Endomorphism Rings and Isogenies Classes for Drinfeld $A$-Modules of Rank 2 over Finite Fields

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Résumé
Soit $\Phi$ un $F_q[T]$-module de Drinfeld de rang 2, sur un corps fini $L$, une extension de degré $n$ d’un corps fini. On abordera plusieurs points d’analogie avec les courbes elliptiques. Nous spécifions les conditions de maximalité et de non maximalité pour l’anneau d’endomorphismes $\text{End}_L \Phi$ en tant que $F_q[T]$-ordre dans l’anneau de division $\text{End}_L \Phi \otimes_{F_q[T]} F_q(T)$, on s’intéressera ensuite aux polynôme caractéristique et par son intermédiaire on calculera le nombre de classes d’iogénies.

Abstract
Let $\Phi$ be a Drinfeld $F_q[T]$-module of rank 2, over a finite field $L$, a finite extension of $n$ degrees of a finite field with $q$ elements $F_q$. Let $m$ be the extension degrees of $L$ over the field $F_q[T]/P$, $P$ is the $F_q[T]$-characteristic of $L$, and $d$ the degree of the polynomial $P$. We will discuss about a many analogies points with elliptic curves. We start by the endomorphism ring of a Drinfeld $F_q[T]$-module of rank 2, $\text{End}_L \Phi$, and we specify the maximality conditions and non maximality conditions as a $F_q[T]$-order in the ring of division $\text{End}_L \Phi \otimes_{F_q[T]} F_q(T)$, in the next point we will interest to the characteristic polynomial of a Drinfeld module of rank 2 and used it to calculate the number of isogeny classes for such module, at last we will interested to the Characteristic of Euler-Poincare $\chi_\Phi$ and we will calculated the cardinal of this ideals.

1 Introduction
Let $E$ be a elliptic curves over a finite field $F_q$, we know, by [9], [12], [13] and [14], that the endomorphism ring of $E$. $\text{End}_{F_q} E$, is a $\mathbb{Z}$—order in a division algebra which is : $\mathbb{Q}$ and in this case $\text{End}_{F_q} E = \mathbb{Z}$, or is complex quadratic field and in this case: $\text{End}_{F_q} E = \mathbb{Z} + c \mathcal{O}_K$ where $c$ is an element of $\mathbb{Z}$ and $\mathcal{O}_K$ is the maximal $\mathbb{Z}$—order in complex quadratic field maximal, or this division algebra is a quaternion algebra $\mathbb{Q}$ in which case $\text{End}_{F_q} E$ is a maximal order in this quaternion algebra. We put $E(F_q)$ the abelian group of $F_q$-rational points of $E$. The cardinal of this abelian group is equal to $N = q + 1 - c$, and by
Hasse-Weil $|c| \leq 2\sqrt{q}$. A morphism of elliptic curve over $F_q$, is an algebraic application $f : E_1 \mapsto E_2$, defined over $F_q$, which respects the law group. An isogeny is a non null morphism. And for one elliptic curve $E$ the set of such morphism $f : E \mapsto E$ forms a ring : the $F_q$-endomorphism ring of $E$, this ring will be noted $\text{End}_{F_q}(E)$, of same the $F_q$-endomorphism ring of $E$ will be noted $\text{End}_{F_q}(E)$, in the case where $\text{End}_{F_q}(E)$ is non commutative the curve $E$ is called supersingular, otherwise it is called ordinary. And by [10], [13], [14]:

**Theorem 1.1.** There are three possibilities for endomorphism ring for an elliptic curve $E$

1. $\text{End}_{F_q}(E) = \mathbb{Z}$
2. $\text{End}_{F_q}(E) = \mathbb{Z} + cO_{\text{max}}$, where $c \in \mathbb{Z}_{>0}$, $p$ not divide $c$, and $O_{\text{max}}$ is a maximal order in complex quadratic field (which is equal to the field of fractions of $\text{End}_{F_q}(E)$) ($c$ is called the conductor of $\text{End}_{F_q}(E)$);
3. $\text{End}_{F_q}(E)$ is a maximal order in the quaternion algebra $\mathbb{Q}_{\infty,p}$.

The cardinal of $E(F_q)$ and endomorphism Frobenius $\varphi$ are given, in [9], [12] and [13], by:

**Theorem 1.2.** Let $E$ be an elliptic curve over $F_q$, $\varphi$ be the endomorphism Frobenius in $\text{End}_{F_q}(E)$ and $p$ the characteristic of $F_q$:

1. The endomorphism $\varphi$ satisfies an unique equation $\varphi^2 - c\varphi + q = 0$ in $\text{End}_{F_q}(E)$, where $c \in \mathbb{Z} \subset \text{End}_{F_q}(E)$,
2. $|c| \leq 2\sqrt{q}$,
3. $|E(F_q)| = q + 1 - c$,
4. $p | c$ if and only if $E$ is supersingular.

The set of isogenies classes for an elliptic curve $E$ over a finite field $F_q$, is given in [10], [13] and [14], by:

**Theorem 1.3.** The set of isogenies classes of an elliptic curve over a finite field $F_q$ is given by a natural bijection between the set of integer $c$ such that $|c| \leq 2\sqrt{q}$ and one of the following conditions is satisfied:

1. $(c, q) = 1$;
2. $q$ is a square and $c = \pm 2\sqrt{q}$;
3. $q$ is a square, $p$ is not congru to $1$ (mod 3), and $c = \pm \sqrt{q}$;
4. $q$ is not a square, $p = 2$ or $3$, and $c = \pm \sqrt{p \cdot q}$;
5. $q$ is not a square and $c = 0$; or $q$ is a square, $p$ not congru to $1$ (mod 4), and $c = 0$. 

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The case 1 correspond to ordinary case and the other cases correspond to the supersingular case.

Our goal here is to give an analog of the previous results in the case of a Drinfeld Module of rank 2. We recall quickly what is it: let \( K \) a non empty global field of characteristic \( p \) (that means a rational functions field of one indeterminate over a finite field) with a constant field the finite field \( \mathbf{F}_q \) with \( p^s \) elements. We fix one place of \( K \), noted \( \infty \) and we call \( A \) the ring of regular elements away from the place \( \infty \). Let \( L \) be a commutator field of characteristic \( p \), and let \( \gamma : A \to L \), be an \( A \)-ring homomorphism, the kernel of this homomorphism is noted \( P \) and \( m = [L, A/P] \) is the extension degrees of \( L \) over \( A/P \).

