Partial Euler Characteristic, Normal Generations and the stable \( D(2) \) problem

Feng Ji and Shengkui Ye

March 9, 2015

Abstract

We obtain relations among normal generation of perfect groups, Swan’s inequality involving partial Euler characteristic, and deficiency of finite groups. The proof is based on the study of a stable version of Wall’s \( D(2) \) problem. Moreover, we prove that a finite 3-dimensional CW complex of cohomological dimension at most 2 with fundamental group \( G \) is homotopy equivalent to a 2-dimensional CW complex after wedging \( n \) copies of the 2-sphere \( S^2 \), where \( n \) depends only on \( G \).

1 Introduction

In this article, we study several classical problems in the low dimension homotopy theory and group theory, focusing on the interplay among these problems.

We start by describing Swan’s problem. Let \( G \) be a group and \( \mathbb{Z}G \) be the group ring. Swan [17] defines the partial Euler characteristic \( \mu_n(G) \) as follows. Let \( F \) be a resolution

\[ \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{Z} \rightarrow 0 \]

of the trivial \( \mathbb{Z}G \)-module \( \mathbb{Z} \) in which each \( F_i \) is \( \mathbb{Z}G \)-free on \( f_i \) generators. If

\[ f_0, f_1, f_2, \cdots, f_n \]

are finite, define

\[ \mu_n(F) = f_n - f_{n-1} + f_{n-2} - \cdots + (-1)^n f_0. \]

If there exists a resolution \( F \) such that \( \mu_n(F) \) is defined, we define \( \mu_n(G) \) as the infimum of \( \mu_n(F) \) over all such resolution \( F \). We call the truncated free resolution

\[ F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{Z} \rightarrow 0 \]

an algebraic \( n \)-complex (following the terminology of Johnson [10]).
On the other hand, we have the following geometric counterpart in the case \( n = 2 \). For a finitely presentable \( G \), the deficiency \( \text{def}(G) \) is the maximum of \( d - k \) over all presentations \( \langle g_1, g_2, \cdots, g_d \mid r_1, r_2, \cdots, r_k \rangle \) of \( G \). It is not hard to see that
\[
\text{def}(G) \leq 1 - \mu_2(G)
\]
([17], Proposition (1)). However, Swan mentioned in [17] that “the problem of determining when \( \text{def}(G) = 1 - \mu_2(G) \) seems very difficult even if \( G \) is a finite \( p \)-group”.

It is a well-known open problem, since the 1970s, whether a finitely generated perfect group can be normally generated by a single element or not. We formulate such a problem in the following conjecture.

**Conjecture 1.1 (Normal Generation Conjecture)** Let \( G \) be any finitely generated perfect group, i.e. \( G = [G, G] \), the commutator subgroup of \( G \). Then \( G \) can be normally generated by a single element.

This conjecture is known to be true when \( G \) is finite (see e.g. [12], 4.2). For infinite groups, it is a long-standing open problem attributed to J. Wiegold (cf. [1], FP14] and [16], 5.52]). One of our results relates Wiegold’s normal generation conjecture to Swan’s inequality of partial Euler characteristic, as follows.

**Theorem 1.2** Assume that Conjecture [17] is true. Then
\[
\text{def}(G) \geq -\mu_2(G)
\]
for any finite group \( G \).

The proof of Theorem 1.2 is based on the study of a stable version of the \( D(2) \) problem (for details, see Section 3). Recall that Wall’s \( D(2) \) problem asks that for a finite 3-dimensional CW complex \( X \) of cohomological dimension \( \leq 2 \), is \( X \) homotopy equivalent to a 2-dimensional CW complex? A positive answer to this problem will imply a finite version of the Eilenberg-Ganea conjecture, which says that a group of cohomological dimension two has a 2-dimensional classifying space. We obtain the following result, which shows that such a CW complex \( X \) with a finite fundamental group is homotopic to a 2-dimensional CW complex after wedging several copies of the sphere \( S^2 \).

**Theorem 1.3** Let \( G \) be a finite group. For a finite 3-dimensional CW complex \( X \) of cohomological dimension at most 2 with fundamental group \( \pi_1(X) = G \), the wedge \( X \vee (S^2)^n \) of \( X \) and \( n \) copies of \( S^2 \) is homotopy to a 2-dimensional CW complex, where \( n \) depends only on \( G \).
The integer $n$ in the previous theorem can be determined completely (see Theorem 4.6).

On the other hand, for groups of low geometric dimensions, we have the following result, which confirms the equality of partial Euler characteristic and deficiency.

**Theorem 1.4** Let $G$ be a group having a finite classifying space $BG$ of dimension at most $2$. Then $\text{def}(G) = 1 - \mu_2(G)$.

Finally, we present an application (cf. Corollary 4.10) of the results to the Whitehead conjecture, which claims that any subcomplex of aspherical complexes is aspherical.

