Some applications of $D\alpha$-closed sets in topological spaces

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ABSTRACT

In this paper, a new kind of sets called $D\alpha$-open sets are introduced and studied in a topological spaces. The class of all $D\alpha$-open sets is strictly between the class of all $\alpha$-open sets and g-open sets. Also, as applications we introduce and study $D\alpha$-continuous, $D\alpha$-open, and $D\alpha$-closed functions between topological spaces. Finally, some properties of $D\alpha$-closed and strongly $D\alpha$-closed graphs are investigated.

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1. Introduction and preliminaries

Generalized open sets play a very important role in General Topology, and they are now the research topics of many topologies worldwide. Indeed a significant theme in General Topology and Real Analysis is the study of variously modified forms of continuity, separation axioms, etc. by utilizing generalized open sets. One of the most well-known notions and also inspiration source are the notion of $\alpha$-open [1] sets introduced by Njastad in 1965 and generalized closed (g-closed) subset of a topological space [2] introduced by Levine in 1970.
Since then, many mathematicians turned their attention to the
generalization of various concepts in General Topology by con-
sidering α-open sets [3–10] and generalized closed sets [11–13].
In 1982 Dunham [14] used the generalized closed sets to deﬁne
a new closure operator, and thus a new topology τ∗, on the
space, and examined some of the properties of this new top-
ology. Throughout the present paper (X, τ), (Y, σ) and (Z, ρ)
deﬁne topological spaces (brieﬂy X, Y and Z) and no separa-
tion axioms are assumed on the spaces unless explicitly stated.
For a subset A of a space (X, τ), Cl(A) and Int(A) denote the
closure and the interior of A, respectively. Since we require the
following known deﬁnitions, notations, and some properties,
we recall in this section.

Definition 1.1. Let (X, τ) be a topological space and A ⊆ X. Then

(i) A is α-open [1] if A ⊆ Int(Cl(αInt(A))) and α-closed [1] if
Int(Cl(αInt(A))) ⊆ A.
(ii) A is generalized closed (brieﬂy g-closed) [2] if Cl(A) ⊆ U
whenever A ⊆ U and U is open in X.
(iii) A is generalized open (brieﬂy g-open) [2] if X \ A is g-closed.

The α-closure of a subset A of X [3] is the intersection of
all α-closed sets containing A and is denoted by Clα(A). The
α-interior of a subset A of X [3] is the union of all α-open sets
contained in A and is denoted by Intα(A). The intersection of
all g-closed sets containing A [14] is called the g-closure of A
and denoted by Cl∗(A), and the g-interior of A [15] is the union
of all g-open sets contained in A and is denoted by Int∗(A).
We need the following notations:

• αO(X) (resp. αC(X)) denotes the family of all α-open sets (resp.
α-closed sets) in (X, τ).
• GO(X) (resp. GC(X)) denotes the family of all generalized open
sets (resp. generalized closed sets) in (X, τ).
• αO(X, x) = {U | x ∈ U & αO(X, τ)} , O(X, x) = {U | x ∈ U & τ}
and αC(X, x) = {U | x ∈ U & αC(X, τ)}.

Definition 1.2. A function f : X → Y is said to be:

(i) α-continuous [16] (resp. g-continuous [17]) if the inverse
image of each open set in Y is α-open (resp. g-open) in X.
(ii) α-open [16] (resp. α-closed [16]) if the image of each open
(resp. closed) set in X is α-open (resp. α-closed) in Y.
(iii) g-open [18] (resp. g-closed [18]) if the image of each open
(resp. closed) set in X is g-open (resp. g-closed) in Y.

Definition 1.3. Let f : X → Y be a function:

(i) The subset {(x, f(x)) | x ∈ X} of the product space X × Y is
called the graph of f [19] and is usually denoted by G(f).
(ii) A closed graph [19] if its graph G(f) is closed sets in the
product space X × Y.
(iii) a strongly closed graph [20] if for each point (x, y) ∈ G(f),
there exist open sets U ⊆ X and V ⊆ Y containing x and
y, respectively, such that (U × Cl(V)) ∩ G(f) = φ.
(iv) A strongly α-closed graph [21] if for each (x, y) ∈ (X × Y) \ G(f),
there exist U ∈ αO(X, x) and V ∈ O(Y, y) such that
(U × Cl(V)) ∩ G(f) = φ.

Definition 1.4. A topological space (X, τ) is said to be:

(i) α-τ, [9] (resp. g-τ, [22]) if for any distinct pair of points
x and y in X, there exist α-open (resp. g-open) set U in
X containing x but not y and an α-open (resp. g-open)
set V in X containing y but not x.
(ii) α-τ, [8] (resp. g-τ, [23]) if for any distinct pair of points
x and y in X, there exist α-open (resp. g-open) sets U and
V in X containing x and y, respectively, such that U ∩ V = φ.

Lemma 1.5. Let A ⊆ X, then

(i) X \ Cl∗(A) = Int∗(X \ A).
(ii) X \ Int∗(A) = Cl∗(X \ A).

Lemma 1.6. A function f : (X, τ) → (Y, σ) has a closed graph [19]
if for each (x, y) ∈ (X × Y) \ G(f), there exist U ∈ O(X, x) and
V ∈ O(Y, y) such that f(U) ∩ V = φ.

Lemma 1.7. The graph G(f) is strongly closed [23] if and only
if for each point (x, y) ∈ O(f), there exist open sets U ⊆ X and
V ⊆ Y containing x and y, respectively, such that f(U) ∩ Cl(V) = φ.

2. Da-closed sets

In this section we introduce Da-closed sets and investigate some
of their basic properties.

Definition 2.1. A subset A of a space X is called Da-closed if
Cl∗(Int(αInt∗(A))) ⊆ A.

The collection of all Da-closed sets in X is denoted by DaC(X).

Lemma 2.2. If there exists an g-closed set F such that
Cl∗(Int(F)) ⊆ A ∩ F, then A is Da-closed.

Proof. Since F is g-closed, Cl∗(F) = F. Therefore, Cl∗(Int(αInt∗(A))) ⊆
Cl∗(Int(Cl∗(F))) = Cl∗(F) ⊆ A. Hence A is Da-closed.

Remark 2.3. The converse of above lemma is not true as shown
in the following example.

Example 2.4. Let (X, τ) be a topological space, where X = {a, b,
c} and τ = {φ, {a}, {a, b}, X}. Then F = {φ, {a}, [b, c], X}, GC(X) =
{a, [c], {a, c}, X}, Cl(X) = φ, {a}, {a, b}, X}, DaC(X) = φ, {a}, {a, c}, {b, c}, X},
DaO(X) = φ, {a}, {b}, X}, Daα(X) = φ, {a}, {a, c}, {b, c}, X}. Therefore
[c] ∈ DaαC(X) and [a, c] ∈ GC(X) but Cl∗(Int([a, c])) = [a, c] ⊆ [a, c].

