Quartic del Pezzo surfaces over function fields of curves

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Abstract: We classify quartic del Pezzo surface fibrations over the projective line via numerical invariants, giving explicit examples for small values of the invariants. For generic such fibrations, we describe explicitly the geometry of spaces of sections to the fibration, and mappings to the intermediate Jacobian of the total space. We exhibit examples where these are birational, which has applications to arithmetic questions, especially over finite fields.

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1. Introduction

The geometry of spaces of rational curves of low degree on Fano threefolds is a very active area of algebraic geometry. One of the main goals is to understand the Abel–Jacobi morphism from the base of a space of such curves to the intermediate Jacobian of the threefold. For instance, does a given space of curves dominate the intermediate Jacobian? If so, what are the fibers of this morphism? Does it give the maximal rationally connected (MRC) quotient? Representative results in this direction are available for:

- cubic threefolds [12, 13, 19, 28];
- Fano threefolds of genus six and degree 10 in $\mathbb{P}^{7}$ [7];

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• Fano threefolds of genus seven and degree 12 in $\mathbb{P}^8$ [20];

• moduli of vector bundles [4] — this case makes clear that one cannot always expect the morphism to the intermediate Jacobian to give the MRC fibration.

At the same time, del Pezzo fibrations $\pi: X \to \mathbb{P}^1$ are equally interesting geometrically. Moreover, the special case where the rational curves happen to be sections of $\pi$ is particularly important for arithmetic applications. It is a major open problem to determine whether or not sections exist over a non-closed ground field. Of course, the Tsen–Lang theorem gives sections when the ground field is algebraically closed. Even when there are rigid sections, these are typically defined over extensions of large degree.

However, suppose that the space of sections of fixed height is rationally connected over the intermediate Jacobian $\text{IJ}(X)$. If the fibration is defined over a finite field and the space of sections descends to this field then $\pi$ has sections over that field. Indeed, this follows by combining a theorem of Lang [25] (principal homogeneous spaces for abelian varieties over finite fields are trivial) and a theorem of Esnault [10] (rationally connected varieties over finite fields have rational points). This point of view was developed for quadric surface bundles in [17], with applications to effective versions of weak approximation.

This paper addresses the case of quartic del Pezzo fibrations $\pi: X \to \mathbb{P}^1$. Throughout, we assume $X$ is smooth and the degenerate fibers of $\pi$ have at worst one ordinary singularity. We classify numerical invariants — the fundamental invariant is the height of the fibration, denoted $h\mathit{h}(X)$. We provide explicit geometric realizations for fibrations of small height. We exhibit families of sections parametrizing the intermediate Jacobian and admitting rationally-connected fibrations over the intermediate Jacobian. The underlying constructions often involve Brill–Noether theory for smooth or nodal curves. Our main result is the comprehensive analysis of spaces of small-height sections for height 12 fibrations in Section 10; here $X$ dominates a nodal Fano threefold of genus seven and degree 12.

2. Basic properties of quartic del Pezzo surfaces

Let $X$ be a quartic del Pezzo surface over an algebraically closed field. $X$ is isomorphic to $\mathbb{P}^2$ blown up at five distinct points such that no three are collinear. Thus we may identify

$$\text{Pic}(X) = \mathbb{Z}L + \mathbb{Z}E_1 + \mathbb{Z}E_2 + \mathbb{Z}E_3 + \mathbb{Z}E_4 + \mathbb{Z}E_5,$$

where $L$ is the pull-back of the hyperplane class on $\mathbb{P}^2$ and $E_i$ are the exceptional divisors. The primitive divisors $\Lambda = K_X^\perp \subset \text{Pic}(X)$ are a lattice under the intersection form; using the basis

$$\{E_1 - E_2, E_2 - E_3, E_3 - E_4, E_4 - E_5, L - E_1 - E_2 - E_3\}$$

we have

$$\Lambda \simeq \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 \\
0 & 0 & 1 & 0 & -2
\end{pmatrix}.$$

Writing $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Z})$, the intersection form gives a surjection $\text{Pic}(X) \twoheadrightarrow \Lambda^*$, with kernel generated by $K_X$. Note that $\Lambda^*/\Lambda$ is a cyclic group of order four.

The action of the monodromy group on $\text{Pic}(X)$ factors through the Weyl group $W(D_5)$ coming from the $D_5$ root system contained in the lattice $(-1)\Lambda$. Abstractly, $W(D_5)$ may be realized as an extension of the symmetric group

$$1 \to (\mathbb{Z}/2\mathbb{Z})^3 \to W(D_5) \to S_5 \to 1;$$

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