ON BLACK HOLE STABILITY
IN MULTIDIMENSIONAL GRAVITY

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ABSTRACT

Exact static, spherically symmetric solutions to the Einstein-Maxwell-scalar equations, with a dilatonic-type scalar-vector coupling, in D-dimensional gravity with a chain of n Ricci-flat internal spaces are considered. Their properties and special cases are discussed. A family of multidimensional dilatonic black-hole solutions is singled out, depending on two integration constants (related to black hole mass and charge) and three free parameters of the theory (the coordinate sphere, internal space dimensions, and the coupling constant). The behaviour of the solutions under small perturbations preserving spherical symmetry, is studied. It is shown that the black-hole solutions without a dilaton field are stable, while other solutions, possessing naked singularities, are catastrophically unstable.

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1 INTRODUCTION

With this paper we continue a series of investigations devoted to multidimensional gravitational models with the aim to obtain exact solutions and to find on their basis some observational consequences of extra (hidden) dimensions in our physical world. We also study the role of different physical fields in a self-consistent manner.

Modern theories of field unification assume in general a greater space-time dimensionality than four. Although multidimensional gravity as an approach to field unification can be traced back to the famous works of Kaluza and Klein [1,2] in the twenties, today’s increased interest to this field is largely stimulated by studies in supersymmetry and some other modern theories [3]; in the field-theoretical limit of such theories gravity is described with reasonable accuracy by multidimensional Einstein equations. Studies of their solutions can lead to predictions of direct observational manifestations of extra dimensions. Thus, cosmological models predict variations of the gravitational constant $G$, so that observational constraints imply certain limits on model parameters [4,5]. In these papers exact solutions as well as relations between the rate of a possible change of $G$ with the Hubble parameter, the deceleration parameter and the mean density of the Universe were obtained. Another possible window to the multidimensional world is opened by an analysis of local effects which could be sensitive to spatial variations of extra-dimension parameters [6]. We shall discuss some effects of this sort, in particular, those connected with electric charges of isolated bodies [6-8]. Quantum variants of all these models [4] deal with problems of quantum wormholes, creation of the Universe via tunnelling, nonsingular models etc.

Here, we consider exact, static, spherically symmetric solutions to the Einstein-Maxwell-scalar field equations, with a dilatonic type coupling between the scalar and vector fields, in $D$-dimensional gravity with a chain of $n$ Ricci-flat internal spaces and an arbitrary dimension of coordinate spheres (orbits of the group of spatial motions) — see Eq.(6) for the space-time structure. In particular, we pay attention to the arbitrariness of $d$, in the spirit of Tangherlini’s generalization of the Schwarzschild metric [9], with the hope to shed some light at the role of the 4-dimensional nature of the physical space-time.

Our approach differs from that applied in some papers on multidimensional black holes in that any solutions, not only black-hole ones, are sought. Consequently, the place of black holes (if any) in the whole set of solutions, the role of fields for the existence of different
types of solutions, as well as the properties of all spherical configurations, become clearer. Here, we call a black hole solution a configuration with the central singularity screened by an event horizon. In addition, we study the stability of these solutions under small perturbations preserving spherical symmetry and arrive at a result of physical interest that from the whole set of static solutions only the black-hole ones are stable.

2 BASIC EQUATIONS AND THE GENERAL STATIC SOLUTION

Let us consider a system with the Lagrangian

\[ L = R^{(D)} + g^{MN} \varphi_M \varphi_N - e^{2\lambda \varphi} F^{MN} F_{MN} \]  

(1)

for the set of interacting fields arising in the field limit of superstring theories. Here \( \varphi \) is a dilaton scalar field and \( F_{MN} = \partial_M A_N - \partial_N A_M \) is an Abelian gauge field interpretable as the electromagnetic one. As pointed out in Ref.[10], an electromagnetic field introduced in the multidimensional action may seem less aesthetic than a purely gravitational (Einstein) action usually considered in Kaluza-Klein theories but it appears that elementary gauge fields are necessary also in higher dimensions in order to obtain a realistic grand unification theory.

The corresponding set of field equations is

\[ \nabla^M \nabla_M \varphi + \lambda e^{2\lambda \varphi} F^{MN} F_{MN} = 0, \]  

(2)

\[ \nabla_N (e^{2\lambda \varphi} F^{NM}) = 0, \]  

(3)

\[ R_{MN} - g_{MN} R_A^A / 2 = -T_{MN} \]  

(4)

where \( T_{MN} \) is the energy-momentum tensor

\[ T_{MN} = \varphi_M \varphi_N - \frac{1}{2} g_{MN} \varphi^A \varphi_A + e^{2\lambda \varphi} [-2 F_M^A F_{NA} + \frac{1}{2} g_{MN} F^{AB} F_{AB}]. \]  

(5)

Capital Latin indices range from 0 to \( D - 1 \), the gravitational constant is put equal to unity.

