On stochastic integrals with controlled growth of their containing range

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Abstract

This short note suggests special examples of stochastic Itô integrals with controlled growth of their containing range. The integrands for this integrals are presented explicitly. The construction does not involve neither stopping times nor forecasting or calculation of the conditional expectations of a contingent claim.

Key words: stochastic integrals, Itô calculus, containing range

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1 Introduction

The paper considers stochastic processes represented as stochastic Itô integrals (possibly, with a drift term). Usually, these integrals have unlimited range of possible values. However, there are special cases of integrals with limited range. These integrals can be obtained, for instance, as conditional expectations of random variables with limited range, or via restriction of the integration interval by a random Markov stopping times preventing the range growth. These approaches may be inconvenient in some cases. For example, calculation of a condition expectation is essentially a forecast of a contingent claim depending on the future values, and this procedure can be difficult. Besides, one would need to specify first this contingent claims. On the other hand, restriction of the integration interval by stopping times leads to stochastic integrals with some paths being frozen at that stopping times. Obviously, this feature could be undesirable.
The present paper suggests special examples of stochastic Itô integrals with controlled growth of their containing range. The integrands for this integrals are presented explicitly. The paper uses an original approach does not involve neither stopping times nor forecasting or calculation of the conditional expectations of a contingent claim. This approach does not involve neither forecasting nor calculation of the conditional expectations of a contingent claim.

2 The main result

We are also given a standard complete probability space \((\Omega, \mathcal{F}, P)\) and a right-continuous filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) of complete \(\sigma\)-algebras of events. In addition, we are given an one-dimensional Wiener process \(w(t)\) such that \(w(0) = 0\) and that \(\mathcal{F}_t\) is independent from \(w(s) - w(q)\) if \(t \geq s > q \geq 0\).

Consider a continuous time one-dimensional random process \(x(t)\) such that
\[
dx(t) = a(t)dt + \sigma(t)dw(t).
\]
Here \(a(t)\) and \(\sigma(t)\) are bounded real-valued one-dimensional \(\mathcal{F}_t\)-adapted processes.

Let \(u(t) : (0, +\infty) \rightarrow \mathbb{R}\) be a real valued random \(\mathcal{F}_t\)-adapted process that is integrable on any finite time interval. For simplicity, we assume that \(\text{ess sup}_{\omega \in \Omega} \int_0^t |u(s)|ds < +\infty\) for any \(t > 0\).

Let \(\{\mathcal{G}_t\}_{t \geq 0}\) be the filtration of complete \(\sigma\)-algebras of events generated by the process \((x(t), u(t))\).

It can be noted that, since the process \(\sigma(t)\) is adapted to the filtration generated by \(x(t)\), it follows that \(\{\mathcal{G}_t\}_{t \geq 0}\) is also the filtration generated by the process \((x(t), u(t), \sigma(t))\); see e.g. Remark 1.1 in [5], p.10, or Proposition 7.1 in [9], where this was shown for a log-normal types of processes which was rather technical.

**Theorem 2.1** Consider processes \(X(t)\) and \(Y(t)\) defined for \(t \in [0, +\infty)\) as
\[
X(t) = \int_0^t \cos(x(t) - x(s))u(s)ds
\]
and
\[
Y(t) = \int_0^t \sin(y(t) - y(s))u(s)ds.
\]
In this case,
\[
\int_0^t X(s)dx(s) = Y(t) + \frac{1}{2} \int_0^t \sigma(s)^2 Y(t)ds.
\]
Then

$$\sqrt{X(t)^2 + Y(t)^2} \leq \int_0^t |u(s)| \, dt \quad \text{a.s.} \quad \forall a(\cdot).$$

Clearly, the process $X(t)$ is bounded uniformly in all $a(\cdot)$ almost surely on any finite time interval and $\mathcal{G}_t$-adapted. Hence the stochastic integral (1) is well defined and is bounded uniformly in all $a(\cdot)$ almost surely on any finite time interval.

Representation (2) in Theorem 2.1 implies that the boundaries for the range of the stochastic integral $Y(t)$ are defined by the choice of the process $u$. Respectively, Theorem 2.1 allows to construct stochastic processes with preselected on a given time interval time depending boundaries for their range.

### 2.1 Proof of Theorem 2.1

The proof follows the idea of the proof of Lemma 3.2 from [11] (see also [12]).

