Relativistic wavefunctions on the Poincaré group

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Abstract

The Biedenharn type relativistic wavefunctions are considered on the group manifold of the Poincaré group. It is shown that the wavefunctions can be factorized on the group manifold into translation group and Lorentz group parts. A Lagrangian formalism and field equations for such factorizations are given. Parametrizations of the functions obtained are studied in terms of a ten-parameter set of the Poincaré group. An explicit construction of the wavefunction for the spin 1/2 is given. A relation of the proposed description with the quantum field theory and harmonic analysis on the Poincaré group is discussed.

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1 Introduction

In 1988, Biedenharn et al. [5] introduced the Poincaré group representations of wavefunctions on the space of complex spinors. The construction presented in [5] is an extension of the Wigner’s group theoretical method [37]. On the other hand, in accordance with basic principles of quantum field theory, a wavefunction of the particle is a solution of some relativistic wave equation. Relativistic wavefunctions of the work [5] were introduced without explicit reference to wave equations and for this reason they represent purely group theoretical constructions. However, if we further develop the Poincaré group representations of wavefunctions with reference to such basic notions of QFT as a Lagrangian and wave equations, then we come to a quantum field theory on the Poincaré group¹ (QFTPG) introduced by Lurçat in 1964 [20] (see also [3, 13, 7, 2, 14, 31, 21, 8, 11, 12] and references therein). In contrast to the standard QFT (QFT in the Minkowski spacetime) case, for QFTPG all the notions and quantities are constructed on a ten-dimensional group manifold $\mathcal{F}$ of the Poincaré group. It should be noted here that a construction of relativistic wave equations theory on the group manifold $\mathcal{F}$ is one of the primary problems in this area, which remains incompletely solved. Wavefunctions and wave equations on a six-dimensional submanifold $\mathcal{L} \subset \mathcal{F}$, which is a group manifold of the Lorentz group, have been studied in the recent work [33].

¹In 1955, Finkelstein [9] showed that elementary particle models with internal degrees of freedom can be described on manifolds larger than Minkowski spacetime (homogeneous spaces of the Poincaré group). A consideration of the field models on the homogeneous spaces leads to a generalization of the concept of wavefunction. One of the first examples of such generalized wavefunctions was studied by Nilsson and Beskow [23].
In the present paper we consider Biedenharn type relativistic wave functions on the group manifold $\mathcal{F}$. It is shown that the general form of the wavefunctions inherits its structure from the semidirect product $SL(2, \mathbb{C}) \circ T_4$ and for that reason the wavefunctions on $\mathcal{F}$ are represented by a factorization $\psi(x)\psi(g)$, where $x \in T_4$, $g \in SL(2, \mathbb{C})$. Using a Lagrangian formalism on the tangent bundle $T\mathcal{F}$ of the manifold $\mathcal{F}$, we obtain field equations separately for the parts $\psi(x)$ and $\psi(g)$. Solutions of the field equations for $\psi(x)$ can be obtained via the usual plane-wave approximation. In turn, solutions of the field equations with $\psi(g)$ have been found in the form of expansions in associated hyperspherical functions $^2$. The wavefunction on the Poincaré group in the case of $(1/2, 0) \oplus (0, 1/2)$ representation space, usually related with electron-positron field, is considered by way of example.

2 Preliminaries

Let us consider some basic facts concerning the Poincaré group $\mathcal{P}$. First of all, the group $\mathcal{P}$ has the same number of connected components as the Lorentz group. Later on we will consider only the component $\mathcal{P}_+^\uparrow$ corresponding the connected component $L_+^\uparrow$ (the so-called special Lorentz group $^2[26]$). As is known, a universal covering $\overline{\mathcal{P}_+^\uparrow}$ of the group $\mathcal{P}_+^\uparrow$ is defined by a semidirect product $\overline{\mathcal{P}_+^\uparrow} = SL(2, \mathbb{C}) \circ T_4 \simeq Spin_+(1,3) \circ T_4$, where $T_4$ is a subgroup of four-dimensional translations. The relations between the groups $\mathcal{P}_+^\uparrow$, $\mathcal{P}_+^\uparrow$, $SL(2, \mathbb{C})$ and $L_+^\uparrow$ are defined by the following diagram of exact sequences:

\[
\begin{array}{ccccccccc}
1 & 1 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
Z_2 & Z_2 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & T_4 & \overline{\mathcal{P}_+^\uparrow} & SL(2, \mathbb{C}) & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & T_4 & \mathcal{P}_+^\uparrow & L_+^\uparrow & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}
\]

The diagram shows that $\overline{\mathcal{P}_+^\uparrow} (\mathcal{P}_+^\uparrow)$ is the semidirect product of $SL(2, \mathbb{C}) (L_+^\uparrow)$ and $T_4$.

