The creation of multiple images
by a gravitational wave

Valerio Faraoni

Department of Physics and Astronomy, University of Victoria, P.O. Box 3055
Victoria, B.C. Canada V8W 3P6

Abstract

We describe gravitational lensing by a gravitational wave, in the regime in
which multiple images of a light source are created. We adapt the vector for-
malism employed for ordinary gravitational lenses to the case of a non–stationary
spacetime, and we derive an approximate condition for multiple imaging. It is
shown that certain astrophysical sources of gravitational waves satisfy this condi-
tion.

To appear in Proceedings of the Pacific Conference on Gravitation and Cosmology,
Seoul, Korea, 1–6 February 1996.
We study gravitational lensing with a gravitational wave acting as the lens. It is well-known that exact gravitational waves deflect light, and light propagation through linearized (cosmological) gravitational waves outside the laboratory has also been studied [1], especially in conjunction with their frequency shift effect on the photons of the cosmic microwave background. It has also been suggested that gravitational waves may create multiple images of distant light sources [2]. We concentrate on the latter aspect of lensing by gravitational waves, for two reasons: i) the possibility of multiple images is associated to strong amplifications of the light source, which makes easier to detect gravitational waves; ii) a previous analysis [3] concluded that the amplification is negligible. This conclusion was based on the Raychaudhuri’s equation, and it is not valid when multiple images are involved.

1 The vector formalism for lensing gravitational waves

We consider a gravitational wave localized in a region of space between a light source and an observer. The spacetime metric is $g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}$, where $\eta_{\mu \nu}$ is the Minkowski metric and $|h_{\mu \nu}| \ll 1$. Let us consider a light ray whose unperturbed path is parallel to the $z$-axis, with 4-momentum $p^\mu = p^\mu(0) + \delta p^\mu = (1 + \delta p^0, \delta p^1, \delta p^2, 1 + \delta p^3)$, where $\delta p^\mu$ are small deflections. We work in the geometric optics approximation, which holds if $\lambda_{g.w.} \gg \lambda_{e.m.}$ and $\lambda_{e.m.} > \lambda_{g.w.}^2 / D_L$ (where $\lambda_{g.w.}$ and $\lambda_{e.m.}$ are the wavelengths of the gravitational wave and of the photon, respectively, and $D_L$ is the observer-lens distance). We set the geometry as customary in gravitational lens theory, using lens and source planes with coordinates $(x, y)$ and $(s_x, s_y)$ respectively. The action of the lens is described by the plane-to-plane mapping $x^A \mapsto s^A (A = 1, 2)$, with $s$ given by the “lens equation”

\[
s^A = x^A + \frac{D_L D_{LS}}{D_S} \delta p^A (\mathbf{x})
\]  

(1.1)

($D_{LS}$ and $D_S$ are the lens-source and the observer-source distances respectively).

The map described by Eq. (1.1) has the Jacobian matrix

\[
J \left( \begin{array}{c} s \\ \mathbf{x} \end{array} \right) = \left( \begin{array}{cc} \frac{\partial s^A}{\partial x^B} & \frac{\partial s^A}{\partial s^B} \\ \frac{D \partial_x (\delta p^x)}{D} & \frac{D \partial_y (\delta p^y)}{1 + D \partial_y (\delta p^y)} \end{array} \right)
\]  

(1.2)

where $D \equiv D_L D_{LS} / D_S$. The inverse matrix $A = J^{-1}$ represents the amplification tensor, while its determinant $A = \text{Det}(J)^{-1}$ is the (scalar) amplification. Since the
surface brightness is conserved during lensing whenever geometric optics holds and
\[ A = \frac{\text{area of an infinitesimal region in the lens plane}}{\text{area of the corresponding region in the source plane}}, \]

(1.3)
\( A \) has also the meaning of the ratio of light intensities with and without the lens. A small circular source will be imaged into a small ellipse whose eccentricity \( e \) is given by the ratio of the eigenvalues \( e_\pm \) of \( A \):
\[ (1 - e^2)^{1/2} = \left| \frac{e_+}{e_-} \right|. \]

(1.4)
The vanishing of the Jacobian \( \text{Det}(J) \) indicates the failure of invertibility of the map (1.1). Therefore, the occurrence of multiple images is signalled by the vanishing of \( \text{Det}(J) \), and this is the condition that we study in the following.

