Mathematical models of transient processes for the distribution of resources in the system

E N Byhovets, S G Klimanov, A V Kryanev and D E Sliva
National Research Nuclear University "MEPhI", Russia

avkryanev@mephi.ru

Abstract. Problems for shares of resources to be distributed in conditions of constraints, including the process of dynamic change of shares are considered. The tasks of forming and dynamically changing the composition of portfolios are considered, in particular, in the presence of group restrictions. A mathematical model for predicting the dynamics of shares for a transition process from one equilibrium state of the system to another equilibrium state is proposed. The case of the presence of a noise chaotic component in such a model is considered. An example of the application of the proposed model of the dynamic process of resource allocation is given.

Keywords: resource allocation, constraints, group constraints, portfolio, effective portfolio, forecasting, equilibrium, transition process.

1. Introduction

In the formulation of many applied problems the unknown functions of time are the shares of the distributed resource over the objects of the system under consideration. Examples of such problems are the problems of restructuring the composition of effective portfolios [1-4] or the dynamics of the distribution of the priority coefficients of the quality indicators of a complex economic or technical system [5]. When solving this type of problem, it is often known, sometimes with some degree of uncertainty, the distribution of shares for the future fixed time. The article considers the problem of determining the transition process from the initial distribution of shares to their final distribution.

2. Schemes for the formation of an effective portfolio of shares of resource allocation in a system under constraints

Let, for example, resources be distributed between objects of the system. Then, based on preliminary estimates, and often on real opportunities that limit the amount of resources that can be distributed in the system under consideration, a priori restrictions are imposed on the volumes of distributed resources, and thus on the shares of the portfolio being formed. Such restrictions have the form of group restrictions:

\[ a_j \leq x_{j1} + \cdots + x_{jm_j} \leq b_j, \quad j = 1, \ldots, m, \quad m_1 + \cdots + m_m = n, \]  
(1)
where \( m \) is the number of a priori groups, \( m_j \) is the number of objects in the \( j \)-th group, \( a_j, b_j, 0 \leq a_j \leq b_j \leq 1 \) is the lower and upper boundaries of the total fraction of the \( j \)-th group.

It is assumed that each object (each index \( i = 1, \ldots, n \)) belongs to one and only one of the groups \((1)\). In order for the system of a priori constraints \((1)\) to be consistent, it is necessary and sufficient that the conditions \( \sum_{j=1}^m a_j \leq 1, \sum_{j=1}^m b_j \geq 1 \), are satisfied. In particular, if \( a_j = b_j, j = 1, \ldots, m, \) then this means fixing the total share of resources included in the \( j \)-th group in the portfolio.

The system of a priori constraints \((1)\) has often the form
\[
a_j \leq x_j \leq b_j, \quad j = 1, \ldots, n, \tag{2}
\]
that is, the constraints are superimposed separately on the share of resources for each object.

If there are a priori constraints to find effective portfolios, we have a two-criteria problem \([6, 7]\):
\[
\begin{align*}
\sigma_p^2 &= (Wx, x) - \min \\
m_p &= (m, x) - \max \\
0 &\leq x_i \leq 1, \quad i = 1, \ldots, n, \quad \sum_{i=1}^n x_i = 1, \\
a_j &\leq x_{j1} + \cdots + x_{jm_j} \leq b_j, \quad j = 1, \ldots, m, \\
m_p = (m, x) \in [m_p^*, m_{p\max}]
\end{align*}
\]
where \( m_p, \sigma_p^2 \) and \((m_{p\max}, \sigma_{p\max}^2)\) are determined by the solutions of the following problems:

problem of forming a portfolio of minimum uncertainty of efficiency:
\[
\sigma_p^2 = (Wx, x) - \min
\]
\[
0 \leq x_i \leq 1, \quad i = 1, \ldots, n, \quad \sum_{i=1}^n x_i = 1, \\
a_j \leq x_{j1} + \cdots + x_{jm_j} \leq b_j, \quad j = 1, \ldots, m,
\]
\[
m_p^* = (m, x^*), \quad \sigma_p^2 = (Wx^*, x^*), \quad x^* \text{ is the solution of problem (4)},
\]
and the problem of forming a portfolio with the maximum average expected value of efficiency:
\[
m_p = (m, x) - \max
\]
\[
0 \leq x_i \leq 1, \quad i = 1, \ldots, n, \quad \sum_{i=1}^n x_i = 1, \\
a_j \leq x_{j1} + \cdots + x_{jm_j} \leq b_j, \quad j = 1, \ldots, m,
\]
\[
m_{p\max} = (m, x_{\max}), \quad \sigma_{p\max}^2 = (Wx_{\max}, x_{\max}), \tag{5}
\]
\( x_{\max} \) is the solution of problem (5).

