Supersymmetric Rényi Entropy

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Abstract: We consider 3d $\mathcal{N} \geq 2$ superconformal field theories on a branched covering of a three-sphere. The Rényi entropy of a CFT is given by the partition function on this space, but conical singularities break the supersymmetry preserved in the bulk. We turn on a compensating $R$-symmetry gauge field and compute the partition function using localization. We define a supersymmetric observable, called the super Rényi entropy, parametrized by a real number $q$. We show that the super Rényi entropy is duality invariant and reduces to entanglement entropy in the $q \to 1$ limit. We provide some examples.

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1 Introduction

Supersymmetric gauge theories provide a vast playground in which to explore aspects of strongly coupled quantum field theory which are not otherwise accessible. These models are generic enough to incorporate the types of fields and interactions included in familiar non-supersymmetric theories, and constrained enough to allow, for example, exact evaluation of special observables even when perturbation theory is inapplicable. Moreover, the ratio in this mixture can frequently be tuned by changing the number of supercharges preserved by the model and the observable.

The entanglement entropy of the vacuum is an example of an observable which is available in any quantum field theory. To define this quantity, one divides a spatial slice into a region \( V \) and its complement \( -V \) (we use the notation of [1]). The Hilbert space is likewise split

\[
\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{-V}. \tag{1.1}
\]

The reduced density matrix associated with \( V \) of the vacuum state \( |0\rangle \) is defined by

\[
\rho_V = \text{tr}_{-V}|0\rangle\langle 0|, \tag{1.2}
\]

where the trace is over a basis in \( \mathcal{H}_{-V} \): the Hilbert space of the states associated with field configurations supported on \( -V \). Normalization of the state \( |0\rangle \) implies

\[
\text{tr}_V (\rho_V) = 1. \tag{1.3}
\]

The entanglement entropy is the von Neumann entropy of \( \rho_V \)

\[
S(V) = -\text{tr}_V (\rho_V \log \rho_V). \tag{1.4}
\]

This provides a measure of the degree to which \( |0\rangle \) departs from a product state in the decomposition defined by (1.1). \( S(V) \) must be regularized and generically contains both universal and non-universal parts. Universal components of \( S(V) \) for a hyper-spherical region have been shown to be related to \( c \)-functions for quantum field theories in various dimensions [2]. A \( c \)-function is an observable which is positive and monotonically decreasing along RG flows, the prototypical example being the central charge of a 2d conformal field theory when appropriately extended away from the fixed points (see [1, 3] and references therein). In 3d, \( S(V) \) has only power law divergences and the finite piece is universal.

One can also consider the Rényi entropies defined by

\[
S_q(V) = \frac{1}{1-q} \log \text{tr}_V \rho_V^q. \tag{1.5}
\]
These contain additional information and can be used to further characterize the ground state. The entanglement entropy can be recovered as

\[ \lim_{q \to 1} S_q(V) = S(V). \]  

(1.6)

Indeed, this is often the easiest way to calculate \( S(V) \). This is because the calculation of \( S_q \), for integer \( q \), can be reduced to the evaluation of a Euclidean partition function \( Z_q \) using the “replica trick”

\[ S_q = \frac{1}{1-q} \log \left( \frac{Z_q}{(Z_1)^q} \right), \]

(1.7)

where the functional path integral is performed on a \( q \)-covering space branched along \( V \). This reproduces both the traces and the powers of the density matrix in (1.5). The \( q \)-covering space has curvature singularities along the boundary \( \partial V \). If the theory has conformal symmetry in \( d \)-dimensions, the calculation can be mapped to a thermal partition function, with temperature set by \( q \), on either \( \mathbb{H}^{d-1} \times \mathbb{R} \) or \( dS_d [4] \). The result is again equivalent to a Euclidean partition function on either \( \mathbb{H}^{d-1} \times S^1 \), where the circumference of the thermal circle is related to the curvature of the hyperbolic space as \( 2\pi R q \), or a branched covering of \( S^d \). The value of the entanglement entropy can be recovered by taking the limit \( q \to 1 \), or directly by evaluating the partition function on either \( S^d \) or the space \( \mathbb{H}^{d-1} \times S^1 \). The calculation of \( S_q \) has been carried out in two-dimensions [5–16] and for free fields in higher-dimensions [17–19]. A calculation of the entanglement entropy for 3d conformal field theories in the large-\( N \) limit was presented in [20, 21] and a proposal for theories with holographic duals was put forth in [22, 23]. In general, however, the calculation of either the entanglement or Rényi entropies for interacting theories, even ones with conformal symmetry, is difficult.

Surprisingly, the Euclidean partition function on \( S^3 \) for a conformal theory with \( N \geq 2 \) supersymmetry in the 3d sense (four supercharges) can be recovered from a supersymmetric calculation. By this we mean that there exists an action on \( S^3 \) for such a theory which preserves a symmetry group that includes fermionic generators [24, 25]. These can be used to localize the theory and reduce the calculation of the partition function, indeed any supersymmetric observable, to a finite matrix model [26–28]. In practice, the finite part of the partition function is independent of many parameters in the effective action such as the D-terms. This means that one can evaluate the partition function of a superconformal field theory by embedding it in the flow from an appropriate UV action. This allows one to compute the entanglement entropy even for strongly coupled IR fixed points of 3d supersymmetric gauge theories as long as one can identify a conserved \( R \)-symmetry current. This has been used to test the conjecture that the entanglement entropy, or equivalently the \( S^3 \) partition function, is a \( c \)-function [29–32]. This conjecture, now known as the \( F \)-theorem, is supported by the \( F \)-maximization principle for \( N = 2 \) supersymmetric field theories [33]. A proof has been presented for general theories with the use of the strong subadditivity property of entanglement entropy [34].

We will explore the possibility of recovering the Rényi entropies of a 3d \( N = 2 \) superconformal theory by calculating a similar Euclidean partition function on the branched covering
of $S^3$. This space is homeomorphic to $S^3$ and also locally isometric to $S^3$ away from a certain co-dimension two singularity. The singularity is supported on a great circle. It is conical in nature and represents a deficit/surplus angle for $q < 1$ and $q > 1$ respectively. The main challenge will be to regulate this singularity in a manner compatible with supersymmetry. We will find that this requires turning on an additional background value for the $R$-symmetry gauge field in the supergravity multiplet. We call the branched sphere with this additional background the singular space.

In section 2, we show how to deal with the singularities of the branched sphere in a supersymmetric way. We define a quantity which is related to the Rényi entropy of a 3d superconformal theory that we call the super Rényi entropy $S_{q}^{\text{susy}}$ by

$$S_{q}^{\text{susy}} = \frac{1}{1-q} \Re \left[ \log \left( \frac{Z_{\text{singular space}}(q)}{(Z_{S^3})^q} \right) \right]. \quad (1.8)$$

In section 3 we show how to compute this quantity for a superconformal $\mathcal{N} = 2$ theory using localization. We also consider more general situation where the branched sphere is parametrized by two parameters $p$ and $q$ whose metric is given by (2.2).

The localized partition function we obtain is summarized in the following form:

$$Z(p,q) = \frac{1}{|W|} \int \prod_{i=1}^{\text{rank } G} d\sigma_i e^{\pi i k \text{Tr}(\sigma^2)} \cdot \prod_{\alpha} \frac{1}{\Gamma_h(\alpha(\sigma))} \cdot \prod_{I} \prod_{\rho \in \mathcal{R}_I} \Gamma_h(\rho(\sigma) + i\omega \Delta_I), \quad (1.9)$$

where

$$\Gamma_h(z) \equiv \Gamma_h(z; i\omega_1, i\omega_2), \quad (1.10)$$

is the hyperbolic gamma function with

$$\omega_1 = \sqrt{\frac{q}{p}}, \quad \omega_2 = \sqrt{\frac{p}{q}}, \quad \omega = \frac{\omega_1 + \omega_2}{2}, \quad (1.11)$$

$k$ is the Chern-Simons level, $I$ labels the types of chiral multiplets and $\rho$ is a weight in a representation $\mathcal{R}_I$ of a gauge group $G$. $\Delta_I$ is the $R$-charge of the scalar field in a chiral multiplet. The super Rényi entropy is obtained by the relation $Z_{\text{singular space}}(q) \equiv Z(1,q)$.

The form of the partition function (1.9) is equivalent to that on the squashed three-sphere $S^3_b$ [35, 36] with squashing parameter $b = \sqrt{q/p}$.

Section 4 contains several checks of the result, especially the duality invariance and the interpretation of the partition function on the branched sphere as insertions of defect operators on the round sphere. Several examples are presented in section 5 to demonstrate how to compute the super Rényi entropy and show how it behaves as a function of $q$.

Our conventions are summarized in appendix A. Appendices B and C contain the details of the branched and resolved three-spheres. The definition of the hyperbolic gamma function and useful formulae are collected in appendix D.

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1We would like to thank Rob Myers for suggesting this possibility.
2 Setup

We are interested in defining a quantity which is as close as possible to the usual Rényi entropy and can be evaluated exactly using localization. To this end we employ the simplification present for conformal field theories which allows us to rewrite the Rényi entropy as a partition function over a curved manifold: the branched three-sphere. We also restrict ourselves to studying superconformal theories with $\mathcal{N} \geq 2$ supersymmetry in the 3d sense. Such theories can be consistently coupled to a fixed gravitational background while preserving supersymmetry. This is accomplished by coupling a special supermultiplet containing the energy-momentum tensor to supergravity and taking an appropriate limit. The procedure is described in detail in [25].

2.1 The branched sphere

The original description of the branched sphere in [18] is with coordinates $\theta, \tau, \phi$ with periodicities $\theta \in [0, \pi/2]$, $\tau \in [0, 2\pi q q')$, and $\phi \in [0, 2\pi)$. We will rescale $\tau$ by $1/q$. The new line element (with the rescaled variable still called $\tau$) is

$$\ell^2 \left( d\theta^2 + q^2 \sin^2 \theta d\tau^2 + \cos^2 \theta d\phi^2 \right), \quad \theta \in [0, \pi/2], \quad \tau \in [0, 2\pi), \quad \phi \in [0, 2\pi), \quad (2.1)$$

and we consider the two parameter generalization

$$\ell^2 \left( d\theta^2 + q^2 \sin^2 \theta d\tau^2 + p^2 \cos^2 \theta d\phi^2 \right), \quad (2.2)$$

Near $\theta = 0$ the Ricci scalar on this space has a delta function singularity as required by the Gauss-Bonnet theorem

$$R \simeq 2 \frac{q - 1}{q} \frac{\delta(\theta)}{\theta}, \quad (2.3)$$

and a similar expression involving $p$ near $\theta = \pi/2$ (see (C.6)). These are co-dimension two conical singularities. The singularities occur on two great circles with linking number 1. Other aspects of this geometry are summarized in appendix B.

2.2 A singular supersymmetric background

The gravitational background described above as the setting for the computation of the Rényi entropy can be embedded into a background supergravity multiplet. It is a priori clear that on the bulk of the space the configuration can be extended to the one that is supersymmetric by turning on appropriate fields in the multiplet as in the usual $S^3$ computation. The curvature singularities can then be viewed as gravitational defects. These, too, can be made supersymmetric by including a compensating gauge defect in the background gauge fields (specifically the $R$-symmetry gauge field). One must still specify boundary conditions on the dynamical fields of the theory as one approaches the singularity. From now on we will set
ℓ = 1. Factors of ℓ can be easily restored, however, the calculation of the super Rényi entropy using localization does not depend on it.

We wish to identify a supersymmetric action on the branched sphere with metric (2.2) for a general \( \mathcal{N} = 2 \) theory with a conserved \( R \)-symmetry. This was carried out for a general background in [25] by coupling the theory to 3d supergravity using the \( \mathcal{R} \)-multiplet of [37]. The supergravity multiplet contains the bosonic fields

\[
g_{\mu\nu}, \quad A_\mu, \quad V_\mu = -i\varepsilon_{\mu\nu\rho} \partial_\nu C_\rho, \quad H = \frac{i}{2} \varepsilon^{\mu\nu\rho} \partial_\mu B_{\nu\rho}. \tag{2.4}
\]

These are defined up to gauge transformations which include diffeomorphisms and background gauge transformations for the \( R \)-symmetry gauge field \( A_\mu \). We refer the reader to [25] for details.

