Convex Hull of Two Orthogonal Disks

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Abstract. Three configurations of two perpendicular disks in \( \mathbb{R}^3 \) are examined, the first in which the disks share centers and the other two in which the disks touch at precisely one point. Volume, surface area and mean width calculations dominate the discussion. Integrated mean curvature also appears as an indirect way to compute mean width.

Our investigation begins with a theoretical question about experimental data. Example 1 is the convex hull of the following two orthogonal disks in \( \mathbb{R}^3 \):

\[
\{(x, y, z) : x^2 + y^2 \leq 1 \& z = 0\} \quad \text{and} \quad \{(x, y, z) : x^2 + z^2 \leq 1 \& y = 0\}.
\]

We can numerically evaluate the volume \( VL \), surface area \( AR \) and mean width \( MW \) of the corresponding solid domain in Figure 1 using [1]:

\[
VL_1 \approx 2.666, \quad AR_1 \approx 10.28, \quad MW_1 \approx 1.869.
\]

Example 2 is the convex hull of the two disks:

\[
\{(x, y, z) : x^2 + y^2 \leq 1 \& z = -1\} \quad \text{and} \quad \{(x, y, z) : x^2 + z^2 \leq 1 \& y = 1\}
\]

with corresponding solid domain in Figure 2 and

\[
VL_2 \approx 3.141, \quad AR_2 \approx 13.92, \quad MW_2 \approx 2.277.
\]

Example 3 is the convex hull of the two disks:

\[
\{(x, y, z) : x^2 + (y + 1)^2 \leq 1 \& z = 0\} \quad \text{and} \quad \{(x, y, z) : (y - 1)^2 + z^2 \leq 1 \& x = 0\}
\]

with corresponding solid domain in Figure 3 and

\[
VL_3 \approx 3.627, \quad AR_3 \approx 15.97, \quad MW_3 \approx 2.645.
\]

Of the nine constants, just two (\( VL_1 = 8/3 \) and \( VL_2 = \pi \)) are readily identifiable. What are exact closed-form expressions for the remaining constants?

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Given \( \Omega \) to be a convex body in \( \mathbb{R}^3 \), a \textit{width} is the distance between a pair of parallel \( \Omega \)-supporting planes. Every unit vector \( v \in \mathbb{R}^3 \) determines a unique such pair of planes orthogonal to \( v \) and hence a width \( w(v) \). Let \( v \) be uniformly distributed on the unit sphere \( S^2 \subset \mathbb{R}^3 \). Then \( w \) is a random variable and its average value is the \textbf{mean width} of \( \Omega \). Three numerical characteristics of \( \Omega \) – volume, surface area and mean width – are central to our study. These quantities, along with the Euler characteristic, form a basis of the space of all additive continuous measures that are invariant under rigid motions in \( \mathbb{R}^3 \).

“The mean width is a new measure on three-dimensional solids that enjoys equal rights with volume and surface area” [2], hence much of this paper is devoted to computing \( MW \) for our three examples. What we call the direct approach is based on the definition of \( MW \); what we call the \textit{indirect} approach utilizes a connection between \( MW \) and integrated mean curvature (often called ”integral” or “total” mean curvature). This connection is suggested in the materials science [3, 4] and astrophysics literature [5, 6]; the closest claim to a proof appears in [7], based chiefly on [8]. Our paper therefore also serves to confirm the validity of the indirect approach for certain non-polyhedral test cases.\(^1\)

1. \textbf{Example 1}

The boundary \( \partial \Omega \) of the convex hull \( \Omega \) here is trivially given by the surface

\[
z = \pm \left( \sqrt{1 - x^2} - |y| \right)
\]

over the planar region \( x^2 + y^2 \leq 1 \). Let

\[
\varphi(x, y) = \sqrt{1 - x^2} - y, \quad x^2 + y^2 \leq 1 \& y \geq 0
\]

then \( \varphi_x, \varphi_y, \varphi_{xx}, \varphi_{xy}, \varphi_{yy} \) denote first/second-order partial derivatives of \( \varphi \) and

\[
VL_1 = 4 \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \varphi(x, y) \, dy \, dx = \frac{8}{3},
\]

\[
AR_1 = \int_{\partial \Omega} dS = 4 \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \sqrt{1 + \varphi_x^2 + \varphi_y^2} \, dy \, dx = 2(2 + \pi)
\]

\[
= 10.2831853071795864769252867....
\]

