The Small Solution Hypothesis for MAPF on Directed Graphs Is True

Bernhard Nebel

Albert-Ludwigs-Universität
Freiburg, Germany
nebel@uni-freiburg.de

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Abstract

The determination of the computational complexity of multi-agent pathfinding on directed graphs has been an open problem for many years. Only recently, it has been established that the problem is NP-hard. Further, it has been proved that it is in NP, provided the short solution hypothesis for strongly connected digraphs holds. In this paper, it is shown that this hypothesis is indeed true.

1 Introduction

The multi-agent pathfinding (MAPF) problem is the problem of deciding the existence of a collision-free movement plan for a set of agents moving on a graph, most often a graph generated from a grid, where agents can move to adjacent grid cells [11]. Two examples are provided in Figure 1.

Figure 1: Multi-agent pathfinding examples

In the left example, the circular agent $C$ needs to move to $v_2$ and the square agent $S$ has to move to $v_3$. $S$ could move first to $v_2$ and then to $v_3$, after which $C$ could move to its destination $v_2$. So, in this case, a collision-free movement plan exists. In the right example, where additionally the triangle agent has to move to $v_1$, there is no possible
way for the square and triangle agent to exchange their places, i.e., there does not exist any collision-free movement plan.

Kornhauser et al. [9] had shown already in the eighties that deciding MAPF is a polynomial-time problem, although it took a while until this result was recognized in the community [18]. The optimizing variant of this problem, assuming that only one agent can move at each time step, had been shown to be NP-complete soon afterwards [7, 17].

Later on, variations of the problem have been studied, such as using simultaneous movements, train-like movements, and synchronous rotations [21, 26, 6], and even distributed and epistemic versions have been considered [15]. Again, optimizing wrt. different criteria turned out to be NP-complete [21, 26], and this holds even for planar and grid graphs [25, 2]. Additionally, it was shown that there are limits to the approximability of the optimal solution for makespan optimizations [12].

The mentioned results all apply to undirected graphs only. However, a couple of years ago, researchers also started to look into the case of directed graphs [4, 3] and proved polynomial-time decidability for MAPF on directed graphs (diMAPF), provided the graph is strongly biconnected and there are at least two unoccupied vertices. Recently, this result has been generalized to general strongly connected directed graphs [1]. The general case for directed graphs is still open, though. It has only been shown that diMAPF is NP-hard [14], but membership in NP was not established.

In general, the state space of (di)MAPF has size $O(n!)$, $n$ being the number of vertices of the graph. However, for directed acyclic graphs, a quadratic upper bound for the plan length is obvious, because steps are not reversible in this case, and so each single agent can perform at most $n$ steps. For strongly connected directed graphs, this kind of reasoning cannot be employed, but it was conjectured that there is nevertheless a polynomial upper bound on plan length [14]. In this paper, we will show that this small solution hypothesis is indeed true.

Using group theoretic arguments, similar to the ones Kornhauser et al. [9] employed, we show that any movement plan on strongly connected directed graphs can be polynomially bounded. At first, it might seem that group theoretic techniques are not applicable [4, 3], because arcs in directed graphs can be traversed only in one direction, and so movements cannot be undone, precluding the existence of an inverse element. However, in strongly connected graphs this problem can be vindicated by using movements on cycles. After all agents in a directed cycle with $k$ occupied nodes and one unoccupied node have been moved one step ahead, one can restore the original configuration by moving all agents $k - 1$ steps ahead.

Using this insight together with a result about the diameter of permutation groups [5] and recent results about MAPF algorithms on strongly connected graphs with two unoccupied nodes [1, 3], we are able to prove that the small solution hypothesis holds. From that, a general polynomial upper bound for movement plans for all directed graphs follows, which implies that diMAPF is NP-complete [14]. Further, this answers a question about robot movement problems on directed graphs [24], and our results give indications whether the diBOX algorithm [3] could be extended to cover also cases with only one free node.

The rest of the paper is structured as follows. In the next section, we will introduce the necessary notation and terminology that is needed for proving the small solution
hypothesis to be true. We will cover basic concepts from graph theory and permutation groups, and will formally introduce MAPF and diMAPF. In the subsequent section, we will establish the correspondence between permutation groups and diMAPF on strongly connected graphs. After that, we will prove the hypothesis for strongly biconnected digraphs with one unoccupied node and generalize to strongly connected graphs in general. Finally, we will conclude and point out remaining open problems.

2 Notation and Terminology

2.1 Graph Theory

A graph \( G \) is a tuple \((V, E)\) with \( E \subseteq \{\{u, v\} \mid u, v \in V\}\). The elements of \( V \) are called nodes or vertices and the elements of \( E \) are called edges. A directed graph or digraph \( D \) is a tuple \((V, A)\) with \( A \subseteq V^2 \). The elements of \( A \) are called arcs. Given a digraph \( D = (V, A) \), the underlying graph of \( D \), in symbols \( G(D) \), is the graph resulting from ignoring the direction of the arcs, i.e., \( G(D) = (V, \{\{u, v\} \mid (u, v) \in A\}) \). We assume all graphs and digraphs to be simple, i.e., not containing any self-loops of the form \( \{u\} \), resp. \((u, u)\).

