Large Time Behavior of Deterministic and Stochastic 3D Convective Brinkman-Forchheimer Equations in Periodic Domains

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Abstract
The large time behavior of deterministic and stochastic three dimensional convective Brinkman-Forchheimer (CBF) equations

\[
\partial_t u - \mu \Delta u + (u \cdot \nabla)u + \alpha u + \beta |u|^{r-1}u + \nabla p = f, \quad \nabla \cdot u = 0,
\]

for \( r \geq 3 (\mu, \beta > 0 \text{ for } r > 3 \text{ and } 2\beta\mu \geq 1 \text{ for } r = 3) \), in periodic domains is carried out in this work. Our first goal is to prove the existence of global attractors for the 3D deterministic CBF equations. Then, we show the existence of random attractors for the 3D stochastic CBF equations perturbed by small additive smooth noise. Furthermore, we establish the upper semicontinuity of random attractors for the 3D stochastic CBF equations (stability of attractors), when the coefficient of random perturbation approaches to zero. Finally, we address the existence and uniqueness of invariant measures of 3D stochastic CBF equations.

Keywords 3D stochastic convective Brinkman-Forchheimer equations · Periodic domains · Cylindrical Wiener process · Random dynamical system · Absorbing sets · Random attractors

Mathematics Subject Classification Primary 35B41 · 35Q35; Secondary 37L55 · 37N10 · 35R60

1 Introduction
The analysis of long time behavior of deterministic evolution equations, especially Navier-Stokes equations (NSE) is well-investigated in [52,57], etc. Attractors are one of the most
important entities in the study of long time behavior of dynamical systems generated by
dissipative evolution equations. When partial differential equations (PDEs) are perturbed by
random noises, which generate random dynamical systems (RDS), it is important to examine
the random attractors for such systems and their stability, that is, the convergence of the
random attractors (cf. [3]). The analysis of long time behavior of infinite dimensional RDS
is also an essential branch in the study of qualitative properties of stochastic PDEs (see
[8,14,16], etc for more details). In the literature, a variety of results on the global and random
attractors for several deterministic and stochastic models are available, the interested readers
are referred to see [1,4,13,25,34,35,40,49,56] etc for global attractors and see [17,23,24,26,
28,29,37,42,43,59–62] etc, for random attractors, and the references therein.

This work is concerned about the long time behavior of three dimensional deterministic and
stochastic convective Brinkman-Forchheimer equations in periodic domains. Let
$T^3 \subset \mathbb{R}^3$ be a periodic domain. Let $u(x,t): T^3 \times (0,\infty) \to \mathbb{R}^3$ denote the velocity field and $p(x,t): T^3 \times (0,\infty) \to \mathbb{R}$ represent the pressure field. We consider the convective Brinkman-
Forchheimer (CBF) equations, which describe the motion of incompressible fluid flows in a
saturated porous medium as:

$$
\begin{cases}
\frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla) u + \alpha u + \beta |u|^{r-1}u + \nabla p = f, \quad \text{in } T^3 \times (0,\infty), \\
\nabla \cdot u = 0, \quad \text{in } T^3 \times (0,\infty), \\
\end{cases}
$$

(1.1)

where $f(x,t): T^3 \times (0,\infty) \to \mathbb{R}^3$ is an external forcing. The positive constants $\mu, \alpha$ and $\beta$ represent the Brinkman coefficient (effective viscosity), Darcy coefficient (permeability of porous medium) and Forchheimer coefficient, respectively. The exponent $r \in [1,\infty)$ is called the absorption exponent and $r = 3$ is known as the critical exponent. For $\alpha = \beta = 0$, we obtain the classical 3D Navier-Stokes equations (NSE). The applicability of CBF equations
(1.1) is limited to flows when the velocities are sufficiently high and porosities are not too
small, that is, when the Darcy law for a porous medium no longer applies (cf. [44]). It should be noted that the critical homogeneous CBF equations (1.1) have the same scaling as the NSE only when $\alpha = 0$ (see Proposition 1.1, [31] and no scale invariance property for other values of $\alpha$ and $r$), which is sometimes referred to as the NSE modified by an absorption term ([2]) or the tamed NSE ([55]). The existence and uniqueness of Leray-Hopf weak solutions satisfying the energy equality as well as strong solutions (in the deterministic sense) for the
3D CBF equations (1.1) for $r \geq 3$ ($2\beta \mu \geq 1$, for $r = 3$) is established in [22,31,44,45], etc. Likewise 3D NSE, the global existence and uniqueness of strong solutions for the 3D CBF equations (1.1) with $r \in [1,3]$ is still an open problem. Therefore, in this work, we consider the cases $r > 3$ (fast growing nonlinearity), for any $\mu$, $\beta > 0$, and $r = 3$, for $2\beta \mu \geq 1$ only.

In the first part of this work, we prove the existence of a global attractor in $H$ and $V$ for the system (1.1) with $r \geq 3$ ($2\beta \mu \geq 1$, for $r = 3$) and $3 \leq r \leq 5$ ($2\beta \mu \geq 1$, for $r = 3$), respectively. The following estimate plays a key role in obtaining such a result (cf. [31]).

$$
\int_{T^3} (-\Delta u(x)) \cdot |u(x)|^{r-1}u(x) dx = \int_{T^3} |\nabla u(x)|^2 |u(x)|^{r-1} dx + 4 \left[ \frac{r-1}{(r+1)^2} \right] \int_{T^3} |\nabla |u(x)|^{\frac{r+1}{2}}|^2 dx.
$$

(1.2)

Note that in the case of bounded domains $\mathcal{P}(|u|^{r-1}u)$ ($\mathcal{P}$ is the Helmholtz-Hodge projection) need not be zero on the boundary, and $\mathcal{P}$ and $-\Delta$ are not necessarily commuting (for a counter example, see Example 2.19, [54]). Thus the equality (1.2) may not be useful in the context.
of bounded domains and we restrict ourselves to periodic domains in this work. In the case of two dimensional bounded and unbounded domains, the existence of global attractors and their properties for the CBF equations (1.1) have been discussed in [47,48], etc. The existence of global attractors for the solutions to 3D viscous primitive equations is obtained in [33].

In the second part of the work, we consider the stochastic convective Brinkman-Forchheimer (SCBF) equations and discuss the long time behavior of its solution and stability results. The 3D SCBF equations in periodic domains are given by

$$
\begin{aligned}
du + [-\mu \Delta u + (u \cdot \nabla)u + \alpha u + \beta |u|^{r-1}u + \nabla p]dt &= fdt + \varepsilon dW(t), \quad \text{in } \mathbb{T}^3 \times (0, \infty), \\
\nabla \cdot u &= 0, \quad \text{in } \mathbb{T}^3 \times (0, \infty), \\
u(0) &= x, \quad \text{in } \mathbb{T}^3,
\end{aligned}
$$

(1.3)

where $W(t), \ t \in \mathbb{R}$, is a two-sided cylindrical Wiener process with its Reproducing Kernel Hilbert Space (RKHS) $K$ satisfying certain assumptions (see Subsect. 4.1 below). Using monotonicity as well as hemicontinuity properties of linear and nonlinear operators, and a stochastic generalization of the Minty-Browder technique, the existence and uniqueness of pathwise strong solutions (in the probabilistic sense) of 3D SCBF equations with nonlinear multiplicative noise is established in [46]. The random attractors and their stability results for 2D SCBF equations in bounded and unbounded domains like Poincaré domains is discussed in [36–38], etc. The local and global solvability of stochastic 3D viscous primitive equations is obtained in [20,21], respectively and the existence of random attractors for 3D viscous primitive equations driven by fractional noises is established in [63]. The existence of a random attractor in $V$ for 3D damped Navier-Stokes equations perturbed by additive noise with $3 < r \leq 5$ in bounded domains is established in [60] by using the equality (1.2) and the method of pullback flattening property. Due to the technical difficulty discussed earlier, it appears to us that the results obtained in [60] are only valid in periodic domains.

Firstly, we establish the existence of a random attractor in $\mathbb{H}$ and $V$ for the system (1.3) with $r \geq 3 (2\beta\mu \geq 1, \text{for } r = 3)$ and $3 \leq r \leq 5 (2\beta\mu \geq 1, \text{for } r = 3)$, respectively. Then, we prove one important property of the random attractors, namely the upper semicontinuity of random attractors, which was introduced in [11]. Roughly speaking, the upper semicontinuity of random attractors means that, if $A$ is a global attractor for the deterministic system and $A_\varepsilon$ is a random attractor for the corresponding stochastic system perturbed by a small noise, we say that these attractors have the property of upper semicontinuity if

$$
\lim_{\varepsilon \to 0} d(A_\varepsilon, A) = 0,
$$

where $d$ is the Hausdorff semidistance given by $d(A, B) = \sup_{y \in A} \inf_{z \in B} \rho(y, z)$, for any $A, B \subset X$, on a Polish space $(X, \rho)$. The upper semicontinuity of random attractors for 2D stochastic NSE and stochastic reaction-diffusion equations is established in [11]. The existence and upper semicontinuity of random attractors for several stochastic models are available in [32,37,41,58], etc. In particular, the upper semicontinuity results for 2D SCBF equations in bounded domains is discussed in [37]. We prove the upper semicontinuity of random attractors for the 3D SCBF equations (1.3) by using the similar techniques available in [11].

The organization of the paper is as follows. In Sect. 2, we provide the functional framework needed for this work. Also, we define linear and nonlinear operators, and discuss their properties. The Sect. 3 is devoted for establishing the existence of global attractors for the deterministic CBF equations (1.1) in $\mathbb{H}$ and $V$ with $r \geq 3 (2\beta\mu \geq 1, \text{for } r = 3)$ (Theorem 3.5) and $3 \leq r \leq 5 (2\beta\mu \geq 1, \text{for } r = 3)$ (Theorem 3.7), respectively. In Sect. 4, we first provide an abstract formulation of the 3D SCBF equations (1.3), the assumption on RKHS
K and the existence and uniqueness results for the system (1.3). Then, we define the metric dynamical system (MDS) and random dynamical system (RDS) for the 3D SCBF equations (1.3). We prove the existence of random attractors for the 3D SCBF equations (1.3) in $H$ and $V$ with $r \geq 3$ ($2\beta \mu \geq 1$, for $r = 3$) (Theorem 4.18) and $3 \leq r \leq 5$ ($2\beta \mu \geq 1$, for $r = 3$) (Theorem 4.20), respectively, in the same section. The upper semicontinuity of random attractors for the 3D SCBF equations (1.3) using the theory given in [11] is discussed in Sect. 5 (Theorem 5.1). In the final section, we discuss the existence and uniqueness of invariant measures for 3D SCBF equations.

2 Mathematical Formulation

In this section, we provide the necessary function spaces needed to obtain the existence of global attractors and random attractors for the equations (1.1) and (1.3), respectively. We consider the problem (1.1) on a three dimensional torus $\mathbb{T}^3 = [0, L]^3$ with the periodic boundary conditions and zero-mean value constraint for the functions, that is, $\int_{\mathbb{T}^3} u(x) dx = 0$. Since, $\alpha$ does not play a major role in our analysis, we fix $\alpha = 0$ in (1.1) and (1.3) for further analysis.

2.1 Function Spaces

Let $\tilde{C}^\infty_p (\mathbb{T}^3; \mathbb{R}^3)$ denote the space of all infinitely differentiable functions ($\mathbb{R}^3$-valued) such that $\int_{\mathbb{T}^3} u(x) dx = 0$ and $u(x + Le_i) = u(x)$, for every $x \in \mathbb{R}^3$ and $i = 1, 2, 3$, where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$. The Sobolev space $\tilde{H}^k_p (\mathbb{T}^3) := \tilde{H}^k_p (\mathbb{T}^3; \mathbb{R}^3)$ is the completion of $\tilde{C}^\infty_p (\mathbb{T}^3; \mathbb{R}^3)$ with respect to the $\tilde{H}^s$ norm

$$\|u\|_{\tilde{H}^s_p} := \left( \sum_{0 \leq |\alpha| \leq s} \|D^\alpha u\|^2_{L^2(\mathbb{T}^3)} \right)^{1/2}.$$

The Sobolev space of periodic functions with zero mean $\tilde{H}^k_p (\mathbb{T}^3)$ is the same as (Proposition 5.39, [52])

$$\left\{ u : u = \sum_{k \in \mathbb{Z}^3} u_k e^{2\pi k \cdot x / L}, u_0 = 0, \tilde{u}_k = u_{-k}, \|u\|_{\tilde{H}^k_p} := \sum_{k \in \mathbb{Z}^3} |k|^{2s} |u_k|^2 < \infty \right\}.$$

From Proposition 5.38, [52], we infer that the norms $\|\cdot\|_{\tilde{H}^s_p}$ and $\|\cdot\|_{\tilde{H}^s_p}$ are equivalent. Let us define

$$V := \{u \in \tilde{C}^\infty_p (\mathbb{T}^3; \mathbb{R}^3) : \nabla \cdot u = 0\},$$

$$\mathbb{H} := \text{the closure of } V \text{ in the Lebesgue space } L^2(\mathbb{T}^3) = L^2(\mathbb{T}^3; \mathbb{R}^3),$$

$$\mathbb{V} := \text{the closure of } V \text{ in the Sobolev space } H^1(\mathbb{T}^3) = H^1(\mathbb{T}^3; \mathbb{R}^3),$$

$$\tilde{\mathbb{H}}_p := \text{the closure of } V \text{ in the Lebesgue space } L^p(\mathbb{T}^3) = L^p(\mathbb{T}^3; \mathbb{R}^3),$$

for $p \in (2, \infty)$. The zero mean condition provides the Poincaré-Wirtinger inequality,

$$\lambda_1 \|u\|^2_{\mathbb{H}} \leq \|u\|^2_{\mathbb{V}}, \quad (2.1)$$

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where \( \lambda_1 = \frac{4\pi^2}{L^2} \) (Lemma 5.40, [52]). Then, we characterize the spaces \( \mathbb{H}, \mathbb{V} \) and \( \mathbb{L}^p \) with the norms

\[
\|u\|_{\mathbb{H}}^2 := \int_{\mathbb{T}^3} |u(x)|^2 \, dx, \quad \|u\|_{\mathbb{V}}^2 := \int_{\mathbb{T}^3} |\nabla u(x)|^2 \, dx \quad \text{and} \quad \|u\|_{\mathbb{L}^p}^p = \int_{\mathbb{T}^3} |u(x)|^p \, dx,
\]

respectively. Let \( (\cdot, \cdot) \) denote the inner product in the Hilbert space \( \mathbb{H} \) and \( (\cdot, \cdot) \) represent the induced duality between the spaces \( \mathbb{V} \) and its dual \( \mathbb{V}' \) as well as \( \mathbb{L}^p \) and its dual \( \mathbb{L}^{p'} \) where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Note that \( \mathbb{H} \) can be identified with its dual \( \mathbb{H}' \). We endow the space \( \mathbb{V} \cap \mathbb{L}^p \) with the norm \( \|u\|_{\mathbb{V}} + \|u\|_{\mathbb{L}^p} \), for \( u \in \mathbb{V} \cap \mathbb{L}^p \) and its dual \( \mathbb{V}' + \mathbb{L}^{p'} \) with the norm

\[
\inf \left\{ \max \left( \|v_1\|_{\mathbb{V}}, \|v_1\|_{\mathbb{L}^{p'}} \right) : v = v_1 + v_2, \ v_1 \in \mathbb{V}', \ v_2 \in \mathbb{L}^{p'} \right\}.
\]

Moreover, we have the continuous embedding

\[
\mathbb{V} \cap \mathbb{L}^p \hookrightarrow \mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{H}' \hookrightarrow \mathbb{V}' \hookrightarrow \mathbb{V}' + \mathbb{L}^{p'}.
\]

We use the following interpolation inequality in the sequel. Assume \( 1 \leq s_1 \leq s \leq s_2 \leq \infty \), \( \theta \in (0, 1) \) such that \( \frac{1}{s} = \frac{\theta}{s_1} + \frac{1-\theta}{s_2} \) and \( u \in \mathbb{L}^{s_1}(\mathbb{T}^3) \cap \mathbb{L}^{s_2}(\mathbb{T}^3) \), then we have

\[
\|u\|_{\mathbb{L}^s(\mathbb{T}^3)} \leq \|u\|_{\mathbb{L}^{s_1}(\mathbb{T}^3)}^{\theta} \|u\|_{\mathbb{L}^{s_2}(\mathbb{T}^3)}^{1-\theta}.
\]  \hfill (2.2)

### 2.2 Linear Operator

Let \( \mathcal{P} : \mathbb{L}^{2}(\mathbb{T}^3) \to \mathbb{H} \) denote the Helmholtz-Hodge (or Leray) projection (section 2.1, [54]). We define the Stokes operator

\[
Au := -\mathcal{P} \Delta u, \ u \in D(A) := \mathbb{V} \cap \mathbb{H}^2_p(\mathbb{T}^3).
\]

Note that \( D(A) \) can also be written as \( D(A) = \{ u \in \mathbb{H}^2_p(\mathbb{T}^3) : \nabla \cdot u = 0 \} \). It should be noted that \( \mathcal{P} \) and \( \Delta \) commutes in periodic domains (Lemma 2.9, [54]). For the Fourier expansion \( u = \sum_{k \in \mathbb{Z}^3} e^{2\pi i k \cdot x/L} u_k \), one obtains

\[
-\Delta u = \frac{4\pi^2}{L^2} \sum_{k \in \mathbb{Z}^3} e^{2\pi i k \cdot x/L} |k|^2 u_k.
\]

It is easy to observe that \( D(A^{s/2}) = \{ u \in \mathbb{H}^s_p(\mathbb{T}^3) : \nabla \cdot u = 0 \} \) and \( \|A^{s/2}u\|_{\mathbb{H}} = C\|u\|_{\mathbb{H}^s_p} \), for all \( u \in D(A^{s/2}), s \geq 0 \) (cf. [52]). Note that the operator \( A \) is a non-negative self-adjoint operator in \( \mathbb{H} \) with a compact resolvent and

\[
\langle Au, u \rangle = \|u\|_{\mathbb{V}}^2, \quad \text{for all} \ u \in \mathbb{V}, \ \text{so that} \ \|Au\|_{\mathbb{V}'} \leq \|u\|_{\mathbb{V}}.
\]  \hfill (2.3)

Since \( A^{-1} \) is a compact self-adjoint operator in \( \mathbb{H} \), we obtain a complete family of orthonormal eigenfunctions \( \{w_i\}_{i=1}^{\infty} \subset \mathbb{C}^\infty_p(\mathbb{T}^3; \mathbb{R}^3) \) such that \( Aw_i = \lambda_i w_i \), for \( i = 1, 2, \ldots \) and \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty \) are the eigenvalues of \( A \). Note that \( \lambda_1 = \frac{4\pi^2}{L^2} \) is the smallest eigenvalue of \( A \) appearing in the Poincaré-Wirtinger inequality (2.1).
2.3 Bilinear Operator

Let us define the trilinear form\( b(\cdot, \cdot, \cdot) : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R} \) by
\[
b(u, v, w) = \int_{\mathbb{T}^3} (u(x) \cdot \nabla)v(x) \cdot w(x)\,dx = \sum_{i,j=1}^{3} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x)\,dx.
\]

If \( u, v \) are such that the linear map \( b(u, v, \cdot) \) is continuous on \( \mathcal{V} \), the corresponding element of \( \mathcal{V}' \) is denoted by \( B(u, v) \). We also denote \( B(u) = B(u, u) = \mathcal{P}[(u \cdot \nabla)u] \). An integration by parts gives
\[
\begin{align*}
\left\{ \begin{array}{l}
b(u, v, w) = -b(u, w, v), \text{ for all } u, v, w \in \mathcal{V}, \\
b(u, v, v) = 0, \text{ for all } u, v \in \mathcal{V}.
\end{array} \right.
\tag{2.4}
\end{align*}
\]

Remark 2.1
1. The following well-known inequality is due to Ladyzhenskaya (Lemma 2, Chapter I. [39]):
\[
\|u\|_{L^4(\mathbb{T}^3)} \leq 2^{1/2} \|u\|_{L^2(\mathbb{T}^3)}^{1/4} \|\nabla u\|_{L^2(\mathbb{T}^3)}^{3/4}, \quad u \in H^1(\mathbb{T}^3).
\tag{2.5}
\]
2. The following inequality is an application of Agmon’s inequality:
\[
\|u\|_{L^\infty(\mathbb{T}^3)} \leq C \|u\|_{H^1(\mathbb{T}^3)}^{1/2} \|u\|_{H^2(\mathbb{T}^3)}^{1/2}, \quad u \in H^2(\mathbb{T}^3).
\tag{2.6}
\]

