Symmetries of deformed supersymmetric mechanics on Kähler manifolds

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Based on the systematic Hamiltonian and superfield approaches we construct the deformed $\mathcal{N} = 4, 8$ supersymmetric mechanics on Kähler manifolds interacting with constant magnetic field, and study their symmetries. At first we construct the deformed $\mathcal{N} = 4, 8$ supersymmetric Landau problem via minimal coupling of standard (undeformed) $\mathcal{N} = 4, 8$ supersymmetric free particle systems on Kähler manifold with constant magnetic field. We show that the initial “flat” supersymmetries are necessarily deformed to $SU(2|1)$ and $SU(4|1)$ supersymmetries, with the magnetic field playing the role of deformation parameter, and that the resulting systems inherit all the kinematical symmetries of the initial ones. Then we construct $SU(2|1)$ supersymmetric Kähler oscillators and find that they include, as particular cases, the harmonic oscillator models on complex Euclidian and complex projective spaces, as well as superintegrable deformations thereof, viz. $\mathbb{C}^N$-Smorodinsky-Winternitz and $\mathbb{C}P^N$-Rosochatius systems. We show that the supersymmetric extensions proposed inherit all the kinematical symmetries of the initial bosonic models. They also inherit, at least in the case of $\mathbb{C}^N$ systems, hidden (non-kinematical) symmetries. The superfield formulation of these supersymmetric systems is presented, based on the worldline $SU(2|1)$ and $SU(4|1)$ superspace formalisms.

I. INTRODUCTION

The models of supersymmetric mechanics were initially introduced as toy models for supersymmetric field theories. However, it was shortly realized that such models are of big interest on their own right. An important feature of the supersymmetric mechanics models is that the main new ingredient they bring in, the fermionic variables, after quantization become the operators representing the spin of particle. As the result, the fermionic parts of the relevant Hamiltonians play the role of generalized Pauli terms describing an interaction of spin with external fields, in particular, with the magnetic field. From this viewpoint, the study of supersymmetric extensions of the mechanical systems interacting with the magnetic field is of obvious importance. However, such systems seem not to have attracted enough attention, despite an enormous number of publications on supersymmetric mechanics.

This is rather surprising, having in mind that the first practical application of ($\mathcal{N} = 2$) supersymmetric mechanics technique was the explanation of the “accidental” double degeneracy of the spectrum of the (planar) Landau problem (see, e.g., [1]), i.e., the problem of the planar motion of non-relativistic electron (charged $\frac{e}{2}$ spin particle) in a constant magnetic field. For a long time it has been one of the central issues treated in the textbooks on quantum mechanics [2]. However, nowadays, saying “Landau problem”, people sometimes ignore the spin of the original system.

The compact (spherical) analog of the planar Landau problem is associated with a particle moving on the two-sphere in the presence of constant magnetic field generated by a Dirac monopole placed in the center of the sphere. The spherical Landau problem enjoys $SO(3)$ invariance which is also characteristic of the “free” particle on the two-sphere. The higher-dimensional generalization of this problem, a particle on $\mathbb{C}P^N$ interacting with a constant magnetic field, inherits $SU(N+1)$ invariance of the relevant free system. Quantum mechanically, the inclusion of constant magnetic field supplies the system with a degenerate ground state, which is just due to the preservation of the symmetries of a free particle. Thanks to this degeneracy, the quantum-mechanical Landau problem constitutes the basis of the theory of quantum Hall effect [3], equally as of its higher-dimensional generalizations to complex projective spaces [4].

It is more or less obvious that the inclusion of constant fields preserves the initial symmetries of the free particle moving on the generic Kähler manifold as well, and (spinless) Landau problem can be defined for any Kähler manifold. In order to restore the initial meaning of the Landau problem in the context of these systems one should try to construct supersymmetric extensions of the (spinless) Landau problem on Kähler manifold, such that they preserve...
the initial kinematical symmetries. However, in the existing literature devoted to supersymmetric extensions of the (generalized) Landau problem, the discussion of symmetry properties of the supersymmetric systems constructed is as a rule left aside (see, e.g., [3, 4]).

While for $\mathcal{N} = 2$ the construction of such supersymmetric extensions is a rather trivial task, it is not the case for $\mathcal{N} \geq 4$ supersymmetric extensions. Generically, one may pose the question:

How should systems on Kähler manifolds in interaction with a constant magnetic fields (in particular, the Landau problem) be supersymmetrized, so that their initial symmetries be preserved?

We guess that the general answer is as follows. Instead of considering $\mathcal{N}, d = 1$ Poincaré supersymmetric extensions of given bosonic systems, one should deal with superextensions based on the proper deformations of standard $d = 1$ Poincaré supersymmetry.

An attempt towards proving this conjecture was performed years ago in [7], where it was observed that the oscillator and the Landau problem on a complex projective space admit the deformed $\mathcal{N} = 4$ supersymmetric extension (later on called “weak $\mathcal{N} = 4$ supersymmetric extension” [8]), which preserves the initial kinematical symmetries of those systems. Departing from this model, the class of systems with non-zero potentials called “Kähler oscillator” was introduced [7, 8], such that they admit similar deformed supersymmetric extensions respecting the inclusion of constant magnetic field. The relevant bosonic Hamiltonian reads

$$H_{osc} = g^{ab} \left( \bar{\pi}_a \pi_b + |\omega|^2 \partial_a K \partial_b K \right),$$

(1)

where $K(z, \bar{z})$ is the Kähler potential.

A few years later, the one-dimensional version of that Kähler superoscillator model was re-derived within a $d = 1$ superfield formalism based on $SU(2|1)$ superalgebra treated as a deformation of $\mathcal{N} = 4, d = 1$ Poincaré superalgebra [10, 11]. Thereby, the “weak $\mathcal{N} = 4$ supersymmetry” was identified with $su(2|1)$ superalgebra (this fact was also independently noticed in the paper [12] treating supersymmetric quantum Landau problem on $\mathbb{C}P^1$). Using similar techniques, the deformed $\mathcal{N} = 8$ one-dimensional Landau problem associated with $su(4|1)$ superalgebra was also defined [13]. This study was to large extent inspired by the activity on building field-theoretical models with the “rigid supersymmetry on curved superspaces” initiated in [14].

Having in mind the “practical importance” of supersymmetrization respecting symmetries of initial system and field-theoretical importance of ”curved superspace approach”, we present here the systematic approach to the deformed supersymmetrization of the systems on Kähler manifolds interacting with the constant magnetic field

Having in mind the “practical importance” of supersymmetrization respecting symmetries of the initial bosonic system and the field-theoretical importance of the “curved superspace approach”, we develop here the systematic approach to the deformed supersymmetrization of systems “living” on Kähler manifolds and interacting with a constant magnetic field by the use of a supersymmetric analog of a minimal coupling. In the superfield formulations, such a coupling naturally comes out under some minimal choice of the related superfield Lagrangians.

Resorting first to the Hamiltonian formalism, we construct in this way the $SU(2|1)$ supersymmetric extensions of the Kähler oscillator (and of the Landau problem) on the generic Kähler space, as well as the $SU(4|1)$ supersymmetric Landau problem on the special Kähler manifolds of the rigid type (that is a Kähler manifold equipped with the holomorphic symmetric tensor of the third rank obeying some compatibility condition [15]). We show that this approach perfectly matches with the requirement that the supersymmetric Landau problem exhibits all the kinematical symmetries of the original system and involves the appropriate spin interaction. It is demonstrated that both $SU(2|1)$ and $SU(4|1)$ supersymmetric Landau problems inherit all the kinematical symmetries of the initial systems. Requiring the Hamiltonian in the $SU(2|1)$ case to commute with all supercharges amounts to adding the appropriate Zeeman term to it. In the superspace language, this means that we should start from the properly central-charge extended superalgebra, with the Hamiltonian being identified with the relevant central charge. Analogously, the general $SU(2|1)$ Kähler superoscillator systems as superextensions of those with the Hamiltonian [11] can be constructed and then reproduced from the superfield approach.

Exemplifying the general analysis, we set up and study $SU(2|1)$ supersymmetric extensions of the following particular superintegrable Kähler oscillator models:

- $\mathbb{C}^N$-oscillator (the sum of $N$ two-dimensional isotropic oscillators);
- $\mathbb{C}^N$-Smorodinsky-Winternitz system (the sum of $N$ copies of two-dimensional isotropic oscillators deformed by ring-shaped potentials) [10];
- $\mathbb{C}P^N$-oscillator [7, 17], i.e. the $\mathbb{C}P^N$- counterpart of $\mathbb{C}^N$-oscillator;
• CP\(^N\)–Rosochatius system \([18]\), i.e. the CP\(^N\)– counterpart of \(CN\)-Smorodinsky-Winternitz system.

We show that these models also inherit all the kinematical symmetries of the initial systems. In addition, we find the explicit expressions for the superanalogs of the hidden symmetry generators of \(CN\)-oscillator and \(CN\)-Smorodinsky-Winternitz system (i.e., of the Fradkin and Uhlenbeck tensors). Unfortunately, we were not yet able to find the superanalogs of such hidden symmetry generators for the CP\(^N\)-oscillator and of the CP\(^N\)-Rosochatius system, though they hopefully exist.

The paper is organized as follows:

In Section 2 we describe the phase superspace as a proper setting for supersymmetrization of systems on Kähler manifolds in interaction with a constant magnetic field. The Legendre transformation relating Hamiltonian and Lagrangian formulations of those systems is given. In Section 3 we present the Hamiltonian formulations of \(SU(2|1)\) and \(SU(4|1)\) supersymmetric Landau problems. The general Hamiltonian formulation of \(SU(2|1)\) Kähler superoscillator is described in Section 4. As an example, we show that this class of Hamiltonians incorporates the supersymmetric version of two-dimensional anisotropic oscillator. In Section 5 the previously considered systems are recovered within the manifestly \(SU(2|1)\) and \(SU(4|1)\) covariant off-shell superfield approaches. Section 6 is devoted to a more detailed discussion of the \(SU(2|1)\) supersymmetric extensions of the oscillator-like systems on \(CN\) and CP\(^N\) that are listed above and to the study of their symmetries.

II. PHASE SUPERSPACE, KINEMATICAL SYMMETRIES, AND LAGRANGIANS

The Kähler manifold \(M\) is the Hermitian manifold with the Hermitian metrics, \(ds^2 = g_{ab}dz^a \wedge d\bar{z}^b\), which also defines the symplectic structure

\[
\omega_M = ig_{ab}dz^a \wedge d\bar{z}^b, \quad dw_M = 0 \quad \Rightarrow \quad g_{ab} = \partial_a \partial_b K, \quad g_{ab} = \frac{\partial}{\partial z^a}, \quad \frac{\partial}{\partial \bar{z}^b},
\]

where the real function \(K(z, \bar{z})\), Kähler potential, is defined up to holomorphic and antiholomorphic functions:

\(K(z, \bar{z}) \rightarrow K(z, \bar{z}) + U(z) + \bar{U}(\bar{z})\).

