On the Virasoro constraints for torus knots

Oleg Dubinkin

Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia
Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Russia

E-mail: dubinkin@itep.ru

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Abstract
We construct a Virasoro algebra of differential operators for the matrix model for torus knots. These operators generate various relations between Wilson loops. Then we discuss the operators constructed and corresponding relations in the stability limit. We also give a series of examples.

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1. Introduction

Matrix models provide a rich set of approaches to physical systems and quantities (see [15] for a review). Constructing a fundamental Virasoro algebra for a given matrix model is one of the most useful methods. This algebra is in itself very fruitful, as it allows us to generate various relations between quantities represented in this model. One can go further—for example, constructing a spectral curve using a Virasoro algebra. But the first step is to derive one algebra, which is the subject of this paper. We will study the matrix model that describes Wilson loops along torus knots. To obtain a Virasoro algebra we will use a standard procedure. First, one should input a generating function, for desirable quantities, into the matrix integral and obtain a Ward identity for this new integral. The second step is to construct differential operators, that can produce such Ward identities on application to the matrix integral with the generating function. This procedure is not always possible, so it can be necessary to input several generating functions in the integrand. We apply this technique to the matrix integral considered and this paper yields generators of the Virasoro algebra for torus knots (27) and identities between Wilson loops, produced by generators of this algebra.

This paper is organized as follows. Section 2 reviews the general algorithm for the matrix integral of the Chern–Simons partition function and obtains a trivial differential equation for it, as an example. In section 3, we will discuss different approaches to the matrix integral for torus knots and obtain two different representations of the Virasoro algebra that correspond to different generating functions. Section 4 discusses the so-called stability limit, where we will
see the decomposition of Virasoro generators into three independent sets of differential operators. Finally, section 5 describes a series of examples.

2. The matrix integral and Ward identities for a torus knot

Let us consider a matrix integral expression for Wilson loops in Chern–Simons theory with gauge group \( G = SU(N) \), written for the case of torus knots. This expression was first obtained for the case of \( SU(2) \) in [3] and then extended to simply laced groups (for additional information see [1, 2, 4, 5, 8, 13, 14]). Also, recently, this matrix integral representation was derived by a localization procedure [6, 7]. The result obtained in these papers reads

\[
W_R(K_{P,Q}) = \frac{1}{Z(K_{P,Q})} \int du e^{-\frac{i\pi}{4} \prod_{\alpha > 0} 4 \sinh \left( \frac{u \cdot \alpha}{2P} \right) \sinh \left( \frac{u \cdot \alpha}{2Q} \right) \chi_R(e^\alpha)}. \tag{1}
\]

In this expression,

\[
Z(K_{P,Q}) = \int du e^{-\frac{i\pi}{4} \prod_{\alpha > 0} 4 \sinh \left( \frac{u \cdot \alpha}{2P} \right) \sinh \left( \frac{u \cdot \alpha}{2Q} \right)}
\]

and the \( \chi_R(e^\alpha) \) are the characters of \( G \), \( g_s \equiv \frac{PQ}{\kappa + h} \), \( \kappa \) is the level of the theory, \( h \) is the dual Coxeter number of \( G \), \( u \) is an element in \( \Lambda_u \otimes \mathbb{R} \) and \( \alpha \) is a set of positive simple roots of \( G \).

A torus knot is specified by a pair of coprime integers, \( P \) and \( Q \). In this section we will work with the integral (2) to develop the necessary technique and show the simplest example of deriving a Ward identity and the corresponding differential operator. Using the well-known fact that the positive roots can be written in an orthonormal basis \( \{e_k\}_{k=1,...,N} \) as

\[
\alpha_{kl} = e_k - e_l, \quad 1 \leq k < l \leq N,
\]

and that \( u = \sum_k \lambda_k e_k \), one can rewrite the integral in a form which is more appropriate for further calculations

\[
Z(K_{P,Q}) = \int \prod_{i=1}^N d\lambda_i e^{-\frac{i\pi}{4} \prod_{k<l} 4 \sinh \left( \frac{\lambda_k - \lambda_l}{2P} \right) \sinh \left( \frac{\lambda_k - \lambda_l}{2Q} \right).} \tag{3}
\]

