Sequential Mechanisms for Multi-type Resource Allocation

Sujoy Sikdar\textsuperscript{1}, Xiaoxi Guo\textsuperscript{2}, Haibin Wang\textsuperscript{2}, Lirong Xia\textsuperscript{3}, and Yongzhi Cao\textsuperscript{4}

\textsuperscript{1}Binghamton University, ssikdar@binghamton.edu
\textsuperscript{2}Peking University, guoxiaoxi@pku.edu.cn, beach@pku.edu.cn
\textsuperscript{3}Rensselaer Polytechnic Institute, xial@cs.rpi.edu
\textsuperscript{4}Peking University, caoyz@pku.edu.cn

February 23, 2021

Abstract

Several resource allocation problems involve multiple types of resources, with a different agency being responsible for “locally” allocating the resources of each type, while a central planner wishes to provide a guarantee on the properties of the final allocation given agents’ preferences. We study the relationship between properties of the local mechanisms, each responsible for assigning all of the resources of a designated type, and the properties of a sequential mechanism which is composed of these local mechanisms, one for each type, applied sequentially, under \textit{lexicographic preferences}, a well studied model of preferences over multiple types of resources in artificial intelligence and economics. We show that when preferences are O-legal, meaning that agents share a common importance order on the types, sequential mechanisms satisfy the desirable properties of anonymity, neutrality, non-bossiness, or Pareto-optimality if and only if every local mechanism also satisfies the same property, and they are applied sequentially according to the order $O$. Our main results are that under $O$-legal lexicographic preferences, every mechanism satisfying strategyproofness and a combination of these properties must be a sequential composition of local mechanisms that are also strategyproof, and satisfy the same combinations of properties.

1 Introduction

Consider the example of a hospital where patients must be allocated surgeons and nurses with different specialties, medical equipment of different types, and a room Huh et al. (2013). This example illustrates multi-type
resource allocation problems (MTRAs), first introduced by Moulin (1995),
where there are \( p \geq 1 \) types of indivisible items which are not interchange-
able, and a group of agents having heterogeneous preferences over receiving
combinations of an item of each type. The goal is to design a mechanism
which allocates each agent with a bundle consisting of an item of each type.

Often, a different agency is responsible for the allocation of each type
of item in a distributed manner, using possibly different local mechanisms,
while a central planner wishes that the mechanism composed of these local
mechanisms satisfies certain desirable properties. For example, different
departments may be responsible for the allocation of each type of medical
resources, while the hospital wishes to deliver a high standard of patient care
and satisfaction given the patients’ preferences and medical conditions; in
an enterprise, clients have heterogeneous preferences over cloud computing
resources like computation and storage Ghodsi et al. (2011, 2012); Bhat-
tacharya et al. (2013), possibly provided by different vendors; in a university,
students must be assigned to different types of courses handled by different
departments; in a seminar class, the research papers and time slots Mackin
and Xia (2016) may be assigned separately by the instructor and a teaching
assistant respectively, and in rationing Elster (1992), different agencies
may be responsible for allocating different types of rations such as food and
shelter.

Unfortunately, as Svensson (1999) shows, even when there is a single type
of items and each agent is to be assigned a single item, serial dictatorships
are the only strategyproof mechanisms which are non-bossy, meaning that
no agent can falsely report her preferences to change the outcome without
also affecting her own allocation, and neutral, meaning that the outcome
is independent of the names of the items. In a serial dictatorship, agents
are assigned their favorite remaining items one after another according to a
fixed priority ordering of the agents. Papai (2001) shows a similar result for
the multiple assignment problem, where agents may be assigned more than
one item, that the only mechanisms which are strategyproof, non-bossy, and
Pareto-optimal are sequential dictatorships, where agents pick a favorite re-
maining item one at a time according to a hierarchical picking sequence,
where the next agent to pick an item depends only on the allocations made
in previous steps. Pareto-optimality is the property that there is no other
allocation which benefits an agent without making at least one agent worse
off. More recently, Hosseini and Larson (2019) show that even under lex-
icographic preferences, the only mechanisms for the multiple assignment
problem that are strategyproof, non-bossy, neutral and Pareto-optimal are
serial dictatorships with a quota for each agent.

Mackin and Xia (2016) study MTRAs in a slightly different setting to
ours: a monolithic central planner controls the allocation of all types of
items. They characterize serial dictatorships under the unrestricted domain
of strict preferences over bundles with strategyproofness, non-bossiness, and
type-wise neutrality, a weaker notion of neutrality where the outcome is independent of permutations on the names of items within each type. Perhaps in light of this and other negative results described above, there has been little further work on strategyproof mechanisms for MTRAs. This is the problem we address in this paper.

We study the design of strategyproof sequential mechanisms for MTRAs with \( p \geq 2 \) types, which are composed of \( p \) local mechanisms, one for each type, applied sequentially one after the other, to allocate all of the items of the type, under the assumption that agents’ preferences are lexicographic and \( O \)-legal.

For MTRAs, lexicographic preferences are a natural, and well-studied assumption for reasoning about ordering alternatives based on multiple criteria in social science Gigerenzer and Goldstein (1996). In artificial intelligence, lexicographic preferences have been studied extensively, for MTRAs Sikdar et al. (2017, 2019); Sun et al. (2015); Wang et al. (2020); Guo et al. (2020), multiple assignment problems Hosseini and Larson (2019); Fujita et al. (2015), voting over multiple issues Lang and Xia (2009); Xia et al. (2011), and committee selection Sekar et al. (2017), since lexicographic preferences allow reasoning about and representing preferences in a structured and compact manner. In MTRAs, lexicographic preferences are defined by an importance order over the types of items, and local preferences over items of each type. The preference relation over any pair of bundles is decided in favor of the bundle that has the more preferred item of the most important type at which the pair of bundles differ, and this decision depends only on the items of more important types.

In several problems, it is natural to assume that every agent shares a common importance order. For example, when rationing Elster (1992), it may be natural to assume that every agent thinks food is more important than shelter, and in a hospital Huh et al. (2013), all patients may consider their allocation of surgeons and nurses to be more important than the medical equipment and room. \( O \)-legal lexicographic preference profiles, where every agent has a common importance order \( O \) over the types, have been studied recently by Lang and Xia (2009); Xia et al. (2011) for the multi-issue voting problem. When agents’ preferences are \( O \)-legal and lexicographic, it is natural to ask the following questions about sequential mechanisms that decide the allocation of each type sequentially using a possibly different local mechanism according to \( O \), which we address in this paper: (1) if every local mechanism satisfies property \( X \), does the sequential mechanism composed of these local mechanisms also satisfy \( X \)?, and (2) what properties must every local mechanism satisfy so that their sequential composition satisfies property \( X \)?
1.1 Contributions

For $O$-legal preferences, a property $X \in \{\text{anonymity, type-wise neutrality, non-bossiness, monotonicity, Pareto-optimality}\}$, and any sequential mechanism $f_O = (f_1, \ldots, f_p)$ which applies each local mechanism $f_i$ one at a time according to the importance order $O$, we show in Theorem 1 and Theorem 2 that $f_O$ satisfies $X$ if and only if every local mechanism it is composed of satisfies $X$.

However, sequential compositions of locally strategyproof mechanisms are not guaranteed to be strategyproof, which raises the question: under what conditions are sequential mechanisms strategyproof? We begin by showing in Proposition 1, that when agents preferences are lexicographic, but agents have different importance orders, sequential mechanisms composed of locally strategyproof mechanisms are, unfortunately, not guaranteed to be strategyproof. In contrast, we show in Proposition 2 that sequential composition of strategyproof mechanisms are indeed strategyproof when either: (1) agents’ preferences are separable and lexicographic, even when different agents may have different importance orders, or (2) agents’ preferences are lexicographic and $O$-legal and all of the local mechanisms are also non-bossy.

Our main results characterize the class of mechanisms that satisfy strategyproofness, along with different combinations of non-bossiness, neutrality, and Pareto-optimality under $O$-legal preferences as $O$-legal sequential mechanisms. We show:

• In Theorem 3, that under $O$-legal lexicographic preferences, the class of mechanisms satisfying strategyproofness and non-bossiness is exactly the class of mechanisms that can be decomposed into multiple locally strategyproof and non-bossy mechanisms, one for each combination of type and allocated items of more important types. This class of mechanisms is exactly the class of $O$-legal conditional rule nets (CR-nets) Lang and Xia (2009);

• In Theorem 4, that a mechanism is strategyproof, non-bossy, and type-wise neutral if and only if it is an $O$-legal sequential composition of serial dictatorships;

• In Theorem 5, that a mechanism is strategyproof, non-bossy, and Pareto-optimal if and only if it is an $O$-legal CR-net composed of serial dictatorships. Finally, we show that despite the negative result in Proposition 1 that when agents’ preferences do not share a common importance order on the types, sequential compositions of locally strategyproof mechanisms may not satisfy strategyproofness, we show in Theorem 6, that computing beneficial manipulations w.r.t. a sequential mechanism is NP-complete.
2 Related Work and Discussion

The MTRA problem was introduced by Moulin (1995). More recently, it was studied by Mackin and Xia (2016), who characterize the class of strategyproof and non-bossy mechanisms under the unrestricted domain of strict preferences over bundles as the class of serial dictatorships. However, as they note, it may be unreasonable to expect agents to express preferences as complete rankings over all possible bundles, besides the obvious communication and complexity issues arising from agents’ preferences being represented by complete rankings.

The literature on characterizations of strategyproof mechanisms Svensson (1999); Pápai (2001); Hosseini and Larson (2019) for resource allocation problems belong to the line of research initiated by the famous Gibbard-Satterthwaite Theorem Gibbard (1973); Satterthwaite (1975) which showed that dictatorships are the only strategyproof voting rules which satisfy non-imposition, which means that every alternative is selected under some preference profile. Several following works have focused on circumventing these negative results by identifying reasonable and natural restrictions on the domain of preferences. For voting, Moulin (1980) provide non-dictatorial rules satisfying strategyproofness and non-imposition under single-peaked Black (1948) preferences. Our work follows in this vein and is closely related to the works by Le Breton and Sen (1999), who assume that agents’ preferences are separable, and more recently, Lang and Xia (2009) who consider the multi-issue voting problem under the restriction of O-legal lexicographic preferences, allowing for conditional preferences given by CP-nets similar to our work. Xia and Conitzer (2010) consider a weaker and more expressive domain of lexicographic preferences allowing for conditional preferences. Here, agents have a common importance order on the issues, and the agents preferences over any issue is conditioned only on the outcome of more important issues. They characterize the class of voting rules satisfying strategyproofness and non-imposition as being exactly the class of all CR-nets. CR-nets define a hierarchy of voting rules, where the voting rule for the most important issue is fixed, and the voting rule for every subsequent issue depends only on the outcome of the previous issues. Similar results were shown earlier by Barbera et al. (1993, 1997, 1991).

In a similar vein, Sikdar et al. (2017, 2019) consider the multi-type variant of the classic housing market Shapley and Scarf (1974), first proposed by Moulin (1995), and Fujita et al. (2015) consider the variant where agents can receive multiple items. These works circumvent previous negative results on the existence of strategyproof and core-selecting mechanisms under the assumption of lexicographic extensions of CP-nets, and lexicographic preferences over bundles consisting of multiple items of a single type respectively. Wang et al. (2020); Guo et al. (2020) study MTRAs with divisible and indivisible items, and provide mechanisms that are fair and efficient under
the notion of stochastic dominance by extending the famous probabilistic serial Bogomolnaia and Moulin (2001) and random priority Abdulkadiroğlu and Sönmez (1998) mechanisms, and show that while their mechanisms do not satisfy strategyproof in general, under the domain restriction of lexicographic preferences, strategyproofness is restored, and stronger notions of efficiency can be satisfied.