Let us choose a \( D \)-dimensional manifold of the structure

\[ M = M^{(3+d)} \times M_1 \times \cdots \times M_n; \quad \text{dim} \, M_i = N_i; \quad D = 3 + d + \sum_{i=1}^n N_i, \]  

(6)
with the signature $(+, - , \ldots , - )$, where $M^{(3+d)}$ plays the role of the ordinary space-time and $M_i$ are Ricci-flat manifolds with the line elements $ds^2_{(i)}$, $i = 1, \ldots , n$. We seek solutions of the field equations such that $M^{(3+d)}$ is a static, generalized spherically symmetric space-time with the metric

$$ds^2_{d+3} = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\beta(u)}d\Omega^2_{d+1}$$

(7)

where $d\Omega^2_{d+1}$ is the line element on a unit $(d + 1)$-dimensional sphere $S^{d+1}$, while all the scale factors $e^{\beta_i}$ of the internal spaces $M_i$ depend on the radial coordinates $u$, i.e., the $D$-metric is

$$ds^2_D = g_{MN}dx^Mdx^N = ds^2_{d+3} - \sum_{i=1}^n e^{2\beta_i(u)}ds^2_{(i)}.$$  

(8)

If we denote $\gamma = \beta_{-1}, N_{-1} = 1, \beta = \beta_0, N_0 = 2$ and choose the harmonic radial coordinate $u$ such that $\alpha = \sum_{i=1}^n \beta_i N_i$, the Ricci tensor components $R^N_M$ can be written in the highly symmetric form ($x^1 = u$)

$$R^1_1 = -e^{-2\alpha} \sum_{i=1}^n N_i [\beta''_i + \beta'_i - \beta'_i \alpha'] ;$$  

$$R^N_\mu = 0 \ (N > d + 2; \ \mu = 0, \ldots , d + 2);$$  

$$R^b_{a_j} = -\delta^b_j \delta^b_{a_i} e^{-2\alpha} \beta''_i \ \ i \neq 0$$  

$$R^2_2 = \ldots = R^{d+2}_{d+2} = (d + 1)e^{-2\alpha} - e^{-2\alpha} \beta'';$$

(10)

where the indices $a_j (b_i)$ refer to the subspace $M_j (M_i)$.

Let us further assume the dilaton scalar field to be $\varphi = \varphi (u)$ and the electromagnetic field to be Coulomb-like: $A_M = \delta^0_M A_0 (u)$. Then the $D$-dimensional Maxwell-like equations give:

$$F^{01} = q e^{-2\lambda \varphi} / \sqrt{g} = q e^{-2\lambda \varphi - 2\alpha}, \ q = \text{const (charge)};$$  

$$g = | \det g_{MN} | = \exp (2\alpha + 2 \sum_{i=1}^n N_i \beta_i) = e^{4\alpha}.$$  

(11)

(the metric determinants of $ds_{(i)}, i = 1, \ldots , n$, never appear in the equations and may be omitted).
Now the scalar and (some linear combinations of) the metric field equations may be written in the form

\[ \phi'' + 2q^2 \lambda e^{2\gamma - 2\lambda \phi} = 0; \quad (12) \]

\[ N\beta_i'' + \gamma'' = 0, \quad i = 1, \ldots, n; \quad N = D - 3; \quad (13) \]

\[ (\alpha - \beta)'' - d^2 e^{2\alpha - 2\beta} = 0; \quad (14) \]

\[ \gamma'' - \frac{2q^2 N}{N + 1} e^{2\gamma - 2\lambda \phi} = 0 \quad (15) \]

\[ \alpha^2 - \sum_{i=-1}^{n} \beta_i'^2 - d(d + 1)e^{2\alpha - 2\beta} = \varphi^2 - 2q^2 e^{2\gamma - 2\lambda \phi} \quad (16) \]

where Eq.(16) is the \((u)\) component of the Einstein equations and forms a first integral of the remaining equations (12-15).

Eqs.(13, 14) are easily integrated to give

\[ \beta_i = -\gamma/N + h_i u, \quad h_i = \text{const, } i = 1, \ldots, n; \quad (17) \]

\[ e^{\beta - \alpha} = d \cdot s(k, u), \quad k = \text{const} \quad (18) \]

where for any choice of variables

\[ s(a, x) = \begin{cases} 
  a^{-1} \sinh ax, & a > 0; \\
  x, & a = 0; \\
  a^{-1} \sin ax, & a < 0.
\end{cases} \quad (19) \]

and inessential integration constants have been eliminated by shifting the origin of \(u\) and rescaling the coordinates in the subspaces \(M_i\) (their true scales are thus hidden in \(ds_{(ij)}^2\)).

