GENERALIZED BERNSTEIN-TYPE APPROXIMATION OF CONTINUOUS FUNCTIONS.

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Abstract.

We derive in this short article the non-asymptotical non-uniform sharp error estimation for the Bernstein’s type approximation of continuous function based on the modern probabilistic apparatus.

Key words and phrases. Generalized Bernstein’s approximation, uniform continuous function, norm, tail function and tail of distribution, absolute tail function, bilateral constants evaluation, random variable (r.v.), slowly and regular varying functions, subgaussian random variables, Ditzian-Totik modulus of continuity, non-asymptotical estimates, generalized Hölder condition, Poisson’s approximation, Hoeffding’s inequality, sharp estimation, trial functions, examples.

1 Introduction. Notations. Statement of problem.

Let $I = (a, b)$ or $I = [a, b)$ or $I = [a, b]$, $a < b$ be a finite or infinite segment on the real axis, $f : I \rightarrow \mathbb{R}$ be uniformly continuous bounded function, $\{\xi_i\} = \xi_i(x)$, $i = 1, 2, \ldots, n$; $\xi = \xi_1 = \xi(x)$, $x \in I$ be a family of independent identically distributed (i., i.d.) random variables (r.v.) which values in the set $I$, and whose distribution dependent on the parameter $x$ such that

$$E\xi_i = x, \ Var \xi_i = \sigma^2(x). \quad (1.1)$$

It will be presumed that $0 < \sigma(x) < \infty$. Evidently, if $-\infty < a < b < \infty$, then $\sigma(x) \leq 0.5(b - a)$.

Denote

$$S_n = n^{-1} \sum_{i=1}^{n} \xi_i, \quad S_n^o = n^{-1} \sum_{i=1}^{n} \xi_i - x. \quad (1.2)$$

Let us introduce the following sequence of approximated linear operators

$$A_n[f](x) \overset{df}{=} Ef(S_n) = Ef(x + S_n^o), \quad (1.3)$$

which are in turn some generalization of the classical Bernstein’s operators. See also [1], [8], [11], [13], [14], [19], [28], [32].

For instance,
be the ordinary Bernstein’s polynomial of degree \( n \), see also [3], [4], [18], [19], [20], [26], [27], [30].

Here

\[
I = [0, 1], \ a = 0, \ b = 1, \ P(\xi = 1) = x,
\]

\[
P(\xi = 0) = 1 - x, \ \sigma(x) = [x(1 - x)]^{1/2}.
\]

Another ("Poisson") example. Here

\[
a = 0, \ b = \infty, \ I = [0, \infty), \ P(\xi = k) = e^{-x} \frac{x^k}{k!}, \ k = 0, 1, 2, \ldots,
\]

so that \( E\xi = x = \text{Var} \xi, \ \sigma(x) = \sqrt{x} \),

\[
S_n[f](x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),
\]

see e.g. [9], [10], [21], [29].

We intend to investigate in this report the non-asymptotical (quality and quantity) error of the uniform approximation properties of introduced operators:

\[
\Delta_n = \Delta_n[f] \overset{def}{=} \sup_{x \in I} | A_n[f](x) - f(x) |.
\]

Note in addition that the standard deviation (weight) \( \sigma(x) \) may has a form

\[
\sigma(x) = C \ x^\alpha (1 - x)^\beta, \ \alpha, \beta = \text{const} \geq 0, \ I = [0, 1];
\]

so that \( \sigma(x) \) is Jacobi weight, see [15].

2 Auxiliary apparatus and notations.

We will use the following concrete Ditzian-Totik [12] modulus of continuity:

\[
\omega_\sigma[f](\delta) \overset{def}{=} \sup_{|h| \leq \delta} \sup_{x \in I} | f(x + h\sigma(x)) - f(x) |,
\]

where we agree to take \( f(x) = f(b) \), if \( x > b \), and \( f(x) = f(a) \) in the case \( x < a \).

As ordinary modulus of continuity, the Ditzian-Totik modulus is continuous monotonically increasing function equal to zero at the origin.

