The quantum brachistochrone problem for two spins-$\frac{1}{2}$ with anisotropic Heisenberg interaction

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Abstract

We study the quantum brachistochrone evolution for a system of two spins-$\frac{1}{2}$ described by an anisotropic Heisenberg Hamiltonian without $x, y$ interacting couplings in magnetic field directed along the $z$-axis. This Hamiltonian realizes quantum evolution in two subspaces spanned by $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ separately and allows us to consider the brachistochrone problem on each subspace separately. Using the evolution operator for this Hamiltonian we generate quantum gates, namely an entangler gate, SWAP gate, iSWAP gate etc.

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1. Introduction

The quantum brachistochrone problem is formulated similarly to the classical brachistochrone [1–6]: what is the optimal Hamiltonian, under a given set of constraints, such that the evolution from a given initial state $|\psi_i\rangle$ to a given final one $|\psi_f\rangle$ is achieved in the shortest time? This problem was considered for the first time by Carlini et al [1]. Using the variational principle, they presented a general framework for finding the time of optimal evolution and the optimal Hamiltonian for a quantum system with a given set of initial and final states. The authors solved this problem for some specific examples of constraints. In [2] it was shown that analogous results as in [1] can be obtained more directly using symmetry properties of the quantum state space. Their approach was based on the idea considered in [7], in which an elementary derivation was provided for the minimum time required to transform an initial quantum state into another orthogonal state. In [8] the quantum brachistochrone problem for mixed states was considered.

Bender et al explored the brachistochrone problem for a PT-symmetric non-Hermitian Hamiltonian [3] and showed that satisfying the same energy constraint for non-Hermitian PT-symmetric Hamiltonians the optimal time of evolution between the two fixed states can be...
made arbitrarily small. For a more detailed discussion on this subject see [4, 5]. The quantum brachistochrone problem for a spin-1 system in the magnetic field was solved in [9].

In [10–13] it was established that the entanglement of quantum states is important in connection with the optimization of quantum evolution. It was discovered by Giovannetti, Lloyd and Maccone [10, 11] that, in certain cases, entanglement enhances the speed of quantum evolution of composite systems. The connection between entanglement and time of evolution in the case of two-qubit and \( n \)-qubit systems was explored in [14]. The authors of [14] showed that, as the number of qubits increases, very little entanglement is needed to reach the quantum speed limit. Also, it was recently discovered that entanglement is an essential resource to achieve the speed limit in the context of the quantum brachistochrone problem [15–17].

In [18] a variational principle for finding the optimal time of realization of a target unitary operation was formulated, when the available Hamiltonians are subject to certain constraints dictated by either experimental or theoretical conditions. This method was illustrated for the case of a two-spin system described by an anisotropic Heisenberg Hamiltonian with \( J_{ij} \) interaction between spins \( (i = x, y, z) \) and magnetic field \( h_\alpha^z \) (\( \alpha = 1, 2 \) for the first and second spin, respectively) which is directed along the \( z \)-axis. Also, the authors of [18] generated three examples of target quantum gates, namely the swap of two qubits \((U_{SWAP})\), the quantum Fourier transformation and the entangler gate \((U_{ENT})\).

The time-optimal generation of quantum gates has already been studied in literature. Recently, speed limits for various unitary quantum operations in multiqubit systems under typical experimental conditions were obtained [19]. References [20–25] contain discussions on the time-optimal generation of unitary operations for a small number of qubits using Lie group methods, the theory of sub-Riemannian geometry and the Pontryagin maximum principle, and on the assumption that one-qubit operations can be performed arbitrarily fast.

The time-optimal synthesis of unitary transformation between two coupled qubits has been discussed in [26–28]. The time-optimal control algorithms used to synthesize arbitrary unitary transformation for the coupled fast and slow qubit system were presented in [29]. The lower bound on the time required to simulate a two-qubit unitary gate using a given two-qubit interaction Hamiltonian and local unitaries is provided in [30], while lower bounds on the time complexity of \( n \)-qubit gates are given in [31] and upper bounds on the time complexity on certain \( n \)-qubit gates are numerically described in [32]. A criterion for optimal quantum computation in terms of a certain geometry in Hamiltonian space was proposed by Nielsen et al [33]. Finally, it was shown in [34] that the quantum gate complexity is related to the optimal control cost problem.

In [35] it was shown that the two-qubit Hamiltonian with the \( J_{xx} \) and \( J_{yy} \) interaction can generate the iSWAP gate. Also, the authors showed that by applying this gate twice, the CNOT operation can be constructed. The quantum CNOT gate is the fundamental two-qubit gate for quantum computation [36]. This gate plays a central role in networks for quantum error correction [37]. The time-optimal implementation of the CNOT gate on indirectly coupled qubits was studied in [38].

