Paraboson quotients. A braided look at Green’s ansatz and a generalization *

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January 28, 2009

Abstract

Bosons and Parabosons are described as associative superalgebras, with an infinite number of odd generators. Bosons are shown to be a quotient superalgebra of Parabosons, establishing thus an even algebra epimorphism which is an immediate link between their simple modules. Parabosons are shown to be a super-Hopf algebra. The super-Hopf algebraic structure of Parabosons, combined with the projection epimorphism previously stated, provides us with a braided interpretation of the Green’s ansatz device and of the parabosonic Fock-like representations. This braided interpretation combined with an old problem leads to the construction of a straightforward generalization of Green’s ansatz.

1 Introduction

For a quantum system with a Hamiltonian of the form $H(p_i, q_i)$ with possibly infinite degrees of freedom $i = 1, 2, \ldots$, the canonical variables $p_i, q_i$ are considered to be generators of a unital associative non-commutative algebra, described in terms of generators and relations by (we have set $\hbar = 1$):

$$[q_i, p_j] = i \delta_{ij} I$$

$$[q_i, q_j] = [p_i, p_j] = 0$$

(1)

*J. Math. Phys. v.48, 113516, (2007)*
I is of course the unity of the algebra, \( i, j = 1, 2, \ldots \) and \([x,y]\) stands for \(xy - yx\). Relations (1) are known in the physical literature as the Weyl algebra, or the Heisenberg-Weyl algebra or more commonly as the Canonical Commutation Relations often abbreviated as CCR. For technical reasons it is common to use -instead of the variables \(p_i,q_i\)- the linear combinations:

\[
    b_j^+ = \frac{1}{\sqrt{2}}(q_j - ip_j), \quad b_j^- = \frac{1}{\sqrt{2}}(q_j + ip_j)
\]

for \(j = 1, 2, \ldots\) in terms of which (1) become:

\[
    [b_i^-, b_j^+] = \delta_{ij}I \quad [b_i^-, b_j^-] = [b_i^+, b_j^+] = 0
\]

for \(i, j = 1, 2, \ldots\). These latter relations are usually called the bosonic algebra, and it is well known that this algebra is the starting point for the second quantisation problem (in the case of the free field theory) leading to the Bose-Einstein statistics.

Parabosonic algebra grew out of the desire to generalize the second quantization method -in the case of the the free field- in a way permitting more general kind of statistics than the Bose-Einstein statistics. It was formally introduced in terms of generators and relations by Green in [8], Greenberg and Messiah in [9] and Volkov in [32]. It is generated by the generators \(B_i^+, B_i^-\) subject to the relations:

\[
    \left[\{B_i^K, B_j^\eta\}, B_k^\xi\right] - (\xi - \eta)\delta_{jk}B_i^\xi - (\varepsilon - \xi)\delta_{ik}B_j^\eta = 0
\]

for all values of \(i, j = 1, 2, \ldots\) and \(\xi, \eta, \varepsilon = \pm 1\). The field theory statistics stemming from such an algebra, has been known with the name “parastatistics” and is still a wide open subject of research. (see [18] and references therein).

This paper consists logically of four parts: The first part consists of Section 2. We state the definitions and derive the \(\mathbb{Z}_2\)-grading of bosonic and parabosonic algebras (in infinite degrees of freedom). Although the bosonic algebra is usually considered to be a quotient of the UEA of the Heisenberg Lie algebra with its generators thus being even elements, we adopt here a totaly different approach: We consider the bosonic algebra to be an associative \(\mathbb{Z}_2\)-graded algebra (associative superalgebra) with its generators being odd elements. This approach is reminiscent -although actually different- of the antisymmetric Clifford-Weyl algebra (see for example [20]). It permits us to express the bosonic algebra as a quotient superalgebra of the parabosonic superalgebra. The projection epimorphism from parabosons to bosons creates thus an immediate link between irreducible representations of the bosonic algebra and irreducible representations of the parabosonic algebra.

The second part of the paper consists of Section 3 and Section 4. In Section 3 the notions of \(\mathbb{Z}_2\)-graded algebra, \(\mathbb{Z}_2\)-graded modules and \(\mathbb{Z}_2\)-graded tensor products [2], are reviewed as special examples of the more general and modern notions
of $G$-module algebras ($G$: a finite abelian group) and of braiding in monoidal categories \cite{17,14,15}. The role of the non-trivial quasitriangular structure of the $\mathbb{C}Z_2$ group Hopf algebra or equivalently the role of the braiding of the category $\mathbb{C}Z_2\mathcal{M}$ of $\mathbb{C}Z_2$ modules is emphasized in order to make clear that similar constructions can be carried out in the same way for more complicated gradings (like for example $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading). In Section 4 the notion of the super-Hopf algebra is reviewed and the super-Hopf algebraic structure of the parabosonic algebra is established. Note that the proof, does not make use of the well known \cite{7} Lie superalgebraic structure of the parabosonic algebra (for the case of the finite degrees of freedom). We conclude the paragraph with an indication of how the super-Hopf algebraic structure can be used to form multiple tensor products of (braided) representations of the original super-Hopf algebra.

The third part of the paper consists of Section 5 and Section 6 and deals with the representation theory of the bosonic and parabosonic algebras. After a review of well known results in Section 5 we apply in Section 6 the techniques developed in the previous sections to the case of the braided representations of the parabosonic super-Hopf algebra. The Green ansatz algebras are shown to be isomorphic to braided tensor product algebras constructed from multiple copies of the bosonic algebra. The $\mathbb{Z}_2$-grading of the bosonic algebra, established in Section 2 plays an essential role in the proof. Furthermore, the Fock-like representations of the parabosonic algebra -which were classified by Greenberg in \cite{9}- are shown to be braided ($\mathbb{Z}_2$-graded), irreducible (simple) submodules of the multiple tensor product module of the first Fock-like representation.

Finally, the last part of this paper consists of Section 7 alone. We combine the isomorphism stated in Section 6 with some partial solutions of an old problem of Ohnuki and Kamefuchi (see \cite{18}): the construction of self contained sets of commutation relations or commutation relations specific to given order $p$. As a result we construct a straightforward generalization of the Green’s ansatz device. Furthermore our construction indicates how a whole family of similar generalizations can be constructed, provided we have (although we don’t) the general solution of Ohnuki’s and Kamefuchi’s problem. Speaking about generalizations of the Green’s ansatz, one should also see \cite{23} for a different approach.

In what follows, all vector spaces and algebras and all tensor products will be considered over the field of complex numbers. Whenever the symbol $i$ enters a formula in another place than an index, it always denotes the imaginary unit $i^2 = -1$. Furthermore, whenever formulas from physics enter the text, we use the traditional convention: $\hbar = m = \omega = 1$. Finally, the Sweedler’s notation for the comultiplication is freely used throughout the text.
2 Bosons, Parabosons and superalgebra quotients

The parabosonic algebra, was originally defined in terms of generators and relations by Green [8] and Greenberg-Messiah [9]. We begin with restating their definition:

Let us consider the vector space $V_X$ freely generated by the elements: $X^+_i, X^-_j$, $i, j = 1, 2, \ldots$. Let $T(V_X)$ denote the tensor algebra of $V_X$. $T(V_X)$ is - up to isomorphism - the free algebra generated by the elements of the basis. In $T(V_X)$ we consider the two-sided ideal $I_B$, generated by the following elements:

$$[X^-_i, X^+_j] = \delta_{ij}X_X$$,  $$[X^+_i, X^-_j]$$,  $$[X^+_i, X^+_j]$$

(5)

for all values of $i, j = 1, 2, \ldots$. $X_X$ is the unity of the tensor algebra. We now have the following:

**Definition 2.1.** The bosonic algebra $B$ is the quotient algebra of the tensor algebra $T(V_X)$ with the ideal $I_B$:

$$B = T(V_X)/I_B$$

We denote by $\pi_B : T(V_X) \rightarrow B$ the canonical projection. The elements $X^+_i, X^-_j$, $I_X$, where $i, j = 1, 2, \ldots$ and $I_X$ is the unity of the tensor algebra, are the generators of the tensor algebra $T(V_X)$. The elements $\pi_B(X^+_i)$, $\pi_B(X^-_j)$, $\pi_B(I_X)$, for all values $i, j = 1, 2, \ldots$ are a set of generators of the bosonic algebra $B$, and they will be denoted by $b^+_i, b^-_j, I$ for $i, j = 1, 2, \ldots$ respectively, from now on. $\pi_B(I_X) = I$ is the unity of the bosonic algebra. In the case of the finite degrees of freedom $i, j = 1, 2, \ldots m$ we have the bosonic algebra in $2m$ generators ($m$ boson algebra) which we will denote by $B^{(m)}$ from now on.

Returning again in the tensor algebra $T(V_X)$ we consider the two-sided ideal $I_{P_B}$, generated by the following elements:

$$[\{X^+_i, X^+_j\}, X^+_k] - (\epsilon - \eta)\delta_{jk}X^+_i - (\epsilon - \xi)\delta_{ik}X^+_j$$

(6)

respectively, for all values of $\xi, \eta, \epsilon = \pm 1$ and $i, j = 1, 2, \ldots$. $X_X$ is the unity of the tensor algebra. We now have the following:

**Definition 2.2.** The parabosonic algebra in $P_B$ is the quotient algebra of the tensor algebra $T(V_X)$ of $V_X$ with the ideal $I_{P_B}$:

$$P_B = T(V_X)/I_{P_B}$$

We denote by $\pi_{P_B} : T(V_X) \rightarrow P_B$ the canonical projection. The elements $\pi_{P_B}(X^+_i)$, $\pi_{P_B}(X^-_j)$, $\pi_{P_B}(I_X)$, for all values $i, j = 1, \ldots$ are a set of generators of the parabosonic algebra $P_B$, and they will be denoted by $B^+_i, B^-_j, I$ for $i, j = 1, 2, \ldots$
respectively, from now on. \( \pi_B(I_X) = I \) is the unity of the parabosonic algebra. In the case of the finite degrees of freedom \( i, j = 1, 2, ..., m \) we have the parabosonic algebra in \( 2m \) generators (\( m \) paraboson algebra) which we will denote by \( P_B^{(m)} \) from now on.

Based on the above definitions we prove now the following proposition which clarifies the relationship between bosonic and parabosonic algebras:

**Proposition 2.3.** The parabosonic algebra \( P_B \) and the bosonic algebra \( B \) are both \( \mathbb{Z}_2 \)-graded algebras with their generators \( B^\pm_i \) and \( b^\pm_i \) respectively, \( i, j = 1, 2, ..., m \), being odd elements. The bosonic algebra \( B \) is a quotient algebra of the parabosonic algebra \( P_B \). The “replacement” map \( \phi: P_B \rightarrow B \) defined by: \( \phi(B^\pm_i) = b^\pm_i \) is a \( \mathbb{Z}_2 \)-graded algebra epimorphism (i.e.: an even algebra epimorphism).