The article is organized as follows. In Section 2, we discuss the Quillen plus construction of 2-dimensional CW complexes. This motivates the stable Wall's $D(2)$ property being discussed in Section 3. In the last section, the Euler characteristics are studied for both finite groups and groups of low geometric dimension.

## 2 Quillen’s plus construction of 2-dimensional CW complexes

Let $X$ be a CW complex with fundamental group $G$ and $P$ a perfect normal subgroup of $G$, i.e. $P = [P, P]$. Quillen shows that there exists a CW complex $X_P^+$, whose fundamental group is $G/P$; and an inclusion $f : X \to X_P^+$ such that

$$H_n(X; f_*M) \cong H_n(X_P^+; M)$$

for any integer $n$ and local coefficient system $M$ over $X_P^+$. Here $X_P^+$ is called the plus-construction of $X$ with respect to $P$ and is unique up to homotopy equivalence. One of the main applications of the plus construction is to define higher algebraic $K$-theory. In general, the space $X_P^+$ is obtained from $X$ by attaching 2-cells and 3-cells. The following discussion shows that for certain particular 2-dimensional CW complex $X$, the Quillen plus construction is homotopy equivalent to a 2-dimensional CW complex. We need to define the following.

**Definition 2.1** The cohomological dimension $\text{cd}(X)$ of a CW complex $X$ is defined as the smallest integer $n$ ($\infty$ is allowed) such that $H^n(G; M) = 0$ for any integer $m > n$ and any local coefficient system $M$.

Clearly, an $n$-dimensional CW complex is of cohomological dimension $\leq n$. We start with a lemma showing a property enjoyed by any 3-dimensional CW complex with cohomological dimension 2.
Lemma 2.2 Suppose that $X$ is a 3-dimensional CW complex and $\tilde{X}$ is the universal cover of $X$. Let $C_\ast(\tilde{X})$ be the cellular chain complex of $\tilde{X}$. Then $X$ is of cohomological dimension 2 if and only if $C_3(\tilde{X})$ is a direct summand of $C_2(\tilde{X})$ as $\mathbb{Z}\pi_1(\tilde{X})$-modules.

This result is well-known and implicit in literature. We include a proof as we are unable to locate a suitable reference.

Proof. If $X$ is of cohomological dimension 2. Consider the $\mathbb{Z}\pi_1(\tilde{X})$-module $C_3(\tilde{X})$ as coefficients. The condition that $H^3(X, C_3(\tilde{X})) = 0$ implies that the identity map $C_3(\tilde{X}) \to C_3(\tilde{X})$ factors throughout $C_2(\tilde{X})$. As all these are free $\mathbb{Z}\pi_1(\tilde{X})$-modules, $C_3(\tilde{X})$ is a direct summand of $C_2(\tilde{X})$.

The converse is true as any homomorphism from $C_3(\tilde{X})$ to a $\mathbb{Z}\pi_1(\tilde{X})$-module factor through $C_2(\tilde{X})$, if $C_3(\tilde{X})$ is a direct summand of $C_2(\tilde{X})$. ■

Theorem 2.3 Let $X$ be a finite 2-dimensional CW complex. Suppose that a perfect normal subgroup $P$ in $\pi_1(X)$ is normally generated by $n$ elements. Then the plus construction $(X \vee (S^2)^n)^+$ taken with respect to $P$ of the wedge of $X$ and $n$ copies of $S^2$, is homotopy equivalent to the 2-skeleton of $X^+$, which is a 2-dimensional CW complex.

Proof. It is not hard to see that the plus construction $(X \vee (S^2)^n)^+$ is the same as the wedge $X^+ \vee (S^2)^n$. Indeed, a simple calculation of the homology groups (for any local coefficient system) shows that $X^+ \vee (S^2)^n$ satisfies the defining properties of $(X \vee (S^2)^n)^+$.

Denote by $Y$ the complex $X^+ \vee (S^2)^n$. Consider the cellular chain complex $C_\ast(\tilde{Y})$ of the universal cover $\tilde{Y}$. By the process of Quillen’s plus construction (cf. the proof of Theorem 1 in [19]), we see that the number of attached 3-cells and the number of attached 2-cells are both $n$. Since $X$ is 2-dimensional, so is $X \vee (S^2)^n$. As the plus construction does not change homology groups (for any local coefficient system), $Y$ is of cohomological dimension 2. This implies that $C_3(\tilde{Y}) \cong \mathbb{Z}\pi_1(Y)^n$ is isomorphic to a direct summand of $C_2(\tilde{X}^+)$ from the previous lemma.