The each transformation $T_\alpha \in \mathcal{P}_+^\uparrow$ is defined by a parameter set $\alpha(\alpha_1, \ldots, \alpha_{10})$, which can be represented by a point of the space $\mathcal{F}_{10}$. The space $\mathcal{F}_{10}$ possesses locally euclidean properties; therefore, it is a manifold, called a group manifold of the Poincare group. It is easy to see that the set $\alpha$ can be divided into two subsets, $\alpha(x_1, x_2, x_3, x_4, g_1, g_2, g_3, g_4, g_5, g_6)$, where $x_i \in T_4$ are parameters of the translation subgroup, $g_j$ are parameters of the group

$^2$Matrix elements of both spinor and principal series representations of the Lorentz group are expressed via the hyperspherical functions $^{32, 34}$.
In turn, the transformation $T_g$ is defined by a set $g(\mathfrak{g}_1, \ldots, \mathfrak{g}_6)$, which can be represented by a point of a six-dimensional submanifold $\mathcal{L}_6 \subset \mathfrak{g}_{10}$, called a group manifold of the Lorentz group.

In the present paper we restrict ourselves to consideration of finite dimensional representations of the Poincaré group. The group $T_4$ of four-dimensional translations is an Abelian group, formed by a direct product of the four one-dimensional translation groups, each of which is isomorphic to an additive group of real numbers. Hence it follows that all irreducible representations of $T_4$ are one dimensional and expressed via the exponential. In turn, as shown by Naimark [22], spinor representations exhaust all the finite dimensional irreducible representations of the group $SL(2, \mathbb{C})$. Any spinor representation of $SL(2, \mathbb{C})$ can be defined in the space of symmetric polynomials in the following form:

$$p(z_0, z_1, \bar{z}_0, \bar{z}_1) = \sum_{(\alpha_1, \ldots, \alpha_k \atop \dot{\alpha}_1, \ldots, \dot{\alpha}_r)} \frac{1}{k!r!} a^{\alpha_1 \ldots \alpha_k \dot{\alpha}_1 \ldots \dot{\alpha}_r} z_{\alpha_1} \cdots z_{\alpha_k} \bar{z}_{\dot{\alpha}_1} \cdots \bar{z}_{\dot{\alpha}_r}$$

where the numbers $a^{\alpha_1 \ldots \alpha_k \dot{\alpha}_1 \ldots \dot{\alpha}_r}$ are unaffected at the permutations of indices. The expressions (1) can be understood as functions on the Lorentz group. When the coefficients $a^{\alpha_1 \ldots \alpha_k \dot{\alpha}_1 \ldots \dot{\alpha}_r}$ in (1) depend on the variables $x_i \in T_4 (i = 1, 2, 3, 4)$, we come to the Biedenharn type functions [5]:

$$p(x, z, \bar{z}) = \sum_{(\alpha_1, \ldots, \alpha_k \atop \dot{\alpha}_1, \ldots, \dot{\alpha}_r)} \frac{1}{k!r!} a^{\alpha_1 \ldots \alpha_k \dot{\alpha}_1 \ldots \dot{\alpha}_r} (x) z_{\alpha_1} \cdots z_{\alpha_k} \bar{z}_{\dot{\alpha}_1} \cdots \bar{z}_{\dot{\alpha}_r}$$

where the numbers $a^{\alpha_1 \ldots \alpha_k \dot{\alpha}_1 \ldots \dot{\alpha}_r}$ are unaffected at the permutations of indices. The expressions (1) can be understood as functions on the Lorentz group. When the coefficients $a^{\alpha_1 \ldots \alpha_k \dot{\alpha}_1 \ldots \dot{\alpha}_r}$ in (1) depend on the variables $x_i \in T_4 (i = 1, 2, 3, 4)$, we come to the Biedenharn type functions [5]:

The functions (2) should be considered as the functions on the Poincaré group. Some applications of these functions are contained in [35, 11]. Representations of the Poincaré group $SL(2, \mathbb{C}) \otimes T(4)$ are realized via the functions (2).

