THE EXISTENCE OF MAXIMUM LIKELIHOOD ESTIMATE IN HIGH-DIMENSIONAL GENERALIZED LINEAR MODELS WITH BINARY RESPONSES

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ABSTRACT. Motivated by recent works on the high-dimensional logistic regression, we establish that the existence of the maximum likelihood estimate (MLE) exhibits a phase transition for a wide range of generalized linear models (GLMs) with binary responses and elliptical covariates. This extends a previous result of Candès and Sur who proved the phase transition for the logistic regression with Gaussian covariates. Precisely, we consider the high-dimensional regime in which the number of observations $n$ and the number of covariates $p$ proportioned, i.e. $p/n \to \kappa$. We provide an explicit threshold $h_{\text{MLE}}$ depending on the unknown regression coefficients and the scaling parameter of covariates such that in high dimensional regime, if $\kappa > h_{\text{MLE}}$, then the MLE does not exist with probability 1, and if $\kappa < h_{\text{MLE}}$, then the MLE exists with probability 1. The main tools for deriving the result are data separation, convex geometry and stochastic approximation. We also conduct simulation studies to corroborate our theoretical findings, and explore other features of the problem.

1. Introduction

Generalized linear models (GLMs) provide a powerful tool in non-linear multivariate statistical analysis, with a wide range of applications including social sciences [30, 41], finance [21, 33], and public health [23, 35]. See also [32] for other applications of GLMs. In this paper, we are concerned with the maximum likelihood estimate (MLE) of GLMs with binary responses. Precisely, we consider $n$ i.i.d. observations $(x_i, y_i), i = 1, \ldots, n$, where the binary response $y_i \in \{-1, 1\}$ is connected to the covariates $x_i \in \mathbb{R}^p$ by the probability model

$$P(y_i = 1 | x_i) = \sigma(\beta_0 + x_i^T \beta).$$

(1.1)

Here $\sigma : \mathbb{R} \to [0, 1]$ is the inverse link function, and $(\beta_0, \beta) \in \mathbb{R}^{p+1}$ are unknown parameters of the model. Popular choices for $\sigma$ are

- *logit link function* [6]: $\sigma(t) = \frac{e^t}{1 + e^t}$.
- *probit link function* [8]: $\sigma(t) = \Phi(t)$, where $\Phi$ is the CDF of standard normal.
- *cloglog link function* [19]: $\sigma(t) = 1 - e^{-e^t}$.

The MLE of $(\beta_0, \beta)$ is any maximizer of the log-likelihood

$$\ell(\beta_0, \beta) := \sum_{y_i = 1} \log(\sigma(\beta_0 + x_i^T \beta)) + \sum_{y_i = -1} \log(\sigma(\beta_0 + x_i^T \beta)).$$

(1.2)

In contrast to the MLE of linear models, the MLE of GLMs does not always exist. This phenomenon is closely related to the separability of observed data, see Section 2.1 for a
review. Classical theory deals with this issue when the number of covariates $p$ is fixed, and the number of observed data $n$ tends to infinity. In the era of big data and data deluge, we are often in a situation where the number of covariates $p$ and the number of observations $n$ are comparable in size. The problems of interest are in high-dimensional asymptotics, in which case the number of parameters $p$ and the number of observations $n$ both tend to infinity, at the same rate.

In a series of papers \cite{11,39,40}, Sur, Chen and Candès developed a theory for the logistic regression with Gaussian covariates in high-dimensional regimes. They studied the asymptotic properties of the MLE when $p/n \to \kappa$, with applications in hypothesis testing. Candès and Sur \cite{11} proved a phase transition for the existence of the MLE in high-dimensional logistic regression with Gaussian covariates. This extends an earlier result of Cover \cite{13} in the context of information theory. Formally, there exists a threshold $h_{\text{MLE}}$, depending on the parameters of the model, such that

- if $\kappa > h_{\text{MLE}}$, then $P(\text{MLE exists}) \to 0$ as $n,p \to \infty$.
- if $\kappa < h_{\text{MLE}}$, then $P(\text{MLE exists}) \to 1$ as $n,p \to \infty$.

This phenomenon is referred to as the phase transition for the MLE existence. The existence of the MLE is crucial to justify the use of large sample approximations to numerous measures of goodness-of-fit, and derive the limiting distribution of the likelihood ratio, as mentioned in \cite{11}. But they only studied the existence of the MLE for Gaussian covariates, which is not always the case in reality. For instance, the covariates are often heavy-tail distributed in financial problems where $p$ and $n$ are large.

The purpose of this paper is to further generalize the results of \cite{11}, proving the phase transition for a large class of GLMs with elliptical covariates which are distribution-free. Here we consider a large number of covariates sampled from elliptical distributions, and predict whether one can expect the MLE to be found or not. Elliptical symmetry is a natural generalization of multivariate normality. The contribution of this paper is twofold.

- **Theoretical justifications.** We give a universal threshold on $p/n$ for the existence of the MLE in the binary classification. Here the word 'universal' refers to a wide class of link functions and covariate distributions. Our work aims to explore to which extent the phase transition occurs in terms of link functions and covariates, including the logit link and Gaussian covariates as a special case. We notice that the projection limit assumption (Assumption 2.6) is essential to our results, which is disguised for the special choice of Gaussian covariates. Without this assumption, the phase transition formula might fail, e.g. log-normal covariates.

- **Novel techniques.** We also bring a few techniques into this field. First, we provide a checkable condition (à la Carleman) to the projection limit assumption. Second, we use a stochastic approximation argument to prove the phase transition formula. Finally, we improve some arguments in \cite{11}, e.g. Proposition 4.2 which reveals additional structure masked by Gaussianity.

We believe that the phase transition also holds for multinomial response models such as the Poisson regression and the log-linear regression. See \cite{14,15} for further discussions. We hope that this work will trigger further research towards a theory of hypothesis testing for GLMs with non-Gaussian covariates.
The rest of the paper is organized as follows. In Section 2, we provide background and state the main result, Theorem 2.7. In Section 3, we use simulations to corroborate our theoretical findings. A proof sketch of Theorem 2.7 is given in Section 4.

2. Background and Main Result

In this section we provide background on the existence of the MLE in GLMs, and the properties of elliptical distributions. Then we present the main result, Theorem 2.7.

2.1. Existence of the MLE and Data Geometry. Often the MLE in the logit model, implemented in many statistical packages, runs smoothly. But sometimes it fails, even when the number of covariates \( p \) is much smaller than the sample size \( n \). One reason for this undesirable phenomenon is that the MLE does not exist. It is a classical problem in statistics to characterize the existence and uniqueness of the MLE in GLMs.

Historically, Haberman [20] and Weddowburn [42] provided general criteria for the MLE to exist. Silvapulle [38], and Albert and Anderson [1] gave conditions for the existence of the MLE in logistic regression via data geometry. Precisely, they classified the data into the following three categories:

- The data points \((x_i, y_i)\) are said to be completely separated if there exists \( b \in \mathbb{R}^p \) such that \( y_i x_i^T b > 0 \) for all \( i \).
- The data points \((x_i, y_i)\) are said to be quasi-completely separated if for each \( b \neq 0 \), \( y_i x_i^T b \geq 0 \) for all \( i \), and equality holds for some \( i \).
- The data points \((x_i, y_i)\) are said to overlap if for each \( b \neq 0 \), there exists one \( i \) such that \( y_i x_i^T b > 0 \), and another \( i \) such that \( y_i x_i^T b < 0 \).

In [1], it was proved that the MLE exists in logistic regression if and only if the data points overlap. See also [36] for a generalization. Later Lesaffre and Kaufmann [29] proposed a necessary and sufficient condition for the existence of the MLE in probit regression, which coincides with that derived in [1] for logistic regression. In fact, their result holds for a general class of GLMs.

**Theorem 2.1.** Consider the GLM defined by (1.1), and assume that \( \sigma(\cdot) \) and \( 1 - \sigma(\cdot) \) are log-concave. Then the MLE exists if and only if the data points overlap.

It is easily seen that the logit, probit and cloglog links all satisfy the log-concavity. Despite this nice characterization, it is not clear how to check these criteria efficiently. See also [12, 15, 28] for algorithmic aspects for detecting separation/overlaps.