Certain combinations of Eqs.(12) and (15) are also easily integrated giving

\[ \varphi = Cu/A - 2\lambda N_+ \omega, \quad (20) \]

\[ \gamma = (\omega + \lambda Cu)/A, \quad (21) \]

with the function \(\omega(u)\) defined by

\[ e^{-\omega} = Qs(h, u + u_1); \quad h, u_1 = \text{const}; \quad Qs(h, u_1) = 1 \quad (22) \]

where \(h\) and \(C\) are integration constants; the other constants are defined by

\[ N = D - 3; \quad A = 1 + \lambda^2 (N + 1)/N; \quad Q^2 = q^2/N_+, \quad N_+ = (N + 1)/(2AN). \quad (23) \]
Since, by our choice of the origin of \( u \), \( e^\beta \to \infty \) for \( u \to 0 \), the value \( u = 0 \) corresponds to spatial infinity. The last condition from (22) is the requirement that \( \gamma = 0 \) at infinity, i.e., \( dt \) is a time interval measured by a distant observer’s clock.

The final form of the \( D \)-dimensional metric is

\[
d s_D^2 = e^{2\gamma} dt^2 - e^{-2\gamma/N} \left\{ \left[ \frac{e^{-Bu}}{d \cdot s(k, u)} \right]^{2/d} \left[ \frac{du^2}{d^2 s^2(k, u)} + d\Omega^2_{d+1} \right] + \sum e^{2h_i u} d s^2_{(i)} \right\}, \tag{24}
\]
\[e^\gamma = \left[ \frac{e^{\lambda Cu}}{Qs(h, u + u_1)} \right]^{1/A}. \tag{25}\]

Here and henceforth \( \sum = \sum_{i=1}^n \). The integration constants are connected by the relation

\[
\frac{d + 1}{d} k^2 \text{sign} k = 2N_+ h^2 \text{sign} h + \frac{C^2}{A} + \frac{B^2}{d} + \sum N_i h_i^2. \tag{26}\]

This general static spherically-symmetric solution has \((n + 3)\) essential integration constants: the scalar charge \( C \), the electric charge \( q \) (or \( Q \)), the “charges” of extra dimensions \( h_i \) and one of the constants \( h \) or \( k \) connected by Eq.(26). For \( d = 1 \) (i.e., conventional spherical symmetry with 2-dimensional spheres) \( h \) along with \( C \) and \( Q \) can be related to the mass \( m \) which is defined by comparing the asymptotic \( u \to 0 \) \( (r \to \infty) \) of \( e^\gamma \) with the Schwarzschild metric:

\[
AGm + \lambda C = (Q^2 + h^2 \text{sign} h)^{1/2}, \tag{27}\]

The coordinate \( u \) is defined in the domain \([0, \infty)\) if \( h \geq 0, u_1 > 0 \) or between zero and some \( u_{\text{max}} > 0 \) in other cases. The scale factors \( e^{\beta_i} = 1 \) for \( u = 0 \).

3 PROPERTIES OF THE SOLUTION

AND SPECIAL CASES

Let us consider some special cases of the solution.

(a) The electromagnetic field is eliminated when \( Q = 0 \). Then we recover the generalized [7] Tangherlini solution [9]; the transformation

\[
e^{-2ku} = 1 - \frac{2k}{R} \equiv f(R); \quad -\frac{h + \lambda C}{A} = ka; \quad h_i = k \left( -a_i + \frac{a}{N} \right), \tag{28}\]
brings it to the form

\[ ds^2_D = f^a dt^2 - \left[ R^2 f^{1-a-b} \right]^{1/d} \left[ \frac{dR^2}{R^2 f} + d\Omega^2_{d+1} \right] - \sum_{i=1}^{n} f^{a_i} ds^2_{(i)}, \]

\[ \varphi = -\frac{C}{2k} \ln f(R), \quad C = \frac{1}{A} (C + 2\lambda N h), \]

\[ (d+1)/d = \sum N_i a_i^2 + a^2 + (a + b)^2 / d + C^2 / k^2 \]  \hspace{1cm} (29)

where \( b = \sum N_i a_i \). The \((d+3)\) dimensional part of (29) coincides with the Schwarzschild solution when \( d = 1 \) (the physical space-time is 4-dimensional) and \( h_i = 0 \).

The scalar field in (29) affects the metric only via the constant \( C \) in the relation among the constants.

The space-time (29) has a horizon (at \( r = 2k \)) only in the simplest case \( a - 1 = a_i = b = C = 0 \). Thus in our model an electrically neutral black hole can exist only with “frozen” extra dimensions \( (e^{\beta_i} = \text{const}) \) and with no scalar field. (The latter result is well known in conventional general relativity: a massless, minimally coupled scalar field is incompatible with an event horizon [11]). In this sense the generalized Schwarzschild-Tangherlini black holes may be called trivial. These are examples of the so called ”spontaneous” compactification of extra dimensions.