In problem (4) there is an effective portfolio of the least uncertainty satisfying a priori constraints \((1)\), and in problem (5) - an effective portfolio with maximally expected efficiency, also satisfying a priori constraints \((1)\). The problem of quadratic programming (4) is solved on the computer by one of the numerical methods well developed in computational mathematics. The problem of linear programming (5) is effectively solved on a computer using a special algorithm that takes into account the specific nature of problem (5), (of course, problem (5) can be solved with the help of the well-known simplex method, but in this case it is inappropriate to apply the simplex method) \([7]\).

As in the Markowitz problem \([1]\), the set of points \((m_p, \sigma_p^2)\) corresponding to the effective Pareto solutions of problem \((3)\) on the plane \((m_p, \sigma_p^2)\) form an increasing convex downward curve.

The next modification of the Markowitz model is related to the situation when there is a strong correlation of capital investment efficiency in the investment objects under consideration. In this case, the covariance matrix \( W \) of the efficiency vector \( \vec{R} \) is ill-conditioned or degenerated and, consequently, the Markowitz problem \([1]\) and the investment portfolio optimization problem in the presence of a priori constraints \((3)\) belong to the class of ill-posed problems which generates instability when solving them \([8]\). The economic reason for the instability of the problem of portfolio optimization is the fact that the risk of the portfolio may vary slightly with the allocation of resources between the objects whose efficiency is highly correlated. The instability of the problem of portfolio optimization from a computational point of view requires its regularization \([8]\).
In addition to the two so far considered criteria - the expected value of efficiency and portfolio uncertainty, it is often desirable to optimize at least one more criterion. The most significant additional criterion is the volume of the transition process associated with the restructuring of the old portfolio composition into a new one. This additional criterion is especially important when the volume of transactions is large (restructuring of a "large" portfolio).

Mathematically, the criterion for the volume of the transition process \( f_{\text{dis}}(x, x_0) \) is described by the "distance" between the old and the new composition of the portfolio \( x_0 = (x_{10}, \ldots, x_{n0})^T \) and \( x = (x_1, \ldots, x_n)^T \):

\[
 f_{\text{dis}}(x, x_0) = \sum_{i=1}^{n} d_i \cdot (x_i \cdot S_p - x_{i0} \cdot S_0)^2 ,
\]

where \( S_0(S_p) \) is invested capital in the old (new) composition of the portfolio, \( d_i \) is positive numbers determined by the resource content of the objects.

It should be mentioned that in general \( S_0 \neq S_p \), since at the date of portfolio restructuring a certain amount of resources can be output from (or into) it.

Taking into account the criterion of the volume of the transition process, the regularized problem of portfolio optimization has the form:

\[
 \sigma_p^2 = (W x, x) + \alpha \cdot f_{\text{dis}}(x, x_0) - \min
\]

\[
 0 \leq x_i \leq 1, \ i = 1, \ldots, n, \ \sum_{i=1}^{n} x_i = 1, \\
 a_j \leq x_{j1} + \cdots + x_{jm} \leq b_j, \ j = 1, \ldots, m, \\
 m_p = (m, x) \in [m_p^*, m_{\text{max}}],
\]

where \( \alpha > 0 \) is the regularization parameter which establishes the level of compromise between the portfolio risk and the criterion for the volume of the transition process.

Mathematically, problem (7) also belongs to the class of problems of quadratic programming and is solved on the computer with the help of a special algorithm, which allows one to choose the appropriate value of the regularization parameter \( \alpha > 0 \). When compiling an portfolio, which includes \( n \) objects, indexed \( i = 1, \ldots, n \), it is necessary to estimate the covariance matrix of a system of random variables

\[
 R_{\text{pre}} = (R_{\text{pre}1}, \ldots, R_{\text{pre}n})^T,
\]

where \( R_{\text{pre}} \) is the predicted effectiveness of the \( i \)-th investment object for the investor under consideration.

Let the investor use the prognostic algorithm (scheme) that can be applied to the already implemented dates in the past (perhaps not all) to obtain predicted values of efficiency. Thus, it is assumed that for the set of past dates the prediction algorithm allows to predict the values of the efficiencies \( R_{\text{pre}}(t_k), k = 1, \ldots, N \) for a future interval of length \( T (t_N = T) \).