2.2. Elliptical Distributions. Elliptical distributions are natural generalizations of multivariate normal, which preserve spherical symmetry. In the sequel, \( S^{p-1} \) denotes the unit sphere in \( \mathbb{R}^p \). The following definition of elliptical distributions is due to Kelker [25], and Cambanis, Huang and Simons [10].

**Definition 2.2.** A random vector \( \mathbf{X} := (X_1, \ldots, X_p) \in \mathbb{R}^p \) is elliptically contoured, or simply elliptical if

\[
\mathbf{X} \overset{(d)}{=} \mu + R \mathbf{A} \mathbf{U},
\]

where \( \mu \in \mathbb{R}^p \), \( \mathbf{A} \in \mathbb{R}^{p \times r} \), \( \mathbf{U} \) is uniformly distributed on \( S^{r-1} \) for some \( r > 0 \), and \( R \) is a non-negative random variable independent of \( \mathbf{U} \). Write \( \mathbf{X} \sim \mathcal{E}_p(\mu, \Sigma, F_R) \), where \( \Sigma = \mathbf{A} \mathbf{A}^T \), and \( F_R \) is the CDF of \( R \).
If \( \mu = 0, \ r = p, \) and \( A \) is orthogonal, then \( X \sim \mathcal{E}_p(0, I_p, F_R) \) is said to be spherically symmetric. For \( X \sim \mathcal{N}(0, I_p) \), the random variable \( R \) is chi-distributed with degree of freedom (df) \( p \). See Appendix A for a collection of properties of elliptical distributions, which explain the assumptions in Section 2.3.

### 2.3. Main Result

Before stating the main result, we make a few assumptions on the link function, the covariate distribution, and model parameters. As seen in Section 2.1, the existence of the MLE in GLMs can be translated into data geometry. Thus, we need the assumptions on the link function in Theorem 2.1.

**Assumption 2.3** (link function). For \( \sigma : \mathbb{R} \to [0, 1], \) both \( \sigma(\cdot) \) and \( 1 - \sigma(\cdot) \) are log-concave.

The phase transition for the existence of the MLE is expected to occur not only with Gaussian distributions but with a broad range of covariate distributions. Here we consider elliptical distributions. To exclude singularity, we assume that the covariate distribution is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^p \).

**Assumption 2.4** (covariate distribution). The covariates \( x_i \sim \mathcal{E}_p(\mu, \Sigma, F_R) \) are of full rank. That is, \( r = p = \text{rank}(\Sigma) \) and \( F_R \) is absolutely continuous on \( \mathbb{R}_+ \).

To get a meaningful result in diverging dimension, we consider a sequence of problems with the intercept \( \beta_0 \) fixed, and \( \text{Var}(x_i^T \beta) \to \gamma_0^2 \). Recall that for \( x_i \sim \mathcal{E}_p(\mu, \Sigma, F_R) \), we have \( \text{Var}(x_i^T \beta) = \frac{\text{Var}(x_i^T \beta)}{p} \). This leads to the following assumption on model parameters.

**Assumption 2.5** (parameter scaling). As \( p \to \infty, \ E R^2 / p \to \alpha_0^2 \) and \( |\beta| \to \gamma_0 / \alpha_0 \).

In the remaining of the paper, we define
\[
(Y^{(p)}, X^{(p)}) \sim F_{\alpha_0, \beta_0, \gamma_0} \quad \text{if} \quad (Y^{(p)}, X^{(p)}) \overset{(d)}{=} (V^{(p)}, V^{(p)} U^{(p)}),
\]
where \( U^{(p)} \) is distributed as any component of \( x_i \sim \mathcal{E}_p(0, I_p, F_R) \), and \( \mathbb{P}(V^{(p)} = 1|U^{(p)}) = 1 - \mathbb{P}(V^{(p)} = -1|U^{(p)}) = \sigma(\beta_0 + \gamma_0 / \alpha_0) \). Here the superscript \( (p) \) emphasizes that the distribution of \( (Y^{(p)}, X^{(p)}) \) or \( (V^{(p)}, U^{(p)}) \) may depend on \( p \). This dependence is easily ignored since for \( x_i \sim \mathcal{N}(0, I_p) \), \( U^{(p)} \) is distributed as standard normal independent of \( p \). For the general elliptical covariates, we make the following technical assumption.

**Assumption 2.6** (projection limit). The projection \( U^{(p)} \) converges in distribution to \( U \). That way, \( (Y^{(p)}, X^{(p)}) \) converges in distribution to \( (Y, X) \).

Now we give a sufficient condition for Assumption 2.6 to hold. Let \( m_{p,k} \) be the \( k^{th} \) moment of \( U^{(p)} \). If for each \( k \geq 1, \ m_{p,k} \to m_k \) as \( p \to \infty \) and
\[
\sum_{k=1}^{\infty} m_{p,k}^{-1} = \infty,
\]
then \( U^{(p)} \) converges in distribution to \( U \) whose distribution is entirely characterized by the moments \( (m_k; k \geq 1) \). The condition (2.3) is referred to as the Carleman’s condition. See Lin [31, Theorem 1] for a list of equivalent conditions.

Define
\[
p_+(x) = \sigma \left( \beta_0 + \frac{\gamma_0}{\alpha_0} x \right) \quad \text{and} \quad p_-(x) := 1 - p_+(x).
\]

### Main Result

Define
\[
(X^{(p)}, Y^{(p)}) \sim \mathcal{E}_p(\mu, \Sigma, F_R) \quad \text{if} \quad (X^{(p)}, Y^{(p)}) \overset{(d)}{=} (V^{(p)}, V^{(p)} U^{(p)}),
\]
where \( U^{(p)} \) is distributed as any component of \( x_i \sim \mathcal{E}_p(0, I_p, F_R) \), and \( \mathbb{P}(V^{(p)} = 1|U^{(p)}) = 1 - \mathbb{P}(V^{(p)} = -1|U^{(p)}) = \sigma(\beta_0 + \gamma_0 / \alpha_0) \). Here the superscript \( (p) \) emphasizes that the distribution of \( (X^{(p)}, Y^{(p)}) \) or \( (V^{(p)}, U^{(p)}) \) may depend on \( p \). This dependence is easily ignored since for \( x_i \sim \mathcal{N}(0, I_p) \), \( U^{(p)} \) is distributed as standard normal independent of \( p \). For the general elliptical covariates, we make the following technical assumption.

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\[
\sum_{k=1}^{\infty} m_{p,k}^{-1} = \infty,
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then \( U^{(p)} \) converges in distribution to \( U \) whose distribution is entirely characterized by the moments \( (m_k; k \geq 1) \). The condition (2.3) is referred to as the Carleman’s condition. See Lin [31, Theorem 1] for a list of equivalent conditions.

Define
\[
p_+(x) = \sigma \left( \beta_0 + \frac{\gamma_0}{\alpha_0} x \right) \quad \text{and} \quad p_-(x) := 1 - p_+(x).
\]
Denote $f_X$ as the density of $X$. Also define

$$G_{p,+}(x) = \int_{z \leq x} p_{+}(z)f_{X(p)}(z)dz \quad \text{and} \quad G_{p,-}(x) = \int_{z \leq x} p_{-}(z)f_{X(p)}(z)dz,$$

$$\overline{G}_{p,+}(x) = \int_{z > x} p_{+}(z)f_{X(p)}(z)dz \quad \text{and} \quad \overline{G}_{p,-}(x) = \int_{z > x} p_{-}(z)f_{X(p)}(z)dz. \quad (2.5)$$

So $G_{p,+}(x) + G_{p,-}(x)$ is the CDF of $X^{(p)}$, and $G_{p,\pm}(x) + \overline{G}_{p,\pm}(x) = EP_{\pm}(X^{(p)})$. The main result is stated as follows.

