ABSTRACT

In a group trellis, the sequence of branches that split from the identity path and merge to the identity path form two normal chains. The Schreier refinement theorem can be applied to these two normal chains. The refinement of the two normal chains can be written in the form of a matrix, called the Schreier matrix form, with rows and columns determined by the two normal chains.

Based on the Schreier matrix form, we give an encoder structure for a group code which is an estimator. The encoder uses the important idea of shortest length generator sequences previously explained by Forney and Trott. In this encoder the generator sequences are shown to have an additional property: the components of the generators are coset representatives in a chain coset decomposition of the branch group $B$ of the code. Therefore this encoder appears to be a natural form for a group code encoder. The encoder has a register implementation which is somewhat different from the classical shift register structure.

This form of the encoder can be extended. We find a composition chain of the branch group $B$ and give an encoder which uses coset representatives in the composition chain of $B$. When $B$ is solvable, the generators are constructed using coset representatives taken from prime cyclic groups.

1. INTRODUCTION

The idea of group codes and group shifts is important in several areas of mathematics and engineering such as symbolic dynamics, linear systems theory, and coding theory. Some of the seminal papers in these areas are the work of Kitchens [1], Willems [2], Forney and Trott [3], and Loeliger and Mittelholzer [4].

Kitchens [1] introduced the idea of a group shift [9] and showed that a group shift has finite memory, i.e., it is a shift of finite type [9]. A group shift is a fundamental example of a time invariant group code [8]. Forney and Trott [3] show that any time invariant group code is equivalent to a labeled group trellis section. They show that any group code that is complete (any global constraints can be determined locally, cf. [3]) is equivalent to a sequence of connected labeled group trellis sections (which may vary in time). They explain the important idea of shortest length code sequences, or generator sequences. They show that at each time epoch, a finite set of generator sequences can be used construct a “local” section of the code. Using the generator sequences, they show that any group code can be mechanized with a minimal encoder which has a shift register structure.

Forney and Trott use a “top down” approach, that is, they start with a group code, a set of sequences with a group property, and then analyze further to determine the state structure, encoder structure, and other properties of the code; the related work of [2] in systems theory also uses a top down approach. The work of Loeliger and Mittelholzer is a “bottom up” approach. They start with a group trellis section, and use it to construct a group trellis, or group code. Among other results, Loeliger and Mittelholzer give an abstract characterization of groups which can be the branch group $B$ of a group code. They also give a shift register structure for a group trellis. The development of their encoder using a bottom up approach is in some sense a mirror image of the Forney and Trott top down encoder construction.

In this paper, we use the bottom up approach of Loeliger and Mittelholzer. We start with a group trellis section and determine properties of the group trellis.

In a group trellis, the sequence of branches that split from the identity path and merge to the identity path form two normal chains, $\{X_j\}$ and $\{Y_k\}$, respectively. These two normal chains were first used in the work of [4]. In this paper we apply the Schreier refinement theorem to $\{X_j\}$ and $\{Y_k\}$. The refinement of the two normal chains can be written in the form of a matrix, called the Schreier matrix form, with rows and columns determined by $\{X_j\}$ and $\{Y_k\}$. When the group trellis is controllable, the Schreier matrix is a triangular form. The Schreier matrix is an echo of matrix ideas used in classical linear systems analysis.

Based on the Schreier matrix form, we give an encoder structure for a group code which is an estimator. Both encoders of [3,4] use shortest length generator sequences. The encoder here also uses shortest length generator sequences, but in this encoder the generator sequences are shown to have an additional property: the components of the generators are coset representatives in a chain coset decomposition of the branch group $B$ of the code. This shows that the generator code sequences are intimately related to the structure of the branch group. Therefore this encoder appears to be a natural form for a group code encoder. In addition, the encoder has a convolution property which is not apparent in the encoders of [3,4]. The encoder has a register implementation which is somewhat
different from the classical shift register structure.

This approach can be iterated. We use properties of the group trellis to find a composition chain of $B$. We insert this composition chain into $\{X_i\}$ to find a refinement of the Schreier matrix which is a composition chain, a Schreier array. Using the Schreier array, we give an encoder which uses coset representatives in the composition chain of $B$. When $B$ is solvable, the generators are constructed using coset representatives taken from prime cyclic groups.

This paper is organized as follows. Section 2 defines a group trellis section and group trellis. We study an $\ell$-controllable group trellis, in which each state can be reached from any other state in $\ell$ branches. Section 3 defines the Schreier matrix and a controllable Schreier matrix, which has a triangular form. In Section 4, we analyze the structure of the controllable Schreier matrix and the controllable group trellis. We focus on the branches that split from the identity path, the sequence $\{X_i\}$. The only technical tool we use is several simple generalizations of the correspondence theorem. Based on the analysis in Section 4, Section 5 gives an encoder for the group trellis. The encoder uses shortest length generator sequences and has a convolution property. The shortest length generator sequences are composed of coset representatives from a chain coset decomposition of $B$. In Section 6, we use properties of the group trellis to give a composition series of $B$. In Section 7, we use the composition series of $B$ and the mechanics of the Schreier matrix to give a novel encoder for the group trellis. The encoder uses shortest length generator sequences whose components are coset representatives in the composition series of $B$.

2. GROUP TRELLIS

As in [S], we say $G$ is a subdirect product of $G_1$ and $G_2$ if it is a subgroup of $G_1 \times G_2$ and the first and second coordinate of $G$ take all values in $G_1$ and $G_2$, respectively; we also say $G$ is a subdirect product of $G_1 \times G_2$.

**Definition.** A group trellis section is a subdirect product $B$ of $S \times S$. (Therefore the left and right coordinates of $B$ use all elements of $S$.) We call $B$ the branch group and $S$ the state group. The elements of $B$ are branches $b$, where $b = (s, s') \in S \times S$.

We can think of a group trellis section $B$ as a bipartite graph with branches in $B$ and vertices in $S$, where there is a branch $(s, s')$ between two vertices $s$ and $s'$ if and only if $(s, s') \in B$.

**Definition.** A labeled group trellis section is a subdirect product $\tilde{B}$ of $S \times A \times S$. We call $A$ the label group or alphabet. The elements of $\tilde{B}$ are labeled branches $\tilde{b}$, where $\tilde{b} = (s, a, s') \in S \times A \times S$.

Note that there is a homomorphism $\omega : B \to A$. Note that if $A = 1$, then the labeled group trellis $\tilde{B}$ is isomorphic to the group trellis $B$.

**Definition.** A group trellis $C$ is the shift of a group trellis section. A labeled group trellis $\tilde{C}$ is the shift of a labeled group trellis section.

We only consider group codes defined on the integers $\mathbb{Z}$.

**Definition.** A group code $C$ is a subgroup of an infinite direct product group $\prod_{\mathbb{Z}} A$, where $\mathbb{Z}$ is the integers.

Consider the projection map $\pi^- : B \to S$ given by the assignment $(s, s') \mapsto s$. This is a homomorphism with kernel $X_0$, the subgroup of all elements of $B$ of the form $(1, s')$. Let $B^-$ be the left states of $B$, so that $B^- = S$. Then

$$\frac{B}{X_0} \simeq B^- = S.$$

Consider the projection map $\pi^+ : B \to S$ given by the assignment $(s, s') \mapsto s'$. This is a homomorphism with kernel $Y_0$, the subgroup of all elements of $B$ of the form $(s, 1)$. Let $B^+$ be the right states of $B$, so that $B^+ = S$. Then

$$\frac{B}{Y_0} \simeq B^+ = S.$$

Together these give

$$\frac{B}{X_0} \simeq B^- = S = B^+ \simeq \frac{B}{Y_0}.$$

**Proposition 1** Let $G$ be a subgroup of $B$. We have $G \cap X_0 \triangleleft G$ and $G \cap Y_0 \triangleleft G$. The branches that split from each left state of $G$, $G^-$, are a coset of $G \cap X_0$. The branches that merge to each right state of $G$, $G^+$, are a coset of $G \cap Y_0$.

We think of a group trellis $C$ as a connected sequence of group trellis sections $B$. The states of the group trellis occur at time epochs $t$, which are integers in the range $-\infty < t < \infty$. The states at time epoch $t$ are $S_t$, where $S_t = S$. The trellis section $B_t$ is the trellis section with left states at time epoch $t$, $S_t$, and right states at time epoch $t + 1$, $S_{t+1}$. The right states of $B_t$, $S_{t+1}$, are the left states of $B_{t+1}$. By the phrase a branch at time epoch $t$ we mean a branch $b_t \in B_t$; we have $b_t = (s, s')$ where $s \in S_t$ and $s' \in S_{t+1}$.

Let $c$ be a trellis path in $C$. Then

$$c = \ldots, b_{-1}, b_0, b_1, \ldots, b_i, \ldots,$$

where component $b_t$ is a branch in $B_t = B$. Define the projection map $\chi_t : C \to B_t$ by the assignment $c \mapsto b_t$. In other words $\chi_t(c) = b_t = (s, s')$ where $s \in S_t$ and $s' \in B_{t+1}$. Define the projection map $\chi_{[t_1, t_2]} : C \to B_{t_1} \times \cdots \times B_{t_2}$ by the assignment $c \mapsto (b_{t_1}, \ldots, b_{t_2})$, the components of $c$ over the time interval $[t_1, t_2]$. For all integers $i$, define $C_{(i, \infty)}$ to be the set of paths $c$ such that components $b_t = 1$ for $t < i$, where $1$ is the identity of $B$. Then $C_{(i, \infty)}$ is the set of paths that are in the identity state at time epochs in the interval $(-\infty, i]$. For
all integers $i$, define $C_{(-i,\infty]}$ to be the set of paths $c$ such that components $b_{t} = 1$ for $t > i$. For all integers $i$, let

$$X_{i} = \{\chi_{0}(c)|c \in C_{(-i,\infty]}\}.$$ 

Note that $X_{i} = 1$ for $i < 0$. For all integers $i$, let

$$Y_{i} = \{\chi_{0}(c)|c \in C_{(-\infty,i]}\}.$$ 

Note that $Y_{i} = 1$ for $i < 0$. As defined, $X_{i}$ and $Y_{i}$ are elements of $B$; a priori there is no intrinsic notion of time associated with subscript $i$. Then for each $i$, $X_{i} \subseteq B_{i}$ and $Y_{i} \subseteq B_{i}$ for all time epochs $t$, $-\infty < t < \infty$. Note that $X_{t}^{+} = X_{t+1}^{+}$ and $Y_{t}^{+} = Y_{t-1}$ for all integers $i$. And $X_{t} < B$ and $Y_{t} \subseteq B$ for all integers $i$.

We say a group trellis is $l$-controllable if for any time epoch $t$, and any pair of states $(s, s') \in S_{t} \times S_{t+1}$, there is a trellis path through these two states. The least integer $l$ for which a group trellis is $l$-controllable is denoted as $\ell$. In this paper, we only study the case $\ell = \ell$.

The next result follows directly from Proposition 7.2 of [4], using our notation.

**Proposition 2** The group trellis $C$ is $l$-controllable if and only if $(X_{\ell-1})^{+} = B^{-}$, or equivalently, if and only if $(Y_{\ell-1})^{-} = B^{+}$. 

**3. SCHREIER MATRIX FORM**

The group $B$ has two normal series (and chief series)

$$1 = X_{-1} \subset X_{0} \subset X_{1} \subset \cdots \subset X_{\ell} = B$$

and

$$1 = Y_{-1} \subset Y_{0} \subset Y_{1} \subset \cdots \subset Y_{\ell} = B.$$ 

We denote these normal series by $\{X_{j}\}$ and $\{Y_{k}\}$. The Schreier refinement theorem used to prove the Jordan-Hölder theorem in [7] shows how to obtain a refinement of $\{X_{j}\}$ by inserting $\{Y_{k}\}$, as shown in equation (3) (see next page). In [3], we have written the refinement as a matrix of $\ell + 1$ columns and $\ell + 2$ rows. Note that the terms in the bottom row form the sequence $X_{-1}, X_{0}, X_{1}, \ldots, X_{\ell-2}, X_{\ell-1}$, and the terms in the top row form the sequence $X_{0}, X_{1}, X_{2}, \ldots, X_{\ell-1}, X_{\ell}$. Thus (3) is indeed a refinement of the normal series $\{X_{j}\}$. We call (3) the Schreier matrix form of $\{X_{j}\}$ and $\{Y_{k}\}$, or (loosely) just the Schreier matrix of $\{X_{j}\}$ and $\{Y_{k}\}$. Since $\{X_{j}\}$ and $\{Y_{k}\}$ are chief series, the Schreier matrix of $\{X_{j}\}$ and $\{Y_{k}\}$ is a chief series.

The diagonal terms of the Schreier matrix are $X_{j-1}(X_{j} \cap Y_{k})$ for $j = 0, \ldots, \ell$. We say the Schreier matrix is $l$-controllable if $X_{j-1}(X_{j} \cap Y_{k}) = X_{j}$ for $j = 0, \ldots, \ell$. This is trivially satisfied for $j = 0$. For $j \in [1, \ell]$, this means all column terms above the diagonal term are the same as the diagonal term. Then we can reduce the Schreier matrix and write the $l$-controllable Schreier matrix as in [4]. Note that the $l$-controllable Schreier matrix is a triangular form. (A triangle can be

formed in two ways, depending on whether the columns in [3] are shifted up or not; we have shifted the columns up since it is more useful here.)

Note that the Schreier matrix term $X_{j-1}(X_{j} \cap Y_{k})$ is in the $j^{th}$ column of $[4]$, for $0 \leq j \leq \ell$, and $(j + k)^{th}$ row, for $0 \leq j + k \leq \ell$ (counting up from the bottom).

**Proposition 3** If the group trellis $C$ is $\ell$-controllable, then the Schreier matrix of $\{X_{j}\}$ and $\{Y_{k}\}$ is $\ell$-controllable.

**Proof.** If the group trellis $C$ is $\ell$-controllable, then from Proposition 7.2 of [4] (using our notation),

$$X_{0} \cap Y_{0} \cap Y_{1} \cap \cdots \cap X_{j} \cap Y_{k} = X_{j}$$

for all $j \geq 0$. This means we can rewrite (1) as

$$X_{j-1}(X_{j} \cap Y_{k}) = X_{j}$$

for all $j \geq 0$. Then the Schreier matrix of $\{X_{j}\}$ and $\{Y_{k}\}$ is $\ell$-controllable.

The Schreier matrix of $\{Y_{k}\}$ and $\{X_{j}\}$ is obtained by interchanging $X$ and $Y$ in (3); it is the dual of the Schreier matrix of $\{X_{j}\}$ and $\{Y_{k}\}$. We say the Schreier matrix of $\{Y_{k}\}$ and $\{X_{j}\}$ is $\ell$-controllable if $Y_{k-1}(X_{j} \cap Y_{k-1}) = Y_{k}$ for $k = 0, \ldots, \ell$.

**Proposition 4** The Schreier matrix $M$ of $\{X_{j}\}$ and $\{Y_{k}\}$ is $\ell$-controllable if and only if the (dual) Schreier matrix $M_{d}$ of $\{Y_{k}\}$ and $\{X_{j}\}$ is $\ell$-controllable.

**Proof.** By the Zassenhaus lemma used to prove the Schreier refinement theorem in [7], we have

$$X_{j-1}(X_{j} \cap Y_{k}) \sim Y_{k-1}(Y_{k} \cap X_{j})$$

for $j$ and $k$ in the range $0 \leq j \leq \ell$, $0 \leq k \leq \ell$. Note that the numerator and denominator on the left are in the same column of $M$ (see [3]) and the numerator and denominator on the right are in the same column of $M_{d}$. Assume the Schreier matrix is $\ell$-controllable. If $j + k - 1 \geq \ell$, then the denominator term on the left is on the diagonal or above, and the left hand side is isomorphic to 1. This means the right hand side is isomorphic to 1. But if $j + k - 1 \geq \ell$, the denominator term on the right is on the diagonal or above, and then the dual Schreier matrix is $\ell$-controllable. The reverse direction is the same proof.

