Vacuum polarization from confined fermions in 3 + 1 dimensions

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Abstract

We study the main properties of the one-loop vacuum polarization function ($\Pi_{\mu\nu}$) for massless spinor QED$_4$ in a slab, namely, with fields defined on $\mathcal{M} \subset \mathbb{R}^{3+1}$, such that $\mathcal{M} = \{(x_0, ..., x_3)|0 \leq x_3 \leq \epsilon\}$, and bag-like boundary conditions on the boundary. We evaluate the induced charge density and current due to an external constant electric field normal to the boundary; we also study the effective action for a purely transverse field, identifying its $\epsilon$-dependent contribution.

In the presence of nontrivial boundary conditions, Quantum Field Theory models may give rise to many interesting effects. Noteworthy examples of them are the Casimir effect [1, 2], the bag model of QCD [3], as well as many others [4].

The common origin of those effects is the presence of boundaries, which strongly affect the structure of the vacuum fluctuations; this fact has relevance not only for global observables, i.e., Casimir energies, but also for local ones, like vacuum energy densities and response functions. The latter, which are determined by the correlation between fluctuations should exhibit a strong dependence with the distance to the boundary (at least for the ones involving degrees of freedom affected by the boundary conditions).

In this letter, we consider the vacuum polarization tensor, $\Pi_{\mu\nu}$, for a Dirac field confined to a slab-shaped region $\mathcal{M}$, defined by the condition $0 \leq x_3 \leq \epsilon$. More concretely, we consider a massless Dirac field in 3+1 dimensions, which satisfies ‘bag’ (i.e., vanishing normal current) boundary conditions on the two static planes $x_3 = 0$ and $x_3 = \epsilon$. Euclidean spacetime coordinates are, in the
conventions that we shall use, denoted by \((x_0, x_1, x_2, x_3)\). Moreover, we shall assume the Dirac field to be confined to the region between those mirrors; hence, \(\Pi_{\mu\nu}\) vanishes identically outside of \(0 < x_3 < \epsilon\). It there are fermions outside of the slab, \(\Pi_{\mu\nu}\) does not vanish there. However, their properties in that region are equivalent to the case of just one boundary; for example, if \(x_3 < 0\), \(\Pi_{\mu\nu}\) in that region is the same as the one for a single wall at \(x_3 = 0\).

In the conventions we shall use, Euclidean coordinates are denoted by \(x^\mu\), \(\mu = 0, 1, 2, 3\), while the metric tensor is given by \(g_{\mu\nu}\).

To account for the effect of the confined fermion fluctuations on the gauge field propagation, one introduces:

\[
e^{-\Gamma_f(A)} = \frac{\int D\bar{\psi} D\psi e^{-S_f(\bar{\psi}, \psi; A)}}{\int D\bar{\psi} D\psi e^{-S_f(\bar{\psi}, \psi; 0)}},
\]

where the fermionic action, \(S_f\) accounts for the (minimal) coupling to the gauge field as well as for the introduction of the bag boundary conditions. To deal with the latter, we follow the approach of representing them by local interaction terms \[5\]. Of course, the resulting propagator should agree with the one one would get by using, for example, the multiple reflection expansion (MRE) \[6, 7, 8\]. That approach has been used in \[9\] for the calculation of \(\Pi_{\mu\nu}\) for a Dirac field in a half-space.

