Phenomenological Theory for Phase Turbulence in Rayleigh-Bénard Convection

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We present a phenomenological theory for phase turbulence (PT) in Rayleigh-Bénard convection, based on the generalized Swift-Hohenberg model. We apply a Hartree-Fock approximation to PT and conjecture a scaling form for the structure factor $S(k)$ with respect to the correlation length $\xi_2$. We hence obtain analytical results for the time-averaged convective current $J$ and the time-averaged mean square vorticity $\Omega$. We also define power-law behaviors such as $J \sim \epsilon^\mu$, $\Omega \sim \epsilon^\lambda$ and $\xi_2 \sim \epsilon^{-\delta}$, where $\epsilon$ is the control parameter. We find from our theory that $\mu = 1$, $\nu \geq 1/2$ and $\lambda = 2\mu + \nu$. These predictions, together with the scaling conjecture for $S(k)$, are confirmed by our numerical results.

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I. INTRODUCTION

Rayleigh-Bénard convection (RBC) has long been a paradigm in the study of pattern formation. This system consists of a thin horizontal layer of fluid heated from below. There are three dimensionless parameters to describe the system: $R = \frac{\alpha g d^3 \Delta T}{\kappa \nu}$ is the control parameter, in which $g$ is the gravitational acceleration, $d$ the layer thickness, $\Delta T$ the temperature gradient across the layer, $\alpha$ the thermal expansion coefficient, $\kappa$ the thermal diffusivity and $\nu$ the kinematic viscosity. Under the Boussinesq approximation, only the density of the fluid is temperature dependent. Then the Prandtl number $\sigma \equiv \nu/\kappa$ is all one needs to specify the fluid properties.

The third parameter is the aspect ratio $\Gamma \equiv L/d$ where $L$ is the horizontal size of the system. When the Rayleigh number $R$ surpasses a critical value $R_c$, the fluid bifurcates from a static conductive state to a convective state, in which the velocity profile $\bar{u} = (\bar{u}_x, \bar{u}_z)$ and the temperature-deviation profile $\theta$ form certain self-organized patterns. These patterns also depend on the boundary conditions (b.c.) at the horizontal surfaces of the container. The two most studied b.c. in the literature are the rigid-rigid b.c., under which the fluid cannot slip, and the free-free b.c., under which the fluid does not experience any stress.

The patterns and the corresponding stability domain in RBC have been studied extensively in the classical work of Busse and his coauthors in the $(R, \sigma, k)$ space with $k$ the wavenumber. For free-free boundaries, Zippelius and Sigga and Busse and Bolton found that the parallel roll state is unstable against the skewed-varicose instability immediately above onset if $\sigma < 0.543$. Busse et al. further investigated the dynamics involved and conjectured a direct transition from conduction to spatiotemporal chaos (STC). This spatiotemporally chaotic state is called phase turbulence (PT). Recently we reported a large scale ($\Gamma = 60$) numerical simulation of the three dimensional hydrodynamical equations for $\sigma = 0.5$ under the free-free b.c. From that simulation, we confirmed the direct transition to PT above onset and studied various properties of it. The patterns we found have very complicated spatial and temporal dependences.

In this paper we develop a theory for PT based on the generalized Swift-Hohenberg (GSH) model of RBC. This model is derived from the three-dimensional hydrodynamic equations, but is much simpler to study both numerically and analytically and is widely used in theoretical studies. Numerical solutions of this model or its modified versions have not only reproduced most patterns observed in experiments but also resembled experimental results relatively well. But there are some shortcomings in the model. The stability boundary of the model does not coincide with that of hydrodynamics for both rigid-rigid and free-free boundaries; it induces an unphysical, short-ranged cross roll instability. Even so, owing to its simplicity and its qualitative resemblance to real systems, this model is very valuable in studying RBC.

There are two coupled equations in two-dimensional space $\tilde{r} = (x, y)$ in the GSH model for two order parameters $\psi(\tilde{r}, t)$ and $\omega_z(\tilde{r}, t)$, which are, respectively, related to the convective current and the vertical vorticity. In this paper we present our analytical calculations of the time-averaged convective current $J = A^{-1} \int d\tilde{r} \psi^2(\tilde{r}, t)$ and the

