Towards the self-adjointness of a Hamiltonian operator in loop quantum gravity

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Abstract

Although the physical Hamiltonian operator can be constructed in the deparametrized model of loop quantum gravity coupled to a scalar field, its property is still unknown. This open issue is attacked in this paper by considering an operator $\hat{H}_v$ representing the square of the physical Hamiltonian operator acting nontrivially on a two-valent vertex of spin networks. The Hilbert space $\mathcal{H}_v$ preserved by the graphing changing operator $\hat{H}_v$ is consist of spin networks with a single two-valent non-degenerate vertex. The matrix element of $\hat{H}_v$ are explicitly worked out in a suitable basis. It turns out that the operator $\hat{H}_v$ is essentially self-adjoint, which implies a well-defined physical Hamiltonian operator in $\mathcal{H}_v$ for the deparametrized model.

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1 Introduction

Loop Quantum Gravity (LQG) is a background independent framework designed for quantization of generally relativistic theories of gravity coupled to other fields [1–4]. In this paper we consider the canonical approach as opposed to the covariant Spin-Foam models. The starting point of the canonical LQG is the standard, torsion free Einstein’s gravity in the Palatini-Holst [1, 5] formulation coupled to the fields of the Standard Model of fundamental interactions. The peculiar step, though, is a reformulation of the canonical classical theory in terms of the Ashtekar-Barbero variables: an SU(2) connection $A$ and the canonically conjugate 3-frame-density variable $E$ [6, 7]. The second peculiar step is the introduction of the parallel transport (holonomy) of $A$ (along all the curves) and flux of $E$ (along all the 2-surfaces) as basic variables subject to the canonical quantization [8]. The quantization provided several important break-throughs. A Hilbert space was defined that carries a unitary action of the spatial diffeomorphisms and quantum representation of the holonomy-flux variables [9, 11]. A family of operators representing geometric observables (2-surface area, 3-region volume, inverse metric tensor) were regularized without need to subtract infinities and their spectra turned out to be discrete [11, 16]. The representation is unique upon the diffeomorphism invariance and the existence of an invariant cyclic state [17]. The Gauss and vector constraint were solved exactly and the ”half-physical” space of solutions was endowed with a natural Hilbert product [18]. Thereon the quantum scalar constraint map was regularized again without emergence of any infinities that would have to be abandoned [1, 19, 20]. The Master Constraint operator that is supposed to capture all the constraints of the vacuum theory was defined [21, 22]. Finally, a
symmetric (in the sense of the Hilbert product) quantum representation for the gravitational part of the scalar constraints was introduced in a suitably defined vertex Hilbert space $[23,25]$. Also matter fields were coupled to LQG and quantizations consistent with the new framework were found. In particular, the Brown-Kuchar model of gravity coupled to dust as well as the Rovelli-Smolin model of gravity coupled to massless Klein-Gordon field were quantized completely $[26,30]$. The mechanisms of swallowing gauge dependent degrees of freedom, constructing and evolving Dirac observables in a relational manner were discussed, understood better and reformulated $[31,34]$.

The curvature of the Ashtekar-Barbero connection present in a quantum gravitational Hamiltonian acts on quantum states of the canonical LQG by attaching loops. A specific way the quantum curvature does it is not determined uniquely, it can be defined in infinitely many different and inequivalent ways $[1]$. That causes a considerable ambiguity in defining a quantum Hamiltonian operator. A rough classification divides the set of all the Hamiltonian operators of LQG into the following two categories: (i) graph preserving, (ii) graph non-preserving. The graph preserving action is natural from the lattice discretization point of view. It makes the action of operators reducible to subspaces corresponding to the graphs. For every graph, analytic properties of operators are much easier to study. There is also a general argument, that if gravity is deparametrized by a coupled dust, then the graph preserving action is the only diffeomorphism invariant option $[28]$. The discretization requires taking a continuum limit. That can be achieved by a suitable renormalization scheme $[35,36]$. Remarkably, the renormalization attempts also to resolve the remaining quantization ambiguities of the Hamiltonian operator. A continuum field theory approach rather than the discretization, leads directly to the second category, that is to the graph-changing action. This is the option our current paper concerns. Several proposals of graph changing quantum Hamiltonian operator were considered in the literature $[11,14,21,28,37,38]$. A requirement that the Hermitian adjoint operator to an operator adding a loop is a well defined operator imposes conditions on admissible ways of attaching loops $[23,25]$. Still, however, nothing was known thus far about the self-adjointness. In quantum mechanics the self-adjointness of quantum observables corresponds to the reality of corresponding classical observables, hence it has a clear physical meaning. The self-adjointness of an operator is equivalent to the spectral decomposition and reality of the spectrum that allows to define a given operator by indicating eigenstates and corresponding eigenvalues. In particular, this is the spectral decomposition of quantum constraints that ensures exact definition of their solutions and endows their space with a natural Hilbert product. The self-adjointness of effective Hamiltonian operators ensures existence of a unitary time evolution of quantum states. The relevance of that property of the quantum Hamiltonian is illustrated in the models of loop quantum cosmology $[39,40]$. The big breakthrough of that theory coming after a genuine self-adjoint quantum Hamiltonian operator was introduced in $[41,42]$.

In the current paper, for the first time, we address the issue of the self-adjointness of the graph changing Hamiltonian in the full theory with the local degrees of freedom. The model we choose to study is LQG coupled to the massless Klein-Gordon field. This is the full set of degrees of freedom version of the Ashtekar-Pawowski-Singh symmetry reduced homogeneous-isotropic model of universe. That is also one of the two known remarkable cases in which the Dirac program of quantum gravity can be completed $[20,30]$ (the second case after the Brown-Kuchar model). Indeed, all the quantum constraints of the canonical General Relativity were solved completely and a general solution was written down explicitly, assuming the existence of certain operators. The physical Hilbert space of the solutions was defined. The general formula for a Dirac observable that commutes with all the constraints was derived. The resulting algebra of the Dirac observables was shown to admit an action of the 1-dimensional group of automorphisms that classically corresponds to the transformations of adding a constant to the scalar field. The generator of those automorphisms was promoted to the physical quantum Hamiltonian operator $\hat{H}_{\text{phys}}$ of the system. An exact derivation of that operator in LQG has become possible with the introduction of the vertex Hilbert spaces $[23,25]$. The advantage of that Hilbert space necessary for our purpose is that it admits quantum operators of the gravitational scalar constraint $\hat{C}^{gr}(x)$ smeared against any suitable test function
An existence of the Hermitian adjoint operator $\left(\hat{C}_{\text{gr}}(N)\right)^\dagger$ is a condition on the quantization. It is satisfied by two alternative proposals. The first one changes the valency of the vertices, but remains the spins on the old edges invariant [23]. The second one preserves the valency of the vertices and changes the spins of the edges in a way controlled to the effect that no spin can be reduced to zero [25]. In the current paper we apply the latter proposal. We combine it with the new idea of quantization of the gravitational scalar constraint [38, 43].

Due to that choice a physical Hamiltonian operator can be constructed without using the quantum volume operator [13]. In the consequence, this physical Hamiltonian even does not annihilate the spin network states with two-valent vertices since it does not contain the volume operator. This helps us to find an example of a simple subspace of the full Hilbert space that is preserved by the action of the quantum Hamiltonian and analyse the restricted operator. The subspace is constructed by introducing a graph with a single two-valent vertex. A quantum state defined by a 2-valent vertex classically corresponds to a degenerate Ashtekar frame such that one of the densitized vectors is zero. The phase space of the classical Ashtekar theory contains points characterized by the lower than 3 rank of the frame. Mathematically, they are regular and make perfect sense [44, 45]. In particular, the rank 2 case is exactly soluble for the vacuum theory [10]. The result is a generalized spacetime foliated by disjoined 2+1 dimensional surfaces. Quantum states defined by graphs containing only 2-valent vertices are quantization of that degenerate sector of Ashtekar’s theory.

The paper is organized as follows. In Sec. 2 we remind the classical and quantum theories of deparametrized model of gravity coupled to massless Klein-Gordon scalar field. Some necessary notions, like kinematical Hilbert space, the vertex Hilbert space and the physical Hilbert space, are included. The key part of this section is the explicit definition of the physical Hamiltonian operator on the vertex Hilbert space. In Sec. 3 we apply the general theory to a simple case, where the Hilbert space is generated from a single bivalent non-degenerate vertex and preserved by the physical Hamiltonian operator. The action of the operator $\hat{H}_v$ representing the square of the physical Hamiltonian on the Hilbert space are derived. In Sec. 4 we study the operator $\hat{H}_v$ on the simple subspace and prove that the restricted operator is well defined and self-adjoint. In Sec. 5 we discuss the issue of eigenvalue problem for $\hat{H}_v$. Conclusions and outlooks are presented in Sec. 6.

