AN EXPONENTIALLY-AVERAGED VASYUNIN FORMULA

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Abstract. We prove a Vasyunin-type formula for an autocorrelation function arising from a Nyman-Beurling criterion generalized to a probabilistic framework. This formula can also be seen as a reciprocity formula for cotangent sums, related to the ones proven in [BC13], [ABB17].

1. Introduction

The main result of this paper is the following identity:

**Theorem 1.** For coprime \( m, n \geq 1 \), we have

\[
mn \int_{0}^{\infty} \left( \frac{1}{mt} - \frac{1}{e^{mt} - 1} \right) \left( \frac{1}{nt} - \frac{1}{e^{nt} - 1} \right) dt = -\frac{1}{2} + C(n + m) + \frac{m - n}{2} \log \left( \frac{n}{m} \right) - \frac{\pi}{2} \sum_{k=1}^{n-1} \frac{mk}{n} \cot \left( \frac{mk \pi}{n} \right) - \frac{\pi}{2} \sum_{l=1}^{m-1} \frac{nl}{m} \cot \left( \frac{nl \pi}{m} \right),
\]

where

\[
C = 1 - \int_{0}^{1} \left( \frac{1}{t(e^t - 1)} - \frac{1}{t^2} + \frac{1}{2t} \right) dt - \int_{1}^{\infty} \frac{dt}{t(e^t - 1)} = \frac{1}{2} (\log 2\pi - \gamma).
\]

Theorem 1 was obtained when studying a probabilistic version of the Nyman-Beurling criterion for the Riemann hypothesis (RH). One of the main results of the deterministic Nyman-Beurling approach for RH (see [Nym50, Beu55]), improved by Báez-Duarte et al. in [BD03], [BDBLS00], is the following:

**Theorem 2.** In \( H = L^2(0, \infty) \), set \( \chi : t \mapsto 1_{[0,1]}(t) \) and \( \rho_n : t \mapsto \{ \frac{1}{nt} \} \), \( n \geq 1 \). Then RH holds if and only if

\[
d_N = d_H(\chi, \text{Span}(\rho_1, \ldots, \rho_N)) \xrightarrow{N \to \infty} 0.
\]

Here, \( \{ \cdot \} \) denote the fractional part, and the notation \( d_F(f, F) \) stands for the distance between the vector \( f \) and the subspace \( F \) in the Hilbert space \( F \).

The squared distance \( d_N^2 \) can be expressed as a quotient of Gram determinants:

\[
d_N^2 = \frac{\det(\text{Gram}(\chi, \rho_1, \ldots, \rho_N))}{\det(\text{Gram}(\rho_1, \ldots, \rho_N))}.
\]

The computation of the coefficients of these Gram matrices is related to the study of the autocorrelation function

\[
\lambda \mapsto A(\lambda) = \int_{0}^{\infty} \left\{ \frac{1}{t} \right\} \left\{ \frac{1}{\lambda t} \right\} dt.
\]

Indeed, for every \( n, m \geq 1 \), a simple change of variables gives

\[
\langle \rho_n, \rho_m \rangle = \int_{0}^{\infty} \left\{ \frac{1}{nt} \right\} \left\{ \frac{1}{mt} \right\} dt = \frac{1}{n} A \left( \frac{m}{n} \right) = \frac{1}{m} A \left( \frac{n}{m} \right).
\]
The autocorrelation function \( A(\lambda) \) has been studied in \[BDBLS05\], where the authors prove in particular that \( A \) is non-differentiable at each rational point. One of the most useful technical tools for the study of \( A \) is the Vasyunin formula \[Vas95\], \[BDBLS00, p.141\]:

\[
\begin{align*}
mn \int_0^\infty \left\{ \frac{1}{nt} \right\} \left\{ \frac{1}{mt} \right\} dt &= C(n + m) + \frac{m - n}{2} \log \left( \frac{n}{m} \right) \\
&\quad - \frac{\pi}{2} \sum_{k=1}^{n-1} \left\{ \frac{mk}{n} \right\} \cot \left( \frac{k\pi}{n} \right) - \frac{\pi}{2} \sum_{l=1}^{m-1} \left\{ \frac{ln}{m} \right\} \cot \left( \frac{l\pi}{m} \right),
\end{align*}
\]

for the same constant \( C = \frac{1}{2} (\log 2 - \gamma) \).

The similarity between Theorem 1 and Vasyunin’s formula is striking. In Section 2, we explain how the left-hand side in Theorem 1 can be seen as an "exponentially-averaged" autocorrelation function coming from a probabilistic Nyman-Beurling criterion. More precisely, it can be written

\[
\begin{align*}
\mathcal{A}(\lambda) &= \int_0^\infty \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) \left( \frac{1}{\lambda t} - \frac{1}{e^{\lambda t} - 1} \right) dt.
\end{align*}
\]

Vasyunin’s formula is one of the motivations for the study of cotangent sums, which has been an area of active research these past years, see for instance the introduction of \[BC13\], or the references \[MR16\], \[Bet15\].