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time-averaged mean square vorticity \( \Omega = A^{-1} \int d\vec{r} \omega^2(\vec{r}, t) \) in PT, where \( \overline{F(t)} \) represents the time-average of \( F(t) \) and \( A \) is the area of the system. We carry out our calculations in Fourier space so the total number of modes \( \psi(\vec{k}, t) \) considered is infinite. We apply a Hartree-Fock approximation (HFA) to PT in which four-point correlation functions are approximated by products of two-point correlation functions. We also assume the time-averaged two-point correlation function \( \overline{\psi(\vec{k}_1, t) \psi(\vec{k}_2, t)} \) obeys a Gaussian distribution, i.e., \( \overline{\psi(\vec{k}_1) \psi(\vec{k}_2)} = J(k_1) \exp[-\frac{1}{2}(\vec{k}_1 + \vec{k}_2)^2 \xi^2_\psi], \) where \( \xi_\psi \) is a length determining the correlation between different modes. This approximation is based on the observation that the patterns in PT are disordered on long length scales and quasi-ordered on short length scales. This suggests that the correlation between different wavevectors \( \vec{k}_1 \) and \( \vec{k}_2 \) is small unless these vectors are close to each other. This approximation seems physically intuitive for PT, in which there is a random superposition of rolls of different orientations. Given the above assumptions, we derive \( J \) and \( \Omega \) in terms of the time-averaged and azimuthally averaged structure factor \( S(k) = \overline{\psi^*(\vec{k}, t) \psi(\vec{k}, t)} / J. \) We further assume that \( S(k) \) obeys a scaling form \( kS(k) = \xi_2 F(|k-k_{max}|) \xi_2 \), in which \( \xi_2 \) is the two-point correlation length, \( F(x) \) is the scaling function and \( k_{max} \) is the peak position of \( kS(k) \). Applying this assumption, we obtain explicit formulas for both \( J \) and \( \Omega \). More precisely, we find that \( J = J_0 \epsilon - J_\xi \xi_2^2 \), which depends on unknown but experimentally measurable parameters, where \( \epsilon = (R - R_c) / R_c \) is the reduced control parameter and \( J_0 \) and \( J_\xi \) are both known. We also find that \( \Omega = \omega J^2 / \xi_2 \) where \( \omega \) is related to the width of the scaling function \( F(x) \). Furthermore, by assuming power law behaviors such that \( J \sim \epsilon^\mu, \xi_2 \sim \epsilon^{-\nu} \) and \( \Omega \sim \epsilon^\lambda \), we predict from our theory that \( \mu = 1, \nu = 1/2 \) and \( \lambda = 2 \mu + \nu \) for PT. This prediction and the scaling assumption for \( S(k) \) have been verified by our numerical solutions.

Our paper is organized as follows. In Sec. [1] we introduce the GSH model in Fourier space and derive the basic formulas governing the time-averaged convective current \( J \) and the time-averaged mean square vorticity \( \Omega \) for any pattern in RBC. In Sec. [II] we introduce the HFA of our theory and use it to calculate explicitly \( J \) and \( \Omega \) for PT. The results are expressed in terms of the structure factor \( S(k) \). Sec. [III] includes two parts. We first postulate the scaling form of the structure factor \( S(k) \) and expand both \( J \) and \( \Omega \) in the leading order of \( 1 / \xi_2 \). We then define power law behaviors for \( J, \xi_2 \) and \( \Omega \) in PT and compare the results from our theory and our numerical work [1], which agree very well for both the exponents and the amplitudes. In the last section, we summarize our results and discuss some related issues.

II. BASIC FORMULAS

In the GSH model, the order parameter \( \psi(\vec{r}, t) \) satisfies [11][12][17]

\[
\tau_0 \left( \partial_t \psi + \vec{U} \cdot \vec{\nabla} \psi \right) = \left[ \epsilon - \left( \xi_0^2 / 4k_c^2 \right) \left( \nabla^2 + k_0^2 \right) \right] \psi - N[\psi].
\]

Here \( N[\psi] \) is the nonlinear term to be specified soon and \( \vec{U}(\vec{r}) \) is the mean flow velocity given by \( \vec{U}(\vec{r}) = \vec{\nabla} \zeta(\vec{r}, t) \times \vec{e}_z \), in which

\[
\left( \partial_t - \sigma \vec{\nabla}^2 \right) \vec{e}_z \zeta = g_m \epsilon_\zeta \cdot \left[ \vec{\nabla}(\vec{\nabla}^2 \psi) \times \vec{\nabla} \psi \right].
\]

In the GSH equations, the reduced Rayleigh number \( \epsilon \equiv (R / R_c) - 1 \) is the control parameter, in which \( R \) and \( R_c \) are the Rayleigh number and its critical value at onset. The Prandtl number \( \sigma \) parameterizes the fluid. While \( k_c \) is the critical wavenumber at onset, other parameters model the properties of the system. More precisely, one takes [11][18]

\( R_c = 27 \pi^4 / 4, k_c = \pi / \sqrt{2}, \tau_0 = 2(1 + \sigma^{-1}) / 3 \pi^2, \xi_0^2 = 8 / 3 \pi^2, \) and \( g_m = 6. \)

It is easier to analyze the GSH equations theoretically in Fourier space than in real space. By convention, we define the Fourier transformation and its inverse transformation of an arbitrary function \( F(\vec{r}) \) as \( \hat{F}(\vec{k}) = \frac{1}{A} \int d\vec{r} e^{-i\vec{k} \cdot \vec{r}} F(\vec{r}) \) and \( F(\vec{r}) = \sum \vec{k} \hat{F}(\vec{k}) e^{i\vec{k} \cdot \vec{r}}, \) where \( A \) is the area of the system. Note that \( \hat{F}^*(\vec{k}) = \hat{F}(\vec{-k}) \) for any real function \( F(\vec{r}). \)

It is easy to check that Eq. [1] can be rewritten in Fourier space as

\[
\tau_0 \hat{\partial}_t \hat{\psi}(\vec{k}) + \hat{\vec{V}}(\vec{k}) = r(\epsilon; k) \hat{\psi}(\vec{k}) - \hat{N}(\vec{k}),
\]

where \( \hat{V}(\vec{r}) = \tau_0 \hat{\vec{U}} \cdot \vec{\nabla} \psi \) and \n
\[
r(\epsilon; k) = \epsilon - \frac{\xi_0^2 (k^2 - k_c^2)^2}{4k_c^2}.
\]

The nonlinear \( \hat{N}(\vec{k}) \) term has been evaluated at onset [1],

\[2\]
where the coupling constant $g(\cos \alpha)$ is given in Ref. 11 with $\alpha$ the angle between $\vec{k}$ and $\vec{k}_2$. Rigorously speaking, the exact forms of Eqs. (1), (2) and (3) are derived near onset and deviations from them in real physical systems are possible for large enough $\epsilon$. But we disregard such complexity and take them as our model for further study.