**Proof.** It is obvious that the tensor algebra \( T(V_X) \) is a \( \mathbb{Z}_2 \)-graded algebra with the monomials being homogeneous elements. If \( x \) is an arbitrary monomial of the tensor algebra, then \( \deg(x) = 0 \), namely \( x \) is an even element, if it constitutes of an even number of factors (an even number of generators of \( T(V_X) \)) and \( \deg(x) = 1 \), namely \( x \) is an odd element, if it constitutes of an odd number of factors (an odd number of generators of \( T(V_X) \)). The generators \( X^+_i, X^-_j, \ i, j = 1, ..., n \) are odd elements in the above mentioned gradation. In view of the above description we can easily conclude that the \( \mathbb{Z}_2 \)-gradation of the tensor algebra is immediately “transfered” to the algebras \( P_B \) and \( B \): Both ideals \( I_{P_B} \) and \( I_B \) are homogeneous ideals of the tensor algebra, since they are generated by homogeneous elements of \( T(V_X) \). Consequently, the projection homomorphisms \( \pi_{P_B} \) and \( \pi_B \) are homogeneous algebra maps of degree zero, or we can equivalently say that they are even algebra homomorphisms. We can straightforwardly check that the bosons satisfy the paraboson relations, i.e:

\[
\pi_B([\{ X^\xi_i X^\eta_j \}, X^\zeta_k] - (\xi - \eta)\delta_{jk}X^\xi_i - (\zeta - \xi)\delta_{ik}X^\eta_j) = \\
= [\{ b^\xi_i b^\eta_j \}, b^\zeta_k] - (\xi - \eta)\delta_{jk}b^\xi_i - (\zeta - \xi)\delta_{ik}b^\eta_j = 0
\]

which simply means that: \( \ker(\pi_{P_B}) \subseteq \ker(\pi_B) \) or equivalently: \( I_{P_B} \subseteq I_B \). By the correspondence theorem for rings, we get that the set \( I_B/I_{P_B} = \pi_{P_B}(I_B) \) is an homogeneous ideal of the algebra \( P_B \), and applying the third isomorphism theorem for rings we get:

\[
P_B/(I_B/I_{P_B}) = (T(V_X)/I_{P_B})/(I_B/I_{P_B}) \cong T(V_X)/I_B = B
\]  

(7)

Thus we have shown that the bosonic algebra \( B \) is a quotient algebra of the parabosonic algebra \( P_B \). The fact that \( I_{P_B} \subseteq I_B \) implies that \( \pi_B \) is uniquely extended to
an even algebra homomorphism $\phi : P_B \rightarrow B$ according to the following (commutative) diagram:

\[
\begin{array}{ccc}
T(V_X) & \xrightarrow{\pi_B} & B \\
\pi_{PB} & & \Downarrow \exists \phi \\
P_B & \xrightarrow{\phi} & B \\
\end{array}
\]

where $\phi$ is completely determined by it’s values on the generators $B_i^\pm$ of $P_B$, i.e.: $\phi(B_i^\pm) = b_i^\pm$. Recalling now that: $\ker \phi = I_B/I_{P_B} = \pi_{P_B}(I_B)$ and using equation (7), we get the $\mathbb{Z}_2$-graded algebra isomorphism:

\[ B \cong P_B/\ker \phi \]

which completes the proof that $\phi$ is an epimorphism of $\mathbb{Z}_2$-graded algebras (or: an even epimorphism).

Note that $\ker \phi$ is exactly the ideal of $P_B$ generated by the elements of the form:

\[ [B_i^-, B_j^+] = \delta_{ij}I, \ [B_i^-, B_j^-], \ [B_i^+, B_j^+] \] for all values of $i, j = 1, 2, \ldots$, and $I$ is the unity of the $P_B$ algebra. We have an immediate corollary:

**Corollary 2.4.** For any vector space $V$, any $B_V$ module of the bosonic algebra $B$ immediately gives rise to a $P_B V$ module of the parabosonic algebra $P_B$ through the replacement map:

\[ B_i^\pm \cdot x = \phi(B_i^\pm) \cdot x = b_i^\pm \cdot x \] (8)

where $x$ is any element of $V$. Furthermore, if $B_V$ is an irreducible representation (a simple module) the fact that $\phi$ is an epimorphism implies that $P_B V$ is also an irreducible representation (a simple module).

At this point we should underline the difference between our approach to the bosonic algebra (or: Weyl algebra or: CCR) as stated in Definition 2.2 and the more mainstream approach which considers the bosonic algebra to be a quotient algebra of the universal enveloping algebra of the Heisenberg Lie algebra: The Heisenberg Lie algebra has -as a complex vector space- the basis: $b_i^\pm, c$ for all values $i = 1, 2, \ldots$ and it’s Lie algebraic structure is determined by the relations:

\[ [b_i^-, b_j^+] = \delta_{ij}c \quad [b_i^-, b_j^-] = [b_i^+, b_j^+] = [b_i^\pm, c] = 0 \] (9)

If we denote the Heisenberg Lie algebra by $L_H$ and it’s universal enveloping algebra by $U(L_H)$, then we can immediately see that we have the associative algebra isomorphism:

\[ B \cong U(L_H)/<c-I> \] (10)
which enables us to consider the bosonic algebra $B$ as a quotient algebra of the universal enveloping algebra of the Lie algebra $L_H$. By $<c - I>$ we denote the ideal of $U(L_H)$ generated by the element $c - I$, where $I$ is the unity of $U(L_H)$. We should mention here that the isomorphism stated in equation (10) is an algebra isomorphism but not a $\mathbb{Z}_2$-graded algebra isomorphism, since in $U(L_H)/<c - I>$ the generators $b^\pm_i$ are considered to be even (ungraded) elements, while according to our approach, in the bosonic algebra $B$ the same generators are considered to be odd elements. It is the $\mathbb{Z}_2$-grading that makes the whole difference, and the reason for this point of view will become clear in the sequel where we are going to discuss tensor products and representations.

Let us stress here, that in the case of finite degrees of freedom, our approach as expressed in Proposition 2.3 combined with the well known results of [7] indicates a Lie superalgebraic rather than a Lie algebraic interpretation of the bosonic algebra $B^{(m)}$: In [7] Ganchev and Palev have shown that the parabosonic algebra $P^{(m)}_B$ is isomorphic to the universal enveloping algebra of the orthosymplectic Lie superalgebra $osp(1/2m)$, with the isomorphism being a superalgebra isomorphism (or equivalently: a $\mathbb{Z}_2$-graded algebra isomorphism). This result combined with Proposition 2.3 says that the $m$ boson algebra is isomorphic to a quotient of $U(osp(1/2m))$: 

$$B^{(m)} \cong U(osp(1/2m))/\ker \phi$$

(11)

where $\phi$ is the replacement map $\phi : P^{(m)}_B \to B^{(m)}$. We stress here that -unlike equation (10)- equation (11) is a superalgebra isomorphism and not merely an algebra isomorphism.

3 Superalgebras, quasitriangularity and braiding

The rise of the theory of quasitriangular Hopf algebras from the mid-80's [5] and thereafter and especially the study and abstraction of their representations (see: [14, 15], [17] and references therein), has provided us with a novel understanding of the notion and the properties of $G$-graded algebras, where $G$ is a finite abelian group:

Restricting ourselves to the simplest case where $G = \mathbb{Z}_2$, we recall that an algebra $A$ being a $\mathbb{Z}_2$-graded algebra (in the physics literature the term superalgebra is also of widespread use) is equivalent to saying that $A$ is a $\mathbb{C}\mathbb{Z}_2$-module algebra, via the $\mathbb{Z}_2$-action determined by: $1 \triangleright a = a$ and $g \triangleright a = (-1)^{|a|}a$ (for any $a$ homogeneous in $A$ and $|a|$ it’s degree). We denote by $1, g$ are the elements of the $\mathbb{Z}_2$ group (written multiplicatively). What we actually mean is that $A$, apart from being an algebra is also a $\mathbb{C}\mathbb{Z}_2$-module and at the same time it’s structure maps (i.e.:
the multiplication and the unity map which embeds the field into the center of the algebra) are $\mathbb{C}Z_2$-module maps which is nothing else but homogeneous linear maps of degree 0 (i.e.: even linear maps). Stated more generally, what we are actually saying is that the $G$-grading of $A$ can be equivalently described in terms of a specific action of the finite abelian group $G$ on $A$, thus in terms of a specific action of the $\mathbb{C}G$ group algebra on $A$. This is not something new. In fact such ideas already appear in works such as [3] and [31].

What is actually new and has been developed from the 90’s and thereafter [15], [17], is the role of the quasitriangularity of the $\mathbb{C}G$ group Hopf algebra (for $G$ a finite abelian group, see [28]) and the role played by the braiding of the $\mathbb{C}G$-modules in the construction of the tensor products of $G$-graded objects. It is well known that for any group $G$, the group algebra $\mathbb{C}G$ equipped with the maps:

$$\Delta(g) = g \otimes g \quad \varepsilon(g) = 1 \quad S(g) = g^{-1}$$

for any $g \in G$, becomes a Hopf algebra. Focusing again in the special case $G = \mathbb{Z}_2$, we can further summarize the description of the preceding paragraph, saying that $A$ is an algebra in the braided monoidal category of $\mathbb{C}Z_2$-modules $\mathbb{C}Z_2\mathcal{M}$. In this case the braiding is induced by the non-trivial quasitriangular structure of the $\mathbb{C}Z_2$ Hopf algebra i.e. by the non-trivial $R$-matrix:

$$R_{Z_2} = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$$

In the above relation $1, g$ are the elements of the $\mathbb{Z}_2$ group (written multiplicatively).

We digress here for a moment, to recall that (see [14,15] or [17]) if $(H, R_H)$ is a quasitriangular Hopf algebra through the $R$-matrix $R_H = \sum R^{(1)}_H \otimes R^{(2)}_H$, then the category of modules $H\mathcal{M}$ is a braided monoidal category, where the braiding is given by a natural family of isomorphisms $\Psi_{V,W} : V \otimes W \cong W \otimes V$, given explicitly by:

$$\Psi_{V,W}(v \otimes w) = \sum (R^{(2)}_H \triangleright w) \otimes (R^{(1)}_H \triangleright v)$$

for any $V, W \in \text{obj}(H\mathcal{M})$. By $v, w$ we denote any elements of $V, W$ respectively. Combining eq. (13) and (14) we immediately get the braiding in the $\mathbb{C}Z_2\mathcal{M}$ category:

$$\Psi_{V,W}(v \otimes w) = (-1)^{|v||w|}w \otimes v$$

In the above relation $|.|$ denotes the degree of an homogeneous element of either $V$ or $W$ (i.e.: $|x| = 0$ if $x$ is an even element and $|x| = 1$ if $x$ is an odd element). This is obviously a symmetric braiding, since $\Psi_{V,W} \circ \Psi_{W,V} = \text{Id}$, so we actually have a symmetric monoidal category $\mathbb{C}Z_2\mathcal{M}$, rather than a truly braided one.
The really important thing about the existence of the braiding (15) is that it provides us with an alternative way of forming tensor products of $\mathbb{Z}_2$-graded algebras: If $A$ and $B$ are superalgebras with multiplications $m_A : A \otimes A \to A$ and $m_B : B \otimes B \to B$ respectively, then the super vector space $A \otimes B$ (with the obvious $\mathbb{Z}_2$-gradation) equipped with the associative multiplication

$$(m_A \otimes m_B)(Id \otimes \Psi_{B,A} \otimes Id) : A \otimes B \otimes A \otimes B \to A \otimes B$$

given equivalently by:

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}ac \otimes bd$$

for $b, c$ homogeneous in $B, A$ respectively, readily becomes a superalgebra (or equivalently an algebra in the braided monoidal category of $\mathbb{CZ}_2$-modules $\mathbb{CZ}_2\mathcal{M}$) which we will denote: $A \underline{\otimes} B$ and call the braided tensor product algebra from now on. Let us close this description with two important remarks: First, we stress that in (16) both superalgebras $A$ and $B$ are viewed as $\mathbb{CZ}_2$-modules and as such we have $B \otimes A \ncong A \otimes B$ through $b \otimes c \mapsto (-1)^{|c||b|}c \otimes b$. Second we underline that the tensor product (16) had been already known from the past [2] but rather as a special possibility of forming tensor products of superalgebras than as an example of the more general conceptual framework of the braiding applicable not only to superalgebras but to any $G$-graded algebra ($G$ a finite abelian group) as long as $\mathbb{C}G$ is equipped with a non-trivial quasitriangular structure or equivalently [17], [28], a bicharacter on $G$ is given.

