Analysis for instability of parameter in quantile regression with Lagrange multiplier: Is the dependent and independent variable relationships have changed?

TJ Parmaningsih\(^1\), S Haryatmi \(^2\), and Danardono \(^2\)

\(^1\) Universitas Sebelas Maret, Jl. Ir. Sutami 36 A Surakarta 57126, Indonesia
\(^2\) Universitas Gadjah Mada, Bulaksumur, Yogyakarta 55281, Indonesia

E-mail: triwik_jt@yahoo.com

Abstract. Quantile regression is a statistical method to estimate the relationship between the dependent variable and independent variables on particular conditional quantile functions. Quantile regression method approach is done by separating the data into two or more groups which are suspected to have a different estimated value of a certain quantile. Test parameter instability and the structural break is a relevant issue. Some tests that have been carried out to verify the existence of structural break have largely focused only on conditional mean, while a structural break that occurs in the conditional distribution or in the conditional quantiles is an important key. This article aims to provide an insight into the test of a structural break in quantile regression with the Lagrange Multiplier. Explanation will be accompanied by the identification of structural break in the analysis of the relationship between household food expenditure and household income in the Belgian community household. The result shows that a structural break occurs on the 10\(^{th}\) and 25\(^{th}\) quantiles.

1. Introduction
Ordinary Least Square (OLS) is a classical method used to build the regression model. The OLS method required four assumptions: normality, constant of error variance, the absence of serial correlation in the residuals, and the multicollinearity. This method is known to be sensitive to violations of the assumptions, for example, if the data does not meet any of the assumptions then OLS estimator is no longer good to be used. One of the most important assumptions is a normal assumption. If the normality assumption has not been fulfilled, the usual solution is to transform the data, but sometimes the transformation that has been done has not been able to meet the assumptions so it gives a biased estimate. From that problem then develops the median regression with Last Absolute Deviation approach developed by replacing the mean approach in the OLS to the median. Sometimes the median approach is less price because it is only based on two groups of data are divided the middle value only. Whereas there is a gap that distribution of the data lies in the particular quantile pieces. Originated from that, quantile regression method developed.

Quantile regression is a statistical method used to estimate the relationship between dependent variable and independent variables on particular conditional quantile functions. Quantile regression
method approach is done by separating the data into two or more groups which are suspected have a different estimated value of a certain quantile. Quantile regression method does not require parametric assumptions [1]. This method can be used to measure the effect of independent variables not only in the center of the data distribution but also on the upper or lower tail of the distribution.

Some test has been carried out to verify the existence of a structural break on quantile regression, an i.e. test based on the estimated objective function and test based on the gradient. Test for instability of parameter and the structural break is a relevant issue. Statement on the null hypothesis, the regression parameter is constant, while on the alternative hypothesis, the regression parameter is allowed to change the response. A structural break is a pattern change that occurs in time series. Hansen [6] revealed that structural break is characterized by changes in the value of the parameter at a time and lead to forecasting the future less accurate.

However the term structural break is used in the realm of quantile regression [11], so that structural break are interpreted as changes in parameter values in certain quantiles or in some quantiles. In the regression model, let Y as the dependent variable and X is the independent variable, then an interesting thing concern is whether Y and X relationships changed or not. Quantile regression is able to answer that question. By paying attention to a certain quantile, we can give an explanation: “Whether the relationship at the 75th percentile changed?” In this case, the statistical test can be constructed to check the stability of the relationship Y and X at a particular quantile or in some quantiles. Quantile regression allows a testing structural break for different quantile. Furthermore, problems may arise that breaks in the distribution tail cannot be detected in the distribution center data or the median. The likelihood ratio test, \( C_1 \), is used for testing the instability of parameter in quantile regression [3]. This test is comparing the objective function on the bounded and unbounded model which is estimated on selected quantile. The bounded model assumes a constant coefficient, while the parameters in the unbounded model can change because of a structural break.

A different test for the instability of parameter, based on the gradient of quantile regression at the bounded model is provided by Qu [11]. This test is comparing the partial and total amount of the gradient. Structural break on the bounded model estimates that the average value before and after the break and the gradient in the subsample will be significantly different with the gradient that is computed on the whole sample. This test does not analyze each regression coefficient. However, this test took the biggest value of the different coefficient and different quantile.

