Bayesian analysis of predictive Non-Homogeneous hidden Markov models using Pólya-Gamma data augmentation*

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Abstract

We consider Non-Homogeneous Hidden Markov Models (NHHMMs) for forecasting univariate time series. We introduce two state NHHMMs where the time series are modeled via different predictive regression models for each state. Also, the time-varying transition probabilities depend on exogenous variables through a logistic function. In a hidden Markov setting, inference for logistic regression coefficients becomes complicated and in some cases impossible due to convergence issues. To address this problem, we use a new latent variable scheme, that utilizes the Pólya-Gamma class of distributions, introduced by Polson et al. [2013]. Given an available set of predictors, we allow for model uncertainty regarding the predictors that affect the series both linearly – in the mean – and non-linearly – in the transition matrix. Predictor selection and inference on the model parameters are based on a MCMC scheme with reversible jump steps. Single-step and multiple-steps-ahead predictions are obtained based on the most probable model, median probability model or a Bayesian Model Averaging (BMA) approach. Simulation experiments, as well as an empirical study on real financial data, illustrate the performance of our algorithm in various setups, in terms of mixing properties, model selection and predictive ability.

Keywords: Non Homogeneous Hidden Markov Models; Model selection; Bayesian model averaging; Forecasting; Pólya-Gamma Data Augmentation, Continuous Ranked Probability Score

1 Introduction

This paper follows and extends the work of Meligkotsidou and Dellaportas [2011] by employing recent methodological advances on Pólya-Gamma data augmentation for predictive discrete-time, finite state-space, Non-Homogeneous Hidden Markov Models (NHHMMs) that can be used for modeling and predicting univariate time-series. We consider two-state NHHMM (easily extended to m-state NHHMMs) where the time series are modeled via different predictive regression models

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for each state, whereas the transition probabilities are modeled via logistic regressions. Given
an available set of predictors we allow for model uncertainty, regarding the predictors that affect
the series both linearly, that is directly in the mean regressions, and non-linearly, that is in the
transition matrix.

Discrete-time finite state-space Homogeneous Hidden Markov models (HHMMs) have been
extensively studied and widely used to model stochastic processes consisting of an observed
process and a latent (hidden) sequence of states which is assumed to affect the observation se-
quence, see for example Cappé et al. [2005] and Billio et al. [1999]. We refer to Scott [2002]
and Rydén [2008] for a review of Bayesian approaches for HHMMs. Bayesian inference, using
Markov chain Monte Carlo (MCMC) techniques, has enhanced the applicability of HHMMs and
has led to the construction of more complex model specifications including NHHMMs. Initially
Diebold et al. [1994] studied the two-state Gaussian NHHMMs where the time varying tran-
sition probabilities were modeled via logistic functions and their approach was based on the
Expectation-Maximization algorithm (EM). Filardo and Gordon [1998] adopted a Bayesian per-
spective to overcome technical and calculation issues of classical approaches. Since then, various
Bayesian methods have been proposed in the literature. For example, Spezia [2006] modeled
the time-varying transition probabilities via a logistic function depending on exogenous variables
and performed model selection based on the Bayes factor. In the same spirit, Meligkotsidou and
Dellaportas [2011] considered a $m$-stage NHMM and assumed that the elements of the transi-
tion matrix are linked through exogenous variables with a multinomial logistic link whereas the
observed process conditional on the unobserved process follows an autoregressive model of order
$p$.

This paper considers inference and variable selection for predictive NHHMMs which exploit
a set of available predictors by allowing them to affect the transition probabilities and/or the state-
specific predictive regressions. We perform Bayesian inference for the proposed models based on
MCMC. Our MCMC scheme aims at overcoming difficulties and solving convergence issues arising
with existing MCMC algorithms, thus offering an improved inferential procedure for estimation
and variable selection in NHHMMs. To this end we exploit the missing data representation
of hidden Markov models and construct an MCMC algorithm based on data augmentation,
consisting of several steps. First, the latent sequence of states is sampled via the Scaled Forward-
Backward algorithm [Scott, 2002], which is a modification of the Forward-Backward algorithm
of [Baum et al., 1970] who used it to implement the classical EM algorithm. Then, following
Meligkotsidou and Dellaportas [2011] we use a logistic regression representation of the transition
probabilities and we simulate the parameters of the mean predictive regression model for each
stage, via Gibbs sampling steps. Using the data augmentation scheme of Holmes and Held [2006],
as in Meligkotsidou and Dellaportas [2011], may result in serious convergence issues, especially
in cases that there exists model uncertainty. Frihwirth-Schnatter and Frühwirth [2010] give
a detailed comparison between various methods for dealing with binary and multinomial logit
data and argue about the efficiency of Holmes and Held [2006] data augmentation scheme. To
deal with this serious issue we use the Pólya-Gamma data augmentation scheme of Polson et al.
[2013] which has a significantly improved performance. The recent work of Holsclaw et al. [2017]
confirms that using Pólya-Gamma data augmentation to parametrize the transition probabilities
of NHHMMs results in an algorithm that mixes well and provides adequate estimates of the
model parameters. Finally, we incorporate variable selection within our MCMC scheme by using
the well-known model selection method of Green [1995], the reversible jump algorithm. Within
our algorithm we perform a couple of reversible jump steps to allow for a different set of covariates
to affect the mean equation and the transition probabilities.

Regarding stochastic variable selection in the transition matrix of the NHHMM we design a
reversible jump step based on the Pólya-Gamma data augmentation representation for the logistic

regression coefficients. Recently, Holsclaw et al. [2017] consider a NHHMM similar to ours for modeling multivariate meteorological time series data. In that paper, the transition probabilities are modeled via multinomial logistic regressions affected by a specific set of exogenous variables. The authors use the BIC criterion for choosing the best model among a specified class of models. We extend this work by considering the problems of statistical inference and variable selection jointly, in a purely Bayesian setting, and we describe the proposed methodology in the context of 2-state NHHMMs, noting though that it can be easily extended to the case of m-state models.

Apart from developing a stable algorithm for inferring NHHMMs in the presence of model uncertainty, another main goal of our work is to explore the advantages of applying the proposed models and methods to realized volatility prediction. Forecasting future volatility accurately is extremely important for asset allocation, portfolio and risk management. A vast amount of literature has investigated the relationship between volatility and macroeconomic and/or financial variables (see for example, Paye [2012], Christiansen et al. [2012], Meligkotsidou et al. [2018] among others). The proposed NHHMMs are able to capture the nonlinear relationship between the logarithm of realized volatility and a set of predictors, as well as other special characteristics of the analyzed series, such as heteroscedasticity and autocorrelation. Our application shows that the proposed model outperforms benchmark alternatives, such as the Homogeneous HMM and the predictive regression model with autoregressive terms, in terms of forecasting ability. We argue that the logarithmic scoring rule for evaluating different models and their predictive accuracy is not appropriate, since the data have multimodal distribution. When data are multimodal, scoring rules that are sensible to distance are preferred (Gneiting and Raftery [2007]). Besides logarithmic scoring rule gives harsh penalty for low probability events (Boero et al. [2011], Gneiting and Raftery [2007]) and prefers the forecast density that is less informative (Machete [2013]).

We propose the use of Continuous Ranked Probability Score, which has gained a lot of interest in meteorological community (Grimit et al. [2006]), as a better alternative for assessing the quality of forecasts as well as for validating the model performance.

The paper proceeds as follows. In Section 2 we briefly describe the proposed model and in Section 3 we present analytically the Bayesian inference for this model, specifically for the model with a fixed number of predictors (covariates), as well as for the model with unknown number of predictors. Then, in Section 4 we present the forecasting criteria we used to assess the predictive ability of our method. To check our methodology we performed extensive experiments with simulated data. We provide in Section 5 some indicative case studies. We illustrate our results regarding the variable selection and forecasting evaluation and we make comparisons with benchmark models. Next, we apply our methodology to monthly realized volatility data set (Section 6). Finally, Section 7 concludes the paper.

2 The Non-Homogeneous Hidden Markov Model

In this section, we present the proposed Non-Homogeneous Hidden Markov Model (NHHMM) for modeling univariate time series. Consider an observed random process \( \{ Y_t \} \) and a hidden underlying process \( \{ Z_t \} \) which is a two-state non-homogeneous discrete-time Markov chain that determines the states of the observed process. Let \( y_t \) and \( z_t \) be the realizations of the observed random process \( \{ Y_t \} \) and of the hidden process \( \{ Z_t \} \), respectively. We assume that at time \( t \), \( y_t \) depends on the current state \( z_t \) and not on the previous states. Consider also a set of available predictors \( \{ X_t \} \) with realization \( x_t = (1, x_{1t}, \ldots, x_{r-1t}) \) at time \( t \). A subset of the predictors \( X_t^{(1)} \subseteq \{ X_t \} \) of length \( r_1 - 1 \) is used in the regression model for the observed process and a subset \( X_t^{(2)} \subseteq \{ X_t \} \) of length \( r_2 - 1 \) is used to model/describe the dynamics of the time-varying transition probabilities. Thus, we allow the covariates to affect the observed process \( \{ Y_t \} \) in a
non-linear fashion.