We note \( L\{\tau\} \) the Ore polynomial ring, that means the polynomial ring of \( \tau \), \( \tau \) is the Frobenius of \( \mathbf{F}_q \), with the usually addition and the product is given by the commutation rule: for every \( \lambda \) of \( L \), \( \tau \lambda = \lambda^q \tau \). We said a \( A \)-Drinfeld module \( \Phi \) for a non trivial ring homomorphism, from \( A \) to \( L\{\tau\} \) such that is different of \( \gamma \). this homomorphism \( \Phi \), once defined, given a \( A \)-module structure over the \( A \)-field \( L \), noted \( L\Phi \), where the name of a Drinfeld \( A \)-module for a homomorphism \( \Phi \). This structure of \( A \)-module is depending on \( \Phi \) and especially on this rank.

Let \( \chi \) be the characteristic of Euler-Poincaré (it is a ideal from \( A \)), so we can speak about the ideal \( \chi(L\Phi) \), will be noted by \( \chi\Phi \), which is by definition a divisor for \( A \), corresponding for the elliptic curves to a number of points of the variety over their basic field. We will work, in this paper, in the special case \( K = \mathbf{F}_q(T), A = \mathbf{F}_q[T] \). Let \( P_\Phi(X) \) be the characteristic polynomial of the \( A \)-module \( \Phi \), it is also a characteristic polynomial of Frobenius \( F \) of \( L \). We can prove that this polynomial can be given by: \( P_\Phi(X) = X^2 - cX + \mu P^m \), such that \( \mu \in \mathbf{F}_q^*, c \in A \) and \( \deg c \leq \frac{m.d}{2} \), the Hasse-Weil analogue in this case. We will interested to the endomorphism ring of a Drinfeld \( A \)-module of rank 2 and we will prove:

**Proposition 1.1.** Let \( \Delta = c^2 - 4\mu P^m \), be the discriminant of \( P_\Phi \), the characteristic polynomial of the endomorphism Frobenius \( F \) of the finite field \( L \), which is \( P_\Phi(X) = X^2 - cX + \mu P^m \), And let \( O_{K(F)} \) the maximal \( A \)-order of the algebra \( K(F) \).

1. For every \( g \in A \) such that \( \Delta = g^2 \omega \), there exists a Drinfeld \( A \)-module \( \Phi \) over \( L \) of rank 2, such that: \( O_{K(F)} = A[\sqrt{\omega}] \) and :
   \[
   \text{End}_L\Phi = A + g.O_{K(F)}.
   \]

2. If there are not a polynomial \( g \) of \( A \) such that \( g^2 \) divide \( \Delta \), then there exists a Drinfeld \( A \)-module \( \Phi \) over \( L \) of rank 2, such that \( \text{End}_L\Phi = O_{K(F)} \).

At next We will proved that the number of isogenies classes of a Drinfeld \( A \)-module of rank 2 is :

**Proposition 1.2.** Let \( \Phi \) be a Drinfeld \( A \)-module of rank 2 over a finite filed \( L = \mathbf{F}_{q^n} \) and let \( P \) the \( A \)-characteristic of \( L \). We put \( m = [L : A/P] \) and \( d = \deg P \):
1. \(m\) is odd and \(d\) odd:

\[
\#\{P_\Phi, \Phi : \text{ordinary}(1)\} = (q - 1)(q^{\frac{m-2}{2}} - q^{\frac{m-2}{2}d+1} + 1).
\]

2. \(m\) is even and \(d\) is odd:

\[
\#\{P_\Phi\} = (q - 1)[\frac{q - 1}{2}q^{\frac{m}{2}d} - q^{\frac{m}{2}d + 1} + q].
\]

3. \(m\) is even and \(d\) is even:

\[
\#\{P_\Phi\} = (q - 1)[\frac{q - 1}{2}q^{\frac{m}{2}d} - q^{\frac{m}{2}d} + 1].
\]

## 2 Drinfeld Modules

Let \(E\) be an extension of \(F_q\), and let \(\tau\) be the Frobenius of \(F_q\). We put \(E\{\tau\}\) the polynomial ring in \(\tau\) with the usual addition and the multiplication defined by:

\[
\forall e \in E, \tau e = e^q \tau.
\]

**Definition 2.1.** Let \(R\) be the \(E\)-linearly polynomials set with the coefficient in \(E\), that means that these elements are on the following form:

\[
Q(x) = \sum_{K > 0} l_k x^q^K,
\]

where \(l_k \in E\) for every \(k > 0\), only a finite number of \(l_k\) is not null. The ring \(R\) is a ring by addition and the polynomials composition.

**Lemma 2.1.** \(E\{\tau\}\) and \(R\) are isomorphic rings.

If we put \(A = F_q[T]\), \(f(\tau) = \sum_{i=0}^n a_i \tau^i \in E\{\tau\}\) and \(Df := a_0 = f'(\tau)\).

It is clear that the application:

\[
E\{\tau\} \ni E
\]

\[
f \mapsto Df,
\]

is a morphism of \(F_q\)-algebra.

**Definition 2.2.** An \(A\)-field \(E\) is a field \(E\) equipped with a fix morphism \(\gamma : A \longrightarrow E\). The prime ideal \(P = \text{Ker} \gamma\) is called the \(A\)-characteristic of \(E\). We say \(E\) has generic characteristic if and only if \(P = (0)\); otherwise (i.e. \(P \neq (0)\)) we said \(E\) has a finite characteristic.

Then we have the following fundamental Definition:
Definition 2.3. Let $E$ be an $A$-field and let $\Phi : A \mapsto E\{\tau\}$ be a homomorphism of algebra. Then $\Phi$ is a $A$-Drinfeld module over $E$ if and only if:

1. $D \circ \Phi = \gamma$;
2. for some $a \in A, \Phi(a) \neq \gamma(a)\tau^0$.

Remark 2.1. 1. the normalization above is analogous to the normalization used in complex multiplication of elliptic curves. The last condition is obviously a non-triviality condition.

2. By $\Phi$, every extension $E_0$ of $E$ became an $A$-module by:

$$E_0 \times A \rightarrow E_0, \quad (k, a) \rightarrow k.a := \Phi(a)(k).$$

We will note this $A$-module by $E_0^\Phi$.

Let $\overline{E}$ a fix algebraic closure of $E$ and $\Phi$ a Drinfeld module over $E$ and $I$ an ideal of $A$. As $A$ is a Dedekind domain, one know that $I$ may be generated by (at most) two elements $\{i_1, i_2\} \subset I$.

Since $E\{\tau\}$ has a right division algorithm, there exists a right greatest common divisor in $E\{\tau\}$. It is the monic generator of the left ideal of $E\{\tau\}$ generated by : $\Phi(i_1)$ et $\Phi(i_2)$.

Definition 2.4. We set $\Phi_I$ to be the monic generator of the left ideal of $E\{\tau\}$ generated by $\Phi(i_1)$ and $\Phi(i_2)$.

Definition 2.5. Let $E_0$ be an extension of $E$ and $I$ an ideal of $A$. We define by : $\Phi[I](E)$ the finite subgroup of $\Phi[E_0]$ given by the roots of $\Phi_I$ in $E$.

