Fractal Geometry For Images Of Continuous Map Of p-Adic Numbers And p-Adic Solenoids Into Euclidean Spaces

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Abstract

Explicit formulas are obtained for a family of continuous mappings of p-adic numbers $\mathbb{Q}_p$ and solenoids $\mathbb{T}_p$ into the complex plane $\mathbb{C}$ and the space $\mathbb{R}^3$, respectively. Accordingly, this family includes the mappings for which the Cantor set and the Sierpinski triangle are images of the unit balls in $\mathbb{Q}_2$ and $\mathbb{Q}_3$. In each of the families, the subset of the embeddings is found. For these embeddings, the Hausdorff dimensions are calculated and it is shown that the fractal measure on the image of $\mathbb{Q}_p$ coincides with the Haar measure on $\mathbb{Q}_p$. It is proved that under certain conditions, the image of the $p$-adic solenoid is an invariant set of fractional dimension for a dynamic system. Computer drawings of some fractal images are presented.

1 Introduction

The hierarchical structure of $p$-adic numbers and fractals, as well as the symmetries of self-similar fractals, point to a close relationship between these objects that has been

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repeatedly noted \[3, 16, 17\]. A clear example of this kind is the homeomorphism of the Cantor set onto the ring \( \mathbb{Z}_2 \). However, as far as we know, explicit formulas for the embeddings of various subsets of \( \mathbb{Q}_p \) into Euclidean spaces and the fractal properties of the corresponding images have received little attention. At the same time, the topology of these objects as strange attractors \[12\] is similar to that of the \( p \)-adic solenoids \( T_p \) and therefore, the construction of the embeddings of \( T_p \), say, into the three-dimensional Euclidean space \( \mathbb{R}^3 \) is also of interest.

In this paper we construct the continuous mappings \( \Upsilon^{(m)}_{s} : \mathbb{Q}_p \mapsto \mathbb{C} \), depending on the parameters \( s, a \in \mathbb{C} \) and the number \( m \), which can be a positive integer or \( \infty \). It turns out that \( \Upsilon^{(0)}_{1/3}(\mathbb{Z}_2) \) is a Cantor set and \( \Upsilon^{(0)}_{1/2}(\mathbb{Z}_3) \) is a Sierpinski carpet. It is shown that for some \( s_0 > 0 \), the condition \( |s| < s_0 \) ( \( |a| < (1 - |s|)^{-1} \) ) determines the sets of mappings \( \Upsilon^{(m)}_{s} \) and \( \Omega^{(m)}_{s,a} \) that are the embeddings.

These sets of embeddings possess the following property. Given \( s \) and \( a \), there exist additively invariant metrics in \( \mathbb{Q}_p \) and \( T_p \) such that the mappings \( \Upsilon^{(m)}_{s} \) and \( \Omega^{(m)}_{s,a} \) preserve the Hausdorff dimensions of arbitrary subsets. This property proves to be convenient when studying the Hausdorff measures of the sets. This property proves to be convenient when studying the Hausdorff measures of the sets \( \Upsilon^{(m)}_{s}(\mathbb{Q}_p) \) and \( \Omega^{(m)}_{s,a}(T_p) \) because the Hausdorff measures in the spaces \( \mathbb{Q}_p \) and \( T_p \), with the corresponding metrics, are simply the Haar measures. Moreover, it turns out that the Haar measure of any set in \( \mathbb{Q}_p \) coincides with the fractal measure of its image (see formula \(22\)). This property and the fact that \( \Upsilon^{(\infty)}_{s} \) is a series of continuous additive characters in \( \mathbb{Q}_p \) make it possible to apply group-theory methods when calculating the integrals with respect to the fractal measure. This can serve as an effective means for solving some problems of quantum mechanics and diffusion or diffraction on such fractals \[17, 18, 14\].

We prove that for an embedding \( \Omega^{(\infty)}_{s,a} \) there is a dynamic system \( 1 \) such that \( \Omega^{(m)}_{s,a}(T_p) \) is an invariant set of this system and any integral trajectory lying in \( \Omega^{(m)}_{s,a}(T_p) \) densely

\[1\]

By a dynamic system in \( \mathbb{R}^n \) we mean an autonomous system of \( n \) first-order equations that satisfies the conditions of the existence and uniqueness theorem
winds around this set. In addition, this mappings Υ_{s}^{(∞)} and Ω_{s,a}^{(∞)}, are inter-related. In the present paper, we construct an injective homomorphism j of the additive group Q_{p} into the group T_{p} such that j(Q_{p}) is dense in T_{p} and prove the existence of a Lipschitz mapping J : C \rightarrow R^{3} commuting with j (see formula (41)) whose restriction to Υ \subseteq \Omega \subseteq Q_{p} is a local isometry.

2 Hausdorff measures (basic definitions)

Here we present some relevant material concerning Hausdorff measures in a form that is suitable for the subsequent presentation. For an arbitrary metric space (M, \rho), we define the δ-dimensional outer Hausdorff measure h_{\delta} by setting ∀A \subseteq M, 

$$h_{\delta}(A) = \lim_{\varepsilon \rightarrow +0} h_{\varepsilon}(A) = \lim_{\varepsilon \rightarrow +0} \inf \{\sum_{i=1}^{\infty} \text{diam}(S_{i})^{\delta} : \bigcup_{i=1}^{\infty} S_{i} \supseteq A, \text{diam}(S_{i}) \leq \varepsilon\},$$

where δ is a fixed positive number and diam(B) ≡ sup{\rho(x,y) : x,y \in B}. Then h_{\delta} is a measure countably additive and regular in the sense of Borel [4]. By definition, the Hausdorff dimension of a subset ∀A \subseteq M is the number [3]

$$D_{h}(A) = \inf \{\delta : h_{\delta}(A) = 0\} = \sup \{\delta : h_{\delta}(A) = \infty\}. \quad (2)$$

In particular, it follows that if h_{\delta}(A) > 0 and the measure h_{\delta A}(\cdot) ≡ h_{\delta}(\cdot \cap A) is σ-finite, then δ = D_{h}(A). By the local Hausdorff dimension at a point x \in A, we mean the number D_{h}^{L}(x) = \inf(D_{h}(A \cap U_{x})), where inf extends over all open neighborhoods U_{x} of the point x.

Convention . Irrespective of the nature of the set X, for arbitrary non-negative real-valued functions F and G on X, we write F \preceq_{L} G or ∀x \in X F(x) \preceq_{L} G(x) whenever there exists C > 0 such that ∀x \in X F(x) \leq CG(x). If the relations F \preceq_{L} G and G \preceq_{L} F hold simultaneously, then F \simeq_{L} G and we say that F and G are equivalent [4]. When interpreting constants as trivial functions, we write c \simeq_{L} 0 instead of 0 < c < \infty. For any
F, G, and \( H \preceq F \). the following elementary relations hold: \( \forall a, b, \alpha > 0 \)

\[
\begin{align*}
\min(F, H) & \preceq H, \max(F, H) \preceq F, \\
aF + bG & \preceq \max(F, G) \preceq \max(F, G \pm H) \preceq (F^\alpha + G^\alpha)\frac{1}{\alpha}. 
\end{align*}
\]

(3)

Let \( \Phi : M \mapsto N \) be a mapping of the metric spaces \((M, \rho)\) and \((N, d)\) and let \( d^\Phi \equiv d(\Phi(\cdot), \Phi(\cdot)) \) Then \( \Phi \) is a Lipschitz mapping if \( d^\Phi \preceq \rho \). In this case, \( \Phi \) is called an \( L \)-contraction. If \( d^\Phi \preceq \rho \), then \( \Phi \) is a Lipschitz embedding and the restriction of \( \Phi^{-1} \) to \( \text{Ran}(\Phi) \) is also a Lipschitz mapping: such a mapping is called a \textit{Lipschitz isometry} or \textit{L-isometry}.