In the case of a univariate random process \( \{Y_t\} \), the NHHMM can be written in the form

\[
Y_t = g(Z_t) + \epsilon_t,
\]

where \( g(Z_t) = X_{t-1}^{(1)} B_{Z_t} \), \( X_{t-1}^{(1)} = (1, x_{1t-1}^{(1)}, \ldots, x_{r_1-1t-1}^{(1)}) \), \( B_{Z_t} = (b_{0Z_t}, b_{1Z_t}, \ldots, b_{r_1-1Z_t})' \) and \( \epsilon_t \sim \mathcal{N}(0, \sigma^2_{Z_t}) \). We use \( \mathcal{N}(\mu, \sigma^2) \) to denote the normal distribution with mean \( \mu \) and variance \( \sigma^2 \). In a less formal way, if \( s \) represents the hidden states, the observed series given the unobserved process has the form

\[
Y_t | Z_t = s \sim \mathcal{N}(X_{t-1}^{(1)} B_s, \sigma^2_s), s = 1, 2.
\]

The dynamics of the unobserved process \( \{Z_t\} \) can be described by the time-varying transition probabilities, which depend on the predictors \( X_t^{(2)} \) and are given by the following relationship

\[
\rho_{ss}^{(t)} = \frac{\exp(x_t^{(2)} \beta_s)}{1 + \exp(x_t^{(2)} \beta_s)}, s = 1, 2,
\]

where \( \beta_s = (\beta_{0s}, \beta_{1s}, \ldots, \beta_{r_2-1s})' \) is the vector of the logistic regression coefficients to be estimated and \( x_t^{(2)} = (1, x_{1t}^{(2)}, \ldots, x_{r_2-1t}^{(2)}) \) is the set of covariates that affect the transition probabilities. The unknown quantities of the NHHMM are \( \{\theta_s = (B_s, \sigma^2_s), s = 1, 2\} \), that is the parameters in the mean predictive regression equation and the parameters in the logistic regression equation for the transition probabilities of the unobserved process \( \{Z_t\}, t = 1, ..., T \). Our model and the methods developed in this paper can easily be generalized into an m-state NHHMM, where the rows of the transition matrix are modeled by multinomial logistic regressions.

### 3 Bayesian Inference and Computational Strategy

This section presents the Bayesian approach to inference for the Non-Homogeneous Hidden Markov model. The key steps in our proposed framework are the following. First, for a given Hidden Markov model with time-varying probabilities, we construct a Markov chain which has as a stationary distribution the posterior distribution of the model parameters. Simulation of this Markov chain provides, after some burn-in period and adequately many iterations, samples from the posterior distribution of interest; see, for details, [Besag et al. 1995]. Second, for a given set of competing models each including a different set of predictors in the mean regression and/or in the transition probabilities equation, we base our inference about the models on their posterior probabilities. Thus, we avoid the usual approach which considers the models separately and chooses the best model via significance tests or via model selection criteria.

#### 3.1 Inference for fixed sets of predictors

Below we provide detailed guidelines on how to estimate the parameters of a given NHHMM, i.e. for fixed sets of predictors used in the mean equation and the transition probabilities \( X^{(1)} \) and \( X^{(2)} \), respectively. The proposed approach to Bayesian inference on the parameters of the NHHMM is based on constructing a Markov chain Monte Carlo algorithm which updates, in turn, the mean regression parameters, the logistic regression coefficients, and the latent variables \( Z \). Let \( Y^t = (Y_1, \ldots, Y_t) \) be the history of the observed process, \( Z^t = (Z_1, \ldots, Z_t) \) the sequence of states up to time \( t \), and let \( f_s(.) \) denote the normal probability density function of \( Y_t | Z_t = s \).
\( s \in S \) and \( \pi_1(z_1) \) the initial distribution of \( Z_1 \). The joint likelihood function of the observed data, \( y^T \), and the unobserved sequence of states, \( z^T \), is given by

\[
\pi(y^T, z^T \mid X, \theta, \beta) = \pi(y^T \mid z^T, X, \theta, \beta)\pi(z^T \mid X, \theta, \beta) = \pi_1(z_1)f_{z_1}(y_1)\prod_{t=2}^{T} \prod_{i=1}^{2} \prod_{j=1}^{2} p_{ij}^{(t-1)} f_j(y_t)
\]

If a prior distribution \( \pi(\theta, \beta) = \pi(\theta)\pi(\beta) \) is specified for the model parameters, then inference on all the unknown quantities in the model is based on their joint posterior distribution

\[
\pi(\theta, \beta, z^T \mid y^T) \propto \pi(\theta, \beta)\pi(y^T, z^T \mid \theta, \beta).
\]

For the parameters in the mean predictive regression equation, we use conjugate prior distributions, i.e.

\[
\sigma^2_s \sim IG(p, q), \quad B_s \mid \sigma^2_s \sim \mathcal{N}(L_0, \sigma^2_s V_0), \quad s = 1, 2,
\]

where \( IG \) denotes the Inverted-Gamma distribution. To make inference about the logistic regression coefficients we use the auxiliary variables method of Polson et al. [2013] as described in Subsection 3.1.1. Given an auxiliary variable \( \omega \), a conjugate prior for the logistic regression coefficients \( \beta_s \), \( s = 1, 2 \) is multivariate normal distribution \( \mathcal{N}(m_\omega, V_\omega) \).

The joint likelihood of the observed data \( y^T \) and the hidden states \( z^T \) is

\[
\mathcal{L}(\theta, \beta) = \pi\left(y^T, z^T \mid X, \theta, \beta\right) = \pi_1(z_1)f_{z_1}(y_1)\prod_{t=2}^{T} \prod_{i=1}^{2} \prod_{j=1}^{2} p_{ij}^{(t-1)} f_j(y_t)
\]

\[
= \prod_{i=1}^{2} \prod_{j=1}^{2} \left[ \prod_{t \neq t_{i,j}}^{T} p_{ij}^{(t-1)} \right] \left( \frac{1}{2\sigma^2_s} \right)^{N_s/2}
\]

\[
\times \exp\left\{ -\frac{1}{2\sigma^2_s} (Y_j - X_j^{(1)'B_j})'(Y_j - X_j^{(1)'B_j}) \right\}
\]

\[
= \prod_{i=1}^{2} \prod_{j=1}^{2} \left[ \prod_{t \neq t_{i,j}}^{T} \exp(x_{t_{i,j}}^{(2)'\beta_s}) \right] \left( \frac{1}{2\sigma^2_s} \right)^{N_s/2}
\]

\[
\times \exp\left\{ -\frac{1}{2\sigma^2_s} (Y_j - X_j^{(1)'B_j})'(Y_j - X_j^{(1)'B_j}) \right\}.
\]

We use the notation \( N_s, \ s = 1, 2 \) for the number of times the chain was in state \( s \), that is \( N_s = \sum_{t=1}^{T} I(Z_t = s) \), the \( N_s \times T \) with \( I \) the indicator function.

The MCMC sampling scheme is constructed with recursive updates of (i) the latent variables \( z^T \) given the current value of the model parameters by using the scaled Forward-Backward algorithm [Scott 2002] (ii) the logistic regression coefficients by adopting the auxiliary variables method of Polson et al. [2013] given the sequence of states \( z^T \), and (iii) the mean regression coefficients conditional on \( z^T \) by using the Gibbs sampling algorithm.

### 3.1.1 Simulation of the logistic regression coefficients

In a two-state NHHMM, as Meligkotsidou and Dellaportas [2011] observe, we can model the two diagonal elements of probability transition matrix by linking them to the set of covariates using a logistic link. However, the algorithm adopting the auxiliary representation of Holmes and Held [2006] in the method of Meligkotsidou and Dellaportas [2011] does not converge in some cases, especially if there exists model uncertainty in the transition probabilities. Many data-augmentation or Metropolis-Hastings algorithms are proposed to model the logistic regression...
model, see for example [O’Brien and Dunson 2004, Fussel et al. 2013, Polson et al. 2013]. We follow Polson et al. [2013] since as shown in their work, using Pólya-Gamma data augmentation gives superior results, in terms of convergence and mixing, among all the competing methods.

Given the unobserved (latent) data \( z^T = (z_1, \ldots, z_T) \) we define \( \tilde{Z}_t^s = I [ Z_{t+1} = Z_t = s ] \). In words \( \tilde{Z}_t^s \) is the number of times where the chain was at the same stage for two consecutive time periods. Then,

\[
p ( \tilde{Z}_t^s ) = p^s_{ss} = \frac{\exp \left( x_t^{(2)} \beta_s \right)}{1 + \exp \left( x_t^{(2)} \beta_s \right)} \implies \text{logit}(p^s_{ss}) = x_t^{(2)} \beta_s, s = 1, 2.
\]

Polson et al. [2013] proved that binomial likelihoods (thus Bernoulli likelihoods in our simpler case) parametrized by log odds can be represented as mixtures of Gaussian distributions with respect to Pólya-Gamma distribution. The main result of Polson et al. [2013] is that letting \( p(\omega) \) be the density of a latent variable \( \omega \) with \( \omega \sim \mathcal{PG}(b,0) \) for \( b > 0 \), the following integral identity holds for all \( a \in \mathbb{R} \):

\[
\frac{\exp (\psi)^a}{1 + \exp (\psi)} = 2^{-b} \exp (k\psi) \int_{0}^{\infty} \exp (-\omega\psi^2/2) p(\omega) d\omega,
\]

where \( k = a - b/2 \). Furthermore, the conditional distribution of \( \omega \mid \psi \) is also Pólya-Gamma, \( \mathcal{PG}(b,\psi) \). Using the previous result and setting \( \Omega_s = \text{diag}(\omega_{1,s}, \ldots, \omega_{N_s,s}) \) as a set of latent variables the likelihood for each state \( s = 1, 2 \) is

\[
\mathcal{L} (\beta, \omega) = \prod_{t=1}^{N_t} \left\{ \frac{\exp \left( x_t^{(2)} \beta_s \right)}{1 + \exp \left( x_t^{(2)} \beta_s \right)} \right\} \frac{\tilde{z}_t}{1} \exp \left( \frac{1}{1 + \exp \left( x_t^{(2)} \beta_s \right)} \right) \]

\[
= \prod_{t=1}^{N_t} \frac{\exp \left( x_t^{(2)} \beta_s \right)}{1 + \exp \left( x_t^{(2)} \beta_s \right)} \exp \left( k_t x_t^{(2)} \beta_s \right) \int_{0}^{\infty} \exp \left\{ -\omega_{t} \left( x_t^{(2)} \beta_s \right)^2 /2 \right\} p(\omega_{t}) d\omega_{t}
\]

