The Hardy-Littlewood conjectures on the twin primes and the binary Goldbach problem are true

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Abstract

A celebrated conjecture of Hardy and Littlewood provides with an asymptotic formula for the counting function of the twin primes. We give an unconditional proof of such a formula by means of a finite Ramanujan expansion of the counting function expressed in terms of the von Mangoldt function and its incomplete form. In a completely analogous way, we solve the conjugate conjecture on the representations of any even integer as the sum of two prime numbers.

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1 Introduction and statement of the results

The von Mangoldt function \( \Lambda \) is defined by \( \Lambda(n) = \log p \) if \( n = p^\alpha \) for some prime number \( p \) and \( \alpha \in \mathbb{N} \), \( \Lambda(n) = 0 \) otherwise. Here and in what follows, the letter \( p \) (with or without subscript) is reserved for the prime numbers, whose set is denoted by \( \mathbb{P} \). By the Möbius inversion formula, the identity [7, §1.4]

\[
\sum_{n \mid q} \Lambda(n) = \log q
\]

is equivalent to

\[
\Lambda(n) = - \sum_{d \mid n} \mu(d) \log d = \sum_{d \mid n} \Lambda'(d),
\]
where we set $\Lambda'(d) = -\mu(d) \log d$ and $\mu$ is the Möbius function defined by

$$
\mu(n) = \begin{cases} 
1, & \text{if } n = 1; \\
(-1)^k, & \text{if } n = p_1 p_2 \cdots p_k \text{ for distinct } p_i \in \mathbb{P}; \\
0, & \text{otherwise}.
\end{cases}
$$

The incomplete $\Lambda$-function of range $z$ is [7, §19.2]

$$
n \in \mathbb{N} \rightarrow \Lambda_z(n) = \sum_{d \leq z, d|n} \Lambda'(d).
$$

Unless otherwise stated, in sums like $\sum_{d \leq z}$ we always assume that $d$ is a positive integer. Note that $\Lambda_z(n) = \Lambda(n)$ for all $n \leq z$. Further, recall that, given $q \in \mathbb{N}$ and $n \in \mathbb{Z}$, the Ramanujan sum evaluated in $n \pmod{q}$ is [9]

$$
c_q(n) = \sum_{j \in \mathbb{Z}^*_q} e(jn/q) = \sum_{j \in \mathbb{Z}^*_q} \cos(2\pi jn/q),
$$

where $\mathbb{Z}^*_q = \{ m \in \mathbb{N} \cap [1, q] : (m, q) = 1 \}$ and $e(x) = \exp(2\pi ix)$ for any real number $x$. Hereafter, the symbol $(m, q)$ denotes the greatest common divisor of the integers $m, q$. Moreover, mainly within formulæ, we often write $m \equiv n \pmod{k}$ to mean that $m \equiv n$ (mod $k$). Well known properties of the Ramanujan sums are [8, Th. 4.1]

$$
\sum_{q|m} c_q(n) = \begin{cases} 
m, & \text{if } m|n; \\
0, & \text{otherwise},
\end{cases} \quad (1.1)
$$

$$
c_q(n) = \varphi(q) \frac{\mu(q/(n,q))}{\varphi(q/(n,q))}, \quad \forall q, n \in \mathbb{N}, \quad (1.2)
$$

where $\varphi(q) = \#\mathbb{Z}^*_q$ is the Euler totient. Since $\mu$ and $\varphi$ are multiplicative, by using (1.2) it is easy to see that if $(q_1, q_2) = 1$, then $c_{q_1 q_2}(n) = c_{q_1}(n)c_{q_2}(n)$ for all $n \in \mathbb{N}$.

Given any $N \in \mathbb{N}$, let us consider the two correlations of $\Lambda$ with itself and with the incomplete $\Lambda$-function of range $N$, respectively given by

$$
h \in \mathbb{N} \rightarrow C_{\Lambda, \Lambda}(N, h) = \sum_{n \leq N} \Lambda(n)\Lambda(n + h),
$$

$$
h \in \mathbb{N} \rightarrow C_{\Lambda, \Lambda_N}(N, h) = \sum_{n \leq N} \Lambda(n)\Lambda_N(n + h).
$$
For any fixed $h \in \mathbb{N}$ and for any sufficiently large $N \in \mathbb{N}$, one has

$$C_{\Lambda,\Lambda}(N, h) = C_{\Lambda,\Lambda_N}(N, h) + O(hL \log(N + h)), \quad (1.3)$$

where we set $L = \log N$ for brevity. Indeed, let us write

$$C_{\Lambda,\Lambda}(N, h) = \sum_{n \leq N} \Lambda(n) \sum_{d \mid n + h} \Lambda'(d),$$

and observe that the conditions $n \leq N$ and $d \mid n + h$ yield $d \leq N + h$ in the second sum, so that $\Lambda(n + h) = \Lambda_{N+h}(n + h)$. Consequently,

$$C_{\Lambda,\Lambda}(N, h) = \sum_{n \leq N} \Lambda(n) \sum_{d \leq N} \Lambda'(d) + \sum_{n \leq N} \Lambda(n) \sum_{N < d \leq N + h} \Lambda'(d) + \sum_{n \leq N} \Lambda(n) \sum_{n \equiv -h} \Lambda(n).$$

Therefore, (1.3) follows immediately after noticing that for $d > N$ there is at most one $n \in \mathbb{N}$ such that $n \leq N$ and $n \equiv -h \pmod{d}$.

