EXTENDED CESÅ®RO COMPOSITION OPERATORS ON WEAK BLOCH-TYPE SPACES ON THE UNIT BALL OF A HILBERT SPACE

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Abstract. Denote by $B_X$ the unit ball of an infinite-dimensional complex Hilbert space $X$. Let $ψ ∈ H(B_X)$, the space of all holomorphic functions on the unit ball $B_X$, $ϕ ∈ S(B_X)$ the set of holomorphic self-maps of $B_X$. Let $C_ψ,ϕ : B_ν(B_X)$ (and $B_ν,0(B_X)$) → $B_µ(B_X)$ (and $B_µ,0(B_X)$) be weighted extended Cesàro operators induced by products of the extended Cesàro operator $C_ϕ$ and integral operator $T_ψ$. In this paper, we characterize the boundedness and compactness of $C_ψ,ϕ$ via the estimates for the restrictions of $ψ$ and $ϕ$ to a $m$-dimensional subspace of $X$ for some $m ≥ 2$. Based on these we give necessary as well as sufficient conditions for the boundedness, the (weak) compactness of $C_ψ,ϕ$ between spaces of Banach-valued holomorphic functions weak-associated to $B_ν(B_X)$ and $B_µ(B_X)$.

1. Introduction

Let $E$ be a space of holomorphic functions on the unit ball $B_X$ in a complex Banach space $X$. Let $H(B_X)$ be the class of all holomorphic functions on $B_X$ and $S(B_X)$ the collection of all the holomorphic self-maps of $B_X$. For a $ϕ ∈ S(B_X)$ and a $ψ ∈ H(B_X)$, the composition operator $C_ϕ$ and the extended Cesàro operator $T_ψ$ are defined by

$$(C_ϕf)(z) := (f ◦ ϕ)(z), \quad (T_ψf)(z) := \int_0^1 f(tz)R_ψ(tz)\frac{dt}{t} \quad ∀f ∈ E, ∀z ∈ B_n.$$ 

where $R_ψ(z) := D_ψ(z)$ is the radial derivative of $ψ$ at $z$ with $D_ψ(z)$ is the Fréchet derivative of $ψ$ at $z$.

The study of composition operators consists in the comparison of the properties of the $C_ϕ$ with that of the function $ϕ$ itself, which is called the symbol of $C_ϕ$. One can characterize boundedness and compactness of $C_ϕ$ and many other properties.

The problem of studying of composition operators on various Banach spaces of holomorphic functions on the unit disk or the unit ball, such as Hardy and Bergman spaces, the space $H^∞$ of all bounded holomorphic functions, the disk algebra and weighted Banach spaces with sup-norm, etc. received a special attention of many authors during the past several decades. The weighted composition operators on these spaces appeared in some works with different applications. There is a great number of topics on operators of such a type: boundedness and compactness, compact differences, topological structure, dynamical and ergodic properties.

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The extended Cesàro operator $T_\psi$ is a natural extension of the Cesàro operator acting $f \in H(B_1)$ where $B_1$ is the unit ball in $\mathbb{C}$,

$$C[f](z) := \sum_{j=0}^\infty \left( \frac{1}{j+1} \sum_{k=0}^j a_k \right)$$

with $f(z) = \sum_{j=0}^\infty a_j z^j$, the Taylor expansion of $f$.

It is well known that $C[\cdot]$ acts as a bounded linear operator on various spaces of holomorphic functions, including the Hardy and Bergman spaces. But it is not bounded on the Bloch space (see [Xi]). A little calculation shows

$$C[f](z) = \frac{1}{z} \int_0^z f(\eta) \frac{1}{1-\eta} d\eta = \int_0^1 f(tz) \left( \log \frac{1}{1-t} \right)' \bigg|_{\eta=1} dt.$$ 

Hence, on most holomorphic function spaces, $C[\cdot]$ is bounded if and only if the integral operator $f \mapsto \int_0^z f(t) \left( \log \frac{1}{1-t} \right)' dt$ is bounded. From this point of view it is natural to consider the extended Cesàro operator with holomorphic symbol $\psi$,

$$T_\psi f(z) := \int_0^z f(\eta) \psi'(\eta) d\eta.$$ 

The boundedness and compactness of this operator on Hardy spaces, Bergman spaces, Bloch-type spaces and Lipschitz spaces have been studied in [AC, AS, WH]. The extended Cesàro operator is a generalization of this operator. It has been well studied in many papers, see, for example, [AS, LS1, Hu] as well as the related references therein.

It is natural to discuss the product of extended Cesàro operator and composition operator. For $\varphi \in S(B_n)$ and $\psi \in H(B_n)$ the product can be expressed as

$$(1.1) \quad C_{\psi, \varphi}(f) := T_\psi C_{\varphi} f(z) = \int_0^1 f(\varphi(tz)) R(\psi(tz)) \frac{dt}{t} \quad f \in H(B_n), z \in B_n.$$ 

This operator is called extended Cesàro composition operator. It is interesting to characterize the boundedness and compactness of the product operator on all kinds of function spaces. Even on the disk of $\mathbb{C}$, some properties are not easily managed; see some recent papers in [LS2, LS3, Ya, LZ, Ta].

Building on those foundations, the present paper continues this line of research and discusses the operator in infinite dimension. Namely, we study the boundedness and the compactness of $C_{\psi, \varphi} : E_1 \to E_2$ between weighted Bloch-type spaces of holomorphic functions on the unit ball of a Hilbert space, which has been investigated by T. T. Quang in [Qu]. Based on these results we give the characterizations of the boundedness and the compactness of the operators $C_{\psi, \varphi} : WE_1(Y) \to WE_2(Y)$ between spaces of holomorphic functions with values in a Banach space $Y$, weak-associated to $E_1, E_2$ in the following sense:

Let $E$ be a space of holomorphic functions on the unit ball $B_X$ of a Banach space $X$ such that it contains the constant functions and its closed unit ball $B_E$ is compact in the compact open topology $\tau_{co}$ of $B_X$. Let $Y$ be a Banach space and $W \subset Y'$ be a separating subspace of the dual $Y'$ of $Y$. We say that the space

$$WE(Y) := \{ f : B_X \to Y : f \text{ is locally bounded and } w \circ f \in E, \forall w \in W \}$$

equipped with the norm

$$\|f\|_{WE(Y)} := \sup_{w \in W, \|w\| \leq 1} \|w \circ f\|_E.$$
is the $Y$-valued Banach space $W$-associated to $E$.

The paper is organized as follows.

Section 2 is devoted to recall some fundamental properties of the Banach space of Banach-valued holomorphic functions $W$-associated to a Banach space of scalar-valued holomorphic functions on the closed unit ball of a Banach space. We also introduce some assumptions on the subspace $W$ of the dual $Y'$ of $Y$ to give necessary as well as sufficient conditions for the boundedness, the (weak) compactness of $\tilde{C}_{\psi,\varphi}$ via the respective properties of $C_{\psi,\varphi}$ (Theorem 2.3). Some fundamental results on weighted Bloch-type space of holomorphic functions on the unit ball of a Hilbert space are also summarized in this section.

The main results are stated in Section 3. Via the estimates for the restrictions of the functions $\psi$ and $\varphi$ to $B_m$ for some $m \geq 2$, we characterize the boundedness and the compactness of $C_{\psi,\varphi}$ between the spaces of (little) Bloch-type $B_{\mu}(B_X)$, $B_{\mu,0}(B_X)$ as well as the equivalent relationships between them. It should be noted that a necessary condition (but not sufficient) and a sufficient condition (but not necessary) for the compactness of $C_{\psi,\varphi}$ are also obtained after any necessary minor modifications for the holomorphic self-map $\varphi$. Finally, by applying the results we give the necessary as well as sufficient conditions for the boundedness and the (weak) compactness of $\tilde{C}_{\psi,\varphi}$.

Several helpful test functions and auxiliary results concerning our computations will be introduced in this section.

The proofs of the main theorems above will be given in Sections 4 and 5.

Throughout this paper, we use the notions $X \lesssim Y$ and $X \simeq Y$ for non negative quantities $X$ and $Y$ to mean $X \leq CY$ and, respectively, $Y/C \leq X \leq CY$ for some inessential constant $C > 0$.

2. Preliminaries and Auxiliary Results

2.1. Weak holomorphic spaces. Let $X, Y$ be complex Banach spaces. Denote by $B_X$ the unit ball of $X$ (we write $B_n$ instead of $B_{C^n}$).

By $H(B_X,Y)$ we denote the vector space of $Y$-valued holomorphic functions on $B_X$. A holomorphic function $f \in H(B_X,Y)$ is called locally bounded holomorphic on $B_X$ if for every $z \in B_X$ there exists a neighbourhood $U_z$ of $0 \in X$ such that $f(U_z)$ is bounded. Put

$$H_{LB}(B_X,Y) = \{ f \in H(B_X,Y) : f \text{ is locally bounded on } B_X \}.$$

Suppose that $E$ is a Banach space of holomorphic functions $B_X \to \mathbb{C}$ such that

(e1) $E$ contains the constant functions,

(e2) the closed unit ball $B_E$ is compact in the compact open topology $\tau_{co}$ of $B_X$.

It is easy to check that the properties (e1)-(e2) are satisfied by a large number of well-known function spaces, such as classical Hardy, Bergman, BMOA, and Bloch spaces.