**Theorem 2.7.** Let $(Y^{(p)}, X^{(p)}) \sim F_{\alpha_0, \beta_0, \gamma_0}$, and $(Y, X)$ be the limit in distribution under Assumption \[2.6\]. Let $Z \sim \mathcal{N}(0, 1)$ be independent of $(Y, X)$. Define

$$h_{\text{MLE}}(\alpha_0, \beta_0, \gamma_0) := \lim_{p \to \infty} \min_{\lambda_0, \lambda_1 \in \mathbb{R}} E(\lambda_0 Y^{(p)} + \lambda_1 X^{(p)} - Z)^2_{+},$$

$$= \min_{\lambda_0, \lambda_1 \in \mathbb{R}} E(\lambda_0 Y + \lambda_1 X - Z)^2_{+}, \quad (2.6)$$

where $x^2_+ := (\max\{x, 0\})^2$. If Assumptions \[2.3, 2.5\] are satisfied, and $\sup_{p} E[(X^{(p)})^8] < \infty$, and

$$E[p_{\pm}(X^{(p)})(G_{p,+}(X^{(p)}) + \overline{G}_{p,+}(X^{(p)}))^{n-1}] = o \left( \frac{1}{n} \right), \quad (2.7)$$

then we have

$$\kappa > h_{\text{MLE}}(\alpha_0, \beta_0, \gamma_0) \implies \lim_{n,p \to \infty} \mathbb{P}(\text{MLE exists}) = 0.$$

$$\kappa < h_{\text{MLE}}(\alpha_0, \beta_0, \gamma_0) \implies \lim_{n,p \to \infty} \mathbb{P}(\text{MLE exists}) = 1.$$

The assumption $\sup_{p} E[(X^{(p)})^8] < \infty$, which is purely technical, is used to prove the law of large numbers for triangle arrays. As suggested by simulations in Section \[3\] the phase transition exists even without this moment condition. The assumption \[2.7\] is used to prove that the probability the data points can be separated via a univariate model is small. A sufficient condition for \[2.7\] to hold is that $G_{p,+} + \overline{G}_{p,+}$ is bounded away from 1. That is, there exists $\epsilon > 0$ independent of $p$ such that

$$G_{p,+}(x) + \overline{G}_{p,+}(x) < 1 - \epsilon \quad \text{for all } x. \quad (2.8)$$

So the term $E[p_{\pm}(X^{(p)})(G_{p,+}(X^{(p)}) + \overline{G}_{p,+}(X^{(p))))^{n-1}]$ is exponentially small in $n$. To illustrate, we check the condition \[2.8\] for the logistic regression with Gaussian covariates. To simplify the discussion, we take $\beta_0 = 0$, and $\alpha_0 = \gamma_0 = 1$. In this case, $p_{+}(x) = 1 - p_{-}(x) = e^x/(1 + e^x)$ and $X^{(p)} = X \sim \mathcal{N}(0, 1)$. Fixing $x \geq 0$, we get

$$G_{p,-}(x) + \overline{G}_{p,+}(x) = \frac{1}{2} + \int_{x}^{\infty} \frac{e^z - 1}{1 + e^z} f_X(z)dz < 1.$$

Similarly, we can prove that $G_{p,-}(x) + \overline{G}_{p,+}(x) < 1$ for $x < 0$. It is also easily checked that all examples in Section \[3\] except the log-normal distribution satisfy the sufficient conditions \[2.3, 2.8\].
3. Empirical Results

In this section we perform experiments to verify the MLE phase transition by (i) computing $h_{\text{MLE}}$ defined by (2.6); (ii) checking whether the data is separated by the linear programming as in [11]:

$$\max_{b_0,b} \sum_{i=1}^{n} y_i (b_0 + x_i^T b)$$

subject to $y_i (b_0 + x_i^T b) \geq 0, \; i = 1, \ldots, n$

$-1 \leq b_0 \leq 1, \; -1 \leq b \leq 1$.

Note that the MLE of the GLM exists if the linear programming (3.1) only has the trivial solution. We compare the theoretical phase transition curve with the empirical observations under several simulation designs as below.

For the link function, we choose the logit function, the cloglog function and the probit function. For the elliptical covariates, we set $\mu = 0, A = I_p$ and consider different distributions for the non-negative variable $R$, including the chi distribution with $df = p$, the Gamma distributions, the Pareto distributions, the half-normal distribution and the log-normal distribution. When generating the binary response by (1.1), we simply take $\beta_0 = 0$. See [11] for results with $\beta_0 \neq 0$. To ensure Assumption 2.5, let

$$E R^2 = p \alpha_0^2 + 1 \quad \text{and} \quad \beta = (\bar{X}/||\bar{X}||_2 + 1/p) \cdot \gamma_0/\alpha_0,$$

where $\bar{X} \sim N(0, I_p)$. We fix $n = 1000, \alpha_0 = 1$, and vary $\gamma_0 \in \{0.01, 0.02, \ldots, 10.00\}$ and $\kappa = p/n \in \{0.005, 0.01, \ldots, 0.6\}$. The parameter $\alpha_0$ is simply set as 1 since we observed that it does not affect the phase transition much. A large $\alpha_0$ slightly shifts the transition curve to the right, and enlarges the uncertain bands. Once the data is generated, we solve the problem (3.1) by checking whether a non-trivial solution exists. We repeat the procedure for 100 times, and get a heat map which indicates the proportion of times that the MLE exists for each pair $(\gamma_0, \kappa)$. See Figure 1 for the chi case with $df = p$. Results for other designs can be found in Appendix C.

![Figure 1](image_url)  
**Figure 1.** Phase transition of the MLE existence for $R \sim$ chi distribution ($df = p$). The red curve is the theoretical $h_{\text{MLE}}$ boundary given by (2.6).
3.1. Multivariate Gaussian covariates with different link functions. We consider the multivariate Gaussian covariates, which have been studied for the logit link in [11]. In our setup, $R$ is sampled from a chi distribution ($\text{df} = p$) and the link function is one of \{$\text{logit}, \text{cloglog}, \text{probit}$\}. Figure 1 displays the phase transition of the MLE existence for different link functions. There is a band in these figures, which indicates that the MLE exists indefinitely when $(\gamma_0, \kappa)$ falls in this band with the given sample size. This region is referred to as the uncertainty band. Observe that for the multivariate Gaussian covariates, as promised, the $h_{\text{MLE}}$ curves lie in the uncertainty bands for different link functions.

3.2. Gamma-distributed $R$. In [11], $U^{(p)}$ defined by (2.2) does not depend on $p$, which is key to their proof of the phase transition for the MLE existence. However, when we go beyond the chi distribution ($\text{df} = p$) for $R$, $U^{(p)}$ depends on $p$ in most cases. We observe that Assumption 2.6 is satisfied for Gamma distributions, and the resulting theoretical phase transition curves agree with the simulations. Precisely, assume $R \sim \text{Gamma}(k, \theta)$ where $k$ is the shape parameter and $\theta$ is the scale parameter. The second moment condition (3.2) gives $\theta = \sqrt{\mathbb{E}R^2/(k^2 + k)}$. When $k = 0.5$, we get $\theta_{0.5} = \sqrt{4(p + 1)/3}$ which corresponds to $\chi$ distribution with $df = 2$ if $\theta_{0.5}$ is an integer; when $k = 1$, it is the Exponential distribution with $\theta_1 = \sqrt{(p + 1)/2}$; when $k = 2$, it is a Gamma distribution with $\theta_2 = \sqrt{(p + 1)/6}$. Figure 2 implies that $h_{\text{MLE}}$ defined by (2.6) converges quickly as $p$ increases. Table 1 indicates that all the theoretical phase transition curves align with the corresponding middle curves of the uncertainty bands.

![Figure 2](image)

**Figure 2.** Convergence of $h_{\text{MLE}}$ by (2.6) for Gamma distributions. $\hat{h}_{\text{MLE}}^{(0)}$ is computed by taking the average of $h_{\text{MLE}}$ for $\kappa \geq 0.3$.

3.3. The moment condition and the tail behavior of $R$. First we explore a case where the eighth moment of $X^{(p)}$ does not exist as required by Theorem 2.7. To this end, we sample $R$ from the Pareto distribution of type I. We specify the shape parameter $\alpha$, and set the scale parameter $x_m = \sqrt{(\alpha - 2)/\alpha \cdot \mathbb{E}R^2}$. Recall that for the Pareto distribution of type I, the fourth moment exists when $\alpha > 4$, and the third moment exists when $\alpha > 3$. From Table 1, we see that the simulation results match the theoretical $h_{\text{MLE}}$ well. This suggests that the moment condition in Theorem 2.7 may be further relaxed.

