Interference of Majorana fermions in NS junctions

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Abstract

We investigate interference of Majorana fermions (MFs) in NS junctions. A general formula of charge conductance \( G \) for NS junctions with two MFs is derived based on the low energy effective model. It is found that \( G \) for two MFs takes various values \( 0 \leq G \leq 4e^2/h \) owing to interference of the MFs, while \( G \) is quantized as \( G = 2e^2/h \) for a single MF. The value of \( G \) is determined by symmetry of the system. As an example, we investigate the complete destructive interference of two degenerate MFs reported by Ii et al. [A. Ii, A. Yamakage, K. Yada, M. Sato, and Y. Tanaka, Phys. Rev. B 86, 174512 (2012)], and identify the symmetry responsible for the destructive interference.

Keywords: Topological superconductor, Majorana fermion, Andreev reflection, NS junction, Quantum anomalous Hall insulator

1. Introduction

Majorana fermion (MF) is a particle long thought to exist since the prediction by Ettore Majorana [1]. In spite of many efforts, MFs have not been identified yet as an elementary particle. On the other hand, in recent years, topological superconductors (TSCs) with a non-trivial bulk topological invariant have been found to host MFs as a collective excitation [2, 3, 4, 5, 6, 7]. In particular, spin-orbit coupled s-wave TSCs have attracted much attention in this context [8, 9, 10, 11, 12, 13]. One of the interesting features of MFs is the non-Abelian anyon statistics [19], which can be applied to the fault tolerant topological quantum computation [20, 21].

To identify MFs in TSCs, NS junctions have been considered. Indeed, if an NS junction supports a single MF, the charge conductance through the NS junction shows a distinct zero-bias conductance peak [22, 23, 24, 25, 26, 27]. However, topological analyses [22, 30, 31, 32] have indicated that multiple MFs can show up in various topological phases if the systems are characterized by an integer topological number \( N \in \mathbb{Z} \geq 2 \). Then a natural question is how to identify multiple MFs? One might expect that \( N \) MFs show a zero-bias conductance peak that is \( N \) times larger than that for a single MF, but this is not always the case. It has been reported that the zero-bias conductance in NS junctions with two MFs can vanish for a quantum anomalous Hall system [33] and for an effective model with single channel in the normal metal [34]. Therefore, more detailed analysis on the charge transport in a system with two MFs is needed.

In this paper, we reveal that two MFs show a variety of the tunneling conductance owing to the destructive interference. The conductance is determined by symmetry of the system. This result implies that symmetry consideration on MFs is essential to understand transport experiments of NS junctions with multiple MFs.

The paper is organized as follows. First, we introduce the effective model and derive the charge conductance of an NS junction with two MFs in Sec. 2. Next, in Sec. 3 we review the NS junction of a superconducting quantum anomalous Hall insulator and apply our effective theory to it. The conductance vanishing is explained in the viewpoint of symmetry of the MFs. Finally, we summarize our results and discuss a future perspective in Sec. 4.

2. Charge conductance for two Majorana fermions

Here we derive a general formula for the charge conductance of the effective model describing a NS junction with two MFs. The behavior of the charge conductance is discussed based on this effective theory.

2.1. Effective model of the NS junction

We extend the effective model [22, 23] of NSN junctions to that of a NS junction with two MFs as an Andreev bound state. In the low energy, quasiparticles in the bulk superconductor are neglected. The Hamiltonian...
reads
\[ H = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} + i E_M \gamma_1 \gamma_2 - i E_M \gamma_2 \gamma_1 \]
\[ + \sum_{k\sigma} \frac{2}{t_{\sigma}} (c_{k\sigma}^\dagger - t_{\sigma} c_{-k\sigma}) \gamma_i, \]
where \( c_{k\sigma} \) is an annihilation operator of an electron in the normal metal with momentum \( k \) and spin \( \sigma = \uparrow, \downarrow \), and \( \gamma_i (i = 1, 2) \) denote MFs satisfying \( \gamma_i = \gamma_i^\dagger \). The first term of the above equation is the kinetic energy of the normal metal, and \( \xi_k \) is the kinetic energy measured from the Fermi level \( \mu \). The second and third terms denote the hybridization between the MFs as \( i E_M \gamma_1 \gamma_2 \) and the relation \( (c_{k\sigma}^\dagger \gamma_i)^\dagger = t_{\sigma}^* \gamma_i c_{k\sigma} \) leads the last term of Eq. \( \text{(1)} \).

