UNBIASED INFERENCE FOR DISCRETELY OBSERVED HIDDEN MARKOV MODEL DIFFUSIONS

JORDAN FRANKS, AJAY JASRA, KODY J. H. LAW & MATTI VIHOLA

Abstract. We develop a Bayesian inference method for diffusions observed discretely and with noise, which is free of discretisation bias. Unlike existing unbiased inference methods, our method does not rely on exact simulation techniques. Instead, our method uses standard time-discretised approximations of diffusions, such as the Euler–Maruyama scheme. Our approach is based on particle marginal Metropolis–Hastings, a particle filter, randomised multilevel Monte Carlo, and importance sampling type correction of approximate Markov chain Monte Carlo. The resulting estimator leads to inference without a bias from the time-discretisation as the number of Markov chain iterations increases. We give convergence results and recommend allocations for algorithm inputs. Our method admits a straightforward parallelisation, and can be computationally efficient. The user-friendly approach is illustrated on two examples, where the underlying diffusion is an Ornstein–Uhlenbeck process or a geometric Brownian motion.

1. Introduction

Hidden Markov models (HMMs) are widely used in real applications, for example, for financial and physical systems modeling [cf. 5]. We focus on the case where the hidden Markov chain arises from a diffusion process that is observed with noise at some number of discrete points in time [cf. 30]. The parameters associated to the model are static and assigned a prior density. Bayesian inference involves expectations with respect to (w.r.t.) the joint posterior distribution of parameters and states, and is important in problems of model calibration and uncertainty quantification. A difficult part of Bayesian inference for these models is simulation of the diffusion dynamics. Except for some special cases where the transition probability is explicitly known [cf. 22, Section 4.4] or exact simulation [3] type methods can be applied [cf. 3, 4, 10, 33], one must time-discretise the diffusion dynamics with an approximation scheme in order to facilitate tractable inference. This is despite the fact that one is ideally interested when there is no time-discretisation: unbiased inference.

Our goal is unbiased inference for HMM diffusions. As previously mentioned, one approach to unbiased inference is based on exact simulation type methods [3, 4, 10, 33]. At the present point in time, exact simulation type methods are mostly only applicable to one-dimensional models where the Lamperti transformation [cf. 24] can be applied (cf. [25, 28, 33] for reviews). In contrast, we proceed with an Euler–Maruyama [cf. 22] (referred henceforth as Euler) or similar time-discretisation of the diffusion, which is generally applicable.

Traditional inference approaches based on time-discretisations face a trade-off between bias and computational cost. Once the user has decided on a suitably fine discretisation size, one can run, for example, the particle marginal Metropolis-Hastings (PMMH) [2]. This algorithm uses a particle filter (PF) [cf. 7], where proposals between time points are generated by the approximation scheme, and ultimately accepted or rejected according to a
Metropolis-Hastings type acceptance ratio [cf. 16]. As the discretisation size adopted must be quite fine, a PMMH algorithm can be computationally intensive.

To deal with the computational cost of PMMH, [19] develop a PMMH based method which uses (deterministic) multilevel Monte Carlo (dMLMC) [14, 17]. The basic premise of MLMC is to introduce a telescoping sum representation of the posterior expectation associated to the most precise time discretisation. Then, given an appropriate coupling of posteriors with ‘consecutive’ time discretisations, the cost associated to a target mean square error is reduced, relative to exact sampling from the most precise (time-discretised) posterior. In the HMM diffusion context, the standard MLMC method is not possible, so based upon a PF coupling approach and PMMH, an MLMC method is devised in [19, 20], which achieves fine-level, though biased, inference.

1.1. Method. The unbiased and computationally efficient inference method suggested in this paper is built firstly on PMMH, using Euler type discretisations, but using a PMMH targeting a coarse-level model, which is less computationally expensive. This does not yield unbiased inference yet, but it can be achieved by an importance sampling (IS) type correction [cf. 32].

We suggest an IS type correction that is based on a single-term (randomised) MLMC type estimator [23, 26] and the PF coupling approach of [19]. The rMLMC correction is based on randomising the running level in the multilevel context of a certain PF, which we refer to as the ‘delta PF (∆PF)’ (Algorithm 3). In short, the ∆PF uses the PF coupling introduced in [19], but here an estimator is used for unbiased estimation of the difference of unnormalised integrals corresponding to two consecutive discretisation levels, over the latent states with parameter held fixed (cf. Section 2), rather than to the difference of self-normalised PMMH averages.

The resulting IS type estimator leads to unbiased inference over the joint posterior distribution, and is highly parallelisable, as the more costly (randomised) ∆PF corrections may be performed independently en masse given the PMMH base chain output. We are also able to suggest optimal choices for algorithm inputs in a straightforward manner (Recommendation 1 and Figure 1). This is because there is no bias, and therefore the difficult cost–variance–bias trade-off triangle associated with dMLMC is not present. Besides being unbiased and efficient, our method is user-friendly, as it is a combination of well-known and relatively straightforward components: PMMH, Euler approximations, PF, rMLMC, and an IS type estimator. For more about the strengths of the method, see Remark 10 later, as well as [12, 32] for more discussion about IS (type) estimators based on approximate Markov chain Monte Carlo (MCMC).

Key to verifying consistency of the method is a finite variance assumption for the r∆PF estimator. We verify a parameter-uniform bound for the variance under a simple set of HMM diffusion conditions in Section 3. Note, however, that consistency of our method is likely to hold more generally. This is in contradistinction to methods based on exact simulation, which require analytically tractable transformations to unit covariance diffusion term and computable bounds in the rejection sampler, in order to even apply the method (see for example the review in the recent preprint [33]).

If an exact simulation method is applicable, the obvious question arises whether our method or the exact simulation method should be applied. The efficiency of exact simulation type methods is dependent upon several and different factors than our method. These factors for exact simulation include proper tuning and tight computable bounds for the rejection sampler. In an ideal scenario for exact simulation, a method based on exact simulation is likely to perform better than our method. However, in the reverse case, our
method can perform better, if the efficiency of exact simulation is poor. For instance, the efficiency of exact simulation decreases to zero as the analytically computed upper bound of the IS weight used in the rejection sampler increases to infinity.

Although we have mostly in mind the case of Euler approximation schemes for the diffusion dynamics approximation, which are generally implementable, other schemes could be possibly be used as well [cf. 13]. However, suitable couplings for these schemes in dimensions greater than one may not be trivial. For the sake of theory and proof of consistency, ideally these would have also known weak and strong order convergence rates [cf. 22]. Indeed, assuming a coupling exists, such higher-order schemes can improve convergence of our method (see Sections 5 and 6). More generally, our approach based on PMMH or other approximate MCMC, increasingly fine families of approximations, MLMC, and IS correction, could be applied beyond the HMM diffusion context, for example, to HMM jump-diffusions [cf. 21].

1.2. Outline. Section 2 introduces the aforementioned ∆PF (Algorithm 2) and subsequently discusses some applications of randomisation techniques. The theoretical properties of the ∆PF in the HMM diffusion context are summarised in Section 3. Section 4 presents the suggested IS type estimator (Algorithm 4), based on PMMH with rMLMC (i.e. r∆PF) correction, and details its consistency and a corresponding central limit theorem (CLT). Section 5 suggests suitable allocations in the ∆PF based on rMLMC efficiency considerations. The numerical experiments in Section 6 illustrate our method in practice in the setting of an Ornstein–Uhlenbeck process and geometric Brownian motion. Proofs for the technical results of Sections 3, 4 and 5 are given in Appendix A, B and C, respectively.

1.3. Notation. Let \((E_n, \mathcal{E}_n)\) be a measurable space. Functions \(\varphi : E_n \to \mathbb{R}\) will be assumed measurable. We denote by \(\mathcal{P}(E_n)\) the collection of probability measures on \((E_n, \mathcal{E}_n)\), and by \(\mathcal{B}_n(E_n)\) the set of \(\varphi : E_n \to \mathbb{R}\) with \(\|\varphi\| := \sup_{x \in X} |\varphi(x)| < \infty\). For a measure \(\mu\) on \((E_n, \mathcal{E}_n)\), we set \(\mu(\varphi) := \int_{E_n} \varphi(x) \mu(dx)\) whenever well-defined. For \(K : E_n \times E_n \to [0,1]\) a Markov kernel and \(\mu \in \mathcal{P}(E_n)\), we set \(\mu K(dy) := \int_{E_n} \mu(dx) K(x, dy)\), and \(K(\varphi)(x) := \int_{E_n} \varphi(y) K(x, dy)\), whenever well-defined. We use the convention \(\prod_{\emptyset} := 1\), and \(p:q := \{r \in \mathbb{Z} : p \leq r \leq q\}\).

2. Delta Particle filter for unbiased estimation of level differences

Consider the (Itô) diffusion process

\[
\text{d}X_t = a_\theta(X_t) \text{d}t + b_\theta(X_t) \text{d}W_t, \quad t \geq 0,
\]

with \(X_t \in X := \mathbb{R}^d\), model parameter \(\theta \in \Theta\), and \(\{W_t\}_{t \geq 0}\) a Brownian motion of appropriate dimension. We suppose that there are data \(\{Y_p = y_p\}_{p=0}^n\), \(y_p \in \mathbb{R}^m\), which are observed at equally spaced discrete times, \(p = 0:n\) for simplicity. The Markov transition between \(X_{p-1}\) and \(X_p\) is given by some kernel \(M^{(\theta, \infty)}(x_{p-1}, dx_p)\). It is assumed that conditional on \(X_p, Y_p\) is independent of random variables \(\{X_i, Y_i\}_{i \neq p}\) and has density \(g_\theta(y_p|x_p) =: G^{(\theta)}(x_p)\). The resulting pair \((M^{(\theta, \infty)}, G^{(\theta)})\) defines the HMM diffusion, and is an example of a so-called Feynman-Kac model [cf. 7] described below. As the results of this section can just as easily be stated in terms of Feynman-Kac models, we do so in the following, which shows the generality of our approach.

2.1. Particle filters. A Feynman–Kac model \((M_n, G_n)\) on spaces \((E_n, \mathcal{E}_n)\) arises when

(i) \(M_n(x_{0:n-1}, dx_n)\) are (regular) probability ‘transition’ kernels from \(E_{0:n-1}\) to \(E_n\) for \(n \geq 1\), and \(M_0(x_{-1:0}, dx_0) := \eta_0(dx_0) \in \mathcal{P}(E_0)\), and

(ii) \(G_n(x_{0:n})\) are \([0, \infty)\)-valued (measurable) ‘potential’ functions for \(n \geq 0\).
Particle filter (Algorithm 1) [cf. 7] generates sets of samples and weights corresponding to the Feynman–Kac model, which for \( \varphi : E_{0:n} \to \mathbb{R} \) lead to an unbiased estimator for the (unnormalised) smoother \( \gamma_n(G_n\varphi) \), defined here in terms of the (unnormalised) predictor

\[
\gamma_n(\varphi) := \int \varphi(x_{0:n}) \left( \prod_{t=0}^{n-1} G_t(x_{0:t}) \right) \eta_0(dx_0) \prod_{t=1}^{n} M_t(x_{0:t-1}, dx_t). 
\]

**Algorithm 1** Particle filter for model \((M_{0:n}, G_{0:n}) := (M_t, G_t)_{t=0:n}\) with \(N\) particles.

In each line, \(i\) takes values 1:N. Do:

(i) Sample \(x_0^{(i)} \sim \eta_0(\cdot)\) and set \(x_0^{(i)} := x_0^{(i)}\).

(ii) Compute \(\omega_0^{(i)} := G_0(x_0^{(i)})\) and set \(\bar{\omega}_0^{(i)} := \omega_0^{(i)}/\omega^*_0\) where \(\omega^*_0 := \sum_{j=1}^{N} \omega_0^{(j)}\).

For \(t = 1:n\), do:

(iii) Given \(\bar{\omega}_{t-1}^{(1:N)}\), sample \(A_{t-1}^{(1:N)}\) satisfying \(E\left[j=1\right] \{ A_{t-1}^{(j)} = k \} \} = N\bar{\omega}_{t-1}^{(k)}\).

(iv) Sample \(x_t^{(i)} \sim M_t(x_{t-1}^{(i)}, \cdot)\) and set \(x_t^{(i)} := (x_{t-1}^{(i)}, x_t^{(i)})\).

(v) Compute \(\omega_t^{(i)} := G_t(x_t^{(i)})\) and set \(\bar{\omega}_t^{(i)} := \omega_t^{(i)}/\omega_t^*\) where \(\omega_t^* := \sum_{j=1}^{N} \omega_t^{(j)}\).

Report \((V^{(1:N)}, X^{(1:N)})\) where \(V^{(i)} := \bar{\omega}_n^{(i)} \prod_{t=0}^{n} \frac{1}{N} \omega_t^*\) and \(X^{(i)} := x_n^{(i)}\).

(In case \(\omega_n^* = 0\), the algorithm is terminated with \(V^{(i)} = 0\) and with arbitrary \(X^{(i)} \in E_{0:n}\)).

**Proposition 1.** Suppose that \(\varphi : E_{0:n} \to \mathbb{R}\) is such that \(\gamma_n(G_n\varphi) < \infty\). Then, the output of Algorithm 1 satisfies

\[
E\left[ \sum_{i=1}^{N} V^{(i)} \varphi(X^{(i)}) \right] = \gamma_n(G_n\varphi).
\]

Proposition 1 is a restatement of [7, Theorem 7.4.2] in case \(A_{t-1}^{(i)}\) are sampled independently (“multinomial resampling”). The extension to the general unbiased case, which covers popular residual, stratified and systematic resampling schemes [cf. 5, 8], is straightforward [cf. 32].