Following \[5\], we include a ‘potential’ \(V\) into the fermionic action, so that:

\[
S_f(\bar{\psi}, \psi; A) = \int d^4x \bar{\psi}(x) \left[ \partial \phi + g V(x_3) + ie \gamma(x) \right] \psi(x)
\]

with

\[
V(x_3) = \left[ \delta(x_3) + \delta(x_3 - \epsilon) \right].
\]

Here, \(g\) is a constant which, in order to enforce bag boundary conditions, has to be equal to 2. The slash notation denotes contraction with the Euclidean \(\gamma\)-matrices which, in our conventions, are all Hermitian, and satisfy:

\[
\{ \gamma_{\mu}, \gamma_\nu \} = 2 \delta_{\mu\nu}, \mu, \nu = 0, 1, 2, 3.
\]

From \(1\), we may write,

\[
\Gamma_f(A) = -\text{Tr} \ln \left[ 1 + ie (\partial + 2V)^{-1} A \right],
\]

which allows us to introduce the vacuum polarization, \(\Pi_{\mu\nu}\), the kernel determining the form of the first non-trivial term for the expansion of \(\Gamma_f\) in powers of \(A\):

\[
\Gamma_f(A) = \frac{1}{2} \int d^4x \int d^4y \, A_\mu(x) \Pi_{\mu\nu}(x, y) A_\nu(y) + \ldots
\]

\[\text{See } [\text{5}]. \text{ Different values would produce 'imperfect' boundary conditions.}\]
From (4), we may give a more explicit, yet formal, expression for $\Pi_{\mu\nu}$ in coordinate space:

$$\Pi_{\mu\nu}(x, y) = -e^2 \text{tr} \left[ S_F(y, x) \gamma_\mu S_F(x, y) \gamma_\nu \right],$$  \hspace{1cm} (6)$$

where $S_F$ denotes the (coordinate space) exact fermion propagator for $e = 0$, namely, the free propagator with bag boundary conditions. This object may be obtained as the matrix elements of the inverse of an operator:

$$S_F(x, y) = \langle x | (\not \partial + 2V)^{-1} | y \rangle, \hspace{1cm} (7)$$

where a ‘bracket’ notation has been used to denote the coordinate space versions of operators (in this case, the inverse of $\not \partial + 2V$). Since the boundary conditions only affect the $x_3$ coordinate, there is symmetry under translations in the ‘parallel’ coordinates $x_i \equiv (x_0, x_1, x_2)$. Thus, $S_F = S_F(x_3 - y_3; x_3, y_3)$, and it is natural to work with $\tilde{S}_F$, a mixed Fourier representation of $S_F$ which results by transforming the parallel coordinates only, so that $S_F(x_3 - y_3; x_3, y_3) \rightarrow \tilde{S}_F(p_3; x_3, y_3)$.

This reduces the problem to a one-dimensional one, where $\tilde{S}_F$ satisfies the equation:

$$[\gamma_3 \partial_{x_3} - i \not \! p_3 + 2V(x_3)] \tilde{S}_F(p_3; x_3, y_3) = \delta(x_3 - y_3).$$  \hspace{1cm} (8)$$

The solution to the inhomogeneous linear equation above may be written as the sum of two terms:

$$\tilde{S}_F(p_3; x_3, y_3) = \tilde{S}_F^{(0)}(p_3; x_3, y_3) + \tilde{U}(p_3; x_3, y_3),$$  \hspace{1cm} (9)$$

where the first of them, $\tilde{S}_F^{(0)}$, is the free fermion propagator in the absence of boundaries, while the other, denoted by $U$ accounts for the boundary (bag) conditions.

The explicit form of $\tilde{S}_F^{(0)}$ may be found quite straightforwardly:

$$\tilde{S}_F^{(0)}(p_3; x_3, y_3) = \frac{1}{2} \left[ \text{sgn}(x_3 - y_3) \gamma_3 - i \frac{\not \! p_3}{|p_3|} \right] e^{-|p_3||x_3 - y_3|}. \hspace{1cm} (10)$$

Since $\tilde{S}_F^{(0)}$ is the Green’s function of the free Dirac operator we derive, from (5), an equation for $\tilde{U}(p_3; x_3, y_3)$:

$$[\gamma_3 \partial_{x_3} + i \not \! p_3] \tilde{U}(p_3; x_3, y_3) = -2V(x_3) \left[ \tilde{S}_F^{(0)}(p_3; x_3, y_3) + \tilde{U}(p_3; x_3, y_3) \right], \hspace{1cm} (11)$$

