ON THE THEOREM OF AMITSUR–LEVITZKI

CLAUDIO PROCESI

Abstract. We present a proof of the Amitsur–Levitzki theorem which is a basis for a general theory of equivariant skew–symmetric maps on matrices.

to the memory of S.A. Amitsur who taught me PI theory

1. Introduction

The Theorem of Amitsur–Levitzki is a cornerstone of the Theory of polynomial identities. It states that the algebra of $n \times n$ matrices over any commutative ring $A$ satisfies the standard polynomial $S_{2n}$. For any integer $h$ the standard polynomial $S_h$ is the element of the free algebra in the variables $x_1, \ldots, x_h$ given by the formula

$$S_h(x_1, \ldots, x_h) := \sum_{\sigma \in S_h} \epsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(h)}$$

where $S_h$ is the symmetric group of permutations on $\{1, \ldots, h\}$ and $\epsilon_\sigma$ denotes the sign of the permutation $\sigma$.

This Theorem has received several proofs, after the original proof [1], which is a direct verification. A proof by Swan using graphs, [7]; one by Kostant relating it to Lie algebra cohomology, [4]; by Formanek as consequence of the Cayley–Hamilton Theorem (see [3] and [4]) and finally a very simple by Rosset using Grassman variables [6]. So why give another proof?

The proof I present shows that the Amitsur–Levitzki Theorem is the Cayley–Hamilton identity for the generic Grassman matrix. Technically it is very similar to Rosset’s proof but while in his proof the Grassman variables are auxiliary, in the present proof these variables are intrinsically embedded in the problem, this is important for applications.

In this formulation the Theorem is the first step for the general Theory of alternating equivariant maps (with respect to conjugation) from matrices to matrices. This theory can be considered as the Grassman analogue of the theory of generic matrices with trace and it is fully explained in a joint paper with Matej Bresar and Spela Spenko [2] but I think it is worth presenting the result on the Amitsur–Levitzki Theorem independently.

I will state at the end of the paper the general Theorem 2.2 from [2] which shows that this algebra is very explicit.

As usual it is enough to prove the Theorem for matrices over a field $F$ of characteristic 0, i.e. $\mathbb{Q}$ so we will assume we are in this setting from now on.

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2. Antisymmetry

2.1. Antisymmetry. By the antisymmetrizer we mean the operator that sends a multilinear expression \( f(x_1, \ldots, x_h) \) into the antisymmetric expression \( \sum_{\sigma \in S_h} \epsilon_\sigma f(x_{\sigma(1)}, \ldots, x_{\sigma(h)}) \); so that, if \( \phi_1, \ldots, \phi_h \) are linear forms on \( V \), antisymmetrizing \( \phi_1(v_1) \cdots \phi_h(v_h) \) one has \( \phi_1 \wedge \cdots \wedge \phi_h \) and, applying the antisymmetrizer to the noncommutative monomial \( x_1 \cdots x_h \) we get the standard polynomial of degree \( h \), \( S_h(x_1, \ldots, x_h) = \sum_{\sigma \in S_h} \epsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(h)} \). Up to a scalar multiple this is the only multilinear antisymmetric noncommutative polynomial of degree \( h \).

Let \( A \) be any \( F \)-algebra (not necessarily associative) with basis \( e_i \), and let \( V \) be a finite dimensional vector space over \( F \). The set of multilinear antisymmetric functions from \( V^k \) to \( A \) is given by \( G(v_1, \ldots, v_k) = \sum_i G_i(v_1, \ldots, v_k) e_i \) with \( G_i(v_1, \ldots, v_k) \) multilinear antisymmetric functions from \( V^k \) to \( F \), moreover if \( A \) is infinite dimensional only finitely many \( G_i(v_1, \ldots, v_k) \) appear for any given \( G \).