Given a graph \( G = (V, E) \), \( G' = (V', E') \) is called sub-graph of \( G \) if \( V \supseteq V' \) and \( E \supseteq E' \). Let \( X \subseteq V \). Then by \( G - X \) we refer to the sub-graph \((V - X, E - \{\{u, v\} \mid u \in X \lor v \in X\})\). Similarly, for digraphs \( D = (V, A) \) and \( D' = (V', A') \), \( D' \) is a sub-digraph if \( V \supseteq V' \) and \( A \supseteq A' \). Let \( D = (V, A) \) and \( X \subseteq V \). Then by \( D - X \) we refer to the sub-digraph \((V - X, A - (X \times V) - (V \times X))\).

A path in a graph \( G = (V, E) \) is a non-empty sequence of vertices and edges of the form \( v_0, e_1, v_1, \ldots, e_k, v_k \) such that \( v_i \in V \), for all \( 0 \leq i \leq k \), \( v_i \neq v_j \) for all \( 0 \leq i < j \leq k \), \( e_j \in E \), and \( \{v_{j-1}, v_j\} = e_j \) for all \( 1 \leq j \leq k \). A cycle in a graph \( G = (V, E) \) is a non-empty sequence of vertices \( v_0, v_1, \ldots, v_k \) such that \( v_0 = v_k \), \( \{v_i, v_{i+1}\} \in E \) for all \( 0 \leq i < k \) and \( v_i \neq v_j \) for all \( 0 \leq i < j \leq k \).

In a digraph \( D = (V, A) \), path and cycle are similarly defined, save that the direction of the arcs have to be respected. This means for a path \( v_0, a_1, v_1, \ldots, a_k, v_k \), we must have \( (v_{j-1}, v_j) = a_j \in A \). And for a cycle \( v_0, v_1, \ldots, v_k = v_0 \), we need \( (v_i, v_{i+1}) \in A \). A digraph that does not contain any cycle is called directed acyclic graph (DAG). A digraph that consists solely of a cycle is called cycle digraph. A digraph that consists of a directed cycle \( v_0, v_1, \ldots, v_k = v_0 \) and any number of arcs connecting adjacent nodes in a backward manner, i.e., \( (v_i, v_{i-1}) \in A \), is called partially bidirectional cycle graph.

A graph \( G = (V, E) \) is connected if there is a path between each pair of vertices. It is biconnected if \( G - \{v\} \) is connected for each \( v \in V \). If a connected graph is not biconnected, then it is separable. This means that there exists a vertex \( v \), called articulation, such that \( G - \{v\} \) is not connected anymore. A connected graph that does not contain a cycle is called tree.

Similarly, a digraph \( D = (V, A) \) is weakly connected, if the underlying graph \( G(D) \) is connected. It is strongly connected, if for every pair of vertices \( u, v \), there is a path in \( D \) from \( u \) to \( v \) and one from \( v \) to \( u \). A digraph is called strongly biconnected if it is strongly connected and the underlying graph \( G(D) \) is biconnected (as, e.g., the digraph
The smallest strongly connected (and strongly biconnected) digraph is the one with one vertex and no arcs. Directed graphs that are strongly connected but not biconnected are called separable (as their undirected counterparts). They possess one or more articulation nodes, which, when removed, lead to disconnected components, as in Figure 2(b).

Figure 2: Strongly biconnected (a) and strongly connected, but separable (b) digraphs

The strongly connected components of a digraph $D = (V, A)$ are the maximal subdigraphs $D_i = (V_i, A_i)$ that are strongly connected. Similarly, a strongly biconnected component of a digraph is a maximal sub-digraph that is strongly biconnected. Note that each strongly connected component can be decomposed into strongly biconnected components that are connected at articulation points. If we now consider the graph formed by the strongly biconnected components as nodes that are connected by edges if the two components share an articulation point, we arrive at the strongly-biconnected-component-graph, which forms a tree [24, Corollary 11].

2.2 Multi-Agent Pathfinding

A multi-agent pathfinding (MAPF) instance $⟨G, R, I, T⟩$ is given by a graph $G = (V, E)$, a set of agents $R$ with $|R| \leq |V|$, an initial state that is an injective function $I: R \rightarrow V$, and a target state that is another injective function $T: R \rightarrow V$. Given an arbitrary state $S: R \rightarrow V$, any node not occupied by an agent, i.e., a node $v \in V - S(R)$, is called blank. A possible successor state $S'$ of $S$ is the function such that one agent $r$ moves from one vertex $u$ to an adjacent vertex $v$ that is a blank in $S$: If $S(r) = u, \{u, v\} \in E$ and there is no $r' \in R$ such that $S(r') = v$, then the successor state $S'$ is identical to $S$ except at the point $r$, where $S'(r) = v$. If there exists a (perhaps empty) sequence of moves that transforms $S$ into $S'$, we say that $S'$ is reachable from $S$. The MAPF problem is then to decide whether $T$ is reachable from $I$.

Often the MAPF problem is defined in terms of parallel movements [19,21], where one step consists of parallel move and wait actions of all agents. However, as long as we are interested only in solution existence, there is no difference between the MAPF problems with parallel and sequential movements. If we allow for simultaneous cyclic rotations [20,26,27], where one assumes that all agents in a fully occupied cycle can move simultaneously, things are a bit different. We will not cover this variation of
permutation is an helpful. For this purpose, we introduce some background on permutation groups.