Moreover, $C_\ast(\tilde{Y})$ is chain homotopy equivalent to the following chain complex

$$0 \to C_2(\tilde{Y})/C_3(\tilde{Y}) \to C_1(\tilde{Y})(= C_1(\tilde{X}^+)) \to \mathbb{Z}\pi_1(Y)(= \mathbb{Z}\pi_1(X^+)) \to \mathbb{Z} \to 0. (*)$$

Notice that $C_2(\tilde{Y})$ is isomorphic to $C_2(\tilde{X}^+) \bigoplus \mathbb{Z}\pi_1(Y)^n$. Therefore, there is an isomorphism

$$C_2(\tilde{Y})/C_3(\tilde{Y}) \cong C_2(\tilde{X}^+).$$

This gives a chain homotopy from $(*)$ to the chain complex of the universal cover of the 2-skeleton of $X^+$. By the following lemma, we see that $Y$ is homotopy equivalent to the 2-skeleton of $X^+$. ■

Lemma 2.4 (Johnson [10], Mannan [15]) Let $Y$ be a finite 3-dimensional CW complex of cohomological dimension 2. If the chain complex

$$0 \to C_2(\tilde{Y})/C_3(\tilde{Y}) \to C_1(\tilde{Y}) \to \mathbb{Z}\pi_1(Y) \to \mathbb{Z} \to 0$$
is homotopy equivalent to the chain complex of a 2-dimensional CW complex 
$X$, $Y$ is homotopy equivalent to a 2-dimensional complex $X$.

3 Wall’s D(2) problem and its stable version

In this section, we apply the results obtained in the previous section to the $D(2)$ problem. Let us recall the $D(2)$ problem raised in [18].

**Conjecture 3.1** (The $D(2)$ problem) If $X$ is a finite 3-dimensional CW complex of cohomological dimension $\leq 2$, then $X$ is homotopy equivalent to a 2-dimensional CW complex.

In [10], Johnson proposes to systematically study the problem by categorizing 3-dimensional CW complexes according to their fundamental groups. For a finitely presentable group $G$, we say the $D(2)$ problem is true for $G$, if any finite 3-dimensional CW complex $X$, of cohomological dimension $\leq 2$ with fundamental group $\pi_1(X) = G$, is homotopy equivalent to a 2-dimensional CW complex.

The $D(2)$ problem is very difficult in general; and it is known to be true for limited amount of groups ([6], [13]). We propose the following stable version by allowing taking wedge with copies of $S^2$.

**Conjecture 3.2** (The $D(2, n)$ problem) For a finitely presentable group $G$ and $n \geq 0$, we say that the $D(2, n)$ problem holds for $G$ if the following is true. If $X$ is a finite 3-dimensional CW complex of cohomological dimension $\leq 2$ with fundamental group $\pi_1(X) = G$, then $X \vee (S^2)^n$ is homotopy equivalent to a 2-dimensional CW complex.

It is immediate that $D(2)$ implies $D(2, n)$ and $D(2, n)$ implies $D(2, n+1)$ for any group $G$ and any integer $n \geq 0$. The $D(2, 0)$ problem is the original $D(2)$ problem.

We first consider CW complexes with finite fundamental groups. In order to relate the problem to the previous section, we record the following observation due to Mannan.

**Lemma 3.3** (Mannan [14]) A finite 3-dimensional CW complex $X$ of cohomological dimension 2 is a Quillen’s plus construction of some 2-dimensional complex $Y$.

Let $G$ be a finitely generated perfect group. It is conjectured that $G$ could be normally generated by one element (cf. Conjecture [1.4]). Assuming Conjecture [1.4] holds, we have the following.

**Theorem 3.4** Let $X$ be a finite 3-dimensional complex of cohomological dimension 2 with $\pi_1(X)$ finite. Suppose that Conjecture [1.4] holds. Then $X \vee S^2$ is homotopy equivalent to a 2-complex, i.e. the $D(2, 1)$ problem holds for any finite group.
**Proof.** By Lemma 3.3, \( X \) is the plus construction of a finite 2-complex \( Z \) with respect to a perfect normal subgroup \( P \leq \pi_1(Z) \). Therefore we have a short exact sequence of groups

\[
1 \to P \to \pi_1(Z) \to \pi_1(X) \to 1.
\]

Since \( \pi_1(Z)/P = \pi_1(X) \) is finite and \( Z \) is finite, we claim that \( P \) is finitely generated. To see this, as \( \pi_1(X) \) is finite, the covering space of \( Z \) with fundamental group \( P \) is again a finite CW complex. Hence \( P \) is finitely generated.

If the normal generation conjecture holds, \( P \) is normally generated by a single element. Theorem 2.3 in the previous section says that \( X \lor S^2 \) is homotopy equivalent to a 2-dimensional CW complex. 

We now study the relation between the stabilization by “wedging” copies of \( S^2 \) with that by “attaching” 3-cells.

**Proposition 3.5** Suppose that \( X \) is a finite 3-dimensional CW complex of cohomological dimension \( \leq 2 \). Then \( X \lor (S^2)^n \) is homotopy equivalent to a finite 2-dimensional CW complex if and only if \( X \) is homotopy equivalent to a 3-dimensional CW complex with \( n \) 3-cells.