**4. STRUCTURE OF THE CONTROLLABLE SCHREIER MATRIX FORM**

We now show that terms in the controllable Schreier matrix in [4] can be related to certain paths in the trellis. Consider paths in the trellis which split from the identity state at time epoch 0. The branches in these paths are $X_{0}$ at time epoch 0, $X_{1}$ at time epoch 1, ..., and $X_{\ell}$ at
\[
\begin{array}{cccccc}
X_{-1}(X_0 \cap Y_\ell) & X_0(X_1 \cap Y_\ell) & U & X_1(X_2 \cap Y_\ell) & \cdots & X_{\ell-2}(X_{\ell-1} \cap Y_\ell) & X_{\ell-1}(X_\ell \cap Y_\ell) \\
\cup & \cup & \cup & \cup & \cdots & \cup & \cup \\
X_{-1}(X_0 \cap Y_{\ell-1}) & X_0(X_1 \cap Y_{\ell-1}) & U & X_1(X_2 \cap Y_{\ell-1}) & \cdots & X_{\ell-2}(X_{\ell-1} \cap Y_{\ell-1}) & X_{\ell-1}(X_\ell \cap Y_{\ell-1}) \\
\cup & \cup & \cup & \cup & \cdots & \cup & \cup \\
X_{-1}(X_0 \cap Y_{\ell-2}) & X_0(X_1 \cap Y_{\ell-2}) & U & X_1(X_2 \cap Y_{\ell-2}) & \cdots & X_{\ell-2}(X_{\ell-1} \cap Y_{\ell-2}) & X_{\ell-1}(X_\ell \cap Y_{\ell-2}) \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
X_{-1}(X_0 \cap Y_2) & X_0(X_1 \cap Y_2) & U & X_1(X_2 \cap Y_2) & \cdots & X_{\ell-2}(X_{\ell-1} \cap Y_2) & X_{\ell-1}(X_\ell \cap Y_2) \\
\cup & \cup & \cup & \cup & \cdots & \cup & \cup \\
X_{-1}(X_0 \cap Y_1) & X_0(X_1 \cap Y_1) & U & X_1(X_2 \cap Y_1) & \cdots & X_{\ell-2}(X_{\ell-1} \cap Y_1) & X_{\ell-1}(X_\ell \cap Y_1) \\
\cup & \cup & \cup & \cup & \cdots & \cup & \cup \\
X_{-1}(X_0 \cap Y_0) & X_0(X_1 \cap Y_0) & U & X_1(X_2 \cap Y_0) & \cdots & X_{\ell-2}(X_{\ell-1} \cap Y_0) & X_{\ell-1}(X_\ell \cap Y_0) \\
\cup & \cup & \cup & \cup & \cdots & \cup & \cup \\
X_{-1}(X_0 \cap Y_{-1}) & X_0(X_1 \cap Y_{-1}) & U & X_1(X_2 \cap Y_{-1}) & \cdots & X_{\ell-2}(X_{\ell-1} \cap Y_{-1}) & X_{\ell-1}(X_\ell \cap Y_{-1}) \\
\end{array}
\]
time epoch \( \ell \). These paths are represented by the upper sloped line in Figure 1 labeled \( X_0, X_1, \ldots, X_\ell \). Now consider those paths that split from the identity at time 1. The branches in these paths are \( X_0 \) at time 1, \( X_j \) at time 2, \ldots, and \( X_\ell -1 \) at time \( \ell \). These paths are represented by the lower sloped line in Figure 1. Define the groups involving the two sets of paths, \( F_0 = (X_0, X_1, \ldots, X_\ell) \) and \( F_1 = (X_0, X_1, \ldots, X_\ell -1) \). It can be seen that \( F_1 \subset F_0 \). The controllable Schreier matrix will be seen to be related to the quotient group \( F_0/F_1 \). Note that the quotient groups of respective components, e.g., \( X_j/X_{j-1} \), contain a complete set of coset representatives of \( X_\ell = B \).

Consider the portion of the trellis given by the following projection:

\[
P_{[0, \ell]} = \chi_{[0, \ell]}(C_{[0,\infty]}).
\]

We call this portion of the trellis, \( P_{[0, \ell]} \), the pletty. The branches at time epoch \( j \) in the pletty, for \( 0 \leq j \leq \ell \), are \( X_j \). We call \( X_j \) the \( j \)th pletty section. Thus a pletty consists of the pletty sections \( X_0, X_1, \ldots, X_\ell \).

**Proposition 5** Any pletty is a group, called the group pletty.

**Proof.** The projection map is a homomorphism, so the image is a group \([3]\).

**Proposition 6** Any pletty section is a group, called the group pletty section.

**Proof.** \( X_j \) is a group.

The elements of \( X_j = X_{j-1}(X_j \cap Y_{\ell-j}) \) form a subgraph of the group trellis section \( B \), where each input state at time \( j \) has \( |X_0| \) splitting branches and each output state at time \( j +1 \) has \( |X_j \cap Y_0| \) merging branches. The right states of \( X_j, X_j^+ \), are isomorphic to cosets of \( X_j \cap Y_0 \). The left states of \( X_j, X_j^- \), are isomorphic to cosets of \( X_0 \). Then \( X_j \) is a subdirect product of \( X_j^- \times X_j^+ \), a subgroup of the subdirect product \( B \).

Of course we know the right states of \( X_j, X_j^+ \), are connected to the left states of \( X_{j+1}, X_j^{j+1} \), in the trellis. Thus the pletty sections connect to form the pletty.

Consider Figure 1 which is a redrawing and refinement of Figure 1. We take the point \( X_0 \) at time 0 on the upper sloped line and pull it vertically so the sloped line becomes horizontal. Now the upper horizontal line in Figure 2 represents the paths that split from the identity state at time 0. The groups below \( X_0 \) at time 0 are subgroups of \( X_0 \); the branches in these subgroups also split from the identity state at time 0. We will show these branches form paths which merge with paths that split from the identity state at time 1.

**Proposition 7** For any subsets \( G, H \) of \( B \), we have

\[
(G \cap H)^+ = G^+ \cap H^+ \quad \text{and} \quad (G \cap H)^- = G^- \cap H^-.
\]

Also

\[
(GH)^+ = G^+H^+ \quad \text{and} \quad (GH)^- = G^-H^-.
\]

**Proposition 8** For \( j \geq 0, k \geq 0, \) such that \( j + k \leq \ell \), we have

\[
(X_j \cap Y_k)^+ = (X_{j+1} \cap Y_{k-1})^{-},
\]

and

\[
(X_j-1(X_j \cap Y_k))^- = (X_{j-1}(X_j \cap Y_k))^-.
\]

**Proof.** Proof of (5). By definition, we have \( X_j^+ = X_{j+1}^- \) and \( Y_k^+ = Y_{k-1}^- \). Then (5) follows from Proposition 6.

The result (6) was previously given in [4]. Each set of paths represented by a line in Figure 2 compares directly to a row in [4]. For the subgroups of branches on a line of Figure 2 are the same as the subgroups in a row of [4]. And as seen from Proposition 5, the right states of any row in column \( j \) of [4] are connected to the left states of the same row in column \( j +1 \), \( 0 \leq j \leq \ell \), in the same way as subgroups on a line of Figure 2 are connected.

It can also be seen that column \( j \) of [4], \( 0 \leq j \leq \ell \), represents the coset decomposition of the quotient group \( X_j/X_{j-1} \) shown in Figure 1 and Figure 2. The right states of branches in column \( j \) are the left states of branches in column \( j +1 \), \( 0 \leq j \leq \ell \). Thus (4) gives a “picture” of the groups that occur in the group pletty, as seen in Figure 2.

The rest of this section contains three subsections. In the first subsection, we study one pletty section, a column of the controllable Schreier matrix in [4], and in the second subsection two pletty sections. In the last subsection we study a sequence of pletty sections. We will show that a rectangle condition holds for the controllable Schreier matrix, in which the quotient group formed using paths from \( H_0 \) to \( H_1 \) in Figure 2 and paths from \( J_0 \) to \( J_1 \), is isomorphic to the quotient group of individual components, e.g., \( H_0/J_0 \) and \( H_1/J_1 \).

### 4.1 One pletty section

We first state the correspondence theorem since it will be useful throughout the remainder of Section 4.

**Theorem 9** (Correspondence theorem [7]) If \( K \) is a normal subgroup of \( G \) and \( \nu : G \to G/K \) is the natural map, then \( \nu \) induces a one to one correspondence between all the subgroups of \( G \) that contain \( K \) and all the subgroups of \( G/K \). If \( J \) is a subgroup of \( G \) that contains \( K \), then \( K \) is a normal subgroup of \( J \), and the corresponding subgroup of \( G/K \) is \( \nu(J) = J/K \).
Figure 1: Trellis paths that split from the identity at time 0 and time 1.

Figure 2: Rearrangement of Figure 1 showing similarity with controllable Schreier matrix (4).

\[ X_{-1}(X_0 \cap Y_\ell) = X_0 \]
\[ X_{-1}(X_0 \cap Y_{\ell-1}) \]
\[ X_{-1}(X_0 \cap Y_1) \]
\[ X_{-1}(X_0 \cap Y_2) \]

**Diagram:**

```
  X_\ell  X_\ell  X_\ell
  \bullet
  X_{\ell-1}

  X_0     X_1     X_2     X_3
  \bullet
  \bullet

  H_0     H_1
  \bullet

  J_0     J_{\ell}
  \bullet

  X_1
  \bullet

  X_0
  \bullet

  \bullet

  1 1 1 1

  time epoch 0 1 2 3 \ell
```

Figure 2: Rearrangement of Figure 1 showing similarity with controllable Schreier matrix (4).
Let $H$ be another subgroup of $G$ that contains $K$. Then

(i) $J \leq H$ if and only if $\nu(J) \leq \nu(H)$, and then $[H : J] = \nu(H) / \nu(J)$;

(ii) $J \lhd H$ if and only if $\nu(J) \triangleleft \nu(H)$, and then $H / J \simeq \nu(H) / \nu(J)$.

In general, for any groups $Q, Q'$ with $Q' \leq Q$, let $Q / Q'$ denote the right cosets of $Q'$ in $Q$. For any set $\chi$ and any group $Q$, we denote a (right) action of $Q$ on $\chi$ by $\{\chi\}_q$, for $\xi \in \chi$ and $q \in Q$.

We now study (i) of the correspondence theorem. Assume that $J \leq H$. The set of right cosets of $J$ in $H$ is denoted by $H / J$. Any right coset of $H / J$ is of the form $Jh$, for $h \in H$. The right cosets of $\nu(J)$ in $\nu(H)$ are $\nu(H) / \nu(J)$. Any right coset of $\nu(H) / \nu(J)$ is of the form $\nu(Jh)$, where $Jh$ is a right coset in $H / J$. Define a function $f : H / J \to \nu(H) / \nu(J)$ by the assignment $Jh \mapsto \nu(Jh)$, for $h \in H$. Let $g, h \in H$. $H$ acts on $H / J$ by the assignment $[Jh]g \mapsto J(hg)$. Let $g, h \in H$. $H$ acts on $\nu(H) / \nu(J)$ by the assignment $[\nu(Jh)]g \mapsto \nu(Jhg)$.

Theorem 10 We can strengthen (i) of the correspondence theorem this way:

(i) $J \leq H$ if and only if $\nu(J) \leq \nu(H)$, and then $f$ is an $H$-isomorphism giving $H / J \simeq \nu(H) / \nu(J)$.

Proof. We first show that $f([Jh]) = [f(Jh)]g$ for $Jh \in H / J$ and $g \in H$. We have

$$f([Jh]) = f(J(hg)) = \nu(Jhg) = \nu(Jh)g = [f(Jh)]g.$$

Thus $f$ is an $H$-map. It is clear that $f$ is a bijection so $f$ is an $H$-isomorphism.

We now study a single group pletty section $X_j$, for $0 \leq j \leq \ell$.

Theorem 11 Consider the $j$th group pletty section $X_j = X_{j-1}(X_j \cap Y_j)$ for $j = 0, 1, \ldots, \ell$. Clearly $X_0 \subset X_j$ for $j > 0$. For $j = 0$, we have $X_0 = X_{-1}(X_0 \cap Y_0)$. Then $X_0 \subset X_j$ for all $j$, $j = 0, 1, \ldots, \ell$. Since $X_0$ is a normal subgroup of $X_j$, there is a natural map

$$\pi_j^*: X_j \to X_j / X_0,$$

and the results of the correspondence theorem apply. Note that when $j = 0$, $\pi_j^*$ trivially maps to the identity.

For all $j$, $j = 0, 1, \ldots, \ell$, $X_j \cap Y_0$ is a normal subgroup of $X_j$. Then there is a natural map

$$\pi_j^*: X_j \to X_j / (X_j \cap Y_0),$$

and the results of the correspondence theorem apply.

Theorem 12 Consider the $j$th group pletty section $X_j = X_{j-1}(X_j \cap Y_{j-1})$ for $j = 0, 1, \ldots, \ell$. Fix $j$, $0 \leq j \leq \ell$. Let $Y'$ and $Y''$ be subgroups of $B$ such that $1 \leq Y' \leq Y'' \leq Y_{j-1}$. Consider the groups $J = X_{j-1}(X_j \cap Y')$ and $H = X_{j-1}(X_j \cap Y'')$, where $X_{j-1} \leq J \leq H \leq X_j$.

(i) Let

$$D = (X_j \cap Y')(X_{j-1} \cap Y'').$$

$D$ is a subgroup of $X_j \cap Y''$. There is a one to one correspondence $\hat{f} : H / J \to (X_j \cap Y'')/D$, and $\hat{f}$ is an $H$-isomorphism giving $H / J \simeq (X_j \cap Y'')/D$.

We can choose a right transversal of $H / J$ using elements of $X_j \cap Y''$ taken from right cosets of $(X_j \cap Y'')/D$.

(ii) Assume that $Y' \triangleleft Y''$. Then $D$ is a normal subgroup of $X_j \cap Y''$. There is a one to one correspondence $\hat{f} : H / J \to (X_j \cap Y'')/D$, and $\hat{f}$ is an isomorphism giving $H / J \simeq (X_j \cap Y'')/D$.

Proof. $J$ is a group since $X_j \cap Y'$ is a group and $X_{j-1} \lhd B$; similarly $H$ is a group. Then $X_{j-1} \leq J \leq H \leq X_j$.

Proof of (i). Let $S = X_j \cap Y'$. Then $H = X_{j-1}(X_j \cap Y'') = JS$. We already know $JS$ is a group. Further $JS$ consists of all cosets of $J$ that have representatives in $S$. The representatives of $S$ in $J$ are $J \cap S$. $J \cap S$ is a group since $J$ and $S$ are groups, and clearly $J \cap S \subset S$. The representatives of right coset $JS$ in $S$, where $s \in S$, are $(J \cap S)s$ (since $s$ takes $J$ to $JS$ and $J \cap S \subset J$ to $J \cap S \subset J$). Then each right coset $JS$ in $H = JS$ contains the elements in right coset $(J \cap S)s$ in $S$. Therefore there is a one to one correspondence $\hat{f} : H / J \to S / (J \cap S)$, where $\hat{f}$ is an $H$-isomorphism giving $H / J \simeq S / (J \cap S)$.

Lastly we evaluate $J \cap S$,

$$J \cap S = X_{j-1}(X_j \cap Y') \cap (X_j \cap Y'') = (X_j \cap Y')(X_{j-1} \cap (X_j \cap Y'')$$

where the last equality follows since $X_{j-1} \lhd B$. Let $M = X_j \cap Y'$, $N = X_{j-1}$, and $L = X_j \cap Y''$. From the Dedekind law, we know $MN \cap L = M(N \cap L)$. Then

$$J \cap S = (X_j \cap Y')(X_{j-1} \cap Y'').$$

Now define $D$ to be $J \cap S$.

Proof of (ii). This is an application of the proof of the Zassenhaus lemma (cf. p. 100 of [7]): see Lemma 13 below.

Lemma 13 (taken from proof of Zassenhaus lemma in [7]) Let $U \triangleleft U^*$ and $V \triangleleft V^*$ be four subgroups of a group $G$. Then $D = (U \cap V)(U \cap V^*)$ is a normal subgroup of $U^* \cap V^*$. If $g \in U(U^* \cap V^*)$, then $g = uu^*$ for $u \in U$ and $u^* \in U^* \cap V^*$. Define function $f : U(U^* \cap V^*) \to (U^* \cap V^*) / D$ by $f(g) = f(uu^*) = Du^*$. Then $f$ is a well defined homomorphism with kernel $U(U^* \cap V)$ and

$$\frac{U(U^* \cap V^*)}{U(U^* \cap V)} \simeq \frac{U^* \cap V^*}{D}.$$
4.2 Two pletty sections

We now study two pletty sections and the Schreier matrix. We give several analogs of the correspondence theorem for two pletty sections. We relate adjacent columns in \( \mathcal{N}(b) \). Fix \( j \) such that \( 0 \leq j < \ell \). Consider the \( j \)th group pletty section \( X_j \) and \( (j+1) \)th group pletty section \( X_{j+1} \).