Remark 2.2
In the trilinear form, an application of Hölder’s inequality yields
\[
|b(u_1, u_2, u_3)| = |b(u_1, u_3, u_2)| \leq \|u_1\|_{L^{r+1}} \|u_2\|_{L^{2(\frac{r+1}{r-1})}} \|u_3\|_{\mathcal{V}},
\]
for all \( u_1 \in \mathcal{V} \cap L^{r+1}, u_2 \in \mathcal{V} \cap \tilde{L}^{2(\frac{r+1}{r-1})} \) and \( u_3 \in \mathcal{V} \), so that we get
\[
\|B(u_1, u_2)\|_{\mathcal{V}'} \leq \|u_1\|_{L^{r+1}} \|u_2\|_{L^{2(\frac{r+1}{r-1})}}.
\tag{2.7}
\]

Hence, the trilinear map \( b : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R} \) has a unique extension to a bounded trilinear map from \( (\mathcal{V} \cap L^{r+1}) \times (\mathcal{V} \cap \tilde{L}^{2(\frac{r+1}{r-1})}) \times \mathcal{V} \) to \( \mathbb{R} \). It can also be seen that \( B \) maps \( \mathcal{V} \cap L^{r+1} \) into \( \mathcal{V}' + \tilde{L}^{\frac{r+1}{r-1}} \) and using interpolation inequality (see (2.2)), we get
\[
|\langle B(u_1), u_2 \rangle| = |b(u_1, u_2, u_1)| \leq \|u_1\|_{L^{r+1}} \|u_2\|_{L^{2(\frac{r+1}{r-1})}} \|u_3\|_{\mathcal{V}} \leq \|u_1\|_{L^{\frac{r+1}{r+1}}} \|u_1\|_{L^{\frac{r+3}{r+1}}} \|u_2\|_{\mathcal{V}},
\tag{2.8}
\]
for \( r \geq 3 \) and all \( u_2 \in \mathcal{V} \). Thus, we have
\[
\|B(u_1)\|_{\mathcal{V}'} \leq \|u_1\|_{L^{\frac{r+1}{r+1}}} \|u_1\|_{L^{\frac{r+3}{r+1}}}.
\tag{2.9}
\]

Remark 2.3
Note that \( \langle B(u_1, u_1 - u_2), u_1 - u_2 \rangle = 0 \) and it implies that
\[
\langle B(u_1) - B(u_2), u_1 - u_2 \rangle = \langle B(u_1 - u_2, u_1 - u_2), u_1 - u_2 \rangle = -\langle B(u_1 - u_2, u_1 - u_2), u_2 \rangle.
\tag{2.10}
\]

Using Hölder’s and Young’s inequalities, we estimate \( |\langle B(u_1 - u_2, u_1 - u_2), u_2 \rangle| \) as
\[
|\langle B(u_1 - u_2, u_1 - u_2), u_2 \rangle| \leq \|u_1 - u_2\|_{\mathcal{V}} \|u_2\|_{L^{2(\frac{r+1}{r-1})}} \|u_2(u_1 - u_2)\|_{H^2}
\leq \frac{\mu}{4} \|u_1 - u_2\|_{\mathcal{V}}^2 + \frac{1}{\mu} \|u_2(u_1 - u_2)\|_{H^2}^2.
\tag{2.11}
\]
We take the term \( \|u_2(u_1 - u_2)\|_{2 \tilde{H}}^2 \) from (2.11) and use Hölder’s and Young’s inequalities to estimate it as (see [46] also)

\[
\int_{\mathbb{T}^3} |u_2(x)|^2 |u_1(x) - u_2(x)|^2 \, dx
\]

\[
= \int_{\mathbb{T}^3} |u_2(x)|^2 |u_1(x) - u_2(x)|^{\frac{4}{r-1}} |u_1(x) - u_2(x)|^{\frac{2(r-3)}{r-1}} \, dx
\]

\[
\leq \left( \int_{\mathbb{T}^3} |u_2(x)|^{r-1} |u_1(x) - u_2(x)|^2 \, dx \right)^{\frac{2}{r}} \left( \int_{\mathbb{T}^3} |u_1(x) - u_2(x)|^2 \, dx \right)^{\frac{r-3}{r}}
\]

\[
\leq \frac{\beta\mu}{8} \left( \int_{\mathbb{T}^3} |u_2(x)|^{r-1} |u_1(x) - u_2(x)|^2 \, dx \right)
\]

\[
+ \frac{r-3}{r-1} \left( \frac{16}{\beta\mu(r-1)} \right)^{\frac{2}{r-3}} \left( \int_{\mathbb{T}^3} |u_1(x) - u_2(x)|^2 \, dx \right), \tag{2.12}
\]

for \( r > 3 \). Using (2.12) in (2.11), we find

\[
\|\langle B(u_1 - u_2, u_1 - u_2), u_2 \rangle\|
\]

\[
\leq \frac{\mu}{4} \|u_1 - u_2\|_{\tilde{V}}^2 + \frac{\beta}{8} \|u_2\|_{\tilde{H}}^{\frac{r-1}{r}} \|u_1 - u_2\|_{\tilde{H}}^2 + \eta_1 \|u_1 - u_2\|_{\tilde{H}}^2, \tag{2.13}
\]

where

\[
\eta_1 = \frac{r-3}{\mu(r-1)} \left( \frac{16}{\beta\mu(r-1)} \right)^{\frac{2}{r-3}}. \tag{2.14}
\]

### 2.4 Nonlinear Operator

Let us now consider the operator \( C(u) := \mathcal{P}(|u|^{r-1} u) \). It is immediate that \( \langle C(u), u \rangle = \|u\|_{\tilde{L}^{r+1}}^{r+1} \) and the map \( C(\cdot) : \tilde{\mathbb{V}} \cap \tilde{\mathbb{H}}^{r+1} \to \mathbb{V} \cap \tilde{\mathbb{H}}^{r+1} \). For all \( u \in \mathbb{V} \cap \tilde{\mathbb{L}}^{r+1} \), the map is Gateaux differentiable with Gateaux derivative

\[
\mathcal{C}'(u)v = \begin{cases} 
\mathcal{P}(v), & \text{for } r = 1, \\
\mathcal{P}(|u|^{r-1} v) + (r-1) \mathcal{P} \left( \frac{u}{|u|^{r-1}} (u \cdot v) \right), & \text{if } u \neq 0, \text{ for } 1 < r < 3, \\
0, & \text{if } u = 0, \text{ for } 3 \leq r, \\
\mathcal{P}(|u|^{r-1} v) + (r-1) \mathcal{P}(u|u|^{r-3} (u \cdot v)), & \text{for } r \geq 3,
\end{cases}
\]

for all \( v \in \mathbb{V} \cap \tilde{\mathbb{L}}^{r+1} \). For any \( r \in [1, \infty) \) and \( u_1, u_2 \in \mathbb{V} \cap \tilde{\mathbb{L}}^{r+1} \), we have (see subsection 2.4, [46]),

\[
\langle C(u_1) - C(u_2), u_1 - u_2 \rangle \geq \frac{1}{2} \|u_1|^{\frac{r-1}{r}} (u_1 - u_2)\|_{\tilde{H}}^2 + \frac{1}{2} \|u_2|^{\frac{r-1}{r}} (u_1 - u_2)\|_{\tilde{H}}^2 \geq 0, \tag{2.16}
\]

and

\[
\|u_1 - u_2\|_{\tilde{L}^{r+1}}^{r+1} \leq 2^{r-2} \|u_1|^{\frac{r-1}{r}} (u_1 - u_2)\|_{\tilde{H}}^2 + 2^{r-2} \|u_2|^{\frac{r-1}{r}} (u_1 - u_2)\|_{\tilde{H}}^2, \tag{2.17}
\]

for \( r \geq 1 \) (replace \( 2^{r-2} \) with 1, for \( 1 \leq r \leq 2 \)).
Lemma 2.4 (Theorem 2.2, [46]) Let \( r > 3 \) and \( u_1, u_2 \in V \cap \tilde{L}^{r+1} \). Then, for the operator \( G(u) = \mu Au + B(u) + \beta C(u) \), we have
\[
\langle G(u_1) - G(u_2), u_1 - u_2 \rangle + \eta_2 \| u_2 - u_2 \|_H^2 \geq 0,
\]
where
\[
\eta_2 = \frac{r - 3}{2 \mu (r - 1)} \left( \frac{2}{\beta \mu (r - 1)} \right) ^{\frac{2}{r - 3}}.
\]

Lemma 2.5 (Theorem 2.3, [46]) For \( r = 3 \) with \( 2 \beta \mu \geq 1 \), the operator \( G(\cdot) : V \cap \tilde{L}^{r+1} \rightarrow \mathbb{V} + \tilde{L}^{r+1} \) is globally monotone, that is, for all \( u_1, u_2 \in V \), we have
\[
\langle G(u_1) - G(u_2), u_1 - u_2 \rangle \geq 0.
\]

3 Global Attractors for 3D Deterministic CBF Equations

In this section, we obtain the existence of global attractors for the 3D deterministic CBF equations (1.1) in periodic domains. We make use of the equality (1.2) to obtain the existence of an absorbing set in \( V \).

3.1 Deterministic CBF Equations

In this subsection, we provide the abstract formulation of deterministic CBF equation (1.1) and discuss the existence and uniqueness of weak solutions.

3.1.1 Abstract Formulation

On taking orthogonal projection \( \mathcal{P} \) onto the first equation in (1.1), we obtain
\[
\left\{ \begin{array}{l}
\frac{du(t)}{dt} + \mu Au(t) + B(u(t)) + \beta C(u(t)) = f, \quad t \geq 0, \\
u(0) = x,
\end{array} \right.
\]
where \( x \in H \) and \( f \in \mathcal{H} \). Now, we shall provide the definition of weak solution of the system (3.1) for \( r \geq 3 \) and discuss global solvability results.

Definition 3.1 Let us assume that \( x \in H \) and \( f \in \mathcal{H} \). Let \( T > 0 \) be any fixed time. Then, the function \( u(\cdot) \) is called a \textit{Leray-Hopf weak solution} of the problem (3.1) on time interval \([0, T] \), if
\[
u(0) = x,\] with \( \partial_t u \in L^2(0, T; \mathbb{V}) + L^{\frac{r+1}{r}}(0, T; \tilde{L}^{r+1}) \) satisfying:
(i) for any \( \psi \in \mathbb{V} \cap \tilde{L}^{r+1} \),
\[
\left\langle \frac{du(t)}{dt}, \psi \right\rangle = -\langle \mu Au(t) + B(u(t)) + \beta C(u(t)), \psi \rangle + (f, \psi),
\]
where \( \mathcal{S} \) Springer
(ii) the initial data is satisfied in the following sense:

$$\lim_{t \to 0} \int_{T^3} u(t, x) \psi(x) dx = \int_{T^3} x(x) \psi(x) dx,$$

for all $\psi \in \mathbb{H}$.

By using the monotonicity property (Theorems 2.4 and 2.5) as well as Minty-Browder technique, the following global solvability result is established in Theorems 3.4 and 3.5, [45] (see [2,31] also).

**Lemma 3.2** (45) For $r \ge 3 (\mu, \beta > 0 \text{ for } r > 3 \text{ and } 2\beta\mu \ge 1 \text{ for } r = 3)$, let $x \in \mathbb{H}$ and $f \in \mathbb{V}'$ be given. Then there exists a unique Leray-Hopf weak solution $u(\cdot)$ to the system (3.1) in the sense of Definition 3.1.

Moreover, the solution $u \in C([0, T]; \mathbb{H})$ and satisfies the following energy equality:

$$\|u(t)\|_{\mathbb{H}}^2 + 2\mu \int_0^t \|u(s)\|_{\mathbb{V}}^2 ds + 2\beta \int_0^t \|u(s)\|_{L_r^{r+1}}^{r+1} ds = \|x\|_{\mathbb{H}}^2 + 2 \int_0^t (f, u(s)) ds, \quad (3.3)$$

for all $t \in [0, T]$. In addition, for $x \in \mathbb{V}$ and $f \in \mathbb{H}$, there exists a unique strong solution $u(\cdot)$ to the system (3.1) satisfying the following regularity:

$$u \in C([0, T]; \mathbb{V}) \cap L^2(0, T; D(A)) \cap L^{r+1}(0, T; \mathbb{L}_r^{3(r+1)}).$$

Without any ambiguity, we fix $\mu, \beta > 0$ for $r > 3$ and $2\beta\mu \ge 1$ for $r = 3$ in the rest of the paper.

### 3.2 Existence of Global Attractors

In this subsection, we establish the existence of a global attractor for the system (3.1). Thanks to the existence and uniqueness of weak solutions to the system (3.1) (Lemma 3.2), one can define a continuous semigroup $\{S(t)\}_{t \ge 0}$ in $\mathbb{H}$ by

$$S(t)x = u(t), \quad t \ge 0,$$

where $u(\cdot)$ is the unique weak solution of (3.1) with $u(0) = x \in \mathbb{H}$. Next we prove the continuity of semigroup $S(t)$ with respect to initial data $x$, when the forcing $f$ depends on time also.

**Theorem 3.3** For $r \ge 3 (2\beta\mu \ge 1 \text{ for } r = 3)$, assume that, for some $T > 0$ fixed, $x_n \to x$ in $\mathbb{H}$ and $f_n \to f$ in $L^2(0, T; \mathbb{H})$. Let us denote by $S(t)x$ for the solution of the system (3.1) and by $S(t)x_n$ for the solution of the system (3.1) with $x$, $f$ being replaced by $x_n$, $f_n$. Then

$$S(t)x_n \to S(t)x \text{ in } C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \cap L^{r+1}(0, T; \mathbb{L}_r^{3(r+1)}).$$

In particular, $S(T)x_n \to S(T)x$ in $\mathbb{H}$.

Furthermore, let $x_n \to x$ in $\mathbb{V}$. Then, for $3 \le r \le 5$,

$$S(t)x_n \to S(t)x \text{ in } C([0, T]; \mathbb{V}) \cap L^2(0, T; D(A)).$$

In particular, $S(T)x_n \to S(T)x$ in $\mathbb{V}$.

**Proof** In order to simplify the proof, we introduce the following notations for $t \in [0, T]$:

$$u_n(t) = S(t)x_n, \quad u(t) = S(t)x, \quad w_n(t) = S(t)x_n - S(t)x, \quad \hat{f}_n(t) = f_n(t) - f(t).$$
It is easy to see that $y_n(\cdot)$ solves the following initial value problem:

$$
\begin{align*}
\frac{dw_n(t)}{dt} &= -\mu A(w_n(t) - B(u_n(t))) - \beta C(u_n(t)) + \beta C(u(t)) + \hat{f}_n(t), \\
w_n(0) &= x_n - x,
\end{align*}
$$

for a.e. $t \in [0, T]$. The system (3.5) is similar to system (4.14) given below and is a particular case of $\varepsilon = 0$ in (4.14). The proof can be carried forward in a similar way as in the proof of Theorems 4.8 and 4.9.

Next, we prove the existence of an absorbing ball in $\mathbb{H}$ as well as in $\mathbb{V}$ for the semigroup $S(t)$, $t \geq 0$ defined in $\mathbb{H}$ for the 3D deterministic CBF equations (3.1) in periodic domains. We provide our estimates in terms of the dimensionless Grashof number, which measures the relative strength of the forcing and viscosity and is defined as

$$
G = \frac{\|f\|_H}{\mu^2 \lambda_1}.
$$

**Theorem 3.4** For $r \geq 3$, let $f \in \mathbb{H}$. Let us define

$$
\varrho_0^2 := \left( \frac{\|f\|_H^2}{\mu^2 \lambda_1^2} \right) = \mu^2 G^2.
$$

Then for any $M_0 > \varrho_0$ and $u \in \mathbb{H}$, there exists a time $t_{M_0}(\|x\|_H)$ such that

$$
\|S(t)x\|_H \leq M_0, \text{ for all } t \geq t_{M_0}(\|x\|_H).
$$

That is, there exists an absorbing set $B_{\mathbb{H}}$ in $\mathbb{H}$ for the semigroup $S(t)$.

For $r > 3$, let

$$
M_1^2 := (2\eta_3 + 1) \frac{M_0^2}{\mu} + \frac{1}{\mu^2 \lambda_1} \|f\|_H^2.
$$

where

$$
\eta_3 = \frac{r - 3}{r - 1} \left[ \frac{4}{\beta \mu (r - 1)} \right]^{\frac{2}{r - 3}}.
$$

Then for any $x \in \mathbb{H}$

$$
\|S(t)x\|_V \leq M_1, \text{ for all } t \geq t_{M_0}(\|x\|_H) + 1.
$$

That is, there exists an absorbing set $B_{\mathbb{V}}$ in $\mathbb{V}$ for the semigroup $S(t)$. Moreover, we have

$$
\limsup_{T \to \infty} \frac{1}{T} \int_t^{t+T} \|Au(s)\|_H^2 ds \leq \left( \frac{\eta_3}{\lambda_1 \mu} + 1 \right) \left[ \frac{2}{\mu} \right] \|f\|_H^2.
$$

For $r = 3$ and $2\beta \mu \geq 1$ also, there exists an absorbing set $B_{\mathbb{V}}$ in $\mathbb{V}$ for the semigroup $S(t)$ with $M_1^2$ replaced by $M_2^2 = \frac{M_0^2}{\mu} + \frac{1}{\mu^2 \lambda_1} \|f\|_H^2$ in (3.7).
Similarly, we obtain

\[ \frac{1}{2} \frac{d}{dt} \| u(t) \|_{H^2}^2 + \mu \| u(t) \|_{V}^2 + \beta \| u(t) \|_{L^{r+1}}^{r+1} = (f, u(t)), \]

for a.e. \( t \in [0, T] \). Using the Cauchy-Schwarz inequality and Young’s inequality, we estimate \(|(f, u)|\) as

\[ |(f, u)| \leq \| f \|_{H^2} \| u \|_{H^2} \leq \frac{\mu}{2} \| u \|_{V}^2 + \frac{1}{2 \mu \lambda_1} \| f \|_{H^2}^2. \]

Thus, it is immediate that

\[ \frac{d}{dt} \| u(t) \|_{H^2}^2 + \mu \lambda_1 \| u(t) \|_{H^2}^2 + 2 \beta \| u(t) \|_{L^{r+1}}^{r+1} \leq \frac{1}{\mu \lambda_1} \| f \|_{H^2}^2. \]

Using the classical Gronwall inequality, we deduce that

\[ \| u(t) \|_{H^2}^2 \leq \| x \|_{H^2}^2 e^{-\mu \lambda_1 t} + \frac{1}{\mu^2 \lambda_1^2} \| f \|_{H^2}^2 (1 - e^{-\mu \lambda_1 t}), \]

for all \( t \geq 0 \). From the above relation, it is clear that for any given \( M_0 > \phi_0 \), there exists a time \( t_0 \), which depends only on \( M_0 \) and \( \| x \|_{H^2} \) such that

\[ \| u(t) \|_{H^2}^2 \leq M_0^2, \quad \text{for all } t \geq t_0. \]

Furthermore, from (3.11), we infer that

\[ \mu \int_t^{t+\theta} \| u(s) \|_{V}^2 ds + 2 \beta \int_t^{t+\theta} \| u(s) \|_{L^{r+1}}^{r+1} ds \leq \| u(t) \|_{H^2}^2 + \frac{\theta}{\mu \lambda_1} \| f \|_{H^2}^2, \]

for all \( \theta > 0 \). The above equality implies that for \( t \geq t_0(\| x \|_{H^2}) \),

\[ \mu \int_t^{t+\theta} \| u(s) \|_{V}^2 ds + 2 \beta \int_t^{t+\theta} \| u(s) \|_{L^{r+1}}^{r+1} ds \leq M_0^2 + \frac{\theta}{\mu \lambda_1} \| f \|_{H^2}^2. \]