Conditioning on \( \Omega \), one can derive the proportion

\[
\pi (\beta \mid z^s, \omega) \propto \prod_{t=1}^{N_t} \exp \left\{ k_t x_t^{(2)} \beta_s - \frac{\omega_{t}}{2} \left( x_t^{(2)} \beta_s \right)^2 \right\} \pi (\beta_s)
\]

\[
\propto \pi (\beta) \prod_{t=1}^{N_t} \exp \left\{ -\frac{\omega_{t}}{2} \left( x_t^{(2)} \beta_s \right)^2 \left[ -2k_t x_t^{(2)} \beta_s \omega_{t} \right] \right\}
\]

\[
\propto \pi (\beta) \prod_{t=1}^{N_t} \exp \left\{ -\frac{\omega_{t}}{2} \left( x_t^{(2)} \beta_s \right)^2 \left[ \frac{2k_t x_t^{(2)} \beta_s + k_t^2}{\omega_{t}} \right] \right\}.
\]

Assuming as prior distributions \( \omega \sim \mathcal{PG}(b,0) \) and \( \beta \sim \mathcal{N}(m_{0\beta}, V_{0\beta}) \), simulation from the posterior distribution can be done iteratively in two steps:

\[
\omega_t | \tilde{z}_t \sim \mathcal{PG} \left(1, x_t^{(2)} \beta_s \right), \quad t = 1 : N_t, \quad s = 1, 2,
\]

\[
\beta_s | \tilde{Z}, \Omega_s \sim \mathcal{N}(m_{\omega}, V_{\omega}),
\]

\[
V_{\omega} = \left(X^{(2)} \Omega \sigma X^{(2)} + m_{0\omega}^{-1}\right)^{-1} \text{ and } m_{\omega} = V_{\omega} \left(X^{(2)'}k + V_{0\omega}^{-1}m_{0\omega} \right),
\]
where \(\mathcal{P}\mathcal{G}\) denotes the Pólya-Gamma distribution and \(k = (\tilde{z}_1 - 1/2, \ldots, \tilde{z}_N - 1/2)\).

### 3.1.2 Simulation of the mean equation parameters using the Gibbs algorithm

We used conditionally conjugate prior distributions on the parameters of the mean predictive distribution, i.e.

\[
\sigma_s^2 \sim IG(p, q) \quad \text{and} \quad B_s \mid \sigma_s^2 \sim N(L_{0s}, \sigma_s^2 V_{0s}) \quad s = 1, 2.
\]

After some straightforward algebra we derive the conditional and the marginal posterior distribution for the state specific parameters \(\sigma_s\) and \(B_s\),

\[
\sigma_s^2 \mid y^T, z^T \sim IG\left(p + \frac{n_s}{2}, q + \frac{1}{2} \left( Y_s - X_s^{(1)} B_s \right)' \left( Y_s - X_s^{(1)} B_s \right) + \frac{1}{2} (B_s - L_{0s})' V_{0s}^{-1} (B_s - L_{0s}) \right)
\]

\[
\sim IG\left(p + \frac{n_s}{2}, q + \frac{1}{2} (L_{0s} V_{0s}^{-1} L_{0s} + Y_s' Y_s - L_s' V_{0s}^{-1} L_s) \right),
\]

\[
B_s \mid \sigma_s^2, z^T, y^T \sim N(L_s, \sigma_s^2 V_s),
\]

with \(V_s = \left( V_{0s}^{-1} + X_s^{(1)' X_s^{(1)}} \right)^{-1}, L_s = V_s \left( V_{0s}^{-1} L_{0s} + X_s^{(1)' Y_s} \right)\).

### 3.2 Inference under model uncertainty

Here, we consider the full model comparison problem where the uncertainty about which predictors should be included in the mean regression model is taken into account together with the uncertainty about the predictors that should be included in the transition probability equation. The proposed model is flexible, since we do not decide a priori which covariates affect the observed or the unobserved process. Instead, we have a common pool of covariates \(\{X\}\) and within the MCMC algorithm we gauge which covariates are included in subset \(\{X^{(1)}\}\), affecting the mean predictive equation of the observed process, and which covariates are included in subset \(\{X^{(2)}\}\), affecting the time-varying transition probabilities. It is worth mentioning that within the framework of the proposed model, a set of \(r - 1\) covariates implies that there are \(2^{r-1} \times 2^{r-1} = 2^{2(r-1)}\) possible models, hence the model selection problem becomes complicated.

Different approaches have been used in the literature to cope with the model selection problem. The use of information criteria, such as the Akaike’s Information Criterion (AIC, \cite{Akaike1973}), the Bayesian Information Criterion (BIC) of \cite{Schwarz1978}, the Deviance Information Criterion (DIC, \cite{Spiegelhalter2002}) or the Widely applicable Bayesian Information Criterion (WBIC, \cite{Watanabe2013}), is another approach to variable selection.

We propose a probabilistic approach to inference, which is based on the calculation of the posterior distribution of different NHHMMs or equivalently on computing the posterior probabilities of different hidden Markov models. Posterior probabilities can be used either for selecting the most probable model (i.e. making inference using the model with the highest posterior probability), or for Bayesian model averaging (i.e. producing inferences averaged over different NHHMM). \cite{Barbieri2004} argue that the optimal predictive model is not necessarily the model with highest posterior probability but the median probability model, which is defined as the model consisting of those covariates which have overall posterior probability of being included in the model (inclusion probability) greater or equal to 0.5. Our method allows us to calculate the posterior probability of the model as well as the probabilities of inclusion. To this end, we develop a Reversible Jump Markov Chain Monte Carlo (RJMCMC) algorithm (\cite{Green1995} \cite{Green2009}) which explores the model space by jumping between different
hidden Markov models. A more applied tutorial based on the lines of [Green 1995], can be found [Waagepetersen and Sorensen 2001]. A study for comparing variable selection methods is well presented in [O’Hara and Sillanpää 2009] whilst [Dellaportas et al. 2002] study the variable selection methods in the context of model choice.

As [Green and Hastie 2009] noticed, reversible jump MCMC is in fact a Metropolis-Hastings algorithm, formulated to allow sampling from a distribution on a union of spaces of differing dimension and to permit state-dependent choice of move type. Suppose that a prior algorithm, formulated to allow sampling from a distribution on a union of spaces of differing dimension and to permit state-dependent choice of move type. Suppose that a prior distribution \( \pi(k) \) is specified over \( k \) models \( (M_1, M_2, \ldots, M_k) \) in a countable set \( K \) and for each \( k \) we are given a prior distribution \( \pi(\theta_k \mid k) \) along with a likelihood \( \mathcal{L}(y \mid k, \theta_k) \) for data \( y \). The joint prior for \( \theta_k \) and \( k \) is \( \pi(k, \theta_k) = \pi(\theta_k \mid k)p(k) \) and obviously the joint posterior distribution is

\[
\pi(k, \theta_k \mid y) = \frac{\pi(k, \theta_k)\mathcal{L}(y \mid k, \theta_k)}{\sum_{k' \in K} \pi(k', \theta_{k'})\mathcal{L}(y \mid k', \theta_{k'})}d\theta_{k'}.
\]

The standard formulation of the Metropolis-Hastings algorithm relies on the construction of a time-reversible Markov chain, that satisfies the detailed balance condition. This condition means that the probability that the state of a chain is in the set \( B \) and moves to set \( A \) is the same with the probability that the state is in the set \( B \) and moves to set \( A \). Note that this is a simple way to ensure that the limiting distribution of the chain is the desired target distribution. Define \( \tilde{x} = (k, \theta_k) \in \bigcup_{k \in K} \{(k) \times A_k\} \) as the state and state space of chain. At the current state \( \tilde{x} \) we generate \( r \) random numbers \( u \) from a known joint density \( g \). The new state of chain \( \tilde{x}' \) is constructed by some suitable deterministic function \( h : R^n \times R^r \to R^{n'} \times R^{r'} \) such that \( \tilde{x}', u' = h(\tilde{x}, u) \). The inverse move is made by generating \( u' \) from a suitable known density \( g' \) and then using the inverse function \( h' : R^{n'} \times R^{r'} \to R^n \times R^r \) of \( h \) to move from \( \tilde{x}' \) to \( \tilde{x} \). In practice, the construction of proposal moves between different models is achieved via the concept of ‘dimension matching’. Specifically, if the dimensions of \( \tilde{x}, \tilde{x}', u, u' \) are \( n, n', r \) and \( r' \) respectively, then \( n + r = n' + r' \). If the move from \( \tilde{x} \) to \( \tilde{x}' \) is accepted with probability \( \alpha(\tilde{x}, \tilde{x}') \) then, the reverse move is accepted with probability \( \alpha(\tilde{x}', \tilde{x}) \). Under this formulation the detailed balance condition is

\[
\int_{(\tilde{x}, \tilde{x}') \in A \times B} \pi(\tilde{x})g(u)\alpha(\tilde{x}, \tilde{x}') \, dx \, du = \int_{(\tilde{x}, \tilde{x}') \in A \times B} \pi(\tilde{x}')g'(u')\alpha(\tilde{x}', \tilde{x}) \, dx' \, du'.
\]

If the transformation \( h \) from \( (x, u) \) to \( (x', u') \) and its inverse \( h' \) are differentiable then the equality holds if

\[
\pi(\tilde{x})g(u)\alpha(\tilde{x}, \tilde{x}') = \pi(\tilde{x}')g'(u')\alpha(\tilde{x}', \tilde{x}) \left| \frac{\partial (x', u')}{\partial (x, u)} \right|,
\]

where \( J = \left| \frac{\partial (x', u')}{\partial (x, u)} \right| \) is the Jacobian of the transformation. Thus, the acceptance probability \( \alpha \), for moving is

\[
\alpha(\tilde{x}, \tilde{x}') = \min \left\{ 1, \frac{\pi(\tilde{x}')g'(u')}{\pi(\tilde{x})g(u)} \left| \frac{\partial (\tilde{x}', u')}{\partial (\tilde{x}, u)} \right| \right\}.
\]

