NLIE of Sausage model

Changrim Ahn
Ewha Womans University
Seoul, South Korea

Based on 1701.08933 (J. Phys. A)
Work with
Janos Balog and Francesco Ravanini
S-matrix program

- Integrable QFT $\rightarrow$ Exact S-matrix $\rightarrow$ TBA
  -(ref) Review lecture by Z. Bajnok

- Main Goal: Find new integrable QFTs
  - One- or two-parameter extensions of known integrable QFTs
  -(ex) Lattice analogy: $XXX \rightarrow XXZ \rightarrow XYZ$
R vs. S

| R       | S        |
|---------|---------|
| YBE     | YBE     |
| Define model Hamiltonian | Result of the int. QFT |

Lagrangian → S-matrix

nontrivial
### “Zoo” of Integrable QFTs

| classes                  | (ex)         | \( \mathcal{L} \) | spectrum       | \( S \)         |
|--------------------------|--------------|-------------------|----------------|-----------------|
| Affine Toda theories     | Sinh-Gordon  | ✓                 | fund. fields   | diago.          |
| NL \( \sigma \) models  | O(3), AdS/CFT | ✓                 | Reps           | non-diago.      |
| Perturbed CFTs           | RSOS         | ✗                 | kinks          | non-diago.      |
Integrable deformations

- Deform S-matrix
  - Drinfeld-Reshetikhin twist
  - Quantum Group

- Deform Lagrangian
  - Classical Yang-Baxter algebra [Klimsik]
  - Discrete symmetries of target manifolds

\[ \text{Lagrangian}_{\nu} \quad \rightarrow \quad S_{q}-\text{matrix} \]

more nontrivial
String manifolds
\( \gamma \)-deform: TsT
\( \eta \)-deform: class. YBE

SYM theories
\( \mathcal{L} = \ldots + [\Phi, \Phi]_\beta^2 \)
Non-Comm. space-time

S-matrix
\( F_\beta S_{su(2|2)} F_\beta \)
q-S_{su(2|2)}
simpler example

\[ O(3) \sigma \text{-model} \rightarrow \text{sausage } \sigma \text{-model} \]
Outline

1. Review of Sausage $\sigma$-model

2. Derive non-linear integral equation

3. UV and IR limits

4. Match Lagrangian with S-matrix
**O(3) σ-model**

\[ |\vec{n}| = 1 \]

\[ A = \frac{1}{2g} \int d^2x (\partial_\mu \vec{n})^2 + iv\vartheta T \]

\[ T = \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n}) \]

**Haldane Conjecture:** O(3) SM is equivalent to spin-s anti-ferro Heisenberg models in large s limit

s=integer \hspace{1cm} \leftrightarrow \hspace{1cm} \vartheta = 0

s=half-integer \hspace{1cm} \leftrightarrow \hspace{1cm} \vartheta = \pi
**S-matrix of O(3) \( \sigma \)-model**

- **Integrable for** \( \vartheta=0, \pi \) \ [Zamolodchikov\(^2\)]

\( \vartheta = 0 \): massive triplet \((+, 0, -)\) \( m \sim e^{-\pi/g} \)

\[
S(\theta) = \frac{\theta + 2\pi i}{\theta - 2\pi i} P_0 + \frac{\theta + 2\pi i}{\theta - 2\pi i} \frac{\theta - \pi i}{\theta + \pi i} P_1 + \frac{\theta - \pi i}{\theta + \pi i} P_2
\]

\( \vartheta = \pi \): massless L-, R- doublet

\[
S_{LL}(\theta) = S_{RR}(\theta) = S_{LR}(\theta) = \frac{\Gamma \left( \frac{1}{2} + \frac{\theta}{2\pi i} \right) \Gamma \left( -\frac{\theta}{2\pi i} \right)}{\Gamma \left( \frac{1}{2} - \frac{\theta}{2\pi i} \right) \Gamma \left( \frac{\theta}{2\pi i} \right)} \frac{\theta 1 - i\pi P}{\theta - i\pi}
\]

- \( \vartheta=\pi \): RG flow to IR CFT = \( \text{su}(2)_1 \) WZW
Sausage SM: defined by S-matrix \( \text{SST}_\lambda^{(\pm)} \)