\(^1\)On page 513 of [7], mean curvature \( K \) is defined as the \textit{average} of the two principal curvatures, but this is inconsistent with [8], which takes \( K \) to be the \textit{sum}. We follow [8], defining \( 2H = K \). Our formula correctly gives \( MW = (\ell + \pi r) / 2 \) for a right circular cylinder of length \( \ell \), radius \( r \) [9, 10].
1.1. **Indirect Approach.** Let $\partial\Omega^+$ denote the upper portion of $\partial\Omega$ and $\partial\Omega^-$ denote the lower portion. On the one hand, the mean curvature of $\partial\Omega^+$ is

$$H(x, y) = -\frac{(1 + \varphi_x^2) \varphi_{yy} - 2\varphi_x \varphi_y \varphi_{xy} + (1 + \varphi_y^2) \varphi_{xx}}{2 (1 + \varphi_x^2 + \varphi_y^2)^{3/2}} = \frac{1}{(2 - x^2)^{3/2}}$$

over the open region $x^2 + y^2 < 1$ & $y > 0$. It follows that

$$H dS = H(x, y) \sqrt{1 + \varphi_x^2 + \varphi_y^2} dy dx = \frac{1}{(2 - x^2)^{3/2}} \sqrt{\frac{2 - x^2}{1 - x^2}} dy dx = \frac{dy dx}{(2 - x^2) \sqrt{1 - y^2}}$$

On the other hand, the exterior dihedral angle on the semicircular edge $x^2 + y^2 = 1$ & $y \geq 0$ is

$$\alpha = 2 \arccos \left( \frac{1}{\sqrt{2 - x^2}} \right)$$

because the unit exterior normal vector to $\partial\Omega^+$ is

$$\frac{1}{\sqrt{1 + \varphi_x^2 + \varphi_y^2}} (-\varphi_x, -\varphi_y, 1) = \sqrt{\frac{1 - x^2}{2 - x^2}} \left( \frac{x}{\sqrt{1 - x^2}}, 1, 1 \right) = \frac{1}{\sqrt{2 - x^2}} (x, y, y)$$

and the unit exterior normal vector to the cylinder $x^2 + y^2 = 1$ is $(x, y, 0)$. The dot product of the two vectors is $1/\sqrt{2 - x^2}$; we multiply the angle by two since the dihedral angle between $\partial\Omega^+$ and $\partial\Omega^-$ is twice the preceding angle. In terms of arclength $s = \theta$, $0 \leq \theta \leq \pi$, we have

$$\alpha ds = 2 \arccos \left( \frac{1}{\sqrt{2 - \cos^2 \theta}} \right) d\theta.$$
where \( \text{Li}_2 \) is the dilogarithm

\[
\text{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = -\int_0^x \frac{\ln(1-t)}{t} dt, \quad |x| \leq 1.
\]

1.2. **Direct Approach.** Consider the portion of \( \partial \Omega \) in the first octant only. In this octant, an \( \Omega \)-supporting plane \( P_d \):

\[
a \frac{dx}{d} + b \frac{dy}{d} + c \frac{dz}{d} = 1 \quad \text{(with coefficients } a > 0, b > 0, c > 0 \text{ and scaling factor } d > 0)\]

has an associated line \( L_{xy}^d \):

\[
a \frac{dx}{d} + b \frac{dy}{d} = 1 \quad \& \quad z = 0
\]

in the \( xy \)-plane and an associated line \( L_{xz}^d \):

\[
a \frac{dx}{d} + c \frac{dz}{d} = 1 \quad \& \quad y = 0
\]

in the \( xz \)-plane. Assume WLOG that \( a^2 + b^2 + c^2 = 1 \). The distance of \( P_d \) from the origin \( O \) is \( d \). Also,

\[
\text{distance of } L_{xy}^d \text{ from } O \text{ is } \frac{d}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \text{distance of } L_{xz}^d \text{ from } O \text{ is } \frac{d}{\sqrt{a^2 + c^2}}.
\]

The largest \( d \) such that

\[
L_{xy}^d \text{ supports } x^2 + y^2 = 1 \quad \text{or} \quad L_{xz}^d \text{ supports } x^2 + z^2 = 1
\]

is thus

\[
d = \max \left\{ \sqrt{a^2 + b^2}, \sqrt{a^2 + c^2} \right\}.
\]

Let

\[
a = \cos \theta \sin \phi, \quad b = \sin \theta \sin \phi, \quad c = \cos \phi
\]

where \( 0 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi/2 \). To ensure uniformity, think of \( (\theta, \phi) \) as possessing joint density \( \frac{2}{\pi} \sin \phi \). We have

\[
MW_1 = 2 \int_0^{\pi/2} \int_0^{\pi/2} \max \left\{ \sin \phi, \sqrt{\cos^2 \theta \sin^2 \phi + \cos^2 \phi} \right\} \frac{2}{\pi} \sin \phi d\phi d\theta
\]

\[
= \frac{8}{\pi} \int_0^{\pi/2} \int_{\xi(\theta)}^{\pi/2} \sin^2 \phi d\phi d\theta = 1.8697727582861870379136441...
\]
where
\[ \xi(\theta) = \arccos \left( \frac{\sin \theta}{\sqrt{2 - \cos^2 \theta}} \right) \]
is the required solution (for \( \phi \) in terms of \( \theta \)) of the equation
\[ \sin \phi = \sqrt{\cos^2 \theta \sin^2 \phi + \cos^2 \phi}. \]

