Control of the geometric phase and pseudo-spin dynamics on coupled Bose-Einstein condensates

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Abstract

We describe the behavior of two coupled Bose-Einstein condensates in time-dependent (TD) trap potentials and TD Rabi (or tunneling) frequency, using the two-mode approach. Starting from Bloch states, we succeed to get analytical solutions for the TD Schrödinger equation and present a detailed analysis of the relative and geometric phases acquired by the wave function of the condensates, as well as their population imbalance. We also establish a connection between the geometric phases and constants of motion which characterize the dynamic of the system. Besides analyzing the affects of temporality on condensates that differs by hyperfine degrees of freedom (internal Josephson effect), we also do present a brief discussion of a one specie condensate in a double-well potential (external Josephson effect).
I. INTRODUCTION

In recent years, concepts which has been restricted to foundation of quantum mechanics have been considerably enlarged by spreading out to different domains of physics. With the introduction of measurements such as separability \[1, 2, 3\] and concurrence \[4, 5\], and the wider understanding that entanglement, and so nonlocality, is at the core of many-body phenomena as quantum-phase transition \[6\], superconductivity \[7\] and Bose-Einstein condensation \[8\], we are considerably far from the time when entanglement and nonlocality were confined to fundamental aspects of quantum mechanics. On the other hand, the experimental techniques developed over the last decades for manipulating atom-field interaction have enabled the building of macroscopic atomic ensembles and the experimental verification of fundamental concepts in macroscopic scales \[9\]. It is worth mentioning the rapid growth of quantum information theory which has conferred to its basic ingredients — the phenomena of superposition of states and decoherence, entanglements and nonlocality — a great deal of advances towards the accomplishments of quantum logical devices.

Among the standard tools to generate and detect multipartite entanglements, experiments in Bose-Einstein condensates (BECs) in dilute gases have deepened our incursion towards the quantum nature of macroscopic systems. In particular, experiments with a trapped gas of \(^{87}\text{Rb}\) atoms with two different hyperfine sublevels prompt the engineering of a Josephson-like coupling between two condensates by a laser-induced Raman transition \[10, 11\]. Such “internal” Josephson effect \[12\] mediates intraspecies collisions apart from interspecies one. These atom-atom interactions empowered the investigation of the dynamics of the relative phase of coupled condensates \[13\] and Rabi oscillations \[14\] on macroscopic systems. Moreover, precise measurements of scattering lengths has also been accomplished \[15\], aiming to quantify properly the non-linear dynamics associated with collisions. In the two-mode approximation, the coupled condensates has been employed to investigate entanglement dynamics \[8, 16, 17\] and the possibility to prepare, control and detect macroscopic superposition states \[18, 19, 20\]. Beyond this achievements, the analysis of the Josephson effect in this two-mode exactly soluble model may provide a clue for the examination of macroscopic coupling arising in less tractable form of the general theory of BECs \[12\].

Whereas in real experiments the trap potential may be considered to be a time-
independent function, excepting for small fluctuations, time-varying scattering lengths are usually produced through Feshbach resonances while, as pointed out in Ref. [12], the amplitude and phase of the laser field may vary in time. In this connection, the present paper is devoted to the TD version of the two-mode Hamiltonian (TMH), where the effective frequency of the trap potential for both atomic species are TD functions, as well as the Rabi frequency and the scattering lengths. A similar approach was employed in Refs. [21, 22] where, however, only the phase of the external field inducing the Raman transition was assumed to be a TD slowly varying function. Instead, our treatment considers time dependence of all Hamiltonian parameters, focusing on two particular subjects: the analyzes of geometric phase acquired by wave function of the whole system and the control of the dynamics of pseudo-spin states governed by the TD TMH. Starting from Bloch states, whose preparation is achieved by applying a laser pulse to atoms condensed in a single hyperfine level [15], we demonstrate that its evolution, visualized as a vector on the Bloch sphere, can be used to control the geometric phase and the population imbalance following from the whole wave function of the condensates. Our treatment also permits a detailed analyzes of the relative phase between the condensed states.

In Ref. [23], the authors studied the dynamics of a strongly driven two-coupled BECs in two spatially localized modes of a double-well potential, where the tunneling coupling between the two modes is periodically modulated. In our work we also study the TD TMH associated to the “external” Josephson effect, analyzing its differences with relation of the “internal” Josephson effect.

Similarly to the above mentioned fundamental phenomena, the geometric phase has overtaken its striking rule on fundamental physics to widening our understanding of phenomena as quantum Hall effect [24, 25, 26] and for the implementation of fault-tolerant quantum gates [27]. After its discovery by Berry on adiabatic processes [28], it has been generalized to nonadiabatic [29], noncyclic [30] and nonunitary [31, 32] quantum evolutions. Recently, it has been investigated in different areas of physics, ranging from BECs [21] and cavity quantum electrodynamics [33, 34] to condensed matter [35] and quantum information theory [27]. In particular, the Berry phase of mesoscopic spin in Bose-Einstein condensates, induced by a TD slowly varying driven field, has been investigated under the TMH [21, 22]. In our treatment, the evolution of the geometric phase of this mesoscopic system is evaluated in a more general scenario, where all the Hamiltonian parameters are assumed to be TD.
The paper is organized as follows. In Sec. II we introduce and solve the Schrödinger equation associated to the TD TMH, presenting the evolution operator. The dynamics of BECs for initial Bloch states is analyzed in Sec. III, where we show that they remain as Bloch states apart from a global phase factor accounting for the elastic collision terms. The geometric phase acquired by the state vector of the system is presented in Sec. IV and a detailed analyzes of its time evolution is found in Sec. V for different regimes of the parameters. In Sec. VI we take the problem of a TMH from a different perspective, considering the external Josephson effect instead of the internal one. Finally, Sec. VII is devoted to our concluding remarks.

**II. THE TIME-DEPENDENT TMH**

Under the two-mode approximation, where the quantum field operators $\Psi_a = \varphi_a (r, t) a$ and $\Psi_b = \varphi_b (r, t) b$ are restricted to the fundamental states $\varphi_\ell (r, t) (\ell = a, b)$ \[18, 36\], the coupled Bose-Einstein condensates are described by the TD Hamiltonian ($\hbar = 1$)

$$H(t) = \sum_{\ell = a, b} \left[ \omega_\ell(t) \ell^\dagger \ell + \gamma_\ell(t) \ell^\dagger \ell \ell^\dagger \ell \right] + \gamma_{ab}(t) a^\dagger b^\dagger b - g(t) \left( e^{-i\delta(t)} a^\dagger b + e^{i\delta(t)} ab^\dagger \right),$$

(1)

where $a$ and $b$ are standard bosonic annihilation operators, associated with condensation in hyperfine levels $|2, 1\rangle$ and $|1, -1\rangle$, respectively \[18, 19, 37\]. The phase $\delta(t)$ is associated to the detuning $\Delta(t)$ from the Raman resonance between the atomic transition $|2, 1\rangle \leftrightarrow |1, -1\rangle$, which may be a TD function (by varying the laser frequency), through the expression $\delta(t) = \int_{t_0}^{t} \Delta(\tau) d\tau + \delta_0$. The TD trap frequencies $\omega_\ell$, the interspecies and intraspecies collision parameters $\gamma_{ab}$ and $\gamma_\ell$, and the Rabi frequency $g$, follow from

$$\omega_\ell(t) = \int d^3r \varphi^*_\ell (r, t) \left[ -\frac{1}{2m} \nabla^2 + V_\ell (r, t) \right] \varphi_\ell (r, t),$$

(2a)

$$\gamma_\ell(t) = \frac{4\pi A_\ell(t)}{2m} \int d^3r |\varphi_\ell (r, t)|^4,$$

(2b)

$$\gamma_{ab}(t) = \frac{4\pi A_{ab}(t)}{m} \int d^3r |\varphi_a (r, t) \varphi_b (r, t)|^2,$$

(2c)

$$g(t) = \frac{\Omega(t)}{2} \int d^3r \varphi^*_a (r, t) \varphi_b (r, t),$$

(2d)

where $m$ is the atomic mass. We assume that the time dependence of the trap potential $V_\ell (r, t)$ is generated by adiabatically varying the trapping magnetic field. Such adiabatic
variation of the trapping field has been assumed to ensure the validity of the two-mode approximation. The time-varying scattering lengths \( A_{ab}(t) \) and \( A_\ell(t) \), are accomplished via Feshbach resonances, by tuning a bias magnetic field \([38]\). Finally, as mentioned above, in real experiments with atomic BECs the Rabi frequency may be a time-varying function since the amplitude and phase of the pumping fields may vary in time \([12]\).

