A PRÉKOPA-LEINDLER TYPE INEQUALITY OF THE $L_p$ BRUNN-MINKOWSKI INEQUALITY

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Abstract. In this paper, we prove a Prékopa-Leindler type inequality of the $L_p$ Brunn-Minkowski inequality. It extends an inequality proved by Das Gupta [8] and Klartag [16], and thus recovers the Prékopa-Leindler inequality. In addition, we prove a functional $L_p$ Minkowski inequality.

1. Introduction

One of the cornerstones of the Brunn-Minkowski theory is the celebrated Brunn-Minkowski inequality (see, e.g., the books by Gardner [12], Gruber [14] and Schneider [26] for references). It has had far reaching consequences for subjects quite distant from geometric convexity. For this, see the wonderful survey by Gardner [11]. By the middle of the last century, the Brunn-Minkowski inequality had been successfully extended to nonconvex sets.

Theorem 1.1. Let $A$ and $B$ be nonempty bounded measurable sets in $n$-dimensional Euclidean space $\mathbb{R}^n$ such that $(1 - \lambda)A + \lambda B$ is also measurable. Then

$$V_n((1 - \lambda)A + \lambda B)^\frac{1}{n} \geq (1 - \lambda)V_n(A)^\frac{1}{n} + \lambda V_n(B)^\frac{1}{n}.$$  \hfill (1.1)

Here, $V_n$ denotes the $n$-dimensional Lebesgue measure and $A + B = \{x + y : x \in A, y \in B\}$ is the Minkowski sum of $A$ and $B$.

Denote by $\int f$ the integral of a function $f$ on its domain with respect to the Lebesgue measure. The following Prékopa-Leindler inequality [22] is a functional type of the Brunn-Minkowski inequality (1.1).

Theorem 1.2. Let $0 < \lambda < 1$ and let $f, g$ and $h$ be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$h((1 - \lambda)x + \lambda y) \geq f(x)^{1 - \lambda}g(y)^\lambda$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int h \geq \left(\int f\right)^{1 - \lambda}\left(\int g\right)^\lambda.$$
The Prékopa-Leindler inequality can quickly imply the Brunn-Minkowski inequality (1.1), see section 7 in [11] for details. This connection helps trigger a fruitful development of functional analogues of several geometric parameters into the class of log-concave functions currently undergoing (see [1, 2, 4, 5, 9, 17, 21, 18]).

The following Borell-Brascamp-Lieb inequality [6, 7] generalizes the Prékopa-Leindler inequality, which is just the case $\alpha = 0$.

**Theorem 1.3.** Let $0 < \lambda < 1, \lambda - 1/n \leq \alpha \leq \infty,$ and let $f, g,$ and $h$ be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$h((1 - \lambda)x + \lambda y) \geq M_\alpha(f(x), g(y), \lambda)$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int h \geq M_{\alpha/(n\alpha + 1)} \left( \int f, \int g, \lambda \right)$$

Here, for $a, b \geq 0,$ if $0 < \lambda < 1$ and $\alpha \neq 0$, we define

$$M_\alpha(a, b, \lambda) = ((1 - \lambda)a^\alpha + \lambda b^\alpha)^{1/\alpha}$$

if $ab \neq 0$ and $M_\alpha(a, b, \lambda) = 0$ if $ab = 0$; we also define

$$M_0(a, b, \lambda) = a^{1-\lambda}b^\lambda$$

$$M_\infty(a, b, \lambda) = \min \{a, b\}, \text{ and } M_\infty(a, b, \lambda) = \max \{a, b\}.$$

Firey [10] generalizes Minkowski addition to $L_p$ addition for convex bodies (compact, convex subsets with nonempty interiors) containing the origin and proves the $L_p$ Brunn-Minkowski inequality, see also Lutwak [19].

If $K, L$ are two convex bodies containing the origin, then the $L_p$ sum $K +_p L$ of $K$ and $L$ is defined by

$$h_{K +_p L}(x)^p = h_K(x)^p + h_L(x)^p$$

for all $x \in \mathbb{R}^n$, where $h_A(x) = \max \{x \cdot y : y \in A\}$ is the support function of $A$. Here, we denote by “.” the standard scalar product.

Lutwak, Yang and Zhang [20] provided the explicit pointwise formula of $L_p$ addition:

$$K +_p L = \{(1 - \lambda)^{1/q}x + \lambda^{1/q}y : x \in K, y \in L, 0 \leq \lambda \leq 1\}$$

where $q$ is the Hölder conjugate of $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. When $p = 1, q = \infty$, and $1/q$ is defined as 0. It is worth to point out that the pointwise $L_p$ addition (1.5) is suited for nonconvex sets.

