Dynamical spin-to-charge conversion on the edge of quantum spin Hall insulator

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We theoretically manifest that the edge of a quantum spin Hall insulator (QSHI), attached to an insulating ferromagnet (FM), can realize a highly efficient spin-to-charge conversion. Based on a one-dimensional QSHI-FM junction, the electron dynamics on the QSHI edge is analyzed, driven by a magnetization dynamics in the FM. Under a large gap opening on the edge from the magnetic exchange coupling, we find that the spin injection into the QSHI edge gets suppressed while the charge current driven on the edge gets maximized, demanded by the band topology of the one-dimensional helical edge states.

Interconversion between spin- and charge-related quantities in materials plays an important role in manipulating spins and magnetism, especially in the context of spintronics [1–3]. In particular, the spin-charge conversion at the interfaces of heterostructures has recently been studied with a great interest, since it can make use of various unconventional spin-dependent properties of the electrons emergent at the interfaces [4]. The conversion phenomena at the Rashba interfaces of oxides, the spin-momentum-locked surfaces states of topological insulators (TIs), etc., have been experimentally investigated [5–9]. The spin-to-charge conversion efficiency \( \lambda_{sc} \equiv J_{sc}/J_{bulk} \), defined as the ratio between the spin current \( J_{sc} \) injected from the bulk of the magnet and the converted charge current \( J_{bulk} \) at the interface, has been reported to reach up to a few nanometers in those systems [10].

In order to improve the efficiency of the spin-to-charge conversion, we need to reduce the spin injection \( J_{bulk} \) and enhance the output current \( J_{sc} \) simultaneously. Here we focus on quantum spin Hall insulator (QSHI), namely a two-dimensional (2D) TI characterized by the \( \mathbb{Z}_2 \) topology, to realize an efficient spin-to-charge conversion. QSHI is advantageous in spin transport in that it exhibits spin-resolved helical edge states, which are free from backscattering by time-reversal-symmetric disorders [11–13]. The spin Hall conductivity of QSHI is quantized to \( e^2/h \), which generates a quantized spin current out of an applied electric field. This effect can be regarded as an ideal charge-to-spin conversion, since it does not suffer from the energy loss by the Joule heating. We can thus assume that QSHI can realize the ideal spin-to-charge conversion as well. So far it has been theoretically seen that magnetization dynamics in a ferromagnet coupled with a QSHI induces a charge current flowing along the junction [14–19]. From the viewpoint of the spin-to-charge conversion, we need to understand whether and how much we can improve the conversion efficiency, by making use of this edge current.

In the present work, we consider a hypothetical lateral junction of a ferromagnet and a QSHI (see Fig. 1), to evaluate the spin-to-charge conversion efficiency of the QSHI. Under a magnetization dynamics in the ferromagnet, we compare the rate of spin injection into the QSHI and the charge current induced on the edge of the QSHI. We evaluate these quantities in terms of the Floquet–Keldysh formalism [20–23], in which many-body dynamics of the electrons, driven by the cyclic dynamics of the magnetization, is imprinted in nonequilibrium Green’s functions. Under a large exchange gap in the edge spectrum, arising from the in-plane component of the magnetization, we find that the spin injection into the QSHI edge is suppressed, whereas the current along the edge reaches its maximum value required by the band topology. This implies that the edge of QSHI can realize a significantly high conversion efficiency \( \lambda_{sc} \), which scales around two orders larger than that observed in 2D interfaces [8, 9], under the exchange gap of \( \sim 10\text{meV} \).

Electrons on the junction. — We start with the model of the electrons residing on the QSHI-ferromagnet junction. The electrons on the helical edge of the QSHI, whose spins are coupled with the magnetization \( \mathbf{n} \) by the proximity exchange coupling \( J \), is described by the Hamiltonian

\[
H(k) = v_F k \sigma_z + J \mathbf{n} \cdot \sigma.
\]

FIG. 1. Schematic picture of the setup of our analysis. A ferromagnet (FM) and a quantum spin Hall insulator (QSHI) are coupled at their 1D boundaries. Phenomenologically, magnetization dynamics in the ferromagnet injects spin into the QSHI \( (J_S) \), which is converted into a transverse charge current on the edge of the QSHI \( (I) \). We introduce a hypothetical metallic reservoir so that the system may maintain a periodic steady state, which corresponds to metallic terminals attached to the edge of the sample in experimental setups.
in momentum space $[14] [17] [19]$. Here $v_F$ is the electron Fermi velocity, $k$ is the electron momentum along the edge, and $\sigma$ is the Pauli matrix for the electron spin. If the precession of the magnetization is kept periodic around $z$-axis, it is written as

$$ n(t) = (\sin \alpha \cos \Omega t, \sin \alpha \sin \Omega t, \cos \alpha), $$

with $\alpha$ the tilting angle from $z$-axis and $\Omega$ the frequency of the precession. Such a steady precession can be maintained, for instance, by tuning an external magnetic field $B_{\text{ext}}$ along $z$-axis and an alternating magnetic field $B_{\text{alt}}(t)$ like a microwave, while we shall not go into details of its mechanism. If there is no magnetization dynamics (i.e. $\Omega = 0$), the edge spectrum obtains a gap $2J \sin \alpha (\equiv 2J')$ corresponding to the in-plane component of the magnetization, with the band dispersion $E(k) = \pm [\sqrt{(v_F k + J \cos \alpha)^2 + J'^2}]^2$. The out-of-plane component $J \cos \alpha$ shifts the momentum homogeneously and gives rise to a steady current in equilibrium, which we shall omit in the present work.

Electron dynamics. — In order to evaluate the dynamically-induced quantities carried by the electrons, we analyze the time-periodic dynamics of the electron ensemble in terms of the nonequilibrium Green’s functions folded within a frequency domain $\Omega$, namely the Keldysh–Floquet formalism $[20] [23]$. The details of the analysis are left for the Appendix. We assume that the electron dynamics reaches a so-called “periodic steady state” after a long time of driving $[24] [25]$, where the retarded/advanced/lesser Green’s functions for the electrons become time-periodic, $G_{R/A/\Sigma}(t, t') = G_{R/A/\Sigma}(t + T, t' + T)$, with $T = 2\pi/\Omega$ the precession cycle. Using the Fourier transform within the frequency domain $\Omega$,

$$ G_{mn}(\omega) = \int_0^T \frac{dt}{T} \int_{-\infty}^{\infty} d\delta t \, e^{i\omega \delta t + i(m-n)\Omega t} G(t_+, t_-), $$

with $t_{\pm} \equiv t \pm \delta t/2$, the expectation value of the (time-independent) operator $O$ is evaluated as

$$ \langle O(t) \rangle = -i \int_0^T \frac{d\omega}{2\pi} \sum_{mn} \text{Tr} \left[ O G_{mn}^{\Sigma}(\omega) \right] e^{i(m-n)\Omega t}, $$

where the trace runs over both the momentum space and the spin space. As the present Hamiltonian can be exactly diagonalized in the Floquet formalism, this analysis does not require any approximations, such as the high-frequency expansion (Floquet–Magnus expansion) or truncation of the Floquet space, which are commonly seen in the analyses of Floquet systems $[20] [27]$. We thus evaluate the physical quantities on the edge of the QSHI, within the whole frequency regime of the magnetization dynamics.