For each branch \( b \in B \), we define the next branch set \( \mathcal{N}(b) \) to be the set of branches that can follow \( b \) at the next time epoch in valid trellis paths. In other words, branch \( e \in \mathcal{N}(b) \) if and only if \( b^+ = e^- \).

For a set \( U \subset B \), define the set \( \mathcal{N}(U) \) to be the union \( \cup_{b \in U} \mathcal{N}(b) \). The set \( \mathcal{N}(U) \) always consists of cosets of \( X_0 \). Note that \( \mathcal{N}(X_j) = X_{j+1} \).

Proposition 14 If \( b^+ = e^- \), the next branch set \( \mathcal{N}(b) \) of a branch \( b \) in \( B \) is the coset \( X_0 e \) in \( B \). If \( b^+ = e^- \), the next branch set \( \mathcal{N}(b) \) of a branch \( b \) in \( B \) is the coset \( X_0 e \) in \( X_{j+1} \).

We know \( X_j \cap Y_0 \) is a normal subgroup of \( X_j \), and so there is a natural map \( \pi^+_j : X_j \rightarrow X_j / X_j \cap Y_0 \). We know \( X_0 \) is a normal subgroup of \( X_{j+1} \), and so there is a natural map \( \pi^+_{j+1} : X_{j+1} \rightarrow X_{j+1} / X_j \cap Y_0 \).

Note that there is an isomorphism \( \alpha_j \).

\[
\alpha_j : \frac{X_j \cap Y_0}{X_j \cap Y_0} \rightarrow \frac{X_j Y_0}{Y_0},
\]
or what is the same,

\[
\alpha_j : \frac{X_j}{X_j \cap Y_0} \rightarrow \frac{X_j Y_0}{Y_0},
\]

where \( X_j Y_0 / Y_0 \) is considered as a subgroup of \( G / Y_0 \). We already know an isomorphism \( \varphi : G / Y_0 \rightarrow G / X_0 \), which gives an isomorphism \( \varphi_j \) restricted to \( X_j Y_0 / Y_0 \),

\[
\varphi_j : \frac{X_j Y_0}{Y_0} \rightarrow \frac{X_{j+1}}{X_j \cap Y_0}.
\]
The composition \( \varphi_j \circ \alpha_j \) is an isomorphism, defined to be \( \Phi_j \).

\[
\Phi_j : \frac{X_j}{X_j \cap Y_0} \rightarrow \frac{X_{j+1} Y_0}{Y_0}.
\]

Then the next branch set \( \mathcal{N}^+ : X_j \rightarrow X_{j+1} \) represents the contraction, isomorphism, and expansion:

\[
X_j \pi^+_j X_j / (X_j \cap Y_0) \cong X_{j+1} / X_j \cap Y_0 \pi^+_{j+1} X_{j+1}.
\]

Specifically, for branches \( b, e \) such that \( b^+ = e^- \), we have \( \mathcal{N}(b) = X_0 e \), given by the assignments:

\[
\pi^+_j (X_j \cap Y_0) b \Phi_j \rightarrow X_0 e.
\]

Proposition 15 There are five results:

(i) Assume \( J \) is a set in \( X_j \). If \( J \) is a group, then \( \mathcal{N}(J) \) is a group. In general, \( \mathcal{N}(J) \) being a group does not mean set \( J \) is a group. But if set \( J \) consists of cosets of \( X_j \cap Y_0 \), then \( J \) is a group if \( \mathcal{N}(J) \) is a group.

(ii) Assume \( J, H \) are sets in \( X_j \). If \( J \subset H \), then \( \mathcal{N}(J) \subset \mathcal{N}(H) \). Assume \( J, H \) are groups in \( X_j \). If \( J \subset H \), then \( \mathcal{N}(J) \subset \mathcal{N}(H) \).

(iii) Assume \( J, H \) are sets in \( X_j \) and \( \mathcal{N}(J), \mathcal{N}(H) \) are groups in \( X_{j+1} \). In general, \( \mathcal{N}(J) \subset \mathcal{N}(H) \) does not mean \( J \subset H \). But if sets \( J, H \) consist of cosets of \( X_j \cap Y_0 \), then \( J \subset H \) if \( \mathcal{N}(J) \subset \mathcal{N}(H) \).

(iv) Assume \( J, H \) are sets in \( X_j \) and \( \mathcal{N}(J), \mathcal{N}(H) \) are groups in \( X_{j+1} \). If sets \( J, H \) consist of cosets of \( X_j \cap Y_0 \), then \( J, H \) are groups. And if \( \mathcal{N}(J) \subset \mathcal{N}(H) \), then \( J \subset H \). And if \( \mathcal{N}(J) \subset \mathcal{N}(H) \), then \( J \subset H \).

(v) Assume \( J, H \) are sets in \( X_j \). Assume that \( X_j \cap Y_0 \subset J \) and \( X_j \cap Y_0 \subset H \). Then we have \( J \subset H \) if and only if \( \mathcal{N}(J) \subset \mathcal{N}(H) \), and \( J \subset H \) if and only if \( \mathcal{N}(J) \subset \mathcal{N}(H) \).

Proof. Proof of (i). First we show that \( \mathcal{N}(J) \) is a group if \( J \) is a group. We know that \( \mathcal{N}(J) = \cup_{g \in J} \mathcal{N}(g) \). Let \( g' \in \mathcal{N}(J) \). Then \( g' \in \mathcal{N}(g) \) for some \( g \). Then \( (g, g') \) is a trellis path segment of length 2. Similarly let \( h' \in \mathcal{N}(J) \).

Then for some \( h \), there is a trellis path segment \( (h, h') \) of length 2. We know \( (g, g') * (h, h') = (gh, gh') \) is a trellis path segment of length 2. But then \( gh' \in \mathcal{N}(gh) \), where \( gh \in J \). This means \( g'h' \in \mathcal{N}(J) \) and shows \( \mathcal{N}(J) \) is a group.

Now assume \( J \) consists of cosets of \( X_j \cap Y_0 \). We show \( J \) is a group if \( \mathcal{N}(J) \) is a group. If \( \mathcal{N}(J) \) is a group, then the left states \( (\mathcal{N}(J))^- \) is a group. Since \( J^+ = (\mathcal{N}(J))^\rightarrow \), the right states of \( J \) are a group. Then if \( J \) consists of cosets of \( X_j \cap Y_0 \), \( J \) is a group.

Proof of (ii). Assume \( J, H \) are groups and \( J \subset H \). We show \( \mathcal{N}(J) \subset \mathcal{N}(H) \). Let \( h' \in \mathcal{N}(H) \) and \( g' \in \mathcal{N}(J) \). We show \( h' g'(h')^{-1} \in \mathcal{N}(J) \). We can find \( h \in H \) and \( g \in J \) such that \( (h, h') \) and \( (g, g') \) are trellis path segments of length 2. But

\[
(h, h') * (g, g') * (h, h')^{-1} = (gh, h'g'(h')^{-1})
\]
is a trellis path segment of length 2. But \( hgh^{-1} \in J \). Therefore \( h'g'(h')^{-1} \in \mathcal{N}(J) \).

Proof of (iii). Assume sets \( J, H \) consist of cosets of \( X_j \cap Y_0 \). We show \( J \subset H \) if \( \mathcal{N}(J) \subset \mathcal{N}(J) \) \( J \subset \mathcal{N}(H) \). If \( \mathcal{N}(J) \subset \mathcal{N}(H) \), then \( \mathcal{N}(J)^\rightarrow \subset \mathcal{N}(H)^\rightarrow \). This means \( J^+ \subset H^+ \). But if sets \( J, H \) consist of cosets of \( X_j \cap Y_0 \), we must have \( J \subset H \).

Proof of (iv). If \( \mathcal{N}(J), \mathcal{N}(H) \) are groups, then \( \mathcal{N}(J)^\rightarrow, \mathcal{N}(H)^\rightarrow \) are groups, and \( J^+ \subset H^+ \) are groups. But if \( J, H \) consist of cosets of \( X_j \cap Y_0 \), this means \( J, H \) are groups.

Assume \( \mathcal{N}(J) \subset \mathcal{N}(H) \). We show \( J \subset H \). Let \( h \in H \) and \( g \in J \). We show \( hgh^{-1} \in J \). We can find \( h' \in \mathcal{N}(H) \) and \( g' \in \mathcal{N}(J) \) such that \( (h, h') \) and \( (g, g') \) are trellis path segments of length 2. But

\[
(h, h') * (g, g') * (h, h')^{-1} = (hgh, h'g'(h')^{-1})
\]
is a trellis path segment of length 2. But \( hgh^{-1} \in J \).

Proof of (v). This follows from (i), (ii), and (iv).
Proposition 16 If $G, H$ are subsets of $B$, then $\mathcal{N}(G) = \mathcal{N}(G)\mathcal{N}(H)$.

Proof. Let $(g, g')$ be a trellis path section of length 2, with $g \in G$, and similarly for $(h, h')$. Then $(g, g') \ast (h, h') = (gh, gh')$ is a trellis path section of length 2, with $gh \in GH$. Then $\mathcal{N}(gh) = \mathcal{N}(g)\mathcal{N}(h)$, so $\mathcal{N}(GH) = \mathcal{N}(G)\mathcal{N}(H)$.

This result means $\mathcal{N}(Gb) = \mathcal{N}(G)\mathcal{N}(b)$ and $\mathcal{N}(bg) = \mathcal{N}(b)\mathcal{N}(G)$ for right coset $Gb$ and left coset $bg$.

The following theorem is an analog of the correspondence theorem, Theorem \ref{correspondence_theorem} and Corollary \ref{correspondence_corollary} with the mapping $\mathcal{N}$ in place of the homomorphism $\nu$.

Theorem 17 Fix $j$ such that $0 \leq j < \ell$. Let $J$ be a subgroup of $X_j$. Then $\mathcal{N}(J)$ is a group, a subgroup of $X_{j+1}$. Let $Jx$ be a right coset of $J$ in $X_j$. Then $\mathcal{N}(Jx)$ is a right coset of $\mathcal{N}(J)$ in $X_{j+1}$, and all right cosets of $\mathcal{N}(J)$ in $X_{j+1}$ are of this form. Define a function $q : X_j/\mathcal{N}(J) \to X_{j+1}/\mathcal{N}(J)$ by the assignment $q : xJ \mapsto \mathcal{N}(xJx)$. In general, the assignment $q$ gives a many to one correspondence between all the right cosets of $X_j/\mathcal{N}(J)$ and all the right cosets of $X_{j+1}/\mathcal{N}(J)$. In case $X_j \cap Y_0 \subseteq J$, this becomes a one to one correspondence. In case $J \subseteq X_j \cap Y_0$, $q$ induces a one to one correspondence between all the subgroups of $X_j$ that contain $X_j \cap Y_0$ and all the subgroups of $X_{j+1}$ that contain $X_0$. If $H$ is a subgroup of $X_j$ that contains $X_j \cap Y_0$, the corresponding subgroup of $X_{j+1}/\mathcal{N}(J)$ is $\mathcal{N}(H)/\mathcal{N}(J)$.

Assume now that $J$ and $H$ are subgroups of $X_j$.

(i) If $J \subseteq H$ then $\mathcal{N}(J) \subseteq \mathcal{N}(H)$, and then $q|_H : H/\mathcal{N}(J) \to \mathcal{N}(H)/\mathcal{N}(J)$ is an H-map ($q|_H$ is the mapping $q$ restricted to $H$). If in addition $X_j \cap Y_0 \subseteq J$, then $q|_H$ is an H-isomorphism.

(ii) Assume that $X_j \cap Y_0 \subseteq J$ and $X_j \cap Y_0 \subseteq H$. We have $J \subseteq H$ if and only if $\mathcal{N}(J) \subseteq \mathcal{N}(H)$, and then $q|_H$ is an H-isomorphism.

(iii) If $J \subseteq H$ then $\mathcal{N}(J) \subseteq \mathcal{N}(H)$, and then $q|_H : H/\mathcal{N}(J) \to \mathcal{N}(H)/\mathcal{N}(J)$ is a homomorphism. If in addition $X_j \cap Y_0 \subseteq J$, then $q|_H$ is an isomorphism.

(iv) Assume that $X_j \cap Y_0 \subseteq J$ and $X_j \cap Y_0 \subseteq H$. We have $J \subseteq H$ if and only if $\mathcal{N}(J) \subseteq \mathcal{N}(H)$, and then $q|_H$ is an isomorphism.

Proof. We know that $\mathcal{N}(J) = \bigcup_{g \in J} N(g)$. For each $g \in J$, let $(g, g')$ be a trellis path segment of length 2. Then $\mathcal{N}(J) = \bigcup_{g \in J} N(g) = \bigcup_{g \in J} X_0 g'$. We show that any right coset of $\mathcal{N}(J)$ in $X_{j+1}$ is of the form $\mathcal{N}(Jx)$ where $Jx$ is a right coset of $J$ in $X_j$. Let $(b, b')$ be a trellis path segment of length 2 in $P_{\ell,j+1}$. Then $(\mathcal{N}(J))b'$ is a right coset of $\mathcal{N}(J)$. But we have

\[
(\mathcal{N}(J))b' = (\bigcup_{g \in J} X_0 g')b' = \bigcup_{g \in J} X_0 gb' = \bigcup_{g \in J} \mathcal{N}(gb) = \mathcal{N}(Jb),
\]

where the second from last equality follows since $\mathcal{N}(gb) = X_0 gb'$. This result means any right coset of $\mathcal{N}(J)$ is of the form $\mathcal{N}(Jx)$, where $Jx$ is a right coset of $J$. This means the function $q$ is well defined.

Proof of (i). Assume that $J \subseteq H$. Define a function $f : H/\mathcal{N}(J) \to \mathcal{N}(H)/\mathcal{N}(J)$ by the assignment $Jh \mapsto \mathcal{N}(Jh)$. Let $g, h \in H$. $H$ acts on $H/\mathcal{N}(J)$ by the assignment $Jhg \mapsto J(hg)$. Let $g, h \in H$. $H$ acts on $\mathcal{N}(J)/\mathcal{N}(J)$ by the assignment $\mathcal{N}(Jh)g \mapsto \mathcal{N}(Jhg)$.

We first show that $f([Jh]g) = [f(Jh)]g$ for $Jh \in H/\mathcal{N}(J)$ and $g \in H$. We have

\[
f([Jh]g) = f(J(hg)) = \mathcal{N}(Jhg) = [\mathcal{N}(J)]g = [f(Jh)]g.
\]

Thus $f$ is an $H$-map. For $h \in H$, note that $q|_H : \mathcal{N}(Jh) \to \mathcal{N}(h)$ is an $H$-map.

Thus $q|_H$ and $f$ are identical, and so $q|_H$ is an $H$-map.

Assume $J \subseteq H$ and $X_j \cap Y_0 \subseteq J$. Then the assignment $f : Jh \mapsto \mathcal{N}(Jh)$ is a bijection. Then $f$ and $q|_H$ are $H$-isomorphisms.

Theorem 18 The chief series $\{X_j\}$ of an $\ell$-controllable group trellis $B$ has a refinement which is a chief series, given by

\[
\begin{align*}
1 = X_{-1} < X_{-1}^* < X_0 < X_0^* < X_1 < X_1^* < \cdots \\
\cdots < X_{j-1} < X_{j-1}^* < X_j < X_j^* < X_{j+1} < \cdots \\
\cdots < X_{\ell-1} < X_{\ell-1}^* < X_\ell = B.
\end{align*}
\]

where $X_{-1}^* = X_{-1}(X_j \cap Y_0)$ for $0 \leq j < \ell$, and each $X_{j-1}^* < \mathcal{N}(X_j)$ for all the right cosets $X_{j-1}^* h$ of $X_j$. This correspondence gives an isomorphism

\[
\frac{X_{\ell}}{X_{j-1}^*} \simeq \frac{\mathcal{N}(X_j)}{\mathcal{N}(X_{j-1}^*)} = \frac{X_{j+1}}{X_j},
\]

where $\mathcal{N}(X_j) = X_{j+1}$ and $\mathcal{N}(X_{j-1}^*) = X_j$.