To determine an appropriate supergravity background, we consider the Killing spinor equation from [25]

\[
(\nabla_\mu - i A_\mu) \zeta = -\frac{1}{2} H \gamma_\mu \zeta - i V_\mu \zeta - \frac{1}{2} \varepsilon_{\mu\nu\rho} V_\nu \gamma^\rho \zeta, \tag{2.5}
\]

where the covariant derivative for a spinor is defined by

\[
\nabla_\mu \zeta = \left( \partial_\mu + \frac{1}{8} \omega_{\mu ij} [\sigma_i, \sigma_j] \right) \zeta.
\]

A solution of this equation, with particular values for the supergravity background fields \( A_\mu, V_\mu, H \), represents a single complex supersymmetry generated by the spinor \( \zeta \) with \( R \)-charge +1. A similar equation exists for spinors of \( R \)-charge −1

\[
(\nabla_\mu + i A_\mu) \zeta = -\frac{1}{2} H \gamma_\mu \zeta + i V_\mu \zeta + \frac{1}{2} \varepsilon_{\mu\nu\rho} V_\nu \gamma^\rho \zeta. \tag{2.6}
\]

On \( S^3 \), one can preserve two spinors of each \( R \)-charge by turning on, in addition to the usual metric for the round sphere, a constant imaginary value for the field \( H = \pm i.2 \)

The usual spinor used in localization on \( S^3 \), \( \varepsilon_L \), satisfies (in the left-invariant basis in appendix B)

\[
\nabla^{S^3}_{\mu} \varepsilon_L = \left( \partial_\mu + \frac{1}{8} \omega_{S^3 \mu ij} [\sigma_i, \sigma_j] \right) \varepsilon_L = \frac{i}{2} (e_{S^3})_\mu \sigma_i \varepsilon_L, \tag{2.7}
\]

because in the left-invariant basis on \( S^3 \) the spin connection is related to the vielbein as

\[
\omega^{ij}|_{q=p=1} = \varepsilon^{ijk} e_k|_{q=p=1}, \tag{2.8}
\]

and \( \varepsilon_L \) is (any) constant spinor in this basis [26]. There are two more spinors (which are constant in the right-invariant basis) that satisfy

\[
\nabla^{S^3}_{\mu} \varepsilon_R = \left( \partial_\mu + \frac{1}{8} \omega_{S^3 \mu ij} [\sigma_i, \sigma_j] \right) \varepsilon_R = -\frac{i}{2} (e_{S^3})_\mu \sigma_i \varepsilon_R. \tag{2.9}
\]

\[2\text{Since } A_\mu = V_\mu = 0 \text{ in this setting, the equations (2.5) and (2.6) coincide. The spinors used to generate supersymmetries with positive and negative } R \text{-charges have the same spatial profile.}\]
On the branched sphere, the spin connection satisfies

$$\omega^{ij} = \varepsilon^{ijk} e_k - \tilde{\omega}^{ij},$$

where

$$\tilde{\omega}^{ij} = \varepsilon^{ij3} ((q - 1) d\tau + (p - 1) d\phi).$$

### 2.2.1 Bulk

For the branched sphere with metric (2.2), the spinor equation can be satisfied in the bulk by taking

$$H = -i, \quad A = 0, \quad V = 0,$$

and the spinors

$$\varepsilon_{\pm}^{\text{branched}} = \begin{pmatrix} c_+ e^{i \frac{q-1}{2} \tau + i \frac{p-1}{2} \phi} \\ c_- e^{-i \frac{q-1}{2} \tau - i \frac{p-1}{2} \phi} \end{pmatrix}.$$

This follows from the fact that this space is (locally) isometric to $S^3$ and these are nothing but the constant spinors of $\varepsilon_L$ after a coordinate transformation.

However, the conical singularities (2.3) make this solution suspect near $\theta = 0, \pi/2$. Note that the equation (5.8) of [25] implies the integrability condition (at constant $H$ and $V = 0$)

$$\left[ \frac{i}{2} (2R_{\mu\nu} - Rg_{\mu\nu}) \gamma^\nu - 2i \varepsilon_{\mu\nu\rho} \nabla^\nu A^\rho + i H^2 \gamma_\mu \right] \zeta = 0,$$

which would include delta function contributions from (2.3).

### 2.2.2 Singularities

A somewhat better solution is to take

$$H = -i, \quad A = \frac{q - 1}{2} d\tau + \frac{p - 1}{2} d\phi, \quad V = 0,$$

so that the field strength

$$F = \frac{q - 1}{2} \delta (\theta) d\phi \wedge d\tau + \frac{p - 1}{2} \delta \left( \frac{\pi}{2} - \theta \right) d\theta \wedge d\phi,$$

compensates for the curvature singularity. Near the singularities we have the identities that satisfy the integrability condition (2.14)

$$\left[ \frac{1 - p}{2} \delta (\theta) \sigma_3 + F_{\theta\phi} \right] \zeta = 0,$$

$$\left[ \frac{1 - q}{2} \delta (\theta) \sigma_3 - F_{\theta\tau} \right] \zeta = 0,$$

and the additional background gauge field cancels the additional term in the spin connection

$$\frac{1}{8} \tilde{\omega}^{ij}_{\mu} [\sigma_i, \sigma_j] = \frac{i}{2} \left( \delta^{\tau}_{\mu} (q - 1) + \delta_{\mu}^\phi (p - 1) \right) \sigma_3,$$
in equations (2.5) and (2.6) provided we choose the $R$-charge correctly. This amounts to performing a singular background gauge transformation for the $R$-symmetry.

This setup preserves two constant spinors, in the basis defined by (B.1), which satisfy the equations

$$\sigma_3 \zeta_\pm = \pm \zeta_\pm,$$  

(2.19)

$$\left( \nabla_\mu \pm iA_\mu \right) \zeta_\pm = -\frac{1}{2} H\gamma_\mu \zeta_\pm,$$  

(2.20)

so that the associated $R$-charges are

$$r (\zeta_\pm) = \mp 1.$$  

(2.21)

There is actually a four complex parameter family of spinors preserved by this background. The basis spinors are given by

$$\zeta_+ = \begin{pmatrix} c_+^1 e^{i(1-q)\tau + i(p+q)\phi} \\ c_+^2 e^{i(1-q)\tau + i(p+q)\phi} \end{pmatrix}, \quad r = -1,$$  

$$\zeta_- = \begin{pmatrix} c_-^1 e^{i(q-1)\tau + i(p-q)\phi} \\ c_-^2 \end{pmatrix}, \quad r = 1.$$  

(2.22)

Note that the spinors parametrized by $c_+^2$ and $c_-^1$ still satisfy (2.14) (taking into account the $R$-charge) only up to delta function contributions.

The fact that this background preserves four independent supercharges is in conflict with expectations. Specifically, a smooth background preserving four supercharges always has constant $H$ and the field strength for the connection $A_\mu - V_\mu$ is flat [25]. The second condition is not satisfied for (2.15), which implies that we may have used the Killing spinor equation (2.5) outside its range of validity. For one, we have ignored possible boundary terms used in its derivation.\(^3\) We could attempt to fix this by excising the loops at $\theta = 0, \pi/2$ and extending the formalism in [25] to include spaces with a boundary. We will instead take a simpler approach which allows us to continue working with a compact space.

### 2.3 The resolved space

We can avoid the pitfalls of the singular background by smoothing out both the curvature of (2.3) and the field strength of (2.16). We do this by introducing a one parameter family of smooth backgrounds (the resolved space) which solve the Killing spinor equation (2.5) and converge to (2.2) and (2.15). This can be achieved by deforming the metric and vielbein as follows

$$ds^2 = \frac{1}{f_\tau (\theta)} d\theta^2 + q^2 \sin^2 \theta d\tau^2 + p^2 \cos^2 \theta d\phi^2,$$  

(2.23)

\(^3\)We would like to thank Thomas Dumitrescu for stressing this to us.
and introducing the following smooth background fields

\[ H = -i \sqrt{f_\epsilon (\theta)} , \quad A = \frac{q \sqrt{f_\epsilon (\theta)} - 1}{2} d\tau + \frac{p \sqrt{f_\epsilon (\theta)} - 1}{2} d\phi , \quad V = 0 , \quad (2.24) \]

where \( f_\epsilon (\theta) \) is a smooth function satisfying (for a small \( \epsilon > 0 \))

\[ f_\epsilon (\theta) = \begin{cases} \\
\frac{1}{q^2} , \quad \theta \to 0 , \\
\frac{1}{p^2} , \quad \theta \to \frac{\pi}{2} , \\
1 , \quad \epsilon < \theta < \frac{\pi}{2} - \epsilon . 
\end{cases} \quad (2.25) \]

As \( \epsilon \to 0 \) this approaches the required background. We will henceforth write simply \( f (\theta) \).

The resolved space still preserves two of the spinors of the singular space

\[ \zeta_+ = \left( \begin{array}{c} c_1^+ \\ 0 \end{array} \right) , \quad r = -1 , \quad (2.26) \]

\[ \zeta_- = \left( \begin{array}{c} 0 \\ c_2^- \end{array} \right) , \quad r = 1 , \quad (2.27) \]

which will be enough to perform the localization on this space in the next section. The boundary conditions for the modes of fields defined on this background are the usual ones which produce non-singular and normalizable field configurations. Since we are interested in the original partition function defining the super Rényi entropy, we will take \( \epsilon \to 0 \) at the end of the localization procedure.

Although the partition function on the resolved space could, in principle, depends on the detailed form of the deformation function \( f (\theta) \), the limit will not. One may still wonder whether there are different supersymmetric backgrounds which could have been used to define the resolved space. For instance, one may consider the more general configuration

\[ H_h = -ih (\theta) , \]

\[ A_h = \frac{1}{4} \left( -2 + q (-1 + 3 \cos (2\theta)) \sqrt{f_\epsilon (\theta)} + 6 q h (\theta) \sin^2 \theta \right) d\tau 
+ \frac{1}{4} \left( -2 - p (1 + 3 \cos (2\theta)) \sqrt{f_\epsilon (\theta)} + 6 p h (\theta) \cos^2 \theta \right) d\phi , \quad (2.28) \]

\[ V_h = q \left( h (\theta) - \sqrt{f_\epsilon (\theta)} \right) \sin^2 \theta dr + p \left( h (\theta) - \sqrt{f_\epsilon (\theta)} \right) \cos^2 \theta d\phi , \]

which represents an infinite class of smooth backgrounds, parametrized by the function \( h (x) \), with which one can take the limit to the branched sphere. The spinors (2.26) and (2.27) are preserved for any \( h (\theta) \) and this is the most general configuration. However, as noted in [25], the scalar \( H \) couples to an operator which becomes redundant in a conformal theory and hence decouples in the IR. The fields \( A \) and \( V \) still couple to the distinguished \( R \)-symmetry current \( j_\mu^{(R)} \) of a superconformal theory which sits in the same multiplet as the energy-momentum tensor. The linearized coupling is of the form

\[ j_\mu^{(R)} \left( A^\mu - \frac{3}{2} V^\mu \right) , \quad (2.29) \]
but we can see that
\[
\frac{\delta}{\delta h(\theta)} \left( A^\mu_h - \frac{3}{2} V^\mu_h \right) = 0 ,
\]
so a small change in \( h(\theta) \) does not affect the result. There may be more general configurations which would yield different partition functions and hence a different type of supersymmetric observable associated with the Rényi entropy. We will not pursue this possibility.

One could try to form this type of a resolved space without the use of the background \( R \)-symmetry gauge field. The Killing spinor equations then imply that \( f(\theta) \equiv 1 \) and hence the deformation does not remove the singularities. We take this to mean that supersymmetry on the branched sphere requires the introduction of the background \( R \)-symmetry gauge field or the equivalent boundary conditions. Hence, the partition function yielding the usual Rényi entropy, with boundary conditions of the type described in [18], is not a supersymmetric observable.

3 Localization

The path integral expression which calculates the partition function on the space defined by (2.23) and (2.24) is invariant under the fermionic symmetries generated by (2.26) and (2.27). Let us call \( Q \) the supercharge generated by \( \zeta \). A standard argument shows that the partition function can be expressed as a sum of contributions from fixed points of \( Q \) along with a determinant (or superdeterminant) representing the equivariant Euler class of the normal bundle [38, 39]. One must also evaluate the classical action on this space. Details are available in [26, 40]. We now carry out this localization calculation by finding the fixed points of the \( Q \) action, the fluctuation determinants and the classical contributions.

3.1 The localizing term

An efficient way of finding the space of fixed points is to add a \( Q \)-exact term \( \{Q, V\} \) to the action \( S \) of the \( \mathcal{N} = 2 \) theory under consideration. The same argument as above shows that the result is independent of this term. The deformed partition function
\[
Z(t) = \int \mathcal{D}\phi e^{-S - t \{Q, V\}} ,
\]
does not depend on the parameter \( t \). We will choose a positive semi-definite term with large \( t \). The semi-classical approximation around the zero locus is then exact.

