An inverse boundary value problem for a linearized Benny–Luc equation with nonlocal boundary conditions

Yashar T. Mehraliyev¹, Bahar K. Valiyeva² and Aysel T. Ramazanova³*

Abstract: The work is devoted to the study of the solvability of an inverse boundary value problem with an unknown time-dependent coefficient for the linearized Benney–Luke equation with non-conjugate boundary conditions and integral conditions. The goal of the paper consists of the determination of the unknown coefficient together with the solution. The problem is considered in a rectangular domain. The definition of the classical solution of the problem is given. First, the given problem is reduced to an equivalent problem in a certain sense. Then, using the Fourier method the equivalent problem is reduced to solving the system of integral equations. Thus, the solution of an auxiliary inverse boundary value problem reduces to a system of three nonlinear integro-differential equations for unknown functions. Concrete Banach space is constructed. Further, in the ball from the constructed Banach space by the contraction mapping principle, the solvability of the system of nonlinear integro-differential equations is proved. This solution is also a unique solution to the equivalent problem. Finally, by equivalence, the theorem of existence and uniqueness of a classical solution to the given problem is proved.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Differential Equations; Applied Mathematics; Inverse Problems

Keywords: inverse value problem; linearized Benney–Luke equation; existence; uniqueness; classical solution

ABOUT THE AUTHORS
Yashar T. Mehraliyev He is working at the Baku State University and he is chief of the Chairprofessor. His research interests include Direct and Inverse Problems for Partial Differential Equations, The Spectral Theory of Differential Equations, Nonlinear Functional Analysis.

Bahar K. Valiyeva She is currently Ph. D. applicant and the main research interests of him include Inverse Problems for Partial Differential Equations.

Aysel T. Ramazanova She works and doing Postdoc Research at the University Duisburg-Essen. Her research interests include Inverse Problems for Partial Differential Equations, optimal control problem in the processes described by elliptic and hyperbolic equations.

PUBLIC INTEREST STATEMENT
Many problems of mathematical physics, continuum mechanics are boundary problems that reduce to the integration of a differential equation or a system of partial differential equations for given boundary and initial conditions. Problems in which, together with the solution of a differential equation, it is also required to determine the coefficient of the equation itself, or the right-hand side of the equation, in mathematics and mathematical modeling are called inverse problems. The theory of inverse problems for differential equations is an actively developing area of modern mathematics. The goal of the paper consists of the determination of the unknown coefficient together with the solution. Our paper establishes existence and uniqueness of the solution to an inverse boundary value problem for the Benny–Luc equation with integral conditions.
1. Introduction

There are many cases where the needs of the practice bring about the problems of determining coefficients or the right-hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, and quality control in industry, which makes them an active field of contemporary mathematics. Inverse problems for various types of have been studied in many papers. Many problems of gas dynamics, theory of elasticity, theory of plates, and shells are reduced to the consideration of differential equations in high-order partial derivatives (Algaizin & Kiyko, 2006). Of particular interest from the point of view of applications are differential equations of the fourth order (Shabrov, 2015), (Benney & Luke, 1964). Partial differential equations of the Benney–Luke type have applications in mathematical physics (Benney & Luke, 1964). Problems in which, together with the solution of a differential equation, it is also required to determine the coefficient of the equation itself, or the right-hand side of the equation, in mathematics and mathematical modeling are called inverse problems. The theory of inverse problems for differential equations is an actively developing area of modern mathematics. Various inverse problems for individual types of partial differential equations have been studied in many papers (Eskin, 2017; Janno & Seletski, 2015; Jiang, Liu, & Yamamoto, 2017; Lavrentyev, Romanov, & Shishatskii, 1980; Nakamura, Watanabe, & Kaltenbacher, 2009; Shcheglov, 2006; Tikhonov, 1963). The theory of inverse boundary value problems for fourth-order equations remains poorly understood. The papers (Kozhanov & Namsaraeva, 2018) and others are devoted to inverse boundary value problems for equations of the fourth order. In (Yuldashev, 2018), the unique solvability of a non-local inverse problem for a fourth-order Benney–Luke integro-differential equation with a degenerate kernel is considered. In contrast to Yuldashev (2018), this paper studies the inverse boundary value problem for the fourth-order Benney–Luke equation with integral conditions of the first kind.

2. Problem statement and its reduction to an equivalent problem

Let \( D_T = \{(x,t): 0 \leq x \leq 1, 0 \leq t \leq T\} \). Consider the following inverse problem. It is required to find a trio \((u(x,t); a(t); b(t))\) of functions \(u(x,t), a(t), b(t)\) connected by Equation [3]:

\[
\begin{align*}
    u_{tt}(x,t) - u_{xxx}(x,t) + a(x,t)u_{xxx}(x,t) - b(x,t)u_{xxtt}(x,t) \\
    = a(t)u(x,t) + b(t)g(x,t) + f(x,t), \quad (x,t) \in D_T,
\end{align*}
\]

in the domain \( D_T \), with the non-local initial conditions

\[
u(x,0) = \int_0^T p(t)u(x,t)dt + \varphi(x), \quad (0 \leq x \leq 1),
\]

the boundary conditions

\[
u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad u_{xxx}(0,t) = 0 \quad (0 \leq t \leq T),
\]

integral conditions

\[
\int_0^1 u(x,t)dx = 0 \quad (0 \leq t \leq T)
\]

and with the overdetermination conditions

\[
u(0,t) = h_1(t) \quad (0 \leq t \leq T),
\]

\[
u(1,t) = h_2(t) \quad (0 \leq t \leq T).
\]
where \( \alpha > 0, \beta > 0, \delta > 0 \) are fixed numbers, \( f(x, t), g(x, t), p(t), \psi(x), \chi(x), h_i(t) (i = 1, 2) \) are the given functions, and \( u(x, t), a(t), b(t) \) are the desired functions. We introduce the notation

\[
\mathcal{C}^4(D_T) = \{ u(x, t) : u(x, t) \in C^4(D_T), u_{xxxx}(x, t) \\
u_{xxxx}(x, t), u_{xxxx}(x, t), u_{xxxxx}(x, t) \in C(D_T) \}.
\]

**Definition 2.1.** Under the classic solution of inverse boundary value problem, we understand the trio \( \{ u(x, t), a(t), b(t) \} \) of functions \( u(x, t) \in \mathcal{C}^4(D_T), a(t) \in C[0, T], b(t) \in C[0, T] \) satisfying Equation (1) and conditions (2)–(6) in the ordinary sense.