2.2. **Level difference estimation.** Suppose that we have two Feynman–Kac models \((M_n^{E_n}, G_n^{E_n})\) and \((M_n^{C_n}, G_n^{C_n})\) defined on common spaces \((E_n, \mathcal{E}_n)\). The models correspond to ‘finer’ and ‘coarser’ Euler type discretised HMM diffusions. We are interested in estimating (unbiasedly) the difference

\[
\gamma_n^{E_n}(G_n^{E_n}\varphi) - \gamma_n^{C_n}(G_n^{C_n}\varphi).
\]

If the models are close to each other, as they will be in the multilevel (diffusion) context, we would like the estimator also to be typically small. In many contexts, if one can estimate the difference using a coupling, it is possible to obtain a variance reduction. The particular coupling approach we use here is based on using a combined Feynman–Kac model as in [19], which provides a simple, general and effective coupling of PFs, and which we will use to estimate the level difference of unnormalised smoother (3).

Hereafter, we denote \(\bar{x}_n = (\bar{x}_n^{E_n}, \bar{x}_n^{C_n}) \in E_n \times E_n\), and for \(\bar{x}_{0:n} = (\bar{x}_0, \ldots, \bar{x}_n) \in E_0^2 \times \ldots \times E_n^2\), we set \(\bar{x}_{0:n}^s := (\bar{x}_0^s, \ldots, \bar{x}_n^s) \in E_{0:n}^s \) for \(s \in \{E, C\}\).

**Assumption 2.** Suppose that \((\bar{M}_t, \bar{G}_t)\) is a Feynman–Kac model on the product spaces \((E_t \times E_t, \mathcal{E}_t \otimes \mathcal{E}_t)\), such that:
Remark

By the unbiasedness property of PF Algorithm 1, we have

\[
\int_{A \times E_t} \tilde{M}_t(\tilde{x}_{0:t-1}, d\tilde{x}_t) = M_t^F(\tilde{x}_{0:t-1}, A), \quad \int_{E_t \times A} \tilde{M}_t(\tilde{x}_{0:t-1}, d\tilde{x}_t) = M_t^C(\tilde{x}_{0:t-1}, A),
\]

and for \( A \in E_0 \), we have \( \tilde{\eta}_0(A \times E_0) = \eta_0^F(A) \) and \( \tilde{\eta}_0(E_0 \times A) = \eta_0^C(A) \).

(ii) \( \tilde{G}_t(\tilde{x}_{0:t}) := \frac{1}{2}\left[ G_t^F(\tilde{x}_{0:t}) + G_t^C(\tilde{x}_{0:t}) \right] \).

Algorithm 2 Delta particle filter (ΔPF) for unbiased estimation of level differences.

(i) Run Algorithm 1 with \((\tilde{M}_{0:n}, \tilde{G}_{0:n}, N)\), outputting \((\tilde{V}^{(1:N)}, \tilde{X}^{(1:N)})\).

(ii) Report \((V^{(1:2N)}, X^{(1:2N)})\) where

\[
(V^{(i)}, X^{(i)}) := \begin{cases} 
\left( \tilde{V}^{(i)}(\tilde{X}^{(i)}), \chi^{(i)}(\tilde{X}^{(i)}) \right) & i = 1;N, \\
\left( -\tilde{V}^{(i-N)}(\tilde{X}^{(i-N)}), \tilde{X}^{(i-N)} \right) & i = (N + 1);2N,
\end{cases}
\]

and where \( w^F(\tilde{x}_{0:n}) := \prod_{i=0}^{n} G_t^F(\tilde{x}_{0:n}) \) and \( w^C(\tilde{x}_{0:n}) := \prod_{i=0}^{n} G_t^C(\tilde{x}_{0:n}) \).

Proposition 3. Under Assumption 2, the output of Algorithm 2 satisfies

\[
\mathbb{E}\left[ \sum_{i=1}^{2N} V^{(i)}(\varphi(X^{(i)})) \right] = \gamma_n^F(G_n^F \varphi) - \gamma_n^C(G_n^C \varphi)
\]

whenever both integrals on the right are well-defined and finite.

Proof. By the unbiasedness property of PF Algorithm 1, we have

\[
\mathbb{E}\left[ \sum_{i=1}^{N} V^{(i)}(\varphi(X^{(i)})) \right] = \int w^F(\tilde{x}_{0:n}) \varphi(\tilde{x}_{0:n}) \left( \prod_{t=0}^{n} G_t(\tilde{x}_{0:t}) \right) \tilde{\eta}(dx_0) \prod_{t=1}^{n} \tilde{M}_t(\tilde{x}_{0:t-1}, d\tilde{x}_t)
\]

\[
= \int \varphi(\tilde{x}_{0:n}) \left( \prod_{i=0}^{n} G_t^F(\tilde{x}_{0:n}) \right) \tilde{\eta}(dx_0) M_t^F(\tilde{x}_{0:t-1}, d\tilde{x}_t) = \gamma_n^F(G_n^F \varphi),
\]

where Assumption 2(ii) guarantees \( \tilde{G}_t > 0 \) whenever \( G_t^F > 0 \), and (i) implies the marginal law of \( \prod_{t=0}^{n} \tilde{M}_t \) is \( \prod_{t=0}^{n} M_t^F \). Similarly, \( \mathbb{E}\left[ \sum_{i=N+1}^{2N} V^{(i)}(\varphi(X^{(i)})) \right] = -\gamma_n^C(G_n^C \varphi). \)

Remark 4. Regarding Algorithm 2:

(i) Typically, in the discretisation of diffusions context [14, 26], the couplings \( \tilde{M}_t \) would be based on using the same underlying Brownian motion [cf. 22]. That is, if

\[
X_{t+hF}^F = X_{t}^F + a_0(X_t^F)h^F + b_0(X_t^F)\delta W_{t+hF}^F \quad \text{and} \quad X_{t+khF}^F = X_{t+hF}^F + a_0(X_{t+hF}^F)h^F + b_0(X_{t+hF}^F)\delta W_{t+2hF}^F
\]

with \( \delta W_{t+khF}^F \sim N(0, h^F) \), \( k = 1, 2, 3, \ldots \), corresponds to two steps of an Euler discretisation with step-size \( h^F \), then we can use

\[
X_{t+hC}^C = X_{t}^C + a_0(X_t^C)h^C + b_0(X_t^C)\left( \delta W_{t+hF}^F + \delta W_{t+2hF}^F \right)
\]

with \( h^C := 2h^F \) for the coarser Euler discretisation. The kernels \( \tilde{M}_t \) on the joint space then move \( N \) particles according to the fine-level discretisation, and \( N \) according to the coarse-level discretisation, both based on the same underlying sequence of standard normals \( (\delta W_{t+khF}^F)_{k \geq 1}. \)
(ii) The choice of \( \tilde{G}_t \) in Assumption 2(ii) provides a safe ‘balance’ in between the approximations, as \( w^F \) and \( w^C \) are upper bounded by \( 2^{n+1} \). Indeed, the coupled Feynman–Kac model can be thought as an ‘average’ of the two extreme cases—with the choice \( G_t(x_{0:t}) = G^F_t(\tilde{x}^F_{0:t}) \) the coupled PF would coincide marginally with the Feynman–Kac model with dynamics \( M^F_t \). What is the optimal choice for \( \tilde{G}_t \) is an interesting question.

(iii) Clearly, the choice of \( \tilde{G}_{0:t} \) can be made also in other ways. It is sufficient for unbiasedness to choose \( \tilde{G}_t(\tilde{x}_{0:t}) \) such that it is strictly positive whenever either the \( G^F_t(\tilde{x}^F_{0:t}) \) or \( G^C_t(\tilde{x}^C_{0:t}) \) product is positive, but choices which make \( w^F \) and \( w^C \) bounded are safer, for instance \( \tilde{G}_{0:t}(\tilde{x}_{0:t}) = \max\{G^F_t(\tilde{x}^F_{0:t}), G^C_t(\tilde{x}^C_{0:t})\} \). This was the original choice made in [19] for approximation of normalised smoother differences. This PF coupling approach based on change of measure and weight corrections \( w^F \) and \( w^C \), has been further used also, for example, in [20].

(iv) Later, in the HMM diffusion context, we set \( G^F_t = G^C_t \), corresponding to common observational densities, but the method is also of interest with differing potentials.

2.3. **Unbiased latent inference.** We show here how the randomisation techniques of [23, 26] can be used with the output of Algorithms 1 and 2 to provide an unbiased estimator according to the true model, even though the PFs are only run according to approximate models. Let us index the transitions \( M^p_\ell \) and potentials \( G^p_\ell \) by \( \ell \geq 0 \). They are assumed throughout to be increasingly refined approximations, in the (weak) sense that

\[
\gamma(n)_\ell(G(n)_\ell \varphi) \rightarrow \gamma(\infty)(G(n)_\infty \varphi), \quad \text{as} \quad \ell \rightarrow \infty,
\]

for all \( \varphi \in \mathcal{B}_b(E_{0:n}) \), where

\[
\gamma(n)_\ell(\varphi) := \int \varphi(x_{0:n}) \left( \prod_{t=0}^{n-1} G^n_\ell(x_{0:t}) \right) \eta_0^n(dx_0) \prod_{t=1}^n M^p_\ell(x_{0:t-1}, dx_t).
\]

In Assumption 2 we set symbols \((F, C)\) to be \((\ell, \ell-1)\) for \( \ell \geq 1 \). Algorithm 3 can then provide unbiased estimation of \( \gamma(\infty)(G(\infty)_N \varphi) \) (Lemma 6), leading to unbiased inference w.r.t. the normalised smoother

\[
\varphi \mapsto \gamma(n)(G_N \varphi)/\gamma(\infty)(G(n)),
\]

which is stated as Proposition 7 below.

**Algorithm 3** Unbiased estimator based on PF and r∆PF; \( N \) particles, probability \( p = (p_\ell)_{\ell \in \mathbb{N}} \).

(i) Run Algorithm 1 with \((M^{(0)}_{0:n}, G^{(0)}_{0:n}, N)\), outputting \((V^{(1:N)'}, X^{(1:N)'})\).

(ii) Sample \( L \sim p \) (independently from the other random variables).

(iii) Run Algorithm 2 with \((M^{(L)}_{0:n}, G^{(L)}_{0:n}, N)\), outputting \((V^{(1:2:N)}, X^{(1:2:N)})\).

Report \( (\{V^{(1:N)'}, X^{(1:N)'}\}, \{V^{(1:2:N)}, X^{(1:2:N), L}\}) \).

**Assumption 5.** Assumption 2 holds, \( p = (p_\ell)_{\ell \in \mathbb{N}} \) is a probability on \( \mathbb{N} := \mathbb{Z}_{\geq 1} \) with \( p_\ell > 0 \) for all \( \ell \geq 1 \), \( g : E_{0:n} \rightarrow \mathbb{R} \) is a function, and

\[
s_g := \sum_{\ell \geq 0} \frac{\mathbb{E} \Delta^2(g)}{p_\ell} < \infty,
\]
where

\begin{equation}
\Delta_\ell(g) := \sum_{i=1}^{2N} V^{(i)} g(X^{(i)})
\end{equation}

is formed from the output \((V^{(1:2N)}, X^{(1:2N)})\) of Algorithm 2 with \((\tilde{M}^{(\ell)}_{0:n}, \tilde{G}^{(\ell)}_{0:n}, N)\).

**Lemma 6.** Under Assumption 5, the estimator

\begin{equation}
\zeta(g) := \sum_{i=1}^{N} V^{(i)} g'(X^{(i)'}) + \frac{1}{p_L} \Delta_L(g)
\end{equation}

formed from the output of Algorithm 3 satisfies

\[ \mathbb{E}[\zeta(g)] = \gamma^{(\infty)}(G^{(\infty)}_n g), \]

whenever \(\gamma^{(0)}(G_n g)\) and \(\gamma^{(\infty)}(G_n g)\) are both finite.

**Proof.** Under Assumption 5, we have [cf. 26, 31]

\[ \mathbb{E}[p_L^{-1} \Delta_L(g)] = \gamma^{(\infty)}(G^{(\infty)}_n g) - \gamma^{(0)}(G^{(0)}_n g), \]

so the result follows by Proposition 1 and linearity of the expectation. \(\square\)

The following suggests a fully parallelisable algorithm for unbiased inference over the normalised smoother, and is an unbiased alternative to the particle independent Metropolis-Hastings (PIMH) [2] run at some fine level of discretisation.

**Proposition 7.** Suppose \(p\) on \(N\) satisfies Assumption 5 for functions \(g \in \{1, \varphi\}\), with \(\gamma^{(0)}(G^{(0)}_n g)\) and \(\gamma^{(\infty)}(G^{(\infty)}_n g)\) finite, and \(\gamma^{(\infty)}(G^{(\infty)}_n g) > 0\). For each \(k \in \{1:m\}\), if one runs independently Algorithm 3, forming \(\zeta_k(g)\) from the output as in (7) for each \(k\), then

\[ E_{m,N,p}(\varphi) := \frac{\sum_{k=1}^{m} \zeta_k(\varphi)}{\sum_{k=1}^{m} \zeta_k(1)} \xrightarrow{m \to \infty} p^{(\infty)}(\varphi) \quad \text{almost surely.} \]

Moreover, with \(\tilde{\varphi} := \varphi - p^{(\infty)}(\varphi)\),

\[ \sqrt{m}[E_{m,N,p}(\varphi) - p^{(\infty)}(\varphi)] \xrightarrow{m \to \infty} \mathcal{N}(0, \sigma^2) \quad \text{in distribution,} \]

where

\[ \sigma^2 = \frac{s_{\tilde{\varphi}} - \left(\gamma^{(\infty)}(G^{(\infty)}_n \tilde{\varphi}) - \gamma^{(0)}(G^{(0)}_n \tilde{\varphi})\right)^2}{\left[\gamma^{(\infty)}(G^{(\infty)}_n)\right]^2}. \]

The above result follows directly from the results of Section 4. It can also be seen as a multilevel version of [32, Proposition 23], with straightforward estimators for \(\sigma^2\). See Section 5 for suggested choices for \(p\) and \(N_{\ell}\).

