The determination of secondary ray-aberration coefficients for axis-symmetrical optical systems

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A B S T R A C T

The ray-aberrations in axis-symmetrical systems are conventionally derived from wavefront functions or characteristic functions using classical approximate partial derivatives. However, the resulting aberrations typically have fifth-order errors, as described by Restrepo et al. [1]. Accordingly, in the present study, the secondary ray-aberration coefficients for object placed at finite distance are determined using the fifth-order Taylor series expansion of a skew ray. Notably, the derived expressions are exact since they are determined without any approximations. It is found that some of the aberration coefficients are not constants, but are functions of the polar angle of the entrance pupil. It is additionally found that, once the required derivative matrices have been generated, determination of the secondary aberration coefficients is straightforward without iteration, and incurs only a low computational cost.

1. Introduction

To explore the monochromatic ray-aberrations of an axis-symmetrical system, the object \( \mathbf{P}_0 = [0 \ h_0 \ P_{0z}]^T \) is assumed to be located on the meridional plane at height \( h_0 \), as shown in Fig. 1. For a ray originating from \( \mathbf{P}_0 \) and passing through the entrance pupil at a point with polar coordinates \( \rho \) and \( \phi \), the intersection point \( \mathbf{P}_a \) of the ray on the image plane is given by (p. 63 of [2])

\[
\mathbf{P}_a = \left[ P_{ax} \ P_{ay} \ P_{az} \right] = \left[ \Delta P_{ax} \ A_2 h_0 + \Delta P_{ay} \ P_{az} \right]^T.
\]

where \( A_2 \) is the lateral magnification and \( \Delta P_{ax} \) and \( \Delta P_{ay} \) are the transverse aberrations of the incidence point in the \( x_\rho \)- and \( y_\rho \)-directions, respectively, and are given by

\[
\Delta P_{ax} = A_1 \rho \phi + B_1 \rho^2 \phi S(2\phi) + B_2(\rho_0 \phi^2)S(2\phi) + (B_3 + B_4)(h_0^2 \rho^2)S\phi + C_1 \rho^2 S\phi + C_3(\rho_0 \phi^2)S(2\phi) + (C_4 + C_6 \phi^2)(h_0^2 \rho^2)S\phi + C_9(\rho_0 \phi^2)S(2\phi) + C_{12}(h_0^2 \rho^2)S\phi,
\]

\[
\Delta P_{ay} = A_1 \rho \phi + B_1 \rho^2 \phi C\phi + B_2(\rho_0 \phi^2)(2 + C(2\phi)) + (B_3 + B_4)(h_0^2 \rho^2)C\phi + C_1 \rho^2 S\phi + C_3(\rho_0 \phi^2)S(2\phi) + (C_4 + C_6 \phi^2)(h_0^2 \rho^2)C\phi + C_9(\rho_0 \phi^2)S(2\phi) + C_{12}(h_0^2 \rho^2)C\phi.
\]

where \( C \) and \( S \) denote cosine and sine, respectively. (Note that the term \( C_7 + C_8 \phi^2 \) of [2] is replaced by \( C_7 + C_8 \phi^2 \) in Eq. (2b) in order to satisfy the Buchdahl-Rimmer formulae [3, 4, 5].) The aberrations, \( \Delta P_{ax} \) and \( \Delta P_{ay} \), represent the distance by which the ray misses the ideal image point \( [0 \ A_2 h_0 \ P_{az}]^T \) on the image plane, as determined by the paraxial raytracing equation. It is noted that Eqs. (2a) and (2b) contain two first-order terms (referred to as A coefficients), five third-order terms (referred to as B coefficients), and twelve fifth-order terms (referred to as C coefficients).

