A list analog of Vizing’s Theorem for simple graphs with triangles but no other odd cycles

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1 Introduction

- **edge-colouring**: assignment of colours to edges so that adjacent edges receive different colours
- **k-edge-colourable**: can be edge-coloured with $k$ colours
- **chromatic index** $\chi'(G) = \min k$ s.t. $G$ is $k$-edge-colourable

\[ \chi'(G) \geq \chi'(G) \geq \Delta \]

- **list-edge-colouring**: edge-colouring where, for each edge $e$, the colour for $e$ must come from a given list $L_e$
- **k-list-edge-colourable**: $|L_e| \geq k \ \forall e \in E(G) \Rightarrow G$ has a list-colouring from the given lists
- **list-chromatic index** $\chi'_l(G) = \min k$ st $G$ is $k$-list-edge-colourable

\begin{array}{ccc}
\text{one} & \{1, 2, 3\} & \{1, 2, 3\} & \{1, 3, 4\} \\
\text{two} & \{1, 2, 3\} & \{1, 2, 3\} & \{1, 3, 4\} \\
\text{three} & \{1, 2, 3\} & \{1, 2, 3\} & \{1, 2, 3\} \\
\text{four} & \{1, 2, 3\} & \{1, 2, 3\} & \{1, 2, 3\} \\
\end{array}
$\chi'_l(G) \geq \chi'(G) \geq \Delta$

**Vizing's Theorem.** $G$ simple $\Rightarrow$ $\chi'(G) \leq \Delta + 1$.

**Theorem (M., 2014+).** $G$ simple, no odd cycles except triangles $\Rightarrow$ $\chi'_l(G) \leq \Delta + 1$.

**Theorem (Galvin, 1995).** $G$ bipartite $\Rightarrow$ $\chi'_l(G) = \chi'(G)$.

**List-edge-colouring Conjecture.** $\chi'_l(G) = \chi'(G)$

*Known about the Conjecture:*
- true for some planar $G$ (Ellingham, Goddyn; Borodin, Kostochka)
- true for $G$ series parallel (Juvan, Mohar, Thomas)
- some results when $G - v$ bipartite (Plantholt, Tipnis)
- $\chi'_l(G) \leq \frac{3\Delta}{2}$ (Borodin, Kostochka)
Vizing’s Theorem. $G$ simple $\Rightarrow \chi'(G) \leq \Delta + 1$.

Theorem (M., 2014+). $G$ simple, no odd cycles except triangles $\Rightarrow \chi'_l(G) \leq \Delta + 1$.

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1. Introduction
2. Galvin’s proof
3. Allowing triangles (at a cost)
4. Wrap-up
2 Galvin’s proof

Theorem (Galvin, 1995). $G$ bipartite $\Rightarrow \chi'_l(G) = \chi'(G)$.

proof.

• $G$ is bipartite so we can $k$-edge-colour $G$ with $k = \Delta = \chi'(G)$.

• Fix a bipartition $(A, B)$. Orient the line graph $L(G)$ as follows:
  - big $\rightarrow$ small for $A$;
  - small $\rightarrow$ big for $B$.

• $d^+_L(v) \leq k - 1$

• Lemma (Bondy, Boppana, Siegel): If a digraph is kernel-perfect, then it is list-vertex-colourable for any lists with $|L_v| \geq d(v)^+ + 1$.

• To complete the proof: show $L(G)$ is kernel-perfect
So far:
- $k$-edge-colour $G = (A, B), \ k = \Delta$
- Orient $L(G)$
  - $A$: high $\rightarrow$ low
  - $B$: low $\rightarrow$ high
- By BBS lemma, $G$ is $k$-edge-colourable if $L(G)$ is kernel-perfect

**kernel-perfect**: every induced subdigraph of has a kernel
(independent set $S$ such that every vertex is either in $S$, or has an edge into $S$)

To prove $L(G)$ is **kernel-perfect**:
- induction on size of subdigraph
  (very dependent on $G$ being bipartite)
  OR
- apply a theorem of Maffray
  (can work provided $G$ has no odd cycles of length 5 or longer)
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  OR
- apply a theorem of Maffray
  (*can work provided $G$ has no odd cycles of length 5 or longer*)
Theorem (Maffray, 1992) A line graph is solvable iff it is perfect.