In order to reach a periodic steady state, the information of the initial condition should be wiped out through dissipation to the environment. Here we set up a hypothetical metallic reservoir coupled with the junction $[23]$, as shown in Fig. 1, which corresponds to metallic terminals in a realistic experimental setup. Once the system reaches a periodic steady state, the Green’s functions satisfy the relation

$$ G^<(k, \omega) = G^R(k, \omega) \Sigma^<(\omega) G^A(k, \omega), $$

where the lesser self energy $\Sigma^<$ contains the information of electron distribution in the periodic steady state. While the electron distribution in driven systems in general depends on microscopic structures of the system Hamiltonian and the dissipation mechanism $[24] [25] [29] [30]$, here we take a simple assumption that the lesser self energy inherits the electron distribution in the reservoir, $\Sigma^<_m(k, \omega) = i\Gamma f(\omega + \nu \Omega)\delta_{mn}$ $[22] [23]$. The parameter $\Gamma$ is related to the coupling between the system and the reservoir, which is derived by integrating out the electron degrees of freedom in the reservoir $[31] [33]$. It arises as the imaginary part of the electrons’ self energy, which is encoded in the Green’s functions $G^{R/A}$ and results in the broadening of the Floquet spectrum. The information of the electron Fermi energy $\mu$ is also included in these Green’s functions. Here we require the temperature of the reservoir to be lower than any other energy scales in the system, so that it can be treated as zero temperature.

Edge current. — Let us first see the electric current driven on the 1D boundary, which emerges as the outcome of the spin-to-charge conversion. With the current operator $I = -e\partial H(k)/\partial k = -ev_F \sigma_z$ on the edge, the edge current can be evaluated by Eq. (4), whose behavior is shown in Fig. 2a). Here we parametrize our result by $J' = J \sin \alpha$, corresponding to the exchange gap from the in-plane component of the magnetization, and make the physical quantities dimensionless by using the precession frequency $\Omega$. The current $I$ is rescaled as $-I(2\pi/e\Omega) = -IT/e$, which corresponds to the num-

FIG. 2. (a) The edge current $I$ and (b) the spin injection rate (z-component) $J_{\chi}$ driven by the magnetization dynamics, parametrized by the in-plane component $J' = J \sin \alpha$ of the exchange energy at the junction. All the physical quantities here are rescaled by the precession frequency $\Omega$ (or the cycle $T = 2\pi/\Omega$) of the magnetization. The solid lines show the values obtained at charge neutrality $\mu = 0$, while the dashed lines are obtained with $\mu$ lifted from charge neutrality.
ber of electrons carried per one cycle $T$. We compare the behavior of this current by varying the dissipation parameter $\Gamma$ and the Fermi energy $\mu$ of the electrons.

We can immediately see from this calculation result that the edge current $I$ reaches a maximum value $I_e = -e/T$ under a large exchange energy $J'$. This behavior is obvious in the dissipationless limit $\Gamma = 0$, where $I_e$ is reached once the exchange gap $2J'$ exceeds the precession frequency $\Omega$. The electron dynamics in this regime can be regarded “adiabatic”, in that an edge electron cannot be excited beyond the exchange gap. In this regime, the induced current $I_e$ can be well described by the adiabatic pumping theory [34], which claims that the Berry phase accumulated by the time evolution of the electron pumps a single electron $-e$ per one cycle of precession $T$ (see Appendix B). Within the adiabatic regime, this quantized pumping behavior was demonstrated by various numerical and analytical schemes in previous literatures [14,36,35]. In the opposite regime $\Omega > 2J'$, the magnetization dynamics can resonantly excite an edge electron from the valence band to the conduction band, which reduces the Berry phase contribution from the valence band and suppresses the edge current $I$ below $I_e$.

The pumping current $I$ gets suppressed once the edge spectrum becomes metallic. In case the Fermi level $\mu$ reaches the valence band, the band becomes partially vacant, leading to reduction of the Berry-phase contribution from the valence band. If $\mu$ comes up to the conduction band, on the other hand, it becomes partially occupied and yields the Berry-phase contribution, which has the sign opposite to the valence-band contribution and thus partially cancels that. Thus the current gets suppressed once the Fermi level $\mu$ is lifted from zero energy, irrespective of its sign. The dissipation effect $\Gamma$ by the reservoir also reduces the pumping current, since the spectral broadening mixes up the Berry-phase contributions from the valence and conduction bands. Such a dissipative correction to the edge current was analytically seen in the context of photo-induced current in QSHI as well [35,36]. From the above calculation results, we can understand that the edge state needs to be insulating, with the exchange gap $2J'$, to maximize the edge current, reaching the adiabatic pumping regime.

**Spin injection rate.** — We next evaluate the rate of spin injection from the ferromagnet into the QSHI edge. In order to understand the spin transfer process at the junction, we focus on the feedback torque exerted on the ferromagnet by the spins of the edge electrons [3,37]. We assume that the ferromagnet is in a thin strip geometry, in which the constituent spins shall feel a uniform effective magnetic field from the electron spins on the QSHI edge. If the ferromagnetic strip consists of $N$ sites per unit length, with spin $S$ for each site, the Landau–Lifshitz–Gilbert (LLG) equation for a single spin $S = -Sn$ reads

$$\dot{S} = \gamma B_{\text{eff}} \times S - \alpha_d S \times \dot{S}/S,$$  \hspace{1cm} (6)

where $\gamma = g\mu_B$ denotes the gyromagnetic ratio and $\alpha_d$ is the Gilbert damping parameter intrinsic to the ferromagnet. The effective magnetic field $B_{\text{eff}}$ for a single spin is given as $B_{\text{eff}} = B_{\text{ext}} + B_{\text{alt}} + B_{\text{d}},$ where $B_{\text{d}} \equiv -J(\sigma)/\gamma NS$ is the contribution from the electron spin accumulation $\langle \sigma \rangle$ on the QSHI edge. The torque from the effective field $B_{\text{el}}$ is

$$t_{\text{el}} = \gamma B_{\text{el}} \times S = (J/N)\langle \sigma \rangle \times n,$$ \hspace{1cm} (7)

which arises as the feedback effect from the edge electrons onto the ferromagnet.

