RATIONAL HOMOTOPY THEORY OF FUNCTION SPACES
AND HOCHSCHILD COHOMOLOGY

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Abstract. Given a map \( f : X \to Y \) of simply connected spaces of finite type such, the space of based loops at \( f \) of the space of maps between \( X \) and \( Y \) is denoted by \( \Omega f \text{Map}(X, Y) \). For \( n > 0 \), we give a model categorical interpretation of the existence (in functorial way) of an injective map of \( \mathbb{Q} \)-vector spaces \( \pi_n \Omega f \text{Map}(X, Y) \text{Q} \to \text{HH}^{-n}(C^*(Y), C^*(X)_f) \), where \( \text{HH}^* \) is the (negative) Hochschild cohomology and \( C^*(X)_f \) is the rational cochain complex associated to \( X \) equipped with a structure of \( C^*(Y) \)-differential graded bimodule via the induced map of differential graded algebras \( f^* : C^*(Y) \to C^*(X) \). Moreover, we identify the image in precise way by using the Hodge filtration on Hochschild cohomology. In particular, when \( X = Y \), we describe the fundamental group of the identity component of the monoid of self equivalence of a (rationalization of) space \( X \) i.e., \( \pi_1 \text{Aut}(X_\mathbb{Q})_{\text{id}} \).

Introduction

Our main goal in this article is the study of the function space \( \text{Map}(X, Y) \) between two rational topological spaces from non-commutative point of view. More precisely, for a fixed map \( f : X \to Y \) we study the homotopy groups of the path connected component \( \text{Map}(X, Y)_f \). It is well known \cite{3} that rationally (under some finiteness conditions) the homotopy groups \( (\pi_n, n > 1) \) of \( \text{Map}(X, Y)_f \) are given by the André-Quillen cohomology \( \text{AQ}^{-n}(C^*(Y), C^*(X)) \), where \( C^*(X) \) is seen as a module over \( C^*(Y) \) via the induced map of differential graded algebras \( f^* : C^*(Y) \to C^*(X) \). The point is that the André-Quillen cohomology is quite complicated to compute. We should notice that we are using the fact that any rational \( E_\infty \)-differential graded algebra is equivalent to a rational commutative differential graded algebra. Let \( k \) be any commutative ring, and denote the model category of \( E_\infty \)-differential graded \( k \)-algebras by \( E_\infty \text{-dgAlg}_k \) and the model category of associative differential graded \( k \)-algebras by \( \text{dgAlg}_k \). The (derived) forgetful functor \( U : E_\infty \text{-dgAlg}_k \to \text{dgAlg}_k \) induces a map of simplicial sets

\[
\alpha : \text{Map}_{E_\infty \text{-dgAlg}_k} (R, S) \to \text{Map}_{\text{dgAlg}_k} (R, S) := \text{Map}_{\text{dgAlg}_k} (UR, US).
\]

In all what follows, we will consider only the positively graded algebras with increasing differentials by degree one. A perfect example is the cochain complex associated to a topological space. The interpretation of the higher homotopy groups is quite

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simple, in fact in \cite{2}, we have shown that for a given map \( f : R \to S \) of differential graded \( k \)-algebras we have
\[
\pi_n \text{Map}_{\text{dgAlg}}(R, S)_f \cong \text{HH}_k^{−n+1}(R, S_f) \quad \forall \ n > 1,
\]
where \( \text{HH}_k \) is the Hochschild cohomology and \( S \) is seen as \( R \)-bimodule via \( f \).