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2.3 Permutation Groups

In order to establish a polynomial bound for diMAPF movement plans, group theory is helpful. For this purpose, we introduce some background on permutation groups.

A permutation over a set $X$ is a bijective function $\sigma : X \to X$. We say that a permutation is an $m$-cycle if it exchanges elements $x_1, \ldots, x_m$ in a cyclic fashion, i.e., $\sigma(x_i) = x_{i+1}$ for $1 \leq i < m$, $\sigma(x_m) = x_1$ and $\sigma(y) = y$ for all $y \notin \{x_1, \ldots, x_m\}$. Such a cyclic permutation is written as a list of elements, i.e., $(x_1 \ x_2 \ \cdots \ x_m)$. A permutation can also consist of different disjoint cycles. These are then written in sequence, e.g., $(x_1 \ x_2)(x_3 \ x_4)$.

The composition of two permutations $\sigma$ and $\tau$, written as $\sigma \circ \tau$ or simply $\sigma \tau$, is the function mapping $x$ to $\tau(\sigma(x))$. The identity, which maps every element to itself, and $\sigma^{-1}$ is the inverse of $\sigma$, i.e., $\sigma^{-1}(y) = x$ if and only if $\sigma(x) = y$. The $k$-fold composition of $\sigma$ with itself is written as $\sigma^k$. We also consider the conjugate of $\sigma$ by $\tau$, written as $\sigma^\tau$, which is defined to be $\tau^{-1}\sigma\tau$. Such conjugations are helpful in creating new permutations out of existing ones. We use exponential notation as in the book by Mulholland [13]: $\sigma^{\alpha + \beta} := \sigma^\alpha \sigma^\beta$ and $\sigma^{\alpha \beta} := (\sigma^\alpha)^\beta$.

Permutations closed under composition and inverse form a permutation group, with $\circ$ as the product operation, $-1$ being the inverse operation and $\epsilon$ being the identity element. Given a set of permutations $\{g_1, \ldots, g_k\}$, we say that $G = \langle g_1, \ldots, g_k \rangle$ is the group generated by $\{g_1, \ldots, g_k\}$ if $G$ is the group of permutations that results from product operations over the elements of $\{g_1, \ldots, g_k\}$. We say that $\sigma \in G = \langle g_1, \ldots, g_k \rangle$ is $k$-expressible if it can be written as a product over the generators using $\leq k$ product operations. The diameter of a group $G = \langle g_1, \ldots, g_k \rangle$ is the least number $k$ such that every element of $G$ is $k$-expressible. Note that this number depends on the generator set.

In our context, two permutation groups are of particular interest. One is $S_n$, the symmetric group over $n$ elements, which consists of all permutations. $S_n$ is, e.g., generated by the set of all 2-cycles. Another group is $A_n$, the alternating group over $n$ elements, which is the set of all permutations generated by compositions of cycles of odd length, so-called even permutations. One generator of this group is the set of all 3-cycles. $A_n$ is a subgroup of $S_n$, written $A_n \leq S_n$, i.e., it contains only permutations from $S_n$ and is closed under composition and inverse.

A permutation group $G$ is $k$-transitive if for all pairs of $k$-tuples $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_k)$, there exists a permutation $\sigma \in G$ such that $\sigma(x_i) = y_i$, $1 \leq i \leq k$. In case of 1-transitivity we simply say that $G$ is transitive.

A block is a subset $B \subseteq X$ such that for each permutation $\sigma$, either $\sigma(B) = B$ or

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1Note that this order of function applications, which is used in the context of permutation groups, is different from ordinary function composition.
\[ \sigma(B) \cap B = \emptyset. \] Singleton sets and the entire set \( X \) are trivial blocks. A permutation group that contains only trivial blocks is called primitive.

3 Permutation Groups and diMAPF

As mentioned in the Introduction, one critical prerequisite for employing group theory in the context of diMAPF is the ability to, perhaps partially, undo movements. In an undirected graph this is straight-forward. In a directed graph, however, it is not always possible. But if movements took place on a directed cycle in a digraph, then such movements can be undone by rotating all the agents further until the initial configuration is reached again.

**Proposition 1.** Let \( \langle D, R, I, T \rangle \) be a diMAPF instance such that \( D \) is a (partially bidirectional) cycle graph and let \( S \) be a reachable state, then one can reach \( I \) from \( S \) in at most \( O(|V|^2) \) moves.

**Proof.** Since the relative order of agents on a cycle cannot be changed by movements, one can always reach the initial state, regardless of what movements have been made in order to deviate from the initial state. Further, the maximal distance of an agent from its position in \( I \) is \( |V| - 1 \), which is the maximal number of moves the agent has to make to reach \( I \). Since there are at most \( |V| \) agents, the stated upper bound follows.

3.1 Reduction to Permutations

In order to apply group theory, we will restrict ourselves to diMAPF instances on strongly connected digraphs \( \langle D, R, I, T \rangle \) such that the set of occupied nodes is identical in the initial and the goal state, i.e., \( I(R) = T(R) \). Such instances can be viewed as permutations, which opens up the possibility of applying results from group theory. This restriction is non-essential since one can polynomially reduce a general diMAPF instance on strongly connected digraphs to such a permutation instance, adding only polynomial overhead for plan length.