3. A variance bound for the delta particle filter

In this section we give theoretical results for the \(\Delta\)PF (Algorithm 2) in the setting of HMM diffusions, which can be used to verify finite variance and therefore consistency of related estimators.
3.1. Hidden Markov model diffusions. We consider an HMM diffusion and corresponding Feynman-Kac model as in Section 2. We omit θ from the notation in the following, which is allowed as the remaining conditions and results in this Section 3 will hold uniformly in θ (i.e. any constants are independent of θ). The following will be assumed throughout.

**Condition (D).** The coefficients $a^j, b^{jk}$ are twice differentiable for $j, k = 1, \ldots, d$, and

- (i) uniform ellipticity: $b(x)b(x)^T$ is uniformly positive definite;
- (ii) globally Lipschitz: there is a $C > 0$ such that $|a(x) - a(y)| + |b(x) - b(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}^d$;
- (iii) boundedness: $\mathbb{E}[X_0^p] < \infty$ for all $p \geq 1$.

Let $M^{(\infty)}(x, dy) =: M^{(\infty)}_p(x, dy)$ for $p = 0; n$ denote the Markov transition of the unobserved diffusion (1), i.e. the distribution of the solution $X_1$ of (1) started at $X_0 = x$. With similar setup from Section 2, with $E_n := X^{n+1}$, we have that (2) takes the form

$$
\gamma_n^{(\infty)}(\varphi) = \int \varphi(x_0) \left( \prod_{p=0}^{n-1} G_p(x_p) \right) \eta_0(dx_0) \prod_{p=1}^{n} M^{(\infty)}(x_{p-1}, dx_p).
$$

In practice one usually must approximate the true dynamics $M^{(\infty)}(x, dy)$ of the underlying diffusion with a simpler transition $M^{(\ell)}(x, dy)$, based on some Euler type scheme using a discretisation parameter $h_\ell = 2^{-\ell}$ for $\ell \geq 0$ [cf. 22]. The scheme allows for a coupling of the diffusions $(X^{(\ell)}_t, X^{(\ell-1)}_t)_{t \geq 0}$ running at discretisation levels $\ell$ and $\ell - 1$ (based on using the same Brownian path $W_t$), such that for some $\beta \in \{1, 2\}$, we have

$$
\mathbb{E}(\mathbb{E} \mathbb{E}(x, y) [\|X^{(\ell)}_1 - X^{(\ell-1)}_1\|^2] \leq M(\|x - y\|^2 + h_\ell^\beta),
$$

where $M < \infty$ does not depend on $\ell \geq 1$. In particular, if the diffusion coefficient $b(X_t)$ in (1) is constant or if a Milstein scheme can be applied otherwise, then $\beta = 2$; otherwise $\beta = 1$ [cf. 18, Proposition D.1].

3.2. Variance bound. Assume we are in the above HMM diffusion setting, and that the coupling of Assumption 2 holds, with symbols $(F, C)$ equal to $(\ell, \ell - 1)$ for $\ell \geq 1$, and $G^{(\ell)}_p = G^{(\ell-1)}_p := G_p$ for $p = 0; n$. Running Algorithm 2, we recall that $\Delta_\ell(\varphi)$, defined in (6), satisfies, by Proposition 3,

$$
\mathbb{E}[\Delta_\ell(\varphi)] = \gamma_\ell^{(\ell)}(G_n \varphi) - \gamma_\ell^{(\ell-1)}(G_n \varphi),
$$

regardless of the number $N \geq 1$ of particles.

Recall that a (measurable) function $\varphi : X \to \mathbb{R}$ is Lipschitz, denoted $\varphi \in \text{Lip}(X)$, if for some $C' < \infty$, $|\varphi(x) - \varphi(y)| \leq C'|x - y|$ for all $x, y \in X$.

**Condition (A).** The following conditions hold for the model $(M_n, G_n)$:

(A1) (i) $\|G_n\| < \infty$ for each $n \geq 0$.
- (ii) $G_n \in \text{Lip}(X)$ for each $n \geq 0$.
- (iii) $\inf_{x \in X} G_n(x) > 0$ for each $n \geq 0$.

(A2) For every $n \geq 1$, $\varphi \in \text{Lip}(X) \cap \mathcal{B}_b(X)$ there exist a $C' < \infty$ such that for $s \in \{F, C\}$, we have for every $(x, y) \in X \times X$ that $|M_n^s(\varphi)(x) - M_n^s(\varphi)(y)| \leq C'|x - y|$.

In the following results for $\Delta_\ell(\varphi)$, the constant $M < \infty$ may change from line-to-line. It will not depend upon $N$ or $\ell$ (or $\theta$), but may depend on the time-horizon $n$ or the function $\varphi$. $\mathbb{E}$ denotes expectation w.r.t. the law associated to the $\Delta$PF started at $(x, x)$, with $x \in X$. Below we only consider multinomial resampling in the $\Delta$PF for simplicity, though Theorem 8 and Corollary 9 can be proved also assuming other resampling schemes.
Theorem 8. Assume (A1-2). Then for any $\varphi \in B_b(X^{n+1}) \cap \text{Lip}(X^{n+1})$, there exists a $M < \infty$ such that
\[
\mathbb{E}\left[ \left( \Delta_t(\varphi) - \mathbb{E}[\Delta_t(\varphi)] \right)^2 \right] \leq \frac{M h_t^{2\wedge \beta}}{N}, \quad \text{with } \beta \text{ as in (8)}.
\]

Corollary 9. Assume (A1-2). Then for any $\varphi \in B_b(X^{n+1}) \cap \text{Lip}(X^{n+1})$, there exists a $M < \infty$ such that
\[
\mathbb{E}\left[ \left( \Delta_t(\varphi) \right)^2 \right] \leq M \left( \frac{h_t^{2\wedge \beta}}{N} + h_t^2 \right), \quad \text{with } \beta \text{ as in (8)}.
\]

The proofs are given in Appendix A.

Based on Corollary 9, Recommendation 1 of Section 5 suggests allocations for $p$ and $N_t$ in the $\Delta$PF (Algorithm 2) to optimally use resources and minimise variance (5).

4. UNBIASED JOINT INFERENCE FOR HIDDEN MARKOV MODEL DIFFUSIONS

We are interested in unbiased inference for the Bayesian model posterior
\[
\pi(\theta, d\gamma_{0:n}) \propto \text{pr}(d\theta) G_n^{(\theta)}(x_n) \gamma_n^{(\theta, \infty)}(dx_{0:n}),
\]
where $\text{pr}(d\theta) = \text{pr}(\theta)d\theta$ is the prior on the model parameters, and
\[
\gamma_n^{(\theta, \infty)}(dx_{0:n}) = \left( \prod_{t=0}^{n-1} G_t^{(\theta)}(x_t) \right) \eta_0^{(\theta)}(dx_0) \prod_{t=1}^n M_t^{(\theta, \infty)}(x_{t-1}, dx_t).
\]

Here, $M_t^{(\theta, \infty)}$ corresponds to the transition density of the diffusion model of interest. The dependence of the HMM on $\theta$ is made explicit in this section. As in Section 3, we assume the transition densities $M_t^{(\theta, \infty)}$ cannot be simulated, but that there are increasingly refined discretisations $M_t^{(\theta, \ell)}$ approximating $M_t^{(\theta, \infty)}$ in the sense of (4) (with $E_{0:n} := X^{n+1}$).

4.1. Randomised MLMC IS type estimator based on coarse-model PMMH. We now consider Algorithm 4 for joint inference w.r.t. the above Bayesian posterior. Algorithm 4 uses the following ingredients:

(i) $M_{0:n}^{(\theta, \ell)}$ satisfying Assumption 2(i) with $M_{0:n}^{F} = M_{0:n}^{(\theta, \ell)}$, and $M_{0:n}^{C} = M_{0:n}^{(\theta, \ell-1)}$.
(ii) $G_t^{(\theta)}$ defined as in Assumption 2(ii), with $G_{0:n}^{F} = G_{0:n}^{C} = G_{0:n}$.
(iii) Metropolis–Hastings proposal distribution $q(\cdot \mid \theta)$ for parameters.
(iv) Algorithm constant $\epsilon \geq 0$.
(v) Number of MCMC iterations $m_{iter} \in \mathbb{N}$ and number of particles $N \in \mathbb{N}$.
(vi) Probability mass $p = (p_\ell)_{\ell \in \mathbb{N}}$ on $\mathbb{N}$ with $p_\ell > 0$ for all $\ell \in \mathbb{N}$.

Remark 10. Before stating consistency and central limit theorems, we briefly discuss various aspects of this approach, which are appealing from a practical perspective, and we also mention certain algorithmic modifications which could be further considered.

(i) The first phase (P1) of Algorithm 4 implements a PMMH type algorithm [2]. If $\epsilon = 0$, this is exactly PMMH targeting the model $\pi^{(0)}(d\theta, dx_{0:n}) \propto \text{pr}(d\theta) G_n^{(\theta)}(x_n) \gamma_n^{(\theta, 0)}(dx_{0:n})$. It is generally safer to choose $\epsilon > 0$ [32], which ensures that the IS type correction in phase (P2) will yield consistent inference for the ideal model $\pi^{(\infty)}(d\theta, dx_{0:n}) \propto \text{pr}(d\theta) G_n^{(\theta)}(x_n) \gamma_n^{(\theta, \infty)}(x_{0:n})$ (Theorem 11). Setting $\epsilon > 0$ may be helpful otherwise in terms of improved mixing, as the PMMH will target marginally an averaged probability between a ‘flat’ prior and a ‘multimodal’ $\ell = 0$ marginal posterior.
Algorithm 4 Randomised multilevel importance sampling type estimator.

(P1) Let \((\Theta_0, V_0^{(1:N)}, X_0^{(1:N)})\) such that \(\sum_{i=1}^{N} V_0^{(i)} > 0\), and for \(k = 1: m_{\text{iter}}\), iterate:
(i) Propose \(\tilde{\Theta}_k \sim q(\cdot | \Theta_{k-1})\).
(ii) Run Algorithm 1 with \((M_{0:n}^{(\Theta_0, L_k)}, G_{0:n}^{(\tilde{\Theta}_k)}, N)\) and call the output \((\hat{V}_k^{(1:N)}, \hat{X}_k^{(1:N)})\).
(iii) With probability
\[
\min \left\{ 1, \frac{\text{pr}(\tilde{\Theta}_k)q(\Theta_{k-1} | \Theta_k)(\sum_{i=1}^{N} \hat{V}_k^{(i)} + \epsilon)}{\text{pr}(\Theta_{k-1})q(\tilde{\Theta}_k | \Theta_{k-1})(\sum_{j=1}^{N} V_{k-1}^{(j)} + \epsilon)} \right\},
\]
accept and set \((\Theta_k, V_k^{(1:N)}, X_k^{(1:N)}) \leftarrow (\hat{\Theta}_k, \hat{V}_k^{(1:N)}, \hat{X}_k^{(1:N)})\); otherwise set \((\Theta_k, V_k^{(1:N)}, X_k^{(1:N)}) \leftarrow (\Theta_{k-1}, V_{k-1}^{(1:N)}, X_{k-1}^{(1:N)})\).

(P2) For all \(k \in \{1: m_{\text{iter}}\}\), independently conditional on \((\Theta_k, V_k^{(1:N)}, X_k^{(1:N)})\):
(i) Set \(X_{k,0} = X_k^{(1:N)}\), and set \(W_k^{(i)} := V_k^{(i)} / (\sum_{j=1}^{N} V_j^{(j)} + \epsilon)\).
(ii) Sample \(L_k \sim p\) (independently from the other random variables).
(iii) Run \(\Delta PF\) (Algorithm 2) with \((M_{0:n}^{(\Theta_k, L_k)}, G_{0:n}^{(\tilde{\Theta}_k)}, N)\), and call the output \((V_k^{(1:2N)}, X_k^{(1:2N)})\). Set \(W_k^{(i)} := V_k^{(i)} / p_{L_k}(\sum_{j=1}^{N} V_j^{(j)} + \epsilon)\).

Report the estimator
\[
E_{m_{\text{iter}}, N, p}(f) := \frac{\sum_{k=1}^{m_{\text{iter}}} \left[ \sum_{i=1}^{N} W_{k,0}^{(i)} f(\Theta_k, X_k^{(i)}) + \sum_{i=1}^{2N} W_{k,L_k}^{(i)} f(\Theta_k, X_k^{(i)}) \right]}{\sum_{k=1}^{m_{\text{iter}}} \left[ \sum_{i=1}^{N} W_k^{(i)} + \sum_{i=1}^{2N} W_{k,L_k}^{(i)} \right]} \approx \pi(\infty)(f).
\]

(ii) It is only necessary to implement PMMH for the coarsest level. This is typically relatively cheap, and therefore allows for a relatively long MCMC run. Consequently, relative cost of burn-in is small, and if the proposal \(q\) is adapted [cf. 1], it has time to converge.

(iii) The (potentially costly) \(\Delta PFs\) are applied independently for each \(\Theta_k\), which allows for efficient parallelisation.

(iv) We suggest that the number of particles \(N_0\) used in the PMMH be chosen based on [9, 29], while the number of particles \(N_\ell\) (and \(p_\ell\)) can be optimised for each level \(\ell\) based on Recommendation 1 of Section 5, or kept constant. One can also afford to increase the number of particles when a ‘jump chain’ representation is used (see the following remark).

(v) The \(\Delta PF\) corrections may be calculated only once for each accepted state [32]. That is, suppose \((\tilde{\Theta}_k, \tilde{V}_k^{(1:N)}, \tilde{X}_k^{(1:N)})^{m_{\text{iter}}}_{k=1}\) are the accepted states, \((D_k)^{m_{\text{iter}}}_{k=1}\) are the corresponding holding times, and \((V_k^{(1:2N_{L_k})}, X_k^{(1:2N_{L_k})})^{m_{\text{iter}}}_{k=1}\) are corresponding \(\Delta PF\) outputs, then the estimator is formed as in Algorithm 4 using \((\tilde{\Theta}_k, \tilde{V}_k^{(1:N)}, \tilde{X}_k^{(1:N)})\), and accounting for the holding times in the weights defined as \(W_k^{(i)} := D_k V_k^{(i)} / (\sum_{j=1}^{N} \tilde{V}_k^{(j)} + \epsilon)\) and \(W_{k,L_k}^{(i)} := \tilde{V}_{k,L_k}^{(i)} / p_{L_k}(\sum_{j=1}^{N} \tilde{V}_k^{(j)} + \epsilon)\).

