Schwinger–Dyson BRST symmetry
and the Batalin–Vilkovisky
Lagrangian Quantisation of Gauge Theories
with Open or Reducible Gauge Algebras

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Abstract

In this short note we extend the results of Alfaro and Damgaard on the origin of antifields to theories with a gauge algebra that is open or reducible.

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1 Introduction

In a recent paper [1], the antifields of the Lagrangian quantisation scheme of Batalin and Vilkovisky [2] were unmasked as being the antighosts of the Schwinger–Dyson BRST symmetry [3]. The latter is implemented most transparently using the so-called collective field formalism. Here it amounts to replacing the fields everywhere by the difference between the field itself and its collective partner. Of course, this introduces a new symmetry, the shift symmetry, where both the field and its collective partner are shifted by an arbitrary amount. The most general Schwinger–Dyson equations, defining the complete quantum theory, can then be obtained as Ward identities of this BRST shift symmetry. Alfaro and Damgaard showed [4] that this BRST shift symmetry can be combined with gauge symmetries originally present in the action. Fixing the collective field to zero and integrating over the ghost field of the shift symmetry, it is seen that the antighost of the shift symmetry is to be identified with the BV antifield.

In [5], it was shown how using the same collective-field technique, also the extended BRST–anti-BRST Lagrangian formalism of [6] can be recovered. There, to every field are associated three antifields, which in the collective-field approach can be identified with the ghosts and antighosts of the shift symmetry, and with the collective field itself, which in this case cannot be integrated out. The Schwinger–Dyson symmetry has also been implemented in the Hamiltonian quantisation scheme of [7], and this way it is possible to prove the equivalence of Hamiltonian and Lagrangian BRST quantisation in a direct and natural way [8].

However, all the above developments were only valid for closed, irreducible algebras. It is our purpose in this short note to show that the treatment of theories with open and/or reducible algebras does not present any particular difficulty. In contrast to the original, very algebraic derivation of BV, a more intuitive introduction of this formidable scheme becomes possible for all types of gauge theories known today.

In section 2, we briefly review the results of [1], in order to make our subsequent treatment of open and reducible algebras more accessible. The former are discussed in section 3, the latter in section 4.

2 Review of the procedure

Here, we will briefly review the steps followed in [1] to develop the antifield formalism. We start from $S_0(\phi)$, the original action. Its gauge invariances are generated by $R^i_\alpha(\phi)$, which satisfy

$$\frac{\delta^\gamma S_0}{\delta \phi^i} R^i_\alpha \epsilon^\alpha = 0. \quad (1)$$

The commutator of two gauge transformations is a linear combination of gauge transformations (see (8) below, with $E_{\alpha \beta} = 0$) for closed algebras. The naïve BRST operator can be constructed

$${\delta_N \phi^i} = R^i_\alpha \epsilon^\alpha \quad \delta_N c^\alpha = T^\alpha_\beta \epsilon^\beta$$

We will refer to this scheme as BV. For recent reviews, see [9].
\[ \delta_N \bar{c}_\alpha = b_\alpha \]
\[ \delta_N b_\alpha = 0, \]
and is nilpotent.

We start by collectively denoting the fields \( \phi^i, c^\alpha, \bar{c}_\alpha \) and \( b_\alpha \) by \( Y^\bar{a} \). The naïve BRST transformations of (2) are then summarized in the statement
\[ \delta_N Y^\bar{a} = \mathcal{R}^\bar{a}(Y^\bar{a}). \] (3)

We now introduce collective fields \( Y^\bar{a} \). The ghosts and antighosts of the corresponding Schwinger–Dyson BRST symmetry, we denote respectively by \( c^\bar{a} \) and \( Y^*_\bar{a} \). Following [1], we organise the combined naïve BRST transformations as follows:
\[ \delta_N Y^\bar{a} = c^\bar{a} \]
\[ \delta_N Y^*_{\bar{a}} = -c^\bar{a} + \mathcal{R}^\bar{a}(Y - Y) \]
\[ \delta_N b^\bar{a} = 0 \]
\[ \delta_N Y^*_\bar{a} = B_\bar{a} \]
\[ \delta_N B_\bar{a} = 0. \] (4)