2.2. Charge conductance

Now we calculate the charge conductance using the effective model. The last term of Eq. \( \text{(1)} \) yields that an electron (hole) is converted to a MF with the transition amplitude \( t_{\sigma} \) (\( -t_{\sigma}^* \)), and the inverse process is also possible. This means that an electron can transit to an hole mediated by a MF. This process, i.e., the Andreev reflection contributes to charge transport with charge flow \( 2e \). The corresponding conductance is given by \( \text{(2)} \).

\[ G(\omega) = \frac{2e^2}{h} \sum_{\sigma\sigma'} |T_{\sigma\sigma'}|^2, \]  
\[ \hat{T} = \begin{pmatrix} T_{\uparrow\uparrow} & T_{\uparrow\downarrow} & T_{\downarrow\uparrow} & T_{\downarrow\downarrow} \\ T_{\uparrow\downarrow} & T_{\uparrow\uparrow} & T_{\downarrow\downarrow} & T_{\downarrow\uparrow} \\ T_{\downarrow\uparrow} & T_{\downarrow\downarrow} & T_{\uparrow\uparrow} & T_{\uparrow\downarrow} \\ T_{\downarrow\uparrow} & T_{\downarrow\downarrow} & T_{\uparrow\downarrow} & T_{\uparrow\uparrow} \end{pmatrix} = \frac{\omega}{\omega - H_M + i\hat{\omega}^\dagger \hat{\omega}}, \]

Here \( \omega = eV \) with the bias voltage \( V \), and the matrices \( \hat{\omega} \) and \( H_M \) are defined by

\[ \hat{\omega} = \begin{pmatrix} \tilde{t}_{11} & \tilde{t}_{12} \\ i\tilde{t}_{11} & -\tilde{t}_{12} \\ -\tilde{t}_{21} & -\tilde{t}_{22} \end{pmatrix}, \]
\[ H_M = \begin{pmatrix} 0 & iE_M \\ -iE_M & 0 \end{pmatrix}, \]

respectively. \( \tilde{t}_{\sigma} \) is defined by \( \tilde{t}_{\sigma} = \sqrt{\mu} \rho_0 t_{\sigma} \) with \( \rho_0 \) being the density of states at the Fermi level. The explicit form of \( G(\omega) \) is easily obtained. (See Appendix A).

First, let us mention that the form of the effective Hamiltonian is severely restricted by the Majorana condition \( \gamma_i = \gamma_i^\dagger \). In the \( s_z \)-conserving system, one of the MFs has spin up while the other has spin down, and the system is decoupled into the up spin and the down spin sectors. Consequently, one obtains \( t_{11}, t_{22} = 0, t_{12} = t_{21} = 0 \). The resulting zero-bias conductance is always given by \( G = 4e^2/h \).

In the following, we focus on several important cases with \( \omega = 0 \) and \( E_M = 0 \). In the below, \( \theta_i \) is the phase of the hybridization amplitude; \( t_{1\uparrow} = t_{\sigma} e^{\theta_{1\uparrow}} \).

I) Unitary case: \( t_{1\uparrow} = \eta_{1\uparrow} t_{1\uparrow} \).

a) same sign case: \( \eta_{1\uparrow} \eta_{2\downarrow} = 1 \).

\[ T_{\sigma\sigma'} = \begin{cases} \frac{i}{2} e^{i2\theta_{1\uparrow}}, & (\theta_1 - \theta_2)/\pi \in \mathbb{Z} \\ 0, & (\theta_1 - \theta_2)/\pi \notin \mathbb{Z} \end{cases}, \]

b) different sign case: \( \eta_{1\uparrow} \eta_{2\downarrow} = -1 \).

\[ T_{\sigma\sigma'} = \begin{cases} \frac{i}{2} e^{i(\theta_1 + \theta_2)} \cos(\theta_1 - \theta_2), & (\theta_1 - \theta_2)/\pi \in \mathbb{Z} \\ 0, & (\theta_1 - \theta_2)/\pi \notin \mathbb{Z} \end{cases}. \]

II) Anti-unitary case: \( t_{1\uparrow} = \eta_{1\uparrow} t_{1\uparrow} \).

a) same sign case: \( \eta_{1\uparrow} \eta_{2\downarrow} = 1 \).

