A NOTE ON THE MODIFIED KP HIERARCHY
AND ITS (YET ANOTHER) DISPERSIONLESS LIMIT

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Abstract. The modified KP hierarchies of Kashiwara and Miwa is formulated in Lax formalism by Dickey. Their solutions are parametrised by flag varieties. Its dispersionless limit is considered.

0. Introduction

The modified KP hierarchy (mKP hierarchy for short) is a system of non-linear differential equations satisfied by the τ function $\tau(s; t) = \langle s | \exp H(t) g | s \rangle$ introduced in early 80’s by [KM], [JM]. Several Lax representations of this system have been proposed (cf., for example, [K1], [MPZ] and references in [Di]). In this paper we take a representation in terms of differential-difference equations by Dickey [Di] in a generalised form, which is suitable for connecting the system to the Toda lattice hierarchy.

As is well-known, the solution space of the KP hierarchy is identified with the Sato Grassmann manifold (cf. [S], [SS], [SN]). The Sato Grassmann manifold consists of subspaces of an infinite dimensional linear space. We shall see that the solution space of the mKP hierarchy is the flag varieties consisting of sequences of subspaces of this linear space.

The dispersionless version of the mKP hierarchy is obtained by the same procedure as that for the KP and the Toda lattice hierarchies in [TT1], [TT2], [TT3], [TT4] and [TT5]. We will discuss several features of the dispersionless mKP hierarchies, including the Riemann-Hilbert type construction (or “twistor construction”) of solutions. We shall also see that any solution of the dispersionless Toda lattice hierarchy is automatically a solution of the modified KP hierarchies via change of variables.

Starting from representation of the mKP hierarchy different from ours, various types of dispersionless mKP hierarchy have been obtained. See, for example, [K1], [CT]. Ours is similar to the system in [K2], but, contrary to Kupershmidt’s system, we take the continuous limit of the lattice.

The author was lead to this subject by a question on Virasoro constraints posed by Anton Zabrodin, although a satisfactory answer to his question has not yet been found. The construction presented in this paper might produce a certain type of Virasoro constraints as in the case of the dispersionless KP hierarchy. See [TT2].

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This paper is organised as follows: in the first section we review the modified KP hierarchies with a slight generalisation and give the description of the solution space. The dispersionless version of the mKP hierarchies is introduced in Section 2. The third section is devoted to the Riemann-Hilbert type construction of solutions of the dispersionless mKP hierarchies. The relation of the dispersionless Toda lattice hierarchy and the dispersionless mKP hierarchies is discussed in Section 4.

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1. Modified KP hierarchy: Review and Generalisation

Fix a non-empty subset \( n \) of \( \mathbb{Z} \). We number the elements of \( n \) in the increasing order as \( \{n_0, n_1, \ldots, n_N\} \) (\( n_0 < n_1 < \cdots < n_N \)) when the cardinality of \( n \) is finite, as \( \{n_0, n_1, \ldots\} \) (\( n_0 < n_{s+1} < \cdots \)) when \( n \) is an infinite set and has the minimum number, and otherwise as \( \{n_s \}_{s \in \mathbb{Z}} \) (\( n_s < n_{s+1} \)). We denote \( m_s = n_{s+1} - n_s \) when both \( s \) and \( s + 1 \) are in \( S \), where \( S \) is the index set of \( n \).

Let us introduce one continuous variable \( x \), a series of continuous independent variables \( t = (t_1, t_2, \ldots) \) and a discrete independent variable \( s \in S \). We denote \( \partial/\partial x \) by \( \partial \). The dependent variables \( u_n(t; s) \) and \( p_k(t; s) \) are encapsulated in the Lax operator,

\[
L(s) = L(t; s) := \partial + u_2(t; s)\partial^{-1} + u_3(t; s)\partial^{-2} + \cdots = \sum_{n=1}^{\infty} u_n(t; s)\partial^{1-n},
\]

with \( u_0 = 1, u_1 = 0 \), and in the auxiliary differential operator,

\[
P(s) = P(t; s) := \partial^{m_s} + p_{m_s-1}(t; s)\partial^{m_s-1} + \cdots + p_1(t; s)\partial + p_0(t; s),
\]

for \( s \). (We do not write the \( x \)-dependence explicitly by the reason we shall mention in the next paragraph.) We always assume \( s, s + 1 \in S \) whenever the operator \( P(s) \) or the number \( m_s \) appear.

The Lax representation of the \( n \)-modified KP hierarchy (\( n \)-mKP hierarchy for short) is the system

\[
\frac{\partial L(s)}{\partial t_n} = [B_n(s), L(s)], \quad B_n(s) := (L(s)^n)_{\geq 0},
\]

\[
L(s + 1)P(s) = P(s)L(s),
\]

\[
(\partial_t - B_n(s + 1))P(s) = P(s)(\partial_t - B_n(s)),
\]

where \( (\cdot)_{\geq 0} \) denotes the usual projection of a micro-differential operator to the differential operator part. The first equation (1.3) is nothing but the Lax representation of the KP hierarchy and the remaining two equations, (1.4) and (1.5), determine the consistency of
The Lax equations (1.3) and (1.5) for $n = 1$ means that $t_1$ and $x$ appear always in a combination $t_1 + x$, so we shall omit $x$ unless it is necessary.

This formulation was introduced by Dickey [Di] when $n = \mathbb{Z}$. When $n$ consists of a single number, e.g., $n = \{0\}$, the $n$-mKP hierarchy is the KP hierarchy. If $n = \{n_s\}_{s \in S} \subset n' = \{n'_s\}_{s \in S'}$, a solution of the $n'$-mKP hierarchy gives a solution of the $n$-mKP hierarchy: let $\iota : S \rightarrow S'$ be an injection such that $n'_{\iota(s)} = n_s$. Then $(L(\iota(s)), \hat{P}(s))_{s \in S}$ is a solution of the $n$-mKP hierarchy for any solution $(L(s), P(s))_{s \in S'}$ of the $n'$-mKP hierarchy. Here $\hat{P}(s)$ is a differential operator defined by $\hat{P}(s) = P(\iota(s + 1) - 1) \cdots P(\iota(s + 1) - 1) P(\iota(s))$.

