On Semiannihilator Supplement Submodules

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Abstract

Let R be associative ring with an identity and let D be unitary left R-module. In this work we present semiannihilator supplement submodule as a generalization of R-annihilator supplement submodule. Let U and V be submodules of an R-module D if D=U+V and whenever Y≤V and D=U+Y, then annY ≪R. We also introduce the concept of semiannihilator supplemented modules and semiannihilator weak supplemented modules, and we give some basic properties of this concept.

Keywords- sa-Small Submodule, sa-Hollow And sa-Lifting Modules, sa-supplemented Modules

Introduction

Throughout this paper all rings are associative ring with identity and modules are unitary left modules, D is named a hollow module if every proper submodule is small in D, where a submodule B of R-module D is named small in D if B + K = D whenever B+K=D, where K a submodule of D implies that ann(K)=0, where ann(K)={r R: r.K=0}. The concept of small submodule has been generalized by some researchers, for this see [2,3]. The authors in [4] introduced the concept of R-annihilator small submodules, that is; a submodule B of an R-module D is called R-annihilator small, if whenever B+K=D, where K a submodule of D, implies that ann(K)=0. In [5] Sahira introduce the concept of semiannihilator small submodules, in case ann(K) ≪R and ann(K) << R where K is a submodule of D whenever B+K=D. Clear that every R-annihilator small submodule is semiannihilator small, but the converse is not true [5]. Recall that a submodule V of M is called a supplement of U in M. If V is a minimal element in the set of submodules L of M with U+L=M.

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M be an R-module, M is called a supplemented module if every submodule of M has a supplement in M. Let U, V be submodules of an R-module M. If M=U+V and U∩V≪M, then V is called a weak supplement of U in M. Let M be an R-module if every submodule of M has a weak supplement in M, then M is called a weakly supplemented module[6]. In this work we present semiannihilator- supplement submodule. Let U and V be submodules of an R-module M if M=U+V and whenever Y≤V and D=U+Y, then annY≪R. We also introduce the the concept of semiannihilator supplemented modules and semiannihilator weak supplemented modules, and we give some basic properties of these concepts.

In section two we introduce the notion of semiannihilator lifting modules and discuss some characteristics of this kind of modules. In part three, we introduce the concept of semiannihilator characteristics lifting submodule and basic properties. We show that if D and D' be R-modules and let f: D→D' be an epimorphism, if D' is semiannihilator supplemented module then D is semiannihilator supplemented module. In part four, the concept of semiannihilator weak supplement submodules with some examples and basic properties was introduced. The below lemma award the characteristics of semiannihilator small submodules.

Lemma[5]:
1- Let D be an R-module with submodules A,N such that A⊆N. If N ≪ sa D then A ≪ sa D.
2- Let D be an R-module with submodules A,N such that A⊆N, if A ≪ sa N then A ≪ sa D.
3- Let D1, D2 be an R-modules. If N1 ≪ sa D1 and N2 ≪ sa D2, then N1 ⊕ N2 ≪ sa D1 ⊕ D2.
4- Let D and S be an R-modules and f: D→S be an epimorphism. If H ≪ sa S, then f⁻¹(H) ≪ sa D.

1. Semiannihilator lifting modules.

An R-module Dis called lifting if for any submodule N of D there exist submodule K of N such that D=K⊗ K' with K'⊆ M and N∩ K' ≪ K. In this part we introduce the notion of semiannihilator lifting modules as generalization of R-Annihilator Lifting modules and discus some properties of this kind of modules.

**Definition 1.1**: An R1-module D is named semiannihilator lifting (sa-lifting) if for any submodule B of D, there exist submodules K, K' of D such that D=K ⊕ K' with K⊆ B and B∩ K' ≪ sa K.

The below theorem gives a characterization of following semiannihilator lifting modules.

**Theorem 1.** For an R-module D the statement are equivalent:
1) M is sa-lifting
2) Every submodule N of D, N can be written as N= A ⊕ B where A is direct summand of D and B ≪ sa D.
3) For every submodule N of D, there exist a direct summand K of D s.t K ≤ N and N/K ≪ sa D/K.

Proof: See proof of lemma2 in [7].

A nontrivial R-module D is called semiannihilator –hollow (sa-hollow) if every proper submodule of D is sa-small in M [5].

**Examples 1.2**: 
1- Z as Z-module is sa-lifting module but it is not lifting.
2- Z1 and Z2 as Z-module are not sa-lifting module.