Subsequently, we study how the tail behavior of the $R$ distribution influences the phase transition curve $h_{\text{MLE}}$. In the previous empirical studies, we consider the chi distributions with $df = p$ and Gamma distributions which have sub-exponential tails; the Pareto distributions have polynomial tails. We also investigate the half-Normal distribution with a sub-Gaussian tail, and the log-Normal distribution with another heavy tail. To ensure
Table 1. Summary of the theoretical $h_{\text{MLE}}$ and the simulations of the MLE phase transition with the logit link function. $h_{0.5}$ is the $\kappa$ such that the proportion of times that the MLE exists is 0.5, and the number inside the bracket is the width of the uncertainty interval (a slice of the uncertainty band) for a given $\gamma_0$. MIW is the mean width of the uncertainty intervals across $\gamma_0 \in (0,10]$; MD is the mean difference between $h_{\text{MLE}}$ and $h_{0.5}$.

| Distribution       | $\gamma_0 = 1$        | $\gamma_0 = 9$        | Overall        |
|--------------------|------------------------|------------------------|----------------|
|                    | $h_{0.5}$  | $h_{\text{MLE}}$ | $h_{0.5}$  | $h_{\text{MLE}}$ | MIW | MD |
| Gamma$(0.5, \theta_{0.5})$ | 0.435 (0.075) | 0.4238 | 0.310 (0.095) | 0.310 | 0.101 | 0.0077 |
| Gamma$(1, \theta_1)$       | 0.450 (0.050) | 0.458 | 0.260 (0.080) | 0.246 | 0.071 | 0.0186 |
| Gamma$(2, \theta_2)$       | 0.450 (0.050) | 0.447 | 0.205 (0.070) | 0.191 | 0.057 | 0.0160 |
| Pareto$(2.5, x_m(2.5))$    | 0.455 (0.045) | 0.458 | 0.165 (0.065) | 0.172 | 0.055 | 0.0045 |
| Pareto$(3.5, x_m(3.5))$    | 0.440 (0.045) | 0.439 | 0.120 (0.060) | 0.137 | 0.057 | 0.0095 |
| Pareto$(4.5, x_m(4.5))$    | 0.435 (0.055) | 0.430 | 0.110 (0.060) | 0.128 | 0.055 | 0.0095 |
| half-normal            | 0.450 (0.050) | 0.448 | 0.240 (0.075) | 0.211 | 0.064 | 0.0215 |
| log-normal             | 0.380 (0.135) | 0.497 | 0.355 (0.150) | 0.450 | 0.132 | 0.1199 |

In this section we sketch a proof of Theorem 2.7.

4. Roadmap to the Proof of Theorem 2.7

In this section we sketch a proof of Theorem 2.7.

Elliptical covariates. Assume that the covariates $x_i \sim \mathcal{E}_p(\mu, \Sigma, F_R)$ with $r = p = \text{rank}(\Sigma)$. It is easily seen that $x_i = \mu + \Sigma^{1/2} z_i$ with $z_i \sim \mathcal{E}_p(0, I_p, F_R)$. Recall from Section 2.1 that for GLMs satisfying Assumption 2.3, the MLE does not exist if and only if there is $(b_0, \tilde{b}) \neq 0$ such that $y_i(b_0 + x_i^T \tilde{b}) \geq 0$ for all $i$. This is equivalent to the existence of $(\tilde{b}_0, \tilde{b}) \neq 0$ such that $y_i(\tilde{b}_0 + z_i^T \tilde{b}) \geq 0$ for all $i$. Without loss of generality, we assume $x_i \sim \mathcal{E}_p(0, I_p, F_R)$ in the sequel.

We are in a situation where the covariates $x_i := (x_{i1}, \ldots, x_ip)$ is spherically symmetric and $\text{Var}(x_i^T \beta) = |\beta|^2 \text{E}R^2/p \to \gamma_0^2$. By rotational invariance, we assume that all the signal is in the first coordinate. That is, $P(y_i = 1|x_i) = \sigma \left( \beta_0 + \frac{x_{i1}}{\alpha_0} x_{i1} \right)$. The results in Appendix A show that

\[ (y_i, y_i x_i) \overset{d}{=} (Y^{(p)}, X^{(p)}, X_2, \ldots, X_p), \]

where $(Y^{(p)}, X^{(p)}) \sim F_{\alpha_0, \beta_0, \gamma_0}$, and $(X_2, \ldots, X_p|x^{(p)} = x) \sim \mathcal{E}_{p-1}(0, I_{p-1}, F_{R-1})$, with

\[ F_{R-1}(r) = \frac{\int_{[0,1]} \left( s - x \right)^{(p-3)/2} s^{-p/2} dF_R(s) \right) \int_{[0,1]} \left( s - x \right)^{(p-3)/2} s^{-p/2} dF_R(s) \right). \]
Now we want to express $\mathbb{P}$(no MLE) via conic geometry. For a fixed space $\mathcal{W} \in \mathbb{R}^n$, let
\[ \mathcal{C}(\mathcal{W}) = \{ \mathbf{w} + \mathbf{u} : \mathbf{w} \in \mathcal{W}, \mathbf{u} \geq 0 \}, \]
be the convex cone generated by $\mathcal{W}$. The following proposition will be proved in Appendix B.1.

**Proposition 4.1.** Let the $n$-dimensional vectors $(\mathbf{Y}^{(p)}, \mathbf{X}^{(p)}, \mathbf{X}_2, \ldots, \mathbf{X}_p)$ be $n$ i.i.d. copies of $(\mathbf{Y}^{(p)}, \mathbf{X}^{(p)}, \mathbf{X}_2, \ldots, \mathbf{X}_p)$ distributed as in (4.1). Let $\mathcal{L} := \text{span}(\mathbf{X}_2, \ldots, \mathbf{X}_p)$ and $\mathcal{W} := \text{span}(\mathbf{Y}^{(p)}, \mathbf{X}^{(p)})$. Let $\{ \text{No MLE Single} \}$ be the event that the data points can be completely or quasi-completely separated by the intercept and the first coordinate only, i.e. $\mathcal{W} \cap \mathbb{R}^n_+ \neq \{0\}$. Then
\[ 0 \leq \mathbb{P}(\text{no MLE}) - \mathbb{P}(\mathcal{L} \cap \mathcal{C}(\mathcal{W}) \neq \{0\}) \leq \mathbb{P}(\text{No MLE Single}). \] (4.2)

By Proposition 4.1, the existence of the MLE boils down to whether $\mathcal{L}$ intersects $\mathcal{C}(\mathcal{W})$ in a non-trivial way. It remains to prove the following: (1) The probability $\mathbb{P}(\text{No MLE Single})$ is relatively small. (2) The probability $\mathbb{P}(\mathcal{L} \cap \mathcal{C}(\mathcal{W}) \neq \{0\})$ exhibits a phase transition through the ratio $\kappa := p/n$, and $h_{\text{MLE}}(\alpha_0, \beta_0, \gamma_0)$ defined by (2.6).

**Separation of data in a univariate model** We aim to prove that $\mathbb{P}(\text{No MLE Single})$ is small. In [11], a sketch of proof is given for the logistic regression with Gaussian covariates. But there seems to be a gap in the proof. They argued that the probability the data can be separated via any fixed $t_0 \in \mathbb{R}$ is exponentially small. However, there are uncountably many such $t_0$ and the union bound does not give a good estimate. In Appendix B.2 we give a proof of this fact in the setting of Theorem 2.7.

**Proposition 4.2.** Under assumptions in Theorem 2.7, the event $\{ \text{No MLE Single} \}$ occurs with small probability. That is, $\mathbb{P}(\text{No MLE Single}) = o(1)$.

**Convex geometry and phase transition** We want to prove the phase transition of $\mathbb{P}(\mathcal{L} \cap \mathcal{C}(\mathcal{W}))$ through the interplay between $\kappa$ and $h_{\text{MLE}}(\alpha_0, \beta_0, \gamma_0)$. The key is to understand when a random subspace $\mathcal{L}$ with uniform orientation intersects $\mathcal{C}(\mathcal{W})$ in a non-trivial way.