2. Example 2

The curved portions of the boundary \( \partial \Omega \) of the convex hull \( \Omega \) here are given by \( z = \varphi(x, y) \) and \( z = \psi(x, y) \), where
\[ \varphi(x, y) = \sqrt{1 - x^2} + y - 1, \quad |x| \leq 1, \ |y| \leq 1 \ \& \ y \geq -\sqrt{1 - x^2}; \]
\[ \psi(x, y) = -\sqrt{1 - x^2} + y - 1, \quad |x| \leq 1, \ |y| \leq 1 \ \& \ y \geq \sqrt{1 - x^2}. \]
The flat portions of \( \partial \Omega \) are the two disks, one of which is given by \( z = -1 \) over \( x^2 + y^2 \leq 1 \). These facts contribute to the following:
\[ \mathcal{V}L_2 = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (\varphi(x, y) + 1) \, dy \, dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{1} (\varphi(x, y) - \psi(x, y)) \, dy \, dx = \pi, \]
\[ \mathcal{A}R_2 = 2\pi + \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + \varphi_x^2 + \varphi_y^2} \, dy \, dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{1} \sqrt{1 + \psi_x^2 + \psi_y^2} \, dy \, dx \]
\[ = 2 \left( \pi + 2\sqrt{2} E \left( \frac{1}{2} \right) \right) \]
\[ = 13.9235808852350105127348109... \]
where
\[ E(\mu) = \int_{0}^{\pi/2} \sqrt{1 - \mu \sin^2 \theta} \, d\theta = \int_{0}^{1} \sqrt{\frac{1 - \mu t^2}{1 - t^2}} \, dt \]
is the complete elliptic integral of the second kind.

2.1. Indirect Approach. Let \( \partial \Omega^+ \) denote the curved portion of \( \partial \Omega \) prescribed by \( \varphi \) and \( \partial \Omega^- \) denote the curved portion prescribed by \( \psi \). We have
\[ -\frac{(1 + \varphi_x^2) \varphi_{yy} - 2\varphi_x \varphi_y \varphi_{xy} + (1 + \varphi_y^2) \varphi_{xx}}{2 \left( 1 + \varphi_x^2 + \varphi_y^2 \right)^{3/2}} = \frac{1}{(2 - x^2)^{3/2}} \]
\[ = \frac{(1 + \psi_x^2) \psi_{yy} - 2\psi_x \psi_y \psi_{xy} + (1 + \psi_y^2) \psi_{xx}}{2 \left( 1 + \psi_x^2 + \psi_y^2 \right)^{3/2}}. \]
everywhere and hence
\[ H \, dS = \frac{dy \, dx}{(2 - x^2) \sqrt{1 - x^2}} \]
as previously. It follows that
\[ \int_{\partial \Omega} H \, dS = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{-\sqrt{1-x^2}} \frac{dy \, dx}{(2 - x^2) \sqrt{1 - x^2}} + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{dy \, dx}{(2 - x^2) \sqrt{1 - x^2}} = \sqrt{2}\pi. \]

Let \( \varepsilon \) denote the circular edge \( x^2 + y^2 = 1 \& z = -1 \). Clearly \( \partial \Omega^+ \cap \varepsilon \) is the semicircle with \( y = -\sqrt{1 - x^2} \) whereas \( \partial \Omega^- \cap \varepsilon \) is the semicircle with \( y = \sqrt{1 - x^2} \). The exterior dihedral angle on \( \varepsilon \) is
\[ \alpha = \arccos \left( \frac{y}{\sqrt{2 - x^2}} \right) \]
because the unit exterior normal vector to \( \partial \Omega^+, \partial \Omega^- \) is
\[ \frac{1}{\sqrt{1 + \varphi_x^2 + \varphi_y^2}} (\varphi_x, -\varphi_y, 1) = \sqrt{\frac{1 - x^2}{2 - x^2}} \left( \frac{x}{\sqrt{1 - x^2}}, -1, 1 \right) = \frac{1}{\sqrt{2 - x^2}} (x, y, -y), \]
\[ \frac{1}{\sqrt{1 + \psi_x^2 + \psi_y^2}} (\psi_x, \psi_y, -1) = \sqrt{\frac{1 - x^2}{2 - x^2}} \left( \frac{x}{\sqrt{1 - x^2}}, 1, -1 \right) = \frac{1}{\sqrt{2 - x^2}} (x, y, -y) \]
respectively and the unit exterior normal vector to the horizontal disk is \((0, 0, -1)\). The dot product of the two vectors is \( y/\sqrt{2 - x^2} \). An identical argument applies for the circular edge \( x^2 + z^2 = 1 \& y = 1 \). In terms of arclength \( s = \theta, 0 \leq \theta \leq 2\pi \), we obtain
\[ 2 \int_{\varepsilon} \alpha \, ds = 2 \int_{0}^{2\pi} \arccos \left( \frac{\sin \theta}{\sqrt{2 - \cos^2 \theta}} \right) d\theta = 2\pi^2 \]
which leads to the conclusion that
\[ MW_2 = \frac{1}{2\pi} \left( \sqrt{2\pi} \right) + \frac{1}{4\pi} (2\pi^2) = \frac{1}{2} \left( \sqrt{2} + \pi \right) = 2.2779031079814441436321660.... \]

