General relativistic magnetospheres of slowly rotating and oscillating magnetized neutron stars

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ABSTRACT

We study the magnetosphere of a slowly rotating magnetized neutron star subject to toroidal oscillations in the relativistic regime. Under the assumption of a zero inclination angle between the magnetic moment and the angular momentum of the star, we analyse the Goldreich–Julian charge density and derive a second-order differential equation for the electrostatic potential. The analytical solution of this equation in the polar cap region of the magnetosphere shows the modification induced by stellar toroidal oscillations on the accelerating electric field and on the charge density. We also find that, after decomposing the oscillation velocity in terms of spherical harmonics, the first few modes with $m = 0, 1$ are responsible for energy losses that are almost linearly dependent on the amplitude of the oscillation and that, for the mode $(l, m) = (2, 1)$, can be a factor $\sim 8$ larger than the rotational energy losses, even for a velocity oscillation amplitude at the star surface as small as $\eta = 0.05 \Omega R$. The results obtained in this paper clarify the extent to which stellar oscillations are reflected in the time variation of the physical properties at the surface of the rotating neutron star, mainly by showing the existence of a relation between $P \dot{P}$ and the oscillation amplitude. Finally, we propose a qualitative model for the explanation of the phenomenology of intermittent pulsars. The idea is that stellar oscillations, periodically excited by star glitches, can create relativistic winds of charged particles because of the additional electric field. When the stellar oscillations damp, the pulsar shifts below the death line in the $P$–$B$ diagram, thus entering the OFF invisible state of intermittent pulsars.

Key words: MHD – plasmas – relativistic processes – stars: neutron – stars: oscillations – pulsars: general.

1 INTRODUCTION

The theoretical study of radio pulsars dates back to the work of Goldreich & Julian (1969) who first suggested the existence of a magnetosphere with a charge-separated plasma around rotating magnetized neutron stars. A spinning magnetized neutron star generates huge potential differences between different parts of its surface. The cascade generation of electron–positron plasma in the polar cap region already proposed by Sturrock (1971) and Ruderman & Sutherland (1975) requires that the magnetosphere of a neutron star is filled with plasma, thus screening the longitudinal electric field and bringing the plasma into corotation with the neutron star. Because corotation is not possible outside the light cylinder (the radius $R_{LC} = c/\Omega$ at which the corotation speed equals the speed of light), essentially two different topologies of the magnetic field lines are naturally produced: closed lines, namely those returning to the stellar surface, and open lines, i.e. those crossing the light cylinder and going to infinity. As a result, plasma may leave the neutron star along the open field lines, and it is generally thought that pulsar radio emission is produced in the region of the open field lines well inside the light cylinder and within a given angle from the polar axis.

Besides the seminal papers by Goldreich & Julian (1969), Sturrock (1971), Ruderman & Sutherland (1975), Mestel (1971) and Arons & Scharlemann (1979), pulsar magnetospheres have been investigated by a large number of authors over the years. We only mention here the

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reviews by Arons (1991), Mestel (1992) and Muslimov & Harding (1997), where subsequent achievements and some new ideas have been presented. Thorough description of known magnetosphere properties may be found, for example, in the book of Beskin (2009). It should also be mentioned that in the last few years, time-dependent numerical simulations of neutron star magnetospheres have been proposed as a new promising tool for investigating the complex physics of these systems. At least qualitatively, the numerical approach has confirmed the most fundamental features of what was expected from the stationary solution of the Grad–Shafranov equation (Contopoulos, Kazanas & Fendt 1999; Gruzinov 2005), such as the existence of closed magnetic field lines up to the light cylinder (Komissarov 2006; McKinney 2006), or the scaling of the spin-down luminosity on the angular velocity and on the inclination angle of the neutron star angular momentum with respect to its magnetic moment (Spitkovsky 2006). In spite of this spectacular progress, however, the numerical approach still suffers from some serious limitations, such as the lack of a unified scheme in which both the force-free regime and the plasma regime of magnetohydrodynamics are simultaneously taken into account, or the lack of a consistent treatment of resistive effects in the current sheet.

The analytic approach, on the other hand, can still provide a deep understanding of pulsar physics. In particular, a lot of attention has been paid to the existence of a strong electric field induced by the rotation of the star, as already noticed by Deutsch (1955). More recently, Beskin (1990) and, independently, Muslimov & Tsygan (1990) were the first to find that the frame dragging induced by general relativistic effects provides a source of additional electric field contributing to particle acceleration in the polar cap region. The accelerating component (parallel to the magnetic field) of the electric field is driven by deviations of the space density charge from the Goldreich–Julian (GJ) charge density, which is determined by the magnetic field geometry. As noted by several authors (Beskin 1990; Muslimov & Tsygan 1990; Muslimov & Harding 1997; Moiz & Ahmedov 2000; Dyks, Rudak & Bulik 2001; Morozova, Ahmedov & Kagramanova 2008), the corrections of general relativity in the plasma magnetosphere of rotating neutron stars are the first-order in the angular velocity of the dragging of inertial frames and have to be carefully included in any self-consistent model of pulsar magnetosphere, especially when computing the resulting electromagnetic radiation.

Tightly related to this aspect is the possibility that neutron star oscillations, most likely excited during a glitch phenomenon (sudden change of the rotational period), propagate into the magnetosphere, thus affecting the acceleration properties in the polar cap region. The first attempt to generalize the GJ formalism to the case of an oscillating neutron star was made by Timokhin, Bisnovatyi-Kogan & Spruit (2000), who developed a general procedure for calculating the GJ charge density in the near zone of an oscillating neutron star. Using this procedure, the GJ charge density and the electromagnetic energy losses were computed for the case of toroidal oscillations at the neutron star surface. A similar approach has been recently extended to the general relativistic context by Abdikamalov, Ahmedov & Miller (2009) who, just like Timokhin et al. (2000), based their results on the so-called low current density approximation, i.e. on the assumption that the magnetic field is mainly produced by volume currents inside the neutron star and by surface currents on its surface, while the magnetic field due to magnetospheric currents can be neglected. In the paper of Abdikamalov et al. (2009), the influence of oscillations to the magnetosphere electrodynamics was considered for the case of a non-rotating Schwarzschild star. In the present paper, we apply some of the results of Abdikamalov et al. (2009) to investigate how oscillations, produced at the star surface, reflect in the energy losses from the polar cap region of the magnetosphere of slowly rotating neutron star. In this respect we extend the work of Muslimov & Harding (1997) by performing a local analysis in the domain of the open magnetic field lines in the inner magnetosphere and taking into account the effects of toroidal oscillations excited at the star surface.

The plan of the paper is as follows. In Section 2, we provide the minimum general relativistic formalism for understanding neutron star electrodynamics, and we perform a detailed analysis of the GJ charge density of the slowly rotating and oscillating neutron star. In Section 3, we derive a version of the Poisson equation that takes into account both the general relativistic effects and the oscillating behaviour of the magnetosphere of the rotating star. Section 4, on the other hand, is devoted to the computation of the energy losses induced by oscillations together with rotation. In Section 5, we propose and motivate a suggestive idea to explain the phenomenology of intermittent pulsars in terms of the excitation of stellar oscillations. Finally, Section 6 contains the conclusions of our work.

Throughout, we assume a signature \{−, +, +, +\} for the space–time metric, and we use Greek letters (running from 0 to 3) for four-dimensional space–time tensor components, while Latin letters (running from 1 to 3) will be employed for three-dimensional spatial tensor components. Moreover, we set \( c = G = 1 \) (however, for those expressions with an astrophysical application we have written the speed of light explicitly).

