Equalities, congruences, and quotients of zeta functions
in Arithmetic Mirror Symmetry

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1 Introduction

The purpose of this note is twofold. First, we demonstrate that under certain conditions we may extend the
Arithmetic Mirror Theorem of [2, Theorems 1.1 and 6.1]. Second, we apply this extension to the study of the
quotient of the zeta functions of $X_\lambda$ and $Y_\lambda$.

With $\lambda \in \mathbb{C}$ we may define a family of complex projective hypersurfaces $X_\lambda$ in $\mathbb{P}^n_C$ by

$$x_1^{n+1} + \cdots + x_{n+1}^{n+1} + \lambda x_1 \cdots x_{n+1} = 0.$$ 

With the group

$$G := \{(\zeta_1, \ldots, \zeta_{n+1}) | \zeta_i \in \mathbb{C}, \zeta_i^{n+1} = 1, \zeta_1 \cdots \zeta_{n+1} = 1\}$$

we may define the (singular) mirror variety $Y_\lambda$ as the quotient $X_\lambda/G$ where $G$ acts by coordinate multiplication.

It turns out that $Y_\lambda$ is a toric hypersurface and may be explicitly described as the projective closure in $\mathbb{P}_\Delta$ of the
affine toric hypersurface

$$g(x_1, \ldots, x_n) := x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda = 0.$$ 

Note, $\mathbb{P}_\Delta$ is the toric variety obtained from the polytope in $\mathbb{R}^n$ with vertices $\{e_1, \ldots, e_n, -(e_1 + \cdots + e_n)\}$, where
the $e_i$ are the standard basis vectors of $\mathbb{R}^n$. From this description of $Y_\lambda$, if we let $F_q$ denote the finite field with
$q$ elements of characteristic $p$, it makes sense to discuss $F_q$-rational points of $X_\lambda$ and its mirror $Y_\lambda$ when the
parameter $\lambda$ lies in $F_q$.

When the $\gcd(n+1, q^k - 1) = 1$, there are no $(n+1)$-roots of unity in the field $F_q$. Viewing $G$ as a group
scheme over $Z$, this means there are no $F_q$-rational points of $G$. This leads one to suspect a direct relation
between the $F_q$-rational points of $X_\lambda$ and $Y_\lambda$.

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Theorem 1.1. For every positive integer $k$ such that $\gcd(n+1,q^k-1) = 1$, we have the equality $\#X_\lambda(\mathbb{F}_{q^k}) = \#Y_\lambda(\mathbb{F}_{q^k})$.

If $W_\lambda$ is a smooth crepant resolution of $Y_\lambda$, then there is a rational map from $W_\lambda$ to $Y_\lambda$ which is injective on rational smooth points. Thus, if none of the $\mathbb{F}_{q^k}$-rational points on $Y_\lambda$ are singular points, we see that $\#Y_\lambda(\mathbb{F}_{q^k}) = \#W_\lambda(\mathbb{F}_{q^k})$. Consequently, we have:

**Corollary 1.2.** Suppose the singular locus of $Y_\lambda$ contains no $\mathbb{F}_{q^k}$-rational points. If $\gcd(n+1,q^k-1) = 1$, then we have $\#X_\lambda(\mathbb{F}_{q^k}) = \#Y_\lambda(\mathbb{F}_{q^k}) = \#W_\lambda(\mathbb{F}_{q^k})$.

Next, when $\gcd(n+1,q^k-1) > 1$ we may prove:

**Theorem 1.3.** Let $d := \gcd(n+1,q^k-1) > 1$. Then

1. $\#X_\lambda(\mathbb{F}_{q^k}) \equiv 0 \mod d$,
2. if $n+1$ is a power of a prime $\ell$, then, writing $\lambda = -(n+1)\psi$ in the new parameter $\psi$, we have

$$\#X_\lambda(\mathbb{F}_{q^k}) \equiv 0 \mod(\ell d) \quad \text{and} \quad \#Y_\lambda(\mathbb{F}_{q^k}) \equiv \begin{cases} 1 & \psi^n+1 = 1 \mod(\ell) \\
0 & \text{otherwise} \end{cases}.$$

Thus, combining [2] Theorems 1.1] and 1.3 with the Chinese Remainder Theorem yields:

**Corollary 1.4.** Suppose $n+1$ is a power of a prime $\ell$ and $\gcd(n+1,q) = 1$. Set $\lambda = -(n+1)\psi$. If $\psi^n+1 \neq 1$, then for every positive integer $k$, we have $\#X_\lambda(\mathbb{F}_{q^k}) \equiv \#Y_\lambda(\mathbb{F}_{q^k}) \mod(\ell q^n)$.

