WEAKLY HAMILTONIAN ACTIONS

DAVID MARTÍNEZ TORRES AND EVA MIRANDA

Abstract. In this paper we generalize constructions of non-commutative inte-
grable systems to the context of weakly Hamiltonian actions. In particular
we prove that abelian weakly Hamiltonian actions on symplectic manifolds
split into Hamiltonian and non-Hamiltonian factors.

1. Introduction

An integrable system on a $2n$-dimensional symplectic manifold is given by $n$
generically independent pairwise commuting functions. More generally, a non-
commutative integrable system is determined by a set of $2n - r$ integrals ($r \leq n$),
out of which $r$ do pairwise commute. Integrable systems come with infinitesimal
abelian actions which are Hamiltonian, in the sense that they have an equivariant
momentum map.

Furthermore, under compactness assumption on the invariant sets these infinitesimal
abelian actions integrate into a torus action for which there is a normal form (action-angle coordinates).

However, some discrete integrable systems [13] do not present commuting first
integrals but rather commuting flows. Moreover, there are systems that become
Hamiltonian after reduction by non-commutative symmetries. This justifies con-
sidering a more general framework where weakly Hamiltonian actions take over
Hamiltonian actions. We look at (infinitesimal) actions of abelian Lie algebras hav-
ing first integrals, but that cannot be arranged into an equivariant momentum map.
We will show that if the action integrates into a group action (i.e if the vector fields
are complete), then we can still find “invariant subsets” where the residual action
is indeed Hamiltonian. More explicitly, we will prove a global splitting theorem for
the action into a Hamiltonian factor and a translational non-Hamiltonian one.

We shall also discuss a version of our results in the Poisson setting [6]. In fact,
we will see that our results have strong reminiscences of the classical Weinstein
splitting theorem in Poisson geometry [14].

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2. Motivating examples

We start by presenting three different types of (complete) weakly Hamiltonian
abelian actions in symplectic and Poisson manifolds, and we close the section with
a related example.
2.1. **Standard action by translations.** The paradigm of abelian symmetries by Hamiltonian vector fields, but not fitting into a Hamiltonian action, is that of a symplectic vector space \((V,\sigma)\) on which \(V\) itself – seen as an abelian Lie group – acts by translations: to each vector \(v \in V\) we can assign a first integral which is the unique linear function \(H_u \in V^*\) such that \(dH_u = i_u \sigma\) (or rather, an affine one with that linear part). Since we have:

\[
\{H_u, H_v\} = \sigma(u,v)
\]

we will never be able to find a basis of first integrals in involution. Notwithstanding, any such choice of first integrals yields a weakly Hamiltonian action (indeed a non-commutative system with constant brackets).

2.2. **The Galilean group.** Let \(G(3)\) be the Galilean group and consider its standard representation (cf. (13.7) in [12]) on \(T^*\mathbb{R}^3\) with position and momentum coordinates \(q_i, p_i\) and particle mass \(m\). Recall that \(G(3)\) is an extension of the Euclidean group \(E(3)\), and the restriction of this representation to \(E(3)\) is Hamiltonian because it is the cotangent lift of its defining action on \(\mathbb{R}^3\). However the Hamiltonian functions corresponding to the Galilean boosts \(mq_i\) and translations in the same direction \(p_i\) do not commute; indeed their Poisson bracket is the mass, and the corresponding cocycle is not exact, bringing in another example of weakly Hamiltonian action.

2.3. **Weakly Hamiltonian actions and nilpotent Lie algebras.** The symplectic form on the symplectic vector space \((V,\sigma)\) can be interpreted as a 2-cocycle, and as such it gives rise to \(g = \mathbb{R} \oplus_v V\) a central extension of the abelian algebra \(V\). This Heisenberg type-Lie algebra is nilpotent with one dimensional center. The coadjoint orbit corresponding to the affine hyperplane:

\[
\{\alpha \in g^* | \alpha(1,0) = 1\}
\]

can be canonically identified with \(V\), and the restriction of the coadjoint action to this orbit is the linear action by translations above (2.1).