## 2 GOLDREICH–JULIAN RELATIVISTIC CHARGE DENSITY

In the slow limit approximation, the space–time around a rotating neutron star of total mass \( M \), angular momentum \( J \) and angular velocity \( \Omega \) is given by (Hartle & Thorne 1968; Landau & Lifshitz 2004)

\[
\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + N^{-2}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\phi^2) - 2\omega_{\phi}r^2 \sin^2 \theta \, \mathrm{d}\phi \, \mathrm{d}t, \tag{1}
\]

where \( N \equiv (1 - 2M/r)^{1/2} \) is the gravitational lapse function, while \( \omega_{\phi} \equiv 2aM/r^3 \) is the Lense–Thirring angular velocity, which represents the angular velocity of a freely falling inertial frame. The specific angular momentum \( a \), on the other hand, is defined as \( a = J/M, \) \( J \) is the angular momentum of the star. We note that the metric (1) is split according to the \( 3 + 1 \) formalism of general relativity (Arnowitt, Deser & Misner 1962), which admits a natural Eulerian observer, also called the zero angular momentum observer (ZAMO), with four-velocity \( n_a \) given by

\[
n_a = \{-N, 0, 0, 0\}. \tag{2}
\]
In the rest of our discussion, when we introduce any three vectors, like for instance the electric field, we assume that it is defined in the locally flat space–time of the ZAMO observer, and we denote its orthonormal components with hat superscripts.

From the system of Maxwell equations and after assuming the magnetic field of a neutron star to be stationary in the corotating frame, Muslimov & Tsygan (1992) derived the following Poisson equation for the scalar potential $\Phi$:

$$\nabla \cdot \left( \frac{1}{N} \nabla \Phi \right) = -4\pi (\rho - \rho_{GJ}),$$

(3)

where $\rho - \rho_{GJ}$ is the effective space charge density responsible for the generation of an unscreened electric field parallel to the magnetic field, while $\rho_{GJ}$ is the Goldreich–Julian charge density that we discuss below. In their pioneering work, Goldreich & Julian (1969) showed that a strongly magnetized and highly conducting neutron star, rotating about the magnetic axis, would spontaneously build up a charged magnetosphere. In a nutshell, the argument is the following: if a magnetized rotating neutron star is placed in vacuum, enormous unbalanced electric forces parallel to the magnetic field $B$ would set up at the surface of the star, extracting charges from the surface into the external vacuum region, thus producing a filled magnetosphere. Therefore, Goldreich & Julian (1969) hypothesized that a far better approximation for the magnetosphere would be obtained by shorting out the component of the electric field $E$ along $B$. The magnetospheric charges that maintain $E \cdot B = 0$ are themselves subject to the $E \times B$ drift that sets them into corotation with the star (Mofiz & Ahmedov 2000). A derivation of the GJ charge density in the presence of oscillations but in the Newtonian framework has been performed by Timokhin (2007). Here we discuss the corrections to the standard GJ charge density when both relativistic effects and stellar oscillations are taken into account.

Our starting point is the general expression for the GJ charge density that takes into account the contribution of the electric field induced by arbitrary stellar oscillations, i.e.,

$$\rho_{GJ} = -\frac{1}{4\pi c} \nabla \cdot \left[ \frac{1}{N} (u - w) \times B + \frac{1}{N} \delta v \times B \right] = -\frac{1}{4\pi c} \nabla \cdot \left[ \frac{1}{N} \left( 1 - \frac{\kappa}{F} \right) u \times B + \frac{1}{N} \delta v \times B \right],$$

(4)

where $u - w = (\Omega - \omega_{l,m}) r \sin \theta e_\phi$, while $\delta v$ is the oscillation velocity. Moreover, $R$ is the star radius, $r = r/R$ is the dimensionless radial coordinate, $\varepsilon = 2M/R$ is the compactness parameter, $\beta = I/I_0$ is the moment of inertia of the star in the units of $I_0 = M R^2$ and $\kappa = \varepsilon \beta$.

We apply expression (4) to the case of toroidal oscillations, whose velocity has components (see e.g. equation 13.71 of Unno et al. 1989)

$$\delta v^i = \left\{ \begin{array}{ll} 0, & \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} Y_{l,m}(\theta, \phi), -\frac{\partial}{\partial \phi} Y_{l,m}(\theta, \phi) \end{array} \right\} \tilde{\eta}(r)e^{i\omega t},$$

(5)

where $\omega$ is the real part of the oscillation frequency, while $\tilde{\eta}$ is the radial eigenfunction expressing the amplitude of the oscillation. We have used multipolars indices $l$ and $m$ to distinguish the harmonic dependence of the velocity perturbations from the harmonic dependence, in terms of $l$ and $m$, of the electromagnetic fields, since these indices are in general distinct. As usual, the spherical orthonormal functions $Y_{l,m}(\theta, \phi)$ are the eigenfunctions of the Laplacian in spherical coordinates. They are given by

$$Y_{l,m}(\theta, \phi) = \frac{1}{\sqrt{2\pi}} e^{i m \phi} \Theta_{l,m}(\cos \theta),$$

(6)

where the functions $\Theta_{l,m}(\cos \theta)$ satisfy the differential equation

$$\frac{1}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{d \Theta_{l,m}(\cos \theta)}{d \theta} \right) - \frac{m^2}{\sin^2 \theta} \Theta_{l,m}(\cos \theta) + (l + 1)\Theta_{l,m}(\cos \theta) = 0,$$

(7)

and can be written as

$$\Theta_{l,m}(\cos \theta) = (-1)^m \sqrt{\frac{2l + 1 (l - m)!}{2 (l + m)!}} P_l^m(\cos \theta),$$

(8)

where $P_l^m(\cos \theta)$ are the Legendre polynomials. For simplicity, we limit our attention to the case in which the magnetic moment $\mu$ of the star is aligned with its angular momentum, and furthermore, we assume that the magnetic field is a dipolar one, with orthonormal components that take the form (Ginzburg & Ozerney 1964; Muslimov & Tsygan 1992)

$$B^i = B_0 \frac{f(\tilde{\varphi})}{f(1)} e^{-3 \cos \theta}, \quad B^j = \frac{1}{2} B_0 N \left[ -2 \frac{f(\tilde{\varphi})}{f(1)} + \frac{3}{(1 - \varepsilon/\tilde{\varphi}) f(1)} \right] \tilde{\varphi}^{-3} \sin \theta,$$

(9)

where

$$f(\tilde{\varphi}) = -3 \left( \frac{\tilde{\varphi}}{\varepsilon} \right)^3 \ln \left(1 - \frac{\tilde{\varphi}}{\varepsilon} \right) + \frac{\tilde{\varphi}}{\varepsilon} \left(1 + \frac{\varepsilon}{2\tilde{\varphi}} \right),$$

