Dynamos driven by poloidal flows in untwisted, curved and flat Riemannian diffusive flux tubes

March 15, 2010

L.C. Garcia de Andrade
Departamento de Física Teórica-IF-UERJ- RJ, Brasil

Recently Vishik anti-fast dynamo theorem, has been tested against non-stretching flux tubes [Phys Plasmas 15 (2008)]. In this paper, another anti-dynamo theorem, called Cowling’s theorem, which states that axisymmetric magnetic fields cannot support dynamo action, is carefully tested against thick tubular and curved Riemannian untwisted flows, as well as thin flux tubes in diffusive and diffusionless media. In the non-diffusive media the Cowling’s theorem is not violated in thin Riemann-flat untwisted flux tubes, where the Frenet curvature is negative. Nevertheless the diffusion action in the thin flux tube leads to a dynamo action driven by poloidal flows as shown by Love and Gubbins (Geophysical Res.) in the context of geodynamos. Actually it is shown that a slow dynamo action is obtained. In this case the Frenet and Riemann curvature still vanishes. In the case of magnetic filaments in diffusive media dynamo action is obtained when the Frenet scalar curvature is negative. Since the Riemann curvature tensor can be expressed in terms of the Frenet curvature of the magnetic flux tube axis, this result can be analogous
to a recent result obtained by Chicone, Latushkin and Smith, which states that geodesic curvature in compact Riemannian manifolds can drive dynamo action in the manifold. It is also shown that in absence of diffusion, magnetic energy does not grow but magnetic toroidal magnetic field can be generated by the poloidal field, what is called a plasma dynamo. Key-words: anti-dynamo theorems, Cowling’s theorem. PACS numbers: 2.40.Hw: differential geometries. 91.25.Cw-dynamo theories.
I Introduction

The most famous anti-dynamo theorems, considered by Zeldovich [1], and Cowling [2], have told us how to recognise dynamo action in the most various types of flows and situations. In the stretch-twist and fold (STF) Vainshtein-Zeldovich [3, 4] technique, besides the fundamental role played by stretching in dynamo action, another topological concept, called twist [5] comes into play. Along with stretch and twist, folding, which is associated with Riemann curvature [6] may provide the doubling of the magnetic field which guarantees dynamo action. Another very recent way to bypass Cowling’s theorem in axisymmetric flows was recently investigated by Gissinger et al [7] and consists of avoiding perturbing the flow non-axisymmetrically. Therefore, is not always clear that the simple lack of the twist ingredient may necessarily leads to a non-dynamo. Actually, untwisting tubes may give rise to an axisymmetric magnetic topology which by Cowling’s anti-dynamo theorem [6], would per se, guarantees the failure of dynamo action. Actually, in the best of hypothesis, this left us with a slow dynamo, which can be detected by performing the non-diffusive medium limit. The presence of diffusion is fundamental, in order to able one to perform the folding stage of the STF dynamo mechanism. The dynamo thickness, maybe understood as proportional to $Rm^{-1}$, which is the magnetic Reynolds number. This means that, large-scale astrophysical dynamos, where $Rm$ acquire very large values, can be considered as thin dynamos. On the other hand, small-scale dynamos such as the Perm toroidal experimental [8] dynamo, can be considered as a thick dynamo. Besides applications in dynamo theory, reconnection of untwisting flux tube gives rise to coalescing twisted magnetic flux tubes. This was shown by Linton and Priest [9] using a numerical simulation. Another interesting anti-fast dynamo has been recently investigated by Klapper and Young [10], who have extended the $C^1$-differentiable anti-fast-dynamo theorem by Vishik [11] to $C^2$. Previously, Arnold, Zeldovich, Ruzmaikin and Sokoloff [12] and Chicone and Latushkin [13] have shown that uniform stretching in flows could lead to fast dynamos in compact Riemannian manifolds. Yet more recently, Garcia de Andraade [14] have presented a new class of conformal Riemannian dynamos in three-dimensional diffusion substrate. In this paper the the zero-resistivity ideal plasma
limit in kinematic dynamos, is used to show that the thin untwisted tubes cannot support
dynamo action in accordance with Cowling’s theorem. Recent investigations by Nunez
[15] and Brandenburg [16] have investigated the testing of Cowling’s theorem in Einsteins
general relativistic black hole accretion disks [17]. Untwisting magnetic flux tube has
been also investigated recently in the context of solar physics by Terradas et al [18]. They
basically show that, untwisted flux tubes possess a kink instability by shearing motion in
diffusive medium. This situation, often takes place in the solar surface, and in conformal
fast dynamos in diffusive media. Diffusion processes in Riemannian manifolds have been
already considered by S. Molchanov [19], where however, the problem of dynamo action
was not addressed. Earlier Soward [20], argued that fast dynamo actions would still be
possible in regions where no non-stretching flows would be presented, such as in some
curved surfaces. Here one takes advantage of Soward’s argument to build these surfaces
as Riemannian surfaces of arbitrarily curvature. The only constraint is that the flux tubes
remains in principle untwisted, and that the curvature be strong in diffusive media, and of
course that the magnetic flux tube axis be planar. To resume it seems that Cowling’s anti-
dynamo theorem is not violated when the Riemann curvature of the flux tube vanishes,
a condition that mathematicians and general relativists call Riemann-flat. The paper is
organized as follows: In section II the the Cowling’s theorem is tested in Riemann-flat
thin tubes in diffusive and non-dissipative media. In section III the Cowling’s theorem
is shown to be violated when the Riemann curvature is turned in diffusive media. In
section IV diffusive fiments are shown to drive dynamo action when normal stretching is
considered. A similar result has been obtained by Love and Gubbins [21] in the case of
gedynamos. Discussions and future prospects are presented in section V.