In this paper we consider the quantum brachistochrone problem for the two-qubit system represented by the Heisenberg Hamiltonian with \( J_{ij} \) \((i = x, y, z)\), \( J_{jk} \) \((j \neq k = x, y)\) interaction between spins and magnetic field \( h_\alpha^z \) (\( \alpha = 1, 2 \) for the first and second spin, respectively) which is directed along the \( z \)-axis. This Hamiltonian realizes quantum evolution in two subspaces spanned by \(|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\) and \(|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\). We solve the quantum brachistochrone problem for each subspace separately and using the evolution operator for this Hamiltonian we construct entangler, SWAP and iSWAP gates. In comparison with the Hamiltonian from [18], our Hamiltonian contains additional \( J_{xy} \) and \( J_{yx} \) couplings which allows us to generate the entangler gate along a geodesic.
Thus, this problem might have an important application in quantum computing, quantum teleportation and quantum cryptography, because, as we will show, it allows one to reach maximally entangled states during the shortest time.

The paper is organized as follows. In section 2 we introduce and discuss the Hamiltonian of a system of two spins-1/2. In section 3 we solve the quantum brachistochrone problem on the subspace spanned by |↑↑⟩, |↑↓⟩ (subsection 3.1) and on the subspace spanned by |↑↓⟩, |↓↓⟩ (subsection 3.2) separately and obtain Hamiltonians which provide optimal evolution on each subspace. We obtain optimal conditions and the optimal time required to generate entangler, SWAP and iSWAP gates in section 4. Finally, the summary and discussion are given in section 5.

2. The Hamiltonian

Our aim is to explore quantum brachistochrone evolution for a two-qubit system. We consider a two-qubit physical system represented by two-spin interaction via anisotropic couplings $J_{jk} (j \neq k = x, y)$ and magnetic field $h^\alpha (\alpha = 1, 2$ for the first and second spin, respectively) which is directed along the z-axis. In other words, we choose the following Hamiltonian:

$$H = \sum_{i, j=x,y,z} J_{ij} \sigma_i^j + \sum_{a=1}^{2} h^a \sigma_a^z,$$

(1)

where $J_{zx} = J_{xz} = J_{zy} = J_{yz} = 0, \sigma_i^1 = \sigma_i \otimes I, \sigma_i^2 = I \otimes \sigma_i, \sigma_i$ are the Pauli matrices. Note that this Hamiltonian does not contain items with $\sigma_i^1 \sigma_j^2, \sigma_i^2 \sigma_j^2$ and $\sigma_i^1 \sigma_j^2, \sigma_i \sigma_j^2$, therefore, it realizes quantum evolution on two subspaces spanned by $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ and does not mix these subspaces. This quality allows us to rewrite Hamiltonian (1) as $H = H_I + H_{II}$, where in the basis labeled as $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$, the Hamiltonians $H_{I/II}$ read:

$$H_I = \begin{pmatrix} h^+ + J_{zz} & 0 & 0 & J_{Re} - i J_{Im} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ J_{Re} + i J_{Im} & 0 & 0 & -h^+ + J_{zz} \end{pmatrix},$$

$$H_{II} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^+ - J_{zz} & J_{Re} + i J_{Im} & 0 \\ 0 & J_{Re} - i J_{Im} & h^+ - J_{zz} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(2)

where we have introduced $h^\pm = h_1^\pm \pm h_2^\pm, J_{Re}^\pm = J_{xx} \pm J_{yy} \pm J_{zz}$ and $J_{Im}^\pm = J_{xy} \pm J_{yx}$. $H_I$ and $H_{II}$ commute ($H_I, H_{II} = 0$). Accordingly, $H_I$ realizes the evolution of a system on the $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$ subspace and $H_{II}$ realizes the evolution on the $|\downarrow\downarrow\rangle$ subspace.

Hamiltonian (1) has four eigenvalues: $E_I^\pm = J_{zz} + \omega_1, E_I^- = J_{zz} - \omega_1, E_{II}^+ = -J_{zz} + \omega_1, E_{II}^- = -J_{zz} - \omega_1$ with the corresponding eigenvectors:

$$|\psi_I^\pm\rangle = \frac{1}{\sqrt{2\omega_1(\omega_1 - h^\pm)}}[(J_{Re} \pm i J_{Im}) |\uparrow\uparrow\rangle + (\omega_1 - h^\pm) |\downarrow\downarrow\rangle],$$

$$|\psi_I^-\rangle = \frac{1}{\sqrt{2\omega_1(\omega_1 + h^+)}}[(J_{Re} \pm i J_{Im}) |\uparrow\uparrow\rangle - (\omega_1 + h^+) |\downarrow\downarrow\rangle],$$

$$|\psi_{II}^\pm\rangle = \frac{1}{\sqrt{2\omega_1(\omega_1 - h^\mp)}}[(J_{Re} \pm i J_{Im}) |\uparrow\downarrow\rangle + (\omega_1 - h^\pm) |\downarrow\uparrow\rangle],$$

$$|\psi_{II}^-\rangle = \frac{1}{\sqrt{2\omega_1(\omega_1 + h^\mp)}}[(J_{Re} \pm i J_{Im}) |\uparrow\downarrow\rangle - (\omega_1 + h^\pm) |\downarrow\uparrow\rangle],$$

(3)

where we introduce $\omega_1 = \sqrt{J_{Re}^2 + J_{Im}^2 + h^\pm}$ and $\omega_1 = \sqrt{J_{Re}^2 + J_{Im}^2 + h^\mp}$. 