Recall that the tail function \( T_\eta(u) \) for each (not necessary to be non-negative) r.v. \( \eta \) is defined as follows:
Definition 2.1. Let \( \eta \) be a centered r.v. with finite variance, and let \( \eta(i) \) be independent copies of the r.v. \( \eta \) defined perhaps on some sufficiently rich probability space. The following numerical function \( Q(u) = Q_\eta(u), \ u \geq 0 \) will be named as an absolute tail function, write \( ATF = ATF(\eta) \) for the r.v. \( \eta \):

\[
Q(u) = Q_\eta(u) \overset{def}{=} \sup_n \mathbf{P} \left( \left| \sum_{i=1}^{n} \eta(i) \right| > u \right), \ u \geq 0. \tag{2.3}
\]

This function was introduced and partially investigated by S.N.Bernstein; we will bring for the estimation of these function the most advanced probabilistic methods.

Definition 2.2. The function \( u \rightarrow T(u), \ u \geq 0 \) is said to be tail function, briefly, TF, if it is right continuous, monotonically non-decreasing, \( T(0) = 1 \), and \( T(\infty) = 0 \). Obviously, every such a function is tail function for some random variable.

It is clear that \( Q_\eta(u) \) is tail function, as long as \( 0 < \text{Var} \eta < \infty \). It follows from the classical CLT that for some positive constant \( K = K(\eta) \)

\[
Q_\eta(u) \geq \exp \left( -K u^2 \right), \ u \geq 1.
\]

Note in addition that if the mean zero r.v. \( \eta \) is bilateral bounded: \( -\infty < a \leq \eta \leq b < \infty \), then

\[
Q_\eta(u) \leq \exp \left( -K_2 u^2 \right), \ u \geq 1,
\]

Hoeffding’s inequality. Moreover, this estimate there holds iff the mean zero r.v. \( \eta \) is subgaussian, see [5] - [7], [16].

Many examples of estimation of ATF function may be found in [17]; [22], chapter 1, sections 1.6., chapter 2.

Further, let us consider the centered and normed variable (variate)

\[
\zeta_i = \zeta_i(x) := \frac{\xi_i - x}{\sigma(x)}; \ \zeta = \zeta(x) := \zeta_1. \tag{2.4}
\]

Define the logarithm of moment generating function for the r.v. \( \zeta \):

\[
\phi(\lambda) = \phi_\zeta(\lambda) \overset{def}{=} \max_{\pm} \sup_{x \in I} \ln \mathbf{E} \exp(\pm \lambda \zeta),
\]

if of course the r.v. \( \zeta \) satisfies the famous Kramer’s condition, which is equal in turn the following implication

\[
\exists \lambda_0 \in (0, \infty], \Rightarrow \forall \lambda : \ |\lambda| < \lambda_0 \Rightarrow \phi(\lambda) < \infty.
\]

This function is also even, convex and generated the so-called Banach space \( B(\phi) \) consisting on the special centered random variables, see [17], [22], chapters 1,2.

Denote

\[
T_\eta(u) = \mathbf{P}(|\eta| > u), \ u \geq 0.
\]
\[ \nu(\lambda) = \sup_n \{ n \phi(\lambda/\sqrt{n}) \}, \]
\[ \nu^*(u) = \sup_\lambda (\lambda u - \nu(\lambda)), \]
the Young-Fenchel transform for the function \( \nu(\cdot) \). Both the introduced functions \( \nu(\cdot), \nu^*(\cdot) \) are correct definite, even and convex. The following Fenchel-Mor aux iden-
tity play a very important role in the theory of random variables with exponential
decreasing tails of distributions: \( \nu^{**}(u) = \nu(u) \).