Except for the Josephson-like coupling, the Fock states are eigenstates of all the terms in Hamiltonian (1). Thus, in order to get rid of this TD coupling in (1), we consider a transformation with the unitary operator

\[
V(t) = \exp \left[ \frac{r(t)}{2} \left( e^{i\phi(t)} ab^\dagger - e^{-i\phi(t)} a^\dagger b \right) \right],
\]

(analogous to that defined in Ref. \([22]\)) to obtain the transformed Hamiltonian

\[
\mathcal{H}(t) = V^\dagger HV - iV^\dagger \partial_t V = \sum_{\ell=a,b} \tilde{\omega}_\ell(t) n_\ell + \mathcal{H}_{el}(t) + \mathcal{H}_{inel}(t),
\]

where \( n_\ell = \ell^\dagger \ell \) is the number operator associated to each condensate having effective frequency

\[
\tilde{\omega}_\ell(t) = \omega_\ell(t) + (2\delta_\ell - 1) g(t) \cos \left[ \phi(t) - \delta(t) \right] \tan \left[ r(t)/2 \right].
\]

In the framework associated to the transformation \([33]\), the system ends up with an inelastic collision term apart from the elastic one already present in (1). The Hamiltonians accounting for such interactions, also weighted by the TD function \( \Lambda(t) = \gamma_a(t) + \gamma_b(t) - \gamma_{ab}(t) \), are given by

\[
\mathcal{H}_{el}(t) = \left\{ \gamma_a(t) \cos^2 \left[ r(t)/2 \right] + \gamma_b(t) \sin^2 \left[ r(t)/2 \right] - \frac{\Lambda(t)}{4} \sin^2 \left[ r(t) \right] \right\} \left( a^\dagger \right)^2 a^2
\]

\[
+ \left\{ \gamma_a(t) \sin^2 \left[ r(t)/2 \right] + \gamma_b(t) \cos^2 \left[ r(t)/2 \right] - \frac{\Lambda(t)}{4} \sin^2 \left[ r(t) \right] \right\} \left( b^\dagger \right)^2 b^2
\]

\[
+ \left\{ \gamma_{ab}(t) + \Lambda(t) \sin^2 \left[ r(t) \right] \right\} a^\dagger ab^\dagger b,
\]

\[
\mathcal{H}_{inel}(t) = \left\{ \frac{[\gamma_b(t) - \gamma_a(t)]}{2} \sin \left[ r(t) \right] - \frac{\Lambda(t)}{4} \sin \left[ 2r(t) \right] \right\} e^{-i\phi(t)} \left( a^\dagger \right)^2 ab
\]

\[
+ \left\{ \frac{[\gamma_b(t) - \gamma_a(t)]}{2} \sin \left[ r(t) \right] + \frac{\Lambda(t)}{4} \sin \left[ 2r(t) \right] \right\} e^{-i\phi(t)} a^\dagger b^\dagger b^2
\]

\[
+ \frac{\Lambda(t)}{4} \sin^2 \left[ r(t) \right] \left( e^{-i\phi(t)} a^\dagger b \right)^2 + h.c..
\]
The form of Hamiltonian (4) is established provided that the TD parameters \( r(t) \) and \( \phi(t) \) satisfy the coupled differential equations

\[
\begin{align*}
\dot{r}(t) &= 2g(t) \sin [\phi(t) - \delta(t)], \\
\dot{\phi}(t) &= \omega(t) + 2g(t) \cot [r(t)] \cos [\phi(t) - \delta(t)],
\end{align*}
\]

(7a)

(7b)

where

\[
\omega(t) = \omega_a(t) - \omega_b(t).
\]

(8)

The expression (8) represents an effective frequency for the system composed by the two-mode condensate, which plays an important role in the solutions of the characteristic equations (7). In the Appendix we present a comprehensive analyzes of the analytical solutions of the Eqs. (7) for the on- and off-resonant regimes which are defined by comparing the effective frequency \( \omega(t) \) with the detuning between the laser field and Raman transition \( \Delta(t) \). The on-resonant regime, where \( \Delta(t) = \omega(t) \), implies that the detuning from Raman transition must equals the effective frequency of the two-mode condensate. Otherwise, we have the off-resonant regime, where \( \Delta(t) = \omega(t) - \varpi \), \( \varpi \) being some constant.

Similar coupled differential equations were obtained by Smerzi et al. \[39\] in a semi-classical treatment of the double-well problem, and by Chen et al. \[22\] in a full quantum approach of the BECs in the two-mode approximation. In both references all the parameters in their Hamiltonians are constants, except for \( \delta(t) \) which, in Ref. \[22\], is an adiabatically time-varying parameter.

After the experiments by Hall and co-workers with \(^{87}\)Rb \[15\], where the scattering lengths satisfy the relation \( A_a : A_{ab} : A_b = 1.03 : 1 : 0.97 \), and consequently \( \gamma_a \simeq \gamma_b \simeq \gamma_{ab}/2 \) (assuming spatial Gaussian function), a number of papers have driven attention to this particular case \[22, 40\] whose Schrödinger equation is exactly soluble. However, if the proportion 1.03 : 1 : 0.97 is broken, the system is not exactly integrable, but admits approximated solutions as shown in Refs. \[21, 22\]. In our paper we are concerned with specific solutions of the characteristic equations (7), under the rotating-wave approximation, which turn negligible the contribution of the inelastic interactions compared to the elastic one. This is done by substituting the solutions for \( r(t) \) and \( \phi(t) \), obtained in the Appendix, into Eqs. (5) and (6), and rewriting Hamiltonian (4) in the interaction picture. Thus, after a time average of the TD parameters appearing in this Hamiltonian, we analyze the conditions leading to the
effective interaction

\[ H_{\text{eff}}(t) \simeq \sum_{\ell=a,b} \tilde{\omega}_\ell(\tau) n_\ell + H_{el}(\tau). \]  

(9)

By adopting this procedure, where the inelastic interactions becomes despicable, we get the evolution operator

\[ U(t, t_0) = \exp \left( -i \int_{t_0}^t H_{\text{eff}}(\tau) d\tau \right) \]

and, consequently, a prepared state \(|\psi(t_0)\rangle\) evolves according to Hamiltonian (1) as

\[ |\psi(t)\rangle = V(t) U(t, t_0) V^\dagger(t_0) |\psi(t_0)\rangle. \]  

(10)

III. DYNAMICS OF BECS FOR INITIAL BLOCH STATES

Following Arecchi et al. [41] and Dowling et al. [42], we recall that the Bloch states (BS), also called atomic coherent states, spanned in the Dicke basis \(|j, m\rangle\), where \(j = N/2\) and \(|m| \leq j\), are obtained through a specific rotation on the reference state \(|j, j\rangle\),

\[ |\alpha, \beta\rangle = e^{\frac{\alpha}{2} (e^{i\beta} J_+ - e^{-i\beta} J_-)} |j, j\rangle, \]  