**Rational homotopy theory.** When \( k = \mathbb{Q} \), Sullivan has proven that there is an \( \infty \)-equivalence between the category of simply connected rational spaces (finite type) and a subcategory of simply connected commutative differential graded \( k \)-algebras (of finite type) \cite{12}. The \( \infty \)-equivalence is given by the cochain functor \( C^*(-, k) \) after strictification. For any \( n > 0 \), \( \text{Map}(\pi_n, \pi_{n+1}) \) is an injective map of abelian groups. Mandell’s fundamental theorem \cite{11} says that the \( \infty \)-category of \( p \)-complete spaces (with some finiteness conditions) is \( \infty \)-equivalent to a full \( \infty \)-subcategory of \( \mathbb{E}_\infty \)-differential graded \( k \)-algebras via the cochain functor \( C^*(-, \mathbb{F}_p) \). Suppose that \( f : X \to Y \) is a map of simply connected spaces (with some finiteness conditions), then the forgetful functor \( U \) induces the following map of \( k \)-vector spaces
\[
\pi_{n+1} \alpha : \pi_{n+1} \text{Map}(X, Y)_f \to \text{HH}_k^{−n}(C^*(Y), C^*(X)_f).
\]
In \cite{3} Theorem 3.8, Block and Lazarev give an explicit formula when \( f \) is homotopy equivalent to a constant map. They (re)proved that (under the convention that the cohomology is negatively graded)
\[
\pi_n \text{Map}(X, Y)_f \cong \prod_{i=1}^{\infty} \pi_i(Y) \otimes H^{−n}(X, \mathbb{Q}).
\]

**\( p \)-Adic homotopy theory.** When \( p \) is a prime number and \( k = \mathbb{F}_p \), the algebraic closure of the field with \( p \)-elements, the forgetful functor \( U \) is the Hochschild cohomology and \( S \) is seen as \( R \)-bimodule via \( f \).

**Theorem 0.1** \cite{22} : Suppose that \( k = \mathbb{Q} \). Let \( f : X \to Y \) be a map of simply connected spaces of (finite type), then the forgetful functor
\[
U : \text{Map}_{\mathbb{E}_\infty \text{-dgAlg}}(C^*(Y), C^*(X)) \to \text{Map}_{\text{dgAlg}}(C^*(Y), C^*(X))
\]
induces a map of \( k \)-vector spaces such that:

1. \( [X, Y] = \pi_0 \text{Map}_{\mathbb{E}_\infty \text{-dgAlg}}(C^*(Y), C^*(X)) \to \pi_0 \text{Map}_{\text{dgAlg}}(C^*(Y), C^*(X)) \) is injective.
2. \( \pi_1 \text{Map}(X, Y)_f = \pi_1 \text{Map}_{\mathbb{E}_\infty \text{-dgAlg}}(C^*(Y), C^*(X))_f \to \pi_1 \text{Map}_{\text{dgAlg}}(C^*(Y), C^*(X))_f \) is injective map of groups.
3. \( \forall n > 0 \), the induced map
\[
\pi_{n+1} \text{Map}(X, Y)_f = \pi_{n+1} \text{Map}_{\mathbb{E}_\infty \text{-dgAlg}}(C^*(Y), C^*(X))_f \to \text{HH}_k^{−n}(C^*(Y), C^*(X)_f),
\]
is injective map of \( Q \)-vector spaces.
4. If \( X = Y \) and \( f = id \), then \( \pi_1 \text{Aut}(X)_{id} \to \text{HH}_k^{0, \infty}(C^*(X), C^*(X)) \) is an injective map of abelian groups.

The space \( X_\mathbb{Q} \) is the rationalization of \( X \), \( \text{Aut}(X) \) is the monoid of self equivalences, and \( \text{HH}_k^{0, \infty}(C^*(X), C^*(X)) \) is the group of invertible elements of the \( k \)-algebra \( \text{HH}_k^{0}(C^*(X), C^*(X)) \).
Warning 0.2. When \( k = \mathbb{F}_p \), the induced maps \( \pi_n \alpha \) are far to be injective in general.

**Theorem 0.3** (Hodge filtration).

With the same assumption as in precedent Theorem, we have the following isomorphism

\[
\pi_{n+1} \text{Map}(X, Y_Q)_f \cong HH_{(1)}(C^*(Y), C^*(X)_f), \quad \forall \ n > 0, \forall \ f.
\]

1. **General framework**

For what follows we fixe \( k = \mathbb{Q} \). Notice that \( \mathbb{E}_\infty \text{-dgAlg}_k \cong \text{dgCAlg}_k \). In the abstract we described only the applications. In order to prove them we pass by the model category of differential graded algebras (commutative and non-commutative).

