Dielectric-branes in Non-supersymmetric SO(3)-invariant Perturbation of Three-dimensional $\mathcal{N} = 8$ Yang-Mills Theory

Changhyun Ahn ¹ and Taichi Itoh ²

Department of Physics, Kyungpook National University, Taegu 702-701, Korea

Abstract

We study non-supersymmetric $SO(3)$-invariant deformations of $d = 3$, $\mathcal{N} = 8$ super Yang-Mills theory and their type IIA string theory dual. By adding both gaugino mass and scalar mass, dielectric D4-brane potential coincides with D5-brane potential in type IIB theory. We find the region of parameter space where the non-supersymmetric vacuum is described by stable dielectric NS5-branes. By considering the generalized action for NS5-branes in the presence of D4-flux, we also analyze the properties of dielectric NS5-branes.

¹Email address: ahn@knu.ac.kr
²Email address: taichi@knu.ac.kr
1. Introduction

The Anti de Sitter(AdS)/Conformal Field Theory(CFT) correspondence (for a review, see [1]) enables us to study not confining theories but conformal $\mathcal{N} = 4$ gauge theories that are dual to type IIB string theory on $\text{AdS}_5 \times S^5$. In order to understand the former, one can perturb by adding mass terms preserving some or none of supersymmetry and gets a confining gauge theory. It is known that in [2], they made a proposal for the dual supergravity description of a four-dimensional confining gauge theory by adding finite mass terms to $\mathcal{N} = 4$ Yang-Mills theory and computed the linearized perturbed background by the presence of non-vanishing boundary conditions on the magnetic three-form. In each of the many vacua, D3-branes were replaced by several five-branes through Myers’ dielectric effect [3]. It turned out that as long as the ratio of five- and three-brane charge densities is very small, the solutions are good near five-brane action minima.

Motivated by the work of [2], the dual M-theory description of a three-dimensional theory living on M2-branes by adding fermion mass terms was found [4]. Similarly, the nonsingular string theory duals corresponding to a perturbed three-dimensional gauge theory on D2-branes was obtained by the polarization of D2-branes into D4-branes and NS5-branes [5]. Moreover, the dual string theory of oblique vacua in the perturbed three-dimensional gauge theory corresponded to the polarization of D2-branes into NS5-branes with D4-brane charge [6]. In [7], $SO(3)$-invariant deformations of four-dimensional $\mathcal{N} = 4$ gauge theory within the context of [2] was studied and the non-supersymmetric vacuum is described by stable dielectric five-branes.

In this paper, we consider perturbed three-dimensional gauge theory living on D2-branes by adding both gaugino mass term and the scalar terms and construct dual string theory corresponding to this $SO(3)$-invariant non-supersymmetric deformation of $d = 3$, $\mathcal{N} = 8$ theory. In section 2, we review three-dimensional Yang-Mills theory and how its $SO(3)$-invariant perturbations appear. In section 3, for given $\mathcal{N} = 2$ supersymmetric gauge theory perturbed by three fermion masses, we go one step further by inserting gaugino mass and scalar mass which are 35 and 27 of $SO(7)_R$ symmetry, respectively. It turns out it is exactly same form of the one of type IIB theory described by D3/D5 potential. Similarly, in section 4, we consider the scalar mass 27 of $SO(7)_R$ symmetry, modify $\mathcal{N} = 2$ dielectric NS5-brane action and study its phase diagram. In section 5, we do the same analysis for generalized NS5-brane action. In section 6, we make our conclusions and future directions.
2. The SO(3)-invariant perturbations in type IIA theory

We start from preliminaries about the type IIA D2-brane solution and its SO(3)-
invariant perturbation both in the bulk and in the boundary. The unperturbed space-time
generated by $N$ coincident D2-branes is obtained as \cite{8, 9}

$$ds^2 = Z^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + Z^{1/2} \delta_{mn} dx^m dx^n,$$

with the R-R three-form potential

$$C_3^0 = -\frac{1}{g_s Z} dx^0 \wedge dx^1 \wedge dx^2,$$

where $\mu, \nu = 0, 1, 2, m, n = 3, \ldots, 9$, and $g_s$ is the string coupling which is related to the
dilaton field $\Phi$ through $e^{\Phi} = g_s Z^{1/4}$. The warp factor $Z$ is given by a harmonic function

$$Z = \frac{R^5}{r^5}, \quad R^5 = 6\pi^2 N g_s \alpha'^{5/2}.$$

The dielectric D4- and NS5-brane configurations are obtained by perturbing the type IIA
D2-brane solution with linearized $H_3 = dB_2$, $F^1_4 = dC_3^1$ field strengths which transform as
35 of $SO(7)_R$ rotation group in transverse seven-dimensions.