Theorem 2.5. Let (X, τ) be a topological space. Then

(i) Every α-closed subset of (X, τ) is Da-closed.
(ii) Every g-closed subset of (X, τ) is Da-closed.

Proof. (i) Since closed set is g-closed, Cl∗(A) ⊆ Cl(A) [14]. Now,
suppose A is α-closed in X, then Cl∗(Int(αInt(A))) ⊆ A. Therefore,
Cl∗(Int(αInt∗(A))) ⊆ Cl(αInt(αInt∗(A))) ⊆ A. Hence A is Da-closed in X.
Remark 2.6. The converse of above theorem is not true as shown in the following example.

(i) $\alpha$-closed set need not be $\alpha$-closed.

(ii) $\alpha$-closed set need not be g-closed.

Example 2.7. Let $(X, \tau)$ be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Then $F_1 = \alpha C(X) = \{\emptyset, \{c\}, X\}$, $\alpha O(X) = \{\emptyset, \{a, b\}, X\}$, $G O(X) = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}$, $\alpha W(X) = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}$. Therefore $\{a\} \in DO(X)$, but $\{a\} \notin \alpha C(X)$ and $\{a\} \notin G C(X)$.

From the above discussions we have the following diagram in which the converses of implications need not be true.

$\alpha$-closed set $\rightarrow$ $\alpha$-closed set $\leftarrow$ g-closed set

Theorem 2.8. Arbitrary intersection of $\alpha$-closed sets is $\alpha$-closed.

Proof. Let $\{F_i : i \in A\}$ be a collection of $\alpha$-closed sets in $X$. Then $\cap \{\alpha C(F) : i \in A\}$ is $\alpha$-closed for each $i$. Since $\cap F_i \subseteq F_i$ for each $i$, $\cap \{\alpha C(F) : i \in A\}$ for each $i$. Hence $\alpha C(\cap F_i) \subseteq \alpha C(\cap F_i)$ for each $i \in A$. Therefore $\alpha C(\cap \{\alpha C(F) : i \in A\}) \subseteq \cap \{\alpha C(F) : i \in A\} \subseteq \alpha C(\cap \{\alpha C(F) : i \in A\}) \subseteq \alpha C(\cap \{\alpha C(F) : i \in A\})$. Hence $\cap F_i$ is $\alpha$-closed.

Remark 2.9. The union of two $\alpha$-closed sets need not to be $\alpha$-closed as shown in Example 2.7, where both $\{a\}$ and $\{b\}$ are $\alpha$-closed sets but $\{a\} \cup \{b\} = \{a, b\}$ is not $\alpha$-closed.

Corollary 2.10. If a subset $A$ is $\alpha$-closed and $B$ is $\alpha$-closed, then $A \cap B$ is $\alpha$-closed.

Proof. Follows from Theorem 2.5 (i) and Theorem 2.8.

Corollary 2.11. If a subset $A$ is $\alpha$-closed and $F$ is g-closed, then $A \cap F$ is $\alpha$-closed.

Proof. Follows from Theorem 2.5 (ii) and Theorem 2.8.

Definition 2.12. Let $A$ be a subset of a space $X$. The $\alpha$-closure of $A$, denoted by $\alpha C(A)$, is the intersection of all $\alpha$-closed sets in $X$ containing $A$. That is $\alpha C(A) = \cap \{F : A \subseteq F \text{ and } F \in DO(X)\}$.

Theorem 2.13. Let $A$ be a subset of $X$. Then $A$ is $\alpha$-closed set in $X$ if and only if $\alpha C(A) = A$.

Proof. Suppose $A$ is $\alpha$-closed set in $X$. By Definition 2.12, $\alpha C(A) = A$. Conversely, suppose $\alpha C(A) = A$. By Theorem 2.8 $A$ is $\alpha$-closed.

Theorem 2.14. Let $A$ and $B$ be subsets of $X$. Then the following results hold.

(i) $A \subseteq \alpha C(B) \subseteq \alpha C(A)$.
(ii) $\alpha C(\emptyset) = \emptyset$ and $\alpha C(A) = X$.
(iii) If $A \subseteq B$, then $\alpha C(A) \subseteq \alpha C(B)$.
(iv) $\alpha C(\alpha C(A)) = \alpha C(A)$.
(v) $\alpha C(A) \cup \alpha C(B) \subseteq \alpha C(A \cup B)$.
(vi) $\alpha C(A \cap B) \subseteq \alpha C(A) \cap \alpha C(B)$.

Proof. (i) Follows from Theorem 2.5 (i) and (ii), respectively.
(ii) and (iii) are obvious.
(iv) If $A \subseteq F$, $F \in DO(X)$, then from (iii) and Theorem 2.13, $\alpha C(A) = \alpha C(F) = F$. Again $\alpha C(\alpha C(A)) = \alpha C(F) = F$. Therefore $\alpha C(\alpha C(A)) = \cap \{F : A \subseteq F, F \in DO(X)\} = \alpha C(A)$.
(v) and (vi) follows from (iii).

Remark 2.15. The equality in the statements (v) of the above theorem need not be true as seen from Example 2.7, where $A = \{a\}, B = \{b\}$, and $A \cup B = \{a, b\}$. Then one can have that, $\alpha C(A) = \{a\}$; $\alpha C(B) = \{b\}$; $\alpha C(A \cup B) = X$; $\alpha C(A) \cup \alpha C(B) = \{a, b\}$. Further more the equality in the statements (iv) of the above theorem need not be true as shown in the following example.

Example 2.16. Let $(X, \tau)$ be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b, c\}, X\}$. Then $F_2 = \alpha C(X) = \{\emptyset, \{c\}, X\}$, $\alpha O(X) = \{\emptyset, \{a, b, c\}, X\}$, $G O(X) = \{\emptyset, \{a\}, \{a, b, c\}, \{b, c\}, X\}$. Therefore $\{a\} \in DO(X)$, but $\{a\} \notin \alpha C(X)$ and $\{a\} \notin G C(X)$.

3. $\alpha$-open sets

In this section we introduce $\alpha$-open sets and investigate some of their basic properties.

Definition 3.1. A subset $A$ of a space $X$ is called an $\alpha$-open if $X \setminus A$ is $\alpha$-closed. Let $DoO(X)$ denote the collection of all an $\alpha$-open sets in $X$.