Furno [4] provided an additional test for a structural break based on the Lagrange Multiplier (LM). LM is implemented by estimating an auxiliary regression to verify whether the residuals of the constant coefficient model can be further explained by the data. The LM is easier to implement when compared to the previous test. The components required in LM are an estimate of the constant coefficient model and an auxiliary regression. Ease of calculation is one of the advantages of the LM test. With these considerations, the test of the parameter instability in quantile regression with Lagrange Multiplier made the subject of this research.

2. Quantile Regression

2.1. Quantile regression model

The summary of the average of the distribution corresponding to x was provided by the regression [7]. Several distinct regression models can be performed at the kinds of percentage distribution points and thus get a more complete picture of the set. Ordinarily, this is not done, and so the regression often gives a rather incomplete picture. Quantile regression is proposed to provide an exhaustive picture of regression.

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method approach is done by separating the data into two or more groups which are suspected has a different estimated value of a certain quantile. This approach allows estimating conditional quantile functions on various desired quantile values. In general, quantile regression is very useful to analyze certain parts of a conditional distribution. For example, there is a policy that aims to help students who have low grades. In this case, attention must focus on the lower quantile so that the impact of the policy can be understood.

Let $Y$ is the dependent variable, $X$ is the independent variable and $F(y) = P(Y \leq y)$ is the distribution function of $Y$, then for $0 < \theta < 1$, the $\theta$th conditional quantile of $Y$ is given by

$$Q_X(\theta) = F^{-1}(\theta) = \inf \left\{ y : F_Y(y) \geq \theta \right\}$$

(2.1)

When evaluating the different quantile values, the function (2.1) gives an exhaustive picture of the effect of $X$ on $Y$. By considering a simple linear regression model

$$y_i = \beta_0 + x_i \beta_1 + \epsilon_i$$

then the quantile regression pioneered by Koenker and Basset [8], formulate a conditional quantile function of $y$ given $x$, $Q_{Y}(\theta|x)$, as the parameter’s linear function and is affected by $m$ structural break :

$$Q_{Y_i}(\theta|x_i) = \begin{cases} 
  x_i \beta_1(\theta), & i = 1, \ldots, n_1 \\
  x_i \beta_2(\theta), & i = n_1 + 1, \ldots, n_2 \\
  \vdots & \vdots \\
  x_i(\beta_{(m+1)}(\theta), & i = n_m + 1, \ldots, n.
\end{cases}$$

(2.2)

where $\theta \in (0,1)$ represent a quantile, $\beta_j(\theta)$ are unknown parameters with $j = 1, \ldots, m+1$ are the segment index, and $x_i(\beta_{(m+1)}(\theta)$ is the independent variable at the $j$th segment, where $i = n_{j-1} + 1, \ldots, n_j$ and $n_j (j = 1, \ldots, m)$ is the unknown breakpoint $n_j = \lambda n$, $0 < \lambda \leq 1$. The parameter vector $\beta(\theta)$ can be estimated by solving the problem of minimization :

$$\min_{b \in \mathbb{R}^n} \sum_{i=1}^{n} \rho_0(y_i - x_i \beta)$$

(2.3)

where $\rho_0(\epsilon_i)$ is the check function given by $\rho_0(\epsilon_i) = \begin{cases} 
  \theta \epsilon_i, & \text{if } \epsilon_i \geq 0 \\
  -(1-\theta) \epsilon_i, & \text{if } \epsilon_i < 0
\end{cases}$

so that

$$\rho_0(\epsilon_i) = \begin{cases} 
  \theta \epsilon_i, & \text{if } \epsilon_i \geq 0 \\
  -(1-\theta) \epsilon_i, & \text{if } \epsilon_i < 0
\end{cases}$$

where $\epsilon_i$ is the error of quantile regression. The value of $\beta$ in (2.3) replaced with the estimator $b$ then

$$\min_{b \in \mathbb{R}} \left[ \sum_{i \in \{i|y_i \geq x_i b\}} \theta(y_i - x_i b) + \sum_{i \in \{i|y_i < x_i b\}} (1-\theta)(y_i - x_i b) \right]$$

(2.4)

This is interesting: whether the $X$ and $Y$ relationships have changed? Quantile regression gives a way to answer it. To see this, let $\left( (x_i, y_i), i = 1, \ldots, n \right)$ represent a size $n$ of a sample and the $\theta$th conditional distribution of $y_i$ given $x_i$ by

$$Q_{Y_i}(\theta|x_i) = x_i \beta(\theta)$$

(2.5)

Then paired of the sample $\left( x_{ik}, y_{ik} \right)$ is different from that $\left( x_{il}, y_{il} \right)$ (hence structural break occurs) if

$$\beta_k(\theta) \neq \beta_l(\theta)$$

for some $\theta \in (0,1)$, where $k, l \in j, j = 1, 2, \ldots, m+1$. 