Now, by using (1.1), we obtain the main tool of the present study, namely the finite Ramanujan expansion of the incomplete $\Lambda$-function,

$$\Lambda_{N}(n) = \sum_{d \leq N \atop d \mid n} \Lambda'(d) = \sum_{d \leq N} \frac{\Lambda'(d)}{d} \sum_{q \mid d} c_q(n) = \sum_{q \leq N} \hat{\Lambda}_{N}(q)c_q(n), \quad (1.4)$$

with [4, §4]

$$\hat{\Lambda}_{N}(q) = -\sum_{d \leq N \atop d \equiv 0 \pmod{q}} \frac{\mu(d) \log d}{d} = -\frac{\mu(q)}{q} \sum_{d \leq N/q \atop (d, q) = 1} \mu(d) \log(dq) \ll \frac{L^2}{q}. \quad (1.5)$$

Assuming the Delange hypothesis [5] on the Eratosthenes transform of $C_{\Lambda,\Lambda_N}(N, h)$, Coppola [3] has recently showed that

$$C_{\Lambda,\Lambda_N}(N, h) = \sum_{n, q \leq N} \Lambda(n) \hat{\Lambda}_{N}(q) \frac{c_q(n)c_q(h)}{\varphi(q)}, \quad \forall N, h \in \mathbb{N}. \quad (1.6)$$
The aforementioned hypothesis demands the absolute convergence of the series
\[
\sum_{m=1}^{\infty} \frac{2^{\omega(m)} C'_{\Lambda,\Lambda_N}(N, m)}{m},
\]
where \(\omega(m)\) is the number of the distinct prime factors of \(m \geq 2\), \(\omega(1) = 0\), and
\[
C'_{\Lambda,\Lambda_N}(N, m) = \sum_{d|m} \mu(d) C_{\Lambda,\Lambda_N}(N, m/d).
\]

Further, Coppola [2] has proved that if \(k \in \mathbb{N}\) is such that \(0 < k < N^{1-\delta}\), with \(\delta \in (0, 1/2)\) fixed, then
\[
\sum_{n,q \leq N} \Lambda(n) \Lambda_N(q) \frac{c_q(n)c_q(2k)}{\varphi(q)} = \mathfrak{S}(2k)N + O\left(N \exp\left(-c\sqrt{L}\right)\right),
\]
where \(c > 0\) is an absolute constant and
\[
\mathfrak{S}(2k) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} c_q(2k) = 2 \prod_{p|k} \left(1 + \frac{1}{p-1}\right) \prod_{(p,2k)=1} \left(1 - \frac{1}{(p-1)^2}\right).
\]

In view of (1.3), this shows that the Delange hypothesis on (1.7) yields the Hardy-Littlewood conjecture for the \(2k\)-twin primes [6, Conjecture B]. On the other hand, unconditionally on (1.7) or else, one has that [3]
\[
C_{\Lambda,\Lambda_N}(N, h) - \sum_{n,q \leq N} \Lambda(n) \Lambda_N(q) \frac{c_q(n)c_q(h)}{\varphi(q)} = \sum_{q \leq N} \mathcal{L}(N, q) c_q(h) - \mathcal{R}(N, h), \forall N, h \in \mathbb{N},
\]
where
\[
\mathcal{L}(N, q) = \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{N < m \leq x} c_q(m) \sum_{d|N \atop d|m} C'_{\Lambda,\Lambda_N}(N, d),
\]
\[
\mathcal{R}(N, h) = \begin{cases} 
\sum_{d|N \atop d|h} C'_{\Lambda,\Lambda_N}(N, d), & \text{if } h > N; \\
0, & \text{otherwise}.
\end{cases}
\]

Therefore, (1.6) is equivalent to
\[
\mathcal{R}(N, h) = \sum_{q \leq N} \mathcal{L}(N, q) c_q(h), \forall N, h \in \mathbb{N}.
\]
In particular, if $h \leq N$, then (1.6) requires that
\[
\sum_{q \leq N} \mathcal{L}(N, q) c_q(h) = 0.
\]
However, in order to solve the twin primes problem plainly it suffices that
\[
\sum_{q \leq N} \mathcal{L}(N, q) c_q(h) - R(N, h) = o(N).
\]
This follows from the next theorem.

**Theorem 1.** Let $h \in \mathbb{N}$ be fixed. For any sufficiently large integer $N$ and for every real number $\varepsilon > 0$, one has
\[
C_{\Lambda, \Lambda_N}(N, h) - \sum_{n, q \leq N} \Lambda(n) \Lambda_N(q) c_q(n)c_q(h) \phi(q) \lesssim (N + h)^{\varepsilon} L^5 \log^2 h.
\]

In view of (1.3) and (1.8), this theorem yields the following consequence unconditionally.

**Corollary 1.** The Hardy-Littlewood conjecture for the $2k$-twin primes is true.

The proof of Theorem 1 is given in the next section together with two lemmata. For the sake of completeness and clarity, in Sect. 3 we provide with the proof of Corollary 1 by essentially repeating Coppola’s proof of (1.8). In the last section, the same arguments are exploited in order to settle the binary Goldbach conjecture [6, Conjecture A], [7, §19].

## 2 The proof of Theorem 1

Let us denote the set of the square-free positive integers by $S = \{q \in \mathbb{N} : \mu(q) \neq 0\}$ and note that if $q \not\in S$, then $\Lambda_N(q) = 0$. The reader is cautioned that most of the next considerations are valid only because we are dealing with square-free integers of the $\Lambda_N$ support. For example, throughout what follows, we shall freely use without explicit mention the fact that if $q \in S$, then $(d, q/d) = 1$ for all $d|q$, so that $f(q) = f(d)f(q/d)$ for any multiplicative arithmetic function involved here.