**Integrability in Low-Dimensional Quantum Systems, MATRIX@Creswick 17.7.5**

\( \text{SST}_\lambda^{(+)} : \) massive triplet \((+, 0, -) m \sim e^{-\pi/g} \)

\[
S_{++}^+(\theta) = S_{+-}^-(i\pi - \theta) = \frac{\sinh (\lambda(\theta - i\pi))}{\sinh (\lambda(\theta + i\pi))},
\]

\[
S_{0+}^0(\theta) = S_{++}^0(i\pi - \theta) = \frac{-i \sin(2\pi \lambda)}{\sinh (\lambda(\theta - 2i\pi))} S_{++}^+(\theta),
\]

\[
S_{0+}^0(\theta) = \frac{\sinh (\lambda \theta)}{\sinh (\lambda(\theta - 2i\pi))} S_{++}^+(\theta),
\]

\[
S_{++}^{-+}(\theta) = -\frac{\sin (\pi \lambda) \sin (2\pi \lambda)}{\sinh (\lambda(\theta - 2i\pi)) \sinh (\lambda(\theta + i\pi))}, \quad S_{00}^{00}(\theta) = S_{0+}^0(\theta) + S_{++}^{-+}(\theta)
\]

**Fateev, Onofri, Zamolodchikov (1993)**

\( 0 < \lambda < 1/2: \) repulsive

\( \lambda > 1/2: \) very complicated

**Quantum group deformation of O(3) S-matrices**

\[
\text{SST}_\lambda^{(-)} : \) massless doublet
\]

\[
U_{++}^{++}(\theta) = U_{--}^{--}(\theta) = U_0(\theta),
\]

\[
U_{+-}^{+-}(\theta) = U_{-+}^{-+}(\theta) = -\frac{\sinh (\lambda \theta / (1 - \lambda))}{\sinh (\lambda(\theta - i\pi) / (1 - \lambda))} U_0(\theta),
\]

\[
U_{++}^{+-}(\theta) = U_{-+}^{-+}(\theta) = -i \frac{\sin (\pi \lambda / (1 - \lambda))}{\sinh (\lambda(\theta - i\pi) / (1 - \lambda))} U_0(\theta),
\]

\[
U_0(\theta) = -\exp \left[ i \int_0^\infty \frac{\sinh ((1 - 2\lambda) \pi \omega / (2\lambda)) \sin (\omega \theta)}{\cosh (\pi \omega / 2) \sinh ((1 - \lambda) \pi \omega / (2\lambda))} \frac{d\omega}{\omega} \right]
\]
Effective Lagrangian of SSM
Fateev, Onofri, Zamolodchikov (1993)

RG analysis for near UV limit ($\nu \to 0, \ t \to -\infty$)

$$\mathcal{A}[\text{SSM}_{\nu}] = \frac{1}{2g(t)} \int d^2 x \frac{(\partial_{\mu} \bar{n})^2}{1 - \frac{\nu^2 n_3^2}{2g(t)^2}} + i\nu T$$

$$g(t) = \frac{\nu}{2} \coth \frac{\nu(t_0 - t)}{4\pi}$$

$$L \approx \frac{\sqrt{2\nu}}{2\pi} (t_0 - t)$$

$$\ell \approx 2\pi \sqrt{\frac{2}{\nu}}$$

- Classical integrability: Bazhanov, Kotousov, Lukyanov 1706.09941
TBA and Y-system of SST(±)λ

- Derived only for
  \[ \lambda = \frac{1}{N}, \quad N = 2, 3, \ldots \]

  \[ \epsilon_a(\theta) = R \ e_a(\theta) - \sum_{b=0}^{N} \phi_{ab} \ast \log(1 + e^{-\epsilon_b})(\theta), \quad \phi_{ab}(\theta) = \frac{\ell_{ab}}{\cosh \theta} \]

- Y-system ("D"-type)
  \[ y_a^+ \ y_a^- = \prod_b Y_{b}^{\ell(D)}_{ab}, \quad y^\pm = y(\theta \pm \frac{i\pi}{2}) \]
**TBA vs. NLIE**

- **TBA:** [Al.B. Zamolodchikov]
  - Direct relations to spectrum and S-matrix
  - Many (even infinite) coupled integral equations

- **NLIE:** [Klumper, Pearce; Destri, de Vega; Ravanini et al.; Dunning, Suzuki, …]
  - Simpler, valid for generic coupling, …
  - Need lattice formulation
  - Not directly related to QFT

- **algebraic method even without lattice formulation**
  - [Balog, Hegedus, …]
  - TBA (Y-system) $\rightarrow$ T-system $\rightarrow$ T-Q system $\rightarrow$ NLIE
Derivation of NLIE for any $\lambda$

