STRUCTURE FORMATION
IN THE QUINTESSENTIAL UNIVERSE * **

EWA L. LOKAS

Nicolaus Copernicus Astronomical Center
Bartycka 18, 00–716 Warsaw, Poland

I review the main characteristics of structure formation in the quintessen-
tial Universe. Assuming equation of state $w = p/\rho = \text{const}$ I provide a brief
description of the background cosmology and discuss the linear growth of
density perturbations, the strongly nonlinear evolution, the power spectra
and rms fluctuations as well as mass functions focusing on the three values
$w = -1, -2/3$ and $-1/3$. Finally I describe the presently available and
future constraints on $w$.

PACS numbers: 95.35.+d, 98.62.-g, 98.62.Ai, 98.62.Ck, 98.80.-k, 98.80.Es

1. Introduction

Our knowledge of background cosmology has recently improved dra-
etically due to new supernovae and cosmic microwave background data.
Current observations favor a flat Universe with matter density $\Omega_0 = 0.3$ [1]
and the remaining contribution in the form of cosmological constant [2, 3]
or some other form of dark energy. The models with cosmological constant
are known, however, to suffer from two major problems. One is related to
the origin of the constant - it cannot be explained in terms of the vacuum
energy since its energy is orders of magnitude smaller. The other is the
lack of explanation why the present densities in matter and cosmological
constant are comparable.

A new class of models that solve these problems and also satisfy present
observational constraints has been proposed a few years ago [4]. In these
models the cosmological constant is replaced with a new energy component,

* Presented at the XXV International School of Theoretical Physics “Particles and
  Astrophysics – Standard Models and Beyond”, Ustroń, Poland, September 10-16,
  2001.
** Work supported in part by the Polish State Committee for Scientific Research (KBN)
  grant No. 2P03D02319.
called quintessence, characterized by the equation of state \( p/\rho = w \neq -1 \). The component can cluster on largest scales and therefore affect the mass power spectrum [5] and microwave background anisotropies [6, 7].

The investigations of the physical basis for the existence of such component are now more than a decade old [8]. One of the promising models is based on so-called “tracker fields” that display an attractor-like behavior causing the energy density of quintessence to follow the radiation density in the radiation dominated era but dominate over matter density after matter-radiation equality [9, 10]. It is still debated, however, how \( w \) should depend on time, and whether its redshift dependence can be reliably determined observationally [11, 12, 13].

A considerable effort has gone into attempts to put constraints on models with quintessence and presently the values of \(-1 < w < -0.6\) seem most feasible observationally [14, 15]. Here I review the main characteristics of structure formation in the quintessential Universe which may provide constraints on the equation of state. In the last Section I discuss the current status of observational limits on \( w \) and future perspectives.

2. Background cosmology

Quintessence obeys the following equation of state relating its density \( \rho_Q \) and pressure \( p_Q \)

\[
p_Q = w \rho_Q, \quad \text{where} \quad -1 \leq w < 0. 
\]

The case of \( w = -1 \) corresponds to the usually defined cosmological constant.

The evolution of the scale factor \( a = R/R_0 = 1/(1 + z) \) (normalized to unity at present, \( z \) is the redshift) in the quintessential Universe is governed by the Friedmann equation

\[
\frac{da}{dt} = \frac{H_0}{u(a)} 
\]

where

\[
u(a) = \left[ 1 + \Omega_0 \left( \frac{1}{a} - 1 \right) + q_0 \left( \frac{1}{a^{1+3w}} - 1 \right) \right]^{-1/2}
\]

and \( H_0 \) is the present value of the Hubble parameter. The quantities with subscript 0 here and below denote the present values. The parameter \( \Omega \) is the standard measure of the amount of matter in units of critical density and \( q \) measures the density of quintessence in the same units

\[
q = \frac{\rho_Q}{\rho_{\text{crit}}}. 
\]
Fig. 1. Left panel: Time evolution of the scale factor in different models. Right panel: The present age of Universe in units of $H_0^{-1}$ in flat models as a function of $\Omega_0$ for different $w$.

