Exact partition functions of Higgsed 5d $T_N$ theories

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ABSTRACT: We present a general prescription by which we can systematically compute exact partition functions of five-dimensional supersymmetric theories which arise in Higgs branches of the $T_N$ theory. The theories may be realized by webs of 5-branes whose dual geometries are non-toric. We have checked our method by calculating the partition functions of the theories realized in various Higgs branches of the $T_3$ theory. A particularly interesting example is the $E_8$ theory which can be obtained by Higgsing the $T_6$ theory. We explicitly compute the partition function of the $E_8$ theory and find the agreement with the field theory result as well as the enhancement of the global symmetry to $E_8$. 
Introduction

String theory is a candidate for the fundamental theory which describes all the forces as well as the matter in the realistic world. However, that is not the only way of the application of string theory, and string theory can be used as a powerful tool to understand exact results of field theories. In particular, the topological vertex method [1, 2] or its refinement [3, 4] can compute the five-dimensional or four-dimensional Nekrasov partition functions of $U(N)$ gauge theories [5–10]. The gauge theories are realized as low energy effective field theories of M-theory or Type IIA compactifications on non-compact toric Calabi–Yau threefolds. The configurations are in fact dual to the webs of $(p, q)$ 5-branes [11], namely the dual web diagram to a toric fan is nothing but a $(p, q)$ 5-brane web. In this case, the five-dimensional gauge theories are
obtained by compactifying the worldvolume theories on 5-branes on segments [12, 13]. The flavor symmetry of the theories can be directly seen by introducing 7-branes [14, 15]. The \((p, q)\) 5-brane realization of the five-dimensional gauge theories can be generalized to certain five-dimensional class \(S\) theories such as the 5d \(T_N\) theory [16] whose four-dimensional versions were originally introduced in [17]. The \(T_N\) theory is a non-Lagrangian theory, and hence it is hard to obtain its partition function from the localization technique like the one in [18, 19]. However, the refined topological vertex may compute the partition function of the \(T_N\) theory since its web diagram is simply toric\(^3\).

Recently, there has been a progress in the computation of the refined topological vertex and it has turned out that the refined topological vertex computation itself automatically contains some factors which are contributions from particles that are decoupled from the theory realized by webs of \((p, q)\) 5-branes [20–23]. Those factors are associated with the contributions of particles coming from strings between parallel external legs in web diagrams. Only after stripping off the factors, we obtain the partition functions of the theories realized by \((p, q)\) 5-brane webs. For example, the refined topological vertex computation from the web diagram which realizes an \(SU(2)\) gauge theory with \(N_f \leq 4\) flavors yields the \(U(2)\) Nekrasov partition functions with \(N_f \leq 4\) flavors [5–10]. Quite interestingly, the instanton partition functions of the \(U(2)\) gauge theories with \(2 \leq N_f \leq 4\) flavors obtained by the refined topological vertex become the instanton partition functions of \(SU(2) \cong Sp(1)\) gauge theories with the same flavors after removing all the factors decoupled from the theories [21–23]. The computation can be further generalized to the partition function of the \(T_N\) theory [21, 22]. One can also check that the enhancement of the global symmetry of some other gauge theories at their superconformal fixed points only arises after removing the decoupled factors [23–25]. Some dualities of five-dimensional \(\mathcal{N} = 1\) supersymmetric theories have been checked at the level of the partition functions, and the agreement occurs after stripping off the decoupled factors [26]. Therefore, by stripping off the decoupled factors, we can obtain partition functions of theories realized by any web diagrams if they are toric.

Let us also mention that the explanation of the similar decoupled factor has been given from the viewpoint of the ADHM quantum mechanics in [27]. For example, the \(Sp(N)\) gauge theory with \(N_f \leq 7\) fundamental and 1 antisymmetric hypermultiplets can be realized on \(N\) D4-branes close to \(N_f\) D8-branes and an O8-plane. The instantons in this system can be considered as D0-branes on D4-branes. However, the ADHM quantum mechanics computation also contains D0-D8-O8 bound states. We need to remove the contributions in order to obtain the partition functions of the \(Sp(N)\) gauge theory with the \(N_f \leq 7\) fundamental and 1 antisymmetric hypermultiplets.

The refined topological vertex computation as well as finding the decoupled factors can be applied to the computation of the partition function of the \(SU(2)\) gauge theory with \(N_f \leq 5\) flavors since the corresponding web diagram is toric. From \(N_f = 6\), the web diagrams

\(^3\)Strictly speaking, toric should be used for geometries in the dual description. The statement that a web diagram is toric means that the web diagram is dual to a toric Calabi–Yau threefold.
are not toric and one cannot simply apply the refined topological vertex computation to the non-toric diagrams\(^2\). However, the \(Sp(1)\) gauge theory with 6 flavors or the \(E_7\) theory can be also obtained as an infrared theory in a Higgs branch of the \(T_4\) theory [16]. In order for hypermultiplets in the theory to get vacuum expectation values, one needs to tune parameters of the theory. The tuning is related to putting two horizontal external 5-branes together on one 7-brane. We can in fact apply the procedure of putting the horizontal external 5-branes together to the computation of the partition function and the partition function of the \(E_7\) theory has been successfully computed in [22]. Note also that the same partition function has been recently obtained in [27] from the D4–D8–O8 system.

Although we have understood how to perform the tuning associated to putting horizontal external 5-branes together in terms of the computation of partition functions, that tuning is not the only tuning by which we obtain all the Higgs branches of the \(T_N\) theory. We may also consider a tuning for putting vertical or diagonal external 5-branes together on one 7-brane since the web diagram for the \(T_N\) theory has three types of external 5-branes, namely horizontal external 5-branes, vertical external 5-branes and diagonal external 5-branes. In order to establish a computational method which can apply to any web diagram yielding any Higgs branch of the \(T_N\) theory, we need to find a prescription which corresponds to the tuning for putting the vertical or diagonal external 5-branes together. In particular, the \(Sp(1)\) gauge theory with 7 flavors or the \(E_8\) theory can be realized by a low energy theory in a Higgs branch of the \(T_6\) theory where the tuning involves putting vertical external 5-branes together. Therefore, we need to understand how to tune the parameters for putting the vertical external 5-branes together to obtain the partition function of the \(E_8\) theory.

The aim of this paper is to find a general procedure of computing the partition functions of IR theories in Higgs branches of the \(T_N\) theory realized by non-toric web diagrams. For that, we will first find a prescription of applying the tuning for putting vertical external 5-branes together on one 7-brane to the computation of partition functions. Originally, tuning for putting horizontal external 5-branes together is described by a simple pole with respect to a flavor fugacity in the superconformal index by applying the method of obtaining the four-dimensional superconformal index of an IR theory in a Higgs branch of a UV theory [30, 31]. However, for the tuning for putting vertical external 5-branes together, we need to find a pole with respect to an instanton fugacity. This is a hard problem from a field theoretic point of view since a partition function obtained by the localization technique is typically expanded by the instanton fugacity. Refined topological vertex is in fact powerful enough to solve this problem. Namely we can change a choice of the preferred direction and then we can sum up the instanton fugacity since the refined topological vertex is conjectured to be invariant under the choice of the preferred direction [4, 32]. We can find the tuning for putting vertical external 5-branes together by relating it with the tuning for putting horizontal external 5-branes together. We will also exemplify the validity of the tuning by applying it to two types

\(^2\)A vertex formalism of unrefined topological string amplitudes which can be applied to certain non-toric geometries has been developed in [28, 29].
of tuning for putting vertical external 5-branes together in the web diagram for the $T_3$ theory. In the Higgs vacuum of the $T_3$ theory, there are some decoupled singlet hypermultiplets. The partition function contains their contributions and we will also discuss how to identify them systematically from the web diagram, which was originally proposed in [22]. The tuning for putting diagonal external 5-branes together will be also found in appendix.

After establishing the tuning for putting vertical external 5-branes together, we will apply the method to the computation of the partition function of the $E_8$ theory which arises in the infrared theory in the Higgs branch of the $T_6$ theory. The partition function should agree with the partition function of the $Sp(1)$ gauge theory with 7 fundamental and 1 antisymmetric hypermultiplets obtained in [27]. Although our method of the computation is completely different from the one in [27], we will find the agreement. This is a very non-trivial check of the claim that the $E_8$ theory arises as an infrared theory in the Higgs branch of the $T_6$ theory [16]. Furthermore, the resulting partition function obtained by our method is written by summations of Young diagrams. Therefore, we can systematically compute higher order terms of the instanton fugacity.

The organization of the paper is as follows. In section 2, we find a prescription of tuning associated to putting parallel vertical external 5-branes together on one 7-brane. We will exemplify the prescription by applying it to two theories in the Higgs branches of the $T_3$ theory. In section 3, we compute the partition function of the $E_8$ theory whose web diagram involves the tuning for putting the parallel external vertical legs together as well as putting parallel external horizontal legs together. We first describe a general procedure to obtain the partition function of an IR theory in a Higgs branch of the $T_N$ theory, and then apply the steps to the computation of the partition function of the $E_8$ theory. Some technical details regarding the computation are relegated to appendix B. In appendix A, we find a prescription of tuning associated to putting parallel diagonal external 5-branes together on one 7-brane.

2 Tuning for coincident vertical 5-branes

Higgs branches of the five-dimensional $T_N$ theory may be geometrically realized by $(p,q)$ 5-brane webs [16]. For that, we consider that a semi-infinite $(p,q)$ 5-brane ends on an orthogonal spacetime filling $(p,q)$ 7-brane at a finite distance. After putting all the semi-infinite $(p,q)$ 5-branes on $(p,q)$ 7-branes, the global symmetry of the theory is realized on the $(p,q)$ 7-branes. The Higgs branch of the theory opens up when we put several parallel external 5-branes on a 7-brane. Then, pieces of 5-branes suspended between the 7-branes can move in directions off the plane of the web. The positions of the 5-branes suspended between the 7-branes as well as a part of the gauge field on the 5-branes parameterize the Higgs branch. When we move the 5-branes between the 7-branes far away from the web, some of the 7-branes are decoupled from the web diagram and the global symmetry is reduced. This procedure corresponds to taking the Higgs vev to be very large. In this infrared limit of the Higgs branch, we obtain a different class $S$ theory. Note also that the the Higgsing may reduce the dimension of the Coulomb branch due to the s-rule [16, 33].
Putting parallel external 5-branes on one 7-brane means the tuning of some parameters of the theory realized from a \((p, q)\) 5-brane web. The tuning can be applied to the computation of partition functions \[22\]. Namely, after inserting the tuning into the partition function of some UV theory such as the \(T_N\) theory, we obtain the partition function of the low energy theory in the Higgs branch of the UV theory. Let us consider putting two parallel horizontal external 5-branes on a single 7-brane as in figure 1. This can be achieved by shrinking the length of the internal 5-branes whose parameters are represented as \(Q_1\) and \(Q_2\). Ref. \[22\] have found that the tuning for putting the parallel horizontal external 5-branes together depicted in figure 1 is

\[ Q_1 = Q_2 = \left(\frac{q}{t}\right)^{\frac{1}{2}}, \tag{2.1} \]

or

\[ Q_1 = Q_2 = \left(\frac{t}{q}\right)^{\frac{1}{2}} \tag{2.2} \]

where \(q\) and \(t\) are associated with the \(\Omega\)-deformation parameters \(q = e^{-i\epsilon_2}, t = e^{i\epsilon_1}\). In fact, both tunings (2.1) and (2.2) give the same result for the examples studied in \[22\].

One can understand the reason why the two tunings (2.1) and (2.2) give the same result in the following way. Let us first parameterize the lengths \(Q_1\) and \(Q_2\) by chemical potentials associated with a gauge symmetry and a global symmetry. From the dual picture in Type IIA string theory, the two external horizontal 5-branes in figure 1 are thought as flavor branes and the one internal horizontal 5-brane in figure 1 is thought as a color brane. In the dual picture, the 5-branes between one external horizontal 5-brane and one internal horizontal 5-brane are strings which yield bi-fundamental hypermultiplets. Then, \(Q_1\) and \(Q_2\) are parameterized as

\[ Q_1 = e^{i(\tilde{\nu}_1 - \nu)}, \quad Q_2 = e^{i(\nu - \tilde{\nu}_2)}. \tag{2.3} \]

Here we choose the orientation of the 5-branes from top to down as in figure 1 and we define positive sign when an arrow of the orientation goes away from the flavor branes or the color.

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\(^3\)One of the charges of the bi-fundamental hypermultiplet here is associated with a global symmetry.
brane. The chemical potential $\tilde{\nu}_1$ is associated with the upper external horizontal 5-brane and the chemical potential $\tilde{\nu}_2$ is associated with the lower external horizontal 5-brane. The chemical potential $\nu$ is the one for the color brane. The exchange between the chemical potentials $\tilde{\nu}_1$ and $\tilde{\nu}_2$ may be a part of the flavor symmetry $U(2)$ associated with the two external horizontal 5-branes. Therefore, the partition function of the theory from the web diagram is invariant under the exchange between the chemical potentials $\tilde{\nu}_1$ and $\tilde{\nu}_2$. If the partition function has the symmetry, the conditions (2.1) and (2.2) yield the same answer after the tuning. In the case of the $T_N$ theory the exchange between the chemical potentials $\tilde{\nu}_1$ and $\tilde{\nu}_2$ is a part of the flavor symmetry $SU(N) \subset SU(N) \times SU(N) \times SU(N)$, and hence we can use either (2.1) and (2.2).4

We can see one of the evidence for (2.1) or (2.2) from the superconformal index computation. In four-dimension, the index of a class $S$ theory may be computed as a residue of the superconformal index of a UV theory which leads to the class $S$ theory in the far infrared [30, 31]. One may apply the same method to the five-dimensional superconformal index, and the superconformal indices studied in [22] indeed have a simple pole at

$$Q_1 Q_2 = \frac{q}{t}, \quad (2.4)$$

and also other simple pole at

$$Q_1 Q_2 = \frac{t}{q}. \quad (2.5)$$

Eq. (2.4) and (2.5) is consistent with (2.1) and (2.2) respectively. Due to the choice of the preferred directions in figure 1, the simple poles are associated with the flavor fugacity $e^{i(\tilde{\nu}_1 - \tilde{\nu}_2)}$. Hence, we can easily identify the location of the pole (2.4) by looking at the perturbative part of the superconformal index.

Note also that shrinking the length of an internal 5-brane which is parameterized by $Q$ does not imply $Q = 1$ in the refined topological vertex computation although that is the case for the unrefined topological vertex. The refined version of the geometric transition suggests $Q = (\frac{q}{t})^{\frac{1}{2}}$ or $Q = (\frac{t}{q})^{\frac{1}{2}}$ [34–37]. The two different results are associated with an overall normalization ambiguity of the partition function of the refined Chern–Simons theory. By combining this result with (2.4) or (2.5), we may obtain the conclusion (2.1) or (2.2).

Let us then consider the case of putting two parallel vertical external 5-branes on a single 7-brane as in the left figure of figure 2. In principle, one may also compute the superconformal index and find a simple pole corresponding to the tuning in the left figure of figure 2. However, the pole is associated with an instanton fugacity and it is technically difficult to identify the location of the pole. This is because the refined topological vertex computation yields the expression expanded by a fugacity assigned along the preferred direction, which is related to an instanton fugacity. When we apply the refined topological vertex computation to the

4The asymmetry under the exchange between $\tilde{\nu}_1$ and $\tilde{\nu}_2$ may arise in the contributions of singlet hypermultiplets in the Higgs vacuum as observed in [22]. The partition function after decoupling the factors of the singlet hypermultiplets is invariant under the exchange.
left figure of figure 2, we obtain an expression which is expanded by $Q'_2$. However, one can circumvent the problem with a different choice of the preferred direction. The refined vertex computation is conjectured to be independent of the choice of the preferred direction [4, 32]. Namely, we obtain the same answer by a different choice of the preferred direction. Therefore, in order to consider the tuning corresponding to the left figure of figure 2, we can use a different choice of the preferred direction like the one in the right figure of figure 2. Then, we can sum up the Young diagrams associated with both $Q'_1$ and $Q'_2$, and we can find a location of the poles. In fact, the structure of the right figure of figure 2 is essentially the same as that of the web diagram in figure 1, and we can use the result of the tuning for putting the parallel external horizontal 5-branes together on one 7-brane. Hence, the tuning prescription associated with figure 2 may be given by

$$Q'_1 = Q'_2 = \left( \frac{q}{t} \right)^{\frac{1}{2}}.$$  \hfill (2.6)

or

$$Q'_1 = Q'_2 = \left( \frac{t}{q} \right)^{\frac{1}{2}}.$$  \hfill (2.7)

The two tunings (2.6) and (2.7) should give the same answer as that was the case for the tunings (2.1) and (2.2). For the later computation, we will use (2.1) for the tuning associated with putting parallel external horizontal 5-branes together on one 7-brane and (2.6) for the tuning associated with putting the parallel external vertical 5-branes together on one 7-brane.