The gauge theory living on $N$ coincident D2-branes is an $\mathcal{N} = 8$ super Yang-Mills \cite{10}. An $\mathcal{N} = 8$ gauge multiplet consists of a gauge field $A_\mu$, eight real fermions $\{\lambda_1, \ldots, \lambda_8\}$ which are in 8 spinor representation of $SO(7)_R$ $R$-symmetry, and seven real scalars $\{\chi_1, \ldots, \chi_7\}$ which are in 7 vector representation of the $SO(7)_R$. The eight fermions are cast into four complex fermions

$$\Lambda_1 = \lambda_1 + i\lambda_2, \quad \Lambda_2 = \lambda_3 + i\lambda_4, \quad \Lambda_3 = \lambda_5 + i\lambda_6, \quad \Lambda_4 = \lambda_7 + i\lambda_8,$$

which transform as 4 of $SU(4) \subset SO(7)_R$, whereas the seven scalars are divided into six scalars of 6 of $SU(4)$ and an $SU(4)$-singlet real scalar. The six scalars are cast into three complex scalars

$$\phi_1 = \chi_1 + i\chi_2, \quad \phi_2 = \chi_3 + i\chi_4, \quad \phi_3 = \chi_5 + i\chi_6,$$

which are combined together with $\Lambda_1, \Lambda_2, \Lambda_3$ into three $\mathcal{N} = 2$ hypermultiplets $\{\phi_i, \Lambda_i\}$, $i = 1, 2, 3$, transforming as 3 of $SU(3) \subset SU(4)$. The $SO(3)$ group considered in this paper is a real subgroup of the $SU(3)$. The gauge boson $A_\mu$ and the singlet scalar $\chi_7$ are made
up into an $\mathcal{N} = 2$ gauge multiplet $\{ A_\mu, \chi_7, \Lambda_4 \}$ together with $\Lambda_4$ including a gaugino field $\lambda_8$.

Both $d = 3, \mathcal{N} = 8$ and $d = 4, \mathcal{N} = 4$ super Yang-Mills theories share sixteen supercharges [10]. The $\mathcal{N} = 2$ gauge (hyper) multiplet in three-dimensions correspond to the $\mathcal{N} = 1$ gauge (chiral) multiplet through the dimensional reduction. The $SU(4)$ subgroup considered above is nothing but the $R$-symmetry of $d = 4, \mathcal{N} = 4$ super Yang-Mills. This implies that the $d = 3, \mathcal{N} = 2$ gauge theory obtained by giving masses $m_1, m_2, m_3$ to the three hypermultiplets corresponds to the $d = 4, \mathcal{N} = 1$ gauge theory [11, 12] considered in [2]. The $\mathcal{N} = 2$ fermion mass terms appear in the Lagrangian

$$\Delta L = \text{Re} \left( m_1 \Lambda_1^2 + m_2 \Lambda_2^2 + m_3 \Lambda_3^2 + m_4 \Lambda_4^2 \right),$$

where the gaugino mass $m_4$ must be zero to obtain the $\mathcal{N} = 2$ gauge theory otherwise the gauge theory becomes non-supersymmetric. If we set $m_1 = m_2 = m_3 \equiv m$, the mass perturbation becomes $SO(3)$-invariant. Both real and imaginary parts of $\{ \Lambda_1, \Lambda_2, \Lambda_3 \}$ are 3 of $SO(3)$, while $\Lambda_4$ is an $SO(3)$-singlet.

In order to make contrast of the type IIA $SO(3)$-invariant perturbation with the one in type IIB case, it may be useful to show the branching rules of $SO(7)_R \rightarrow SU(4)_R$:

$$\begin{align*}
8 & \rightarrow 4 + \overline{4}, \\
7 & \rightarrow 6 + 1, \\
35 & \rightarrow 10 + \overline{10} + 15, \\
27 & \rightarrow 20' + 6 + 1.
\end{align*}$$

Each of the $H_3$ and $F_4^1$ perturbations corresponds to 35 of $SO(7)_R$, that is, a traceless $8 \times 8$ fermion mass matrix $m_{\alpha\beta} \lambda_\alpha \lambda_\beta$. The $\mathcal{N} = 2$ $SO(3)$-invariant mass term is a specific choice of $m_{\alpha\beta}$ and is given by setting $m_1 = m_2 = m_3 \equiv m, m_4 = 0$ in Eq. (2). The branching rules imply that the 35 is a counter part of $d = 4$, fermion masses in $\text{TO}$ of $SU(4)_R$. The $\mathcal{N} = 0$ $SO(3)$-invariant perturbation in four dimensions considered in [4] consists of a gaugino mass included in the $\text{TO}$ and the $6 \times 6$ symmetric traceless scalar mass matrix in $20'$ of $SU(4)_R$. The branching rules therefore tell us that the corresponding IIA scalar mass term is a $7 \times 7$ symmetric traceless mass matrix $\mu_{ij}^2 \chi_i \chi_j$ in 27 of $SO(7)_R$. The $\mathcal{N} = 0$ $SO(3)$ invariant scalar mass term considered in this paper is a specific choice of $\mu_{ij}^2$ and is given by $\text{Re} (\mu^2 \phi_i \phi_i)$. 

4
3. Dielectric D4-branes wrapping on $S^2$

The action for the dielectric D4-branes consists of the Born-Infeld action and the Wess-Zumino action and is given by

$$S_{D4} = -\mu_4 \int d^5\xi \ g_s^{-1}Z^{-1/4} \sqrt{-\det G_\parallel \det (G_\perp + 2\pi\alpha' F_2)}$$

$$-\mu_4 \int (C_5 + 2\pi\alpha' F_2 \wedge C_3),$$

where $G_\parallel$ stands for the metric along D2-brane world volume $\{x^0, x^1, x^2\}$ and $G_\perp$ is a metric on 2-sphere $S^2$ in seven transverse dimensions. The 2-form field living on the D4-brane is $2\pi\alpha' F_2 \equiv 2\pi\alpha' F_2 - B_2$. The D$p$-brane tension is given by $\mu_p = \alpha'^{(p-1)/2}/(2\pi\alpha')^p$ which reproduces the string tension $1/2\pi\alpha'$ when $p = 1$.