Hamiltonians $H_I$ and $H_{II}$ have a common set of eigenvectors (3). $H_I$ has two eigenvalues $E^+_I, E^-_I$ with corresponding eigenvectors $|\psi^+_I\rangle, |\psi^-_I\rangle$ and one two-fold degenerate eigenvalue 0 with $|\psi^+_{II}\rangle$ and $|\psi^-_{II}\rangle$ eigenvectors. A similar situation is found for the case of $H_{II}$. It has two eigenvalues $E^+_II, E^-_II$ with eigenvectors $|\psi^+_II\rangle, |\psi^-_II\rangle$, respectively and one two-fold degenerate eigenvalue 0 with two eigenvectors $|\psi^+_I\rangle$ and $|\psi^-_I\rangle$.

By simply reordering the basis states as $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$, the Hamiltonian can be rewritten as $H = H_I \oplus H_{II}$, where

\[
H_{I,II} = \begin{pmatrix}
\hbar^{\pm} \pm J_{zz} & J_{Re} \mp i J_{Im} \\
J_{Re} \pm i J_{Im} & -\hbar^{\pm} \pm J_{zz}
\end{pmatrix}.
\]

In this basis, the further notation can be simplified. However, below we will use a standard basis labeled as $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ as we will compare our results with previous papers, in particular with [18] where the standard basis was used. Also, we will consider quantum gates which are represented in the standard basis.

3. The quantum brachistochrone

The brachistochrone problem is as follows: what is the optimal Hamiltonian, with the finite energy condition, such that the evolution from a given initial state $|\psi_i\rangle$ to a given final one $|\psi_f\rangle$ is achieved in the shortest possible time? We must assume the finite energy condition because the physical systems do not have an unbounded energy resource. For instance, the energy resource for a spin-$\frac{1}{2}$ system in the magnetic field is fixed by a value of this field. A simple finite energy condition is to assume that the difference between the largest and the smallest eigenvalues has a fixed value of $2\omega$:

\[
\Delta E = 2\omega,
\]

where $\omega$ is a constant. We assume the finite energy condition as in [18]:

\[
\text{Tr} H^2 - 2\omega^2 = 0.
\]

For a two-level system with energies $\Delta E$ and $-\Delta E$ conditions (4) and (5) coincide.

Let us provide some introduction on the quantum brachistochrone problem before we consider the quantum brachistochrone for Hamiltonian (1). In [2] a problem on an $n$-dimensional Hilbert space was considered. The authors have shown that the shortest path joining $|\psi_i\rangle$ and $|\psi_f\rangle$ should lie on the two-dimensional subspace spanned by $|\psi_i\rangle$ and $|\psi_f\rangle$. This subspace is represented by the Bloch sphere (the Bloch sphere is a sphere of the unit radius which represents the state space of a two-level system). The shortest distance $s_{min}$ between these states is given by:

\[
s_{min} = 2 \arccos(|\langle \psi_i | \psi_f \rangle|).
\]

The speed $v$ of quantum evolution is given by the Anandan–Aharonov relation [39]:

\[
v = 2\sqrt{\langle \psi(t) | \Delta H^2 | \psi(t) \rangle},
\]

where the energy uncertainty $\Delta H$ is bounded by $\omega$. We take $\hbar = 1$. The maximum speed of quantum evolution is given by:

\[
v_{max} = 2\omega.
\]

By using the result in (6) and (7) we obtain that the minimal time required to realize the unitary transportation $|\psi_i\rangle \rightarrow |\psi_f\rangle$ is given by the ratio:

\[
t_{min} = \frac{s_{min}}{v_{max}} = \frac{\arccos(|\langle \psi_i | \psi_f \rangle|)}{\omega}.
\]

The optimal Hamiltonian that generates this unitary transportation is given in [2].
Let us revert to our problem. The finite energy condition (5) for Hamiltonian (1) is:

\[ \omega_I^2 + \omega_{II}^2 + 2J_{zz}^2 = \omega^2. \] (9)

As mentioned earlier, Hamiltonian (1) realizes evolution in two subspaces separately and does not mix these subspaces, due to which we cannot observe evolution between states from different subspaces. Consequently, we can consider the quantum brachistochrone problem on each subspace separately.

Let us consider the evolution operator with Hamiltonian (1):

\[ U(t) = e^{-iHt} = e^{-iH_I t} e^{-iH_{II} t} = U_I U_{II}, \] (10)

where we use that \( H_I \) and \( H_{II} \) commute, and \( U_I(t) = e^{-iH_I t}, U_{II}(t) = e^{-iH_{II} t} \). \( U_I \) realizes the transformation \( |\psi_f⟩ = U_I |\psi_i⟩ \) on the subspace spanned by \(|↑↑⟩, |↓↓⟩\) and acts as a unit operator on another subspace (the subspace spanned by \(|↑↓⟩, |↓↑⟩\)). Contrary to \( U_I \), \( U_{II} \) realizes similar transformation \( |\psi_f⟩ = U_{II} |\psi_i⟩ \) on the subspace spanned by \(|↑↓⟩, |↓↑⟩\) and acts as a unit operator on the subspace spanned by \(|↑↑⟩, |↓↓⟩\). Let us consider the quantum brachistochrone problem on each subspace in detail.