Hereafter we fix $n$ and omit it unless mentioned otherwise.

Remark 1.1. The Lax equations (1.3) are redundant since we can recover all of them from the equation (1.3) for $s = 0$, (1.4) and (1.5). See Proposition 2.2 of [Di].

Remark 1.2. If we rewrite the commutation relation (1.4) as

\[(L(s + 1) - L(s)) P(s) = [P(s), L(s)],\]

we can formally interpret it as a Lax type equation, “$L(s + 1) - L(s) = [\log P(s), L(s)]$”.

The mKP hierarchy is the consistency condition of the following system of linear equations for $w(t; s; \lambda)$:

\[
\begin{align*}
L(s)w(t; s; \lambda) &= \lambda w(t; s; \lambda), \quad \frac{\partial}{\partial t_n} w(t; s; \lambda) = B_n(s)w(t; s; \lambda), \\
P(s)w(t; s; \lambda) &= w(t; s + 1; \lambda).
\end{align*}
\]

By the argument well-known in the theory of the KP hierarchy, we have a monic 0-th order micro-differential operator, $W(s) = W(t; s) = 1 + w_1(t; s)\partial^{-1} + w_2(t; s)\partial^{-2} + \cdots$, which satisfies

\[
L(S) = W(s)\partial W(s)^{-1}, \quad \frac{\partial W(s)}{\partial t_n} = -(L(s)^n)_{<0} W(s),
\]

where $(\cdot)_{<0}$ is the projection to the negative order part. Moreover, the equations (1.4) and (1.5) assures that we can adjust $W(s)$ so that $P(s)$ is expressed as

\[
P(s) = W(s + 1)\partial^{n_s} W(s)^{-1}.
\]

Hence, the linear problem (1.7) has a solution (the wave function) of the form,

\[
\begin{align*}
w(t; s; \lambda) := W(s)\partial^{n_s} e^{\xi(t; \lambda)} &= \hat{w}(t; s; \lambda)\lambda^{n_s} e^{\xi(t; \lambda)}, \\
\hat{w}(t; s; \lambda) := 1 + w_1(t; s)\lambda^{-1} + w_2(t; s)\lambda^{-2} + \cdots,
\end{align*}
\]

where $\xi(t; \lambda) = \sum_{n=1}^{\infty} t_n \lambda^n$. The adjoint wave function is defined by

\[
\begin{align*}
w^*(t; s; \lambda) := (W(s)^*)^{-1}(-\partial)^{n_s} e^{-\xi(t; \lambda)} &= \hat{w}^*(t; s; \lambda)\lambda^{n_s} e^{-\xi(t; \lambda)}, \\
\hat{w}^*(t; s; \lambda) := 1 + w_1^*(t; s)\lambda^{-1} + w_2^*(t; s)\lambda^{-2} + \cdots,
\end{align*}
\]
where $W(s)^* = 1 + (-\partial) \circ w_1^*(t; s) + (-\partial)^2 \circ w_2^*(t; s) + \cdots$ is the formal adjoint operator of $W(s)$.

In terms of the wave operator $W(s)$, the mKP hierarchy is rewritten as

$$\frac{\partial W(s)}{\partial t_n} = -(W(s)\partial^n W(s)^{-1})_{<0} W(s), \quad (W(s + 1)\partial^{m*} W(s)^{-1})_{<0} = 0. \tag{1.12}$$

This system is expressed as the bilinear residue identity, which corresponds to (1.5.1) of [DJKM] for the KP hierarchy: two functions $w(t; s; \lambda)$ and $w^*(t; s; \lambda)$ of the form (1.10) and (1.11) are a wave function and its adjoint if and only if they satisfy

$$\oint_{\lambda=\infty} w(t; s; \lambda) w^*(t'; s'; \lambda) d\lambda = 0, \tag{1.13}$$

for any $t, t'$ and $s \geq s'$. The proof is essentially the same as that in §1.5 of [DJKM] or the proof of Proposition 2.6 and 2.7 of [Di], which rewrites (1.12) by Lemma 1.1 of [DJKM] cited as Lemma 2.5 in [Di].

The Orlov-Schulman operator (cf. [DS], [TT5]) is defined by

$$M(t; s) := W(t; s) \left( \sum_{n=1}^{\infty} nt_n \partial^{n-1} + x + n_s \partial^{-1} \right) W(t; s)^{-1}$$

$$= \sum_{n=1}^{\infty} nt_n L^{n-1} + x + n_s L^{-1} + \sum_{n=1}^{\infty} v_n(t; s) L^{-n-1} \tag{1.14}$$

and satisfies

$$\frac{\partial M}{\partial t_n} = [B_n, M], \quad M(s + 1) P(s) = P(s) M(s), \quad [L(s), M(s)] = 1. \tag{1.15}$$

The wave function defined by (1.10) satisfies

$$M(s) w(t; s; \lambda) = \frac{\partial}{\partial x} w(t; s; \lambda), \tag{1.16}$$

The Orlov-Schulman operator plays an important role when we consider the dispersionless limit. As in [DS], we can describe the symmetries of the mKP hierarchy by using $L$ and $M$ but we do not go to this direction here.

There exists the $\tau$ function, $\tau(t; s)$, which satisfies

$$\hat{w}(t; s; \lambda) = \frac{\tau(t - [\lambda^{-1}]; s)}{\tau(t; s)}. \quad \hat{w}^*(t; s; \lambda) = \frac{\tau(t + [\lambda^{-1}]; s)}{\tau(t; s)}. \tag{1.17}$$

This is the direct consequence of the theory of the KP hierarchy. See p. 269 of [SS] or §1.6 of [DJKM]. Note that we have a gauge freedom,

$$\tau(t; s) \mapsto \tau(t; s) e^{\Lambda(s)}, \tag{1.18}$$

where $\Lambda(s)$ is an arbitrary function of $s$ and independent of $t_n (n = 1, 2, \ldots)$. We can write down the bilinear equations characterising the $\tau$ function by substituting the expressions
The representation theoretical description of the $\tau$ function of the modified KP hierarchy goes back to [KM] and [JM].