3 Preliminaries

A multi-type resource allocation problem (MTRA) Mackin and Xia (2016), is given by a tuple \((N, M, P)\). Here, (1) \(N = \{1, \ldots, n\}\) is a set of agents, (2) \(M = D_1 \cup \cdots \cup D_p\) is a set of items of \(p\) types, where for each \(i \leq p\), \(D_i\) is a set of \(n\) items of type \(i\), and (3) \(P = (\succ_j)_{j \leq n}\) is a preference profile, where for each \(j \leq n\), \(\succ_j\) represents the preferences of agent \(j\) over the set of all possible bundles \(D = D_1 \times \cdots \times D_p\). For any type \(i \leq p\), we use \(k_i\) to refer to the \(k\)-th item of type \(i\), and we define \(T = \{D_1, \ldots, D_p\}\). We also use \(D_{<i}\) to refer to the set of \(\{D_1, \ldots, D_{i-1}\}\), \(D_{>i}\) refers to \(\{D_{i+1}, \ldots, D_p\}\), and \(D_{\leq i}, D_{\geq i}\) are in the same manner. For any profile \(P\), and agent \(j \leq n\), we define \(P^{-j} = (\succ_k)_{k \leq n, k \neq j}\), and \(P = (P^{-j}, \succ_j)\).

Bundles. Each bundle \(x \in D\) is a \(p\)-vector, where for each type \(i \leq p\), \([x]_i\) denotes the item of type \(i\). We use \(a \in x\) to indicate that bundle \(x\) contains item \(a\). For any \(S \subseteq T\), we define \(D_S = \times_{D \in S} D\), and \(-S = T \setminus S\). For any \(S \subseteq T\), any bundle \(x \in D_S\), for any \(D \in -S\), and item \(a \in D\), \((a, x)\) denotes the bundle consisting of \(a\) and the items in \(x\), and similarly, for any \(U \subseteq -S\), and any bundle \(y \in D_U\), we use \((x, y)\) to represent the bundle consisting of the items in \(x\) and \(y\). For any \(S \subseteq T\), we use \(x_{\perp S}\) to denote the items in \(x\) restricted to the types in \(S\).

Allocations. An allocation \(A : N \rightarrow D\) is a one-to-one mapping from agents to bundles such that no item is assigned to more than one agent. \(A\) denotes the set of all possible allocations. Given an allocation \(A \in A\), \(A(j)\) denotes the bundle allocated to agent \(j\). For any \(S \subseteq T\), we use \(A_{\perp S} : N \rightarrow D_S\) to denote the allocation of items restricted to the types in \(S\).

CP-nets and \(O\)-legal Lexicographic Preferences. An acyclic CP-net Boutilier et al. (2004) \(N\) over \(D\) is defined by (i) a directed graph \(G = (T, E)\) called the dependency graph, and (ii) for each type \(i \leq p\), a conditional preference table \(CPT(D_i)\) that contains a linear order \(\succ^x\) over \(D_i\) for each \(x \in D_{Pa(D_i)}\), where \(Pa(D_i)\) is the set of types corresponding to the parents of \(D_i\) in \(G\). A CP-net \(N\) represents a partial order over \(D\) which is the transitive closure of the preference relations represented by all
Figure 1: An $O$-legal lexicographic preference with an underlying CP-net, where $O = [D_1 \triangleright D_2]$. of the CPT entries which are $\{(a_i, u, z) \succ (b_i, u, z) : i \leq p; a_i, b_i \in D_i, u \in D_{Pa(D_i)}; z \in D_{-Pa(D_i)}\{D_i}\}$. Let $O = [D_1 \triangleright \cdots \triangleright D_p]$ be a linear order over the types. A CP-net is $O$-legal if there is no edge $(D_i, D_l)$ with $i > l$ in its dependency graph. A lexicographic extension of an $O$-legal CP-net $N$ is a linear order $\succ$ over $D$, such that for any $i \leq p$, any $x \in D_{D_{<i}^D}$, any $a_i, b_i \in D_i$, and any $y, z \in D_{D_{\geq i}^D}$, if $a_i \succ^x b_i$ in $N$, then, $(x, a_i, y) \succ (x, b_i, z)$. The linear order $O$ over types is called an importance order, and $D_1$ is the most important type, $D_2$ is the second most important type, etc. We use $O$ to denote the set of all possible importance orders over types.

Given an important order $O$, we use $L_O$ to denote the set of all possible linear orders that can be induced by lexicographic extensions of $O$-legal CP-nets as defined above. A preference relation $\succ \in L_O$ is said to be an $O$-legal lexicographic preference relation, and a profile $P \in L_O^p$ is an $O$-legal lexicographic profile. An $O$-legal preference relation is separable, if the dependency graph of the underlying CP-net has no edges. We will assume that all preferences are $O$-legal lexicographic preferences throughout this paper unless specified otherwise.

Example 1. Here we show how to compare bundles under an $O$-legal lexicographic preference with CP-net. In Figure 1(a) is a dependency graph which shows that $D_2$ depends on $D_1$. Figure 1(b) is the CPT for both types, which implies $(1_1, 2_2) \succ (1_1, 2_2), (2_1, 2_2) \succ (2_1, 2_2)$. Figure 1(c) gives the importance order $O = [D_1 \triangleright D_2]$. With $O$ we can compare some bundles directly. For example, $(1_1, 2_2) \succ (2_1, 1_2), (1_1, 1_2) \succ (2_1, 2_2)$ because the most important type with different allocations is $D_1$ and $1_1 \succ^\emptyset 2_1$. Finally, Figure 1(d) shows the relations among all the bundles.

We note that any lexicographic extension of an $O$-legal CP-net according to the order $O$ does not violate any of the relations induced by the original CP-net, and always induces a linear order over all possible bundles unlike
CP-nets which may induce partial orders.

For any $O$-legal lexicographic preference relation $\succ$ over $D_i$ and given any $x \in D_{D_{\leq i}}$, we use $\succ_{\perp D_i,x}$ as the projection of the relation $\succ$ over $D_i$ given $x$, and $\succ_{\perp D_{\geq i},x}$ as the projection of $\succ$ over $\{ (x,z) : z \in D_{D_{\geq i}} \}$. For convenience, given an allocation $A$, for any $i \leq p$, we define $\succ_{\perp D_i,A}$ and $\succ_{\perp D_{\geq i},A}$ similarly, where the preferences are projected based on the allocation of items of types that are more important than $i$, and given an $O$-legal lexicographic profile $P$, we define $P_{\perp D_i,A}$ and $P_{\perp D_{\geq i},A}$ similarly, by projecting the preferences of every agent. We just leave out $x$ (and similarly, $A$) if $i = 1$. We use $D_{-i}$ to stand for the set of all types except $D_i$.

**Sequential and Local Mechanisms.** An allocation mechanism $f : \mathcal{P} \rightarrow \mathcal{A}$ maps $O$-legal preference profiles to allocations. Given an importance order $O = [D_1 \triangleright \cdots \triangleright D_p]$, an $O$-legal sequential mechanism $f_O = (f_1, \ldots, f_p)$ is composed of $p$ local mechanisms, that are applied one after the other in $p$ rounds, where in each round $i \leq p$, a local mechanism $f_i$ allocates all of the items of $D_i$ given agents’ projected preferences over $D_i$ conditioned on the partial allocation in previous rounds.

**Desirable Properties.** An allocation mechanism $f$ satisfies:

- *anonymity*, if for any permutation $\Pi$ on the names of agents, and any profile $P$, $f(\Pi(P)) = \Pi(f(P))$;

- *type-wise neutrality*, if for any permutation $\Pi = (\Pi_1, \ldots, \Pi_p)$, where for any $i \leq p$, $\Pi$ only permutes the names of the items of type $i$ according to a permutation $\Pi_i$, and any profile $P$, $f(\Pi(P)) = \Pi(f(P))$;

- *Pareto-optimality*, if for every allocation $A$ such that there exists an agent $j$ such that $A(j) \succ_j f(P)(j)$, there is another agent $k$ such that $f(P)(k) \succ_j A(k)$.

- *non-bossiness*, if no agent can misreport her preferences and change the allocation of other agents without also changing her own allocation, i.e. there does not exist any pair $(P, \succ_j')$ where $P$ is a profile and $\succ_j'$ is the misreported preferences of agent $j$ such that $f(P)(j) = f(P_{-j}, \succ_j')(j)$ and for some agent $k \neq j$, $f(P)(k) \neq f(P_{-j}, \succ_j')(k)$.

- *non-bossiness of more important types*, if no agent $j$ can misreport her local preferences for less important types and change the allocation of more important types to other agents without also changing her own allocation of more important types. i.e. for every profile $P$, every agent $j \leq n$, every type $D_i, i \leq p$, and every misreport of agent $j$’s preferences $\succ_j'$ where for every $h < i$, every $u \in Par(D_h)$, $\succ_j'_{\perp D_h,u} = \succ_j_{\perp D_h,u}$, it holds that if for some agent $k \neq j$, $f(P_{-j}, \succ_j')(k)_{\perp D_{\leq i}} \neq f(P)(k)_{\perp D_{\leq i}}$, then $f(P_{-j}, \succ_j')(j)_{\perp D_{\leq i}} \neq f(P)(j)_{\perp D_{\leq i}}$.
**monotonicity**, for any agent \( j \), any profile \( P \), let \( \succ'_j \) be a misreport preference such that if \( Y \subseteq D \) is the set of all bundles whose ranks are raised and it holds that for every \( x, z \in Y, x \succ_j z \implies x \succ'_j z \), then, \( f(P_{-j}, \succ'_j)(j) \in \{ f(P)(j) \} \cup Y \).

- **strategyproofness**, if no agent has a beneficial manipulation, i.e. there is no pair \((P, \succ'_j)\) where \( P \) is a profile and \( \succ'_j \) is a manipulation of agent \( j \)'s preferences such that \( f(P_{-j}, \succ'_j)(j) \succ_j f(P)(j) \).

### 4 Properties of Sequential Mechanisms Under Lexicographic Preferences

**Theorem 1.** For any importance order \( O \in \mathcal{O} \), any \( X \in \{ \text{anonymity, type-wise neutrality, non-bossiness, monotonicity, Pareto-optimality} \} \), and \( f_O = (f_1, \ldots, f_p) \) be any \( O \)-legal sequential mechanism. Then, for \( O \)-legal preferences, if for every \( i \leq p \), the local mechanism \( f_i \) satisfies \( X \), then \( f_O \) satisfies \( X \).

**Proof.** (Sketch) Throughout, we will assume that \( O = [D_1 \succ \cdots \succ D_p] \), and that \( P \) is an arbitrary \( O \)-legal preference profile over \( p \) types. For any \( i \leq p \), we define \( g_i \) to be the sequential mechanism \( (f_1, \ldots, f_i) \). The proofs of anonymity and type-wise neutrality are relegated to the appendix.

**non-bossiness.** Let us assume for the sake of contradiction that the claim is false, i.e. there exists a profile \( P \), an agent \( j \) and a misreport \( \succ'_j \) such that for \( P' = (\succ_{-j}, \succ'_j) \), \( f_O(P)(j) = f_O(P')(j) \), and \( f_O(P) \neq f_O(P') \). Then, there is a type \( i \leq p \) such that, \( f_O(P)\bot_{D_{<i}} = f_O(P')\bot_{D_{<i}} \) and \( f_O(P)\bot_{D_i} \neq f_O(P')\bot_{D_i} \). Let \( A = f_O(P)\bot_{D_{<i}} \). Then, there is an agent \( k \) such that \( f_i(P\bot_{D_{<i}})(k) \neq f_i(P'\bot_{D_{<i}})(k) \). By the choice of \( i \), and the assumption that every other agent reports preferences truthfully, \( \succ_{-j\bot_{D_i}} \neq \succ'_j\bot_{D_i} \). Then, \( f_i(\succ_{-j\bot_{D_i}})(j) = f_i(\succ_{-j\bot_{D_i}})(j) = f_i(\succ_{-j\bot_{D_i}})(j) \neq f_i(\succ_{-j\bot_{D_i}})(j) \), a contradiction to our assumption that \( f_i \) is non-bossy.