We denote the pointed model category of augmented (resp. commutative and \( \mathbb{E}_\infty \)) differential graded \( k \)-algebras by \( \text{dgAlg}_k^+ \) (resp. \( \text{dgCAlg}_k^+ \) and \( \mathbb{E}_\infty \text{-dgAlg}_k^+ \)). Notice that the model structure in the commutative case make sense when \( k \) is of characteristic 0. For some technical reasons, we define the functor of cochain complexes \( C^*(-, k) = C^*(-) : \text{sSet}^{op} \to \mathbb{E}_\infty \text{-dgAlg}_k \). In this section, a space means a simplicial set.

**Notation 1.1.** All differential graded algebras are non-negatively graded and the differentials increase the degree by +1. Consider the map of operads (in the differential graded context) \( \text{Ass} \to \text{Com} \), since the category of operad is a model category we have the factorization \( \text{Ass} \to \mathbb{E}_\infty \to \text{Com} \), where the first map is a cofibration and the second map is a trivial fibration. We have shown in [1, Lemma 1.1], that \( \mathbb{E}_\infty \) is admissible and the forgetful functor \( U : \mathbb{E}_\infty \text{-dgAlg}_k^+ \to \text{dgAlg}_k^+ \) preserves cofibrant objects and cofibration between cofibrant objects. That is the reason why we work with \( \mathbb{E}_\infty \)-operad instead of the operad \( \text{Com} \).

Recall that we have a following diagram of (Quillen) adjunctions:

\[
\begin{array}{ccc}
\text{dgAlg}_k & \xrightarrow{F} & \mathbb{E}_\infty \text{-dgAlg}_k \\
\downarrow \oplus k & & \downarrow U \\
\text{dgAlg}_k^+ & \xrightarrow{F} & \mathbb{E}_\infty \text{-dgAlg}_k^+
\end{array}
\]

where, \( F \) and \( U \) are left adjoints and \( U, \oplus k \) are right adjoints.

**Warning 1.2.** In what follows, we took the liberty to not denote the forgetful functors i.e., when \( R \) is an (augmented) \( \mathbb{E}_\infty \)-differential graded algebra we consider it also as an (augmented) associative differential graded algebra without mentioning the forgetful functor.

**Theorem 1.3.** [1 Theorem 3.1] Let \( k = \mathbb{Q} \), for any \( R \) and \( S \) augmented commutative differential graded \( k \)-algebras, the forgetful functor \( U : \mathbb{E}_\infty \text{-dgAlg}_k^+ \to \text{dgAlg}_k^+ \) induces a map \( \alpha : \text{Map}_{\mathbb{E}_\infty \text{-dgAlg}_k^+}(R, S) \to \text{Map}_{\text{dgAlg}_k^+}(R, S) \) such that

\[
\pi_n \text{Map}_{\mathbb{E}_\infty \text{-dgAlg}_k^+}(R, S) \to \pi_n \text{Map}_{\text{dgAlg}_k^+}(R, S)
\]

is injective map of groups for \( n > 0 \). Moreover, the map \( \Omega \text{Map}_{\mathbb{E}_\infty \text{-dgAlg}_k^+}(R, S) \to \Omega \text{Map}_{\text{dgAlg}_k^+}(R, S) \) has a functorial retract with respect to the target argument \( S \).
Lemma 1.4. Let $k$ be any field. The (derived) functor $C^*(-) : sSet^{op} \rightarrow \mathcal{E}_\infty \text{dgAlg}_k^*$ commutes with homotopy limits.

Proof. The functor $C^*$ has a left adjoint (cf [11, Proposition 4.2]), they form a Quillen pair. The homotopy limits in $sSet^{op}$ are the homotopy colimits in $sSet$, it follows that for any diagram $J \rightarrow sSet$ we have an isomorphism $C^*(\text{hocolim}_{j \in J} X_j) \cong \text{holim}_{j \in J} C^*(X_j)$ in the homotopy category $\text{Ho}(sSet)$. □

Notation 1.5. We denote the simplicial sphere of dimension $n$ by $S^n$.