Lemma 3.2. Let $A \subseteq X$, then

(i) $X \setminus \alpha C(X \setminus A) = \alpha C(A)$.
(ii) $X \setminus \alpha C(X \setminus A) = \alpha C(A)$.

Proof. Obvious.

Theorem 3.3. A subset $A$ of a space $X$ is $\alpha$-open if and only if $A \subseteq \alpha C(\int \alpha C(\int A))$.

Proof. Let $A$ be $\alpha$-open set. Then $X \setminus A$ is $\alpha$-closed and $\alpha C(\int \alpha C(X \setminus A)) \subseteq X \setminus A$. By Lemma 3.2 $A \subseteq \alpha C(\int \alpha C(\int A))$. Conversely, suppose $A \subseteq \alpha C(\int \alpha C(\int A))$. Then $X \setminus \alpha C(\int \alpha C(\int A)) \subseteq X \setminus A$. Hence $\alpha C(\int \alpha C(X \setminus A)) \subseteq X \setminus A$. This shows that $X \setminus A$ is $\alpha$-closed. Thus $A$ is $\alpha$-open.

Lemma 3.4. If there exists g-open set $V$ such that $V \subseteq A \subseteq \int \alpha C(\int V)$, then $A$ is $\alpha$-open.
Proof. Since $V$ is $g$-open, $X \setminus V$ is $g$-closed and $X \setminus Cl^*(Cl(V)) \subseteq X \setminus A \subseteq X \setminus V$. Therefore From Lemma 3.2, $Cl^*(Int(X \setminus V)) \subseteq X \setminus A \subseteq X \setminus V$. From Lemma 2.2 we have $X \setminus A$ is $D\alpha$-closed. Hence $A$ is $D\alpha$-open.

Remark 3.5. The converse of Lemma 3.4 need not to be true as seen from Example 2.4, where $\{a, b\} \in DaO(X)$ and $\{b\} \in GO(X)$ but $\{b\} \subseteq \{a, b\} \nsubseteq \{b\}$.

Theorem 3.6. Let $(X, \tau)$ be a topological space. Then

(i) Every $\alpha$-open subset of $(X, \tau)$ is $D\alpha$-open.

(ii) Every $g$-open subset of $(X, \tau)$ is $D\alpha$-open.

Proof. From Theorem 2.5, the proof is obvious.

Remark 3.7. The converse of the above theorem is not true as seen from Example 2.7, where $\{b, c\} \in DaO(X)$ but $\{b, c\} \notin aO(X)$ and $\{b, c\} \notin GO(X)$.

From the above discussions we have the following diagram in which the converses of implications need not be true.

$\alpha$-open set $\rightarrow$ $D\alpha$-open set $\leftarrow$ $g$-open set

Theorem 3.8. Arbitrary union of $D\alpha$-open set is $D\alpha$-open.

Proof. Follows from Theorem 2.8.

Remark 3.9. The intersection of two $D\alpha$-open sets need not be $D\alpha$-open as seen from Example 2.7, where both $\{b, c\}$ and $\{a, c\}$ are $D\alpha$-open sets but $\{b, c\} \cap \{a, c\} = \{c\}$ is not $D\alpha$-open.

Corollary 3.10. If a subset $A$ is $D\alpha$-open and $B$ is $\alpha$-open, then $A \cup B$ is $D\alpha$-open.

Proof. Follows from Theorem 3.6 (i) and Theorem 3.8.

Corollary 3.11. If a subset $A$ is $D\alpha$-open and $U$ is $g$-open, then $A \cup U$ is $D\alpha$-open.

Proof. Follows from Theorem 3.6 (ii) and Theorem 3.8.

Definition 3.12. Let $A$ be a subset of a space $X$. The $D\alpha$-interior of $A$ is denoted by $Int^\alpha(A)$, is the union of all $D\alpha$-open sets in $X$ contained in $A$. That is $Int^\alpha(A) = \cup \{U : U \subseteq A, \ U \in DaO(X)\}$.

Lemma 3.13. If $A$ is a subset of $X$, then

(i) $X \setminus CE(A) = Int^\alpha(X \setminus A)$.

(ii) $X \setminus Int^\alpha(A) = CE(A \setminus X \setminus A)$.

Proof. Obvious.

Theorem 3.14. Let $A$ be a subset of $X$. Then $A$ is $D\alpha$-open if and only if $Int^\alpha(A) = A$.

Proof. Follows from Theorem 2.13 and Lemma 3.13.

Theorem 3.15. Let $A$ and $B$ be subsets of $X$. Then the following results hold.

(i) $Int^\alpha(A \cap Int^\alpha(A)) \subseteq A \cap Int^\alpha(A)$.

(ii) $Int^\alpha(\phi) = \phi$ and $Int^\alpha(X) = X$.

(iii) If $A \subseteq B$, then $Int^\alpha(A) \subseteq Int^\alpha(B)$.

(iv) $Int^\alpha(A \cup Int^\alpha(B)) \subseteq Int^\alpha(A \cup B)$.

(vi) $Int^\alpha(A \cap B) \subseteq Int^\alpha(A) \cap Int^\alpha(B)$.

Proof. Obvious.

Remark 3.16. The equality in the statements (v) of Theorem 3.15 need not be true as seen from Example 2.7, where $A = \{b, c\}$, $B = \{a, c\}$, and $A \cup B = X$. Then one can have that, $Int^\alpha(A) = \{b, c\}$; $Int^\alpha(B) = \{c\}$; $Int^\alpha(A \cup B) = \{b, c\}$; $Int^\alpha_\alpha(A \cup B) = X$. Furthermore the equality in the statements (iv) of the above theorem need not be true as seen from Example 2.7, where $A = \{b, c\}$, $B = \{a, c\}$, and $A \cap B = \{c\}$. Then one can have that, $Int^\alpha(A) = \{b, c\}$; $Int^\alpha(B) = \{a, c\}$; $Int^\alpha(A \cap B) = \phi$; $Int^\alpha(A) \cap Int^\alpha_\alpha(B) = \{c\}$.

Theorem 3.17. Let $x \in X$. Then $x \in CE(A)$ if and only if $U \cap A \neq \phi$ for every $D\alpha$-open set $U$ containing $x$.

