Synchronization of degree correlated physical networks

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We propose that negative degree correlation among nodes in a network of nonlinear oscillators, often detected in real world networks, is motivated by its positive effects on synchronizability. In so doing, we use a novel methodology to characterise degree correlation based on clustering the network vertices in $p$ classes according to their degrees. Using this strategy we construct networks with an assigned power law distribution but changing degree correlation properties. We find that the network synchronizability improves when the network becomes disassortative, i.e. when nodes with low degree are more likely to be connected to nodes with higher degree. Our numerical observations are confirmed by the analytical estimates found in this letter using an innovative approach based on the use of graph theoretic results.

Networks of oscillators abound in physics, biology and engineering. Examples include communication networks, sensor networks, neuronal connectivity networks, biologic networks and food webs. Under certain conditions such networks are known to synchronize on a common evolution with all the oscillators exhibiting the same asymptotic trajectories. Moreover synchronization was observed to play an important role in many different problems of a most diverse nature (physical, ecological, physiological, meteorological, to name a few); see for example [1–8].

Recently, it has been proposed that the network topology, i.e. the way in which the oscillators are mutually coupled between themselves, has an important effect on its synchronization properties. In [9], it was shown that the eigenratio $R = \frac{\lambda_1}{\lambda_2}$ between the highest eigenvalue $\lambda_1$ and the lowest eigenvalue $\lambda_2$ of the Laplacian associated to the network structure is an essential measure of the network synchronizability, i.e. the smaller the eigenratio, the larger the interval of the values of the coupling gain, say $\sigma$, for which the stability of the synchronous state is achieved. It is therefore important to characterise how the network topological features affects the Laplacian eigenratio. For example, scale free networks, which are common in nature, were found to show better synchronizability for increasing value of the power law exponent $[10], [11]$.

Another important topological property of physical and biological networks is that often their nodes show preferential attachment to other nodes in the network according to their degree $[12], [13]$. According to this property, networks are said to exhibit assortative mixing (or positive correlation) if nodes of a given degree tend to be attached with higher likelihood to nodes with similar degree. (Similarly disassortative networks are those with nodes of higher degree more likely to be connected with nodes of lower degree.)

The presence of correlation has been detected experimentally in many real-world networks. Interestingly, from the analysis of real networks, it was noticed that social networks are characterized by positive correlation, while physical and biological networks show typically a disassortative structure $[12]$. For instance, in $[14]$, Internet was found to exhibit disassortative mixing at the AS level. A pressing open problem is to understand why negative correlation is an emerging property of physical and biological network.

In this Letter, we study the effects of degree correlation on the network synchronizability properties both analytically and numerically. Our main finding is that disassortative networks of oscillators synchronize better. These findings have immediate theoretical and experimental relevance for understanding the properties of real-world networks. Because of its effects on synchronizability, we conjecture that disassortative mixing has played the role of a self organizing principle in leading the formation of many physical networks as the Internet, the World Wide Web, protein interactions, neural and metabolic networks.

We apply our analysis to scale-free networks in which the probability of finding a node having degree $k$ scales as $k^{-\gamma}$. The network generation model is the configuration model as described in $[15] [10]$. The overall results are summarized in Fig. 1(a) where the effects of varying the degree correlation (measured through the index $r$ defined below) on the Laplacian eigenratio $R$ are shown for different values of the degree distribution exponent $\gamma$. For all values of $\gamma$, we observe a reduction of $R$ for decreasing values of $r$. This means that disassortative mixing enhances the network synchronizability. Interestingly, as depicted in Fig. 1(b) and Fig. 1(c), we observe that, under variations of the correlation parameter, the changes in $R$ seem to be mainly due to variations of $\lambda_2$ while $\lambda_1$, the largest eigenvalue of the Laplacian, is found to be practically independent from $r$. As discussed in $[10]$, in the case of uncorrelated networks ($r = 0$), synchronizability improves for increasing values of $\gamma$. 

\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \]
Here we will give evidence to the fact that tools from algebraic graph theory can be conveniently used for the analysis of network dynamical properties, such as the network synchronizability. The analytical framework we propose to capture the essence of the observed phenomenon is based on the following steps: (i) we introduce a novel index to estimate network assortativity and use such a quantity to construct networks with fixed degree distribution but different (desired) degree correlation properties; (ii) using Cheeger inequalities from algebraic graph theory, we derive novel bounds on the Laplacian eigenvalues of interest and hence new analytical bounds on the eigenratio \( R \); (iii) using the new bounds, we validate our numerical observations.