Let $W \subset Y'$ be a separating subspace of the dual $Y'$ of $Y$. Consider the $Y$-valued Banach space $W$-associated to $E$

$$WE(Y) := \{ f : B_X \to Y : f \text{ is locally bounded and } w \circ f \in E, \forall w \in W \}$$

equipped with the norm

$$\| f \|_{WE(Y)} := \sup_{w \in W, \| w \| \leq 1} \| w \circ f \|_E.$$

The following proposition introduce a relation between $E$ and $WE(Y)$:
Proposition 2.1 ([Qu], Proposition 2.1). Let \( X, Y \) be complex Banach spaces and \( W \subset Y' \) be a separating subspace. Let \( E \) be a Banach space of holomorphic functions \( B_X \rightarrow \mathbb{C} \) satisfying (e1)-(e2) and \( WE(Y) \) be the \( Y \)-valued Banach space \( W \)-associated to \( E \). Then, the following assertions hold:

1. (we1) \( f \mapsto f \otimes y \) defines a bounded linear operator \( P_y : E \rightarrow WE(Y) \) for any \( y \in Y \), where \( (f \otimes y)(z) = f(z) y \) for \( z \in B_X \).
2. (we2) \( g \mapsto w \circ g \) defines a bounded linear operator \( Q_w : WE(Y) \rightarrow E \) for any \( w \in W \).
3. (we3) For all \( z \in B_X \) the point evaluations \( \delta_z : WE(Y) \rightarrow (Y, \sigma(Y, W)) \), where \( \delta_z(g) = g(z) \), are continuous.

In the case the hypothesis "separating" of \( W \) is replaced by a stronger one that \( W \) is "almost norming", we obtain the assertion (we3') below instead of (we3):

4. (we3') For all \( z \in B_X \) the point evaluations \( \tilde{\delta}_z : WE(Y) \rightarrow Y \), are bounded.

Here, the subspace \( W \) of \( Y' \) is called almost norming if

\[
g_{W}(x) = \sup_{w \in W, \|w\| \leq 1} |w(x)|
\]

defines an equivalent norm on \( Y \).

Theorem 2.2 (Linearization, [Qu], Theorem 2.2). Let \( X, Y \) be complex Banach spaces and \( W \subset Y' \) be a separating subspace. Let \( E \) be a Banach space of holomorphic functions \( B_X \rightarrow \mathbb{C} \) satisfying (e1)-(e2). Then there exist a Banach space \( *E \) and a mapping \( \delta_X \in H(B_X, *E) \) with the following universal property: A function \( f \in WE(Y) \) if and only if there is a unique mapping \( T_f \in L(*E, Y) \) such that \( T_f \circ \delta_X = f \). This property characterizes \( *E \) uniquely up to an isometric isomorphism.

Moreover, the mapping

\[
\Phi : f \in WE(Y) \mapsto T_f \in L(*E, Y)
\]

is a topological isomorphism.

2.2. The extended Cesàro composition operators. Given \( \psi \in H(B_X) \) and \( z \in B_X \).

Then, we denote by \( D\psi(z) \) the Fréchet derivative of \( \psi \) at \( z \) and \( R\psi(z) := D\psi(z)z \) the radial derivative of \( \psi \) at \( z \). Note that, in the case \( X \) is a complex Hilbert space with an orthonormal basis \( (e_k)_{k \in \Gamma} \) we have

\[
R\psi(z) = \langle \nabla\psi(z), z \rangle
\]

where \( \nabla\psi(z) = \left( \frac{\partial\psi}{\partial x_k}(z) \right)_{k \in \Gamma} \) is the gradient of \( \psi \) at \( z \), and \( z = \sum_{k \in \Gamma} z_k e_k \in B_X \). It is obvious that \( |R\psi(z)| \leq |\nabla\psi(z)||z| < |\nabla\psi(z)| \)

for every \( z \in B_X \).

Let \( E_1 \) and \( E_2 \) be Banach spaces of holomorphic functions \( B_X \rightarrow \mathbb{C} \) satisfying the conditions (e1) and (e2). Let \( \psi \in H(B_X) \) and \( \varphi \in S(B_X) \), the set of holomorphic self-maps of \( B_X \). Consider the operators \( C_{\psi, \varphi} : E_1 \rightarrow E_2 \) and \( \tilde{C}_{\psi, \varphi} : WE_1(Y) \rightarrow WE_2(Y) \) given by

\[
C_{\psi, \varphi}(f) = \int_0^1 f(\varphi(tz))R(\psi(tz))\frac{dt}{t} \quad f \in E_1, z \in B_X,
\]

\[
\tilde{C}_{\psi, \varphi}(g) = \int_0^1 g(\varphi(tz))R(\psi(tz))\frac{dt}{t} \quad g \in WE_1(Y), z \in B_X.
\]
Now, let \( y \in Y, w \in W \) such that \( \|y\| = \|w\| = 1 \) and \( w(y) = 1 \). Consider the maps \( P_y \) and \( Q_w \) as in Proposition 2.1. It is easy to check that
\[
C_{\psi, \varphi} = Q_w \circ \tilde{C}_{\psi, \varphi} \circ P_y.
\]
Then, by an argument analogous to that used for the proof of Proposition 2.1 in [Qu], we get the following:

**Theorem 2.3.** Let \( X, Y \) be complex Banach spaces and \( W \subset Y' \) be a subspace. Let \( E_1 \) and \( E_2 \) be Banach spaces of holomorphic functions \( B_X \to \mathbb{C} \) satisfying (e1)-(e2). Let \( \psi \in H(B_X) \) and \( \varphi \in S(B_X) \).

1. If \( W \) is separating then \( C_{\psi, \varphi} \) is bounded if and only if \( \tilde{C}_{\psi, \varphi} \) is bounded;
2. If \( W \) is almost norming and \( C_{\psi, \varphi} \) compact then:
   a. \( \tilde{C}_{\psi, \varphi} \) is compact if and only if the identity map \( I_Y : Y \to Y \) is compact, i.e., \( \dim Y < \infty \);
   b. \( C_{\psi, \varphi} \) is weakly compact if and only if the identity map \( I_Y : Y \to Y \) is weakly compact.

**Remark 2.1.** In the case \( W \) is separating and \( Y \) is separable, we have \( W \) is almost norming and \( I_Y \) is weakly compact. Then as in (b) we get \( C_{\psi, \varphi} \) is weakly compact if \( C_{\psi, \varphi} \) is compact.

2.3. The Bloch-type spaces on the unit ball of a Hilbert space. Throughout the forthcoming, unless otherwise specified, we shall denote by \( X \) a complex Hilbert space with the open unit ball \( B_X \) and \( Y \) a Banach space. Denote
\[
H^\infty(B_X, Y) = \left\{ f \in H(B_X, Y) : \sup_{z \in B_X} \|f(z)\| < \infty \right\}.
\]
It is easy to check that \( H^\infty(B_X, Y) \) is Banach under the sup-norm
\[
\|f\|_\infty := \sup_{z \in B_X} \|f(z)\|.
\]

Let \((e_k)_{k \in \Gamma}\) be an orthonormal basis of \( X \) that we fix at once. Then every \( z \in X \) can be written as
\[
z = \sum_{k \in \Gamma} z_k e_k, \quad \overline{z} = \sum_{k \in \Gamma} \overline{z_k} e_k.
\]
For \( \varphi \in S(B_X) \), the set of holomorphic self-maps on \( B_X \) we write \( \varphi(z) = \sum_{k \in \Gamma} \varphi_k(z) \) and \( \varphi'(z) : X \to X \) its derivative at \( z \), and \( R \varphi(z) = \langle \varphi'(z), \overline{z} \rangle \) its radial derivative at \( z \).

Given \( f \in H(B_X, Y) \) and \( z \in B_X \). We define
\[
\|\nabla f(z)\| := \sup_{u \in Y': \|u\| = 1} \|\nabla(u \circ f)(z)\|,
\]
\[
\|R f(z)\| := \sup_{u \in Y': \|u\| = 1} |R(u \circ f)(z)|,
\]
where
\[
R(u \circ f)(z) = \langle \nabla(u \circ f)(z), \overline{z} \rangle.
\]
As above, it is obvious that \( \|R f(z)\| \leq \|\nabla f(z)\||z| < \|\nabla f(z)\| \) for every \( z \in B_X \).

**Definition 2.1.** A positive continuous function \( \mu \) on the interval \([0, 1]\) is called normal if there are three constants \( 0 \leq \delta < 1 \) and \( 0 < a < b < \infty \) such that
\[
(W_1) \quad \frac{\mu(t)}{(1-t)^a} \ \text{is decreasing on} \ [\delta, 1), \quad \lim_{t \to 1} \frac{\mu(t)}{(1-t)^a} = 0,
\]
(W₂) \( \frac{\mu(t)}{(1-t)^b} \) is increasing on \([\delta, 1)\), \( \lim_{t \to 1} \frac{\mu(t)}{(1-t)^b} = \infty \).

If we say that a function \( \mu : B_X \to [0, \infty) \) is normal, we also assume that it is radial, that is, \( \mu(z) = \mu(\|z\|) \) for every \( z \in B_X \).

Note that, since \( \mu \) is positive, continuous, \( m_\mu := \min_{t \in [0,\delta]} \mu(t) > 0 \). Moreover, it follows from (W₁) that \( \mu \) is increasing on \([\delta, 1)\), hence, we obtain that \( \max_{t \in [0,1]} \mu(t) =: M_\mu < \infty \).

Then, it is easy to check that

\[
\mu(z) \int_0^{\|z\|} \frac{dt}{\mu(t)} < R_\mu := \delta \frac{M_\mu}{m_\mu} + 1 - \delta < \infty \quad \text{for every } z \in B_X.
\]

Throughout this paper, the weight \( \mu \) always is assumed to be normal. In the sequel, when no confusion can arise, we will use the symbol \( ♦ \) to denote either \( \nabla \) or \( R \).