More generally, let \(g\) be a nilpotent Lie algebra such that \([g,g] \subset g^\circ\) (a 2-step nilpotent Lie algebra). As brackets lie in the center, they become Casimirs as functions on \(g^\circ\), and therefore constants on coadjoint orbits. Then any subspace of \(g\) intersecting trivially with the center provides an abelian Lie algebra acting in a weakly Hamiltonian fashion (which is not Hamiltonian provided that some of the brackets are non-trivial) on any coadjoint orbit (and in fact on the whole \(g^\circ\)). We illustrate this with the following low dimensional example (additional ones can be found by inspecting the list of low dimensional nilpotent Lie algebras up to dimension 7 ([8], [9], [10])):

The nilpotent Lie algebra of dimension 6, \(A_{6,5}^a\) (for \(a \neq 0\)) for which the non-vanishing relations on a base (see table III in [8]) are \([e_1,e_2] = ce_3, [e_1,e_4] = e_6, [e_2,e_3] = ae_6, [e_2,e_4] = e_5\). In this case, the symplectic foliation by coadjoint orbits is given by \(e_2^*\) and \(e_4^*\), thus defining a foliation with regular 4-dimensional symplectic leaves away from zero. The subspace spanned by \(e_1, e_2, e_3\) and \(e_4\) acts by commuting Hamiltonian vector fields but without momentum map, providing an example of weakly Hamiltonian action on a Poisson manifold.

2.4. **Related examples.** In [13], motivated by the study of discrete integrable systems, the “multi-time” Legendre transform is applied to multi-time Euler-Lagrange equations to obtain a system of commuting Hamiltonian flows. As observed in [13] this situation corresponds to having functions with constant Poisson brackets but
these brackets are not necessarily zero\(^1\), thus providing an extra motivation to consider weakly Hamiltonian actions.

3. Weakly Hamiltonian actions and real analytic functions

In this section we discuss how real analytic functions become a tool to study weakly Hamiltonian actions.

**Definition 1.** Let \((M, \omega)\) be a symplectic manifold and let 
\[ \rho : g \rightarrow \text{symp}(M, \omega) \]
\[ u \mapsto X_u \]
be an action by symplectic vector fields. The action is weakly Hamiltonian if the fundamental vector fields for the action are Hamiltonian vector fields. The action is called complete if all fundamental vector fields are complete.

If \(\rho : g \rightarrow \text{ham}(M, \omega)\) is a weakly Hamiltonian action, then it can always be lifted to a linear map (for which we use the same notation):
\[ \rho : g \rightarrow C^\infty(M) \]
\[ u \mapsto H_u. \]
As it is well-known, the defect from the action being Hamiltonian is measured by the 2-cocycle:
\[ c : g \times g \rightarrow \mathbb{R} \]
\[ (u, v) \mapsto \{H_u, H_v\} - H_{[u, v]}. \]
(1)

More precisely, the 2-cocycle \(c \in \Lambda^2 g^*\) is:
- zero iff the chosen lift defines a Lie algebra morphism (this is equivalent to the mapping \(\rho\) being equivariant);
- exact iff there exists a choice of lift which is a morphism of Lie algebras.

**Definition 2.** To any complete weakly Hamiltonian action \(\rho : g \rightarrow C^\infty(M)\), we assign its flow evaluation map:
\[ \zeta : g \times g \times M \rightarrow C^\infty(\mathbb{R}) \]
\[ (u, v, x) \mapsto H_u(\phi^s_v(x)), \]
(2)

where \(\phi^s_v\) denotes the flow of \(X_v\).

Note that if the action \(\rho\) is Hamiltonian with momentum map \(\mu\), then the flow evaluation map is the result of pulling back via the momentum map the flow evaluation map for the coadjoint action: indeed, because the momentum map is \(g\)-equivariant we have
\[ \zeta_{u,v,x}(s) = H_u(\phi^s_v(x)) = H_u(\phi^s_v \cdot x) = (\text{Ad}^*(\Phi^s_v)(\mu(x)), u) \]
(3)
where \(\Phi^s_v \in G(g)\) – the simply connected Lie group integrating \(g\) – and \(\text{Ad}^*(\Phi^s_v)(\mu(x))\) is the corresponding coadjoint flow of \(v\) starting at \(\mu(x)\). Therefore, the \(\zeta_{u,v,x}\)’s generalize the linear projections of the coadjoint flow to the complete weakly Hamiltonian case.