Similarly, we obtain

\[ \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| u(s) \|_{V}^2 ds \leq \frac{1}{\mu^2 \lambda_1^2} \| f \|_{H^2}^2 \]

and

\[ \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| u(s) \|_{L^{r+1}}^{r+1} ds \leq \frac{1}{\beta \mu \lambda_1} \| f \|_{H^2}^2. \]

**Step II: Absorbing ball in \( \mathbb{V} \).** Let us now take the inner product with \( Au(\cdot) \) to first equation of (3.1) to get

\[ \frac{1}{2} \frac{d}{dt} \| u(t) \|_{V}^2 + \mu \| Au(t) \|_{H^2}^2 + \beta \| C(u(t)) \|_{H^2}^2 \leq -(B(u(t)), Au(t)) + (f, Au(t)). \]
Case I: \( r > 3 \). From (1.2), we have
\[
(C(u), Au) = \|\nabla u\|u\|^{r-1}\|u\|_{H^1}^2 + 4 \left[ \frac{r-1}{(r+1)^2} \right] \|\nabla u\|^{r+1}\|u\|_{H^1}^2. \tag{3.19}
\]

Once again the Cauchy-Schwarz and Young’s inequalities yield
\[
|(f, Au)| \leq \|f\|_{H^1} \|Au\|_{H^1} \leq \frac{\mu}{4} \|Au\|_{H^1}^2 + \frac{1}{\mu} \|f\|_{H^1}^2. \tag{3.20}
\]

We estimate \(|(B(u), Au)|\) using Hölder’s and Young’s inequalities as
\[
|(B(u), Au)| \leq \|u\|\|\nabla u\|\|Au\|_{H^1} \leq \frac{\mu}{4} \|Au\|_{H^1}^2 + \frac{1}{\mu} \|u\|\|\nabla u\|_{H^1}^2. \tag{3.21}
\]

We estimate the final term from (3.21) using Hölder’s and Young’s inequalities as (similarly as in (2.12))
\[
\int_{\mathbb{T}^d} |u(x)|^2 |\nabla u(x)|^2 \, dx
\leq \frac{\beta \mu}{2} \left( \int_{\mathbb{T}^d} |u(x)|^{r-1} |\nabla u(x)|^2 \, dx \right) + \frac{r-3}{r-1} \left[ \frac{4}{\beta \mu (r-1)} \right]^{\frac{r-3}{2}} \left( \int_{\mathbb{T}^d} |\nabla u(x)|^2 \, dx \right)^{\frac{r-3}{2}}. \tag{3.22}
\]

Making use of the estimate (3.22) in (3.21), we find
\[
|(B(u), Au)| \leq \frac{\mu}{4} \|Au\|_{H^1}^2 + \frac{\beta}{2} \|\nabla u\|\|u\|^{r-1}\|u\|_{H^1}^2 + \eta_3 \|u\|_{\tilde{V}}^2. \tag{3.23}
\]

Using (3.19)–(3.20) and (3.23) in (1.8), we obtain
\[
\frac{d}{dt} \|u(t)\|_{\tilde{V}}^2 + \mu \|Au(t)\|_{H^1}^2 + \beta \|\nabla u(t)\|\|u(t)\|^{r-1}\|u(t)\|_{H^1}^2 + 8\beta \left[ \frac{r-1}{(r+1)^2} \right] \|\nabla u(t)\|^{r+1}\|u(t)\|_{H^1}^2
\leq 2\eta_3 \|u(t)\|_{\tilde{V}}^2 + \frac{2}{\mu} \|f\|_{H^1}^2. \tag{3.24}
\]

We use the double integration trick used in [53] to obtain an absorbing ball in \( \tilde{V} \). Dropping the terms \( \mu \|Au(t)\|_{H^1}^2 + \beta \|\nabla u(t)\|\|u(t)\|^{r-1}\|u(t)\|_{H^1}^2 + 8\beta \left[ \frac{r-1}{(r+1)^2} \right] \|\nabla u(t)\|^{r+1}\|u(t)\|_{H^1}^2 \) from the left hand side of the inequality (3.24) and then integrating from \( s \) to \( t + 1 \), with \( t \leq s < t + 1 \), we find
\[
\|u(t+1)\|_{\tilde{V}}^2 \leq \|u(s)\|_{\tilde{V}}^2 + 2\eta_3 \int_s^{t+1} \|u(\xi)\|_{\tilde{V}}^2 d\xi + \frac{2}{\mu} \|f\|_{H^1}^2
\leq \|u(s)\|_{\tilde{V}}^2 + 2\eta_3 \int_t^{t+1} \|u(\xi)\|_{\tilde{V}}^2 d\xi + \frac{2}{\mu} \|f\|_{H^1}^2,
\]
where \( \eta_3 \) is defined in (3.8). Let us now integrate both sides of the above inequality with respect to \( s \) between \( t \) and \( t + 1 \) to obtain
\[
\|u(t+1)\|_{\tilde{V}}^2 \leq (2\eta_3 + 1) \int_t^{t+1} \|u(s)\|_{\tilde{V}}^2 ds + \frac{2}{\mu} \|f\|_{H^1}^2
\leq (2\eta_3 + 1) \frac{\bar{M}_0^2}{\mu} + \frac{1}{\mu \lambda_1} \|f\|_{H^1}^2, \tag{3.25}
\]
for all $t \geq t_{M_0}(\|x\|_{\mathbb{H}})$. Integrating the inequality (3.24) from $t$ to $t + T$, we find

$$
\|u(t + T)\|_{\mathcal{V}}^2 + \mu \int_t^{t+T} \|Au(s)\|_{\mathbb{H}}^2 ds \\
\leq \|u(t)\|_{\mathcal{V}}^2 + 2\eta_3 \int_t^{t+T} \|u(s)\|_{\mathcal{V}}^2 ds + \frac{2T\|f\|_{\mathbb{H}}^2}{\mu}.
$$

Dividing by $T$, taking the limit supremum and then using (3.16) and (3.25), we finally obtain (3.10).

**Case II:** $r = 3$ and $2\beta\mu \geq 1$. Using (1.2), the Cauchy-Schwarz and Young’s inequalities, we find

$$
|(B(u), Au)| \leq \|u\|_{\mathcal{V}} \|\nabla u\|_{\mathbb{H}} \|Au\|_{\mathbb{H}} \leq \frac{1}{4\beta} \|Au\|_{\mathbb{H}}^2 + \beta \|u\|_{\mathcal{V}} \|\nabla u\|_{\mathbb{H}}^2,
$$

(3.26)

$$
(C(u), Au) = \|\nabla u\|_{\mathcal{V}}^2 + \frac{1}{2} \|\nabla u\|_{\mathbb{H}}^2,
$$

(3.27)

$$
|(f, Au)| \leq \|f\|_{\mathbb{H}} \|Au\|_{\mathbb{H}} \leq \frac{\mu}{2} \|Au\|_{\mathbb{H}}^2 + \frac{1}{2\mu} \|f\|_{\mathbb{H}}^2.
$$

(3.28)

Using (3.26)–(3.28) in (3.18), we obtain

$$
\frac{d}{dt}\|u(t)\|_{\mathcal{V}}^2 + \left(\mu - \frac{1}{2\beta}\right) \|Au(t)\|_{\mathbb{H}}^2 + \beta \|\nabla u\|_{\mathbb{H}}^2 \leq \frac{\mu}{2} \|f\|_{\mathbb{H}}^2.
$$

(3.29)

For $2\beta\mu \geq 1$, performing calculations similar to the case of $r > 3$, we have

$$
\|u(t)\|_{\mathcal{V}}^2 \leq \frac{M_0^2}{\mu} + \frac{\theta}{\mu^2\lambda_1} \|f\|_{\mathbb{H}}^2 =: M_2^2,
$$

(3.30)

for all $t \geq t_{M_0}(\|x\|_{\mathbb{H}})$, and hence the conclusions of Theorem follows.

Thanks to the compactness of $\mathcal{V}$ in $\mathbb{H}$, the operators $\{S(t)\}$ are uniformly compact. From Theorem 3.4 and the abstract theory of global attractors (Theorem 1.1, Chapter I, [57]), we immediately conclude the following result.

**Theorem 3.5** The dynamical system associated with the 3D deterministic CBF equations (3.1) possesses an attractor $\mathcal{A}$ that is compact, connected, and maximal in $\mathbb{H}$. $\mathcal{A}$ attracts the bounded sets of $\mathbb{H}$ and $\mathcal{A}$ is also maximal among the functional invariant bounded sets in $\mathbb{H}$.

**Theorem 3.6** For $3 \leq r \leq 5$ ($2\beta\mu \geq 1$ for $r = 3$), the solution operator $S(t)$ defined by (3.4) is compact from $\mathcal{V}$ into itself.

**Proof** Consider the solution $S(t)\cdot = u(t, \cdot)$ of (3.1) for $t \in [0, T]$, where $T > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{V}$. We know that

$$
\{u(\cdot, x_n)\}_{n\in\mathbb{N}} \text{ is bounded in } L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; D(A)) \cap L^{r+1}(0, T; \widetilde{L}^{3(r+1)}).
$$
For any arbitrary element $\nu \in \mathbb{H}$, using Hölder’s inequality, Sobolev’s embedding and interpolation inequality (see (2.2)), we have

$$\left| \frac{d}{dt} u(t, x_n), \nu \right| \leq \mu |(Au(t, x_n), \nu)| + |b(u(t, x_n), u(t, x_n), \nu)| + \beta |(C(u(t, x_n)), \nu)| + |(f, \nu)|$$

$$\leq \left[ \mu \|Au(t, x_n)\|_{\mathbb{H}} + \|u(t, x_n)\|_{L^\infty} \|u(t, x_n)\|_V + \beta \|u(t, x_n)\|_{L^{r/2}} + \|f\|_{\mathbb{H}} \right] \|\nu\|_{\mathbb{H}}$$

$$\leq C \left[ \|Au(t, x_n)\|_{\mathbb{H}} + \|Au(t, x_n)\|_{\mathbb{H}} \|u(t, x_n)\|_V + \|u(t, x_n)\|_{L^{r/2}} + \|f\|_{\mathbb{H}} \right]. \quad (3.31)$$

We infer from (3.31) that

$$\left\| \frac{d}{dt} [u(t, x_n)] \right\|_{L^2(0, T; \mathbb{H})} \leq C \left[ \left\| u(t, x_n) \right\|_{L^2(0, T; D(A))} + \left\| u(t, x_n) \right\|_{L^2(0, T; D(A))} \left\| u(t, x_n) \right\|_{L^\infty(0, T; V)}$$

$$+ T^{\frac{3-r}{2}} \left\| u(t, x_n) \right\|_{L^r(0, T; V)} + \left\| f \right\|_{L^2(0, T; \mathbb{H})} \right]$$

$$\leq \tilde{C} < \infty, \quad (3.32)$$

and the constant $\tilde{C}$ is independent of $n$. Since $u(\cdot, x_n) \in L^2(0, T; D(A))$, $\frac{d}{dt} [u(\cdot, x_n)] \in L^2(0, T; \mathbb{H})$, $D(A) \subset V \subset \mathbb{H}$ and $D(A)$ is compactly embedded in $V$, by the Aubin-Lions compactness lemma, there exists a subsequence (using the same notation) and $\tilde{u} \in L^2(0, T; V)$ such that

$$u(\cdot, x_n) \to \tilde{u}(\cdot) \text{ strongly in } L^2(0, T; V). \quad (3.33)$$

Again, choosing one more subsequence (again not relabeling), we infer from (3.33) that

$$u(\tau, x_n) \to \tilde{u}(\tau) \text{ in } V, \text{ for almost all } \tau \in (0, T). \quad (3.34)$$

Since $0 < t < T$, we obtain from (3.34) that there exists $\tau \in (0, t)$ such that (3.34) holds true for this particular $\tau$. Then by Theorem 3.3, we obtain

$$S(t)x_n = S(t - \tau)S(\tau)x_n = S(t - \tau)u(\tau, x_n) \to S(t - \tau)\tilde{u}(\tau) \text{ in } V,$$

which completes the proof. \qed

Again, thanks to the abstract theory of global attractors (Theorem 1.1, Chapter I, [57]), we immediately conclude the following result.
Theorem 3.7 For $3 \leq r \leq 5$ ($2\beta \mu \geq 1$ for $r = 3$), the dynamical system associated with the 3D deterministic CBF equations (3.1) possesses an attractor $\hat{A}$ that is compact, connected, and maximal in $V$. $\hat{A}$ attracts the bounded sets of $V$ and $\hat{A}$ is also maximal among the functional invariant bounded sets in $V$.

Remark 3.8 Note that using the method described in Theorem 3.6, one can prove the existence of global attractor for (3.1) in $\mathbb{H}$, with $f \in V'$ also.

4 Random Attractors for 3D Stochastic CBF Equations

In this section, we establish the existence of a random attractor for 3D stochastic CBF equations perturbed by a small additive smooth Gaussian noise.

4.1 Abstract Formulation

Taking orthogonal projection $\mathcal{P}$ onto the first equation in (1.3), we obtain the following abstract formulation of the 3D SCBF equations:

$$
\begin{cases}
\frac{du_e(t)}{dt} + \{\mu A u_e(t) + B(u_e(t)) + \beta C(u_e(t))\}dt = f(t)dt + \varepsilon dW(t), \quad t \geq 0, \\
u_e(0) = x,
\end{cases}
$$

for $r \geq 3$ and $\varepsilon \in (0, 1]$, where we assume that $x \in \mathbb{H}$, $f \in \mathbb{H}$ and $W(t)$, $t \in \mathbb{R}$, is a two-sided cylindrical Wiener process with its Reproducing Kernel Hilbert Space (RKHS [18]) $K$ satisfying Assumption 4.1 below defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$.

Assumption 4.1 $K \subset \mathbb{V} \cap \mathbb{H}_p^2(T^3)$ is a Hilbert space such that for some $\delta \in (0, 1/2)$,

$$
A^{-\delta} : K \rightarrow \mathbb{V} \cap \mathbb{H}_p^2(T^3) \text{ is } \gamma\text{-radonifying.}
$$

Remark 4.2 1. Let $\mathcal{H}$ be a real separable Hilbert space, $\{w_k\}$ be an orthonormal basis of $\mathcal{H}$, and $\{\beta_k\}$ be a system of independent normal real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a real Banach space $(E, \|\cdot\|_E)$, a bounded linear operator $\Psi : \mathcal{H} \to E$ is called $\gamma$-radonifying if and only if the series $\sum_{k=1}^{\infty} \beta_k \Psi w_k$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; E)$.

The set of all $\gamma$-radonifying operators from $\mathcal{H}$ into $E$ is denoted by $\gamma(\mathcal{H}; E)$, and the space $\gamma(\mathcal{H}; E)$ is the norm defined on $\gamma(\mathcal{H}; E)$ is a separable Banach space. If $E$ is a separable Hilbert space, then we know that $\mathcal{L}_2(\mathcal{H}; E)$, the class of all Hilbert-Schmidt operators from $\mathcal{H}$ into $E$ endowed with the norm

$$
\|\Psi\|_{\mathcal{L}_2(\mathcal{H}; E)} = \left( \sum_{k=1}^{\infty} \|\Psi w_k\|_E^2 \right)^{1/2}
$$

is same as that of $\gamma(\mathcal{H}; E)$.

2. Since $D(A) = \mathbb{V} \cap \mathbb{H}_p^2(T^3)$, one can reformulate Assumption 4.1 in the following way also (see [9]). $K$ is a Hilbert space such that $K \subset D(A)$ and, for some $\delta \in (0, 1/2)$, the map

$$
A^{-\delta -1} : K \to \mathbb{H} \text{ is } \gamma\text{-radonifying.}
$$

The condition (4.3) also says that the mapping $A^{-\delta -1} : K \to \mathbb{H}$ is Hilbert-Schmidt. Since $T^3$ is a periodic domain, then $A^{-\delta} : \mathbb{H} \to \mathbb{H}$ is Hilbert-Schmidt if and only if $\sum_{j=1}^{\infty} \lambda_j^{-2\delta} < \infty$,
where $A w_j = \lambda_j w_j$, $j \in \mathbb{N}$ and $w_j$ is an orthogonal basis of $\mathbb{H}$. In periodic domains, we know that $\lambda_j \sim j^{2/3}$, for large $j$ (growth of eigenvalues, [25]) and hence $A^{-s}$ is Hilbert-Schmidt if and only if $s > \frac{3}{4}$. In other words, with $K = D(A^{s+1})$, the embedding $K \hookrightarrow \mathbb{V} \cap \mathbb{H}^2_p(T^3)$ is $\gamma$-radonifying if and only if $s > \frac{3}{4}$. Thus, Assumption 4.1 is satisfied for any $\delta > 0$. In fact, the condition (4.2) holds if and only if the operator $A^{-(s+1+\delta)} : \mathbb{H} \to \mathbb{V} \cap \mathbb{H}^2_p(T^3)$ is $\gamma$-radonifying. We emphasize that the requirement of $\delta < \frac{1}{2}$ in Assumption 4.1 is necessary because we need the corresponding Ornstein-Uhlenbeck process has to take values in $\mathbb{V} \cap \mathbb{H}^2_p(T^3)$ (see Subsect. 4.2).

### 4.2 Ornstein-Uhlenbeck Process

We define Ornstein-Uhlenbeck processes under Assumption 4.1 and discuss its properties in this subsection (for more details see section 3, [37]).

Let us define $X := \mathbb{V} \cap \mathbb{H}^2_p(T^3)$ and let $E$ be the completion of $A^{-\delta}(X)$ with respect to the image norm $\|x\|_E = \|A^{-\delta}x\|_X$, for $x \in X$, where $\|\cdot\|_X = \|\cdot\|_\mathbb{V} + \|\cdot\|_\mathbb{H}^2_p$. For $\xi \in (0, 1/2)$, we define

$$C_{1/2}^\xi(\mathbb{R}, E) = \left\{ \omega \in C(\mathbb{R}, E) : \omega(0) = 0, \sup_{t \neq s \in \mathbb{R}} \frac{\|\omega(t) - \omega(s)\|_E}{|t - s|^\xi (1 + |t| + |s|)^{1/2}} < \infty \right\},$$

$$\Omega(\xi, E) = \text{the closure of } \{ \omega \in C_0^\infty(\mathbb{R}, E) : \omega(0) = 0 \} \text{ in } C_{1/2}^\xi(\mathbb{R}, E).$$

The space $\Omega(\xi, E)$ is a separable Banach space. Let us denote $\mathcal{F}$ for the Borel $\sigma$-algebra on $\Omega(\xi, E)$. For $\xi \in (0, 1/2)$, there exists a Borel probability measure $\mathbb{P}$ on $\Omega(\xi, E)$ (see [6]). For $t \in \mathbb{R}$, let $\mathcal{F}_t := \sigma \{ w_s : s \leq t \}$, where $w_t$ is the canonical process defined by the elements of $\Omega(\xi, E)$. Then there exists a family $\{ W(t) \}_{t \in \mathbb{R}}$, which is $\mathbb{H}$-cylindrical Wiener process on a filtered probability space $(\Omega(\xi, E), \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$.

On the space $\Omega(\xi, E)$, we consider a flow $\theta = (\theta_t)_{t \in \mathbb{R}}$ defined by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \ \omega \in \Omega(\xi, E), \ t \in \mathbb{R}.$$

**Lemma 4.3** (Proposition 6.10, [10]) The process $z_\alpha(t), \ t \in \mathbb{R}$, is stationary Ornstein-Uhlenbeck process on $(\Omega(\xi, E), \mathcal{F}, \mathbb{P})$. It is a solution of the equation

$$dz_\alpha(t) + (\mu A + \alpha I)z_\alpha(t)dt = dW(t), \ t \in \mathbb{R},$$

that is, for all $t \in \mathbb{R}$,

$$z_\alpha(t) = \int_{-\infty}^{t} e^{-(t-s)(\mu A+\alpha I)}dW(s),$$

$\mathbb{P}$-a.s., where the integral is the Itô integral on the $M$-type 2 Banach space $X$ (cf. [7]). In particular, for some $C$ depending on $X$,

$$\mathbb{E} \left[ \|z_\alpha(t)\|_X^2 \right] \leq C \int_0^\infty e^{-2\alpha s}e^{-\mu s}A^2_{\gamma(K, X)}ds.$$  

Moreover, $\mathbb{E} \left[ \|z_\alpha(t)\|_X^2 \right] \to 0$ as $\alpha \to \infty$. 