**monotonicity.** Let \( P' = (P_{-j}, \succ'_j) \) be an \( O \)-legal profile obtained from \( P \) and \( Y \subseteq D \) is the set of bundles raising the ranks in \( P' \) such that the relative rankings of bundles in \( Y \) are unchanged in \( P \) and \( P' \). For any \( Y \subseteq D \), and any \( u \in D_{<i} \), let \( Y^{D_i|u} = \{ x_i : x \in Y, x_h = u_h \text{ for all } h \leq i \} \). It is easy to see that if \( x_1 = f_O(P')(j)\bot_{D_{<i}} \), then it follows from strong monotonicity of \( f_1 \) that \( x_1 \in f_O(P)(j)\bot_{D_{<i}} \cup Y^{D_1} \). Now, either \( x_1 \neq f_O(P)(j)\bot_{D_{<i}} \), or \( x_1 = f_O(P)(j)\bot_{D_{<i}} \). Suppose \( x_1 \neq f_O(P)(j)\bot_{D_{<i}} \). Then, by strong monotonicity of \( f_1 \), \( x_1 \succ f_O(P)(j)\bot_{D_{<i}} \). Then, by our assumption of \( O \)-legal lexicographic preferences, for any \( z \in D_{<i} \), \((x_1, z) \in Y \). Therefore, \( f_O(P')(j) \in Y \). Suppose \( x_1 = f_O(P)(j)\bot_{D_{<i}} \),
then by a similar argument, \( f_O(P')(j)_{\perp \{D_2\}} \in \{ f_O(P)(j)_{\perp \{D_2\}} \} \cup Y_{\lceil x \rceil}^{D_2} \). Applying our argument recursively, we get that \( f_O(P')(j) \in \{ f_O(P)(j) \} \cup Y \).

**Pareto-optimality.** Suppose the claim is true for \( p \leq k \) types. Let \( P \) be an \( O \)-legal lexicographic profile over \( k + 1 \) types, and \( f_O = (f_i)_{i \leq k+1} \) is a sequential composition of Pareto-optimal local mechanisms. Suppose for the sake of contradiction that there exists an allocation \( B \) such that some agents strictly better off compared to \( f_O(P) \), and no agent is worse off. Then, by our assumption of lexicographic preferences, for every agent \( k \) who is not strictly better off, \( B(k) = f_O(P)(k) \), and for every agent \( j \) who is strictly better off, one of two cases must hold. (1) \( B(j)_{\perp D_1} \succ_j f_O(P)(j)_{\perp D_1} \), or (2) \( B(j)_{\perp D_1} = f_O(P)(j)_{\perp D_1} \). (1): If there exists an agent such that \( B(j)_{\perp D_1} \succ_j f_O(P)(j)_{\perp D_1} \), this is a contradiction to our assumption that \( f_1 \) is Pareto-optimal. (2): Suppose \( B(j)_{\perp D_1} = f_O(P)(j)_{\perp D_1} \) for all agents who are strictly better off. Let \( g = (f_2, \ldots, f_{k+1}) \). W.l.o.g. let agent 1 strictly prefer \( B(1) \) to \( f_O(P)(1) \). Then, \( g(P_{\perp D_{\leq k+1}} \setminus D_1, f_O(P)_{\perp D_1})_{\perp D_1} \succ 1 B(1)_{\perp D_{\leq k+1}} \setminus D_1 \). and for every other agent \( l \neq 1 \), either \( g(P_{\perp D_{\leq k+1}} \setminus D_1, f_O(P)_{\perp D_1})_{\perp D_1} \succ l B(l)_{\perp D_{\leq k+1}} \setminus D_1 \), or \( g(P_{\perp D_{\leq k+1}} \setminus D_1, f_O(P)_{\perp D_1})_{\perp D_1} = B(l)_{\perp D_{\leq k+1}} \setminus D_1 \), which is a contradiction to our induction assumption.

\( \square \)

**Theorem 2.** For any importance order \( O \in \mathcal{O} \), \( X \in \{ \text{anonymity, type-wise neutrality, non-bossiness, monotonicity, Pareto-optimality} \} \), and \( f_O = (f_1, \ldots, f_p) \) be any \( O \)-legal sequential mechanism. For \( O \)-legal preferences, if \( f_O \) satisfies \( X \), then for every \( i \leq p \), \( f_i \) satisfies \( X \).

**Proof.** (Sketch) We only provide the proof of non-bossiness here. The rest of the proofs are in the appendix.

**non-bossiness.** Assume for the sake of contradiction that \( k \leq p \) is the most important type such that \( f_k \) does not satisfy non-bossiness. Then, there exists a preference profile \( Q = (\succ^k)_{j \leq n} \) over \( D_k \), and a bossy agent \( l \) and a misreport \( Q' = (\succ^k, \succ^k_{l < l}) \), such that \( f_k(Q')(l) = f_k(Q)(l) \), but \( f_k(Q') \neq f_k(Q) \). Now, consider the \( O \)-legal separable lexicographic profile \( P \), where for any type \( i \leq p \), the preferences over type \( D_i \) is denoted \( P_{\perp D_i} \) and \( P_{\perp D_k} = Q \), and the profile \( P' \) obtained from \( P \) by replacing \( \succ_i \) with \( \succ_i^l \), which in turn is obtained from \( \succ_i \) by replacing \( \succ_{l \perp D_k} \) with \( \succ_{l i}^k \). It is easy to see that \( f_O(P')_{\perp D_{\leq k}} = f_O(P)_{\perp D_{< k}} \), and \( f_O(P')(l)_{\perp D_k} = f_O(P)(l)_{\perp D_k} \), but \( f_O(P')_{\perp D_k} \neq f_O(P)_{\perp D_k} \), and by our assumption of separable preferences, \( f_O(P')(l) \neq f_O(P)(l) \), implying that \( f_O \) does not satisfy non-bossiness, which is a contradiction. \( \square \)

## 5 Strategyproofness of Sequential Mechanisms

A natural question to ask is whether it is possible to design strategyproof sequential mechanisms when preferences are lexicographic, but each agent
$j \leq n$ may have a possibly different importance order $O_j \in \mathcal{O}$ over the types, and their preference over $D$ is $O_j$-legal and lexicographic. A sequential mechanism applies local mechanisms according to some importance order $O \in \mathcal{O}$ and is only well defined for $O$-legal preferences. When preferences are not $O$-legal, it is necessary to define how to project agents’ preferences given a partial allocation when a sequential mechanism is applied. Consider an agent $j$ with $O_j$-legal lexicographic preferences, and a partial allocation $A \perp S$ for some $S \subseteq T$, which allocates $x \in D_S$ to $j$. A natural question to ask is how should agent $j$’s preferences be interpreted over a type $D_i$ which has not been allocated yet. We define two natural ways in which agents may wish their preferences to be interpreted. We say that an agent is optimistic, if for any type $D_i \notin S$, and any pair of items $a_i, b_i \in D_i$, $a_i \succ b_i$ if and only if according to their original preferences $\sup \{y \in D : y_k = x_k \text{ for every } D_k \in S, y_i = a_i\} \succ \sup \{y \in D : y_k = x_k \text{ for every } D_k \in S, y_i = b_i\}$. Similarly, an agent is pessimistic, if for any type $D_i \notin S$, and any pair of items $a_i, b_i \in D_i$, $a_i \succ b_i$ if and only if $\inf \{y \in D : y_k = x_k \text{ for every } D_k \in S, y_i = a_i\} \succ \inf \{y \in D : y_k = x_k \text{ for every } D_k \in S, y_i = b_i\}$.

**Proposition 1.** For any importance order $O \in \mathcal{O}$, when the preferences are not $O$-legal, and agents are either optimistic or pessimistic, a sequential mechanism $f_O$ composed of strategyproof mechanisms is not necessarily strategyproof.

**Proof.** When preferences are lexicographic, and not $O$-legal, a sequential mechanism composed of locally strategyproof mechanisms is not necessarily strategyproof, when agents are either optimistic or pessimistic, as we show with counterexamples. Consider the profile with two agents and two types $H$ and $C$. Agent 1’s importance order is $H \triangleright C$, preferences over $H$ is $1\_H \succ 2\_H$ and over $C$ is conditioned on the assignment of house $1\_H : 1\_C \succ 2\_C, 2\_H : 2\_C \succ 1\_C$. Agent 2 has importance order $C \triangleright H$ and separable preferences with order on cars being $2\_C \succ 1\_C$, and order on houses $1\_H \succ 2\_H$. Consider the sequential mechanism composed of serial dictatorships where $H \triangleright C$ and for houses the picking order over agents is $(2,1)$, and for cars $(1,2)$. When agents are truthful and either optimistic or pessimistic, the allocation is $2\_H 2\_C$ and $1\_H 1\_C$ respectively to agents 1 and 2. When agent 2 misreports her preferences over houses as $2\_H \succ 1\_H$, and agent 1 is truthful and either optimistic or pessimistic, the allocation is $1\_H 1\_C$ and $2\_H 2\_C$ to agents 1 and 2 respectively, a beneficial misreport for agent 2.

In contrast, sequential mechanisms composed of locally strategyproof mechanisms are guaranteed to be strategyproof under two natural restrictions on the domain of lexicographic preferences: (1) when agents’ preferences are lexicographic and separable, but not necessarily $O$-legal w.r.t. a common importance order $O$, and (2) when agents have $O$-legal lexicographic preferences, and the local mechanisms also satisfy non-bossiness.
Proposition 2. For any importance order $O \in \mathcal{O}$, a sequential mechanism composed of strategyproof local mechanisms is strategyproof,

(1) when agents are either optimistic or pessimistic, and their preferences are separable and lexicographic, or

(2) when agents’ preferences are lexicographic and $O$-legal and the local mechanisms also satisfy non-bossiness.

Proof. (1): Let $P$ be a profile of separable lexicographic preferences. Suppose for the sake of contradiction that an agent $j$ has a beneficial misreport $\succ'_j$, and let $P' = (P_{-j}, \succ'_j)$. Let $k$ be the type of highest importance to $j$ for which $[f_O(P')(j)]_k \neq [f_O(P)(j)]_k$. Then, by our assumption that preferences are lexicographic, $k$ being the most important type for $j$ where her allocated item differs, and that $P'$ is a beneficial manipulation, it must hold that $[f_O(P')(j)]_k \succ [f_O(P)(j)]_k$. Since, preferences are separable, $[f(P')]_k = f_k(P'_\perp(D_k))$. Since every other agent is truthful, $P'_\perp(D_k) = (P_{-j\perp(D_k)}, \succ'_j \perp D_k)$, and $\gamma'_j \perp D_k \neq \gamma_j \perp D_k$ is a beneficial manipulation, which implies that $f_k$ is not strategyproof, a contradiction to our assumption.

(2) Now, we consider the case where the profile of truthful preferences $P$ is an arbitrary $O$-legal and lexicographic profile of preferences that may not be separable, and the local mechanisms are non-bossy and strategyproof. Suppose for the sake of contradiction that an agent $j$ has a beneficial misreport $\succ'_j$, and let $P' = (P_{-j}, \succ'_j)$. W.l.o.g. let $O = [1 \triangleright \cdots \triangleright p]$.