The coadjoint flow is real analytic, and each of its projections (3) is a real analytic function which extends to an entire function of exponential type (growth). For example, this can be checked by noting that to analyze the coadjoint action it suffices to use matrix groups. For complete weakly Hamiltonian actions this

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\(^1\)As observed by the author in [13] indeed the vanishing of these Poisson brackets is equivalent to Lagrangian 1-form employed in the construction being closed on the solutions of the Euler-Lagrange equations.
property still holds (cf. [4], lemma 6, where real analyticity is studied is the setting of quasi-representations):

**Proposition 1.** For any triple \((u, v, x) \in \mathfrak{g} \times \mathfrak{g} \times M\) the corresponding flow evaluation \(\zeta_{u,v,x}\) is real analytic with expansion at zero:

\[
\zeta_{u,v,x}(s) = \sum_{j=0}^{\infty} \text{Ad}^j(v)(x) \frac{s^j}{j!} + \sum_{j=1}^{\infty} e_{\text{Ad}^{j-1}(v)(u), v} \frac{s^j}{j!}.
\]

Moreover, it extends to an entire function of exponential type.

**Proof.** The fundamental theorem of calculus yields:

\[
\zeta_{u,v,x}(s) = \zeta_{u,v,x}(0) + \int_0^s \zeta_{u,v,x}'(t) dt.
\]

The formula in (4) follows by induction. In order to check convergence of the power series expansion fix any metric in \(\mathfrak{g}\), and pick a neighborhood of the origin \(W \subset \mathfrak{g}\) so that \(|[u, v]| \leq C|u||v|, \forall u, v \in W\).

The linear maps \(u \mapsto \text{Ad}^j(v)(x)\) are continuous, and also vary continuously with \(y\). The bilinear map \((u, v) \mapsto e_{u,v}\) is also continuous. Therefore the norm of the remainder of the Taylor expansion can be bounded in any compact subset as follows:

\[
\left| \int_0^{s_1} \cdots \int_0^{s_{j+1}} \text{Ad}^j(v)(x) \, ds_{j+1} \cdots ds_1 + e_{\text{Ad}^{j+1}(v)(u), v} \, \frac{s^{j+1}}{(j+1)!} \right| \leq K_1|u|\left| C|v|s^{j+1} \frac{1}{(j+1)!} \right| + K_2|u|\left| C|v|s^{j+1} \frac{1}{(j+1)!} \right|
\]

Then uniform convergence on any compact subset is straightforward.

Nearly the same proof shows that the power expansion has coefficients whose norm is dominated by those of an exponential, and therefore the desired result follows.

One can roughly rephrase Proposition 1 by saying that a complete weakly Hamiltonian action on a symplectic manifold gives a large supply of very special analytic functions (non-trivial except for abelian Hamiltonian actions).

**Remark 1.** In [5] it is shown that any action of a compact group on a symplectic manifold is equivalent to an analytic one (also the symplectic form being analytic). Thus if the action is weakly Hamiltonian one obtains automatically that the functions \(\zeta_{u,v,x}\) are analytic. Our result is more general in the sense that it only requires a complete Lie algebra action, and it makes clear the dependence of the expansion of \(\zeta_{u,v,x}\) in terms of the Lie algebra structure and the cocycle \(c\).

### 4. The splitting theorem

In this section we show that complete weakly Hamiltonian actions split into a translational standard component and a Hamiltonian component. This makes precise the idea that in some systems after reducing by translations (or restricting to appropriate slices to the translation action), what we get is a Hamiltonian system (and sometimes a completely integrable one).
Let us introduce the following notation for the standard action by translations on a symplectic vector space \((V, \sigma)\):
\[
\rho_{\text{std}} : g \longrightarrow V^* \\
u \longmapsto \sigma^\#(u)
\]
characterized by the equation
\[
\rho_{\text{std}}(u)(v) = \sigma(u, v).
\]
By construction the cocycle is the symplectic form itself:
\[
c_{u,v} = \{\sigma^\#(u), \sigma^\#(v)\} = \sigma(u, v).
\]
With these provisions in place we can now state our main result:

**Theorem 1.** Let \(\rho : g \rightarrow C^\infty(M)\) be a complete weakly Hamiltonian action of an abelian Lie algebra, and let \(n \subset g\) be the kernel of the associated cocycle \(c \in \wedge^2 g^*\). Then a choice of \(U \subset g\) a complementary subspace to \(n\) canonically defines a symplectic splitting:
\[
(M, \omega) \cong (U \times N, c|_U \oplus \omega|_N),
\]
so that the representation also splits as:
\[
\rho = \rho_{\text{std}} \times \rho_n,
\]
where \(\rho_n\) is the restriction of \(\rho\) to \(n\) acting on the symplectic submanifold \((N, \omega|_N)\), which is Hamiltonian, and \(\rho_{\text{std}}\) is the standard translational action on the symplectic vector space \((U, c|_U)\) in (2.1).