**Proof.** Assume that \( X \) is homotopy equivalent to a 3-dimensional CW complex with \( n \) 3-cells. By Lemma 3.3, \( X \) is the plus construction of a 2-complex \( Y \) with respect to a perfect normal subgroup \( K \) of \( \pi_1(Y) \). We thus have a short exact sequence

\[
1 \to K \to \pi_1(Y) \to \pi_1(X) \to 1.
\]

Moreover, \( K \) is normally generated by \( n \)-elements. Therefore \( X \lor (S^2)^n \) is homotopy equivalent to a 2-dimensional CW complex by Theorem 2.3.

Conversely, suppose that \( X \lor (S^2)^n \) is homotopy equivalent to a finite 2-complex \( Y \) via \( f : X \lor (S^2)^n \to Y \). It is clear that

\[
\pi_1(X) = \pi_1(X \lor (S^2)^n) \cong \pi_1(Y).
\]

Let \( G = \pi_1(X) \) and \( \hat{X}, \hat{Y} \) be the universal covers of \( X, Y \) respectively. We proceed as follows. By the Hurewicz theorem, we have isomorphisms

\[
\pi_2(Y) \cong \pi_2(\hat{Y}) \cong H_2(\hat{Y}) \cong \pi_2(\hat{X}) \oplus ZG^n.
\]

Therefore, there are \( n \) maps \( f_i : S^2 \to Y, 1 \leq i \leq n \) correspond to the following inclusion onto the second factor (for a fixed basis of \( ZG^n \))

\[
ZG^n \to H_2(\hat{Y}) \cong \pi_2(\hat{X}) \oplus ZG^n.
\]

Using these \( f_i, 1 \leq i \leq n \) as the attaching maps, we obtain a 3-dimensional CW complex \( Y \cup_{f_i, 1 \leq i \leq n} e_2^n \). Let \( i : X \hookrightarrow X \lor (S^2)^n \) be the natural inclusion. By our construction, the canonical composition

\[
f' : X \hookrightarrow X \lor (S^2)^n \xrightarrow{i} Y \to Y \cup_{f_i, 1 \leq i \leq n} e_2^n
\]
induces isomorphisms on both $\pi_1$ and $\pi_2$ (the same as the second homology groups of the universal covers). It is not hard to see that $H_3(\tilde{X}) = H_3(Y \cup \cup_{f_i, 1 \leq i \leq n} e_i^n) = 0$. Therefore, $f'$ induces a homotopy equivalence between the chain complexes of the universal covering spaces. By the Whitehead theorem, $f'$ is a homotopy equivalence. 

If we combine this proposition with Theorem 3.4 we obtain the following immediately.

**Corollary 3.6** Suppose that a group $G$ satisfies $D(2, n)$ for some $n \geq 0$. Then any finite 3-dimensional CW complex of cohomological dimension 2 with fundamental group $G$ is homotopy equivalent to a complex with at most $n$ 3-cells.

In particular, if Conjecture 1.1 holds, then any finite 3-dimensional CW complex of cohomological dimension 2 with finite fundamental group is homotopy equivalent to a complex with at most a single 3-cell.

We remark that in fact, we shall see in the next section that each finite group $G$ satisfies $D(2, n)$ for some $n$, even if we do not assume the normal generation conjecture.

Other examples of groups satisfying $D(2, n)$ for some $n$ include groups with both cohomological dimension $\leq 2$ and cancellation property.

### 4 Partial Euler characteristic of groups and $(G, n)$-complexes

Recall definitions of $\mu_n(F)$ for an algebraic $n$-complex $F_\ast$ and $\mu_n(G)$ from Introduction. For a finitely presentable group $G$, we start with the following lemma. It follows from Swan [17] easily, although it is not stated explicitly.

**Lemma 4.1** Assume that $G$ is finitely presentable. The invariant $\mu_2(G)$ can be realized by an algebraic 2-complex. In other words, there exists an algebraic 2-complex

$$F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

such that

$$\mu_2(G) = \dim_{\mathbb{Z}G} F_2 + \dim_{\mathbb{Z}G} F_0 - \dim_{\mathbb{Z}G} F_1.$$

**Proof.** Let $M = \mathbb{Q}$ in Theorem 1.2 of [17]. As $G$ is finitely presentable, all the Betti numbers are finite for $n \leq 2$. Therefore Theorem 1.2 of [17] asserts $\mu_2(G)$ is finite and hence realizable by an algebraic 2-complex. 

**Lemma 4.2** (Johnson [10] Theorem 60.2) Every algebraic 2-complex is geometric realizable by a 3-dimensional CW complex. In other words, for every algebraic 2-complex

$$(F_\ast) : F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$
over a finitely presentable group \( G \), there is a finite 3-dimensional CW complex of cohomological dimension 2 such that the reduced chain complex

\[
C_2(\tilde{Y})/C_3(\tilde{Y}) \to C_1(\tilde{Y}) \to \mathbb{Z}\pi_1(Y) \to \mathbb{Z} \to 0
\]

is homotopy equivalent to \((F_\ast)\).