Not: Every sa-hollow is sa-lifting

Proof: - For a submodule B of D if B ≠ D, B ≪ sa D, B = (0) ∅ B the result go after, directly by theorem 1

**Proposition 1.3**: Let D be indecomposable module then D is sa-hollow module if and only if D is sa-lifting.

Proof: Let D be sa-hollow then D is sa-lifting. Conversely suppose that D is sa-lifting and A a proper submodule of D, by Theorem 1. We have A= N⊗ K where N is a direct summand in D and K ≪ sa D, but D is indecomposable. Therefore either N = (0) or N=D then D= N⊗ A which attend that A=D which is contradiction, if N= (0), so A=K ≪ sa D, and D is sa-hollow.

**Proposition 1.4**: Let D=H1 ⊕ H2 be duo module. If H1 and H2 are sa-lifting modules, then D is sa-lifting module.

Proof: Let H1 and H2 be sa-lifting modules, and N submodule of D, then N= (N∩ H1 ) ⊕ (N∩ H2). For each i∈ {1,2}, there exists a direct summand Ki of Hi, such that Hi= Ki⊗ Li with Di ⊆ N∩ Hi and N∩
\textbf{Corollary 1.5}: Let $D = H_1 \oplus H_2$ be a module such that $R = \text{ann}(H_1) + \text{ann}(H_2)$. If $H_1$ and $H_2$ are sa-lifting modules, then $D$ is a lifting module.

\textbf{Proposition 1.6}: Let $D$ be a multiplication $R$-module, if $D$ is a lifting module, then $R$ is a sa-lifting ring.

\textbf{Proof}: Assume that $D$ is a lifting module. Where $I$ is an ideal in $R$. $D$ is a multiplication module. Then $\mathcal{N} = \text{id}(D)$ is a submodule of $D$, thus there exist submodules $K$ and $K'$ in $M$ with $K \subseteq \mathcal{N}, D = K \oplus K'$ and $(\mathcal{N} \cap K') \triangleleft \triangleleft D$. $D$ is a multiplication $R$-module, so there are ideals $J$ and $J'$ of $R$ such that $K = JD$ and $K = J'D$. Since $K \subseteq \mathcal{N}$ then $J \subseteq I$. We have $D = K \oplus K' = JD \oplus J'D = (J \oplus J')D$ implies that $J = \left( J \cup J' \right)$.

Next, $\mathcal{N} \cap K' = (J \cap J'D) \triangleleft \triangleleft \triangleleft D$ and since $(J \cap J')D \subseteq \text{id}(J \cap J'D)$ it follows that $(J \cap J')D \triangleleft \triangleleft D$ and according to [3] we get $\left( J \cap J' \right) \triangleleft \triangleleft D$. But $\left[ (J \cap J')D : D \right] = I \cup J'$, then $(I \cap J') \triangleleft \triangleleft D$ and $R$ is a sa-lifting ring.

2. \textbf{Semiannihilator suplemente submodule.}

In this section we present the definition of semiannihilator (sa-supplement) supplement class. Then, some basic properties of this class are presented. In addition, several examples are given to illustrate the results.

\textbf{Definition 2.1}: Let $V$ and $U$ be submodules of an $R$-module $D$. We say that $V$ is a semiannihilator supplement (sa-supplement) of $U$ in $D$ if $D = U + V$ and whenever $Y \subseteq V$ and $D = U + Y$, then $\text{ann}Y \triangleleft \triangleleft \triangleleft R$.

Let $D$ be an $R$-module. We say that $D$ is semiannihilator supplemented (sa-supplemented) module if every proper submodule of $D$ has a sa-supplement. Let $R$ be a commutative ring and let $I$ be an ideal of $R$. We say that $R$ is a sa-supplemented ring of $R$ is sa-supplemented as an $R$-module.

The below proposition gives a characterization of sa-supplement submodule.

\textbf{Proposition 2.2}: For submodules $U$ and $V$ of an $R$-module $D$. Then $V$ is a sa-supplement of $U$ if and only if $D = U + V$ and $U \cap V \triangleleft \triangleleft \triangleleft V$.

\textbf{Proof}: Let $V$ be a sa-supplement of $U$. To show that $U \cap V$ is a submodule of $V$, let $V = (U \cap V) + Y$. Now $D = U + V = U + (U \cap V) + Y = U + Y$. But $Y \subseteq V$, therefore $\text{ann}Y \triangleleft \triangleleft \triangleleft R$, and $U \cap V \triangleleft \triangleleft \triangleleft V$.

