MÖBIUS RANDOMNESS LAW FOR FROBENIUS TRACES

MIN SHA AND IGOR E. SHPARLINSKI

Abstract. Recently E. Bombieri and N. M. Katz (2010) have demonstrated that several well-known results about the distribution of values of linear recurrence sequences lead to interesting statements for Frobenius traces of algebraic curves. Here we continue this line of study and establish the Möbius randomness law quantitatively for the normalised form of Frobenius traces.

1. Introduction

1.1. Background on Frobenius traces. Throughout the paper, $C$ denotes a smooth projective curve over a finite field $\mathbb{F}_q$ of $q$ elements. Following Bombieri and Katz [4], we consider the sequence $A(n)$ of Frobenius traces defined by

$$\#C(\mathbb{F}_{q^n}) = q^n + 1 - A(n)$$

where $C(\mathbb{F}_{q^n})$ is the set of $\mathbb{F}_{q^n}$-rational points on $C$.

Let $g$ be the genus of $C$, and assume $g \geq 1$. Since by the Weil bound (see [16, Section VIII.5.9]), we have

$$|A(n)| \leq 2gq^{n/2},$$

it is convenient to normalise the sequence $A(n)$ as

$$a(n) = \frac{A(n)}{2gq^{n/2}},$$  \hspace{1cm} (1.1)

which is called the normalised Frobenius trace.

1.2. Some previous results. Here we recall some previous results on the distribution of the sequence $a(n)$ given by (1.1). First we recall that Bombieri and Katz [4], using an interpretation of $A(n)$ as a linear recurrence sequence of order $2g$, have showed that $|a(n)|$ is not too small. More precisely, by [4, Theorem 3.1], for any $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that either $a(n) = 0$ or $|a(n)| \geq c(\varepsilon)q^{-n\varepsilon}$.

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However, unless $g = 1$, that is, if $C$ is an elliptic curve, this bound is not effective and the constant $c(\varepsilon)$ is not effectively computable. Using the argument of [6, Theorem 2.6], one can give a stronger and fully effective bound of the form

$$|a(n)| \geq n^{-\psi(n)},$$

which for any function $\psi(z) \to \infty$ as $z \to \infty$, holds for almost all $n$ in the sense of asymptotic density.

The asymptotic distribution of the values $a(n)$ in the interval $[-1,1]$ has been studied in [1], where it is shown that this distribution differs from the usually expected Sato–Tate law [12, 13]. On the other hand, an asymptotic formula for the average $\ell$-adic order of $A(n)$ for a prime $\ell \nmid q$ has been given in [20, Theorem 4].

Several related results about the distribution of Kloosterman and Birch sums have recently been given by Perret-Gentil [18].

We also note that using the upper bound of van der Poorten and Schlickewei [19, Theorem 1], on the number of zeros of linear recurrence sequences, one can estimate the number of zero values $a(n) = 0$ in a better way which is outlined in [4, Section 5] (via uniform bounds on the number of zeros of linear recurrence sequences such as in [2, 7]).

The second part of our motivations comes from the so-called Möbius randomness law (see, for example, [11, Section 13.1], and also Sarnak’s conjecture [22]) which roughly asserts that for any bounded sequence $s(n)$ of complex numbers, defined in terms which are not directly related to $\mu(n)$, we have

$$\sum_{n=1}^{N} \mu(n)s(n) = o(N), \quad \text{as } N \to \infty.$$  

Here, we establish quantitatively the Möbius randomness law for the sequence $a(n)$ defined in (1.1).

1.3. Our results. We recall that the Möbius function is defined as $\mu(n) = 0$ if an integer $n$ is divisible by a prime square and $\mu(n) = (-1)^r$ if $n$ is a product of $r$ distinct primes.

**Theorem 1.1.** For any $B > 0$, for every integer $N \geq 2$ we have

$$\left| \sum_{n=1}^{N} \mu(n)a(n) \right| \leq c(B)N(\log N)^{-B},$$

where $c(B) > 0$ is a constant depending only on $B$.