(11)

where \(N\) is the number of condensed particles, \(J_+, J_-(\text{together with } J_z)\), are the generators of the \(su(2)\) algebra, and \((\alpha, \beta)\) are the polar and the azimuthal angles, respectively, defined on the Bloch sphere. The Heisenberg angular-momentum uncertainty relation for the BS reduces to

\[ \langle(\Delta J_{x'})^2\rangle \langle(\Delta J_{y'})^2\rangle = \frac{1}{4} |\langle J_{z'}\rangle|^2, \]  

(12)

with the mean values being calculated in a rotated coordinate system \(x', y', z'\), where \(z'\) is an axis in the \((\alpha, \beta)\) direction through the center of the Bloch sphere. Therefore, the Bloch vector is defined as the unit vector

\[ n = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha), \]  

(13)

in \(z'\)-axis. Using the Schwinger relations

\[ J_x = \frac{1}{2} (a^\dagger b + ab^\dagger), \]  

(14a)

\[ J_y = \frac{1}{2i} (a^\dagger b - ab^\dagger), \]  

(14b)

\[ J_z = \frac{1}{2} (a^\dagger a - b^\dagger b). \]  

(14c)

with \(J_\pm = J_x \pm iJ_y\) and the basis states \(\{|j, m\rangle = |N/2, (N_a - N_b) / 2\}\) \equiv \(\{|N_a\} |N_b\rangle \equiv |N_a, N_b\rangle\), where \(N_a\) and \(N_b\) \((N = N_a + N_b)\) stand for the number of atoms in
the condensates, such that $|j,j\rangle \Leftrightarrow |N,0\rangle$, it is straightforward to check that the BS can be defined through bosonic operators as

$$|\alpha,\beta\rangle = \frac{1}{\sqrt{N!}} \left[ \cos \left( \frac{\alpha}{2} \right) a^\dagger + \sin \left( \frac{\alpha}{2} \right) e^{i\beta} b^\dagger \right]^N |0,0\rangle . \quad (15)$$

This state has a well-defined relative phase $\beta$ between the two bosonic modes.

Now, it is evident from relations (14) that the unitary transformation $V(t)$ turns to be exactly the rotation operator $e^{iN\phi(t) (\hat{J}_x - e^{-i\phi(t)} \hat{J}_y)}$, if one considers $r(t) = \alpha$ and $\phi(t) = \beta$. Therefore, it is easy to demonstrate through Eq. (10) that an initial BS $|\psi(t_0)\rangle = |\alpha_0,\beta_0\rangle = |r_0,\phi_0\rangle$ evolves to another BS

$$|\psi(t)\rangle = \frac{e^{-iN\varphi_N(t)}}{\sqrt{N!}} \left[ \cos \left( \frac{r(t)}{2} \right) a^\dagger + \sin \left( \frac{r(t)}{2} \right) e^{i\varphi(t)} b^\dagger \right]^N |0,0\rangle ,$$

apart from the global phase factor $e^{-iN\varphi_N(t)}$, where

$$\varphi_N(t) = \int_{t_0}^{t} \left\{ \tilde{\omega}_a(\tau) + (N - 1) \left[ \gamma_a(\tau) \cos^2 [r(\tau)/2] 
+ \gamma_b(\tau) \sin^2 [r(\tau)/2] - \frac{\Lambda(\tau)}{4} \sin^2 [r(\tau)] \right] \right\} d\tau. \quad (17)$$

The relative phase between the condensates $\phi(t)$ is associated to the mean values $\langle J_x(t) \rangle = N \sin [r(t)] \cos [\phi(t)] / 2$ and $\langle J_y(t) \rangle = N \sin [r(t)] \sin [\phi(t)] / 2$, whereas $r(t)$ is related to the population imbalance, $\Delta N(t) = \langle N_a - N_b \rangle = 2 \langle J_z(t) \rangle$, through the relation

$$\Delta N(t) = N \cos [r(t)]. \quad (18)$$

We stress that according to the evolution operator, Eq. (10), the evolved state (16) remains a BS (apart from a global phase factor), since $\alpha_0 = r_0$ and $\beta_0 = \phi_0$. Note that the collision parameters are restricted to the global phase $e^{-iN\varphi_N(t)}$, being irrelevant to the analyzes developed below of the nonadiabatic geometric phases acquired by the state vector. On the other hand, collisions become relevant when considering other initial states as the product of Glauber’s coherent states $|\alpha_0\rangle |\beta_0\rangle$ instead of the BS [40].

IV. GEOMETRIC PHASES OF THE BS

To study the evolution of the geometric phase in the two-mode BECs we use the kinematic approach developed by Mukunda and Simon [43], where the geometric phase $\phi_G$ is obtained
as the difference between the total phase \( \phi_T(t) = \arg(\langle \psi(t_0) | \psi(t) \rangle) \) and the dynamical phase \( \phi_D(t) = -i \int_{t_0}^{t} \langle \psi(\tau) | \frac{\partial}{\partial \tau} | \psi(\tau) \rangle d\tau \), resulting in

\[
\phi_G(t) = \arg(\langle \psi(t_0) | \psi(t) \rangle) + i \int_{t_0}^{t} \langle \psi(\tau) | \frac{\partial}{\partial \tau} | \psi(\tau) \rangle d\tau.
\] (19)

The expressions for \( \phi_G(t) \) and \( \phi_D(t) \) as function of the system parameters are, respectively,

\[
\phi_G(t) = N \arg\{ \cos (r_0/2) \cos [r(t)/2] + e^{i[\phi(t)-\phi_0]} \sin (r_0/2) \sin [r(t)/2] \}
- \frac{N}{2} \int_{t_0}^{t} \phi(\tau) \{ 1 - \cos [r(\tau)] \} d\tau,
\] (20)

and

\[
\phi_D(t) = -N [\varphi_N(t) - \varphi_N(t_0)] + \frac{N}{2} \int_{t_0}^{t} \phi(\tau) \{ 1 - \cos [r(\tau)] \} d\tau,
\] (21)

where \( \varphi_N(t) \) was given in Eq. (17). In Ref. [44], the authors also obtain Eq. (20), through the Gross-Pitaevskii equation, for the geometric phase acquired by the wave function of a BEC in the double-well problem, under the two-mode approximation.

The time evolution of the BS can be followed on the Bloch sphere through the vector

\[
n(t) = (\sin[r(t)] \cos[\phi(t)], \sin[r(t)] \sin[\phi(t)], \cos[r(t)])
\]

for the different solutions \( r(t) \) and \( \phi(t) \) presented in the Appendix. To follow such evolution and, consequently, to analyze its geometric phase \( \phi_G(t) \), we must estimate the integrals in Eq. (2), i.e., the typical values for trap frequencies, Josephson-like coupling, intraspecies, and interspecies collision rates. For the sake of simplicity we model the effective frequency for both atomic species as harmonic trap potentials where the TD distribution for each condensate \( \varphi_\ell (r, t) \) can be approximated by a stationary Gaussian function such that

\[
\varphi_\ell(r) = \left( \frac{1}{2\pi x_\ell^2} \right)^{3/4} e^{-r^2/4x_\ell^2},
\] (22)

where \( x_\ell = \sqrt{\hbar/2m\omega_\ell} \) stands for the position uncertainty in each harmonic oscillator ground state [23]. With this assumption the integrals in Eq. (2) are immediately estimated using typical physical parameters of the experiments with \(^{87}\text{Rb} \) atoms [15, 45, 46]: \( m = 1.4 \times 10^{-25} \) Kg, \( \omega_\ell \sim 10^{-2} \) Hz, \( A_\ell \sim 5 \) nm, and \( \Omega \sim 10^3 \) Hz. To obtain some insights of the pseudo-spin dynamics under the TD Hamiltonian parameters, we consider the trap and Rabi frequencies as harmonic functions, oscillating around the typical constant values, as follow