II Cowling theorem in thin untwisted Riemann-flat
magnetic flux tubes

Since we are dealing with Cowling theorem, we repeat here, for reader convenience, the
Cowling theorem. This theorem states that, dynamo action cannot be supported in
systems with axisymmetric flows and magnetic fields. In this section, a thin untwisted
Riemann-flat flux tube is shown to present a magnetic field decay or at best a slow dynamo action, showing that Cowling’s theorem is not violated in these substrate. In this case one shall drop the Cartesian coordinates used here and consider the Riemannian tubular coordinates, in principle curved Riemannian flux tube. In the next section we shall observe that the absence of twisting only can be substitute by strong curvature (folding) and stretching in the metric, with the aid of a expressive thickness in the tube in diffusionless case. Actually in the best hypothesis a slow dynamo develops. To start with let us consider the Riemannian geometry of flux tubes as considered by Ricca [22] and magnetic flux tube coordinates \((r, \theta_R, s)\), yields, which is also used in plasma torus called tokamaks, by plasma physicists. Actually since the folding in flux tubes maybe be represented by the Riemann curvature tensor, destructive folding that leads to fast dynamos, can be obtained by the vanishing of folding or by vanishing of the Riemann curvature tensor. The general flux tube Riemannian metric is

\[
ds_0^2 = dr^2 + r^2 d\theta^2 + K^2(r, s) ds^2
\]  

(II.1)

The thin Riemann-flat in principle twisted magnetic flux tube metric is obtained by making the \(K^2 := (1 - r\kappa(s)\cos\theta)\) equal to one. This is clearly obtained when the coordinate \(r\) approaches zero. This situation happens when one approaches the torsioned flux tube axis. Here coordinate \(\theta(s)\) is one of the Riemannian curvilinear coordinates \((r, \theta_R, s)\) and \(\theta(s) = \theta_R - \int \tau(s) ds\), \(\tau\) being the Frenet torsion. Note that the thin tube metric is

\[
ds_0^2 = dr^2 + r^2 d\theta^2 + ds^2
\]  

(II.2)

