Jain hierarchy for the Fractional Quantum Hall Effect.

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Abstract

We propose the effective hierarchical partition function which is able to describe both the Jain states and the Jain-type hierarchical states. Using this partition function (effective Lagrangian) we calculate the charge of the quasiparticle excitations. We show that the Jain-type hierarchical states are equivalent to the system of anyons in the external magnetic field.

1. Introduction

At present time it is well known that almost all of the fractional quantum Hall effect (FQHE) states observed experimentally (for example see the recent papers [1]) are the members of the Jain sequence [2] of FQH states. The rest of the states are the so called hierarchical states which is nothing else but the FQH states of quasiparticles i.e. the Jain-type eigenstates of quasiparticles (or composite fermions). Theoretically although the corresponding filling fractions are known [3] their derivation remains somewhat obscure. It is the goal of the present letter to make these questions more clear. In particular we show that the effective hierarchical Lagrangian is nothing else but the Lagrangian of the system of anyons in the external magnetic field. Although quasiparticle (vortex) states for the Jain states cannot be constructed explicitly its quantum numbers and the external magnetic field can be evaluated. Using these quantities the hierarchical FQHE filling fractions can be predicted. To calculate them we propose the effective hierarchical partition function (effective Lagrangian) which is able to describe the Jain states and the Jain-type hierarchical states. In particular we calculate the charge of quasiparticles for the states of the Jain sequence. Although the value of this charge is known (for example see [4]) its derivation remains obscure. Another goal is to present the simple derivation of the formula for the effective Lagrangian of fermions in the external magnetic field [5]. Finally the third goal is to clarify the simple picture of anyons in the external magnetic field behind the hierarchical Jain-type FQHE states and calculate the conductivity by means of filling the integer number of Landau levels in the effective magnetic field. The FQHE is then follows from the non-zero contribution of the Chern-Simons term to the effective Lagrangian [6],[7]. We believe all these points are an interesting contribution to the theory of FQHE.
2. Jain-type hierarchy for the Jain FQHE states.

The object to be studied is the partition function corresponding to the eigenstate which consists of the stable FQHE parent state of the Jain sequence and the number of quasiparticles (Vortices) on the top of the parent state. The Jain sequence [2] can be obtained in the framework of the Fermionic second-quantized approach [6],[7]. The starting point of this approach is the anyonic transformation of the electron wave function of the form:

\[
\Psi(x_1, \ldots, x_N) = \prod_{i<j} \left( \frac{z_i - z_j}{|z_i - z_j|} \right) \Psi'(x_1, \ldots, x_N),
\]

where \(z_i = x_i + iy_i\) and \(k\) is an integer. After that in the second-quantized language the Lagrangian takes the form:

\[
L = i\psi^+(\partial_0 - ia_0)\psi + \psi^+(\partial - ia - iA_1)^2\psi + \frac{1}{2k} \frac{1}{4\pi} \epsilon^{\mu\nu\alpha\beta} a_\mu \partial_\nu a_\alpha,
\]

where \(\psi^\dagger\psi\) is the fermionic field and the effective magnetic field \(H_1 = H/(2nk + 1)\) for the filling factor \(\nu = \nu_0 + \nu_1 = n/(2nk + 1) + \nu_1\). Equation (2) is the starting point of our derivation of the FQHE hierarchical partition function. Roughly speaking we write \(\psi \simeq \psi_0 + V\), where the field \(\psi_0\) corresponds to the Fermions in the parent Jain state and the field \(V\) corresponds to the quasiparticles moving in the medium of the parent state. Clearly the resulting effective Lagrangian takes the following form:

\[
L_{\text{eff}} = i\psi_0^+(\partial_0 - ia_0 - iA_0)\psi_0 + \psi_0^+(\partial - ia - iA - iA_1)^2\psi_0 + \frac{1}{2k} \frac{1}{4\pi} \epsilon^{\mu\nu\alpha\beta} a_\mu \partial_\nu a_\alpha +

iV^+(\partial_0 - ia_0 - iA_0)V + V^+(\partial - ia - iA - iA_1)^2V,
\]

where the external effective magnetic field \(H_1 = H/(2nk + 1)\) is the same for both type of Fermions \(\psi_0\) and \(V\). The partition function for the Lagrangian (3) is constructed with the help of the factors which fix the number of particles (for example see eq.(4) below). Thus we get the effective partition function for the Jain-type hierarchical states for the states of the Jain sequence. This is the main result of the present letter. To proceed further we should integrate out the Fermi-field \(\psi_0\). This procedure was performed in the number of papers [5],[7]. Here we propose the simple way to obtain the effective action for the gauge fields. In fact one can see that the following equation is valid:

\[
\int D\psi_0^+ D\psi_0 \delta(\int dx \psi_0^+ \psi_0 - \rho_0 - \frac{1}{2\pi n h}) e^{i\psi_0^+ \partial_0 \psi_0 + a_0(\psi_0^+ \psi_0 - \rho_0) + \psi_0^+(\partial - ia - iA_1)^2\psi_0} =

