Rapid LISA Astronomy

Neil J. Cornish

Department of Physics, Montana State University, Bozeman, MT 59717

A simple method is presented for removing the amplitude, frequency and phase modulations from the Laser Interferometer Space Antenna (LISA) data stream for sources at any sky location. When combined with an excess power trigger or the fast chirp transform, the total demodulation procedure allows the majority of LISA sources to be identified without recourse to matched filtering.

A revolution is underway that will transform the field of astronomy. An array of ground and space based gravitational wave detectors are in various stages of design, construction and operation. The first space based detector, the Laser Interferometer Space Antenna [1], is being developed for a launch in 2011. These gravitational wave detectors will open a new window on the Universe that complements traditional forms of astronomy [2].

However, before we can talk about gravitational wave astronomy, a method must be found to turn a gravitational wave detector into a gravitational wave telescope - that is, a device that is capable of locating individual sources on the sky and determining their physical characteristics. In traditional electromagnetic astronomy the conversion of a detector - such as the human eye or a photographic plate - into an observational instrument is achieved by building a device that collects and focuses the incident radiation onto the detector. Such an approach is impractical for gravitational radiation as the weak coupling of the radiation to matter prevents the incident radiation onto the detector. Such an approach is impractical for gravitational radiation as the weak coupling of the radiation to matter prevents the conversion of a detector - such as the human eye or a photographic plate - into an observational instrument. Instead, a gravitational wave detector is transformed into a gravitational wave telescope by data analysis algorithms.

In principle we would like to use matched filtering of the data against suitable waveform templates as this delivers the highest fidelity results, but in practice the computational cost of matched filtering can be prohibitive. The solution is to adopt a hierarchical approach where a fast, but sub-optimal, method is used to find a good initial solution, which can then be refined using matched filtering. The excess power statistic [3] used in ground based gravitational wave astronomy is a good example of this approach. However, we cannot simply take the ground based data analysis algorithms and apply them to the space based systems as the high frequency, ground based detectors primarily look for short lived, burst like signals, whereas the low frequency space based detectors primarily look for long lived, continuous sources. One of the key differences between ground and space based gravitational wave detection is ratio of the observation time to the orbital period of the detector. The ratio is small for most high frequency sources, and of order one for most low frequency sources. This means that LISA data analysis algorithms have to deal with orbital signal modulation and multiple overlapping sources.

Here we describe a simple method for converting the output of a space based gravitational wave detector, such as LISA, into useful astronomical information about the location and physical characteristics a gravitational wave source. The method works by removing the amplitude, phase and frequency modulation imparted by the orbital motion of the detector. For monochromatic sources, the demodulation procedure focuses the signal into a narrow frequency band, which creates a spike in the demodulated power spectrum. By searching for sky locations that yield the largest spikes it is possible to locate the brightest sources on the sky. These can then be subtracted and the procedure iterated until all resolvable sources have been found. For chirping sources the demodulation can be followed by a fast chirp transform [4], which effectively re-concentrates the power that is spread by the frequency evolution of the source. Once again the brightest sources on the sky can be identified and iteratively subtracted to expose the weaker sources.

The demodulation procedure is remarkably simple. It utilizes the fact that space based detectors such as LISA are able to return several interferometer outputs [5]. A general gravitational wave coming from a source in the \( \hat{n} \) direction can be expressed in barycentric coordinates as

\[
h_{\mu\nu}(t, \hat{n}) = h_+^{}(t, \hat{n}) \epsilon_{\mu\nu}^+(\hat{n}) + h_\times(t, \hat{n}) \epsilon_{\mu\nu}^\times(\hat{n}) \tag{1}
\]

where \( h_+^{}(t, \hat{n}) \) and \( h_\times(t, \hat{n}) \) describe the two polarization states with basis tensors \( \epsilon_{\mu\nu}^+(\hat{n}) \) and \( \epsilon_{\mu\nu}^\times(\hat{n}) \). The polarization states \( h_+^{} \), \( h_\times \) are related to the principal polarization states \( h_+^P \), \( h_\times^P \) by the polarization angle \( \psi \):

\[
\begin{align*}
h_+ & = h_+^P \cos 2\psi + h_\times^P \sin 2\psi, \\
h_\times & = h_+^P \cos 2\psi - h_\times^P \sin 2\psi. \tag{2}
\end{align*}
\]