As is discussed in [UT], a solution of the Toda lattice hierarchy gives a series of solutions of the KP hierarchy parametrised by a discrete variable $s$, if one fixes the half set of the continuous independent variables. In fact, this series is a solution of the $n$-mKP hierarchy defined above where $n = \mathbb{Z}$. The differential operator $P(t; s)$ is the operator $\partial - b_0(s)$ at the end of §1.2 of [UT].

While each solution of the KP hierarchy corresponds to a point on the Sato Grassmann manifold, the solution of the mKP hierarchy corresponds to a point on an infinite dimensional flag variety. More precise statement is as follows. Let $V$ be an infinite dimensional linear space and $V^0$ is its subspace defined by $V = \bigoplus_{\nu \in \mathbb{Z}} \mathbb{C}e_\nu$, $V^{(0)} = \bigoplus_{\nu \geq 0} \mathbb{C}e_\nu$. (Actually we have to take the completion of $V$, but details are omitted.) The Sato Grassmann manifold of charge $n$, $SGM(n)$, is defined by

\[(1.19) \quad SGM(n) = \{ U \subset V \mid \text{index of } U \to V/V^{(0)} \text{ is } n \} \]

The solution space of the KP hierarchy is $SGM(0)$ as is shown in [S], [SS] or [SN].

**Proposition 1.3.** Each solution of the $n$-mKP hierarchy for $W(t; s)$, \( (1.12) \), is parametrised by the flag variety,

\[(1.20) \quad \text{Flag}_n := \{ (U_s)_{s \in \mathbb{S}} \mid U_s \in SGM(n_s), U_s \subset U_{s+1} \} \]

The proof is the same as that of Corollary 3.4 of [Take].

## 2. Dispersionless limit of modified KP hierarchy

The dispersionless version of the mKP hierarchy is obtained by the same procedure as that for the KP and the Toda lattice hierarchies in [TT1], [TT2], [TT3], [TT4] and [TT5]. Since the discrete parameter $s$ of the mKP hierarchy turns into a continuous parameter, we need to start from the $n$-mKP hierarchy with constant $m_s = n_{s+1} - n_s$, which is the order of the differential operator $P(t; s)$. We denote this number by $N$ and assume $n = N\mathbb{Z} = \{ n_s = Ns \}_{s \in \mathbb{Z}}$.

First, let us introduce a parameter $\hbar$ and rewrite the equations \((1.3), (1.4) \) or \((1.6), (1.3) \) and \((1.15) \), rescaling the independent variables $t$ and $s$, as follows:

\[(2.1) \quad \hbar \frac{\partial L}{\partial t_n} = [B_n, L], \quad \hbar \frac{\partial M}{\partial t_n} = [B_n, M], \quad B_n := (L^n)_{\geq 0}, \]

\[(2.2) \quad [L(s), M(s)] = \hbar, \]

\[(2.3) \quad (L(s + \hbar) - L(s))P(s) = [P(s), L(s)], \]

\[(2.4) \quad (M(s + \hbar) - M(s))P(s) = [P(s), M(s)], \]

\[(2.5) \quad -(B_n(s + \hbar) - B_n(s))P(s) = [P(s), \hbar \partial_{t_n} - B_n(s)], \]
where the Lax operator $L(t; s)$, the Orlov-Schulman operator $M(t; s)$ and the differential operator $P(t; s)$ have the following form:

\begin{equation}
L(t; s) = \sum_{n=1}^{\infty} u_n(t; s) (\hbar \partial)^{1-n},
\end{equation}

\begin{equation}
M(t; s) = \sum_{n=1}^{\infty} n t_n L^{n-1} + x + N s L^{-1} + \sum_{n=1}^{\infty} u_n(t; s) L^{-n-1}
\end{equation}

\begin{equation}
P(t; s) = (\hbar \partial)^N + p_{N-1}(t; s)(\hbar \partial)^{N-1} + \cdots + p_0(t; s).
\end{equation}

The leading terms of (2.1), (2.2), (2.3), (2.4) and (2.5) with respect to the order $(\text{ord} \ h = -1, \ \text{ord} \ \partial/\partial x = \text{ord} \ \partial/\partial t_n = 1)$ give the dispersionless $N$-modified KP hierarchy. The operators $L(t; s)$, $M(t; s)$ and $P(t; s)$ are replaced by formal series in variable $k$,

\begin{equation}
\mathcal{L}(t; s) = \sum_{n=1}^{\infty} u_n(t; s) k^{1-n} = k + u_2(t; s)k^{-1} + \cdots,
\end{equation}

\begin{equation}
\mathcal{M}(t; s) = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + x + N s \mathcal{L}^{-1} + \sum_{i=1}^{\infty} v_i(t; s) \mathcal{L}^{-i-1},
\end{equation}

\begin{equation}
P(t; s) = k^N + p_{N-1}(t; s)k^{N-1} + \cdots + p_0(t; s).
\end{equation}

(We set $v_0(t; s) = N s$.) The system of differential-difference equations (2.1) etc. become the differential equations with respect to $t$ and $s$:

\begin{equation}
\frac{\partial \mathcal{L}}{\partial t_n} = \{ \mathcal{B}_n, \mathcal{L} \}, \quad \frac{\partial \mathcal{M}}{\partial t_n} = \{ \mathcal{B}_n, \mathcal{M} \}, \quad \mathcal{B}_n := (\mathcal{L}^n)_{\geq 0},
\end{equation}

\begin{equation}
\{ \mathcal{L}(s), \mathcal{M}(s) \} = 1,
\end{equation}

\begin{equation}
\frac{\partial \mathcal{L}}{\partial s} \mathcal{P}(s) = \{ \mathcal{P}(s), \mathcal{L}(s) \}, \quad \frac{\partial \mathcal{M}}{\partial s} \mathcal{P}(s) = \{ \mathcal{P}(s), \mathcal{M}(s) \},
\end{equation}

\begin{equation}
- \frac{\partial \mathcal{B}_n}{\partial s} \mathcal{P}(s) = - \frac{\partial \mathcal{P}}{\partial t_n} - \{ \mathcal{P}(s), \mathcal{B}_n(s) \},
\end{equation}

where $\{ , \}$ is the projection of power series in $k$ to the polynomial part and $\{ , \}$ is the same Poisson bracket as the Poisson bracket for the dispersionless KP hierarchy:

\begin{equation}
\{ f(k, x), g(k, x) \} = \frac{\partial f}{\partial k} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial k}.
\end{equation}

