CLASSIFICATION AND BIFURCATION OF A CLASS OF SECOND-ORDER ODES AND ITS APPLICATION TO NONLINEAR PDES

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Abstract. In this paper, by using dynamical system theorems we study the bifurcation of a second-order ordinary differential equation which can be obtained from many nonlinear partial differential equations via traveling wave transformation and integrations. We present all the bounded exact solutions of this second-order ordinary differential equation which contains four parameters by normalization and classification. As a result, one can obtain all possible bounded exact traveling wave solutions including solitay waves, kink and periodic wave solutions of many nonlinear wave equations by the formulas presented in this paper. As an example, all bounded traveling wave solutions of the modified regularized long wave equation are obtained to illustrate our approach.

1. Introduction. A variety of nonlinear partial differential equations (NLPDEs) have been used to model a majority of real-world physical phenomena of fluid mechanics, elasticity, mathematical biology, and many more. Therefore it is of utmost importance that we look for exact solutions of such NLPDEs. Unfortunately, there are no systematic mathematical tools that can be used to obtain exact solutions of NLPDEs. Despite of this fact, in recent years, many researchers have developed several methods of obtaining exact solutions of such equations. Most of these methods involve change of variables, usually the traveling wave variable, which transforms the given NLPDE to a nonlinear ordinary differential equation (ODE). Then the ODE is solved and one obtains the traveling wave solutions of the PDE. Traveling wave solutions are of great importance when one is dealing with nonlinear wave equations arising in the context of hydrodynamics.

To obtain traveling wave solutions of a PDE the important step is to find exact solutions of the corresponding ODE. Also, one can see that during the process of the transformation, different PDEs may lead to same type of ODE but with different
coefficients. Some ODEs associated with certain higher-order NPDs can be written in the form $F(u, u'^2, u'', u^{(4)}) = 0$ [10, 16, 17, 18] whose solutions may be constructed by a class of second-order ODE of the form

$$Q'' = P_m(Q).$$

(1.1)

Here $F(u, u'^2, u'', u^{(4)})$ is a polynomial function and $P_m(Q)$ is also a polynomial function in $Q$ of degree $m$. The bifurcation and exact solutions of (1.1) with $m = 2$, i.e., $P_2(Q) = a_2 Q^2 + a_1 Q + a_0$ were studied in [16] and these results have been well applied to study the traveling wave solutions of many nonlinear wave equations [16, 17, 18, 11].

In the present paper, we study the bifurcation and exact solutions of (1.1) with $m = 3$, that is, we study the second-order ODE given by

$$Q'' = a_3 Q^3 + a_2 Q^2 + a_1 Q + a_0,$$

(1.2)

where $Q'' = \frac{d^2 Q}{d \xi^2}$ and $a_i, i = 0, 1, 2, 3$ are constants with $a_3 \neq 0$. Our motivation stems from the fact that many nonlinear wave model equations, for instance, the KdV equation, the long wave equation and Kaup-Boussinesq system, are reducible to ODEs of this form by traveling wave transformation and integrations. Clearly, equation (1.2) is equivalent to the following planar dynamical system:

$$\begin{cases} Q' = G, \\ G' = a_3 Q^3 + a_2 Q^2 + a_1 Q + a_0, \end{cases}$$

(1.3)

which is a Hamiltonian system with Hamiltonian

$$H(Q, G) = \frac{G^2}{2} - \left[ \frac{1}{4} a_3 Q^4 + \frac{1}{3} a_2 Q^3 + \frac{1}{2} a_1 Q^2 + a_0 Q \right].$$

(1.4)

By dynamical system theorems [3, 6, 12], we know that only bounded orbits of system (1.3) correspond to the bounded solutions of equation (1.2). However, the bounded orbits of Hamiltonian system could only be periodic orbits surrounding center, heteroclinic orbits or homoclinic orbits which are boundary curves of a family of closed orbits and connected saddle points. Therefore, we just need to consider the case when it has at least one center if we only focus on the bounded solutions of system (1.3). However, it is not easy to study the bifurcation and phase portraits of system (1.3) because of the four coefficients involved in this system.

The paper is organized as follows: In Section 2, we show that system (1.3) can be classified into eight different classes in terms of the reduced forms after rescaling or transformation of dependent or independent variables. All bounded solutions and bifurcation of system (1.3) are derived by the exact bounded solutions of their reduced equations in Section 3. In Section 4, we apply the results of Section 2 and Section 3 to investigate the bounded traveling wave solutions of the modified regularized long wave equation as an example to illustrate our approach. Some discussion and conclusion are presented in Section 5.

2. Classification of system (1.3) and its bifurcations. To study the bifurcation of (1.3), we first show that system (1.3) can be classified into eight special classes by rescaling or transformation of dependent or independent variables.
2.1. Classification of (1.3). Let \( f(Q) = a_3 Q^3 + a_2 Q^2 + a_1 Q + a_0 \) with \( a_3 \neq 0 \) and suppose that \( Q_0 \) is a real root of \( f(Q) \) when \( a_0 \neq 0 \) and that \( Q_0 = 0 \) when \( a_0 = 0 \). Let \( \tilde{Q} = Q - Q_0 \). Then system (1.3) can be written in the form:

\[
\begin{aligned}
\dot{\tilde{Q}} &= G \\
G' &= a_3 \tilde{Q}^3 + b_2 \tilde{Q}^2 + b_1 \tilde{Q},
\end{aligned}
\]  
\tag{2.1}

where \( b_2 = 3a_3 Q_0 + a_2 \) and \( b_1 = 3a_3 Q_0^2 + 2a_2 Q_0 + a_1 \).