The following considerations are very similar to those in [11]. To obtain Ward identities, one should perform the following substitution

\[
\lambda_i = \lambda_i + \epsilon f_i, \quad \lambda_i = \lambda_i - \epsilon f_i. \tag{4}
\]

Considering \( \epsilon \) as an infinitesimal, we can decompose the integrals over each \( \lambda_i \) in \( Z_{P,Q} \) as follows: \( \int f(\lambda) d\lambda = \int f(\lambda) d\lambda + \epsilon \int g(\lambda) d\lambda + o(\epsilon^2) \). In the first order in \( \epsilon \), this gives us

\[
\int g(\lambda) d\lambda = 0. \]

Performing such a procedure for each \( \lambda_i \) in \( Z_{P,Q} \), one obtains the following Ward identity

\[
\left( \sum_{i=1}^N \frac{\partial f_i}{\partial \lambda_i} - \frac{\lambda f_i}{g_s} \right) + \frac{f_k - f_l}{2P} \coth \left( \frac{\lambda_k - \lambda_l}{2P} \right) + \frac{f_k - f_l}{2Q} \coth \left( \frac{\lambda_k - \lambda_l}{2Q} \right)
\]

\[
\times e^{\frac{i\pi}{4} \prod_{k<l} 4 \sinh \left( \frac{\lambda_k - \lambda_l}{2P} \right) \sinh \left( \frac{\lambda_k - \lambda_l}{2Q} \right)} = 0. \tag{5}
\]

Now we put \( f_i = \lambda_i \) and, to shorten the expression, we replace integrals with brackets (this notation will be used throughout this paper)
\[
\left\{ \sum_{i=1}^{N} \left( 1 - \frac{\lambda_i^2}{6i} \right) \right\} + \left\{ \frac{\lambda_k - \lambda_l}{2Q} \coth \left( \frac{\lambda_k - \lambda_l}{2Q} \right) \right\} + \left\{ \frac{\lambda_k - \lambda_l}{2P} \coth \left( \frac{\lambda_k - \lambda_l}{2P} \right) \right\} = 0. \quad (6)
\]

The resulting identity can be obtained by applying the following differential operator to \(Z_{P,Q}\)
\[
\hat{L} = N - P \frac{\partial}{\partial P} - Q \frac{\partial}{\partial Q}, \quad \hat{L}Z_{P,Q} = 0. \quad (7)
\]
The last equation simply states that \(Z_{P,Q}\) is a homogeneous function of degree \(N\) with respect to variables \(P\) and \(Q\). This statement is easily verified by taking the integral.

### 3. Elements of Virasoro algebra

Now we will take a look at the observables in Chern–Simons theory—Wilson loops. In terms of this matrix integral, Wilson loops for torus knots \(K_{Q,P}\) are the averages of the characters of \(G\), which can be expressed through Schur polynomials by the Weyl determinant formula as \(\chi_k(t) = \det(x_{i,j-1}(t))\) (for additional information see [12])
\[
W_k(K_{P,Q}) = \frac{1}{Z_{P,Q}} \int \prod_{i=1}^{N} dx_i \, e^{-\frac{P Q}{2} \sum_{k<l} 4 \sinh \left( \frac{\lambda_k - \lambda_l}{2P} \right) \sinh \left( \frac{\lambda_k - \lambda_l}{2Q} \right) \chi_k(e^t)} = \left\{ \chi_k(e^t) \right\}. \quad (8)
\]