(vi) In case the Markov chain in (P1) phase is slow mixing, (further) thinning may be applied (to the jump chain) before the (P2) phase.

(vii) In practice, Algorithm 4 may be implemented in an on-line fashion w.r.t. the number of iterations \(m_{\text{iter}}\), and by progressively refining the estimator \(E_{m_{\text{iter}}, N, p}(f)\). The \(\Delta PF\) corrections may be calculated in parallel with the Markov chain.

(viii) In Algorithm 4, the \(\Delta PFs\) depend only on \(\Theta_k\). They could depend also on \(V_k^{(i)}\) and \(X_k^{(i)}\), but it is not clear how such dependence could be used in practice to achieve better...
Then, the estimator of Algorithm 4 is strongly consistent: \( \bar{\sigma} \) is finite. Here, \( \sigma(9) \)

Remark Suppose that the conditions of Theorem 11 hold. Suppose additionally that
Proposition 13.

Equation denoted by \( \Theta \) Markov chain 
Assume that the algorithm constant \( \epsilon \) is chosen positive, and that the base chain \( (\Theta_k, X_k^{(1:N)}, V_k^{(1:N)})_{k \geq 1} \) is \( \psi \)-irreducible, and that \( \pi(0)(f) \) and \( \pi(\infty)(f) \) are finite. For each \( \theta \in \mathcal{T} \), suppose Assumption 5 holds for \( g := 1 \) and \( g = f(\theta) := f(\theta, \cdot) \), with \( M(\ell) := M(\ell, 0) \) and \( C(\ell) := C(\ell, 0) \). Assume

\[
\int pr(\theta)(\sqrt{s_1(\theta)} + \sqrt{s_f(\theta)}) \, d\theta < \infty.
\]

Then, the estimator of Algorithm 4 is strongly consistent:

\[
E_{m_{iter}, N, p}(f) \xrightarrow{m_{iter} \to \infty} \int \pi(\infty)(d\theta, dx_{0:n}) f(\theta, x_{0:n}) \quad (a.s.)
\]

Remark 12. Regarding Theorem 11, whose proof is given in Appendix B:

(i) If all potentials \( G_i \) are strictly positive, the algorithm constant \( \epsilon \) may be taken to be zero. However, if \( \epsilon = 0 \) and Algorithm 1 with \( (M(0,0) := M(\theta, 0), C(\theta), N) \) can produce an estimate with \( \sum_{i=1}^N V(i) = 0 \) with positive probability, the consistency may be lost [32].

Proposition 13. Suppose that the conditions of Theorem 11 hold. Suppose additionally that \( \pi(\infty)(f^2) < \infty \) and that the base chain \( (\Theta_k, V_k^{(1:N)}, X_k^{(1:N)})_{k \geq 1} \) is aperiodic, with transition probability denoted by \( P \). Then,

\[
\sqrt{m_{iter}} [E_{m_{iter}, N, p}(f) - \pi(\infty)(f)] \xrightarrow{m_{iter} \to \infty} \mathcal{N}(0, \sigma^2), \quad \text{in distribution,}
\]

whenever the asymptotic variance

\[
(9) \quad \sigma^2 = \frac{\text{var}(P, M(f) + \Pi(\sigma^2))}{c^2}
\]

is finite. Here, \( \bar{f} := f - \pi(\infty)(f), c > 0 \) is a constant (equal to \( \Pi(M_1) \)), and

\[
\sigma^2(\theta, v^{(1:N)}, x^{(1:N)}) := \text{var}(\xi_k(\bar{f}) | \Theta_k = \theta, V_k^{(1:N)} = v^{(1:N)}, X_k^{(1:N)} = x^{(1:N)})
\]

\[
= s_f(\theta) - \left( \gamma_n(\theta) (G_n f(\theta) - \gamma_n(\theta)) + \gamma_n(\theta) (G_n f(\theta)) \right)^2
\]

\[
\left( \sum_{i=1}^N v(i) + \epsilon \right)^2.
\]
Remark 14. Proposition 13 follows from [32, Theorem 7]. We suggest that $N = N_0$ for (P1) be chosen based on [9, 29] to minimise $\text{var}(P, \mu \bar{f})$, and that $(p_\ell)$ and $N = N_\ell$ in (P2) for the rΔPF be chosen as in Recommendation 1 of Section 5, to minimise $\sigma^2$, subject to cost constraints, in order to jointly minimise $\sigma^2$. However, the question of the optimal choice for $N_0$ in the IS context is not yet settled.

5. Asymptotic efficiency and randomised multilevel considerations

We summarise the results of this section by suggesting the following safe allocations for probability $p = (p_\ell)_{\ell \in \mathbb{N}}$ and number $N = N_\ell \in \mathbb{N}$ in ΔPF (Algorithm 2) used in Algorithm 3 and 4, and Proposition 7, with $\beta$ given in (8) in the HMM diffusion context of Section 3, or, indeed, with $\beta$ given in the abstract framework under Assumption 18 given later. See also Figure 1 for the recommended allocations.

**Recommendation 1.** With strong error convergence rate $\beta$ given in (8), we suggest the following for $p = (p_\ell)_{\ell \in \mathbb{N}}$ and $N_\ell \in \mathbb{N}$ in ΔPF (Algorithm 2):

$(\beta = 1)$ (e.g. Euler scheme). Choose $p_\ell \propto (\frac{1}{2})^\ell$ and $N_\ell \propto 1$ constant.

$(\beta = 2)$ (e.g. Milstein scheme). Choose $p_\ell \propto 2^{-(1+\rho)\ell} \approx (\frac{1}{2})^\ell$ and $N_\ell \propto 2^\rho \ell$.

The suggestions are based on Corollary 9 of Section 3, and Propositions 20 ($\beta = 2$) and 26 ($\beta = 1$) given below (with weak convergence rate $\alpha = 1$; see Figure 1 for general $\alpha$). In the Euler case, although the theory below gives the same computational complexity order by choosing any $\rho \in [0, 1]$ and setting $p_\ell \propto 2^{-(1+\rho)\ell}$ and $N_\ell \propto 2^\rho \ell$, the experiment in Section 6 gave a better result using simply $\rho = 0$, corresponding to no scaling.

5.1. Efficiency framework. The asymptotic efficiency of Monte Carlo was considered theoretically in [15]; see [14] in the dMLMC context. The developments of this section follow [26] for rMLMC (originally in the i.i.d. setting), while also giving some extensions (also applicable to that setting). We will see that the basic rMLMC results carry over to our setting involving MCMC and randomised estimators based on PF outputs. Proofs are given in Appendix C.

We are interested in modeling the computational costs involved in running Algorithm 4; the algorithm of Proposition 7 is recovered with $T = \{\theta\}$. Let $\tau_{\Theta_k, L_k}$ represent the combined cost at iteration $k$ of the base Markov chain and weight calculation in Algorithm 4, so that the total cost $C(m)$ of Algorithm 4 with $m$ iterations is

$$C(m) := \sum_{k=1}^{m} \tau_{\Theta_k, L_k}.$$ 

The following assumption seems natural in our setting.

Assumption 15. For $\Theta_k \in \mathcal{T}$, a family $\{\tau_{\Theta_k, \ell}\}_{k, \ell \geq 1}$ consists of positive-valued random variables that are independent of $\{L_k\}_{k \geq 1}$, where $L_k \sim p$ i.i.d., and that are conditionally independent given $\{\Theta_k\}_{k \geq 1}$, such that $\tau_{\Theta_k, \ell}$ depends only on $\Theta_k \in \mathcal{T}$ and $\ell \in \mathbb{N}$.

Under a budget constraint $\kappa > 0$, the realised length of the chain is $\mathcal{L}(\kappa)$ iterations, where

$$\mathcal{L}(\kappa) := \max\{m \geq 1 : C(m) \leq \kappa\}.$$ 

Under a budget constraint, the CLT of Proposition 13 takes the following altered form, where here $\Pi_\theta(d\theta)$ denotes the $\theta$-marginal of the invariant probability measure (given as (25) in Appendix B) of the base Markov chain (equal to the $\theta$-marginal posterior of the $\ell = 0$ model).
**Proposition 16.** If the assumptions of Theorem 13 hold with \( \sigma^2 < \infty \), and if \( \mathbb{E}[\tau] := \mathbb{E}_{\mathbb{P}_m \circ \mathbb{P}[\tau]} < \infty \) with \( \tau(\theta, \ell) := \mathbb{E}[\tau_{\theta_k \ell_k} | \Theta_k = \theta, L_k = \ell] \), then

\[
\sqrt{\tau} \left[ E_{X^{(\ell)}(\cdot), N, \mathbb{P}}(f) - \pi^{(\infty)}(f) \right] \xrightarrow{\kappa} \mathcal{N}(0, \mathbb{E}[\tau] \sigma^2), \quad \text{in distribution.}
\]

**Remark 17.** The quantity \( \mathbb{E}[\tau] \sigma^2 \) is called the ‘inverse relative efficiency’ by \([15]\), and is considered a more accurate quantity than the asymptotic variance \( \sigma^2 \) here for comparison of Monte Carlo algorithms run on the same computer, as it takes into account also the average computational time.

In the following we consider (possibly) variance reduced (if \( \rho > 0 \)) versions of \( \Delta(\gamma) \) of Assumption 5, denoted \( \Delta_\ell \), where \( \gamma = f(\theta) \), based on running the \( \Delta \text{PF} \) (Algorithm 2) with parameters \( \theta, \ell \) fixed. The constant \( C < \infty \) may change line-to-line, but does not depend on \( N, \ell, \) or \( \theta \), but may depend on the time-horizon \( n \) and function \( f \).

**Assumption 18.** Assumption 15 holds, and constants \( 2\alpha \geq \beta > 0, \gamma > 0 \), and \( \rho \geq 0 \) are such that the following hold:

(i) (Mean cost) \( \mathbb{E}[\tau_{\theta, \ell}] \leq C2^{2\gamma\ell(1+\rho)} \)
(ii) (Strong order) \( \mathbb{E}[\Delta_\ell^2] \leq C2^{-\ell(\beta+\rho)} + C2^{-2\alpha\ell} \)
(iii) (Weak order) \( |\mathbb{E}\Delta_\ell| \leq C2^{-\alpha\ell} \)

**Remark 19.** Regarding Assumption 18:

(i) We only assume bounded mean cost in (i), rather than the almost sure cost bound commonly used. This generalisation allows for the setting where occasional algorithmic runs may take a long time.

(ii) In the original MLMC setting, the cost scaling \( \gamma \) in (i) is taken to be \( \gamma = 1 \) \([14, 26]\). However, in settings involving uncertainty quantification, and where the forward solver may involve non-sparse matrix inversions, often \( \gamma \geq 1 \) \([6, 18, 20]\).

(iii) We assume in (i) that the mean cost to form \( \Delta_\ell \) is bounded by the \( \gamma \)-scaled product of the number of samples or particles \( N_\ell \) times the number of Euler time steps \( 2^\ell + 2^{\ell-1} \) together with the \( O(N_\ell) \)-resampling cost, where there are \( N_\ell \propto 2^\ell \) particles at level \( \ell \). Here, we recall that the stratified, systematic, and residual resampling algorithms have \( O(N_\ell) \) cost, as does an improved implementation of multinomial resampling \([cf. 5, 8]\).

(iv) With \( \rho = 0 \), by Jensen’s inequality one sees why \( \alpha \geq \beta/2 \) can be assumed, and that (ii) becomes \( \mathbb{E}\Delta_\ell^2 \leq C2^{-\alpha\ell} \).

(v) \( \rho \geq 0 \) in (i) and (ii) corresponds to using an average of \( N_\ell := [2^{\rho\ell}] \) i.i.d samples of \( \Delta_\ell \), i.e. \( \Delta_\ell = \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} \Delta_\ell^{(i)} \), or, of more present interest to us, to increasing the number of particles used in a PF by a factor of \( N_\ell \) instead of the default lower number. The former leads to \( \mathbb{E}\Delta_\ell^2 = \frac{1}{N_\ell} \text{var}(\Delta_\ell^{(1)}) + \mathbb{E}[\Delta_\ell^{(1)}]^2 \), justifying (ii), as does Corollary 9, with \( \beta \in \{1, 2\} \) and \( \alpha = 1 \), for the \( \Delta \text{PF} \) (Algorithm 2) in the HMM diffusion context (Section 3).

**Proposition 20.** Suppose Assumption 18 and the assumptions of Proposition 13 hold, with \( \text{var}(P, \mu_f) < \infty \). If \( p_f \propto 2^{-r\ell} \) for some \( r \in (\gamma(1+\rho), \min(\beta+\rho, 2\alpha)) \), then (10) holds, i.e.

\[
\sqrt{\tau} \left[ E_{X^{(\ell)}(\cdot), N, \mathbb{P}}(f) - \pi^{(\infty)}(f) \right] \xrightarrow{\kappa} \mathcal{N}(0, \mathbb{E}[\tau] \sigma^2), \quad \text{in distribution.}
\]

**Remark 21.** Regarding Proposition 20, in the common case \( \gamma = 1 \) for simplicity:

(i) If \( \beta > 1 \) (‘canonical convergence regime’) and \( \rho = 0 \), then a choice for \( r \) exists. See also \([26, \text{Theorem 4}]\) for a discussion of the theoretically optimal \( p \).