A remarkable property of cotangent sums has been unveiled by Bettin and Conrey in \[BC13\], who obtained a so-called reciprocity formula. Following their notations, we set

\[
c(x) = - \sum_{a=1}^{k-1} \frac{a}{k} \cot \left( \frac{\pi ah}{k} \right),
\]

where \( x = h/k, k > 0 \) and \( \gcd(h, k) = 1 \). The reciprocity formula states that

\[
x c(x) + c \left( \frac{1}{x} \right) - \frac{1}{\pi k} = g(x),
\]

for a smooth function \( g \) defined from Eisenstein series. What makes the reciprocity formula interesting is the contrast with the fact that the function \( c \), defined on the set of rational numbers, cannot be extended into a continuous function on \( \mathbb{R}^*_+ \).

Theorem 1 can be seen as another formulation of the reciprocity formula for cotangent sums. Indeed, with the notations of \[BC13\], Theorem 1 can be rewritten as

\[
x c(x) + c \left( \frac{1}{x} \right) - \frac{1}{\pi k} = \frac{1}{\pi} \left( 2xA(x) - (1 + x)C + (x - 1) \log(x) \right).
\]

Both results combined give a simple representation formula for the function \( g \).

Actually, the reciprocity formula is stated and proved in \[BC13\] for more general cotangent sums. More precisely, given \( a > 0 \), let us consider the arithmetic sum

\[
c_a \left( \frac{h}{k} \right) = k^a \sum_{k=1}^{m-1} \cot \left( \frac{\pi mh}{k} \right) \zeta \left( -a, \frac{m}{k} \right),
\]

where \( \zeta(a, x) \) denotes the Hurwitz zeta function. For such sums, Bettin and Conrey proved that the function

\[
g_a \left( \frac{h}{k} \right) = c_a \left( \frac{h}{k} \right) - \left( \frac{k}{h} \right)^{1+a} c_a \left( -\frac{k}{h} \right) + \frac{k^a \zeta(1-a)}{\pi h}
\]

can be extended from \( \mathbb{Q} \) to an analytic function on \( \mathbb{C} - \mathbb{R}_{\leq 0} \).

In the article \[ABB17\] by Auli, Bayad and Beck, the authors give a more explicit expression for the function \( g_a \), written as a contour integral.
2. An exponentially-averaged autocorrelation function

Although they are not needed for the proof of Theorem 1, the results stated in this paragraph explain how the function $A$ arises from a Nyman-Beurling criterion set in a probabilistic framework, and how the study of reciprocity formulas for cotangent sums is related to RH, see [DH18].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider the Hilbert space $H = L^2(\Omega \times [0, \infty))$ endowed with the scalar product

$$\langle X, Y \rangle_H = \int_0^\infty \int_{\Omega} X(\omega, t)Y(\omega, t)d\mathbb{P}(\omega)dt.$$ 

Let $(X_k)_{k \geq 1}$ be an independent sequence of random variables, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with exponential distribution $X_k \sim \mathcal{E}(1)$. For each $k \geq 1$, the mapping $R_k : (\omega, t) \mapsto \{X_k(\omega)\}$ belongs to $H$. We also define $\chi \in H$ by $\chi(\omega, t) = 1_{[0,1]}(t)$.

In [DH18], we prove the following:

**Theorem 3.** If $D_N = d_H(\chi, \text{Vect}(R_1, \ldots, R_N)) \xrightarrow{N \to \infty} 0$, then RH holds.

As in the deterministic case, the squared distance $D^2_N$ can be expressed as a quotient of Gram determinants, which is a motivation for the computation of the scalar products $\langle R_n, R_m \rangle_H$, for $m, n \geq 1$. In order to compute these scalar products, we use the following fact: if $Z$ is a random variable with exponential distribution $\mathcal{E}(\alpha)$, $\alpha > 0$, then

$$\mathbb{E}[[Z]] = \frac{1}{\alpha} - \frac{1}{e^\alpha - 1},$$

where $\mathbb{E}[X]$ is the expectation of the random variable $X$. Such a formula is obtained by straightforward calculations.

As $\frac{X_n}{nt}$ (resp. $\frac{X_m}{mt}$) follows the exponential distribution $\mathcal{E}(nt)$ (resp. $\mathcal{E}(mt)$) we obtain, by independence of $R_n$ with $R_m$:

$$\langle R_n, R_m \rangle_H = \int_0^\infty \left( \frac{1}{mt} - \frac{1}{e^{mt} - 1} \right) \left( \frac{1}{nt} - \frac{1}{e^{nt} - 1} \right) dt = \frac{1}{n}A\left(\frac{m}{n}\right) = \frac{1}{m}A\left(\frac{n}{m}\right).$$

This formula explains the terminology "exponentially-averaged autocorrelation function" for $A(\lambda)$ and "exponentially-averaged Vasyunin formula" for Theorem 1.

We draw the graph of $\lambda \mapsto A(1/\lambda)$, which has been obtained using Theorem 1. This graph can be compared with the one of the deterministic autocorrelation function $A$ as seen in [BDBLS05]. Although there are some global similarities in the behaviour of both functions, the function $A$ is analytic whereas $A$ is not differentiable (cf Eq.(2) in [BDBLS05]).
Figure 1. The exponentially-averaged autocorrelation function $\lambda \mapsto A(1/\lambda)$, $\lambda > 0$.

3. Computation of $C$

Numerical evidence suggested that $C = \frac{1}{2} (\log(2\pi) - \gamma)$, where $\gamma$ is the Euler constant. Balazard actually proved this identity $\text{[Bal18]}$. We thank Michel Balazard for his proof and for authorizing us to reproduce it below.