Let us introduce the complex coordinates $z^i = \frac{1}{\sqrt{2}}(x^i + 2i + i x^6)$ ($i = 1, 2, 3$) with $x^6$ as a moduli direction. Suppose that the D2-branes polarize into a noncommutative $S^2$ under the perturbation, then the $S^2$ is specified by a single complex coordinate $z$ through $z^i = ze^i$ with a real unit vector $e^i$ and its radius is given by $|z|$. The metric of the $S^2$ couples with the two-form field strength $F_2$ which measures the D2-brane charge $n$:

$$\int_{S^2} F_2 = 2\pi n.$$  

The same $F_2$ also arises in the Wess-Zumino action through the term $F_2 \wedge C_3$. If we suppose the large D2-brane charge density $n \gg N^{1/2}$, the $F_2$-dependent terms in both Born-Infeld and Wess-Zumino action cancel each other and yield a term quartic in $|z|$. By taking the Poincaré dual of the IIA field equation

$$d(*F_4 + B_2 \wedge F_4) = 0,$$

the linear perturbation of $H_3$ and $F_4$ arises in the $C_5$ term of the Wess-Zumino action and provides a term cubic in $|z|$. Thus, after dividing by the D2-brane world volume $V$, we obtain the D2/D4-brane action

$$\frac{-S_{D4}}{V} = \frac{2\mu_4}{g_s n\alpha'} \left[ |z|^4 - 2\pi n\alpha' \text{Im} (mzz\bar{z}) + (\pi n\alpha')^2 m^2 |z|^2 \right],$$

which describes the dielectric D4-brane action where $n$ D2-branes polarize into the noncommutative $S^2$ so that the D4-brane world volume has a geometry $R^3 \times S^2$. Due to the $\mathcal{N} = 2$ supersymmetry, the third term in the action was added so as to complete the square and to obtain a supersymmetric minimum at $z = i\pi n\alpha'm$. 

5
Now we move on our main goal of this paper, that is the $\mathcal{N} = 0$ $SO(3)$-invariant deformation of 3-dimensional super Yang-Mills, by adding $SO(3)$-invariant perturbations which fully breaks $\mathcal{N} = 2$ supersymmetry in 3-dimensions. They correspond to the gaugino mass $m_4$ in 35 and the scalar mass which is given by a traceless $7 \times 7$ matrix and transforms as 27 of $SO(7)_R$ $R$-symmetry. In the D4-brane action (6), this procedure is achieved by shifting

$$\text{Im}(mzz\bar{z}) \rightarrow \text{Im}\left(mzz\bar{z} + \frac{m_4}{3}zzz\right),$$

$$m^2|z|^2 \rightarrow \left(m^2 + \frac{|m_4|^2}{3}\right)|z|^2 + \text{Re}(\mu^2 zz).$$

For simplicity, we rescale the complex coordinate $z$ and introduce the dimensionless parameters $\tilde{b}$ and $c$ such that

$$z = i\pi n\alpha' m x e^{i\varphi}, \quad \tilde{b} \equiv -\frac{m_4}{m}, \quad c \equiv \frac{\mu^2}{m^2}. \quad (7)$$

Then we obtain the D4-brane potential with the $SO(3)$-invariant non-supersymmetric perturbation

$$V_{D4}(x, \tilde{b}, c) = 2\pi \sqrt{\alpha'} \left(\frac{m^4 n^2}{16\pi}\right) \frac{n}{g_s} x^2 \left[x^2 - 2x \text{Re}\left(e^{-i\varphi} + e^{3i\varphi}\frac{\tilde{b}}{3}\right) + 1 + \frac{|	ilde{b}|^2}{3} - \text{Re}\left(e^{2i\varphi}c\right)\right], \quad (8)$$

which coincides with the D3/D5 potential in 7 except for the ratio of D-brane tensions $\mu_4/\mu_5 = 2\pi \sqrt{\alpha'}$. Note that 7, 2 introduced fermions which transform as 4 of $SU(4)_R$ $R$-symmetry in $d = 4, \mathcal{N} = 4$ super Yang-Mills, whereas we have used fermions in 5 which transform as $3 + 1$ of $SU(3) \subset SU(4)_R$.