3 Field equations on the Poincaré group

Let $\mathcal{L}(\alpha)$ be a Lagrangian on the group manifold $\mathfrak{g}$ of the Poincaré group (in other words, $\mathcal{L}(\alpha)$ is a ten-dimensional point function), where $\alpha$ is the parameter set of this group. Then we will call an integral for $\mathcal{L}(\alpha)$ on some 10-dimensional volume $\Omega$ of the group manifold an action on the Poincaré group:

$$A = \int_{\Omega} d\alpha \mathcal{L}(\alpha),$$

where $d\alpha$ is a Haar measure on the group $\mathfrak{g}$.

Let $\psi(\alpha)$ be a function on the group manifold $\mathfrak{g}$ (now it is sufficient to assume that $\psi(\alpha)$ is a square integrable function on the Poincaré group) and let

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial \alpha} \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

$^3$Antisymmetrizing the variables $z$ and $\bar{z}$ in (2), we come to the functions on the supergroup. In particular case of Grassmann variables [4] we have the functions on a so-called super-Poincaré group.
be Euler-Lagrange equations on $\mathfrak{F}$ (more precisely speaking, the equations act on the tangent bundle $T\mathfrak{F} = \bigcup_{\alpha \in \mathfrak{F}} T\alpha \mathfrak{F}$ of the manifold $\mathfrak{F}$; see [1]). Let us introduce a Lagrangian $\mathcal{L}(\alpha)$ depending on the field function $\psi(\alpha)$ as follows:

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left( \psi^*(\alpha) B_\mu \frac{\partial \psi(\alpha)}{\partial \alpha_\mu} - \frac{\partial \psi^*(\alpha)}{\partial \alpha_\mu} B_\mu \psi(\alpha) \right) - \kappa \psi^*(\alpha) B_{11} \psi(\alpha),$$

where $B_\nu (\nu = 1, 2, \ldots, 10)$ are square matrices. The number of rows and columns in these matrices is equal to the number of components of $\psi(\alpha)$; $\kappa$ is a non-null real constant.

Further, if $B_{11}$ is non-singular, then we can introduce the matrices

$$\Gamma_\mu = B^{-1}_{11} B_\mu, \quad \mu = 1, 2, \ldots, 10,$$

and represent the Lagrangian $\mathcal{L}(\alpha)$ in the form

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left( \psi(\alpha) \Gamma_\mu \frac{\partial \psi(\alpha)}{\partial \alpha_\mu} - \frac{\partial \psi(\alpha)}{\partial \alpha_\mu} \Gamma_\mu \psi(\alpha) \right) - \kappa \psi(\alpha) \psi^*(\alpha),$$

where

$$\psi^*(\alpha) = \psi^* (\alpha) B_{11}.$$

As a direct consequence of (2), the relativistic wavefunction $\psi(\alpha)$ on the group manifold $\mathfrak{F}$ is represented by the following factorization:

$$\psi(\alpha) = \psi(x) \psi(g) = \psi(x_1, x_2, x_3, x_4) \psi(\varphi, \epsilon, \theta, \tau, \phi, \varepsilon),$$

where $\psi(x_i)$ is a function depending on the parameters of the subgroup $T_4, x_i \in T_4 (i = 1, \ldots, 4)$, and $\psi(g)$ is a function on the Lorentz group, where six parameters of this group are defined by the Euler angles $\varphi, \epsilon, \theta, \tau, \phi, \varepsilon$ which compose complex angles of the form $\varphi^c = \varphi - i\epsilon, \theta^c = \theta - i\tau, \phi^c = \phi - i\varepsilon$.