Consider a process

$$dZ(t) = iZ(t) dx(t) - \frac{1}{2} \sigma(t)^2 Z(t) dt + u(t) dt, \quad t \in (0, \infty),$$

$$Z(0) = 0.$$  \hspace{1cm} (4)

In (4), $i = \sqrt{-1}$ is the imaginary unit.

**Lemma 2.2** For any $T > 0$, we have that

$$Z(t) = i \int_0^t e^{i[x(t)-x(s)]} u(s) ds$$

and

$$\text{Im } Z(t) = \int_0^t \text{Re } Z(s) dx(s) \quad \forall t > 0.$$

The process $Z(s)$ is $\mathcal{G}_t$-adapted and such that

$$|Z(t)| \leq \int_0^t |u(s)| ds \quad \forall t > 0.$$  \hspace{1cm} (5)

**Proof of Lemma 2.2.** Let $F(t,s)$ be defined as the solution of the Itô equation

$$dF(t,s) = iF(t,s)[a(t)dt + \sigma(t)d\omega(t)] - \frac{1}{2} F(t,s)\sigma(t)^2 dt, \quad t > s \geq 0,$$

$$F(s,s) = 1.$$
By the Itô formula,

\[ F(t, s) = F(s, s) \exp \left( i \int_s^t a(r)dr + i \int_s^t \sigma(r)dw(r) - \frac{i^2}{2} \int_s^t \sigma(r)^2 dr - \frac{1}{2} \int_s^t \sigma(r)^2 dr \right) \]

\[ = \exp \left( i \int_s^t a(r)dr + i \int_0^t \sigma(r)dw(r) \right) \]

\[ = \exp (i[y(t) - y(s)]) \text{ a.s.} \]

In particular, we have that

\[ |F(t, s)| = 1 \text{ a.s.} \]

Direct differentiation gives that

\[ Z(t) = \int_0^t F(t, s)u(s)ds. \]

Hence

\[ |Z(t)| \leq \int_0^t |F(t, s)||u(s)|ds \leq \int_0^t |u(s)|ds. \]

Let \( X(t) = \text{Re } Z(t) \) and \( Y(t) = \text{Im } Z(t) \). Then (1) and (2) hold. Further, we have from (4) that

\[ dX(t) = -Y(t)dx(t) + u(t)dt - \frac{1}{2} \sigma(t)^2 X(t)dt, \]

\[ dY(t) = X(t)dx(t) - \frac{1}{2} \sigma(t)^2 Y(t)dt. \]

Then the proof of Lemma 2.2 follows. \( \square \)

**Proof of Theorem 2.1** follows from this and from (5). \( \square \)

**Remark 2.3** Consider the case where \( a(t) \equiv 0 \). In this case, it follows from the proof above that, for any \( \mathcal{F}_t \)-adapted process \( \sigma(\cdot) \), there exists a \( \mathcal{F}_t \)-adapted process \( U : [0, T] \times \Omega \to \mathbb{C} \) such that \( |U(t)| = 1 \) and that the integral \( \int_0^T \sigma(t)U(t)dw(t) \) has a limited range in \( \mathbb{C} \). To see this, it suffices to select \( U(t) = F(t, 0) \) and observe that \( i \int_0^T \sigma(t)F(t, 0)dw(t) = F(T, 0) - 1 + \frac{1}{2} \int_0^T \sigma(t)^2 F(t, 0)dt \) and that

\[ \left| F(T, 0) - 1 + \frac{1}{2} \int_0^T \sigma(t)^2 F(t, 0)dt \right| \leq |F(T, 0)| + 1 + \frac{1}{2} \int_0^T \sigma(t)^2 |F(t, 0)|dt \]

\[ \leq 2 + \frac{1}{2} \int_0^T \sigma(t)^2 dt. \]
3 Some modifications

The approach demonstrated above allows many modifications. Let us provide one of possible modifications,

**Theorem 3.1** Consider a process \( \tilde{Y}(t) \) defined as

\[
\tilde{Y}(t) = \int_0^t \cos(x(t) - x(s))u(s)e^{\frac{1}{2} \int_s^t \sigma(r)^2 dr} ds,
\]

where \( t \in [0, +\infty) \). Further, let a process \( \tilde{X}(t) \) be defined as the stochastic integral

\[
\tilde{X}(t) = - \int_0^t \tilde{Y}(s)dx(s),
\]

Then \( \tilde{X}(t) \) can be represented as

\[
\tilde{X}(t) = - \int_0^t \sin(y(t) - y(s))e^{\frac{1}{2} \int_s^t \sigma(r)^2 dr} u(s)ds.
\]

Clearly, the process \( \tilde{Y}(t) \) is bounded uniformly in all \( a(\cdot) \) almost surely on any finite time interval and \( \mathcal{G}_t \)-adapted. Hence the stochastic integral (7) is well defined.