It is known, see [22], chapter 1, that
\[ Q_\zeta(u) \leq 2 \exp(-\nu^*(u)), \quad u \geq 0. \quad (2.5) \]

3 Main result.

Theorem 3.1.
\[ \Delta_n[f] \leq \int_0^\infty \omega_\sigma[f \{ z/\sqrt{n} \}] \, |dQ_\zeta(z)|. \quad (3.0) \]

Proof. We have using the direct definition of Ditzian-Totik modulus of contin-
uuity
\[ \Delta_n[f] \leq \sup_x E \left| f(x + \zeta_n \sigma(x)/\sqrt{n}) - f(x) \right| \leq \]
\[ E \sup_x \left| f(x + \zeta_n \sigma(x)/\sqrt{n}) - f(x) \right| \leq E \omega_\sigma[f \{ |\zeta_n|/\sqrt{n} \}] = \]
\[ \int_0^\infty \omega_\sigma[f \{ z/\sqrt{n} \}] \, |dT_\zeta_n(z)|. \quad (3.1) \]

Lemma 3.1. Let \( g = g(z), \quad z \geq 0 \) be a continuous monotonically increasing
function equal to zero at the origin. If \( \xi, \eta \) are two non-negative r.v. such that
\[ T_\xi(z) \leq T_\eta(z), \quad z \geq 0, \]
then
\[ E g(\xi) \leq E g(\eta). \quad (3.2) \]

Proof of lemma 3.1. We can and will assume without loss of generality that
all the functions \( g(x), T_\xi(z) \) and \( T_\eta(z) \) are continuous differentiable. We deduce by
means of integration by parts:
\[ E g(\xi) = - \int_0^\infty g(x) \, dT_\xi(x) = -g(x) \left. T_\xi(x) \right|_0^\infty + \int_0^\infty T_\xi(x) \, g'(x) \, dx = \]
\[
\int_0^\infty T_\xi(x) \, g'(x) \, dx \leq \int_0^\infty T_\eta(x) \, g'(x) \, dx = -\int_0^\infty g(x) \, dT_\eta(x) = Eg(\eta).
\]

There is another proof. Namely, we can realize both the r.v. \( \xi, \eta \) on at the same probability space, say \([0,1]\), so that
\[
\eta = T_\eta^{-1}(\tau), \quad \xi = T_\xi^{-1}(\tau),
\]
where the r.v. \( \tau \) has an uniform distribution on the unit interval \([0,1]\), if for definiteness both the tail functions \( T_\eta(\cdot) \) and \( T_\xi(\cdot) \) are continuous and strictly decreasing.

Therefore \( \xi \leq \eta \) almost everywhere in this realization and following \( Eg(\xi) \leq Eg(\eta) \) under arbitrary realization.

It is no hard to finish the proof proof of theorem 3.1. Since \( T_\zeta_n(z) \leq Q_\zeta(z) \), we conclude on the basis of lemma 3.1
\[
\Delta_n[f] \leq \int_0^\infty \omega_\sigma[f] \left( z/\sqrt{n} \right) |dQ_\zeta(z)|,
\]
Q.E.D.

**Remark 3.1.** It follows immediately from the proposition of theorem 3.1 by virtue of Lebesgue dominated convergence theorem that under formulated above conditions

\[
\lim_{n \to \infty} \Delta_n[f] = 0,
\]
as long as \( \omega_\sigma[f](\delta) \leq 2 \sup_x |f(x)| \).

**Example 3.1.** Suppose that the (centered normed) variable \( \zeta \) has a following tail function
\[
T_\zeta(u) \leq \exp(-u^p), \ u \geq 0
\]
for some constant \( p > 0 \). Denote \( q = q(p) = \min(p,2) \). It is known, see [17], [22], chapters 1,2 that
\[
Q_\zeta(u) \leq \exp(-K(p) \ u^q), \ u \geq 0, \ K(p) = \text{const} \in (0, \infty),
\]
and the last estimate is essentially non-improvable.

We get relying on the theorem 3.1
\[
\Delta_n[f] \leq K(p) \int_0^\infty \omega_\sigma[f] \left( \frac{z}{\sqrt{n}} \right) z^{q-1} \exp(-K(p) \ z^q) \, dz.
\]
4 Some examples.

We will consider in this section some examples in order to make sure the result of theorem 3.1. Note at first that the case of the classical Bernstein's approximation in this spirit was considered in [25].

**Definition 4.1.** The (continuous) function \( f : I \to \mathbb{R} \) belongs by definition to the Hölder-Ditzian-Totik class, write \( f \in HDT(\sigma, \alpha) \), iff

\[
\omega[\sigma][f](\delta) \leq H \cdot \delta^\alpha, \; \delta \geq 0,
\]

for some constants \( 0 \leq H < \infty, \; \alpha \in (0, 1] \).