\textbf{Definition 1} For given pseudometrics \( \rho_1 \) and \( \rho_2 \) on a set \( M \), the quantity

\[
\kappa_{1,2} = \sup_{x,y \in M} \left( \frac{\rho_1(x, y) - \rho_2(x, y)}{\rho_1(x, y) + \rho_2(x, y)} \right)
\]

(4) (where we set \( 0/0 = 0 \)) is called the divergence of \( \rho_1 \) and \( \rho_2 \).

Obviously, \( \kappa_{1,2} \leq 1 \) and, moreover, it can be shown that \( \kappa_{1,2} < 1 \) if and only if \( \rho_1 \preceq L \rho_2 \).

It is easy to prove that the following assertion holds.

\textbf{Theorem 1} Let \( \rho_1 \) and \( \rho_2 \) - be metrics on \( M \), let \( \kappa_{1,2} \) be their divergence, and \( h^\delta_1, D_1, D^L_1 \) and \( h^\delta_2, D_2, D^L_2 \) denote the corresponding \( \delta \)-dimensional measures and global and local Hausdorff dimensions. Then the inequality

\[
\left( \frac{1 - \kappa_{1,2}}{1 + \kappa_{1,2}} \right) h^\delta_1(F) \leq h^\delta_2(F) \leq \left( \frac{1 + \kappa_{1,2}}{1 - \kappa_{1,2}} \right) h^\delta_1(F),
\]

(5) holds, whence follows that if \( \rho_1 \preceq L \rho_2 \), then \( h^\delta_1 \preceq L h^\delta_2 \) and we have \( \forall F \subset M, D_{h,1}(F) = D_{h,2}(F) \) and \( D^L_{h,1}(x) = D^L_{h,2}(x) \forall x \in F \).

In view of [1], Theorem 1 and Theorem 2.10.45 of [4] imply the next theorem.

\textbf{Theorem 2} Let \((M, \rho)\) be a metric space. Introduce a metric \( d \) in \( \mathbb{R} \times M \) by the formula

\[
d((x, a), (y, b)) = \max(|x - y|, \rho(a, b)), \forall x, y \in \mathbb{R}, \forall a, b \in M.
\]

Then for every Borel set \( A \subset \mathbb{R} \) and every \( \forall B \subset M, \text{ with } h^\delta(B) < \infty \), we have

\[
h^{\delta+1}(A \times B) \preceq h^1(A)h^\delta(B).
\]
3 Hausdorff measures in $\mathbb{Q}_p$ and $\mathbb{T}_p$

In this section, $\mathbb{Q}_p$ and $\mathbb{T}_p$ are regarded as completions of $\mathbb{Q}$ with respect to the corresponding additively invariant metrics. Each element $x$ of the $p$-adic number field $\mathbb{Q}_p$ is uniquely representable as a formal series \[ x = \sum_{n=0}^{\infty} a_n p^n = -\sum_{n=0}^{v} a_n p^n + \sum_{n=v}^{\infty} a_n p^n \] with coefficients $a_n \in \{0, 1, ..., p-1\}$ where $v < \infty$ and $p$ is a fixed prime number. This series absolutely converges in the $p$-adic norm defined $\forall x$ by the relation $\|x\| = p^{-\alpha v(x)}$ for some $\alpha > 0$ and $v(x) = v$ is called the logarithmic norm of $x$. The first sum on the right-hand side of (6) is denoted as $\{x\}_p$ and is the fractional part of $x$, and the other is denoted as $[x]_p$ and is the integral part of $x$. In this case, $\{x\}_p \in \mathbb{Q} \cap [0, 1)$ and $[x]_p \in \mathbb{Q} \cap [0, 1)$, where $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : \|x\| \leq 1 \}$ is the ring of integer $p$-adic numbers. Any number $q \in \mathbb{Q}$ can be expanded uniquely as a series (6) and $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$. The norm $\| \cdot \|$ possesses the property of ultrametricity, $\forall x, y \in \mathbb{Q}_p$, we have
\[ \|x - y\| \leq \max(\|x\|, \|y\|). \] (7)

The norm with $\alpha = 1$ is denoted by $| \cdot |_p$ and the others are denoted simply as $| \cdot |_p^\alpha$. All of these norms depending on $\alpha$ are topologically equivalent \[ \text{[1]} \]; however, the corresponding Hausdorff measures are different (see below). The standard Haar measure $\chi$ in $\mathbb{Q}_p$ is chosen such that \[ \chi(\mathbb{Z}_p) = \int_{\mathbb{Z}_p} d\chi = 1. \] (8)

**Theorem 3** The Hausdorff measure $h_1^{1/\alpha}$ on $(\mathbb{Q}_p, | \cdot |_p^\alpha)$ is the standard regular Haar measure $\forall \alpha > 0$ and, consequently, the local and global Hausdorff dimensions of $(\mathbb{Q}_p, | \cdot |_p^\alpha)$ coincide and are equal to $1/\alpha$.

\[ \text{[3]} \] Actually, the part of $p$ can be played by any positive integer because we do not use the existence of inverse elements the ring $\mathbb{Q}_p$ anywhere.
Proof. By construction, \( h^\delta \) is an invariant measure \( \forall \delta > 0 \) and, therefore, the uniqueness of \( \chi \) implies that it suffices to show that, for instance, \( h^{1/\alpha}(\mathbb{Z}_p) = \chi(\mathbb{Z}_p) = 1 \). Because \( \mathbb{Z}_p \) consists of exactly \( p^M \) disjoint balls of diameter \( p^{-M\alpha} \forall M \in \mathbb{N} \), we have \( h^{1/\alpha}(\mathbb{Z}_p) \leq 1 \).

We now show that \( h^{1/\alpha}(\mathbb{Z}_p) \geq 1 \). Indeed, by the semiadditivity of \( \chi \forall A \subset \mathbb{Q}_p, \forall \varepsilon > 0 \), the inequality

\[
\chi(A) \leq \inf \{ \sum_{i=1}^{\infty} \chi(U_i) : \bigcup_{i=1}^{\infty} U_i \supseteq A, \text{diam}(U_i) \leq \varepsilon \},
\]

holds, where \( U_i \), are open balls in \( \mathbb{Q}_p \). On the other hand, it follows from the properties of norm (4) that any subset \( B \subset \mathbb{Q}_p \) \( \text{diam}(B) = r \) is contained in the open (and, simultaneously, closed) ball \( U = \{ x : | x - x_0 |_p \leq r, x_0 \in B \} \) with \( \text{diam}(U) = r \) and this, together with the relation \( \chi(U) = \text{diam}^{1/\alpha}(U) \), which is valid for any ball \( U \), implies that \( h^{1/\alpha} \) coincides with the right-hand side of inequality (4) and, consequently, we have \( h^{1/\alpha}(\mathbb{Z}_p) \geq \chi(\mathbb{Z}_p) \).