(ii) If \( \beta \leq 1 \) (‘subcanonical convergence regime’), then \( \beta + \rho \leq 1 + \rho \) and so no choice for \( r \) exists.
5.2. Subcanonical convergence. When $\beta > 1$, within the framework above we have seen that a canonical convergence rate holds (Proposition 20) because $\mathbb{E}[\tau] < \infty$ and $\sigma^2 < \infty$. When $\beta \leq 1$, this is no longer the case, and one must choose between a finite asymptotic variance and infinite expected cost, or vice versa. Assuming the former, and that a CLT holds (Proposition 13), for $\epsilon > 0$ and $0 < \delta < 1$ the Chebyshev inequality implies that the number of iterations of Algorithm 4 so that

\[
\mathbb{P}[|E_{m,N,\rho}(f) - \pi^{(\infty)}(f)| \leq \epsilon] \geq 1 - \delta
\]

holds implies that $m$ must be of the order $O(\epsilon^{-2})$. The question is then how to minimise the total cost $\mathcal{C}(m)$, or computational complexity, involved in obtaining the $m$ samples. This will involve optimising for $(p_{\ell})$ and $N_{\ell}$ to minimise $\mathcal{C}(m)$, while keeping the asymptotic variance finite.

**Proposition 22.** Suppose that the assumptions of Proposition 13 hold with $\sigma^2 < \infty$, and Assumption 18 holds with $\beta \leq 1$, where moreover $\tau_{\Theta,\ell} \leq C2^\gamma(1 + \rho)$ almost surely, uniformly in $\Theta_k = \theta \in T$. For all $q > 2$ and $\eta > 1$, the choice of probability

\[
p_{\ell} \propto 2^{-2\beta(\log_2(\ell + 1))^{\eta}}
\]

with $a_k = O(k^{c_0}(\log_2k)^{c_1})$ for some constants $c_0 > 0$ and $c_1 \geq 1$, then (11) can be obtained with computational complexity

\[
O(\epsilon^{-2c_1}\log_2\epsilon) \quad \text{as} \quad \epsilon \to 0.
\]

**Remark 23.** The above result shows that even for costs with unbounded tails, reasonable confidence intervals and complexity order may be possible. This may be the case for example when a rejection sampler or adaptive resampling mechanism is used within Algorithm 1 or 4, which may lead to large costs for some $\Theta_k$, for example a cost with a geometric tail.

The next results are as in [26, Proposition 4 and 5] in the standard rMLMC setting, and shows how one can choose $p_{\ell}$, assuming an additional almost sure cost bound, so that $\sigma^2 < \infty$, with reasonable complexity.

**Proposition 24.** Suppose that the assumptions of Proposition 13 hold with $\operatorname{var}(P,\mu_f) < \infty$, and that Assumption 18 holds with $\beta \leq 1$, where moreover $\tau_{\Theta,\ell} \leq C2^\gamma(1 + \rho)$ almost surely, uniformly in $\Theta_k = \theta \in T$. For all $q > 2$ and $\eta > 1$, the choice of probability

\[
p_{\ell} \propto 2^{-2\beta(\log_2(\ell + 1))^{\eta}}
\]
where \( b := \min((\beta + \rho)/2, \alpha) \), leads to \( \sigma^2 < \infty \), and (11) can be obtained with computational complexity
\[
O\left(\epsilon^{-\gamma(\frac{1+\rho}{3})} |\log_2 \epsilon|^{\gamma(\frac{1+\rho}{2\alpha})}\right) \quad \text{as} \quad \epsilon \to 0.
\]

Remark 25. Regarding Proposition 24, with \( \gamma = 1 \):

(i) Under Assumption 18 with \( \rho = 0 \), the usual setup in MLMC before variance reduced estimators are used, the above proposition shows that finite variance and (11) can be obtained without increasing the number of particles at the higher levels, even in the subcanonical regime. We have in this case \( b = \beta/2 \leq \alpha \) and complexity
\[
O\left(\epsilon^{-\frac{1}{2}} |\log_2 \epsilon|^{\frac{1}{2}}\right).
\]
When \( \beta = 1 \) (borderline case), dMLMC gives complexity \( O(\epsilon^{-2} |\log_2 \epsilon|^2) \) [14, 18], which is negligibly better (recall \( q > 2 \)), but is biased inference.

(ii) When \( \alpha > \beta/2 \), which is the usual case in the subcanonical regime \( (\beta \leq 1) \) [cf. 22], a more efficient use of resources can be obtained by increasing the number of particles (see Proposition 26 below).

Proposition 26. Suppose the assumptions of Proposition 24 hold, where moreover \( \rho \geq 0 \) may vary as a free parameter without changing the constant \( C > 0 \). Then, for all \( q > 2 \), \( \eta > 1 \) constants, the choice \( \rho = 2\alpha - \beta \) and probability
\[
p_\ell \propto 2^{-2\alpha\ell}(\log_2(\ell + 1))^{\eta}
\]
leads to \( \sigma^2 < \infty \), and (11) can be obtained with computational complexity
\[
O\left(\epsilon^{\gamma(\frac{1}{2} - \frac{1-\beta}{\alpha})} |\log_2 \epsilon|^{\gamma(\frac{1}{2} - \frac{1-\beta}{2\alpha})}\right) \quad \text{as} \quad \epsilon \to 0.
\]

6. Numerical simulations

Now the theoretical results relating to the method herein introduced will be demonstrated on two examples. We will consider one example in the canonical regime, and one in the sub-canonical, both of which have likelihoods that can be computed exactly, so that the ground truth \( \pi(\infty)(f) \) can be easily calculated to arbitrary precision. We run each example with 100 independent replications, and calculate the MSE when the chain is at length \( m \) as
\[
\text{MSE}(m) = \frac{1}{100} \sum_{i=1}^{100} |E_m^{(i)}(f) - \pi(\infty)(f)|^2,
\]
which is depicted as the thick red line, average of the thin lines, in Figure 2 below. The error decays with the optimal rate of cost\(^{-1}\) and \( \log(\text{cost})\text{cost}^{-1} \) in the canonical and sub-canonical cases, respectively, where cost is the realised cost of the run, \( C(m) \) from Section 5, measured in seconds, with \( m \) iterations of the Markov chain.

6.1. Ornstein–Uhlenbeck process. Consider the Ornstein–Uhlenbeck (OU) process
\[
\begin{align*}
\frac{dX_t}{dt} &= -a X_t dt + b dW_t, \\
X_0 &= 0,
\end{align*}
\]
with initial condition \( x_0 = 0 \), model parameter \( \theta = (\theta_1, \theta_2) \sim \mathcal{N}(0, \sigma^2 I) \), and \( a := a_\theta = \exp(\theta_1) \) and \( b := b_\theta = \exp(\theta_2) \). The process is discretely observed for \( k = 1, \ldots, n \),
\[
Y_k = X_k + \xi_k,
\]
where \( \xi_k \sim \mathcal{N}(0, \gamma^2) \) i.i.d. Therefore,
\[
G_k(x) = \exp\left(-\frac{1}{2\gamma^2} |x - y_k|^2\right).
\]
The marginal likelihood is given by
\[
\mathbb{P}[y_{1:n}\mid \theta] = \prod_{k=1}^{n} \mathbb{P}[y_k\mid y_{1:k-1}, \theta],
\]
and each factor can be computed as the marginal of the joint on the prediction and current observation, i.e.
\[
P[y_k\mid y_{1:k-1}, \theta] = \int_{\mathbb{R}} \mathbb{P}[y_k\mid x_k, \theta] \mathbb{P}[x_k\mid y_{1:k-1}, \theta] dx_k.
\]

In this example the ground truth can be computed exactly via the Kalman filter. In particular, the solution of (12) is given by
\[
X_1 = e^{-a}X_0 + W_1, \quad W_1 \sim \mathcal{N} \left(0, \frac{b^2}{2a} \left(1 - e^{-2a}\right)\right).
\]
The filter at time \(k\) is given by the following simple recursion
\[
m_k = c_k \left(\frac{y_k}{\gamma^2} + \frac{\hat{m}_k}{\hat{c}_k}\right), \quad c_k = (\gamma^{-2} + \hat{c}_k^{-1})^{-1}, \quad \hat{m}_k = e^{-a}m_{k-1}, \quad \hat{c}_k = e^{-2a}c_{k-1} + \frac{b^2}{2a} (1 - e^{-2a}).
\]

Additionally, the incremental marginal likelihoods (14) can be computed exactly
\[
\mathbb{P}[y_k\mid y_{1:k-1}, \theta] = \sqrt{\frac{c_k}{2\pi \hat{c}_k \gamma^2}} \exp \left\{-\frac{1}{2} \left[\frac{y_k^2}{\gamma^2} + \hat{m}_k^2 \frac{\gamma^2}{\hat{c}_k} - c_k \left(\frac{y_k}{\gamma^2} + \frac{\hat{m}_k}{\hat{c}_k}\right)^2 \right]\right\}.
\]

The parameters are chosen as \(\gamma = 1, \sigma^2 = 0.1, n = 5\), and the data is generated with \(\theta = (0,0)^T\). Our aim is to compute \(\mathbb{E}(\theta\mid y_{1:n})\) (or \(\mathbb{E}[(a, b)^T\mid y_{1:n}]\), etc., but we will content ourselves with the former). This is done via a brute force random walk MCMC for \(m = 10^8\) steps using the exact likelihood \(\mathbb{P}[y_{1:n}\mid \theta]\) as above. The IACT is around 10, so this gives a healthy limit for MSE computations.

For the numerical experiment, we use Euler-Maruyama method at resolution \(h_\ell = 2^{-\ell}\) to solve (12) as follows
\[
X_{k+1} = (1 - a h_\ell) X_k + b B_{k+1}, \quad B_{k+1} \sim \mathcal{N}(0, h_\ell) \text{ i.i.d.}
\]
for \(k = 1, \ldots, K_\ell, h_\ell = h^{-1}\). Levels \(\ell\) and \(\ell - 1\) are coupled in the simulation of \(\Delta_\ell\) by defining \(B_{1:K_{\ell/2}}^C = B_{1:K_{\ell-1}}^F + B_{2:K_{\ell}}^F\). Algorithm 2 is then run using the standard bootstrap particle filter (Algorithm 1) with \(N = 20\) particles and \(O(N)\)-complexity multinomial resampling [cf. 5]. Theorem 8 provides a rate of \(\beta = 2\) for Algorithm 2, because the diffusion coefficient is constant, which implies we are essentially running a Milstein scheme (cf. (8) and [22]).

Recommendation 1 (or Proposition 20) of Section 5 suggests arbitrary precision can be obtained by Algorithm 4 with \(p_\ell \propto \gamma^{-d/2}\) and no scaling of particle numbers based on \(\ell\) in this canonical \(\beta = 2\) regime (with weak rate \(\alpha = 1\)). We choose a positive PMMH algorithm constant \(\epsilon = 10^{-6}\) (cf. Remark 10(i)). We run Algorithm 4 for \(10^4\) steps, with 100 replications. The results are presented in Figure 2, where it is clear that the theory holds and the MSE decays according to 1/cost. The variance of the run-times is very small over replications.

6.2. Geometric Brownian motion. We next consider the following stochastic differential equation
\[
dX_t = aX_t dW_t,
\]
with initial condition \(x_0 = 1\), and \(a := a_\theta = \exp(\theta)\) with \(\theta \sim \mathcal{N}(0, \sigma^2)\). This equation is
analytically tractable as well, and the solution of the transformed equation \( Z = \log X \) is given via Itô’s formula by

\[
dZ_t = -\frac{a^2}{2} \, dt + a \, dW_t.
\]

Defining \( W_k \sim \mathcal{N}(0, 1) \) i.i.d., one has that

\[
Z_{k+1} = Z_k + \frac{a^2}{2} + a \, W_k, \quad \text{with} \quad z_0 = \log x_0 = 0,
\]

and the solution of (16) can be obtained via exponentiation: \( X_k = e^{Z_k} \). Moreover, noisy observations are introduced on the form

\[
Y_k = \log(X_k) + \xi_k = Z_k + \xi_k,
\]

where \( \xi_k \sim \mathcal{N}(0, \gamma^2) \) i.i.d. as above. Therefore we have

\[
G_k(z) = \exp\left(-\frac{1}{2\gamma^2} |z - y_k|^2\right).
\]

Again \( \mathbb{P}[y_{1:n} | \theta] \) can be computed analytically by integrating over \((z_1, \ldots, z_n)\).

In order to investigate the theoretical sub-canonical rate, we return to (16) and approximate this directly using Euler-Maruyama method (15), which introduces artificial approximation error. This problem suffers from stability problems when \( X < 0 \), so we take \( h_\ell = 2^{-5-\ell} \). Algorithm 1 is then used along with the selection functions

\[
G_k(x) = \exp\left(-\frac{1}{2\gamma^2} |\log(x) - y_k|^2\right).
\]

Here the diffusion coefficient is not constant, and Euler-Maruyama method yields a rate of \( \beta = 1 = \alpha \), the borderline case, which is expected to give a logarithmic penalty. Based on Recommendation 1 (or Proposition 26) of Section 5, we consider scaling the particles as \( 2^\rho \) with \( \rho = 2\alpha - \beta = 1 \) and \( \rho = 0 \), with \( p_\ell \propto 2^{-2\ell} \log(\ell)^2 \) in both cases. Again we let \( \epsilon = 10^{-6} \). Again the standard bootstrap particle filter is used, with \( N = 20 \times 2^\rho \) particles. Algorithm 4 is run for \( 10^5 \) steps, with 100 replications. The results are presented in Figure 2, and they show good agreement with the theory, in terms of rate. On the other hand, the cost for \( \rho = 0 \) is apparently smaller than that of \( \rho = 1 \) by a factor of approximately 100.
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APPENDIX A. ANALYSIS OF THE DELTA PARTICLE FILTER

We now give our analysis that is required for the proofs of Theorem 8 and Corollary 9 of Section 3 regarding the ∆PF (Algorithm 2) for HMM diffusions. The structure of the appendix is as follows. In Section A.1 we introduce some more Feynman–Kac notations, following [7, 18], emphasising that here we consider standard HMMs that can be coupled. In Section A.2 we recall the ∆PF stated earlier. A general variance bound for quantities such as ∆_t(ϕ) is given in Section A.3. This is particularised to the HMM diffusion case in Section A.4, where we supply the proofs for the results of Section 3.