(b) when \( \lambda = 0 \), we obtain the generalized Reissner-Nordström-Tangherlini solution for linear scalar and electromagnetic fields discussed in Ref.[8].

(c) The scalar field is “switched off” when \( \lambda = C = 0 \), which yields a special case of item (b).

A new feature implied by a nonzero electric charge as compared with item (a) is that the constants \( k \) and \( h \) can have any sign and the function \( s(h, u + u_1) \) can be sinusoidal, which yields \( u_{\text{max}} < \infty \). Physically that means the appearance of a Reissner-Nordström repelling singularity at the center of the configuration.

(d) When extra dimensions are absent, we arrive at a solution for interacting scalar and electromagnetic fields in \((d+3)\)-dimensional general relativity; for the conventional case \( d = 1 \) it was obtained in Ref.[12] (s. also [14]).
For the case of 4-dimensional physical space-time (d=1, i.e., in Eq.(5) the space $M^{3+d} = M^4 = R^1 \times R^1 \times S^2$), the corresponding solution was obtained in Refs.[6,7,13].

Comparing the solutions with and without extra dimensions, one sees that indeed the scale factors $\beta_i$ of the Ricci-flat spaces $M_i$ behave like additional minimally coupled scalar fields.

For nonzero electric charge, i.e., in the general case, there is no choice of integration constants such that the extra dimensions are frozen ($\beta_i = \text{const}$). So, in this case we have no solutions with spontaneous compactification. The behaviour of the metric coefficients for different combinations of integration constants is rather various: thus, for $u \to \infty$ (provided $h \geq 0$)

$$
\beta \sim \left[-h + \frac{\lambda C}{AN} - B - k\right] u, \quad \gamma \sim \frac{-h + \lambda C}{A} u, \quad \beta_i \sim \left[-h + \frac{\lambda C}{AN} + h_i\right] u, \quad (30)
$$

so they may be finite or infinite. Calculations show that the solution has a naked singularity at $u = u_{\text{max}}$ or $u = \infty$ in all cases except

$$
h_i = -k/N; \quad h = k; \quad C = -\lambda k(N + 1)/N, \quad (31)
$$

when the sphere $u = \infty$ is a Schwarzschild-like event horizon: at finite radius $r = e^\beta$ of a coordinate sphere the integral $\int \exp(\alpha - \gamma) du$ for the light travel time diverges. So, (31) is the black hole solution we are looking for.

In general, four types of field behaviors may be singled out:

A. If $h < 0$ or (and) $u_1 < 0$, which may take place only when there is a nonzero charge $q$, the coordinate $u$ is defined between 0 and a certain finite value $u_{\text{max}} = \text{zero}$.

$s(h, u + u_1) < \infty$ corresponds to a repelling (since $g_{00} \to \infty$) naked Reissner-Nordström-like singularity where the electric field tends to infinity while the scalar field $\varphi$ and the extra-dimension scale factors $\beta_i$ remain finite.

B. The case $h > 0$, $u_1 > 0$; $r \to 0$ as $u \to \infty$. The value $u \to \infty$ corresponds to an attracting ($g_{00} \to 0$ scalar-field dominated naked central singularity where the dilaton field $\varphi$ and the “scalar fields” $\beta_i$ (at least some of them) become infinite.

C. The case $h > 0$, $u_1 > 0$; $r \to \infty$ as $u \to \infty$. This configuration, sometimes called a space pocket (“Raumtasche” by P.Jordan), possesses an attracting naked
singularity located at a sphere of infinite radius hidden beyond a regular “throat”, i.e., a minimum of the coordinate sphere radius $e^\beta$ which is reached at some finite value of the $u$ coordinate. Again at least some of the “scalar fields” are infinite at the singularity. The system geometry would be like that of a wormhole if it were regular at $u = \infty$.

D. The special case (31), which is intermediate between cases B and C, corresponds to a configuration called a dilatonic black hole. In General Relativity ($D = 4, d = 1$) such a solution was obtained for the first time in Ref.[12] (see also [14]) and was recently widely discussed in various aspects (see, e.g. [15] and references therein). We will also pay some attention to it.