Proof. For $0 \leq j < \ell$, note that $X_{j-1} \not< B$ and $X_j \cap Y_0 < B$. Therefore $X_{j-1}^* = X_{j-1}(X_j \cap Y_0) < B$.

Fix $j$ such that $0 \leq j < \ell$. We apply Theorem \ref{correspondence_theorem} with $J = X_{j-1}^*$ and $H = X_j$.

Define a mapping $q' : H/\mathcal{N}(J) \to \mathcal{N}(H)/\mathcal{N}(J)$ by the assignment $q' : Jh \mapsto \mathcal{N}(Jh)$. The quotient group $\mathcal{N}(H)/\mathcal{N}(J)$ is well defined since $\mathcal{N}(J) < \mathcal{N}(H)$ if $J < H$. Since $X_j \cap Y_0 < X_{j-1}^* = J$, from Theorem \ref{correspondence_theorem} the assignment $q'$ gives a one to one correspondence.
between all the cosets $X_{j-1}h$ of $X_j/X_{j-1}$ and all cosets $\mathcal{N}(X_{j-1}h) = X_j\mathcal{N}(h)$ of $X_{j+1}/X_j$.

We have

$$\mathcal{N}(J) = \mathcal{N}(X_{j-1}) = \mathcal{N}(X_j \cap Y_0) = X_j.$$ 

Then $q'$ is the mapping

$$q' : \frac{X_j}{X_{j-1}} \rightarrow \frac{X_{j+1}}{X_j}.$$

Since $X_j \cap Y_0 \subset X_{j-1} = J$, the condition in (iii) of Theorem 17 is met. Therefore $q'$ is an isomorphism giving $X_j/X_{j-1} \simeq X_{j+1}/X_j$. This is $\mathbb{S}$. $\bullet$

**Theorem 19** The Schreier matrix $J$ is a chief series of $B$ which is a refinement of the chief series in $\mathcal{N}(J)$. Fix $j$ such that $0 \leq j < \ell$. Let $J, H$ be groups such that $X_{j-1} \leq J \leq H \leq X_j$. Define function $\psi : H/J \rightarrow \mathcal{N}(H)/\mathcal{N}(J)$ by the assignment $\psi : Jh \mapsto \mathcal{N}(Jh)$. The assignment $\psi$ gives a one to one correspondence between all the right cosets $Jh$ of $H/J$ and all the right cosets $\mathcal{N}(Jh)$ of $\mathcal{N}(H)/\mathcal{N}(J)$. Moreover $\psi$ is an $H$-isomorphism giving

$$H/J \simeq \mathcal{N}(H)/\mathcal{N}(J).$$

By (v) of Proposition 12 we have $J < H$ if and only if $\mathcal{N}(J) < \mathcal{N}(H)$. Thus if $J < H$ or $\mathcal{N}(J) < \mathcal{N}(H)$, then $\psi$ is an isomorphism giving $H/J \simeq \mathcal{N}(H)/\mathcal{N}(J)$. In particular if $H = X_{j-1}(X_j \cap Y_k)$ and $J = X_{j-1}(X_j \cap Y_{k-m})$, where $k \geq 1$ such that $j + k \leq \ell$ and $m \geq 1$ such that $k - m \geq 0$, then $\psi$ gives an isomorphism

$$\frac{X_{j-1}(X_j \cap Y_k)}{X_{j-1}(X_j \cap Y_{k-m})} \simeq \frac{\mathcal{N}(X_{j-1}(X_j \cap Y_k))}{\mathcal{N}(X_{j-1}(X_j \cap Y_{k-m}))} = \frac{X_j(X_{j+1} \cap Y_{k-m})}{X_j(X_{j+1} \cap Y_{k-m-1})}. \quad (9)$$

This is an isomorphism between two adjacent columns of the Schreier matrix, for numerator (denominator) terms in the same row.

**Proof.** The equality in (9) follows from Proposition 11 and Proposition 11. $\bullet$

For a set $U \subset B$, define $U \Join \mathcal{N}(U)$ to be the subset of $U \times \mathcal{N}(U)$ consisting of all possible trellis path segments $(b, b')$ of length 2 that start with branches in $U$. In other words, $(b, b') \in U \Join \mathcal{N}(U)$ if and only if if $b \in U$, $b' \in \mathcal{N}(U)$, and $(b')^{-1} = b^+$. Define $X_j \Join \mathcal{N}(X_j) = X_j \Join X_{j+1}$ to be $P_{[j,j+1]}$.

**Theorem 20** Fix $j$ such that $0 \leq j < \ell$. Let $J$ be a subgroup of $X_j$. Then $J \Join \mathcal{N}(J)$ is a group, a subgroup of $X_j \Join X_{j+1} = P_{[j,j+1]}$. Let $Jx$ be a right coset of $J$ in $X_j$. Then $Jx \Join \mathcal{N}(Jx)$ is a right coset of $J \Join \mathcal{N}(J)$ in $P_{[j,j+1]}$, and all right cosets of $J \Join \mathcal{N}(J)$ in $P_{[j,j+1]}$ are of this form. Define a function $q : X_j/J \rightarrow P_{[j,j+1]}/(J \Join \mathcal{N}(J))$ by the assignment $q : Jx \mapsto Jx \Join \mathcal{N}(Jx)$. The assignment $q$ gives a one to one correspondence between all the right cosets of $X_j/J$ and all the right cosets of $P_{[j,j+1]}/(J \Join \mathcal{N}(J))$. In case $J = X_j \cap Y_0$, $q$ induces a one to one correspondence between all the subgroups of $X_j$ that contain $X_j \cap Y_0$ and all the subgroups of $P_{[j,j+1]}$ that contain $(X_j \cap Y_0) \Join X_0$. If $H$ is a subgroup of $X_j$ that contains $X_j \cap Y_0$, the corresponding subgroup of $P_{[j,j+1]}/(X_j \cap Y_0) \Join X_0)$ is $q(H)/(X_j \cap Y_0) = H \Join \mathcal{N}(H)/(X_j \cap Y_0) \Join X_0$.

Assume now that $J$ and $H$ are subgroups of $X_j$.

(i) We have $J \leq H$ if and only if $J \Join \mathcal{N}(J) \leq H \Join \mathcal{N}(H)$, and then $q_{JH} : H/J \rightarrow H \Join \mathcal{N}(H)/(J \Join \mathcal{N}(J))$ is an $H$-isomorphism ($q_{JH}$ is the mapping $q$ restricted to $H$).

(ii) We have $J \leq H$ if and only if $J \Join \mathcal{N}(J) \leq H \Join \mathcal{N}(H)$, and then $q_H : H/J \rightarrow H \Join \mathcal{N}(H)/(J \Join \mathcal{N}(J))$ is an isomorphism.

**Proof.** First we show that $J \Join \mathcal{N}(J)$ is a group. It is clear that $J \Join \mathcal{N}(J) = \cup_{g \in J}(g \Join \mathcal{N}(g))$. Then any element of $J \Join \mathcal{N}(J)$ is of the form $(g, g')$ where $g \in J$, and $(g, g')$ is a trellis path segment of length 2. Let $(e, e')$ and $(s, s')$ be two elements of $J \Join \mathcal{N}(J)$. Then $(e, e') \cdot (s, s') = (es, e's')$. But $es \in J$, and $e's' \in \mathcal{N}(es)$. Then $(es, e's') \in \cup_{g \in J}(g \Join \mathcal{N}(g)) = J \Join \mathcal{N}(J)$, and so $J \Join \mathcal{N}(J)$ is a group.

For each $g \in J$, let $(g, g')$ be a trellis path segment of length 2. Then $J \Join \mathcal{N}(J) = \cup_{g \in J}(g \Join \mathcal{N}(g)) = \cup_{g \in J}(g \Join X_{0g})$.

We show that any right coset of $J \Join \mathcal{N}(J)$ in $P_{[j,j+1]}$ is of the form $Jx \Join \mathcal{N}(Jx)$ where $Jx$ is a right coset of $J$ in $X_j$. Let $(b, b')$ be a trellis path segment of length 2 in $P_{[j,j+1]}$. Then $(J \Join \mathcal{N}(J)) \ast (b, b')$ is a right coset of $J \Join \mathcal{N}(J)$. But we have

$$(J \Join \mathcal{N}(J)) \ast (b, b') = (\cup_{g \in J}(g \Join X_{0g})) \ast (b, b') = \cup_{g \in J}(gb \Join X_{0g}b') = \cup_{g \in J}(gb \Join \mathcal{N}(gb)).$$

where the last equality follows since $\mathcal{N}(gb) = X_{0g'}$. We show

$$\cup_{g \in J}(gb \Join \mathcal{N}(gb)) = Jb \Join \mathcal{N}(Jb).$$

First we show LHS $\subset$ RHS. Fix $g \in J$. Then $gb \Join \mathcal{N}(gb) \in Jb \Join \mathcal{N}(Jb)$. Now we show RHS $\subset$ LHS. Any element of $Jb \Join \mathcal{N}(Jb)$ is of the form $(gb, r')$ for some $g \in J$ and $r' \in \mathcal{N}(gb)$. But then $(gb, r') \in gb \Join \mathcal{N}(gb) \subset \cup_{g \in J}(gb \Join \mathcal{N}(gb))$.

Combining the above results gives

$$(J \Join \mathcal{N}(J)) \ast (b, b') = Jb \Join \mathcal{N}(Jb).$$

This result means any right coset of $J \Join \mathcal{N}(J)$ is of the form $Jx \Join \mathcal{N}(Jx)$, where $Jx$ is a right coset of $J$. This means the function $q$ is well defined. Further it is easy to see the assignment $Jx \mapsto Jx \Join \mathcal{N}(Jx)$ gives a one to one correspondence between all the right cosets of $X_j/J$ and all the right cosets of $P_{[j,j+1]}/(J \Join \mathcal{N}(J))$. 

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Proof of (i). Let \( J, H \) be subgroups of \( X_j \). Clearly \( H \ltimes N(H) \) is a group in the same way as \( J \ltimes N(J) \) is a group. Assume that \( J \leq H \). Then clearly \( J \leq N(J) \leq H \ltimes N(H) \). Conversely if \( J \ltimes N(J) \leq H \ltimes N(H) \), then we must have \( J \leq H \).

Assume that \( J \leq H \). Let \( h \in H \). Define a function \( f : H//J \to H \ltimes N(H)//(J \ltimes N(J)) \) by the assignment \( Jh \mapsto Jh \ltimes N(J) \). Let \( g, h \in H \). \( H \) acts on \( H//J \) by the assignment \( [Jh]g \mapsto J(hg) \). Let \( g, h \in H \). \( H \) acts on \( H \ltimes N(H)//(J \ltimes N(J)) \) by the assignment \( [Jh \ltimes N(J)]g \mapsto J(hg) \ltimes N(J(hg)) \).

We show that \( f([Jh]g) = [f(Jh)]g \) for \( Jh \in H//J \) and \( g \in H \):

\[
\begin{align*}
f([Jh]g) &= f(J(hg)) = Jhg \ltimes N(J(hg)) \\
&= [Jh \ltimes N(J)]g = [f(Jh)]g.
\end{align*}
\]

Thus \( f \) is an \( H \)-map. It is clear from the assignment \( Jh \mapsto Jh \ltimes N(J) \) that \( f \) is a bijection. Thus \( f \) is an \( H \)-isomorphism. For \( h \in H \), note that

\[
q_H(Jh) = Jh \ltimes N(J) = f(Jh).
\]

Thus \( q_H \) and \( f \) are identical, and so \( q_H \) is an \( H \)-isomorphism.

Proof of (ii). Assume \( J \leq H \). Let \( (g, g') \in J \ltimes N(J) \) and \( (h, h') \in H \ltimes N(H) \). Then

\[
(h, h') * (g, g') * (h, h')^{-1} = (hgh^{-1}, h'g(h')^{-1}).
\]

But \( hgh^{-1} \in J \), and therefore \( (hgh^{-1}, h'g(h')^{-1}) \) is an element of \( J \ltimes N(J) \). Then \( J \ltimes N(J) \leq H \ltimes N(H) \). Conversely if \( J \ltimes N(J) \leq H \ltimes N(H) \), it is easy to see \( J \leq H \).

**Lemma 21.** Let \( J, H \) be groups such that \( J \leq H \leq X_j \). Define function \( \psi : H//J \to H \ltimes N(H)//(J \ltimes N(J)) \) by the assignment \( \psi : Jh \mapsto Jh \ltimes N(J) \). Then \( \psi \) is a one to one correspondence between all the right cosets \( Jh \) of \( H//J \) and all the right cosets \( Jh \ltimes N(J) \) of \( H \ltimes N(H)//(J \ltimes N(J)) \). The function \( \psi \) induces a mapping \( \psi' : H//J \to N(H)//N(J) \) between all the right cosets \( Jh \) of \( H//J \) and all the right cosets \( N(J) \) of \( N(H)//N(J) \). From Theorem 7 we in general the mapping \( \psi' \) is many to one. But if \( X_j \cap Y_0 \subset J \), then \( \psi' \) is one to one, and in this case, both \( \psi \) and \( \psi' \) are one to one.

We can use Lemma 21 in the following way. Let \( J, H \) be groups such that \( J \leq H \leq X_j \). We give a procedure to find right transversals of \( H//J \times N(H)//N(J) \), and \( H \ltimes N(H)//(J \ltimes N(J)) \). Let \( Jh \) be a right coset of \( J \) in \( H \). Let \( b \) be a coset representative of \( Jh \). Let \( b' \) be any branch in \( N(b) \). Then \( b' \) is a coset representative of \( N(Jh) \), a right coset of \( N(J) \) in \( N(H) \). In addition, \( (b, b') \) is a trellis path segment of length 2, and \( (b, b') \) is a coset representative of \( Jh \ltimes N(Jh) \), a right coset of \( J \ltimes N(J) \) in \( H \ltimes N(H) \).

In general the mapping \( \psi' \) in Lemma 22 is many to one. Then there exists coset \( Jh^* \) such that \( N(Jh^*) = N(Jh) \). Let \( g \) be a coset representative of \( Jh^* \). Let \( g' \) be any branch in \( N(g) \). Then \( g' \) is a coset representative of \( N(Jh^*) = N(Jh) \). And \( (g, g') \) is a coset representative of \( Jh^* \ltimes N(Jh^*) = Jh^* \ltimes N(Jh) \). Then this procedure finds two coset representatives \( b', g' \) of \( N(Jh) \). Note however that if \( X_j \cap Y_0 \subset J \), the map \( \psi' \) is one to one and the procedure obtains just one coset representative of \( N(Jh) \).

This gives the following result.

**Theorem 22.** Let \( J, H \) be groups such that \( J \leq H \leq X_j \). Let \( T \) be a right transversal of \( H//J \). For each \( b \in T \), pick one and only one element \( b' \in N(b) \). Then \( \{b, b' \} \mid b \in T \) is a collection of trellis path segments of length 2 that form a right transversal of \( H \ltimes N(H)//(J \ltimes N(J)) \). Let \( T' = \cup_{b \in T} \{b, b' \} \in N(b) \) \( \). In general \( T' \) contains more than one coset representative from each coset of \( N(H)//N(J) \). We can always parse \( T' \) so that only one element is taken from each right coset of \( N(H)//N(J) \), such that the parsed \( T' \) is a right transversal of \( N(H)//N(J) \). If \( X_j \cap Y_0 \subset J \), then \( T' \) is a right transversal of \( N(H)//N(J) \) without parsing.

**Remark.** Note that we can find transversals in a more judicious way. By going backwards in time, first finding a coset representative \( b' \) of \( N(Jh^*) = N(Jh) \), and then coset representatives \( b, b' \) of \( Jh^* \), respectively, such that \( (b, b') \) and \( (g, b') \) are trellis path segments of length 2, we can always construct a transversal \( T' \) that has only one representative in each coset, such that \( T' \) is a transversal without parsing.