The three equations, (2.14) and (2.13) can be formally interpreted as follows (cf. Remark 1.2):

\begin{equation}
\frac{\partial \mathcal{L}}{\partial s} = \{ \log \mathcal{P}(s), \mathcal{L}(s) \}, \quad \frac{\partial \mathcal{M}}{\partial s} = \{ \log \mathcal{P}(s), \mathcal{M}(s) \},
\end{equation}

\begin{equation}
\frac{\partial}{\partial t_n} \log \mathcal{P} - \frac{\partial \mathcal{B}_n}{\partial s} + \{ \log \mathcal{P}, \mathcal{B}_n \} = 0.
\end{equation}
We can also define the dispersionless $N$-mKP hierarchy as the system for $\mathcal{L}$ and $\mathcal{P}$ only. Namely, it is enough to take the equations (2.12), (2.14) and (2.15) for $\mathcal{L}$ and $\mathcal{P}$. It is possible to introduce the dressing operator $\exp \text{ad} \varphi$ as in [TT5]. (\text{ad} f(g) = \{f, g\}.) The series $\varphi(t; s) = \sum_{n=1}^{\infty} \varphi_n(t) k^{-n}$ satisfies
\begin{equation}
\begin{aligned}
\nabla_{t_n} \varphi &= \mathcal{B}_n - e^{\text{ad} \varphi} k^n = -(e^{\text{ad} \varphi}(k^n))_{<0}, \\
\nabla_{s, \varphi} \varphi &= \log \mathcal{P} - e^{\text{ad} \varphi} \log k^N,
\end{aligned}
\end{equation}

(2.19)

where $\nabla_{u, \psi} \varphi = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} (\text{ad} \psi)^m \frac{\partial \varphi}{\partial u}$, for series $\psi$ and $\varphi$ and a variable $u$. The symbol $(\cdot)_{<0}$ denotes the projection to the negative power part as usual. Equations (2.12) (for $\mathcal{L}$), (2.14) (for $\mathcal{L}$) and (2.15) are the compatibility conditions of (2.19). To define the Orlov-Schulman function $\mathcal{M}$ we have only to put
\begin{equation}
\mathcal{M} = e^{\text{ad} \varphi} \left( \sum_{n=1}^{\infty} n t_n k^{n-1} + x + N s k^{-1} \right).
\end{equation}

(2.20)

**Lemma 2.1.** The coefficient $v_n$ in (2.10) satisfies
\begin{equation}
\begin{aligned}
\frac{\partial v_n}{\partial t_m} &= \text{Res}_{k=\infty} \mathcal{L}^n d \mathcal{B}_m, \\
\frac{\partial v_n}{\partial s} &= \text{Res}_{k=\infty} \mathcal{L}^n d \log \mathcal{P}.
\end{aligned}
\end{equation}

(2.21)

The first equation is Proposition 4 of [TT2] and the second equation is proved in the same way. Using these relations, we can show that $\mathcal{B}_n$ and $\mathcal{P}$ are expanded with respect to $\mathcal{L}$ as follows:
\begin{equation}
\mathcal{B}_n = \mathcal{L}^n + \sum_{m=1}^{\infty} \frac{-1}{m} \frac{\partial v_m}{\partial t_n} \mathcal{L}^{-m}, \quad \mathcal{P} = \mathcal{L}^N \exp \left( \sum_{m=1}^{\infty} \frac{-1}{m} \frac{\partial v_m}{\partial s} \mathcal{L}^{-m} \right).
\end{equation}

(2.22)

The first equation is (4.7) of [TT2], while the second is proved by applying the similar argument to $\log(\mathcal{P}/\mathcal{L}^N)$ instead of $\mathcal{B}_n$.

It follows from Lemma 2.1 and Proposition 6 of [TT2] (a consequence of (2.21); $\partial v_n/\partial t_m = \partial v_m/\partial t_n$) that the system for a function $\phi(t; s)$
\begin{equation}
\frac{\partial \phi}{\partial t_n} = \text{Res}_{k=\infty} \mathcal{L}^n d \log \mathcal{P},
\end{equation}

(2.23)

is consistent. The $s$-dependence of $\phi$ is not fixed by these equations and we may add any function of $s$ independent of $t_n$ to $\phi(t; s)$.

The $\tau$ function for the dispersionless mKP hierarchy is defined by
\begin{equation}
d \log \tau_{\text{dmKP}}(t; s) = \sum_{n=1}^{\infty} v_n(t; s) dt_n + \phi(t; s) ds,
\end{equation}

(2.24)
where the exterior differentiation \( d \) is taken with respect to the variables \( t_n (n = 1, 2, \ldots) \) and \( s \). Because of the additive ambiguity of \( \phi(t; s) \) mentioned above, \( \log \tau_{\text{dmKP}}(t; s) \) has the gauge freedom:

\[
(2.25) \quad \log \tau_{\text{dmKP}}(t; s) \mapsto \log \tau_{\text{dmKP}}(t; s) + \Lambda(s),
\]

for an arbitrary function \( \Lambda(s) \) of \( s \). This corresponds to the gauge freedom of the \( \tau \) function of the mKP hierarchy, (1.18).

