A Schwartz-Zippel Type Estimate for Homogenous Finite Field Polynomials

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Abstract

In this paper, we obtain a Schwartz-Zippel type estimate for homogenous finite field polynomials. Specifically, we use a probabilistic recursion technique to find upper and lower bounds for the number of zeros of a homogenous polynomial and illustrate our result with two examples involving perfect matching in bipartite graphs and common zeros in a collection of polynomials, respectively.

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1. Introduction

The Schwartz-Zippel bound \textsuperscript{5} and combinatorial nullstellensatz \textsuperscript{1} are important tools from both theoretical and application perspectives. The Schwartz-Zippel bound provides an estimate on the number of zeros of a polynomial whose coefficients take values in a finite field and has applications in polynomial comparison, primality testing, perfect matching in graphs etc. The combinatorial nullstellensatz determines conditions under which a polynomial contains a non-zero in a given set whose cardinality is larger than the overall degree of the polynomial. We refer to Chapter 7, \textsuperscript{3} for more material. Recently \textsuperscript{2} have used algebraic techniques to obtain Schwartz-Zippel type bounds for intersection of algebraic varieties with Cartesian products of two-dimensional sets.

In this paper, we use probabilistic methods and recursive techniques to find upper and lower bounds the number of zeros of a homogenous polynomial. As an illustration of the upper bound, we apply our bounds to estimate the error probability in the randomized algorithm that determines the presence of a perfect matching in a bipartite graph. We use the lower bound on the number of zeros to determine the presence of common zeros in a set of polynomials.
Homogenous polynomials

For integers \( l \geq 1, q = p^l, p \text{ prime}, \) let \( \mathbb{F}_q \) be the finite field consisting of \( q \) elements with characteristic \( p \). A homogenous polynomial in \( \mathbb{F}_q \) in the variables \( x_1, \ldots, x_m \) and having degree \( k \leq m \) is of the form

\[
Q(x_1, \ldots, x_m) = \sum_{I: \#I = k} \delta_I \cdot \prod_{i \in I} x_i \tag{1.1}
\]

where \( \delta_I \in \mathbb{F}_q \) are not all zero and the sum is over all subsets \( I \subseteq \{1, 2, \ldots, m\} \) of cardinality \( \#I = k \).

We say that the \( m \)-tuple \( (y_1, \ldots, y_m) \in \mathbb{F}_m^q \) is a zero for \( Q \) if

\[
Q(y_1, \ldots, y_m) = 0.
\]

**Theorem 1** If \( Z \subseteq \mathbb{F}_q^m \) denotes the set of zeros of \( Q \), then

\[
q^{m-k}(q-1)^{k-1} \leq \#Z \leq q^{m-k}(q^k - (q-1)^k)
\]

where \( \#Z \) denotes the cardinality of the set \( Z \).

Using \( (1+t)^k - t^k = \sum_{l=0}^{k-1} \binom{k}{l} t^l \leq k^{k-1}(2^k - 1) \) with \( t = q-1 \), we get from (1.2) that

\[
q^{m-k}(q-1)^{k-1} \leq \#Z \leq q^{m-k}(q-1)^{k-1}(2^k - 1).
\]

Thus the number of zeros is of the order of \( q^{m-k}(q-1)^{k-1} \).

We use recursion to prove Theorem 1 in the next Section. If \( k \leq q \), we use \( \left( 1 - \frac{1}{q} \right)^k \geq 1 - \frac{k}{q} \) to get from (1.2) that \( \#Z \leq k \cdot q^{m-1} \), the Schwartz-Zippel bound (3). We remark that in Theorem 1, we can choose the field size \( q \) independent of the degree \( k \) of the polynomial and the number of variables \( m \). In Section 3, we describe applications for both the lower and the upper bounds.

The paper is organized as follows. In Section 2, we prove Theorem 1 and in Section 3, we describe applications that use the bounds in (1.2).

### 2. Proof of Theorem 1

We use the probabilistic method analogous to the proof of Schwartz-Zippel Lemma and obtain a recursion relation involving the probability that the set of randomly chosen values indeed form a zero of the polynomial. We then evaluate the recursive relation to obtain the desired bounds on the corresponding probability.