2.2. Direct Approach. Consider the curved portion of \( \partial \Omega \) in the halfspace \( x \geq 0 \) only. In this halfspace, an \( \Omega \)-supporting plane \( P^d : \)
\[ \frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z = 1 \quad (\text{with coefficients } a > 0, b, c \text{ and scaling factor } d > 0) \]
has associated lines \( L_{xy}^d, L_{xz}^d : \)
\[ \frac{a}{d}x + \frac{b}{d}y = \frac{d + c}{d} \& z = -1, \quad \frac{a}{d}x + \frac{c}{d}z = \frac{d - b}{d} \& y = 1. \]
Assume WLOG that \( a^2 + b^2 + c^2 = 1 \). The distance of \( P_d \) from the origin \( O \) is \( d \). Also,

\[
\text{distance of } L_{xy}^d \text{ from } O_z = \frac{d + c}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \text{distance of } L_{xz}^d \text{ from } O_y = \frac{d - b}{\sqrt{a^2 + c^2}}
\]

where \( O_z = (0, 0, -1) \) and \( O_y = (0, 1, 0) \). The largest \( d \) such that one of the unit circles is supported is thus

\[
d = \max \left\{ \sqrt{a^2 + b^2} - c, \sqrt{a^2 + c^2} + b \right\}.
\]

We introduce spherical coordinates as before, but with \(-\pi/2 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi\) instead. To ensure uniformity, think of \((\theta, \phi)\) as possessing joint density \( \frac{1}{2\pi} \sin \phi \).

We have

\[
MW_2 = 2 \int_{-\pi/2}^{\pi/2} \int_{0}^{\pi/2} \int_{0}^{\xi(\theta)} \max \left\{ \sin \phi - \cos \phi, \sqrt{\cos^2 \theta \sin^2 \phi + \cos^2 \phi + \sin \theta \sin \phi} \right\} \frac{1}{2\pi} \sin \phi \, d\phi \, d\theta \]

\[
= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \int_{0}^{\pi/2} \sin \phi \, d\phi \, d\theta = 2.277903107981441436321660...
\]

where

\[
\xi(\theta) = \frac{\pi}{2} + \arctan(\sin \theta)
\]

is the required solution (for \( \phi \) in terms of \( \theta \)) of the equation

\[
\sin \phi - \cos \phi = \sqrt{\cos^2 \theta \sin^2 \phi + \cos^2 \phi + \sin \theta \sin \phi}.
\]

### 3. Example 2 (Again)

Vinzant, using techniques in her thesis \[11\], computed that \( \partial \Omega \) is given implicitly by the equation

\[
0 = -x^2 + 2y + x^2y - 3y^2 + y^3 - 2z - x^2z + 6yz + x^2yz - 5y^2z + y^3z - 3z^2 + 5y^2z^2 - 2y^3z^2 - 3y^3z + y^2z^3
\]

\[
= (y - 1)(z + 1)(x^2 - 2y + y^2 + 2z - 2yz + z^2)
\]

\[
= (y - 1)(z + 1)(z - \varphi(x, y))(z - \psi(x, y))
\]

verifying what we already know. She additionally gave an elegant parametric representation of the curved portion in \( x \geq 0 \):

\[
x = \sqrt{1 - v^2}, \quad y = 1 - u + uv, \quad z = -u - v + uv, \quad 0 \leq u \leq 1 \text{ & } -1 \leq v \leq 1
\]
which deserves further attention. In the following, we reproduce our results from the preceding section. The purpose in doing so is not to torture the reader, but rather to set the stage for Example 3 (for which a parametric representation is the only workable method available.) The Jacobian determinant

\[
\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{(-1 + v)v}{\sqrt{1 - v^2}}
\]

allows us to evaluate

\[
VL_2 = 2 \int_0^1 \int_{-1}^1 ((-u - v + uv) + 1) \frac{(-1 + v)v}{\sqrt{1 - v^2}} \, dv \, du = \pi.
\]

Defining

\[
E = (x_u, y_u, z_u) \cdot (x_u, y_u, z_u), \quad G = (x_v, y_v, z_v) \cdot (x_v, y_v, z_v),
\]

\[
F = (x_u, y_u, z_u) \cdot (x_v, y_v, z_v)
\]

we have

\[
AR_2 = \int_{\partial \Omega} dS = 2\pi + 2 \int_0^1 \int_{-1}^1 \sqrt{EG - F^2} \, dv \, du
\]

\[
= 2\pi + 2 \int_0^1 \int_{-1}^1 \frac{(1 - v)(1 + v^2)}{1 + v} \, dv \, du
\]

\[
= 2 \left( \pi + 2\sqrt{2}E \left( \frac{1}{2} \right) \right).
\]