4. Dielectric NS5-branes wrapping on $E^3$

The NS5-brane action with $n$ D2-branes polarized into a 3-ellipsoid $E^3$ has been studied in 5 based on the type IIA NS5-brane action formulated in 13. The action is quite similar to the M5-brane action which couples with the self-dual 4-form field strength in M-theory 14, 15. It consists of the Born-Infeld term, the Wess-Zumino term and the mixed term which is necessary to build in the self-dual field strength in manifestly 6-dimensional covariant way by invoking a certain auxiliary fields. After eliminating the auxiliary field by choosing for example $x^2$ as a special direction, the action becomes similar to the M5-brane
action which shows only 5-dimensional covariance \[15\] and is given by \[5 \]

\[S_{NS5} = S_{BI} + S_{mix} + S_{WZ},\]
\[S_{BI} = -\mu_5 \int d^6\xi \, g_s^{-2} Z^{-1/2} \sqrt{-\det (G_{mn} + ig_s Z^{1/4} D_{mn})},\]
\[S_{mix} = -\mu_5 \int d^6\xi \, \frac{1}{4} \sqrt{-G} \, D_{mn} D_{mn2},\]
\[S_{WZ} = -\mu_5 \int \left( B_6 - \frac{1}{2} F_3 \wedge C_3 \right),\]

(9)

where \(G\) is a determinant of a 6-dimensional metric \(G_{\mu\nu}\), \(\mu, \nu = 0, 1, 2, 3, 4, 5\) and its 5-dimensional restriction is \(G_{mn}\), \(m, n = 0, 1, 3, 4, 5\). The 2-form field \(D_{mn}\) is given by

\[D_{mn} = \frac{\sqrt{G_{22}}}{3! \sqrt{-G}} \epsilon^{2mpqr} D_{pqr},\]

where \(D_{pqr}\) is a 5-dimensional component of a 3-form \(D_3 \equiv F_3 - C_3\). The 6-dimensional self-dual constraints is obtained as the equation which determines \(D_{mn2}\) in terms of \(D_{mn}\) by solving the 6-dimensional Euler-Lagrange equations of motion.

When the 3-ellipsoid is situated in the 3456-plane, the nonzero components of \(D_3\) are \(D_{345} = F_{345} - C_{345}\) and its permutations. The 3-form field \(F_3\) is determined by the quantization of the D2-brane charge along the 3-ellipsoid

\[\mu_2 \int_{E^3} F_3 = 2\pi n.\]

The 3-form potential \(C_3\) is given by solving the IIA field equations (3) and

\[2d(e^{-2\Phi} \ast H_3) = F_4 \wedge F_4,\]

(10)

and depends on the fermion mass perturbation (4) with setting \(m_1 = m_2 = m_3 \equiv m\) on the D2-branes. The 6-form potential \(B_6\) in the Wess-Zumino action can be shown to be zero by taking the Poincaré dual of Eq. (10).

In the limit when D2-brane charge is much bigger than NS5-brane charge, say \(n \gg N^{1/2}\), the action can be expanded with respect to \(D_{345}\). In this approximation, the Wess-Zumino action is fully given by the interaction of the dissolved D2-branes and canceled by those in the Born-Infeld and the mixed actions to yield the simplified action \[5\]

\[-\frac{S_{NS5}}{2\pi^2 \mu_5 V} = \frac{3}{16 g_s^3 A} \left( 3|z|^4|w|^2 + |z|^6 \right) - \frac{1}{4g_s^2} \text{Re} \left( 3m wzzz \bar{z} + m_4 zzz \bar{w} \right) + \frac{A}{12g_s} \left( 3m_2 |z|^2 + m_4^2 |w|^2 \right),\]

(11)
where $A \equiv 4\pi n(n')^{3/2}$ and $w = x^6$ corresponds to the fourth complex coordinate $z^4 = x^6 + ix^{10}$ in M-theory. The first term of the action is the gravitational energy of the NS5-brane and is attractive. The second term is proportional to the linear perturbation of $C_{345}$ and is repelling. The balance between the two terms determines a finite size 3-ellipsoid. The last term is introduced to complete the square in the action in the following sense. Since we are interested in the gauge theory on $n$ D2-branes, we give a mass $m_4$ only to a gaugino field in the $\mathcal{N} = 1$ gauge multiplet and supersymmetry is fully broken. On the other hand, the D2/NS5 bound state in the theory is a descendant of an M2/M5 bound state in parent M-theory. The field theory on $n$ M2-branes is a super conformal fixed point of $d = 3, \mathcal{N} = 8$ super Yang-Mills and the $\mathcal{N} = 2$ gauge multiplet turns to an $\mathcal{N} = 2$ hypermultiplet at the fixed point. The $SO(3)$-invariant configuration corresponds to the $\mathcal{N} = 2$ supersymmetric configuration in M-theory where three of hypermultiplets have the same mass $m$ and the fourth hypermultiplet has a mass $m_4$. Although supersymmetry is fully broken by the gaugino mass $m_4$ in IIA theory, we can complete the square in the NS5-brane action due to the hidden $\mathcal{N} = 2$ supersymmetry of parent M-theory and can find supersymmetric minimum at $[3]$

$$z^2 = \frac{2Ag_s}{3}mx^2, \quad x_6^2 = \frac{2Ag_s}{3}m\sqrt{\frac{m}{m_4}}. \quad (12)$$