[Holsclaw et al. 2017] recently introduced NHHMMs using the Pólya-gamma latent data method. As a model selection strategy, they use BIC values to choose the best model among a prespecified set of models. We apply a reversible jump algorithm to account model uncertainty. Due to the fact that in the proposed model there is uncertainty both in the mean predictive equation, as well as in the transition probability equation, we will have to perform two reversible jump steps within our MCMC algorithm. Let us now assume that in a reversible jump step we propose a move type \( m \) from \( \tilde{x} = (k, \theta_k) \) to \( \tilde{x}^* = (k^*, \theta_{k^*}) \). Let \( j_m(\tilde{x}) \) denote the probability that
move \( m \) is attempted at state \( \tilde{x} \) and \( j_m, (\tilde{x}^*) \) the probability of the reverse move attempted at state \( \tilde{x}^* \). We accept the proposed move with probability \( \alpha_m (\tilde{x}, \tilde{x}^*) = \min \{ 1, A_m(\tilde{x}, \tilde{x}^*) \} \) where

\[
A_m(\tilde{x}, \tilde{x}^*) = \frac{L(y^T | \tilde{x}^*) p(\theta^* | k^*) p(k^*) j_m, (\tilde{x}^*) g_m^{\star}(u^* | \tilde{x}^*, k) \partial (\theta^*, u^*)}{L(y^T | \tilde{x}) p(\theta | k) p(k) j_m(\tilde{x}) g_m(u | \tilde{x}, k^*) \partial (\theta, u)}.
\]

We adopt the proposal of Meligkotsidou and Dellaportas (2011) to implement the above algorithm. In each step, we choose to add or remove one covariate with probability 0.5 and then we randomly choose which covariate we will add/remove. We propose a new value for the mean equation coefficients \( \beta \) or for the regression equation coefficients \( \beta^* \) from the full conditional posterior density, conditionally on the other coefficients, thus the Jacobian of transformation will be equal to unity. To be more specific, if we want to update the covariates in the mean equation, the proposal distribution \( g^* \) is just the product of the two conditional posterior distributions for \( s = 1, 2 \) derived in Subsection 3.1.2 given \( \sigma \) and if we want to update the covariates that affect the transition matrix, the proposal distribution is the product of conditional normal distributions derived in Subsection 3.1.1 given \( \Omega \) for \( s = 1, 2 \). With some straightforward matrix algebra, the acceptance probability for the mean equation is \( \alpha_{\beta} = \min \{ 1, A_{\beta} \} \) and the acceptance probability for the transition matrix is \( \alpha_{\beta} = \min \{ 1, A_{\beta} \} \) where

\[
A_B = \frac{j_m, (k^*)}{j_m (k)} \prod_{s=1}^{2} \left| V_s^{\beta} \right|^{1/2} \left| V_0^{\beta} \right|^{1/2} \left| V_s^{\beta} \right|^{1/2} \left| V_0^{\beta} \right|^{1/2} \times \exp \left\{ -\frac{1}{2\sigma^2} \left( L_0^s V_0^{s^{-1}} L_0^s - L_s^s V_s^{s^{-1}} L_s^s - L_0^s V_0^{s^{-1}} L_0^s + L_s^s V_s^{s^{-1}} L_s \right) \right\},
\]

and

\[
A_{\beta} = \frac{j_m, (k^*)}{j_m (k)} \prod_{s=1}^{2} \left| V_s^{\beta} \right|^{1/2} \left| V_0^{\beta} \right|^{1/2} \left| V_s^{\beta} \right|^{1/2} \left| V_0^{\beta} \right|^{1/2} \times \exp \left\{ -\frac{1}{2\sigma^2} \left( L_0^s V_0^{s^{-1}} L_0^s - L_s^s V_s^{s^{-1}} L_s^s - L_0^s V_0^{s^{-1}} L_0^s + L_s^s V_s^{s^{-1}} L_s \right) \right\}.
\]

### 3.3 MCMC Sampling Scheme

In the next lines, we summarize the MCMC algorithm that we have constructed for joint inference on model specification and model parameters.

1. Start with initial values of \( \beta, \theta = [B, \sigma] \).
2. Calculate the probabilities of time-varying transition matrix for \( t = 1 : T \).
3. Run a Scaled Forward-Backward (Scott (2002)) algorithm to simulate the hidden states given the parameters of the model.
4. Simulate the parameters of the regression model for the mean via a Gibbs sampler method.
5. Simulate the coefficients \( \beta \) using the Pólya-gamma representation by Polson et al. (2013).
6. Do a double reversible jump algorithm to define which covariates will affect the transition matrix and which covariates will affect the mean regression model.
7. Make one-step-ahead predictions conditional on the simulated unknown quantities.

Repeat steps 3-6 until convergence and then repeat steps 3-7.
4 Bayesian Forecasting and Scoring rules

4.1 One-step-ahead predictions

The proposed modeling and inferential approach is used for forecasting. The posterior predictive density can not be found in closed form, but it can instead be evaluated numerically. Given model $M$, the predictive distribution of $y_{T+1}$ is

$$f_p(y_{T+1} \mid y^T) = \int f(y_{T+1} \mid y^T, z^T, M, \beta_M, \theta_M) \pi(\beta_M, \theta_M \mid y^T) d\beta_M d\theta_M,$$

where

$$f(y_{T+1} \mid y^T, z^T, \beta_M, \theta_M) = \sum_{s=1}^2 P(Z_T = z_T \mid Z_T = s) f_s(y_{T+1}).$$

In practice we follow an iterative procedure within the MCMC algorithm to draw a sample from the posterior predictive distribution. At the $r$-th iteration of our algorithm, the algorithm chooses model $M$. Also, the hidden states and the unknown parameters $\beta^{(r)}_M, \theta^{(r)}_M$ are simulated as described Subsection 3.1. Let $z^{(r)}_T$ be the hidden state at time $T$. To make an one-step-ahead prediction (i.e. simulate $y_{T+1}$), we first simulate the hidden state for time $T+1$, from the discrete distribution based on the transition probabilities $P(Z_{T+1} = s \mid Z_T = z^{(r)}_T)$, $s = 1, 2$ and then conditional on the hidden state we draw a value $y^{(r)}_{T+1}$ from

$$Y^{(r)}_{T+1} \mid Z^{(r)}_{T+1} = s \sim N(X^{(1)}_T B_s, \sigma^2_s), s = 1, 2.$$

Given $Y_{T+l}$, $l = 1, \ldots, L$, the hidden state $Z_{T+l}$ and the covariates $X_{T+l-1}$, we may also update the transition matrix $P^{T+l-1}$, simulate $Z_{T+l+1}$ and then simulate the prediction $y_{T+l+1}$ from its respective predictive distribution.

4.2 Forecasting criteria

An issue of major importance in forecasting experiments is to evaluate the quality of the obtained forecasts. Moreover, one way to evaluate a model under consideration is through the accuracy of its forecasts (Geweke and Whiteman [2006]). As noted by Gelman et al. [2014], predictive accuracy is valued not only for its own sake but also for comparing different models. Advances in numerical integration via MCMC algorithms made probabilistic forecasts possible, which are in most cases, preferable. Besides, having the posterior predictive distribution, one can obtain point effective forecasts, using suitable scoring functions (Gneiting [2011]).

Scoring rules provide summary measures for the evaluation of probabilistic forecasts, by assigning a numerical score based on the forecast and on the event or value that it materializes. We refer to Gneiting and Raftery [2007] for a review on the theory and properties of scoring rules. If the forecaster quotes the predictive distribution $P$ and the event $x$ materializes, then his reward is $S(P, x)$. If $Q$ is the forecaster judgment then $S(P, Q)$ is the expected value of his reward under $Q$. A scoring rule is said to be proper if $S(Q, Q) \geq S(P, Q)$ and if the equality $S(Q, Q) = S(P, Q)$ holds if and only if $Q = P$, then the rule is strictly proper. Intuitively, proper scoring rules encourage a forecaster to report the truth about his judgment distribution and they are strictly proper if the issued forecasts are the same with the forecaster’s judgment.

There are various proper scoring rules for continuous variables, such as the probabilistic score, the Linear Score (LinS), the Quadratic Score (QS), the Logarithmic Score (LogS) and the Continuous Ranked Probability Score (CRPS), among others [Machete 2013, Gneiting and]
Probabilistic and Linear score are often used, nevertheless they are improper and can have misleading results. A widely used and quite powerful criterion is the Logarithmic Score. It is based on the posterior predictive density evaluated at the observed value and there is a strong connection with the classical Kullback-Leibler divergence, which is one of the reasons for being studied quite intensively (see for example Gelman et al. [2014], Gschlößl and Czado [2007]). The Logarithmic Score is defined as follows: Let $T$ be the total sample size and $N$ the number of out-of-sample forecasts. For $i = T - N, T - N + 1, \ldots, T$ let $y_i$ be the real observed values of the forecasts, $\hat{y}_i$ the estimated forecasts and $\hat{\theta}$ the unknown quantities. Finally, let $D$ be the number of draws of the simulation. Using the notation $f_p(y_i)$ for the posterior predictive density of the new data, $f$ for the distribution of the true model and $\pi(y \mid \theta)$ for the likelihood of the data, the predictive fit for the $i$-th observation would be,

$$\log(f_p(\hat{y}_i)) = \log \int \pi(\hat{y}_i \mid \hat{\theta}) \pi(\hat{\theta}) d\hat{\theta}. $$

