such that $i \leq \eta j$. This comes with projections

$$
I \mid J \xrightarrow{\pi_I} I \xleftarrow{\pi_J} J,
$$

the left-hand projection being a Cartesian fibration and the right-hand one co-Cartesian. We restrict $U$ to a map $I \mid J \to \mathcal{U}(X)$ by pulling back along the projection $\pi_I$.

**Proof.**
- By left exactness of $\eta$, $I = \bigcup_{j : J} \eta j$, hence $X = \bigcup_{j : J} U_j$. Therefore the $U_j$ cover $X$ as $i$ ranges over $I \mid J$.
- If $j \leq j'$ in $J$, then $I \mid j \subseteq I \mid j'$. Intersection with $I \mid j$ defines a lattice homomorphism $\eta j : \mathcal{U}(I \mid j') \to \mathcal{U}(I \mid j)$ on the attendant lattices of sieves. These pullbacks also commute with the natural extensions $\mathcal{U}(I \mid j) \to \mathcal{U}(X)$: if $K : \mathcal{U}(I \mid j')$ is a sieve, then $U_K \cap U_j = U_{K \cup j}$ by the site property.
- Let $(i_0, j_0), (i_1, j_1) : I \mid J$ be two indices. Then $i_0 \land i_1 \leq \eta j_0 \land \eta j_1$ by left exactness of $\eta$, that is,

$$
i_0 \land i_1 = \bigcup_{j \leq j_0, j_1} (\eta j \land i_0 \land i_1).
$$

Since for each $j \leq j_0, j_1$, $U \mid I \mid j$ is an atlas, the restriction of $U$ to the filter $\eta j \land i_0 \land i_1 \subseteq I \mid j$ covers $U_j \cap U_{i_0} \cap U_{i_1}$.

Because $U \mid I \mid J$ is obtained by pullback from $U \mid I$, the latter is also an atlas for $X$.

3.16 Example (Direct composition of atlases). Suppose we are given a factorisation

$$
I \xleftarrow{V} J \xrightarrow{U} \mathcal{U}(X)
$$

where:
- $V \mid J$ is an atlas;
- for each $j : J$, $U \mid I \mid j$ is an atlas for $V_j$. 

10
Then the preimage map $J \to \mathcal{V}(I)$ is left exact, and we are in the situation of 3.15, so that $U \mid I$ is an atlas.

3.17 Example (Subordination of a site to an atlas). Let $X$ be a topological space, $U \mid I$ a site, and $V \mid J$ an atlas. Then by composition with the right adjoint to $\mathcal{V}$, we obtain a left exact map $J \to \mathcal{V}(I)$ satisfying the hypotheses of lemma 3.15.

The construction of loc. cit. therefore provides us with a new site $U \mid I \mid J$ pulled back from $U \mid I$ and such that for each $k : I \mid J$, $U_k \subseteq V_j$ for some $j : J$.

4 Geometries

In this paper, we will formalise the idea of ‘patching’ or ‘glueing’ geometric objects modelled on those of a given category by the notions of spatial geometry 4.1 and scheme and manifold objects 5.1, 5.9. For the sake of intuition, I have chosen a formalism that resembles that appearing in Spivak’s axioms (2-3) [Spi10, def. 2.1]; see 5.11.

This approach is certainly not the most general possible; it may not even be the most efficient in the cases that it does apply. At the end of this section, I’ve collected some remarks 4.17-4.19 comparing the theory here to some others appearing in the literature, including an outline of a generalisation to orbifolds or smooth stacks.

4.1 Definition (Spatial geometry). Let $C$ be a category equipped with a functor

$$| - |_C : C \to \text{Top}$$

into the category of topological spaces, called the underlying space functor. A morphism $U \to X$ in $C$ is said to be an open immersion if it is Cartesian over an open immersion of topological spaces; that is, $|U|$ is the underlying space of $U$, and $U$ is terminal among objects $U'$ of $C$ fitting into a diagram

$$\begin{array}{ccc}
U' & \longrightarrow & X \\
|U'| & \downarrow & |X| \\
|U| & \downarrow & |X| \\
\end{array}$$

Since they are defined by a Cartesian property, open immersions in $C$ are stable for composition. Moreover, the set of open immersions is stable under right cancellation, in that $f$ and $fg$ being open immersions entails that $g$ is too.

If an open subset $|U| \subseteq |X|$ is the underlying space of an open subobject of $X$, we say that it is representable or principal. The slice category of open immersions into $X$ is denoted $\mathcal{V}(X)$. Via the underlying space functor, it is equivalent to the poset of representable open subsets of $|X|$ — in particular, it is a preorder, and $\mathcal{V}(X) \subseteq \mathcal{V}(|X|)$.

The pair $(C, | - |_C)$ is called a spatial geometry if for each object $X : C$, the set of representable open subsets of $|X|$ forms a sub-basis for the topology.

A set $\{U_i \to X\}_{i : I}$ of open subobjects of $X$ is said to be covering if $\bigcup_{i : I} |U_i| = |X|$.

4.2 Example (Trivial example). The category of topological spaces is a geometry with respect to the identity functor.

4.3 Aside. In complete generality, it is technically better to use the category of locales - which prioritises the lattice of open subspaces over the underlying point set - in place of topological spaces as a recipient for the underlying space functor, since this theory is better adapted to the theories of Grothendieck sites and topoi hiding behind the constructions here.

In practice, this distinction will make little difference, as the theories of topological spaces and locales agree on a subcategory sufficiently large to include all reasonable examples. For example, there is no ambiguity between these theories for locally compact Hausdorff spaces.

4.4 Definition (Morphisms of geometries). A geometric functor $f^* : C_1 \to C_2$ of spatial geometries is a pair $(f^*, f_*)$, where:

- $f^*$ is a functor of the underlying categories $C_1 \to C_2$ that takes open immersions to open immersions;
- $f_* : |f^*| \to | - |$ is a natural transformation of functors $C_1 \to \text{Top}$;
which satisfy the following condition:

i) \( f^* \) is compatible with \( f_- \) in the sense that for any open immersion \( U \to X \) in \( \mathbf{C}_1 \),

\[
\begin{array}{ccc}
|f^* U| & \to & |f^* X| \\
\downarrow f_U & & \downarrow f_X \\
|U| & \to & |X|
\end{array}
\]

is a pullback square. Given that \( f^* \) preserves open immersions, it is enough to check that for each member of an open cover \( \{V_i \to U\} \), \( f_X^{-1}|V_i| \subseteq |f^* U| \).

When the underlying spaces are locales (cf. remark 4.3), the transformation \( f_- \) is completely determined by \( f^* \).

4.5 (The category of geometries). Let us write \( \text{Cat}^{\text{ lax}}_{\text{Top}} \) for the ‘lax slice’ of categories over \( \text{Top} \) — that is, the total space of the Cartesian fibration associated to the functor \( \mathbf{C} \to \text{Fun}(\mathbf{C}, \text{Top}) \). The data of a functor of geometries is that of an arrow in this category satisfying certain conditions, and these conditions are stable under composition. Hence, geometries and geometric functors assemble to form a faithfully embedded subcategory \( \text{Geo} \) of \( \text{Cat}^{\text{ lax}}_{\text{Top}} \).

4.6 (Functors that preserve underlying spaces). A particular class of geometric functors that are easy to characterise are those that induce homeomorphisms on the underlying space — that is, those maps \( f^*: \mathbf{C}_0 \to \mathbf{C}_1 \) for which the triangle

\[
\begin{array}{ccc}
\mathbf{C}_0 & \xrightarrow{\sim} & \mathbf{C}_1 \\
\downarrow f_- & & \downarrow f_-
\end{array}
\]

commutes, rather than lax commutes. Such functors are exactly those that are Cartesian with respect to the forgetful functor \( \text{Geo} \to \text{Cat} \).

If \( (f^*, f_-) \) is any functor over \( \text{Top} \), the compatibility squares in definition 4.4 are trivially pullback squares. Thus \( (f^*, f_-) \) is geometric if and only if \( f^* \) preserves open immersions.

4.7 (Full geometric subcategories). Let \( \mathbf{C} \) be a spatial geometry, \( \mathbf{C}_0 \subseteq \mathbf{C} \) a full subcategory.

If \( U \to X \) is an open immersion in \( \mathbf{C} \), then it remains so in \( \mathbf{C}_0 \) with respect to this structure. Hence, by 4.6, if \( \mathbf{C}_0 \) is a geometry, then the inclusion functor is geometric. Let us call \( \mathbf{C}_0 \) in this case a full geometric subcategory.

4.8 Example. If \( (\mathbf{C}, |\cdot|) \) is any geometry, the underlying space functor \( |\cdot|: \mathbf{C} \to \text{Top} \) is a geometric functor with respect to the tautological geometric structure on \( \text{Top} 4.2 \).

\[\text{A functor } F: \mathbf{C} \to \mathbf{D} \text{ of higher categories is called faithful if the induced maps}
\[\text{Map}(\cdot, \cdot) \to \text{Map}(F\cdot, F\cdot)\]
\[\text{on mapping spaces are monomorphisms of types — that is, they are unions of connected components. In other words, the induced functor on homotopy categories is faithful in the sense of 1-category theory, and the square}
\[\begin{array}{ccc}
\mathbf{C} & \to & \mathbf{D} \\
\downarrow & & \downarrow \\
\text{HoC} & \to & \text{HoD}
\end{array}\]
\[\text{is a pullback of categories. In this case, we may also say that } \mathbf{C} \text{ is faithfully embedded in } \mathbf{D}. \text{ It makes sense to define faithfully embedded subcategories by specifying a subcategory of the homotopy category — see [HTT, §1.2.11].} \]
4.9 (Base change of open immersions). Let \( X' \to X \), and write \(|U'| := |U| \times_X |X'| \to |X'|\). If \(|U'|\) is representable, then the Cartesian property implies that

\[
\begin{array}{ccc}
U' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
U & \longrightarrow & X
\end{array}
\]

is a pullback square in \( C \). Hence:

- a pullback of an open immersion, when it exists, is an open immersion;
- the underlying space functor preserves pullbacks of open immersions.

It follows that geometric functors also preserve pullbacks of open immersions.

4.10 Proposition. Geometric functors preserve pullbacks of open immersions and transform covers into covers.

Proof. Fix a geometric functor \( f^* : C_1 \to C_2 \).

Pullbacks of open immersions. Let \( U \to Y \) be an open immersion in \( C_1 \), \( X \to Y \) a map. In the commutative cube

\[
\begin{array}{ccc}
|f^*(U \times_Y X)| & \longrightarrow & |f^*X| \\
\downarrow & & \downarrow \\
|U \times_Y X| & \longrightarrow & |X| \\
\downarrow & & \downarrow \\
|U| & \longrightarrow & |Y|
\end{array}
\]

the upper and lower faces are Cartesian by the compatibility condition for the functor, as is the obverse face by 4.9. Hence the reverse face is also Cartesian.

Covers. By the universality of coproducts in \( \text{Top} \), for any set \( \{U_i \to X\}_{i \in I} \) of open subobjects of \( X : C_1 \) the square

\[
\begin{array}{ccc}
\bigsqcup_{i \in I} |f^*U_i| & \longrightarrow & |f^*X| \\
\downarrow & & \downarrow \\
\bigsqcup_{i \in I} |U_i| & \longrightarrow & |X|
\end{array}
\]

is a pullback. Hence by universality of coverings in \( \text{Top} \), if \( \{U_i \to X\} \) is covering, then so is \( \{f^*U_i \to f^*X\} \).

4.11 Definitions (C-atlases and covers). Let \((C, |-|)\) be a spatial geometry. An abstract \( C \)-atlas is a functor \( U_1 : I \to C \) such that

i) \(|U_I| := |-| \circ U_1 : I \to \text{Top} \) is an atlas in the sense of §3 of its colimit \(|X|\);

ii) for each \( i \to j \) in \( I \), \( U_i \to U_j \) is Cartesian over \(|U_i| \to |U_j|\); in particular, it is an open immersion.

4.12 Definition (Effectivity). If \( I \) is a category, we denote by \( I^e \) the right cone over \( I \), that is, \( I \) with a disjoint terminal object \( e \) attached.

An effective \( C \)-atlas (for descent of objects) is an abstract \( C \)-atlas \( U : I^e \to C \) such that \(|U_I| := |-| \circ U_1 |U| \) is an atlas for the topological space \(|U_I|\) in the sense of definition 3.2. In this case, we may call \( U_e \) the target or effect of the atlas, and say that \( U_I \) is an atlas for \( U_e \). If an abstract atlas \( U_1 |U| \) admits an extension to an effective atlas \( U^e |I^e| \), then we say that it is potentially effective.

An effective atlas \( U : I^e \to C \) is said to be effective for descent of morphisms, or subcanonical, if \( U_e = \text{colim}_I U \) in \( C \).
4.13 Example (Topological spaces). The geometry of topological spaces is effective.

4.14 Example. The category Sch\(_Z\) of schemes in the usual sense of algebraic geometry is an effective spatial geometry with respect to the standard underlying space functor.

4.15 Proposition. Geometric functors transform (effective) atlases into (effective) atlases.

Proof. Let \( f : U \mid I \) be an abstract atlas in \( C \). By the compatibility condition for geometric functors, \( f^* U : f^* I \to U I \) is a Cartesian transformation of diagrams. By universality of atlases 3.4, \( f^* U \mid I \) is an atlas of its colimit.

4.16 Definition. A spatial geometry \((C, \mid - \mid)\) is said to be effective, or a category of schemes, if its objects form a stack over \( \text{Top} \) in the sense that every \( C \)-valued atlas is potentially effective. The full subcategory of \( \text{Geo} \) spanned by the effective geometries is denoted \( \text{Geo}^{\text{ef}} \).

Comparisons

4.17 (Geometry via charts). A spatial geometry in the sense of 4.1 is in a natural way a Grothendieck site, in which the generating covering families are those representing an open covering of the underlying topological space. Conversely, the data of the covering condition, together with the class of open immersions \( \mathcal{U} \subseteq C \), can be used to characterise the geometry. This gives a road to a more general definition of geometry in which the underlying space is implicit, rather than explicit.

A convenient formalisation of this idea is J. Lurie’s notion of an admissibility structure \([\text{DAGV}, \text{def. 1.2.1}]\), which is a class of morphisms \( \mathcal{U} \) in \( C \) with certain stability properties. To recover spatial (localic) geometry, we add the hypothesis that the admissible morphisms are monic. A Grothendieck site equipped with an admissibility structure such that covering families are generated by admissible morphisms is called a geometry.\(^4\)

The class \( \mathcal{U} \) extends naturally to the sheaf category \( \text{Sh}C \), and associates to each object \( X : \text{Sh}C \) a small site \( \mathcal{U} \text{Sh}C \mid X \) which, when the terms of \( \mathcal{U} \) are monic, is localic. This defines the underlying space functor

\[ |-| : C \to \text{Sh}C \to \text{Loc}. \]

Atlases and effectivity can then be defined as in definition 5.1, except with ‘open immersion’ replaced by ‘admissible morphism’ - so an atlas is a functor from \( J \) into the category of admissible morphisms.

4.18 (Comparison). Except for the other comparison statement 9.15, the results of this paper do not depend on comparing our approach with the more abstract one of 4.17. Hence, this paragraph only sketches the relationships without proof.

If \( (C, |-|) \) is a spatial geometry in our sense 4.1, and pullbacks of open immersions are representable, then \( (C, \mathcal{U}) \) is a Lurie pregeometry (less finite products), and the induced underlying space functor described in 4.17 coincides with the given one. Moreover, we have already observed that geometric functor between spatial geometries 4.4 preserves covers and pullbacks of open immersions; hence it induces a transformation of geometries in the sense of \([\text{DAGV}, \text{def. 3.2.1}]\) (bar products).

Conversely, if \( (C, \mathcal{U}) \) is a subcanonical Lurie pregeometry, then it is automatic from the construction described above that open immersions in \( C \) generate the topology of any underlying space, and hence that \( (C, |-|) \) is a spatial geometry in the sense of definition 4.1. In this case \( \mathcal{U} \) is precisely the set of open immersions in the sense of definition 4.1.

For non-subcanonical topologies, our spatial geometries form only a full subcategory of Lurie’s pregeometries. For example, a topology in which coverings are generated by those consisting of just one morphism - that is, one whose associated sheaves are just presheaves on a left exact localisation of \( C \) - cannot arise from an underlying space functor on \( C \).

\(^4\)This is actually weaker than Lurie’s definitions: his admissibility structures are supposed to have pullbacks of admissible morphisms, and geometries, resp. pregeometries are also required to admit finite limits, resp. products. Since these conditions do not appear to be relevant to the matter of defining ‘scheme objects’, this and the subsequent paragraph are written as though there were no such clauses.
4.19 (Orbifolds). By replacing the underlying topological space, as in [DAGV], with a more general site or topos, definition 4.1 can be adapted to define contexts for orbifolds (or perhaps even more general geometric stacks). The comparison arguments continue to be valid in this extension.

5 Schemes

The arguments of this section make use of the results of section 7 - to wit, proposition 5.2 and lemma 5.4.

5.1 Definition (Schemes). An effective geometry $\text{Sch}_C$ (def. 4.16) is said to be an effective envelope of a given geometry $C$, or a category of $C$-schemes, if it is universal (initial) among effective geometries equipped with a geometric functor $C \to \text{Sch}_C$.