We define Bloch-type spaces on the unit ball \( B_X \) as follows:

\[
\mathcal{B}_{\mu}^{\circ}(B_X, Y) := \left\{ f \in H(B_X, Y) : \|f\|_{s\mathcal{B}_{\mu}^{\circ}(B_X, Y)} := \sup_{z \in B_X} \mu(z)\|\nabla f(z)\| < \infty \right\}.
\]

It is easy to check \( \| \cdot \|_{s\mathcal{B}_{\mu}^{\circ}(B_X, Y)} \) is a semi-norm on \( \mathcal{B}_{\mu}^{\circ}(B_X, Y) \) and this space is Banach under the sup-norm

\[
\|f\|_{\mathcal{B}_{\mu}^{\circ}(B_X, Y)} := \|f(0)\| + \|f\|_{s\mathcal{B}_{\mu}^{\circ}(B_X, Y)}.
\]

We also define little Bloch-type spaces on the unit ball \( B_X \) as follows:

\[
\mathcal{B}_{\mu}^{\circ,0}(B_X, Y) := \left\{ f \in \mathcal{B}_{\mu}^{\circ}(B_X, Y) : \lim_{\|z\| \to 1} \mu(z)\|\nabla f(z)\| = 0 \right\}
\]

endowed with the norm induced by \( \mathcal{B}_{\mu}^{\circ}(B_X, Y) \).

In the case \( Y = \mathbb{C} \) we write \( \mathcal{B}_{\mu}^{\circ}(B_X) \), \( \mathcal{B}_{\mu,0}^{\circ}(B_X) \) instead of the respective notations.

It is clear that for every separating subspace \( W \) of \( Y \) we have

\[
\mathcal{B}_{\mu}^{\circ}(B_X, Y) \subset WB_{\mu}^{\circ}(B_X)(Y), \quad \mathcal{B}_{\mu,0}^{\circ}(B_X, Y) \subset WB_{\mu,0}^{\circ}(B_X)(Y),
\]

For \( \mu(z) = 1 - \|z\|^2 \) we write \( \mathcal{B}^{\circ}(B_X, Y) \) instead of \( \mathcal{B}_{\mu}^{\circ}(B_X, Y) \) and when \( \dim X = m \), \( Y = \mathbb{C} \) we obtain correspondingly the classical Bloch space \( \mathcal{B}^{\circ}(\mathbb{B}_m) \).

We will show below that the study of Bloch-type spaces on the unit ball can be reduced to studying functions defined on finite dimensional subspaces.

For each \( m \in \mathbb{N} \) we denote

\[ z[m] := (z_1, \ldots, z_m) \in \mathbb{B}_m \]

where \( \mathbb{B}_m \) is the open unit ball in \( \mathbb{C}^m \). For \( m \geq 2 \) by

\[ OSM_m := \{ x = (x_1, \ldots, x_m), \ x_k \in X, \langle x_k, x_j \rangle = \delta_{kj} \} \]

we denote the family of orthonormal systems of order \( m \).

It is clear that \( OSM_1 \) is the unit sphere of \( X \).

For every \( x \in OSM_m \), \( f \in H(B_X, Y) \) we define

\[ f_x(z[m]) = f\left( \sum_{k=1}^{m} z_k x_k \right). \]

Then

\[ \nabla (u \circ f_x)(z[m]) = \left( \frac{\partial (u \circ f_x)}{\partial z_j} \left( \sum_{k=1}^{m} z_k x_k \right) \right)_{j \in \Gamma} \quad \text{for every } u \in Y', \]
and hence
\[(2.2) \quad \|\nabla f(x(z_{[m]}))\| = \|\nabla f \left(\sum_{k=1}^{m} z_k e_k\right)\|.
\]

Now, for each finite subset \(F \subset \Gamma\), in symbol \(|F| < \infty\), we denote by \(\mathbb{B}_F\) the unit ball of span\(\{e_k, k \in F\}\) and \(f_F = f_x\) where \(x = \{e_k, k \in F\}\). For each \(z \in B_X\) and each \(F \subset \Gamma\) finite we write
\[
z_F = \sum_{k \in F} z_k e_k \in \mathbb{B}_F.
\]

**Definition 2.2.** Let \(\mathbb{B}_1\) be the open unit ball in \(C\) and \(f \in H(B_X, Y)\). We define an affine semi-norm as follows
\[
\|f\|_{\mathcal{B}_{\mu}^{aff}(B_X, Y)} := \sup_{\|x\|=1} \|f(\cdot x)\|_{\mathcal{B}_{\mu}(\mathbb{B}_1, Y)}
\]
where \(f(\cdot x) : \mathbb{B}_1 \to Y\) given by \(f(\cdot x)(\lambda) = f(\lambda x)\) for every \(\lambda \in \mathbb{B}_1\), and
\[
\|f(\cdot x)\|_{\mathcal{B}_{\mu}^{aff}(\mathbb{B}_1, Y)} = \sup_{\lambda \in \mathbb{B}_1} \mu(\lambda x)\|f'(\cdot x)(\lambda)\|.
\]

It is easy to see that \(\|\cdot\|_{\mathcal{B}_{\mu}^{aff}(B_X, Y)}\) is a semi-norm on \(\mathcal{B}_{\mu}(B_X, Y)\). We denote
\[
\mathcal{B}_{\mu}^{aff}(B_X, Y) := \{f \in \mathcal{B}_{\mu}(B_X, Y) : \|f\|_{\mathcal{B}_{\mu}^{aff}(B_X, Y)} < \infty\}.
\]

It is also easy to check that \(\mathcal{B}_{\mu}^{aff}(B_X, Y)\) is Banach under the norm
\[
\|f\|_{\mathcal{B}_{\mu}^{aff}(B_X, Y)} := \|f(0)\| + \|f\|_{\mathcal{B}_{\mu}^{aff}(B_X, Y)}.
\]

We also define little affine Bloch-type spaces on the unit ball \(B_X\) as follows:
\[
\mathcal{B}_{\mu,0}^{aff}(B_X, Y) := \{f \in \mathcal{B}_{\mu}^{aff}(B_X, Y) : \lim_{|\lambda| \to 0} \sup_{\|x\|=1} \mu(\lambda x)\|f'(\cdot x)(\lambda)\| = 0\}.
\]

As the above, for \(\mu(z) = 1 - \|z\|^2\) we use notation \(\mathcal{B}\) instead of \(\mathcal{B}_{\mu}\).

Now, let us recall a number of results proved in [Qu].

**Proposition 2.4 ([Qu], Proposition 4.1).** Let \(f \in H(B_X, Y)\). The following are equivalent:

1. \(f \in \mathcal{B}_{\mu}^{\nabla}(B_X, Y)\);
2. \(\sup_{\|F\| < \infty} \|f_F\|_{\mathcal{B}_{\mu}^{\nabla}(\mathbb{B}_F, Y)} : F \subset \Gamma, |F| < \infty < \infty\);
3. \(\sup_{x \in \mathbb{B}_m} \|f_F\|_{\mathcal{B}_{\mu}^{\nabla}(\mathbb{B}_m, Y)} < \infty\) for every \(m \geq 2\);
4. There exists \(m \geq 2\) such that \(\sup_{x \in \mathbb{B}_m} \|f_F\|_{\mathcal{B}_{\mu}^{\nabla}(\mathbb{B}_m, Y)} < \infty\).

Moreover, for each \(m \geq 2\)
\[(2.3) \quad \|f\|_{\mathcal{B}_{\mu}^{\nabla}(B_X, Y)} = \sup_{|F| < \infty} \|f_F\|_{\mathcal{B}_{\mu}^{\nabla}(\mathbb{B}_F, Y)} = \sup_{x \in \mathbb{B}_m} \|f_F\|_{\mathcal{B}_{\mu}^{\nabla}(\mathbb{B}_m, Y)}.
\]

**Remark 2.2.** The proposition is not true for the case \(m = 1\).

**Proposition 2.5 ([Qu], Proposition 4.2).** Let \(f \in H(B_X, Y)\). The following are equivalent:

1. \(f \in \mathcal{B}_{\mu,0}^{\nabla}(B_X, Y)\);
2. \(\forall \varepsilon > 0 \exists \theta > 0 \forall z \in B_X\) with \(\|z_{[F]}\| > \theta\) for every \(F \subset \Gamma, |F| < \infty\)
\[
\sup_{F \subset \Gamma, |F| < \infty} \mu(\|z_{[F]}\|)\|\nabla f_F(z_{[F]}\|) < \varepsilon;
\]
Moreover, (Qu, Theorem 4.9) of the norms in associated Bloch-type spaces:

Proposition 2.8 (Qu, Proposition 4.11)

We denote $H$ be a Banach space and $f \in H(B_X, Y)$. The invariant gradient norm $\varphi_a(z) = \frac{a - P_a(z) - s_aQ_a(z)}{1 - \langle z, a \rangle}$, $z \in B_X$ where $s_a = \sqrt{1 - \|a\|^2}$, $P_a$ is the orthogonal projection from $X$ onto the one dimensional subspace $[a]$ generated by $a$, and $Q_a$ is the orthogonal projection from $X$ onto $X \ominus [a]$.

When $a = 0$, we simply define $\varphi_a(z) = -z$. It is obvious that each $\varphi_a$ is a holomorphic mapping from $B_X$ into $X$.