The Kähler manifold can be equipped with the Poisson brackets associated with the above symplectic structure

\[
\{f, g\}_M = ig^{\bar{a}b} \left( \frac{\partial f}{\partial z^a} \frac{\partial g}{\partial \bar{z}^b} - \frac{\partial f}{\partial \bar{z}^b} \frac{\partial g}{\partial z^a} \right) , \quad g^{\bar{a}b}g_{bc} = \delta^\alpha_c .
\]

Therefore, the isometries of Kähler structure should both preserve both complex and symplectic structures, i.e., they are generated by the holomorphic Hamiltonian vector fields,

\[
V_\mu = \{h_\mu, \}\_M = V_\mu^a(z) \frac{\partial}{\partial z^a} + V_\mu^{\bar{a}}(\bar{z}) \frac{\partial}{\partial \bar{z}^{\bar{a}}}, \quad V_\mu^a = ig^{\bar{b}a} \partial_{\bar{b}} h_\mu (z, \bar{z}) , \quad V_\mu^{\bar{a}} = \bar{V}_\mu^\alpha ,
\]

where the real function \(h_\mu(z, \bar{z})\) is a momentum map sometimes called Killing potential. The holomorphicity of the vector field yields the following equation to the Killing potential

\[
\frac{\partial^2 h_\mu}{\partial z^a \partial \bar{z}^b} - \Gamma_{ab}^c \frac{\partial h_\mu}{\partial \bar{z}^c} = 0 ,
\]

with \(\Gamma_{ab} = g^{cd} \partial_a g_{bd}\). \(^1\) The same result can be obtained by the direct solving of the Killing equations

\[(a) \ V_{\mu a; b} + V_{\mu b; a} = 0 , \quad (b) \ V_{\mu a; b} + V_{\mu b; a} = 0 , \quad \text{with} \quad V_{\mu a} = g_{ab} V_\mu^b .\]

The action of the vector field \(V_\mu\) on an arbitrary function \(f(z, \bar{z})\) can be expressed through the Poisson bracket with the Killing potential

\[
V_\mu \ f = \{h_\mu, f\}_M .
\]

\(^1\) The only non-vanishing components of the Christoffel symbol in the Kähler geometry are \(\Gamma_{ab}^c\) and \(\Gamma_{ab}^\alpha = g^{bc} \partial_a g_{\alpha \bar{b}}\).
Thus, the requirement that the vector fields $V_\mu$ form Lie algebra amounts to the same Lie algebra relations for Killing potentials

$$[V_\mu, V_\nu] = C^\lambda_{\mu\nu} V_\lambda,$$

where the constant term either corresponds to co-circle in that Lie algebra or can be absorbed by the appropriate constant shift of Killing potentials.

Let us consider the electrically charged particle moving on a Kähler manifold and interacting with the constant magnetic field of strength $B$, i.e. the $U(1)$-Landau problem on Kähler manifold. For this aim we equip the cotangent bundle of the Kähler manifold with the following symplectic structure and Hamiltonian

$$\omega_B = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a - iBg_{ab}dz^a \wedge d\bar{z}^b, \quad H_0 = g^{ab}\pi_a\bar{\pi}_b.$$

The corresponding Poisson brackets are given by

$$\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \bar{\pi}_b\} = iBg_{ab}.$$

The isometries of a Kähler structure discussed earlier define the Noether constants of motion

$$J_\mu = V^a_\mu \pi_a + \dot{\bar{\pi}}_{\bar{a}} - B\eta_\mu (z, \bar{z}), \quad V^a_\mu = ig^{ba}\partial_\mu \eta_a(z, \bar{z}) : \begin{cases} \{H_0, J_\mu\} = 0 \\ \{J_\mu, J_\nu\} = C^\lambda_{\mu\nu} J_\lambda \end{cases},$$

where the brackets $\{\cdot, \cdot\}_B$ are calculated according to (8). Notice that the vector fields generated by $J_\mu$ are independent of $B$,

$$\tilde{V}_\mu = \{J_\mu, \cdot\}_B = V^a_\mu(z) \frac{\partial}{\partial z^a} - V^a_{\mu,b} \pi_a \frac{\partial}{\partial \bar{\pi}_b} + \text{c.c.}.$$

Hence, coupling to a constant magnetic field preserves the whole symmetry algebra of a free particle moving on a Kähler manifold. This implies that the Landau problem can be properly defined on any Kähler manifold.

To construct fermionic extensions of the systems on Kähler manifolds interacting with constant magnetic field we define the $(2N|MN)_C$-dimensional phase superspace equipped with the symplectic structure

$$\Omega = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a - i(Bg_{ab} - R_{abcd}\eta^{ac}\eta^{bd})dz^a \wedge d\bar{z}^b + ig_{ab}\eta^{ac}\eta^{bd} \wedge d\bar{\pi}_a \wedge d\bar{\pi}_b,$$

where $a = 1, \ldots, M$ are spinorial indices, $D\eta^{ac} = D\eta^{ba} + \Gamma^{ac}_{be}\eta^{ba}dz^e$, and $\Gamma^{ac}_{be}, R_{abcd} = g_{eb}(\Gamma^c_{ac})_d$ are, respectively, the components of connection and curvature of the Kähler structure.

The Poisson brackets corresponding to the symplectic structure (11) amount to the relations

$$\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \eta^{ba}\} = -\Gamma^{b}_{ac}\eta^{ca}, \quad \{\pi_a, \bar{\pi}_b\} = i(Bg_{ab} - R_{abcd}\eta^{ac}\eta^{bd}), \quad \{\eta^{ac}, \eta^{bd}\} = ig^{ab}\delta_3$$

and their complex conjugates. They induce the following generic Poisson bracket for the functions on the phase superspace

$$\{f, g\} = \frac{\partial f}{\partial \pi_a} \wedge \nabla_a g + \frac{\partial f}{\partial \bar{\pi}_a} \wedge \nabla_a g + i(Bg_{ab} - R_{abcd}\eta^{ac}\eta^{bd}) \frac{\partial f}{\partial \pi_a} \wedge \frac{\partial g}{\partial \bar{\pi}_b} + ig^{ab} \left( \frac{\partial f}{\partial \eta^{ca}} \wedge \frac{\partial g}{\partial \eta^{ba}} \right),$$

where $A \wedge B = AB - (-1)^{p(A)p(B)}BA$ and

$$\nabla_a \equiv \frac{\partial}{\partial z^a} - \Gamma^{b}_{ac}\eta^{ca} \frac{\partial}{\partial \eta^{ba}}.$$

The extended symplectic structure (11) and Poisson brackets (13) are manifestly covariant with respect to the transformation

$$z^a = \tilde{z}^a(z), \quad \tilde{\pi}_a = \frac{\partial z^b}{\partial \tilde{z}^a} \pi_b, \quad \bar{\eta}^{ca} = \frac{\partial \tilde{z}^a}{\partial \bar{\eta}^{cb}} \eta^{ba}.$$

Hence, we can lift the isometries (10) to the whole phase superspace and define the respective super-Hamiltonian vector fields as

$$V_\mu = \{J_\mu, \cdot\} = V^a_\mu(z) \frac{\partial}{\partial z^a} - V^a_{\mu,b} \pi_a \frac{\partial}{\partial \bar{\pi}_b} + V^a_{\mu,c} \eta^{ba} \frac{\partial}{\partial \eta^{ca}} + \text{c.c.},$$

where $\{\cdot, \cdot\}_B$ are calculated according to (8). Notice that the vector fields generated by $J_\mu$ are independent of $B$. 
where
\[ J_\mu = J_\mu + \frac{\partial^2 h}{\partial z^c \partial \bar{z}^d} \eta^{ca} \bar{\eta}_d, \] (17)
with \( J_\mu \) defined by (9).

Note that the symplectic structure [11] can be represented as a locally exact one-form,
\[ \Omega = dA \quad A = \pi_a dz^a + \bar{\pi}_a d\bar{z}^a + i \frac{B}{2} (\partial_a K dz^a - \partial_{\bar{a}} K d\bar{z}^{\bar{a}}) + \frac{i}{2} g_{ab} (\eta^{ca} D_t \bar{\eta}_a^b + \bar{\eta}_a^b D_t \eta^{ca}). \] (18)

Then, by the Hamiltonian
\[ H = g^{ab} \pi_a \pi_b + U(z, \bar{z}, \eta, \bar{\eta}), \] (19)
where the potential term \( U(z, \bar{z}, \eta, \bar{\eta}) \) will be defined later for each specific system, we can immediately write down the first-order-Lagrangian with the action
\[ S = \int A - H dt. \] (20)

Eliminating cyclic variables \( \pi_a, \bar{\pi}_a \), we arrive at the second-order Lagrangian
\[ \mathcal{L} = g_{ab} \frac{dz^a \cdot \cdot \cdot}{\cdot \cdot \cdot} + i \frac{B}{2} (\partial_a K \cdot \cdot \cdot - \partial_{\bar{a}} K \cdot \cdot \cdot) + i \frac{1}{2} g_{ab} (\eta^{ca} \cdot \cdot \cdot \bar{\eta}_a^b + \bar{\eta}_a^b D_t \cdot \cdot \cdot \eta^{ca}) - \mathcal{U}(z, \bar{z}, \eta, \bar{\eta}) \] with \( D_t \eta^a = \eta^a_t + \Gamma_{bc}^a \eta^b \cdot \cdot \cdot \bar{\eta}_c \cdot \cdot \cdot \). (21)

Now we can re-derive (and so check) all the previous formulas by applying the standard Legendre transformation just to this Lagrangian. We define the canonical bosonic momenta
\[ P_a := \frac{\partial \mathcal{L}}{\partial \dot{z}^a} = g_{ab} \cdot \cdot \cdot + i \frac{B}{2} \partial_b K \cdot \cdot \cdot - i \frac{1}{2} \partial_c g_{ab} (\eta^{ca} \cdot \cdot \cdot \bar{\eta}_a^b), \quad P_a := \frac{\partial \mathcal{L}}{\partial \dot{\bar{z}}^a} = \bar{z}^b g_{ab} - i \frac{B}{2} \partial_b K + i \frac{1}{2} \partial_c g_{ab} (\eta^{ca} \cdot \cdot \cdot \bar{\eta}_a^b), \] (22)
and the canonical fermionic ones
\[ P_{a\alpha} := \frac{\partial \mathcal{L}}{\partial \dot{\eta}^a} = i \frac{1}{2} g_{ab} \eta^b, \quad P_{\bar{a}\alpha} := \frac{\partial \mathcal{L}}{\partial \dot{\bar{\eta}}^a} = i \frac{1}{2} g_{ab} \eta^b. \] (23)

The above expressions indicate the appearance of second-class constraints
\[ \phi_{a\alpha} = P_{a\alpha} - i \frac{1}{2} g_{ab} \cdot \cdot \cdot \bar{\eta}_a^b \approx 0, \quad \phi_{\bar{a}\alpha} = P_{\bar{a}\alpha} - i \frac{1}{2} g_{ab} \eta^b \approx 0. \] (24)

Thus, for the Hamiltonian formulation we need to eliminate these constraints in accordance with the Dirac’s method. The standard procedure yields the following non-vanishing Dirac brackets (and their c.c.)
\[ \{ P_a, \cdot \cdot \cdot \} = \delta_a^b, \quad \{ P_a, \eta^{b\alpha} \} = -\frac{1}{2} g^{ab} \eta \eta^{\alpha}, \quad \{ P_a, \bar{\eta}^a \} = -\frac{1}{2} \partial_c g_{cd} \bar{\eta} \eta^d, \quad \{ \eta^{a\beta}, \bar{\eta}^a \} = i g^{ab} \delta^\beta_\alpha, \quad \{ P_a, P_b \} = -\frac{1}{4} \left[ \partial_a g_{cd} \partial_b g_{ef} - (a \leftrightarrow b) \right] g^{ce} (\eta^{f\alpha} \bar{\eta}_d^a), \quad \{ P_a, P_b \} = \frac{1}{4} \left[ \partial_a g_{cd} \partial_b g_{ef} - (a \leftrightarrow b) \right] g^{ce} (\eta^{f\alpha} \bar{\eta}_d^a). \] (25)

Introducing the non-canonical bosonic momenta \( \pi_a = g_{ab} \cdot \cdot \cdot \bar{\eta}_a^b, \bar{\pi}_a = \cdot \cdot \cdot \bar{z}^b g_{ab} \) and taking into account the relations between the momenta \( P_a, P_b, \pi_a, \bar{\pi}_a \) in (22) it is straightforward to recover the brackets involving \( \pi_a, \bar{\pi}_a \) and defined earlier in eqs. (12). In particular, it is easy to show that \( \{ \pi_a, \pi_b \} = \{ \bar{\pi}_a, \bar{\pi}_b \} = 0 \). It is also straightforward, applying the Noether procedure directly to (21) and assuming that the potential term \( \mathcal{U} \) is invariant, to reproduce the conserved isometry current \( J_\mu \) defined in (17). With all these ingredients at hand, we are prepared to turn to supersymmetrizing the Landau problem on Kähler manifold.