(10)

and $B_0 \equiv 2\mu/R^3$ is the Newtonian value of the magnetic field at the pole of the star. When the oscillation velocity and the magnetic field of the star are given, respectively, by equations (5) and (9), then equation (4) provides the following modified GJ charge density:

$$\rho_{GJ} = -\Omega B_0 \frac{f(\tilde{\varphi})}{2\pi c N^2 \tilde{\varphi}^3 f(1)} \left[ 1 - \frac{\kappa}{\tilde{\varphi}} \right]$$

$$-\frac{1}{4\pi c R^3} B_0 \rho e^{-i\omega t} \left\{ -\frac{1}{N} \frac{f(\tilde{\varphi})}{f(1)} \tilde{\eta}(\tilde{\varphi}) \cot \theta \left[ \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} Y_{l,m}(\theta, \phi) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} Y_{l,m}(\theta, \phi) \right] \right. $$

$$+ \left. N \left[ -2 \frac{f(\tilde{\varphi})}{f(1)} + \frac{3}{(1 - \varepsilon/\tilde{\varphi}) f(1)} \right] \sin \theta \frac{\partial}{\partial \theta} Y_{l,m}(\theta, \phi) \frac{\partial}{\partial \tilde{\varphi}} \frac{\tilde{\varphi}}{N} \tilde{\eta}(\tilde{\varphi}) \right\}.$$
From the point of view of the emission of energy, which we consider with greater detail in Section 4, the most interesting region of the magnetosphere is the so-called polar cap region, i.e. the region where the magnetic field lines remain open and at distances from the star surface much smaller than the light cylinder radius. If we denote by \( \Theta_0 \) (computed as explained in Section 4) the colatitude of the last closed magnetic field line at the star surface, then the angle \( \Theta \) of the last closed magnetic field line as a function of \( r \) is well approximated as (Muslimov & Tsygan 1992)
\[
\Theta(r) \cong \sin^{-1} \left( \sqrt{\frac{\int f(1)}{f'(r)}} \right)^{1/2} \sin \Theta_0.
\]  
\( (12) \)

It should be remarked that on the surface of the star, and even within reasonably large distances far from it, the aperture angles of the open magnetic field lines remain small. For example, for a neutron star with \( \epsilon = 0.3 \) and \( \Theta_0 = 0.087 \approx 5^\circ \), we find \( \Theta = 0.197 \approx 11.3^\circ \) at \( r = 4R \).

As a result, in this approximation the last term in the curl brackets on the right-hand side of equation (11) can be neglected, since it contains \( \theta \), with \( \theta \leq \Theta(r) \), to a second power larger than other terms in the same brackets. Taking into account equations (6) and (7), in the limit of small angles \( \theta \), we obtain from (11) the following equation for the GJ charge density:
\[
\rho_{\text{GJ}} = \rho_{\text{GJ,0}} + \delta \rho_{\text{GJ,m'}} = -\frac{\Omega B_0}{2\pi c} \frac{\int f(1)}{N^2 f(1)} \left( 1 - \frac{\kappa}{R^2} \right) - \frac{1}{4\pi c} \frac{B_0 e^{-i\omega t}}{R^4} \frac{f(1)}{N^2 f(1)} \tilde{\eta}(1') \left( 1' + 1 \right) Y_{lm'},
\]  
\( (13) \)

where \( \rho_{\text{GJ,0}} \) is the Goldreich–Julian charge density of a slowly rotating neutron star while \( \delta \rho_{\text{GJ,m'}} \) is the correction induced by oscillations. We are here interested in analysing the GJ charge density of the first few modes, namely those with \( (m', 0) \) given by \( m' = 0, 1, 2, \ldots \).

To this extent, however, we greatly simplify our calculations by approximating \( Y_{lm'}(\theta, \phi) \approx A_{lm'}(\phi) \theta^m \), where the terms \( A_{lm'}(\phi) \) have real parts given by
\[
A_{00} = 0, \quad A_{10} = \frac{3}{4\pi}, \quad A_{11} = -\frac{3}{8\pi} \cos \phi, \quad A_{20} = \frac{5}{4\pi}, \quad A_{21} = -\frac{3}{24} \cos \phi.
\]  
\( (14) \)

From (13) we can compute the ratio
\[
\frac{\delta \rho_{\text{GJ,m'}}}{\rho_{\text{GJ,0}}} = \frac{K}{2\pi^{2-m/2}} \left( \frac{\int f(1)}{N^2 f(1)} \right)^{(2-m)/2} \frac{1}{\left( 1 - \frac{\kappa}{R^2} \right)} \left( 1' + 1 \right) A_{lm'}(\phi),
\]  
\( (15) \)

where we have posed \( \tilde{\eta}(1') \approx \tilde{\eta}(1) \), which amounts to the assumption that the oscillation amplitude maintains the value it has on the surface of the star, at least within small distances far from it, as we are considering here. Moreover, we have introduced the small number \( K = \tilde{\eta}(1)/\Omega R \) to parametrize the amplitude of the oscillation. Finally, we have considered \( \Theta \approx \theta \). One can easily see that \( \delta \rho_{\text{GJ,m'}} = 0 \) for the mode \( m = 0 \).

Fig. 1, on the other hand, shows the ratios \( \delta \rho_{\text{GJ,m'}}/\rho_{\text{GJ,0}} \) for the other four modes, computed at \( t = 0 \). When plotting these graphs, we have used the following typical set of parameters: \( \kappa = 0.15, \epsilon = 1/3, K = 0.01, \Theta_0 = 0.008, \Omega = 1 \text{ rad s}^{-1} \).

As it is clear from Fig. 1, the oscillation induced GJ charge density not only can be a significant part of \( \rho_{\text{GJ,0}} \), but can even prevail several hundred times over it, for example, for the mode \( m = 2 \).

An exception is represented by the mode \( (1, 1) \), for which \( |\delta \rho_{\text{GJ,m'}}| < |\rho_{\text{GJ,0}}| \) even very close to the star. The influence of oscillations is greater near the surface of the star, which is the most interesting region of the magnetosphere, while it decreases far from it. On the other hand, the relativistic charge density \( \rho \) that enters equation (3) is proportional to the intensity of the magnetic field through a proportionality coefficient that is constant along the given magnetic field line (Muslimov & Tsygan 1991), i.e.
\[
\rho = \frac{\Omega B_0}{2\pi c} \frac{\int f(1)}{N^2 f(1)} \left( \tilde{A}(\xi) + e^{-i\omega t} \tilde{a}(\xi, \phi) \right),
\]  
\( (16) \)

where we have introduced the variable \( \xi = \theta/\Theta(r) \), and where \( \tilde{A}(\xi) \) and \( \tilde{a}(\xi, \phi) \) are unknown functions to be specified from the boundary conditions. The computation of \( \tilde{A}(\xi) \), which corresponds to the case of pure rotation, has already been performed by Muslimov & Harding (1997) (see their equation 58), showing that \( \tilde{A}(\xi) \approx \frac{\kappa}{1 - \kappa} \). The computation of \( \tilde{a}(\xi, \phi) \), on the other hand, is discussed in Section 3.1.

### 3 POISSON EQUATION

After inserting (13) and (16) into equation (3), we find the following expression for the Poisson equation:
\[
R^{-2} \left\{ \frac{N}{R^2} \frac{\partial^2}{\partial \xi^2} \left( \frac{\partial}{\partial \xi} \right) \Phi + \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \xi} \left( \frac{\partial}{\partial \xi} \right) + \frac{1}{\partial \xi^2} \right] \right\} \Phi = -4\pi \frac{\Omega B_0}{2\pi c} \frac{\int f(1)}{N^2 f(1)} \left[ \tilde{A}(\xi) + e^{-i\omega t} \tilde{a}(\xi, \phi) \right] - 4\pi \frac{\Omega B_0}{2\pi c} \frac{\int f(1)}{N^2 f(1)} \left( 1 - \frac{\kappa}{R^2} \right) - \frac{1}{c} \frac{B_0 e^{-i\omega t}}{\Omega^2(1')} \frac{f(1)}{N} \frac{f(1)}{f(1)} \tilde{\eta}(1') \left( 1' + 1 \right) Y_{lm'}.
\]  
\( (17) \)

Solving such an equation represents a complicated task, and our strategy consists in, first of all, expanding the scalar potential \( \Phi \) as
\[
\Phi(t, \bar{r}, \xi, \phi) = \Phi_0(t, \xi) + e^{-i\omega t} \delta \Phi(t, \xi, \phi),
\]  
\( (18) \)

