Orientational probability distribution of an active Brownian particle: an analytical study

Supurna Sinha
Raman Research Institute, Bangalore 560080, India

Received 4 May 2020
Accepted for publication 23 June 2020
Published 3 August 2020

Online at stacks.iop.org/JSTAT/2020/083201
https://doi.org/10.1088/1742-5468/aba497

Abstract. We use the Fokker–Planck equation as a starting point for studying the orientational probability distribution of an active Brownian particle (ABP) in \((d + 1)\) dimensions (i.e. \(d\) angular dimensions and 1 radial dimension). This Fokker–Planck equation admits an exact solution in series form which is, however, unwieldy to use because of poor convergence for short and intermediate times. We present an analytical closed form expression, which gives a good short time approximate orientational probability distribution. The analytical formula is derived using the saddle point method for short times. However, it works well even for intermediate times. We also present a simple analytical form for the long time limit of the orientational probability distribution. Thus, we have obtained simple analytical forms for the orientational probability distribution of an ABP for the \(entire\) range of time scales. Our predictions can be tested against future experiments and simulations probing orientational probability distribution of an ABP.

Keywords: active matter, Brownian motion, probability distribution

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1. Introduction

The study of active matter has been the focus of current research. Active particles are self-propelled particles which generate dissipative directed motion by consuming energy from the environment. There are many examples of active matter in soft matter and biological systems like bacterial run and tumble motion, swimming microbes, schools of fish, swarms of birds, driven granular matter and so on.

There has been a considerable amount of theoretical, simulational and experimental work in this area both at the large scale hydrodynamic level [1, 2] and at the level of single particle dynamics [3, 4]. While there has been a fair amount of research in studying the position distribution function of a single active Brownian particle (ABP), there are fewer studies addressing the orientational probability distribution of an ABP.

In this paper we focus on the orientational probability distribution of an ABP using the Fokker–Planck equation describing the orientational probability distribution of an ABP as a starting point. As is well known, the exact solution to this equation can be expressed in terms of the eigenvalues and eigenfunctions of the Laplacian [5, 6]. Our main results are the following: (a) an approximate short time closed form analytical expression which, in fact turns out to be an effective approximation even at intermediate times. (b) A long time approximate form which works well at long times. Thus, we have obtained simple analytical forms for the orientational probability distribution for the entire range of time scales.

The orientational probability distributions predicted in our analysis can be probed experimentally via experiments on Janus particles, for instance [5].

The paper is organized as follows. In section 2 we discuss the Fokker–Planck equation for the orientational dynamics of an ABP in \((d + 1)\) dimensions (\(d\) angular dimensions and 1 radial dimension) which admits an exact series solution for the distribution in three (i.e. \(2 + 1\)) dimensions. In section 3 we derive a short time approximate orientational probability distribution in \((d + 1)\) dimensions and discuss the particular case of
three dimensions. In section 4 we present a long time approximate form for the orientational probability distribution. In section 5 we compare the short time and the long time approximate orientational probability distributions in three dimensions, against the exact orientational probability distribution in three dimensions. The short time approximation works very well at short and intermediate times. As expected, the exact orientational probability distribution deviates from the short time approximate orientational probability distribution at very long times. In the long time domain, the long time approximate form works very well. The predictions that stem out of our analysis, which are discussed in this section, can be tested against simulations and future experiments. We finally conclude with some discussions in section 6.

2. Fokker–Planck equation for orientational dynamics of an ABP

The speed of an ABP is fixed but its direction is a vector diffusing on the unit sphere [4]. The Fokker–Planck equation describing the orientational probability distribution of an ABP in \((d+1)\) dimensions [6] is [14]:

\[
\frac{\partial P}{\partial t} = D_R \left[ \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial \xi^i} (\sqrt{\det g} g^{ij} \frac{\partial P}{\partial \xi^j}) \right]
\]

(1)

where we have considered \(d\) arbitrary curvilinear coordinates \((\xi^1, \xi^2, \xi^3, \ldots \xi^d)\) on the surface of the unit sphere. \(g^{ij}\) is the inverse metric tensor and \(D_R\) is the rotational diffusion coefficient.