Let us consider \( \mathbb{R} \times \mathbb{Z}_p \) as a direct product of additive groups and introduce a metric in this group by fixing some \( \alpha > 0 \) and setting \( \forall a, b \in \mathbb{R} \) and \( \forall x, y \in \mathbb{Z}_p \)

\[
\hat{\rho}_\alpha((a, x), (b, y)) = \max(| a - b |, | x - y |_p^\alpha).
\]

(10)

It is clear that \( \hat{\rho}_\alpha \) is an invariant metric and the topology generated by it coincides with that of the direct, product of groups.

Definition 2 Let \( B \) denote the subgroup \( \{(n, -n) : n \in \mathbb{Z} \} \) of the group \( \mathbb{R} \times \mathbb{Z}_p \). Then the quotient group \( \mathbb{T}_p = (\mathbb{R} \times \mathbb{Z}_p) / B \) is called a \( p \)-adic solenoid.

It can be shown \( [5] \) that \( \mathbb{T}_p \) is a connected compact Abelian group. We can define an invariant metric \( \rho_\alpha \) on \( \mathbb{T}_p \) that is compatible with the topology as the quotient metric according to the standard scheme\( [3] \). namely, \( \forall f, g \in \mathbb{T}_p \), we set

\[
\rho_\alpha(f, g) = \inf \{ \hat{\rho}_\alpha((a, x), (b, y)) : (a, x) \in f, (b, y) \in g \}.
\]

(11)

The construction below gives a concrete realization of \( \mathbb{T}_p \).

\footnote{This definition differs from the one in \( [3] \), where \( B = \{(n, n) : n \in \mathbb{Z} \} \). However, since the mapping \( x \mapsto -x \) is automorphism of the additive group \( \mathbb{Z}_p \), the corresponding quotient groups are isomorphic.}
Theorem 4 Consider the product \([0, 1) \times \mathbb{Z}_p\) and define addition in it by the following rule:

\[
f + g = (\xi + \eta - \lfloor \xi + \eta \rfloor, x + y + \lfloor \xi + \eta \rfloor),
\]

where \([\xi + \eta]\) is the integral part of the real number \(\xi + \eta\). For a fixed \(\alpha > 0\), we define a metric \(\rho_\alpha\) by setting

\[
\rho_\alpha(f, g) = \min(\ell_\alpha(f - g), \ell_\alpha(g - f)),
\]

where \(\ell_\alpha(f) = \max(\xi, |x|_p^\alpha)\). Then the resulting Abelian group with the topology induced by the metric is algebraically and isometrically isomorphic \(T_p\) with metric\((12)\).

Proof. The algebraic isomorphism is established in practically the same way as in the proof of theorem 10.15\(^5\). It follows from \((10), (11)\) and the definition of subgroup \(B\) that the metric on \(T_p\) satisfies the relation

\[
\rho_\alpha(f, 0) = \inf \{\max(\xi - n, |x + n|_p^\alpha) : n \in \mathbb{Z}\},
\]

However, because \(\forall x \in \mathbb{Z}_p\), the inequality \(|x + n|_p^\alpha \leq 1\) holds, the infimum in the above formula can be extended over the set \(n = 0, 1\). Taking into account that \((-f) = (1 - \xi, -x - 1)\) for \(\xi \neq 0\), we can easily show that the metrics do, in fact, coincide. \(\square\)

Using Theorem \((4)\) and the facts that \(\forall x \in \mathbb{Z}_p\) \(\rho_\alpha((0, x), 0) = |x|_p^\alpha\) and \(\{x\}_p = 0\), and, also, that the set \(\{\{x\}_p : x \in \mathbb{Q}_p\}\) is dense in \([0, 1)\) (in the usual topology), we can prove the following theorem.

Theorem 5 The mapping \(j : \mathbb{Q}_p \mapsto T_p\), associating the element \((\{x\}_p, [x]_p) \in T_p\) with each \(x \in \mathbb{Q}_p\), is an injective homomorphism of the additive group \(\mathbb{Q}_p\) into \(T_p\) and is also a local isometry from \((\mathbb{Q}_p, | \cdot |_p^\alpha)\) into \((T_p, \rho_\alpha)\) \(\forall \alpha > 0\). Furthermore, \(\text{Ran}(\mathbb{Q}) \subset \text{Ran}(\mathbb{Q}_p)\) is dense in \(T_p\). Because \(T_p\) is complete (in view of the compactness \((9)\)), \(T_p\) is the completion of \(\mathbb{Q}\) with respect to the metric \(\rho_\alpha\).\(^5\)

\(^5\) Note that although \(j\) is not an embedding (\(\text{Ran}(\mathbb{Q}_p) \subset T_p\)), the restriction of \(j\) to \(p^M\mathbb{Z}_p\) is an (L-) isometry \(\forall M \in \mathbb{N}(\mathbb{Z})\).
The $p$-adic solenoid $T_p$ is a compact Abelian group \([5]\) and, therefore, there is a finite Haar measure $\chi$ on it that is unique and invariant. Let us show that the Hausdorff measure $h_\delta$ on $(T_p, \rho_\alpha)$ for $\delta = \alpha^{-1} + 1$ coincides (to within a finite nonzero multiplier) with $\chi$ on all Borel subsets of $T_p$.

Since $\rho_\alpha$ is an invariant metric, it suffices to prove that $U_{1/2} = \{y \in \mathbb{R} \times \mathbb{Z}_p : \rho_\alpha(y, 0) < 1/2\}$ $h_\delta(\phi(U_{1/2})) \overset{L}{\simeq} 1$, where $(\phi$ is the canonical projection of $\mathbb{R} \times \mathbb{Z}_p$ onto $T_p$.

However, because $\forall a, b \in B \subset \mathbb{R} \times \mathbb{Z}_p$ we have $\hat{\rho}_\alpha(a, b) \geq 1$ for $a \neq b$, the restriction of $\phi$ to $U_{1/2}$ is an isometry. Therefore, applying the relation $U_{1/2} = (-\frac{1}{2}, \frac{1}{2}) \times p^{n_\alpha} \mathbb{Z}_p$ for some $n_\alpha > 0$, we conclude, in view of Theorem (2), that $h_\delta(\phi(U_{1/2})) = \hat{h}_\delta((-1/2, 1/2) \times \{x \in \mathbb{Z}_p : |x|_p \leq p^{-1}\}) \overset{L}{\simeq} p^{-n_\alpha} \tilde{h}_{1/\alpha}(\mathbb{Z}_p)$.

Here $\hat{h}_\delta$ and $\tilde{h}_\delta$ denote the Hausdorff measures in $\mathbb{R} \times \mathbb{Z}_p$ and $T_p$, respectively. Thus, as in the case of $\mathbb{Q}_p$, the local and global Hausdorff dimensions of $(T_p, \rho_\alpha)$ coincide and are equal to $\alpha^{-1} + 1$.

4 Continuous mappings of $\mathbb{Q}_p$ into $\mathbb{C}$

Given $\forall n \in \mathbb{Z}$ and $\forall m \in \overline{\mathbb{N}} \equiv \mathbb{N} \cup \{\infty\}$, we define the complex-valued functions $\chi_n^{(m)}(\cdot)$ on $\mathbb{Q}_p$ by the formula

$$\chi_n^{(m)}(x) = \exp\left(\frac{i 2\pi}{p} \sum_{k=0}^{m} x_{n-k} p^{-k}\right), \quad (14)$$

where $x_n$ is the $n$th coefficient in the expansion of $x$ into series \([\mathbb{R}]\). It is easy to show that $\chi_n^{(m)}(\cdot)$ are continuous. Note that $\chi_n^{(\infty)}(\cdot)$ coincides with the continuous additive character $\chi_{\frac{1}{p^{1+\epsilon}}}(\cdot)$ on $\mathbb{Q}_p$ \([3, 5]\).