If $a \in A$, then we set $\Phi[a] := \Phi([a])$. We can see it as : $\Phi[a] = \{ \text{set of roots of } \Phi(a) \text{ in } \overline{K} \}$, and $\Phi_I = \cap_{a \in I} \Phi[a]$. Then :

$$\Phi_a(\overline{E}) := \Phi[a](\overline{E}) = \{ x \in \overline{E}, \Phi_a(x) = 0 \}$$

For every ideal $Q \subset A$,

$$\Phi_Q(\overline{E}) := \Phi[Q](\overline{E}) = \cap_{a \in Q} \Phi_a(\overline{E}).$$

Remark 2.2. The groups : $\Phi[I](E)$ and $\Phi[I](\overline{E})$ are clearly stable under $\{\Phi_a\}_{a \in A}$.

Definition 2.6. Let $\Phi$ be a Drinfeld $A$-module over an $A$-field $E$. We say that $\Phi$ is supersingular, if and only if, the $A$-module constituted by a $P$-division points $\Phi_P(\overline{E})$ is trivial.
2.1 The Height and Rank of a Drinfeld Module $\Phi$

Let $\Phi$ be a Drinfeld $A$-module over the $A$-field $E$. We note by $\deg_\tau$ the degree in indeterminate $\tau$.

**Definition 2.7.** An element of $E\{\tau\}$ is called separable, if the coefficient of its constant element is no null. It called purely inseparable if it is on the form $\lambda\tau^n$, $n > 1$ and $\lambda \in E$, $\lambda \neq 0$.

Let $H$ be a global field of characteristic $p > 0$, and let $\infty$ one place (a Prime ideal ) of $H$, we will note by $H_\infty$ the completude of $H$ at the place $\infty$. We define the degree of function over $A$ by :

**Definition 2.8.** Let $a \in A$, $\deg a = \dim_{F_q} A_{aA}$ if $a \neq 0$ and $\deg 0 = -\infty$.

We extend $\deg$ to $K$ by putting $\deg x = \deg a - \deg b$ if $0 \neq x = \frac{a}{b} \in K$. If $A = F_q[T]$, then the degree function is the usual polynomial degree. Let $Q$ be a no null ideal of $A$, we define the ideal degrees of $Q$, noted $\deg Q$, by :

$$\deg Q = \dim_{F_q} A_{Q}.$$ 

**Lemma 2.2.** there exists a rational number $r$ such that :

$$\deg_\tau \Phi_a = r \deg a.$$ 

**Proof.** It is easy to see that $\Phi$ is an injection, otherwise since $K\{\tau\}$ is an integre ring, $\text{Ker } \Phi$ is a prime ideal non null, so maximal in $A$ and by consequence $\text{Im } \Phi$ is a field, so $\Phi = \gamma$. Since $-\deg_\tau$ define a no trivial valuation over $\text{Frac}(\Phi(A))$ ( the fractions field of $\Phi(A)$ ) which is isomorphic to $K$, so $-\deg_\tau$ and $-\deg$ are equivalent valuations over $K$. There is rational number $r > 0$, such that :

$$r \deg = \deg_\tau.$$ 

**Corollary 2.1.** Let $\Phi : A \mapsto E\{\tau\}$ be an $A$-Drinfeld module, so $\Phi$ is injective.

**Proposition 2.1.** The number $r$ is a positive integer.

**Definition 2.9.** The number $r$ is called the rank of the Drinfeld $A$-module $\Phi$.

For example if $A = F_q[T]$, an $A$-Drinfeld module of rank $r$ is on the form :

$$\Phi(T) = a_1 + a_2\tau + \ldots + a_r\tau^r,$$

where $a_i \in E$, $1 \leq i \leq r - 1$ and $a_r \in E^*$.  

In this case $\text{char } E = P \neq (0)$ we can define the notion of height of a Drinfeld module $\Phi$.

For this, we suppose that $\text{char}(E) = P \neq 0$. We put $v_P : K \mapsto \mathbb{Z}$, an associate normalized valuation at $P$, this means, if $a \in K$ has a root over $P$ of order $t$, we have $v_P(a) = t$.

For every $a \in A$, let $w(a)$ the most small integer $t > 0$, where $\tau^t$ occurs at $\Phi_a$ with a non null coefficient.
Lemma 2.3. There exists a rational number $h$ such that:

$$w(a) = hv_P(a) \deg P.$$ 

Proposition 2.2. The number $h$ is a positive integer.

Definition 2.10. The integer $h$ is called the height of $\Phi$.

For example if $A = \mathbb{F}_q[T]$, a Drinfeld $A$-module of height $h$ of rank $r$ is of the form:

$$\Phi(T) = a_0 + a_1 T + \cdots + a_r T^r,$$

where $a_i \in E$, $0 \leq i \leq r - 1$ and $a_r \in E^*$. 

2.2 Endomorphism Ring of a Drinfeld module

Let $E$ be an $A$-field and let $\overline{E}$ a fix algebraic closure. Let $\Phi$ and $\Psi$ two Drinfeld $A$-modules over $E$ of rank $r > 0$. We define a morphism $\Phi$ to $\Psi$ over $E$ by:

Definition 2.11. Let $\Phi$ and $\Psi$ two Drinfeld modules over an $A$-field $L$. A morphism of $\Phi$ to $\Psi$ over $E$ is an element $p(\tau) \in L\tau$ such that:

$$p \Phi a = \Psi p, \forall a \in A.$$

An null morphisme is called an isogeny. We note that is possible only between two Drinfeld modules of same rank.

A invertible isogeny $u$ (i.e : $\deg_{\tau} u = 0$) is called a isomorphism and the module became isomorphic. The set of the morphisms form an $A$–module noted by $\text{Hom}_E(\Phi, \Psi)$.

We can see $\text{Hom}_E(\Phi, \Psi)$ by the fact that : a morphism (or $E$-morphisme) $p$: $\Phi \mapsto \Psi$ from $\Phi$ to $\Psi$ is a morphism of $A$-modules $p$: $(E, \Phi) \mapsto (L, \Psi)$ where $(E, \Phi)$ (respectively $(E, \Psi)$) is $E$ with the structure of $A$-module given by $\Phi$ (resp $\Psi$).

such morphism is also a morphism of additive groups of $E$. Then $E\tau$ is a finie $E[F]$–module so $E\tau$ is integral over $E[F]$. So :

$$E(\tau) = E\tau \otimes_{E[F]} E(F) = L\tau \otimes_A K,$$

The fraction ring of $E\tau$ is noted $E(\tau)$ ( $E(\tau)$ is a no commutative field called left field of fractions of $E\tau$).