Before discussing the proofs of Theorems 1.1 and 1.3 let us apply Theorem 1.1 to the quotient of the zeta functions of $X_\lambda$ and $Y_\lambda$.

2 Application to zeta functions

From [2] Theorem 7.3], when $(n+1)|q-1$ then the quotient of the zeta functions of $X_\lambda$ and $Y_\lambda$, when raised to the $(-1)^n$ power, is a polynomial of specified degree. We suspect that the divisibility $(n+1)|q-1$ is unnecessary and may be removed without disturbing the conclusion. Evidence for this is the following:

**Theorem 2.1.** Let $n+1$ be a prime such that $\gcd(n+1,q) = 1$. Let $k$ be the smallest positive integer such that $q^k \equiv 1 \mod(n+1)$. Assume $X_\lambda$ is non-singular and $X_{\psi+1} \neq -(n+1)\psi^n+1$. Then there are positive integers $\rho_1, \ldots, \rho_s$, each divisible by $k$, and polynomials $Q_1, \ldots, Q_s \in 1+T\mathbb{Z}[T]$ which are pure of weight $n-3$ and irreducible over $\mathbb{Z}$, such that

$$\left( \frac{Z(X_\lambda/\mathbb{F}_{q^k}, T)}{Z(Y_\lambda/\mathbb{F}_{q^k}, T)} \right)^{(-1)^n} = Q_1(q^kT)Q_2(q^kT)Q_3(q^kT)$$

Furthermore, $\rho_1 + \cdots + \rho_s = \frac{n(n^n-(-1)^n)}{n+1} - n$. (Note, the polynomials $Q_i$ depend on $n$ and $\lambda$.)

**Proof.** For every nonnegative integer $s$ and $j = 1, \ldots, k-1$, we have $\gcd(n+1, q^{sk+j}-1) = 1$. So, by Theorem 1.1 we have $\#X_\lambda(\mathbb{F}_{q^{sk+j}}) = \#Y_\lambda(\mathbb{F}_{q^{sk+j}})$ for every $s \geq 0$ and $j = 1, \ldots, k-1$. This implies

$$\frac{Z(X_\lambda/\mathbb{F}_{q^k}, T)}{Z(Y_\lambda/\mathbb{F}_{q^k})} = \exp \sum_{s \geq 1} \frac{\#X_\lambda(\mathbb{F}_{q^{sk}}, T)}{\#Y_\lambda(\mathbb{F}_{q^{sk}}, T)} = \left( \frac{Z(X_\lambda/\mathbb{F}_{q^k}, T)}{Z(Y_\lambda/\mathbb{F}_{q^k}, T)} \right)^{1/k}$$

(2.1)

where the first equality uses the previous sentence and the second equality is simply definition. By [2] Theorem 7.3], there exists a polynomial $R_n(\lambda, T) \in 1+T\mathbb{Z}[T]$ of degree $\frac{n(n^n-(-1)^n)}{n+1} - n$, pure of weight $n-3$, such that

$$\left( \frac{Z(X_\lambda/\mathbb{F}_{q^k}, T)}{Z(Y_\lambda/\mathbb{F}_{q^k}, T)} \right)^{(-1)^n} = R_n(\lambda, q^kT).$$
Combining this with \([2.1]\) shows us that

$$
\left( \frac{Z(X_\lambda/\mathbb{F}_q, T)}{Z(Y_\lambda/\mathbb{F}_q, T)} \right)^{(−1)^n} = R_n(\lambda, q^kT^k)^{1/k}.
$$

Therefore, factorizing \(R_n(\lambda, T) = Q_1(T)^{\rho_1} \cdots Q_s(T)^{\rho_s}\) into irreducibles over \(\mathbb{Z}\) proves the theorem. \(\square\)

As a side remark, for \(n + 1 = 5\), Theorem \([1,1]\) explains the form of the zeta function of \(Z(X_\lambda/\mathbb{F}_q, T)\) found in \([1]\).