Defining

\[
N = \frac{(x_u, y_u, z_u) \times (x_v, y_v, z_v)}{|(x_u, y_u, z_u) \times (x_v, y_v, z_v)|} = \left( \sqrt{\frac{1 - v^2}{1 + v^2}}, \frac{v}{\sqrt{1 + v^2}}, -\frac{v}{\sqrt{1 + v^2}} \right),
\]

\[
L = -(x_u, y_u, z_u) \cdot N_u, \quad N = -(x_v, y_v, z_v) \cdot N_v, \quad M = -\frac{1}{2} ((x_u, y_u, z_u) \cdot N_v + (x_v, y_v, z_v) \cdot N_u)
\]

\[\text{The fact that this has indefinite sign doesn’t affect the volume calculation.}\]
we have
\[ \int_{\partial \Omega} H \, dS = \int_{\partial \Omega} \frac{EN - 2FM + GL}{2(EG - F^2)} \, dS \]
\[ = 2 \int_{0}^{1} \int_{-1}^{1} \frac{EN - 2FM + GL}{2(EG - F^2)} \sqrt{EG - F^2} \, dv \, du \]
\[ = 2 \int_{0}^{1} \int_{-1}^{1} \frac{1}{(1 + v^2)^{3/2}} \sqrt{(1 - v)(1 + v^2)} \, dv \, du \]
\[ = 2 \int_{0}^{1} \int_{-1}^{1} \frac{1}{1 + v^2} \sqrt{\frac{1 - v}{1 + v}} \, dv \, du = \sqrt{2\pi}. \]

The semicircular edge \( x^2 + z^2 = 1, y = 1, x \geq 0 \) corresponds to \( u = 0, -1 \leq v \leq 1 \). Call this \( \varepsilon \). The exterior dihedral angle is
\[ \alpha = \arccos (\mathbf{N} \cdot (0, 1, 0)) = \arccos \left( \frac{v}{\sqrt{1 + v^2}} \right) \]
and arclength \( s \) satisfies
\[ ds = \sqrt{x_v^2 + z_v^2} \, dv = \frac{1}{\sqrt{1 - v^2}} \, dv. \]
Consequently
\[ 4 \int_{\varepsilon} \alpha \, ds = 4 \int_{-1}^{1} \frac{1}{\sqrt{1 - v^2}} \arccos \left( \frac{v}{\sqrt{1 + v^2}} \right) \, dv = 2\pi^2 \]
and
\[ MW_2 = \frac{1}{2\pi} \left( \sqrt{2\pi} \right) + \frac{1}{4\pi} \left( 2\pi^2 \right) = \frac{1}{2} \left( \sqrt{2} + \pi \right) \]
as was to be shown.

4. Example 3
Vinzant, using techniques in her thesis [11], computed that \( \partial \Omega \) here is given implicitly by the equation
\[ 0 = -4x^4 + 8x^6 - 16x^2y + 36x^4y - 16y^2 + 40x^2y^2 + 15x^4y^2 + 36x^2y^3 + 8y^4 + 6x^2y^4 - y^6 - 8x^2z^2 + 24x^4z^2 + 16yz^2 + 40y^2z^2 - 78x^2y^2z^2 - 36y^3z^2 + 6y^4z^2 - 4z^4 + 24x^2z^4 - 36yz^4 + 15y^2z^4 + 8z^6. \]
One could solve for this cubic (in $z^2$) and proceed as earlier, laboring against the weight of complicated expressions. We prefer, however, to exploit another of her elegant parametric representations:

$$x = u\sqrt{-v(2 + v)}, \quad y = \frac{v(1 + 2uv)}{1 + 2v}, \quad z = \frac{-(1 + u)\sqrt{v(2 + 3v)}}{1 + 2v}$$

for $0 \leq u \leq 1$ & $-2 \leq v \leq -2/3$. The Jacobian determinant

$$\frac{\partial (x, y)}{\partial (u, v)} = \sqrt{-v^2 + v^2 + v + 6uv + 6uv^2}$$

allows us to evaluate

$$VL_3 = 4 \int_0^{1-2/3} \int_{-2}^{-2/3} \frac{-(1 + u)\sqrt{v(2 + 3v)}}{1 + 2v} \sqrt{-v^2 + v^2 + 6uv + 6uv^2} dv \, du \approx \frac{2\pi}{\sqrt{3}} = 3.6275987284684357011881565\ldots$$