Before going to the proof of Theorem 1, we give two lemmata.
Lemma 1. Let $h, q, r \in \mathbb{N}$ and set $q' = q/(q, r)$, $r' = r/(q, r)$, $m = qr/(q, r)$. If $r, q \in \mathbb{S}$, then for any $s \in \{1, \ldots, m - 1\}$ one has

$$
\sum_{a \in \mathbb{Z}_r^+, b \in \mathbb{Z}_q^+} \left( e\left(\frac{bh}{q}\right) - \frac{c_q(h)}{\varphi(q)} \right) = \frac{\varphi(q)\varphi(r)}{(q, r)} \sum_{d\mid (q, r), t\mid q} \mu(d)\mu(t) \sum_{x \in \mathbb{Z}_r^+, y \in \mathbb{Z}_q^+, z \in \mathbb{Z}} e\left(\frac{sx}{d}\right) - \frac{c_q(h)}{\varphi(q)} = \frac{c_q(h')\varphi(r)}{(q, r)} \sum_{d\mid (q, r)} \frac{c_d(s)}{\varphi(d)^2},
$$

(2.1)

where $h' \in \{1, \ldots, q\}$ is such that $r'h' \equiv h \pmod{q}$. If, in addition, $qr \geq 2$ and $(q, r) = 1$, then

$$
\sum_{a \in \mathbb{Z}_r^+, b \in \mathbb{Z}_q^+} \left( e\left(\frac{bh}{q}\right) - \frac{c_q(h)}{\varphi(q)} \right) = 0, \ \forall s \in \{1, \ldots, qr - 1\}.
$$

(2.2)

Proof. Since $(q', r') = 1$, it is easily seen that the hypothesis $r, q \in \mathbb{S}$ yields $(r, q') = (q, r') = 1$. In particular, from $(q, r') = 1$ it follows that there exists a unique $h' \in \{1, \ldots, q\}$ such that $r'h' \equiv h \pmod{q}$. Consequently, $c_q(h) = c_q(r'h') = c_q(h')$. Further, one has that $a \in \mathbb{Z}_r^*$ if, and only if, $aq' \in \mathbb{Z}_r^*$, and $b \in \mathbb{Z}_q^*$ if, and only if, $br' \in \mathbb{Z}_q^*$. Thus, we can write

$$
\sum_{a \in \mathbb{Z}_r^+, b \in \mathbb{Z}_q^+} \left( e\left(\frac{bh}{q}\right) - \frac{c_q(h)}{\varphi(q)} \right) = \sum_{a, b = 1, a + b \equiv s \pmod{m}}\left( e\left(\frac{bh'}{q}\right) - \frac{c_q(h')}{\varphi(q)} \right)1_r(a)1_q(b),
$$

where $1_r, 1_q$ are the characteristic functions of $\mathbb{Z}_r^*, \mathbb{Z}_q^*$, respectively. As such functions are even $(\pmod{r})$ and $(\pmod{q})$, respectively, by applying a theorem of Cohen [1, Th. 1], [10, Th. 139], and using (1.2), we have

$$
1_r(a) = \frac{1}{r} \sum_{d\mid r} c_r(r/d)c_d(a) = \frac{\varphi(r)}{r} \sum_{d\mid r} \frac{\mu(d)}{\varphi(d)} c_d(a),
$$

$$
1_q(b) = \frac{1}{q} \sum_{t\mid q} c_q(q/t)c_q(b) = \frac{\varphi(q)}{q} \sum_{t\mid q} \frac{\mu(t)}{\varphi(t)} c_t(b).
$$
Hence,

$$
\sum_{a \in \mathbb{Z}_{\ast}^d, b \in \mathbb{Z}_{\ast}^t \atop a \equiv b (m)} \left( e \left( \frac{bh}{q} \right) - \frac{c_q(h)}{\varphi(q)} \right) =
\frac{\varphi(q) \varphi(r)}{qr} \sum_{d \mid (q, t)} \mu(d) \mu(t) \sum_{x \in \mathbb{Z}_{d}^*} \sum_{y \in \mathbb{Z}_{t}^*} \left( e \left( \frac{bh'}{q} \right) - \frac{c_q(h')}{\varphi(q)} \right) e \left( \frac{ax}{d} + \frac{by}{t} \right).
$$

Now, recalling that $m = qr'$, we evaluate

$$
\sum_{a, b = 1 \atop a \equiv b (m)}^m e \left( \frac{bh'}{q} + \frac{ax}{d} + \frac{by}{t} \right) = e \left( \frac{sx}{d} \right) \sum_{b = 1}^m e \left( \frac{r'h' + ym/t - xm/d}{m} \right) =
\begin{cases}
  e(sx/d), & \text{if } r'h' \equiv xm/d - ym/t \pmod{m}; \\
  0, & \text{otherwise}.
\end{cases}
$$

Analogously,

$$
\sum_{a, b = 1 \atop a \equiv b (m)}^m e \left( \frac{ax}{d} + \frac{by}{t} \right) =
\begin{cases}
  e(sx/d), & \text{if } xm/d \equiv ym/t \pmod{m}; \\
  0, & \text{otherwise}.
\end{cases}
$$

Thus, we write

$$
\sum_{a \in \mathbb{Z}_{\ast}^d, b \in \mathbb{Z}_{\ast}^t \atop a \equiv b (m)} \left( e \left( \frac{bh}{q} \right) - \frac{c_q(h)}{\varphi(q)} \right) =
\frac{\varphi(q) \varphi(r)}{(q, r)} \sum_{d \mid (q, t)} \mu(d) \mu(t) \sum_{x \in \mathbb{Z}_{d}^*} \sum_{y \in \mathbb{Z}_{t}^*} \frac{c_q(h') \varphi(r)}{(q, r)} e \left( \frac{sx}{d} \right) -
\sum_{d \mid (q, t)} \mu(d) \mu(t) \sum_{x \in \mathbb{Z}_{d}^*} \sum_{y \in \mathbb{Z}_{t}^*} \frac{c_q(h') \varphi(r)}{(q, r)} e \left( \frac{sx}{d} \right).
$$