Since $\tilde{S}_F^{(0)}$ is the Green’s function of the free Dirac operator we derive, from (5), an equation for $\tilde{U}(p_3; x_3, y_3)$:

$$[\gamma_3 \partial_{x_3} + i \not \! p_3] \tilde{U}(p_3; x_3, y_3) = -2V(x_3) \left[ \tilde{S}_F^{(0)}(p_3; x_3, y_3) + \tilde{U}(p_3; x_3, y_3) \right], \hspace{1cm} (11)$$
which implies:

\[ \tilde{U}(p_i; x_3, y_3) = -2 \left[ S_F^{(0)}(p_i; x_3, 0) \tilde{S}^{(0)}(p_i; 0, y_3) + \tilde{S}_F^{(0)}(p_i; x_3, 0) \tilde{U}(p_i; 0, y_3) \right. \\
\left. + \tilde{S}_F^{(0)}(p_i; x_3, \epsilon) \tilde{S}_F^{(0)}(p_i; \epsilon, y_3) + \tilde{S}_F^{(0)}(p_i; x_3, \epsilon) \tilde{U}(p_i, \epsilon, y_3) \right] \tag{12} \]

One can then obtain \( \tilde{U}(p_i; x_3, y_3) \) from the equation above, for example by first finding \( \tilde{U}(p_i, 0, y_3) \) and \( \tilde{U}(p_i, \epsilon, y_3) \); these can be found by evaluating (12) at \( x_3 = 0 \) and \( x_3 = \epsilon \), respectively, and then solving the resulting system of equations.

Once those objects are found, the outcome of this procedure is an explicit expression for \( \tilde{U}(p_i; x_3, y_3) \), which can be expressed as a sum of terms, distinguished according to its \( \gamma \)-matrix content:

\[ \tilde{U}(p_i; x_3, y_3) = \tilde{U}_0(p_i; x_3, y_3) I + \tilde{U}_1(p_i; x_3, y_3) \left( -i \frac{p_i}{|p_i|} \right) + \tilde{U}_2(p_i; x_3, y_3) \left( -i \frac{p_i}{|p_i|} \gamma_3 \right) + \tilde{U}_3(p_i; x_3, y_3) \gamma_3 , \tag{13} \]

where we have introduced four functions \( \tilde{U}_a \) (\( a = 0, 1, 2, 3 \)). The explicit form of these functions, for \( 0 < x_3 < \epsilon \) and \( 0 < y_3 < \epsilon \), is the following:

\[ \tilde{U}_0(p_i; x_3, y_3) = \frac{1}{2(e^2|p_i|\epsilon + 1)} \left[ e^{\left| p_i \right| (x_3 + y_3)} + e^{-\left| p_i \right| (x_3 + y_3 - 2\epsilon)} \right] , \tag{14} \]

\[ \tilde{U}_1(p_i; x_3, y_3) = -\frac{1}{2(e^2|p_i|\epsilon + 1)} \left[ e^{\left| p_i \right| (x_3 - y_3)} + e^{-\left| p_i \right| (x_3 - y_3)} \right] , \tag{15} \]

\[ \tilde{U}_2(p_i; x_3, y_3) = -\frac{1}{2(e^2|p_i|\epsilon + 1)} \left[ e^{\left| p_i \right| (x_3 + y_3)} - e^{-\left| p_i \right| (x_3 + y_3 - 2\epsilon)} \right] , \tag{16} \]

\[ \tilde{U}_3(p_i; x_3, y_3) = \frac{1}{2(e^2|p_i|\epsilon + 1)} \left[ e^{\left| p_i \right| (x_3 - y_3)} - e^{-\left| p_i \right| (x_3 - y_3)} \right] . \tag{17} \]