Likewise,

$$AR_3 = 4 \int_0^{-2/3} \int_{-2}^{-2/3} \sqrt{EG - F^2} \, dv \, du = 4 \int_0^{1-2/3} \int_{-2}^{-2/3} \frac{2 + v + 6uv + 6uv^2}{(1 + 2v)^2} \sqrt{-\frac{1 + 2v + 3v^2}{(2 + v)(2 + 3v)}} \, dv \, du \approx 4 \int_0^{1-2/3} \int_{-2}^{-2/3} \frac{2 + 4v + 3v^2}{(1 + 2v)^2} \sqrt{-\frac{1 + 2v + 3v^2}{(2 + v)(2 + 3v)}} \, dv \, du.$$

Gosper & Bickford [12] conjectured that $AR_3 = 4\sigma/\tau$, where

$$\sigma = -81i E\left(\frac{1}{81} \left(17 - 56i\sqrt{2}\right)\right) + 9\sqrt{-17 + 56i\sqrt{2}} E\left(\frac{1}{81} \left(17 + 56i\sqrt{2}\right)\right) + 108\sqrt{2} K\left(\frac{1}{9}\right) - 72\sqrt{2} K\left(\frac{1}{81} \left(17 + 56i\sqrt{2}\right)\right) - (56i + 32\sqrt{2}) \Pi\left(\frac{1}{27} \left(7 - 4i\sqrt{2}\right), \frac{1}{81} \left(17 - 56i\sqrt{2}\right)\right) + (56i + 32\sqrt{2}) \Pi\left(\frac{1}{3} \left(7 - 4i\sqrt{2}\right), \frac{1}{81} \left(17 - 56i\sqrt{2}\right)\right) + 72i \Pi\left(\frac{1}{27} \left(7 + 4i\sqrt{2}\right), \frac{1}{81} \left(17 + 56i\sqrt{2}\right)\right) - 72i \Pi\left(\frac{1}{3} \left(7 + 4i\sqrt{2}\right), \frac{1}{81} \left(17 + 56i\sqrt{2}\right)\right)$$
and \( \tau = 18(-i + 2\sqrt{2}) \), where \( i \) is the imaginary unit, \( E(\mu) \) was defined earlier,

\[
K(\mu) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \mu \sin^2 \theta}} \, d\theta = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-\mu t^2)}} \, dt
\]

is the complete elliptic integral of the first kind and

\[
\Pi(\nu, \mu) = \int_0^{\pi/2} \frac{1}{(1 - \nu \sin^2 \theta)\sqrt{1 - \mu \sin^2 \theta}} \, d\theta = \int_0^1 \frac{1}{(1 - \nu t^2)\sqrt{(1-t^2)(1-\mu t^2)}} \, dt
\]

is the complete elliptic integral of the third kind. A simplification of \( \sigma \) would be good to see someday.

### 4.1. Indirect Approach.

As before,

\[
\mathcal{N} = \frac{(x_u, y_u, z_u) \times (x_v, y_v, z_v)}{|(x_u, y_u, z_u) \times (x_v, y_v, z_v)|} = \left( \sqrt{-v(2+v)}, \frac{1+v}{1+2v+3v^2}, \frac{v(2+3v)}{1+2v+3v^2} \right)
\]

and

\[
\int_{\partial \Omega} H \, dS = 4 \int_0^{1} \int_{-2}^{-2/3} \frac{EN - 2FM + GL}{2(EG - F^2)} \sqrt{EG - F^2} \, dv \, du
\]

\[
= -4 \int_0^{1} \int_{-2}^{-2/3} \frac{3v(1+2v)^2}{(1+2v+3v^2)^{3/2}(2+v+6uv+6uv^2)} \frac{2+v+6uv+6uv^2}{(1+2v)^2} \sqrt{-\frac{1+2v+3v^2}{(2+v)(2+3v)}} \, dv \, du
\]

\[
= -4 \int_{-2}^{-2/3} \frac{3v}{1+2v+3v^2} \sqrt{-\frac{1}{(2+v)(2+3v)}} \, dv = 2\sqrt{2}\pi.
\]

Let \( \varepsilon \) denote the arc of the semicircle \( x^2 + (y + 1)^2 = 1, z = 0 \) \& \( x \geq 0 \) that runs counterclockwise from points \((0, -2, 0)\) to \((2\sqrt{2}/3, -2/3, 0)\); this corresponds to \( u = 1 \) \& \(-2 \leq v \leq -2/3 \). The exterior dihedral angle is

\[
\alpha = 2 \arccos \left( \mathcal{N} \cdot \left( \frac{\sqrt{-v(2+v)}, 1+v, 0}{\sqrt{1+2v+3v^2}} \right) \right) = 2 \arccos \left( \frac{1}{\sqrt{1+2v+3v^2}} \right)
\]
because the unit exterior normal vector to the cylinder \( x^2 + (y+1)^2 = 1 \) is \((x, y+1, 0)\); the arclength \( s \) satisfies

\[
ds = \sqrt{x^2 + y^2} \, dv = \frac{1}{\sqrt{-v(2+v)}} \, dv.
\]