Gauge-fixing the collective field to zero and gauge-fixing the original symmetries is then done by adding \( \delta_N (Y^*_\bar{a} Y^\bar{a} + \Psi(Y)) = \delta_N X \) to the original action. It was then recognised that
\[ S_{gf} = S_0(\phi - \varphi) + \delta_N X = S_{BV}(Y - Y, Y^*) + (-1)^\bar{a} B_\bar{a} Y^\bar{a} + Y^*_\bar{a} c^\bar{a} + \frac{\delta^r \Psi}{\delta Y_{\bar{a}}} c^\bar{a}. \] (5)
The statement that \( \delta_N S_{gf} = 0 \) then immediately implies that \( S_{BV} \) satisfies the BV classical master equation:
\[ \frac{\delta^r S_{BV}}{\delta Y^\bar{a}} \cdot \frac{\delta^s S_{BV}}{\delta Y^*_\bar{a}} = 0. \] (6)
The antighosts act as sources for the BRST transformation of the original symmetries, so that also the BV boundary conditions are satisfied.

Of course, when considering the path integral of the theory, the classical action might have to be supplemented by terms of higher order in \( \hbar \), because the possible non-invariance of the measure under BRST transformations may spoil the usual derivation of Ward identities etc. For closed algebras, one can invoke the formal argument that because of the invariance of the measure for the classical fields \( \phi^i \) the only possible Jacobian can come from the ghost measure. It vanishes due to the fact that \( T^\alpha_{\beta\bar{a}} = 0 \) for most of the interesting theories. All such statements, however, presuppose the introduction of a suitable regularisation scheme. The choice of the scheme has a large influence on the actual form of these contributions from the measure. We will not discuss this issue, but only restrict ourselves to the classical master equation. For a detailed treatment of the use of Pauli–Villars regularisation for studying BV, we refer to [1]. Care should be taken when following the formal arguments, which was exemplified e.g. in [10].

By integrating over the Nakanishi–Lautrup field \( B_\bar{a} \), the collective fields are fixed to zero and disappear upon integration. Finally, integrating over the ghosts \( c^\bar{a} \) of the shift symmetry the conventional gauge fixing delta-function of BV is recovered:
\[ \delta \left( \frac{\delta^r \Psi}{\delta Y^\bar{a}} + Y^*_\bar{a} \right). \] (7)
3 Open algebras

By definition, one speaks of an open algebra when the field equations of the original action $S_0$ are needed to obtain the usual structure that the commutator of two gauge transformations is a linear combination of gauge transformations:

$$\frac{\delta^r R^i_\alpha}{\delta \phi^{ij}} R^j_\beta - (-1)^{\epsilon_{\alpha \beta}} \frac{\delta^r R^i_\beta}{\delta \phi^{ij}} R^j_\alpha = 2 R^i_\gamma T^\gamma_{\alpha \beta} (a) - \frac{4}{\delta \phi^{ij}} F^{ij}_{a \beta} (a) \epsilon (a) + 1,$$  \hspace{1cm} (8)

We will consider a set of gauge generators that is irreducible. For open algebras, however, the naïve BRST operator (2) fails to be nilpotent off-shell, owing to the term proportional to field equations in (8). This prevents us from following the usual gauge-fixing procedure. Nilpotency on all the fields only holds when imposing the field equations of $S_0$. Open algebras were first encountered in the study of supergravity theories [11], and it proved possible in these cases to introduce extra auxiliary fields to construct an off-shell nilpotent BRST operator. After integrating out these auxiliary fields, one ended up with terms quartic in the ghosts, which one would not expect when applying straightforwardly the Faddeev–Popov procedure. As the existence of these auxiliary fields cannot be guaranteed in general, the quantisation procedure based on their existence may be felt as unsatisfactory.

In [12], de Wit and van Holten proposed a general method for quantising open algebras, without the need of having auxiliary fields at one's disposal. Their basic observation is the following. In order to derive, for example, Ward identities, one needs an action $S_{gf}$, which is invariant under some global transformation reflecting the original gauge invariances, the BRST transformation $\delta$. Usually, this is achieved by constructing $\delta$ such that

$$\delta S_0 = 0$$

and

$$\delta^2 = 0.$$  \hspace{1cm} (9)

It is then clear that any $S_{gf} = S_0 + \delta X$ satisfies $\delta S_{gf} = 0$, and that BRST invariant observables will be independent of $X$. Both requirements (1) can be dropped however, if one can just define an operator $\delta$ and a gauge-fixed action $S_{gf}$ such that $\delta S_{gf} = 0$, but where the former is not necessarily nilpotent and the latter need not be decomposable in $S_0 + \delta X$. This still allows for the derivation of the fundamental property $\langle \delta Y \rangle = 0$, where $\langle Z \rangle$ stands for the vacuum expectation value of an arbitrary operator $Z$.