A lengthy but otherwise straightforward calculation shows that the bag boundary conditions are fulfilled, namely, the following equations are satisfied:

\[ \lim_{x_3 \to 0^+} (I - \gamma_3) \tilde{S}_F(p_i; x_3, y_3) = 0 \]
\[ \lim_{x_3 \to \epsilon^-} (I + \gamma_3) \tilde{S}_F(p_i; x_3, y_3) = 0 \tag{18} \]
We now analyze the UV properties of the propagator $\tilde{S}_F$, as presented in (9). We see that the two terms have a quite different behaviour. Indeed, the first one, $\tilde{S}_F^{(0)}$, being the free propagator in the absence of boundaries, does have the well-known UV behaviour ($\sim p^{-1}$ in momentum space). In the ‘mixed’ Fourier representation, where it depends on $p_\parallel$, $x_3$, $y_3$, that behaviour translates into a $\sim |p_\parallel|0$ behaviour when $x_3 = y_3$, and an exponential decay when $x_3 \neq y_3$. The $\tilde{U}$ term, on the other hand, decays exponentially everywhere, except when both $x_3$ and $y_3$ approach one (the same) boundary, namely, when either $x_3 + y_3 \to 0$ or $x_3 + y_3 \to 2\epsilon$. Moreover, we can trace the origin of that behaviour to see that it comes from the $\tilde{U}_0$ and $\tilde{U}_2$ terms ($\tilde{U}_1$ and $\tilde{U}_3$ always decrease exponentially, regardless of the values of $x_3$ and $y_3$).

As a consistency check for the expression of $S_F$, one can find an alternative representation, obtained by expressing $(e^{2|p_\parallel|\epsilon} + 1)^{-1}$, which appears as a common factor in the expressions for $\tilde{U}_a$, as a series:

$$\frac{1}{e^{2|p_\parallel|\epsilon} + 1} = \sum_{n=0}^{\infty} e^{-2(n+1)|p_\parallel|\epsilon}.$$  

As a result, and after some algebra, we obtain:

$$\tilde{S}_F(p_\parallel; x_3, y_3) = \sum_{n=-\infty}^{+\infty} (-1)^n \left[ \tilde{S}_F^{(0)}(p_\parallel; x_3, y_3 + 2n\epsilon) + \gamma_3 \tilde{S}_F^{(0)}(p_\parallel; x_3, -y_3 + 2n\epsilon) \right],$$  

which may naturally be thought of as an MRE representation of $\tilde{S}_F$. It is possible to pinpoint also here the part of the propagator that control its UV behaviour. Since all the terms in the sum can be written as functions of $\tilde{S}_F^{(0)}$, it is quite straightforward to see that the terms that control the large-$|p_\parallel|$ regime are: $n = 0$ (for $x_3 = y_3$ or $x_3 + y_3 = 0$) and $n = 1$ (for $x_3 + y_3 = 2\epsilon$).

To obtain a more explicit expression for $\Pi_{\mu\nu}$, we insert the results derived previously for $\tilde{S}_F$ into (6). We first note that, since $\Pi_{\mu\nu}$ shall also be a function of $(x_\parallel - y_\parallel; x_3, y_3)$, we may write its Fourier transform with respect to the parallel variables, as follows:

$$\tilde{\Pi}_{\mu\nu}(k_\parallel; x_3, y_3) = -e^2 \int \frac{d^3p_\parallel}{(2\pi)^3} \text{tr} \left[ \tilde{S}_F(p_\parallel; y_3, x_3) \gamma_\mu \tilde{S}_F(p_\parallel + k_\parallel; x_3, y_3) \gamma_\nu \right].$$  