**Proposition 4.3** If a finitely presentable group \( G \) satisfies the \( D(2, n) \) problem, then \( \text{def}(G) \geq (1 - n) - \mu_2(G) \).

Theorem 1.2 follows immediately from this proposition in view of Theorem 3.4.

**Proof.** By Lemma 4.1, we can choose an algebraic 2-complex

\[
F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0
\]

such that

\[
\mu_2(G) = \dim_{\mathbb{Z}G} F_2 + \dim_{\mathbb{Z}G} F_0 - \dim_{\mathbb{Z}G} F_1.
\]

By Lemma 4.2, there is a finite 3-dimensional CW complex of cohomological dimension 2 such that the reduced chain complex

\[
0 \to C_3(\tilde{Y}) \to C_2(\tilde{Y}) \to C_1(\tilde{Y}) \to \mathbb{Z}\pi_1(Y) \to \mathbb{Z} \to 0
\]

is homotopy equivalent to \((F_\ast)\). Assuming that \( G \) satisfies the \( D(2, n) \) problem, the wedge \( X \vee (S^2)^n \) is homotopy equivalent to a 2-dimensional CW complex, which gives a presentation of \( G \). This implies that \( \mu_2(G) + n \geq 1 - \text{def}(G) \), i.e. \( \text{def}(G) \geq (1 - n) - \mu_2(G) \).

It is possible to place \( \mu_2(G) \) in a broader setting following [9].

**Definition 4.4** We define a \( (G, n) \)-complex as a finite \( n \)-dimensional CW complex with fundamental group \( G \) and vanishing higher homotopy groups up to dimension \( n - 1 \).

In particular, a \( (G, 2) \)-complex is a usual finite 2-dimensional CW complex.

**Definition 4.5** Let \( G \) be a finitely presentable group. Define

\[
\mu_n^G(G) = \min\{(-1)^n \chi(X) \mid X \text{ is a } (G, n)\text{-complex}\}.
\]

If there is no such \( X \) exists, define \( \mu_n^G(G) = +\infty \). We call a \( (G, n) \)-complex \( X \) with \( (-1)^n \chi(X) = \mu_n^G(G) \) realize \( \mu_n^G(G) \).

A few observations are immediate. It is clearly true that \( \mu_n(G) \leq \mu_n^G(G) \). Therefore, \( \mu_n^G(G) > -\infty \) since \( \mu_n(G) > -\infty \) (cf. Swan [17]). Moreover, \( \mu_2(G) = \mu_2^G(G) \) if and only if \( \mu_2(G) = 1 - \text{def}(G) \). We can use this language to discuss the \( D(2, n) \) problem for finite groups without assuming the normal generation conjecture.
**Theorem 4.6** If $G$ is a finite group, then the $D(2,n)$ problem holds for $G$ when $n = 2 - \text{def}(G) - \mu_2(G)$.

**Proof.** The key point here is that algebraic $m$-complexes of $ZG$ are classified according to the partial Euler characteristics, when $G$ is finite. More precisely, for any two algebraic $m$-complexes $F$ and $F'$ with $\mu_m(F) = \mu_m(F') \neq \mu_m(G)$, we have that $F$ and $F'$ are chain homotopic. For a proof, see [3], for example.

Apply this result to the case $m = 2$. As $G$ is finitely presentable, we fix a $(G,2)$-complex $X$ with the Euler characteristic $1 - \text{def}(G)$, where $\text{def}(G)$ is the deficiency of $G$. We claim that $G$ satisfies $D(2,n)$ for $n = (1 - \text{def}(G)) - \mu_2(G) + 1$.

To see this, any algebraic 2-complex $F$ such that $\mu_2(F) > 1 - \text{def}(G)$ is homotopy equivalent to the chain complex of the universal covering space of the wedge product $X \vee (S^2)^{\mu_2(F) + \text{def}(G) - 1}$, since both complexes have the same Euler characteristics (cf. [3]). For any finite 3-complex $X'$ of cohomological dimension 2, the chain complex of the universal covering space of the wedge $X' \vee (S^2)^n$, denoted by $Y$, is homotopic to the algebraic 2-complex

$$F : C_2(\tilde{Y})/C_3(\tilde{Y}) \to C_1(\tilde{Y}) \to ZG \to Z \to 0.$$ 

Clearly, $\mu_2(F) = \chi(X') + n - 1 > 1 - \text{def}(G)$. By Lemma 2.3 and Lemma 4.2, the complex $X' \vee (S^2)^n$ is homotopy equivalent to a 2-dimensional CW complex. The proof is finished.

Theorem 1.3 is an easy corollary of the previous theorem.