### 4.3 A sequence of pletty sections

Line (1) of Theorem 19 shows that if we take successive elements in two different rows of \( \mathcal{U} \) and form quotient groups using elements in the same column (e.g., the four elements form a rectangle), the quotient groups are isomorphic. This can be regarded as a correspondence theorem for two group pletty sections. We now use the preceding results to give a more general version of the correspondence theorem for a sequence of pletty sections.

For a set \( U \subset B \) and integer \( i > 0 \), define \( N^0(U) \) to be the \( i \)-fold composition \( N^i(U) = N \circ N \circ \cdots \circ N(U) \). For \( i = 0 \), define \( N^0(U) = N(U) \) to be just \( U \).

For a set \( U \subset B \), define \( U \ltimes N^i(U) \) to be \( U \ltimes N^2(U) \) to be \( (U \ltimes N^2(U)) \ltimes N^3(U) \), the subset of \( (U \ltimes N^2(U)) \ltimes N^3(U) \) consisting of all possible trellis path segments of length 3 that start with trellis path segments of length 2 in \( U \ltimes N^2(U) \). This is just the set of all possible trellis path segments of length 3 that start with branches in \( U \).

For a set \( U \subset B \), in a similar way define \( U \ltimes N^i(U) \) consisting of all possible trellis path segments of length \( i + 1 \) that start with branches in \( U \).

The next result follows from Proposition 8.
Proposition 23  Fix integer $k$, $0 \leq k \leq \ell$. For $0 \leq j \leq k$, we have
\[(X_j \cap Y_{k-j})^+ = (X_{j+1} \cap Y_{k-j-1})^- .\] (10)

This result means the sequence of groups
\[
\ldots, 1, X_0 \cap Y_k, \ldots, X_j \cap Y_{k-j}, X_{j+1} \cap Y_{k-j-1}, \ldots, X_k \cap Y_0, 1, \ldots
\]
consists of paths which split from the identity state at time epoch 0 and merge to the identity state at time epoch $k+1$.

We choose $H_0$ such that $H_0 \leq X_1(X_0 \cap Y_\ell)$ and $J_0$ such that $J_0 \geq X_1(X_0 \cap Y_0)$. Thus fix integer $k$, $0 < k < \ell$. Choose integer $m \geq 1$ such that $k - m > -1$. Define
\[
H_0 = X_1(X_0 \cap Y_k), \\
J_0 = X_1(X_0 \cap Y_{k-m}).
\]

Choose integer $l$ such that $l > 0$ and $l \leq k - m + 1$. For $0 < j \leq l$, recursively define
\[
H_j = N(H_{j-1}), \\
J_j = N(J_{j-1}).
\]
Then using Proposition 23 and Proposition 14
\[
H_j = X_{j-1}(X_j \cap Y_{k-j}), \\
J_j = X_{j-1}(X_j \cap Y_{k-j-m}).
\]

For $0 \leq j \leq l$, the groups $H_j$ and $J_j$ correspond to rows in 23 or horizontal lines in Figure 2. Note that for $l = k - m + 1$, $J_l = X_{l-1}(1) = X_{k-m}(1)$ is the group on the diagonal of 23 or the diagonal of Figure 2.

For $0 \leq j \leq l$, $H_j$ and $J_j$ are subgroups of $X_j$. For $0 < j \leq l$, $H_j$ and $J_j$ contain $X_0$. For $0 \leq j \leq l$, $H_j$ contains $X_{j+1} \cap Y_0$. For $0 < l < k - m + 1$ and $0 \leq j < l$, $J_j$ contains $X_j \cap Y_0$. For $l = k - m + 1$ and $0 \leq j < l$, $J_j$ contains $X_j \cap Y_0$.

Define
\[
P_{[0,l]} = X_0 \ltimes N(X_0) \ltimes N^2(X_0) \ltimes \cdots \ltimes N^l(X_0), \\
H_{[0,l]} = H_0 \ltimes N(H_0) \ltimes N^2(H_0) \ltimes \cdots \ltimes N^l(H_0),
\]
and
\[
J_{[0,l]} = J_0 \ltimes N(J_0) \ltimes N^2(J_0) \ltimes \cdots \ltimes N^l(J_0).
\]
These are just
\[
P_{[0,l]} = X_0 \ltimes X_1 \ltimes X_2 \ltimes \cdots \ltimes X_l, \\
H_{[0,l]} = H_0 \ltimes H_1 \ltimes H_2 \ltimes \cdots \ltimes H_l,
\]
and
\[
J_{[0,l]} = J_0 \ltimes J_1 \ltimes J_2 \ltimes \cdots \ltimes J_l.
\]

In Theorem 24 we give an extension of the correspondence theorem to a controllable Schreier matrix. This result can be regarded as a rectangle criterion for a controllable Schreier matrix, with $H_0$, $J_0$, $H_1$, and $J_1$ as the corners of a rectangle, as shown in Figure 2. It is similar in spirit to a quadrangle criterion for a Latin square and a configuration theorem for a net.

Theorem 24  We have $P_{[0,l]}, H_{[0,l]}$, and $J_{[0,l]}$ are groups with $H_{[0,l]} < H_{[0,l]} \ltimes J_{[0,l]}$.

Let $J_0 h_0$ be a right coset of $J_0$ in $H_0$. Then
\[
J_0 h_0 \ltimes N(J_0 h_0) \ltimes N^2(J_0 h_0) \ltimes \cdots \ltimes N^l(J_0 h_0)
\]
is a right coset of $J_{[0,l]}$ in $H_{[0,l]}$. And for $j$ such that $0 \leq j \leq l$, $N^j(J_0 h_0)$ is a right coset of $J_j$ in $H_j$. There are 4 results:

(i) The function $f : H_0/J_0 \to H_{[0,l]}/J_{[0,l]}$ defined by the assignment
\[
f : J_0 h_0 \mapsto J_0 h_0 \ltimes N(J_0 h_0) \ltimes N^2(J_0 h_0) \ltimes \cdots \ltimes N^l(J_0 h_0)
\]
gives a one to one correspondence between all the right cosets of $H_0/J_0$ and all the right cosets of $H_{[0,l]}/J_{[0,l]}$.

We have $J_0 \leq H_0$ if and only if $J_{[0,l]} \leq H_{[0,l]}$, and then $f$ is an $H_0$-isomorphism giving
\[
H_0/J_0 \simeq H_{[0,l]}/J_{[0,l]}.
\]

We have $J_0 \ltimes H_0$ if and only if $J_{[0,l]} \ltimes H_{[0,l]}$, and then $f$ is an isomorphism giving
\[
H_0/J_0 \simeq H_{[0,l]}/J_{[0,l]}.
\]

(ii) Provided $l > 0$, fix $j$ such that $0 < j \leq l$. The function $f_{0,j} : H_0/J_0 \to H_j/J_j$ defined by the assignment
\[
f_{0,j} : J_0 h_0 \mapsto N^j(J_0 h_0)
\]
gives a one to one correspondence between all the right cosets of $H_0/J_0$ and all the right cosets of $H_j/J_j$.

We have $J_0 \leq H_0$ if and only if $J_j \leq H_j$, and then $f_{0,j}$ is an $H_0$-isomorphism giving
\[
H_0/J_0 \simeq H_j/J_j.
\]

We have $J_0 \ltimes H_0$ if and only if $J_j \ltimes H_j$, and then $f_{0,j}$ is an isomorphism giving
\[
H_0/J_0 \simeq H_j/J_j.
\]

(iii) Provided $l > 0$, fix $j, k$ such that $0 \leq j < k \leq l$. The function $f_{j,k} : H_j/J_j \to H_k/J_k$ defined by the assignment
\[
f_{j,k} : N^j(J_0 h_0) \mapsto N^k(J_0 h_0)
\]
gives a one to one correspondence between all the right cosets of $H_j/J_j$ and all the right cosets of $H_k/J_k$.

We have $J_j \leq H_j$ if and only if $J_k \leq H_k$, and then $f_{j,k}$ is an $H_j$-isomorphism giving
\[
H_j/J_j \simeq H_k/J_k.
\]
We have $J_j \triangleleft H_j$ if and only if $J_k \triangleleft H_k$, and then $f_{j,k}$ is an isomorphism giving

$$H_j/J_j \cong H_k/J_k.$$  

(iv) Fix $j$ such that $0 \leq j \leq l$. The function $f_j^*: H_{[0,j]}/J_{[0,j]} \to H_j/J_j$ defined by the assignment

$$f_j^*: J_{0j} \ncong N(J_{0j}) \ncong N^2(J_{0j}) \ncong \cdots \ncong N^l(J_{0j})$$



\(\mapsto N^2(J_{0j})\)

gives a one to one correspondence between all the right cosets of $H_{[0,j]}/J_{[0,j]}$ and all the right cosets of $H_j/J_j$. We have $J_j \leq H_j$ if and only if $J_{[0,j]} \n H_{[0,j]}$, and then $f_j^*$ is an $H_{[0,j]}$-isomorphism giving

$$H_{[0,j]}/J_{[0,j]} \cong H_j/J_j.$$  

We have $J_j \triangleleft H_j$ if and only if $J_{[0,j]} \triangleleft H_{[0,j]}$, and then $f_j^*$ is an isomorphism giving

$$H_{[0,j]}/J_{[0,j]} \cong H_j/J_j.$$  

Proof. First we show that $J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)$ is a group. It is clear that

$$J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0) = \cup_{g \in J_0} (g \ncong N(g) \ncong \cdots \ncong N^l(g)).$$  

Then any element of $J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)$ is of the form $(g, g', \ldots, g^n)$ where $g \in J_0$ and $(g', g'', \ldots, g^n)$ is a treillis path segment of length $l + 1$. Let $(e, e', \ldots, e^n)$ and $(s, s', \ldots, s^n)$ be two elements of $J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)$. Then $(e, e', \ldots, e^n) \ast (s, s', \ldots, s^n) = (es, e's', \ldots, e^n s^n)$. But $es \in J_0$, $e's' \in N(es)$, and $e^n s^n \in N^l(es)$. Then $(es', e's', \ldots, e^n s^n) \in \cup_{g \in J_0} (g \ncong N(g) \ncong \cdots \ncong N^l(g))$, so $J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)$ is a group.  

For each $g \in J_0$, let $(g, g', \ldots, g^n)$ be a treillis path segment of length $l + 1$. Then

$$J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0) = \cup_{g \in J_0} (g \ncong N(g) \ncong \cdots \ncong N^l(g)) = \cup_{g \in J_0} (g \ncong X_0 g' \ncong \cdots \ncong X_{l-1} g^n).$$  

We show that any right coset of $J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)$ in $P_{[0,l]}$ is of the form $J_{0x} \ncong N(J_{0x}) \ncong \cdots \ncong N^l(J_{0x})$ where $J_{0x}$ is a right coset of $J_0$ in $X_0$. Let $(b, b', \ldots, b^n)$ be a treillis path segment of length $l + 1$ in $P_{[0,l]}$. Then $(J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)) \ast (b, b', \ldots, b^n)$ is a right coset of $J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)$. But we have

$$(J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)) \ast (b, b', \ldots, b^n) = (\cup_{g \in J_0} (g \ncong X_0 g' \ncong \cdots \ncong X_{l-1} g^n)) \ast (b, b', \ldots, b^n) = \cup_{g \in J_0} (g \ncong X_0 g' \ncong \cdots \ncong X_{l-1} g^n) \ast (b, b', \ldots, b^n) = \cup_{g \in J_0} (gb \ncong X_0 g' b' \ncong \cdots \ncong X_{l-1} g^n b^n) = \cup_{g \in J_0} (gb \ncong N(gb) \ncong \cdots \ncong N^l(gb)).$$  

where the last equality follows since $N(gb) = X_0 g' b'$ and $N^l(gb) = X_{l-1} g^n b^n$.  

We now show

$$\cup_{g \in J_0} \ncong N(gb) \ncong \cdots \ncong N^l(gb) = J_0 \ncong N(J_0b) \ncong \cdots \ncong N^l(J_0b).$$  

First we show $LHS \subset RHS$. Fix $g \in J_0$. Then $gb \ncong N(gb) \ncong \cdots \ncong N^l(gb) \in J_0b \ncong N(J_0b) \ncong \cdots \ncong N^l(J_0b)$. Now we show $RHS \subset LHS$. Any element of $J_0 \ncong N(J_0b) \ncong \cdots \ncong N^l(J_0b)$ is of the form $(gb, r', \ldots, r''')$ for some $g \in J_0$, $r' \in N(gb)$, and $r'' \in N^l(gb)$. But then $(gb, r', \ldots, r''') \in \cup_{g \in J_0} (gb \ncong N(gb) \ncong \cdots \ncong N^l(gb)) \subset \cup_{g \in J_0} \ncong N(gb) \ncong \cdots \ncong N^l(gb))$. Combining the above results gives

$$(J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)) \ast (b, b', \ldots, b^n) = J_0b \ncong N(J_0b) \ncong \cdots \ncong N^l(J_0b).$$  

This result means any right coset of $J_0 \ncong N(J_0) \ncong \cdots \ncong N^l(J_0)$ is of the form $J_0x \ncong N(J_0 x) \ncong \cdots \ncong N^l(J_0 x)$, where $J_0 x$ is a right coset of $J_0$. This means the function $f$ is well defined. Further it is easy to see the function $f$ gives a one to one correspondence between all the right cosets of $H_0/J_0$ and all the right cosets of $H_{[0,l]}/J_{[0,l]}$. This proves the first part of (i). The proof of the remainder of (i) is similar to the proof of (i) and (ii) of Theorem 20.  

Proof of (ii). We proceed by induction on $n$, $n = 1, \ldots, l$. Consider the following hypothesis (*):

(*) The function $f_{0,n}: H_0/J_0 \to H_n/J_n$ defined by the assignment

$$f_{0,n}: J_0h_0 \mapsto N^n(J_0h_0)$$  

gives a one to one correspondence between all the right cosets of $H_0/J_0$ and all the right cosets of $H_{[0,n]}/J_{[0,n]}$. We have $J_0 \leq H_0$ if and only if $J_n \leq H_n$, and then $f_{0,n}$ is an $H_0$-isomorphism giving

$$H_0/J_0 \cong H_n/J_n.$$  

We have $J_0 \leq H_0$ if and only if $J_n \leq H_n$, and then $f_{0,n}$ is an isomorphism giving

$$H_0/J_0 \cong H_n/J_n.$$  

We know that hypothesis (*) is true for $n = 1$, by Theorem 17. Thus assume hypothesis (*) holds for $n$, $1 \leq n \leq l$; we show it holds for $n + 1$.  

Define a function $q: H_n/J_n \to N^n(J_0h_0)/N^l(J_0h_0)$ by the assignment $q: J_nx \mapsto N^n(J_0 x)$, for $x \in H_n$. By Theorem 17 $q$ is a one to one correspondence between all right cosets $J_nx$ of $H_n/J_n$, and all right cosets of $N^n(J_0h_0)/N^l(J_0h_0) = H_{n+1}/J_{n+1}$. By hypothesis (*) all right cosets of $H_n/J_n$ are of the form $N^n(J_0h_0)$, $h_0 \in H_n$. Thus there is a one to one correspondence between all right cosets written as $J_n x$ and all right cosets written as $N^n(J_0h_0)$. Then $q$ gives the assignment

$$q: N^n(J_0h_0) \mapsto N^n(J_0h_0) = N^{n+1}(J_0h_0).$$
Thus all right cosets of $H_{n+1}/J_{n+1}$ are of the form $N^{n+1}(J_0b_0)$. Define $f_{0,n+1} : H_0/J_0 \rightarrow H_{n+1}/J_{n+1}$ by $q \circ f_{0,n}$. Then $f_{0,n+1}$ gives the assignment $f_{0,n+1} : J_0b_0 \mapsto N^{n+1}(J_0b_0)$, which is a one to one correspondence between all the right cosets of $H_0/J_0$ and all right cosets of $H_{n+1}/J_{n+1}$.

By hypothesis we have $J_0 \leq H_0$ if and only if $J_n \leq H_n$, and then $f_{0,n}$ is an $H_0$-isomorphism. From Theorem 17 since $X_j \cap Y_0 \subset J_n$, we have $J_n \leq H_n$ if and only if $J_{n+1} \leq H_{n+1}$, and $q$ is an $H_0$-isomorphism. Then $J_0 \leq H_0$ if and only if $J_{n+1} \leq H_{n+1}$. Since $f_{0,n}$ is an $H_0$-isomorphism, and $q$ is an $H_0$-isomorphism, it can be shown $f_{0,n+1} = q \circ f_{0,n}$ is an $H_0$-isomorphism.