We follow the supersymmetry transformation rules in [25] to obtain an appropriate localizing term for vector and chiral multiplets. A general superfield \( S \) whose bottom component is a complex scalar \( C \) of \( R \)-charge \( r \) and central charge \( z \) has \( 16 + 16 \) bosonic and fermionic components
\[
S = (C, \chi_\alpha, \tilde{\chi}_\alpha, M, \bar{M}, a_\mu, \sigma, \lambda_\alpha, \bar{\lambda}_{\bar{\alpha}}, D) .
\]
The supersymmetry transformation rules with respect to the Killing spinor $\zeta$ of $R$-charge $+1$ are given by

$$
\delta \zeta C = i \zeta \chi , \\
\delta \zeta \chi = \zeta M , \\
\delta \zeta \tilde{\chi} = -\zeta (\sigma - (z - rH)C) - \gamma^\mu \zeta (D_\mu C - ia_\mu) , \\
\delta \zeta M = 0 , \\
\delta \zeta \tilde{M} = 2\zeta \lambda - 2i(z - (r + 2)H)\zeta \tilde{\chi} - 2iD_\mu (\zeta \gamma^\mu \chi) , \\
\delta \zeta a_\mu = -i \zeta \gamma_\mu \tilde{\lambda} + D_\mu (\zeta \chi) , \\
\delta \zeta \sigma = -\zeta \tilde{\lambda} + i(z - rH)\zeta \chi , \\
\delta \zeta \lambda = i (D + \sigma H) - i \varepsilon^{\mu \nu \rho} \gamma_\rho D_\mu a_\nu - \gamma^\mu \zeta ((z - rH) a_\mu + i D_\mu \sigma - V_\mu \sigma) , \\
\delta \zeta \tilde{\lambda} = 0 , \\
\delta \zeta D = D_\mu (\zeta \gamma^\mu \tilde{\lambda}) - i V_\mu \zeta \gamma^\mu \tilde{\lambda} - H \zeta \lambda + (z - rH) \left( \zeta \tilde{\lambda} - i H \zeta \chi \right) + \frac{ir}{4} (R - 2V^\mu V_\mu - 6H^2) \zeta \chi , 
$$

(3.3)

and for the Killing spinor $\tilde{\zeta}$ of $R$-charge $-1$

$$
\delta \tilde{\zeta} C = i \tilde{\zeta} \tilde{\chi} , \\
\delta \tilde{\zeta} \chi = -\tilde{\zeta} (\sigma + (z - rH)C) - \gamma^\mu \tilde{\zeta} (D_\mu C + ia_\mu) , \\
\delta \tilde{\zeta} \tilde{\chi} = \tilde{\zeta} \tilde{M} , \\
\delta \tilde{\zeta} M = -2\tilde{\zeta} \tilde{\lambda} + 2i(z - (r - 2)H)\tilde{\zeta} \chi - 2iD_\mu (\tilde{\zeta} \gamma^\mu \chi) , \\
\delta \tilde{\zeta} \tilde{M} = 0 , \\
\delta \tilde{\zeta} a_\mu = -i \tilde{\zeta} \gamma_\mu \lambda - D_\mu (\tilde{\zeta} \chi) , \\
\delta \tilde{\zeta} \sigma = \tilde{\zeta} \lambda - i(z - rH)\tilde{\zeta} \chi , \\
\delta \tilde{\zeta} \lambda = 0 , \\
\delta \tilde{\zeta} \tilde{\lambda} = -i \tilde{\zeta} (D + \sigma H) - i \varepsilon^{\mu \nu \rho} \gamma_\rho D_\mu a_\nu + \gamma^\mu \tilde{\zeta} ((z - rH) a_\mu + i D_\mu \sigma + V_\mu \sigma) , \\
\delta \tilde{\zeta} D = -D_\mu (\tilde{\zeta} \gamma^\mu \lambda) - i V_\mu \tilde{\zeta} \gamma^\mu \lambda + H \tilde{\zeta} \lambda + (z - rH) \left( \tilde{\zeta} \lambda + i H \tilde{\zeta} \chi \right) - \frac{ir}{4} (R - 2V^\mu V_\mu - 6H^2) \tilde{\zeta} \chi , 
$$

(3.4)

where the covariant derivative is defined by

$$
D_\mu = \nabla_\mu - ir \left( A_\mu - \frac{1}{2} V_\mu \right) - iz C_\mu .
$$

(3.5)

A supersymmetric action can be obtained from the D-term (plus terms coupled to the background supergravity fields) of a general superfield $S$ of $r = z = 0$,

$$
\mathcal{L}_D = -\frac{1}{2} (D - \sigma H - a_\mu V^\mu) ,
$$

(3.6)
whose supersymmetry transformation is a total derivative. One can show that this action is $Q$-exact when $V_\mu = 0$. More explicitly, one can derive by using the transformation rules (3.3) and (3.4)

$$
\delta \zeta \delta \tilde{\zeta} (i \sigma + 2iHC) = \zeta \tilde{\zeta} (D - \sigma H) - 2iHK^\mu a_\mu - \varepsilon^{\mu \nu \rho} a_\nu \nabla_\mu K_\rho + iV^\mu K_\mu \sigma \\
+ \nabla_\mu (\varepsilon^{\mu \nu \rho} a_\nu K_\rho + K^\mu \sigma + 2HC),
$$

(3.7)

where we used the notation

$$K^\mu \equiv \zeta \gamma^\mu \tilde{\zeta},$$

(3.8)

that satisfies

$$\nabla^\mu K_\mu = 0.$$  

(3.9)

The total derivative term in the second line of (3.7) vanishes inside the spacetime integral. One can show that the second and third terms in the first line vanish due to the following identity derived from the Killing spinor equations (2.5) and (2.6)

$$\nabla_\mu K_\rho = iH \varepsilon_{\mu \rho \kappa} K_\kappa + \varepsilon_{\mu \rho \kappa} V^\kappa \zeta \tilde{\zeta}. 
$$

(3.10)

Thus we have shown that the D-term of a general superfield of $r = z = 0$ is always $Q$-exact as long as the background gauge field $V_\mu$ vanishes$^5$

$$\zeta \tilde{\zeta} L_D|_{r=z=0} = \delta \zeta \delta \tilde{\zeta} \left( -\frac{i}{2} \sigma - iHC \right). 
$$

(3.11)

The relation (3.11) ensures that the Yang-Mills and matter actions given as D-terms are $Q$-exact and can be used to localize the partition function (see [25] for the vector and chiral multiplets and their transformation rules). This also guarantees that the resulting partition function is independent of $\ell$, as a change in $\ell$ can be absorbed, by rescaling the dynamical fields, into the overall normalization of one of the $Q$-exact terms.

### 3.2 Zero modes

The Yang-Mills term is $Q$-exact and can be used to localize the gauge sector

$$
\mathcal{L}_{YM} = \text{Tr} \left[ \frac{1}{4} f_{\mu \nu} f^{\mu \nu} + \frac{1}{2} D_\mu \sigma D^\mu \sigma - i \lambda \gamma^\mu \left( D_\mu + \frac{i}{2} V_\mu \right) \lambda \\
+ \frac{i}{2} \sigma \varepsilon^{\mu \nu \rho} V_\mu f_{\nu \rho} - \frac{1}{2} V_\mu V^\mu \sigma^2 - \frac{1}{2} (D + \sigma H)^2 - i \lambda [\sigma, \lambda] + \frac{i}{2} H \lambda \lambda \right],
$$

(3.12)

$^4$ Spinors are commuting variables and the supersymmetry transformation is anti-commuting in [25]. In this convention, two spinors $\xi$ and $\eta$ satisfy $\xi \eta = -\eta \xi$, $\xi \gamma_\mu \eta = \eta \gamma_\mu \xi$ and so on. We would like to thank Guido Festuccia for clarifying this point for us.

$^5$ The overall normalization is immaterial as long as the function $\zeta \tilde{\zeta} \neq 0$. In our case $\zeta \tilde{\zeta} \neq 1$. 


where

\[
\begin{align*}
    f_{\mu\nu} &= \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu], \\
    D_\mu \sigma &= \partial_\mu \sigma - i[a_\mu, \sigma], \\
    D_\mu \lambda &= (\nabla_\mu + iA_\mu)\lambda - i[a_\mu, \lambda],
\end{align*}
\]

(3.13)

and \(a_\mu\) is a connection on a (trivial) principle bundle for a Lie group \(G\) with Lie algebra \(\mathfrak{g}\) and all other fields are valued in the adjoint bundle. We can fix the gauge freedom by adding the gauge fixing term

\[
L_{\text{g.f.}} = \bar{c} \nabla_\mu D^\mu c + b \nabla^\mu a_\mu,
\]

(3.14)

which imposes the covariant gauge \(\nabla_\mu a_\mu = 0\). For \(V_\mu = 0\), the bosonic part of the above action is a sum of positive semi-definite terms. It is easy to see that it can vanish only when

\[
\begin{align*}
    f_{\mu\nu} &= 0, \\
    D_\mu \sigma &= 0, \\
    D &= -H \sigma.
\end{align*}
\]

(3.15)

The first condition leads to a flat connection which is equivalent to \(a_\mu = 0\) on a smooth simply connected compact space. It follows that the second condition gives the constant configuration of the adjoint scalar

\[
\sigma = \sigma_0,
\]

(3.16)

and the third yields

\[
D = -H \sigma_0.
\]

(3.17)

We refer to the Lie algebra valued variable \(\sigma_0\) as a zero mode.

The \(Q\)-exact term used to localize the matter fields of \(R\)-charge \(\Delta\) is

\[
\begin{align*}
L_{\text{matter}} &= D^\mu \bar{\phi} D_\mu \phi - i\bar{\psi} \gamma^\mu D_\mu \psi - \bar{F}F \\
&+ \left[ D + \left( \sigma + \left( \Delta - \frac{1}{2} \right) H \right)^2 - \frac{\Delta}{4} R + \frac{1}{2} \left( \Delta - \frac{1}{2} \right) \left( V^\mu V_\mu + H^2 \right) \right] \bar{\phi} \phi \\
&- \left( \sigma + \left( \Delta - \frac{1}{2} \right) H \right) i\bar{\psi} \psi + \sqrt{2} i (\bar{\phi}\lambda \psi + \phi \bar{\lambda} \bar{\psi}),
\end{align*}
\]

(3.18)

where the covariant derivatives are defined by

\[
\begin{align*}
    D_\mu \phi &= \left( \nabla_\mu - i\Delta A_\mu - i a_\mu + i \frac{\Delta + 1}{2} V_\mu \right) \phi, \\
    D_\mu \psi &= \left( \nabla_\mu - i(\Delta - 1) A_\mu - i a_\mu + i \frac{\Delta + 1}{2} V_\mu \right) \psi.
\end{align*}
\]

(3.19)

\[\text{We actually need to deal with a combined supersymmetry and BRST complex. This extension is explained in [26].}\]
This is non-vanishing for $V_\mu = 0$ and the gauge multiplet configuration above. Hence, there are no additional zero modes coming from the dynamical chiral multiplets.\footnote{Actually, this is only true for $0 < \Delta < 2$. However, at a superconformal fixed point, $\Delta$ coincides with the conformal dimension of $\phi$. Unitarity then restrict a charged multiplet to have $\Delta > 1/4$.}

Finally the measure of the integral over the zero modes is given by the flat measure of $\sigma_0$

$$[d\sigma] = \frac{1}{\text{Vol}(G)} \prod_{\alpha \in \mathfrak{g}} d\sigma_\alpha = \frac{1}{|W| \text{Vol}(T)} \prod_{i=1}^{\text{rank } G} d(\sigma_0)_i \prod_{\alpha > 0} \alpha(\sigma_0)^2 , \quad (3.20)$$

where $(\sigma_0)_i$ are the Cartan parts of $\sigma_0$ and

$$\prod_{\alpha > 0} \alpha(\sigma_0)^2 , \quad (3.21)$$

is the Vandermonde determinant that arises when $\sigma_0$ is diagonalized. $|W|$ is the order of the Weyl group $W$ and $\text{Vol}(T)$ is the volume of the maximal torus.

### 3.3 Fluctuation determinants

A straightforward way of evaluating the equivariant Euler class appearing in the localization formula is to compute the fluctuation (super)determinant of the dynamical fields around the space of zero modes. This is done by solving the eigenvalue equation for the quadratic part of the $Q$-exact term. We do so below for the gauge and chiral multiplets.\footnote{Our convention for the Ricci scalar, which appears in the action for a chiral multiplet, is that of [25]. With this convention, $R$ on $S^3$ is negative.}

#### 3.3.1 Matter one-loop determinant

We will calculate the one-loop determinant of the chiral multiplet around the background $V_\mu = 0$ and $A_\mu$ given by (2.24). The quadratic parts of the Lagrangian for the scalars and spinors in (3.18) consist of the following operators

$$\Delta_\phi = -\nabla^2 + \Delta^2 A_\mu A^\mu + ((\Delta - 1)H + \sigma_0)^2 - H^2 - \frac{\Delta}{4}(R - 6H^2) ,$$

$$\Delta_\psi = -i\gamma^\mu (\nabla_\mu - i(\Delta - 1)A_\mu) - i \left( (\Delta - \frac{1}{2})H + \sigma_0 \right) . \quad (3.22)$$

where we must use the Ricci scalar calculated in appendix C.