Let \( R_{\text{pre}} \) be the fulfilled values of the efficiencies. Then, assuming the stationarity of the covariance matrix \( W \), for the vector of predictive efficiencies \( R_{\text{pre}} \) we obtain an estimation of its elements \( W_{ij} \):

\[
 W_{ij} = \frac{1}{N} \cdot \sum_{k=1}^{N} (R_{\text{pre}}(t_k) - R_{\text{pre}}(t_k)), \quad i, j = 1, \ldots, n.
\]

Let, for example, the predicted efficiencies of \( R_{\text{pre}} \) for the current date belong to finite intervals:

\[
 R_{\text{imin}} < R_{\text{pre}} < R_{\text{imax}}, \quad i = 1, \ldots, n
\]

and the random variables \( R_{\text{pre}} \) do not pairwise correlate. Then, if the normal distribution law is postulated for each \( R_{\text{pre}} \), we have \( W = \text{diag} (\sigma^2 (R_{\text{pre}})) = \frac{1}{N} \cdot \sum_{k=1}^{N} (R_{\text{pre}}(t_k) - R_{\text{pre}}(t_k))^2, \ i = 1, \ldots, n.\)

According to the introduced definition of predictive efficiencies and risks for the task of optimizing the portfolio, the estimation of the vector of expected mean values \( M_{\text{pre}} = (M_{\text{pre}1}, \ldots, M_{\text{pre}n})^T \) is taken as the calculated values of efficiencies and the estimation of the covariance matrix of the vector \( R_{\text{pre}} \) is taken as the matrix \( W \).
3. The scheme for calculating the dynamics of the transition process for the shares of resources allocation in the system

The predicted dynamics of the transition process for the shares of the distributed resource is described by the following composite system of equations:

$$\frac{dx_i(t)}{dt} = a_i - \alpha \cdot x_i, \quad 0 < t < t^*, i = 1, ..., n$$  \hspace{1cm} (8)

with the initial condition

$$x_i(t) = x_{i0} \geq 0, i = 1, ..., n, \quad \sum_{i=1}^{n} x_{i0} = 1.$$  \hspace{1cm} (9)

In system (8) the parameters $a_i, i = 1, ..., n$ are unknown and are to be determined.

Let's consider the following inverse problem for the system of differential equations (8).

It is required to find parameters $a_i, i = 1, ..., n$ so that the normalization conditions $\sum_{i=1}^{n} x_i(t) = 1, \quad 0 < t < t^*, x_i(t) \geq 0, i = 1, ..., n$, and the solution of the Cauchy problem (8) - (9) must satisfy the finite condition:

$$x_i(t^*) = x_i^* \geq 0, \quad i = 1, ..., n, \quad \sum_{i=1}^{n} x_i^* = 1.$$  \hspace{1cm} (10)

The solution of the inverse problem (8) - (10) has the form:

$$x_i(t) = x_i^* + (x_{i0} - x_i^*) \cdot \exp(-\alpha \cdot t), \quad i = 1, ..., n.$$  \hspace{1cm} (11)

If the additive chaotic component $\varepsilon(t), M(t) = 0$ is present on the right-hand side of the original system (8), then the unbiased estimate for the solution of the inverse problems (8) - (10) will also be (11) and the estimate of the parameter $\alpha$ can be found from of the dynamics carrying out of the change $x_i(t), i = 1, ..., n$ by the method of least squares.

Below it is presented a numerical example of the dynamics of the shares of a distributed resource for a system of objects into which these resources are distributed.

Table 1 presents the results of forecasting the shares of the distributed resource using the above scheme (the shares in Table 1 are given in percentages). Here the initial shares are defined by the equalities:

$x_{10} = 0.04, x_{20} = 0.21, x_{30} = 0.75, x_{40} = 0.00$, and the shares of the final state by the equalities

$x_1^* = 0.01, x_2^* = 0.03, x_3^* = 0.21, x_4^* = 0.75$.

Table 1 shows the fulfilled values of the shares of resource allocation at regular intervals in four objects (in parentheses are projected values of shares).

| Table 1. The forecasted and the realized shares of the resource |
|---------------------------------------------------------------|
| 4%   | 21% | 75% | 0%      |
| 3% (3%) | 15% (15%) | 61% (58%) | 21% (24%) |
| 3% (2.8%) | 12% (12.2%) | 50% (51%) | 35% (34%) |
| 2% (2%) | 10% (10%) | 43% (42%) | 45% (46%) |
| 2% (2%) | 7% (7%) | 35% (32%) | 56% (59%) |
| 2% (1.5%) | 6% (6.5%) | 29% (30%) | 63% (62%) |
| 1% (1.5%) | 5% (5.5%) | 25% (28%) | 69% (65%) |
| 2% (1%) | 4% (4%) | 24% (24%) | 70% (71%) |
| 1% (1%) | 4% (4%) | 22% (22%) | 73% (73%) |
| 1% (1%) | 3% (3%) | 21% (21%) | 75% (75%) |
| 1% (1%) | 3% (3%) | 21% (21%) | 75% (75%) |

4. Conclusion

The article presents mathematical models of transient processes for the distribution of resources in the system including the forecasting of the dynamic process of resource allocation in the transition from one equilibrium state to another equilibrium state. The mathematical model of predicting a dynamic process is based on the inverse problem for a system of ordinary differential equations taking into account the normalization and non-negativity of the shares for the entire considered period of time. An
explicit solution of the inverse problem is obtained. It is specified that the obtained solution can be used as an estimation of the forecast in a dynamic mode with an unbiased chaotic component.

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