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Remark 4.4 By Proposition 4.1, [36], we obtain the following result for the Ornstein-Uhlenbeck process given in Lemma 4.3:

$$z_{\alpha} \in L^q(a, b; X),$$  \hspace{1cm} (4.7)

where $q \in [1, \infty]$.

4.3 Metric Dynamical System

Since $z_{\alpha}(t)$ is a Gaussian random vector, by the Burkholder inequality (see [51]), for each $p \geq 2$, there exists a constant $C_p > 0$ such that

$$\mathbb{E}[\|z_{\alpha}(t)\|_X^p] \leq C_p (\mathbb{E}[\|z_{\alpha}(t)\|_X^2])^{p/2}. \hspace{1cm} (4.8)$$

In particular, for $p = 4$, we obtain

$$\mathbb{E}[\|z_{\alpha}(t)\|_X^4] \leq C (\mathbb{E}[\|z_{\alpha}(t)\|_X^2])^2. \hspace{1cm} (4.9)$$

Moreover,

$$\mathbb{E}[\|z_{\alpha}(t)\|_X^4] \to 0 \text{ as } \alpha \to \infty. \hspace{1cm} (4.10)$$

By Lemma 4.3, the process $z_{\alpha}(t)$, $t \in \mathbb{R}$ is an $X$-valued stationary and ergodic. Hence, by the strong law of large numbers (see [19]), we have

$$\lim_{t \to \infty} \frac{1}{t} \int_{-t}^{0} \|z_{\alpha}(s)\|_X^4 ds = \mathbb{E}[\|z_{\alpha}(0)\|_X^4]. \hspace{1cm} \mathbb{P}\text{-a.s. on } C_{1/2}(\mathbb{R}; X). \hspace{1cm} (4.9)$$

Therefore by Lemma 4.3, we find $\alpha_0$ such that

$$\mathbb{E}[\|z_{\alpha}(0)\|_X^4] \leq \frac{\mu^4 \lambda_1}{432}, \hspace{1cm} (4.10)$$

for all $\alpha \geq \alpha_0$, where $\lambda_1$ is the Poincaré-Wirtinger constant.

Let us denote

$$\Omega_{\alpha}(\xi, E) = \{\omega \in \Omega(\xi, E) : \text{the equality (4.9) holds true}\}.$$ 

Therefore, we fix $\xi \in (\delta, 1/2)$ and set

$$\Omega := \hat{\Omega}(\xi, E) = \bigcap_{n=0}^{\infty} \Omega_n(\xi, E).$$

Lemma 4.5 The quadruple $(\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\theta})$ is a metric DS, where $\hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\theta}$ are respectively the natural restrictions of $\mathcal{F}, \mathbb{P},$ and $\theta$ to $\Omega$. For each $\omega \in \Omega$, the limit in (4.9) exists.

The following result is the important consequence of (4.9) and (4.10).

Corollary 4.6 For each $\omega \in \Omega$, there exists $t_0(t_0(\omega)) \geq 0$ such that

$$\frac{216}{\mu^3} \int_{-t}^{0} \|z_{\alpha}(s)\|_X^4 ds \leq \frac{216}{\mu^3} \int_{-t}^{0} \|z_{\alpha}(s)\|_X^4 ds \leq \frac{\mu \lambda_1 t}{2}, \hspace{1cm} t \geq t_0. \hspace{1cm} (4.11)$$
4.4 Random Dynamical System

Let us recall that Assumption 4.1 is satisfied and that \( \delta \) has the property stated there. We also take fixed \( \mu, \beta > 0 \) and some parameter \( \alpha \geq 0 \). We also fix \( \xi \in (\delta, 1/2) \).

Denote by \( v^\varepsilon_\alpha(t) = u_\varepsilon(t) - \varepsilon z_\alpha(\omega)(t) \), then \( v^\varepsilon_\alpha(t) \) satisfies the following abstract random dynamical system (for convenience, we write \( v^\varepsilon_\alpha(t) = v_\varepsilon(t) \) and \( z_\alpha(\omega)(t) = z(t) \)):

\[
\begin{align*}
\begin{cases}
\frac{dv_\varepsilon(t)}{dt} &= -\mu A v_\varepsilon(t) - B(v_\varepsilon(t) + \varepsilon z(t)) - \beta C(v_\varepsilon(t) + \varepsilon z(t)) + \varepsilon a z(t) + f, \\
v_\varepsilon(0) &= x - \varepsilon z(0).
\end{cases}
\end{align*}
\]

(4.12)

Because \( z \in C_{1/2}(\mathbb{R}, X) \), \( z(0) \) is a well defined element of \( \mathbb{H} \). In what follows, we provide the definition of weak solution of the system (4.12).

**Definition 4.7** Assume that \( x \in \mathbb{H} \) and \( f \in \mathbb{V} \). Let \( T > 0 \) be any fixed time, the function \( v_\varepsilon(\cdot) \) is called a weak solution of the problem (4.12) on time interval \([0, T]\), if

\[
v_\varepsilon \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \cap L^{r+1}(0, T; \tilde{\mathbb{L}}^{r+1})
\]

and it satisfies

(i) for any \( \psi \in \mathbb{V} \cap \tilde{\mathbb{L}}^{r+1} \),

\[
\begin{align*}
\left\langle \frac{dv_\varepsilon(t)}{dt}, \psi \right\rangle &= -\left\langle \mu A v_\varepsilon(t) + B(v_\varepsilon(t) + \varepsilon z(t)) + \beta C(v_\varepsilon(t) + \varepsilon z(t)), \psi \right\rangle + \langle f + \varepsilon z(t), \psi \rangle.
\end{align*}
\]

(ii) \( v_\varepsilon(\cdot) \) satisfies the following initial data

\[
v_\varepsilon(0) = x - \varepsilon z(0).
\]

The system (4.12) is a pathwise deterministic system. By a standard Galerkin method (see [45,46]), one can obtain that if \( z \in L^\infty(0, T; \mathbb{V}) \cap L^{r+1}(0, T; D(A)) \), then for all \( t > 0 \), and for every \( x \in \mathbb{H} \) and \( \omega \in \Omega, (4.12) \) has a unique solution in the sense of Definition 4.7.

In addition, for \( x \in \mathbb{V} \) and \( f \in \mathbb{H} \), there exists a unique strong solution \( v_\varepsilon(\cdot) \) to the system (4.12) satisfying the following regularity:

\[
v_\varepsilon \in C([0, T]; \mathbb{V}) \cap L^2(0, T; D(A)) \cap L^{r+1}(0, T; \tilde{\mathbb{L}}^{3(r+1)}).
\]

In the next few results, we assume that the forcing \( f \) depends on \( t \).

**Theorem 4.8** For \( r \geq 3 \) \( (2\beta\mu \geq 1 \text{ for } r = 3) \), assume that, for some \( T > 0 \) fixed, \( x_n \rightarrow x \) in \( \mathbb{H} \),

\[
z_n \rightarrow z \text{ in } L^\infty(0, T; \mathbb{V}) \cap L^{r+1}(0, T; D(A)) \text{ and } f_n \rightarrow f \text{ in } L^2(0, T; \mathbb{H}).
\]

Let us denote by \( v_\varepsilon(t, z)x \) for the solution of the problem (4.12) and by \( v_\varepsilon(t, z_n)x_n \) for the solution of the problem (4.12) with \( z, f, x \) being replaced by \( z_n, f_n, x_n \). Then

\[
v_\varepsilon(\cdot, z_n)x_n \rightarrow v_\varepsilon(\cdot, z)x \text{ in } C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}).
\]

In particular, \( v_\varepsilon(T, z_n)x_n \rightarrow v_\varepsilon(T, z)x \) in \( \mathbb{H} \).

**Proof** In order to simplify the proof, we introduce the following notations for \( t \in [0, T] \):

\[
\begin{align*}
v_n(t) &= v_\varepsilon(t, z_n)x_n, & v(t) &= v_\varepsilon(t, z)x, & y_n(t) &= v_\varepsilon(t, z_n)x_n - v_\varepsilon(t, z)x, \\
\dot{z}_n(t) &= z_n(t) - z(t), & \dot{f}_n(t) &= f_n(t) - f(t).
\end{align*}
\]
It is easy to see that \( y_n(\cdot) \) solves the following initial value problem:

\[
\begin{aligned}
\frac{\text{d}y_n(t)}{\text{d}t} &= -\mu A y_n(t) - B(v(t) + \varepsilon z_n(t)) + B(v(t) + \varepsilon z(t)) \\
&\quad - \beta C(v(t) + \varepsilon z_n(t)) + \beta C(v(t) + \varepsilon z(t)) + \varepsilon \alpha \hat{z}_n(t) + \hat{f}_n(t),
\end{aligned}
\]  

\( (4.14) \)

Taking the inner product with \( y_n(\cdot) \) to the first equation of \( (4.14) \), we get

\[
\begin{aligned}
\frac{1}{2} \frac{\text{d}}{\text{d}t} \| y_n(t) \|_{E_1}^2 &\leq -\mu \| y_n(t) \|_{E_1}^2 - \langle B(v(t) + \varepsilon z_n(t)) - B(v(t) + \varepsilon z(t)), y_n(t) \rangle \\
&\quad - \beta \| C(v(t) + \varepsilon z_n(t)) - C(v(t) + \varepsilon z(t)), y_n(t) \rangle \\
&\quad + \varepsilon \alpha \| \hat{z}_n(t), y_n(t) \rangle + \langle \hat{f}_n(t), y_n(t) \rangle,
\end{aligned}
\]  

\( (4.15) \)

for a.e. \( t \in [0, T] \). We consider the cases \( r > 3 \) and \( r = 3 \) separately.

**Case 1:** \( r > 3 \). Using \( 0 < \varepsilon \leq 1 \) and \( (2.8) \), we have

\[
\begin{aligned}
\| (B(v(t) + \varepsilon z_n(t), \varepsilon \hat{z}_n(t)) \| \leq &\| v_n + \varepsilon z_n \|_{E_{r+1}} \| v_n + \varepsilon z_n \|_{E_1} \| \hat{z}_n \|_V \\
&\leq C \| v_n + \varepsilon z_n \|_{E_{r+1}} \| v_n + \varepsilon z_n \|_{E_1} \| \hat{z}_n \|_V, \\
\| (B(v + \varepsilon z), \varepsilon \hat{z}_n(t)) \| \leq &\| v + \varepsilon z \|_{E_{r+1}} \| v + \varepsilon z \|_{E_1} \| \hat{z}_n \|_V \\
&\leq C \| v + \varepsilon z \|_{E_{r+1}} \| v + \varepsilon z \|_{E_1} \| \hat{z}_n \|_V.
\end{aligned}
\]  

\( (4.16) \) \( (4.17) \)

Using \( 0 < \varepsilon \leq 1 \), \( (2.10) \) and \( (2.13) \), we obtain

\[
\begin{aligned}
\| (B(v(t) + \varepsilon z_n(t) - B(v + \varepsilon z), (v_n + \varepsilon z_n(t) - (v + \varepsilon z)) \| &\leq \frac{\mu}{4} \| (v_n + \varepsilon z_n(t) - (v + \varepsilon z) \|_{E_1}^2 + \frac{\beta}{8} \| (v + \varepsilon z \|_{E_1}^{r-1} \| (v_n + \varepsilon z_n(t) - (v + \varepsilon z)) \|_{E_1}^2 \\
&\quad + \eta_1 \| (v_n + \varepsilon z_n(t) - (v + \varepsilon z) \|_{E_1}^2 \\
&\leq \frac{\mu}{2} \| y_n \|_{E_1}^2 + \frac{\mu}{2} \| \hat{z}_n \|_{E_1}^2 + \frac{\beta}{8} \| (v + \varepsilon z \|_{E_1}^{r-1} \| (v_n + \varepsilon z_n(t) - (v + \varepsilon z)) \|_{E_1}^2 \\
&\quad + 2\eta_1 \| y_n \|_{E_1}^2 + 2\eta_1 \| \hat{z}_n \|_{E_1}^2 \\
&\leq \frac{\mu}{2} \| y_n \|_{E_1}^2 + C \| \hat{z}_n \|_{E_1}^2 + \frac{\beta}{8} \| (v + \varepsilon z \|_{E_1}^{r-1} \| (v_n + \varepsilon z_n(t) - (v + \varepsilon z)) \|_{E_1}^2 \\
&\quad + 2\eta_1 \| y_n \|_{E_1}^2.
\end{aligned}
\]  

\( (4.18) \)
where $\eta_1 = \frac{r-3}{\mu(r-1)} \left( \frac{16}{\beta \mu(r-1)} \right)^{\frac{2}{r-3}}$. Using Taylor’s formula (Theorem 7.9.1, [12]), we find

$$
\beta \|C(v_n + \varepsilon z_n) - C(v + \varepsilon z), \varepsilon \hat{z}_n)\ |
= \beta \left( \int_0^1 [C'(\theta (v_n + \varepsilon z_n)) + (1 - \theta)(v + \varepsilon z)((v_n + \varepsilon z_n) - (v + \varepsilon z))]d\theta, \varepsilon \hat{z}_n) \right)
\leq r \beta 2^{r-2} \|(|v_n + \varepsilon z_n)|^{r-1} ((v_n + \varepsilon z_n) - (v + \varepsilon z))\|y_1 \|v_n + \varepsilon z_n\|^{r-1} \|\hat{z}_n\|\|y_1 + \varepsilon z\|^{r-1} \|\hat{z}_n\|^{r-1}
+ r \beta 2^{r-2} \|(|v_n + \varepsilon z_n)|^{r-1} ((v_n + \varepsilon z_n) - (v + \varepsilon z))\|y_1 \|v_n + \varepsilon z_n\|^{r-1} \|\hat{z}_n\|^{r-1}
\leq \frac{\beta}{4} \|(|v_n + \varepsilon z_n)|^{r-1} ((v_n + \varepsilon z_n) - (v + \varepsilon z))\|y_1 \|v_n + \varepsilon z_n\|^{r-1} \|\hat{z}_n\|^{r-1}
+ \frac{\beta}{8} \|(|v_n + \varepsilon z_n)|^{r-1} ((v_n + \varepsilon z_n) - (v + \varepsilon z))\|y_1 \|v_n + \varepsilon z_n\|^{r-1} \|\hat{z}_n\|^{r-1}. \tag{4.19}
$$

Making use of (2.16), we get

$$
-\beta \|C(v_n + \varepsilon z_n) - C(v + \varepsilon z), (v_n + \varepsilon z_n) - (v + \varepsilon z)\|
\leq -\frac{\beta}{2} \|v_n + \varepsilon z_n|^{r-1} ((v_n + \varepsilon z_n) - (v + \varepsilon z))\|y_1 \|v_n + \varepsilon z_n\|^{r-1} \|\hat{z}_n\|^{r-1}
- \frac{\beta}{2} \|v + \varepsilon z|^{r-1} ((v_n + \varepsilon z_n) - (v + \varepsilon z))\|y_1 \|v_n + \varepsilon z_n\|^{r-1} \|\hat{z}_n\|^{r-1}. \tag{4.20}
$$

Using Hölder’s and Young’s inequalities, we have

$$
\varepsilon \alpha |(\hat{z}_n, y_n)| \leq \alpha \|y_n\|_y \|\hat{z}_n\|_y \leq \frac{1}{2} \|y_n\|_y^2 + \frac{\alpha^2}{2} \|\hat{z}_n\|_y^2
\leq \frac{1}{2} \|y_n\|_y^2 + \frac{\alpha^2 C}{2} \|\hat{z}_n\|_y^2, \tag{4.21}
$$

$$
|(\hat{f}_n, y_n)| \leq \|y_n\|_y \|\hat{f}_n\|_y \leq \frac{1}{2} \|y_n\|_y^2 + \frac{1}{2} \|\hat{f}_n\|_y^2. \tag{4.22}
$$

Combining (4.16)–(4.22), substituting it in (4.15), and then using (2.17), we deduce that

$$
\frac{d}{dt} \|y_n(t)\|_y^2 + \mu \|y_n(t)\|_y^2 + \frac{\beta}{2r-1} \|y_n(t) + \varepsilon z_n(t)\|^{r-1}_y
\leq C \left\{ \|v(t) + \varepsilon z_n(t)\|^{r-1}_y \|v(t) + \varepsilon z_n(t)\|^{r-1}_y \|\hat{z}_n(t)\|_y + \|v(t) + \varepsilon z(t)\|^{r-1}_y + \|v(t) + \varepsilon z(t)\|^{r-1}_y \right\}
\|\hat{z}_n(t)\|_y + C \|\hat{z}_n(t)\|_y^2 + \tilde{\eta}_1 \|y_n(t)\|_y^2
+ C \left\{ \|v(t) + \varepsilon z_n(t)\|^{r-1}_y + \|v(t) + \varepsilon z(t)\|^{r-1}_y \right\} \|\hat{z}_n(t)\|_y^2 + \|\hat{f}_n(t)\|_y^2. \tag{4.23}
$$

where $\tilde{\eta}_1 = 2(2\eta_1 + 1)$, for a.e. $t \in [0, T]$. By integrating the above inequality from 0 to $t$, for $t \in [0, T]$, we get

$$
\|y_n(t)\|_y^2 + \mu \int_0^t \|y_n(s)\|_y^2 ds + \frac{\beta}{2r-1} \int_0^t \|y_n(s) + \varepsilon z_n(s)\|^{r-1}_y ds
\leq \|y_n(0)\|_y^2 + \int_0^t \tilde{\eta}_1 \|y_n(s)\|_y^2 ds + \int_0^t K_n(s) ds, \tag{4.24}
$$
for all \( t \in [0, T] \), where
\[
K_n = C \left\{ \| v_n + \varepsilon z_n \|_{L^{r+1}} \| v_n + \varepsilon z_n \|_{\frac{r-3}{r+1}} + \| v + \varepsilon z \|_{L^{r+1}} + \| v + \varepsilon z \|_{\frac{r-3}{r+1}} \right\} \| \hat{z}_n \|_V
\]
\[
+ C \left\{ \| v_n + \varepsilon z_n \|_{L^{r+1}} \| v_n + \varepsilon z_n \|_{\frac{r-3}{r+1}} \right\} \| \hat{z}_n \|_V^2 + C \| \hat{z}_n \|_V^2 + \| \hat{f}_n \|_{\tilde{H}^1}^2.
\]
Then by the classical Gronwall inequality, we arrive at
\[
\| y_n(t) \|_{\tilde{H}^1}^2 \leq \left( \| y_n(0) \|_{\tilde{H}^1}^2 + \int_0^T K_n(t) \, dt \right) e^{\tilde{\eta} T},
\]
for all \( t \in [0, T] \). On the other hand, we have
\[
\int_0^T K_n(t) \, dt \leq 0 + \int_0^T \left[ C \left\{ \| v_n + \varepsilon z_n \|_{L^{r+1}(0, T; \tilde{L}^{r+1})} \| v_n + \varepsilon z_n \|_{\frac{r-3}{r+1}} \right\} \| \hat{z}_n \|_V
\]
\[
+ \| v + \varepsilon z \|_{L^{r+1}(0, T; \tilde{L}^{r+1})} + \| v + \varepsilon z \|_{L^{r+1}(0, T; \tilde{L}^{r+1})} \right\} \| \hat{z}_n \|_V^2 + \| \hat{f}_n \|_{L^2(0, T; \tilde{H}^1)}^2.
\]
Since, \( z_n \to z \) in \( L^{r+1}(0, T; D(A)) \) and by the Sobolev inequality, we find
\[
\| \hat{z}_n \|_{L^{r+1}(0, T; \tilde{L}^{r+1})} \leq C \| \hat{z}_n \|_{L^{r+1}(0, T; D(A))},
\]
which implies that \( z_n \to z \) in \( L^{r+1}(0, T; \tilde{L}^{r+1}) \). Therefore, \( \int_0^T K_n(t) \, dt \to 0 \) as \( n \to \infty \).