If $C$ is itself effective, then it is its own scheme category. The existence in general of effective envelopes is unlikely to surprise:

5.2 Proposition. Every geometry admits a category of schemes. Moreover:

i) Every object of $\text{Sch}_C$ admits an atlas in $C$.

ii) If $C$ is finitely complete, then so is $\text{Sch}_C$, and the structural functor $C \to \text{Sch}_C$ is left exact. The functor $C \to \text{Sch}_C$ is fully faithful if and only if $C$ is subcanonical.

Proof. In §6 it is shown (theorem 6.14) that an explicit candidate category of $C$-schemes obeys the universal property. It follows from the definition 6.10 that every object of this category has a $C$-valued atlas. The statements about fibre products are proved in 6.12.

5.3 (Naturality). By the universal property, a geometric functor $C_1 \to C_2$ extends uniquely to a functor of scheme objects, so that the square

\[
\begin{array}{ccc}
C_1 & \to & C_2 \\
\downarrow & & \downarrow \\
\text{Sch}_{C_1} & \to & \text{Sch}_{C_2}
\end{array}
\]

commutes. By geometricity, the extension can be calculated on an atlas. More precisely, taking effective envelopes determines a left adjoint reflector $\text{Sch} : \text{Geo} \to \text{Geo}_{\text{eff}}$ to the inclusion functor of effective geometries into all geometries.

5.4 Lemma (Recognition of effective envelopes). Let $C$ be an effective geometry, $C_0 \subseteq C$ a full geometric subcategory. If every object of $C$ admits an open covering by objects of $C_0$, then $C$ is an effective envelope of $C$.

Proof. This is a special case of corollary 6.15.

5.5 Example (Schemes). Let $\text{Sch}_Z$ denote the category of schemes in the usual sense. Then $\text{Sch}_Z$ is an effective geometry. By lemma 5.4, it is an effective envelope of the category of affine schemes.

5.6. In this section, we will refer to a full geometric subcategory $P$ as a property of topological spaces. Suitable examples are the following:

- Hausdorff (or any separation axiom);
- paracompact ([HTT, Prop. 7.1.1.1]);
- finite covering dimension;
- coherent;
- locally compact;
- second countable (this is hereditary);
- combinations of the above.
We say that an object of a spatial geometry is \( P \), or \( \text{satisfies } P \), if its underlying space is an object of \( P \). It is said to be \( \text{locally } P \) if its underlying space admits a covering by objects of \( P \).

**Proposition.** Let \( C \) be a spatial geometry all of whose objects are \( P \). The following conditions are equivalent:

i) If \( U \mid I \) is an abstract \( C \)-atlas such that \( |U_I| \) is an object of \( P \), then \( U \mid I \) can be rendered effective (for descent of objects and morphisms).

ii) The square

\[
\begin{array}{ccc}
C & \to & \mathbf{Sch}_C \\
\downarrow & & \downarrow \\
P & \to & \mathbf{Top}
\end{array}
\]

is Cartesian (in \( \mathbf{Cat} \), or equivalently in \( \mathbf{Geo} \)).

**Proof.** Suppose that \( C \cong \mathbf{Sch}_C \times_{\mathbf{Top}} P \), and let \( U \mid I \) be an atlas whose target satisfies \( P \). Then the \( C \)-scheme \( U_I \) lies in \( C \).

Conversely, suppose that the condition is satisfied. Then every effective atlas is effective for descent of morphisms, so that \( C \) is a full geometric subcategory of \( \mathbf{Sch}_C \) — indeed, by definition exactly that subcategory whose objects are \( P \).

**Definition.** A spatial geometry is said to be \( P \)-**effective** if it satisfies the conditions of the proposition.

The full subcategory of \( \mathbf{Geo} \) spanned by the \( P \)-effective geometries is denoted \( \mathbf{Geo}_P^{\text{eff}} \). From the second criterion it follows that fibre product with \( P \) over \( \mathbf{Top} \) determines a functor

\[
(-) \times_{\mathbf{Top}} P : \mathbf{Geo}_P^{\text{eff}} \to \mathbf{Geo}_P^{\text{eff}}
\]

which we call \( \text{restriction to } P \).

**5.7 Proposition.** The functors of effective envelope and restriction to \( P \) are in adjunction \( \mathbf{Geo}_P^{\text{eff}} \rightleftarrows \mathbf{Geo}_P^{\text{eff}} \). The right adjoint is fully faithful: it embeds the left hand side as the category of effective geometries whose objects are locally \( P \).

**Proof.** The various universal properties yield isomorphisms

\[
\mathbf{Geo}(-,-) \cong \mathbf{Geo}(-,-) \cong \mathbf{Geo}(\mathbf{Sch}_-,-)
\]

as bifunctors \( \mathbf{Geo}_P^{\text{eff}} \times \mathbf{Geo}_P^{\text{eff}} \to \mathbf{S} \), putting \( - \times_{\mathbf{Top}} P \) and \( \mathbf{Sch}_- \) in adjunction. Secondly, by lemma 4.2, if \( C \) is effective and its objects are locally in \( P \), then it is an effective envelope of \( C \times_{\mathbf{Top}} P \).

**5.8 Corollary.** Let \( C \) be a geometry, and suppose that the underlying topological space of each object of \( C \) satisfies a topological property \( P \). Then \( C \) admits a category of \( P \)-schemes. Every \( P \) open subset is representable.

**Proof.** By definition, \( C \)-schemes are locally in \( P \). Hence, \( \mathbf{Sch}_C \) is the (fully) effective envelope of its restriction to \( P \).

Enveloping \( \mathbf{Geo}_P^{\text{eff}} \to \mathbf{Geo}_P^{\text{eff}} \) creates colimits. Thus \( \mathbf{Sch}_C \) being initial in \( \mathbf{Geo}_P^{\text{eff}} \uparrow C \), it follows that \( \mathbf{Sch}_C^P \) is initial in \( \mathbf{Geo}_P^{\text{eff}} \uparrow C \).

The motivating example of this paper is that of manifolds:

**5.9 Definition** (Manifolds). A geometry \( C \) is said to be effective for descent of manifolds, or a category of manifolds, if it is \( P \)-effective with \( P \) the category of Hausdorff, locally compact, second countable spaces of finite covering dimension.

If \( C \) is any geometry whose objects are \( P \), then a \( P \)-effective envelope of \( C \) may also be called a category of \( C \)-manifolds, and denoted \( \text{Man}_C \).
5.10 Example ($C^k$ manifolds). Denote by $\text{Man}_{C^k}$ the category of $C^k$ manifolds. Then the usual underlying space functor gives $\text{Man}_{C^k}$ the structure of a geometry in which every open subset is representable. Moreover, it is effective for descent of manifold objects.

In fact, $\text{Man}_{C^k}$ is the effective envelope of the category of open balls in $\mathbb{R}^n$ and $C^\infty$ maps, as can be seen from the fact that for every manifold, the poset of balls contained in $M$ is an atlas. Note that this latter site does admit pullbacks of open immersions, hence strictly speaking does not admit a representation in the theory of [DAGV].

Note that several variations of definition 5.9 are possible without affecting this key example. Since I do not imagine the scope of this definition to extend much beyond classical manifolds and their derived counterparts, which particular variant is chosen would seem to be primarily a matter of taste.

The special case $k = 0$ is even a ‘property of topological spaces’ in the sense of 5.6.

5.11 (Spivak). Spivak’s axioms (2-3) of [Spi10, def. 1.2] say that the pair $(C, |−|)$, with $|−|$ valued in compactly generated Hausdorff spaces, is a spatial geometry in which every open subset of a representable object is representable and closed under glueing finite atlases.

In other words, it is equivalent to our definition of a category of manifolds except that we in principle allow glueing along countably infinite atlases. In light of Whitney embedding (conjecture 7.11), this should not make any difference for the $C^k$ setting.

5.12 (Stacks). I expect that given a suitable generalisation of geometry (cf. 4.17), definition 5.1 can be used to define reasonable categories of geometric stacks of any flavour. That is, it seems likely that the already-published [HAGII] definitions of Simpson-style geometric n-stacks (def. 1.3.3.1) and Deligne-Mumford stacks, algebraic spaces, and schemes (def. 2.1.1.4) satisfy precisely the same universal property of our categories of schemes, except with ‘open immersion’ everywhere replaced by ‘étale’, ‘smooth’, etc. (The definitions of op. cit. are given in the authors’ HAG context, but they make sense for more general Lurie geometries — although the expression ‘algebraic space’ hardly seems appropriate outside the algebraic world).

6 Sheaves

Let $\text{Sh}C \subseteq \text{PSh}C$ denote the full subcategory spanned by the functors that transform effective atlases into colimits. In other words, $\text{Sh}C$ is the category of hypercomplete sheaves appearing in [HAGI] and [HTT, §6, 2.2 & 5.2]. For a proof of this statement and further details, see the appendix B.

Along the lines of [TV09, §2.4], we will locate an effective envelope of $C$ inside $\text{Sh}C$.

6.1 (Geometry of sheaves). The universal property B.2 of $\text{Sh}C$ yields a unique colimit-preserving extension $|−| : \text{Sh}C \to \text{ShTop}$ of the underlying space functor of $C$. We can use this extension to put the structure of a geometry on $\text{Sh}C$ — more precisely, a variant of the spatial geometries of §4 in which the ‘underlying space’ of an object can be a sheaf rather than a space — see 4.19). Moreover, the Yoneda functor $C \to \text{Sh}C$ can be uniquely upgraded to a geometric functor with respect to this structure.

Technically, our notions of geometry in §4 were based on an underlying space functor with values in $\text{Top}$, rather than the sheaf category thereupon. However, if we define an open immersion in $\text{ShTop}$ to be any morphism that becomes an open immersion of topological spaces after base change to any topological space, then all notions in §4 continue to make sense in this mildly generalised setting.

Alternatively, it will be harmless to restrict attention to those objects whose underlying object is an honest topological space, since of course in the end our scheme objects will all have this property.

The purpose of this section is to prove the following statement:

6.2 Proposition. The category of sheaves is an effective geometry with ‘underlying space’ functor given by Yoneda extension of that of $C$. All open subsets of all objects are representable. The Yoneda functor $C \to \text{Sh}C$ is a transformation of geometries.

5 Up to a hypercompletion issue.
Proof. That open immersions in $\mathbf{Sh}C$ are always represented, and hence that $\mathbf{Sh}C$ is a geometry, is 6.3. This paragraph also argues that Yoneda is a geometric functor. Descent for morphisms is 6.4, while effectivity of atlases is 6.7.

6.3 (Open subfunctors). Let $X$ be a topological space. Then the functor

$$S \mapsto C^0(|S|, X)$$

is a sheaf on $C$. Indeed, if $V : I \to \mathcal{V}(S)$ is an atlas, then the underlying space $|V|$ is an atlas for $|S|$, and so $|S|$ is a colimit of $|V|$ by descent for morphisms in $\mathbf{Top}$. This construction is natural in $X$, hence defines a functor $\mathbf{Top} \to \mathbf{Sh}C$, right adjoint to the underlying space functor.

Using this adjoint, we can reformulate the property of a morphism $X \to Y$ in $C$ to be Cartesian over a continuous map $X' \to Y'$ of topological spaces (as in definition 4.1) to be that these fit into a pullback square

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{i'} & Y'
\end{array}$$

in $\mathbf{PSh}C$ and hence in $\mathbf{Sh}C$. In particular, $X \to Y$ is an open immersion if $|X| \to |Y|$ is an open immersion and $X \cong |X| \times |Y|$ in $\mathbf{Sh}C$.

Similarly, if $X : \mathbf{Sh}C$ and $|U| \subseteq |X|$ is an open subset, then the functor $|U| \times |X| \to X$ is Cartesian over $|U| \subseteq |X|$. Indeed, this is true by construction for representable objects, and therefore in general because colimit-preserving functors on $\mathbf{Sh}C$ are determined by their restriction to $C$. In particular, every open subset of an object of $\mathbf{Sh}C$ is representable — whence $\mathbf{Sh}C$ is a geometry.

Moreover, the Yoneda functor $(\cdot)^* : C \to \mathbf{Sh}C$ preserves open immersions. Indeed, suppose $U \to X$ in $C$ is an open immersion. Since Yoneda commutes with underlying space, the diagram

$$\begin{array}{ccc}
U & \xrightarrow{\iota} & X \\
|U| \times |X| & \xrightarrow{\iota^*} & X^* \\
|U| & \xrightarrow{\iota} & |X|
\end{array}$$

is commutative. Moreover, as we have just observed, the outer square is Cartesian; hence, by the two out of three property, the upper right square is Cartesian.

Because local isomorphisms are stable for base change, $U \to |U| \times |X| \to X$ is a local isomorphism, that is, $U^* = |U| \times |X|$. It follows from the argument of example 4.6 that the Yoneda functor is geometric.

6.4 (Atlases are colimits). An effective atlas in $\mathbf{Sh}C$ is also effective for descent of morphisms. Indeed, if $U \mid I'$ is an effective atlas for $X : \mathbf{Sh}C$ and $S : X$, then by B.6 $\mathcal{V}(S'; \mathbf{Sh}C)|_X U_I$ is a site for $S^*$, hence the induced diagram $\mathcal{V}(S; \mathbf{C})|_X U_I \to \mathcal{V}(S; \mathbf{C})$ is a $\mathbf{C}$-site by 3.17.

6.5 (Unions of opens). Since as we have seen 6.1 every open subset of a sheaf $X$ is representable, the underlying space map $\mathcal{V}(X) \to \mathcal{V}(|X|)$ is bijective. In particular, it commutes with unions: $|\bigcup_i U_i| = \bigcup_i |U_i|$.

In fact, this notion of union agrees with the notion of supremum in the full subobject lattice of $X$. Denote by $\text{Sub}X$ the preorder of monomorphisms $Z \to X$ in $\mathbf{Sh}C$. This is a complete, distributive lattice containing $\mathcal{V}(X)$.

Indeed, if $U_i \to X$ is a family of open subobjects, then the sieve in $\mathcal{V}(X)$ generated by the $U_i$ is an atlas for $U = \bigcup_i U_i$. By proposition 6.4, $U$ is a colimit of the $U_i$. Therefore it is equal to the supremum in $\text{Sub}X$. See part one of [TV09, Def. 2.12].

6.6 (Descent for open immersions). Let $X : \mathbf{Sh}C$, and let $V : I \to \mathbf{Sh}C$ be an effective atlas of $X$. Let $i : U \to V$ be a Cartesian transformation of functors $I \to \mathbf{C}$ (see 3.4) such that for each $i_\iota : U_\iota \to V_\iota$ is an open immersion. By universality of atlases, $U \mid I$ is an atlas of its colimit $U_I$. Moreover, $U_I \to X$ is an open immersion.
Proof. $|V \times_X U|$ is an atlas for $\bigcup_i |U_i|$. Indeed, since $U_i \to V_i \to X$ are open immersions, we only need to check the condition for binary overlaps; this is satisfied because unions distribute over intersections. Moreover, for each $i$

$$|U_i| = \left( \bigcup_i |U_i| \right) \cap |V_i|.$$ 

Now let $\bar{U} \to X$ be the open subobject representing $\bigcup_i |U_i| \subseteq |X|$. By stability for pullback of open immersions, the square

\[
\begin{array}{ccc}
U_i & \to & V_i \\
\downarrow & & \downarrow \\
\bar{U} & \to & X
\end{array}
\]

is Cartesian for each $i$. Since colimits in ShC are universal, $\bar{U}$ is a colimit of $U$. □

6.7 (Colimits are atlases). Let $U : I \to \text{Sch}_C$ be a diagram of open immersions such that $|U|$ is an atlas of its colimit $|X|$ — that is, a ShC-valued atlas. Let $X$ be a colimit of $U$. Because $\{\cdot\}$ commutes with colimits, $|X|$ is the underlying space of $X$.

We will show that for each $i : I$ the map $U_i \to X$ is Cartesian over $|U_i| \subseteq |X|$. In particular, $U$ is an atlas for $X$.

Proof. If $K \subseteq I$ is a sieve, then we write $U_K$ for the colimit of $U|K$. By the discussion in 6.5, if $K$ is contained in (the sieve generated by) $i : I$, then $U_K \to U_i$ is an open immersion over $|U_K| \subseteq |U_i|$ (notation from somewhere in §3).

Let $0 : I$, and for each $i : I$ let $0 \wedge i \subseteq I$ be the intersection of the sieves generated by 0 and $i$. Then $U_{0 \wedge i} \subseteq U_i$ is an open immersion, and for $i \leq j$ in $I$, the square

\[
\begin{array}{ccc}
U_{0 \wedge i} & \to & U_i \\
\downarrow & & \downarrow \\
U_{0 \wedge j} & \to & U_j
\end{array}
\]

is a pullback, because by the atlas property, it is a pullback on the level of underlying spaces. Thus by descent for open immersions 6.6, $U_i \to X$ is an open immersion. □

6.8 Corollary (to Prop. 6.2). A geometry $C$ is effective for descent of morphisms if and only if its Yoneda embedding $C \to \text{ShC}$ is fully faithful. It is moreover effective for descent of objects if and only if it is closed in $\text{ShC}$ under colimits of atlases.