We note that, for the quintic \((n + 1 = 5)\), it follows from \([1\text{, Equation 10.3}]\) and \([1\text{, Equation 10.7}]\), in which they empirically compute the zeta functions of \(X_\lambda\) and \(W_\lambda\), that for \(X_\lambda\) smooth,

$$
\frac{Z(X_\lambda/\mathbb{F}_q, T)}{Z(W_\lambda/\mathbb{F}_q, T)} = R_\mathcal{A}(q^kT^k, \lambda)^{20/k} R_\mathcal{B}(q^kT^k, \lambda)^{30/k}
$$

where \(k\) is the smallest positive integer such that \(q^k \equiv 1 \mod 5\) and the \(R\)'s are quartic polynomials over \(\mathbb{Z}\) which are not necessarily irreducible. Note, \(k = 1, 2\) or \(4\). Furthermore, they have constructed auxiliary curves \(\mathcal{A}\) and \(\mathcal{B}\), both of genus \(4\), whose zeta functions experimentally correspond to \(R_\mathcal{A}\) and \(R_\mathcal{B}\), respectively. It would be interesting to find these “auxiliary varieties” for general \(n + 1\) and see how they fit into the framework of mirror symmetry (if at all).

\section{The proof of Theorem \([1\text{,1}]\)}

Without loss, we will write \(q\) instead of \(q^k\) in the following proof.

\subsection{Formulas for \(X_\lambda(\mathbb{F}_q)\) and \(Y_\lambda(\mathbb{F}_q)\) in terms of Gauss sums}

Define the Gauss sums \(G(k)\) as in Section 2. Also, let \(M\) be the \((n+2) \times (n+2)\)-matrix defined in \([2\text{, Section 3}]\).

Define the set

\[ E := \{ k \in \mathbb{Z}^{n+2} | 0 \leq k_i \leq q - 1 \text{ and } Mk \equiv 0 \mod(q - 1) \}. \]

For each \(k \in \mathbb{Z}^{n+2}\), define \(s(k)\) as the number of non-zero entries in \(Mk \in \mathbb{Z}^{n+2}\). Next, define

\[ E_1 := \{ k \in E | \text{not all } k_1, \ldots, k_{n+1} \text{ are the same, but } 0 \leq k_{n+2} \leq q - 1 \} \]
\[ E_2 := \{ k \in E | k_1 = k_2 = \cdots = k_{n+1}, 0 \leq k_{n+2} \leq q - 1 \} \]
\[ E_2^* := \{ k \in E_2 | 0 < k_1 < q - 1, s(k) = n + 2 \} \]
\[ S_k := \frac{q^{n+1-s(k)}}{(q-1)^{n+3-s(k)}} \left( \prod_{j=1}^{n+2} G(k_j) \right) \chi(\lambda)^{k_{n+2}}. \]

Now, \([2\text{, Section 3}]\) demonstrated that

\[ \#X_\lambda(\mathbb{F}_q) = \frac{-1}{q-1} + \sum_{k \in E_1} S_k + \sum_{k \in E_2} S_k. \]

Consider \(k \in E\). Suppose \(k_1 = \cdots = k_{n+1} = 0\). If \(k_{n+2} = 0\), then \(S_k = q^{n+1}/(q-1)\), else, if \(k_{n+2} = q - 1\), then \(S_k = -(q-1)^n\). Similarly, suppose \(k_1 = \cdots = k_{n+1} = q - 1\). If \(k_{n+2} = 0\), then \(S_k = (-1)^{n+1}q^n\), else, if \(k_{n+2} = q - 1\), then \(S_k = (-1)^nq^{n+1}/(q-1)\).