When $C$ is an ordinary curve, we can get a better result. We recall that $C$ is called ordinary if and only if the number of $p$-torsion points
on the Jacobian of \( C \) is exactly \( p^g \), where \( p \) is the characteristic of \( \mathbb{F}_q \) (see [10, Definition 3.1] for several equivalent definitions).

Recall that the assertion \( U \ll V \) is equivalent to the inequality \( |U| \leq cV \) with some absolute constant \( c > 0 \).

**Theorem 1.2.** If \( C \) is an ordinary curve of genus \( g \geq 1 \), for any integer \( N \geq 3 \) we have

\[
\sum_{n=1}^{N} \mu(n)a(n) \ll N^{1-1/(4\kappa(q,g)+4)}(\log N)^4,
\]

where

\[
\kappa(q,g) = 2^{31}3^3\pi^3(\pi + \log q) \log(16g).
\]

We prove the above results by interpreting the normalised Frobenius traces as linear recurrence sequences via the zeta function (see (3.4)).

We remark that full analogues of our results also hold for similar sums with the sequences of Kloosterman sums

\[
K(n) = \frac{1}{q^n} \sum_{x \in \mathbb{F}_q^n} \psi \left( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q}(ax + x^{-1}) \right), \quad n = 1, 2, \ldots,
\]

where \( \text{Tr}_{\mathbb{F}_q^n/\mathbb{F}_q} \) is the trace function from \( \mathbb{F}_q^n \) to \( \mathbb{F}_q \), \( \psi \) is a fixed additive character of \( \mathbb{F}_q \) and \( a \in \mathbb{F}_q^* \) is a fixed element. This setting is dual to that of [8, 14] where Kloosterman sums modulo a large fixed prime \( p \) (and more general trace functions) are ordered by the coefficient \( n \) in the exponent \( nx + x^{-1} \).

2. **Diophantine properties of arguments of algebraic numbers and exponential sums with Möbius function**

2.1. **Linear form in the logarithms of algebraic numbers.** The main tool in this paper is Baker’s theory of linear forms in the logarithms of algebraic numbers. Here we restate one of its explicit forms due to Baker and Wüstholz [3].

First, recall that for a non-zero complex number \( z \), the principal value of the natural logarithm of \( z \) is

\[
\log z = \log |z| + i \cdot \text{Arg}(z),
\]

where as usual, \( i \) is the imaginary unit, and \( \text{Arg}(z) \) is the principal value of the arguments of \( z \) (0 \( \leq \text{Arg}(z) < 2\pi \)).

Let

\[
\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + \cdots + b_n \log \alpha_n,
\]
where \( n \geq 2, b_1, \ldots, b_n \in \mathbb{Z}, \) and \( \alpha_1, \ldots, \alpha_n \) are non-zero elements of a number field \( K. \) Let \( d = [K : \mathbb{Q}] \) and \( B = \max\{|b_1|, \ldots, |b_n|\}. \) For any \( 1 \leq j \leq n, \) choose a real number \( A_j \) such that

\[
A_j \geq \max\{h(\alpha_j), |\log \alpha_j|/d, 1/d\},
\]

where \( h \) stands for the logarithmic absolute Weil height (note that the height function used in [3] is different from ours).

Suppose that \( \Lambda \neq 0. \) Then, we have

\[
(2.1) \quad \log |\Lambda| > -C(n,d)A_1 \cdots A_n \cdot \max\{\log B, 1/d\},
\]

where

\[
C(n,d) = 18(n+1)!n^{n+1}(32d)^{n+2}\log(2nd).
\]

We remark that we in fact only need a lower bound on linear forms in two logarithms which are also treated in [15, Théorème 3]. However, the lower bound in [15, Théorème 3] essentially has the term \((\log B)^2\). This is not sufficient for our purpose. Alternatively, using [9, Theorem 2.1] one may obtain such a lower bound having the term \(\log B\).