In the black-hole case (31) only two independent integration constants remain, say, $k$ and $Q$, and the transformation (28) brings the solution to the form

$$ds^2_D = \frac{(1 - 2k/R)dt^2}{(1 + p/R)^{2/4}} - (1 + p/R)^{2/AN} \left[ \left( \frac{R^1 - d}{d+1} \right)^{2/d} \frac{dR^2}{1 - 2k/R} + \left( \frac{R}{d} \right)^{2/d} d\Omega_{d+1}^2 + \sum_i ds_i^2 \right], \quad (32)$$

$$\varphi = \frac{\lambda N + 1}{A N} \ln(1 + p/R), \quad (33)$$

$$p = \sqrt{Q^2 + k^2 - k}. \quad (34)$$

It is a generalization of the Reissner-Nordström solution. For $d = 1$ the metric takes the form

$$ds^2_D = \frac{(1 - 2k/R)dt^2}{(1 + p/R)^{2/4}} - (1 + p/R)^{2/AN} \left[ \frac{dR^2}{1 - 2k/R} + R^2 d\Omega^2 + \sum_i ds_i^2 \right], \quad (35)$$

considered in Refs. [16] (in other notation) and [13] and is in turn reduced to the genuine Reissner-Nordström metric if one puts, in addition, $D = 4$ (no extra dimension), $\lambda = 0$ (no scalar field). For $d = 1$, analysing the asymptotic $R \to \infty$, one naturally introduces the so-called Schwarzschild mass $m$ expressible in terms of $Q$ and $k$ (or $k$ is expressed in terms of $Q$ and $m$): now Eq.(34) is supplemented by $p = A(Gm - k)$.

In this family of black-hole solutions a nonzero scalar dilaton field (33) exists solely due to the interaction ($\lambda \neq 0$). When $\lambda = 0$, i.e., the $\varphi$ field becomes minimally coupled, a horizon is compatible only with $\varphi =$const. This conforms with the well-known “no-hair” theorems and the properties of the general-relativistic scalar-vacuum and scalar-electrovacuum configurations [11,14].
It should be pointed out that the solution (32-34) is a very special case of the general solution (2 integration constants instead of \((n + 3)\)). So there should exist very strong arguments for an assertion that in multidimensional gravity an actual collapse of a spherical body should lead to black hole (BH) formation. Such arguments do arise when we investigate the stability of the static solutions.

It can be demonstrated that one of the specific potentially observable effects for multidimensional systems is the violation of the Coulomb law. Indeed, the electric field strength \(E = (F^{01}F_{10})^{1/2}\) reads for our general solution (10) in the physical case \(d = 1\)

\[
E = \frac{|q|}{r^2} e^{-\sigma - 2\lambda \varphi}
\]

where \(r(u) = e^{\beta}\) is (as before) the curvature radius of a coordinate sphere (the curvature coordinate) and

\[
\sigma \equiv \sum_{i=1}^{n} N_i \beta_i.
\]

As seen from Eq.(35), the deviations from the Coulomb law are both due to extra dimensions and due to the scalar-electromagnetic interaction.

In the BH case for \(d = 1\) from the asymptotic \((r \to \infty)\) of the metric (35) we have:

\[
E = \frac{|q|}{r^2} \left\{ 1 - \frac{1}{r} \left[ (Gm - k) \frac{N - 1}{N} + 2\lambda^2(Gm + k) \frac{N + 1}{N} \right] + O \left( \frac{1}{r^2} \right) \right\}
\]

One can conclude that the magnitude of Coulomb law violation is of the order of the gravitational field strength characterized by the ratio \(Gm/r\) and depends also on \(N\) and \(\lambda\).

4 STABILITY ANALYSIS

Now we would like to investigate the stability of our static solutions under perturbations preserving spherical symmetry (i.e., monopole modes).

As verified by experience, although the D-dimensional setting of the problem is quite convenient for finding the static solutions, it makes a stability investigation very awkward. The latter is better carried out in a \((d + 3)\)-dimensional formulation.

From the viewpoint of the physical space-time \(V_{d+3}\) the scale factors \(\beta_i\) are additional scalar fields. Thus in the Lagrangian (1) the \(D\)-curvature and the metric are to be written...
explicitly in terms of $\beta_i$, after which the Lagrangian acquires the characteristic form for a scalar-tensor theory of gravity:

$$L = e^\sigma \left[ R^{(d+3)} - \sigma^\mu \sigma_\mu + \sum \beta_i^{\mu} \beta_i^{\mu} + \varphi^\mu \varphi_\mu - e^{2\lambda \varphi} F^{\mu\nu} F_{\mu\nu} \right]$$  \hspace{1cm} (39)

where $R^{(d+3)}$ corresponds to the metric $g_{\mu\nu}$, $\sigma$ is defined in (37) and it is assumed that

$$\varphi = \varphi(x^\mu), \quad F_{\mu\nu} = F_{\mu\nu}(x^\alpha); \quad F_{MN} = 0 \text{ for } M \text{ or } N > d + 2.$$

The factor $e^\sigma$ emerges from the $D$-dimensional determinant.

