THE FLEX DIVISOR OF A K3 SURFACE

VALERY ALEXEEV AND PHILIP ENGEL

ABSTRACT. The flex divisor $R_{\text{flex}}$ of a primitively polarized K3 surface $(X, L)$ is, generically, the set of all points $x \in X$ for which there exists a pencil $V \subset |L|$ whose base locus is $\{x\}$. We show that if $L^2 = 2d$ then $R_{\text{flex}} \in |n_dL|$ with

$$n_d = \frac{(2d)!((2d+1)!)}{d^2(d+1)!^2} = (2d+1)C(d)^2,$$

where $C(d)$ is the Catalan number. We also show that there is a well-defined notion of flex divisor over the whole moduli space $F_{2d}$ of polarized K3 surfaces.

1. Introduction

Let $(X, L)$ be a primitively polarized K3 surface of degree $2d$. Recent work of the authors on compactification of the moduli space $F_{2d}$ of such surfaces has highlighted the importance of a canonical choice of polarizing divisor: An algebraically varying choice of divisor $R \in |nL|$ on the generic polarized K3 surface. If this choice of divisor extends over all of $F_{2d}$ then it gives rise to a modular compactification

$$F_{2d} \hookrightarrow \overline{F}_{2d}^R.$$

The compactification is constructed by taking the closure of the space of pairs $(X, \epsilon R)$ in the moduli space of stable slc pairs, for some small $\epsilon > 0$.

By the main theorem of [AE21], the normalization of $\overline{F}_{2d}^R$ is semitoroidal whenever $R$ satisfies a property called recognizability. Thus, the search for modular toroidal compactifications of $F_{2d}$ is intimately related to finding canonical choices of polarizing divisor, and verifying that those choices are recognizable.

One infinite series of divisors, ranging over all degrees $2d$, is the rational curve divisor. On a generic $F_{2d}$ it can be concretely thought of as the set of points $x \in X \subset \mathbb{P}^g$ for which there exists a flex space: A codimension 2 linear subspace of $\mathbb{P}^g$ intersecting $X$ at only the point $x$. 

Claire Voisin suggested to the authors a second series of divisors, which we call here the flex divisor $R_{\text{flex}}$. It was first considered by Welters [Wel81] for a quartic K3 surface, who called it the curve of hyperflexes. On the generic $(X, L)$ it is defined as the set of all points $x \in X$ for which there exists a pencil $V \in |L|$ whose set-theoretic base locus is $\{x\}$. When $|L|$ defines an embedding $X \hookrightarrow \mathbb{P}^g$ with $g = d + 1$, which is the case for generic $(X, L) \in F_{2d}$ when $d \geq 2$, the flex divisor can be concretely thought of as the set of points $x \in X \subset \mathbb{P}^g$ for which there exists a flex space: A codimension 2 linear subspace of $\mathbb{P}^g$ intersecting $X$ at only the point $x$. 

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Our first result hints towards a positive answer on the question of whether $R_{\text{flex}}$ is recognizable. Concretely, we show:

**Theorem 1.1.** There is a canonical choice of divisor $R_{\text{flex}}$ varying algebraically over all of $\mathbb{F}_{2^d}$ and giving the flex divisor on the generic K3 surface $(X, L)$.

The theorem is not obvious, because it is not clear if the flex points form a subvariety of $X$ of the expected dimension, which is one. Additionally, the flex divisor may have multiple components and one must determine their multiplicities. Perhaps most importantly, sometimes points in $R_{\text{flex}}$ as in Theorem 1.1 are not flex under the naive definition! This occurs on a quartic surface containing a line—the points on the line are not flex according to the naive definition because the relevant pencil $V$ contains the whole line as a base curve. But the line appears as a component of the flex divisor, see Example 3.14.

The flex divisor is notably an example of a constant cycle curve [Huy14]: One whose points all have the same class in the Chow group of zero-cycles $\text{CH}_0(X)$. The method of proof of Theorem 1.1 suggests strongly:

**Conjecture 1.2.** Let $R$ be a canonical choice of polarizing divisor for $\mathbb{F}_{2^d}$. If $R$ is a constant cycle curve, then it is recognizable.

A resolution of this conjecture would unify various results about recognizable divisors, such as [AET19], [ABE20], [AE21], and [AEH21].

Our second result is an analogue of the Yau-Zaslow formula. That is, we determine in what multiple of the polarization the flex divisor lives, generalizing known results in the cases $d = 1, 2$.

**Theorem 1.3.** Let $(X, L)$ be a K3 surface of degree $2d$. Then $R_{\text{flex}} \in |n_dL|$ with

$$n_d = \frac{(2d)!(2d+1)!}{d!(d+1)!^2} = (2d+1)C(d)^2,$$

where $C(d)$ is the Catalan number.

| $d$  | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $n_d$| 3   | 20  | 175 | 1764| 19404| 226512| 2760615| 34763300| 449141836|

Table 1. Flex divisor classes

Table 1 tabulates the first nine values of $n_d$. The value $n_1 = 3$ is well-known, see Example 3.13 while the value $n_2 = 20$ has been computed by various authors [Huy14 Prop. 8.8], [Wit14 Cor. 2.4.6].