\[ T_{\sigma\sigma'} = \begin{cases} \frac{i}{2} e^{i2\theta_{1\uparrow}}, & (\theta_1 - \theta_2)/\pi \in \mathbb{Z} \\ 0, & (\theta_1 - \theta_2)/\pi \notin \mathbb{Z} \end{cases}. \]

b) different sign case: \( \eta_{1\uparrow} \eta_{2\downarrow} = -1 \).

\[ T_{\sigma\sigma'} = -T_{\sigma\sigma'}^* = \begin{cases} \frac{i}{2} e^{i(\theta_1 + \theta_2)} \cos(\theta_1 - \theta_2), & (\theta_1 - \theta_2)/\pi \in \mathbb{Z} \\ 0, & (\theta_1 - \theta_2)/\pi \notin \mathbb{Z} \end{cases}. \]
Eqs. (11) and (12) are modified as

\[ t_{\tilde{\tau}_i} = \eta_1 t_{\tilde{\tau}_i} \quad t_{\tilde{\tau}_i} = \eta_2^* t_{\tilde{\tau}_i} \]

with \( \eta_1 \eta_2 \neq 1 \) and \( \eta_1 \eta_2 = -1 \) are satisfied. In the case of the Hamiltonian of a superconducting QAH insulator \([39]\), the conductance vanishes. This peculiar behavior is robust against the hybridization between the MFs \((E_{3\mathcal{M}} \neq 0)\). When \( E_{3\mathcal{M}} \neq 0 \), Eqs. (13) and (14) are modified as

\[ T_{\tilde{\tau}_i, \tilde{\tau}_j} = -T_{\tilde{\tau}_j, \tilde{\tau}_i} = -\frac{i\kappa \cos(\theta_1 - \theta_2)}{1 + \kappa^2} \]

\[ T_{\tilde{\tau}_i, \tilde{\tau}_j} = T_{\tilde{\tau}_j, \tilde{\tau}_i} = \frac{-i\kappa \cos(\theta_1 - \theta_2)}{1 + \kappa^2} \]

with \( \kappa = E_{3\mathcal{M}}/(4t_{\tilde{\tau}_0}) \), but the resulting conductance remains the same, i.e., \( 4\cos^2(\theta_1 - \theta_2)e^2/h \). Therefore, the conductance vanishes again when \( (\theta_1 - \theta_2)/\pi \in \mathbb{Z} + 1/2 \). We will see that the last case is realized in a superconducting quantum anomalous Hall insulator in the next Section.

### Conductance vanishing in a superconducting quantum anomalous Hall insulator

In the previous section, we present general results of the charge conductance for NS junctions with two MFs. The conductance depends on the phase differences of the couplings between the electron in the normal metal and the MFs on the interface. The phase differences are restricted by symmetry of the system. Here, we apply the results to the superconducting quantum anomalous Hall (QAH) insulator, in which the destructive interference occurs.

#### 3.1 Model

Based on the Hamiltonian of a QAH insulator,

\[ H_{\text{QAH}}(k_x, k_y) = m(k_x, k_y)\sigma_z + A(k_x, \sigma_x + k_y, \sigma_y) \]

Table 1: Zero-bias charge conductance for NS junctions with two Majorana fermions. \( \eta_1 \) is the phase of the hybridization amplitude; \( t_{\tilde{\tau}_i} = t_{\tilde{\tau}_i}e^{\theta_i} \). Here, we assume that \( (\theta_1 - \theta_2)/\pi \notin \mathbb{Z} \) for \( \eta_1 \eta_2 = +1 \) in both the unitary and anti-unitary cases.

| \( \eta_1 \eta_2 \) | Unitary | Anti-unitary |
|-----------------|---------|-------------|
| +1 | 0 | \( 4e^2/h \) |
| -1 | \( 4e^2/h \) | \( 2\cos^2(\theta_1 - \theta_2)2e^2/h \) |

#### Table 2: Topological phase of the normal state \( H_{\text{QAH}} \)

| Mass term | Chern number |
|-----------|-------------|
| \( m_0 > +\sqrt{\Delta^2 + \mu^2} \) | \( N = 0 \) |
| \( m_0 < +\sqrt{\Delta^2 + \mu^2} \) | \( N = 1 \) |