The reduction of diMAPF to a permutation instance is simple. We modify the goal state in a way such that the blanks are moved to the blanks in the initial state. Such a state can be easily reached from the original goal state and, moreover, the original goal state can be easily reached from the modified goal state.

**Lemma 1.** Given a diMAPF instance \( \langle D, R, I, T \rangle \), with \( D \) a strongly connected digraph, an instance \( \langle D, R, I, T' \rangle \) can be computed in polynomial time such that \( I(R) = T'(R) \), and \( \langle D, R, T', T' \rangle \) and \( \langle D, R, T, T' \rangle \) are both solvable using plans of polynomial length.

**Proof.** In order to construct \( \langle D, R, I, T' \rangle \), generate an arbitrary mapping from blanks in \( T \), so called source nodes, to blanks in \( I \), the target nodes. Then move the blanks from the source nodes to the target nodes (against the direction of the arcs) by moving the appropriate agents. This is always possible because \( D \) is strongly connected.

If source and target nodes are both on one cycle, then the movement of the blank can be done by movements of agents on this cycle. If source and target are on different
cycles, then there is must exist a sequence of cycles connected by shared nodes and the blank needs to be moved in each cycle to the shared nodes of the next cycle until the target node is reached.

The new configuration is $T'$ and clearly $T'(R) = I(R)$. Further, $T'$ is obviously reachable from $T$ in at most $O(n^2)$ moves, $n$ being the number of nodes.

Reaching $T$ from $T'$ is possible by undoing each movement of a blank in the opposite order. Undoing such a movement can be done by applying Proposition 1. Note that in case more than cycle has been used, undoing might involve movements on different cycles. Undoing the movement of one blank needs at most $O(n^2)$ moves. Since there can be at most $n$ blanks, one needs overall $O(n^3)$ moves.

In other words, solving $⟨D,R,I,T⟩$ is equivalent to solving $⟨D,R,I,T'⟩$ with, perhaps, some additional overhead. This means, we can view diMAPF instances on strongly connected digraphs as permutations.

The interesting question is whether solvable diMAPF instances on strongly connected graphs induce permutation groups, i.e., sets of permutations closed under composition and inverse. Composition is straightforward, because it simply amounts to solving two diMAPF instances.

**Proposition 2.** If $⟨D,R,I,T⟩$ and $⟨D,R,T,S⟩$ are both solvable diMAPF instances on strongly connected graphs using $k$ and $\ell$ moves, respectively, then $⟨D,R,I,S⟩$ is solvable using $k + \ell$ moves.

When we turn to the inverse operation, Proposition 1 is helpful. This proposition gives a recipe for undoing movements that have taken place when moving agents on a cycle, i.e., providing the inverse for a rotation on a cycle. Since general movements on a strongly connected component can be viewed as a sequence of rotations on cycles, the next proposition follows immediately.

**Proposition 3.** If $⟨D,R,I,T⟩$ is a solvable diMAPF instance with $k$ moves on a strongly connected graph, then $⟨D,R,T,I⟩$ is solvable in $O(k \times n^2)$ moves.

Given a diMAPF instance $⟨D,R,I,T⟩$ with $D$ a strongly connected digraph, let us now consider the set of permutations corresponding to rotations on cycles that can be implemented by movements of agents. We call the group generated by this permutations the permutation group induced by the diMAPF instance. From the above, the following Proposition is immediate.

**Proposition 4.** Let $⟨D,R,I,T⟩$ with $D$ a strongly connected digraph be a diMAPF instance. Then the induced permutation group has a polynomial diameter if and only if for each $S$, $⟨D,R,I,S⟩$ admits a polynomial movement plan or is unsolvable.

Initial state and terminal state are not relevant for the induced permutation group. For this reason, we will often talk about the group generated by a digraph and the number of blanks in the following, meaning the group generated by diMAPF instances with such a digraph and number of blanks.
3.2 Strongly Connected Digraphs With One Blank

In case a diMAPF instance contains only one blank, the movements of agents are somewhat restricted. In particular, no agent can cross an articulation point using a rotation on a cycle as demonstrated in Figure 3.

![Figure 3: Strongly connected digraph with articulation node](image)

Note that when moving an agent from $a_1$ to the blank on node $z$, the only way to get the blank back to node $z$ is to move agents left of the articulation node. So, the blank may freely move, but no agent can cross an articulation point.

**Proposition 5.** A diMAPF instance on a strongly connected digraph with one blank has a solution if and only if the strongly biconnected components have each a solution in isolation.

3.3 Strongly Biconnected Digraphs

Proposition 5 implies that we can concentrate on strongly biconnected digraphs. For this reason, we will have a closer look at them and extract what can be considered as the minimal structure. If it is not a partially bidirectional cycle graph, then it can be decomposed into a basic cycle and one or more open ears [24, 26]. This means that each such graph contains at least a sub-graph consisting of a directed cycle (e.g., as $c_1, c_2, c_3, c_4, c_5$ in Figure 4 which is drawn solidly), and an open ear (as, e.g., $e_1, e_2$, drawn in a dotted way). An open ear is a directed path that starts at some node of the cycle (in this case, $c_2$) and ends at a different node ($c_4$). The ear could either be oriented in the same direction as the cycle, providing a detour or short-cut as in Figure 4(a), or it points back, as in Figure 4(b).

