2 + 1 Covariant Lattice Theory and t’Hooft’s Formulation

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Abstract

We show that ’t Hooft’s representation of (2+1)-dimensional gravity in terms of flat polygonal tiles is closely related to a gauge-fixed version of the covariant Hamiltonian lattice theory. ’t Hooft’s gauge is remarkable in that it leads to a Hamiltonian which is a linear sum of vertex Hamiltonians, each of which is defined modulo $2\pi$. A cyclic Hamiltonian implies that “time” is quantized. However, it turns out that this Hamiltonian is constrained. If one chooses an internal time and solves this constraint for the “physical Hamiltonian”, the result is not a cyclic function. Even if one quantizes a la Dirac, the “internal time” observable does not acquire a discrete spectrum. We also show that in Euclidean 3-d lattice gravity, “space” can be either discrete or continuous depending on the choice of quantization. Finally, we propose a generalization of ’t Hooft’s gauge for Hamiltonian lattice formulations of topological gravity dimension 4.
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I. INTRODUCTION

Because (2+1)-dimensional gravity is closely related to the Chern-Simons topological invariant for the gauge group $ISO(2, 1)$, an exact lattice version of the theory exists [1]. This in turn has led to a convenient parametrization of the reduced phase space, the moduli space of flat $ISO(2, 1)$ connections, known as the “polygon representation” of (2+1)-dimensional gravity. In both the lattice theory and the polygon representation, the translations of the vertices are generated by first-class constraints, so the gauge is never fixed – rather, the reduced variables are independent of the foliation of spacetime. Having thus decoupled the physical degrees of freedom from the gauge, the canonical quantization programme can be carried out explicitly [3], leading to a proposal for the modular-invariant wave function, in terms of a sum of plane waves analogous to diffraction by a grating [4].

Proceeding from a different angle, ‘t Hooft has proposed a gauge-fixed representation of (2+1)-dimensional gravity based on a foliation with a patchwork of flat polygons [5]. With this choice of gauge, the Hamiltonian becomes a linear sum of the deficit angles at the vertices, rather than the usual trace of a product of non-commuting Lorentz matrices. The cyclic nature of the Hamiltonian, which is then only defined modulo $2\pi$, indicates that time is quantized [6].

In this letter, we throw a bridge between the exact lattice theory and ‘t Hooft’s flat polygons. To do so, we solve the $SO(2, 1)$ Gauss constraint at the lattice faces by changing to scalar variables. The resulting symplectic structure agrees with that proposed by ‘t Hooft, and the dynamical evolution is generated by an arbitrary linear combination of first-class translation constraints. On a tessellation in terms of flat polygons, these constraints reduce to conditions on the deficit angles; the sum of which is the Hamiltonian proposed by ‘t Hooft.

The canonical gauge fixing procedure, however, would require solving the constraints together with the gauge conditions, or alternatively computing Dirac brackets which cancel the fluctuations of the gauge variables. This would lead to non-local Dirac brackets for the scalar variables, so it seems preferable to invoke instead the relation between ‘t Hooft’s scalar variables and the covariant lattice theory. Carrying on with the canonical gauge fixing, one would choose an “internal time” to compute the physical Hamiltonian, by solving the Hamiltonian constraint. Although this constraint is a cyclical function, its solution for the momentum canonically conjugate to the internal time is not. Therefore, even if one considers time as an observable and quantizes a la Dirac, there is no prediction of a discrete spectrum. Rather, the original argument for the discretization of time appears to be an artifice of the procedure whereby the Hamiltonian was postulated by demanding that it generate the correct dynamics.

One only has quantization of time if one does not fix the gauge, but nonetheless uses an initial condition which satisfies the gauge conditions and notes that the Hamiltonian preserves these conditions – seen from this perspective, it appears that ‘t Hooft’s proposal is not a canonical quantization of a diffeomorphism-invariant gravity theory, but the quantization of a lattice model which represents the same classical solutions – the conclusion, that classically equivalent formalisms differ as quantum theories, should come as no surprise.