We will show in Section 4 below that \( K \) and \( \Theta_0 \) are not independent of each other and that \( \Theta_0 \) does also depend on the indices \( l' \) and \( m' \). However, we have chosen to use a single value \( \Theta_0 = 0.008 \) for all of the plots reported in Fig. 1 as this nevertheless represents a mean value typical of a standard astrophysical situation.
where the first term $\Phi_0(\tilde{r}, \xi)$ corresponds to the case of a non-oscillating but rotating star. In this case, in fact, we know that $\Phi_0$ must satisfy the following equation (see e.g. equation 37 of Muslimov & Tsygan 1992):

$$R^{-2} \left\{ N \frac{1}{\tilde{r}^2} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r}^2 \frac{\partial}{\partial \tilde{r}} \right) + \frac{1}{N \tilde{r}^2} \frac{\partial}{\partial \theta} \left( \theta \frac{\partial}{\partial \theta} \right) \right\} \Phi_0 = \frac{2\Omega B_0}{c \tilde{r}^3} \frac{f(\tilde{r})}{f(1)} \left\{ \frac{1}{N} A(\xi) + \frac{1}{N} \left( 1 - \frac{\kappa}{\tilde{r}^3} \right) \right\}. \quad (19)$$

The next step consists in expanding in terms of spherical harmonics both the second term of (18), representing the perturbation to the non-oscillating case, and the function $\delta(\xi, \phi)$, which contains the $\phi$ dependence of the charge density (16). Namely, we write

$$\delta \Phi(\tilde{r}, \xi, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \delta \Phi_{lm}(\tilde{r}) Y_{lm}(\xi, \phi), \quad (20)$$

$$\delta(\xi, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \delta\bar{a}_{lm}(\tilde{r}) Y_{lm}(\xi, \phi). \quad (21)$$

At this point, lengthy but straightforward calculations allow to derive from equation (17) the following equation for the perturbation $\delta \Phi(\tilde{r}, \xi, \phi)$:

$$R^{-2} N \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \frac{1}{\tilde{r}^2} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r}^2 \frac{\partial}{\partial \tilde{r}} \right) \frac{l(l+1)}{N \tilde{r}^2 \tilde{\Omega}^2(\tilde{r})} \right\} \delta \Phi_{lm} Y_{lm} = -4\pi \frac{\Omega B_0}{2\pi c} \frac{1}{N^3} \frac{f(\tilde{r})}{f(1)} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \tilde{\delta}_{lm} Y_{lm}$$

$$- \frac{1}{c \tilde{r}^3} \frac{B_0}{N} \frac{1}{f(1)} \frac{f'(\tilde{r})}{f(\tilde{r})} \left( l'(l'+1) \right) Y_{l'm'} \quad (22).$$

We now introduce $\delta F_{lm}(\tilde{r}) = \tilde{r} \delta \Phi_{lm}(\tilde{r})$, and we exploit the fact that the spherical harmonics $Y_{lm}$ form a set of orthogonal basis functions. This means that in equation (22), we must first fix $l' = l$ and $m' = m$, and we then equal the coefficients of each basis function $Y_{lm}$ on the left- and right-hand side. In this way we finally obtain the Poisson equation in the form

$$R^{-2} N \left\{ \frac{d^2}{d\tilde{r}^2} - \frac{l(l+1)}{N \tilde{r}^2 \tilde{\Omega}^2(\tilde{r})} \right\} \delta F_{lm}(\tilde{r}) = - \frac{B_0}{c N^3} \frac{f(\tilde{r})}{f(1)} \frac{2\Omega \tilde{\delta}_{lm} + \frac{1}{\tilde{r}^3} \frac{f'(\tilde{r})}{f(\tilde{r})} \left( l(l+1) \right)}{\tilde{\Omega}^2(\tilde{r})}. \quad (23)$$

It is worth stressing that equation (23) is valid for small polar angles $\theta$, but for any distance $\tilde{r}$ from the star surface. The limit of equation (23) for small $\tilde{r}$ is considered in the rest of this section.
3.1 Solution close to the star surface

As a first example, we wish to compute the solution of the Poisson equation close to the star surface, where z = r̃ - 1 ≪ 1, while imposing no restrictions on the aperture angle Θo of the last closed magnetic field line. In this case equation (23) for the unknown δFim becomes

\[
\frac{d^2}{dz^2} - \frac{l(l+1)}{(1 - \epsilon)\Theta_o^2} \delta F_{lm}(z) = -\frac{2\Omega B_0 R^2}{c} \left( 1 - 2z \right) \bar{a}_{lm} + \frac{B_0 R}{\Theta_o^2} \left( l + 1 \right) l(l+1) \epsilon \Theta_o \epsilon \Theta_o, \tag{24}
\]

which represents the extension of equation (44) of Muslimov & Tsygan (1992) to account for the presence of oscillations. The general solution of equation (24), which is an inhomogeneous differential equation, is

\[
\delta F_{lm} = \tilde{C} e^{-\epsilon/\Theta_o} \epsilon \Theta_o \epsilon \Theta_o + \frac{\Theta_o^2}{l(l+1)} \frac{B_0 R}{c} \left( 1 - 2z \right) \bar{a}_{lm} + \frac{l(l+1)}{(1 - \epsilon)\Theta_o^2} \eta(1) l(l+1). \tag{25}
\]

The constant \( \tilde{C} \) and the coefficients \( \bar{a}_{lm} \) may be found after imposing physically motivated boundary conditions. In particular, at the star surface we require both equipotentiality and absence of a steady state electric field, which amounts to the two conditions:

\[
\delta F_{lm}|_{z=0} = 0, \quad \frac{d\delta F_{lm}}{dz}|_{z=0} = 0. \tag{26}
\]

From them, simple calculations allow us to derive

\[
\bar{a}_{lm} = -\frac{l(l+1)}{2\Omega R \Theta_o^2} \eta(1) \left( 1 - \frac{\Theta_o \sqrt{1 - \epsilon}}{\sqrt{l(l+1) - 2\Theta_o \sqrt{1 - \epsilon}}} \right), \quad \tilde{C} = -\frac{B_0 R}{c} \eta(1) \left( \frac{\Theta_o \sqrt{1 - \epsilon}}{\sqrt{l(l+1) - 2\Theta_o \sqrt{1 - \epsilon}}} \right). \tag{27}
\]

The final solution for the scalar potential near the surface of the oscillating rotating neutron star has therefore the form

\[
\Phi(t, \bar{r}, \bar{\xi}, \bar{\phi}) = \Phi_0(\bar{r}, \bar{\xi}) + e^{-i\omega t} \frac{B_0 R}{c} \eta(1) \left( \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\Theta_o \sqrt{1 - \epsilon}}{\sqrt{l(l+1) - 2\Theta_o \sqrt{1 - \epsilon}}} \left( -e^{-i\epsilon/\Theta_o} \epsilon \Theta_o \epsilon \Theta_o + 1 - \frac{\sqrt{l(l+1)}}{\sqrt{1 - \epsilon}} \eta(1) \right) Y_{lm}(\bar{\xi}, \bar{\phi}), \tag{28}
\]

and the corresponding expression for the accelerating component of the electric field is

\[
E_1 = E_0 - e^{-i\omega t} \frac{B_0 R}{c} \eta(1) \left( \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\Theta_o \sqrt{1 - \epsilon}}{\sqrt{l(l+1) - 2\Theta_o \sqrt{1 - \epsilon}}} \left( -e^{-i\epsilon/\Theta_o} \epsilon \Theta_o \epsilon \Theta_o + 1 - \frac{\sqrt{l(l+1)}}{\sqrt{1 - \epsilon}} \eta(1) \right) Y_{lm}(\bar{\xi}, \bar{\phi}), \tag{29}
\]

where

\[
E_0 = -\frac{1}{R} \frac{\partial \Phi_0}{\partial \bar{r}} \tag{30}
\]

is the accelerating field that is present even in the absence of oscillations.