In other words, the structural break occurs when there is a difference in the parameter values of each segment or each subsample.

Koenker [7] expressed that the problem of finding the \( \theta \)th sample quantile, a problem that might seem inherently tied to the notion of an ordering of the sample observations, as the solution to a simple optimization problem. In effect, we have replaced sorting by optimizing. Quantile regression problem in (2.3) can be reformulated as a linear program:

\[
\min_{(\beta, u, v) \in \mathbb{R}^p \times \mathbb{R}^n_+} \{ \theta \mathbf{1}_n^T u + (1 - \theta) \mathbf{1}_n^T v | X\beta + u - v = y \} 
\]

(2.6)

where \( \mathbf{1}_n \) denote an \( n \) vector of 1, \( \{u_i, v_i : i=1, ..., n\} \) represent the positive and negative parts of the vector residuals \( y - X\beta \). The equation (2.6) is called the primal problem. The primal quantile regression problem has the corresponding dual problem:

\[
\max \left\{ y^T a | X^T a = (1 - \theta) X^T \mathbf{1}_n, a \in [0,1]^n \right\} 
\]

(2.7)

The primal problem (2.6) may be viewed as generating the sample quantiles, the corresponding dual problem may be seen to generate the order statistics, or perhaps more precisely the ranks of the observations [7]. The dual problem in (2.7) will have a solution as proposed by Gutenbrunner and Jurecková [5] as \( \hat{a}_i(\theta) \) was defined as

\[
\hat{a}_i(\theta) = \begin{cases} 
0, & \text{if } \theta > \frac{R_i}{n}; \\
\frac{R_i - 1}{n}, & \text{if } \frac{R_i - 1}{n} \leq \theta \leq \frac{R_i}{n}; \\
1, & \text{if } \theta < \frac{R_i - 1}{n}.
\end{cases} 
\]

(2.8)

where \( R_i \) is the rank of \( i^{th} \) observation, \( Y_i \) in the sample \( y_1, ..., y_n \).

2.2 The Hypotheses Testing of Structural Break

The statistical test can be constructed to check the stability of the equation (2.5) at a particular quantile or in some quantiles [11].

We are interested in testing two hypotheses under model (2.5). The first is concerned with testing for a parameter instability in a prespecified quantile with the null \( H_0 \) and alternative \( H_1 \) hypotheses are given by

\[
H_0 : \beta_i(\theta) = \beta_0(\theta)
\]

for all \( i \), for a given \( \theta \in (0,1) \)

\[
H_1 : \beta_i(\theta) = \begin{cases} 
\beta_1(\theta), & i = 1, 2, ..., n_i; \\
\beta_2(\theta), & i = n_i + 1, ..., n.
\end{cases}
\]

for a given \( \theta \in (0,1) \).

The second is concerned with testing for structural break across multiple quantiles, say quantiles contained in a set \( \mathcal{L} \), with the null \( H_0^* \) and alternative hypotheses \( H_1^* \) given by

\[
H_0^* : \beta_i(\theta) = \beta_0(\theta)
\]

for all \( I \) and for all \( \theta \in \mathcal{L} \).
\[
H_1^* : \beta_i(\theta) = \begin{cases} 
\beta_i(\theta), & i = 1,2,\ldots,n_i; \\
\beta_i(\theta), & i = n_i + 1,\ldots,n. 
\end{cases}
\]

for some \( \theta \in \mathcal{L} \).

3. Test for Instability of Parameter with Lagrange Multiplier

3.1 Structural Break Test

Test parameter instability and the structural break is a relevant issue. Hansen [6] said that structural break characterized by changes of parameter values at any given time. Structural break by [11] used in the studies of quantile regression, so that structural break is defined as the difference in the value of the parameter in certain quantile or in some quantile.