Representation (8) in Theorem 3.1 implies that the boundaries for the range of the stochastic integral \( \tilde{X}(t) \) are defined by the choice of the process \( u \). Respectively, Theorem 3.1 allows to construct stochastic processes with preselected on a given time interval time depending boundaries for their range.

**Corollary 3.2** Let \( T > 0 \) be fixed, and let \( \psi(t) : (0, T) \to \mathbb{R} \) be a integrable function. Let \( u(t) = e^{-\frac{1}{2} \int_t^T \sigma(s)^2 ds} \psi(t) \). Then

\[
\tilde{X}(t) = - \int_0^t \sin(y(t) - y(s))\psi(s)ds,
\]

\[
\tilde{Y}(t) = \int_0^t \cos(y(t) - y(s))\psi(s)ds, \quad t \in [0, T], \quad T \in (0, \infty),
\]

and

\[
|\tilde{X}(t)| \leq \sqrt{\tilde{X}(t)^2 + \tilde{Y}(t)^2} \leq \int_0^t |\psi(s)|dt \quad a.s. \quad \forall a(\cdot).
\]

**Example 3.3** (i). if \( |\psi(t)| = \alpha^{-1} t^{\alpha-1} \) for \( \alpha > 1/2 \), then \( \sqrt{\tilde{X}(t)^2 + \tilde{Y}(t)^2} \leq t^\alpha \) a.s. for all \( a(\cdot) \).

(ii). If \( |\psi(t)| = 1/(a + t) \) for some \( q > 0 \), then \( \sqrt{\tilde{X}(t)^2 + \tilde{Y}(t)^2} \leq \ln(q + t) - \ln q \) a.s. for all \( a(\cdot) \).
3.1 Proof of Theorem 3.1

The proof is similar to the proofs of Theorem 2.1; we provide it for completeness.

Consider a process

\[ d\tilde{Z}(t) = i[\tilde{Z}(t)dx(t) + u(t)dt], \quad t \in (0, \infty), \]
\[ \tilde{Z}(0) = 0. \]  \hspace{1cm} (11)

In (11), \( i = \sqrt{-1} \) is the imaginary unit.

Lemma 3.4 For any \( T > 0 \), we have that

\[ \tilde{Z}(t) = i \int_0^t e^{i[x(t) - x(s)]} e^{\frac{1}{2} \int_s^t \sigma(r)^2 dr} u(s) ds \]

and

\[ \text{Re} \tilde{Z}(t) = \int_0^t \text{Im} \tilde{Z}(s)dx(s) \quad \forall t > 0. \]

The process \( \tilde{Z}(s) \) is \( \mathcal{G}_t \)-adapted and such that

\[ |\tilde{Z}(t)| \leq e^{\frac{1}{2} \int_0^T \sigma(s)^2 ds} |u(t)| \quad \forall T > 0, \quad \forall t \in [0, T]. \]

Proof of Lemma 3.1. Let \( \tilde{F}(t, s) \) be defined as the solution of the Itô equation

\[ d\tilde{F}(t, s) = i\tilde{F}(t, s)[a(t)dt + \sigma(t)dw(t)], \quad t > s \geq 0, \quad \tilde{F}(s, s) = 1. \]

By the Itô formula,

\[ \tilde{F}(t, s) = \tilde{F}(s, s) \exp \left( i \int_s^t a(r) dr + i \int_s^t \sigma(r)dw(r) - \frac{i^2}{2} \int_s^t \sigma(r)^2 dr \right) \]
\[ = \exp \left( i \int_s^t a(r) dr + i \int_0^t \sigma(r)dw(r) + \frac{1}{2} \int_s^t \sigma(r)^2 dr \right) \]
\[ = \exp \left( i[y(t) - y(s)] + \frac{1}{2} \int_s^t \sigma(r)^2 dr \right) \quad \text{a.s.} \]