Let us here consider the net feedback torque on the spins within a unit length of the strip, given by $T_{\text{el}} = Nt_{\text{el}} = J(\sigma) \times n$. This torque can be separated into the field-like component $T^f \propto e_z \times n$ and the damping-like component $T^d \propto \dot{n} \times n$. Whereas the field-like component $T^f$ gives a correction to the external magnetic field $B_{\text{ext}} \parallel z$ that maintains the spin precession around $z$-axis, the damping-like component $T^d$ yields a correction to the Gilbert damping parameter $\alpha_d$. $T^d$ leads to a negative angular momentum transfer from the conduction electrons to the ferromagnet, which conversely corresponds to the spin injection from the ferromagnet into the conduction electrons [38,69]. Therefore, in order to understand the spin injection behavior, we need to evaluate the damping-like torque $T^d$.

Among the three components of the electron spin accumulation $\langle \sigma_{x,y,z}(t) \rangle$, the damping-like torque $T^d \propto \dot{n} \times n$ comes from the component parallel to $\dot{n}(t) = \Omega \sin \alpha (\dot{e}_z \sin \Omega t + e_y \cos \Omega t)$. By denoting this component in $\langle \sigma(t) \rangle$ as $\sigma_d(t) = \sigma_d (-e_z \sin \Omega t + e_y \cos \Omega t)$, the time average of $T^d$ is given as

$$T^d = \frac{1}{T} \int_0^T dt [J\sigma_d(t) \times n(t)] = -J\sigma_d \sin \alpha e_z,$$ \hspace{1cm} (8)

from which we obtain the spin angular momentum $J_S = -T^d = J'\sigma_d e_z$ transferred from the ferromagnet to the QSHI edge, per unit time and unit length in average. We can thus straightforwardly calculate the spin injection rate $J_S$, by evaluating the spin accumulation $\langle \sigma(t) \rangle$ based on the Keldysh–Floquet formalism (see Appendix for details).

The behavior of $J_S$ parametrized by $J' = J \sin \alpha$ is shown in Fig. 2 (b). Here $J_S$ having the dimension of $\text{[time]}^{-1} \times \text{[length]}^{-1}$, is rescaled by multiplying the time scale $T$ and the length scale $v_F T$. If the system is isolated from the environment, corresponding to the dissipationless limit $\Gamma \rightarrow 0$, $J_S$ vanishes: since the edge electrons do not lose spin angular momentum in this limit, the periodic steady state is maintained without injecting spin continuously. The spin injection $J_S$ arises due to the loss of spin in the reservoir. We should note here that the spin injection is suppressed in the “adiabatic” regime $J' \gtrsim \Omega/2, \mu, \Gamma$, with its asymptotic behavior

$$J'_S \approx \frac{\Omega^2}{8\pi v_F J'}.$$ \hspace{1cm} (9)
Since the edge electron can hardly be excited beyond the exchange gap in this regime, we can understand that the spin injection process, accompanied with a spin flip of the edge electron, is suppressed. On the other hand, in case the Fermi energy \( \mu \) or the spectral broadening \( \Gamma \) exceeds the exchange energy \( J' \), the system becomes metallic and thus admits a large spin injection.

Spin torque is accompanied with energy transfer as well. The damping-like torque by the effective field \( \mathbf{B}_{\text{eff}} \) exerts a negative work on the spins, with its power

\[-P = -NP = -J' \langle \sigma \rangle \cdot \dot{n} = -\Omega J \sigma_3 \sin \alpha. \tag{10}\]

Therefore, we can see that energy \( P \equiv \Omega J' \sigma_3 \) is injected from the ferromagnet to the edge electrons of the QSHI per unit time and unit length. The spin injection rate \( J_3^s \) and the energy injection rate \( P \) satisfy a simple relation

\[P \equiv \Omega J_3^s. \tag{11}\]

This relation can be attributed to the magnon exchange picture: if we consider the constituent spins in the ferromagnet as quantum spins, their precession modes can be quantized as magnons, where the uniform (Kittel) mode carries spin 1 and energy \( \Omega \). In this picture, the spin injection can be regarded as the flow of magnons from the ferromagnet into the QSHI edge, which requires the proportionality between the injected spin \( J_3^s \) and energy \( P \). Once we attach a reservoir, or terminals, to the system to extract the transport properties, there arises a loss of spin and energy in the reservoir, leading to a continuous injection of spin \( J_3^s \) and energy \( P \) that maintains the periodic steady state.

Spin-to-charge conversion efficiency. — Finally, we evaluate the efficiency of the spin-to-charge conversion by comparing the z-component of the spin injection rate \( J_3^s \) and the induced edge current \( I \). The conversion efficiency

\[\lambda_{sc} \equiv -I/e J_3^s \tag{12}\]

at the present 1D junction has the dimension of length, which is the same as that defined at 2D interfaces. This ratio \( \lambda_{sc} \) accounts for the energy efficiency for inducing the edge current, namely \( I/P \), as well, due to the proportionality \( P = \Omega J_3^s \) stated above. By fixing the exchange energy \( J' \) and varying the Fermi energy \( \mu \) of the electrons, the spin-to-charge conversion rate \( \lambda_{sc} \) on the QSHI edge is obtained as shown in Fig. 3. We can see from this calculation result that the conversion on the edge becomes highly efficient if the Fermi level \( \mu \) is deeply inside the exchange gap, i.e., \( |\mu| \ll J' \), since the current \( I \) reaches the constant value \(-e/T\) demanded by the adiabatic pumping theory, whereas the spin injection \( J_3^s \) gets suppressed by the exchange gap. This behavior becomes significant if the magnetization dynamics is adiabatic, i.e., \( \Omega < 2J' \), as shown in Fig. 3(a), so that it may not excite an electron beyond the exchange gap. We also need a low dissipation effect \( \Gamma \) by the reservoir (terminals) to achieve the highly efficient spin-to-charge conversion, as seen from Fig. 3(b).

The asymptotic behavior of \( \lambda_{sc} \) under a large exchange energy \( J' \) is given as

\[\lambda_{sc} \xrightarrow{J' \to \infty} 4v_F J'/\Gamma. \tag{12}\]

By using the typical scales \( v_F = 10^6 \text{m/s} \) and \( J' = \Gamma = 10 \text{meV} \) common in graphene and TIs, we can estimate \( \lambda_{sc} \approx 1-2 \times 10^2 \text{nm} \), which is two orders larger than that reported in 2D Rashba and TI interfaces experimentally.