Z \exp(i\pi \int dx \frac{1}{4\pi} \epsilon^{\mu\nu\alpha\beta} a_\mu \partial_\nu a_\alpha + \ldots),
\]

where the dots stand for the terms with two derivatives. Here the argument of the \(\delta\)-function which is introduced to fix the number of particles is chosen in such a way that
at arbitrary $a_\mu$ (including non-zero value of the magnetic field $h = \epsilon_{ij} \partial_i a_j$) the number of the completely filled Landau levels is equal to $n$. Differentiating both sides of eq.(4) over $a_0$ we see that the result is correct which means that eq.(4) is correct. So we have the simple way to derive the equation (4). The same result could be obtained by means of fixing the number of particles with the help of the chemical potential. Thus according to eq.(4) after integration out of the Fermi-field $\psi_0$ we obtain the following effective Lagrangian depending on the statistical gauge field $a_\mu$ and the Vortex Fermi-field $V$:

$$L(a, A) = n \frac{1}{4\pi} \epsilon^{\mu\nu\alpha} (a + A)_\mu \partial_\nu (a + A)_\alpha + \frac{1}{2k} \frac{1}{4\pi} \epsilon^{\mu\nu\alpha} a_\mu \partial_\nu a_\alpha +$$

$$V^+(\partial - ia - iA - iA_1)^2V + (a_0 + A_0)V^+V + A_0 \rho_0,$$

where $A_\mu$ is the external electromagnetic field used to probe the system. Here the number of the completely filled electromagnetic field after the transformation (1) equals $n$. Since the effective magnetic field $H_1 = H/(2nk + 1)$ the density $\rho_0$ corresponding to the field $\psi_0$ corresponds to the filling factor equal to the Jain sequence $\nu_0 = n/(2nk + 1)$. If the quasiparticles are absent at $n = 1$ the Lagrangian (5) coincides with the dual effective Lagrangian for the Laughlin $1/m$ states proposed by Wen [8]. To represent the field $V$ as anyons we introduce the new statistical gauge field $b_\mu$ according to

$$a_\mu + A_\mu = b_\mu + qA_\mu,$$

where the charge of the vortex $q$ equals

$$q = \frac{1}{2nk + 1}. \tag{6}$$

Rewriting the Lagrangian (5) we obtain the Lagrangian

$$L(b, A) = \nu_0 \frac{1}{4\pi} \epsilon^{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha + \frac{1}{\alpha} \frac{1}{4\pi} \epsilon^{\mu\nu\alpha} b_\mu \partial_\nu b_\alpha +$$

$$V^+(\partial - ib - iqA - iA_1)^2V + (b_0 + qA_0)V^+V,$$

where the statistics parameter $\alpha$ (with respect to the Fermions) equals

$$\alpha = \frac{2k}{2nk + 1}. \tag{8}$$

We see from (7) that the field $V$ in fact represents the particles (anyons) with the charge $q$ and statistics $\alpha$. Differentiating the partition function corresponding to (7) over $A_0$ we obtain the additional part of the density $\rho_1 = qn_q$, where $n_q$ is the density corresponding to the field $V$ (the density of Vortices). One could take the derivative over $A_0$ before integration over the field $\psi_0$. In this case we would obtain $\rho = \langle \psi_0^+ \psi_0 + V^+V \rangle$, where the integration measure includes the $\delta$-functions of the form (4). The density $\rho$ takes
the form \( \rho = \rho_0 + 2n\langle \epsilon_{ij}\partial_i a_j \rangle + \langle V^+V \rangle \), where according to eq.(5) \( \epsilon_{ij}\partial_i a_j = -(\alpha/2)V^+V \). Thus we would obtain \( \rho = \rho_0 + \langle V^+V \rangle(-n\alpha + 1) = \rho_0 + q\langle V^+V \rangle \), which gives the same result \( \rho = \rho_0 + qn_q \).