The remainder of the proof, that $J_0 < H_0$ if and only if $J_{n+1} < H_{n+1}$, and $f_{0,n+1}$ is an isomorphism, is similar.

Proof of (iii). The function $f_{0,j}$ in (ii) is a bijection and so the inverse $f^{-1}_{0,j} : H_j/J_j \rightarrow H_0/J_0$ exists. It is easy to show $f^{-1}_{0,j}$ is an $H_j$-isomorphism. Define the function $f_{j,k} : H_j/J_j \rightarrow H_k/J_k$ by $f_{j,k} = f_{0,k} \circ f^{-1}_{0,j}$. Then $f_{j,k}$ gives a one to one correspondence between all the right cosets of $H_j/J_j$ and all the right cosets of $H_k/J_k$, and $f_{j,k}$ is an $H_j$-isomorphism.

Proof of (iv). The function $f$ in (i) is a bijection and so the inverse $f^{-1} : H_{[0,l]}/J_{[0,l]} \rightarrow H_0/J_0$ exists, and $f^{-1}$ is an $H_{[0,l]}$-isomorphism. Define the function $f^*_j$ by $f^*_j = f_{0,j} \circ f^{-1}$. Then $f^*_j$ gives a one to one correspondence between right cosets in $H_{[0,l]}/J_{[0,l]}$ and $H_j/J_j$, and $f^*_j$ is an $H_{[0,l]}$-isomorphism.

Remark: The theorem holds for a more general rectangle, using $H_j'$ in place of $H_0$ and $J_j'$ in place of $J_0$, for $0 < l' < l$. This more general result is not needed.

We can diagram Theorem 23 as shown in Figure 3. This diagram shows some formal similarity with the trellis analog of the code granule theorem (cf. Theorem 8.1 of [1]).

---

**Theorem 25** Fix $l$ such that $0 < l < k - m + 1$. There are two results:

(i) We can select a right transversal of $H_{[0,l]}/J_{[0,l]}$ such that for $j = 0, 1, \ldots, l$, component $j$ forms a right transversal of $H_j/J_j$.

(ii) Assume that $J_0 < H_0$. Then $J_i < H_0$, $i > 0$, and $J_{[0,l]} < H_{[0,l]}$. We can select a transversal of $H_{[0,l]}/J_{[0,l]}$ such that for $j = 0, 1, \ldots, l$, component $j$ forms a transversal of $H_j/J_j$.

We now give an extension of Theorem 25. First we give a useful extension of Figure 3. Fix $j$, $0 \leq j < l$. From Theorem 12 we know there is a one to one correspondence $f_j : H_j/J_j \rightarrow (X_j \cap Y_{l-j})/D_j$, where

$$D_j = (X_j \cap Y_{l-j-m})(X_{j-1} \cap Y_{l-j})$$

Also $f_j$ is an $H_j$-isomorphism. Under the one to one correspondence, the elements of a right coset of $(X_j \cap Y_{l-j})/D_j$ are contained in a right coset of $H_j/J_j$.

Similarly there is a one to one correspondence $f_{j+1} : H_{j+1}/J_{j+1} \rightarrow (X_{j+1} \cap Y_{l-j-1})/D_{j+1}$, where

$$D_{j+1} = (X_{j+1} \cap Y_{l-j-m-1})(X_j \cap Y_{l-j-1})$$

Also $f_{j+1}$ is an $H_{j+1}$-isomorphism. Under the one to one correspondence, the elements of a right coset of $(X_{j+1} \cap Y_{l-j-1})/D_{j+1}$ are contained in a right coset of $H_{j+1}/J_{j+1}$. These relations are shown in Figure 4.

Now note further that

$$(X_j \cap Y_{l-j})^+ = (X_{j+1} \cap Y_{l-j-1})^-$$

and

$$D_j^+ = D_{j+1}^-.$$
For each the one to one correspondence $f$ isomorphism $\hat{H}$ right transversal of $H$ transversal of $H$.

$$f_{j,j+1} : (X_j \cap Y_{i-j})/D_j \rightarrow (X_{j+1} \cap Y_{i-j-1})/D_{j+1},$$

and $f_{j,j+1}$ is an $(X_j \cap Y_{i-j})$-isomorphism (see Figure 3).

Assume we have chosen a right transversal $T$ of $H_j/J_j$ using elements of $X_j \cap Y_{i-j}$ taken from right cosets of $(X_j \cap Y_{i-j})/D_j$; we can use function $\hat{f}_j$ to do this. For each $b \in T$, choose a $b' \in N(b)$ such that $b' \in X_{j+1} \cap Y_{i-j-1}$; by (13) this is always possible. Under the one to one correspondence $f_{j,j+1}$, we know the collection $T' = \cup_{b \in T}\{b' \mid b' \in N(b)\}$ is a right transversal of $(X_{j+1} \cap Y_{i-j-1})/D_{j+1}$. Further under the $H_{j+1}$-isomorphism $\hat{f}_{j+1}$, we know $T'$ is also a right transversal of $H_{j+1}/J_{j+1}$. Lastly under the $H_j$-isomorphism $\hat{f}_{j,j+1}$, we know $(b,b')|b \in T$ is a collection of trellis path segments of length 2 that form a right transversal of $H_{j,j+1}/J_{j,j+1}$.

Now applying Figure 3 repeatedly as we did with Figure 3 we can obtain the following extension of Theorem 26.

**Theorem 26** Fix $l$ such that $0 < l \leq k - m + 1$. There are two results:

(i) We can select a right transversal of $H_{[0,l]}/J_{[0,l]}$ such that for $j = 0, 1, \ldots, l$, component $j$ forms a right transversal of $H_{j}/J_{j}$. In particular we can choose the right transversal of $H_{[0,l]}/J_{[0,l]}$ so that component $j$ uses elements of $X_j \cap Y_{i-j}$ taken from right cosets of

$$(X_j \cap Y_{i-j})/D_j.$$  

(ii) Assume that $J_0 < H_0$. Then $J_i < H_i$, $i > 0$, and $J_{[0,l]} < H_{[0,l]}$. We can select a transversal of $H_{[0,l]}/J_{[0,l]}$ such that for $j = 0, 1, \ldots, l$, component $j$ forms a transversal of $H_{j}/J_{j}$. In particular we can choose the transversal of $H_{[0,l]}/J_{[0,l]}$ so that component $j$ uses elements of $X_j \cap Y_{i-j}$ taken from cosets of

$$(X_j \cap Y_{i-j})/D_j.$$  

5. **ENCODER FOR $\{X_j\}$ AND $\{Y_k\}$**

Fix $j$ and $i$ such that $j \geq 0$, $i \geq 0$, and $j + i \leq \ell$. From Theorem 12 we know that we can find coset representatives of

$$\frac{X_{j-1}(X_j \cap Y_j)}{X_{j-1}(X_j \cap Y_{i-j})}$$  

(13)

using elements of $X_j \cap Y_j$. Thus let $q_{j,i}$ be an element of $X_j \cap Y_j$ and $[q_{j,i}]$ be a transversal of (13). We can arrange the $[q_{j,i}]$ as shown below:

$$\begin{array}{cccccc}
[q_0,0] & [q_0,1] & [q_0,2] & \cdots & [q_{0,e},1] & [q_{0,e},0] \\
[q_0,1-t] & [q_0,1-t+1] & [q_0,1-t+2] & \cdots & [q_{0,e},t-1] & [q_{0,e},t] \\
[q_0,2] & [q_1,1] & [q_1,2] & \cdots & [q_{1,e},1] & [q_{1,e},0] \\
[q_0,1] & [q_1,0] & [q_2,0] & \cdots & [q_{2,e},0] & \cdots \\
[q_0,0] & [q_1,0] & [q_2,0] & \cdots & [q_{2,e},0] & \cdots \\
\end{array}$$  

(14)

Note that $[q_{0,0}]$ is just the elements of $X_0 \cap Y_0$, i.e., $[q_{0,0}] = X_0 \cap Y_0$. It can be seen the array (14) corresponds to the controllable Schreier matrix (1) and $[q_{j,i}]$ corresponds to terms in the controllable Schreier matrix.

The controllable Schreier matrix gives a chain coset decomposition of $B$, and the elements of (14) form a chain of coset representatives. Thus any element $b \in B$ can be written using elements of (14) as

$$b = q_0q_0q_0 \cdots q_0q_1q_1 \cdots q_1q_2q_2 \cdots \cdots q_{j,i-q}q_{j,i-q-1}q_{j,i}q_{j,i-1}q_{j,i-2}q_{j,i-3} \cdots q_{j,0} \cdots q_{j,i} \cdots q_{j,0} \cdots [q_{j,i}] \cdots [q_{j,0}] \cdots [q_{j,0}]$$  

(15)

where the terms in product $(q_{j,i} \cdots q_{j,0} \cdots q_{j,0})$ are coset representatives from the $j$-th column of (14), $0 \leq j \leq \ell$.

We now show how a similar result can be obtained using Theorem 26. Fix integer $k$, $0 < k \leq \ell$. Choose $m = 1$. Choose $l = k - m + 1$; then $l = k$. Then for $j$, $0 \leq j \leq k$, we have

$$H_j = X_{j-1}(X_j \cap Y_{i-j}),$$  

$$J_j = X_{j-1}(X_j \cap Y_{i-j-1}).$$

A transversal of $H_{[0,k]}/J_{[0,k]}$ consists of a trellis path segment of length $k + 1$:

$$r_{0,k}, r_{1,k-1}, \ldots, r_{j,k-j}, \ldots, r_{k,0}.$$  

(16)
By Theorem 26 we can select $r_{j,k-j}$ that is an element of $X_j \cap Y_{k-j}$ and a representative of $H_j/J_j$, 

$$X_j^{-1}(X_j \cap Y_{k-j}) \over X_j^{-1}(X_j \cap Y_{k-j-1}).$$  

(17)

Thus the transversals of $H_{[0,k]}/J_{[0,k]}$ give a set of transversals 

$$[r_{0,0},[r_{1,1},\ldots,[r_{j,j},\ldots,[r_{k,0}],0],$$

where $[r_{j,j-k}]$ is a transversal of $H_j/J_j$, for $j = 0, 1, \ldots, k$. Since this is true for each $k, 0 < k < \ell$, we can obtain the following matrix, similar to (14), which gives a chain of coset representatives of $B$: 

$$
\begin{bmatrix}
[r_{0,0}] & [r_{1,1}] & \cdots & [r_{j,j}] & \cdots & [r_{k,0}] \\
[r_{0,\ell-1}] & [r_{1,\ell-2}] & \cdots & [r_{j-1,1}] & \cdots & [r_{\ell,0}] \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
[r_{0,1}] & [r_{1,0}] \\
[r_{0,0}] \\
\end{bmatrix}
$$

(18)

We define $[r_{0,0}]$ to be $X_0 \cap Y_0$. Comparing (18) and (14), we see that representatives in $[r_{j,j-k}]$ in (18) are in the same quotient group as representatives in $[q_{j,i}]$ in (14) when $k - j = i$. Therefore as in (15), we can write $b \in B$, and thus any $b_i \in B_i$, as 

$$b_i = r_{0,0} r_{0,1} r_{0,2} \cdots r_{0,\ell-1} r_{1,0} \cdots r_{1,\ell-1} r_{2,0} \cdots \cdots r_{j,0} \cdots r_{j,k-j} \cdots r_{\ell,j-\ell} \cdots r_{\ell,0},$$

(19)

where the terms in product $(r_{j,0} \cdots r_{j,k-j} \cdots r_{\ell,j-\ell})$ are coset representatives from the $j$-th column of (18), $0 \leq j \leq \ell$.

Consider again a representative (10) in the transversal of $H_{[0,k]}/J_{[0,k]}$. Since $[r_{k,0}] \subset X_k \cap Y_0$, any representative $r_{k,0} \in [r_{k,0}]$ merges to the identity state. Thus the components (10) of a transversal of $H_{[0,k]}/J_{[0,k]}$ can be extended with the identity element to form a path 

$$\ldots, 1, 1, r_{0,0}, r_{1,0}, \ldots, r_{j,j-k}, \ldots, r_{k,0}, 1, 1, \ldots$$

in the trellis. We call this path a generator $g_{[0,k]}$; generator $g_{[0,k]}$ is the same as the identity path outside the time interval $[0, k]$. Let $A_{[0,k]}$ be the set of generators formed by transversals of $H_{[0,k]}/J_{[0,k]}$, $0 < k \leq \ell$. For each $g_{[0,k]} \in A_{[0,k]}$ and each time $t$, let $g_{[t,t+k]}$ be the time shift of $g_{[0,k]}$: $g_{[t,t+k]}$ is the same as the identity path outside the time interval $[t, t+k]$. We say $g_{[t,t+k]}$ is a generator at time $t$. Let $A_{[t,t+k]}$ be the set of generators $g_{[t,t+k]}$, $0 < k \leq \ell$. Let $A_{[0,0]}$ be the set of generators formed by elements of $[r_{0,0}] = X_0 \cap Y_0$. These are generators which are the same as the identity path, except that time 0 branch $b_0 \in X_0 \cap Y_0$. Let $A_{[0,t]}$ be the time shift of these generators.

Fix time $t$. We now show how to realize any element in (19) at time $t$. Clearly any $r_{0,0}$ can be realized at time $t$ by $\chi_t(g_{[0,t]})$ for some $g_{[t,t]} \in A_{[0,t]}$. For any element $r_{j,k-j}$ in (19), $0 < k \leq \ell$ and $0 \leq j \leq k$, there is a $g_{[t-j,t-j+k]} \in A_{[t-j,t-j+k]}$ such that 

$$\chi_t(g_{[t-j,t-j+k]}) = r_{j,k-j}.$$ 

We can rewrite this as 

$$\chi_t(g_{[t-j,t+(j-k)]}) = r_{j,k-j}.$$ 

Using index $i$ in place of $k - j$, we can rewrite $b_t$ in (19) as 

$$b_t = \prod_{j=0}^{\ell} \left( \prod_{i=0}^{t-j} \chi_t(g_{[t-j,t+i]}) \right).$$

(20)

Thus we have shown any element $b_t \in B_t$ can be written using generators selected at times in the interval $[t-\ell,t]$, i.e., generators

$$g_{[t-j,t+i]},$$

for $j = 0, \ldots, \ell$, for $i = 0, \ldots, \ell - j$.

An encoder of the group trellis is a finite state machine that, given a sequence of inputs, can produce any path (any sequence of states and branches) in the group trellis. We assume the encoder has the same number of states as $B$. An encoder can help to explain the structure of a group trellis. We give an encoder here which has a register structure and uses shortest length sequences as in [3, 4], but the encoder is different.

We will define an encoder based on (20). We can rewrite $b_t$ to separate out the $j = 0$ term as 

$$b_t = \prod_{i=0}^{\ell} \chi_t(g_{[t,t+i]}),$$

(21)

$$\prod_{j=1}^{\ell} \left( \prod_{i=0}^{t-j} \chi_t(g_{[t-j,t+i]}) \right).$$

(22)

The term in (21) is just an element $x_t \in X_0$. Note that $x_t$ is a function of generators at time $t$. We will think of $x_t$ as an input of the encoder. The term in (22) is just a branch $b_t \in B_t$; it corresponds to the branch obtained when $x_t = 1$. Note that $b_t$ is a function of generators at times $t', t' < t$. We can rewrite $b_t$ as $b_t = x_t \hat{b}_t$, where $\hat{b}_t \in B_t$ and 

$$\hat{b}_t = \prod_{j=1}^{\ell} \left( \prod_{i=0}^{t-j} \chi_t(g_{[t-j,t+i]}) \right).$$

Note that $\hat{b}_t$ can take on $|B/X_0|$ different values, and each value is in a distinct coset of $B/X_0$. Thus there is a one to one correspondence between the set of $\hat{b}_t$, $\{\hat{b}_t\}$, and $B/X_0$, $\{\hat{b}_t\} \leftrightarrow B/X_0$. Thus we will think of $b_t$ as a state of the encoder.