2 A general work on deparametrized model

2.1 The classical theory

Considering gravity minimally coupled to a massless Klein-Gordon field in the ADM formalism with Ashtekar-Barbero variables, we have a totally constrained system with the standard canonical variables $(A^i_a(x), E^a_i(x))$ for gravity and $(\phi(x), \pi(x))$ for scalar field defined at every point $x$ of an underlying 3-dimensional manifold $\Sigma$. The diffeomorphism and scalar constraints are respectively

\[ C_a(x) = C_{\text{gr}}^a(x) + \pi(x)\phi_a(x) = 0, \]

\[ C(x) = C^{\text{gr}}(x) + \frac{1}{2} \frac{\pi(x)^2}{\sqrt{|\det E(x)|}} + \frac{1}{2} q^{ab}(x)\phi_a(x)\phi_b(x)\sqrt{|\det E(x)|} = 0, \]

where $C_{\text{gr}}^a$ and $C^{\text{gr}}$ are the vacuum gravity constraints and $q^{ab} = \frac{E^a_i E^b_i}{|\det E|}$. 

\[ \hat{C}_{\text{gr}}(N) = \int_\Sigma d^3x N(x)\hat{C}_{\text{gr}}(x). \]
The deparametrized procedure starts with assuming that the constraints (1) are satisfied. By replacing $\phi, a$ by $-C_{gr} a / \pi$, the constraints (2) are rewritten as

$$\pi^2 = \sqrt{|\det E|} \left( -C_{gr} \pm \sqrt{(C_{gr})^2 - q_{ab} C_{gr}^a C_{gr}^b} \right).$$  

(3)

The sign ambiguity is solved depending on a quarter of the phase space. We choose the one that contains the homogeneous cosmological solutions [38]. In that part of the phase space, the scalar constraint $C(x)$ can be replaced by,

$$C'(x) = \pi(x) \pm \sqrt{h(x)},$$  

(4)

where

$$h = \sqrt{|\det E|} \left( -C_{gr} \pm \sqrt{(C_{gr})^2 - q_{ab} C_{gr}^a C_{gr}^b} \right).$$  

(5)

2.2 The structure of the quantum theory

For the deparametrized theory, the Dirac quantization scheme can be implemented and performed to the end [29, 30]. The result is a physical Hilbert space of solutions to the constraints, together with algebra of quantum Dirac observables endowed with one dimensional group of automorphisms generated by a quantum Hamiltonian operator. This resulted structure is equivalent to the following model that is expressed in a derivable way by elements of the framework of LQG:

- The physical Hilbert space $H$ is the space of the quantum states of the vacuum (matter free) gravity in the Ashtekar-Barbero connection-frame variables that satisfy the vacuum quantum vector constraint and the vacuum quantum Gauss constraint. In other words, in the connection representation, the states are constructed from functions $A \mapsto \Psi(A)$ invariant with respect to the diffeomorphism gauge transformations

$$A' = f^* A, \quad f \in \text{Diff}(\Sigma)$$

and to the Yang-Mills gauge transformations

$$A' = g^{-1} A g + g^{-1} dg, \quad g \in C(\Sigma, G).$$  

(6)

They are not assumed to satisfy the vacuum scalar constraint, though. That Hilbert space is available in the LQG framework.

- The Dirac observables are represented by the set of operators $\{\hat{O}\}$ in $H$. When the scalar field transforms as $\phi \mapsto \phi + t$ with a constant $t$, the observables transform as

$$\hat{O} \mapsto e^{i \hat{H} t} \hat{O} e^{-i \hat{H} t}$$

Therefore the quantum dynamics in the Schrödinger picture is given by

$$i \frac{d}{dt} \Psi = \hat{H} \Psi,$$  

(8)

$\hat{H}$ is called the quantum Hamiltonian.

- The quantum Hamiltonian

$$\hat{H} = \int d^3 x \sqrt{-2 \sqrt{|\det E(x)|} C_{gr}^v(x)},$$

(9)

is a quantum operator corresponding to the classical physical Hamiltonian

$$H = \int d^3 x \sqrt{-2 \sqrt{|\det E(x)|} C_{gr}^v(x)}.$$
The classical Hamiltonian $H$ is manifestly spatial diffeomorphism invariant, the same is expected about a quantum Hamiltonian operator $\hat{H}$. There seems to be a perfect compatibility between the diffeomorphism invariance of the quantum Hamiltonian operator and the diffeomorphism invariance of the quantum states, elements of the Hilbert space $\mathcal{H}$. However, the integrand $\sqrt{-2\sqrt{\det E(x)}C^{gr}(x)}$ involves the square root of an expression assigned to each point $x$. In order to quantize it, we should know the operator corresponding to the expression of $-2\sqrt{\det E(x)}C^{gr}(x)$ under the square root at first. However, on one hand $-2\sqrt{\det E(x)}C^{gr}(x)$ itself is not diffeomorphism invariant, which leads to the fact that the corresponding operator can not be well defined within the diffeomorphism invariant Hilbert space. On the other hand, group averaging with respect to diffeomorphism transformation is necessary because the operator corresponding to the expression of $-2\sqrt{\det E(x)}C^{gr}(x)$ is defined by taking some limit of holonomies along a sequence of closed loops nearby the vertices of spin networks. The convergence of such limits requires partial diffeomorphism invariance nearby each vertex. Therefore, the kinematical Hilbert space is not a suitable choice either. The idea to solve the contradiction is to introduce the vertex Hilbert space $\mathcal{H}_{\text{vtx}}$, of partially diffeomorphism invariant states in which an operator $-2\sqrt{\det E(x)}C^{gr}(x)$ is well defined for each $x \in \Sigma$ [23, 25, 38]. We can finally pass to the operators in $\mathcal{H}$ from those in $\mathcal{H}_{\text{vtx}}$ by the dual action naturally since $\mathcal{H}$ is a dual space of $\mathcal{H}_{\text{vtx}}$ as shown in the following context.

2.3 The Hilbert spaces $\mathcal{H}_{\text{kin}}$, $\mathcal{H}_{\text{vtx}}$ and $\mathcal{H}$

The kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ of the vacuum LQG consists of functions

$$\Psi_\gamma(A) = \psi_\gamma(h_{e_1}(A), \ldots, h_{e_n}(A)),$$

where $e_1, \ldots, e_n$ are the edges of a graph embedded in $\Sigma$, and $h_e(A) \in SU(2)$ is the parallel transport along a path $e$ in $\Sigma$ with respect to a given connection 1-form $A$,

$$h_e(A) = \mathcal{P} \exp(- \int_e A).$$

In the LQG framework those functions of the variable $A$ are called cylindrical functions. It may be also used to define a multiplication operator, given a representation $D^{(l)}$ of $SU(2)$,

$$(D^{(l)}m_n(h_p))\Psi(A) = D^{(l)}m_n(h_p(A))\Psi(A),$$

where $m, n$ label an entry of the matrix.

The kinematical space can be decomposed into the orthogonal sum

$$\mathcal{H}_{\text{kin}} = \bigoplus_\gamma \mathcal{H}_\gamma,$$

where $\gamma$ runs through the set of embedded graphs in $\Sigma$ (un-oriented, and without removable vertices).

We also use a basis $\tau_1, \tau_2, \tau_3 \in \mathfrak{su}(2)$ such that

$$[\tau_i, \tau_j] = \epsilon_{ijk} \tau_k.$$

Another operator we will apply in the current paper is defined in $\mathcal{H}_\gamma$. Given a graph $\gamma$, a vertex $v$, and an edge $e_0$ at $v$, and $\tau_i$, it acts on function $\Psi_\gamma$ of (10) as follows,

$$(J^\gamma_{v, e_0} \Psi_\gamma)(A) = \begin{cases} i \frac{d}{dt} \bigg|_{t=0} \psi_\gamma(\ldots, h_e(A), e^{-\tau_3} h_{e_0}(A), g_e(A), \ldots), & v = t(e) \\ i \frac{d}{dt} \bigg|_{t=0} \psi_\gamma(\ldots, h_e(A), h_{e_0}(A)e^{\tau_3}, h_e(A), \ldots), & v = s(e). \end{cases}$$

see [1] for more details.
In this paper we restrict to functions invariant with respect to the Yang-Mills gauge transformations \[^1\]. An orthonormal basis can be constructed from the spin-network states. Given a graph \(\gamma\) in \(\Sigma\), we denote by \(V(\gamma)\) the set of the vertices and \(E(\gamma)\) the set of the edges. A vertex \(v \in V(\gamma)\) is called degenerate if all of the edges \(e\) at \(v\) are tangent to each other. We denote the set of all non-degenerate vertices by \(V_{nd}(\gamma)\), the diffeomorphisms acting trivially on \(V_{nd}(\gamma)\) by \(\text{Diff}^{\omega}_{nd}(\gamma)\) with \(\omega\) representing that the diffeomorphism is semi-analytic \[^1\]7\), and the elements of \(\text{Diff}^{\omega}_{nd}(\gamma)\) preserving every edge of \(\gamma\) by \(\text{TDiff}(\gamma)\). For any cylindrical function \(\eta_{\gamma} \in \mathcal{H}_{\gamma}\), the map \(\eta\) is defined as

\[
\eta : \Psi_{\gamma} \mapsto \sum_{f \in \text{Diff}^{\omega}_{nd}(\gamma)/\text{TDiff}(\gamma)} \langle \hat{U}_f \cdot \Psi_{\gamma} \rangle,
\]

where \(\hat{U}_f\) denotes the unitary operator corresponding to the diffeomorphism transformation \(f\) on \(\Sigma\) \[^1\]H\). \(\eta\) maps all elements in \(\mathcal{H}_{\gamma}\) into the algebraic dual \((\bigoplus_{\gamma} \mathcal{H}_{\gamma})')\'. The inner product in \(\eta \left( \bigoplus_{\gamma} \mathcal{H}_{\gamma} \right)\) is defined naturally by

\[
\left( \eta(\Psi_{\gamma}), \eta(\phi_{\gamma^\prime}) \right) = \eta(\Psi_{\gamma})[\phi_{\gamma^\prime}].
\]

The resulting space \(\eta \left( \bigoplus_{\gamma} \mathcal{H}_{\gamma} \right)\) admits the natural orthogonal decomposition

\[
\eta \left( \bigoplus_{\gamma} \mathcal{H}_{\gamma} \right) = \bigoplus_{[\gamma]} \eta(\mathcal{H}_{\gamma}),
\]

where \([\gamma]\) stands for the set of all the graphs \(\gamma^\prime\) such that

\[
\eta(\mathcal{H}_{\gamma^\prime}) = \eta(\mathcal{H}_{\gamma}).
\]

The vertex Hilbert space \(\mathcal{H}_{vtx}\) is the completion of \(\eta \left( \bigoplus_{\gamma} \mathcal{H}_{\gamma} \right)\) under this inner product. One can conclude easily that for every graph \(\gamma\), every element \(\eta(\Psi_{\gamma}) \in \eta(\mathcal{H}_{\gamma})\) is a partial solution to the quantum diffeomorphism constraint invariant with respect to all the diffeomorphisms contained in \(\text{Diff}^{\omega}_{nd}(\gamma)\). It can be turned into a full solution of the quantum diffeomorphism constraint by a similar averaging with respect to the remaining quotient space \(\text{Diff}_{\omega}(\Sigma)/\text{Diff}^{\omega}_{nd}(\gamma)\), which equals to the set of embeddings of \(V(\gamma)\) in \(\Sigma\). In this way the Hilbert space \(\mathcal{H}\) mentioned above is defined as a dual space of \(\mathcal{H}_{vtx}\). Passing a diffeomorphism invariant operator from \(\mathcal{H}_{vtx}\) to \(\mathcal{H}\) is naturally realized by the dual action. Therefore, without losing the generality, we will study the quantum Hamiltonian operator \(\hat{H}\) in the Hilbert space \(\mathcal{H}_{vtx}\).