Smith [2], Zemax [6] and Johanson [7] use different terminologies for describing ray-aberrations. For example, Smith [2] utilizes \( B_1, B_2, B_3, B_4 \) and to refer to primary spherical aberration, coma, astigmatism, field curvature and distortion, respectively. By contrast, Zemax [6] and Johanson [7] use \( B_1 \rho_{\text{max}}^3, B_2 \rho_{\text{max}}^5, B_3 \rho_{\text{max}}^7, B_4 \rho_{\text{max}}^9 \) and \( B_5 \rho_{\text{max}}^5 \) where \( \rho_{\text{max}} \) is the maximum entrance pupil radius, as the coefficients of spherical aberration, coma, astigmatism, field curvature and distortion, respectively. The aberration coefficients given in [6, 7] are defined as the aberrations at \( \rho = \rho_{\text{max}} \) for a given \( h_0 \) since the aberrations at any other value of \( \rho \) can be simply determined using an appropriate scaling factor (i.e., \( \rho/\rho_{\text{max}} \)). The present study adopts the notations employed in [6, 7]. Consequently, \( B_1 \rho_{\text{max}}^3 \) is the secondary spherical aberration coefficient; \( C_1 \rho_{\text{max}}^4 \) and \( C_2 \rho_{\text{max}}^6 \) are the linear coma coefficients; \( C_3 \rho_{\text{max}}^8 \) and \( C_4 \rho_{\text{max}}^8 \) are the oblique spherical aberration coefficients; \( C_5 \rho_{\text{max}}^8 \), \( C_6 \rho_{\text{max}}^8 \) and \( C_7 \rho_{\text{max}}^8 \) are the ellip-

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tical coma coefficients; $C_{12h}^r r_{p_{\max}}$ and $C_{12h}^s s_{\max}$ are the Petzval and astigmatism coefficients; and $C_{12h}^a a_{\max}$ is the distortion coefficient.

Aberrations result in significant blurring or distortion of the image. Consequently, effective methods for quantifying their effects during the optical design stage are of paramount importance. The literature contains many approaches for determining the Seidel primary aberration coefficients [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. One of the most widely used methods in optical software (e.g., Zemax [6]) is that proposed by Buchdahl [8]. However, an alternative approach was recently proposed by Lin and Johnson [23, 24] based on Taylor series expansion of a skew ray. The numerical results of the primary ray-aberrations were shown to be in remarkable agreement with those obtained from Zemax simulations. Consequently, the present study extends this approach to determine the secondary ray-aberration coefficients of an axis-symmetrical system when the object is placed at finite distance. In doing so, a general skew ray $\overline{R}_0$ is expanded with respect to the source ray variable vector $\vec{X}_0$ by a Taylor series expansion up to the fifth order. Notably, the obtained equations are exact formulae without approximation. The validity of the proposed method is demonstrated by comparing the computation results with those obtained from Zemax simulations and raytracing, respectively.

2. Variable vectors of source ray

Fig. 2 shows an illustrative axis-symmetrical system with $n$ boundaries when the object is placed at finite position. In accordance with convention, the label $i=0$ is assigned to the source ray $\overline{R}_0 = \mathbf{[} \overline{P}_0 \overline{r}_0 \mathbf{]}^T$ originating from the object $\overline{P}_0$ with height $h_0$, i.e.,

$$\overline{P}_0 = [P_{0x} \ h_0 \ P_{0y}]^T.$$

The unit directional vector of $\overline{R}_0$ can be defined as

$$\overline{r}_0 = [Sa_0C\beta_0 \ S\beta_0C \ a_0C\beta_0]^T.$$

From Eqs. (3) and (4), the independent variable vector of ray $\overline{R}_0$ is given by

$$\overline{X}_0 = [P_{0x} \ h_0 \ P_{0y} \ a_0 \ \beta_0]^T.$$

In conventional aberration determination, the unit directional vector, $\overline{r}_0$, is defined in terms of the in-plane coordinates $[x_a \ y_a]^T$ of the entrance pupil (Fig. 3). This representation causes the independent variable vector to be different from that given in Eq. (5), i.e., (Eq. (5) of [24]):

$$\overline{Y}_0 = [P_{0x} \ h_0 \ P_{0y} \ x_a \ (y_a - h_0)]^T.$$

In exploring the aberrations of object $\overline{P}_0 = [0 \ h_0 \ P_{0y}]^T$ in Fig. 2, a general skew ray $\overline{R}_i$ should be expanded with respect to the ray originating from point $\overline{P}_{0axis} = [0 \ 0 \ P_{0z}]^T$ with unit directional vector $\overline{r}_{0axis} = [0 \ 0 \ 1]^T$ (see Fig. 4). The two independent variables of $\overline{r}_{0axis}$ are thus given as $\alpha_0 = \beta_0 = x_a = y_a = 0$. For convenience, this on-axis ray $[\overline{P}_{0axis} \overline{r}_{0axis}]$ can be denoted by the following vector:

$$\overline{δ}_{axis} ≡ [0 \ 0 \ P_{0z} \ 0 \ 0]^T.$$

It was shown in the Appendix of [24] that the conversion between Eqs. (5) and (6) for the ray defined by Eq. (7) can be achieved by Eq. (8)

$$\frac{\partial^g s^h \overline{R}_{i}}{\partial h_{i} \partial x_{i}^j \partial (y_{a} - h_0)_i} = \frac{\partial^g s^h \overline{R}_{i}}{\partial h_{i} \partial x_{i}^j \partial (y_{a} - h_0)_i} \left( \frac{\partial \delta_{0}}{\partial \alpha_{0}} \right) \left( \frac{\partial \delta_{0}}{\partial \beta_{0}} \right)^h,$$

where $\partial \alpha_{0} / \partial x_{a} = \partial \beta_{0} / \partial (y_{a} - h_0) = 4.502409965 \times 10^{-3}$. (Note that all lengths and angles given in this study have units of mm and degrees, respectively.)

3. Taylor series expansion of skew ray

In geometrical optics, a general ray $\overline{R}_i$ is a composite function of $\overline{X}_0$ (or $\overline{Y}_0$). The most elegant solution in mathematics for hard problems of this type is the Taylor series expansion since it can provide the polynomial expression of $\overline{R}_i = \overline{R}_i(\overline{X}_0)$ in terms of $\overline{X}_0 - \overline{δ}_{axis}$, i.e.,
Due to the axis-symmetrical nature of the considered system, various terms in Eq. (11) disappear (as confirmed by our numerical results), and hence the expressions of $\Delta P_{ax/3h}$ and $\Delta P_{oy/3h}$ become

$$\Delta P_{ax/3h} = \frac{1}{120} \left[ \frac{\partial^3 P_{ax}}{\partial x^3} x^3 + 10 \frac{\partial^2 P_{ax}}{\partial x^2} x^2 (y_a - h_0)^2 + \frac{\partial P_{ax}}{\partial x} (y_a - h_0)^3 \right]$$

$$\Delta P_{oy/3h} = \frac{1}{120} \left[ \frac{\partial^3 P_{oy}}{\partial y^3} y^3 + 10 \frac{\partial^2 P_{oy}}{\partial y^2} y^2 (x_a - h_0)^2 + \frac{\partial P_{oy}}{\partial y} (x_a - h_0)^3 \right]$$

As shown, Eqs. (12a) and (12b) are both expressed with respect to the object height $h_0$ and in-plane Cartesian coordinates $(x_a, y_a)$ of the entrance pupil. However, aberration equations are usually expressed in terms of the polar coordinates of the entrance pupil, i.e., $(r, \phi)$ (Fig. 3). Hence, it is necessary to replace $(x_a, y_a)$ in Eqs. (12a) and (12b) with $(r, \phi)$. The following equations are thus obtained:

$$\Delta P_{ax/3h} = \frac{1}{120} \left[ \frac{\partial^3 P_{ax}}{\partial x^3} r^3 + 10 \frac{\partial^2 P_{ax}}{\partial x^2} r^2 \cos^2 \phi + \frac{\partial P_{ax}}{\partial x} \cos \phi \sin \phi \right]$$

$$\Delta P_{oy/3h} = \frac{1}{120} \left[ \frac{\partial^3 P_{oy}}{\partial y^3} r^3 + 10 \frac{\partial^2 P_{oy}}{\partial y^2} r^2 \sin^2 \phi + \frac{\partial P_{oy}}{\partial y} \sin \phi \cos \phi \right]$$