Proposition 3.1 ($\text{[Bal18]}$). One has

$$1 - \int_0^1 \left( \frac{1}{t(e^t - 1)} - \frac{1}{t^2} + \frac{1}{2t} \right) dt - \int_1^\infty \frac{dt}{t(e^t - 1)} = \frac{1}{2} (\log 2\pi - \gamma).$$

Proof. (translated from $\text{[Bal18]}$) One has

$$\int_0^1 \left( \frac{1}{t(e^t - 1)} - \frac{1}{t^2} + \frac{1}{2t} \right) dt + \int_1^\infty \frac{dt}{t(e^t - 1)} = \lim_{x \to 0} \left( \int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-tx} \frac{dt}{t} + \int_1^\infty \frac{e^{-tx}}{e^t - 1} \frac{dt}{t} \right).$$

But, for $x > 0$,

$$I(x) = \int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-tx} \frac{dt}{t} + \int_1^\infty \frac{e^{-tx}}{e^t - 1} \frac{dt}{t}$$

$$= \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-tx} \frac{dt}{t} + \int_1^\infty \left( \frac{1}{t} - \frac{1}{2} \right) e^{-tx} \frac{dt}{t}.$$ 

On one hand,

$$\int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-tx} \frac{dt}{t} = \log \Gamma(x) - (x - 1/2) \log x + x - \frac{1}{2} \log 2\pi,$$

(Binet, 1839, cf. $\text{[WW27]}$, §12.31, p. 249).

On the other hand, as $x$ tends to 0,

$$\int_1^\infty \left( \frac{1}{t} - \frac{1}{2} \right) e^{-tx} \frac{dt}{t} = -\frac{1}{2} \int_1^\infty e^{-tx} \frac{dt}{t} + 1 + o(1)$$
with
\[ \int_{1}^{\infty} e^{-i t} \frac{dt}{t} = \int_{x}^{\infty} e^{-i t} \frac{dt}{t} = \int_{x}^{1} (e^{-t} - 1) \frac{dt}{t} - \log x + \int_{1}^{\infty} e^{-i t} \frac{dt}{t} = -\log x - \gamma + o(1), \]
by virtue of a classical formula for \( \gamma \) (cf. [WW27], §12·2, Example 4, p. 243).

Finally,
\[ I(x) = \log \Gamma(x) - (x - 1/2) \log x + x - \frac{1}{2} \log 2\pi - \frac{1}{2} \left( -\log x - \gamma + o(1) \right) + 1 + o(1), \]
where one used \( x\Gamma(x) = \Gamma(x + 1) \), \( x > 0 \), and \( \Gamma(1) = 1 \). This completes the proof. \( \square \)

4. PROOF OF THEOREM 1

We need several technical lemmas to proceed the proof of Theorem 1.

**Lemma 1.** We have the following expansions

\[ (4.1) \quad \frac{1}{z^n - 1} - \frac{1}{nz - 1} = \frac{1}{2n} \sum_{k=1}^{n-1} \frac{2 \cos \left( \frac{2k\pi}{n} \right) z - 2}{z^2 - 2 \cos \left( \frac{2k\pi}{n} \right) z + 1}, \]
\[ (4.2) \quad \frac{z^{n-1}}{z^n - 1} - \frac{1}{nz - 1} = \frac{1}{2n} \sum_{k=1}^{n-1} \frac{-2 \cos \left( \frac{2k\pi}{n} \right) + 2z}{z^2 - 2 \cos \left( \frac{2k\pi}{n} \right) z + 1}, \]
\[ (4.3) \quad \frac{z^n + 1}{z^n - 1} - \frac{1}{n z - 1} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{z^2 + \frac{z^{n-1}}{z^{n-2}} - \cos \left( \frac{2k\pi}{n} \right)}. \]

**Proof.** We consider the \( n \)-th roots of unity \( \omega_{k,n} = e^{\frac{2i\pi k}{n}} \) for \( k \in \{0, \ldots, n-1\} \). We have
\[ \frac{1}{z^n - 1} = \prod_{k=0}^{n-1} \left( z - \omega_{k,n} \right) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\omega_{k,n}}{z - \omega_{k,n}} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{\omega_{k,n}}{z - \omega_{k,n}}. \]
We sum in both directions to obtain
\[ \frac{1}{z^n - 1} - \frac{1}{nz - 1} = \frac{1}{2n} \sum_{k=1}^{n-1} \frac{\omega_{k,n}}{z - \omega_{k,n}} + \frac{\omega_{n-k,n}}{z - \omega_{n-k,n}} = \frac{1}{2n} \sum_{k=1}^{n-1} \frac{2 \cos \left( \frac{2k\pi}{n} \right) z - 2}{z^2 - 2 \cos \left( \frac{2k\pi}{n} \right) z + 1}, \]
which is exactly Equation (4.1).