In the linear regression model with i.i.d. errors \( \varepsilon_i \), a test for instability of parameter takes into account, under the null, the bounded equation

\[
y_i = \alpha + \beta x_i + \varepsilon_i, \quad i = 1,2,\ldots,n \tag{3.1}
\]

where the coefficients remain constant in the whole sample. Under the alternative, the unbounded is given by

\[
y_i = \alpha + \beta x_i + \xi z_i + \varepsilon_i, \quad i = 1,2,\ldots,n \tag{3.2}
\]

where \( z_i = \delta_i x_i \) and \( \delta_i \) is a dummy variable having a unit value after the break. Thus \( \xi \) quantify the difference, increase or decrease, of the parameter regression after the break. To replace the independent variable \( z_i \), the sample is divided into two groups (before and after the break) and each of them is estimated as a regression model:

\[
\begin{align*}
y_i &= \alpha + \beta x_i + \varepsilon_i, \quad i = 1,2,\ldots,\lambda n \\
y_i &= \alpha + \beta x_i + \varepsilon_i, \quad i = \lambda n + 1,2,\ldots,n
\end{align*} \tag{3.3}
\]

where \( 0 < \lambda \leq 1 \).

The stability of parameter was assumed in the model (3.1) and the alternative permits the regression parameters to change after the break as in (3.2). It shows that the parameters estimates in the two groups differ from the parameters computed in the whole sample.

When considering quantile regressions, the objective function for the chosen quantile \( \theta \) is:

\[
V(\theta) = \sum_{y_i, \alpha + \beta x_i} (1 - \theta)(y_i - \alpha(\theta) - \beta(\theta)x_i) \\
+ \sum_{y_i, \alpha + \beta x_i} \theta(y_i - \alpha(\theta) + \beta(\theta)x_i) \\
= \sum_{i} \rho_{\theta}(y_i - \alpha(\theta) + \beta(\theta)x_i) \\
= \sum_{i} \rho_{\theta}(e_i)
\]

with \( \rho \) being the check function \( \rho_{\theta}(e_i) = e_i(\theta - 1(e_i < 0)) \) and \( e_i \) being the quantile regression residuals [8].

The test for an instability of parameter based on \( V(\theta) \) is given by [3]:

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\( H_1^* : \beta_i(\theta) = \begin{cases} 
\beta_i(\theta), & i = 1,2,\ldots,n_i; \\
\beta_i(\theta), & i = n_i + 1,\ldots,n. 
\end{cases} \)

for some \( \theta \in \mathcal{L} \).

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\begin{align*}
y_i &= \alpha_i + \beta x_i + \varepsilon_i, \quad i = 1,2,\ldots,\lambda n \\
y_i &= \alpha + \beta x_i + \varepsilon_i, \quad i = \lambda n + 1,2,\ldots,n
\end{align*} \tag{3.3}
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The stability of parameter was assumed in the model (3.1) and the alternative permits the regression parameters to change after the break as in (3.2). It shows that the parameters estimates in the two groups differ from the parameters computed in the whole sample.

When considering quantile regressions, the objective function for the chosen quantile \( \theta \) is:

\[
V(\theta) = \sum_{y_i, \alpha + \beta x_i} (1 - \theta)(y_i - \alpha(\theta) - \beta(\theta)x_i) \\
+ \sum_{y_i, \alpha + \beta x_i} \theta(y_i - \alpha(\theta) + \beta(\theta)x_i) \\
= \sum_{i} \rho_{\theta}(y_i - \alpha(\theta) + \beta(\theta)x_i) \\
= \sum_{i} \rho_{\theta}(e_i)
\]

with \( \rho \) being the check function \( \rho_{\theta}(e_i) = e_i(\theta - 1(e_i < 0)) \) and \( e_i \) being the quantile regression residuals [8].

The test for an instability of parameter based on \( V(\theta) \) is given by [3]:
\[ C^i = \left[ \tilde{V}(\theta) - \tilde{V}(\theta) \right] / d_i = \left( \frac{\tilde{V}(\theta)}{\tilde{V}(\theta)} - 1 \right) \frac{d_2}{d_1} \]  

(3.4)

The function of \( C^i \) has \( F_{d_i,d_2} \) distributed.