Varying $\psi(x)$ and $\bar{\psi}(x)$ independently, we obtain from (4) in accordance with (3) the following equations:

$$\Gamma_i \frac{\partial \psi(x)}{\partial x_i} + \kappa \psi(x) = 0, \quad (i = 1, \ldots, 4)$$

$$\Gamma^T_i \frac{\partial \bar{\psi}(x)}{\partial x_i} - \kappa \bar{\psi}(x) = 0.$$

Analogously, varying $\psi(g)$ and $\bar{\psi}(g)$ independently, one gets

$$\Gamma_k \frac{\partial \psi(g)}{\partial g_k} + \kappa \psi(g) = 0, \quad (k = 1, \ldots, 6)$$

$$\Gamma^T_k \frac{\partial \bar{\psi}(g)}{\partial g_k} - \kappa \bar{\psi}(g) = 0,$$

where

$$\psi(g) = \left( \begin{array}{c} \psi(g) \\ \dot{\psi}(g) \end{array} \right), \quad \Gamma_k = \left( \begin{array}{cc} 0 & \Lambda^*_k \\ \Lambda_k & 0 \end{array} \right).$$

The doubling of representations, described by a bispinor $\psi(g) = (\psi(g), \dot{\psi}(g))^T$, is the well known feature of the Lorentz group representations [10, 22]. The structure of the matrices
$\Lambda_k$ and $\Lambda_k^*$ is studied in details in [33]. Since a universal covering $SL(2, \mathbb{C})$ of the proper orthochronous Lorentz group is a complexification of the group $SU(2)$ (see, for example, [36] [32]), it is more convenient to express the six parameters $g_k$ of the Lorentz group via the three parameters $a_1, a_2, a_3$ of the group $SU(2)$. It is obvious that $g_1 = a_1, g_2 = a_2, g_3 = a_3, g_4 = ia_1, g_5 = ia_2, g_6 = ia_3$. Then the first equation from (17) can be written as

$$\sum_{j=1}^{3} \Lambda_j \frac{\partial \psi}{\partial a_j} + i \sum_{j=1}^{3} \Lambda_j^* \frac{\partial \psi}{\partial a_j^*} + \kappa \psi = 0,$$

$$\sum_{j=1}^{3} \Lambda_j \frac{\partial \psi}{\partial a_j} - i \sum_{j=1}^{3} \Lambda_j^* \frac{\partial \psi}{\partial a_j^*} + \kappa \psi = 0,$$

where $a_1^* = -ig_4, a_2^* = -ig_5, a_3^* = -ig_6$, and $\tilde{a}_j, \tilde{a}_j^*$ are the parameters corresponding the dual basis. In essence, the equations (8) are defined in a three-dimensional complex space $\mathbb{C}^3$. In turn, the space $\mathbb{C}^3$ is isometric to a six-dimensional bivector space $\mathbb{R}^6$ (a parameter space of the Lorentz group [17] [21]). The bivector space $\mathbb{R}^6$ is a tangent space of the group manifold $\mathcal{L}$ of the Lorentz group; that is, the manifold $\mathcal{L}$ at each point is equivalent locally to the space $\mathbb{R}^6$. Thus, for all $g \in \mathcal{L}$ we have $T_g \mathcal{L} \simeq \mathbb{R}^6$. General solutions of the system [8] have been found in the work [33] on the tangent bundle $T \mathcal{L} = \bigcup_{g \in \mathcal{L}} T_g \mathcal{L}$ of the group manifold $\mathcal{L}$. A separation of variables in (8) is realized via the following factorization:

$$\psi_{lm, \hat{l}m}^k = f_{lm}^0(r) M_{lm}^m(\varphi, \epsilon, \theta, \tau, 0, 0),$$

$$\psi_{\hat{l}m, lm}^k = f_{\hat{l}m}^0(r^*) M_{\hat{l}m}^m(\varphi, \epsilon, \theta, \tau, 0, 0),$$

where $l_0 \geq l, -l_0 \leq \hat{l}, -\hat{l}_0 \leq \hat{\hat{l}}, M_{lm}^m(\varphi, \epsilon, \theta, \tau, 0, 0)$ are associated hyperspherical functions defined on the surface of the two-dimensional complex sphere of the radius $r$, $f_{lm}^0(r)$ and $f_{\hat{l}m}^0(r^*)$ are radial functions (for more details on two-dimensional complex sphere see [15] [29] [16] [33]). The associated hyperspherical function $M_{lm}^m$ has a form