### III. SUPERSYMMETRIC LANDAU PROBLEM

To define the (deformed) \( \mathcal{N} = 2M \) supersymmetric extension of Landau problem (i.e. of the free particle interacting with a constant magnetic field) we make use of the strategy similar to symplectic coupling in the pure bosonic case.
The starting point is some supersymmetric Hamiltonian system supplied by supercharges $Q^\alpha$ and $\overline{Q}_\alpha$ which close on a Hamiltonian $\mathcal{H}_0$,

$$\{Q^\alpha, Q^\beta\}_0 = \{\overline{Q}_\alpha, \overline{Q}_\beta\}_0 = 0, \quad \{Q^\alpha, \overline{Q}_\beta\}_0 = i\delta^\alpha_\beta \mathcal{H}_0, \quad \{Q^\alpha, \mathcal{H}_0\}_0 = \{\overline{Q}_\alpha, \mathcal{H}_0\}_0 = 0,$$

(26)

where the Poisson brackets are defined by [12] with zero magnetic field, $B = 0$.

To introduce interaction with an external magnetic field, we deform the supersymplectic structure, still preserving the form of the supercharges: $\langle \Omega_B = 0 \rangle \to \langle \Omega_B, Q^\alpha, \overline{Q}_\alpha \rangle$. Now, the graded Poisson bracket $\{\cdot, \cdot\}$ is defined through the symplectic form $\Omega_B$ defined in [11], and one has to check whether the supersymmetry algebra (26) remains unaltered.

If this is the case, then the Hamiltonian can be defined as $\mathcal{H}_0 := \frac{\mathcal{H}_0}{\pi} \{Q^\alpha, \overline{Q}_\alpha\}$. Otherwise we end up with some deformed superalgebra which is different from the standard $d = 1, \mathcal{N} = 2M$ super Poincaré algebra [20], and we have to select there the generator admitting an interpretation as the appropriate Hamiltonian, i.e.

$$\{Q^\alpha, Q^\beta\} = 0 + iB \ldots, \quad \{Q^\alpha, \overline{Q}_\beta\} = i\delta^\alpha_\beta \mathcal{H}_0 + iB \ldots$$

(27)

Here, dots stand for some possible extra generators, which should be further commuted with supercharges and among themselves in order to obtain a closed superalgebra.

Below we will show that this program works perfectly well for the cases of (deformed) $\mathcal{N} = 4, 8$ supersymmetric Landau problems.

### A. The $SU(2|1)$ (deformed $\mathcal{N} = 4$) supersymmetric Landau Problem

In order to set up $\mathcal{N} = 4$ Landau problem we choose the standard “chiral” supercharges $Q^\alpha, \overline{Q}_\alpha (\alpha = 1, 2)$ with the same ansatz for them as in the absence of magnetic field, and introduce the charges generating the $SU(2)$ $R$-symmetry

$$Q^\alpha = \pi_a \eta^{\alpha a}, \quad \overline{Q}_\alpha = \overline{\pi}_a \overline{\eta}^a, \quad R^a_\beta = g_{ab} \eta^{\alpha a} \eta^{b}_\beta - \frac{1}{2} g^{\alpha \gamma} g_{ab} \eta^{b \gamma} \overline{\eta}^a_\gamma.$$  

(28)

The closure of their Poisson brackets yields the superalgebra

$$\{Q^\alpha, Q^\beta\} = 0, \quad \{R^a_\beta, R^b_\gamma\} = -i\delta^a_\gamma R^b_\beta + i\delta^a_\beta R^b_\gamma, \quad \{Q^\alpha, R^a_\beta\} = i\delta^\alpha_\beta Q^a - \frac{1}{2} \delta^a_\beta Q^a,$$

$$\{Q^\alpha, \overline{Q}_\beta\} = i\delta^\alpha_\beta \mathcal{H}_0 + iB R^a_\beta, \quad \{Q^\alpha, \mathcal{H}_0\} = i\frac{B}{2} Q^\alpha, \quad \{R^a_\beta, \mathcal{H}_0\} = 0,$$

(29)

where

$$\mathcal{H}_0 = g^{\bar{a}b} \pi_a \overline{\pi}_b - \frac{1}{2} g_{abcd} \eta^{\alpha a} \eta^{\alpha b} \eta^{\gamma c} \eta^{\beta d} + \frac{B}{2} g_{ab} \eta^{\alpha a} \eta^{\gamma b}.$$  

(30)

Extending the set [20] by the generator [30] we arrive at the $su(2|1)$ superalgebra (or “weak $\mathcal{N} = 4$ superalgebra” in the terminology of [8]). We observe, however, that the supercharges do not commute with the Hamiltonian. This drawback can be remedied via the appropriate modification of the Hamiltonian:

$$\tilde{\mathcal{H}}_0 = \mathcal{H}_0 - \frac{B}{2} g_{ab} \eta^{\alpha a} \eta^{\gamma b} \eta^{\alpha b} \eta^{\gamma a} - \frac{1}{2} g_{abcd} \eta^{\alpha a} \eta^{\alpha b} \eta^{\gamma c} \eta^{\beta d} + B g_{ab} \eta^{\alpha a} \eta^{\alpha b} : \{Q^\alpha, \tilde{\mathcal{H}}_0\} = 0.$$  

(31)

The last term in the Hamiltonians [30], [31] is obviously Zeeman term describing interaction of spin with an external magnetic field. From the mathematical point of view, the shift in [31] is the new $R$-symmetry $U(1)$ generator $R = \frac{1}{2} g_{ab} \eta^{\alpha a} \eta^{\gamma b}$. It extends $SU(2)$ $R$-symmetry generated by $R^a_\beta$ to $U(2)$ $R$-symmetry. Since $\tilde{\mathcal{H}}_0$ commutes with all other generators of the extended superalgebra, it can be interpreted as the central charge generator promoting the standard $su(2|1)$ superalgebra to its central extension $\tilde{su}(2|1)$ [11].

All the generators of $su(2|1)$ superalgebra (and of its central extension) are manifestly invariant under the action of the isometry current [17]:

$$\{Q^\alpha, J_\mu\} = \{\overline{Q}_\alpha, J_\mu\} = \{R^a_\beta, J_\mu\} = \{\mathcal{H}_0, J_\mu\} = 0.$$

(32)

This means that the supersymmetric system constructed inherits all the kinematical symmetries of the initial system. In particular, in the case of $\mathbb{C}P^N$-Landau problem the extended system respects $SU(N + 1)$ symmetry.

Thus we have accomplished the well defined “weak $\mathcal{N} = 4$ supersymmetrization” of the Landau problem on a generic Kähler manifold and found that its supersymmetry algebra is $\tilde{su}(2|1)$. 
Finally, it is straightforward to write down the Lagrangian corresponding to (33),
\[
\mathcal{L}_0 = g_{ab} \dot{z}^a \dot{z}^b + \frac{B}{2} (\partial_a K \dot{z}^a - \partial_a \dot{z}^a) + \frac{i}{2} g_{ab} (\eta^{\alpha 
abla_0 D_{\alpha} \bar{\eta}_{\beta}^a + \bar{\eta}_{\beta}^a D_{\alpha} \eta_{\alpha}) + \frac{1}{2} R_{abcd} \eta^{\alpha a} \bar{\eta}_{\beta}^b \eta^{\beta d} \bar{\eta}_{\gamma} - \frac{B}{2} g_{ab} \eta^{\alpha a} \bar{\eta}_{\beta}^b. (33)
\]

The Lagrangian corresponding to the shifted Hamiltonian \([31]\) is obviously \(\tilde{\mathcal{L}}_0 = \mathcal{L}_0 - \frac{1}{2} B g_{ab} \eta^{\alpha a} \bar{\eta}_{\beta}^b\). These Lagrangians provide a higher-dimensional generalization of those constructed in \([19, 10]\), using the superfield techniques. The superfield derivation of (33) will be given in Sect. VI. The relevant \(SU(2|1)\) off-shell multiplet content is \(N\) chiral multiplets \((2, 4, 2)\). Note that the Lagrangian and Hamiltonian \(\mathcal{L}_0\) and \(\mathcal{H}_0\) coincide with the previously derived general expressions \([21]\) and \([19]\) for \(\alpha = 1, 2\) and the choice \(\mathcal{U} = \frac{1}{2} R_{abcd} \eta^{\alpha a} \bar{\eta}_{\beta}^b \eta^{\beta d} \bar{\eta}_{\gamma} - B g_{ab} \eta^{\alpha a} \bar{\eta}_{\beta}^b\).

**B. \(SU(4|1)\) (deformed \(N = 8\)) supersymmetric Landau problem**

In the previous subsection we considered the coupling of \(N = 4\) supersymmetric particle on Kähler manifold to a constant magnetic field and showed that the resulting system yields the deformed \(SU(2|1)\) supersymmetric Landau problem and that the latter inherits the whole isometry group of the original system. Now we perform a similar construction for \(N = 8\) supersymmetric mechanics on the special Kähler manifolds of the rigid type \([20]\).

The special Kähler manifold of the rigid type is the Kähler manifold equipped with the symmetric tensor \(f_{abc} dz^a dz^b dz^c\) and its complex conjugate which obey the following compatibility conditions:
\[
\frac{\partial}{\partial z^d} f_{abc} = 0, \quad f_{abc; d} = f_{ab; c d}, \quad R_{abcd} = -\tilde{f}_{abcd} g^{am} f_{mac}, (34)
\]
where \(f_{abc; d} = f_{abc, d} - \Gamma_{da} f_{bce} - \Gamma_{db} f_{ace} - \Gamma_{dc} f_{abe}\) is the covariant derivative of the third-rank covariant tensor. The special Kähler manifolds of the rigid type are widely known because of their close relevance to T-duality that relates the UV and IR limits of \(N = 2, d = 4\) super Yang-Mills theory \([21]\).