It is helpful to pass from $g_{\mu\nu}$ to the conformal metric $\overline{g}_{\mu\nu}$ defined by

$$\overline{g}_{\mu\nu} = e^{-2\sigma/(d+1)} g_{\mu\nu}$$  \hspace{1cm} (40)

The Lagrangian acquires the form

$$\overline{L} = \overline{R}^{(4)} + \frac{1}{d'} \sigma^\alpha \sigma_\alpha + \sum N_i \beta_i^{\mu} \beta_i^{\mu} + \varphi^\mu \varphi_\mu - e^{2\sigma/d' + 2\lambda \varphi} F^{\alpha\beta} F_{\alpha\beta},$$  \hspace{1cm} (41)

where we have denoted $d' = d + 1$, $\overline{R}^{(d+3)}$ corresponds to $\overline{g}_{\mu\nu}$ and the indices are raised and lowered using $\overline{g}_{\mu\nu}$.

Taking $\overline{g}_{\mu\nu}$ in the standard static, spherically symmetric form

$$ds_{d+3}^2 = \overline{g}_{\mu\nu} dx^\mu dx^\nu = e^{2\gamma} dt^2 - e^{2\delta} du^2 - e^{2\beta} d\Omega_{d+1}^2$$  \hspace{1cm} (42)

and choosing the harmonic coordinate $u$, i.e., putting $\overline{\alpha} = (d + 1) \overline{\beta} + \overline{\gamma}$, one rather easily restores the general static solution (20)-(24). It is noteworthy that the coordinate $u$ is harmonic with respect to both the $D$-dimensional metric $g_{MN}$ and the $(d+3)$-dimensional metric $\overline{g}_{\mu\nu}$, i.e., $\Box_D u = \Box_{d+3} u = 0$, but not with respect to the $(d+3)$-dimensional part $g_{\mu\nu}$ of the $D$-metric.

The explicit form of the metric (42) is easily obtained from (24), (25) using the transformation (40).

Now let us investigate small deviations from the static configurations

$$\delta \varphi(u, t), \quad \delta \beta_i(u, t), \quad \delta \overline{g}_{\mu\nu}(u, t), \quad \delta F_{\mu\nu}(u, t),$$  \hspace{1cm} (43)

preserving spherical symmetry, i.e. monopole modes. Thus dynamical degrees of freedom are restricted to the scalar field and the scale factors $\beta_i$ which in the 4-dimensional representation behave as effective scalar fields. For simplicity let us assume that there is only one internal space:

$$N_1 = D - d - 3 = N - d > 0; \quad N_2 = N_3 = \cdots = 0.$$  \hspace{1cm} (44)
In what follows, with no risk of confusion, we will omit the overbars at the symbols \(\alpha, \beta, \gamma\).

The next step is to choose the frame of reference and the coordinates in the perturbed space-time. Evidently this choice may be carried out by prescribing certain relations among the perturbations. Following [17], we would like to choose the so-called central frame of reference, where coordinate spheres of fixed radii are at rest, and the radial coordinate is taken such that the numerical values of these radii are the same as those in the static background configuration with the metric (42). Thus we postulate \(\delta\beta \equiv 0\), or, as it is sometimes more convenient to use the radius \(r \equiv e^\beta\) (coinciding with the Schwarzschild radial coordinate), \(\delta r \equiv 0\).

The perturbed metric functions \(\tilde{\gamma}(u, t)\) and \(\tilde{\alpha}(u, t)\) are taken in the form

\[
\tilde{\gamma}(u, t) = \gamma(u) + \delta\gamma(u, t); \quad \tilde{\alpha}(u, t) = \alpha(u) + \delta\alpha(u, t),
\]

and similarly for \(\tilde{\varphi}(u, t), \tilde{\beta}_1(u, t) \equiv \tilde{\mu}(u, t)\) and \(\tilde{F}_{\mu\nu}(u, t)\). The perturbed Maxwell-like field is defined by \(\tilde{A}_0(u, t)\).

Integrating Eq.(3), we get

\[
\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta} = -2q^2e^{-4\lambda\beta-2(N-1)\tilde{\mu}r^{-2d'}}
\]

where \(q\) is unperturbed since we study only dynamical perturbations rather than changes of integration constants and \(r\) is unperturbed by the choice of the frame of reference. From the metric field equations we obtain equations for \(\delta\mu\) and \(\delta\varphi\):

\[
r^{2d'}\delta\tilde{\mu} - \delta\mu'' - \mu' (\delta\gamma - \delta\alpha') + 2\mu'' (\delta\alpha - w) = 0,
\]

\[
r^{2d'}\delta\tilde{\varphi} - \delta\varphi'' - \varphi' (\delta\gamma - \delta\alpha') + 2\varphi'' (\delta\alpha - w) = 0,
\]

\[w \equiv \lambda\delta\varphi + (N - d)\delta\mu/d'
\]