In three dimensions it takes the following familiar form in polar coordinates:

\[
\frac{\partial P(\theta, \phi, t)}{\partial t} = D_R \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial P}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2 P}{\partial \phi^2} \right]
\]

(2)

This Fokker–Planck equation admits an exact solution as a kernel in series form [5]:

\[
K(\theta_0, \phi_0, 0, \theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-D_R l(l+1)} Y_l^m(\theta_0, \phi_0) Y_l^m(\theta, \phi)
\]

(3)

An azimuthally symmetric initial distribution \(P(\theta, 0)\) leads to a simpler form for the series solution [6]:

\[
P_{\text{exact}}(\theta, t) = \frac{\sin \theta}{2} \sum_{l=0}^{\infty} (2l + 1) e^{-\frac{l(l+1)}{2} D_R t} P_l(\cos \theta)
\]

(4)

where we have included the measure \(\sin \theta\) and set \(D_R = \frac{1}{2}\). Although the solution in equation (4) is exact, poor convergence at short times is expected to come in the way of an effective implementation in the short time domain.
3. Orientational probability distribution of an ABP: a short time approximation

In this section we arrive at a short time approximate form for the orientational probability distribution of an ABP which can be implemented effectively at short and intermediate times.

The solution to the Fokker–Planck equation (1) on the surface of a \((d+1)\) dimensional unit sphere is given formally by the Wiener integral kernel [7]. This kernel \(K(\hat{n}_1,0,\hat{n}_2, t)\) is the probability of the particle to be at \(\hat{n}_2\) at time \(t\) given that it was at \(\hat{n}_1\) at time 0.

\[
K(\hat{n}_1,0,\hat{n}_2, t) = \int D[\hat{n}(t)] \exp[-S[\hat{n}(t)]]
\]

where \(S[\hat{n}(t)] = \frac{1}{2} \int_0^t \frac{d\hat{n}(t)}{dt} \cdot \frac{d\hat{n}(t)}{dt} dt\).

At short times, this kernel is dominated by the classical path connecting \(\hat{n}_1\) to \(\hat{n}_2\) and is given by [7, 8]:

\[
K(\hat{n}_1,0,\hat{n}_2, t) \sim \exp -S_{cl}[\hat{n}_1, \hat{n}_2, t]
\]

where \(S_{cl}[\hat{n}_1, \hat{n}_2, t]\) is the classical action pertaining to the least action path connecting \(\hat{n}_1\) and \(\hat{n}_2\) in time \(t\).

Taking into consideration the quadratic fluctuations around the classical path we get [9, 15]:

\[
K(\hat{n}_1,0,\hat{n}_2, t) \sim \sqrt{\det V} \exp -S_{cl}[\hat{n}_1, \hat{n}_2, t]
\]

\(\det V\) is the Van Vleck determinant given by the determinant of the \(d \times d\) Hessian matrix:

\[
V_{ij} = \frac{\partial^2 S_{cl}[\hat{n}_1, \hat{n}_2, t]}{\partial \hat{n}_1^i \partial \hat{n}_2^j}
\]

Finally, incorporating the normalisation \(N(t)\) we arrive at [10, 11]:

\[
K(\hat{n}_1,0,\hat{n}_2, t) = N(t)\sqrt{\det V} \exp -S_{cl}[\hat{n}_1, \hat{n}_2, t]
\]

Varying the action in equation (5) we find the classical path governed by the equation:

\[
\frac{\partial^2 \hat{n}}{\partial t^2} = \lambda \hat{n}
\]

with \(\lambda\) is a Lagrange multiplier enforcing the constraint \(\hat{n}_1 \cdot \hat{n}_2 = 1\). The solution of equation (10) is the unique great circle passing through \(\hat{n}_1^1\) and \(\hat{n}_2^2\) (we are assuming here that \(\hat{n}_1\) and \(\hat{n}_2\) are not collinear). Thus the classical action is given by:

\[
S_{cl} = \frac{\theta^2}{2t}
\]

\[
\theta = \cos^{-1}(\hat{n}_1, \hat{n}_2).
\]

We can formally extend the function \(S_{cl}\) to a neighbourhood of the unit sphere by letting \(n_1\) and \(n_2\) be unnormalised, which makes the operation of differentiation more tractable. Here \(\cos \theta = \frac{\hat{n}_1 \cdot \hat{n}_2}{||\hat{n}_1|| ||\hat{n}_2||}\).

https://doi.org/10.1088/1742-5468/aba497
Thus our fluctuation determinant is the determinant of a \((d + 1) \times (d + 1)\) matrix \(\tilde{V}'_{ij}\), where we have removed the zero eigenvalue pertaining to the radial coordinate.

\[
\tilde{V}'_{ij} = \frac{\partial^2 S_{cl}}{\partial n_1^i \partial n_2^j}
\]

(11)

Consider Cartesian coordinates so that \(n_1\) and \(n_2\) lie in the \(x-z\) plane. In addition there are \((d - 1)\) transverse dimensions.