So, we will be able to always find two connected cycles. In order to be able to deal with only one kind of cycle pair, we will view a graph as in Figure 4(a) as one in Figure 4(b). This can be accomplished by considering the outer, larger cycle as the basic cycle and the path with $c_3$ as an ear, starting at $c_2$ and ending at $c_4$, as illustrated in Figure 5. Note that in the extreme one may have an ear with no additional nodes (which technically would not be called ear).

This means that in a strongly biconnected digraph one can always find two directed cycles that share at least two nodes (otherwise it would not be biconnected) and at least one of the cycles contains more nodes than just the shared nodes (otherwise we would not have two cycles). Further, in such graphs, where one cycle consists only of shared
nodes, there must be at least three shared nodes, because otherwise we would not have two directed cycles, but the structure would be a partially bidirectional cycle.

**Proposition 6.** A strongly biconnected digraph that is not a partially bidirectional cycle graph contains two directed cycles that share at least 2 nodes, where at least one cycle contains more nodes than the shared ones. Further, if one cycle contains only shared nodes, then there are at least three shared nodes.

## 4 The Small Solution Hypothesis

The state space of diMAPF has size $O(n!)$, where $n$ is the number of nodes of the graph. However, it is by no means obvious that one has to go through a significant part of the state space in order to arrive at the goal configuration. Since for DAGs the movement plans are polynomially bounded [14, Proposition 3], we only have to consider the movements in strongly connected components.

We know from the work of Botea et al. [3] that for all instances on strongly biconnected digraphs that are not partially bidirectional cycle graphs, and where we have at least two unoccupied vertices, all configurations can be reached using only cubicly many moves. The recent result by Ardizzoni et al. [1] generalizes this result to general strongly connected graphs. If we allow for only one unoccupied vertex, solution existence cannot be any longer guaranteed [3]. Moreover, it is not any longer clear whether a polynomial long sequence suffices, although the latter sounds very plausible, which
led to the formulation of the following hypothesis [14].

**Hypothesis 1** (Short Solution Hypothesis for diMAPF). *For each solvable diMAPF instance on strongly connected digraphs, there exists a movement plan of polynomial length.*

### 4.1 General Approach to Proving the Hypothesis

In order to show that the permutation groups induced by diMAPF instances have polynomial diameter (which implies that the instance have polynomial plans), we will use the following result by Driscoll and Furst [5, Theorem 3.2].

**Theorem 1** (Driscoll & Furst). *If $G$ is a primitive group containing a polynomially expressible 3-cycle, then the diameter of $G$ is polynomially bounded.*

Incidentally, if the conditions of the Theorem are satisfied, the group contains the alternating group $A_n$, as is obvious from the proof of the Theorem.

It should be noted that 2-transitivity implies primitiveness, because for a non-trivial block $Y$ there would exist one permutation that fixes an element (staying in the block) and moves another element out of the block, which contradicts that $Y$ is a non-trivial block.

**Proposition 7.** *Every 2-transitive permutation group is primitive.*

### 4.2 2-Transitivity in Strongly Biconnected Digraphs With One Blank

Almost all permutation groups induced by diMAPF instances on strongly biconnected digraphs with one blank are 2-transitive, as shown next. Intuitively, it means that we can move any two agents to any two places in the digraph—moving perhaps other agents around as well.

**Lemma 2.** *Permutation groups induced by diMAPF instances with at least one blank on strongly biconnected digraphs that are not partially bidirectional cycle graphs are 2-transitive.*

**Proof.** Such permutation groups are transitive, i.e., we can send any agent $a_x$ on node $x$ to any place $y$. This holds, because for each pair of nodes, there exists a sequence of cycles that are connected by shared nodes, so that $x$ is on the first cycle, $y$ is on the last cycles, and $a_x$ can be sent to $y$ by a sequence of rotations.

Further, since we assumed that the graph is not a partially bidirectional cycle graph, the graph must contain at least two cycles, as depicted in Figure 6. Figure 6(a) covers the case that both cycles contain nodes that are not shared nodes, while Figure 6(b) displays the case where the left cycle consists only of shared nodes. In both cases, we will assume that node $z$ is a blank. The dashed structures signify possible additional nodes. The dotted arcs exemplify potential connections to other nodes in the graph.

By transitivity, we can move any agent $a_x$ to node $x$. After that, we can move any agent $a_y$ to the node $y$. This may lead to rotating the agent $a_x$ out of the left cycle. In order to prohibit that, we modify the movement plan as follows. As long as $a_y$ has not
entered one of the two cycles yet, every time \(a_x\) is threatened to be rotated out of the left cycle in the next move, we rotate the entire cycle so that \(a_x\) is moved to a node that will not lead to rotating \(a_x\) out of the left cycle. Depending on the structure, we proceed differently after \(a_y\) has entered one of the cycles.

In case (a), if \(a_y\) arrived at the right cycle and not on the shared nodes, we rotate \(a_x\) to \(x\) and then \(a_y\) to \(y\). If \(a_y\) arrived at the left cycle or the shared nodes, we rotate the left cycle until \(a_y\) is on node \(u\), then we rotate the right cycle moving \(a_y\) to \(y\) and afterwards \(a_x\) can be rotated to \(x\).