In addition, there is another explicit form of Baker’s theory due to Matveev [17, Corollary 2.3]. However, for our purpose the one of Baker and Wüstholz [3] gives a slightly better result.

2.2. Lower bounds for Diophantine approximations. The famous theorem of Roth [21] states that given an irrational algebraic number \( \alpha \) and \( \varepsilon > 0, \) there exists a constant \( c(\alpha, \varepsilon) > 0 \) such that for any integers \( r, s \) with \( s > 0 \) we have

\[
(2.2) \quad \left| \alpha - \frac{r}{s} \right| > \frac{c(\alpha, \varepsilon)}{s^{2+\varepsilon}}.
\]

Note that the above lower bound is for irrational algebraic numbers. It is natural to pursue a lower bound for real transcendental numbers. However, so far there is no lower bound for all real transcendental numbers. For any positive algebraic number \( \alpha \neq 1, \) using Baker’s theory one can get such a lower bound for \( \log \alpha. \)

Here, we want to obtain such a lower bound for another kind of real transcendental numbers by using Baker’s theory.

Now, let \( \alpha \) be an irrational number. Define

\[
e(\alpha) = e^{2\pi i\alpha},
\]

where as usual \( e \) is the base of the natural logarithm. If \( e(\alpha) \) is an algebraic number, then by the Gelfond–Schneider theorem we know that \( \alpha \) must be a transcendental number. Indeed, assume that \( \alpha \) is an algebraic number, then by the Gelfond-Schneider theorem \( 1 = e(\alpha)^{1/\alpha} \) is a transcendental number, which is impossible.
The following result gives such a lower bound for any irrational number $\alpha$ when $e(\alpha)$ is an algebraic number. This can be viewed as a Diophantine property of the arguments of algebraic numbers. In fact, $1 + \kappa(\alpha)$ is an upper bound of the irrationality exponent of $\alpha$.

**Lemma 2.1.** Let $\alpha$ be an irrational number. Assume that $e(\alpha)$ is an algebraic number. Then, for any integers $r, s$ with $s \geq 1$, we have
\[
|\alpha - \frac{r}{s}| > \frac{1}{\pi(2s)^{1+\kappa(\alpha)}},
\]
where
\[
\kappa(\alpha) = 2^{25}3^3 \pi d^3 \log(4d) A_1,
\]
\[
d = [\mathbb{Q}(e(\alpha)) : \mathbb{Q}],
\]
\[
A_1 = \max\{h(e(\alpha)), 2\pi \alpha/d, 1/d\}.
\]

**Proof.** We can always replace $\alpha$ with its fractional part $\{\alpha\}$. Hence, without loss of generality, we assume that $0 < \alpha < 1$ and $\gcd(r, s) = 1$. Then, if $|r| > s$, we have
\[
|\alpha - \frac{r}{s}| > \frac{1}{s},
\]
which is better than the desired result. In the sequel, we assume $|r| \leq s$.

Denote
\[
\Delta = \alpha - \frac{r}{s}.
\]
Then, since $0 < \alpha < 1$, we have
\[
\log e(\alpha) = 2\pi i\alpha = 2\pi i(\Delta + \frac{r}{s}),
\]
and so
\[
2s\pi i \Delta = s \log e(\alpha) - 2r\pi i = s \log e(\alpha) - 2r \log(-1).
\]
Denote
\[
\Lambda = s \log e(\alpha) - 2r \log(-1).
\]
So,
\[
(2.3) \quad |\Delta| = \frac{|\Lambda|}{2\pi s}.
\]
Recall that $\alpha$ is an irrational number. So, $e(\alpha)$ is not a root of unity, and thus $\Lambda \neq 0$. Using (2.1) with $n = 2$, we obtain
\[
(2.4) \quad \log |\Lambda| > -C(d) A_1 A_2 \cdot \max\{\log B, 1/d\},
\]
where
\[ C(d) = 2^{25}3^3d^4 \log(4d), \]
\[ d = [\mathbb{Q}(e(\alpha)) : \mathbb{Q}], \]
\[ A_1 = \max\{h(e(\alpha)), 2\pi \alpha/d, 1/d\}, \quad A_2 = \pi/d, \]
\[ B = \max\{s, 2|r|\}. \]