For further calculations it is convenient to perform the following change of variables [8–10]
\[
\lambda_i = PQ \log x_i. \quad (9)
\]
The expression for a Wilson loop will then take the following form
\[
W_k(K_{P,Q}) = \frac{(P Q)^N}{Z_{P,Q}} \int_{\mathbb{R}^N} \prod_{i=1}^{N} dx_i \, \exp \left\{ -\frac{P Q (\log x_i)^2}{2\tilde{g}_i} \right\} - \left( \frac{P + Q}{2} (N - 1) + 1 \right) \log x_i \right\} \prod_{k<l} \left( x_k^p - x_l^p \right) \left( x_k^Q - x_l^Q \right) \chi_k(x) \quad (10)
\]
where \(\tilde{g}_i = \frac{2\pi i}{k + \hbar}\). Let us replace the characters by their generating function
\[
\sum_{R} \chi_R(t) \chi_k(x) = \exp \left\{ \sum_{k=1}^{\infty} \sum_{i} \lambda_i \right\}, \quad \text{where} \quad p_k = \sum_{i=1}^{\infty} \lambda_i; \quad (11)
\]
the resulting integral is
\[
W = \sum_{R} \chi_R(t) W_k(K_{P,Q}). \quad (12)
\]
The next step consists of obtaining Ward identities for \(W\), performing a change of variables as in the previous section. Then we will construct differential polynomials with respect to \(\frac{\partial}{\partial x_i}\) that generate these identities. Applying them to \(W\) and fixing the infinite set of parameters \(\{\lambda_i\}\), one will obtain various relations between Wilson loops for the same knot in different representations. Unfortunately, since a logarithm cannot be represented as a power series of \(x_i\) that converges on the whole positive semi-axis, we cannot construct a term with
The most obvious way to solve this problem comprises inserting another generating function into the integral
\[
\exp \left( \sum_{k=1}^{\infty} l_k \left( \sum_{i=1}^{N} x_i^k \log x_i \right) \right)
\]
This function also should be interpreted as a character decomposition. From now on we will be considering the following integral
\[
\int \prod_{N_{PQ}} \exp \log (x_i^{PQ} - x_n^{PQ}) \exp \left( \sum_{k=1}^{\infty} l_k p_k \right) \exp \left( \sum_{i=1}^{\infty} l_k \sum_{i=1}^{N} x_i^k \log x_i \right)
\]
Repeating the procedure described in the previous section, we obtain a Ward identity
\[
\left\langle \sum_{i=1}^{N} \left( \frac{\partial f_i}{\partial x_i} - \frac{PQ f_i \log x_i}{x_i} - \left( \frac{P + Q}{2} (N - 1) + 1 \right) \frac{f_i}{x_i} \right) \right\rangle + Q \sum_{k=1}^{\infty} k f_k \left( \sum_{i=1}^{N} x_i^{N-k-1} f_i \right) + \left( \sum_{i=1}^{\infty} l_k \left( \sum_{i=1}^{N} x_i^{N-k-1} (1 + k \log x_i) \right) \right) = 0
\]
Note that we have omitted generating functions inside the brackets to make the expression shorter. This notation with skipped generating functions will be used throughout this paper.
Now we have to choose the actual form of the deformations \( f_i \). The most common way to do this is to put \( f_i = x_i^{nPQ+1} \). The resulting identities can be obtained by applying the following differential operators to \( \hat{W} \)
\[
\hat{M}_{nPQ} = (nPQ - (P + Q)(N - 1)) \frac{\partial}{\partial n_{PQ}} - \frac{PQ}{g_{\nu}} \frac{\partial}{\partial l_{nPQ}} + P \sum_{a+b=n_{PQ}} \frac{\partial}{\partial a_{PQ}} \frac{\partial}{\partial b_{PQ}} + Q \sum_{a+b=n_{PQ}} \frac{\partial}{\partial a_{PQ}} + \sum_{k=1}^{\infty} k f_k \frac{\partial}{\partial n_{PQ+k}} + \sum_{k=1}^{\infty} l_k \left( \frac{\partial}{\partial n_{PQ+k}} + k - \frac{\partial}{\partial l_{nPQ+k}} \right)
\]
These operators form the Virasoro algebra. Applying them to \( \hat{W} \) and defining two infinite sets of parameters \( \{ t_k \} \) and \( \{ l_k \} \), one can generate identities between Wilson loops in different representations and averages of the form \( \langle \chi_{R_k} (p_k) \chi_{R_k} (\hat{p}_k) \rangle \), where \( p_k = \sum_{i=1}^{N} x_i^{k} \) and \( \hat{p}_k = \sum_{i=1}^{N} x_i^{k} \log x_i \). For example, if we put all of the \( t_k \) and the \( l_k \) equal to zero, we will obtain relations between the average \( \langle \sum_{i=1}^{N} x_i^{nPQ} \log x_i \rangle \) and Wilson loops, taken in different representations with the same number of boxes in the corresponding Young tableau. For more details, see section 5.