The Jacobian determinant \( \text{Det}(J) \) can be written as follows:
\[
\text{Det}(J) = 1 + \sqrt{f(\alpha)} D_S \frac{\partial (\delta p^A)}{\partial x^A} + f(\alpha) D^2_S \left[ \partial_x (\delta p^x) \cdot \partial_y (\delta p^y) - \partial_y (\delta p^x) \cdot \partial_x (\delta p^y) \right] \equiv 1 + J_1 + J_2, \]

(1.5)
where \( \alpha \equiv D_{LS}/D_S \) and \( f(\alpha) = \alpha^2(1 - \alpha)^2 \) is symmetric about \( \alpha = 1/2 \), where it assumes its maximum value \( 1/16 \). The deflection \( \delta p^\mu \), computed from the equation of null geodesics, is
\[ \delta p^A = \frac{1}{2} \int_{\text{Observer}}^{\text{Source}} dz \left( h_{00} + 2h_{03} + h_{33} \right)^A + O(h^2). \]

(1.6)
Moreover, \( \partial_A (\delta p^A) = 0 \) to first order and \( J_1 \sim h^2 D/P \ll J_2 \sim h^2 (D/P)^2 \) for large values of \( D/P \). Thus, in order to have multiple imaging, it must be \( J_2 < 0 \) and \( f(\alpha) \left[ D_S \partial_A (\delta p^B) \right] \approx 1 \). For ordinary gravitational lenses, the probability of lensing of a distant source is maximum when the lens is halfway between the source and the observer, hence we take \( f(\alpha) \) in the range \( \frac{1}{100} - \frac{1}{16} \), near its maximum. Then, in order to have multiple imaging, it must be
\[ \frac{h}{P} \geq \mathcal{S}_c, \]

(1.7)
where \( P \) is the period of the gravitational wave, \( \mathcal{S}_c \equiv (4 - 10) c/D \). The approximate condition for multiple imaging (1.7) involves the “strength” \( h \) and the “size” \( P \) of gravitational waves, and the geometry of the problem (through \( D \)). (1.7) is somewhat analogous to the well-known condition for multiple imaging by ordinary gravitational lenses, \( \Sigma \geq \Sigma_c \equiv \frac{c^2}{4\pi G} \frac{D_S}{D_L D_{LS}} \).
2 Lensing by gravitational waves and the Fermat principle

The vector formalism for ordinary gravitational lenses requires the stationarity of the lens potential; clearly this hypothesis is not satisfied in our case, and a new approach is needed. We substitute the true photon path in the 3-dimensional space with a zig–zag path composed of two straight lines from the source to the lens, and from the lens to the observer. By applying a new version of the Fermat principle valid for non(conformally)–stationary spacetimes [4], it was shown in Ref. [5] that the zig–zag paths are extrema of the travel time functional

\[
\tilde{t} = \text{constant} + \frac{1}{2c} \left[ \frac{D_S}{D_LD_{LS}} (\vec{x} - \vec{s})^2 + \int_{S}^{O} dz (h_{00} + 2h_{03} + h_{33}) \right].
\] (2.1)

The lens equation (1.1) and the deflection (1.6) are obtained by requiring stationarity of this functional: \(\nabla \tilde{t} = 0\). Therefore, the conclusions of the previous section are justified in the context of a rigorous formalism.