#### Table 3: Topological phase of the superconducting state \( H_{\text{SQAH}} \)

| Mass term | Chern number |
|-----------|-------------|
| \( m_0 > +\sqrt{\Delta^2 + \mu^2} \) | \( N = 0 \) |
| \( m_0 < +\sqrt{\Delta^2 + \mu^2} \) | \( N = 1 \) |

the Hamiltonian of a superconducting QAH insulator \([35, 23, 24, 25]\) is given by

\[ H_{\text{SQAH}}(k_x, k_y) = \left( H_{\text{QAH}}(k_x, k_y) - \mu \right) - \frac{i\sigma_y \Delta}{\mu \sigma_y \Delta} - \frac{i\sigma_y \Delta}{\mu \sigma_y \Delta} - \frac{i\sigma_y \Delta}{\mu \sigma_y \Delta} - \Delta \sigma_y \tau_y \]

with \( m(k_x, k_y) = m_0 + B(k_x^2 + k_y^2) \). Here \( m_0 \) is the band gap between the conduction and the valence bands, \( 1/2B \) is the effective masses of these bands, which are the same in this model, for simplicity, \( \mu \) is the chemical potential controlled by the carrier doping, and \( \Delta \) is the pair potential induced by an s-wave superconductor attached to the QAH insulator. \( \tau_i \) and \( \sigma_i \) are the Pauli matrices in the Nambu space and the spin space, respectively. Hereafter we assume \( B > 0 \) without loss of generality.

The topological phase of this system is characterized by the Chern numbers \([39]\). For the QAH system \( H_{\text{QAH}} \), the Chern number \( N \) is defined by

\[ N = \frac{1}{2\pi} \int d^2k \left( \frac{\partial a_y(k)}{\partial k_x} - \frac{\partial a_x(k)}{\partial k_y} \right) \]

where \( a_y(k) = -i(k)\partial/\partial k_y |k\rangle \) and \( |k\rangle \) is the eigenvector of \( H_{\text{QAH}}(k) \) for the valence band. The Chern number is given by \( N = 0 \) for \( m_0 > 0 \) (nontopological insulator) and \( N = 1 \) (QAH insulator) for \( m_0 < 0 \) (Table 2). In a similar manner, the Chern number \( N \) for the superconducting state \( H_{\text{SQAH}} \) is defined. Note that when \( \Delta = 0 \), \( N \) reduces to \( 2N \) since the hole component in the BdG Hamiltonian redundantly contributes to the Chern number. In the case of \( \Delta \neq 0 \), the Chern number \( N \) is obtained to be \( N = 0 \) for \( m_0 > \sqrt{\Delta^2 + \mu^2} \), \( N = 1 \) for \( |m_0| < \sqrt{\Delta^2 + \mu^2} \), and \( \Delta = 2 \) for \( m_0 < -\sqrt{\Delta^2 + \mu^2} \) (Table 3). From the bulk-edge correspondence, there are \( N \) Majorana edge states in each phase.

The bulk and edge energy dispersions are shown in Figs (a)-(f) for the \( N = 2, 1, \) and 0 phases. For \( \mu = 0 \), while the systems with \( N = 2 \) and \( N = 1 \) (Figs. (a) and (b)) have the similar energy dispersions, two MFs are degenerated in the momentum space in the case of \( N = 2 \).
On the other hand, for $\mu \neq 0$, the degeneracy of the MFs in the case of $N = 2$ is lifted (Fig. 1) since the chemical potential gives rise to an additional coupling between the MFs.

3.2. Mapping to the effective model

The electronic and transport properties of the bulk and the edge in the superconducting QAH insulator are extensively studied. Below, we discuss why the conductance vanishes in the $N = 2$ phase from the viewpoint of the effective theory discussed in the previous section. In the low transmissivity limit of the NS junction, the Andreev reflection occurs only through MFs localized at the interface. The NS junction can be mapped to the effective model in this limit. For instance, such a situation is realized for NS junctions with a thick oxide layer between the normal metal and superconductor.

For this mapping, let us first examine the symmetry of the QAH insulator and its superconducting state. Because of an internal magnetization given by $m(k_x, k_y) \sigma_z$, neither the time-reversal symmetry or the two-fold rotational symmetry with respect to the $x$-axis is preserved in the QAH insulator. Nevertheless, the combination of them is preserved, which gives a hidden time-reversal symmetry of the system. Correspondingly, the superconducting QAH insulator also has the same hidden time-reversal symmetry since the $s$-wave gap function does not break any symmetry. As a result, the BdG Hamiltonian of the superconducting QAH insulator satisfies $\tilde{\Theta} H_{\text{QAH}}(k_x, k_y) \tilde{\Theta}^\dagger = H_{\text{SQAH}}(-k_x, k_y)$ with $\tilde{\Theta} = \sigma_z \tau_y$. In addition, the BdG Hamiltonian has the particle-hole symmetry, $\tau_z H_{\text{QAH}}(k) \tau_z - H_{\text{SQAH}}(-k)$ as an intrinsic symmetry of a superconductor. Moreover, to discuss the effective model, it is convenient to consider the hidden chiral symmetry that is obtained by combining the hidden time-reversal symmetry with the particle-hole symmetry. The explicit form of the hidden chiral symmetry is given by $(\Gamma, H_{\text{QAH}}(k)) = 0$ at $k_y = 0$ with $\Gamma = \tilde{\Theta} \tau_x = \sigma_y \tau_y$. We dub the eigenvalue of $\Gamma = \pm 1$ as chirality.