The authors of [12] succeeded in constructing such a $\delta$ and $S_{gf}$ for open algebras, generalising the results of [13] for supergravity. Consider the fermionic function $F = \bar{c}_\alpha F^\alpha(\phi)$, where the functions $F^\alpha$ are the gauge conditions. We will refer to $F$ as the gauge fermion. Define $F_i = \frac{\delta F}{\delta \phi^{ij}}$. Then $S_{gf}$ is expanded as a power series in these $F_i$, where the linear term is given by the Faddeev–Popov quadratic ghost action. The BRST transformation that leaves this $S_{gf}$ invariant is obtained by again adding an expansion in $F_i$ to the naïve transformation laws. All coefficients in these expansions are determined by demanding $\delta S_{gf} = 0$, order by order in the $F_i$.

We will now show how the combination of this procedure with the demand that the symmetry algebra includes the Schwinger–Dyson BRST symmetry, leads straightforwardly to the antifield formalism of Batalin and Vilkovisky for open algebras. The most important aspect is the appearance of terms of order higher than one in the antifields, which of course corresponds to the expansion in powers of $F_i$ mentioned above.
We start by repeating the steps of the closed algebra case. That is, we construct the naïve BRST operator \((2)\) and extend it to include the shift symmetry, leading to \((4)\). Notice that the introduction of collective fields in no way influences the answer to the question whether the gauge algebra is open or closed, as can be seen from calculating \(\delta^2 N(Y^\bar{a} - Y^{\bar{a}})\).

Consider now
\[
F = Y^s_{\bar{a}}Y^{\bar{a}} + \Psi(Y) \tag{10}
\]
as gauge fermion. The first term is clearly such that it fixes the collective field to zero, while the second term is there to fix the original gauge symmetry. This choice for \(F\) gives
\[
\frac{\delta r F}{\delta Y^{\bar{a}}} = Y^s_{\bar{a}} \tag{11}
\]
and
\[
\frac{\delta r F}{\delta Y^\bar{a}} = \frac{\delta \Psi(Y)}{\delta Y^\bar{a}} \tag{12}
\]
as the quantities \(F_i\) in which we have to expand \(S_{gf}\) and the BRST transformations.

Following de Wit and van Holten \([12]\), we then conjecture that quantities \(M_{\bar{a}_1...\bar{a}_n}^{\bar{a}_1...\bar{a}_n}\) exist, with the properties
\[
M_{\bar{a}_1...\bar{a}_i\bar{a}_{i+1}...\bar{a}_n} = (-1)^{(\bar{a}_{i+1} + 1)(\bar{a}_i + 1)} M_{\bar{a}_1...\bar{a}_{i+1}\bar{a}_i...\bar{a}_n} \tag{13}
\]
and
\[
\epsilon(M_{\bar{a}_1...\bar{a}_n}^{\bar{a}_1...\bar{a}_n}) = \sum_i (\epsilon_{\bar{a}_i} + 1), \tag{14}
\]
such that
\[
S_{gf} = S_0(\phi - \varphi) + (-1)^{\bar{a}_{i+1}} B_{\bar{a}} Y^{\bar{a}} + Y^s_{\bar{a}} c^{\bar{a}} - Y^s_{\bar{a}} R^{\bar{a}} (Y - Y) + \sum_{n \geq 2} \frac{1}{n} Y^s_{\bar{a}_1} ... Y^s_{\bar{a}_n} M_{\bar{a}_1...\bar{a}_n} Y^s_{\bar{a}_{n+1}} (Y - Y) \tag{15}
\]
\[
+ \sum_{n \geq 2} \frac{\delta^r \Psi(Y)}{\delta Y^{\bar{a}}} c^{\bar{a}}
\]
is invariant under the BRST transformations
\[
\delta Y^{\bar{a}} = c^{\bar{a}}
\]
\[
\delta Y^\bar{a} = c^{\bar{a}} - R^{\bar{a}} (Y - Y) + \sum_{n \geq 2} Y^s_{\bar{a}_1} ... Y^s_{\bar{a}_n} M_{\bar{a}_1...\bar{a}_n} Y^s_{\bar{a}_{n+1}} (Y - Y)
\]
\[
\delta c^{\bar{a}} = 0
\]
\[
\delta Y^s_{\bar{a}} = B_{\bar{a}}
\]
\[
\delta B_{\bar{a}} = 0. \tag{16}
\]