Definition 2.3. Let $X$ be a complex Hilbert space, $Y$ be a Banach space and $f \in H(B_X, Y)$. The invariant gradient norm $\|\nabla f(z)\| := \|\nabla (f \circ \varphi_a)(0)\|$ for any $z \in B_X$.

We define invariant semi-norm as follows

$$\|f\|_{sB^\text{inv}(B_X, Y)} := \sup_{z \in B_X} \|\nabla f(z)\| = \sup_{z \in B_X} \sup_{u \in Y', \|u\| \leq 1} \|\nabla (u \circ f)(z)\|.$$

We denote $B^\text{inv}(B_X, Y) := \{f \in B(B_X, Y) : \|f\|_{sB^\text{inv}(B_X, Y)} < \infty\}$.

It is also easy to check that $B^\text{inv}(B_X, Y)$ is Banach under the norm

$$\|f\|_{B^\text{inv}(B_X, Y)} := \|f(0)\| + \|f\|_{sB^\text{inv}(B_X, Y)}.$$

Theorem 2.7 (Qu, Theorem 4.9). The spaces $B^\nabla(B_X, Y)$, and $B^\text{inv}(B_X, Y)$ coincide. Moreover:

$$\|f\|_{B^\text{inv}(B_X, Y)} \leq \|f\|_{B^\text{inv}(B_X, Y)} \leq \|f\|_{B^\nabla(B_X, Y)}.$$

Now let $W \subset Y'$ be a separating subspace of the dual $Y'$, then we obtain the equivalence of the norms in associated Bloch-type spaces:

$$\|\cdot\|_{W^R(B_X)} \cong \|\cdot\|_{W^R(B_X)} \cong \|\cdot\|_{W^\text{inv}(B_X)}.$$

Hence, for the sake of simplicity, from now on we write $B_\mu$ instead of $B^\mu_\mu$.

Finally, the following shows that $WB_\mu(B_X)(Y), WB_\mu(0)(B_X)(Y)$ satisfy (we1)-(we3).

Proposition 2.8 (Qu, Proposition 4.11). Let $W \subset Y'$ be a separating subspace. Let $\mu$ be a normal weight on $B_X$. Then $B_\mu(B_X), B_\mu(0)(B_X)$ satisfy (e1) and (e2), and hence, $WB_\mu(B_X)(Y)$ and $WB_\mu(0)(B_X)(Y)$ satisfy (we1)-(we3).
Remark 2.3. (1) In the proof of this proposition we used the following estimate

\begin{equation}
|f(z)| \leq \max \left\{ 1, \int_0^{\|z\|} \frac{dt}{\mu(t)} \right\} \|f\|_{B_\nu(B_X)} \quad \forall f \in B_\mu(B_X), \forall z \in B_X.
\end{equation}

In fact, the estimate \((2.5)\) can be written as follows

\begin{equation}
|f(z)| \leq |f(0)| + \int_0^{\|z\|} \frac{dt}{\mu(t)} \|f\|_{sB_\nu}.
\end{equation}

(2) Since \(\mu\) is non-increasing, it is easy to see that \(M_\mu := \sup_{t \in [0,1]} \mu(t) < \infty\). Then, from \((2.5)\) we obtain that

\begin{equation}
\mu(z)|f(z)| \leq \max\{1, M_\mu\} \|f\|_{B_\nu(B_X)} \quad \forall z \in B_X.
\end{equation}

3. The Main Results and Test Functions

This section introduces the main results of the paper and test functions that are useful for the proofs.

In this section we consider \(\nu\) is a non-increasing, normal weight on \(B_X\) and \(\varphi \in S(B_X)\). 

We begin this section by constructing test functions that are useful for the proofs of our main results.

3.1. The main results. Let \(\nu, \mu\) be normal weights on \(B_X\). Let \(\varphi \in S(B_X)\), the set of holomorphic self-maps on \(B_X\) and \(\psi \in H(B_X)\). For each \(F \subset \Gamma\) finite and \(m \in \mathbb{N}\) we write

\[ \varphi^{[F]} = \varphi|_{\text{span}\{e_k, k \in F\}}, \quad \varphi^{[m]} = \varphi|_{\text{span}\{e_1, \ldots, e_m\}} \]

and

\[ \psi^{[F]} = \psi|_{\text{span}\{e_k, k \in F\}}, \quad \psi^{[m]} = \psi|_{\text{span}\{e_1, \ldots, e_m\}}. \]

For each \(j \in \Gamma\) and \(k \geq 1\) we denote

\[ \varphi_j(\cdot) := \langle \varphi(\cdot), e_j \rangle, \quad \varphi_k(\cdot) := (\varphi_1(\cdot), \ldots, \varphi_k(\cdot)). \]

In this section we investigate the boundedness and the compactness of the operators \(W_{\psi, \varphi}\) between the (little) Bloch-type spaces \(B_\nu\) \((B_\nu, 0)\) and \(B_\mu\) \((B_\mu, 0)\) via the estimates of \(\psi^{[m]}, \varphi^{[m]}\) and \(\varphi_k\). Hence, by Theorem 2.3 some characterizations for the boundedness and compactness of the operators \(C_{\psi, \varphi}\) between spaces \(W_{B_\nu}(B_X, Y)\), \((W_{B_\nu, 0}(B_X, Y))\) and \(W_{B_\mu}(B_X, Y)\), \((W_{B_\mu, 0}(B_X, Y))\) will be obtained from these results.

By Theorem 2.4 in the paper we will only present the results for the spaces \(B_\mu(B_X)\).

First we investigate the boundedness of extended Cesàro composition operators. We use there certain quantities, which will be used in this work. We list them below:

\begin{align}
(3.1a) & \mu^{[m]}(y)\|R\psi^{[m]}(y)\| \max \left\{ 1, \int_0^{\|\varphi^{[m]}(y)\|} \frac{dt}{\nu(t)} \right\}, \\
(3.1b) & \mu^{[F]}(y)\|R\psi^{[F]}(y)\| \max \left\{ 1, \int_0^{\|\varphi^{[F]}(y)\|} \frac{dt}{\nu(t)} \right\}, \\
(3.1c) & \mu(y)\|R\psi(y)\| \max \left\{ 1, \int_0^{\|\varphi(y)\|} \frac{dt}{\nu(t)} \right\}, \\
(3.1d) & \mu(y)\|R\psi(y)\| \max \left\{ 1, \int_0^{\|\varphi(y)\|} \frac{dt}{\nu(t)} \right\}.
\end{align}
We also use the notations $C_{\psi, \varphi}^m : B_\nu \to B_\mu$, $C_{\psi, \varphi}^0 : B_{\nu, 0} \to B_\mu$, $C_{\psi, \varphi}^{0, 0} : B_{\nu, 0} \to B_{\mu, 0}$ to denote the extended Cesàro composition operators.

Now we are ready to state the first main result. The first ones propose to the boundeness of $C_{\psi, \varphi}^m$, $C_{\psi, \varphi}^0$, $C_{\psi, \varphi}^{0, 0}$ and the equivalent relationships between them.

**Theorem 3.1.** Let $\psi \in H(B_X)$, $\varphi \in S(B_X)$ and $\mu, \nu$ be normal weights on $B_X$. Then the following are equivalent:

1. $M_{\psi, \varphi}^{[m]} := \sup_{y \in B_m} \| \psi(y) \| < \infty$ for some $m \geq 2$;
2. $M_{\psi, \varphi}^{\langle F \rangle} := \sup_{y \in B_{\Gamma \varphi}} \sup_{F \subset \Gamma \varphi} \| \psi(y) \| < \infty$ for every $F \subset \Gamma$ finite;
3. $M_{\psi, \varphi}^{(k)} := \sup_{y \in B_X} \| \psi(y) \| < \infty$ for every $k \geq 1$;
4. $M_{\psi, \varphi} := \sup_{y \in B_X} \| \psi(y) \| < \infty$;
5. $C_{\psi, \varphi}^m : B_\nu(B_X) \to B_\mu(B_X)$ is bounded;
6. $C_{\psi, \varphi}^0 : B_{\nu, 0}(B_X) \to B_{\mu, 0}(B_X)$ is bounded.

Moreover, if $C_{\psi, \varphi}$ is bounded, the following asymptotic relation holds for some $m \geq 2$:

$$\| C_{\psi, \varphi} \| \asymp M_{\psi, \varphi}^{[m]}.$$

Next, we will touch the characterizations for the boundedness of the operators from $B_{\nu, 0}(B_X)$ to $B_{\mu, 0}(B_X)$.

**Theorem 3.2.** Let $\psi \in H(B_X)$, $\varphi \in S(B_X)$ and $\mu, \nu$ be normal weights on $B_X$. Then the following are equivalent:

1. $\psi[m] \in B_{\mu, 0}(B_m)$ and $M_{\psi, \varphi}^{[m]} := \sup_{y \in B_m} \| \psi(y) \| < \infty$ for some $m \geq 2$;
2. $\psi[F] \in B_{\mu, 0}(B_{\Gamma \varphi})$ and $M_{\psi, \varphi}^{\langle F \rangle} := \sup_{y \in B_{\Gamma \varphi}} \| \psi(y) \| < \infty$ for every $F \subset \Gamma$ finite;
3. $\psi \in B_{\mu, 0}(B_X)$ and $M_{\psi, \varphi}^{(k)} := \sup_{y \in B_X} \| \psi(y) \| < \infty$ for every $k \geq 1$;
4. $\psi \in B_{\mu, 0}(B_X)$ and $M_{\psi, \varphi} := \sup_{y \in B_X} \| \psi(y) \| < \infty$;
5. $C_{\psi, \varphi}^0 : B_{\nu, 0}(B_X) \to B_{\mu, 0}(B_X)$ is bounded.