We begin with the eigenvalue problem of the scalar Laplacian

$$\Delta_\phi \phi = \lambda_\phi \phi . \quad (3.23)$$

We Fourier decompose the scalar field as

$$\phi(\theta, \tau, \phi) = e^{i(m\tau + n\phi)}\phi(\theta) , \quad m, n \in \mathbb{Z} , \quad (3.24)$$
which leads to the second order ordinary differential equation

\[
\phi''(\theta) + \left(2 \cot(2\theta) + \frac{f'(\theta)}{2f(\theta)}\right) \phi'(\theta) + \left(\Lambda_1 - \frac{\Lambda_2^2}{\sin^2 \theta} - \frac{\Lambda_3^2}{\cos^2 \theta}\right) \phi(\theta) = 0 ,
\]

(3.25)

where

\[
\Lambda_1 \equiv \frac{\lambda_s}{f(\theta)} - 1 - \frac{(\sigma_0 - i(\Delta - 1)\sqrt{f(\theta)})^2}{f(\theta)} + \frac{f'(\theta)}{2f(\theta)} \Delta \cot(2\theta) ,
\]

\[
\Lambda_2 \equiv \frac{2m - \Delta(q\sqrt{f(\theta)} - 1)}{2q} ,
\]

\[
\Lambda_3 \equiv \frac{2n - \Delta(p\sqrt{f(\theta)} - 1)}{2p} .
\]

The eigenvalue problem for the Dirac operator acting on the spinor is

\[
\Delta \psi = \lambda_f \psi .
\]

(3.27)

We decompose \(\psi\) as

\[
\psi(\theta, \tau, \phi) = e^{i(m\tau + n\phi)} \begin{pmatrix} \psi_1(\theta) \\ e^{i(\tau + \phi)} \psi_2(\theta) \end{pmatrix} , \quad m, n \in \mathbb{Z} ,
\]

(3.28)

in the basis defined by the vielbein (C.1). This results in a set of coupled equations

\[
\psi_1' + (\Delta \cot(2\theta) + c_1 \tan \theta - c_2 \cot \theta) \psi_1 + \left(\frac{\lambda_f + i\sigma_0}{\sqrt{f(\theta)}} + c_1 + c_2\right) \psi_2 = 0 ,
\]

\[
\psi_2' + ((2 - \Delta) \cot(2\theta) - c_1 \tan \theta + c_2 \cot \theta) \psi_2 + \left(-\frac{\lambda_f + i\sigma_0}{\sqrt{f(\theta)}} + 2(1 - \Delta) + c_1 + c_2\right) \psi_1 = 0 ,
\]

(3.29)

where

\[
c_1 = \frac{2n + \Delta}{2p\sqrt{f(\theta)}} , \quad c_2 = \frac{2m + \Delta}{2q\sqrt{f(\theta)}} .
\]

(3.30)

Solving for \(\psi_2\) yields the second order differential equation for \(\psi_1\)

\[
\psi_1'' + \left(2 \cot(2\theta) + \frac{f'(\theta)}{2f(\theta)}\right) \psi_1' + \left(\bar{\Lambda}_1 - \frac{\Lambda_2^2}{\sin^2 \theta} - \frac{\Lambda_3^2}{\cos^2 \theta}\right) \psi_1 = 0 ,
\]

(3.31)

where

\[
\bar{\Lambda}_1 \equiv \frac{(\lambda_f + (\Delta - 1)\sqrt{f(\theta)} + i\sigma_0)^2}{f(\theta)} - 1 + \frac{f'(\theta)}{2f(\theta)} \Delta \cot(2\theta) .
\]

(3.32)
The second order differential equations for the scalar and spinor (3.25) and (3.31) have the same form up to the difference of \( \Lambda_1 \) and \( \tilde{\Lambda}_1 \). To zeroth order in \( \epsilon \) they can be equal by identifying \( \Lambda_1 = \tilde{\Lambda}_1 \) and \( f(\theta) \equiv 1 \). Comparing (3.26) and (3.32), we obtain

\[
\lambda_s = \lambda_f(\lambda_f + 2(\Delta - 1) + 2i\sigma_0) .
\] (3.33)

This relation between the eigenvalues of the scalar and spinor gives rise to the cancellation in the one-loop determinant of the matter sector: given \( \lambda_s \), there are two solutions of the spinor eigenfunction \( \lambda(\pm) \) satisfying 

\[
\lambda(+)f\lambda(-)f = \lambda_s
\]

as long as \( \psi_1 \neq 0 \) and \( \psi_2 \neq 0 \). Alternatively, we could view this cancellation as happening in the singular space in section 2.2 whose supersymmetry preserving boundary conditions we are after.

The only contributions to the one-loop determinant thus come from the modes with either \( \psi_1 \neq 0, \psi_2 = 0 \) or \( \psi_1 = 0, \psi_2 \neq 0 \). When \( \psi_1 \neq 0, \psi_2 = 0 \), the eigenvalue of \( \lambda_f \) has to be (again to zeroth order in \( \epsilon \))

\[
\lambda_f^{(1)} = \frac{n}{p} + \frac{m}{q} - i\sigma_0 + \frac{\Delta}{2} \left( \frac{1}{p} + \frac{1}{q} \right) - 2(\Delta - 1) ,
\] (3.34)

and we must solve the equation

\[
\psi'_1(\theta) + \left( \Delta \cot(2\theta) + \frac{2n + \Delta}{2p\sqrt{f(\theta)}} \tan\theta - \frac{2m + \Delta}{2q\sqrt{f(\theta)}} \cot\theta \right) \psi_1(\theta) = 0 .
\] (3.35)

To obtain the boundary condition at \( \theta = 0 \) and \( \theta = \pi/2 \), we solve it near the singularities

\[
\psi_1(\theta) = \begin{cases} 
\sin^m \theta + \cdots, & \theta = 0 , \\
\cos^n \theta + \cdots, & \theta = \pi/2 .
\end{cases}
\] (3.36)

We need to impose \( m > -1 \) and \( n > -1 \) for the solution to be integrable \( \int d\theta \sin \theta \cos \theta |\psi(\theta)|^2 < \infty \).

When \( \psi_1 = 0, \psi_2 \neq 0 \), the eigenvalue of \( \lambda_f \) has to be

\[
\lambda_f^{(2)} = -\frac{n}{p} - \frac{m}{q} - i\sigma_0 - \frac{\Delta}{2} \left( \frac{1}{p} + \frac{1}{q} \right) ,
\] (3.37)

and we solve the equation

\[
\psi'_2(\theta) + \left( (2 - \Delta) \cot(2\theta) - \frac{2n + \Delta}{2p\sqrt{f(\theta)}} \tan\theta + \frac{2m + \Delta}{2q\sqrt{f(\theta)}} \cot\theta \right) \psi_2(\theta) = 0 .
\] (3.38)

To obtain the boundary condition at \( \theta = 0 \) and \( \theta = \pi/2 \), we solve it near the singularities

\[
\psi_2(\theta) = \begin{cases} 
\sin^{-m-1} \theta + \cdots, & \theta = 0 , \\
\cos^{-n-1} \theta + \cdots, & \theta = \pi/2 .
\end{cases}
\] (3.39)

and we need to impose \( m < 0 \) and \( n < 0 \) for the solution to be integrable.
In total, once we take the $\epsilon \to 0$ limit, there is complete cancellation between $\lambda_s$ and $\lambda_f$ (and $\lambda_f + 2(\Delta - 1) + 2i\sigma_0$) for the modes with $\psi_1 \neq 0, \psi_2 \neq 0$. The modes of $\lambda_f^{(2)}$ with $\psi_1 = 0, \psi_2 \neq 0$ have no bosonic counterparts, while the modes of $\lambda_f^{(1)}$ with $\psi_1 \neq 0, \psi_2 = 0$ do not have pairing modes of $\lambda_f + 2(\Delta - 1) + 2i\sigma_0$. Taking this into account, we obtain the one-loop partition function of the matter sector

$$Z_{\text{matter}}^{1\text{-loop}} = \frac{\det \Delta_\psi}{\det \Delta_\phi} = \prod \frac{\lambda_f^{(2)}}{\lambda_f^{(1)} + 2(\Delta - 1) + 2i\sigma_0} ,$$

where the final result is expressed in terms of the hyperbolic gamma function defined in appendix D.

### 3.3.2 Gauge one-loop determinant

The quadratic part of the Yang-Mills Lagrangian (3.12) is

$$L_{\text{fluc}} = \text{Tr} \left[ a_\mu \Delta_v a^\mu - [a_\mu, \sigma_0]^2 + \partial_\mu \sigma'^\mu \sigma' + (D' + \sigma')^2 - i\tilde{\lambda} \gamma^\mu (\nabla_\mu + iA_\mu) \lambda - i\tilde{\lambda}[\sigma_0, \lambda] + \frac{1}{2} \tilde{\lambda} \lambda + \partial_\mu \bar{c} \partial^\mu c \right].$$

The D-term fluctuation $D'$ can be integrated out easily. The vector Laplacian can be written as

$$\Delta_v = \star d \star d + d \star d \star ,$$

and we integrate out $b$ in (3.14) to impose the covariant gauge $d \star a = 0$.

The vector potential one-form can be decomposed as

$$a = B + d\chi ,$$

where $B$ is a divergence-less one-form

$$d \star B = 0 .$$

The covariant gauge guarantees that $\chi$ does not couple to $\sigma_0$

$$\star d \star d\chi = \Delta_{\text{scalar}} \chi = 0 .$$

The remaining gauge fixed action for the divergence-less vectors and the gauginos becomes

$$L_{\text{fluc}} = \text{Tr} \left[ B_\mu \Delta_v B^\mu - [B_\mu, \sigma_0]^2 - i\tilde{\lambda} \gamma^\mu (\nabla_\mu + iA_\mu) \lambda - i\tilde{\lambda}[\sigma_0, \lambda] + \frac{1}{2} \tilde{\lambda} \lambda \right] .$$
We decompose all adjoint fields with respect to the Cartan-Weyl basis

\[ L_{\text{fluc}} = \sum_i \left[ B^\mu_i \Delta v B^\mu_i + \tilde{\lambda}_i \Delta \psi |_{\Delta=0,\sigma_0=0} \right] \]

\[ + \sum_{\alpha} \left[ B^\mu_{\alpha} (\Delta v + \alpha(\sigma_0)^2) B^\mu_{\alpha} + \tilde{\lambda}_{-\alpha} (\Delta \psi |_{\Delta=0,\sigma_0=0} - i\alpha(\sigma_0)) \lambda_\alpha \right], \]

where we now assume that \( \sigma_0 \) is in the Cartan subalgebra. We already know the spectrum of the Dirac operator for the adjoint fermion. We need to determine the spectrum of the vector Laplacian \( \Delta v \).

Consider the eigenvalue problem of the form as in [18]

\[ d \ast B = 0, \quad \ast dB = \lambda_v B. \]  \hspace{1cm} (3.48)

From the explicit form of the vector Laplacian, the solution of the eigenvalue problem yields the eigenfunction of the vector Laplacian

\[ \Delta_v B = \lambda^2_v B. \]  \hspace{1cm} (3.49)

To solve (3.48), we expand \( B \) in terms of the left-invariant one-forms on the resolved space (C.1)

\[ B = e^{i(mq+n\phi)} \left[ e^{-i(\phi+\tau)} b_+ (\theta) e^+ + b_0 (\theta) e^3 + e^{i(\phi+\tau)} b_- (\theta) e^- \right], \]

where we defined \( e^\pm = e^1 \pm ie^2 \). Then we obtain four ordinary differential equations which can be solved by

\[ b_\pm (\theta) = \left( \frac{m}{q} + \frac{n}{p} \pm \lambda_v \right) \left[ \pm \frac{\sqrt{f(\theta)} B'(\theta)}{2} + \frac{B(\theta)}{2} \left( \frac{m}{q} \cot \theta - \frac{n}{p} \tan \theta \right) \right], \]

\[ b_0 (\theta) = \left( \frac{m}{q} + \frac{n}{p} + \lambda_v \right) \left( \frac{m}{q} + \frac{n}{p} - \lambda_v \right) B(\theta), \]  \hspace{1cm} (3.51)

where \( B(\theta) \) satisfies the second order differential equation

\[ B''(\theta) + \left( 2 \cot (2\theta) + \frac{f'(\theta)}{2f(\theta)} \right) B'(\theta) + \frac{1}{f(\theta)} \left( \lambda^2_v - 2\sqrt{f(\theta)} \lambda_v - \frac{m^2}{q^2 \sin^2 \theta} - \frac{n^2}{p^2 \cos^2 \theta} \right) B(\theta) = 0. \]  \hspace{1cm} (3.52)

The eigenfunction of the Dirac operator satisfying

\[ \Delta \psi |_{\Delta=0,\sigma_0=0} = \lambda_v f \psi \],

can be obtained by solving the second order differential equation (3.31) with \( \Delta = 0 \) and \( \sigma_0 = 0 \). Comparing the two differential equations (3.31) and (3.52), there is a one-to-one correspondence between the eigenfunctions of the divergence-less vector field and the adjoint fermion given either \( \lambda_v = \lambda_v f \) or \( \lambda_v = 2 - \lambda_v f \) to zeroth order of \( \epsilon \). This map does not exist.
when one of the component of the spinor vanishes as we saw in the matter sector. The gaugino
modes which have bosonic counterparts, but do not pair up with \(2 - \lambda_{vf}\) have eigenvalues of
\[
\lambda_{vf}^{(1)} = \frac{n}{p} + \frac{m}{q} + 2 ,
\]
with \(m > -1\) and \(n > -1\). This would include non-normalizable modes for the vector field
and the actual ranges turn out to be \(m > 0\) and \(n > 0\). The fermionic modes which have no
bosonic counterparts have eigenvalues of
\[
\lambda_{vf}^{(2)} = -\frac{n}{p} - \frac{m}{q} ,
\]
with \(m < 0\) and \(n < 0\).

Taking into account the non-canceling modes, the one-loop partition function of the gauge
sector is given by
\[
Z_{\text{1-loop}} = \prod_{i=1}^{\text{rank } G} \frac{\lambda_{vf}^{(2)}}{2 - \lambda_{vf}^{(1)}} \prod_{\alpha > 0} \left( (\lambda_{vf}^{(2)})^2 + \alpha(\sigma_0)^2 \right) ,
\]
\[
= \prod_{i=1}^{\text{rank } G} \prod_{m,n < 0} \frac{n - m}{p - q} \prod_{m,n > 0} \frac{n - m}{p - q} \prod_{\alpha > 0} \left( \frac{n - m}{p - q} \right)^2 + \alpha(\sigma_0)^2 ,
\]
\[
= \left( \frac{\pi \sqrt{pq}}{\text{rank } G} \right)^{\text{rank } G} \prod_{\alpha > 0} \frac{1}{\alpha(\sigma_0)^2} \Gamma_h \left( \alpha(\sigma_0); \frac{i}{p}, \frac{i}{q} \right) \Gamma_h \left( -\alpha(\sigma_0); \frac{i}{p}, \frac{i}{q} \right) ,
\]
(3.56)

where the result has again been expressed in terms of the hyperbolic gamma function.