Proof. The condition of subcanonicity is precisely that each object be a sheaf under the embedding $C \to \text{PShC}$. This implies that Yoneda is fully faithful. Conversely, each effective atlas of $C$ becomes a colimit in ShC. Therefore, if Yoneda is fully faithful, effective atlases are colimits already in $C$.

The second statement follows from the fact that any $C$-atlas admits a colimit in ShC; since Yoneda is geometric, the atlas can be rendered effective in $C$ only by this target, that is, if and only if it belongs to $C$. □

6.9 Corollary. Let $C$ be an effective geometry. Then every open subset of every object of $C$ is representable. In particular, pullbacks of open immersions are representable.

Open immersions in $C$ are stable for descent in the sense of 6.6.

Proof. Let $X : C$, $|U| \subseteq |X|$ an open subset. Since $C$ is a geometry, the sieve in $\text{P}(X)$ generated by $|U|$ is an atlas for the representing sheaf $U : \text{ShC}$. Therefore by corollary 6.8, $U \in C$. The second statement follows by essentially the same argument. □
**Locally representable sheaves** In the remainder of this section, we will identify the effective envelope of $\mathcal{C}$ with the full subcategory of $\mathcal{C}$ spanned by $\mathcal{C}$-atlases.

**6.10 (Sheaves that admit an atlas).** Let $X : \mathcal{C}$. We define the category $\mathcal{V}(X; C)$ of principal open subobjects of $X$ as follows:

- Objects of $\mathcal{V}(X; C)$ are objects $U$ of $C$ equipped with an open immersion $U^* \hookrightarrow X$.
- A morphism in $\mathcal{V}(X; C)$ is a morphism in $C_X$ whose action on the underlying objects is an open immersion in $C$.

More precisely, it is defined by these criteria as a faithfully embedded subcategory of $C_X$.

This $\mathcal{V}(X; C)$ is a preorder. Indeed, if $U \to V$ in $\mathcal{V}(X; C)$, then the map is determined uniquely by its being Cartesian over $|U| \subseteq |V| \subseteq |X|$.

**Proposition.** The following conditions on an object $X : \mathcal{C}$ are equivalent:

i) $\mathcal{V}(X; C) \to \text{Sh} C$ is a site for $|X|$;
ii) $\mathcal{V}(X; C) \to \text{Sh} C$ is an atlas for $|X|$;
iii) $X$ admits an atlas factoring through $\mathcal{V}(X; C)$;
iv) $|X| = \bigcup_{U : \mathcal{V}(X; C)} |U|$;
vi) $X$ is a colimit of $\mathcal{V}(X; C) \to \text{Sh} C$;
vi) $X = \bigcup_{U : \mathcal{V}(X; C)} U^*$.

Compare vi) with [TV09, Déf. 2.15].

**Proof.** The implications i)⇒ii)⇒iii)⇒iv) follow straight from the definitions. Meanwhile iv)⇒vi) by 6.5. Evidently vi)⇒vi), while i/ii)⇒vi) is the statement 6.4 that $\text{Sh} C$ is subcanonical.

The remaining implication is iv)⇒i). We have to check the condition for binary overlaps in definition 3.2, which is a consequence of the following lemma:

**Lemma (Overlaps of representable charts).** Let $U : \mathcal{V}(X; C)$ and $\xi : S \to X$ in $C_X$. Then $\xi^{-1}|U| \subseteq |S|$ is covered by the objects of the set $\mathcal{V}(S; C)|_X U$.

**Proof.** We may factor $V \to X$ in $\text{PSh} C$ into a local isomorphism $V \to V^*$ followed by an open immersion $V^* \hookrightarrow X$ in $\text{Sh} C$ and prove the result for these stages separately. The first stage is a consequence of lemma B.6.

For the second, note that by the construction 6.3 of open immersions in $\text{Sh} C$ the desired slice is precisely the subset of $W : \mathcal{V}(S)$ satisfying $|W| \to |U| \subseteq |X|$, which by the site axiom for a geometry covers $\xi^{-1}|U|$.

Applying the lemma to the case $S : \mathcal{V}(X; C)$ proves that $\mathcal{V}(X; C) \to \text{Sh} C$ is an atlas. It is even a site: if now $U \to X$ is an open subobject, then $|U|$ is covered by the sets $|U| \cap |V|$ as $V$ ranges over $\mathcal{V}(X; C)$, which in turn is covered by the elements of $\mathcal{V}(V; C)|_U \subseteq \mathcal{V}(X; C)$.

**Definition.** A $C$-sheaf is said to be a $C$-scheme if it satisfies the equivalent conditions of the proposition.

**6.11 Corollary.** Every open subobject of a $C$-scheme is a $C$-scheme. In particular, pullbacks of open immersions are representable.

**Proof.** Indeed, if as in part i) of proposition 6.10 $\mathcal{V}(X; C)$ is a site for $X$ and $U \to X$ is an open subobject, then $\mathcal{V}(U; C) \subseteq \mathcal{V}(X; C)$ is the sieve generated by $|U|$, which is also a site.

**6.12 (An atlas for the fibre product).** In this paragraph, we will assume that $\mathcal{C}$ admits fibre products. Let $I = [\cdot \to \cdot \to \cdot]$ denote the index category for fibre products, and let

$$
\begin{array}{c}
X_0 \\
X_1 \xrightarrow{f_1} X_{01}
\end{array}
$$
be an $I$-indexed diagram in $\text{Sch}_C$. Let’s write $Z := X_0 \times_{X_0} X_1$ for its fibre product in $\text{Sh} C$. We will show that $Z$ is covered by elements of $\mathcal{U}(Z; C)$.

Since pullbacks of open immersions are representable, $\mathcal{U}(\_; C)$ is a contravariant functor from $\text{Sch}_C$ to posets. Applying the Grothendieck construction, we obtain a Cartesian fibration

$$\int_{I} \mathcal{U}(-; C) \to I.$$ 

We’ll construct a covering for the fibre product indexed by the poset $K$ of sections of this family. An element of $K$ is a triple $(U_0, U_1, U_{01})$ with $U_i \subseteq X_i$ and the images of $U_0, U_1$ in $X_{01}$ contained in $U_{01}$. Since open immersions are stable for base change in $C$, taking the fibre product defines a functor $V : K \to \mathcal{U}(C; Z)$.

It is elementary to see that these sets cover $Z$. Write $p_i$ for the projection $|Z| \to |X_i|$.

- Since $X_{01}$ is covered by $U \mid \mathcal{U}(X_{01}; C)$, it will suffice to show that for each $U_{01}, V \mid K \mid U_{01}$ covers $p^{-1}_{01}|U_{01}|$.
- For each $U_{01} : \mathcal{U}(X_{01}; C)$ and $i : \{0, 1\}$, $f^{-1}_{i} U_{01} \hookrightarrow X_i$ is a $C$-scheme. Therefore $f^{-1}_{i} |U_{01}|$ is covered by sets of the form $|U_i|$ with $U_i : \mathcal{U}(X_i; C) \mid X_{01}$ $U_{01}$. Hence, $p^{-1}_{01}|U_{01}| \subseteq |Z|$

is covered by sets of the form $p^{-1}_{0}|U_0|$ with $U_0$ in this range. It will therefore suffice to show that for each such $U_0, V \mid K \mid U_0$ covers $p^{-1}_{0}|U_0|$.

- Let $U_0 : \mathcal{U}(X_0; C)$. Now $p^{-1}_{0}|U_0| \subseteq |Z|$

is covered by sets of the form $p^{-1}_{0}|U_0| \cap p^{-1}_{1}|U_1|$ for $U_1 : \mathcal{U}(X_1; C)$.

6.13 Aside. A more complicated, inductively defined version of this construction may of course be applied to produce atlases for limits over arbitrary finite directed diagrams.

6.14 Theorem. The Yoneda functor $C \to \text{Sch}_C$ exhibits $\text{Sch}_C$ as an effective envelope of $C$.

Proof of 6.2. We have already shown that $\text{Sh} C$ is effective for descent of objects and morphisms. Moreover, $\text{Sch}_C$ is stable in $\text{Sh} C$ for descent along atlases. Indeed, if $X : \text{Sh} C$ admits an atlas in $\text{Sch}_C$, then in particular it admits a covering in $\text{Sch}_C$ and hence in $C$, whence $X$ is a $C$-scheme by criterion iv) of proposition 6.10. Moreover, $X$ is a colimit of its atlas in $\text{Sh} C$ and therefore also in $\text{Sch}_C$, because this is a full subcategory. Therefore $\text{Sch}_C$ is effective.

It remains to show that $\text{Sch}_C$ is universal. By construction, $\text{Sch}_C$ is a $\text{F}_R\text{HC}$ in the sense of appendix A, with

- $R$ the set of effective atlases in $C$,
- $K$ the set of all abstract $C$-atlases.

Hence, by proposition A.9 it is universal among.

It will therefore suffice to prove that for each $f^* : C \to D$ with $D$ effective, the unique (left Kan) extension Lan$f^* : \text{Sch}_C \to D$ transforming atlases to colimits belongs to a unique morphism of geometries.

Lan$f^*$ preserves open immersions. If $U \hookrightarrow X$ is an open immersion of $C$-schemes, then as observed in 6.5, an atlas for $U$ is given by $\mathcal{U}(U; C) \subseteq \mathcal{U}(X; C)$. It follows that in fact, $f^*$ transforms atlases into atlases.

Unicity of the extension of $f_*$. By descent for morphisms between topological spaces, there is for each $X : \text{Sch}_C$ a unique $f_X : |f^* X| \to |X|$ such that for all $U : \mathcal{U}(X; C)$ the square

$$\begin{array}{ccc}
|f^* U| & \to & |f^* X| \\
\downarrow & & \downarrow \\
|U| & \to & |X|
\end{array}$$

in which the left-hand vertical arrow is given by the $f_U$ component of $C \to D$, commutes. We must show that this functor is geometric.

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Lan\( f^* \) is compatible with \( f_- \) (representable target). Suppose \( X : C \) and let \( U \hookrightarrow X^* \) be an open immersion. Then \( U = \bigcup U_i \) is a union of representables. For each \( i \), Therefore \( \bigcup_i [f_X^{-1}]U_i = [f^*U] \); that is,

\[
\begin{array}{ccc}
\text{Lan}f^*U & \longrightarrow & f^*X \\
\downarrow & & \downarrow \\
U & \longrightarrow & X
\end{array}
\]

is a pullback square.

Lan\( f^* \) is compatible with \( f_- \) (general case). Let \( U \hookrightarrow X \) be an open immersion in \( \text{Sch}_C \), and denote by \( V | \mathcal{V}(X; C) \) the tautological principal atlas. Consider the following commutative cube:

\[
\begin{array}{ccc}
|f^*(U \cap V_i)| & \longrightarrow & |f^*V_i| \\
|f^*U| & \longrightarrow & |f^*X| \\
\downarrow & & \downarrow \\
|V_i \cap U| & \longrightarrow & |V_i|
\end{array}
\]

We will show that the obverse face in this cube is a Cartesian square.

By compatibility of Lan\( f^* \) with \( f_- \) in the case of a representable target, the reverse face is Cartesian. The same goes for the lower square because \(|-|\) preserves pullbacks in \( \text{Sch}_C \).

Now if the reverse face is replaced with its disjoint union over \( i : \mathcal{V}(X; C) \), then the forward pointing arrows become surjective. Since disjoint unions commute with fibre products in \( \text{Top} \), this preserves the exactness of the squares. It therefore remains to show that the upper square is a pullback.

**Lemma.** For each \( i : \mathcal{V}(X; C) \), the square

\[
\begin{array}{ccc}
f^*(U \cap V_i) & \longrightarrow & f^*V_i \\
\downarrow & & \downarrow \\
f^*U & \longrightarrow & f^*X
\end{array}
\]

is Cartesian.

**Proof.** For \( i \leq j \) in \( \mathcal{V}(Y; C) \), it follows from the compatibility of Lan\( f^* \) with \( f_- \) in the case of a representable target that

\[
\begin{array}{ccc}
f^*(U \cap V_i) & \longrightarrow & f^*V_i \\
\downarrow & & \downarrow \\
f^*(U \cap V_j) & \longrightarrow & f^*V_j
\end{array}
\]

is a pullback square. The claim then follows by descent for open immersions 6.6.

Using again the fact that \(|-|\) and \( \bigcup \) commute with pullbacks, this completes the proof of compatibility.

Rephrasing provides a certain recognition principle for effective envelopes:

**6.15 Corollary** (Recognition principle). Let \( i^* : C \to D \) be a geometric functor commuting with the underlying space functor, and suppose that the Yoneda extension \( \text{Sh}_C \to \text{Sh}_D \) is fully faithful — for example, that \( C \to D \) is fully faithful. Suppose further that

- every object of \( D \) admits a covering by objects that lift to \( C \), and
• **D** is effective.

Then it exhibits **D** as an effective envelope of **C**.

**Proof.** By the universal property, it will suffice to provide a functor **D** → Sch**C** under **C**. Now by the same argument as that used to prove proposition 6.10, the covering condition actually implies that every object of **D** admits an atlas in **C**. Therefore every object of **D** is actually a member of Sch**C** ⊂ Sh**D**. □

6.16 (Structured spaces). The purpose of this paragraph is to sketch the relation of our procedure to an alternative approach, espoused in [DAGV]. For this to make sense, we embed **C** as a colimit-closed subcategory of some **ÎC** whose opposite is compactly generated (for example we can use Fun(**C,S**)op). This entails that there is a reasonable theory of sheaves with coefficients in **C**op.

In this case, for each topological space **X** the sheaf condition B.2 defines a subcategory Sh(**X; ÎC**op) of Fun(Σ(**X**), ÎC) whose inclusion admits a left exact reflector, and pullback and pushforward of sheaves is defined so that Sh(−, **C**op) defines a bivariant functor Top → Cat. The total space of this functor is what we call the category of ÎC-structured spaces, denoted TopÎC.6 This category is cocomplete.

Let **X** : Sch**C**. Then taking the sheaf on |**X**| associated to Σ(**X**; **C**) → **C** defines an object Σ**X** : Sh(|**X**|, ÎCop), the structure sheaf of **X**, and hence a ÎC-structured space. With a little technical work, this construction may be upgraded to a faithful, conservative functor

\[(|−|, Σ) : Sch**C** → TopÎC\]

that lifts the underlying space functor.

**Proof.** Let **X** : **C**. Pullback of open sets defines a map −, Σ : **C** | **X** × Σ(**X**) → Σ(|−|) and hence by composition with the Kan extension **C** | **X** → Fun(Σ(|**X**|), ÎC). If f : **X** → **Y**, then there is a natural isomorphism making the square

\[
\begin{array}{ccc}
**C** | **X** & → & Fun(Σ(|**X**|), ÎC) \\
\downarrow & & \downarrow \\
**C** | **Y** & → & Fun(Σ(|**Y**|), ÎC)
\end{array}
\]

commute. In fact, functoriality of pushforward means that this defines a natural transformation

\[**C** | (−) → Fun(Σ(|−|), ÎC)) → Sh(−, ÎCop)\]

of covariant functors **C** → Cat. Because **C** | − is left adjoint to f, this yields a functor **C** → TopÎC that lifts the underlying space functor. □

Its image can be described as follows:

• objects are structured spaces (**X**, Σ**X**) for which the diagram Σ(**X**; **C**) → Σ(**X**) defined below is a site for **X**. Indeed, in this case Σ**X** | Σ(**X**; **C**) is an atlas for a scheme object of **C**.

Objects are pairs consisting of an open subset **U** ⊆ **X** and a homeomorphism φ(U) : **U** ∼= |Σ**X**(U)| and whose maps are inclusions **U** ⊆ **V** for which the attented square

\[
\begin{array}{ccc}
SpecΣ**X**(U) & → & SpecΣ**X**(V) \\
\downarrow_φ & & \downarrow_φ \\
**U** & → & **V
\end{array}
\]

commutes and Σ**X**(U) → Σ**X**(V) is Cartesian.

6 Again, for this paragraph to make sense in full generality we ought strictly speaking to be working with locales rather than topological spaces; see 4.3.
• Let \( f : (X, O_X) \to (Y, O_Y) \) be a map in \( \textbf{Top}_C \). Let \( \Psi(\psi; f) \) be defined as follows: for all \( V \) in \( \Psi(\psi) \mid_Y U \), there is a map \( O_Y(U) \to O_X(V) \), and the resulting square

\[
\begin{array}{ccc}
\text{Spec} O_X(V) & \to & V \\
\downarrow & & \downarrow \\
\text{Spec} O_Y(U) & \to & U
\end{array}
\]

should commute.

In other words, these conditions may be used to construct \( \text{Sch}_C \) directly.

6.17 Corollary. The subcategory of the category \( \textbf{Top}_C \) of \( \hat{C} \)-structured spaces spanned by the locally \( C \)-representable objects and locally \( C \)-representable morphisms is an effective envelope of \( C \).

7 Derived manifolds

7.1 Definitions. Let \( E \) be a category with finite products. A product-preserving extension of \( E \) is a category \( \tilde{E} \) equipped with a product-preserving functor

\[
i : E \to \tilde{E}.
\]

All extensions considered in this section will be product-preserving, hence we will usually omit this qualifier.

Let \( \tilde{E} \) be an extension of \( E \). An object of \( \tilde{E} \) is said to be:

• \textit{presentable} if it is a limit of objects of \( E \);
• \textit{finitely presented} if it is a finite limit of objects of \( E \);
• \textit{quasi-smooth} if it is a fibre product of objects of \( E \);
• \textit{split} finitely presented, resp. quasi-smooth, if it is a retract of a finitely presented, resp. quasi-smooth object.