**Definition 3** For every $s \in U_1 \equiv \{z \in \mathbb{C} : |z| < 1\}$ and $\forall m \in \overline{\mathbb{N}}$, we define a continuous mapping $\forall m \in \overline{\mathbb{N}}$ by setting $\Upsilon_s^{(m)} : \mathbb{Q}_p \mapsto \mathbb{C}$

$$\Upsilon_s^{(m)}(x) = \frac{1 - s v(x)}{1 - s} + \sum_{n=v(x)}^{\infty} s^n \chi_n^{(m)}(x) = [\Upsilon_s^{(m)}]_{}(x) + \{\Upsilon_s^{(m)}\}_{}(x), \quad \forall x \in \mathbb{Q}_p. \quad (15)$$
Here \( \{ \Upsilon_s^{(m)}(x) \} = \sum_{n=0}^{\infty} s^n \chi_n^{(m)}(x) \) is the "integral part" \( \Upsilon_s^{(m)}(x) \) and \( \{ \Upsilon_s^{(m)} \} = \Upsilon_s^{(m)}(x) - \lfloor \Upsilon_s^{(m)}(x) \rfloor \) is the "fractional part" of \( \Upsilon_s^{(m)}(x) \).

Obviously, the mapping \( \Upsilon_s^{(m)} \) is well defined and for any fixed \( x \in \mathbb{Q}_p \), formula (13) defines a function of \( s \) which is holomorphic in the circle \( U_1 \). In addition, taking into account that the coefficients of series (13) are zero (periodic (13)) beginning with some \( \forall q \in \mathbb{N}(\mathbb{Q}) \), we can show that the following interesting property holds: \( \Upsilon_s^{(m)}(q) - \Upsilon_s^{(m)}(0) \) is a polynomial (a rational function) for \( m < \infty \) and \( g(\cdot) \cdot \Upsilon_s^{(\infty)}(q) \) is an entire function (a meromorphic function, i.e., a ratio of entire functions) for \( m = \infty \), where

\[
g(s) = \prod_{k=0}^{\infty} (1 - p^{-k}s)
\]

In the case \( m < \infty \), the proof of this assertion follows from the periodicity of \( \chi_n^{(m)}(q) \) with respect to \( n \) for \( n > n_q + m \). If \( m = \infty \), we have

\[
\chi_n^{(\infty)}(q) = \exp(\pi \{ q/p^{n+1} \}_p) = \exp(\pi q/p^{n+1}) \exp(-i \pi [q/p^{n+1}]_p)
\]

, \( \forall q \in \mathbb{Q} \) and, therefore, it is possible to construct the analytic continuation with the aid of the Taylor expansion of \( \exp(\pi q/p^{n+1}) \), using the periodicity of \( \exp(-i \pi [q/p^{n+1}]_p) \) with respect to \( n \) for \( n > n_q \). [4] It is easy to prove that \( \forall x \in \mathbb{Q}_p \), we have the relations

\[
\{ \Upsilon_s^{(m)} \} = \{ \Upsilon_s^{(m)} \}(\{ x \}_p), \ \Upsilon_s^{(m)}([x]_p) = [\Upsilon_s^{(m)}([x]_p), \ [\Upsilon_s^{(m)}(p^{m}x) = [\Upsilon_s^{(m)}(p^{m}][x]_p), (16)
\]

The following basic property of the mapping (the scaling) \( \Upsilon_s^{(m)} \) also holds: \( \Upsilon_s^{(m)} \)-scaling:

\[
\forall x \in \mathbb{Q}_p \quad \Upsilon_s^{(m)}(px) = s \Upsilon_s^{(m)}(x) + 1 = p^{-D_s} e^{\text{arg}(s) \Upsilon_s^{(m)}(x) + 1}, \quad (17)
\]

where \( D_s = -\log_p^{-1}(s) \) is called the scaling dimension of \( \Upsilon_s^{(m)} \). This relation follows from the fact that \( \chi_n^{(m+k)}(p^kx) = \chi_n^{(m)}(x), \forall k \in \mathbb{Z} \). Furthermore, by applying [10], (17), expansion (13), say, we set \( x_n = \sum_{k=0}^{\infty} \delta_{n,2^k} \) for \( x \in \mathbb{Q}_2 \), then \( T_s^{(0)}(x) = -2 \sum_{k=0}^{\infty} s^{2^k} \) and it can be shown [11] that \( U_1 \) is a holomorphy domain of \( \Upsilon_s^{(0)}(x) \).
we can show that the set $\text{Ran} \Upsilon_s^{(m)}$ is self-similar $\text{Ran} \Upsilon_s^{(m)}$. More precisely, let $B_l^n \equiv \Upsilon_s^{(m)}(\{x \in \mathbb{Q}_p : |x - l|_p \leq p^{-n}\})$, then

$$B_l^n = \bigcup_{l=0}^{p^{n-1}} \{z^i_{l,i} + e^{i\arg(s)m}p^{-\frac{m}{2}}B_0^0\},$$

\[ (18) \]

$\forall n \in \mathbb{Z}, l \in \mathbb{Q}_p$. Here $z^i_{l,i}$ are shifts of $\mathcal{C}$ depending on $l, \bar{l}$ and $n$. Thus, every $B_l^n$ can be obtained from $p^n$ sets $B_0^0$ by means of continuous motions of the plane $\mathcal{C}$ (shifts and rotations) and a scaling transformation.

Now, we find the conditions under which the mappings $\Upsilon_s^{(m)}$ become embeddings. To this end, we define the number

$$\Delta_s^{(m)} = \inf \{ |\Upsilon_s^{(m)}(x) - \Upsilon_s^{(m)}(y)| : \forall x, y \in \mathbb{Q}_p : |x - y|_p = 1 \}.$$  

\[ (19) \]

Using the fact that $|\chi_0^{(m)}(x) - \chi_0^{(m)}(y)| \geq 2\sin(\pi/p)$ for $|x - y|_p = 1$. we can derive the inequality

$$\frac{\Delta_s^{(m)}}{2} \geq \sin \left( \frac{\pi}{p} \right) - \frac{|s|}{1 - |s|}.$$  

\[ (20) \]

It follows that $\Delta_s^{(m)} > 0$ for

$$|s| < s_0 = \frac{\sin(\pi/p)}{1 + \sin(\pi/p)}.$$

**Theorem 6** Let $s$ and $m$ be such that $\Delta_s^{(m)} > 0$, then $\Upsilon_s^{(m)}$ is an $L$-isometry from $(\mathbb{Q}_p, |\cdot|_{p^{-1}})$ into $(\mathcal{C}, |\cdot|)$ and, therefore, $\Upsilon_s^{(m)}$ is an embedding.