Next, notice

\[ Mk = \begin{pmatrix} k_1 + \cdots + k_{n+2} \\ (n+1)k_1 + k_{n+2} \\ (n+1)k_2 + k_{n+2} \\ \vdots \\ (n+1)k_{n+1} + k_{n+2} \end{pmatrix} \in \mathbb{Z}^{n+2}. \] (3.1)
If one of the rows equals zero, then we must have \( k_i = 0 \) for some \( 1 \leq i \leq n + 1 \). Thus, if \( k \in E_2 \) such that \( 0 < k_1 < q - 1 \), then all the rows of \( Mk \) must be non-zero; that is, \( s(k) = n + 2 \). Putting this together with the last paragraph, we find that for \( \lambda \neq 0 \), then

\[
#X_\lambda(F_q) = \frac{q^{n+1} + (-1)^n q^n - 1 - (q-1)^{n+1}}{q-1} + \sum_{k \in E_1} S_k + \sum_{k \in E_2} S_k. \tag{3.2}
\]

If \( \lambda = 0 \), then Section 2 tells us that \( k_{n+2} \) is forced to equal zero. Thus, in the above calculations, we need to neglect all terms in which \( k_{n+2} \neq 0 \). Doing this, we obtain

\[
#X_0(F_q) = \sum_{k \in E_1} S_k + N_0^* + \frac{q^{n+1} - 1}{q-1} + (-1)^n q^n + \frac{(-1)^n - (q-1)^n}{q}. \tag{3.3}
\]

Let \( N_\lambda^* \) denote the number of \( F_q \)-rational points on the affine (toric) variety defined by

\[
g(x_1, \ldots, x_n) := x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda = 0.
\]

In the proof of \cite[Theorem 5.1]{2}, we saw that for \( \lambda \neq 0 \)

\[
N_\lambda^* = \frac{(q-1)^n}{q} + \frac{(-1)^n}{q(q-1)} + \sum_{k \in E_2} S_k. \tag{3.3}
\]

Also, if \( \lambda = 0 \), we may calculate that

\[
N_0^* = \frac{(q-1)^n}{q} + \frac{(-1)^n+1}{q} + \sum_{k \in E_2} S_k.
\]

For ease of reference, recall from \cite[Section 2]{2} that, for all \( \lambda \in F_q \), we have

\[
#Y_\lambda(F_q) = N_\lambda^* - \frac{(q-1)^n}{q} + \frac{(-1)^n}{q} + \frac{q^n - 1}{q-1}. \tag{3.4}
\]

### 3.2 Finishing the proof of Theorem 1.1

For \( \lambda \neq 0 \), combining equations \(3.2\), \(3.3\), and \(3.4\) yields

\[
#X_\lambda(F_q) - #Y_\lambda(F_q) = \sum_{k \in E_1} S_k - (q-1)^n + \frac{q^{n+1} + (-1)^n q^n + (-1)^{n+1} - q^n}{q-1}.
\]

Similarly, if \( \lambda = 0 \), then

\[
#X_0(F_q) - #Y_0(F_q) = q^n(-1)^{n+1} + 1 + \sum_{k \in E_1} S_k.
\]

We may now prove Theorem 1.1 by demonstrating that the right-hand sides of the above two formulas equal zero when \( \gcd(n+1, q-1) = 1 \).

**Lemma 3.1.** If \( \gcd(n+1, q-1) = 1 \), then the right-hand sides are zero.

**Proof.** Let \( k \in E \). Suppose \( k_i \neq 0 \) for \( 1 \leq i \leq q-1 \). Then, by \(3.1\), we see that \( (n+1)k_i \equiv (n+1)k_j \) modulo \( q-1 \) for every \( 1 \leq i, j \leq q-1 \). By hypothesis, \( n+1 \) is invertible in \( \mathbb{Z}/(q-1) \), and so \( k_i \equiv k_j \). This means that, if \( k \in E_1 \) then at least one of the first \( n+1 \) coordinates must be zero.