### 3. Twistor construction of solutions of the dmKP hierarchy

Any solution \((\mathcal{L}, \mathcal{M})\) of the dispersionless KP hierarchy is obtained from a pair of functions \((f(k, x), g(k, x))\) ("twistor data") with the canonical Poisson relation \(\{f, g\} = 1\) by requiring

\[
(3.1) \quad (f(\mathcal{L}, \mathcal{M}))_{<0} = 0, \quad (g(\mathcal{L}, \mathcal{M}))_{<0} = 0,
\]

which was shown in [TT2] and [TT5]. We call this method "the twistor construction" or "the Riemann-Hilbert type construction" of solutions.

This theorem implies that any solution of the dispersionless mKP hierarchy \((\mathcal{L}(s), \mathcal{M}(s), P(s))\) should be associated to twistor data \((f(k, x, s), g(k, x, s))\) depending on the variable \( s \), satisfying

\[
(3.2) \quad (f(\mathcal{L}(s), \mathcal{M}(s), s))_{<0} = 0, \quad (g(\mathcal{L}(s), \mathcal{M}(s), s))_{<0} = 0.
\]

The dependence of the series \( f(k, x, s) \) and \( g(k, x, s) \) on \( s \) is given by the following proposition. Hereafter we use the notations like

\[
\frac{\partial f}{\partial s}(\mathcal{L}(s), \mathcal{M}(s), s) := \left. \frac{\partial f}{\partial s}(k, x, s) \right|_{k=\mathcal{L}(s), x=\mathcal{M}(s)},
\]

\[
\frac{\partial}{\partial s} f(\mathcal{L}(s), \mathcal{M}(s), s) := \frac{\partial}{\partial s} \left( f(\mathcal{L}(s), \mathcal{M}(s), s) \text{ as a function of } s \right).
\]

We define

\[
(3.3) \quad \hat{\mathcal{L}}(s) = f(\mathcal{L}(s), \mathcal{M}(s), s), \quad \hat{\mathcal{M}}(s) = g(\mathcal{L}(s), \mathcal{M}(s), s),
\]

which consists of positive powers of \( k \) if \((3.2)\) is satisfied.

**Proposition 3.1.** If the series \( \mathcal{L}(s), \mathcal{M}(s) \) and the polynomial \( P(s) \) of the form \((2.9), (2.10)\) and \((2.11)\) satisfy \((3.2)\) and

\[
(3.4) \quad \left( P(s) \frac{\partial}{\partial s} \hat{\mathcal{L}}(s) + \{P(s), \hat{\mathcal{L}}(s)\} - P(s) \frac{\partial f}{\partial s}(\mathcal{L}(s), \mathcal{M}(s), s) \right)_{<N-1} = 0,
\]

\[
\left( P(s) \frac{\partial}{\partial s} \hat{\mathcal{M}}(s) + \{P(s), \hat{\mathcal{M}}(s)\} - P(s) \frac{\partial g}{\partial s}(\mathcal{L}(s), \mathcal{M}(s), s) \right)_{<N-1} = 0,
\]

\[\text{[For recent developments of twistor construction of solutions of the dispersionless KP equations, see [DMT] and [Du].}\]
then \((\mathcal{L}(s), \mathcal{M}(s), \mathcal{P}(s))\) is a solution of the dispersionless mKP hierarchy. Here \((\cdot)_{N-1}\) is the projection to the Laurent series in \(k\) with powers less than \(N-1\).

Note that the contents in the parentheses of (3.4) vanish for any function \(f\) and \(g\) if \((\mathcal{L}, \mathcal{M}, \mathcal{P})\) is a solution of the dispersionless mKP hierarchy.

The conditions (3.4) are not so neat as (3.2), but when \(N=1\), they reduce to the following due to (3.2):

\[
(3.5) \quad \left( \mathcal{P}(s) \frac{\partial f}{\partial s}(\mathcal{L}(s), \mathcal{M}(s), s) \right)_{<0} = 0, \quad \left( \mathcal{P}(s) \frac{\partial g}{\partial s}(\mathcal{L}(s), \mathcal{M}(s), s) \right)_{<0} = 0.
\]

**Proof of Proposition 3.4.** The conditions (3.2) assure that \((\mathcal{L}(s), \mathcal{M}(s))\) are solutions of the dispersionless KP hierarchy, (2.12) and (2.13) by virtue of Proposition 7 of [TT2]. Therefore we have only to check the equations (2.14) and (2.15).

Differentiating (3.3) with respect to \(s\), we have

\[
(3.6) \quad \left( \frac{\partial \hat{\mathcal{L}}}{\partial s} - \frac{\partial f}{\partial s}(\mathcal{L}, \mathcal{M}, s) \right) = \left( \frac{\partial f(\mathcal{L}, \mathcal{M}, s)}{\partial \hat{\mathcal{L}}} \frac{\partial f(\mathcal{L}, \mathcal{M}, s)}{\partial \hat{\mathcal{M}}} \right) \left( \frac{\partial \hat{\mathcal{L}}}{\partial \hat{\mathcal{M}}} \right).
\]

According to the equation (6.5) of [TT2], the \(2\times2\) matrix in the right hand side is decomposed as

\[
(3.7) \quad \left( \begin{array}{cc}
\frac{\partial f(\mathcal{L}, \mathcal{M}, s)}{\partial \hat{\mathcal{L}}} & \frac{\partial f(\mathcal{L}, \mathcal{M}, s)}{\partial \hat{\mathcal{M}}} \\
\frac{\partial g(\mathcal{L}, \mathcal{M}, s)}{\partial \hat{\mathcal{L}}} & \frac{\partial g(\mathcal{L}, \mathcal{M}, s)}{\partial \hat{\mathcal{M}}}
\end{array} \right) = \left( \begin{array}{cc}
\frac{\partial \hat{\mathcal{L}}}{\partial \hat{\mathcal{M}}} & \frac{\partial \hat{\mathcal{L}}}{\partial \hat{s}} \\
-\frac{\partial \hat{\mathcal{M}}}{\partial \hat{s}} & -\frac{\partial \hat{\mathcal{M}}}{\partial \hat{s}}
\end{array} \right) \left( \begin{array}{cc}
\frac{\partial \hat{\mathcal{L}}}{\partial \hat{\mathcal{M}}} & \frac{\partial \hat{\mathcal{L}}}{\partial \hat{s}} \\
-\frac{\partial \hat{\mathcal{M}}}{\partial \hat{s}} & -\frac{\partial \hat{\mathcal{M}}}{\partial \hat{s}}
\end{array} \right)^{-1}
\]