For any fixed subspace $\mathcal{W} \in \mathbb{R}^p$, the approximate kinematic formula [2, Theorem I] shows that for any $\varepsilon \in (0, 1)$, there exists $a_\varepsilon > 0$ such that
\[ p - 1 + \delta(\mathcal{C}(\mathcal{W})) > n + a_\varepsilon \sqrt{n} \implies \mathbb{P}(\mathcal{L} \cap \mathcal{C}(\mathcal{W}) \geq 1 - \varepsilon), \]
\[ p - 1 + \delta(\mathcal{C}(\mathcal{W})) < n - a_\varepsilon \sqrt{n} \implies \mathbb{P}(\mathcal{L} \cap \mathcal{C}(\mathcal{W}) \leq \varepsilon), \] (4.3)

Here $\delta(\mathcal{C})$ is the statistical dimension of the convex cone $\mathcal{C}$ defined by $\delta(\mathcal{C}) := n - \mathbb{E}|\mathbf{Z} - \Pi_{\mathcal{C}}(\mathbf{Z})|^2$, where $\mathbf{Z} \sim \mathcal{N}(0, \mathbf{I}_n)$ and $\Pi_{\mathcal{C}}$ is the projection onto $\mathcal{C}$. An elementary argument shows that
\[ \delta(\mathcal{C}(\mathcal{W})) = n - \mathbb{E}\left(\min_{\mathbf{w} \in \mathcal{W}} |\mathbf{w} - \mathbf{Z}_+|^2\right). \] (4.4)

Theorem 2.7 can be derived from the formulas (4.3)-4.4 and the following theorem.

**Theorem 4.3.** Let $(\mathbf{Y}^{(p)}, \mathbf{X}^{(p)})$ be $n$ i.i.d. samples from $F_{\alpha_0, \beta_0, \gamma_0}$ satisfying Assumption 2.6, and $\sup_{p} \mathbb{E}[(X^{(p)})^8] < \infty$. Let
\[ Q_{p,n} := \min_{\lambda_0, \lambda_1 \in \mathbb{R}} \frac{1}{n} |(\lambda_0 \mathbf{Y}^{(p)} + \lambda_1 \mathbf{X}^{(p)} - \mathbf{Z})_+|^2. \]
Then $Q_{p,n}$ converges in probability to $\min_{\lambda_0, \lambda_1 \in \mathbb{R}} \mathbb{E}(\lambda_0 Y + \lambda_1 X - Z)_+^2$ as $n, p \to \infty$. 

In [11], the authors proved Theorem 4.3 in the setting of the logistic regression by a bare-hands argument. Now we show how this result follows from stochastic approximation.

**Stochastic approximation** We sketch a proof of Theorem 4.3 via a stochastic approximation. In the stochastic approximation literature [26, 34], people seek to approximate the optimization problem

$$\min_{\lambda \in S} G(\lambda), \quad G(\lambda) := \mathbb{E} g(\lambda, \xi),$$

where $S \subset \mathbb{R}^k$ for some $k > 0$ and $\xi$ is a generic random vector, by a sequence of stochastic optimization problems $\min_{\lambda \in S} \hat{G}_n(\lambda; \xi_1, \ldots, \xi_n)$, $\hat{G}_n(\lambda; \xi_1, \ldots, \xi_n) := \frac{1}{n} \sum_{i=1}^n g(\lambda, \xi_i)$ where $(\xi_i; i \geq 1)$ are i.i.d. copies of $\xi$.

A deep connection between stochastic approximation and convergence of random closed sets was established by Attouch and Wets [5] via the epi-convergence of functions. A sequence of lower semi-continuous functions $f_n : \mathbb{R}^k \to (\infty, \infty]$ is said to epi-converges to $f$ if for each $x \in \mathbb{R}^k$,

- $\liminf f_n(x_n) \to f(x)$ if $x_n \to x$,
- $\lim f_n(x_n) \to f(x)$ for at least one sequence $x_n \to x$.

See [3, 4, 16, 22, 27, 37] for further development on epi-convergence.

Here we consider a sequence of stochastic optimization problems with triangle arrays

$$\min_{\lambda \in S} \hat{G}_n(\lambda; \xi_{1,n}, \ldots, \xi_{n,n}), \quad \hat{G}_n(\lambda; \xi_{1,n}, \ldots, \xi_{n,n}) := \frac{1}{n} \sum_{i=1}^n g(\lambda, \xi_{i,n}),$$

where $(\xi_{i,n}; 1 \leq i \leq n)$ are i.i.d. copies of $\xi_n$, and $\xi_n$ converges in distribution to $\xi$. Let $\nu_n$, $\hat{\nu}_n$, and argmin $G$, argmin $\hat{G}_n$ be optimal values, and optimal solutions to the problems (4.5)-(4.6). Note that argmin $G$ and argmin $\hat{G}_n$ are set-valued. The following result gives asymptotic inference of $\hat{\nu}_n$ as $n \to \infty$. The proof will be given in Appendix B.3.

**Lemma 4.4.** Assume that $g(\cdot, \cdot)$ is measurable and bounded from below, and $g(\cdot, \xi)$ is convex. Assume that $\sup_n \mathbb{E} g^2(\lambda, \xi_n) < \infty$ and $\mathbb{E} g(\lambda, \xi_n) \to \mathbb{E} g(\lambda, \xi)$ for all $\lambda$. Further assume that argmin $\hat{G}_n$, $n \geq 1$ are non-empty and bounded in probability. Then argmin $G \neq \emptyset$, and

$$\hat{\nu}_n \to \nu \quad \text{in probability.}$$

To prove Theorem 4.3 we need to show that the set of minimizers argmin $\hat{G}_n$ is non-empty and bounded in probability. In Appendix B.3, we prove that under the assumptions in Theorem 2.7

- The problem (4.5) has a unique minimizer $\lambda_0$.
- For any minimizer $\hat{\lambda}_n \in$ argmin $\hat{G}_n$, $|\hat{\lambda}_n - \lambda_0| = O_p(1)$. Here we use the Chebyshev inequality instead of the concentration inequality for sub-exponential variables in [11].

**Proof of Theorem 2.7** The proof goes along the same line as [11], with two modifications. Assume that $\kappa > h_{MLE}(\alpha_0, \beta_0, \gamma_0)$.

- Given $(Y^{(p)}, X^{(p)})$, the random vector $(X_2, \ldots, X_p)$ is also elliptical whose distribution is given by (4.1). By the geometric characterization (4.3), we get

$$\mathbb{P}(\mathcal{L} \cap \mathcal{C}(\mathcal{W}) \neq \{0\}) \geq \mathbb{P}(p/n > \mathbb{E}(Q_{p,n}|X, Y) + a_n n^{-1/2}) - \varepsilon_n,$$
for some $\varepsilon_n \to 0$.

- The random variable $Q_{p,n}$ is uniform integrable, since $Q_{p,n} \leq |Z_+|^2/n$ which is sub-exponential. This implies that $E(Q_{p,n}|X,Y)$ converges in probability to $h_{\text{MLE}}(\alpha_0, \beta_0, \gamma_0)$.

Thus, $P(\mathcal{L} \cap \mathcal{C}(\mathcal{W}) \neq \{0\}) \to 1$ and $P(\text{MLE exists}) \to 0$. Similarly, we can prove if $\kappa < h_{\text{MLE}}(\alpha_0, \beta_0, \gamma_0)$, then $P(\text{MLE exists}) \to 1$. 
Appendix A. Elliptical Distributions

Recall the definition of elliptical distributions from Section 2.2. Below we list a few useful properties of elliptical distributions, see [9, 10, 17] for further development.

(1) The random vector \( \mathbf{X} \sim \mathcal{E}_p(\mu, \Sigma, F_R) \) has finite moments of order \( k > 0 \) if and only if \( \mathbb{E}R^k < \infty \). If the first two moments exist, then \( \mathbb{E}\mathbf{X} = \mu \) and \( \text{Var} \mathbf{X} = \mathbb{E}R^2 \Sigma \).

(2) The distribution of \( \mathbf{X} \sim \mathcal{E}_p(\mu, \Sigma, F_R) \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^p \) if and only if \( r = p = \text{rank}(\Sigma) \), and the distribution of \( R \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}_+ \).