Since, \( \| y_n(0) \|_{\tilde{H}^1} = \| x_n - x \|_{\tilde{H}^1} \to 0 \) and \( \int_0^T K_n(s) \, ds \to 0 \) as \( n \to \infty \), by (4.25), we infer that \( \| y_n(t) \|_{\tilde{H}^1} \to 0 \) as \( n \to \infty \) uniformly in \( t \in [0, T] \). In other words,
\[
v_x(\cdot, z_n)x_n \to v_x(\cdot, z)x \quad \text{in} \quad C([0, T]; \tilde{H}) \quad \text{as} \quad n \to \infty.
\]
From the inequality (4.24), we also obtain
\[
\mu \int_0^T \| y_n(t) \|_V^2 \, dt + \frac{\beta}{2^{r-1}} \int_0^T \| y_n(t) \|_{L^{r+1}}^2 \, dt
\]
\[
\leq \mu \int_0^T \| y_n(t) \|_V^2 \, dt + \frac{\beta}{2^{r-1}} \int_0^T \| y_n(t) + \varepsilon z_n(t) \|_{L^{r+1}}^2 \, dt + \frac{\beta}{2^{r-1}} \int_0^T \| \hat{z}_n(t) \|_{L^{r+1}}^2 \, dt
\]
\[
\leq \| y_n(0) \|_{\tilde{H}^1}^2 + \tilde{\eta} T \sup_{t \in [0, T]} \| y_n(t) \|_{\tilde{H}^1}^2 + \int_0^T K_n(t) \, dt + \frac{\beta}{2^{r-1}} \int_0^T \| \hat{z}_n(t) \|_{L^{r+1}}^2 \, dt.
\]
Hence, \( \int_0^T \| y_n(t) \|^2 \, dt, \int_0^T \| y_n(t) \|^{r+1}_{L^{r+1}} \, dt \to 0 \) as \( n \to \infty \) and therefore,

\[
v_z(\cdot, z_n) x_n \to v_z(\cdot, z) x \text{ in } L^2(0, T; V) \cap L^{r+1}(0, T; \tilde{L}^{r+1}) \text{ as } n \to \infty,
\]

which completes the proof for \( r > 3 \).

**Case II:** \( r = 3 \) and \( 2\mu \geq 1 \). By Hölder’s inequality, (2.5) and (2.1), we find

\[
\begin{align*}
|\langle B(v_n + \varepsilon z_n), \varepsilon \hat{z}_n \rangle| & \leq \| v_n + \varepsilon z_n \|^2_{L^2} \| \hat{z}_n \|_V \leq C \| v_n + \varepsilon z_n \|^2_V \| \hat{z}_n \|_V, \\
|\langle B(v + \varepsilon z), \varepsilon \hat{z}_n \rangle| & \leq \| v + \varepsilon z \|^2_{L^2} \| \hat{z}_n \|_V \leq C \| v + \varepsilon z \|^2_V \| \hat{z}_n \|_V.
\end{align*}
\]

(4.26), (4.27)

Using (2.10) and Hölder’s inequality, we get

\[
\begin{align*}
& \| (B(v_n + \varepsilon z_n) - B(v + \varepsilon z), (v_n + \varepsilon z_n) - (v + \varepsilon z)) \| \\
& \leq \| (v_n + \varepsilon z_n) - (v + \varepsilon z) \|^2_V + \frac{\beta}{2} \| v + \varepsilon z \|^2_H \| \hat{z}_n \|_V \\
& \leq \frac{\beta}{2} \| v \|^2_V + \frac{\beta}{2} \| \hat{z}_n \|^2_V + \frac{\beta}{2} \| v + \varepsilon z \|^2_H \| \hat{z}_n \|_V \\
& \leq \frac{\beta}{2} \| v \|^2_V + \frac{\beta}{2} \| \hat{z}_n \|^2_V + \frac{\beta}{2} \| v + \varepsilon z \|^2_H \| \hat{z}_n \|_V.
\end{align*}
\]

(4.28)

Using Hölder’s inequality, (2.5) and (2.1), we have

\[
\begin{align*}
& |\langle C(v_n + \varepsilon z_n), \varepsilon \hat{z}_n \rangle| \\
& \leq \left\{ \| v_n + \varepsilon z_n \|^3_{L^3} + \| v + \varepsilon z \|^3_{L^3} \right\} \| \hat{z}_n \|_{L^4} \\
& \leq C \left\{ \| v_n \|^3_{L^3} + \| v \|^3_{L^3} + \| z_n \|^3_{L^3} + \| z \|^3_{L^3} \right\} \| \hat{z}_n \|_V.
\end{align*}
\]

(4.29)

By (2.16), we obtain

\[
\begin{align*}
- \beta |\langle C(v_n + \varepsilon z_n) - C(v + \varepsilon z), (v_n + \varepsilon z_n) - (v + \varepsilon z) \rangle| \\
& \leq - \beta \left\{ \| v_n + \varepsilon z_n \|^2_{L^3} + \| v + \varepsilon z \|^2_{L^3} \right\} \\
& - \beta \left\{ \| v + \varepsilon z \|^2_{L^3} + \| v_n + \varepsilon z_n \|^2_{L^3} \right\}.
\end{align*}
\]

(4.30)

Using (4.26)–(4.30) along with (4.21)–(4.22), (2.17) in (4.15), we deduce that

\[
\begin{align*}
\frac{d}{dt} \| y_n(t) \|^2_H + 2\left( \mu - \frac{1}{2\beta} \right) \| y_n(t) \|^2_V \\
\leq C \left\{ \| v_n(t) + \varepsilon z_n(t) \|^2_V + \| v(t) + \varepsilon z(t) \|^2_V + \| v_n(t) \|_V + \| v(t) \|_V \\
+ \| v_n(t) \|^3_{L^3} + \| v(t) \|^3_{L^3} + \| z_n(t) \|^3_{L^3} + \| z(t) \|^3_{L^3} \right\} \| \hat{z}_n(t) \|_V \\
+ 2\| y_n(t) \|^2_H + C \| \hat{z}_n(t) \|^2_V + \| \hat{f}_n(t) \|^2_H.
\end{align*}
\]

(4.31)
for a.e. \( t \in [0, T] \). Integrating the above inequality from 0 to \( t \), \( t \in [0, T] \), we get
\[
\| y_n(t) \|_{\mathbb{H}}^2 + 2 \left( \mu - \frac{1}{2\beta} \right) \int_0^t \| y_n(s) \|_{\mathbb{V}}^2 \, ds \\
\leq \| y_n(0) \|_{\mathbb{H}}^2 + 2 \int_0^t \| y_n(s) \|_{\mathbb{H}}^2 \, ds + \int_0^t \tilde{K}_n(s) \, ds,
\] (4.32)
for all \( t \in [0, T] \), where
\[
\tilde{K}_n = C \left\{ \| v_n + \varepsilon z_n \|_{\mathbb{V}} + \| v + \varepsilon z \|_{\mathbb{V}} + \| v_n \|_{\mathbb{V}} + \| v \|_{\mathbb{V}} + \| v_n \|_{L^4}^3 + \| v \|_{L^4}^3 \\
+ \| z_n \|_{L^3}^3 + \| z \|_{L^3}^3 \right\} \| \hat{z}_n \|_{\mathbb{V}} + C \| \hat{z}_n \|_{\mathbb{H}}^2 + \| \hat{f}_n \|_{\mathbb{H}}^2.
\]

Then, for \( 2\beta\mu \geq 1 \), by the classical Gronwall inequality, we obtain
\[
\| y_n(t) \|_{\mathbb{H}}^2 \leq \left( \| y_n(0) \|_{\mathbb{H}}^2 + \int_0^T \tilde{K}_n(t) \, dt \right) e^{2T},
\] (4.33)
for all \( t \in [0, T] \). On the other hand, we have
\[
\int_0^T \tilde{K}_n(t) \, dt \to 0 \text{ as } n \to \infty. \quad \text{Since, } \| y_n(0) \|_{\mathbb{H}} = \| x_n - x \|_{\mathbb{H}} \to 0 \text{ and }
\int_0^T \tilde{K}_n(t) \, dt \to 0 \text{ as } n \to \infty\), by (4.33) we infer that \( \| y_n(t) \|_{\mathbb{H}} \to 0 \text{ as } n \to \infty \) uniformly in \( t \in [0, T] \).

In other words,
\[
v_x(\cdot, z_n)x_n \to v_x(\cdot, z)x \text{ in } C([0, T]; \mathbb{H}).
\]

From inequality (4.32), we also have
\[
2 \left( \mu - \frac{1}{2\beta} \right) \int_0^T \| y_n(t) \|_{\mathbb{V}}^2 \, dt \leq \| y_n(0) \|_{\mathbb{H}}^2 + 2T \sup_{t \in [0, T]} \| y_n(t) \|_{\mathbb{H}}^2 + \int_0^T \tilde{K}_n(t) \, dt.
\]

Thus, for \( 2\beta\mu > 1 \), it is immediate that \( \int_0^T \| y_n(t) \|_{\mathbb{V}}^2 \, dt \to 0 \text{ as } n \to \infty \) and therefore, we arrive at
\[
v_x(\cdot, z_n)x_n \to v_x(\cdot, z)x \text{ in } L^2(0, T; \mathbb{V}),
\]
which completes the proof for the case \( r = 3 \).

**Theorem 4.9** For \( 3 \leq r \leq 5 \ (2\beta\mu \geq 1 \text{ for } r = 3) \), assume that, for some \( T > 0 \) fixed, \( x_n \to x \text{ in } \mathbb{V} \),
\[
z_n \to z \text{ in } L^\infty(0, T; \mathbb{V}) \cap L^{r+1}(0, T; D(A)), \quad f_n \to f \text{ in } L^2(0, T; \mathbb{H}).
\]
Let us denote by $v_e(t, z)x$, the solution of the problem (4.12) and by $v_e(t, z_n)x_n$, the solution of the problem (4.12) with $z, f, x$ being replaced by $z_n, f_n, x_n$, respectively. Then

$$v_e(\cdot, z_n)x_n \to v_e(\cdot, z)x \text{ in } C([0, T]; V) \cap L^2(0, T; D(A)).$$

In particular, $v_e(T, z_n)x_n \to v_e(T, z)x$ in $V$.

**Proof** Taking the inner product with $A_y(t)$ to the first equation in (4.14), and then using (2.3) and (2.4), we get

$$\frac{1}{2} \frac{d}{dt} \| y_n(t) \|_V^2 = -\mu \| Ay(t) \|_H^2 - b(y_n(t), v_n(t) + \varepsilon z_n(t), Ay(t))$$

$$- \varepsilon b(\hat{z}_n(t), v_n(t) + \varepsilon z_n(t), Ay(t)) - b(v(t) + \varepsilon z(t), y_n(t), Ay(t))$$

$$- \beta (C(v(t) + \varepsilon z(t)) - C(v(t) + \varepsilon z(t)), Ay(t))$$

$$+ \varepsilon \alpha (\hat{z}_n(t), Ay(t)) + (\hat{f}(t), Ay(t)),$$

for a.e. $t \in [0, T]$. Now, using (2.1), (2.6), $0 < \varepsilon \leq 1$, Hölder’s and Young’s inequalities, we have

$$|b(y_n, v_n + \varepsilon z_n, Ay_n)| \leq \| y_n \|_{L^\infty} \| v_n + \varepsilon z_n \|_V \| Ay_n \|_H$$

$$\leq C \| y_n \|_V^{1/2} \| Ay_n \|_H^{1/2} \| v_n + \varepsilon z_n \|_V \| Ay_n \|_H$$

$$\leq \frac{\mu}{14} \| Ay_n \|_H^2 + C \| y_n \|_V^2 \| v_n + \varepsilon z_n \|_V^2,$$

(4.35)

$$|\varepsilon b(\hat{z}_n, v_n + \varepsilon z_n, Ay_n)| \leq C \| \hat{z}_n \|_V^{1/2} \| A\hat{z}_n \|_H^{1/2} \| v_n + \varepsilon z_n \|_V \| Ay_n \|_H$$

$$\leq \frac{\mu}{14} \| Ay_n \|_H^2 + C \| A\hat{z}_n \|_H^2 \| v_n + \varepsilon z_n \|_V^2,$$

(4.36)

$$|b(v + \varepsilon z, y_n, Ay_n)| \leq C \| v + \varepsilon z \|_V^{1/2} \| v + \varepsilon z \|^{D(A)}_H \| y_n \|_V \| Ay_n \|_H$$

$$\leq \frac{\mu}{14} \| Ay_n \|_H^2 + C \| v + \varepsilon z \|^{D(A)}_H \| y_n \|_V^2,$$

(4.37)

By using Taylor’s formula, Hölder’s inequality, Sobolev’s embedding, Young’s and interpolation (see (2.2)), inequalities, we find, for $3 \leq r \leq 5$,

$$\beta |(C(v_n + \varepsilon z_n) - C(v + \varepsilon z), Ay_n)|$$

$$\leq C \left( \| v_n + \varepsilon z_n \|_V^{1/2} \| v + \varepsilon z \|_V \| Ay_n \|_H \right) \| y_n + \varepsilon \hat{z}_n \|_V \| Ay_n \|_H$$

$$\leq C \left( \| v_n + \varepsilon z_n \|_V^{1/2} \| v + \varepsilon z \|_V \| Ay_n \|_H \right) \| y_n + \varepsilon \hat{z}_n \|_V \| Ay_n \|_H$$

$$\leq C \left[ \| v_n + \varepsilon z_n \|_V^{1/2} \left( 1 + \| v_n + \varepsilon z_n \|_V \right) \right] \| v + \varepsilon z \|_V^{1/2} \left[ 1 + \| v + \varepsilon z \|_V \right]$$

$$\leq \frac{\mu}{14} \| Ay_n \|_H^2 + C \left[ \| v_n + \varepsilon z_n \|_V^{1/2} \left( 1 + \| v_n + \varepsilon z_n \|_V \right) \right] \| v + \varepsilon z \|_V^{1/2},$$

(4.38)
Lemma 4.11

Let \( \varepsilon \in (0, 1) \) be a constant. Then for almost every \( t \in [0, T] \), we have

\[
\begin{align*}
&\|y_n(t)\|_{\mathbb{V}}^2 + \mu \|y_n(t)\|_{\mathbb{H}}^2 \\
\leq &\quad C \left( \|v_n(t) + \varepsilon z_n(t)\|_{\mathbb{V}}^2 + \|v(t) + \varepsilon z(t)\|_{D(A)}^2 \\
&\quad + \|v_n(t) + \varepsilon z_n(t)\|_{\mathbb{V}}^{r+1} \left( 1 + \|y_n(t)\|_{\mathbb{V}}^{r+1} \right) \\
&\quad + \|v(t) + \varepsilon z(t)\|_{\mathbb{V}}^{r+1} \left( 1 + \|y(t)\|_{\mathbb{V}}^{r+1} \right) \right) \|y_n(t)\|_{\mathbb{V}}^2 \\
&\quad + C \left( \|v(t) + \varepsilon z(t)\|_{D(A)}^2 + \|v_n(t) + \varepsilon z_n(t)\|_{\mathbb{V}}^{r+1} \left( 1 + \|y_n(t)\|_{\mathbb{V}}^{r+1} \right) \\
&\quad + \|v(t) + \varepsilon z(t)\|_{\mathbb{V}}^{r+1} \left( 1 + \|y(t)\|_{\mathbb{V}}^{r+1} \right) \right) \|y_n(t)\|_{\mathbb{V}}^2 \\
&\quad + C \|A\hat{z}_n(t)\|_{\mathbb{H}}^2 \|v_n(t) + \varepsilon z_n(t)\|_{\mathbb{V}}^2 + C \|\hat{f}_n(t)\|_{\mathbb{H}}^2,
\end{align*}
\]

for almost every \( t \in [0, T] \). Now arguing similarly as in the proof of Theorem 4.8 and using the fact that

\[
v_n, v \in C([0, T]; \mathbb{V}) \cap L^2(0, T; D(A)) \cap L^{r+1}(0, T; \mathbb{H}^{3(r+1)}),
\]

\( x_n \to x \) in \( \mathbb{V} \), \( z_n \to z \) in \( L^\infty(0, T; \mathbb{V}) \cap L^{r+1}(0, T; D(A)) \) and \( f_n \to f \) in \( L^2(0, T; \mathbb{H}) \), one can complete the proof. \( \square \)

**Definition 4.10** We define a map \( \varphi^\alpha_e : \mathbb{R}^+ \times \Omega \times \mathbb{H} \to \mathbb{H} \) by

\[
(t, \omega, x) \mapsto \varphi_e(t) + \varepsilon z_\alpha(\omega)(t) \in \mathbb{H}.
\]

**Lemma 4.11** If \( \alpha_1, \alpha_2 \geq 0 \), then \( \varphi^\alpha_{\varepsilon_1} = \varphi^\alpha_{\varepsilon_2} \).

**Proof** Using inequality (2.18) (for \( r > 3 \)) and inequality (2.20) (for \( r = 3 \)), one can complete the proof similarly as in Proposition 4.11, [36]. \( \square \)

Invoking Lemma 4.11, we denote \( \varphi^\alpha_e \) by \( \varphi_e \).

**Lemma 4.12** \( (\varphi_e, \theta) \) is an RDS.

**Proof** All the properties with the exception of the cocycle one of an RDS follow from Theorem 4.8. Hence we only need to show that for any \( x \in \mathbb{H} \),

\[
\varphi_e(t + s, \omega)x = \varphi_e(t, \theta_s \omega) \varphi_e(s, \omega)x, \quad t, s \in \mathbb{R}^+.
\]

Remaining proof is similar to that of Theorem 6.15, [10] and hence we omit it here. \( \square \)
If we define, for \( u_s \in \mathbb{H}, \omega \in \Omega, \) and \( t \geq s, \)
\[
 u_s(t, \omega; s, u_s) := \varphi_e(t - s; \theta_s \omega) u_s = v_e(t, \omega; s, u_s - \varepsilon(z(s)) + \varepsilon(t),
\]
then for each \( s \in \mathbb{R} \) and each \( u_s \in \mathbb{H}, \) the process \( u(t), t \geq s, \) is a solution to the problem (4.1).