### 3.4 Classical contributions

To complete the localization calculation, we must evaluate the classical action on the space
of zero modes found in section 3.2. As in the \(S^3\) calculation, only the Chern-Simons and
Fayet-Iliopoulos (FI) terms give these classical contributions. Their action is given by

\[
\mathcal{L}_{\text{cl}} = \frac{k}{4\pi} \text{Tr} \left[ i\varepsilon^{\mu\nu\rho} \left( a_\mu \partial_\nu a_\rho + \frac{2i}{3} a_\mu a_\nu a_\rho \right) - 2D\sigma + 2i\lambda \lambda \right] + \frac{\xi}{2\pi} \text{Tr} (D - \sigma H - a_\mu V^\mu) ,
\]
(3.57)

where \(k\) is the Chern-Simons level and \(\xi\) the FI parameter. They give rise to classical con-
tributions of the form

\[
Z_{\text{classical}} = \exp \left[ pq \left( i\pi k \text{Tr} (\sigma_0^2) - 2\pi i\xi \text{Tr} (\sigma_0) \right) \right] .
\]
(3.58)
3.5 The matrix model

Combining the results of sections 3.3 and 3.4 we obtain the result for the partition function on the singular space (2.2) in terms of the $\epsilon \to 0$ limit of the one on the resolved space (2.23). The result is a familiar matrix model. The integration is over the Lie algebra $g$ of the gauge group $G$. At this point it is advantageous to rescale the integration variables by a factor of $1/\sqrt{pq}$ and use the scale invariance of the hyperbolic gamma function (see appendix D) to set

$$
Z_{\text{matter}}^{1\text{-loop}} = \Gamma_h \left( -\sigma_0 + \frac{i\Delta}{2} \left( b + b^{-1} \right) ; ib, \frac{i}{b} \right), \quad b = \sqrt{\frac{q}{p}}.
$$

This expression coincides with the one for the squashed sphere $S^3_b$ with squashing parameter $b$ [35, 36]. The same applies to the expression for the fluctuation determinant of the vector multiplet. The rescaled contribution of a Chern-Simons term is

$$
Z_{CS} = \exp \left( i\pi k \text{Tr} (\sigma_0)^2 \right), \quad (3.59)
$$

which also coincides with the squashed sphere.

We may also consider FI ($\xi$) and real mass terms ($m$). Combining the results above for an arbitrary theory we get

$$
Z_{\text{singular space}} (\{\xi_i\}, \{m_j\}|p, q) = Z_{S^3_b} \left( \left\{ \frac{\xi_i}{\sqrt{pq}} \right\}, \left\{ \frac{m_j}{\sqrt{pq}} \right\} | b = \sqrt{\frac{q}{p}} \right), \quad (3.60)
$$

and note that

$$
Z_{S^3_b} (\{\xi_i\}, \{m_j\}| b = 1) = Z_{S^3} (\{\xi_i\}, \{m_j\}). \quad (3.61)
$$

The total partition function is

$$
Z_{\text{singular space}} (\{\xi_i\}, \{m_j\}|p, q) = \frac{1}{|W|} \int \prod_{i=1}^{\text{rank} G} d\sigma_i \, e^{\pi i k \text{Tr}(\sigma^2) - 2\pi i \tilde{\xi} \text{Tr}(\sigma)} \cdot \prod_{\alpha} \frac{1}{\Gamma_h (\alpha(\sigma))} \cdot \prod_I \prod_{\rho \in R_I} \Gamma_h (\tilde{\rho}(\sigma) + \tilde{m}_I + i\omega \Delta_I), \quad (3.62)
$$

where

$$
\Gamma_h (z) \equiv \Gamma_h (z; i\omega_1, i\omega_2), \quad \omega_1 = \sqrt{\frac{q}{p}}, \quad \omega_2 = \sqrt{\frac{p}{q}}, \quad (3.63)
$$

is the hyperbolic gamma function in appendix D and

$$
\omega = \frac{\omega_1 + \omega_2}{2}. \quad (3.64)
$$

$k$ is the Chern-Simons level and

$$
\tilde{\xi} = \sqrt{pq} \xi, \quad \tilde{m} = \sqrt{pq} m, \quad (3.65)
$$

are the FI and real mass parameters. $I$ labels the types of chiral multiplets and $\rho$ is a weight in a representation $R_I$ of a gauge group $G$. $\Delta_I$ is the $R$-charge of the scalar field in a chiral multiplet.
3.6 Supersymmetric Rényi entropy

As explained in [33], the partition function of an $\mathcal{N} = 2$ theory on $S^3$ can receive contributions from certain non-universal terms involving the fields in the supergravity multiplet and, possibly, background flavor symmetry vector multiplets. One can check, however, that the real part of the free energy is universal and an intrinsic observable of the SCFT to which the theory flows. The partition function (3.62) shares these properties. This is because the field $A_\mu$ is real and, as on $S^3$, $H = -i$. With these values, the contribution of all non-universal terms specified in [33] remains purely imaginary, while the free energy of a conformal theory on the singular space, as the limit from the smooth resolved space, is expected to be real.

We define the super Rényi entropy to be

$$S^{\text{susy}}_q = \frac{1}{1-q} \Re \left[ \log \left( \left( \begin{array}{c} |Z_{\text{singular space}}(\{\xi_i\} = \{m_j\} = 0|1,q)\rangle \\ (Z_{S^3})^q \end{array} \right) \right) \right], \quad (3.66)$$

hence our main result is that

$$S^{\text{susy}}_q = \frac{1}{1-q} \Re \left[ \log \left( \frac{Z_{S^3}^q(b = \sqrt{q})}{(Z_{S^3})^q} \right) \right]. \quad (3.67)$$

4 Checks

We have embedded the metric of the branched sphere into a supergravity background. In order to preserve supersymmetry on this (singular) space, we were forced to turn on an additional background $R$-symmetry gauge field. Since this deformation is not part of the original definition of the Rényi entropy we should explain its appearance. Note that the deformation is pure gauge away from the singularity and can, hence, be traded for boundary conditions on the $R$-charged fields by using a singular background $R$-symmetry gauge transformation. The space is then the original branched sphere, except for the effects associated with the imaginary value of $H$, with supersymmetry preserving boundary conditions for all fields.

4.1 Limits and behavior

An immediate consequence of the formula in section 3.6 is that for a topological theory the super Rényi entropy is $q$-independent. This is because it is numerically equal to the partition function of a smooth space: the squashed sphere.\(^{10}\)

The behavior around $q = 1$ for a general theory is such that

$$S^{\text{susy}}_q \longrightarrow_{q \to 1} S = -F, \quad (4.1)$$

where $S$ is the entanglement entropy and $F$ is the free energy on $S^3$, which implies

$$\partial_q Z_{\text{singular space}}(\{\xi_i\} = \{m_j\} = 0|1,q) |_{q=1} = 0, \quad (4.2)$$

\(^{9}\)This is no longer true for the resolved space.

\(^{10}\)Of course the usual framing dependence of the Chern-Simons theory applies.
and this statement in turn follows from the more general observation that

\[ Z_{\text{singular space}} (\{\xi_i\} = \{m_j\} = 0 | 1, q) = Z_{\text{singular space}} (\{\xi_i\} = \{m_j\} = 0 | 1, \frac{1}{q}) . \quad (4.3) \]

The next term in the expansion around \( q = 1 \) is also interesting. The squashed sphere partition function has been shown to have the following expansion around \( b = 1 \) \[25\]

\[ F_b = - \log Z_{S^3_b} , \]
\[ \partial_b F_b |_{b=1} = 0 , \quad (4.4) \]
\[ \Re (\partial_b^2 F_b) |_{b=1} = \frac{\pi^2}{2} \tau_{rr} , \]

where \( \tau_{rr} \) is a constant which appears in the SCFT flat space correlation functions at separated points

\[ \langle j^{(R)}_{\mu}(x) j^{(R)}_{\nu}(0) \rangle = \frac{\tau_{rr}}{16\pi^2} (\delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) \frac{1}{x^2} , \]
\[ \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = - \frac{\tau_{rr}}{64\pi^2} (\delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) (\delta_{\rho\sigma} \partial^2 - \partial_\rho \partial_\sigma) \frac{1}{x^2} \]
\[ + \frac{\tau_{rr}}{64\pi^2} (\delta_{\mu\rho} \partial^2 - \partial_\mu \partial_\rho) (\delta_{\nu\sigma} \partial^2 - \partial_\nu \partial_\sigma) + (\mu \leftrightarrow \nu) \frac{1}{x^2} . \quad (4.5) \]

Note that

\[ \Re (\partial_b^2 F_b) |_{b=1} = - \Re \left( \frac{\partial^2_b Z_{S^3_b}}{Z_{S^3_b}} \right) |_{b=1} , \quad (4.6) \]

which implies

\[ S_q^{\text{susy}} = S + \frac{\pi^2}{16} \tau_{rr} (q - 1) + O \left((q - 1)^2\right) . \quad (4.7) \]

4.2 Free fields

We can compare the results of section 3 to the previous results for free field theories \[18\]. Free (complex) scalars have canonical dimension \( \Delta = 1/2 \). Consider the action for a single uncharged chiral multiplet with \( \Delta = 1/2 \) coupled to the supergravity background of the singular space given in section 2.2. There are two major differences between this action and the one used to compute the Rényi entropies of free scalars and fermions in \[18\]

1. The supersymmetric action (3.18) includes a coupling to a background \( R \)-symmetry gauge field \( A_\mu \). This background gauge field is flat away from the singularities.

2. The action (3.18) includes a coupling of the scalars to the Ricci scalar \( R \). Excising the loop at \( \theta = 0 \) means ignoring the delta function supported there in the expression for \( R \) given by (C.6) taken at \( p = 1 \). We have, instead, regularized this contribution by smoothing out the space and then taking an appropriate limit.

\[ \text{The possibility of } \tau_{rr} \text{ being a } \delta \text{-function in three-dimensions has been recently tested and ruled out by } [41] . \]
Both of the deformations above can be reformulated as alternative boundary conditions for bosons and fermions. A choice of such boundary conditions is already implicit in the flat space calculation of the Rényi entropy. Interestingly, for odd integer \( q \), one can still rewrite the results of [18] adapted to a free conformally coupled chiral multiplet in a way that resembles a supersymmetric calculation

\[
Z_{\text{non-susy}}(q) = \Gamma_h \left( \frac{i}{2}; i, i \right) \prod_{r=1}^{q-1} \Gamma_h \left( \frac{i}{2} + \frac{r}{q}; i, i \right)^2.
\]

(4.8)

4.3 Duality

If the super Rényi entropy is computed by embedding a 3d SCFT into a flow from a UV action, which is used in the localization, we expect the result to be an intrinsic (scheme independent) observable of the IR fixed point. This is in line with the definition of the usual Rényi entropy as an operation on a density matrix obtained from the ground state wave function.

One way to test these properties of the super Rényi entropy is to compare its value for a pair of IR dual quantum field theories. Examples of such dualities abound in 3d gauge theories with \( \mathcal{N} \geq 2 \) supersymmetry. These are roughly classified into 3d mirror symmetry [42–46] and Seiberg-like duality [47–49]. Extensive comparisons have been made for the partition functions of the theories involved: for \( S^3 \) in [50–52]; the squashed sphere, \( S^3_b \), in [53, 54]; for \( S^2 \times S^1 \) (the superconformal index) in [55–63].

The results in section 3 relate the super Rényi entropy to the \( S^3_b \) partition function. This makes such a duality comparison for the super Rényi entropy, or equivalently the branched sphere partition function, fairly trivial. We note that, as in the case of other partition functions, we can further deform the computation by real mass and FI terms and compare the result as a function of them. Equation (3.60) guarantees that the deformed expression is duality invariant. Note that one could naively use the expression (4.8) as the one-loop determinant for a chiral superfield. Such an approach does not lead to a duality invariant expression.

4.4 The defect operator interpretation

The Rényi entropy associated with a region \( V \) at integer \( q \) can be computed using the replica trick. This involves defining the theory on a \( q \)-covering space with conical singularities. If the theory in question is free, there is an alternative formulation of this procedure [1]. One first splits each field \( \Phi \) on the covering space into \( q \) fields

\[
\{ \Phi_n \}_{n=1}^q ,
\]

(4.9)
each defined on a single sheet. The Rényi entropy is computed by the partition function of this theory with boundary conditions such that in crossing $V$ the vector

$$\vec{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_q \end{pmatrix},$$

gets multiplied by the matrices $T$ or $T^{-1}$ given by (take $\pm 1$ for bosons and fermions respectively)

$$T = \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ (\pm 1)^{q+1} & 0 & & & \end{pmatrix}.$$

Equivalently,

$$\Phi_n|_{V_+} = \Phi_{n+1}|_{V_-},$$

where $V_\pm$ denote the two “sides” of the $(d-1)$-dimensional region $V$, whose boundary $\partial V$ is the “entangling surface”, and the condition relates $n = q$ to $n = 1$ (possibly with a sign). This condition is diagonalized by fields $\tilde{\Phi}_n$ with monodromy

$$\beta_n = e^{2\pi i \frac{n}{q}}, \quad \begin{cases} n \in \{0, \cdots, q-1\}, & \text{bosons,} \\ n \in \{-\frac{q-1}{2}, \cdots, \frac{q-1}{2}\}, & \text{fermions,} \end{cases}$$

around the singularity at $\partial V$.