For the case when \( b_2 \neq 0 \), the scaling \( \tilde{Q} = \frac{b_2}{a_3} \tilde{Q} \), \( G = \frac{b_2^2}{a_3^3 |a_3|} \tilde{G} \) and \( \xi = \frac{\sqrt{|a_3|}}{b_2} \eta \)

transforms system (2.1) into

\[
\begin{aligned}
\dot{\tilde{Q}} &= \tilde{G} \\
\tilde{G}' &= sgn(a_3)(\tilde{Q}^3 + \tilde{Q}^2) + c_1 \tilde{Q},
\end{aligned}
\]  
\tag{2.2}

where \( c_1 = \frac{|a_3| b_2}{b_2} \).

When \( b_2 = 0 \) and \( b_1 \neq 0 \), the scaling \( \tilde{Q} = \sqrt{|a_3|} \tilde{Q} \), \( G = \frac{|a_3|}{\sqrt{|a_3|}} \tilde{G} \) and \( \xi = \frac{1}{\sqrt{|a_3|}} \eta \)

transforms system (2.1) into

\[
\begin{aligned}
\dot{\tilde{Q}} &= \tilde{G} \\
\tilde{G}' &= sgn(a_3) \tilde{Q}^3 + sgn(b_1) \tilde{Q},
\end{aligned}
\]  
\tag{2.3}

In the case when \( b_2 = 0 \) and \( b_1 = 0 \), the scaling \( \tilde{Q} = \frac{1}{\sqrt{|a_3|}} \tilde{Q} \) and \( G = \frac{1}{\sqrt{|a_3|}} \tilde{G} \)

transforms system (2.1) into

\[
\begin{aligned}
\dot{\tilde{Q}} &= \tilde{G} \\
\tilde{G}' &= sgn(a_3) \tilde{Q}^3.
\end{aligned}
\]  
\tag{2.4}

Clearly, systems (2.2)-(2.4) are special cases of (1.3) with coefficients \( a_0 = 0 \), \( a_1, a_2 \) and \( a_3 \) satisfying one of the following eight cases:

1. \( a_3 = 1, a_2 = 0, a_1 = 1 \);
2. \( a_3 = 1, a_2 = 1, a_1 \) arbitrary;
3. \( a_3 = 1, a_2 = 0, a_1 = -1 \);
4. \( a_3 = -1, a_2 = 0, a_1 = -1 \);
5. \( a_3 = -1, a_2 = -1, a_1 \) arbitrary;
6. \( a_3 = -1, a_2 = 0, a_1 = 1 \);
7. \( a_3 = 1, a_2 = a_1 = 0 \);
8. \( a_3 = -1, a_2 = a_1 = 0 \).

For now, we have shown that system (1.3) with \( a_3 \neq 0 \) can be fully classified into eight different classes. Thus, any system (1.3) can be transformed into a system whose coefficients are one of the above eight cases by rescaling or transformation of dependent or independent variables. Therefore, to investigate the bifurcation and exact solutions of system (1.3), we only need to study the eight special cases of (1.3), that is, system (1.3) with coefficients \( a_0 = 0, a_1, a_2 \) and \( a_3 \) satisfying one of forgoing eight standard cases.

2.2. Bifurcations of system (1.3) for eight standard cases. In this subsection, we investigate the bifurcations of system (1.3) for above eight cases, that is,
system (1.3) with \( a_i, i = 0, 1, 2, 3 \) satisfying one of the eight restricted conditions given in subsection 2.1.

**Case (1).** \( a_3 = 1, a_2 = 0, a_1 = 1 \)

Clearly, system (1.3) under this restriction has only one equilibrium point which is a saddle and thus has no bounded solutions.

**Case (2).** \( a_3 = 1, a_2 = 1, a_1 \) arbitrary

In this case, system (1.3) becomes

\[
\begin{align*}
Q' &= G \\
G' &= (Q^2 + Q + a_1)Q.
\end{align*}
\]  

(2.5)

Obviously, it has only one equilibrium point which is a saddle if \( a_1 > \frac{1}{4} \). It has one saddle and one cusp if \( a_1 = \frac{1}{4} \) or \( a_1 = 0 \). So for this case, system (1.3) has no bounded solutions if \( a_1 \geq \frac{1}{4} \) or \( a_1 = 0 \). However, when \( a_1 < \frac{1}{4} \) and \( a_1 \neq 0 \), it has three equilibrium points \((0,0), (Q_+, 0)\) and \((Q_-, 0)\), where \( Q_\pm = \left(-1 \pm \sqrt{1 - 4a_1}\right)/2 \). The point \((0,0)\) is a center but \((Q_+, 0)\) and \((Q_-, 0)\) are saddle if \( a_1 < 0 \). Since \( H(0,0) < H(Q_+, 0) < H(Q_-, 0) \), there is a homoclinic orbit connecting \((Q_+, 0)\), which is the boundary of a family of closed orbits surrounding the center \((0,0)\) if \( a_1 < 0 \).

The point \((Q_+, 0)\) is a center whereas \((0,0)\) and \((Q_-, 0)\) are saddle if \( 0 < a_1 < \frac{1}{4} \).

For \( a_1 = \frac{7}{4} \), since \( H(Q_+, 0) < H(0,0) = H(Q_-, 0) \), there are two heteroclinic orbits connecting \((0,0)\) and \((Q_-, 0)\), which are the boundary of a family of closed orbits surrounding the center \((Q_+, 0)\); for \( 0 < a_1 < \frac{2}{3} \), since \( H(Q_+, 0) < H(0,0) < H(Q_-, 0) \), there is a homoclinic orbit connecting \((0,0)\), which is the boundary of a family of closed orbits surrounding the center \((Q_+, 0)\); for \( \frac{2}{3} < a_1 < \frac{1}{2} \), since \( H(Q_+, 0) < H(Q_-, 0) < H(0,0) \), there is a homoclinic orbit connected \((Q_-, 0)\), which is the boundary of a family of closed orbits surrounding the center \((Q_+, 0)\).