(3) The marginal and conditional distributions of \( \mathbf{X} \sim \mathcal{E}_p(\mu, \Sigma, F_R) \) are also elliptical. For the sake of simplicity, assume that \( r = p = \text{rank}(\Sigma) \). Let \( \mathbf{X} := (\mathbf{X}_1, \mathbf{X}_2) \), with \( \mathbf{X}_1 \in \mathbb{R}^{p_1} \) and \( \mathbf{X}_2 \in \mathbb{R}^{p_2} \) \( (p_1 + p_2 = p) \). Let \( \mu := \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \) and \( \Sigma := \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \), with \( \mu_1 \in \mathbb{R}^{p_1} \), \( \mu_2 \in \mathbb{R}^{p_2} \), \( \Sigma_{11} \in \mathcal{M}_{p_1}(\mathbb{R}) \), \( \Sigma_{12} = \Sigma_{21}^T \in \mathcal{M}_{p_1,p_2}(\mathbb{R}) \), and \( \Sigma_{22} \in \mathcal{M}_{p_2}(\mathbb{R}) \). Then \( \mathbf{X}_1 \sim \mathcal{E}_{p_1}(\mu_1, \Sigma_{11}, F_{R}) \), and \( (\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2) \sim \mathcal{E}_{p_1}(\mu_{1|2}, \Sigma_{1|2}, F_{R|2}) \) where

\[
\begin{align*}
\mu_{1|2} := & \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \mu_2), \\
\Sigma_{1|2} := & \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21},
\end{align*}
\]

and

\[
F_{R|2}(r) = \frac{\int_{d_{\Sigma_{22}}(\mathbf{x}_2, \mu_2)}^{\sqrt{\mathbf{s}^2 + d_{\Sigma_{22}}^2(\mathbf{x}_2, \mu_2)}} \left( s^2 - d^2_{\Sigma_{22}}(\mathbf{x}_2, \mu_2) \right)^{p_1/2-1} s^{-p+2} dF_R(s)}{\int_{d_{\Sigma_{22}}(\mathbf{x}_2, \mu_2)}^{\infty} \left( s^2 - d^2_{\Sigma_{22}}(\mathbf{x}_2, \mu_2) \right)^{p_1/2-1} s^{-p+2} dF_R(s)},
\]

with \( d_{\Sigma_{22}}(\mathbf{x}_2, \mu_2) := \sqrt{(\mathbf{x}_2 - \mu_2)^T\Sigma_{22}^{-1}(\mathbf{x}_2 - \mu_2)} \) the Mahalanobis distance between \( \mathbf{x}_2 \) and \( \mu_2 \) in the metric associated with \( \Sigma_{22} \).

Appendix B. Roadmap to the Proof of Theorem 2.7

B.1. Proof of Proposition 4.1. We aim to prove that

\[
P(\text{No MLE}) = P(\text{No MLE Single}) + P(\mathcal{L} \cap \mathcal{C}(\mathcal{W}) \neq \{0\} \text{ and } \{\text{No MLE Single}\}^c),
\]

from which the result follows. If \( \{\text{No MLE Single}\} \) occurs, there is no MLE. Assume that \( \{\text{No MLE Single}\} \) does not occur. If

\[
P(\text{no MLE}) = P(\text{Span}(\mathbf{Y}^{(p)}, \mathbf{X}^{(p)}, \mathbf{X}_2, \ldots, \mathbf{X}_p) \cap \mathbb{R}_+^n \neq \{0\}),
\]

then there is no MLE if and only if there is a non-zero vector \((b_0, \ldots, b_p)\) such that \(b_0\mathbf{Y}^{(p)} + b_1\mathbf{X}^{(p)} + \cdots + b_p\mathbf{X}_p = \mathbf{u}, \mathbf{u} \geq 0, \mathbf{u} \neq 0\). By assumption, \(b_0\mathbf{Y}^{(p)} + b_1\mathbf{X}^{(p)} \neq u\) so \(b_2\mathbf{X}_2 + \cdots + b_p\mathbf{X}_p\) is a non-zero element of \(\mathcal{C}(\mathcal{W})\). This leads to (B.3). Note that there is no MLE if and only if there is a non-zero vector \((b_0, \ldots, b_p)\) such that \(b_0\mathbf{Y}^{(p)} + b_1\mathbf{X}^{(p)} + \cdots + b_p\mathbf{X}_p \geq 0\). The identity in law (9) implies that the equality occurs with probability 0, which proves (B.2).

B.2. Proof of Proposition 4.2. Let \( (X_1, \ldots, X_n) \) be i.i.d. samples with density \( f_{X^{(p)}} \). It is well known that the distribution of the order statistics \((X^{(1)}, \ldots, X^{(n)})\) is given by

\[
\prod_{i=1}^{n} f_{X^{(p)}}(x_i) \text{ for } x_1 < \cdots < x_n. \]

Note that there exists \( t \in \mathbb{R} \) separating \( X^{(1)} < \cdots < X^{(n)} \)
if and only if for some \( k \in \{0, \ldots, n\} \), the responses corresponding to \( X^{(1)}, \ldots, X^{(k)} \) is of the same sign, and those corresponding to \( X^{(k+1)}, \ldots, X^{(n)} \) is of the opposite sign. Consequently,

\[
P(\exists t : \text{separate } X_1, \ldots, X_n)
= \int_{x_1 < \ldots < x_n} P(\exists t : \text{separate } X_1, \ldots, X_n | X_i = x_i \ \forall i \cdot n! \prod_{i=1}^{n} f_{X^{(p)}}(x_i) dx_1 \ldots dx_n
\]

Moreover,

\[
\int_{x_1 < \ldots < x_n} n! \prod_{i=1}^{n} f_{X^{(p)}}(x_i) \prod_{i=1}^{n} p_{\pm}(x_i) dx_1 \ldots dx_n = \left( \int_{\mathbb{R}} f_{X^{(p)}}(x)p_{\pm}(x)dx \right)^n
= (E_{p_{\pm}}(X^{(p)}))^n.
\]

For \( 1 \leq k \leq n - 1 \),

\[
\int_{x_1 < \ldots < x_n} n! \prod_{i=1}^{n} f_{X^{(p)}}(x_i) \prod_{i=1}^{k} p_{-}(x_i) \prod_{i=k+1}^{n} p_{+}(x_i) dx_1 \ldots dx_n
= n! \int_{x_{k+1} = -\infty}^{\infty} f_{X^{(p)}}(x_{k+1})p_{+}(x_{k+1})dx_{k+1} \left( \int_{x_1 < \ldots < x_{k+1}} \prod_{i=1}^{k} f_{X^{(p)}}(x_i)p_{-}(x_i) dx_1 \ldots dx_k \right)
= \frac{n!}{k! (n-k-1)!} \int_{x_{k+1} = -\infty}^{\infty} f_{X^{(p)}}(x_{k+1})p_{+}(x_{k+1})^{k}p_{-}(x_{k+1})G_{p_{\pm}^{n-k-1}}(x_{k+1})dx_{k+1}
= n \binom{n-1}{k} E[p_{+}(X^{(p)})G_{p_{-}^{k}}(X^{(p)})G_{p_{\pm}^{n-k-1}}(X^{(p)})].
\]

Combining the above identities yields

\[
P(\exists t : \text{separate } X_1, \ldots, X_n)
= (E_{p_{-}}(X^{(p)}))^n + n \sum_{k=1}^{n} \binom{n-1}{k} E[p_{+}(X^{(p)})G_{p_{-}^{k}}(X^{(p)})G_{p_{\pm}^{n-k-1}}(X^{(p)})]
+ n \sum_{k=1}^{n} \binom{n-1}{k} E[p_{-}(X^{(p)})G_{p_{+}^{k}}(X^{(p)})G_{p_{\pm}^{n-k-1}}(X^{(p)})] + (E_{p_{+}}(X^{(p)}))^n
= (E_{p_{-}}(X^{(p)}))^n + nE[p_{+}(X^{(p)})(G_{p_{-}^{k}(X^{(p)}) + G_{p_{+}^{k}(X^{(p)})})^{n-1}]
+ nE[p_{-}(X^{(p)})(G_{p_{+}^{k}(X^{(p)}) + G_{p_{-}^{k}(X^{(p)})})^{n-1}] + (E_{p_{+}}(X^{(p)}))^n. \quad (19)
\]

Finally, the condition (7) together with (19) lead to the desired result.
B.3. Proof of Theorem 4.3. We start with the proof of Lemma 4.4.