In view of the results of the previous section, for each finite group $G$, there is an integer $n$, which depends only on $G$, such that each 3-complex with fundamental group $G$ and cohomological dimension $\leq 2$ is homotopy equivalent to a 3-complex with at most $n$ 3-cells (Corollary 3.6).

The argument of the proof of Theorem 4.6 does not work for an infinite finitely presentable group. For example, for the trefoil knot group, Euler characteristic is not enough to classify the chain homotopy classes of algebraic 2-complexes (see [5]).

The $D(2,n)$ problem stems from the $D(2)$ problem. On the other hand, the question on the equality $\mu_2(G) = 1 - \text{def}(G)$ is a second “face” of the $D(2)$ problem. We can say something on this question when $G$ is torsion free of low geometric dimension.

Recall that for a group $G$, the classifying space $BG$ of $G$ is defined as the connected CW complex with $\pi_1(BG) = G$ and $\pi_i(BG) = 0$, $i \geq 2$. It is unique up to homotopy. Theorem 1.4 is a special case of (i) in the following theorem.

**Theorem 4.7** Let $G$ be a group having a finite $n$-dimensional classifying space $BG$. We have the following.

(i) $\mu_n(G) = \mu_n^0(G)$; In particular, $\mu_2(G) = 1 - \text{def}(G)$ if $G$ has a finite 2-dimensional $BG$. 

9
(ii) Any finite CW complex $X$ with the following property:

a) the dimension is at most $n + 1$;
b) the cohomological dimension $\text{cd}(X)$ is at most $n$;
c) if $n \geq 3$, the homotopy group $\pi_i(X) = 0$ for $2 \leq i \leq n - 1$;
d) $(-1)^n \chi(X) = \mu_n(G)$,

is homotopy equivalent to $BG$.

**Proof.** Let $EG$ be the universal cover of $BG$. Since $EG$ is contractible, one obtains the exact cellular chain complex of $EG$:

$$C_*(EG) : 0 \to C_n(EG) \to C_{n-1}(EG) \ldots \to ZG \to 0.$$  

This gives a (truncated) free resolution of $G$. In order to prove (i), it suffices to show that this resolution gives the minimal Euler characteristic $\mu_n(G)$ (as we notice earlier that $\mu_n(G) \leq \mu_n^g(G)$).

Suppose that $\mu_n(G)$ is obtained from the following partial resolution of finitely generated free $ZG$-modules:

$$F_* : F_n \xrightarrow{d} F_{n-1} \ldots \to F_1 \to ZG \to 0.$$  

We claim that $F_*$ is exact at $d : F_n \to F_{n-1}$. Once this is proved $C_*(EG)$ and $F_*$ are chain homotopic to each other; and hence have the same Euler characteristic.

To prove the claim, let $J$ be the kernel of $d$. By Schanuel’s lemma, there is an isomorphism

$$J \oplus C_n(EG) \oplus F_{n-1} \ldots \cong F_n \oplus C_{n-1}(EG) \ldots.$$  

Applying the functor $- \otimes_{ZG} Z$ to both sides of this isomorphisms, we see that $\mu_n(F) = (-1)^n \chi(BG)$ and $J \otimes_{ZG} Z = 0$ by noticing the fact that the complex $F_*$ attains minimal Euler characteristic multiplying $(-1)^n$ among all the algebraic $n$-complexes. This implies that $C_n(EG) \oplus F_{n-1} \ldots$ and $F_n \oplus C_{n-1}(EG) \ldots$ have the same finite free $ZG$-rank. By Kaplansky’s theorem, $J$ is the trivial $ZG$-module (cf. [11], p. 328). This proves (i).

We now prove (ii). Let $C_*(X)$ be the chain complex of the universal covering space of $X$. Since $\text{cd}(X) \leq n$, $C_{n+1}(\tilde{X})$ is direct summand of $C_n(\tilde{X})$, by the same argument as Lemma 2.2. Let $F$ be the chain complex

$$F_* : C_n(\tilde{X})/C_{n+1}(\tilde{X}) \xrightarrow{d} C_{n-1}(\tilde{X}) \to \cdots \to C_1(\tilde{X}) \to ZG \to 0.$$  

It is not hard to see that $\pi_n(X) \cong \ker d$. Note that

$$\mu_n(F) = (-1)^n \chi(X) = \mu_n(G).$$  

By the same argument as the first part of the proof, we get that $\ker d = 0$. This implies that $\tilde{X}$ is $n$-connected. Since $H_{n+1}(\tilde{X}) = 0$, $\tilde{X}$ is contractible and $X$ is homotopy equivalent to $BG$. $\blacksquare$
Remark 4.8 Under the condition of Theorem 4.7, Harlander and Jensen [9] already prove that a \((G, n)\)-complex realizing \(\mu_2^n(G)\) is homotopy equivalent to \(BG\). Note that a \((G, n)\)-complex is a special case of \(X\) in Theorem 4.7.

