Quantum monotone metrics induced from trace non-increasing maps and additive noise

Koichi Yamagata*
The University of Electro-Communications Department of Informatics,
1-5-1, Chofugaoka, Chofu, Tokyo 182-8585, Japan

Abstract
Quantum monotone metric was introduced by Petz, and it was proved that quantum monotone metrics on the set of quantum states with trace one were characterized by operator monotone functions. Later, these were extended to monotone metrics on the set of positive operators whose traces are not always one based on completely positive, trace preserving (CPTP) maps. It was shown that these extended monotone metrics were characterized by operator monotone functions continuously parameterized by traces of positive operators, and did not have some ideal properties such as monotonicity and convexity with respect to the positive operators. In this paper, we introduce another extension of quantum monotone metrics which have monotonicity under completely positive, trace non-increasing (CPTNI) maps and additive noise. We prove that our extended monotone metrics can be characterized only by static operator monotone functions from few assumptions without assuming continuities of metrics. We show that our monotone metrics have some natural properties such as additivity of direct sum, convexity and monotonicity with respect to positive operators.

1 Introduction

In classical statistics, Cencov [3] proved that the Fisher information metric is the only Riemannian metric on families of probabilities up to rescaling that has monotonicity under Markov maps. Petz extended Cencov’s theorem to families of quantum states, and it was revealed that there is a one-to-one correspondence between quantum monotone metrics and operator monotone functions [7]. To introduce Petz’s characterization of monotone metrics, let us define some notations.

Let \( \mathcal{B}(\mathbb{C}^n) \) be the set of all linear operators on \( \mathbb{C}^n \), and let \( \mathcal{S}^{++}(\mathbb{C}^n) = \{ \rho \in \mathcal{B}(\mathbb{C}^n) \mid \text{Tr} \rho = 1, \rho > 0 \} \) be the set of all strictly positive quantum states on \( \mathbb{C}^n \). The quantum analog of Markov map can be expressed as completely positive, trace preserving map (CPTP map) \( T : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^m) \), and it can be represented by operator sum representation

\[
T(X) = \sum_{i=1}^{k} A_i X A_i^* \]

with linear maps \( A_1, \ldots, A_k \) from \( \mathbb{C}^n \) to \( \mathbb{C}^m \) such that

\[
\sum_{i=1}^{k} A_i^* A_i = I.
\]

We denote by \( \mathcal{C}(\mathbb{C}^n, \mathbb{C}^m) \) the set of all CPTP maps from \( \mathcal{B}(\mathbb{C}^n) \) to \( \mathcal{B}(\mathbb{C}^m) \). Let \( L_\rho \) and \( R_\rho \) be super operators on \( \mathcal{B}(\mathbb{C}^n) \) defined by

\[
L_\rho(X) = \rho X, \\
R_\rho(X) = X \rho,
\]

*koichi.yamagata@uec.ac.jp
with a strictly positive operator \( \rho \).

Petz defined quantum monotone metric as follows.

**Definition 1.1.** A family of functions \( \{ K^{(n)}(\cdot, \cdot) \}_{n \in \mathbb{N}^+} \) from \( \mathcal{S}^+ (\mathbb{C}^n) \times \mathcal{B} (\mathbb{C}^n) \times \mathcal{B} (\mathbb{C}^n) \) to \( \mathbb{C} \) is a family of monotone metrics, if the following conditions hold:

(a) For every \( n \in \mathbb{N}^+ \) and \( \rho \in \mathcal{S}^+ (\mathbb{C}^n) \) the map

\[
K^{(n)}_\rho : \mathcal{B} (\mathbb{C}^n) \times \mathcal{B} (\mathbb{C}^n) \to \mathbb{C} \quad (X, Y) \mapsto K^{(n)}_\rho (X, Y)
\]

is an inner product.

(b) For every \( n, m \in \mathbb{N}^+ \), \( X \in \mathcal{B} (\mathbb{C}^n) \), CPTP map \( T \in \mathcal{C} (\mathbb{C}^n, \mathbb{C}^m) \), \( \rho \in \mathcal{S}^+ (\mathbb{C}^n) \) such that \( T (\rho) \in \mathcal{S}^+ (\mathbb{C}^m) \), the inequality

\[
K^{(m)}_{T (\rho)} (T (X), T (X)) \leq K^{(n)}_\rho (X, X)
\]

holds.

(c) For every \( n \in \mathbb{N}^+ \) and \( X \in \mathcal{B} (\mathbb{C}^n) \), the map \( \rho \mapsto K^{(n)}_\rho (X, X) \) is continuous.

Any monotone metric can be characterized by operator monotone functions as follows. (See Appendix \( \square \) for a brief account of operator monotone functions. See \( \square \) for more details.)

**Theorem 1.2 (Petz\[11\])**. \( \{ K^{(n)}(\cdot, \cdot) \}_{n \in \mathbb{N}^+} \) is a family of monotone metrics if and only if there exists an operator monotone function \( f : (0, \infty) \to (0, \infty) \) and a non-negative constant \( c \in \mathbb{R} \) such that

\[
K^{(n)}_\rho (X, Y) = \text{Tr} X^* \left[ (\mathbf{R}_\rho f (\mathbf{L}_\rho \mathbf{R}_\rho^{-1}))^{-1} Y \right] + c (\text{Tr} X^* (\text{Tr} Y)) \tag{1.1}
\]

for \( n \in \mathbb{N}^+ \), \( \rho \in \mathcal{S}^+ (\mathbb{C}^n) \), and \( X, Y \in \mathcal{B} (\mathbb{C}^n) \).