where \(\mu', \mu'', \varphi'\) and \(\varphi''\) are static functions from the background solution and \(\delta\gamma'\) and \(\delta\alpha\) are defined in (45). The \(\binom{1}{0}\)-component of the metric field equation is easily integrated in \(t\):

\[
d'\beta'\delta\alpha = \frac{(N - d)(N + 1)}{d'}\mu'\delta\mu + \varphi'\delta\varphi + F(u),
\]

and the difference of the \(\binom{0}{0}\) and \(\binom{1}{1}\) components gives:

\[
\frac{d'}{2} \beta' (\delta\alpha' + \delta\gamma') = \frac{(N - d)(N + 1)}{d'}\mu'\delta\mu + \varphi'\delta\varphi.
\]
Taking $\delta \alpha$ and $\delta \gamma'$ from (50) and (51) and substituting them into (47) and (48), we arrive at coupled wave equations for $\delta \mu$ and $\delta \varphi$:

\[
\begin{align*}
    r^{2d} \delta \ddot{\varphi} - \delta \varphi'' + \frac{2}{d'} \left[ \left( \frac{r \varphi'^2}{r'} \right)' \delta \varphi + \frac{(N - d)(N + 1)}{d'} \left( \frac{r \varphi' \mu'}{r'} \right)' \delta \mu \right] &= 2 \varphi'' w; \quad (52) \\
    r^{2d} \delta \ddot{\mu} - \delta \mu'' + \frac{2}{d'} \left[ \left( \frac{r \mu' \varphi'}{r'} \right)' \delta \varphi + \frac{(N - d)(N + 1)}{d'} \left( \frac{r \mu'^2}{r'} \right)' \delta \mu \right] &= 2 \mu'' w. \quad (53)
\end{align*}
\]

Our static system is unstable if there exist physically allowed solutions to Eqs. (52) and (53) growing at $t \to \infty$. A separate problem is to define which solutions should be accepted as physically allowed ones. Let us join the approach of Ref. [17] and require

\[
\delta \mu \to 0, \quad \delta \varphi \to 0 \quad \text{for} \quad u \to 0 \quad (54)
\]

at spatial infinity $r \to \infty$ and

\[
|\delta \mu/\mu| < \infty, \quad |\delta \varphi/\varphi| < \infty \quad (55)
\]

at singularities and horizons. These requirements provide the validity of linear perturbation theory over the whole space-time, including the neighbourhood of the singularities. We must in addition forbid energy fluxes to our system from outside; however [17], such a requirement constrains only the signs of the integration constants and does not affect any conclusions.

There are some cases when the set of equations (52), (53) simplifies, i.e., decouples or reduces to a single equation with one unknown function:

1. the dilaton field is absent: $\lambda = 0, \varphi \equiv \delta \varphi \equiv 0$;
2. there are no extra dimensions: $\mu \equiv \delta \mu \equiv 0, N = 1$;
3. some combinations of (52) and (53) decouple.

Due to [18], the third possibility is realized for black-hole solutions when $d = 1$; however, this is probably not the case for arbitrary $d$. Here we confine the study to the first variant when the system dynamics is driven by extra dimensions. We shall verify that the arbitrariness of $d$ does not affect the stability conclusions.

Separating variables in (53) and transforming $\mu$ and $u$ to obtain the normal Liouville form of the mode equation,
\[ \delta \mu = e^{i\Omega t} y(x)/r^{d/2}, \quad x = -\int r^{d'}(u) du, \]  

we obtain the Schrödinger-like equation

\[ y_{xx} + [\Omega^2 - V(x)] y = 0 \]  

with the effective potential

\[ V(x) = \frac{2}{r^{2d}} \left[ \frac{(N-d)(N+1)}{d^2} \left( \frac{r\mu'^2}{r'} \right)' + \frac{d'r^{d'/2}}{4} \left( \frac{r'}{r^{(d+3)/2}} \right)' - \frac{N-d}{d'} \mu'' \right] \]

where \( d' = d + 1 \), again.

Our static system is unstable if there exist physically allowed solutions of (57) with \( \Omega^2 < 0 \) (negative energy levels in the language of the Schrödinger equation).

\( V(x) \) is a rather complicated function of \( x \), undetermined explicitly, so that Eq.(57) can hardly be solved exactly. However, the main conclusions on the system stability can be drawn on the basis of the qualitative behavior of \( V(x) \) and the asymptotic forms of the solutions to Eq.(57). The latter can be classified according to different behaviors of the background static solutions for different values of the integration constants (the notations A-D coincide with those in Section 3).