Since the cohomological dimension of a finite group \(G\) is always infinity, the finite group \(G\) cannot have a finite dimensional \(BG\). The previous theorem is thus a discussion on torsion free “low dimension” groups. The trefoil knot group is an example in the case \(n = 2\).

We conclude with an application to another situation in low dimensional homotopy theory. Suppose that \(G\) is a finitely presentable group and

\[
P = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle
\]

is a presentation of \(G\). Call

\[
P' = \langle y_1, \cdots, y_{n'} \mid s_1, \cdots, s_{m'} \rangle
\]

a sub-presentation of \(P\) if

\[
\{y_1, \cdots, y_{n'}\} \subseteq \{x_1, \cdots, x_n\}
\]

and

\[
\{s_1, \cdots, s_{m'}\} \subseteq \{r_1, \cdots, r_m\}.
\]

Denote by \(G_{P'}\) the group given by the presentation \(P'\).

From each finite 2-complex, one obtains a finite presentation of \(\pi_1(X)\). Namely, the 1-cells correspond one-one to a set of generators; while the 2-cells correspond one-one to a set of relators.

Lemma 4.9 Suppose that

\[
P' = \langle y_1, \cdots, y_{n'} \mid s_1, \cdots, s_{m'} \rangle
\]

is a sub-presentation of

\[
P = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle
\]

of \(G\) as above. If \(P''\) is another finite presentation of \(G_{P'}\), then one can obtain a presentation of \(G\) from \(P\) by adding \(n - n'\) generators and \(m - m'\) relations.

In particular, if \(P\) realizes \(\mu_2^n(G)\), then \(P'\) realizes \(\mu_2^n(G_{P'})\).

Proof. Re-indexing and re-naming if necessary, we assume that

\[
y_1 = x_1, \cdots, y_{n'} = x_{n'}, n' \leq n
\]

and

\[
s_1 = r_1, \cdots, s_{m'} = r_{m'}, m' \leq m.
\]

It is clear that the words corresponding to \(s_1, \cdots, s_{m'}\) do not involve \(x_{n'+1}, \cdots, x_n\).

If

\[
P'' = \langle y_1', \cdots, y_{u'} \mid s_1', \cdots, s_{v'} \rangle
\]

then the presentation \(P''\) is obtained from \(P'\) by adding \(u' - n'\) generators and \(v' - m'\) relations.
is another presentation of $G_{P'}$, we form a group $G''$ with the presentation

$$\langle y'_1, \cdots, y'_n, x_{n'+1}, \cdots, x_n \mid s'_1, \cdots, s'_v \rangle$$

by adding $n-n'$ free generators to $P''$. For each $1 \leq i < m'$, the letter $x_i$, viewed as an element in $G_{P'}$, has a lifting $w_i$ in the free group $\langle y'_1, \cdots, y'_n \rangle$. For each $1 \leq i \leq n$, define the word $\omega_i$ of $\{y'_1, \cdots, y'_n, x_{n'+1}, \cdots, x_n\}$ as

$$\omega_i = \begin{cases} w_i, & 1 \leq i < m'; \\ x_i, & m' < i \leq m. \end{cases}$$

Denote by $\phi$ the bijection $\phi : \{x_1, \cdots, x_n\} \to \{\omega_1, \cdots, \omega_n\}$ given by $x_i \mapsto \omega_i$. For each $m' < i \leq m$, write $r_i = \Pi_{j=1}^{k_i} x_{ij}$ as a reduced word of $\{x_1, \cdots, x_n\}$. Let $r'_i = \Pi_{j=1}^{k_i} \phi(x_{ij})$ be the corresponding word of $\{y'_1, \cdots, y'_n, x_{n'+1}, \cdots, x_n\}$.

Let $K$ be the normal subgroup of $G''$ normally generated by the $m'-m$ elements $r'_{m'+1}, \cdots, r'_m$. We obtain a short exact sequence of groups

$$1 \to K \to G'' \to G \to 1,$$

where the third arrow is induced by the map $G_{P'} \to G$ from the natural inclusions of generators and relators. From this exact sequence, obtains a desired presentation

$$\langle y'_1, \cdots, y'_n, x_{n'+1}, \cdots, x_n \mid s'_1, \cdots, s'_v, r'_{m'+1}, \cdots, r'_m \rangle$$

of $G$.

Assume that $P$ realizes $\mu_2^2(G)$, while $P'$ does not realize $\mu_2^2(G_{P'})$. We apply the above construction to a presentation $P''$ of $G_{P'}$ realizing $\mu_2^2(G')$. In doing so, we obtain of presentation of $G$ with Euler characteristic strictly smaller than that of $P$. This gives a contradiction. ■

Recall that the famous Whitehead conjecture says that any subcomplex of $X'$ of an aspherical complex $X$ is aspherical as well (for more details, see the survey article [3]). As an application of results proved above, we gives an equivalent condition of asphericity of $X'$, as follows.