The phase portraits of each case are shown in Figure 1.

**Case (3).** \( a_3 = 1, a_2 = 0, a_1 = -1 \)

In this case, system (1.3) is given by

\[
\begin{align*}
Q' &= G \\
G' &= (Q^2 - 1)Q.
\end{align*}
\]  

(2.6)

Obviously, (2.6) has a center \((0,0)\) and two saddle points \((1,0)\) and \((-1,0)\) with \( H(0,0) < H(1,0) = H(-1,0) \). Consequently there are two heteroclinic orbits connecting the two saddle points \((1,0)\) and \((-1,0)\), which are the boundary of a family of closed orbits surrounding the center \((0,0)\).

**Case (4).** \( a_3 = -1, a_2 = 0, a_1 = -1 \)

In this case, system (1.3) has only one equilibrium point \((0,0)\), which is a center and thus all the orbits of system (1.3) are periodic.

**Case (5).** \( a_3 = -1, a_2 = -1, a_1 \) arbitrary

For this case, system (1.3) can be written as

\[
\begin{align*}
Q' &= G \\
G' &= -(Q^2 + Q - a_1)Q.
\end{align*}
\]  

(2.7)

Clearly, it has only one equilibrium point \((0,0)\), which is a center if \( a_1 < -\frac{1}{4} \). Thus if \( a_1 < -\frac{1}{4} \), all the orbits of system (1.3) are closed curves, which correspond
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(a) (b)

(c) (d)

Figure 1. The phase portrait of system (1.3) with $a_3 = 1$, $a_2 = 1$, $a_0 = 0$ and (a) $a_1 = -1$; (b) $a_1 = \frac{2}{5}$; (c) $a_1 = \frac{1}{5}$; (d) $a_1 = \frac{17}{72}$.

to the periodic solutions of this system. It has a cusp and a center if $a_1 = 0$ or $a_1 = -\frac{1}{4}$ and thus all the closed orbits not passing through the cusp correspond to the periodic solutions and the orbits passing through the cusp correspond to solitary wave form solutions. For $a_1 > -\frac{1}{4}$ and $a_1 \neq 0$, system (2.7) has two center points and a saddle. There are two homoclinic orbits connecting the saddle which are the boundary curves of the two families of closed orbits surrounding the two centers.

Case (6). $a_3 = -1$, $a_2 = 0$, $a_1 = 1$

For this case, system (1.3) becomes

\[
\begin{cases}
Q' = G \\
G' = -(Q^2 - 1)Q.
\end{cases}
\]

(2.8)

Obviously, system (2.8) has a saddle (0,0) and two centers (1,0) and (-1,0). Hence there are two homoclinic orbits connecting the saddle point (0,0) which are the boundary curves of two families of closed orbits surrounding the centers (1,0) and (-1,0) respectively.

(7) $a_3 = 1$, $a_2 = a_1 = 0$

In this case, system (1.3) has only one equilibrium point (0,0) which is a saddle.

(8) $a_3 = -1$, $a_2 = a_1 = 0$

For this case, system (1.3) has only one equilibrium point (0,0) which is a center.

3. The exact bounded solutions of ODE (1.2). In this section, we present the exact bounded solutions of (1.2) corresponding to the eight cases of Section 2.
To derive the exact bounded solutions of equation (1.2) which is equivalent to system (1.3), we have to study the bounded orbits of (1.3) which actually are determined by the Hamiltonian (1.4) \( H(Q,G) = h \) for different values of \( h \). For a given \( h \) corresponding to a bounded orbit, (1.4) leads to

\[
\frac{dQ}{d\xi} = \pm \sqrt{\frac{1}{2}a_3Q^4 + \frac{2}{3}a_2Q^3 + a_1Q^2 + a_0Q + 2h}, \tag{3.1}
\]

via solving \( H(Q,G) = h \) for \( G \) and substituting it into the first equation of (1.3). However, the difficulty to solve (3.1) is that five coefficients are involved in system (1.3). Note that system (1.3) can be simplified into eight different special classes as what was shown in the foregoing section. Therefore, in what follows, we only need to study system (1.3) with coefficients of eight special cases.

**Case (1).** \( a_3 = 1, a_2 = 0, a_1 = 1 \)

In this case, system (1.3) has no bounded solutions because it has only one equilibrium point which is a saddle.

**Case (2).** \( a_3 = 1, a_2 = 1, a_1 \) arbitrary

For this case, system (1.3) is reduced to system (2.5) and (3.1) is written as

\[
\frac{dQ}{d\xi} = \pm \sqrt{\frac{1}{2}Q^4 + \frac{2}{3}Q^3 + a_1Q^2 + 2h}. \tag{3.2}
\]

From the bifurcation analysis of Section 2, we find that system (2.5) has bounded solutions only for the following four subcases:

**Case (2.1).** \( a_1 < 0 \)

By the analysis in Section 2.2, we know that corresponding Hamiltonian of the homoclinic orbit is given by \( h = H(Q_+, 0) \), where \( Q_+ = (-1 + \sqrt{1 - 4a_1})/2 \). By substituting \( h = H(Q_+, 0) \) in (3.2), one can derive

\[
\frac{dQ}{d\xi} = \pm (Q - Q_+) \sqrt{\frac{1}{2}(Q - r_+)(Q - r_-)}, \tag{3.3}
\]

where \( r_\pm = -\frac{1}{2} \sqrt{1 - 4a_1} - \frac{1}{6} \pm \frac{1}{3} \sqrt{1 + 3\sqrt{1 - 4a_1}} \).

Solving equation (3.3)(refer to the formula in [5]) leads to the following bounded solution of (2.5):

\[
Q(\xi) = A - \frac{6A(2A + 1)}{2 + 6A + \sqrt{4 + 6A \cosh(A(2A + 1)(\xi - \xi_0))}}, \tag{3.4}
\]

where \( A = -\frac{1 + \sqrt{1 - 4a_1}}{2} \), which corresponds to the homoclinic orbit.