One may take an adiabatic approximation ($\partial_t = 0$) in Eq. (3) by neglecting the first term on the left-hand side. This term is small in comparison with the other terms, which can be verified by applying the same perturbation as that in phase dynamics [19]. With this approximation, it now is easy to solve Eq. (2) for $\hat{\zeta}(\vec{r})$. We are also interested in the vertical vorticity $\omega_z(\vec{r}) = -\nabla^2 \zeta(\vec{r})$. From Eq. (2), it is straightforward to get that

$$\hat{\omega}_z(\vec{k}) = k^2 \hat{\zeta}(\vec{k}) = \sum_{\vec{k}_2} f(\vec{k}; \vec{k}_2) \hat{\psi}(\vec{k}_2) \hat{\psi}(\vec{k} - \vec{k}_2),$$

where, with an exchange of index $\vec{k}_2 \rightarrow \vec{k} - \vec{k}_2$,

$$f(\vec{k}; \vec{k}_2) = \frac{g_m}{2\sigma k^2} (k^2 - 2\vec{k} \cdot \vec{k}_2)(\vec{e}_z \cdot \vec{k}_2 \times \vec{k}).$$

Applying these results, one may easily evaluate the mean-flow contribution to Eq. (3), which is given by

$$\hat{V}(\vec{k}) = \sum_{\vec{k}_2, \vec{k}_3} v(\vec{k}; \vec{k}_2; \vec{k}_3) \hat{\psi}^*(\vec{k}_2) \hat{\psi}(\vec{k}_3) \hat{\psi}(\vec{k} + \vec{k}_2 - \vec{k}_3),$$

where

$$v(\vec{k}; \vec{k}_2; \vec{k}_3) = \frac{g_m \tau_0 |\vec{e}_z \cdot \vec{k} \times (\vec{k}_3 - \vec{k}_2)| |\vec{e}_z \cdot \vec{k}_2 \times \vec{k}_3 - k^2_3 - k^2_2|}{|\vec{k}_3 - \vec{k}_2|^4}.$$

Notice that the coupling constant $v(\vec{k}; \vec{k}_2; \vec{k}_3)$ is zero under two conditions: (1) If all $\vec{k}$ allowed in $\hat{\psi}(\vec{k})$ point at one single direction, say $\hat{k}$; or, (2) if all $\vec{k}$ lie on one single ring, say $|\vec{k}| = k$. For this reason, ordered states such as parallel rolls, hexagons, concentric rings, etc., do not have significant mean-flow couplings. Furthermore, the coupling constant $v(\vec{k}; \vec{k}_2; \vec{k}_3)$ seems to have a pole at $\vec{k}_2 = \vec{k}_3$. Assume that $\vec{k}_3 = \vec{k}_2 + \vec{q}$ with $\vec{q}$ very small; then $v \sim (\vec{e}_z \cdot \vec{k} \times \vec{q})(\vec{e}_z \cdot \vec{k}_2 \times \vec{q})/(\vec{k}_2 \cdot \vec{q})/q^4$. A pole normally exists (since $v \sim 1/q$) unless $\vec{q}||\vec{k}_2$, or $\vec{q}||\vec{k}_2$, or $\vec{q} \perp \vec{k}_2$.

In this paper, we will mainly focus on two global quantities: One is the total convective current defined by

$$J(t) = \frac{1}{A} \int d\vec{r} \hat{\psi}^2(\vec{r}, t) = \sum_{\vec{k}} \hat{J}(\vec{k}, t) \quad \text{with} \quad \hat{J}(\vec{k}, t) = \hat{\psi}^*(\vec{k}, t)\hat{\psi}(\vec{k}, t);$$

the other is the mean square vorticity defined by

$$\Omega(t) = \frac{1}{A} \int d\vec{r} \omega^2(\vec{r}, t) = \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4} f(\vec{k}_1 + \vec{k}_2; \vec{k}_3)f(\vec{k}_1 + \vec{k}_2; \vec{k}_3)
\times \hat{\psi}^*(\vec{k}_1)\hat{\psi}^*(\vec{k}_2)\hat{\psi}(\vec{k}_3)\hat{\psi}(\vec{k}_4)\delta_{\vec{k}_1 + \vec{k}_2, \vec{k}_3 + \vec{k}_4}.$$