Here the torsion term is responsible for the twist of the tube. Since the tube is untwisted we assume here that the torsion vanishes and coordinate \(\theta(s) = \theta_R\). In general even the solar flux tubes are closed in the inner parts of the Sun, in the background of this compact Riemannian manifold, the Riemannian gradient compact operator is given in general diffusive media by

\[
\nabla = e_r \partial_r + e_\theta \frac{1}{r} \partial_\theta + t \frac{1}{K} \partial_s
\]  

(II.3)

\[
d_t B = \eta \Delta B + B \cdot \nabla v
\]  

(II.4)
where $\Delta = \nabla^2$ is the Laplacian in general curvilinear coordinates and $\eta = 0$ in this section, is the plasma resistivity or dissipation coefficient. By considering that the magnetic field is strictly confined along and inside the tube, and that the resistivity free case leads to the frozen condition, the flow is also confined on the tube which allows us not to stretch the magnetic flux tube without stretching the magnetic field. Since the tube is untwisted the following relation is simplified

$$\partial_s e_\theta = -\tau \sin \theta t$$  \hspace{1cm} (II.5)

since the torsion vanishes

$$\partial_\theta e_r = e_\theta$$  \hspace{1cm} (II.6)

and

$$\partial_r e_\theta = -e_r$$  \hspace{1cm} (II.7)

Together with the expression

$$\partial_t B = \gamma B - \omega_0 B_\theta e_\theta$$  \hspace{1cm} (II.8)

whose extra term is a non-inertial term similar to one that is introduced into a inertial frame by the use of curvilinear coordinates, \((\text{Coriolis force in the frame})\) along with two last equations helps to simplify analytical computations on the self-induction equation, which can be splitted into the equation for the general untwisted tubes

$$B. \nabla v = \frac{1}{K}[\partial_\theta e_\theta + v_s \kappa n_s] B_s + \frac{1}{r} B_\theta \partial_\theta e_\theta$$  \hspace{1cm} (II.9)

which applied to the thin tubes along with expression (II.8) yields the following self-induction scalar components along the Frenet frame \((t, n, b)\), yields

$$-\gamma B_\theta \sin \theta - \omega_0 B_\theta \cos \theta = B_s v_s \kappa - \omega_0 B_\theta \cos \theta$$  \hspace{1cm} (II.10)

$$-\gamma B_\theta \cos \theta - \omega_0 B_\theta \sin \theta = -\omega_0 B_\theta \sin \theta$$  \hspace{1cm} (II.11)

$$\gamma B_s = 0$$  \hspace{1cm} (II.12)

Thus the last equation shows immediatly that no dynamo action can be supported in this thin tube in diffusionless plasma medium. The remaining equations show that the Frenet curvature $\kappa(s)$ vanishes. Note that from equation (II.8) one obtains

$$B_s v_s \kappa = 0$$  \hspace{1cm} (II.13)
This shows that the tube is untwisted, or better torsion-free, since twisting possesses a non-torsion contribution, the toroidal flow does not necessarily vanishes, but can vanish as well, which is the result given by Love and Gubbins in the context of geodynamos. Love and Gubbins have obtained this result numerically and not analytically as here. Since, as one shall see in the next section, the Riemann curvature is related to the Frenet curvature, which allows us to say that the tube is Riemann-flat. The solenoidal condition

\[ \text{div}\mathbf{B} = 0 \]  
\[(\text{II.14)}\]
implies that

\[ \partial_s B_\theta = \kappa \tau r \sin \theta B_\theta \]  
\[(\text{II.15)}\]
which in the absence of torsion shows that the poloidal component of the magnetic field does not depend on the toroidal coordinate, or

\[ \partial_s B_\theta = 0 \]  
\[(\text{II.16)}\]
To simplify matters, this expression has been also used in the above computations. One has also used the same equation for the incompressible plasma flow to obtain

\[ \partial_s v_\theta = 0 \]  
\[(\text{II.17)}\]
which allows us to define the vorticity \( \omega_0 \) as

\[ v_\theta = \omega_0 r \]  
\[(\text{II.18)}\]
This also simplify the above expressions. Other set of equations used in the computations are, the Frenet steady equations