Note that the second term in \( (1.1) \) does not appear when \( X \) and \( Y \) are derivatives of parameterized density operators whose traces are fixed to one, and \( \text{Tr} X = \text{Tr} Y = 0 \). However, if we consider a extended parametric family containing states whose traces are not fixed to one, the traces of derivatives are not always zero.

In a previous study, monotone metrics on the quantum state family \( \mathcal{S}^+ (\mathbb{C}^n) \) are extended to strictly positive operators \( \mathcal{B}^+ (\mathbb{C}^n) \) in Ref. \( \square \). In their study, a family of functions \( \{ K^{(n)}(\cdot, \cdot) \}_{n \in \mathbb{N}^+} \) from \( \mathcal{B}^+ (\mathbb{C}^n) \times \mathcal{B} (\mathbb{C}^n) \times \mathcal{B} (\mathbb{C}^n) \) to \( \mathbb{C} \) is a family of monotone metrics if following conditions hold:

(a) For every \( n \in \mathbb{N}^+ \) and \( \rho \in \mathcal{S}^+ (\mathbb{C}^n) \) the map

\[
K^{(n)}_\rho : \mathcal{B} (\mathbb{C}^n) \times \mathcal{B} (\mathbb{C}^n) \to \mathbb{C} \quad (X, Y) \mapsto K^{(n)}_\rho (X, Y)
\]

is an inner product.

(b) For every \( n, m \in \mathbb{N}^+ \), \( X \in \mathcal{B} (\mathbb{C}^n) \), CPTP map \( T \in \mathcal{C} (\mathbb{C}^n, \mathbb{C}^m) \), \( \rho \in \mathcal{S}^+ (\mathbb{C}^n) \) such that \( T (\rho) \in \mathcal{S}^+ (\mathbb{C}^m) \), the inequality

\[
K^{(m)}_{T (\rho)} (T (X), T (X)) \leq K^{(n)}_\rho (X, X)
\]

holds.

(c) For every \( n \in \mathbb{N}^+ \) and \( X \in \mathcal{B} (\mathbb{C}^n) \), the map \( \rho \mapsto K^{(n)}_\rho (X, X) \) is continuous.

These conditions are same as Definition \( \square \) except that \( \mathcal{S}^+ (\mathbb{C}^n) \) is replaced by \( \mathcal{B}^+ (\mathbb{C}^n) \). They proved that \( \{ K^{(n)}(\cdot, \cdot) \}_{n \in \mathbb{N}^+} \) is a family of monotone metrics if and only if there exist a continuous function \( b : \mathbb{R}^+ \to \mathbb{R}^+ \) and a continuous family of operator monotone functions \( \{ f_t : \mathbb{R}^+ \to \mathbb{R}^+ \}_{t \in \mathbb{R}^+} \) such that

\[
K^{(n)}_\rho (X, Y) = \text{Tr} X^* \left[ (\mathbf{R}_\rho f_t (\mathbf{L}_\rho \mathbf{R}_\rho^{-1}))^{-1} Y \right] + b (\text{Tr} \rho) (\text{Tr} X^* (\text{Tr} Y)) \tag{1.2}
\]
with \( f_t(1) - t b(t) > 0 \), for \( n \in \mathbb{N}^+ \), \( \rho \in \mathcal{B}^{++}(\mathbb{C}^n) \), and \( X, Y \in \mathcal{B}(\mathbb{C}^n) \). To distinguish this metric from ours defined later, we call it CPTP monotone metric. It can be seen that this CPTP monotone metric does not have some desirable properties. For example, \( K^{(\sigma)}_\rho(X, X) \) is not convex with respect to \( \rho \) in general. Further, it does not have the additivity with respect to direct sum:

\[
K^{(\sigma_1 + \sigma_2)}_{\rho_1 \oplus \rho_2}(X_1 \oplus X_2, X_1 \oplus X_2) = K^{(\sigma_1)}_{\rho_1}(X_1, X_1) + K^{(\sigma_2)}_{\rho_2}(X_2, X_2).
\]

This means that the inner product structures are different between whole and part.

In this study, we introduce another extension of quantum monotone metrics which have monotonicity under completely positive, trace non-increasing maps. We prove that our extended monotone metrics can be characterized by fixed operator monotone functions from few assumptions without assuming continuities of metrics. We show that our monotone metrics have some natural properties such as additivity of direct sum, convexity and monotonicity with respect to unnormalized states.

## 2 Quantum monotone metrics induced from CPTNI maps and additive noise

In quantum mechanics, a quantum operation \( T \) is used to describe transformations of quantum states, and it must satisfy \( 0 \leq \text{Tr} \, T(\rho) \leq 1 \) for any quantum state \( \rho \) to be physical\cite{6}. A quantum operations can be expressed as a completely positive, trace non-increasing (CPTNI) maps \( T : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^m) \), and it can be represented by operator sum representation

\[
T(X) = \sum_{i=1}^k A_i X A_i^*.
\]

with linear maps \( A_1, \ldots, A_k \) from \( \mathbb{C}^n \) to \( \mathbb{C}^m \) such that

\[
\sum_{i=1}^k A_i^* A_i \leq I.
\]

If there is a state \( \rho \) such that \( \text{Tr} \,[T(\rho)] < 1 \), then the quantum operation \( T \) does not provide a complete description of processes that may occur in a system, and \( \text{Tr} \,[T(\rho)] \) is equal to the probability that \( T \) occurs\cite{6}. In this sense, unnormalized states having traces less than one can be interpreted as results of incomplete quantum operations. We denote by \( \mathcal{T}(\mathbb{C}^n, \mathbb{C}^m) \) the set of all CPTNI maps from \( \mathcal{B}(\mathbb{C}^n) \) to \( \mathcal{B}(\mathbb{C}^m) \). We denote by \( \mathcal{T}^{++}(\mathbb{C}^n) := \{ \rho \in \mathcal{B}(\mathbb{C}^n) \mid \text{Tr} \rho \leq 1, \rho > 0 \} \) and \( \mathcal{T}(\mathbb{C}^n) := \{ \rho \in \mathcal{B}(\mathbb{C}^n) \mid \text{Tr} \rho \leq 1, \rho \geq 0 \} \) the set of all strictly and non-strictly positive operators with traces less than one. To extend monotone metric, it is natural to consider a condition \( K_{\mathcal{T}(\rho)}(T(X), T(\rho)) \leq K_\rho(X, X) \) for every CPTNI map \( T \).