By similar calculation and same procedure, one can drive all the bounded solutions of (2.5) with \( a_1 \) for other cases. Here we omit the details.

**Case (2.2).** \( 0 < a_1 < \frac{2}{3} \)

The bounded solution of (2.5) corresponding to the homoclinic orbit is

\[
Q(\xi) = \frac{-6a_1}{2 + \sqrt{4 - 18a_1 \cosh(\sqrt{a_1}(\xi - \xi_0))}}. \tag{3.5}
\]

**Case (2.3).** \( a_1 = \frac{2}{3} \)

In this case, the solutions of (2.5) corresponding to the two heteroclinic orbits are

\[
Q(\xi) = \pm \frac{1}{3} \tanh \left( \frac{\sqrt{2}}{6}(\xi - \xi_0) \right) - \frac{1}{3}. \tag{3.6}
\]
Case (2.4). \( \frac{2}{3} < a_1 < \frac{1}{4} \)

The bounded solution of (2.5) corresponding to the homoclinic orbit can be expressed as (3.4) with \( A = -\frac{1 + \sqrt{\frac{4}{9} - 4a_1}}{2} \).

System (2.5) has a family of periodic solutions

\[
Q(\xi) = Q_1 + \frac{(Q_2 - Q_1)(Q_0 - Q_1)}{Q_0 - Q_1 - (Q_0 - Q_2)\text{sn}^2(\Omega(\xi - \xi_0), q)},
\]

where \( Q_1 < Q_2 < Q_3 \) are three roots of equation \( Q^3 + (\frac{4}{3} + Q_0)Q^2 + (\frac{2}{3}Q_0 + Q_0^2 + 2a_1)Q + 2a_1Q_0 + \frac{4}{3}Q_0^2 + Q_0^3 = 0 \). \( \Omega = \frac{\sqrt{2}}{4} \sqrt{(Q_3 - Q_2)(Q_0 - Q_1)} \) and 

\[
q = \sqrt{\frac{(Q_2 - Q_1)(Q_0 - Q_1)}{(Q_3 - Q_2)(Q_0 - Q_1)}}. \]

For \( a_1 < 0, Q_0 \in (Q_-, 0) \); for \( 0 < a_1 < \frac{2}{3} \), \( Q_0 \in (0, Q_+) \); but for \( \frac{2}{3} < a_1 < \frac{1}{4} \), \( Q_0 \in (Q_-, Q_+) \). Note here \( Q_{\pm} = (-1 \pm \sqrt{1 - 4a_1})/2 \).

Case (3). \( a_3 = 1, a_2 = 0, a_1 = -1 \)

The bounded solutions of (2.6) corresponding to the two heteroclinic orbits are

\[
Q(\xi) = \pm \text{tanh}(\frac{\sqrt{2}}{2}(\xi - \xi_0)).
\]

The bounded periodic solutions of (2.6) corresponding to the periodic orbits are

\[
Q(\xi) = -\sqrt{2 - Q_0^2} + \frac{2 - 2Q_0^2}{Q_0 + \sqrt{2 - Q_0^2} - 2Q_0sn^2(\Omega(\xi - \xi_0), q)},
\]

where \( \Omega = \frac{\sqrt{2}}{4}(Q_0 + \sqrt{2 - Q_0^2}) \) and 

\[
q = \frac{2}{Q_0 + \sqrt{2 - Q_0^2}} \sqrt{Q_0\sqrt{2 - Q_0^2}}
\]

for arbitrary \( 0 < Q_0 < 1 \).

Case (4). \( a_3 = -1, a_2 = 0, a_1 = -1 \)

The bounded periodic solutions of (1.2) corresponding to the periodic orbits are

\[
Q(\xi) = \frac{2Q_0}{1 + (ns(\Omega(\xi - \xi_0), k) + cs(\Omega(\xi - \xi_0), k))^2} - Q_0,
\]

where \( \Omega = \sqrt{1 + Q_0^2}, k = \frac{Q_0}{\sqrt{2 + 2Q_0^2}} \) for arbitrary \( Q_0 \).

Case (5). \( a_3 = -1, a_2 = -1 \)

We have the following five subcases:

Case (5.1). For any \( a_1 < -\frac{1}{4} \), the phase orbits of system (1.3) are periodic solutions and the corresponding periodic solutions are

\[
Q(\xi) = \frac{q(Q_0 - Q_1)}{a + p(ns(\Omega(\xi - \xi_0), k) + cs(\Omega(\xi - \xi_0), k))^2} + Q_1,
\]

where \( Q_0 \) is an arbitrary constant and \( Q_1 \) is the real root of the equation

\[
Q_1^2 + (Q_0 + \frac{4}{3})Q_1^2 + (Q_0^2 + \frac{4}{3}Q_0^2 - 2a_1)Q_1 + (Q_0^2 + \frac{4}{3}Q_0^2 - 2a_1)Q_0 = 0,
\]

\[
p = \frac{1}{4} \sqrt{9Q_0^2 + Q_1(18Q_0 + 12) + 27Q_0^2 + 24Q_0 - 18a_1}, \quad k = \frac{1}{2} \sqrt{\frac{(Q_0 - Q_1)^2 - (p - q)^2}{pq}},
\]

\[
q = \frac{1}{4} \sqrt{27Q_1^2 + Q_1(18Q_0 + 24) + 9Q_0^2 + 12Q_0 - 18a_1} \quad \text{and} \quad \Omega = \frac{\sqrt{2}}{4}.
\]