A.1. Models. Let (X, ℳ) be a measurable space and \{G_n\}_{n≥0} a sequence of non-negative, bounded and measurable functions such that \(G_n : X → \mathbb{R}_+\). Let \(η_0^F, η_0^C ∈ ℳ(X)\) and \(\{M_n^F\}_{n≥1}, \{M_n^C\}_{n≥1}\) be two sequences of Markov kernels, i.e. \(M_n^F : X → ℳ(X), M_n^C : X → ℳ(X)\). Set \(E_n := X^{n+1}\) for \(n ≥ 0\), and for \(x_{0:n} ∈ E_n\),

\[G_n(x_{0:n}) = G_n(x_n)\]

and for \(n ≥ 1, s ∈ \{F, C\}, x_{0:n-1} ∈ E_{n-1}\)

\[M_n^s(x_{0:n-1}, dx_{0:n}) = δ_{x_{0:n-1}}(dx_{0:n-1})M_n^s(x_{n-1}, dx_n).\]

Define for \(s ∈ \{F, C\}, ϕ ∈ B_b(E_n), u_n ∈ E_n\)

\[γ_n^s(ϕ) = \int_{E_0×⋯×E_n} ϕ(u_n) \left(\prod_{p=0}^{n-1} G_p^s(u_p)\right) η_0^p(du_0) \prod_{p=1}^n M_p^s(u_{p-1}, du_p)\]

and

\[η_n^s(ϕ) = γ_n^s(ϕ) γ_n^s(1).\]

Throughout this appendix, we assume (D), and that Assumption 2(i) holds, i.e. there exists \(η_b ∈ ℳ(X × X)\) such that for any \(A ∈ ℳ\)

\[η_0^F(A × X) = η_0^C(A × X) = η_0^C(A)\]

and moreover for any \(n ≥ 1\) there exists Markov kernels \(\{\tilde{M}_n\}\), \(\tilde{M}_n : X × X → ℳ(X × X)\) such that for any \(A ∈ ℳ, (x, x') ∈ X × X\):

\[\tilde{M}_n(A × X)(x, x') = M_n^F(A)(x) \tilde{M}_n(X × A)(x, x') = M_n^C(A)(x').\]

A.2. Delta particle filter. Define \(x_p = (x_p^F, x_p^C) ∈ X × X\) and

\[\hat{G}_p(x_p) = \frac{1}{2}(G_p(x_p^F) + G_p(x_p^C)),\]

as in Assumption 2(ii). Set, for \(n ≥ 0, x_{0:n} ∈ X^{2(n+1)}\)

\[\hat{G}_n(x_{0:n}) = \hat{G}_n(x_n)\]

and for \(n ≥ 1, x_{0:n-1} ∈ X^{2n}\)

\[\tilde{M}_n(x_{0:n-1}, dx_{0:n}) = δ_{x_{0:n-1}}(dx_{0:n-1})\tilde{M}_n(x_{n-1}, dx_n),\]
Note that coupling assumption (17) for \( \hat{M}_n \) can be equivalently formulated for \( \hat{M}_n \).

For \( n \geq 0, \varphi \in \mathcal{B}_b(E_n \times E_n), u_n \in E_n \times E_n, \) we have

\[
\hat{\gamma}_n(\varphi) = \int \varphi(u_n) \left( \prod_{p=0}^{n-1} \hat{G}_p(u_p) \right) \hat{\eta}_0(du_0) \prod_{p=1}^n \hat{M}_p(u_{p-1}, du_p)
\]

and

\[
\hat{\eta}_n(\varphi) = \frac{\hat{\gamma}_n(\varphi)}{\hat{\gamma}_n(1)}.
\]

As noted in [19] it is simple to establish that for \( \varphi \in \mathcal{B}_b(E_n), \) if

\[
\psi(x_{0:n}) = \mathcal{G}_n(x_{0:n}) \left( \varphi(x_{0:n}^F) \prod_{p=0}^n \mathcal{G}_p(x_{0:p}^F) - \varphi(x_{0:n}^C) \prod_{p=0}^n \mathcal{G}_p(x_{0:p}^C) \right)
\]

then

\[
\hat{\gamma}_n(\psi) = \hat{\gamma}_n(1) \hat{\eta}_n(\psi) = \gamma_n^F(G_n \varphi) - \gamma_n^C(G_n \varphi).
\]

Note

\[
\hat{\gamma}_n(1) = \prod_{p=0}^{n-1} \hat{\eta}_p(\hat{G}_p).
\]

In order to approximate \( \hat{\gamma}_n(\psi) \) one can run the following abstract version of Algorithm 2 (recall from Section 3 that we will only consider multinomial resampling). Define for \( n \geq 1, \mu \in \mathcal{P}(E_{n-1} \times E_{n-1}), \varphi \in \mathcal{B}_b(E_n \times E_n) \)

\[
\hat{\phi}_n(\mu)(\varphi) = \frac{\mu(\hat{G}_{n-1} \hat{M}_n(\varphi))}{\mu(\hat{G}_{n-1})}.
\]

The algorithm begins by generating \( u_0^i \in E_0 \times E_0, i \in \{1, \ldots, N\} \) with joint law

\[
\prod_{i=1}^N \hat{\eta}_0(du_0^i) = \prod_{i=1}^N \hat{\eta}_0(du_0^i).
\]

Defining

\[
\hat{\eta}_0^N(du_0) = \frac{1}{N} \sum_{i=1}^N \delta_{u_0^i}(du_0)
\]

we then generate \( u_1^i \in E_1 \times E_1, i \in \{1, \ldots, N\} \) with joint law

\[
\prod_{i=1}^N \hat{\phi}_1(\hat{\eta}_0^N)(du_1^i).
\]

This proceeds recursively, so the joint law of the particles up to time \( n \) is

\[
\left( \prod_{i=1}^N \hat{\eta}_0(du_0^i) \right) \left( \prod_{p=1}^n \prod_{i=1}^N \hat{\phi}_p(\hat{\eta}_0^N)(du_p^i) \right).
\]

Hence we have the estimate

\[
\hat{\gamma}_n^N(\psi) = \left( \prod_{p=0}^{n-1} \hat{\eta}_p^N(\hat{G}_p) \right) \hat{\eta}_n^N(\psi).
\]

Remark 27. Note that \( \hat{\gamma}_n^N(\psi) \) corresponds to the quantity \( \Delta_f(\varphi) \) in (6) from the \( \DeltaPF \) output (Algorithm 2).
A.3. General hidden Markov model case. Define for \( p \geq 1 \) the semigroup
\[
\mathcal{Q}_p(x_{0:p-1}, dx'_{0:p}) = \mathcal{G}_{p-1}(x_{0:p-1}) \mathcal{M}_p(x_{0:p-1}, dx'_{0:p})
\]
with the definition for \( 0 \leq p \leq n \), \( \varphi \in \mathcal{B}_b(E_n \times E_n) \)
\[
\mathcal{Q}_{p,n}(\varphi)(u_p) = \int \varphi(u_n) \prod_{j=p+1}^n \mathcal{Q}_j(u_{j-1}, du_j)
\]
if \( p = n \) clearly \( \mathcal{Q}_{n,n} \) is the identity operator. For any \( 0 \leq n \), \( \varphi \in \mathcal{B}_b(E_n \times E_n) \) we set \( \mathcal{Q}_{-1,n}(\varphi)(u_{-1}) = 0 \).

Now following [7, Chapter 7] we have the following martingale (w.r.t. the natural filtration of the particle system), \( \varphi \in \mathcal{B}_b(E_n \times E_n) \):
\[
\tilde{\gamma}_n^N(\varphi) - \hat{\gamma}_n(\varphi) = \sum_{p=0}^n \tilde{\gamma}_p^N(1) [\hat{\eta}_p^N - \hat{\phi}_p(\hat{\eta}_{p-1}^N)](\mathcal{Q}_{p,n}(\varphi)) 
\]
with the convention that \( \hat{\phi}_p(\hat{\eta}_{p-1}^N) = \hat{\eta}_0 \) if \( p = 0 \). The representation immediately establishes that
\[
\mathbb{E}[\tilde{\gamma}_n^N(\varphi)] = \hat{\gamma}_n(\varphi)
\]
where the expectation is w.r.t. the law associated to the particle system. We will use the following convention that \( C' \) is a finite positive constant that does not depend upon \( n, N \) or any of the \( G_n, M_n^s (s \in \{F, C\}, \), \( M_n \). The value of \( C' \) may change from line-to-line. Define for \( 0 \leq p \leq n < \infty \)
\[
\overline{G}_{p,n} = \prod_{q=p}^n \|G_q\|
\]
with the convention that if \( p = 0 \) we write \( \overline{G}_n \). We have the following result.

**Proposition 28.** Suppose that \( \|G_n\| < \infty \) for each \( n \geq 0 \). Then there exist a \( C' < \infty \) such that for any \( n \geq 0 \), \( \varphi \in \mathcal{B}_b(E_n \times E_n) \)
\[
\mathbb{E} \left[ \left( \tilde{\gamma}_n^N(\varphi) - \hat{\gamma}_n(\varphi) \right)^2 \right] \leq \frac{C' n^2}{N^2} \sum_{p=0}^n \overline{G}_{p-1}^2 \mathbb{E}[\mathcal{Q}_{p,n}(\varphi)(u_p^1)^2].
\]

**Proof.** Set
\[
\tilde{S}_{p,n}^N(\varphi) = \tilde{\gamma}_p^N(1) [\hat{\eta}_p^N - \hat{\phi}_p(\hat{\eta}_{p-1}^N)](\mathcal{Q}_{p,n}(\varphi))
\]
By (20), one can apply the Burkholder-Gundy-Davis inequality to obtain
\[
\mathbb{E} \left[ \left( \tilde{\gamma}_n^N(\varphi) - \hat{\gamma}_n(\varphi) \right)^2 \right] \leq C' \sum_{p=0}^n \mathbb{E}[\tilde{S}_{p,n}^N(\varphi)^2].
\]
Now, we have that
\[
\mathbb{E}[\tilde{S}_{p,n}^N(\varphi)^2] \leq C_{p-1}^2 \mathbb{E}[\hat{\eta}_p^N - \hat{\phi}_p(\hat{\eta}_{p-1}^N)](\mathcal{Q}_{p,n}(\varphi))^2.
\]
Application of the (conditional) Marcinkiewicz-Zygmund inequality yields
\[
\mathbb{E}[\tilde{S}_{p,n}^N(\varphi)^2] \leq \frac{C_n C_{p-1}^2}{N} \mathbb{E} \left[ \left( \mathcal{Q}_{p,n}(\varphi)(u_p^1) - \hat{\phi}_p(\hat{\eta}_{p-1}^N)(\mathcal{Q}_{p,n}(\varphi)) \right)^2 \right].
\]
After applying \( C_2 \) and Jensen inequalities, we then conclude by (21).
A.4. Diffusion case. We now consider the model of Section 3, where we recall that \( \theta \) is omitted from the notation. A series of technical results are given and the proofs for Theorem 8 and Corollary 9 are given at the end of this section.

We recall that the joint probability density of the observations and the unobserved diffusion at the observation times is given by

\[
\prod_{p=0}^{n} G_p(x_p)Q^{(\infty)}(x_{p-1}, x_p).
\]

As the true dynamics can not be simulated, in practice we work with

\[
\prod_{p=0}^{n} G_p(x_p)Q^{(\ell)}(x_{p-1}, x_p).
\]

Recall an (Euler) approximation scheme with discretisation \( h_\ell = 2^{-\ell}, \ell \geq 0 \). In our context then, \( M_n^F \) corresponds \( Q^{(\ell)} (\ell \geq 1) \) and \( M_n^C \) corresponds \( Q^{(\ell-1)} \). The initial distribution \( \eta_0 \) is simply the (Euler) kernel started at some given \( x_0 \). As noted earlier in Remark 4(i), a natural coupling of \( M_n^F \) and \( M_n^C \) (and hence of \( \eta_0 \)) exists. As established in [18, eq. (32)] one has (uniformly in \( \theta \) as (D) holds with \( \theta \) independent constants) for \( C' < \infty \)

\[
\sup_{A} \sup_{x \in X} |M_n^F(\varphi)(x) - M_n^C(\varphi)(x)| \leq C'h_\ell
\]

where \( A = \{ \varphi \in B_b(X) \cap \text{Lip}(X): ||\varphi|| \leq 1 \} \). We also recall that (8) holds (recall (D) is assumed).

We will use \( M < \infty \) to denote a constant that may change from line-to-line. It will not depend upon \( \theta \) nor \( N, \ell \), but may depend on the time parameter or a function. The following result will be needed later on. The proof is given after the proof of Lemma 30 below.

**Proposition 29.** Assume (A1(i)-(ii),2). Then for any \( n \geq 0 \) and \( \varphi \in B_b(X^{n+1}) \cap \text{Lip}(X^{n+1}) \) there exists a \( M < \infty \) such that

\[
|\gamma_n^F(G_n\varphi) - \gamma_n^C(G_n\varphi)| \leq Mh_\ell
\]

We write expectations w.r.t. the time-inhomogeneous Markov chain associated to the sequence of kernels \( (M_p^F)_{p \geq 1} \) (resp. \( (M_p^C)_{p \geq 1} \)) as \( \mathbb{E}^F \), (resp. \( \mathbb{E}^C \)).

**Lemma 30.** Assume (A1(i)-(ii),2). Let \( s \in \{ F, C \} \) and \( \varphi \in B_b(X^{n+1}) \cap \text{Lip}(X^{n+1}) \), then, define the function for \( 0 \leq p \leq n \)

\[
\varphi_{s,p,n}(x_0:p) := \mathbb{E}^s[\varphi(x_0:p, X_{p+1:n}) \prod_{q=p+1}^{n} G_q(X_q)|x_p].
\]

Then we have that \( \varphi_{s,p,n} \in B_b(X^{p+1}) \cap \text{Lip}(X^{p+1}) \).