3.1. Quantum evolution on the subspace spanned by \(|↑↑⟩, |↓↓⟩\)

We consider the quantum brachistochrone problem of two spins-\( \frac{1}{2} \) represented by Hamiltonian (1) on the \(|↑↑⟩, |↓↓⟩\) subspace. As we noticed earlier Hamiltonian \( H \) (1) acts on any state \(|ψ⟩ = a|↑↑⟩ + b|↓↓⟩\) (where the normalization condition is the following: \( |a|^2 + |b|^2 = 1 \)) as:

\[ H|ψ⟩ = H_I |ψ⟩. \]

Therefore, we consider the quantum brachistochrone problem on this subspace using Hamiltonian \( H_I \) from (2). Let us introduce the following operators:

\[ \sigma_z^I = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_y^I = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \sigma_z^I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad I^I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (11)

These operators satisfy the properties of ordinary Pauli matrices \( \sigma_z, \sigma_y, \sigma_z \) and unit matrix \( I \) for \(|↑↑⟩, |↓↓⟩\) states when we denote them as follows: \(|↑↑⟩ = |↑⟩, |↓↓⟩ = |↓⟩\).

With the help of these introduced operators (11) Hamiltonian \( H_I \) from (2) can be written in the form:

\[ H_I = \sigma_z^I \cdot h^I + J_{zz}^I I^I, \] (12)

where \( h^I = (J_{Rz}, J_{Lz}, h^+) \). This Hamiltonian is similar to the Hamiltonian of one spin-\( \frac{1}{2} \) in the magnetic field. In the case of one spin-\( \frac{1}{2} \), vector \( h^I \) is vector of the magnetic field. The brachistochrone problem for the spin-\( \frac{1}{2} \) in the magnetic field was considered in the paper [6]. We cannot use the result from this paper because we assume another finite energy condition (5) when comparing to paper [6]. In [6] the author fixed the largest and the smallest eigenvalues of the Hamiltonian.

Now, using (12) we represent the evolution operator \( U_I = e^{-iH_{II}} \) as follows:

\[ U_I = \left( I - I^I + \cos(\omega_I t) I^I - \frac{i}{\omega_I} \sin(\omega_I t) \sigma_z^I \cdot h^I \right) A^I, \] (13)
where
\[
A^1 = \begin{pmatrix}
    e^{-i\omega t} & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & e^{-i\omega t}
\end{pmatrix}.
\]

Here we use that \((\sigma^1 \cdot \mathbf{h})^2n = \omega t 2^n I^1\) and \((\sigma^1 \cdot \mathbf{h})^{2n+1} = \omega t 2^n \sigma^1 \cdot \mathbf{h}^1\) where \(n = 1, 2, 3, \ldots\). In a matrix form (13) reads:
\[
U_1 = \begin{pmatrix}
    (\cos(\omega t) - \frac{i}{\omega} \sin(\omega t) h^+)^e^{-i\omega t} & 0 & 0 & -\frac{i}{\omega} \sin(\omega t)(J_{Re}^- - i J_{Im}^+) e^{-i\omega t} \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    -\frac{i}{\omega} \sin(\omega t)(J_{Re}^- + i J_{Im}^+) e^{-i\omega t} & 0 & 0 & (\cos(\omega t) + \frac{i}{\omega} \sin(\omega t) h^+) e^{-i\omega t}
\end{pmatrix}.
\] (14)

Let us put the initial state as |ψ⟩ = |↑↑⟩ and the final one as |ψ⟩ = |↓↓⟩. Then, using the matrix representation for the evolution operator \(U_1\) (14), the relation \(|ψ⟩ = U_1|ψ⟩\) takes the form:
\[
\begin{pmatrix}
a \\
0 \\
0 \\
b
\end{pmatrix} = e^{-i\omega t} \begin{pmatrix}
  \cos(\omega t) - \frac{i}{\omega} \sin(\omega t) h^+ & 0 & 0 & -\frac{i}{\omega} \sin(\omega t)(J_{Re}^- - i J_{Im}^+) e^{-i\omega t} \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  -\frac{i}{\omega} \sin(\omega t)(J_{Re}^- + i J_{Im}^+) e^{-i\omega t} & 0 & 0 & (\cos(\omega t) + \frac{i}{\omega} \sin(\omega t) h^+) e^{-i\omega t}
\end{pmatrix} \begin{pmatrix}
a \\
0 \\
0 \\
b
\end{pmatrix}.
\] (15)

From the fourth component of (15) we obtain the necessary condition that the initial state reaches the final one:
\[
\frac{1}{\omega t} \sin(\omega t) \sqrt{J_{Re}^2 + J_{Im}^2} = |b|.
\] (16)

From equation (16) we obtain the time required to transform the initial state |↑↑⟩ into the final one \(a|↑↑⟩ + b|↓↓⟩\): \(t = \frac{1}{\omega t} \sin \left( \frac{1}{\omega t} \sqrt{J_{Re}^2 + J_{Im}^2} |b| \right)\). Now, using the finite energy condition (9) we optimize the time of evolution if we put the following conditions:
\[
\omega t = J_{zz} = h^+ = 0.
\] (17)

The optimal time is thus:
\[
\tau = \frac{1}{\omega t} \arcsin |b|.
\] (18)

If the final state is |ψ⟩ = |↓↓⟩ then we obtain the following passage time \(\tau = \frac{\pi}{2\sigma}\) (the shortest time of evolution between the two fixed orthogonal states is called the passage time [7]).