Lutwak, Yang and Zhang [20] established the following $L_p$ Brunn-Minkowski inequality for compact sets:

**Theorem 1.4.** Suppose $p \geq 1$. If $K$ and $L$ are nonempty compact sets in $\mathbb{R}^n$, then

$$V(K +_p L)^{\frac{1}{p}} \geq V(K)^{\frac{1}{p}} + V(L)^{\frac{1}{p}}.$$
We consider the following problem: is there a Prékopa-Leindler type inequality that can simply imply (1.6)?

For any nonnegative function \( f \) on \( \mathbb{R}^n \), we denote \( \text{supp} f \) by the support of \( f \), i.e., the closure of \( \{ x \in \mathbb{R}^n : f(x) > 0 \} \). In this paper, we will prove the following theorem:

**Theorem 1.5.** Let \( p \geq 1 \), let \( s, \mu, \omega > 0 \), and let \( f, g, h : \mathbb{R}^n \rightarrow [0, \infty) \) be integrable with nonempty supports. If for all \( x \in \text{supp} f, y \in \text{supp} g \) and \( \lambda \in [0, 1] \),

\[
h((1 - \lambda)^{\frac{1}{s}} \mu^{\frac{1}{p}} x + \lambda^{\frac{1}{s}} \omega^{\frac{1}{p}} y)^{\frac{q}{p}} \geq (1 - \lambda)^{\frac{1}{s}} \mu^{\frac{1}{p}} f(x)^{\frac{q}{p}} + \lambda^{\frac{1}{s}} \omega^{\frac{1}{p}} g(y)^{\frac{q}{p}},
\]

where \( q \) is the Hölder conjugate of \( p \). Then,

\[
\left( \int h \right)^{\frac{p}{n+s}} \geq \mu \left( \int f \right)^{\frac{p}{n+s}} + \omega \left( \int g \right)^{\frac{p}{n+s}}.
\]

Let \( \mu = \omega = 1 \) and \( f = \chi_K, g = \chi_L, h = \chi_{K+pL} \), where \( \chi_E \) denotes the characteristic function of \( E \). It follows from (1.5) that (1.7) holds. Thus, we obtain (1.6) by letting \( s \rightarrow 0^+ \) in (1.8). This shows that Theorem 1.5 can be viewed as a functional generalization of Theorem 1.4.

Theorem 1.5 for \( p = 1 \) is proved by Das Gupta [8] and Klartag [16], which recovers the Prékopa-Leindler inequality, see Corollary 2.2 in [16] for details.

Let \( p = 1, s = \frac{1}{\alpha} \) and \( \mu + \omega = 1 \) in Theorem 1.5. When \( f, g \) are positive in their supports and have positive integrals, Theorem 1.5 and Theorem 1.3 coincide with each other for \( \alpha \in (0, \infty) \).

This paper was completed and submitted to a Journal on February 18, 2020. We recently found that [25] was published on Arxiv on April 20, 2020. Proposition 3.1 of [25] is the same as our Theorem 1.5.

This paper is organized as follows: In section 2, some basic facts and definitions for quick reference are provided. In section 3, some useful lemmas are given. In Section 4, we prove Theorem 1.5 and a functional \( L_p \) Minkowski inequality.

## 2. Preliminaries

In this section, we collect some terminologies and notations. We recommend the books of Gardner [12], Gruber [14] and Schneider [26] as excellent references on convex geometry.

For a nonempty set \( M \subset \mathbb{R}^2 \), the \( M \)-addition of two sets \( K, L \subset \mathbb{R}^n \) is defined as

\[
K \oplus_M L = \{ ax + by : (a, b) \in M, x \in K, y \in L \}.
\]

If \( M = \{(1,1)\} \), then \( M \)-addition is the classical Minkowski addition. For \( p \geq 1 \) and its Hölder conjugate \( q \), if \( M = \{(a,b) : a^q + b^q = 1, a \geq 0, b \geq 0 \} \), then \( M \)-addition is the explicit pointwise formula of \( L_p \) addition (1.5).
For \(0 < s < \infty\), we say that \(f : \mathbb{R}^n \to [0, \infty)\) is \(s\)-concave if suppf is nonempty, compact and convex, and \(f^{\frac{1}{s}}\) is concave, i.e., for all \(x, y \in \text{suppf}\) and \(0 \leq \lambda \leq 1\), we have
\[
f(\lambda x + (1 - \lambda)y) \geq \left[\lambda f(x)^{\frac{1}{s}} + (1 - \lambda)f(y)^{\frac{1}{s}}\right]^s.
\]

\(s\)-concave function has been studied by Avriel \[3\], Borell \[6\], Brascamp and Lieb \[6, 7\], Rotem \[23, 24\].