**Proof.** The case \( p = n \) follows immediately from \( \varphi \in B_b(X^{n+1}) \cap \text{Lip}(X^{n+1}) \). We will use a backward inductive argument on \( p \). Suppose \( p = n - 1 \) then we have for any \( (x_{0:n-1}, x'_{0:n-1}) \in X^n \times X^n \)

\[
|\varphi_{s,n-1,n}(x_{0:n-1}) - \varphi_{s,n-1,n}(x'_{0:n-1})| = \\
|\mathbb{E}^s[\varphi(x_{0:n-1}, X_n)G_n(X_n)|x_{n-1}] - \mathbb{E}^s[\varphi(x'_{0:n-1}, X_n)G_n(X_n)|x'_{n-1}]| \leq \\
|\mathbb{E}^s[\varphi(x_{0:n-1}, X_n)G_n(X_n)|x_{n-1}] - \mathbb{E}^s[\varphi(x'_{0:n-1}, X_n)G_n(X_n)|x_{n-1}]| + \\
|\mathbb{E}^s[\varphi(x'_{0:n-1}, X_n)G_n(X_n)|x_{n-1}] - \mathbb{E}^s[\varphi(x'_{0:n-1}, X_n)G_n(X_n)|x'_{n-1}]|
\]
By \( \varphi \in \text{Lip}(X^{n+1}) \) it easily follows via (A1(i)) that
\[
\left| E^* \left[ \varphi(x_{0:n-1}, X_n) G_n(X_n) | x_{n-1} \right] - E^* \left[ \varphi(x_{0:n-1}', X_n) G_n(X_n) | x_{n-1} \right] \right| \leq M \sum_{j=0}^{n-1} |x_j - x_j'|.
\]

By (A1(ii)) and \( \varphi \in \text{Lip}(X^{n+1}) \), \( \varphi(x_{0:n}) G_n(x_n) \) is Lipschitz in \( x_n \) and hence by (A2)
\[
| E^* \left[ \varphi(x_{0:n-1}, X_n) G_n(X_n) | x_{n-1} \right] - E^* \left[ \varphi(x_{0:n-1}', X_n) G_n(X_n) | x_{n-1}' \right] | \leq M | x_{n-1} - x_{n-1}' |.
\]

Hence it follows
\[
| \varphi_{n-1,n}(x_{0:n-1}) - \varphi_{n-1,n}(x_{0:n-1}') | \leq M \sum_{j=0}^{n-1} |x_j - x_j'|.
\]

The induction step follows by almost the same argument as above and is hence omitted. \( \square \)

**Proof of Proposition 29.** We have the following standard collapsing sum representation:
\[
\gamma_n^F(G_n \varphi) - \gamma_n^C(G_n \varphi) = \sum_{p=0}^{n} \left( E^F \left[ \prod_{q=0}^{p} G_q(X_q) E^C[\varphi(x_{0:n}) \prod_{q=p+1}^{n} G_q(X_q) | X_p] \right] - E^F \left[ \prod_{q=0}^{p-1} G_q(X_q) E^C[\varphi(x_{0:n}) \prod_{q=p}^{n} G_q(X_q) | X_{p-1}] \right] \right)
\]

The summand is
\[
T_p := E^F \left[ \left( \prod_{q=0}^{p-1} G_q(X_q) \right) \left( E^F - E^C \right) \left( E^C[\varphi(x_{0:n}) \prod_{q=p+1}^{n} G_q(X_q) | X_p] G_p(X_p) | X_{p-1} \right) \right].
\]

By Lemma 30, \( E^C[\varphi(x_{0:p}, X_{p+1:n}) \prod_{q=p+1}^{n} G_q(X_q) | x_p] \in B_0(X^{p+1}) \cap \text{Lip}(X^{p+1}) \) and by (A1) (i) and (ii) \( G_p \in B_0(X) \cap \text{Lip}(X) \). So by (22)
\[
\left| (E^F - E^C) \left( E^C[\varphi(x_{0:n}) \prod_{q=p+1}^{n} G_q(X_q) | X_p] G_p(X_p) | X_{p-1} \right) \right| \leq M h_\ell \sup_{x_{0:p} \in X^{p+1}} \left| E^C[\varphi(x_{0:p}, X_{p+1:n}) \prod_{q=p+1}^{n} G_q(X_q) | x_p] G_p(x_p) \right|
\]

and hence
\[
| T_p | \leq M h_\ell E^F \left[ \prod_{q=0}^{p-1} G_q(X_q) \right] \sup_{x_{0:p} \in X^{p+1}} \left| E^C[\varphi(x_{0:p}, X_{p+1:n}) \prod_{q=p+1}^{n} G_q(X_q) | x_p] G_p(x_p) \right|.
\]

Application of (A1) (i) gives \( | T_p | \leq M h_\ell \) and the proof is hence concluded. \( \square \)

**Lemma 31.** Assume (A1). Then for any \( n \geq 0 \) there exists a \( M < \infty \) such that for any \( x_{0:n} \in X^{2(n+1)} \)
\[
\left| \prod_{p=0}^{n} \frac{G_p(x_p^F)}{G_p(x_p)} - \prod_{p=0}^{n} \frac{G_p(x_p^C)}{G_p(x_p)} \right| \leq M \sum_{p=0}^{n} | x_p^F - x_p^C |.
\]

**Proof.** The proof is by induction. The case \( n = 0 \):
\[
\left| \frac{G_0(x_0^F)}{G_0(x_0)} - \frac{G_0(x_0^C)}{G_0(x_0)} \right| = \frac{1}{G_0(x_0)} | G_0(x_0^F) - G_0(x_0^C) |.
\]
Application of (A1) (ii) and (iii) yield that
\[
\left| \frac{G_0(x^F_0)}{G_0(x_0)} - \frac{G_0(x^C_0)}{G_0(x_0)} \right| \leq M|x^F_0 - x^C_0|.
\]

The result is assumed to hold at rank \( n - 1 \), then
\[
\left| \prod_{p=0}^{n} \frac{G_p(x^F_p)}{G_p(x_p)} - \prod_{p=0}^{n} \frac{G_p(x^C_p)}{G_p(x_p)} \right| \leq
\]
\[
\left| \frac{G_n(x^F_n)}{G_n(x_n)} - \frac{G_n(x^C_n)}{G_n(x_n)} \right| \cdot \left| \prod_{p=0}^{n-1} \frac{G_p(x^F_p)}{G_p(x_p)} + \prod_{p=0}^{n-1} \frac{G_p(x^C_p)}{G_p(x_p)} \right| \cdot \frac{G_n(x^C_n)}{G_n(x_n)}.
\]

For the first term of the R.H.S. one can follow the argument at the initialisation and apply (A1) (i) and (iii). For the second term of the R.H.S., the induction hypothesis and (A1) (i) and (iii) can be used. That is one can deduce that
\[
\left| \prod_{p=0}^{n} \frac{G_p(x^F_p)}{G_p(x_p)} - \prod_{p=0}^{n} \frac{G_p(x^C_p)}{G_p(x_p)} \right| \leq M \sum_{p=0}^{n} |x^F_p - x^C_p|.
\]

Recall (18) for the definition of \( \psi \) and that \( x_p = (x^F_p, x^C_p) \in \mathbb{X} \times \mathbb{X} \).

**Lemma 32.** Assume (A1-2). Then for any \( 0 \leq p < n \), \( \varphi \in \mathcal{B}_b(\mathbb{X}^{n+1}) \cap \text{Lip}(\mathbb{X}^{n+1}) \) there exists a \( M < \infty \) such that for any \( x_{0:p} \in \mathbb{E}_p \times \mathbb{E}_p \)
\[
|\tilde{Q}_{p,n}(\psi)(x_{0:p})| \leq M \left( \sum_{j=0}^{p} |x^F_j - x^C_j| + h_t \right)
\]

**Proof.** We have
\[
\tilde{Q}_{p,n}(\psi)(x_{0:p}) = \tilde{G}_p(x_p) \times \left( \prod_{q=0}^{p} \frac{G_q(x^F_q)}{G_q(x_q)} \right) \mathbb{E}^F[\varphi(x_{0:p}, Y_{p+1:n}) \prod_{s=p+1}^{n} G_s(X^F_s)|x_p]
\]
\[
- \prod_{q=0}^{p} \frac{G_q(x^C_q)}{G_q(x_q)} \mathbb{E}^C[\varphi(x_{0:p}, Y_{p+1:n}) \prod_{s=p+1}^{n} G_s(X^C_s)|x_p]]
\]

It then follows that \( \tilde{Q}_{p,n}(\psi)(x_{0:p}) = \tilde{G}_p(x_p)(T_1 + T_2) \) where
\[
T_1 = \left( \prod_{q=0}^{p} \frac{G_q(x^F_q)}{G_q(x_q)} - \prod_{q=0}^{p} \frac{G_q(x^C_q)}{G_q(x_q)} \right) \mathbb{E}^F[\varphi(x_{0:p}, Y_{p+1:n}) \prod_{s=p+1}^{n} G_s(X^F_s)|x_p]
\]
\[
T_2 = \prod_{q=0}^{p} \frac{G_q(x^C_q)}{G_q(x_q)} \left( \mathbb{E}^F[\varphi(x_{0:p}, Y_{p+1:n}) \prod_{s=p+1}^{n} G_s(X^F_s)|x_p] - \mathbb{E}^C[\varphi(x_{0:p}, Y_{p+1:n}) \prod_{s=p+1}^{n} G_s(X^C_s)|x_p] \right).
\]

By Lemma 31, \( \varphi \in \mathcal{B}_b(\mathbb{X}^{n+1}) \cap \text{Lip}(\mathbb{X}^{n+1}) \) and (A1) (i)
\[
|T_1| \leq M \sum_{j=0}^{p} |x^F_j - x^C_j|.
\]
Now $T_2 = T_3 + T_4$ where

$$T_3 = \prod_{q=0}^{p} \frac{G_q(x_q^C)}{G_q(x_q)} \left( \mathbb{E}^F[\varphi(x_{0:p}, Y_{p+1:n})] \prod_{q=p+1}^{n} G_s(X_s^F)|x_p^F] - \mathbb{E}^F[\varphi(x_{0:p}, Y_{p+1:n})] \prod_{s=p+1}^{n} G_s(X_s^F)|x_p^F] \right)$$

$$T_4 = \prod_{q=0}^{p} \frac{G_q(x_q^C)}{G_q(x_q)} \left( \mathbb{E}^F[\varphi(x_{0:p}, Y_{p+1:n})] \prod_{s=p+1}^{n} G_s(X_s^F)|x_p^F] - \mathbb{E}^C[\varphi(x_{0:p}, Y_{p+1:n})] \prod_{s=p+1}^{n} G_s(X_s^C)|x_p^F] \right).$$

For $T_3$ one can use Lemma 30 (along with (A1) (i) and (iii)) to get that

$$|T_3| \leq M \sum_{j=0}^{p} |x_j^F - x_j^C|.$$ 

For $T_4$ a similar collapsing sum argument that is used in the proof of Proposition 29 can be used to deduce that

$$|T_4| \leq Mh_{\ell}.$$ 

One can then conclude the proof via the above bounds (along with (A1) (i)).

Below $\mathbb{E}$ denotes expectation w.r.t. the particle system described in Section A.2 started at position $(x, x)$ at time $n = 0$ with $x \in \mathbb{X}$, in the diffusion case of Section A.4. Recall the particle $U_n^s \in \mathbb{E}_n \times \mathbb{E}_n$ at time $n \geq 0$ in path space. We denote by $U_n^s(j) \in \mathbb{X}$ as the $j \in \{0, \ldots, n\}$ component of particle $i \in \{1, \ldots, N\}$ at time $n \geq 0$ of $s \in \{F, C\}$ component. Recall $(U_{n}^{i,F}(n), U_{n}^{i,C}(n))$ for $n \geq 1$ is sampled from the kernel $M_{\ell}((\bar{u}_{n-1}^{i,F}(n-1), \bar{u}_{n-1}^{i,C}(n-1)), \cdot)$ where the $\bar{u}$ denotes post-resampling and the component $(U_{n}^{i,F}(j), U_{n}^{i,C}(j)) = (\bar{u}_{n-1}^{i,F}(j), \bar{u}_{n-1}^{i,C}(j))$ for $j \in \{0, \ldots, n-1\}$ is kept the same for the earlier components of the particle.

**Lemma 33.** Assume (A1 (i) (iii), 2). Then for any $n \geq 0$ there exists a $M < \infty$ such that

$$\mathbb{E} \left[ \sum_{j=0}^{n} |U_{n}^{1,F}(j) - U_{n}^{1,C}(j)|^2 \right] \leq M h_{\ell}^\beta.$$

where $\beta$ is as in (8).