A different interpretation of the procedure above is the evaluation (in the free theory) of the product of expectation values of certain defect operators. These defects are supported on $\partial V$ and are defined to reproduce the monodromies $\beta_n$. In a 2d CFT these would be twist operators defined at the boundary of the entangling interval. In a 3d theory, we expect line (or loop) operators. Since the computation of the expectation value of a $Q$-closed observable in a supersymmetric theory using localization is essentially (ignoring the moduli) equivalent to a free field computation, we might expect that a similar result holds for the super Rényi entropy.

Consider the subset of the partition functions in section 3 for parameters $p = 1, q \in \mathbb{N}$. We can rewrite the fluctuation determinant (3.40) of a chiral field with the help of the identity
The terms appearing in the final product can be reinterpreted as coming from the introduction of a supersymmetric abelian vortex loop of charge
\[
q_{\text{vortex}}^r = \frac{\Delta}{2} \left( \frac{1}{q} - 1 \right) + \frac{r}{q},
\]
 supported on the great circle \( \theta = 0 \). These affect the fluctuation determinant of a chiral multiplet on the round \( S^3 \) in exactly this way \cite{64,65}.

When considering the gauge sector we must deal with the moduli. We do not expect to be able to express the contribution of these modes as the determinant arising from a free theory in the presence of a defect. Rather, we should try to apply the splitting (4.9) and the boundary conditions (4.12) directly. The boundary conditions (4.12) imply that the moduli defined on different sheets, which we denote by \( a_n = (\sigma_0)_n \), are related by
\[
a_n = a_{n+1}.
\]

There are therefore \( q - 1 \) such “delta function” insertions in the matrix model. This would be somewhat ill-defined if the \( a_n \) each transformed in the adjoint representation of a different copy of the gauge group. We therefore consider a single gauge group with \( q \) independent adjoint valued fields \( \sigma_n \) of which the \( a_n \) are the constant modes. We expect the part of the matrix model coming from the gauge multiplets to be of the form
\[
\frac{1}{\text{Vol} (G)} \int \left( \prod_{m=1}^{q-1} \delta (a_m - a_{m+1}) \right) \prod_{r=0}^{q-1} Z_{\text{vector}}^{1\text{-loop}} (a_r | q_{\text{vortex}}^r) da_r ,
\]
 where, following either the expansion in section 3.3.2 or the evaluation in \cite{65}
\[
Z_{\text{vector}}^{1\text{-loop}} (a | q_{\text{vortex}}) = \left\{ \begin{array}{ll}
(2\pi)^{\text{rank} G} \prod_{\alpha > 0} \frac{1}{\text{rank} G} \Gamma_h \left( \frac{1}{\alpha (a) + i q_{\text{vortex}}^r}; i, i \right) \\
\Gamma_h \left( i q_{\text{vortex}}^r; i, i \right) \right\} \left( \begin{array}{l}
\prod_{\alpha > 0} \Gamma_h \left( \alpha (a); i, i \right) \\
\prod_{\alpha > 0} \Gamma_h \left( -\alpha (a); i, i \right) \\
\prod_{\alpha > 0} \Gamma_h \left( \alpha (a) + i q_{\text{vortex}}^r; i, i \right) \\
\prod_{\alpha > 0} \Gamma_h \left( -\alpha (a) + i q_{\text{vortex}}^r; i, i \right) \\
q_{\text{vortex}} = 0 , \quad q_{\text{vortex}} \neq 0.
\end{array} \right.
\]

We set
\[
q_{\text{vortex}}^r = \frac{r}{q}.
\]
Using the identities
\[ \prod_{r=1}^{q-1} \frac{1}{\Gamma_h \left( i \frac{\pi}{q} ; i, i \right)} = \sqrt{q} , \]
and (D.12) we recover the one-loop determinant of the gauge sector (3.56)
\[ \frac{1}{\text{Vol} (G)} \int \left( \prod_{m=1}^{q-1} \delta (a_m - a_{m+1}) \right)^{q-1} \prod_{r=0}^{q-1} Z_{\text{vector}}^{\text{1-loop}} (a_r | q_r^{\text{vortex}}) \ da_r \rightarrow \frac{1}{\text{Vol} (G)} \int Z_{\text{vector}}^{\text{1-loop}} (a, q) \ da . \]

The contribution of a classical Chern-Simons term is similarly
\[ \exp \left( i \pi k \sum_{n=1}^{q} \text{Tr} (a_n)^2 \right) \rightarrow \exp \left( i \pi q k \text{Tr} (a)^2 \right) . \]

The upshot is that the supersymmetric Rényi entropy can indeed be re-expressed in terms of (supersymmetric) defect operators.\(^{12}\) Note that the defect operator charge for a chiral superfield is not simply the monodromy implied by (4.13). The extra \( \Delta \) dependent charge reflects the partial twisting inherent in the definition of the supersymmetric Rényi entropy.

### 5 Examples

In this section, we present several examples to show how to compute the super Rényi entropy and show how it behaves as a function of \( q \). To this end, we rewrite the super Rényi entropy (3.67) in terms of the free energy \( F(q) \equiv - \log |Z_{\text{singular space}}(1, q)| \) on the branched sphere
\[ S_{\text{susy}}^q = \frac{1}{1 - q} (q F(1) - F(q)) . \]

The computation is carried out in two steps:

- compute the value of \( R \)-charge \( \Delta \) where the free energy \( F(1) \) on the round sphere is extremized,
- evaluate (5.1) with the obtained \( R \)-charge.

We are mostly interested in the limits \( q \to 0, q \to 1 \) where
\[ S_{\text{susy}}^q \xrightarrow{q \to 0} -F(0) , \]
\[ S_{\text{susy}}^q \xrightarrow{q \to 1} -F(1) , \]
and \( q \to \infty \). Analytic evaluation of (5.1) is still difficult. We will mostly compute it numerically except in the large-\( N \) limit. We will show, for instance, that the super Rényi entropy is a monotonically decreasing function of \( q \).

\(^{12}\)The physical defects are in the gravitational and \( R \)-symmetry backgrounds. We have re-expressed their effect as (fictitious) flavor/gauge defects acting on free fields.
5.1 $\mathcal{N} = 4$ SQED with one flavor

$\mathcal{N} = 4$ SQED with one hypermultiplet flavor is dual to the free theory of a twisted hypermultiplet [66]. In terms of an $\mathcal{N} = 2$ theory, an $\mathcal{N} = 4$ hypermultiplet is a pair of $\mathcal{N} = 2$ chiral multiplets with $R$-charge $1/2$ and in conjugate representations of the gauge group.

We first consider a free chiral multiplet of $R$-charge $\Delta$ whose partition function is given by

$$F_{\text{chiral}}(q, \Delta) = -\log \Gamma_h(i\omega\Delta),$$

where $b = \sqrt{q}$ and then $\omega = (b + b^{-1})/2$. For $q = 1$ and $\Delta = 1/2$, the integral can be performed

$$F_{\text{chiral}}(1, 1/2) = \frac{\log 2}{2},$$

and

$$S_1^{\mathcal{N}=4\text{SQED}} = -\log 2.$$  

In the $q \to \infty$ limit

$$F_{\text{chiral}}(q, \Delta) \xrightarrow{q \to \infty} q \int_0^\infty \frac{dx}{2x^2} \left( 1 - \frac{\sinh((1 - \Delta)x)}{\sinh(x)} \right),$$

$$= \frac{iq}{4\pi} \left( \text{Li}_2(e^{-\pi i\Delta}) - \text{Li}_2(e^{\pi i\Delta}) \right),$$

and the super Rényi entropy becomes

$$S_q^{\mathcal{N}=4\text{SQED}} \xrightarrow{q \to \infty} \frac{G}{\pi} - \log 2 \simeq -0.402,$$

where $G$ is the Catalan’s constant. Since the free energy is invariant under $q \to 1/q$, the entropy near $q = 0$ behaves as

$$S_q^{\mathcal{N}=4\text{SQED}} \xrightarrow{q \to 0} -\frac{G}{\pi q}.$$  

We compute the super Rényi entropy of generic $q$ numerically. The plot of the ratio between $S_q$ and $S_1$ is shown in figure 1. It becomes $-F(1) = -\log 2$ for $q = 1$ and monotonically decreases and asymptotes to the value of (5.7) divided by (5.5) in the large-$q$ limit.

5.2 $\mathcal{N} = 2$ SQED with one flavor

We consider $\mathcal{N} = 2$ SQED with two chiral multiplets of opposite charges. This is dual to the $XYZ$ model with $R$-charges $\Delta = 2/3$ at the IR fixed point [67]. The super Rényi entropy at $q = 1$ is minus the free energy on a round three-sphere

$$S_1^{\mathcal{N}=2\text{SQED}} = -F_{XYZ}(1) = -3F_{\text{chiral}}(1, 2/3) \simeq -0.872,$$  

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The super Rényi entropy of $\mathcal{N} = 4$ SQED with one hypermultiplet.

while the value in the large-$q$ limit is

$$S_{q}^{\mathcal{N}=2\text{SQED}} \xrightarrow{q \to \infty} S_{1}^{\mathcal{N}=2\text{SQED}} + 3F_{\text{chiral}}(q,2/3)q \approx -0.549 .$$

(5.10)

The $q \to 0$ limit follows from (5.6)

$$S_{q}^{\mathcal{N}=2\text{SQED}} \xrightarrow{q \to 0} \frac{\psi^{(1)}(2/3) - \psi^{(1)}(1/3)}{4\sqrt{3}\pi q} \approx -\frac{0.323}{q},$$

(5.11)

where $\psi^{(n)}(z)$ is the polygamma function. The numerical plot is shown in figure 2, which asymptotes to $0.63 \sim (-0.549)/(-0.872)$ in the large-$q$ limit as expected.

A related simple example is a theory of one chiral multiplet $\Phi$ with a superpotential

$$W = \frac{g}{6} \Phi^{3} .$$

(5.12)
The $R$-charge of $\Phi$ is canonical $\Delta = 1/2$ in the UV, but becomes $\Delta = 2/3$ in the IR fixed point. The super Rényi entropy is given by $1/3$ of that of the $XYZ$ model in the previous section. This theory is known to emerge on the surface of topological insulators in four-dimensions [68, 69].

5.3 ABJM model

As a more elaborate example, we consider the ABJM model with $U(N)_{k} \times U(N)_{-k}$ groups [70]. There are four bifundamental chiral multiplets, two of which are in $N \times \bar{N}$ representation and the other two are in $\bar{N} \times N$ representation. We study the simplest case of $N = 1$ where the partition function is given by double integrals

\[
Z_{\text{ABJM}}(1, k, b) = \int_{-\infty}^{\infty} d\sigma d\tilde{\sigma} e^{\pi i k (\sigma^2 - \tilde{\sigma}^2)} \Gamma_h \left[ \sigma - \tilde{\sigma} + \frac{i\omega}{2} \right] \Gamma_h \left[ \tilde{\sigma} - \sigma + \frac{i\omega}{2} \right],
\]

resulting in the super Rényi entropy of the ABJM model with $N = 1$

\[
S_{q}^{\text{ABJM}, N=1} = -\log k - \frac{2}{1 - q} \int_{0}^{\infty} \frac{dx}{x} \left( \frac{\omega - q}{x} - \frac{\sinh(\omega x)}{\sinh(bx) \sinh(x/b)} \right).
\]

The entanglement entropy is obtained in $q \to 1$ limit

\[
S_1 = -\log k - 2 \log 2.
\]

In the $q \to \infty$ limit, it becomes

\[
S_{q}^{\text{ABJM}, N=1} \xrightarrow{q \to \infty} -\log k - \int_{0}^{\infty} \frac{dx}{x^2} \left( 1 - \frac{2(x - \sinh(x/2))}{\sinh(x)} \right),
\]

while it behaves near $q = 0$ as

\[
S_{q}^{\text{ABJM}, N=1} \xrightarrow{q \to 0} \frac{1}{q} \int_{0}^{\infty} \frac{dx}{x^2} \left( \frac{1}{\cosh(x/2)} - 1 \right) = -\frac{2G}{\pi q}.
\]

The numerical plot for $k = 2$ is given in figure 3, that is also monotonically decreasing with respect to $q$ and approaches to the critical value $\simeq 0.72$ in the large-$N$ limit.

5.4 Large-$N$ limit

The partition function can be solved in the large-$N$ limit as in [35]

\[
F(q) = \frac{(q + 1)^2}{4q} F(1),
\]
and the super Rényi entropy takes a simple form

\[ S_{q}^{\text{Large-}\ N} = -\frac{3q + 1}{4q} F(1) = \frac{3q + 1}{4q} S_{1}^{\text{Large-}\ N}. \]  

Figure 3. The super Rényi entropy of the ABJM model of \( N = 1 \) and \( k = 2 \).

The \( q \)-dependence of the super Rényi entropy in the large-\( N \) limit is shown in figure 4. It is monotonically decreasing and asymptotes to \( 3/4 \) in \( q \to \infty \) limit. The dependence on \( q \) is quite similar to those of the previous sections as well as the holographic results given in [4]. The asymptotic value of \( S_{q}^{\text{Large-}\ N}/S_{1}^{\text{Large-}\ N} \), however, does not agree with those of [4] since their gravity solution is not supersymmetric. It would be interesting to construct a supersymmetric gravity background dual to our theory on the \( q \)-covering space by introducing a background bulk \( U(1) \) gauge field dual to the \( R \)-current in a similar manner to [71].