Let us proceed to the analysis of $\mathcal{N} = 0$ $SO(3)$-invariant deformation, which is our main goal in this paper. We introduce the same $SO(3)$-invariant scalar mass term, which is in $27$ of $SO(7)_R$ symmetry, as in the D4-brane action. We only have to shift the quadratic term as

$$m^2|z|^2 \longrightarrow m^2|z|^2 + \text{Re} \left( \mu^2 zz \right).$$

In contrast with D2/D4 potential, the mass ratio $m_4/m$ arises at the supersymmetric minimum as the aspect ratio of 3-ellipsoid so that it must be always positive. We will therefore use the parameters $b \equiv m_4/m$ instead of $\tilde{b}$ itself and the same parameter $c$ as before. Rescaling the coordinates such that

$$z^2 = \frac{2Ag_s}{3}m\sqrt{b}x^2, \quad x_6^2 = \frac{2Ag_s}{3}m\frac{1}{\sqrt{b}}y^2, \quad (13)$$

the D2/NS5-brane action becomes

$$U(x, y, b, c) = \frac{A^2m^3}{18}\sqrt{b}\left[ bx^6 + 3gy^2x^4 - 2(3 + b)yx^3 + 3(1 + c)x^2 + by^2 \right], \quad (14)$$
which is a two-dimensional potential depending on two coordinates $x$ and $y$. In order to find out the local minima of the potential, we first solve the equation $\partial U/\partial y = 0$. Then we obtain a trajectory

$$y = \left(\frac{3 + b}{3x^4 + b}\right)x^3,$$

along which the potential remains flat. Substitution of this equation back into the potential (14) gives us one-dimensional potential

$$V_{NS5}(x, b, c) = \frac{A^2m^3}{18}b^{3/2}\left[\frac{3x^2}{3x^4 + b}\right]\left[(x^4 - 1)^2 + c + \frac{3c}{b}x^4\right],$$

of which local minima can be identified with those of the original potential (14). Note that when $c = 1$ the potential has a zero at $x = 1$ corresponding to the supersymmetric minimum of Eq. (12).

Let us find out the regions in $(b, c)$-plane where we have a finite size 3-ellipsoid. Differentiating the potential (16) with respect to $x$, one finds that the equation which determines local minima of the potential is given by the cubic equation

$$f(X) \equiv X^3 + c_1X^2 + c_2X + c_3 = 0,$$

Figure 1: The phase diagram of D2/NS5-branes. The thick line is the critical line $c = h(b)$ for a D2/NS5-minimum without D4-brane charge. The allowed region in the left hand side of the critical line is separated into phase I and phase II. The local maximum disappears in phase II. The dashed line is the critical line for a D2/NS5-branes in the presence of D4-flux which will be discussed in Section 5.
where $X \equiv x^4$ and $c_1$, $c_2$, and $c_3$ are determined as
\[
c_1 = \frac{5b^2 - 6b + 9c}{9b}, \quad c_2 = -\frac{2b - 2c + 1}{3}, \quad c_3 = \frac{b(1 + c)}{9}.
\]
The extrema of the cubic function $f(X)$ are located at
\[
X_\pm = \frac{1}{3} \left( -c_1 \pm \sqrt{c_1^2 - 3c_2} \right).
\] (18)

Since $c_1^2 - 3c_2$ takes positive values, we always have two extrema. Furthermore, we can easily see that $X_-$ is always negative in the whole of $(b, c)$-plane. The allowed region of $(b, c)$-plane, where the cubic equation (17) has at least one solution, is therefore given by the inequality
\[
f(X_+) \leq 0 \leftrightarrow \left( 2c_1^3 - 9c_1c_2 + 27c_3 \right)^2 \leq 4 \left( c_1^2 - 3c_2 \right)^3,
\] (19)
which can be solved with respect to $c$ such that
\[
c \leq h(b) \equiv \frac{1}{216} \left[ 9 + 150b + 97b^2 + H(b)^{1/3} + \frac{(3 + b)^2 (9 + 6b - 2591b^2)}{H(b)^{1/3}} \right],
\] (20)
where the function $H(b)$ is given by
\[
H(b) \equiv (3 + b)^2 \left[ 81 + 108b + 58374b^2 + 38892b^3 - 833327b^4 + 144b \left( 9 + 6b + 325b^2 \right)^{3/2} \right].
\]

The corresponding phase diagram is shown in Fig. 11. Since the parameter $b$ turns to the aspect ratio of 3-ellipsoid in the supersymmetric limit $c \to 0$ and therefore should be positive, the negative half of $b$ axis is forbidden. The critical line $c = h(b)$ approaches to $4/5$ when $b \to \infty$ which is in perfect agreement with the upper-bound of $c$ for the D3/NS5 bound state in [7]. The critical line intersects the $c$ axis at $c = 1/8$ which is also the same as in the D3/NS5 bound state. The cubic function in Eq. (17) has two extrema corresponding to a local maximum and a local minimum of the potential (16) when $c_3 > 0$ ($c > -1$), whereas it has only one minimum when $c_3 < 0$ ($c < -1$). Hence the allowed region in $(b, c)$-plane was separated into phase I with $c > -1$ and phase II with $c < -1$. Again, the critical line $c = -1$ coincides with the upper-bound of the region where two D3/NS5 minima coexist in [7].

Now let us look at some aspects of the potential (16) with varying $c$ for a fixed value of $b$. We set $b = 2$ for simplicity so that the critical value of $c$ is given by $c^* = h(b = 2) \approx 0.556281$. In Fig. 2, five aspects of the potential are depicted. Each line corresponds to
Figure 2: The $D2/NS5$ potential \(^{(16)}\) when $b = 2$ with varying $c$. The horizontal axis $x$ denotes the rescaled $S^2$ radius \(^{(13)}\) of a 3-ellipsoidal shell for a $D2/NS5$-brane. Each line corresponds to $c = 0.6, 0.25, 0, -0.4, \text{ and } -1.1$ in order from above.