The summary of the paper is as follows: Section 2 shows that the flex divisor is well-defined and extends to a divisor over all of $\mathbb{F}_{2^d}$ and Section 3 computes the multiple $n_d$ of the primitive polarization in which the flex divisor lives, using intersection theory on the Hilbert scheme $X^{[2d]}$ of the K3 surface.

2. WELL-DEFINEDNESS

**Definition 2.1.** We say $(X, L)$ is a polarized K3 surface of degree $2d$ if $X$ is a K3 surface with ADE singularities, and $L \to X$ is a primitive, ample line bundle satisfying $L^2 = 2d$.

Let $F_{2d}$ denote the moduli stack of polarized K3 surfaces over $\mathbb{C}$. It is a smooth, irreducible, Deligne-Mumford stack of dimension 19.
Definition 2.2. A point $x \in X$ is flex if there exists a pencil $V \subset |L|$ whose base locus is the singleton $\{x\}$.

Let $L_{K3}$ denote the unique even, unimodular lattice of signature $(3,19)$ and fix a primitive vector $v \in L_{K3}$ of norm $v^2 = 2d$. Define

$$
\mathbb{D} := \mathbb{P}\{x \in v^\perp \otimes \mathbb{C} \mid x \cdot x = 0, x \cdot \overline{x} > 0\} \text{ and}
$$

$$
\Gamma := \{\gamma \in O(L_{K3}) \mid \gamma(v) = v\}.
$$

By the Torelli theorem, the coarse space of $F_{2d}$ is the arithmetic quotient $\mathbb{D}/\Gamma$.

Definition 2.3. A Heegner divisor in $\mathbb{D}/\Gamma$ is the image of a hyperplane section $w^\perp \cap \mathbb{D}$ for some vector $w \in L_{K3} \setminus \mathbb{Z}v$.

Proposition 2.4. Let $S \subset F_{2d}$ denote the polarized K3 surfaces $(X,L)$ for which there exists a pencil $V \subset |L|$ containing a base curve. Then $S$ is contained in a finite union of Heegner divisors.

Proof. The condition that $|L|$ contain a pencil with a base curve is an algebraic condition, which is easily seen to be closed on $F_{2d}$.

Let $C$ be the base curve of a pencil $V \subset |L|$. Fix $H \in V$ and note $H = C + D$ for some non-empty effective divisor $D$. Since $L$ is ample, we have $0 < L \cdot C < 2d$. Thus $|C| \notin ZL$. By the primitivity of $L$, the rank of $\text{Pic}(X)$ is least 2. Hence any point of $S$ lies in some Heegner divisor. Since $S$ is algebraic, we conclude that $S$ is contained in a finite union of Heegner divisors.

Lemma 2.5. Let $(X,L)$ be a polarized K3 surface. The flex points $\{x \in X \mid x \text{ flex}\}$ form a constructible subset of $X$ of dimension at most 1.

Proof. Constructibility is elementary. To show the second statement, it suffices to make the following observation: Any flex point $x \in X$ lies in the Beauville-Voisin class $[x] = c_X \in CH_0(X)$, defined in [BV04] as the class of any point on a rational curve in $X$. This follows because:

1. $2d[x] = H_1 \cdot H_2$ for hyperplane sections $H_1, H_2$ spanning the pencil $V$,
2. the intersection of two curves is some multiple of $c_X$ [BV04] Thm. 1, and
3. $CH_0(X)$ is torsion-free [Roj80].

If a Zariski-open subset of points of $X$ were flex, we would have that $CH_0(X) = \mathbb{Z}$, contradicting Mumford’s theorem [Mum69] on the uncountability of the Chow group. So the constructible set of flex points has dimension at most 1.

Given a smooth surface $X$, denote by $X^{[k]}$ the Hilbert scheme of $k$ points on $X$. Let $F_{2d}^{\text{sing}}$ denote the substack of $F_{2d}$ parameterizing singular ADE K3 surfaces, which is also a finite union of Heegner divisors.

Definition 2.6. Define the Zariski open subset $T := F_{2d} \setminus (S \cup F_{2d}^{\text{sing}})$. We assume for the remainder of the text that $(X,L) \in T$, unless otherwise stated.

Let $G := \text{Gr}(g-1, g+1)$ be the Grassmannian of codimension 2 linear spaces in $H^0(X,L)^*$, or equivalently pencils in $|L|$. Consider the map

$$
i : G \to X^{[2d]}, \quad V \mapsto PV \cap X$$

sending a codimension 2 linear space to its scheme-theoretic intersection with $X$, or equivalently sending a pencil to its scheme-theoretic basic locus.
Proposition 2.7. The mapping \( i : G \to X^{[2d]} \) is a closed immersion.