In other words \( G_i(v_1, \ldots, v_k) \in \bigwedge^k V^* \) and this space can be identified with \( \bigwedge^k V^* \otimes A \). Using the algebra structure of \( A \) we have a wedge product of these functions:

for \( G \in \bigwedge^h V^* \otimes A, \quad H \in \bigwedge^k V^* \otimes A \) we define

\[
(G \wedge H)(v_1, \ldots, v_{h+k}) := \frac{1}{h!k!} \sum_{\sigma \in S_{h+k}} \epsilon_\sigma G(v_{\sigma(1)}, \ldots, v_{\sigma(h)}) H(v_{\sigma(h+1)}, \ldots, v_{\sigma(h+k)})
\]

As an example we have \( S_a \wedge S_b = S_{a+b} \).

With this multiplication the algebra of multilinear antisymmetric functions from \( V \) to \( A \) is isomorphic to the tensor product algebra \( \bigwedge V^* \otimes A \). We shall denote by \( \wedge \) the product in this algebra.

Assume now that \( A \) is an associative algebra and \( V \subset A \). The inclusion map \( X : V \to A \) is of course antisymmetric, since the symmetric group on one variable is trivial, hence \( X \in \bigwedge V^* \otimes A \). By iterating the definition of wedge product we have the important fact:

**Proposition 2.1.** As a multilinear function, each power \( X^a := X^{\wedge a} \) equals the standard polynomial \( S_a \) computed in \( V \).

We apply this to \( V = A = M_n(F) := M_n \); the group \( G = PGL(n, F) \) acts on this space and hence on functions by conjugation and it is interesting to study the invariant algebra, i.e. the algebra of \( G \)-equivariant maps

\[
A_n := (\bigwedge M_n^* \otimes M_n)^G.
\]

This among other topics is discussed in [2].

In the natural basis \( e_{ij} \) of matrices and the coordinates \( x_{ij} \) the element \( X \in A_n \) (cf. (11)) is the generic Grassman matrix \( X = \sum_{h,k} x_{hk} e_{hk} \).

Hence in this language the Amitsur–Levitzki Theorem is the single identity \( X^{2n} = 0 \).

**Proof of Amitsur–Levitzki** \( X^{2n} = 0 \). Since \( X \) is an element of degree 1 we have that \( X^2 \) is in \( \bigwedge^2 M_n^* \otimes M_n \subset M_n(\bigwedge^{even} M_n^*) \). We have that \( X^2 \) is an \( n \times n \) matrix over a commutative ring, the even part of the Grassman algebra, hence in order to prove that \( X^{2n} = (X^2)^n = 0 \) we need to show that \( tr((X^2)^i) = tr(X^{2i}) = 0 \) for \( i \leq n \). Now the fact that the trace of an
even standard polynomial vanishes is an easy exercise and it appears also in Kostant and in Rosset’s proof so I will not reproduce it here. Thus the Theorem is proved.

As I already mentioned this formulation can be taken as basis of the description of the algebra $A_n$ (Formula (1)) of equivariant multilinear antisymmetric maps from matrices to matrices. In [2] we prove among other results

**Theorem 2.2.** The algebra $A_n$ is generated by $X$ and the elements $\text{tr}(X^{2i-1})$, $i = 1,\ldots,n$. All these elements anti commute.

$A_n$ is a free module with basis $X^i$, $i = 0,\ldots,2n-1$ over the Grassman algebra in the elements $\text{tr}(X^{2i-1})$, $i = 1,\ldots,n-1$ and we have the two defining identities

$$X^{2n} = 0, \quad \text{tr}(X^{2n-1}) = -\sum_{i=1}^{n-1} X^{2i} \wedge \text{tr}(X^{2(n-i)-1}) + nX^{2n-1}.$$

This Theorem is based on the first and second fundamental Theorem for matrix invariants and in turn gives rise to a series of questions on Lie algebras which will be treated elsewhere.

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C. Procesi, Dipartimento di Matematica, Sapienza Università di Roma, Italy

E-mail address: procesi@mat.uniroma1.it