For any function \(f : \mathbb{R}^n \to [0, \infty)\) and any integer \(s > 0\), we define
\[
\mathcal{K}_f = \left\{(x, y) \in \mathbb{R}^{n+s} = \mathbb{R}^n \times \mathbb{R}^s : x \in \text{suppf}, |y| \leq f(x)^{\frac{1}{s}}\right\}. \tag{2.1}
\]
where, for given \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^s\), \((x, y)\) are coordinates in \(\mathbb{R}^{n+s}\). This set \(\mathcal{K}_f\) is nonempty and convex if and only if \(f\) is \(s\)-concave. The volume of \(\mathcal{K}_f\) can be computed as
\[
V_{n+s}(\mathcal{K}_f) = \int_{\text{suppf}} \kappa_s \cdot \left(f^{\frac{1}{s}}(x)^s\right)^s = \kappa_s \int f, \tag{2.2}
\]
where \(\kappa_s = \frac{\pi^{s/2}}{\Gamma(s+1)}\) is the volume of the \(s\)-dimensional Euclidean unit ball.

For positive number \(s\), two functions \(f, g : \mathbb{R}^n \to [0, \infty)\) with nonempty supports and nonempty set \(M \subset \mathbb{R}^2\) with nonnegative coordinates, we define the function \(f \oplus_{M,s} g\) as
\[
[f \oplus_{M,s} g](z) = \sup \left\{ \left(af(x)^{\frac{1}{s}} + bg(y)^{\frac{1}{s}}\right)^s : x \in \text{suppf}, y \in \text{suppg}, z = ax + by, (a, b) \in M\right\}
\]
when \(z \in \text{suppf} \oplus_{M} \text{suppg}\). If \(z \notin \text{suppf} \oplus_{M} \text{suppg}\), we set \([f \oplus_{M,s} g](z) = 0\). This definition is motivated by \[13\] and Lemma 3.2.

For \(s > 0\), two functions \(f, g : \mathbb{R}^n \to [0, \infty)\) with nonempty supports, \(p \geq 1\) and its Hölder conjugate \(q\), we define \(f \oplus_{p,s} g\) as \(f \oplus_{M,s} g\) by taking \(M = \{(a, b) : a^q + b^q = 1, a, b \geq 0\}\), i.e.,
\[
[f \oplus_{p,s} g](z) = \sup \left\{ \left((1 - \lambda)^{\frac{1}{q}}f(x)^{\frac{1}{q}} + \lambda^{\frac{1}{q}}g(y)^{\frac{1}{q}}\right)^s : x \in \text{suppf}, y \in \text{suppg}, \lambda \in [0, 1], z = (1 - \lambda)^{\frac{1}{q}}x + \lambda^{\frac{1}{q}}y\right\}
\]
when \(z \in \text{suppf} \oplus_{p} \text{suppg}\). If \(z \notin \text{suppf} \oplus_{p} \text{suppg}\), we set \([f \oplus_{p,s} g](z) = 0\).

For \(s > 0\), \(p \geq 1\), \(\lambda > 0\) and \(f : \mathbb{R}^n \to [0, \infty)\), we define the function \(\lambda \times_{p,s} f : \mathbb{R}^n \to [0, \infty)\) as
\[
[\lambda \times_{p,s} f](x) = \lambda^{\frac{1}{q}}f(\lambda^{-\frac{1}{q}}x). \tag{2.3}
\]
Note that, condition (1.7) implies
\[
h \geq [\mu \times_{p,s} f] \oplus_{p,s} [\omega \times_{p,s} g] \tag{2.4}
\]
pointwise.