**Proof.** The proof can be derived from the inequality below, which is a consequence of (17), and definition (14), namely, $\forall x, y \in \mathbb{Q}_p$ we have

$$\Delta_s^{(m)} |s|^v(x - y) \leq |\Upsilon_s^{(m)}(x) - \Upsilon_s^{(m)}(y)| \leq \frac{2}{1 - |s|} |s|^v(x - y).$$

\[ (21) \]

\[ \Box \]

Using the compactness of $\mathbb{Z}_p$, property (17) and the completeness of $\mathbb{Q}_p$ and $\mathcal{C}$, we can show that if $\Upsilon_s^{(m)}$ is an injective mapping, then it is an embedding.
It follows from Theorems 1 and 6 that if $\Delta_s^{(m)} > 0$, then the local and global Hausdorff dimensions of $\Upsilon_s^{(m)}(Q_p)$ are equal to $D_s$. These theorems also imply that $h^{D_s}(\Upsilon_s^{(m)}(Z_p)) \lesssim 1$. Therefore, we can introduce a fractal measure $\mu_f$ on $Q_p$, by the formula

$$\mu_f(B) = \frac{h^\delta(B \cap \Upsilon_s^{(m)}(Q_p))}{h^\delta(\Upsilon_s^{(m)}(Z_p))} \quad \forall B \subset C.$$  

(22)

The restriction of this measure to $\Upsilon_s^{(m)}(Z_p)$ coincides with the multifractal measure $\mu_0^f$ of the set $\Upsilon_s^{(m)}(Z_p)$. Moreover, the assertion below holds.

**Theorem 7** Let $\chi(\cdot)$ be the standard Haar measure in $Q_p$. In this case, if $\Delta_s^{(m)} > 0$, then

$$\mu_f(\cdot) = \chi((\Upsilon_s^{(m)})^{-1}(\cdot)).$$  

(23)

**Proof.** Because $\Upsilon_s^{(m)}$ is an embedding, the clusters $B^n_l$ defined in (18) do not intersect. Therefore, if $m < \infty$, then the self-similarity of the set $\Upsilon_s^{(m)}(Q_p)$ implies that $h^{D_s}(\Upsilon_s^{(m)}(B)) = h^{D_s}(\Upsilon_s^{(m)}(Z_p))\chi(B)$ for all open sets $B$. The proof of the theorem in the case $m = \infty$ follows from Theorem (1) and the lemma below.

**Lemma 1** Let $\rho_m$ be pseudometrics on $Q_p$, denoted by the formula $\rho_m(x, y) = | \Upsilon_s^{(m)}(x) - \Upsilon_s^{(m)}(y) | \forall x, y \in Q_p$ and let $\kappa_{m, \infty}$ be the divergence of $\rho_m$ and $\rho_\infty$. Then

$$\kappa_{m, \infty} < \frac{4\pi}{(1 - | s |)(\Delta_s^{(m)} + \Delta_s^{(\infty)})p^{-m}}.$$  

Proof of the lemma. Note that in view of scaling (17), the supremum in formula (2) can be bounded over all $x, y$ such that $| x - y |_p = 1$ and, therefore, the denominator in (4) is greater than or equal to $\Delta_s^{(m)} + \Delta_s^{(\infty)}$. On the other hand, since $\forall x \in Q_p$

$$| \chi_n^{(\infty)}(x) - \chi_n^{(m)}(x) | = | 1 - \exp\left(\frac{i2\pi}{p} \sum_{k=m+1}^{\infty} x_{n-k}p^{-k}\right) | < 2\pi p^{-m}$$  

(24)

it is easy to show that the numerator in (4) is always less than $4\pi p^{-m}/(1 - | s |)$. Thus, Theorem 7 permits the integration technique in $Q_p$ to be applied for calculating integrals with respect to the fractal measure $\mu_f$ or $\mu_0^f$. For instance, it is possible to
calculate the integrals of arbitrary polynomials in $z$ and $\bar{z}$ with respect to $\mu_f^0$, for $z \in \mathbb{C}$. Indeed, it can be proved

$$< z^L \bar{z}^\bar{L} > \equiv \int_\mathbb{C} z^L \bar{z}^\bar{L} d\mu_f^0(x) = \int_\mathbb{Z}_p (\Upsilon_s^{(m)}(x))^L (\overline{\Upsilon_s^{(m)}(x)})^\bar{L} d\chi(x) = \sum_{n,\bar{n}=0}^{\infty} C_{n,\bar{n}}^{L,\bar{L}} s^{n} \bar{s}^{\bar{n}}. \quad (25)$$

holds. Furthermore, $C_{n,\bar{n}}^{L,\bar{L}} \in \mathbb{N}$ for $m = 0, \infty$. In particular, for $m = \infty$, the expression $C_{n,\bar{n}}^{L,\bar{L}}$ is the number of representations $n = n_0 + \ldots + n_L$ and $\bar{n} = \bar{n}_0 + \ldots + \bar{n}_{\bar{L}}$ such that $\sum_{k=0}^{L} p^{-n_k} = \sum_{k=0}^{\bar{L}} p^{-\bar{n}_k} \mod(p)$. Applying this formula, we can show that, say, $< z^p > = < \bar{z}^p > = 1$ and $< z^L \bar{z}^\bar{L} > = (1 - |s|^2)^{-L} \delta_{L,\bar{L}}$ for $L, \bar{L} < p$.

Laying aside the problem of strict mathematical justification, we indicate one more possible application of Theorem 7. Consider a quantum particle on the fractal $F = \text{Ran}(\Upsilon_s^{(m)})$. In this case, $L^2(\mathbb{C}, \mu_f)$ can be regarded as a Hilbert space of quantum state. Assume that the Schrodinger equation for energy eigenvalues $E$ can be written in the form

$$E \Psi(z) = \int K(|z - \hat{z}|) \Psi(\hat{z}) d\mu_f(\hat{z}), \quad (26)$$

Using Theorem 7, we can pass to an equivalent equation in $L^2(Q_p, \chi)$ with the kernel $K(|\Upsilon_s^{(m)}(x) - \Upsilon_s^{(m)}(y)|)$ for $x, y \in Q_p$. Evidently, in the general case, this does not give any new results because the kernel $K$ is translation-invariant in $\mathbb{C}$, whereas $\mu_f$ is not. Conversely, $\mu_f$ is invariant in $Q_p$, whereas $K$ is not. However, applying (17), we can write the kernel in the form

$$K(|x - y| \sigma_p \cdot |(\chi_s^{(m)}(x)(x - y) - (1 - |s|^2)^{-L} \delta_{L,\bar{L}})|), \quad (27)$$

where $|O^{x,y}(s)| \leq |s|/(1 - |s|) \forall x, y \in Q_p$. Hence, for $|s| \ll 1$ the kernel depends solely on $x - y$ and, therefore, the Fourier transformation for $Q_p$ brings the Hamiltonian to a diagonal form. In addition for $p = 2, 3$, the relation $|\chi_s^{(m)}(x) -$
1 \mid 2 \sin(\pi/p) \text{ can be applied to prove that the spectrum of the Hamiltonian (in this approximation) has the form } \{ E_n = \tilde{K}(p^n) : n \in \mathbb{Z} \} \text{ and all eigenfunctions, except for the ground state, are strictly localized (i.e., they are compactly supported \[3\]). Here } \tilde{K}(\cdot) \text{ is the Fourier image of the function } K(2 \sin(\pi/p) \mid \cdot \mid_p^a) \text{. Applying formulas(14,15) and the } p\text{-adic integration technique, and consecutively expanding (27) into power series in } s \text{ (but for fixed } a \text{), we can find the Fourier images of the coefficients of the series at least in the case } m = \infty \text{ and } p = 2, 3 \text{. Thus, there appears to be a possibility of calculating subsequent corrections to the spectrum and wave functions using perturbation theory.}

Note that the foregoing equally applies to the diffusion problem on the fractal } F.