Proof. Let $x \in CE(A)$ and there exists $D\alpha$-open set $U$ containing $x$ such that $U \cap A = \phi$. Then $A \subseteq X \setminus U$ and $X \setminus U$ is $D\alpha$-closed. Therefore $CE(A) \subseteq CE(X \setminus U) = X \setminus U$. This implies $x \in CE(A)$, which is a contradiction. Conversely, assume that $x \notin CE(A)$, and let $U \cap A = \phi$ for every $D\alpha$-open set $U$ containing $x$ and $x \notin CE(A)$. Then there exists $D\alpha$-closed subset $F$ containing $A$ such that $x \notin F$. Hence $x \in X \setminus F$ and $X \setminus F$ is $D\alpha$-open. Therefore $A \subseteq F$, $(X \setminus F) \cap A = \phi$. This is a contradiction to our assumption.

Lemma 3.18. Let $A$ be any subset of $(X, \tau)$. Then

(i) $A \cap Int^\alpha(Cl(\alpha)) = DaO(A)$.

(ii) $A \cup Cl^*(Int^\alpha(A))$ is $D\alpha$-closed.

Proof.

(i) $Int^\alpha(Cl(A \cap Int^\alpha(Cl(\alpha)))) = Int^\alpha(Cl(A) \cap Int^\alpha(Cl(\alpha)))$. This implies that $A \cap Int^\alpha(Cl(\alpha)) = A \cap Int^\alpha(Cl(A \cap Int^\alpha(Cl(\alpha)))) \subseteq Int^\alpha(Cl(A \cap Int^\alpha(Cl(\alpha))))$. Therefore $A \cap Int^\alpha(Cl(\alpha))$ is $D\alpha$-open.

(ii) From (i) we have that $X \setminus (A \cup Cl^*(Int^\alpha(A))) = (X \setminus A) \cap Cl^*(Int^\alpha(X \setminus A))$ is $D\alpha$-open that further implies $A \cup Cl^*(Int^\alpha(A))$ is $D\alpha$-closed.

Theorem 3.19. If $A$ is a subset of a topological space $X$,

(i) $Int^\alpha(A) = A \cap Int^\alpha(Cl(\alpha))$.

(ii) $CE(A) = A \cup Cl^*(Int^\alpha(A))$.

Proof.

(i) $Int^\alpha(A) = A \cap Int^\alpha(Cl(\alpha))$. Clearly $B$ is $D\alpha$-open and $B \subseteq A$. Since $B$ is $D\alpha$-open, $B \subseteq Int^\alpha(Cl(\alpha)) \subseteq Int^\alpha(Cl(A))$. This proves that $B \subseteq A \cap Int^\alpha(Cl(A))$. By Lemma 3.18,
A \cap \text{Int}'(\text{Cl}(\text{Int}'(A))) \text{ is } Da\alpha-\text{open}. By the definition of \text{Int}'(\text{Cl}(\text{Int}'(A))) \text{ is } Da\alpha-\text{open}. Therefore \text{Int}'(A) = A \cap \text{Int}'(\text{Cl}(\text{Int}'(A))).

(ii) By Lemma 3.13 we have $CE'(X) = X \cup \text{Int}'(X \setminus A)$, $X \setminus (X \setminus A) \cap \text{Int}'(\text{Cl}(X \setminus A))$, using (i) $X \setminus (X \setminus A) \cup (X \setminus \text{Int}'(\text{Cl}(X \setminus A))) = A \cap \text{Int}'(\text{Cl}(A)))$.

4. \textbf{Da\alpha-continuous functions}

In this section we introduce Da\alpha-continuous functions and investigate some of their basic properties.

Definition 4.1. A function $f : X \to Y$ is called Da\alpha-continuous if the inverse image of each open set in $Y$ is Da\alpha-open in $X$.

Theorem 4.2.

(i) Every $\alpha$-continuous function is Da\alpha-continuous.

Proof. It is obvious from Theorem 3.6.

Remark 4.3.

(i) Da\alpha-continuous function need not be $\alpha$-continuous. (see Example 4.4 (i) below)

(ii) Da\alpha-continuous function need not be $g$-continuous. (see Example 4.4 (ii) below)

Example 4.4. (i) Let $X = \{a, b, c\}$ associated with the topology $\tau = \{\emptyset, \{a\}, X\}$ and $Y = \{x, y, z\}$ associated with the topology $\sigma = \{\emptyset, \{x, y\}, \{x, z\}, Y\}$. Let $f : X \to Y$ be a function defined by $f(a) = f(b) = x$, $f(c) = z$. One can have that $F_2 = \{a, b, c\}, X$, $GO(X) = \{a, b, c, \{a, b, c, X\} \setminus \{a, b, c\}\}$, $GO(X) = \{a, b, c, \{a, b, c\} \setminus \{a, b, c\}\}$. Thus $f$ is Da\alpha-continuous but not $\alpha$-continuous.

(ii) Let $(X, \tau)$ and $(Y, \sigma)$ be the topological spaces in (i) and $f : X \to Y$ be a function defined by $f(a) = x, f(b) = f(c) = z$. Since $\{x\}$ is open in $Y$, $f^{-1}(\{x\}) = \{a, b\} \in \text{Im}(f)$, but $\{x\} \notin \text{GO}(X)$. Therefore $f$ is Da\alpha-continuous but not $g$-continuous.

From the above discussions we have the following diagram in which the converses of implications need not be true.

\[ \alpha\text{-continuity} \to Da\alpha\text{-continuity} \to g\text{-continuity} \]

Theorem 4.5. Let $f : X \to Y$ be a function. Then the following are equivalent:

(i) $f$ is Da\alpha-continuous.

(ii) For each $x \in X$ and each open set $V \subset Y$ containing $f(x)$, there exists Da\alpha-open set $W \subset X$ containing $x$ such that $f(W) \subset V$.

(iii) The inverse image of each closed set in $Y$ is Da\alpha-closed in $X$.

(iv) $f(\text{Cl}(A)) \subset \text{Cl}(f(A))$ for every subset $A$ of $X$.

(v) $f(\text{Cl}(B)) \subset \text{Cl}(f(B))$ for every subset $B$ of $Y$.

(vi) $f^{-1}(\text{Int}(B)) \subset \text{Int}(f^{-1}(B))$ for every subset $B$ of $Y$.

Proof. (i)$\Rightarrow$(ii) Since $V \subset Y$ containing $f(x)$ is open, then $f^{-1}(V) = \text{Da}(X)$. Let $W = f^{-1}(V)$ which contains $x$, therefore $f(W) \subset V$.

(ii)$\Rightarrow$(i) Let $V \subset Y$ be open, and let $x \in f^{-1}(V)$, then $f(x) \in V$ and thus there exists $W_x \in \text{Da}(X)$ such that $x \in W_x$ and $f(W_x) \subset V$. Then $x \in W_x \subset f^{-1}(V)$, and so $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} W_x$ but $\bigcup_{x \in f^{-1}(V)} W_x \in \text{Da}(X)$ by Theorem 3.8. Hence $f^{-1}(V) \in \text{Da}(X)$, and therefore $f$ is Da\alpha-continuous.