The Einstein equation for acceleration $\frac{d^2a}{dt^2} = -4\pi Ga(p + \varrho/3)$ shows that $w < -1/3$ is needed for the accelerated expansion to occur. The left panel of Fig. 1 shows the evolution of scale factor in different models. The right panel presents the dependence on $w$ of the present age of Universe

$$t_0 = \frac{1}{H_0} \int_0^1 u(a) da.$$  

(5)

Solving the equation for the conservation of energy $d(\varrho_Q a^3)/da = -3p_Q a^2$ with condition (1) we get the evolution of the density of quintessence which for $w = \text{const}$, the case considered in this paper, reduces to

$$\varrho_Q = \varrho_{Q,0} a^{-3(1+w)}.$$  

(6)

The evolution of $\Omega$ and $q$ with scale factor is given by

$$\Omega(a) = \frac{\Omega_0 u^2(a)}{a}, \quad q(a) = \frac{q_0 u^2(a)}{a^{1+3w}}$$  

(7)

while the Hubble parameter itself evolves so that $H(a) = H_0/[a u(a)]$.

3. Linear growth of perturbations

The linear evolution of the matter density contrast $\delta = \delta\varrho/\varrho$ is governed by equation [16]

$$\ddot{\delta} + 2\frac{\dot{a}}{a} \dot{\delta} - 4\pi G \varrho \ddot{\delta} = 0$$  

(8)
Fig. 2. The linear growth rate of density fluctuations for $\Omega_0 = 0.3$, $q_0 = 0.7$ in three cases of $w = -1, -2/3$ and $-1/3$.

where dots represent derivatives with respect to time. For flat models and arbitrary $w$ an analytical expression for $D(a)$, the growing mode of the time-dependent part of $\delta$, was found \[17\]. With our notation and the normalization of $D(a) = a$ for $\Omega = 1$ and $q = 0$ it becomes

$$D(a) = a \, {}_2F_1 \left[ -\frac{1}{3w}, \frac{w-1}{2w}, 1 - \frac{5}{6w}, -a^{-3w} \frac{1 - \Omega_0}{\Omega_0} \right]$$

(9)

where $\, {}_2F_1$ is a hypergeometric function. The solutions (9) for different $w$ and cosmological parameters $\Omega_0 = 0.3$ and $q_0 = 0.7$ are plotted in Fig. 2.

The peculiar velocity field in linear perturbation theory is obtained from \[16\]

$$v = \frac{2f}{3\Omega H} g$$

(10)

where $g = -\nabla \phi / a$ is the peculiar gravitational acceleration and $f$ is the dimensionless velocity factor

$$f = \frac{a \, \dot{D}}{\dot{a} \, D}.$$  

(11)

For flat models this formula can be evaluated analytically using Eq. (9). The dependence of $f$ on $\Omega_0$ at present ($z = 0$) for flat models with different $w$ is
Fig. 3. Left panel: The velocity factor $f$ at present as a function of $\Omega_0$ for flat models with different $w$. Right panel: The redshift dependence of $f$ for $\Omega_0 = 0.3$, $q_0 = 0.7$ and different $w$.

shown in the left panel of Fig. 3. We see immediately that the dependence on $w$ is very weak. However, as shown in the right panel of Fig. 3, when going to higher redshifts we find that the velocity factor is much more sensitive to $w$ which gives some hope for applying it to determine $w$ from local peculiar velocity field.

4. Strongly nonlinear evolution

The simplest model of formation of bound objects (called the spherical or top hat model) [18, 19, 20, 21] describes the nonlinear evolution of a uniform spherical density perturbation in an otherwise homogeneous Universe. If the overdensity is big enough the perturbed region will slow down the expansion and eventually turn around and collapse. The evolution of the size of such a perturbation can be described [22] by the Einstein equation

$$\ddot{r}/r = -4\pi G \left[ \left( w + \frac{1}{3} \right) \rho_Q + \frac{1}{3} \rho \right].$$

(12)

Together with the Friedmann equation (2) it can be solved to obtain the ratio of the cluster density to the background density at the time of turn-around

$$\zeta = \frac{\rho}{\rho_b}(z_{ta}) = \left( \frac{3\pi}{4} \right)^2 \Omega^{-0.79+0.26\Omega-0.06w}|_{z_{ta}},$$