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Thus,
\[
\Pi_{\mu\nu}(k_i; x_3, y_3) = \Pi^{LL}_{\mu\nu}(k_i; x_3, y_3) + \Pi^{LU}_{\mu\nu}(k_i; x_3, y_3) + \Pi^{UL}_{\mu\nu}(k_i; x_3, y_3) + \Pi^{UU}_{\mu\nu}(k_i; x_3, y_3),
\]
(22)
where:
\[
\Pi^{LL}_{\mu\nu}(k_i; x_3, y_3) = -\epsilon^2 \int \frac{d^3p_i}{(2\pi)^3} \text{tr} \left[ S_F^{(0)}(p_i; y_3, x_3) \gamma_\mu S_F^{(0)}(p_i + k_i; x_3, y_3) \gamma_\nu \right],
\]
(23)
\[
\Pi^{LU}_{\mu\nu}(k_i; x_3, y_3) = -\epsilon^2 \int \frac{d^3p_i}{(2\pi)^3} \text{tr} \left[ S_F^{(0)}(p_i; y_3, x_3) \gamma_\mu \tilde{U}(p_i + k_i; x_3, y_3) \gamma_\nu \right],
\]
(24)
\[
\Pi^{UL}_{\mu\nu}(k_i; x_3, y_3) = -\epsilon^2 \int \frac{d^3p_i}{(2\pi)^3} \text{tr} \left[ \tilde{U}(p_i; y_3, x_3) \gamma_\mu S_F^{(0)}(p_i + k_i; x_3, y_3) \gamma_\nu \right],
\]
(25)
\[
\Pi^{UU}_{\mu\nu}(k_i; x_3, y_3) = -\epsilon^2 \int \frac{d^3p_i}{(2\pi)^3} \text{tr} \left[ \tilde{U}(p_i; y_3, x_3) \gamma_\mu \tilde{U}(p_i + k_i; x_3, y_3) \gamma_\nu \right].
\]
(26)

Inserting the explicit expressions for \( S_F^{(0)} \) and \( \tilde{U} \) presented in the Appendix, one sees, after evaluating the traces, that \( \Pi^{LU} + \Pi^{UL} \) vanishes identically. Thus,
\[
\Pi_{\mu\nu}(k_i; x_3, y_3) = \Pi^{LL}_{\mu\nu}(k_i; x_3, y_3) + \Pi^{UU}_{\mu\nu}(k_i; x_3, y_3),
\]
(27)
where: \( \Pi^{L}_{\mu\nu} \equiv \Pi^{LL}_{\mu\nu} \) and \( \Pi^{U}_{\mu\nu} \equiv \Pi^{UU}_{\mu\nu} \).

Before evaluating the above for some particular cases, we calculate a magnitude corresponding to a related effect: the boundary conditions at 0 and \( \epsilon \) do break the chiral symmetry of the (massless) unconfined theory. A quantitative and local measure of that violation, based on the expectation value of a bilinear observable is the fermion condensate \( \rho(x) \equiv \langle \bar{\psi}(x)\psi(x) \rangle \).

In terms of the fermion propagator, we see that \( \rho \) may be expressed as follows:
\[
\rho(x) = -\text{tr} \left[ S_F(x, x) \right].
\]
(28)
Furthermore, taking into account translation invariance in the parallel coordinates, and the specific form of \( S_F \):
\[
\rho(x) = \rho(x_3) = -\int \frac{d^3p_i}{(2\pi)^3} \text{tr} \left[ \tilde{S}_F(p_i; x_3, x_3) \right],
\]
(29)
which may be exactly calculated:
\[
\rho(x_3) = -\frac{\pi^4}{2\epsilon^3} \frac{3 + \cos\left(\frac{2\pi x_3}{\epsilon}\right)}{\sin^3\left(\frac{\pi x_3}{\epsilon}\right)},
\]
(30)
which is of course symmetric with respect to $x_3 = \frac{\epsilon}{2}$, and finite everywhere, except at $x_3 = 0, \epsilon$.