Note that if \( \tilde{E} \) is an \( n \)-category for finite \( n \), then split finitely presented implies finitely presented. Beware that for finitely presented objects, \textit{finite} means \textit{homotopically finite}: so, for example, fixed points sets for finite group actions — limits over the category \( BG \) associated to a finite group \( G \) — are not considered finite unless \( \tilde{E} \) is \( n \)-truncated for some finite \( n \).

7.2 Definitions. Suppose now \( E \) carries an underlying space functor \( |−|_E \). A geometric extension is a product-preserving extension for which \( \tilde{E} \) is a spatial geometry and \( i \) a geometric functor. Note that we do not at this stage assume that \( E \) is itself a spatial geometry, though we are allowing this extra generality only to admit example 7.14. The methods of §9 do not, in general, apply without this assumption.

An object of a geometric extension is said to be \textit{locally} presentable, (split) finitely presented, (split) quasi-smooth, if it admits a covering/atlas/site by objects with this property. (In the case of quasi-smooth, I'll drop the prefix 'locally'.)

7.3 Definition. A geometric extension of \( E \) is a category of derived manifolds relative to \( E \) if it is universal among split finitely complete categories of manifolds (def. 5.9) extending \( E \). A category of derived \( E \)-manifolds is denoted \( \text{DMang}_E \).

7.4 (Variant definitions). By substituting various words in the definition, we obtain related categories.

• Substituting for the word ‘manifold’ some other type of space, one may equally define categories of derived scheme objects, étale-spaces, Deligne-Mumford stacks, orbifolds — see 4.19 — and so on. Such categories are related to one another by universal properties; see, for example, 5.7.
• The words ‘split’ or ‘finitely complete’ may be removed or replaced with other finiteness conditions.

For example, the category $\text{QSM} \text{an}_E$ of quasi-smooth derived $E$-manifolds is obtained by replacing ‘finitely complete’ with ‘admitting fibre products of objects of $E$’. It is precisely the full subcategory of (locally) quasi-smooth objects of $\text{DM} \text{an}_E$ — see A.7.

• All of the above statements may be interpreted in $\infty$-category theory, or in $n$-categories for some finite $n$, yielding an $n$-category $\text{DMan}^{\leq n}_E$ of $n$-truncated derived manifolds. If $m \leq n$, then the universal property implies that there are left exact truncation functors

$$\text{DMan}^{\leq n}_E \to \text{DMan}^{\leq m}_E.$$ 

These truncation functors fit into the general framework of truncations in presentable categories, as described in A.11 (and [HTT, §5.5.6]). It follows that if we remove finiteness conditions from the definition, an $m$-truncated derived manifold can also be regarded as an $n$-truncated or fully derived manifold.

In particular, for $E = C^\infty$, the case $n = 1$ yields a version of classical $C^\infty$-schemes as studied in [MR91], while $n = 2$ ought to be related to Joyce’s d-manifolds [Joy12], as in example 7.9. See also example 2.11.

7.5. It is possible to deduce some preliminary properties of categories of derived manifolds immediately from the definition:

i) Because it is defined using a universal property, categories of derived $E$-manifolds are unique. Therefore, we say from now on the category of derived $E$-manifolds.

The existence can be deduced from fairly general results about completions of categories, which also anticipate that the inclusion functor $E \to \text{DMan}_E$ will be fully faithful; this is the approach taken in [DAGV, Prop. 3.4.5]. We will see a more explicit description in §9.

ii) Applying the universal property, categories of derived manifolds are functorial for product-preserving geometric functors of $E$.

iii) Suppose $E$ is a spatial geometry. The category of derived manifolds attached to $E$ is the same as the one attached to its category $\text{Man}_E$ of manifolds (def. 5.9). Indeed, the universal property of $\text{Man}_E$ yields a unique product-preserving geometric functor $\text{Man}_E \to \text{DMan}_E$, which extends uniquely to an equivalence $\text{DMan}_E=\text{DMan}_E$.

It follows that any site for $\text{Man}_E$ yields the same category of derived manifolds.

iv) If $\hat{E}$ is a finitely complete extension of $E$, then so is the full subcategory spanned by the finitely presented objects. Similar remarks apply to the quasi-smooth objects, and to the local variants. The subcategory of quasi-smooth objects satisfies Spivak’s axiom (5) 2.2.

It follows that every object of $\text{DM} \text{an}_E$ is locally split finitely presented.

v) I don’t know whether or not assuming $E$ is idempotent complete means that ‘split’ may be omitted from definition 7.3 without changing the resultant category (cf. question 9.9).

In the remainder of this section we collect a farrago of geometries to which the above definition can be applied and easy applications of the universal property.

Derived differential manifolds

7.6 Examples. The main examples for this paper are the categories $\text{Man}_{C^k}$ of manifolds of class $C^k$, resp. $\text{Man}_{\text{PL}}$ of piecewise-linear manifolds. Naturally, these are stable under the formation of manifold objects 5.9.

It’s more common to start with a site for the category of manifolds, for instance, the category $C^k$, resp. PL, with objects $[\mathbb{R}^n]_{n \in \mathbb{N}}$ and maps of class $C^k$, resp. PL. These are all spatial geometries because open subsets of $\mathbb{R}^n$ are generated by bounded rectangles, and there exist isomorphisms $\mathbb{R} \cong (0,1)$ in any of these classes. Other options include the category of all open sets in $\mathbb{R}^n$, all open rectangles, all bounded open sets or rectangles, and so on.

By part ii) of 7.5, the forgetful functors $C^k \to C^\ell$ for $k > \ell$ and $\text{PL} \to C^0$ induce forgetful functors $\text{DMan}_{C^k} \to \text{DMan}_{C^\ell}, \text{DMan}_{\text{PL}} \to \text{DMan}_{C^0}$.
In fact, it was the introduction to this work that drew my attention to Lurie’s formulation of the universal property.

7.1 Example. The category Man\textsubscript{C} of complex manifolds, or a site such as the category of Stein spaces or of open polydiscs. There is a forgetful functor DM\textsubscript{C} \rightarrow DM\textsubscript{C}^{\omega}. This category (or more precisely, a version based on [DAGV]) was studied extensively in [Por16].

7.7 Example. The category Man\textsubscript{C} of complex manifolds is left exact. If all inclusions of points are permitted, then these axioms also imply that the inclusions \(\mathbb{R}^n \subseteq \mathbb{R}^{n+k}, k_1 < k_2\) of boundary strata are also in the category.

7.8 Example (Corners). Manifolds with corners are locally modelled on the spaces \(\mathbb{R}^n_k := \mathbb{R}^n \times \mathbb{R}^k_{\geq 0}\). Unfortunately, there does not appear to be any general consensus on how best to define smooth maps between these objects. As long as a set of \(C^0\) maps satisfies the three conditions

i) the set includes enough open immersions to form a basis for the topology (for example, it includes the exponential \(\mathbb{R} \rightarrow \mathbb{R}\));

ii) every product projection is in the set;

iii) if \(X \rightarrow Y\) and \(X \rightrightarrows Z\) is in the class, then so is \(X \rightarrow Y \times Z\);

it comes with its own attendant category of derived manifolds with corners whose underlying space functor is left exact. If all inclusions of points are permitted, then these axioms also imply that the inclusions \(\mathbb{R}^n_{k_1} \subseteq \mathbb{R}^n_{k_2}, k_1 < k_2\) of boundary strata are also in the category.

A reasonable theory of derived manifolds with corners is likely to require an extension of the corner stratification of \(\mathbb{R}^n_k\); that is, the assignment to each object \(U\) a meet semilattice \(\text{Strat}(U)\) together with its characteristic functor \(\text{Strat}(U) \rightarrow [\mathbb{R}^n_k]\). In order for this construction to extend, it must be functorial for all maps and moreover compatible with products and covers, i.e.

- \(\text{Strat}(U \times V) = \text{Strat}(U) \times \text{Strat}(V)\).
- \(\text{Strat}(\bigsqcup V_i) = \text{colim} \{ \prod V_i, \text{Strat}(U_{ij}) = [\prod, \text{Strat}(U_j)]\}\).

Hence, maps between \(\mathbb{R}^n_k\) should be stratified in the sense that the preimage of each stratum in the target is a union of strata in the source. It is not hard to see that the set of all stratified \(C^0\) maps satisfies the conditions i-iii) above.

Differentiability conditions introduce yet more complications, for example: do we require maps to be differentiable in the normal directions at the boundary, are maps allowed to ramify at the boundary? The answers to these questions will affect the functoriality of the tangent complex at the boundary.

Some of the subtleties are discussed in [Joy09] and its sequels, which provide various candidate definitions.

7.9 (Joyce d-manifolds). By the universal property of derived \(C^\infty\)-manifolds, there is a unique left exact functor from DM\textsubscript{C}^{\omega} into Joyce’s category of d-spaces [Joy12], restricting to a functor from quasi-smooth derived manifolds to d-manifolds.

This second functor has been studied in detail in [Bor14], where it is shown that the induced functor of homotopy categories is full and essentially surjective, but not an equivalence.

Likewise, starting from a site \(C^\infty\) for manifolds with corners with Joyce’s notion of smooth map between such, one obtains a similar category of derived manifolds with corners and a functor from the quasi-smooth objects to Joyce’s d-manifolds with corners.

7.10 Example (Tameness). Useful theories of singular or noncompact manifolds tend to include some kind of tameness criteria, usually encapsulated in a semi- or subanalytic site or an o-minimal structure on \(\mathbb{R}^n\). There are many ways that such criteria can be embedded into a spatial geometry:

- Let \(\mathcal{S}\) be an o-minimal structure. We denote also by \(\mathcal{S}\) the category whose objects are \(\mathbb{R}^n\) and whose maps are continuous and locally \(\mathcal{S}\)-definable. Then \(\text{PL} \subset \mathcal{S}\), and so \(\mathcal{S}\) is a spatial geometry.
- The subcategory \(\mathcal{S} \cap C^\omega\) of analytic definable functions is a spatial geometry as long as the exponential function is definable, since this, together with linear functions, is enough to make \(\mathbb{R}\) isomorphic to any open interval. Otherwise, the category suffers from the same disability as the algebraic category; see remark 7.16.

\footnote{In fact, it was the introduction to this work that drew my attention to Lurie’s formulation of the universal property.}
• One also often works with a site consisting of a class \( \mathcal{U} \) of relatively compact open subsets of \( \mathbb{R}^n \) with only finite coverings. The underlying space of any such object is coherent with quasi-compact opens in one-to-one correspondence with open subsets in class \( \mathcal{U} \). For example, \( \mathcal{U} \) could be semi- or subanalytic open sets with \( C^\omega \) functions, or \( J \)-definable open sets with \( J \)-definable functions.

In any case, the underlying space of an \( J \)-definable derived manifold (resp. \( J \)-definable derived analytic manifold) is locally an \( J \)-definable set (resp. a definable analytic set). Thus by general facts about o-minimal structures, \( J \)-definable derived manifolds are locally contractible, admit triangulations and Whitney stratifications, and so on. See the ‘analytic-geometric’ and ‘geometric’ categories of [DM96].

**Whitney theorems**  Classical manifold theory is simplified by famous results of Whitney, which we may optimistically conjecture continue to hold in the derived setting. I admit that my evidence for these conjectures is rather thin.

The most well-known result is the Whitney embedding theorem, which was proved for compact quasi-smooth \( C^\infty \) derived manifolds by Spivak [Spi10, Prop. 3.3]. The \( C^\omega \) version of this statement is due to Morrey and Grauert.

**7.11 Conjecture** (Whitney-Morrey-Grauert embedding). Let \( k > 0 \) (including \( k = \omega \)). Each derived \( C^k \)-manifold may be embedded as a closed submanifold of some \( \mathbb{R}^n \).

In other words, the criterion of 7.3 that DMan be effective for descent of manifolds is satisfied automatically in the case of \( C^k \).

**7.12 Aside.** The Whitney embedding theorem even allows us to embed our manifold in a Euclidean space of twice the dimension. Any extension of this dimension bound to the derived setting would of course have to depend on the local embedding dimensions of the derived manifold.

**7.13 Conjecture** (Whitney). Let \( l \geq k > 0 \), and fix an o-minimal structure. Every derived \( C^k \)-manifold is \( C^l \)-diffeomorphic to a derived \( C^l \)-manifold. This derived \( C^l \)-manifold is unique up to \( C^l \)-diffeomorphism.

In the classical case, this fact follows from the embedding theorem and the Weierstrass approximation theorem, which allows us to find a \( C^k \)-small perturbation of the embedding to a \( C^l \)-one. In the derived setting, however, underlying spaces may be singular and so a small perturbation needn’t be topologically trivial; hence a proof of this statement would need a more refined method of ‘equisingular’ perturbation. For this reason, it is likely to be true only with some ‘tameness’ hypotheses in the vein of 7.10.

**Derived algebraic varieties**  The case of derived algebraic varieties was treated in some detail in [DAGV, §4.1.2].

**7.14 Example** (DAG). Let \( R \) be a commutative ring, and denote by \( \mathbb{A}^n_R \) the category whose objects are the affine spaces \( \mathbb{A}^n_R \) for \( n \in \mathbb{N} \) and morphisms polynomial maps. We equip \( \mathbb{A}^n_R \) with an underlying space functor via the prime spectrum. Then \( (\mathbb{A}^n_R, |-|) \) is *not* a spatial geometry, because open subsets of affine space are not usually algebraically isomorphic to affine spaces.

Nonetheless, our construction of the category of \( \mathbb{A}^n_R \)-algebras (§9), together with the extension of the underlying space functor still makes sense; in fact, \( \mathbb{A}^n_R \) is nothing but the \( \infty \)-category of simplicial commutative \( R \)-algebras. (Note that if \( Q \not\subseteq R \), there is another variant of derived \( R \)-algebras via \( E_\infty \)-rings.)

In particular, \( (\text{Alg}_{\mathbb{A}^n_R})^{op} \) is a spatial geometry: the Zariski site of affine derived schemes of finite presentation over \( R \). Thus the category of derived \( R \)-schemes of finite presentation is a category of derived schemes for \( \mathbb{A}^n_R \) in the sense of definition 7.3.

**7.15 Aside.** Of course, if we enlarge \( \mathbb{A}^n_R \) to include a basis of Zariski-open subsets of affine space as objects, then it is a spatial geometry and the general construction of §9 applies. Lurie has shown in his context (prop. 4.2.3 of *op. cit.*) that this actually yields the same result.

---

\[^8\]For the definition of embeddings, cf. §10.
7.16 Aside. It seems likely that the same logic applies to the category of analytic functions defineable with respect to an o-minimal structure (cf. 7.10) possibly without the exponential (equipped with the topology generated by open sets defineable by these functions).

7.17 Example (Real and complex points). Suppose that \( R \to C \). The functor of \( C \)-points extends to a functor

\[
-(C) : \text{DSch}_R \to \text{DMan}_C
\]

to a functor of derived affine schemes with values in derived complex manifolds. This functor extends moreover to a full subcategory of the category of derived \( R \)-schemes whose objects admit a Zariski-open covering by open subsets of affine space.

The same logic applies to taking real points when \( R \to \mathbb{R} \), yielding a functor

\[
-(\mathbb{R}) : \text{DSch}_R \to \text{DMan}_{C^\omega}
\]

A more sophisticated construction using the étale site over \( R \) should allow us to extend these functors to all locally finitely presented derived Deligne-Mumford stacks over \( R \).

8 Transversality

In part (2) of Spivak’s notion of geometricity, he requires that the inclusion

\[
i : \text{Man} \to \text{dM}
\]

preserves transverse fibre products of manifolds. Some version of this feature is common to all approaches to derived geometry.

8.1 Definition. A diagram \( J \to \text{Man}_E \) is said to be categorically transverse if its limit exists in \( \text{Man}_E \) and is preserved by the functor \( \text{Man} \to \text{DMan} \) - or equivalently, if it is preserved by any product-preserving geometric extension.

A pair of maps \( X, Y \to Z \) with common codomain is said to be categorically transverse if the cospan

\[
\begin{array}{ccc}
Y & \to & Z \\
\downarrow & & \downarrow \\
X & \to & Z
\end{array}
\]

has this property.

Fibre products of categorically transverse maps preserve ‘classiciy’ — membership of \( \text{Man}_E \) — in any reasonable derived theory. On the other hand, it is conceivable that there are pairs of morphisms in \( E \) that admit fibre products and that we would like to consider as transverse, but which are not categorically transverse. One would then have to impose a localisation of \( \text{DMan} \) for these to be preserved.

The point of this section is to show that maps between \( C^k \) manifolds are formally transverse if and only if they are transverse in the usual sense. Conversely, example 2.6 shows that in general, categorical transversality can be quite unpredictable.

8.2 Proposition. Let \( k \geq 1 \). A pair of maps of \( C^k \)-manifolds is transverse if and only if it is categorically transverse.

8.3 (Submersions). A map \( f : X \to Y \) in \( E \) is said to be a submersion if, locally on \( X \), it is isomorphic to the projection on product

\[
f \simeq \pi_Y : V_X \times Y \to Y
\]

for some local identification \( X \simeq V_X \times Y \). By the implicit function theorem, a map of \( C^\infty \)-manifolds is a submersion if and only if its differential is everywhere surjective.