The objective function at the quantile \( \theta \) of the bounded model (3.1) is estimated by \( \tilde{V}(\theta) \) and the objective function at the \( \theta \) quantile of the unbounded model (3.2) is estimated by \( \tilde{V}(\theta) \). The gradient of the objective function’s quantile regression for the restricted model \( \tilde{V}(\theta) \) is the building block of the Qu \[11\] test:

\[ S_n(\theta) = n^{-\frac{1}{2}} \sum_{i=1}^{n} x_i \psi_\theta \left( y_i - a(\theta) - b(\theta) x_i \right) \]

\[ = n^{-\frac{1}{2}} \sum_{i=1}^{n} x_i \psi_\theta \left( \tilde{e}_i \right) \]  

(3.5)

where \( \psi_\theta(\tilde{e}_i) = \rho_\theta'(\tilde{e}_i) = \theta - 1(\tilde{e}_i < 0) \) is the sign function. Now consider to evaluate the gradient in a subset of size \( \lambda n \), \( S_n(\lambda, \theta) \) for \( 0 < \lambda \leq 1 \), and define a function of it, \( H_{\lambda,n}(\theta) \) as follows:

\[ S_n(\lambda, \theta) = n^{-\frac{1}{2}} \sum_{i=1}^{\lambda n} x_i \psi_\theta \left( y_i - a(\theta) - b(\theta) x_i \right) \]

\[ = n^{-\frac{1}{2}} \sum_{i=1}^{\lambda n} x_i \psi_\theta \left( \tilde{e}_i \right) \]

\[ H_{\lambda,n}(\theta) = \left( X^T X \right)^{-\frac{1}{2}} \sum_{i=1}^{\lambda n} x_i \psi_\theta \left( \tilde{e}_i \right) \]  

(3.6)

(3.7)

where \( X \) is the \( (n,p) \) matrix of independent variables. Qu [11] compares \( H_{\lambda,n}(\theta) \) and \( H_{1,n}(\theta) \) in the following test function:

\[ Q_\theta = \sup_\lambda \left\| \left( \theta (1-\theta) \right)^{-\frac{1}{2}} \left[ H_{\lambda,n}(\theta) - \lambda H_{1,n}(\theta) \right] \right\| \]  

(3.8)

3.2 The Lagrange Multiplier Test for Parameter Instability in Quantile Regression

Because only requires a bounded model estimation and an auxiliary regression that is convenient, then the LM is a fairly easy test in the calculation. On the quantile regression, the auxiliary or artificial regression is:

\[ \psi_\theta(\tilde{e}_i) = \tilde{\xi} \tilde{z}_i + \varepsilon_i \]  

(3.9)

where \( \psi_\theta(\tilde{e}_i) = \rho_\theta'(\tilde{e}_i) = \theta - 1(\tilde{e}_i \leq 0) \) is the sign function which is the first derivative of the \( \rho_\theta(\tilde{e}_i) \). The LM test function for parameter instability in quantile regression is represented as

\[ LM^1 = nR(\theta) \] , where \( R(\theta) \) represents the goodness of fit index of (3.9). The goodness of fit index on quantile regression is the comparison among the estimated objective function divided by estimates of the objective function coefficients of the regression model assumes constant in the whole sample. Koenker and Machado [9] explained that the goodness of index formula on quantile regression is
\[ R(\theta) = 1 - \frac{\bar{V}(\theta)}{V(\theta)} \quad (3.10) \]

In (3.4) assumed that the breakpoint is known. When this is not the case, a search on all the possible breakpoints \( \lambda \) has to be implemented as in (3.8).

4. The Application of Test for Instability of Parameter

Test for parameter instability in quantile regression with lagrange multiplier will apply to the data Engel food expenditure presented in Koenker [7]. The data consists of 235 observations on two variables to analyze the food expenditure and income relationships on the Belgian households. Estimation of a regression model with ordinary least square is given as follows:

Food expenditure = 147.475 + 0.485 income \quad (4.1)

The estimation quantile regression on \( \theta = (0.25; 0.50; 0.75) \) are given as follows:

Food expenditure = 95.484 + 0.474 income \quad (4.2)
Food expenditure = 81.482 + 0.560 income \quad (4.3)
Food expenditure = 62.397 + 0.644 income \quad (4.4)

It is possible to check the stability of the coefficient in (4.2), (4.3), and (4.4) to examine whether the food expenditure and income relationships have changed in the 25\(^{th}\), 50\(^{th}\), or 75\(^{th}\) quantile? This involves the implementation of the test Lagrange Multiplier.