Definition 1.6. Let $R$ be an augmented $E_\infty$-differential graded $k$-algebra, we say that $R$ connected if $\pi_0 \text{Map}_{\text{dgAlg}_k^*}(R, k \oplus k) = \pi_0 \text{Map}_{\mathcal{E}_\infty \text{dgAlg}_k^*}(R, k \oplus k) = \ast$.

Lemma 1.7. Let $X$ be a pointed connected simplicial set, and let $R \in \mathcal{E}_\infty - \text{dgAlg}_k^*$ be connected (cofibrant). Then the induced map by the forgetful functor

$$\text{Map}_{\mathcal{E}_\infty - \text{dgAlg}_k^*}(R, C^*(X)) \rightarrow \text{Map}_{\text{dgAlg}_k^*}(R, C^*(X))$$

has a functorial (depending on $X$) retract in $\text{Ho}(sSet_*)$.

Proof. We define two functors $\Psi, \Phi : sSet_\ast^{op} \rightarrow sSet_*$ as follows

1. $\Psi(X) = \text{Map}_{\mathcal{E}_\infty - \text{dgAlg}_k^*}(R, C^*(X))$ and
2. $\Phi(X) = \text{Map}_{\text{dgAlg}_k^*}(R, C^*(X))$.

These functors verify the following properties

1. They send a weak equivalence $X \rightarrow Y$ to a weak equivalence since the functor $C^*(-)$ preserves weak between cofibrant objects and $\text{Map}_{\mathcal{E}_\infty - \text{dgAlg}_k^*}(R, -)$, $\text{Map}_{\mathcal{E}_\infty - \text{dgAlg}_k^*}(R, -)$ preserves weak equivalence between fibrant objects since $R$ is cofibrant as $\mathcal{E}_\infty$-algebra and as associative algebra cf [11].
2. The functors $\Psi$ and $\Phi$ take homotopy limits to homotopy colimits, it follows that the mapping spaces of a model category commutes with homotopy limits in the second argument and the fact that $C^*(-)$ takes homotopy colimits to homotopy limits [13]. Moreover the forgetful functor $U : \mathcal{E}_\infty - \text{dgAlg}_k^* \rightarrow \text{dgAlg}_k^*$ commutes with homotopy limits.
3. $\Psi(\ast)$ and $\Phi(\ast)$ are contractible since $k$ is a terminal object in $\mathcal{E}_\infty - \text{dgAlg}_k^*$ and $\text{dgAlg}_k^*$.

It follows form [11, Theorem 16], that $\Psi(-)$ and $\Phi(-)$ are representable i.e., there exists two simplicial sets $C$ and $A$ such that $\Psi(-) \simeq \text{Map}_{sSet_\ast}(-, C)$ and $\Phi(-) \simeq \text{Map}_{sSet_\ast}(-, A)$ in $\text{Ho}(sSet_\ast)$, the natural transformation $\Psi(-) \rightarrow \Phi(-)$ is represented by a map $C \rightarrow A$. By theorem [13] we know that the map $\Omega \Psi(-) \rightarrow \Omega \Phi(-)$ has a functorial retract (in $\text{Ho}(sSet_\ast)$), it follows that the map $\Omega \text{Map}_{sSet_\ast}(-, C) \rightarrow \Omega \text{Map}_{sSet_\ast}(-, A)$ has a functorial retract, it implies that $\Omega C \rightarrow \Omega A$ has a retract. On another hand $R$ is connected, it follows that $A$ and $C$ are connected, hence, the induced map $A \rightarrow C$ has a retract in $\text{Ho}(sSet_\ast)$. We conclude that $\text{Map}_{\mathcal{E}_\infty - \text{dgAlg}_k^*}(R, C^*(X)) \rightarrow \text{Map}_{\text{dgAlg}_k^*}(R, C^*(X))$ has a functorial retract in $\text{Ho}(sSet_\ast)$ for any simplicial set $X$. □