Lemma. Pullbacks of submersions exist in \( E \) and are submersions. Submersions are categorically transverse to any map.
Proof. Indeed, if \( f \) is globally a product projection, then the two-out-of-three property for Cartesian squares applied to the diagram

\[
\begin{array}{c}
V_X \times Y' \longrightarrow V_X \times Y \longrightarrow V_X \\
\downarrow \quad \downarrow \quad \downarrow \\
Y' \longrightarrow Y \longrightarrow 1
\end{array}
\]

implies that the left-hand square is Cartesian and remains so in any geometric completion. This implies the result because fibre products and \( i \) are calculated locally in \( E \).  

See also definition 7 and proposition 8 of [DAGV, §3.1].

**8.4 (Immersions).** A map \( f : X \to Y \) in \( E \) is a normally nonsingular immersion if, locally on \( X \), it may be written as the fibre of a submersion over a point \( 1 \to Y, \) \(^9\) that is, it locally has the form

\[ X \cong X \times 1 \to X \times N_{X/Y} = Y \]

for some product decomposition \( X \times N_{X/Y} \) of \( Y \). We then write \( X \to Y \). The pullback of an immersion along a submersion is an immersion; indeed,

\[
\begin{array}{c}
Z \times X \longrightarrow Z \times X \times N_X \\
\downarrow \quad \downarrow \\
X \longrightarrow X \times N_X
\end{array}
\]

is Cartesian, by the calculation of lemma 8.3.

A map \( X \to Z \) is said to be transverse to a normally nonsingular immersion \( Y \to Z \) if, locally on \( Y \), there exists a normal space \( N_{Y/Z} \) such that the composite \( X \to Z \xrightarrow{\pi} N_{Y/Z} \) is a submersion. Any submersion is transverse to any immersion in this sense.

**Lemma.** A pair of maps consisting of a normally nonsingular immersion \( Y \to Z \) and a transverse map \( Y \to Z \) is categorically transverse.

**Proof.** Locally on \( Y \) we have decompositions \( Z \cong Y \times N_Y \) and \( X \cong V_X \times N_Y \). Hence we have a diagram

\[
\begin{array}{c}
V_X \longrightarrow Y \longrightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \\
V_X \times N_Y \longrightarrow Y \times N_Y \longrightarrow N_Y.
\end{array}
\]

The right-hand and outer squares are Cartesian and categorically transverse because they are pullbacks of submersions, cf. 8.3. Hence the left-hand square is Cartesian and categorically transverse.

**Proof of 8.2.** Let \( X \to Z \to Y \) be maps of manifolds. The fibre product \( X \times_Z Y \) can also be represented as the pullback of the diagram

\[
\begin{array}{c}
Z \longrightarrow \\
\downarrow \quad \downarrow \\
X \times Y \longrightarrow Z \times Z
\end{array}
\]

where the diagonal \( Z \to Z \times Z \) is a normally nonsingular immersion. Then \( X \) and \( Y \) are transverse if and only if \( X \times Y \) is transverse to the diagonal.

For the converse, note that the tangent space 2.5 at a point of the fibre product is a cone over a map \( TX \oplus TY \to TZ \). If this map is not surjective, then the cone is not concentrated in degree zero.

\(^9\)Our assumptions on \( E \) do not imply that \(|1|\) is the one-point topological space. Of course in derived differential geometry \(|X| = \text{Map}(1,X)\) as sets.
8.5 Aside. The diagonal being a normally nonsingular immersion is special to the category of manifolds - for instance, as D. Joyce has pointed out to me, this fails in the category of manifolds with corners. Thus, given a suitable definition of transversality in this setting, alternative arguments would be needed.

9 Lawvere algebras

Lurie’s approach to defining the geometric envelope [DAGV, Prop3.4.5] is to appeal directly to general principles about attaching colimits to categories. Broadly speaking, the arguments of this section must rest on those same principles, which are detailed in appendix A.

9.1 (Layout for the construction of DMan). Since it is defined using a universal property, the category of derived manifolds is unique. A construction proceeds as follows:

- First, we will discuss the ∞-category Alg_E of algebras for the Lawvere algebraic theory E. There is a fully faithful functor E → Alg_E^{op} that exhibits this latter as universal among complete extensions of E.

- The above inclusion fails to be compatible with the geometric structure |−|_E of E. Rather, there is a reflective subcategory Alg_{E,|−|_E} = Alg_E through which i factors, and such that open immersions in E remain open immersions in Alg_E^{op}(|−|_E). Hence, this latter is universal among complete geometric extensions of E.

- The full subcategories of split finitely presented - equivalently, compact - objects Alg_{E,|−|_E}^{fp} → Alg_{E,|−|_E} are universal among split finitely complete extensions, resp. geometric extensions, of E.

- Hence, the category of manifold objects modelled on the opposite category of split finitely presented E-algebras is a category of derived manifolds for E.

9.2 Theorem. The category of manifold objects modelled on the opposite category of split finitely presented E-algebras is a category of derived E-manifolds.

Proof. By combining the universal properties of:

i) Alg_{E,|−|_E}^{fp} as a split finite completion of E (9.8);

ii) Alg_{E,|−|_E}^{fp} as a localisation of Alg_E^{fp} admitting the weak topology (9.12-9.13);

iii) the universal property of the weak topology (9.10);

iv) the universal property of the category of manifold objects. □

By replacing the appropriate terms, the same reasoning applies to give constructions of categories of quasi-smooth derived manifolds, scheme objects, orbifolds, etc..

9.3 Corollary. The extension functor Man_E → DMan_E is fully faithful.

9.4 Aside. The site of principal derived C^∞ manifolds is not subcanonical; for example, the fibre in DMan_{C^∞} of a submersion is a manifold if and only if it is globally a product projection, while by 8.2 it is always locally equivalent to the fibre in Man_{C^∞}.

9.5 Aside. The approach of Spivak (and Lurie in [DAGV, §2]) proceeds via a structured spaces formalism that condenses ii-iv) into one step, so that a derived E-manifold is defined to be an E-structured space locally equivalent to a finite limit of E-schemes. See 9.14.

E-algebras

9.6. Let E be a category with finite products, and define

Alg_E := Fun^*(E, S)

to be the category of product-preserving presheaves of spaces on E^{op}. Note that this is not a sheaf condition; the localisation PSh(E^{op}) → PSh^*(E^{op}) is not left exact. Objects of Alg_E are
models of the Lawvere (higher) algebraic theory \( \mathbf{E} \). In particular, \( \text{Alg}_{C^\infty} \) (see example 7.6) is a higher version of what are usually called \( C^\infty \)-rings.

The category of \( \mathbf{E} \)-algebras has been studied in [HTT, §5.5.8], where it is called the non-Abelian derived category of \( \mathbf{E}^{op} \) and denoted \( \mathcal{P}_2(\mathbf{E}^{op}) \). In remark 16 of loc. cit. Lurie notes two universal properties:

- among cocomplete categories with a coproduct-preserving functor from \( \mathbf{E}^{op} \);
- among categories stable under sifted colimits (def. 1 of loc. cit.) with a functor from \( \mathbf{E}^{op} \);

(established in propositions 10 and 15 of loc. cit.). The first of these exhibits it as a localisation of the presheaf category, the second as a full subcategory. In particular, the Yoneda embedding

\[
\mathbf{E} \hookrightarrow (\text{Alg}_E)^{op}, \quad X \mapsto \mathbf{E}(X,-)
\]

is fully faithful.

9.7. A finite coproduct of \( \mathbf{E} \)-algebras \( \mathbf{E}(X), \mathbf{E}(Y) \) is usually written with a tensor product notation \( \mathbf{E}(X) \otimes_E \mathbf{E}(Y) \); a pushout over a third \( \mathbf{E}(Z) \) with a relative tensor product \( \mathbf{E}(X) \otimes_{\mathbf{E}(Z)} \mathbf{E}(Y) \).

For the case \( \mathbf{E} = C^\infty \), the fact that the Yoneda embedding preserves products - i.e. that for \( M, N : \mathbf{E} \),

\[
C^\infty(M) \otimes C^\infty(N) \cong C^\infty(M \times N)
\]

appears as proposition 2.5 of [MR91, Chap. I].

9.8 (Finitely presented and quasi-smooth \( \mathbf{E} \)-algebras). The compact objects of \( \text{Alg}_E \) are the split finitely presented ones. Similarly, quasi-smooth \( \mathbf{E} \)-algebras are those that may be written as a pushout of objects of \( \mathbf{E}^{op} \). The full subcategories spanned by these objects have the following universal properties:

- \( (\text{Alg}_{E}^{fp})^{op} \) is universal among finitely complete and idempotent-closed product-preserving extensions of \( \mathbf{E} \);
- \( (\text{Alg}_{E}^{qs})^{op} \) is universal among product-preserving extensions of \( \mathbf{E} \) that admit fibre products of objects in \( \mathbf{E} \).

Indeed, this follows from the general argument of appendix A.

9.9 Question. Let \( k \geq 1 \). The categories \( C^k \) and \( \text{Man}_C^k \) of example 7.6 are idempotent complete. Are the categories of finitely presented, or quasi-smooth \( C^k \)-algebras automatically idempotent complete? Is Spivak’s category \( \text{dM} \) idempotent complete?

The weak topology

9.10 (Weak topology). Let \( \tilde{\mathbf{E}} \) be a finitely complete geometric extension of \( \mathbf{E} \). Then for any open immersion \( U \hookrightarrow Y \) in \( \mathbf{E} \) and object \( X : \mathbf{E} \mid Y \), its base change

\[
\begin{array}{ccc}
U \times_Y X & \longrightarrow & X \\
\downarrow & & \downarrow \\
U & \longrightarrow & Y
\end{array}
\]

exists in \( \tilde{\mathbf{E}} \) and is an open immersion. It is said to be weakly topologised (with respect to \( i \)) if open immersions of this form generate the topology of each object \( X : \tilde{\mathbf{E}} \); that is, if \( \dashv \) is a right Kan extension of \( \dashv \) along \( i \).

The weak topology on \( \tilde{\mathbf{E}} \) satisfies the following universal property: if \( (\mathcal{D}, i) \) is any geometric extension of \( \mathbf{E} \) and \( F : \tilde{\mathbf{E}} \to \mathcal{D} \) a functor, then \( F \) is automatically a functor of spatial geometries with respect to the weak topology on \( \tilde{\mathbf{E}} \).

We may always use right Kan extension to define an underlying space functor on an extension \( \tilde{\mathbf{E}} \) of \( \mathbf{E} \); if this defines a spatial geometry structure on \( \tilde{\mathbf{E}} \) such that \( i \) is a continuous functor, then we say that \( \mathbf{E}/\tilde{\mathbf{E}} \) admits the weak topology.

---

10This construction is a limit and so commutes with passing from locales to topological spaces, but not necessarily the other way - the locale \( \dashv \) may fail to have enough points even if \( \dashv \) does. In our main case of interest, we will only need to take finite limits of locally compact Hausdorff spaces, which is known to be preserved by passing back to locales (lemma 2.1 and prop. 2.13 of [Joh86, chap. II])
Forcing the weak topology

It is not usually the case that Alg\(_E\) can be given a geometric structure making the inclusion \(\mathfrak{E} \rightarrow Alg\_E\) a map of spatial geometries. The problem is that using (1) as a definition of open immersions retains too much ‘virtual’ information about the topology.

For instance, suppose that \(X\) is the equaliser of some fork \(X_0 \rightrightarrows X_1\) in \(\mathfrak{E}\). Then if \(U_0 \hookrightarrow X_0\) is an open neighbourhood of \(|X_0|\), the canonical map \(X \rightarrow X_0\) should factor uniquely through \(U_0\), making it the equaliser of \(U_0 \rightrightarrows X_1\). However, this won’t usually be the case in Alg\(_E\), since its definition makes no account of the open immersions of \(\mathfrak{E}\).

9.12 (Admitting the weak topology). If \(\tilde{\mathfrak{E}}\) admits the weak topology, then in particular open immersions in \(\mathfrak{E}\) must remain Cartesian in the extension. Let us say that an object \(X : \tilde{\mathfrak{E}}\) is actual with respect to \(|-|_\mathfrak{E}\) if for every open immersion \(U \hookrightarrow Y\) in \(\mathfrak{E}\), the square

\[
\begin{array}{ccc}
\text{Map}_\mathfrak{E}(X,U) & \longrightarrow & \text{Map}_\mathfrak{E}(X,Y) \\
\downarrow & & \downarrow \\
C^0(|X|,|U|) & \longrightarrow & C^0(|X|,|Y|)
\end{array}
\]

is Cartesian. That is, any \(X \rightarrow Y\) such that \(|X| \rightarrow |Y|\) factors through \(|U|\) factors uniquely through \(U\). By definition:

**Lemma.** The following are equivalent:

i) every open immersion in \(\mathfrak{E}\) remains an open immersion in \(\tilde{\mathfrak{E}}\) with respect to the weak topology;

ii) each object \(X : \tilde{\mathfrak{E}}\) is actual with respect to \(|-|_\mathfrak{E}\).

These conditions are necessary for \(\tilde{\mathfrak{E}}\) to admit the weak topology. Conversely, if \(\tilde{\mathfrak{E}}\) satisfies the conditions of the lemma, and moreover pullbacks in \(\tilde{\mathfrak{E}}\) of open immersions in \(\mathfrak{E}\) are representable, then \(\tilde{\mathfrak{E}}\) admits the weak topology.

9.13 (Forcing the weak topology). Suppose that every open subset of every object of \(\mathfrak{E}\) is representable. This can be achieved by replacing \(\mathfrak{E}\) by its category of scheme or manifold objects. Let \(\mathfrak{E}\) be an extension of \(\mathfrak{E}\), and suppose that pullbacks of open immersions in \(\mathfrak{E}\) exist in \(\tilde{\mathfrak{E}}\) - for instance, if the latter is finitely complete.

Then \(\tilde{\mathfrak{E}}\) admits a localisation universal among those satisfying the equivalent conditions of the lemma. This localisation is generated by inverting morphisms of the form

\[
X \times_Y U \rightarrow X
\]

(3)

where \(X \rightarrow Y\) is a morphism with \(Y : \mathfrak{E}\) and \(U \hookrightarrow Y\) is an open subobject such that \(|X| \rightarrow |U| \rightarrow |Y|\); such morphisms fix the underlying space. Indeed, such morphisms must become invertible in any extension of \(\mathfrak{E}\) under \(\tilde{\mathfrak{E}}\) that is compatible with \(|-|_\mathfrak{E}\).

**Proof.** We must show that inverting these morphisms is enough to make \(\tilde{\mathfrak{E}}\) compatible with \(|-|_\mathfrak{E}\). Write \(\mathcal{M}(X \mid \mathfrak{E})\) for the category whose objects are triples \((X \rightarrow Y \leftarrow U)\) such that \(|X| \rightarrow |Y|\) factors through \(|U|\). Now define

\[
\text{Loc}_\mathfrak{E} X := \lim_{U \subseteq Y : \mathcal{M}(X \mid \mathfrak{E})} X \times_Y U
\]

as a pro-object of \(\tilde{\mathfrak{E}}\). This limit is equivalent to a limit over a cofiltered set because:

i) \(\mathcal{M}(X \mid \mathfrak{E})\) admits binary products;

ii) any two maps \(Y_1 \rightrightarrows Y_2\) in \(\mathcal{M}(X \mid \mathfrak{E})\) induce equal maps \(X \times_{Y_1} U \rightrightarrows X \times_{Y_2} U\) in \(\tilde{\mathfrak{E}}\).
Suppose that $\text{Loc}_E X \to Y$ factors through $X$, and that $|X|$ factors through $|U|$, $U \hookrightarrow Y$. Then by construction, $\text{Loc}_E X \to Y$ factors uniquely through $U$ such that the square

$$
\begin{array}{c}
\text{Loc}_E X \\
\downarrow \\
U \\
\downarrow \\
Y
\end{array}
$$

is commutative. Since every morphism $\text{Loc}_E X \to Y$ factors through some $X$, uniquely up to further refinement, it follows that $\text{Loc}_E X$ is compatible with $|-|_E$. \hfill \Box

In particular, the full subcategory of $\text{Alg}_E$ whose objects are actual with respect to $|-|_E$, denoted $\text{Alg}_{E,|-}$, admits a reflector. The essential image of $\text{Alg}_{E,|-}^p$ in this category is denoted $\text{Alg}_{E,|-}^p$; note that the objects of this category are not finitely presented as $E$-algebras (unless they are representable). The diagram

$$
\begin{array}{c}
\text{Alg}_{E,|-}^p \\
\downarrow \\
\text{Alg}_{E,|-}^p
\end{array}
\quad \text{commutes, since by 9.11, all the morphisms (3) into a finitely presented object also have finitely presented domain.}

9.14 (Spivak). The localisation construction of 9.13 is implicit in the passage from $\text{Alg}_E$ to the category of $\text{Alg}_{E,|-}^p$-structured spaces described in 6.16 — note that $\text{Alg}_E$ is compactly generated so no auxiliary completion is needed in this case — and hence in Spivak’s definition 6.15.