There is a way to rewrite this matrix model without the logarithmic term. This will allow us to obtain a Virasoro algebra using only one generating function, which is our final goal in this section. Let us consider the following simple identity of integrals (as in [16], appendix B)
\[
\frac{1}{\sqrt{2\pi g}} \int_{0}^{\infty} dx \, x^{d} \exp \left( \frac{\log x}{2g} \right) = q^{(d+1)/2} = \frac{1}{2\pi} \oint_{|z|=1} dz \, z^{d} \theta_{q}(z) \tag{17}
\]

where \( q = e^{\delta} \) and \( \theta_{q}(z) = \sum_{k=1}^{\infty} q^{k^2/2} z^{k} \) is the Jacobi theta function. This allows us to rewrite the matrix integral for Wilson loops (10) in the following way

\[
W_k(\mathcal{K}_{P,Q}) = \left( \frac{2\pi \delta_{P,Q}}{Z_{P,Q}} \right)^{N/2} \oint_{|z|=1} d\zeta \, \zeta_{i}^{-\frac{1}{2}} \theta_{q}(z_{i}) \times \prod_{k<l} \left( z_{k}^{P} - z_{l}^{P} \right) \left( z_{k}^{Q} - z_{l}^{Q} \right) \chi_{R}(z), \tag{18}
\]

with \( q = \exp \left( \frac{\delta}{PQ} \right) \). We will make a small shift of coupling constant, \( \delta_{i} \to \delta_{i} - \epsilon \), where \( 0 < \epsilon \ll 1 \). Since \( \text{Re} \, \delta_{i} = 0 \) and \( P, Q \in \mathbb{Z} \), we get \( |\epsilon| < 1 \). Also note that \( |\epsilon| = 1 \). As usual, we will work with the modified integral, where characters are replaced with their generating function

\[
W = \left( \frac{2\pi \delta_{P,Q}}{Z_{P,Q}} \right)^{N/2} \oint_{|z|=1} d\zeta \, \zeta_{i}^{-\frac{1}{2}} \theta_{q}(z_{i}) \times \prod_{k<l} \left( z_{k}^{P} - z_{l}^{P} \right) \left( z_{k}^{Q} - z_{l}^{Q} \right) \exp \left( \sum_{k=-\infty}^{\infty} t_{k} p_{k} \right) \tag{19}
\]

where \( p_{k} = \sum_{i=1}^{N} \zeta_{i}^{k} \). Note that the summation in the generating function is taken over all integers including negative values. This means that we are actually using two generating functions for positive and negative values of \( k \). Performing a transformation of variables, as in the previous case

\[
\tilde{z}_{j} = z_{j} + \epsilon \zeta_{j}^{n_{P}Q} + 1, \quad \tilde{z}_{j} = z_{j} - \epsilon \zeta_{j}^{n_{P}Q} + 1, \tag{20}
\]

we obtain the following identity for the \( W \) integral in the first order in \( \epsilon \):

