We present progress in the QCD analysis of the Bjorken sum rule at low momentum transfers. We study asymptotic structure of the perturbative QCD expansion at low $Q^2$ scales based on analysis of recent accurate data on the Bjorken sum rule and available now four-loop expression for the coefficient-function $C_{Bj}(Q^2)$. We demonstrate that the standard perturbative series for $C_{Bj}(Q^2)$ gives a hint to its asymptotic nature manifesting itself in the region $Q^2 \lesssim 1$ GeV$^2$. It is confirmed by the considered integral model for the perturbative QCD correction. We extract a value of higher-twist $\mu_4$ coefficient and study the interplay between higher orders and higher-twist contributions. Results of other approaches to the description of Bjorken sum rule data are discussed.

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I. INTRODUCTION

Perturbation theory (PT), supplemented by the renormalization group method and the renormalization procedure, is a main tool of QCD calculations. Conventional perturbative expansions for hadronic observables are the power series in the strong coupling $\alpha_s$. Owing to the property of asymptotic freedom in QCD, the perturbative description becomes more reliable at high energy region. On the contrary, at low momentum transfer, $Q^2 \lesssim 1 - 2$ GeV$^2$, the use of the perturbative power series may be questionable. Therefore, it is important to know at least a few terms of PT series to estimate theoretical uncertainties associated with a truncation of the perturbative expansion. Currently, the perturbative QCD analysis is aimed at the perturbative calculations of new terms of PT series. Only recently up to the fourth-order perturbative approximation for some physical processes became available [1, 2]. The number of these processes includes the fundamental sum rule for polarized deep inelastic scattering which is the Bjorken sum rule $\Gamma_1^{p-n}(Q^2)$ defined as a difference between the first moments of the proton $g_1^p$ and the neutron $g_n^p$ spin structure functions. The Bjorken sum rule has been measured in polarized deep inelastic lepton scattering at SLAC, CERN, DESY [4–9]. At low $Q^2$, in a $Q^2$-range from $0.05 < Q^2 < 3.2$ GeV$^2$, high-precision data on the $\Gamma_1^{p-n}$ has been presented by the Jefferson Lab [10] and at 3.0 GeV$^2$ by the COMPASS collaboration [11].

At low momentum scales, the QCD analysis of the Bjorken sum rule includes both the perturbative and non-perturbative higher-twist (HT) components related to each other. It is clear that the reliability of extracting information on the HT effects is connected to the indeterminacy in the description of the PT series. As new perturbative correction of an order of $\alpha_s^2$ to the Bjorken sum rule became available [2], the opportunity for new researches opened.

In this report, we present results of the four-loop QCD analysis of the Bjorken sum rule at low momentum scales, continuing our investigations started in Refs. [12, 13]. We consider the features of the four-loop PT series and the interplay between the higher PT order and the HT contributions. We discuss results of application of other approaches which were applied to the description of low $Q^2$ Jefferson Lab data.

II. STANDARD APPROACH

Away from the large $Q^2$ limit, the polarized Bjorken sum rule is given by a double series in powers of $\alpha_s$ and in powers of $1/Q^2$ (nonperturbative HT corrections):

$$\Gamma_1^{p-n}(Q^2) = \frac{|g_A|}{6} \left[ 1 - \Delta_{Bj}^{PT}(Q^2) \right] + \sum_{i=2}^{\infty} \frac{\mu_{2i}}{Q^{2i-2}} ,$$

where $|g_A| = 1.2701 \pm 0.0025$ [14] is the nucleon axial charge, $\mu_4, \mu_6, \ldots$ are the HT coefficients; $\Delta_{Bj}^{PT}(Q^2)$ is the perturbative correction which is defined by the coefficient-function $C_{Bj}(Q^2)$, $\Delta_{Bj}^{PT}(Q^2) \equiv 1 - C_{Bj}(Q^2)$. Note that until

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very recently $\Delta_{PT}^{Bj}(Q^2)$ has been known up to the third order in $\alpha_s$. The corresponding expression was used in many studies (see, e.g., Refs. [12, 15–17]), in particular, to extract a value of $\alpha_s$ from experimental data. Comparison of these values with other accurate $\alpha_s$ values, such as those obtained from the $\tau$-lepton and the $Z$-boson into hadrons width decays, is an important test of the consistency of QCD [14, 18].