Since \(|r| \leq s\), we have \(B \leq 2s\). In view of \(s \geq 1\) and \(d \geq 2\), we get
\[
\max\{\log B, 1/d\} \leq \log(2s).
\]

Hence, the inequality (2.4) becomes
\[
\log |\Lambda| > -2^{25}3^3\pi d^3 \log(4d) A_1 \log(2s) = -\kappa(\alpha) \log(2s),
\]
which, together with (2.3), implies the desired result. \(\blacksquare\)

We remark that Lemma 2.1 is essentially a variant of [4, Theorem 4.1].

2.3. Dirichlet’s theorem. We first recall Dirichlet’s theorem in Diophantine approximation; see, for example [11, Equation (20.29)].

**Lemma 2.2.** Let \(\alpha\) be an irrational number. Then, for any integer \(N \geq 2\) there are two integers \(r, s\) such that
\[
0 < \left| \alpha - \frac{r}{s} \right| \leq \frac{1}{sN}, \quad 1 \leq s \leq N, \quad \gcd(r, s) = 1.
\]

In Lemma 2.2, if \(N\) tends to infinity, then \(s\) also goes to infinity. It is natural to ask how large \(s\) can be. If \(\alpha\) is an irrational algebraic number, combining Lemma 2.2 with the bound (2.2) we have
\[
(2.5) \quad s > (c(\alpha, \varepsilon)N)^{1/(1+\varepsilon)}.
\]

The next result follows directly from Lemmas 2.1 and 2.2.

**Lemma 2.3.** Let \(\alpha\) be an irrational number. Assume that \(e(\alpha)\) is an algebraic number. Then, for any integer \(N \geq 2\), there are integers \(r, s\) such that
\[
0 < \left| \alpha - \frac{r}{s} \right| \leq \frac{1}{sN}, \quad 1 \leq s \leq N, \quad \gcd(r, s) = 1,
\]
and
\[
s > \frac{1}{2} \left( \frac{N}{(2\pi)} \right)^{1/\kappa(\alpha)},
\]
where \(\kappa(\alpha)\) has been defined in Lemma 2.1.
2.4. **Bounds of exponential sums with Möbius function.** Recall the following bound of exponential sums with Möbius function, which depends on the Diophantine properties of the exponent $\alpha$; see [11, Theorem 13.9].

**Lemma 2.4.** Suppose that the real $\alpha$ satisfies
\[ |\alpha - \frac{r}{s}| \leq \frac{1}{s^2} \]
for some integers $r, s$ with $s > 0$ and $\gcd(r, s) = 1$. Then, for any integer $N \geq 2$, we have
\[ \sum_{n=1}^{N} \mu(n)e(n\alpha) \ll \left( s^{1/4}N^{1/4} + s^{-1/4}N^{1/2} + N^{2/5} \right) N^{1/2}(\log N)^4. \]

We remark that Davenport [5] has established the following general result: for any real number $\alpha$ and $N \geq 2$, we have
\[ \left| \sum_{n=1}^{N} \mu(n)e(n\alpha) \right| \leq c(B)N(\log N)^{-B} \]
for any $B > 0$, where $c(B) > 0$ is a constant depending only on $B$. The upper bound in (2.6) has a very attractive feature that it is independent of $\alpha$.

Involving the dependence on $\alpha$, the bound (2.6) can be improved for some special cases.

**Lemma 2.5.** Let $\alpha$ be an irrational number. Assume that $e(\alpha)$ is an algebraic number. Then, for any integer $N \geq 3$, we have
\[ \sum_{n=1}^{N} \mu(n)e(n\alpha) \ll N^{1-1/(4\kappa(\alpha)+4)}(\log N)^4, \]
where $\kappa(\alpha)$ is defined as in Lemma 2.1.