The leading coefficients of Eqs. (13a) and (13b) are given respectively by

$$g_{r/0} = \frac{1}{120} \left( \frac{\partial^3 P_{ax}}{\partial x^3} r^3 + 10 \frac{\partial^2 P_{ax}}{\partial x^2} r^2 \cos^2 \phi + \frac{\partial P_{ax}}{\partial x} \cos \phi \sin \phi \right)$$

$$g_{r/0} = \frac{1}{120} \left( \frac{\partial^3 P_{oy}}{\partial y^3} r^3 + 10 \frac{\partial^2 P_{oy}}{\partial y^2} r^2 \sin^2 \phi + \frac{\partial P_{oy}}{\partial y} \sin \phi \cos \phi \right)$$
\[ f_{x/2} = \frac{1}{4} \left( \frac{\partial^2 P_{ax}}{\partial x_0 \partial x_0} (y_a - h_0)^3 - 2 \frac{\partial P_{ax}}{\partial x_0} \frac{\partial^2 P_{ax}}{\partial x_0^2} \phi + \frac{\partial^3 P_{ax}}{\partial x_0^3} \right) \]
\[ + \frac{1}{12} \left( \frac{\partial^2 P_{ax}}{\partial x_0^2} \phi + \frac{\partial^3 P_{ax}}{\partial x_0^3} \right) S^2 \phi. \]
\[ f_{y/3} = \frac{1}{12} \left( \frac{\partial^2 P_{ax}}{\partial x_0 \partial x_0} (y_a - h_0)^3 + 3 \frac{\partial P_{ax}}{\partial x_0} \frac{\partial^2 P_{ax}}{\partial x_0^2} \phi + \frac{\partial^3 P_{ax}}{\partial x_0^3} \right) S^2 \phi. \]
\[ f_{y/4} = \frac{1}{24} \left( \frac{\partial^2 P_{ax}}{\partial x_0 \partial x_0} (y_a - h_0)^3 - 4 \frac{\partial P_{ax}}{\partial x_0} \frac{\partial^2 P_{ax}}{\partial x_0^2} \phi + 6 \frac{\partial^3 P_{ax}}{\partial x_0^3} \right) \]
\[ - \frac{1}{12} \left( \frac{\partial^2 P_{ax}}{\partial x_0^2} \phi + \frac{\partial^3 P_{ax}}{\partial x_0^3} \right) S^2 \phi. \]

Equations (10a) and (10b) are derived independently from other aberration theory. However, Eqs. (2a) and (2b) are obtained from the wavefront aberration function \( W(h_0, \rho, \phi) \) by using the following approximate partial derivatives [1]:

\[ \Delta P_{\text{in}} \approx \frac{R_{\text{reference}}}{\partial x_0} \frac{\partial W}{\partial x_0}. \]
\[ \Delta P_{\text{ny}} \approx \frac{R_{\text{reference}}}{\partial y_0} \frac{\partial W}{\partial y_0}. \]
Table 1. Specification of illustrative rotationally-symmetric optical system shown in Fig. 2.

| Surface | Radius | Separation | Refractive index |
|---------|--------|------------|------------------|
| 0 (object) | ∞ | | 1.00000 |
| 1 | 38.22190 | 15.84960 | 1.65000 |
| 2 | -56.08570 | 5.96900 | 1.71736 |
| 3 | -590.68200 | 3.02260 | 1.00000 |
| 4 (aperture) | ∞ | | 14.02080 |
| 5 | -41.79570 | 2.514600 | 1.52583 |
| 6 | 29.34460 | 7.924800 | 1.00000 |
| 7 | 63.56350 | 6.09600 | 1.65000 |
| 8 | -56.86550 | 92.088474 | 1.00000 |
| 9 (image plane) | ∞ | | |

Table 2. Values of aberration coefficients obtained from Zemax simulations.

| C, C′ | | | |
|-------|---|---|---|
| A1 = 0.000000 | A2 = -0.660413 | B1, ρmax = -1.533486 |
| B1, hρmax = -0.112776 | B1, hρmax = 8.16154 × 10−3 | B1, hρmax = -0.035624 |
| C1, hρmax = -0.075594 | C1, hρmax = -0.090476 | C1, hρmax = -0.067381 |
| C1, hρmax = 0.090556 | C1, hρmax = -0.037370 | C1, hρmax = -0.067592 |
| C1, hρmax = -0.030459 | C1, hρmax = -0.032128 | C1, hρmax = -0.099668 |
| C1, hρmax = 0.001088 | C1, hρmax = 0.001406 | C1, hρmax = 0.0006322 |