When the system has such a chiral symmetry, the BdG Hamiltonian becomes off-diagonal if we take the basis where $\Gamma$ is diagonal. This restricts possible couplings of the system. Indeed, only couplings between states with opposite chiralities are possible.

Since the effective model of NS junctions we considered is one-dimensional, we perform the dimensional reduction of the superconducting QAH insulator by fixing $k_y$ as $k_y = 0$. The resultant one-dimensional system also has the hidden chiral symmetry. It also has the quasiparticle spectrum of the superconducting QAH insulator at $k_y = 0$. Therefore, as illustrated in Fig. 1(a), there are two degenerate Majorana zero modes $\gamma_1$ and $\gamma_2$ in the $N = 2$ phase when $\mu = 0$.

When $\mu \neq 0$, the two Majorana zero modes are gapped, as is seen in Fig. 1(d). Therefore, the chemical potential $\mu$ induces the mass term, $iE_M |\gamma_1\gamma_2 - iE_M |\gamma_2\gamma_1|$, in Eq. (1).

As was mentioned in the above, since the hidden chiral symmetry admits only the coupling between states with opposite chiralities, this means that $\gamma_1$ and $\gamma_2$ have opposite chiralities. Without loss of generality, we assume that $\gamma_1$ has the chirality $\Gamma = +1$, and $\gamma_2$ has the chirality $\Gamma = -1$.

Using the hidden chiral symmetry, we can determine the couplings between $c_{k\sigma}$ and $\gamma_i$: In the Nambu basis, $(c_{k\uparrow}, c_{-k\downarrow}, c_{-k\uparrow}, c_{k\downarrow})^T$, the two independent eigenstates with $\Gamma = 1$ are given by $(1, 0, 0, 0)^T$ and $(0, 1, 0, -i)^T$, which correspond to the operators, $c_{k\uparrow} + ic_{-k\uparrow}$ and $c_{k\downarrow} + ic_{-k\downarrow}$, respectively, and the eigenstates with $\Gamma = -1$ are $(1, 0, -i, 0)^T$ and $(0, 1, 0, i)^T$, which correspond to $c_{k\uparrow} + ic_{-k\uparrow}$ and $c_{k\downarrow} - ic_{-k\downarrow}$, respectively. Thus if the hidden chiral symmetry is preserved, the possible couplings are $(c_{k\uparrow} + ic_{-k\uparrow})\gamma_1$, $(c_{k\downarrow} - ic_{-k\downarrow})\gamma_1$, $(c_{k\uparrow} + ic_{-k\uparrow})\gamma_2$, and $(c_{k\downarrow} + ic_{-k\downarrow})\gamma_2$.

In terms of $t_{\sigma i}$ in Eq. (4), these couplings imply that

$$
\begin{align*}
t_{11} &= -it_{11}^\ast, \quad t_{12} = it_{12}^\ast, \\
t_{22} &= it_{22}^\ast, \quad t_{12} = -it_{12}^\ast.
\end{align*}
$$

From these relations, we obtain

$$
G = G^{(\pm)} = \frac{4e^2}{h} \left[ 1 - \frac{(|t_{11}|^2 + |t_{12}|^2)^2}{\Gamma_1 \Gamma_2} \right],
$$

$$
= \frac{4e^2}{h} \left[ |t_{11}|^2 + |t_{12}|^2 \right] \frac{|t_{11}|^2}{\Gamma_1 \Gamma_2}
$$

with $\Gamma_i = \sum_\sigma |t_{\sigma i}|^2$. Here note that only $G^{(\pm)}$ can be zero smoothly with finite $t_{\sigma i}$. This enables us to choose $G^{(\pm)}$ as the conductance for the NS junction of the superconducting QAH insulator. Indeed, as is shown below, $G$ in the superconducting QAH insulator can go to zero smoothly as $\mu \to 0$.