Having in mind though that the new data are unknown Gelman et al. [2014] defined the expected out-of-sample log predictive density for the $i$-th observation as

$$E_f(\log(f_p(\hat{y}_i))) = \int (\log(f_p(\hat{y}_i))) f(\hat{y}_i) d\hat{y}_i. $$

To compute the predictive density for $N$ estimated forecasts in practice, since $\hat{\theta}$ is unknown, we shall calculate

$$LogS = \sum_{i=T-N}^{T} \log \left( \frac{1}{D} \sum_{j=1}^{D} \pi(\hat{y}_i \mid \hat{\theta}^j) \right). $$

Although the Logarithmic Score is a strictly proper rule, it lacks robustness as it involves harsh penalty for low probability events and thus is sensitive to extreme cases (Boero et al. [2011]). Besides, comparing the entropies of the forecasts, Machete [2013] showed that LS prefers the density of the new data, the predictive fit for the $i$-th observation would be,

$$\sum_{i=T-N}^{T} \sum_{j=1}^{D} \pi(\hat{y}_i \mid \hat{\theta}^j) \frac{1}{D} \sum_{j=1}^{D} \pi(\hat{y}_i \mid \hat{\theta}^j). $$

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$$LogS = \sum_{i=T-N}^{T} \log \left( \frac{1}{D} \sum_{j=1}^{D} \pi(\hat{y}_i \mid \hat{\theta}^j) \right). $$
\[ H(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
1 & \text{otherwise}. 
\end{cases} \]

As described above, in our setting, at each iteration of the MCMC algorithm, we obtain a sample of length \( L \) from the predictive distributions of \( y_{T+1}, \ldots, y_{T+L} \). Hence we can evaluate both the posterior predictive probability and cumulative distribution function. Instead, there is an easier way of evaluating CRPS using the identity of Székely and Rizzo [2005].

\[
CRPS(F, y_i) = \frac{1}{2} E_{F} |Y - Y'| - E_{F} |Y - y_i|,
\]

were \( Y, Y' \) are independent copies of a random variable with distribution function \( F \) (see also Gschlößl and Czado [2007]).

Finally, for the sake of completeness we estimate the mean, mode and median of the predictive distribution and compare them using two well known point forecasting criteria, the Mean Square Forecast Error, \( MSFE = \frac{1}{N} \sum_{i=T-N}^{T} (y_i - \hat{y}_i)^2 \) and the Mean Absolute Forecast Error, \( MAFE = \frac{1}{N} \sum_{i=T-N}^{T} |y_i - \hat{y}_i| \). Based on the MCMC sample, the computed point forecasting criteria is just the average score of MSFE or MAFE for all the sample values.

5 Simulation Study

We have conducted a series of simulation experiments to assess the performance of the proposed approach in terms of inference, model selection and predictive ability. We have tested quite extensively our algorithms, accounting for model uncertainty or not, using different sample sizes and assigning various values to the parameters. We compared our model with a Homogeneous Hidden Markov model and a linear model with autoregressive terms (LR). Although considered standard, for the sake of completeness, we give the definitions of HHMM and LR models. In the Homogeneous Hidden Markov model, covariates affect the mean equation but the transition probability matrix is constant. That is,

\[
Y_t \mid Z_t = s \sim N(X_{t-1}^{(1)} B_s, \sigma_s^2), \ s = 1, 2, \ t = 1, \ldots, T,
\]

\[
P(z_t = j \mid z_{t-1} = i) = p_{ij}, \ i, j = 1, 2 \ \forall \ t = 1, \ldots, T.
\]

In the linear regression with autocovariates, besides covariates \( X_{t-1}, Y_{t-1} \) lagged values of \( Y_t \) are assumed to affect the mean equation. Specifically,

\[
Y_t = [X_{t-1} Y_{t-1}]B + \epsilon_t , \ t = 1, \ldots, T \text{ and } \epsilon_t \sim N(0, \sigma^2).
\]

The data were generated either from a Homogeneous Hidden Markov model (HHMM) or from a NHHMM model with covariates simulated from independent normal distributions. We found that the mean equation parameters converge rapidly whereas the logistic regression coefficients needed some burn in period to converge. The hidden chain \( Z_t \) was well estimated. For each iteration we kept a replication of the hidden chain (thus the number of the hidden-chain replications is the same with the MCMC sample size) and compared it with the real simulated hidden chain, using a 1-0 loss function. Also the reversible jump algorithm behaved really well in all of our simulations. Even in the case where the data were homogeneous, no covariates were selected in the logistic regression equation, thus the transition probability matrix remained constant through time. Furthermore, in order to test the predictive ability of our model we kept \( N \) out-of-sample observations and we calculated the CRPS for every observation. In all our experiments we found that our model outperforms the competing models in forecasting the observed process \( y^T \). A summary of our findings is presented and discussed in the following subsections.
5.1 The fixed model

Below, we present two experiments with simulated data from the fixed NHHMM. For all our experiments we used non-informative priors for the unknown parameters $\sigma_s, B_s, \beta_{ss}$, $s = 1, 2$, that is $\sigma^2_s \sim IG(0.1, 0.1), B_s \mid \sigma^2_s \sim N(0, 100\sigma^2_s)$ and finally $\beta_{ss} \sim N(0, 80 \times I)$. Inferences are based on an MCMC sample of 15000 iterations after a burn in period of 10000 iterations.

In the first experiment we used a sample of $T = 1000$ observations. From a common pool of covariates $X = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$ we chose $X^{(1)} = \{X_1, X_2, X_3, X_7\}$ to affect the mean equation and $X^{(2)} = \{X_1, X_3, X_4, X_5, X_6\}$ to affect the transition matrix. The predictors, as already mentioned, were simulated from independent normal distributions with means $\mu_x = [4, 3, -2, -5, 7, -1, 0.6]$ and variance $\sigma^2_x = [1, 1, 0.3, 0.8, 1, 0.25]$. The mean equation parameters were $B_1 = [5, 1, 3, 2, 4]'$, $\sigma^2_1 = 0.8$ and $B_2 = [3, -4, 3, 1, 2]'$, $\sigma^2_2 = 1.3$ whereas the logistic regression coefficients where $\beta_1 = [-2, 3, 2, 4, 1, -2]'$ and $\beta_2 = [2, 4, 1, 2, -1, 3]'$ for states $s = 1, 2$ respectively. Secondly, we run another experiment with sample size $T = 1500$. From a common pool of independently normally distributed covariates $X = \{X_1, X_2, X_3, X_4\}$ with means $\mu_x = [4.3, -2, -5]$ and variance $\sigma^2_x = [1, 1, 0.25, 1]$ we used 3 covariates $X^{(1)} = \{X_1, X_2, X_3\}$ affecting the mean equation and $X^{(2)} = \{X_1, X_2, X_3\}$ the transition matrix. The mean equation parameters were $B_1 = [2, -0.3, 2, 2]'$, $\sigma^2_1 = 0.15$ and $B_2 = [1, 3, 4, 3]'$, $\sigma^2_2 = 0.8$ whereas the logistic regression coefficients where $\beta_1 = [1.5, 1, 2, 3]'$ and $\beta_2 = [3, -2.5, 4, 1]'$ for states $s = 1, 2$ respectively. For both experiments, we kept 10 out-of-sample observations and we computed a sequence of one-step-ahead forecasts of the real observed process. Our algorithm gives accurate estimations and has excellent convergence performance. Also, comparing our forecasts with the forecasts of the two aforementioned simpler models, we found that our methodology outperforms the two models. For completeness we present in Section 8 various plots that confirm the estimation performance and convergence of our methodology. Parameter estimates and summary statistics are presented in Table 7 and a comparison of the three competing models in terms of forecasting ability in Table 8 for the first experiment and in Tables 9 and 10 for the second experiment study.
Figure 1: Experiment 1. The upper panel shows a comparison between the actual (simulated) transition probabilities $p_{11}^t, p_{22}^t$ and the corresponding estimated transition probabilities. True probability values are marked with red dots whereas the posterior mean estimated transition probabilities are marked with blue crosses. The lower panel plots the difference between actual transition probabilities and the estimated ones.

Figure 2: Forecasts of experiment 1. Empirical continuous approximation of the posterior predictive distribution (based on a normal kernel function) using the NHHMM (blue continuous line) and the HHMM (red dotted line). Actual out-of-sample values are marked with yellow asterisks.
Figure 3: Experiment 1. Quantiles of the predictive distribution. The ten one-step-look-ahead actual values (as taken from the sample) are marked with dashed blue line. The gray line represents the median (the gray dots represent the point forecasts) of the predictive distribution and the gray area shows how the 95% credible interval (CI) of forecasts evolves in time. The CI at each point is defined by the 2.5% and 97.5% quantiles of the respective predictive distributions. The red dots represent the median values of the predictive distribution using the HHMM.
Figure 4: Experiment 2. The upper panel shows a comparison between the actual (simulated) transition probabilities $p_{11}^t, p_{22}^t$ and the corresponding estimated transition probabilities. True probability values are marked with red dots whereas the posterior mean estimated transition probabilities are marked with blue crosses. The lower panel plots the difference between actual transition probabilities and the estimated ones.

Figure 5: Forecasts of experiment 2. Empirical continuous approximation of the posterior predictive distribution (based on a normal kernel function) using the NHHMM (blue continuous line) and the HHMM (red dotted line). Actual out-of-sample values are marked with yellow asterisks.
5.2 Model selection

In this section we present two simulation experiments (experiments 3 and 4) regarding the problem of variance selection. We compared results from the three models (NHHMM, HHMM, LR) subject to model uncertainty. We performed stochastic variable selection with reversible jump for all the competing models. We used the same non-informative priors and burn-in period, as in Subsection 5.1.