Note that the congruence $xm/d \equiv ym/t \pmod{m}$ is equivalent to $xqr/d \equiv yqr/t \pmod{mrq}$. Since $xqr/d \leq qr$, $yqr/t \leq qr$ for $x \in \mathbb{Z}_{d}^*, y \in \mathbb{Z}_{t}^*$, from $xqr/d \equiv yqr/t \pmod{mrq}$ it follows that $xqr/d = yqr/t$, i.e., $xt = yd$. Hence, such a congruence has solutions in $x \in \mathbb{Z}_{d}^*, y \in \mathbb{Z}_{t}^*$ if, and only if, $t = d$ and $x = y$. Consequently, for any $s \in \{1, \ldots, m - 1\}$ one has

$$
\sum_{x \in \mathbb{Z}_{d}^* \atop xm/d \equiv ym/t (m)} e \left( \frac{sx}{d} \right) =
\begin{cases}
  c_d(s), & \text{if } t = d; \\
  0, & \text{otherwise},
\end{cases}
$$
thereby showing that
\[
\sum_{d|\frac{m}{d},t|q} \mu(d)\mu(t) \frac{\varphi(d)\varphi(t)}{\varphi(d)} \sum_{x \in \mathbb{Z}_d^*, y \in \mathbb{Z}_d^*} e\left(\frac{sx}{d}\right) = \sum_{d|(q,r), t|q, (q/t,q/d)/h'} \mu(d)\mu(t) \frac{c_d(s)}{\varphi(d)^2}.
\]

Let us turn our attention to the congruence $x m/d - y m/t \equiv r' h' \mod m$. First, notice that it can be equivalently written as $x r q/d - y r q/t \equiv r q h' \mod q r$, i.e., $x q - y d q/t \equiv d h' \mod q d$. As $x \in \mathbb{Z}_d^*$, it follows immediately that a necessary condition is $d|q$. Therefore, the previous congruence becomes $x q/d - y q/t \equiv h' \mod q$, yielding $x q/d - y q/t = \pm h'$, because $1 \leq h' \leq q$ and $1 \leq x q/d \leq q$, $1 \leq y q/t \leq q$ for any $x \in \mathbb{Z}_d^*$, $y \in \mathbb{Z}_d^*$. Thus, we conclude that
\[
\sum_{a \in \mathbb{Z}_d^*, b \in \mathbb{Z}_d^*} \left(e\left(\frac{bh}{q}\right) - \frac{c_q(h)}{\varphi(q)}\right) = \varphi(q) \varphi(r) \sum_{d|(q,r), t|q, (q/t,q/d)/h'} \mu(d)\mu(t) \varphi(d) \varphi(t) \sum_{x \in \mathbb{Z}_d^*, y \in \mathbb{Z}_d^*} e\left(\frac{sx}{d}\right) \frac{c_q(h') \varphi(r)}{(q,r)} \sum_{d|(q,r)} c_d(s) \frac{1}{\varphi(d)^2},
\]
that is (2.1). In particular, assuming that $q r \geq 2$ and $(q,r) = 1$, for any $s \in \{1, \ldots, qr - 1\}$ this formula becomes
\[
\sum_{a \in \mathbb{Z}_d^*, b \in \mathbb{Z}_d^*} \left(e\left(\frac{bh}{q}\right) - \frac{c_q(h)}{\varphi(q)}\right) = \varphi(r) \left(\sum_{\frac{q h'}{(h' t/q, t)} = 1} \varphi(q/t) \mu(t) - c_q(h')\right).
\]

Since $q/(q, h')$ is the only divisor of $q$ fulfilling the conditions on the right-hand side, one has
\[
\sum_{a \in \mathbb{Z}_d^*, b \in \mathbb{Z}_d^*} \left(e\left(\frac{bh}{q}\right) - \frac{c_q(h)}{\varphi(q)}\right) = \varphi(r) \left(\varphi\left((q, h')\right) \mu(q/(q, h')) - c_q(h')\right) = 0,
\]
after applying (1.2). Hence, the identity (2.2) is proved as well. \qed

**Lemma 2.** Let $h, q, r \in \mathbb{N}$. If $r, q \in \mathbb{S}$ and $(q,r) = 1$, then
\[
\sum_{n \leq N \atop (n,q)=1} c_r(n)c_q(n+h) = \frac{c_q(h)}{\varphi(q)} \sum_{n \leq N \atop (n,q)=1} c_r(n)c_q(n).
\]
Proof. It is plain that the equation is trivially true for $q = 1$. Thus, let us assume that $q \geq 2$ and consider the sum

$$\sum_{n \leq N \atop (n,q)=1} c_r(n)c_q(n + h) = \sum_{b \in \mathbb{Z}_q^*} e\left(\frac{bh}{q}\right) \sum_{a \in \mathbb{Z}_r^*} \sum_{n \leq N \atop (n,q)=1} e\left(n\left(\frac{a}{r} + \frac{b}{q}\right)\right).$$

Note that $a/r + b/q \in \mathbb{Z}$ implies that $aq + br = dqr$ for some $d \in \mathbb{Z}$, i.e., $r = q$ and $a + b \equiv 0 \pmod{q}$. Since $q \geq 2$, the hypothesis $(q,r) = 1$ yields $a/r + b/q \notin \mathbb{Z}$ for all $a \in \mathbb{Z}_r^*$, $b \in \mathbb{Z}_q^*$. Therefore,

$$\sum_{n \leq N \atop (n,q)=1} c_r(n)c_q(n + h) - \frac{c_q(h)}{\varphi(q)} \sum_{n \leq N \atop (n,q)=1} c_r(n)c_q(n)$$

$$= \sum_{b \in \mathbb{Z}_q^*} \left(e\left(\frac{bh}{q}\right) - \frac{c_q(h)}{\varphi(q)}\right) \sum_{a \in \mathbb{Z}_r^*} \sum_{n \leq N \atop (n,q)=1} e\left(n\left(\frac{a}{r} + \frac{b}{q}\right)\right)$$

$$= \sum_{b \in \mathbb{Z}_q^*} \left(e\left(\frac{bh}{q}\right) - \frac{c_q(h)}{\varphi(q)}\right) \sum_{a \in \mathbb{Z}_r^*} \sum_{n \leq N \atop (n,q)=1} e\left(n\left(\frac{aq + br}{qr}\right)\right)$$

$$= \sum_{s \leq qr - 1} \frac{e\left(\frac{ns}{qr}\right)}{\varphi(q)} \sum_{a \in \mathbb{Z}_r^*, h \in \mathbb{Z}_q^* \atop aq + br \equiv s \pmod{qr}} \left(e\left(\frac{bh}{q}\right) - \frac{c_q(h)}{\varphi(q)}\right).$$

