ESTIMATES AND IDENTITIES FOR THE AVERAGE DISTORTION OF A LINEAR TRANSFORMATION

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Abstract. Let $A$ be a linear transformation $A: \mathbb{R}^n \to \mathbb{R}^n$. We give sharp estimates on

$$\int_{S^{n-1}} \log \|Au\| du$$

We also show asymptotic results (for large $n$) and evaluate a class of integrals over the sphere, including the integral of the logarithm of absolute value of one coordinate projection.

Introduction

Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be a non-singular linear transformation. In this note we estimate the “average distortion” of $A$. More precisely, we estimate

(1) $$I(A) = \int_{S^{n-1}} \log \|Au\| du.$$ 

We remind the reader that $\int$ denotes the mean of the integrand with respect to the measure denoted by $du$, which, in our case, is the standard rotationally invariant measure on the sphere. Another way of putting it is that the measure is normalized to be a probability measure. This average is of considerable interest in dynamics (see the author’s paper [1] and references cited therein). A somewhat related problem of estimating the average

(2) $$\int_{S^{n-1}} \|Au\| du$$

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is one of the principal problems addressed in the paper [2], and we happily carry over some of the techniques and observations to the current setting.

The main estimate is summarized in the following

**Theorem 1.** Let the singular values of the matrix $A$ be $\sigma_1, \ldots, \sigma_n$. Further, let

$$\Xi: \mathbb{N} \rightarrow \mathbb{R}$$

be defined as follows:

$$\Xi(n) = \begin{cases} 
- \log 2 \left(1 - \frac{\sum_{k=1}^{n/2} \frac{1}{k}}{2^{n/2}}\right) & n \text{ even}, \\
- \sum_{k=1}^{(n-1)/2} \frac{1}{2k-1} & n \text{ odd}.
\end{cases}$$

Then

$$-\frac{1}{2} \log n \geq \int_{S^{n-1}} \log \|Au\| du - \frac{1}{2} \log \left(\sum_{i=1}^{n} \sigma_i^2\right) \geq \Xi(n).$$

This will be shown by way of Theorem 2 in Section 1 followed by the explicit computation of the lower bound in Sections 2 and 3.

In Section 4 we will indicate asymptotic results (a “law of large numbers”) which indicates that the upper bound in Theorem 1 is a better guess for reasonably well-conditioned matrices $A$. Note, however, that the difference between the lower and upper bounds is asymptotic to $\gamma + \log 2$, and so the gap between the two bounds is dimension independent, which indicates that $\frac{1}{2} \log \left(\sum_{i=1}^{n} \sigma_i^2\right)$ is the “right” approximation to $I(A)$.

1. **A sharp inequality**

Our first observation is that we can assume that the matrix $A$ (as in eq. (1)) can be assumed to be diagonal, since the average in eq. (1) does not change if the matrix $A$ is replaced by the diagonal matrix of its singular values $\sigma_1, \ldots, \sigma_n$. (see [2] [1] for more discussion of this), so we can replace the integral to be estimated by

$$I_\Xi = \int_{S^{n-1}} \log \left(\sum_{i=1}^{n} \sigma_i^2 x_i^2\right) du = \frac{1}{2} \int_{S^{n-1}} \log \left(\sum_{i=1}^{n} \sigma_i^2 x_i^2\right) du.$$ 

We will estimate the quantity

$$J_\Xi = 2I_\Xi - \log \sum_{i=1}^{n} \sigma_i^2.$$
Since \( J \) is scale invariant, it will be enough to estimate it (or, what is the same, \( I \)) on the simplex \( H : \sum_{i=1}^{n} \sigma_i^2 \). Let us change variables, so that \( s_i = \sigma_i^2 \). By the concavity of the logarithm function, the integrand of \( I \) is concave (in the \((s_1, \ldots, s_n)\) variables), and hence so is \( I \) itself. Since, in addition, \( \Sigma \) is symmetric, it follows that the maximum of \( I \) on \( H \) is attained when \( s_1 = \cdots = s_n = 1/n \), while the minimum is attained at (any) vertex of \( H \), for example when \( s_1 = 1 \), while \( s_i = 0 \), \( i \neq 1 \).

We have just proved the following

**Theorem 2.**

\[
\frac{-1}{2} \log n \geq \frac{1}{2} \int_{S^{n-1}} \log \left( \sum_{i=1}^{n} \sigma_i^2 x_i^2 \right) du - \frac{1}{2} \log \left( \sum_{i=1}^{n} \sigma_i^2 \right) \geq \int_{S^{n-1}} \log |x_1| du.
\]

2. How to integrate over the sphere

**Theorem 3.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) have the property that

\[
f(ax) = g(a) + f(x).
\]

Let \( \mu \) be the standard measure on \( S^{n-1} \). Then

\[
2^{n/2-1} \Gamma(n/2) \int_{S^{n-1}} f(x) d\mu = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \exp \left( \frac{-x_1^2 - \cdots - x_n^2}{2} \right) dx_1 \cdots dx_n - \omega_{n-1} \int_{0}^{\infty} g(r) r^{n-1} e^{-r^2/2} dr,
\]

where

\[
\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)},
\]

is the area of \( S^{n-1} \).

