Finite-Dimensional Reduction of Systems of Nonlinear Diffusion Equations

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Abstract—We present a class of one-dimensional systems of nonlinear parabolic equations for which the phase dynamics at large time can be described by an ODE with a Lipschitz vector field in $\mathbb{R}^n$. In the considered case of the Dirichlet boundary value problem, the sufficient conditions for a finite-dimensional reduction turn out to be much wider than the known conditions of this kind for a periodic situation.

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1. INTRODUCTION

One of the main problems in the study of evolution equations is related to the description of the final (long-time) behavior of their solutions. We consider systems of diffusion equations with the Dirichlet boundary condition

$$\frac{\partial}{\partial t} u = D \frac{\partial^2}{\partial x^2} u + f(x, u) \frac{\partial}{\partial x} u + g(x, u), \quad u(0) = u(1) = 0 \quad (1.1)$$

on the interval $J = [0, 1]$. Here $u = (u_1, \ldots, u_m)$, and $f$ and $g$ are sufficiently regular matrix functions and vector functions, respectively. We assume that the constant coefficient matrix $D$ is similar to a diagonal matrix with positive eigenvalues. In the case of $D = \text{diag}\{d_1, \ldots, d_m\}$ with $d_j > 0$, we deal with reaction-diffusion-convection equations. Under appropriate conditions on $f$ and $g$, system (1.1) induces a smooth dissipative semilow $\Phi_t \geq 0$ in the phase space $X^\alpha \subset C^1(J, \mathbb{R}^m)$ with an appropriate $\alpha > 0$, where $\{X^\alpha\}_{\alpha \geq 0}$ is the Hilbert half-scale [1] generated by the linear sectorial operator $u \rightarrow -Du_{xx}$ in $X = L^2(J, \mathbb{R}^m)$. In this situation, there exists a global attractor [2]–[4] (simply an attractor below), i.e., a connected compact invariant set $\mathcal{A} \subset X^\alpha$ of finite Hausdorff dimension which uniformly attracts bounded subsets $X^\alpha$ as $t \to +\infty$.

Our goal is to find conditions under which the dynamics on attractor (final dynamics) of the parabolic system (1.1) is finite-dimensional in the sense of [5]. This means that, for some ODE $\frac{\partial}{\partial t} \xi = h(\xi)$ in $\mathbb{R}^N$ with Lipschitz vector field $h$, resolving flow $\{\Theta_t\}$, and invariant compact set $\mathcal{K} \subset \mathbb{R}^N$, the phase semilows $\Phi_t \geq 0$ on $\mathcal{A}$ and $\Theta_t \geq 0$ on $\mathcal{K}$ are Lipschitz conjugate. In this connection, one can speak [6] about a finite reduction of the evolution problem (1.1).

The main result of the paper (Theorem 4.3) ensures that the final phase dynamics of system (1.1) is finite under the matching condition

$$Df(x, u) = f(x, u)D, \quad (x, u) \in J \times \text{co} \mathcal{A}, \quad (1.2)$$

where $\text{co} \mathcal{A}$ is the convex hull of $\mathcal{A}$.

It is known [7] that, in the case of scalar diffusion ($D = dE$ with unit matrix $E$) and sufficiently regular $f = f(u)$ and $g = g(u)$, there exists an inertial manifold (IM) which is a finite-dimensional
invariant $C^1$–surface in the phase space containing an attractor and exponentially attracting (with an asymptotic phase) all trajectories of the system as $t \to +\infty$. The presence of IM implies the finite-dimensionality of the final dynamics; an extensive literature is devoted to the existence of such manifolds (see, e.g., [3], [4], [6], [8]). An original approach to such problems is presented in recent Anikushin’s works (see [9] and the references therein).

For the periodic case ($J$ is a circle of length 1), conditions for the finite-dimensionality of the final dynamics of system (1.1) with $D = \text{diag}$ were obtained by the author in [10, p. 13409]. We note that, in the class of periodic systems (1.1) with scalar diffusion, the first example of semilinear parabolic equation of mathematical physics that does not exhibit such a dynamics was constructed in [11, Theorem 1.2].