(iii)$\Rightarrow$(iv) Let $A \subset Y$ and $B$ be a closed set in $Y$ containing $f(A)$. Then by (iii), $f^{-1}(B)$ is Da\alpha-closed set containing $A$. It follows that $CE'(A) \subset \text{Cl}(f^{-1}(B)) = f^{-1}(B)$ and hence $f(\text{CE}(A)) \subset f^{-1}(B)$. Therefore $f(\text{CE}(A)) \subset \text{Cl}(f(A))$.

(iv)$\Rightarrow$(v) Let $B \subset Y$ and $A = f^{-1}(B)$. Then by assumption, $f(\text{CE}(A)) \subset \text{Cl}(f(A))$. This implies that $\text{CE}(A) \subset f^{-1}(B)$. Hence $\text{CE}(A) \subset f^{-1}(B)$.

(v)$\Rightarrow$(vi) Let $B \subset Y$. By assumption, $\text{CE}(f^{-1}(V \cap B)) \subset f^{-1}(B)$. Thus there exists $f^{-1}(V \cap B) \subset X \setminus \text{Int}(B)$. By taking complement on both sides we get $f^{-1}(\text{Int}(B)) \subset \text{Int}(f^{-1}(B))$.

(vi)$\Rightarrow$(i) Let $U$ be any open set in $Y$. Then $\text{Int}(U) = U$. By assumption, $f^{-1}(\text{Int}(U)) \subset \text{Int}(f^{-1}(U))$ and hence $f^{-1}(U) \subset \text{Int}(f^{-1}(U))$. Therefore by Theorem 3.14, $f^{-1}(U)$ is Da\alpha-open in $X$. Thus $f$ is Da\alpha-continuous.

Theorem 4.6. Let $f : X \to Y$ be Da\alpha-continuous and let $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is Da\alpha-continuous.

Proof. Obvious.

Remark 4.7. Composition of two Da\alpha-continuous functions need not be Da\alpha-continuous as seen from the following example.

Example 4.8. Let $X = \{a, b, c\}$ associated with the topology $\tau = \{\emptyset, \{a, b, c\}, X\}$, $Y = \{x, y, z\}$ associated with the topology $\sigma = \{\emptyset, \{x, y\}, \{x, z\}, Y\}$ and $Z = \{p, q, r\}$ associated with the topology $\nu = \{\emptyset, \{p, q\}, \emptyset\}$ and $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = y, f(b) = x, f(c) = z$. Define $g : (Y, \sigma) \to (Z, \nu)$ by $g(a) = y, g(b) = p, g(c) = r$. One can have that $F_3 = \{a, b, c, X\}$, $GO(X) = \{a, b, c, \{a, b, c\} \setminus \{a, b, c\}\}, GO(y) = \{a, b, c\} \setminus \{a, b, c\} \setminus \{a, b, c\}$. Thus $f$ is Da\alpha-continuous but not $g$-continuous.

$\alpha$-continuity $\Rightarrow$ Da\alpha-continuity $\Rightarrow$ $g$-continuity

5. \textbf{Da\alpha-open functions and Da\alpha-closed functions}

In this section we introduce Da\alpha-open functions and Da\alpha-closed functions and investigate some of their basic properties.
Definition 5.1. A function $f : X \to Y$ is said to be $\alpha$-open (resp. $\alpha$-closed) if the image of each open (resp. closed) set in $X$ is $\alpha$-open (resp. $\alpha$-closed) in $Y$.

Theorem 5.2.

(i) Every $\alpha$-open function is $\alpha$-open.
(ii) Every $g$-open function is $\alpha$-open.

Proof. It is obvious from Theorem 3.6.

Remark 5.3.

(i) $\alpha$-open function need not be $\alpha$-open.
(see Example 5.4 below)
(ii) $\alpha$-open function set need not be $g$-open.
(see Example 5.5 below)

Example 5.4. (i) Let $X = \{x, y, z\}$ associated with the topology $\tau = \{\emptyset, \{x\}, X\}$ and $Y = \{a, b, c\}$ associated with the topology $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(x) = a$, $f(y) = b$ and $f(z) = c$. One can have that $F_\tau = \emptyset$, $f_\tau = \emptyset$, $G_\sigma(Y) = \{a\}$, $G_\sigma(Y) = \{a\}$ and $D_\sigma(Y) = D_\sigma(Y) = P(X)$. Since $x$ is open in $X$, $f(x) = a \in D_\sigma(Y)$, but $\{a\} \notin D_\sigma(Y)$. Therefore $f$ is $\alpha$-open function but not $g$-open.

From the above discussions we have the following diagram in which the converses of implications need not be true.

$\alpha$-open function $\implies D\alpha$-open function $\leftarrow g$-open function

Theorem 5.6. Let $f : X \to Y$ be a function. The following statements are equivalent.

(i) $f$ is $D\alpha$-open.
(ii) For each $x \in X$ and each neighborhood $U$ of $x$, there exists $D\alpha$-open set $W \subseteq Y$ containing $f(x)$ such that $W \subseteq f(U)$.

Proof. (i) $\implies$ (ii) Let $x \in X$ and $U$ is a neighborhood of $x$, then there exists an open set $V \subseteq X$ such that $x \in V \subseteq U$. Set $W = f(V)$. Since $f$ is $D\alpha$-open, $f(V) = W \in D_\alpha(Y)$ and so $f(x) \in W \subseteq f(U)$.

(ii) $\implies$ (i) Obvious.

Theorem 5.7. Let $f : X \to Y$ be $D\alpha$-open (resp. $D\alpha$-closed) function and $W \subseteq Y$. If $A \subseteq X$ is a closed (resp. open) set containing $f^{-1}(W)$, then there exists $D\alpha$-closed (resp. $D\alpha$-open) set $H \subseteq Y$ containing $W$ such that $f^{-1}(H) \subseteq A$.

Proof. Let $H = Y \setminus f(X \setminus A)$. Since $f^{-1}(W) \subseteq A$, we have $f(X \setminus A) \subseteq Y \setminus W$. Since $f$ is $D\alpha$-open (resp. $D\alpha$-closed), then $H$ is $D\alpha$-closed (resp. $D\alpha$-open) set and $f^{-1}(H) = X \setminus f^{-1}(f(X \setminus A)) \subseteq X \setminus (X \setminus A) = A$.