Condition (16) does not account for the phase of \(b\). Let us find the condition that allows us to reach the phase of components of the final state during the optimal time (18). For this we insert the optimal time of evolution (18) and conditions (17) in equation (15). From the fourth component of (15) we obtain the following equation: \(-\frac{i}{\omega t} J_{Re}^+ = J_{Im}^+ = b\), which allows us to find another condition for optimal evolution: \(J_{Re} = -\omega t h^+, J_{Im} = \omega t h^+\), where \(b = \omega t h^+\). Hence, all the necessary conditions for the optimal evolution in the subspace spanned by |↑↑⟩, |↓↓⟩ read:
\[
\begin{aligned}
J_{xx} &= -J_{yy} = -\omega t \frac{3b}{2 |b|}, & J_{xy} &= J_{yx} = \omega t \frac{3b}{2 |b|}, & J_{zz} &= 0, & h_1^z &= h_2^z = 0.
\end{aligned}
\] (19)

The initial state can reach maximally entangled states \(\frac{1}{\sqrt{2}}(|↑↑⟩ + e^{i\phi} |↓↓⟩)\) (\(\phi \in [0, 2\pi]\)) during the minimal time \(\tau = \frac{\pi}{2\sigma}\).
If we put optimal conditions (19) on Hamiltonian $H (1)$ we obtain a Hamiltonian which provides optimal evolution on the subspace spanned by $|↑↑⟩, |↓↓⟩$. In the matrix form it reads:

$$H = \begin{pmatrix} 0 & 0 & 0 & -i\omega b^\dagger \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\omega b^\dagger & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (20)

The Hilbert space of the $|↑↑⟩, |↓↓⟩$ subspace is two-dimensional and is represented by the Bloch sphere. Any pure state can be identified as a point on this sphere. The shortest distance between $|ψ_i⟩$ and $|ψ_f⟩$ is a large circle arc on the sphere (geodesic path). The optimal way of transporting $|ψ_i⟩$ into $|ψ_f⟩$ is therefore to rotate the sphere around the axis orthogonal to the large circle. The axis of rotation passes along two quantum states

$$\sqrt{2} \left[ |↑↑⟩ + i b |b⟩ |↓↓⟩ \right],$$

and

$$\sqrt{2} \left[ |↑↑⟩ - i b |b⟩ |↓↓⟩ \right]$$

which are eigenvectors of optimal Hamiltonian $H (20)$ that correspond to the largest $ω$ and the smallest $−ω$ eigenvalues.

### 3.2. Quantum evolution on the subspace spanned by $|↑↓⟩, |↓↑⟩$

In this section we consider the quantum brachistochrone problem of two spin-$\frac{1}{2}$ represented by Hamiltonian (1) on the subspace spanned by $|↑↓⟩, |↓↑⟩$. The Hamiltonian (1) acts on any state $|ψ⟩ = a |↑⟩ + b |↓⟩$ as:

$$H |ψ⟩ = H_{II} |ψ⟩.$$  \hspace{1cm} (21)

Let us introduce the following analogues of Pauli matrices and unit matrix for the basis states $|↑⟩ ≡ |↑⟩, |↓⟩ ≡ |↓⟩$:

$$\sigma^I_{x} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma^I_{y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma^I_{z} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I^I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (21)

Using (21) and following the same steps as in the previous case we rewrite Hamiltonian $H_{II}$ from (2) as:

$$H_{II} = \sigma^II \cdot h^II - J_{zz} I^II.$$  \hspace{1cm} (22)

The components of vector $h^II$ read $h^II = (J^+_R, -J^-_{in}, h^-)$. The evolution operator for the case of $U_{II} = e^{-i\omega t}$ can be represented as:

$$U_{II} = \left( I - I^II + \cos(\omega t) I^II - i \frac{\omega}{\omega^II} \sin(\omega t) \sigma^II \cdot h^II \right) A^II,$$  \hspace{1cm} (23)

where

$$A^II = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{iJ^+_R t} & 0 & 0 \\ 0 & 0 & e^{iJ^-_{in} t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Here we use that $(\sigma^I, h^I)_{2n} = \omega h^I_{2n} h^I$ and $(\sigma^I, h^I)_{2n+1} = \omega h^I_{2n} h^I$ where $n = 1, 2, 3, \ldots$.

In the matrix representation $U_\Pi$ can be read as:

$$U_\Pi = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos(\omega t) - \frac{i}{\omega} \sin(\omega t) h^- & 0 & 0 \\
0 & \frac{i}{\omega} \sin(\omega t)(J_{re}^+ - iJ_{im}^-) e^{i J_z t} & \cos(\omega t) + \frac{i}{\omega} \sin(\omega t) h^- & 0 \\
0 & 0 & 0 & 1 
\end{pmatrix}. \quad (24)$$

Let us put the initial state as $|\psi_i\rangle = |\uparrow \downarrow\rangle$ and the final one as $|\psi_f\rangle = a|\uparrow \downarrow\rangle + b|\downarrow \uparrow\rangle$.