3.2 Solution in the polar cap region

The solution in the polar cap region, namely when \( \Theta_o \ll \bar{r} - 1 \ll R \), can be obtained from equations (28) and (29) in the limit of small \( \Theta_o \), or directly from equation (23). In this case, in fact, \( |d^2\delta F_{lm}/d\bar{r}^2| \ll |l(l+1)|\delta F_{lm}/N^22^2\Theta^2(\bar{r}) \), and equation (23) therefore reduces to

\[
\frac{l(l+1)}{R^2N^2\Theta^2(\bar{r})} \delta F_{lm} = \frac{2\Omega B_0 R}{c} \frac{f(\bar{r})}{f(1)} \bar{a}_{lm} + \frac{B_0}{c \Theta^2(\bar{r})} \frac{f(\bar{r})}{R N^2 f(1)} \eta(1) l(l+1). \tag{31}
\]

After using \( \Theta \) as given by equation (12) and with the same coefficients \( \bar{a}_{lm} \) expressed by equation (27), we can obtain

\[
\delta F_{lm} = \frac{B_0 R}{c} \left[ -\bar{r} \eta(1) \sqrt{l(l+1) - 3\sqrt{1 - \epsilon} \Theta_o} + \frac{f(\bar{r}) \eta(1) f(1)}{f(1) \bar{r}} \right]. \tag{32}
\]

This allows us to derive both the electric potential and the accelerating component of the electric field as\(^2\)

\[
\Phi(t, \bar{r}, \bar{\xi}, \bar{\phi}) = \Phi_0(\bar{r}, \bar{\xi}) + \frac{B_0 R}{c} \left[ -\eta(1) + \frac{f(\bar{r}) \eta(1) f(1)}{f(1) \bar{r}^2} \right] \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\bar{\xi}, \bar{\phi}), \tag{33}
\]

\[
E_1 = E_0 - \frac{B_0}{c} \frac{d}{d\bar{r}} \left( \frac{f(\bar{r}) \eta(1)}{f(1) \bar{r}^2} \right) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\bar{\xi}, \bar{\phi}). \tag{34}
\]

It is interesting to compare the second term on the right-hand side of equation (34), namely the contribution due to the oscillations, with \( E_0 \) as reported, for example, in equation (55) of Muslimov & Tsygan (1992). We find

\[
\frac{\delta E_{lm}}{E_0} = e^{-i\omega t} \frac{2}{3} \frac{\eta(1)}{\Omega R \bar{r}} \left[ \frac{d}{d\bar{r}} \left( \frac{f(\bar{r}) 1}{f(1) \bar{r}^2} \right) \right] \bar{r}^2 \Theta_o^{-2} \frac{\xi}{1 - \epsilon} A_{lm}(\bar{\phi}), \tag{35}
\]

that we plot in Fig. 2 for the different modes considered so far and computed at \( t = 0 \) using the same set of parameters as for the plots in Fig. 1. Moreover, we have posed \( \xi = 1/2 \), i.e. we have considered the middle magnetic field line between the polar axis and the edge of the

\(^2\)For small values of \( \Theta_o \) we use the approximation \( (\sqrt{l(l+1)} - 3\sqrt{1 - \epsilon} \Theta_o)/(\sqrt{l(l+1)} - 2\sqrt{1 - \epsilon} \Theta_o) \sim 1.\)
polar cap. The plots demonstrate the significant influence of oscillations on the electric field of a pulsar. In particular, because $\Theta_0$ is supposed to be small, the ratio $\delta E_{\text{lm}}/E_0$ can be very large for modes with $m < 2$. Indeed, the modes $(l, m) = (0, 0), (2, 0)$ induce an electric field in the opposite direction to $E_0$, which is three orders of magnitude larger in modulus than $E_0$. In general, the ratio $|\delta E_{\text{lm}}/E_0|$ increases when increasing the distance from the star surface.

In a recent work aimed at quantifying the impact of neutron star oscillations on the accelerating electric field, Timokhin (2007) considered the whole range of angles $\theta$ and more complicated cases of spheroidal oscillations. He found that the contribution of oscillations to such electric field may be either positive or negative and that this contribution is substantial for values of $l$ of several hundreds. Indeed, as shown in the Section 4, the appearance of the small parameter $\Theta_0$ in the computation of the energy losses makes astrophysically relevant even the modes with small values of $l$ and $m$.

4 ENERGY LOSSES

In this section we calculate the energy losses from the polar cup region of a rotating and oscillating neutron star, using several results presented in Abdikamalov et al. (2009) for the spherical Schwarzschild star. The total energy loss from the open field lines region, averaged over an oscillation period and carried out by the outflowing plasma, is determined as (Timokhin et al. 2000)

$$L_{\text{lm}} = \frac{1}{\tau} \int_0^\tau dt \int_0^{2\pi} d\phi \int_0^{\Theta_0} d\theta |j_{\text{lm}}(R, \theta, \phi) \Delta \epsilon_{\text{lm}}(\theta, \phi)| R^2 \sin \theta,$$  \hspace{1cm} (36)

where $\Delta \epsilon(\theta, \phi)$ is the work done by the electric field to move a unit charge to the point with coordinate $(R, \theta, \phi)$:

$$\Delta \epsilon_{\text{lm}} = RN_R^2 \int_0^{\Theta_0} \epsilon_{\text{GJ},\text{lm}}(R, \theta', \phi) d\theta',$$  \hspace{1cm} (37)

while $j_{\text{lm}}$ is the electric current density and $\tau$ is the period of oscillations. It should be noted that despite some terms in the expression for the energy losses will contain the factor $e^{-i\omega t}$, which mathematically gives a zero net result when averaged over time, physically these terms will give a non-zero net contribution. That is because the average crossing time of the acceleration zone by particles which are responsible for the energy losses is much shorter than the oscillation period (typical frequencies of oscillations are of the order of 10–50 kHz). In practice, therefore, the particles do not return back to the star (see also the discussion of Timokhin et al. 2000). If the polar cap region is in the conditions of complete charge separation, the current density is well approximated as (Ruderman & Sutherland 1975; Timokhin et al. 2000)

$$j_{\text{lm}} \simeq \rho_{\text{GJ}} c.$$  \hspace{1cm} (38)

In order to compute the integral in (36), we need to provide an estimate of the angle $\Theta_0$ of the last closed magnetic field line at the surface of the star. After using $R_*$ to denote the radial coordinate of the point where the last closed magnetic field line crosses the equatorial plane (note
that $R_c > R$, we derive $\Theta_0$ from the condition

$$\epsilon_{\phi}(R_c, \pi/2, \phi) = \epsilon_{cem}(R_c, \pi/2, \phi),$$

(39)

where $\epsilon_{\phi}$ and $\epsilon_{cem}$ are the kinetic energy density of the outflowing plasma and the energy density of the magnetic field, respectively. The underlining idea, in fact, is that at the last closed magnetic field line equipartition of energy exists between the magnetic field and the plasma, and, for convenience, the condition is evaluated at the equator. The two energies $\epsilon_{\phi}$ and $\epsilon_{cem}$ have already been computed by Abdikamalov et al. (2009) (see their equations 100 and 101) and are

$$\epsilon_{\phi}(R_c, \pi/2, \phi) = \frac{1}{2f(1)} \frac{N_c}{R_a} R_0^3 \Delta E f^2,$$