Notice that $f(\vec{k}_1 + \vec{k}_2; \vec{k}_3) \sim (k_1^2 - k_2^2)(\vec{e}_z \cdot \vec{k}_2 \times \vec{k}_1)$. So $\Omega(t) = 0$ if all the wavenumbers allowed in $\hat{\psi}(\vec{k})$ point at one single direction $\pm \vec{k}$ or lie on one single ring $|\vec{k}| = k$. In other words, the vorticity must be generated by couplings between modes of different $k$ and $\vec{k}$. From Eq. (3), it is easy to derive that

$$\tau_0 \partial_t \hat{J}(\vec{k}, t) = 2r(c; k)\hat{J}(\vec{k}, t) - \sum_{\vec{k}_2, \vec{k}_3} \left[ g(\vec{k} \cdot \vec{k}_2) + v(\vec{k}; \vec{k}_2; \vec{k}_3) \right]
\times \left[ \hat{\psi}^*(\vec{k})\hat{\psi}^*(\vec{k}_2)\hat{\psi}(\vec{k}_3)\hat{\psi}(\vec{k} + \vec{k}_2 - \vec{k}_3) + c.c. \right].$$

In principle, this is the equation determining the structure of the convective current $\hat{J}(\vec{k}, t)$ which, however, is beyond our present goal. Now applying the relations $\hat{\psi}^*(\vec{k}) = \hat{\psi}(\vec{k})$ and $v(\vec{k}; \vec{k}_2; \vec{k}_3) = -v(\vec{k}; \vec{k}_3; \vec{k}_2)$, and, exchanging the
summation indices \( \vec{k} \to -\vec{k}, \vec{k}_{2,3} \to -\vec{k}_{2,3} \) for the \( g \) terms and \( \vec{k} \to \vec{k} + \vec{k}_2 - \vec{k}_3, \vec{k}_2 \leftrightarrow \vec{k}_3 \) for the \( v \) terms, one obtains from the above equation and the definition of \( J(t) \) that

\[
\frac{1}{2} \tau_0 \partial_t J(t) = \sum_{\vec{k}} r(c; k) \dot{J}(\vec{k}, t) - \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} g(\vec{k}_1 \cdot \vec{k}_2) \tilde{\psi}^*(\vec{k}_1) \tilde{\psi}^*(\vec{k}_2) \tilde{\psi}(\vec{k}_3) \tilde{\psi}(\vec{k}_4) \delta_{\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4}. \tag{13}
\]

This is the equation determining the total convective current \( J(t) \). Notice that the \( v \) terms vanish from this equation, which can also be derived directly from Eq. (1) by converting the corresponding integral in Eq. (13) into a surface term. In general, the \( v \) couplings affect \( J(t) \) implicitly by modifying its structure \( \dot{J}(\vec{k}, t) \) unless, of course, \( v \equiv 0 \).

For stationary states, the convective current and the mean square vorticity are time-independent. This, however, is no long true if the state is spatiotemporal chaotic. For a spatiotemporal chaotic state, these two quantities normally fluctuate in time around some well-defined averaged values: see Refs. [8][9]. While the fluctuations appear chaotic in time, they are relatively small in comparison with their averaged values. For simplicity, we only consider the two corresponding time-averaged quantities in our theory. We now introduce the time-average operator \( \mathcal{T} \) defined by \( \mathcal{T} F(t) \equiv \overline{F(t)} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T dt F(t) \). Applying \( \mathcal{T} \) to Eq. (13) yields

\[
\sum_{\vec{k}} r(c; k) \overline{J(\vec{k}, t)} - \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4} \overline{g(\vec{k}_1 \cdot \vec{k}_2) \tilde{\psi}^*(\vec{k}_1) \tilde{\psi}^*(\vec{k}_2) \tilde{\psi}(\vec{k}_3) \tilde{\psi}(\vec{k}_4) \delta_{\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4}} = 0. \tag{14}
\]

In the next section, we show how, under a certain assumption, to calculate the time-averaged convective current of PT from this equation. The time-averaged mean square vorticity can be obtained with \( \mathcal{T} \) acting on Eq. (11). For simplicity, we denote from now on \( \overline{\dot{J}(\vec{k})} = \dot{J}(\vec{k}, t), J = \overline{J(t)} \) and \( \Omega = \overline{\Omega(t)} \).

Finally we introduce the time-averaged structure factor defined by \( \overline{S(\vec{k})} = \overline{\dot{J}(\vec{k})}/J \) with \( \sum_{\vec{k}} \overline{S(\vec{k})} = 1 \), and the corresponding averages \( \langle \overline{F} \rangle_{\vec{k}} = \sum_{\vec{k}} \overline{\dot{S}(\vec{k})} \overline{F(\vec{k})} \). With this notation, the first term in Eq. (14) can be rewritten as \( \langle r(c) \rangle \overline{J} \). If the \( k \)-dependence and the angular dependence in \( \overline{S(\vec{k})} \) can be separated, then it is more convenient to define \( \overline{\dot{S}(\vec{k})} = (2\pi)^2 A^{-1} S(k) \Phi(\alpha) \) with \( \int_0^{2\pi} dk k S(k) = 1 \) and \( \int_0^{2\pi} d\alpha \Phi(\alpha) = 1 \), where \( \alpha \) is the angle between \( \vec{k} \) and some reference direction. Here the discrete \( \vec{k} \) lattice has been converted into a continuous one. So a proper phase factor has been taken into account. Notice also that \( \Phi(\pi + \alpha) = \Phi(\alpha) \) since \( \overline{S(\vec{k})} = S(\vec{k}) \). Now the corresponding averages with respect to \( S(k) \) and \( \Phi(\alpha) \) are defined as \( \langle F \rangle_k = \int_0^{2\pi} dk k S(k) F(k) \) and \( \langle F \rangle_\alpha = \int_0^{2\pi} d\alpha \Phi(\alpha) F(\alpha) \).