Let \( 1 \leq i \leq n \). Suppose \( k \in E_1 \) and its first \( i \) coordinates are zero. Then \(3.1\) tells us that \( k_{n+2} \) is either zero or \( q-1 \). In the first case, we have \( k_{i+1} = \cdots = k_{n+1} = q-1 \) and \( s(k) = n+2 \). In the second case, again we have \( k_{i+1} = \cdots = k_{n+2} = q-1 \), but \( s(k) = (n+2) - i \). This leads to the following formulas:

- When \( k_{n+2} = 0 \) (first case): \( S_k = (-1)^{(n+1)-i}q^n \)
- When \( k_{n+2} = q-1 \) (second case): \( S_k = (-1)^{n-i}(q-1)^i q^{(n+1)-i} \).
(Note, if $\lambda = 0$, then $k_{n+2}$ must be zero, and so, the second case never occurs.) Permuting these zeros around in \( \binom{n+1}{i} \) ways among the first $n + 1$ coordinates gives us all possible points in $E_1$. That is, if we set

$$A := \sum_{i=1}^{n} \binom{n+1}{i} (-1)^{(n+1)-i} q^n \quad \text{(counts first case)}$$

and

$$B := \sum_{i=1}^{n} \binom{n+1}{i} (-1)^{(n+1)-i} q^n \quad \text{(counts second case)},$$

then we have $\sum_{k \in E_1} S_k = A + B$ for $\lambda \neq 0$, else $\sum_{k \in E_1} S_k = A$ for $\lambda = 0$. Now, by the binomial theorem, we see that

$$A = q^n[(1 - 1)^n - (-1)^{n+1} - 1] = q^n[(-1)^n]$$

and

$$B = (q - 1)^{-1}(-1)^n[(-q - q)^n - q^{n+1} - (-1)^n(q - 1)^{n+1}]$$

$$= (q - 1)^{-1}[-n + (-1)^{n+1}q^{n+1} + (q - 1)^{n+1}].$$

Thus, for $\lambda \neq 0$, we have

$$\sum_{k \in E_1} S_k = q^n[(-1)^n] + \frac{(-1)^n + (-1)^{n+1}q^{n+1} + (q - 1)^{n+1}}{q - 1}$$

which proves the lemma.

\[ \square \]

### 4 The proof of Theorem 1.3

Let us recall what we will prove.

**Theorem 1.3** Let $d := \gcd(n + 1, q^k - 1) > 1$. Then

1. $\#X_\lambda(\mathbb{F}_{q^k}) \equiv 0 \mod d$.
2. Writing $\lambda = -(n + 1)\psi$ in the new parameter $\psi$, if $n + 1$ is a power of a prime $\ell$, then

$$\#X_\lambda(\mathbb{F}_{q^k}) \equiv 0 \mod(\ell d) \quad \text{and} \quad \#Y_\lambda(\mathbb{F}_{q^k}) \equiv \begin{cases} 1 & \psi^{n+1} \equiv 1 \mod(\ell) \\ 0 & \text{otherwise} \end{cases}.$$

**Proof.** Without loss, we will write $q$ instead of $q^k$ in the following proof. First, let us prove the congruences on $X_\lambda$. We do this by gathering all the points of $X_\lambda$ that have the same number of coordinates zero. For each $1 \leq i \leq n - 1$, define $M^*_i$ as the number of $\mathbb{F}_q$-rational points in $\mathbb{P}^{n-i}_{\mathbb{F}_q}$ which lie on the diagonal hypersurface

$$x_{i+1}^{n+1} + \ldots + x_{n+1}^{n+1} = 0.$$

Notice that the group

$$G_i(\mathbb{F}_q) := \{(\zeta_{i+1}, \ldots, \zeta_{n+1})|\zeta_j \in \mathbb{F}_q, \zeta_j^{n+1} = 1, \zeta_j \cdot \zeta_{n+1} = 1\}/\{(\zeta_{i+1}, \ldots, \zeta_{n+1})|\zeta_j^{n+1} = 1\}$$

acts freely on the set of points defining $M^*_i$. Since there are $d := \gcd(n + 1, q - 1)$ many $(n + 1)$-roots of unity in $\mathbb{F}_q$, we have $\#G_i(\mathbb{F}_q) = d^{n+1-i}/d = d^{n-i}$. Consequently, $d^{n-i}$ divides $M^*_i$. Next, let $M^*_0$ be the number of $\mathbb{F}_q$-rational points in $\mathbb{P}^n_{\mathbb{F}_q}$ which lie on $X_\lambda$. The group