Conclusion. — In this article, we have theoretically investigated the dynamical spin-to-charge conversion phenomenon on the edge of a 2D QSHI. By taking a hypothetical lateral junction of a 2D ferromagnet and a QSHI, we have evaluated the spin-to-charge conversion efficiency on the edge of the QSHI, driven by magnetization dynamics in the ferromagnet. The main finding in this article is that the conversion efficiency is highly enhanced, under a large exchange gap on the edge spectrum induced by the in-plane component of the magnetization. In contrast to the conventional spin pumping phenomena in metals, the edge state should be insulating to idealize the spin-to-charge conversion, since the converted charge current is based on the topological origin, namely the adiabatic charge pumping. The electrons on the 1D helical edge states is completely free from scattering by charged disorders, which minimizes the leakage of spin and energy in this spin-to-charge conversion process as long as the coupling to the terminals or environment is weak enough. In order to make the best of the topological characteristics of the QSHI edge for realizing the ideal spin-to-charge conversion, we find the following criteria from our calculations: (i) the Fermi level \( \mu \) should...
lie inside the exchange gap ($|\mu| < J'$), (ii) the precession frequency $\Omega$ of the magnetization should not exceed the exchange gap ($\Omega < 2J'$), and (iii) the exchange gap should be well resolved against the spectral broadening ($\Gamma < J'$).

Our findings imply that a 2D QSHI can serve as an efficient detector of a spin current. Using layered QSHI materials, such as the transition metal dichalcogenide $\text{Te}^n - \text{WTe}_2$ [40–42], monolayer germanene (Ge) or stanene (Sn) [43–46], reported in recent studies, one can expect a flexible design of highly integrated spin-charge devices. Topological Dirac semimetals (e.g. Na$_3$Bi), which are characterized by the $\mathbb{Z}_2$ topology and are seen to exhibit the QSHI phase in a thin film geometry [17,48], may also exhibit this type of spin-to-charge conversion on the surface [49].

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[1] Edited by M. I. Dyakonov, Spin Physics in Semiconduc-
tors (Springer, 2008).

[2] S. Takahashi and S. Maekawa, Sci. Technol. Adv. Mater. 9, 014105 (2008).

[3] Edited by S. Maekawa, S. O. Valenzuela, E. Saitoh, and
T. Kimura, Spin Current (Oxford University Press, New York, 2012).

[4] W. Han, Y. Otani, and S. Maekawa, npj Quant. Mater. 3, 27 (2018).

[5] J. C. Rojas Sánchez, L. Vila, G. Desfonds, S. Gambarelli,
J. P. Attané, J. M. D. Teresa, C. Magat, and A. Fert, Nat. Commun. 4, 2944 (2013).

[6] Y. Shiomi, K. Nomura, Y. Kajiwara, K. Eto, M. Novak,
K. Segawa, Y. Ando, and E. Saitoh, Phys. Rev. Lett. 113, 196601 (2014).

[7] K. Shen, G. Vignale, and R. Raimondi, Phys. Rev. Lett. 112, 096601 (2014).

[8] E. Lesne, Y. Fu, S. Oyarzún, J. C. Rojas-Sánchez,
D. C. Vaz, H. Naganuma, G. Sicoli, J.-P. Attané,
M. Jamet, E. Jacquet, J.-M. George, A. Barthélémy,
H. Jaffrès, A. Fert, M. Bibes, and L. Vila, Nat. Mater. 15, 1261 (2016).

[9] H. Wang, J. K ally, J. S. Lee, T. Liu, H. Chang,
D. R. Hickey, K. A. Mikhailov, M. Wu, A. Richardella,
and N. Samarth, Phys. Rev. Lett. 117, 076601 (2016).

[10] The conversion efficiency $\lambda_{\text{sc}}$ has the dimension of length
due to the mismatch of dimensionalities between $J_{\text{int}}^2$ and $J_{\text{bulk}}^2$.

[11] C. L. Kane, and E. J. Mele, Phys. Rev. Lett. 95, 226801
(2005).

[12] B. A. Bernevig, and S.-C. Zhang, Phys. Rev. Lett. 96, 106802 (2006).

[13] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science 314, 1757 (2006).

[14] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Nat. Phys. 4, 273 (2008).

[15] S.-H. Chen, B. K. Nikolić, and C.-R. Chang, Phys. Rev. B 81, 035428 (2010).

[16] F. Mahfouzi, B. K. Nikolić, S.-H. Chen, and C.-R. Chang,
Phys. Rev. B 82, 195440 (2010).

[17] K. Hattori, J. Phys. Soc. Jpn. 82, 024708 (2013).

[18] W. Y. Deng, W. Luo, H. Geng, M. N. Chen, L. Sheng,
and D. Y. Xing, New J. Phys. 17, 103018 (2015).

[19] M.-J. Wang, J. Wang, and J.-F. Liu, Sci. Rep. 9, 3378 (2019).

[20] F. H. M. Faisal, Comp. Phys. Rep. 9,57 (1989).

[21] T. Oka and H. Aoki, Phys. Rev. B 79, 081406 (2009).

[22] N. Tsuji, T. Oka, and H. Aoki, Phys. Rev. Lett. 103, 047403 (2009).

[23] H. Aoki, N. Tsuji, M. Eckstein, M. Kollar, T. Oka,
and P. Werner, Rev. Mod. Phys. 86, 779 (2014).

[24] H. Dehghani, T. Oka, and A. Mitra, Phys. Rev. B 90, 195429 (2014).

[25] Effective FloquetGibbs states for dissipative quantum
systems T. Shirai, J. Thingna, T. Mori, S. Denisov,
P. Hänggi, and S. Miyashita, New J. Phys. 18, 053008 (2016).

[26] F. Casas, J. A. Oteo, and J. Ros, J. Phys. A: Math. Gen. 34, 3379 (2001).

[27] E. S. Mananga and T. Charpentier, J. Chem. Phys. 135, 044109 (2011).

[28] J. E. Han, Phys. Rev. B 87, 085119 (2013).

[29] T. Iadecola, T. Neupert, and C. Chamon, Phys. Rev. B 91, 235133 (2015).

[30] K. I. Seetharam, C.-E. Bardyn, N. H. Lindner,
M. S. Rudner, and G. Refael, Phys. Rev. X 5, 041050 (2015).

[31] A. O. Caldeira, and A. J. Leggett, Phys. Rev. Lett. 46, 211 (1981).

[32] M. Büttiker, Phys. Rev. B 32, 1846(R) (1985).

[33] M. Büttiker, Phys. Rev. B 33, 3020 (1986).

[34] D. J. Thouless, Phys. Rev. B 27, 6083 (1983).

[35] B. Dóra, J. Cayssol, F. Simon, and R. Moessner,
Phys. Rev. Lett. 108, 056602 (2012).

[36] S. Vajna, B. Horovitz, B. Dóra, and G. Zaráng, Phys. Rev. B 94, 115145 (2016).