Proof of Lemma 4.4. By law of large numbers of triangle arrays, the condition $\sup_i E g^4(\lambda, \xi_{i,n}) < \infty$ implies that $\frac{1}{n} \sum_{i=1}^n g(\lambda, \xi_{i,n}) - E g(\lambda, \xi_n) \to 0$ a.s. It follows from [3] Theorem 2.3 that

$$\frac{1}{n} \sum_{i=1}^n g(\lambda, \xi_{i,n}) - E g(\lambda, \xi_n) \text{ epi-converges to } 0 \ a.s.$$ 

It is well known that if a sequence of convex functions converge pointwise, then they converge uniformly on compact sets. Therefore, $E g(\lambda, \xi_n) \to E g(\lambda, \xi)$ for all $\lambda$ and $g(\cdot, \xi)$ is convex, the convergence is uniform on compact sets. This implies the epi-convergence. Therefore,

$$\frac{1}{n} \sum_{i=1}^n g(\lambda, \xi_{i,n}) \text{ epi-converges to } E g(\lambda, \xi) \ a.s.$$ 

Combining with [10] Proposition 3.3 yields the desired result. $\square$

Now we are ready to prove Theorem 4.3. We specialize to $\lambda = (\lambda_0, \lambda_1)$, $\xi_n = (Y^{(p)}, X^{(p)}, Z)$ with $(Y^{(p)}, X^{(p)}) \sim F_{\alpha_0, \beta_0, \gamma_0}$, $\xi = (Y, X, Z)$, and $Z \sim N(0,1)$ independent of $\{(Y^{(p)}, X^{(p)}), (Y, X)\}$, and

$$g(\lambda, \xi) = (\lambda_0 Y + \lambda_1 X - Z)^2_+.$$ (20)

It is clear that the function $g$ defined by (20) is measurable and non-negative, and $g(\cdot, \xi)$ is convex. It follows from $\sup_p E[(X^{(p)})^2] < \infty$ that $\sup_n g^4(\lambda, \xi_n) < \infty$. By Assumption 2.5, $E[(X^{(p)})^2]$ converges to $\alpha_0^2$, and by Assumption 2.6, $(Y^{(p)}, X^{(p)})$ converges in distribution to $(Y, X)$. Now by [7] Lemma 8.3, we get $E g(\lambda, \xi_n) \to E g(\lambda, \xi)$ for all $\lambda$.

Let $\lambda_{\min}$ be a minimum of $E g(\lambda, \xi)$. By convexity of $\lambda \to E g(\lambda, \xi)$, there exists $r > 0$ such that $E g(\lambda_{\min}, \xi) < \min_{|\lambda| \leq r+1} E g(\lambda, \xi)$, and $\min_\lambda E g(\lambda, \xi) = \min_{|\lambda| \leq r+1} E g(\lambda, \xi)$. Note that $E g(\lambda, \xi_n)$ converges uniformly to $E g(\lambda, \xi)$ on $\{\lambda : r \leq |\lambda| \leq r+1\}$. So for $p$ large enough, $E g(\lambda_{\min}, \xi_n) < \min_{|\lambda| \leq r+1} E g(\lambda, \xi_n)$ and $\min_\lambda E g(\lambda, \xi_n) = \min_{|\lambda| \leq r+1} E g(\lambda, \xi_n)$. Now by [21] Theorem 2.1, we have

$$\min_\lambda E g(\lambda, \xi_n) = \min_{|\lambda| \leq r+1} E g(\lambda, \xi_n) \to \min_{|\lambda| \leq r+1} E g(\lambda, \xi) = \min_\lambda E g(\lambda, \xi) \text{ as } p \to \infty.$$ 

By Lemma 4.4, it suffices to prove that the set of minimizers $\arg\min \hat{G}_n$ is non-empty and bounded in probability. We aim to show that under the assumptions in Theorem 2.7, the problem (13) has a unique minimizer $\lambda_0$, and for any minimizer $\hat{\lambda}_n \in \arg\min \hat{G}_n$, $|\hat{\lambda}_n - \lambda_0| = O_P(1)$.

We prove these statements in the next two lemmas. It is easily seen that $G(\lambda)$ and $\hat{G}_n(\lambda)$ are convex. The following lemma shows that the function $G$ is strongly convex, which was stated in [11] without proof. Here we give a complete proof.

Lemma B.1. Under the assumptions in Theorem 2.7, the function $\lambda \to G(\lambda)$ with $g(\cdot)$ defined by (20) is strongly convex. That is, there exists $\alpha_1 > \alpha_0 > 0$ such that

$$\alpha_1 I_2 \preceq \nabla^2 G(\lambda) \preceq \alpha_0 I_2.$$ (21)
Proof. Elementary analysis shows that
\[
\nabla^2 G(\lambda) = \begin{pmatrix}
\mathbb{E}[Y^2 \Phi(\lambda_0 Y + \lambda_1 X)] & \mathbb{E}[YX \Phi(\lambda_0 Y + \lambda_1 X)] \\
\mathbb{E}[YX \Phi(\lambda_0 Y + \lambda_1 X)] & \mathbb{E}[X^2 \Phi(\lambda_0 Y + \lambda_1 X)]
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\mathbb{E}[\Phi(\lambda_0 V + \lambda_1 VU)] & \mathbb{E}[U \Phi(\lambda_0 V + \lambda_1 VU)] \\
\mathbb{E}[U \Phi(\lambda_0 V + \lambda_1 VU)] & \mathbb{E}[U^2 \Phi(\lambda_0 V + \lambda_1 VU)]
\end{pmatrix},
\]
where \(\Phi(\cdot)\) is the CDF of standard normal, \(U\) is defined in Assumption 2.6, and \(\mathbb{P}(V = 1|U) = 1 - \mathbb{P}(V = -1|U) = p_+(U)\). The r.h.s. of (21) is clear. By Cauchy-Schwarz inequality, \(\det \nabla^2 G(\lambda) \geq 0\). If \(\det \nabla^2 G(\lambda) = 0\), then \(U\) is constant almost surely which violates the non-degeneracy of \(U\). Thus, \(G(\lambda)\) is strictly convex.

Note that \(\mathbb{E}[\Phi(\lambda_0 V + \lambda_1 VU)] = \mathbb{E}[\Phi(\lambda_0 + \lambda_1 X)p_+(X)] + \mathbb{E}[\Phi(-\lambda_0 - \lambda_1 X)p_-(X)]\), and the decomposition holds for other terms. So
\[
\nabla^2 G(\lambda) = \begin{pmatrix}
\mathbb{E}[\Phi(\lambda_0 + \lambda_1 U)p_+(U)] & \mathbb{E}[U \Phi(\lambda_0 + \lambda_1 U)p_+(U)] \\
\mathbb{E}[U \Phi(\lambda_0 + \lambda_1 U)p_+(U)] & \mathbb{E}[U^2 \Phi(\lambda_0 + \lambda_1 U)p_+(U)]
\end{pmatrix}
\]
\[
+ \begin{pmatrix}
\mathbb{E}[\Phi(\lambda_0 + \lambda_1 U)p_-(U)] & \mathbb{E}[U \Phi(\lambda_0 + \lambda_1 U)p_-(U)] \\
\mathbb{E}[U \Phi(\lambda_0 + \lambda_1 U)p_-(U)] & \mathbb{E}[U^2 \Phi(\lambda_0 + \lambda_1 U)p_-(U)]
\end{pmatrix}.
\]

Without loss of generality, consider \(\lambda_1, \lambda_2 > 0\). For \(\lambda_0, \lambda_1\) sufficiently large,
- if \(\lambda_0/\lambda_1\) is large, then \(\mathbb{E}[\Phi(\lambda_0 + \lambda_1 U)p_+(U)]\) can be approximated by \(\mathbb{E}p_+(U)\).
- if \(\lambda_0/\lambda_1\) is small, then \(\mathbb{E}[\Phi(\lambda_0 + \lambda_1 U)p_+(U)]\) can be approximated by \(\mathbb{E}[1(U > -\lambda_0/\lambda_1 + \eta)p_+(U)]\), where \(\eta\) is a fixed small value.