### III. HARTREE-FOCK APPROXIMATION FOR PT

#### A. Convective Current

In order to calculate the time-averaged convective current and the time-averaged mean square vorticity from Eqs. (14) and (13), it is obvious that more information about the corresponding state is needed. For this purpose, we notice that the instantaneous patterns in PT are irregular and random in space [8]. So, if we write \( \tilde{\psi}(\vec{k}, t) = \hat{\rho}(\vec{k}, t) e^{i\hat{\phi}(\vec{k}, t)} \) with both \( \hat{\rho}(\vec{k}, t) \) and \( \hat{\phi}(\vec{k}, t) \) real, the phase \( \hat{\phi}(\vec{k}, t) \) seems to have a rather complicated, irregular time-dependence. This suggests that two modes of different \( \vec{k} \)'s are poorly time-correlated. Because of this, we think it is justified to adapt a Hartree-Fock Approximation (HFA) to PT in which a four-point correlation function is approximated by products of two-point correlation functions, i.e.,

\[
\tilde{C}_{\psi}^{(4)}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \equiv \tilde{\psi}(\vec{k}_1, t) \tilde{\psi}(\vec{k}_2, t) \tilde{\psi}(\vec{k}_3, t) \tilde{\psi}(\vec{k}_4, t)
\]

\[
\simeq \tilde{C}_{\psi}(\vec{k}_1, \vec{k}_2) \tilde{C}_{\psi}(\vec{k}_3, \vec{k}_4) + \tilde{C}_{\psi}(\vec{k}_1, \vec{k}_3) \tilde{C}_{\psi}(\vec{k}_2, \vec{k}_4) + \tilde{C}_{\psi}(\vec{k}_1, \vec{k}_4) \tilde{C}_{\psi}(\vec{k}_2, \vec{k}_3), \tag{15}
\]

where we denote

\[
\tilde{C}_{\psi}(\vec{k}_1, \vec{k}_2) \equiv \tilde{\psi}(\vec{k}_1, t) \tilde{\psi}(\vec{k}_2, t). \tag{16}
\]
Apparently $\hat{C}_\psi(\vec{k}, -\vec{k}) = \hat{J}(\vec{k})$.

Now it is essential to know the behavior of $\hat{C}_\psi(\vec{k}_1, \vec{k}_2)$ in PT. We notice that the patterns in PT are disordered at large scales but quasi-ordered inside some smaller domains. From these, we speculate that the correlation between two modes of different $\vec{k}$'s are poor if their $\vec{k}$'s are quite different but good if their $\vec{k}$'s are very close to each other. Hence, we postulate a Gaussian form for $\hat{C}_\psi(\vec{k}_1, \vec{k}_2)$ in PT, i.e.,

$$\hat{C}_\psi(\vec{k}_1, \vec{k}_2) = \hat{J}(\vec{k}_1) \exp[-\frac{1}{2}(\vec{k}_1 + \vec{k}_2)^2\xi_\psi^2],$$  \hspace{1cm} (17)

where $\xi_\psi$ is a length determining the correlation between different modes. We expect $\xi_\psi$ to be large.

We now calculate the convective current $J$ from Eqs. (15), (17) and (7). One may rewrite Eq. (14) in a continuous Fourier space as

$$\langle r(\epsilon, k) \rangle_k J = \frac{A^4}{(2\pi)^8} \int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \frac{(2\pi)^2}{A} \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) g(\vec{k}_1 \cdot \vec{k}_2) \times \hat{C}_\psi(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4).$$  \hspace{1cm} (18)

Inserting Eqs. (14) and (17) into this equation, one finds after some algebra that

$$J = \frac{4\pi \xi_\psi^2 \langle r(\epsilon, k) \rangle_k}{g_{PT}} A,$$  \hspace{1cm} (19)

in addition to the conduction solution $J = 0$. Here $\hat{S}(\vec{k})$ is assumed to be azimuthally uniform, and $g_{PT}$ is defined as

$$g_{PT} = g(-1) + \frac{2}{\pi} \int_0^\pi d\alpha g(\cos\alpha).$$  \hspace{1cm} (20)

Using the explicit formula given in Ref. [11] for free-free boundaries, one has that

$$g_{PT} = 0.855922 + 0.0458144\sigma^{-1} + 0.0709326\sigma^{-2},$$  \hspace{1cm} (21)

where $\sigma$ is the Prandtl number.