[37] Y. Tserkovnyak, A. Brataas, G. E. W. Bauer, and
B. I. Halperin, Rev. Mod. Phys. 77, 1375 (2005).

[38] E. Lesne, Y. Fu, S. Oyarzun, J. C. Rojas-Sánchez,
M. Jamet, E. Jacquet, J.-M. George, A. Barthélémy,
H. Jaffrès, A. Fert, M. Bibes, and L. Vila, Nat. Mater. 15, 1261 (2016).

[39] Y. Tserkovnyak, A. Brataas, and G. E. W. Bauer,
Phys. Rev. Lett. 103, 096601 (2014).

[40] F. Mahfouzi, B. K. Nikolić, S.-H. Chen, and C.-R. Chang,
Phys. Rev. B 82, 195440 (2010).

[41] K. Hattori, J. Phys. Soc. Jpn. 82, 024708 (2013).

[42] W. Y. Deng, W. Luo, H. Geng, M. N. Chen, L. Sheng,
and D. Y. Xing, New J. Phys. 17, 103018 (2015).
Appendix A: Keldysh–Floquet treatment of nonequilibrium physical quantities

In this part of the Appendix, we give a detailed explanation on our analytical treatment of the dynamics of the ensemble of edge electrons, based on the Keldysh–Floquet formalism.

1. One-particle states in Floquet picture

Let us start with the dynamics of one-particle state. Since the edge Hamiltonian under this precession is time-periodic, i.e., $H(k, t) = H(k, t + T)$ (with the periodicity $T = 2\pi/\Omega$), the electron dynamics can be treated in terms of the Floquet theory. The time dependence of the solution of the Schrödinger equation $H(k, t)|\Psi_\alpha(k, t)\rangle = i\hbar\partial_t|\Psi_\alpha(k, t)\rangle$ is expanded as

$$|\Psi_\alpha(k, t)\rangle = e^{-i\varepsilon_\alpha(k)t}\sum_{n \in \mathbb{Z}} e^{-in\Omega t}|\phi_\alpha^m(k)\rangle,$$

where its quasienergy $\varepsilon_\alpha(k)$ and the expanded components $|\phi_\alpha^m(k)\rangle$ are related by the Floquet equation

$$\sum_n H_{mn}(k)|\phi_\alpha^m(k)\rangle = \varepsilon_\alpha(k)|\phi_\alpha^n(k)\rangle,$$

with the “Floquet Hamiltonian”

$$H_{mn}(k) = \frac{1}{T} \int_0^T dt e^{i(m-n)\Omega t} H(k, t) - n\Omega \delta_{mn}. \quad (A3)$$

(It should be noted that some literatures use the terminology “Floquet Hamiltonian” for $H_{mn} = H_{mn} + n\Omega \delta_{mn}$.) Thus the time-dependent solution $|\Psi_\alpha(k, t)\rangle$ based on the original Hilbert space $\mathbb{H}$ is mapped to the time-independent wave function $|\Phi_\alpha(k)\rangle \equiv \{|\phi_\alpha^m(k)\rangle\}_{n \in \mathbb{Z}}$ based on the “extended” Hilbert space $\mathbb{H} \times \mathbb{Z}$, which is required to satisfy the infinite-dimensional eigenvalue equation $H|\Phi_\alpha(k)\rangle = \varepsilon_\alpha|\Phi_\alpha(k)\rangle$. The Floquet Hamiltonian $H$ here reads

$$H_{mn}(k) = \left( \begin{array}{c|c} \varepsilon_0(k) - m\Omega \delta_{mn} & J' \delta_{m,n+1} \\ \hline J' \delta_{m,n-1} & -\varepsilon_0(k) + m\Omega \delta_{mn} \end{array} \right), \quad (A4)$$

with $\varepsilon_0(k) = v_F k + J \cos \alpha$, $J' = J \sin \alpha$.

The Floquet index $n$, related with the phase factor $e^{-in\Omega t}$ in the time-dependent solution, accounts for the number of “energy quanta” of $\Omega$ arising from the time-periodic dynamics in the system; here the energy quantum is a magnon (with spin 1) arising from the precession of magnetization. The off-diagonal components in the Floquet Hamiltonian [Eq. (A4)] couples the neighboring magnon number sectors. The top-right component in Eq. (A4) adds one magnon and flips the electron spin from $\downarrow$ to $\uparrow$, and vice versa for the bottom-left component. Thus the Floquet matrix $H$ becomes block diagonal: each subspace $\mathbb{H}_\nu(n \in \mathbb{Z})$ spanned by the basis $\{|n = \nu - \frac{1}{2}; \sigma_z = \downarrow\}, |n = \nu + \frac{1}{2}; \sigma_z = \uparrow\rangle$ gets decoupled from one another (see Fig. 4). The block in this subspace $\mathbb{H}_\nu$ reads

$$H_\nu(k) = -\nu \Omega + \left( \begin{array}{c|c} -\varepsilon_0(k) + \frac{\Omega}{2} & J' \\ \hline J' & \varepsilon_0(k) - \frac{\Omega}{2} \end{array} \right), \quad (A5)$$

where the upper and lower components correspond to $|\nu - \frac{1}{2}; \sigma_z = \downarrow\rangle$ and $|\nu + \frac{1}{2}; \sigma_z = \uparrow\rangle$, respectively. Omitting the constant shift of the energy, $H_\nu(k)$ is exactly the same as the equilibrium edge Hamiltonian under a Zeeman field,

$$\bar{H}(k) = \left( \begin{array}{c|c} -\varepsilon_0(k) + \frac{\Omega}{2} & J' \\ \hline J' & \varepsilon_0(k) - \frac{\Omega}{2} \end{array} \right). \quad (A6)$$

Therefore, the Floquet Hamiltonian $H(k)$ can be exactly diagonalized, yielding the quasienergies

$$\varepsilon_{\nu \pm}(k) = -\nu \Omega \pm \sqrt{\varepsilon_0(k) - \frac{\Omega^2}{4} + J'^2}. \quad (A7)$$
We denote the corresponding eigenstates as $|\Phi_{\nu}^\pm(k)\rangle$, which are based on the subspace $\mathbb{H}$. Due to the redundancy of the extended Hilbert space, it is just enough to focus on one of the subspaces $\{\mathbb{H}_\nu\}$. The helical edge states with $| \pm 1, \downarrow \rangle$ and $| 0, \uparrow \rangle$ get hybridized by the magnon exchange, leading to gap opening $2J'$ at $\epsilon_0(k) = \frac{\Omega}{2}$, i.e. $k = k_0 \equiv v_F^{-1} \left( \frac{\Omega}{2} - J' \cos \alpha \right)$. It should be noted that the present Floquet Hamiltonian can be thus exactly diagonalized regardless of the precession periodicity, in contrast to the adiabaticity (low frequency) needed for the Thouless pumping theory, or the Magnus expansion (high frequency expansion) applied to ordinary Floquet systems.