The factor \(\frac{1}{n}\) was introduced in \((13)\) in order to make all \(B\)-dependent terms cancel each other in \(\delta S_{gf}\), as
\[
\delta \left[ \frac{1}{n} Y^s_{\bar{a}_1} ... Y^s_{\bar{a}_n} M_{\bar{a}_1...\bar{a}_n} \right] = \frac{1}{n} Y^s_{\bar{a}_1} ... Y^s_{\bar{a}_n} \delta M_{\bar{a}_1...\bar{a}_n} + (-1)^{\bar{a}_{i+1}} B_{\bar{a}_1} Y^s_{\bar{a}_2} ... Y^s_{\bar{a}_n} M_{\bar{a}_1...\bar{a}_n} \tag{17}
\]
owning to the permutation property \((13)\) of the \(M_n\). The term in \(\delta S_{gf}\) that depends on \(\Psi\) vanishes trivially, so that no non-linear terms in \(\frac{\delta \Psi}{\delta Y^\bar{a}}\) are needed. In fact, this shows that our
procedure is independent of the choice of Ψ. This is a consequence of the fact that the set of transformation rules (10) is nilpotent on \(Y^\bar{a}\) and \(c^\alpha\). Taking these two facts into account, the condition \(\delta S_{gf} = 0\) leads to the following conditions, obtained by equating order by order in the antifields \(Y^*\) to zero:

\[
\begin{align*}
(Y^*)^0 & : \frac{\delta^r S_0(Y - Y)}{\delta Y^\bar{a}} R^\bar{a}(Y - Y) = 0 \\
(Y^*)^1 & : \frac{\delta^r R^\bar{a}(Y - Y)}{\delta Y^a} R^a(Y - Y) + (-1)^{(\bar{a} + 1)} \frac{\delta^r S_0(Y - Y)}{\delta Y^\bar{a}} M_2^{\bar{a}0}(Y - Y) = 0 \\
(Y^*)^2 & : \ldots
\end{align*}
\]

In principle, this gives equations at each order in the antifields, which allow the construction of the \(M_n\). Let us only study the two above relations. The term independent of the antifields expresses the invariance of the classical action. Considering the contribution to \(\delta S_{gf}\) linear in the antifields leads to two conditions, obtained by taking for the \(\bar{a}\)-index \(\phi^i\) and \(c^\alpha\). In both cases \(Y^b\) runs over \(\phi^j\) and \(c^\beta\), so we get:

\[
\begin{align*}
0 &= \frac{\delta^r R^\bar{a}}{\delta \bar{a}} c^\alpha R^\bar{a} j^\alpha c^\beta + R^\bar{a} T^\gamma_\beta c^\gamma c^\beta + (-1)^{(\bar{a} + 1)} \frac{\delta^r S_0}{\delta \bar{a}} M_2^{\bar{a}i} \\
0 &= \frac{\delta^r T^\alpha_\beta c^\gamma c^\beta}{\delta \bar{a}} R^\alpha_i c^\mu + 2T^\alpha_\beta c^\gamma T^\mu_\gamma c^\beta + (-1)^{\alpha j} \frac{\delta^r S_0}{\delta \bar{a}} M_2^{\bar{a}j}.
\end{align*}
\]

From (8) and (20) it follows that

\[
M_2^{ij} = 2E^{ij}_{\alpha \beta} c^\beta c^\alpha.
\]

We thus see that the coefficient of the term quadratic in the antifields of the fields \(\phi^i\) is completely determined by the non-closure functions \(E^{ij}_{\alpha \beta}\) of the algebra. Furthermore, we see from (21) that

\[
M_2^{\bar{a} \alpha} = D^{\bar{a} \alpha}_{\mu \nu \sigma} c^\sigma c^\nu c^\mu.
\]