\[
\omega_\ell(t) = \omega_{\ell 0} + \tilde{\omega}_\ell \sin (\chi_\ell t + \xi_\ell),
\] (23a)

\[
g(t) = g_0 + \tilde{g} \sin(\mu t),
\] (23b)
with the parameters $\omega_0, \tilde{\omega}_\ell, \chi_\ell, \xi_\ell, g_0, \tilde{g}$, and $\mu$ being constant. Since the elastic collisions contribute only to a global phase factor, they will be assumed as the standard constant parameters in the literature

$$\gamma_\ell = \frac{4\pi A_\ell}{2m},$$

$$\gamma_{ab} = \frac{4\pi A_{ab}}{m}. \quad (24a)$$

$$\gamma_{ab} = \frac{4\pi A_{ab}}{m}. \quad (24b)$$

V. NONADIABATIC GEOMETRIC PHASES AND PSEUDO-SPIN DYNAMICS IN BECS

In this section we present a detailed study of the geometric phase acquired by the whole wave function of the two-mode condensates. To this end, we plot the time evolution of the geometric phase (20), and analyze its behavior through the evolution of Bloch vector (a map of the wave function of the BECs on Bloch sphere). This procedure allows for a better understanding and visualization of the concept of geometric phase for open trajectories introduced by Samuel and Bhandari [30]. In particular, we are interested in the dependence of the geometric phase on the constant of motions coming from the solutions of the characteristic equations (7) and also on the time dependence of the Hamiltonian parameters. In spite of the general solutions presented in the Appendix for these equations, we have assumed, for the analysis developed below, $N = 1$, $\delta_0 = 0$, and $g_0 = 625\pi$ Hz.

A. Solutions for $r$ constant

Before analyzing the geometric phases for the on- and off-resonant solutions, it is instructive to present their evolutions for the simple case where the parameter $r$ is kept constant while $\phi(t)$ obeys Eq.(37) (since $g = 0$). In this case, the expression for the geometric phase coming from Eq. (20), becomes

$$\phi_G(t) = N \left\{ \arg \left\{ \cos^2 \left( \frac{r}{2} \right) + e^{-i[\phi(t)-\phi_0]} \sin^2 \left( \frac{r}{2} \right) \right\} - \frac{1 - \cos r}{2} [\phi(t) - \phi_0] \right\}. \quad (25)$$

In Fig.1 the absolute value for $\phi_G(t)$ is plotted against the dimensionless $\tau = \omega_{a0}t$, assuming typical values $\omega_{a0} = 2\omega_{b0} = 62.5\pi$ Hz. For $r = \pi/2$, with the Bloch vector standing...
on the equatorial plane, and \( \tilde{\omega}_a = \tilde{\omega}_b = 0 \), the geometric phase evolves by jumps, as indicated by the thick solid line. These jumps occur every time the relative phase \( \phi \) connecting the final to the initial Bloch vectors equals \( (2n + 1)\pi \), \( n \) being an integer. The jump discontinuities occur because there are an infinite number of small geodesic-lengths connecting the vectors extremities, rendering the geometric phase undefined \[48\]. On the other hand, we observe that before jumping to \( \phi_G(\tau) = \pi \), i.e., for \( \tau < 2\pi \), the geometric phase remain null since the small geodesic-length connecting the extremities equals the Bloch-vector trajectory itself. As soon as the Bloch vector acquires a relative phase larger than \( \pi \), the small geodesic-length connecting the extremities completes a loop over the equator, making the acquired geometric phase proportional to \( nN\pi \), where \( n \), as defined above, turns out to be the winding number, i.e., the number of loops around the \( z \)-axis of the sphere.

The same interpretation given above for the geometric phase holds for the two other curves obtained for \( r = \pi/2.1 \), except that the jump discontinuities are substituted by high-slope curves around the points where \( \phi = (2n + 1)\pi \). Moreover, the net effect coming from the TD parameters of the Hamiltonian is to delay or advance the sequential increments of the relative phase \( \phi \) and, consequently, of the geometric phase, as observed from Fig.1. The solid line, obtained for \( r = \pi/2.1 \) and \( \tilde{\omega}_a = \tilde{\omega}_b = 0 \), shows that the increments of the geometric phase, besides being smaller, present the same rate of variation when compared to the case \( r = \pi/2 \). When the trap frequencies are oscillating functions, with \( \tilde{\omega}_a = \tilde{\omega}_b = \omega_{a0}/4 \), \( \chi_a = \chi_b = \omega_{a0}/2 \), \( \xi_a = 0 \), and \( \xi_b = \pi/2 \), the time-dependence shown by the dashed line, results in the oscillations of the time intervals between the increments of the geometric phase.

To better visualize the above discussion about geometric phases acquired in open trajectories, in Fig.2 we plot the evolution of the Bloch vectors for the cases \( r = \pi/2 \) and \( r = \pi/2.1 \), with \( \tilde{\omega}_a = \tilde{\omega}_b = 0 \), considering the same time interval \( \tau = 4\pi \). The black and grey vectors indicate the coincident positions of the initial and final Bloch vectors, for the cases \( r = \pi/2.1 \) and \( r = \pi/2 \), respectively, after a complete rotation around the sphere whose directions are indicated by the arrows. The trajectories described by the black and grey vectors are indicated by solid and dashed curves, respectively. Evidently, the geometric phase acquired during the evolution of the Bloch vector in the case \( r = \pi/2.1 \) (the solid angle comprehended by the semi-hemisphere above the solid circumference), is smaller than that for the case \( r = \pi/2 \) (the solid angle corresponding to the north hemisphere, equal to \( 2\pi \)).
The solution with \( r \) constant means steady population imbalance \( \Delta N \), whereas the relative phase \( \phi(t) \), another parameter examined by experimentalists and necessary to completely define the BS, is a linear function of time when \( \tilde{\omega}_a = \tilde{\omega}_b = 0 \) or an oscillating function when \( \tilde{\omega}_a = \tilde{\omega}_b = \omega_{a0}/4 \). Note that the dynamics of the population imbalance and the relative phase may be followed through the projection of the Bloch vector trajectory on \( z \)-axis and \( x-y \) plane, respectively.

**B. On-resonant solution**

As indicated in the Appendix, through the constant of motion \( C = \sin[r(t)] \cos[\phi(t) - \delta(t)] \) we obtain the solution of the characteristic equations (7) in the on-resonant regime where \( \Delta(t) = \omega(t) \). All the possible trajectories in the portrait space \( r(t) \times (\phi(t) - \delta(t)) \), are restrained to the level curves obtained as projection of the surface plotted in Fig.3, which follows from \( C \).