Now one can easily calculate the FQHE filling fractions corresponding to the Jain-type hierarchy from the simple picture of anyons in the magnetic field. We perform the transformation of the type (1) with the integer parameter \( k_1 \) for the anyon wave function and demand that the integer number \( n_1 \) of the Landau levels in the effective magnetic field are completely filled. Thus we obtain the following equation for the density of Vortices \( n_q \):

\[
\frac{2\pi n_q}{H_1 - 2\pi n_q(\alpha - 2k_1)} = n_1.
\]

From this equation we obtain the following equation for the Anyon filling fraction \( \nu_a = 2\pi n_q/H_1 \):

\[
\frac{1}{1/\nu_a - (\alpha - 2k_1)} = n_1.
\]

One can easily see that the total filling fraction \( \nu \) equals

\[
\nu = \nu_0 + q^2 \nu_a,
\]

where \( q \), eq.(6), is the charge of the Vortices. Substituting the values of \( \alpha \), \( q \), \( \nu_a \) into the equation (9) we obtain the following filling factors of Jain-type hierarchy:

\[
\nu = \frac{n(2n_1k_1 + 1) + n_1}{(2nk + 1)(2n_1k_1 + 1) + 2kn_1},
\]

where the numbers \( n,k \ (n_1,k_1) \) are integers (positive or negative) with the allowed values given by the condition \( \nu > 0 \). For \( n = 1 \ (m = 2k + 1) \) the sequence (10) was first presented long time ago in ref.[9].

Let us derive the same result directly from the Lagrangian (5) using the other method. This method does not rely on the values of the charge \( q \), statistics \( \alpha \), and the effective magnetic field \( H_1 \), and is based only on the general form of eq.(5). In fact we know that after the transformation (1) with the parameter \( k_1 \) integrating out the Vortex field \( V \) in eq.(5) since \( n_1 \) Landau levels in the effective magnetic field are filled we obtain the effective Lagrangian of the form:

\[
L_{\text{eff}} = \frac{n}{4\pi} \epsilon^{\mu\nu\alpha}(a + A)_\mu \partial_\nu(a + A)_\alpha + \frac{1}{2k} \frac{1}{4\pi} \epsilon^{\mu\nu\alpha} a_\mu \partial_\nu a_\alpha + \frac{1}{2k_1} \frac{1}{4\pi} \epsilon^{\mu\nu\alpha} c_\mu \partial_\nu c_\alpha + n_1 \frac{1}{4\pi} \epsilon^{\mu\nu\alpha}(a + A + c)_\mu \partial_\nu(a + A + c)_\alpha.
\]

Here the gauge field \( c_\mu \) corresponds to the transformation (1) for the Vortex field. Integrating out the gauge fields \( a_\mu \) and \( c_\mu \) in eq.(11) we obtain the effective Lagrangian of the form \( \nu \frac{1}{4\pi} \epsilon^{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha \) where the conductivity (or the filling factor) \( \nu \) is given exactly
by the equation (10). Thus we see that under the certain assumptions the filling factor \(\nu\) for the Jain-like hierarchy can be obtained even without the knowledge of the parameters \(\alpha, q, H_1\) of the parent state. Presumably the Lagrangian (11) is analogous to the effective Lagrangian proposed in ref.\[3\],\[4\].

3. Conclusion.

In conclusion we proposed the effective Lagrangian which contains both the Fermi- and Bose- fields to describe the Jain-type hierarchical FQHE states. Using this effective Lagrangian (partition function) the charge of the quasiparticle excitations for the states of the Jain sequence was calculated. In contrast to the previous works \[3\],\[4\] the conductivity \(\nu\) was calculated by means of considering the system of anyons in the external magnetic field, and the filling fraction was calculated by means of filling of the Landau levels in the effective magnetic field. The FQHE filling fractions are found from the condition of the non-zero coefficient of the Chern-Simons term in the effective Lagrangian. The comparison of the prediction (10) with the results of the experiments is given in ref.\[3\].

References

[1] W.Pan et.al. Phys.Rev.Lett. 90 (2003) 016801; Phys.Rev.B 77 (2008) 075307.

[2] J.K.Jain, Phys.Rev.Lett. 63 (1989) 199.

[3] A.Lopez, E.Fradkin, Phys.Rev.B 69 (2004) 214519.

[4] A.Lopez, E.Fradkin, Phys.Rev.B 59 (1999) 1532.

[5] S.Ranjbar-Daemi, A.Salam, J.Strathdee, Nucl.Phys.B 340 (1990) 403.

[6] A.A.Ovchinnikov, Pisma Zh.Eksp.Teor.Fiz. 54 (1991) 579; Mod.Phys.Lett. A7 (1992) 611.

[7] A.Lopez, E.Fradkin, Phys.Rev.B 44 (1991) 5246.

[8] X.G.Wen, Adv.Phys. 44 (1995) 405.

[9] A.A.Ovchinnikov, Pisma Zh.Eksp.Teor.Fiz. 55 (1992) 252.