3 Comparison between a gravitational wave and an ordinary gravitational lens

We compare the action of a gravitational wave with that of an ordinary gravitational lens. The latter is a mass distribution described by a Newtonian potential \(\Phi\) (satisfying the Poisson equation \(\nabla^2 \Phi = 4\pi \rho\), where \(\rho\) is the lens mass density). The plane–to–plane map describing the lens action is given by the lens equation and the Jacobian matrix can be written as

\[
J = \begin{pmatrix} 1 - \chi - \Lambda & -\mu \\ -\mu & 1 - \chi + \Lambda \end{pmatrix},
\] (3.1)

with

\[
\chi \equiv \frac{\Sigma}{\Sigma_c},
\] (3.2)

\[
\Lambda \equiv \frac{D}{c^2} \int_{-\infty}^{+\infty} dl \left( \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} \right),
\] (3.3)

\[
\mu \equiv \frac{D}{c^2} \int_{-\infty}^{+\infty} dl \frac{\partial^2 \Phi}{\partial x \partial y},
\] (3.4)

3
where $D$ and $\Sigma_c$ have been defined in the previous sections and

$$\Sigma \equiv \int_{-\infty}^{+\infty} dl \rho .$$  \hspace{1cm} (3.5)$$

The Jacobian determinant is given by

$$\text{Det}(J) = (1 - \chi)^2 - (\Lambda^2 + \mu^2) .$$  \hspace{1cm} (3.6)$$

The convergence $\chi$ describes the action of matter, while $\Lambda$ and $\mu$ describe the action of shear. For a lensing gravitational wave one obtains

$$J_{gw} = \begin{pmatrix} 1 - \Lambda_1 & -\mu_1 \\ -\mu_2 & 1 - \Lambda_2 \end{pmatrix} .$$  \hspace{1cm} (3.7)$$

where, to first order, one has

$$\Lambda_1 \equiv -\frac{D}{2} \int_S^O d\lambda \ h_{\lambda\lambda}^{x,x} ,$$  \hspace{1cm} (3.8)$$

$$\Lambda_2 \equiv -\frac{D}{2} \int_S^O d\lambda \ h_{\lambda\lambda}^{y,y} ,$$  \hspace{1cm} (3.9)$$

$$\mu_1 \equiv -\frac{D}{2} \int_S^O d\lambda \ h_{\lambda\lambda}^{x,y} ,$$  \hspace{1cm} (3.10)$$

$$\mu_2 \equiv -\frac{D}{2} \int_S^O d\lambda \ h_{\lambda\lambda}^{y,x} ,$$  \hspace{1cm} (3.11)$$

where the integrals are computed along the photon’s path from the source to the observer. To first order, $\Lambda_2 = -\Lambda_1 \equiv -\Lambda_{gw}$ and $\mu_1 = \mu_2 \equiv \mu_{gw}$. To the lowest order, we have

$$J_{gw} = \begin{pmatrix} 1 - \Lambda_{gw} & -\mu_{gw} \\ -\mu_{gw} & 1 + \Lambda_{gw} \end{pmatrix} .$$  \hspace{1cm} (3.12)$$

The convergence term is absent and hence the lens action is due only to the shear. This result was derived in Ref. [3] using the Raychaudhuri equation and the optical scalars formalism, and is now recovered in the vector formalism.

The deflection angle does not depend on the frequency of the light: gravitational waves are achromatic lenses, like ordinary gravitational lenses. However, while the latter do not shift the frequency of the photons propagating through them, lensing gravitational waves do. In addition, gravitational waves do not rotate the polarization plane of the electromagnetic field, to first order $[6]$. In this aspect, they behave like ordinary gravitational lenses.
4 Order of magnitude estimates

In order to apply the previous theory, and for the multiple images to be detectable, the following conditions must be satisfied:

1. the scale of separation $\delta \approx h$ between different images must not be smaller than $10^{-3}$ arcseconds, which gives
   \[ h \geq 5 \cdot 10^{-9} \]  \hspace{1cm} (4.1)
   at the deflection place;

2. the impact parameter $r$ must satisfy
   \[ r > \lambda_{g.w.}; \]  \hspace{1cm} (4.2)

3. the lens must not be exceptionally rare;

4. the period of the lensing gravitational wave must not be too short (let us say $P < 10^8$ s), in order to appreciate variability of the images.

Multiple imaging by gravitational waves is possible, and it occurs when the waves satisfy the approximate inequality $(1.7)$. We examine the validity of this condition for the most common astrophysical sources of gravitational radiation.