2. PRELIMINARIES

In what follows, if necessary, we will use the technique developed in [10]. All preliminary constructions in Secs. 2, 3 are performed for the case $D = \text{diag}$. We write system (1.1) as a semilinear parabolic equation (SPE)

$$\partial_t u = -Au + F(u) \quad (2.1)$$

in the real Hilbert space $X = L^2(J, \mathbb{R}^m)$ with norm $\| \cdot \|$. Here $A: u \to -Du_{xx}$ with the Dirichlet boundary condition and nonlinearity $F: u \to f(x, u) \partial_x u + g(u)$. For a linear positive definite operator $A$, we set $X^\alpha = \mathcal{D}(A^\alpha)$ with $\alpha \geq 0$ and $X_0 = X$; then $\| u \|_\alpha = \| A^\alpha u \|$. We will say that a function $F$ belongs to the class $W^2(X^\alpha, X)$ if

$$F \in C^2(X^\alpha, X) \cap \text{Lip}(X^\alpha, X) \quad \text{and} \quad \| F(u) \| \leq M \quad \text{for } u \in X^\alpha \quad (2.2)$$

for some $\alpha \in [0, 1)$. In this case, SPE (2.1) generates [1] a smooth compact resolving semilow $\{ \Phi_t \}_{t \geq 0}$ in the phase space $X^\alpha$. Assumption (2.2) implies [8, Lemma 1.1] the $X^\alpha$–dissipativity of (2.1):

$$\limsup_{t \to +\infty} \| \Phi_t u \|_\alpha \leq r$$

for some $r > 0$ uniformly in $u \in \text{balls in } X^\alpha$. Under these conditions, there exists [2]–[4] a compact attractor $\mathcal{A} \subset X^\alpha$ consisting of all bounded complete trajectories $\{ u(t) \}_{t \in \mathbb{R}} \subset X^\alpha$. In fact, $\mathcal{A} \subset X^1$ owing to the smoothing action of the parabolic equation [1]. Simple reasoning [10, p. 13410] shows that, in all constructions related to SPE (2.1), the nonlinearity exponent $\alpha$ can be replaced by any value $\alpha_1 \in (\alpha, 1)$, and if condition (2.2) is satisfied in the two spaces $(X^\theta, X^{\theta + \alpha})$ with $\theta > 0$ instead of $(X, X^\alpha)$, then all the above-listed properties of the dynamics remain valid for the phase space $X^{\theta + \alpha}$. In what follows, we will use functions $Y_1 \to Y_2$ of class (2.2) for some Banach spaces $Y_1$ and $Y_2$.

As in [10], we will use sufficient conditions for the final finite-dimensionality of the dynamics [12]. Let $G(u) = F(u) - Au$ be the vector field (2.1), and let $\mathcal{N} = \mathcal{A} \times \mathcal{A}$ and $Y$ be Banach spaces.

**Definition 2.1** [12]. A continuous field $\Pi: \mathcal{N} \to Y$ is said to be regular if, for any $u, v \in \mathcal{A}$, the function $\Pi(\Phi_t u, \Phi_t v): [0, +\infty) \to Y$ is of class $C^1$ with the derivative $\partial_t \Pi(u, v)$ at zero uniformly bounded in $(u, v) \in \mathcal{N}$.

The smoothness of the semilow $\{ \Phi_t \}$ and the invariance of the compact set $\mathcal{A} \subset X^\alpha$ imply the regularity of the identical embedding $\mathcal{N} \to X^\alpha \times X^\alpha$, and hence of any field $\Pi: \mathcal{N} \to Y$ that can be continued to a $C^1$-mapping in the $(X^\alpha \times X^\alpha)$-neighborhood of the set $\mathcal{N}$. In this situation, $\partial_t \Pi(u, v) = \Pi'(u, v)(G(u), G(v))$, where $(\cdot)'$ is the Fréchet differentiation. Under condition (2.2) on the nonlinearity $F$, the function $u \to G(u)$ on $\mathcal{A}$ is continuous and even Hölder [5] in the $X^\alpha$-metric. The regular fields $\mathcal{N} \to Y$ form a linear structure as well as a multiplicative one if $Y$ is a Banach algebra. In the last case, if all elements $\Pi(u, v) \in Y$ are invertible, then the field $\Pi^{-1}$ also turns out to be regular.