$c = 0.6, 0.25, 0, -0.4, \text{ and } -1.1$ in order from above. When $c > c^*$ the potential has no local minima except for the origin and therefore any finite size ellipsoid does not exist (the first line with $c = 0.6 > c^*$ in Fig. 2). We find a local minimum other than the origin when $c = 0.25$ though the potential has positive energy at the point. In the supersymmetric limit $c = 0$, we find a local minimum at $x = 1$ and the potential energy becomes zero as required by supersymmetry. When the parameter $c$ becomes negative (the fourth line with $c = -0.4$ in Fig. 2), the potential energy at the finite size local minimum turns to negative so that we can identify the minimum point as a stable finite size solution. Finally, if the parameter $c$ becomes smaller than $-1$ and enters into phase II (the fifth line with $c = -1.1$ in Fig. 2), the local maximum point disappears. The trivial solution at the origin becomes unstable and the vacuum necessarily goes to formation of the finite size 3-ellipsoid.

5. Dielectric NS5-branes wrapping on $E^3$ with D4-brane charge

The general action for NS5-branes in the type IIA theory was found recently in \(^{[13]}\). The Wess-Zumino term of the action contains, other than the NS-NS six-form potential $B_6$, the coupling between the bulk R-R five-form potential $C_5$ and the one-form field strength $\mathcal{F}$ living on the NS5-brane. Since a nonzero $\mathcal{F}$ corresponds to a nonzero D4-brane charge,
we have to take it into account when we analyze D2/D4/NS5 bound states and its action is obtained by shifting $B_6 \rightarrow B_6 + C_5 \wedge F$ in the Wess-Zumino term of the D2/NS5-brane action (9) to yield
\[ \delta S_{\text{WZ}} = -\mu_5 \int C_5 \wedge F. \] (21)

Let us derive the D2/D4/NS5-brane action explicitly from the D2/NS5-brane action [6]. First, the nonzero D4-brane charge may possibly cause a rotation of the 3-ellipsoid with an aspect ratio $\alpha$ in 3-7, 4-8, 5-9 planes at an angle $\gamma$. This rotation is achieved by setting $z = re^{i\gamma}$, and $w = \alpha r$ in the D2/NS5-brane action (11) to obtain
\[ -S_{\text{NS5}} = \frac{3r^6}{16g_s^2 A} (3\alpha^2 + 1) - \frac{\alpha r^4}{4g_s^2} (3m \cos \gamma + m_4 \cos 3\gamma) \]
\[ + \frac{Ar^2}{12g_s} (3m^2 + \alpha^2 m_4^2 + 3\mu^2 \cos 2\gamma), \] (22)
where the $SO(3)$-invariant scalar mass $\mu^2$ is accompanied by a factor of $\cos 2\gamma$ induced by the rotation. Then, we evaluate the D4-brane charge contribution (21) on the 3-ellipsoid and minimize it to obtain [6]
\[ -\frac{\delta S_{\text{WZ}}}{2\pi^2 \mu_5 V} = -\frac{Ar^2}{4g_s} \left( m \sin \gamma + \frac{m_4}{3} \sin 3\gamma \right)^2. \]

Finally, the generalized NS5-brane action is given by
\[ S_{\text{GNS5}} \equiv S_{\text{NS5}} + \delta S_{\text{WZ}}, \]
which still has a supersymmetric minimum of (12) at $\gamma = 0$ when $c = 0$. Again, rescaling the coordinates such that
\[ r^2 = \frac{2Ag_s}{3} m \sqrt{b} x^2, \quad \alpha r^2 = \frac{2Ag_s}{3} m \frac{1}{\sqrt{b}} y^2, \] (23)
the action $S_{\text{GNS5}}$ turns into the D2/D4/NS5-brane potential
\[ U(x, y, \gamma, b, c) = \frac{A^2 m^3}{18} \sqrt{b} \left[ bx^6 + 3y^2 x^4 - 2(3 \cos \gamma + b \cos 3\gamma)y x^3 \right. \]
\[ + 3 \left[ 1 + c \cos 2\gamma - \left( \sin \gamma + \frac{b}{3} \sin 3\gamma \right)^2 \right] x^2 + by^2 \]. (24)

which reproduces the D2/NS5-brane potential (14) when $\gamma = 0$ as expected.
Figure 3: The critical line (a dashed line in Fig. [1]) for a stable D2/NS5 minimum \((u = 1)\) in the presence of D4-flux. The maximum point is at \((b, c) \approx (0.468, 0.0922)\) which is below the critical line (a thick line in Fig. [1]) for a D2/NS5 minimum without D4-brane charge. The line intersects the \(b\)-axis at \((b, c) = (0, 0)\) and \((1, 0)\), and goes down to \(c \approx -0.601\) at \(b = 2\).

