Normal automorphisms of the metabelian product of free abelian Lie algebras

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Abstract. Let $M$ be the metabelian product of free abelian Lie algebras of finite rank. In this study we prove that every normal automorphism of $M$ is an IA-automorphism and acts identically on $M'$.

1. Introduction

Let $L$ be a Lie algebra over a field $K$. An automorphism $\varphi$ of $L$ is called a normal automorphism if $\varphi(I) = I$ for every ideal $I$ of $L$. The set of normal automorphisms of $L$ is a normal subgroup of the automorphism group of $L$.

Automorphisms and more particularly normal automorphisms have a very important place in groups and Lie algebras. Let $G$ be a soluble product of class $n \geq 2$ of nontrivial free abelian groups. In [5] it is shown that the subgroup of all normal automorphisms of $G$ coincides with the subgroup of all inner automorphisms. In [4] Romankov showed that if $S$ is a free non-abelian soluble group, then the subgroup of normal automorphisms of $S$ is the subgroup of inner automorphisms of $S$. In [1] it is studied normal automorphisms of a free metabelian nilpotent group. Let $L_{m,c}$ be the free $m$-generated metabelian nilpotent of class $c$ Lie algebra over a field of characteristic zero. In [2] it is shown that the group of normal automorphisms of $L_{m,c}$ is contained by the group of IA-automorphisms of $L_{m,c}$ for $m \geq 3$, $c \geq 2$.

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For an arbitrary variety of Lie algebras, the metabelian product of Lie algebras $F_i$, $i = 1, ..., m$ is defined as
\[
\left( \prod^* F_i \right) / \left( D \cap F'' \right),
\]
where $F = \prod^* F_i$ is the free product of the Lie algebras $F_i$ and $D$ is the cartesian subalgebra of $\prod^* F_i$. If the algebras $F_i$ are non-trivial free abelian Lie algebras then the metabelian product of them is isomorphic to $F/F''$, where $F'' = [F', F']$ and $F' = [F, F]$ is the derived subalgebra.

Let $M$ be the metabelian product of free abelian Lie algebras of finite rank. In this study it is shown that every normal automorphism of $M$ is an IA-automorphism and acts identically on $M'$. In proving this result we inspired by the result of Timoshenko in the case of groups [5].

Let $L$ be a Lie algebra and $B$ any subset of $L$. We show that by $\langle B \rangle$ the ideal of $L$ generated by the set $B$.

2. Normal Automorphisms of Metabelian Product

Let $A_i$, $i = 1, ..., m$, be free abelian Lie algebras of finite rank and $F = \prod^* A_i$ is the free product of the abelian Lie algebras $A_i$, $i = 1, ..., m$. If $M$ is the metabelian product of the algebras $A_i$, $M$ is isomorphic to $F/F''$.

**Definition 1.** Let $L$ be a Lie algebra. An automorphism $\varphi$ of $L$ is called a normal automorphism if $\varphi(I) = I$ for every ideal $I$ of $L$.

**Theorem 1.** Let $A_i$, $i = 1, ..., m$, be free abelian Lie algebras of finite rank and let $M$ be their metabelian product. If $\varphi$ is a normal automorphism of $M$ then $\varphi$ is an IA-automorphism.

**Proof.** Let $\varphi$ be a normal automorphism of $M$. The algebra $M$ can be considered as $M = F/F''$. Let denote by $\hat{v} = v + F''$, where $v \in F$. Then by [3] there exist $u_i \in F'$, $1 \leq i \leq m$, such that

\[
\varphi(\hat{a}_i) = \alpha a_i + \hat{u}_i,
\]

where $a_i \in A_i$. Consider the ideal $\langle \hat{a}_1 \rangle$ of $M$. Since $\varphi$ is normal we have $\varphi(\hat{a}_1) \in \langle \hat{a}_1 \rangle$ and so $\hat{u}_1 \in \langle \hat{a}_1 \rangle$ and similarly, for the ideal $\langle [a_2, a_3] \rangle$ of $M$ we have $\varphi([a_2, a_3]) \in \langle [a_2, a_3] \rangle$. Then for an element $\hat{y}$ of $\langle [a_2, a_3] \rangle$ we have

\[
\varphi([a_2, a_3]) = \alpha^2[a_2, a_3] + \hat{y}.
\]
Now consider the ideal \( \langle a_1 + [a_2, a_3] \rangle \) of \( M \). Since \( \varphi \) is normal we have \( \varphi \left( a_1 + [a_2, a_3] \right) \in \langle a_1 + [a_2, a_3] \rangle \) and for an element \( \hat{z} \) of \( \langle a_1 + [a_2, a_3] \rangle \)

\[
\varphi \left( a_1 + [a_2, a_3] \right) = c \left( a_1 + [a_2, a_3] \right) + \hat{z}
\]

where \( c \in K \). From the last equality we have

\[
(\alpha - c) \hat{a}_1 + (\alpha^2 - c) [a_2, a_3] = \hat{0}.
\]

Then we get \( c = \alpha \) and \( c = \alpha^2 \), that is, \( \alpha^2 = \alpha \). Hence \( \alpha = 1 \) and \( \varphi \) is an IA-automorphism.