In order to investigate problem (1)–(5), we first consider the following problem:

\[
y''(t) = a(t)y(t) \quad (0 \leq t \leq T), \tag{7}
\]

\[
y(0) = \int_0^T p(t)y(t)dt, \quad y'(0) + \delta y(T) = 0. \tag{8}
\]

where \( \delta \) is a given number, \( a(t) \in C[0, T], b(t) \in C[0, T] \) are the given functions, \( y = y(t) \) is a desired function, under the solution of problem (7),(8) we understand the function \( y(t) \) from \( C[0, T] \) and satisfying conditions (6),(7) in the ordinary sense. The following lemma is proved:

**Lemma 2.2.** Let \( p(t) \in C[0, T], a(t) \in C[0, T] \) and

\[
\| a(t) \|_{C[0, T]} \leq R = \text{const.}
\]

\[
(\| p(t) \|_{C[0, T]} + 2RT)T < 1. \tag{9}
\]

Then problem (7),(8) has only a trivial solution.

**Proof.** It is known that the boundary value problem (7), (8) is equivalent to the integral equation

\[
y(t) = \int_0^T \left( \frac{1 + \delta(T - t)}{1 + \delta t} p(r) - \frac{\delta t(T - r)}{1 + \delta t} a(r) \right) y(r)dr \\
+ \int_0^t (t - r) a(r) y(r) dr. \tag{10}
\]

Having denoted

\[
Ay(t) = \int_0^T \left( \frac{1 + \delta(T - t)}{1 + \delta t} p(r) - \frac{\delta t(T - r)}{1 + \delta t} a(r) \right) y(r)dr \\
+ \int_0^t (t - r) a(r) y(r) dr.
\]

and we write (10) in the form of an operator equation:

\[
y(t) = Ay(t).
\]

Equation (11) will be studied in the space \( C[0, T] \).

It is easy to see that the operator \( A \) is continuous in the space \( C[0, T] \).
Let us show that $A$ is a contraction mapping in $C[0, T]$. Indeed, for any $y(t), \tilde{y}(t)$ from $C[0, T]$ we have:

$$\|Ay(t) - A\tilde{y}(t)\|_{C[0, T]} \leq \left(\|p(t)\|_{C[0, T]} + 2RT\right)\|y(t) - \tilde{y}(t)\|_{C[0, T]}.$$  \hfill (12)

Then, using (9) in (12), we obtain $A$ is contraction mapping in the space $C[0, T]$. Therefore, in the space $C[0, T]$, the operator $A$ has a single fixed point $y(t)$ which is a solution of Equation (11). Thus, integral equation (10) has a unique solution in $C[0, T]$ and consequently, boundary value problem (7), (8) also has a unique solution in $C[0, T]$. Since $y(t) = 0$ is the solution of boundary value problem (7), (8), then it has only trivial solution.

The lemma is proved. \hfill $\square$

Along with problem (1)–(6), we consider the following auxiliary inverse boundary value problem.

It is required to determine a triple $\{u(x, t), a(t), b(t)\}$ functions $u(x, t) \in \mathcal{C}^{4,2}(D)$, $a(t) \in C[0, T]$, $b(t) \in C[0, T]$ from relations (1)–(3).

\begin{align}
\tag{13}
u_{xxx}(1, t) &= 0 (0 \leq t \leq T), \\
h_1''(t) - u_{xx}(0, t) + au_{xxx}(0, t) - \beta u_{xxxx}(0, t) &= a(t)h_1(t) + b(t)g(0, t) + f(0, t) (0 \leq t \leq T), \tag{14} \\
h_2''(t) - u_{xx}(1, t) + au_{xxx}(1, t) - \beta u_{xxxx}(1, t) &= a(t)h_2(t) + b(t)g(1, t) + f(1, t) (0 \leq t \leq T), \tag{15}
\end{align}

where

$$h(t) = h_1(t)g(1, t) - h_2(t)g(0, t) \neq 0 (0 \leq t \leq T).$$

The following theorem is valid.

**Theorem 2.3.** Let $\varphi(x), \psi(x) \in C[0, 1], p(t) \in C[0, T], \ h_i(t) \in C^2[0, T]$ $(i = 1, 2)$, $f(x, t) \in C(D)$, $g(x, t) \in C(D)$, $h(t) = h_1(t)g(1, t) - h_2(t)g(0, t) \neq 0$ $\int_0^1 f(x, t)dx = 0$, $\int_0^1 g(x, t)dx = 0$ $(0 \leq t \leq T)$ and the consistency conditions

$$\int_0^1 \varphi(x)dx = 0, \int_0^1 \psi(x)dx = 0,$$

$$\varphi(0) = h_1(0) - \int_0^T p(t)h_1(t)dt, \ \psi(0) = h_1''(0) + \beta h_1(T),$$

$$\varphi(1) = h_2(0) - \int_0^T p(t)h_2(t)dt, \ \psi(1) = h_2''(0) + \beta h_2(T).$$

be satisfied. Then the following statements are valid:

- Each classical solution $\{u(x, t), a(t), b(t)\}$ of problem (1)–(6) is the solution of problem (1)–(3), (13)–(15)

- Each solution $\{u(x, t), a(t), b(t)\}$ of problem (1)–(3), (13)–(15), is a classical solution of the problem (1)–(3), if
\[
\left( \|p(t)\|_{C(0,T)} + 2T\|a(t)\|_{C(0,T)} \right)T < 1
\]  \hspace{1cm} (16)

**Proof.** Let \( \{u(x,t), a(t), b(t)\} \) be a solution of problem (1)–(6). Integrating Equation (1) over \( x \) from 0 to 1, we have:

\[
\frac{d^2}{dt^2} \int_0^1 u(x,t)dx - u_x(1,t) + u_x(0,t) \\
+ a(u_{xxx}(1,t) - u_{xxx}(0,t)) - \beta(u_{ext}(1,t) - u_{ext}(0,t)) \\
= a(t) \int_0^1 u(x,t)dx + b(t) \int_0^1 g(x,t)dx + \int_0^1 f(x,t)dx \quad (0 \leq t \leq T).
\]

Assuming that \( \int_0^1 f(x,t)dx = 0, \int_0^1 g(x,t)dx = 0 \quad (0 \leq t \leq T) \), in view of (3), (4), we arrive at fulfillment (13).

Substituting \( x = 0 \) and \( x = 1 \) in Equation (1), respectively, we find:

\[
u_{tt}(0,t) - u_{xx}(0,t) + a_u u_{xxx}(0,t) - \beta u_{extx}(1,t) \\
= a(t) u(0,t) + b(t) g(0,t) + f(0,t)(0 \leq t \leq T),
\]  \hspace{1cm} (18)

\[
u_{tt}(1,t) - u_{xx}(1,t) + a_u u_{xxx}(1,t) - \beta u_{extx}(1,t) \\
= a(t) u(1,t) + b(t) g(1,t) + f(1,t)(0 \leq t \leq T),
\]  \hspace{1cm} (19)

Under the assumption \( h_i(t) \in C^2[0,T] (i = 1, 2) \) and differentiating two times (6) we have:

\[
u_t(0,t) = h'_1(t), \quad u_{tt}(0,t) = h''_1(t)(0 \leq t \leq T),
\]

\[
u_t(1,t) = h'_2(t), \quad u_{tt}(1,t) = h''_2(t)(0 \leq t \leq T).
\]

Considering these relations, from (18) and (19), taking into account (6), the fulfillment of (14) and (15) follows, respectively.