### 2.1 $T_3$ theory revisited

We will first exemplify the validity of the tuning (2.6) by applying it to the two parallel vertical legs of the $T_3$ theory. The infrared theory in the Higgs branch is a free theory with nine hypermultiplets\(^5\). For that, we will first review the partition function of the $T_3$ theory.

The web diagram for the $T_3$ theory is depicted in figure 3. The web diagram in fact can be interpreted as the dual toric diagram of a toric Calabi–Yau threefold [11]. In the dual picture, the five-dimensional theory is obtained by an M-theory compactification on

\(^5\)The partition function of the free theory in the Higgs branch of the $T_3$ theory by putting two parallel horizontal external 5-branes on one 7-brane has been already obtained in [22].
Figure 3. The web diagram for the $T_3$ theory. $Q_i, (i = 1, 2, 3, 4, 5)$ and $Q_b, Q_f$ parameterize the lengths of the corresponding internal 5-branes.

the Calabi–Yau threefold. In that viewpoint, the finite length internal 5-branes are compact two-cycles in the Calabi–Yau threefold. The particle in the five-dimensional theory may be understood as M2-branes wrapping two-cycles. A compact face corresponds to a compact divisor and a non-compact face corresponds to a non-compact divisor. The existence of a divisor yields a gauge field by expanding a three-form in M-theory by the Poincaré dual two-form. A compact divisor implies a gauge symmetry, and a certain non-compact divisor implies a global symmetry. The rank of the group associated with the symmetry is the number of the divisors. In figure 3, we depict a compact divisor by $D$ and six non-compact divisors by $D_a, (a = 1, \cdots , 6)$. $D$ is associated with the Cartan generator of the gauge group $U(1) \subset SU(2)$ and $D_a, (a = 1, \cdots , 6)$ are associated with the Cartan generators of the global symmetry group of the $T_3$ theory, namely $SU(3) \times SU(3) \times SU(3) \subset E_6$ which is explicitly realized in the web diagram in figure 3. The intersection number between the divisor and a two-cycle gives a charge of the particle under the corresponding symmetry. Since the Calabi–Yau threefold picture and the 5-brane web picture are dual to each other, we will use both terminology interchangeably.

By utilizing the picture of the dual Calabi–Yau threefold, one can compute the exact partition function of the five-dimensional $T_3$ theory from the refined topological vertex. The partition function of the $T_3$ theory has been obtained in [21, 22]

$$Z_{T_3} = Z_0 \cdot Z_{\text{inst}} \cdot Z_{\text{dec}}^{-1},$$

(2.8)
\[ Z_0 = \prod_{i,j=1}^{\infty} \left[ \frac{1 - e^{-i\lambda + i m_0 q^{-\frac{1}{2}} t^{-\frac{1}{2}}} (1 - e^{-i\lambda - im_0 q^{-\frac{1}{2}} t^{-\frac{1}{2}}})}{(1 - q^{-i t^{-1}})^{\frac{1}{2}} (1 - q^{-i t^{-1}})^{\frac{1}{2}} (1 - e^{-2i\lambda q^{-i t^{-1}}}) (1 - e^{-2i\lambda q^{-i t^{-1}}})} \right] \]

\[ (1 - e^{i\lambda + im_2 q^{-\frac{1}{2}} t^{-\frac{1}{2}}}) (1 - e^{-i\lambda + im_2 q^{-\frac{1}{2}} t^{-\frac{1}{2}}}) (1 - e^{i\lambda - im_3 q^{-\frac{1}{2}} t^{-\frac{1}{2}}}) (1 - e^{-i\lambda - im_3 q^{-\frac{1}{2}} t^{-\frac{1}{2}}}) \right] \]

(2.9)

\[ Z_{\text{inst}} = \sum_{\nu_1, \nu_2, \mu_5} u_2^{[\nu_1] + [\nu_2]} u_1^{[\mu_5]} \left[ \prod_{\alpha=1}^{2} \prod_{s \in \nu_0} \left( \prod_{a=1}^{2} \frac{2 i \sin E_{a \beta - m_3 + 2i \gamma_1}}{2} \right) \left( 2 i \sin \frac{E_{a \beta - m_3 + 2i \gamma_1}}{2} \right) \prod_{\beta=1}^{2} \prod_{y \in \mu_5} \left( \prod_{s \in \mu_5} \frac{2 i \sin E_{a \beta - m_3 + 2i \gamma_1}}{2} \right) \prod_{s \in \mu_5} \frac{2 i \sin E_{a \beta - m_3 + 2i \gamma_1}}{2} \right] \]

(2.10)

\[ Z_{\text{dec}}^{-1} = \prod_{i,j=1}^{\infty} \left[ (1 - u_1 e^{im_4 q^{-\frac{1}{2}} t^{-1}}) (1 - u_2 e^{-im_4 q^{-\frac{1}{2}} t^{-1}}) (1 - u_1 u_2 e^{-im_4 q^{-\frac{1}{2}} t^{-1}}) (1 - u_1 u_2 e^{im_4 q^{-\frac{1}{2}} t^{-1}}) \right] \]

(2.11)

where the notation follows the ones in [22], namely

\[ q = e^{-\gamma_1 + \gamma_2}, \quad t = e^{\gamma_1 + \gamma_2}, \]

\[ E_{\alpha \beta} = \lambda_{\alpha} - \lambda_{\beta} + i (\gamma_1 + \gamma_2) l_{\nu_0} (s) - i (\gamma_1 - \gamma_2) (a_{\nu_0} (s) + 1), \]

(2.12)

and \( l_{\nu} (i, j) = \nu_i - j, a_{\nu} (i, j) = \nu_j^i - i \). \( \gamma_1, \gamma_2 \) are related to the \( \Omega \)-deformation parameters as \( i \epsilon_1 = \gamma_1 + \gamma_2, i \epsilon_2 = \gamma_1 - \gamma_2 \). \( \lambda \) is a Coulomb branch modulus and we set \( \lambda_0 = \lambda_5 = 0 \) and \( \lambda_1 = -\lambda_2 = \lambda \). The relations between the Kähler parameters and the parameters appearing in (2.9)–(2.11) are

\[ Q_6 Q_1^{\frac{1}{2}} Q_3^{\frac{1}{2}} Q_2^{\frac{1}{2}} Q_4^{-\frac{1}{2}} = u_2, \quad Q_f = e^{-2i\lambda}, \quad Q_5 = e^{i\lambda} u_1, \]

(2.13)

\[ Q_1 = e^{-i\lambda + im_1}, Q_2 = e^{i\lambda + im_2}, Q_3 = e^{i\lambda - im_3}, Q_4 = e^{-i\lambda - im_4} \]

(2.14)

The convention of the computation by the refined topological vertex used here is summarized in [22].

The partition function (2.8) has been shown to be equal to the partition function of the \( Sp(1) \) gauge theory with 5 flavors under the reparameterization \( u_1 = e^{-im_5} \) and

\[ u = u_2 e^{-\frac{i}{2} m_5} \]

(2.15)

in [22] up to the 3–instanton order, where \( u \) is now the instanton fugacity of the \( Sp(1) \) gauge theory.
In order to reproduce the partition function of the $T_3$ theory, it is important to subtract $Z_{\text{dec}}$ in (2.11). The partition function (2.11) is the contribution of particles which are decoupled from the $T_3$ theory. The contribution is nicely encoded in the web diagram, and it is associated with the contribution of strings between the parallel external legs [20–23]. We will call this factor as a decoupled factor\(^6\). Only after subtracting the decoupled factor, the refined topological vertex computation yield the partition function of the $T_3$ theory.

We can also understand the reason of the shift of the instanton fugacity (2.15) from the global symmetry enhancement to $E_6$. The $Sp(1)$ gauge theory with 5 flavors perturbatively has an $SO(10) \times U(1)$ global symmetry where the $U(1)$ is the global symmetry associated with the instanton current. The global symmetry is enhanced to $E_6$ at the superconformal fixed point. On the other hand, from the web diagram of figure 3, the $SU(3) \times SU(3) \times SU(3)$ global symmetry is manifestly seen. The relation between the Lie algebras is depicted in figure 4. The Cartan generators of $SU(3) \times SU(3) \times SU(3)$ are associated with the non-compact divisors $D_a, a = 1, \cdots, 6$. The simple roots of $SU(3) \times SU(3) \times SU(3)$ correspond to the two-cycles parameterized by

$$\{Q_1 Q_3, Q_2 Q_1 Q_4^{-1}\} = \{e^{i(m_1-m_3)}, e^{i(m_2-m_1)}\},$$  \hspace{1cm} (2.16)

$$\{Q_4 Q_5, Q_6 Q_1 Q_4^{-1}\} = \{e^{-i(m_4+m_5)}, u_2 e^{\frac{i}{2}(m_1+m_2+m_3+m_4)}\},$$  \hspace{1cm} (2.17)

$$\{Q_f Q_4^{-1} Q_5, Q_b Q_3\} = \{e^{i(m_4-m_5)}, u_2 e^{-\frac{i}{2}(m_1+m_2+m_3+m_4)}\}.$$  \hspace{1cm} (2.18)

Since $SU(3) \times SU(3) \times SU(3) \subset E_6$, the roots of $SU(3) \times SU(3) \times SU(3)$ can be also understood as the roots of $E_6$ which are

$$\pm e_i \pm e_j \quad (1 \leq i \neq j \leq 5),$$  \hspace{1cm} (2.19)

\(^6\)The same factor was called as a non-full spin content in [21] or a $U(1)$ factor in [22]. The decoupled factor which we need to subtract from the index computation of the ADHM quantum mechanics was called $Z_{\text{string}}$ indicating extra string theory states in [27].
Figure 5. Left: The web diagram of the first kind of the Higgsed $T_3$ theory. Right: The dot diagram corresponding to the web on the left. The red line shows the new external leg.

and

$$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm \sqrt{3}e_6).$$  \hfill (2.20)

with the number of the minus signs even. $e_i, (i = 1, \cdots 6)$ is the orthonormal basis of $\mathbb{R}^6$. In order to match the charges of the particles from M2-branes wrapping the two-cycles (2.16)–(2.18) with the charges of the roots (2.19) and (2.20), one has to shift the instanton fugacity $u_2 = ue^{\frac{1}{2}m_5}$. Then we can also see that the particles of M2-branes wrapping the two-cycles (2.16)–(2.18) are in the roots or spinor weights of $SO(10)$.

2.2 Higgsed $T_3$ theory I

Let us then consider a Higgs branch arising by putting two parallel vertical external 5-brane on one 7-brane. We will call the web diagram as the Higgsed $T_3$ web diagram and the infrared theory realized by the diagram as the Higgsed $T_3$ theory. There are two ways to do that, and we first consider putting the two leftmost parallel vertical external 5-branes together as in figure 5. We use the tuning (2.6), and in this case it corresponds to the tuning of the Kähler parameters

$$Q_2 = \left(\frac{q}{t}\right)^{\frac{1}{2}}, \quad Q_bQ_4^{-1} = \left(\frac{q}{t}\right)^{\frac{1}{3}}. \hfill (2.21)$$

By inserting the conditions (2.21), the partition function of the low energy theory arising in the Higgs branch of the $T_3$ theory becomes\footnote{To get the partition function (2.22), we erase $m_3, u_3$ by using the equations (2.21). We can make a choice of erasing other parameters by using (2.21), which does not affect any physics.}

$$Z_{TIR} = Z_0 \cdot Z_{\text{inst}} \cdot Z_{\text{dec}}^{-1}, \hfill (2.22)$$
\[ Z_0 = \prod_{i,j=1}^{\infty} \left[ \frac{\prod_{\alpha=1,4} (1 - e^{-i\lambda + im_\alpha q^i - \frac{1}{2} t^j - \frac{1}{2}})(1 - e^{-i\lambda - im_\alpha q^i + \frac{1}{2} t^j + \frac{1}{2}})}{(1 - q^t t^{-1})^{\frac{1}{2}}(1 - q^{t^{-1}} t)\frac{1}{2}}(1 - e^{-2i\lambda q^i + 1 t^j}) \times (1 - e^{i\lambda - im_3 q^i - \frac{1}{2} t^j - \frac{1}{2}})(1 - e^{-i\lambda - im_3 q^i + \frac{1}{2} t^j + \frac{1}{2}}) \right], \tag{2.23} \]

\[ Z_{\text{inst}} = \sum_{\nu_1, \mu_5} \left( e^{\frac{\pi}{4}(m_1 + m_3 + m_4)} \left( q \right)^{\frac{3}{2}} \right)^{|\nu_1|} \nu_1^{[\mu_5]} \left[ \prod_{s \in \nu_1} \frac{(2i \sin \frac{E_{1s} - m_4 + t\gamma_1}{2})(2i \sin \frac{E_{1s} - m_4 + t\gamma_1}{2})}{(2i)^2 \sin \frac{E_{1s} + 2t\gamma_1}{2} \sin \frac{E_{1s} + 2t\gamma_1}{2}} \right], \tag{2.24} \]

\[ Z_{\text{dec}}^{-1} = \prod_{i,j=1}^{\infty} \left[ (1 - u_1 e^{-im_4 q^i - 1 t^j} - q^t t^{-1})(1 - u_1 e^{-im_4 q^i + 1 t^j t^{-1}})(1 - u_1 e^{-i\lambda - i(m_1 + m_3 + m_4) q^i + \frac{1}{2} t^j + \frac{3}{2}})(1 - u_1 e^{i\lambda - i(m_1 + m_3) q^i - \frac{1}{2} t^j - \frac{3}{2}}) \right], \tag{2.25} \]

where \( T_{\text{IR}} \) represents the low energy theory which arises in the Higgs branch of the \( T_3 \) theory. Note that the Young diagram summation of \( \nu_2 \) disappears due to the first tuning of \( (2.21) \). After the first tuning of \( (2.21) \), the factor \( \sin \left( \frac{E_{1s} + t}{2} \right) \) appears. The term \( \sin \left( \frac{E_{1s} + t}{2} \right) \) always contains zero in the product of the Young diagram \( \nu_2 \), and hence the Young diagram summation of \( \nu_2 \) vanishes.

In fact, the instanton partition function \( (2.24) \) can be written by the product of the Pllythystic exponentials

\[ Z_{\text{inst}} = \prod_{i,j=1}^{\infty} \left[ (1 - Q_1 (Q_3 Q_5 Q_6 q^i \frac{1}{2} t^j - \frac{1}{2}))(1 - Q_1 Q_3 Q_5 Q_6 Q_f Q_f q^i \frac{1}{2} t^j - \frac{1}{2})(1 - Q_6 q^i \frac{1}{2} t^j - \frac{1}{2})(1 - u_1 e^{-im_4 q^i - 1 t^j - 1})(1 - u_1 e^{-im_4 q^i t^j - 1})(1 - u_1 e^{im_4 q^i t^j t^{-1}}) \right] \]

\[ \times \left[ (1 - Q_4^{-1} Q_5 Q_6 Q_f q^i \frac{1}{2} t^j - \frac{1}{2})(1 - Q_5 q^i \frac{1}{2} t^j - \frac{1}{2})(1 - Q_6 Q_f^{-1} Q_3 q^{-1} t^j \frac{1}{2})(1 - u_1 e^{i\lambda - i(m_1 + m_3 + m_4) q^i + \frac{1}{2} t^j + \frac{3}{2}})(1 - u_1 e^{-i\lambda - im_3 q^i + \frac{1}{2} t^j + \frac{3}{2}}) \right] \]

\[ \times \left[ (1 - Q_4 Q_5 q^{-1} \frac{1}{2} t^j - \frac{1}{2})(1 - e^{-2i\lambda q^{-1} t^j}) \right], \tag{2.26} \]

Checking the equality \( (2.26) \) is not straightforward since, in the instanton partition function \( (2.24) \), the original instanton fugacity \( u_2 \) is replaced with other parameters \( \lambda, m_1, m_3, m_4 \) by which the original instanton partition function is not expanded. However, we can still check the equality by carefully choosing expansion parameters. For doing it, we need to choose expansion parameters so that we can use the expression \( (2.24) \) truncated at some finite order

\(^8\)When we use the tuning \( (2.7) \), the simplification of the disappearance of the Young diagram summation of \( \nu_2 \) does not happen.
\[|\nu_1|\]. We first rewrite the equations on both sides of (2.24) by \(Q_1, Q_3, Q_4, Q_f\). In fact, at the zeroth order of \(u_1\), both Eq. (2.24) and the right-hand side of (2.26) can be expanded by \(Q_f\) and \(Q_4\) and there are no poles with respect to \(Q_f\) and \(Q_4\). Furthermore, if we expand (2.24) until \(k = |\nu_1|\), the expression (2.24) is exact until \(O(Q_f^k Q_4^l)\) with \(a+b=k\). Therefore, we can check the equality (2.26) by truncating the Young diagram summation of \(|\nu_1|\) at finite order. We have checked the equality (2.26) until \(k = 3\) order. As for the equality of the order \(O(u_1^l)\), the negative power of \(Q_f\) and \(Q_4\) appears. However, when one factors out \(Q_f^{-\frac{1}{2}} Q_4^{-l}\) at each order of \(O(u_1^l)\), the expression of (2.24) is exact until \(O(Q_f^k Q_4^l)\) with \(a+b=k\) if we include the Young diagram summation \(|\nu_1|\) until \(|\nu_1| = k\). Therefore, we can include the expansion until \(|\nu_1| = k\), and check the equality (2.26). We have checked it up to \((l,k) = (2,2)\).