To start, we assume the network consists of \( N \) identical oscillators coupled through the edges of the network itself [9], [10], [11]. Moreover, we suppose each oscillator is characterized by its own dynamics, \( \{ x_i(t), i=1,...,N \} \), described by a nonlinear vector field, say \( f = f(x) \), perturbed by the output function of its neighbors represented by another nonlinear term, say \( h = h(x) \). The equations of motion for each oscillator can then be given as follows:

\[
\frac{dx_i}{dt} = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} h(x_j), \quad i = 1, 2,...,N, \tag{1}
\]

where \( \sigma \) is the overall coupling strength, and \( L = \{ L_{ij} \} \) is the Laplacian associated to the network topology [16]. Given a network of interest, we start by dividing the network vertices in \( p \) classes such that each one contains \( n_1, n_2,...,n_p \) nodes and \( \sum_{i} n_i = N \). In particular, we assign network vertices to each class in order of increasing (decreasing) degree. Hence, if we label as \( k_i \) the mean degree of vertices belonging to the \( i \)-th class, then \( k_i < k_{i+1} \) \( (i = 1, 2,...,p-1) \).

The probability, \( p(i) \), that a generic vertex belongs to class \( i \) is then given by \( p(i) = n_i/N \). Thus, considering all the vertices in each class \( i \) as they had mean degree \( k_i \), we can extend the usual definition of degree distribution, assuming that \( p(k_i) = n_i/N \). Analogously the probability of finding a vertex belonging to class \( i \) at the end of a randomly chosen edge within the network is given by \( q(i) = n_i k_i/\sum_i n_i k_i \).

Then, following [12], the presence of degree correlation with respect to the subdivision in \( p \) classes proposed above, can be estimated by using a new coefficient, \( r_p \) defined as:

\[
r_p = \frac{k^T(E - qq^T)k}{\sigma_q^2}, \tag{2}
\]

where \( \sigma_q \) is the standard deviation of the distribution \( q_k, \ k = (k_1,k_2,...,k_p)^T, \ q = (q_1,q_2,...,q_p)^T \) and \( E = \{ e_{ij} \} \in \mathbb{R}^{p \times p} \), with \( e_{ij} \) being the probability that a randomly chosen edge in the network connects nodes belonging to class \( i \) and \( j \) (note that in the particular case where each class contains exactly only the vertices having a given degree, \( r_p \) coincides with the coefficient \( r \) as defined in [12], [13].)

From (2), it is possible to derive the distribution of edges among the network vertices as a function of \( r_p \) as follows:

\[
E = qq^T + r_p \sigma_q^2 M, \tag{3}
\]

where \( M \) is a symmetric matrix having all row sums equal to zero and appropriately normalized such that \( k^TMk = 1 \). Specifically, we can express \( M \) as follows:

\[
M = \frac{mm^T}{(k^Tm)^2},
\]

where \( m = (m_1,m_2,...,m_p)^T \) is a vector such that \( \sum_i m_i = 0 \) (for instance we can choose \( m \) such that \( m_i \leq m_{i+1} \) for \( i = 1, 2,...,p-1 \) in order to have a convenient form of the matrix \( M \) with positive values near the main diagonal and negative values far away from it).

As explained in [17], it is possible to devise a strategy similar to the one presented in [12], [13], using the new coefficient \( r_p \), to generate networks with a given degree

![FIG. 1. Synchronizability of degree correlated scale free networks of size 10^3 nodes. Behavior of the eigenratio \( \lambda_N/\lambda_2 \) (a), of the second lowest eigenvalue \( \lambda_2 \) (b) and of the highest one \( \lambda_N \) (c), as functions of the correlation coefficient \( r \) defined in [11], for \( r \) varying from 2 (top line) to 5 (bottom line) in steps of 0.2. (d) The eigenratio as function of \( r \) from \(-0.3 \) (bottom line) to 0.3 (top line) in steps of 0.1. The lines are guided by the eye.](image-url)
distribution and a desired value of the degree correlation coefficient \( r_p \). Such a strategy was used to carry out the computations depicted in Fig. 1.