For the case $A = B$ we get the braided tensor algebra $A \underline{\otimes} A$. Of course we can form longer -braided- tensor product algebras $A \underline{\otimes} \ldots \underline{\otimes} A$ between any (finite) number of copies of $A$. In computing the multiplication in the braided tensor product algebra $A \underline{\otimes} \ldots \underline{\otimes} A$ the only thing we have to keep in mind is that each time two copies of $A$ have to interchange their order, the suitable sign factor (due to the braiding (15)) has to be inserted. Finally, due to Proposition 2.3 we immediately check that the above description fits perfectly the superalgebras $P_B$ and $B$, and offers us a method for constructing the braided tensor product algebras $P_B \underline{\otimes} \ldots \underline{\otimes} P_B$ and $B \underline{\otimes} \ldots \underline{\otimes} B$ respectively.

A similar discussion applies to the case of $\mathbb{Z}_2$-graded $A$-modules where $A$ is a $\mathbb{Z}_2$-graded algebra. We recall that [2], [27], $V$ being a $\mathbb{Z}_2$-graded $A$-module means that first of all $V$ is a $\mathbb{Z}_2$-graded vector space: $V = V_0 \oplus V_1$. Furthermore the $A$-action is such that the even elements of $A$ act as homogeneous linear maps of degree zero: $a \cdot V_0 \subseteq V_0$ and $a \cdot V_1 \subseteq V_1$ for every $a \in A_0$ (even linear maps) and the odd elements of $A$ act as homogeneous linear maps of degree one: $a \cdot V_0 \subseteq V_1$ and $a \cdot V_1 \subseteq V_0$ for every $a \in A_1$ (odd linear maps).

It is again the abstraction in the representation theory of the quasitriangular Hopf
algebras that provides us with an equivalent description of the above mentioned ideas: We can equivalently say that \( V \) is a \( \mathbb{CZ}_2 \)-module and at the same time an \( A \)-module, such that the \( A \)-action \( \phi_V : A \otimes V \rightarrow V \) is also a \( \mathbb{CZ}_2 \)-module map. This is equivalent to saying that \( V \) is a (left) \( A \)-module in the braided monoidal category of \( \mathbb{CZ}_2 \)-modules \( \mathbb{CZ}_2 \mathcal{M} \), or a braided \( A \)-module, where the braiding is given by equation (15).

The braiding (15), provides us with an alternative way of forming tensor products of braided modules (tensor product of \( \mathbb{Z}_2 \)-graded modules) through the same “mechanism” which led us to the braided tensor product algebra (16): If \( A, B \) are algebras in \( \mathbb{CZ}_2 \mathcal{M} \), \( V \) is a (left) \( A \)-module in \( \mathbb{CZ}_2 \mathcal{M} \), \( W \) is a (left) \( B \)-module in \( \mathbb{CZ}_2 \mathcal{M} \), via the actions:

\[
\phi_V : A \otimes V \rightarrow V \quad \phi_W : B \otimes W \rightarrow W
\]

respectively, then the vector space \( V \otimes W \) becomes a (left) \( A \otimes B \)-module in \( \mathbb{CZ}_2 \mathcal{M} \) via the action:

\[
\phi_{V \otimes W} = (\phi_V \otimes \phi_W) \circ (\text{Id} \otimes \Psi_{B,V} \otimes \text{Id}) : A \otimes B \otimes V \otimes W \rightarrow V \otimes W \quad (17)
\]

given equivalently by:

\[
(a \otimes b) \cdot (v \otimes w) = (-1)^{|b||v|} a \cdot v \otimes b \cdot w
\]

for \( a,b,v,w \) homogeneous in \( A,B,V,W \) respectively. See also the discussion in [27]. Once more, we underline the fact that in (17), both the superalgebra \( B \) and the superspace \( V \) are considered as \( \mathbb{CZ}_2 \) modules and as such: \( B \otimes V \cong V \otimes B \) through \( b \otimes v \mapsto (-1)^{|b||v|} v \otimes b \), which is the \( \mathbb{Z}_2 \)-module isomorphism “reflected” by the braiding (15) which is the same thing as the \( R_{\mathbb{Z}_2} R_{\mathbb{Z}_2} \)-matrix (13).

4 Super-Hopf algebraic structure of Parabosons: a braided group

The notion of \( G \)-graded Hopf algebra, for \( G \) a finite abelian group, is not a new one neither in physics nor in mathematics. The idea appears already in the work of Milnor and Moore [16], where we actually have \( \mathbb{Z} \)-graded Hopf algebras. On the other hand, universal enveloping algebras of Lie superalgebras are widely used in physics and they are examples of \( \mathbb{Z}_2 \)-graded Hopf algebras (see for example [12], [27]). These structures are strongly resemblant of Hopf algebras but they are not Hopf algebras at least in the ordinary sense.

Restricting again to the simplest case where \( G = \mathbb{Z}_2 \) we briefly recall this idea: An algebra \( A \) being a \( \mathbb{Z}_2 \)-graded Hopf algebra (or super-Hopf algebra) means
first of all that $A$ is a $\mathbb{Z}_2$-graded associative algebra (or: superalgebra). We now consider the braided tensor product algebra $A \otimes A$. Then $A$ is equipped with a coproduct

$$\Delta : A \rightarrow A \otimes A$$

which is an superalgebra homomorphism from $A$ to the braided tensor product algebra $A \otimes A$:

$$\Delta(ab) = \sum (-1)^{|a_2||b_1|} a_1 b_1 \otimes a_2 b_2 = \Delta(a) \cdot \Delta(b)$$

for any $a, b$ in $A$, with $\Delta(a) = \sum a_1 \otimes a_2$, $\Delta(b) = \sum b_1 \otimes b_2$, and $a_2, b_1$ homogeneous. We emphasize here that this is exactly the central point of difference between the “super” and the “ordinary” Hopf algebraic structure: In an ordinary Hopf algebra $H$ we should have a coproduct $\Delta : H \rightarrow H \otimes H$ which should be an algebra homomorphism from $H$ to the usual tensor product algebra $H \otimes H$.

Similarly, $A$ is equipped with an antipode $S : A \rightarrow A$ which is not an algebra anti-homomorphism (as in ordinary Hopf algebras) but a superalgebra anti-homomorphism (or: “twisted” anti-homomorphism or: braided anti-homomorphism) in the following sense (for any homogeneous $a, b \in A$):

$$S(ab) = (-1)^{|a||b|} S(b) S(a)$$

The rest of the axioms which complete the super-Hopf algebraic structure (i.e.: coassociativity, counity property, and compatibility with the antipode) have the same formal description as in ordinary Hopf algebras.

Once again, the abstraction of the representation theory of quasitriangular Hopf algebras provides us with a language in which the above description becomes much more compact: We simply say that $A$ is a Hopf algebra in the braided monoidal category of $\mathbb{C}\mathbb{Z}_2$-modules $\mathbb{C}\mathbb{Z}_2\mathcal{M}$ or: a braided group where the braiding is given in equation (15). What we actually mean is that $A$ is simultaneously an algebra, a coalgebra and a $\mathbb{C}\mathbb{Z}_2$-module, while all the structure maps of $A$ (multiplication, comultiplication, unity, counity and the antipode) are also $\mathbb{C}\mathbb{Z}_2$-module maps and at the same time the comultiplication $\Delta : A \rightarrow A \otimes A$ and the counit are algebra morphisms in the category $\mathbb{C}\mathbb{Z}_2\mathcal{M}$ (see also [14, 15] or [17] for a more detailed description).

We proceed now to the proof of the following proposition which establishes the super-Hopf algebraic structure of the parabosonic algebra $P_B$:

**Proposition 4.1.** The parabosonic algebra equipped with the even linear maps

$$\Delta : P_B \rightarrow P_B \otimes P_B \quad S : P_B \rightarrow P_B \quad \varepsilon : P_B \rightarrow \mathbb{C}$$

determined by their values on the generators:

$$\Delta(B_i^\pm) = B_i^\pm \otimes B_i^\pm + 1 \otimes B_i^\pm \quad \varepsilon(B_i^\pm) = 0 \quad S(B_i^\pm) = -B_i^\mp$$

for $i = 1, 2, \ldots$, becomes a super-Hopf algebra.
Proof. Recall that by definition $P_B = T(V_X)/I_{P_B}$. Consider the linear map:

$$\Delta^T : V_X \rightarrow P_B \otimes P_B$$

determined by it's values on the basis elements specified by:

$$\Delta^T(X_i^\pm) = I \otimes B_i^\pm + B_i^\pm \otimes I$$

By the universality of the tensor algebra this map is uniquely extended to a superalgebra homomorphism: $\Delta^T : T(V_X) \rightarrow P_B \otimes P_B$. Now we compute:

$$\Delta^T \left( \left[ \{X_i^{\xi}, X_j^{\eta} \}, X_k^{\zeta} \right] - (\epsilon - \eta) \delta_{jk} X_i^{\xi} - (\epsilon - \xi) \delta_{ik} X_j^{\eta} \right) = 0$$

for all values of $\xi, \eta, \epsilon = \pm 1$ and $i, j = 1, 2, \ldots$. This means that $I_{P_B} \subseteq ker \Delta^T$, which in turn implies that $\Delta^T$ is uniquely extended as a superalgebra homomorphism: $\Delta : P_B \rightarrow P_B \otimes P_B$, according to the following (commutative) diagram:

$$
\begin{array}{ccc}
T(V_X) & \xrightarrow{\Delta} & P_B \otimes P_B \\
\pi \downarrow & & \exists! \Delta \\
& P_B & 
\end{array}
$$

with values on the generators determined by (20).