In the idealized limit of a single low frequency source and a noise-free detector, the interferometer outputs \( s(t) \) can be expressed as

\[
s(t) = \bar{h}_+^{}(t, \hat{n}) D^+(t, \hat{n}) + \bar{h}_\times(t, \hat{n}) D^\times(t, \hat{n}) \tag{3}
\]

where \( D^+(t, \hat{n}) \) and \( D^\times(t, \hat{n}) \) are the sky-location dependent interferometer response functions [6] and \( \bar{h}_+^{}(t, \hat{n}) \) and \( \bar{h}_\times(t, \hat{n}) \) are the Doppler shifted gravitational wave strains

\[
\bar{h}(t, \hat{n}) = h(t + \frac{R}{c} \sin \theta \cos(2\pi f_m t - \phi), \hat{n}). \tag{4}
\]

Here \( \hat{n} \to (\theta, \phi) \) denotes the location of the source in ecliptic coordinates, \( R \) is the Earth-Sun distance and \( f_m = 1/\text{year} \) is the modulation frequency. A simple,
but impractical, approach for demodulating the signal is to form the combinations

\[ \bar{h}_X(t, \hat{n}) = \frac{s_1(t)D^X_{11}(t, \hat{n}) - s_1(t)D^X_{11}(t, \hat{n})}{D^X_{11}(t, \hat{n})D^X_{11}(t, \hat{n}) - D^X_{11}(t, \hat{n})D^X_{11}(t, \hat{n})} \]  

(5)

and

\[ \bar{h}_+(t, \hat{n}) = \frac{s_1(t)D^+_{11}(t, \hat{n}) - s_1(t)D^+_{11}(t, \hat{n})}{D^+_{11}(t, \hat{n})D^+_{11}(t, \hat{n}) - D^+_{11}(t, \hat{n})D^+_{11}(t, \hat{n})} \]  

(6)

from interferometer outputs I and II, then perform the coordinate transformation

\[ t = t' - \frac{R}{c} \sin \theta \cos(2\pi f_m t' - \phi) \]  

(7)

to remove the leading order Doppler modulation \[ \hat{t} \], thereby recovering the demodulated strains \[ \bar{h}_+(t') \] and \[ \bar{h}_X(t') \]. The main problem with this approach is that the denominators in Eqs. (5) and (6) vanish at certain times of the year for sources within 60 degrees of the ecliptic plane. When detector noise or other sources are present the inversion becomes singular. Although this simple demodulation procedure does not work in the time domain, it can be made to work in the frequency domain.

The key advantage of moving to the frequency domain is that the singularities in the temporal inversion do not correspond to singularities in the frequency space inversion. By Fourier transforming the data over a finite time interval, the demodulation procedure can be re-cast as a linear algebra problem that can be solved using the robust technique of singular value decomposition.

To simply our expressions we assume that the observation time \( T \) is equal to one year so that the frequency resolution \( \Delta f = 1/T \) is equal to the modulation frequency \( f_m \) (the general expressions are not hard to derive, but they are more complicated). The Fourier space equivalents of (5) and (6) become

\[ s_k = A_{kl}^+(\hat{n}) \bar{h}_l^+(\hat{n}) + A_{kl}^X(\hat{n}) \bar{h}_l^X(\hat{n}), \]  

(8)

and

\[ \bar{h}_l^+ = B_{ln}(\hat{n}) h_{l+n}^+{\hat{n}}, \quad \bar{h}_l^X = B_{ln}(\hat{n}) h_{l+n}^X{\hat{n}}, \]  

(9)

respectively. The matrices \( A \) and \( B \) are given by

\[ A_{kl}^+(\hat{n}) = D^+_{k-l}(\hat{n}), \quad A_{kl}^X(\hat{n}) = D^X_{k-l}(\hat{n}) \]  

(10)

and

\[ B_{ln}(\hat{n}) = J_{l-n}(2\pi n f_m \frac{R}{c} \sin \theta) e^{i(l-n)(\pi/2-\phi)}. \]  

(11)

Here \( J_q(x) \) is a Bessel function of the first kind of order \( q \) and we have used the Einstein summation convention for repeated indices. It is important to note that the matrices \( A \) and \( B \) are both band diagonal as \( D^+_{l} \) \( \hat{n} \) \( \approx \) 0 \( \approx D^X_{l} \) \( \hat{n} \) for \( |j| > 4 \) and \( J_q(x) \approx 0 \) for \( |q| > x \). The next step is to interleave the Fourier coefficients of the two polarizations \( h^+ (t) \) and \( h^X (t) \) into a single column vector \( s \) such that \( h_{2n} = h^+_{2n} \) and \( h_{2n+1} = h^X_{2n} \). Similarly, the Fourier coefficients of the two interferometer outputs \( s_1(t) \) and \( s_1(t) \) are interleaved into a single column vector \( s \). The detector response can then be written as

\[ s = M(\hat{n})h(\hat{n}), \]  