**Proof.** By construction \((U, c|_U)\) is a symplectic vector space.

For each \(u \in U\) we claim that \(H_u\) can be used as a coordinate function; moreover a basis of \(U\) defines the corresponding number of (independent) coordinates: for each \(u \in U\) we pick \(v \in U\) for which \(c_{u,v} \neq 0\). Because
\[
H_u(X_v) = \omega(X_u, X_v) = \{H_u, H_v\} = \{H_u, H_v\} - H_{[u,v]} = c_{u,v},
\]
\(H_u\) is a submersion. Moreover, in the abelian case the formula for the flow evaluation (4) becomes:
\[
c_{u,v,x}(s) = H_u(x) + c_{u,v}s, \tag{5}
\]
implying that \(H_u\) is surjective. Similarly, a basis \(b\) of \(U\) gives rise to functionally independent functions (one can just follow the steps of the construction of a Darboux basis for example), thus fitting into a surjective submersion:
\[
H_b : M \rightarrow \mathbb{R}^{2d}
\]
Next, we define \(N\) to be the fibre over zero. Note that the definition is independent of the fixed base \(b\), as \(N\) is the common zero set of all Hamiltonian functions of vectors in \(U\).

Because \(\omega(X_u, X_v) = c_{u,v}\), for \(u \neq 0\) the symplectic vector field \(X_u\) is nowhere zero. Hence the action of \(U\) on \(M\) as a Lie group is clearly locally free. As we shall see now, \(N\) provides a full slice for the \(U\)-action on \(M\) so that each \(U\)-orbit intersects \(N\) at a single point: let \(x \in M\) and consider for any non-zero \(u \in U\) the value \(H_u(x)\). In case it is non-zero we select \(v \in U\) with \(c_{u,v} = 1\).

Since
\[
H_u(\phi^s_u(x)) = H_u(x) + c_{u,v}s = H_u(x) + s, \tag{6}
\]
we can follow the flow of \(-v\) to reach a point which is a zero of \(H_u\). This can be done for a basis of \(U\), and therefore \(x\) is in the \(U\)-orbit of some \(y \in N\). Furthermore, the point \(y \in N\) we can reach is unique. Otherwise an orbit of some \(X_v\) would intersect \(N\) more than once, but this is not possible since by (6) for a given flow there exists a Hamiltonian which is a non-trivial affine function on it, and therefore cannot take the value 0 more than once.
The outcome of the preceding discussion is the construction of a diffeomorphism:

$$\Phi: U \times N \longrightarrow M$$

$$(u, y) \mapsto x := \phi^1_u(y) (= u \cdot y)$$  \hspace{1cm} (7)

The symplectic structure also splits: the tangent bundle to (all) $U$-orbits has a frame given by symplectic vector fields, and one has the canonical identification $X_u(x) \mapsto u \in U$. Because

$$\omega(X_u, X_v) = c_{u,v}$$

this identification sends the restriction of $\omega$ to the restriction of $c$.

Observe that under the product structure in (7) the fibers of the first projection (copies of $N$) are the fibers of $H_b$. Now if $z$ is a vector tangent to a fiber of the first projection, then

$$dH_u(z) = 0, \, \forall u \in U,$$

or equivalently:

$$\omega(X_u, z) = 0, \, \forall u \in U.$$  \hspace{1cm} (8)

Thus $z$ is in the symplectic orthogonal to the $U$-orbit. Because the latter is symplectic, by counting dimensions it follows that the tangent space to the fiber of the first projection is exactly the symplectic annihilator of the $U$-orbit, thus also symplectic.