If one takes the limit $\xi_\psi \rightarrow +\infty$, the two-point correlation function $\hat{C}_\psi(\vec{k}_1, \vec{k}_2)$ in Eq. (17) reduces to $(A/2\pi \xi_\psi^2)\hat{J}(\vec{k}_1)\delta_{\vec{k}_1, -\vec{k}_2}$. This, by applying the inverse Fourier transformation, leads to the translation invariance of the two-point correlation function $C(\vec{r}_1, \vec{r}_2) \equiv \hat{\psi}(\vec{r}_1, t)\hat{\psi}(\vec{r}_2, t)/J$, i.e., $C(\vec{r}_1, \vec{r}_2) = C(\vec{r}_1 - \vec{r}_2)$. Under the same limit, one also has $\xi_\psi = [A/2\pi]^{1/2}$ since $\hat{C}_\psi(\vec{k}, -\vec{k}) = \hat{J}(\vec{k})$. In this case, one has

$$J = \frac{2\langle r(\epsilon, k) \rangle_k}{g_{PT}}.$$  \hspace{1cm} (22)

From now on, we simply choose $\xi_\psi = [A/2\pi]^{1/2}$.

**B. Mean Square Vorticity**

We now use the HFA to calculate the time-averaged mean square vorticity. From Eq. (11), one finds that

$$\Omega = \frac{A^4}{(2\pi)^8} \int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \frac{(2\pi)^2}{A} \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) f(\vec{k}_1 + \vec{k}_2; \vec{k}_3) f(\vec{k}_1 + \vec{k}_2; \vec{k}_3) \times \hat{C}_\psi^4(-\vec{k}_1, -\vec{k}_2, \vec{k}_3, \vec{k}_4),$$  \hspace{1cm} (23)

where, from Eq. (6),

$$f(\vec{k}_1 + \vec{k}_2; \vec{k}_3) f(\vec{k}_1 + \vec{k}_2; \vec{k}_3) \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) = \frac{g_m^2}{4\sigma^2} \frac{(k_1^2 - k_2^2)(k_3^2 - k_4^2)(\vec{e}_z \cdot \vec{k}_1)(\vec{e}_z \cdot \vec{k}_2)(\vec{e}_z \cdot \vec{k}_3)(\vec{e}_z \cdot \vec{k}_4)}{|\vec{k}_1 + \vec{k}_2|^2 |\vec{k}_3 + \vec{k}_4|^2} \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4).$$  \hspace{1cm} (24)
Notice that there is a singular point \( \vec{k}_1 + \vec{k}_2 = 0 \) in the above expression: its value depends on how the point is approached.

Following the same calculation as for \( J \) and assuming that \( \hat{S}(\vec{k}) \) is azimuthally uniform, one finds after some algebra that

\[
\Omega = \frac{g_0^2 J^2}{4 \sigma^2} \int dk_1 k_1 S(k_1) \int dk_2 k_2 S(k_2) \Delta(k_1; k_2),
\]

(25)

where we have chosen \( \xi_0 = [A/2\pi]^{1/2} \), and

\[
\Delta(k_1; k_2) = \frac{1}{4} |k_1^2 - k_2^2| [k_1^2 + k_2^2 - |k_1^2 - k_2^2|],
\]

(26)

which has a second-order singularity at \( k_1 = k_2 \) and is due to the singularity in Eq. (24).

IV. SCALING RELATIONS IN PT

A. General

To evaluate the convective current \( J \) and the mean square vorticity, \( \Omega \), one must know the structure factor \( S(k) \) which, however, is beyond our present theory. We thus turn to phenomenological arguments. We define a two-point correlation length as

\[
\xi_2 = \left[ \langle k^2 \rangle_k - \langle k \rangle_k^2 \right]^{-1/2}.
\]

(27)

Then we assume that the structure factor satisfies the following scaling form

\[
kS(k) = \xi_2 \mathcal{F}((k - k_{\text{max}})\xi_2),
\]

(28)

where \( k_{\text{max}} \) is the peak position of \( kS(k) \) and \( \mathcal{F}(x) \) is the scaling function satisfying \( \int_{-\infty}^{\infty} dx \mathcal{F}(x) = 1 \). [Since \( k \geq 0 \) in \( kS(k) \), the lower limit for \( \mathcal{F}(x) \) is \( -k_{\text{max}} \xi_2 \), which we approximate by \( -\infty \).] Inserting \( k = k_{\text{max}} + x\xi_2^{-1} \) and Eq. (23) into Eq. (27), one gets that \( \langle x^2 \rangle_x - \langle x \rangle_x^2 = 1 \), where we have used the notation \( \langle F(x) \rangle_x = \int_{-\infty}^{\infty} dx \mathcal{F}(x)F(x) \). It is also easy to see that \( \langle k \rangle_x = k_{\text{max}} + \xi_2^{-1}\langle x \rangle_x \).

For very large \( \xi_2 \), one may take \( k = k_{\text{max}} + x\xi_2^{-1} \) in Eqs. (22) and (23) and expand the results in order of \( 1/\xi_2 \). It is easy to find from Eqs. (4) and (22), and from \( k_{\text{max}} = k_c + O(\epsilon) \) that

\[
J = \frac{2}{g_{\text{PT}}} \left[ \epsilon - \langle x^2 \rangle_x \frac{\xi_2^2}{\xi_2^2} \right] + O \left( 1/\xi_2^2, \epsilon/\xi_2, \epsilon^2 \right),
\]

(29)

for small enough \( \epsilon \) and large enough \( \xi_2 \). This expression depends on two unknown but experimentally measurable quantities \( \xi_2 \) and \( \langle x^2 \rangle_x = 1 + \langle x \rangle_x^2 \). If \( \mathcal{F}(x) \) is symmetric, then \( \langle x \rangle_x = 0 \) and \( \langle x^2 \rangle_x = 1 \). In general, however, one has \( \langle x^2 \rangle_x \geq 1 \). Notice also that although mean-flow couplings are not explicitly present in Eq. (4), they affect the value of the convective current via the structure factor \( \hat{S}(\vec{k}) \) [see Eq. (22)] and the two-point correlation length \( \xi_2 \). For PT, since \( \xi_2 \approx \frac{\epsilon}{\sqrt{\epsilon}} \), the contribution from \( \xi_2 \) reduces the value of \( J \) quite significantly. This feature is absent in the simple theory presented earlier [4].