Case (5.2). For \( a_1 = -\frac{1}{4} \), the phase orbits of system are all periodic orbits and the corresponding periodic solutions except the one passing through the singular point
(-1/2, 0) can be expressed as (3.11) with \( Q_0 \notin \{-\frac{1}{2}, \frac{1}{5}\} \). For the orbits connecting the singular point \((-1/2, 0)\) the corresponding solution is
\[
Q(\xi) = \frac{12}{(\xi - \xi_0)^2 + 18} - \frac{1}{2}.
\] (3.13)

**Case (5.3).** For \( a_1 = 0 \), the phase orbits of system are all periodic orbits and the corresponding periodic solutions except the one passing through the singular point \((0, 0)\) can be expressed same as (3.11) with \( Q_0 \notin \{-\frac{4}{3}, 0\} \). For the orbits connecting the singular point \((0, 0)\) the corresponding solution is
\[
Q(\xi) = -\frac{12}{2(\xi - \xi_0)^2 + 9}.
\] (3.14)

**Case (5.4).** For \(-\frac{1}{4} < a_1 < 0\), the solutions corresponding to the two homoclinic orbits are
\[
Q_{\pm}(\xi) = \frac{1}{2}(\sqrt{1 + 4a_1} - 1) - \frac{6\Omega^2}{1 - 3\sqrt{1 + 4a_1} \pm \sqrt{1 + 3\sqrt{1 + 4a_1}\cosh(\Omega(\xi - \xi_0))}}.
\] (3.15)

where \( \Omega = \frac{\sqrt{2}}{2}\sqrt{1 + 4a_1} - 1 - 4a_1 \).

The bounded periodic solutions of (2.7) corresponding to the periodic orbits inside the right homoclinic orbits are
\[
Q(\xi) = Q_3 - \frac{(Q_3 - Q_1)(Q_3 - Q_2)}{(Q_1 - Q_2)\sn^2(\Omega\xi, k) + (Q_3 - Q_1)}.
\] (3.16)

and part of the periodic orbits inside the left homoclinic orbits are
\[
Q(\xi) = Q_4 - \frac{(Q_4 - Q_2)(Q_4 - Q_1)}{(Q_2 - Q_1)\sn^2(\Omega\xi, k) + (Q_4 - Q_2)}.
\] (3.17)

for arbitrary \(\frac{1}{2}(-1 + \sqrt{1 + 4a_1}) < Q_3 < 0\). Here \( \Omega = \sqrt{\frac{2(Q_1 - Q_2)(Q_2 - Q_4)}{Q_1 - Q_3)(Q_2 - Q_4)}}, Q_1, Q_2 and Q_4 are three roots of equation
\[
Q^3 + \left(Q_3 + \frac{4}{3}Q_3\right)^2 + (Q_3^2 + \frac{4}{3}Q_3 - 2a_1)Q + Q_3^3 + \frac{4}{3}Q_3^2 - 2a_1Q_3 = 0,
\] (3.18)

where \( Q_1 < Q_2 < Q_4 \).

The bounded periodic solutions corresponding to another part of the periodic orbits inside the left homoclinic orbits and the periodic orbits outside the homoclinic orbits are given by (3.11) for \(\frac{1}{2}(-1 - \sqrt{1 + 4a_1}) < Q_0 < \frac{1}{2}(-2 + \sqrt{4 + 18a_1})\) and \( Q_0 > -\frac{1}{2}\sqrt{1 + 4a_1} - \frac{1}{6} + \frac{3}{2}\sqrt{1 + 3\sqrt{1 + 4a_1}} \) respectively.

One bounded periodic solutions corresponding to one periodic orbit inside the left homoclinic orbits can be given by
\[
Q(\xi) = \frac{6a_1}{2 + \sqrt{4 + 18a_1}\cosh(\sqrt{a_1}(\xi - \xi_0))}.
\] (3.19)

**Case (5.5).** For any \( a_1 > 0 \), the solutions of (2.6) corresponding to the two homoclinic orbits are
\[
Q_{\pm}(\xi) = \frac{6a_1}{2 \pm \sqrt{4 + 18a_1}\cosh(\sqrt{a_1}(\xi - \xi_0))}.
\] (3.20)
A periodic solution is given by

\[ Q(\xi) = \frac{-6\Omega^2}{3\sqrt{1+4a_1} - 1 + \sqrt{1+3\sqrt{1+4a_1}} \cos(\Omega(\xi - \xi_0))} + \frac{1}{2}(\sqrt{1+4a_1} - 1), \]  

(3.21)

where \( \Omega = \frac{\sqrt{2}}{2} \sqrt{1 - \sqrt{1+4a_1} + 4a_1} \).

The bounded periodic solutions of (2.6) corresponding to the periodic orbits inside the homoclinic orbits are given by (3.16) and (3.17) for any 0 < \( Q_3 < -\frac{1}{2} \frac{1}{1 + \sqrt{1+4a_1}} \).

The bounded periodic solutions corresponding to another part of the periodic orbits inside the left homoclinic orbits and the periodic orbits outside the homoclinic orbits are given by (3.11) for \(-\frac{1}{2}(1 + \sqrt{1+4a_1}) < Q_0 < \frac{1}{2}(1 + 3\sqrt{1+4a_1}) - \sqrt{1+4a_1} \) and \( Q_0 > \frac{1}{2}(-2 + \sqrt{1+18a_1}) \) respectively.