Proof. First, we show that \( i \) is a set-theoretic embedding. Suppose, for the sake of contradiction, that two pencils \( PV_1 \cap X = PV_2 \cap X \) intersect at the same length \( 2d \) subscheme \( Z \subset X \). Consider the set of codimension 2 linear spaces

\[ P := \{ V \in G \mid V \supset V_1 \cap V_2 \}. \]

We necessarily have that \( PV \cap X = Z \) for all \( V \in P \) because \( Z \subset PV_1 \cap PV_2 \cap X \). Hence \( i(P) \) consists of a single point. Since \( P \) contains a curve, we conclude that \( i \) contracts a curve. But any morphism from a Grassmannian to a projective variety contracting a curve must be constant. So \( i \) is constant, which is absurd.

Next, we show that the differential \( di \) is injective. Recall that the tangent space \( TV_G = \text{Hom}(V, H^0(X, L)^*/V) \) whereas \( T[Z]X^{[2d]} = \text{Hom}(I_Z/I^2_Z, O_Z) \). We may write \( PV = \{ x \in \mathbb{P}^g \mid s_1(x) = s_2(x) = 0 \} \) for two sections \( s_1, s_2 \in H^0(X, L) \). A tangent vector \( TV_G \) can be represented as the vanishing locus of \( (s_1 + et_1, s_2 + et_2) \), where \( t_1, t_2 \in H^0(X, L)/\mathbb{C}s_1 \oplus \mathbb{C}s_2 \). Then \( di \) maps it to \( (s_1 \mapsto t_1|_Z, s_2 \mapsto t_2|_Z) \), which uniquely determines an element of \( \text{Hom}(I_Z/I^2_Z, O_Z) \) because \( I_Z = (s_1, s_2) \).

Supposing some nonzero \( \phi \in TV_G \) satisfies \( di(\phi) = 0 \), at least one of \( t_1, t_2 \in H^0(X, L)/\mathbb{C}s_1 \oplus \mathbb{C}s_2 \) is nonzero and satisfies \( t_i|_Z = 0 \). So \( Z \) is contained in the codimension 3 linear space \( \{ x \in \mathbb{P}^g \mid s_1(x) = s_2(x) = t_i(x) = 0 \} \). But then the argument of the first paragraph applies to show \( i \) is constant. Contradiction. \( \square \)

Consider the Hilbert-Chow morphism \( HC : X^{[2d]} \to X^{(2d)} \). Let \( \Delta \subset X^{(2d)} \) be the small diagonal of effective zero cycles of the form \( 2d[x] \) for some \( x \in X \). Define a subscheme \( P_{2d} \subset X^{[2d]} \) as the scheme-theoretic fiber \( P_{2d} := HC^{-1}(\Delta) \). Let

\[ \text{supp} : P_{2d} \to X \]

be the support morphism, sending a scheme to the point at which it is supported. Finally, let \( i(G) \subset X^{[2d]} \) denote the image of the Grassmannian \( G = \text{Gr}(g - 1, g + 1) \) under the morphism \( i \), endowed with its natural structure of a reduced, smooth subscheme. Finally, we may now describe the flex divisor, at least generically.

Definition 2.8. The flex divisor on a K3 surface \((X, L) \in T\) is the algebraic cycle

\[ R_{\text{flex}} := \text{supp}_* [P_{2d} \cap i(G)] \]

Here \( \text{supp}_* \) denotes the proper pushforward of algebraic cycles, and the brackets \([\cdot]\) denote the effective algebraic cycle underlying a subscheme.

Note that the cycle class is being taken in \( P_{2d} \) to make \( \text{supp}_* \) sensible.

Lemma 2.9. The subschemes \( P_{2d} \) and \( i(G) \) intersect properly in \( X^{[2d]} \), i.e. their intersection has pure dimension 1. Furthermore, \( [P_{2d}] \cdot [i(G)] = [P_{2d} \cap i(G)]_{X^{[2d]}} \).

Proof. We have that \( i(G) \subset X^{[2d]} \) is a smooth subscheme of dimension \( 2d \). By a result of Haïm [Hai08, Prop. 2.10], \( P_{2d} \subset X^{[2d]} \) is a reduced and irreducible Cohen-Macaulay scheme of dimension \( 2d + 1 \). Hence, each component of their intersection has dimension at least 1. We claim additionally that each component has dimension at most 1. Note \( \text{supp}(P_{2d} \cap i(G)) \subset X \) by Lemma 2.5.