To construct the relevant supersymmetric Landau problem we choose the symplectic structure \([11]\) and Poisson brackets \([13]\) with \(su(4)\) spinor indices \(\alpha, \beta = 1, \ldots, 4\). To avoid possible confusion, we relabel them by the capital Latin letters \(I, J, K, L\). With this notation, the “flat” \(N = 8\) supersymmetry algebra reads
\[
\{Q^I, Q^J\} = \{\overline{Q}_I, \overline{Q}_J\} = 0, \quad \{Q^I, \overline{Q}_J\} = i \delta^I_J \mathcal{H}_{\text{SY}}. (35)
\]

Following \([20]\) we define the supercharges as
\[
Q^I = \pi_a \eta^{aI} + \frac{i}{3} f_{abc} \bar{T}^{abc} J, \quad \overline{Q}_I = \bar{\pi}_a \bar{\eta}^{aI} + \frac{i}{3} f_{abc} \bar{T}^{abc} J, \quad T^{abc} = \frac{1}{2} \bar{\epsilon}_{IJKL} \eta^{aJ} \eta^{bK} \eta^{cL}, (36)
\]
where the symmetric tensor \(f_{abc}\) obeys the relations \([31]\). 2 Also, we introduce the following deformation of the Poisson brackets used in \([20]\):
\[
\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \eta^b\} = -\Gamma^b_{ac} \eta^c, \quad \{\pi_a, \bar{\pi}_b\} = i(B g_{ab} - R_{abcd} \eta^c \eta^d), \quad \{\eta^a, \bar{\eta}^b\} = i g^{ab} \delta^I_J. (37)
\]
Then we can construct \(R\)-symmetry charges forming \(su(4)\) algebra by the same relations as in the undeformed case,
\[
R^I_J = \eta^a g_{ab} \bar{\eta}^b_{J} - \frac{\delta^I_J}{4} \eta^{aK} g_{ab} \bar{\eta}^b_{K}, \quad \{R^I_J, R^K_L\} = i \left( \delta^I_K R^J_L - \delta^J_K R^I_L \right). (38)
\]

Calculating the modified Poisson brackets between the supercharges and \(R\)-charges, we arrive at the generators \(\mathcal{H}_{\text{SY}}, Q^I, R^I_J\) which form the superalgebra \(su(4|1)\)
\[
\{Q^I, Q^J\} = \{\overline{Q}_I, \overline{Q}_J\} = 0, \quad \{Q^I, \overline{Q}_J\} = i \delta^I_J \mathcal{H}_0 + i B R^I_J, \quad \{R^I_J, Q^K\} = i \delta^K_J Q^I - \frac{4}{3} \delta^I_J Q^K, \quad \{\mathcal{H}_0, Q^K\} = -\frac{3B}{4} Q^K. (39)
\]

---

2 Here we introduced the antisymmetric symbol \(\epsilon^{IJKL}\) satisfying the following identities:
\[
\epsilon^{1234} = \epsilon^{1234} = 1, \quad \epsilon^{IJKL} \epsilon_{JIKL} = 24, \quad \epsilon^{IJLK} \epsilon_{IJKM} = 6 \delta^L_M, \quad \epsilon^{IJLK} \epsilon_{IJMN} = 2 \left( \delta^K_M \delta^L_N - \delta^K_N \delta^L_M \right), \quad \epsilon^{IJLK} \epsilon_{IMNP} = \delta^L_M \delta^J_N \delta^K_P - \delta^L_M \delta^K_P \delta^J_N + \delta^K_N \delta^K_P \delta^L_M - \delta^K_N \delta^L_M \delta^K_P - \delta^K_P \delta^L_M \delta^K_N.
\]
The highest-degree monomial of the Grassmann variables can be represented as \(\psi^I \psi^J \psi^K \psi^L = \frac{1}{24} \epsilon^{IJKL} \epsilon_{MNPR} \psi^M \psi^N \psi^P \psi^R\).
Here,
\[ H_0 = g^{ab} \bar{\pi}_a \pi_b + R_{abcd} \Lambda_0^{abcd} + \frac{B}{4} \eta^{aK} g_{ab} \eta_b^K - \frac{1}{3} f_{abcd} \Delta^{abcd} - \frac{1}{3} \bar{f}_{abcd} \bar{\Delta}^{abcd}, \]
where, as before, \( f_{abcd} \) is the covariant derivative of the third-rank covariant symmetric tensor, and
\[ \Lambda^{abcd} := -\frac{1}{8} \epsilon_{ijkl} \eta^{aj} \eta^{bj} \eta^{cK} \eta^{dL}, \quad \Lambda_0^{abcd} := \frac{1}{2} \eta^{aj} \eta^{bj} \eta^{K} \eta^{d}. \]
We observe that the inclusion of constant magnetic field \( B \) deforms \( \mathcal{N} = 8, d = 1 \) Poincaré superalgebra to the \( su(4|1) \) superalgebra.

Let us require that the isometry of Kähler structure given by the vector field \( \mathbf{V}_\mu \) preserves as well the third-order tensor \( f_{abc}dz^adz^bdz^c \), i.e. that the Lie derivative of the latter along this field equals to zero:
\[ \mathbf{L}_{\mathbf{V}_\mu} f_{abc}dz^adz^bdz^c = 0 \iff 3V^{\nu}_{\mu,(\beta} f_{\gamma\delta\lambda)\nu} + V^{\nu}_{\mu} f_{abc,\nu} = 0. \]
Using these additional relations, one can check that the isometry generator \( \mathbf{J}_\mu \) commutes with all elements of \( SU(4|1) \) superalgebra:
\[ \{ \mathbf{J}_\mu, Q_I \} = \{ \mathbf{J}_\mu, \bar{Q}_{\bar{I}} \} = \{ \mathbf{J}_\mu, \mathcal{H}_{\text{Lan}} \} = 0. \]
Thus we managed to define the consistent \( SU(4|1) \) Landau problem on special Kähler manifolds of the rigid type.

In contrast to \( SU(2|1) \) Landau problem we cannot bring the Hamiltonian to the form in which it commutes with the supercharges, except for the trivial case \( f_{abc} = 0 \).

Finally, taking into account the correspondence \[21], we can write the expression for the relevant Lagrangian
\[ \mathcal{L}_0 = g^{ab} \bar{z}_a \dot{z}_b + \frac{i}{2} \left( \partial_a K \bar{z}_b - \partial_b K \bar{z}_a \right) + \frac{i}{2} g^{ab} (\eta^{aj} \bar{D}_a \bar{\eta}^b_j + \bar{\eta}^a_j D_b \eta^{aj}) - \frac{B}{4} \eta^{aK} g_{ab} \bar{\eta}_b^K \]
\[ + \frac{1}{3} (f_{abcd} \Delta^{abcd} + \bar{f}_{abcd} \bar{\Delta}^{abcd}) + f_{abc} g^{cc'} \bar{f}_{cc'd} \partial_{c'} \Lambda^{abc}. \]

The re-derivation of this Lagrangian from the appropriate off-shell \( SU(4|1) \) superfield formalism is given in Sect. \[17] where the conditions \[43] are resolved, in the special coordinate frame, through the single holomorphic function \( \mathcal{F}(z) \) known as Seiberg-Witten prepotential:
\[ g_{ab} = \frac{\partial^2 \mathcal{F}(z)}{\partial z^a \partial z^b} + c.c., \quad \Gamma_{abc} = \frac{\partial^3 \mathcal{F}(z)}{\partial z^a \partial z^b \partial z^c}, \quad f_{abc} = e^{i\omega} \frac{\partial^3 \mathcal{F}(z)}{\partial z^a \partial z^b \partial z^c}. \]
Clearly, the function \( \mathcal{F}(z) \) is defined up to redefinition
\[ \mathcal{F}(z) \rightarrow \mathcal{F}(z) + i c_{ab} z^a z^b + c a z^a + c, \]
where \( c_a, c \) being arbitrary complex constants, and \( c_{ab} \) are real ones, \( \bar{c}_{ab} = c_{ab} \).

The corresponding Kähler potential is given by the expression
\[ K(z, \bar{z}) = z^a \frac{\partial \mathcal{F}(z)}{\partial z^a} + \bar{z}^a \frac{\partial \mathcal{F}(\bar{z})}{\partial \bar{z}^a}. \]
In these coordinates, the T-duality transformation is realized as follows \[21]
\[ (z^a, \mathcal{F}(z)) \rightarrow \left( u_a = \frac{\partial \mathcal{F}(u)}{\partial z^a}, \mathcal{F}(u) \right), \quad \text{where} \quad \frac{\partial^2 \mathcal{F}(u)}{\partial u_a \partial u_c} \frac{\partial \mathcal{F}(u)}{\partial z^b} = -\delta^a_b, \quad \mathcal{F}(u) = (u_a z^a - \mathcal{F}(z))|_{u_a = \partial_a \mathcal{F}(z)}. \]

\[ \text{IV. SU(2|1) KÄHLER SUPEROSCILLATOR} \]

The Kähler oscillator is defined by the symplectic structure \[17\] and the Hamiltonian \[8\]
\[ H_{osc} = g^{ab} \left( \bar{\pi}_a \pi_b + |\omega|^2 \partial_b K \partial_b K \right), \]
where $K(z, \bar{z})$ is the Kähler potential.

This system is distinguished in that it is "friendly" to supersymmetrization: the addition of the potential $\Theta$ amounts to minor changes in the procedure of $SU(2|1)$ supersymmetrization of the Landau problem described in the previous section. Namely, we can preserve the form \[\Theta = \pi_a \eta^{a\alpha} + i \bar{\omega} \bar{\pi}_a K \epsilon^{a\alpha} \bar{\eta}_a \] of $SU(2)$ $R$-charges and adopt the following slightly modified expressions for the supercharges \[\Theta^\alpha = \pi_a \eta^{a\alpha} + i \bar{\omega} \bar{\pi}_a K \epsilon^{a\alpha} \bar{\eta}_a, \quad \Theta^\alpha = \pi_a \eta^{a\alpha} + i \omega \partial_a K \epsilon_{a\alpha} \eta^\beta. \] 

Calculating their Poisson brackets, we obtain

\[\{\Theta^\alpha, \Theta^\beta\} = i \delta^\alpha_\beta \mathcal{H}_{osc} + i B R^\beta_\alpha, \quad \{\Theta^\alpha, \Theta_\beta\} = 2i \bar{\omega} R^\alpha_\beta, \quad \{\Theta^\alpha, \Theta^\beta\} = -i \delta^\alpha_\gamma \Theta^\beta + \frac{i}{2} \delta^\alpha_\gamma \Theta^\alpha, \] 

where the Hamiltonian is now given by the expression

\[\mathcal{H}_{osc} = \bar{g}^{ab}(\pi_a \pi_b + [\omega]^2 \partial_a K \partial_b K) - \frac{1}{2} R_{a\beta c\delta} \bar{\eta}^{a\alpha} \bar{\eta}_\alpha \bar{\eta}^{\beta} \bar{\eta}_\beta - \frac{1}{2} \omega K_{a\beta} \eta^{a\alpha} \eta^\beta - \frac{1}{2} \bar{\omega} \bar{K}_{a\alpha} \eta^\alpha \eta^\beta + \frac{B}{2} g_{ab} \eta^{a\alpha} \eta^b. \] 

To close the superalgebra, we have to complete \[\Theta^\alpha \] by the $SU(2)$ algebra relations between $R$-charges as is given in \[23\], and by the full set of Poisson brackets involving the supercharges $\Theta^\alpha$.

In order to bring this superalgebra into the conventional form it is convenient to rotate the supercharges as

\[Q^\alpha = e^{i\nu/2} \cos \lambda \Theta^\alpha + e^{-i\nu/2} \sin \lambda \epsilon^{a\alpha} \Theta^\alpha, \quad \bar{Q}^a = e^{-i\nu/2} \cos \lambda \Theta^\alpha - e^{i\nu/2} \sin \lambda \epsilon^{a\alpha} \Theta^\alpha, \]

where

\[\cos 2\lambda = \frac{B}{\sqrt{4|\omega|^2 + B^2}}, \quad \sin 2\lambda = -\frac{2|\omega|}{\sqrt{4|\omega|^2 + B^2}}, \quad \omega = |\omega| e^{i\nu}. \]

In terms of these newly defined quantities the symmetry algebra is rewritten as

\[\{Q^\alpha, \bar{Q}^\beta\} = i \delta^\beta_\alpha \mathcal{H}_{osc} + \frac{1}{2} \sqrt{4|\omega|^2 + B^2} R^\alpha_\beta, \quad \{Q^\alpha, \mathcal{H}_{osc}\} = \frac{i}{2} \sqrt{4|\omega|^2 + B^2} Q^\alpha, \quad \{Q^\alpha, Q^\beta\} = \{\bar{Q}^\alpha, \bar{Q}^\beta\} = 0, \]

\[\{Q^\alpha, R^\beta_\gamma\} = -i \delta^\beta_\gamma Q^\alpha + \frac{i}{2} \delta^\alpha_\gamma Q^\beta - i \delta^\beta_\alpha R^\beta_\gamma = \{\bar{R}^\alpha_\beta, \bar{R}^\beta_\gamma\} = 0. \]

Comparing these relations with those of the supersymmetric $N = 4$ Landau problem \[29\], we can identify them as defining $SU(2|1)$ superalgebra with the deformation parameter $m = \sqrt{4|\omega|^2 + B^2}$.