4. Results of secondary ray-aberration coefficients

In this section, the values of the secondary ray-aberration coefficients are determined for the axis-symmetrical system shown in Fig. 2 (with the object positioned at P0 = −200, the image plane coinciding with the Gaussian image plane, and the maximum radius of the entrance pupil being ρmax = 21). The symbols in Eqs. (2a) and (2b) are still used under the assumption that they are functions of ψ. One has the following equations from Eqs. (2a) and (2b) by taking only the terms with C coefficients:

\[
\Delta P_{\text{ax}}(\text{5th}) = C_{1} \rho^{2} S \phi + C_{3} (h_{0} \rho^{3}) S(2 \phi) + (C_{5} + C_{2} C_{3}^{2} \phi) (h_{0} \rho^{3})^{2} S \phi + C_{1} (h_{0} \rho^{3}) S(2 \phi) + C_{1} (h_{0} \rho^{3}) S \phi,
\]

\[
\Delta P_{\text{ny}}(\text{5th}) = C_{1} \rho^{2} C \phi + (C_{2} + C_{3} C_{2}^{2} \phi) (h_{0} \rho^{3}) + (C_{4} + C_{2} C_{3}^{2} \phi) (h_{0} \rho^{3}) S \phi + (C_{3} + C_{4} C_{2}^{2} \phi) (h_{0} \rho^{3})^{2} + C_{1} (h_{0} \rho^{3}) C \phi + C_{1} \rho^{2} S \phi + C_{3} (h_{0} \rho^{3})^{2} S(2 \phi).
\]

It is noted that coefficients \( C_{2} \) and \( C_{3} \) are used in Eq. (17b) to take account of possible differences in the expressions and possible errors arising from the approximation of Eqs. (16b). One thus has the following equations by comparing Eqs. (13a) and (13b) with Eqs. (17a) and (17b), respectively:

\[
C_{1} = f_{x/2}, \quad (18a)
\]

\[
C_{3} = f_{x/3}, \quad (18b)
\]

\[
C_{3} + C_{2} C_{3}^{2} \phi = f_{x/2}, \quad (18c)
\]

\[
C_{3} = f_{x/3}, \quad (18d)
\]

\[
C_{1} = f_{x/4}, \quad (18e)
\]

\[
C_{4} + C_{2} C_{3}^{2} \phi = f_{x/2}, \quad (19a)
\]

\[
C_{2} + C_{3} C_{2}^{2} \phi = f_{x/2}, \quad (19b)
\]

\[
C_{4} + C_{6} C_{3}^{2} \phi = f_{x/2}, \quad (19c)
\]

\[
C_{2} + C_{3} C_{2}^{2} \phi = f_{x/3}, \quad (19d)
\]

\[
C_{10} = f_{x/4}, \quad (19e)
\]

\[
C_{12} = f_{x/3}, \quad (19f)
\]

Since Eqs. (16a) and (16b) are not exact equations and possess fifth-order errors [1], it is possible that the values obtained in this study are different from those of Zemax. Therefore, the following percentage difference metric is introduced to compare the results obtained using the proposed method with those obtained from Zemax simulations:
The values of the linear coma aberration coefficient are obtained from Eq. (22e) as $C_{2/zmax} = C_{12h5} = -0.018219$. The other two linear coma aberration coefficients are found from Eq. (22a) with $\phi = 90^\circ$ and Eq. (22d) with $\phi = 0^\circ$ to be $C_{3/this} = (C_{3zero} + C_{3ywo})^{\phi}_{max} = -0.062674$ and $C_{3/this} = C_{3/this} = -0.108368$, respectively. These values deviate by $C_5 \% = 53.5 \%$, $C_6 \% = 25.4 \%$ and $C_4 \% = 51.8 \%$ from the corresponding Zemax results (i.e., $C_{2/zmax} = -0.090476$ and $C_{1/zmax} = -0.067381$).