We proceed to the analysis of local minima of the potential in three-dimensional coordinate space \((x, y, \gamma)\). The trajectory along which the \(y\)-derivative of the potential is always zero is given by

\[
y = \left( \frac{3 \cos \gamma + b \cos 3\gamma}{3x^4 + b} \right) x^3,
\]

which reproduces Eq. (15) when we turn off the D4-brane charge \((\gamma = 0)\). Substitution of this equation back into the potential (24) provides two-dimensional potential

\[
V_{\text{GNS5}}(x, u, b, c) = \frac{A^2 m^3}{18} b^{3/2} \left[ \frac{3x^2}{3x^4 + b} \right] \left[ (x^4 - u)^2 + cu + \frac{3cu}{b} x^4 \right.
\]

\[
+ \frac{(2u + 1)(u - 1)}{18} \left[ (2u + 1)b^2 + 6b - 9 \right],
\]

where \(u \equiv \cos 2\gamma\) was introduced as a new coordinate. We notice that the potential (26) coincides with the D2/NS5-brane potential (16) when we turn off the D4-brane charge. In contrast with the D2/NS5 case, we have to solve both \(x\)- and \(u\)-flatness conditions in order to determine the allowed region for the stable D2/D4/NS5 minima in the \((b, c)\)-plane.

The \(x\)-flatness condition is given by the same cubic equation (17) as before except that its coefficients are modified to be

\[
c_1 = \frac{5b^2 - 6bu + 9cu}{9b},
\]
Figure 4: The trajectory of a D2/D4/NS5 minimum when $b = 2$. The D2/NS5 minimum continues to exist even under non-zero D4-flux when $c < c_\approx -0.601$. At $c = c_\approx$, the trajectory starts to move along $u$-direction to yield a D2/D4/NS5 minimum. Finally, the minimum disappears at $(u, c) \approx (-0.50, 1.455)$.

\begin{align*}
    c_2 &= -\frac{1}{54} \left[ (2u + 1)^2(u - 1)b^2 + 6(2u^2 + 5u - 1)b + 9(1 + u) - 36cu \right], \\
    c_3 &= \frac{b}{162} \left[ (2u + 1)(u - 1) \left[(2u + 1)b^2 + 6b\right] + 9(1 + u + 2cu) \right].
\end{align*}

We again see that $c_2^2 - 3c_2$ takes positive values and $X_\approx < 0 < X_+$ so that the condition which restricts the allowed region for finite size minima is given by the same inequality (19) as before. The inequality is solved with respect to $cu$ to yield $cu \leq h(b, u)$ which coincides with the inequality (20) when D4-brane charge is turned off at $u = 1$ ($\gamma = 0$). However, the function $h(b, u)$ is no longer bounded from above for generic values of $u \neq 1$. This simply means that the $x$-flatness is not enough to determine the local minima of the potential once we turn on the D4-brane charge. The correct procedure is first we solve the cubic equation (17) and find its largest solution $X_{\text{max}}$ corresponding to the potential minimum along $x$-axis, then substitute it into the $u$-flatness condition $(\partial/\partial u)V_{GNS5} = 0$ to yield

\begin{equation}
    X_{\text{max}} = -\frac{b \left[ b^2(1 - 4u^2) + 2b(1 - 4u) - 3b(1 + 2c) \right]}{6(2b - 3c)},
\end{equation}

which determines the critical surface $g(u, b, c) = 0$ giving a foliation of trajectories of a stable D2/D4/NS5 minimum in $(u, c)$-plane along $b$.

Let us analyze the D2/D4/NS5 constraint (27) precisely. The cross section of the surface
at $u = 1$, say $g(1, b, c) = 0$, provides a new critical line for a stable D2/NS5 minimum in the presence of D4-flux. The new critical line denoted as a dashed line is located in the left hand side of the critical line denoted as a thick line for a D2/NS5-minimum without D4-brane charge as shown in Fig. 1. The numerical plot of that line is again shown in Fig. 3. The maximum point is at $(b, c) \approx (0, 0.468, 0.0922)$ which is below the critical line for a D2/NS5 minimum without D4-brane charge. The line intersects the $b$-axis at $(b, c) = (0, 0)$ and $(1, 0)$, and goes down to $c = c_- \approx -0.601$ at $b = 2$. To look at the upper and lower bounds of $c$ for a specific value of $b$, let us choose for example $b = 2$ as before. The constraint is now $g(u, 2, c) = 0$ and provides the trajectory of a D2/D4/NS5 minimum in $(u, c)$-plane as shown in Fig. 4. As we go up along the D2/NS5 line $u = 1$ from infinitely below, the D2/NS5 minimum continues to stay on $u = 1$ even under non-zero D4-flux. When $c$ reaches the lower critical value of $c_- \approx -0.601$, the trajectory starts to move along $u$-direction so that we necessarily have a D2/D4/NS5 minimum. Finally, at the maximum point $(u, c) \approx (-0.50, 1.455)$, the minimum disappears to yield the upper critical value of $c_+ \approx 1.455$. We can also demonstrate the behavior of D2/D4/NS5 minimum by using the contour plots of the potential (26) with $b = 2$. In Fig. 5, we can see that the stable D2/NS5 ground state with the minimum radius $x \approx 1.077$ ceases to stay on $u = 1$ and begins to move along $u$-direction.

Figure 5: The D2/D4/NS5 minimum when $(b, c) = (2, -0.601)$. It ceases to be a D2/NS5 minimum and starts to move along $u$-direction.