For the first experiment we simulated data of size \( T = 1200 \). We used the same two covariates affecting the mean equation and transition matrix via the logistic regression. To check the variable selection ability of our method we included three more covariates in the common pool of predictors. We found that our approach was able to identify the correct data generating process. In Table 1 we show that the both the most probable model and the Median probability model are the same as the true model. Additionally, as in the fixed model case experiments we kept 10 out-of-sample observations for evaluating the forecasting ability of our approach. We confirmed that our model performed better than the other two models in terms of the CRPS, MAFE and MSFE (see Table 2). Finally, for each MCMC iteration we kept a replicated chain of the hidden process and we compared it with the true simulated chain. Using the 0-1 Loss function we computed the mean mis-estimated states in each chain. On average from the chain with 1200 hidden states of the first experiment, our model failed to recognize 22 states per iteration. We note that we did not encounter any label switching problems.

Next, we present the mean estimated probabilities (for remaining at the same stage two consecutive time periods, that is \( p_{11} \) or \( p_{22} \)) of the transition matrix and the differences with the true simulated probabilities (Figure 7). A realization of the observed process and hidden process as well as one realization of the simulated hidden process are presented in Figure 8. Finally,
the predictive distribution using a kernel density function is plotted, for both NHHM and HHM models, and are shown in Figure 10, while the quantiles of the predictive distributions are shown in Figure 9.

Moreover, we present a second, more complicated, experiment. In that we used 3 covariates affecting the mean equation and 5 covariates affecting the transition probability matrix. The results coincide with the previous experiment. We identified the true generating process, using either the most probable model or the median probability model (Table 3) and the forecasts of our model were better than the benchmark models (Table 4). As far as the hidden states, on average from the chain of the 1500 hidden states we failed to recognize only 8 states. The mean estimated probabilities $p_{11}$ or $p_{22}$ of the transition matrix are shown in Figure 11. Figure 12 shows a realization of the simulated hidden process against the true observed process and finally as in experiment 3, Figures 13 and 14 are the plots of the predictive distribution and its quantiles. It can been seen that our approach performs very well in terms of predictive ability in both simulation experiments.

Figure 7: Experiment 3. The upper panel shows a comparison between the actual (simulated) transition probabilities $p_{11}^t, p_{22}^t$ and the corresponding estimated transition probabilities. True probability values are marked with red dots whereas the posterior mean estimated transition probabilities are marked with blue crosses. The lower panel plots the difference between actual transition probabilities and the estimated ones.
Figure 8: Observed process (black dotted line) and hidden process. True hidden states are marked with blue x and a realized simulated states are marked with red dots.

| True model                  |                               |
|-----------------------------|-------------------------------|
| Included Covariates, Mean   | $X_1, X_2$                    |
| Included Covariates, Transition Matrix | $X_1, X_2$                  |
| Total Covariates            | $X_1, X_2, X_3, X_4, X_5$    |

| Highest Posterior Probability model |                               |
|-------------------------------------|-------------------------------|
| Included Covariates, Mean           | $X_1, X_2$                    |
| Included Covariates, Transition Matrix | $X_1, X_2$                  |
| Posterior probability               | 0.9958                        |

| Median Probability model           |                               |
|------------------------------------|-------------------------------|
| Included Covariates, Mean          | $X_1, X_2$                    |
| Included Covariates, Transition Matrix | $X_1, X_2$                  |

Table 1: Experiment 3. Highest posterior probability model and median probability model.
Figure 9: Experiment 3. Empirical posterior predictive distribution for the 10 out-of-sample forecasts using the NHHMM (blue line) and the HHMM (red dotted line). Actual out-of-sample values are marked with yellow asterisks.

Figure 10: Experiment 3. Quantiles of the predictive distribution. The ten one-step-look-ahead actual values (as taken from the sample) are marked with dashed blue line. The gray line represents the median (the gray dots represent the point forecasts) of the predictive distribution and the gray area shows how the 95% credible interval (CI) of forecasts evolves in time. The CI at each point is defined by the 2.5% and 97.5% quantiles of the respective predictive distributions. The red dots represent the median values of the predictive distribution using the HHMM.
| \( y \) | \( y_1 \) | \( y_2 \) | \( y_3 \) | \( y_4 \) | \( y_5 \) | \( y_6 \) | \( y_7 \) | \( y_8 \) | \( y_9 \) | \( y_{10} \) | \( E(CRPS(y_i)) \) | \( MAFE \) | \( MSFE \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| NHHMM | -0.2428 | -0.7669 | -0.9958 | -2.3435 | -2.0090 | -0.9844 | -1.4488 | -2.4239 | -1.0679 | -2.6806 | -1.5163 | 3.5105 | 22.2409 |
| HHMM | -0.2735 | -5.5975 | -2.6073 | -5.1726 | -4.6137 | -1.15774 | -2.4993 | -5.5700 | -0.7877 | -1.3794 | -2.9957 | 4.8270 | 37.0315 |
| LR | -6.0555 | -6.2297 | -6.0018 | -6.6054 | -6.4204 | -5.8633 | -6.0581 | -6.6718 | -6.1227 | -5.6358 | -6.1728 | 20.5394 | 665.1931 |

Table 2: Experiment 3: Predictive ability of the three competing models.

Figure 11: Experiment 4. The upper panel shows a comparison between the actual (simulated) transition probabilities \( p_{11}, p_{22} \) and the corresponding estimated transition probabilities. True probability values are marked with red dots whereas the posterior mean estimated transition probabilities are marked with blue crosses. The lower panel plots the difference between actual transition probabilities and the estimated ones.
Figure 12: Experiment 4: Thinned observed process (black dotted line) and hidden process. True hidden states are marked with blue x and a realized simulated states are marked with red dots.

| True model |  |
|------------|--|
| Included Covariates, Mean | $X_1, X_2, X_3$ |
| Included Covariates, Transition Matrix | $X_1, X_2, X_5, X_6, X_7$ |
| Total Covariates | $X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8$ |

| Highest Posterior Probability model |  |
|------------------------------------|--|
| Included Covariates, Mean | $X_1, X_2, X_3$ |
| Included Covariates, Transition Matrix | $X_1, X_2, X_5, X_6, X_7$ |
| Posterior probability | 0.9775 |

| Median Probability model |  |
|--------------------------|--|
| Included Covariates, Mean | $X_1, X_2, X_3$ |
| Included Covariates, Transition Matrix | $X_1, X_2, X_5, X_6, X_7$ |

Table 3: Experiment 4. Highest posterior probability model and median probability model.
Figure 13: Experiment 4. Empirical posterior predictive distribution for the 10 out-of-sample forecasts using the NHHMM (blue line) and the HHMM (red dotted line). Actual out-of-sample values are marked with yellow asterisks.

Figure 14: Experiment 4. Quantiles of the predictive distribution. The ten one-step-look-ahead actual values (as taken from the sample) are marked with dashed blue line. The gray line represents the median (the gray dots represent the point forecasts) of the predictive distribution and the gray area shows how the 95% credible interval (CI) of forecasts evolves in time. The CI at each point is defined by the 2.5% and 97.5% quantiles of the respective predictive distributions. The red dots represent the median values of the predictive distribution using the HHMM.
6 Empirical Application: Realized volatility data

Financial volatility has been extensively studied in the literature, due to its crucial role in various financial fields, such as asset pricing, risk management, investment and asset allocation among others. Many models have been proposed, exploring different methodologies. A review on the realized volatility literature can be found in McAleer and Medeiros [2008]. Several studies have considered predicting realized stock volatility using various financial and/or economic predictors see, for example Mittnik et al. [2015], Meligkotsidou et al. [2018], Christiansen et al. [2012], Paye [2012]. We use the described model and methodology to asses the predictive ability of 13 financial variables in forecasting future volatility.

6.1 The data

We applied our method to realized stock market volatility data, as described in detail by Christiansen et al. [2012]. Specifically, we use the 'long' sample of U.S. equity market of the S&P500. The realized volatility is the squared root of the realized variance for asset class $i$ in month $t$ expressed as the sum of squared intra-period (daily) returns

$$RV_{i,t} = \sqrt{\sum_{\tau=1}^{M_t} r_{i,t,\tau}^2}, \ t = 1,\ldots,T,$$

where $r_{i,t,\tau}$ is the $r$th daily continuously compounded return of month $t$ for asset $i$ and $M_t$ denotes the trading days during month $t$. Thus $\sum_{\tau=1}^{M_t} r_{i,t,\tau}^2$ is the realized variance for asset class $i$ in month $t$. The distribution of the realized daily variances are highly non-normal and skewed to the right, but the logarithms of the realized variances are approximately normal and thus they have better behavior (see Andersen et al. [2003]). Hence, in the following analysis we study the natural logarithm of the realized volatility series,

$$RV_{i,t} = \ln \left( \sum_{\tau=1}^{M_t} r_{i,t,\tau}^2 \right), \ t = 1,\ldots,T.$$
The data are observed in a monthly basis, starting from December 1926 to October 2002. The out-of-sample forecast evaluation period is set to two years, i.e. 24 observations from October of 2000 until October 2002. We had a burn-in period of 20000 and we generated 20000 MCMC samples. We used non-informative priors for the unknown parameters $\sigma_s, B_s, \beta_{ss}$, $s = 1, 2$, that is $\sigma_s^2 \sim IG(0.1, 0.1)$, $B_s \mid \sigma_s^2 \sim N(0, 1000\sigma_s^2)$ and finally $\beta_{ss} \sim N(0, 100 \times I)$. Following Christiansen et al. [2012] and Meligkotsidou et al. [2018] we took into account 13 macroeconomic and financial predictive covariates. Particularly, from a list of equity market variables and risk factors we considered dividend price ratio (DP) and earnings price ratio (EP) (Welch and Goyal [2008]), lagged equity market returns (MKT), in order to capture the leverage effect, that is the asymmetric response of volatility to positive and negative returns (Nelson [1991]). We also used the risk factors of Fama and French [1993], that is, the size factor (SMB), value factor (HML), a short-term reversal factor (STR). From the set of interest rates, spreads and bond market factors we included the treasury bill rate (TBL), i.e. the interest rate on a three-month Treasury bill, the long-term return (LTR) on long-term government bonds, the term spread (TMS), i.e. the difference between the log-term yield and Treasury bill rate, the relative T-bill rate (RTB) as the difference between, T-bill rate and its 12-month moving average and the relative bond rate (RBR), as the difference between LTR and its 12 month moving average (Welch and Goyal [2008]). To proxy for weighted credit risk we also used the default spread (DEF) defined as the yield spread between BAA and AAA rated bonds. Lastly we consider the macroeconomic variable, inflation rate (INF), the monthly growth rate of CPI. The strong contemporaneous relation between the volatility and the business conditions implies that lagged volatility plays important role in forecasting (see Paye [2012]). We run a series of experiments for this data. Specifically we performed our analysis using the predictors described and then we repeated the analysis using the predictors plus autoregressive terms (AR) of lag 1, 2 and lag 3 to investigate if there is additional predictive content of the macroeconomic and financial variables that go beyond the information contained in time-series history of volatility.