Conversely, let $D = U + V$ and $U \cap V \triangleleft \triangleleft \triangleleft V$. We want to show that $V$ is a sa-supplement of $U$. Let $Y \subseteq V$ such that $D = U + Y$. By (Modular law), $V = (U \cap V) + Y$. But $U \cap V \triangleleft \triangleleft \triangleleft V$, therefore $\text{ann}Y \triangleleft \triangleleft \triangleleft R$. Thus $V$ is a sa-supplement of $D$.

\textbf{Examples and Remarks 2.3}

1. sa-supplement submodule not supplement submodule to see that consider $Z$ as $Z$-module. For every proper submodule $nZ$ of $Z$, $Z = nZ + Z$ and $nZ \cap Z = nZ \triangleleft \triangleleft \triangleleft Z$. Then $Z$ is a sa-supplement of $nZ$. Thus every proper submodule of $Z$ has a sa-supplement. But it is known that every non trivial submodule of $Z$ has no supplement in $Z$. Where $Z$ is indecomposable and $\{ 0 \}$ is the only small submodule of $Z$.

2. A supplement submodule need not be a sa-supplement submodule. For example, let $Z_4$ as $Z$-module. $Z_4$ is a supplement of $\{ 0, 2 \}$. And $Z_4$ is not a sa-supplement of $\{ 0, 2 \}$, where $\{ 0, 2 \} \cap Z_4 = \{ 0, 2 \}$ is not sa-small in $Z_4$. Since $Z_4 = \{ 0, 2 \} + Z_4$ and ann $Z_4 = \{ n \in Z ; n \cdot Z_4 = 0 \} = 4Z$ not small in $Z$.

3. Let $D$ be an $R$-module. Then every sa-small submodule of $D$ has a sa-supplement in $D$. That is if $N$ be sa-small submodule of $D$. Then $D = N + D$ and $N \cap D = N$ is sa-small submodule of $D$. Thus $D$ is sa-supplement of $N$ in $D$.

4. Let $U$ and $V$ be two submodules in an $R$-module $D$ such that $V$ is a sa-supplement in $U$. If $D = W + V$, where $W$ a submodule in $U$, then $V$ is a sa-supplement in $W$.

\textbf{Proof}: Since $V$ is a sa-supplement of $U$, then $D = U + V$ and $U \cap V \triangleleft \triangleleft \triangleleft V$. Since $W \subseteq U$. Then $W \cap V \triangleleft \triangleleft \triangleleft V$, by prop. (2.4) in [5]. Thus $V$ is a sa-supplement of $W$ in $D$.

Let $D$ be an $R_I$-module its known that every direct summand of $D$ has a supplement in $D$. But this is not true for sa-supplement as the below examples shows:

\textbf{Example 2.4}: Let $Z_6$ as $Z$-module. and $U = \{ 0, 2, 4 \}$, $V = \{ 0, 3 \}$, $Z_6 = U \oplus V$. $U$ and $V$ are supplement of each others. But each of $U$ and $V$ has no sa-supplement in $Z_6$, where ann $Z_6 = \{ n \in Z ; n \cdot Z_6 = 0 \} = 6Z$ not small in $Z$. Hence $Z_6$ has no sa-small submodule [5]. Thus every submodule of $Z_6$ has no sa-supplement in $Z_6$. 

\textbf{Helal and Yaseen} 
\textbf{Iraqi Journal of Science, 2020, Vol. 61, Special Issue, pp: 16-20}
Proposition 2.5: For a finitely generated D in R-module and For submodules U and V in D such that V sa-supplement of U in D. Thus there exists a finitely generated sa-supplement W in U s.t W ⊆ V.

Proof: Let M=Rx₁+Rx₂+...+Rxn, where xi∈D, Vi=1,2,...,n. Since D=U+V, then xi=ui+vi, where ui∈U, vi∈V, Vi=1,2,...,n. Now let W=RV₁+RV₂+...+Rvn. Clearly that D=U+W. Since V sa-supplement of U and W⊆V, then annW ⊆ R. is clear that W is sa-supplement of U.

Proposition 2.6: Let D and N be R₁-modules and let f: D→N an epimorphism, if N is sa-supplemented module, then D is sa-supplemented module.