$$M_{lm}^m(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{-m(\epsilon+\varphi)} Z_m^l(\theta, \tau),$$

where the function $Z_m^l$ can be represented by a product of the two hypergeometric functions:

$$Z_m^l(\theta, \tau) = \cos^2 \frac{\theta}{2} \cosh^2 \frac{\tau}{2} \sum_{k=-l}^{l} i^{m-k} \tan^m \tan^k \frac{\theta}{2} \tanh \frac{\tau}{2} \times$$

$$2F_1 \left( m - l + 1, l - k \atop m - k + 1 \right) 2F_1 \left( -l + 1, l - k \atop -k + 1 \right) \tanh^2 \frac{\tau}{2}. $$

4 The field $(1/2, 0) \oplus (0, 1/2)$

Let us consider now an explicit construction of the relativistic wavefunction $\psi(\alpha)$ on the Poincaré group for the first nontrivial case described by the field $(1/2, 0) \oplus (0, 1/2)$ (electron-positron field or Dirac field). In this case, the first equation from (10) coincides with the Dirac equation

$$i \gamma_n \frac{\partial \psi(x)}{\partial x_n} - m \psi(x) = 0,$$
where $\gamma$-matrices are defined in the standard form, that is, in the Weyl basis:

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},$$

where $\sigma_i$ are the Pauli matrices.

As is known, solutions of the equation \[10\] are found in the plane-wave approximation, that is, in the form \[32\]

$$\psi^+(x) = u(p)e^{-ipx},$$
$$\psi^-(x) = v(p)e^{ipx},$$

where the solutions $\psi^+(x)$ and $\psi^-(x)$ correspond to positive and negative energy, respectively, and the amplitudes $u(p)$ and $v(p)$ have the following components:

$$u_1(p) = \left( \frac{E + m}{2m} \right)^{1/2} \begin{pmatrix} 1 \\ \frac{p_x}{E+m} \frac{p_y}{E+m} \end{pmatrix}, \quad u_2(p) = \left( \frac{E + m}{2m} \right)^{1/2} \begin{pmatrix} 0 \\ \frac{p_x}{E+m} \frac{p_y}{E+m} \end{pmatrix},$$
$$v_1(p) = \left( \frac{E + m}{2m} \right)^{1/2} \begin{pmatrix} \frac{p_x}{E+m} \\ \frac{p_y}{E+m} \end{pmatrix}, \quad v_2(p) = \left( \frac{E + m}{2m} \right)^{1/2} \begin{pmatrix} 0 \\ \frac{p_x}{E+m} \frac{p_y}{E+m} \end{pmatrix},$$

where $p_{\pm} = p_x \pm ip_y$.

Let us consider now solutions of the system \[31\] for the spin $l = 1/2$, that is, when the field is defined by a $P$-invariant direct sum $(1/2,0) \oplus (0,1/2)$. In this case the matrices $\Lambda_i$ and $\Lambda_i^*$ have the form

$$\Lambda_1 = \frac{1}{2} c_{1\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda_2 = \frac{1}{2} c_{1\frac{1}{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Lambda_3 = \frac{1}{2} c_{1\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\Lambda_1^* = \frac{1}{2} c_{1\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda_2^* = \frac{1}{2} c_{1\frac{1}{2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \Lambda_3^* = \frac{1}{2} c_{1\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to see that these matrices coincide with the Pauli matrices $\sigma_i$ when $c_{1\frac{1}{2}} = 2$. The system \[31\] at $l = 1/2$ and $c_{1\frac{1}{2}} = \hat{c}_{1\frac{1}{2}}$ takes the form