6.2 Results

Our in-sample analysis revealed some interesting findings. Based on the posterior probabilities of inclusion we see that if we do not include any AR terms in the pool of the predictors, then the median NHHMM model has 9 predictors that affect the series linearly or not. When we added the AR(1) term, the included predictors in the median probability model immediately reduced to 6. Adding more AR terms (of lag 2 and lag 3), the median probability model remained almost the same as in the case of the model with one autoregressive term. Furthermore, in our out-of-sample analysis, we did not encounter any significant improvement in the forecasting ability of the models with AR(2) and AR(3) terms. We note that this result confirms the findings of Christiansen et al. [2012], who also used only one AR term of lag 1 in their analysis. Hereafter, even though based on the CRPS the model with the best performance was the one with AR(1) term, we present both results of the model with no autoregressive terms (NHHMM$_0$) and the model with one autoregressive term (NHHMM$_1$), for the sake of completeness. Also, we compare our results with two benchmark models, namely the Homogeneous model with one autoregressive term (HMM) and the linear model with autoregressive terms (LR), as described in Section 5.

Figure 15 shows a plot of the realized volatility data (blue line), the probability of staying at the same stage (e.g. if at time $t$ we are on state 1 then the red dot at time $t$ shows the value of probability $p_{11}$) and the shaded bars represent the time period that the chain was in state 2, based on the smoothed probabilities of being above 0.5 for NHHMM$_1$. We observe that both states are highly persistent, i.e. probabilities of staying at the same state are high. Furthermore, in Figure 17 we present a thinned in-sample realization of the observed process inferred by our
algorithm along with the real data.

6.2.1 Model Selection

Our model selection algorithm, did not assign high probability to any specific model. Instead, there was a large amount of model uncertainty. In we summarize Table 6 the posterior probabilities of inclusion for each predictor, both for the mean equation and for the transition matrix for the NHHMM$_0$ and NHHMM$_1$. Our methodology is not only able to identify which covariates affect the realized volatility series but also to decide how the covariates affect the series, i.e. linearly or non-linearly. Based on the median model of NHHMM$_0$ we found that predicting realized volatility series can be improved using several predictors. Earnings price ratio (EP), lagged equity market returns (MKT), value factor HML, relative T-bill rate (RTB), long-term government bonds (LTR), relative T-bill rate (RBR) affect the series linearly, term spread (TMS), difference between LTR and TBL (RBR), inflation (INF) affect the series non-linearly and term spread (TMS), default spread (DEF) affect the series in a linear and non-linear fashion. The median probability NHHMM with an AR(1) term, NHHMM$_1$, has notably less predictors affecting the RV-series. Specifically we found that MKT affect the series linearly, TBL, RTB, LTR, TMS affect the series non-linearly and DEF both linearly and non-linearly. Our analysis agrees with the analysis of Christiansen et al. [2012] who find that MKT, TBL, LTR, DEF improve volatility forecasts, but we also find significant two more predictors (TMS, RTB), whereas we do not include the variables STR and LTR that Christiansen et al. [2012] found also significant.

6.2.2 Forecasting

As far as forecasting is concerned, we report in Table 5 the values of the forecasting criteria that we used for all the competing models. We confirm that NHHMM$_1$ performs better than all the other models, since it has higher scores in Mean Ranked Probability Score (E(CPRS)) and also in MAFE and MSFE. However, we would like to note that, eventhough the NHHMM$_0$ performs worse than other models in terms of the forecasting criteria considered, it is able to predict the moves of the series better.

![Figure 15: Time series of the monthly realized volatility of the Standard & Poor (S&P) 500 index (in logarithmic scale) for the period 1926-2002, using the NHHMM$_0$ (top figure) and NHHMM$_1$ (lower figure). Red dots are the posterior mean probabilities of staying at the same stage. We calculated the smoothed probability of staying at state 1. If the probability is above 0.5 we marked this event with gray-shaded bar.](image-url)
Table 5: Out-of-Sample results: Predictive ability of the competing models (NHHMM\textsubscript{0}, NHHMM\textsubscript{1}, HHMM, LR(3)) for the log-realized volatility data. For each of the competing models the table presents the mean Continuous Ranked Probability Score E(CRPS). Also Mean Absolute Forecasting Error and Mean Squared Forecasting Error are reported. Better performance by means of forecasting criteria is marked with asterisks.
| Covariates | \( \text{NHHM}_0 \) | \( \text{NHHM}_1 \) | HHM | AR |
|------------|-----------------|-----------------|-----|-----|
|            | Mean Equation   | Transition Matrix | Mean Equation | Transition Matrix | Mean Equation | Mean Equation |
| DP         | 0.0016           | 0.0452           | 0.0051 | 0.0592 | 0.0099 | 0.0083 |
| EP         | 1.0000*          | 0.0629           | 0.4709 | 0.2565 | 0.9991* | 0.0182 |
| MKT        | 1.0000*          | 0.3372           | 0.9867* | 0.3162 | 1.0000* | 0.3297 |
| SMB        | 0.0182           | 0.4228           | 0.0153 | 0.4452 | 0.1444 | 0.0367 |
| HML        | 0.6249*          | 0.4304           | 0.0156 | 0.4397 | 0.6519* | 0.0297 |
| STR        | 0.0013           | 0.4322           | 0.0908 | 0.4438 | 0.0059 | 0.4087 |
| TBL        | 0.0173           | 0.3896           | 0.0124 | 0.5300* | 0.0581 | 0.0466 |
| RTB        | 0.9733*          | 0.4663           | 0.0786 | 0.5145* | 0.8265* | 0.1485 |
| LTR        | 1.0000*          | 0.4799           | 0.0494 | 0.5014* | 0.9431* | 0.1467 |
| RBR        | 1.0000*          | 0.4731           | 0.0589 | 0.4860 | 0.9448* | 0.1119 |
| TMS        | 0.9241*          | 0.5082*          | 0.0517 | 0.5018* | 0.6313 | 0.0961 |
| DEF        | 1.0000*          | 0.5102*          | 1.0000* | 0.5025* | 1.0000* | 1.0000* |
| INF        | 0.4130           | 0.5074*          | 0.3018 | 0.4964 | 0.5444* | 0.1669 |

Table 6: Posterior probabilities of inclusion for the competing models. Predictors with inclusion probability above 0.5 (median probability model) are marked with asterisks.
7 Conclusions

In this paper we have considered inference on predictive Non-Homogeneous Hidden Markov models. We allowed different covariates affecting the mean equation and the time-varying transition probabilities. Transition probabilities were parametrized via a logistic link. Eventhough inference for logistic regression coefficients is considered to be an easy task in the case of using logistic functions for modeling the transition probabilities of NHHMMs, many proposed algorithms face serious convergence issues. Applying the recently proposed Pólya-Gamma data augmentation scheme of Polson et al. [2013] we were able to build a stable and accurate MCMC scheme to make inference on the parameters of our model. Furthermore, we performed stochastic variable selection using a double reversible jump step in order to determine which covariates affect the series linearly and/or in a non-linear fashion. To summarize our approach, we simulated the hidden states using the scaled Forward-Backward algorithm of Scott [2002], simulated the mean equation parameters using a Gibbs step, then we used the Pólya-Gamma augmentation scheme to simulate the logistic regression coefficients for the transition probabilities and performed a double reversible jump step to choose the covariates that affect the mean equation and the transition probabilities. When convergence was achieved we then made one-step-look-ahead predictions.

To assess the performance of the proposed algorithm, as well as the predictive ability of our model and methods, we conducted a number of simulation experiments. Our results have confirmed that our algorithm mixes and converges well and provides accurate estimates of the model’s parameters. Moreover using the continuous ranked probability score we showed that our model outperforms simpler competing models used as benchmarks in terms of forecasting ability. The proposed methodology was applied to realized volatility dataset for predicting future observations and for predictor selection. In addition, we identify which predictors affect the series linearly and/or non-linearly. The median probability model that is derived with our methodology has 6 predictors. In a comparison with the included variables of Christiansen et al. [2012] we find that 4 out of 6 predictors of our model are in common, whereas we do not include 2 predictors that the authors of Christiansen et al. [2012] find significant. We reported that using our model we had better predictive ability. We believe our empirical findings pave the way for applying complicated Non-Homogeneous Hidden Markov models for predicting financial and economic univariate and possibly multivariate time series.

8 Appendix

8.1 Simulation studies’ complementary tables and plots

In this section we present complementary results, of the experiments described in Subsection 5.1 on the accuracy of the obtained estimates and on the mixing properties of our algorithm. Summary statistics for the parameters of the first experiment are presented in Table 7 and the assessment of the forecasting ability of our methodology (in correspondence of the experiments 3 and 4 of Subsection 5.2) is shown Table 2. Also, we provide the histograms of the mean equation parameters and logistic regression parameters of aforementioned experiment in Figures 18 and 20. The convergence of the algorithm is shown in Figures 19 and 21. Analogously, summary statistics, histograms and convergence plots for the second experiment are shown in Table 9 and figs. 22 to 25. Ultimately, we demonstrate the forecasting ability of our methodology for the second experiment in Table 10.