**Proof.** By Lemma 2.3, for any integer $M \geq 2$, there are integers $r, s$ such that
\[ 0 < \left| \alpha - \frac{r}{s} \right| \leq \frac{1}{sM}, \quad 1 \leq s \leq M, \quad \gcd(r, s) = 1, \]
and
\[ s > \frac{1}{2}(M/(2\pi))^{1/\kappa(\alpha)}, \]
where $\kappa(\alpha)$ has been defined in Lemma 2.1. Then, by Lemma 2.4, we have
\[ \sum_{n=1}^{N} \mu(n)e(n\alpha) \ll \left( s^{1/4}N^{1/4} + s^{-1/4}N^{1/2} + N^{2/5} \right) N^{1/2}(\log N)^4. \]
Note that
\[
\frac{1}{2} \left( \frac{M}{2\pi} \right)^{1/\kappa(\alpha)} < s \leq M,
\]
hence
\[
s^{1/4}N^{1/4} + s^{-1/4}N^{1/2} \ll M^{1/4}N^{1/4} + M^{-1/(4\kappa(\alpha))}N^{1/2},
\]
which with
\[
M = \left\lceil N^{\kappa(\alpha)/(\kappa(\alpha)+1)} \right\rceil \geq 2
\]
becomes
\[
s^{1/4}N^{1/4} + s^{-1/4}N^{1/2} \ll N^{1/2 - 1/(4\kappa(\alpha)+4)}.
\]
Substituting this into (2.7), we see that the term \(N^{9/10}\) never dominates, and we obtain the desired result. \(\square\)

From the above proof, one can see that if an irrational number \(\alpha\) has a Diophantine property as in Lemma 2.3, then the upper bound (2.6) can be improved similarly. For example, this can be done for irrational algebraic numbers by (2.5).

3. Diophantine properties of Frobenius eigenvalues

3.1. Frobenius eigenvalues and angles. We refer to [16] for a background on curves and their zeta-functions.

For a smooth projective curve \(C\) over the finite field \(\mathbb{F}_q\), we define the zeta-function of \(C\) as
\[
Z(T) = \exp \left( \sum_{n=1}^{\infty} \frac{\#C(\mathbb{F}_{q^n})T^n}{n} \right).
\]
It is well-known that if \(C\) is of genus \(g \geq 1\) then
\[
Z(T) = \frac{P(T)}{(1-T)(1-qT)},
\]
where
\[
P(T) = \prod_{j=1}^{2g} (1 - \beta_j T)
\]
is a polynomial of degree \(2g\) with integer coefficients, and \(\beta_1, \beta_2, \ldots, \beta_{2g}\) are algebraic integers, called the Frobenius eigenvalues, which satisfy
\[
|\beta_j| = q^{1/2}, \quad j = 1, 2, \ldots, 2g;
\]
see [16, Section VIII.5]. Then, for each \(\beta_j\), since all its conjugates have absolute value \(q^{1/2}\), we have (via the Mahler measure, see [23, Lemma 3.10])
\[
h(\beta_j) = \frac{1}{2} \log q, \quad j = 1, 2, \ldots, 2g.
\]
Furthermore, in view of (3.1) we write

\[ \beta_j = q^{1/2}e(\alpha_j), \]

with some \( \alpha_j \in [0,1) \), \( j = 1, 2, \ldots, 2g \). Usually, these \( 2\pi\alpha_j \) are called Frobenius angles. We then call \( \alpha_j \) normalised Frobenius angles. Now simple combinatorial arguments lead to the well-known identity

\[ \#C(\mathbb{F}_{q^n}) = q^n + 1 - \sum_{j=1}^{2g} \beta_j^n, \]

which implies

\[ a(n) = \frac{1}{2g} \sum_{j=1}^{2g} e(n\alpha_j). \]

This is crucial for our approach.

3.2. Diophantine properties of normalised Frobenius angles.

We recall the following irrationality property of normalised Frobenius angles given by [1, Lemma 8]

**Lemma 3.1.** Suppose that \( C \) is an ordinary smooth projective curve of genus \( g \geq 1 \) over \( \mathbb{F}_q \). Then all normalised Frobenius angles \( \alpha_j \), \( j = 1, 2, \ldots, 2g \), are irrational.