Figure 4. The super Rényi entropy in the large-\( N \) limit.

It follows from (5.19) that the ratio of the super Rényi entropy \( H_q \equiv S_{q}^{\text{susy}}/S_{1}^{\text{susy}} \) in the
large-$N$ limit satisfies the following inequalities

\[
\begin{align*}
\partial_q H_q & \leq 0 , \\
\partial_q \left( q - \frac{1}{q} H_q \right) & \geq 0 , \\
\partial_q ((q - 1) H_q) & \geq 0 , \\
\partial_q^2 ((q - 1) H_q) & \leq 0 .
\end{align*}
\] (5.20)

These are the same inequalities satisfied by the usual Rényi entropy (see for instance [72]). We however note that our $S_{\text{susy}}^1$ is always negative. The sign change may come from the renormalization of the UV divergences that could make the Rényi entropy negative.

6 Conclusion

We have defined a quantity, which we call the super Rényi entropy, for a 3d SCFT with $\mathcal{N} \geq 2$ by considering the partition function in a specific supergravity background. We have shown that the super Rényi entropy can be computed exactly using the method of localization. This quantity differs from the Rényi entropy of the SCFT by a twist involving the $R$-symmetry of the theory. However, it shares many of the properties of the usual Rényi entropy

- it is an intrinsic observable of a 3d CFT,
- it is independent of $q$ for topological theories,
- it can be used to recover the entanglement entropy, although in a rather trivial way,
- it has nice analytic properties in the large-$N$ limit,

and has additional desirable properties

- deformation invariance makes it possible to calculate the super Rényi entropy of a SCFT arising as an IR fixed point by localizing the UV action,
- the result can sometimes be written down exactly, even for strongly coupled theories, using special functions,
- duality invariance can be checked explicitly,
- analytic continuation to non-integer $q$ is automatic.

We have seen that the expansion of the super Rényi entropy includes the usual entanglement entropy and also a coefficient of the two-point function of the energy-momentum tensors or $R$-symmetry currents. Both quantities have been previously calculated using localization. It would be interesting to see if additional simple SCFT observables arise as limits.
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A Conventions

We use a Clifford algebra with the Pauli matrices

\[ [\sigma_i, \sigma_j] = 2i \varepsilon_{ij}^k \sigma_k, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}, \]  

(A.1)

where indices \( i, j \) are raised and lowered by \( \delta^{ij} \) and

\[ \gamma_\mu := e_\mu^i \sigma_i. \]  

(A.2)

Our convention is such that

\[ \omega^{i\mu}_{\ j} = e_\nu^i \nabla_\mu e_\nu^j, \]  

(A.3)

and that\(^{13}\)

\[ \nabla_\mu \varepsilon = \left( \partial_\mu + \frac{1}{8} \omega^{i\mu}_{\ j} [\sigma_i, \sigma_j] \right) \varepsilon. \]

B The branched sphere

For the branched sphere (2.2) we choose a vielbein

\[ e^1 = \sin (\tau + \phi) d\theta + q \frac{\sin 2\theta}{2} \cos (\tau + \phi) d\tau - p \frac{\sin 2\theta}{2} \cos (\tau + \phi) d\phi, \]
\[ e^2 = -\cos (\tau + \phi) d\theta + q \frac{\sin 2\theta}{2} \sin (\tau + \phi) d\tau - p \frac{\sin 2\theta}{2} \sin (\tau + \phi) d\phi, \]
\[ e^3 = q \sin^2 \theta d\tau + p \cos^2 \theta d\phi. \]  

(B.1)

with a spin connection

\[ \omega^1_{\ 2} = (1 - q \cos^2 \theta) d\tau + (1 - p \sin^2 \theta) d\phi, \]
\[ \omega^1_{\ 3} = \cos(\tau + \phi) d\theta - \sin \theta \cos \theta \sin(\tau + \phi)(qd\tau - pd\phi), \]
\[ \omega^2_{\ 3} = \sin(\tau + \phi) d\theta + \sin \theta \cos \theta \cos(\tau + \phi)(qd\tau - pd\phi). \]  

(B.2)

One can check

\[ g^{\mu\nu} e^i_\mu e^j_\nu = \delta^{ij}, \quad \delta_{ij} e^i_\mu e^j_\nu = g_{\mu\nu}, \]  

(B.3)

\(^{13}\)The convention in [25] for the spin connection differs by a sign from (A.3). However, the covariant derivative is also defined with an additional sign.
\[ \omega^{ij} = \varepsilon^{ijk} e_k - \tilde{\omega}^{ij}, \quad (B.4) \]

where
\[ \tilde{\omega}^{ij} = \varepsilon^{ij3} ((q - 1) d\tau + (p - 1) d\phi), \quad (B.5) \]

and
\[ \nabla_{\mu} \varepsilon = \left( \partial_{\mu} + \frac{1}{8} \omega^{ij}_{\mu} [\sigma_i, \sigma_j] \right) \varepsilon = \left( \partial_{\mu} + \frac{i}{2} \gamma_{\mu} - \frac{i}{2} \left( (q - 1) \delta^{\tau}_{\mu} + (p - 1) \delta^{\phi}_{\mu} \right) \sigma_3 \right) \varepsilon. \quad (B.6) \]

When \( p = q = 1 \) we recover the round three-sphere. The usual left-invariant basis is then defined by (B.1).

C The resolved space

For the resolved space (2.23) with \( f(\theta) \equiv f_{\epsilon}(\theta) \) we choose a vielbein
\[ e^1 = \frac{1}{\sqrt{f(\theta)}} \sin(\tau + \phi) d\theta + \cos(\tau + \phi) \sin \theta \cos \theta (qd\tau - pd\phi), \]
\[ e^2 = -\frac{1}{\sqrt{f(\theta)}} \cos(\tau + \phi) d\theta + \sin(\tau + \phi) \sin \theta \cos \theta (qd\tau - pd\phi), \]
\[ e^3 = q \sin^2 \theta d\tau + p \cos^2 \theta d\phi, \quad (C.1) \]

with a spin connection
\[ \omega^{12} = (1 - q \sqrt{f(\theta)} \cos^2 \theta) d\tau + (1 - p \sqrt{f(\theta)} \sin^2 \theta) d\phi, \]
\[ \omega^{13} = \cos(\tau + \phi) d\theta - \sqrt{f(\theta)} \sin \theta \cos \theta \sin(\tau + \phi) (qd\tau - pd\phi), \]
\[ \omega^{23} = \sin(\tau + \phi) d\theta + \sqrt{f(\theta)} \sin \theta \cos \theta \cos(\tau + \phi) (qd\tau - pd\phi). \quad (C.2) \]

The Ricci tensor of the resolved space is given by\(^\text{14}\)
\[ R_{\theta\theta} = -2 + \frac{\cot(2\theta) f'(\theta)}{f(\theta)}, \]
\[ R_{\tau\tau} = -\frac{q^2}{2} \sin(4 \sin \theta f(\theta) - \cos \theta f'(\theta)), \quad (C.3) \]
\[ R_{\phi\phi} = -\frac{p^2}{2} \cos(4 \cos \theta f(\theta) + \sin \theta f'(\theta)), \]

and the Ricci scalar follows as
\[ R = -6 f(\theta) + 2 \cot(2\theta) f'(\theta). \quad (C.4) \]

\(^\text{14}\)Our convention for the Ricci scalar is that of [25]. With this convention, \( R \) on \( S^3 \) is negative.
We can read off the form of the Ricci scalar in $\epsilon \to 0$ limit by integrating it on the resolved space

$$\int d^3x \sqrt{g} R = 4\pi^2 p q \int^\pi d\theta \sin \theta \cos \theta \frac{1}{f^{1/2}(\theta)} (-6 f(\theta) + 2 \cot(2\theta) f'(\theta)),$$

$$= 4\pi^2 p q \left[ \int_0^\epsilon d\theta \left( 6 f_{1/2}(\theta) - \frac{f'}{f^{1/2}} \right) + \int^\pi d\theta \cos \theta \cdot 6 \right. \left. + \int^{\pi/2-\epsilon}_\epsilon \left( 6 \left( \frac{\pi}{2} - \theta \right) f^{1/2}(\theta) - \frac{f'}{f^{1/2}} \right) \right],$$

$$= 4\pi^2 p q \left[ 3 + 2 \left( \frac{1}{q} - 1 \right) - 2 \left( \frac{1}{p} - 1 \right) + O(\epsilon) \right]. \quad (C.5)$$

It follows that there are delta functional terms at $\theta = 0$ and $\theta = \pi/2$ in the Ricci scalar

$$R = -6 - \frac{2}{\sin \theta \cos \theta} \left( \frac{1}{q} - 1 \right) \delta(\theta) + \frac{2}{\sin \theta \cos \theta} \left( \frac{1}{p} - 1 \right) \delta \left( \frac{\pi}{2} - \theta \right). \quad (C.6)$$

Similarly, the Ricci tensor has delta functional terms

$$R_{\theta\theta} = -2 - \frac{1}{\sin \theta \cos \theta} \left( \frac{1}{q} - 1 \right) \delta(\theta) + \frac{1}{\sin \theta \cos \theta} \left( \frac{1}{p} - 1 \right) \delta \left( \frac{\pi}{2} - \theta \right),$$

$$R_{\tau\tau} = -2q^2 \sin^2 \theta - q^2 \tan \theta \left( \frac{1}{q} - 1 \right) \delta(\theta),$$

$$R_{\phi\phi} = -2p^2 \cos^2 \theta + p^2 \cot \theta \left( \frac{1}{p} - 1 \right) \delta \left( \frac{\pi}{2} - \theta \right). \quad (C.7)$$

## D Special functions

The hyperbolic gamma function is a meromorphic function of a single complex variable with two parameters defined in [73]

$$\Gamma_h(z;\omega_1,\omega_2) = \prod_{n_1,n_2 \geq 0} \frac{(n_1+1)\omega_1 + (n_2+1)\omega_2 - z}{n_1\omega_1 + n_2\omega_2 + z},$$

$$= \exp \left( i \int_0^\infty \frac{dx}{x} \left( \frac{z - \omega}{\omega_1 \omega_2 x} - \frac{\sin(2x(z - \omega))}{2 \sin(\omega_1 x) \sin(\omega_2 x)} \right) \right), \quad (D.1)$$

with the integral defined for

$$0 < \Im(z) < \Im(\omega_1 + \omega_2), \quad (D.2)$$

and then analytically continued to the entire complex plain. The poles are at

$$\Lambda = -\omega_1 Z_{\geq 0} - \omega_2 Z_{\geq 0}, \quad (D.3)$$

and the zeros at

$$\omega_1 + \omega_2 - \Lambda. \quad (D.4)$$
We will sometimes suppress $\omega_{1,2}$ and denote $\Gamma(z) \equiv \Gamma(z; \omega_1, \omega_2)$. We also define

$$\omega = \frac{\omega_1 + \omega_2}{2}.$$  \hfill (D.5)

The function satisfies

$$\Gamma_h(z + \omega_1) = 2 \sin \left( \frac{\pi z}{\omega_2} \right) \Gamma_h(z),$$  \hfill (D.6)

$$\Gamma_h(z + \omega_2) = 2 \sin \left( \frac{\pi z}{\omega_1} \right) \Gamma_h(z),$$  \hfill (D.7)

$$\Gamma_h(\omega_1 + \omega_2 - z) = \Gamma_h(z)^{-1},$$  \hfill (D.8)

$$\Gamma_h(z; \omega_1, \omega_2) = \Gamma_h(\alpha z; \alpha \omega_1, \alpha \omega_2), \quad \alpha \in \mathbb{C} \setminus \{0\},$$  \hfill (D.9)

and hence

$$\Gamma_h(\pm z) := \Gamma_h(z) \Gamma_h(-z) = \frac{-1}{4 \sin(\pi z/\omega_1) \sin(\pi z/\omega_2)}. \hfill (D.10)$$

There are also multiple-angle formulas

$$\Gamma_h(Nz) = \prod_{k_1, k_2 = 0, \ldots, N-1} \Gamma_h \left( z + \frac{k_1 \omega_1 + k_2 \omega_2}{N} \right). \hfill (D.11)$$

For an integer $N \geq 1$, it satisfies

$$\Gamma_h \left( z; i, \frac{i}{N} \right) = \prod_{k=0}^{N-1} \Gamma_h \left( z + \frac{ik}{N}; i, i \right). \hfill (D.12)$$

There is a product formula of dividing $N$

$$\prod_{k_1, k_2 = 0, \ldots, N-1} \Gamma_h \left( \frac{k_1 \omega_1 + k_2 \omega_2}{N} \right) = \frac{1}{N}. \hfill (D.13)$$

Some of the values at special points are

$$\Gamma_h(\omega) = 1, \quad \Gamma_h(0) = 0, \quad \Gamma_h \left( \frac{\omega_1}{2} \right) = \frac{1}{\sqrt{2}}, \quad \Gamma_h \left( \omega + \frac{\omega_1}{2} \right) = \sqrt{2},$$ \hfill (D.14)

$$\Gamma_h(\omega_1) = \sqrt{\frac{\omega_1}{\omega_2}}, \quad \Gamma_h(\omega_2) = \sqrt{\frac{\omega_2}{\omega_1}}.$$

$\Gamma_h$ is related to the double sine function $S_2$ and to the double gamma function $\Gamma_2$ defined in [74] by

$$\Gamma_h(z; \omega_1, \omega_2) = S_2 \left( z; \omega_1, \omega_2 \right) \Gamma_2 \left( \frac{2 \omega - z}{\omega_1, \omega_2} \right),$$ \hfill (D.15)

and to the Ruijsenaars’ hyperbolic gamma function $G$ defined in [75] by

$$\Gamma_h(z; \omega_1, \omega_2) = G \left( -i \omega_1, -i \omega_2; z - \omega \right).$$ \hfill (D.16)
References

[1] H. Casini and M. Huerta, Entanglement Entropy in Free Quantum Field Theory, J. Phys. A42 (2009) 504007, [0905.2562].