Corollary 1.8. For any connected augmented $E_\infty$-differential graded algebra, and any pointed simplicial set $X$, the natural map

$$\text{Map}_{\mathcal{E}_\infty - \text{dgAlg}_k^*}(R, C^*(X)) \rightarrow \text{Map}_{\text{dgAlg}_k^*}(R, C^*(X))$$

induces an injective map on homotopy groups.
Theorem 1.9. Let $R$ be a connected augmented $E_\infty$-differential graded algebra, with augmentation $\nu : R \to k$. Let $X$ be any pointed simplicial set, let $f : R \to C^*(X)$ be any map of augmented $E_\infty$-differential graded algebras. Then the induced map by the forgetful functor 
\[ \alpha : \text{Map}_{E_\infty - \text{dgAlg}}(R, C^*(X)) \to \text{Map}_{\text{dgAlg}}(R, C^*(X)) \]
has a functorial retract (on the variable $X$), in particular $\forall f, \forall n > 0$:
- $\pi_0 \alpha : \pi_0 \text{Map}_{E_\infty - \text{dgAlg}}(R, C^*(X)) \to \pi_0 \text{Map}_{\text{dgAlg}}(R, C^*(X))$ and
- $\pi_n \alpha : \pi_n \text{Map}_{E_\infty - \text{dgAlg}}(R, C^*(X)) \to \pi_n \text{Map}_{\text{dgAlg}}(R, C^*(X))$ are injective maps.

Proof. First of all, notice that we have an obvious cofiber sequence of pointed simplicial sets
\[ S^0 \xrightarrow{i} X_+ \xrightarrow{p} X \]
where $X_+$ is the pointed simplicial set $X \coprod_* X$. It is enough to notice that
\[ \text{Map}_{E_\infty - \text{dgAlg}}(R, C^*(X_+)) \simeq \text{Map}_{E_\infty - \text{dgAlg}}(R, C^*(X)) \]
and
\[ \text{Map}_{\text{dgAlg}}(R, C^*(X_+)) \simeq \text{Map}_{\text{dgAlg}}(R, C^*(X)), \]
then the result follows from [1.7].

\[ \square \]

2. Main Theorems and Applications

Proposition 2.1. Suppose that $k = \mathbb{Q}$. Let $R$ be an augmented $E_\infty$-differential graded $k$-algebra of finite type (i.e. dim$_k H^i(R) < \infty \ \forall i$) such that $H^0(R) = k$ and $H^1(R) = 0$ then $R$ is connected in the sense of [1.6].

Proof. First of all, by adjunction $\text{Map}_{E_\infty - \text{dgAlg}}(R, k \oplus k) \simeq \text{Map}_{E_\infty - \text{dgAlg}}(R, k)$. Without losing generality we can suppose that $R$ is cofibrant as $E_\infty - \text{dgAlg}$, hence $R$ is cofibrant as $\text{dgAlg}_k$ (by construction of the operad $E_\infty$ cf [11]). By Sullivan Theorem $\pi_0 \text{Map}_{E_\infty - \text{dgAlg}}(R, k) = \ast$. It follows that for any maps $\nu : R \to k$ and $\mu : R \to k$ are homotopic in $E_\infty - \text{dgAlg}_k$. According to [8], we have a commutative diagram in $E_\infty - \text{dgAlg}_k$

\[ \begin{array}{ccc}
R & \xrightarrow{\mu} & P(R) \\
\downarrow{\nu} & & \downarrow{\nu} \\
k & & k
\end{array} \]

where $P(R)$ is a path object associated to $R$. Notice that the path object is the same for graded differential associative algebras if we consider $R \in \text{dgAlg}_k$. Since $H^0(R) = k$ any map $R \to k$ in $\text{dgAlg}_k$ is actually a map in $E_\infty - \text{dgAlg}_k$. We conclude that $\pi_0 \text{Map}_{\text{dgAlg}}(R, k) = \ast = \pi_0 \text{Map}_{\text{dgAlg}}(R, k \oplus k)$.

\[ \square \]
Theorem 2.2 (Main Theorem). Suppose that $k = \mathbb{Q}$. Let $f : X \to Y$ be a map of simply connected spaces of (finite type), then the forgetful functor

$$U : \text{Map}_{E_\infty \text{-dgAlg}}(C^*(Y), C^*(X)) \to \text{Map}_{\text{dgAlg}}(C^*(Y), C^*(X))$$

induces a map of $k$-vector spaces such that:

1. $[X, Y] = \pi_0 \text{Map}_{E_\infty \text{-dgAlg}}(C^*(Y), C^*(X)) \to \pi_0 \text{Map}_{\text{dgAlg}}(C^*(Y), C^*(X))$ is injective.
2. $\pi_1 \text{Map}(X, Y)_f = \pi_1 \text{Map}_{E_\infty \text{-dgAlg}}(C^*(Y), C^*(X))f \to \pi_1 \text{Map}_{\text{dgAlg}}(C^*(Y), C^*(X))f$ is injective.
3. $\forall n > 0$,

$$\pi_{n+1} \text{Map}(X, Y)_f = \pi_{n+1} \text{Map}_{E_\infty \text{-dgAlg}}(C^*(Y), C^*(X))f = \text{AQ}^{-n-1}(C^*(Y), C^*(X))f$$

and the induced map

$$\pi_{n+1} \text{Map}(X, Y)_f \to \pi_{n+1} \text{Map}_{E_\infty \text{-dgAlg}}(C^*(Y), C^*(X))f = \text{HH}^{-n}_k(C^*(Y), C^*(X))f$$

is an injective map of $Q$-vector spaces.
4. If $X = Y$ and $f = \text{id}$, then $\pi_1 \text{Aut}(X)_\text{id} \to \text{HH}^{0, \infty}(C^*(X), C^*(X))$ is an injective map of abelian groups.

Proof. By hypothesis $X$ and $Y$ are of finite type, we deduce by \cite{12} that

$$\text{Map}_{E_\infty \text{-dgAlg}}(C^*(Y), C^*(X))$$

is equivalent to $\text{Map}(X, Y)_\mathbb{Q}$, on the other hand by Theorem 1.9 the forgetful functor $U : E_\infty \to \text{dgAlg}_k$ induces an injective map

$$\alpha : \pi_i \text{Map}_{E_\infty \text{-dgAlg}}(C^*(Y), C^*(X))f \to \pi_i \text{Map}_{\text{dgAlg}}(C^*(Y), C^*(X))f$$

for all $i \geq 0$. Moreover if $i > 1$, Block-Lazarev theorem gives us the isomorphism

$$\pi_i \text{Map}_{E_\infty \text{-dgAlg}}(C^*(Y), C^*(X))f \cong \text{AQ}^{-i}(C^*(Y), C^*(X))f,$$

and by \cite{2},

$$\pi_i \text{Map}_{\text{dgAlg}}(C^*(Y), C^*(X))f \cong \text{HH}^{-i+1}_k(C^*(Y), C^*(X))f.$$

Hence, the induced map $\alpha$ is exactly $\text{AQ}^{-i}(C^*(Y), C^*(X))f \to \text{HH}^{-i+1}_k(C^*(Y), C^*(X))f$, which is injective map of $Q$-vector spaces. Applying Sullivan theorem we deduce that $\pi_1 \text{Map}(X, Y)_f \cong \text{AQ}^{-i}(C^*(Y), C^*(X))f$ for $i > 1$. In particular, when $X = Y$ and $f = \text{id}$, $\text{Map}(X, X)_\text{id} = \text{Aut}(X)_\text{id}$ and

$$\text{Map}(X, X)_\text{id} = \text{Aut}(X)_\text{id}.$$

Therefore, $\pi_1 \text{Aut}(X)_\text{id} \cong \pi_1 \text{Map}_{E_\infty \text{-dgAlg}}(C^*(X), C^*(X))_\text{id}$. In \cite{2} Corollary 3.6], we have shown that $\pi_1 \text{Map}_{\text{dgAlg}}(C^*(X), C^*(X))_\text{id}$ is isomorphic to the kernel of the natural map of (abelian) groups $\text{HH}^{0, \infty}_k(C^*(X), C^*(X)) \to H^{0, \infty}(C^*(X)) = \mathbb{Q}^\times$. The result follows for Theorem 1.9.