We notice that on reflecting in the \(x-z\) plane the \((d - 1) \times 2\) matrix block \(\tilde{V}'_{a\beta}\) and the \(2 \times (d - 1)\) matrix block \(\tilde{V}'_{ab}\) in the \((d + 1) \times (d + 1)\) matrix \(\tilde{V}'\) change sign. The requirement of invariance tells us that the entries in these two blocks are all zero. Thus we are left with two blocks with nonzero entries: the \(2 \times 2\) matrix \(\tilde{V}'_{ab}\) pertaining to the \(x-z\) plane and the \((d - 1) \times (d - 1)\) matrix block \(\tilde{V}'_{a\beta}\). Taking into consideration the rotational invariance of the \((d - 1) \times (d - 1)\) matrix block \(\tilde{V}'_{a\beta}\) in \(\tilde{V}'\), we find that it is of the form \(\tilde{V}'_{a\beta} = \tilde{V}'\delta_{a\beta}\). Thus the determinant pertaining to the \((d + 1) \times (d + 1)\) matrix \(\tilde{V}'\) is \(\det \tilde{V}' = \det \tilde{V}'_{ab} \det \tilde{V}'_{a\beta}\).

We first compute the determinant corresponding to the \(2 \times 2\) matrix \(\tilde{V}'_{ab}\) [16]. This determinant can be expressed as follows:

\[
\det \tilde{V}'_{ab} = \hat{n}_{1a} e^{\alpha c} \hat{n}_{2b} e^{\beta d} \tilde{V}'_{cd}
\]

(12)

An explicit computation of this \(2 \times 2\) determinant gives us:

\[
\det \tilde{V}'_{ab} = \frac{1}{t^2}.
\]

Thus, the total propagator in 2 dimensions (i.e.\((d + 1)\) dimensions with \(d = 1\)) is given by [12]

\[
K(\hat{n}_1, 0, \hat{n}_2, t) = N(t) \sqrt{\frac{1}{t}} \exp \left[ -\frac{\theta^2}{2t} \right]
\]

(13)

The normalisation \(N(t)\) can be fixed by integrating the kernel \(K(\hat{n}_1, 0, \hat{n}_2, t)\) over initial conditions to get \(P(\theta, t)\) and then by numerically imposing the normalisation condition \(\int P(\theta, t) d\theta = 1\).

The fact that this expression is correct can be seen by specialising to \(d = 1\) and noticing that diffusion on a circle can be well approximated by diffusion on a line (\(x\) axis) for short times:

\[
K = \frac{1}{\sqrt{4\pi Dt}} \exp \left[ -\frac{x^2}{4Dt} \right]
\]

(14)

where \(D\) is the diffusion constant.

The remaining \((d - 1) \times (d - 1)\) matrix \(\tilde{V}'_{a\beta}\) has a diagonal form and corresponds to a determinant \(\det \tilde{V}'_{a\beta} = \left(\frac{\theta}{\sin \theta}\right)^{(d-1)}\).

Thus the determinant pertaining to the \((d + 1) \times (d + 1)\) matrix \(\tilde{V}'\) is \(\det \tilde{V}' = \det \tilde{V}'_{ab} \det \tilde{V}'_{a\beta} = \frac{1}{t} \left(\frac{\theta}{\sin \theta}\right)^{(d-1)}\). (See appendix for a more explicit and algebraic derivation.)
Thus, the final expression for the kernel in \((d + 1)\) dimensions is \([12]\):

\[
K(\hat{n}_1, 0, \hat{n}_2, t) = N(t)\sqrt{\frac{1}{t}} \left(\frac{\theta}{t \sin \theta}\right)^{(d-1)} \exp\left[-\frac{\theta^2}{2t}\right]
\]  

(15)

As a special case, we can consider the three dimensional determinant which gives us, on setting \(d = 2\) (the number of angular dimensions in three dimensions): \(\det \tilde{V} = \frac{1}{2^2 \sin \theta}\).