To better understand the geometric phase acquired by the state vector \( |\Psi(t)\rangle \) in the on-resonant solutions, we consider two different cases, \( \Delta = 0 \) and \( \Delta \neq 0 \), and analyze its dependence on the constant \( C \), through the relation

\[
\phi_G(t) = N \arg \left\{ \cos \left( \frac{r_0}{2} \right) \cos \left( \frac{r(t)}{2} \right) + e^{-i[\phi(t) - \phi_0]} \sin \left( \frac{r_0}{2} \right) \sin \left( \frac{r(t)}{2} \right) \right\} \\
- \frac{N}{2} \int_{t_0}^{t} dt' \left\{ \omega(t') \left\{ 1 - \cos \left[ \frac{r(t')}{2} \right] \right\} + \frac{2Cg(t') \cos \left[ \frac{r(t')}{2} \right]}{1 + \cos \left[ \frac{r(t')}{2} \right]} \right\}.
\]

(26)

1. **The case \( \Delta = 0 \)**

In Fig.4 we plot the evolution of the geometric phase against \( \tau = g_0 t \), considering \( \tilde{\omega}_a = \tilde{\omega}_b = \tilde{g} = 0 \). The thick solid line on the abscissa axis corresponds to the choice \( r_0 = \pi \) and \( \phi_0 = \pi/2 \), leading to \( C = 0 \), under which the geometric phase is null or undefined as indicated by the open dots over the abscissa axis. Note that for \( r_0 = \pi \) we get, at \( t = 0 \), an undetermined equation (31b) for \( \phi(t) \). To circumvent such indetermination we impose on Eq.(30) the constraint \( \phi(t) - \delta(t) = (2n + 1)\pi/2 \) over any time interval, to determine \( \phi(t) \) independently of Eq.(31b). Since for \( \Delta = 0 \) it follows that \( \delta(t) = \delta_0 \), implying that \( \phi(t) = \delta_0 + (2n + 1)\pi/2 \), the geometric phase for the case \( C = \Delta = 0 \) simplifies to \( \phi_G(t) = N \arg \left\{ \cos \left[ \frac{(r(t) - r_0)}{2} \right] \right\} \) and, consequently, \( \phi_G(t) \) is null for \( |r(t) - r_0| \leq \pi \).
and undefined for \(|r(t) - r_0| = \pi\). Still in Fig. 4, the solid and dashed lines, associated to the pairs \((r_0, \phi_0) = (\pi/5, \pi/4)\) and \((\pi/4, 3\pi/10)\), respectively, correspond to the same constant \(C \simeq 0.41\). These curves exhibits similar behaviors due to the fact that, with the same constant \(C\), they present the same trajectory on the portrait space of Fig.3, despite starting from different initial conditions. The dotted and dashed-dotted lines, associated to the pairs \((\pi/3, 0)\) and \((\pi/3, \pi)\), and corresponding to the constants \(C \simeq 0.87\) and \(C \simeq -0.87\), respectively, are symmetric around the abscissa axis \(\tau\). Such property of symmetry reflection of the geometric phase \(\phi_G \to -\phi_G\), is consequence of the change \(\phi_0 \to \phi_0 \pm \pi\), implying that \(C \to -C\). It is worth noting that the larger the absolute value of \(C\) the smaller the acquired geometric phase and vice-versa.

In Fig.5 we plot the evolution of the Bloch vectors coming from the on-resonant solution with the initial conditions \((\pi, \pi/2)\) and \((\pi/3, 0)\) corresponding to \(C = 0\) and \(C \simeq 0.87\), whose initial and final positions are represented by the black and grey vectors, respectively. As in Fig.2, we consider the evolution of both vectors during the same time interval \(\tau = \pi\). Through the solid line trajectory described by the black vector, which oscillates between the north and south poles, it is straightforward to conclude that the geometric phase is null during the whole time evolution, except when the vector reaches the north pole, where the geometric phase becomes undetermined. As the dashed trajectory of the grey vector is not restricted to a meridian, as in the case \(C = 0\), the geometric phase acquired is evidently non-null.

2. The case \(\Delta \neq 0\)

As we are interested in the dependence of the geometric phase on constant \(C\) and, now, on the effective frequency of the two-mode condensate \(\Delta(t) = \omega(t)\), we consider all the parameters of the Hamiltonian being time-independent, \(\tilde{\omega}_a = \tilde{\omega}_b = \tilde{g} = 0\), except for \(\delta(t) = \delta_0 + \int_0^t \omega(t') dt'\). In Fig.6 we plot the geometric phase against \(\tau = g_0 t\) for different initial conditions \((r_0, \phi_0)\) and effective frequencies \(\Delta\). As indicated by the thick solid line associated to the initial conditions \((\pi, \pi/2)\), corresponding to \(C = 0\), with \(\omega_a = 2\omega_b = g_0/10\), the geometric phase is not null, differently from the case \(\Delta = 0\). The discontinuity exhibited by this curve follows from the arg function, whose characteristic jumps occurs whenever \(\tau_n = [(2n - 1) \pi g_0]/2\omega\), \(n\) being a positive integer. As indicated by the solid and dashed
lines, the property of symmetry reflection of the geometric phase in the abscissa axis still follows when changing, simultaneously, \( \phi_0 \rightarrow \phi_0 \pm \pi \), and \( \omega \) to \(-\omega\). In fact, the solid line corresponds to the initial condition \((\pi/3, 0)\) with \( C \simeq 0.87 \) and \( \omega_a = 2\omega_b = g_0/10 \), while the dashed line corresponds to \((\pi/3, \pi)\), with \( C \simeq -0.87 \) and \( \omega_a = 2\omega_b = g_0/10 \). When the change \( \phi_0 \rightarrow \phi_0 \pm \pi \), are not followed by that \( \omega \rightarrow -\omega \), such a symmetry is not accomplished as indicated by the dotted line corresponding to the initial conditions \((\pi/3, \pi)\) with \( C \simeq -0.87 \) and, now, \( \omega_a = 2\omega_b = g_0/10 \).

To visualize the acquisition of the geometric phase we again return to the evolution of the Bloch vector. As observed from Fig.7, the solid trajectory described by the vector associated to \( C = 0 \), whose coincident initial and final positions are indicated by the black vector, leads to a finite solid angle and, consequently, a finite geometric phase (during the time evolution \( \tau = \pi \) considered for both cases presented). The control of this solid angle may be accomplished through the parameter \( \Delta \) — the larger \( \Delta \) the larger the solid angle and vice-versa — as demonstrated experimentally through the polarization vector of a photon undergoing a Mach-Zehnder interferometer [47]. The evolution of the grey vector, associated to \( C \simeq 0.87 \), exhibits a dashed trajectory where the initial and final positions are slightly different. As the time evolution proceeds, such trajectory leads to a geometric phase which easily exceed that of the case \( C = 0 \), as indicated in Fig.6.

The population imbalance \( \Delta N \) for the cases \( \Delta = 0 \) and \( \Delta \neq 0 \) is a time-oscillating function strictly dependent on the shape of Rabi frequency \( g(t) \), thus exhibiting a strong connection between the dynamics of population inversion in two-level systems and population imbalance of BECs. In fact, when written the Hamiltonian (1) through the quase-spin operators in Eqs.(13), we obtain, apart from the collision terms, a driven interaction. On the other hand, the relative phase \( \phi(t) \) depends also on the detuning \( \delta(t) \) besides \( g(t) \), as shown by Eqs.(31).

C. Off-resonant solution

To analyze the off-resonant solution, where the detuning \( \Delta(t) = \omega(t) - \varpi \) is controlled by adjusting the parameter \( \varpi \), we impose a constant Rabi frequency \( g(t) = g_0 \) which implies a constant of motion \( C = \eta \sin [r(t)] \cos [\phi(t) - \delta(t)] - \cos [r(t)] \). Similarly to the on-resonant case, all the possible trajectories for the off-resonant \( r(t) \) and \( \phi(t) \) are restrained to the
level curves of the surface following from $C$, presented in Fig.8. When $\eta = 2g_0/\varpi \gg 1$ it is verified that the constant $C$ reduces to that of the on-resonant solutions, unless for the multiplicative factor $\eta$, and the surface presented in Fig.8 also reduces to that of Fig.3. However, for $\eta \ll 1$ we obtain an approximated constant value of $r$. Finally, when $\eta \sim 1$, we obtain the surface whose level curves encapsulate all the possible trajectories in the portrait space $r(t) \times \phi(t) - \delta(t)$, as shown in Fig.8.