If one considers the gauge group $ISO(3)$ rather than $ISO(2, 1)$, both the exact lattice theory and the polygon representation predict a discrete spectrum for the links, as these are
represented by rotation operators. In contrast, if one follows ‘t Hooft’s procedure one finds that the cyclic form of the spatial translation generators imply the discretization of the link lengths as \( l_i = k \), rather than \( l_i^2 = k(k+1) \). Finally, in the scalar theory which results from fixing the SO(3) symmetry and the translation symmetry before quantizing, the lattice link lengths have a continuous spectrum.

These results indicate that the prediction of a discrete structure for space and time can depend on the choice of quantization, particularly on whether one quantizes before or after fixing the gauge.

II. FROM THE COVARIANT LATTICE TO ‘T HOOFT’S POLYGONS

Now we briefly review the covariant lattice theory \([1]\) based on a lattice version of the Ashtekar-Witten variables \([7,8]\). To each lattice face we assign a reference frame which we denote by an index \((i,j,\ldots)\). The role of the connection is played by three-dimensional Lorentz matrices \( M_{ij} \) that define parallel transport from face \((j)\) to face \((i)\). If face \((i)\) is surrounded by faces \((j),(k),(l),\ldots\), the boundary of face \((i)\) is described in the frame assigned to \((i)\) by 3-vectors \( E_{ij}, E_{ik}, E_{il},\ldots\) (figure 1). The following identities hold (we will use parentheses for the lattice indices to avoid confusion with the frame indices).

\[
E_{(ij)}^a = -M_{(ij)}^a E_{(ji)}^b \tag{2.1}
\]
\[
M_{(ij)}^a M_{(ji)}^c b = \delta^a_b \tag{2.2}
\]
\[
M_{(ij)}^a M_{(ji)}^b c = \eta^{ab} c \tag{2.3}
\]

The symplectic structure is defined by the Poisson brackets

\[
\{ E_{(ij)}^a, E_{(ij)}^b \} = \varepsilon^{ab}_d E_{(ij)}^d \tag{2.4}
\]
\[
\{ E_{(ij)}^a, M_{(ij)}^b c \} = \varepsilon^{ab}_d M_{(ij)}^d c \tag{2.5}
\]
\[
\{ E_{(ij)}^a, M_{(ji)}^b \} = -\varepsilon^{ad}_c M_{(ji)}^b d \tag{2.6}
\]

which were originally derived from a Chern-Simons action \([1]\) (all other brackets are null).

**Fig. 1** A triangular lattice. The labels \(i, j, k,\ldots\) etc denote faces of the lattice. The vector \( E_{ji} \), in the frame which is associated to face \(j\), represents the boundary between faces \(j\) and \(i\); this same vector points from the vertex \(J\) to the vertex \(I\).

We require that each face \((i)\) closes, that the curvature vanishes at vertices \((I)\) with no particles and that spacetime has a conical singularity of deficit angle \(2\pi m_p\) at vertices \((p)\) with particles of mass \(m_p\).

\[
J_{(i)}^a = E_{(ij)}^a + E_{(ik)}^a + \ldots \approx 0 \tag{2.7}
\]
\[
W_{(j)}^b c = (M_{ij} M_{jk} \ldots M_{ni})^b c \approx \delta^b_c \tag{2.8}
\]
\[
\text{tr}(W_{(p)}) \approx 1 + 2 \cos(m_p) \tag{2.9}
\]

where the vertex \((I)\) is shared by faces \((i),(j),\ldots(n)\). The constraints \((2.8), (2.9)\) can be replaced by \( P_{(i)}^a := \frac{1}{2} \varepsilon^{ac}_{\ b} W_{(i)}^b c \approx 0 \) and \( P^2(p) - m_p^2 \approx 0 \). These constraints are first-class.
and generate Lorentz transformations of the frame at \((i)\), translations of vertex \((I)\), and time reparametrizations of particle \((p)\)'s world line respectively:

\[
\begin{align*}
\{J^a_{(i)}, E^b_{(ij)}\} &= \varepsilon^{ab} d E^d_{(ij)} \\
\{J^a_{(i)}, M^b_{(ij)} c\} &= \varepsilon^{ab} d M^d_{(ij)} c \\
\{\xi^a P_{(I) a}, E^b_{(ij)}\} &\approx \xi^b \\
\{P^a_{(p) a}, E^b_{(ij)}\} &\approx P^b_{(p)} .
\end{align*}
\] (2.10-2.13)

(The weak equality indicates that the constraints have been used in simplifying the last brackets.)