Let $k$ be the most important type for which agent $j$ receives a different item. We begin by showing that by our assumption that the local mechanisms are non-bossy, and our assumption of $O$-legal lexicographic preferences, it holds that for every $i < k$ according to $O$, $f_i(P'_\perp D_h) = f_i(P_\perp D_h)$. For the sake of contradiction, let $h < k$ be the first type for which some agent $l$ receives a different item, i.e. $[f(P')(l)]_h \neq [f(P)(l)]_h$, and $f(P')_\perp D_{ch} = f(P)_\perp D_{ch}$. Then, by our assumption of $O$-legal lexicographic preferences, and every other agent reporting truthfully, $P'_\perp D_h = f(P')_\perp D_{ch} = (P_{-j\perp D_h} f(P')_\perp D_{ch}, \succ'_j \perp D_h, f(P')_\perp D_{ch})$. By minimality of $k$, we know that $f_h(P')(j)_{\perp D_h} = f_h(P)(j)_{\perp D_h}$. But, $f_h(P')(l)_{\perp D_h} \neq f_h(P)(l)_{\perp D_h}$, which implies that $f_h$ does not satisfy non-bossiness, which is a contradiction.

Now, by minimality of $k$ and our assumption that preferences are $O$-legal and lexicographic and that $k$ is the most important type for which any agents’ allocation changes as we just showed, it must hold that $[f(P')(j)]_k = f_k(P'_\perp D_h f(P')_\perp D_{ch})(j) \succ f_k(P_{-j\perp D_h} f(P)_\perp D_{ch})(j) = [f(P)(j)]_k$. However, $f(P')_\perp D_{ch} = f(P)_\perp D_{ch}$, and $P'_j \perp D_h f(P')_\perp D_{ch} = P_{-j \perp D_h} f(P'_\perp D_{ch})$. This implies that $f_k$ is not strategyproof, which is a contradiction. 

Having established that it is possible to design strategyproof sequential mechanisms, we now turn our attention to strategyproof sequential
mechanisms that satisfy other desirable properties such as non-bossiness, neutrality, monotonicity, and Pareto-optimality. In Theorem 3, we show that under $O$-legal preferences, a mechanism satisfies strategyproofness and non-bossiness of more important types if and only if it is an $O$-legal CR-net composed of mechanisms that satisfy the corresponding counterparts of these properties for allocating items of a single type, namely, local strategyproofness and non-bossiness.

**Definition 1.** (CR-net) A (directed) conditional rule net (CR-net) $M$ over $D$ is defined by

(i) a directed graph $G = (\{D_1, ..., D_p\}, E)$, called the dependency graph, and

(ii) for each $D_i$, there is a conditional rule table $CRT_i$ that contains a mechanism denoted $M_{\perp D_i, A}$ for $D_i$ for each allocation $A$ of all items of types that are parents of $D_i$ in $G$, denoted $Pa(D_i)$.

Let $O = [D_1 \triangleright \cdots \triangleright D_p]$, then a CR-net is $O$-legal if there is no edge $(D_i, D_l)$ in its dependency with $i > l$.

**Example 2.** We note that the local mechanisms in a CR-net may be any mechanism that can allocate $n$ items to $n$ agents given strict preferences. In Figure 2, we show a CR-net $f$ where all the local mechanisms are serial dictatorships. The directed graph is shown in Figure 2(a), which implies $D_2$ depends on $D_1$. Figure 2(b) shows the CRT of $f$. In the CRT, $f_1 : (b, a)$ means that in the serial dictatorship $f_1$, agent $b$ picks her most preferred item first followed by agent $a$, and it is similar for $f_2, f'_2$. The conditions in the CR-net, which are partial allocations are represented by mappings, for example, $(a \rightarrow 1_1, b \rightarrow 2_1)$ means agent $a$ gets $2_1$. Figure 2 (c) and (d) are the $O$-legal preferences of agents $a$ and $b$, respectively, where $O = [D_1 \triangleright D_2]$. According to $f$, first we apply $f_1$ on $D_1$, and we have $a \rightarrow 1_1, b \rightarrow 2_1$. Then,
by CRT of \( f \) we use \( f_2 \) for \( D_2 \), and we have \( a \to 1_2, b \to 2_2 \). Therefore \( f \) outputs an allocation where \( a \to (1_1, 1_2), b \to (2_1, 2_2) \).

**Lemma 1.** When agents’ preferences are restricted to the O-legal lexicographic preference domain, for any strategyproof mechanism \( f \), any profile \( P \), and any pair \((P_{-j}, \succ'_j)\) obtained by agent \( j \) misreporting her preferences by raising the rank of \( f(P)(j) \) such that for any bundle \( b \), \( f(P)(j) \succ_j b \implies f(P)(j) \succ'_j b \), it holds that \( f(P_{-j}, \succ'_j)(j) = f(P)(j) \).

**Proof.** Suppose for the sake of contradiction that \( f \) is a strategyproof mechanism that does not satisfy monotonicity. Let \( P = (\succ_j)_{j \leq n} \) be a profile, \( j \) be an agent who misreports her preferences as \( \succ'_j \) obtained from \( \succ_j \) by raising the rank of \( f(P)(j) \), specifically, for any bundle \( b \), \( f(P)(j) \succ_j b \implies f(P)(j) \succ'_j b \). Then, either: (1) \( f(P_{-j}, \succ'_j)(j) \succ'_j f(P)(j) \), or (2) \( f(P)(j) \succ'_j f(P_{-j}, \succ'_j)(j) \).

(1) Suppose \( f(P_{-j}, \succ'_j)(j) \succ'_j f(P)(j) \). First, we claim that if \( f(P_{-j}, \succ'_j)(j) \succ'_j f(P)(j) \), then \( f(P_{-j}, \succ'_j)(j) \succ_j f(P)(j) \). Suppose for the sake of contradiction that this were not true, then \( f(P)(j) \succ_j f(P_{-j}, \succ'_j)(j) \) and \( f(P_{-j}, \succ'_j)(j) \succ'_j f(P)(j) \). This is a contradiction to our assumption on \( \succ'_j \). This implies that \( f(P_{-j}, \succ'_j)(j) \succ_j f(P)(j) \) and \( \succ'_j \) is a beneficial misreport for agent \( j \), a contradiction to our assumption that \( f \) is strategyproof.

(2) If \( f(P)(j) \succ'_j f(P_{-j}, \succ'_j)(j) \), then \( \succ_j \) is a beneficial misreport for agent \( j \) w.r.t. \( P' \), a contradiction to our assumption that \( f \) is strategyproof. \( \square \)

**Theorem 3.** For any importance order \( O \), a mechanism satisfies strategyproofness and non-bossiness of more important types under the O-legal lexicographic preference domain if and only if it is an O-legal locally strategyproof and non-bossy CR-net.

**Proof.** The if part is obvious (and is proved in Proposition 2). We prove the only if part by induction.

**Claim 1.** If an allocation mechanism satisfies non-bossiness of more important types and strategyproofness, then it can be decomposed into a locally strategyproof and non-bossy CR-net.

Proof by induction on the number of types. The claim is trivially true for the base case with \( p = 1 \) type. Suppose the claim holds true for \( p = k \) types i.e. when there are at most \( k \) types, if an allocation mechanism is non-bossy in more important types and strategyproof, then it can be decomposed into locally strategyproof and non-bossy mechanisms.

When \( p = k + 1 \), we prove that any non-bossy and strategyproof allocation mechanism \( f \) for a basic type-wise allocation problem can be decomposed into two parts by Step 1:
1. Applying a local allocation mechanism $f_1$ to $D_1$ to compute allocation $[A]_1$.

2. Applying an allocation mechanism $f_{\perp D_{-1},[A]_1}$ to types $D_{-1}$.

- **Step 1.** For any strategyproof allocation mechanism satisfying non-bossiness of more important types, allocations for type 1 depend only on preferences restricted to $D_1$.

**Claim 2.** For any pair of profiles $P = (\succ_j)_{j \leq n}, Q = (\succ_j')_{j \leq n}$, and $P_{\perp D_1} = Q_{\perp D_1}$, we must have that $f(P)_{\perp D_1} = f(Q)_{\perp D_1}$.

**Proof.** Suppose for sake of contradiction that $f(P)_{\perp D_1} \neq f(Q)_{\perp D_1}$. For any $0 \leq j \leq n$, define $P_j = (\succ_1, \ldots, \succ_j', \succ_{j+1}, \ldots, \succ_n)$ and suppose $f(P_j)_{\perp D_1} \neq f(P_{j+1})_{\perp D_1}$ for some $j \leq n - 1$. Let $[A]_1 = f(P_j)(j+1)_{\perp D_1}$ and $[B]_1 = f(P_{j+1})(j+1)_{\perp D_1}$. Now, suppose that

Case 1: $[A]_1 = [B]_1$, but for some other agent $\hat{j}$, $f(P_j)(\hat{j})_{\perp D_1} \neq f(P_{j+1})(\hat{j})_{\perp D_1}$. This is a direct violation of non-bossiness of more important types because $P_{j \perp D_1} = P_{j+1 \perp D_1}$ by construction.

Case 2: $[A]_1 \neq [B]_1$. If $[B]_1 \succ_{j+1 \perp D_1} [A]_1$, then $(P_j, \succ_{j+1})$ is a beneficial manipulation due to agents’ lexicographic preferences. Otherwise, if $[A]_1 \succ_{j+1 \perp D_1} [B]_1$, then $(P_{j+1}, \succ_{j+1})$ is a beneficial manipulation due to our assumption that $\succ_{j+1 \perp D_1} = \succ_{j+1 \perp D_1}$ and agents’ lexicographic preferences. This contradicts the strategyproofness of $f$.

- **Step 2.** Show that $f_1$ satisfies strategyproofness and non-bossiness.

First, we show that $f_1$ must satisfy strategyproofness by contradiction. Suppose for the sake of contradiction that $f$ is strategyproof but $f_1$ is not strategyproof. Let $P = (\succ_j)_{j \leq n}$ be a profile of agents’ preferences over $D_1$. Then, there exists an agent $j^*$ with a beneficial manipulation $\succ_{j^*}$. Now, consider a profile $Q = (\succ_j)_{j \leq n}$ where for every agent $j, \succ_{j \perp D_1} = \succ_j$ and the mechanism $f$ whose local mechanism for $D_1$ is $f_1$. We know from Step 1 that $f(Q)_{\perp D_1} = f_1(Q_{\perp D_1}) = f_1(P)$. However, in that case, because agents’ preferences are lexicographic with $D_1$ being the most important type, agent $j^*$ has a successful manipulation $\succ_{j^*}'$, where $\succ_{j^*} \perp D_1 = \succ_{j^*}'$, since the resulting allocation of $f_1(\succ_{j^*}, \succ_{j^*}')$ is a strictly preferred item of type $D_1$. This is a contradiction to our assumption on the strategyproofness of $f$.

Then, we also show that $f_1$ satisfies non-bossiness. Suppose for the sake of contradiction that $f_1$ is not non-bossy. Let $P = (\succ_j)_{j \leq n}$ be a profile of agents’ preferences over $D_1$. Then, there exists an agent $j^*$ with a bossy preference $\succ_{j^*}'$, such that for $P' = (\succ_{-j^*}, \succ_{j^*}')$, $f_1(P')(j^*) = f_1(P')(j^*)$ while $f_1(P)(j) \neq f_1(P')(j)$ for some $j$. Now, consider a profile $Q = (\succ_j)_{j \leq n}$ where for every agent $j, \succ_{j \perp D_1} = \succ_j$ and the mechanism $f$ whose local mechanism for $D_1$ is $f_1$. We know from Step 1 that $f(Q)_{\perp D_1} = f_1(Q_{\perp D_1}) = f_1(P)$. However, in that case, because agents’ preferences are lexicographic with
$D_1$ being the most important type, agent $j^*$ has a bossy preference $\succ'_j$ where $\succ'_j_{\perp D_1} \Rightarrow \succ'_j$ such that $f(Q)(j^*)_{\perp D_1} = f(\succ'_j, \succ'_j)(j^*)_{\perp D_1}$ while $f(Q)(j)_{\perp D_1} \neq f(\succ'_j, \succ'_j)(j)_{\perp D_1}$ for some $j$. This is a contradiction to our assumption that $f$ satisfies non-bossiness of more important types.