2. Keldysh–Floquet treatment

We now move on to the many-body dynamics. The many-body dynamics in the periodic steady state is described by the nonequilibrium Green’s functions based on the extended Hilbert space $\mathbb{H} \times \mathbb{Z}$, defined by Eq. (\ref{eq:GreenFunction}). If the periodic steady state is ensured by the dissipation into the metallic reservoir (fermionic heat bath), the system inherits the electron distribution in the reservoir, and the lesser Green’s function satisfies Eq. (\ref{eq:GreenFunction}):

$$\mathcal{G}^<(k, \omega) = \mathcal{G}^R(k, \omega) \Sigma^<(\omega) \mathcal{G}^A(k, \omega),$$

(A8)

with $\Sigma^<(\omega) = i\Gamma f(\omega + n\Omega)\delta_{nn}$. The retarded and advanced Green’s functions for the dissipative time evolution are given by

$$\mathcal{G}^{R/A}(k, \omega) = \left[\omega + \mu - \mathcal{H}(k) \pm i\frac{\Gamma}{2}\right]^{-1},$$

(A9)

with the chemical potential $\mu$. Here the parameter $\Gamma$ can be microscopically obtained by integrating out the electron degrees of freedom in the reservoir; we regard $\Gamma$ as a phenomenological parameter in our analysis here.

In the present case, since the Floquet Hamiltonian $\mathcal{H}(k)$ is block diagonal within each subspace $\mathbb{H}_\nu$, spanned by $\{|\nu - \frac{1}{2}, \downarrow\}, |\nu + \frac{1}{2}, \uparrow\}$, the Green’s functions $\mathcal{G}^{R/A/<}(k, \omega)$ are also block diagonal. The Green’s functions in the subspace $\mathbb{H}_\nu$ read

$$\mathcal{G}^{R/A}(k, \omega) = \left[\omega + \mu + \nu\Omega - \mathcal{H}(k) \pm i\frac{\Gamma}{2}\right]^{-1},$$

(A10)

$$\mathcal{G}^{<}(k, \omega) = i\Gamma \mathcal{G}^{R}(k, \omega) \mathcal{F}_\nu(\omega) \mathcal{G}^{A}(k, \omega),$$

(A11)

with

$$\mathcal{F}_\nu(\omega) \equiv \text{diag} \left\{ f(\omega + (\nu - \frac{1}{2})\Omega), f(\omega + (\nu + \frac{1}{2})\Omega) \right\}.$$  

(A12)

We should note here that, similarly to the Floquet quasienegeries $\mathcal{E}(k)$, these Green’s functions also show redundancy in energy: they can be written with the $2 \times 2$ matrices $\mathcal{G}^{R/A/<}(k, \omega)$ as

$$\mathcal{G}^{R/A/<}(k, \omega) = \mathcal{G}^{R/A/<}(k, \omega + \nu\Omega),$$

(A13)

where

$$\mathcal{G}^{R/A}(k, \omega) = \left[\omega + \mu - \mathcal{H}(k) \pm i\frac{\Gamma}{2}\right]^{-1},$$

(A14)

$$\mathcal{G}^{<}(k, \omega) = i\Gamma \mathcal{G}^{R}(k, \omega) \mathcal{F}(\omega) \mathcal{G}^{A}(k, \omega)$$

(A15)

$$\mathcal{F}(\omega) = \text{diag} \left\{ f(\omega - \frac{\Omega}{2}), f(\omega + \frac{\Omega}{2}) \right\}.$$  

(A16)

In particular, each component in these Green’s functions is given as

$$\mathcal{G}^{R/A}(k, \omega) = \frac{1}{D^{R/A}_+ D^{R/A}_- - J'^2} \left[ D^{R/A}_+ J' D^{R/A}_- \right]$$

(A17)

$$\mathcal{G}^{<}(k, \omega) = \frac{i\Gamma}{D^{R}_+ D^{R}_- - J'^2} \left[ D^{A}_- D^{A}_+ - J'^2 \right]$$

(A18)

$$\times \left( f_{\downarrow}^2 D^{R}_+ D^{A}_+ + f_{\downarrow} J'^2 f_{\downarrow}^2 J'^2 + f_{\downarrow} J'^2 f_{\uparrow} J'^2 + f_{\uparrow} J'^2 f_{\uparrow} J'^2 \right),$$

where we use the notations

$$D^{R/A}_+(k, \omega) = \omega + \frac{\Omega}{2} + \epsilon_0(k) \pm i\frac{\Gamma}{2}$$

(A19)

$$D^{R/A}_-(k, \omega) = \omega + \frac{\Omega}{2} - \epsilon_0(k) \pm i\frac{\Gamma}{2}$$

(A20)

$$f_{\downarrow}(\omega) = f(\omega - \frac{\Omega}{2}), \quad f_{\uparrow}(\omega) = f(\omega + \frac{\Omega}{2}).$$

(A21)

3. Edge current

We are now ready to evaluate the physical quantities carried by the electrons on the edge of QSHI. The current along the edge is given by

$$I = i e \nu F \int_0^\Omega \frac{d\omega}{2\pi} L \sum_k \text{Tr} [\sigma_z \mathcal{G}^{<}(k, \omega)],$$

(A22)

where $\text{Tr}'$ denotes the trace over the extended Hilbert space $\mathbb{H} \times \mathbb{Z}$. By evaluating the trace, we can transform the zone of the $\omega$-integral from a folded zone into an infinite zone,

$$I = i e \nu F \int_0^\Omega \frac{d\omega}{2\pi} L \sum_k \sum_{\nu \in \mathbb{Z} + 1/2} \left[ \mathcal{G}^{<}_{\uparrow \downarrow}(k, \omega + \nu\Omega) - \mathcal{G}^{<}_{\downarrow \uparrow}(k, \omega + \nu\Omega) \right]$$

(A23)

$$= i e \nu F \sum_k \int_{\nu\Omega}^{\nu\Omega+1/2} \frac{d\bar{\omega}}{2\pi} L \sum_k \left[ \mathcal{G}^{<}_{\uparrow \downarrow}(k, \bar{\omega}) - \mathcal{G}^{<}_{\downarrow \uparrow}(k, \bar{\omega}) \right]$$

(A24)

$$= i e \nu F \int_{-\infty}^{\infty} \frac{d\bar{\omega}}{2\pi} L \sum_k i\Gamma \left[ \mathcal{G}^{<}_{\uparrow \downarrow}(k, \bar{\omega}) - \mathcal{G}^{<}_{\downarrow \uparrow}(k, \bar{\omega}) \right]$$