We begin with the upper bound in (1.2). Let \( r_1, \ldots, r_m \) be independently and uniformly randomly chosen from \( \mathbb{F}_q \) and let

\[
p_{k,m} := \max_{q} \mathbb{P}(Q(r_1, \ldots, r_m) = 0) \tag{2.1}
\]
where the maximum is over all homogenous polynomials in \( m \) variables and with degree \( k \). To prove the upper bound in (1.2) it suffices to see that

\[
p_{k,m} \leq 1 - \left( 1 - \frac{1}{q} \right)^k.
\]

(2.2)

We first prove (2.2) for \( k = 1 \). Consider the polynomial

\[
Q(x_1, \ldots, x_m) = \sum_{i=1}^{m} \alpha_i \cdot x_i,
\]

where \( \alpha_1 \in \mathbb{F}_q \setminus \{0\} \) and \( \alpha_i \in \mathbb{F}_q \) for \( 2 \leq i \leq m \). Letting \( Q_0 := Q(r_1, \ldots, r_m) \), we have

\[
P(Q_0 = 0) = \sum_{j=2}^{m} \sum_{a_j \in \mathbb{F}_q} P(Q_0 = 0 \mid r_2 = a_2, \ldots, r_m = a_m)P(r_2 = a_2, \ldots, r_m = a_m)
\]

where

\[
P(Q_0 = 0 \mid r_2 = a_2, \ldots, r_m = a_m) = P \left( r_1 = -\frac{1}{\alpha_1} \left( \sum_{i=2}^{m} \alpha_i \cdot a_i \right) \right) = \frac{1}{q}.
\]

This implies that \( P(Q_0 = 0) = \frac{1}{q} \) and consequently we get

\[
p_{1,m} = \frac{1}{q}
\]

(2.3)

proving (2.2) for \( k = 1 \).

To estimate \( p_{k,m} \) for larger values of \( k \), we obtain a recursive relation for \( p_{k,m} \) as follows. Let \( Q = Q(x_1, \ldots, x_m) \) be a homogenous polynomial of degree \( k \) and write

\[
Q = x_1 \cdot R + S,
\]

(2.4)

where \( R = R(x_2, \ldots, x_m) \) is a homogenous polynomial of degree \( k-1 \) in the variables \( x_2, \ldots, x_m \) and \( S = S(x_2, \ldots, x_m) \) is a homogenous polynomial of degree \( k \) in the variables \( x_2, \ldots, x_m \). If \( Q_0 := Q(r_1, \ldots, r_m) = 0 \) then either \( R_0 := R(r_2, \ldots, r_m) = 0 \) and \( S_0 := S(r_2, \ldots, r_m) = 0 \) or \( R_0 \neq 0 \) and so we have

\[
P(Q_0 = 0) = P(R_0 = 0, S_0 = 0) + P(R_0 \neq 0, Q_0 = 0).
\]

(2.5)

The term

\[
P(Q_0 = 0, R_0 \neq 0) = \sum_{a \in \mathbb{F}_q \setminus \{0\}} \sum_{b \in \mathbb{F}_q} P(Q_0 = 0 \mid R_0 = a, S_0 = b)P(R_0 = a, S_0 = b),
\]

(2.6)

and so for any \( a \in \mathbb{F}_q \setminus \{0\}, b \in \mathbb{F}_q \) we have from (2.4) that

\[
P(Q_0 = 0 \mid R_0 = a, S_0 = b) = P \left( r_1 = -\frac{b}{a} \right) = \frac{1}{q}.
\]

(2.7)

Consequently

\[
P(R_0 \neq 0, Q_0 = 0) = \frac{1}{q}P(R_0 \neq 0).
\]

(2.8)
To obtain the lower bound in (1.2), we use (2.5) and (2.8) to get that
\[
P(Q_0 = 0) \leq P(R_0 = 0) + \frac{1}{q} P(R_0 \neq 0)
\]
\[
= \frac{1}{q} + \left(1 - \frac{1}{q}\right) P(R_0 = 0)
\]
\[
\leq \frac{1}{q} + \left(1 - \frac{1}{q}\right) p_{k-1,m-1}.
\]

From (2.1) we therefore get that
\[
p_{k,m} \leq \frac{1}{q} + \left(1 - \frac{1}{q}\right) p_{k-1,m-1}.
\]

Letting \(\beta = 1 - \frac{1}{q}\) and applying the recursion (2.9) repeatedly \(i\) times, we get
\[
p_{k,m} \leq (1 - \beta)(1 + \beta + \beta^2 + \ldots + \beta^i) + \beta^{i+1} \cdot p_{k-1-i,m-1-i}
\]
\[
= (1 - \beta^{i+1}) + \beta^{i+1} \cdot p_{k-1-i,m-1-i} \tag{2.10}
\]

Setting \(i = k - 2\) in (2.10) and using (2.3) we then get (2.2). This obtains the upper bound in (1.2).

For the lower bound in (1.2), we again use (2.5) and get
\[
P(Q_0 = 0) \geq P(R_0 \neq 0, Q_0 = 0) = \frac{1}{q} P(R_0 \neq 0)
\]
by (2.8). Thus
\[
P(Q_0 = 0) \geq \frac{1}{q} (1 - p_{k-1,m-1}) \geq \frac{1}{q} \left(1 - \frac{1}{q}\right)^{k-1}
\]
by (2.2) and this proves the lower bound in (1.2).