- **Step 3.** The allocations for the remaining types only depend on the allocations for $D_1$.

**Claim 3.** Consider any pair of profiles $P_1, P_2$ such that $[A]_1 = f_1(P_{1\perp D_1}) = f_1(P_{2\perp D_1})$, and $P_{1\perp D_{-1}[A]_1} = P_{2\perp D_{-1}[A]_1}$, then $f(P_1) = f(P_2)$.

*Proof.* We prove the claim by constructing a profile $P$ such that $f(P) = f(P_1) = f(P_2)$.

Let $P_1 = (\succ_j)_{j \leq n}$, $P_2 = (\tilde{\succ}_j)_{j \leq n}$ and $P = (\tilde{\succ}_j)_{j \leq n}$. Let $\tilde{\succ}_j$ be obtained from $\succ_j$ by changing the preferences over $D_1$ by raising $[A]_1(j)$ to the top position. Agents’ preference over $D_{-1}$ are $\tilde{\succ}_{j \perp D_{-1}[A]_1} = \succ_{j \perp D_{-1}[A]_1}$ ( RestrictedAgents’ preferences to $D$ ). It is easy to check that for every bundle $b$, $f(P)(j) \succ_j b \iff f(P)(j) \succ_j b$. By applying Lemma 1 sequentially to every agent, $f(P) = f(P_1)$. Similarly, $f(P) = f(P_2)$. It follows that for any allocation $[A]_1$ of items of type $D_1$, there exists a mechanism $f_{\perp D_{-1}[A]_1}$ such that for any profile $P$, we can write $f(P)$ as $(f_1(P_{1\perp D_1}), f_{\perp D_{-1}[A]_1}(P_{2\perp D_{-1}[A]_1}))$.

- **Step 4.** Show that $f_{\perp D_{-1}[A]_1}$ satisfies strategyproofness and non-bossiness of important types for any allocation $[A]_1$ of $D_1$.

Suppose for the sake of contradiction that $f_{\perp D_{-1}[A]_1}$ is not strategyproof for some profile $P_{\perp D_{-1}[A]_1}$. Then, for $P = (\succ_j)_{j \leq n}$ there is an agent $j^*$ with a beneficial manipulation w.r.t. $P$ and $[A]_1$, $(\succ'_j_{\perp D_{-1}[A]_1} \neq \succ'_{j_{\perp D_{-1}[A]_1}})$ and $\succ'_{j_{\perp D_1}} = \succ'_{j_{\perp D_1}}$. Let $Q = (\succ'_{-j}, \succ'_j)$. Then, $f(Q)(j) = ([A]_1, f_{\perp D_{-1}[A]_1}(P_{\perp D_{-1}[A]_1}))(j) \succ_j ([A]_1, f_{\perp D_{-1}[A]_1}(P_{\perp D_{-1}[A]_1}))(j) = f(P)(j)$. This is a contradiction to the strategyproofness of $f$.

Suppose for sake of contradiction that $f_{\perp D_{-1}[A]_1}$ does not satisfy non-bossiness of important types. Then, there is a profile $P = (\succ_j)_{j \leq n}$, and an agent $j^*$ with a bossy manipulation of her preferences $\succ'_{j_{\perp D_{-1}[A]_1}}$. Then, it is easy to verify that $f$ also does not satisfy non-bossiness of important types.

In Step 1, we showed that the allocation for $D_1$ only depends on the restriction of agents’ preferences to $D_1$ i.e. over $P_{\perp D_1}$. In Step 3 we showed that $f(P)$ can be decomposed as $(f_1(P_{\perp D_1}), f_{\perp D_{-1}[A]_1}(P_{\perp D_{-1}[A]_1}))$ where $[A]_1 = f_1(P_{\perp D_1})$. In Steps 2 we showed that $f_1$ must be strategyproof and non-bossy. In Step 4, we showed that for any output $[A]_1$ of $f_1$, the mechanism $f_{\perp D_{-1}[A]_1}$ satisfies both strategyproofness and non-bossiness of important types i.e. that we can apply the induction assumption that $f_{\perp D_{-1}[A]_1}$ is a locally strategyproof and non-bossy CR-net of allocation mechanisms. Together with the statement of Step 2, this completes the inductive argument. 

□
In Theorem 4, we characterize the class of strategyproof, non-bossy, and type-wise neutral mechanisms under $O$-legal lexicographic preferences, as the class of $O$-legal sequential compositions of serial dictatorships. The proof relies on Theorem 3 and Claim 4, where we show that any CR-net mechanism that satisfies type-wise neutrality is an $O$-legal sequential composition of neutral mechanisms, one for each type.

**Claim 4.** For any importance order $O$, an $O$-legal CR-net with type-wise neutrality is an $O$-legal sequential composition of neutral mechanisms.

**Proof.** We prove the claim by induction. Suppose $f$ is such a CR-net. From the decomposition in the proof of Claim 1, we observe that the mechanism used for type $i$ depends on $f(P)_{\perp D_{\leq i}}$. From this observation, and the importance order $O$, we can deduce that the mechanism for type 1 depends on no other type, and therefore there is only one mechanism for type 1, say, $f_1$. First we show that $f_1$ is neutral. Otherwise, there exists a permutation $\Pi$ over $D_1$, $f_1(\Pi(P_{\perp D_1})) \neq \Pi(1(P_{\perp D_1}))$. Let $I = (I_i)_{i \leq p}$ where $I_i$ is the identity permutation for type $i$. Then for $\Pi = (\Pi_1, I_{-1})$, we have $f(\Pi(P))_{\perp D_1} = f_1(\Pi(P_{\perp D_1})) \neq \Pi_1(f_1(P_{\perp D_1})) = \Pi(f(P))_{\perp D_1}$, a contradiction.

Now, suppose that for a given $i$, there is only one mechanism $f_i'$ for each type $i' \leq i$, and each $f_i'$ is neutral. Let $\Pi = (\Pi_{\leq i}, I_{>i})$ and we have $f(\Pi(P))_{\perp D_{\leq i}} = \Pi(f(P))_{\perp D_{\leq i}}$. Let $A = f(P)_{\perp D_{\leq i}}$ and $B = f(\Pi(P))_{\perp D_{\leq i}} = \Pi_{<i}(A)$. Because $P$ is chosen arbitrarily, $A$ and $B$ are also arbitrary outputs of mechanism $f$ over $D_{\leq i}$. Let $f_{i+1} = f_{\perp D_{i+1}, A}$, and $f'_{i+1} = f_{\perp D_{i+1}, B}$. Similarly both $f_{i+1}$ and $f'_{i+1}$ are arbitrary mechanisms in $CRT$. Because $f$ is neutral, we have $f(\Pi(P))_{\perp D_{i+1}} = \Pi(f(P))_{\perp D_{i+1}}$, i.e. $f_{i+1}(P_{\perp D_{i+1}, A}) = f'_{i+1}(M(P)_{\perp D_{i+1}, B})$. By assumption we know that $\Pi_{i+1} = I_{i+1}$, so $P_{\perp D_{i+1}, A} = \Pi(P)_{\perp D_{i+1}, B}$. That means $f_{i+1}$ and $f'_{i+1}$ can replace each other in $CRT$ of $f$ for type $i+1$. Therefore in fact there is only one mechanism $f_{i+1}$ for type $i+1$ in $CRT$.

Moreover $f_{i+1}$ must be neutral. Otherwise, there must be some permutation $\Pi_{i+1}$ over $D_{i+1}$, $f_{i+1}(\Pi_{i+1}(P_{\perp D_{i+1}, A})) \neq \Pi_{i+1}(f_{i+1}(P_{\perp D_{i+1}, A}))$. Then for $\Pi = (\Pi_{\leq i+1}, I_{>i+1})$, we have $f(\Pi(P))_{\perp D_{i+1}} = f_{i+1}(\Pi(P)_{\perp D_{i+1}, B}) = f_{i+1}(\Pi_{i+1}(P_{\perp D_{i+1}, A})) \neq f_{i+1}(\Pi_{i+1}(P_{\perp D_{i+1}, A})) = \Pi_{i+1}(f(P))_{\perp D_{i+1}}$, a contradiction. \hfill \Box

**Theorem 4.** For any importance order $O$, under the $O$-legal lexicographic preference domain, an allocation mechanism satisfies strategyproofness, non-bossiness of more important types, and type-wise neutrality if and only if it is an $O$-legal sequential composition of serial dictatorships.

**Proof.** Let $O = [D_1 \triangleright D_2 \triangleright \cdots \triangleright D_p]$. When $p = 1$, we know that serial dictatorship is characterized by strategyproofness, non-bossiness, and neu-
trality Mackin and Xia (2016). Let $P = (\succ_j)_{j \leq n}$ be an arbitrary $O$-legal lexicographic preference profile.

$\Rightarrow$: Let $f_O = (f_1, \ldots, f_p)$. It follows from Theorem 3 that if each $f_i$ satisfies strategyproofness and non-bossiness, then $f_O$ satisfies strategyproofness and non-bossiness of more important types, because $f_O$ can be regarded as a CR-net with no dependency among types. If each $f_i$ satisfies neutrality, then by Theorem 1 we have that $f$ satisfies type-wise neutrality. Therefore, since each $f_i$ is a serial dictatorship, which implies that it satisfies strategyproofness, non-bossiness, and neutrality, we have that $f_O$ satisfies strategyproofness, non-bossiness of more important types, and type-wise neutrality.

$\Leftarrow$: We now prove the converse. Let $f$ be a strategyproof and non-bossy mechanism under $O$-legal lexicographic preferences. Then by Theorem 3, we have that $f$ is an $O$-legal strategyproof and non-bossy CR-net. The rest of the proof depends on the following claim:

Claim 4 implies that there is only one mechanism $f_i$ for each type $i$ in $CRT$, and $f_i$ is neutral. Therefore with Theorem 3 and Claim 4, if $f$ satisfies strategyproofness, non-bossiness of more important types, and type-wise neutrality, we have that $f$ is an $O$-legal sequential composition of local mechanisms that are strategyproof, non-bossy, and neutral, which implies that they are serial dictatorships Mackin and Xia (2016).

**Theorem 5.** For any arbitrary importance order $O$, under the $O$-legal lexicographic preference domain, an allocation mechanism satisfies strategyproofness, non-bossiness of more important types, and Pareto-optimality if and only if it is an $O$-legal CR-net composed of serial dictatorships.

**Proof.** (Sketch) For a single type, we know that serial dictatorship is characterized by strategyproofness, Pareto-optimality, and non-bossiness Pápai (2001). The proof is similar to Theorem 4, and uses a similar argument to Theorems 1 and 2, to show that an $O$-legal CR-net is Pareto-optimal if and only if every local mechanism is Pareto-optimal. The details are provided in the appendix. □

Finally, we revisit the question of strategyproofness when preferences are not $O$-legal w.r.t. a common importance order. We show in Theorem 6 that even when agents’ preferences are restricted to lexicographic preferences, there is a computational barrier against manipulation; determining whether there exists a beneficial manipulation w.r.t. a sequential mechanism is NP-complete for MTRAs, even when agents’ preferences are lexicographic. Details and the full proof are relegated to the appendix.