The Lagrangian of $SU(2|1)$ supersymmetric Kähler oscillator is given by the general expression \[21\], with

\[U = |\omega|^2 g^{ab} \partial_a \partial_b U \frac{1}{2} R_{a\beta c\delta} \bar{\eta}^{a\alpha} \bar{\eta}_\alpha \bar{\eta}^{\beta} \bar{\eta}_\beta + \frac{w}{2} K_{a\beta} \bar{\eta}^{a\alpha} \eta^\beta + \frac{\bar{\omega}}{2} \bar{K}_{a\beta} \eta^\alpha \eta^\beta + \frac{B}{2} g_{ab} \bar{\eta}^{a\alpha} \eta^b. \]

The supersymmetrization procedure described above is capable to produce a family of non-equivalent Hamiltonians parameterized by an arbitrary holomorphic function. Namely, replacing the initial Kähler potential $K$ by the gauge-equivalent one,

\[K(z, \bar{z}) \to K(z, \bar{z}) + \frac{1}{\omega} U(z) + \frac{1}{\omega} \bar{U}(\bar{z}), \]

we obtain the class of Hamiltonians parameterized by an arbitrary holomorphic function $U(z)$,

\[\mathcal{H}_{osc} \to \mathcal{H}_{osc} = g^{ab}(\pi_a \pi_b + \partial_a \bar{\partial}_b U) - \frac{1}{2} R_{a\beta c\delta} \bar{\eta}^{a\alpha} \bar{\eta}_\alpha \bar{\eta}^{\beta} \bar{\eta}_\beta - \frac{w}{2} K_{a\beta} \bar{\eta}^{a\alpha} \eta^\beta + \frac{\bar{\omega}}{2} \bar{K}_{a\beta} \eta^\alpha \eta^\beta + \frac{B}{2} g_{ab} \bar{\eta}^{a\alpha} \eta^b + |\omega|^2 g^{ab} \partial_a K \partial_b K + |\omega|^2 g^{ab} (\partial_a \bar{K} \partial_b U + \partial_a \bar{U} \partial_b K) - \frac{\omega}{2} K_{a\beta} \bar{\eta}^{a\alpha} \eta^\beta - \frac{\bar{\omega}}{2} \bar{K}_{a\beta} \eta^\alpha \eta^\beta. \]

In the limit $\omega = 0$ we arrive at the well-known Hamiltonian which admits, in the absence of magnetic field, the "flat" $N = 4$ supersymmetry (see, e.g. \[22\]). It is given by the first line in the above expression with $B = 0$.

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3 We use here the following rules for complex conjugation and raising and lowering of $SU(2)$ spinor indices

\[\epsilon_{a\beta} = -\epsilon^{a\beta}, \quad \epsilon^{a\beta} = -\epsilon_{a\beta}, \quad \epsilon_{12} = \epsilon^{21} = 1, \quad \epsilon^{a\alpha} \epsilon_{a\beta} = \delta^a_\delta \delta^\beta_\gamma - \delta^a_\gamma \delta^\beta_\delta. \]
A. Two-dimensional anisotropic oscillator

The supersymmetrization procedure outlined above makes it possible to extend the class of the known systems admitting such a supersymmetrization. Here we illustrate this on the case of two-dimensional harmonic oscillator which is the simplest system possessing the conventional $\mathcal{N} = 4, d = 1$ “Poincaré” supersymmetric extension. Take the one-dimensional complex space $(\mathbb{C}, ds^2 = dz^2)$ and consider on it the Kähler oscillator defined by the potential

$$K(z, \bar{z}) = z\bar{z} + \frac{i g z^2}{2\omega} - \frac{ig \bar{z}^2}{2\bar{\omega}}. \quad (60)$$

It gives rise to the following Kähler-oscillator system

$$H = \pi\bar{\pi} + (\omega\bar{\omega} + g\bar{g})z\bar{z} + i\omega g z^2 - i\omega \bar{g} \bar{z}^2, \quad \{\pi, z\} = \{\bar{\pi}, \bar{z}\} = 1, \quad \{\pi, \bar{\pi}\} = iB. \quad (61)$$

Diagonalizing this potential, we arrive at the two-dimensional anisotropic oscillator system with frequencies

$$\omega^\pm = \left| |\omega| \pm |g| \right|. \quad (62)$$

For the choice $\omega = 0$ it yields the two-dimensional isotropic oscillator with the frequency $|g|$, which admits, in the absence of magnetic field, the standard $\mathcal{N} = 4, d = 1$ supersymmetrization. In the presence of magnetic field this supersymmetry is deformed to $SU(2|1)$. In the opposite limit, at $g = 0$, we once again obtain some $SU(2|1)$ supersymmetric extension of two-dimensional isotropic oscillator, but different from the first option. In the generic case of $g \neq 0, \omega \neq 0$ the procedure proposed allows to construct $SU(2|1)$ superextension of the two-dimensional anisotropic oscillator interacting with a constant magnetic field perpendicular to the plane. Enlarging the above set of Poisson brackets by the relation $\{\eta^\alpha, \bar{\eta}_\beta\} = i\delta^\alpha_\beta$, we can write down the Hamiltonian of the supersymmetric extension of this system as

$$\mathcal{H}_{\text{anosc}} = \pi\bar{\pi} + (\omega\bar{\omega} + g\bar{g})z\bar{z} + i\omega g z^2 - i\omega \bar{g} \bar{z}^2 - \frac{i g}{2} \eta^\alpha \eta_\alpha + \frac{i \bar{g}}{2} \bar{\eta}_\alpha \bar{\eta}_\bar{\alpha} + \frac{B}{2} \eta^\alpha \bar{\eta}_\bar{\alpha}. \quad (63)$$

The relevant supercharges and $R$-charges have the following simple form

$$\Theta^\alpha = \pi \eta^\alpha + (i\omega z + \bar{g} \bar{z})e^{\alpha \bar{\beta}} \bar{\eta}_\beta \quad R^\alpha_\beta = \eta^\alpha \bar{\eta}_\beta - \frac{1}{2} \delta^\alpha_\beta \eta^\gamma \bar{\eta}_\gamma. \quad (64)$$

It is straightforward to extend this model to $N$-dimensional complex Euclidian space $\mathbb{C}^N$ (see Section VI).

V. SUPERFIELD FORMULATION

The one-particle (i.e. one-(complex)dimensional) versions of the Lagrangians presented above were derived from the $SU(2|1)$ and $SU(4|1)$ superfield approaches in [11] and [13]. The generalization of these models to the $N$-dimensional case is straightforward. We briefly describe it below.

A. $SU(2|1)$ case

As the first step, we reproduce the Lagrangian of $SU(2|1)$ Kähler superoscillator corresponding to [62], and its particular case, the Lagrangian of $SU(2|1)$ supersymmetric Landau problem [63].

In [10] and [11] the coset method was used to define the world-line realizations of the supergroup $SU(2|1)$ on the $d = 1$ superspace $(t, \theta_\alpha, \bar{\theta}\bar{\beta})$ identified with the coset of $SU(2|1)$ over its $R$-symmetry subgroup $SU(2)$. The basic objects of this realization are covariant spinor derivatives

$$D^\alpha = e^{-\frac{i m t}{2}} \left( \left[ 1 + \frac{m}{2} \bar{\theta}\beta_\bar{\beta} \right] - \frac{3m^2}{16} \left( \bar{\theta}\beta_\bar{\beta} \right)^2 \right) \frac{\partial}{\partial \theta_\alpha} - \frac{m}{2} \bar{\theta}\beta_\bar{\beta} \frac{\partial}{\partial \theta_\alpha} - \frac{i}{2} \bar{\theta}\alpha_\beta \frac{\partial}{\partial \theta_\alpha} \right),$$

$$\bar{D}_\alpha = e^{\frac{i m t}{2}} \left[ \left( 1 + \frac{m}{2} \bar{\theta}\beta_\bar{\beta} \right) - \frac{3m^2}{16} \left( \bar{\theta}\beta_\bar{\beta} \right)^2 \right] \frac{\partial}{\partial \bar{\theta}^\alpha} + \frac{m}{2} \bar{\theta}\beta_\bar{\beta} \frac{\partial}{\partial \bar{\theta}^\alpha} + \frac{i}{2} \theta_\alpha \frac{\partial}{\partial \bar{\theta}^\alpha} \right], \quad (65)$$

which, in the contraction limit $m = 0$, become standard covariant spinor derivatives of flat $\mathcal{N} = 4, d = 1$ supersymmetry. The chiral $SU(2|1)$ superfields $\Phi^a(t, \theta, \bar{\theta})$ satisfy the generalized $SU(2|1)$ covariant chirality constraints

$$(\cos \lambda \bar{D}_\alpha - \sin \lambda D^\alpha) \Phi^a = 0. \quad (66)$$
In the appropriate superspace basis the conditions \((66)\) become “short” up to an overall factor,
\[
\left(\cos \lambda \dot{D}_a - \sin \lambda D_a\right) \Phi^a = \left[1 + \frac{B}{4} \tilde{\theta} \dot{\theta} + \frac{\omega}{4} \left(\dot{\theta} \beta \dot{\beta} + \tilde{\theta} \beta \tilde{\beta}\right) - \frac{m^2}{32} \left(\tilde{\theta} \beta \dot{\theta} \beta \right)^2 \right] \left[-\frac{\partial}{\partial \theta^a} + \frac{i}{2} \dot{\theta}_a \partial_t\right] \Phi^a ,
\]
and are solved by the expressions
\[
\Phi^a(t_L, \dot{\theta}_a) = z^a + \dot{\theta}_a \eta^{\alpha a} + \frac{1}{2} \dot{\theta}_a \dot{\alpha} A^a , \quad t_L = t + \frac{i}{2} \tilde{\theta} \dot{\theta}_a .
\]

The dependence on the new parameter \(\lambda\) is now hidden in the definition of the superspace coordinates \(t_L\) and \(\dot{\theta}_a\), which have the following \(SU(2|1)\) transformation properties
\[
\delta \dot{\theta}_a = \cos \lambda \left(\epsilon_\alpha \epsilon^\alpha e^z m t_L + \frac{m}{2} \tilde{\epsilon}_\beta \dot{\beta} \dot{\alpha} e^{-z} m t_L\right) + \sin \lambda \left(\tilde{\epsilon} \alpha e^{-z} m t_L + \frac{m}{2} \epsilon \beta \dot{\beta} \dot{\alpha} e^z m t_L\right) ,
\]
\[
\delta t_L = i \cos \lambda \tilde{\epsilon}_\beta \dot{\beta} e^{-z} m t - i \sin \lambda \epsilon_\beta \dot{\beta} e^z m t.
\]

These coordinate transformations induce the off-shell \(SU(2|1)\) supersymmetry transformation of chiral superfields. On the component fields they are realized as
\[
\delta \dot{\alpha} = -\left(\cos \lambda \epsilon_\alpha \epsilon^\alpha + \sin \lambda \tilde{\epsilon}_\alpha \tilde{\epsilon}^\alpha \right) \eta^{\alpha a} ,
\]
\[
\delta \dot{\eta}^{\alpha a} = \tilde{\epsilon}_\alpha \left(i \cos \lambda \dot{z} - \sin \lambda A^a \right) e^{z} m t - \epsilon_\alpha \left(i \sin \lambda \dot{z} + \cos \lambda A^a \right) e^{-z} m t ,
\]
\[
\delta \dot{A}^a = -\cos \lambda \tilde{\epsilon}_\alpha \left(i \eta^{\alpha a} + \frac{m}{2} \eta^{\alpha a} \right) e^{-z} m t + \sin \lambda \epsilon_\alpha \left(i \eta^{\alpha a} - \frac{m}{2} \eta^{\alpha a} \right) e^z m t ,
\]
where \(\epsilon_\alpha\) are “infinitesimal” Grassmann parameters.