We propose an encoder based on (20) which has the same form as (20) for each time epoch. Accordingly, the form of the encoder for time $t + 1$ is 

$$b_{t+1} = \prod_{j=0}^{\ell} \prod_{i=0}^{t-j} \chi_{t+1}(g'_{[t+1-j,t+i]}).$$
We can rewrite this to separate out the \( j = 0 \) term:

\[
\hat{b}_{t+1} = \sum_{i=0}^{\ell} x_{t+1} (g'_{[t+1,t+1+i]})
\]

The term in (23) uses \( \ell + 1 \) new generators at time \( t+1 \), \( g'_{[t+1,t+1+i]} \) for \( i = 0, \ldots, \ell \). It is the new input \( x_{t+1} \in X_0 \). Let \( \hat{b}_{t+1} \) be the term in (24). We see that \( \hat{b}_{t+1} \) uses generators \( g'_{[t+1-j,t+1+i]} \) at times in the interval \([t, t - \ell + 1]\).

We complete the specification of the encoder by requiring that the generators in (24), or \( \hat{b}_{t+1} \), used by the encoder at time \( t + 1 \) be the same as the generators used by the encoder at time \( t \). In other words, we require

\[
g'_{[t+1-j,t+1+i]} = g_{[t+1-j,t+1+i]}
\]

for \( j = 1, \ldots, \ell \), for \( i = 0, \ldots, \ell - j \). Then

\[
\hat{b}_{t+1} = \sum_{j=1}^{\ell} x_{t+1} (g_{[t+1-j,t+1+i]}).
\]

The encoder output at time \( t + 1 \) is \( \hat{x}_{t+1} = x_{t+1} \hat{b}_{t+1} \). Thus the encoder uses a sliding block encoding of the past, given by \( b_{t+1} \), with new inputs at each time epoch, given by \( x_{t+1} \). With this definition of the encoder, note that for any input \( x_{t+1} \) to the encoder, \( (b_t, b_{t+1}) \) is a trellis path segment of length two. In particular, for \( x_{t+1} = 1 \), we have \( (b_t, b_{t+1}) \) is a trellis path segment of length two.

We now verify that given an arbitrary path \( c^* \) in the group trellis, the encoder can track it perfectly, i.e., produce the same path. First we give a useful lemma. We think of the encoder as an estimator.

Lemma 27 Let \( c \) and \( c^* \) be two paths in the group trellis \( C \):

\[
c = \ldots, b_t, b_{t+1}, b_{t+2}, \ldots
\]

\[
c^* = \ldots, b_t^*, b_{t+1}^*, b_{t+2}^*, \ldots
\]

The two trellis paths are arbitrary except that at time \( t \), \( b_t = b_t^* \). Then at time \( t + 1 \), \( b_{t+1} \) is in the same coset of \( B/X_0 \) as \( b_{t+1}^* \), i.e., \( b_{t+1}^* \in X_0 b_{t+1} \). At time \( t + 2 \), \( b_{t+2} \) is in the same coset of \( B/X_1 \) as \( b_{t+2}^* \). In general at time \( t + j \), \( j \leq \ell \), \( b_{t+j} \) is in the same coset of \( B/X_{t+1} \) as \( b_{t+j}^* \). For time \( t + j \), \( j > \ell \), \( b_{t+j} \) is in the same coset of \( B/X_{t+j} \) as \( b_{t+j}^* \); this coset is just \( B \). In a similar way, at time \( t - 1 \), \( b_{t-1} \) is in the same coset of \( B/Y_0 \) as \( b_{t-1}^* \).

Proof. We have \( \mathcal{N}(b_t) = \mathcal{N}(b_t^*) \), so \( b_{t+1} \) and \( b_{t+1}^* \) are in the same coset of \( X_0 \). The rest of the proof is analogous.

We can think of \( c \) as a path which estimates \( c^* \). The lemma shows how the estimation degrades as time goes on when it is perfect at \( t = 0 \), for two otherwise arbitrary paths. Note that the conclusion of the lemma is unchanged if, instead of one path \( c \), we use a finite number of paths to estimate \( c^* \).

We now show that the encoder can track arbitrary path \( c^* \). Fix time \( t \). The branch in \( c^* \) at time \( t \) is \( b_t^* \). First, we show that the encoder can produce the initial condition \( b_t^* \). But we already know the encoder can produce any branch \( b_t = b_t^* \) (see (20)). Thus at time \( t \), the encoder can produce the branch \( b_t^* \) in \( c^* \).

Now we show the encoder can track path \( c^* \). First, we show the encoder can track \( c^* \) at time \( t + 1 \). First find \( b_{t+1} \) as given in (24). We know \( (b_t, b_{t+1}) \) is a trellis path segment of length two. Therefore we know from Lemma 27 that \( b_{t+1}^* \in X_0 b_{t+1} \). Thus there exists \( x_{t+1} \in X_0 \) such that \( b_{t+1}^* = x_{t+1} b_{t+1} \). Now select \( \ell + 1 \) new generators at time \( t + 1 \), \( g_{[t+1,t+1+i]} \), for \( i = 0, \ldots, \ell \), such that

\[
\prod_{i=0}^{\ell} x_{t+1} (g_{[t+1,t+1+i]}) = x_{t+1} \hat{b}_{t+1}.
\]

We know this is possible since \( x_{t+1} \in X_0 \) and the first column of (18) is a coset decomposition of \( X_0 \). We get a path \( c^* \) at time \( t + 1 \), the encoder output is given by \( \hat{x}_{t+1} \hat{b}_{t+1} = \hat{x}_{t+1} \hat{b}_{t+1} \). Thus we see that proper input \( x_{t+1} \), the encoder can track path \( c^* \) at time \( t + 1 \). We can repeat this same argument for succeeding time epochs.

We can observe several features of the encoder. The term in the parentheses of (20) is some function of \( t - j \), say \( h_{t-j} \). Thus \( b_t = \prod_{j=0}^{\ell} h_{t-j} \). Thus the encoder has the form of a convolution, reminiscent of a linear system. The Forney-Trott encoder [3] and Loeliger-Mittelholzer encoder [4] do not have the convolution analogoy.

Note that if we apply input \( x_t = \chi_t (g_{[t,t+k]} \) at time \( t \) to an encoder in the identity state, the output is just

\[
\chi_{t+n} (g_{[t,t+k]} \) for \( n = 0, 1, \ldots, \infty \).
\]

Thus the output response of the encoder is the input generator. This is reminiscent of the impulse response of a linear system.

In view of Figure 2 the controllable Schreier matrix (1), and (13), a natural interpretation of the encoder is shown in Figure 3. The encoder consists of \( \ell + 1 \) registers, where the \( j \)th register, \( j = 0, 1, \ldots, \ell \), contains the information (coset representatives) in quotient group \( X_j/X_{j-1} \). At the next time epoch, the register containing \( X_j/X_{j-1} \) is shifted or “shoved” into the register containing \( X_{j+1}/X_j \). The encoder shown in Figure 3 is somewhat different from the classic interpretation of a shift register.

Any arbitrary group \( G \) has a chain coset decomposition. Any element \( g \in G \) can be specified by the set \( R \) of chain coset representatives.

Theorem 28 Let \( G \) be an arbitrary group. Let \( G \) have two chain coset decompositions and two sets of chain coset representatives, \( R \) and \( R' \), respectively. Then \( |R| = |R'| \).
Proof. Any chain coset decomposition of $G$ can be refined into a composition chain. Then the two chain coset decompositions with sets of chain coset representatives $R$ and $R'$ can be refined into two composition chains with sets of chain coset representatives $R_c$ and $R'_c$, respectively. But we must have $|R| = |R_c|$ and $|R'| = |R'_c|$. But by the Jordan-Hölder theorem, any two composition chains are equivalent. Thus $|R_c| = |R'_c|$ and so $|R| = |R'|$. 

Thus we can associate with any group $G$ a unique number $\eta_G$, such that $\eta_G$ elements of $G$ are necessary and sufficient to specify an arbitrary element of $G$, irrespective of the chain coset decomposition used.

At each time $t$, our encoder outputs an element $b_t \in B$. We say the encoder is minimal because at each time epoch $t$, it uses $\eta_B$ elements to specify $b_t$. This definition of minimality is different from the approach in [3, 4, 5]. However the number of elements and number of states used in the encoder here and the encoders in [3, 4] are the same.

6. COMPOSITION SERIES OF $B$

For each branch $b \in B$, we define the previous branch set $\mathcal{P}(b)$ to be the set of branches that can precede $b$ at the previous time epoch in valid trellis paths. In other words, branch $e \in \mathcal{P}(b)$ if and only if $b^- = e^+$.

For a set $U \subset B$, define the set $\mathcal{P}(U)$ to be the union $\cup_{b \in U} \mathcal{P}(b)$. The set $\mathcal{P}(U)$ always consists of cosets of $Y_0$.

Note that $\mathcal{P}(Y_j) = Y_{j+1}$.

Proposition 29 If $b^- = e^+$, the previous branch set $\mathcal{N}(b)$ of a branch $b$ in $B$ is the coset $Y_{0e}$ in $B$. If $b^- = e^+$, the previous branch set $\mathcal{P}(b)$ of a branch $b$ in $Y_j$ is the coset $Y_{0e}$ in $Y_{j+1}$.

Note that $\mathcal{P}(X_{j+1}) \cap X_j = X_j$. For a set $U' \subset X_{j+1}$, define $\mathcal{P}_j(U') = \mathcal{P}(U') \cap X_j$.

Proposition 30 If $U' \subset X_{j+1}$ is a group, then $\mathcal{P}_j(U')$ is a group and $\mathcal{P}_j(U') \subset X_j$. If $U' \subset X_{j+1}$ is a set, then $\mathcal{P}_j(U')$ consists of cosets of $X_j \cap Y_0$. If set $U \subset X_j$ consists of cosets of $X_j \cap Y_0$, then $U = \mathcal{P}_j(\mathcal{N}(U))$. 

Proof. If $U'$ is a group, then $\mathcal{P}(U')$ is a group so $\mathcal{P}(U') \cap X_j$ is a group. This proves the first statement.

Assume $U' \subset X_{j+1}$ is a set. Let state $s \in (\mathcal{P}_j(U'))^+$. Then $s \in (\mathcal{P}(U') \cap X_j)^+ = (\mathcal{P}(U'))^+ \cap X_j$. Therefore $s \in (\mathcal{P}(U'))^+$ and $s \in X_j^+$. But branches in $\mathcal{P}(U')$ that merge to $s$ are a coset of $Y_0$, and branches in $X_j$ that merge to $s$ are a coset of $X_j \cap Y_0$ (by Proposition 1). Then branches in $\mathcal{P}(U') \cap X_j$ that merge to $s$ are a coset of $X_j \cap Y_0$. This proves the second statement.

The third statement follows from the second. 

The following proposition is similar to (iv) of Proposition 13.

Proposition 31 Assume $J', H'$ are subgroups of $X_{j+1}$ and $X_0 \leq J' \leq H'$. Then there exist subsets $J, H \subset X_j$ such that $\mathcal{N}(J) = J'$ and $\mathcal{N}(H) = H'$. In particular, we can choose $J = \mathcal{P}_j(J')$ and $H = \mathcal{P}_j(H')$. Then $J, H$ are groups and $J \leq H$. And if $J' \triangleleft H'$, then $J \triangleleft H$.

We want to find a refinement of the Schreier matrix [4], which we denote by $\{X_{j,k}(\sigma)\}$. The refinement has terms $X_{j,k}(\sigma)$, for $j \geq 0$, $k \geq -1$, and $\sigma$ such that $0 \leq \sigma \leq \sigma_{j+k}$, as shown here:
for \(0 \leq j \leq \ell\).

In general the subchain
\[
X^{(\sigma)}_{j,k} \subset X^{(\sigma)}_{j,k-1} \subset \cdots \subset X^{(\sigma)}_{j,k} \subset \cdots \subset X^{(\sigma_{j+k})}_{j,k}
\]
(36)
in \(\mathbb{F}_1\) is a refinement of the subchain \(X_{j-1}(X_j \cap Y_k) \subset X_{j-1}(X_j \cap Y_{k+1})\) in the Schreier matrix \([4]\). The groups in subchain \([33]\) are indexed by \(\sigma\). We show in Theorem \([32]\) that we can find a refinement which is a composition chain in which the number of groups in subchain \([30]\) just depends on the sum \(j + k\); then index \(\sigma\) runs from 0 to parameter \(\sigma_{j+k}\) as shown in \([31]\). In fact Theorem \([32]\) shows that for \(k \geq 0\),
\[
\mathcal{N}(X^{(\sigma)}_{j,k}) = X^{(\sigma)}_{j+1,k-1},
\]
for \(\sigma = 0, 1, \ldots, \sigma_{j+k}\).

**Theorem 32** Let \(B\) be an \(\ell\)-controllable group trellis section with normal chains \(\{X_j\}\) and \(\{Y_k\}\). There is a refinement \([X^{(\sigma)}_{j,k}]\) of the Schreier matrix \([4]\), given by \([29]\)-\([32]\), which is a composition chain of \(\{X_j\}\) (and \(B\)). The terms \(X^{(\sigma)}_{j,k}\) exist for \(j \geq 0, k \geq -1\) such that \(j + k < \ell\), for \(0 \leq \sigma \leq \sigma_{j+k}\). For \(j \geq 0, k \geq 0\) such that \(j + k < \ell\), for \(0 \leq \sigma \leq \sigma_{j+k}\), if \(X^{(\sigma)}_{j,k}\) is a group in the composition chain, then \(\mathcal{N}(X^{(\sigma)}_{j,k})\) is the group \(X^{(\sigma)}_{j+1,k-1}\) in the composition chain. Lastly, fix \(k\) such that \(0 \leq k < \ell\). Then for \(j\) such that \(0 \leq j \leq k + 1\), and \(\sigma\) such that \(0 < \sigma \leq \sigma_k\), we have isomorphisms
\[
\frac{X^{(\sigma)}_{0,k}}{X^{(\sigma-1)}_{0,k}} \simeq \frac{\mathcal{N}(X^{(\sigma)}_{j,k})}{\mathcal{N}(X^{(\sigma-1)}_{j,k})} = \frac{X^{(\sigma)}_{j,k-1}}{X^{(\sigma-1)}_{j,k-1}}.
\]

**Proof.** First we find a refinement of \([1, X_{-1}(X_0 \cap Y_1)] = [X^{(0)}_{0,-1}, X^{(\sigma)}_{0,-1}],\) the first column of \([4]\) (for a normal chain containing groups \(G, H\), the notation \([G, H]\) means the subchain starting with group \(G\) and ending with group \(H\), i.e., the subchain with groups in the interval \([G, H]\)). We select the refinement so it is a composition chain; this refinement is given in \([29]-[32]\). In general the groups in the composition chain of subchain \([X^{(0)}_{0,k}, X^{(\sigma)}_{0,k}]\) are indexed by \(\sigma, \sigma = 0, 1, \ldots, \sigma_k\), for \(k = -1, 0, \ldots, \ell - 1\),

\[
\begin{align*}
X^{(0)}_{0,k} & \subset X^{(1)}_{0,k} \subset \cdots \subset X^{(\sigma)}_{0,k} \subset \cdots \subset X^{(\sigma_k)}_{0,k}.
\end{align*}
\]

Next we show how to find a refinement of \([X_0, X_0(X_1 \cap Y_{-1})] = [X^{(0)}_{1,-1}, X^{(\sigma)}_{1,-1}],\) the second column of \([4]\). The refinement of \([X^{(0)}_{1,-1}, X^{(\sigma)}_{1,-1}]\) is found by taking the refinement of \([X^{(0)}_{0,0}, X^{(\sigma)}_{0,0}]\) previously found and applying \(\mathcal{N}\) to each of the terms. For example for the subchain \([X^{(0)}_{0,k}, X^{(\sigma)}_{0,k}],\) \(0 \leq k \leq \ell - 1\), of \([X^{(0)}_{0,0}, X^{(\sigma)}_{0,0}]\) shown in \([38]\), the corresponding subchain of \([X^{(0)}_{1,1}, X^{(\sigma)}_{1,1}]\) is
\[
\mathcal{N}(X^{(0)}_{0,k}) \subset \mathcal{N}(X^{(1)}_{0,k}) \subset \cdots \subset \mathcal{N}(X^{(\sigma-1)}_{0,k}) \subset \mathcal{N}(X^{(\sigma)}_{0,k}) \subset \cdots \subset \mathcal{N}(X^{(\sigma_k)}_{0,k}).
\]
Using Proposition 8 and Proposition 13 we have $N(X_{0,k}^{(0)}) = X_{1,k-1}^{(0)}$ and $N(X_{0,k}^{(a)}) = X_{1,k-1}^{(a)}$. Then (39) becomes

$$X_{1,k-1}^{(0)} \subset N(X_{0,k}^{(1)}) \subset \ldots \subset N(X_{0,k}^{(a-1)}) \subset N(X_{0,k}^{(a)}) \subset \ldots \subset X_{1,k-1}^{(a)}.$$ (40)

Since $X_{0,k}^{(a-1)} < X_{0,k}^{(a)}$, we have by Proposition 13 that $N(X_{0,k}^{(a-1)}) < N(X_{0,k}^{(a)})$. Then (40) becomes

$$X_{1,k-1}^{(a)} \subset N(X_{0,k}^{(1)}) \subset \ldots \subset N(X_{0,k}^{(a-1)}) \subset N(X_{0,k}^{(a)}) \subset X_{1,k-1}^{(a)}.$$ (41)

and so we have found a refinement of $[X_{1,k-1}^{(0)}, X_{1,k-1}^{(a-1)}]$.