$$-\frac{1}{2} \frac{\partial \dot{\psi}_1}{\partial a_1} + i \frac{\partial \dot{\psi}_2}{\partial a_2} - \frac{1}{2} \frac{\partial \dot{\psi}_1}{\partial a_3} - i \frac{\partial \dot{\psi}_2}{\partial a_3} - \frac{1}{2} \frac{\partial \dot{\psi}_2}{\partial a_1} - i \frac{\partial \dot{\psi}_1}{\partial a_2} - \kappa \hat{c} \psi_1 = 0,$$
$$-\frac{1}{2} \frac{\partial \dot{\psi}_1}{\partial a_1} + i \frac{\partial \dot{\psi}_2}{\partial a_2} + \frac{1}{2} \frac{\partial \dot{\psi}_1}{\partial a_3} + i \frac{\partial \dot{\psi}_2}{\partial a_3} + \frac{1}{2} \frac{\partial \dot{\psi}_2}{\partial a_1} + i \frac{\partial \dot{\psi}_1}{\partial a_2} - \kappa \hat{c} \psi_2 = 0,$$
$$-\frac{1}{2} \frac{\partial \dot{\psi}_1}{\partial a_1} + i \frac{\partial \dot{\psi}_2}{\partial a_2} - \frac{1}{2} \frac{\partial \dot{\psi}_1}{\partial a_3} - i \frac{\partial \dot{\psi}_2}{\partial a_3} - \frac{1}{2} \frac{\partial \dot{\psi}_2}{\partial a_1} - i \frac{\partial \dot{\psi}_1}{\partial a_2} - \kappa \hat{c} \dot{\psi}_1 = 0,$$
$$-\frac{1}{2} \frac{\partial \dot{\psi}_1}{\partial a_1} - i \frac{\partial \dot{\psi}_2}{\partial a_2} + \frac{1}{2} \frac{\partial \dot{\psi}_1}{\partial a_3} + i \frac{\partial \dot{\psi}_2}{\partial a_3} + \frac{1}{2} \frac{\partial \dot{\psi}_2}{\partial a_1} + i \frac{\partial \dot{\psi}_1}{\partial a_2} - \kappa \hat{c} \dot{\psi}_2 = 0, \quad (11)$$
This system is defined on the tangent bundle $T\mathcal{L}$ of the group manifold $\mathcal{L}$. Let us find solutions of the system \(11\) in terms of the functions on the Lorentz group:

\[
\begin{align*}
\psi_1(g) &= f^i_{\frac{1}{2}, \frac{1}{2}}(r)M_{\frac{1}{2}}^\frac{1}{2}(\varphi, \epsilon, \theta, \tau, 0, 0), \\
\psi_2(g) &= f^i_{\frac{1}{2}, -\frac{1}{2}}(r)M_{\frac{1}{2}}^{-\frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0), \\
\dot{\psi}_1(g) &= f^i_{\frac{1}{2}, \frac{1}{2}}\left(\varphi, \epsilon, \theta, \tau, 0, 0\right), \\
\dot{\psi}_2(g) &= f^i_{\frac{1}{2}, -\frac{1}{2}}\left(\varphi, \epsilon, \theta, \tau, 0, 0\right),
\end{align*}
\]

Substituting these functions into \(11\) and separating the variables with the aid of recurrence relations of the system \(11\) in terms of the functions on the Lorentz group:

For the brevity of exposition we suppose $f^i_{\frac{1}{2}, \frac{1}{2}}(r)$, $f^i_{\frac{1}{2}, -\frac{1}{2}}(r)$, $f^i_{\frac{1}{2}, \frac{1}{2}}(r)$, and $f^i_{\frac{1}{2}, -\frac{1}{2}}(r)$.

Let us assume that $f_1 = f^i_{\frac{1}{2}, \frac{1}{2}}(r)$, $f_2 = f^i_{\frac{1}{2}, -\frac{1}{2}}(r)$, $f_3 = f^i_{\frac{1}{2}, \frac{1}{2}}(r)$, $f_4 = f^i_{\frac{1}{2}, -\frac{1}{2}}(r)$. Then

\[
\begin{align*}
-2\frac{df_3}{dr} + \frac{1}{r^*}f_3 &= \frac{2\left(\frac{l}{r} + \frac{1}{2}\right)}{r^*}f_4 - 4\kappa f_1 = 0, \\
2\frac{df_4}{dr} &= \frac{1}{r^*}f_4 + \frac{2\left(\frac{l}{r} + \frac{1}{2}\right)}{r^*}f_3 - 4\kappa f_2 = 0, \\
2\frac{df_1}{dr} &= \frac{1}{r}f_1 - \frac{2\left(\frac{l}{r} + \frac{1}{2}\right)}{r}f_2 - 4\kappa f_3 = 0, \\
-2\frac{df_2}{dr} &= \frac{1}{r}f_2 + \frac{2\left(\frac{l}{r} + \frac{1}{2}\right)}{r}f_1 - 4\kappa f_4 = 0.
\end{align*}
\]