### 4.5 Existence of Random Attractors

In this subsection, we consider the RDS \( \varphi_e \) over the metric \( (\Omega, \mathcal{F}, \mathbb{P}, \hat{\theta}). \)

**Lemma 4.13** For each \( \omega \in \Omega, \)
\[
 \limsup_{t \to -\infty} \|z(\omega)(t)\|_{H}^2 e^{\mu \lambda_1 t + \frac{216}{\mu^3} \int_{-\infty}^{0} \|z(\omega)(s)\|_{V}^{4} ds} = 0.
\]

**Proof** For a proof, see Lemma 4.1 [37].

**Lemma 4.14** For each \( \omega \in \Omega, \)
\[
 \int_{-\infty}^{0} \left( 1 + \|z(t)\|_{\mathbb{H}}^4 + \|A z(t)\|_{H}^{r+1} + \|z(t)\|_{V}^2 \right) e^{\mu \lambda_1 t + \frac{216}{\mu^3} \int_{-\infty}^{0} \|z(\omega)(s)\|_{V}^{4} ds} dt < \infty.
\]

**Proof** For a proof, see Lemma 4.2 [37].

**Definition 4.15** A function \( \kappa : \Omega \to (0, \infty) \) belongs to class \( \hat{\mathcal{R}} \) if and only if
\[
 \limsup_{t \to -\infty} [\kappa(\theta_{-t} \omega)]^2 e^{-\mu \lambda_1 t + \frac{216}{\mu^3} \int_{-\infty}^{0} \|z(\omega)(s)\|_{V}^{4} ds} = 0,
\]
where \( \lambda_1 \) is the first eigenvalue of the Stokes operator \( A. \)

We denote by \( \hat{D} \hat{\mathcal{R}}, \) the class of all closed and bounded random sets \( D \) on \( \mathbb{H} \) such that the radius function \( \Omega \ni \omega \mapsto \kappa(D(\omega)) := \sup\{\|x\|_{\mathbb{H}} : x \in D(\omega)\} \) belongs to the class \( \hat{\mathcal{R}}. \) Also, we denote by \( \hat{\mathcal{R}}, \) the class of all closed and bounded random sets \( D(\omega) \) on \( V \) such that the radius function \( \Omega \ni \omega \mapsto \kappa(D(\omega)) := \sup\{\|x\|_{V} : x \in D(\omega)\} \) belongs to the class \( \hat{\mathcal{R}}. \)

By Corollary 4.6, we infer that the constant functions belong to \( \hat{\mathcal{R}}. \) The class \( \hat{\mathcal{R}} \) is closed with respect to sum, multiplication by a constant and if \( \kappa \in \hat{\mathcal{R}}, \) \( 0 \leq \bar{\kappa} \leq \kappa, \) then \( \bar{\kappa} \in \hat{\mathcal{R}}. \)

**Lemma 4.16** Define functions \( \kappa_i : \Omega \to (0, \infty), i = 1, 2, 3, 4, 5, 6, \) by the following formulae, for \( \omega \in \Omega, \)
\[
 \begin{align*}
 [\kappa_1(\omega)]^2 & := \|z(\omega)(0)\|_{\mathbb{H}}^2, \\
 [\kappa_2(\omega)]^2 & := \sup_{s \leq 0} \|z(\omega)(s)\|_{\mathbb{H}}^2 e^{\mu \lambda_1 s + \frac{216}{\mu^3} \int_{-\infty}^{0} \|z(\omega)(\zeta)\|_{V}^{4} d\zeta}, \\
 [\kappa_3(\omega)]^2 & := \int_{-\infty}^{0} \|z(\omega)(t)\|_{V}^4 e^{\mu \lambda_1 t + \frac{216}{\mu^3} \int_{-\infty}^{0} \|z(\omega)(\zeta)\|_{V}^{4} d\zeta} dt, \\
 [\kappa_4(\omega)]^2 & := \int_{-\infty}^{0} \|A z(\omega)(t)\|_{H}^{r+1} e^{\mu \lambda_1 t + \frac{216}{\mu^3} \int_{-\infty}^{0} \|z(\omega)(\zeta)\|_{V}^{4} d\zeta} dt, \\
 [\kappa_5(\omega)]^2 & := \int_{-\infty}^{0} \|z(\omega)(t)\|_{V}^4 e^{\mu \lambda_1 t + \frac{216}{\mu^3} \int_{-\infty}^{0} \|z(\omega)(\zeta)\|_{V}^{4} d\zeta} dt, \\
 [\kappa_6(\omega)]^2 & := \int_{-\infty}^{0} e^{\mu \lambda_1 t + \frac{216}{\mu^3} \int_{-\infty}^{0} \|z(\omega)(\zeta)\|_{V}^{4} d\zeta} dt.
\end{align*}
\]

Then all these functions belongs to class \( \hat{\mathcal{R}}. \)
Theorem 4.17 Assume that $f \in \mathbb{H}$ and Assumption 4.1 holds. Then there exists a family $\hat{B}_0 = \{B_0(\omega) : \omega \in \Omega\}$ of $\mathcal{D}\mathcal{K}$-random absorbing sets in $\mathbb{H}$ and a family $\hat{B} = \{B(\omega) : \omega \in \Omega\}$ of $\mathcal{D}\mathcal{K}$-random absorbing sets in $\mathbb{V}$ corresponding to the RDS $\varphi_\varepsilon$.

Proof We divide the proof into the following steps.

Step I: Absorbing set in $\mathbb{H}$. Let $D$ be a random set from the class $\mathcal{D}\mathcal{K}$. Let $\kappa_D(\omega)$ be the radius of $D(\omega)$, that is, $\kappa_D(\omega) := \sup\{\|x\|_\mathbb{H} : x \in D(\omega)\}, \omega \in \Omega$.

Let $\omega \in \Omega$ be fixed. For given $s \leq 0$ and $x \in \mathbb{H}$, let $v(\cdot)$ be the solution of (4.12) on time interval $[s, \infty)$ with the initial condition $v_\varepsilon(s) = x - \varepsilon z(s)$. Multiplying the first equation of (4.12) by $v_\varepsilon(\cdot)$ and then integrating the resulting equation over $T^3$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|v_\varepsilon(t)\|_\mathbb{H}^2 = -\mu \|v_\varepsilon(t)\|_\mathbb{V}^2 - b(v_\varepsilon(t) + \varepsilon z(t), v_\varepsilon(t) + \varepsilon z(t), v_\varepsilon(t)) - \beta(C(v_\varepsilon(t) + \varepsilon z(t)), v_\varepsilon(t) + \varepsilon z(t), v_\varepsilon(t))$$

for a.e. $t \in [0, T]$. By using $0 < \varepsilon \leq 1$, Hölder’s and Young’s inequalities, (2.1), (2.5) and Sobolev’s inequality, we have

$$|\varepsilon b(v_\varepsilon, z, v_\varepsilon)| \leq \|v_\varepsilon\|^2_{L^2} |z| |v| \leq 2 \|v_\varepsilon\|^2_{\mathbb{H}} \|v_\varepsilon\|^3 \|z\| \|v\|
\leq \frac{\mu}{8} \|v_\varepsilon\|^2_{\mathbb{V}} + \frac{108}{\mu^3} \|z\|^4_{\mathbb{V}} \|v_\varepsilon\|^2_{\mathbb{H}},$$

$$|\varepsilon^2 b(z, z, v_\varepsilon)| \leq \|z\|^2_{H^2} \|v_\varepsilon\| \|v\| \leq 2 \|z\|^2_{\mathbb{H}} \|v_\varepsilon\|^3 \|v_\varepsilon\| \|v\|
\leq \frac{8}{\mu^{1/2}} \|z\|^2_{H} + \frac{\mu}{8} \|v_\varepsilon\|^2_{\mathbb{V}},$$

$$\beta(C(v_\varepsilon + \varepsilon z), v_\varepsilon + \varepsilon z) = \beta \|v_\varepsilon + \varepsilon z\|_{L^r+1}^{r+1},$$

$$\|\beta(C(v_\varepsilon + \varepsilon z), \varepsilon z)\| \leq \beta \|v_\varepsilon + \varepsilon z\|_{L^r+1}^{r+1} \|z\|_{L^r+1}$$

$$\leq \frac{\beta}{2} \|v_\varepsilon + \varepsilon z\|_{L^r+1}^{r+1} + \frac{\beta(2r)^r}{(r+1)(r+1)} \|z\|_{L^r+1}^{r+1}$$

$$\leq \frac{\beta}{2} \|v_\varepsilon + \varepsilon z\|_{L^r+1}^{r+1} + C \|A z\|_{L^r}^{r+1},$$

$$|\varepsilon \alpha(z, v_\varepsilon)| \leq \alpha \|z\|_{L^2} \|v_\varepsilon\|_{L^2} \leq \frac{\alpha}{\lambda_1} \|z\| \|v\| \|v_\varepsilon\| \|v\|
\leq \frac{\mu}{8} \|v_\varepsilon\|_{\mathbb{V}}^2 + \frac{4\alpha^2}{\mu \lambda_1^2} \|z\|_{\mathbb{V}}^2,$$

$$|(f, v_\varepsilon)| \leq \|f\|_{L^2} \|v_\varepsilon\|_{L^2} \leq \frac{1}{\lambda_1^{1/2}} \|f\|_{L^2} \|v_\varepsilon\|_{L^2} \|v\| \|v_\varepsilon\| \|v\|
\leq \frac{\mu}{8} \|v_\varepsilon\|_{\mathbb{V}}^2 + \frac{4}{\mu \lambda_1^2} \|f\|_{L^2}^2.$$

Proof For a proof, see Proposition 4.4 [37].
Combining the above estimates and substituting it in (4.45), we get
\[
\frac{d}{dt} \| v_x(t) \|^2_{\mathbb{H}} + \mu \lambda_1 \| v_x(t) \|^2_{\mathbb{H}} \\
\leq \frac{216}{\mu^3} \| z(t) \|^4_{V} + \frac{16}{\mu \lambda_1^{1/2}} \| z(t) \|^4_{V} + C \| A z(t) \|_{H^1}^4 \\
+ \frac{8\alpha^2}{\mu \lambda_1^2} \| z(t) \|^2_{V} + \frac{8}{\mu \lambda_1} \| f \|^2_{\mathbb{H}},
\]
for a.e. \( t \in [0, T] \). We infer from the classical Gronwall inequality that
\[
\| v_x(0) \|^2_{\mathbb{H}} \leq 2 \| x \|^2_{\mathbb{H}} e^{\mu \lambda_1 s + \frac{216}{\mu^3} \int_s^0 \| z(\zeta) \|^4_{V} d\zeta} + \frac{16}{\mu \lambda_1^{1/2}} \| z(t) \|^4_{V} + C \| A z(t) \|_{H^1}^4 + \frac{8\alpha^2}{\mu \lambda_1^2} \| z(t) \|^2_{V} + \frac{8}{\mu \lambda_1} \| f \|^2_{\mathbb{H}},
\]
for all \( t \in [0, T] \), since \( \| v_x(s) \|_{\mathbb{H}} \leq \| x \|_{\mathbb{H}} + \| z(s) \|_{\mathbb{H}} \). For \( \omega \in \Omega \), let us set
\[
[\kappa_{11}(\omega)]^2 = 2 + 2 \sup_{s \leq 0} \left\{ \| z(s) \|^2_{\mathbb{H}} e^{\mu \lambda_1 s + \frac{216}{\mu^3} \int_s^0 \| z(\zeta) \|^4_{V} d\zeta} \right\} + \int_{-\infty}^0 \left\{ \frac{16}{\mu \lambda_1^{1/2}} \| z(t) \|^4_{V} \right\} \text{e}^{\mu \lambda_1 t + \frac{216}{\mu^3} \int_t^0 \| z(\zeta) \|^4_{V} d\zeta} dt,
\]
and hence it follows that
\[
[\kappa_{12}(\omega)]^2 = \| z(\omega)(0) \|^2_{\mathbb{H}}.
\]
By Lemmas 4.14 and 4.16, we infer that both \( \kappa_{11} \) and \( \kappa_{12} \) belongs to class \( \mathcal{K} \) and also that \( \kappa_{13} := \kappa_{11} + \kappa_{12} \) belongs to class \( \mathcal{K} \) as well. Therefore the random set \( B_0(\cdot) \) defined by
\[
B_0(\omega) := \{ u \in \mathbb{H} : \| u \|_{\mathbb{H}} \leq \kappa_{13}(\omega) \}
\]
belongs to the family \( \mathcal{D}\mathcal{K} \).

Next, we show that \( B_0 \) absorbs \( D \). Let \( \omega \in \Omega \) be fixed. Since \( \kappa_D(\omega) \in \mathcal{K} \), there exists \( t_D(\omega) \geq 0 \), such that
\[
[\kappa_D(\theta_{-t}\omega)]^2 e^{-\mu \lambda_1 t + \frac{216}{\mu^3} \int_{-t}^0 \| z(\omega)(s) \|^4_{V} ds} \leq 1, \text{ for } t \geq t_D(\omega).
\]
Thus, if \( x \in D(\theta_{-t}\omega) \) and \( s \leq -t_D(\omega) \), then by (4.47), we have
\[
\| v_x(0, \omega; s, x - \varepsilon z(s)) \|_{\mathbb{H}} \leq \kappa_{11}(\omega).
\]
Hence, we deduce that
\[
\| u_x(0, \omega; s, x) \|_{\mathbb{H}} \leq \| v_x(0, \omega; s, x - \varepsilon z(s)) \|_{\mathbb{H}} + \| z(\omega)(0) \|_{\mathbb{H}} \leq \kappa_{13}(\omega).
\]
This implies that \( u_x(0, \omega; s, x) \in B_0(\omega) \), for all \( s \leq -t_D(\omega) \), and hence it follows that \( B_0 \) absorbs \( D \).
Moreover, integrating (4.46) over \((-1, 0)\), we find for any \(\omega \in \Omega\), there exists a \(K_{14}(\omega) \geq 0\) such that

\[
\int_{-1}^{0} \left[ \|v_\epsilon(t)\|_V^2 + \|v_\epsilon(t) + \epsilon z(t)\|_{L^{r+1}}^{r+1} \right] dt \leq K_{14}(\omega), \tag{4.50}
\]

for all \(s \leq -t_D(\omega)\).

**Step II: Absorbing set in \(V\).** Multiplying the first equation in (4.12) by \(A v_\epsilon(\cdot)\) and then integrating the resulting equation over \(T^3\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|v_\epsilon(t)\|_V^2 = -\mu \|A v_\epsilon(t)\|_V^2 - b(v_\epsilon(t) + \epsilon z(t), v_\epsilon(t) + \epsilon z(t), A v_\epsilon(t)) - \beta(C(v_\epsilon(t) + \epsilon z(t)), A v_\epsilon(t)) + \alpha(\epsilon z(t), A v_\epsilon(t)) + (f, A v_\epsilon(t))
\]

\[
= -\mu \|A v_\epsilon(t)\|_V^2 - b(u_\epsilon(t), u_\epsilon(t), A v_\epsilon(t)) - \beta(C(u_\epsilon(t)), A u_\epsilon(t))
\]

\[
+ \beta(C(u_\epsilon(t)), \epsilon A z(t)) + \alpha(\epsilon z(t), A v_\epsilon(t)) + (f, A v_\epsilon(t)), \tag{4.51}
\]

where \(u_\epsilon(t) = v_\epsilon(t) + \epsilon z(t)\), for a.e. \(t \in [0, T]\). We consider the cases \(r > 3\) and \(r = 3\) separately.

**Case I: \(r > 3\).** From (1.2), we have

\[
(C(u_\epsilon), A u_\epsilon) = \|\nabla u_\epsilon\|_{L^{r+1}} \geq C\|\nabla u_\epsilon\|_{L^{r+1}}^2 \leq C\|\nabla u_\epsilon\|_{L^{r+1}}^2 \leq C\|\nabla u_\epsilon\|_{L^{r+1}}^2 + \frac{C}{2}\|\nabla u_\epsilon\|_{L^{r+1}}^2.
\]

We estimate \(|b(u_\epsilon, u_\epsilon, A v_\epsilon)|\) using Hölder’s, and Young’s inequalities as

\[
|b(u_\epsilon, u_\epsilon, A v_\epsilon)| \leq \|u_\epsilon\|_{L^{r+1}} \|\nabla u_\epsilon\|_{L^{r+1}} \leq \frac{\mu}{6} \|A v_\epsilon\|_{L^{r+1}}^2 + \frac{3}{2\mu} \|u_\epsilon\|_{L^{r+1}}^2. \tag{4.56}
\]

Let us estimate the final term from (4.56) using Hölder’s and Young’s inequalities as (similarly as in (2.12))

\[
\int_{T^3} |u_\epsilon(x)|^2 |\nabla u_\epsilon(x)|^2 dx \leq \frac{\beta \mu}{3} \left( \int_{T^3} |u_\epsilon(x)|^{r-1} |\nabla u_\epsilon(x)|^2 dx \right)
\]

\[
+ \frac{r-3}{r-1} \left[ \frac{6}{\beta \mu (r-1)} \right]^{\frac{2}{r-3}} \left( \int_{T^3} |\nabla u_\epsilon(x)|^2 dx \right). \tag{4.57}
\]
Making use of the estimate (4.57) in (4.56), we find
\[ |b(u_x, u_s, Av_x)| \leq \frac{\mu}{6} \|Av_x\|^2_{E^2} + \frac{\beta}{2} \|\nabla u_x\| |u|^2_{E^2} + \frac{3(r-3)}{2\mu(r-1)} \left[ \frac{6}{\beta \mu (r-1)} \right]^{\frac{2}{r-3}} \|u_x\|^2_{V}\]
\[ \leq \frac{\mu}{6} \|Av_x\|^2_{E^2} + \frac{\beta}{2} \|\nabla u_x\| |u|^2_{E^2} + C \|v_x\|^2_{V} + \varepsilon^2 C \|z\|^2_{V}. \quad (4.58) \]

For any \( \omega \in \Omega \), we infer from the inequalities (4.52)–(4.58) that
\[ \frac{d}{dt} \|v_x(t)\|^2_{V} + \mu \|Av_x(t)\|^2_{E^2} + \beta \|\nabla u_x(t)\| |u_x(t)|^{\frac{r-2}{r}} \|u_x\|^2_{E^2} \leq C \|v_x(t)\|^2_{V} + \varepsilon^2 C \|z(t)\|^2_{E^2} + C \|f\|^2_{E^2}, \quad (4.59) \]
for a.e. \( t \in [0, T] \). From the uniform Gronwall lemma (Lemma 1.1, [57]) and (4.50), we deduce that for any \( \omega \in \Omega \), there exists \( \kappa_{15}(\omega) \geq 0 \) and \( \kappa_{16}(\omega) \geq 0 \) such that
\[ \|v_x(0, \omega; s, x - \varepsilon z(s))\|_{V} \leq \kappa_{15}(\omega) \quad (4.60) \]
and
\[ \mu \int_{-1}^{0} \|Av_x(s)\|^2_{E^2} ds + \beta \int_{-1}^{0} \|\nabla u_x(t)\| |u_x(t)|^{\frac{r-2}{r}} \|u_x\|^2_{E^2} ds \leq \kappa_{16}(\omega), \quad (4.61) \]
for any \( s \leq -(t_{D}(\omega) + 1) \), which completes the proof for the case \( r > 3 \).

**Case II:** \( r = 3 \) and \( 2 \beta \mu \geq 1 \). From (1.2), we have
\[ (C(u_x), Av_x) = |||\nabla u_x||| |u_x|^2_{E^2} + \frac{1}{2} \|\nabla u_x\|^2_{E^2}, \quad (4.62) \]
\[ \|C(u_x, \varepsilon Az)\| \leq \|u_x\|^3_{E^2} \|Az\|_{E^2} \leq C \|u_x\|^3_{E^2} \|Az\|_{E^2} \]
\[ = C \|u_x\|^3_{E^2} \|Az\|_{E^2} \leq C \|\nabla u_x\|^2_{E^2} \|Az\|_{E^2} \]
\[ \leq \frac{1}{4} \|\nabla u_x\|^2_{E^2} + \varepsilon^4 C \|Az\|_{E^2}^4, \quad (4.63) \]
\[ \alpha |(\varepsilon z, Av_x)| \leq \alpha \|z\|_{E^2} \|Av_x\|_{E^2} \leq \frac{\mu}{4} \|Av_x\|^2_{E^2} + \varepsilon^2 C \|z\|^2_{E^2}, \quad (4.64) \]
\[ \|f, Av_x\| \leq \|f\|_{E^2} \|Av_x\|_{E^2} \leq \frac{\mu}{4} \|Av_x\|^2_{E^2} + C \|f\|^2_{E^2}. \quad (4.65) \]

We estimate \( |b(u_x, u_s, Av_x)| \) using Hölder’s and Young’s inequalities as
\[ |b(u_x, u_s, Av_x)| \leq \|u_x\| \|\nabla u_x\| \|Av_x\| \|z\|^2_{E^2} \|f\|^2_{E^2}. \quad (4.66) \]
For any \( \omega \in \Omega \), we infer from the inequalities (4.62)–(4.66) and (4.51) that
\[ \frac{d}{dt} \|v_x(t)\|_{V} + \left( \mu - \frac{1}{2\beta} \right) \|Av_x(t)\|^2_{E^2} + \beta \|\nabla u_x(t)\|^2_{E^2} \leq \varepsilon^4 C \|Az(t)\|^4_{E^2} + \varepsilon^2 C \|z(t)\|^2_{E^2} + C \|f\|^2_{E^2}, \quad (4.67) \]
for a.e. $t \in [0, T]$. From the uniform Gronwall lemma and (4.50), we deduce that for any $\omega \in \Omega$, there exists $\kappa_{17}(\omega) \geq 0$ and $\kappa_{18}(\omega) \geq 0$ such that
\[
\|v_{\varepsilon}(0, \omega; s, x - \varepsilon z(s))\|_V \leq \kappa_{17}(\omega)
\] (4.68)
and
\[
\left(\mu - \frac{1}{2\beta}\right) \int_{-1}^{0} \|A_{\varepsilon}v(s)\|_{\mathbb{H}}^2 ds + \frac{\beta}{2} \int_{-1}^{0} \|\nabla u_{\varepsilon}(t)\|_{L_2}^2 ds \leq \kappa_{18}(\omega).
\] (4.69)
for any $s \leq -(t_0(\omega) + 1)$, which completes the proof for $r = 3$ and $2\beta \mu \geq 1$. \hfill \Box

Thanks to the compactness of $V$ in $\mathbb{H}$, from Theorem 4.17 and the abstract theory of random attractors (Theorem 3.11, [16]), we immediately conclude the following result.

**Theorem 4.18** Assume that $f \in \mathbb{H}$ and Assumption 4.1 holds. Then, the cocycle $\varphi_{\varepsilon}$ corresponding to the 3D stochastic CBF equations with small additive noise (4.1) for $r \geq 3$ has a random attractor $\mathcal{A}_{\varepsilon} = \{A_{\varepsilon}(\omega) : \omega \in \Omega\}$ in $\mathbb{H}$.