The corresponding off-shell superfield Lagrangian is as follows (see \([11]\) for one-particle case)
\[
\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \left[1 + \frac{B}{2} \tilde{\theta} \dot{\theta} + \frac{\omega}{2} \left(\dot{\theta} \beta \dot{\beta} + \tilde{\theta} \beta \tilde{\beta}\right)\right] K \left(\Phi^a, \bar{\Phi}^b\right) ,
\]
where
\[
B = m \cos 2\lambda , \quad \omega = -\frac{m}{2} \sin 2\lambda .
\]

It is straightforward to check that the transformation of the factor within the square brackets in \((72)\) precisely cancels the transformation of the integration measure \(dt_L d^2 \theta d^2 \bar{\theta}\). Integrating in \((72)\) over \(\dot{\theta}, \bar{\theta}\) and eliminating the auxiliary fields \(A^a\), we recover the on-shell Lagrangian \((21)\) with the expression \((57)\) for \(U\). In the particular case \(\lambda = 0\) \((\omega = 0)\), we arrive at the Lagrangian \((33)\) of Landau problem. Holomorphic terms \((58)\) can be naturally inserted in \((72)\) with \(\omega \neq 0\) through the shift
\[
K \left(\Phi^a, \bar{\Phi}^b\right) \rightarrow K \left(\Phi^a, \bar{\Phi}^b\right) + \frac{1}{\omega} U(\Phi^a) + \frac{1}{\omega} \bar{U}(\bar{\Phi}^b) ,
\]
which amounts to introduction of the additional superpotential terms which, in components, induce the modified potential \(U\), as in \((59)\).

It is instructive to see how the phenomenon of preserving the isometries under the deformation manifests itself in the superfield language. For this purpose we need to know how the Kähler potential itself transforms under isometry of Kähler structure given by \((3)\). To this end, we rewrite the equation \((6)\) in the equivalent form as
\[
\partial_z \partial_{\bar{z}} \left\{ V^a_{\mu} (z) \partial_a + V^a_{\mu} (\bar{z}) \partial_{\bar{a}} \right\} K(z, \bar{z}) = 0 ,
\]
whence
\[
\left[ V^a_{\mu} (z) \partial_a + V^a_{\mu} (\bar{z}) \partial_{\bar{a}} \right] K(z, \bar{z}) = \varphi_{\mu}(z) + \bar{\varphi}_{\mu}(\bar{z}) .
\]

\(^4\) We limit our attention to real frequencies \(\omega = |\omega|\) in order to match the superfield approach elaborated in \([11]\). In fact, one can easily generalize this consideration to \(\omega \in \mathbb{C}\).
The holomorphic function $\varphi_\mu(z)$, in each specific case, can be defined up to a constant by differentiating (70) with respect to $z^h$.

The isometry transformations of the Kähler manifold in the superfield coordinates are obtained just by the changes $z^a \to \Phi^a$, $\bar{z}^b \to \bar{\Phi}^b$ in the relevant holomorphic Hamiltonian vector fields. Recalling the transformation (70) of $K(z, \bar{z})$ under isometry, we see that the superfield Lagrangian in (72) is transformed as

$$
\delta^* K = b^\mu \varphi(\Phi^a) + b^\mu \bar{\varphi}(\bar{\Phi}^a),
$$

where $b_\mu$, $\bar{b}_\mu$ are constant isometry parameters. Taking the bar-spinor derivatives from the integration measure and making use of the chirality of $\Phi^a$, it is easy to see that the holomorphic term in (77) does not contribute at $\omega = \lambda = 0, B = m$. The vanishing of the contribution from the conjugated antiholomorphic term in (77) can be proved after passing to the right-chiral basis in the $SU(2|1)$ superspace. This is the superfield proof of the property that the $SU(2|1)$ super Landau model inherits all isometries of the undeformed case $\omega = \lambda = m = 0$. The isometries are not generically inherited by the Kähler superscillator, when $\omega \neq 0$.

It should be pointed out that the input parameters of the above superfield formalism are just the contraction mass-dimension parameter $m$ coming from the (anti)commutation relations of the $su(2|1)$ algebra and the angle $\lambda$ coming from the chirality constraint (60). The physical meaning of these parameters as the strength of the external magnetic field and the oscillator frequency is revealed at the level of the component Lagrangians and Hamiltonians.

**B. $SU(4|1)$ case**

Next, let us present the $SU(4|1)$ superfield formulation for the Lagrangian of the $N = 8$ Landau problem [11], based on the superspace approach developed in [27]. This superfield Lagrangian is written in terms of chiral (2,8,6) superfields as follows (its one-particle case was constructed in [13]):

$$
S = \int dt \mathcal{L} = -\int dt_L d^4 \theta e^{-3i m t_L} \mathcal{F}(\Phi^a) - \int dt_R d^4 \theta e^{3i m t_R} \mathcal{F}(\bar{\Phi}^a), \quad m = |B|.
$$

(78)

Here $\mathcal{F}(z)$ is Seiberg-Witten prepotential [15], while the $\theta$-expansion of the superfields $\Phi^a$ reads

$$
\Phi^a (t_L, \theta_I) = z^a + \theta_I K \eta^a K e^{3i m t_L/4} + \frac{1}{2} \theta_I \theta_J A^{IJ} e^{3i m t_L/2} - \frac{1}{6} \varepsilon^{IJK} \theta_I \theta_J \theta_K \left( i \eta^a_L - \frac{m}{4} \bar{\eta}^a_L \right) e^{9i m t_L/4}
$$

$$
+ \frac{1}{24} \varepsilon^{IJKL} \theta_I \theta_J \theta_K \theta_L \left( \bar{z}^a + i m \bar{z}^a \right) e^{3i m t_L},
$$

(79)

with the following conjugation rules

$$
(A^{aIJ}) = A^{a}_{IJ} = \frac{1}{2} \varepsilon_{IJKL} A^{aK}_{L}, (\eta^a_I) = \bar{\eta}^a_I.
$$

The coordinate set $\{t_L, \theta^I\}$ is closed under the $SU(4|1)$ transformations

$$
\delta \theta_I = \epsilon_I + m \varepsilon K \theta_K \theta_I, \quad \delta t_L = i \varepsilon K \theta_K.
$$

(80)

The corresponding off-shell supersymmetry transformations of the component fields read

$$
\delta z^a = -\epsilon_K \eta^a K e^{-3i m t/4}, \quad \delta \bar{z}^a = \bar{\epsilon}_K \bar{\eta}^a_K e^{-3i m t/4},
$$

$$
\delta A^{aIJ} = 2 \varepsilon^{I} \left( i \bar{\eta}^{aJ} + \frac{m}{4} \eta^a_j \right) e^{-3i m t/4} + \varepsilon^{IJKL} \epsilon_K \left( i \eta^a_L - \frac{m}{4} \bar{\eta}^a_L \right) e^{3i m t/4},
$$

$$
\delta \eta^a_I = \bar{\epsilon}_I \left( i \bar{z}^a \right) e^{-3i m t/4} - \epsilon_K A^{aK}_{L} e^{3i m t/4}, \quad \bar{\delta} \eta^a_I = -\epsilon_I \left( i \bar{z}^a \right) e^{3i m t/4} - \bar{\epsilon}_K A^{aK}_{L} e^{-3i m t/4}.
$$

(81)

Integration in (79) over $\theta$, $\bar{\theta}$ gives the off-shell Lagrangian

$$
\mathcal{L}_{\text{off-shell}} = g_{ab} \left[ z^a \bar{z}^b - \frac{1}{4} A^{aIJ} A^{bIJ} + \frac{i}{2} \left( \eta^a K \bar{\eta}^b_K - \eta^a K \bar{\eta}^b_K - \frac{m}{4} \eta^a K \bar{\eta}^b_K \right) \right] - \frac{i}{2} \left( \bar{z}^a \partial_c g_{ab} - \bar{\bar{z}}^a \partial_c g_{ab} \right) \eta^a K \bar{\eta}^b_K
$$

$$
+ im \left( \bar{z}^a \partial_a \mathcal{F} - \bar{\bar{z}}^a \partial_a \mathcal{F} \right) + \frac{1}{2} A^{aIJ} \eta^a I \eta^{cJ} \partial_c g_{ab} - \frac{1}{2} A^{aIJ} \bar{\eta}^a I \bar{\eta}^{cJ} \bar{\partial}_c g_{ab} - \frac{1}{2} \varepsilon^{IJKL} \bar{\eta}^a I \eta^a K \eta^b J \partial_c g_{ab} + \varepsilon^{IJKL} \bar{\eta}^a I \bar{\eta}^a K \bar{\eta}^b J \bar{\partial}_c g_{ab},
$$

(82)

where the metric $g_{ab}$ is identified with the metric defined in [15]. The subsequent elimination of the auxiliary fields $A^{aIJ}$ yields just the on-shell Lagrangian [14].
It is important that the superfield action (78) is invariant under the transformations corresponding to (83) (see [28])
\[
\mathcal{F}(\Phi^a) \rightarrow \mathcal{F}(\Phi^a) + ic_{ab}\Phi^a\Phi^b + c_a\Phi^a + c, \quad \mathcal{F}(\bar{\Phi}^a) \rightarrow \mathcal{F}(\bar{\Phi}^a) - ic_{ab}\bar{\Phi}^a\bar{\Phi}^b + \bar{c}_a\bar{\Phi}^a + \bar{c},
\]
where \(c, c_a\) are complex numbers, and \(c_{ab}\) are real ones.

These transformations are just the \(\mathcal{N} = 8\) superfield version of the general transformations of the holomorphic prepotential \(\mathcal{F}(z)\) under an arbitrary isometry of the special Kähler structure, i.e. of the isometry of Kähler structure preserving holomorphic third-order tensor \(\mathcal{H}\) (see Appendix A). Hence, the invariance of (78) under (83) explicitly demonstrates that the deformed \(\mathcal{N} = 8\) supersymmetric mechanics we are considering inherits the full set of isometries of the undeformed case.

The proof of this superfield invariance is not so easy. To this end, one needs to represent the invariant chiral measure \(d^4\theta e^{-3imt_L}\) in the action (78) in terms of covariant derivatives (up to total time derivatives) as\(^5\)
\[
d^4\theta e^{-3imt_L} = \frac{1}{24} e^{-3imt_L} \varepsilon_{IJKL} \partial^I \partial^J \partial^K \partial^L = \frac{1}{24} \varepsilon_{IJKL} D^I D^J D^K D^L.
\]

Covariant derivatives anticommute as
\[
\{\bar{D}_I, D_J\} = 0, \quad \{D^I, D^J\} = 0, \quad \{D^I, \bar{D}_J\} = \delta^I_J \mathcal{H}_0 + m\bar{R}_I^J, \quad \bar{R}_I^J D^K = \frac{1}{4} \delta^I_J \delta^K L - \delta^I_J \delta^K D^L,
\]
where \(\bar{R}_I^J\) are \(SU(4)\) matrix generators acting on external indices of superfields and covariant derivatives. The chiral superfield \(\Phi^a (a = 1, \ldots, N)\) describing \(N\) multiplets \((2,8,6)\) satisfies the constraints [28]
\[
D^I \Phi^a = 0, \quad \bar{D}_K \Phi^a = 0, \quad D^J \bar{D}^K \Phi^a = \frac{1}{2} \varepsilon_{IJKL} \bar{D}_K \bar{D}_L \Phi^a.
\]

Exploiting (84) - (86) for the action (78), one can show its invariance under the transformations (83). Another, more direct proof is to substitute the explicit expressions (79) for \(\Phi^a\) and the conjugated ones for \(\bar{\Phi}^a\) into (83) and to be convinced that the coefficients of the higher-order monomials in \(\theta_I(\bar{\theta}^I)\) in the holomorphic(antiholomorphic) shifts (83) either are combined into total \(t\)-derivatives or just vanish. Note that the reality condition for the coefficient \(c_{ab}\) in (83) is essential for ensuring the properties just mentioned.

Derivation of the purely bosonic counterpart of the transformations (83) from the isometry condition (42) is discussed in Appendix A.