\[ \frac{dt}{ds} = \kappa(s) \mathbf{n} \]  
\[(\text{II.19)}\]
\[ \frac{dn}{ds} = -\kappa(s) \mathbf{t} + \tau \mathbf{b} \]  
\[(\text{II.20)}\]
\[ \frac{db}{ds} = -\tau(s) \mathbf{n} \]  
\[(\text{II.21)}\]
Note that, to introduce diffusion now, the Laplacian term is needed. This can be written as

\[ \Delta \mathbf{B} = [t(\partial_r^2 + \frac{1}{r} \partial_r - \kappa^2)B_s + e_\theta(\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2})B_\theta + + (\frac{d}{ds})\kappa B_s \mathbf{n}] \]  
\[(\text{II.22)}\]
Thus, for thin tubes the above Riemannian Laplacian operators, inside brackets is substituted into the self-induction equation yielding the following differential equations in the approximation of linear Frenet curvature, yields

\[
\partial_r^2 \frac{1}{r} \partial_r \gamma \eta B_s = 0 \tag{II.23}
\]

\[
\partial_r^2 \frac{1}{r} \partial_r \left[ \frac{\gamma}{\eta} + \frac{1}{r^2} \right] B_\theta = 0 \tag{II.24}
\]

where second order terms like \(\kappa^2\) have been dropped. The PDEs can be solved to yields the following solutions

\[
B_s = J_0 \left( \sqrt{\frac{\gamma}{\eta}} r \right) e^{-\gamma t} \tag{II.25}
\]

This toroidal field would decay in time. When \(\gamma_1 = \gamma\) vanishes, due to the presence of the Bessel function \(J_0\), the toroidal field oscillates radially. As consequence, the magnetic field does not grow in time, but yet could be called a marginal dynamo or non-dynamo.

The poloidal component \(B_\theta\) can be easily computed in the case the second- variations of \(B_\theta\) decays very fast. This reduces its PDE above to

\[
\partial_r - \left[ \frac{\gamma r}{r^2} + \frac{1}{r} \right] B_\theta = 0 \tag{II.26}
\]

making the approximations \(\frac{\gamma r}{r^2} \ll \frac{1}{r}\) and \(r \approx 0\) to be valid close to the magnetic axis, this equation is reduces to

\[
\partial_r - \frac{1}{r} B_\theta = 0 \tag{II.27}
\]

which immeadiatly yields \(B_s = B_0 r e^{-\gamma t} = B_0 r\) in the marginal dynamo case \(\gamma_1 = 0\).

## III Riemannian untwisted flux tube slow dynamos in diffusive media

In the first part of this section, it is shown that there is a relation between Riemann curvature, which justifies to call these tubes with Frenet curved untwisted and untorsioned Riemannian tubes. The Riemann curvature \(R_{ijkl}\) where, \((i,j = 1,2,3)\), can be expressed as

\[
R_{ijkl} = \partial_i \Gamma_{jkl} + \Gamma^p_{ij} \Gamma_{p/kl} \tag{III.28}
\]
where $\Gamma_{ijk}$ is the Riemann-Christoffel symbols which can be expressed in terms of the
Riemann metric components $g_{ij}$ of the line element

$$ds^2 = g_{ij}dx^i dx^j \quad (III.29)$$

by

$$\Gamma_{ijk} = \frac{1}{2}[g_{ij,k} + g_{ik,j} - g_{ij,k}] \quad (III.30)$$

Here, the line element of the untwisted thick flux tube above one obtains

$$R_{1313} = R_{rsrs} = -\frac{K^4}{2r^2} = -\frac{1}{2}r^2\kappa^4 \cos^2 \theta \quad (III.31)$$

$$R_{2323} = -K^2 \quad (III.32)$$

This shows that the argument above is valid. Back to the self-induction equation, for
thick tubes in diffusive medium, one notes that the only difference now resides on the first
term of the equation (II.10). Due to thickness hypothesis this leads to the extra equation