\[
\begin{split}
\left\langle \left( nPQ - \frac{P + Q}{2} (N - 1) \right) \sum_{i=1}^{N} \zeta_{i}^{n_{P}Q} \right\rangle &+ \left\langle \sum_{i=1}^{N} \zeta_{i}^{n_{P}Q} \sum_{k=-\infty}^{\infty} k q^{k^{2}/2} z_{i}^{k} \right\rangle \theta_{q}(z_{i}) \\
&+ \left\langle \sum_{k<l} \left( z_{k}^{P} - z_{l}^{P} \right) \left( z_{k}^{Q} - z_{l}^{Q} \right) \right\rangle + \left\langle \sum_{k<l} \left( z_{k}^{P(n+1)} - z_{l}^{P(n+1)} \right) \frac{z_{k}^{P} - z_{l}^{P}}{z_{k}^{Q} - z_{l}^{Q}} \right\rangle \\
&+ \left( \sum_{k=-\infty}^{\infty} k t_{k} \left( \sum_{i=1}^{N} \zeta_{i}^{n_{P}Q+k} \right) \right) = 0. \tag{21}
\end{split}
\]

The second term in the above expression can be transformed as follows

\[
\left\langle \sum_{i=1}^{N} \zeta_{i}^{n_{P}Q} \sum_{k=-\infty}^{\infty} k q^{k^{2}/2} z_{i}^{k} \right\rangle = \sum_{i=1}^{N} \zeta_{i}^{n_{P}Q} \frac{\theta_{q}(z_{i})}{\theta_{q}(z_{i})} = \sum_{i=1}^{N} \zeta_{i}^{n_{P}Q} + \left( \log \theta_{q}(z_{i}) \right) \zeta_{i}. \tag{22}
\]

We will use the next representation for the Jacobi theta function, which will help us to rewrite the logarithmic term in a more convenient way.
\[
\theta_q(z_i) = \prod_{k=1}^{\infty} \left(1 - q^k \left(1 + q^k \frac{z_i}{\sqrt{q}} \right) \left(1 + q^k \frac{1}{z_i \sqrt{q}} \right) \right)
\]  

(23)

\[
z_i \left( \log \theta_q(z_i) \right)' = z_i \left( \sum_{k=1}^{\infty} \left( \log \left(1 - q^k \right) + \log \left(1 + q^k \frac{z_i}{\sqrt{q}} \right) + \log \left(1 + q^k \frac{1}{z_i \sqrt{q}} \right) \right) \right)'_{z_i}
\]

(24)

Now, using that \(|q| < 1\) and \(|z_i| = 1\), we can decompose these two fractions into the Laurent power series

\[
\sum_{k=1}^{\infty} \left( \sum_{l=1}^{\infty} (-1)^{l+k} \left( z_i q^{k-l} \right)^l \right) = \sum_{l=1}^{\infty} (-1)^{l+k} \left( \frac{1}{z_i} q^{k-l} \right)^l
\]

\[
= \sum_{l=1}^{\infty} (-1)^{l+k} \left( \frac{1}{z_i} q^{k-l} \right)^l
\]

Finally, we can write the second term in the Ward identity in a more appropriate form

\[
\left\langle \sum_{i=1}^{N} \frac{\sum_{k=-\infty}^{\infty} k q^{k/2} z_i^k}{\theta_q(z_i)} \right\rangle = \left\langle \sum_{i=1}^{\infty} (-1)^l \left( \frac{\sqrt{q}}{1 - q} \right)^l \left( z_i^{n_{PQ} - l} - z_i^{n_{PQ} + l} \right) \right\rangle.
\]  

(26)

Thus we can finally write down differential operators corresponding to the Ward identity (21)

\[
\hat{L}_{n_{PQ}} = (n_{PQ} - (P + Q)(N - 1)) \frac{\partial}{\partial n_{PQ}} + \sum_{l=1}^{\infty} (-1)^l \left( \frac{\sqrt{q}}{1 - q} \right)^l \left( \frac{\partial}{\partial n_{PQ} - l} - \frac{\partial}{\partial n_{PQ} + l} \right)
\]