\hfill \Box

Corollary 2.3. Let $M$ be a simply connected orientable closed manifold of dimension $d$, for all $i > 0$, we have an injective map of $Q$ vector spaces

$$\pi_1 Q_\text{id Aut}(M) \otimes Q \to H_{i+d}(\mathcal{L}M, \mathbb{Q}),$$

where $\mathcal{L}M$ is the space of free loops on $M$, i.e., $\text{Map}(S^1, M)$.

Proof. Since $M$ is a finite CW-complex, it is a direct consequence of Theorem 2.2 and the fact that $\text{HH}^*_k(C^*(M), C^*(M)) \cong H_{i+d}(\mathcal{L}M, \mathbb{Q})$ \cite{4}.

\hfill \Box
Remark 2.4. Corollary 2.3 was also proven in [5, Theorem 2 (1)] using a different method.

2.1. Hodge filtration on Hochschild cohomology over a field of characteristic zero. In our main Theorem 2.2 we have identified the higher homotopy groups of $\text{Map}(X, Y \mathbb{Q})_f$ based at some continuous map $f : X \to Y$ as a sub $\mathbb{Q}$-vector space of the (negative) Hochschild cohomology. According to [6, Theorem 3.1], there exists a Hodge decomposition on the Hochschild cohomology $\text{HH}^*_{(1)}(R, S)$ for any differential graded $\mathbb{Q}$-algebra $R$ and any differential graded $R$-bimodule $S$. More precisely Ginot has proved in [6], the following formula in the rational case:

$$\text{HH}^*(R, S) \cong \prod_{n \geq 0} \text{HH}^*_{(n)}(R, S),$$

where the $\mathbb{Q}$-vector spaces $\text{HH}^*_{(n)}(R, S)$ are eigenspaces for an iterated power of some operator.

**Theorem 2.5.** With the same assempion as in Theorem 2.2 we have the following isomorphism

$$\pi_{n+1}\text{Map}(X, Y \mathbb{Q})_f \cong \text{HH}^*_{(1)}(C^*(Y), C^*(X)_f), \forall n > 0, \forall f.$$ 

**Proof.** First of all, we notice that $\pi_n\text{Map}(X, Y \mathbb{Q})_f \cong \text{AQ}^*_{n-1}(C^*(Y), C^*(X)_f)$ for all $n > 1$ (cf. [3]), where $\text{AQ}^*$ is the André-Quillen cohomology. On another hand $\text{HH}^*_{(1)}(C^*(Y), C^*(X)_f) = \text{Harr}^*_{-n}(C^*(Y), C^*(X)_f)$, where $\text{Harr}^*$ is the Harrison cohomology, cf. [6, Theorem 3.1]. Since we work in characteristic zero, Harrison cohomology and André-Quillen cohomology agree up to a shift, more precisely $\text{AQ}^{n-1} = \text{Harr}^n$. It follows that

$$\pi_{n+1}\text{Map}(X, Y \mathbb{Q})_f \cong \text{AQ}^{n-1}_{-1}(C^*(Y), C^*(X)_f)$$

$$\cong \text{Harr}^n_{-1}(C^*(Y), C^*(X)_f)$$

$$\cong \text{HH}^*_{(1)}(C^*(Y), C^*(X)_f), \forall n > 0, \forall f.$$ 

\[\square\]

Remark 2.6. Theorem 2.5 is a generalization of [5, Theorem 2 (2)].

**APPENDIX**

There is a class of model categories called simplicial model categories [7], roughly speaking a simplicial model category is tensored, cotensored and enriched over the model category of simplicial sets in a compatible way (adjunction compatibility, and model structure compatibility). In general a model category $\mathcal{C}$ do not need to be simplicial model category. Moreover, a Quillen adjunction between simplicial model categories

$$\mathcal{C} \xrightarrow{F} \mathcal{D},$$

is not a simplicial adjunction in general. In [9, Chapter 5, 6], Hovey introduced a notion of module category. We will need a more richer structure and we will call it **enriched module structure**. In the classical context any ordinary category with product and coproduct is an enriched Set-module. More precisely, suppose that $\mathcal{D}$ is an ordinary category with products and coproducts, we can define the following functors:
1. \(- \otimes - : \text{Set} \times D \to D\) such that for any set \(X\) and any object \(D \in D\) we have \(X \otimes D = \bigsqcup_{x \in X} D\).
2. \(A(-, -) : \text{Set}^{op} \times D \to D\) such that for any set \(X\) and for any \(D \in D\) we define \(A(X, D) = \prod_{x \in X} D\).