(A25)

$$= i e \nu F \int_{-\infty}^{\infty} \frac{d\bar{\omega}}{2\pi} L \sum_k i\Gamma \left[ f_{\downarrow}^2 D^{R}_+ D^{A}_+ + f_{\downarrow} J'^2 f_{\downarrow}^2 J'^2 + f_{\uparrow} J'^2 f_{\uparrow} J'^2 + f_{\uparrow} J'^2 f_{\uparrow} J'^2 \right]$$

(A26)

$$\times \left[ D^{R}_+ D^{R}_- - J'^2 \right] \left[ D^{A}_- D^{A}_+ - J'^2 \right].$$


In order to evaluate this integral, we decompose the integrand into partial fractions. Here we define the single-band Green’s functions

\[
g_{\pm}(k, \omega) = \left(\frac{\omega}{\mu + \xi(k) + i\frac{\pi}{2}}\right)^{-1}, \quad g_{\pm}^\alpha = [g_{\pm}]^*, \quad (A27)
\]

with

\[
\epsilon(k) = v_F k + J \cos \alpha - \frac{\Omega}{2}, \quad (A28)
\]
\[
\xi(k) = \sqrt{\epsilon^2(k) + J^2}. \quad (A29)
\]

In terms of these single-band Green’s functions, the integrand can be decomposed as

\[
\left|\frac{D_{\pm}^R}{D_{\pm}^R - J'^2}\right|^2 = \frac{i}{2\pi} \sum_{\pm} \left[ 1 \pm \frac{\epsilon}{\xi} - \frac{J'^2}{\xi(\xi + i\frac{\pi}{2})} \right] g_{\pm}^R + c.c. \quad (A30)
\]
\[
\left|\frac{D_{\pm}^R}{D_{\pm}^R - J'^2}\right|^2 = \frac{i}{2\pi} \sum_{\pm} \left[ 1 \pm \frac{\epsilon}{\xi} - \frac{J'^2}{\xi(\xi + i\frac{\pi}{2})} \right] g_{\pm}^R + c.c. \quad (A31)
\]

By using the indefinite integrals

\[
\int d\omega \; g_{\pm}^R = \ln \left[ \frac{\omega + \mu + \xi + i\frac{\pi}{2}}{\omega + \mu + \xi} \right], \quad (A32)
\]
\[
\int d\omega \; g_{\pm}^R g_{\pm}^R = \frac{2}{\pi} \arctan \frac{\omega + \mu + \xi}{\xi/2}, \quad (A33)
\]

the \(\omega\)-integrals in Eq. (A26) can be exactly evaluated as

\[
I = -\frac{e v_F}{2\pi L} \sum_{\nu, \nu' = \pm} \left\{ \frac{\cos \omega}{4\xi(\xi^2 + \xi^2)} \ln \left[ \left( \xi - \nu \Omega \right)^2 + \nu^2 \right]^2 + \frac{r^2}{4} \right\} \nu.
\]
\[
+ \left[ 1 - \frac{\nu e}{\xi} - \frac{J'^2}{\xi^2 + \xi^2} \right] \arctan \frac{\xi - \nu \Omega - \nu' \mu}{\Gamma/2} \nu',
\]
\[
(A34)
\]

By evaluating the \(k\)-integral numerically, we obtain the current \(I\) in the presence of the dissipation effect \(\Gamma\), as shown in Fig. 2(a) in the main text.

In the dissipationless limit \(\Gamma \rightarrow 0\) with charge neutrality \(\mu = 0\), Equation (A34) can be further reduced as

\[
I = \frac{e v_F}{L} \sum_k \left[ \frac{\epsilon}{\xi} \theta(\xi - \Omega/2) + \frac{\epsilon^2}{\xi^2} \theta(\xi/2 - \xi) \right],
\]
\[
(A35)
\]

where we have used \(\arctan(x/\Gamma) \rightarrow \frac{\pi}{2} \theta(x) - \theta(-x)\) and \(2\xi + \Omega > 0\). By evaluating the \(k\)-integral over \(k \in [-k_e, k_e]\) with the cutoff \(k_e\), we obtain

\[
I = \frac{e}{2\pi} \left[ \sqrt{\epsilon^2_L + J^2} - \sqrt{\epsilon^2_L + J'^2} \right] + \frac{e}{\pi} \left[ \epsilon_0 - J' \arctan \frac{\epsilon_0}{J} \right].
\]
\[
(A36)
\]

with

\[
-\epsilon_L = -v_F k_e + J \cos \alpha - \Omega/2 \quad (A37)
\]
\[
\epsilon_R = v_F k_e + J \cos \alpha - \Omega/2 \quad (A38)
\]
\[
\epsilon_0 = \theta(\Omega/2 - J') \sqrt{\Omega^2/4 - J'^2}. \quad (A39)
\]

Taking the limit \(v_F k_e \gg J, \Omega\), we obtain the form

\[
I = -\frac{e \Omega}{2\pi} \left[ 1 - \theta(1 - \delta) \left( \sqrt{1 - \delta^2} - \delta \arctan \sqrt{\delta^2 - 1} \right) \right].
\]
\[
(A40)
\]

This is the result shown by the black dashed lines in Fig. 2(a).

4. Spin injection rate

In order to estimate the spin injection rate from the ferromagnet into the QSHI edge, we need to evaluate the in-plane components of the electron spin accumulation on the edge, as discussed in the main text. By using Eq. (1), each component can be given in the time-dependent form,

\[
\langle \sigma_x(t) \rangle = -i \int_0^\Omega \frac{d\omega}{2\pi L} \sum_{\nu, \nu'} \left[ g_{\nu + \nu'}^e e^{-i\Omega t} + g_{\nu - \nu'}^e e^{i\Omega t} \right] \quad (A41)
\]
\[
\langle \sigma_y(t) \rangle = -i \int_0^\Omega \frac{d\omega}{2\pi L} \sum_{\nu, \nu'} \left[ g_{\nu + \nu'}^e e^{-i\Omega t} - g_{\nu - \nu'}^e e^{i\Omega t} \right] \quad (A42)
\]
\[
\langle \sigma_y(t) \rangle = -i \int_0^\Omega \frac{d\omega}{2\pi L} \sum_{\nu, \nu'} \left[ g_{\nu + \nu'}^e e^{-i\Omega t} - g_{\nu - \nu'}^e e^{i\Omega t} \right] \quad (A44)
\]
\[
\langle \sigma_y(t) \rangle = \int_{-\infty}^\infty \frac{d\omega}{2\pi L} \sum_{\nu, \nu'} \left[ g_{\nu + \nu'}^e e^{-i\Omega t} - g_{\nu - \nu'}^e e^{i\Omega t} \right] \quad (A45)
\]
\[
\langle \sigma_y(t) \rangle = \int_{-\infty}^\infty \frac{d\omega}{2\pi L} \sum_{\nu, \nu'} \left[ -\Re g_{\nu + \nu'}^e \cos \Omega t + \Im g_{\nu + \nu'}^e \sin \Omega t \right],
\]
\[
(A46)
\]

where we have used the relation \( g_{\nu + \nu'}^e = -g_{\nu + \nu'}^e \). Therefore, the component parallel to \( \mathbf{n}(t) \parallel \mathbf{n}(t) = -e_x \sin \Omega t + e_y \cos \Omega t \) is given as

\[
\sigma_d = -\int_{-\infty}^\infty \frac{d\omega}{\pi L} \sum_k \Re g_{\nu + \nu'}^e (k, \omega).
\]
\[
(A47)
\]