The restriction of \( \text{supp} \) to \( P_{2d} \cap i(G) \) contracts no curves because no flex point \( x \in X \) has a curve-worth of flex spaces: If \( x \in X \) supported a curve-worth of flex spaces, the morphism \( HC \circ i : G \to X^{(2d)} \) would contract a curve and hence, as
before, $G$ would collapse to a point in $X^{(2d)}$. This is absurd. So $\text{supp}|_{P_{2d} \cap i(G)}$ is finite onto its image in $X$, which has dimension at most 1.

Hence, each component of $P_{2d} \cap i(G)$ has dimension exactly 1, that is, $P_{2d}$ and $i(G)$ intersect properly. Since $i(G)$ is smooth and $P_{2d}$ is Cohen-Macaulay, [Ful16 Prop. 7.1] gives the second statement. \hfill \Box

**Remark 2.10.** The proof of Lemma 2.9 implies that every component of the scheme $P_{2d} \cap i(G)$ contributes nontrivially to $R_{\text{flex}}$. Hence, for $(X, L) \in T$, $R_{\text{flex}}$ is, as a set, exactly the set of flex points.

**Proposition 2.11.** Let $u : \mathfrak{X} \to T$ be the restriction of the universal family of polarized K3 surfaces. Then the flex divisors $R_{\text{flex}} \subset \mathfrak{X}$ form a flat subfamily of curves, specializing to $R_{\text{flex}}$ on any fiber $X = \mathfrak{X}_t$.

**Proof.** It suffices to relativize the construction of $R_{\text{flex}}$ and check flatness of the resulting family of algebraic cycles.

Let $\mathfrak{G}$ be the relative Grassmannian of codimension 2 linear subspaces of $\mathbb{P}(u_*, \Sigma)^*$ where $\Sigma \to \mathfrak{X}$ is the universal polarization. Let $\mathfrak{X}^{[2d]}$ be the relative Hilbert scheme of $2d$ points, and let $\mathfrak{P}_{2d}$ be the subfamily of the relative Hilbert scheme consisting of schemes supported at a single point of the fiber and $i$ the relative inclusion $\mathfrak{G} \hookrightarrow \mathfrak{X}^{[2d]}$. Let $\text{supp} : \mathfrak{P}_{2d} \to \mathfrak{X}$ be the relative support morphism. Consider the algebraic cycle

$$R_{\text{flex}} := \text{supp}_* [\mathfrak{P}_{2d} \cap i(\mathfrak{G})] \subset \mathfrak{X}.$$  

This cycle is a divisor in the smooth DM stack $\mathfrak{X}$. Any fiber $X = \mathfrak{X}_t \hookrightarrow \mathfrak{X}$ intersects $R_{\text{flex}}$ properly by Lemma 2.9. Hence $R_{\text{flex}}$ forms a flat family of divisors in $\mathfrak{X}$.

It remains to show that $R_{\text{flex}}$ specializes to $R_{\text{flex}}$ as defined above on a fiber $X = \mathfrak{X}_t$. The pushforward $\text{supp}_*$ of algebraic cycles and the cycle class map $[\cdot]$ commute with taking fibers over $t$ because the fibers $\mathfrak{X}^{[2d]}_{\mathfrak{X}_t} \hookrightarrow \mathfrak{X}^{[2d]}$ are smoothly immersed and properly intersecting the cycles $\mathfrak{G}$ and $\mathfrak{P}_{2d}$. Hence, $(R_{\text{flex}})_t = R_{\text{flex}}$. \hfill \Box

**Question 2.12.** For a sufficiently generic $(X, L) \in F_{2d}$ is $R_{\text{flex}}$ an irreducible divisor? What is its geometric genus, generically?

**Remark 2.13.** Based off [Wel81], Huybrechts [Huy14 Prop. 8.8] shows that when $L^2 = 4$, $R_{\text{flex}} \in [20L]$ is generically irreducible of geometric genus 201. Strangely, this is the genus of a smooth element of $[10L]$. This is not an error: $R_{\text{flex}}$ is generically singular for a quartic surface.

We recall now the notion of a constant cycle curve:

**Definition 2.14.** Let $X$ be a smooth K3 surface, and let $R \subset X$ be a curve. We say that $R$ is a constant cycle curve if every point $p \in R$ represents the same class in $\text{CH}_0(X)$. This definition extends to curves $R \subset X$ in an ADE K3 surface by taking the inverse image of $R$ in the minimal resolution of $X$.

It is known that if $R$ is constant cycle, then $[p] = c_X \in \text{CH}_0(X)$ for all $p \in R$.