(40)

$$\epsilon_{cem}(R_c, \pi/2, \phi) = \frac{N_c}{32 \pi R_0^6} n_0^2,$$

(41)

where $N_c = \sqrt{1 - 2M/R}$ and $N_a = \sqrt{1 - 2M/R_a} \approx 1$. The angle $\Theta_0$ does not explicitly appear in (40) and (41), but rather implicitly. In fact, for a dipole magnetic field $f(r) \sin^2 \theta/r = \text{const}$ and therefore

$$\frac{R}{R_a} = \frac{f(1)}{f(R_a/R)} \Theta_0^2,$$

(42)

where $f(R_a/R)$ is close to unity. In order to proceed with the computation of $\Theta_0$, and hence of the energy loss, we need the electric field component

$$(E_0)_{Gl,lm} = - \frac{1}{N_c} \left[ (1 - \kappa f) \mathbf{u} + \delta \mathbf{v} \right] \delta \eta,$$

(43)

from which we can compute $\Delta \epsilon_{lm}$ through (37). This allows us to obtain (for small angles $\theta$)

$$\Delta \epsilon_{lm} = - \frac{R N_a B_0}{c} \left[ \Omega R(1 - \kappa) \frac{\theta^2}{2} - A_{lm} \theta^m \tilde{\eta}(1) \right] \text{ for } m \neq 0,$$

(44)

$$\Delta \epsilon_{lm} = - \frac{R N_a B_0}{c} \frac{\theta^2}{2} \Omega R(1 - \kappa - 2A_{00} \tilde{\eta}(1)) \text{ for } m = 0.$$

(45)

Taking into account equations (39)–(41), we can obtain the expression for the angle $\Theta_0$, which is an algebraic equation in the case $m \neq 0$ while it has a closed form in the case $m = 0$, i.e.

$$\frac{R N_a}{c} \left[ \Omega R(1 - \kappa) \frac{\theta^2}{2} - A_{lm} \theta^m \tilde{\eta}(1) \right] \left\{ \frac{\Omega(1 - \kappa)}{c} + \tilde{\eta}(1) \frac{l(l + 1)A_{lm}}{2c} \right\} = \frac{1}{8} f^4(1) \Theta_0^6 \text{ for } m \neq 0,$$

(46)

$$\Theta_0 = \frac{2}{f(1)} \sqrt[4]{\left[8 \frac{R N_a}{c} \Omega R(1 - \kappa - 2A_{00} \tilde{\eta}(1)) \left\{ \frac{\Omega(1 - \kappa)}{c} + \frac{1}{4c} \tilde{\eta}(1) l(l + 1)A_{00} \right\} \right]^{1/4} \text{ for } m = 0.}\]$$

(47)

When $m = 0$ and $\Omega = 0$, hence in the axisymmetric case with no rotation, we get the expression

$$\Theta_{0,lm=0,\Omega=0} = \left[ \frac{A_{00}^{2}}{f^4(1)} \right]^{1/2} \tilde{\eta}(1).$$

(48)

This estimate is slightly different from that obtained by Abdikamalov et al. (2009):

$$\Theta_0 = \frac{2N_a^{1/2}}{c} \left[ \frac{\tilde{\eta}(1) A_{00}^{2}}{f(1)} \right]^{1/2},$$

(49)

first, because of the different expansion of the spherical harmonics, which in Abdikamalov et al. (2009) was chosen to be $Y_{lm}(\theta, \phi) \approx A_{00}^{ij}(\phi) \phi^m + A_{lm}^{ij}(\phi) \phi^{m+2}$, and secondly, because of the different determination of $B_0$ (equations 73–75 for magnetic field in the paper of Abdikamalov et al. 2009 do not contain $f(1)$ in the denominator). On the other hand, when $m = 0$ and $\tilde{\eta} = 0$, hence in the axisymmetric case with no oscillations, we get the expression

$$\Theta_{0,lm=0,\Omega=0} = \sqrt{4 \frac{4N_a R_0^2}{f^4(1)}} \tilde{\eta}(1) (1 - \kappa)^2,$$

(50)

which represents the correct general relativistic extension of the expression reported by Muslimov & Tsygan (1992). Inserting the angles $\Theta_0$ defined by the equations (46) and (47) into (36), we derive the total energy losses from the polar cap region for a rotating and oscillating magnetized neutron star as

$$L_{lm \neq 0} = \int_0^{2\pi} d\phi \frac{R_0^4 B_0^2}{2\pi} \left\{ \frac{\Omega^2 R_0^2}{2c N_a} (1 - \kappa) \tilde{\eta}(1) \frac{\Theta_0^m}{3} + \frac{\Omega}{4c N_a} (1 - \kappa) \frac{l(l + 1)A_{lm} \tilde{\eta}(1) \Theta_0^{m+4}}{m + 2} \right\} \text{ for } m \neq 0,$$

(51)

$$L_{lm = 0} = \int_0^{2\pi} d\phi \frac{R_0^4 B_0^2}{2\pi} \left\{ \frac{\Omega^2 R_0^2}{2c N_a} (1 - \kappa) \tilde{\eta}(1) \frac{\Theta_0^m}{3} + \frac{\Omega}{4c N_a} (1 - \kappa) \frac{l(l + 1)A_{lm} \tilde{\eta}(1) \Theta_0^{m+4}}{m + 2} \right\} \text{ for } m = 0.$$
\[ L_{I_{\text{tot}}} = \int_{0}^{2\pi} d\phi \frac{R_{1}^{2} N_{R} B_{0}^{2} \Theta_{0}^{2}}{2\pi} \left[ \frac{\Omega R (1 - \kappa) - 2 A_{0} (\phi) \eta (1)}{\Omega R (1 - \kappa) + 1 - \frac{1}{2 \kappa N_{R}} \eta (1) \mu (l + 1) A_{l m} (\phi)} \right] \text{ for } m = 0. \] (52)

Equations (51) and (52) just derived are astrophysically very relevant and deserve some comments. In the first place, it is interesting to note that (52) takes a simpler form if only the linear terms in the amplitude of the stellar oscillation are retained. In this case, in fact, we find

\[ L_{I_{\text{tot}}} = \frac{R_{1}^{2} B_{0}^{2} \Omega_{1}^{2}}{8 c} \left[ 1 + \frac{\eta (1) A_{0} (\phi) \mu^{2} + l - 2}{\Omega R (1 - \kappa)} \right]. \] (53)

We can highlight the corrections with respect to the energy losses in the Newtonian case and in the absence of oscillations if we replace \( \Theta_{0} \) in (53) with one of the expressions computed above. For simplicity, we consider the case given by (50), namely the case of pure rotation, and we obtain

\[ L_{I_{\text{tot}}} = 3 (1 - \kappa)^{4} N_{R} f_{4} (1) \left[ 1 + \frac{\eta (1) A_{0} (\phi) \mu^{2} + l - 2}{\Omega R (1 - \kappa)} \right] (E_{\text{rot}})^{\text{Newt}}, \] (54)

where \((E_{\text{rot}})^{\text{Newt}}\) is the standard Newtonian expression for the magnetodipole losses in flat space–time approximation:

\[ (E_{\text{rot}})^{\text{Newt}} = \frac{1}{6} \frac{\Omega_{1}^{2} B_{0}^{2} R_{1}^{6}}{c^{3}}. \] (55)

The astrophysical relevance of equation (54) becomes even more transparent when it is rewritten in terms of the pulsar observables \( P \), i.e. the period, and \( \dot{P} = dP/dt \). To this extent we first recall two standard relations of pulsar physics. The first one is the relation between luminosity and (2, 1). Equation (60) contains two major contributions. The first contribution includes the rotational energy losses plus terms linear in \( \kappa \), which are neither present in pure rotational regime nor in pure oscillatory regime. The second contribution, on the other hand, is the only one present in the pure oscillatory regime. More specifically, in the pure rotation regime (\( \eta = 0 \), \( \Omega \neq 0 \)), only the first term in the curl brackets of equation (60) survives, and we simply recover \( L = L_{\text{rot}} \). In the opposite regime of pure oscillation (\( \eta \neq 0 \), \( \Omega = 0 \)), the whole first contribution to the right-hand side of equation (60) vanishes and only the second contribution proportional to \( K^{2} \) survives. Finally, in the mixed regime with both \( \eta \neq 0 \) and \( \Omega \neq 0 \), all of the terms must be included.