When the trace of an unnormalized quantum state \( \rho \) is less than one, \( \rho + \sigma \) may also physical with \( \sigma \geq 0 \) such that \( \text{Tr} \,(\rho + \sigma) < 1 \). In this case, \( \sigma \) is considered to be noise for \( \rho \). Therefore a metric on \( \mathcal{T}^{++}(\mathbb{C}^n) \) which has monotonicity under noise should satisfy \( K_{\rho + \sigma}(X, X) \leq K_\rho(X, X) \). In this inequality, \( X \) does not need to be changed under this kind of noise because the derivative of \( \rho + iX + \sigma \) with respect to \( i \) is \( X \). Note that this kind of noise is not necessary to be considered when only normalized states and CPTP maps are treated because \( \rho \to T(\rho) + (\text{Tr} \rho - \text{Tr} T(\rho))\sigma \) is a CPTP map for \( \sigma \in \mathcal{T}^{++}(\mathbb{C}^m) \) and \( T \in \mathcal{T}(\mathbb{C}^n, \mathbb{C}^m) \), in fact, it has an operator sum representation

\[
\sum_{i=1}^k A_i \rho A_i^* + \sum_{m} B_m \rho B_m^*.
\]

where

\[
B_m = \sqrt{\lambda_m} |e_m\rangle \langle e_m| \sqrt{1 - \sum_{i=1}^k A_i^* A_i},
\]
with the spectral decomposition $\sigma = \sum \lambda_i |e_i\rangle \langle e_i|$ and the operator sum representation $T(X) = \sum_{i=1}^k A_i X A_i^*$.

Based on the above considerations, we define quantum monotone metrics which have monotonocity under CPTNI maps as follows.

**Definition 2.1.** A family of functions \( \{K^{(n)}(\cdot, \cdot)\}_{n \in \mathbb{N}^+} \) from $\mathcal{F}^{++}(\mathbb{C}^n) \times \mathcal{B}(\mathbb{C}^n) \times \mathcal{B}(\mathbb{C}^n)$ to $\mathbb{C}$ is a family of monotone metrics, if the following conditions hold:

(a) For every $n \in \mathbb{N}^+$ and $\rho \in \mathcal{F}^{++}(\mathbb{C}^n)$ the map

\[
K^{(n)}_\rho : \mathcal{B}(\mathbb{C}^n) \times \mathcal{B}(\mathbb{C}^n) \to \mathbb{C} \quad (X, Y) \mapsto K^{(n)}_\rho(X, Y)
\]

is an inner product.

(b) For every $n,m \in \mathbb{N}^+$, CPTNI map $T \in \mathcal{F}(\mathbb{C}^n, \mathbb{C}^m)$, $X \in \mathcal{B}(\mathbb{C}^n)$, $\rho \in \mathcal{F}^{++}(\mathbb{C}^n)$, and $\sigma \in \mathcal{F}(\mathbb{C}^m)$ such that $T(\rho) + \sigma \in \mathcal{F}^{++}(\mathbb{C}^m)$, the inequality

\[
K^{(m)}_{T(\rho)+\sigma}(T(X), T(X)) \leq K^{(n)}_\rho(X, X)
\]

holds.

We call these metrics CPTNI monotone metrics to distinguish them from metrics based on CPTP maps \cite{12}. We prove the following Theorem.

**Theorem 2.2.** \( \{K^{(n)}(\cdot, \cdot)\}_{n \in \mathbb{N}^+} \) is a family of CPTNI monotone metrics if and only if there exists an operator monotone function $f : (0, \infty) \to (0, \infty)$ such that

\[
K^{(n)}_\rho(X, Y) = \text{Tr} X^* [(R_\rho f(L_\rho R_\rho^{-1}))^{-1} Y]
\]

for $n \in \mathbb{N}^+$, $\rho \in \mathcal{F}^{++}(\mathbb{C}^n)$, and $X, Y \in \mathcal{B}(\mathbb{C}^n)$.

Note that the continuous condition is not necessary to characterize CPTNI monotone metrics unlike Definition\cite{11}. Further, the second term in RHS of \cite{11} does not appear in this theorem, and the operator monotone $f$ does not depend on $\text{Tr} \rho$ unlike CPTP monotone metrics \cite{12}.

**The proof of “only if” part.** The inequality (2.1) implies the unitary covariance

\[
K^{(n)}_\rho(X, X) = K^{(n)}_{U \rho U^*}(UXU^*, UXU^*)
\]

for any unitary operator $U$ because $U$ is an invertible CPTNI map, and

\[
K^{(n)}_\rho(X, Y) = K^{(n)}_{U \rho U^*}(UXU^*, UYU^*)
\]

due to the polarization identity of the inner product. So we can assume $\rho = \text{Diag}(p_1, \ldots, p_n)$ is diagonal, since we can freely choose such basis for further computations which consists of the eigenvectors of $\rho$. We denote by $E_{ij}^{(n)}$ the matrix unit in $\mathcal{B}(\mathbb{C}^n)$. We characterize all elements $K^{(n)}_\rho(E_{ij}^{(n)}, E_{kl}^{(n)})$ of $K^{(n)}_\rho$ for $i, j, k, l \in \{1, \ldots, n\}$ by characterizing the following four types of elements:

(i) $K^{(n)}_\rho(E_{12}^{(n)}, E_{ij}^{(n)})$ \( (k \neq 1) \)

(ii) $K^{(n)}_\rho(E_{11}^{(n)}, E_{22}^{(n)})$

(iii) $K^{(n)}_\rho(E_{12}^{(n)}, E_{12}^{(n)})$

(iv) $K^{(n)}_\rho(E_{11}^{(n)}, E_{11}^{(n)})$
Other elements of $K_{\rho}^{(n)}$ can be obtained by replacing bases of the Hilbert space $\mathbb{C}^n$.