Given an operator \(\hat{O}\) on \(\mathcal{H}_{kin}\), the corresponding operator \(\hat{O}'\) on \(\mathcal{H}_{vtx}\) is defined by the duality

\[
\left( \hat{O}'\eta(\Psi_{\gamma}) \right)[f] := \eta(\Psi_{\gamma})[\hat{O}f], \quad \forall f \in \bigoplus_{\gamma} \mathcal{H}_{\gamma}.
\]

### 2.4 The physical quantum Hamiltonian operator

In the current paper we combine the general regularization scheme for the operator \(\sqrt{-2\sqrt{\det E(x)} C^{\tiny\text{gr}}(x)}\) introduced in \[^3\]8\) with the vertex valency preserving proposal for the loop assignment in \[^2\]5\). Thus the resulting operator is defined in the version of \[^2\]5\) in the vertex Hilbert space introduced above.

According to the framework, the integrant in the formula for the physical Hamiltonian operator is defined on the dual space \((\bigoplus_{\gamma} \mathcal{H}_{\gamma})')\' and takes the following form,

\[
\sqrt{-2\sqrt{\det E(x)} C^{\tiny\text{gr}}(x)} = \sum_{v \in \Sigma} \delta(v, x) \sqrt{\hat{H}_v}
\]

\[^1\]This definition is from our quantization of the physical Hamiltonian in the following.
where $\delta(v, x)$ is the Dirac distribution. The sum seems to be awfully infinite. However, for every subspace $\eta(\mathcal{H}_\gamma)$, the only non-zero terms correspond to the vertices of the underlying graph $\gamma$ of a cylindrical function. For every vertex $v \in V_{\text{nd}}(\gamma)$ the operator $\hat{H}_v$ is defined first as an operator $^{\text{kin}}\hat{H}_v$ in the kinematical Hilbert subspace $\mathcal{H}_\gamma$, next pulled back by $\eta$ to a well-defined operator in $\eta(\mathcal{H}_\gamma)$, and finally symmetrized. The issue is self-adjointness. $^{\text{kin}}\hat{H}_v$ takes the form of the sum with respect to pairs of edges of $\gamma$ that meet at $v$,

$$^{\text{kin}}\hat{H}_v = \sum_{e, e' \text{ at } v} \epsilon(e, e')^{\text{kin}}\hat{H}_{v, ee'}$$

(16)

where $\epsilon(e, e')$ equals to 0 if $e$ and $e'$ are tangent at $v$ or 1 otherwise. The operator consists of two parts,

$$^{\text{kin}}\hat{H}_{v, ee'} = (1 + i^2)^{\text{kin}}\hat{H}_{v, ee'}^{L} + ^{\text{kin}}\hat{H}_{v, ee'}^{E},$$

where the operators $^{\text{kin}}\hat{H}_{v, ee'}^{E}$ and $^{\text{kin}}\hat{H}_{v, ee'}^{L}$ act as follows:

- $^{\text{kin}}\hat{H}_{v, ee'}^{E}$ creates a pair of vertices $v_L$ and $v_R$ that split $e$, and $e'$, respectively, and attaches a new edge $\ell$ connecting the new vertices,

$$^{\text{kin}}\hat{H}_{v, ee'}^{E} = \epsilon(e, e')$$

(17)

A new element in this definition is that the new edge $\ell$ is tangent to both, $e$ at $v_L$, and $e'$ at $v_R$. More specifically

$$^{\text{kin}}\hat{H}_{v, ee'}^{E} = \kappa_1 \epsilon_{ijk}(h_{\alpha_{ee'}}^i)^{(l)} j_{v,e}^i j_{v,e'}^k,$$

(18)

where $\alpha_{ee'}$ is the loop passing through the vertices $v, v_L, v_R, v$ in the given order and along the segments of $e$ and $e'$, and along $\ell$, and

$$(h_{\alpha_{ee'}}^i)^{(l)} := -\frac{3}{l(l+1)(2l+1)} \text{Tr}^{(l)}(h_{\alpha_{ee'}} \tau^i),$$

(19)

$$\text{Tr}^{(l)}(h_{\alpha_{ee'}} \tau^i) := \text{Tr} \left( D^{(l)}(h_{\alpha_{ee'}}) D^{(l)}(\tau^i) \right).$$

(20)

The factor $\kappa_1$ is arbitrary, representing a residual ambiguity of the quantization. The spin $l$ is introduced in such a way, that for every spin-network state $\Psi_\gamma$ the spin-network decomposition of the state $\epsilon_{ijk}(h_{\alpha_{ee'}}^i)^{(l)} j_{v,e}^i j_{v,e'}^k \Psi_\gamma$ does not contain a component of the zero spin at a segment of $e$ or $e'$ [25]. In other words, neither edge $e$ nor edge $e'$ can even partially disappear as the effect of the action. We achieve that goal by fixing

$$l = \frac{1}{2}, \quad \text{provided} \quad j_e, j_{e'} > \frac{1}{2},$$

$$l = \frac{3}{2}, \quad \text{provided} \quad j_e = \frac{1}{2}, j_{e'} = 1 \text{ or } j_{e'} = \frac{1}{2}, j_{e} = 1,$$

$$l = 1, \quad \text{otherwise.}$$

- $^{\text{kin}}\hat{H}_{v, ee'}^{L}$ does not change any given graph $\gamma$, and even commutes with each of the operators $j_{v,e}^i$,

$$^{\text{kin}}\hat{H}_{v, ee'}^{L} := \left[ \sqrt{\delta_{ii'}} \left( \epsilon_{ijk} j_{v,e}^i j_{v,e'}^k \right) \left( \epsilon_{i'j'k'} j_{v,e}^i j_{v,e'}^k \right) \left( \frac{2\pi}{\alpha} - \pi + \arccos \left[ \frac{\delta_{kk'} j_{v,e}^k j_{v,e'}^{k'}}{\sqrt{\delta_{kk'} j_{v,e}^k j_{v,e}^{k}}} \right] \right) \right],$$

(22)

The factor $\alpha$ is arbitrary, representing another residual ambiguity of the quantization.
Next, we pass the operator to $\mathcal{H}_{\text{vtx}}$, by the duality (14),

$$
\hat{H}'_{\nu}\eta(\Psi_\gamma) := \eta(\Psi_\gamma)^{\text{kin}}\hat{H}_{\nu},
$$

for every subspace $\eta(\mathcal{H}_\gamma)$. Finally, in the Hilbert space $\mathcal{H}_{\text{vtx}}$ we turn it into a symmetric operator

$$
\hat{H}_{\nu} := \frac{1}{2}\left(\hat{H}'_{\nu} + (\hat{H}'_{\nu})^\dagger\right) = (1 + \beta^2)\hat{H}_v^L + \frac{1}{2}\left(\hat{H}'_v^E + (\hat{H}'_v^E)^\dagger\right) = (1 + \beta^2)\hat{H}_v^L + \hat{H}_v^E
$$

(24)

If we considered $(\text{kin}\hat{H}_{\nu})^\dagger$ and do symmetrization in the kinematical Hilbert space, the resulting operator would break the diffeomorphism invariance.

In order to implement (15), one has to find a basis in $\mathcal{H}_{\text{vtx}}$ that consists of eigenstates of $\hat{H}_{\nu}$,

$$
\hat{H}_{\nu}|v,\lambda\rangle = \lambda|v,\lambda\rangle,
$$

restrict the Hilbert space to the physical sector defined by the non-negative eigenvalues, and consider an operator

$$
\sqrt{-2\sqrt{|\det E(x)|C^{gr}(x)}|v,\lambda\rangle} = \sum_{v\epsilon \Sigma} \delta(v,x)\sqrt{\lambda}|v,\lambda\rangle.
$$

(25)

For the time being, we do not even know a single non-trivial eigenstate of the operator $\hat{H}_{\nu}$, except for the subspaces $\eta(\mathcal{H}_\gamma)$ given by graphs $\gamma$ that have no non-degenerate vertices. What we do in the next section is to consider a simplest subspace of $\mathcal{H}_{\text{vtx}}$, which contains states of non-degenerate vertices and is preserved by the action of the operator $\hat{H}_{\nu}$. We study the properties of the operator $\hat{H}_{\nu}$ therein. We prove that it is self-adjoint. Hence, that subspace does admit an orthogonal decomposition into the eigenstates of $\hat{H}_{\nu}$.

3 Restriction to a subspace of $\mathcal{H}_{\text{vtx}}$ preserved by $\hat{H}_{\nu}$

3.1 The subspace

In this section we construct a subspace $\mathcal{H}_{\nu} \subset \mathcal{H}_{\text{vtx}}$ from a single loop with a kink at a point $v$ (there is only one possibility, up to the diffeomorphisms), which is denoted by graph $\gamma_0$ as follows.