VI. EXAMPLES OF SUPERINTEGRABLE KÄHLER OSCILLATOR MODELS

In the previous sections we dealt with two classes of models admitting deformed supersymmetry: the Landau problems, and the Kähler oscillators. In the case of Landau problem we found that the supersymmetric extensions preserve all (kinematical) symmetries of the initial systems. But we were not able to prove the similar general proposition for the Kähler oscillators. In this section we present supersymmetric extensions of two particular types of the Kähler oscillator systems which possess kinematical symmetries and the hidden symmetries generated by the constants of motion quadratic in momenta. These two types are encompassed by the following models

- \(\mathbb{C}^N\)-oscillator (the sum of \(N\) two-dimensional isotropic oscillators) and \(\mathbb{C}^N\)-Smorodinsky-Winternitz system (the sum of \(N\) copies of two-dimensional isotropic oscillators deformed by ring-shaped potentials).
- \(\mathbb{CP}^N\)-oscillator and \(\mathbb{CP}^N\)-Rosochatius system, which are superintegrable counterparts of \(\mathbb{C}^N\)-oscillator and \(\mathbb{C}^N\)-Smorodinsky-Winternitz systems on the complex projective spaces.

Our main goal will be to inspect whether \(SU(2|1)\) supersymmetric extensions of these systems inherit their hidden symmetries.

---

\(^5\) Though expressions for \(SU(4|1)\) covariant derivatives were not calculated, the function \(D^I D^J D^K D^L \mathcal{F}(\Phi^a)\) is \(SU(4|1)\) invariant. Hence, it must give the same invariant action (78).
A. Euclidean spaces

We start by considering the Kähler oscillators on the complex Euclidean space \((\mathbb{C}^N, ds^2 = \sum_{a=1}^{N} dz^a d\bar{z}^a)\). The relevant phase space is defined by the Poisson brackets

\[
\{\pi_a, z^b\} = \delta_b^a, \quad \{\bar{\pi}_a, \bar{z}^b\} = \delta_b^a, \quad \{\pi_a, \bar{\pi}_b\} = iB\delta_{ab}.
\] (87)

The set of symmetries of this space is constituted by the \(SU(N)\) generators

\[
J_{ab} = i\pi_a z^b - i\bar{\pi}_b \bar{z}^a - Bz^b \bar{z}^a : \{J_{ab}, J_{cd}\} = i\delta_{ad}J_{bc} - i\delta_{cb}J_{ad},
\] (88)

and the translation generators

\[
J_a = i\pi_a - Bz^a : \{J_a, J_b\} = \{J_a, J_b\} = 0, \quad \{J_a, J_{bc}\} = -i\delta_{ab}J_{ac}.
\] (89)

For the construction of \(SU(2|1)\) supersymmetric Kähler oscillator models on this space we have to complete the Poisson brackets by the following ones

\[
\{\eta^{\alpha a}, \bar{\eta}^{b}_\beta\} = i\delta^a_\beta \delta^{n}_\alpha,
\] (90)

with \(\alpha, \beta = 1,2\). Then we should perform the \(SU(2|1)\) supersymmetrization procedure described above, for the appropriate choice of the initial bosonic Kähler oscillator model.

**Harmonic oscillator**

We define the \(\mathbb{C}^N\)-harmonic oscillator defined as a Kähler oscillator with \(K(z, \bar{z}) = \sum_{a=1}^{N} z^a \bar{z}^a\) and \(\omega = \bar{\omega}\):

\[
H_{osc} = \sum_{a=1}^{N} \left(\pi_a \bar{\pi}_a + \omega^2 z^a \bar{z}^a\right).
\] (91)

This system possesses \(SU(N)\) kinematical symmetry generated by the generators, and hidden symmetries defined by the so-called Fradkin tensor

\[
I_{ab} = \pi_a \bar{\pi}_b + \omega^2 z^a \bar{z}^b : \{I_{ab}, I_{cd}\} = i\delta_{ad}I_{bc} - i\delta_{cb}I_{ad}, \quad \{I_{ab}, J_{cd}\} = i\omega \delta_{ad}I_{bc} - i\omega \delta_{cb}I_{ad}.
\] (92)

In the \(SU(2|1)\) supersymmetric extension of this system, the Hamiltonian, dynamical supercharges and \(R\)-charges are determined by those of the two-dimensional isotropic oscillator

\[
\mathcal{H} = \sum_{a=1}^{N} \mathcal{H}_a, \quad \Theta^\alpha = \sum_{a=1}^{N} \Theta^{\alpha a}, \quad R^{\alpha}_{\beta} = \sum_{a=1}^{N} R^{\alpha a}_{\beta}.
\] (93)

with

\[
\mathcal{H}_a = \pi_a \bar{\pi}_a + \omega^2 z^a \bar{z}^a + \frac{B}{2} \eta^{\alpha a} \bar{\eta}^{\alpha a}, \quad \Theta^{\alpha a} = \pi_a \eta^{\alpha a} + i\omega z^a \bar{\epsilon}^{\alpha \beta} \bar{\eta}^{\beta}, \quad R^{\alpha a}_{\beta} = \eta^{\alpha a} \bar{\eta}^{\beta} - \frac{1}{2} \delta^{a}_{\beta} i\eta^{\alpha \gamma} \bar{\eta}^{\gamma a}.
\] (94)

All constants of motion of the bosonic Hamiltonian become those of the supersymmetrized one, since all these quantities are just sums of bosonic and fermionic parts. Moreover, in the supersymmetric system there appear additional symmetry generators acting on the fermionic variables only. Thus, the system with the Hamiltonian inherits kinematical \(SU(N)\) symmetries of the bosonic sector, hidden symmetries generated by the Fradkin tensor, and reveals an additional \(U(N)\) symmetry realized in the fermionic sector:

\[
R_{ab} = \sum_{\alpha} \eta^{\alpha b} \bar{\eta}^{\alpha a} : \{R_{ab}, R_{cd}\} = i\delta_{ad}R_{cb} - i\delta_{cb}R_{ad}.
\] (95)

Now we turn to considering less trivial example of \(SU(2|1)\) supersymmetric Kähler oscillator with hidden symmetries.
\[ H_{SW} = \sum_{a=1}^{N} I_a, \quad I_a = \pi_a \bar{\pi}_a + |\omega|^2 \frac{z^a \bar{z}^a}{z^a \bar{z}^a} + \frac{|g_a|^2}{z^a \bar{z}^a}. \]

(96)

It has \( N \) manifest \( U(1) \) symmetries \( z^a \rightarrow e^{i\alpha} z^a \), with the generators \( J_{a\bar{a}} \), and the hidden symmetries spanned by the above generators \( I_a \), as well as by the following ones (the so-called Uhlenbeck tensor)

\[ I_{ab} = J_{ab} J_{\bar{a} \bar{b}} - \frac{1}{2} J_{a\bar{a}} J_{b\bar{b}} + \frac{g_a}{\omega} \frac{z^b \bar{z}^b}{z^a \bar{z}^a} + \frac{|g_a|^2}{2} \frac{z^a \bar{z}^a}{z^a \bar{z}^a}, \quad \{ I_{ab}, H_{SW} \} = 0, \]

(97)

where \( J_{ab} \) are \( u(N) \) generators defined in \( \mathbb{S} \).

This system can be identified as a Kähler oscillator with the following Kähler potential

\[ K = z \bar{z} + \frac{g_a}{\omega} \log z^a + \frac{\bar{g}_a}{\omega} \log \bar{z}^a, \quad \arg \omega = \arg \sum_{a=1}^{N} g_a + \pi/2. \]

(98)

Its \( SU(2|1) \) supersymmetric extension is found to be associated with the Hamiltonian

\[ \mathcal{H}_{SW} = \sum_{a=1}^{N} \mathcal{I}_a, \quad \mathcal{I}_a = \pi_a \bar{\pi}_a + |\omega|^2 \frac{z^a \bar{z}^a}{z^a \bar{z}^a} + \frac{|g_a|^2}{2} \frac{z^a \bar{z}^a}{z^a \bar{z}^a} + \frac{\bar{g}_a}{2} \eta_a \bar{\eta}_a + \frac{B}{2} \eta_a \bar{\eta}_a \]

(99)

and the supercharges

\[ \Theta^{a\alpha} = \pi_a \eta^{a\alpha} + i \omega \varepsilon^{a\alpha \beta} \bar{\eta}_b \left( z^a + \frac{g_a}{\omega} \right). \]

(100)

Clearly, the generators \( \mathcal{I}_a \) commute with each other, and so they are the constants of motion of the supersymmetric \( \mathbb{C}^N \)-Smorodinsky-Winternitz system. This supersymmetric system possesses \( N \) manifest \( U(1) \) symmetries \( z^a \rightarrow e^{i\alpha} z^a \), \( \eta_a \rightarrow e^{i\alpha} \eta_a \), with the generators

\[ J_{a\bar{a}} = J_{a\bar{a}} + \eta^{a\beta} \bar{\eta}_b : \{ J_{a\bar{a}}, J_{b\bar{b}} \} = 0. \]

(101)

The extensions of the hidden symmetry generators \( I_a, I_{ab} \) are given, respectively, by the generators \( \mathcal{I}_a \) defined in (99) and by the following ones

\[ \mathcal{I}_{ab} = I_{ab} + \frac{g_a}{2} \frac{z^b \bar{z}^b}{z^a \bar{z}^a} \eta^{a\alpha} \eta_a + \frac{\bar{g}_a}{2} \frac{z^b \bar{z}^b}{z^a \bar{z}^a} \eta^{\beta a} \eta_b + \frac{g_a}{2} \frac{z^a \bar{z}^a}{z^b \bar{z}^b} \eta^{\alpha b} \eta_b + \frac{\bar{g}_a}{2} \frac{z^a \bar{z}^a}{z^b \bar{z}^b} \eta^{b a} \eta_b : \{ I_{ab}, \mathcal{H}_{SW} \} = 0. \]

(102)

Thus \( SU(2|1) \) supersymmetric extension of \( \mathbb{C}^N \)-Smorodinsky-Winternitz system inherits all its hidden symmetries.

The conclusion is that the “Kähler superoscillator approach” yields the well defined superextensions of both the isotropic oscillator and the Smorodinsky-Winternitz system on \( \mathbb{C}^N \).

### B. Complex projective spaces

In this Section we will deal with superintegrable systems on complex projective spaces \( \mathbb{CP}^N \) which are specified by the presence of constant magnetic field and belong to the class of the Kähler oscillator models.

Consider the complex projective space equipped with \( su(N+1) \)-invariant Fubini-Study metrics

\[ g_{ab} d\bar{z}^a d\bar{z}^b, \quad \text{with} \quad g_{ab} = \frac{\log(1 + z \bar{z})}{\partial z^a \partial \bar{z}^b} = \frac{\delta_{ab}}{1 + z \bar{z}} - \frac{z^a \bar{z}^b}{(1 + z \bar{z})^2}. \]

(103)

The inverse metrics, non-zero Christoffel symbols and Riemann tensor are defined by the expressions

\[ g^{ab} = (1 + z \bar{z})(\delta^{ab} + z^a \bar{z}^b), \quad \Gamma^a_{bc} = -\frac{g^{a}}{\delta_{b}} \frac{z^c}{1 + z \bar{z}} + \frac{\delta_{c}}{\delta_{b}} \frac{z^b}{1 + z \bar{z}}, \quad R_{abcd} = g_{ab} g_{cd} + g_{bc} g_{ad}, \]

(104)
The Killing potentials of \( su(N + 1) \) isometry algebra are of the form
\[
h_{ab} = \frac{z^b z^a}{1 + z \bar{z}}, \quad h_a = \frac{z^a}{1 + z \bar{z}}. \tag{105}
\]

Equipping the cotangent bundle of \( \mathbb{CP}^N \) with the twisted symplectic structure \(^7\) and the related Poisson brackets, we obtain the mechanics systems involving an interaction with a constant magnetic field.