Consequently

\[
4 \int_{\varepsilon}^{\alpha} ds = 8 \int_{-2}^{-2/3} \frac{1}{\sqrt{-v(2+v)}} \arccos \left( \frac{1}{\sqrt{1+2v+3v^2}} \right) \, dv = 4\pi \arccos(3)
\]

and therefore

\[
MW_3 = \frac{1}{2\pi} \int_{\partial \Omega} H \, dS + \frac{1}{\pi} \int_{\varepsilon}^{\alpha} ds
\]

\[
= \sqrt{2} + \arccos(3)
\]

\[
= 2.6451729797138697309366179\ldots
\]

All nine constants exhibited (at the beginning) possess closed-form expressions, although the result for \( AR_3 \) is partly conjectural. We had expected that there might be required “more time to develop the languages, functions, symmetries, etc., to express the constants more naturally” [13], but this belief turned out to be overly cautious.

4.2. Direct Approach. Consider the portion of \( \partial \Omega \) in the first octant only. In this octant, an \( \Omega \)-supporting plane \( P^d \):

\[
\frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z = 1 \quad \text{(with coefficients } a > 0, b > 0, c > 0 \text{ and scaling factor } d > 0)
\]

has associated lines \( L^d_{xy}, L^d_{yz} \):

\[
\frac{a}{d}x + \frac{b}{d}y = 1 \& z = 0, \quad \frac{b}{d}y + \frac{c}{d}z = 1 \& x = 0.
\]

Assume WLOG that \( a^2 + b^2 + c^2 = 1 \). The distance of \( P^d \) from the origin \( O \) is \( d \). Also,

- distance of \( L^d_{xy} \) from \( O_+ \) is \( \frac{d + b}{\sqrt{a^2 + b^2}} \) and distance of \( L^d_{yz} \) from \( O_+ \) is \( \frac{d - b}{\sqrt{b^2 + c^2}} \)
where \( O_- = (0, -1, 0) \) and \( O_+ = (0, 1, 0) \). The largest \( d \) such that one of the unit circles is supported is thus

\[
d = \max \left\{ \sqrt{a^2 + b^2 - b}, \sqrt{b^2 + c^2 + b} \right\}.
\]

We introduce spherical coordinates with \( 0 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi/2 \), obtaining

\[
MW_3 = 2 \int_0^{\pi/2} \int_0^{\pi/2} \max \left\{ (1 - \sin \theta) \sin \phi, \sin \theta \sin \phi + \sqrt{\sin^2 \theta \sin^2 \phi + \cos^2 \phi} \right\} \frac{2}{\pi} \sin \phi d\phi d\theta
\]

\[
= \frac{4}{\pi} \int_0^{\kappa} \int_0^{\xi(\theta)} \left( \sin \theta \sin \phi + \sqrt{\sin^2 \theta \sin^2 \phi + \cos^2 \phi} \right) \sin \phi d\phi d\theta
\]

\[
+ \frac{4}{\pi} \int_0^{\kappa} \int_0^{\pi/2} (1 - \sin \theta) \sin^2 \phi d\phi d\theta
\]

\[
+ \frac{4}{\pi} \int_0^{\kappa} \int_0^{\pi/2} \left( \sin \theta \sin \phi + \sqrt{\sin^2 \theta \sin^2 \phi + \cos^2 \phi} \right) \sin \phi d\phi d\theta
\]

\[
= 2.6451729797138697309366179...
\]

where \( \kappa = \arccsc(3) \) and

\[
\xi(\theta) = \arccos \left( \sqrt{\frac{1 - 4 \sin \theta + 3 \sin^2 \theta}{2 - 4 \sin \theta + 3 \sin^2 \theta}} \right), \quad 0 \leq \theta \leq \kappa
\]

is the required solution (for \( \phi \) in terms of \( \theta \)) of the equation

\[
(1 - \sin \theta) \sin \phi = \sin \theta \sin \phi + \sqrt{\sin^2 \theta \sin^2 \phi + \cos^2 \phi}.
\]

5. Related Topics

Dirnböck & Stachel [14] studied the convex hull of the two disks:

\[
\left\{(x, y, z) : x^2 + (y + \frac{\delta}{2})^2 \leq 1 \& z = 0 \right\} \quad \text{and} \quad \left\{(x, y, z) : (y - \frac{\delta}{2})^2 + z^2 \leq 1 \& x = 0 \right\}
\]

when \( \delta = 1 \) and Ira [15] studied the same when \( \delta = \sqrt{2} \). These are intermediate cases relative to our Example 3 (for which \( \delta = 2 \)) and what is essentially Example 1.
Convex Hull of Two Orthogonal Disks