Proceeding the same way we construct the maps $\xi^T, S^T$, as determined in (20). Note here that in the case of the antipode $S$ we need the notion of the $\mathbb{Z}_2$-graded opposite algebra (or: opposite superalgebra) $P_B^{op}$, which is a superalgebra defined as follows: $P_B^{op}$ has the same underlying super vector space as $P_B$, but the multiplication is now defined as: $a \cdot b = (-1)^{|a||b|} ba$, for all $a, b \in P_B$. (In the right hand side, the product is of course the product of $P_B$). We start by defining a linear map

$$S^T : V_X \rightarrow P_B^{op}$$

determined by:

$$S^T(X_i^\pm) = -B_i^\pm$$

This map is (uniquely) extended to a superalgebra homomorphism: $S^T : T(V_X) \rightarrow P_B^{op}$. We proceed by showing that:

$$S^T \left( \left[ \{X_i^{\xi}, X_j^{\eta} \}, X_k^{\zeta} \right] - (\epsilon - \eta) \delta_{jk} X_i^{\xi} - (\epsilon - \xi) \delta_{ik} X_j^{\eta} \right) = 0$$

(21)

for all values of $\xi, \eta, \epsilon = \pm 1$ and $i, j = 1, 2, \ldots$, or in other words: $I_{P_B} \subseteq ker S^T$, which in turn implies that $S^T$ is uniquely extended to a superalgebra homomorphism $S : P_B \rightarrow P_B^{op}$, thus to a superalgebra anti-homomorphism: $S : P_B \rightarrow P_B$ with values on the generators determined by (20).
Now it is sufficient to verify the rest of the super-Hopf algebra axioms (coassociativity, counity and the compatibility condition for the antipode) on the generators of $P_B$. This can be done with straightforward computations.

Let us note here, that the above proposition generalizes a result which -in the case of finite degrees of freedom- is a direct consequence of the work in [7].

Finally, before closing this paragraph we should mention an important consequence of the super-Hopf algebraic structure at the level of representations. If $A$ is a super-Hopf algebra, the existence of the comultiplication (18) permits us to construct tensor product of braided representations (tensor product of $\mathbb{Z}_2$-graded representations), through the mechanism provided by the braiding and developed at the end of Section 3. If $A$ is a Hopf algebra in $\mathbb{CZ}_2 \mathcal{M}$ and $V$, $W$, are (left) $A$-modules in $\mathbb{CZ}_2 \mathcal{M}$, via the actions:

$$\phi_V : A \otimes V \rightarrow V \quad \phi_W : A \otimes W \rightarrow W$$

then the vector space $V \otimes W$ becomes a (left) $A$-module in $\mathbb{CZ}_2 \mathcal{M}$, via the action:

$$\phi_{V \otimes W} = (\phi_V \otimes \phi_W) \circ (Id \otimes \Psi_{A,V} \otimes Id) \circ (\Delta \otimes Id \otimes Id) : A \otimes V \otimes W \rightarrow V \otimes W \quad (22)$$

given equivalently by:

$$a \cdot (v \otimes w) = \sum (-1)^{|a_2||v|} a_1 \cdot v \otimes a_2 \cdot w$$

for $a, v, w$ homogeneous in $A, V, W$ respectively, and $\Delta(a) = \sum a_1 \otimes a_2$ for any $a \in A$.

The preceding construction, can be generalized for longer tensor products of braided representations: If $(V_k)_{k \in I}$ ($I$: some finite set) is a collection of (left) $A$-modules in $\mathbb{CZ}_2 \mathcal{M}$ for $A$ some super-Hopf algebra, then the vector space $\otimes_{i \in I} V_i$ readily becomes an $A \otimes \ldots \otimes A$-module ($I$-copies of $A$) according to the discussion at the end of Section 2. The only thing we have to keep in mind is that each time $A$ and $V_i$ have to interchange their order in the tensor product $A \otimes \ldots \otimes A \otimes \otimes_{i \in I} V_i$, in order for the action to be computed, the suitable sign factor due to the braiding (15) has to be inserted. (We underline the fact that in such permutations between a copy of $A$ and $V_i$ both of $A$ and $V_i$ are considered as $\mathbb{CZ}_2$-modules). Repeated use of the comultiplication of the super-Hopf algebra $A$ may then be used to turn this $A \otimes \ldots \otimes A$-module to an $A$-module. We will come back in Section 6 to discuss some applications and a more detailed formulation of these ideas in the case of the tensor product representations of the parabosonic algebra $P_B$. 

13
5 Bosonic and Parabosonic modules

5.1 Fock representation for Bosons

We start our discussion by considering for simplicity, the case of one degree of freedom. We thus consider the Weyl algebra (or : CCR) determined by

\[ [q, p] = i\hbar I \]  

The generalization to the case of the finite degrees of freedom (finite number of bosons) is straightforward as we shall see.

Since the early days of quantum theory, the analysis of the representations of such an algebra underlies much of the foundations of quantum mechanics. It was in the mid-twenties that Heisenberg himself and Schrödinger stated explicitly the first representation of such an algebra: Guided by physical requirements, they considered representations of on a complex, separable, infinite dimensional Hilbert space, where and act as unbounded self-adjoint operators. Of course such a Hilbert space is unique and we can regard as -isomorphic- copies of it, either the space of all square integrable complex sequences i.e. sequences \((x_1, x_2, ..., x_n, ...)\) such that \(\sum_{i=1}^{\infty} |x_i|^2 < \infty\), or the space of all square integrable complex valued functions i.e. functions \(f(x)\) such that \(\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\), where \(x\) is a real variable.

Heisenberg made the first choice and using as “basis” of the total orthonormal set consisting of the elements \(e_i = (0, ..., 0, 1, 0, ...)\) with 1 on the \(i\)-th place and zero everywhere else \((i = 1, 2, ...)\) represented the generators \(q, p\), in the form of infinite dimensional matrices. That was the origin of the so-called matrix mechanics formalism.

On the other hand Schrödinger made use of the second option and represented the generators \(q, p\), as the multiplication and the differentiation operators \(x\) and \(-i\frac{d}{dx}\) in \(L_2(-\infty, \infty)\) respectively, resulting thus with what is now known as the wave mechanics formalism.

These two representations are (unitarily) equivalent: If we choose a total orthonormal set in \(L_2(-\infty, \infty)\) consisting of the functions \(e_n(x) = e^{-x^2/2}H_n(x)/\pi^{1/4}\sqrt{2^n n!}\) where \(H_n(x)\) are the Hermite polynomials \((n = 0, 1, 2, ...)\) we can straightforwardly check that the matrix forms of the operators \(x\) and \(-i\frac{d}{dx}\) in \(L_2(-\infty, \infty)\) coincide with the infinite dimensional matrices stated by Heisenberg. The corresponding

\footnote{Generally speaking, an old result \([33]\) (but see also in \([22]\)) states that when considering representations of the CCR in a Hilbert space, the generators \(p\) and \(q\) cannot be both represented by bounded operators: at least one of them has to be unbounded. This fact automatically raises delicate questions about the domains of such operators and at the same time indicates the importance of the use of unbounded operators in the mathematical formulation of quantum mechanics. An analogous result for the parabosonic algebra has been obtained in \([26]\).}
modules are thus isomorphic.

We must point out here that the Heisenberg - Schrödinger representation described above is not a representation in the strict algebraic sense. The multiplication and differentiation operators are obviously not defined on the entire of \( L_2(-\infty, \infty) \). Hopefully the situation is easily cured: Neither the sequences \( e_i, i = 1, 2, \ldots \) nor the functions \( e_n(x), n = 0, 1, 2, \ldots \) constitute bases -in the sense of algebra- of the spaces \( l_2 \) and \( L_2(-\infty, \infty) \) respectively. In other words they are not Hammel bases but simply total orthonormal sequences. This means that their span is not the entire space but only a dense subspace of it. This subspace is no longer a Hilbert space, since it is not complete. It is merely a euclidean (or: pre-Hilbert) space. Restricting our attention to this space, the Heisenberg - Schrödinger representation is actually well-defined (in the strict algebraic sense) and it is a well-known fact for physicists that it is an irreducible representation (i.e: a simple module). If we use the basis given in (2) instead of (1) we have -for the case of a single degree of freedom- the CCR in the form of the single boson algebra:

\[
[b^-, b^+] = I \tag{24}
\]

Following a widespread notation of the physical literature, we shall denote the \( e_i, i = 1, 2, \ldots \) or the \( e_n(x), n = 0, 1, 2, \ldots \) total orthonormal sets of \( l_2 \) or \( L_2(-\infty, \infty) \) respectively by \( |n> \) for \( n = 0, 1, 2, \ldots \). Using this formalism it is straightforward to check that the Heisenberg - Schrödinger representation can be equivalently described in terms of the following action:

\[
b^+|n> = \sqrt{n+1}|n+1> \quad b^-|n> = \sqrt{n}|n-1>
\tag{25}
\]

where the elements \( b^+, b^- \), act as Hilbert-adjoint operators, i.e.:

\[
(b^-)^\dagger = b^+
\]

The carrier space is the pre-Hilbert space generated by the set \( \{ |n> / n = 0, 1, 2, \ldots \} \). This is a dense subspace of the complex separable infinite dimensional Hilbert space \( H \). We can easily check using the above described formalism that:

\[
|n> = \frac{1}{\sqrt{n!}}(b^+)^n|0> \quad b^-|0> = 0
\tag{26}
\]

which enables us to conclude that the Heisenberg - Schrödinger representation is a cyclic module generated by any of its elements, which in turn implies that it is a simple module (or: an irreducible representation). The vector \( |0> \) is called the “ground state” or the “vacuum” of the system and it’s corresponding wavefunction (under the above mentioned isomorphism) is: \( \pi^{-1/4}e^{-x^2/2} \).
The generalization of the above mentioned representation for the general case of the bosonic algebra is a well known fact for physicists:

In the case of a finite number of bosons we have the algebra $B^{(m)}$ (bosonic algebra in $2m$ generators or $m-$boson algebra) described in terms of generators and relations by (3) for $i, j = 1, 2, ..., m$. We construct a representation which is uniquely determined (up to unitary equivalence) by the demand for the existence of a unique vacuum vector belonging in a Hilbert space and annihilated by all $b_i^-$'s, together with the Hilbert adjointness condition:

$$b_i^- |0> = 0 \quad (b_i^-)^\dagger = b_i^+$$  \hspace{1cm} (27)

for every value $i = 1, 2, ..., m$. The carrier space is generated by elements of the form:

$$|k_1, k_2, \cdots, k_i, \cdots, k_m> = \frac{(b_1^+)^{k_1}(b_2^+)^{k_2} \cdots (b_i^+)^{k_i} \cdots (b_m^+)^{k_m}}{\sqrt{k_1!k_2! \cdots k_i! \cdots k_m!}}|0>$$  \hspace{1cm} (28)

where $i = 1, 2, ..., m$, the $k_i$'s are non-negative integers. The corresponding actions of the elements of $B^{(m)}$ can be calculated to be:

$$b_i^+ |k_1, k_2, \cdots, k_i, \cdots, k_m> = \sqrt{k_i + 1} |k_1, k_2, \cdots, k_i + 1, \cdots, k_m>$$

$$b_i^- |k_1, k_2, \cdots, k_i, \cdots, k_m> = \sqrt{k_i} |k_1, k_2, \cdots, k_i - 1, \cdots, k_m>$$  \hspace{1cm} (29)

This is again a cyclic module generated by any of it’s elements, thus a simple module. This representation is usually called the Fock representation or the Fock-Cook representation. We are going to denote it by $B^{(m)} F$. Certain subspaces of this vector space contain the physically realizable states i.e. vectors which are symmetric under the exchange of particles leading thus to the Bose-Einstein statistics. The Fock representation $B^{(m)} F$ can be constructed as the tensor product of $m-$copies of the 1–particle Heisenberg - Schröedinger representation $B^{(1)} F$ described by (25), (26), where we do not take into account the $\mathbb{Z}_2-$grading when constructing the tensor product. We will not pursue these subjects further here but instead we refer to the classic works [4, 30].