where we used the canonical Poisson relations \(\{\mathcal{L}(s), \mathcal{M}(s)\} = \{\hat{\mathcal{L}}(s), \hat{\mathcal{M}}(s)\} = 1\) (cf. (6.8) of [TT2]). Substituting (3.7) into (3.6) and multiplying \(\mathcal{P}(s)\), we have

\[
(3.8) \quad \left( \begin{array}{cc}
\frac{\partial \hat{\mathcal{M}}}{\partial \hat{x}} & -\frac{\partial \hat{\mathcal{L}}}{\partial \hat{x}} \\
\frac{\partial \hat{\mathcal{M}}}{\partial \hat{k}} & \frac{\partial \hat{\mathcal{L}}}{\partial \hat{k}}
\end{array} \right) \left( \begin{array}{c}
\mathcal{P}(s) \frac{\partial \hat{\mathcal{L}}}{\partial \hat{s}} - \mathcal{P}(s) \frac{\partial f}{\partial \hat{s}}(\mathcal{L}, \mathcal{M}, s) \\
\mathcal{P}(s) \frac{\partial \hat{\mathcal{M}}}{\partial \hat{s}} - \mathcal{P}(s) \frac{\partial g}{\partial \hat{s}}(\mathcal{L}, \mathcal{M}, s)
\end{array} \right) = \mathcal{P}(s) \left( \begin{array}{c}
\frac{\partial \mathcal{M} \partial \mathcal{L}}{\partial \hat{k} \partial \hat{s}} - \frac{\partial \mathcal{M} \partial \mathcal{L}}{\partial \hat{k} \partial \hat{s}} \\
-\frac{\partial \mathcal{L} \partial \mathcal{M}}{\partial \hat{k} \partial \hat{s}} + \frac{\partial \mathcal{L} \partial \mathcal{M}}{\partial \hat{k} \partial \hat{s}}
\end{array} \right).
\]
The upper component of the vector in the right hand side is, by the computation similar to (6.12), (6.13) of [LT2], rewritten as follows:

\[
\frac{\partial M}{\partial x} \frac{\partial \mathcal{L}}{\partial s} - \frac{\partial \mathcal{L}}{\partial x} \frac{\partial M}{\partial s} = \left( \frac{\partial \mathcal{L}}{\partial s} + \sum_{n=1}^{\infty} \frac{1}{-n} \frac{\partial \mathcal{L}^{-n}}{\partial x} \right) - \left( N \frac{\partial \log \mathcal{L}}{\partial x} + \sum_{n=1}^{\infty} \frac{1}{-n} \frac{\partial \mathcal{L}^{-n}}{\partial x} \right)
\]

\[
= \frac{\partial}{\partial s} \left( \mathcal{L} + \sum_{n=1}^{\infty} \frac{1}{-n} \frac{\partial \mathcal{L}^{-n}}{\partial x} \right) - \frac{\partial}{\partial x} \left( N \log \mathcal{L} + \sum_{n=1}^{\infty} \frac{1}{-n} \frac{\partial \mathcal{L}^{-n}}{\partial s} \right).
\]

Thus we have

\[
\frac{\partial M}{\partial x} \frac{\partial \mathcal{L}}{\partial s} - \frac{\partial \mathcal{L}}{\partial x} \frac{\partial M}{\partial s} = - \frac{\partial}{\partial x} \log Q,
\]

due to (4.9) and (5.6) of [LT2], where

\[
Q = \mathcal{L}^N \exp \left( \sum_{n=1}^{\infty} \frac{1}{-n} \frac{\partial \mathcal{L}^{-n}}{\partial s} \right).
\]

(Reminding (2.22), one would expect that Q should be P, which will turn out to be true.)

The lower component of the vector in the right hand side of (3.8) is, by the same computation,

\[
\frac{\partial M}{\partial k} \frac{\partial \mathcal{L}}{\partial s} - \frac{\partial \mathcal{L}}{\partial k} \frac{\partial M}{\partial s} = - \frac{\partial}{\partial k} \log Q.
\]

Thus, substituting (3.10) and (3.12) into (3.8) and adding the equation

\[
\left( \frac{\partial \hat{M}}{\partial x} \frac{\partial \hat{L}}{\partial s} - \frac{\partial \hat{L}}{\partial x} \frac{\partial \hat{M}}{\partial s} \right) \left\{ \log P(s), \hat{\mathcal{L}}(s) \right\} = \left( \frac{\partial}{\partial x} \log P \right) \left\{ \frac{\partial}{\partial k} \log P \right\},
\]

we obtain

\[
\left( \frac{\partial \hat{M}}{\partial x} \frac{\partial \hat{L}}{\partial k} - \frac{\partial \hat{L}}{\partial x} \frac{\partial \hat{M}}{\partial k} \right) \left( \frac{\partial}{\partial x} P(s) \frac{\partial \hat{L}}{\partial s} + \left\{ P(s), \hat{\mathcal{L}}(s) \right\} - P(s) \frac{\partial f}{\partial s} (\mathcal{L}, \mathcal{M}, s) \right)
\]

\[
= P(s) \left( - \frac{\partial}{\partial x} \log(\mathcal{Q}/\mathcal{P}) \right) \left( - \frac{\partial}{\partial k} \log(\mathcal{Q}/\mathcal{P}) \right).
\]

This is the point where the condition (3.4) plays the role. According to this condition, the left hand side does not contain \( k^m \) with \( m < N - 1 \). Because of the normalisation (2.11) of \( \mathcal{P} \) and the definition (3.11) of \( Q \), the second component of the right hand side of (3.13) is a Laurent series of order not greater than \( N - 1 \), which proves \( \frac{\partial}{\partial k} \log(\mathcal{Q}/\mathcal{P}) = 0 \). Hence
\( P = Q \times (\text{constant independent of } k) \). The normalisations of \( P \) and \( Q \) implies \( P = Q \). Thus the equations (3.10) and (3.13) take the form

\[
\begin{pmatrix}
\frac{\partial \hat{M}}{\partial x} & -\frac{\partial \hat{L}}{\partial x} \\
\frac{\partial \hat{M}}{\partial k} & -\frac{\partial \hat{L}}{\partial k}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial L}{\partial s} \\
\frac{\partial M}{\partial s}
\end{pmatrix}
= \begin{pmatrix}
-\frac{\partial}{\partial x} \log P \\
-\frac{\partial}{\partial k} \log P
\end{pmatrix},
\]

from which follows (2.14) by the unimodularity of the matrix in the left hand side.