In particularly if $\Phi = \Psi$, the $E$-endomorphism ring ($\text{End}_E\Phi = \text{Hom}_E(\Phi, \Phi)$) is a subring of $E\tau$ and an $A$-module contained $\Phi(A)$:

$$\text{End}_E\Phi = \{ u \in E\tau / \forall a \in A, u \Phi a = \Psi a u \}.$$

Since $\Phi$ is an injection, $\Phi$ can be naturally prolonged to an injection $\Phi$: $K \mapsto E(\tau)$. By this injection we identify on in $E(\tau)$, $A$ and $\Phi(A)$ of same $K$ and $\Phi(K)$.

Let $F$ the Frobenius of $E$ we have : $\Phi(A) \subset \text{End}_E\Phi, F \in \text{End}_E\Phi$.

Definition 2.12. Let $\Phi$ and $\Psi$ two Drinfeld $A$-modules over an $A$-field $E$ and $p$ an isogeny over $E$ from $\Phi$ to $\Psi$.

1. We say that $p$ is separable if and only if $p(\tau)$ is separable.

2. We say that $p$ is purely no separable if and only if $p(\tau) = \tau^j$ for one $j > 0$. 

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2.3 Norm of Isogeny

**Definition 2.13.** Let $F$ an integer over a ring $A$, with fractions field $K$, we note by $N_{K/K(F)}$ the determinant of the $K$-linearly application of multiplication by $F$ to $K(F)$ (it is the usual norm if the extension $K(F)/K$ is separable.

We can see that there is a morphism $N_{K/K(F)} : I_A \rightarrow I_A$ from the fractional ideals groups of $A$ to functionary ideals group of $A$, by this morphism we have:

**Proposition 2.3.** The norm of an isogeny is a principal ideal.

**Proposition 2.4.** Let $M_{fin}(A)$ the category of primes ideals of $A$ and let $D(A)$ the mono iđe of ideal of $A$. There exists a unique function :

$$
\chi : M_{fin}(A) \mapsto D(A),
$$

multiplicative over the exact sequence and such that $\chi(0) = 1$ and $\chi(A/\wp) = \wp$ for every prime ideal $\wp$ of $A$.

**Definition 2.14.** The function $\chi$ is called the Euler-Poincare characteristics.

We can see $\chi(E^\Phi)$ and we note it by $\chi_\Phi$.

**Proposition 2.5.** The ideals $\chi_\Phi$ and $P^m$ are principals (in $A$), and more clearly $\chi_\Phi = (P^\Phi(1))$ and $P^m = P^\Phi(0)$.

1. We know that the norm of an isogeny is a principal ideal, indeed $N(F) = P^\Phi(0)$ and $N(1 - F) = (P^\Phi(1))$ since $F$ and $1 - F$ are a $K$-isogenys.
2. We can call $\chi_\Phi$ the divisor of $E$-points, this divisor is analogue at the number of $E$-points for elliptic curves.
3. $\chi_\Phi$ is the annulator of $A$-module $E^\Phi$. We can deduct that : $E^\Phi \subset (\frac{A}{\chi_\Phi})^\tau$.
4. The structure of $A$-module $E^\Phi$ is stable by the Frobenius endomorphism $F$.

**Corollary 2.2.** If there are a Drinfeld $A$-module $\Phi$, over a field $E$, of characteristic $P$ and of degree $m$ over $A/P$, then the ideal $P^m$ is a principal ideal.

**Remark 2.3.** The above Corollary shows that there exists a restriction of the existence of Drinfeld $A$-modules.

3 Drinfeld Modules Over Finite Fields

We substitute the Field $E$, by $L$ a finite extension of degree $n$ of the finite fields $F_q$. The Frobenius $F$ of $L$ is $F = \tau^n$, so $F_q[F]$ is the center of $L\{\tau\}$. We put $m = [L : A/P]$ and $d = \deg P$, then $n = m.d$. The function $-\deg$ define a valuation over $K$, the field of fractions of $A$. Let $\tau : x \mapsto x^q$ the Frobenius of $F_q$.
and let $L$ a finite extension of $F_q$. We put $r_1 = [K(F) : K]$ and $r_2^2$ the degree of left field $\text{End}_L \Phi \otimes_A K$ over this center $K(F)$.

So a Drinfeld $A$-modules $\Phi$ over $L$ give a structure of $A$-module over the additive finite group $L$, this structure will be noted $L^\Phi$. Let $\gamma$ the application of $A$ to $L$ which an element $a$ for $A$ associate the constant of $\Phi_a$, then it is easy to see that $\gamma$ is a ring homomorphism, and that $\Phi$ and $\gamma$ are equal over $A^* = F_q^*$ the set of reversible elements of $A$.

**Definition 3.1.** Let $\Phi$ be a Drinfeld $A$-module over a finite field $L$. We note by $M_\Phi(X)$ the unitary minimal polynomial of $F$ over $K$.

**Proposition 3.1.** With the above notations : $M_\Phi(X)$ is an element of $A[X]$, equal to $P_\Phi^{r_2}$.

**Corollary 3.1.** For two Drinfeld $A$-modules $\Phi$ and $\Psi$, of rank $r$ over a finite field $L$, then the following are equivalent :

1. $\Phi$ and $\Psi$ are isogenous,
2. $M_\Phi(X) = M_\Psi(X)$,
3. $P_\Phi = P_\Psi$.

**Proposition 3.2.** Let $L$ be a finite extension of degree $n$ over a finite field $F_q$, let $F$ the Frobenius of $L$. Then $L(\tau)$ is a central division algebra over $F_q(F)$ of dimension $n^2$.

**Definition 3.2.** Every $u \in L\{\tau\}$ can be writing on this form $u = \tau^h u'$ ( since $L$ is a perfect field) where $u' \in L\{\tau\}$ separable. The integer $h$ is called the height of $u$ and will be noted by $ht u$.

In the finite fields case, we can see the height of a Drinfeld $A$-module $\Phi$ over finite field $L$, the integer $H_\Phi$ as been :

$$H_\Phi = \frac{1}{r} \inf \{ht \Phi_a, 0 \neq a \in P\}.$$

**Remark 3.1.** It is easy to see that $H_\Phi$ is invariant under isogeny and that

$$1 \leq H_\Phi \leq r.$$

**Proposition 3.3.** Let $\Phi$ be a Drinfeld $A$-module of rank $r$ over a finite field $L$, the following assertions are equivalent :

1. There exists a finite extension $L'$ of $L$, such that the endomorphism ring $\text{End}_{L'} \Phi \otimes_A K$, has dimension $r^2$ over $K$.
2. Some power of the Frobenius $F$ of $L$ lies in $A$.
3. $\Phi$ is supersingular.
4. The field \( K(F) \) has only one prime above \( P \).