Proof: For a submodule K of D, then f(K) submodule in N. Since N is sa-supplemented module, then there exists a supplement L in N s.t N=f(K)+L and f(K)∩L is sa-small in L. M=1(N)=f'(f(K)+L)= f'(f(K))+f'(L)=Kker f+f'(L)=K+f'(L).Claim that K∩f'(L) is sa-small module of f'(L). Since f(K)∩L≤saL, then by prop (2-7) in [3], f'(f(K)∩L)≤saf'(L). But f'(f(K)∩L)=f'(f(K))∩f'(L)=(K+Kerf)∩f'(L)=kerf+(K∩f'(L)), by (Modular Law). ker f+(K∩f'(L)) £ sa f'(L). By Prop (2-4) in [5], K∩f'(L) ≤ sa f'(L). So f'(L) is sa-supplement of K in D. Then M is sa-supplemented module.

Proposition 2.7: Let D be a finitely generated faithful multiplication module over a commutative ring R and let I be an ideal of R. if ID has sa-supplement in D, so I has sa-supplement in R.

Proof: For an ideal I in R₁ such that ID has sa-supplement in D. Then there exists a submodule N in D s.t D=ID+N and ID∩N £ saN. D a multiplication module, thus N=JD, for some ideal J of D. Now D=RID=ID(ID∩J)=I(J)+D. But D is finitely generated faithful multiplication module and hence R=I+J[8]. ID∩N=ID∩J=(ID∩J)D £ sa JD. To show that I∩J£ saJ. Let J=(I∩J)+L, where L an ideal of R. Then JD=(I∩J)+L=(I∩J)D+L. Therefore ann (LD)£ R. ann L£ ann LD. Thus ann L£R and I∩J£ saJ. Then J is sa-supplement of I.

3. Semiannihilator weakly supplemented modules .

In this part, we introduce the definition of semiannihilator weakly supplemented modules. And we introduce some basic characterization of this modules.

Definition 3.1: For submodules U and V of an R-module D. We say that V semiannihilator -weak supplement (sa-supplement) of U in D if D=U+V and U∩V £ sa D.

We say that D is semiannihilator weakly (sa-)supplemented module if every submodule of D has sa-supplement in D.

Remarks and examples 3.4:

1. semiannihilator weak supplement submodule not be weak supplement submodule. For example, consider Z as Z module. Clearly that Z=2Z+3Z and 2Z∩3Z=6Z £ sa Z. Thus 3Z is sa-supplement of 2Z. But {0} is the only small submodule of Z and hence 3Z is not weak supplement of 2Z.

2. For an ideal I in R₁ the module that ID has sa-supplement in D. To show that, let D be an sa-supplemented module and let U be a proper submodule of D, then there exists a submodule V of D, such that D=U+V and U∩V £ saV. By prop (2-3) in [8], U∩V £ sa D. Hence V is sa-supplement of U. Clearly that D=ID+0 and D∩{0}=0 £ sa D. So {0} is sa-supplement of D. Thus D is sa-supplemented module.

3. sa-supplement submodule need not be sa-supplement submodule. For example, let D be a faithful R-module. Then D=0+D and D∩{0}=0 £ sa D. Thus {0} is sa-supplement of D. Now D∩{0}=0 is not sa-small in 0, when 0=0+0 and 0 not small in R.

4. Let X and Y be submodules of R₁-module D if X is sa-supplement of Y, then Y is sa-supplement of X, where X=D+Y and X∩Y £ sa D.

Proposition 3.5: For N, K and L submodules in an R-module D such that L£ N. If K is sa-supplement of N and D=L+K, there K is sa-supplement of L in D.

Proof: Since K sa-supplement of N in D, then D=N+K and N∩K £ sa D. Now D=L+K and L∩K £ N∩K £ sa D. Hence L∩K £ sa D by prop (2-4) in [5]. Thus K is sa-supplement of L in D.

Proposition 3.6: For submodules N and L of a finitely generated R-module D if L sa-supplement of N, then L contains a finitely generated sa-supplement of N.

Proof: Let D=Rx₁+Rx₂+...+Rxₙ, xi∈D, for some xi∈D, Vi=1,2,...,n. Since M=N+L, then xi=aᵢ+bᵢ, where aᵢ∈N, bᵢ∈K, Vi=1,2,..., n. Now let L'=Rb₁+Rb₂+...+Rbₙ, D=N+K'. Clearly that K' £ K. But N∩L'=N∩L £ sa D, therefore N∩L' £ sa D, by prop (2-4) in [5]. Thus L' is a finitely generated sa-supplement of N.
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