The conclusion follows immediately by applying (2.2) of Lemma 1. \qed

Proof of Theorem 1. For any $h \in \mathbb{N}$ let us write

$$C_{\Lambda,\Lambda_N}(N, h) - \sum_{n,q \leq N} \Lambda(n)\Lambda_{\tilde{N}}(q) \frac{c_q(n)c_q(h)}{\varphi(q)} =$$

$$\sum_{n,q \leq N} \Lambda(n)\Lambda_{\tilde{N}}(q) \left(c_q(n + h) - \frac{c_q(n)c_q(h)}{\varphi(q)}\right) = \sum_{I} + \sum_{II},$$

where

$$\sum_{I} = \sum_{n,q \leq N \atop (n,q)=1} \Lambda(n)\Lambda_{\tilde{N}}(q) \left(c_q(n + h) - \frac{c_q(n)c_q(h)}{\varphi(q)}\right),$$

$$\sum_{II} = \sum_{n,q \leq N \atop (n,q)>1} \Lambda(n)\Lambda_{\tilde{N}}(q) \left(c_q(n + h) - \frac{c_q(n)c_q(h)}{\varphi(q)}\right).$$
Let us show that $\sum_I = 0$. First, after recalling (1.4) and (1.5), note that for any integer $n \in \mathbb{N}$ such that $n \leq N$ and $(n, q) = 1$, we have

$$\Lambda(n) = \Lambda_n(n) = \sum_{d \leq N \atop (d, n, (d, q) = 1)} \Lambda'(d) = \sum_{d \leq N \atop (d, q) = 1} \sum_{r \mid d} \Lambda_N(r) c_r(n) = \sum_{r \leq N \atop (r, q) = 1} \Lambda_N(r) c_r(n).$$

Hence, the conclusion follows immediately from Lemma 2 because

$$\sum_I = \sum_{q \leq N} \Lambda_N(q) \sum_{n \leq N \atop (n, q) = 1} \left( c_q(n + h) - \frac{c_q(n)c_q(h)}{\varphi(q)} \right) \sum_{r \leq N \atop (r, q) = 1} \Lambda_N(r) c_r(n)$$

$$= \sum_{q \leq N} \Lambda_N(q) \sum_{r \leq N \atop (r, q) = 1} \Lambda_N(r) \left( \sum_{n \leq N \atop (n, q) = 1} c_r(n)c_q(n + h) - \frac{c_q(h)}{\varphi(q)} \sum_{n \leq N \atop (n, q) = 1} c_r(n)c_q(n) \right).$$

Thus, in view of (1.3) and (2.3), the theorem is proved once we show that

$$\sum_{I} \ll (N + h)^{\varepsilon} L^2 \log^2 h.$$

For this purpose, let us note first that, since $\Lambda(n) = 0$, unless $n = p^\alpha$ with $p \in \mathbb{P}$, $\alpha \in \mathbb{N}$, the condition $(q, n) > 1$ for $q \in \mathbb{S}$ in $\sum_I$ reduces to $(q, n) = (q, p^\alpha) = p$ with $p \mid q$, i.e., $p \mid q$ and $p^2 \nmid q$. Then, we can also assume that $q \in \mathbb{S} \setminus \mathbb{P}$ because, taking $q = p$ and $n = p^\alpha$ in $\sum_I$, we immediately see that

$$c_p(p^\alpha + h) - \frac{c_p(p^\alpha)c_p(h)}{\varphi(p)} = c_p(h) - \frac{\varphi(p)c_p(h)}{\varphi(p)} = 0.$$

Hereafter, in sums like $\sum^*$ we mean that $q \in \mathbb{S} \setminus \mathbb{P}$. Further, we set $S_d = \{q \in \mathbb{S} : (q, d) = 1\}$.

Thus, using (1.2), let us write

$$\sum_{I} = \sum_{p \leq N} \log p \sum_{q \leq N \atop \mu(q) = 0 (p)}^* \Lambda_N(q) \sum_{\alpha \leq \log_p N} \left( c_q(p^\alpha + h) - \frac{c_q(p^\alpha)c_q(h)}{\varphi(q)} \right)$$

$$= \sum_{p \leq N} \log p \sum_{q \leq N \atop \mu(q) = 0 (p)} \left( \sum_{t \mid p^\alpha + h} \sum_{\mu(q) \neq 0 (p)} \Lambda_N(q) \left( \varphi(q) \frac{\mu(q/t)}{\varphi(q/t)} - \frac{c_q/p(p^\alpha)c_p(p^\alpha)c_q(h)}{\varphi(q)} \right) \right)$$

$$= \sum_{p \leq N} \log p \sum_{q \leq N \atop \mu(q) = 0 (p)} \left( \sum_{t \mid p^\alpha + h} \sum_{\mu(q) \neq 0 (p)} \Lambda_N(q) \left( \varphi(t) \mu(q/t) - \frac{\mu(q/p)c_q(h)}{\varphi(q/p)} \right) \right).$$
Note that if \(t \in S\), \(v_p(t) = 1\), and \(v_p(h) \neq 0\), then the condition \((pq, p^\alpha + h) = t\), with \(q \in S_p\), becomes \((q, p^\alpha + h) = t/p\), with \(v_p(t/p) = 0\). On the other hand, if \(v_p(h) = 0\), then \((pq, p^\alpha + h) = t\) is equivalent to \((q, p^\alpha + h) = t\). Consequently,

\[
\sum_{\alpha \leq \log p, N} \sum_{t \in S, v_p(t) = 1} \sum_{q \in S_p, q \leq N/p} \sum_{(pq, p^\alpha + h) = t} \hat{N}(pq) \mu(q)\phi(q) = \sum_{III} - \sum_{IV},
\]