Let $\omega|_N$ denote the restriction of $\omega$ to $N$. Because the action of $U$ is by symplectic vector fields we have the symplectomorphism:

$$\Phi: (U \times N, \omega|_N \oplus c|_U) \cong (M, \omega).$$

We now check that the representation also splits: indeed, for each $u \in U$ by (8) the Hamiltonian $H_u$ does not depend on the coordinates of $N$. Therefore $\rho|_U$ splits as the trivial representation on $N$ times a representation on $(U, c|_U)$. Also, by (6) the latter representation is the standard one.

The restriction $\rho|_n$ is by definition a Hamiltonian representation on $(M, \omega)$. We need to check that the corresponding Hamiltonians are independent of the $U$-coordinates. But for $z \in n, \, u \in U$, we have:

$$H_z(\phi^a_u(x)) = H_z(x) + c_{z,u} = H_z(x),$$

and this proves the desired result. \hspace{1cm} \square

Remark 2. Theorem 1 is a generalization of a (local) result in [11]. There, one assumes the existence of a what is called semicanonical system of functions in a $2n$ dimensional symplectic manifold. In our language these is a weakly Hamiltonian action of an $n$-dimensional abelian Lie algebra, so that the differentials of the associated Hamiltonian functions are linearly independent. It turns out that neither hypothesis on the number of functions nor on the rank of differentials of the functions in involution are necessary.

4.1. Poisson manifolds. In this section we state some natural extensions to Poisson geometry of the results in the previous section.

A weakly Hamiltonian representation on a Poisson manifold $(M, \pi)$ is a representation by Hamiltonian vector fields:

$$\rho: \mathfrak{g} \rightarrow \text{ham}(M, \pi)$$

As in the symplectic setting, for such an action the exists lifts to a linear map:

$$\rho: \mathfrak{g} \rightarrow C^\infty(M).$$
The defect from being Hamiltonian is now measured by a Casimir valued 2-cocycle \( c \) (a 2-cocycle with values in “smooth functions” on the leaf space of \((M, \pi)\)). Exactly the same formula (2) defines the flow evaluation map:

\[
\zeta_{u,v,x} = H_u(\phi_v^s(x)),
\]

Proposition 1 holds also in this setting, where \( c \) is the restriction of the Casimir valued 2-cocycle to a usual cocycle on the symplectic leaf \( F \in \mathcal{F}_\pi \) containing the point \( x \).

If for example \((M, \pi)\) supports only trivial Casimirs (constants), then the splitting theorem holds word by word. As examples of Poisson manifolds with trivial Casimirs we may consider, for instance, the Reeb foliation of \( S^3 \) with leafwise area form, compact cosymplectic manifolds with non-compact leaves endowed with natural Poisson structures [3], and other Poisson manifolds constructed out of them via products, surgeries, etc.

More generally, one can look at then collection of kernels \( \{n_F\}_{F \in \mathcal{F}_\pi} \) of the action on each leaf. If there exists \( U \) a subspace of \( \mathfrak{g} \) which intersects trivially all \( \{n_F\}_{F \in \mathcal{F}_\pi} \), then we have a Poisson splitting:

\[
(M, \pi) \cong (U \times N, c|_U^{-1} \times \pi|_N),
\]

given by the action of \( U \). However, the whole action need not split. It is worth pointing out that this splitting result can be seen as a global version of Weinstein splitting theorem. The original proof of the Weinstein’s splitting theorem amounts to constructing a weakly Hamiltonian abelian action of maximal rank with symplectic cocycle.

It is also possible to reinterpret other splitting results in the Poisson setting using this language. For example, in [7, 2] an equivariant Weinstein splitting theorem is proved at a fixed point of a Poisson for an action of a compact Lie group. For a Hamiltonian action of \( G \) compact, this result can be restated saying that the (local) Hamiltonian action of \( \mathfrak{g} \) can be extended to a weakly Hamiltonian action of an extension \( \mathfrak{g} \rtimes \mathbb{R}^{2d} \), where \( 2d \) is the rank of the Poisson structure at \( x \) (the cocycle has kernel \( \mathfrak{g} \)). From this perspective, the equivariant Poisson splitting follows trivially.

### 4.2. Nilpotent actions

The real analyticity of the flow evaluation map is rather powerful, and it has been used (under a different guise) to draw consequences on complete actions of nilpotent and semisimple Lie algebras on compact manifolds [1].