Case (6). \( a_3 = -1, a_2 = 0, a_1 = 1 \)

The bounded solution of (2.8) corresponding to the two homoclinic orbits are

\[ Q(\xi) = \pm \sqrt{2} \text{sech}(\xi - \xi_0). \]  

(3.22)

The bounded periodic solutions corresponding to the periodic orbits inside the homoclinic orbits are

\[ Q(\xi) = \pm \left[ -\sqrt{2 - Q_0^2} + \frac{2Q_0^2 - 2Q_0\sqrt{2 - Q_0^2} - 4}{(\sqrt{Q_0^2 - 2Q_0^2})sn^2(\Omega \xi, k) - (Q_0 + \sqrt{2 - Q_0^2})} \right], \]  

(3.23)

where \( \Omega = \frac{1}{2}\sqrt{1 + Q_0\sqrt{2 - Q_0^2}} \), \( k = \frac{\sqrt{2(1 - \sqrt{2 - Q_0^2})}}{Q_0 + \sqrt{2 - Q_0^2}} \) and 1 < \( Q_0 < \sqrt{2} \) is an arbitrary constant.

The bounded periodic solutions corresponding to the periodic orbits outside the homoclinic orbits are

\[ Q(\xi) = \frac{2Q_0}{1 + (ns(\Omega \xi, k) + cs(\Omega \xi, k))^2} - Q_0, \]  

(3.24)

where \( \Omega = \sqrt{Q_0^2 - 1} \), \( k = \frac{Q_0}{\sqrt{2Q_0^2 - 2}} \) and \( Q_0 > \sqrt{2} \) is an arbitrary constant.

Case (7). \( a_3 = 1, a_2 = a_1 = 0 \)

For this case, system (1.3) has no bounded solutions because it has only one equilibrium point which is a saddle.

Case (8). \( a_3 = -1, a_2 = a_1 = 0 \)

For this case, the bounded periodic solutions of system (1.3) corresponding to the periodic orbits are

\[ Q = -Q_0 + \frac{2Q_0}{1 + \left( ns \left( Q_0^2, \frac{Q_0^2}{2} \right) + cs \left( Q_0^2, \frac{Q_0^2}{2} \right) \right)^2}, \]  

(3.25)

where \( Q_0 \) is an arbitrary constant.

4. The modified regularized long wave equation. The modified regularized long wave equation is given by

\[ u_t + u_x + 6u^2u_x - u_{xxt} = 0, \]  

(4.1)
which has been studied numerically in [4]. By supposing \( \xi = x - ct \), equation (4.1) transforms to the ODE
\[
(1 - c)u' + 6u^2u' + cu'' = 0. \tag{4.2}
\]
Integrating once (4.2) gives
\[
(1 - c)u + 2u^3 + cu'' = g, \tag{4.3}
\]
where \( g \) is an integration constant.

For the case when \( g \neq 0 \), let \( g = 2u_0^3 + (1 - c)u_0 \) and \( Q = u - u_0 \), where \( u_0 \) is a nonzero constant. Clearly, for any \( c \neq 0 \), equation (4.3) is equivalent to system (2.1) with \( a_3 = -2/c \), \( b_2 = -6u_0/c \) and \( b_1 = -(6u_0^2 - c + 1)/c \). The rescaling \( \bar{Q} = 3u_0\tilde{Q}, \bar{G} = -9u_0^2\sqrt{2/c}\tilde{G}/c \) and \( \xi = -\frac{c}{3u_0\sqrt{2/c}}\eta \) transform system into
\[
\begin{aligned}
\dot{Q} &= \bar{G} \\
\dot{G} &= -\sgn(c) \left( \bar{Q}^3 + \bar{G}^2 + \frac{6u_0^2 - 1}{18u_0^2} \bar{Q} \right). \tag{4.4}
\end{aligned}
\]
According to Section 2, we know that for any arbitrary constant \( u_0 \neq 0 \)
\[
u(\xi) = u_0 + 3u_0Q\left(-\frac{3u_0\sqrt{2/c}}{c}\xi\right) \tag{4.5}
\]
is the solution of (4.2) if \( Q(\eta) \) satisfies (4.4).

**Theorem 4.1.** Denote that \( a_1 = -(6u_0^2 + 1 - c)/(18u_0^2) \). The following conclusion holds for the traveling wave solutions of the modified regularized long wave equation (4.1) for arbitrary constant \( \xi_0 \).

1. For arbitrary constant \( u_0 \), equation (4.1) has the solitary wave solutions
\[
u(x, t) = u_0 \left( \frac{6u_0^2 + 4}{2u_0^2(x - (\frac{3}{2}u_0^2 + 1)t - \xi_0)^2 + 3u_0^2 + 2} - \frac{1}{2} \right), \tag{4.6}
\]
and
\[
u(x, t) = u_0 \left( 1 - \frac{4(6u_0^2 + 1)}{4u_0^2(x - (6u_0^2 + 1)t - \xi_0)^2 + 6u_0^2 + 1} \right). \tag{4.7}
\]

For arbitrary constant \( u_0 \), \( Q_0 \not\in \{-\frac{1}{2}, \frac{1}{2}\} \) and \( c = \frac{3}{2}u_0^2 + 1 \) or \( Q_0 \not\in \{0, -\frac{3}{2}\} \) and \( c = 6u_0^2 + 1 \), equation (4.1) has a family of periodic traveling wave solutions
\[
u_\pm(\xi) = u_0 + 3u_0\left( \frac{q(Q_0 - Q_1)}{q + p(ns(\Omega(x - ct - \xi_0), k) + cs(\Omega(x - ct - \xi_0), k))^2 + Q_1} \right) \tag{4.8}
\]
Here \( Q_1 \) is a real root of (3.12), \( p = \frac{1}{2}\sqrt{9Q_1^2 + Q_1(18Q_0 + 12) + 27Q_0^2 + 24Q_0 - 18a_1} \), \( q = \frac{1}{2}\sqrt{27Q_1^2 + Q_1(18Q_0 + 24) + 9Q_0^2 + 12Q_0 - 18a_1} \), \( \Omega = 3u_0\sqrt{pq/c} \) and \( k = \frac{1}{2}\sqrt{(Q_0 - Q_1)^2 - (p - q)^2} \).