Test for parameter instability in quantile regression allows for testing any desired quantile value. In this paper, will be tested on \( \theta = (0.1, 0.25, 0.5, 0.75, 0.9) \), so it can know whether the food expenditure and income relationships in the pre-specified quantile has changed or not. There, the quantiles 0.1 and 0.9 were used to examine the tails of the distribution, the quantile 0.5 was used to quantify the central tendency while 0.25 and 0.75 examined the intermediate cases. Test for parameter instability with Lagrange Multiplier requires a dummy variable having a unit value after the break, so it is necessary to know the location of the breakpoint in order to give value to a dummy variable. A searching process on all possible breakpoints \( \lambda \) has to be implemented as in (3.8). The candidate breakpoint estimates are the value of \( \lambda \) which is consistent with the observation that produces the biggest value of Qu test.

The process of structural break test performed on \( \theta = (0.1, 0.25, 0.5, 0.75, 0.9) \) begins by detecting whether a structural break occurs in the data or not. The detection is done by Qu test and the resulting of the test statistic compared to the critical value in Qu [11]. When \( H_0: \xi = 0 \) is rejected (there is a parameter instability in a particular quantile) then continued with track the breakpoint value that produces the largest value of Qu test. Next, the value of \( \lambda \) and the corresponding observation will be used for defining a dummy variable to the calculation of the Lagrange Multiplier test.

After getting the breakpoint estimation and the corresponding observation, followed by estimating quantile regression models on the interested quantile, \( \theta = (0.1, 0.25, 0.5, 0.75, 0.9) \). Quantile regression parameter estimation is done with R on the quantreg package.

The next step is checked whether there has been a parameter instability of each quantile \( \theta = (0.1, 0.25, 0.5, 0.75, 0.9) \) through the auxiliary regression estimation in (4.2), (4.3), and (4.4). Auxiliary regression is the regression between residual and the independent variables. From the Qu test result at \( \theta = 0.1 \) was concluded that structural break occurred so that a breakpoint was obtained on the 142\(^{nd}\) observation. From these result, it could be defined as a dummy variable for the calculation of
Lagrange multiplier. Note that the LM test formula, $LM^1 = nR(\theta)$, with $\hat{V}(\theta)$ and $\hat{V} (\theta)$ as a supporting component of the goodness of fit $R(\theta)$. The estimation of objective function at bounded model, $\hat{V}(\theta)$ obtained from quantile regression model estimates on $\theta = (0.1, 0.25, 0.5, 0.75, 0.9)$, while the objective function estimation on unbounded model $\check{V} (\theta)$, obtained from the estimated regression model auxiliary on $\theta = (0.1, 0.25, 0.5, 0.75, 0.9)$. From the results of calculations with $R$, we obtained $\check{V} (0.1) = 20.77641$ and $\check{V} (0.1) = 3869.9327$, also $R(0.1) = 0.9946313$, so that $LM^1_{0.1} = 233.7384$. The value of $LM^1_{0.1} = 233.7384 > \chi^2_{(i)} = 3.184$ make $H_0$ be rejected which indicates there is a structural break and validity of the unbounded model. This means that $\beta_k(\theta) = \beta_i(\theta)$, $k$ is the first segment of the observation $i = 1, 2, ..., 142$ and $l$ is the second segment of the observation $I = 143, ..., 235$. The change in structure at $\theta = 0.1$ with a breakpoint at $n=142$ which is worth $(829.40, 630.8)$ shows that in Belgian society there are two economic level, namely first level (lower class economy with low consumptive level) and the second level (community upper class economy with high consumption level). Estimation of the model for each level is as follows:

$$\hat{y}_{level 1} = 70.416 + 0.444\hat{x} \quad (4.5)$$

$$\hat{y}_{level 2} = 507.681 + 0.140\hat{x} \quad (4.6)$$

Furthermore, it can be seen how far the consumptive level differs between the first level and second level. Suppose given income $x = 700$ euros for both strata, then using equation $(4.5)$ and $(4.6)$ it will be seen that the first level and second level will each spend 381,216 euros and 605,681 euros for their consumption. It can be resumed that the consumptive of the second level is 1.6 times more than the first consumptive level.

5. Conclusion

Quantile regression is a statistical method used to estimate relationships between dependent variable and independent variables in certain conditional quantile functions. The LM test based on the residual quantile regression is a test to examine the presence of a parameter instability. Tests for parameter instability in quantile regression allow for testing in each desired quantile value. The Lagrange Multiplier test requires a bounded estimation model in which the residuals from the bounded model are regressed on dummy variables that have unit values after the break. This test is quite easy to process but tends to have almost the same value in all quantiles because there is one element in the $LM^1$ function, that is, the sample size, which does not change in different quantiles.

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