(13)

where the last formula is the approximation given in [22].
One of the most useful quantities one can derive from the spherical top hat model is the density of the virialized halo in units of the critical density at the time of collapse corresponding to redshift $z_{\text{coll}}$

$$\Delta_c = \frac{\rho}{\rho_{\text{crit}}}(z_{\text{coll}}) = \zeta(z_{\text{ta}})\Omega(z_{\text{coll}}) \left(\frac{r_{\text{ta}}}{r_{\text{coll}}}\right)^3 \left(\frac{1 + z_{\text{ta}}}{1 + z_{\text{coll}}}\right)^3. \quad (14)$$

The redshifts $z_{\text{coll}}$ and $z_{\text{ta}}$ are related by the assumption that the collapse time $t_{\text{coll}}$, the time corresponding to $r \to 0$, is twice the turn-around time

$$t_{\text{ta}} = \frac{1}{H_0} \int_0^{r_{\text{ta}}} u(a) da. \quad (15)$$

In reality the object does not decrease its size to zero but virializes to reach the effective final radius $r_{\text{coll}}$. The ratio by which the halo collapses is estimated from the virial theorem which gives [22]

$$\frac{r_{\text{coll}}}{r_{\text{ta}}} = \frac{1 - \eta_v/2}{2 + \eta_t - 3\eta_v/2}, \quad (16)$$

where

$$\eta_t = \frac{2}{\zeta \Omega(z_{\text{ta}})} \frac{q(z_{\text{ta}})}{q(z_{\text{coll}})} \left(\frac{1 + z_{\text{ta}}}{1 + z_{\text{coll}}}\right)^3. \quad (17)$$

$$\eta_v = \frac{2}{\zeta \Omega(z_{\text{coll}})} \left(\frac{1 + z_{\text{coll}}}{1 + z_{\text{ta}}}\right)^3. \quad (18)$$

Figure 4 shows how $\Delta_c$ depends on $\Omega_0$, $w$ and the redshift of collapse.
5. Power spectrum of density fluctuations

Power spectrum $P(k, a)$ is defined as the Fourier transform of the correlation function of density fluctuations

$$P(k, a) = \int \xi(r, a) e^{-ik\cdot r} d^3r.$$  \hspace{1cm} (19)

The spectra for Universe dominated by cold dark matter (CDM) have been widely discussed in the literature, e.g. [23]. For the present time ($a = 1$) the power spectrum is usually written in the form

$$P(k) = A k^n T^2(k)$$  \hspace{1cm} (20)

where $n$ measures the slope of the primordial power spectrum (we will assume $n = 1$), $T$ is the transfer function and $A$ is a normalization constant.

In the presence of cosmological constant ($\Lambda$CDM) the transfer function $T_\Lambda$ can be approximated by [23]

$$T_\Lambda^2(p) = \ln^{\frac{1}{2}}[1 + 2.34p]^3 \frac{[1 + 3.89p + (16.1p)^2 + (5.46p)^3 + (6.71p)^4]^{-1/2}}{(2.34p)^2},$$  \hspace{1cm} (21)

where $p = k/(\Gamma h \text{Mpc}^{-1})$, $\Gamma = \Omega_0 h \exp[-\Omega_b(1 + \sqrt{2h}/\Omega_0)]$ and $h = H_0/[100 \text{ km}/(\text{s Mpc})] = 0.7$. We assume the barion contribution to the density parameter $\Omega_b = 0.04$.

For flat models with quintessence the modification of the transfer function has been proposed by Ma et al. [5]. The transfer function in equation (20) is then $T_Q = T_Q \Lambda T_\Lambda$, where $T_Q \Lambda = T_Q / T_\Lambda$ is approximated by fits given in [5]. The present linear power spectra obtained for $w = -1, -2/3$ and $-1/3$ assuming COBE normalization are shown in the left panel of Fig. 5.