**Lemma 2.15.** For $(X, L) \in T$, the divisor $R_{\text{flex}}$ is a constant cycle curve.

**Proof.** This follows immediately from Remark 2.10 and items (1), (2), (3) in the proof of Lemma 2.9. \hfill \Box

**Lemma 2.16.** Let $X \to (C, 0)$ be a family of polarized K3 surfaces and let $R \subset X$ be a flat family of curves over $C$. Suppose that $R_t$ is a constant cycle curve for all $t \neq 0$. Then $R_0 \subset X_0$ is also a constant cycle curve.
Proof. Replacing $\mathcal{X}$ with a finite base change, there is a simultaneous resolution of singularities which is the minimal resolution on any fiber. So we may assume $\mathcal{X} \to (C, 0)$ is smooth. Any two points $p, q \in R_0$ can be realized as specializations of points over a finite extension of $\mathbb{C}(C)$. The lemma follows because the specializations of rationally equivalent cycles are rationally equivalent [Ful16 Cor. 20.3]. □

Theorem 2.17. Let $u : \mathcal{X} \to F_{2d}$ be the universal K3 surface, $T \subset F_{2d}$ a Zariski open subset, and let $R^* \subset \mathcal{X}^* := \mathcal{X}|_T$ be a flat family of divisors, which is a constant cycle curve $R = R_t$ on every fiber $X = \mathcal{X}_t$. Then $R^*$ extends to a flat family of divisors $R$ over the universal K3 surface $\mathcal{X} \to F_{2d}$.

Proof. Let $\mathcal{L}$ be an extension of $\mathcal{O}_{\mathcal{X}^*}(R^*)$ to $\mathcal{X}$ and define the projective bundle $\mathbb{P}(\mathcal{L}) \to F_{2d}$. By assumption, we have a section of $\mathbb{P}(\mathcal{L})$ over the open subset $T$ defined by $R^*$. Let $0 \in F_{2d} \setminus T$. Given any arc $(C, 0) \subset F_{2d}$ with $C \setminus \{0\} \subset T$, there is a unique flat family of curves $R \subset \mathcal{X}_{\{0\}}$ extending $R^*|_{C \setminus \{0\}}$.

By Lemma 2.16 the central fiber $R_0$ is constant cycle. As noted in [Huy14 Sec. 2.3], Mumford’s theorem [Mum69] implies constant cycle curves are rigid. So the flat limit $R_0$ doesn’t deform as the arc $(C, 0)$ deforms. Since $F_{2d}$ is smooth, in particular normal, we conclude by a well-known argument [AE19 Lem. 3.16] that the section of $\mathbb{P}(\mathcal{L})$ over $T$ extends, as a morphism, over 0. The result follows. □

Corollary 2.18. $R_{\text{flex}}$ extends to a flat family of divisors in the universal K3 surface over $F_{2d}$.

3. Degree of the Flex Divisor

In this section, we compute the degree of the flex divisor. We follow [EG00] as a general reference on the cohomology of Hilbert schemes.

Definition 3.1. Let $n > 0$ be a positive integer and let $\alpha \in H^*(X)$ be a cohomology class of pure degree. Define

$$\mathbb{L} := \bigoplus_{m, k \geq 0} H^m(X^{[k]})$$

The Nakajima (raising) operator $q_{-n}(\alpha) : \mathbb{L} \to \mathbb{L}$ is defined by the following correspondence: Let $a \geq 0$ and define $b := a + n$. Let $X^{[a, b]}$ be the incidence correspondence of nested pairs of zero-dimensional subschemes $Z_1 \subset Z_2 \subset X$ for which $\text{len}
Z_1 = a$ and $\text{len}
Z_2 = b$, and let $\pi_a$ and $\pi_b$ be the projections to $X^{[a]}$ and $X^{[b]}$. Let $S$ be the residual support morphism $X^{[a, b]} \to X^{(n)}$ sending

$$S : (Z_1, Z_2) \mapsto \supp(Z_2) - \supp(Z_1)$$

and let $W_{a, b} \subset S^{-1}(\Delta)$ be the irreducible component of $S^{-1}(\Delta)$ which is the Zariski closure of the $Z_1 \subset Z_2$ for which $\supp(Z_1)$ and $\supp(Z_2) - \supp(Z_1)$ are disjoint. Let $s : W_{a, b} \to \Delta \cong X$ denote the restriction of $S$ and let $t : W_{a, b} \to X^{[a]}$ be the inclusion. Then for any $c \in H^r(X^{[a]})$ we define

$$q_{-n}(\alpha)(c) := (\pi_b)_* (\pi_a^* c \cdot t_* s^* \alpha) \in H^{r+2n-2+\deg \alpha}(X^{[b]}).$$

By definition, we declare $H^*(X^{[0]}) = \mathbb{C}1$ where 1 is called the vacuum element.

The bidegree of the operator $q_{-n}(\alpha)$ is $(2n - 2 + \deg \alpha, n)$, where the first degree is cohomological degree, and the second is number of points.
Remark 3.2. Definition \([3.1]\) can be intuitively rephrased as follows: The operator \(q_{-n}(\alpha)\) takes a family of subschemes of length \(n\) (i.e. a cycle in \(X^{[n]}\)) and tacks on a subscheme of length \(n\) supported at a single point lying on the cycle \(\alpha\).