+ \[P \sum_{a+b=n_{PQ}} \frac{\partial}{\partial a} \frac{\partial}{\partial b} + Q \sum_{a+b=n_{PQ}} \frac{\partial}{\partial a} \frac{\partial}{\partial b} + \sum_{k=-\infty}^{\infty} k \partial \frac{\partial}{\partial n_{PQ} + k}.
\]  

(27)

As mentioned already, we have used two generating functions. The first one is \(\exp(\sum_{k=1}^{\infty} t_k p_k)\), which is the generating function for the characters of \(G\). The second generating function is \(\exp(\sum_{k=-\infty}^{\infty} t_k p_k)\). It can be represented as a character decomposition, but in this case, variables \(p_k = \sum_{i=1}^{N} z_i^k\) in the character functions have negative values of \(k\). We denote characters with negative \(k\) as \(\chi_{-k}(p_-)\) and the character functions with positive \(k\) as \(\chi_{+k}(p_+)\). Considering the integral (19), it is easy to see that \(\langle \chi_{k}(p_-) \rangle_{P,Q} = \langle \chi_{k}(p_-) \rangle_{-P,-Q} = \langle \chi_{k}(p_+) \rangle_{P,Q}\). Thus the integral (19) can be written as

\[
W = \sum_{k} \sum_{\pm} \chi_{k}(p_+) \chi_{k}(p_-) \left( \chi_{k}(p_+) \chi_{k}(p_-) \right).
\]  

(28)

After we apply the differential operator (27) to the expression above and put all of the \(t_k\) equal to zero, we obtain relations of the form
\[
\sum_{R} A_R \langle \chi_R(p_{\perp}) \rangle + \sum_{R} B_R \langle \chi_R(p_{\parallel}) \rangle = \sum_{R} (A_R + B_R) \langle \chi_R(p_{\perp}) \rangle = 0, \quad (29)
\]

where \(A_R\) and \(B_R\) are some constants.

4. The stability limit

Now let us consider the stability limit, i.e.,

\[
P \to \infty, \quad Q \to \infty, \quad \text{while } \frac{P}{Q} = \text{const.} \quad (30)
\]

In this and the following sections we will carry out calculations for the \(\hat{W}\) integral (14) and the corresponding \(\hat{M}_{nPQ}\) operators (16), but all of the following conclusions can easily be rewritten for the \(\hat{W}\) integral and the \(\hat{L}\) operators (27). First, our Ward identity (15) with \(f_i = x_i^{nPQ+1}\) must be resolved in all orders of \(P\) and \(Q\). Thus we end up with three independent equations:

\[
\left\langle \sum_{i=1}^{N} nPQ x_i^{nPQ} \right\rangle - \left\langle \sum_{i=1}^{N} P Q x_i^{nPQ} \log x_i \right\rangle = 0
\]

\[
\left\langle \sum_{i=1}^{N} P + Q (N - 1) x_i^{nPQ} \right\rangle = \left\langle \sum_{k<i} Q \frac{x_k^{Q+nPQ} - x_i^{Q} + nPQ}{x_k^Q - x_i^Q} \right\rangle \\
\times \left\langle \sum_{k<i} P \frac{x_k^{P+nPQ} - x_i^{P} + nPQ}{x_k^P - x_i^P} \right\rangle \quad (31)
\]

And, correspondingly, every single Virasoro algebra generator factorizes into three simple operators:

\[
\hat{e}_n = nPQ \frac{\partial}{\partial_{nPQ}} - \frac{PQ}{\delta_x} \frac{\partial}{\partial_{nPQ}}
\]

\[
\hat{j}_n = - (P + Q)(N - 1) \frac{\partial}{\partial_{nPQ}} + P \sum_{a+b=nQ} \frac{\partial}{\partial_{aQ}} \frac{\partial}{\partial_{bQ}} + Q \sum_{a+b=nP} \frac{\partial}{\partial_{aP}} \frac{\partial}{\partial_{bP}} \quad (32)
\]

\[
\hat{h}_n = \sum_{k=0}^{\infty} k \delta_k \frac{\partial}{\partial_{nPQ+k}} + \sum_{k=0}^{\infty} \left( \frac{\partial}{\partial_{nPQ+k}} + k \frac{\partial}{\partial_{nPQ+k}} \right)
\]

Thus, in this limit, we get much simpler expressions for the Ward identities and corresponding differential operators. As we will see, this simplifies things greatly when we want to get actual relations for Wilson loops.