We mention that the Fock representation $B^{(m)} F$ is a $\mathbb{Z}_2-$graded $B^{(m)}-$module (or equivalently a $B^{(m)}-$module in the braided monoidal category of the $\mathbb{C}\mathbb{Z}_2-$modules $\mathbb{C}\mathbb{Z}_2 M$). This can be easily seen since the vector space spanned by elements of the form (28) is a $\mathbb{Z}_2-$graded vector space. It’s even subspace is spanned by elements with $\sum_{i=1}^{m} k_i = even$ and it’s odd subspace is spanned by elements with $\sum_{i=1}^{m} k_i = odd$. The action given in (29) describes how the (odd) generators of $B^{(m)}$ act on the above described subspaces.
In the case of infinite degrees of freedom, we have the general case of the bosonic algebra $B$ described in terms of generators and relations by (3) for $i, j = 1, 2, ...$ (see also Definition 2.2). In this case the Fock space is generated by elements of the form:

$$|k_1, k_2, \ldots, k_i, \ldots> = \frac{(b_1^+)^{k_1}(b_2^+)^{k_2} \cdots (b_i^+)^{k_i}\cdots}{\sqrt{k_1!k_2!\cdots k_i!\cdots}}|0>$$

and the Fock representation $\mathcal{F}$ is uniquely determined (up to unitary equivalence) by the demand for the existence of a unique vacuum together with the Hilbert adjointness condition:

$$b_i^-|0> = 0 \quad (b_i^-)^\dagger = b_i^+$$

which in turn imply:

$$b_i^+|k_1, k_2, \ldots, k_i, \ldots> = \sqrt{k_i+1}|k_1, k_2, \ldots, k_i + 1, \ldots>$$

$$b_i^-|k_1, k_2, \ldots, k_i, \ldots> = \sqrt{k_i}|k_1, k_2, \ldots, k_i - 1, \ldots>$$

where $i = 1, 2, ...$ and the $k_i$’s are non-negative integers. The monomials $(b_1^+)^{k_1}(b_2^+)^{k_2} \cdots (b_i^+)^{k_i}\cdots$ stated in the right hand side of (30) contain of course a finite number of factors only. In other words for any vector of the form (30) we have: $\sum_{j=1}^{\infty} k_j < \infty$. We have an irreducible, $\mathbb{Z}_2$-graded, $B-$module which we will denote by $\mathcal{F}$. This is actually the starting point for the study of the second quantization problem (see: [4, 30]).

5.2 The Green’s Ansatz and Fock-like representations for Para-bosons

When Green first introduced the parabosonic algebra in infinite degrees of freedom $\mathcal{P}_B$ in [8], one of his first tasks was the investigation of it’s representations and more specifically the construction of representations which might serve as a generalization of the bosonic Fock representation previously described. For this purpose, he introduced a device which has been known since then in the bibliography of mathematical physics as the “Green’s ansatz”.

The Green’s ansatz $(G(p))_{p=1,2,\ldots}$ is an infinite family of associative algebras labelled by a positive integer $p$. For each specific value of $p$ we have a different algebra denoted by $G(p)$ described by means of generators and relations: It is the associative algebra generated by the elements $b_i^\alpha$, $b_i^{\alpha+}$ and the unity $I$, for all values $i = 1, 2, \ldots$ and $\alpha = 1, 2, \ldots, p$. For a specific value of $\alpha$ the generators satisfy the usual CCR:

$$[b_i^\alpha^-, b_j^{\alpha^+}] = \delta_{ij}I \quad [b_i^{\alpha+}, b_j^{\alpha^-}] = [b_i^{\alpha^+}, b_j^{\alpha^-}] = 0$$  \quad (33)
while for values $\alpha \neq \beta$ we have the “anomalous” (anticommutation) relations:

$$\{b_\alpha^\alpha - , b^\beta_j + \} = \{b_\alpha^\alpha - , b_j^\beta - \} = \{b_\alpha^\alpha + , b^\beta_j + \} = 0$$  \hspace{1cm} (34)$$

The above relations hold for all values of $i, j = 1, 2, \ldots$ and $\alpha = 1, 2, \ldots, p$ completely describe the algebra $G(p)$. In a terminology more familiar to physicists, we can say that $G(p)$ describes $p$ anticommuting bosonic fields. We note here, that for $p = 1$, the corresponding algebra of Green’s ansatz is just the familiar bosonic algebra: $G(1) = B$.

Green showed that for every specific algebra $G(p)$ we pick, among the ones constituting the Green’s ansatz, i.e. for every specific choice we make for the value of the positive integer $p$, we have an associative algebra homomorphism $\pi_p : P_B \rightarrow G(p)$ stated explicitly by:

$$\pi_p(b^\pm) = \sum_{\alpha=1}^p b_\alpha^\pm$$  \hspace{1cm} (35)$$

We remark here, that for $p = 1$, we have: $\pi_1 = \phi : P_B \rightarrow B = G(1)$. This is exactly the replacement map of Proposition 2.3 which is nothing else but the algebra projection of $P_B$ onto $B$.

Green’s idea was to make use of the above mentioned homomorphisms, in order to construct representations of the parabosonic algebra $P_B$ initiating from representations of $G(p)$. However his idea was not fully exploited until a few years later by Greenberg and Messiah, in [9], where they virtually classified all the Fock-like representations of $P_B$. Their method was -shortly- as follows: For a specific value of $p$ they considered a representation of the $G(p)$ in (a subspace of) a Hilbert space, determined by the demand for the existence of a unique “vacuum” vector $|0 > :$

$$b_\alpha^\alpha - |0 > = 0$$  \hspace{1cm} (36)$$

under the Hilbert-adjointness condition:

$$(b_\alpha^\alpha - )^\dagger = b_\alpha^\alpha +$$  \hspace{1cm} (37)$$

for all values $i = 1, 2, \ldots$ and $\alpha = 1, 2, \ldots, p$. They showed that the cyclic $G(p)$-module generated by $|0 >$, is an irreducible representation of the $G(p)$ algebra and is specified -up to unitary equivalence- by the conditions (36), (37). This representation has as carrier space the vector space spanned by vectors of the form: $\mathcal{P}(b_\alpha^\pm)|0 >$, where $\mathcal{P}$ is an arbitrary polynomial of the generators $b_\alpha^\pm$ for all values $i = 1, 2, \ldots$ and $\alpha = 1, 2, \ldots, p$. Utilizing the algebra homomorphism (35) they turned the above constructed vector space into a $P_B$-module through: $B_\alpha^\pm |.. > = \pi_p(B_\alpha^\pm)|.. >$ for all values $i = 1, 2, \ldots$. They showed that this $P_B$-module
is reducible, and that it contains an irreducible \( P_B \)-submodule which is cyclic and generated (as a submodule) by \( |0> \). This irreducible \( P_B \)-submodule is generated (as a vector space) by all the elements of the form: \( \mathcal{P}(B_i^\pm)|0> \) for \( \mathcal{P} \) an arbitrary polynomial of the generators. Furthermore they showed that in this (irreducible) \( P_B \)-submodule we have the relations:

\[
B_i^-|0> = 0, \quad B_i^- B_j^+|0> = p\delta_{ij}|0> \quad (B_i^-)^\dagger = B_i^+
\]

for all values \( i, j = 1, 2, \ldots \), which specify it up to unitary equivalence.

Apart from utilizing Green’s ansatz, Greenberg and Messiah also studied the representations of the parabosonic algebra in more abstract terms. Inspired by the case of the bosonic algebra (CCR), they studied Fock-like representations of the parabosonic algebra i.e. representations of \( P_B \) in a (suitably chosen subspace of a) Hilbert space possessing a unique “vacuum” which means a unique vector satisfying: \( B_i^-|0> = 0 \) under the condition that \( B_i^-, B_i^+ \), act as Hilbert adjoint operators: \((B_i^-)^\dagger = B_i^+ \) for all values \( i = 1, 2, \ldots \). They found that unlike the boson case where such requirements specify a unique up to isomorphism irreducible representation, the Fock-representation \( P_B F \), in the paraboson case these requirements specify -up to isomorphism- an infinite collection of irreducible representations labelled by a positive integer \( p \). We are going to denote these Fock-like representations by \( P_B F(p) \). Their central results are summarized in the following theorem:

**Theorem 5.1.** Any representation of the parabosonic algebra \( P_B \) in a Hilbert space, possessing a unique “vacuum” vector satisfying: \( B_i^-|0> = 0 \) for all values \( i = 1, 2, \ldots \) and for which: \((B_i^-)^\dagger = B_i^+ \) \( \forall \ i = 1, 2, \ldots \), has also the following properties:

- It also satisfies: \( B_i^- B_j^+|0> = p\delta_{ij}|0> \) for all values \( i, j = 1, 2, \ldots \), where \( p \) is an arbitrary positive integer (\( p = 1, 2, \ldots \)).

- For a fixed value of \( p \), it is an irreducible representation (i.e.: the corresponding module is simple).

- For a fixed value of \( p \) this irreducible representation is unique up to unitary equivalence (and thus is isomorphic with the \( p \)-submodule contained in the reducible module produced by Green’s ansatz previously described).

- The carrier space is spanned by vectors of the form: \( \mathcal{P}(B_i^+)|0> \) where \( \mathcal{P} \) an arbitrary monomial of the generators \( B_i^+ \) for all values \( i = 1, 2, \ldots \), and it is an infinite dimensional vector space.
Thus: the simple $P_B$-modules satisfying the “vacuum” state (or: “ground” state) condition $B_i^- |0> = 0$, are labelled by a positive integer $p$. In the physics literature this integer is called “the order of the parastatistics”. These are the Fock-like representations $p_B F(p)$ of the parabosonic algebra $P_B$ generalizing the bosonic Fock representation $B F$ described in the previous paragraph. For a detailed proof of the theorem without the use of Green’s ansatz see also [18] chapter 4, while the proof via the Green’s ansatz can be also found in chapter 5 of the same reference.

For $p = 1$ we get the first Fock-like representation $p_B F(1)$ of the parabosonic algebra, which we will denote by $p_B F$ from now on. We can further check that for $p = 1$ we have $G(1) = B$ and the Green homomorphism $\pi_1$ of (35) coincides with the replacement map $\phi(B_i^+) = b_i^+$ of Proposition [2,3]. This implies, that the representation $p_B F$ coincides with the representation of $P_B$ constructed via the bosonic Fock representation $B F$ and Corollary [2,4].