(i) Let $U = \text{Diag}(c, 1, \ldots, 1) \in \mathcal{B}(\mathbb{C}^n)$ be an unitary operator with $|c| = 1$. When $k \neq 1$, we have

$$K_{U\rho U^*}^{(n)}(E_{12}^{(n)}, U E_{kl}^{(n)} U^*) = \begin{cases} c^2 K_{\rho}^{(n)}(E_{12}^{(n)}, E_{kl}^{(n)}) & \text{if } l = 1, \\ c K_{\rho}^{(n)}(E_{12}^{(n)}, E_{kl}^{(n)}) & \text{if } l \neq 1. \end{cases}$$

Therefore $K_{\rho}^{(n)}(E_{12}^{(n)}, E_{kl}^{(n)}) = 0$.

(ii) Let $T \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n)$ be a CPTNI map defined by

$$T(E_{ij}^{(n)}) = \begin{cases} E_{11}^{(n)} & \text{if } (i, j) = (1, 1), \\ 0 & \text{if } (i, j) \neq (1, 1), \end{cases}$$

and let $\sigma = \text{Diag}(0, \lambda_2, \ldots, \lambda_n) \in \mathcal{F}(\mathbb{C}^n)$. By the definition of CPTNI monotone metrics, for any $\lambda \in \mathbb{R}$,

$$K_{\rho}^{(n)}(E_{11}^{(n)} + \lambda E_{22}^{(n)}, E_{11}^{(n)} + \lambda E_{22}^{(n)}) \geq K_{T(\rho) + \sigma}^{(n)}(T(E_{11}^{(n)} + \lambda E_{22}^{(n)}), T(E_{11}^{(n)} + \lambda E_{22}^{(n)})) = K_{\rho}^{(n)}(E_{11}^{(n)}, E_{11}^{(n)}).$$

This means $E_{11}^{(n)}$ and $E_{22}^{(n)}$ are orthogonal with respect to the inner product $K_{\rho}^{(n)}$, that is,

$$K_{\rho}^{(n)}(E_{11}^{(n)}, E_{22}^{(n)}) = 0.$$

Note that the second terms of RHSs of (1.1) and (1.2) are vanished here by using CPTNI maps and additive noise.

(iii) By using a CPTNI map $T_1 \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^2)$ defined by

$$T_1(E_{ij}^{(n)}) = \begin{cases} E_{ij}^{(2)} & \text{if } i, j \leq 2, \\ O & \text{otherwise}, \end{cases}$$

we have

$$K_{\rho}^{(n)}(E_{12}^{(n)}, E_{12}^{(n)}) \geq K_{T_1(\rho)}^{(n)}(T_1(E_{12}^{(n)}), T_1(E_{12}^{(n)})) = K_{\text{Diag}(\rho)}^{(2)}(E_{12}^{(2)}, E_{12}^{(2)}).$$

By using a CPTNI map $T_2 \in \mathcal{B}(\mathbb{C}^2, \mathbb{C}^n)$ defined by

$$T_2(E_{ij}^{(2)}) = E_{ij}^{(n)}$$

and $\sigma = \text{Diag}(0, 0, \lambda_3, \ldots, \lambda_n) \in \mathcal{F}(\mathbb{C}^n)$, we have

$$K_{\text{Diag}(\rho)}^{(2)}(E_{12}^{(2)}, E_{12}^{(2)}) \geq K_{T_2(\text{Diag}(\rho)) + \sigma}^{(2)}(T_2(E_{12}^{(2)}), T_2(E_{12}^{(2)})) = K_{\rho}^{(n)}(E_{12}^{(n)}, E_{12}^{(n)}).$$

Therefore

$$K_{\rho}^{(n)}(E_{12}^{(n)}, E_{12}^{(n)}) = K_{\text{Diag}(\rho)}^{(2)}(E_{12}^{(2)}, E_{12}^{(2)}) = \lambda_g(p_1, p_2)$$

depends only on $p_1, p_2$. Note that this fact is different from CPTP monotone metrics (1.2) which depends on a trace of $\rho$.

Let $S_1 \in \mathcal{B}(\mathbb{C}^2, \mathbb{C}^2 \otimes \mathbb{C}^m)$ be a CPTNI map defined by

$$S_1(E^{(2)}) = E^{(2)} \otimes I^{(m)}/m$$

for any $E^{(2)} \in \mathcal{B}(\mathbb{C}^2)$ with an identity operator $I^{(m)} \in \mathcal{B}(\mathbb{C}^m)$, and let $S_2 \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^m, \mathbb{C}^2)$ be a CPTNI map defined by

$$S_2(E^{(2m)}) = \text{Tr}_2 E^{(2m)}$$

(2.4)
for any $E^{(2n)} \in \mathcal{B}(C^2 \otimes C^n)$ where $\text{Tr}_2$ is a partial trace with respect to $C^n$. Because $S_2(S_1(E^{(2)})) = E^{(2)}$, we have