A. \( h < 0 \) or (and) \( u_1 < 0 \), the case of a repelling Reissner-Nordström singularity. In the limit \( u \to u_{\text{max}} [18] \)

\[ x \sim (u_{\text{max}} - u)^{2+1/N}, \]  

\[ V(x) = -\frac{(N+1)(3N+1)}{4(2N+1)^2x^2} (1 + o(1)). \]  

so that \( V \) has a negative pole. The boundary conditions (54), (55) take the form

\[ y \to 0 \quad \text{at} \quad x \to \infty, \]  

\[ x^{-(N+1)/(4N+2)} |y| < \infty \quad \text{at} \quad x \to 0. \]

At spatial infinity \( (x \to \infty) \) \( V(x) \) behaves like \( \text{const}/x^{d+2} \) (in this and all the other cases).

We see that both the potential and the boundary condition near \( x = 0 \) are independent of \( d \) and coincide with those derived in Refs. [6,18]. Therefore the conclusion is also the same. It looks as follows.
The asymptotic forms of the solutions to Eq.(57) for \( \Omega^2 < 0 \) at \( x \to 0 \) and \( x \to \infty \) are

\[
x \to \infty : \quad y = C_1 e^{-|\Omega|x} + C_2 e^{|\Omega|x},
\]

\[
x \to 0 : \quad y = C_3 x^{s_1} + C_4 x^{s_2}, \quad s_{1,2} = \frac{1}{2} \left( 1 \pm \frac{N}{2N+1} \right).
\]

The condition (63) is satisfied for any \( C_3 \) and \( C_4 \). Therefore the solution \( y(x) \) chosen on the basis of (61), i.e., that with \( C_2 = 0 \), is admissible and realizes the instability of our static system.

In other words, in the potential well corresponding to the bare singularity there exist arbitrarily low “energy levels” for the perturbations. Thus the system is unstable and the instability is of a catastrophic nature since the increment \( |\Omega| \) has no upper bound.

B. The case \( h > 0, u_1 > 0; r \to 0 \) at \( u \to \infty \), a scalar-type central singularity where the “scalar field” \( \mu \) is infinite. At \( u \to \infty \)

\[
x \to 0, \quad V(x) = -\frac{1}{4x^2}(1 + o(1)).
\]

The boundary condition at \( x \to 0 \) is

\[
|y|/\left(\sqrt{x} \ln x\right) < \infty
\]

while the solution has the following asymptotic form at \( x \to 0 \):

\[
y = \sqrt{x}(C_5 + C_6 \ln x).
\]

Again both the potential and the boundary conditions are \( d \)-independent. Considerations similar to those in item A lead to the same conclusion, namely, that the system is catastrophically unstable.

C. For the \((d + 3)\)-dimensional section of the D-dimensional metric there is a range of integration constant values when the geometry has the form of a “space pocket” (see item C in Section 3). On the contrary, in our \((d + 3)\)-dimensional metric (42), (43), containing an additional conformal factor, such a possibility is not realized. Indeed, using the expression (43) for \( \beta = \ln r \) at the asymptotic \( u \to \infty \) and the relation (26) among the constants, one can verify that \( r \) can tend either to zero, or to a nonzero finite value corresponding to a black hole.

D. Let us now consider perturbations of multidimensional black holes described by (20)-(24) under the conditions (31). The effective potential (59) may be written in an explicit
form using the transformation (28) in terms of the $R$ coordinate:

$$V(x) = V_1(x) + V_2(x),$$

$$V_1(x) = \frac{d + 1}{2d r^{2(d+1)}} \frac{R - 2k}{(R + p)^2} \{(R + pN_-)[R(2k + p - pN_-) + 2kpN_-] + (R - 2k)[Rp(1 - N_-) + \frac{d - 1}{2d} (R + pN_-)^2]\},$$

$$V_2(x) = \frac{2pN_- (2k + pN_-) R(R - 2k)}{(R + pN_-)^2 r^{2(d+1)}}.$$

(68)

where $N_- = (N - d)/[N(d + 1)]$.

As follows from (68), $V_1 > 0$ and $V_2 > 0$ when $R > 2k$. The boundary conditions (54), (55) correspond to those conventional in quantum mechanics and thus the positiveness of $V(x)$ means that solutions to (57) with $\Omega^2 < 0$ are absent. Consequently, multidimensional BHs are stable under monopole perturbations. Other types of multidimensional spherically symmetric solutions are strongly unstable.

This generalizes the conclusions of Refs. [6,18] for the case of arbitrary $d$.

5 CONCLUSIONS

We have seen that multidimensional black holes are stable under monopole perturbations, while other types of (electro)vacuum spherically symmetric solutions are strongly unstable. This property distinguishes the black hole (BH) solutions from all the other possible solutions to the multidimensional field equations and supports the view that even if space, time and gravity are described by some multidimensional model, realistic collapse of isolated bodies should lead to black hole formation, as it is conventionally asserted in general relativity. In particular, this favours the models with a great number of primordial BHs present in the Early Universe, at epochs when there was no crucial distinction between the physical and internal dimensions.

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