Let us assume that $f_3 = \pm f_4$ and $f_2 = \pm f_1$; then the first equation coincides with the second, and the third equations coincides with the fourth. Therefore,

\[
\begin{align*}
\frac{df_4}{dr} &= \frac{i}{r^*}f_4 - 2\kappa f_1 = 0, \\
\frac{df_1}{dr} &= \frac{1}{r}f_1 + 2\kappa f_4 = 0.
\end{align*}
\]

Let us consider a real part $\text{Re} r$ of the radius of a complex sphere. It is obvious that $\text{Re} r = \text{Re} r^*$. Writing $z = \text{Re} r = \text{Re} r^*$ and excluding the function $f_4$ at $l = \hat{l}$, we come to
the following differential equation:

$$z^2 \frac{d^2 f_1}{dz^2} - z \frac{df_1}{dz} - (l^2 - 1 - 4\kappa^c \kappa^c z^2) f_1 = 0. \quad (12)$$

The latter equation is solvable in the Bessel functions of half-integer order:

$$f_1(z) = C_1 \sqrt{\kappa^c \kappa^c} z J_l \left( \sqrt{\kappa^c \kappa^c} z \right) + C_2 \sqrt{\kappa^c \kappa^c} z J_{l-1} \left( \sqrt{\kappa^c \kappa^c} z \right).$$

Further, using recurrence relations between Bessel functions, we find

$$f_4(z) = \frac{1}{2\kappa^c} \left( \frac{l+1}{z} f_1(z) - \frac{df_1}{dz} \right) = C_1 \frac{\kappa^c}{2} \sqrt{\kappa^c} z J_{l+1} \left( \sqrt{\kappa^c \kappa^c} z \right) - C_2 \frac{\kappa^c}{2} \sqrt{\kappa^c} z J_{l-1} \left( \sqrt{\kappa^c \kappa^c} z \right).$$

Therefore,

$$f_{\frac{l}{2}, \frac{l}{2}}(\text{Re} r) = C_1 \sqrt{\kappa^c \kappa^c} \text{Re} r J_l \left( \sqrt{\kappa^c \kappa^c} \text{Re} r \right) + C_2 \sqrt{\kappa^c \kappa^c} \text{Re} r J_{l-1} \left( \sqrt{\kappa^c \kappa^c} \text{Re} r \right),$$

$$f_{\frac{l}{2}, -\frac{l}{2}}(\text{Re} r^*) = C_1 \frac{\kappa^c}{2} \sqrt{\kappa^c} \text{Re} r^* J_{l+1} \left( \sqrt{\kappa^c \kappa^c} \text{Re} r^* \right) - C_2 \frac{\kappa^c}{2} \sqrt{\kappa^c} \text{Re} r^* J_{l-1} \left( \sqrt{\kappa^c \kappa^c} \text{Re} r^* \right).$$

In this way, solutions of the system (11) are defined by the following functions:

$$\psi_1(r, \varphi^c, \theta^c) = f_{\frac{l}{2}, \frac{l}{2}}(\text{Re} r) \mathcal{M}_{l}^{\frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

$$\psi_2(r, \varphi^c, \theta^c) = \pm f_{\frac{l}{2}, -\frac{l}{2}}(\text{Re} r) \mathcal{M}_{l}^{-\frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

$$\dot{\psi}_1(r^*, \varphi^c, \dot{\theta}^c) = \mp f_{\frac{l}{2}, -\frac{l}{2}}(\text{Re} r^*) \mathcal{M}_{l}^{\frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

$$\dot{\psi}_2(r^*, \varphi^c, \dot{\theta}^c) = f_{\frac{l}{2}, \frac{l}{2}}(\text{Re} r^*) \mathcal{M}_{l}^{-\frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0),$$

where

$$l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots;$$

$$\dot{l} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots;$$

$$\mathcal{M}_{l}^{\pm \frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{\pm \frac{1}{2}(k+i\varphi)} Z_{l}^{\pm \frac{1}{2}}(\theta, \tau),$$