In case (b), the plan for moving \(a_y\) is changed in the following way. After \(a_y\) arrived in the right cycle, we rotate the right cycle iteratively. Whenever \(a_x\) is placed on \(x\) and \(a_y\) has not yet arrived at \(y\), we rotate on the left cycle. Otherwise, we stop and are done. When \(a_y\) is placed on \(x\), then in the next move, we rotate the right cycle and \(a_y\) is placed on \(y\). After that, we can rotate the left cycle until \(a_x\) is arrives at \(x\). This is possible, because \(a_x\) never leaves the left cycle.

Similarly, we could move two agents \(a_v\) and \(a_w\) to \(x\) and \(y\). Now, by combining the first movement plan for \(a_x\) and \(a_y\) with the reversal of the plan for moving \(a_v\) and \(a_w\), one can move any pair of agents to any desired two places, which means that the induced permutation group is 2-transitive.

\[\square\]

### 4.3 3-Cycles in Strongly Biconnected Graphs

The construction of 3-cycles will be shown by a case analysis over the possible forms of two connected cycles. By Proposition 6, we know that every strongly biconnected digraph contains a subgraph as shown in Figure 6, where we will assume that the node \(z\) is a blank.

We characterize such connected cycles by the three parameters \((n_1, n_2, n_3)\) and will talk about cycle pairs of type \((n_1, n_2, n_3)\), assuming wlg. \(n_1 \leq n_3\). Below, we will show that for almost all cycle pairs one can construct a 3-cycle, save for two exceptions, namely, for cycle pairs of type \((2, 2, 2)\) and \((1, 3, 2)\). These are the directed counterparts of Kornhauser et al.’s \([9]\) \(T_0\)- and Wilson’s \([23]\) \(\theta_0\)-graph. We will therefore call such
cycle pairs $T_0$-pairs.

**Lemma 3.** Each permutation group induced by a cycle pair with one blank that is not a $T_0$-pair contains a polynomially expressible 3-cycle.

**Proof.** The 3-cycles will be constructed from $\alpha = (a_1 \ldots a_{n_1} b_{n_2} \ldots b_1)$ and $\beta = (c_1 \ldots c_{n_3} b_{n_2} \ldots b_1)$. We proof the claim by case analysis over the parameters $(n_1, n_2, n_3)$.

1. $(0,1,\_):$ This implies $n_2 \geq 2$ and $n_3 \geq 1$ and we have $\beta \alpha^{-1} \beta^{-1} \alpha = (b_{n_2} c_{n_3} b_1)$ as a 3-cycle.

2. $(\geq 1,1,\_):$ $\beta \alpha^{-1} \beta^{-1} \alpha = (b_1 a_1 c_{n_3})$ is the desired 3-cycle.

3. $(1,2,\_):$ $\alpha$ is a 3-cycle.

4. $(1,\geq 3,1):$ $\alpha^{-1} \beta = (b_{n_2} a_1 c_1)$ is the desired 3-cycle.

5. $(1,3,2):$ This case is excluded, so there is nothing to prove here. However, it can be shown by exhaustive enumeration that the induced group does not contain 3-cycles.

6. $(1,3,\geq 3):$ $\beta \alpha^{-1} \beta^{-1} \alpha = (b_{n_2} c_{n_3})(b_1 a_1)$. Consider now $\chi = \beta^2 (\alpha^{-1} \beta)^2 \beta^{-2}$. This permutation fixes $b_{n_2}$ and $c_{n_3}$ and moves $c_{n_3}^{-2}$ to $a_1$ and $a_1$ to $b_1$ while also moving other things around. This means:

$$
(\beta \alpha^{-1} \beta^{-1} \alpha)^{\chi^{-1}} = \chi \beta \alpha^{-1} \beta^{-1} \alpha \chi^{-1} = \chi (b_{n_2} c_{n_3})(b_1 a_1)^{\chi^{-1}} = (b_{n_2} c_{n_3})(a_1 c_{n_3}^{-2}).
$$

Composing the result with the original permutation is now what results in a 3-
That is, \( \lambda = (\beta \alpha^{-1} \beta^{-1} \alpha)^{r+1} \) is the permutation we looked for:

\[
\lambda = (\beta \alpha^{-1} \beta^{-1} \alpha)^{r+1} \\
= ((b_{n_2} c_{n_3})(b_1 a_1)) \circ ((b_{n_2} c_{n_3})(b_1 a_1))^{x^{-1}} \\
= ((b_{n_2} c_{n_3})(b_1 a_1)) \circ ((b_{n_2} c_{n_3})(a_1 c_{n_3-2})) \\
= (b_1 c_{n_3-2} a_1).
\]

\((1, \geq 4, \geq 2):\) In this case, \( \xi = (\alpha \beta^{-1} \alpha^{-1} \beta)^{y(\epsilon+\alpha^{-2})} \) is the claimed 3-cycle:

\[
\xi = (\alpha \beta^{-1} \alpha^{-1} \beta)^{y(\epsilon+\alpha^{-2})} \\
= ((b_{n_2} a_{n_1})(b_1 c_1))^{y(\epsilon+\alpha^{-2})} \\
= ((b_{n_2-1} a_1)(c_1 c_2))^{y+\alpha^{-2}} \\
= ((b_{n_2-1} a_{n_1})(c_1 c_2)) \circ ((b_{n_2-1} a_1)(c_1 c_2))^{\alpha^{-2}} \\
= ((b_{n_2-1} a_{n_1})(c_1 c_2)) \circ ((b_2 a_{n_1})(c_1 c_2)) \\
= (b_{n_2-1} b_2 a_{n_1}).
\]

\((2, 2, 2):\) This is the other case that is excluded in the claim and the same comment as in case \((1, 3, 2)\) applies.