Indeed, suppose that $|-|_E$ commutes with products. Then by 6.16, there is a limit-preserving functor

$$
\begin{array}{c}
\text{Alg}_{E,|-}^p \\
\downarrow \\
\text{Alg}_E \\
\downarrow \\
\text{Alg}_{E,|-}
\end{array}
$$

commutes, since by 9.11, all the morphisms (3) into a finitely presented object also have finitely presented domain.

9.15 (Lurie). Lurie’s notion of a geometric envelope of a pregeometry [DAGV, §3.4] is at least as fine as our construction. Indeed, Lurie replaces our conditions on functors of spatial geometries with the condition that they preserve pullbacks of open immersions; in other words, this functor factors through the category $\text{Alg}_{E,|-}^p$ of actual $E$-algebras.

Hence, the $\text{Alg}_{E,|-}^p$-schemes are precisely the $\text{Alg}_E$-structured spaces that are locally equivalent to a limit of $E$-schemes, as was claimed in the introduction.

Note the similarity with the criterion (2) for tangibility.

When $E$ is a spatial geometry in which pullbacks of open immersions are representable, it defines also a Lurie pregeometry as in 4.17. Moreover, our geometric functors 4.4 are morphisms of pregeometries in Lurie’s sense. Hence, Lurie’s geometric envelope admits a unique left exact functor to our $(\text{Alg}_{E,|-}^p)^{op}$.

When embeddings are closed, the converse is true:

**Proposition.** Suppose that every retract in $E$ has closed image. Then the opposite category to $\text{Alg}_{E,|-}^p$ is a Lurie geometric envelope of $E$. 

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Proof. The Lurie geometric envelope is certainly a localisation of \((\text{Alg}_{\mathbb{E}}^{\text{fp}})^{\text{op}}\). It will suffice to prove that it is a spatial geometry. We continue with our earlier assumption that all open subsets of representable objects are representable.

Let \(X\) be an object of \((\text{Alg}_{\mathbb{E}}^{\text{fp}})^{\text{op}}\). Then \(X\) may be written as the limit of a cosimplicial object of \(\mathbb{E}\); let us write the first two terms as \(s, t : Y \to E\). The underlying space of \(X\) is the equaliser

\[
|X| \xrightarrow{i} |Y| \rightrightarrows |E|.
\]

Let \(U \hookrightarrow Y\) be an open immersion such that \(|X| \subseteq |U|\). We will show that \(X \to U\) in the envelope, which is therefore also a spatial geometry.

Indeed, let \(t_*[U] \subseteq |E|\) equal the union of \([U] \subseteq |X|\) and the complement of \(t[Y]\) in \(|E|\). Then \(t_*[U]\) is open, and \(s^*t_*[U] = |Y|\), while \(t^* t_*[U] = |U|\). This is an instance of the proper base change formula for open subsets for the square

\[
|X| \xrightarrow{i} |Y| \\
i |Y| \xrightarrow{s} |E|.
\]

Thus \(X \to Y \xrightarrow{s} t_*U\); meanwhile, the square

\[
U \xrightarrow{t} Y \xrightarrow{t} E
\]

must remain Cartesian in the geometric envelope, and so \(X \to U\).

The assumption that \(|Y|\) is closed in \(|E|\) is probably not needed; the point is to find an element of \(\mathcal{U}(E)\) that connect different open subsets of \(Y\) in the coequaliser diagram

\[
\mathcal{U}(E) \rightrightarrows \mathcal{U}(Y) \to \mathcal{U}(X).
\]

For instance, the same argument would also run through if \(\mathcal{U}(E)\) were to surject onto \(\mathcal{U}(Y) \times \mathcal{U}(X)\) \(\mathcal{U}(Y)\) - that is, if the topological pushout \(|Y| \sqcup |X|\) \(|Y|\) embeds into \(|E|\).

Undoubtedly, for orbifolds or higher stacks a more sophisticated argument involving further terms of the resolution would be required.

10 Embeddings

In this section, we expand upon the notion of model imbedding introduced by Spivak. This theory is subsumed in that of \([\text{DAGIX}]\).

10.1. Let \(C\) be a (higher) category. I gather here a few facts about epimorphisms in \(C\):

- A morphism \(f : X \to Y\) is a regular epimorphism in \(C\) if it exhibits \(Y\) as the colimit of some simplicial object of \(C\) with \(X\) as its object of 0-simplices.
- A pushout of a regular epimorphism is always regular epic.
- It is effective epimorphic if it is regular and its Čech nerve \(Cf\) is representable, in which case \(Y\) is a colimit of \(Cf\). Clearly, if \(C\) admits fibre products these two concepts are equivalent.
- If \(C\) is a presheaf category, then regular epimorphisms are detected pointwise; i.e. \(f\) is regular epic if and only if for every representable object \(S, f : \text{Map}(S, X) \to \text{Map}(S, Y)\) is a regular epimorphism of spaces.
  In this case, a pullback of a regular epimorphism is regular epic.
- In particular, a morphism between representable objects is regular epic in the presheaf category if and only if it is split epic, i.e. admits a section.
The following fact is used, but not made explicit, in the proof of [Spi10, prop. 9.4].

10.2 Lemma. Let $\mathbf{E}$ be a category with finite products. A morphism $\varphi_X \to \varphi_Y$ in $\text{Alg}_E$ is regular epic if and only if it is regular epic in $\text{PSh}(\mathbf{E}^{\text{op}})$, that is, it induces regular epimorphisms $\varphi_X(Z) \to \varphi_Y(Z)$ for every $Z : \mathbf{E}$.

Proof. Part one follows because this is true in the presheaf category and colimits are preserved by localisation onto $\text{Alg}_E$. The second follows because sifted colimits, and hence regular epimorphisms, are preserved by the inclusion.

10.3 Definition. A map $X \to Y$ of derived $\mathbf{E}$-manifolds is called an embedding if it is a topological embedding, representable by principal derived manifolds and the dual map of $\mathbf{E}$-algebras is an effective epimorphism; i.e.

$$\check{C}[\mathbf{E}(Y,-) \to \mathbf{E}(X,-)] \to \mathbf{E}(X,-)$$

is a colimit locally on $|Y|$.

Since underlying spaces are Hausdorff, an embedding is automatically closed.

10.4 Aside. In the geometry of [Mac15, §3.1] I found it necessary to define non-principal embeddings, i.e. ones that are possibly not representable in the site. Since this is not the case for the primary examples of this paper, I only refer the reader there for further details.

10.5 Proposition (Properties of embeddings). Embeddings are stable for:

i) composition;

ii) base change (if $| - |$ is left exact);

iii) right cancellation (i.e. if $fg$ is an embedding then $g$ is an embedding).

iv) descent on the codomain;

Any injective immersion in the sense of 8.4 is an embedding.

If $Z \to X$ is an embedding, then $\text{Hom}(X,M) \to \text{Hom}(Z,M)$ is a regular epimorphism for any manifold $M$. In particular, it induces surjections on $\pi_0$.

Proof. Properties i-iii) are general properties of regular epimorphisms, while iv) is true because the definition is inherently local.\footnote{Actually, iii) holds with a weaker hypothesis than exactness for $| - |$: Lurie’s notion of unramfiedness, which holds in the algebraic category. See def. 1.3 and corollary 1.7 of [DAGIX].}

The last part is an application of lemma 10.2.

10.6 Example. In the category of $C^k$ manifolds, every split monic is an immersion. Hence, embeddings between manifolds are precisely the injective immersions.

11 Mapping spaces

The mapping spaces in the category of derived $\mathbf{E}$-manifolds may in principle be computed in terms of mapping spaces in $\mathbf{E}$.

11.1. By definition, an object $X : (\text{Alg}_E)^{op}$ is determined by the functor

$$\text{Map}(X,-) : \mathbf{E} \to \mathbf{S}$$

it corepresents. If these spaces are computed, then $\text{Map}(X,Y)$ may be computed for any $Y$ in terms of a presentation of $Y$ as a limit of objects in $\mathbf{E}$,

$$\text{Map}(X,Y) = \lim_i \text{Map}(X,Y_i).$$

On the whole, there is no obvious way to compute the mapping space out of a limit, and so a limit presentation for $X$ does not help us in such a naïve way.
However, the Yoneda embedding $\text{Alg}_E \to \text{PSh}(E^{op})$ preserves sifted colimits (9.6, [HTT, §5.5.8]), and colimits in the presheaf category are computed pointwise. Hence, if $X = \lim_i X_i$ is a limit over a cosifted diagram - for example, a cofiltered diagram or a cosimplicial object - then

$$\text{Map}(X, -) = \lim_i \text{Map}(X_i, -).$$

In particular, if $i \to X^i$ is a cosimplicial object of $E$, then the diagram $\text{Map}(X^i, -)$ defines a functor from $E$ valued in the category of simplicial sets. If $M : E$, then $\text{Map}(X, M)$ is the homotopy type of the simplicial set $\text{Map}(X^i, M)$.

11.2 (Localisation). Suppose now that $E$ is a spatial geometry, and let $X^\ast$ be a cosimplicial manifold object of $E$. Then $\lim X^i$ is a manifold object of $\text{Alg}_E^{op}$, with underlying space $|X| = \lim |X^i|$.

Recall that if $f : X \to Y$ is a continuous map, the pullback functor

$$f^{-1} : \text{Sh}(Y, \text{Alg}_E) \to \text{Sh}(X, \text{Alg}_E)$$

is defined for the category of $E$-algebra sheaves and preserves $n$-truncated objects. Denote by $a_n : |X| \to |X[^n]|$ the $n$th canonical map of the colimit. Then if $E$ is a 1-category, then each $a_n^{-1}\theta_X$ is a set-theoretic $E$-algebra sheaf, and $a_n^{-1}\theta_X$ is a simplicial set-theoretic $E$-algebra sheaf.

By the argument of 11.1, the space of maps $X \to M$ with fixed underlying space map $f : |X| \to |M|$ is modelled by the simplicial set valued sheaf $\text{Hom}(f^{-1}\theta_M, a_n^{-1}\theta_X)$ on $X$, and $\text{Map}(X, M)$ is the space of derived global sections of this simplicial sheaf. This latter can be calculated by a hypercohomology spectral sequence.

11.3 Example. In the case $C^k$ for $k \leq \infty$, the sheaves $a_n^{-1}\theta_X$ are soft and therefore have no cohomology. Hence $\text{Map}(X, -)$ is nothing but the set of global sections of $\pi_k(a_n^{-1}\theta_X)$. Indeed, it is known by results of Cartan and Grauert that the sheaf of $C^\infty$ functions on any real analytic manifold has vanishing higher cohomology. Since $a_n$ is a closed embedding into a real analytic manifold, $a_n^{-1}C^\infty(X[^n])$ also has vanishing higher cohomology.

11.4. Suppose we have a derived $E$-manifold with an expression as a finite limit $X = \lim_i X_i$ of $E$-manifolds. In order to calculate $\text{Map}(X, -)$ as a functor of $E$, we must convert the diagram $F : I \to E$ into a cosifted diagram.

A standard way to do this is via the nerve functor that attaches to $I$ the simplicial set

$$NI[^k] = \text{Map}(\Delta^k, I).$$

Interpreting the nerve as a category fibred in sets over the simplex category $\Delta$, this gives a correspondence

$$\begin{array}{ccc}
I & \xleftarrow{r} & \Delta \\
\pi & & \\
\end{array}$$

where the left side of the roof sends a $k$-simplex $\sigma : \Delta^k \to I$ to $\sigma(0) : I$. Then:

- $r$ exhibits $I$ as a deformation retract of $NI$, and hence is in particular coinitial; therefore $\lim F = \lim r \ast F$.
- The limit of an $NI$-indexed diagram may be computed in two steps: by first multiplying out along the discrete Cartesian fibration $NI \to \Delta$, then by taking the limit over the resultant $\Delta$-indexed diagram; because the extension $E \to \text{DMan}_E$ preserves products, the first step can be computed in $E$.

Hence, the cosifted diagram $\pi \ast r \ast F$ represents the same derived manifold as $F$; it can therefore be used to calculate the structure sheaf of $X$ by the prescription of 11.2.
11.5 (Hochschild complex). The nerve construction generally gives a rather unwieldy simplicial object. Sometimes a simpler construction is available.

Consider the standard quasi-smooth case, a pullback $X \times_Z Y$. For example, by \cite{Spi10}, every quasi-smooth derived $C^\infty$ manifold can be written this way. This diagram is indexed by the power set $\mathcal{P}(\Delta^1)$ of the 1-simplex. Then we can use the correspondence

$$
\Delta \mid \Delta^1 \xrightarrow{\mathcal{P}(\Delta^1)} \Delta
$$

to convert $\mathcal{P}(\Delta^1) \to E$ into a diagram indexed by $\Delta$.

Indeed, as in 11.4,
- taking the image $\Delta \mid \Delta^1 \to \Delta$ is a strong deformation retract, hence in particular coinitial;
- the forgetful functor $\Delta \mid \Delta^1 \to \Delta$ is a Cartesian fibration with discrete fibre $[-, \Delta^1]$.

The fibres of the second map are relatively simple: $[\Delta_n, \Delta_1]$ is just the set $[n + 1]$ of partitions of $[n]$ into (up to) two parts, including the two degenerate partitions with one part empty corresponding to 0 and $n + 1$. With respect to this identification, the $i$th face map $[n + 1] \to [n]$ identifies $i$ and $i + 1$, while the $j$th degeneracy map $[n] \to [n + 1]$ skips $j + 1$.

Hence, the simplicial object obtained by convolution along this roof is the Hochschild complex defined by the formulae

$$
CH(X \times_Z Y)_{[n]} = \prod_{\delta : \Delta_n \to \Delta_1} \delta^* F
$$

$$
d_i f(x, z_0, \ldots, z_n, y) = f(x, z_0, \ldots, z_i, z_{i+1}, \ldots, z_n, y)
$$

$$
s_j f(x, z_0, \ldots, z_n, y) = f(x, z_0, \ldots, z_j, \ldots, z_n, y)
$$

defining a simplicial $E$-algebra.

If $X \times Z Y$ is considered as a derived $E$-manifold, then this construction must be localised, yielding a simplicial $E$-algebra sheaf $\mathcal{E}_\mathcal{H}(X \times_Z Y)$ on $|X| \times_Z |Y|$.

11.6 Proposition. Let $X$ be a pullback in $\text{DMang}$, $M$ a manifold. Then $\text{Map}(X, M)$ is the space of derived global sections on $|X \times_Z Y|$ of the simplicial sheaf $\mathcal{E}_\mathcal{H}(X \times_Z Y, M)$.

11.7 Corollary. Let $W$ be a quasi-smooth derived $C^k$-manifold, $k : [0, \ldots, \infty, \omega)$. Then $\text{Map}(W, M)$ is the homotopy type of the simplicial set of germs along $|X \times_Z Y|$ of elements of $CH_* (X \times Z Y, M)$.

Proof. By acyclicity of sheaves of $C^k$ functions.

11.8 Example. We reprise the example of the introductions, and try to compute the pullback

$$
\begin{array}{ccc}
X & \to & \mathbb{R}^0 \\
\downarrow & & \downarrow \\
\mathbb{R}^0 & \to & \mathbb{R}^1
\end{array}
$$

in $\text{DMang}_{C^k}$ by applying corollary 11.7. Clearly, $\pi_0 \text{Map}(X, M) = M$.

The complex at $\pi_1$ (based at a map $p : \mathbb{R}^0 \to M$) looks like

$$
d_0 f(x) = f(x, 0), \quad d_1 f(x) = f(x, x), \quad d_2 f(x) = f(0, x).
$$

A triple of (germs of) functions $h_0, h_1, h_2$ on $\mathbb{R}$ can be interpolated only if their derivatives at 0 satisfy

$$
dh_1 = dh_0 + dh_2.
$$

Conversely, if the derivatives satisfy this equation, there are enough degrees of freedom to interpolate the $f_i$ to all orders - since $\dim_{\mathbb{R}} \text{Sym}^k \mathbb{R}^2 \geq 3$ for $k \geq 2$.

Therefore, the image of $d : CH_2(X, M) \to CH_1(X, M)$ consists of those functions with zero derivative. Hence the homology is just $T_f / M$. Note that it is an $\mathbb{R}$-vector space, even though $M$ was arbitrary.

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11.9 Aside. By analogy with algebraic geometry, it seems likely that the space of $C^\infty$ functions on the self-intersection $Y \times_Z Y$ of an more general embedding $Y \hookrightarrow Z$ will be computed, as a commutative dga, by the Koszul algebra $\Lambda^* N_{Y/Z}$.

11.10 Aside. D. Joyce has shown me a draft of a new book on Kuranishi structures, defining a general context in which one can ‘do differential geometry’. The axioms are an extension of the ones used in this paper.

This work defines the tangent sheaf of a generalised manifold $M$ via an interpolation problem on $M \times \mathbb{R}^2$ for retractions $M \times \mathbb{R} \rightarrow M$ of the zero section, similar to the calculation of 11.8. Indeed, it seems that in our language, Dominic’s tangent sheaf is nothing but $\pi_1 \text{Map}_{M/}(M \times \Omega_{\mathbb{R}}, \mathcal{M})$. 

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A Presentations of categories

The major definitions of this paper — of sheaves, schemes, and Lawvere algebras — all concern categories satisfying a universal property with respect to functors from a provided input category. The constructions of categories defined in this way fit into a general pattern.