By applying (1.5) and the inequality for the divisor function, \(d(n) \ll n^\varepsilon [8, \S 2.3]\), we get

\[
\sum_{\alpha \leq \log p, N} \sum_{t \in S, v_p(t) = 0} \sum_{q \in S_p, q \leq N/(pt)} \sum_{(pq, p^\alpha + h) = t} \hat{N}(pq) \mu(q/t) \phi(q/t) = \sum_{\alpha \leq \log p, N} \sum_{t \in S, v_p(t) = 0} \sum_{q \in S_p, q \leq N/(pt)} \sum_{(pq, p^\alpha + h) = t} \hat{N}(pq) \mu(q/t)
\]

\[
\ll L^2 \sum_{\alpha \leq \log p, N} \sum_{t \in S, v_p(t) = 0} \sum_{q \in S_p, q \leq N/(pt)} \frac{1}{t} \sum_{q \leq N} \frac{1}{q} \ll L^3 \sum_{\alpha \leq \log p, N} \sum_{t \in S, v_p(t) = 0} \sum_{q \in S_p, q \leq N/(pt)} \sum_{(pq, p^\alpha + h) = t} \hat{N}(pq) \mu(q/t)
\]

\[
\ll \varepsilon (N + h)\varepsilon L^4 \sum_{\alpha \leq \log p, N} \sum_{t \in S, v_p(t) = 0} \sum_{q \in S_p, q \leq N/(pt)} \sum_{(pq, p^\alpha + h) = t} \hat{N}(pq) \mu(q/t) \ll \varepsilon (N + h)\varepsilon L^4 \sum_{\alpha \leq \log p, N} \sum_{t \in S, v_p(t) = 0} \sum_{q \in S_p, q \leq N/(pt)} \sum_{(pq, p^\alpha + h) = t} \hat{N}(pq) \mu(q/t)
\]

Analogously, we see that

\[
\ll \varepsilon (N + h)\varepsilon L^4 \sum_{\alpha \leq \log p, N} \sum_{t \in S, v_p(t) = 0} \sum_{q \in S_p, q \leq N/(pt)} \sum_{(pq, p^\alpha + h) = t} \hat{N}(pq) \mu(q/t) \ll \varepsilon (N + h)\varepsilon L^4 \sum_{\alpha \leq \log p, N} \sum_{t \in S, v_p(t) = 0} \sum_{q \in S_p, q \leq N/(pt)} \sum_{(pq, p^\alpha + h) = t} \hat{N}(pq) \mu(q/t)
\]
Since

\[
\sum_{p \leq N \text{ } v_p(h) \geq 1} \varphi(p) \log p \sum_{\alpha \leq \log p N} \sum_{t \in \mathbb{S}, \varphi(t) = 1} \sum_{q \in S_p, q \leq N / p} \varphi(t/p) \mu(t/p) \sum_{q \in S_p, q \leq N / t} \sum_{(q, p^\alpha + h) = 1} \widehat{\Lambda}_N(pq) \mu(q)
\]

Thus, by using the inequality \(|c_q(n)| \leq (q, n)| [4, Lemma A.1], [8, §4.1.1, Ex.3], we infer

\[
\sum_{p \leq N \text{ } v_p(h) = 0} \log p \sum_{\alpha \leq \log p N} \sum_{t \in \mathbb{S}, \varphi(t) = 1} \sum_{q \in S_p, q \leq N / p} \mu(t) \varphi(t) \sum_{q \in S_p, q \leq N / (p^\alpha + h)} \sum_{(q, p^\alpha + h) = 1} \widehat{\Lambda}_N(pqt) \mu(q) c_q(h) / \varphi(q)
\]

\[
\ll L^2 \sum_{p \leq N \text{ } v_p(h) = 0} \log p \sum_{\alpha \leq \log p N} \sum_{t \in \mathbb{S}, \varphi(t) = 1} \sum_{q \in S_p, q \leq N / t} \sum_{(q, p^\alpha + h) = 1} \frac{(t, h) \sum_{q \leq N} (q, h)}{q \varphi(q)}.
\]
Now, let us apply the inequality \( \varphi(n) \gg n / \log \log n \) [8, Th. 2.9] to see that

\[
\sum_{q \leq N} \frac{(q, h)}{q \varphi(q)} \ll \log L \sum_{q \leq N} \frac{(q, h)}{q^2} = \log L \sum_{d|h} \frac{1}{d} \sum_{q \leq N} \frac{1}{q^2} = \log L \sum_{d|h} \frac{1}{d} \sum_{m \leq N/d \atop (dm, h)=d} \frac{1}{m^2} \ll \log h \log L.
\]

Further, note that if \( t \in \mathcal{S}, t|p^\alpha + h, \) and \((t, h) = d > 1,\) then necessarily \( d = p.\) Consequently, recalling that \( v_p(h) = 0,\) one has

\[
\sum_{t \in \mathcal{S}, v_p(t)=0} \sum_{q \leq N \atop (q, p\alpha + h) = t} q \varphi(q) \ll \log L \sum_{t \in \mathcal{S}, v_p(t)=0} \sum_{q \leq N \atop (q, p\alpha + h) = t} q \varphi(q) \ll \log h \log L.
\]

Hence,

\[
\sum_{p \leq N \atop v_p(h) = 0} \frac{\log p}{\alpha \leq [\log_p N]} \frac{\sum_{t \in \mathcal{S}, v_p(t)=0} \sum_{q \leq N \atop (q, p\alpha + h) = t} \Lambda_N(pq) \mu(q) c_q(h) \varphi(q)}{\varphi(q)} \ll L^3 \log h \log^3 L.
\]