In order to obtain the second identity, we write
\[ \frac{1}{z - 1} + \sum_{k=1}^{n-1} \frac{1}{z - \omega_{k,n}} = \sum_{k=0}^{n-1} \frac{1}{z - \omega_{k,n}} = \frac{P'(z)}{P(z)}, \]
where \( P(z) = \prod_{k=0}^{n-1} (z - \omega_{k,n}) = z^n - 1 \). We then have
\[ \frac{n z^{n-1}}{z^n - 1} - \frac{1}{z - 1} = \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{z - \omega_{k,n}} + \frac{1}{z - \omega_{n-k,n}} = \frac{1}{2} \sum_{k=1}^{n-1} \frac{-2 \cos \left( \frac{2k\pi}{n} \right) + 2z}{z^2 - 2 \cos \left( \frac{2k\pi}{n} \right) z + 1}. \]
In order to obtain (4.3), we notice that
\[ \frac{z^n + 1}{z^n - 1} - \frac{1}{nz - 1} = z \left( \frac{z^{n-1}}{z^n - 1} - \frac{1}{nz - 1} \right) + \left( \frac{1}{z^n - 1} - \frac{1}{nz - 1} \right), \]
and we use Equations (4.1) and (4.2). \( \square \)
Lemma 2. For any \( n \geq 1 \) and \( a \notin \frac{2m\pi}{n} \), we have
\[
\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{\cos(a) - \cos(\frac{2k\pi}{n})} = \frac{1}{\sin(a)} \left( \frac{1}{n} \cot \left( \frac{a}{2} \right) - \cot \left( \frac{na}{2} \right) \right).
\]

Proof. We evaluate equation (4.3) at \( z = e^{ia} \):
\[
\frac{e^{ian} + 1}{e^{ian} - 1} \cdot \frac{1}{n} \frac{e^{ia} + 1}{e^{ia} - 1} = \frac{1}{n} \frac{e^{2ia} - 1}{2e^{ia} - \cos(\frac{2k\pi}{n})}.
\]
The right-hand side reads
\[
\frac{1}{n} \frac{e^{2ia} - 1}{2e^{ia} - \cos(\frac{2k\pi}{n})} \cdot \sum_{k=1}^{n-1} \frac{1}{\cos(a) - \cos(\frac{2k\pi}{n})} = \frac{i}{n} \sin(a) \sum_{k=1}^{n-1} \frac{1}{\cos(a) - \cos(\frac{2k\pi}{n})},
\]
and the left-hand side:
\[
\frac{e^{ian} + 1}{e^{ian} - 1} \cdot \frac{1}{n} \frac{e^{ia} + 1}{e^{ia} - 1} = \frac{2\cos(na/2)}{2i\sin(na/2)} \cdot \frac{1}{n} \frac{2\cos(a/2)}{2i\sin(a/2)} = -i \left( \cot \left( \frac{na}{2} \right) - \frac{1}{n} \cot \left( \frac{a}{2} \right) \right).
\]

We then obtain the desired result.

Lemma 3. For any \( a \in (0, 2\pi) \), \( a \neq \pi \),
\[
\int_1^\infty \frac{1}{z^2 - 2 \cos(a)z + 1} \, dz = \frac{1}{\sin(a)} \left( \frac{\pi}{2} - \frac{a}{2} \right).
\]

Proof. We have, for \(|a| < 1\),
\[
\int_1^\infty \frac{1}{z^2 - 2az + 1} \, dz = \frac{1}{\sqrt{1 - a^2}} \arctan \left( \sqrt{\frac{1 + a}{1 - a}} \right),
\]
which is obtained using the change of variables \( x = \sqrt{\frac{1 + a}{1 - a}} \) and \( 2 - \arctan(x) = \arctan(1/x) \). If \( \alpha = \cos(a) \) for some \( a \in (0, 2\pi) \), \( a \neq \pi \), the above expression simplifies into
\[
\frac{1}{\sqrt{1 - a^2}} \arctan \left( \sqrt{\frac{1 + a}{1 - a}} \right) = \frac{1}{\sin(a)} \arctan \left( \frac{2\cos(a/2)^2}{2\sin(a/2)^2} \right)
= \frac{1}{\sin(a)} \arctan \left( \frac{\tan(a/2)}{\tan(a/2)} \right)
= \frac{1}{\sin(a)} \left( \frac{\pi}{2} - \frac{a}{2} \right) = \frac{1}{\sin(a)} \left( \frac{\pi}{2} - \frac{a}{2} \right),
\]
the last identity being obtained by studying separately the cases \( 0 < a < \pi \) and \( \pi < a < 2\pi \).

For \( m, n \geq 1 \), we set
\[
I(m, n) = \int_1^\infty \frac{1}{e^{mx} - 1} \, dx.
\]
The change of variables \( z = e^{x} \) gives
\[
I(m, n) = \int_1^\infty \frac{1}{z^n - 1} \, dz.
\]

Lemma 4. Let \( m, n \geq 2 \) be coprime numbers. Then
\[
I(m, n) = \frac{m - 1 \log(n)}{2m} + \frac{n - 1 \log(m)}{2n} + \frac{1}{2n} \sum_{k=1}^{n-1} \left( \frac{1}{m} \cot \left( \frac{mk\pi}{n} \right) - \frac{1}{n} \cot \left( \frac{k\pi}{n} \right) \right) \left( \frac{\pi}{2} - \frac{k\pi}{m} \right)
+ \frac{1}{2m} \sum_{l=1}^{m-1} \left( \frac{1}{n} \cot \left( \frac{nl\pi}{m} \right) - \frac{1}{l} \cot \left( \frac{l\pi}{m} \right) \right) \left( \frac{\pi}{2} - \frac{l\pi}{m} \right).
\]