5 \text{ Continuous mappings of } T_p \text{ into } \mathbb{R}^3

For arbitrary } \forall n \in \mathbb{Z} \text{ and } \forall m \in \bar{\mathbb{N}} \equiv \mathbb{N} \cup \{ \infty \} \text{, we define some complex-valued continuous functions } \tilde{\chi}_n^{(m)}(\cdot) \text{ on } \mathbb{R} \times \mathbb{Z}_p \text{ by setting } \forall (\xi, x) \in \mathbb{R} \times \mathbb{Z}_p

\tilde{\chi}_n^{(m)}(\xi, x) = \exp(i2\pi\xi \frac{\theta(p^{m-n} - \mid x \mid_p)}{p^{\min(n,m)+1}})\chi_n^{(m)}(x).

(28)

Here the Heaviside function } \theta(\cdot) \text{ is the indicator of the set } [0, \infty]. \text{ It is easy to show that the functions } \tilde{\chi}_n^{(m)}(\cdot) \text{ satisfy the relation}

\tilde{\chi}_n^{(m)}(\xi + l, x - l) = \tilde{\chi}_n^{(m)}(\xi, x),

(29)

\forall l \in \mathbb{Z}, (\xi, x) \in \mathbb{R} \times \mathbb{Z}_p, \text{ i.e., they are constant on the cosets of group } B \text{ (see Definition } 2\text{). Note that } \forall f = (\xi, x) \in T_p \tilde{\chi}_n^{(\infty)}(\xi, x) = \tilde{\chi}_{(-p^{-n}-1)}(f) \text{, where } \tilde{\chi}_{(q)}(\cdot) \text{ are continuous additive characters on } T_p \text{ (in view of Remark 4).}

\text{Definition 4 For every } s \in U_1 \text{ and } \forall m \in \bar{\mathbb{N}} \text{, we define the continuous mapping}\n
\omega_s^{(m)} : \mathbb{R} \times \mathbb{Z}_p \mapsto \mathbb{C}, \text{ by the formula}

\omega_s^{(m)}(\xi, x) = \sum_{n=0}^{\infty} s^n \tilde{\chi}_n^{(m)}(\xi, x). \forall (\xi, x) \in \mathbb{R} \times \mathbb{Z}_p

(30)

Note that } \forall x \in \mathbb{Z}_p \text{ the relation}

\gamma_s^{(m)}(x) = \omega_s^{(m)}(0, x).

(31)
holds. For given \( a \in \mathbb{C} \), we introduce the mapping \( \sigma_a : S_1 \times \mathbb{C} \mapsto \mathbb{R}^3 \) by defining the correspondence \( S_1 \times \mathbb{C} \ni (e^{i2\pi \xi}, z) \mapsto (x_1, x_2, x_3) \in \mathbb{R}^3 \) according to the formulas
\[
x_1 + ix_3 = e^{i2\pi \xi} | a | \left( 1 + \text{Re} \left( \frac{z}{a} \right) \right), \quad x_2 = | a | \text{Im} \left( \frac{z}{a} \right),
\]
where \( S_1 \) is the unit circle in \( \mathbb{C} \). We now define a mapping from \( T_p \) into \( \mathbb{R}^3 \) as the composition of \( \sigma_a \) a mapping from \( T_p \times S_1 \times \mathbb{C} \) is constructed below. To this end, we consider \( T_p \) and \( S_1 \times \mathbb{C} \) as quotient spaces with respect to the action of the group \( \mathbb{Z} \) on \( R \times \mathbb{Z} \) and \( R \times \mathbb{C} \), namely, \( R \times \mathbb{Z} \) and \( R \times \mathbb{C} \) is defined as shifts by the elements \( (n, -n) \in B \) and \( (n, 0) \in \mathbb{Z} \times \{0\} \), respectively. Let \( \otimes \omega_s^{(m)} \equiv id \times \omega_s^{(m)} : R \times \mathbb{Z} \mapsto R \times \mathbb{C} \), where \( id \) is the identity mapping. It is easy to see that property (29) guarantees the existence of the continuous quotient mapping \( \Upsilon^{(m)}_{s,a} : T_p \mapsto S_1 \times \mathbb{C} \), as well as a mapping \( \Omega^{(m)}_{s,a} : T_p \mapsto \mathbb{R}^3 \), such that the diagram
\[
\begin{array}{ccc}
R \times \mathbb{Z} & \xrightarrow{\otimes \omega_s^{(m)}} & R \times \mathbb{C} \\
\downarrow \phi & & \downarrow \varphi \\
T_p & \xrightarrow{\Upsilon^{(m)}} & S_1 \times \mathbb{C} \\
\downarrow \Omega^{(m)}_{s,a} & & \downarrow \sigma_a \\
\mathbb{R}^3 & & \mathbb{R}^3
\end{array}
\]
is commutative. Here \( \phi \) and \( \varphi \) are the corresponding canonical projections. Moreover, it can be shown that the actions of the group \( \mathbb{Z} \) on \( R \times \mathbb{Z} \) and \( R \times \mathbb{C} \) are compatible in the sense that \( \forall f \in T_p \) the relation
\[
\otimes \omega_s^{(m)}(\phi^{-1}(f)) = \varphi^{-1}(\Upsilon^{(m)}(f)).
\]
holds. We introduce an invariant metric \( \hat{d} \) on \( R \times \mathbb{C} \) such that
\[
\hat{d}((\xi_1, z_1), (\xi_2, z_2)) = \max \{ | \xi_1 - \xi_2 |, | z_1 - z_2 \} \forall (\xi_1, z_1), (\xi_2, z_2) \in R \times \mathbb{C}
\]
The metric \( d \) on \( S_1 \times \mathbb{C} \) is defined as the quotient metric of \( \hat{d} \) following the same scheme as in (14). Then it is not difficult to prove that \( \forall (e^{i2\pi \xi_1}, z_1), (e^{i2\pi \xi_2}, z_2) \in S_1 \times \mathbb{C} \) (here we assume that \( \xi_1, \xi_2 \in [0, 1) \)),
\[
d((e^{i2\pi \xi_1}, z_1), (e^{i2\pi \xi_2}, z_2)) = \max \left( \inf_{n=0, \pm 1} \{ | \xi_1 - \xi_2 + n |, | z_1 - z_2 | \} \right).
\]
Because $\phi, \varphi$ and the restriction of $\sigma_a$ to any bounded set in $S_1 \times C$ are L-contractions, if we show that $\omega_{s}^{(m)}$ is an L-contraction, then this implies that $U_s^{(m)}$ and $\Omega_{s,a}^{(m)}$ are L-contractions as well. Moreover, if $\omega_{s}^{(m)}$ is an L-isometry, then so is $U_s^{(m)}$. Using (33), we can show that $\forall f, g \in T_p$, there are $(\xi, x) \in f$, $(\eta, y) \in g$ such that, $\forall \varepsilon > 0$, the chain of inequalities

$$\rho_a(f, g) \leq \hat{\rho}_a((\xi, x), (\eta, y)) \leq \hat{d}(\omega_{s}^{(m)}(\xi, x), \omega_{s}^{(m)}(\eta, y)) \leq d(U_s^{(m)}(f), U_s^{(m)}(g)) + \varepsilon.$$ 

holds. It is also obvious that if the restriction of $\sigma_a$ to $\text{Ran}(\varphi \circ \omega_{s}^{(m)})$ an L-isometry, then the mapping $\Omega_{s,a}^{(m)}$ is also. To find the conditions under which $\omega_{s}^{(m)}$ is an L-contraction or an L-isometry, we introduce the following two numbers:

$$\Delta_s^m = \inf_{x, y \in Z_p, \xi \in R} (| s^{v(x-y)}(\omega_s^{(m)}(\xi, x) - \omega_s^{(m)}(\xi, y)) |),$$