Figure 6: The D2/D4/NS5 minimum when $(b, c) = (2, 0)$. A stable minimum appears away from $u = 1$ and becomes a D2/D4/NS5-minimum.
shift in the D4-brane charge direction. Then, the D2/D4/NS5 minimum continues moving to the ending point at \((u, x) \approx (-0.50, 0.71)\) (see Fig. 6 and Fig. 7) and finally merges into the \(u = -1\) edge and disappears as shown in Fig. 8.

Though the above analysis is just for \(b = 2\), it seems to reflect the generic feature of D2/D4/NS5 minimum. As we increase the value of \(b\), the ending point of D2/D4/NS5 minimum approaches to \(u = 0\) and the upper- and lower-bounds in \(c\) blow up as \(b\) goes to infinity.

6. Conclusions

We have studied the type IIA dielectric D4- and NS5-branes in the bulk corresponding to the \(SO(3)\)-invariant \(\mathcal{N} = 2, 0\) deformations of three-dimensional \(\mathcal{N} = 8\) super Yang-Mills. Specifically, we added the \(SO(3)\)-invariant fermion masses in 35 of \(SO(7)_R\) in \(\mathcal{N} = 2\) case, and the gaugino mass in 35 as well as the scalar masses in 27 of \(SO(7)_R\) in the \(\mathcal{N} = 0\) case. We find that the D2/D4 bound states show exactly same phase structure as in the D3/D5 bound states in type IIB theory as expected from T-duality. Moreover, the D2/NS5
bound states show the same critical values $c = 4/5, 1/8$ and $-1$ as those for D3/NS5 bound states although the critical lines are deformed. This seems to be consistent with $T$-duality between IIA and IIB confining vacua. We have also examined the D2/D4/NS5 bound states and find the phase for a stable D2/NS5 minimum in the presence of D4-flux. The corresponding critical line is obtained as a $u = 1$ cross section of the critical surface $g(u, b, c) = 0$. If we fix the parameter $b = 2$ as a specific value, the surface provides the trajectory of a D2/D4/NS5 minimum in $(u, c)$-plane. The minimum continues staying on the D2/NS5 axis $u = 1$ until the parameter $c$ reaches to the critical value given by $g(1, b, c_-(b)) = 0$, then it starts to shift along $u$-direction to become a D2/D4/NS5 minimum. Finally, the minimum disappears at the maximum point of the trajectory when $c$ gets to the upper critical value determined by $g(u_{\text{min}}, b, c_+(b)) = 0$ and $(\partial g/\partial u)(u_{\text{min}}, b, c_+(b)) = 0$. This suggests that in gauge theory side oblique confining vacua may exist in some particular regions of the scalar mass $\text{Re} (\mu^2 \phi_i \phi_i)$ bounded from above and below for a given gaugino mass $m_4$. There exist many supersymmetric or non-supersymmetric vacua preserving a particular symmetry in the four-dimensional gauged supergravity [16]. It would be interesting to study corresponding dual gauge theory side by looking at the perturbations in the supergravity side.

C.A. was supported by Korea Research Foundation Grant (KRF-2000-003-D00056). T.I. was supported by the grant of Post-Doc. Program, Kyungpook National University (2000).
References

[1] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. 323 (2000) 183 (hep-th/9905111).

[2] J. Polchinski and M. Strassler, hep-th/0003136.

[3] R.C. Myers, J. High Energy Phys. 9912 (1999) 022 (hep-th/9910053).

[4] I. Bena, Phys. Rev. D 62 (2000) 126006 (hep-th/0004142).

[5] I. Bena and A. Nudelman, Phys. Rev. D 62 (2000) 086008 (hep-th/0005163).

[6] I. Bena and A. Nudelman, Phys. Rev. D 62 (2000) 126007 (hep-th/0006102).

[7] F. Zamora, J. High Energy Phys. 0012 (2000) 021 (hep-th/0007082).

[8] G. Horowitz and A. Strominger, Nucl. Phys. B360 (1991) 197.

[9] N. Itzhaki, J. Maldacena, J. Sonnenschein, and S. Yankielowicz, Phys. Rev. D 58 (1998) 046004 (hep-th/9802042).

[10] N. Seiberg, Nucl. Phys. Proc. Suppl. 67 (1998) 158 (hep-th/9705117).

[11] C. Vafa and E. Witten, Nucl. Phys. B431, (1994) 3 (hep-th/9408074).

[12] R. Donagi and E. Witten, Nucl. Phys. B460 (1996) 299 (hep-th/9510101).

[13] I. Bandos, A. Nurmagambetov, and D. Sorokin, Nucl. Phys. B586 (2000) 315 (hep-th/0003169).

[14] P. Pasti, D. Sorokin, and M. Tonin, Phys. Lett. B398 (1997) 41 (hep-th/9701037).

[15] M. Perry and J. H. Schwarz, Nucl. Phys. B489 (1997) 47 (hep-th/9611065); J. H. Schwarz, Phys. Lett. B395 (1997) 191 (hep-th/9701008).

[16] C. Ahn and S.-J. Rey, Nucl. Phys. B572 (2000) 188 (hep-th/9911199); C. Ahn and J. Paeng, Nucl. Phys. B595 (2001) 119 (hep-th/0008065); C. Ahn and K. Woo, Nucl. Phys. B599 (2001) 83 (hep-th/0011121).