Now, we can use Lemma 2.1 to obtain a Diophantine property for the normalised Frobenius angles.

**Lemma 3.2.** Suppose that \( C \) is an ordinary smooth projective curve of genus \( g \geq 1 \) over \( \mathbb{F}_q \). Let \( \alpha \) be an arbitrary normalised Frobenius angle of \( C \). Then, for any integers \( r, s \) with \( s \geq 1 \)

\[ |\alpha - \frac{r}{s}| > \frac{1}{\pi(2s)^{1+\kappa(q,g)}}, \]

where

\[ \kappa(q,g) = 2^{31/3}q^3(\pi + \log q)\log(16g). \]

**Proof.** Let \( \beta \) be the Frobenius eigenvalue corresponding to \( \alpha \) as defined in (3.3). That is, \( \beta = q^{1/2}e(\alpha) \). Hence, \( e(\alpha) \) is an algebraic number. Besides, by Lemma 3.1, \( \alpha \) is an irrational number. Then, applying Lemma 2.1, we have that for any integers \( r, s \) with \( s \geq 2 \)

\[ |\alpha - \frac{r}{s}| > \frac{1}{\pi(2s)^{1+\kappa(\alpha)}}, \]
where
\[ \kappa(\alpha) = 2^{25}3^3 \pi d^3 \log(4d)A_1, \]
\[ d = [\mathbb{Q}(e(\alpha)) : \mathbb{Q}], \]
\[ A_1 = \max\{h(e(\alpha)), 2\pi \alpha/d, 1/d\}. \]

Since \( \deg \beta \leq 2g \), we have
\[ d = [\mathbb{Q}(e(\alpha)) : \mathbb{Q}] = [\mathbb{Q}(\beta q^{-1/2}) : \mathbb{Q}] \leq 2 \deg \beta \leq 4g. \]

Using (3.2), we obtain
\[ h(e(\alpha)) = h(\beta q^{-1/2}) \leq h(\beta) + h(q^{1/2}) = \log q. \]

Note that we must have \( d \geq 2 \). So, we have
\[ A_1 \leq \pi + \log q. \]

Hence, we get
\[ \kappa(\alpha) \leq 2^{31}3^3 \pi g^3 (\pi + \log q) \log(16g). \]

This completes the proof. \( \square \)

4. PROOFS OF THE MAIN RESULTS

4.1. Proof of Theorem 1.1. By (3.4), we have
\[ \left| \sum_{n=1}^{N} \mu(n)a(n) \right| \leq \frac{1}{2g} \sum_{j=1}^{2g} \left| \sum_{n=1}^{N} \mu(n)e(n\alpha_j) \right|. \]

Then, the desired result follows directly from the bound (2.6).

4.2. Proof of Theorem 1.2. Let \( \alpha_1, \alpha_2, \ldots, \alpha_{2g} \) be the normalised Frobenius angles of \( C \). For each \( \alpha_j \), using Lemmas 2.5 and 3.2, we have that for any integer \( N \geq 3 \), we have
\[ \sum_{n=1}^{N} \mu(n)e(n\alpha_j) \ll N^{1-1/(4\kappa(q,g) + 4)} (\log N)^4, \]

where \( \kappa(q,g) \) has been defined in Lemma 3.2.
So, combining (3.4) with (4.1), for any integer \( N \geq 3 \), we have

\[
\left| \sum_{n=1}^{N} \mu(n)a(n) \right| = \frac{1}{2g} \left| \sum_{n=1}^{N} \mu(n) \sum_{j=1}^{2g} e(n\alpha_j) \right|
\leq \frac{1}{2g} \sum_{j=1}^{2g} \left| \sum_{n=1}^{N} \mu(n)e(n\alpha_j) \right|
\ll N^{1-1/(4\kappa(q,g)+1)}(\log N)^4,
\]

which concludes the proof.

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