A. Four-loop analysis

At the four-loop ($N^3$LO) level, the perturbative QCD correction to the Bjorken sum rule in the case of massless quarks in the $\overline{\text{MS}}$ renormalization scheme for $f = 3$ light quarks flavors reads as

$$\Delta_{PT}^{Bj} = 0.318 \alpha_s + 0.363 \alpha_s^2 + 0.652 \alpha_s^3 + 1.804 \alpha_s^4. \quad (2)$$

We use the PT running coupling $\alpha_s(Q^2)$ obtained by integration of the renormalization group equation with the four-loop $\beta$-function (see Ref. [13] for additional details).

Let us discuss the convergence properties of the PT power series (2) at low momentum transfers. Figure 1 shows the relative contribution of the $i$-th term of this series as a function of $Q^2$. As can be seen from this figure, in the region of small $Q^2 \lesssim 1$ GeV$^2$ the dominant contribution comes from the $\alpha_s^4$-term. This may be considered as an indication of the transition of PT series to the asymptotic regime and one can estimate the value $\alpha_s(1 \text{ GeV}^2) \simeq 0.5$ as a critical. In the region $Q^2 > 2$ GeV$^2$ the situation changes – the major contribution comes from the first and second terms.

To specify discussed above convergence properties of series (2), we consider the exact example following from the integral model [19] (see Ref. [20] for more details):

$$C(g) = \frac{1}{3\pi} \int_{-\infty}^{\infty} e^{-x^2(1-\frac{x^2}{2})^2} dx = 1 + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} C_k g^k, \quad C_k = \frac{\Gamma(2k + 1/2)}{4^k \Gamma(k + 1)}. \quad (3)$$

Using this expression, one can build series which is close to the series (2):

$$\Delta(g) = \frac{16}{3\pi} [C(g) - 1] = 0.318 g + 0.348 \ g^2 + 0.718 \ g^3 + 2.188 \ g^4 + \ldots. \quad (4)$$

In Fig. 2 we compare the finite sum approximations $\Delta_{[n]}(g) = \sum_{i=1}^{n} C_i g^i$, $n = 1, \ldots, 4$ both with the exact result and with the Bjorken series (2). This figure demonstrates the closeness of the 4-term approximation (dashed curve) and the $N^3$LO approximation (2) (dotted curve). It is necessary to note, that 1-, 2-, and 3-order approximations to $\Delta(g)$ and $\Delta_{Bj}$ practically coincide with each other. This figure shows that the 2-term approximant is good up only to $g = 0.15 - 0.20$ and the 3-term one up to $g \sim 0.33$ while the 4-term sum starts to deviate from exact $C(g)$-curve at $g \sim 0.27$. We interpret this observation as asymptotic structure manifestation. This model confirms that the asymptotic structure of the pQCD expansions manifests. Therefore, the application of standard PT greater than $N^3$LO approximation can not allows to extract accurate information in the low-energy domain.

**FIG. 1:** The relative contribution $N_i(Q^2) = c_i \alpha_s(Q^2)/\Delta_{Bj}(Q^2)$ for the $i$-th term of series (2) as a function of the $Q^2$.

**FIG. 2:** Comparison of the 1-, 2-, 3-, and 4-term approximants of series (2) with the exact result (2) and with $N^3$LO Bjorken sum rule series (2).
B. Higher twist contribution

We expand our consideration, including the HT part which is presented in expression \([1] \). In Table \([1] \) we show our results for values of the coefficient \(\mu_4\) (the errors are statistical only) fitted to the low \(Q^2\) data \([10, 11] \) in different PT orders. One can see that \(\mu_4\)-values extracted changes rather strongly between different PT orders. The absolute value of \(\mu_4\) decreases with the order of PT and just at \(N^3\)LO becomes compatible to zero. It can be interpreted as a manifestation of duality between higher orders and HT (see Ref. \([21] \)). Note that a value extracted in the leading-order (LO) is consistent with a value \(\mu_4 = -0.047 \pm 0.025\) GeV\(^2\) presented in Ref. \([22] \) as well as with a value \(\mu_4 = -0.028 \pm 0.019\) GeV\(^2\) obtained from the next-to-leading-order (NLO) fit based on the \(x\)-dependent structure functions data \([23] \).

III. OTHER APPROACHES

As shown above, use of the conventional PT series even with the HT component does not allow to describe the Jefferson Lab data down to the infrared region (see Fig. \([3] \)). Let us consider some other approaches which are more successful in this direction.