(3) $C_4$, $C_5$ and $C_6$ coefficients: From Eqs. (18c) and (14c), one has the following expressions for $C_4$ and $C_6$:

$$C_5 = \frac{1}{12} \left( \frac{\partial^3 P_{ny}}{\partial x^2 \partial (y_2 - h_0)} - \frac{2}{\partial h_0 \partial x \partial (y_2 - h_0)} + \frac{\partial P_{ny}}{\partial x^2 \partial (y_2 - h_0)} \right),$$

(23a)

$$C_6 = \frac{1}{4} \left( \frac{\partial^3 P_{ny}}{\partial x \partial (y_2 - h_0)} - \frac{2}{\partial x \partial h_0 \partial (y_2 - h_0)} + \frac{\partial P_{ny}}{\partial x \partial h_0 \partial (y_2 - h_0)} \right) - \frac{1}{12} \left( \frac{\partial^2 P_{ny}}{\partial h_0 \partial (y_2 - h_0)} - \frac{2}{\partial h_0 \partial x \partial (y_2 - h_0)} + \frac{\partial P_{ny}}{\partial h_0 \partial x \partial (y_2 - h_0)} \right).$$

(23b)

Moreover, the following expressions for $C_3$ and $C_4$ are obtained from Eqs. (19c) and (15c):

$$C_3 = \frac{1}{12} \left( \frac{\partial^3 P_{ny}}{\partial x^2 \partial (y_2 - h_0)} - \frac{2}{\partial h_0 \partial x \partial (y_2 - h_0)} + \frac{\partial P_{ny}}{\partial x^2 \partial (y_2 - h_0)} \right),$$

(23c)

$$C_4 = \frac{1}{4} \left( \frac{\partial^3 P_{ny}}{\partial x^2 \partial (y_2 - h_0)} - \frac{2}{\partial h_0 \partial x \partial (y_2 - h_0)} + \frac{\partial P_{ny}}{\partial x^2 \partial (y_2 - h_0)} \right).$$

(23d)

From Eqs. (23c), (23a), (23b) and (23d), coefficients $C_3$, $C_4$, $C_5$, $C_6$, $C_7$, $C_8$, $C_9$ and $C_{10}$ have values of $C_3 = 3.13908676 \times 10^{-10}$, $C_3 = 128.793228 \times 10^{-10}$, $C_4 = 228.793228 \times 10^{-10}$ and $C_4 = 194.675353 \times 10^{-10}$, respectively. Consequently, the oblique spherical aberration coefficients have values of $C_{2/this} = C_{8h5}^{\phi}_{max} = 0.053951$, $C_{3/this} = C_{8h3}^{\phi}_{max} = 0.02441$, $C_{4/this} = C_{h2}^{\phi}_{max} = 0.005021$ and $C_{5/this} = 0.005021$, respectively. These values deviate from the corresponding Zemax results (i.e., $C_{2/this} = 0.04697$, $C_{3/this} = 0.04697$, $C_{4/this} = 0.04697$ and $C_{5/this} = 0.04697$), respectively.

(4) $C_7$, $C_8$, $C_9$ and $C_{10}$ coefficients: The expressions of $C_7$, $C_8$, $C_9$ and $C_{10}$ are obtained from Eqs. (19d) and (15d), respectively, as

$$C_7 = \frac{1}{12} \left( \frac{\partial^3 P_{ny}}{\partial x^2 \partial (y_2 - h_0)} - \frac{2}{\partial h_0 \partial x \partial (y_2 - h_0)} + \frac{\partial P_{ny}}{\partial x^2 \partial (y_2 - h_0)} \right) + \frac{3}{\partial f_{max} \partial h_0 \partial (y_2 - h_0)} \frac{\partial^2 P_{ny}}{\partial x \partial (y_2 - h_0)} - \frac{3}{\partial f_{max} \partial x \partial (y_2 - h_0)} \frac{\partial P_{ny}}{\partial x \partial (y_2 - h_0)}.$$
been generated, determination of the aberration coefficients is straightforward without iteration, and hence incurs only a low computational cost.

**Declarations**

**Author contribution statement**

Psang Dain Lin: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

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**Data availability statement**

Data included in article/supp. material/referenced in article.

**Declaration of interests statement**

The authors declare no conflict of interest.

**Additional information**

No additional information is available for this paper.

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