As it can be inferred from Figure 1, for finite values of $\epsilon$, $\rho(x_3)$ diverges on the boundaries, and has a maximum of $-\pi^4/\epsilon^3$ when $x_3 = \epsilon/2$. It can also be shown that it tends to 0 when $\epsilon \to \infty$ and $x_3$ is far from the borders. This concentration around the boundaries does also show up in the calculation of the induced vacuum current, $j_\mu$, resulting from some special electromagnetic field configurations. Those induced currents are, in this linear response approximation, determined by $\Pi_{\mu\nu}$ and the gauge field $A_\mu$ corresponding to the given electromagnetic field. In our conventions:

\begin{align}
    j_\mu(x) &= e \langle \bar{\psi}(x) \gamma_\mu \psi(x) \rangle \\
    &= -i \int d^4y \Pi_{\mu\nu}(x, y) A_\nu(y),
\end{align}

(31)

and in the Fourier representation:

\begin{align}
    \tilde{j}_\mu(k_3; x_3) &= -i \int dy_3 \tilde{\Pi}_{\mu\nu}(x_3, y_3) \tilde{A}_\nu(k_3; y_3) .
\end{align}

(32)

We have evaluated the current for the particular case of a static electric field with normal incidence ($x_3$ direction). This corresponds to $F_{01} = E$, where $E$ is a constant. In the $A_3 = 0$ gauge: $A_0 = -Ex_3$, and $A_\mu = 0$ for $\mu \neq 0$. 

Figure 1: $\epsilon^3 \rho$ as a function of $x_3/\epsilon$. 

Thus, \[ j_\mu = j_\mu(x_3) = iE \int dy_3 \tilde{\Pi}_{\mu 0}(0; x_3, y_3) y_3, \] and its usually more convenient real-time (Minkowski) version, \( j_\mu^M \), becomes: \[ j_\mu^0(x_3) = -E \int dy_3 \tilde{\Pi}_{00}(0; x_3, y_3) y_3, \] and, (with \( l \neq 0 \)): \[ j_\mu^l(x_3) = +E \int dy_3 \tilde{\Pi}_{l0}(0; x_3, y_3) y_3. \] Since the system is parity conserving, \( j_1 = j_2 = 0 \) for this external field. Thus, we concentrate on \( j_\mu^0 \) and \( j_\mu^3 \). Recalling (27), we see that these currents receive two contributions:

\[ j_\mu^0(x_3) = -E \int dy_3 \left[ \tilde{\Pi}_{L00}^U(0; x_3, y_3) y_3 \right. \]

\[ + \tilde{\Pi}_{00}^L(0; x_3, y_3) y_3, \]

and analogously for \( j_\mu^3 \).

To obtain \( j_\mu^0 \), we note first that:

\[ \tilde{\Pi}_{L00}^U(0; x_3, y_3) = - \frac{e^2}{(2\pi)^3} \int d^3 p \, e^{2i p \cdot |x_3 - y_3|} 2 \left( 1 - \frac{p_0^2}{p^2} \right) = - \frac{e^2}{6\pi^2 |x_3 - y_3|^3}, \]

which is finite for all \( x_3 \neq y_3 \). A divergence appear when integrating over \( y_3 \), to calculate the current. However, the usual renormalization conditions imply that one has to use Hadamard’s finite part in that integral.

On the other hand, for \( \tilde{\Pi}_{00}^U \) we find the expression:

\[ \tilde{\Pi}_{00}^U(0; x_3, y_3) = \mathcal{M}_1 \left( \frac{x_3 - y_3}{\epsilon} \right) + \mathcal{M}_2 \left( \frac{x_3 + y_3}{\epsilon} \right), \]

where

\[ \mathcal{M}_1(u) = - \frac{e^2}{\epsilon^3} \left\{ -\frac{1}{72} + \frac{\zeta(3)}{8\pi^2} + F_1(u) \right\} \]

\[ \mathcal{M}_2(v) = - \frac{e^2}{\epsilon^3} \left[ \frac{5}{72} + F_2(v) \right] \]
In Figure 2, we plot the profile of $\frac{\epsilon j_M^0}{e^2 E}$ as a function of $x_3/\epsilon$ that follows from the previous results.