(35)

$$\gamma_{s,a}^m = - \inf_{(\xi, x) \in R \times Z_p} (\text{Re}(\omega_s^{(m)}(\xi, x)/a)).$$

(36)

It is easy to show that $\Delta_s^m$ satisfies the same inequality as the one in (24) and, hence, $\Delta_s^m > 0$ for $|s| < s_0$. Furthermore, it can be proved that $\Delta_s^\infty = \Delta_s^\infty$ (see below). The above-mentioned conditions follow from the chain of inequalities below:

$$\Delta_s^m \max(|\xi - \eta|, |x - y| \frac{2\pi}{p}) \leq \Delta_s^m \max(|\xi - \eta|, |x - y| \frac{2\pi}{p} - \frac{2\pi}{p(1 - |s|)} |\xi - \eta|)$$

$$\leq \max(|\xi - \eta|, |\omega_s^{(m)}(\xi, x) - \omega_s^{(m)}(\eta, y)|)$$

$$\leq \max(|\xi - \eta|, \frac{1}{1 - |s|} (|x - y| \frac{2\pi}{p} + \frac{2\pi}{p} |\xi - \eta|)) \leq \Delta_s^\infty \max(|\xi - \eta|, |x - y| \frac{2\pi}{p}),$$

the first and last being the consequences of relations (3); the others were obtained analogously to (21). An elementary geometric consideration implies that the restriction of $a a$ to any compact set in $\pi_a \equiv \{z \in C : \text{Re}(z/a) > 1\}$ is an L-isometry and, furthermore, $\text{Ran}(U_s^{(m)}) \subset \pi_a$ for $\gamma_{s,a}^m < 1$. Thus, we have proved the following theorem.

**Theorem 8** $\hat{\rho}_a, \rho_a, \hat{d}$ and $d$ be the metrics in $R \times Z_p, T_p, R \times C$ and $S_1 \times C$, respectively. Let $R^3$ be endowed with the standard Euclidean metric and let $\alpha = D_s^{-1}$. Then for any $m \in N$ and $s \in U_1$, the following assertions hold:

1) $\omega_{s}^{(m)}$ is an L-contraction and, therefore, $U_s^{(m)}$ and $\Omega_{s,a}^{(m)}$ are also L-contractions:
2) if \( \tilde{\Delta}_s^m > 0 \), then \( \otimes \omega_s^{(m)} \) and \( \mathcal{U}_s^{(m)} \) are L-isometries and addition \( \gamma_{s,a}^m < 1 \), then \( \Omega_{s,a}^{(m)} \) is an L-isometry.

It follows immediately that if \( s \) and \( a \) are such that \( \tilde{\Delta}_s^m > 0 \) and \( \gamma_{s,a}^m < 1 \), then the mapping \( \Omega_{s,a}^{(m)} \) a continuous embedding of \( T_p \) in \( \mathbb{R}^3 \) and the local and global Hausdorff dimensions \( \Omega_{s,a}^{(m)}(T_p) \) are equal to \( D_s + 1 \).

Next, we describe the geometric structure of the set \( \Omega_{s,a}^{(m)}(T_p) \). To this end, we note that \( (T_p, \rho_a) \) can be regarded as the total space of the locally trivial fiber bundle \( (T_p, S_1) \) with the projection \( \chi_0 : T_p \mapsto S_1 \), such that \( \chi_0(\xi, x) = \exp(i2\pi \xi) \in S_1 \), \( \forall (\xi, x) \in T_p \) and fibers \( \chi_0^{-1}((\xi, Z_p)) = (\xi, Z_p) \) isometric to \( (Z_p, | \cdot |^2_p) \). It is easily seen that the mapping \( \mathcal{U}_s^{(m)} \) is a fiber morphism of the bundle \( (T_p, S_1) \) into the trivial bundle \( (S_1 \times D(r_s), S_1) \subset (S_1 \times C, S_1) \) (where \( D(r_s) \) is a closed disk of radius \( r_s = (1 - |s|)^{-1} \)). In this case, the fibers are the sets \( \omega_s^{(m)}(\xi, Z_p) \subset D(r_s) \) for every fixed \( \xi \) and if \( \mathcal{U}_s^{(m)} \) is an L-isometry, then these sets are fractals with global and local dimensions equal to \( D_s \). Moreover, it follows from (11) that the fiber with \( \xi = 0 \) is isometric to \( \Upsilon_{s,a}^{(m)}(Z_p) \). For simplicity, we assume that \( a > r_s \). Then \( \sigma_a \) is the standard embedding of \( (S_1 \times D(r_s), S_1) \) in a solid torus in \( \mathbb{R}^3 \) (a torus together with its interior) and the fibers are mapped isometrically onto disks lying in the plane turned around the \( x_2 \)-axis through an angle \( 2\pi \xi \) relative to the plane \( x_3 = 0 \).

Let us define the action of an element \( t \) of the group \( R \) on \( f \in T_p \) as the shift \( f \mapsto f_t = f + \phi(t, 0) \) and consider the orbits of this group in \( T_p \). It can be shown (11) that each orbit is a dense subset in \( T_p \). Because the mappings \( \mathcal{U}_s^{(m)} \) and \( \Omega_{s,a}^{(m)} \) are L-contractions, the images of these orbits are continuous curves that also wind densely around \( \mathcal{U}_s^{(m)}(T_p) \subset S_1 \times D(r_s) \) and \( \Omega_{s,a}^{(m)} \subset \mathbb{R}^3 \), respectively.

It turns out that in the case \( m = \infty \) the images of the orbits of the group \( R \) are smooth curves that are trajectories of a dynamic system.

**Theorem 9** Let \( \Omega_{s,a}^{(\infty)} \) be an L-isometry. Then there is a global Lipschitz vector field \( \Gamma : \mathbb{R}^3 \mapsto \mathbb{R}^3 \) such that \( \forall f \in T_p \),

\[
\Gamma(\Omega_{s,a}^{(\infty)}(f)) = -2\pi \left( L(2)\Omega_{s,a}^{(\infty)}(f) + p^{-1}\text{Re}\left( \bar{\chi}_{-1}^{(\infty)}(f)(L(3) + iL(4)) \right) \Omega_{s,a}^{(\infty)}(f) \right).
\]
The action of the one-parameter homeomorphism group \( U_t : \mathbb{R}^3 \mapsto \mathbb{R}^3 \), generated by the equation
\[
\frac{d}{dt} r = \Gamma(r),
\]
is compatible with the action of \( \mathbb{R} \) in \( \mathbb{T}_p \) in the sense that \( \forall f \in \mathbb{T}_p, \forall t \in \mathbb{R}, \)
\[
\Omega_{s,a}^{(\infty)}(f_t) = U_t \Omega_{s,a}^{(\infty)}(f).
\]
Here \( (L_{(k)})_{i,j} = \varepsilon_{i,j,k} \), where \( \varepsilon_{i,j,k} \) is the Levi-Civita symbol and \( \Omega_{s/p,\varepsilon}^{(\infty)}(f) \equiv \lim_{\varepsilon \to 0^+} \Omega_{s/p,\varepsilon}^{(\infty)}(f) \).