$$\frac{v_s}{r \cos \theta} = \eta \frac{d}{ds} \kappa \quad (III.33)$$

If one assumes, as before, that the curvature is constant, the dynamo action is not present.
Nevertheless, if one assumes a more general non-vanishing curvature, the solution can be
obtained as follows: Since by assumption the toroidal flow depends only on the radial
coordinate $r$, the only way out, equation (III.33) is to impose the following constraint

$$v_s = \eta r \quad (III.34)$$

$$\frac{d}{ds} \kappa = \frac{1}{\cos \theta} \quad (III.35)$$

Expression (III.34) shows that the Riemannian flow is shear flow, since the flow is toroidal and
depends linearly on another direction. As long as the tube is untwisted, the trigonometric
function does not depend upon the toroidal coordinate $s$. This in turn, yields the solution
of equation (III.34) as

$$\kappa = \frac{s}{\cos \theta} + c_0 \quad (III.36)$$

From this expression, one is able to compute the Riemann curvature components for
this slow dynamo flow as

$$R_{1313} = R_{rsrs} = -\frac{1}{2}r^2 s^4 \cos^6 \theta \quad (III.37)$$
\[ R_{2323} = -r^2 \]  
(III.38)

Note that component \( R_{2323} \) grows very fast, when one walks along the magnetic flux tube axis toroidal direction.

IV Magnetic diffusive filaments with normal stretching and dynamo action

Let us define the poloidal flow \( v_P \) in terms of Frenet frame of the magnetic filament \((t, n, b)\) as

\[ v_P = v_n n + v_b b \]  
(IV.39)

and the toroidal flow as

\[ v_T = v_s t \]  
(IV.40)

Substitution of these expressions into the diffusion-free self-induction equation

\[ \frac{d}{dt} B = \nabla \times (v \times B) \]  
(IV.41)

where the poloidal and toroidal components of the magnetic field can be expressed in analogy to the components of the flow, results in the following equation

\[ \gamma B_s = 0 \rightarrow \gamma = 0 \]  
(IV.42)

when a toroidal component of the magnetic field is present. This implies that the remaining equation is

\[ v_s \tau (B_n + B_b) = 0 \]  
(IV.43)

In the absence of toroidal flow, \( v_s = 0 \) yields

\[ B_n = -B_b \]  
(IV.44)

which considers also that the torsion is present in the filaments. In the absence of toroidal component of the magnetic field \( B_s = 0 \) it is clear that dynamo action is possible, since equation (IV.42) does not necessarily implies that vanishes. This is called a marginal dynamo II.21 in the dynamo theory literature. This case yields

\[ \gamma = v_s \tau \]  
(IV.45)
which shows that only poloidal flow \(v_s = 0\) cannot drive dynamo action as in Love Gubbins case. Before introducing diffusion, one must note that the presence of normal flows \(v_n\) on filaments leads necessarily to the compressibility of the flow. This can be easily shown by considering the computation of the divergence of the flow as

\[
div \mathbf{v} = v_n \kappa
\]  

(IV.46)

which is distinct from zero for curved filaments such appears in the turbulent regime. In the diffusive case one has

\[
div \mathbf{v} = v_n \kappa
\]  

(IV.47)

where one has considered a helical filament here the torsion coincides with the curvature and are constants. The growth rate of magnetic fields are given by

\[
\gamma = \tau_0 [1 + \eta \tau_0]
\]  

(IV.48)

Since \(\tau_0 = \kappa_0\), this expression shows that in the diffusion-free limit, the fast dynamo action \(\gamma > 0\) is driven by curves on negative scalar curvatures \(\kappa_0 < 0\), in strong analogy to the Chicone et al discover that the geodesic of negative curvature compact Riemannian manifolds may drive fast dynamo action.