For $j_M^3$, under the same external field configuration, we have found a vanishing result, namely, $j_M^3 = 0$. It should not be surprising that the normal current vanishes at the borders, since the bag conditions should do precisely that. The vanishing of the current inside the slab, however, should be regarded as a steady state configuration feature, reached after the charge density has adopted the profile above.

The profile for the charge density above corresponds to the case of a normal electric field; to explore a situation that is, somehow, opposite to

$$F_1(u) = \frac{1}{24\pi^2u^2} - \frac{\cos(\pi u)}{24 \sin^2(\pi u)} + \frac{1}{384\pi^2} \left[ (1 - u)\psi^{(2)}(1 + \frac{u}{2}) - (1 + u)\psi^{(2)}(1 - \frac{u}{2}) \right]$$

$$F_2(v) = \frac{1}{48} \csc^3(\pi v) \left[ 2\sin(2\pi v) + \pi(1 - v) \left( 3 + \cos(2\pi v) \right) \right]. \quad (40)$$

where $\zeta$ is Riemann’s zeta function and \( \psi^{(2)}(z) = \frac{d^2(\ln \Gamma(z))}{dz^2} \).
that one, we presenting here an exact expression for the effective action, $\Gamma_f$, in the case where the gauge field depends only on the parallel coordinates, and moreover that field is a function of only the parallel coordinates. Following the approach of [5] for the case at hand, an entirely analogous procedure to the one used there, yields now:

$$e^{-\Gamma_f(A_\parallel)} = \det \mathcal{K}$$  \hspace{1cm} (41)

where $\mathcal{K}$ is the object

$$\mathcal{K} = \begin{bmatrix} 1 + V & (V - \gamma_3)e^{-\epsilon \mathcal{H}} \\ (V + \gamma_3)e^{-\epsilon \mathcal{H}} & 1 + V \end{bmatrix}$$  \hspace{1cm} (42)

with

$$\mathcal{H} \equiv \sqrt{-D_\parallel^2}, \quad V \equiv -\frac{D_\parallel}{\mathcal{H}}.$$  \hspace{1cm} (43)

Here, $D_\parallel$ is the ‘parallel Dirac operator’: $D_\parallel = \gamma_\alpha D_\alpha$, $\alpha = 0, 1, 2$, and $D = \partial_\parallel + A_\parallel(x_\parallel)$.

Using some algebra, the determinant above may also be written in the equivalent form:

$$\det \mathcal{K} = \left[ \det(1 + V) \right]^2 \det \left[ 1 - \frac{1 + \gamma_3}{2}(1 + V)e^{-2\epsilon \mathcal{H}} \right].$$  \hspace{1cm} (44)

As a consequence, $\Gamma_f(A_\parallel)$ receives two contributions, one coming from the determinant of $(1+V)$, which is essentially two dimensional and independent of $\epsilon$, plus an extra term $\Gamma_\epsilon$, which does depend on $\epsilon$, it is finite (because of the exponential factor), and tends to zero when $\epsilon \to \infty$:

$$\Gamma_\epsilon(A_\parallel) = -\text{Tr} \log \left[ 1 - \frac{1 + \gamma_3}{2}(1 + V)e^{-2\epsilon \mathcal{H}} \right].$$  \hspace{1cm} (45)

We end this note by presenting our conclusions: We have found the form of the induced vacuum current due to a normal electric field, which shows that the induced charge density distributes itself in order to counterbalance the external electric field. We obtained an expression for the effective action that follows from considering a parallel gauge field configuration. It contains a term which represents the finite-width contribution to the effective action. It is finite, and it contains a $\frac{1+\gamma_3}{2}$ factor, which is a projector which accounts for the suppression of part of the fermion modes because of the boundary conditions.
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