We will understood as a capacity of the value \( H \) in (4.1) its minimal value, namely

\[
H = H_{\alpha, \sigma}[f] \overset{\text{def}}{=} \sup_{\delta > 0} \left[ \frac{\omega[\sigma][f](\delta)}{\delta^\alpha} \right].
\]

Evidently, the functional \( f \to H_{\alpha, \sigma}[f] \) is (complete) semi-norm relative the function \( f, \; f \in HDT(\sigma, \alpha) \), as in the case of classical Hölder's norm, in which \( \sigma = 1 \).

**Example 4.1.** It is easily to compute by means of theorem 3.1, that if \( f \in HDT(\sigma, \alpha) \), then

\[
\Delta_n[f] \leq H_{\alpha, \sigma}[f] \cdot n^{-\alpha/2} \cdot \int_0^\infty z^\alpha |dQ_\zeta(z)| = 
\]

\[
\alpha \cdot H_{\alpha, \sigma}[f] \cdot n^{-\alpha/2} \cdot \int_0^\infty z^{\alpha-1} Q_\zeta(z) \, dz. \tag{4.2}
\]

If in addition the r.v. \( \zeta \) satisfies the condition of the example 3.1, then

\[
\Delta_n[f] \leq K^{-\alpha/q}(p) \cdot \alpha \cdot H_{\alpha, \sigma}[f] \cdot n^{-\alpha/2} \cdot \Gamma (\alpha/q). \tag{4.3}
\]

The case \( p = q = 2 \) and \( K(p) = 1/2 \) correspondent to the classical Bernstein's case, see [25].

**Remark 4.1.** The case when \( \omega[\sigma][f](\delta) \) is (continuous) non-negative regular varying at the origin function:

\[
\omega[\sigma][f](\delta) \leq H_L \cdot \delta^\alpha \cdot L(\delta), \; \delta \geq 0, \; \alpha = \text{const} \in (0, 1]
\]

where \( L = L(\delta) \) is non-negative continuous in the set \((0, b-a)\) slowly varying at the origin function, that is

\[
\forall z > 0 \Rightarrow \lim_{\delta \to 0^+} \frac{L(\delta z)}{L(\delta)} = 1,
\]

may be considered analogously. Indeed:

\[
\Delta_n[f] \leq H_L \cdot \int_0^\infty L \left( \frac{z}{\sqrt{n}} \right) \left[ \frac{z}{\sqrt{n}} \right]^\alpha |dQ_\zeta(z)| \sim
\]
\[ H_L \cdot n^{-\alpha/2} \cdot L \left( \frac{1}{\sqrt{n}} \right) \int_0^\infty z^\alpha |dQ_\xi(z)|, \ n \to \infty. \]

**Example 4.2; ”Poisson” case.** Suppose now that the r.v. \( \xi \) has a Poisson distribution with a parameter \( x \). Here \( I = [0, \infty) \), and let \( x = \lambda \geq 1 \).

\[ \mathbb{P}(\xi = k) = e^{-x} \frac{x^k}{k!}, \ k = 0, 1, \ldots. \]

Recall that ”Poisson” approximation of an uniform continuous function \( f \) has a following form

\[ S_{n,P} = S_n[f](x) = e^{-nx} \sum_{k=0}^\infty \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right), \]

and was investigated, e.g. in the articles [9], [21], [29].

Let us estimate the moment generating function for the centered and normed variable

\[ \eta = \frac{\eta_0}{\sqrt{x}} = \frac{\xi - x}{\sqrt{x}}. \]

We have for the values \( z > 0 \) and \( \lambda = x \geq 1 \):

\[ E e^{z\lambda} = \sum_{k=0}^\infty e^{kz} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^\infty \frac{(\lambda e^z)^k}{k!} = \exp \left( \lambda(e^z - 1) \right); \]

\[ \ln E e^{z\eta} = -z \sqrt{\lambda} + \lambda \left( e^{z/\sqrt{\lambda}} - 1 \right) = \frac{z^2}{2} + \frac{z^3}{3! \sqrt{\lambda}} + \ldots + \frac{z^k}{k! \lambda^{k/2-1}} + \ldots; \quad (4.4) \]