If \(s\) is an integer, it is easy to see that \(\mathcal{K}_{\lambda \times_{p,s} f} = \lambda^{\frac{1}{s}}\mathcal{K}_f = \{\lambda^{\frac{1}{s}}y : y \in \mathcal{K}_f\}\). Thus,
\[
V_{n+s}(\mathcal{K}_{\lambda \times_{p,s} f}) = \lambda^{\frac{n+s}{p}}V_{n+s}(\mathcal{K}_f). \tag{2.5}
\]
Let \( p \geq 1 \). If \( f \) is an \( s \)-concave function, so is \( \lambda \times_p s f \) for \( \lambda > 0 \). In addition, if \( f, g \) are \( s \)-concave functions containing the origin in their supports, the function \( f \oplus_p s g \) is also \( s \)-concave and contain the origin in its support, which can be deduced from Lemma 3.1 by taking \( M = \{(a, b) : a^q + b^q = 1, a, b \geq 0\} \).

**Lemma 3.1.** Let \( M \subset \mathbb{R}^2 \) be a nonempty compact set with nonnegative coordinates and \( M \neq \{(0, 0)\} \). Let \( f, g \) be \( s \)-concave functions where \( s > 0 \). Then \( f \oplus_M s g \) is \( s \)-concave if one of the following conditions holds:

(i) \( M \) is convex.

(ii) \( \text{supp } f \) and \( \text{supp } g \) contain the origin.

**Proof.** Set \( h = [f \oplus_M s g] \). Since \( M \neq \{(0, 0)\} \), we get that \( h \) is not identically zero. This gives that \( \text{supp } h \) is nonempty.

We turn to prove the compactness of \( \text{supp } h \). It is equivalent to proving that \( \{z : h(z) > 0\} \) is bounded. By the definition of \( h \), we have

\[
\{z : h(z) > 0\} \subset \text{supp } f \oplus_M \text{supp } g. \tag{3.1}
\]

Since \( \text{supp } f, M \) and \( \text{supp } g \) are all compact, \( \text{supp } f \oplus_M \text{supp } g \) is compact. Therefore, we obtain that \( \{z : h(z) > 0\} \) is bounded. Therefore, \( \text{supp } h \) is compact.

We will prove that

\[
\text{supp } h = \text{supp } f \oplus_M \text{supp } g. \tag{3.2}
\]

The compactness of \( \text{supp } f, M \) and \( \text{supp } g \) gives that \( \text{supp } f \oplus_M \text{supp } g \) is the closure of \( \{x : f(x) > 0\} \oplus_M \{y : g(y) > 0\} \). It follows from our assumption of \( M \) that

\[
\{x : f(x) > 0\} \oplus_M \{y : g(y) > 0\} \subset \{z : h(z) > 0\}. \tag{3.3}
\]

Taking closure on both side gives

\[
\text{supp } f \oplus_M \text{supp } g \subset \overline{\{z : h(z) > 0\}} = \text{supp } h. \tag{3.4}
\]

Now, (3.1) implies (3.2).

Let \( z_1, z_2 \in \text{supp } h \) and \( \theta \in (0, 1) \). For given \( \varepsilon > 0 \), there exist \((a_1, b_1), (a_2, b_2) \in M, x_1, x_2 \in \text{supp } f, y_1, y_2 \in \text{supp } g\) with

\[
z_1 = a_1 x_1 + b_1 y_1, z_2 = a_2 x_2 + b_2 y_2 \tag{3.5}
\]

such that

\[
h(z_1)^\theta - \varepsilon \leq a_1 f(x_1)^\theta + b_1 g(y_1)^\theta, \tag{3.6}
\]

\[
h(z_2)^\theta - \varepsilon \leq a_2 f(x_2)^\theta + b_2 g(y_2)^\theta. \tag{3.7}
\]
It remains to prove that \( \text{supp} \ h \) is convex and \( h^\frac{1}{\theta} \) is concave in its support. By (3.2), the former is equivalent to proving that there exist \( (a, b) \in M, x \in \text{supp} \ f, y \in \text{supp} \ g \) such that

\[
(1 - \theta)z_1 + \theta z_2 = ax + by. \tag{3.5}
\]

The latter is equivalent to proving

\[
h((1 - \theta)z_1 + \theta z_2)^\frac{1}{\theta} \geq (1 - \theta)h(z_1)^\frac{1}{\theta} + \theta h(z_2)^\frac{1}{\theta}. \tag{3.6}
\]

(i) Since \( \text{supp} \ f \) and \( \text{supp} \ g \) are convex, let

\[
x = (1 - \lambda)x_1 + \lambda x_2, \quad y = (1 - \mu)y_1 + \mu y_2, \tag{3.7}
\]

where \( \lambda, \mu \in [0, 1] \) are to be determined. Then, by (3.3), (3.5) becomes

\[
\begin{align*}
(1 - \theta)a_1 &= (1 - \lambda)a, \\
\theta a_2 &= \lambda a, \\
(1 - \theta)b_1 &= (1 - \mu)b, \\
\theta b_2 &= \mu b.
\end{align*} \tag{3.8}
\]

This is equivalent to solve the system

\[
\begin{align*}
(1 - \theta)a_1 + \theta a_2 &= a, \\
(1 - \theta)b_1 + \theta b_2 &= b.
\end{align*}
\]

Since \( M \) is convex and \( (a_1, b_1), (a_2, b_2) \in M \), one can find \( (a, b) \in M \) that satisfies this system. Thus, (3.5) holds.