$$g(p_1, p_2) = K_{\text{Diag}(p_1, p_2)}^{(2)}(E^{(2)}_{12}, E^{(2)}_{12}) = K_{\text{Diag}(p_1, p_2)}^{(2n)}(E^{(2)}_{12} \otimes I^{(m)} / m, E^{(2)}_{12} \otimes I^{(m)} / m) = \frac{1}{m} g\left(\frac{p_1}{m}, \frac{p_2}{m}\right),$$

for any $m \in \mathbb{N}$. From this, we have, for any rational number $q = \frac{m_1}{m_2}$ with $m_1, m_2 \in \mathbb{N}^+$ such that $0 < qp_1 + qp_2 \leq 1$,

$$g(qp_1, qp_2) = g\left(\frac{m_1}{m_2} p_1, \frac{m_1}{m_2} p_2\right) = \frac{m_1}{m_2} g\left(\frac{1}{m_2} p_1, \frac{1}{m_2} p_2\right) = \frac{m_2}{m_1} g(p_1, p_2) = \frac{1}{q} g(p_1, p_2).$$

Because a function $q \mapsto g(qp_1, qp_2)$ monotonically decreases due to the definition 2.1 we have

$$g(qp_1, qp_2) = \frac{1}{q} g(p_1, p_2) \quad \text{(2.5)}$$

for any real number $q \in (0, \frac{1}{p_1 + p_2}]$. Note that we don’t require the continuity of $K_{\rho}^{(n)}$ to obtain 2.5.

We can define a function $f : (0, \infty) \rightarrow (0, \infty)$ such that

$$g(p_1, p_2) = \frac{1}{p_2 f(p_1/p_2)}$$

because $p_2 g(p_1, p_2)$ depends only on $p_1/p_2$ due to 2.5. We prove $f$ is an operator monotone function. Let

$$\bar{X} = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \in \mathcal{B}(C^{2m})$$

and

$$\bar{\rho} = \begin{pmatrix} \epsilon I^{(m)} & 0 \\ 0 & \rho \end{pmatrix} \in \mathcal{F}^{(+)}(C^{2m})$$

are an observable and an unnormalized state represented by block matrices with a positive real number $\epsilon$. It follows that

$$K_{\rho}^{(2m)}(\bar{X}, \bar{\rho}) = \text{Tr} X^{*} \left[ \epsilon f(\frac{\rho}{\epsilon}) \right]^{-1} X.$$

Because of definition 2.1 $\rho \leq \rho’$ implies $f(\frac{\rho}{\epsilon}) \leq f(\frac{\rho’}{\epsilon})$. Therefore $f$ is an operator monotone function. Note that every operator monotone function on $(0, \infty)$ is continuous and operator concave. (See Appendix A).

(iv) By a similar discussion as (iii), $K_{\rho}^{(n)}(E^{(n)}_{11}, E^{(n)}_{11}) = g(p_1)$ depends only on $p_1$. It follows that

$$g(p_1) = K_{\frac{1}{2} f^{(2)}}^{(2)}(\frac{1}{2} f^{(2)}, \frac{1}{2} f^{(2)}) = K_{\frac{1}{2} f^{(1)}}^{(2)}(\frac{1}{2} X, \frac{1}{2} X) = \frac{1}{p_1 f(1)} X,$$

with $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, because eigenvalues of $X$ are $\pm 1$.}

Note that the above proof derives continuity of the metric $K_{\rho}^{(n)}(X, X)$ with respect to $\rho$ from only a few assumptions. Also note that, unlike 1.2, the variety of CPTNI monotone metrics depends only on an operator monotone function $f$.

The proof of “if” part. When $f(x) = 1$, we prove a metric defined by

$$K_{\rho}^{(n)}(X, X) = \text{Tr} X^{*} \left[ R_{\rho}^{-1} X \right] = \text{Tr} X \rho^{-1} X^{*} \quad \text{(2.6)}$$


is a CPTNI monotone metric. Because
\[
\begin{pmatrix}
\rho & X^* \\
X & X\rho^{-1}X^*
\end{pmatrix} \geq 0,
\]
it follows
\[
\begin{pmatrix}
T(\rho) + \sigma & T(X^*) \\
T(X) & T(X\rho^{-1}X^*)
\end{pmatrix} \geq \begin{pmatrix}
T(\rho) & T(X^*) \\
T(X) & T(X\rho^{-1}X^*)
\end{pmatrix} \geq 0
\]
for any CPTNI map \( \mathcal{T} \in \mathcal{C}(\mathbb{C}^m, \mathbb{C}^m) \) and a positive operator \( \sigma \). Then it follows
\[
T(X) (T(\rho) + \sigma)^{-1} T(X^*) \leq T(X\rho^{-1}X^*)
\]
by considering the Schur complement of the block matrix. Because \( T \) is trace non-increasing,
\[
K^{(m)}_{T(\rho)+\sigma}(T(X), T(X)) = \text{Tr} T(X) (T(\rho) + \sigma)^{-1} T(X^*) \\
\leq \text{Tr} T(X\rho^{-1}X^*) \leq \text{Tr} X\rho^{-1}X^* \\
= K^{(n)}(X,X).
\]
Therefore \( K^{(n)}(\cdot, \cdot)_{n \in \mathbb{N}^+} \) is a family of CPTNI monotone metrics. Similarly, when \( f(x) = x \),
\[
K^{(n)}_{\rho}(X,X) = \text{Tr} X^* \left[ (L_{\rho}^{-1} m_{\rho} R_{\rho}^{-1}) X \right],
\]
is also CPTNI operator metric.