Let us finally emphasize that Greenberg’s work allows one to specify a much more general class of representations than the Fock-like representations described in Theorem [5,1]: If the requirement of the positivity of the inner product is dropped, i.e. if we consider more general spaces than Hilbert spaces (for example in Krein spaces), then only the first and the last conclusions of the above theorem hold, but with $p$ now being an arbitrary positive number. We won’t deal with such representations here.

We should mention here that the carrier space for every Fock-like representation of the parabosonic algebra (this means: for every specific value of the positive integer $p$) has a much more complicated structure than the corresponding carrier space of the bosonic Fock representation: In the latter case, the fact that the generators $b_i^+$ commute with each other for every value of $i = 1, 2, \ldots$ means that an arbitrary monomial of the generators $b_i^+$ is of the form $\left( b_1^+ \right)^{k_1} \left( b_2^+ \right)^{k_2} \cdots \left( b_i^+ \right)^{k_i} \cdots \left( b_i^+ \right)^0$ (with a finite number of factors only, as has already been stated). This in turn implies that the positive integers $k_1, k_2, \ldots, k_i, \ldots (k_i \neq 0$ for a finite number of values of $i$ only) uniquely determine the vectors spanning the bosonic Fock space. Consequently, the notation used in equations (28) and (30) is well defined. It is obvious however that this is not the case for the parabosonic Fock-like representations: The more complicated nature of the relations in the parabosonic algebra imply that in general, even vectors of the form: $B_i^+ B_j^+ |0> \; \text{and} \; B_j^+ B_i^+ |0>$ for $i \neq j$ may be linearly independent. From the physicist’s point of view, one may say that permutations of the $b_i^+, b_j^+, \ldots$ correspond to permutations of particles (bosons) while in general permutations of the $B_i^+, B_j^+, \ldots$ do not correspond to permutations of particles (parabosons). Since the construction of the Fock space is ultimately connected with the statistics of the particles, this immediately draws the line between ordinary Bose-Einstein statistics and parastatistics, but this is an
entirely different subject and we will not discuss it here.

A consequence of the preceding conversation, is that the computation of the explicit form of the action of the generators $B^+_i, B^-_j$, on vectors of the form $\mathcal{P}(B^+_i)|0>$ which span the carrier space for the parabosonic Fock-like representation, is not a trivial matter at all. Up to our knowledge, it has been done only for the case of a single paraboson (i.e.: the parabosonic algebra $P^{(1)}_B$ generated by the elements $B^+, B^-$ subject to the relations (6) for $i = 1$). Such computations can be found in [18]. They lead to the following relations:

$$B^+|2n> = \sqrt{2n+p} |2n+1> \quad B^+|2n+1> = \sqrt{2n+2} |2n+2>$$

$$(39)$$

$$(B^- |2n> = \sqrt{2n} |2n-1> \quad B^- |2n+1> = \sqrt{2n+p} |2n>$$

for $n = 0, 1, 2, ...$ and $|n > \sim (B^+)n|0>$, which explicitly describe the action of the single paraboson algebra on its Fock-like representation determined by the specific value of $p$.

Finally it is worth noticing before closing this paragraph, that the Fock-like representations $p_B F(p)$ of the parabosonic algebra $P_B$ are $\mathbb{Z}_2-$graded $P_B-$modules (or equivalently $P_B-$modules in the braided monoidal category of the $\mathbb{C}\mathbb{Z}_2$-modules $\mathbb{C}\mathbb{Z}_2 M$).

6 Braided bosons: The Green’s ansatz revisited

We turn in this paragraph to an application of the braided tensor product “technology” in the theory of the representations of the parabosonic algebra $P_B$. We prove the following proposition which gives a “braided” interpretation of the Green’s ansatz device:

**Proposition 6.1.** For every specific choice of the positive integer $p$, the corresponding algebra $G(p)$ in Green’s ansatz, which is described in terms of generators and relations in equations (33), (34), is isomorphic to the braided tensor product algebra, of $p$ copies of the bosonic algebra $B$. We have the superalgebra isomorphism:

$$G(p) \cong B \otimes \ldots \otimes B$$

(40)

where we have $p$-copies of the bosonic algebra $B$ in the right hand side of the above relation.

**Proof.** Let us use -for the elements of the braided tensor product algebra $B \otimes \ldots \otimes B$ -the notation:

$$I \otimes \ldots \otimes I \otimes b_i^{(r)} \otimes I \otimes I \otimes ... \otimes I \equiv b_i^{(r)}$$
where in the left hand side of the above there are $p$-factors, $b_i^\pm$ is placed on the $r$-th place and the identity element $I$ of the bosonic algebra $B$ is everywhere else, for all values of $i = 1, 2, \ldots$ and $r = 1, 2, \ldots, p$. The elements $b_i^{(r)\pm}$ together with the identity $I \otimes I \otimes \ldots \otimes I$ constitute a set of generators of the braided tensor product algebra $B \otimes \ldots \otimes B$ ($p$-copies of $B$). We proceed now to compute the multiplication in the braided tensor product algebra. We have for all values of $i, j = 1, 2, \ldots$ and $r = 1, 2, \ldots, p$

\[
b_i^{(r)-} b_j^{(r)+} = I \otimes \ldots \otimes I \otimes b_i^{-} b_j^{+} \otimes I \otimes \ldots \otimes I
\]

\[
b_j^{(r)+} b_i^{(r)-} = I \otimes \ldots \otimes I \otimes b_j^{+} b_i^{-} \otimes I \otimes \ldots \otimes I
\]

where on the right hand side of the above equations $b_i^{-} b_j^{+}$ and $b_j^{+} b_i^{-}$ respectively, lie on the $r$-th place. Subtracting the above equations by parts finally gives:

\[
[b_i^{(r)-}, b_j^{(r)+}] = I \otimes \ldots \otimes I \otimes [b_i^{-}, b_j^{+}] \otimes I \otimes I \otimes \ldots \otimes I = 0
\]

\[
= I \otimes \ldots \otimes I \otimes \delta_{ij} I \otimes I \otimes I \otimes \ldots \otimes I = \delta_{ij}(I \otimes \ldots \otimes I)
\]

In the right hand side of the above $I \otimes \ldots \otimes I$ is the identity element of the braided tensor product algebra $B \otimes \ldots \otimes B$. In the same way we also have:

\[
[b_i^{(r)+}, b_j^{(r)+}] = [b_i^{(r)-}, b_j^{(r)-}] = 0
\]

in $B \otimes \ldots \otimes B$.

Proceeding in the same way, we compute for all values of $i, j = 1, 2, \ldots$ and $r, s = 1, 2, \ldots, p$ with $r < s$:

\[
b_i^{(r)-} b_j^{(s)+} = I \otimes \ldots \otimes I \otimes b_i^{-} \otimes I \otimes \ldots \otimes I \otimes b_j^{+} \otimes I \otimes \ldots \otimes I
\]

\[
b_j^{(s)+} b_i^{(r)-} = -I \otimes \ldots \otimes I \otimes b_i^{-} \otimes I \otimes \ldots \otimes I \otimes b_j^{+} \otimes I \otimes \ldots \otimes I
\]

On the right hand side of the above equations, the elements $b_i^{-}$ and $b_j^{+}$ lie on the $r$-th and the $s$-th places respectively. The minus sign in the second of the above equations is inserted due to the braiding (15) of Section 2. The reason is that since $r < s$, when computing the product $b_j^{(s)+} b_i^{(r)-}$ in $B \otimes \ldots \otimes B$, the elements $b_j^{+}$ and $b_i^{-}$ have to interchange their order according to the braiding (15):

\[
... \otimes \Psi_{B,B} \otimes \ldots (\ldots \otimes b_j^{+} \otimes b_i^{-} \otimes \ldots) = -\ldots \otimes b_i^{-} \otimes b_j^{+} \otimes \ldots
\]

In the above relation: $|b_i^{-}| = |b_j^{+}| = 1$, since the elements $b_i^{-}$ and $b_j^{+}$ are odd elements in the $\mathbb{Z}_2$-gradation of the bosonic algebra $B$. Addition of the above
mentioned equations by parts produces:

\[ \{ b_i^{(r)-}, b_j^{(s)+} \} = 0 \]  

(43)

In exactly the same way we compute:

\[ \{ b_i^{(r)-}, b_j^{(s)+} \} = \{ b_i^{(r)+}, b_j^{(s)+} \} = 0 \]  

(44)

for all values of \( r, s = 1, 2, ..., p \) and \( r \neq s \).

Now it is straightforward to check that relations (41) and (42) coincide with relations (33) of Section 5.2 under the mapping \( b_i^{(r)\pm} \leftrightarrow b_\alpha^{\pm} \) for \( r = \alpha \), while at the same time relations (43) and (44) coincide with relations (34) of Section 5.2 under the same mapping for \( r = \alpha \) and \( s = \beta \).

This completes the proof.

We should underline here that the above mentioned isomorphism is actually much different than Green’s original idea [8] about the nature of his ansatz: By that time the idea of \( \mathbb{Z}_2 \)-graded algebras and \( \mathbb{Z}_2 \)-graded tensor products was a well known fact for mathematicians (see [2]) -although rather as a special possibility of forming superalgebras and tensor products than as an example of the more general conceptual framework of the braiding- but it was not a mainstream idea in physics. Green conjectured that his ansatz should be a subalgebra of the usual tensor product algebra between \( p \)-copies of the bosonic algebra \( B \) and a copy of a Clifford-like algebra. For a formulation of Green’s original interpretation one should also see [26].

The above proposition permits us a braided reinterpretation of the Green’s ansatz device. Let us start with a restatement of the associative superalgebra homomorphism stated in equation (35) of Section 5.2. We start by inductively defining:

\[ \Delta^{(1)} = \Delta \] where \( \Delta : P_B \to P_B \otimes P_B \) is the coproduct of the Parabosonic super-Hopf algebra \( P_B \) stated in Proposition 4.1 and:

\[ \Delta^{(p)} = (\Delta \otimes Id \otimes ... \otimes Id) \circ \Delta^{(p-1)} \]  

(45)

for any integer \( p \geq 2 \). \( Id : P_B \to P_B \) is the identity map and there are \( p - 1 \) copies of it on the tensor product map of the right hand side of the above equation. It is obvious that \( \Delta^{(p)} : P_B \to P_B \otimes ... \otimes P_B \) (\( P_B \) appearing \( p + 1 \) times) is an associative superalgebra homomorphism. Furthermore we can inductively prove the following Lemma:

**Lemma 6.2.** For any \( p \geq 2 \) and for any \( i = 0, 1, ..., p - 1 \) we have the following:

\[ \Delta^{(p)} = (Id^i \otimes \Delta \otimes Id^{p-i-1}) \circ \Delta^{(p-1)} \]
What the above lemma implies is that the definition of $\Delta^{(p)}$ stated in equation (45) is actually independent of the position where $\Delta$ is placed in the tensor product of the right hand side. Now we can straightforwardly compute:

$$\Delta^{(p-1)}(B_i^\pm) = \sum_{r=1}^{p} I \otimes \ldots \otimes I \otimes B_i^\pm \otimes I \otimes \ldots \otimes I$$  \hspace{1cm} (46)$$

where there are $p$-factors in the tensor product of the right hand side and the generator $B_i^\pm$ of $P_B$ is placed in the $r$-th place for $r = 1, 2, \ldots, p$ and $i = 1, 2, \ldots$.