Now, suppose that \( \{u(x,t), a(t), b(t)\} \) is a solution to problem (1)–(3), (13)–(15), and (16) is satisfied. Then from (17) and (13), we find:

\[
\frac{d^2}{dt^2} \int_0^1 u(x,t)dx = a(t) \int_0^1 u(x,t)dx(0 \leq t \leq T).
\]  \hspace{1cm} (20)

From (22) and \( \int_0^1 \psi(x)dx = 0, \int_0^1 \phi(x)dx = 0 \) we have:

\[
\int_0^1 u(x,0)dx - \int_0^1 p(t) \left( \int_0^1 u(x,t)dx \right)dt \\
= \int_0^1 \left( u(x,0) - \int_0^1 p(t)u(x,t)dt \right)dx = \int_0^1 \psi(x)dx = 0,
\]

\[
\int_0^1 u_t(x,0)dx + \delta \int_0^1 u(x,T)dx = \int_0^1 \psi(x)dx = 0.
\]  \hspace{1cm} (21)
Since, by Lemma 2.2, problem (20), (21) has only a trivial solution, \[ \int_{0}^{1} u(x,t)dx = 0 \ (0 \leq t \leq T), \] i.e. conditions (4) are satisfied.

Further, from (14) and (18), (15) and (19) we get:

\[ \frac{d^2}{dt^2} (u(0,t) - h_1(t)) = a(t)(u(0,t) - h_1(t))(0 \leq t \leq T), \] \[ \frac{d^2}{dt^2} (u(1,t) - h_2(t)) = a(t)(u(1,t) - h_2(t))(0 \leq t \leq T). \] \[ (22) \]

From (2) and consistency conditions \[ \varphi(0) = h_1(0) - \int_{0}^{T} p(t)h_1(t)dt, \quad \psi(0) = h_1'(0) + \delta h_1(T), \]

\[ \varphi(1) = h_2(0) - \int_{0}^{T} p(t)h_2(t)dt, \quad \psi(1) = h_2'(0) + \delta h_2(T), \] we have:

\[ u(0,0) - h_1(0) - \int_{0}^{T} p(t)(u(0,t) - h_1(t))dt = u(0,0) - \int_{0}^{T} p(t)u(0,t)dt - \] \[ = \varphi(0) - \left( h_1(0) - \int_{0}^{T} p(t)h_1(t)dt \right) = 0, \] \[ u_1(0,0) - h_1'(0) + \delta (u(0,T) - h_1(T)) = \psi(0) - h_1'(0) - \delta h_1(T) = 0. \] \[ (24) \]

\[ u(1,0) - h_2(0) - \int_{0}^{T} p(t)(u(1,t) - h_2(t))dt = u(1,0) - \int_{0}^{T} p(t)u(1,t)dt - \] \[ = \varphi(1) - \left( h_2(0) - \int_{0}^{T} p(t)h_2(t)dt \right) = 0, \] \[ u_1(1,0) - h_2'(0) + \delta (u(1,T) - h_2(T)) = \psi(1) - h_2'(0) - \delta h_2(T) = 0. \] \[ (25) \]

From (22), (24), and also from (23), (25) by virtue of Lemma 2.2, we conclude that the conditions (5) and (6) is obtained.

The theorem is proved.

\[ \square \]

3. **Existence and uniqueness of the classical solution of the inverse problem**

The first component \( u(x,t) \) of the solution \( \{u(x,t), a(t), b(t)\} \) to problem (1)–(3), (13)–(15) will be sought in the form:

\[ u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = k\pi). \] \[ (26) \]

where

\[ u_k(t) = m_k \int_{0}^{1} u(x,t) \cos \lambda_k xdx \quad (k = 0, 1, 2, \ldots), \]

and

\[ m_k = \begin{cases} 1, k = 0, \\ 2, k = 1, 2, \ldots. \end{cases} \]

Then, applying the formal scheme of the Fourier method, from (1), (2), we get:
\[(1 + \beta_k^2)u_k'(t) + \lambda_k^2(1 + \alpha_k^2)u_k(t) = F_k(t; u, a, b) \quad 0 \leq t \leq T; k = 0, 1, \ldots \] (27)

\[u_k(0) = \int_0^T p(t)u_k(t)dt + \varphi_k; u_k'(0) + \delta u_k(T) = \psi_k (k = 0, 1, \ldots). \] (28)

where

\[F_k(t; u, a, b) = f_k(t) + a(t)u_k(t) + b(t)g_k(t); f_k(t) = 2 \int_0^1 f(x, t) \cos \lambda_k x dx; g_k(t) = 2 \int_0^1 g(x, t) \cos \lambda_k x dx; \]

\[\varphi_k = 2 \int_0^1 \varphi(x) \cos \lambda_k x dx; \psi_k = 2 \int_0^1 \psi(x) \cos \lambda_k x dx (k = 0, 1, \ldots). \]

Solving the problem (27), (28) we find:

\[u_0(t) = \left(1 - \frac{\delta t}{1 + \delta t}\right) \left(\int_0^T p(t)u_0(t)dt + \varphi_0\right) + \frac{t}{1 + \delta t}\psi_0 \]

\[- \frac{\delta t}{1 + \delta t} \int_0^T (T - r)F_0(r; u, a, b)dr + \int_0^t (t - r)F_0(r; u, a, b)dr. \] (29)

\[u_k(t) = \frac{\beta_k \cos \beta_k t + \delta \sin \beta_k (T - t)}{\beta_k + \delta \sin \beta_k T} \left(\int_0^T p(t)u_k(t)dt + \varphi_k\right) + \frac{\sin \beta_k t}{\beta_k + \delta \sin \beta_k T} \psi_k \]

\[- \frac{\delta \sin \beta_k t}{\beta_k (\beta_k + \delta \sin \beta_k T)(1 + \beta_k^2)} \int_0^T F_k(r; u, a, b) \sin \beta_k (T - r)dr \]

\[+ \frac{1}{\beta_k (1 + \beta_k^2)} \int_0^T F_k(r; u, a, b) \sin \beta_k (T - r)dr. \] (30)

where

\[\beta_k = \lambda_k \sqrt{\frac{1 + \alpha_k^2}{1 + \beta_k^2}} (k = 1, 2, \ldots). \]