![Graph $\gamma_0$](image)

The assumption that $\mathcal{H}_{\nu}$ contains the $\gamma_0$ and is preserved by the operator $\hat{H}_{\nu}$ determines its construction. Fix a point $v \epsilon \Sigma$. For every integer $n$ (including $n = 0$) consider the following graph $\gamma_n$

![Graph $\gamma_n$](image)

(26)
The edges $\ell_1, \ldots, \ell_n$ are defined such that they could have been attached by the operator $k^\text{kin}_\mathbf{H}^E$ introduced above and acting $n$ times in a row at the non-degenerate vertex $v$. The vertices $v_{L1}, \ldots, v_{Rn}$ are ignored by the operator, because they are degenerate. Each of the graphs defines a subspace, $\mathcal{H}_\gamma \in \mathcal{H}_\text{kin}$ of the kinematical Hilbert space (11) and a subspace, $\eta(\mathcal{H}_\gamma) \subset \mathcal{H}_\text{vtx}$, of the space of diffeomorphism (preserving $v$) invariant states. Consider the subspace $\mathcal{H}_\gamma^G$ of the Yang-Mills gauge (6) invariant elements of $\mathcal{H}_\gamma$ and denote $\mathcal{H}_\gamma[n] := \eta(\mathcal{H}_\gamma^G)$.

The subspace $\mathcal{H}_v$ of $\mathcal{H}_\text{vtx}$ we are constructing will be contained in the subspace $\bigoplus_n \mathcal{H}_\gamma[n] \subset \mathcal{H}_\text{vtx}$. (28)

For every graph $\gamma_n$, consider a spin-network state $|\gamma_n, \vec{j}, \vec{l}\rangle$, where $\vec{j} = (j_1, j_2, \ldots, j_{n+1})$ and $\vec{l} = (l_1, l_2, \ldots, l_n)$ are the spins assigned to edges of $\gamma_n$ as shown in the following equation (referring to Appendix A for the graph notion)

$$|\gamma_n, \vec{j}, \vec{l}\rangle := N_n$$

where $N$ is a real and positive normalization factor. If we act on any of those states $|\gamma_n, \vec{j}, \vec{l}\rangle$ with operator $\hat{H}_v$, we obtain linear combination of states $|\gamma_{n\pm 1}, \vec{j}', \vec{l}'\rangle$ and $|\gamma_n, \vec{j}, \vec{l}\rangle$ (see details below). Therefore, Hilbert space $\mathcal{H}_v$ is preserved by the action of operator $\hat{H}_v$. Hence we do not need to introduce other states.

Because all the vertices of $\gamma_n$ are at most 3-valent the spin-network $|\gamma_n, \vec{j}, \vec{l}\rangle$ is determined by the spins up to a phase factor. One can fix the intertwiners to be the 3-$j$ symbols $\begin{pmatrix} j_n & j_{n+1} & l_n \\ m_n & m_{n+1} & \mu_n \end{pmatrix}$ at 3-valent vertices and the 2-$j$ symbol $\epsilon_{mn}^j := (-1)^{j+m} \delta_{m,-n}$ at the 2-valent vertex. The existence condition is that $\vec{j}$ satisfies $|j_m - l_m| \leq j_{m+1} \leq j_m + l_m$, $m = 1, 2, \ldots, n$.

The next step in the construction is application of the rigging map $\eta$ (13). For every $n = 0, 1, \ldots$ the corresponding graph $\gamma_n$ has a symmetry, $f_{\gamma_n} \in \text{Diff}_v$, such that

$$f_{\gamma_n}(v_{Lk}) = v_{Rk}, \quad f_{\gamma_n}(v_{Rk}) = v_{Lk}. \quad (30)$$

Because of the symmetry of the graph $\gamma_n$, every $\Psi_{\gamma_n} \in \mathcal{H}_\gamma$ may have symmetric part $\psi_{\gamma_n}^+$ and antisymmetric part $\psi_{\gamma_n}^-$ with respect to the transformation (30), i.e. $\psi_{\gamma_n}^\pm = \pm \psi_{\gamma_n}^\mp$, while only the symmetric part contributes to $\eta(\Psi_{\gamma_n})$. We will show now, that each state $|\gamma_n, \vec{j}, \vec{l}\rangle$ defined above is invariant with respect to the symmetry (30). Consider the function $\psi_{\gamma_n}$ (10) corresponding to the state $|\gamma_n, \vec{j}, \vec{l}\rangle$,

$$\psi_{\gamma_n}(g_{L1}, \ldots, g_{Ln+1}, g_{R1}, \ldots, g_{Rn+1}, h_1, \ldots, h_n) = \langle \tilde{g}_L \tilde{g}_R \tilde{h} | \gamma_n, \vec{j}, \vec{l} \rangle.$$
Then, the cylindrical function \( \psi' \) states

\[
\langle \bar{g}_L \bar{g}_R \bar{h} | \gamma_n, \bar{j}, \bar{l} \rangle = N_n
\]

which is defined graphically as

Then, the cylindrical function \( \psi'_n \) corresponding to the flipped state \( U_{F_n} | \bar{j}, \bar{l} \rangle \) is

\[
\psi'_n(g_{L1}, \ldots, g_{Ln+1}; g_{R1}, \ldots, g_{Rn+1}, h_1, \ldots, h_n) = \psi_n(g_{R1}, \ldots, g_{Rn+1}; g_{L1}, \ldots, g_{Ln+1}, (h_1)^{-1}, \ldots, (h_n)^{-1}).
\]

We show below that indeed,

\[
\psi_n(g_{R1}, \ldots, g_{Rn+1}, g_{L1}, \ldots, g_{Ln+1}, (h_1)^{-1}, \ldots, (h_n)^{-1}) = \psi_n(g_{L1}, \ldots, g_{Ln+1}, g_{R1}, \ldots, g_{Rn+1}, h_1, \ldots, h_n).
\]

Using the following properties,

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}, \text{ and, } \epsilon_{mn} = (-1)^{2j} \epsilon_{mn},
\]

we can get

\[
\langle \bar{g}_L \bar{g}_R \bar{h} | \bar{U}_{F_n} | \gamma_n, \bar{j}, \bar{l} \rangle = (-1)^{2j_1+2l_1+2l_2+\ldots+2l_n+2j_{n+1}} \langle \bar{g}_L \bar{g}_R \bar{h} | \gamma_n, \bar{j}, \bar{l} \rangle.
\]

Because \( j_m + l_m + j_{m+1} \) is integer, we have

\[
(-1)^{2j_1+2l_1+2l_2+\ldots+2l_n+2j_{n+1}} = (-1)^{2j_2+2l_2+\ldots+2l_n+2j_{n+1}} = \ldots = (-1)^{4j_{n+1}} = 1.
\]

Therefore

\[
\langle \bar{g}_L \bar{g}_R \bar{h} | \bar{U}_{F_n} | \gamma_n, \bar{j}, \bar{l} \rangle = \langle \bar{g}_L \bar{g}_R \bar{h} | \gamma_n, \bar{j}, \bar{l} \rangle.
\]

This ensures that \( \langle \bar{g}_L \bar{g}_R \bar{h} | \gamma_n, \bar{j}, \bar{l} \rangle \) will not be annihilated by the rigging map \( \eta \). We denote

\[
\left( [\gamma_n], \bar{j}, \bar{l} \right) := \eta(\gamma_n, \bar{j}, \bar{l}).
\]

We further restrict the set of states that will define the subspace \( \mathcal{H}_v \), by adjusting the spins \( l_1, \ldots, l_n \) to the spins \( j_1, \ldots, j_n \) in the way corresponding to the action of the operator \( ^{\text{kin}} \hat{H}^E \) of (18). Hence, \( l_n \) becomes a function of \( j_n \) as that in (21). It is simplified in our case as

\[
l_n = l(j_n) = \begin{cases} 1 & , j_n = 1/2, \\ 1/2 & , j_n \neq 1/2. \end{cases}
\]

Then \( \left( [\gamma_n], \bar{j}, \bar{l} \right) \) can be abbreviated to \( \left( [\gamma_n], \bar{j} \right) \). Finally, the Hilbert space \( \mathcal{H}_v \) is defined as

\[
\mathcal{H}_v = \text{Span}\{ \left( [\gamma_n], \bar{j} \right) | \text{with } |j_m - l_m(j_m)| \leq j_{m+1} \leq j_m + l_m(j_m), \forall m \leq n \}.
\]

The natural domain \( \mathcal{F} \subset \mathcal{H}_v \) for our operators will be the space of the finite linear combinations of the states \( \left( [\gamma_n], \bar{j} \right) \),

\[
\mathcal{F} := \text{Span}\{ \left( [\gamma_n], \bar{j} \right) | \text{with } |j_m - l_m(j_m)| \leq j_{m+1} \leq j_m + l_m(j_m), \forall m \leq n \}.
\]
3.2 The action of the operator $\hat{H}_v$ on $\mathcal{H}_v$

We calculate now the action of the operator $\hat{H}_v$ defined in the section 2.4 on the space $\mathcal{H}_v$. Following the framework, we start with the operators $\hat{H}_v^{\text{kin}}$, $\hat{H}_v^{\text{L}}$, and $\hat{H}_v^{\text{E}}$. For every graph $\gamma_n$ of (26) the vertex $v$ is the only non-degenerate vertex such that $\hat{H}_v^{\text{kin}} \neq 0$, and there is only one pair of edges meeting at $v$ and featuring in (16). They are

$$(e, e') = (e_L, e_R)$$

connecting the vertex $v$ with the vertex $v_{Ln}$ and with the vertex $v_{Rn}$, respectively. Therefore

$$\hat{H}_v^{\text{kin}} = \hat{H}_v^{\text{kin}, e_L e_R} = (1 + \beta^2)\hat{H}_v^{\text{kin}, e_L e_R} + \hat{H}_v^{\text{kin}, e_L e_R}.$$

(39)

For the Lorentz part, which is shown to be diagonalized under the basis, we have

$$\hat{H}_v^{\text{L}}|\gamma_n, \vec{j}, \vec{l}\rangle = \frac{\pi}{\alpha} \sqrt{j_{n+1}} |\gamma_n, \vec{j}, \vec{l}\rangle.$$

(40)

That formula passes to the dual states, elements of the Hilbert space $\mathcal{H}_v$, simply as

$$\hat{H}_v^{\text{L}}(\gamma_n, \vec{j}) = (\gamma_n, \vec{j})^{\text{kin}} \hat{H}_v^{\text{L}} = \frac{\pi}{\alpha} \sqrt{j_{n+1}} (\gamma_n, \vec{j}, \vec{l}).$$