The equality (2.26) enables us to write (2.22) by the product of Plethystic exponentials

\[
Z_{T\bar{T}} = \prod_{i,j=1}^{\infty} \left[ (1 - Q_1 Q_3 Q_5 q^{i-\frac{1}{2}} j^{i-\frac{1}{2}}) (1 - Q_1 Q_3 Q_4 Q_5 Q_f q^{i-\frac{1}{2}} j^{i-\frac{1}{2}}) (1 - Q_4 q^{i-\frac{1}{2}} t^{i-\frac{1}{2}}) \right] \\
\times \left[ (1 - Q_4^{-1} Q_5 Q_6 q^{i-\frac{1}{2}} j^{i-\frac{1}{2}}) (1 - Q_4^{-1} Q_5 Q_f q^{i-\frac{1}{2}} j^{i-\frac{1}{2}}) \right] \\
\times \left[ (1 - e^{-i\lambda + i m_1} q^{i-\frac{1}{2}} j^{i-\frac{1}{2}}) (1 - e^{-i\lambda - i m_2} q^{i-\frac{1}{2}} j^{i-\frac{1}{2}}) \right] \\
\times \left[ (1 - q^{i-1} j^{i-1})^{\frac{1}{2}} (1 - q^{i-1} t^{i-1})^{\frac{1}{2}} (1 - Q_4 Q_5 q^{i-\frac{1}{2}} j^{i-\frac{1}{2}}) \right], \tag{2.27}
\]

where the factors in the first big bracket in (2.27) correspond to nine hypermultiplets of the infrared theory in the Higgs branch of the \(T_3\) theory. On the other hand, the factors in the last big bracket in (2.27) correspond to singlet hypermultiplets as a result of the Higgsing.

The physical meaning of the partition function (2.27) becomes more clear when one use the parameters associated with the unbroken global symmetry \(SU(3) \times SU(3) \times U(1)\). Originally, the generators of the global symmetry are \(D_a, (a = 1, \ldots , 6)\) and we define the parameters \(\mu_a, (a = 1, \ldots , 6)\) as

\[
t_{SU(3) \times SU(3) \times SU(3)} = \mu_1 D_1 + \mu_2 D_2 + \mu_3 D_3 + \mu_4 D_4 + \mu_5 D_5 + \mu_6 D_6. \tag{2.28}
\]

Due to the tuning (2.21), the generators of the unbroken flavor symmetry in the Higgsed vacuum is determined such that \(Q_2\) and \(Q_6 Q_1 Q_4^{-1}\) do not have any charge under the unbroken global symmetry. Then the generators of the unbroken global symmetry after the Higgsing can be chosen as

\[
t_{SU(3) \times SU(3) \times U(1)} = \mu_1 D_1 + \mu_2 (D_2 + D) + \mu_3 D_5 + \mu_2 D_6 + \mu (D_3 + 2D_4 + D). \tag{2.29}
\]

By using the generators (2.29), the chemical potentials assigned to the two-cycles in figure 5 are then

\[
Q_1 = e^{i(-\nu_1 - \bar{\nu}_1 - \mu)}, \quad Q_3 = e^{i(\nu_2 + \bar{\nu}_3 + \mu)}, \quad Q_4 = e^{i(\nu_4 + \bar{\nu}_2 - 2\mu)}, \quad Q_5 = e^{i(-\nu_3 - \bar{\nu}_1 - \mu)}, \quad Q_b = e^{i(-\nu_2 - \bar{\nu}_3 - \mu)}, \quad Q_f = e^{i(2\nu_3 - \bar{\nu}_1 - \mu)}. \tag{2.30}
\]
The two factors in the first curly bracket in (2.35) simply come from the Cartan parts of the original $T_3$ theory. The two factors in the second curly bracket can be thought as the contributions from M2-branes wrapping the two-cycle with the Kähler parameter $Q_2$ and $Q_b Q_4^{-1}$ respectively. This is essentially the same situation as the case of putting two horizontal external 5-branes together discussed in [22]. At the computational level, one factor of the two factors in the second curly bracket comes from one of the decoupled factor in (2.25).

This is because the instanton summation from the refined topological vertex automatically contains the decoupled factor $Z_{dec}$. Therefore, the singlet hypermultiplet contribution from the M2-brane wrapping the two-cycle with the Kähler parameter $Q_b Q_1 Q_4^{-1}$ is automatically canceled in the instanton summation of $\nu_1$ when one does not take into account the decoupled factor. Then, the factor $(1 - q^i j^{i-1})$ in (2.25) recovers the the contribution from the M2-brane wrapping $Q_b Q_1 Q_4^{-1}$ which was canceled in the computation of the instanton summation of $\nu_1$.

There is also another factor of a singlet hypermultiplet which depends on the parameters associated with the flavor symmetry of the theory in (2.35). The singlet hypermultiplet which is in the third curly bracket in (2.35) may be inferred from the web diagram of figure 5. Since it is a contribution of a singlet which is decoupled from the infrared theory in the Higgs
branch of the $T_3$ theory, it is associated with the contribution from new parallel external legs which only appear after the Higgsing as considered in [22]. This is analogous to the decoupled factor (2.11) before the Higgsing, which is the contribution from the parallel external legs in the $T_3$ web diagram. After the Higgsing of the first kind, an internal line becomes an external line. The new external line can be easily identified from the dot diagram depicted in the right figure of figure 5. The dot diagram was introduced in [16], and it is the dual diagram of the web diagram corresponding to a theory in a Higgs branch. The dual of the usual web diagram is a toric diagram with all the dots are denoted by black dots. The dot diagram introduces a white dot which implies the 5-branes which are separated by the white dot are on top of each other. Then, if an external line of the web diagram after a tuning crosses a line which is not on boundaries of the dot diagram, then the external line corresponds to a new external leg. For the current example, the new external leg is depicted in red color in the dot diagram of figure 5. Then we have new parallel external legs whose distance is parameterized by $Q_4Q_5$. Therefore, M2-branes wrapping the two-cycle whose Kähler parameter is $Q_4Q_5$ gives a singlet hypermultiplet contribution. The contribution is nothing but the very last factor in (2.35). Note also that in this case, the factors in the second curly bracket in (2.35) may be regarded as the contributions from new parallel external legs where the parallel external legs are on top of each other in figure 5.

## 2.3 Higgsed $T_3$ theory II

We then consider the different Higgs branch realized by putting the two rightmost parallel vertical legs together, corresponding to the figure 6. By applying (2.6) again, the Higgs branch
can be achieved by choosing the following tuning of the Kähler parameters

\[ Q_4 = \left( \frac{q}{t} \right)^{\frac{1}{2}}, \quad Q_5 = \left( \frac{q}{t} \right)^{\frac{1}{2}}. \]  \tag{2.36}

The partition function of the infrared theory in this Higgs branch of the \( T_3 \) theory becomes

\[ Z_{IR} = Z_0 \cdot Z_{\text{inst}} \cdot Z_{\text{dec}}^{-1} \]  \tag{2.37}

\[ Z_0 = \prod_{i,j=1}^{\infty} \frac{(1 - e^{-i\lambda + im_2} q^{-\frac{1}{2}} t^{i - \frac{1}{2}})(1 - e^{-i\lambda - im_1} q^{-\frac{1}{2}} t^{i - \frac{1}{2}})}{(1 - q^{i j - 1})^{-\frac{1}{2}}(1 - q^{-i j - 1})^{\frac{1}{2}}(1 - e^{-2i\lambda} q^{i j - 1})}
\]

\[ (1 - e^{i\lambda + im_2} q^{-\frac{1}{2}} t^{i - \frac{1}{2}})(1 - e^{-i\lambda + im_2} q^{-\frac{1}{2}} t^{i - \frac{1}{2}})(1 - e^{i\lambda - im_1} q^{-\frac{1}{2}} t^{j - \frac{1}{2}})(1 - e^{-i\lambda - im_1} q^{-\frac{1}{2}} t^{j - \frac{1}{2}})(1 - e^{-i\lambda - im_1} q^{-\frac{1}{2}} t^{j - \frac{1}{2}}) \],

\[ Z_{\text{inst}} = \sum_{\nu_1, \nu_2, \mu_5} u_2^{[\nu_1] + [\nu_2]} \left( e^{-i\lambda} \left( \frac{q}{t} \right)^{\frac{1}{2}} \right)^{[\mu_5]} \prod_{\alpha=1}^{2} \prod_{\alpha=1}^{3} \frac{2i \sin \frac{E_{\alpha \beta} - m_\alpha + \gamma_1}{2}}{2i \sin \frac{E_{\alpha \beta} + 2\gamma_1}{2}} \frac{2i \sin \frac{E_{\alpha \beta} - m_\alpha + \gamma_1}{2}}{2i \sin \frac{E_{\alpha \beta} + 2\gamma_1}{2}} \right) \tag{2.38}
\]

\[ \prod_{s \in \mu_5} \frac{\prod_{\alpha=1}^{2} 2i \sin \frac{E_{s \alpha} - \lambda}{2}}{(2i)^2 \sin \frac{E_{s \alpha} + 2\gamma_1}{2}} , \]

\[ Z_{\text{dec}}^{-1} = \prod_{i,j=1}^{\infty} \frac{(1 - e^{-2i\lambda} q^{i j - 1})(1 - u_2 e^{-\frac{3i\lambda}{2}} (m_1 m_2 + m_3) q^{-\frac{1}{2}} t^{i j - \frac{1}{2}})(1 - u_2 e^{-\frac{3i\lambda}{2}} (m_1 m_2 + m_3) q^{i j - \frac{1}{2}} t^{i j - \frac{1}{2}})}{(1 - q^{i j - 1})(1 - u_2 e^{-\frac{3i\lambda}{2}} (m_1 m_2 + m_3) q^{-\frac{1}{2}} t^{i j + \frac{1}{2}})(1 - u_2 e^{-\frac{3i\lambda}{2}} (m_1 m_2 + m_3) q^{i j + \frac{1}{2}} t^{i j + \frac{1}{2}})} \]  \tag{2.39}

The tuning of \( Q_4 \) by (2.36) simplifies the Young diagram summation of \( \mu_5 \). The instanton partition function (2.38) can be non-zero if \( \mu_5, \nu_2 \) for all \( i \). Here \( \nu_1 \) implies an \( i \)-th row of the Young diagram \( \nu \). Namely, the summation of \( \mu_5 \) vanishes if any \( i \)-th row of \( \mu_5 \) is not greater than the \( i \)-th row of \( \nu_2 \). This is due to the term \( \sin \left( \frac{E_{s \alpha} - \lambda}{2} \right) \) in (2.38). If \( \mu_5 > \nu_2 \), then the function \( \sin \left( \frac{E_{s \alpha} - \lambda}{2} \right) \) at \( (1, |\mu_5, i|) \in \mu_5 \) gives zero. If we then assume \( \mu_5, \nu_2 \leq \nu_2 \) and \( \nu_5, \nu_2 > \nu_2 \), the function \( \sin \left( \frac{E_{s \alpha} - \lambda}{2} \right) \) at \( (2, |\mu_5, i|) \in \mu_5 \) yields zero. In this way, the term \( \sin \left( \frac{E_{s \alpha} - \lambda}{2} \right) \) gives zero unless \( \mu_5, \nu_2 \leq \nu_2 \) for all \( i \). Therefore, until the order \( \mathcal{O}(u_1^{[\nu_1] + [\nu_2]}) \) with \( |\nu_1| + |\nu_2| = k \), the expansion by \( u_2 \) is exact when one includes the expansion regarding \( \mu_5 \). Note also that there are non-zero contributions from \( |\mu_5| = |\nu_2| \leq k \). Although the two-cycles associated with the Young diagrams \( \mu_5 \) and \( \nu_2 \) are connected with each other in the web diagram.
The instanton partition function (2.38) again can be written by the product of Plethystic exponentials

\[
Z_{\text{inst}} = \prod_{i,j=1}^{\infty} \frac{(1 - Q_{b}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - Q_{1}Q_{3}Q_{b}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})}{(1 - u_{2}e^{\frac{\lambda + \frac{1}{2}(m_{1} + m_{2} + m_{3})}{4}(1 - u_{2}e^{-\frac{\lambda}{2}(m_{1} + m_{2} + m_{3})}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})}(1 - Q_{2}Q_{3}Q_{b}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - Q_{1}Q_{2}Q_{b}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})}{(1 - u_{2}e^{-\frac{\lambda + \frac{1}{2}(m_{1} + m_{2} + m_{3})}{4}(1 - u_{2}e^{-\frac{\lambda}{2}(m_{1} + m_{2} + m_{3})}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}))},
\]

(2.40)

The equality of (2.40) has been checked up to \(\mathcal{O}(u_{2}^{2})\). Then, the partition function (2.37) becomes

\[
Z_{TZR} = \prod_{i,j=1}^{\infty} \left[ (1 - Q_{b}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - Q_{1}Q_{3}Q_{b}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - Q_{2}Q_{3}Q_{b}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{-i\lambda + im_{1}}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i\lambda + im_{2}}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}) \right. \\
\times (1 - e^{-i\lambda + im_{3}}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i\lambda + im_{1}}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i\lambda + im_{2}}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}) \\
\times (1 - e^{-i\lambda + im_{3}}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i\lambda + im_{1}}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i\lambda + im_{2}}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}) \\
\times \left. \left[ (1 - q^{i}t^{j-1})^{\frac{3}{2}}(1 - q^{i-1}t^{j})^{\frac{3}{2}}(1 - Q_{1}Q_{2}Q_{b}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}) \right) \right].
\]

(2.41)

As with the case of (2.27), the factors in the first big bracket stand for the nine free hypermultiplets and the factors in the last big bracket represent the singlet hypermultiplet contributions.

One can again rewrite the partition function (2.41) by the parameters associated with the global symmetry \(SU(3) \times SU(3) \times U(1)\). The generators of the unbroken global symmetry can be found by requiring that \(Q_{2}\) and \(Q_{b}Q_{1}Q_{4}^{-1}\) have no charge under the unbroken global symmetry in the Higgs branch. Then the generators are

\[
t_{SU(3) \times SU(3) \times U(1)} = \mu_{1}D_{1} + \mu_{2}D_{2} + \tilde{\mu}_{1}(D_{5} + D) + \tilde{\mu}_{2}D_{6} + \mu(2D_{3} + D_{4} + D).
\]

(2.42)

The parameterization of the two-cycles with finite size in figure 6 is

\[
Q_{1} = e^{i(\nu_{2} + \nu_{3} - \mu)}, \quad Q_{2} = e^{i(\nu_{3} + \tilde{\nu}_{1} - \mu)}, \quad Q_{3} = e^{i(-\nu_{1} - \tilde{\nu}_{2} + \mu)}, \quad Q_{b} = e^{i(\nu_{1} + \tilde{\nu}_{3} - \mu)}, \quad Q_{f} = e^{i(-\tilde{\nu}_{1} + \tilde{\nu}_{2})},
\]

(2.43)

where we again use (2.31) and (2.32). By using the parameters (2.43), the partition function (2.41) can be written

\[
Z_{TZR} = \prod_{i,j=1}^{\infty} \left[ (1 - e^{i(\nu_{1} + \tilde{\nu}_{3} - \mu)}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i(\nu_{2} + \tilde{\nu}_{3} - \mu)}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i(\nu_{3} + \tilde{\nu}_{3} - \mu)}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}) \\
\times (1 - e^{i(\nu_{2} + \tilde{\nu}_{3} - \mu)}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i(-\nu_{1} - \tilde{\nu}_{2} + \mu)}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i(\nu_{3} + \tilde{\nu}_{1} - \mu)}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}) \\
\times (1 - e^{i(\nu_{3} + \tilde{\nu}_{2} - \mu)}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i(-\nu_{1} - \tilde{\nu}_{2} + \mu)}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{i(-\nu_{1} - \tilde{\nu}_{2} + \mu)}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}) \\
\times \left. \left[ (1 - q^{i}t^{j-1})^{\frac{3}{2}}(1 - q^{i-1}t^{j})^{\frac{3}{2}}(1 - q^{i-3\mu}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}) \right) \right].
\]

(2.44)
Therefore, we can explicitly see that the partition function describes the free 9 hypermultiplets associated with the global symmetry $SU(3) \times SU(3) \times U(1)$ plus singlet hypermultiplets in this case also.