In this case, the following asymptotic relation holds for some $m \geq 2$:

$$\| C_{\psi, \varphi}^0 \| \asymp M_{\psi, \varphi}^{[m]}.$$

Now the following are characterizations of the compactness of extended Cesàro composition operators $C_{\psi, \varphi}$.

**Theorem 3.3.** Let $\psi \in H(B_X)$, $\varphi \in S(B_X)$ and $\mu, \nu$ be normal weights on $B_X$ such that $\int_0^1 \frac{d\nu}{\nu(y)} = \infty$.

(A) The following are equivalent:

1. $\| \varphi[k](y) \| \to 1$ or every $k \geq 1$;
2. $C_{\psi, \varphi}^m : B_\nu(B_X) \to B_\mu(B_X)$ is compact;
3. $C_{\psi, \varphi}^0 : B_{\nu, 0}(B_X) \to B_{\mu, 0}(B_X)$ is compact.

(B) Under the additional assumption that there exists $m \geq 2$ such that

$B[m, r] := \{ \varphi[m](y) : \| \varphi[m](y) \| < r, y \in B_m \}$

is relatively compact for every $0 \leq r < 1$.

the assertions (2), (3) and following are equivalent:
Theorem 3.7. Let \( \phi \) the function (5) \( C(4) \). It is easy to check that \( \phi \) is compact. Then (2) \( \psi \) is compact. Next, we discuss the compactness of the operator \( C_{\psi,\varphi}^0 : B_\varphi(B_X) \to B_\mu(B_X) \). Theorem 3.4. Let \( \psi \in H(B_X) \) and \( \varphi \in S(B_X) \). Assume that \( C_{\psi,\varphi} : B_\varphi(B_X) \to B_\mu(B_X) \) is compact. Then (3.5) \( \varphi(rB_X) \) is relatively compact for every \( 0 \leq r < 1 \).

Example 3.1. Let \( \{e_j\}_{j \geq 1} \) be an orthonormal sequence in a Hilbert space \( X \). Consider the function \( \varphi \in S(B_X) \) given by

\[
\varphi(z) := \sum_{n=1}^{\infty} \langle z, e_n \rangle^n e_n \quad \forall z \in B_X.
\]

It is easy to check that \( \varphi(rB_X) \) is relatively compact for every \( 0 < r < 1 \). Now we show that \( B[\varphi, \frac{1}{2}] \) is not relatively compact. Consider the sequence \( \{z_k\}_{k \geq 1} \subset B_X \) given by

\[
z_k = \frac{1}{\sqrt{4}} e_k \quad \forall k \geq 1.
\]

It is obvious \( \|\varphi(z_k)\| < \frac{1}{2} \) for every \( k \geq 1 \). Then for every \( k \geq 1 \) and \( s > 1 \) we have

\[
\|\varphi(z_k) - \varphi(z_{k+s})\| = \frac{\sqrt{2}}{4}.
\]

Thus, we get the desired claim.

Corollary 3.5. Assume that \( \sup_{z \in X} \|\varphi(z)\| < \infty \). Then \( C_{\psi,\varphi}, C_{\psi,\varphi}^0 \) are compact if and only if \( \varphi(B_X) \) is relatively compact.

Indeed, in this case, by the hypotheses, (3.12) \( \to 0 \) as \( \|\varphi(y)\| \to 1 \). and (3.3), (3.5) always hold.

Theorem 3.6. Let \( \psi \in H(B_X) \) and \( \varphi \in S(B_X) \) such that (3.4) holds for some \( m \geq 2 \). Let \( \mu, \nu \) be normal weights on \( B_X \) such that \( \int_0^1 \frac{dt}{\nu(t)} < \infty \). Then the following are equivalent:

1. \( \psi^{[m]} \in B_{\mu^{[m]}}(B_m) \);
2. \( \psi^{[F]} \in B_{\mu^{[F]}}(B_{[F]}) \) for every \( F \subset \Gamma \) finite;
3. \( \psi \in B_{\mu}(B_X) \);
4. \( C_{\psi,\varphi} : B_\varphi(B_X) \to B_\mu(B_X) \) is compact;
5. \( C_{\psi,\varphi}^0 : B_{\nu,0}(B_X) \to B_\mu(B_X) \) is compact.

Next, we discuss the compactness of the operator \( C_{\psi,\varphi}^0 : B_{\nu,0}(B_X) \to B_{\mu,0}(B_X) \).

Theorem 3.7. Let \( \psi \in H(B_X) \), \( \varphi \in S(B_X) \) and \( \mu, \nu \) be normal weights on \( B_X \). Then

(A) The following are equivalent:
1. (3.12) \( \to 0 \) as \( \|\varphi(k)\| \to 1 \) for every \( k \geq 1 \);
Corollary 3.8. Let $\psi \in H(B_X)$, $\varphi \in S(B_X)$ and $\mu, \nu$ be normal weights on $B_X$ such that $\int_0^1 \frac{dt}{\rho(t)} = \infty$. Then

(A) The following are equivalent:

1. $\int_0^1 \frac{dt}{\rho(t)} - \int_0^1 \frac{dt}{\rho(t)} = -\infty$;
2. $C_{\psi,\varphi}^0 : B_{\nu,0}(B_X) \to B_{\mu,0}(B_X)$ is compact;
3. $C_{\psi,\varphi} : B_{\nu,0}(B_X) \to B_{\mu,0}(B_X)$ is compact;
4. $C_{\psi,\varphi} : B_{\nu,0}(B_X) \to B_{\mu,0}(B_X)$ is compact;

(B) Under the additional assumption that there exist $m \geq 2$ such that $\int_0^1 \frac{dt}{\rho(t)} = \infty$, $\psi, \varphi$ are replaced by similar ones but with $\nu, \omega$.

Corollary 3.9. Let $\psi \in H(B_X)$, $\varphi \in S(B_X)$ and $\mu, \nu$ be normal weights on $B_X$ such that $\int_0^1 \frac{dt}{\rho(t)} < \infty$ and $\int_0^1 \frac{dt}{\rho(t)} = \infty$. Then, the following are equivalent:

1. $\psi \in B_{\nu,0}^m(\mathbb{B}_1)$;
2. $\psi \in B_{\nu,0}^m(\mathbb{B}_1)$, $\nu \in B_{\mu,0}^m(B_X)$, $\mu \in B_{\mu,0}^m(B_X)$;
3. $\psi \in B_{\nu,0}(B_X)$;
4. $C_{\psi,\varphi}^0 : B_{\nu,0}(B_X) \to B_{\mu,0}(B_X)$ is compact,

Remark 3.2. Under the additional condition that $\varphi(0) = 0$, the limits in Theorems 3.3 and 3.7 are replaced by similar ones but with $\int_0^1 \frac{dt}{\rho(t)}$, $\int_0^1 \frac{dt}{\rho(t)}$, $\nu$ and $\mu$, respectively. Indeed, in a more general framework, it suffices to show $\int_0^1 \frac{dt}{\rho(t)} \leq 1$ for every $\nu \in B_X$. That means we have to give an infinite version of Schwarz’s lemma.

For each $z \in X$, $z \neq 0$ and $w \in \overline{B_X}$, applying classical Schwarz’s lemma to the functions $\phi_{z,w} : \mathbb{B}_1 \to \mathbb{B}_1$ given by

$$\phi_{z,w}(t) := \langle \varphi(tz), w \rangle, \quad \forall t \in \mathbb{B}_1,$$

we have

$$|\phi_{z,w}(t)| \leq |t|.$$

Then, choosing $t = \|z\| \|w\|$ and $w = \varphi(z)/\|\varphi(z)\|$, we get the desired inequality.

To finish this paper, combining Theorem 2.3 with the main results in Section 6, we state the characterizations for the boundedness and the compactness of the operators $C_{\psi,\varphi}^0, \overline{C}_{\psi,\varphi}, \overline{C}_{\psi,\varphi}.$

Theorem 3.10. Let $W \subset Y'$ be a separating subspace. Let $\psi \in H(B_X)$, $\varphi \in S(B_X)$ and $\mu, \nu$ be normal weights on $B_X$. Then

1. The following are equivalent:
   - $C_{\psi,\varphi}$ is bounded;
   - $\overline{C}_{\psi,\varphi}$ is bounded;

2. Under the additional assumption that $\varphi(0) = 0$, the limits in Theorems 3.3 and 3.7 are replaced by similar ones but with $\int_0^1 \frac{dt}{\rho(t)}$, $\int_0^1 \frac{dt}{\rho(t)}$, $\nu$ and $\mu$, respectively. Indeed, in a more general framework, it suffices to show $\int_0^1 \frac{dt}{\rho(t)} \leq 1$ for every $\nu \in B_X$. That means we have to give an infinite version of Schwarz’s lemma.

For each $z \in X$, $z \neq 0$ and $w \in \overline{B_X}$, applying classical Schwarz’s lemma to the functions $\phi_{z,w} : \mathbb{B}_1 \to \mathbb{B}_1$ given by

$$\phi_{z,w}(t) := \langle \varphi(tz), w \rangle, \quad \forall t \in \mathbb{B}_1,$$
• One of the assertions (1)-(4) in Theorem 3.1.

(2) The following are equivalent:
• \( \tilde{C}_{\psi,\varphi}^{0,0} \) is bounded;
• One of the assertions (1)-(4) in Theorem 3.3.