In particular, we have that

\[ |\tilde{F}(t, s)| = \exp \left( \frac{1}{2} \int_s^t \sigma(r)^2 dr \right) \quad \text{a.s.} \]

Direct differentiation gives that

\[ \tilde{Z}(t) = i \int_0^t \tilde{F}(t, s)u(s)ds. \]
Hence
\[ |\tilde{Z}(t)| \leq \int_0^t \tilde{F}(t,s)|u(s)|ds \leq \int_0^t e^{\frac{1}{2} \int_0^s \sigma(r)^2 dr} |u(s)|ds. \]

Let \( \tilde{X}(t) = \text{Re} \tilde{Z}(t) \) and \( \tilde{Y}(t) = \text{Im} \tilde{Z}(t) \). We have that
\[ d\tilde{X}(t) = -\tilde{Y}(t) dx(t), \]
\[ d\tilde{Y}(t) = \tilde{X}(t) dx(t) + u(t). \]

It follows that
\[ X(T) = \int_0^T \gamma(t) dx(t), \]
where \( \gamma(t) = -\tilde{Y}(t) \). We have that
\[ \int_0^t |\gamma(s)|ds \leq \int_0^t |\tilde{Z}(s)|ds \leq \int_0^t e^{\frac{1}{2} \int_0^s \sigma(r)^2 dr} |u(s)|ds \leq e^{\frac{1}{2} \int_0^T \sigma(r)^2 dr} \int_0^t |u(s)|ds \quad \forall T > 0, \quad t \in [0,T]. \]

Hence \( |\gamma(t)| \leq |\tilde{Z}(t)| \leq e^{\frac{1}{2} \int_0^T \sigma(r)^2 dt} |u(t)| \) for any \( T > 0 \) and \( t \in [0,T] \).

Then the proof of Lemma follows. \( \square \)

The proof of Theorem 3.1 follows from the lemma. The proof of Corollary 3.2 follows from the theorem applied to the corresponding choice of \( u \).

Remark 3.5 For the case where \( a(t) \equiv 0 \), similarly Remark 2.3, it can be shown that, that, for any \( \mathcal{F}_t \)-adapted process \( \sigma(\cdot) \), there exists a \( \mathcal{F}_t \)-adapted process \( U : [0,T] \times \Omega \rightarrow \mathbb{C} \) such that \( |U(t)| = \exp \left( \frac{1}{2} \int_0^t \sigma(r)^2 dr \right) \) and that the integral \( \int_0^T \sigma(t) U(t) dw(t) \) has a limited range in \( \mathbb{C} \). To see this, it suffices to select \( U(t) = i\tilde{F}(t,0) \) and observe that \( \int_0^T \sigma(t)i\tilde{F}(t,0) dw(t) = \tilde{F}(T,0) - 1. \)

4 Possible applications for financial modelling

One of core problems of financial mathematics is the portfolio selection problem. Application of classical methods of optimal stochastic control for portfolio optimization problems requires forecasting of market parameters. This forecasting is usually difficult. This problem is related to the open problem of validation of the so-called technical analysis methods that offer trading strategies based on historical observations. There are many different strategies suggested in this framework (see, e.g., [1, 2, 4, 14, 15, 5, 6] and the references therein. It is known that mean-reverting market models and market models with bounded range for the prices generate some special speculative opportunities (see, e.g., [3, 4, 7, 8, 10, 13, 16, 17, 18]). Theorems 2.1-3.1 give
a possibility to convert a stock price process \( x(t) \) into processes \( Y(t) \) or \( X(t) \) that could have features similar to mean-reverting market models and market models with bounded range for the prices. For this new artificial asset, one can apply strategies from \([3, 4, 7, 8, 10, 13]\). We leave this for the future research.

Figures 1-6 shows sample paths of processes introduced above and obtained via Monte-Carlo simulation under the assumption that \( T = 5 \), \( \sigma(t) \equiv \sigma = 1 \), \( a(t) \equiv a = 2 \), and \( u(t) \equiv 1 \). We used natural discretization in time with \( 10^5 \) grid point on the interval \([0, T]\). Calculations were executed using R and RStudio programmes.

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Figure 1: $x(t)$.

Figure 2: $X(t)$.

Figure 3: $Y(t)$. 

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Figure 4: $Z(t) = X(t) + iY(t)$.

Figure 5: $|Z(t)|$.

Figure 6: $\int_0^t X(s) dx(s)$.