We start from the decomposition

$$G(u) - G(v) = (T_0(u, v) - T(u, v))(u - v), \quad (u, v) \in \mathcal{N}, \quad (2.3)$$
where $T_0 \in \mathcal{L}(X^\alpha)$ and $T \in \mathcal{L}(X^1, X)$ are unbounded linear operators in $X$ similar to positive definite ones. By

$$\Sigma_T = \bigcup_{u,v \in \mathcal{A}} \text{spec} T(u,v)$$

we denote the total spectrum of the operators $T$.

We will need the special case of [12, Theorem 2.8] in the situation $\Sigma_T \subset \mathbb{R}^+$.  

**Theorem 2.2.** Assume that $F \in W^2(X^\alpha, X)$ and

$$T(u,v) = S^{-1}(u,v)H(u,v)S(u,v)$$  \tag{2.4}$$
on $\mathcal{N}$, where the unbounded self-adjoint linear operators $H(u,v)$ are positive definite in $X$, the fields $S, S^{-1}: \mathcal{N} \to \mathcal{L}(X)$ and $T_0: \mathcal{N} \to \mathcal{L}(X^\alpha, X)$ are regular, and the field $T_0: \mathcal{N} \to \mathcal{L}(X^\alpha)$ is bounded. Moreover, if the set $\mathbb{R}^+ \setminus \Sigma_T$ contains intervals $(a_k - \xi_k, a_k + \xi_k)$ with $a_k > \xi_k > 0$ such that

$$\xi_k \to \infty, \quad a_k^{1/2} = o(\xi_k)$$  \tag{2.5}$$
as $k \to +\infty$, then the final $X^\alpha$-dynamics of SPE (2.1) is finite-dimensional.

We further assume that the matrix function $f = f(x, u)$ and the vector function $g = g(x, u)$ in (1.1) satisfy the following regularity conditions.

**Condition (H).** The functions $f$ and $g$ of the class $C^\infty$ on $J \times \mathbb{R}^m$ are compactly supported in $u$, and $f(x,0) = g(x,0) = 0$ for $x = 0, 1$.

By $\mathcal{H}^s = \mathcal{H}^s(J)$ we denote generalized Sobolev $L^2$-spaces (spaces of Bessel potentials [1], [13]) of scalar functions on $J$ with arbitrary $s \geq 0$. If $s > 1/2$, then $\mathcal{H}^s \subset C(J)$ and $\mathcal{H}^s$ is a Banach algebra [13, Sec. 2.8.3]. The differentiation operator acts in the spaces $\partial_x \in \mathcal{L}(\mathcal{H}^{s+1}, \mathcal{H}^s)$. In fact, the $X^s$ are closed subspaces (with equivalent norm) in the spaces $\mathcal{H}^{2s}(J, \mathbb{R}^m)$ of vector-functions, and $X^s = \mathcal{H}^{2s}(J, \mathbb{R}^m)$ for $s \leq 1/4$. For $s > 1/4$, the space $X^s$ consists of elements $u \in \mathcal{H}^{2s}(J, \mathbb{R}^m)$ with $u(0) = u(1) = 0$.

Now fix an arbitrary $x \in (3/4, 1)$; then $\mathcal{H}^{2s} \to C^1(J)$ and $X^\alpha \hookrightarrow C^{1}(J, \mathbb{R}^m)$, where the symbol $\hookrightarrow$ denotes a linear continuous embedding of function spaces. Let us use necessary embedding theorems [1], [13]. For an arbitrary $C^\infty$-function $z: J \times \mathbb{R}^m \to \mathbb{R}$, the mapping $\psi: u \to z(x, u)$ is a function of class $W^2$ (see (2.2)) from $C^s(J, \mathbb{R}^m)$ to $C^s(J)$ for all $s \in \mathbb{N}$. This implies that $\psi \in W^2(\mathcal{H}^{2s}(J, \mathbb{R}^m), C^s(J))$. Using the embedding $\mathcal{H}^{s+1} \hookrightarrow C^s(J) \hookrightarrow \mathcal{H}^s$, we can conclude that $\psi \in W^2(\mathcal{H}^s(J, \mathbb{R}^m), \mathcal{H}^s(J))$. So $F \in W^2(X^1, X^{1/2})$ for the nonlinear part $F: u \to f(x, u) \partial_x u + g(u)$ of system (1.1). Moreover, $X^\alpha \hookrightarrow C^1(J, \mathbb{R}^m) \hookrightarrow C(J, \mathbb{R}^m)$ $\to X$, and hence $F \in W^2(X^\alpha, X)$. We also note that $X^{3/2} \hookrightarrow C^2(J, \mathbb{R}^m)$.