After substituting expressions \[u_0(t)\] from (29), \[u_k(t)(k = 1, 2, \ldots)\] from (30) into (26), to determine the component \[u(x, t)\] of the solution of problem (1)—(3), (13)—(15), we get:
\[ u(x, t) = \left(1 - \frac{\delta t}{1 + \delta t}\right) \left( \int_0^t p(t)u_0(t)dt + \varphi_0 \right) + \frac{t}{1 + \delta t} \psi_0 \\
- \frac{\delta t}{1 + \delta t} \left( \int_0^t (T - r)F_0(r; u, a, b)dr + \int_0^t (t - r)F_0(t; u, a, b)dr \right) \\
+ \sum_{k=1}^\infty \left( \frac{\beta_k \cos \beta_k t + \delta \sin \beta_k (T - t)}{\beta_k^2} \left( \frac{1}{T} \int_0^t p(t)u_k(t)dt + \varphi_k \right) + \frac{\sin \beta_k t}{\beta_k} \right) \\
+ \frac{1}{\beta_k^2} \left( \frac{1}{1 + \beta_k^2} F_{2k-1}(r; u, a, b) \sin \beta_k (T - r)dr \right) \cos \lambda_k x. \quad (31) \]

Now from (14) and (15), taking into account (30), we obtain:
\[ a(t)h_1(t) + b(t)g(0, t) = h_1(t) - f(0, t) \]
\[ + \sum_{k=1}^\infty \left( \int_0^t (1 + \alpha_k^2)u_k(t) + \beta_k^2 u_k(t) \right), \]
\[ a(t)h_2(t) + b(t)g(1, t) = h_2(t) - f(1, t) \]
\[ + \sum_{k=1}^\infty (-1)^k \left( \int_0^t (1 + \alpha_k^2)u_k(t) + \beta_k^2 u_k(t) \right) \]
or considering that
\[ u_k(t) = \int_0^t \left( \frac{\beta_k^2}{1 + \beta_k^2} F_k(t; u, a, b) \right) \]

we have:
\[ a(t)h_1(t) + b(t)g(0, t) = h_1(t) - f(0, t) \]
\[ + \sum_{k=1}^\infty \left( \frac{\beta_k^2}{1 + \beta_k^2} u_k(t) + \frac{\beta_k^2}{1 + \beta_k^2} F_k(t; u, a, b) \right). \quad (32) \]
\[ a(t)h_2(t) + b(t)g(1, t) = h_2(t) - f(1, t) \]
\[ + \sum_{k=1}^\infty (-1)^k \left( \frac{\beta_k^2}{1 + \beta_k^2} u_k(t) + \frac{\beta_k^2}{1 + \beta_k^2} F_k(t; u, a, b) \right). \quad (33) \]

Assume that
\[ h(t) = h_1(t)g(1, t) - h_2(t)g(0, t) \neq 0(0 \leq t \leq T) \]

Then from (32) and (33) we find:
\[ a(t) = [h(t)]^{-1} \left\{ h_1(t) - f(0, t)g(1, t) - h_2(t) - f(1, t)g(0, t) \right\} \]
\[ + \sum_{k=1}^\infty \left( \int_0^t (1 + \alpha_k^2) u_k(t) + \frac{\beta_k^2}{1 + \beta_k^2} F_k(t; u, a, b) \right) \right\}. \quad (34) \]
\[ b(t) = [h(t)]^{-1} \left\{ h_1(t)h_2(t) - f(1, t) - h_2(t)h_1(t) - f(0, t) \right\} \]
\[ + \sum_{k=1}^\infty (-1)^k \left( \int_0^t (1 + \alpha_k^2) u_k(t) + \frac{\beta_k^2}{1 + \beta_k^2} F_k(t; u, a, b) \right) \right\}. \quad (35) \]

Further, after substituting the expression \( u_k(t) \) (\( k = 1, 2, \ldots \)) from (30) into (34), (35), respectively, we have:
\[ a(t) = (h(t))^{-1} \left( (h_1^2(t) - f(0, t))g(1, t) - (h_2^2(t) - f(0, t))g(0, t) \right) + \sum_{k=1}^{\infty} \left( g(1, t) - (-1)^k g(0, t) \right) \mu_k \begin{pmatrix} \beta_k & \delta \sin \beta_k \sin \beta_k \\ \beta_k \frac{\cos \beta_k t + \delta \sin \beta_k T}{\beta_k + \delta \sin \beta_k T} & \psi_k \end{pmatrix} \\
+ \int_0^t \beta(t) u_k(t) dt + \frac{\delta \sin \beta_k T}{\beta_k \left( \beta_k + \delta \sin \beta_k T \right)} \int_0^T F_k(r; a, b) \sin \beta_k (T - r) dr \]
\[ b(t) = (h(t))^{-1} \left( h_1^2(t) - f(1, t) - h_2^2(t) - f(0, t) \right) + \sum_{k=1}^{\infty} \left( h_1^2(t) - h_2^2(t) \right) \mu_k \begin{pmatrix} \beta_k & \delta \sin \beta_k \sin \beta_k \\ \beta_k \frac{\cos \beta_k t + \delta \sin \beta_k T}{\beta_k + \delta \sin \beta_k T} & \psi_k \end{pmatrix} \\
+ \int_0^t \beta(t) u_k(t) dt + \frac{\delta \sin \beta_k T}{\beta_k \left( \beta_k + \delta \sin \beta_k T \right)} \int_0^T F_k(r; a, b) \sin \beta_k (T - r) dr \]

Thus, the solution of the problem (1)-(3), (13)-(15) was reduced to the solution of the system (31), (36), (37) with respect to the unknown functions \( u(x, t), a(t) \) and \( b(t) \).

To study the question of the uniqueness of the solution of problem (1)-(3), (13)-(15), the following Lemma is important:

**Lemma 3.1.** If \( \{u(x, t), a(t), b(t)\} \) is any solution of problem (1)-(3), (13)-(15), then the functions \( u_k(t) \ (k = 0, 1, 2, \ldots) \), defined by
\[ u_k(t) = m_k \int_0^t u(x, t) \cos \lambda_k x dx \ (k = 0, 1, 2, \ldots), \]

satisfy system (29) and (30) on \([0, T]\).

**Proof.** Let \( \{u(x, t), a(t), b(t)\} \) be any solution of (1)-(3), (13)-(15). Then, multiplying both sides of Equation (1) by the function \( m_k \cos \lambda_k x \ (k = 0, 1, \ldots) \), integrating the obtained equality over \( x \) from 0 to 1 and using the relations,
\[ m_k \int_0^1 u(x, t) \cos \lambda_k x dx = \frac{d^2}{dt^2} \left( m_k \int_0^t u(x, t) \cos \lambda_k x dx \right) = u_k''(t) (k = 0, 1, \ldots), \]
The space $C^k([0,T])$ is a Banach space.