Here we want to point out yet another global results for nilpotent actions:

**Corollary 1.** Let \( \rho: \mathfrak{g} \to C^\infty(M) \) be a complete weakly Hamiltonian effective representation of a nilpotent non-abelian Lie algebra on a (non-compact) symplectic manifold. Let \( v \) be a vector not in the center of \( \mathfrak{g} \). Then the any periodic of \( X_v \) must be contained in the level set of a non-constant function (in general different from \( H_v \)).

**Proof.** By our assumptions we can find \( u \in \mathfrak{g} \) such that according to (4) for all \( x \in M \), \( \zeta_{u,v,x}(s) \) is a polynomial with linear coefficient \( H_{[v,u]}(x) - c_{u,v} \), which is a non-constant function on \( x \). Hence if we have a periodic orbit of \( X_v \), by compactness \( \zeta_{u,v,x}(s) \) must be constant and therefore must be in the zero subset of \( H_{[v,u]}(x) - c_{u,v} \) (and also in the zero set of functions corresponding to the coefficients of higher order in (4)).

As for an analog of the splitting theorem for abelian complete weakly Hamiltonian actions, the situation in the nilpotent case is much more complicated. One may take a basis of \( \mathfrak{g} \) and use the corresponding Hamiltonians to arrange a function to
Euclidean space, and one can obtain the following information: for a given point \( x \) and \( u, v \in \mathfrak{g} \):

- either \( H_{\text{adj}v(u)}(x) - c_{\text{adj}^{-1}v(u),v} = 0 \) for all \( j \), in which case the orbit of \( X_v \) through \( x \) will either not intersect \( H^{-1}_u(a) \), \( a \in \mathbb{R} \), or be contained in it;
- or some \( H_{\text{adj}v(u)}(x) - c_{\text{adj}^{-1}v(u),v} \neq 0 \), in which case for all \( a \in \mathbb{R} \) the orbit of \( X_v \) through \( x \) intersects \( H^{-1}_u(a) \) a finite number of times.

References

[1] T. Delzant, *Sous-algèbres de dimension finie de l’algèbre des champs Hamiltoniens*. Preprint available on the author’s webpage.

[2] P. Frejlich and I. Marcut, *The Normal Form Theorem around Poisson Transversals*, arXiv:1306.6055.

[3] V. Guillemin, E. Miranda, and A. Pires, *Codimension one symplectic foliations and regular Poisson structures*. Bulletin of the Brazilian Mathematical Society, 42(4):607–625, 2011.

[4] V. Humilière, *Hamiltonian pseudo-representations*. Comment. Math. Helv. 84 (2009), no. 3, 571–585.

[5] F. Kutzschebauch and F. Loose, *Real analytic structures on a symplectic manifold*. Proc. Amer. Math. Soc. 128 (2000), no. 10, 3009–3016.

[6] C. Laurent-Gengoux, E. Miranda, and P. Vanhaecke. *Action-angle coordinates for integrable systems on Poisson manifolds*. Int. Math. Res. Not. IMRN 2011, no. 8, 1839–1869.

[7] E. Miranda and N. T. Zung, *A note on equivariant normal forms of Poisson structures*. Math. Res. Lett. 13 (2006), no. 5-6, 1001–1012.

[8] J. Patera, R. T. Sharp, P. Winternitz and H. Zassenhaus, *Invariants of real low dimension Lie algebras*. J. Mathematical Phys. 17 (1976), no. 6, 986–994.

[9] A. Ooms, *The Poisson center and polynomial, maximal Poisson commutative subalgebras, especially for nilpotent Lie algebras of dimension at most seven*. J. Algebra 365 (2012), 83–113.

[10] A. Ooms, *Computing invariants and semi-invariants by means of Frobenius Lie algebras*. J. Algebra 321 (2009), no. 4, 1293–1312.

[11] J. Roels and A. Weinstein, *Functions whose Poisson brackets are constants*. J. Mathematical Phys. 12 1971 1482–1486.

[12] J.-M., Souriau, *Structure des systèmes dynamiques*, Dunod, Paris 1969.

[13] Y. Suris, *Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms*. J. Geom. Mech. 5 (2013), no. 3, 365–379.

[14] A. Weinstein, *The local structure of Poisson manifolds*. J. Differential Geom. 18 (1983), no. 3, 523–557.