When \( f : (0, \infty) \rightarrow (0, \infty) \) is any operator monotone function, \( h(x) = \frac{x}{T^{1/2}} \) is also operator monotone (See Appendix A), and \( K^{(n)}_{\rho}(X,X) \) can be rewritten to
\[
K^{(n)}_{\rho}(X,X) = \text{Tr} X^* \left[ (L_{\rho}^{-1} m_{\rho} R_{\rho}^{-1}) X \right],
\]
where
\[
A m_{\rho} B = A^{1/2} h(A^{-1/2} B A^{-1/2}) A^{1/2}
\]
is operator mean of strictly positive operators \( A, B \) on a Hilbert space \( \mathcal{H} \). Operator mean of non-negative operators \( A \) and \( B \) is \( A m_{\rho} B = \lim_{\varepsilon \rightarrow 0} (A + \varepsilon I) m_{\rho} (B + \varepsilon I) \). It is known that operator means fulfill inequalities
\[
A m_{\rho} B \leq A' m_{\rho} B'
\]
if \( A \leq A' \) and \( B \leq B' \), and
\[
C(A m_{\rho} B) C^* \leq (CAC^*) m_{\rho} (CBC^*)
\]
for any operator \( C \). (See Appendix B) By using them, we have
\[
K^{(m)}_{T(\rho)+\sigma}(T(X), T(X)) = \text{Tr} T(X^*) \left[ (L_{T(\rho)+\sigma} m_{\rho} R_{T(\rho)+\sigma}^{-1}) T(X) \right] \\
= \text{Tr} X^* \left[ T^* (L_{T(\rho)+\sigma} m_{\rho} R_{T(\rho)+\sigma}^{-1}) T \right] X \\
\leq \text{Tr} X^* \left[ T^* (L_{T(\rho)+\sigma} m_{\rho} R_{T(\rho)+\sigma}) T \right] X \\
\leq \text{Tr} X^* \left[ L_{\rho}^{-1} m_{\rho} R_{\rho}^{-1} \right] X = K^{(n)}_{\rho}(X,X),
\]
where CPTNI monotonicity of (2.6) and (2.7) are used in the last inequality. This proves \( K^{(n)}(\cdot, \cdot)_{n \in \mathbb{N}^+} \) is a family of CPTNI monotone metrics for any operator monotone function \( f \).
Corollary 2.3. For any CPTNI monotone metric,
\[ K_{\rho_1 \oplus \rho_2}^{(n)}(X_1 \oplus X_2, Y_1 \oplus Y_2) = K_{\rho_1}^{(n)}(X_1, Y_1) + K_{\rho_2}^{(n)}(X_2, Y_2) \]
with \( \rho_i \in \mathcal{F}^+(\mathbb{C}^n) \) and \( X_i, Y_i \in \mathcal{B}(\mathbb{C}^n) \) for \( i = 1, 2 \).

This is a natural property meaning that the inner product structure of the whole and part is the same, while CPTP monotone metrics do not have this property. The following two corollaries about monotonicity and convexity are also natural consequences which CPTP monotone metrics do not have.

Corollary 2.4. For any CPTNI monotone metric,
\[ K_{\rho_1 \oplus \rho_2}^{(n)}\left(\frac{X_1 + X_2}{2}, \frac{X_1 + X_2}{2}\right) \leq \frac{1}{2} \left\{ K_{\rho_1}^{(n)}(X_1, X_1) + K_{\rho_2}^{(n)}(X_2, X_2) \right\}. \]

Proof. By using the above corollary and a partial trace, for \( \rho_1, \rho_2 \in \mathcal{F}^+(\mathbb{C}^n) \) and \( X_1, X_2 \in \mathcal{B}(\mathbb{C}^n) \),
\[
\frac{1}{2} \left\{ K_{\rho_1}^{(n)}(X_1, X_1) + K_{\rho_2}^{(n)}(X_2, X_2) \right\} = K_{\rho_1}^{(n)}\left(\frac{X_1 + X_2}{2}, \frac{X_1 + X_2}{2}\right) + K_{\rho_2}^{(n)}\left(\frac{X_1 + X_2}{2}, \frac{X_1 + X_2}{2}\right)
\geq K_{\rho_1 \oplus \rho_2}^{(n)}\left(\frac{X_1 + X_2}{2}, \frac{X_1 + X_2}{2}\right).
\]

Corollary 2.5. For any CPTNI monotone metric \( K_{\rho}^{(n)} \), a function \( \rho \mapsto K_{\rho}^{(n)}(X, X) \) is monotonically decreasing and convex with respect to \( \rho \in \mathcal{F}(\mathbb{C}^n) \).

Proof. The monotonicity is obvious from the definition of CPTNI monotone metric. The convexity is also obvious form the above corollary.

In this section, we consider CPTNI monotone metric \( K_{\rho}^{(n)} \) for physical unnormalized state \( \rho \) which is restricted to \( \mathcal{F}^+(\mathbb{C}^n) = \{ \rho \in \mathcal{B}(\mathcal{H}) \mid \text{Tr} \rho \leq 1, \rho > 0 \} \). However, it is easy to generalize \( \mathcal{F}^+(\mathbb{C}^n) \) to \( \mathcal{F}^+(\mathbb{C}^n) := \{ \rho \in \mathcal{B}(\mathcal{H}) \mid \rho > 0 \} \). In fact, \( \mathcal{F}^+(\mathbb{C}^n) \) can be replaced by \( \mathcal{F}^+(\mathbb{C}^n) \) in this section without any restrictions.

3 Conclusion

In the present paper, we introduced CPTNI monotone metrics which have monotonicity under CPTNI maps and additive noise. It is a natural generalization of quantum monotone metrics introduced by Petz which have monotonicity under CPTP maps. We prove the CPTNI monotone metrics can be characterized only by operator monotone functions from few assumptions without assuming continuities of metrics. It was shown that CPTNI monotone metrics have some natural properties such as additivity of direct sum (Corollary 2.3), convexity (Corollary 2.4), monotonicity with respect to unnormalized state (Corollary 2.5). These properties did not appear in monotone metrics based on CPTP maps.