**Theorem 3.3** (Nakajima \cite{Nak97}, Grojnowski \cite{Gro96}). Let \(\{e_i\}_{i=1}^{24}\) be a basis of \(H^*(X)\). Then \(q_{-n_1}(e_{i_1}) \cdots q_{-n_k}(e_{i_k}) \mathbf{1}\) (up to reordering operators) are a basis of \(L\).

More precisely, these Nakajima operators extend to an action of the Heisenberg algebra of \(H^*(X)\) on \(L\), which becomes identified with the bosonic Fock space.

**Remark 3.4.** It follows directly from the definition of the Nakajima operators that \([P_{2d}] = q_{-2d}(\mathbf{1})\). Similarly, the schemes supported on a single point of a hyperplane section \(H \subset X\) have class \(q_{-2d}(h)\mathbf{1}\), with \([H] = h \in H^2(X)\).

**Lemma 3.5.** The degree of the flex divisor is \(\deg(i^*q_{-2d}(h))\).

**Proof.** By push-pull formula,

\[
\deg(R_{\text{flex}}) = R_{\text{flex}} \cdot X H := \text{supp}_*[P_{2d} \cap i(G)] \cdot X H = [P_{2d} \cap i(G)] \cdot P_{2d} \supp^* H
\]

Thus \(i^*(q_{-1}(1)2d-1) = 2dq_{-2d}(h)\mathbf{1}\). Then we can apply Lemma \([3.3]\). The first step is set-theoretically clear; the intersection multiplicity \(2d\) follows quickly from the description of the ring structure on \(H^*(X^{[2d]})\) due to Lehn and Sorger \cite{LS03} Thm. 1.1 and Prop. 2.13.

To verify the second step, note that \(q_{-1}(1)2d-1\) is represented by the divisor \(D_H \subset X^{[2d]}\) of schemes whose support intersects \(H \subset X\). Thus \([i^{-1}(D_H)]\) represents \(i^*(q_{-1}(1)2d-1)\). But \(i^{-1}(D_H)\) is simply the locus of codimension 2 linear spaces passing through some point of \(H\). Since \([H]\) is \(2d\) where \(\ell\) is the line class in \(\mathbb{P}^9\), we conclude that \([i^{-1}(D_H)] = 2d\sigma_1\).

Let \(Z \subset X^{[2d]} \times X\) denote the universal subscheme of length \(2d\). Let \(Z_G \subset G \times X\) denote the restriction of this subscheme to \(G\) (along the inclusion \(i\)). Let \(\pi_{X^{[2d]}}\) and \(\pi\) denote the projections from \(X^{[2d]} \times X\) and \(G \times X\) to the first factor, respectively. The tautological bundle \(\mathcal{O}^{[2d]} \rightarrow X^{[2d]}\) is the pushforward \((\pi_{X^{[2d]}})_*\mathcal{O}_Z\) and is a vector bundle of rank \(2d\) on \(X^{[2d]}\). Let \(\mathcal{O}^{[2d]}_G := i^*\mathcal{O}^{[2d]}\) denote the restriction of this vector bundle to the Grassmannian \(G\).

**Proposition 3.6.** The degree of the flex divisor is \(\sigma_1 \cdot i^*q_{-2d}(\mathbf{1})\).

**Proof.** The first step is to verify the intersection product

\[
q_{-1}(h)q_{-1}(1)2d-1 \cdot q_{-2d}(1)\mathbf{1} = 2dq_{-2d}(h)\mathbf{1}
\]
on \(X^{[2d]}\) and the second step is to verify that \(i^*(q_{-1}(h)q_{-1}(1)2d-1) = 2d\sigma_1\). Then we can apply Lemma \([3.5]\).

Let \(Z \subset X^{[2d]} \times X\) denote the universal subscheme of length \(2d\). Let \(Z_G \subset G \times X\) denote the restriction of this subscheme to \(G\) (along the inclusion \(i\)). Let \(\pi_{X^{[2d]}}\) and \(\pi\) denote the projections from \(X^{[2d]} \times X\) and \(G \times X\) to the first factor, respectively. The tautological bundle \(\mathcal{O}^{[2d]} \rightarrow X^{[2d]}\) is the pushforward \((\pi_{X^{[2d]}})_*\mathcal{O}_Z\) and is a vector bundle of rank \(2d\) on \(X^{[2d]}\). Let \(\mathcal{O}^{[2d]}_G := i^*\mathcal{O}^{[2d]}\) denote the restriction of this vector bundle to the Grassmannian \(G\).