Analogously, we obtain

\[
\sum_{p \leq N \atop v_p(h) \geq 1} \frac{\varphi(p) \log p}{p} \sum_{\alpha \leq [\log_p N]} \sum_{t \in \mathcal{S}, v_p(t)=0} \sum_{q \leq N \atop (q, p\alpha + h) = t} \Lambda_N(pq) \mu(q) c_q(h) \varphi(q)
\]

\[
= \sum_{p \leq N \atop v_p(h) \geq 1} \frac{\varphi(p) \log p}{p} \sum_{\alpha \leq [\log_p N]} \sum_{t \in \mathcal{S}, v_p(t)=0} \sum_{q \leq N \atop (q, p\alpha + h) = t} \frac{\mu(t)}{\varphi(t)} \sum_{q \leq N \atop (q, p\alpha + h) = t} \Lambda_N(pqt) \mu(q) c_q(h/p) \varphi(q)
\]

\[
\ll L^2 \sum_{p \leq N \atop v_p(h) \geq 1} \frac{\varphi(p) \log p}{p} \sum_{\alpha \leq [\log_p N]} \sum_{t \in \mathcal{S}, v_p(t)=0} \sum_{q \leq N \atop (q, h/p) = t} \frac{(t, h/p)}{t \varphi(t)} \sum_{q \leq N \atop q \varphi(q)} (q, h/p) \varphi(q)
\]

\[
\ll L^3 \log^2 h \log^2 L.
\]
and

\[
\sum_{p \leq N} \varphi(p) \log p \sum_{\alpha \leq \log p} \sum_{t \in S, v_p(t) = 1} \sum_{t|p^\alpha + h} \frac{\Lambda_N(pq) \mu(q)c_q(h)}{\varphi(q)}
\]

\[
= \sum_{p \leq N} \varphi(p) \log p \sum_{\alpha \leq \log p} \sum_{t \in S, v_p(t) = 1} \frac{\mu(t/p)c_{t/p}(h)}{\varphi(t/p)} \sum_{q \leq N/t} \frac{\Lambda_N(qt) \mu(q)c_q(h)}{\varphi(q)}
\]

\[\ll L^2 \sum_{p \leq N} \varphi(p) \log p \sum_{\alpha \leq \log p} \sum_{t \in S, v_p(t) = 1} \frac{(h, t/p)}{t\varphi(t/p)} \sum_{q \leq N} \frac{(q, h)}{q\varphi(q)}\]

\[\ll L^2 \sum_{p \leq N} \frac{\varphi(p)}{p} \log p \sum_{\alpha \leq \log p} \sum_{t \in S, v_p(t) = 1} \frac{1}{\varphi(t/p)t/p} \sum_{q \leq N} \frac{(q, h)}{q\varphi(q)}\]

\[\ll L^{3} \log^{2} h \log^{2} L.\]

Therefore, we can write

\[\sum_{IV} \ll L^{3} \log^{2} h \log^{3} L.\]

Hence, \[\sum_{II} = \sum_{III} - \sum_{IV} \ll \varepsilon (N + h)^\varepsilon L^{4} \log^{3} L \log^{2} h.\]

3 The proof of Corollary 1

As a consequence of Theorem 1, following closely Coppola’s arguments [2], let us prove the Hardy-Littlewood conjecture for the \(2k\)-twin primes unconditionally. More precisely, we prove that if the integer \(2k\) is such that \(0 < k < N^{1-\delta}\) for a fixed \(\delta \in (0, 1/2)\), then (1.8) holds with an absolute constant \(c > 0\) and with \(\mathcal{S}_{\lambda, \lambda}(2k)\) as in (1.9).

**Proof of Corollary 1.** First, note that if \(q \leq \sqrt{N}\), then from (1.5) one has that there exists an absolute constant \(c > 0\) such that [4, §4], [8, §6.2.1, Ex.17],

\[
\Lambda_N(q) = -\frac{\mu(q)}{q} \sum_{d \leq N/q} \frac{\mu(d) \log d}{d} - \frac{\mu(q) \log q}{q} \sum_{d \leq N/q} \frac{\mu(d)}{d} = \frac{\mu(q)}{\varphi(q)} + O(q^{-1} \exp(-c\sqrt{L})),
\]

where we set \(L = \log N\) as before. Recall also that \(\Lambda_N(q)\) is supported on the set \(S\) of the square-free numbers. Thus, let us take \(h = 2k\) in Theorem 1. By using (1.3), the
above formula for $\hat{\Lambda}_n(q)$, with $q \leq \sqrt{N}$, the bound (1.5), and the aforementioned inequality $|c_q(n)| \leq (q, n)$, we write

$$C_{\Lambda, \Lambda}(N, 2k) = \sum_{q \in S} \frac{\mu(q)}{\varphi^2(q)} c_q(2k) \sum_{n \leq N} \Lambda(n)c_q(n) + O \left( \frac{1}{e^{c\sqrt{L}}} \sum_{q \in S} \frac{(2k, q)}{q\varphi(q)} \sum_{n \leq N} \Lambda(n, q) \right) + O \left( \frac{L^2}{\sqrt{N}} \sum_{q \leq \sqrt{N}} \frac{(2k, q)}{q\varphi(q)} \sum_{n \leq N} \Lambda(n, q) \right) + O(N^{1-\delta}).$$