A Operator monotone and operator concave functions

A function \( f : (0, \infty) \to (0, \infty) \) is said to be operator monotone if for all positive operators \( A \) and \( B \), \( 0 < A \leq B \) implies \( 0 < f(A) \leq f(B) \).
Theorem A.1. For a function $f : (0, \infty) \to (0, \infty)$, the following statements are equivalent:

(i) $f$ is operator monotone.

(ii) $\tilde{f}(C^*AC) \geq C^*\tilde{f}(A)C$ for any positive operator $A$, and any operator $C$ such that $\|C\| \leq 1$, where $\tilde{f} : [0, \infty) \to [0, \infty)$ is a function such that $\tilde{f}(x) = f(x)$ for $x > 0$ and $\tilde{f}(0) = \limsup_{\varepsilon \searrow 0} f(\varepsilon)$.

(iii) $f$ is operator concave, i.e., $f(pA + (1-p)B) \geq pf(A) + (1-p)f(B)$ for any strictly positive Hermitian operators $A, B$ and any real number $0 \leq p \leq 1$.

Proof. (ii)$\Rightarrow$(iii): Consider operators $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $U = \begin{pmatrix} \sqrt{pI} & \sqrt{1-p}I \\ \sqrt{1-p}I & -\sqrt{p}I \end{pmatrix}$, $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Because $\|UP\| \leq 1$, 

$$\left(\tilde{f}(pA + (1-p)B) \ 0 \ \tilde{f}(0)\right) = \tilde{f}(PUXUP) \geq PU\tilde{f}(X)UP = \begin{pmatrix} p\tilde{f}(A) + (1-p)\tilde{f}(B) & 0 \\ 0 & 0 \end{pmatrix}.$$ 

This proves $\tilde{f}$ and $f$ are operator concave.

(iii)$\Rightarrow$(i): For operators $A$ and $B$ such that $0 < A \leq A + B$,

$$f(A + B) = f\left(p\frac{1}{p}A + (1-p)\frac{1}{1-p}B\right) \geq pf\left(\frac{1}{p}A\right) + (1-p)f\left(\frac{1}{1-p}B\right).$$

Because every concave function is continuous, $\lim_{p \to 1} pf\left(\frac{1}{p}A\right) = f(A)$. This proves $f(A + B) \geq f(A)$.

(i)$\Rightarrow$(ii): Without loss of generality, we can assume $0 \leq C \leq I$ because any operator $C$ such that $\|C\| \leq 1$ has a singular value decomposition $C = SW$ with $0 \leq S \leq I$ and an unitary operator $W$. Let $U = \begin{pmatrix} C & D \\ D & -C \end{pmatrix}$ be an unitary operator with $D = \sqrt{I-C^2}$. For any real number $\varepsilon > 0$, there exists $\mu > 0$ such that

$$\begin{pmatrix} CAC + 2\varepsilon I & 0 \\ 0 & \mu I \end{pmatrix} \geq \begin{pmatrix} CAC + \varepsilon I & CAD \\ DAC & DAD + \varepsilon I \end{pmatrix} = U \begin{pmatrix} A + \varepsilon I & 0 \\ 0 & \varepsilon I \end{pmatrix} U.$$

Then

$$\begin{pmatrix} f(CAC + 2\varepsilon I) & 0 \\ 0 & f(\mu I) \end{pmatrix} \geq U \begin{pmatrix} f(A + \varepsilon I) & 0 \\ 0 & f(\varepsilon I) \end{pmatrix} U \geq U \begin{pmatrix} f(A + \varepsilon I) & 0 \\ 0 & 0 \end{pmatrix} U$$

$$= \begin{pmatrix} Cf(A + \varepsilon I)C & Cf(A + \varepsilon I)D \\ Df(A + \varepsilon I)C & Df(A + \varepsilon I)D \end{pmatrix}.$$ 

Therefore

$$f^+(CAC) \geq Cf^+(A)C,$$

where $f^+(x) = \limsup_{\varepsilon \searrow 0} f(x + \varepsilon)$ for $x \in [0, \infty)$. Due to the proof of (ii)$\Rightarrow$(iii), $f^+$ is operator concave and continuous at every $x > 0$. Hence, $f^+(x) = f(x)$ for $x > 0$. \qed

Theorem A.2. If a function $f : (0, \infty) \to (0, \infty)$ is operator monotone, 

$$f^+(x) = x/f(x)$$

and 

$$f^+(x) = x f(1/x)$$

are also operator monotone.
Proof. Let $A, B$ be positive operators such that $0 < A \leq B$. Because a operator $C = B^{-\frac{1}{2}}A^2$ satisfies $\|C\| \leq 1$,
\[
f(A) = f(C^*BC) \geq C^*f(B)C = A^{\frac{1}{2}}B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}A^{\frac{1}{2}}
\]
due to the above theorem. Therefore
\[
B^{\frac{1}{2}}f(B)^{-1}B^{\frac{1}{2}} \geq A^{\frac{1}{2}}f(A)^{-1}A^{\frac{1}{2}}.
\]
This proves $f^\perp$ is operator monotone. Further, $f'$ is also operator monotone because $f'(x) = 1/f^\perp(1/x)$.

B Operator means

Operator mean of strictly positive operators $A$ and $B$ with respect to an operator monotone function is defined by
\[
Am_fB = A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.
\]
Later, this is extended to non-negative operators.

Theorem B.1. For strictly positive operators $A, B$ on a Hilbert space $\mathbb{C}^n$, $Am_fB = Bm_fA$.