**Corollary 4.10** Suppose that $X$ is a finite aspherical 2-complex and $X'$ is a subcomplex of $X'$. We have the following.

(i) The complex $X'$ realizes $\mu_2^2(\pi_1(X'))$;

(ii) The complex $X'$ is aspherical if and only if the fundamental group $\pi_1(X'$) is of geometric dimension at most 2.
Proof. As $X$ is aspherical, it realizes $\mu_2^g(\pi_1(X))$ by Theorem 4.7. Notice that $X'$ gives a subpresentation of $\pi_1(X')$ of the presentation of $\pi_1(X)$, which corresponds to $X$. Lemma 4.9 implies that $X'$ realizes $\mu_2^g(\pi_1(X'))$. This proves part (i).

If $X'$ is aspherical, it is $B\pi_1(X')$ and hence $\pi_1(X')$ is of geometric dimension at most 2. Conversely, assume that $\pi_1(X')$ is of geometric dimension at most 2. By Theorem 4.7 all the $(\pi_1(X'), 2)$-complexes realizing $\mu_2^g(\pi_1(X'))$ are homtopic to $B\pi_1(X')$. Therefore $X'$ is aspherical by part (i).

We remark that in a recent article [7], Gersten obtains a result stronger than Corollary 4.10 (ii) using the method of $L_2$-theory. Namely, he is able to replace the condition “geometric dimension 2” by “cohomological dimension 2”. These two condition are equivalent to each other if the Eilenberg-Ganea conjecture ([2] VIII. 7) is true.

References

[1] G. Baumslag, A. G. Myasnikov, and V. Shpilrain, Open problems in combinatorial group theory. Second edition. In Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), volume 296 of Contemp. Math., pages 1–38. Amer. Math. Soc., Providence, RI, 2002.

[2] K. Brown, Cohomology of groups, GTM 87 (1982), Springer-Verlag.

[3] W. A. Bogley, J.H.C. Whitehead’s asphericity question, in: “Two-dimensional Homotopy and Combinatorial Group Theory”, eds. C. Hog-Angeloni, A. Sieradski and W. Metzler, LMS Lecture Notes 197, Cambridge Univ Press (1993), 309-334.

[4] J. Cohn, Aspherical 2-complexes, Journal of Pure and Applied Algebra, 12(1978), Pages 101–110

[5] M. J. Dunwoody, Relation modules, Bull. London Math. Soc., 4 (1972), 151-155.

[6] T. Edwards, Generalized Swan modules and the D(2) problem, Algebraic & Geometric Topology, 2006, 6: 71-89.

[7] S. Gersten, Asphericity for certain groups of cohomological dimension 2, arXiv: 1501.06875v2, 2015.

[8] K. W. Gruenberg, Homotopy classes of truncated resolutions, Comment. Math. Helvetici 68 (1993) 579-598.

[9] J. Harlander and J. Jensen, On the homotopy type of CW complexes with aspherical fundamental group, Topology and its Applications 153 (2006) 3000–3006.
[10] F. E. A. Johnson, Stable modules and the D(2)-problem, London Mathematical Society Lecture Notes Series 301, Cambridge University Press 2003.

[11] C. Hog-Angeloni, W. Metzler, A.J. Sieradski (Eds.), Two-Dimensional Homotopy and Combinatorial Group Theory, Cambridge University Press, 1993.

[12] J. C. Lennox and J. Wiegold, Generators and killers for direct and free products. Arch. Math. (Basel), 34(4):296-300, 1980.

[13] W. Mannan, The D(2) property for $D_8$, Algebraic & Geometric Topology, 2007, 7: 517–528.

[14] W. Mannan, Quillen's plus-construction and the D(2) problem, Algebraic & Geometric Topology, 2009, 9: 1399–1411.

[15] W. Mannan, Realizing algebraic 2-complexes by cell complexes, Math. Proc. Camb. Phil. Soc., 2009, 146: 671-673.

[16] V. D. Mazurov and E. I. Khukhro, editors. The Kourovka notebook. Russian Academy of Sciences Siberian Division Institute of Mathematics, Novosibirsk, augmented edition, 1999. Unsolved problems in group theory

[17] R. G. Swan, Minimal resolutions for finite groups, Topology 4 (1965), 193–208.

[18] C. T. C. Wall, Finiteness conditions for CW complexes, Ann. of Math. 81 (1965), 55-69.

[19] S. Ye, A unified approach to the plus-construction, Bousfield localization, Moore spaces and zero-in-the-spectrum examples, Israel Journal of Mathematics. 192 (2012), 699-717.

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076. E-mail: matjfeng@nus.edu.sg

Department of Mathematical Sciences, Xi’an Jiaotong-Liverpool University, 111 Ren Ai Road, Suzhou, Jiangsu, China 215123. E-mail: Shengkui.Ye@xjtlu.edu.cn