2. For arbitrary constant \( u_0 \) and the wave speed \( c \) satisfying \( \frac{3}{2}u_0^2 + 1 < c < 6u_0^2 + 1 \), equation (4.1) has the two solitary wave solutions given by
\[
u(\xi) = -\frac{18Q_0^2u_0}{1 - 3\sqrt{1 + 4a_1} \pm \sqrt{1 + 3\sqrt{1 + 4a_1}}\cosh(\Omega_1(x - ct - \xi_0))} + \frac{1}{2}u_0(3\sqrt{1 + 4a_1} - 1), \tag{4.9}
\]
where \( \Omega = \sqrt{-1 - 4a_1 + \sqrt{1 + 4a_1}} \) and \( \Omega_1 = 3u_0\Omega\sqrt{c}/c \).
It has two families of periodic wave solutions

\[ u_1(\xi) = u_0 + 3u_0 \left( Q_3 - \frac{(Q_1 - Q_3)(Q_2 - Q_3)}{(Q_1 - Q_2)\text{sn}^2(\Omega(x - ct - \xi_0), k) - (Q_1 - Q_3)} \right) \]  
(4.10)

and

\[ u_2(\xi) = u_0 + 3u_0 \left( Q_4 - \frac{(Q_2 - Q_4)(Q_1 - Q_4)}{(Q_2 - Q_1)\text{sn}^2(\Omega(x - ct - \xi_0), k) - (Q_2 - Q_4)} \right). \]  
(4.11)

Here \( \Omega = \frac{3u_0}{2c} \sqrt{c(Q_1 - Q_3)(Q_2 - Q_4)} \), \( k = \sqrt{\frac{(Q_1 - Q_2)(Q_2 - Q_4)}{(Q_1 - Q_3)(Q_2 - Q_4)}} \), \( Q_3 \) is an arbitrary constant satisfying \( -3 - \sqrt{2c - 3u_0^2 - \frac{2}{6|u_0|}} < Q_3 < -3 + \sqrt{2c - 3u_0^2 - \frac{2}{6|u_0|}} \), \( Q_1, Q_2 \) and \( Q_4 \) are three roots of equation (3.18) and \( Q_1 < Q_2 < Q_4 \).

It has a family of periodic wave solutions given by (4.8). Here \( Q_0 > -\frac{1}{\sqrt{6}} \sqrt{1 + 4a_1 - \frac{1}{6} + \frac{1}{3} \sqrt{1 + 4a_1}}, Q_1 \) is a real root of (3.12), \( k = \frac{1}{2} \sqrt{(-p - q + (Q_0 - Q_1)^2)}, p = \frac{1}{3} \sqrt{9Q_1^2 + Q_1(18Q_0 + 24) - 27Q_0^2 + 24Q_0 - 18a_1}, Q = 3u_0 \sqrt{pq}/c \) and \( q = \frac{1}{3} \sqrt{27Q_1^2 + Q_1(18Q_0 - 24) + 9Q_0^2 + 12Q_0 - 18a_1}. \)

(3) For arbitrary constant \( u_0 \) and the wave speed \( c \) satisfying \( 0 < c < \frac{3}{2}u_0^2 + 1 \), equation (4.1) has the periodic wave solutions expressed by (4.8) for arbitrary constant \( Q_0 \).

(4) For arbitrary constant \( u_0 \) and the wave speed \( c \) satisfying \( c > 6u_0^2 + 1 \), equation (4.1) has two solitary wave solutions and three families of periodic solutions: solitary wave solutions given by

\[ u_{\pm}(\xi) = u_0 \left( 1 \pm \frac{18a_1}{2 \pm \sqrt{4 + 18a_1} \cosh \left( \frac{3u_0 \sqrt{2u_0^2 + c}}{c} (x - ct - \xi_0) \right) } \right); \]  
(4.12)

periodic wave solutions given by (4.8) for any \( Q_0 > \frac{1}{2} \left( -2 + \sqrt{4 + 18a_1} \right) \) and by (4.10) and (4.11) for any \( Q_3 \) satisfying \( \frac{1}{2} \left( -1 + \sqrt{4 + 18a_1} \right) < Q_3 < \frac{1}{2} \left( -2 + \sqrt{4 + 18a_1} \right) \).

Proof. For the case when \( g \neq 0 \), equation (4.3) can be transformed into system (4.4) by rescaling. Clearly, system (4.4) is equivalent to (2.5) with \( a_1 = (6u_0^2 + 1 - c)/(18u_0^2) \) for \( c < 0 \). Note that \( a_1 > \frac{1}{3} \) if \( c < 0 \), so system (4.4) and equation (4.3) have no bounded solutions for \( c < 0 \).

However, system (4.4) is equivalent to (2.7) with \( a_1 = -6u_0^2 + 1 - c)/(18u_0^2) \) when \( c > 0 \). Solving \( a_1 = -\frac{1}{2} \) for \( c \) yields \( c = \frac{3}{2}u_0^2 + 1 \). From (3.13), we get the solution (4.6). While solving \( a_1 = 0 \) for \( c \) gives \( c = 6u_0^2 + 1 \). From (3.14), we get the solution (4.7). From (3.11), we get the periodic wave solutions (4.8). When solving \(-\frac{1}{4} < a_1 < 0 \) for \( c \) gives \( \frac{3}{2}u_0^2 + 1 < c < 6u_0^2 + 1 \). From (3.15)-(3.18) and (4.5), we get two solitary wave solutions (4.9) and the three families of periodic wave solutions (4.8), (4.10) and (4.11).

When \( a_1 < -\frac{1}{4} \), one has \( 0 < c < \frac{3}{2}u_0^2 + 1 \). From (3.11) and (4.5), we know that (4.8) is a family of periodic solutions of (4.1) for arbitrary constant \( u_0 \).