\((2, \geq 3, 2):\) For this case, the sought 3-cycle is \( \zeta = (\beta \alpha^{-1} \beta^{-1} \alpha)^{y(\epsilon+\beta^{-2})}:\)

\[
\zeta = (\beta \alpha^{-1} \beta^{-1} \alpha)^{y(\epsilon+\beta^{-2})} \\
= ((b_{n_2} c_{n_3})(a_1 b_1))^{y(\epsilon+\beta^{-2})} \\
= ((b_{n_2-1} c_{n_3})(a_1 a_2))^{y+\beta^{-2}} \\
= ((b_{n_2-1} c_{n_3})(a_1 a_2)) \circ ((b_{n_2-1} c_{n_3})(a_1 a_2))^{\beta^{-2}} \\
= ((b_{n_2-1} c_{n_3})(a_1 a_2)) \circ ((b_1 c_{n_3})(a_1 a_2)) \\
= (b_{n_2-1} b_1 c_{n_3}).
\]

\((\geq 3, \geq 2, \geq 3):\) Interestingly, the above permutation works for the general case, when a cycle pair is “large enough,” as well. Because of the different structure of the cycle pairs, the result is slightly different, though (differences are underlined):

\[
\zeta = (\beta \alpha^{-1} \beta^{-1} \alpha)^{y(\epsilon+\beta^{-2})} \\
= ((b_{n_2} c_{n_3})(a_1 b_1))^{y(\epsilon+\beta^{-2})} \\
= ((b_{n_2-1} c_{n_3})(a_1 a_2))^{y+\beta^{-2}} \\
= ((b_{n_2-1} c_{n_3})(a_1 a_2)) \circ ((b_{n_2-1} c_{n_3})(a_1 a_2))^{\beta^{-2}} \\
= ((b_{n_2-1} c_{n_3})(a_1 a_2)) \circ ((c_{n_3-2} c_{n_3})(a_1 a_2)) \\
= (b_{n_2-1} c_{n_3-2} c_{n_3}).
\]

\(^2\)This construction is similar to one used by Kornhauser in the proof of Theorem 1 for \( T_2 \)-graphs.
This covers all possible cases. So, the claim holds. □

In order to be able to do away with $T_0$-pairs, we will assume that our digraphs contain at least 8 nodes. For all smaller digraphs, the diameter of the induced group is constant. One only has then to show that strongly biconnected digraphs with 8 or more nodes admit for the generation of a 3-cycle.

**Lemma 4.** Each strongly biconnected digraph with at least eight nodes, where one node is blank, induces a permutation group containing a polynomially expressible 3-cycle.

**Proof.** By Lemma 3, it is enough to prove the claim for digraphs that contain a $T_0$-pair. In order to do so, all possible extensions of $T_0$-pairs with one additional ear have to be analyzed. One then has to show that by adding the ear, a new cycle pair is created that is not a $T_0$-pair, in which case Lemma 3 is applicable. If the newly created cycle pairs are all $T_0$-pairs, one has to demonstrate that by the addition of the ear a new permutation is added that can be used to create a 3-cycle.

Because the longest ear in $T_0$-pairs has a length of 2, we consider ears up to length 3. Since the $T_0$-pairs contain seven nodes each, and we consider ears of length one to three, we need to analyze $2 \times (7^2 - 7) \times 3 = 252$ cases. This has been done using a SageMath [22] script, which is listed in the appendix. This script identified seven extensions of $(2, 2, 2)$-type cycle pairs and two extensions of $(1, 3, 2)$-type pairs which contain only $T_0$-pairs. These are shown in Figure 8, whereby we have left out symmetric cases. For instance, the case with an ear from $a_1$ to $c_2$ was left out because we already consider the case with an ear from $c_1$ to $a_2$. It is now an easy exercise to identify 3-cycles for all cases:

(a) & (b): $\alpha^{-1} \gamma^{-1} \beta^{-1} \alpha^2 \gamma^{-1} \beta^{-1} \gamma^{-1}$,

(c): $\alpha \beta \gamma' \alpha^{-1} \beta^{-1} \gamma'^{-1}$,

(d): $\gamma'' \beta^{-1} \gamma'^{-1} \alpha$,

(e): $\delta \beta^{-1} \delta^{-1} \alpha$,

(f): $\beta^{-1} \delta'^{-1} \beta \alpha$.

So, the claim holds for all cases, therefore it is true for all digraphs with eight nodes or more. □

**Theorem 2.** Each permutation group induced by a strongly connected digraph containing one blank has a polynomial diameter.