5. A trefoil example

Now let us consider a trefoil knot and generate relations between its Wilson loops, or HOMFLY (Hoste–Ocneanu–Millett–Freyd–Lickorish–Yetter) polynomials, written in
different representations. To do so, one should consider the generalized matrix integral (14) and put $P = 2$, $Q = 3$. This matrix integral can be written as

$$
\bar{W} = \sum_R \sum_{R'} \chi_R(t_k) \chi_{R'}(l_k) \left\{ \chi_R(p_k) \chi_{R'}(\tilde{p}_k) \right\}
$$

(33)

where $p_k = \sum_{i=1}^N x_i^k$ and $\tilde{p}_k = \sum_{i=1}^N x_i^k \log(x_i)$. We are interested in the $\langle \chi_R(p_k) \rangle$, as they appear to be $W_R(K_{PQ})$; this is why we should put all of the $l_k$ and the $t_k$ equal to zero—that will leave us only separate averages of $\chi_R(p_k)$ and $\chi_R(\tilde{p}_k)$ in (33) and put generating functions to 1 in the Ward identities. Now let us write an exact form of the $n = 1$ element of the Virasoro algebra for the trefoil

$$
\tilde{M}_{n'PQ} = \tilde{M}_0 = (6 - 5(N - 1)) \frac{\partial}{\partial t_6} - \frac{6}{g_s} \frac{\partial}{\partial t_6} + 2 \sum_{a+b=3} \frac{\partial}{\partial t_{2a}} \frac{\partial}{\partial t_{2b}} + 3 \sum_{a+b=2} \frac{\partial}{\partial t_{3a}} \frac{\partial}{\partial t_{3b}} + \sum_{k=0}^\infty k t_k \frac{\partial}{\partial t_{6+k}} + \sum_{k=0}^\infty l_k \left( \frac{\partial}{\partial t_{6+k}} + k \frac{\partial}{\partial t_{6+k}} \right).
$$

(34)

Next, one should apply this differential polynomial to $\bar{W}$ and put all of the $\{t_k\}$ and $\{l_k\}$ equal to zero, as was discussed above, and take, for example, the simplest case of a $2 \times 2$ matrix

$$
\bar{M}_0 \bar{W} \bigg|_{t_k = l_k = 0, \forall k} = 0
$$

(35)

$$
\int \frac{6}{g_s} \left( \sum_{i=1}^N x_i^6 \log x_i \right) = 5W_{(6)} - 5W_{(5,1)} + 2W_{(4,1,1,1)} - 3W_{(2,1,1,1,1)} + 3W_{(1,1,1,1,1,1)} + 3W_{(2,2,1,1)} - 2W_{(3,2,2)}
$$

(36)

In the considered case, $N = 2$, and the average of $\sum x_i^6 \log (x_i)$ will be

$$
\left\langle \sum_{i=1}^N x_i^6 \log x_i \right\rangle = 48\pi g_s^2 \exp \left( \frac{37 \tilde{g}_s}{24} \right) \left( 7 - 11 e^{\tilde{g}_s} - 13 e^{2\tilde{g}_s} + 17 e^{5\tilde{g}_s} \right).
$$

(37)

It is easy to see that in the LHS of equation (37) we always have a term proportional to $\left\langle \sum x_i^6 \log (x_i) \right\rangle$. That means that one can easily rewrite this identity as the identity between Wilson loops in different representations for two different knots if it is possible to make such choices of $n$ that the products of $n$, $P$ and $Q$ for these knots are equal. For example, we can compare Wilson loops for the trefoil knot with $n = 2$ and the torus knot with $P = 4$, $Q = 3$, $n = 1$, but the result is too lengthy to present here.