$$G(\mathbb{F}_q) := \{(\zeta_1, \ldots, \zeta_{n+1})|\zeta_j \in \mathbb{F}_q, \zeta_j^{n+1} = 1, \zeta_1 \cdots \zeta_{n+1} = 1\}/\{(\zeta_1, \ldots, \zeta)|\zeta_{n+1}^{n+1} = 1\}$$

acts freely on the points defining $M^*_0$, and so $\#G(\mathbb{F}_q) = d^n$ divides $M^*_0$. Putting this together, we have

$$\#X_\lambda(\mathbb{F}_q) = M^*_0 + \sum_{i=1}^{n-1} \binom{n+1}{i} M^*_i.$$
This proves the first part of the theorem since each \( M_i^* \) is divisible by \( d \). If \( n + 1 \) is a power of a prime \( \ell \), then not only are the \( M_i^* \) divisible by \( d \), but each of the binomial factors are divisible by \( \ell \); this proves the congruence on \( X_\lambda \) in the second part of the theorem.

We now assume \( n + 1 \) is a power of a prime \( \ell \). Let us prove the congruence on \( Y_\lambda \). With \( \lambda = -(n + 1)\psi \), recall from (3.4) that

\[
\# Y_\lambda(F_q) = N_\lambda^* - \frac{(q - 1)^n}{q} + \frac{(-1)^n}{q} + \frac{q^n - 1}{q - 1}
\]

where \( N_\lambda^* \) is the number of \( F_q \)-rational points in \( \mathbb{A}^{n+1}_q \) that satisfy

\[
\begin{align*}
x_1 + \cdots + x_{n+1} - (n + 1)\psi &= 0 \\
x_1 \cdots x_{n+1} &= 1
\end{align*}
\]

(4.1)

We claim that \( \# Y_\lambda(F_q) \equiv N_\lambda^* \mod \ell \). Since \( \gcd(n + 1, q - 1) > 1 \), \( q \equiv 1 \mod \ell \). Using the fact that \( \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1 \), we have: if \( \ell \) is an odd prime (the even case is similar), then

\[
-\frac{(q - 1)^n}{q} + \frac{(-1)^n}{q} + \frac{q^n - 1}{q - 1} \equiv -0 + 1 + n \equiv 0 \mod(\ell).
\]

This proves the claim.

Since we now have \( \# Y_\lambda(F_q) \equiv N_\lambda^* \mod \ell \), we will concentrate on computing \( N_\lambda^* \). Consider counting the points on (4.1) as follows: suppose a point \( x := (x_1, \ldots, x_{n+1}) \in A^{n+1}_q \) has two coordinates equal. Then we may permute these two around in \( \binom{n+1}{2} \) ways without changing the order of the other coordinates. Thus, the orbit of the point \( x \) under this type of permutation contains \( \binom{n+1}{2} \) points contained in the affine toric variety defined by (4.1). Note that we are not overcounting the points which have multiple pairs of coordinates being the same, like \( (1,1,2,2,2) \). If all the coordinates of \( x \) are different then we may permute these around in \( (n + 1)! \) ways.

Putting this together, we find, modulo \( \ell \):

\[
N_\lambda^*(F_q) \equiv \# \{ x \in A^{n+1}_q | \text{all coordinates are equal and } x \text{ satisfies (4.1)} \}.
\]

If all the coordinates are equal then we have the system \((n + 1)x - (n + 1)\psi = 0\) and \(x^{n+1} = 1\). By hypothesis, \( n + 1 \) is invertible in \( F_q \), thus, we have \( x = \psi \) for the first equation, and so \( \psi^{n+1} = 1 \) for the second. Therefore,

\[
N_\lambda^* \equiv \begin{cases} 1 & \psi^{n+1} = 1 \\ 0 & \text{otherwise} \end{cases} \mod(\ell).
\]

\[\square\]

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