To study the effect of $\eta$ on the off-resonant geometric phase, given by

$$
\phi_G(t) = \arg \left\{ \cos \left( \frac{r_0}{2} \right) \cos \left[ \frac{r(t)}{2} \right] + e^{-i(\phi(t)-\phi_0)} \sin \left( \frac{r_0}{2} \right) \sin \left[ \frac{r(t)}{2} \right] \right\} - \frac{N}{2} \int_{t_0}^{t} dt' \left\{ \omega(t') \left\{ 1 - \cos \left[ \frac{r(t')}{} \right] \right\} + \varpi \cos \left[ \frac{r(t')}{2} \right] \frac{C + \cos \left[ \frac{r(t')}{2} \right]}{1 + \cos \left[ \frac{r(t')}{2} \right]} \right\}
$$

we plot in Fig.9 the evolution of $\phi_G(t)$ for different values of $\eta$, all starting from the point $(\pi/3, 0)$, with $\omega_a = 2\omega_b = g_0/10$ and $\tilde{\omega}_a = \tilde{\omega}_b = 0$. The dotted line, following from the case where $\eta = 40$, indicates a similar behavior to the corresponding case of the on-resonant solution (represented by the solid line in Fig. 6). For $\eta = 0.1$, the solid line shows that the geometric phase is a strongly oscillating function, a behavior that is better visualized through the evolution of the corresponding vector in Bloch sphere, Fig.10. Finally, when $\varpi \sim g_0$, as for $\eta = 2$, the geometric phase shows discontinuities, as indicated by the thick solid line, which turns out to be a signature of the off-resonant solution. We note that the property of reflection exhibited by the geometric phase coming from the on-resonant solution is also present here when substituting, simultaneously, $\phi_0 \to \phi_0 \pm \pi$, $\omega \to -\omega$, and $\eta \to -\eta$.

In Fig.10 we present the evolution of the Bloch vectors in the time interval $\tau \simeq 9\pi/10$, for the cases $\eta = 0.1$ and $\eta = 2$, both starting from the common point $(\pi/3, 0)$. The black vector for $\eta = 0.1$, presents a behavior limited to the north hemisphere, described by the solid trajectory, which exhibits periodic up and down motions on the parallels, responsible for the oscillations of the geometric phase showed in Fig.9. The grey vector for $\eta = 2$, by its turn, indicates a rather complicated dashed trajectory which descends to the south hemisphere and goes back to the north.

Besides depending on the Rabi frequency, as in the on-resonant case, the population imbalance for the off-resonant solution also depends on the detuning $\varpi$ which makes its mean value not null. The relative phase depends on $\delta(t)$, $g(t)$, and the detuning $\varpi$. 
D. Time-Dependent Effects on the Geometric Phases

The behavior of the geometric phase when both, the trap and Rabi frequencies are TD harmonic functions, as described by Eq.(23), is analyzed in Fig.11, where we plot $\phi_G(\tau)$ against $\tau = g_0t$, considering the same initial conditions $(\pi/3, 0)$ for all the curves. Starting with the resonant solution with $\Delta = 0$, we obtain the dotted curve for the parameters $\tilde{\omega}_a = \tilde{\omega}_b = 0$ and $\tilde{g} = \mu = g_0$, to be compared with the dotted curve of Fig.4. We observe that both dotted curves are very close to each other, with the the increasing rate of the geometric phase being modulated by the TD Rabi frequency in Fig.11. The solid line corresponds to the on-resonant solution with $\Delta \neq 0$, for the parameters $\omega_{a0} = 2\omega_{b0} = g_0/10$, $\tilde{\omega}_a = \tilde{\omega}_b = 0$, and $\tilde{g} = \mu = g_0$. This curve is to be compared with the solid line in Fig.6, showing again that the increasing rate of $\phi_G$ can be controlled through the TD Rabi frequency. The dashed line, also corresponding to $\Delta \neq 0$, with $\omega_{a0} = 2\omega_{b0} = g_0$, $\tilde{\omega}_a = \tilde{\omega}_b = 0$, and $\tilde{g} = \mu = g_0$, shows that the increasing rate of the geometric phase may also be controlled through the trap frequencies. Finally, the thick solid line, corresponding to the off-resonant solution, with $\omega_{a0} = 2\omega_{b0} = g_0$, $\tilde{\omega}_a = \tilde{\omega}_b = \chi_a = \chi_b = g_0/2$, $\xi_a = 0$, $\xi_b = -\pi/2$, and $\tilde{g} = 0$, to be compared with the thick solid line of Fig.9, also indicates the important role played by the time dependence of the trap frequency in the geometric phase. We finally observe that, evidently, these TD effects have direct implications on the population imbalance and relative phase between the condensates.

VI. GEOMETRIC PHASES AND THE EXTERNAL JOSEPHSON EFFECT

In this section we analyze the two-mode condensates from a different perspective: as a single atomic specie trapped in a symmetric or asymmetric double-well potential. As the internal Josephson effect is here substituted by the tunneling interaction, the laser pumping becomes unnecessary and we impose that $\Delta = 0$, such that $\delta = 0$. Moreover, in the external Josephson effect the interspecies collision rate correspond to a second order correction compared to the intraspecies collision rates, justifying the assumption that $\gamma_{ab} \simeq 0$ [36, 49].

The Hamiltonian (1) applied to this different physical situation leads, under the above restrictions, to a similar transformed interaction (4) and characteristic equations (7). There-
fore, the different solutions of the coupled differential equations apply directly to the external Josephson effect, with the on- and off-resonant processes describing the symmetric (\( \omega = 0 \)) and asymmetric (\( \omega \neq 0 \)) wells solutions, respectively.

A. Symmetric wells

From the above discussion we readily verify that the solutions for \( r(t) \) and \( \phi(t) \), coming from the symmetric wells, are give by Eqs. (31) and (32) with the constant of motion \( C = \sin[r(t)] \cos[\phi(t)] \). We observe that the TD tunneling rate \( g(t) \) is accomplished by modulating the amplitude of the counter propagating classical fields that generate the barrier. Similarly to the internal Josephson effect, all the possible trajectories for \( r(t) \) and \( \phi(t) \) follow from the level curves of the surface plotted in Fig.8, assuming \( \delta = 0 \). The same level curves were obtained, by numerical methods, in Ref. [44]. The geometric phases acquired by the evolution of the state vector of the BECs are given by Fig.4 and the Bloch-vector trajectories by Fig.5, both obtained from the on-resonant solution of the internal Josephson effect with \( \delta = 0 \).

B. Asymmetric wells

The solutions of the characteristic equations for the asymmetric wells follow by imposing constant values for the effective frequency \( \omega = \varpi \) (since \( \Delta = 0 \)) and the tunneling rate \( g = g_0 \). We thus obtain the solutions (34) with \( \eta \) replaced by \( \tilde{\eta} = 2g_0/\omega \) and \( \delta = 0 \). The constant of motion becomes \( C = \tilde{\eta} \cos[\phi(t)] \sin[r(t)] - \cos[r(t)] \). The phase-space portrait \( r(t) \times \phi(t) \) is given by Fig.8 (with \( \delta = 0 \)), whose level curves indicate all the possible trajectories for \( r(t) \) and \( \phi(t) \). The same level curves were obtained by numerical methods in Ref. [44]. As an example of the geometric phase acquired by the evolution of the state vector in the asymmetric wells, we take the dotted line curve of Fig.9, corresponding to the case \( \omega = \varpi = g_0/20 \). (The other two curves in Fig.9 do not satisfy the condition \( \omega = \varpi \).) The trajectory of the Bloch vector associated to \( \omega = g_0/20 \) is approximately give by the dashed curve in Fig.7.
VII. CONCLUDING REMARKS

In the present work we analyze the dynamics of two interacting condensates, with a full TD Hamiltonian. Starting from the Hamiltonian (1) under the two-mode approximation, an effective interaction is established under the RWA, provided that the polar $r(t)$ and azimuthal $\phi(t)$ angles, which define a Bloch state, satisfy coupled differential equations. This procedure enable us to define a detuning $\Delta(t)$ from the Raman resonance between the atomic transition, together with an effective frequency for the condensates $\omega(t)$, as in Eq. (8). Thus, two different solutions arise for the differential equations coupling the parameters $r(t)$ and $\phi(t)$, the on-resonant solution, where $\Delta(t) = \omega(t)$, and the off-resonant solution where $\Delta(t) = \omega(t) + \varpi$, $\varpi$ being a constant. After solving analytically the coupled equations for both regimes, we present a detailed analyzes of the geometric phases acquired by the Bloch state of the system, also discussing the relative phase and population imbalance.