In [10] analytical bounds were given to explain the changes observed in the eigenratio \( R \) as the parameter \( \gamma \) was varied in a scale-free network topology. We found that, although these bounds should hold for any generic network topology, they seem to be inappropriate to account for the changes in \( R \) observed in the network when degree correlation is introduced. In particular, as shown in Fig. 2, the values of the upper bounds on \( R \) computed according to the formulas given in [10] give estimates which do not reproduce the behavior of the eigenratio under variation of the network degree correlation.

Therefore we shall seek to define new analytical bounds based on the mathematical theory of graph spectra. In particular, as \( \lambda_N \) was found to be almost independent from the correlation coefficient \( r \) (see Fig.1(c)), we will focus on estimating the effects of correlation on the eigenvalue \( \lambda_2 \), the parameter known as algebraic connectivity of graphs [16]. Specifically, given a graph, consider a subset of vertices in \( S \), then, the noticeable features of the network are only the deformation we have on the network. We will assume that the Cheeger constant of a graph is given by \( h_G = \min_S h_G(S) \) and the so-called Cheeger inequality states that \( \lambda_2 \leq h_G \) [18].

We will show below that (4) can be successfully used to compute an upper bound on \( \lambda_2 \). To overcome the limitations due to the computation of the subset \( S \) that minimizes \( h_G(S) \), we will follow a stochastic approach in order to estimate \( h_G(S) \), starting from the available information we have on the network. We will assume that the noticeable features of the network are only the degree distribution and the correlation specified; all other aspects being completely random. Note that finding the subset \( S \) such as to achieve the minimization of \( h_G(S) \) is known to be an NP-hard problem [19].

Then, we can give a full characterization of a randomly chosen subset \( S \) in terms of the number of nodes in it, say \( x_i \), belonging to each class \( i \) \((i = 1, 2, ..., p)\) and the network correlation measure \( r_p \). Analogously let us term as \( y_i = \frac{x_i}{n_i} \) the fraction of nodes in \( S \) drawn from each class \( i \) \((i = 1, 2, ..., p)\). Note that the subset \( S \) is not supposed to satisfy any particular condition, not even of being connected.

Now, we observe that the number of edges in the boundary, say \( D(S) \), is given by the total number of edges starting from the vertices in \( S \), less the ones, say \( I(S) \), that are contained in \( S \), i.e. having both endpoints in \( S \). Thus we can estimate \( I \) and \( D \) as follows:

\[
I(S) = \sum_{i=1}^{p} \sum_{j=1}^{p} y_i n_i x_j n_j, \\
D(S) = \sum_{i=1}^{p} \sum_{j=1}^{p} y_i x_j n_i n_j - 2I(S) = 2\sum_{i=1}^{p} \sum_{j=1}^{p} y_i n_i x_j n_j - \sum_{i=1}^{p} x_i^2 n_i^2 - \sum_{j=1}^{p} y_j n_j^2, \\
\]

where \( \mathbf{x} = (x_1, x_2, ..., x_p)^T \), \( \mathbf{y} = (y_1, y_2, ..., y_p)^T \) and \( E \) is the total number of edges in the network.

Thus, \( h_G(S) \) becomes:

\[
h_G(S) = \frac{D(S)N}{|S|(|N - |S|)}.
\]

under the constraint that \( \mathbf{n}^T \mathbf{y} < N/2 \), where \( \mathbf{n} \) is the vector \((n_1, n_2, ..., n_p)^T\). A numerical optimization algorithm can then be used to find the subset \( S \) that minimizes \( h_G(\mathbf{y}, r_p) \) in terms of \( y_1, y_2, ..., y_p \) and, consequently \( r_p \) and, in turns, an upper bound for \( \lambda_2 \).

Also, from (3) and (5), we get:

\[
\frac{\partial h_G(S)}{\partial r_p} \propto \frac{\partial D(S)}{\partial r_p} = -2\sum_{i=1}^{p} \sum_{j=1}^{p} y_i n_i x_j n_j = -2E \sigma^2 \mathbf{m}^T (\mathbf{m})^T \leq 0.
\]

Since, for any vector \( \mathbf{y} \) (6) is satisfied, then we have that \( \frac{\partial h_G(S)}{\partial r_p} \leq 0 \). Therefore, we can predict analytically that \( h_G \) and hence \( \lambda_2 \) will be decreasing as the degree correlation is increased and, as a consequence, the degree eigenratio \( \lambda_N/\lambda_2 \) will increase for higher values of the correlation coefficient.