One may evaluate the mean square vorticity in the same way. From Eqs. (25) and (26), one gets for free-free boundaries that

\[
\Omega \approx \frac{g_0^2 J^2}{4 \sigma^2} \langle |x_1 - x_2| \rangle_{x_1,x_2} \frac{J^2}{\xi_2},
\]

(30)

where we have used \( k_{\text{max}} \approx k_c \). The quantity \( \langle |x_1 - x_2| \rangle_{x_1,x_2} \) is related to the width of the scaling function \( \mathcal{F}(x) \) and, since \( \langle |x_1 - x_2| \rangle_{x_1,x_2} \leq \sqrt{\langle (x_1 - x_2)^2 \rangle_{x_1,x_2}} = \sqrt{2} \), we expect \( \langle |x_1 - x_2| \rangle_{x_1,x_2} \) to be of order of unity. Clearly, by phenomenological arguments, we can express \( J \) and \( \Omega \) in PT in terms of measurable quantities.
B. Power Laws

For PT, we further assume power law behaviors for the two-point correlation length, the convective current and the mean square vorticity such as
\[ \xi_2 \approx \xi_{2,0} e^{-\nu}, \quad J \approx J_0 e^{\mu} \quad \text{and} \quad \Omega \approx \Omega_0 e^{\lambda}. \]  

(31)

Then, from Eq. (31), we find the following scaling relation
\[ \lambda = 2\mu + \nu. \]

(32)

Recalling Eq. (22), one obtains that
\[ J \approx \frac{2}{g_{PT}} \left[ e - \langle x^2 \rangle_x \frac{\xi_2^2}{\xi_{2,0}^2} e^{2\nu} \right] \approx J_0 e^{\mu}. \]

(33)

Since \( J \) is positive by definition, the values of the exponents satisfy
\[ \mu = 1, \quad \nu \geq 1/2 \quad \text{and} \quad \lambda = 2 + \nu \geq 5/2. \]

(34)

It is very likely that \( \nu = 1/2 \), hence, \( \lambda = 5/2 \). If so, then one finds from Eqs. (29) and (30) that
\[ J_0 = \frac{2}{g_{PT}} \left[ 1 - \langle x^2 \rangle_x \frac{\xi_0^2}{\xi_{2,0}^2} \right] \quad \text{and} \quad \Omega_0 = \frac{g_m k_c^3}{4 \sigma^2 \xi_{2,0}^2} \langle |x_1 - x_2| \rangle_{x_1,x_2}, \]

(35)

which depend on three phenomenological parameters \( \xi_{2,0}, \langle x^2 \rangle_x \) and \( \langle |x_1 - x_2| \rangle_{x_1,x_2} \). If \( \nu > 1/2 \), then \( J_0 = 2/g_{PT} \) since the \( e^{2\nu} \) term in Eq. (33) contributes only to the leading correction to scaling. It is interesting to notice that the amplitude equations coupled with mean-flow [12] predicts for free-free boundaries that \( \Omega \sim e^{5/2} \) for almost perfect parallel rolls.

We now verify our predictions for the power laws in PT by our numerical solutions. We have carried out large-scale numerical calculations of the three-dimensional Boussinesq equations under free-free boundaries for fluids of \( \sigma = 0.5 \). We have confirmed in Ref. [8] that the structure factor in PT satisfies the scaling form (28). From Table 1, one can see that our theoretical and our numerical results are in very good agreement for the exponents. The scaling relation Eq. (22) is confirmed within our numerical uncertainties. The comparison between the corresponding amplitudes, however, is only moderately successful. Calculations of \( \xi_{2,0}, \langle x^2 \rangle_x \) and \( \langle |x_1 - x_2| \rangle_{x_1,x_2} \) are obviously beyond the present theory, so we take our numerical result for \( \xi_{2,0} \). Since our numerical results for \( \langle x^2 \rangle_x \) and \( \langle |x_1 - x_2| \rangle_{x_1,x_2} \) are too sensitive to the large value cutoff to be meaningful, we assume equalities in \( \langle x^2 \rangle_x = 1 + \langle x \rangle_x^2 \geq 1 \) and \( \langle |x_1 - x_2| \rangle_{x_1,x_2} \leq 4 \langle (x_1 - x_2)^2 \rangle_{x_1,x_2} = \sqrt{2} \). From Eqs. (33) and (21), one gets \( J_0 \approx 0.972 \), which is about 20% larger than the numerical value. A non-zero value of \( \langle x \rangle_x \) will apparently reduce the theoretical value of \( J_0 \) in the right direction. On the other hand, one finds that \( \Omega_0 \approx 454.7 \langle |x_1 - x_2| \rangle_{x_1,x_2} = 643.0 \) as an upper bound, which is about ten times larger than our numerical result. Nevertheless, we note that while our theory is based on the two-dimensional GSH equations, our numerical calculations are done for the three-dimensional Boussinesq equations. Although the former is very good in reproducing qualitative features of RBC, it may not be quantitatively accurate in modeling RBC [10]. So one should be cautious in comparing the results from the GSH equations with those from real experiments or those from numerical calculations with hydrodynamical equations.