**Proposition 3.7.** We have \(i^*q_{-2d}(1) = -2dq_{2d-1}(\mathcal{O}^{[2d]}_G)\).

**Proof.** Applying \([EG00]\) Thm. 12.4 to the line bundle \(\mathcal{O}\) gives the formula

\[
\sum_n c(\mathcal{O}^{[n]}) = \exp \left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} q_m(c(\mathcal{O})) \right).
\]
Note $c(O) = 1$ and that $q_{-m}(1)$ has bidegree $(2m - 2, m)$. So the only term on the right-hand side landing in $H^{2n-2}(X^{[n]})$ is $(-1)^{n-1}q_{-n}(1)$. We conclude

$$t^*q_{-2d}(1) = -2d t^*c_{2d-1}(O^{[2d]}) = -2d c_{2d-1}(O^{[2d]}_G)$$

which follows via commutativity of taking Chern classes with pullback. \[\square\]

Let $Q$ denote the rank 2 universal quotient bundle on $G$. To compute the Chern class $c_{2d-1}(O^{[2d]}_G)$ we make use of the following exact sequence:

**Proposition 3.8.** There is a resolution of $O_{Z_G}$ by vector bundles on $G \times X$:

$$0 \to \det(Q^*) \boxtimes (-2L) \to Q^* \boxtimes (-L) \to O \to O_{Z_G} \to 0.$$  

**Proof.** This exact sequence is simply the global version of the Koszul resolution of $O_{X \cap FV}$ where $\mathbb{F}V = \{x \in \mathbb{F}^g \mid s_1(x) = s_2(x) = 0\}$ is a codimension 2 linear space:

$$0 \to (s_1s_2) \to (s_1) \oplus (s_2) \to O_X \to O_{X \cap FV} \to 0.$$

On a given fiber of $\pi$ the restrictions of $\det(Q^*) \boxtimes (-2L)$ and $Q^* \boxtimes (-L)$ are $(s_1s_2)$ and $(s_1) \oplus (s_2)$ respectively, because $Q^* = \mathbb{C}s_1 \oplus \mathbb{C}s_2$.

Let $r_1$ and $r_2$ denote the Chern roots of $Q$.

**Proposition 3.9.** $\ch(O^{[2d]}_G) = 2 - (d + 2)e^{-r_1} - (d + 2)e^{-r_2} + (4d + 2)e^{-r_1 - r_2}$.

**Proof.** Consider the (derived) pushforward $R\pi_*$. of the exact sequence of Proposition 3.8. Computing the derived pushforward sheaves of each term gives

$$R^i\pi_*O_{Z_G} = \begin{cases} O^{[2d]}_G & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \quad R^i\pi_*O = \begin{cases} O & \text{if } i = 0, 2 \\ 0 & \text{if } i = 1 \end{cases}$$

$$R^i\pi_*(Q^* \boxtimes (-L)) = \begin{cases} 0 & \text{if } i = 0, 1 \\ Q^* \otimes H^0(X, L)^* & \text{if } i = 2 \end{cases}$$

$$R^i\pi_*(\det(Q^*) \boxtimes (-2L)) = \begin{cases} 0 & \text{if } i = 0, 1 \\ \det(Q^*) \otimes H^0(X, 2L)^* & \text{if } i = 2. \end{cases}$$

The first equation follows from the definition of $O^{[2d]}_G$ and that $Z_G$ is finite over $G$, and the last three equations all follows from relative Serre duality applied to $\pi$. From these computations, and the fact that $h^0(X, L) = d + 2$ and $h^0(X, 2L) = 4d + 2$, we get the following equality in the $K$-group of $G$:

$$[O^{[2d]}_G] - 2[O] + (d + 2)[Q^*] - (4d + 2)[\det(Q^*)] = 0.$$  

Since the Chern character $\ch$ is a homomorphism from $K$-theory to cohomology, the proposition follows from the equalities $\ch(O) = 1$, $\ch(Q^*) = e^{-r_1} + e^{-r_2}$, $\ch(\det(Q^*)) = e^{-r_1 - r_2}$. \[\square\]

**Corollary 3.10.** The total Chern character of $O^{[2d]}_G$ is

$$c(O^{[2d]}_G) = \frac{(1 - r_1 - r_2)^{4d+2}}{(1 - r_1)^{d+2}(1 - r_2)^{d+2}} = \frac{(1 - \sigma_1)^{4d+2}}{(1 - \sigma_1 + \sigma_2)^{d+2}}.$$  

**Proof.** Since $O^{[2d]}_G$ is a vector bundle, we can compute the total Chern character using the splitting principle and the set of “virtual Chern roots”

$$\{-r_1 - r_2, 0, 0\} - \{-r_1, -r_2\}.\quad \{\underbrace{-r_1 - r_2, 0, 0}_{4d+2}, \underbrace{-r_1, -r_2}_{d+2}\}.$$  

The theorem then follows from the equalities $r_1 + r_2 = \sigma_1$ and $r_1r_2 = \sigma_2$. \[\square\]
Remark 3.11. Let $X \subset \mathbb{P}^g$ be Cohen-Macaulay of degree $d$ and codimension $r$. If $X$ intersects any $r$-plane in $\mathbb{P}^g$ properly, there is a map $\text{Gr}(r+1, g+1) \rightarrow X[4]$ which sends an $r$-plane to its intersection with $X$. There is a rank $d$ tautological vector bundle $\mathcal{O}_d \rightarrow X[4]$ and the Chern classes of its pullback to $\text{Gr}(r+1, g+1)$ can be computed in the same manner as above, via the Koszul resolution.