$$Z_{l}^{\pm \frac{1}{2}}(\theta, \tau) = \cos^2 \frac{\theta}{2} \cosh^2 \frac{\tau}{2} \sum_{k=-l}^{l} i^{\pm \frac{1}{2} - k} \tan^{\pm \frac{1}{2} - k} \frac{\theta}{2} \tanh^{-k} \frac{\tau}{2} \times$$

$$2F_1 \left( \pm \frac{1}{2} - l + 1, 1 - l - k \left. i^2 \tan^2 \frac{\theta}{2} \right| \frac{-l + 1, 1 - l - k}{-k + 1} \right) \tanh^2 \frac{\tau}{2},$$

$$\mathcal{M}_{l}^{\pm \frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{\pm \frac{1}{2}(k-i\varphi)} Z_{l}^{\pm \frac{1}{2}}(\theta, \tau),$$

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\[ Z^\pm_\frac{1}{2}(\theta, \tau) = \cos^2 \frac{\theta}{2} \cosh 2i \tau \frac{\tau}{2} \sum_{k=-l}^i i^{\pm i-k} \tan^{\pm \frac{i}{2} - k} \theta \frac{\tau}{2} \times \]
\[ 2F_1 \left( \pm \frac{1}{2} - i + 1, -i - k \right) \right) \left( \frac{i^2 \tan^2 \theta}{2} \right) \right) \phi \left( -i + 1, 1 - k \right) \right) \tanh^2 \frac{\tau}{2} \right). \]

Therefore, in accordance with the factorization, an explicit form of the relativistic wavefunction \( \psi(\alpha) = \psi(x)\psi(g) \) on the Poincaré group in the case of \((1/2, 0) \oplus (0,1/2)\)-representation is given by the following expressions:

\[
\begin{align*}
\psi_1(\alpha) &= \psi_1^+(x)\psi_1(g) = u_1(p)e^{-ipx} f^{l \frac{1}{2}, \frac{1}{2}}( \text{Re} r) M_{l \frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0), \\
\psi_2(\alpha) &= \psi_2^+(x)\psi_2(g) = \pm u_2(p)e^{-ipx} f^{l \frac{1}{2}, \frac{1}{2}}( \text{Re} r) M_{l \frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0), \\
\psi_1(\alpha) &= \psi_1^-(x)\psi_1(g) = \mp v_1(p)e^{ipx} f^{l \frac{1}{2}, \frac{1}{2}}( \text{Re} r^*) M_{l \frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0), \\
\psi_2(\alpha) &= \psi_2^-(x)\psi_2(g) = v_2(p)e^{ipx} f^{l \frac{1}{2}, \frac{1}{2}}( \text{Re} r^*) M_{l \frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0), \quad (13)
\end{align*}
\]

The quantities \((13)\) form a bispinor on the Poincaré group, \( \psi(\alpha) = (\psi_1(\alpha), \psi_2(\alpha), \dot{\psi}_1(\alpha), \dot{\psi}_2(\alpha))^T \).

It is obvious that, solving the equations for \( \psi(x) \) and \( \psi(g) \) separately, we can find in like manner all the parametrized forms of the functions \((2)\) for any spin. The important case of the field \((1, 0) \oplus (0,1)\) will be considered in a separate paper.

In conclusion, it should be noted the following circumstance. As is known, in the standard QFT, solutions of relativistic wave equations are found in the plane-wave approximation (it is hardly too much to say that these solutions are strongly degenerate) and field operators are defined in the form of Fourier expansions (or Fourier integrals) in such solutions. In passing to the functions on the Poincaré group and their parametrized forms, the Fourier expansions are replaced by more general transformations; that is, we come to an expansion of \( \psi(\alpha) \) on the group \( \mathcal{P} \) or harmonic analysis of the functions on the groups (see, for example, \( [25, 13, 14, 30, 28] \)). In this way, usual Fourier analysis of the standard QFT is replaced by harmonic analysis in the case of QFTPG. The field operators on the functions \((4)\) and their parametrizations of the form \((13)\) will be studied in the future work in terms of harmonic analysis on the Poincaré group.

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