1. Map “D”-type to “A”-type Y-system

$$y_a^+ y_a^- = \prod_{b} Y_{ab}^{(D)}$$

$$z_a^+ z_a^- = \prod_{b} Z_{ab}^{(A)} = Z_{a+1} Z_{a-1}$$

$$z_k^+ z_k^- = Z_{k-1} Z_{k+1}, \quad z_k \equiv y_k, \quad k = 2, \ldots, N - 2,$$

$$Z_1 = Y_0 Y_1,$$

$$Z_{N-1} = Y_{N-1} Y_N$$

Integrability in Low-Dimensional Quantum Systems, MATRIX@Creswick
2. T-system

\[ T_k^+ T_k^- = 1 + T_{k-1} T_{k+1}, \quad k = 2, \ldots, N - 2 \]

\[ z_k = T_{k-1} T_{k+1}, \quad T_k^+ T_k^- = Z_k, \quad k = 1, \ldots, N - 1 \]

with extra condition

\[ Z_{N-2} = T_{N-2}^+ T_{N-2}^- = Y_{N-2} = y_N^+ y_N^- = y_{N-1}^+ y_{N-1}^- \quad \rightarrow \quad y_N = y_{N-1} = T_{N-2} \]

\[ Z_{N-1} = T_{N-1}^+ T_{N-1}^- = Y_{N-1} Y_N = (1 + T_{N-2})^2 = 1 + T_{N-2} T_N \quad \rightarrow \quad T_N = 2 + T_{N-2} \]

3. T-Q system

\[ T_{k+1} Q^{[k]} - T_k^- Q^{[k+2]} = \overline{Q}^{[-k-2]}, \quad T_k^- \overline{Q}^{[-k]} - T_{k-1} \overline{Q}^{[-k-2]} = Q^{[k]} \]

with \[ \overline{Q} = Q^{[2N]} \]
4. From T-Q to NLIE

\[
b_k = \frac{Q^{[k+2]} T_k}{Q^{[-k-2]}}, \quad B_k = 1 + b_k = \frac{Q^{[k]} T_{k+1}}{Q^{[-k-2]}}
\]

Fourier transform of logarithmic derivatives

\[
\tilde{b}_k = p^{k+2} \tilde{Q} + p^{-1} \tilde{T}_k - p^{-k-2} \tilde{Q},
\]

\[
\tilde{B}_k = p^k \tilde{Q} + \tilde{T}_{k+1} - p^{-k-2} \tilde{Q}
\]

Eliminate Q using

\[
\tilde{Q} = p^{2N} \tilde{Q}
\]

\[
\tilde{b} = \tilde{K} (\tilde{B} - \tilde{\bar{B}}) + p^{-1} \tilde{s} \tilde{Y}_1 \tilde{Y}_0,
\]

\[
\tilde{b} = \tilde{K} (\tilde{B} - \tilde{\bar{B}}) + p \tilde{s} \tilde{Y}_1 \tilde{Y}_0,
\]

\[
\tilde{y} = p \tilde{s} \tilde{B} + p^{-1} \tilde{s} \tilde{\bar{B}}
\]

Analytically continue: \( N \to 1/\lambda \)

\[
\tilde{s} = \frac{1}{p + p^{-1}} = \frac{1}{2 \cosh \frac{\omega \pi}{2}}
\]

\[
\tilde{K} = \frac{\sinh \left( \frac{\omega \pi (1-3\lambda)}{2\lambda} \right)}{2 \sinh \left( \frac{\omega \pi (1-2\lambda)}{2\lambda} \right) \cosh \frac{\omega \pi}{2}}
\]
NLIE in rapidity space

**SST**(+)\
\[
\log a = K \star \log(1 + a) - K^{[2\alpha]} \star \log(1 + \bar{a}) + s^{[\alpha-1]} \star [\log(1 + y) + \log(1 + \xi y)],
\]
\[
\log \bar{a} = K \star \log(1 + \bar{a}) - K^{[-2\alpha]} \star \log(1 + a) + s^{[1-\alpha]} \star [\log(1 + y) + \log(1 + \xi y)],
\]
\[
\log y = s^{[1-\alpha]} \star \log(1 + a) + s^{[\alpha-1]} \star \log(1 + \bar{a})
\]

**SST**(-)\
\[
\log a = K \star \log(1 + a) - K^{[2\alpha]} \star \log(1 + \bar{a}) + s^{[\alpha-1]} \star [\log(1 + \xi^+ y) + \log(1 + \xi^- y)],
\]
\[
\log \bar{a} = K \star \log(1 + \bar{a}) - K^{[-2\alpha]} \star \log(1 + a) + s^{[1-\alpha]} \star [\log(1 + \xi^+ y) + \log(1 + \xi^- y)],
\]
\[
\log y = s^{[1-\alpha]} \star \log(1 + a) + s^{[\alpha-1]} \star \log(1 + \bar{a})
\]