To a certain extent, the constructions of this section are discussed in [HTT, §5.3.6]. However, the methods loc. cit. are not entirely satisfactory for our purposes for two reasons:

- The conditional completion constructed in proposition 2 of loc. cit. is defined as a localisation of a presheaf category. In this paper, it will be more useful to have an explicit description of the subcategory of local objects.
- The constraints used in loc. cit. are to be closed under colimits of a certain shape, whereas here a constraint is allowed to be closure under a specific set of diagrams in the starting category. This generality is necessary if, for example, we wish to obtain universal properties for categories of ‘quasi-smooth’ derived manifolds.

Of course, solutions to these problems can be extracted from Lurie’s methods, but it is almost as straightforward to redo it all from scratch.

A.1 (Relations). Let C be some category, and let R be a set of relations on C — that is, pairs consisting of a category I and a diagram u : I^R → C indexed by the category I^R obtained from I by freely adjoining a terminal vertex e.

The relation set R is said to be subcanonical if each u : I^R → C in R is a colimit diagram; that is, if u(e) is a colimit in C of u|I. In this case, (C, R) is what is known to category theorists as a colimit sketch.

A.2 Definition. An R-continuous cocompletion of C is a presentable category D equipped with a functor C → D that takes diagrams in R to colimits in D. That is, if u : I^R → C is in R, then Fu(e) is a colimit of Fu(I) in D.

The set of R-continuous cocompletions forms an (∞,2)-category, which in the notation of [HTT, §5.5.3] is the full subcategory of

Pr^L × Cat_C/

whose objects have structure map R-continuous.

A.3 (Continuous presheaves). Denote by PR(C) ⊆ PSh(C) the full subcategory of C spanned by the presheaves that take diagrams I^R → C in R to limit diagrams of types. That is, PR(C) is the category of R-continuous cocompletions of C whose underlying category is identified with the opposite category S^op to types.

If X : J^R → C is an element of R with vertex X in C, then for any F : PR(C),

\[
\text{Map}_{PR(C)}(X, F) = \lim_{j \in J} F(X_j)
\]

defining property of PR

\[
= \lim_{j \in J} \text{Map}_{PR(C)}(X_j, F)
\]

Yoneda

that is, the colimit X in C remains a colimit in PR(C).

Finally, if F : C → D is any R-continuous cocompletion, there is an induced adjunction

\[
\text{Ran}_F : PR(C) \simeq D : F^\dagger
\]

where the right adjoint is Yoneda pullback F^\dagger : Y → Map_D(F -, Y).

A.4 (Presentability). This condition of R-continuity is stable under limits, hence assuming that size issues can be managed, PR(C) is a reflective subcategory; that is, it is presentable.

There are two ways to manage the size issues:

- Assuming a large cardinal axiom — the natural choice being the weak Vopěnka principle.
- Manually construct a reflection functor by a construction of size bounded by some fixed cardinal — for instance, the cardinality of the set R (if infinite) suffices.
In both applications §§9, B of this construction to the present paper, a specific construction is available and detailed in the corresponding sections of [HTT]. In this section we will not concern ourselves with such matters.

**A.5 Proposition.** For each \( R \), \( C \to P_R(C) \) is universal — i.e. initial — among \( R \)-continuous cocompletions of \( C \). The structural functor is fully faithful if and only if \( R \) is subcanonical.

*Proof.* Let \( D : I \to P_R \) be a diagram of cocomplete categories and colimit-preserving functors, \( F : C \to D \) a cone that transforms elements of \( R \) into colimits. A colimit-preserving functor \( F : P_R C \to D \) must satisfy, in particular,

\[
F(X) = \text{colim}_{U \in C_X} FU;
\]

that is, \( F \) is a left Kan extension of \( F \) in the sense of [HTT, Def. 4.3.3.2]. By proposition 2.15 of *loc. cit.* this property determines \( F \) uniquely. \( \square \)

**A.6 Aside.** For the trivial case \( R = \emptyset \), this result was proved in [HTT, §5.1.5] (specifically theorem 6). It is also possible to deduce proposition A.5 from this together with the existence of the reflector A.4.

**A.7 (Transitivity).** Now let \( K \) be a set of constraints on \( C \) compatible with \( R \), that is, a set of diagrams in \( C \) such that if \( u : I^P \to C \) is in \( R \), then \( uI \) is in \( K \). Denote by \( P^K R \subseteq P_R \) the full subcategory spanned by \( C \) and the objects that can be exhibited as colimits of elements of \( K \).

By construction, if \( u : I \to C \) is in \( K \), then the composite of \( u \) with the Yoneda functor admits a colimit extension \( u^P : I^P \to P^K R C \). In this way, \( K \) may also be interpreted as a set of relations on \( P_R C \).

**Lemma.** There is a unique equivalence \( P_R C \cong P_K P^K R C \) under \( C \).\(^{12}\)

*Proof.*

- \( P_K P^K R \) is cocomplete because \( P_K \) is, and the inclusion \( C \to P_R C \to P_K P^K R \) preserves \( R \)-colimits, hence by the universal property of \( P_R \) there is a unique colimit-preserving functor \( P_R \to P_K P^K R \) under \( C \). This functor moreover commutes with the inclusions of \( P^K R \).

- \( P_R \) is cocomplete and the inclusion \( P^K R \subseteq P_R \) preserves \( R \)-limits, so by the universal property of \( P_K \) there is a unique colimit-preserving functor \( P_K P^K R \to P_R \) under \( P^K R \).

The compositions of these functors in each direction are colimit-preserving endofunctors of \( P_R C \), resp. \( P_K P^K R C \), under \( P^K R C \); hence must be the identity by the respective universal properties. \( \square \)

**A.8 Definition.** A \( K \)-conditional \( R \)-continuous cocompletion of \( C \) is an \( R \)-continuous functor. A \( K \)-continuous functor of \( K \)-conditional cocompletions is a functor preserving elements of \( K \) (understood as colimit diagrams in the target).

Note that even if \( K \) is the set of all diagrams in \( C \), the condition for a functor of cocompletions of \( C \) to be \( K \)-continuous is weaker than preserving colimits, unless \( C \) is dense in \( D \).

**A.9 Proposition.** The category \( C \to P^K R C \) is initial among \( K \)-conditional, \( R \)-continuous cocompletions of \( C \).

*Proof.* Let \( C \to D \) be a cone of \( K \)-conditional, \( R \)-continuous cocompletions. Then \( P_R D \) is cocomplete, hence there is a unique colimit-preserving functor \( P_R C \to P_R D \). By transitivity, \( P^K R C \subseteq P_R C \) and this subcategory factors through \( D \). \( \square \)

**A.10.** Suppose \( K \) is the set of all diagrams in \( C \) whose index category belongs to some specified set \( \mathcal{K} \) of categories. Suppose further that \( \mathcal{K} \) is closed under composition in the sense that \( P^K R \) is closed in \( P_R \) under all colimits indexed by members of \( \mathcal{K} \). (An example of this would be \( \mathcal{K} \) the set of categories cofinal with a finite category.) This is closer to the situation considered in [HTT, §5.3.6].

\(^{12}\) See also proposition 11 of [HTT, §5.3.6].
We may replace \( \mathcal{K} \) with its saturation (a large set) \( \overline{\mathcal{K}} \), consisting of all categories cofinal with an element of \( \mathcal{K} \). In particular, if \( X : P^K \mathcal{C} \), then \( C_X \) is a member of \( \overline{\mathcal{K}} \). Evidently, \( P^K \mathcal{C} = P^K \mathcal{F} \), whence \( \overline{\mathcal{K}} \) is also closed under composition. Moreover, \( \mathcal{C} \to P^K \mathcal{C} \) is \( \overline{\mathcal{K}} \)-continuous.

**Lemma.** A \( \overline{\mathcal{K}} \)-continuous functor \( P^K \mathcal{C} \to \mathcal{D} \) automatically commutes with all \( \overline{\mathcal{K}} \)-indexed colimits.

If \( I : \mathcal{K} \) and \( u : I \to P^K \mathcal{C} \) is a colimit diagram, then the index category \( \text{colim}_{i \in I} C_{u(i)} \) belongs to \( \overline{\mathcal{K}} \). Hence \( \text{colim} \ u \) is also a colimit of the source functor \( \mathcal{C} \vdash u : \mathcal{C} \to P^K \mathcal{C} \), hence remains so in \( \mathcal{D} \).

In this case, the universal property of \( P^K \mathcal{C} \) may be restated as follows: \( P^K \mathcal{C} \) is universal among \( \overline{\mathcal{K}} \)-conditional, \( R \)-continuous cocompletions of \( \mathcal{C} \) and functors that preserve colimits indexed by elements of \( \mathcal{K} \).

**A.11 (Truncations).** In the preceding paragraphs, the objects of the presheaf category \( \text{PSh}_{\mathcal{C}} \) can be understood as taking values in spaces, or in \( n \)-types for some finite \( n \). In the latter case, we can declare \( n \) explicitly by writing \( (P^K) \leq^n \) in place of \( P^K \); these satisfy universal properties with respect to the category of \( n \)-truncated cocompletions of \( \mathcal{C} \).

If \( n \leq m \) (where \( m \) may equal \( \infty \)), then the universal property provides a unique colimit-preserving functor

\[
(P^K) \leq^m \mathcal{C} \to (P^K) \leq^n \mathcal{C}
\]

under \( \mathcal{C} \). This functor may also be thought of as exhibiting \( (P^K) \leq^n \) as universal among \( n \)-truncated \( \overline{\mathcal{K}} \)-continuous extensions of \( (P^K) \leq^m \).

In the constraint-free case where \( K \) is the set of all diagrams in \( \mathcal{C} \), this functor admits a right adjoint, which includes \( (P_R) \leq^n \) as the reflective subcategory of \( n \)-truncated \( R \)-continuous presheaves [HTT, Def. 5.5.6.1]. Indeed, the condition for an \( n \)-truncated presheaf to be \( R \)-continuous does not depend on whether it is interpreted in \( n \) or higher category theory, because limits preserve truncatedness (cf. proposition 5 of loc. cit.).

**B Sheaves**

In this paper we use a non-traditional approach to descent theory in which the essential notion is that of an *atlas*, rather than *covers* or *hypercovers*. The point of this section is to prove proposition B.4, which equates several notions of sheaf.

The downside to this is that when we require certain properties of topoi, we are forced to either reprove them in the new language or obtain a comparison with a more traditional approach, as exemplified in, say, [HAGI] or [HTT, §6]. The technical part of a comparison result was contained in appendix C.

**B.1 (Continuity).** Let \( \mathcal{C} \) be a spatial geometry, \( F : \mathcal{C} \to \mathcal{T} \) a functor. The following conditions on \( F \) are equivalent:

- \( F \) takes atlases to colimits;
- \( F \) takes hypercovers to colimits.

The proof of this statement is the content of appendix C: proposition C.4 shows that i)\( \Rightarrow \)ii); while by theorem C.6, every atlas is cofinal with a hypercover, and so the converse holds as well.

A *continuous cocompletion* of \( \mathcal{C} \) is a functor \( F \) as above with \( \mathcal{T} \) presentable. A morphism of continuous cocompletions is a colimit-preserving functor of categories under \( \mathcal{C} \). Continuous cocompletions of \( \mathcal{C} \) form a category, which in the notation of [HTT] is calculated as a fibre product \( \text{PRCat}^L \times_{\text{Cat}} \text{Cat}_{\mathcal{C}} \).

**B.2 Definition (Sheaves).** A *category of sheaves* on \( (\mathcal{C},|\_\_|) \) is a category under \( \mathcal{C} \) universal among continuous cocompletions of \( (\mathcal{C},|\_\_|) \). That is, it is an initial object in the category of continuous cocompletions of \( \mathcal{C} \).

**B.3 (Local isomorphisms).** Let \( \psi : X \to Y \) be a morphism of presheaves on a geometry \( \mathcal{C} \). We write

\[
\mathcal{U}(S) \downarrow X := \mathcal{U}(S) \times_{\mathcal{C} \mid Y} (\mathcal{C} \mid X)
\]
for the set of open subobjects of $S$ over $X \to Y$. Its objects are commuting squares

$$
\begin{array}{ccc}
U & \rightarrow & S \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
$$

in $\text{PShC}$ with $U \to S$ an open immersion. Via the natural map $\text{pr}_S : \mathcal{V}(S) \mid_Y X \to \mathcal{V}(S)$, this category indexes a slice diagram of open subobjects of $S$.

The slice diagram behaves predictably with respect to pullback and composition:

- **Pullback.** If $W \to Y$ and $S \to W$, then the data of an open subobject of $S$ over $X \times_Y W \to W$ is exactly that of an open subobject over $X \to Y$; that is, $\mathcal{V}(S) \mid_W (X \times_Y W) = \mathcal{V}(S) \mid_Y X$ compatibly with the projection to $\mathcal{V}(S)$.

- **Composition.** If $Y \to Z$ is another map, then for any $S$ over $Z$ composition with $\psi$ induces a map $\text{pr}_Y : \mathcal{V}(S) \mid_Z X \to \mathcal{V}(S) \mid_Z Y$ commuting with $\text{pr}_S$. Moreover, the slice over an element $V : \mathcal{V}(S) \mid_Z Y$ is naturally identified with $\mathcal{V}(V) \mid_Y X$ so that the corresponding square

$$
\begin{array}{ccc}
\mathcal{V}(V) \mid_Y X & \rightarrow & \mathcal{V}(V) \\
\downarrow & & \downarrow \\
\mathcal{V}(S) \mid_Z X & \rightarrow & \mathcal{V}(S),
\end{array}
$$

where of course $\mathcal{V}(V) \subseteq \mathcal{V}(S)$ is the slice inclusion, commutes.

**Definition.** A morphism $X \to Y$ of presheaves is called a *local isomorphism* if for all $S : C \mid Y$ the diagram $\text{pr}_S \mid \mathcal{V}(S) \mid_Y X$ is a site for $S$.

**Lemma.** Local isomorphisms are stable for pullback and composition.

**Proof.** *Pullback.* Let $\psi : X \to Y$ be a local isomorphism, $W \to Y$ a map. By the compatibility of slice diagrams with pullback, for each $S$ over $W$ the slice diagram over $X \times_Y W \to W$ is isomorphic to $\mathcal{V}(S) \mid_Y X$, which is a site by hypothesis. Therefore the pullback of $\psi$ is a local isomorphism.

*Composition.* If $X \to Y \to Z$ are local isomorphisms, then for each $S$ over $Z$ the transformations $\text{pr}_Y$ and $\text{pr}_V \mid V : \mathcal{V}(S) \mid_Z Y$ appearing in the composition of the slice diagrams are sites. Hence by direct induction 3.16 of sites, $\mathcal{V}(S) \mid_Z Y$ is a site, that is, $X \to Z$ is a local isomorphism.

The set of local isomorphisms is generated by atlases in the following sense:

**B.4 Proposition.** The following are equivalent for a presheaf $F : \text{PShC}$:

i) $F$ is local for the class of local isomorphisms;

ii) $F$ is local for the class of effective atlases;

iii) $F$ is local for the class of hypercovers.

**Proof.** As observed above B.1, the equivalence ii)⇔iii) follows from the results of appendix C: by proposition C.4, every hypercover is an atlas, and by theorem C.6, every atlas is cofinal with a hypercover. Hence, it remains to show that i)⇔ii) B.5, B.6.

**B.5 (Atlases generate local isomorphisms).** Suppose that $X$ is local for effective atlases, and let $\psi : F \to G$ be a local isomorphism, $f : F \to X$ a morphism. We must show that there is a unique extension

$$
\begin{array}{ccc}
F & \rightarrow & G \\
\downarrow & \nearrow & \downarrow \\
X & \rightarrow & X
\end{array}
$$
of $f$ along $\psi$.

For each representable $S \to G$ there is a unique map $S \to X$ such that for all $S'$ over $S \times_G F$, the diagram

$$
\begin{array}{ccc}
S' & \longrightarrow & S \\
\downarrow & & \downarrow \\
F & \longrightarrow & G \\
\downarrow & & \downarrow \\
\psi & \longrightarrow & X
\end{array}
$$

is commutative. Indeed, applying this criterion to the case $S' \to S$ an open immersion, we see that in particular

$$
\colim_{(S) \times G \pr} S \longrightarrow S
$$

must commute. Because $\psi$ is a local isomorphism, the horizontal map is an effective at last, therefore by the locality property of $X$ there is indeed a unique extension with this property.

The unicity now implies that the induced constructions $G(S) \to X(S)$ fit together into a natural transformation, unique with respect to the property of extending $f$ along $\psi$.

B.6 (Atlases are local isomorphisms). Let $U \mid I$ be an atlas for $X$. Then evaluating the colimit of $U \mid I$ in $\text{PSh} C$ defines a presheaf, $U \mid I$, and a morphism $U \mid I \to X$. This morphism is a local isomorphism.

Denote by $\tilde{U}$ the colimit of $U \mid I$ in the category of presheaves. Then the statement is equivalent to the claim that $\tilde{U} \to X$ is a local isomorphism in the sense of B.3.

For each $i : I$, the data of an element

$$
\begin{array}{ccc}
V & \longrightarrow & S \\
\downarrow & & \downarrow \\
U_i & \longrightarrow & X
\end{array}
$$

of $\mathcal{U}(S) \times_X U_i$ is simply that of an open immersion $V \hookrightarrow S$ such that $|V| \subseteq \xi^{-1}|U_i|$. In other words, $\mathcal{U}(S) \times_X U_i \subseteq \mathcal{U}(S)$ is the filter generated by $\xi^{-1}|U_i|$.