Once we know the variables, restrictions, and symmetries we can compute the dimension of the physical phase space. The lattice has \(N\) particles, \(N_0 - N\) vertices with no particles, \(N_1\) links, and \(N_2\) faces. There are \(6N_1\) phase space variables \(E_{ij}\) and \(M_{ij}\), and \(3(N_0 - N + N_2) + N\) constraints and symmetries, so that the dimension of the phase space is

\[
6N_1 - 2(3N_0 + 3N_2) + 4N = 4N - 6\chi = 4N + 12g - 12
\] (2.14)

where \(\chi\) is the Euler number and \(g\) the genus of the surface.

Now we solve the Gauss constraint \((\ref{eq:gauss})\) by changing to scalar variables. For the triangular lattice, a good set of variables is the link lengths \(l_{(ij)}\) and the (hyperbolical)angles \(\eta_{(ij)}\) between neighboring faces

\[
l^2_{(ij)} := E^a_{(ij)} E^a_{(ij)} \\
\cosh(\eta_{(ij)}) := \frac{1}{N_{(i)} N_{(j)}} N^a_{(i)} (M_{(ij)} N_{(j)})_a .
\] (2.15-2.16)

where \(N^a_{(ij)} = \varepsilon^{abc} E^a_{(ij)} E^b_{(ik)} \). These \(2N_1\) variables encode all the scalar information given in the \(6N_1\) covariant variables \(E_{ij}\), \(M_{ij}\). For a triangular lattice the number of scalar variables \((2N_1 = 3N_2)\) equals the number of covariant variables minus constraints and symmetries \((6N_1 - 3N_2 - 3N_2 = 2N_1)\). Thus, the dimension of the phase space after reducing the Lorentz gauge freedom is \(2N_1\). We could check from the geometry of the lattice that the scalar variables are independent. Instead, we see that \(l_{(ij)}\) and \(\eta_{(ij)}\) are canonically conjugated. Their independence in the reduced phase-space is then guaranteed.

After reducing the phase space by gauge fixing and solving the Gauss law, the relevant brackets of the theory are the Dirac brackets. However, an immediate consequence of working with scalar variables is that their Dirac brackets coincide with their Poisson brackets. By definition, if a pair of functions \(f, g\) is scalar \(\{f, J(i)^a\} = \{g, J(i)^a\} = 0\) for all \((i)\), then

\[
\{f, g\}_{DB} = \{f, g\} - \{f, G_A\} M^{-1}_{D AB} \{G_B, g\} \\
= \{f, g\}
\] (2.17)

where some gauge fixing conditions and the gauge generators \(J(i)^a \approx 0\) were written in a collective fashion as \(G_A \approx 0\), and \(M_{D AB} := \{G_A, G_B\}\).
The problem is now to find the brackets of the scalar variables. As an intermediate step we calculate \( \{ E^b_{(ij)}, \cosh(\eta_{(ij)}) \} \) by calculating the brackets of \( E_{(ij)} \) with both sides of \( N^a_{(i)} (M_{(ij)} N_{(j)})_a = N_{(i)} N_{(j)} \cosh(\eta_{(ij)}) \); our result is

\[
\{ E^b_{(ij)}, \cosh(\eta_{(ij)}) \} = \frac{1}{N_{(i)} N_{(j)}} N^a_{(i)} \varepsilon^a_{cb} (M_{(ij)} N_{(j)})_c
\]

which has as consequence

\[
\{ l_{(ij)}, \eta_{(ij)} \} = 1 \quad . (2.18)
\]