Next we show the refinement of $[X_{1,k-1}^{(0)}, X_{1,k-1}^{(a-1)}]$ is a composition chain. If the refinement is not a composition chain, then we can insert a group, say $J'$, in the chain which gives a nontrivial refinement. Without loss of generality assume we can insert $J'$ in (41) so that

$$N(X_{0,k}^{(a-1)}) \subset J' \subset N(X_{0,k}^{(a)}).$$ (42)

is a nontrivial chain. Since $N(X_{0,k}^{(a-1)}) \subset X_{1,k}^{(a-1)} = X_1$, and $X_0 = X_{1,k-1}^{(0)} \subset N(X_{0,k}^{(a-1)})$, we can apply Proposition 30. Then there exists groups $P_0, N(X_{0,k}^{(a-1)}), J = P_0(J'),$ and $P_0(N(X_{0,k}^{(a)}))$ in $X_0$ such that

$$P_0(N(X_{0,k}^{(a-1)})) \subset J \subset P_0(N(X_{0,k}^{a})).$$ (43)

Since $X_0 \cap Y_0 = X_{0,0}^{(0)} \subset X_{0,k}^{(a-1)}$, by Proposition 30 (43) is the same as

$$X_{0,k}^{(a-1)} \subset J \subset X_{0,k}^{(a)}.$$ (44)

It is clear (44) is a nontrivial chain since (42) is a nontrivial chain. Thus we have found a nontrivial refinement of the composition chain of $[X_{0,k}^{(0)}, X_{0,k}^{(a-1)}]$, a contradiction.

This shows the above procedure gives a composition chain of $[X_{1,k-1}^{(0)}, X_{1,k-1}^{(a-1)}]$. Finally, define terms of the composition chain $(X_{1,k-1}^{(0)}, X_{1,k-1}^{(a-1)})$ by

$$X_{1,k-1}^{(a)} = N(X_{0,k}^{(a)})$$

for $0 < a < a_k$, $-1 \leq k \leq \ell - 2$. This gives the refinement in (20) (30).

The theorem assertion that $N(X_{0,k}^{(a)}) = X_{1,k-1}^{(a)}$ follows by construction. Note that $X_{0,k}^{(0)} = X_{1,k-1}$. Therefore from (8) of Theorem 13 $N$ defines an isomorphism

$$\frac{X_{1,k-1}^{(a)} \cap N(X_{0,k}^{(a)})}{X_{0,k}^{(0)}} \simeq \frac{N(X_{0,k}^{(a-1)})}{N(X_{0,k}^{(a))})}. $$

Therefore the construction procedure applying $N$ to the terms of $[X_{0,k}^{(0)}, X_{0,k}^{(a-1)}]$ gives (37) for $j = 1$.

Next using the same approach we find a composition chain of $[X_{2,k-2}^{(0)}, X_{2,k-2}^{(a-1)}]$. Continuing in this way gives a composition chain $(X_{j,k}^{(a)})$ of (41), and in like manner the other theorem assertions hold.

**Theorem 33** Let $B$ be an $\ell$-controllable group trellis section with normal chains $(X_j)$ and $(Y_k)$. Then $B$ is solvable if and only if $X_0$ is solvable.

**Proof.** If $G$ is solvable, then every subgroup is solvable, so $X_0$ is solvable. For the converse result, assume that $X_0$ is solvable. Then the composition chain $[1,X_0]$ constructed in Theorem 32 is solvable. By Theorem 32 this means the chain in $[X_j,X_{j+1}]$ is solvable for $0 \leq j < \ell$. Then the entire chain $[1,X_\ell]$ is solvable and $B$ is solvable.

7. **ENCODER FOR $(X_j)$ AND $(Y_{k,\rho})$**

It is clear that $(Y_k)$ is the dual of $(X_j)$, and $P$ is the dual of $N$. Then we can obtain the following dual result to Theorem 32 giving the dual composition chain $(Y_{k,\rho})$ of $(X_{j,\rho})$.

For a set $U \subset B$ and integer $i > 0$, define $P^i(U)$ to be the $i$-fold composition $P^i(U) = P \circ P \circ \ldots \circ P(U)$. For $i = 0$, define $P^i(U) = P^0(U)$ to be just $U$.

**Theorem 34** Let $B$ be an $\ell$-controllable group trellis section with normal chains $(X_j)$ and $(Y_k)$. There is a refinement $(Y_{k,\rho})$ of the Schreier matrix of $(X_j)$ and $(Y_k)$ which is a composition chain of $(Y_{k,\rho})$ (and $B$). The terms $Y_{k,\rho}^{(a)}$ exist for $k \geq 0$, $j \geq -1$ such that $j + k < \ell$, for $0 \leq \sigma \leq \sigma_{j+k}$. For $k \geq 0$, $j \geq 0$ such that $j + k < \ell$, for $0 \leq \sigma \leq \sigma_{j+k}$, if $Y_{k,\rho}^{(a)}$ is a group in the composition chain, then $P(Y_{k,\rho}^{(a)})$ is the group $Y_{k,\rho}^{(a-1,j-k+1)}$ in the composition chain. Lastly, fix $j$ such that $0 \leq j < \ell$. Then for $k$ such that $0 \leq k \leq j + 1$, and $\sigma$ such that $0 < \sigma \leq \sigma_j$, we have isomorphisms

$$\frac{Y_{k,\rho}^{(a)}}{Y_{k,\rho}^{(a-1,j-k)}} \simeq \frac{P^k(Y_{k,\rho}^{(a)})}{P^{k-1}(Y_{k,\rho}^{(a-1,j-k-1)})} = \frac{Y_{k,j-k}^{(a)}}{Y_{k,j-k}^{(a-1,j-k-1)}}. $$ (45)

Find a composition chain $(Y_{k,\rho}^{(a)})$ of $(Y_k)$, where we have used index $\rho$ in place of index $j$. Insert the composition chain $(Y_{k,\rho}^{(a)})$ into the normal series $(X_j)$, and vice versa. This gives a refinement of $(X_j)$ and a refinement of $(Y_{k,\rho}^{(a)})$. The two refinements are equivalent by the Schreier refinement theorem. Since the refinement of $(Y_{k,\rho}^{(a)})$ is a composition series, we know both refinements are composition series.

The refinement of $(X_j)$ contains terms of the form

$$X_{j-1}(X_j \cap Y_{k,\rho}^{(a)})$$

for $0 \leq j \leq \ell$, for $k \geq 0$, $\rho \geq 1$ such that $k + \rho < \ell$, for $0 \leq \sigma \leq \sigma_{k+\rho}$. We can think of the refinement or composition series of $(X_j)$ as a 4-dimensional array with indices $j$, $k$, $\rho$, and $\sigma$. We call this the Schreier array form of $(X_j)$ and $(Y_{k,\rho}^{(a)})$.

We think of a Schreier array as composed of pages.

- For fixed $j$ such that $0 \leq j \leq \ell$, and fixed $k$ such that
0 ≤ k ≤ ℓ, page Ωj,k of the Schreier array is the set containing all terms of the form

\[ X_{j-1}(X_j \cap Y_{k,\rho}) \]

for ρ ≥ −1 such that k + ρ < ℓ, for σ such that 0 ≤ σ ≤ σk,ρ. We can write page Ωj,k as a subchain of groups

\[ X_{j-1}(X_j \cap Y_{k-1,0}) \subset \cdots \subset X_{j-1}(X_j \cap Y_{k-1,0}^{(\sigma_k,1)}) \]
\[ \subset X_{j-1}(X_j \cap Y_{k,0}) \subset \cdots \subset X_{j-1}(X_j \cap Y_{k,0}^{(\sigma_k,0)}) \]

\[ \cdots \]
\[ \subset X_{j-1}(X_j \cap Y_{k,\ell-k-1}) \subset \cdots \subset X_{j-1}(X_j \cap Y_{k,\ell-k-1}^{(\sigma_{\ell-k},1)}). \]

Using the dual result to (35), we have \( Y_{k-1}^{(\sigma_{k-1})} = Y_k \), and so the first term of subchain (46) is \( X_{j-1}(X_j \cap Y_k) \).

Using the dual result to (34), we have \( Y_{k-1}^{(\sigma_{k-1})} = Y_k \), and so the last term of subchain (46) is \( X_{j-1}(X_j \cap Y_k) \).

Since the Schreier matrix of \( \{X_j\} \) and \( \{Y_k\} \) in (33) is \( \ell \)-controllable, we know that for 0 ≤ j ≤ ℓ, 0 ≤ k ≤ ℓ, and \( j + k \geq \ell \), \( X_{j-1}(X_j \cap Y_k) \). Then using (46) and its endpoint conditions, for 0 ≤ j ≤ ℓ, 0 ≤ k ≤ ℓ, and \( j + k > \ell \),

\[ \Omega_{j,k} = X_j. \]

Eliminating the trivially redundant pages satisfying (47), we can arrange the remaining pages into an inclusion chain of sets, as shown in (48):

\[
\begin{array}{cccccc}
\Omega_{0,\ell} & \Omega_{1,\ell-1} & \Omega_{2,\ell-2} & \cdots & \Omega_{\ell-1,1} & \Omega_{\ell,0} \\
\Omega_{0,\ell-1} & \Omega_{1,\ell-2} & \cdots & \cdots & \cdots & \\
\Omega_{0,2} & \Omega_{1,1} & \Omega_{2,0} & \\
\Omega_{0,1} & \Omega_{1,0} & \\
\Omega_{0,0} & \\
\end{array}
\]

The indices j, k of \( \Omega_{j,k} \) in (48) satisfy j ≥ 0, k ≥ 0, and j + k ≤ ℓ.

The notation

\[ V = U \cup U \]

(also written \( U \cup U \)) means set \( U \) is contained in \( V \), and sets \( U \) and \( V \) share a common element. For \( j \geq 0, k \geq 0 \) such that \( j + k \leq \ell \), the common element in \( \Omega_{j,k,\ell} \cup U \Omega_{j,k,\ell+1} \) is \( X_{j-1}(X_j \cap Y_k) \). For \( j \geq 0, k > 0, j + k = \ell, j \neq \ell \), the common element in \( \Omega_{j,\ell-j} \cup U \Omega_{j,\ell+1,0} \) is \( X_{j-1}(X_j \cap Y_{\ell-j}) = X_j \).

If we arrange the elements in each page into a subchain as in (46), then (48) represents a chain of groups. Regarded as a chain of groups, the page matrix (48) contains all terms in the \( \ell \)-controllable Schreier matrix (4), i.e., the terms \( X_{j-1}(X_j \cap Y_k) \) for \( j \geq 0, k \geq -1, j + k \leq \ell \). Therefore as a chain of groups, the page matrix (48) is a refinement of the Schreier matrix (4). We now show groups in (48) are related in a way similar to the Schreier matrix (4).

**Theorem 35** The Schreier array of \( \{X_j\} \) and \( \{Y_{k,\rho}\} \) is a composition chain of \( B \) which is a refinement of the Schreier matrix of \( \{X_j\} \) and \( \{Y_k\} \) given in (4). Fix j such that 0 ≤ j ≤ ℓ. Let \( L, H \) be groups such that \( X_{j-1} \leq J \leq H \leq X_j \). Define function \( \psi : H/J \rightarrow N(H)/N(J) \) by the assignment \( \psi : Jh \mapsto N(Jh) \). The assignment \( \psi \) gives a one to one correspondence between all the right cosets \( Jh \) of \( H/J \) and all the right cosets \( N(Jh) \) of \( N(H)/N(J) \). Moreover \( \psi \) is an \( H \)-isomorphism giving

\[ H/J \approx N(H)/N(J). \]

By (v) of Proposition 15, we have \( J < H \) if and only if \( N(J) < N(H) \). Thus if \( J \subset H \) or \( N(J) < N(H) \), then \( \psi \) is an isomorphism giving \( H/J \approx N(H)/N(J) \). In particular let

\[ H = X_{j-1}(X_j \cap Y_{k,\rho}) \]

and

\[ J = X_{j-1}(X_j \cap Y_{k,\sigma-1}) \]

where \( k > 0 \) and \( \rho \geq -1 \), such that \( k + \rho < \ell \), and \( 0 < \sigma \leq \sigma_{k,\rho} \). Then \( \psi \) gives an isomorphism

\[
\frac{X_{j-1}(X_j \cap Y_{k,\rho})}{X_{j-1}(X_j \cap Y_{k,\rho}^{(\sigma-1)})} \cong \frac{N(X_{j-1}(X_j \cap Y_{k,\rho}^{(\sigma-1)}))}{N(X_{j-1}(X_j \cap Y_{k,\rho}^{(\sigma-1)}))} \]

\[
= \frac{X_{j-1}(X_j \cap Y_{k,\rho}^{(\sigma-1)})}{X_{j-1}(X_j \cap Y_{k,\rho}^{(\sigma-1)})} \]

This is an isomorphism between terms at the intersection of two lines with two pages in the Schreier array.

**Proof.** We have already shown that the Schreier array of \( \{X_j\} \) and \( \{Y_{k,\rho}\} \) is a composition chain and a refinement of the Schreier matrix (4).

To show (49) we use the isomorphism in (iii) of Theorem 17. We only need to verify that \( X_j \cap Y_0 \subset J \). But \( X_j \cap Y_0 \subset J \) if \( Y_0 \subset Y_{k,\rho}^{(\sigma-1)} \). Since \( Y_0 = Y_{k,0}^{(\sigma-1)} \), the inclusion is satisfied if \( k > 0 \) and \( \rho \leq -1 \), as assumed.

We now show (50). We first prove the numerator portion of the equality in (50). We know \( X_j^+ = X_j^- \). From Theorem 33 we have

\[ (Y^{(\sigma)}_{k,\rho})^+ = (Y^{(\sigma)}_{k-1,\rho+1})^- \]

Therefore

\[ (X_j \cap Y_{k,\rho})^+ = (X_j^+ \cap Y_{k-1,\rho+1}^-) \]

and

\[ N(X_j \cap Y_{k,\rho}) = X_0(X_{j-1} \cap Y_{k-1,\rho+1}^-) \]

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From $\mathcal{N}(GH) = \mathcal{N}(G)\mathcal{N}(H)$, we get

$$\mathcal{N}(X_{j-1}(X_j \cap Y_{k,\rho}^{(\sigma)})) = X_j(X_{j+1} \cap Y_{k-1,\rho+1}^{(\sigma)}).$$

The proof of the denominator portion of the equality in (50) is analogous.

Theorem 35 corresponds to Theorem 19 in the development of an encoder for $\{X_j\}$ and $\{Y_k\}$. It follows that we can obtain an encoder for $\{X_j\}$ and $\{Y_k\}$ in a similar way as for $\{X_j\}$ and $\{Y_k\}$. Instead of using generators with components from $X_j \cap Y_k$ that are representatives of

$$\frac{X_{j-1}(X_j \cap Y_k)}{X_{j-1}(X_j \cap Y_{k-1})},$$

we use generators with components from $X_j \cap Y_{k,\rho}^{(\sigma)}$ that are representatives of

$$\frac{X_{j-1}(X_j \cap Y_{k,\rho}^{(\sigma)})}{X_{j-1}(X_j \cap Y_{k,\rho}^{(\sigma-1)}).}$$

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