The norm in this set is defined as follows:

$$
\|u(x, t)\|_{C^k([0,T])} = \sum_{k=0}^{\infty} \rho_k \|u_k(t)\|_{C^k([0,T])},
$$

where $u(x, t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x$.

The norm of element $z = \{u, p, q\}$ is determined by the formula:

$$
\|z\|_{E^3_T} = \|u(x, t)\|_{B^3_{2,T}} + \|p(x, t)\|_{C[0,T]} + \|q(x, t)\|_{C[0,T]}.
$$

It is obvious that $B^3_{2,T}$ and $E^3_T$ are Banach spaces.
Now, in the space $E^k_T$ consider the operator

$$\Phi(u, a) = \{\Phi_1(u, a), \Phi_2(u, a), \Phi_3(u, a)\},$$

where

$$\Phi_1(u, a, b) = \tilde{u}(x, t) = \sum_{k=0}^{\infty} \tilde{u}_k(t)X_k(x).$$

$$\Phi_2(u, a, b) = \tilde{a}(t), \Phi_3(u, a, b) = \tilde{b}(t).$$

$\tilde{u}_0(t), \tilde{u}_k(t), \tilde{a}(t)$ and $\tilde{b}(t)$ are equal to the right-hand sides of (29), (30), (36) and (37).

It is easy to see that

$$1 + \beta h^2_s > \beta h^2_s, \quad \frac{1}{1 + \beta h^2_s} < \frac{1}{\beta h^2_s}.$$

$$\sqrt{\frac{\alpha}{1 + \beta h^2_s}} \leq \beta_k \leq \sqrt{\frac{\beta}{1 + \alpha \beta h^2_s}}, \quad \frac{1}{\sqrt{1 + \alpha \beta h^2_s}} \leq \beta_k \leq \sqrt{\frac{\beta}{1 + \alpha \beta h^2_s}}.$$

Taking into account these relations and $0 < \sqrt{1 + \alpha \beta h^2_s} - \delta \leq \beta_k - \delta \leq \beta_k + \delta \sin \beta h_T$, with the help of simple transformations we find:

$$\|\tilde{u}_0(t)\|_{C[0, T]} \leq \left(1 + \frac{T}{1 + \beta h^2_s}\right)\|\varphi_0\| + T\|p(t)\|_{C[0, T]}\|u_0(t)\|_{C[0, T]} + \frac{T}{1 + \beta h^2_s}\|\varphi_0\|$$

$$+ \left(1 + \frac{T}{1 + \beta h^2_s}\right)T \left[\sqrt{T} \left(\int_0^T |f_0(r)|^2 \, dr\right)^{1/2} + T\|a(t)\|_{C[0, T]}\|u_0(t)\|_{C[0, T]}\right]$$

$$+ \|b(t)\|_{C[0, T]} \sqrt{T} \left(\int_0^T |g_0(r)|^2 \, dr\right)^{1/2},$$

$$\left(\sum_{k=1}^{\infty} \left(\lambda_k^2 \|u_k(t)\|_{C[0, T]}\right)^2\right)^{1/2} \leq \sqrt{6(1 + \beta h^2_s)} \sup_k \left(\frac{\beta_k}{\beta_k - \delta}\right)\left(\sum_{k=1}^{\infty} \left(\lambda_k^2 \|\varphi_k\|\right)^2\right)^{1/2}$$

$$+ T\|p(t)\|_{C[0, T]} \left(\sum_{k=1}^{\infty} \left(\lambda_k^2 \|u_{2k}(t)\|_{C[0, T]}\right)^2\right)^{1/2}$$

$$+ \sqrt{\frac{6(1 + \beta h^2_s)}{\beta h^2_s}} \sup_k \left(\frac{\beta_k}{\beta_k - \delta}\right)\left(\sum_{k=1}^{\infty} \left(\lambda_k^2 \|\varphi_{2k}\|\right)^2\right)^{1/2}$$

$$+ \frac{1}{\beta h^2_s} \sqrt{\frac{6(1 + \beta h^2_s)}{\beta h^2_s}} \left(1 + \frac{1}{\sqrt{1 + \alpha \beta h^2_s}} - \delta\right)\left[\sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^2 \|f_k(r)\|\right)^2 \, dr\right)^{1/2}\right]$$

$$+ T\|a(t)\|_{C[0, T]} \times \left(\sum_{k=1}^{\infty} \left(\lambda_k^2 \|u_k(t)\|_{C[0, T]}\right)^2\right)^{1/2}$$

$$+ \sqrt{T}\|b(t)\|_{C[0, T]} \left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^2 \|g_k(r)\|\right)^2 \, dr\right)^{1/2},$$

$$\|\tilde{a}(t)\|_{C[0, T]} \leq \left\|h(t)\right\|_{C[0, T]}^{-1}\left[\left\|h(t)^{-1}\right\|_{C[0, T]}^{1/2}\right.$$
\[ + \|g(1, t)| + |g(0, t)\|_{C^{0, \alpha}} \left( \sum_{k=1}^{\infty} \lambda_k^2 \right) \]
\[ + \left( \frac{1 + \alpha}{\beta} \sup_k \left( \frac{\beta_k}{\beta_k - \delta} \right) \left( \sum_{k=1}^{\infty} \left( \lambda_k \phi_k \right)^2 \right) \right)^{\frac{1}{2}} \]
\[ + \|p(t)\|_{C^{0, \alpha}} \left( \sum_{k=1}^{\infty} \left( \lambda_k^2 \|u_k(t)\|_{C^{0, \alpha}} \right)^2 \right)^{\frac{1}{2}} \]
\[ + \sqrt{\frac{1 + \beta}{\alpha}} \sup_k \left( \frac{\beta_k}{\beta_k - \delta} \right) \left( \sum_{k=1}^{\infty} \left( \lambda_k \phi_k \right)^2 \right)^{\frac{1}{2}} \]
\[ + \frac{1}{\beta} \sqrt{\frac{1 + \beta}{\alpha}} \left( 1 + \frac{1}{\sqrt{\frac{1}{T} + \delta}} \right) \left[ \sqrt{T} \left( \sum_{k=1}^{\infty} (\lambda_k^2 f_k(t))^2 \right) \right]^{\frac{1}{2}} \]
\[ + \|a(t)\|_{C^{0, \alpha}} \left( \sum_{k=1}^{\infty} \left( \lambda_k^2 \|u_k(t)\|_{C^{0, \alpha}} \right)^2 \right)^{\frac{1}{2}} \]
\[ + \|b(t)\|_{C^{0, \alpha}} \left( \sum_{k=1}^{\infty} \left( \lambda_k^2 \|g_k(t)\|_{C^{0, \alpha}} \right)^2 \right)^{\frac{1}{2}} \right] \}
\[ \|b(t)\|_{C^{0, \alpha}} \leq \|h(t)^{-1}\|_{C^{0, \alpha}} \]
\[ + \left\{ \|h_1(t)(h_2^\prime(t) - f(1, t)) - h_2(t)(h_1^\prime(t) - f(0, t))\|_{C^{0, \alpha}} \right\} + \|h_1(t)\| + \|h_2(t)\|_{C^{0, \alpha}} \left( \sum_{k=1}^{\infty} \lambda_k^2 \right)^{\frac{1}{2}} \left[ \frac{1 + \alpha}{\beta} \sup_k \left( \frac{\beta_k}{\beta_k - \delta} \right) \right] \]
\[ \left[ \left( \sum_{k=1}^{\infty} \left( \lambda_k \phi_k \right)^2 \right) \right]^{\frac{1}{2}} + \|p(t)\|_{C^{0, \alpha}} \left( \sum_{k=1}^{\infty} \left( \lambda_k^2 \|u_k(t)\|_{C^{0, \alpha}} \right)^2 \right)^{\frac{1}{2}} \]
\[ + \sqrt{\frac{1 + \beta}{\alpha}} \sup_k \left( \frac{\beta_k}{\beta_k - \delta} \right) \left( \sum_{k=1}^{\infty} \left( \lambda_k \phi_k \right)^2 \right)^{\frac{1}{2}} \]
\[ + \frac{1}{\beta} \sqrt{\frac{1 + \beta}{\alpha}} \left( 1 + \frac{1}{\sqrt{\frac{1}{T} + \delta}} \right) \left[ \sqrt{T} \left( \sum_{k=1}^{\infty} (\lambda_k^2 f_k(t))^2 \right) \right]^{\frac{1}{2}} \]
Suppose that the data of the problem (1)–(3), (13)–(15) satisfy the following conditions:

1. \( \alpha > 0, \beta > 0, 0 \leq \delta < \sqrt{\frac{\pi}{2}} \), \( \mathbf{p}(t) \in C[0, T] \).

2. \( \mathbf{q}(x) \in C^4[0, 1], \mathbf{q}^{(5)}(x) \in L_2(0, 1), \mathbf{q}(0) = \mathbf{q}'(1) = \mathbf{q}''(0) = \mathbf{q}'''(1) = 0 \).

3. \( \mathbf{w}(x) \in C^4[0, 1], \mathbf{w}^{(4)}(x) \in L_2(0, 1), \mathbf{w}(0) = \mathbf{w}'(1) = \mathbf{w}''(0) = \mathbf{w}'''(1) = 0 \).

4. \( f(x, t), f_\alpha(x, t) \in C(D_T), f_\alpha(x, t) \in L_2(D_T), f_\alpha(0, t) = f_\alpha(1, t) = 0 \quad (0 \leq t \leq T) \).

5. \( g(x, t), g_\alpha(x, t) \in C(D_T), g_\alpha(x, t) \in L_2(D_T), g_\alpha(0, t) = g_\alpha(1, t) = 0 \quad (0 \leq t \leq T) \).

6. \( h_i(t) \in C^3[0, T] (i = 1, 2), h_i(t) = h_i(t)g_i(1, t) - h_i(t)g_i(0, t) \neq 0 \quad (0 \leq t \leq T) \).

Then, considering (38)—(39), (40) and (41) we get:

\[
\| \hat{u}_0(t) \|_{C[0, T]} \leq A_1(T) + B_1(T) \| a(t) \|_{C[0, T]} \| u(x, t) \|_{L^2},
\]

\[
+ C_1(T) \| u(x, t) \|_{L^2} + D_1(T) \| b(t) \|_{C[0, T]} ;
\]

\[
\left\{ \sum_{k=1}^T \left( \lambda_k^2 \| \hat{u}_k(t) \|_{C[0, T]} \right) \right\}^\frac{3}{2} \leq A_2(T) + B_2(T) \| a(t) \|_{C[0, T]} \| u(x, t) \|_{L^2},
\]

\[
+ C_2(T) \| u(x, t) \|_{L^2} + D_2(T) \| b(t) \|_{C[0, T]} ;
\]

\[
\| \hat{a}(t) \|_{C[0, T]} \leq A_3(T) + B_3(T) \| a(t) \|_{C[0, T]} \| u(x, t) \|_{L^2},
\]

\[
+ C_3(T) \| u(x, t) \|_{L^2} + D_3(T) \| b(t) \|_{C[0, T]} ;
\]

\[
\| \hat{b}(t) \|_{C[0, T]} \leq A_4(T) + B_4(T) \| a(t) \|_{C[0, T]} \| u(x, t) \|_{L^2},
\]

\[
+ C_4(T) \| u(x, t) \|_{L^2} + D_4(T) \| b(t) \|_{C[0, T]} ;
\]

where

\[
A_1(T) = \left( 1 + \frac{T}{1 + \delta} \right) \| \mathbf{q}(x) \|_{L^2[0, 1]}
\]