(41)

The Euclidean part (18) is more complicated. The straightforward calculation in Appendix B shows that

$$\hat{H}_v^{\text{E}}|\gamma_n, \vec{j}, \vec{l}\rangle = \frac{-3}{l_{n+1}(l_{n+1} + 1)(2l_{n+1} + 1)} \kappa_1 \sum_{j_{n+2}} \frac{\sqrt{2j_{n+2} + 1}}{\sqrt{2j_{n+1} + 1}} \left( \vec{j}_{n+1} \cdot \vec{l}_{n+1} \right) |\gamma_{n+1}, (\vec{j}, j_{n+2}, \vec{l}, l_{n+1})\rangle,$$

(42)

where we have denoted (see (12))

$$\vec{j}_{n+1} = (J_{v_{Ln}, e_L}^1, J_{v_{Ln}, e_L}^2, J_{v_{Ln}, e_L}^3),$$

(43)

$$\vec{l}_{n+1} = (J_{v_{Ln}, e_{Ln+1}}^1, J_{v_{Ln}, e_{Ln+1}}^2, J_{v_{Ln}, e_{Ln+1}}^3),$$

(44)

and given $\vec{j} = (j_1, \ldots, j_{n+1})$ the symbol $(\vec{j}, j_{n+2})$ standing for $(j_1, \ldots, j_{n+1}, j_{n+2})$. Because of (36), we can explicitly show out the factor

$$\frac{-3}{l_{n+1}(l_{n+1} + 1)(2l_{n+1} + 1)} = \begin{cases} -2, & j_{n+1} / 2, \\ 1 / 2, & j_{n+1} = 1 / 2 =: \omega(j_{n+1}). \end{cases}$$

By defining the following function

$$\eta(x) := \begin{cases} 1, & x < 1, \\ \frac{1}{x}, & x \geq 1. \end{cases}$$

(45)

it follows from Eq. (36) immediately that

$$j_{n+1} = \eta(j_n) \pm \frac{1}{2}.$$

(46)

In the following, we frequently use $\pm$ instead of $\eta(j_n) \pm 1/2$ in the following. For example, we may write $(j_1, j_2, \ldots, j_n, -)$ rather than $(j_1, j_2, \ldots, j_n, \eta(j_n) - 1/2)$, and $(j_1, -, +, \ldots, -)$ can also be used instead of $(j_1, \eta(j_1) - 1/2, \eta(j_1) - 1/2 + 1/2, \ldots, -)$. Then (42) can be rewritten as

$$\hat{H}_v^{\text{E}}|\gamma_n, \vec{j}\rangle = \kappa_1 \omega(j_{n+1}) \left( \frac{\eta(j_{n+1}) \sqrt{\eta(j_{n+1}) + 1}}{2 \sqrt{2\eta(j_{n+1}) + 1}} |\gamma_{n+1}, (\vec{j}, +)\rangle - \frac{[\eta(j_{n+1}) + 1] \sqrt{\eta(j_{n+1})}}{2 \sqrt{2\eta(j_{n+1}) + 1}} |\gamma_{n+1}, (\vec{j}, -)\rangle \right),$$

(47)
By the definition, we have
\[ \hat{H}_v^E \cdot \left( [\gamma_n], \vec{j} \right) = \left( [\gamma_n], \vec{j} \right) \text{kin} \hat{H}_v^E = \kappa_1 \frac{\sqrt{\eta(j_n)(\eta(j_n) + 1)}}{2\sqrt{2\eta(j_n) + 1}} \Theta(j_{n+1}, j_n) \left( [\gamma_{n-1}], (j_1, j_2, \cdots, j_n) \right), \] (48)
where
\[ \Theta(j_{n+1}, j_n) = \begin{cases} \omega(j_n) \sqrt{\eta(j_n)} & \text{if } j_{n+1} = \eta(j_n) + 1/2, \\ -\omega(j_n) \sqrt{\eta(j_n)} + 1 & \text{if } j_{n+1} = \eta(j_n) - 1/2. \end{cases} \] (49)

Hence, in our subspace \( \mathcal{H}_v \), the operator \( \hat{H}_v^E \) just annihilates the edges \( \ell_n \) of the graphs \( \gamma_n \). On the other hand, the adjoint operator \( (\hat{H}_v^E)^\dagger \) acts by creating new edges \( \ell \) by a formula very similar to that of \( \text{kin} \hat{H}_v^E \), namely
\[ (\hat{H}_v^E)^\dagger \cdot \left( [\gamma_n], \vec{j} \right) = \kappa_1 \omega(j_{n+1}) \frac{\eta(j_{n+1}) \sqrt{\eta(j_{n+1}) + 1}}{2\sqrt{2\eta(j_{n+1}) + 1}} \left( [\gamma_{n+1}], (\vec{j}, +) \right) - \kappa_1 \omega(j_{n+1}) \frac{\eta(j_{n+1}) + 1 \sqrt{\eta(j_{n+1})}}{2\sqrt{2\eta(j_{n+1}) + 1}} \left( [\gamma_{n+1}], (\vec{j}, -) \right). \] (50)

Finally, we obtain a formula for the symmetric part, and the action of the operator \( \hat{H}_v^E \) in \( \mathcal{H}_v \),
\[ \left( [\gamma_n], \vec{j} \right) \hat{H}_v^E = \frac{1}{2} (\hat{H}_v^E + (\hat{H}_v^E)^\dagger) \cdot \left( [\gamma_n], \vec{j} \right) = \kappa_1 \sum_{j_{n+2} = \eta(j_{n+1}) \pm 1/2} \zeta(j_{n+1}) \Theta(j_{n+2}, j_{n+1}) \left( [\gamma_{n+1}], (\vec{j}, j_{n+2}) \right) + \zeta(j_n) \Theta(j_{n+1}, j_n) \left( [\gamma_{n-1}], (j_1, j_2, \cdots, j_n) \right), \] (51)
where
\[ \zeta(j) := \frac{\sqrt{\eta(j)(\eta(j) + 1)}}{2\sqrt{2\eta(j) + 1}}. \] (52)

In summary, we have given the action of the operator \( H_v \) defined in the domain \( \mathcal{F} \) of the subspace \( \mathcal{H}_v \) of the vertex Hilbert space \( \mathcal{H}_{\text{vtx}} \). It can be repressed as
\[ \hat{H}_v = (1 + \beta^2) \hat{H}_v^L + \hat{H}_v^E, \] (53)
where the terms of the right hand side are defined in (41) and (51). There are present several arbitrary constant factors. The first term involving \( \hat{H}_v^L \) is proportional to a positive constant factor \( \frac{1 + \beta^2}{\alpha} \), while the second one to a positive constant factor \( \kappa_1 \). The factors represent ambiguity of the quantization [38]. For the analysis of the problem of self-adjointness of the operator we can fix one of those factors arbitrarily. Hence we set
\[ \kappa_1 = 2. \]

4 Self-adjointness of the operators

In this section we will prove the following result:

**Theorem 4.1.** On the Hilbert space \( \mathcal{H}_v \), the operator \( \hat{H}_v \) defined by (53) in Sec. 3.2 in the domain \( \mathcal{F} \) is essentially self-adjoint.

First, we sketch the proof. Consider the following operator \( \hat{Z} \) defined in \( \mathcal{F} \),
\[ \left( [\gamma_n], \vec{j} \right) \hat{Z} := \zeta(j_{n+1}) \left( [\gamma_n], \vec{j} \right), \] (54)
and introduce the operator
\[ \hat{N} := \hat{Z}^2. \]

By definition, \( \hat{N} \), the closure of \( \hat{N} \), is self-adjoint. \( \mathcal{F} \) is a core of \( D(\hat{N}) \).

The key part of the proof is the following Lemma:

**Lemma 4.0.1.** There exist \( c,d \in \mathbb{R}^+ \) such that for every \( \psi \in \mathcal{F} \) the following two inequalities are true:

\[
\left\| \left( \psi \right| \hat{H}_v \left| \psi \right\|^2 \leq c \left\| \left( \psi \right| \hat{N} \left| \psi \right\|^2, \right.
\]

\[
\left| \left( \psi \left| \left[ \hat{H}_v, \hat{N} \right] \left| \psi \right| \right\right| \right| \leq d \left\| \left( \psi \right| \hat{N}^{1/2} \left| \psi \right\| \right|^2. \right.
\]

(55)

It turns out (see Appendix C) that Theorem 4.1 follows directly from Lemma 4.0.1.

Note that in the calculations proving Lemma 4.0.1 it is convenient to express the Euclidean part \( \hat{H}_v^E \) of the operator \( \hat{H}_v \) by the following ”creation” and ”annihilation” operators \( \hat{a}^\dagger \) and \( \hat{a} \), respectively:

\[
\left( \left[ \gamma_n \right], \vec{j} \right| \hat{a} := \Theta(j_{n+1}, j_n) \left( \left[ \gamma_{n-1} \right], (j_1, \cdots, j_n) \right) \right.
\]

(56)

\[
\left( \left[ \gamma_n \right], \vec{j} \right| \hat{a}^\dagger := \sum_{j_{n+2} = \eta(j_{n+1} + 1/2)} \Theta(j_{n+2}, j_{n+1}) \left( \left[ \gamma_{n+1} \right], (\vec{j}, j_{n+2}) \right). \right.
\]

(57)

Of course \( \hat{a}^\dagger \) is adjoint to \( \hat{a} \) and restricted to \( \mathcal{F} \). In terms of the operators \( \hat{Z}, \hat{a} \) and \( \hat{a}^\dagger \), we have

\[ \hat{H}_v^E = \hat{a} \hat{Z} + \hat{Z} \hat{a}^\dagger. \]

(58)