**Theorem 2.** Every normal automorphism of \( M \) acts identically on \( M' \).

**Proof.** The algebra \( M \) can be considered as \( M = F/F'' \). Let denote by \( \hat{v} = v + F'' \), where \( v \in F \). Let \( \varphi \) be a normal automorphism of \( M \). By theorem 1 we have that \( \varphi \) is an IA-automorphism. Then there is an element \( \hat{v} \) of \( M' \) such that

\[
\varphi \left( [a_1, a_2] \right) = [a_1, a_2] + \hat{v},
\]

where \( a_1 \in A_1, a_2 \in A_2 \). Let \( H \) be the ideal of \( M' \) generated by the element \([a_1, a_2]\). It is clear that

\[
H = \left\{ c[a_1, a_2] : c \in K \right\}.
\]

Now suppose that \( \hat{v} \neq \hat{0} \). Consider the homomorphism \( \theta : M' \rightarrow M'/H \) which is defined \( \theta (\hat{u}) = \varphi (\hat{u}) + H \) for every element \( \hat{u} \in M' \). Since \( \varphi \) is a normal automorphism of \( M \) it is clear that \( \theta \) is an epimorphism. Let \( \hat{u} \in Ker\theta \). Consider the ideal \( \langle \hat{u} \rangle \) of \( M \). Since \( \varphi \) is normal we have \( \varphi (\hat{u}) \in \langle \hat{u} \rangle \). Then we have \( \varphi (\hat{u}) = \beta \hat{u} + \hat{w} \), where \( \hat{w} \in \langle \hat{u} \rangle \), \( \beta \in K \). Since \( \hat{u} \in Ker\theta \), we have \( \varphi (\hat{u}) \in H \), that is,

\[
\beta \hat{u} + \hat{w} \in \left\{ c[a_1, a_2] : c \in K \right\}.
\]

Thus we have \( \hat{u} = d[a_1, a_2] \), where \( d \in K \). Then we get

\[
\varphi (\hat{u}) = d[a_1, a_2] + d\hat{v} \in H.
\]

If \( \hat{v} \neq \hat{0} \) we get \( d = 0 \) and \( \hat{u} = \hat{0} \). Hence we obtain that \( \theta \) is an isomorphism. Since \( \varphi (M') = M' \) and \( \hat{v} \in M' \) there exist an element \( \hat{g} \) of \( M' \) such that \( \varphi (\hat{g}) = \hat{v} \). By the definition of \( \theta \) we have

\[
\theta (\hat{g}) = \hat{v} + H.
\]
We also have that
\[ \theta \left( \widehat{[a_1, a_2]} \right) = \widehat{v} \mathbf{H}. \]

Since \( \theta \) is an isomorphism we get
\[ \widehat{g} = \widehat{[a_1, a_2]}. \]

Thus we have
\[ \varphi \left( \widehat{[a_1, a_2]} \right) = \varphi \left( \widehat{g} \right) = \widehat{v} \]
and
\[ [a_1, a_2] + \widehat{v} = \widehat{v}. \]

We obtain that \( \widehat{[a_1, a_2]} = 0 \). This is a contradiction. Thus we get \( \widehat{v} = 0 \)
and
\[ \varphi \left( \widehat{[a_1, a_2]} \right) = \widehat{[a_1, a_2]} \]

Similarly, we obtain that
\[ \varphi \left( \widehat{[a_i, a_j]} \right) = \widehat{[a_i, a_j]}, \]
where \( a_i \in A_i, a_j \in A_j, 1 \leq i < j \leq m \). Let \( \widehat{u} \in M' \). Then \( \widehat{u} \) is a linear combinations of some elements of \( M \) of the form
\[ \ldots \widehat{[a_{j_1}, a_{j_2}, a_{j_3}, \ldots, a_{j_n}]}, \]
where \( a_{j_1}, a_{j_2}, \ldots, a_{j_n} \in \bigcup_{i=1}^{m} A_i, n \geq 2 \). we know that
\[ \varphi \left( \widehat{[a_{j_1}, a_{j_2}]} \right) = \widehat{[a_{j_1}, a_{j_2}]} \]

Since \( \varphi \) is an IA-automorphism there exist some elements \( u_{j_1}, \ldots, u_{j_n} \in F' \)
such that
\[ \varphi \left( \widehat{a_{j_k}} \right) = \widehat{a_{j_k}} + \widehat{u_{j_k}}, \ k \geq 3. \]

Then
\[ \varphi \left( \ldots \widehat{[a_{j_1}, a_{j_2}, a_{j_3}, \ldots, a_{j_n}]} \right) = \ldots \left[ \varphi \left( \widehat{[a_{j_1}, a_{j_2}]} \right), \varphi \left( \widehat{a_{j_3}} \right), \ldots, \varphi \left( \widehat{a_{j_n}} \right) \right] \\
= \ldots \left[ \widehat{[a_{j_1}, a_{j_2}, a_{j_3} + \widehat{u_{j_3}}], \ldots, a_{j_n} + \widehat{u_{j_n}}} \right] \\
= \ldots \left[ \widehat{[a_{j_1}, a_{j_2}, a_{j_3}, \ldots, a_{j_n}]} \right]. \]

Hence we get \( \varphi \left( \widehat{u} \right) = \widehat{u} \) for all \( \widehat{u} \in M' \). Therefore \( \varphi \) acts identically on \( M' \). \( \square \)
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