The curvature constraints (2.8), (2.9) induce constraints on the scalar phase space spanned by \( l_{ij}, \eta_{(ij)} \). These are first-class constraints that generate translations of the vertices; however, in terms of the scalar variables, the translation generators are generally complicated non-local expressions. To determine the explicit form of the induced constraints, we go back to the covariant description by choosing a gauge. We choose auxiliary reference frames at all the faces to embed the vectors \( E_{ij} \), with the restriction that they form closed triangles with edge lengths \( l_{ij} \). Once the vectors \( E \) are set in local frames, we prescribe the relative orientation between neighboring frames with the parallel transport matrices \( M \). These matrices are subject to two requirements, e.g. in the case of the matrix \( M_{ij} \), the hyperbolical angle between face \((i)\) and face \((j)\) has to be \( \eta_{(ij)} \) and the identity \( E_{(ij)} = -M_{(ij)} E_{(ji)} \) must hold.

To recover a simple form of the curvature constraints, we have to choose a gauge that is well-tailored for the scalar variables. In regions of the lattice where the extrinsic curvature vanishes, i.e., where \( \eta_{(ij)} = 0 \), we know that the parallel transport around a vertex, with no particle sitting on it, is given by the matrix

\[
W(v)^a_b = \exp(\alpha_v \hat{N}_{(v)}^c \varepsilon^a_{cb})
\]

where \( \hat{N}_{(v)} \) is the unit normal to any of the faces containing \((v)\), and \( \alpha_v \) is the deficit angle at \( v \)

\[
\alpha_v[\ell's] = 2\pi - \sum \alpha_i \approx \alpha^*_v[\eta's] = 0 \quad . (2.21)
\]

In the zero extrinsic curvature case, the only part of the parallel transport matrix that is not automatically equal to the identity matrix is the rotational part. The weak equality (2.21), that sets the deficit angle \( \alpha(v) \) to zero holds only for the zero extrinsic curvature case, and is what the generator of translations of \((v)\) in the direction normal to the lattice. Obviously, orthogonal translations of all vertices in an extrinsically flat region of the lattice leave the links’ lengths invariant.

Zero extrinsic curvature and normal translations produce trivial dynamics; therefore, the strategy is to choose a gauge in which the lattice is extrinsically flat in as many links as possible. Following ‘t Hooft, we consider a slicing of spacetime with a lattice that enjoys the following properties:
1. Zero extrinsic curvature polygons cover the lattice. The parallel transport between two neighboring cells maps the normal to the normal if their boundary belongs to one of the polygons, i.e. \( \eta_{(ij)} = 0 \) if the link \( (ij) \) is in one polygon.

2. The boundaries between polygon (that are the union of several lattice links) are straight lines called bones; bones meet at vertices named joints. Exactly three bones meet at each joint.

3. Particles are located at vertices inside the polygons and are allowed to move. Hence, a bone, where the extrinsic curvature does not vanish, ends at the particle’s vertex.

Note that the dual to the lattice of bones and joints is a triangular lattice; since every surface \( \Sigma \) can be triangulated this gauge induces no restriction in \( \Sigma \)'s topology.

Normal translation preserves this gauge, but it is not the only evolution that preserves it. However, normal translations have the advantage that they are generated by local expressions in terms of the scalar variables.

As seen above, orthonormal translations of vertices in the polygons are generated by the deficit angles, which are local functions. Now we show how local expressions for the generators of normal translations of vertices at joints, bones or particles are derived. At a joint \((J)\) three bones, with extrinsic curvatures \( \eta_{1,2}, \eta_{2,3}, \eta_{3,1} \), meet making angles \( \alpha_{1,2}, \alpha_{2,3}, \alpha_{3,1} \); these angles are functions of the links’ lengths \( \alpha = \alpha[l] \). Since we have only three bones going to a joint, the relation \( W(J) \approx 1 \) determines implicitly the angles \( \alpha(l) \) in terms of the extrinsic curvatures \( \eta \)

\[
W(J)[\alpha_i, \eta_{ij}] \approx 1 \Rightarrow \alpha_i[l']s \approx \alpha_i^*[\eta_{ij}] . \tag{2.22}
\]