The rms density fluctuation, $\sigma$, at comoving smoothing scale $R$ is given by

$$\sigma^2 = \frac{1}{(2\pi)^3} \int d^3k P(k) W_{TH}^2(kR)$$  \hspace{1cm} (22)

where the smoothing is performed with the top hat filter

$$W_{TH}^2(kR) = \frac{3}{(kR)^2} \left( \frac{\sin kR}{kR} - \cos kR \right).$$  \hspace{1cm} (23)

The dependence of $\sigma$ on smoothing scale for flat models with different $w$ is shown in the right panel of Fig. 5. A particularly useful quantity, constrained by cluster abundance is the rms fluctuation at the scale of $8h^{-1} \text{ Mpc}$. Due to COBE normalization its values turn out to depend strongly on $w$ and we get $\sigma_8 = 0.96, 0.80$ and $0.46$ for $w = -1, -2/3$ and $-1/3$ respectively.
Fig. 5. The linear power spectra (left panel) and rms fluctuation as a function of smoothing scale (right panel) for flat models with different $w$.

6. Mass functions

One of the most important measures of structure formation is provided by the mass function of bound objects. Using the analytical prescription of Press and Schechter [24], we can estimate the cumulative mass function (the comoving number density of objects of mass greater than $M$)

$$N(> M) = \int_M^{\infty} n(M) dM$$

where $n(M)$ is the number density of objects with mass between $M$ and $M + dM$

$$n(M) = -\left(\frac{2}{\pi}\right)^{1/2} \frac{\varrho_b \delta_c}{M \sigma^2} \frac{d\sigma}{dM} \exp\left(-\frac{\delta_c^2}{2\sigma^2}\right).$$

In the expression above, $\varrho_b$ is the background density, $\sigma$ is the rms density fluctuation at comoving smoothing scale $R$ described in Section 5. The mass is related to the smoothing scale by $M = 4\pi \varrho_b R^3/3$. The parameter $\delta_c$ is the halo density at collapse as predicted by linear theory. We adopt here its standard value $\delta_c = 1.69$ (exact for the Einstein-de Sitter Universe) since it depends very weakly on $\Omega_0$ and $w$ [22].

Figure 6 shows the cumulative mass functions calculated from equation (25) with $\Omega_0 = 0.3$ and $q_0 = 0.7$ for three models with $w = -1, -2/3$ and $-1/3$. The strong dependence on $w$ is due to the COBE normalization of the power spectra. The Figure also shows data for rich clusters of galaxies from [25]. For our choice of cosmological parameters the value of $w \approx -1$ is
preferred. However, the Press-Schechter formulae are known to underestimate the mass functions on the scale of clusters of galaxies when compared to N-body simulations. The mass functions are also very sensitive to the value of $\Omega_0$. Therefore when more exact predictions are used and $\Omega_0$ is measured to be close to 0.3 with a small error bar, we can expect higher values of $w$ to be preferred by the cluster data.

7. Constraints on $w$

The primary constraint on the value of $w$ comes from the observations of accelerating expansion, which can only be obtained in models with $w < -1/3$. Age of Universe is quite sensitive to $w$ and increases for lower $w$, however the accuracy of our knowledge of both $H_0$ and $t_0$ is not good enough to put strong constraints on $w$. Current estimates are consistent with $w < -0.5$.

One of the strongest arguments for the existence of dark energy comes from the studies of cosmic microwave background (CMB). Although its power spectrum is weakly sensitive to $w$ (e.g. the height of the first acoustic peak is somewhat increased and its position is shifted to higher multipoles for lower $w$), it has already provided some limits. Combining the data from COBE and recent balloon experiments Balbi et al. [6] find $-1 < w < -0.6$.
while Baccigalupi et al. [7] estimate the best-fitting value of $w$ to be $w = -0.8$. The ongoing and future satellite experiments are expected to put even stronger limits on $w$.

Among the promising probes of dark energy are also supernovae Ia. The comoving distance they serve to measure is sensitive only to the interesting cosmological parameters and the errors related to supernova evolution or extinction are estimated to be small. The existing data restrict $w$ only weakly [26], but future experiments like The Supernova Acceleration Probe are expected to measure the value of $w$ with a few percent accuracy [27].

Structure formation also offers methods to constrain the cosmic equation of state. The suppression of linear growth of density fluctuations for higher $w$ alone shows that only for $w < -1/2$ the structure observed today could have evolved from small initial perturbations deduced from CMB observations. The same range of acceptable $w$ values follows from the behaviour of $\sigma_8$ which is a strongly decreasing function of $w$.