Now we come back to the proof of the lemma 4.0.1. Given \( \left( \psi \right| = \sum_{n, \vec{j}} \beta \left( \left[ \gamma_n \right], \vec{j} \right) \right| \), we have

\[
\left( \psi \right| \hat{H}_v = \sum_{m, \vec{i}} \left( \left[ \gamma_m \right], \vec{i} \right| \sum_{n, \vec{j}} \beta \left( \left[ \gamma_n \right], \vec{j} \right) \left| \hat{H}_v \right| \left[ \gamma_m \right], \vec{i} \right) \right.
\]

which gives us

\[
\left\| \left( \psi \right| \hat{H}_v \left| \psi \right\|^2 = \sum_{m, \vec{i}} \left| \sum_{n, \vec{j}} \left( \left[ \gamma_n \right], \vec{j} \right| \left( \hat{H}_v \right| \left[ \gamma_m \right], \vec{i} \right) \beta \left( \left[ \gamma_n \right], \vec{j} \right) \right\|^2 \right.
\]

\[
\leq 4 \sum_{m, \vec{i}} \left| \sum_{n, \vec{j}} \left( \left[ \gamma_n \right], \vec{j} \right| \hat{H}_v \left| \left[ \gamma_m \right], \vec{i} \right) \right|^2 \left| \beta \left( \left[ \gamma_n \right], \vec{j} \right) \right|^2 \right.
\]

\[
= 4 \sum_{n, \vec{j}} \left( \left| \sum_{m, \vec{i}} \left( \left[ \gamma_n \right], \vec{j} \right| \hat{H}_v \left| \left[ \gamma_m \right], \vec{i} \right) \right|^2 \right) \left| \beta \left( \left[ \gamma_n \right], \vec{j} \right) \right|^2 ; \right.
\]

(59)

where the factor of 4 is due to the fact that there are only 4 non-vanishing entries in each row of the matrix of \( \hat{H}_v \). By 41 and 51, we get

\[
\sum_{m, \vec{i}} \left| \left( \left[ \gamma_n \right], \vec{j} \right| \hat{H}_v \left| \left[ \gamma_m \right], \vec{i} \right) \right|^2 \right.
\]

\[=(1 + \beta^2) \frac{\pi}{\alpha} j_{n+1} (j_{n+1} + 1) + \sum_{j_{n+2} = \eta(j_{n+1}) + 1} \left| \zeta(j_{n+1}) \Theta(j_{n+2}, j_{n+1}) \right|^2 + \left| \zeta(j_n) \Theta(j_{n+1}, j_n) \right|^2 \]

13
When \( j_{n+1} \to \infty \), the right hand side, as well as \( \zeta(j_{n+1})^4 \), increases as \( \sim j_{n+1}^2 \). Hence there must exist a number \( c \in \mathbb{R}^+ \) such that
\[
\sum_{m,i} \left| \left( [\gamma_n, \vec{j}] \hat{H}_v |\gamma_m, \vec{i} \right) \right|^2 \leq c \zeta(j_{n+1})^4,
\]
for all \( j_{n+1} \geq \frac{1}{2} \). Therefore we have
\[
||\hat{H}_v \psi||^2 \leq 4c \sum_{n,j} \zeta(j_{n+1})^4 |\beta_{n,j}|^2 = 4c \sum_{n,j} |\zeta(j_{n+1})^2 \beta_{n,j}|^2 = 4c ||N \psi||^2.
\]
(60)

For the second equation in (55), we define
\[
\hat{C} := i \frac{1}{\hat{Z}} [\hat{H}_v, \hat{N}] \frac{1}{\hat{Z}} = i(\frac{1}{\hat{Z}} \hat{a} \hat{Z}^2 - \hat{Z} \hat{a} + \hat{a}^\dagger \hat{Z} - \hat{Z}^2 \hat{a}^\dagger \frac{1}{\hat{Z}}).
\]

Playing the same game as we did for \( \hat{H}_v \) in (59), we obtain
\[
||\left( \psi \right| \hat{C} \left| \psi \right||^2 \leq 3 \sum_{n,j} \left| \sum_{m,i} \left( [\gamma_n, \vec{j}] \hat{C} |\gamma_m, \vec{i} \right) \right|^2 |\beta_{n,j}|^2,
\]

Because of
\[
\left( [\gamma_n, \vec{j}] \hat{C} = i \sum_{j_{n+2}=\eta(j_{n+1}) \pm 1/2} \Theta(j_{n+2} \pm 1/2) \left( \frac{\zeta(j_{n+1})^2}{\zeta(j_{n+2})} - \zeta(j_{n+2}) \right) \left( [\gamma_{n+1}, \vec{j}, j_{n+2}] \right) + i \Theta(j_{n+1}, j_n) \left( \zeta(j_{n+1}) - \zeta(j_{n+1}) \right) \left( [\gamma_{n-1}, \vec{j}, \cdots, \vec{j}] \right),
\]
we have
\[
\sum_{m,i} \left| \left( [\gamma_n, \vec{j}] \hat{C} |\gamma_m, \vec{i} \right) \right|^2 = \sum_{j_{n+2}=\pm} \left( \frac{\zeta(j_{n+1})^2}{\zeta(j_{n+2})} - \zeta(j_{n+2}) \right)^2 \Theta(j_{n+2})^2 + \left( \zeta(j_{n+1}) - \zeta(j_{n+1}) \right)^2 \Theta(j_{n+1})^2.
\]

It is easy to check that the function
\[
\left( \frac{\zeta(j)^2}{\zeta(\eta(j) \pm 1/2)} - \zeta(\eta(j) \pm 1/2) \right)^2 \Theta(\eta(j) \pm 1/2, j)
\]
is bounded for \( j \geq 0 \). Hence there exists a \( d > 0 \) such that
\[
||\left( \psi \right| \hat{C} \left| \psi \right||^2 \leq d ||\psi||^2,
\]
which means that \( \hat{C} \) is bounded. Because \( \left( \psi \right| \hat{Z} \in \mathcal{F} \) is well defined for all \( \left( \psi \right| \in \mathcal{F} \), we obtain
\[
\left| \left( \psi \right|[\hat{H}_v, \hat{N}] |\psi \right) = \left| \left( \psi \right| \hat{Z} \hat{C} \hat{Z} |\psi \right) \leq ||\hat{C}|| \left| \left| \left( \psi \right| \hat{Z} \right| \right|^2 = ||\hat{C}|| \left| \left( \psi \right| \hat{N}^{1/2} \right| \right|^2, \forall \left( \psi \right| \in \mathcal{F}.
\]
(61)

This completes the proof of the lemma 4.0.1. In conclusion, according to Theorem C.1 in Appendix C and the Lemma 4.0.1, the operator \( H_v \) defined in the domain \( \mathcal{F} \) is essentially self-adjoint on \( \mathcal{H}_v \). Because the above proof is valid also for the case when \( (1 + \beta^2) = 0 \), the Euclidean part \( \hat{H}^E_v \) of \( \hat{H}_v \) is also essentially self-adjoint by itself.
5 Eigenvalue problem

Let $|\gamma_n,j\rangle$ be an eigenstate of $\hat{H}_v$ with the eigenvalue $\lambda$. By definition we have

$$\sum_{j_{n+2}=\eta(j_{n+1})\pm 1/2} \zeta(j_{n+1})\Theta(j_{n+2}, j_{n+1})\psi_{n+1,\{j,j_{n+2}\}} + \zeta(j_n)\Theta(j_{n+1}, j_n)\psi_{j_{n-1},\{j_1,\ldots, j_n\}}$$ \hspace{1cm} (62)

$$+ \frac{\pi(1 + \beta^2)}{\alpha} \sqrt{j_{n+1}(j_{n+1} + 1)}\psi_{n,j} = \lambda\psi_{n,j}.$$ 

To understand the recurrence equations for all coefficients $\psi_{n,j}$, we introduce a triangle array of the coefficients of $|\psi\rangle$ as follows. In the array, the rows are conventionally enumerated starting with $n = 0$ at the top. There are $2^n$ entries in the nth row. The entries in the nth row are the coefficients with index $n$ (i.e. $|\psi|\{\gamma_n,j\}_n\rangle$ for various $j$). The coefficients $\left( |\psi|\{\gamma_n+1,j,j_{n+1}\}_n\rangle \right)$ are listed below to the left and right of $\left( |\psi|\{\gamma_n,j\}_n\rangle \right)$, i.e.,

$$\psi_{n+1,\{j,\ldots,j\}_n} \psi_{n,\{j\}_n} \psi_{n-1,\{j_1,\ldots,j_{n}\}_n}$$

We call the array as coefficient triangle. In a given recurrence equation (62), involved block looks like\(^2\)

$$\begin{array}{cc}
\psi_{n-1,\{j_1,\ldots,j_{n}\}_n} & \psi_{n+1,\{j,\ldots,j\}_n} \\
\psi_{n,\{j\}_n} & \psi_{n+1,\{j\}_n} \\
\psi_{n+1,\{j\}_n} & \psi_{n+1,\{j\}_n} \\
\end{array}$$

The two coefficients $\psi_{n-1,\{j_1,j_2,\ldots,j_n\}_n}$ and $\psi_{n+1,\{j\}_n}$ on the top are either fixed, or derived by previous recurrence equations. Thus, for instance, in order to determine $\psi_{n+1,\{j\}_n}$ by (62), the other coefficient $\psi_{n+1,\{j\}_n}$ in the same row must be fixed by hand. It follows immediately that $1 + \sum_{n=0}^{\infty} 2^n$ initial data should be fixed to solve all the recurrence equations in (62). One choice of the initial data, for instance, is to fix $\psi_{0,j_1}$ and $\psi_{0,\{j_1,\ldots,j_{n}\}_n}$. The degrees of degeneracy of eigenstates for a given eigenvalue is therefore $\sum_{n=0}^{\infty} 2^n$. We leave the resolution of this complicated eigenvalue problem for further study.