\[
+ \frac{T}{1 + \delta} \| \mathbf{q}(x) \|_{L^2[0, 1]} + \left( 1 + \frac{T}{1 + \delta} \right) T \sqrt{T} \| f(x, t) \|_{L^2(D_T)} ;
\]
\[
B_1(T) = \left(1 + \frac{T}{1 + \delta t}\right)T^2, \quad C_1(T) = \left(1 + \frac{T}{1 + \delta t}\right)T\|p(t)\|_{C[0, T]},
\]
\[
D_1(T) = \left(1 + \frac{T}{1 + \delta t}\right)T\sqrt{T}\|g(x, t)\|_{L_2(0, T)},
\]
\[
A_2(T) = \sqrt{6}(1 + \delta) \sup_k \left(\frac{\beta_k}{\beta_k - \delta}\right) \|\psi^{(5)}(x)\|_{L_2(0, 1)}
\]
\[
+ \sqrt{6(1 + \beta)} \sup_k \left(\frac{\beta_k}{\beta_k - \delta}\right) \|\psi^{(4)}(x)\|_{L_2(0, 1)}
\]
\[
+ \frac{1}{\sqrt{T}} \sqrt{6(1 + \beta)} \left(1 + \frac{1}{\sqrt{T} + \delta} \right) \|f_{xx}(x, t)\|_{L_2(0, T)},
\]
\[
B_2(T) = \frac{1}{\sqrt{T}} \sqrt{6(1 + \beta)} \left(1 + \frac{1}{\sqrt{T} + \delta} \right) T,
\]
\[
C_2(T) = \sqrt{6}(1 + \delta) \sup_k \left(\frac{\beta_k}{\beta_k - \delta}\right) T\|p(t)\|_{C[0, T]},
\]
\[
D_2(T) = \frac{1}{\sqrt{T}} \sqrt{6(1 + \beta)} \left(1 + \frac{1}{\sqrt{T} + \delta} \right) \|g_{xx}(x, t)\|_{L_2(0, T)},
\]
\[
A_3(T) = \left\|\|h(t)\|^{-1}\right\|_{C[0, T]}
\]
\[
\left\{\left(\|h_1''(t) - f(0, t)\| g(1, t) - (h_1''(t) - f(1, t)) g(0, t)\|_{C[0, T]}
\right.
\right.
\]
\[
+ \|g(1, t) + g(0, t)\|_{C[0, T]} \left(\sum_{k=1}^\infty \lambda_k^2\right)^{\frac{1}{2}}
\]
\[
\times \left(1 + \alpha(1 + \delta) \right) \sup_k \left(\frac{\beta_k}{\beta_k - \delta}\right) \|\psi^{(5)}(x)\|_{L_2(0, 1)}
\]
\[
+ \sqrt{6(1 + \beta)} \sup_k \left(\frac{\beta_k}{\beta_k - \delta}\right) \|\psi^{(4)}(x)\|_{L_2(0, 1)}
\]
\[
+ \frac{1}{\sqrt{T}} \sqrt{6(1 + \beta)} \left(1 + \frac{1}{\sqrt{T} + \delta} \right)
\]
\[
\times \left\|f_{xx}(x, t)\|_{L_2(0, T)} + \left\|f_{xx}(x, t)\|_{C[0, T]} \left(\sum_{k=1}^\infty \lambda_k^2\right)^{\frac{1}{2}} \right\}
\]
\[
B_3(T) = \left\|\|h(t)\|^{-1}\right\|_{C[0, T]} \|g(1, t) + g(0, t)\|_{C[0, T]} \left(\sum_{k=1}^\infty \lambda_k^2\right)^{\frac{1}{2}}
\]
\[
C_3(T) = \left\| \left | \frac{1}{h(t)} \right| + \left | \frac{g(0, t)}{\int_{0}^{T} h(t) \, dt} \right| \right\|_{C[0, T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \\
\times \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) + 1, \\
D_3(T) = \left\| \left | \frac{1}{h(t)} \right| + \left | \frac{g(0, t)}{\int_{0}^{T} h(t) \, dt} \right| \right\|_{C[0, T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \\
\times \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) + 1, \\
A_4(T) = \left\| \left | \frac{1}{h(t)} \right| + \left | \frac{g(0, t)}{\int_{0}^{T} h(t) \, dt} \right| \right\|_{C[0, T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \\
\times \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) + 1, \\
B_4(T) = \left\| \left | \frac{1}{h(t)} \right| + \left | \frac{g(0, t)}{\int_{0}^{T} h(t) \, dt} \right| \right\|_{C[0, T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \\
\times \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) + 1, \\
C_4(T) = \left\| \left | \frac{1}{h(t)} \right| + \left | \frac{g(0, t)}{\int_{0}^{T} h(t) \, dt} \right| \right\|_{C[0, T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \\
\times \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) \left( 1 + \frac{1}{\sqrt{\lambda_k}} \right) + 1.
\]
\[
\times \frac{(1 + \alpha)(1 + \delta)}{\beta} \sup_k \left( \frac{b_k}{b_k - \delta} \right) T \| p(t) \|_{C(0,T)},
\]

\[
D_4(T) = \left\| [h(t)]^{-1} \right\|_{C(0,T)} [\| h_1(t) \| + \| h_2(t) \|]_{C(0,T)} \left( \sum_{k=1}^{\infty} \frac{1}{\kappa_k^2} \right)^{\frac{1}{2}} \times \left( \frac{1}{\beta} \sqrt{\frac{T(1 + \beta)}{\alpha}} \right) \left( 1 + \frac{1}{\sqrt{\frac{\alpha}{T \beta^{\sigma}}}} \right) \| g_{xx}(x, t) \|_{L^2(0,T)} + \left\| g_{xx}(x, t) \right\|_{C(0,T)} \right),
\]

From inequalities (45)–(48) we deduce:

\[
\| \ddot{u}(x, t) \|_{\mathbb{B}^1} + \| \ddot{\theta}(t) \|_{C(0,T)} \leq A(T)
\]

\[
+ B(T) \| a(t) \|_{C(0,T)} \| u(x, t) \|_{\mathbb{B}^2} + C(T) \| u(x, t) \|_{\mathbb{B}^2} + D(T) \| b(t) \|_{C(0,T)},
\]

where

\[
A(T) = A_1(T) + A_2(T) + A_3(T) + A_4(T),
\]

\[
B(T) = B_1(T) + B_2(T) + B_3(T) + B_4(T),
\]

\[
C(T) = C_1(T) + C_2(T) + C_3(T) + C_4(T),
\]

\[
D(T) = D_1(T) + D_2(T) + D_3(T) + D_4(T).
\]

So, we can prove the following theorem:

**Theorem 3.2.** Let conditions 1) – 6) be satisfied, and

\[
(B(T)(A(T) + 2) + C(T) + D(T))(A(T) + 2) < 1
\]

(46)

then problem (1)–(3), (13)–(15) has a unique solution in the sphere \( K = K_R(\| z \|_{E^1} \leq R \leq A(T) + 2) \) of the space \( E^1_2 \).

**Proof.** In the space \( E^1_2 \) consider the equation

\[
z = \Phi z,
\]

(47)

where \( z = \{u, a, b\} \), the components \( \Phi_i(u, a, b)(i = 1, 2, 3) \), of the operator \( \Phi(u, a, b) \), are determined by the right-hand sides of Equations (31), (36), and (37).

Consider the operator \( \Phi(u, a, b) \) in the sphere \( K = K_R \) from \( E^1_2 \). Similar to (45) we get that for any \( z, z_1, z_2 \in K_R \) the following estimates are valid:

\[
\| \Phi z \|_{E^1_2} \leq A(T) + B(T) \| a(t) \|_{C(0,T)} \| u(x, t) \|_{\mathbb{B}^2} + C(T) \| u(x, t) \|_{\mathbb{B}^2} + D(T) \| b(t) \|_{C(0,T)} \leq A(T) + B(T)(A(T) + 2) + C(T)(A(T) + 2) + D(T)(A(T) + 2),
\]

(48)
\[ \|\Phi z_1 - \Phi z_2\|_{\mathcal{F}_1} \leq B(T) R \left( \|\omega_1(t) - \omega_2(t)\|_{C[0,T]} \right) \]
\[ + \|u_1(x,t) - u_2(x,t)\|_{\mathcal{B}_T} + C(T) \|u_1(x,t) - u_2(x,t)\|_{\mathcal{B}_T} \]
\[ + D(T) \|b_1(t) - b_2(t)\|_{C[0,T]} \]
(49)

Then, using (46), from (48) and (49), it follows that the operator \( \Phi \) acts in the sphere \( K = K_\delta \) and it is contraction mapping. Therefore, in the sphere \( K = K_\delta \), the operator \( \Phi \) has a unique fixed point \( \{z\} = \{u, a, b\} \), that is a solution of Equation (47).