For \( a_1 > 0 \), we have \( c > 6u_0^2 + 1 \). From (3.11), (3.16)-(3.20) and (4.5), we get two solitary wave solutions (4.12) and three families of periodic solutions (4.8), (4.10) and (4.11).
For the case when \( g = 0 \) and \( c \neq 1 \), (4.3) can be transformed into (2.3). Consequently, one knows that \( u = \sqrt{\frac{1-c^2}{c}} Q(\sqrt{\frac{1-c^2}{c}} \xi) \) is a solution of (4.3) and thus \( u = \sqrt{\frac{1-c^2}{c}} Q(\sqrt{\frac{1-c^2}{c}} (x - ct)) \) is a traveling wave solution of (4.1) provided \( Q(\xi) \) satisfies (2.3). Especially, if \( c < 0 \), (4.3) has no bounded solution since (2.3) has no bounded solutions. However, for the case \( 0 < c < 1 \), (3.11) are a family of periodic solutions of system (2.3). For the case \( c > 1 \), (3.22), (3.23) and (3.24) are solutions of system (2.3). So we can obtain all the bounded traveling wave solutions of equation (4.1) as follows.

**Theorem 4.2.** (1) For arbitrary nonzero \( u_0 \), equation (4.1) has a family of bounded periodic solutions with wave speed \( c = 1 \)

\[
u(x,t) = \sqrt{2u_0} \left[ \begin{array}{c} 1 \\ 1 + (ns(u_0(x - t) - \xi_0, \frac{\sqrt{2}}{2}) + cs(u_0(x - t) - \xi_0, \frac{\sqrt{2}}{2}))^2 - \frac{1}{2} \end{array} \right]
\]

and a family of bounded periodic solutions with wave speed \( 0 < c < 1 \)

\[
u(x,t) = \sqrt{2(1-c)u_0} \left[ \begin{array}{c} 1 \\ 1 + (ns(\Omega(x - ct) - \xi_0, k) + cs(\Omega(x - ct) - \xi_0, k))^2 - \frac{1}{2} \end{array} \right]
\]

where \( \Omega = \sqrt{\frac{1-c}{c} (1 + u_0^2)}, \ k = \frac{u_0}{\sqrt{2+2u_0^2}} \).

(2) Equation (4.1) has the following traveling wave solutions with wave speed \( c > 1 \): two solitary wave solutions

\[
u(x,t) = \pm \sqrt{c-1} \sech \left( \sqrt{\frac{c-1}{c}} (x - ct) - \xi_0 \right);
\]

a family of bounded periodic solutions

\[
u(x,t) = \pm \sqrt{\frac{c-1}{2}} \left( \begin{array}{c} 2u_0^2 - 4 - 2u_0 \sqrt{2 - u_0^2} \\ (u_0 - \sqrt{2 - u_0^2})sn^2(\Omega(x - ct) - \xi_0, k) - (u_0 + \sqrt{2 - u_0^2}) \\ - \sqrt{2 - u_0^2} \end{array} \right),
\]

where \( \Omega = \frac{1}{2} \sqrt{\frac{c-1}{c} (1 + u_0 \sqrt{2 - u_0^2})}, \ k = \frac{\sqrt{2(1-\sqrt{2-u_0^2})}}{u_0 + \sqrt{2-u_0^2}} \) and \( 1 < u_0 < \sqrt{2} \) is an arbitrary constant; and the bounded periodic solutions

\[
u(x,t) = u_0 \sqrt{2(c-1)} \left[ \begin{array}{c} 1 \\ 1 + (ns(\Omega(x - ct) - \xi_0, k) + cs(\Omega(x - ct) - \xi_0, k))^2 - \frac{1}{2} \end{array} \right],
\]

where \( \Omega = \sqrt{(c-1)(u_0^2 - 1)/c}, \ k = \frac{u_0}{\sqrt{2u_0^2 - 2}} \) and \( u_0 > \sqrt{2} \) is an arbitrary constant.

It is easy to find from the analysis above that (4.3) has no bounded solutions when \( c < 0 \) for arbitrary value of \( g \). The modified regularized long wave solution (4.1) has no bounded traveling wave solutions with negative wave speed since the traveling wave equation of (4.1) is equivalent to equation (4.3). We conclude it in the following theorem.
**Theorem 4.3.** Equation (4.1) has no bounded left-going traveling wave solution, that is, it has no bounded traveling wave solution with wave speed \( c < 0 \).

5. **Conclusion and discussion.** It is shown in this paper that in order to determine traveling wave solutions of some nonlinear PDEs, one can study the bifurcation and exact bounded solutions of a general second-order ODE (1.2) using bifurcation and qualitative analysis. We proved that such family of equations can be classified into eight classes. By studying the bifurcation and exact bounded solutions of these eight normalized equations, we obtained all exact bounded solutions of equation (1.2). To illustrate our approach, we studied the modified regularized long wave (MRLW) equation as an example. As a result, bifurcation and all bounded exact traveling wave solutions, including solitary waves, kink and periodic wave solutions of the MRLW equation were derived by using the formulas presented in this paper. We also showed that the MRLW equation has no bounded left-going traveling wave solution.

The bifurcation and exact bounded traveling wave solutions of some other nonlinear PDEs, such as the integrable version of the Boussinesq system [14], the mKdV equation, the combined KdV-mKdV equation [1], Kaup-Boussinesq system [4, 7] and BBM equation [8] can be investigated completely similarly. The solutions of some special cases of equation (1.2) are even used as auxiliary differential equation to study the generalized mKdV equation with variable coefficients [15] and a generalized Benjamin-Bona-Mahony equation [9]. We would like to point out that the time fractional Kaup-Boussinesq system with time dependent coefficient was transformed into a similar second-order ODE (refer to equation (4.3) in [13]) with variant coefficients. It would be feasible to obtain all possible bounded solutions by using the method presented in this paper, which will be investigated in our next work.

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