Theorem 3.11. Let \( W \subset Y' \) be a almost norming subspace. Let \( \psi \in H(B_X), \varphi \in S(B_X) \) and \( \mu, \nu \) be normal weights on \( B_X \). Assume that one of the following is satisfied:
(a) \( \int_0^1 \frac{dt}{\nu(t)} = \infty \) and the assertion (1) in Theorem 3.3;
(b) \( \int_0^1 \frac{dt}{\nu(t)} = \infty \) and one of the assertions (4)-(6) in Theorem 3.3 with the additional assumption that (3.4) holds for some \( m \geq 2 \);
(c) \( \int_0^1 \frac{dt}{\nu(t)} < \infty \) and one of the assertions (1)-(3) in Theorem 3.6 holds under the additional assumption that (3.4) holds for some \( m \geq 2 \).

Then the following are equivalent:
(1) \( \tilde{C}_{\psi,\varphi}^{0,0} \) is (resp. weakly) compact;
(2) \( \tilde{C}_{\psi,\varphi}^{0,0} \) is (resp. weakly) compact;
(3) The identity map \( I_Y : Y \to Y \) is (resp. weakly) compact.

Theorem 3.12. Let \( W \subset Y' \) be a almost norming subspace. Let \( \psi \in H(B_X), \varphi \in S(B_X) \) and \( \mu, \nu \) be normal weights on \( B_X \). Assume that one of the following is satisfied:
(a) The assertion (1) in Theorem 3.7;
(b) One of the assertions (3)-(5) in Theorem 3.7 with the additional assumption that (3.4) holds for some \( m \geq 2 \).

Then the following are equivalent:
(1) \( \tilde{C}_{\psi,\varphi}^{0,0} \) is (resp. weakly) compact,
(2) The identity map \( I_Y : Y \to Y \) is (resp. weakly) compact.

3.2. The test functions. Let \( \nu \) be a normal weight on \( B_X \). First we consider the holomorphic function

\[
g(z) := 1 + \sum_{k > k_0} 2^k z^{n_k} \quad \forall z \in \mathbb{B}_1
\]

where \( k_0 = \lfloor \log_2 \frac{1}{\nu(\delta)} \rfloor, \ n_k = \lfloor \frac{1}{r_k} \rfloor \) with \( r_k = \nu^{-1}(1/2^k) \) for every \( k \geq 1 \), and \( \delta \) is the constant in Definition 2.1. Here the symbol \( \lfloor x \rfloor \) means the greatest integer not more than \( x \). By [HW] Theorem 2.3, \( g(t) \) is increasing on \( [0, 1) \) and

\[
|g(z)| \leq g(|z|) \in \mathbb{R} \quad \forall z \in \mathbb{B}_1,
\]

\[
0 < C_1 := \inf_{t \in (0,1)} \nu(t)g(t) \leq \sup_{t \in (0,1)} \nu(t)g(t) \leq C_2 := \sup_{z \in \mathbb{B}_1} \nu(z)|g(z)| < \infty.
\]

Moreover, there exists a positive constant \( C_3 \) such that the inequality

\[
\int_0^r g(t)dt \leq C_3 \int_0^{r^2} g(t)dt
\]

holds for all \( r \in [r_1, 1) \), where \( r_1 \in (0,1) \) is a constant such that \( \int_0^{r_1} g(t)dt = 1 \) (see [Qu], Proposition 5.1).

Now, for \( w \in B_X \) fixed consider put the test functions

\[
\beta_w(z) := \int_0^{(z,w)} g(t)dt, \quad z \in B_X,
\]
and for \( \|w\| > r \) for some \( r > 0 \), put
\[
\gamma_w(z) := \frac{1}{\int_0^{\|w\|} g(t) dt} \left( \int_0^{(z, w)} g(t) dt \right)^2, \quad z \in B_X,
\]

\[\text{Proposition 3.13 (\cite{Qu}, Propositions 5.2 and 5.3).} \]
We have

(1) \( \beta_w, \gamma_w \in B_{\nu, 0}(B_X) \) and
\[
\|\beta_w\|_{B_{\nu}(B_X)} \leq C_2, \quad \|\gamma_w\|_{B_{\nu}(B_X)} \leq 2C_2.
\]

(2) The sequences \( \{\beta_n\}_{n \geq 1}, \{\gamma_n\}_{n \geq 1} \) are bounded in \( B_{\nu}(B_X) \);

(3) \( \gamma_n \to 0 \) uniformly on any compact subset of \( B_X \) if \( \int_0^1 \frac{dt}{\nu(t)} = \infty \).

We also need the following lemma:

\[\text{Lemma 3.14.} \quad \text{For every } f, \psi \in H(B_X), \text{ and } \varphi \in S(B_X) \text{ we have}\]
\[
R(C_{\psi, \varphi} f)(z) = f(\varphi(z))R\psi(z) \quad \forall z \in B_X.
\]

\[\text{Indeed, assume that } \sum_{n=1}^{\infty} P_n(z) \text{ is the Taylor series of } (f \circ \varphi)(z)R\psi(z) \text{, where } P_n \text{ is a homogeneous polynomial of degree } n. \text{ Then we have}\]
\[
R(C_{\psi, \varphi}) (z) = R \int_0^1 \sum_{n=1}^{\infty} P_n(z)^n \frac{dt}{t} = R \sum_{n=1}^{\infty} \frac{P_n(z)}{n} = \sum_{n=1}^{\infty} P_n(z) = f(\varphi(z))R\psi(z).
\]

\[\text{4. PROOF OF THEOREMS 3.1 AND 3.2}\]

\[\text{4.1. Proof of Theorem 3.1}\]

\[\text{Proof of Theorem 3.1.} \quad \text{It is clear that } (4) \Rightarrow (2) \Rightarrow (1) \text{ and } (4) \Rightarrow (3) \text{ and, since } B_{\nu, 0}(B_X) \subset B_{\nu}(B_X), \text{ the implication } (5) \Rightarrow (6) \text{ is obvious.}\]

(1) \( \Rightarrow (5) \): Assume that \( M_{R\psi, \varphi}^{[m]} < \infty \) for some \( m \geq 2 \). For each \( x \in OS_m \) we write
\[
z_x := \sum_{k=1}^{m} z_k x_k. \text{ Note that } \|z_x\| = \|z_{[m]}\| \text{ and hence } \mu^{[m]}(z_{[m]}) = \mu^{[m]}(z_x). \text{ By (2.3) and (3.12), for every } f \in B_{\nu}(B_X) \text{ and for every } x \in OS_m \text{ we have}\]
\[
\|((C_{\psi, \varphi}(f))_x)_{x \in OS_m} \|= \sup_{z_x \in B_m} \|\mu^{[m]}(z_x)R(C_{\psi, \varphi}(f))(z_{[m]}))\|
\]
\[
= \sup_{z_x \in B_m} \mu^{[m]}(z_x)\|f(\varphi(z_{[m]}))\|\|R\psi^{[m]}(z_x)\|
\]
\[
\leq \sup_{z_x \in B_m} \mu^{[m]}(z_x)\|R\psi^{[m]}(z_x)\|\max\left\{1, \int_0^{\varphi^{[m]}(z_x)} \frac{dt}{\nu(t)}\right\}\|f\|_{B_{\nu}(B_X)}
\]
\[
\leq M_{R\psi, \varphi}^{[m]}\|f\|_{B_{\nu}(B_X)}.
\]

Consequently, by (2.3) and \( C_{\psi, \varphi}(0) = 0 \), we have
\[
\|C_{\psi, \varphi}(f)\|_{B_{\nu}(B_X)} = \sup_{x \in OS_m} \|((C_{\psi, \varphi}(f))_x)_{x \in OS_m}\| \leq M_{R\psi, \varphi}^{[m]}\|f\|_{B_{\nu}(B_X)}.
\]

Thus, (5) is proved.

(3) \( \Rightarrow (5) \): Use an argument analogous and an estimate to the previous one.

(6) \( \Rightarrow (4) \): Suppose \( C_{\psi, \varphi}^{0} : B_{\nu, 0}(B_X) \to B_{\mu}(B_X) \) is bounded.
First, it is obvious that $\psi \in B_\mu(B_X)$ because $1 \in B_\mu(B_X)$ and

\begin{equation}
\psi(z) = \psi(0) + \int_0^1 R\psi(tz)\frac{dt}{t} = \psi(0) + (C_{\psi,\varphi}1)(z).
\end{equation}

Fix $z \in B_X$, put $w = \varphi(z)$ and consider the test function $\beta_w \in B_{w,0}(B_X)$ defined by (3.10). Then, by (3.12)

\begin{equation}
\|C_{\psi,\varphi}\beta_w\|_{B_{w,0}(B_X)} = \mu(z)\|R(\beta_w(\varphi(z)))\|
= \mu(z)\|R\psi(\varphi(z))\|
\leq C_2\|C_{\psi,\varphi}0\| < \infty.
\end{equation}

Let $r_1 = \nu^{-1}(1/2)$. If $\|w\| \geq r_1$ by (3.3), (3.9) and (4.3) we have

\begin{align*}
\mu(z)\|R\psi(z)\|\int_0^{\|w\|} \frac{dt}{\nu(t)} &\leq \mu(z)\|R\psi(z)\|\int_0^{\|w\|} \frac{g(t)}{C_1} dt \\
&\leq \frac{C_3}{C_1}\mu(z)\|R\psi(z)\|\int_0^{\|w\|} g(t) dt \\
&\leq \frac{C_3C_2}{C_1}\|C_{\psi,\varphi}0\| < \infty.
\end{align*}

If $\|w\| < r_1$ by (3.8) again we obtain

\begin{align*}
\nu(z)\|R\psi(z)\|\int_0^{\|w\|} \frac{dt}{\nu(t)} &\leq \mu(z)\|R\psi(z)\|\int_0^{\|w\|} \frac{g(t)}{C_1} dt \\
&\leq \frac{1}{C_1}\mu(z)\|R\psi(z)\| \\
&= \frac{1}{C_1}\|C_{\psi,\varphi}0\|_{B_{w,0}(B_X)} \\
&\leq \frac{1}{C_1}\|C_{\psi,\varphi}0\| < \infty.
\end{align*}

Hence we get $M_{R\psi,\varphi} < \infty$.