We can again understand the origin of the singlet hypermultiplets from the web diagram of figure 6 as in section 2.3. The singlet hypermultiplets contribution which only depend on the $Ω$–deformation parameters come from the Cartan parts of the original $T_3$ theory and also M2-branes wrapping the two-cycle with the Kähler parameter $Q_4$ or $Q_5$. The total contributions explain the factors $(1 - q^i t^j)^{3/2} (1 - q^{-1} t^j)^{-3/2}$ in (2.44). Also, the contribution of the very last factor of (2.44) comes from the new parallel external legs after the Higgsing. The new external leg in the dot diagram is depicted in red color in the right figure of figure 6, which corresponds to the two-cycle with the Kähler parameter $Q_f Q_4^{-1}$. Then the distance between the new parallel external legs is parameterized by $Q_1 Q_2 Q_b$. Hence, the singlet hypermultiplet from M2-branes wrapping the two-cycle yields the contribution which is nothing but the very last factor in (2.44).

3 The partition function of the $E_8$ theory

We will apply the prescription of the tuning discussed in section 2 to the procedure for obtaining the partition function of the $E_8$ theory. The $E_8$ theory can arise in the infrared limit of a Higgs branch of the $T_6$ theory [16]. In section 2, we have seen how to obtain the partition functions of the free theory in the Higgs branch of the $T_3$ theory. We propose here the general procedure to obtain a partition function of an infrared theory realized in a Higgs branch by putting several 5-branes on a 7-brane.

1. We first compute the partition function of a UV theory by the refined topological vertex method. It is important to remove the decoupled factors which are associated with the parallel external legs.

2. For the tuning of putting several parallel external 5-branes on one 7-brane, we impose a condition (2.1) or (2.2) in the case of horizontal 5-branes, or (2.6) or (2.7) in the case of vertical 5-branes. The tuning of the Kähler parameters for the two-cycles inside the web diagram is determined by the consistency of the geometry.

3. We parameterize the lengths of internal 5-branes or the Kähler parameters of compact two-cycles by the chemical potentials associated with unbroken gauge symmetries and those of unbroken global symmetries. The unbroken symmetries can be determined by requiring that the tuned two-cycles have no charge under the unbroken symmetries in the Higgs vacuum. Linear combinations of the Cartan generators of the unbroken global symmetries are associated with masses and instanton fugacities in the perturbative regime.

\footnote{For the tuning of putting the parallel diagonal external 5-branes together, we can use the condition (A.1) or (A.2).}
After inserting the tuning conditions as well as the new parameterization, we almost obtain the partition function of the low energy theory in the Higgs branch of the UV theory. However, there can be still some contributions from singlet hypermultiplets. We need to remove such contributions. The singlet hypermultiplet factor which depends on some parameters associated with flavor symmetries in the theory may be inferred from the web diagram. The contribution of such a singlet hypermultiplet is associated with strings between new parallel external 5-branes which only appear after moving to the Higgs branch. Note that such a singlet hypermultiplet contribution can depend on an instanton fugacity. The other singlet hypermultiplet factor which only depends on the Ω–deformation parameters appears in the perturbative part, namely the zero-th order of the instanton fugacities. Once we obtain the perturbative part, we can identify those contributions.

After eliminating the singlet hypermultiplet contributions, we finally obtain the partition function of the infrared theory in the Higgs branch.

In this section, we will obtain the partition function of the \( E_8 \) theory by following the above steps.

### 3.1 \( T_6 \) partition function

In this section we review the partition function of the \( T_6 \) theory. This theory can be obtained by compactifying M-theory on the blow-up of \( \mathbb{C}^3/(\mathbb{Z}_6 \times \mathbb{Z}_6) \) whose toric diagram we show in figure 7. In the figure we also show how the fugacities \( P_k^{(n)} \), \( Q_k^{(n)} \) and \( R_k^{(n)} \) are associated to the two cycles present in the geometry. Note that the geometry imposes some conditions on these fugacities

\[
Q_k^{(n)} P_k = Q_k^{(n+1)} P_k^{(n+1)}, \quad R_k^{(n+1)} Q_k = R_k^{(n+1)} Q_k^{(n)},
\]

so that the actual number of Kähler parameters is 25. The partition function of this theory was computed in [21, 22]

\[
Z_{T_6} = (M(t, q) M(q, t))^5 Z_0 Z_{\text{inst}} Z_{\text{dec}}^{-1},
\]

\[
M(t, q) = \prod_{i,j=1}^{\infty} (1 - q^i t^j)^{-1},
\]

\[
Z_0 = \prod_{i,j=1}^{\infty} \left\{ \frac{\prod_{a \leq b} (1 - e^{-i\lambda_{a,b} + i\tilde{m}_a q^{i-\frac{1}{2} i j - \frac{1}{2}}}) \prod_{b < a} (1 - e^{i\lambda_{b,a} - i\tilde{m}_a q^{i-\frac{1}{2} i j - \frac{1}{2}}})}{\prod_{n=1}^{5} \prod_{a < b} (1 - e^{i\lambda_{a,b}} q^{i-\frac{1}{2} i j - \frac{1}{2}})(1 - e^{i\lambda_{a,b}} q^{i-\frac{1}{2} i j - \frac{1}{2}})} \right\}
\]

\[
\times \prod_{n=2}^{5} \prod_{a \leq b} (1 - e^{i\lambda_{a,b}} q^{i-\frac{1}{2} i j - \frac{1}{2}})(1 - e^{i\lambda_{a,b}} q^{i-\frac{1}{2} i j - \frac{1}{2}}),
\]
Figure 7. The web diagram for the $T_6$ theory.

$$Z_{\text{inst}} = \sum_{Y_1, \ldots, Y_5} \left\{ \prod_{n=1}^{4} u_{n}^{Y_n} \prod_{\alpha=1}^{n} \prod_{s \in Y_{n, \alpha}} \left[ \prod_{\beta=1}^{n+1} 2i \sin \frac{E_{\alpha \beta} - \hat{m}_{n+1} + i \gamma_1}{2} \right] \prod_{\beta=1}^{n} (2i)^2 \sin \frac{E_{\alpha \beta} + 2i \gamma_1}{2} \right\}$$

$$\times \left\{ \prod_{\alpha=1}^{5} \prod_{s \in Y_{5, \alpha}} \left[ \prod_{k=1}^{6} 2i \sin \frac{E_{\alpha \beta} - \hat{m}_k + i \gamma_1}{2} \right] \prod_{\beta=1}^{4} 2i \sin \frac{E_{\alpha \beta} + \hat{m}_k + i \gamma_1}{2} \right\},$$

$$Z_{\text{dec}}^{-1} = \prod_{i,j=1}^{\infty} \prod_{1 \leq a < b \leq 6} \left( 1 - \left( \prod_{n=a}^{b-1} R_{1}^{(n)} P_{1}^{(n)} \right) q^{i-1} t^j \right) \left( 1 - \left( \prod_{n=a}^{b-1} R_{n}^{(n)} Q_{n}^{(n)} \right) q^{i} t^{j-1} \right).$$

In writing the partition function we have used the Coulomb branch moduli $\lambda_{n,k}$ with $1 \leq k \leq n = 2, \ldots, 5$ defined by

$$P_{k}^{(n-1)} Q_{k}^{(n-1)} = \exp(-i \lambda_{n,k+1} + i \lambda_{n,k}).$$
and subject to the condition $\sum_{k=1}^{n} \lambda_{n;k} = 0$. Moreover the parameters $\tilde{m}_n$ with $n = 2, \ldots, 5$ are defined by
\[
P_{k}^{(n-1)} = \exp(i \lambda_{n;k} - i \lambda_{n-1;k} + i \tilde{m}_n),
\] (3.8)
and the parameters $\tilde{m}_k$ with $k = 1, \ldots, 6$ by
\[
P_{k}^{(5)} Q_{k}^{(5)} = \exp(-i \tilde{m}_{k+1} + i \tilde{m}_k), \quad P_{k}^{(5)} = \exp(i \tilde{m}_k - i \lambda_{5;k}).
\] (3.9)
Moreover the parameters $u_k$ with $k = 1, \ldots, 5$ are defined as
\[
u_k = \sqrt{R_1^{(k)} P_1^{(k)} R_k^{(k)} Q_k^{(k)}}.
\] (3.10)

3.2 The $E_8$ theory from $T_6$ theory

It was argued in [16] that it is possible to engineer a theory with an $E_8$ global symmetry in the Higgs branch of the $T_6$ theory and we show in figure 8 the web diagram that realizes this theory. The resulting theory has a manifest $SU(6) \times SU(3) \times SU(2)$ global symmetry which is believed to enhance to $E_8$. A similar story happens for a 5d $Sp(1)$ gauge theory with $N_f = 7$ fundamental flavors whose manifest global $SO(14) \times U(1)$ symmetry enhances to $E_8$ as well at the conformal point [38–41]. The relation between these Lie algebras is shown in

\[\text{Figure 8. Higgsed } T_6 \text{ diagram. On the left the original diagram, in green the curves whose Kähler parameters are restricted to engineer the } E_8 \text{ theory and in red the curves whose Kähler parameters are restricted because of the geometric constraint (3.1). On the right the resulting web diagram after the Higgsing.}\]
Figure 9. The Dynkin diagram of the affine $E_8$ Lie algebra. The nodes in the dotted line represent the Dynkin diagram of $SO(14)$. The nodes in the solid lines denote the Dynkin diagram of $SU(6) \times SU(3) \times SU(2)$.

Furthermore these theories have Coulomb branch and Higgs branch with the same dimensions, namely $\dim_C(M_C) = 1$ and $\dim_H(M_H) = 29$. As we will see later the partition function will have $E_8$ symmetry providing further evidence for the enhancement of the global symmetry. In order to achieve this diagram from the web diagram of the $T_6$ theory it is necessary to perform a tuning of the Kähler parameters of some of the curves in the diagram in order to group some of the external 5-branes on a single 7-brane. From figure 8 we see that we need to group the three upper left legs, the three lower left legs, the two leftmost lower legs, the two central lower legs and the two rightmost lower legs. To group the three upper left legs we need to impose

$$Q_5^{(5)} = P_5^{(5)} = Q_4^{(5)} = P_4^{(5)} = \left(\frac{q}{t}\right)^{\frac{1}{2}},$$

and to group the three lower left legs the conditions are

$$P_1^{(5)} = Q_1^{(5)} = P_2^{(5)} = Q_2^{(5)} = \left(\frac{q}{t}\right)^{\frac{1}{2}},$$

by using (2.1). Finally for the leftmost lower legs we impose

$$P_1^{(5)} = R_1^{(5)} = \left(\frac{q}{t}\right)^{\frac{1}{2}},$$

for the central ones we impose

$$P_1^{(3)} = R_1^{(3)} = \left(\frac{q}{t}\right)^{\frac{1}{2}},$$

and finally for the rightmost lower legs we impose

$$P_1^{(1)} = R_1^{(1)} = \left(\frac{q}{t}\right)^{\frac{1}{2}},$$

by using (2.6). While these conditions are sufficient to realize the desired pattern for external legs we also need to take into account the geometric constraints of the web diagram (3.1) and in the end some additional Kähler parameters will be restricted. Quite interestingly applying
these geometric constraints appears to be equivalent to the propagation of the generalized s-rule presented in [16]. In the end we will have that the geometric constraints (3.1) will imply the following conditions on Kähler parameters

\[ Q_4^{(4)} = P_4^{(4)} = Q_1^{(4)} = P_1^{(4)} = R_2^{(5)} = R_3^{(5)} = Q_2^{(4)} = \left( \frac{q}{l} \right)^2, \]  

(3.16)

### 3.3 \textit{Sp}(1) gauge theory parametrization

In this section we describe how to define the instanton fugacity of the \textit{Sp}(1) gauge theory analyzing the global \( SU(6) \times SU(3) \times SU(2) \) symmetry inside \( E_8 \). The first step is to determine the unbroken generators of the unbroken flavor symmetry \( SU(6) \times SU(3) \times SU(2) \). In the original \( T_6 \) theory there are 25 generators, 10 of these generators are associated to compact divisors in the geometry and are parameterized by the Coulomb branch moduli while the remaining 15 are associated to non-compact divisors and realize the \( SU(6) \times SU(6) \times SU(6) \) flavor symmetry.

After fixing some Kähler parameters to realize the \textit{Sp}(1) with 7 flavors gauge theory only a reduced number of generators will be unbroken, namely there will be a single Coulomb branch modulus and the generators of the \( SU(6) \times SU(3) \times SU(2) \) flavor symmetry. The unbroken generators are easily identified as the linear combinations of compact and non-compact divisors of the geometry that do not intersect any of curves whose Kähler parameter is restricted. This procedure yields as expected 9 linearly independent generators which we wish to identify with the generators of \( SU(6) \times SU(3) \times SU(2) \) and the generator associated to the Coulomb branch modulus. First we label the divisors in the geometry as in figure 7. Naively we would associate the generators of the \( SU(6) \) part of the flavor symmetry with the non-compact divisors \( D_{11}, D_{16}, D_{20}, D_{23}, \) and \( D_{25} \), the generators of the \( SU(3) \) part of the flavor symmetry with the non-compact divisors \( D_{12} \) and \( D_{21} \), and the generator of the \( SU(2) \) part of the flavor symmetry with the non-compact divisors \( D_3 \) while the generator associated with the Coulomb branch modulus with \( D_{19} \). This allows us to identify one of the generators of \( SU(6) \) as the linear combination of unbroken generators that contains \( D_{11} \) with coefficient 1 but does not contain any of the other flavor generators and the gauge generator. A similar procedure can be applied to the other generators as well allowing the identifications of the generators of the flavor symmetry. For concreteness, we list up the Cartan generators for \( SU(6) \times SU(3) \times SU(2) \) in appendix B.

Let us first define the mass parameters \( m_i, (i = 1, \cdots, 7) \) as follows,

\[ Q_3^{(3)} = e^{i\lambda - im_1}, \quad P_3^{(3)} = e^{-i\lambda + im_2}, \quad R_3^{(4)} = e^{i\lambda + im_3}, \quad P_2^{(3)} = e^{i\lambda + im_4}, \]
\[ Q_1^{(2)} = e^{i\lambda + im_5}, \quad P_2^{(2)} = e^{-i\lambda - im_6}, \quad R_2^{(2)} = e^{i\lambda - im_7}, \quad R_3^{(3)} = e^{i\tilde{u} - i\lambda}. \]  

(3.17)

The dependence of the Coulomb branch modulus \( \lambda \) is determined by the intersection between the compact divisor \( D_{19} \) and two-cycles. The two-cycles in (3.17) are the ones which have non-zero intersection number with \( D_{19} \). We also introduced \( \tilde{u} \) whose linear combination with \( m_i, (i = 1, \cdots, 7) \) eventually becomes a chemical potential for the instanton fugacity of the
$Sp(1)$ gauge theory. By using the parameters in (3.17), we find that the fugacities for particles in the canonical simple roots of the flavor symmetry are

\[
\begin{align*}
SU(6) : & \{e^{im_2-im_4}, e^{-im_2-im_3}, e^{im_1-i\bar{u}}, e^{-im_6+im_7}, e^{-im_5+im_6}\}, \\
SU(3) : & \{e^{-im_3-im_5-im_6-\bar{u}}, e^{-im_2-im_4+im_7-\bar{u}}\}, \\
SU(2) : & \{e^{im_1-im_2-im_4-im_5-im_6-\bar{u}}\}. 
\end{align*}
\tag{3.18}
\]

As in section 2.1 we would like the simple roots of $SU(6) \times SU(3) \times SU(2)$ to be understood as roots of $E_8$. Recalling that the roots of $E_8$ are 

\[
\pm (e_i \pm e_j),
\tag{3.19}
\]

with $i, j = 1, \cdots, 8$ and 

\[
\frac{1}{2} (\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8),
\tag{3.20}
\]

with an even number of minus signs, we see that the chemical potentials for the particles in the simple roots of $SU(6) \times SU(3) \times SU(2)$ fit in the $E_8$ root system if we choose 

\[
\bar{u} = \frac{1}{2} m_8 + \frac{1}{2} (m_1 - m_2 - m_3 - m_4 - m_5 - m_6 + m_7).
\tag{3.21}
\]

Writing the instanton fugacity of the $Sp(1)$ gauge theory as 

\[
u = e^{i \bar{u} m_8}
\tag{3.22}
\]

we find that 

\[
R_3^{(3)} = u e^{-i \lambda + \frac{1}{2} (m_1 - m_2 - m_3 - m_4 - m_5 - m_6 + m_7)} \equiv u e^{-i \lambda + i f(m)}.
\tag{3.23}
\]

where for later purposes we have defined a particular linear combination of masses $f(m)$.

In the perturbative regime of the $Sp(1)$ gauge theory with 7 flavors, the mass parameters are associated with the $SO(14)$ flavor symmetry and the instanton current supplies another $U(1)$ symmetry. However, not all the simple roots of $SO(14)$ inside $E_8$ as in figure 9 are written by $\pm m_i \pm m_j, (i, j = 1, \cdots, 7)$ in (3.18). This is because we are in a different Weyl chamber of the $E_8$ Cartan subalgebra. If we perform a sequence of Weyl reflections, we can write the mass parameters of the particles in the simple roots of $SO(14)$ inside $E_8$ as $m_i - m_{i+1}, m_6 + m_7, (i = 1, \cdots, 6)$.