In order to appreciate the dependence of the energy losses on the oscillation amplitudes, we have computed the ratio \( L_{\text{rot}} / L_{\text{rot}} \) as a function of \( K \) for all of the modes mentioned above. As typical and representative parameters of the rotating neutron star, we have chosen...
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Figure 3. The angle (in °) of the last open magnetic field line $\Theta_0$ as a function of the parameter $K = \tilde{\eta}/\Omega R$ for modes (1, 1) (continuous red line) and (2, 1) (dotted blue line). The representative parameters of the star are $R = 10$ km, $\Omega = 1$ rad s$^{-1}$ and $\epsilon = 1/3$.

Figure 4. Left-hand panel: the ratio $L_{m \neq 0}/L_{\text{rot}}$ as a function of parameter $K = \tilde{\eta}/\Omega R$ for modes (1, 1) (continuous red line) and (2, 1) (dotted blue line). Right-hand panel: the ratio $L_{m=0}/L_{\text{rot}}$ as a function of parameter $K = \tilde{\eta}/\Omega R$ for modes (0, 0) (continuous red line) and (2, 0) (dotted blue line).

$R = 10$ km, $\Omega = 1$ rad s$^{-1}$, $\epsilon = 1/3$. We first computed the angle $\Theta_0$ after applying a standard root solver to equation (46). Graphs of $\Theta_0$ as a function of the parameter $K$ are presented in Fig. 3 for the modes (1, 1) and (2, 1), and they show that the size of the polar cap increases with the amplitude of the stellar oscillations. Fig. 4, on the other hand, reports the ratio $L_{m \neq 0}/L_{\text{rot}}$. Interestingly, it follows from equation (60) that modes with $m = 1$ have the small parameter $\Theta_0$ to a negative power. Therefore, for such modes the energy losses due to oscillations may exceed significantly the energy losses due to pure rotation, even for relatively small $K$. This effect is indeed shown in the left-hand panel of Fig. 4 which reports the ratio $L_{m \neq 0}/L_{\text{rot}}$ for the modes (1, 1) and (2, 1). The right-hand panel of Fig. 4, on the other hand, reports the ratio $L_{m=0}/L_{\text{rot}}$ for the modes (0, 0) and (2, 0)$^3$ and they turn out to be a factor of 10 smaller than the ones reported in the left-hand panel, for the reason explained above. We note, on the contrary, that the energy losses due to oscillations are smaller than those due to rotation for the mode (0, 0). Modes with $m > 1$ do not contain $\Theta_0$ to a negative power. Although modes with higher $m$ practically do not give any contribution to the energy losses, we should remark that for modes with $m > 3$ the angle $\Theta_0$ may not be small, thus requiring an alternative approach to the one presented in this paper.

$^3$ The mode (1, 0) does not have linear contributions in $K$ to the energy losses, as it is follows from (54).
Since the energy losses of the pulsar are proportional to the spin-down rate, as it is clear from equation (56), the same graphs obtained for the ratios $L_{\text{me}}/L_{\text{rot}}$ describe also the ratio $\Omega_{\text{me}}/\Omega_{\text{rot}}$, where $\Omega_{\text{me}}$ denotes the time derivative of the pulsar rotation frequency when the star oscillates with the mode $(l, m)$, while $\Omega_{\text{rot}}$ corresponds to the case of pure rotation. Thus, it follows from Fig. 4 that an observable value of $\Omega_{\text{me}}/\Omega_{\text{rot}} \sim 1.5$ may be reached for a value of $K$ as small as $K \sim 0.03$ for the mode $(1, 1)$ and for $K \sim 0.01$ for the mode $(2, 1)$.

Finally, it is interesting to note that the comparison of our results with those of Timokhin et al. (2000) can be done only with caution, since he considered the whole range of angles $\theta$, more complicated cases of spheroidal oscillations, and he did not show the ratio $L_{\text{me}}/L_{\text{rot}}$, but rather the ratio between plasma energy losses and vacuum energy losses, both in the presence of oscillations. It should also be noted that Timokhin et al. (2000) performed estimations rather than exact calculations.

5 CONNECTION TO THE PHENOMENOLOGY OF PART-TIME PULSARS

Recently, Rea et al. (2008) and Lyne (2009) reported that a previously known pulsar, PSR B1931+24, with a spin period of 813 ms at the relatively large distance of ~4.6 kpc, when monitored long enough, shows an intermittent radio emission, consisting of an active ON state, lasting for 5–10 d, followed by a sharp transition (happening in less than 10 s) to an OFF state during which the pulsar remains undetectable for 25–35 d. More interestingly, the spin-down rate during the ON state is $\dot{\nu}_{\text{ON}} = -16.3(4) \times 10^{-15} \text{Hz s}^{-1}$, while, when measured over longer periods, the average value of the spin-down is sensibly smaller. This is compatible with a picture in which the spin-down rate during the OFF state is different from that in the ON state, and, in particular, it is a 50 per cent factor smaller, namely $\dot{\nu}_{\text{OFF}} = -10.8(2) \times 10^{-15} \text{Hz s}^{-1}$. A further search for similar objects from the Parkes multibeam survey data revealed at least four additional objects presenting properties similar to those of PSR B1931+24 (Lyne 2009), like, for instance, PSR J1832+0031 with an ON state of ~ 300 d and an OFF state of ~ 700 d. Understanding the physical mechanism responsible for such a remarkable phenomenology as well as the relationship between intermittent (or ‘part-time’) pulsars and conventional radio pulsars is of course of great interest, especially as it may help clarifying aspects that are still obscure about pulsar radio emission.

All of the (few) models presented so far for explaining the phenomenology of intermittent radio pulsars are based on the common idea that intermittent pulsars are isolated neutron stars, similar to conventional radio pulsars. One of the first ideas was that this effect could be similar to nulling, already reported several years ago by Backer (1970). However, the nulling phenomenon lasts only for a few pulse periods and not on a time-scales of tens of days as detected for intermittent pulsars. A second argument that was proposed is that the intermittent phenomenology could be due to precession, which is an effect by which the pulsar undergoes a slow periodic wobble, thus moving the beams of radio radiation out of our line of sight. What ruled out this idea, however, is that precession certainly cannot produce a transition from the ON to the OFF state in less than 10 s.

A more convincing explanation proposed by Lyne (2009) and Gurevich & Istomin (2007) is that there is a global failure of charge particle currents in the magnetosphere. In particular, the changes in the radio emission would be due to the presence or absence of a plasma whose current flow provides the expected extra torque on the star. In this model, the open field lines above the magnetic pole become depleted of the charged radiating particles during the OFF states, and the rotational slow down, $\dot{\nu}_{\text{OFF}}$, is produced by a torque dominated by magnetic dipole radiation. On the contrary, when the pulsar is ON, $\dot{\nu}_{\text{ON}}$ is enhanced by an additional torque provided by the outflowing plasma. In other words, during the ON state the energy release is due to the current losses only, while during the OFF state it is due to the magnetodipole vacuum radiation (in this case, it is not the plasma-filled magnetosphere). In spite of its plausibility, this idea suffers from the lack of a physical mechanism for changing the plasma flow in the magnetosphere in such a drastic way.