As in the previous case, using the matrix representation for $U_\Pi$ (24), we write the relation $|\psi_f\rangle = U_\Pi |\psi_i\rangle$ in the form:

$$\begin{pmatrix}
a \\
b \\
0 
\end{pmatrix} = e^{i J_z t} \begin{pmatrix}
\cos(\omega t) - \frac{i}{\omega} \sin(\omega t) h^- \\
- \frac{i}{\omega} \sin(\omega t)(J_{re}^+ - iJ_{im}^-) \\
0 
\end{pmatrix}. \quad (25)$$

In this case conditions for optimal evolution read:

$$J_{xx} = J_{yy} = -\frac{\omega \hbar^2}{2 |b|}, \quad J_{xy} = -J_{yx} = -\frac{\omega \hbar^2}{2 |b|}, \quad J_{zz} = 0, \quad h_x^i = h_x^f = 0, \quad h_z^i = h_z^f = 0, \quad (26)$$

and the optimal time is (18). Also, in this case, the initial state $|\uparrow \downarrow\rangle$ can reach maximally entangled states $\frac{1}{\sqrt{2}} (|\uparrow \downarrow\rangle + e^{i \phi} |\downarrow \uparrow\rangle)$ ($\phi \in [0, 2\pi]$) during the minimal time $\tau = \frac{\pi}{\omega}$.

If we put conditions (26) on Hamiltonian $H$ (1) we obtain a Hamiltonian which provides optimal evolution on the subspace spanned by $|\uparrow \downarrow\rangle, |\downarrow \uparrow\rangle$. In the matrix form this Hamiltonian reads:

$$H = \begin{pmatrix}
0 & 0 & -i \omega \hbar^2 |b| & 0 \\
0 & 0 & 0 & 0 \\
i \omega \hbar^2 |b| & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix}. \quad (27)$$

The Hilbert space of the $|\uparrow \downarrow\rangle, |\downarrow \uparrow\rangle$ subspace is two-dimensional and is represented by the Bloch sphere. In this case the axis of rotation passes along two quantum states $\frac{1}{\sqrt{2}} \left( |\uparrow \downarrow\rangle + i \frac{\hbar^2 |b|}{\omega} |\downarrow \uparrow\rangle \right), \frac{1}{\sqrt{2}} \left( |\uparrow \downarrow\rangle - i \frac{\hbar^2 |b|}{\omega} |\downarrow \uparrow\rangle \right)$ which are eigenvectors of the optimal Hamiltonian $\hat{H}$ (27) that correspond to the largest $\omega$ and the smallest $-\omega$ eigenvalues.

4. Realization of quantum gates by two interacting spins

The evolution operator can be associated with some quantum gates. The target gate $U_{\text{target}}$ equals $U(T)$ modulo a global phase as:

$$U(T) = e^{i \lambda} U_{\text{target}}, \quad (28)$$

where $T$ is the optimal generation time of the target gate and $\lambda$ is some real number.

We now demonstrate this explicitly using a few examples. Let us consider the entangler gate:

$$U_{\text{ENT}} = \begin{pmatrix}
\cos \phi & 0 & 0 & \sin \phi \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sin \phi & 0 & 0 & \cos \phi 
\end{pmatrix}. \quad (29)$$
where $\phi \in [0, \pi]$. If this gate acts on the state $|\uparrow\uparrow\rangle$, it will produce the $\phi$-dependent entangled state $\cos \phi |\uparrow\uparrow\rangle - \sin \phi |\downarrow\downarrow\rangle$. The comparison of (10) with (29) using (14) and (24) leads to the following set of parameters: $J_{xx} = J_{yy} = J_{zz} = 0$, $J_{xy} = J_{yx} = -\frac{\omega}{2}$, $h^1_x = h^2_x = 0$ and $t = \frac{\phi}{\sqrt{2}}$. If we compare $t = \frac{\phi}{\sqrt{2}}$ with optimal time (18) we can see that this is the optimal time of evolution along the brachistochrone on the subspace spanned by $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ and the connection between parameter $\phi$ and $b$ is as follows: $\phi = \arcsin |b|$ ($\phi$ is called the Wooters distance).

Entangled gates allow us to reach Bell states from the nonentangled states. When $\phi = \frac{\pi}{4}$, this allows us to reach the maximally entangled Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$ from the initial state $|\psi_i\rangle = |\downarrow\downarrow\rangle$ and Bell state $|\Phi^-\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$ from the initial state $|\uparrow\uparrow\rangle$. The time of evolution between these pairs of states is the shortest possible time $t = \frac{\phi}{\sqrt{2}}$ which is allowed by Hamiltonian (1) and finite energy condition (9). The initial state $|\uparrow\uparrow\rangle$ can reach orthogonal final states $|\downarrow\downarrow\rangle$ during the time $t = \frac{\pi}{4}$, which is the passage time. In [18] the entangler gate is generated from the evolution operator which is obtained from the Heisenberg Hamiltonian with $J_{xy}$ interaction between spins ($i = x, y, z$) and magnetic field $h^\alpha$ ($\alpha = 1, 2$ for the first and second spin, respectively) which is directed along the $z$-axis. In addition to this Hamiltonian, our Hamiltonian (1) contains items with $J_{xy}$ and $J_{yx}$ interaction which allow us to generate the entangler gate along a geodesic in the subspace spanned by $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$.