Proof. We deduce from equation (4.1) that
\[
I(m, n) = \frac{1}{nm} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \int_1^\infty \frac{\cos \left( \frac{2k\pi}{n} \right) z - 1}{z^2 - 2 \cos \left( \frac{2k\pi}{n} \right) z + 1} \frac{\cos \left( \frac{2\pi}{m} \right) z - 1}{z^2 - 2 \cos \left( \frac{2\pi}{m} \right) z + 1} \frac{1}{z} \, dz.
\]
For \( k \in \{1, \ldots, n-1\} \) and \( l \in \{1, \ldots, m-1\} \), we set \( \alpha_k = \cos \left( \frac{2k\pi}{n} \right) \), \( \beta_l = \cos \left( \frac{2\pi}{m} \right) \), and
\[
F_{k,l}(z) = \frac{\alpha_k z - 1}{z^2 - 2\alpha_k z + 1} \frac{\beta_l z - 1}{z^2 - 2\beta_l z + 1}.
\]
As \( m \) and \( n \) are coprime numbers, we have \( \alpha_k \neq \beta_l \). Therefore,
\[
F_{k,l}(z) = \frac{1}{z} + \frac{-\frac{1}{2} z + \frac{1-2\alpha_k^2 + \alpha_k \beta_l}{2(\beta_l - \alpha_k)}}{z^2 - 2\alpha_k z + 1} + \frac{-\frac{1}{2} z - \frac{1-2\beta_l^2 + \alpha_k \beta_l}{2(\beta_l - \alpha_k)}}{z^2 - 2\beta_l z + 1}.
\]
\[
= \frac{1}{z} - \frac{1}{4} \frac{2z - 2\alpha_k}{z^2 - 2\alpha_k z + 1} + \frac{1}{2} \frac{1 - \alpha_k^2}{z^2 - 2\alpha_k z + 1} - \frac{1}{4} \frac{2\beta_l - 2\alpha_k}{z^2 - 2\beta_l z + 1} + \frac{1}{2} \frac{1 - \beta_l^2}{z^2 - 2\beta_l z + 1}.
\]
We use equation (4.2) to write
\[
\sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{1}{z} - \frac{1}{4} \frac{2z - 2\alpha_k}{z^2 - 2\alpha_k z + 1} - \frac{1}{4} \frac{2z - 2\beta_l}{z^2 - 2\beta_l z + 1} = \frac{(m-1)(n-1)}{z} - \frac{(m-1)n}{2} \left( \frac{z^{-1}}{z^{-1} - \frac{1}{n}} \right) - \frac{(n-1)m}{2} \left( \frac{z^{m-1}}{z^{m-1} - \frac{1}{m}} \right) = \frac{1}{2} \partial \frac{1}{dz} \log \left( \frac{z^{2(m-1)(n-1)}(z-1)^{m-1+n-1}}{(z^n-1)^{m-1}(z^{m-1}-1)^n} \right).
\]
From the limits
\[
\lim_{z \to 1} \log \left( \frac{z^{2(m-1)(n-1)}(z-1)^{m-1+n-1}}{(z^n-1)^{m-1}(z^{m-1}-1)^n} \right) = 0,
\]
\[
\lim_{z \to \infty} \log \left( \frac{z^{2(m-1)(n-1)}(z-1)^{m-1+n-1}}{(z^n-1)^{m-1}(z^{m-1}-1)^n} \right) = \log \left( \frac{1}{n^{m-1} m^{n-1}} \right),
\]
we deduce
\[
(4.4) \quad \frac{1}{mn} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \int_1^\infty \frac{1}{z} - \frac{1}{4} \frac{2z - 2\alpha_k}{z^2 - 2\alpha_k z + 1} - \frac{1}{4} \frac{2z - 2\beta_l}{z^2 - 2\beta_l z + 1} \, dz = \frac{m-1}{2m} \log(n) + \frac{n-1}{2n} \log(m).
\]
In order to compute the integral of the remaining terms, we first use Lemma 2 to write
\[
\frac{1}{mn} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{1 - \alpha_k^2}{\beta_l - \alpha_k} \frac{1}{z^2 - 2\alpha_k z + 1} = \frac{1}{n} \sum_{k=1}^{n-1} \left( \frac{1}{m} \sum_{l=1}^{m-1} \left( \frac{1}{\cos \left( \frac{2k\pi}{n} \right) - \cos \left( \frac{2\pi}{m} \right)} \right) \right) \frac{\sin^2 \left( \frac{2k\pi}{n} \right)}{z^2 - 2 \cos \left( \frac{2\pi}{m} \right) z + 1} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{m} \cot \left( \frac{k\pi}{n} \right) - \cot \left( \frac{mk\pi}{n} \right) \frac{\sin^2 \left( \frac{2k\pi}{n} \right)}{z^2 - 2 \cos \left( \frac{2\pi}{m} \right) z + 1} = \frac{1}{n} \sum_{k=1}^{n-1} \left( \frac{1}{m} \cot \left( \frac{k\pi}{n} \right) - \cot \left( \frac{mk\pi}{n} \right) \right) \frac{\sin^2 \left( \frac{2k\pi}{n} \right)}{z^2 - 2 \cos \left( \frac{2\pi}{m} \right) z + 1}.
\]
Lemma 3 yields
\[
\int_1^\infty \frac{\sin \left( \frac{2k\pi}{n} \right)}{z^2 - 2 \cos \left( \frac{2\pi}{m} \right) z + 1} \, dz = \left( \frac{\pi}{2} - \frac{k\pi}{n} \right).
\]
We thus have shown that
\[ (4.5) \]
\[
\int_1^\infty \frac{n^{-1} m^{-1}}{t} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{1}{1 - \frac{\alpha_k^2}{t}} \frac{1}{z^2 - 2\alpha_k z + 1} \, dz = -\frac{1}{n} \sum_{k=1}^{n-1} \left( \frac{1}{m} \cot \left( \frac{k\pi}{n} \right) - \cot \left( \frac{mk\pi}{n} \right) \right) \left( \frac{\pi}{2} - \frac{k\pi}{m} \right).
\]