We take $X^\alpha$ as the phase space of system (1.1). Following [7], we can show that the phase dynamics of (1.1) in $X^\alpha$ is dissipative and there exists a global attractor $\mathcal{A} \subset X^\alpha$. Since $F \in W^2(X^1, X^{1/2})$, system (1.1) also generates a smooth dissipative phase semilow in the space $X^1$, and the attractor $\mathcal{A}$ is compact in $X^{3/2}$. As above, we denote $\mathcal{N} = \mathcal{A} \times \mathcal{A}$.

**Remark 2.3.** The phase dynamics of system (1.1) has the following property: if $Y$ is a Banach space, then each vector field $\Pi: \mathcal{N} \to Y$ continuous in the $(X^\alpha \times X^\alpha)$-metric and extendable to a $C^1$-mapping $X^1 \times X^1 \to Y$ is regular in the sense of Definition 2.1.

Indeed, the smoothness of a semiflow in $X^1$ means the smoothness of the mapping

$$(t, u) \to \Phi_t u: (0, +\infty) \times X^1 \to X^1.$$ 

This ensures the regularity of the identity mapping $\mathcal{N} \to X^1 \times X^1$ and hence the regularity of the field $\Pi$ on $\mathcal{N}$.
3. DECOMPOSITION OF THE VECTOR FIELD ON THE ATTRACTOR

We will apply Theorem 2.2 to SPE (1.1) with $D = \text{diag}$ and the phase spaces $X^\alpha$, $\alpha \in (3/4, 1)$. By $M^m$ we denote the algebra of numerical $(m \times m)$-matrices with Euclidean norm, and by $Y(J, M^m)$, the linear spaces of such matrices with elements from some Banach space $Y$ of scalar functions on $J = [0, 1]$. Following [10, pp. 13412–13413], we set

$$
B_0(x; u, v) = \int_0^1 (f_u(x, w(x))w_x(x) + g_u(x, w(x)) \, d\tau,
$$

$$
B(x; u, v) = \int_0^1 f(x, w(x)) \, d\tau
$$

for $u, v \in X^\alpha$, where $w(x) = \tau u(x) + (1 - \tau)v(x)$, $x \in J$. The entries of the matrices $B_0$ and $B$ are continuous functions, and for $u, v \in \mathcal{A}$ they are functions of the class $C^2$ on $J$. Using the $C^1$-smoothness of the mappings $(u, v) \mapsto f_u(x, w)w_x + g_u(x, w)$, $(u, v) \mapsto f(x, w)$, $X^\alpha \times X^\alpha \to C(J, M^m)$ for a fixed $\tau \in [0, 1]$ and differentiating the expressions for $B_0$ and $B$ with respect to the parameter $(u, v)$, we conclude that the mappings

$$
(u, v) \mapsto B_0(\cdot; u, v), \quad (u, v) \mapsto B(\cdot; u, v)
$$

are of the class $C^1(X^\alpha \times X^\alpha, C(J, M^m))$. We use the integral mean value theorem for nonlinear operators to write the decomposition of the vector field (1.1) on the attractor $\mathcal{A} \subset X^\alpha$ in the form

$$
G(u) - G(v) = -Ah + \left( \int_0^1 F'(\tau u + (1 - \tau)v) \, d\tau \right) h
$$