The proof of Theorem is completed. \qed

4.2. Proof of Theorem 3.2

Proof of Theorem 3.2 As in the previous one, it is obvious that (4) $\Rightarrow$ (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3).

(1) $\Rightarrow$ (5): Assume that $\psi \in B_\mu,0$ and $M_{B_{\psi,\varphi}}^{[m]} < \infty$ for some $m \geq 2$. By Theorem 3.1 $C_{\psi,\varphi}0 : B_{e,0}(B_X) \to B_\mu(B_X)$ is bounded. It suffices to show that $C_{\psi,\varphi}0f \in B_\mu(B_X)$ for every $f \in B_{e,0}(B_X)$.

Let $f \in B_{e,0}(B_X)$ be arbitrarily fixed. Let $\varepsilon > 0$ be fixed. Then there exists $r_0 \in (1/2, 1)$ such that

\begin{equation}
\nu(z)|Rf(z)| < \frac{\varepsilon}{4M_{R\psi,\varphi}}, \quad \|z\| \geq r_0.
\end{equation}

By (2.5) we have

\begin{equation}
K := \sup_{\|w\| \leq r_0} |f(w)| < \infty.
\end{equation}
Since $\psi^{[m]} \in B^{[m]}_{\mu}(\mathbb{B}_m)$ we can find $\theta \in (0, 1)$ such that for every $x \in OS_m$

\begin{equation}
\mu^{[m]}(z_x)\|R\psi^{[m]}(z_x)\| < \frac{\varepsilon}{2K} \quad \text{whenever } \|z_x\| < 1.
\end{equation}

For $\theta < \|z_x\| < 1$ we consider two cases:

- The case $\|u^{[m]}\| := \|\varphi^{[m]}(z_x)\| > r_0$: Let $\tilde{u}^{[m]} = r_0\frac{u^{[m]}}{\|u^{[m]}\|}$. We have

\[ |f(\varphi^{[m]}(z_x)) - f(\tilde{u}^{[m]})| = |f(u^{[m]}) - f(\tilde{u}^{[m]})| \leq \left| \frac{1}{r_0}\int_{r_0}^{1} Rf(tu^{[m]})\|tu^{[m]}\|\|\psi,\phi\| dt \right| \leq \frac{\varepsilon\|u^{[m]}\|}{4M^{[m]}_{\psi,\phi}} \left( \int_{r_0}^{1} \frac{1}{\|tu^{[m]}\|\|\psi,\phi\|} dt \right). \]

Combining (4.6) with (1.0), and by (3.12), for $\theta < \|z_x\| < 1$ we obtain

\[ \mu^{[m]}(z_x)\|RC_{\psi,\phi}^{[m]}f(z_x)\| = \mu^{[m]}(z_x)\|f(\varphi^{[m]}(z_x))\| \leq \mu^{[m]}(z_x)\|f(\varphi^{[m]}(z_x))\| + \mu^{[m]}(z_x)\|R\psi^{[m]}(z_x)\| \leq M^{[m]}_{\psi,\phi}\frac{\varepsilon}{2M^{[m]}_{\psi,\phi}} + K\frac{\varepsilon}{2K} = \varepsilon. \]

- The case $\|u^{[m]}\| := \|\varphi^{[m]}(z_x)\| \leq r_0$: We have

\[ \mu^{[m]}(z_x)\|RC_{\psi,\phi}^{[m]}f(z_x)\| = \mu^{[m]}(z_x)\|f(\varphi^{[m]}(z_x))\| \leq \mu^{[m]}(z_x)\|f(\varphi^{[m]}(z_x))\| < K\frac{\varepsilon}{2K} < \varepsilon. \]

Consequently, by (2.3) we have

\[ \mu(z)\|RC_{\psi,\phi}^{[0]}f(z)\| = \sup_{x \in OS_m} \mu^{[m]}(z_x)\|RC_{\psi,\phi}^{[m]}f(z_x)\| \leq \varepsilon \quad \text{whenever } \theta < \|z\| < 1. \]

Thus, $C_{\psi,\phi}^{[0]}f \in B_{\mu,0}(B_X)$.

(3) $\Rightarrow$ (5): By an estimate analogous to that used for the proof of (1) $\Rightarrow$ (5) we get $C_{\psi,\phi}^{[0]}f \in B_{\mu}(B_X)$. Then, since

\[ \mu(z)\|RC_{\psi,\phi}^{[0]}f(z)\| = \lim_{m \to \infty} \mu(z)\|RC_{\psi,\phi}^{[m]}f(z)\| \]

and $B_{\mu,0}(B_X)$ is closed in $B_{\mu}(B_X)$, we infer that $C_{\psi,\phi}^{[0]}f \in B_{\mu,0}(B_X)$.

(5) $\Rightarrow$ (4): Assume that $C_{\psi,\phi}^{[0]}f$ is bounded. By (4.2), $\psi \in B_{\mu,0}(B_X)$. And then, we obtain $M_{\psi,\phi} < \infty$ by the proof of Theorem 3.11.

5. Proof of Theorems 3.3, 3.4, 3.6 and 3.7

In order to study the compactness of the operators $W_{\psi,\phi}$, as in [1] we can prove the following.

Lemma 5.1. Let $E, F$ be two Banach spaces of holomorphic functions on $B_X$. Suppose that

(1) The point evaluation functionals on $E$ are continuous;
(2) The closed unit ball of $E$ is a compact subset of $E$ in the topology of uniform convergence on compact sets.

(3) $T : E \rightarrow F$ is continuous when $E$ and $F$ are given the topology of uniform convergence on compact sets.

Then, $T$ is a compact operator if and only if given a bounded sequence $\{f_n\}$ in $E$ such that $f_n \rightarrow 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of $F$.

We can now combine this result with Montel theorem and (2.5) to obtain the following proposition. The details of the proof are omitted here.

**Proposition 5.2.** Let $\psi \in H(BX)$ and $\varphi \in S(BX)$. Then $W_{\psi, \varphi} : B_\nu(BX) \rightarrow B_\mu(BX)$ is compact if and only if given a bounded sequence $\{f_n\}$ in $B_\nu(BX)$ converging to 0 uniformly on compact sets in $BX$.

**5.1. Proof of Theorem 3.3**

Proof of Theorem 3.3. First, we prove (B).

(B) The implications (6) $\Rightarrow$ (5) $\Rightarrow$ (4) and (2) $\Rightarrow$ (3) are obvious.

(4) $\Rightarrow$ (2): For each $x \in OS_m$ we write $z_x := \sum_{k=1}^{m} z_k x_k$. Denote $w^{[m]} = \varphi^{[m]}(z_x)$. Since $\int_0^1 \frac{dt}{\nu(t)} = \infty$ and (3.1a) $\rightarrow 0$ as $\|\varphi^{[m]}(y)\| \rightarrow 1$, $\psi^{[m]} \in B_{\mu,0}(B_m)$ and for any $\varepsilon > 0$, there exists $r_0 \in (1/2, 1)$ such that

\begin{equation}
\mu^{[m]}(z_x) \|R\psi^{[m]}(z_x)\| \int_{r_0}^{\varepsilon} \frac{dt}{\nu^{[m]}(t)} < \frac{\varepsilon}{3C_2} \quad \text{for } r_0 < \|w^{[m]}\| < 1.
\end{equation}

Let $\{f_n\}_{n \geq 1}$ be a bounded sequence in $B_\nu(BX)$ converging to 0 uniformly on compact subsets of $BX$ and fix an $\varepsilon > 0$. We may assume that $\|f_n\|_{B_\nu(BX)} \leq 1$ for every $n \geq 1$. By the hypothesis on the sequence $\{f_n\}_{n \geq 1}$, there exists a positive integer $n_0$ such that

\begin{equation}
|f_n(w)| \leq \frac{\varepsilon}{3\|\psi^{[m]}\|_{B_\mu,0}(B_m)}, \quad n \geq n_0, \quad w \in B[\varphi^{[m]}, r_0].
\end{equation}