Proof. By using a singular value decomposition $A^{1/2}B^{-1/2} = S^{1/2}W$ with $0 < S$ and an unitary operator $W$,
\[
Am_fB = A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}
= B^{1/2}W^*S^{1/2}f(S^{-1})S^{1/2}WB^{1/2}
= B^{1/2}W^*f(S)WB^{1/2}
= B^{1/2}f'(B^{-1/2}AB^{-1/2})B^{1/2}.
\]

Theorem B.2. For strictly positive operators $A_1, A_2, B_1, B_2$ on a Hilbert space $\mathbb{C}^n$ such that $0 < A_1 \leq A_2$ and $0 < B_1 \leq B_2$,
\[
A_1m_fB_1 \leq A_2m_fB_2.
\]

Proof. Because $f$ and $f'$ are operator monotone,
\[
A_1m_fB_1 = A_1^{1/2}f\left(A_1^{-1/2}B_1A_1^{-1/2}\right)A_1^{1/2}
\leq A_1^{1/2}f\left(A_1^{-1/2}B_2A_1^{-1/2}\right)A_1^{1/2}
= B_2^{1/2}f'(B_2^{-1/2}A_2B_2^{-1/2})B_2^{1/2}
\leq B_2^{1/2}f'(B_2^{-1/2}A_2B_2^{-1/2})B_2^{1/2} = B_2m_fA_2.
\]

Due to this theorem, operator mean of non-negative operators $A$ and $B$ can be defined by
\[
Am_fB = \lim_{\epsilon \downarrow 0}(A + \epsilon I)m_f(B + \epsilon I)
\]
because this monotonically decreases, as $\epsilon \downarrow 0$, and the limit exists.

Theorem B.3. For non-negative operators $A$ and $B$ on $\mathbb{C}^n$ and a linear map $C$ from $\mathbb{C}^n$ to $\mathbb{C}^m$,
\[
C(Am_fB)C^* \leq (CAC^*)m_f(CBC^*).
\]
\textit{Proof.} When $A > 0$, $B > 0$, and $C$ is invertible, by using a singular value decomposition $CA^{1/2} = WS^{1/2}$ with $S > 0$ and an unitary operator $W$,

\[(CAC^*)m_f(CBC^*) = (WSW^*)m_f(CBC^*)\]
\[= WS^{1/2}f(WS^{-1/2}W^*CBC^*WS^{-1/2}W^*)WS^{1/2}W^*\]
\[= WS^{1/2}f(S^{-1/2}W^*CBC^*WS^{-1/2})S^{1/2}W^*\]
\[= CA^{1/2}f(A^{-1/2}C^{-1}CBC^*C^{-1}A^{-1/2})A^{1/2}C^*\]
\[= CA^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}C^* = C(Am_fB)C^* .\]

When $A \geq 0$, $B \geq 0$, and $C$ is any linear map, $A, B, C$ can be embedded to $B(\mathbb{C}^l)$ with $l = \max(n,m)$. Therefore we can assume $A, B, C \in B(\mathbb{C}^l)$ without loss of generality. For any $\varepsilon > 0$, by using a singular value decomposition $C = WC_S$ with $S \geq 0$ and an unitary operator $W_C$,

\[C(Am_fB)C^* = \lim_{\varepsilon \downarrow 0} W_C [(S_{C+\varepsilon I}) (A+\varepsilon I) (S_{C+\varepsilon I})] W_C^*\]
\[= \lim_{\varepsilon \downarrow 0} W_C [(S_{C+\varepsilon I}) (A+\varepsilon I) (S_{C+\varepsilon I})] m_f [(S_{C+\varepsilon I}) (B+\varepsilon I) (S_{C+\varepsilon I})] W_C^*.\]

Here,

\[(S_{C+\varepsilon I}) (A+\varepsilon I) (S_{C+\varepsilon I}) \leq S_CAS_C + \delta_A(\varepsilon)I\]

with $\delta_A(\varepsilon) = \| (S_{C+\varepsilon I}) (A+\varepsilon I) (S_{C+\varepsilon I}) - S_CAS_C \|$. Similarly, $(S_{C+\varepsilon I}) (B+\varepsilon I) (S_{C+\varepsilon I}) \leq \delta_B(\varepsilon)I$. Let $\delta(\varepsilon) = \max \{ \delta_A(\varepsilon), \delta_B(\varepsilon) \}$. Then

\[C(Am_fB)C^* \leq \lim_{\varepsilon \downarrow 0} W_C [(S_CAS_C + \delta(\varepsilon)I) m_f (S_CBS_C + \delta(\varepsilon)I)] W_C^*\]
\[= \lim_{\varepsilon \downarrow 0} [(CAC^* + \delta(\varepsilon)I) m_f (CBC^* + \delta(\varepsilon)I)] = (CAC^*) m_f (CBC^*) .\]

\[\square\]

\textbf{References}

[1] Bhatia, R. Matrix Analysis, Graduate Texts in Mathematics 169, Springer, New York (1997).

[2] C. M. Caves, Quantum error correction and reversible operations, Journal of Superconductivity, 12(6), 707–718 (1999).

[3] N.N. Čencov, Statistical decision rules and optimal inferences, Transl. Math. Monogr. Amer. Math. Soc. 53 (1982)

[4] F. Kubo and T. Ando, Means of positive linear operators, Math. Ann. 246:205-224 (1980).

[5] W. Kumagai, A characterization of extended monotone metrics, Linear Algebra and its Applications 434(1) 224-231 (2011)

[6] M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information (10th ed.). Cambridge: Cambridge University Press. ISBN 9781107002173. OCLC 665137861, (2010).

[7] D. Petz, Monotone metrics on matrix spaces, Linear Algebra Appl. 244 (1996) 81–96.