By taking \(\bar{c}^\alpha\) and \(b^\alpha\) for \(Y^\bar{a}\), we find that the corresponding \(M_2^{\bar{a} \alpha}\) and \(M_2^{\beta \alpha}\) are zero. This was to be expected, as they are introduced as trivial pairs, decoupled from the original gauge algebra. Let us now define \(S_{BV}\) and \(S_{AD}\) by writing

\[
S_{gf}(Y - Y) = S_{BV}(Y - Y, Y^*) + (-1)^{\bar{a}} B_0 Y^\bar{a} + Y^*_{\bar{a}} c^\bar{a} + \frac{\delta^r \Psi}{\delta Y^\bar{a}} c^\bar{a}
\]

\[
= S_{AD}(Y - Y, Y^*) + \frac{\delta^r \Psi}{\delta Y^\bar{a}} c^\bar{a}.
\]

It is then clear that we recover the familiar form (14) for \(S_{BV}\):

\[
S_{BV} = S_0(\phi) - Y^*_{\bar{a}} R^\bar{a}(Y) + \phi^j_{\bar{a}} \phi^i_{\bar{a}} E^{ji}_{\alpha \beta} c^\beta c^\alpha + \frac{1}{2} \phi^j_{\bar{a}} \phi^i_{\bar{a}} D^{ji}_{\mu \nu \sigma} c^\sigma c^\nu c^\mu + \ldots,
\]

where the \(\ldots\) stand for possible terms of more than quadratic order in the antifields. Notice that the terms non-linear in the antifields always contain at least one antifield \(\phi^i\), because \(S_0\) only depends on the fields \(\phi^i\).

We now remark that we can again decompose \(S_{gf} = S_{AD} + \delta \Psi(Y)\), where \(\delta S_{AD} = 0\) and because \(\delta^2 Y^\bar{a} = 0\). Notice also that

\[
\delta Y^\bar{a} = c^\bar{a} + \frac{\delta i S_{BV}}{\delta Y^\bar{a}} = \frac{\delta i S_{AD}}{\delta Y^\bar{a}}.
\]
Taking all this into account, it becomes trivial to see that

\[ 0 = \delta S_{AD} = -\frac{\delta r S_{BV}}{\delta Y^a} \frac{\delta l S_{BV}}{\delta Y_a^*}. \]  

(28)

So also in the case of open algebras, we recover the classical master equation of Batalin and Vilkovisky. The gauge-fixing prescription is again recovered. Notice that the extra terms in the BRST transformation rules invalidate even the formal arguments on the absence of possible Jacobians. Again, we will not discuss this important topic and refer to [9].

Introducing collective fields after all leads to the existence of a nilpotent BRST operator. In [1] it was shown that integrating out \( B \bar{a}, Y \bar{a} \) and \( c \bar{a} \) leads to the so-called quantum BRST operator \( \sigma X = (X, S_{BV}) - i\hbar \Delta X \), acting on quantities \( X(Y, Y^*) \). This quantum BRST operator is nilpotent since \( S_{BV} \) satisfies the master equation and because of properties of \( \Delta \). This comes as no surprise, as the BRST operator (16) is nilpotent on \( Y, Y^*, c \) and \( B \). At the classical level, this gives as BRST operator the antibracket with \( S_{BV} \), which is nilpotent. The existence of a nilpotent BRST operator is of course a less trivial result for open algebras.

Above we applied the procedure suggested by de Wit and van Holten [12] for the quantisation of open algebras. This leads us to an explicit construction of \( S_{BV} \) in (24), which is then easily seen to satisfy the classical master equation of the BV formalism. However, turning the argument around, one can see that their procedure follows uniquely when starting from the requirement of invariance of the gauge-fixed action under the Schwinger–Dyson BRST symmetry. After integrating out the collective field, the latter is given by

\[ \delta Y^{\bar{a}} = c^{\bar{a}} \]
\[ \delta c^{\bar{a}} = 0 \]
\[ \delta Y^*_{\bar{a}} = \frac{\delta l S_{gf}}{\delta Y^{\bar{a}}}. \]

(29)

From the study of the closed, irreducible gauge algebras the form of \( S_{gf} \) is then generalized to be always

\[ S_{gf} = S_{BV}(Y, Y^*) + Y^*_a c^{\bar{a}} + \frac{\delta r \Psi}{\delta Y^a} c^{\bar{a}}. \]

(30)

The requirement \( \delta S_{gf} = 0 \) then leads immediately, as shown above, to the classical master equation for \( S_{BV} \). Together with the boundary condition that the term of \( S_{BV} \) linear in the antifields acts as a source for the naive BRST transformations, leads uniquely to the de Wit–van Holten quantisation for open algebras.