A main result of our work is the connection between geometric phases and constant of motions of the interacting condensates which are identified through the analytical solutions of the coupled differential equations for $r(t)$ and $\phi(t)$. For each on- or off-resonant solution we assign a constant of motion which determine the dynamical behavior of the state of the system and, consequently, its geometric phase. We also note that these constants of motions follow from level curves in the portrait space which are also obtained analytically.

To better visualize the time-evolution of the geometric and relative phases acquired by the state vector, together with population imbalance, we also analyze the trajectory of the state vector mapped on Bloch sphere. Finally, we present a brief discussion of the TD effects of the trap and Rabi frequencies in the geometric phases together with the connection between its evolution in both cases of internal and external Josephson coupling.

As in this work we studied only the evolution of an initial Bloch state, the collision parameters were restricted to the global phase of the evolved state, which remains a Bloch state under the two-mode approximation. Therefore, collisions terms, which are also assumed as TD parameters, do not play a decisive role in our present analyzes. It is worth to consider distinct initial state to analyze the effects of the collisions parameters in the dynamics of the geometric phase and population imbalance.

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VIII. APPENDIX

A. Analytical solutions of the characteristic equations (7)

In this Appendix we present some specific solutions of the characteristic equations (7), following a more detailed treatment in [50]. We investigate two different regimes of the laser field amplification, the on-resonant and off-resonant regimes, which are defined comparing the effective frequency of the two-mode condensate, $\omega(t)$, with the detuning between the laser field and the Raman transition $\Delta(t)$. As mentioned above, in the on-resonant regime, where $\Delta(t) = \omega(t)$, the rate of time variation of the laser field equals the effective frequency of the two-mode condensate. Otherwise, we have the off-resonant regime.

B. On-resonant process

Defining $\chi(t) \equiv \phi(t) - \delta(t)$, the characteristic equations (7) becomes

$$
\dot{r}(t) = 2g(t) \sin[\chi(t)], \quad \text{(28a)}
$$

$$
\dot{\chi}(t) = \omega(t) - \Delta(t) + 2g(t) \cos[\chi(t)] \cot[r(t)], \quad \text{(28b)}
$$

such that, in the on-resonant regime we are left with the first-order differential equation

$$
\frac{dr}{d\chi} = \tan \chi \tan r. \quad \text{(29)}
$$

After integrating Eq. (29) we obtain the constant of motion

$$
\sin[r(t)] \cos[\phi(t) - \delta(t)] = C, \quad \text{(30)}
$$

with $C$ depending on the initial values $r_0$, $\phi_0$, and $\delta_0$. Thus, the resonant solutions of Eqs. (28), are given by

$$
\cos[r(t)] = \sqrt{1 - C^2} \sin \left[ u(t, t_0) + \arcsin \left( \frac{\cos r_0}{\sqrt{1 - C^2}} \right) \right], \quad \text{(31a)}
$$

$$
\phi(t) = \delta(t) + \arccos \left( \frac{C}{\sin |r(t)|} \right), \quad \text{(31b)}
$$

where

$$
u(t, t_0) = -2 \int_{t_0}^{t} g(\tau) d\tau. \quad \text{(32)}$$
C. Off-resonant process

Considering the off-resonant regime, where \( \Delta(t) = \omega(t) - \varpi \), \( \varpi \) being a constant, Eqs. (28) can again be solved by quadrature as far as we assume the Rabi frequency \( g \) to be also a constant \( g_0 \). In this regime, defining \( \eta = 2g_0/\varpi \), Eq. (30) is replaced by

\[
\eta \cos [\phi(t) - \delta(t)] \sin [r(t)] - \cos [r(t)] = C,
\]

which again depends on the initial values \( r_0, \phi_0, \delta_0 \). The solutions for this case are given by

\[
\cos [r(t)] = \frac{\eta \sqrt{1 + \eta^2 - C^2}}{1 + \eta^2} \sin \left\{ -\varpi \sqrt{1 + \eta^2} (t - t_0) \right\} \\
+ \arcsin \left\{ \frac{(1 + \eta^2) \cos r_0 + C}{\eta \sqrt{1 + \eta^2 - C^2}} \right\} - \frac{C}{1 + \eta^2},
\]

\[
\phi(t) = \delta(t) + \arccos \left\{ \frac{C + \cos [r(t)]}{\eta \sin [r(t)]} \right\}.
\]

D. A constant solution for \( r \)

Another solution for \( r(t) \) and \( \phi(t) \) arises when we impose that \( r \) remains constant in time. Through this solution, given by

\[
r(t) = r_0, \tag{35}
\]

\[
\phi(t) = \delta(t) + n\pi = \phi_0 + \int_{t_0}^{t} [\omega(\tau) + 2(-)^n g(\tau) \cot(r_0)] d\tau, \tag{36}
\]

with \( r_0 \neq n\pi \) and \( n \) being an integer, the state vector of the system acquire only relative phase \( \phi(t) \). This formal solution for \( \phi(t) \) can be even simplified noting that the physical implementation of this regime requires, necessarily, that \( g \simeq 0 \). In fact, for the population imbalance to be null, due to the constant value for \( r \), the Rabi frequency must also be null. therefore, Eq. (36) simplifies to

\[
\phi(t) = \phi_0 + \int_{t_0}^{t} \omega(\tau) d\tau. \tag{37}
\]

[1] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[2] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).
[3] R. Simon, Phys. Rev. Lett. 84, 2726 (2000).
[4] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[5] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, Phys. Rev. Lett. 91, 107901 (2003).
[6] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
[7] J. Bardeen, L. N. Cooper, and J. R. Schriffer, Phys. Rev. 108, 1175 (1957).
[8] F. Pan and J. P. Draayer, Phys. Lett. A 339, 403 (2005).
[9] J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, Rev. Lett. 83, 1319 (1999); J. M. Geremia, J. K. Stockton, and H. Mabuchi, Science 304, 270 (2004); J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Rev. Lett. 95, 120502 (2005).
[10] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell, and C. E. Wieman, Phys. Rev. Lett. 78, 586 (1997).
[11] M. R. Matthews, D. S. Hall, D. S. Jin, J. R. Ensher, C. E. Wieman, E. A. Cornell, F. Dalfovo, C. Minniti, and S. Stringari, Phys. Rev. Lett. 81, 243 (1998).
[12] A. J. Leggett, Rev. Mod. Phys. 73, 307 (2001).
[13] D. S. Hall, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 81, 1543 (1998).
[14] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, M. J. Holland, J. E. Williams, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 83, 3358 (1999).
[15] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 81, 1539 (1998).
[16] A. P. Hines, R. H. McKenzie, and G. J. Milburn, Phys. Rev. A 67, 013609 (2003).
[17] L Sanz, R. M. Angelo, and K. Furuya, J. Phys. A: Math. Gen. 36, 9737 (2003).
[18] J. I. Cirac, M. Lewenstein, K. Mølmer, and P. Zoller, Phys. Rev. A 57, 1208 (1998).
[19] D. Gordon and C. M. Savage, Phys. Rev. A 59, 4623 (1999).
[20] J. A. Dunningham and K. Burnett, J. Mod. Opt. 48, 1837 (2001).
[21] I. Fuentes-Guridi, J. Pachos, S. Bose, V. Vedral, and S. Choi, Phys. Rev. A 66, 022102 (2002).
[22] Z.-D. Chen, J.-Q. Liang, S.-Q. Shen, and W.-F. Xie, Phys. Rev. A 69, 023611 (2004).
[23] G. L. Salmond, C. A. Holmes, and G. J. Milburn, Phys. Rev. A 65, 033623 (2002).
[24] S. A. Baily and M. B. Salamon, Phys. Rev. B 71, 104407 (2005).
[25] P. Bruno, V. K. Dugaev, and M. Taillefumier, Phys. Rev. Lett. 93, 096806 (2004).