Corollary 5.8. If $f : X \to Y$ is $D\alpha$-open, then $f^{-1}(\overline{C_\alpha(B)}) \subseteq \overline{C_\alpha(f^{-1}(B))}$ for each set $B \subseteq Y$.

Proof. Since $\overline{C_\alpha(f^{-1}(B))}$ is closed in $X$ containing $f^{-1}(B)$ for a set $B \subseteq Y$. By Theorem 5.7, there exists $D\alpha$-closed set $H \subseteq Y$, $B \subseteq H$ such that $f^{-1}(H) \subseteq \overline{C_\alpha(f^{-1}(B))}$. Thus, $f^{-1}(C_\alpha(B)) \subseteq f^{-1}(C_\alpha(f^{-1}(B))) \subseteq f^{-1}(H) \subseteq \overline{C_\alpha(f^{-1}(B))}$.

Theorem 5.9. A function $f : X \to Y$ is $D\alpha$-open if and only if $f(\text{int}(A)) \subseteq \text{int}_\alpha(f(A))$ for every subset $A$ of $X$.

Proof. Suppose $f : X \to Y$ is $D\alpha$-open function and $A \subseteq X$. Then $\text{int}(A)$ is open set in $X$ and $f(\text{int}(A))$ is $D\alpha$-open set contained in $f(A)$. Therefore $f(\text{int}(A)) \subseteq \text{int}_\alpha(f(A))$. Conversely, let be $f(\text{int}(A)) \subseteq \text{int}_\alpha(f(A))$ for every subset $A$ of $X$ and $U$ is open set in $X$. Then $\text{int}(U) = U$, $f(U) \subseteq \text{int}_\alpha(f(U))$. Hence $f(U) = \text{int}_\alpha(f(U))$. By Theorem 3.14 $f(U)$ is $D\alpha$-open.

Theorem 5.10. For any bijective function $f : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent.

(i) $f^{-1}$ is $D\alpha$-continuous function.
(ii) $f$ is $D\alpha$-open function.
(iii) $f$ is $D\alpha$-closed function.

Proof. (i)$\implies$(ii) Let $U$ be an open set in $X$. Then $X \setminus U$ is closed in $X$. Since $f^{-1}$ is $D\alpha$-continuous, $(f^{-1})^{-1}(\overline{U})$ is $D\alpha$-closed in $Y$. That is $f(X \setminus U) = Y \setminus f(U)$ is $D\alpha$-closed in $Y$. This implies $f(U)$ is $D\alpha$-open in $Y$. Hence $f$ is $D\alpha$-open function.

(ii)$\implies$(iii) Let $F$ be a closed set in $X$. Then $X \setminus F$ is open in $X$. Since $f$ is $D\alpha$-open, $f(X \setminus F)$ is $D\alpha$-open in $Y$. That is $f(X \setminus F) = Y \setminus f(F)$ is $D\alpha$-open in $Y$. This implies $f(F)$ is $D\alpha$-closed in $Y$. Hence $f$ is $D\alpha$-closed function.

(iii)$\implies$(i) Let $F$ be closed set in $X$. Since $f$ is $D\alpha$-closed function, $f(F)$ is $D\alpha$-closed in $Y$. That is $(f^{-1})^{-1}(F)$ is $D\alpha$-closed in $Y$. Hence $f^{-1}$ is $D\alpha$-continuous function.

Remark 5.11. Composition of two $D\alpha$-open functions need not be $D\alpha$-open as seen from the following example.

Example 5.12. Let $X = \{x, y, z\}$ associated with the topology $\tau = \{\emptyset, \{x, y\}, \{x\}, X\}$, $Y = \{p, q, r\}$ associated with the topology $\sigma = \{\emptyset, \{p\}, \{p, q\}, Y\}$. $Z = \{a, b, c\}$ associated with the topology $\nu = \{\emptyset, \{a\}, \{a, b\}, Z\}$. Define $g : (X, \tau) \to (Y, \sigma)$ by $f(x) = p$, $f(y) = q$, $f(z) = r$ and $g : (Y, \sigma) \to (Z, \nu)$ by $g(p) = b$, $g(q) = a$, $g(r) = c$. One can have that $F_\tau = \emptyset$, $f_\tau = \emptyset$, $G_\sigma(Y) = \{a\}$, $G_\sigma(Y) = \{a\}$ and $D_\sigma(Y) = D_\sigma(Y) = P(X)$. Since $x$ is open in $X$, $f(x) = a \in D_\sigma(Y)$, but $\{a\} \notin D_\sigma(Y)$. Therefore $f$ is $\alpha$-open function but not $g$-open.

Definition 6.1. A function $f : X \to Y$ is said to be $\alpha$-open (resp. $\alpha$-closed) if the image of each open (resp. closed) set in $X$ is $\alpha$-open (resp. $\alpha$-closed) in $Y$.

Theorem 5.7. Let $f : X \to Y$ be $D\alpha$-open (resp. $D\alpha$-closed) function and $W \subseteq Y$. If $A \subseteq X$ is a closed (resp. open) set containing $f^{-1}(W)$, then there exists $D\alpha$-closed (resp. $D\alpha$-open) set $H \subseteq Y$ containing $W$ such that $f^{-1}(H) \subseteq A$.

Proof. Let $H = Y \setminus f(X \setminus A)$. Since $f^{-1}(W) \subseteq A$, we have $f(X \setminus A) \subseteq Y \setminus W$. Since $f$ is $D\alpha$-open (resp. $D\alpha$-closed), then $H$ is $D\alpha$-closed (resp. $D\alpha$-open) set and $f^{-1}(H) = X \setminus f^{-1}(f(X \setminus A)) \subseteq X \setminus (X \setminus A) = A$.

6. $D\alpha$-closed graph and strongly $D\alpha$-closed

In this section we introduce $D\alpha$-closed graph and strongly $D\alpha$-closed and investigate some of their basic properties.
Definition 6.1. A function \( f : X \to Y \) has \( \Delta \alpha \)-closed graph if for each \( (x, y) \in (X \times Y) \setminus G(f) \), there exists \( U \in \Delta \alpha \text{O}(X, x) \) and \( V \in \text{GO}(Y, y) \) such that \( (U \times \text{Cl}^\alpha(V)) \cap G = \emptyset \).

Remark 6.2. Evidently every closed graph is \( \Delta \alpha \)-closed. That the converse is not true is seen from the following example.