### 3.4 Singlets in the Higgs vacuum

As already noted in [22] and explained in section 2 applying the tuning to the $T_6$ partition function will not give simply the partition function of $Sp(1)$ gauge theory with $N_f = 7$ fundamental flavors as there will be additional contributions coming from singlet hypermultiplets. Therefore the actual partition function of the $E_8$ theory will be 

\[
Z_{E_8} = Z_{T_6}^H / Z_{extra},
\tag{3.24}
\]

- 24 -
where we called $Z_{T_6}^H$ the $T_6$ partition function after tuning the Kähler parameters and gathered in $Z_{extra}$ the contributions due to singlet hypermultiplets. In this section, we identify $Z_{extra}$ for the infrared theory in the Higgs branch of the $T_6$ theory corresponding to figure 7.

We will start by explaining how to identify the singlet hypermultiplets factors that only depend on the $\Omega$–deformation parameters. This kind of singlets originate from M2-branes wrapping two cycles and linear combinations of two cycles whose Kähler parameter is $(q/t)^{\frac{1}{2}}$ and their contributions to the partition function can be understood locally in the diagram. This allows us to split the discussion in six different parts: looking at figure 8 we see that the kind of curves we are interested in appear in the upper left part, in the bottom left part, in the middle top part, in the middle bottom part and in the bottom right part of the diagram. We will now discuss all these contributions separately. In the upper left part the contribution involves the curves $Q_5^{(5)}$, $P_5^{(5)}$, $Q_4^{(5)}$, $P_4^{(5)}$, and the contribution due to singlet hypermultiplets and vector multiplets is

$$Z_{singl}^{(1)} = \prod_{i,j=1}^{\infty} (1 - q^i t^{-1})^3(1 - q^{i+1} t^{-1})^2.$$ \hspace{1cm} (3.25)

In the bottom left part the contribution is a bit more involved, but being the contribution local we can select a part of the diagram that looks like the higgsed $T_3$ diagram of section 2. Being careful not to subtract the decoupled factor from parallel diagonal legs that are not
external in this case we get the following contribution

\[ Z_{\text{singl}}^{(2)} = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^6 (1 - q^{i+1} t^{j-2}). \] (3.26)

In the middle left part we have only the curves \( Q_2^{(4)} \) and \( R_3^{(5)} \). In this case, we need to be careful of subtracting a part of the vector multiplet coming from M2-branes wrapping the two-cycle whose Kähler parameter is \( Q_2^{(4)} R_3^{(5)} \). Then, the final contribution is simply

\[ Z_{\text{singl}}^{(3)} = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1}). \] (3.27)

Finally we have the contributions in the middle top part (that involves the curves \( Q_4^{(4)} \) and \( P_4^{(4)} \)), in the middle bottom part (that involves the curves \( P_1^{(3)} \) and \( R_1^{(3)} \)) and in the bottom right part (that involves the curves \( P_1^{(1)} \) and \( R_1^{(1)} \)). These contributions are identical and are

\[ Z_{\text{singl}}^{(4)} = Z_{\text{singl}}^{(5)} = Z_{\text{singl}}^{(6)} = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^2. \] (3.28)

We are thus able to write the contribution to the partition function coming from decoupled hypermultiplets that only depend on the \( \Omega \)-deformation parameters

\[ Z_{\text{singl}} = \prod_{k=1}^{6} Z_{\text{singl}}^{(k)} = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^{16} (1 - q^{i+1} t^{j-2})^2. \] (3.29)

Next we turn to the discussion of decoupled hypermultiplets that depend on the parameters associated with the flavor symmetry. In [22] this contribution was identified with the perturbative part of the partition function of hypermultiplets and vector multiplets which come from strings stretching between parallel branes that become external after Higgsing. However while in the examples presented in [22] the identification of branes becoming external after Higgsing presented no difficulty in the case of \( E_8 \) theory this identification is a bit more subtle because of the propagation of the generalized s-rule inside the diagram, and we will apply a rule used in 2.2 and 2.3 to identify new external legs after Higgsing using the dot diagrams introduced in [16]. We briefly describe the rule here again. We identify a new horizontal external leg with a pair of vertical segments in the dot diagram, one external and one internal, that can be connected with a horizontal line without crossing any diagonal line in the dot diagram. A similar identification of parallel external legs works for vertical and diagonal legs in the diagram. Using this procedure we can identify which legs are external for the dot diagram of \( E_8 \) theory, and we show in figure 10 the result. In the result of the computation we need to discard the hypermultiplets that only depend on the \( \Omega \)-deformation parameters as these have already been included in \( Z_{\text{singl}} \). Including also the contributions
due to the higgsed Cartan part as well as \((3.29)\) we find that the total contribution is

\[
Z_{\text{extra}} = (M(q, t)M(t, q))^{\frac{q}{2}} \prod_{i, j=1}^{\infty} (1 - q^{i+1} t^{j-2})^2 (1 - q^i t^j - 1)^{16} \times \\
(1 - ue^{-im_1+im_2+im_3+im_4+im_5+im_6+if(m)} q^{i} t^{j-1})^2 \\
(1 - ue^{-im_1+im_2+im_3+im_4+im_5+im_6+if(m)} q^{i-1} t^{j})^2 \\
(1 - ue^{-im_1+im_2+im_3+im_4+im_5+im_6+im_7+if(m)} q^{i-2} t^{j+1}) \\
(1 - ue^{-im_1+im_2+im_3+im_4+im_5+im_6+im_7+2if(m)} q^{i-1} t^{j}) \\
(1 - u^2 e^{im_2+im_3+im_4+im_5+im_6+im_7+2if(m)} q^i t^j) \\
(1 - u^2 e^{im_2+im_3+im_4+im_5+im_6+im_7+2if(m)} q^i t^j - 1).
\]

\[\text{(3.30)}\]

3.5 The partition function of \(Sp(1)\) with \(N_f = 7\) flavors

Here we write the resulting partition function of the \(E_8\) theory. We recall from the previous section that

\[
Z_{E_8} = \frac{Z_{E_8}^{T_6}}{Z_{\text{extra}}},
\]

where \(Z_{E_8}^{T_6}\) is the \(T_6\) partition function after tuning the Kähler parameters and \(Z_{\text{extra}}\) includes the contributions of singlet hypermultiplets. Before writing the result some comments are needed regarding the instanton summation in \((3.4)\) as the tuning of some Kähler parameters greatly simplifies it. This happens for the same reasons as in sections 2.2 and 2.3, namely the tuning of the Kähler parameters will imply the appearance of terms of the form \(\sin \left( \frac{E_{\alpha \beta} - \lambda_\alpha + \lambda_\beta}{2} \right)\) giving a zero in the instanton summation whenever \(Y_\alpha > Y_\beta\).\(^{11}\) As in some cases the Young diagram \(Y_\beta\) is trivial this implies that the only possible diagram contributing to the instanton summation is \(Y_\alpha = \emptyset\). In the end only 8 Young diagram summations will be non-trivial, and we will call the non-trivial Young diagrams as

\[
R_3^{(5)} \to Y_1 \quad R_2^{(4)} \to Y_2 \quad R_3^{(4)} \to Y_3 \quad R_2^{(3)} \to Y_4 \\
R_3^{(3)} \to Y_5 \quad R_1^{(2)} \to Y_6 \quad R_2^{(2)} \to Y_7 \quad R_1^{(1)} \to Y_8.
\]

\[\text{(3.32)}\]

and moreover the result will vanish if \(Y_1 > Y_2\) and \(Y_8 > Y_6\).

We write the \(T_6\) partition function after tuning the Kähler parameters as

\[
Z_{T_6}^H = (M(q, t)M(t, q))^{\frac{q}{2}} Z_0^H Z_{\text{inst}}^H (Z_{\text{dec}}^H Z_{\text{dec}}^{\|})^{-1},
\]

\[\text{(3.33)}\]
where
\[
Z^H_0 = \prod_{i,j=1}^{\infty} \frac{(1 - q^{i+1}t^{i-2})(1 - q^{j}t^{j-1})^{13}(1 - e^{-i\lambda + im_2 q^{i-\frac{1}{2}t^{i-\frac{1}{2}}}})(1 - e^{-2i\lambda t^{j-1}})(1 - e^{-im_5 - im_6 q^{i}t^{j-1}})(1 - e^{im_4 - im_2 q^{i}t^{j-1}})}{(1 - e^{-2i\lambda} q^{i}t^{j-1})(1 - e^{-i\lambda + im_4 q^{i-\frac{1}{2}t^{i-\frac{1}{2}}}})(1 - e^{-im_5 - im_6 q^{i}t^{j-1}})(1 - e^{im_4 - im_2 q^{i}t^{j-1}}) \times (1 - e^{-i\lambda + im_2 q^{i-\frac{1}{2}t^{i-\frac{1}{2}}}})(1 - e^{im_5 - im_6 q^{i}t^{j-1}})(1 - e^{im_4 - im_2 q^{i}t^{j-1}}) \times (1 - e^{im_5 - im_6 q^{i}t^{j-1}})(1 - e^{im_4 - im_2 q^{i}t^{j-1}}) \times (1 - e^{-i\lambda + im_2 q^{i-\frac{1}{2}t^{i-\frac{1}{2}}}})(1 - e^{-im_5 - im_6 q^{i}t^{j-1}})(1 - e^{im_4 - im_2 q^{i}t^{j-1}}) \times \frac{(1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})(1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})}{(1 - u e^{im_1 + im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})(1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})}.
\]

(3.34)

\[
\frac{1}{Z^H_{dec}} = \prod_{i,j=1}^{\infty} (1 - e^{-im_4 - im_2 q^{i}t^{j-1}})(1 - e^{im_3 + im_4 q^{i}t^{j-1}})(1 - e^{im_2 + im_3 q^{i}t^{j-1}}) \times (1 - e^{-im_7 + im_6 q^{i}t^{j-1}})(1 - e^{im_5 - im_7 q^{i}t^{j-1}})(1 - e^{im_5 - im_6 q^{i}t^{j-1}}) \times (1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_7 + if(m) q^{i}t^{j-1}})(1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_7 + if(m) q^{i}t^{j-1}}) \times (1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_7 + if(m) q^{i}t^{j-1}})(1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_7 + if(m) q^{i}t^{j-1}}) \times \frac{(1 - u e^{-im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})(1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_7 + if(m) q^{i}t^{j-1}})}{(1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_7 + if(m) q^{i}t^{j-1}})(1 - u e^{-im_1 + im_2 + im_4 + im_5 + im_7 + if(m) q^{i}t^{j-1}})}.
\]

(3.35)

\[
\frac{1}{Z^H_{dec}} = \prod_{i,j=1}^{\infty} (1 - q^{i}t^{j-1})^3(1 - u e^{im_2 + im_4 + im_7 + if(m) q^{i}t^{j-1}})(1 - u e^{im_2 + im_4 + im_7 + if(m) q^{i}t^{j-1}}) \times (1 - u e^{im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})(1 - u e^{im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}}) \times \frac{(1 - u e^{im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})(1 - u e^{im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})}{(1 - u e^{im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})(1 - u e^{im_2 + im_4 + im_5 + im_6 + if(m) q^{i}t^{j-1}})}.
\]

(3.36)

\[
Z^H_{inst} = \sum_{Y_1, \ldots, Y_8} u_3^{[Y_1]+[Y_5]} Z_L(Y_4, Y_5) Z_M Z_R(Y_4, Y_5).
\]

(3.37)
\[ Z_L(Y_4, Y_5) = \prod_{\alpha=2,3} \left[ \prod_{s \in Y_\alpha} \left( \frac{2i \sin \frac{E_{\alpha \phi} - m^L_s + i \gamma_1}{2} - 2i \sin \frac{E_{\alpha \phi} - m^L_s + i \gamma_1}{2} - 2i \sin \frac{E_{\alpha \phi} - m^L_s + i \gamma_1}{2} \right) \prod_{\beta=2,3} \left( 2i \right)^2 \sin \frac{E_{\alpha \beta} + 2i \gamma_1}{2} \right] \right. \\
\left. \prod_{s \in Y_4} \frac{(2i)^2 \sin \frac{E_{\alpha \phi} + \lambda_3}{2} \sin \frac{E_{\alpha \phi} + \lambda_3}{2}}{(2i)^2 \sin \frac{E_{\alpha \phi} + \lambda_3}{2} \sin \frac{E_{\alpha \phi} + \lambda_3}{2}} \right] u_4 \left| Y_2 \right| \left| Y_3 \right| \left| Y_1 \right| \right. \\
\left. \prod_{\alpha=4,5} \left( 2i \right)^2 \sin \frac{E_{\alpha 6} + m^L_1 + i \gamma_1}{2} \sin \frac{E_{\alpha 7} + m^L_1 + i \gamma_1}{2} \right) , \\
\quad (3.38) \\
\]

\[ Z_M = \prod_{\alpha=4,5} \prod_{s \in Y_\alpha} \frac{2i \sin \frac{E_{\alpha \phi} - m^L_s + i \gamma_1}{2}}{(2i)^2 \sin \frac{E_{\alpha \phi} + \lambda_3}{2} \sin \frac{E_{\alpha \phi} + \lambda_3}{2}} \prod_{\beta=4,5} \left( 2i \right)^2 \sin \frac{E_{\alpha \beta} + 2i \gamma_1}{2} , \\
\quad (3.39) \\
\]

\[ Z_R(Y_4, Y_5) = \prod_{\alpha=6,7} \left[ \prod_{s \in Y_\alpha} \left( \frac{2i \sin \frac{E_{\alpha \phi} - m^R_s + i \gamma_1}{2} - 2i \sin \frac{E_{\alpha \phi} - m^R_s + i \gamma_1}{2} - 2i \sin \frac{E_{\alpha \phi} - m^R_s + i \gamma_1}{2} \right) \prod_{\beta=6,7} \left( 2i \right)^2 \sin \frac{E_{\alpha \phi} + \lambda_3}{2} \sin \frac{E_{\alpha \phi} + \lambda_3}{2} \right] \right. \\
\left. \prod_{s \in Y_8} \frac{(2i)^2 \sin \frac{E_{\alpha \phi} + \lambda_3}{2} \sin \frac{E_{\alpha \phi} + \lambda_3}{2}}{(2i)^2 \sin \frac{E_{\alpha \phi} + \lambda_3}{2} \sin \frac{E_{\alpha \phi} + \lambda_3}{2}} \right] u_8 \left| Y_6 \right| \left| Y_7 \right| \left| Y_8 \right| \\
\left. \prod_{\alpha=4,5} \left( 2i \right)^2 \sin \frac{E_{\alpha 6} + m^R_1 + i \gamma_1}{2} \sin \frac{E_{\alpha 7} + m^R_1 + i \gamma_1}{2} \right) , \\
\quad (3.40) \\
\]

where we defined the parameters

\[ \lambda_2 = -\lambda_3 = -\frac{1}{2} (m_2 - m_4) , \quad \lambda_6 = -\lambda_7 = -\frac{1}{2} (m_6 - m_5) , \quad \lambda_1 = \lambda_8 = \lambda_{\phi} = 0 , \quad \lambda_4 = -\lambda_5 = -\lambda , \]

\[ m^L_1 = m^L_3 = -\frac{1}{2} (m_4 + m_2) , \quad m^L_2 = -i \log u + \frac{1}{2} (m_4 + m_2) + m_5 + m_6 + f(m) - i \gamma_1 , \]

\[ m^R_1 = m^R_3 = -\frac{1}{2} (m_5 + m_6) , \quad m^R_2 = -i \log u + \frac{1}{2} (m_5 + m_6) + m_2 + m_4 + f(m) - i \gamma_1 , \]

\[ (3.41) \]

and the instanton fugacities

\[ u_5 = e^{i(m_4 - m_2)/2} , \quad u_4 = u^{1/2} e^{\gamma_1} e^{[2m_3 + m_5 + g(m)]/2} , \quad u_3 = u^{1/2} e^{-\gamma_1} e^{[-m_1 + f(m)]/2} , \]

\[ u_2 = u^{1/2} e^{\gamma_1} e^{[m_4 + m_5 + g(m)]/2} , \quad u_1 = e^{-\gamma_1} e^{(m_5 - m_6)/2} . \]

\[ (3.42) \]

The \( Z_{inst}^H \) part in (3.33) has a peculiar structure. It is written by gluing \( Z_L(Y_4, Y_5) \) and \( Z_R(Y_4, Y_5) \) with \( Z_M \). This is almost identical to gluing the two \( Z_{inst} \) parts of the Higgsed \( T_3 \) theory in section 2.3 with additional bi-fundamental hypermultiplets and \( U(2) \) vector multiplet along the two-cycles whose the Kähler parameters are \( R_{3} \) and \( R_{2} \). The difference only appears in \( Z_M \) where a hypermultiplet contribution from the two-cycle with the Kähler
parameter \( Q_3^{(3)} \) in the numerator, and also the remnant of the \( U(3) \) vector multiplet due to the tuning of the two-cycles whose Kähler parameters are \( P_1^{(3)} \) and \( R_1^{(3)} \) in the denominator.