Now that we have a collective-field formalism at our disposal for the derivation of the BV formalism for the quantisation of gauge theories with open algebras, we can try to extend the results of [4] to include the extended BRST-invariant quantisation of this kind of theories [4]. The natural generalization is then to allow for terms proportional to an arbitrary power in the antifields \( Y^*_{\bar{a}, a} \) both in the transformation law for the collective field \( Y^{\bar{a}} \) and in the gauge-fixed action. In the latter, also terms with an arbitrary power in the collective field can be allowed.

In [5], it was found that the collective field formalism leads to an unusual way of removing the antifield-dependent terms from the path integral when one imposes extended BRST

\(^2\)The extra index \( a \) takes the values 1, 2. It distinguishes the antifields associated with the BRST symmetry and those associated with the anti-BRST symmetry, see [6].
symmetry. In the ordinary BV scheme and in [6], this is done by just putting the antifields (including the collective field) to zero after gauge-fixing. In contrast, in the collective-field approach to extended BRST symmetry, the collective field itself is fixed to zero (this is at the heart of this approach), but the two antifields $Y^*_{\bar{a}a}$ are removed by a Gaussian integration. This is possible because the antifields only appear as linear source terms for the extended BRST transformations. We try to maintain this procedure for open algebras, i.e. we try the decomposition

$$S_{gf} = S_{BLT} + B\bar{a}(-1)^{\bar{a}+1}M_{\bar{a}b}Y^b - \frac{1}{2} e^{ab}Y^*_{\bar{a}a}M_{\bar{a}b}Y^*_{\bar{b}b} - \frac{1}{2} e^{ab}\mathcal{R}^\alpha_{\bar{a}}M_{\bar{a}b}\mathcal{R}^b_\alpha. \tag{31}$$

For closed algebras, the variation of the last term under a (anti-)BRST transformation is cancelled by the variation of the term linear in the collective field in $S_{BLT}$. The latter is the source term for the composition of a BRST transformation with an anti-BRST transformation. For open algebras this is no longer true, a term proportional to the non-closure functions of the algebra appears in the variation of the former. Also, owing to the non-linear terms, the gauge-fixing of the $Y^*_{\bar{a}a}$ would also no longer be a Gaussian integral. Both symptoms seem to indicate that still more terms need to be introduced, even terms independent of antifields. One, however, has no guiding principle when doing so. It thus seems very difficult to make contact with [6] in the case of an open gauge algebra.

4 Reducible gauge algebras

In [4], Batalin and Vilkovisky gave the first complete prescription for the quantisation of gauge theories with arbitrary reducible gauge algebras. Nevertheless, for the sake of completeness, we will also discuss this case using the collective-field approach. There are two aspects to the problem of quantising reducible gauge theories. One is the construction of the BRST operator and the ghost spectrum, and the other is the judicious choice of the gauge fermion. We will only discuss the former, the latter being extensively treated in [4]. The collective field formalism has no bearing on the construction of the gauge fermion, which is always considered to be available.

We start again from a classical action $S_0(\phi)$, which has gauge symmetries generated by $m$ operators $R^i_\alpha$. Suppose now that $k$ quantities $Z^\alpha_{\bar{a}}(\phi)$ exist, such that

$$R^i_\alpha Z^\alpha_{\bar{a}} = 0. \tag{32}$$

This is just the expression that the original set of generators was redundant, i.e. that not all $R^i_\alpha$ are independent. If the $k Z^\alpha_{\bar{a}}$ are all independent, then one speaks of a first-stage-reducible theory. Effectively, there are then $m - k$ independent gauge symmetries. The functions $Z^\alpha_{\bar{a}}$ are not necessarily independent, however, leading to second-stage-reducible theories and so on. Here, we will only treat the first-stage theories explicitly, higher-state theories allow for analogous constructions.