Similarly,
\[ (4.6) \]
\[
\int_1^\infty \frac{1}{mn} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{1 - \beta_k^2}{\alpha_k - \beta_1^2 z^2 - 2\beta_1 z + 1} \, dz = -\frac{1}{m} \sum_{l=1}^{m-1} \left( \frac{1}{n} \cot \left( \frac{l\pi}{m} \right) - \cot \left( \frac{nkl\pi}{m} \right) \right) \left( \frac{\pi}{2} - \frac{l\pi}{m} \right).
\]

Lemma 4 is then proven by summing the identities (4.4), (4.5) and (4.6).

**Lemma 5.** Let \( n \geq 1 \). There exists \( C > 0 \), which does not depend on \( n \), such that
\[ \int_0^1 \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) \left( \frac{1}{nt} - \frac{1}{e^{nt} - 1} \right) \, dt = C \left( 1 + \frac{1}{n} \right) - \frac{1}{2n} - \frac{n-1}{2n} \log(n) + \frac{1}{2n} \sum_{k=1}^{n-1} \cot \left( \frac{k\pi}{n} \right) \left( \frac{\pi}{2} - \frac{k\pi}{n} \right). \]

**Proof.** It suffices to compute the integrals
\[
I_1 = \int_e^{\infty} \frac{dt}{nt^2}, \quad I_2 = -\int_e^{\infty} \frac{dt}{t(e^t - 1)}, \quad I_3 = -\int_e^{\infty} \frac{dt}{nt(e^t - 1)}, \quad I_4 = \int_e^{\infty} \frac{dt}{e^t - 1 - e^{nt} - 1},
\]

to sum them up and let \( \varepsilon \to 0 \).

We have \( I_1 = \frac{1}{ne} \). In order to compute \( I_2 \) and \( I_3 \), we notice that the function
\[
t \mapsto \frac{1}{t(e^t - 1)} - \frac{1}{t^2} + \frac{1}{2t}
\]
can be continuously extended at \( t = 0 \), so we have
\[
\int_e^{\infty} \frac{dt}{t(e^t - 1)} = \frac{1}{\varepsilon} + \frac{1}{2} \log(\varepsilon) - C + o(\varepsilon).
\]

Recall that
\[
C = 1 - \int_0^1 \left( \frac{1}{t(e^t - 1)} - \frac{1}{t^2} + \frac{1}{2t} \right) \, dt - \int_1^{\infty} \frac{dt}{t(e^t - 1)}.
\]

Hence,
\[
I_2 = -\frac{1}{n\varepsilon} - \frac{1}{2} \log(n\varepsilon) + C + o(\varepsilon),
\]
\[
I_3 = -\frac{1}{n\varepsilon} - \frac{1}{2n} \log(\varepsilon) + \frac{C}{n} + o(\varepsilon).
\]

It remains to compute \( I_4 \). The change of variables \( x = e^t \) gives, with \( c = \exp(\varepsilon) \),
\[
I_4 = \int_c^{\infty} \frac{1}{x^2 - 1} \frac{1}{x^n - 1} \, dx.
\]

Still using \( \omega_{k,n} = e^{\frac{2k\pi i}{n}} \) and \( \alpha_k = \cos \left( \frac{2k\pi}{n} \right) \), we perform the partial fraction expansion of \( \frac{1}{x^n - 1} \):
\[
\frac{n}{x^n - 1} = \sum_{k=0}^{n-1} \frac{\omega_{k,n}}{x - \omega_{k,n}} = \frac{1}{x - 1} + \sum_{k=1}^{n-1} \frac{x\alpha_k - 1}{x^2 - 2\alpha_k x + 1}.
\]

We compute
\[
\int_c^{\infty} \frac{dx}{x(x-1)^2} = \int_c^{\infty} \left( \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2} \right) \, dx
\]
\[= \left[ \log \left( \frac{x}{x-1} \right) \right]_c^{\infty} + \left[ -\frac{1}{x-1} \right]_c^{\infty}
\]
\[= \log \left( \frac{c-1}{c} \right) + \frac{1}{c - 1}
\]
\[= \frac{1}{\varepsilon} + \log(\varepsilon) - \frac{1}{2} + o_{\varepsilon \to 0}(1).
\]
For \(1 \leq k \leq n - 1\), we have
\[
\frac{1}{x} x \alpha_k - 1 = \frac{1}{x} - \frac{1}{2x} - \frac{1}{4x^2 - 2\alpha_k x + 1} + \frac{1}{2x^2 - 2\alpha_k x + 1}.
\]