We would like to extract the perturbative part of the partition function, namely we would like to take the limit \( \lim_{u \to 0} Z_{E_8} \) and see if this correctly reproduces the perturbative part of \( Sp(1) \) gauge theory with \( N_f = 7 \) fundamental flavors. We start by taking the terms in \( Z_{\text{inst}}^H \) with \( Y_4 = Y_5 = \emptyset \) because taking these Young diagrams to be non-trivial only adds terms that vanish in the limit \( u \to 0 \). Doing this the instanton summation becomes the product of two factors

\[
\left( \sum_{Y_1,Y_2,Y_3} Z_L(0,0) \right) \left( \sum_{Y_6,Y_7,Y_8} Z_R(0,0) \right).
\]  

Using the definitions of \( Z_L \) and \( Z_R \) it is easy to see that \( Z_L(0,0) \) and \( Z_R(0,0) \) are simply the instanton part of the Higgsed \( T_3 \) diagram described in section 2.3. Knowing the result of the summation it is quite easy to extract from it the perturbative part and the result is

\[
\left( \sum_{Y_1,Y_2,Y_3} Z_L(0,0) \right) \bigg|_{u=0} = \prod_{i,j=1} \frac{(1 - e^{i\lambda + im_3 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda + im_3 q^i - 1/2 tj^{-1/2}})}{(1 - e^{im_2 + im_3 q^i tj^{-1}})(1 - e^{im_4 + im_3 q^i tj^{-1}})}, \tag{3.43}
\]

\[
\left( \sum_{Y_6,Y_7,Y_8} Z_R(0,0) \right) \bigg|_{u=0} = \prod_{i,j=1} \frac{(1 - e^{i\lambda - im_7 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda - im_7 q^i - 1/2 tj^{-1/2}})}{(1 - e^{im_6 + im_7 q^i tj^{-1}})(1 - e^{im_5 + im_7 q^i tj^{-1}})}. \tag{3.44}
\]

We are now able to write the partition function as

\[
Z_{E_8} = Z_{\text{pert}} Z_{n.p.}, \tag{3.46}
\]

where

\[
Z_{\text{pert}} = (M(q,t)M(t,q))^{1/2} \prod_{i,j=1} \frac{(1 - e^{i\lambda + im_2 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda - im_2 q^i - 1/2 tj^{-1/2}})}{(1 - e^{-2i\lambda q^i tj^{-1}})(1 - e^{-2i\lambda q^i tj^{-1}})} \times
\]

\[
\times (1 - e^{i\lambda + im_4 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda + im_4 q^i - 1/2 tj^{-1/2}})(1 - e^{i\lambda - im_4 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda - im_4 q^i - 1/2 tj^{-1/2}}) \times
\]

\[
\times (1 - e^{i\lambda - im_1 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda - im_1 q^i - 1/2 tj^{-1/2}})(1 - e^{i\lambda - im_1 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda - im_1 q^i - 1/2 tj^{-1/2}}) \times
\]

\[
\times (1 - e^{i\lambda + im_5 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda + im_5 q^i - 1/2 tj^{-1/2}})(1 - e^{i\lambda - im_5 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda - im_5 q^i - 1/2 tj^{-1/2}}) \times
\]

\[
\times (1 - e^{i\lambda - im_7 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda - im_7 q^i - 1/2 tj^{-1/2}})(1 - e^{i\lambda - im_7 q^i - 1/2 tj^{-1/2}})(1 - e^{-i\lambda - im_7 q^i - 1/2 tj^{-1/2}}), \tag{3.47}
\]

\[
-
\]
\[ Z_{n.p.} = Z_{inst}^H \prod_{i,j=1}^{\infty} \frac{(1 - u e^{i m_2 + i m_3 + i m_6 + i f(m) q^{i-1} j})(1 - u e^{i m_4 + i m_5 + i m_6 + i f(m) q^{i-1} j})}{(1 - u e^{i \lambda + i m_4 + i m_5 + i g(m) q^{i-1} j})(1 - u e^{-i \lambda + i m_4 + i m_5 + i g(m) q^{i-1} j})} \times \\
\times \frac{(1 - u e^{-i m_1 + i f(m) q^{i-1} j})(1 - u e^{-i m_1 - i m_7 + i f(m) + i m e q^{i-1} j})(1 - u e^{-i m_1 + i m_3 + i m_7 + i f(m) q^{i-1} j})}{(1 - u e^{-i m_1 + i m_5 + i m_7 + i f(m) q^{i-1} j})(1 - u e^{-i m_1 + i m_3 + i m_7 + i f(m) q^{i-1} j})} \times \\
\times \frac{(1 - u e^{i m_2 + i m_4 + i m_6 + i f(m) q^{i-1} j})(1 - u e^{-i m_1 + i m_3 + i m_7 + i f(m) q^{i-1} j})}{(1 - u e^{i m_2 + i m_4 + i m_7 + i f(m) q^{i-1} j})(1 - u e^{-i m_1 + i m_3 + i m_7 + i f(m) q^{i-1} j})} \times \\
\times \frac{(1 - u e^{i m_2 + i m_4 + i m_6 + i f(m) q^{i-1} j})(1 - u e^{-i m_1 + i m_3 + i m_7 + i f(m) q^{i-1} j})}{(1 - u e^{i m_2 + i m_4 + i m_7 + i f(m) q^{i-1} j})(1 - u e^{-i m_1 + i m_3 + i m_7 + i f(m) q^{i-1} j})} \times \\
\times \frac{(1 - u e^{i m_3 + i m_5 + i m_7 + i f(m) q^{i-1} j})(1 - u e^{i m_2 + i m_3 + i m_7 + i f(m) q^{i-1} j})}{(1 - u e^{i m_3 + i m_5 + i m_7 + i f(m) q^{i-1} j})(1 - u e^{i m_2 + i m_3 + i m_7 + i f(m) q^{i-1} j})} \times \\
\times \frac{(1 - u e^{i m_2 + i m_4 + i m_7 + i f(m) q^{i-1} j})(1 - u e^{i m_3 + i m_4 + i m_7 + i f(m) q^{i-1} j})}{(1 - u e^{i m_2 + i m_4 + i m_7 + i f(m) q^{i-1} j})(1 - u e^{i m_3 + i m_4 + i m_7 + i f(m) q^{i-1} j})} \times \\
\times \frac{(1 - u e^{i m_3 + i m_5 + i m_7 + i f(m) q^{i-1} j})(1 - u e^{i m_2 + i m_3 + i m_7 + i f(m) q^{i-1} j})}{(1 - u e^{i m_3 + i m_5 + i m_7 + i f(m) q^{i-1} j})(1 - u e^{i m_2 + i m_3 + i m_7 + i f(m) q^{i-1} j})}. \tag{3.48} \]

With this choice we have that \( Z_{n.p.}|_{u=0} = 1. \)

### 3.6 Partition function at 1-instanton level

Having successfully reproduced the perturbative part of the partition function of \( Sp(1) \) with 7 flavors we would like now to discuss the partition function at 1-instanton level. In order to compute it (and also the partition function at higher instanton level) we will need to take the Young diagrams to be non-trivial and perform the instanton summation for the remaining Young diagrams. We have already noticed the equality between the instanton part of the Higgsed \( T_3 \) diagram in section 2.3 and the contributions \( Z_L(\emptyset, \emptyset) \) and \( Z_R(\emptyset, \emptyset) \), and somehow taking \( Y_4 \) or \( Y_5 \) to be non-trivial is related somehow to the instanton part of a Higgsed \( T_3 \) diagram with non-trivial representation on an external leg with some additional hypermultiplets. We can consider the following quantity

\[ \tilde{Z}_L(Y_4, Y_5) \equiv \frac{\sum_{Y_1, Y_2, Y_3} Z_L(Y_4, Y_5)}{\sum_{Y_1, Y_2, Y_3} Z_L(\emptyset, \emptyset)}, \tag{3.49} \]

and a similar quantity involving \( Z_R(Y_4, Y_5) \). Knowing the result of the summation for \( Z_L(\emptyset, \emptyset) \) if we are able to compute \( \tilde{Z}_L(Y_4, Y_5) \) we automatically have the result of the summation for \( Z_L(Y_4, Y_5) \). We have observed that expressing \( \tilde{Z}_L(Y_4, Y_5) \) as a series in the instanton fugacity \( u_4 \) the series stops at a finite order. More specifically we expect at level \( k = |Y_4| + |Y_5| \) the series terminates at order \( u_4^k \) with higher order terms vanishing. We have checked this explicitly up to \( k = 2 \) for higher orders of \( u_4 \). We emphasize that the termination of the series happens separately for each choice of \( Y_4 \) and \( Y_5 \) in the external legs, not only for the sum of all contributions with fixed \( k \). Using this it is possible to compute explicitly the partition...
function at 1-instanton level and the result matches with field theory one \cite{42}

\[ Z_{Sp(1)}^{k=1} = \frac{1}{32} \left[ \prod_{a=1}^{7} \frac{2i \sin \frac{\gamma_a}{2}}{\sinh \frac{\gamma_a + \gamma_b}{2}} + \prod_{a=1}^{7} \frac{2 \cos \frac{\gamma_a}{2}}{\sin \frac{\gamma_a + \gamma_b}{2} \cos \frac{\gamma_a + \gamma_b}{2}} \right], \tag{3.50} \]

where we used the notation \( \sin(a \pm b) = \sin(a + b)\sin(a - b) \).

### 3.7 2-instanton order and the comparison with field theory result

We would like to understand if \( Z_{E_8} \) correctly reproduces the partition function of an \( Sp(1) \) gauge theory with 7 fundamental flavors at 2-instanton level, however it is first useful to review how the computation of the instanton partition function is performed in field theory. It is possible to engineer 5d \( Sp(N) \) gauge theory with \( N_f \leq 7 \) in string theory on the worldvolume of \( N \) D4-branes in the proximity of \( N_f \) D8-branes and an O8-plane. In this system instantons in the 5d gauge theory are D0-branes and as we will discuss later the partition function at \( k \) instanton level can be computed as a Witten index in the ADHM quantum mechanics on the worldvolume of \( k \) D0-branes. Note that in this system an additional hypermultiplet in the antisymmetric representation of \( Sp(N) \) is present which originates from strings stretching between the \( N \) D4-branes and the orientifold plane (or the mirror \( N \) D4-branes). The presence of the antisymmetric hypermultiplet is important even for the case of \( N = 1 \) where the antisymmetric representation is trivial for it changes the instanton calculation providing non-perturbative couplings due to small instantons. Even the naive expectation that in the final result for \( N = 1 \) the contribution due to the antisymmetric representation simply factors out of the partition function is not true for \( N_f = 7 \) as noted in \cite{27, 42} and the computation performed without including the antisymmetric representation does not give the correct partition function (for instance the superconformal index does not respect the \( E_8 \) symmetry). However it is important to note that the computation will contain the contributions of additional states that are present in the string theory realization but are not present in the field theory, states that can be interpreted as due to strings in the system D0-D8-O8, and once the contributions due to these states are canceled the 5d partition function is correctly reproduced.

The quantity we would like to discuss is a Witten index \( Z_{QM}^k \) for the ADHM quantum mechanics on the worldvolume of \( k \) D0-branes

\[ Z_{QM}^k(\epsilon_1, \epsilon_2, \alpha_1, z) = \text{Tr} \left[ (-1)^F e^{-\beta \{Q,Q^\dagger\}} e^{-i\epsilon_1 (J_1 + J_R)} e^{-i\epsilon_2 (J_2 + J_R)} e^{-i\lambda_i \Pi_i} e^{-izF} \right], \tag{3.51} \]

where \( Q \) and its conjugate \( Q^\dagger \) are a couple of supercharges, \( J_1 \) and \( J_2 \) are the Cartan generators of \( SO(4) \) symmetry rotating in two orthogonal planes, \( J_R \) is the Cartan generator of the \( SU(2)_R \) R-symmetry group, \( \lambda_i \) are the Coulomb branch moduli and \( z \) generically denote other chemical potentials. Knowing the index \( Z_{QM} = \sum_k \eta^k Z_{QM}^k \) it is possible to compute the instanton part of the 5d partition function as \( Z_{inst} = Z_{QM}/Z_{string} \) where \( Z_{string} \) contains the contributions of additional states that are present in the string theory realization but not present in the field theory. We will write its explicit expression later, but first we will discuss
how to compute $Z_{QM}^k$. The result can be expressed as a contour integral in the space of zero modes given by the holonomies of the gauge field and the scalar in the vector multiplet in the ADHM quantum mechanics. Since the gauge group $\hat{G}$ of the ADHM quantum mechanics is compact the holonomies of the vector field actually live in a compact space and the space of zero modes will be the product of $r$ cylinders where $r$ is the rank of $\hat{G}$.

For the case of $Sp(N)$ gauge theories some additional care is needed for $\hat{G} = O(k)$ which is not connected and the $k$ instanton index is

$$Z_{QM}^k = \frac{1}{2}(Z_+^k + Z_-^k) \tag{3.52}$$

where $Z_{\pm}^k$ is the index for the $O(k)_\pm$ component. The correct definition of the contour of integration is discussed in [27] and here we will simply state the result for the case we are interested in. The rank of $O(2)_+$ is 1 so that the moduli space is a cylinder and we have that

$$Z_2^+ = \int_C \frac{[d\phi]}{2\pi} Z_{vec}^+ Z_{anti}^+ (m) \prod_{i=1}^7 Z_{fund}^+(m_i),$$

$$Z_{vec}^+ = \frac{1}{2^g} \frac{\sinh \frac{\gamma_2 + \gamma_1}{2}}{\sinh \frac{\gamma_2 - \gamma_1}{2} \sinh \frac{2\phi + \gamma_2 + \gamma_1}{2} \sinh \frac{\phi + i\lambda + \gamma_1}{2}}, \tag{3.53}$$

$$Z_{anti}^+ (m) = \frac{\sinh \frac{im - \gamma_2}{2}}{\sinh \frac{im - \gamma_1}{2} \sinh \frac{2\phi + im - \gamma_1}{2}},$$

$$Z_{fund}^+(m_i) = 2 \sinh \frac{\phi + im_i}{2},$$

where $m$ is the mass of the hypermultiplet in the antisymmetric representation. Moreover the measure of integration is simply $[d\phi] = \frac{1}{2\pi} d\phi$. As we see the integrand has simple poles at the zeroes of the hyperbolic sines with the general form

$$\frac{1}{\sinh \frac{Q\phi + \gamma}{2}}. \tag{3.54}$$

The contour of integration $C$ is defined to surround the poles with $Q > 0$, or alternatively we can define the contour of integration as the unit circle in the variable $z = e^\phi$ and substitute $t = e^{-\gamma_1}$ in $Z_{vec}^+$ and $T = e^{-\gamma_1}$ in $Z_{anti}^+$ and taking $t < 1$ and $T > 1$. The two procedures are equivalent for the poles with $Q > 0$ will lay inside the unit circle in $z$ if $t$ is taken sufficiently small and $T$ sufficiently large. In our case the contour $C$ will surround 10 poles, 6 of which will come from $Z_{vec}^+$ and 4 from $Z_{anti}^+$, we choose not to write the result of the computation here being it quite long. The situation is much simpler for $Z_2^-$ for the rank of $O(2)_-$ is 0 and
no integration is needed. The result is

$$Z_2 = Z_{vec} Z_{anti}(m) \prod_{i=1}^{7} Z_{fund}(m_i),$$

$$Z_{vec} = \frac{1}{32} \frac{\cosh \gamma_1}{\sinh \frac{x_2 + \gamma_1}{2} \sinh((\pm \gamma_1 + \gamma_2) \sinh(\pm i \lambda + \gamma_1))},$$

$$Z_{anti}(m) = -\frac{\cosh \frac{\pm im - \gamma_2}{2} \sin(\pm \lambda + m)}{\sinh \frac{m \pm \gamma_1}{2} \sinh(i m \pm \gamma_1)},$$

$$Z_{fund}(m_i) = 2i \sin m_i.$$  

(3.55)

The only last piece necessary for the computation of the partition function is the factor $Z_{string}$ that as explained before will cancel from $Z_{QM}$ will cancel the contributions due to additional states present in the string theory realization of $Sp(1)$ gauge theory. This contribution was computed in [27] and the result for $N_f = 7$ is

$$Z_{string} = \text{PE}[f_7(x, y, v, w_i, u)],$$

(3.56)

where $x = e^{\gamma_1}$, $y = e^{-\gamma_2}$, $v = e^{-im}$, $u$ is the instanton fugacity of $Sp(1)$ gauge theory and $w_i = e^{\frac{\pi}{2}m_i}$ with $i = 1, \ldots, 7$. In (3.56) we also defined the Plethystic exponential of a function $f(x)$ as

$$\text{PE}[f(x)] = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f(x^n) \right].$$

(3.57)

Finally in (3.56) $f_7$ is

$$f_7 = \frac{u x^2}{(1 - xy)(1 - x/y)(1 - xv)(1 - x/v)} \left[ \chi(w_i)^{SO(14)}_{64} + u \chi(w_i)^{SO(14)}_{14} \right].$$

(3.58)

Knowing this it is possible to extract the instanton partition function of $Sp(1)$ with 7 flavors and one anti-symmetric hypermultiplet at instanton level 2 and check whether there is agreement with the result coming from $Z_{E_8}$. While it has not been possible so far to check agreement between the two expressions because of computational difficulties however it has been possible to check that the two expressions agree in the special limit where all but two masses of the fundamental hypermultiplets are taken to zero. Moreover expanding the two expressions in the fugacity $x = e^{\gamma_1}$ we have found complete agreement between the two expression up to order $x^3$.