**Definition 2.7.** An enriched \(\text{Set}\)-module \(D\) is a category with all products and coproducts such that we have natural isomorphism for any \(X, Y \in \text{Set}\) and any \(C, D \in D\):

\[
\begin{align*}
\bullet & \quad (X \times Y) \otimes D \cong X \otimes (Y \otimes D). \\
\bullet & \quad \text{hom}_D(C, A(X, D)) \cong \text{hom}_D(X \otimes C, D).
\end{align*}
\]

A simplicial category \(D\) in the sense of [7] is an enriched \(\text{sSet}\)-module in the sense of 2.7, where we replace \(\text{hom}_D\) by the natural enrichment of \(D\) denoted by \(\text{Map}_D\) (simplicial set).

**Theorem 2.8.** Let \(D\) be any (pointed) model category, then the homotopy category \(\text{Ho}(D)\) is an enriched \(\text{Ho}(\text{sSet})\)-module (enriched \(\text{Ho}(\text{sSet}^\ast)\)-module).

**Proposition 2.9.** Given any Quillen adjunction between model categories \(C \xrightarrow{F} D\), it induces the following isomorphisms:

\[
\begin{align*}
\bullet & \quad \text{Map}_C(X, U(Y)) \cong \text{Map}_D(F(X), Y) \text{ in } \text{Ho}(\text{sSet}) \\
\bullet & \quad F(X \otimes C) \cong X \otimes F(C) \text{ in } \text{Ho}(D) \text{ for any } X \in \text{sSet} \text{ and any } C \in C. \\
\bullet & \quad U(A(X, D)) \cong A(X, UD) \text{ in } \text{Ho}(\text{C}) \text{ for all } X \in \text{sSet} \text{ and any } D \in D
\end{align*}
\]

The proof of the precedent theorem and proposition can be deduced from [9]. The involved mapping spaces tensors and cotensors are defined in the derived sense, we took the liberty to not specify the derived symbols (e.g. \(\mathcal{R}\) and \(\mathcal{L}\)).

**Notation 2.10.** If \(D\) is a pointed model category, we denote by \(\Omega D\) the object \(\Omega S^1, D\) and \(\Sigma D\) the object \(S^1 \otimes D\).

2.2. Complement to Theorem 1.3. We explain, the cited Theorem using the previous language. Let \(R\) be cofibrant an augmented \(E_\infty\)-differential graded \(Q\)-algebras. Considering the adjunction

\[
\begin{array}{c}
\text{dgAlg}_k \xrightarrow{F} E_\infty - \text{dgAlg}_k, \\
\xrightarrow{U} \end{array}
\]

our theorem says that we have a natural map \(S^1 \otimes R \to F(S^1 \otimes UR) \simeq S^1 \otimes FUR\) which has a retract in \(\text{Ho}(E_\infty - \text{dgAlg}_k)\). In other words, suppose that \(S \in E_\infty - \text{dgAlg}_k\), we have a retract in \(\text{Ho}(\text{sSet}_\ast)\) of the map

\[
\ker \text{Map}_{E_\infty - \text{dgAlg}_k}(S^1 \otimes R, S) \to \text{Map}_{E_\infty - \text{dgAlg}_k}(S^1 \otimes FUR, S)
\]

which can be rewritten by using adjunctions as:

\[
h : \Omega \text{Map}_{E_\infty - \text{dgAlg}_k}(R, S) \to \Omega \text{Map}_{\text{dgAlg}_k}(UR, US),
\]

such that, there is an induced left inverse map \(r\), i.e., \(r \circ h = id\) and it is functorial with respect to \(S\).
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