Lastly we prove (2.15). By decomposing \( L \) as \( B_n + (L)_{<0} \), this equation is rewritten as

\[
\frac{\partial}{\partial t_n} \log \left( \frac{P}{L^N} \right) + \frac{\partial}{\partial s} (L^n)_{<0} - \{ \log (P/L^n), (L^n)_{<0} \} = 0.
\]

Note that the relations (2.22) hold for our \( L \) and \( P \), since the first equation is the result of the theory of the dispersionless KP hierarchy only and the second equation is the result of (3.11) and \( P = Q \). Hence we may substitute them into (3.13). The rest of the proof is a straightforward computation using the Lax equation (2.12).

4. Relation with the dispersionless Toda lattice hierarchy

In this section we show that any solution of the dispersionless Toda lattice hierarchy provides a solution of the dispersionless mKP hierarchy.

Before discussing this relation, recall that the relation among the (dispersionful) mKP hierarchies mentioned in §1: if \( n \subset n' \subset \mathbb{Z} \), then a solution of the \( n' \)-mKP hierarchy gives a solution of the \( n \)-mKP hierarchy. The dispersionless version of this relation is almost trivial: suppose \( N = mN' \) for a positive integer \( m \). If \((L(s), P(s))\) is a solution of the dispersionless \( N \)-mKP hierarchy, setting \( s' = ms \) and \( \tilde{P}(s') = P(s)^m \), we have

\[
\frac{\partial L}{\partial s'} = \{ \log \tilde{P}(s'), L \}, \quad \frac{\partial}{\partial t_n} \log \tilde{P}(s') - \frac{\partial B_n}{\partial s'} + \{ \log \tilde{P}(s'), B_n \} = 0,
\]

which means that \((L(s'), \tilde{P}(s'))\) is a solution of the dispersionless \( N' \)-mKP hierarchy. Hence to show that the dispersionless \( N \)-mKP hierarchy is embedded in the dispersionless Toda lattice hierarchy, it suffices to show the case \( N = 1 \), which we always assume in this section.

The dispersionless Toda lattice hierarchy is defined as follows: It is a system of differential equations with two sets of infinite variables \( t = (t_1, t_2, \ldots) \) and \( \bar{t} = (\bar{t}_1, \bar{t}_2, \ldots) \) and one more variable \( s \). The Lax representation is:

\[
\begin{align*}
\frac{\partial L}{\partial t_n} &= \{ B_n, L \}_\text{Toda}, \quad \frac{\partial L}{\partial \bar{t}_n} = \{ \bar{B}_n, L \}_\text{Toda}, \\
\frac{\partial \bar{L}}{\partial t_n} &= \{ B_n, \bar{L} \}_\text{Toda}, \quad \frac{\partial \bar{L}}{\partial \bar{t}_n} = \{ \bar{B}_n, \bar{L} \}_\text{Toda}, \quad n = 1, 2, \ldots,
\end{align*}
\]
where $\mathcal{L}$ and $\mathcal{L}$ are Laurent series
\begin{equation}
\mathcal{L} = p + \sum_{n=0}^{\infty} u_{n+1}(t, \bar{t}, s)p^{-n}, \quad \mathcal{L}^{-1} = \bar{u}_0(t, \bar{t}, s)p^{-1} + \sum_{n=0}^{\infty} \bar{u}_{n+1}(t, \bar{t}, s)p^n,
\end{equation}
of a variable $p$, and $B_n$ and $\bar{B}_n$ are given by $B_n = (\mathcal{L}^n)_{\geq 0}$, $\bar{B}_n = (\mathcal{L}^{-n})_{< 0}$. Here $(\cdot)_{\geq 0}$ and $(\cdot)_{< 0}$ denote the projection of power series to a positive power part and a negative power part respectively.

The Poisson bracket $\{\cdot, \cdot\}_\text{Toda}$ is defined by
\begin{equation}
\{A(p, s), B(p, s)\}_\text{Toda} = p \frac{\partial A(p, s)}{\partial p} \frac{\partial B(p, s)}{\partial s} - p \frac{\partial A(p, s)}{\partial s} \frac{\partial B(p, s)}{\partial p}.
\end{equation}
The Orlov-Schulman functions $M$ and $\bar{M}$ are of the form
\begin{equation}
M = \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + s + \sum_{n=1}^{\infty} v_n(t, \bar{t}, s) \mathcal{L}^{-n},
\end{equation}
\begin{equation}
\bar{M} = -\sum_{n=1}^{\infty} n \bar{t}_n \mathcal{L}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n(t, \bar{t}, s) \mathcal{L}^n.
\end{equation}
They satisfy the Lax equations (4.1) with $\mathcal{L}$ and $\mathcal{L}$ replaced by $M$ and $\bar{M}$, and the canonical Poisson relations $\{\mathcal{L}, M\}_\text{Toda} = \mathcal{L}$, $\{\mathcal{L}, \bar{M}\}_\text{Toda} = \mathcal{L}$.