$$
= Dh_{xx} + B_0(x; u, v)h + B(x; u, v)h_x, \quad u, v \in \mathcal{A},
$$

where $h = u - v$, $\tau u + (1 - \tau)v \in \text{co} \mathcal{A}$, and $(\cdot)'$ is the Frechet differentiation. To eliminate the dependence on $h_x$, we (following [14]) apply the transformation $h = U\eta$, where the $(m \times m)$-matrix function $U(x) = U(x; u, v)$, $x \in [0, 1]$, is the solution of the Cauchy problem

$$
U_x = -\frac{1}{2} D^{-1} B(x)U, \quad U(0) = E.
$$

As a result, we obtain relation (2.3) with linear operators

$$
T_0(u, v)h = \left( B_0(x) - \frac{1}{2} B_x(x) - \frac{1}{4} B(x)D^{-1}B(x) \right) h,
$$

$$
T(u, v)h = -DU\partial_{xx}U^{-1}h.
$$

Note that, under the change of variable $h = U\eta$, the Dirichlet boundary conditions for the linear part of (1.1) are preserved. When writing the matrices $B_0, B, U, U^{-1}$, we often omit the dependence on $u, v$, and sometimes, on $x$.

**Lemma 3.1.** The field of the operator $T_0$ on $\mathcal{N}$ is regular with values in $\mathcal{L}(X^\alpha, X)$ and bounded with values in $\mathcal{L}(X^\alpha)$.

**Proof.** We set $T_0h = Q(x; u, v)h$ in (3.5) with $h \in \mathcal{A} - \mathcal{A} \subset X^\alpha$. The convex hull of the attractor $\mathcal{A}$ is bounded in the $X^{3/2}$-norm equivalent to the norm in $H^3(J, \mathbb{R}^m)$, and therefore, the matrix functions $B, BD^{-1}B$, and $B_0$ are bounded uniformly in $(u, v) \in \mathcal{N}$ in $H^3(J, M^m)$ and $H^2(J, M^m)$, respectively. Thus, the matrix functions $B_x$ and $Q$ are bounded on $\mathcal{N}$ in the norm of $H^2(J, M^m)$, and $T_0$ is the operator of multiplication of vector functions in $X^\alpha \subset H^{2\alpha}(J, \mathbb{R}^m)$ by the matrix $Q \in H^{2\alpha}(J, M^m)$ with $2\alpha \in (3/2, 2)$. Since $H^{2\alpha}(J)$ is a Banach algebra, we see that $T_0(u, v) \in \mathcal{L}(X^\alpha)$ and $\|T_0(u, v)\|_\alpha \leq \text{const} \text{ on } \mathcal{N}$.

In view of Remark 2.3 and the above-noted smoothness of the mapping (3.3), the regularity of the field of the operator $T_0: \mathcal{N} \to \mathcal{L}(X^\alpha, X)$ can be established exactly as for the case of periodic boundary conditions in [10, Lemma 3.3].

[Proof completed]
The matrix function $U(x)$ in the Cauchy problem (3.4) can be treated as a bounded linear operator in $X$.

**Lemma 3.2.** The fields of the operators $U, U^{-1}: \mathcal{N} \to \mathcal{L}(X)$ are regular.

**Proof.** For the field $U$, this can proved as the similar assertion in the periodic case [10, Lemma 3.4]. At the same time, the regularity of $U$ implies the regularity of the field of inverse operators $U^{-1}$.

Now let $d_- = \min_{1 \leq j \leq m} d_j$ and $d_+ = \max_{1 \leq j \leq m} d_j$ for $D = \text{diag}(d_j)$. Also let $\{\lambda_n : \lambda_1 < \lambda_2 < \cdots \}$ be the eigenvectors of the linear operator $A = -D \partial_{xx}$. Since

$$\text{spec}(A) = \{d_j \pi^2 \nu^2, \nu \in \mathbb{N}, j \in 1, \ldots, m\},$$

we have $\lambda_n \leq \pi^2 d_+ n^2$. Using the counting function for $\text{spec}(A)$, we obtain

$$n \leq \sum_{j=1}^{m} \frac{\sqrt{\lambda_n}}{\pi \sqrt{d_j}} \leq \frac{m}{\pi \sqrt{d_-}} \sqrt{\lambda_n},$$

and hence

$$\frac{\pi^2 d_-}{m^2} n^2 \leq \lambda_n \leq \pi^2 d_+ n^2, \quad n \in \mathbb{N}.$$ (3.8)