**Proof.** Our proof is by induction, the case $n = 0$ following by (8). Assuming the result at $n - 1$ we have

$$\mathbb{E} \left[ \sum_{j=0}^{n} |U_{n}^{1,F}(j) - U_{n}^{1,C}(j)|^2 \right] = \mathbb{E} \left[ \sum_{j=0}^{n-1} |\bar{U}_{n-1}^{1,F}(j) - \bar{U}_{n-1}^{1,C}(j)|^2 + |U_{n}^{1,F}(n) - U_{n}^{1,C}(n)|^2 \right].$$

Now

$$\mathbb{E} \left[ \sum_{j=0}^{n-1} |\bar{U}_{n-1}^{1,F}(j) - \bar{U}_{n-1}^{1,C}(j)|^2 \right] = N \sum_{j=0}^{n-1} \mathbb{E} \left[ \frac{G_{n-1}(U_{n-1}^{1,F}(n-1), U_{n-1}^{1,C}(n-1))}{\sum_{j=1}^{N} G_{n-1}(U_{n-1}^{1,F}(n-1), U_{n-1}^{1,C}(n-1))} \times |\bar{U}_{n-1}^{1,F}(j) - \bar{U}_{n-1}^{1,C}(j)|^2 \right] \leq M \mathbb{E} \left[ \sum_{j=0}^{n-1} |U_{n-1}^{1,F}(j) - U_{n-1}^{1,C}(j)|^2 \right]$$

where we have used (A1) (i) and (iii). Applying the induction hypothesis along with (8) yields

$$\mathbb{E} \left[ \sum_{j=0}^{n} |U_{n}^{1,F}(j) - U_{n}^{1,C}(j)|^2 \right] \leq M \left( h_{\ell}^\beta + \mathbb{E} [|\bar{U}_{n-1}^{1,F}(n-1) - \bar{U}_{n-1}^{1,C}(n-1)|^2] \right)$$
Now
\[ \mathbb{E}[\hat{U}_{n-1}^{1,F}(n - 1) - \bar{U}_{n-1}^{1,C}(n - 1)]^2 = \]
\[ N \mathbb{E}\left[ \frac{\hat{G}_{n-1}(U_{n-1}^{1,F}(n - 1), U_{n-1}^{1,C}(n - 1))}{\sum_{j=1}^{N} \hat{G}_{n-1}(U_{n-1}^{1,F}(n - 1), U_{n-1}^{1,C}(n - 1))} \right] \]
\[ \frac{1}{\sum_{j=1}^{N} \hat{G}_{n-1}(U_{n-1}^{1,F}(n - 1), U_{n-1}^{1,C}(n - 1))} |U_{n-1}^{1,F}(n - 1) - U_{n-1}^{1,C}(n - 1)|^2 \]
Then by (A1) (i) and (iii)
\[ \mathbb{E}\left[ \frac{\hat{G}_{n-1}(U_{n-1}^{1,F}(n - 1), U_{n-1}^{1,C}(n - 1))}{\sum_{j=1}^{N} \hat{G}_{n-1}(U_{n-1}^{1,F}(n - 1), U_{n-1}^{1,C}(n - 1))} \right] \leq \frac{M}{N} \mathbb{E}[|U_{n-1}^{1,F}(j) - U_{n-1}^{1,C}(j)|^2]. \]
Hence via the induction hypothesis, one has
\[ \mathbb{E}[|\hat{U}_{n-1}^{1,F}(n - 1) - \bar{U}_{n-1}^{1,C}(n - 1)|^2] \leq MH_{\ell}^2 \]
and the proof is concluded. \( \square \)

Recall Remark 27.

Proof of Theorem 8. This follows first by applying Proposition 28, followed by Lemma 32 and some standard calculations followed by Lemma 33. \( \square \)

Proof of Corollary 9. Easily follows by adding and subtracting \( \check{\gamma}_n(\psi) \) the \( C_2 \) inequality along with Theorem 8, and then using (19) combined with Proposition 29. \( \square \)

Appendix B. Proof of consistency of the Markov chain Monte Carlo

Proof of Theorem 11. Denote
\[ \xi_k(g) := \left( \sum_{i=1}^{N} V_k^{(i)} + \epsilon \right)^{-1} \left[ \sum_{i=1}^{N} V_k^{(i)} g(\Theta_k, X_k^{(i)}) + \tilde{\Delta}_k(g(\Theta_k)) \right], \]
where \( g(\theta)(x) := g(\theta, x) \) and \( \tilde{\Delta}_k(g(\theta)) := p_{L_k} \sum_{i=1}^{2N} V_k^{(i)} g(\Theta_k, X_k^{(i)}) \). Then \( E_{\text{iter},N,p}(f) = \sum_{k=1}^{\text{iter}} \xi_k(f) \). Furthermore, by Assumption 5 [cf. 26, 31], we have
\[ \mathbb{E}[\tilde{\Delta}_k^2(g) \mid \Theta_k = \theta] = s_g(\theta), \]
\[ \mathbb{E}[\tilde{\Delta}_k(g) \mid \Theta_k = \theta] = \gamma_n^{(0,\infty)}(G_n g) - \gamma_n^{(0,0)}(G_n g) \]
for \( g = 1 \) and \( g = f(\theta) \). This implies for \( g = f \) and \( g = 1 \),
\[ \mu_g(\theta, v^{(1:N)}, x^{(1:N)}) := \mathbb{E}[\xi_k(g) \mid (\Theta_k, V_k^{(1:N)}, X_k^{(1:N)}) = (\theta, v^{(1:N)}, x^{(1:N)})] \]
\[ = \frac{1}{\sum_{j=1}^{N} v^{(j)} + \epsilon} \left[ \sum_{i=1}^{N} v^{(i)} g(\theta, x^{(i)}) - \gamma_n^{(0,0)}(G_n g) + \gamma_n^{(0,\infty)}(G_n g) \right], \]
\[ m_g^{(1)}(\theta, v^{(1:N)}, x^{(1:N)}) := \mathbb{E}[|\xi_k(g)| \mid (\Theta_k, V_k^{(1:N)}, X_k^{(1:N)}) = (\theta, v^{(1:N)}, x^{(1:N)})] \]
\[ \leq \frac{1}{\sum_{j=1}^{N} v^{(j)} + \epsilon} \left[ \sum_{i=1}^{N} v^{(i)} |g(\theta, x^{(i)})| + \sqrt{s_g(\theta)} \right]. \]
It is direct to check that the PMMH type chain \((\Theta_k, X_k^{(1:N)}, V_k^{(1:N)})\) is reversible with respect to the probability
\[
\Pi(d\theta, dx^{(1:N)}, dv^{(1:N)}) = c_0 \text{pr}(\theta)d\theta F_0^{(0)}(dx^{(1:N)}, dv^{(1:N)}) \left( \sum_{i=1}^{N} v^{(i)} + \epsilon \right),
\]
where \(c_0 > 0\) is a normalisation constant and \(F_0^{(0)}(\cdot)\) stands for the law of the output of Algorithm 1 with \((M_{0:n}^{(0)}, G_{0:n}^{(0)}, N)\), and therefore is Harris recurrent as a full-dimensional Metropolis–Hastings that is \(\psi\)-irreducible [cf. 27, Theorem 8]. It is direct to check that \(\Pi(m^{(1)}_f) < \infty, \Pi(m^{(1)}_f) < \infty, \Pi(\mu_f) = c\pi^{(\infty)}(f)\) and \(\Pi(\mu_1) = c\), where \(c > 0\) is a constant, so the result follows from [32, Theorem 3].

### Appendix C. Proofs about asymptotic efficiency and allocations

**Proof of Proposition 16.** By Harris ergodicity, \(m^{-1}\mathcal{C}(m) \to \mathbb{E}[\tau]\) almost surely. Dividing the inequality
\[
\mathcal{C}(\mathcal{L}(\kappa)) \leq \kappa < \mathcal{C}(\mathcal{L}(\kappa) + 1)
\]
by \(\mathcal{L}(\kappa)\) and taking the limit \(\kappa \to \infty\), which implies \(\mathcal{L}(\kappa) \to \infty\), we get that \(\kappa/\mathcal{L}(\kappa) \to \mathbb{E}[\tau]\) almost surely. Also, by Proposition 13,
\[
\sqrt{\mathcal{L}(\kappa)} \left[ E_{\mathcal{L}(\kappa), N}(f) - \pi^{(\infty)}(f) \right] \xrightarrow{\kappa \to \infty} \mathcal{N}(0, \sigma^2),
\]
in distribution, so the result follows by Slutsky’s theorem.

**Proof of Proposition 20.** We have that
\[
\mathbb{E}[\mathcal{C}(m)] = \sum_{k=1}^{m} \mathbb{E}[\tau_{\Theta_k, L_k}] = \sum_{k=1}^{m} \sum_{\ell=1}^{\infty} \mathbb{E}[\tau_{\Theta_k, \ell}] p_\ell \leq Cm \sum_{k=1}^{\infty} p_\ell 2^{\gamma(1+\rho)},
\]
by Assumption 18(i), which is finite if \(r > \gamma(1+\rho)\). Also,
\[
s_\rho(\theta) = \mathbb{E}[(\Delta^2_k(g)|\Theta_k = \theta)] = \sum_{\ell \geq 1} \frac{\mathbb{E}[\Delta_k^2]}{p_\ell} \leq C \sum_{\ell \geq 1} \left( 2^{-\ell(\beta+\rho-r)} + 2^{-\ell(2\alpha-r)} \right),
\]
which is finite if \(r < \min(\beta+\rho, 2\alpha)\). Therefore, \(\sigma^2 < \infty\), and the CLT follows by Proposition 16.

**Lemma 34.** Let \(\{X_k\}_{k \geq 1}\) be a sequence of independent random variables with \(\mathbb{E}[X_{k_0}] = \infty\) for at least one \(k_0\), and let \(\{a_k\}_{k \geq 1}\) be a sequence of monotonically increasing real numbers with \(a_k/k \to \infty\). Suppose one of the following assumptions holds:

(i) \(\sum_{k \geq 1} \mathbb{P}[X_k > a_k] < \infty\), and \(\{X_k\}_{k \geq 1}\) are also identically distributed, or

(ii) \(\sum_{k \geq 1} \sup_{m \geq 1} \mathbb{P}[X_m > a_k] < \infty\).

Then
\[
\mathbb{P}\left[ \sum_{k=1}^{m} X_k > a_m \right. \text{ infinitely many } m \in \mathbb{N} \left. \right] = 0.
\]

**Proof.** (i) is [11, Theorem 2] since \(\mathbb{E}[X_{k_0}] = \infty\) implies \(\mathbb{E}[X_k] = \infty\) for all \(k \geq 1\) as \(\{X_k\}_{k \geq 1}\) are i.i.d. For (ii), note that if \(X_k\) has c.d.f. denoted \(F_k\), then it is straightforward to check that
\[
F^*(x) := \inf_{k \geq 1} F_k(x)
\]
is a c.d.f. also. With \(X_k^* \sim F^*\) i.i.d. for \(k \geq 1\), we have
\[
\mathbb{P}[X_k^* > a_k] = 1 - F^*(a_k) = \sup_{m \geq 1} 1 - F_m(a_k) = \sup_{m \geq 1} \mathbb{P}[X_m > a_k].
\]
Summing over $k \geq 1$, we obtain $\sum_{k \geq 1} \mathbb{P}[X_k^* > a_k] < \infty$. In addition, 
\[ \mathbb{E}[X_k^*] = \int \mathbb{P}[X_k^* > x] dx \geq \int \mathbb{P}[X_{k_0} > x] dx = \infty, \]
for all $k \geq 1$. Hence, we can apply (i) for i.i.d. random variables, obtaining
\[ 0 = \mathbb{P}[\sum_{k=1}^m X_k^* > a_m \text{ infinitely many } m] \geq \mathbb{P}[\sum_{k=1}^m X_k > a_m \text{ infinitely many } m], \]
where the first equality comes from (i), and so we conclude. \( \square \)

**Proof of Proposition 22.** Conditional on output $\{\Theta_k\}_{k \geq 1}$ of Algorithm 4, $\{\tau_{\Theta_k, L_k}\}_{k \geq 1}$ are independent random variables. Our assumptions imply Lemma 34(ii) holds, so
\[ \mathbb{P}[\mathcal{C}(m) > a_m \text{ infinitely many } m] = 0, \]
which means that $\mathcal{C}(m)$ is asymptotically bounded by $a_m$. Setting $m = O(\epsilon^{-2})$ allows us to conclude. \( \square \)

The proofs below of Proposition 24 and 26 are similar to that of [26, Proposition 4 and 5].

**Proof of Proposition 24.** With the prescribed choice of $p_\ell$ we have finite variance, as
\[ s_g(\theta) = \sum_{\ell \geq 1} \frac{\mathbb{E} \Delta_\ell^2}{p_\ell} \leq C \sum_{\ell \geq 1} \frac{1}{\ell [\log_2 (\ell + 1)]^\eta} < \infty, \]
uniformly in $\theta \in \Theta$. To determine the order of complexity, we would like to apply Lemma 34(i) to the i.i.d sequence $\{\tau^*_k\}_{k \geq 1}$, where $\tau^*_k := C 2^{\gamma \ell (1 + \rho)}$. For any $k \geq 1$, where $a_k > 0$ is some positive real number, we have,
\[ \mathbb{P}[\tau^*_k > a_k] = \sum_{\ell \geq 1} \mathbb{P}[\tau^*_k > a_k | p_\ell] = \sum_{\ell \geq 1} \mathbb{1} \left\{ \ell > \frac{1}{\gamma (1 + \rho)} \log_2 \frac{a_k}{C} \right\} p_\ell. \]
Because $\sum_{\ell \geq 1} p_\ell = 1$ and $p_\ell$ is monotonically decreasing, we have $\sum_{\ell \geq \ell_*} p_\ell$ is $O(p_{\ell_*})$. Setting $\ell_* = \lfloor \frac{1}{\gamma (1 + \rho)} \log_2 \frac{a_k}{C} \rfloor$, we therefore obtain that (26) is of order
\[ a_k^{-\frac{3b}{\gamma(1+\rho)}} \left( \log_2 a_k \right) \left( \log_2 \log_2 a_k \right)^\eta. \]
Setting
\[ a_k := [k (\log_2 k)^\eta]^{\frac{1}{\gamma(1+\rho)}} \]
then ensures that $\sum_{k \geq 1} \mathbb{P}[\tau^*_k > a_k] < \infty$. As $\beta \leq 1$, it is easy to check that $\mathbb{E}[\tau^*_k] = \infty$. We then apply Lemma 34(i), obtaining
\[ 0 = \mathbb{P}[\sum_{k=1}^m \tau^*_k > a_m \text{ infinitely many } m] \geq \mathbb{P}[\sum_{k=1}^m \tau_{\Theta_k, L_k} > a_m \text{ infinitely many } m], \]
and conclude as before, by using that $\mathcal{C}(m)$ is asymptotically bounded by $a_m$ and setting $m = O(\epsilon^{-2})$. \( \square \)

**Proof of Proposition 26.** We are in the basic setting of Proposition 24 as before, but additionally may choose $\rho \geq 0$ as we please. The growth of $a_k$ given in (27) is essentially determined by $\gamma (1 + \rho)/2b$, which can be made small when $\rho = 2\alpha - \beta$, implying $b = \alpha$. \( \square \)
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