[26] Y. Kats, I. Genish, L. Klein, J. W. Reiner, and M. R. Beasley, Phys. Rev. B 70, 180407(R) (2004).

[27] P. Solinas, P. Zanardi, N. Zanghi, and F. Rossi, Phys. Rev. A 67, 052309 (2003).

[28] M. V. Berry, Proc. R. Soc. London A 392, 45 (1984).

[29] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593 (1987).

[30] J. Samuel and R. Bhandari, Phys. Rev. Lett. 60, 2339 (1988).

[31] D. M. Tong, E. Sjöqvist, L. C. Kwek, and C. H. Oh, Phys. Rev. Lett. 93, 080405 (2004).

[32] J. G. P. de Faria, A. F. R. D. Piza, M. C. Nemes, Europhys. Lett. 62, 782 (2003).

[33] A. Carollo, M. F. Santos, and V. Vedral, Phys. Rev. A 67, 063804 (2003).

[34] E. I. Duzzioni, C. J. Villas-Bôas, S. S. Mizrahi, M. H. Y. Moussa, and R. M. Serra, Europhys. Lett. 72, 21 (2005).

[35] K. Y. Bliokh and Y. P. Bliokh, Annals of Physics 319, 13 (2005).

[36] G.J. Milburn, J. Corney, E.M. Wright, and D.F. Walls, Phys. Rev. A 55, 4318 (1997).

[37] P. Villain and M. Lewenstein, Phys. Rev. A 59, 2250 (1999).

[38] J. M. Vogels, C. C. Tsai, R. S. Freeland, S. J. J. M. F. Kokkelmans, B. J. Verhaar, and D. J. Heinzen, Phys. Rev. A 56, R1067 (1997); J. P. Burke, Jr., J. L. Bohn, B. D. Esry, and C. H. Greene, Phys. Rev. Lett. 80, 2097 (1998).

[39] A. Smerzi, S. Fantoni, S. Giovanazzi, and S.R. Shenoy, Phys. Rev. Lett. 79, 4950 (1997).

[40] L. Sanz, M. H. Y. Moussa, K. Furuya, Ann. Phys. 321, 1206 (2006).

[41] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A 6, 2211 (1972).

[42] J. P. Dowling, G. S. Agarwal, and W. P. Schleich, Phys. Rev. A 49, 4101 (1994).

[43] N. Mukunda and R. Simon, Ann. Phys. 228, 205 (1993).

[44] R. Balakrishnan and M Mehta, Eur. Phys. J. D 33, 437 (2005).

[45] M. Albiez, R. Gati, J. Folling, S. Hunsmann, M. Cristiani, and M.K. Oberthaler, Phys. Rev. Lett. 95, 010402 (2005).

[46] D. Gordon and C.M. Savage, Phys. Rev. A 58, 1440 (1998).

[47] M. Ericsson, D. Achilles, J. T. Barreiro, D. Branning, N. A. Peters, and P. G. Kwiat, Phy. Rev. Lett. 94, 050401 (2005).

[48] G. G. Polavieja and E. Sjöquist, Am. J. Phys. 66, 431 (1998).

[49] M.R. Andrews, C.G. Townsend, H.-J. Miesner, D.S. Durfee, D.M. Kurn, and W. Ketterle,
Science 275, 637 (1997).

[50] S. S. Mizrahi, M. H. Y. Moussa, and B. Baseia, Int. J. Mod. Phys. B 8, 1563 (1994); C. J. Villas-Boas, F. R. de Paula, R. M. Serra, and M. H. Y. Moussa, Phys. Rev. A 68, 053808 (2003).
Figure captions

Fig. 1 Absolute value of geometric phase $\phi_G(t)$ against the dimensionless $\tau = \omega_{a0} t$, for $r$ constant, with $\omega_{a0} = 2\omega_{b0} = 62.5\pi$ Hz.

Fig. 2 Evolution of the Bloch vectors coming from constant solutions of $r$ in the time interval $\tau = 4\pi$. The grey and black vectors correspond to the initial conditions $(\pi/2, 0)$ and $(\pi/2.1, 0)$, respectively, with $\omega_{a0} = 2\omega_{b0} = 62.5\pi$ Hz and $\tilde{\omega}_a = \tilde{\omega}_b = 0$.

Fig. 3 The portrait space $r(t) \times \phi(t) - \delta(t)$ obtained as projection of the surface which follows from the on-resonant constant of motion $C = \sin[r(t)] \cos[\phi(t) - \delta(t)]$.

Fig. 4 Evolution of the geometric phase $\phi_G(t)$ against $\tau = g_0 t$, for on-resonant solutions of the characteristic equations (7), with $\Delta = 0$ and $\tilde{\omega}_a = \tilde{\omega}_b = \tilde{g} = 0$.

Fig. 5 Evolution of the Bloch vectors coming from the on-resonant solutions of Eqs. (7) in the time interval $\tau = \pi$. The black and grey vectors correspond to the initial conditions $(\pi, \pi/2)$ and $(\pi/3, 0)$, with $\Delta = 0$ and $\tilde{\omega}_a = \tilde{\omega}_b = \tilde{g} = 0$.

Fig. 6 Evolution of the geometric phase $\phi_G(t)$ against $\tau = g_0 t$, for on-resonant solutions of the characteristic equations (7), with $\Delta \neq 0$ and $\tilde{\omega}_a = \tilde{\omega}_b = \tilde{g} = 0$.

Fig. 7 Evolution of the Bloch vectors coming from the on-resonant solutions of Eqs. (7) in the time interval $\tau = \pi$. The black and grey vectors correspond to the initial conditions $(\pi, \pi/2)$ and $(\pi/3, 0)$, with $\Delta \neq 0$ and $\tilde{\omega}_a = \tilde{\omega}_b = \tilde{g} = 0$.

Fig. 8 The portrait space $r(t) \times \phi(t) - \delta(t)$ obtained as projection of the surface which follows from the off-resonant constant of motion $C = \eta \sin[r(t)] \cos[\phi(t) - \delta(t)] - \cos[r(t)]$.

Fig. 9 Evolution of geometric phase $\phi_G(t)$ against $\tau = g_0 t$, for off-resonant solutions of the characteristic equations (7) and different values of $\eta$, all starting from the point $(\pi/3, 0)$, with $\omega_{a0} = 2\omega_{b0} = g_0/10$ and $\tilde{\omega}_a = \tilde{\omega}_b = 0$.

Fig. 10 Evolution of the Bloch vectors coming from the off-resonant solutions of Eqs. (7) in the time interval $\tau \approx 9\pi/10$. The black and grey vectors, corresponding to $\eta = 0.1$ and $\eta = 2$, respectively, both start from the common point $(\pi/3, 0)$, with $\omega_{a0} = 2\omega_{b0} = g_0/10$ and $\tilde{\omega}_a = \tilde{\omega}_b = 0$.

Fig. 11 Evolution of geometric phase $\phi_G(t)$ against $\tau = g_0 t$, for on- and off-resonant solutions of the characteristic equations (7), considering the same initial conditions $(\pi/3, 0)$ for all the curves.