### 3. Applications of Theorem 1

#### Perfect matching

In this subsection, we illustrate an application for the upper bound in (1.2). We recall the polynomial identity testing procedure to determine the presence of a perfect matching in a bipartite graph. Let \(G = (X \cup Y, E)\) be a bipartite graph where \(#X = #Y = k\). A matching in \(G\) is a set of edges that share no endpoint. A perfect matching in \(G\) is a set of edges \(M \subseteq E\) such that every vertex is contained in exactly one edge of \(M\).

Let \(A = [a_{i,j}]\) be a matrix whose rows are indexed by vertices in \(X\) and whose columns are indexed by vertices in \(Y\) with entries
\[
a_{u,v} := \begin{cases} x_{u,v} & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases} \tag{3.1}
\]
where \( \{x_{u,v}\} \) are distinct variables and \((u, v)\) denotes the edge with endvertices \( u \) and \( v \). It is well known (Theorem 7.3, (3)) that the determinant of \( A \) is zero if and only if the graph \( G \) does not have a perfect matching.

Suppose \( G \) has a perfect matching and we would like to use the above determinant criterion to devise a randomized algorithm for determining whether \( G \) has a perfect matching or not. The Schwartz-Zippel procedure is as follows. Assuming that \( \det(A) \) is a polynomial of degree \( k \), we first choose a field size \( q \geq k + 1 \). Fixing such a \( q \), we then choose \( \{x_{u,v}\} \) independent and identically distributed (i.i.d.) in \( \mathbb{F}_q \) and compute the random determinant \( \det(A) \). If \( \det(A) = 0 \) we say that \( G \) does not have a perfect matching else we say that \( G \) has a perfect matching.

From the Schwartz-Zippel lemma we get that the probability that \( \det(A) \) is zero is at most \( \frac{k}{q} < 1 \) strictly and so the probability that our decision is wrong is at most \( \frac{k}{q} \). To reduce the probability of a wrong decision, we run the above procedure \( n \) times using fresh independently random values for \( \{x_{u,v}\} \) each time. If we get \( \det(A) = 0 \) all the \( n \) times, we output “\( G \) has no perfect matching” else we output “\( G \) has a perfect matching”. The probability that our decision is wrong in this case is at most \( \left( \frac{k}{q} \right)^n \) which is small for all large \( n \), provided the field size \( q \geq k + 1 \).

Using Theorem 1 we now perform the above procedure using binary random variables. We first see that the determinant \( \det(A) \) of \( A \) is a homogenous polynomial of degree \( k \) in the variables \( \{x_{u,v}\} \). We then set \( x_{u,v} \) to be independent random binary values satisfying

\[
P(x_{u,v} = 0) = \frac{1}{2} = P(x_{u,v} = 1)
\]

and evaluate \( \det(A) \). If \( \det(A) = 0 \), we output the statement “\( G \) has no perfect matching”; else we output the statement “\( G \) has a perfect matching”. From (1.2), the probability that \( \det(A) \) equals zero is at most \( 1 - \frac{1}{2^k} \) and so the probability that we output the wrong decision is at most \( 1 - \frac{1}{2^k} \).

As before, to reduce the probability of a wrong decision, we run the above procedure \( n \) times using fresh independently random values for \( \{x_{u,v}\} \) each time. If we get \( \det(A) = 0 \) all the \( n \) times, we output “\( G \) has no perfect matching” else we output “\( G \) has a perfect matching”. Again using (1.2), we get that the probability that our decision is wrong is at most \( \left( 1 - \frac{1}{2^k} \right)^n \), which decays exponentially with \( n \).

Of course the tradeoff involved in the above procedure is the running time: our algorithm requires \( n \times \text{poly}(k) \) running time since we need to compute \( n \) determinants, each of size \( k \times k \). It would be interesting to design algorithms that require lesser computation and this would be a potential direction for future study.

**Common zeros**

We illustrate the lower bound in (1.2) with an example involving common zeros of polynomials. We have the following result.
Proposition 1 Suppose $Q_1, Q_2, \ldots, Q_N$ are $N$ homogeneous polynomials in the variables $x_1, \ldots, x_m$, each with degree $k$. If $N \geq 1 + q \cdot \left( \frac{q}{q-1} \right)^{k-1}$, then there are indices $1 \leq i \neq j \leq N$ such that $Q_i$ and $Q_j$ have a common zero. $\square$

We remark here that the Chevalley-Warning theorem (4) is used to describe conditions under which a set polynomials whose sum degree is smaller than the number of variables, all have more than one common root. In Proposition 1 we require that the number of polynomials is sufficiently large in order that at least two of the polynomials have a common root.

Proof of Proposition 1 We use the lower bound in (1.2) and prove by contradiction. Let $Z_i$ be the set of zeros of the polynomial $Q_i$. If the sets $\{ Z_i \}$ are all mutually disjoint, then by the lower bound in (1.2), the total number of elements in $\bigcup_{i=1}^{N} Z_i$ is at least $N \cdot q^{m-k} (q-1)^{k-1} > q^m$ strictly, a contradiction.

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