(12)

where the modulation matrix \( M(\hat{n}) \) is the interleaved product of \( A^+ \) \( \hat{n} \), \( A^X \) \( \hat{n} \) and \( B(\hat{n}) \):

\[ M_{2k} = A_{k,j}^+ B_{j}, \quad M_{2k+1} = A^X_{k,j} B_{j}, \]  

(13)

If the interferometer outputs are sampled \( N \) times in a year, \( M(\hat{n}) \) will be a \( 2 \times 2 \) dimensional, complex, band diagonal matrix. Multiplying \( s \) by the inverse of the modulation matrix demodulates any sources located in the \( \hat{n} \) direction. In practice, \( s \) will contain contributions from instrument noise \( n \), and multiple sources \( h^i(\hat{n}_i) \):

\[ s = \sum_i M^i(\hat{n}_i)h(\hat{n}_i) + n. \]  

(14)

We can estimate the contribution to the signal \( s \) from a source located in the \( \hat{n} \) direction by solving for the vector \( h^\text{eff}(\hat{n}) \) that satisfies the equation

\[ s = M(\hat{n})h^\text{eff}(\hat{n}). \]  

(15)

The best fit solution is found using a singular value decomposition. Suppose that source \( j \) happens to lie in the \( \hat{n} \) direction. Then

\[ h^\text{eff}(\hat{n}) \approx h^j(\hat{n}) + \sum_{i \neq j} M^{-1}_i(\hat{n})M^j(\hat{n}_i)h^i(\hat{n}_i) + M^{-1}(\hat{n})n. \]  

(16)

The key to the method is that sources at other sky locations are not demodulated, and the noise is merely reshuffled, not amplified. Thus, we can use the solution vector \( h^\text{eff}(\hat{n}) \) as an estimate for the source vector \( h^j(\hat{n}) \). Another key point is that the modulation matrix is band diagonal, which allows us to perform the demodulation across limited bandwidths using small modulation sub-matrices. If this were not the case we would be faced with the problem of inverting a \( \sim 10^8 \times 10^8 \) matrix. Moreover, a source only has to compete with other sources that overlap with it in frequency space.

The demodulation procedure works for any type of gravitational wave source, but it is especially useful when the source is a mildly eccentric, nearly Newtonian binary as the demodulated signal is then approximately monochromatic. We can search for these sources by looking for spikes in the the power spectrum of \( h^\text{eff}(\hat{n}) \) for different sky locations \( \hat{n} \). A similar approach was employed in Ref. [7] using Doppler demodulation of a single interferometer output. Since the vast majority of LISA sources are non-chirping, circular Newtonian binaries, the combined demodulation, power spike search goes a
long way toward solving the LISA data analysis problem. After establishing the sky location and frequency of a source, the next step is to refine the frequency measurement and extract the Fourier amplitudes of the two polarizations $\tilde{h}^+$ and $\tilde{h}^\times$. Unless the source frequency $f$ is an integer multiple of the sample frequency $\Delta f = f_m$, the power will be shared by several discrete Fourier coefficients:

$$h^+_n \simeq \tilde{h}^+ e^{i\pi x_n} e^{i\pi x_n} \quad \text{where} \quad x_n = f/f_m - n, \quad (17)$$

and similarly for $h^\times_n$. Now suppose that the integration period $T$ is changed slightly to $T' = T(1 + \epsilon)$ where $\epsilon \ll 1$. The Fourier coefficients of $h(t)$ evaluated over the interval $T'$ are related to those evaluated over the interval $T$ by

$$h'_n = \sum_j h_j e^{i\pi x_j} = \sum_j h_j e^{i\pi x_{jn}}, \quad (18)$$

where

$$x_{jn} = j \left(\frac{T'}{T}\right) - n. \quad (19)$$

If the source frequency $f$ is an integer multiple of the new sample frequency $\Delta f' = 1/T'$, all the power will be concentrated in a single Fourier mode $h'_n$. Thus, the frequency spreading can be removed by finding the value of $\epsilon$ that maximizes the spike in the stretched power spectra of $h^+(t)$ and $h^\times(t)$. This procedure yields an improved estimate of the source frequency, and a direct estimate of the Fourier amplitudes of the gravitational wave:

$$\tilde{h}^+ \simeq h^+_n e^{i\delta} \quad \text{and} \quad \tilde{h}^\times \simeq h^\times_n e^{i\delta}. \quad (20)$$