**Proof.** We prove the claim by case analysis.

1. The digraph is a strongly biconnected:

(a) The digraph has less than 8 nodes: There exist only finitely many permutation groups induced by such graphs. The diameter is therefore $O(1)$ in this case.
Figure 8: Extensions of $T_0$-pairs containing only $T_0$-pairs

(b) The digraph has 8 nodes or more:

i. The digraph is a partially bidirectional cycle of $n$ nodes: In this case, each possible permutation can be expressed by at most $O(n)$ compositions of the permutation induced by rotating all agents by one place, i.e., the claim holds.

ii. The digraph is a strongly biconnected digraph that is not a partially bidirectional cycle: For this case, we use Theorem 1. It is enough to show that a group is 2-transitive and contains a polynomially expressible 3-cycle. By Proposition 6, we know that the digraph contains a cycle pair. Now, 2-transitivity follows from Lemma 2. The existence of a polynomially expressible 3-cycle follows from Lemma 4. So, in this case, the claim holds as well.

2. The digraph is a strongly connected, but not a biconnected digraph: According to Proposition 5, such a graph can be decomposed into strongly biconnected com-
ponents, each of which can be solved in isolation. This means, we can consider the permutation groups that are induced by each strongly biconnected component in isolation. As is shown in case 1 strongly biconnected digraphs have a polynomial diameter, i.e., the claim is true.

This covers all possible cases, so the claim holds.

4.4 General Case

Now we can put everything together and address the original question.

**Theorem 3.** The Small Solution Hypothesis for diMAPF is true.

*Proof.* We have to show that the diMAPF problem on strongly connected digraphs admits polynomial plans. If there are no blanks, no agent can be moved and for this reason the plan length is always 0. If there is one blank, the claim follows from Theorem 2 and Proposition 4. If there are two or more blanks, the claim follows from the results of Ardizzoni et al. [1].

As mentioned above, this result enables us to finally settle the question of the computational complexity of diMAPF. Using Theorem 4 from the paper establishing NP-hardness of diMAPF [14], the next Theorem is immediate.

**Theorem 4.** diMAPF is NP-complete.

5 Conclusion and Outlook

This paper provides an answer to a longstanding open question about the computational complexity of the multi-agent pathfinding problem on directed graphs. Together with the results from an earlier paper [14], we can conclude that diMAPF is NP-complete. At the same time, this answers a question about the generalization of the robot movement problem [16] to directed graphs with more than one robot and no movable obstacles [24].

While the result might have only a limited impact on practical applications, it nevertheless provides some surprising insights. First of all, it shows that group theory is applicable for the analysis of diMAPF, something that was not obvious previously [4, 3]. Second, in proving the result, some unforeseen obstacles popped up, such as that Lemma 1 from the paper by Kornhauser et al. [9] is not applicable to directed graphs and that there are two different directed counterparts to the $T_0$-graph. Third, the result could be taken as a suggestion to extend the diBOX algorithm [3] in order to deal with the one-blank case. Although the proof of the small solution hypothesis is in principle a constructive one, it probably does not provide hints for an efficient implementation. Interesting questions in this context are to come up with an efficient decision criterion for feasibility and a good bound for the generated movement plans.

Finally, there may be the question of how simultaneous cyclic rotations [20, 26] would fit into the picture. It looks as if the proofs of the Lemmas 3 and 4 could be adapted to the case of no blanks in a strongly connected digraph, i.e., it is very likely that the result still holds.
A  SageMath Script for the Proof of Lemma 4

```python
def sharednodes(c1, c2):
    # computes number of shared nodes of two simple cycles
    return len(set(c1) & set(c2))

def pairtype(c1, c2):
    # computes the type of a pair of cycles that share some nodes
    if len(c1) > len(c2):
        c1, c2 = c2, c1
    if sharednodes(c1, c2) < 2:
        return None
    else:
        return (len(c1) - sharednodes(c1, c2) - 1,
                sharednodes(c1, c2) - 1,
                len(c2) - sharednodes(c1, c2) - 1)

def checkforpair(dig):
    # iterate over all cycle pairs in a digraph
    # and try to identify a non-T0-pair
    for c1 in dig.all_simple_cycles():
        for c2 in dig.all_simple_cycles():
            if c1 != c2:
                if pairtype(c1, c2) not in [(2, 2, 2), (1, 3, 2), None]:
                    return pairtype(c1, c2)
    return None

T0adict = { 'a1': ['a2'], 'a2': ['z'], 'z': ['b2'], 'b2': ['b1'], 'b1': ['a1', 'c1'], 'c1': ['c2'], 'c2': ['z'] }
T0bdict = { 'a1': ['z'], 'z': ['b3'], 'b3': ['b2'], 'b2': ['b1'], 'b1': ['a1', 'c1'], 'c1': ['c2'], 'c2': ['z'] }
Ears = [ ['e1'], ['e1', 'e2'], ['e1', 'e2', 'e3'] ]

for graph, ptype in ( (T0adict, (2, 2, 2)), (T0bdict, (1, 3, 2)) ):
    for head in graph.keys():
        for tail in graph.keys():
            if tail == head:
                continue
            for ear in Ears:
                T0 = DiGraph(graph)
                T0.add_path([head] + ear + [tail])
                if not checkforpair(T0):
                    print("Only T0s in pair of type", ptype,
                          "for ear:", [head] + ear + [tail])
```

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