**Proposition 3.4.** Let \( \Phi \) be a Drinfeld \( A \)-module of rank \( r \) and let \( Q \) be an ideal from \( A \) prime with \( P \), then :

\[
\Phi_Q(L) = \left( \frac{A}{Q} \right)^r.
\]

**Corollary 3.2.** Then we can deduct that : \( \Phi_P(L) = \left( \frac{A}{P} \right)^{r-H_\Phi} \).

We can deduct from above mentioned Proposition the following important result, which characterize the supersingularity :

**Proposition 3.5.** A Drinfeld \( A \)-module \( \Phi \) is supersingular (\( \Phi_P(L) = 0 \)), if and only if, \( r = H_\Phi \).

**Definition 3.3.** We say that the field \( L \) is so big if all endomorphism rings defined over \( L \) are also defined over \( L \), i.e : \( \text{End}_L \Phi = \text{End}_L \Phi \).

Two Drinfeld modules \( \Phi \) and \( \Psi \) are isomorphic, if and only if, there exists an \( a \in L \) such that : \( a^{-1} \Phi = \Psi_a \).

**Lemma 3.1.** Let \( \Phi \) be a Drinfeld \( A \)-module of rank \( r \), over a finite field \( L \) which is a finite extension of degree \( n \) of \( F_q \). The characteristic polynomial of Frobenius endomorphisms \( F \) is :

\[
P_{\Phi}(X) = X^r + c_1 X^{r-1} + ... + c_{r-1} X + \mu P^m, c_1,...c_{r-1} \in A \text{ et } \mu \in F_q^*.
\]

**Remark 3.2.** The fact that constant of the polynomial \( P \) is \( \mu P^m \) comes from the fact that \( \Phi(0) = P^m \) in \( A \).

The following Proposition is an analogue of the Riemann’s hypothesis for elliptic curves :

**Proposition 3.6.** Let \( \Phi \) be a Drinfeld \( A \)-module of rank \( r \) over finite field \( L \) which is a finite extension of degree \( n \) of \( F_q \). Then \( \deg (w) = \frac{r}{P} \) for every root \( w \) of characteristic polynomial \( P_\Phi(X) \).

The following result is the Hasse-Weil’s analogue for elliptic curves :

**Corollary 3.3.** Let \( P_{\Phi}(X) = X^r + c_1 X^{r-1} + ... + c_r X + \mu P^m \) the characteristic polynomial of a Drinfeld Module \( \Phi \), of rank \( r \), over a finite field \( L \). Then:

\[
\forall 1 \leq i \leq r - 1, \deg c_i \leq \frac{i}{r} m \deg P.
\]

**Proof.** The proof can be deducted immediately by the above Proposition. \( \square \)

\section{4 Drinfeld Modules of rank 2}

In all next of this paper, \( \Phi \) will be considered a Drinfeld \( A \)-module of rank 2, And \( A = F_q[T] \) for proof and more details see [1], [12] and [6].

Our interest for the arithmetic of such modules motive us to interesting to their endomorphism rings and their isogenies classes:
4.1 Endomorphism Ring

We start by giving the following result which characterized the order in quadratic extension, to proof see [12]:

**Proposition 4.1.** Let $O$ be an $A$-order in a quadratic extension $K(F)$, and let $O_{K(F)}$ the maximal $A$-order in $K(F)$, then the an $A$-order $O$ of $K(F)$ is on the form:

$$O = A + gO_{K(F)},$$

where $g$ is unitary element of $A$.

**Definition 4.1.** The element $g$ of $A$ in the above proposition is called the conductor of $A$-order $O$.

**Proposition 4.2.** Let $\Phi$ be a Drinfeld $A$-module of rank 2, over a finite field $L$ of $m$, degrees over $A/P$ and the Frobenius $F$, and let $P$ be the $A$-characteristic of $L$ and $D_P$ the completude, at the place $P$, of $\text{End}_L \Phi \otimes A K$.

1. If $F$ is on the form $kP^\frac{1}{m}$ ($k \in F_q^*$), then the ring $\text{End}_L \Phi$ can be identified with a maximal $A$-order in $D_P$, conversely every maximal $D_P$ can be obtained by this way.

2. Otherwise, the ring $\text{End}_L \Phi$ can be identified with an $A$-order in an imaginary quadratic field $K(F)$. An $A$-order $O$ of $K(F)$ occur on this way if and only if $F \in O$, and the conductor of $O$ is prime with $P$ in two following cases:

   (a) $F$ is on the form $\sqrt{\mu P^{\frac{m}{2}}}$ with $\mu \in F_q^*$ if $m$ is odd and $\sqrt{\mu P^{\frac{m}{2}}}$ is imaginary quadratic,

   (b) $F$ is in the form $\sqrt{kP^\frac{m}{2}}$ and $m$ is even and $\text{deg}P$ is odd.

**Corollary 4.1.** If the conductor of $O$ is prime with $P$, then $O$ is a maximal $A$-order in the algebra $\text{End}_L \Phi \otimes_A K$.

For the Drinfeld modules of rank 2, we can specify $\text{End}_L \Phi \otimes_A K$ as been in the ordinary case equal to $K(F)$.

**Proposition 4.3.** Let $\Phi$ be a Drinfeld $A$-module of rank 2 over $L$:

1. Let $\Phi$ is supersingulary module,

2. Let $\text{End}_L \Phi \otimes_A K = K(F)$.

**Proof.** Let $r_1 = [K(F) : K]$ and $r_2^2$ is the degree of left field $\text{End}_L \Phi \otimes_A K$ over this center $K(F)$. Since $2 = r_1, r_1$, we have two cases ($r_1 = 1$ and $r_2 = 2$) or ($r_1 = 2$ and $r_2 = 1$), then in the where case($r_1 = 1$ and $r_2 = 2$) we have a supersingulary Drinfeld since : $r_1 = [K(F) : K] = 1$ and :

$$F \in \overline{A} = \overline{A} = A.$$

Otherwise (i.e : $r_1 = 2$ and $r_2 = 1$), we will have $\text{End}_L \Phi \otimes_A K = K(F)$ and $\text{End}_L \Phi$ is $A$-order in the quadratic field $K(F)$. □
Remark 4.1. The above result show that the ordinary case and since $\text{End}_L \Phi$ is $A$-order contained $A[F]$, so to study the maximality of the endomorphism ring $\text{End}_L \Phi$, we can satisfy by study the existence of the $A$-order containing $A[F]$ and contained in the maximal $A$-order $O_{K(F)}$ of the algebra $K(F)$.

Now we have a no supersingulary Drinfeld $A$-module and $O_{K(F)}$ is the maximal $A$-order in the algebra $K(F)$, we interest to know : when there is an $A$-order $O$ such :

$$A[F] \subset O \subset O_{K(F)}?$$

To answer to the above question we have the following result:

**Proposition 4.4.** Let $\Delta = c^2 - 4\mu P^m$, the discriminat of $P_F$, the characteristic polynomial of $F$ the of the finite field $L$, which is : $P_F(X) = X^2 - cx + \mu P^m$, and let $O_{K(F)}$ the maximally $A$-order of the algebra $K(F)$.