**Vacuum energy**\
\[
E(r) = -\frac{m}{2\pi} \int_{-\infty}^{\infty} \cosh \theta \log(1 + \xi y), \quad \xi = e^{-mr \cosh \theta}
\]
\[
E(r) = -\frac{m}{4\pi} \int_{-\infty}^{\infty} \left[ e^\theta \log(1 + \xi^+ y) + e^{-\theta} \log(1 + \xi^- y) \right], \quad \xi^\pm = e^{-mr \exp(\pm \theta)/2}
\]
IR limit of SST$^{(+)}$

- **Linearized NLIE** ($\xi<<1$)

  \[ a = z(1 + w + \ldots), \quad y = h(1 + u + \ldots), \quad z = 2, \quad h = 3 \]

  \[ w = \frac{2}{3} K \star w - \frac{2}{3} K^{[2\alpha]} \star \bar{w} + s^{[\alpha-1]} \star \left( 3\xi + \frac{3}{4}u \right), \]

  \[ \bar{w} = \frac{2}{3} K \star \bar{w} - \frac{2}{3} K^{[2\alpha]} \star w + s^{[1-\alpha]} \star \left( 3\xi + \frac{3}{4}u \right), \]

  \[ u = \frac{2}{3} s^{[1-\alpha]} \star w + \frac{2}{3} s^{[\alpha-1]} \star \bar{w} \]

- **Solutions by F.T.**

  \[ \tilde{u} = \frac{1}{3} \xi \tilde{\varphi}, \quad \tilde{\varphi} = 8 \frac{\sinh \left[ \pi \omega \left( \frac{1}{2\lambda} - 1 \right) \right]}{\sinh \frac{\pi \omega}{2\lambda}} - 4 \frac{\sinh \left[ \pi \omega \left( \frac{1}{2\lambda} - 2 \right) \right]}{\sinh \frac{\pi \omega}{2\lambda}} \]
• Virial expansion

\[ E = E^{(1)} + E_1^{(2)} + E_2^{(2)} + \mathcal{O}(e^{-3mr}), \]
\[ E^{(1)} = -\frac{e_1 m}{2\pi} \int_{-\infty}^{\infty} \cosh \theta \, e^{-mr \cosh \theta} \, d\theta, \]
\[ E_1^{(2)} = \frac{e_2 m}{4\pi} \int_{-\infty}^{\infty} \cosh \theta \, e^{-2mr \cosh \theta} \, d\theta, \]
\[ E_2^{(2)} = -\frac{m}{2\pi} \int_{-\infty}^{\infty} d\theta \cosh \theta \, e^{-mr \cosh \theta} \int_{-\infty}^{\infty} d\theta' \varphi(\theta - \theta') \, e^{-mr \cosh \theta'} \, d\theta', \]
\[ e_1 = 3, \quad e_2 = 9 \]
\[ \varphi(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \log \det S^{(+)}_{\lambda}(\theta) \]
UV limit

- NLIE as infinite order DE [Al.B.Zamolodchikov]
  \[ \delta_{a0} m r \cosh \theta = \varepsilon_a(\theta) + \sum_{b} \sum_{n=0}^{\infty} \tilde{\Psi}_{ab,n} L_b^{(n)}(\theta) \]

- Zero-mode dynamics from Lagrangian [FOZ]
  
  "Cigar" or sine-Liouville

\[ c(r) \approx 2 - \frac{3\pi^2}{2} \frac{\lambda^{-1} - 2}{(\log(mr) + C)^2} + \ldots \]

\[ c(r) = 2 - \frac{4\pi}{\nu} \frac{3\pi^2}{2(\log(mr) + \delta)^2} \]
Parametric relation: L vs. S

\[ \frac{\nu}{4\pi} = \frac{\lambda}{1 - 2\lambda} \]
Summary and Discussion

• Derived NLIE for any $\lambda (<1/2)$ from TBA by taking analytic continuation
• Satisfy all consistent checks
• Parametric relation between $\nu$ and $\lambda$

• Bazhanov, Kotousov, Lukyanov 1706.09941 proposed another NLIE which looks similar but not exactly same.
  – But generates same numerics
  – Need to be clarified
• Attractive regime ($\lambda >1/2$)?
• Applicable to other deformed NLSM?