The \( su(N + 1) \) isometry generators are given by the expressions of the form
\[
J_{ab} = i(z^b \pi_a - \bar{\pi}_b \bar{z}^a) - B \frac{z^a z^b}{1 + z \bar{z}}, \quad J_a = i(\pi_a + \bar{z}^a (\bar{z} \pi)) - B \frac{\bar{z}^a}{1 + z \bar{z}};
\tag{106}
\]
\[
\{J_{ab}, J_{cd}\} = i \delta_{ad} J_{bc} - i \delta_{eb} J_{ad}, \quad \{J_a, \bar{J}_b\} = i J_{ab}, \quad \{J_a, J_b\} = \mp i J_{a \bar{c}}.\]

Extending these generators to this phase superspace as in \(^7\), we obtain
\[
\mathcal{J}_{\bar{a}b} = J_{\bar{a}b} + \frac{\partial^2 h_{ab}}{\partial \bar{z} \partial z} \eta^c \bar{a}_b^d, \quad J_a = J_a + \frac{\partial^2 h_a}{\partial \bar{z} \partial z} \eta^c \bar{a}_a^d. \tag{107}
\]

With these expressions at hand we can construct superintegrable models admitting weak \( SU(2|1) \) supersymmetry.

\[\mathbb{CP}^N\text{-oscillator}\]

The oscillator on a complex projective space is defined by the Hamiltonian \(^6\)
\[
H_{osc} = g^{a\bar{b}} \bar{\pi}_a \pi_b + |\omega|^2 z \bar{z}. \tag{108}
\]

The constants of motion of this system are given by the \( u(N) \)-generators \( J_{a\bar{b}} \) \(^{106}\) and by the analog of “Fradkin tensor”
\[
I_{ab} = J_a \bar{J}_b + |\omega|^2 \bar{z}^a z^b. \tag{109}
\]

This system belongs to the class of “Kähler oscillators” \(^1\) with \( K = \log(1 + z \bar{z}) \), and hence admits \( SU(2|1) \) supersymmetric extension. The relevant Hamiltonian and supercharges read
\[
\mathcal{H}_{osc} = g^{a\bar{b}} \bar{\pi}_a \pi_b + |\omega|^2 z \bar{z} - \frac{1}{2} (g_{ab} g_{\bar{c}d} + g_{d\bar{b}} g_{a\bar{c}}) \eta^a \bar{a}_b^c \eta^d \bar{a}_d^c - \frac{\omega z^a \bar{z}^b \eta^a \bar{a}_b^c \eta^d \bar{a}_d^c}{2 (1 + z \bar{z})^2} - \frac{\omega \bar{z}^a z^b \eta^a \bar{a}_b^c \eta^d \bar{a}_d^c}{2 (1 + z \bar{z})^2} + \frac{B}{2} g_{ab} \eta^a \bar{a}_b^c \eta^d \bar{a}_d^c, \tag{110}
\]
\[
\Theta^a = \pi_a \eta^a + i \omega \frac{z^a}{1 + z \bar{z}} \epsilon^{a \beta} \eta^\beta, \quad \Theta_a = \bar{\pi}_a \eta_a + i \omega \frac{\bar{z}^a}{1 + z \bar{z}} \epsilon_{a \beta} \eta^\beta. \tag{111}
\]

This system has the manifest \( u(N) \) symmetry defined by the generators \( \mathcal{J}_{a\bar{b}} \): \( \{\mathcal{J}_{a\bar{b}}, \mathcal{H}_{osc}\} = 0 \).

One could expect that the appropriate generalization of the Fradkin tensor should still have the form \(^{109}\), with \( J_a \) replaced by \( \mathcal{J}_a \), and that just this minimal modification yields constants of motion of the super-oscillator. However, one can check that it is not the case. So, for the time being, it is an open question whether a supersymmetric counterpart of the Fradkin tensor exists.

\[\mathbb{CP}^N\text{-Rosochatius system}\]

The \( \mathbb{CP}^N \)-Rosochatius system is defined by the symplectic structure \(^7\) and by the Hamiltonian \(^{18}\)
\[
H_{Ros} = (1 + z \bar{z}) \left( \pi \bar{\pi} + (z \pi)(\bar{z} \bar{\pi}) + |\omega_0|^2 + \sum_{a=1}^N \frac{\omega_a^2}{z^a \bar{z}^a} \right) - \sum_{i=0}^{N} |\omega_i|^2. \tag{112}
\]

\(^6\) Hereafter we use the notation \( z^c \equiv \sum_{c=1}^N z^c \bar{z}^c \), \( (\pi z) = \sum_{c=1}^N \pi_c z^c \) etc.
The second equation has the simple graphical illustration: it defines the planar polygon with the edges \( \omega \) the special choice of the parameters \( | \omega | \leq | \omega_i | \), where, without loss of generality, we assume that \( | \omega_0 | \geq | \omega_1 | \geq \ldots \geq | \omega_N | \). In this case we arrive at the well-known \( N = 4 \) supersymmetric mechanics on Kähler manifold with the holomorphic prepotential \( U(z) = \sum_{a=1}^{N} \omega_a \log z^a \) (see, e.g., [22]).

Finally, we note that all symmetries respected by the systems considered in this section are symmetries of the appropriate superfield Lagrangians (72) at \( B \neq 0, \omega \neq 0 \), with \( \Phi^a, \Phi^b \) standing for \( z^a, \bar{z}^b \).

VII. DISCUSSION AND OUTLOOK

In this paper we presented the systematic combined Hamiltonian and superfield approach to the construction of the multi-particle models of deformed \( N = 4, 8 \) supersymmetric mechanics on Kähler manifolds in interaction with a constant magnetic fields. The latter are introduced via a supersymmetric version of minimal coupling. We applied this approach to the various (super)integrable models and demonstrated that such superextensions preserve all kinematical symmetries of the initial bosonic systems (and some hidden symmetries in a few particular cases). One of the basic features of our approach is that diverse isometries are realized on the \( SU(2|1) \) multiplets of the same sort, without introducing any extra multiplet. This is a crucial difference of our approach from the models of Refs. [23], [24], [25] in which similar isometries were realized within the standard \( N = 4 \) supersymmetric mechanics at cost of introducing extra degrees of freedom (coming back to the spin variables introduced in [20]) 7.

The next obvious task is the study of the quantum mechanical properties (spectra, etc) of the \( SU(2|1) \) supersymmetric Landau problem on \( \mathbb{CP}^N \), as well as of the \( SU(2|1) \) supersymmetric oscillator-like models on \( \mathbb{C}^N \) and \( \mathbb{C}P^N \).

Some other tasks are:

7 Applications of the spin variables in the models of \( SU(2|1) \) mechanics were considered, e.g., in [24].
• Coupling, to a constant magnetic field, of “flat” $\mathcal{N} = 8$ supersymmetric mechanics with non-zero potential on special Kähler manifolds suggested in [30] and studying the new deformed $\mathcal{N} = 8$ mechanics models obtained in this way;

• The construction of the deformed supersymmetric extensions of the Landau problem on quaternionic manifolds and, in particular, on quaternionic projective spaces $\mathbb{HP}^N$, having in mind their relevance to the so-called high-dimensional Hall effect [31];

• The construction of the $\mathbb{HP}^N$-Rosochatius system and studying the symmetry properties of it and of the $\mathbb{HP}^N$-oscillator’s [32], as well as of their supersymmetric extensions.

• Introducing the notion of quaternionic oscillator, by analogy with the Kähler one, and the study of its possible deformed supersymmetric extensions.

We plan to address this circle of problems in a not distant future.

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Appendix A: Isometries of special Kähler structure in the local coordinates

In this Appendix we formulate the conditions [12] defining the isometries of special Kähler structure in the local coordinate frame, in which the Kähler metric and the tensor $f_{abc}(z)$ take the form [17]. The equation [12] expresses, in the special coordinate frame, via Seiberg-Witten prepotential $F(z)$ as follows

$$3\partial_c V^d a\partial_b \partial_c \partial_d F + V^d a\partial_a \partial_b \partial_c \partial_d F = 0, \quad (A1)$$

with $V^a a, V^a a$ being the components of the holomorphic Hamiltonian vector field [33].

To extract the necessary corollaries of this equation, we first act by the derivative $\partial_a$ on (75), where the Kähler potential is defined by [17]. Step by step it yields

$$\partial_a \partial_b \partial_c \left[V^d a\partial_d + V^d a\partial_d F + V^d a\partial_d \partial_d F\right] = 0 \Rightarrow$$
$$\partial_a \partial_b \partial_c \left[V^d a\partial_d (\partial_d F + V^d a\partial_d \partial_d F)\right] = 0 \Rightarrow$$
$$\partial_a \partial_c \left[V^d a\partial_d \partial_d F + V^d a\partial_d \partial_d F\right] = 0 \Rightarrow$$
$$\partial_a \partial_c \left[V^d a\partial_d \partial_d F - \partial_a \partial_c \partial_d F\right] = 0. \quad (A2)$$

Using the last condition, we become able to rewrite (A1) as

$$g\partial_c \partial_a \partial_b V^d a - \partial_a \partial_b \partial_d F = 0. \quad (A3)$$

Next, taking $\partial_c$ derivative of this relation, we obtain

$$\partial_c \partial_a \partial_b \partial_d F V^d a = - \partial_a \partial_b \partial_d F \partial_c \partial_c V^d a. \quad (A4)$$
The left- and right-hand sides of this relation are products of holomorphic and antiholomorphic functions. Obviously, the factors of the same holomorphicity should be equal, which yields

\[ \partial_a \partial_b V^c_\mu = i C^c_{\mu} \partial_a \partial_b D F, \quad C^c_{\mu} = \bar{C}^{dc}_\mu, \]  

(A5)

where \( C^c_{\mu} \) are some complex constant parameters.

Taking also into account (A3), the solution of (A5) can be written as

\[ V^c_\mu = i C^c_{\mu} \partial_c F + \beta^c_{\mu a} z^a + \alpha^c_{\mu}, \quad V^d_\mu = -i C^d_{\mu} \partial_c F + \beta^d_{\mu a} z^a + \alpha^d_{\mu}, \]  

(A6)

where \( \beta^c_{\mu a} \) and \( \alpha^c_{\mu} \) are, respectively, real and complex constant parameters. From (A3) and (A5), it follows that \( C^c_{\mu} \) is a symmetric real matrix, \( C^c_{\mu} = \bar{C}^{dc}_\mu \).

The variation of \( F \) is then equal to

\[ \delta_\mu F \equiv V^c_\mu \partial_c F = (i C^c_{\mu} \partial_c F + \beta^c_{\mu a} z^a + \alpha^c_{\mu}) \partial_c F. \]  

(A7)

Inserting this solution in (A1) yields the condition

\[ \partial_a \partial_b \partial_c \left( \delta_\mu F \right) = 0, \]  

(A8)

having the obvious general solution

\[ \delta_\mu F = c_\mu + c_{a \mu} z^a + c_{ab \mu} z^a z^b, \]  

(A9)

where \( c_\mu \), \( c_{a \mu} \) and \( c_{ab \mu} \) are complex parameters.

Next we insert the solution (A6) in the Killing equation (5) (b), with the metric defined by (45), and derive the additional condition on \( \delta_\mu F \):

\[ \partial_a \partial_b \left( \delta_\mu F \right) + \partial_a \partial_b \left( \delta_\mu F \right) = 0. \]  

(A10)

This equation amounts to the reality condition \( \{c_{ab \mu}\} = -c_{ab \mu} \).

The superfield transformations (83) have precisely the form of the general isometry \( \delta_\mu F \), with the complex coordinates \( z^a, \bar{z}^a \) being replaced by the chiral \( SU(4|1) \) superfields \( \Phi^a \) and their anti-chiral counterparts.

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