Let us construct a similar gate which produces the entangled states on the subspace spanned by $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$. This gate is constructed with the entangler gate ($U_{\text{ENT}}$) and the NOT gate ($U_{\text{NOT}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) as:

$$
U_{\text{ENT}} = U_{\text{NOT}}^\dagger U_{\text{ENT}} U_{\text{NOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi & 0 \\ 0 & \sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

where $U_{\text{NOT}}^\dagger = U_{\text{NOT}} \otimes I$. If gate (30) acts on the initial state $|\uparrow\downarrow\rangle$ we get entangled state $\cos \phi |\uparrow\downarrow\rangle + \sin \phi |\downarrow\uparrow\rangle$. Similarly to the previous example we compare unitary operator (10) with quantum gate (30) and obtain the following set of parameters: $J_{xx} = J_{yy} = J_{zz} = 0$, $J_{xy} = -J_{yx} = -\frac{\omega}{2}$, $h^1_x = h^2_x = 0$ and $t = \frac{\phi}{\sqrt{2}}$. The generation time of $U_{\text{ENT}}$ is the optimal time of evolution along the brachistochrone (18) ($\phi = \arcsin |b|$) in the subspace spanned by $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$.

Let us insert $\phi = \frac{\pi}{4}$ in $U_{\text{ENT}}$ (30) and act by $U_{\text{ENT}}$ on the initial nonentangled state $|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$. We reach the maximally entangled Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ or the state that describes the EPR pair $|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, respectively. In this case, the time of evolution between these states is $t = \frac{\pi}{4\sqrt{2}}$. Evolution between two orthogonal states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ is realized within the passage time $t = \frac{\pi}{4\sqrt{2}}$.

The next important gate which we consider is the SWAP gate:

$$
U_{\text{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

which exchanges the states of two qubits. Using similar steps as in the case for the entangler gate, we obtain the following set of parameters: $J_{xx} = J_{yy} = J_{zz} = (-1)^p \frac{\omega}{\sqrt{2}}$, $J_{xy} = J_{yx} = 0$, $h^1_x = h^2_x = 0$, $t = \frac{\pi}{4\sqrt{2}}$ and $\chi = -\frac{\pi}{4} (-1)^p$ (where $p = 0, 1$). These parameters are the optimal conditions for $H$ (1) to generate the SWAP gate. In other words, they allow us to provide the unitary transformation between $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ states during the time $t = \frac{\pi}{4\sqrt{2}}$. Similarly as in [18]
the time-optimal generation of \( U_{\text{SWAP}} \) is along a geodesic on the subspace spanned by \(|\uparrow\downarrow\rangle\), \(|\downarrow\uparrow\rangle\).

As in the previous example we can generate the gate which exchanges the \(|\uparrow\uparrow\rangle\) state into \(|\downarrow\downarrow\rangle\) and vice versa:

\[
U_{\text{SWAP}} = U_{\text{NOT}}^1 U_{\text{SWAP}} U_{\text{NOT}}^1 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

(32)

when we put the following conditions: \( J_{xx} = -J_{yy} = -J_{zz} = (-1)^p \frac{\sqrt{\pi}}{\sqrt{6}}, J_{xy} = J_{yx} = 0, h_1^z = h_2^z = 0, t = \frac{\sqrt{6}}{2\pi} \) on unitary operator \((10)\). Here \( \chi = -\frac{\pi}{4} (-1)^p \) (where \( p = 0, 1 \)). In this case, these conditions allow us to provide the unitary transformation between \(|\uparrow\uparrow\rangle\), \(|\downarrow\downarrow\rangle\) states during \( t = \frac{\sqrt{6}}{2\pi} \). The time-optimal evolution is along a geodesic on the subspace which is spanned by \(|\uparrow\uparrow\rangle\), \(|\downarrow\downarrow\rangle\).

As a last example, we consider the unitary operator \( U_{\text{iSWAP}} \):

\[
U_{\text{iSWAP}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(33)

It can be obtained from the time unitary operator \((10)\) if we put the following conditions: \( J_{xx} = J_{yy} = -\frac{\omega}{2}, J_{zz} = J_{xy} = J_{yx} = 0, h_1^z = h_2^z = 0 \) and \( t = \frac{\pi}{2} \).

Also, we can obtain the following gate:

\[
U_{\text{iSWAP}} = U_{\text{NOT}}^1 U_{\text{iSWAP}} U_{\text{NOT}}^1 = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix},
\]

(34)

which is constructed using \( U_{\text{SWAP}} \) and \( U_{\text{NOT}} \). This gate can be obtained if we put the following conditions: \( J_{xx} = -J_{yy} = -\frac{\omega}{2}, J_{zz} = J_{xy} = J_{yx} = 0, h_1^z = h_2^z = 0 \) and \( t = \frac{\pi}{2} \) on unitary operator \((10)\).