6 Summary and discussion

The general issue addressed in our paper is to understand the analytic properties of Hamiltonian operators of LQG that follow from attaching and removing loops. Particularly, the interesting case for us is when the action of the Hamiltonian operator produced by the creation and annihilation of the loops at a single pair of edges intersecting at a vertex $v$. Therefore, we constructed a smallest subspace $\mathcal{H}_v$ of the vertex Hilbert space $\mathcal{H}_{vtx}$, which has the following desired properties: (i) being preserved by the quantum Hamiltonian operator, and (ii) containing a spin-network state defined by the graph $\gamma_0$ depicted at the beginning of Sec.\[^3\] colored by a non-trivial representation $j_1$. The subspace is still infinite dimensional, and for arbitrary integer $n$ it contains a spin-network state (29). It properly captures the properties of the Hamiltonian operator we wanted to know. We have restricted our study to this subspace $\mathcal{H}_v$. Therein, we considered the operator $\hat{H}_v$ defined by (24), which was employed in the construction of the physical Hamiltonian operator (15). The action of $\hat{H}_v$ in $\mathcal{H}_v$ is analysed in detail, and the explicit formulae for its matrix elements in a suitable normalized basis $|\gamma_n,j\rangle$ are derived. It turns out that the operator $\hat{H}_v$ possess the following properties, which are relevant for further analysis,

\[^2\]By definition, $\psi_{n,j}$ in the block should be located below to the left or right depending on the sign of $j_{n+1}$.
• each state \( (\gamma_n, j) \) is mapped by \( H_v \) into a linear combination of at most 4 elements of that basis;
• the coefficients depend on \( j_{n+1} \) only, and are of the order of \( j_{n+1} \) in the limit \( j_{n+1} \to \infty \).

These properties are crucially used in the proof of the lemma 4.0.1 which ensure the self-adjointness of \( \hat{H}_v \) on \( \mathcal{H}_v \). Since \( \hat{H}_v \) is self-adjoint, the physical Hamiltonian operator \( \hat{H}_{\text{phy}} := \sqrt{\hat{H}_v} \) can be well defined in \( \mathcal{H}_v \) by restricting to the non-negative part of the spectrum of \( \hat{H}_v \). Moreover, our analysis gives insight into the eigenvector problem of \( \hat{H}_v \). However we have not found a normalizable solution.

It is desirable to further generalize the above result of \( \hat{H}_v \) to the vertex Hilbert space \( \mathcal{H}_{\text{vtx}} \). If the matrix elements of \( \hat{H}_v \) increased linearly with the spins like the second property above, the Theorem 4.1 could still be employed for the generalization. However, a tentative calculation shows that it is not the case for vertices of arbitrary valency. For instance, some quadratic terms of spins will appear in the case of 3-valent vertices. Thus, the generalization of our result to \( \mathcal{H}_{\text{vtx}} \) is not straightforward. This issue has to be left for further study.

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A Graphical Calculation method

In the appendix, we give some notations about the graphical method. For detail, we refer to [38, 47–49] and references therein. The 2-j symbol \( \epsilon^j_{mn} \) and the 3-j symbol are represented as

\[
\epsilon^{(j)}_{nm} = (-1)^{j+n} \delta(m, n) \quad m \rightarrow n
\]  
(63)

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  m_1 & m_2 & m_3
\end{pmatrix} = \begin{pmatrix}
  j_1 \\
  j_2 \\
  j_3
\end{pmatrix}
\]  
(64)

For the Wigner D matrix, we define

\[
m \rightarrow j \quad g \quad n =: D^j(g)^m_n
\]  
(65)

Define \( J_{\pm} = \mp \frac{1}{\sqrt{2}} (J_x \pm i J_y) \) and \( J_0 := J_z \). Let \( |jm\rangle \) be the usual basis of the \( j \)-irreducible representation space of \( SU(2) \). Graphically we have

\[
\langle jm | J_\mu | jn \rangle = -W_j \begin{pmatrix}
  j & 1 & j \\
  n & \mu & -m
\end{pmatrix} = -W_j
\]  
(66)
Let \( \epsilon_{\mu \nu \sigma} \) be the totally antisymmetric matrix with \( \epsilon_{-1,0,1} = 1 \), then

\[
\epsilon_{\mu \nu \sigma} = \sqrt{6} \begin{pmatrix} 1 & 1 & 1 \\ \mu & \nu & \sigma \end{pmatrix} = \sqrt{6}
\]  

(67)

By the formula

\[
D^{j_1} (g)_{m_1}^{n_1} D^{j_2} (g)_{m_2}^{n_2} = \sum_{J = |j_1 - j_2|}^{j_1 + j_2} d_J (-1)^{M-N} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} j_1 & j_2 & J \\ n_1 & n_2 & -N \end{pmatrix} D^J (g)_{M}^{N} 
\]

we have

(68)

For the operator \( J_{v,e} \) in (12), if \( t(e) = v \), we have

\[
J^{v,e}_{\mu} D^j_{mn} (g) = J^{(R)}_{\mu} D^j_{mn} (g) = \sum_{k=-j}^{j} \langle jm|\mu|jk \rangle D^j_{kn} (g) = W_j 
\]

(70)

If \( s(e) = v \), we get

\[
J^{v,e}_{\mu} D^j_{mn} (g) = J^{(L)}_{\mu} D^j_{mn} (g) = D^j_{mk} (g) \langle jk|\mu|jn \rangle = -W_j 
\]

(71)
B  Detail calculation of $\text{kin} H^E_v |n, \vec{j}, \vec{l}\rangle$

We show the detail calculation of $\text{kin} H^E_v |n, \vec{j}, \vec{l}\rangle$ by the graphical method introduced above. It reads

\[
\text{kin} H^E_v |n, \vec{j}, \vec{l}\rangle = -\kappa_1 (-1)^{2j_{n+1}+3} \sqrt{6} W_{j_{n+1}} \frac{W_{j_{n+2}}^2}{W_{l_{n+1}}}
\]

By definition, we have

\[
\left( \begin{array}{ccc} 
\hat{j}_{n+1} & \hat{j}_{n+1} & 1 \\
1 & 1 & \hat{j}_{n+1} \\
1 & 1 & \hat{j}_{n+1} \\
\end{array} \right) = (-1)^{2j_{n+1}} \frac{1}{\sqrt{6} j_{n+1} (j_{n+1} + 1)(2j_{n+1} + 1)} = \frac{(-1)^{2j_{n+1}}}{\sqrt{6} W_{j_{n+1}}}.
\]

\[
\left( \begin{array}{ccc} 
\hat{j}_{n+1} & \hat{l}_{n+1} & \hat{j}_{n+2} \\
l_{n+1} & l_{n+1} & \hat{j}_{n+2} \\
\hat{j}_{n+1} & \hat{j}_{n+2} & \hat{j}_{n+2} \\
\end{array} \right) = (-1)^{j_{n+1} + l_{n+1} + j_{n+2}} \left( \begin{array}{ccc} 
\hat{l}_{n+1} & \hat{l}_{n+1} & 1 \\
\hat{j}_{n+1} & \hat{j}_{n+1} & \hat{j}_{n+2} \\
\hat{j}_{n+1} & \hat{j}_{n+2} & \hat{j}_{n+2} \\
\end{array} \right)
\]

\[
= - \frac{[j_{n+1}(j_{n+1} + 1) + l_{n+1}(l_{n+1} + 1) - j_{n+2}(j_{n+2} + 1)]}{2 W_{j_{n+1}} W_{l_{n+1}}}
\]

\[
= \frac{\hat{j}_{n+1} \cdot \hat{l}_{n+1}}{W_{j_{n+1}} W_{l_{n+1}}}.
\]
Finally, because $j_{n+1} + l_{n+1} + j_{n+2} \in \mathbb{N}$, we get

$$
\kappa \hat{H}_{v}^{E} = \sum_{j_{n+2}} -3\kappa_{1} \frac{d_{j_{n+2}}}{W_{n+1}^{2}} \left( \vec{J}_{n+1} \cdot \vec{L}_{n+1} \right),
$$

(73)

Because of

$$
\langle \gamma_{n+1}, \vec{j}, \vec{l} \rangle = \frac{1}{(d_{j_{1}} d_{j_{2}} \cdots d_{j_{n}})^{2} d_{j_{n+1}} d_{l_{1}} \cdots d_{l_{n}}},
$$

(74)

we get

$$
\kappa \hat{H}_{v}^{E}|_{\gamma_{n}, \vec{j}, \vec{l}} = \frac{-3\kappa_{1}}{l_{n+1} (l_{n+1} + 1)(2l_{n+1} + 1)} \sum_{j_{n+2}} \frac{\sqrt{2j_{n+2} + 1}}{\sqrt{(2j_{n+1} + 1)(2l_{n+1} + 1)}} \left( \vec{J}_{n+1} \cdot \vec{L}_{n+1} \right) |_{\gamma_{n+1}, (\vec{j}, j_{n+2}), (\vec{l}, l_{n+1})}. \quad (75)
$$

C The underlying theorem

The underlying theorem can be found in [50, 51] for more details.

**Theorem C.1.** Let $\hat{N}$ be a self-adjoint operator with $\hat{N} \geq 1$. Let $\hat{A}$ be a symmetric operator with domain $D$ which is a core for $\hat{N}$. Suppose that:

(i) For some $c$ and all $\psi \in D$, one has

$$
||\hat{A}\psi|| \leq c||\hat{N}\psi||. \quad (76)
$$

(ii) For some $d$ and all $\psi \in D$, one has

$$
|(\hat{A}\psi, \hat{N}\psi) - (\hat{N}\psi, \hat{A}\psi)| \leq d||\hat{N}^{1/2}\psi||^{2}. \quad (77)
$$

Then $\hat{A}$ is essential self-adjoint on $D$ and its closure is essentially self-adjoint on any core for $\hat{N}$.

In our case, $\hat{H}_{v}$ is the operator $\hat{A}$ in the theorem and $D = \mathcal{F}$ (38). By definition of $\hat{N}$ in the present work, $[\hat{H}_{v}, \hat{N}]$ is well defined on $\mathcal{F}$. Then (77) can be rewritten as

$$
|(\psi, [\hat{H}_{v}, \hat{N}]\psi)| \leq d||\hat{N}^{1/2}\psi||^{2}. \quad (78)
$$

References

[1] A. Ashtekar and J. Lewandowski. Background independent quantum gravity: a status report. Classical and Quantum Gravity, 21(15):R53, 2004.