The most promising tests are based on the number counts of galaxy clusters. It turns out [22] that the slope of comoving abundance as a function of redshift depends sensitively on $w$ and therefore can be used to break degeneracies between $w$ and other cosmological parameters that appear e.g. in the analysis of CMB. Such measurements are expected to be performed using the proposed new X-ray and Sunyaev-Zeldovich effect surveys [28] and the ongoing DEEP Redshift Survey [29]. The constraints from structure formation appear to be complementary to those from supernovae and CMB measurements.

REFERENCES

[1] S. M. Harun-or-Rashid, M. Roos, Astron. & Astrophys., 373, 369 (2001)
[2] S. M. Carroll, W. H. Press, E. L. Turner, Annual Review of Astron. & Astrophys., 30, 499 (1992)
[3] O. Lahav, P. B. Lilje, J. R. Primack, M. J. Rees, MNRAS, 251, 128 (1991)
[4] R. R. Caldwell, R. Dave, P. J. Steinhardt, Phys. Rev. Lett., 80, 1582 (1998)
[5] C. P. Ma, R. R. Caldwell, P. Bode, L. Wang, Astrophys. J., 521, L1 (1999)
[6] A. Balbi, C. Baccigalupi, S. Matarrese, F. Perrotta, N. Vittorio, Astrophys. J. Lett., 547, 89 (2001)
[7] C. Baccigalupi, A. Balbi, S. Matarrese, F. Perrotta, N. Vittorio, Phys. Rev. D, 65, 63520 (2002)
[8] B. Ratra, P. J. E. Peebles, Phys. Rev. D, 37, 3406 (1988)
[9] I. Zlatev, L. Wang, P. J. Steinhardt, Phys. Rev. Lett., 82, 896 (1999)
[10] P. J. Steinhardt, L. Wang, I. Zlatev, Phys. Rev. D, 591, 270 (1999)
[11] V. Barger, D. Marfatia, *Phys. Lett. B*, 498, 67 (2001)
[12] I. Maor, R. Brustein, P. J. Steinhardt, *Phys. Rev. Lett.*, 86, 6 (2001)
[13] J. Weller, A. Albrecht, *Phys., Rev. Lett.*, 86, 1939 (2001)
[14] L. Wang, R. R. Caldwell, J. P. Ostriker, P. J. Steinhardt, *Astrophys. J.*, 530, 17 (2000)
[15] D. Huterer, M. S. Turner, *Phys. Rev. D*, 64, 123527 (2001)
[16] P. J. E. Peebles, *Principles of Physical Cosmology*, Princeton Univ. Press, Princeton 1993
[17] V. Silveira, I. Waga, *Phys. Rev. D*, 50, 4890 (1994)
[18] J. E. Gunn, J. R. Gott, *Astrophys. J.*, 176, 1 (1972)
[19] J. E. Gunn, *Astrophys. J.*, 218, 592 (1977)
[20] E. L. Lokas, Y. Hoffman, *The Identification of Dark Matter*, Proc. 3rd International Workshop, eds N. J. C. Spooner, V. Kudryavtsev, World Scientific, Singapore 2001, p. 121
[21] E. L. Lokas, Y. Hoffman, astro-ph/0108283
[22] L. Wang, P. J. Steinhardt, *Astrophys. J.*, 508, 483 (1998)
[23] N. Sugiyama, *Astrophys. J.*, 471, 542 (1995)
[24] W. H. Press, P. Schechter, *Astrophys. J.*, 187, 425 (1974)
[25] M. Girardi, S. Borgani, G. Giuricin, F. Mardroossian, M. Mezzetti, *Astrophys. J.*, 506, 45 (1998)
[26] S. Perlmutter, M. S. Turner, M. White, *Phys., Rev. Lett.*, 83, 670 (1999)
[27] S. C. C. Ng, D. L. Wiltshire, *Phys. Rev. D*, 64, 123519 (2001)
[28] Z. Haiman, J. J. Mohr, G. P. Holder, *Astrophys. J.*, 553, 545 (2001)
[29] J. A. Newman, M. Davis, *Astrophys. J. Lett.*, 534, L11 (2000)