## V Conclusions

Anti-dynamo theorems, such as Cowling’s can be bypass in certain physical situations in turbulent flows. In this paper it is shown that the Cowling’s theorem is not violated when the flux tube is thin and untwisting, whose curvature is Riemann-flat in diffusionless case. Riemann metric curvature of thick flux tubes is shown to be related to the Frenet curvature. Cowling’s theorem is shown to be bypassed in a sort of violation in the diffusive case. Actually, when the tube is thin, and the medium is diffusive the Cowling’s theorem is not applied even if the tube is axisymmetric as the example of Gissinger et al. The dynamo action is however slow in this case, and a marginal dynamo is obtained. In this case the tube is also Riemann-flat. In a Riemannian general case the Riemann curvature does not vanish for a slow dynamo action as well for an untwisting thick tube. At the end one
must conclude that the absence of twist along with the presence of stretching and folding, nay be enough to get a slow dynamo action in the untwisted flux tube. Analogy between the geodesic fast dynamo action in compact constant negative curvature Riemannian manifolds and the Frenet negative constant curvature slow dynamo, in magnetic curved filaments in turbulent regime.

VI Acknowledgements

I am deeply grateful to Andrew Soward, Jean Luc Thiffeault, Andrew D. Gilbert and Renzo Ricca for their extremely kind attention and discussions on the subject of this paper. Thanks are also due to I thank financial supports from Universidade do Estado do Rio de Janeiro (UERJ) and CNPq (Brazilian Ministry of Science and Technology).

References

[1] Ya B Zeldovich, JETP Phys 4, 460 (1957).
[2] S. Cowling, Magnetohydrodynamics (1964) Oxford.
[3] S. I. Vainshtein, Ya B Zeldovich, Sov Phys Usp 15, 159 (1972).
[4] S. Childress, A. Gilbert, Stretch, Twist and Fold: The Fast Dynamo (1996), Springer, Berlin. (2007).
[5] V. Arnold, Ya B. Zeldovich, A. Ruzmaikin and D.D. Sokoloff, JETP 81,n. 6, 2052 (1981). V. Arnold, Ya B. Zeldovich, A. Ruzmaikin and D.D. Sokoloff, Doklady Akad. Nauka SSSR 266, n6, 1357 (1982). V Arnold and B Khesin, Topological methods in Hydrodynamics, Springer (1990).
[6] I. Klapper and L. Young, Comm Math Phys. 173 623 (1995).
[7] C. Gissinger, E Dormy and S Fauve, Phys Rev Lett (2008).
[8] I. Klapper and D. Longcope, Evolution Equations of Thin Twisted Flux Tubes, in
“Workshop on Stellar Dynamos ASP conference series No. 178, p.79-Astronomical
society of Pacific (1999) M. Nunez and Ferriz-Mas eds.

[9] L. C. Garcia de Andrade, Physics of Plasmas 14 (2007).

[10] V. Arnold, Appl Math and Mech 36, 236 (1972).

[11] M. Vishik, Sov. Phys. Dokl. 33, 192 (1988).

[12] V. Arnold, Ya B. Zeldovich, A. Ruzmaikin and D.D. Sokolo, JETP 81, n. 6, 2052
(1981).

[13] C. Chicone, Yu Latushkin and Montgomery, Comm. in Mathematical Physics (1995)
1. C. Chicone and Yu Latushkin, Proc of the American Mathematical Society 125,
Providence, Rhode Island, N. 11, (1997) 3391.

[14] L. C. Garcia de Andrade, Physics of Plasmas 14 (2007).

[15] M Nunez, Phys Rev Lett, 79796 (1997).

[16] A Brandenburg, ApJ Letters L111, 465 (2003).

[17] J. Terradas, J Andrias, M Goosens, J Aguerri, R Oliver and J Ballester, Nonlinear
instability of kink oscillations due to shearing motions, astro-ph/0809.3664 (2008).

[18] F. Krause and K-H. Rdler, Mean eld Magnetohydrodynamics and Dynamo Theory,
(1980) NY, Pergamon Press.

[19] S Molchanov, Russian mathematical surveys 30:1,1 (1975).

[20] A M Soward, Geophys Astrophys Fluid Dyn 53,81 (1990).

[21] Love and D Gubbins, Geophysical res. (1973).

[22] L. C. Garcia de Andrade, Non-holonomic dynamo filaments as Arnolds map in Rie-
mannian space, Astronomical notes (2008) in press.