1. For every $g \in A$ such that $\Delta = g^2 \cdot \omega$, there is a Drinfeld $A$-module $\Phi$ over $L$ of rank 2, such that : $O_{K(F)} = A[\sqrt{\omega}]$ and $\text{End}_L \Phi = A + g.A[\sqrt{\omega}]$.

2. there is not a polynomial $g$ of $A$ such that $g^2$ divide $\Delta$, there is a ordinary Drinfeld $A$-module $\Phi$ over $L$ of rank 2, such that :

$$\text{End}_L \Phi = O_{K(F)}.$$

**Proof.** 1) We suppose that there is $g \in A$, such that $\Delta = g^2 \cdot \omega$, where $F$ is a root of characteristic polynomial $P_\Phi$, then we can put : $F = -c/2 + \sqrt{\Delta}/2 = -c/2 + g.\sqrt{\omega}/2$, then $A[F] = A[-c/2 + g.\sqrt{\omega}/2] = A[ g\sqrt{\omega}/2 ] \subseteq A + g.A[\sqrt{\omega}]$ and it is easy to see that in this case the $A$-order $O_{K(F)} = A[\sqrt{\omega}]$ is a maximal $A$-order, and by the proposition 4.2, there is a Drinfeld $A$-module $\Phi$ such that: $\text{End}_L \Phi = A + g.O_{K(F)}$.

2) In the other case : there is not a polynomial $g \in A$ such that $g^2$ | $\Delta$, and by the proposition 4.2, there is a Drinfeld $A$-module $\Phi$ such that the $A$-order $\text{End}_L \Phi$ can not be written on the form $A + g.O_{K(F)}$ and in this case it will be certainly equal to $O_{K(F)}$. 

4.2 Isogenies Classes

Let $\Phi$ be a Drinfeld $A$-module of rank 2 over a finite field $L$ with the characteristic polynomial $P$, and let $m = \deg P$. The characteristic polynomial $P_\Phi$ can be given by the unitary minimal polynomial of $F$ in $A[X]$, $M_\Phi$, and with the relation $P_\Phi = M_\Phi^2$, $r_2$ is a root of degree of the left field $\text{End}_L \Phi \otimes_A K$ over the center $K(F)$.

Let $\overline{K}$ be an algebraic closure of $K$, and $\infty$ a place of $K$ which divide $\frac{1}{r_2}$, and let $K_\infty = F_{\infty}((\frac{1}{r_2}))$, and $C_\infty$ the completude of the algebraic closure of $K_{\infty}$.

We fix a plongement $\overline{K} \hookrightarrow C_\infty$.  

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For every \( \alpha \in \mathbb{C}_\infty \), \( | \alpha |_\infty \) is normalized valuation of \( \alpha \) (\( | \frac{1}{\alpha} |_\infty = \frac{1}{q} \)).

Let \( \theta \in \overline{K} \), we say that \( \theta \) is an ordinary number of:

1. \( \theta \) is integral over \( A \),
2. \( | \theta |_\infty = q^{\frac{md}{2}} \)
3. \( K(\theta)/K \) is imaginary, and \( [K(\theta), K] = 2 \);
4. there is only one place of \( K(\theta) \) which divide \( \theta \) and \( \text{Tr}_{K(\theta)/K}(\theta) \neq 0(P) \).

Let \( \theta \) an ordinary Weil number \( \forall \sigma \in \text{col}(K/K), \theta^\sigma \). We will noted by \( \text{Word} \) the set of conjugacy classes of an ordinary Weil numbers of rank 2. Then:

**Theorem 4.1.** There is a bijection between \( \text{Word} \) and isogeny classes of ordinary Drinfeld \( A \)-modules of rank 2 defined over \( L \).

To prove the theorem above: let \( \theta \) an ordinary Weil number, we put:

\[
P(x) = Hr(\theta, K; x).
\]

Then by (1), (2), (3) and (4):

\[
P(x) = x^2 - cx + \mu P^m,
\]

with \( \mu \in \mathbb{F}_q^* \) and \( c \in A, c \neq 0(P) \), and \( \text{deg}_T c \leq \frac{md}{2} \).

We put \( \Gamma = \{ c \in A, c \neq 0(P), \text{deg}_T c \leq \frac{md}{2} \} \).

**Lemma 4.1.** Let \( \mu \in \mathbb{F}_q^*, c \in \Gamma \), and \( E \) the filed of decomposition of \( P(x) = x^2 - cx + \mu P^m \) over \( K \). Let \( \theta \) a root of \( P(x) \). Then \( \theta \) verify (1), (2), and (4)

\[
\text{and } [K(\theta), K] = 2.
\]

**Proof.** Let \( B \) be the integral closure of \( A \) in \( E \). We suppose that there is \( \theta \) a root of \( P(x) \) with \( \theta \in B^* \). Since the constant coefficient of \( P(x) \) is \( \mu P^m \), we have \( \theta \in \mathbb{F}_q^* \). Then: \( v_\infty(\theta^2 - c\theta) = -\text{deg}_T c \geq -md \), et \( \theta^2 - c\theta = -\mu P^m \) where the contradiction. We fix then \( \theta \) a root of \( P(x) \), we have \( \theta \notin B^* \) and \( (\theta - c) \notin B^* \). Or \( \theta(\theta - c) = -\mu P^m \). Since \( c \neq 0(P) \). There exists exactly two primes \( \beta_1, \beta_2 \) of \( B \) above \( P \) and \( \beta_1 \mid \theta, \beta_2 \mid \theta - c \). In particularly \( [E : K] = 2 \).

We work in \( \mathbb{C}_\infty \), we have:

\[
v_\infty(\theta) + v_\infty(\theta - c) = -md.
\]

Since \( v_\infty(c) = -\text{deg}_T c \geq -\frac{md}{2} \), we have \( v_\infty(\theta) < 0 \). We suppose that \( v_\infty(\theta) \) or \( v_\infty(\theta - c) \neq -\frac{md}{2} \). Even we replace \( \theta \) by \( \theta - c \), we can suppose:

\[
v_\infty(\theta) < -\frac{md}{2}.
\]

Then:

\[
v_\infty(\theta - c) = \inf(v_\infty(\theta), v_\infty(c)) = v_\infty(\theta).
\]

Where from the contradiction.
Corollary 4.2. 1) Let \( \mu \in F_q^* \) and \( c \in \Gamma \), then if \( \theta \) is a root of \( x^2 - cx + \mu P_m \), \( \theta \) is a ordinary Weil number, if and only if \( K(\theta)/K \) is imaginary.

2) If \( md \equiv 1(2) \), then \( \forall \mu \in F_q^* \) and \( \forall c \in \Gamma \), the roots of \( x^2 - cx + \mu P_m \) are the Weil numbers.

To simplify our speech, we suppose \( p \neq 2 \). And we put \( md \equiv 0(2) \).