As shown in Fig. 2, this is indeed what is observed with the new bound on \( \lambda_2 \) giving a much better estimate of the behavior of the eigenratio with respect to both changes in the degree distribution and the degree correlation. (Note that \( \lambda_N \) is found to be almost independent from \( r \) in Fig. 1.)
where \( \lambda_N \) was found to be practically independent from variations of \( r \), we use a simulated annealing meta-heuristic technique [21], to solve the problem of maximising \( \lambda_2 \) while keeping unchanged the network degree distribution. Namely, given a network with a certain degree distribution, we perform the following iterative procedure. At each iteration \( t \), the endpoints of a randomly selected pair of edges are exchanged if \( \exp \left( \frac{\Delta(\lambda_2)}{T} \right) > z \); \( z \) being an uniformly distributed random variable between 0 and 1, \( \Delta(\lambda_2) \) the variation achieved in the objective function \( \lambda_2 \) before and after the execution of the move and \( T \) the system temperature.

As shown in Fig. 3, under the effects of this iterative procedure aimed at maximising \( \lambda_2 \), we observe the spontaneous emergence in the network of interest of negative degree correlation, namely the increasing of its disassortativity. Thus, following an entirely different approach, we come to the conclusion that if the degree distribution of a given network is fixed, then to improve its synchronizability one has to introduce negative degree correlation among its nodes.

In conclusion, we have shown that degree correlation among the nodes of a network of nonlinear oscillators does indeed have an effect on the Laplacian eigenratio and hence its synchronizability. Using a novel correlation index, we were able to show both analytically and numerically that the eigenratio is lower in disassortative networks. Hence we conjectured that in many evolutionary physical and biological networks of nonlinear oscillators, negative correlation among nodes can be an emerging property aimed at facilitating the synchronization process. An iterative nonlinear optimization technique was used to illustrate this latter point.

\[ \lambda_2 \geq k_{max} - \sqrt{k_{max}^2 - h_G'^2}, \quad (7) \]

where \( h'_G = \min_S \frac{D(S)}{|S|} = 2\mathcal{E}(y^Tq - y^TEy)/(n^Ty) \). Using (7), we can also get a lower bound on \( \lambda_2 \). Then following an approach similar to the one used to compute the upper bound, it is easy to show that the lower bound in (7) has to decrease with \( r \) (note that when making the correlation change, the degree distribution is fixed and thus, \( k_{max} \) cannot vary with \( r \)). Then since both the upper and the lower bounds have to decrease with \( r \), \( \lambda_2 \) is also expected to have the same trend.

In order to derive bounds on \( R \), we can use the novel upper and lower bounds on \( \lambda_2 \) and the bounds on \( \lambda_N \) (that was observed numerically not to depend on \( r \)) given in [20]:

\[ \frac{N}{N-1} k_{max} \leq \lambda_N \leq 2k_{max}. \quad (8) \]

In so doing, we easily get the following analytical bounds on the Laplacian eigenratio:

\[ \frac{N}{N-1} k_{max} \leq \lambda_N \leq \frac{2}{1 - \sqrt{1 - \frac{h_G'^2}{k_{max}^2}}}. \quad (9) \]

The comparison between these bounds computed as explained above and those proposed in [10] is shown in Fig. 2. We observe that the upper bounds in (9) provide better estimates of changes in the eigenratio under variations of \( \gamma \) and more importantly \( r \).

The main result of the derivation presented above is the finding that disassortative networks synchronize better. Note that such a derivation can be easily extended to weighted topologies, i.e. to the case in which the strength of the coupling of each vertex on its neighbors is rescaled by the vertex degree. We wish now to assess whether negative correlation can be thought of as an emerging property of networks with an assigned degree distribution in order to improve their synchronization. Since \( \lambda_N \) was found to be practically independent from variations of \( r \), we use a simulated annealing meta-heuristic technique [21], to solve the problem of maximising \( \lambda_2 \) while keeping unchanged the network degree distribution. Namely, given a network with a certain degree distribution, we perform the following iterative procedure. At each iteration \( t \), the endpoints of a randomly selected pair of edges are exchanged if \( \exp \left( \frac{\Delta(\lambda_2)}{T} \right) > z \); \( z \) being an uniformly distributed random variable between 0 and 1, \( \Delta(\lambda_2) \) the variation achieved in the objective function \( \lambda_2 \) before and after the execution of the move and \( T \) the system temperature.

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