[2] R. C. Myers and A. Sinha, Holographic C-Theorems in Arbitrary Dimensions, 1011.5819.

[3] H. Casini, M. Huerta, and R. C. Myers, Towards a Derivation of Holographic Entanglement Entropy, JHEP 05 (2011) 036, [1102.0440].

[4] L.-Y. Hung, R. C. Myers, M. Smolkin, and A. Yale, Holographic Calculations of Renyi Entropy, JHEP 1112 (2011) 047, [1110.1084].

[5] P. Calabrese and J. L. Cardy, Entanglement Entropy and Quantum Field Theory, J. Stat. Mech. 0406 (2004) P06002, [hep-th/0405152].

[6] H. Casini, C. D. Fosco, and M. Huerta, Entanglement and Alpha Entropies for a Massive Dirac Field in Two Dimensions, J. Stat. Mech. 0507 (2005) P07007, [cond-mat/0505563].

[7] H. Casini and M. Huerta, Entanglement and Alpha Entropies for a Massive Scalar Field in Two Dimensions, J. Stat. Mech. 0512 (2005) P12012, [cond-mat/0511014].

[8] T. Azeyanagi, T. Nishioka, and T. Takayanagi, Near Extremal Black Hole Entropy as Entanglement Entropy via AdS2/CFT1, Phys.Rev. D77 (2008) 064005, [0710.2956].

[9] P. Calabrese, J. Cardy, and E. Tonni, Entanglement Entropy of Two Disjoint Intervals in Conformal Field Theory, J.Stat.Mech. 0911 (2009) P11001, [0905.2069].

[10] P. Calabrese, J. Cardy, and E. Tonni, Entanglement Entropy of Two Disjoint Intervals in Conformal Field Theory II, J.Stat.Mech. 1101 (2011) P01021, [1011.5482].

[11] M. Headrick, Entanglement Renyi Entropies in Holographic Theories, Phys.Rev. D82 (2010) 126010, [1006.0047].

[12] M. Headrick, A. Lawrence, and M. Roberts, Bose-Fermi Duality and Entanglement Entropies, J.Stat.Mech. 1302 (2013) P02022, [1209.2428].

[13] A. Lewkowycz, R. C. Myers, and M. Smolkin, Observations on Entanglement Entropy in Massive Qft’s, JHEP 1304 (2013) 017, [1210.6858].

[14] C. P. Herzog and T. Nishioka, Entanglement Entropy of a Massive Fermion on a Torus, JHEP 1303 (2013) 077, [1301.0336].

[15] T. Hartman, Entanglement Entropy at Large Central Charge, 1303.6955.

[16] T. Faulkner, The Entanglement Renyi Entropies of Disjoint Intervals in AdS/CFT, 1303.7221.

[17] H. Casini and M. Huerta, Entanglement Entropy for the N-Sphere, Phys.Lett. B694 (2010) 167–171, [1007.1813].

[18] I. R. Klebanov, S. S. Pufu, S. Sachdev, and B. R. Safdi, Renyi Entropies for Free Field Theories, JHEP 1204 (2012) 074, [1111.6290].

[19] D. Fursaev, Entanglement Renyi Entropies in Conformal Field Theories and Holography, JHEP 1205 (2012) 080, [1201.1702].

[20] M. A. Metlitski, C. A. Fuertes, and S. Sachdev, Entanglement entropy in the o(n) model, Physical Review B 80 (2009), no. 11 115122.
I. R. Klebanov, S. S. Pufu, S. Sachdev, and B. R. Safdi, Entanglement Entropy of 3-D Conformal Gauge Theories with Many Flavors, 1112.5342.

S. Ryu and T. Takayanagi, Holographic Derivation of Entanglement Entropy from AdS/CFT, Phys. Rev. Lett. 96 (2006) 181602, [hep-th/0603001].

S. Ryu and T. Takayanagi, Aspects of Holographic Entanglement Entropy, JHEP 08 (2006) 045, [hep-th/0605073].

G. Festuccia and N. Seiberg, Rigid Supersymmetric Theories in Curved Superspace, JHEP 06 (2011) 114, [1105.0689].

C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, Supersymmetric Field Theories on Three-Manifolds, 1212.3388.

A. Kapustin, B. Willett, and I. Yaakov, Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter, JHEP 03 (2010) 089, [0909.4559].

D. L. Jafferis, The Exact Superconformal R-Symmetry Extremizes Z, 1012.3210.

N. Hama, K. Hosomichi, and S. Lee, Notes on SUSY Gauge Theories on Three-Sphere, JHEP 03 (2011) 127, [1012.3512].

D. L. Jafferis, I. R. Klebanov, S. S. Pufu, and B. R. Safdi, Towards the F-Theorem: $\mathcal{N} = 2$ Field Theories on the Three-Sphere, JHEP 06 (2011) 102, [1103.1181].

I. R. Klebanov, S. S. Pufu, and B. R. Safdi, F-Theorem without Supersymmetry, 1105.4598.

A. Amariti and M. Siani, Z-Extremization and F-Theorem in Chern-Simons Matter Theories, JHEP 10 (2011) 016, [1105.0933].

T. Morita and V. Niarchos, F-Theorem, Duality and SUSY Breaking in One-Adjoint Chern-Simons-Matter Theories, Nucl.Phys. B858 (2012) 84–116, [1108.4963].

C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, Contact Terms, Unitarity, and F-Maximization in Three-Dimensional Superconformal Theories, JHEP 1210 (2012) 053, [1205.4142].

H. Casini and M. Huerta, On the RG Running of the Entanglement Entropy of a Circle, Phys.Rev. D85 (2012) 125016, [1202.5650].

Y. Imamura and D. Yokoyama, $\mathcal{N} = 2$ Supersymmetric Theories on Squashed Three-Sphere, 1109.4734.

N. Hama, K. Hosomichi, and S. Lee, SUSY Gauge Theories on Squashed Three-Spheres, JHEP 05 (2011) 014, [1102.4716].

Z. Komargodski and N. Seiberg, Comments on Supercurrent Multiplets, Supersymmetric Field Theories and Supergravity, JHEP 1007 (2010) 017, [1002.2228].

E. Witten, Supersymmetric Index of Three-Dimensional Gauge Theory, hep-th/9903005.

V. Pestun, Localization of Gauge Theory on a Four-Sphere and Supersymmetric Wilson Loops, 0712.2824.

M. Mariño, Lectures on Localization and Matrix Models in Supersymmetric Chern-Simons-Matter Theories, 1104.0783.
[41] T. Nishioka and K. Yonekura, *On RG Flow of $\tau_{RH}$ for Supersymmetric Field Theories in Three-Dimensions*, 1303.1522.

[42] K. A. Intriligator and N. Seiberg, *Mirror Symmetry in Three Dimensional Gauge Theories*, Phys. Lett. B387 (1996) 513–519, [hep-th/9607207].

[43] J. de Boer, K. Hori, H. Ooguri, Y. Oz, and Z. Yin, *Mirror Symmetry in Three-Dimensional Theories, SL(2, Z) and D-Branes Moduli Spaces*, Nucl. Phys. B493 (1997) 148–176, [hep-th/9612131].

[44] J. de Boer, K. Hori, H. Ooguri, and Y. Oz, *Mirror Symmetry in Three-Dimensional Gauge Theories, Quivers and D-Branes*, Nucl. Phys. B493 (1997) 101–147, [hep-th/9611063].

[45] J. de Boer, K. Hori, Y. Oz, and Z. Yin, *Branes and Mirror Symmetry in $\mathcal{N} = 2$ Supersymmetric Gauge Theories in Three-Dimensions*, Nucl. Phys. B502 (1997) 107–124, [hep-th/9702154].

[46] K. Jensen and A. Karch, *Abjm Mirrors and a Duality of Dualities*, JHEP 0909 (2009) 004, [0906.3013].

[47] O. Aharony, *IR Duality in $D = 3$ $\mathcal{N} = 2$ Supersymmetric $U(2N_c)$ and $U(N_c)$ Gauge Theories*, Phys. Lett. B404 (1997) 71–76, [hep-th/9703215].

[48] A. Giveon and D. Kutasov, *Seiberg Duality in Chern-Simons Theory*, Nucl. Phys. B812 (2009) 1–11, [0808.0360].

[49] V. Niarchos, *Seiberg Duality in Chern-Simons Theories with Fundamental and Adjoint Matter*, JHEP 0811 (2008) 001, [0808.2771].

[50] A. Kapustin, B. Willett, and I. Yaakov, *Tests of Seiberg-Like Duality in Three Dimensions*, 1012.4021.

[51] A. Kapustin, B. Willett, and I. Yaakov, *Nonperturbative Tests of Three-Dimensional Dualities*, JHEP 1010 (2010) 013, [1003.5694].

[52] B. Willett and I. Yaakov, *$\mathcal{N} = 2$ Dualities and Z Extremization in Three Dimensions*, 1104.0487.

[53] F. Benini, C. Closet, and S. Cremonesi, *Comments on 3D Seiberg-Like Dualities*, JHEP 10 (2011) 075, [1108.5373].

[54] I. Yaakov, *Redeeming Bad Theories*, 1303.2769.

[55] Y. Imamura and S. Yokoyama, *Index for Three Dimensional Superconformal Field Theories with General R-Charge Assignments*, JHEP 04 (2011) 007, [1101.0557].

[56] H. Kim and J. Park, *Aharony Dualities for 3D Theories with Adjoint Matter*, 1302.3645.

[57] D. Bashkirov, *Aharony Duality and Monopole Operators in Three Dimensions*, 1106.4110.

[58] D. Bashkirov and A. Kapustin, *Dualities Between $\mathcal{N} = 8$ Superconformal Field Theories in Three Dimensions*, JHEP 1105 (2011) 074, [1103.3548].

[59] A. Kapustin and B. Willett, *Generalized Superconformal Index for Three Dimensional Field Theories*, 1106.2484.

[60] C. Krattenthaler, V. Spiridonov, and G. Vartanov, *Superconformal Indices of Three-Dimensional Theories Related by Mirror Symmetry*, JHEP 1106 (2011) 008, [1103.4075].
[61] A. Kapustin, H. Kim, and J. Park, *Dualities for 3D Theories with Tensor Matter*, JHEP **1112** (2011) 087, [1110.2547].

[62] C. Hwang, K.-J. Park, and J. Park, *Evidence for Aharony Duality for Orthogonal Gauge Groups*, JHEP **1111** (2011) 011, [1109.2828].

[63] C. Hwang, H. Kim, K.-J. Park, and J. Park, *Index Computation for 3D Chern-Simons Matter Theory: Test of Seiberg-Like Duality*, JHEP **1109** (2011) 037, [1107.4942].

[64] A. Kapustin, B. Willett, and I. Yaakov, *Exact Results for Supersymmetric Abelian Vortex Loops in 2+1 Dimensions*, 1211.2861.

[65] N. Drukker, T. Okuda, and F. Passerini, *Exact Results for Vortex Loop Operators in 3D Supersymmetric Theories*, 1211.3409.

[66] A. Kapustin and M. J. Strassler, *On Mirror Symmetry in Three Dimensional Abelian Gauge Theories*, JHEP **04** (1999) 021, [hep-th/9902033].

[67] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg, and M. J. Strassler, *Aspects of \( N = 2 \) Superconformal Chern-Simons-Matter Theories in Three Dimensions*, Nucl. Phys. B**499** (1997) 67–99, [hep-th/9703110].

[68] T. Grover and A. Vishwanath, *Quantum Criticality in Topological Insulators and Superconductors: Emergence of Strongly Coupled Majoranas and Supersymmetry*, 1206.1332.

[69] S.-S. Lee, *Emergence of Supersymmetry at a Critical Point of a Lattice Model*, Phys.Rev. B**76** (2007) 075103, [cond-mat/0611658].

[70] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, \( N = 6 \) Superconformal Chern-Simons-Matter Theories, M2-Branes and Their Gravity Duals, JHEP **10** (2008) 091, [0806.1218].

[71] D. Martelli and J. Sparks, *The Gravity Dual of Supersymmetric Gauge Theories on a Biaxially Squashed Three-Sphere*, Nucl.Phys. B**866** (2013) 72–85, [1111.6930].

[72] K. Życzkowski, Rényi extrapolation of shannon entropy, Open Systems & Information Dynamics **10** (2003), no. 03 297–310.

[73] F. J. van de Bult, Hyperbolic hypergeometric functions, .

[74] N. Kurokawa and S.-y. Koyama, *Multiple sine functions*, in Forum Mathematicum, vol. 15, pp. 839–876, Berlin; New York: De Gruyter, c1989-, 2003.

[75] S. N. Ruijsenaars, *On barnes’ multiple zeta and gamma functions*, Advances in Mathematics **156** (2000), no. 1 107–132.