Let us now consider the case of $\mu = 0$. When $\mu = 0$, the BdG Hamiltonian $H_{\text{SQAH}}(k)$ has an accidental symmetry, $[H_{\text{SQAH}}(k), \sigma_y \tau_x] = 0$, while this symmetry is broken by the chemical potential $\mu$. Combining this with hidden chiral symmetry, we can obtain another hidden chiral symmetry, $(H_{\text{SQAH}}(k), \Gamma') = 0$ with $\Gamma' = \sigma_y \tau_z$, at $k_y = 0$ when $\mu = 0$. In contrast to $\Gamma$, the second chiral symmetry $\Gamma'$ is broken by $\mu$, and thus the chemical potential term is diagonal in the basis where $\Gamma'$ is diagonal. This implies that $\gamma_1$ and $\gamma_2$ have the same chirality $\chi$ of $\Gamma'$, because the mass term $iE_M (\tau_2 \gamma_1 - i\tau_1 \gamma_2)$ corresponding to the chemical potential term $\mu$ also should be diagonal in the basis where $\Gamma'$ is diagonal.

Here we assume that the interface between the normal metal and the superconducting QAH insulator preserves this accidental chiral symmetry. Actually, in our numerical calculations in Ref. 53 and in the next subsection, any symmetry is not broken by the boundary condition at the interface. Under this assumption, we can determine possible couplings between $c_{k\sigma}$ and $\gamma_i$ in a manner similar to the $\Gamma$ case: In the Nambu basis, $(c_{k\uparrow}, c_{-k\downarrow}, c_{-k\uparrow}, c_{k\downarrow})^T$, the two independent eigenstates with $\Gamma' = 1$ are given by

...
These relations yield \( \chi_{ic} \). Therefore, the possible couplings are (1, \( \gamma \)), (0, \( \gamma \)), respectively, and the eigenstates with \( \Gamma' = 1 \) are (1, \( \gamma \)), (0, \( \gamma \)), which correspond to \( c_k + ic_k \) and \( c_k^\dagger - ic_k^\dagger \), respectively. Therefore, the possible couplings are \( c_k + ic_k \), \( c_k^\dagger - ic_k^\dagger \), respectively. In terms of \( t_{\sigma i} \) in Eq. (11), these couplings give

\[
t_{11} = i\chi t_{11}, \quad t_{12} = i\chi t_{12}.
\]

From Eqs. (13) and (21), we can see that \( G = 0 \) when \( \mu = 0 \) in the effective model of the superconducting QAH insulator: These relations yield

\[
t_{11} = -\chi t_{11}, \quad t_{12} = \chi t_{12}, \quad \theta_1 - \theta_2 = \frac{\pi}{2} + n\pi,
\]

with \( n \in \mathbb{Z} \), and thus the present case reduces to the anti-unitary case with \( \eta_1 \theta_2 = -1 \) in Sec. 2.2. Therefore, the conduction, \( G = (4e^2/h)\cos^2(\theta_1 - \theta_2) \), vanishes.

Before closing this subsection, we would like to make some comments. (1) First, the above discussions reveal that the complete destructive interference of MFs in the \( \mathcal{N} = 2 \) phase is not robust. While the accidental symmetry is needed to obtain \( G = 0 \), this symmetry is easily broken by the chemical potential \( \mu \). To make things worse, the normal metal attached to the NS junction does not have such an accidental symmetry. Thus an ignored interaction in a real system also may break the accidental symmetry. (2) In spite that the complete destructive interference cannot be expected in a real system, as we mentioned above, the destructive interference of MFs can be expected in the

\[
\mathcal{N} = 2 \text{ phase of the superconducting QAH insulator. Since the hidden chiral symmetry } \Gamma \text{ is originated from the remnant of the two-fold rotation symmetry, it is not broken as far as the NS junction respects the two-fold rotation symmetry. In this sense, } \Gamma \text{ is intrinsic. Because the conductance } G = G^{(1)} \text{ is less than } 4e^2/h, \text{ our effective model predicts that the interference of two MFs is always destructive in the } \mathcal{N} = 2 \text{ phase of the superconducting QAH insulator.}
\]