**Definition 2.** Given an MTRA $(N,M,P)$, where $P$ is a profile of lexicographic preferences, and a sequential mechanism $f_O$, in Beneficial Ma
we are asked whether there exists an agent $j$ and an $O$-legal lexicographic preference relation $\succ'_j$ such that $f_O((P_{-j}, \succ'_j))(j) \succ_j f_O(P)(j)$.

**Theorem 6.** **BeneficialManipulation** is NP-complete when preferences are not $O$-legal.

### 6 Conclusion and Future Work

We studied the design of strategyproof sequential mechanisms for MTRAs under $O$-legal lexicographic preferences, and showed the relationship between properties of sequential mechanisms and the local mechanisms that they are composed of. In doing so, we obtained strong characterization results showing that any mechanism satisfying strategyproofness, and combinations of appropriate notions of non-bossiness, neutrality, and Pareto-optimality for MTRAs must be a sequential composition of local mechanisms. This decomposability of strategyproof mechanisms for MTRAs provides a fresh hope for the design of decentralized mechanisms for MTRAs and multiple assignment problems. Going forward, there are several interesting open questions such as whether it is possible to design decentralized mechanisms for MTRAs that are fair, efficient, and strategyproof under different preference domains.

### Acknowledgments

LX acknowledges support from NSF #1453542 and #1716333 and a gift fund from Google. YC acknowledges NSFC under Grants 61772035 and 61932001.

### References

Atila Abdulkadiroğlu and Tayfun Sönmez. Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems. *Econometrica*, 66(3):689–702, 1998. (Cited on page 6)

Salvador Barbera, Hugo Sonnenschein, and Lin Zhou. Voting by committees. *Econometrica*, 59(3):595–609, 1991. (Cited on page 5)

Salvador Barbera, Faruk Gul, and Ennio Stacchetti. Generalized median voter schemes and committees. *Journal of Economic Theory*, 61(2):262–289, 1993. (Cited on page 5)

Salvador Barbera, Jordi Masso, and Alejandro Neme. Voting under constraints. *Journal of Economic Theory*, 76(2):298–321, 1997. (Cited on page 5)
Arka A. Bhattacharya, David Culler, Eric Friedman, Ali Ghodsi, Scott Shenker, and Ion Stoica. Hierarchical scheduling for diverse datacenter workloads. In Proceedings of the 4th Annual Symposium on Cloud Computing, pages 4:1–4:15, Santa Clara, CA, USA, 2013. (Cited on page 2)

Duncan Black. On the rationale of group decision-making. Journal of Political Economy, 56(1):23–34, 1948. (Cited on page 5)

Anna Bogomolnaia and Hervé Moulin. A New Solution to the Random Assignment Problem. Journal of Economic Theory, 100(2):295–328, 2001. (Cited on page 6)

Craig Boutilier, Ronen Brafman, Carmel Domshlak, Holger Hoos, and David Poole. CP-nets: A tool for representing and reasoning with conditional ceteris paribus statements. Journal of Artificial Intelligence Research, 21:135–191, 2004. (Cited on page 6)

Jon Elster. Local justice: How institutions allocate scarce goods and necessary burdens. Russell Sage Foundation, 1992. (Cited on pages 2 and 3)

Etsushi Fujita, Julien Lesca, Akihisa Sonoda, Taiki Todo, and Makoto Yokoo. A Complexity Approach for Core-Selecting Exchange with Multiple Indivisible Goods under Lexicographic Preferences. In Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, pages 907–913, 2015. (Cited on pages 3 and 5)

Ali Ghodsi, Matei Zaharia, Benjamin Hindman, Andy Konwinski, Scott Shenker, and Ion Stoica. Dominant Resource Fairness: Fair Allocation of Multiple Resource Types. In Proceedings of the 8th USENIX Conference on Networked Systems Design and Implementation, pages 323–336, Boston, MA, USA, 2011. (Cited on page 2)

Ali Ghodsi, Vyas Sekar, Matei Zaharia, and Ion Stoica. Multi-resource Fair Queueing for Packet Processing. In Proceedings of the ACM SIGCOMM 2012 conference on Applications, technologies, architectures, and protocols for computer communication, volume 42, pages 1–12, Helsinki, Finland, 2012. (Cited on page 2)

Allan Gibbard. Manipulation of voting schemes: A general result. Econometrica, 41:587–601, 1973. (Cited on page 5)

Gerd Gigerenzer and Daniel G. Goldstein. Reasoning the fast and frugal way: Models of bounded rationality. Psychological Review, 103(4):650–669, 1996. (Cited on page 3)

Xiaoxi Guo, Sujoy Sikdar, Haibin Wang, Lirong Xia, Yongzhi Cao, and Hanpin Wang. Probabilistic serial mechanism for multi-type resource allocation. arXiv preprint arXiv:2004.12062, 2020. (Cited on pages 3 and 5)
Hadi Hosseini and Kate Larson. Multiple assignment problems under lexicographic preferences. In Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, pages 837–845, 2019. (Cited on pages 2, 3, and 5)

Woonghee Tim Huh, Nan Liu, and Van-Anh Truong. Multiresource Allocation Scheduling in Dynamic Environments. Manufacturing and Service Operations Management, 15(2):280–291, 2013. (Cited on pages 1 and 3)

Jéréôme Lang and Lirong Xia. Sequential composition of voting rules in multi-issue domains. Mathematical Social Sciences, 57(3):304–324, 2009. (Cited on pages 3, 4, and 5)

Michel Le Breton and Arunava Sen. Separable preferences, strategyproofness, and decomposability. Econometrica, 67(3):605–628, 1999. (Cited on page 5)

Erika Mackin and Lirong Xia. Allocating Indivisible Items in Categorized Domains. In Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence (IJCAI-16), pages 359–365, 2016. (Cited on pages 2, 5, 6, and 18)

Hervé Moulin. On strategy-proofness and single peakedness. Public Choice, 35(4):437–455, 1980. (Cited on page 5)

Hervé Moulin. Cooperative Microeconomics: A Game-Theoretic Introduction. Prentice Hall, 1995. (Cited on pages 2 and 5)

Szilvia Pápai. Strategyproof and nonbossy multiple assignments. Journal of Public Economic Theory, 3(3):257–71, 2001. (Cited on pages 2, 5, 18, and 26)

Mark Satterthwaite. Strategy-proofness and Arrow’s conditions: Existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory, 10:187–217, 1975. (Cited on page 5)

Shreyas Sekar, Sujoy Sikdar, and Lirong Xia. Condorcet consistent bundling with social choice. In Proceedings of the 2017 International Conference on Autonomous Agents and Multiagent Systems. International Foundation for Autonomous Agents and Multiagent Systems, 2017. (Cited on page 3)

Lloyd Shapley and Herbert Scarf. On cores and indivisibility. Journal of Mathematical Economics, 1(1):23–37, 1974. (Cited on page 5)

Sujoy Sikdar, Sibel Adali, and Lirong Xia. Mechanism Design for Multi-Type Housing Markets. In Proceedings of the 31st AAAI Conference on Artificial Intelligence, 2017. (Cited on pages 3 and 5)
Sujoy Sikdar, Sibel Adalı, and Lirong Xia. Mechanism design for multi-type housing markets with acceptable bundles. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 2165–2172, 2019. (Cited on pages 3 and 5)

Zhaohong Sun, Hideaki Hata, Taiki Todo, and Makoto Yokoo. Exchange of Indivisible Objects with Asymmetry. In *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence (IJCAI 2015)*, pages 97–103, 2015. (Cited on page 3)

Lars-Gunnar Svensson. Strategy-proof allocation of indivisible goods. *Social Choice and Welfare*, 16(4):557–567, 1999. (Cited on pages 2 and 5)

H. Wang, Sujoy Sikdar, Xiaoxi Guo, Lirong Xia, Yongzhi Cao, and Hanpin Wang. Multi-type resource allocation with partial preferences. In *AAAI*, 2020. (Cited on pages 3 and 5)

Lirong Xia and Vincent Conitzer. Strategy-proof voting rules over multi-issue domains with restricted preferences. In *Proceedings of the Sixth Workshop on Internet and Network Economics (WINE)*, pages 402–414, Stanford, CA, USA, 2010. (Cited on page 5)

Lirong Xia, Vincent Conitzer, and Jérôme Lang. Strategic sequential voting in multi-issue domains and multiple-election paradoxes. In *Proceedings of the ACM Conference on Electronic Commerce (EC)*, pages 179–188, San Jose, CA, USA, 2011. (Cited on page 3)
7 Appendix

7.1 Proof of Theorem 1

**Theorem 1.** For any importance order \( O \in \mathcal{O} \), any \( X \in \{\text{anonymity, type-wise neutrality, non-bossiness, monotonicity, Pareto-optimality} \} \), and \( f_O = (f_1, \ldots, f_p) \) be any \( O \)-legal sequential mechanism. Then, for \( O \)-legal preferences, if for every \( i \leq p \), the local mechanism \( f_i \) satisfies \( X \), then \( f_O \) satisfies \( X \).

**Proof.** Throughout, we will assume that \( O = [D_1 \triangleright \cdots \triangleright D_p] \), and that \( P \) is an arbitrary \( O \)-legal preference profile over \( p \) types. For any \( i \leq p \), we define \( g_i \) to be the sequential mechanism \( (f_1, \ldots, f_i) \).

*anonymity.* It is easy to see that the claim is true when \( p = 1 \). Now, suppose that claim is true for all \( p \leq k \). Let \( P \) be an arbitrary preference profile over \( k + 1 \) types. Let \( g = (f_2, \ldots, f_{k+1}) \), and \( \Pi_1 = (\Pi_2, \ldots, \Pi_{k+1}) \). Let \( A_1 = \Pi_1(f_O(P_{\perp D_1})) \) and \( B_1 = f_1(\Pi_1(P_{\perp D_1})) \). Now, \( \Pi(f_O(P)) = (\Pi_1(f_1(P_{\perp D_1})), \Pi(g(P_{\perp D \leq k+1} \setminus D_1 \Pi_1(f_1(P_{\perp D_1})))) \), and \( f_O(\Pi(P)) = (f_1(\Pi(P_{\perp D_1})), g(\Pi(P_{\perp D \leq k+1} \setminus D_1 \Pi_1(f_1(P_{\perp D_1})))) \). Since \( f_1 \) is anonymous, \( \Pi(f_1(P_{\perp D_1})) = f_1(\Pi(P_{\perp D_1})) \). Therefore, \( P_{\perp D \leq k+1} \setminus D_1 \Pi_1(f_1(P_{\perp D_1})) = P_{\perp D \leq k+1} \setminus D_1 f_1(\Pi(P_{\perp D_1})) \). Then, by the induction assumption, \( g \) satisfies anonymity, and we have \( \Pi(g(P_{\perp D \leq k+1} \setminus D_1 \Pi_1(f_1(P_{\perp D_1})))) = g(\Pi(P_{\perp D \leq k+1} \setminus D_1 \Pi_1(f_1(\Pi(P_{\perp D_1})))) \). It follows that \( \Pi(f_O(P)) = f_O(\Pi(P)) \).

*type-wise neutrality.* We show only the induction step. Suppose that the claim is always true when \( p \leq k \). Let \( P \) be an arbitrary preference profile over \( p + 1 \) types. Let \( g = (f_2, \ldots, f_{p+1}) \), and \( \Pi_{p+1} = (\Pi_2, \ldots, \Pi_{p+1}) \). Let \( A_1 = \Pi_1(f_O(P_{\perp D_1})) \) and \( B_1 = f_1(\Pi_1(P_{\perp D_1})) \). Now, \( \Pi(f_O(P)) = (A_1, \Pi_{p+1}(g(P_{\perp D \leq p+1} \setminus D_1, A_1)), f_O(\Pi(P)) = (B_1, g(\Pi_{p+1}(P_{\perp D \leq p+1} \setminus D_1, B_1))) \). Since \( f_1 \) is neutral, \( A_1 = B_1 \). Then, \( P_{\perp D \leq p+1} \setminus D_1, A_1 = P_{\perp D \leq p+1} \setminus D_1, B_1 \). Then, by the induction assumption, \( g \) satisfies type-wise neutrality, and \( \Pi_{p+1}(g(P_{\perp D \leq p+1} \setminus D_1, A_1)) = g(\Pi_{p+1}(P_{\perp D \leq p+1} \setminus D_1, B_1))) \). It follows that \( \Pi(f_O(P)) = f_O(\Pi(P)) \).