We first focus on the first three terms. Equation (4.2) allows us to write
\[
\sum_{k=1}^{n-1} \frac{1}{x} - \frac{1}{2x} - \frac{1}{4x^2 - 2\alpha_k x + 1} = \frac{n-1}{x} - \frac{n-1}{2(x-1)} - \frac{1}{2} \left( \frac{nx^{n-1} - 1}{x^{n-1} - 1} \right) = \frac{n-1}{x} - \frac{n-2}{2(x-1)} - \frac{1}{2} \frac{x^{n-1}}{2^n - 1}.
\]

and so
\[
\frac{1}{n} \int_c^\infty \sum_{k=1}^{n-1} \frac{1}{x} - \frac{1}{2x} - \frac{1}{4x^2 - 2\alpha_k x + 1} \, dx = \frac{1}{2n} \left[ \log \left( \frac{z^{2(n-1)}}{(z-1)^{n-2}(z^n-1)} \right) \right]_c
= \frac{n-2}{2n} \log(c-1) + \frac{1}{2n} \log(c^n-1) - \frac{2n-1}{2n} \log(c)
= \frac{n-1}{2n} \log(\varepsilon) + \frac{\log(n)}{2n} + o(1).
\]

We also compute, with Lemma 3,
\[
\frac{1}{n} \int_c^\infty \sum_{k=1}^{n-1} \frac{1}{2x^2 - 2\alpha_k x + 1} \, dx = \frac{1}{2n} \sum_{k=1}^{n-1} \cos \left( \frac{2k\pi}{n} \right) + 1 \int_c^\infty \log \left( \frac{1}{x^2 - 2x \cos \left( \frac{2k\pi}{n} \right) + 1} \right) \, dx
= \frac{1}{2n} \sum_{k=1}^{n-1} \cos \left( \frac{2k\pi}{n} \right) + 1 \cos \left( \frac{\pi}{2} - k\pi \right)
= \frac{1}{2n} \sum_{k=1}^{n-1} \cot \left( \frac{k\pi}{n} \right) \left( \frac{\pi}{2} - k\pi \right).
\]

The statement of Lemma 4 and Lemma 5 can be simplified by using the following

Lemma 6. If \(m, n \geq 1\) are coprime, then \(\sum_{k=1}^{n-1} \cot \left( \frac{mk\pi}{n} \right) = 0\).

Proof. As \(m, n\) are coprime, we know that the map \(k \mapsto mk\), i.e. the multiplication by \(m\) modulo \(n\), is one-to-one from \(\{1, \ldots, n-1\}\) onto itself, so
\[
\sum_{k=1}^{n-1} \cot \left( \frac{mk\pi}{n} \right) = \sum_{l=1}^{n-1} \cot \left( \frac{l\pi}{n} \right).
\]

The change of indices \(l \mapsto n - l\) and the formula \(\cot(\pi - x) = -\cot(x)\) allow us to conclude. \(\square\)

We can now complete the proof of Theorem 1. For \(n \geq 1\), we set \(E_n(t) = \frac{1}{mt} - \frac{1}{e^{m/t} - 1}\). We have
\[
I(m, n) = \int_0^\infty \left( E_m(t) - \frac{1}{m} E_1(t) \right) \left( E_n(t) - \frac{1}{n} E_1(t) \right) \, dt,
\]
and Lemma 4 gives a formula for \(I(m, n)\). Moreover, Lemma 5 gives a formula for \(J(n) = \int_0^\infty E_n(t) E_1(t) \, dt\). In particular \(J(1) = 2\). We now notice that
\[
mn \int_0^\infty E_n(t) E_m(t) \, dt = mnI(m, n) + mJ(n) + nJ(m) - mnJ(1).
\]
Adding up everything and using Lemma 6 conclude the proof.
5. Theorem 1 for \( \gcd(m, n) \neq 1 \)

If \( m, n \) are not coprime, the statement of Theorem 1 cannot be correct, because the term \( \cot \left( \frac{mk\pi}{n} \right) \) is not defined if \( \frac{mk}{n} \) is an integer. However, we have the following:

**Lemma 7.** Given integers \( a, p, q \geq 1 \), we have

\[
\lim_{\lambda \to \frac{1}{q}} \cot(a\lambda q\pi)a\lambda q\pi + \cot \left( \frac{1}{\lambda} \right) a\lambda p\pi = 1.
\]

**Proof.** This limit comes from the expansion \( \cot(x) = \frac{1}{x} + o(1) \) when \( x \to 0 \). We set \( h = \lambda q - p \). Then \( \frac{h}{p} - q = -\frac{1}{h} \), so we have:

\[
\cot(a\lambda q\pi)a\lambda q\pi + \cot \left( \frac{1}{\lambda} \right) a\lambda p\pi = \cot(ah\pi)a\lambda q\pi + \cot \left( -\frac{1}{\lambda} \right) a\lambda p\pi
\]

\[
= \frac{a\lambda q\pi}{ah\pi} + \frac{a\lambda p\pi}{-a\lambda h\pi} + o_{h\to 0}(1)
\]

\[
= 1 + o_{h\to 0}(1).
\]

\[\Box\]