Example 6.3. Let \( X = \{a, b, c\} \) associated with the topology \( \tau = \{\emptyset, \{a, b\}, X\} \) and \( Y = \{x, y, z\} \) associated with the topology \( \sigma = \{\emptyset, \{x, y, z\}, Y\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be a function defined by \( f(a) = f(c) = x, f(b) = y \). One can check that \( F_\tau = \emptyset, [\emptyset, \{a, c\}, X] \), \( \text{GO}(X) = \emptyset, \{a, [b, \{a, b, X\}], \Delta \alpha \text{O}(X) = \emptyset, \{a, [b, \{a, b, X\}], \text{GO}(Y) = \emptyset, \{x, \{y, \{x, y, \{x, \{y, z\}, Y\}\}\}\}. \) Since \( \{a, c\} \in \Delta \alpha \text{O}(X) \) and \( \{y\} \in \text{GO}(Y) \) but \( \{a, c\} \in \text{O}(X) \) and \( \{y\} \in \text{O}(Y) \). Therefore \( G(f) \) is \( \Delta \alpha \)-closed but not closed.

Theorem 6.4. Let \( f : (X, \tau) \to (Y, \sigma) \) be a function and

\( \text{(i)} \) \( f \) is \( \Delta \alpha \)-closed graph;
\( \text{(ii)} \) For each \( (x, y) \in (X \times Y) \setminus G(f) \), there exist \( U \in \Delta \alpha \text{O}(X, x) \) and \( V \in \text{GO}(Y, y) \) such that \( (U \times \text{Cl}^\alpha(V)) \cap G = \emptyset \).
\( \text{(iii)} \) For each \( (x, y) \in (X \times Y) \setminus G(f) \), there exist \( U \in \Delta \alpha \text{O}(X, x) \) and \( V \in \text{GO}(Y, y) \) such that \( (U \times \text{Cl}^\alpha(V)) \cap G = \emptyset \).
\( \text{(iv)} \) For each \( (x, y) \in (X \times Y) \setminus G(f) \), there exist \( U \in \Delta \alpha \text{O}(X, x) \) and \( V \in \text{GO}(Y, y) \) such that \( (U \times \text{Cl}^\alpha(V)) \cap G = \emptyset \).

Proof. \( \text{(i)} \Rightarrow \text{(ii)} \) Suppose \( f \) is \( \Delta \alpha \)-closed graph. Then for each \( (x, y) \in (X \times Y) \setminus G(f) \), there exists \( U \in \Delta \alpha \text{O}(X, x) \) and \( V \in \text{GO}(Y, y) \) such that \( (U \times \text{Cl}^\alpha(V)) \cap G = \emptyset \). This implies that for each \( f(x) \in f(U) \) and \( y \in \text{Cl}^\alpha(V) \). Since \( y \neq f(x) \), \( f(U) \cap \text{Cl}^\alpha(V) = \emptyset \). \( \text{(ii)} \Rightarrow \text{(i)} \) Let \( (x, y) \in (X \times Y) \setminus G(f) \). Then there exists \( U \in \Delta \alpha \text{O}(X, x) \) and \( V \in \text{GO}(Y, y) \) such that \( (U \times \text{Cl}^\alpha(V)) \cap G = \emptyset \). This implies that \( f(x) \neq y \) for each \( x \in U \) and \( y \in \text{Cl}^\alpha(V) \). Therefore \( (U \times \text{Cl}^\alpha(V)) \cap G = \emptyset \).

\( \text{(iii)} \Rightarrow \text{(i)} \) Suppose \( f \) is \( \Delta \alpha \)-closed graph. Then for each \( (x, y) \in (X \times Y) \setminus G(f) \), there exists \( U \in \Delta \alpha \text{O}(X, x) \) and \( V \in \text{GO}(Y, y) \) such that \( (U \times \text{Cl}^\alpha(V)) \cap G = \emptyset \). Since \( g \)-open set is \( \Delta \alpha \)-open, \( \text{CE}(V) \subset \text{Cl}^\alpha(V) \). Therefore \( (U \times \text{CE}(V)) \cap G = \emptyset \).

\( \text{(iv)} \Rightarrow \text{(i)} \) Let \( (x, y) \in (X \times Y) \setminus G(f) \). Then there exists \( U \in \Delta \alpha \text{O}(X, x) \) and \( V \in \text{GO}(Y, y) \) such that \( f(U) \cap \text{Cl}^\alpha(V) = \emptyset \). Since \( \text{CE}(V) \subset \text{Cl}^\alpha(V) \), \( f(U) \cap \text{CE}(V) \subset f(U) \cap \text{Cl}^\alpha(V) = \emptyset \).

Remark 6.7. The converse of the above theorem is not true as seen from Example 2.7.

Theorem 6.8. Let \( f : X \to Y \) be any surjection with \( G(f) \) \( \Delta \alpha \)-closed. Then \( Y \) is \( g\text{-T}_1 \).

Proof. Let \( y_1, y_2 \neq y_1 \in Y \). The subjectivity of \( f \) gives the existence of an element \( x_1 \in X \) such that \( f(x_1) = y_1 \). Now \( f(x_1) \neq f(x_2) \). The \( \Delta \alpha \)-closedness of \( G(f) \) provides \( U_1 \in \Delta \alpha \text{O}(X, x_1), V_1 \in \text{GO}(Y, y_1) \) such that \( f(U_1) \cap \text{Cl}^\alpha(V_1) = \emptyset \). Now \( x_1 \in U_1 \Rightarrow f(x_1) = y_1 \neq f(U_1) \). This and the fact that \( f(U_1) \cap \text{Cl}^\alpha(V_1) = \emptyset \) guarantee that \( y_1 \notin V_1 \). Again from the subjectivity of \( f \) gives a \( x_2 \in X \) such that \( f(x_2) = y_1 \). Now \( x_2 \in U_2 \Rightarrow f(x_2) = y_1 \neq f(U_2) \). Thus we obtain sets \( V_1, V_2 \subset \text{GO}(Y, y_1) \) such that \( y_1 \notin V_1 \) but \( y_2 \notin V_1 \) while \( y_2 \notin V_2 \). Hence \( Y \) is \( g\text{-T}_1 \).

Corollary 6.9. Let \( f : X \to Y \) be any surjection with \( G(f) \) \( \Delta \alpha \)-closed. Then \( Y \) is \( \Delta \alpha \text{-T}_1 \).

Theorem 6.10. Let \( f : X \to Y \) be any injective with \( G(f) \) \( \Delta \alpha \)-closed. Then \( X \) is \( \Delta \alpha \text{-T}_1 \).

Proof. It readily follows from Theorems 6.6 (i) and 6.8.