**perfect**: $\chi(H) = \omega(H) \ \forall \ \text{induced} \ H \subseteq G$

**solvable**: there is a kernel in any orientation with the property that every clique has a sink.

- In our $L(G)$, every clique has a sink (also true for induced subdigraphs)
- Line graphs of bipartite graphs (and their subgraphs) are perfect.
- $L(G)$ is kernel-perfect.

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Theorem (Trotter, 1977) A line graph $L(G)$ is perfect iff $G$ has no odd cycles of length 5 or longer.

To extend Galvin’s result to all $G$ without odd cycles of length 5 or longer, we need only to define an orientation of $L(G)$ where:

- every clique has a sink
- $d^+(e) \leq k - 1$.

*To do this we will need $k = \Delta + 1$, $G$ simple.*
Theorem (Maffray, 1992) A line graph is \textit{solvable} iff it is \textit{perfect}.

\textit{perfect}: \( \chi(H) = \omega(H) \ \forall \) induced \( H \subseteq G \)

\textit{solvable}: there is a kernel in any orientation with the property that every clique has a sink.

- In our \( L(G) \), every clique has a sink (also true for induced subdigraphs)
- Line graphs of bipartite graphs (and their subgraphs) are perfect.
- \( L(G) \) is \textit{kernel-perfect}. \hfill \square

Theorem (Trotter, 1977) A line graph \( L(G) \) is perfect iff \( G \) has no odd cycles of length 5 or longer.

To extend Galvin’s result to all \( G \) without odd cycles of length 5 or longer, we need only to define an orientation of \( L(G) \) where:
- every clique has a sink
- \( d^+(e) \leq k - 1 \). *To do this we will need \( k = \Delta + 1 \), \( G \) simple.
3 Allowing triangles (at a cost)

Theorem (M., 2014+). $G$ simple, no odd cycles except triangles \[ \Rightarrow \chi'_l(G) \leq \Delta + 1. \]

proof sketch.

It suffices to define an orientation of $L(G)$ where:
(1) every clique has a sink
(2) $d^+(e) \leq k - 1$ (we let $k = \Delta + 1$).

We define a partition $(D, U)$ of $V(G)$ and a $k$-edge-colouring $c$ of $G$, so that the following orientation works:
• incidences in $D$ go down, big $\rightarrow$ small colour
• incidences in $U$ go up, small $\rightarrow$ big colour

For (2):
• edges between $D$ and $U$ are good.
  • edges between $Ds$ must be low – in $\{1, 2, \ldots, \lfloor \frac{k+1}{2} \rfloor \}$
  • edges between $Us$ must be high – in $\{\lceil \frac{k+1}{2} \rceil, \ldots, k \}$

For (1):
• All cliques not arising from triangles are good.
  • For every triangle, we must ensure a sink (source)
Theorem (Maffray, 1992) A simple graph has no odd cycles of length 5 or longer iff every block $B$ satisfies one of:

1. $B$ is bipartite
2. $B = K_4$
3. $B$ is ...

Define a $k$-edge-colouring $c$ and partition $(U, D)$ one block $B$ at a time:

At each step, we assume one vertex $b \in B$ has already been assigned to $D$ or $U$ and that all its incident edges not in $B$ have been coloured by $c$.

We must ensure $B$ satisfies the rules:

- edges between $D$s must be low – in \{1, 2, \ldots, \lceil \frac{k+1}{2} \rceil \}
- edges between $U$s must be high – in \{\lceil \frac{k+1}{2} \rceil, \ldots, k \}
- For every triangle, we must ensure a sink (source)
  (incidences in $D$ go down, big$\rightarrow$small colour)
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It can be done! (for $k = \Delta + 1$)
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It can be done! (for $k = \Delta + 1$)
Q: What prevents using $k = \Delta$?

The rules for assigning $c$ and $(U, D)$ in $B$, given pre-determined $b$:

- edges between $Ds$ must be **low** – in $\{1, 2, \ldots, \left\lfloor \frac{k+1}{2} \right\rfloor \}$
- edges between $Us$ must be **high** – in $\{\left\lceil \frac{k+1}{2} \right\rceil, \ldots, k\}$
- For every triangle, we must ensure a sink (source)
  - (incidences in $D$ go **down**, big $\rightarrow$ small colour)
  - (incidences in $U$ go **up**, small $\rightarrow$ big colour)

A: We might have no choice in colouring edges incident to $b$ in $B$, and get one of these, which can’t be completed:
4 Wrap-up

\[ \chi'_l(G) \geq \chi'(G) \geq \Delta \]

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Thank-you