(for which $\delta = 0$). The former case, called an **oloid**, has volume

\[
VL = \frac{2}{3} \int_0^{\pi/2} \frac{(2 + \cos \theta)^2}{(1 + \cos \theta) \sqrt{1 + 2 \cos \theta}} d\theta
\]

\[
= \frac{2}{3} \left(-1 + 2\sqrt{3}E \left(\frac{\pi}{4}, \frac{4}{3}\right) + 2\sqrt{3}F \left(\frac{\pi}{4}, \frac{4}{3}\right)\right)
\]

\[
= 3.0524184684243748566972005...
\]

and surface area $AR = 4\pi$, where $E$ & $F$ are incomplete elliptic integral of the second & first kinds respectively:

\[
E(\phi, \mu) = \int_0^{\phi} \sqrt{1 - \mu \sin^2 \theta} d\theta = \int_0^{\phi} \sqrt{\frac{1 - \mu t^2}{1 - t^2}} dt,
\]

\[
F(\phi, \mu) = \int_0^{\phi} \frac{1}{\sqrt{1 - \mu \sin^2 \theta}} d\theta = \int_0^{\phi} \frac{1}{\sqrt{(1 - t^2)(1 - \mu t^2)}} dt
\]

(of course, $F(\pi/2, \cdot) = K(\cdot)$). A feature of the oloid is that each of the two interlocking orthogonal circles $C_{\text{left}}$ & $C_{\text{right}}$ intersects the center of the other. Define constants $\omega = \arcsin \left(\sqrt{\frac{2}{\sqrt{2} - 1}}\right)$ and

\[
\gamma = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{(2 + \sqrt{2} \cos \theta)^2}{(1 + \sqrt{2} \cos \theta)^2 \sqrt{1 + 2\sqrt{2} \cos \theta + \cos^2 \theta}} d\theta
\]

\[
= \frac{\sqrt{2} - 1}{2} + E(\omega, -1) + \sqrt{2} \left(\Pi(-\sqrt{2} - 1, \omega, -1) + \Pi(\sqrt{2} - 1, \omega, -1) - F(\omega, -1)\right)
\]

where $\Pi$ is the incomplete elliptic integral of the third kind:

\[
\Pi(\nu, \phi, \mu) = \int_0^{\phi} \frac{1}{(1 - \nu \sin^2 \theta)^2} d\theta = \int_0^{\phi} \frac{1}{(1 - \nu t^2)^2 \sqrt{(1 - t^2)(1 - \mu t^2)}} dt.
\]

The latter case, called a **two-circle roller**, has volume

\[
VL = \frac{8}{3\sqrt{2}} \gamma = 3.2818194874496894190321933...
\]
and surface area

$$AR = 8\gamma = 13.9235808852350105127348109...$$

As the name “roller” suggests, the authors of [14, 15] were raising issues of a physical/mechanical nature. We suspect that an algebraic approach based on [11] could supplant some of their technical arguments leading to $VL & AR$. Exact expressions for $MW$ remain open, as far as is known.

A picture in [16] depicts the circles $C_{\text{left}} & C_{\text{right}}$ but not within our context of convex hulls (taut rubber sheets spanning wire frames). The context instead is about minimal surfaces (elastic soap films spanning the same), for which mean curvature is zero everywhere. Many questions suggest themselves.

6. Acknowledgements

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7. Addendum

Axel Vogt improved upon the conjecture in [12] and successfully obtained

$$AR_3 = \frac{4}{3} \left[ 9E \left( \frac{1}{9} \right) - 8K \left( \frac{1}{9} \right) + 8\Pi \left( -\frac{1}{3}, \frac{1}{9} \right) \right]$$

via a change of variables and Maple. It is possible to reduce the complicated expression $4\sigma/\tau$ further:

$$\sigma = -243iE \left( \frac{1}{81} \left( 17 - 56i\sqrt{2} \right) \right) + 27\sqrt{-17 + 56i\sqrt{2}}E \left( \frac{1}{81} \left( 17 + 56i\sqrt{2} \right) \right)$$

$$+ 162\sqrt{2}K \left( \frac{1}{9} \right) - 162\sqrt{2}K \left( \frac{1}{81} \left( 17 + 56i\sqrt{2} \right) \right)$$

$$+ 6 \left( 8i - 7\sqrt{2} \right) \Pi \left( \frac{1}{27} \left( 7 - 4i\sqrt{2} \right), \frac{1}{81} \left( 17 - 56i\sqrt{2} \right) \right)$$

$$+ 2 \left( 104i + 71\sqrt{2} \right) \Pi \left( \frac{1}{3} \left( 7 - 4i\sqrt{2} \right), \frac{1}{81} \left( 17 - 56i\sqrt{2} \right) \right),$$

$$\tau = 54 \left( -i + 2\sqrt{2} \right)$$

using numerics and Mathematica, but Bill Gosper’s question (on symbolic transformations between elliptic integrals to link these) remains unanswered.
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Figure 1: Centers coincide

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Figure 2: Centers on diagonal
Figure 3: Centers on bisecting line
This figure "Example1.jpg" is available in "jpg" format from:

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