Next, we shall prove the existence of random attractors in a more regular space $V$. To prove the existence of random attractors in $V$, it is only remain to show that our RDS $\varphi_{\varepsilon}$ is compact operator from $V$ into itself.

**Theorem 4.19** For $3 \leq r \leq 5$ ($2\beta \mu \geq 1$ for $r = 3$), the solution operator $\varphi_{\varepsilon}$ given in Definition (4.10) is compact from $V$ into itself.

**Proof** Consider the solution $\varphi_{\varepsilon}(t - s, \theta_{s}\omega) \cdot v_{\varepsilon}(t, \omega; s, \cdot - \varepsilon z(s)) + \varepsilon z(t)$ of (4.1) for $t \in [0, T]$, where $T > 0$. For simplicity, we write $\varphi_{\varepsilon}(t, \omega; s, \cdot - \varepsilon z(s)) = \varphi_{\varepsilon}(t, s, \cdot)$. Assume that the sequence $\{x_n\}_{n \in \mathbb{N}} \subset V$ is bounded. We have that, for $z \in L^\infty(-1, 0; \mathbb{V}) \cap L^{r+1}(-1, 0; \mathbb{D}(\mathbb{A}))$,
\[
\{v(\cdot, -1, x_n)\}_{n \in \mathbb{N}} \subset L^\infty(-1, 0; \mathbb{V}) \cap L^2(-1, 0; \mathbb{D}(\mathbb{A})) \cap L^{r+1}(-1, 0; \tilde{\mathbb{L}}^{3(r+1)}).
\]
For any arbitrary element $v \in \mathbb{H}$, using Hölder’s inequality, Sobolev embeddings and interpolation inequality, we have
\[
\left| \left( \frac{d}{dt} v(t, -1, x_n), v \right) \right| 
\leq \mu \left| (Av(t, -1, x_n), v) \right| + |b(v(t, -1, x_n) + z(t), v(t, -1, x_n) + z(t), v)|
+ \beta \left| (C(v(t, -1, x_n) + z(t)), v) \right| + |\alpha(z(t), v)| + |(f, v)|
\leq C \left[ \|Av(t, -1, x_n)\|_{\mathbb{H}} + \|Av(t, -1, x_n) + \varepsilon Az(t)\|_{\mathbb{H}} \|v(t, -1, x_n) + \varepsilon z(t)\|_V
+ \|v(t, -1, x_n) + \varepsilon z(t)\|_{\mathbb{H}}^\frac{r+3}{2} \|v(t, -1, x_n) + \varepsilon z(t)\|_{\mathbb{H}}^{\frac{(r+1)(r-3)}{2(r+1)}}
+ \alpha \|z(t)\|_{\mathbb{H}} + \|f\|_{\mathbb{H}} \right] \|v\|_{\mathbb{H}},
\]
which implies that (for details of calculations, see Theorem 3.6)
\[
\left\| \frac{d}{dt} [v(\cdot, -1, x_n)] \right\|_{L^2(-1,0;\mathbb{H})} \leq \hat{C} < \infty,
\] (4.70)
and the constant $\hat{C}$ is independent of $n$. Since $v(\cdot, -1, x_n) \in L^2(-1, 0; \mathbb{D}(\mathbb{A}))$, $\frac{d}{dt} [v(\cdot, -1, x_n)] \in L^2(-1, 0; \tilde{\mathbb{L}})$, $\mathbb{D}(\mathbb{A}) \subset \mathbb{V} \subset \mathbb{H}$ and $\mathbb{D}(\mathbb{A})$ is compactly embedded in $\mathbb{V}$, by the
Aubin-Lions compactness lemma, there exists a subsequence (using the same notation) and \( \tilde{v} \in L^2(\mathbb{R}^N, \mathbb{V}) \) such that
\[
\psi(-1, x_n) \to \tilde{v} \quad \text{strongly in } L^2(\mathbb{R}^N, \mathbb{V}).
\] (4.71)

Again, choosing one more subsequence (again not relabeling), we infer from (4.71) that
\[
\psi(t, -1, x_n) \to \tilde{v}(t) \quad \text{in } \mathbb{V}, \quad \text{for a.a. } t \in (-1, 0).
\] (4.72)

Since \(-1 < t < 0\), we obtain from (4.72) that there exists \( \tau \in (-1, t) \) such that (4.72) holds true for this particular \( \tau \). Then by Theorem 4.9, we obtain
\[
\phi(\tau, -1, x_n; -1) \to \tilde{v}(\tau) \quad \text{in } \mathbb{V},
\]
in \( \mathbb{V} \), which completes the proof. \( \square \)

Thanks to the abstract theory of random attractors (Theorem 2.2, [15]), we immediately conclude the following result.

**Theorem 4.20** Assume that \( f \in H \) and Assumption 4.1 holds. Then, the cocycle \( \phi \) corresponding to the 3D stochastic CBF equations with small additive noise (4.1) for \( 3 \leq r \leq 5 \) has a random attractor \( G = \{ G(\omega) : \omega \in \Omega \} \) in \( \mathbb{V} \).

**Remark 4.21** Note that, using the method adopted in Theorem 4.19, one can prove the existence of random attractor for (4.1) in \( \mathbb{H} \), with \( f \in \mathbb{V}' \) and \( K \subset \tilde{L}^{r+1} \) such that \( K \) is a Hilbert space and for some \( \delta \in (0, 1/2) \),
\[
A^{-\delta} : K \to \tilde{L}^{r+1} \quad \text{is } \gamma\text{-radonifying}.
\]

## 5 Upper Semi-Continuity of \( DR \)-Random Attractors in \( \mathbb{H} \)

The aim of this section is to prove the upper semicontinuity of random attractors in \( \mathbb{H} \). The existence of random attractors for the stochastic system (4.1) in \( \mathbb{H} \) is proved in Theorem 4.18 and the existence of global attractors for the deterministic system (3.1) in \( \mathbb{H} \) is proved in Theorem 3.5. Upper semicontinuity results for several infinite dimensional stochastic models is obtained in [11,37,42] etc. In particular, authors in [37] obtained the upper semicontinuity results for 2D SCBF equations with \( r \in [1, 3] \). Using similar techniques used in the work [11], we state and prove the following result on the upper semicontinuity of the random attractors:

**Theorem 5.1** For \( r \geq 3 \) (\( 2\beta \mu \geq 1 \) for \( r = 3 \)), assume that \( f \in \mathbb{H} \) and Assumption 4.1 is satisfied. Also, assume that the deterministic system (3.1) has a global attractor \( \mathcal{A} \) and its small random perturbed dynamical system (4.1) possesses a \( DR \)-random attractor \( \mathcal{A} = \{ A(\omega) : \omega \in \Omega \} \), for any \( \epsilon \in (0, 1] \). Let the following conditions hold:

\( (K_1) \) For each \( t_0 \geq 0 \) and for \( \omega \in \Omega \),
\[
\lim_{\epsilon \to 0^+} d(\phi(\omega, \theta_{-t_0} \omega)x, S(t)x) = 0,
\]
uniformly on bounded sets of \( \mathbb{H} \), where \( \phi \) is a RDS and \( S(t) \) is a semigroup generated by (4.1) and (3.1), respectively with the same initial condition \( x \in \mathbb{H} \).
(K2) There exists a compact set $K \subset \mathbb{H}$ such that

$$\lim_{\varepsilon \to 0^+} d(A_\varepsilon(\omega), K) = 0,$$

for $\omega \in \Omega$.

Then $A_\varepsilon$ and $A$ have the property of upper semicontinuity, that is,

$$\lim_{\varepsilon \to 0^+} d(A_\varepsilon(\omega), A) = 0,$$

for $\omega \in \Omega$.

Furthermore, if for $\varepsilon_0 \in (0, 1]$, we have that for $\omega \in \Omega$ and all $t_0 > 0$

$$\varphi_\varepsilon(t_0, \theta_{-t_0})x \to \varphi_{\varepsilon\eta}(t_0, \theta_{-t_0})x \text{ as } \varepsilon \to \varepsilon_0,$$

uniformly on bounded sets of $\mathbb{H}$, then the convergence (5.2) is upper semicontinuous in $\varepsilon$, that is,

$$\lim_{\varepsilon \to \varepsilon_0} d(A_\varepsilon(\omega), A_{\varepsilon_0}(\omega)) = 0,$$

for $\omega \in \Omega$.

**Proof** To prove the property of upper semicontinuity for our system, we only need to verify the conditions (K1) and (K2).

**Step I. Verification of (K2):** Let us introduce

$$v_\varepsilon(t, \omega) = u_\varepsilon(t, \omega) - \varepsilon z(t, \omega),$$

where $u_\varepsilon(t, \omega)$ and $z(t, \omega)$ are the unique solutions of (4.1) and (4.4), respectively. Also from (4.7), we have $z \in L^\infty_{i0}(I_{0}, \infty); \mathbb{V}) \cap L^{r + 1}_{i0}(I_{0}, \infty); D(A)).$ Clearly, $v_\varepsilon(\cdot)$ satisfies

$$\frac{dv_\varepsilon(t)}{dr} = -\mu A v_\varepsilon(t) - B(v_\varepsilon(t) + \varepsilon z(t)) - \beta C(v_\varepsilon(t) + \varepsilon z(t)) + \varepsilon a z(t) + f,$$

in $\mathbb{V} + \mathbb{V}^{r + 1}$. From Theorem 4.17, we observe that there exists $\hat{k}_e(\omega) \in \mathbb{R}$ such that

$$\|v_\varepsilon(0)\|_{\mathbb{V}} \leq \hat{k}_e(\omega).$$

If we call $K_\varepsilon(\omega)$, the ball in $\mathbb{V}$ of radius $\hat{k}_e(\omega) + \varepsilon \|z(0)\|_{\mathbb{V}}$, we have a compact (since $\mathbb{V} \hookrightarrow \mathbb{H}$ is compact) $\mathbb{D}\mathbb{R}$-absorbing set in $\mathbb{H}$ for $\varphi_\varepsilon$. Furthermore, there exists a $\hat{k}_d$ independent of $\omega \in \Omega$ such that

$$\lim_{\varepsilon \to 0^+} \hat{k}_e(\omega) \leq \hat{k}_d,$$

which easily verifies Lemma 1, [11] and hence (K2) follows.

**Step II. Verification of (K1):** In order to verify the assertion (K1), it is enough to prove that the solution $\varphi_\varepsilon(t, \omega)x$ of system (4.1) $\hat{P}$-a.s. converges to the solution $S(t)x$ of the unperturbed system (3.1) in $\mathbb{H}$ as $\varepsilon \to 0^+$ uniformly on bounded sets of initial conditions. That is, for $\hat{P}$-a.e $\omega \in \Omega$, any $t_0 \geq 0$ and any bounded subset $G \subset \mathbb{H}$, we have

$$\lim_{\varepsilon \to 0^+} \sup_{x \in G} \|\varphi_\varepsilon(t_0, \theta_{-t_0})x - S(t_0)x\|_{\mathbb{H}} = 0.$$

(5.5)

For any $x \in G$, let $u_\varepsilon(t) = \varphi_\varepsilon(t + t_0, \theta_{-t_0})x$ and $u(t) = S(t + t_0)x$ respectively, be the unique weak solutions of the systems (4.1) and (3.1) with the initial condition $x$ at $t = -t_0$. Also, for $T \geq 0$, let

$$\eta_\varepsilon(t) = u_\varepsilon(t) - u(t), t \in [-t_0, T].$$
Clearly $\eta_\varepsilon(\cdot)$ satisfies
\[
\begin{cases}
\frac{d\eta_\varepsilon}{dt} + \{\mu \Lambda \eta_\varepsilon + B(\eta_\varepsilon + u) - B(u) + \beta C(\eta_\varepsilon + u) - \beta C(u)\}dt = \varepsilon dW(t), \\
\eta_\varepsilon(-t_0) = 0,
\end{cases}
\tag{5.6}
\]
in $V' + \widetilde{L}^{-\frac{1}{2}}$, for a.e. $t \in [-t_0, T]$. Let us introduce $\rho_\varepsilon(\cdot) = \eta_\varepsilon(\cdot) - \varepsilon z(\cdot)$, where $z(\cdot)$ is the solution of (4.4), then $\rho_\varepsilon(\cdot)$ satisfies the following equation in $V' + \widetilde{L}^{-\frac{1}{2}}$:
\[
\begin{cases}
\frac{d\rho_\varepsilon}{dt} = -\mu A \rho_\varepsilon - B(\rho_\varepsilon + \varepsilon z + u) + B(u) - \beta C(\rho_\varepsilon + \varepsilon z + u) + \beta C(u) + \varepsilon \alpha z, \\
\rho_\varepsilon(-t_0) = -\varepsilon z(-t_0),
\end{cases}
\tag{5.7}
\]
for a.e. $t \in [-t_0, T]$. Taking the inner product of the first equation of (5.7) with $\rho_\varepsilon(\cdot)$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\rho_\varepsilon(t)\|^2_{H^1} = -\mu \|\rho_\varepsilon(t)\|^2_V - \{B(\rho_\varepsilon(t) + \varepsilon z(t) + u(t)) - B(u(t)), \rho_\varepsilon(t)\}
- \beta \{C(\rho_\varepsilon(t) + \varepsilon z(t) + u(t)) - C(u(t)), \rho_\varepsilon(t)\} + \alpha(\varepsilon z(t), \rho_\varepsilon(t)),
\]
\[
= -\mu \|\rho_\varepsilon(t)\|^2_V + \frac{1}{2} \{B(\rho_\varepsilon(t) + \varepsilon z(t) + u(t)), \varepsilon z(t)\} - \{B(u(t)), \varepsilon z(t)\}
- \{B(\rho_\varepsilon(t) + \varepsilon z(t) + u(t)) - B(u(t)), (\rho_\varepsilon(t) + \varepsilon z(t) + u(t)) - (u(t))\}
+ \beta \{C(\rho_\varepsilon(t) + \varepsilon z(t) + u(t)) - C(u(t)), \varepsilon z(t)\}
- \beta \{C(\rho_\varepsilon(t) + \varepsilon z(t) + u(t)) - C(u(t)), (\rho_\varepsilon(t) + \varepsilon z(t) + u(t)) - (u(t))\}
+ \alpha(\varepsilon z(t), \rho_\varepsilon(t)),
\tag{5.8}
\]
for a.e. $\in [-t_0, T]$.

**Case I:** $r > 3$. Using (2.8), we have
\[
\|B(\rho_\varepsilon + \varepsilon z + u), \varepsilon z\| \leq \|\rho_\varepsilon + \varepsilon z + u\|_{\tilde{L}^{-1}}^\frac{r+1}{2} \|\rho_\varepsilon + \varepsilon z + u\|^\frac{r-3}{2} \|\varepsilon z\|_V
\leq C \|\rho_\varepsilon + \varepsilon z + u\|_{\tilde{L}^{-1}}^\frac{r+1}{2} \|\rho_\varepsilon + \varepsilon z + u\|^\frac{r-3}{2} \|\varepsilon z\|_V.
\tag{5.9}
\]
\[
\|B(u), \varepsilon z\| \leq \|u\|_{\tilde{L}^{-1}}^\frac{r+1}{2} \|\varepsilon z\|_V \leq C \|u\|_{\tilde{L}^{-1}}^\frac{r+1}{2} \|\varepsilon z\|_V.
\tag{5.10}
\]
Using (2.10) and Young's inequality (as in equation (2.13)), we deduce that
\[
\|B(\rho_\varepsilon + \varepsilon z + u) - B(u), (\rho_\varepsilon + \varepsilon z + u) - (u)\|
\leq \frac{\mu}{4} \|\rho_\varepsilon + \varepsilon z + u - (u)\|_V^2 + \frac{\beta}{4} \|u\|_{L^{r+1}}^\frac{r-1}{2} \|\rho_\varepsilon + \varepsilon z + u - (u)\|_V^\frac{r-3}{2}
+ \eta_4 \|\rho_\varepsilon + \varepsilon z + u - (u)\|_{H^1}^2
\leq \frac{\mu}{2} \|\rho_\varepsilon\|_V^2 + C \|\varepsilon z\|_V^2 + \frac{\beta}{4} \|u\|_{L^{r+1}}^\frac{r-1}{2} \|\rho_\varepsilon + \varepsilon z + u - (u)\|_V^\frac{r-3}{2} + 2 \eta_4 \|\rho_\varepsilon\|_{H^1}^2,
\tag{5.11}
\]
where $\eta_4 = \frac{r^{-3}}{(\mu r)^{-1}} \left( \frac{8}{\beta \mu r^{-1}} \right)^{\frac{2}{3}}$. Now, using Taylor’s formula, we obtain

$$
\beta \left| C(\rho_\varepsilon + \varepsilon z + u) - C(u), \varepsilon z \right|
\leq \beta \left| \int_0^1 \left[C'(\theta(\rho_\varepsilon + \varepsilon z + u) + (1 - \theta)u)((\rho_\varepsilon + \varepsilon z + u) - (u))\right] d\theta, \varepsilon z \right|
\leq r \beta 2^{r-2} \|(\rho_\varepsilon + \varepsilon z + u)\|^\frac{r-1}{r} \|(\rho_\varepsilon + \varepsilon z + u)\|_{L^{r+1}} \|\varepsilon z\|_{L^{r+1}}
+ r \beta 2^{r-2} \|u\|^\frac{r-1}{r} \|(\rho_\varepsilon + \varepsilon z)\|_{L^{r+1}} \|\varepsilon z\|_{L^{r+1}}
\leq \frac{\beta}{2} \|(\rho_\varepsilon + \varepsilon z + u)\|^\frac{r-1}{r} \|(\rho_\varepsilon + \varepsilon z + u)\|_{L^{r+1}}^2 + C \|\rho_\varepsilon + \varepsilon z + u\|_{L^{r+1}} \|\varepsilon z\|_{L^{r+1}}^2
+ \frac{\beta}{4} \|u\|^\frac{r-1}{r} \|(\rho_\varepsilon + \varepsilon z)\|_{L^{r+1}}^2 + C \|u\|^\frac{r-1}{r} \|\varepsilon z\|_{L^{r+1}}^2. \quad (5.12)
$$

By (2.16), we have

$$
-\beta \left| C(\rho_\varepsilon + \varepsilon z + u) - C(u), (\rho_\varepsilon + \varepsilon z + u) - (u) \right|
\leq -\frac{\beta}{2} \|(\rho_\varepsilon + \varepsilon z + u)\|^\frac{r-1}{r} \|(\rho_\varepsilon + \varepsilon z + u) - (u)\|_{L^{r+1}}^2
- \frac{\beta}{2} \|u\|^\frac{r-1}{r} \|(\rho_\varepsilon + \varepsilon z + u) - (u)\|_{L^{r+1}}^2. \quad (5.13)
$$

Using Hölder’s and Young’s inequalities, we get

$$
\alpha \left| (\varepsilon z, \rho_\varepsilon) \right| \leq \alpha \|\rho_\varepsilon\|_\mathbb{H} \|\varepsilon z\|_\mathbb{H} \leq \frac{1}{2} \|\rho_\varepsilon\|_{L^2}^2 + \frac{\alpha^2}{2} \|\varepsilon z\|_{L^2}^2 \leq \frac{1}{2} \|\rho_\varepsilon\|_{L^2}^2 + \frac{\alpha^2C}{2} \|\varepsilon z\|_{\mathbb{V}}^2. \quad (5.14)
$$