Here the integrand reads

\[
\Re g_{\nu + \nu'}^e = \frac{J' T^2}{2} \left[ f_l - f_l \right] \quad (A36)
\]
\[
\left[ D_{\pm}^R D_{\pm}^R - J'^2 \right] \left[ D_{\pm}^A D_{\pm}^A - J'^2 \right]
\]
\[
(A48)
\]
obtained from Eq. (A18). By using the decomposition by partial fractions

$$\frac{1}{D^R_k D^R_L - J^2} = \frac{i}{4\xi^2} \sum_{\pm} \frac{\xi \pm i\tau}{\xi^2 + \frac{\tau^2}{4}} g_k + \text{c.c.,}$$  \hspace{1cm} \text{(A49)}

the $\omega$-integral in Eq. (A47) can be evaluated at zero temperature as

$$\sigma_d = -\frac{J^d}{4\pi L} \sum_k \sum_{\nu, \nu'} \left\{ \frac{\Gamma}{4\xi} \ln \left[ (\xi - \nu \Omega/2 - \nu' \mu)^2 + \frac{\tau^2}{4} \right] \right. $$
$$+ \arctan \frac{\xi - \nu \Omega/2 - \nu' \mu}{\Gamma/2} \left\} \nu - \frac{\nu \Omega/2}{\Gamma/2} \right.$$  \hspace{1cm} \text{(A50)}

In the regime $J' \gg \Omega, \Gamma$, this quantity gets suppressed. In particular, this quantity completely vanishes in the limits $\Gamma \to 0$, where the heat reservoir is detached from the system, or $\Omega \to 0$, where the magnetization is fixed. Therefore, we should take finite orders in $\Omega$ and $\Gamma$ to evaluate the asymptotic behavior of $\sigma_d$ in the limit $J' \to \infty$. At charge neutrality $\mu = 0$, the asymptotic behavior becomes

$$\sigma_d \approx -\frac{J^d}{2\pi L} \sum_k \sum_{\nu} \nu \left[ \frac{\Gamma}{2\xi} \ln \left( \xi - \frac{\nu \Omega}{2} \right) + \arctan \frac{\xi - \nu \Omega/2}{\Gamma/2} \right] $$
$$\approx -\frac{J^d \Omega^2}{2\pi L} \sum_k \sum_{\nu} \nu \frac{1}{\xi^2(k)}$$
$$= \frac{J^d \Omega^2}{(2\pi)^2} \int_{-k_c}^{k_c} \frac{dk}{(v_F k + J \cos \alpha - \frac{\Omega}{2})^2 + J'^2}.$$  \hspace{1cm} \text{(A53)}

Since this $k$-integral is free from the ultraviolet divergence, we can safely extend its zone to $(-\infty, \infty)$ to evaluate its asymptotic behavior. As a result, the integral reads

$$\sigma_d \approx -\frac{J^d \Omega^2}{(2\pi)^2 v_F} \int_{-\infty}^{\infty} \frac{d\xi}{(\xi^2 + J'^2)^2} = \frac{\Omega^2}{8\pi v_F J^2}.$$  \hspace{1cm} \text{(A55)}

By taking a time average over a cycle, the averaged damping-like torque, corresponding to the spin injection rate, reads

$$-J_S = \overline{T^d} = \int_0^T \frac{dt}{T} \int \mathbf{T}^d(t) = -J_s \sigma \sin \alpha \mathbf{e}_z.$$  \hspace{1cm} \text{(A59)}

By substituting Eq. (A55), its asymptotic behavior for $J' \to 0$ is given as

$$J_S \approx \frac{\Omega^2}{8\pi v_F J^2} \mathbf{e}_z.$$  \hspace{1cm} \text{(A60)}

Thus the asymptotic behavior of the spin-to-charge conversion rate $\lambda_{sc}$ reaches

$$\lambda_{sc} = \frac{J_S}{e} \approx \frac{\Omega^2}{2 \pi v_F J} \approx \frac{4 \pi v_F J}{\Gamma^2},$$  \hspace{1cm} \text{(A61)}

as mentioned in the main text.

**Appendix B: Adiabatic pumping picture**

The maximum current $I_e = -e/T$, which implies that a single electron is pumped along the edge during one cycle of precession, can be described as the adiabatic (Thouless) pumping. If the magnetization is static and pointing in the direction

$$\mathbf{n} = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha),$$  \hspace{1cm} \text{(B1)}

the edge mode opens a gap $2J \sin \alpha$. The eigenstate wave function in the valence band reads

$$u(k) = \left( \sin \frac{\theta(k)}{2} e^{-i\beta/2}, -\cos \frac{\theta(k)}{2} e^{i\beta/2} \right),$$  \hspace{1cm} \text{(B2)}

where $\theta(k)$ is the polar angle of the eigenstate spin, defined by

$$\cos \theta(k) = \frac{v_F k + J \cos \alpha}{\sqrt{(v_F k + J \cos \alpha)^2 + (J \sin \alpha)^2}}.$$  \hspace{1cm} \text{(B3)}

If $\beta$ precesses slowly from 0 to $2\pi$, we can define the Berry curvature in $(k, t)$-space,

$$\Omega_{kt} = 2 \text{Im} \left( \frac{\partial u}{\partial t} \frac{\partial u}{\partial k} \right) = \frac{1}{2} \sin \frac{\theta}{2} \frac{d\theta}{dt} \frac{d\beta}{dt}.$$  \hspace{1cm} \text{(B4)}

The number of the electrons pumped per one cycle is given by integrating this Berry curvature over the $(k, t)$-plane, yielding

$$n_{\text{pump}} = -\int_0^T dt \int \frac{dk}{2\pi} \Omega_{kt} = 1.$$  \hspace{1cm} \text{(B5)}

This discussion applies as long as an electron cannot be excited by a single magnon with the energy $\Omega$, from the hole band to the electron band separated by the gap $2J'$.