The quantity $\delta = 2\pi f(T' - T)$ accounts for the phase rotation imparted by the change in integration period. Once the Fourier amplitudes of the two polarizations have been estimated they can be used to calculate the amplitude $A$, inclination $i$, polarization angle $\psi$ and orbital phase $\varphi_0$ of the binary using the relations

$$\psi = \frac{1}{4} \left( \arctan \left( \frac{\tilde{h}^+_R + \tilde{h}^\times_R}{\tilde{h}^+_R - \tilde{h}^\times_R} \right) + \arctan \left( \frac{\tilde{h}^+_I + \tilde{h}^\times_I}{\tilde{h}^+_I - \tilde{h}^\times_I} \right) \right), \quad (21)$$

$$\varphi_0 = \frac{1}{2} \left( \arctan \left( \frac{\tilde{h}^+_R - \tilde{h}^\times_R}{\tilde{h}^+_R + \tilde{h}^\times_R} \right) - \arctan \left( \frac{\tilde{h}^+_I + \tilde{h}^\times_I}{\tilde{h}^+_I - \tilde{h}^\times_I} \right) \right), \quad (22)$$

$$A = \frac{A_+ + \sqrt{A_+^2 - A_\times^2}}{2}, \quad (23)$$

$$i = \arccos \left( \frac{-A_\times}{A_+ + \sqrt{A_+^2 - A_\times^2}} \right), \quad (24)$$

and

$$A_+ = \frac{4(\cos 2\psi \tilde{h}^+_R - \sin 2\psi \tilde{h}^\times_R)}{\cos \varphi_0}, \quad (25)$$

$$A_\times = \frac{4(\sin 2\psi \tilde{h}^\times_R + \cos 2\psi \tilde{h}^\times_R)}{\sin \varphi_0}$$

FIG. 1: The RMS strain spectral density in LISA channels $s_1$ (upper panel) and $s_{11}$ (lower panel) in units of $10^{-22}$.\[1\text{kHz}^{-1/2}].$
The RMS strain spectral density of the demodulated plus $\tilde{h}$ (upper panel) and cross $\tilde{h} \times$ polarizations (lower panel) in units of $10^{-22}$.

definitions we find that $\text{SNR}_I = 5.4$, $\text{SNR}_II = 5.2$, and the optimal signal to noise is $\text{SNR}_m = 49.9$. Applying the demodulation procedure yields the two gravitational wave polarizations shown in Fig. 2. The source parameters are recovered to an accuracy of $\Delta f = 0.008 f_m$, $\Delta \theta = 1.4^\circ$, $\Delta \phi = 0.2^\circ$, $\Delta A/A = 0.054$, $\Delta \psi = 0.04^\circ$, $\Delta \psi = 16.4^\circ$ and $\Delta \phi_0 = 8.6^\circ$. These uncertainties are comparable to those found using matched filtering, so the demodulation method is not only extremely fast, but also close to optimal.

The method continues to perform well when the SNR ratios are smaller, or when there are multiple sources in the same frequency band. Even with multiple sources the procedure delivers a good initial fit for the parameters describing each source. By combining the demodulation procedure with the gCLEAN algorithm described in Ref. [8], it will be possible to extract thousands of individual sources from the galactic background at a modest computational cost.

The simple power spike search will not work for rapidly evolving systems, such as supermassive black hole binaries, as even the demodulated power spectrum will be spread over thousands of frequency bins. If one thinks of a Fourier transform as a matched filter using sines and cosines, we see that we get a good match for monochromatic sources (the power spike), but a poor match for chirping sources. The solution is to employ a different kind of transform that uses filter functions that evolve in amplitude and frequency - the Fast Chirp Transform [4]. Just as a Fast Fourier Transform (FFT) allows us to detect signals with a constant frequency, the Fast Chirp Transform (FCT) allows us to detect signals with variable frequency. By applying a FCT rather than a FFT to the demodulated signal, we can search for the sky locations that harbour the brightest chirping sources and extract their physical characteristics.

Much remains to be done in the field of LISA data analysis, but fast non-template based methods such as the total demodulation procedure and the Fast Chirp Transform will likely play a key role in future developments.

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