Lemma 7 is only used to explain the abusive notations below:

**Theorem 4.** The Vasyunin-type formula stated in Theorem 1 is valid for any integers \( m, n \geq 1 \) with the following abuse of notations: if \( k \in \{1, \ldots, n-1\} \) and \( l \in \{1, \ldots, m-1\} \) such that \( k/l = n/m \), we set:

(5.7)

\[
\cot \left( \frac{mk\pi}{n} \right) \frac{mk\pi}{n} + \cot \left( \frac{nl\pi}{m} \right) \frac{nl\pi}{m} = 1.
\]

**Proof.** We only need to consider the case where \( m, n \) are not coprime. We set \( r = \gcd(m, n) \), \( m = rp \) and \( n = rq \). The following is obvious:

\[
mn \int_0^\infty \left( \frac{1}{mt} - \frac{1}{e^{mt} - 1} \right) \left( \frac{1}{nt} - \frac{1}{e^{nt} - 1} \right) dt = rpq \int_0^\infty \left( \frac{1}{pt} - \frac{1}{e^{pt} - 1} \right) \left( \frac{1}{qt} - \frac{1}{e^{qt} - 1} \right) dt,
\]

\[
\frac{m-n}{2} \log \left( \frac{n}{m} \right) + C(n+m) = r \left( \frac{p-q}{2} \log \left( \frac{q}{p} \right) + C(q+p) \right)
\]

We then notice that there exist exactly \( r-1 \) couples \( (k, l) \in \{1, \ldots, n-1\} \times \{1, \ldots, m-1\} \) such that \( k/l = n/m \). Such couples are given by \( (k, l) = (aq, ap) \) with \( a \in \{1, \ldots, r-1\} \), and equation (5.7) gives:

\[
-\frac{1}{2} \sum_{a=1}^{r-1} \cot \left( \frac{maq\pi}{n} \right) \frac{maq\pi}{n} + \cot \left( \frac{narp\pi}{m} \right) \frac{narp\pi}{m} = -\frac{r-1}{2}
\]

It remains to compute the sum \(-\frac{m}{2} \sum_{k \in K} \cot \left( \frac{mk\pi}{n} \right) \frac{kn}{n} \) when \( K \) is the set of \( k \in \{1, \ldots, n-1\} \) which are not multiples of \( q \). Any \( k \) in this set is written in a unique way under the form \( k = aq + i \), with \( a \in \{0, \ldots, r-1\} \) and \( i \in \{1, \ldots, p-1\} \), so we have

\[
-\frac{m}{2} \sum_{k \in K} \cot \left( \frac{mk\pi}{n} \right) \frac{kn}{n} = -\frac{m}{2} \sum_{i=1}^{p-1} \sum_{a=0}^{r-1} \cot \left( \frac{m(aq+i)\pi}{n} \right) \frac{(aq+i)\pi}{n}
\]

\[
= -\frac{p}{2} \sum_{i=1}^{p-1} \sum_{a=0}^{r-1} \cot \left( \frac{pi\pi}{q} \right) \frac{(aq+i)\pi}{q}
\]

\[
= -\frac{p}{2} \sum_{i=1}^{p-1} \cot \left( \frac{pi\pi}{q} \right) \frac{ar(r-1)\pi}{2} + ri\pi
\]

\[
= -\frac{p}{2} \sum_{i=1}^{p-1} \cot \left( \frac{pi\pi}{q} \right) \frac{i\pi}{q}
\]
We used the $\pi$-periodicity of the cotangent function and Lemma 6. Similarly, we have

$$\frac{n}{2} \sum_{l \in L} \cot \left( \frac{nl\pi}{m} \right) \frac{l\pi}{m} = -r \frac{q}{2} \sum_{j=1}^{q-1} \cot \left( \frac{jq\pi}{p} \right) \frac{j\pi}{p},$$

where $L$ is the set of $l \in \{1, \ldots, m-1\}$ which are not multiples of $p$. We finally obtain that

$$m - n \frac{2}{2} \log \left( \frac{n}{m} \right) + C(n + m) - \frac{1}{2} \frac{m}{2} \sum_{k=1}^{n-1} \cot \left( \frac{mk\pi}{n} \right) \frac{k\pi}{n} - \frac{n}{2} \sum_{l=1}^{m-1} \cot \left( \frac{nl\pi}{m} \right) \frac{l\pi}{m} = r \left( \frac{p-q}{2} \log \left( \frac{q}{p} \right) + C(q + p) - \frac{1}{2} - \frac{p}{2} \sum_{i=1}^{p-1} \cot \left( \frac{pi\pi}{q} \right) \frac{i\pi}{q} - \frac{q}{2} \sum_{j=1}^{q-1} \cot \left( \frac{qj\pi}{p} \right) \frac{j\pi}{p} \right),$$

$$= rpg \int_{0}^{\infty} \left( \frac{1}{pt} - \frac{1}{e^{pt} - 1} \right) \left( \frac{1}{qt} - \frac{1}{e^{qt} - 1} \right) dt = mn \int_{0}^{\infty} \left( \frac{1}{mt} - \frac{1}{e^{mt} - 1} \right) \left( \frac{1}{nt} - \frac{1}{e^{nt} - 1} \right) dt,$$

which is the general Vasyunin-type formula we wanted. □

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