What we propose here is indeed an alternative idea based on our results about the oscillating magnetospheres. As we have shown in Section 4, the energy loss by the pulsar can be significantly altered by the stellar oscillations. Therefore, it is reasonable to assume that during the ON state the stellar oscillations create relativistic wind of charged particles by virtue of the additional accelerating electric field. In a period of about 10 d the stellar oscillations are damped and the OFF period starts. The quasi-periodic glitch, whose driving mechanism is still largely obscure, is the only plausible excitation mechanism of oscillations for isolated pulsars, and it would also be responsible for the emergence of new ON states with a certain periodicity. It is well known that the radio emission of pulsar is negligible in the overall energy budget, usually constituting a very tiny fraction of the pulsar spin-down rate, less then $10^{-3}$. Most of the energy flux is carried away by the relativistic pulsar wind and does not reveal itself during pulsar emission. Therefore, switching the radio emission on or off cannot change the pulsar spin-down significantly enough to be detected by observations. As a result, we propose that $\dot{\nu}_{\text{OFF}} \approx \dot{\nu}_{\text{ON}}$, and that pulsar quasi-periodic glitch is the real reason for (i) the sudden increase of rotational energy at the end of the OFF state and (ii) the excitation of the stellar oscillations which switches pulsar radio emission on.

As suggested by Zhang, Gil & Dyks (2007), the transition from the OFF to the ON state of intermittent pulsar would correspond to the reactivation of a dead pulsar above the ‘death line’ in the $P$–$B$ diagram, thus becoming occasionally active only when the conditions for pair production and coherent emission are satisfied. It is worth stressing that it is very difficult to define an exact line in the $P$–$B$ diagram for rotating neutron stars. However, in a recent investigation Ahmedov & Morozova (2009) showed that oscillating but non-rotating neutron star remain below the death line for the majority of known radio pulsars. That result, when combined to the findings of the present study, suggests that oscillations in a rotating star could be the key ingredient for explaining the transient reactivation of intermittent pulsars.

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4 What he found is that for some modes, like for instance $(1, 0)$ or $(l \geq 2, m = 0, 1)$ the plasma energy losses are larger than the vacuum ones even for small oscillation amplitudes.
The connection that we have proposed here between the intermittent pulsar phenomenology and the presence of oscillations in the magnetosphere of rotating magnetized neutron stars will be exhaustively investigated in a more quantitative way in a forthcoming paper.

Making contribution to the accelerating electric field and the total energy losses of the pulsar, oscillations should also have influence on various characteristics of pulses, such as the shape of the profiles, spectral properties, polarization and a few others as described in the works of Taylor & Manchester (1977) and Flanagan (1990). As the oscillatory contribution to quantities described in our paper is periodic, it is natural to think that the modification of the profile due to oscillations will also have a periodic character, as it was indeed suggested by Timokhin (2007). Therefore, the first task in exploring the pulses modifications due to oscillations is to look for periodic perturbations following the glitches. One may also try to predict the modification of pulse arrival times due to oscillations using equation (59) and more or less accurate approximate values of the moment of inertia of the star. The question concerning the influence of oscillations on the pulses arrival time and on the other pulse characteristics is rather interesting, and it will become the subject of our future research.

6 CONCLUSIONS

In this paper we have studied the astrophysical processes in the polar cap region of the magnetosphere of an oscillating neutron star. The background space–time is given by the metric of Hartle & Thorne (1968) within the slow rotation approximation. The novelties of our analysis consist in quantifying the contributions of stellar oscillations in a general relativistic framework. In particular, we have computed the general-relativistic corrections to the GJ charge density to the electrostatic scalar potential and to the component of the electric field parallel to the magnetic field lines in the polar cap region when toroidal stellar oscillations are present. As already remarked by Timokhin (2007), the effective electric charge density, i.e. the difference between the GJ charge density \( \rho_{\text{GJ}} \) (proportional to \( \Omega \) \( \cdot \) \( \mathbf{B} \) for rotating stars and to \( \omega \) \( \cdot \) \( \mathbf{B} \) for oscillating stars in the flat space–time case) and the electric charge density (proportional to \( \mathbf{B} \)) in the oscillating star magnetosphere is responsible for the generation of an electric field parallel to the magnetic field lines. Such difference vanishes only at the surface of the star while in general it becomes significantly large at some distance \( r \) from the surface, due to the fact that \( \rho \) cannot compensate \( \rho_{\text{GJ}} \). As already pointed out by Muslimov & Tsygan (1992), general relativistic terms arising from the dragging of inertial frames give very important additional contribution to this difference. These terms depend on the radial distance from the star as \( 1/r^3 \) and have important influence on the value of accelerating electric field generated in the magnetosphere near the surface of the neutron star.

Our solutions for the accelerating electric field for the oscillating and the rotating magnetized neutron stars may have some significant implications for the pulsar polar cap models. These models assume that charged particles are accelerated above the polar caps, initiating pair cascades through one-photon pair creation of photons. The electric field induced by the stellar oscillations becomes therefore very important. Thus, the potential drop at the pair formation front, and the total energy gained by particles in the open field region is larger for the oscillating star while in general it becomes significantly large at some distance \( r \) from the surface, due to the fact that \( \rho \) cannot compensate \( \rho_{\text{GJ}} \). As already pointed out by Muslimov & Tsygan (1992), general relativistic terms arising from the dragging of inertial frames give very important additional contribution to this difference. These terms depend on the radial distance from the star as \( 1/r^3 \) and have important influence on the value of accelerating electric field generated in the magnetosphere near the surface of the neutron star.

(i) The oscillation regime of particle ejection from the stellar surface increases the total power carried away by relativistic primary particles relative to the purely rotating regime. Moreover, the fluctuation of the charge density of particles ejected from the stellar surface modulates the particle energy along a field line.

(ii) The energy losses along the open magnetic field lines in the polar cap region and due to toroidal oscillations are significantly larger than the rotational energy losses for the \( m = 1 \) modes of oscillation. In particular, the energy losses of the mode \( (l, m) = (2, 1) \) can be a factor of 8 larger than the rotational energy losses, even for an oscillation amplitude at the star surface as small as \( \eta = 0.05 \Omega R \).

(iii) The oscillation-induced inhomogeneity of the physical conditions at the stellar surface may substantially affect the global electrodynamics within the inner magnetosphere of a neutron star.

(iv) The new dependence obtained for the energy losses on the oscillating behaviour reflects in a new relation, namely equation (59), between the product \( P \dot{P} \) and the amplitude of the oscillation at the star surface. In cases when the moment of inertia of the star is known with good accuracy, such a relation will allow to fully appreciate the effects of oscillations on pulsar magnetospheres.

Finally, we have proposed a connection between the phenomenology of intermittent pulsars, characterized by the periodic transition from active to dead periods of radio emission in few observed sources, with the presence of an oscillating magnetosphere. In particular, we propose that, during the active state, star oscillations induced by periodic glitches of the neutron star create relativistic wind of charged particles by virtue of the additional accelerating electric field. After a timescale of the order of tens of days stellar oscillations are damped, and the pulsar shifts below the death line in the \( P-B \) diagram, thus entering the OFF invisible state of intermittent pulsars. This seminal idea, proposed here on a qualitative level, will be further explored in a future work.

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