Definition 6.12. A topological space \((X, \tau)\) is said to be \( \Delta \alpha \text{-T}_1 \) if for any distinct pair of points \( x \) and \( y \) in \( X \), there exist \( \Delta \alpha \)-open sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \), respectively, such that \( U \cap V = \emptyset \).

Theorem 6.13. \( \text{(i)} \) Every \( \alpha\text{-T}_2 \) space is \( \Delta \alpha \text{-T}_2 \).
\( \text{(ii)} \) Every \( g\text{-T}_1 \) space is \( \Delta \alpha \text{-T}_2 \).

Proof. Obvious.

Remark 6.14. The converse of the above theorem is not true as seen from Example 2.7.

Theorem 6.15. Let \( f : X \to Y \) be any surjection with \( G(f) \) \( \Delta \alpha \)-closed. Then \( Y \) is \( g\text{-T}_2 \).
Proof. Let y₁, y₂ ∈ Y. The subjectivity of f gives a x₁ ∈ X such that f(x₁) = y₁. Now (x₁, y₁) ∈ (X × Y) \ G(f). The Dα-closedness of G(f) provides U ∈ DαO(X, x₁) such that f(U) \ ∩ Cl'(V) = φ. Now x₁ ∈ U = f(x₁) = y₁ ∈ f(U). This and the fact that f(U) \ ∩ Cl'(V) = φ guarantee that y₁ \ ∉ Cl'(V). This mean that there exists W ∈ GO(Y, y₁) such that W ∩ V = φ. Hence Y is g-T₂.

Corollary 6.16. Let f : X → Y be any surjection with G(f) Dα-closed. Then Y is Dα-T₂.

Proof. Follows from Theorems 6.13 (ii) and 6.15.

Definition 6.17. A function f : X → Y has a strongly Dα-closed graph if for each (x, y) ∈ (X × Y) \ G(f), there exist U ∈ DαO(X, x) and V ∈ O(Y, y) such that (U × Cl(V)) \ G(f) = φ.

Corollary 6.18. A strongly Dα-closed graph is Dα-closed. That the converse is not true is seen from Example 6.3, where \{y\} ∈ GO(Y, y) but \{y\} ∉ O(Y). Therefore G(f) is Dα-closed but not strongly Dα-closed.

Remark 6.19. Evidently every strongly α-closed graph (resp. strongly closed graph) is strongly Dα-closed graph. That the converse is not true is seen from the following example.

Example 6.20. Let X = [a, b, c] associated with the topology τ = {[a, b], [a], [b], [a, b, X]} and Y = [x, y, z] associated with the topology σ = {[x, y], [y], [z], [x, y, z]}. Let f : (X, τ) → (Y, σ) be a function defined by f(a) = f(c) = x, f(b) = y. One can have that F₁ = {c, [x, X], GC(X) = {c, [c, x], [a, c, [b, c, X]], GO(X) = {a, [a, b], [a, b, X], αO(X) = {a, [a, b], x}, DαO(X) = {a, [a], [b], [a, b], [a, c], [b, c, X]}. Since [a, c] ∈ DαO(X, c) and [a] ∉ O(Y, z) but [a, c] ∉ O(X) (resp. [a, c] ∉ O(X)). Therefore G(f) is strongly Dα-closed but not strongly α-closed (resp. strongly closed).

Theorem 6.21. For a function f : (X, τ) → (Y, σ), the following properties are equivalent:

(i) f has strongly Dα-closed graph.
(ii) For each (x, y) ∈ (X × Y) \ G(f), there exist U ∈ DαO(X, x) and V ∈ O(Y, y) such that f(U) \ ∩ Cl(V) = φ.
(iii) For each (x, y) ∈ (X × Y) \ G(f), there exist U ∈ DαO(X, x) and V ∈ O(Y, y) such that (U \ × Cl(V)) \ G(f) = φ.
(iv) For each (x, y) ∈ (X × Y) \ G(f), there exist U ∈ DαO(X, x) and V ∈ O(Y, y) such that f(U) \ ∩ Cl(V) = φ.

Proof. Similar to the proof of Theorem 6.4.

Theorem 6.22. If f : X → Y is a function with a strongly Dα-closed graph, then for each x ∈ X, f(x) = α Cl(f(U)) : U ∈ DαO(X, x))

Proof. Suppose the theorem is false. Then there exists a y \∉ f(x) such that y \ ∈ α Cl(f(U)) : U ∈ DαO(X, x)). This implies that y ∈ Cl(f(U)) for every U ∈ DαO(X, x). So V \ ∩ f(U) = φ for every V ∈ αO(Y, y). This, in its turn, indicates that Cl(V) \ ∩ f(U) \ V \supset f(U) \neq φ, which contradicts the hypothesis that f is a function with Dα-closed graph. Hence the theorem holds.

Theorem 6.23. If f : X → Y is Dα-continuous function and Y is T₂. Then G(f) is strongly Dα-closed.

Proof. Let (x, y) ∈ (X × Y) \ G(f). Since Y is T₂, there exists a set V ∈ O(Y, y) such that f(x) \ ∈ Cl(V). But Cl(V) is closed. Now Y \ \supset Cl(V) ∈ O(Y, f(x)). By Theorem 4.5 there exists U ∈ DαO(X, x) such that f(U) \ \supset Y \ \supset Cl(V). Consequently, f(U) \ \supset Cl(V) = φ and therefore G(f) is strongly Dα-closed.

Theorem 6.24. Let f : X → Y be any surjection with G(f) strongly Dα-closed. Then Y is T₁ and α-T₁.

Proof. Follows from Theorem 6.8 and T₁-ness always guarantees α-T₁-ness. Hence Y is α-T₁.

Corollary 6.25. Let f : X → Y be any surjection with G(f) strongly Dα-closed. Then Y is Dα-T₁.

Proof. Follows from Corollary 6.25 and Theorem 6.26.

Theorem 6.26. Let f : X → Y be any injective with G(f) strongly Dα-closed. Then X is Dα-T₁.

Proof. Similar to the proof of Theorem 6.10.

Corollary 6.27. Let f : X → Y be any bijection with G(f) strongly Dα-closed. Then both X and Y are Dα-T₁.

Proof. It readily follows from Corollary 6.25 and Theorem 6.26.

Theorem 6.28. Let f : X → Y be any surjection with G(f) strongly Dα-closed. Then Y is T₂ and α-T₂.

Proof. Similar to the proof Theorem 6.15 and T₂-ness always guarantees α-T₂-ness. Hence Y is α-T₂.

Corollary 6.29. Let f : X → Y be any surjection with G(f) strongly Dα-closed. Then Y is Dα-T₂.

Proof. Follows from Theorems 6.13 (i) and 6.28.

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