The function \( u(x,t) \), as the element of the space \( \mathcal{B}_T^2 \), has continuous derivatives \( u_k(x,t), u_{xx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t) \) in \( D_T \).

Now, differentiating two times (29), (30), we get:
\[ u_0''(t) = -\frac{\delta}{1 + \delta T} \left( \int_0^T p(t)u_0(t)dt + \varphi_0 \right) + \frac{1}{1 + \delta T} \varphi_0 \]
\[ - \frac{\delta}{1 + \delta T} \left( \int_0^T (T - \tau)F_0(\tau; u, a, b)d\tau + \int_0^T F_0(\tau; u, a, b)d\tau \right) \]
\[ + \frac{\beta}{\beta_k + \delta \sin \beta_k T} \left( \int_0^T p(t)u_k(t)dt + \varphi_k \right) \]
\[ + \frac{\delta \cos \beta_k t}{\beta_k + \delta \sin \beta_k T} \varphi_k - \frac{\delta \cos \beta_k t}{(\beta_k + \delta \sin \beta_k T)(1 + \delta T)} \]
\[ \times \left( \int_0^T F_k(\tau; u, a, b) \sin \beta_k (T - \tau)d\tau \right) \]
\[ + \frac{1}{1 + \beta_k^2} \int_0^T F_k(\tau; u, a, b) \cos \beta_k (t - \tau)d\tau. \]
(50)

It is clear that \( u_0''(t) \in C[0,T] \) \( (k = 0, 1, \ldots) \). Further, from (50) and (51), we obtained:
\[ \|u_0''(t)\|_{C[0,T]} \leq \frac{\delta}{1 + \delta T} \left( \|\varphi(x)\|_{L_2(0,1)} + T \|p(t)\|_{C[0,T]} \|u(x,t)\|_{\mathcal{B}_T} \right) \]
\[ + \frac{\delta}{1 + \delta T} \|\varphi(x)\|_{L_2(0,1)} + \left( 1 + \frac{\delta T}{1 + \delta T} \right) \]
\[ \times \left( \sqrt{T} \|f(x,t)\|_{L_2(0,1)} + T \|a(t)\|_{C[0,T]} \|u(x,t)\|_{\mathcal{B}_T} + \sqrt{T} \|b(t)\|_{C[0,T]} \|g(x,t)\|_{L_2(0,1)} \right) \]
\[ \left( \sum_{k=1}^{\infty} \left( \frac{\beta}{\beta_k + \delta} \right) \right)^{\frac{1}{2}} \leq (1 + \delta) \sqrt{\frac{6(1 + \beta)}{\alpha} \sup_k \left( \frac{\beta_k}{\beta_k - \delta} \right)} \]
\[ \left( \|\varphi(\delta x)\|_{L_2(0,1)} + T \|p(t)\|_{C[0,T]} \|u(x,t)\|_{\mathcal{B}_T} \right) \]
\[ + \sup_k \left( \frac{\beta_k}{\beta_k - \delta} \right) \|\varphi(\delta x)\|_{L_2(0,1)} + \frac{\sqrt{6}}{\beta} \left( 1 + \frac{1}{1 + \frac{1}{\sqrt{T} \delta - \delta}} \right) \]
\[ \times \left( \sqrt{T} \|f(x,t)\|_{L_2(0,1)} + \sqrt{T} \|a(t)\|_{C[0,T]} \|u(x,t)\|_{\mathcal{B}_T} + \sqrt{T} \|b(t)\|_{C[0,T]} \|g(x,t)\|_{L_2(0,1)} \right) \]
\[ + T \|a(t)\|_{C[0,T]} \|u(x,t)\|_{\mathcal{B}_T} + \sqrt{T} \|b(t)\|_{C[0,T]} \|g(x,t)\|_{L_2(0,1)} \].

It is seen that \( u_k(x,t) \in C(D_T) \).

From (26) it is easy to see that \( u_k''(t) \in C[0,T] \) \( (k = 0, 1, \ldots) \) and the validity of the estimates:
Let all the conditions of theorem 2 has a δ₀ \sum_{k=1}^{∞} \frac{\lambda_k^2}{\beta} \left(\frac{\sum_{k=1}^{∞} \lambda_k^2 \|u_k(t)\|_{C(0, T)}}{\beta} \right)^{\frac{1}{2}} + \frac{\sqrt{2}}{\beta} \left\|a(t)u_x(x, t) + b(t)g_x(x, t) + g_x(x, t)\|_{L_2(0, 1)} \right\|.

Then, it follows that \( u_1(x, t), u_{tx}(x, t), u_{txx}(x, t) \in C(D_T) \).

It is easy to verify that Equation (1) and conditions (2), (3), (13)–(15) are satisfied in the ordinary sense. Consequently, \( \{u(x, t), a(t), b(t)\} \) is a solution of problem (1)–(3), (13)–(15) and by Lemma 3 it is unique in the sphere \( K = K_2 \). Theorem is proved.

The following theorem is proved by means of Theorem 3.2

**Theorem 3.3.** Let all the conditions of theorem 2,

\[
\int_0^1 f(x, t)dx = 0, \int_0^1 g(x, t)dx = 0 (0 \leq t \leq T)
\]

and consistency conditions

\[
\int_0^1 \psi(x)dx = 0, \int_0^1 \psi(x)dx = 0.
\]

\[
\psi(0) = h_1(0) - \int_0^T p(t)h_1(t)dt, \psi(0) = h_1'(0) + \delta h_1(T),
\]

\[
\psi(1) = h_2(0) - \int_0^T p(t)h_2(t)dt, \psi(1) = h_2'(0) + \delta h_2(T).
\]

be satisfied. Then in the sphere \( K = K_2 ||z||_{E_0} \leq R = A(T) + 2 \) of the space \( E_0^3 \), problem (1)–(6) has a unique classical solution.

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**Author details**

Yashar T. Mehraliyev
E-mail: yashar.aze@mail.ru

Bahar K. Valiyeva
E-mail: aristokratka-1988@mail.ru

Aysel T. Ramazanova
E-mail: aysel.ramazanova@uni-de.de

ORCID ID: http://orcid.org/0000-0003-0166-6018

1 Faculty of Mechanics and Mathematics, Department of Differential and Integral equation, Baku State University, Baku, Azerbaijan.

2 Faculty of Mathematics and Informatik, Department of Algebra und Geometry, Ganja State University, Ganja, Azerbaijan.

3 Faculty of Mathematics, Department of Nonlinear Optimization, University Duisburg-Essen, Essen, Germany.

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