[2] M. Han, Y. Ma, and W. Huang. Fundamental structure of loop quantum gravity. International Journal of Modern Physics D, 16(09):1397–1474, 2007.

[3] C. Rovelli. *quantum gravity*. Cambridge University Press, 2005.

[4] T. Thiemann. *Modern canonical quantum general relativity*. Cambridge University Press, 2007.
[5] S. Holst. Barbero’s hamiltonian derived from a generalized hilbert-palatini action. *Physical Review D*, 53(10):5966, 1996.

[6] A. Ashtekar and R. S. Tate. *Lectures on non-perturbative canonical gravity*, volume 6. World Scientific, 1991.

[7] J. F. Barbero G. Real ashtekar variables for lorentzian signature space-times. *Phys. Rev. D*, 51:5507–5510, May 1995. doi: 10.1103/PhysRevD.51.5507. URL [https://link.aps.org/doi/10.1103/PhysRevD.51.5507](https://link.aps.org/doi/10.1103/PhysRevD.51.5507)

[8] C. Rovelli and L. Smolin. Knot theory and quantum gravity. *Physical Review Letters*, 61(10):1155, 1988.

[9] A. Ashtekar and J. Lewandowski. Representation theory of analytic holonomy c*-algebras. In *Knots and Quantum Gravity*, page 21, 1994.

[10] J. Lewandowski. Topological measure and graph-differential geometry on the quotient space of connections. *International Journal of Modern Physics D*, 3(01):207–210, 1994.

[11] A. Ashtekar and J. Lewandowski. Differential geometry on the space of connections via graphs and projective limits. *Journal of Geometry and Physics*, 17(3):191 – 230, 1995. ISSN 0393-0440. doi: [https://doi.org/10.1016/0393-0440(95)00028-G](https://doi.org/10.1016/0393-0440(95)00028-G). URL [http://www.sciencedirect.com/science/article/pii/039304409500028G](http://www.sciencedirect.com/science/article/pii/039304409500028G)

[12] C. Rovelli and L. Smolin. Discreteness of area and volume in quantum gravity. *Nuclear Physics B*, 442(3):593–619, 1995.

[13] A. Ashtekar and J. Lewandowski. Quantum theory of geometry: I. area operators. *Classical and Quantum Gravity*, 14(1A):A55, 1997.

[14] A. Ashtekar and J. Lewandowski. Quantum theory of geometry ii: Volume operators. *Advances in Theoretical and Mathematical Physics*, 1(2):388–429, 1997.

[15] Y. Ma and Y. Ling. Q operator for canonical quantum gravity. *Phys. Rev. D*, 62:104021, Oct 2000. doi: 10.1103/PhysRevD.62.104021. URL [https://link.aps.org/doi/10.1103/PhysRevD.62.104021](https://link.aps.org/doi/10.1103/PhysRevD.62.104021)

[16] J. Yang and Y. Ma. New volume and inverse volume operators for loop quantum gravity. *Phys. Rev. D*, 94:044003, Aug 2016. doi: 10.1103/PhysRevD.94.044003. URL [https://link.aps.org/doi/10.1103/PhysRevD.94.044003](https://link.aps.org/doi/10.1103/PhysRevD.94.044003)

[17] J. Lewandowski, A. Okolow, H. Sahlmann, and T. Thiemann. Uniqueness of diffeomorphism invariant states on holonomy-flux algebras. *Commun. Math. Phys.*, 267:703–733, 2006. doi: 10.1007/s00220-006-0100-7.

[18] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, and T. Thiemann. Quantization of diffeomorphism invariant theories of connections with local degrees of freedom. *Journal of Mathematical Physics*, 36(11):6456–6493, 1995.

[19] T. Thiemann. Quantum spin dynamics (qsd). *Classical and Quantum Gravity*, 15(4):839, 1998.

[20] T. Thiemann. Quantum spin dynamics (qsd): Ii. the kernel of the wheeler-dewitt constraint operator. *Classical and Quantum Gravity*, 15(4):875, 1998.

[21] T. Thiemann. The phoenix project: master constraint programme for loop quantum gravity. *Classical and Quantum Gravity*, 23(7):2211, 2006.
[22] M. Han and Y. Ma. Master constraint operators in loop quantum gravity. *Physics Letters B*, 635(4):225–231, 2006.

[23] J. Lewandowski and H. Sahlmann. Symmetric scalar constraint for loop quantum gravity. *Physical Review D*, 91(4):044022, 2015.

[24] M. Assanioussi, J. Lewandowski, and I. Mäkinen. New scalar constraint operator for loop quantum gravity. *Phys. Rev. D*, 92:044042, Aug 2015. doi: 10.1103/PhysRevD.92.044042.

[25] J. Yang and Y. Ma. New hamiltonian constraint operator for loop quantum gravity. *Physics Letters B*, 751:343–347, 2015.

[26] C. Rovelli and L. Smolin. The physical hamiltonian in nonperturbative quantum gravity. *Physical review letters*, 72(4):446, 1994.

[27] J. D. Brown and K. V. Kuchař. Dust as a standard of space and time in canonical quantum gravity. *Physical Review D*, 51(10):5600, 1995.

[28] K. Giesel and T. Thiemann. Algebraic quantum gravity (aqg): Iv. reduced phase space quantization of loop quantum gravity. *Classical and Quantum Gravity*, 27(17):175009, 2010.

[29] M. Domagala, K. Giesel, W. Kamiński, and J. Lewandowski. Gravity quantized: Loop quantum gravity with a scalar field. *Phys. Rev. D*, 82:104038, Nov 2010.

[30] J. Lewandowski, M. Domagala, and M. Dziendzikowski. The dynamics of the massless scalar field coupled to lqg in the polymer quantization. *Proc. Sci. QGQGS*, 25, 2011.

[31] C. Rovelli. What is observable in classical and quantum gravity? *Classical and Quantum Gravity*, 8(2):297, 1991.

[32] B. Dittrich. Partial and complete observables for canonical general relativity. *Classical and Quantum Gravity*, 23(22):6155, 2006.

[33] T. Thiemann. Reduced phase space quantization and dirac observables. *Classical and Quantum Gravity*, 23(4):1163, 2006.

[34] A. Dapor, W. Kamiński, J. Lewandowski, and J. m. k. Świeżewski. Relational evolution of observables for hamiltonian-constrained systems. *Phys. Rev. D*, 88:084007, Oct 2013. doi: 10.1103/PhysRevD.88.084007. URL https://link.aps.org/doi/10.1103/PhysRevD.88.084007.

[35] T. Lang, K. Liegener, and T. Thiemann. Hamiltonian renormalisation i: Derivation from osterwalder-schrader reconstruction. *arXiv preprint arXiv:1711.05685*, 2017.

[36] T. Lang, K. Liegener, and T. Thiemann. Hamiltonian renormalisation iv. renormalisation flow of d+1 dimensional free scalar fields and rotation invariance. *arXiv preprint arXiv:1711.05695*, 2017.

[37] E. Alesci and C. Rovelli. Regularization of the hamiltonian constraint compatible with the spinfoam dynamics. *Physical Review D*, 82(4):044007, 2010.

[38] E. Alesci, M. Assanioussi, J. Lewandowski, and I. Mäkinen. Hamiltonian operator for loop quantum gravity coupled to a scalar field. *Phys. Rev. D*, 91:124067, Jun 2015. doi: 10.1103/PhysRevD.91.124067.

[39] A. Ashtekar and P. Singh. Loop quantum cosmology: a status report. *Classical and Quantum Gravity*, 28(21):213001, 2011.
[40] A. Ashtekar, M. Bojowald, J. Lewandowski, et al. Mathematical structure of loop quantum cosmology. *Advances in Theoretical and Mathematical Physics*, 7(2):233–268, 2003.

[41] A. Ashtekar, T. Pawłowski, and P. Singh. Quantum nature of the big bang: Improved dynamics. *Phys. Rev. D*, 74:084003, Oct 2006. doi: 10.1103/PhysRevD.74.084003.

[42] W. Kamiński and J. Lewandowski. The flat FRW model in LQC: self-adjointness. *Classical and Quantum Gravity*, 25(3):035001, 2008. URL [http://stacks.iop.org/0264-9381/25/i=3/a=035001](http://stacks.iop.org/0264-9381/25/i=3/a=035001).

[43] M. Domagala. *On quantum model of the massless Klein-Gordon field coupled to gravity*. PhD thesis, Warsaw U., 2015. URL [https://depotuw.ceon.pl/handle/item/1147](https://depotuw.ceon.pl/handle/item/1147).

[44] Y. Ma and C. Liang. The degenerate sector of Ashtekar’s phase space. *Modern Physics Letters A*, 13(35):2839–2843, 1998.

[45] Y. Ma and C. Liang. Causal structure and degenerate phase boundaries. *Physical Review D*, 59(4):044008, 1999.

[46] J. Lewandowski and J. Wisniewski. 2+ 1 sector of 3+ 1 gravity. *Classical and Quantum Gravity*, 14(3):775, 1997.

[47] J. Yang and Y. Ma. Graphical method in loop quantum gravity: I. derivation of the closed formula for the matrix element of the volume operator. *arXiv preprint arXiv:1505.00223*, 2015.

[48] J. Yang and Y. Ma. Graphical method in loop quantum gravity: II. the hamiltonian constraint and inverse volume operators. *arXiv preprint arXiv:1505.00225*, 2015.

[49] J. Yang and Y. Ma. Graphical calculus of volume, inverse volume and hamiltonian operators in loop quantum gravity. *The European Physical Journal C*, 77(4):235, 2017.

[50] W. G. Faris and R. B. Lavine. Commutators and self-adjointness of hamiltonian operators. *Communications in Mathematical Physics*, 35(1):39–48, 1974.

[51] M. Reed and B. Simon. *Methods of modern mathematical physics II: Fourier Analysis, Self-Adjointness*, volume 2. Elsevier, 1975.