These relations between \( \alpha's \) and \( \eta's \) at a joint are local. For a particle at the end of a bone we can produce a similar expression of the deficit angle \( \beta_p = \beta_p[l'ls] \) in terms of the extrinsic curvature at the bone \( \eta_p \) and the mass of the particle \( m(v) \)

\[
tr(W(p)[\beta_p, \eta_p]) \approx 1 + 2 \cos(m(p)) \Rightarrow \beta_p[l'ls] \approx \beta_p^*[\eta_p, m(p)] . \tag{2.23}
\]

’t Hooft derived the explicit formulas \((2.22), (2.23)\) for the angles in terms of the extrinsic curvatures. He also proved that the generator of normal translations at \((v)\) \( H(v) \) was simply the deficit angle at \((v)\)

\[
H(v) = 2\pi - \sum_{i \rightarrow v} \alpha_i^* \text{ if there is no particle at } (v) \\
H(v) = \beta_p^* \text{ if there is a particle at } (v). \tag{2.24}
\]

By \((2.22), (2.23)\), we see that these are linear combinations of the translation constraints. The local Hamiltonian constraints are

\[
\mathcal{H}(v) = H(v) - (2\pi - \sum_{i \rightarrow v} \alpha_i[l's]) \approx 0 \text{ if there is no particle at } (v) \\
\mathcal{H}(v) = H(v) - \beta_p[l'ls] \approx 0 \text{ if there is a particle at } (v). \tag{2.25}
\]
Previously we saw that the generator of normal translations of vertices in the interior of the domains is also the deficit angle; for calculational purposes we can place the vertices at bones on the same footing as joints. To do that, we consider one of the links connected to the vertex to be a zero extrinsic curvature “bone”. Thus, the generator of normal translations of every vertex is the deficit angle at the vertex, and the total deficit angle

\[ H = \sum_v H(v) \]  

(2.26)
can be regarded as a "Hamiltonian". Rather than in deriving 't Hooft's results [5,6] by an alternative method, we are interested in analyzing 't Hooft’s quantization procedure using the covariant lattice theory as a framework. We follow this procedure because in the covariant theory the structure of the constrained system is transparent. In particular, we have just found that the Hamiltonian (2.26) is constrained, by (2.25). In the next section, we compare the results of different choices of quantization for the lattice theory.

### III. FOUR QUANTIZATIONS

1. **'t Hooft quantization.** One assumes that the lattice is organized as a patchwork of extrinsically flat polygons, bones (polygon’s boundaries) and joints (vertices where bones meet) as described above; we will refer to these conditions hereafter as ’t Hooft’s gauge. The Hamiltonian constraints then fix the values of the deficit angles. If one forms the sum of these Hamiltonians over all lattice sites, the resulting function generates precisely the dynamical evolution which preserves ’t Hooft’s gauge. This suggests the following quantum theory: one considers the phase space spanned by the link lengths and boosts \( l_{ij}, \eta_{ij} \), with the canonical brackets, and the Hamiltonian operator which corresponds from the sum of deficit angles at the vertices. If the initial wave function has support only over \( l'_{ij} \)'s which respect ’t Hooft’s gauge, the same will hold at all times until the gauge crashes because a particle collides with a polygon’s wall. Then one uses ’t Hooft’s prescription for ”transitions” to get the initial conditions after the gauge’s crash [6].

2. **Canonical Quantization in ’t Hooft’s Gauge.** One assumes a simplicial lattice described in terms of the scalar variables and Poisson brackets described above. The first class curvature constraints are complicated non-local functions of these variables; then one goes to ’t Hooft’s gauge and fixes all the translational gauge freedom in the direction non-normal to the polygons. After gauge fixing, one expects the Dirac brackets to be complicated non-local expressions. One then chooses an internal time which labels the Cauchy surfaces, such as the length of a link which connects two particles with a non-zero relative velocity. The momentum conjugate to this internal time is the corresponding boost parameter. The Hamiltonian constraint (sum of deficit angles) with the gauge conditions can be solved for this momentum, leading to the physical Hamiltonian. Unlike the previous case, this Hamiltonian (a boost parameter) is not a cyclic function so there is no reason to expect that time would be quantized.