Thus we get the following short time approximate kernel in three dimensions:

\[
K(\hat{n}_1, 0, \hat{n}_2, t) = \frac{N(t)}{t} \sqrt{\frac{\theta}{\sin \theta}} \exp\left[-\frac{\theta^2}{2t}\right]
\]  

(16)

Multiplying the above expression for \(K(\hat{n}_1, 0, \hat{n}_2, t)\) by the measure \(\sin \theta\) we arrive at the following approximate short time probability distribution in three dimensions:

\[
P_{\text{approx}}^S(\theta, t) = \frac{N(t)}{t} \sqrt{\frac{\theta}{\sin \theta}} \exp\left[-\frac{\theta^2}{2t}\right]
\]  

(17)

Notice that in contrast to \(P(\theta, t)\) where time \(t\) appears in the numerator of the argument of the exponential, the approximate propagator \(P_{\text{approx}}^S(\theta, t)\) has much better convergence properties at short times because of the appearance of time in the denominator of the argument of the exponential.

### 4. Orientational probability distribution of an ABP: a long time approximation

In this section we restrict to \(d = 2\) and discuss an approximate analytic form for the orientational probability distribution of an ABP. This can be obtained simply by noticing that at long times, the expression for the orientational probability distribution given by the series in equation (4) is dominated by the first few terms. Truncating the series and retaining the first three terms in the summation, we get in three dimensions (i.e. in \((d + 1)\) dimensions with \(d = 2\)):

\[
P_{\text{approx}}^L(\theta, t) = \frac{1}{2} \left[ \sin \theta \left(1 - \frac{5}{2} e^{-3t}\right) + \left(\frac{3 \sin 2\theta}{2}\right) \left(e^{-t} + \frac{5}{2} \cos \theta e^{-3t}\right)\right]
\]  

(18)

Thus equation (18) gives us an analytical form for the long time orientational probability distribution.

### 5. Orientational probability distribution of an ABP: comparison of the exact solution with approximate analytical forms

In this section, we restrict to three dimensions (i.e. consider \(d = 2\)) and graphically compare the exact series solution for the orientational probability distribution of an ABP with the approximate analytical forms obtained in the short time regime and
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Figure 1. Comparison of the short time approximate probability distribution (equation (17)) (red dotted line) and the exact probability distribution (equation (4)) (blue dashed line) at time $t = 0.3$.

Figure 2. Comparison of the short time approximate probability distribution (equation (17)) (red dotted line) and the exact probability distribution (equation (4)) (blue dashed line) at time $t = 2$.

the long time regime. We first compare the exact series solution with the short time approximate solution obtained by using the saddle point method (sometimes called Laplace’s method). We notice that at short and intermediate times the two probability distributions agree very well (figures 1 and 2). The two probability distributions deviate from one another at long times, as expected (figure 3).

Thus, we conclude that the short time approximate form works well at short and intermediate times. Here by short time we mean times shorter than the time scale $\tau$ associated with rotational diffusion, by long time we mean times longer than $\tau$ and by intermediate time we mean time scales comparable to $\tau$. In our analysis we have taken $\tau$ to be equal to 2.

We then compare the exact series solution with the long time approximate solution obtained in section 4 and the short time approximate solution obtained in section 3.
Figure 3. Comparison of the short time approximate probability distribution (equation (17)) (red dotted line) and the exact probability distribution (equation (4)) (blue dashed line) at time $t = 10$.

Figure 4. Comparison of the short time approximate probability distribution (equation (17)) (green dashed line), the exact probability distribution (equation (4)) (blue line) and the long time approximate probability distribution (equation (18) (red dotted line) at time $t = 0.6$.

notice that even at relatively short times ($t = 0.6$), the short time approximate distribution, the long time approximate distribution and the exact distribution agree very well (see figure 4). At an intermediate time ($t = 1.5$) the short time approximate probability distribution, the long time approximate probability distribution and the exact probability distribution merge (see figure 5). At long times, the long time approximation works very well (see figure 6) whereas the short time approximation deviates considerably from the exact distribution. At very short times (say $t = 0.2$), the long time approximate distribution shows oscillations, indicating a breakdown of the long time approximation at very short times, stemming from truncation errors.

We also display a comparison of the three distributions (the short time, the long time and the exact) using the Kullback–Leibler divergence. This measure, also known as the
relative entropy, is widely used in information theory to measure the extent of deviation between a trial distribution and a fiducial one. In our case the fiducial distribution is the exact distribution and the trial one is the approximate form (the short time approximate form or the long time approximate form, as the case may be).

\[ D_{KL} := \int_{0}^{\pi} d\theta P_{\text{approx}}(\theta) \log \left( \frac{P_{\text{approx}}(\theta)}{P_{\text{exact}}(\theta)} \right) \]

In the table below, the relative entropy of the long time approximate distribution with respect to the exact distribution is denoted by \( D_{KL,\text{long}} \) and the relative entropy of the short time approximate distribution with respect to the exact distribution is denoted by \( D_{KL,\text{short}} \).
This table shows that the short time approximate analytical form works well at short and intermediate times and the long time approximate analytical form works well at long times. Notice that at an intermediate value \( t = 1.5 \) of time \( D_{KL\text{long}} \) and \( D_{KL\text{short}} \) are both quite small and comparable, indicating that at \( t = 1.5 \), both approximate forms are good approximations to the exact distribution.