In the stability limit, as was discussed above, we get the following two identities (the third one vanishes as we put the $t_k$ and the $l_k$ equal to zero) for Wilson loops, when $\frac{P}{Q} = \frac{3}{2}$

$$
\int \frac{6}{g_s} \left( \sum_{i=1}^N x_i^6 \log x_i \right) = W_6 - W_{5,1} + W_{4,1,1,1} - W_{3,2,1,1,1} + W_{2,1,1,1,1} - W_{1,1,1,1,1,1,1} \equiv 6 \left\langle \sum_{i=1}^N x_i^6 \right\rangle
$$

(38)
\[ W_{(3,3)} + 3W_{(4,2)} + W_{(2,2,2)} - 4W_{(3,2,1)} - 3W_{(4,1,1)} + 3W_{(2,2,1,1)} + 7W_{(3,1,1,1,1)} - 10W_{(2,1,1,1,1,1)} + 10W_{(1,1,1,1,1,1,1,1,1)} = 0. \]  (39)

This identities correspond to the \( \hat{e}_i \) and \( \hat{f}_i \) operators in (32).

6. Discussion

We obtained the Virasoro constraints for two different cases. The resulting Virasoro generators are obviously connected. Such a connection can be seen after applying them to the integral for torus knots with the corresponding generating function(s) and putting all additional parameters equal to zero. Namely, we can make the following statement

\[
\left( \sum_{l=0}^{\infty} (-1)^{l+H(l)} \left( \frac{\sqrt{q}}{1-q} \right)^{H(l)} \frac{\partial}{\partial t_{lPQ}} - \frac{\partial}{\partial t_{nPQ}} \right) W \bigg|_{t_{lPQ}=0,\forall k} = \left( \frac{PQ}{\hat{g}_s} \frac{\partial}{\partial l_{PQ}} \right) W \bigg|_{t_{lPQ}=0,\forall k}. \]  (40)

Perhaps it would look better if we rewrote this equality in terms of averages

\[
\left( \sum_{l=-\infty}^{\infty} (-1)^{l+H(l)} \left( \frac{\sqrt{q}}{1-q} \right)^{H(l+\frac{1}{2})} - x_i^{nPQ} \right) = - \frac{PQ}{\hat{g}_s} \sum_{i=1}^{N} x_i^{nPQ} \log (x_i). \]  (41)

Note that in the LHS we have replaced \( z_i \) with \( x_i \). This is a valid operation, as was discussed in section 3 (see (17)). However, such relations are yet to be discussed. Direct application of Virasoro generators gives us identities between Wilson loops, or HOMFLY polynomials. These identities contain either an undesirable term \( \langle \sum x_i^{nPQ} \log (x_i) \rangle \) or an infinite sum of Wilson loops in different representations that prevents us from getting closed recurrent relations. On the other hand, the logarithmic term can be directly computed; for example, in the \( N = 2 \) case,

\[
\left\langle \sum x_i^{nPQ} \log (x_i) \right\rangle = 8\hat{g}_s^2 \pi e^{\frac{\delta}{PQ} (P-Q)^2 + 2nPQ-PQ} \left( -e^{\delta P} (P + Q - 2nPQ + 1) + e^{\delta (1+nP)} (P + Q + 2nPQ) + e^{\delta nP(Q-P - 2nPQ)} + e^{\delta nP(Q-P - 2nPQ)} \right). \]  (42)

or it can be used to compare compositions of colored Wilson loops for two different knots, with equal products of \( n, P \) and \( Q \). The Virasoro generators obtained are also useful for studying these relations in the so-called stability limit, where they are significantly simplified.

Though there is a working formula for HOMFLY polynomials or Wilson loops for all torus knots, the relations between these objects are not fully understood and this example of Virasoro algebra can give some idea of the general structure of such algebras for all knots.

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