**Proof.** First, we show that \( \Gamma \big|_{\text{Ran}(\Omega_{s,a}^{(\infty)})} \) is an L-contraction. Obviously, it is sufficient to prove that \( (\Omega_{s,a}^{(\infty)})^{-1} \circ \Omega_{s/p,\varepsilon}^{(\infty)} \big|_{\text{Ran}(\Omega_{s,a}^{(\infty)})} \) is an L-contraction. However, this is a direct consequence of the fact that \( (\Omega_{s,a}^{(\infty)})^{-1} \circ \Omega_{s/p,\varepsilon}^{(\infty)} \) are L-contractions for \( (\mathbb{T}_p, \rho_{\alpha}) \) if \( \alpha = D_{s}^{-1} \Omega_{s/p,\varepsilon}^{(\infty)} \) is an L-contraction since \( \rho_{\alpha} + 1 \preceq \rho_{\alpha} \). The existence of a Lipschitz mapping \( \Gamma \) defined throughout \( \mathbb{R}^3 \) and satisfying (37) is now ensured by the Kirszbraun theorem [4]. It follows immediately from formulas (28,30) that \( \forall f \in \mathbb{T}_p, \forall t \in \mathbb{R} \) the equation
\[
\frac{d}{dt} \omega_{s}^{(\infty)}(t, x) = i2\pi p \omega_{s/p}^{(\infty)}(t, x).
\]
hold. With the help of (32), it can be deduced from this equation that \( \forall f \in \mathbb{T}_p \) and \( \forall t \in \mathbb{R} \) we have
\[
\frac{d}{dt} \Omega_{s,a}^{(\infty)}(f_t) = \Gamma(\Omega_{s,a}^{(\infty)}(f_t)).
\]
According to theory of differential equations [13], this implies formula (38). \( \square \)

Let us mention another specific peculiarity of the case \( m = \infty \). It follows from formulas (14,15,28,30) that \( \forall x \in \mathbb{Q}_p, \)
\[
[\Upsilon_{s}^{(\infty)}](x) = \omega_{s}^{(\infty)}([x]_p),
\]
(40)
This implies the above-mentioned relation \( \tilde{\Delta}_{s}^{\infty} = \Delta_{s}^{\infty} \). Moreover, using Theorem [3] and the Kirszbraun theorem [4], we can now prove that if \( \Omega_{s,a}^{(\infty)} \) is an L-isometry, then there exists an L-contraction \( J : \mathbb{C} \mapsto \mathbb{R}^3 \) such that \( J \) isometrically maps every cluster \( B_{l}^{0} = \Upsilon_{s}^{(\infty)}(\{ x \in \mathbb{Q}_p : |x - l|_p < 1 \}) \) onto the image of the \( \{l\}_p \)-fiber \( \Omega_{s,a}^{(\infty)}(\{\{l\}_p, \mathbb{Z}_p\}) \) and we have
\[
\Omega_{s,a}^{(\infty)} \circ j = J \circ \Upsilon_{s}^{(\infty)}.
\]
(41)
6 Explanation of the figures

Due to the rapid convergence of series (6) and (15) and, also, because even not very large positive integers form a sufficiently dense network in $\mathbb{Z}_p$, the computer construction of the sets $\Upsilon_s^{(m)}(\mathbb{Z}_p)$ and $\Omega^{(m)}_{s,a}(\mathbb{Z}_p)$ encounters no fundamental difficulties. Figure 1.1 represents a Cantor set and, for the sake of clarity, a segment, whose length is proportional to one of the numbers $e_i = 1, 3, 5, 7, 2, 6, 10, 14$ is associated with each point if the point of the image has the form $\Upsilon_{1/3}^{(0)}(e_i \cdot y^2)$ for $y \in \mathbb{Z}_p$ (for these numbers, see [3]). The set in
Figure 2: Embeddings in $\mathbb{R}^3$ a) $T_2$, $(m = 0)$; b) $T_3$ (by fibers) or $3^{-4}Z_3$ ,$(m = \infty)$.

Fig. 1.10 is obviously the Sierpiriski triangle. It can be seen from Fig. 1.9 that the boundaries of the connected components of the set $C \setminus C_1^{(0)}(Z_6)$ consist of Koch curves (note that $p$ is not a prime number: see footnote 3). Finally, Fig. 1.4 can serve as an illustration of the fact that $Z_4$ is homeomorphic to $Z_2$. Figure 2a demonstrates an embedding of $T_2$ into $\mathbb{R}^3$ with the parameters $s = 1/2.2$, $a = i2$ and $m = 0$. and the Cantor structure of this set can be distinctly seen. Figure 2b illustrates an embedding of $T_3$ $(s = s_0 - 0.02 \approx 0.46$, $a = 5/2$ and $m = \infty$ ). Fibers for which the values of $\xi$ are multiples of 1/81 are represented and, therefore, Fig. 2b can simultaneously be regarded as the image of $Y_s^{(\infty)}(3^{-4}Z_3)$ under the mapping $J$ (see formulas (40, 41)); this image is, in fact, shown in Fig. 1.12. In conclusion, we note that additional general constructions, such as $a$-adic numbers and solenoids (see [11]), can also be treated in a similar way, at least for $m = \infty$. It seems, in this case, that constructing an embedding of $A/Q$ in $\mathbb{R}^3$, for instance (where $A$ is the adele ring $\mathbb{N}$), could be interesting.

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References
[1] N. Koblitz p-Adic Numbers, p-Adic Analysis, and Zeta-Functions, Springer, New York Heidelberg Berlin (1977).

[2] I. M Gel’fand, M. I. Graev. and L. I. Pyatetskii-Shapiro Representation Theory and Automorpttic Functions. Saunders. Philadelphia (1969).

[3] V. S. Vladimirov, I. V. Volovich, and E. I. Zeienov p-Adic Analysis and Mathematical Physics, World Scientific, Singapore-New Jersey-London-Hong Kong (1994).

[4] H. Federer Geometric Measure Theory, Springer, New York-Heidelberg-Berlin (1969).

[5] E. Hewitt and K. Ross Abstract Harmonic Analysis, Vol. I, Springer, New York-Heidelberg-Berlin (1965).

[6] P. Billingsley Ergodic Theory and Information, Wiley, New York-London-Sidney (1965).

[7] J. Feder Fractals, Plenum, New York (1988).

[8] D. Mumford. Tata Lectures Notes on Theta Functions, Vols. I, II, Birkhauser, Boston-Basel-Stuttgart (1983. 1984).

[9] J. L. Kelley General Topology. Van Nostrand, Princeton, New Jersey (1957).

[10] B. V. Shabat Complex Analysis [in Russian], Vol. 1, Nauka, Moscow (1985).

[11] J. M. Ziman Models of Disorder. The Theoretical Physics of Homogeneously Disordered Systems. Cambridge University Press, Cambridge-London-New York-Melbourne (1979).

[12] A. J. Lichtenberg and M. A. Lieberman Regular and Stochastic Motion, Springer, New York- Heidelberg Berlin (1983).

[13] V. I. Arnold Ordinary Differential Equations, MIT Press, Cambridge (1978).
[14] K. Alien and M. Kluater, Optical Transformations in Fractals. Fractals in Physics, North Holland. Amsterdam Oxford-New York-Tokyo (1986).

[15] Zelenov E.I. // J.Math.Phys. V32.147-152. 1991.

[16] Pitkanen M. // p-adic Physics ?. Department of Theoretical Physics, University of Helsinki, SF-00170 Helsinki, Finland. 8. September 1994.

[17] Havlin S., Weissman N. // J.Phys.A 19. L1021-1026.1986.

[18] Ogielski A.T., Stein D.L. // Phys. Rev. Let. V.55. N15. 1985.