Now, we have that

$$\sum_{n \leq N} \Lambda(n)c_q(n) = \sum_{n \leq N} \Lambda(n)c_q(n) + \sum_{n \leq N} \Lambda(n)c_q(n)$$

$$= \mu(q) \sum_{n \leq N} \Lambda(n) + \sum_{p|q} \log p \sum_{\alpha \leq \log_p N} c_q(p^\alpha)$$

$$= \mu(q) \sum_{n \leq N} \Lambda(n) + \sum_{p|q} \log p \sum_{\alpha \leq \log_p N} \left( c_q(p^\alpha) - \mu(q) \right)$$

$$= \mu(q)\psi(N) + O \left( L\varphi(q) \sum_{p|q} 1 \right)$$

$$= \mu(q)N + O \left( N\exp(-c\sqrt{L}) \right) + O(L\varphi(q)\log q),$$

where we have applied the Prime Number Theorem [8, Th. 6.9], i.e.,

$$\psi(N) = \sum_{n \leq N} \Lambda(n) = N + O \left( N\exp(-c\sqrt{L}) \right),$$

and the inequality $\sum_{p|q} 1 \leq \log q$. Further, for any $q \in S$ such that $q \leq N$, Chebyshev’s inequality $\psi(N) \ll N$, [8, Th. 2.4], yields

$$\sum_{n \leq N} \Lambda(n)(n, q) = \sum_{d|q} \frac{d}{n \leq N} \Lambda(n) \leq \sum_{d|q} \frac{d}{n \leq N} \Lambda(n) = \psi(N) + \sum_{p|q} \log p \sum_{\alpha \leq \log_p N} 1 \ll NL.$$
Thus, using the inequality \( \varphi(q) \gg q / \log \log q \) and suitably changing \( c > 0 \), we get

\[
C_{\Lambda,\Lambda}(N, 2k) = N \sum_{q \leq \sqrt{N}} \frac{\mu^2(q)}{\varphi^2(q)} c_q(2k) +
O\left( N \exp(-c\sqrt{L}) \sum_{q \leq \sqrt{N}} \frac{(2k, q)}{q^2} + NL^4 \sum_{\sqrt{N} < q \leq N} \frac{(2k, q)}{q^2} + N^{1-\delta} \right)
= \mathcal{S}_{\Lambda,\Lambda}(2k) N + O\left( N \sum_{q > \sqrt{N}} \frac{(2k, q)}{q^2} \log^2 q \right) +
O\left( N \exp(-c\sqrt{L}) \sum_{q \leq \sqrt{N}} \frac{(2k, q)}{q^2} + NL^4 \sum_{\sqrt{N} < q \leq N} \frac{(2k, q)}{q^2} + N^{1-\delta} \right),
\]

where \( \mathcal{S}_{\Lambda,\Lambda}(2k) \) is the singular series given in (1.9). Finally, the \( O \)-terms are estimated uniformly with respect to \( 2k \) as it follows. Since \( \log^2(dm) \ll \log^2 d + \log^2 m \ll L^2 \) for all \( d, m \in \mathbb{N} \), by applying the aforementioned inequality for the divisor function, \( d(n) \ll \varepsilon n^{\varepsilon} \), we can write

\[
\sum_{q > \sqrt{N}} \frac{(2k, q)}{q^2} \log^2 q = \sum_{d \mid 2k} d \sum_{q > \sqrt{N} \atop (q, 2k) = d} \frac{\log^2 q}{q^2} = \sum_{d \mid 2k} \frac{1}{d} \sum_{m > \sqrt{N}/d \atop (dm, 2k) = d} \frac{\log^2(dm)}{m^2}
\ll \sum_{d \mid 2k} \frac{1}{d} \sum_{m > \sqrt{N}/d} \frac{\log^2(dm)}{m^2} + \sum_{d \mid 2k} \frac{1}{d} \sum_{m = 1}^{\infty} \frac{\log^2(dm)}{m^2} \ll \varepsilon \frac{k^\varepsilon L^2}{\sqrt{N}}.
\]

Analogously,

\[
\sum_{\sqrt{N} < q \leq N} \frac{(2k, q)}{q^2} \ll \varepsilon \frac{k^\varepsilon}{\sqrt{N}}, \quad \sum_{q \leq \sqrt{N}} \frac{(2k, q)}{q^2} \ll L.
\]

The corollary is completely proved.

\[\square\]

4 The proof of the Goldbach conjecture

Given any \( 2k \in \mathbb{N} \), with \( k \geq 2 \), the correlation of \( \Lambda \) associated to the Goldbach problem is [7, §19]

\[
\sum_{n_1 + n_2 = 2k} \Lambda(n_1)\Lambda(n_2) = \sum_{n \leq K} \Lambda(n)\Lambda(2k - n),
\]
where we have set $K = 2k - 2$. Since $\Lambda(2k - n) = \Lambda_K(2k - n)$ for $2 \leq n \leq K$, and $c_q(2k - n) = c_q(n - 2k)$, by using (1.4) we see that such a correlation is equal to

$$\sum_{n \leq K} \Lambda(n)\Lambda_K(2k - n) = \sum_{n, q \leq K} \Lambda(n)\hat{\Lambda}_K(q)c_q(n - 2k).$$

It is also plain that both Lemmata 1 and 2 still hold by taking $N = K$, and replacing $h$ by $-2k$ in $e(bh/q)$ and $c_q(n + h)$. Therefore, arguing as in the proof of Theorem 1 we obtain the following analogous result.

**Theorem 2.** For every sufficiently large $2k \in \mathbb{N}$ and for every real number $\varepsilon > 0$, one has

$$\sum_{n_1 + n_2 = 2k} \Lambda(n_1)\Lambda(n_2) - \sum_{n, q \leq K} \Lambda(n)\hat{\Lambda}_K(q)c_q(n)\phi(q) \ll \varepsilon k^\varepsilon,$$

where $K = 2k - 2$.

As a consequence, by the same proof of Corollary 1 we get that the Hardy-Littlewood conjecture on the binary Goldbach problem is true [6, Conjecture A]:

**Corollary 2.** For every sufficiently large $2k \in \mathbb{N}$, one has

$$\sum_{n_1 + n_2 = 2k} \Lambda(n_1)\Lambda(n_2) = 2k\Theta(2k) + O\left(k \exp\left(-c\sqrt{\log(2k)}\right)\right),$$

where $c > 0$ is an absolute constant and $\Theta(2k)$ is defined in (1.9).

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