**Lemma 3.3.** The following estimate holds:

$$\limsup_{n \to \infty} n^{-1} (\lambda_{n+1} - \lambda_n) > 0.$$

**Proof.** If, on the contrary, $\lambda_{n+1} - \lambda_n = \beta_n n$ with $\beta_n \overset{n \to \infty}{\to} 0$, then

$$n^{-2} \lambda_n = n^{-2} \left( \lambda_1 + \sum_{k=1}^{n-1} (\lambda_{k+1} - \lambda_k) \right) = n^{-2} \left( \lambda_1 + \sum_{k=1}^{n-1} \beta_k k \right) \leq n^{-2} \left( \lambda_1 + \sum_{k=1}^{n-1} \beta_k n \right) \leq n^{-2} \lambda_1 + n^{-1} \sum_{k=1}^{n} \beta_k.$$

But this implies the relation $\lambda_n = o(n^2)$, which contradicts the left inequality in (3.8). □

4. MAIN RESULTS

By the assumptions of Theorem 2.2, it is necessary, for the operators $T(u, v)$ in (3.6), to establish “uniform” similarity of the positive definite form (2.4) as well as the required sparsity (2.5) of their total spectrum $\Sigma_T$. We assume that the regularity Condition (H) is satisfied for the functions $f$ and $g$ in (1.1).

**Theorem 4.1.** If the matrix $D$ has the form $D = \text{diag}(d_j)$ with $d_j > 0$ and the matching condition (1.2) is satisfied, then the phase dynamics on the attractor is finite-dimensional.

**Proof.** The operator $A = -D \partial_{xx}$ with the Dirichlet condition is self-adjoint and positive definite in $X$. Assumption (1.2) implies (for any $x \in J$ and $u, v \in \mathcal{A}$) the relation $DB(x) = B(x)D$ for the matrices $B(x) = B(x; u, v)$ in (3.2). Thus, the matrices $B(x)$ and $D^{-1}B(x)$ inherit the block (with respect to the same $d_j$) structure of the diffusion matrix $D = \text{diag}(d_1, \ldots, d_m)$. Therefore, this is also true for the solutions $U(x)$ of the Cauchy problem (3.4), and hence $DU(x) = U(x)D$, $x \in J$, and

$$T(u, v) = U(u, v)(-D \partial_{xx}) U^{-1}(u, v)$$
in (3.6). Thus, for \( T(u, v) \) we have the representation (2.4) with \( S(u, v) = U^{-1}(u, v) \) and \( H(u, v) \equiv A \).

The total spectrum \( \Sigma_T \) coincides with \( \text{spec}(A) \) in (3.7). By Lemma 3.3, there exists an \( \varepsilon > 0 \) and an increasing sequence of indices \( n(k) \) such that \( \lambda_{n(k)+1} - \lambda_{n(k)} > \varepsilon n(k) \) for \( k \geq k_0 \). Set

\[
a_k = \frac{\lambda_{n(k)+1} + \lambda_{n(k)}}{2}, \quad \xi_k = \frac{\lambda_{n(k)+1} - \lambda_{n(k)}}{3}, \quad M = \pi^2 d_+.
\]

From the right inequality in (3.8), we obtain

\[
a_k \leq M \left( n^2(k) + n(k) + \frac{1}{2} \right) \leq 3Mn^2(k) \leq \frac{3M}{\varepsilon^2} (\lambda_{n(k)+1} - \lambda_{n(k)})^2 \leq \frac{27M}{\varepsilon^2} \xi_k^2
\]

for \( k \geq k_0 \); i.e., \( a_k = O(\xi_k^2) \) as \( k \to \infty \). Since \( a_k^{\alpha/2} = o(\xi_k) \) for \( \alpha \in (3/4, 1) \) as \( k \to \infty \), we see that the desired assertion follows from Lemmas 3.1 and 3.2 and Theorem 2.2.

\[\square\]

**Remark 4.2.** Parabolic systems (1.1) with \( D = \text{diag} \) demonstrate a finite-dimensional dynamics on the attractor for any admissible nonlinearities \( f \) and \( g \) in the case of scalar diffusion and under the condition \( f = \text{diag} \) in the case of \( m \) distinct diffusion coefficients \( d_j \). In the case of \( s \) distinct diffusion coefficients with \( 1 < s < m \), the dynamics on the attractor is finite-dimensional under the condition that the matrix function \( f \) inherits the block structure (with the same \( d_j \) of the matrix \( D = \text{diag}\{d_j\} \)).