Theorem 3.12. Let $X \subset \mathbb{P}^g$ be a smooth K3 surface embedded by a primitive ample line bundle $L$ of square $L^2 = 2d = 2g - 2$, for which no pencil in $|L|$ has a base curve. Then, the flex divisor satisfies $R_{\text{flex}} \subset |n_dL|$ where

$$n_d = \frac{(2d)!(2d+1)!}{d!^2(d+1)!^2}.$$

Proof. By Propositions 3.6 and 3.7 we have the formula

$$n_d = -\sigma_1 \cdot c_{2d-1}(\mathcal{O}_G^{2d}).$$

From the formula of Corollary 3.10 for $c(\mathcal{O}_G^{2d})$, plus the fact that the minus signs cancel in any contribution to top degree, we conclude

$$n_d = \left[ \sigma_1 \cdot \frac{(1 + \sigma_1)^{4d+2}}{(1 + \sigma_1 + \sigma_2)^{d+2}} \right]_{\text{top}}.$$

The Pieri and Giambelli formulae imply that

$$\sigma_1^m \cdot \sigma_2^n = \frac{m!}{(m/2)!(m/2+1)!}$$

when $m + 2n = 2d$ add up to the correct dimension to give a top class on $G$. After performing binomial expansion in $\sigma_1$ then $\sigma_2$, collecting terms of top degree, and plugging in the above formula, we get the ugly expression

$$n_d = \sum_{j=0}^{d} \sum_{\ell=1}^{d-j} (-1)^{j+1} \binom{4d+2}{j} \binom{3d-j}{2d+\ell} \binom{2d+\ell}{2(\ell-1)} \binom{2\ell}{\ell+1} \frac{1}{\ell+1}.$$

Applying automated choose identity verification gives the result.

Example 3.13. Let $(X, L)$ be any ADE K3 surface of degree $L^2 = 2$. The linear system $|L|$ defines a $2:1$ morphism from $X$ onto either $\mathbb{P}^2$ or $\mathbb{P}^1$ and $R_{\text{flex}}$ is naturally the ramification divisor of this map. The double cover of $\mathbb{P}^2$ is branched in a sextic $B$. One has $R_{\text{flex}}^2 = B^2/2 = 18 = (3L)^2$, so $n_1 = 3$.

Example 3.14. For a quartic surface, one can compute the flex divisor directly from the definition. Here are some results:

The Fermat quartic $X = V(x_0^4 + x_1^4 + x_2^4 + x_3^4) \subset \mathbb{P}^3$ contains 48 lines. Each line appears with multiplicity one in $R_{\text{flex}}$. The intersections of $X$ with the coordinate hyperplanes $x_i = 0$ appear with multiplicity 2 in $R_{\text{flex}}$. So $R_{\text{flex}}$ is cut out by $(x_0^4 + x_1^4)(x_0^4 + x_2^4)(x_0^4 + x_3^4)x_0^2x_1^2x_2^2x_3^2$.

The maximal number of 64 lines on a smooth quartic surface is realized by the Schur quartic $X = V(x_0^4 - x_0x_1^3 + x_2x_3^3 - x_4^3) \subset \mathbb{P}^3$. These lines come in two types. The first type, of which there are 16, are the lines joining the 4 + 4 points lying on the skew lines $V(x_0, x_1), V(x_2, x_3)$. They appear in $R_{\text{flex}}$ with multiplicity two, while the remaining 48 lines of the second type appear with multiplicity one. So $R_{\text{flex}}$ consists only of lines. Thus $X$ has no “flex points” in the naive sense.
Remark 3.15. Based on the $d = 1$ case, the authors hoped that $R_{\text{flex}}$ would be a canonical choice of polarizing divisor living in a reasonably small multiple of the polarization class, at least compared to the rational curve divisor $R_{rc}$. But in fact, the formula of Theorem 3.12 grows significantly faster than the Yau-Zaslow formula, with the switch occurring between $d = 8$ and $d = 9$. Asymptotically, $n_d \sim 2^{4d+1}/\pi d^2$ while Yau-Zaslow $\sim e^{4\sqrt{d}/\sqrt{2d^2/4}}$.

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Email address: valery@uga.edu

Department of Mathematics, University of Georgia, Athens GA 30602, USA

Email address: philip.engel@uga.edu

Department of Mathematics, University of Georgia, Athens GA 30602, USA