Another check is to see the perturbative flavor symmetry $SO(14)$ at each instanton level. We have checked that the 2-instanton part we obtained is indeed invariant under the Weyl symmetry of $SO(14)$. This is also a non-trivial evidence that our calculation yields the correct result of the 2-instanton part of the $E_8$ theory. Further check will be discussed in the next section and involves the computation of the superconformal index.
3.8 Superconformal index of the $E_8$ theory

Knowledge of the 5d Nekrasov partition function allows us to perform the computation of the superconformal index which will allow us to verify explicitly the non-perturbative enhancement of the flavor symmetry. The superconformal index for a 5d theory (or equivalently the partition function on $S^1 \times S^4$) is defined as

$$I(\gamma_1, \gamma_2, m_i, u) = \text{Tr} \left[ (-1)^F e^{-\beta(Q, Q')} e^{-2(i_j + j_R)\gamma_1} e^{-2i j_r e^{-i \sum_i H_i m_i} u^k} \right], \quad (3.59)$$

where $j_r$ and $j_l$ are the Cartan generators of $SU(2)_r \times SU(2)_l \subset SO(5)$ with $j_r = \frac{j_l + j_{\lambda}}{2}$ and $j_l = \frac{j_l - j_{\lambda}}{2}$, $j_R$ is the Cartan generator of the $SU(2)_R$ R-symmetry group, $H_i$ are the flavor charges and $k$ is the instanton number. The computation of the superconformal index can be performed using localization techniques and the result is [42]

$$I(\gamma_1, \gamma_2, m_i, u) = \int [d\lambda]_H \text{PE} \left[ f_{\text{mat}}(x, y, e^{i\lambda}, e^{i m_i}) + f_{\text{vec}}(x, y, e^{i\lambda}) \right] \left| \int_{\text{inst}} (x, y, e^{i\lambda}, e^{i m_i}, u) \right|^2, \quad (3.60)$$

where $f_{\text{mat}}$ and $f_{\text{vec}}$ take into account the perturbative contributions given by hypermultiplets and vector multiplets and they are

$$f_{\text{mat}}(x, y, e^{i\lambda}, e^{i m_i}) = \frac{x}{(1 - xy)(1 - x/y)} \sum_{w \in W} \sum_{i=1}^{N_f} (e^{-i w \cdot \lambda - i m_i} + e^{i w \cdot \lambda + i m_i}) \quad (3.61)$$

$$f_{\text{vec}}(x, y, e^{i\lambda}) = -\frac{x y + x/y}{(1 - xy)(1 - x/y)} \sum_{R} e^{-i R \cdot \lambda} \quad (3.62)$$

where $R$ is the set of all roots of the Lie algebra of the gauge group and $W$ is the weight system for the representation of the hypermultiplets. Moreover in (3.60) $[d\lambda]_H$ denotes the the Haar measure of the gauge group which for $Sp(N)$ is equal to

$$[d\lambda]_H = \frac{2^N}{N!} \left[ \prod_{i=1}^{N} \frac{d \lambda_i}{2\pi} \sin^2 \lambda_i \right] \prod_{i<j}^{N} \left[ 2 \sin \left( \frac{\lambda_i - \lambda_j}{2} \right) 2 \sin \left( \frac{\lambda_i + \lambda_j}{2} \right) \right]^2, \quad (3.63)$$

and $\left| \int_{\text{inst}} (x, y, e^{i\lambda}, e^{i m_i}, u) \right|^2$ includes the contributions due to instantons and is given by

$$\left| \int_{\text{inst}} (x, y, e^{i\lambda}, e^{i m_i}, u) \right|^2 = \left[ \int_{\text{north}}^\text{north} (x, y, e^{i\lambda}, e^{i m_i}, u) \int_{\text{south}}^\text{south} (x, y, e^{i\lambda}, e^{i m_i}, u) \right] = \left[ \sum_{k=0}^{\infty} u^{-k} I_k(x, y, e^{i\lambda}, e^{i m_i}) \right] \left[ \sum_{k=0}^{\infty} u^k I_k(x, y, e^{i\lambda}, e^{i m_i}) \right]. \quad (3.64)$$

In (3.64) $I_{\text{north}}(x, y, e^{i\lambda}, e^{i m_i}, u)$ contains the contributions due to anti-instantons localized at the north pole of $S^4$ and $I_{\text{south}}(x, y, e^{i\lambda}, e^{i m_i}, u)$ contains the contributions of instantons localized at the south pole of $S^4$. It was noticed in [43] that for the case of an $SU(2)$ gauge group the index can be also computed as

$$I(\gamma_1, \gamma_2, m_i, u) = \int [d\lambda]_H Z_{\text{Nekra}}(\lambda, \gamma_1, \gamma_2, m_i, u) Z_{\text{Nekra}}(-\lambda, \gamma_1, \gamma_2, -m_i, u^{-1}), \quad (3.65)$$
where $Z_{\text{Nekra}}$ implies the whole partition function obtained by the refined topological vertex after removing decoupled factors. In this case the integration measure does not contain the Haar measure of the gauge group for this is already included in the perturbative contributions due to vector multiplets in the Nekrasov partition function, and so the resulting measure for $SU(2)$ is simply $[d\lambda] = \frac{1}{2\pi} d\lambda$.

We have been able to compute the superconformal index using $Z_{E_8}$ expanding it in the fugacity $x$ up to order $x^3$ and the result is\footnote{$\chi_2(y) = y + 1/y$ is the character of the fundamental representation of $SU(2)$.}

$$I = 1 + (1 + \chi_{91}^{SO(14)} + u \chi_{64}^{SO(14)} + u^{-1} \chi_{64}^{SO(14)} + u^2 \chi_{14}^{SO(14)} + u^{-2} \chi_{14}^{SO(14)}) x^2$$

$$+ \chi_2(y)(1 + 1 + \chi_{91}^{SO(14)} + u \chi_{64}^{SO(14)} + u^{-1} \chi_{64}^{SO(14)} + u^2 \chi_{14}^{SO(14)} + u^{-2} \chi_{14}^{SO(14)}) x^3 + \ldots$$

$$= 1 + \chi_{248}^{E_8} x^2 + \chi_2(y)(1 + \chi_{248}^{E_8}) x^3 + \ldots$$

which is expected from the branching

$$E_8 \supset SO(14) \times U(1)$$

$$248 \rightarrow 1_0 + 91_0 + 64_1 + \overline{64}_{-1} + 14_2 + \overline{14}_{-2}.$$  

In (3.66) we have assumed that contributions with higher instanton number will appear in the superconformal index only with higher powers of $x$. Finally let us mention that we have expanded the partition function $Z_{E_8}$ at order $x^4$ and found the following contributions to the superconformal index

$$1 + \chi_{3080}^{SO(14)} + u^2 \chi_{1716}^{SO(14)} + u^{-2} \chi_{1716}^{SO(14)} + \chi_2(y)(1 + \chi_{248}^{E_8})$$

which again is consistent with the results of [27, 42]. However the complete expression at order $x^4$ has not been reproduced because part of the expression involves contributions at 3 and 4 instanton number. A similar computation has been performed using the field theory result for the Nekrasov partition function [27] and the same result has been obtained. This provides further evidence for the equality of the partition function at instanton level 2 computed from $Z_{E_8}$ and the field theory result.

4 Conclusion and discussion

In this paper, we have obtained the prescription to compute partition functions of five-dimensional class $S$ theories which are realized as low energy theories in Higgs branches of the $TN$ theory. Although the web diagrams of the resulting theories are non-toric, one can obtain their exact partition functions by inserting the conditions of the tunings of the parameters in the theories corresponding to putting parallel horizontal external 5-branes together, putting parallel vertical external 5-branes together or putting parallel diagonal 5-branes together. The first type of the tuning was found in [22], and we have further extended the
result including the latter two tunings. Their validity has been exemplified by applying them to the theories in the corresponding Higgs branches of the $T_3$ theory. The tunings inside the web diagrams are determined by consistency conditions from the geometry. The three types of the tunings are enough for moving to any Higgs branch of the $T_N$ theory.

With this general prescription, we have computed the exact partition function of the $E_8$ theory which arises in the far infrared of a Higgs branch of the $T_6$ theory. In the Higgs vacuum, there are singlet hypermultiplets which are decoupled from the $E_8$ theory. We have determined their contributions, and in particular we propose that the singlet hypermultiplets which depend on the parameters associated with flavor symmetries can be understood as the decoupled factor associated with new parallel external legs of the web diagram in the Higgs branch. Identifying the singlet hypermultiplets is important since their contributions depend on the instanton fugacity of $Sp(1)$. The proposal works perfectly for the examples we have computed. The final expression of the partition function is written by the summation of the eight Young diagrams. We observed that the six Young diagrams summations terminate at finite order with the fixed order for the other two Young diagrams. The other two Young diagrams are related to the summation with respect to the instanton fugacity. Therefore, we can evaluate the partition function exactly at some order of the instanton fugacity. We have also compared the our result with the partition function of the $Sp(1)$ gauge theory with 7 fundamental and 1 anti-symmetric hypermultiplets obtained in [27]. Although the method we obtained the partition is completely different from the one in [27], we found the quite non-trivial agreement as expected.

In the computation of the $E_8$ theory, the singlet hypermultiplets contributions which depend on the parameters associated with the flavor symmetry were totally determined by the factors coming from the new parallel external legs in the Higgs branch of the web diagram. However, not all the singlet hypermultiplets contributions which only depend on the $\Omega$-deformation parameters are interpreted in this way. It is interesting to find a method which can determine the total contribution of singlet hypermultiplets in a Higgs branch purely from a web diagram. Practically, the singlet hypermultiplets contributions which only depend on the $\Omega$-deformation parameters are all contained in the perturbative part of the partition function. Therefore, we can identify them easily once we obtain the perturbative part.

In the computation of a partition function of a theory from a web diagram or a web diagram for a Higgsed $T_N$ theory, we often end up with a partition function with Young diagrams summations related to flavor fugacities. For the partition function of the $T_3$ theory and the $E_7$ theory, we essentially need the exact partition function of the $T_2$ theory where a Young diagram is assigned to each horizontal external legs, with some additional hypermultiplets which are bi-fundamental between the $Sp(1)$ and the flavor symmetry associated to the Young diagram summation of the partition function of the $T_2$ theory. For the partition function of the $E_8$ theory, we need the exact partition function of the Higgsed $T_3$ theory where a Young diagram is assigned to two upper horizontal external legs, with some additional hypermultiplets which are bi-fundamental between the $Sp(1)$ and the flavor symmetry associated to the Young diagram summation of the partition function of the Higgsed $T_3$ theory. Since the
Young diagram summation is related to a summation of a flavor fugacity, the summation may terminate at finite order. Indeed we have observed the termination of the summation in the case of the computation of the $E_8$ theory in this paper as well as in the case of the $E_7$ theory and the $E_6$ theory in [22]. It is interesting to show and explore the origin of the termination of the Young diagrams summations. This computation can be also used for a prediction of the exact partition function of the theory in the Higgs branch of the $T_3$ theory where non-trivial Young diagrams are assigned to two upper horizontal external legs. Since the Young diagram summation of a flavor fugacity often occurs in the computation from a web diagram, other computation using some web diagram may predict exact results for some Young diagram summation in other theories.

Finally, since our prescription can be used for web diagrams of any Higgs branch, it is interesting to compute partition functions of other theories realized by some non-toric diagrams. Particular examples are higher rank $E_{6,7,8}$ theories discussed in [16].

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A Tuning for coincident diagonal 5-branes

The Higgs branch of the $T_N$ theory opens up when we put parallel external 5-branes on a 7-brane. So far, we have discussed the tuning associated with putting the parallel vertical external 5-branes on one 7-brane (2.6) and (2.7) as well as the tuning associated with putting the parallel horizontal external 5-branes on one 7-brane (2.1) and (2.2). We then find a similar tuning for putting parallel two diagonal external 5-branes on one single 7-brane as in figure 11. As with the case of putting two parallel vertical external 5-branes on one 7-brane, a pole in the superconformal index computation in this case is associated with an instanton fugacity. One can again change the preferred direction into the diagonal direction. Then we can sum up $\tilde{Q}_2$ as well as $\tilde{Q}_1$ and find a location of the poles. In fact, the tuning is essentially the same.
Figure 11. The process of putting parallel diagonal external 5-branes together on one 7-brane.

as the other ones associated with the parallel horizontal external legs or the parallel vertical external legs. We then propose that we can obtain the partition function of an infrared theory in the Higgs branch associated with the web in figure 11 by requiring

$$\tilde{Q}_1 = \tilde{Q}_2 = \left( \frac{q}{t} \right)^{\frac{1}{2}},$$ \hspace{1cm} (A.1)

or

$$\tilde{Q}_1 = \tilde{Q}_2 = \left( \frac{t}{q} \right)^{\frac{1}{2}}.$$ \hspace{1cm} (A.2)

We will again apply the prescription (A.1) or (A.2) to the partition functions of two Higgsed $T^3$ theories in order the exemplify the prescription. We have two types of the tuning associated with putting two parallel diagonal external 5-branes on one 7-brane. We will exemplify the prescription for each Higgs branch.

A.1 Higgsed $T^3$ theory III

We first consider putting two leftmost parallel diagonal external 5-branes on one 7-brane as in figure 12. In order to obtain the partition function of the infrared theory in the Higgs branch arising from figure 12, we adopt the tuning (A.1) to Kähler parameters in figure 12

$$Q_3 = \left( \frac{t}{q} \right)^{\frac{1}{2}}, \quad Q_b = \left( \frac{t}{q} \right)^{\frac{1}{2}}.$$ \hspace{1cm} (A.3)

By inserting the conditions (A.3) to the partition function of the $T_3$ theory (2.8), one obtains

$$Z_{T_3} = Z_0 \cdot Z_{\text{inst}} \cdot Z_{\text{dec}}^{-1}$$ \hspace{1cm} (A.4)

$$Z_0 = \prod_{i,j=1}^{\infty} \left[ \frac{1 - e^{-i\lambda + ima} q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}(1 - e^{-i\lambda - ima} q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})(1 - e^{-2i\lambda} q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}q^{i-\frac{1}{2}t^{j-\frac{1}{2}}})}{(1 - q^{i}t^{j-1})^{\frac{1}{2}}(1 - q^{i-1}t^{j})^{-\frac{1}{2}}(1 - e^{-2i\lambda} q^{i}t^{j-1})} \times (1 - e^{i\lambda + ima} q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}(1 - e^{-i\lambda + ima} q^{i-\frac{1}{2}t^{j-\frac{1}{2}}}) \right],$$ \hspace{1cm} (A.5)
Plethystic exponentials obtain (A.5)–(A.7), we erased coincident diagonal external 5-branes. Right: The corresponding dot diagram of the web diagram on the left. The red line shows the new external leg.

\[
\begin{aligned}
Z_{\text{inst}} &= \sum_{\nu_2, \mu_5} \left( e^{\frac{1}{2} \nu_2 \mu_5 (m_1 + m_2 + m_4)} \left( \frac{t}{q} \right)^{\frac{3}{2}} \right) |\nu_2| \prod_{a=1,2} \left( \frac{2i \sin \frac{E_{2a} - m_a + i\gamma_1}{2}}{(2i)^2 \sin \frac{E_{2a} + 2i\gamma_1}{2}} \frac{2i \sin \frac{E_{2a} - m_a + i\gamma_1}{2}}{(2i)^2 \sin \frac{E_{2a} - \lambda + 2i\gamma_1}{2}} \right) \frac{\sin \frac{E_{2a} - m_a + i\gamma_1}{2}}{(2i)^2 \sin \frac{E_{2a} - \lambda + 2i\gamma_1}{2}} \\
\prod_{s \in \mu_5} \frac{(2i \sin \frac{E_{2s} + m_a + i\gamma_1}{2})(2i \sin \frac{E_{2s} - \lambda + m_a + i\gamma_1}{2})}{(2i)^2 \sin \frac{E_{2s}}{2} \sin \frac{E_{2s} + 2i\gamma_1}{2}} \right],
\end{aligned}
\]  
(A.6)

\[
Z_{\text{dec}}^{-1} = \prod_{i,j=1}^{\infty} \left[ (1 - u_1 e^{im_1 q^i} t^j)(1 - q^{i-1} q)^{-1}(1 - u_1 e^{im_1 q^{i-1} t^j}) \\
(1 - u_1 e^{-im_1 q^{i-1} t^j})(1 - e^{\lambda + i(m_1 + m_2 + m_4)} q^{i-1} q^{i-1} t^j)^{-1}(1 - u_1 e^{\lambda + i(m_1 + m_2) q^{i-1} t^j}) \right],
\]  
(A.7)

Due to the first tuning of (A.3), Young diagram summation of \(\nu_1\) vanishes unless \(\nu_1 = 0\). To obtain (A.5)–(A.7), we erased \(m_3\) and \(u_2\) by using (A.3).