Now we state the main result. We assume that the matrix \( D \) in system (1.1) has the form \( D = C \overrightarrow{D} C^{-1} \), where the matrix \( C \) is nonsingular and \( \overrightarrow{D} = \text{diag}\{d_1, \ldots, d_m\} \) with \( d_j > 0 \). The linear operator \( -D \partial_{xx} = -C(\overrightarrow{D} \partial_{xx}) C^{-1} \) is sectorial in \( X = L^2(J, \mathbb{R}^m) \). The linear change of variables \( u = CV \) reduces (1.1) to the system of equations

\[
\begin{align*}
\partial_t v &= \overrightarrow{D} \partial_{xx} v + \overrightarrow{f}(x, v) \partial_x v + \overrightarrow{g}(x, v), \quad v(0) = v(1) = 0, \\
\overrightarrow{f}(x, v) &= C^{-1} f(x, Cv) C, \quad \overrightarrow{g}(x, v) = C^{-1} g(x, Cv).
\end{align*}
\] (4.1)

The matrix function \( \overrightarrow{f} \) and the vector function \( \overrightarrow{g} \) inherit the regularity properties (H) of the original functions \( f \) and \( g \). The phase semilows of systems (4.1) and (1.1) are linearly conjugate. System (4.1) is dissipative in \( X^\alpha \), and hence this is also true for system (1.1). The attractors \( \mathcal{A} \) of system (1.1) and \( \overrightarrow{\mathcal{A}} \) of system (4.1) are related by the expression \( \mathcal{A} = C \overrightarrow{\mathcal{A}} \). According to the definition of the finite-dimensionality of the phase dynamics (Sec. 1), systems (4.1) and (1.1) demonstrate this property simultaneously.

**Theorem 4.3** (main). If the matrix \( D \) is similar to \( \text{diag}\{d_j\} \) with \( d_j > 0 \) and the matching condition (1.2) is satisfied, then the phase dynamics of system (1.1) is finite-dimensional.

**Proof.** Since \( D f(x, u) = f(x, u) D \) on \( J \times \text{co} \mathcal{A} \), it follows that

\[
\begin{align*}
\overrightarrow{D} f(x, v) &= C^{-1} DC \cdot C^{-1} f(x, Cv) C = C^{-1} D f(x, u) C \\
&= C^{-1} f(x, u) DC = C^{-1} f(x, Cv) C \overrightarrow{D} = \overrightarrow{f}(x, v) \overrightarrow{D}
\end{align*}
\]

on \( J \times \text{co} \overrightarrow{\mathcal{A}} \). Here \( u \in \text{co} \mathcal{A} \) and \( v \in \overrightarrow{\mathcal{A}} \). As we can see, condition (1.2) is satisfied for the matrix function \( \overrightarrow{f} \), and the dynamics of system (4.1) on the attractor \( \overrightarrow{\mathcal{A}} \subset X^\alpha \) is finite-dimensional by Theorem 4.1. This also implies the finite-dimensionality of the dynamics of system (1.1) on the attractor \( \mathcal{A} \subset X^\alpha \). \[\square\]

**Remark 4.4.** Under the matching condition (1.2), the final dynamics of system (1.1) is finite-dimensional if all eigenvalues of the matrix \( D \) are distinct and positive. Condition (1.2) is satisfied, in particular, for \( f = D_1 \varphi \), where the numerical matrix \( D_1 \) commutes with \( D \) and \( \varphi = \varphi(x, u) \) is a smooth scalar function compactly supported in \( u \).
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