3.3. Comparison between the original and effective models

To confirm the validity of the effective model, we compare the conductance obtained from the effective model and from those form the original model described by \( H_{SQAH} \).
The NS junction is illustrated in Fig. 2. For comparison, we consider the one-dimensional NS junction ($k_y = 0$) as in the previous subsection. The Hamiltonian of the NS junction is given by

$$H(x < 0) = \left( \frac{k_x^2}{2m_N} - \mu N \right) \tau_z,$$

(23)

$$H(x > 0) = H_{SQAH}(k_x \rightarrow -i\partial_x, k_y = 0),$$

(24)

with $m_N$ and $\mu N$ being the effective mass and the chemical potential in the normal metal ($x < 0$), respectively. We calculate the charge conductance in the NS junction developing the Blonder–Tinkham–Klapwijk theory [33]. The obtained data are shown in Fig. 3. The left panel shows the conductance as a function of bias voltage $V$ for the $\mathcal{N} = 2$, 1, and 0 phases. The chemical potential is set to $\mu = 0$. At the zero bias voltage $V = 0$, one can clearly see $G = 2e^2/h$ for the $\mathcal{N} = 1$ phase while $G = 0$ for the $\mathcal{N} = 0$ and $\mathcal{N} = 2$ phases. Note that the conductance vanishes even though two-fold degenerated MFs exists at the zero energy. The conductance vanishing at $V = 0$ for $\mathcal{N} = 2$ is consistent with the effective theory.

As illustrated in the right panel of Fig. 3, the zero-bias conductance in the original model takes a nonzero value for a non-zero $\mu$ and a small $\mu N$. This behaviour is also consistent with our consideration in the previous section: When $\mu \neq 0$, the relation Eq. (21) does not hold since the accidental chiral symmetry $\Gamma'$ is broken by $\mu$. Therefore, the conductance $G$ in the effective model can be nonzero. In addition, when $\mu N$ is small, the Andreev reflection can occur directly without the mediation of MFs, because the decrease of $\mu N$ increases the transmissivity of the NS junction. This effect is beyond the range of our effective model.

4. Discussion

In this paper we have shown that MFs can cause destructive interference, based on the effective theory focusing on the MFs. We have derived a general formula of the conductance for NS junctions with two MFs. In particular, for a quantum anomalous Hall system, the conductance completely vanishes when the phase difference of the MFs is given by $\pm \pi/2$ although the density of states of the MFs is nonzero. This is the direct consequence of interference of MFs. It is worth mentioning that this interference is robust against perturbations since it stems from the chiral symmetry of the system.

One of the systems exhibiting the interference of MFs is a superconducting QAH system, which can be realized in magnetically doped/ordered topological insulators [42, 43, 44, 45, 46]. Recently, the QAH effect in such a system has been experimentally observed [47]. Our theory will be relevant to future experiments on superconducting proximity effect for QAH systems.

The present results suggest that the conductance in systems with $N$-fold degenerated MFs is not given by $2Ne^2/h$ (This will be discussed elsewhere). Generally, there are even-odd effects by the number of MFs on the conductance: if the total number of MFs is even, the resulting conductance at zero energy may vanish, but if it is odd, the conductance must be finite. This is because an unpaired MF exists in the latter case. However, this does not mean that the even-odd effects always occur. Indeed, if there are additional symmetries which stabilize MFs, even-odd effects can be obscure. For example, if the system supports time-reversal symmetry, the tunneling conductance cannot be zero in spite of two MFs. A clear experimental signal for MFs is the quantized conductance $G = 2Ne^2/h$, when the $N$ MFs are somehow divided into $N$ independent sectors. Otherwise, the MFs cause interference ($G \neq 2Ne^2/h$), thus they can form a Dirac fermion rather than MFs. Then a careful analysis, e.g., symmetry consideration, is needed to correctly understand the zero-bias conductance for multiple MFs systems.

Note added. Upon completing the manuscript, we became aware of a work that discusses the complete destructive interference of MFs in the SQAH insulator by J. J. He, J. Wu, T. P. Choy, X.-J. Liu, Y. Tanaka, and K. T. Law [48].