V. DISCUSSION

Our phenomenological theory for PT in RBC depends on two basic assumptions. In Sec. [14] we adapt a Hartree-Fock approximation to PT and postulate a Gaussian form for the time-averaged two-point correlation function. In Sec. [15], we further assume that the structure factor satisfies a scaling form such as \( kS(k) = \xi_2 F(k - k_{max}) \xi_2 \). In comparison with similar scaling forms in critical phenomena, critical dynamics and phase ordering, we find it necessary to replace \( k \) with \( k - k_{max} \) in the scaling form. The physical origin of this replacement is due to the fact that patterns in RBC have an intrinsic wavenumber, which is close to \( k_c \). By the same reason, we find it necessary to seek the scaling form of \( kS(k) \) instead of \( S(k) \), where the \( k \) factor comes from \( dk = k \, dk \, d\alpha \) in two-dimensional \( k \)-space. The physical origin of this scaling invariance in PT is yet unknown. In Sec. [15], we have confirmed the scaling form of
S(k) within our numerical accuracy. It is not clear currently whether the scaling form breaks down for small or large k.

In summary, we present a phenomenological theory for PT in RBC. We calculate analytically the time-averaged convective current J and the time-averaged mean square vorticity Ω as functions of ε and ξ2. We believe that our theoretical results will be useful in understanding the complicated behavior of STC in RBC. We also believe that our theory provides a new approach to STC and also raises some interesting questions. For example, how can one calculate the structure factor S(k) and the two-point correlation length ξ2 analytically? Is it possible that certain global quantities in STC form a complete set in the same way as temperature, pressure and density do for thermodynamic systems? Can we derive some effective variational principle in terms of global quantities? How far can we apply the ideas in critical phenomena to study STC? Since our assumptions are quite general, it will also be interesting to see whether our theory can be generalized to STC in other systems

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[17] Near the conduction to convection onset, the velocity field \(\vec{u}(\vec{r}, z, t) = (u_1, u_2)\) and the temperature-deviation field \(\theta(\vec{r}, z, t)\) in RBC can be approximated by \(\vec{u}_1(\vec{r}, z, t) \approx w_0(z) \nabla \psi(\vec{r}, t) + \zeta_0(z) \nabla \zeta(\vec{r}, t) \times \vec{e}_z\), \(u_2(\vec{r}, z, t) \approx w_0(z) \psi(\vec{r}, t)\), and \(\theta(\vec{r}, z, t) \approx \theta_0(z) \psi(\vec{r}, t)\), where \(\nabla\) is the gradient operator in two-dimensional space \(\vec{r} = (x, y)\) and \(\psi(\vec{r}, t)\) is the order parameter \[\mathbb{1}\] .

For free-free boundaries at \(z = 0, 1\), the explicit forms of \(w_0(z)\), \(\psi(z)\), and \(\theta(z)\) are given in Ref. \[\mathbb{1}\]; one takes \(\zeta_0(z) = 1\). Notice that the vertical vorticity \(\zeta_0(z) \omega_z(\vec{r}, t)\) is now replaced by \(\zeta_0(z) \omega_z(\vec{r}, t)\) in which \(\omega_z(\vec{r}, t) = -\nabla^2 \zeta(\vec{r}, t)\), where \(\zeta(\vec{r}, t)\) is the mean flow field. Inserting these approximations into the three-dimensional hydrodynamical equations in RBC and applying a few more approximations, one ends up with the two-dimensional generalized Swift-Hohenberg (GSH) model of RBC \[\mathbb{1}\].

[18] One may take \(g_n = -R_l \zeta_0(z) w_0(z) \partial^2 w_0(z) / \partial z^2 \), \(k_4 w_0(z) \theta_0(z) \zeta_0^2(z) \), and \(c^2 = - \zeta_0(z) \partial^2 \zeta_0(z) / \partial z^2 \), where \([\cdot]\) means the average over the vertical direction.
TABLE I. Time-averaged convective current $J \approx J_0 \epsilon^\mu$, time-averaged mean square vorticity $\Omega \approx \Omega_0 \epsilon^\lambda$ and two-point correlation length $\xi_2 \approx \xi_{2,0} \epsilon^{-\nu}$ in PT with $\sigma = 0.5$. For theoretical result of $\nu$, we assume equality in Eq. (34). See also discussions in Sec. IV(B).

|     | $\mu$    | $\nu$    | $\lambda$ | $\xi_{2,0}$ | $J_0$    | $\Omega_0$ |
|-----|----------|----------|-----------|-------------|----------|------------|
| Numerics | 1.034 ± 0.025 | 0.472 ± 0.016 | 2.55 ± 0.10 | 0.82 ± 0.04 | 0.787 ± 0.019 | 70.1 ± 1.0 |
| Theory  | 1        | 1/2      | 5/2       | —           | 0.972    | 643.0      |