It is clear that the right-hand side of the relation (4.4) attains the maximal value relative the variable \( \lambda \) at the point \( \lambda = 1 \). Thus,

\[ \phi(z) = \phi_P(x) \overset{def}{=} \sup_{\lambda \geq 1} E e^{z\eta} = e^z - 1 - z. \quad (4.5) \]

The Young-Fenchel transform of these function has a form

\[ \ln \phi_P^*(u) = u \ln(1 + u) - u + \ln(1 + u), \ u > 0, \]

following \( \ln \phi_P^*(u) \sim u \ln(1 + u), \ u \to \infty \) and analogously

\[ Q_P^*(u) \sim \{\ln(1 + u) + u/(u + 1)\} \cdot e^{-u \ln(1 + u)}, \ u \to \infty, \]

the index ”P” correspondent the name ”Poisson.” It remains to use theorem 3.1:

\[ \Delta_{n,P}[f] \leq \int_0^\infty \omega_\sigma[f] \left( \frac{z}{\sqrt{n}} \right) Q_P^*(z) \, dz. \quad (4.6) \]

Here \( \sigma = \sigma(x) = \sqrt{x}, \ x \geq 1. \)
Of course, if the function \( f(\cdot) \) satisfies Hölder’s, more exactly, HDT condition (4.1), then the integral in the right-hand side of inequality (4.6) converges and we conclude as before

\[
\Delta_{n,p}[f] \leq C_P \cdot H_{\alpha,\sigma}[f] \cdot n^{-\alpha/2}, \ n \geq 1. \tag{4.7}
\]

5 Low bounds. Exactness of our estimates.

Let us consider the following example. \( I := [0, 1] \), the distribution \( \xi \) is such that

\[
\sigma(x) = \text{Var} \xi \text{ is continuous and denote } \overline{\sigma} = \max \sigma(x), \ x_0 = \argmax \sigma(x) \in (0, 1).
\]

Note that the Bernstein’s case \( \sigma(x) = \sqrt{x(1-x)} \), \( x_0 = 1/2 \) is suitable for us.

Let also \( g = g(x), \ x \in [0, 1] \) be non-negative trial continuous function from the set \( HDT(\sigma, \alpha) \), \( 0 < \alpha \leq 1 \) for which

\[
g(x_0 + \delta) \geq H_{\sigma,\alpha}[g] \cdot \delta^\alpha = H_- \cdot \delta^\alpha, \ \delta \in [0, \min(x_0, 1 - x_0)]. \tag{5.1}
\]

We deduce

\[
\Delta_n[g] \geq E g(x_0 + \zeta_n \overline{\sigma}/\sqrt{n}) \geq H_- \cdot n^{-\alpha/2} \cdot E |\zeta_n|^\alpha. \tag{5.2}
\]

It follows from the (another) Bernstein’s theorem that

\[
\lim_{n \to \infty} E |\zeta_n|^\alpha = E |\tau|^\alpha = 2^{\alpha/2} \pi^{-1/2} \Gamma((\alpha + 1)/2) \overset{def}{=} G(\alpha),
\]

where the r.v. \( \tau \) has a standard normal distribution.

To summarize, we have proved in fact the following low bound the generalized Bernstein’s approximation.

\[\text{Proposition 5.1.}\]

\[
\sup_{\text{const} \neq g \in H(\alpha, \sigma)} \lim_{n \to \infty} \left[ \frac{\Delta_n[g]}{H_{\alpha,\sigma}[g] \cdot n^{-\alpha/2}} \right] \geq G(\alpha). \tag{5.3}
\]

See also [31].

6 Concluding remarks.

A. It is no hard perhaps to generalize by our opinion obtained results into the ”more” multivariate case \( d = 2, 3, 4, 5, \ldots \) as well as into other methods of approximation, if only they had a probabilistic representation.

B. One can also investigate and improve the rate of convergence of partial derivatives for the multivariate Bernstein’s polynomials, in the spirit of the article [33].
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