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Appendix A. General formula of charge conductance for the effective model

The T-matrix Eq. (23) is given by

$$T_{\sigma e, \sigma h}(\omega) = \left[ -2(i\Gamma_{12} + i\Gamma_{21}) \omega - i4(\tilde{\Gamma}_{\sigma 1}\Gamma_{22} + i\tilde{\Gamma}_{\sigma 2}\Gamma_{11}) \right] / X, \quad (A.1)$$

$$T_{\sigma e, -\sigma h}(\omega) = \left[ -2(\tilde{\Gamma}_{\sigma 1}\tilde{\Gamma}_{-\sigma 1} + \tilde{\Gamma}_{\sigma 2}\tilde{\Gamma}_{-\sigma 2}) \omega - i4(\tilde{\Gamma}_{\sigma 1}\tilde{\Gamma}_{-\sigma 1}\Gamma_{22} + \tilde{\Gamma}_{\sigma 2}\tilde{\Gamma}_{-\sigma 2}\Gamma_{11}) \right] / X, \quad (A.2)$$

where the denominator $X$ of the T-matrix is given by

$$X = (\omega + i2\Gamma_{11})(\omega + i2\Gamma_{22}) + (\Gamma_{11} + \Gamma_{22})^2 - E_M^2, \quad (A.3)$$

with $\Gamma_{ij} = \sum_\sigma \tilde{\Gamma}_{\sigma 1}t_{\sigma j}$. 

6
Figure 3: Charge conductance $G$ in unit of $e^2/h$ for a one-dimensional NS junction of SQAH insulator for the $N = 2, 1, 0$ phases. The left panel shows $G$ as a function of the bias voltage $V$ for $\mu = 0$ and $\mu_N = 100$. The right panel shows $G$ as a function of the chemical potential $\mu_N$ of the normal metal for $V = 0$ and $N = 2$. $E_g$ is the energy gap of the SQAH insulator and $\Delta$ is the pair potential. The mass term is set to $m_0 = -0.2$ for $N = 2$, $m_0 = 0$ for $N = 1$, and $m_0 = 0.2$ for $N = 0$. The other parameters are taken as follows: $A = B = 1$, $\Delta = 0.1$, $m_N B = 1$. In $\mu_N \to \infty$ limit, which corresponds to the low transmissivity limit, the effective theory proposed in the paper becomes exact.

Appendix B. Tunneling conductance of the NS junction

In this Appendix, we explain how to calculate the tunneling conductance for a NS junction of the superconducting QAH insulator. $N$ ($S$) is located in $x < 0$ ($x > 0$). The Hamiltonian of the NS junction is given by Eqs. (23) and (24). The wave functions $\psi$ of the scattering state for $x < 0$ and $x > 0$ has the following form,

$$\psi(x < 0) = \chi_{\sigma e} e^{ik_x x} + \sum_{\sigma' r} r_{\sigma \sigma' r} \chi_{\sigma' e} e^{-ik_x x},$$

(B.1) \hspace{1cm} $$\psi(x > 0) = \sum_{\mu} t_{\sigma \mu} u_{\mu}(q_{\mu}) e^{iq_{\mu} x},$$

(B.2)

where $\sigma$ is spin of the incident electron, $k_x = \tau \sqrt{2m_N (\mu_N + \tau E)}$ and $\chi_{\sigma e}$ and $\chi_{\sigma' h}$ are eigenvectors for electron and hole states in $N$ with spin $\sigma'$, and $\tau = e = +1$, $\tau = h = -1$. $u_{\mu}(q_{\mu})$ is the eigenvector of $H_{\text{SQAH}}$:

$$H_{\text{SQAH}}(q_{\mu}) u_{\mu}(q_{\mu}) = E_{\mu}(q_{\mu}) u_{\mu}(q_{\mu}).$$

(B.3)

The energy satisfies $E_{\mu}(q_{\mu}) = E$, where $\mu$ is the band index. The momentum is determined by $\text{Im}(q_{\mu}) > 0$ for an evanescent state, $\partial E_{\mu}(q_{\mu})/\partial q_{\mu} > 0$ and $\text{Im}(q_{\mu}) = 0$ for a propagating state. Note that there are four states satisfying the above conditions.

The tunneling conductance is given by

$$G = \frac{e^2}{h} \left[ 2 - \sum_{\sigma \sigma'} \left( |r_{\sigma \sigma' e}|^2 + |r_{\sigma \sigma' h}|^2 \right) \right].$$

(B.4)

The reflection coefficient $r_{\sigma \sigma' \tau}$ is deduced by applying the continuity condition on the wave function:

$$\psi_{\sigma}(-0) = \psi_{\sigma}(+0),$$

(B.5) \hspace{1cm} $$v^N \psi_{\sigma}(-0) = v_{\text{SQAH}} \psi_{\sigma}(+0),$$

(B.6)

being the velocity operators in $N$ and $S$, respectively.

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