**Proof.** Let us evaluate the first integral on the right hand side. First, let us transform to polar coordinates:

\[
I = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \exp \left( \frac{-x_1^2 - \cdots - x_n^2}{2} \right) dx_1 \cdots dx_n = \int_{0}^{\infty} r^{n-1} e^{-r^2/2} dr \int_{S^{n-1}} f(rx) d\mu = \int_{0}^{\infty} r^{n-1} e^{-r^2/2} dr \int_{S^{n-1}} (f(x) + g(r)) d\mu = \int_{0}^{\infty} r^{n-1} e^{-r^2/2} dr \int_{S^{n-1}} f(x) + \omega_{n-1} \int_{0}^{\infty} r^{n-1} g(r) e^{-r^2/2} dr,
\]
where we have used Eq. (3), and have denoted the area of the unit sphere $S^{n-1}$ by $\omega_{n-1}$. To evaluate $\omega_{n-1}$ (and, at the same time, the first integral in the right line) let $f(x) = 1$. Then $g(r) = 0$, and we have the equation:

$$\int_{\mathbb{R}^n} \exp \left( -\frac{x_1^2 + \cdots + x_n^2}{2} \right) \, dx_1 \ldots dx_n = \omega_{n-1} \int_0^\infty r^{n-1} e^{-r^2/2} \, dr.$$  

The left hand side equals

$$\left( \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \right)^n = (2\pi)^{n/2}.$$  

The integral on the right hand side can be evaluated by changing variables to $u = r^2/2$, thus getting:

$$\int_0^\infty r^{n-1} e^{-r^2/2} \, dr = 2^{(n/2)-1} \int_0^\infty u^{n/2-1} e^{-u} \, du = 2^{n/2-1} \Gamma(n/2).$$  

Finally obtaining:

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

\[\square\]

3. An extended example

Let $f(x_1, \ldots, x_n) = \log(|x_1|)$. Then $f(a(x)) = \log(a) + f((x))$, so $g(r) = \log(r)$. Let us evaluate the integrals on the right hand side of (4). First,

$$\int_{\mathbb{R}^n} \log(|x_1|) e^{-\left(x_1^2 + \cdots + x_n^2\right)/2} \, dx_1 \ldots dx_n =$$

$$\int_{\mathbb{R}^{n-1}} e^{-\left(x_1^2 + \cdots + x_{n-1}^2\right)/2} \, dx_1 \ldots dx_{n-1} \int_{-\infty}^\infty \log(|x|) e^{-x^2/2} \, dx =$$

$$2^{(n+1)/2} \pi^{(n-1)/2} \int_{0}^{\infty} \log(x) e^{-x^2/2} \, dx =$$

$$-2^{n/2-1} \pi^{n/2}(\gamma + \log(2)),$$

where $\gamma$ is Euler’s constant. The second integral is also simply evaluated in terms of special functions:

$$\int_{0}^{\infty} \log(r) r^{n-1} e^{-r^2/2} = 2^{n/2-2} \Gamma(n/2)(\log(2) + \psi(n/2)),$$

where $\psi$ is the digamma function.
where $\psi(x)$ is the logarithmic derivative of the $\Gamma$ function. Putting everything together, we get:

$$
\int_{S^{n-1}} f(x) d\mu = \frac{1}{2^{n/2-1}\Gamma\left(\frac{n}{2}\right)} \times \\
\left\{ -2^{(n/2-1)}\pi^{n/2}(\gamma + \log(2)) - \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}2^{n/2-2}\Gamma\left(\frac{n}{2}\right)(\log(2) + \psi(n/2)) \right\} = \\
- \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}(\gamma + \log(2)) - \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}(\log(2) + \psi(n/2)).
$$

It is sometimes more useful to compute the mean of a function over the sphere, and we can do that too:

$$
\int_{S^{n-1}} f(x) d\mu = \\
\frac{1}{2} (-\gamma - \log(2)) - (\log(2) + \psi(n/2)) = \\
\frac{1}{2} (-2\log 2 - \gamma - \psi(n/2)).
$$

Lest the reader is discomfitted by the appearance of the digamma function $\psi$, we note the following simple formula for its special values at integer and half-integer points:

$$
\psi(n/2) = \begin{cases} 
-\gamma + \sum_{i=1}^{n/2-1} \frac{1}{i}, & n \text{ even}, \\
-\gamma - 2\log(2) + 2\sum_{k=1}^{(n-1)/2} \frac{1}{2k-1}, & \text{otherwise}
\end{cases}
$$

Curiously, this indicates the the mean of the log $|x_1|$ over the unit sphere is either rational or in $\mathbb{Q}[\log(2)]$, since $\gamma$ always cancels.\footnote{Good thing, since it is not known whether Euler’s constant is rational.}

4. Laws of large numbers

The methods of \cite{2} go through without change to show the following results:
Theorem 4. Let \( \sigma_1, \ldots, \sigma_n, \ldots \) be a sequence of positive numbers such that
\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} \sigma_i^4}{\left( \sum_{i=1}^{n} \sigma_i^2 \right)^2} = 0.
\]

Let \( A_n \in \text{GL}(n, \mathbb{R}) \) be a matrix with singular values \( \sigma_1, \ldots, \sigma_n \). Then
\[
\lim_{n \to \infty} \int_{S^{n-1}} \|Au\| du - \frac{1}{2} \log \left( \sum_{i=1}^{n} \sigma_i^2 \right) + \frac{1}{2} \log n = 0.
\]

Corollary 5. The conclusion of Theorem 4 holds under the assumption that there exists a constant \( c \) such that \( \sigma_i/\sigma_j < c \) for all pairs \( i \neq j \).

Remark 6. The hypothesis of Corollary 5 says that the condition numbers of the matrices \( A_n \) is uniformly bounded.

References

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