**Corollary 6.3.** Under the isomorphism of Proposition 6.1 the homomorphism $\pi_p : P_B \to G(p)$ stated in equation (35) is actually given by:

$$\pi_p = (\phi \otimes \ldots \otimes \phi) \circ \Delta^{(p-1)} = (\pi_1 \otimes \ldots \otimes \pi_1) \circ \Delta^{(p-1)}$$  \hspace{1cm} (47)$$

where $p \geq 2$, $\phi$ is the replacement map defined in Proposition 2.3 and there are $p$-copies of it on the right hand side of the above formula.

If we consider the bosonic Fock representation $B_F$, this is a $\mathbb{Z}_2$-graded complex vector space and at the same time a $\mathbb{Z}_2$-graded $B$-module (we can equivalently say: a braided $B$-module, where the braiding is given by (15)). Applying the discussion at the end of Section 3 we immediately get that the tensor product vector space $B_F \otimes \ldots \otimes B_F$ ($p$-copies) becomes a braided ($\mathbb{Z}_2$-graded) module over the braided tensor product algebra $B \otimes \ldots \otimes B$ ($p$-copies). We can straightforwardly check that it is a cyclic module generated by any of it’s elements, thus a simple module (an irreducible representation). Under the isomorphism of Proposition 6.1 this irreducible, braided, ($\mathbb{Z}_2$-graded) $B \otimes \ldots \otimes B$-module coincides with the $G(p)$-module used by Greenberg and Messiah and specified by equations (36), (37). Furthermore, the $\mathbb{Z}_2$-graded complex vector space $B_F \otimes \ldots \otimes B_F$ readily becomes a braided ($\mathbb{Z}_2$-graded) module over the braided tensor product algebra $P_B \otimes \ldots \otimes P_B$ ($p$-copies) through the superalgebra homomorphism:

$$\phi \otimes \ldots \otimes \phi : P_B \otimes \ldots \otimes P_B \to B \otimes \ldots \otimes B$$  \hspace{1cm} (48)$$

We note here that since $\phi$ is an epimorphism of superalgebras (see Proposition 2.3) $\phi \otimes \ldots \otimes \phi$ is also an epimorphism of superalgebras. This in turn implies that since $B_F \otimes \ldots \otimes B_F$ is an irreducible $B \otimes \ldots \otimes B$ module, it is also an irreducible $P_B \otimes \ldots \otimes P_B$ module (the situation being analogous to the one described in Corollary 2.4).

Finally, the space $B_F \otimes \ldots \otimes B_F$ becomes a braided ($\mathbb{Z}_2$-graded) module over the super-Hopf algebra $P_B$ via the superalgebra homomorphism:

$$\Delta^{(p-1)} : P_B \to P_B \otimes \ldots \otimes P_B$$  \hspace{1cm} (49)$$
This is a reducible representation of the parabosonic algebra $P_B$, which obviously coincides with the tensor product of $p$-copies of the braided $P_B$-module $p_B F$, where the tensor product is constructed according to the method described in the end of Section 4.

What the above discussion finally implies is that Greenberg’s and Messiah’s use of the Green’s ansatz which led to the classification of the Fock-like representations $p_B F(p)$ of the parabosonic algebra (see Theorem 5.1) was in fact nothing else than a systematic study of the reduction of the tensor product representations of the parabosonic algebra. This is summarized in the following corollary:

**Corollary 6.4.** For every specific value of $p = 1, 2, \ldots$, the vector space

$$p_B F \otimes \ldots \otimes p_B F$$

($p$-copies of $p_B F$), becomes a braided ($\mathbb{Z}_2$-graded) $P_B$-module through the iterated comultiplication (49). This is a reducible $P_B$-module. It contains an irreducible braided ($\mathbb{Z}_2$-graded) $P_B$-submodule, which is generated (as a submodule) by the vacuum vector $|0> \otimes \ldots \otimes |0>$. This submodule is exactly the parabosonic Fock-like representation $p_B F(p)$.

For a similar discussion, but for the case of the finite degree’s of freedom i.e. for the algebra $P_B^{(n)}$, see also [20], [1].

Let us note here, that the systematic study of the decomposition of the $p_B F \otimes \ldots \otimes p_B F$ $P_B$-module, i.e. the investigation of it’s structure and any other submodules it may contain, is an unsolved problem until nowadays.

7 Self-contained sets and generalizations of Green’s ansatz

As we have already explained in Section 5.2 Theorem 5.1, Greenberg’s and Messiah’s study of the parabosonic Fock-like representations concluded to the fact that for the parabosonic algebra $P_B$, the requirements

$$B_i^- |0> = 0 \quad (B_i^-)^\dagger = B_i^+$$

for all $i = 1, 2, \ldots$, specify -up to isomorphism- an infinite collection of irreducible representations, labelled by a positive integer $p$ and denoted by $p_B F(p)$. Each of them is further specified by the condition

$$B_i^- B_j^+ |0> = p \delta_{ij} |0>$$

for all $i, j = 1, 2, \ldots$ and for any positive, integer value of $p$. Each one of these representations is isomorphic to the corresponding one constructed -for the same
value of $p$- through the homomorphism: \( \pi_p : P_B \to G(p) \) described by equation (35) or equivalently Corollary [6.3].

In [18], Ohnuki and Kamefuchi virtually inverted the problem and posed the following question: Instead of looking for representations of the \( P_B \) algebra constructed through homomorphisms to the Green’s ansatz algebras \( G(p) \), they looked for algebras \( \Gamma(p) \) (where \( p \) is a positive integer) satisfying the following conditions:

(1). For each specific value of \( p \), the corresponding algebra \( \Gamma(p) \) must be generated by a set of generators denoted by \( b_i^\pm (i = 1, 2, \ldots) \), satisfying all the relations of the parabosonic algebra (among any other relations).

(2). Every \( \Gamma(p) \) algebra must possess the following property: Any \( \Gamma(p) \)-module subject for any \( i, j = 1, 2, \ldots \) to the relation

\[
\langle b_i^- b_j^+ |0\rangle = q \delta_{ij} |0\rangle
\]

(50)

for some unique vector \( |0\rangle \) of the representation space, with \( q \) a complex number, must imply: \( q = p \).

The first of the above conditions virtually means that each one of the \( \Gamma(p) \) algebras (for every positive integer value of \( p \)) must be a quotient of the parabosonic algebra \( P_B \), while the second condition means that the order of the representation of the parabosonic algebra \( P_B \) must be “absorbed” into the algebra \( \Gamma(p) \). The last statement is the reason that the \( \Gamma(p) \) algebras (according to our notation) were named in [18] as “self-contained sets of commutation relations” or “commutation relations specific to given order \( p \)”.

In other words: we are looking for an infinite family of quotients of the parabosonic algebra \( P_B \), labelled by a positive integer \( p \), each one of them admitting only one Fock-like representation.

Ohnuki and Kamefuchi, solved the problem for small positive integer values of \( p \) for the parafermionic algebra. In the case of the parabosonic algebra, they only solved it for \( p = 1 \) concluding that \( \Gamma(1) = B \) is no other than the familiar bosonic algebra. They also conjectured the solution for the values \( p = 2 \) and \( p = 3 \) in the case of the parabosonic algebra.

We state here their conjecture for the case of the \( \Gamma(2) \) algebra: It is specified in terms of generators and relations by the following:

\[
\langle b_k^-, b_1^+ b_m^- \rangle_\pm = 2 \delta_{kl} b_m^- - 2 \delta_{lm} b_k^- , \quad \langle b_m^+, b_1^- b_k^+ \rangle_\pm = 2 \delta_{kl} b_m^+ - 2 \delta_{lm} b_k^+ \\
\langle b_k^-, b_1^+ b_m^+ \rangle_\pm = 2 \delta_{lm} b_k^- , \quad \langle b_m^+, b_1^- b_k^+ \rangle_\pm = 2 \delta_{lm} b_k^+ \\
\langle b_k^-, b_1^- b_m^- \rangle_\pm = 0 , \quad \langle b_k^+, b_1^+ b_m^+ \rangle_\pm = 0
\]

(51)
for all values of $k, l, m = 1, 2, ...$ and $\langle A, B, C \rangle$ stands for $ABC - CBA$.

We are now going to prove that the $\Gamma(2)$ algebra, specified in terms of generators and relations by (51), is actually the solution to the Ohnuki’s-Kamefuchi’s question, as this is specified by conditions (1) and (2) stated above. In other words we are going to prove that the $\Gamma(2)$ algebra is the self contained set of commutation relations for $p = 2$ or the commutation relations specific to given order $p = 2$.

**Proposition 7.1.** The $\Gamma(2)$ algebra specified in terms of generators and relations by (51), is a $\mathbb{Z}_2$-graded algebra with it’s generators $b^{\pm}_i$ ($i = 1, 2, ...$), being odd elements. The $\Gamma(2)$ algebra is a quotient algebra of the parabosonic algebra $P_B$. The “replacement” map $f_2 : P_B \rightarrow \Gamma(2)$ defined by: $f_2(B^\pm_i) = b^\pm_i$ is a $\mathbb{Z}_2$-graded algebra epimorphism (i.e.: an even algebra epimorphism).

**Proof.** Let us consider the $\Gamma(2)$ algebra as a quotient of the tensor algebra $T(V_X)$ via it’s ideal generated by the elements specified by relations (51). Let us denote $I_{\Gamma(2)}$ the corresponding ideal of $T(V_X)$ and $\pi_{\Gamma(2)}$ the natural projection:

$$\pi_{\Gamma(2)} : T(V_X) \rightarrow \Gamma(2) = T(V_X) / I_{\Gamma(2)}$$

It is immediate that since relations (51) are homogeneous relations, the $I_{\Gamma(2)}$ ideal is generated by homogeneous elements, thus $\Gamma(2)$ is a $\mathbb{Z}_2$-graded algebra with it’s generators being odd elements. We can now straightforwardly check that the generators of the $\Gamma(2)$ superalgebra satisfy the paraboson relations, i.e.:

$$\pi_{\Gamma(2)}([\{X^{\xi}_i,X^{\eta}_j\},X^{\xi}_k] - (\epsilon - \eta)\delta_{jk}X^{\xi}_i - (\epsilon - \xi)\delta_{ik}X^{\eta}_j) =$$

$$= \{[b^{\xi}_i,b^{\eta}_j],b^{\xi}_k\} - (\epsilon - \eta)\delta_{jk}b^{\xi}_i - (\epsilon - \xi)\delta_{ik}b^{\eta}_j = 0 \tag{52}$$

In other words, the above computation shows that, given generators satisfying relations (51), the same generators have to satisfy the relations of the parabosonic algebra $P_B$ also. In a more mathematical statement we can equivalently say that $\ker(\pi_{P_B}) \subseteq \ker(\pi_{\Gamma(2)})$ or equivalently: $I_{P_B} \subseteq I_{\Gamma(2)}$. The rest of the proof flows exactly as in the case of Proposition 2.3. $I_{P_B} \subseteq I_{\Gamma(2)}$ implies that $\pi_{\Gamma(2)}$ is uniquely extended to an even algebra epimorphism $f_2 : P_B \rightarrow \Gamma(2)$ through the commutative diagram:
where $f_2$ is completely determined by its values on the generators $B_i^{\pm}$ of $P_B$, i.e.: $f_2(B_i^{\pm}) = b_i^{\pm}$. We also mention that: $\ker f_2 = I_{\Gamma(2)}/I_{P_B} = \pi_{P_B}(I_{\Gamma(2)})$ and we finally have the $\mathbb{Z}_2$-graded algebra isomorphism:

$$\Gamma(2) \cong P_B / \ker f_2$$  \hspace{1cm} (53)

The confirmation of the second of the conditions mentioned in page 26 is an easy exercise and can be found in chapter 5 of [18].