Since maps from objects of $C$ into presheaves commute with colimits in the target,

$$
\colim_i \mathcal{U}(S) \times_X U_i \subseteq \mathcal{U}(S) \times_X \tilde{U},
$$

and the projection $\mathcal{U}(S) \times_X \tilde{U} \to \mathcal{U}(S)$ is precisely the atlas obtained by subordinating $\mathcal{U}(S)$ to the pullback atlas $\xi^{-1}|U|$ in the sense of 3.17. By loc. cit. this is a site, that is, $\alpha$ is a local isomorphism.

This completes the proof that i)$\Leftrightarrow$ii) in proposition B.4. \hfill \Box

B.7 Aside (Size). Proposition B.4 implies that the size of the sheafification of a presheaf is ‘controlled’ by the cardinality of the set of atlases, which in turn may be derived from the cardinality and compactness properties of the underlying spaces of objects of $C$. I invite the reader to consult the construction appearing in the proof of proposition 7 of [HTT, §6.2.2] to see how the cardinality of the set of atlases comes into play.

B.8 Corollary. The category $\text{ShC}$ is equivalent to the hypercompletion of Lurie’s category of sheaves on $C$.\[13]\[14]
Proof (sketch). By definition, the category of hypercomplete sheaves is the full subcategory of presheaves on $\mathcal{C}$ that become $\infty$-connected in the non-hypercomplete sheaf category. By the argument of [HAGI, Lemma 3.4.3] (in which $\infty$-connected morphisms are referred to as $\pi_*$-local equivalences) this set of maps is generated by a set $H$ of hypercovers between objects of $\mathcal{C}$. See also remark 6 of loc. cit.

B.9 Aside. Suitably interpreted, this also means that our category of sheaves may be modelled by the categories of simplicial prestacks considered in [HAGI]. Since it is the policy of this paper not to discuss models, I will defer the details to a sequel — but note that the case of $\mathcal{C}$ a 1-category appears already as proposition 14 of [HTT, §6.5.2].

B.10 Corollary. Suppose that for each $X : \mathcal{C}$, $|X|$ has finite Lebesgue covering dimension. Then $\text{Sh}(\mathcal{C})$ is equivalent to Lurie’s category of sheaves on $\mathcal{C}$.

Proof (sketch). By [HTT, Cor. 7.2.3.7], the condition is equivalent to the condition that for each $X$, $\text{Sh}(|X|)$ has finite homotopy dimension in the sense of [HTT, §7.2.1]. We will argue that in this case, $\text{Sh}(\mathcal{C})/X$ also has finite homotopy dimension (this being the sketchy part).

Indeed, in this case the right adjoint $a_* \to \text{inclusion} a^* : \text{Sh}(X) \hookrightarrow \text{Sh}(\mathcal{C})/X$ admits a further right adjoint, and hence preserves connectedness of objects by part (4) of [HTT, Prop. 6.5.1.16]. Roughly speaking, this is because the covering condition for the terminal object $X$ is generated by maps coming from $\text{Sh}(X)$. Thus if $Y : \text{Sh}(\mathcal{C})/X$ is fin-connected, then so is $a^* a_* F$, and a section of the latter yields a section of the former by postcomposition with the counit $a^* a_* \to 1$.

It follows that $\text{Sh}(\mathcal{C})$ locally has finite homotopy dimension (def. 6 of loc. cit.), hence it is hypercomplete by corollary 12. □

C Exchanging atlases for hypercovers

Hypercovers are defined as simplicial objects. Hence, in order to translate between the language of atlases used in this paper and the more classical language of hypercovers, we must in this section temporarily suspend our abnegation of simplicial techniques.

C.1 (Simplices). We denote by $\Delta$ the usual geometric simplex category of inhabited totally ordered sets, $\Delta^+$ the subcategory consisting of injective maps.

We denote by $i : \text{sSet} \to \text{Dia}$ the 1-categorical Grothendieck construction functor $F \mapsto \Delta^+ F$. Its right adjoint

$$i^* : I \mapsto \text{Hom}(\Delta^+ | - |, I)$$

to $i_\Delta$ is the tautological test functor attached to the Grothendieck test category. Using this adjunction, we can make sense of the notion of a morphism from a simplicial set to a 1-category; the notation

$$\text{Hom}_\text{sSet}(F, i^* J) \equiv \text{Hom}(F, J) \equiv \text{Fun}(i_* F, J)$$

is unambiguous.

A similar story is true for $\Delta_+$. Note that as a category, $i_\Delta \Delta_+^k$ is the power set of $[k + 1]$ less the empty set.

(Note that we are not considering $\Delta$, as would be more usual, as the full subcategory of $\text{Cat}$ spanned by the inhabited, finite, totally ordered sets. The associated adjoint is the functor of nerve.)

The counit of the adjunction is the map

$$\epsilon_\Delta : i_\Delta \epsilon_* \to I$$

that evaluates a map $\Delta/\Delta^k \to I$ at the terminal object. By construction, $i_\Delta \epsilon_* I$ is final among categories equipped with a left Grothendieck fibration to $\Delta^0$ and a map to $I$.

C.2 (Index set of a hypercover). According to [SGA4, Exp. V, §7.3], a hypercover in a site $\mathcal{C}$ is a certain simplicial object of the category $\text{SRC}$ of ‘semi-representable’ presheaves on $\mathcal{C}$. Each object of $\text{SRC}$ can be uniquely represented as a coproduct $\biguplus_{i \in \pi_0 X} X_i$ of objects of $\mathcal{C}$, the connected components of $X$. The index set $\pi_0 X$ is covariantly functorial in $X$. 44
Let $U : \Delta^{op} \to \mathbf{SRC}$ be any simplicial semi-representable presheaf. By naturality, its set of connected components is a simplicial set $\pi_0U : \Delta^{op} \to \mathbf{Set}$ with $k$-simplices $\pi_0(U_k)$. The data of $U$ can be equivalently - and uniquely - presented as a diagram in $\mathbf{C}$ indexed by the category $I \to \Delta^{op}$ left fibred over the opposite to the simplex category attached by the Grothendieck construction to $\pi_0U$. In this section, we will consider hypercovers in terms of these data.

C.3 (Joins of simplices). Let $I \to \Delta^{op}$ be a left fibration. An element $i : I$ sitting over $\Delta^k$ is called a $k$-simplex of $I$.

The standard $i$-simplex and $j$-simplex are considered to sit inside their join $\{0, \ldots, i, 0, \ldots, j\} \simeq \Delta^{i+j+1}$ in the manner implied by the notation:

$$\Delta^i \hookrightarrow \Delta^i \star \Delta^j \hookrightarrow \Delta^j.$$

Let $\sigma, \tau : I$ be an $i$-, resp. $j$-simplex. We write $I_{\sigma \star \tau} \subseteq I_{i+j+1}$ for the set of $i+j+1$-simplices $v$ for which $v|\Delta^i = \sigma$ and $v|\Delta^j = \tau$.

C.4 Proposition (Hypercovers are atlases). Let $U : I \to \mathcal{U}(X)$ be a hypercover in the format of C.2. Then $U$ is an atlas of $X$.

Proof. That the $U_i$ cover $X$ is explicitly part of the definition of a hypercover, so the meat of the argument lies in the case of binary intersections. Let $\sigma : I_i, \tau : I_j$. We will show by induction that $U|I_{\sigma \star \tau}$ covers $U_\sigma \cap U_\tau$.

The hypercover axiom says that the $i+j+1$-simplices of $U_{\sigma \star \tau}$ cover the intersection

$$\bigcup_{k=0}^{i+j+1} U_{\sigma \star \tau \setminus k}$$

of the open sets $U_{\sigma \star \tau \setminus k} \subseteq X$ indexed by the $(i+j+1)$ facets of $\sigma \star \tau$.

By the induction hypothesis, $U|I_{(\sigma \setminus k) \star \tau}$ covers $U_{\sigma \setminus k} \cap U_\tau$, and similarly for $k : \Delta^j$. (Unless $\sigma = [k]$, in which case the result is self-evident rather than true by induction.) In particular, it contains $U_\sigma \cap U_\tau$. Hence the above intersection equals

$$\bigcap_{k=0}^{i} U_{\sigma \setminus k} \cap U_\tau \bigcap_{j=0}^{j} U_\sigma \cap U_{\tau \setminus k} = U_\sigma \cap U_\tau;$$

indeed, the rightward inclusion holds because both $U_\sigma$ and $U_\tau$ appear in the intersection, and the opposite inclusion because $U_\sigma \subseteq U_{\sigma \setminus k}$ for any $k$. \[ \square \]

Now we will go the other way, and outline a procedure to convert any atlas into a hypercover.

C.5 (Atlas to hypercover). Let $U_I : I \to \mathcal{U}(X)$ be any diagram. The $\Delta$-refinement of $U_I$ is $\epsilon^*_\Delta U_I$. This refinement is ‘equivalent’ to the $U_I$ in the following sense:

Lemma. The counit $\epsilon^*_\Delta$ is coinitial (as an $\infty$-functor).

Proof. Let $i : I$. The slice category $(i_! \Delta^i) \downarrow i$ may be identified with the refinement $i_! \Delta^i(I \downarrow i)$ of the slice. But this category is contractible because $I \downarrow i$ is and because $\Delta$ is a test category. \[ \square \]

Applying opposites, $(i_* \Delta^i(I^{op}))^{op} \to I$ is cofinal; that is, the natural morphism of diagrams $\epsilon^*_\Delta U_I \to U_I$ identifies their colimits in any category.

C.6 Theorem. Let $U_I : I \to \mathcal{U}(X)$ be a diagram. The following are equivalent:

i) $U_I$ is an atlas.

ii) $i_* U_I$ is a hypercover.

Before passing to the proof, let us introduce a little more simplicial notation.
C.7 (Semi-simplices). Maps in $\Delta$ can be naturally factored into a surjection followed by an injection. Hence for each $k$ there is a natural functor
\[
\text{Im} : \Delta^k \to \Delta_+^k
\]
of image factorisation, left inverse to the inclusion, and a right deformation retraction $\text{id}_{\Delta^k} \to \text{Im}$ that is surjective on each term. It follows in particular that $\Delta_+^k$ is cofinal in $\Delta^k$ (even in the sense of higher category theory, though we won’t use this).

C.8 (Boundaries). For each $k$, let $\partial \Delta^k$ resp. $\partial \Delta_+^k$ be the standard simplicial, resp. semi-simplicial, set modelling the boundary of $\Delta^k$; that is, the subfunctor of $\Delta^k$ obtained by deleting (all degeneracies of) the unique non-degenerate $k$-simplex.

- As a category, $\partial \Delta^k$ is the partially ordered set $\mathcal{P}(k+1) \setminus \{\varnothing, [k+1]\}$ of inhabited proper subsets of $[k+1]$.
- In particular, $\Delta_+^k \cong (\partial \Delta^k)^\circ$ is the categorical right cone over $\partial \Delta_+^k$.
- Similarly, $\partial \Delta^k = \Delta^k \times_{\Delta^1} \partial \Delta^1$ is the category of simplices equipped with a non-surjective map to $\Delta^k$. It is a sieve in $\Delta^k$.
- Finally, the image factorisation C.7 for $\Delta^k$ restricts to a right deformation retraction $\partial \Delta^k \rightleftarrows \partial \Delta_+^k$.

By general principles,
\[
\text{colim}_{\Delta^0, \partial \Delta^0} \Delta^k \rightleftarrows \partial \Delta^n
\]
in the category of (higher) categories, whence
\[
J(\partial \Delta^{n+1}) \cong \lim_{\Delta^0, \partial \Delta^{n+1}} J(\Delta^k)
\]
for any ($\infty$-)category $F$; the same with subscript $+$ throughout. Moreover, these identifications are compatible with the retractions
\[
\xymatrix{J(\partial \Delta^{n+1}) \ar[r] & \lim_{\Delta^0, \partial \Delta^{n+1}} J(\Delta^k) \\
J(\partial \Delta^{n+1}) \ar[r] & \lim_{\Delta^0, \partial \Delta^{n+1}} J(\Delta^k)
}
\]
induced by $\Delta^k \rightleftarrows \Delta_+^k$ and $\partial \Delta^k \rightleftarrows \partial \Delta_+^k$.

By cofinality of $\partial \Delta_+^n+1$ in $\partial \Delta^{n+1}$, the upper-right term may also be written $\lim_{\Delta^0, \partial \Delta^{n+1}} J(\Delta^k)$.

C.9 (Fillings). Let $J$ be a category, $s : J(\partial \Delta^k)$. The fibre over $s$ of
\[
J(\Delta^k) \to J(\partial \Delta^k)
\]
is the set $J(\Delta^k)_s$ of fillings of $s$. Similar terminology is in force with subscripts $+$ throughout.

Image factorisation C.7 is compatible with fillings in the sense that the two squares
\[
\xymatrix{J(\Delta^k) \ar[r] & J(\partial \Delta^k) \\
J(\Delta_+^k) \ar[r] & J(\partial \Delta_+^k)
}
\]
commute. In fact:

**Lemma.** For any $s : J(\partial \Delta^k)$, the set of fillings of $s$ surjects onto the set of fillings of $s_+$.

**Proof.** Let $\bar{s}_+ : \Delta^k_+ \to J$ be a filling of $s_+$. Set $\bar{s} = s(\sigma)$ if $\sigma : \partial \Delta^k$ and $\bar{s} = \bar{s}_+(\Delta_k)$ otherwise. For $\sigma' \not\sim \sigma$ with $\sigma' \not\in \partial \Delta^k$ we put $\bar{s}(f) = \bar{s}_+(\text{Im}(f))$; in particular, if $\sigma \to \Delta^k$ then $\bar{s}(f)$ is the identity. Naturality of taking the image implies that this rule is compatible with composition.  

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Proof of C.6. hypercover condition states that given \( s \) in the index set

\[
(\cosk_n t^*_\Delta I)_{n+1} := \lim_{\Delta^s \in \Delta^{n+1}} I^{op}(\Delta^k),
\]

the restriction of \( U \) to the set \( I^{op}(\Delta^{n+1})_s \) of fillings of \( s \) covers

\[
U_s := \lim_{\sigma: \Delta^s \in \Delta^{n+1}} U_{s(\sigma)}
\]
in the sense that \( U_s = \bigcup_{\Delta^s \in \Delta^{n+1}} U_{s(\sigma)} \). Here, recall that \( U \) is defined on the simplex \( s(\sigma) : I(\Delta^k) \) by evaluating at the final object of \( \Delta^k \).

Reduction to semisimplicial setting. By mutual cofinality C.7, the limits in the preceding paragraph may equivalently be indexed over \( \partial \Delta^{n+1}_s \) or \( \partial \Delta^{n+1} \). In particular, by C.8,

\[
(\cosk_n I^{op})_{n+1} = \lim_{\Delta^k \in \partial \Delta^{n+1}} I^{op}(\Delta^k) = I^{op}(\partial \Delta^{n+1}).
\]

In these terms, the hypercover axiom asks that the restriction of \( U \) to \( I^{op}(\partial \Delta^{n+1}) \) covers

\[
U_s = \lim_{\sigma: \partial \Delta^{n+1}} U_{s(\sigma)} = \lim_{\sigma: \Delta^{n+1}} U_{s(\sigma)};
\]

since \( \mathcal{V}(X) \) is a poset, this limit is simply an intersection \( \bigcap_{j=0}^n s(\Delta^j) \) where \( \Delta^j \subset \Delta^{n+1} \) denotes the \( j \)th facet. In particular, \( U_s = U_{s^1} \).

Now for given \( s \), compare the following two criteria:

i) the restriction of \( U \) to \( I^{op}(\Delta^{n+1})_s \) covers \( U_s \);

ii) the restriction of \( U \) to \( I^{op}(\Delta^{n+1})_s \) covers \( U_s = U_{s^1} \).

By definition, \( U \) is pulled back along

\[
I^{op}(\Delta^{n+1})_s \to I^{op}(\Delta^{n+1})_{s^1} \to I^{op}
\]

where the second arrow is evaluation on the final object. Therefore i)\( \Rightarrow \)ii). By lemma C.9, the first map is surjective and so the converse implication holds too.

Conclusion. Because \( I \) is a poset, \( I^{op}(\Delta^{n+1})_s \) is nothing but the set of indices \( i : I \) that are dominated by the final object of \( \Delta^s \to I \) for each \( j = 0, \ldots, n \). This is precisely the set of indices appearing in the atlas condition 3.2. Therefore, if \( U_I \) is an atlas, then \( t^*_{\Delta} U_I \) is a hypercover.

Conversely, if \( t^*_{\Delta} U_I \) is a hypercover, then the hypercover condition for fillings of the form \( \partial \Delta^1 \subset \Delta^1 \) implies the binary atlas condition; whence \( U_I \) is an atlas.

C.10 Aside. Because the indexing categories are assumed to be posets, the preceding proof only makes essential use of fillings \( \partial \Delta^1 \subset \Delta^1 \). If \( I \) were allowed to be a more general 1-category, then fillings one dimension up \( \partial \Delta^2 \subset \Delta^2 \) would intervene to handle forks in \( I \). Indeed, a fork is obtained from \( \Delta^2 \) by collapsing two edges of the triangle.

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