The value of the global phase \( \chi \) for \( U_{\text{ENT}}, U_{\text{ENT}'} \) and \( U_{\text{SWAP}} \) and \( U_{\text{iSWAP}} \) gates is equal to \( 2\pi p \), where \( p \) is the arbitrary integer.

5. Summary and discussion

We have studied the quantum brachistochrone problem for a two-qubit system represented by two spins interacting via anisotropic couplings \( J_{ii} \) (\( i = x, y, z \)), \( J_{jk} \) (\( j \neq k = x, y \)) and magnetic fields \( h_\alpha^z \) (\( \alpha = 1, 2 \) for the first and second spin, respectively) which is directed along the z-axis \((1)\). This Hamiltonian realizes quantum evolution in two subspaces spanned by \(|\uparrow\uparrow\rangle\), \(|\downarrow\downarrow\rangle\) and \(|\uparrow\down\rangle\), \(|\down\uparrow\rangle\) and does not mix these subspaces because it does not contain items with \( \sigma_i^x \sigma_i^z, \sigma_i^z \sigma_i^x \), \( \sigma_i^y \sigma_i^z \), and \( \sigma_i^z \sigma_i^y \).

We solved this problem with the finite energy condition \((9)\) for each subspace separately. We obtained the conditions for optimal quantum evolution in each subspace and calculated the shortest possible time for evolution from the initial state \(|\psi_i\rangle\) to the final one \(|\psi_f\rangle\). Also, we obtained Hamiltonians \((20)\) and \((27)\) which provide optimal evolution on the subspaces spanned by \(|\uparrow\uparrow\rangle, |\downarrow\down\rangle\) and \(|\up\down\rangle, |\down\up\rangle\), respectively.

The Hilbert space of each subspace is two dimensional and is represented by the Bloch sphere. The optimal way of transporting \(|\psi_i\rangle\) into \(|\psi_f\rangle\) is therefore to rotate the sphere around the axis which passes along the eigenvectors that correspond to the largest \( \omega \) and the smallest
−ω eigenvalues of the optimal Hamiltonian. Thus, the shortest distance between \(|ψ_i⟩\) and \(|ψ_f⟩\) is a large circle arc on the sphere (geodesic path).

We used our result for important examples of \(U_{\text{ENT}}\), \(U_{\text{ENT}}\), \(U_{\text{SWAP}}\), \(U_{\text{iSWAP}}\) and \(U_{\text{iSWAP}}\) gates. The authors of [18] studied the time-optimal evolution of a unitary operator in the context of the variational principle. In the two-qubit demonstration of their method they obtain the optimal solution for the generation of some target quantum gates, namely the swap of qubits, the quantum Fourier transformation and the entangler gate. In [18] time-optimal generation of the entangler gate does not occur along a geodesic on the subspace spanned by \(|↑↑⟩\), \(|↓↓⟩\). In addition to the Hamiltonian, which is considered in [18], our Hamiltonian (1) contains items with \(J_{xy}\) and \(J_{yx}\) interaction which allow us to generate the entangler gate along a geodesic. Also, we considered gate \(U_{\text{ENT}}\) (30) which allows us to produce the entangler states on the subspace \(|↑↓⟩\), \(|↓↑⟩\). These gates allow us to reach maximally entangled Bell states from the nonentangled ones during the shortest time \(t = \frac{π}{ω}\).

Similarly as in [18] we obtained the optimal parameters for \(H\) (1) which allowed us to generate the SWAP gate. Also, we generated \(\text{SWAP}\) gate (32) which exchanges \(|↑↑⟩\) into \(|↓↓⟩\) and vice versa.

The Hamiltonian for generation of the iSWAP gate was proposed in [35]. The authors showed that the iSWAP operation can be obtained by applying an \(XY\) Hamiltonian with \(-\frac{Δ}{ω}\) interacting couplings for a time \(t = \frac{π}{ ω}\). Recently, the authors of [19] used analytical and numerical calculations to obtain the minimal times required for various quantum gates such as iSWAP, the controlled-π-phase (or CZ), CNOT, \(\sqrt{\text{SWAP}}\) and Toffoli gates under typical experimental conditions. For the creation of the iSWAP gate they consider a two-qubit physical system described by the Heisenberg Hamiltonian with \(J_{zz}\) and \(J_{yy}\) interacting couplings, which are equal \(J_{zz} = J_{yy} = \frac{J}{2}\) in magnetic field \(-\frac{Δ}{ω}\) directed along the \(x\)-axis. This Hamiltonian effects an iSWAP gate, in addition to two single-qubit rotations, during the time \(t = \frac{π}{J}\). In addition to the results of previous works we obtained the conditions for \(H\) (1) to generate iSWAP gates (34). This gate does not change states from \(|↑↓⟩\), \(|↓↑⟩\) subspace and realizes the following transformations: \(U_{\text{iSWAP}}|↑↑⟩\) = \(|↓↓⟩\), \(U_{\text{iSWAP}}|↓↓⟩\) = \(|↑↑⟩\) with states from another subspace.

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