Let us stress here, that according to the preceding discussion, the $\Gamma(p)$ algebras arise as generalizations of the bosonic algebra rather than the parabosonic algebra. The $\Gamma(p)$ algebra, admits at most one Fock-like representation for the specific value of $p$, just as in the case of the bosonic algebra there is only one Fock representation corresponding to $p = 1$ and described in Section 5.1.

Based on the above interpretation of the $\Gamma(p)$ algebras, we can proceed in constructing a straightforward generalization of the Green’s ansatz device: Inspired by Proposition 6.1 and Corollary 6.3, we can carry the corresponding construction using now the $\Gamma(2)$ algebra -described in Proposition 7.1- and the $f_2$ epimorphism, instead of the bosonic algebra $B$ and the replacement map $\phi$ used in the mainstream idea of the Green’s ansatz device. Our generalization of the Green’s ansatz consists of the algebra:

$$\Gamma(2) \otimes \Gamma(2) \otimes ... \otimes \Gamma(2)$$ \hspace{1cm} (54)

where there are $p$ copies of $\Gamma(2)$ in the braided ($\mathbb{Z}_2$-graded) tensor product and the homomorphism:

$$(f_2 \otimes ... \otimes f_2) \circ \Delta^{(p-1)} : P_B \rightarrow \Gamma(2) \otimes \Gamma(2) \otimes ... \otimes \Gamma(2)$$ \hspace{1cm} (55)

where $p \geq 2$, $f_2$ is the projection epimorphism defined in Proposition 7.1, and there are $p$-copies of it on the above formula.

We are now going to describe the braided ($\mathbb{Z}_2$-graded) tensor product algebra (54) in terms of generators and relations. Let us use -for the elements of $\Gamma(2) \otimes ... \otimes \Gamma(2)$ - a notation analogous to the one used in the proof of Proposition 6.1, i.e. we denote the elements of (54) as:

$$I \otimes ... \otimes I \otimes b_i^{\pm} \otimes I \otimes I \otimes ... \otimes I \equiv b_i^{(r)}$$ \hspace{1cm} (56)

where in the left hand side of the above there are $p$-factors, $b_i^{\pm} \in \Gamma(2)$ is placed on the $r$-th place and the identity element $I$ of the $\Gamma(2)$ algebra is everywhere else, for
all values of $i = 1, 2, \ldots$ and $r = 1, 2, \ldots, p$. The elements $b_i^{(r)\pm}$ together with the identity $I \otimes I \otimes \ldots \otimes I$ constitute a set of generators of the braided tensor product algebra $\Gamma(2) \otimes \ldots \otimes \Gamma(2)$ (p-copies of $\Gamma(2)$). Using the notation specified in (56) and the braiding (15) of Section 2, we have the following relations:

$$\langle b_k^{(r)-}, b_l^{(r)+}, b_m^{(r)-} \rangle_- = 2\delta_{kl}b_{m}^{(r)-} - 2\delta_{lm}b_{k}^{(r)}- , \quad \langle b_k^{(r)-}, b_l^{(r)+}, b_m^{(r)+} \rangle_- = 2\delta_{lm}b_{k}^{(r)+}$$

$$\langle b_m^{(r)+}, b_l^{(r)-}, b_k^{(r)}- \rangle_- = 2\delta_{kl}b_{m}^{(r)+} - 2\delta_{lm}b_{k}^{(r)+} , \quad \langle b_m^{(r)-}, b_l^{(r)+}, b_k^{(r)}+ \rangle_- = 2\delta_{lm}b_{k}^{(r)-}$$

$$\langle b_k^{(r)-}, b_l^{(r)-}, b_m^{(r)-} \rangle_- = 0 , \quad \langle b_k^{(r)+}, b_l^{(r)+}, b_m^{(r)+} \rangle_- = 0$$

$$\{ b_k^{(r)-}, b_l^{(r)} \} = \{ b_k^{(r)+}, b_l^{(r)} \} = 0$$

which for all values of $k, l = 1, 2, \ldots$, and $r, s = 1, 2, \ldots p$ with $r \neq s$ completely specify the braided $(\mathbb{Z}_{2p\text{-}}$-graded) tensor product algebra (54) in terms of generators and relations. It is worth noting that we have constructed an algebra which mixes both bilinear and trilinear relations. Finally we can immediately conclude that the homomorphism (55) has the same functional form as in the usual Green ansatz

$$(f_2 \otimes \ldots \otimes f_2) \circ \Delta^{(p-1)}(B_i^{\pm}) = \sum_{\alpha=1}^{p} b_i^{\alpha \pm}$$

but the elements on the right hand side of the above are no more “anticommuting” bosons -as in the usual Green’s ansatz case- but elements of the $\Gamma(2) \otimes \ldots \otimes \Gamma(2)$ algebra satisfying relations (57).

8 Discussion

The explicit construction of the parabosonic Fock-like representation $p_nF(p)$ for $p > 1$ is an unsolved problem until nowadays. The problem is yet unsolved even for the case of the finite degrees of freedom, i.e. even for the algebra $P_B^{(n)}$ for $n > 1$ and $p > 1$ there are no explicit expressions in the bibliography for the matrix elements of the $p_n^{(s)}F(p)$ module. Even the -simpler- problems of the construction of an orthonormal basis for the $p_nF(p)$ or the $p_n^{(s)}F(p)$ modules are open for the case $p > 1$ and $n > 1$. We should note here that a very interesting paper has appeared [13] -by the time the writing of this work was almost completed- where some of the above problems are dealt with, for the case of finite degrees of freedom: The authors consider the module $p_n^{(s)}F(p)$ for $n > 1$ and $p > 1$. They use techniques of
induced representations, the well known \[7\] isomorphism of \(P_B^{(n)}\) with the Lie superalgebra \(osp(1/2n)\) and elements from the representation theory of \(osp(1/2n)\).

They construct an orthogonal Gelfand-Zetlin basis of \(P_B^{(n)}F(p)\) and they calculate explicitly the corresponding matrix elements. However, their techniques do not give an answer for the general case of the \(P_B F(p)\) module with \(p > 1\) since the representation theory of the infinite dimensional Lie superalgebras is yet an unexplored subject. On the other hand, the braided interpretation of the Green’s ansatz device presented here indicates a possible way for a general solution of the above mentioned problem. Our approach indicates that the Green’s ansatz algebras \(G(p)\) for \(p = 1, 2, \ldots\), should be “utilized” as \(\mathbb{Z}_2\)-graded algebras (with their generators being odd elements). Their braided \((\mathbb{Z}_2\text{-graded})\), irreducible modules have been shown to give rise to braided \((\mathbb{Z}_2\text{-graded})\), reducible tensor product modules of the parabosonic algebra \(P_B\). The role of the super-Hopf structure of the \(P_B\) algebra is essential in this process. These \(P_B\)-modules finally must be reduced to their irreducible constituents. The isomorphism of proposition Proposition 6.1 provides us with an analytic tool for such a calculation to be performed. It will be a very interesting thing to proceed with the explicit computations, to compare the results -for the case of the finite degrees of freedom- with the corresponding results of \[13\] and to extract the matrix elements for the general case of the \(P_B\) algebra with an infinite number of parabosons. This will be the subject of a forthcoming work.

Furthermore, we stress here that the above mentioned approach admits straightforward generalisations for the case of algebras which describe mixed systems of paraparticles such as the relative parabose or the relative parafermi sets (see \[9\] for their description and for generalized versions of the Green’s ansatz for these algebras): The relative parafermi set has been shown to be a \(\mathbb{Z}_2\)-graded algebra \[21\] while the relative parabose set has been shown to be a \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-graded \[34\] algebra. It would thus be an interesting idea to apply similar techniques to these algebras and obtain results about their graded Hopf structure, their braided representations and their braided tensor products. Such results, combined with suitable generalizations of Proposition 6.1 can lead us to the explicit construction (matrix elements, orthonormal basis, character formulas) of Fock-like modules for mixed parafields. Of course such questions inevitably involve questions of pure mathematical interest, such as the possible quasitriangular structures (and thus the possible braidings) for the \(\mathbb{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)\) group Hopf algebra, which up to our knowledge have not yet been solved. (see \[28\] for a relevant discussion).

Let us close this discussion with some comments regarding the generalization of Green’s ansatz presented in Section\[7\] An obvious question which arises at first glance is the study of the representations of the \(\Gamma(2) \otimes \ldots \otimes \Gamma(2)\) superalgebras and specifically the determination of whether they can help us “build” -through the homomorphism \[55\]— essentially new representations of the parabosonic algebra.
Of course the relation -if any- of the $\Gamma(2)$ superalgebra with the $B \otimes B$ or even the $\text{End}(B^F \otimes B^F)$ superalgebras (or any of it’s subalgebras) plays an essential part in answering this question and should be the starting point of such an investigation. More generally, we have already mentioned that the general solution to the problem of the construction of the commutation relations specific to given order $p$ or equivalently the determination of the $\Gamma(p)$ algebra for an arbitrary positive integer value of $p$ is yet an unsolved problem. In [18] solutions to the problem are conjectured for the cases of $p = 2, 3$. But a general solution to this problem does not exist in the bibliography. Provided such a general solution, it is easy to see that our method will then provide a whole family of generalizations to the Green’s ansatz. They will be of the form:

$$\Gamma(p) \otimes \ldots \otimes \Gamma(p)$$

with $q$ factors of $\Gamma(p)$ appearing on the above braided ($\mathbb{Z}_2$-graded) tensor product (Note that in this case $q$ is an arbitrary positive integer irrelevant of the value of $p$). Homomorphisms of the form

$$(f_p \otimes \ldots \otimes f_p) \circ \Delta^{(q-1)} : P_B \rightarrow \Gamma(p) \otimes \ldots \otimes \Gamma(p)$$

will provide the suitable link between the representationstheories of $\Gamma(p) \otimes \ldots \otimes \Gamma(p)$ and the parabosonic algebras as long as suitable generalizations of the Proposition[7,1] are stated.

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