Therefore, by (3.5), the condition that \( a, b \geq 0, (3.7), (3.8), (3.3) \) and (3.4), we get

\[
h((1 - \theta)z_1 + \theta z_2)^\frac{1}{\theta} \\
\geq a f(x)^\frac{1}{\theta} + b g(y)^\frac{1}{\theta} \\
\geq a(1 - \lambda)f(x_1)^\frac{1}{\theta} + a\lambda f(x_2)^\frac{1}{\theta} + b(1 - \mu)f(y_1)^\frac{1}{\theta} + b\mu f(y_2)^\frac{1}{\theta} \\
= (1 - \theta)a_1 f(x_1)^\frac{1}{\theta} + (1 - \theta)b_1 f(y_1)^\frac{1}{\theta} + \theta a_2 f(x_2)^\frac{1}{\theta} + \theta b_2 f(y_2)^\frac{1}{\theta} \\
\geq (1 - \theta)h(z_1)^\frac{1}{\theta} + \theta h(z_2)^\frac{1}{\theta} - \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, (3.6) holds.

(ii) Since \( \text{supp} \ f \) and \( \text{supp} \ g \) are convex and contain the origin, let

\[
x = \alpha \left( (1 - \lambda)x_1 + \lambda x_2 \right), \quad y = \beta \left( (1 - \mu)y_1 + \mu y_2 \right), \tag{3.9}
\]
where \( \lambda, \mu, \alpha, \beta \in [0, 1] \) are to be determined. Similarly, (3.5) becomes

\[
\begin{aligned}
(1 - \theta)a_1 &= (1 - \lambda)a a, \\
\theta a_2 &= \lambda\alpha a, \\
(1 - \theta)b_1 &= (1 - \mu)\beta b, \\
\theta b_2 &= \mu\beta b.
\end{aligned}
\]  

(3.10)

This is equivalent to solve the system

\[
\begin{aligned}
(1 - \theta)a_1 + \theta a_2 &= \alpha a, \\
(1 - \theta)b_1 + \theta b_2 &= \beta b.
\end{aligned}
\]

Set \( a = \max\{a_1, a_2\}, b = \max\{b_1, b_2\} \), then one can find \( \alpha, \beta \in [0, 1] \) that satisfy this system. Thus, (3.5) holds.

It follows from \( f, g \) are s-concave and the assumption of (ii) that

\[
f(\alpha x)^{\frac{1}{s}} \geq \alpha f(x)^{\frac{1}{s}} + (1 - \alpha)f(0)^{\frac{1}{s}} \geq \alpha f(x)^{\frac{1}{s}},
\]

and

\[
g(\beta y)^{\frac{1}{s}} \geq \beta g(y)^{\frac{1}{s}} + (1 - \beta)g(0)^{\frac{1}{s}} \geq \beta g(y)^{\frac{1}{s}}.
\]

Together with (3.5), the condition that \( a, b \geq 0, (3.9), (3.10), (3.3) \) and (3.4), we obtain

\[
\begin{aligned}
h((1 - \theta)z_1 + \theta z_2)^{\frac{1}{s}} \\
\geq &af(x)^{\frac{1}{s}} + bg(y)^{\frac{1}{s}} \\
\geq &\lambda a f(\alpha x_1)^{\frac{1}{s}} + \lambda a f(\alpha x_2)^{\frac{1}{s}} + \lambda a f(\beta y_1)^{\frac{1}{s}} + \lambda a f(\beta y_2)^{\frac{1}{s}} + \mu b f(\beta y_1)^{\frac{1}{s}} + \mu b f(\beta y_2)^{\frac{1}{s}} \\
\geq &\lambda a f(\alpha x_1)^{\frac{1}{s}} + \lambda a f(\alpha x_2)^{\frac{1}{s}} + \lambda a f(\beta y_1)^{\frac{1}{s}} + \lambda a f(\beta y_2)^{\frac{1}{s}} + \mu b f(\beta y_1)^{\frac{1}{s}} + \mu b f(\beta y_2)^{\frac{1}{s}} \\
= &\lambda a f(\alpha x_1)^{\frac{1}{s}} + (1 - \theta)\beta f(y_1)^{\frac{1}{s}} + \theta a_2 f(x_2)^{\frac{1}{s}} + \theta b_2 f(y_2)^{\frac{1}{s}} \\
\geq &\lambda a f(\alpha x_1)^{\frac{1}{s}} + (1 - \theta)\beta f(y_1)^{\frac{1}{s}} + \theta a_2 f(x_2)^{\frac{1}{s}} + \theta b_2 f(y_2)^{\frac{1}{s}} - \varepsilon.
\end{aligned}
\]