Therefore, for every $n \geq n_0$ and $\|w^{[m]}\| \leq r_0$ we have

\begin{equation}
\mu^{[m]}(z_x) \|R\psi^{[m]}(z_x)\| |f_n(w^{[m]})| < \frac{\varepsilon}{3}.
\end{equation}
Now, for every $n \geq n_0$ and $r_0 < \|w^{[m]}\| < 1$, with noting that $r_0 \frac{w^{[m]}}{\|w^{[m]}\|} \in B[\varphi^{[m]}, r_0]$, by (5.8), (5.7) and (5.8) we have
\[
\mu^{[m]}(z_x)\|R\varphi^{[m]}(z_x)\|f_n(w^{[m]}) \leq \mu^{[m]}(z_x)\|R\varphi^{[m]}(z_x)\| \left| f_n(w^{[m]}) - f_n\left( r_0 \frac{w^{[m]}}{\|w^{[m]}\|} \right) \right| + \mu^{[m]}(z_x)\|R\varphi^{[m]}(z_x)\| \left| f_n\left( r_0 \frac{w^{[m]}}{\|w^{[m]}\|} \right) \right|
\]
\[
\leq \mu^{[m]}(z_x)\|R\varphi^{[m]}(z_x)\| \int_{r_0/\|w^{[m]}\|}^{1} \|Rf_n(tw^{[m]})\| \frac{dt}{t} + \frac{\varepsilon}{3}
\]
\[
\leq \frac{1}{C_2} \mu^{[m]}(z_x)\|R\varphi^{[m]}(z_x)\| \int_{r_0/\|w^{[m]}\|}^{1} \frac{1}{\nu(t)} \frac{dt}{\|w^{[m]}\|} + \frac{\varepsilon}{3}
\]
\[
\leq \frac{1}{C_2} \mu^{[m]}(z_x)\|R\varphi^{[m]}(z_x)\| \int_{r_0}^{\|w^{[m]}\|} \frac{1}{\|w^{[m]}\|} \frac{dt}{\nu(t)} + \frac{\varepsilon}{3}
\]
\[
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.
\]
Consequently, it follows from (5.8) and (5.8) that
\[\mu^{[m]}(z_x)\|R\varphi^{[m]}(z_x)\|f_n(w^{[m]}) < \varepsilon\]
for every $z_x \in B_m$ and every $n \geq n_0$. This means $\|C_{\psi, \varphi}f_n\|_{B_n(B_X)} \to 0$ as $n \to \infty$.

Finally, with the note that the above estimates is independent of $x \in OS_m$, we obtain
\[\|C_{\psi, \varphi}f_n\|_{B_n(B_X)} = \sup_{x \in OS_m} \|C_{\psi, \varphi}f_n\|_{B_n(B_X)} \to 0 \text{ as } n \to \infty.
\]

Hence, by Lemma 5.2 $C_{\psi, \varphi}$ is compact.

(3) \Rightarrow (6): Suppose $C_{\psi, \varphi}^{0}$ is compact. Then clearly, $C_{\psi, \varphi}^{0}$ is bounded.

Firstly, assume that $\gamma_{w} \neq 0$ as $\langle \varphi(y) \rangle \to 1$. Then we can take $\varepsilon_0 > 0$ and a sequence $\{z^n\}_{n \geq 1} \subset B_X$ such that $\|w^n\| := \|\varphi(z^n)\| \to 1$ but
\[\mu(z^n)\|R\varphi(z^n)\| \int_{0}^{\|w^n\|} \frac{dt}{\nu(t)} \geq \varepsilon_0 \text{ for every } n = 1, 2, \ldots\]
We may assume that $\|z^n\| > r_1 := \nu^{-1}(1/2)$.

Consider the sequence $\{\gamma_w\}_{n \geq 1}$ defined by (5.11). By Proposition ??, this sequence is bounded in $B_{\varphi}(B_X)$ and converges to 0 uniformly on compact subsets of $B_X$. Then $\|C_{\psi, \varphi}^{0}\gamma_w\|_{B_n(B_X)} \to 0$ as $n \to \infty$ by Lemma 5.2.

On the other hand, by (5.8), (5.9) and (5.10) we have
\[\|C_{\psi, \varphi}^{0}\gamma_w\|_{B_n(B_X)} = \sup_{z \in B_X} \mu(z)\|R\varphi(z)\|\|\gamma_w(\varphi(z))\|
\]
\[\geq \mu(z^n)\|R\varphi(z^n)\|\|\gamma_w(\varphi(z^n))\|
\]
\[\geq \mu(z^n)\|R\varphi(z^n)\|\int_{0}^{\|w^n\|^2} g(t)dt
\]
\[\geq \frac{C_1}{C_3} \mu(z^n)\|R\varphi(z^n)\|\int_{0}^{\|w^n\|} \frac{dt}{\nu(t)}
\]
\[\geq \frac{C_1}{C_3}\varepsilon_0.
\]
Thus, we get a contradiction. And the proof of (B) is complete.

Finally, we prove (A).

(A) Since (2) ⇔ (3) ⇔ (6) and (6) ⇒ (1) is obvious, it suffices to prove (1) ⇒ (2). We obtain this proof by an estimate analogous to that used for the proof of (4) ⇒ (2) by using the known fact that

\[
B[\varphi(k),r_0] = \{\varphi(k)(z) : \|\varphi(k)(z)\| \leq r_0\} \subset \mathbb{B}_k \subset \mathbb{C}^k
\]

is relatively compact for every \(0 \leq r_0 < 1\) and \(k \geq 1\), instead of (3.4). \(\Box\)

5.2. Proof of Theorem 3.4

Proof of Theorem 3.4. For every \(z \in B_X\), consider the function \(\delta_z\) given by

\[
\delta_z(f) = f(z)
\]

for every \(f \in B_\mu(B_X)\). By (2.5), it is clear that \(\delta_z \in (B_\mu(B_X))^\prime\). Moreover, we have

\[
\frac{1}{2}\|z-w\| \leq \|\delta_z - \delta_w\| \quad \forall z, w \in B_X.
\]

Indeed, it is easy to check by direct calculation that

\[
\frac{1}{2}\|z-w\| \leq \sqrt{1 - \frac{(1-\|z\|^2)(1-\|w\|^2)}{1-\langle z,w \rangle^2}} = \varrho_X(z,w)
\]

where \(\varrho_X\) is the pseudohyperbolic metric in \(B_X\) (see [GR, p.99]). On the other hand, we also have

\[
\varrho_X(z,w) = \sup \{\varrho(f(z), f(w)) : f \in H^\infty(B_X) \text{ with } \|f\|_\infty \leq 1\}
\]

(see (3.4) in [BGM]), where \(\varrho(x,y) = \frac{|x-y|}{1-\langle x, y \rangle}\) is the pseudohyperbolic metric in \(\mathbb{B}_1\). Note that, since the function \(\eta \mapsto \frac{\eta}{1-f(z)f(w)}\) is holomorphic from \(\mathbb{B}_1\) into \(\mathbb{B}_1\) and \(f(z) - f(w) \to 0\), it follows from Schwarz’s lemma that \(\varrho(f(z), f(w)) \leq |f(z) - f(w)|\) for every \(z, w \in B_X\). Consequently,

\[
\varrho_X(z,w) \leq \sup \{|f(z) - f(w)| : f \in H^\infty(B_X) \text{ with } \|f\|_\infty \leq 1\}
\]

\[
\leq \sup \{|\delta_z(f) - \delta_w(f)| : f \in H^\infty(B_X) \text{ with } \|f\|_\infty \leq 1\}
\]

\[
= \|\delta_z - \delta_w\|.
\]

Hence, (5.12) is proved.

For \(0 < r < 1\), the set \(V_r := \{z : \|z\| \leq r\} \subset (B_\mu(B_X))^\prime\) is bounded. Then, by the compactness of \(C_{\psi,\varphi}\) the set

\[
(C_{\psi,\varphi})^\ast(V_r) = \{\psi(z)\delta_{\varphi(z)} : \|z\| \leq r\}
\]

is relatively compact in \((H^\infty(B_X))^\prime\).

It should be noted that, for every subset \(K\) of the dual of a Banach space \(E\) and every bounded subset \(D \subset \mathbb{C}\), if the set \(\{t\eta : t \in D, \eta \in A\}\) is relatively compact in \(E\) then \(A \subset E'\) is relatively compact. With this fact in hand, since the set \(\{\psi(z) : \|z\| \leq r\}\) is bounded, the set \(\{\delta_z, \|z\| \leq r\}\) is relatively compact. Then, it follows from the inequality (5.12) that \(\varphi(rB_X)\) is relatively compact. \(\Box\)
5.3. Proof of Theorem 3.6

Proof of Theorem 3.6. Since the implications (3) ⇒ (2) ⇒ (1) and (4) ⇒ (5) are clear, it suffices to prove (1) ⇒ (4) and (5) ⇒ (3).

(1) ⇒ (4): Since \( \int_0^1 \frac{dt}{\nu(t)} < \infty \) and \( \psi[m] \in B_{\mu[m]}(B_m) \) we have \( M_{R\psi[m]} < \infty \), therefore, \( C_{\psi,\varphi} \) is bounded and for any \( \varepsilon > 0 \), there exists \( r_0 \in (1/2, 1) \) such that (5.7) holds. The rest of the proof is similar to the case (1) ⇒ (4) of Theorem 3.1.

(5) ⇒ (3): Suppose \( C_{\psi,\varphi}^0 : B_{\nu,0}(B_X) \to B_{\mu}(B_X) \) is compact, it is bounded. As in the proof (5) ⇒ (3) of Theorem 3.1 we get \( \psi \in B_{\mu}(B_X) \).

This concludes the proof of the theorem. \( \square \)

5.4. Proof of Theorem 3.7

Proof of Theorem 3.7. First, we prove (B).

It is clear that (5) ⇒ (4) ⇒ (3).

(3) ⇒ (2): By Theorems 3.2 and 3.3(B) we obtain (2) from (3).

(2) ⇒ (5): By the hypothesis and Theorem 3.2 \( \psi \in B_{\mu}(B_X) \). Therefore, (5) holds if \( \int_0^1 \frac{dt}{\nu(t)} < \infty \), in the case \( \int_0^1 \frac{dt}{\nu(t)} = \infty \), in the same way as in the proof of (3) ⇒ (6) of Theorem 3.1 we also obtain that (5) holds.

Finally, the proof of (A) is similar to the one of (B) but here we use Theorem 3.3(A) instead of Theorem 3.3(B). \( \square \)

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