Lemma 4.2. Let \( \mu \in F_q^* \) and \( c \in \Gamma \) with \( \deg c \leq \frac{md}{2} \). Let \( \theta \) a root of \( x^2 - cx + \mu P_m \). Then \( \theta \) is a Weil number, if and only if, \( -\mu \notin (F_q^*)^2 \).

Proof. By the Hensel lemma : \( P_m \in (K_{\infty}^*)^2 \). We have :

\[
v_\infty \left( \frac{c}{\sqrt{P_m}} \right) = \frac{md}{2} - \deg c > 0.
\]

Or \( \frac{\theta}{\sqrt{P_m}} \) is a root:

\[
x^2 - \frac{c}{\sqrt{P_m}} x + \mu = 0.
\]

Or : \( x^2 - \sqrt{P_m} \mu + \mu \equiv x^2 + \mu \left( \frac{1}{T} \right) \).

By the Hensel lemma \( \theta \notin (F_q^*)^2 \iff -\mu \notin (F_q^*)^2 \). \( \square \)

Lemma 4.3. Let \( \mu \in F_q^* \) and \( c \in \Gamma \) with \( \deg c \leq \frac{md}{2} \), and we note \( c_0 \) the term of more high degree of \( c \). We suppose \( c_0^2 \neq -4\mu \). Let \( \theta \) a root of \( x^2 - cx + \mu P_m \). Then \( \theta \) is a Weil number, if and only if, \( x^2 - c_0 x + \mu \) is irreducible in \( F_q[X] \).

Proof. In this time we choose, \( \sqrt{P_m} \) such that :

\[
\sqrt{P_m} \left( \frac{1}{T} \right)^{\frac{md}{2}} \equiv 1 \left( \frac{1}{T} \right)
\]

Then : \( \frac{c}{\sqrt{P_m}} \equiv 0 \left( \frac{1}{T} \right) \). Then :

\[
x^2 - \frac{c}{\sqrt{P_m}} x + \mu \equiv x^2 - c_0 x + \mu \left( \frac{1}{T} \right).
\]

We apply the Hensel lemma since \( x^2 - c_0 x + \mu \) have two simple roots. \( \square \)

If \( c_0^2 = -4\mu \), we put \( \Delta = c^2 - 4\mu \). Then \( \theta \) is a Weil number if and only if \( \deg \tau \Delta \equiv 1(2) \). If \( \deg \tau \Delta \equiv 0 \ (2) \) and the term of more high degree in \( \Delta \) is not a square in \( F_q^* \).

We note some remarks about the Weil numbers:

1. The Weil numbers are defined for all the rank,

2. The Frobenius \( F \) over \( L \) is a Weil number.

We know, By [7] and [12] that the characteristic polynomial \( P_\Phi \) of a Drinfeld \( A \)-module of rank 2, is one of the four following form :
Proposition 4.5. Let $\Phi$ be a Drinfeld $A$-module of rank 2 over the finite field $L = F_{q^n}$ and let $P$ the characteristic of $L$. We put $m = [L : A/P]$ and $d = \deg P$. The characteristic polynomial $P_\Phi$ is on the form :

1) $P_\Phi(X) = X^2 - cX + \mu P^m$, where $c^2 - 4\mu P^m$ is imaginary, $c \in A$, $(c, P) = 1$ and $\mu \in F_q^*$, if $\Phi$ is an ordinary module.

And the characteristic polynomial $P_\Phi$ for a supersingulary case is on the form :

2) $P_\Phi(X) = X^2 + \mu P^m$, with $\mu \in F_q^*$, if $m$ is odd,

3) $P_\Phi(X) = X^2 + c_0X + \mu P^m$, if $m$ is even and $d = \deg P$ is odd, $\mu \in F_q^*$ and $c_0 \in F_q$.

4) $P_\Phi(X) = (X + \mu P^{\frac{m}{2}})^2$ if $m$ is even.

We can resumed all the cases in following manner :

1. For the ordinary case, the characteristic polynomial is on the form :

   $$P_\Phi(X) = X^2 - cX + \mu P^m,$$

   such that : $2 \deg c < \deg P,m$ or $2 \deg c = \deg P,m$ and $X^2 - a_0X + \mu$ is irreducible over $F_{q^n}$ where $a_0$ is the coefficient of more big degree of $c$. For the supersingular case, we have the two following cases :

2. $\deg P$ is even or $-\mu \not\in (F_q^*)^2$.

3. $X^2 + c_0X + \mu$ is irreducible over $F_q$.

Then we can now calculate the number of characteristic polynomial which corresponding of the number of isogeny classes :

Lemma 4.4.

$$\#\{\text{classes d'isogénies}\} = \#\{P_\Phi\}.$$

We start by calculate the ordinary case. The case(1) give us the generally number $c$, which corresponding the number of isogeny classes for an ordinary Drinfeld module, and by using the 2,3 and 4 we find the number of isogeny classes for an supersingular Drinfeld module. Indeed in this case 1, the principal condition la condition that we have above $c$, other than this prime with $P$, is the Riemann condition which certify that $c^2 - 4\mu P^m$ is imaginary which condition can be wrote by : $\deg c \leq \frac{m.d}{2}$.

We distinguish, two cases :

1) The case where the number $m.d$ is odd, that means $m$ and $d$ are odds. We will have $q^{[\frac{m.d}{2}]+1}$ polynomials of degree less or equal than $[\frac{m.d}{2}]$ where $[.]$ is the partial entire ). In next, we eliminate the polynomials $c$ which are not primes with $P$, that means divisible by $P$. We can remark that for each $c$ divisible by $P$, there is a polynomial $Q$ such that $c = Q.P$ then the cardinal of such polynomials $c$ which are divisible by $P$ is equal to the cardinal of the set of $Q$ which is the order of $q^{\frac{m.d}{2}d+1}$ (since $\deg Q \leq \frac{m-2}{2}d$). If we consider than $\mu \in F_q^*$, we will have :

$$\#\{P_\Phi, \Phi : \text{ordinary(1)}\} = (q - 1)(q^{[\frac{m.d}{2}]+1} - q^{[\frac{m-2}{2}d]+1}).$$
2) For the case where the number \( \frac{m}{d} \) is even that means that at least one of the \( m \) or \( d \) is even, we will exclude the minimal polynomial associated to the corresponding modules which are not irreducibles and the condition over \( c \) became then:

\[ \text{deg} c < m, \quad \text{and the polynomial } X^2 - c_0 X + c \text{ is irreducible where } c_0 \text{ is the coefficient of more big degree of } c, \text{ with the prime condition of } c \text{ and } P \text{ we will have in this case:} \]

\[ \#\{P_\Phi, \Phi \text{ ordinary (1)}\} = (q - 1)(q^{\frac{m}{2}d} - q^{\frac{m-2}{2}d+1}). \]