Since \( \varepsilon \) is arbitrary, (3.6) holds.

\[\square\]

**Lemma 3.2.** Let \( s > 0 \) be an integer and let \( M \subset \mathbb{R}^2 \) be a nonempty set with nonnegative coordinates. Then, for any two functions \( f, g : \mathbb{R}^n \to [0, \infty) \) with nonempty supports,\n
\[\mathcal{K}_f \oplus_M \mathcal{K}_g \subset \mathcal{K}_{f \oplus_M s,g}.\]

In addition, if \( M \subset \{(a, b) : 0 \leq a \leq 1, 0 \leq b \leq 1\} \), then\n
\[\text{int} (\mathcal{K}_f \oplus_M \mathcal{K}_g) = \text{int} \mathcal{K}_{f \oplus_M s,g},\]

where \( \text{int} A \) denotes the interior of a set \( A \).
Proof. First, we prove
\[ K_f \oplus_M K_g \subset K_{f \oplus_M s g}. \] (3.11)

Let \((x, x') \in K_f, (y, y') \in K_g, (a, b) \in M.\) Then
\[ |x'| \leq f(x)^{\frac{1}{s}}, |y'| \leq g(y)^{\frac{1}{s}}, a, b \geq 0. \]
By the definition of \(f \oplus_M s g\), we obtain
\[ [f \oplus_M s g](az + by)^{\frac{1}{s}} \geq af(x)^{\frac{1}{s}} + bg(y)^{\frac{1}{s}} \]
\[ \geq a|x'| + b|y'| \]
\[ \geq |ax' + by'|. \]
That is, \((ax + by, ax' + by') \in K_{f \oplus_M s g}.\) Thus, (3.11) holds.

It remains to prove
\[ \text{int } K_{f \oplus_M s g} \subset K_f \oplus_M K_g. \] (3.12)
Without loss of generality, we can assume that \(\text{int } K_{f \oplus_M s g}\) is nonempty. Let \((z, z') \in \text{int } K_{f \oplus_M s g}.\) Then, there exists \(\varepsilon > 0,\) such that
\[ [f \oplus_M s g](z)^{\frac{1}{s}} - \varepsilon > |z'|. \]
By the definition of \(f \oplus_M s g,\) there exist \((a, b) \in M, x \in \text{supp } f, y \in \text{supp } g\) with \(z = ax + by\) such that
\[ af(x)^{\frac{1}{s}} + bg(y)^{\frac{1}{s}} > [f \oplus_M s g](z)^{\frac{1}{s}} - \varepsilon. \]
Therefore, we get
\[ af(x)^{\frac{1}{s}} + bg(y)^{\frac{1}{s}} > |z'|. \] (3.13)
Set
\[ x' = \frac{af(x)^{\frac{1}{s}}}{af(x)^{\frac{1}{2}} + bg(y)^{\frac{1}{2}}} z', y' = \frac{bg(y)^{\frac{1}{s}}}{af(x)^{\frac{1}{2}} + bg(y)^{\frac{1}{2}}} z'. \]
Since \(0 \leq a, b \leq 1,\)
\[ (x, x') \in K_f, (y, y') \in K_g. \]
Thus
\[ (z, z') = a(x, x') + b(y, y') \in K_f \oplus_M K_g. \]
Therefore, (3.12) holds. \(\square\)

Remark 3.1. Let \(p \geq 1.\) By Lemma 3.2, we can conclude that for any \(\mu, \omega > 0\) and integer \(s > 0,\)
\[ K_{[\mu \times p, s f]} + p K_{[\omega \times p, s g]} \subset K_{[\mu \times p, s f] \oplus_p s [\omega \times p, s g]} \]
and
\[ \text{int } (K_{[\mu \times p, s f]} + p K_{[\omega \times p, s g]}) = \text{int } K_{[\mu \times p, s f] \oplus_p s [\omega \times p, s g]}. \] (3.14)
4. PROOF OF MAIN THEOREMS

We turn to prove Theorem 1.5.