3. **Covariant Lattice Quantization.** One can quantize the lattice theory prior to fixing the frames. The link vectors become operators in the Lie algebra – in the Euclidian gravity
case the algebra is \( so(3) \) and one has the prediction that the lattice links have a discrete spectrum, \( l_{ij}^2 = l(l+1) \). This leads to a picture which is reminiscent of the original proposal of Ponzano and Regge, only with discrete space and continuous time.

4. Quantization in the Polygon Representation. One eliminates the pure gauge lattice structure down to the minimal lattice where all links are either basis loops of the homotopy group or separate two particles. The resulting link vectors have an \( SO(2,1) (SO(3)) \) algebra as in the covariant lattice, with the same remark with regard to the quantization of space. An internal time parameter can be identified and the physical Hamiltonian can be computed explicitly \[2\], leading to a quantization in the Schroedinger picture (with a continuous time parameter) \[3,4\].

IV. DISCUSSION – 't HOOFT'S GAUGE IN 3+1 DIMENSIONS

We have shown that the representation of (2+1)–dimensional gravity in terms of 't Hooft’s polygons can be derived from the exact lattice formulation of 2+1 gravity as the Chern-Simons theory of \( ISO(2,1) \) connections. However, the vertex Hamiltonians proposed by 't Hooft turn out to be first-class constraints, which must be solved together with the gauge conditions that specify a foliation by Cauchy surfaces that are patchworks of plat polygons.

If this gauge-fixing is carried out and the constraints are solved, one loses the cyclic form of the Hamiltonian and the conclusion that time is quantized. Vice-versa, if one chooses not to fix the gauge then this is not the correct form of the Hamiltonian when the quantized polygons fluctuate away from 't Hooft’s gauge.

Besides the issue of quantization of time, we considered also the question of whether space is quantized in the Euclidian version of the theory. Again, the answer depends crucially on the choice of quantization. If the frames are fixed prior to quantizing, one finds a continuous spectrum for the link lengths – but if the theory is quantized in a covariant manner then the link lengths acquire the spectra of rotation operators.

There is an extension of these results and of 't Hooft’s gauge to the lattice theory of topological gravity in 3+1 dimensions: an exact lattice theory has been proposed \[4\] based on the similarity between the topological gravity and \( B \wedge F \) theory \[5\]. In this theory the \( SO(3,1) \) matrices \( M_{(ij)} \) define parallel-transport between neighboring simplicial cells, and \( E_{(ij)} \) are the face bivectors in the local frames. One can choose frames where the matrices are boosts which leave the separating face between two cells fixed. Similar arguments to those followed here then show that the product of Lorentz matrices around a bone of the lattice is a pure rotation in a plane orthogonal to the bone; if one is considering the case of topological gravity with no matter sources, this would be constrained to be weakly equal to zero. The possibility of allowing a linear distribution of mass on the lattice bones and letting the deficit angle be non-zero is intriguing; perhaps such a construction can lead to a lattice theory for cosmic strings.

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REFERENCES

[1] Waelbroeck, H 1990 Class. Quantum Grav. 7 751
[2] Waelbroeck, H and Zertuche, F 1994 Phys. Rev. D 50 4966
[3] Waelbroeck, H 1994 Phys. Rev. D 50 4982
[4] Criscuolo, A Quevedo, H and Waelbroeck H 1995: “Quantization of 2+1 Gravity on the Torus”. Proceedings of the Canadian Association for Physics meeting, Quebec 1995, to be published
[5] ’t Hooft G. 1992 Class. Quantum Grav. 9 1335
[6] ’t Hooft G. 1993 Class. Quantum Grav. 10 1653
[7] Ashtekar A 1986 Phys. Rev.Lett. 57 2244
   Ashtekar A 1987 Phys. Rev. D 36 1587
   Renteln P and Smolin L 1989 Class. Quantum Grav. 6 275
[8] Witten E 1988 Nucl. Phys. B 311 46
[9] Waelbroeck H and Zapata J A Class. Quant. Grav. 11 989
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