For the case where the characteristic polynomial is on the form

\[ P_\Phi(X) = X^2 + \mu P^n, \] where \( \mu \in F_q^n \) if \( m \) is even. We will have \( q - 1 \) possibilities, and \( q^2 - q \) possibilities for the case 3, and finally we will have \( q - 1 \) possibilities for the case 4. So we can calculate the cardinal of the isogeny classes of a Drinfeld module of rank 2:

**Proposition 4.6.** Let \( \Phi \) a Drinfeld \( A \)-module of rank 2 over a finite field \( L = F_{q^n} \) and let \( P \) be the \( A \)-characteristic of \( L \). We put \( m = [L : A/P] \) and \( d = \text{deg } P \):

1. \( m \) is odd and \( d \) is odd:
   \[ \#\{P_\Phi, \Phi \text{ ordinary (1)}\} = (q - 1)(q^{\frac{m}{2}d} + 1 - q^{\frac{m-2}{2}d+1} + 1). \]

2. \( m \) is even and \( d \) is odd:
   \[ \#\{P_\Phi\} = (q - 1)[q^{\frac{1}{2}d} - q^{\frac{m-2}{2}d+1} + 1]. \]

3. \( m \) is even and \( d \) is even:
   \[ \#\{P_\Phi\} = (q - 1)[q^{\frac{1}{2}d} - q^{\frac{m-2}{2}d} + 1]. \]

### 4.2.1 Characteristic of Euler-Poincare

Let \( \Phi \) be a Drinfeld \( A \)-module of rank 2, over a finite field \( L = F_{q^n} \) and the polynomial characteristic \( P_\Phi \). We have seen above that \( \chi_\Phi = (P_\Phi(1)) \), this give us the possibility to deduct that if \( \Psi \) is an other Drinfeld \( A \)-module of rank 2 over finite field \( L \) of the polynomial characteristic \( P_\Psi \) and the Euler-Poincare characteristic \( \chi_\Psi \), then:

\[ \chi_\Phi = \chi_\Psi \iff \exists \lambda \in F_q^n : P_\Phi(1) = \lambda P_\Psi(1). \]

that means that the cardinal of the set of characteristic of the Euler-Poincare, can be deduced by the cardinal of the set of characteristic polynomial, and we have:

\[ \#\{\chi_\Phi\} \leq \frac{\#\{P_\Phi\}}{q^n - 1}. \]

About the characteristic of the Euler-Poincare we can enumerate the following remark:


1. The characteristic of the Euler-Poincare is the analogue of the number of points of the elliptic curve over this finite field.

2. For two elliptic curves, it is sufficient to have the same number of points to be isogenous, but for two Drinfeld modules it is no sufficient for two Drinfeld modules $\Phi$ and $\Psi$ are isogenous, since this two modules are isogenous if and only if $P_\Phi = P_\Psi$, or the fact that $\chi_\Phi = \chi_\Psi$ implies only that there is $\lambda \in F_{q^m}^*$ such that:

$$P_\Phi(1) = \lambda P_\Psi(1).$$

Then we can have a expression for the cardinal of the set of characteristic of Euler-Poincare:

**Proposition 4.7.** Let $\Phi$ be a Drinfeld $A$-module of rank 2 over the finite field $L = F_{q^n}$ and let $P$ be the characteristic of $L$. We put $m = [L : A/P]$ and $d = \deg P$. There exists $H, B \in L$, such that:

$$\#\{\chi_\Phi\} = H + B$$

where $H$ and $B$ verifies:

$$\#\{P_\Phi\} = (q - 1)H + (q - 2)B.$$

**Proof.** Let $\Phi$ and $\Psi$, two Drinfeld $A$-modules over $F_{q^n}$, $P_\Phi(1) = 1 - c + mp^m$ and $P_\Psi(1) = 1 - c' + \mu^m$. Then $\chi_\Phi = \chi_\Psi$, if and only if, $\lambda \in F_{q^n}$, such that $P_\Phi(1) = \lambda P_\Psi(1)$ then $1 - c + \mu^m = \lambda - \lambda c' + \lambda^m$. that means that : $\mu = \mu'$ and $c' = \lambda^{-1}(1 - c + \lambda)$, then the $\lambda$ are of order of $q - 2$ ( car $\lambda \in F_q - \{0, 1\}$).

At past the non prime condition with $P$, we will have, if such divisor $Q$ exists, $Q.P = 1 + \lambda + \lambda c'$ and then $\deg Q = -d + \deg c' \leq \frac{(n-2)d}{2}$. Then the cardinal of these $Q$ is equal to the cardinal of these $c'$, which is $q^{\frac{d}{2}}d + 1 - q^{\frac{m-2}{2}}d + 1$ which is $B$, and the couple $(\lambda, t')$ are of the order $(q - 2)(q^{\frac{d}{2}}d + 1 - q^{\frac{m-2}{2}}d + 1)$.

We can have $H$ by the equation: $\#\{P_\Phi\} = (q - 1)H + (q - 2)B \implies H = \frac{1}{q-1}(\#\{P_\Phi\} - (q - 2)B)$: we start by the case: 1) $m$ is odd and $d$ is odd:

$$H = \frac{1}{q-1}q^{\frac{d}{2}}d + 1 - \frac{1}{q-1}q^{\frac{m-2}{2}d + 1} + 1.$$

2) $m$ is even and $d$ is odd:

$$H = \frac{1 + 2q - q^2}{2q - 2}q^{\frac{d}{2}}d - \frac{1}{q-1}q^{\frac{m-2}{2}d + 1} + q.$$

3) $m$ is even and $d$ is even:

$$H = \frac{1 + 2q - q^2}{2q - 2}q^{\frac{d}{2}}d - \frac{1}{q-1}q^{\frac{m-2}{2}d + 1} + 1.$$
Finally we recuperate the values of $\#\{\chi_\Phi\}$:

**Proposition 4.8.** Let $\Phi$ be a Drinfeld $A$-module of rank 2 over a finite field $L = \mathbb{F}_{q^n}$ and let $P$ the $A$-characteristic of $L$. We put $m = [L : A/P]$ and $d = \deg P$:

1. $m$ is odd and $d$ is odd:

$$\#\{\chi_\Phi\} = \frac{q}{q - 1}q^{\frac{m}{2}d} + 1 - \frac{q}{q - 1}q^{\frac{m - 2}{2}d} + 1$$

2. $m$ is even and $d$ odd:

$$\#\{\chi_\Phi\} = \frac{q^2 + 1}{2q - 2}q^{\frac{m}{2}d} - \frac{q}{q - 1}q^{\frac{m - 2}{2}d} + 1 + q$$

3. $m$ is even and $d$ is even:

$$\#\{\chi_\Phi\} = \frac{q^2 + 1}{2q - 2}q^{\frac{m}{2}d} - \frac{q}{q - 1}q^{\frac{m - 2}{2}d} + 1.$$

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