*non-bossiness.* Let us assume for the sake of contradiction that the claim is false, i.e. there exists a profile \( P \), an agent \( j \) and a misreport \( \succ_j^f \) such that for \( P' = (\succ_j^f, f_O(P))(j) = f_O(P')(j) \), and \( f_O(P) \neq f_O(P') \). Then, there is a type \( i \leq p \) such that, \( f_O(P)_{\perp D_{\leq i}} = f_O(P')_{\perp D_{\leq i}} \) and \( f_O(P)_{\perp D_i} \neq f_O(P')_{\perp D_i} \). Let \( A = f_O(P)_{\perp D_{\leq i}} \). Then, there is an agent \( k \) such that \( f_i(P_{\perp D_i, A})(k) \neq f_i(P'_{\perp D_i, A})(k) \). By the choice of \( i \), and the assumption that every other agent reports preferences truthfully, \( \succ_j^{D_i, A} \neq \succ_j^{D_i, A'} \). Then, \( f_i(\succ_j^{D_i, A}, \succ_{j}^{D_i, A})(j) = f_i(\succ_j^{D_i, A'}, \succ_{j}^{D_i, A})(j) \), but \( f_i(\succ_j^{D_i, A}, \succ_{j}^{D_i, A})(k) \neq f_i(\succ_j^{D_i, A'}, \succ_{j}^{D_i, A})(k) \), a contradiction to our assumption that \( f_i \) is non-bossy.

*monotonicity.* Let \( P' = (P, \succ_j^f) \) be an \( O \)-legal profile obtained from \( P \) and \( Y \subseteq D \) is the set of bundles raising the ranks in \( P' \) such that the relative rankings of bundles in \( Y \) are unchanged in \( P \) and \( P' \). For any
\(Y \subseteq D\), and any \(u \in D_{D_i}\), let \(Y^{D_i\mid u} = \{x_i : x \in Y, x_h = u_h \text{ for all } h \leq i - 1\}\). It is easy to see that if \(x_1 = f_O(P^\prime)(j)_{\perp\{D_i\}}\), then it follows from strong monotonicity of \(f_1\) that \(x_1 \in f_O(P)(j)_{\perp\{D_i\}} \cup Y^{D_i}\). Now, either \(x_1 \neq f_O(P)(j)_{\perp\{D_i\}}\) or \(x_1 = f_O(P)(j)_{\perp\{D_i\}}\). Suppose \(x_1 \neq f_O(P)(j)_{\perp\{D_i\}}\). Then, by strong monotonicity of \(f_1\), \(x_1 > f_O(P)(j)_{\perp\{D_i\}}\). Then, by our assumption of \(O\)-legal lexicographic preferences, for any \(z \in D_{\{D_2, \ldots, D_p\}}\), \((x_1, z) \in Y\). Therefore, \(f_O(P^\prime)(j) \in Y\). Suppose \(x_1 = f_O(P)(j)_{\perp\{D_i\}}\). then by a similar argument, \(f_O(P)(j)_{\perp\{D_i\}} \in \{f_O(P)(j)_{\perp\{D_j\}}\} \cup Y^{D_j\mid x_1}\).

Applying our argument recursively, we get that \(f_O(P^\prime)(j) \in \{f_O(P)(j)\} \cup Y\).

**Pareto-optimality.** Suppose the claim is true for \(p \leq k\) types. Let \(P\) be an \(O\)-legal lexicographic profile over \(k + 1\) types, and \(f_O = (f_i)_{i \leq k+1}\) is a sequential composition of Pareto-optimal local mechanisms. Suppose for the sake of contradiction that there exists an allocation \(B\) such that some agents strictly better off compared to \(f_O(P)\), and no agent is worse off. Then, by our assumption of lexicographic preferences, for every agent \(k\) who is not strictly better off, \(B(k) = f_O(P)(k)\), and for every agent \(j\) who is strictly better off, one of two cases must hold. (1) \(B(j)_{\perp D_1} > j f_O(P)(j)_{\perp D_1}\), or (2) \(B(j)_{\perp D_1} = f_O(P)(j)_{\perp D_1}\). (1): If there exists an agent such that \(B(j)_{\perp D_1} > j f_O(P)(j)_{\perp D_1}\), this is a contradiction to our assumption that \(f_1\) is Pareto-optimal. (2): Suppose \(B(j)_{\perp D_1} = f_O(P)(j)_{\perp D_1}\) for all agents who are strictly better off. Let \(g = (f_2, \ldots, f_k+1)\). W.l.o.g. let agent 1 strictly prefer \(B(1)\) to \(f_O(P)(1)\). Then, \(g(P_{\perp D_{\leq k+1}\mid D_1} f_O(P)_{\perp D_1})(1) \succ_1 B(1)_{\perp D_{\leq k+1}\mid D_1}\), and for every other agent \(l \neq 1\), either \(g(P_{\perp D_{\leq k+1}\mid D_1} f_O(P)_{\perp D_1})(l) \succ_l B(l)_{\perp D_{\leq k+1}\mid D_1}\), or \(g(P_{\perp D_{\leq k+1}\mid D_1} f_O(P)_{\perp D_1})(l) = B(l)_{\perp D_{\leq k+1}\mid D_1}\), which is a contradiction to our induction assumption.

### 7.2 Proof of Theorem 2

**Theorem 2.** For any importance order \(O \in \mathcal{O}\), \(X \in \{\text{anonymity, type-wise neutrality, non-bossiness, monotonicity, Pareto-optimality}\}\), and \(f_O = (f_1, \ldots, f_p)\) be any \(O\)-legal sequential mechanism. For \(O\)-legal preferences, if \(f_O\) satisfies \(X\), then for every \(i \leq p\), \(f_i\) satisfies \(X\).

**Proof.** **anonymity.** Suppose that for some \(k \leq p\), \(f_k\) does not satisfy anonymity. Then, there exists a profile \(P_k\) on \(D_k\) such that for some permutation \(\Pi\) on agents \(f_k(\Pi(P_k)) \neq \Pi(f_k(P_k))\). Now, consider the \(O\)-legal separable lexicographic profile \(P\), where for any type \(i \leq p\), the preferences over type \(D_i\) is denoted \(P_{\perp D_i}\) and \(P_{\perp D_k} = P_k\). It is easy to see that, \(f_O(\Pi(\Pi)) = (f_1(\Pi(\Pi P_{\perp D_1})), \ldots, f_p(\Pi(\Pi P_{\perp D_p})) = (\Pi(f_1(\Pi P_{\perp D_1})), \ldots, \Pi(f_p(\Pi P_{\perp D_p}))).\) By anonymity of \(f\), \(f_O(\Pi(\Pi)) = f_O(\Pi(\Pi))\), which implies that \(f_k(\Pi(\Pi P_{\perp D_k}))) = f_k(\Pi(\Pi P_{\perp D_k}))\), which is a contradiction.

**type-wise neutrality.** Suppose that some \(k \leq p\), \(f_k\) does not satisfy neutrality. Then, there exists a profile \(P_k\) on \(D_k\) such that for some permutation
$\Pi_k$ on $D_k$, $f_k(\Pi_k(P_k)) \neq \Pi_k(f_k(P_k))$. Now, consider the $O$-legal separable lexicographic profile $P$, where for any type $i \leq p$, the preferences over type $D_i$ is denoted $P_{\bot D_i}$ and $P_{\bot D_k} = P_k$, and let $\Pi = (\Pi_1, \ldots, \Pi_k, \ldots, \Pi_p)$ be a permutation over $D$ by applying $P_{i_k}$ on $D_i$ for each type $i \leq p$. $f_O(\Pi(P)) = (f_1(\Pi_1(P_{\bot D_1})), \ldots, \Pi_p(f_p(P_{\bot D_p})))$. By type-wise neutrality of $f_O$, $f_O(\Pi(P)) = \Pi(f_O(P))$. This implies that $f_k(\Pi_k(P_{\bot D_k})) = \Pi_k(f_k(P_{\bot D_k}))$, where $P_{\bot D_k} = P_k$, which is a contradiction.

**non-bossiness.** Assume for the sake of contradiction that $k \leq p$ is the most important type such that $f_k$ does not satisfy non-bossiness. Then, there exists a preference profile $Q = (\succ^k)_{j \leq n}$ over $D_k$, and a bossy agent $l$ and a misreport $Q' = (\succ^k_{-l, \succ^k_l})$, such that $f_k(Q')(l) = f_k(Q)(l)$, but $f_k(Q') \neq f_k(Q)$. Now, consider the $O$-legal separable lexicographic profile $P$, where for any type $i \leq p$, the preferences over type $D_i$ is denoted $P_{\bot D_i}$ and $P_{\bot D_k} = Q$, and the profile $P'$ obtained from $P$ by replacing $\succ_l$ with $\succ'_l$, which in turn is obtained from $\succ$ by replacing $\succ_{l \bot D_k}$ with $\succ^k_{l \bot D_k}$. It is easy to see that $f_O(P')_{\bot D_{<k}} = f_O(P)_{\bot D_{<k}}$, and $f_O(P')(l)_{\bot D_k} = f_O(P)(l)_{\bot D_k}$, but $f_O(P')_{\bot D_{\neq k}} \neq f_O(P)_{\bot D_{\neq k}}$, and by our assumption of separable preferences, $f_O(P')_{\bot D_{\neq k}} = f_O(P)_{\bot D_{\neq k}}$. This implies that $f_O(P')(l) \neq f_O(P)(l)$, but $f_O(P') \neq f_O(P)$, implying that $f_O$ does not satisfy non-bossiness, which is a contradiction.

**monotonicity.** Suppose for the sake of contradiction that $k$ is the most important type for which $f_k$ does not satisfy monotonicity. Then, there exists a profile $Q = (\succ^k)_{j \leq n}$ of linear orders over $D_k$, such that for some agent $j$, $\succ^k_j$ obtained from $\succ^k_j$ by raising the rank of a set of items $Z \subseteq D_k$ without changing their relative order, $f_k(Q)(j) \not\in \{f_k(Q)(j)\} \cup Z$. Now, consider the $O$-legal separable lexicographic profile $P$, where for any type $i \leq p$, the preferences over type $D_i$ is denoted $P_{\bot D_i}$ and $P_{\bot D_k} = Q$, and the profile $P'$ obtained from $P$ by replacing $\succ_l$ with $\succ'_l$, which in turn is obtained from $\succ$ by replacing $\succ_{l \bot D_k}$ with $\succ^k_{l \bot D_k}$. It is easy to see that $f_O(P')_{\bot D_{<k}} = f_O(P)_{\bot D_{<k}}$, and $f_O(P')(l)_{\bot D_k} \neq f_O(P)(l)_{\bot D_k} \cup Z$. By our assumption of $O$-legal separable lexicographic preferences, this implies that $f_O$ does not satisfy monotonicity, which is a contradiction.