**Proof of Theorem 1.5.** First assume that \( s \) is an integer.

The \( L_p \) Brunn-Minkowski inequality (1.6) for \((n + s)\)-dimensional sets, (2.5) and Remark 3.1 implies

\[
V_{n+s}^*(K) \geq V_{n+s}^*(K) + \mu V_{n+s}^*(K) + \omega V_{n+s}^*(K)
\]

where \( V_{n+s}^* \) stands for outer Lebesgue measure (the set \( K \) may be non-measurable). By (2.2), this is equivalent to

\[
\left( \int_{\mathbb{R}^n} f \right)^{\frac{\nu}{n+s}} \geq \mu \left( \int_{\mathbb{R}^n} f \right)^{\frac{\nu}{n+s}} + \omega \left( \int_{\mathbb{R}^n} g \right)^{\frac{\nu}{n+s}},
\]

where \( f \) is rational. Thus, it follows from (2.4) and (4.1) that (1.8) holds.

Next assume that \( s = \frac{t}{l} \) is rational.

Note that, by Hölder’s inequality (See [15]) and (1.7), for any \( x_1, \cdots, x_t, y_1, \cdots, y_t \in \mathbb{R}^n,

\[
(1 - \lambda)^\frac{1}{t} \mu^\frac{1}{t} \prod_{i=1}^{t} f(x_i)^{\frac{1}{t}} + \lambda^\frac{1}{t} \omega^\frac{1}{t} \prod_{i=1}^{t} g(y_i)^{\frac{1}{t}} \leq \left( \prod_{i=1}^{t} \left( (1 - \lambda)^\frac{1}{t} \mu^\frac{1}{t} f(x_i)^{\frac{1}{t}} + \lambda^\frac{1}{t} \omega^\frac{1}{t} g(y_i)^{\frac{1}{t}} \right) \right)^{\frac{1}{t}}
\]

\[
\leq \prod_{i=1}^{t} h \left( (1 - \lambda)^\frac{1}{t} \mu^\frac{1}{t} x_i + \lambda^\frac{1}{t} \omega^\frac{1}{t} y_i \right)^{\frac{1}{ts}}.
\]

(4.2)

For a function \( r : \mathbb{R}^n \to [0, \infty) \), we define \( \tilde{r} : \mathbb{R}^{nt} \to [0, \infty) \) by

\[
\tilde{r}(x) = \tilde{r}(x_1, \cdots, x_t) = \prod_{i=1}^{t} r(x_i)
\]

where \( x = (x_1, \cdots, x_t) \in (\mathbb{R}^n)^t \) are coordinates in \( \mathbb{R}^{nt} \). Thus, (4.2) implies that for \( x, y \in \mathbb{R}^{nt},

\[
\tilde{h}((1 - \lambda)^\frac{1}{t} \mu^\frac{1}{t} x + \lambda^\frac{1}{t} \omega^\frac{1}{t} y)^\frac{1}{ts} \geq (1 - \lambda)^\frac{1}{t} \mu^\frac{1}{t} f(x)^\frac{1}{t} + \lambda^\frac{1}{t} \omega^\frac{1}{t} g(y)^\frac{1}{t}.
\]

Now, \( ts = l \) is integer. This gives that

\[
\left( \int_{\mathbb{R}^n} h \right)^{\frac{p}{n+s}} = \left( \int_{\mathbb{R}^n} f \right)^{\frac{p}{n+s}} + \omega \left( \int_{\mathbb{R}^n} g \right)^{\frac{p}{n+s}}
\]

\[
= \mu \left( \int_{\mathbb{R}^n} f \right)^{\frac{p}{n+s}} + \omega \left( \int_{\mathbb{R}^n} g \right)^{\frac{p}{n+s}}.
\]

The case that \( s \) is irrational follows by a standard approximation argument. \( \square \)
Using Theorem 1.5, we will prove a functional $L_p$ Minkowski inequality. For $p \geq 1, s > 0$ and two functions $f, g : \mathbb{R}^n \to [0, \infty)$ with nonempty support, we define

$$\tilde{S}_{p,s}(f; g) = \frac{p}{n + s} \lim_{\varepsilon \to 0^+} \int \frac{f \oplus_{p,s} (\varepsilon \times_{p,s} g)}{\varepsilon} - \int f$$

whenever the integrals are defined and the limit exists. The motivation of this definition is from the definition of $L_p$ mixed volume, see [19].