If one has not noticed the redundancy in the set of gauge generators, the naïvely gauge fixed action

$$S_{gf} = S_0 + \bar{c}_\alpha \delta^\alpha f^\alpha(\phi)R^i_\alpha c^i + f^\alpha b_\alpha \tag{33}$$

has to be modified to

$$S_{gf} = S_0 + \bar{c}_\alpha \delta^\alpha f^\alpha(\phi)R^i_\alpha c^i + f^\alpha b_\alpha + \frac{1}{2} e^{ab}\mathcal{R}^\alpha_{\bar{a}}M_{\bar{a}b}\mathcal{R}^b_\alpha. \tag{31}$$

The matrix $M_{\bar{a}b}$ is needed for gauge-fixing purposes, as is described in extenso in [4]. Again, the unbarred indices $a, b$ take the values 1, 2; $R^i_a$ denotes the BRST transformation of $Y^a$, while $\mathcal{R}^\alpha_2$ denotes its anti-BRST transformation.
turns out to have a gauge symmetry in the ghost sector:

\[ \frac{\delta^r S_0}{\delta c^\alpha} Z_\alpha^\alpha c^\alpha = 0, \] (34)

owing to the reducibility relations (32). Of course, this symmetry can be fixed by introducing so-called ghosts for ghosts \( \eta^\alpha \), adding to the BRST transformations of \( c^\alpha \) a term \( Z_\alpha^\alpha \eta^\alpha \), as the functions \( Z_\alpha^\alpha \) are the gauge generators of this gauge symmetry in the ghost action.

The ghost action in (33) has however another gauge symmetry, where the \( \check{c}_\alpha \) are the gauge fields. This symmetry is generated by the left zero modes of \( \frac{\delta f^\alpha}{\delta \phi^i} R_\alpha^i \). These generators clearly depend on the choice of the \( f^\alpha \), and they do not enter the gauge algebra. Their gauge fixing does not require more than trivial modifications \(^4\) of the BRST transformations.

So, the situation can be summarized by saying that a nilpotent BRST operator exists:

\[
\begin{align*}
\delta \phi^i & = R_\alpha^i c^\alpha \\
\delta c^\alpha & = T_\alpha^\beta c^\beta + Z_\alpha^\alpha \eta^\alpha \\
\delta \eta^\alpha & = A_{\mu \nu \sigma}^\alpha c^\mu c^\nu c^\sigma,
\end{align*}
\] (35)

supplemented with some trivial systems, introducing antighosts, such that a suitable gauge fermion can be constructed, satisfying the requirements of [14]. The functions \( F \) and \( A \) are determined from the nilpotency requirement of \( \delta \) on \( c^\alpha \). Everything is then just the same as for the case of irreducible, closed algebras, as far as the collective field-formalism is concerned. Specifically, an extended action linear in the antifields is obtained, where the antifields act as sources for (33).

However, even if the gauge algebra itself is closed, the BRST operator (33) may not necessarily be nilpotent off-shell, because terms proportional to the field equations of \( S_0 \) can appear in the reducibility relations (32):

\[ R_\alpha^i Z_\alpha^\alpha - 2 \frac{\delta^r S_0}{\delta \phi^i} B^{ij}_{\alpha} (-1)^i = 0. \] (36)

The procedure to be followed is then the same as for open algebras, as described above. For instance, (20) gets an extra term \( R_\alpha^i Z_\alpha^\alpha \eta^\alpha \), leading to an extra contribution \( -2 B^{ij}_{\alpha} \eta^\alpha \) to \( M_{ij}^i \), so that finally a term \(-\phi^i_{\alpha} \phi^j_{\beta} B^{ij}_{\alpha} \eta^\alpha \) appears in \( S_{BV} \). Other extra terms follow in exactly the same way from (21).

5 Conclusion

We have thus shown that a collective-field derivation exists of the BV quantisation scheme for theories with open and/or reducible gauge algebras. The presence of terms proportional to field equations in the original algebra and/or the reducibility relations necessitates an approach like in [12]. Demanding Schwinger–Dyson BRST, which leads to the BV classical master-equation straightforwardly, leads to this generalisation of classical BRST in a natural way.

\(^4\) By trivial modifications of the BRST transformations we mean that one can always add pairs of fields \( A \) and \( B \), such that \( \delta A = B \) and \( \delta B = 0 \), without changing the physical content of the BRST cohomology. Such a pair of fields is called a trivial system.
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