Combining (5.9)–(5.14) and then substituting it in (5.8), we find

$$
\frac{d}{dt} \|\rho_\varepsilon(t)\|_{L^2}^2 + \mu \|\rho_\varepsilon(t)\|_{\mathbb{V}}^2
\leq C \left\{ \|\rho_\varepsilon(t) + \varepsilon z(t) + u(t)\|_{L^{r+1}} \|\rho_\varepsilon(t) + \varepsilon z(t) + u(t)\|_{L^2}^{\frac{r-1}{r}} + \|u(t)\|_{L^{r+1}} \|u(t)\|_{L^2}^{\frac{r-1}{r}} \right\}
\times \|\varepsilon z(t)\|_{L^2} + C \|\varepsilon z(t)\|_{L^2}^2 + \widetilde{\eta}_4 \|\rho_\varepsilon(t)\|_{L^2}^2
+ C \left\{ \|\rho_\varepsilon(t) + \varepsilon z(t) + u(t)\|_{L^{r+1}} \|\rho_\varepsilon(t) + \varepsilon z(t) + u(t)\|_{L^2}^{\frac{r-1}{r}} \right\} \|\varepsilon z(t)\|_{L^2}^2
\leq C \left\{ \|u(t)\|_{L^{r+1}} \|u(t)\|_{L^2}^{\frac{r-1}{r}} + \|u(t)\|_{L^{r+1}} \|u(t)\|_{L^2}^{\frac{r-1}{r}} \right\} \|\varepsilon z(t)\|_{L^2}^2
+ C \|\varepsilon z(t)\|_{L^2}^2 + \widetilde{\eta}_4 \|\rho_\varepsilon(t)\|_{L^2}^2 + C \left\{ \|u(t)\|_{L^{r+1}} \|u(t)\|_{L^2}^{\frac{r-1}{r}} \right\} \|\varepsilon z(t)\|_{L^2}^2, \quad (5.15)
$$

where $\widetilde{\eta}_4 = 4\eta_4 + 1$, for a.e. $t \in [-t_0, T]$. By integrating the above inequality from $-t_0$ to $t$, $t \in [-t_0, T]$, we get

$$
\|\rho_\varepsilon(t)\|_{L^2}^2 + \mu \int_{-t_0}^t \|\rho_\varepsilon(s)\|_{\mathbb{V}}^2 ds \leq \|\rho_\varepsilon(-t_0)\|_{L^2}^2 + \int_{-t_0}^t \widetilde{\eta}_4 \|\rho_\varepsilon(s)\|_{L^2}^2 ds + \int_{-t_0}^t \alpha_\varepsilon(s) ds, \quad (5.16)
$$
where
\[
\alpha_\varepsilon = C \left\{ \| u_\varepsilon \|_{L^1_{r+1}}^{\frac{r+1}{r}} \| u_\varepsilon \|_{\ell^1}^{\frac{r-3}{2}} + \| u \|_{L^1_{r+1}}^{\frac{r+1}{r}} \| u \|_{\ell^1}^{\frac{r-3}{2}} \right\} \| \varepsilon z \|_V
+ C \left\{ \| u_\varepsilon \|_{L^1_{r+1}}^{\frac{r-1}{r}} + \| u \|_{L^1_{r+1}}^{\frac{r-1}{r}} \right\} \| \varepsilon z \|_{L^1_{r+1}}^{\frac{r}{r}} + C \| \varepsilon z \|_V^2.
\]

Then by the classical Gronwall inequality, we deduce that
\[
\| \rho_\varepsilon (t) \|_{H^1}^2 \leq \left( \frac{\varepsilon^2}{\beta \lambda_1} \| z (-t_0) \|_V^2 + \int_{-t_0}^t \alpha_\varepsilon (s) ds \right) e^{\beta (t-t_0)},
\]
for all \( t \in [-t_0, T] \). On the other hand, we have
\[
\int_{-t_0}^t \alpha_\varepsilon (s) ds \leq \int_{-t_0}^t \varepsilon C \left\{ \| u_\varepsilon (s) \|_{L^1_{r+1}((-t_0, 0); \ell^1)}^{\frac{r+1}{r}} \| u_\varepsilon (s) \|_{\ell^1}^{\frac{r-3}{2}} + \| u (s) \|_{L^1_{r+1}((-t_0, 0); \ell^1)}^{\frac{r+1}{r}} \| u (s) \|_{\ell^1}^{\frac{r-3}{2}} \right\} \| z (s) \|_V
+ \varepsilon^2 C \left\{ \| (u_\varepsilon (s)) \|_{L^1_{r+1}((-t_0, 0); \ell^1)}^{\frac{r-1}{r}} + \| (u (s)) \|_{L^1_{r+1}((-t_0, 0); \ell^1)}^{\frac{r-1}{r}} \right\} \| Az (s) \|_{H^1}^2 + \varepsilon^2 C \| z (s) \|_V^2 \] ds,
\[
\leq \varepsilon C (t + t_0)^{1/2} \left[ \| u_\varepsilon \|_{L^1_{r+1}((-t_0, 0); \ell^1)}^{\frac{r+1}{r}} \| u_\varepsilon \|_{L^1_{r+1}((-t_0, 0); \ell^1)}^{\frac{r-3}{2}} \right] \| z \|_{L^1 ((-t_0, 0); V)}
+ \varepsilon^2 C \left[ \| u_\varepsilon \|_{L^1_{r+1}((-t_0, 0); \ell^1)}^{\frac{r-1}{r}} + \| u \|_{L^1_{r+1}((-t_0, 0); \ell^1)}^{\frac{r-1}{r}} \right] \| z \|_{L^1 ((-t_0, 0); D(A))}
+ \varepsilon^2 C (t + t_0)^{1/2} \| z \|_{L^1 ((-t_0, 0); V)}^2.
\]

Since the processes \( u, u_\varepsilon, z \in H^1 ((-t_0, 0); \mathbb{H}) \cap \cap L^1_{loc} ((-t_0, 0); V) \), and \( \varepsilon \in L^\infty ((-t_0, 0); \mathbb{V}) \), and \( \varepsilon z \in L^\infty ((-t_0, 0); \mathbb{V}) \), therefore \( \int_{-t_0}^t \alpha_\varepsilon (s) ds \to 0 \) as \( \varepsilon \to 0 \).

Hence, by (5.17), we arrive at
\[
\lim_{\varepsilon \to 0^+} \| \rho_\varepsilon (t) \|_{H^1}^2 = 0,
\]
which completes the proof of (5.5) by taking \( t = 0 \). Hence \( (K_2) \) is verified for \( r > 3 \).

**Case II:** \( r = 3 \) and \( 2\beta \mu_a \geq 1 \). By Hölder’s inequality, (2.5) and (2.1), we have
\[
\| [B(\rho_\varepsilon + \varepsilon z + u), \varepsilon z] \| \leq \| \rho_\varepsilon + \varepsilon z + u \|_{L^4_{\alpha+1}} \| \varepsilon z \|_V \leq C \| \rho_\varepsilon + \varepsilon z + u \|_V \| \varepsilon z \|_V,
\]
\[
\| [B(u), \varepsilon z] \| \leq \| u \|_{L^4_{\alpha+1}} \| \varepsilon z \|_V \leq C \| u \|_V \| \varepsilon z \|_V.
\]

Using (2.10) and Hölder’s inequality, we find
\[
\| [B(\rho_\varepsilon + \varepsilon z + u) - B(u), (\rho_\varepsilon + \varepsilon z + u) - (u)] \|
\leq \| \rho_\varepsilon + \varepsilon z \|_V \| | u | (\rho_\varepsilon + \varepsilon z) | | H \leq \frac{1}{2\beta} \| \rho_\varepsilon + \varepsilon z \|_V^2 + \frac{\beta}{2} \| u \|_V (\rho_\varepsilon + \varepsilon z) \|_{H^1}^2
\leq \frac{1}{2\beta} \| \rho_\varepsilon \|_V^2 + \frac{3}{2\beta} \| \varepsilon z \|_V^2 + \frac{1}{\beta} | u_e \| V \| \varepsilon z \|_V + \frac{1}{\beta} | u \| V \| \varepsilon z \|_V + \frac{\beta}{2} \| u \|_V (\rho_\varepsilon + \varepsilon z) \|_{H^1}^2.
\]

Making use of Hölder’s inequality, (2.5) and (2.1), we obtain
\[
\| C(\rho_\varepsilon + \varepsilon z + u) - C(u), \varepsilon z \| \leq \left\{ \| \rho_\varepsilon + \varepsilon z + u \|_{L^4_{\alpha+1}}^3 + \| u \|_{L^4_{\alpha+1}}^3 \right\} \| \varepsilon z \|_{L^4_{\alpha+1}}
\leq C \left\{ \| u_e \|_{L^4_{\alpha+1}}^3 + \| u \|_{L^4_{\alpha+1}}^3 \right\} \| \varepsilon z \|_V.
\]
By (2.16), we have
\[-\beta(C(\rho_\varepsilon + \varepsilon z + u) - C(u), (\rho_\varepsilon + \varepsilon z + u) - (u)) \leq -\frac{\beta}{2} \|u(\rho_\varepsilon + \varepsilon z)\|_{\mathbb{H}}^2. \tag{5.22}\]

Using (5.18)–(5.22) with (5.14) in (5.8), we arrive at
\[
\frac{d}{dt} \|\rho_\varepsilon(t)\|_{\mathbb{H}}^2 + 2\left(\mu - \frac{1}{2\beta}\right) \|\rho_\varepsilon(t)\|_{V}^2 \\
\leq C \left\{ \|u_\varepsilon(t)\|_V^2 + \|u(t)\|_V^2 + \|u_\varepsilon(t)\|_V + \|u(t)\|_V + \|u_\varepsilon(t)\|_{L^2}^3 + \|u(t)\|_{L^2}^3 \right\} \|\varepsilon z(t)\|_V \\
+ \|\varepsilon z(t)\|_{L^2}^2 + C \|\varepsilon z(t)\|_{V}^2, \tag{5.23}\]
for a.e. \( t \in [-t_0, T] \). By integrating the above inequality from \(-t_0\) to \( t \), we get
\[
\|\rho_\varepsilon(t)\|_{\mathbb{H}}^2 \leq \|\rho_\varepsilon(-t_0)\|_{\mathbb{H}}^2 + 2\int_{-t_0}^t \|\rho_\varepsilon(s)\|_{\mathbb{H}}^2 ds + \int_{-t_0}^t \beta_\varepsilon(s) ds, \tag{5.24}\]
where
\[
\beta_\varepsilon = C \left\{ \|u_\varepsilon\|_V^2 + \|u\|_V^2 + \|u_\varepsilon\|_V + \|u\|_V + \|u_\varepsilon\|_{L^2}^3 + \|u\|_{L^2}^3 \right\} \|\varepsilon z\|_V + C \|\varepsilon z\|_{V}^2.
\]
Then by the classical Gronwall inequality, we deduce that
\[
\|\rho_\varepsilon(t)\|_{\mathbb{H}}^2 \leq \left( \varepsilon^2 \|z(-t_0)\|_{\mathbb{H}}^2 + \int_{-t_0}^t \beta_\varepsilon(s) ds \right) e^{2(t+0)} \\
\leq \left( \frac{\varepsilon^2}{K_1} \|z(-t_0)\|_V^2 + \int_{-t_0}^t \beta_\varepsilon(s) ds \right) e^{2(t+0)}, \tag{5.25}\]
for all \( t \in [-t_0, T] \). On the other hand, we have
\[
\int_{-t_0}^t \beta_\varepsilon(s) ds \\
\leq \varepsilon C \left\{ \|u_\varepsilon\|_{L^2([-t_0,t]; V)} + \|u\|_{L^2([-t_0,t]; V)} + (t + t_0)^{1/2} \|u_\varepsilon\|_{L^2([-t_0,t]; V)} \\
+ (t + t_0)^{1/2} \|u\|_{L^2([-t_0,t]; V)} + (t + t_0)^{1/4} \|u_\varepsilon\|_{L^2([-t_0,t]; \mathbb{H}^4)} \\
+ (t + t_0)^{1/4} \|u\|_{L^2([-t_0,t]; \mathbb{H}^4)} \right\} \|z\|_{L^2([-t_0,t]; V)} + \varepsilon^2 (t + t_0) \|z\|_{L^2([-t_0,t]; V)}.
\]
Since the processes \( u_\varepsilon, u \in L_{loc}^2([-t_0, \infty); \mathbb{H}) \cap L_{loc}^4([-t_0, \infty); V) \) and \( z \in L_{loc}^2([-t_0, \infty); V) \cap L_{loc}^4([-t_0, \infty); \mathbb{H}^4) \), and \( D(A) \) is a bounded subset \( G \subset \mathbb{H} \), we have
\[
\lim_{\varepsilon \to 0^+} \|\rho_\varepsilon(t)\|_{\mathbb{H}}^2 = 0,
\]
which completes the proof of (5.5) by taking \( t = 0 \). Hence \((K_2)\) is verified for \( r = 3 \).

Step III. Proof of (5.2): In order to prove (5.2), it is enough to show that for any bounded subset \( G \subset \mathbb{H} \), we have
\[
\lim_{\varepsilon \to \varepsilon_0} \sup_{x \in G} \|\varphi_\varepsilon(t_0, \theta_{-t_0} \omega)x - \varphi_{\varepsilon_0}(t_0, \theta_{-t_0} \omega)x\|_{\mathbb{H}} = 0. \tag{5.26}\]
For any $x \in G$, let us take $u_\varepsilon(t) = \varphi_\varepsilon(t + t_0, \theta_{-t_0})x$ and $u_{\varepsilon_0}(t) = \varphi_{\varepsilon_0}(t + t_0, \theta_{-t_0})x$. Let $u_\varepsilon(\cdot)$ be the unique weak solution of the system (4.1) and $u_{\varepsilon_0}(\cdot)$ be the unique weak solution of the system (4.1) when $\varepsilon$ is replaced by $\varepsilon_0$, with initial condition $x$ at $t = -t_0$. Also, let

$$y(t) = u_\varepsilon(t) - u_{\varepsilon_0}(t), \quad t \in [0, T].$$

Clearly $y(\cdot)$ satisfies

$$\left\{ \begin{align*}
dy + (\mu A y + B(y + u_\varepsilon) - B(u_{\varepsilon_0}) - \beta C(y + u_{\varepsilon_0}) - \beta C(u_{\varepsilon_0}))dt &= \varepsilon^* dW(t), \\
y(-t_0) &= 0,
\end{align*} \right.$$  \hspace{1cm} (5.27)

in $V' + \widetilde{L}^{r+1}_\varepsilon$ for a.e. $t \in [-t_0, T]$, where $\varepsilon^* = \varepsilon - \varepsilon_0$. Let us introduce $w(\cdot) = y(\cdot) - \varepsilon^* z(\cdot)$, where $z(\cdot)$ is the unique solution of (4.4). Then $w(\cdot)$ satisfies the following equation in $V' + \widetilde{L}^{r+1}_\varepsilon$:

$$\left\{ \begin{align*}
\frac{dw}{dt} &= -\mu A w - B(w + \varepsilon^* z + u) + B(u) - \beta C(w + \varepsilon^* z + u) \\
&\quad + \beta C(u) + \varepsilon^* \alpha z, \\
w(-t_0) &= -\varepsilon^* z(-t_0),
\end{align*} \right.$$  \hspace{1cm} (5.28)

for a.e. $t \in [-t_0, T]$. The above system is similar to (5.7) and a calculation similar to (5.16) (for $r > 3$) and (5.24) (for $r = 3$ and $2\beta \mu \geq 1$) yields (5.3).

\section{Invariant Measures}

In this section, we discuss the existence of an invariant measure for the 3D SCBF equations (4.1), which is a direct consequence of Corollary 4.4, [16] along with Theorems 4.18 and 4.20.

Let $\varphi_\varepsilon$ be the RDS corresponding to the 3D SCBF equations (4.1), which is defined by (4.43). Let us define the transition operator $\{P_t\}_{t \geq 0}$ by

$$P_t f(x) = \int_{\Omega} f(\varphi_\varepsilon(t, \omega, x))d\mathbb{P}(\omega) = \mathbb{E} [f(\varphi_\varepsilon(t, x))],$$  \hspace{1cm} (6.1)

for all $f \in B_b(\mathbb{H})$, where $B_b(\mathbb{H})$ is the space of all bounded and Borel measurable functions on $\mathbb{H}$. Similar to Proposition 3.8, [10], we have the following result:

\textbf{Lemma 6.1} The family $\{P_t\}_{t \geq 0}$ is Feller, that is, $P_t f \in C_b(\mathbb{H})$ if $f \in C_b(\mathbb{H})$, where $C_b(\mathbb{H})$ is the space of all bounded and continuous functions on $\mathbb{H}$. Furthermore, for any $f \in C_b(\mathbb{H})$, $P_t f(x) \to f(x)$ as $t \downarrow 0$.

Using similar arguments as in the proof of Theorem 5.6, [16], one can prove that $\varphi_\varepsilon$ is a Markov RDS, that is, $P_{t+s} = P_t P_s$, for all $t, s \geq 0$. Since, we know by Corollary 4.4, [16] that if a Markov RDS on a Polish space has an invariant compact random set, then there exists a Feller invariant probability measure $\nu_\varepsilon$ for $\varphi_\varepsilon$.

\textbf{Definition 6.2} A Borel probability measure $\nu$ on $\mathbb{H}$ is called an invariant measure for a Markov semigroup $\{P_t\}_{t \geq 0}$ of Feller operators on $C_b(\mathbb{H})$ if and only if

$$P_t^* \nu = \nu, \quad t \geq 0,$$

where $(P_t^* \nu)(\Gamma) = \int_{\mathbb{H}} P_t(x, \Gamma) \nu(dx)$, for $\Gamma \in B(\mathbb{H})$ and the $P_t(x, \cdot)$ is the transition probability, $P_t(x, \Gamma) = P_t(\chi_\Gamma)(x), \quad x \in \mathbb{H}$. 

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In Theorems 4.18 and 4.20, we have proved the existence of random attractors in \( \mathbb{H} \) and \( \mathbb{V} \), respectively. By the definition of random attractors, it is immediate that there exists an invariant compact random set in \( \mathbb{H} \) as well as in \( \mathbb{V} \). A Feller invariant probability measure for a Markov RDS \( \varphi_t \) on \( \mathbb{H} \) is, by definition, an invariant probability measure for the semigroup \( \{ P_t \}_{t \geq 0} \) defined by (6.1). Hence, we have the following result on the existence of invariant measures for the 3D SCBF equations (4.1).

**Theorem 6.3** For \( r \geq 3 \) (\( 2\beta \mu \geq 1 \) for \( r = 3 \)), there exists an invariant measure for the 3D SCBF equations (4.1) in \( \mathbb{H} \).

**Remark 6.4**

1. In Theorem 4.20, we have also proved that there exists a random attractor in \( \mathbb{V} \) for \( 3 \leq r \leq 5 \) (\( 2\beta \mu \geq 1 \) for \( r = 3 \)) and hence there exists an invariant compact random set in \( \mathbb{V} \) for \( 3 \leq r \leq 5 \). Invoking Corollary 4.4 in [16], we obtain the existence of an invariant measure for the 3D SCBF equations (4.1) in \( \mathbb{V} \) for \( 3 \leq r \leq 5 \) as well.

2. In this work, \( W(t) \) is a Wiener process with RKHS \( K \) satisfying Assumption 4.1. In particular, \( K \subset \mathbb{H} \) and the natural embedding \( i : K \hookrightarrow \mathbb{H} \) is a Hilbert-Schmidt operator. For a fixed orthonormal basis \( \{ e_k \}_{k \in \mathbb{N}} \) of \( K \) and a sequence \( \{ \beta_k \}_{k \in \mathbb{N}} \) of independent Brownian motions defined on some filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P}) \) such that \( W(t) \) can be written in the following form

\[
W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k, \quad t \geq 0.
\]

Moreover, there exists a covariance operator \( J \in \mathcal{L}(\mathbb{H}) \) associated with \( W(t) \) defined by

\[
\langle Jh_1, h_2 \rangle = \mathbb{E} \left[ \langle h_1, W(1) \rangle_{\mathbb{H}} \langle W(1), h_2 \rangle_{\mathbb{H}} \right], \quad h_1, h_2 \in \mathbb{H}.
\]

It is well known from [18] that \( J \) is a non-negative self-adjoint and trace class operator in \( \mathbb{H} \). Furthermore, \( J = ii^* \) and \( K = R(J^{1/2}) \), where \( R(J^{1/2}) \) is the range of the operator \( J^{1/2} \) (see [5]). Note that

\[
\sum_{k=1}^{\infty} ||ie_k||^2_{\mathbb{H}} = tr [J] < \infty.
\]

The existence of a unique ergodic and strongly mixing invariant measure for 3D SCBF subjected to non-degenerate additive noise (Assumption 4.1) is proved in [46] by using the exponential stability of solutions. The uniqueness of invariant measure for 3D SCBF equations driven by additive degenerate noise via the asymptotic coupling method ([27,30]) is established in [50].

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