Thus we conclude that we have excellent analytical forms for the entire range of time scales.

### 6. Conclusion

We study the orientational probability distribution of an ABP via the Fokker–Planck equation. Our starting point is the orientational Fokker–Planck equation of an ABP in \((d + 1)\) dimensions. It is well known that the Fokker–Planck equation for orientational dynamics admits an exact solution in series form which, however, has poor convergence for short and intermediate times.

We present an analytical closed form expression, which gives a good short time approximate orientational probability distribution. In some earlier papers [3, 13] methods used in the study of equilibrium properties of semiflexible polymers have been incorporated to study ABP dynamics. Here we use such methods to analyse the orientational probability distribution of an ABP at short and intermediate times. The analytical formula is derived using the saddle point method for short times. In computing the approximate orientational probability distribution we have considered a situation where the particle winds around the great circle once. We have neglected higher order windings. The excellent agreement between our approximate probability distribution and the exact probability distribution shows that higher order windings are negligible and our restriction to only one winding is indeed a very good approximation. We also present a long time approximate analytical form for the orientational probability distribution of an ABP. Thus we have obtained explicit analytical forms for the orientational probability distribution of an ABP in the entire range of time scales.

Restricting to three dimensions, we graphically compare the exact formula with the short time and the long time approximate probability distributions. We show that the short time approximate formula works well at short as well as at intermediate times. The short time approximate formula deviates from the exact one at long times, as expected. In the long time domain, our long time approximate analytical expression for the orientational probability distribution works well.
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Our predictions can be tested against future experiments and simulations probing orientational probability distribution of an ABP.

Acknowledgments

We acknowledge Urna Basu, Abhishek Dhar and Satya Majumdar for discussions and Baruch Meerson for drawing our attention to reference [8] and the theory of geometric diffusion.

Appendix. Computation of the Van Vleck determinant

Here we present an explicit computation of the Van Vleck determinant, which has been schematically outlined in the main body of the paper. We use explicit notation, writing \( \hat{r}_1 \) and \( \hat{r}_2 \) for \( \hat{n}_1 \) and \( \hat{n}_2 \) and setting \( r \) to \( n \) after differentiation. Consider two vectors \( \hat{r}^L(\gamma) \) and \( \hat{r}^T(\psi) \). \( \hat{r}^L \) lies in the \( \hat{x} - \hat{z} \) plane and \( \hat{r}^T \) is in the transverse direction. We consider variations of \( \hat{n} \) in the transverse direction described by \( \hat{r}^T \)

\[
\hat{r}^T(\psi) = (\cos \psi)\hat{n} + (\sin \psi)\hat{y}^\alpha
\]  (20)

(\text{where } \alpha = 1, 2, 3, \ldots, (d-1)) and variations in the \( x-z \) plane described by \( \hat{r}^L \).

\[
\hat{r}^L(\gamma) = (\cos \gamma)\hat{x} + (\sin \gamma)\hat{z}
\]  (21)

We first compute the determinant of the \( 2 \times 2 \) matrix in the \( x-z \) plane. Consider \( \hat{r}^L_1(\gamma_1) \) and \( \hat{r}^L_2(\gamma_2) \) which can be expressed as follows in the \( x-z \) plane:

\[
\hat{r}^L_1(\gamma_1) = \cos \gamma_1\hat{x} + \sin \gamma_1\hat{z}
\]  (22)

\[
\hat{r}^L_2(\gamma_2) = \cos \gamma_2\hat{x} + \sin \gamma_2\hat{z}
\]  (23)

The dot product \( \hat{r}^L_1.\hat{r}^L_2 = \cos(\theta) \) where \( \theta = (\gamma_1 - \gamma_2) \).

Thus the \( 1 \times 1 \) determinant \( \left. \frac{\partial S_{cl}}{\partial \psi} \right|_{\gamma_1=\theta_1, \gamma_2=\theta_2} = -\frac{1}{\theta} \).