**Pareto-optimality.** Suppose that some $k \leq p$, $f_k$ does not satisfy Pareto-optimality. Then, there exists a profile $P_k$ such that $f_k(P_k)$ is Pareto-dominated by an allocation $B$ of $D_k$. Now, consider the $O$-legal separable lexicographic profile $P$, where for any type $i \leq p$, the preferences over type $D_i$ is denoted $P_{\bot D_i}$ and $P_{\bot D_k} = P_k$. Then, $f_O(P) = (f_i(P_{\bot D_i}))_{i \leq p}$ is Pareto-dominated by the allocation $B$ of all types, where for all types $i \neq k$, $B_{\bot D_i} = f_i(P_{\bot D_i})$, and $B_{\bot D_k} = A$, which is a contradiction to the assumption that $f_O$ is Pareto-optimal. \qed
7.3 Proof of Theorem 5

**Theorem 5.** For any arbitrary importance order $O$, under the $O$-legal lexicographic preference domain, an allocation mechanism satisfies strategyproofness, non-bossiness of more important types, and Pareto-optimality if and only if it is an $O$-legal CR-net composed of serial dictatorships.

**Proof.** Let $O = [D_1 \triangleright D_2 \triangleright \cdots \triangleright D_p]$. Under single type, we know that serial dictatorship is characterized by strategyproofness, Pareto-Optimality, and non-bossiness Pápai (2001). Let $P = (\succ_j)_{j \leq n}$ be an arbitrary $O$-legal lexicographic preference profile.

$\Rightarrow$: Let $f$ be an $O$-legal CR-net. From Theorem 3 we know that if each local mechanism of $f$ satisfies strategyproofness and non-bossiness, then $f$ satisfies strategyproofness and non-bossiness of more important types.

We now prove that if each local mechanism is Pareto-Optimal, then $f$ is Pareto-optimal, similarly to Theorem 1. Suppose for the sake of contradiction that $f$ is not Pareto-optimal, i.e. for some $P$, the allocation $B = (B_i)_{i \leq p}$ Pareto-dominates $f(P) = A = (A_i)_{i \leq p}$. Let $i$ be the most important type that $A$ and $B$ and different allocuation, and we have $A_{<i} = B_{<i}$ and $B_i$ Pareto-dominates $A_i$. Let $P_i = P \perp D_i, A_{<i}$. However, by the assumption that $f$ is a CR-net, we know that $A_i = f \perp D_i, A_{<i}(P_i)$ is Pareto-optimal, i.e. $A_i$ does not Pareto-dominated by $B_i$, which is a contradiction.

Therefore if each local mechanism of $f$ is a serial dictatorship, which implies that it satisfies strategyproofness, non-bossiness, and Pareto-optimality, then $f$ satisfies strategyproofness, non-bossiness of more important types, and Pareto-optimality.

$\Leftarrow$: Let $f$ be a mechanism for $O$-legal lexicographic preferences. With Theorem 3, we have that if $f$ satisfies strategyproofness and non-bossiness of more important types, then it is an $O$-legal strategyproof and non-bossy CR-net. We also have that if $f$ is a CR-net satisfying Pareto-optimality, then each local mechanism is also Pareto-optimal with a similar proof to Theorem 2. Together we have that if $f$ satisfies strategyproofness, non-bossiness of more important types, and Pareto-optimality, then $f$ is an $O$-legal CR-net and each local mechanism satisfies strategyproofness, non-bossiness, and Pareto-Optimality, which implies that it is a serial dictatorship Pápai (2001).

7.4 Proof of Theorem 6

**Theorem 6.** BeneficialManipulation is NP-complete when preferences are not $O$-legal.

**Proof.** We show a reduction from 3-SAT. In an instance $I$ of 3-SAT involving $s$ Boolean variables $\{x_1, \ldots, x_s\}$, and a formula $F$ involving $t$ clauses $\{c_1, \ldots, c_t\}$ in 3-CNF, we are asked if $F$ is satisfiable. Given such an arbitrary instance $I$ of 3-SAT, we construct an instance $J$ of BeneficialManipulation in polynomial time. We will show that $I$ is a Yes instance of
3-SAT if and only if $J$ is a Yes instance of $\text{BENEFICIALMANIPULATION}$. For each $j \leq t$, we label the three literals in clause $j$ as $l_{j1}^j, l_{j2}^j, \text{ and } l_{j3}^j$ where $j_1 < j_2 < j_3$. We construct instance $J$ of $\text{BENEFICIALMANIPULATION}$ to have:

**Types:** $s + 1$ types.

**Agents:**

- For every variable $i \leq s$, and every clause $j \leq t$, two agents $0_i^j, 1_i^j$, and a dummy agent $d_i^j$.
- For every clause $j$, an agent $c_j$.
- A special agent $0$.

**Items:** For every agent $a$ and every type $k \leq s + 1$, an item named $[a]_k$.

**Preferences:** For some agents, we only specify their importance orders (or local preferences) over types (or items) that are important for this proof, and assume that their preferences are an arbitrary linear order with the specified preferences over the top few types (or items).

- **agent** $0$ has importance order $s + 1 > \text{others}$, with local preferences:
  - type $s + 1$: $[0]_{s+1} > [c_1]_{s+1} > \text{others}$
  - every other type $k < s + 1$: $[0]_k > \text{others}$

- **agents** $l_i^j, l \in \{0, 1\}$ have importance order $i > \text{others}$.
  - type $i$: $\text{NEXT}^j_i > l_i^j > 0_i > \text{others}$, where $\text{NEXT}^j_i = [l_i^{j+1}]_i$ if $j < t$, and $\text{NEXT}^j_i = [l_i^t]_i$ if $j = t$.
  - type $s + 1$ preferences are conditioned on assignment on type $i$.
    - $D_i \setminus \{\text{NEXT}^j_i\}$: $[d_i^j]_i > \text{others}$.
    - $\text{NEXT}^j_i$: $[l_i^j]_i > \text{others}$.
  - every other type $k$: $[l_i^j]_k > \text{others}$.

- **agents** $d_i^j$ have importance order $i > \text{others}$.
  - type $i$: $[d_i^j]_i > \text{others}$.

- **agents** $c_j$ have importance order $s + 1 > \text{others}$.
  - type $s + 1$: $[l_{j1}]_{s+1} > [l_{j2}]_{s+1} > [l_{j3}]_{s+1} > [0]_{s+1} > \text{others}$.
  - every other type $k$: $[c_j]_k > \text{others}$.

**Sequential mechanism:** composed of serial dictatorships applied in the order $O = 1 > \cdots > s + 1$, where the priority orders over agents are:

- for types $i \leq s$: $(\text{others}, 0, 0_i^1, \ldots, 0_i^t, 1_i^1, \ldots, 1_i^t)$.
• type $s + 1$: $(0_1^1, \ldots, 0_1^t, 0_1^{s+1}, \ldots, 0_s^1, 1_1^1, \ldots, 1_1^t, 1_2^1, \ldots, 1_s^1, c_1, \ldots, c_t, 0, \text{others})$.

Similar to preferences, we only specify the part of the priority orderings over the agents for each serial dictatorship that is relevant to the proof, and assume that the priority orderings are linear orders over the agents, where the specified part holds.

The main idea is that if the 3-SAT instance is satisfiable, special agent 0 enables every $c_j$ agent to get an item of type $s + 1$ corresponding to a literal $l_j^i$ that satisfies the clause $j$ by a beneficial manipulation which results in agents $l_j^i$ corresponding to literals in clause $j$ being allocated their favorite item of type $i$.

When agents report preferences truthfully and are either optimistic or pessimistic, it is easy to check $f_0$ allocates items as follows: for types $i < s+1$ agent 0 gets $0_i^1$ and every agent $l_j^i$ gets $\text{NEXT}_j^i$. Then, for type $s + 1$, for any $l \in \{0, 1\}$, every agent $l_j^i$ gets the item $[l_j^i]_{s+1}$. This in turn makes it so that for every $i \leq s, j \leq t$, the items $[l_j^i]_{s+1}$ unavailable to the agent $c_j$. Then, $c_1$ gets $[0]_{s+1}$, and finally, 0 gets $[c_1]_{s+1}$.

Upon examining agent 0’s preferences, it is easy to check that the only way for agent 0 to improve upon this allocation is to receive a better item of type $s + 1$, specifically, item $[0]_{s+1}$.

$\Rightarrow$ Let $\phi$ be a satisfying assignment for instance $I$. Consider the manipulation where agent 0 reports her top item of type $i$ to be $[0_1^i]$ if $\phi_i = 0$, and $[1_1^i]$ if $\phi_i = 1$. Now, suppose that every other agent reports preferences truthfully.

$\Rightarrow$ Let us consider the case where for some $i \leq s$, $\phi_i = 0$. It is easy to check that for type $i$, agents’ allocations are as follows: Agent 0 gets $[0_1^i]$ if $\phi_i = 0$, and in the sequence $j = t \ldots 2$ agents $0_j^i$ get items $[0_j^i]$, respectively, and agent $0_1^i$ gets $[0_i^1]$, i.e. none of the agents $0_j^i$ gets their corresponding top item $\text{NEXT}_j^i$. Now, for type $s + 1$, agent $0_1^i$ gets $[d_1^i]_{s+1}$ according to their true preferences since they did not receive their item $\text{NEXT}_i^i$ of type $i$, leaving item $[0_1^i]_{s+1}$ available. Then, agents $1_j^i$ get the items $[1_j^i]_{s+1}$, crucially, before agents $c_j$ get to choose any item. Then for every agent $c_j$, if $0_j^i$ is the literal with the lowest index $i^*$ such that $\phi_i$ corresponds to a satisfying assignment of clause $c_j$, $[0_i^{i^*}]_{s+1}$ must be available when $c_j$ gets her turn to pick an item, and gets it. Moreover, since $\phi$ is a satisfying assignment, there is such an item for every $c_j$. This leaves $[0]_{s+1}$ available when it is agent 0’s turn to pick an item. Thus, special agent 0 prefers the resulting allocation to her allocation when she picked items truthfully, and the manipulation was beneficial, irrespective of whether agent 0 is optimistic or pessimistic.

$\Leftarrow$ Suppose agent 0 has a beneficial manipulation. Then, as we have already established, agent 0 must get item $[0]_{s+1}$ as a result of the manipulation. Now, agents $c_j$ get their turn to pick an item before agent 0 in the serial
dictatorship for type $s + 1$. Then, since they are truthful, each agent $c_j$ receives an item $[l^i_{i^*}]_{s+1}$ corresponding to a satisfying assignment. Otherwise one of them must get $[0]_{s+1}$, a contradiction.

Let us construct an assignment $\phi$ as follows: if $c_j$ gets item $[0^i_{i^*}]_{s+1}$ in the final allocation, set $\phi_i = 0$, and set $\phi_i = 1$ otherwise. We will show that $\phi$ is a satisfying assignment for $I$. Since agents $0^j_i$ and $1^j_i$ come before agents $c_j$ in the serial dictatorship, and are also truthful, it must be that for every item $[l^i_{i^*}]_{s+1}$ allocated to agent $c_j$ in the final allocation, the corresponding agent $l^i_{i^*}$ does not get $\text{NEXT}^j_i$ of type $i^*$.

By construction of the preferences over type $i^*$ and the serial dictatorship for type $i^*$, it must be that special agent 0 picks an item $[l^k_{i^*}]_{i^*}$, where either $k > j$ or $k = 1$. It is easy to check from the construction that if this is not the case, agent $l^j_i$ can pick item $\text{NEXT}^j_i$ when every agent other than 0 picks truthfully. Further, if agent 0 picks some item $[0^j_{i^*}]_{i^*}$, it is easy to check that by the construction of the preferences, every agent other than $1^j_i$ gets their top item. Thus, none of the agents $c_j$ may receive the item $[1^j_{i^*}]_{i^*}$ since it must already have been picked by the agent $1^j_i$ in the serial dictatorship. Together with the fact that every agent $c_j$ receives an item that corresponds to a satisfying assignment, $\phi$ constructed above is a satisfying assignment for instance $I$. $\square$