The approximation also holds for other terms. By the strict convexity and the approximation, we can show that there exist \(N_+ > 0\) such that for \(\lambda_0, \lambda_1 > N_+\),
\[
\begin{pmatrix}
\mathbb{E}[\Phi(\lambda_0 + \lambda_1 U)p_+(U)] & \mathbb{E}[U \Phi(\lambda_0 + \lambda_1 U)p_+(U)] \\
\mathbb{E}[U \Phi(\lambda_0 + \lambda_1 U)p_+(U)] & \mathbb{E}[U^2 \Phi(\lambda_0 + \lambda_1 U)p_+(U)]
\end{pmatrix} \geq \epsilon_+ I_2,
\]
for some \(\epsilon_+ > 0\). Similarly, there exist \(N_- > 0\) such that for \(\lambda_0, \lambda_1 > N_-\), we get a bound \(\epsilon_- I_2\) for the second term on the r.h.s. of (21). Thus, for \(\lambda_1, \lambda_2 > \max(N_+, N_-)\),
\(\nabla^2 G(\lambda) \succeq \min(\epsilon_-, \epsilon_+) I_2\). By continuity of \(\det \nabla^2 G(\lambda)\), we get \(\nabla^2 G(\lambda) \succeq \epsilon I_2\) for \(\lambda_0, \lambda_1 < \max(N_+, N_-)\). It suffices to take \(\alpha_0 = \min(\epsilon_+, \epsilon_-, \epsilon)\) to conclude.

Finally, we prove that the set of minimizers \(\arg\min \hat{G}_n\) is bounded in probability. The argument can be used to show that \(\hat{\lambda}_n - \lambda_0 = O_P(n^{-1/4})\), where \(\hat{\lambda}_n\) is any minimizer of \(\hat{G}_n\). The proof is adapted from [11] Lemma 4, which we include for completeness.

Lemma B.2. Under the assumptions in Theorem 2.7, we have \(|\hat{\lambda}_n - \lambda_0| = O_P(1)\), where \(\hat{\lambda}_n \in \arg\min \hat{G}_n\).

Proof. For any \(\lambda \in \mathbb{R}^2\), the strong convexity [21] gives that
\[
G(\lambda) \geq G(\lambda_0) + \frac{\alpha_0}{2} |\lambda - \lambda_0|^2.
\]
Fix \(x \geq 1\). For any \(\lambda\) on the circle \(C(x) := \{\lambda : |\lambda - \lambda_0| = x\}\), we have
\[
G(\lambda) \geq G(\lambda_0) + 3y, \quad y := \frac{\alpha_0 x^2}{6}, \tag{23}
\]
Fix $z = G(\lambda_0) + y$, and consider the event
\[ E := \left\{ \hat{G}_n(\lambda_0) < z \text{ and } \inf_{\lambda \in C(x)} \hat{G}_n(\lambda) > z \right\}. \]

By convexity of $\hat{G}_n$, when $E$ occurs, $\hat{\lambda}_n$ must lie in the circle. Hence, $|\hat{\lambda}_n - \lambda_0| \leq x$.

Next we prove that the event $E$ occurs with high probability. Fix $d$ equi-spaced point \( \{\lambda_i\}_{i=1}^d \) on the set $C(x)$. Take any point $\lambda$ on the circle, and let $\lambda_i$ be its closest point. So $|\lambda - \lambda_i| \leq \pi x/d$. By convexity of $\hat{G}_n$,
\[ \hat{G}_n(\lambda) \geq \hat{G}_n(\lambda_i) - |\nabla \hat{G}(\lambda_i)||\lambda - \lambda_i|. \]  

Define the event
\[ B := \left\{ \max_i |\nabla \hat{G}(\lambda_i) - \nabla G(\lambda_i)| \leq x \right\}. \]

By Chebyshev inequality and union bound, we get
\[ \mathbb{P}(B^c) \leq \frac{d\sigma^2}{nx^2}, \]  

where $\sigma^2 := \sup_n \text{Var}[g(\lambda, \xi_n)] < \infty$. As $|\nabla^2 G|$ is bounded and $\nabla G(\lambda_0) = 0$, $|\nabla G(\lambda_i)| \leq \alpha_1|\lambda_i - \lambda_0| = \alpha_1 x$.

Now for $n$ sufficiently large, on $B$ we have $|\nabla \hat{G}_n(\lambda_i)||\lambda - \lambda_i| \leq \pi(1 + \alpha_1)x^2/d$. Choose $d > 6\pi(1 + \alpha_1)/\alpha_0$ so that $|\nabla \hat{G}_n(\lambda_i)||\lambda - \lambda_i| \leq y$. Then it follows from (24) that on $B$, $\inf_{\lambda \in C(x)} \hat{G}_n(\lambda) \geq \min_i \hat{G}(\lambda_i) - y$.

Observe that $\hat{G}_n(\lambda_i) > G(\lambda_i) - y \implies \hat{G}_n(\lambda_i) - y > G(\lambda_0) + y = z$, since $G(\lambda_i) \geq G(\lambda_0) + 3y$ by (23). Consequently,
\[ \mathbb{P}(E^c) \leq \mathbb{P}(B^c) + \mathbb{P}(\hat{G}_n(\lambda_0) \geq G(\lambda_0) + y) + \sum_{i=1}^{d} \mathbb{P}(\hat{G}_n(\lambda_i) \leq G(\lambda_i) - y), \]

where $\mathbb{P}(B^c)$ is controlled by (25), and the last two terms can also be bounded by Chebyshev inequality.

\section*{Appendix C. Empirical Results}

To study whether the MLE exists for a given pair of $(\gamma_0, \kappa)$ with some distribution for $R$, we generate the data by the mechanism described in Section 3. We fix the sample size at $n = 1000$, and vary $\gamma_0 \in \{0.01, 0.02, \ldots, 10.00\}$ and $\kappa = p/n \in \{0.005, 0.01, \ldots, 0.6\}$. We generate 100 such datasets, and for each dataset we solve the linear programming (3.1) by checking whether there is a nonzero solution using the package CVXOPT\footnote{https://cvxopt.org/documentation/index.html} in python.

On the other hand, the optimization problem (2.6) does not have a closed-form solution. Here we solve numerically this convex optimization. Using the same data generation mechanism but with a sample size $n = 4000$, we compute $h_{\text{MLE}}$ by CVXOPT as well. We repeat
the procedure for 100 times and take the average of these replicates as the reported $h_{\text{MLE}}$. Here we take $\gamma_0 \in \{0.5, 1.0, \ldots, 10.00\}$ and $\kappa = p/n \in \{0.02, 0.04, \ldots, 0.6\}$.

**Figure 3.** Phase transition of the MLE existence for $R \sim \text{chi distribution (df} = p)$ with different link functions. Upper: The value of each grid in the heatmap is the proportion of times that the MLE exists across the 100 replicates. The red curve is the theoretical $h_{\text{MLE}}$ boundary given by (2.6). Bottom: The heatmap for the $h_{\text{MLE}}$. Each grid is the averaged value $h_{\text{MLE}}$ across 100 replicates.
Figure 4. Phase transition of the MLE existence for $R \sim \text{Gamma}$ distributions with different parameters and the logit link (see Section 3.2). Upper: The value of each grid in the heatmap is the proportion of times that the MLE exists across the 100 replicates. The red curve is the theoretical $h_{\text{MLE}}$ boundary given by (2.6). Bottom: The heatmap for the $h_{\text{MLE}}$. Each grid is the averaged value $h_{\text{MLE}}$ across 100 replicates.
Figure 5. Phase transition of the MLE existence for the Pareto distributions with different parameters and the logit link (see Section 3.3). Upper: The value of each grid in the heatmap is the proportion of times that the MLE exists across the 100 replicates. The red curve is the theoretical $h_{\text{MLE}}$ boundary given by (2.6). Bottom: The heatmap for the $h_{\text{MLE}}$. Each grid is the averaged value $h_{\text{MLE}}$ across 100 replicates.
Figure 6. Phase transition of the MLE existence for the half-normal distribution and the logit link function. Left: The value of each grid in the heatmap is the proportion of times that the MLE exists across the 100 replicates. The red curve is the theoretical $h_{\text{MLE}}$ boundary given by (2.6). Right: The heatmap for the $h_{\text{MLE}}$. Each grid is the averaged value $h_{\text{MLE}}$ across 100 replicates.

Figure 7. Phase transition of the MLE existence for the log-Normal distribution and the logit link function. Left: The value of each grid in the heatmap is the proportion of times that the MLE exists across the 100 replicates. The red curve is the theoretical $h_{\text{MLE}}$ boundary given by (2.6). Right: The heatmap for the $h_{\text{MLE}}$. Each grid is the averaged value $h_{\text{MLE}}$ across 100 replicates.
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