Let us now consider \( \vec{V}_{a\beta} \) and \( \vec{V}_{ab} \), the off diagonal matrix blocks of the \((d+1) \times (d+1)\) matrix \( \vec{V} \).

The Van Vleck determinant pertaining to \( \vec{V}_{a\beta} \) can be computed as follows. Consider

\[
\hat{r}^T_1(\psi_1) = (\cos \psi_1)\hat{n}_1 + (\sin \psi_1)\hat{y}^\alpha
\]  (24)

where \( \alpha = 1, 2, 3, \ldots, (d-1) \).

\[
\hat{r}^L_2(\gamma_2) = (\cos \gamma_2)\hat{x} + (\sin \gamma_2)\hat{z}
\]  (25)

Notice that \( \left. \frac{\partial S_{cl}}{\partial \psi_1} \right|_{\gamma_1=\theta_1, \gamma_2=\theta_2} = -\frac{1}{\theta} \frac{\partial \theta}{\partial \psi_1} \). Taking the dot product of \( \hat{r}^T_1(\psi_1) \) and \( \hat{r}^L_2(\gamma_2) \) and taking the derivative with respect to \( \psi_1 \) we get:

\[
-\sin \theta \frac{\partial \theta}{\partial \psi_1} = (\cos \gamma_2)(\hat{n}_1.\hat{x}) - (\sin \psi_1)(\sin \gamma_2)(\hat{n}_1.\hat{z})
\]  (26)
Clearly, computation of the Van Vleck determinant gives us
\[
\left. \frac{\partial^2 S_{cl}}{\partial \gamma_2 \partial \psi_1} \right|_{\psi_1=0,\gamma_2=\theta_2} = 0
\]
(27)
in accord with the symmetry argument presented in the main body of the paper. Following similar steps it can be shown that the Van Vleck determinant pertaining to \( \vec{V}_{ab} \) is:
\[
\left. \frac{\partial^2 S_{cl}}{\partial \gamma_1 \partial \psi_2} \right|_{\psi_2=0,\gamma_1=\theta_1} = 0
\]
(28)

Finally we compute the determinant of the remaining \((d-1) \times (d-1)\) matrix block \( \vec{V}_{\alpha\beta} \) in \( \vec{V} \).
Consider two vectors \( \hat{r}_1^T \) and \( \hat{r}_2^T \).
\[
\hat{r}_1^T(\psi_1) = (\cos \psi_1)\hat{n}_1 + (\sin \psi_1)\hat{y}^\alpha
\]
(29)
\[
\hat{r}_2^T(\psi_2) = (\cos \psi_2)\hat{n}_2 + (\sin \psi_2)\hat{y}^\beta
\]
(30)
As before, taking the dot product of \( \hat{r}_1^T(\psi_1) \) and \( \hat{r}_2^T(\psi_2) \) and taking the derivative with respect to \( \psi_1 \) we get:
\[
-\sin \theta \frac{\partial \theta}{\partial \psi_1} = (-\sin \psi_1)(\cos \psi_2)(\hat{n}_1.\hat{n}_2) + (\cos \psi_1)(\sin \psi_2)(\hat{y}^\alpha.\hat{y}^\beta)
\]
(31)
Now
\[
\frac{\partial S_{cl}}{\partial \psi_1} = \theta \frac{\partial \theta}{\partial \psi_1} = - \frac{\theta}{l \sin \theta} \frac{1}{l} \left[ (-\sin \psi_1 \cos \psi_2)(\hat{n}_1.\hat{n}_2) + (\cos \psi_1)(\sin \psi_2)(\hat{y}^\alpha.\hat{y}^\beta) \right]
\]
(32)
Thus we finally get:
\[
\tilde{V}_{\alpha\beta} = \frac{\theta}{l \sin \theta} \delta^{\alpha\beta}
\]
(33)
Each diagonal entry of the \((d-1) \times (d-1)\) matrix block \( \tilde{V}_{\alpha\beta} \) is \( \frac{\theta}{l \sin \theta} \) which gives the final expression for the determinant of \( \tilde{V}_{\alpha\beta} \) as \( \left( \frac{\theta}{l \sin \theta} \right)^{(d-1)} \).

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[15] In [8] the authors derive a short time approximate formula, which, however, does not incorporate the quadratic fluctuation corrections that have been taken into consideration in our analysis
[16] This is effectively a $1 \times 1$ determinant since the matrix has a zero eigenvalue pertaining to the radial coordinate

https://doi.org/10.1088/1742-5468/aba497