**Corollary 4.1.** Let $s > 0$ and $f, g : \mathbb{R}^n \to [0, \infty)$ be integrable functions with nonempty support such that $\tilde{S}_{p,s}(f; g)$ exists. Then,

$$\tilde{S}_{p,s}(f; g) \geq \left( \int f \right)^{1 - \frac{p}{n + s}} \left( \int g \right)^{\frac{p}{n + s}}. \quad (4.3)$$

If $s$ is an integer, $f = \lambda \times_{p,s} g$, where $\lambda > 0$, and $g$ is $s$-concave such that supp $g$ has nonempty interior and contains the origin, then equality holds.

**Proof.** By (4.1),

$$\int \frac{f \oplus_{p,s} (\varepsilon \times_{p,s} g)}{\varepsilon} \geq \left( \left( \int f \right)^{\frac{p}{n + s}} + \varepsilon \left( \int g \right)^{\frac{p}{n + s}} \right)^{\frac{n + s}{p}} \quad (4.4)$$

$$\geq \left( \int f \right) + \varepsilon \cdot \frac{n + s}{p} \left( \int f \right)^{1 - \frac{p}{n + s}} \left( \int g \right)^{\frac{p}{n + s}}.$$

Since $\tilde{S}_{p,s}(f; g)$ exists, the definition of $\tilde{S}_{p,s}(f; g)$ implies the desired inequality.

If $s$ is an integer, $f = \lambda \times_{p,s} g$ and $g$ is $s$-concave such that supp $g$ has nonempty interior and contains the origin, then $K_f$ and $K_{[\varepsilon \times_{p,s} g]}$ are convex bodies containing the origin for $\varepsilon > 0$, and $K_f = \lambda^{\frac{1}{p}} K_g$.

By (1.4) and the homogeneity of support function,

$$h_{K_f + pK_{[\varepsilon \times_{p,s} g]}}^p(x) = h_{K_f}^p(x) + p h_{K_{[\varepsilon \times_{p,s} g]}}^p(x) = h_{\lambda^{\frac{1}{p}} K_g}^p(x) + p h_{\varepsilon^{\frac{1}{p}} K_g}^p(x) = (\lambda + \varepsilon) h_{K_g}^p(x)$$

for all $x \in \mathbb{R}^n$, which shows that $K_f + pK_{[\varepsilon \times_{p,s} g]} = (\lambda + \varepsilon)^{\frac{1}{p}} K_g$ is a convex body. It follows from Remark 3.1 that

$$\text{int } K_{f \oplus_{p,s} g} = \text{int } (K_f + pK_{[\varepsilon \times_{p,s} g]}) = \text{int } ((\lambda + \varepsilon)^{\frac{1}{p}} K_g).$$

Since these sets are convex, we get that

$$V_{n+s} (K_{f \oplus_{p,s} g}) = (\lambda + \varepsilon)^{\frac{n + s}{p}} V_{n+s} (K_g).$$
Therefore, by (2.2),
\[
\int [f \oplus_{p,s} (\varepsilon \times_{p,s} g)] - \int f = \frac{V_{n+s} (K_{f \oplus_{p,s} [\varepsilon \times_{p,s} g]}) - V_{n+s} (K_f)}{K_s} \\
= \left( \lambda + \varepsilon \right)^{\frac{n+s}{p}} - \lambda^{\frac{n+s}{p}} \frac{V_{n+s} (K_g)}{K_s} \\
= \left( \lambda + \varepsilon \right)^{\frac{n+s}{p}} - \lambda^{\frac{n+s}{p}} \int g.
\]

Now, by the definition $\tilde{S}_{p,s}(f; g)$ and the condition that $f = \lambda \times_{p,s} g = \lambda \tilde{V}_f (\frac{g}{\lambda})$, we have
\[
\tilde{S}_{p,s}(f; g) = \frac{p}{n+s} \lim_{\varepsilon \to 0^+} \left( \lambda + \varepsilon \right)^{\frac{n+s}{p}} - \frac{\varepsilon}{p} \int g \\
= \lambda^{\frac{n+s}{p} - 1} \int g = \left( \int f \right)^{1 - \frac{p}{n+s}} \left( \int g \right)^{\frac{p}{n+s}}.
\]

\[\square\]

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