A. APT

In Ref. \([24] \) the conventional method of the renormalization group improvement of the perturbative expansions was modified by requiring K"allen-Lehmann analyticity, which reflects the principle of causality. In the framework of this approach \([24] \) called as the analytic perturbation theory (APT) the ghost pole and corresponding branch points, which appear in higher PT orders, are absent (see, e.g., Ref. \([25] \)). As the moments of the structure functions should be analytic functions in the complex \(Q^2\) plane with a cut along the negative real axis (see Ref. \([26] \) for more details), the standard description of the \(\Gamma_{1}^{-n}(Q^2)\) violates analytic properties due to the unphysical singularities of perturbative running coupling. On the other hand, the APT support these analytic properties (see Ref. \([27] \) as review). The perturbative correction to the Bjorken sum rule can be written in the form of a spectral representation \([16] \), and at four-loop level is

\[
\Delta_{\text{Bj}}^{\text{APT}} = 0.318\mathcal{A}_1 + 0.363\mathcal{A}_2 + 0.652\mathcal{A}_3 + 1.804\mathcal{A}_4.
\]  

The power coefficients in this expression are the same as in series \([2] \), and the functions \(\mathcal{A}_k(Q^2)\) are defined through the spectral density \(g_k(\sigma) \equiv \text{Im} [\alpha_k(\sigma + i\epsilon)]\) by the spectral integral (see Ref. \([12] \) for additional details). At large momentum transfers, all the functions \(\mathcal{A}_k(Q^2)\) become proportional to the \(k\)-th power of the usual perturbative coupling \([\alpha_k(Q^2)]^k\) and the expansion \([5] \) reduces to the power series \([2] \). However, at small enough \(Q \lesssim 1 - 2\) GeV the properties of the non-power expansion \([5] \) become considerably different from the PT power series \([2] \).

Figure \([3] \) shows the results of \(\mu_4\)-fits in different PT orders both in the PT and APT approaches. From this figure follows that in the framework APT including \(\mu_4\)-coefficient allows one to move further down to \(Q^2 \sim 0.1\) GeV\(^2\) \([13] \) in description of the Jefferson Lab data. It is important to note that the APT leads to higher-loops stability of the HT-extraction: \(\mu_4 = -0.044 \pm 0.004\) GeV\(^2\) in all loop approximations.

B. MPT

In context of data on the \(\Gamma_{1}^{-n}\) analysis it should be noted the MPT approach \([28] \). Basis of the MPT is a simple idea to change the usual logarithm in the expression for the QCD running coupling, \(1/\ln(Q^2/\Lambda^2)\), that is singular in the infrared region, on the “long logarithm” \(\ln(\xi + Q^2/\Lambda^2)\), where parameter \(\xi\) corresponds to the “effective gluonic

| PT order | LO | NLO | \(N^2\)LO | \(N^4\)LO |
|----------|----|-----|----------|----------|
| \(\mu_4\), GeV\(^2\) | \(-0.037 \pm 0.003\) | \(-0.025 \pm 0.004\) | \(-0.012 \pm 0.006\) | \(0.005 \pm 0.008\) |

TABLE I: Results of \(\mu_4\)-extraction from the data on the Bjorken sum rule in different PT orders.
masse" $m_{gl} = \sqrt{\xi} \Lambda$:

$$A_{1,MPT}(Q^2) = \frac{1}{\beta_0 \ln(\xi + Q^2/\Lambda^2)}.$$  

(6)

Result of MPT application can be interpreted as a transition to the new momentum-transfer scale both in the PT and the HT contributions: $\Delta_{Bj}^{MPT} \sim \Delta_{Bj}^{PT}(Q^2 + m_{gl}^2)$ and $\mu_4/Q^2 \sim \mu_4/(Q^2 + m_{ht}^2)$. The MPT curve is shown in Fig. 4 as solid line (preliminary result Ref. [29]). From this figure one can see that the MPT approach allows us to describe the Bjorken sum rule data down the infrared region $Q^2 \to 0$.

Note that very close description of the Jefferson Lab data was obtained in Ref. [30], where was used a modified expression for the coefficient-function $C_{Bj}$ in a combination with “frozen” behavior of running coupling $\alpha_s(Q^2)$.

IV. SUMMARY

We have presented the QCD analysis of the Bjorken sum rule up to four-loop level in $\alpha_s$ at low momentum transfers. It has been shown that the asymptotic nature of the perturbative series manifested itself at the four-loop level in the region $Q^2 \lesssim 1$ GeV$^2$. Therefore, at low $Q^2$ the inclusion of the four-loop term does not improve the precision of theoretical predictions. It was confirmed by the considered integral model for the perturbative QCD correction.

Using the recent low $Q^2$-data from the Jefferson Lab and COMPASS experiments, we have extracted a value of higher-twist $\mu_4$ coefficient and have shown the interplay between higher orders and higher-twist contributions. Results of other approaches to the description of Bjorken sum rule data have discussed.

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