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| States | True Values | Posterior Mean | Std | 2.5% | Median | 97.5% |
|--------|-------------|----------------|-----|------|--------|-------|
| s = 1  |             |                |     |      |        |       |
| b_0 = 3 | 3.5566      | 0.3282         |     | 2.9187 | 3.5566 | 4.2020 |
| b_1 = -4 | -4.0533     | 0.0481         | -4.1483 | -4.0533 | -4.0524 | -3.9592 |
| b_2 = 3 | 2.9584      | 0.0491         | 2.8619 | 2.9584 | 3.0544 |
| b_3 = 1 | 1.0855      | 0.0978         | 0.8929 | 1.0855 | 1.2748 |
| b_4 = 2 | 1.8942      | 0.0954         | 1.7083 | 1.8942 | 2.0809 |
| \sigma^2 = 1.3 | 1.4228 | 0.0829 | 1.2709 | 1.4202 | 1.5947 |
| \beta_0 = 2 | 3.1800 | 1.8851 | -0.4722 | 3.1685 | 6.9587 |
| \beta_1 = 4 | 3.7009 | 0.4236 | 2.9081 | 3.6835 | 4.3753 |
| \beta_2 = 1 | 0.7587 | 0.3742 | 0.0439 | 0.7527 | 1.5111 |
| \beta_3 = 2 | 1.1884 | 0.2659 | 1.3518 | 1.8290 | 2.3900 |
| \beta_4 = -1 | -1.1579 | 0.2074 | -1.5772 | -1.1528 | -0.7683 |
| \beta_5 = 3 | 3.2933 | 0.3852 | 2.5767 | 3.2777 | 4.0901 |
| s = 2  |             |                |     |      |        |       |
| b_0 = 5 | 4.8797      | 0.2422         |     | 4.4067 | 4.8797 | 5.3552 |
| b_1 = 1 | 1.0220      | 0.0382         | 0.9467 | 1.0220 | 1.0971 |
| b_2 = 3 | 2.9519      | 0.0366         | 2.8802 | 2.9519 | 3.0234 |
| b_3 = 2 | 1.9104      | 0.0688         | 1.7739 | 1.9104 | 2.0448 |
| b_4 = 4 | 3.9987      | 0.0744         | 3.8555 | 3.9987 | 4.1465 |
| \sigma^2 = 0.8 | 0.7598 | 0.0446 | 0.6766 | 0.7589 | 0.8515 |
| \beta_0 = -2 | 3.0154 | 0.3732 | 2.3360 | 2.9994 | 3.7981 |
| \beta_1 = 3 | 2.0436 | 0.4490 | 1.1941 | 2.0327 | 2.9592 |
| \beta_2 = 2 | 4.2320 | 0.4990 | 3.3177 | 4.2092 | 5.2799 |
| \beta_3 = 4 | 0.6664 | 0.1944 | 0.2906 | 0.6627 | 1.0621 |
| \beta_4 = -2 | -2.4036 | 0.3300 | -3.0887 | -2.3927 | -1.7888 |

Table 7: The fixed model: Summary statistics for the explanatory variables of experiment 1.

| Out-of-sample Forecasting Criteria | NHMM | HHMM | LR |
|----------------------------------|------|------|----|
| y_1 = -1.3404                   | -1.5227 | -2.7418 | -22.8871 |
| y_2 = 25.0334                   | -0.9262 | -2.9887 | -22.9191 |
| y_3 = 3.7751                    | -0.9208 | -2.2213 | -21.9447 |
| y_4 = 12.4843 \ CRPS(y_i)       | -0.3128 | -4.4827 | -25.5160 |
| y_5 = -7.8430                   | -5.5556 | -6.8169 | -23.7165 |
| y_6 = 13.8031                   | -3.2179 | -4.7337 | -23.2957 |
| y_7 = 1.4307                    | -2.4628 | -2.9928 | -22.7315 |
| y_8 = -11.1139                  | -0.3740 | -7.4199 | -22.5383 |
| y_9 = 16.5539                   | -0.8481 | -3.6279 | -22.6431 |
| y_{10} = 24.4482                | -2.3927 | -5.4199 | -22.7568 |
| \ E(\text{CRPS}) \ MAFE         | 3.9173 | 9.3282 | 77.2418 |
| \ MSFE                           | 62.0849 | 187.9507 | 9.4 × 10^4 |

Table 8: Comparing the forecasting ability of the three competing models of experiment 1.
Figure 18: Experiment 1: Histograms of the mean equation parameters.

Figure 19: Experiment 1: Convergence plots of the mean equation parameters.
Figure 20: Experiment 1: Histograms of the logistic regression coefficients.

Figure 21: Experiment 1: Convergence plots of the logistic regression coefficients.
Table 9: The fixed model. Summary statistics for the explanatory variables of the experiment 2 using the NHMM model.

| States | True Values | Posterior Mean | Std   | 2.5%   | Median | 97.5%   |
|--------|-------------|----------------|-------|--------|--------|---------|
| \( s = 1 \) | \( b_0 = 2 \) | 1.7134 | 0.3147 | 1.0985 | 1.7147 | 2.3336 |
| \( b_1 = -0.3 \) | -0.2569 | 0.0538 | -0.3632 | -0.2572 | -0.1536 |
| \( b_2 = 2 \) | 2.0721 | 0.0501 | 1.9730 | 2.0729 | 2.1696 |
| \( b_3 = 2 \) | 2.0488 | 0.0975 | 1.8604 | 2.0481 | 2.2406 |
| \( \sigma^2 = 1.5 \) | 1.0245 | 0.0822 | 0.8012 | 1.2312 | 1.5556 |
| \( \beta_0 = 1.5 \) | 1.3815 | 1.1805 | 0.0472 | 2.2870 | 4.6590 |
| \( \beta_1 = 1 \) | 1.0364 | 0.1902 | 0.6770 | 1.0314 | 1.4203 |
| \( \beta_2 = 2 \) | 2.0923 | 0.2343 | 1.6540 | 2.0840 | 2.5682 |
| \( \beta_3 = 3 \) | 3.2644 | 0.3325 | 2.6567 | 3.2515 | 3.9610 |

| States | True Values | Posterior Mean | Std   | 2.5%   | Median | 97.5%   |
|--------|-------------|----------------|-------|--------|--------|---------|
| \( s = 2 \) | \( b_0 = 1 \) | 1.1955 | 0.1878 | 0.8263 | 1.1935 | 1.5650 |
| \( b_1 = 3 \) | 2.9608 | 0.0298 | 2.9018 | 2.9608 | 3.0192 |
| \( b_2 = 4 \) | 3.9925 | 0.0302 | 3.9333 | 3.9927 | 4.0514 |
| \( b_3 = 3 \) | 3.0312 | 0.0595 | 2.9142 | 3.0315 | 3.1471 |
| \( \sigma^2 = 0.8 \) | 0.7655 | 0.0359 | 0.6968 | 0.7626 | 0.8369 |
| \( \beta_0 = 3 \) | 3.5477 | 0.9022 | 1.8020 | 3.5363 | 5.3628 |
| \( \beta_1 = -2.5 \) | -2.5752 | 0.2103 | -3.0022 | -2.5683 | -2.1785 |
| \( \beta_2 = 4 \) | 3.9736 | 0.2917 | 3.4352 | 3.9674 | 4.5709 |
| \( \beta_3 = 1 \) | 1.0602 | 0.1405 | 0.7397 | 1.0573 | 1.3453 |

Table 10: Comparing the forecasting ability of the three competing models of experiment 2.

| Out-of-sample | Forecasting Criteria | NHMM | HHMM | LR |
|---------------|----------------------|------|------|----|
| \( y_1 = 19.4094 \) | \( CRPS(y_1) \) | -0.2674 | -0.8738 | -27.3658 |
| \( y_2 = 2.6234 \) | \( CRPS(y_2) \) | -0.6003 | -4.9020 | -26.5102 |
| \( y_3 = 12.3667 \) | \( CRPS(y_3) \) | -0.4427 | -1.7485 | -28.4347 |
| \( y_4 = 13.3616 \) | \( CRPS(y_4) \) | -0.2889 | -1.7146 | -26.5481 |
| \( y_5 = 22.12014 \) | \( CRPS(y_5) \) | -0.4673 | -2.5203 | -27.2373 |
| \( y_6 = -0.1348 \) | \( CRPS(y_6) \) | -1.2171 | -4.1459 | -26.6438 |
| \( y_7 = 12.7453 \) | \( CRPS(y_7) \) | -0.3824 | -1.7933 | -26.6438 |
| \( y_8 = 0.9820 \) | \( CRPS(y_8) \) | -1.0587 | -10.5293 | -26.6438 |
| \( y_9 = 20.1527 \) | \( CRPS(y_9) \) | -0.1921 | -2.6289 | -26.6438 |
| \( y_{10} = 0.3442 \) | \( CRPS(y_{10}) \) | -1.2822 | -8.0522 | -26.6438 |
| \( E(CRPS) \) | \( E(CRPS) \) | -0.6199 | -3.8929 | -27.1233 |
| \( MAPE \) | \( MAPE \) | 1.8977 | 7.5656 | 91.8549 |
| \( MSFE \) | \( MSFE \) | 15.0261 | 123.1742 | 1.3255×10^4 |

Table 10: Comparing the forecasting ability of the three competing models of experiment 2.
Figure 22: Experiment 2. Histograms of the mean equation parameters

Figure 23: Experiment 2. Convergence plots of the mean equation parameters
Figure 24: Experiment 2. Histograms of the logistic regression coefficients

Figure 25: Experiment 2. Convergence plots of the logistic regression coefficients
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