The instanton partition function (A.6) can be again written by the products of the Plethystic exponentials

\[
\begin{aligned}
Z_{\text{inst}} &= \prod_{i,j=1} \left[ \frac{(1 - Q_5 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})(1 - Q_b Q_1 Q_4^{-1} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})(1 - Q_b Q_1 Q_5 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})}{(1 - u_1 e^{im_1 q^{i} t^{j-1}})(1 - u_1 e^{im_1 q^{i-1} t^j})(1 - u_1 e^{-im_1 q^{i-1} t^j})} \\
&\times \frac{(1 - Q_b Q_1 Q_2 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})(1 - Q_b Q_2 Q_4^{-1} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})(1 - Q_b Q_2 Q_5 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})}{(1 - e^{i\lambda + i(m_1 + m_2 + m_4)} q^{i-\frac{1}{2}} t^{j+\frac{1}{2}})(1 - u_1 e^{i\lambda + i(m_1 + m_2) q^{i-\frac{1}{2}} t^{j+\frac{1}{2}}})(1 - e^{-i\lambda + im_1 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}})} \\
&\times \frac{(1 - Q_b Q_4^{-1} Q_5 Q_f q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})(1 - e^{-2i\lambda q^{i} t^{j-1}})}{(1 - e^{-i\lambda + im_1 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}})(1 - e^{-i\lambda + im_1 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}})} \right].
\end{aligned}
\]  
(A.8)
The equality of (A.8) can be checked in the same way as we have checked (2.26) in section 2.2. We first write the equations on both sides of (A.8) by the variables $Q_1, Q_2, Q_3, Q_f$. Let us then focus on the order $O(u_0^a)$. If we compute (A.6) until the order $|\nu_2| = k$, the result is exact until the order $O(Q_1^a Q_2^b)$ with $a + b = k$. Therefore, we can compare (A.6) with (A.8) until the order $O(Q_1^a Q_2^b)$ with $a + b = k$. We have checked the equality until $k = 3$.

When $|\mu_5| = l$, we multiply (A.6) by $Q_4^a Q_5^b$ and then the result is exact until $O(Q_1^a Q_2^b)$ with $a + b = k$ when we include the Young diagram summation of $\nu_2$ until $|\nu_2| = k$. We have checked the equality (A.8) until $(l, k) = (2, 2)$.

By combining (A.8) with (A.5)–(A.7), we finally obtain the partition function of the infrared theory of in the Higgs branch of the $T_3$ theory corresponding to figure 12

$$Z_{T_3} = \prod_{i,j=1}^{\infty} \left[ (1 - e^{i(\nu_1 + \nu_2 - \mu)q^{i-\frac{1}{2}t_j-\frac{1}{2}}})(1 - e^{i(\nu_2 + \nu_3 - \mu)q^{i-\frac{1}{2}t_j-\frac{1}{2}}})(1 - e^{i(\nu_3 - \mu)q^{i-\frac{1}{2}t_j-\frac{1}{2}}}) \times (1 - e^{i(-\nu_1 + \nu_3 + \mu)q^{i-\frac{1}{2}t_j-\frac{1}{2}}})(1 - e^{i(\nu_2 - \nu_1 + \mu)q^{i-\frac{1}{2}t_j-\frac{1}{2}}})(1 - e^{i(\nu_3 - \mu)q^{i-\frac{1}{2}t_j-\frac{1}{2}}}) \times (1 - q^{i-1}t_j)^{\frac{1}{2}}(1 - q^{i-1}t_j)^{-\frac{1}{2}} \times (1 - e^{-3i\mu q^{i-\frac{1}{2}t_j-\frac{1}{2}}}) \right], \tag{A.9}$$

where we rewrite the parameters by the chemical potentials associated with the unbroken global symmetry $SU(3) \times SU(3) \times U(1)$ in the Higgs branch. The generator of the unbroken global symmetry is

$$t_{SU(3) \times SU(3) \times U(1)} = \mu_1(D_1 + D) + \nu_2 D_2 + \mu_1' D_3 + \mu_2' D_4 + \mu(D + 2D_5 + D_6), \tag{A.10}$$

and we further defined $\nu_i, \nu'_i, (i = 1, 2, 3)$ by (2.31) and

$$\nu'_1 = \mu'_1, \quad \nu'_2 = -\nu'_1 + \mu'_2, \quad \nu'_3 = -\mu'_2 \tag{A.11}$$

The explicit parameterization is

$$Q_1 = e^{i(-\nu_1 + \nu_2)}, \quad Q_2 = e^{i(-\nu_2 - \nu_1 + \mu)}, \quad Q_3 = e^{i(-\nu_1 - \nu'_2 + \mu)}, \quad Q_4 = e^{i(\nu_1 + \nu'_3 - \mu)}, \quad Q_f = e^{i(-2\nu_2 + \nu'_1 - \mu)}. \tag{A.12}$$

The factors in the last line of (A.9) correspond to the singlet hypermultiplets in the Higgs branch. Those factors can be understood from the web diagram 12 as in the examples of section 2.2 and 2.3. In particular, the very last factor in the last line of (A.9) may come from the contribution of strings between the new parallel external leg after the Higgsing. The new external leg is depicted in the red line in the dot diagram of figure 12.

**A.2 Higgsing $T_3$ theory IV**

We then consider the second type of tuning associated with putting the two rightmost parallel diagonal external 5-branes together on one 7-brane as in figure 13. For that, we adopt the
\[ Q_5 = \left( \frac{t}{q} \right)^{\frac{1}{2}}, \quad Q_4^{-1}Q_f = \left( \frac{t}{q} \right)^{\frac{1}{2}}. \]  
(A.13)

With the conditions (A.13), the partition function of (2.8) becomes

\[ Z_{TIR} = Z_0 \cdot Z_{\text{inst}} \cdot Z_{\text{dec}}^{-1} \quad (A.14) \]

\[ Z_0 = \prod_{i,j=1}^{\infty} \left[ \frac{(1 - e^{-i\lambda + i\mu_1 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}})(1 - e^{-i\lambda - i\mu_1 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}})}{(1 - q^{i-1}t^{j-1})^{\frac{1}{2}}(1 - q^{i-1}t^{j-1})^{\frac{1}{2}}(1 - e^{-2i\lambda q^{i-1}t^{j-1}})} \right] \]

\[ Z_{\text{inst}} = \sum_{\nu_1, \nu_2, \mu_5} u_2^{\vert \nu_1 \vert + \vert \nu_2 \vert} \left( e^{-i\lambda \left( \frac{t}{q} \right)^{\frac{1}{2}}} \right)^{\vert \mu_5 \vert} \prod_{\alpha=1}^{2} \prod_{s \in \nu_\alpha} \left[ \prod_{a=1}^{3} \frac{2i \sin \frac{E_{\alpha\beta - m_a + i\gamma_1}}{2}}{2i \sin \frac{E_{\alpha\beta} + 2i\gamma_1}{2}} \right] \prod_{\beta=1}^{2} \left( \frac{E_{\alpha\beta} + \frac{\lambda}{2}}{2} \right) \frac{2i \sin \frac{E_{\alpha\beta}}{2} \sin \frac{E_{\alpha\beta + 2i\gamma_1}}{2}}{2i \sin \frac{E_{\alpha\beta + 2i\gamma_1}}{2}} \left( \frac{2i \sin \frac{E_{\alpha\beta - m_a + i\gamma_1}}{2}}{2i \sin \frac{E_{\alpha\beta} + 2i\gamma_1}{2}} \right), \quad (A.16) \]
\[ Z_{\text{dec}}^{-1} = \prod_{i,j=1}^{\infty} \left[ (1 - q^{i-1}t^j)(1 - u_2e^{-\frac{i}{2}\lambda - \frac{j}{2}(m_1 + m_2 + m_3)}q^{i+\frac{1}{2}t^j - \frac{1}{4}})(1 - u_2e^{-\frac{i}{2}\lambda - \frac{j}{2}(m_1 + m_2 + m_3)}q^{-i \frac{3}{4}t^j + \frac{1}{4}}) 
\right. \\
(1 - e^{-2i\lambda}q^{i-1}\nu^j)(1 - u_2e^{\frac{i}{2}\lambda + \frac{j}{2}(m_1 + m_2 + m_3)}q^{i+\frac{1}{2}t^j + \frac{1}{4}})(1 - u_2e^{-\frac{i}{2}\lambda + \frac{j}{2}(m_1 + m_2 + m_3)}q^{-i \frac{3}{4}t^j - \frac{1}{4}}) \right], \\
(A.17) \]

where we erased \( u_1 \) and \( m_4 \) to obtain \((A.15)-(A.17)\). As discussed in section 2.3, not all the Young diagram summations with respect to \( \mu_5 \) contribute for a fixed order of \( |\nu_1| = k \). The the contribution is non-zero if \( \mu_{5,i} \leq \nu_{1,i} \) for all \( i \).

The instanton partition function \((A.16)\) turns out to be the product of the Plethystic exponentials

\[ Z_{\text{inst}} = \prod_{i,j=1}^{\infty} \left[ \frac{(1 - Q_6Q_4^{-1}q^{i-1}t^j - \frac{1}{2})q^{\frac{1}{2}t^j - \frac{1}{2}}(1 - Q_6Q_2Q_4^{-1}q^{i-1}t^j - \frac{1}{2})}{(1 - u_2e^{-\frac{i}{2}\lambda - \frac{j}{2}(m_1 + m_2 + m_3)}q^{i+\frac{1}{2}t^j - \frac{1}{4}})(1 - u_2e^{-\frac{i}{2}\lambda - \frac{j}{2}(m_1 + m_2 + m_3)}q^{-i \frac{3}{4}t^j + \frac{1}{4}})} \right. \\
\times \left. \frac{(1 - Q_6Q_1Q_4^{-1}q^{i-1}t^j - \frac{1}{2})q^{\frac{1}{2}t^j - \frac{1}{2}}(1 - Q_6Q_3Q_4^{-1}q^{i-1}t^j - \frac{1}{2})}{(1 - u_2e^{-\frac{i}{2}\lambda + \frac{j}{2}(m_1 + m_2 + m_3)}q^{i+\frac{1}{2}t^j + \frac{1}{4}})(1 - u_2e^{-\frac{i}{2}\lambda + \frac{j}{2}(m_1 + m_2 + m_3)}q^{-i \frac{3}{4}t^j - \frac{1}{4}})} \right] \quad (A.18) \]

We have checked the equality \((A.18)\) until \( \mathcal{O}(u_2^2) \).

By combining the result \((A.18)\) with \((A.15)-(A.17)\), we obtain the partition function of the infrared theory in the Higgs branch of the \( T_3 \) theory

\[ Z_{\text{IR}} = \prod_{i,j=1}^{\infty} \left[ (1 - e^{i(\nu_2 + \nu_3 + \mu)}q^{i-1}t^j - \frac{1}{2})q^{i-1}t^j - \frac{1}{2})(1 - e^{i(\nu_3 + \nu_4 + \mu)}q^{i-1}t^j - \frac{1}{2}) \right. \\
\times \left. (1 - e^{i(\nu_2 + \nu_4 + \mu)}q^{i-1}t^j - \frac{1}{2})q^{i-1}t^j - \frac{1}{2})(1 - e^{i(-\nu_1 - \nu_4 - \mu)}q^{i-1}t^j - \frac{1}{2}) \right] \\
\times \left[ (1 - q^{i-1}t^j)^{\frac{3}{2}}(1 - q^{i-1}t^j)^{-\frac{1}{2}}(1 - e^{-3i\mu}q^{i-1}t^j - \frac{1}{2}) \right] \], \quad (A.19) \]

which can be explicitly seen as the partition function of nine free hypermultiplets up to singlet hypermultiplets. We have also parameterized the Kähler parameters by the chemical potentials associated with the unbroken flavor symmetry \( SU(3) \times SU(3) \times U(1) \) as

\[ Q_1 = e^{i(\nu_2 + \nu_3 + \mu)}, \quad Q_2 = e^{i(\nu_2 + \nu_4 + \mu)}, \quad Q_3 = e^{i(-\nu_1 - \nu_4 - \mu)}, \quad Q_4 = e^{i(-\nu_2 + \nu_4)}, \quad Q_b = e^{i(\nu_1 + \nu_4 - 2\mu)}, \quad Q_f = e^{i(\nu_3 + \nu_4)}. \quad (A.20) \]

The generator of the unbroken global symmetry is

\[ t_{SU(3) \times SU(3) \times U(1)} = \mu_1 D_1 + \mu_2 D_2 + \mu_1' D_3 + \mu_2' (D_4 + D) + \mu (D_5 + 2D_6 + D). \quad (A.21) \]

The relations between \( \mu_i, \mu_i', (i = 1, \cdots, 3) \) and \( \nu_i, \nu_i', (i = 1, \cdots, 3) \) are \((2.31)\) and \((A.11)\).

The factors in the last big bracket of \((A.19)\) are the contributions from the singlet hypermultiplets in the Higgs branch. Those factors again have the interpretation from the web diagram as discussed in 2.2 and 2.3. In particular, the very last factor can be understood from the contribution of strings between the new parallel diagonal external leg. The new diagonal external leg after the tuning is depicted in the dot diagram of figure 13.
B Cartan generators of $SU(6) \times SU(3) \times SU(2)$

We list up the Cartan generators which correspond to the $SU(6) \times SU(3) \times SU(2)$ in the Higgs vacuum of the $T_6$ theory corresponding to the web 8. As discussed in section 3.3, the generators for $SU(6)$ can be determined as

$$
\begin{align*}
\mathbf{1}^{SU(6)} &= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
\mathbf{2}^{SU(6)} &= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
\mathbf{3}^{SU(6)} &= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
\mathbf{4}^{SU(6)} &= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
\mathbf{5}^{SU(6)} &= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
\end{align*}
$$

Similarly, the generators for $SU(3)$ and $SU(2)$ are

$$
\begin{align*}
\mathbf{1}^{SU(3)} &= \left\{0, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0\right\}, \\
\mathbf{2}^{SU(3)} &= \left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\},
\end{align*}
$$

and

$$
\mathbf{1}^{SU(2)} = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0\right\}
$$

respectively. The gauge generator is

$$
t_{\text{gauge}} = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}. 
$$

We then define parameters associated with the generators (B.1)–(B.9) as

$$
t = \mu_1 \mathbf{1}^{SU(6)} + \mu_2 \mathbf{2}^{SU(6)} + \mu_3 \mathbf{3}^{SU(6)} + \mu_4 \mathbf{4}^{SU(6)} + \mu_5 \mathbf{5}^{SU(6)} + \mu_1' \mathbf{1}^{SU(3)} + \mu_2' \mathbf{2}^{SU(3)} + \tilde{\mu} \mathbf{SU(2)} + \lambda_{\text{gauge}}.
$$

By using the definition of the masses and the tentative instanton fugacity (3.17), we find their relation with the chemical potentials for particles in the canonical simple roots of $SU(6)$

$$
\begin{align*}
2\mu_1 - \mu_2 &= m_2 - m_4, \\
-\mu_1 + 2\mu_2 - \mu_3 &= -m_2 - m_3, \\
-\mu_2 + 2\mu_3 - \mu_4 &= m_1 - \tilde{u}, \\
-\mu_3 + 2\mu_4 - \mu_5 &= -m_6 + m_7, \\
-\mu_4 + 2\mu_5 &= -m_5 + m_6.
\end{align*}
$$

Similarly, the chemical potentials for particles in the canonical simple roots of $SU(3)$ are

$$
\begin{align*}
2\mu'_1 - \mu'_2 &= -m_3 - m_5 - m_6 + \tilde{u}, \\
-\mu'_1 + 2\mu'_2 &= -m_2 - m_4 + m_7 - \tilde{u}.
\end{align*}
$$

The chemical potential for a particle in the canonical simple root of $SU(2)$ is

$$
2\tilde{\mu} = m_1 - m_2 - m_4 - m_5 - m_6 - \tilde{u}.
$$

Eq. (B.12)–(B.18) yield (3.18) in section 3.3.
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