Stellar core collapse: using the values of $h$ and $P$ predicted in studies of collapsing homogeneous ellipsoids, (4.1) and (4.2) allow only a rather narrow range for the impact parameter $r$, for which $Dh/P \sim 10$ if $D \sim 6 \cdot 10^{15}$ cm. The late phase when the ellipsoid has settled down as a rapidly rotating neutron star does not give appreciable lensing. Asymmetry due to the core’s bouncing gives $Dh/P \sim 10$ if $D \sim 10^{12}$ cm, $r \sim 3 \cdot 10^7$ cm, or if $D \sim 6 \cdot 10^{16}$ cm, $r \sim 2 \cdot 10^{12}$ cm. Extrapolating the results obtained in studies of the perturbations of pressureless spherical collapse leading to the formation of a black hole, one obtains $Dh/P \sim 10$ if $D \sim 3 \cdot 10^{16}$ cm (with a rather narrow range of values of $r$).

Final decay of a neutron star/neutron star binary: rough estimates for the final decay of a binary system composed of two neutron stars give $Dh/P \sim 10$ if $D \sim 3 \cdot 10^{16}$ cm, $r \sim 5 \cdot 10^{10}$ cm, or if $D \sim 6 \cdot 10^{7}$ cm, $r \sim 6 \cdot 10^{7}$ cm.

Black hole collisions: if two black holes collide with enough angular momentum to go into orbit before coalescing, one has $Dh/P \sim 10$ if $D \sim 3 \cdot 10^{17}$ cm, $r \sim 3 \cdot 10^{10}$ cm, or if $D \sim 6 \cdot 10^{19}$ cm, $r \sim 10^{12}$ cm.

The binary pulsar: the binary pulsar PSR 1913+16 is believed to radiate gravitational
waves in a continuous way with amplitude given, in order of magnitude, by $h \sim \dot{Q}/r \sim Ma^2 \omega^2/r$, where $Q$ and $M$ are the quadrupole moment and the mass of the pulsar, and $a$ is the semimajor axis of the binary system. Conditions (4.1) and (4.2) are incompatible, hence multiple imaging is not possible in this case.

The gravitational wave background: one has $Dh/P \approx \sqrt{\Omega_{g.w.}}D/R$, where $R$ is the radius of the universe and $\Omega_{g.w.}$ is the density of gravitational waves (in units of the critical density). Upper bounds on $\Omega_{g.w.}$ give $\Omega_{g.w.} < 1$ for all frequencies, and $\Omega_{g.w.} \ll 1$ in many bands. Moreover, since $D/R \ll 1$, also $Dh/P \ll 1$, and multiple imaging by the gravitational wave background is, on average, impossible.

As a conclusion, the creation of multiple images by a gravitational wave is possible and it is expected to occur. There are high amplification events associated to multiple images, in contrast with previous conclusions based on an inadequate formalism. The details of this work, and the study of the probability of observing multiple images and strong amplifications, and the details of the phenomenon will be the subject of a future publication [7].

Acknowledgments

The author is indebted to Prof. B. Bertotti for suggesting the possibility of lensing by gravitational waves and the approach used in this paper, and to Prof. G.F.R. Ellis for stimulating discussions.
References

[1] D.M. Zipoy 1966, *Phys. Rev.* **142**, 825; G. Dautcourt 1975, *Astron. Astrophys.* **38**, 344; R. Fakir 1993, *Astrophys. J.* **418**, 202; 1994, *Astrophys. J.* **426**, 74; 1994, *Phys. Rev. D* **50**, 3795.

[2] B. Bertotti, private communication; J.A. Wheeler 1961, in *Rendiconti della Società Italiana di Fisica, 11th Course of the Varenna Summer School* (Academic Press, New York); A. Labeyrie 1993, *Astron. Astrophys.* **268**, 823.

[3] B. Bertotti 1971, in *General Relativity and Cosmology*, R.K. Sachs ed. (Academic Press, New York).

[4] V. Perlick 1990, *Class. Quant. Grav.* **7**, 1319.

[5] V. Faraoni 1992, *Astrophys. J.* **398**, 425.

[6] V. Faraoni 1993, *Astron. Astrophys.* **272**, 385.

[7] V. Faraoni 1996 (submitted for publication).