Proposition 2.8.1 of [TT5] asserts that, if $(\mathcal{L}, \mathcal{L})$ is a solution of the dispersionless Toda hierarchy (4.1), then $L(t, \bar{t}, s)$ is a solution of the dispersionless KP hierarchy (2.12) when we identify $B_1 = p + u_1(t, \bar{t}, s)$ with $k$:
\begin{equation}
\frac{\partial L}{\partial t_n} \bigg|_{k: \text{fixed}} = \{B_n, \mathcal{L}\}.
\end{equation}
Here $\{\cdot, \cdot\}$ is the Poisson bracket for the dispersionless KP hierarchy, (2.16). Note that the projection onto a polynomial in $p$ is equal to the projection onto a polynomial in $k$, since $k = B_1 = p + u_1(t, \bar{t}, s)$. Thus $B_n = (\mathcal{L}^n)_{\geq 0}$ in the sense of (2.12) as well as in the sense of (1.1).

In fact, it is a direct computation to check that the above solution of the dispersionless KP hierarchy $\mathcal{L}(s)$ together with $\mathcal{P} = p - k - u_1(t, \bar{t}, s)$ gives a solution of the dispersionless mKP hierarchy (2.12), (2.14) and (2.15).

As is shown in [ITT1], if there is a pair of functions $(f_{\text{Toda}}(p, s), g_{\text{Toda}}(p, s))$ in $(p, s)$ which satisfies $\{f, g\}_\text{Toda} = f$, and the series $\mathcal{L}$, $\mathcal{L}$, $M$, $\bar{M}$ of the form (1.2) and (4.4) satisfy
\begin{equation}
f_{\text{Toda}}(\mathcal{L}_{\text{Toda}}, M_{\text{Toda}}) = \mathcal{L}_{\text{Toda}}, \quad g_{\text{Toda}}(\mathcal{L}_{\text{Toda}}, M_{\text{Toda}}) = \mathcal{M}_{\text{Toda}},
\end{equation}
then $(\mathcal{L}_{\text{Toda}}, \mathcal{L}_{\text{Toda}}, M_{\text{Toda}}, \bar{M}_{\text{Toda}})$ gives a solution of the dispersionless Toda lattice hierarchy (“twistor construction of solutions”).
Proposition 4.1. Let \((f_{\text{Toda}}(p, s), g_{\text{Toda}}(p, s))\) be the above twistor data for the dispersionless Toda lattice hierarchy and \((\mathcal{L}_{\text{Toda}}, \mathcal{M}_{\text{Toda}}, \mathcal{M}_{\text{Toda}})\) be the corresponding solution. Then \((f_{\text{dmKP}}(k, x, s), g_{\text{dmKP}}(k, x, s))\) defined by

\[
\begin{align*}
    f_{\text{dmKP}}(k, x, s) &:= f_{\text{Toda}}(k, kx), \\
    g_{\text{dmKP}}(k, x, s) &:= (g_{\text{Toda}}(k, kx) - s) f_{\text{Toda}}(k, kx)^{-1},
\end{align*}
\]

satisfies the canonical Poisson relation.

Let us define the triplet \((\mathcal{L}_{\text{dmKP}}, \mathcal{M}_{\text{dmKP}}, \mathcal{P}_{\text{dmKP}})\) by

\[
\begin{align*}
    \mathcal{L}_{\text{dmKP}} &= \mathcal{L}_{\text{Toda}}, \\
    \mathcal{M}_{\text{dmKP}} &= \mathcal{M}_{\text{Toda}} \mathcal{L}_{\text{Toda}}^{-1}, \\
    \mathcal{P}_{\text{dmKP}} &= k - u_0(t; 0; s),
\end{align*}
\]

with all \(p\) in \(\mathcal{L}_{\text{Toda}}, \mathcal{M}_{\text{Toda}}\) replaced by \(k - u_0(t; 0; s)\). Then it gives a solution of the dispersionless mKP hierarchy corresponding to \((f_{\text{dmKP}}, g_{\text{dmKP}})\) by Proposition 3.1.

The proof of the canonical Poisson relation is nothing more than an elementary computation. The fact that the triplet \((\mathcal{L}_{\text{dmKP}}, \mathcal{M}_{\text{dmKP}}, \mathcal{P}_{\text{dmKP}})\) satisfies (3.2) and (3.5) is trivial by the definition of \((\mathcal{L}_{\text{dmKP}}, \mathcal{M}_{\text{dmKP}}, \mathcal{P}_{\text{dmKP}})\) and \((f_{\text{dmKP}}, g_{\text{dmKP}})\).

For example, the twistor data of the dispersionless Toda lattice hierarchy applied to the two-dimensional string theory in [Taka] and to the interface dynamics in [MWZ] is

\[
f_{\text{Toda}}(p, s) = ps^{-1}, \quad g_{\text{Toda}}(p, s) = s.
\]

Proposition 4.1 says that the solution of the dispersionless mKP hierarchy obtained from this solution by the change of variables solves the Riemann-Hilbert type problem (3.2) and (3.4) for

\[
\begin{align*}
    f_{\text{dmKP}}(k, x, s) &= x^{-1}, \\
    g_{\text{dmKP}}(k, x, s) &= (kx - s)x.
\end{align*}
\]

5. Concluding remarks

We have shown that the basic properties of the dispersionless KP and Toda hierarchies hold also for the dispersionless mKP hierarchies. The \(w_{\infty+1}\)-symmetries in the dispersionless KP and Toda hierarchies should also be found easily in the modified case, which should be inherited from the \(W_{\infty+1}\)-symmetries of the mKP hierarchies (or the Toda lattice hierarchy), though we did not discuss them in this paper.

Apparently small but maybe essential discrepancy with the KP/Toda case arises when we consider solutions. The last example (4.10) in Section 4 shows that the twistor data for the dispersionless mKP hierarchies can be rather complicated. (Recall that the analysis of the Virasoro constraint was possible in [TT2] because the twistor data are at most linear in \(\mathcal{M}\).)

Problems like construction of the \((f, g)\)-pair for the \(M\)-reduction \(\mathcal{L}^M = \text{a polynomial of } k\), namely, \(f(k, x, s) = k^M\) and analysis of the Virasoro constraints for such solutions still remain open. One of the reason why this is not so trivial is the normalisation \(v_0(t; s) = Ns\) in (2.10). To attack such problems, it might be necessary to refine Proposition 3.1.


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