FRIEZE PATTERNS OVER INTEGERS AND OTHER SUBSETS OF THE COMPLEX NUMBERS

MICHAEL CUNTZ AND THORSTEN HOLM

Abstract. We study (tame) frieze patterns over subsets of the complex numbers, with particular emphasis on the corresponding quiddity cycles. We provide new general transformations for quiddity cycles of frieze patterns. As one application, we present a combinatorial model for obtaining the quiddity cycles of all tame frieze patterns over the integers (with zero entries allowed), generalising the classic Conway-Coxeter theory. This model is thus also a model for the set of specializations of cluster algebras of Dynkin type $A$ in which all cluster variables are integers.

Moreover, we address the question of whether for a given height there are only finitely many non-zero frieze patterns over a given subset $R$ of the complex numbers. Under certain conditions on $R$, we show upper bounds for the absolute values of entries in the quiddity cycles. As a consequence, we obtain that if $R$ is a discrete subset of the complex numbers then for every height there are only finitely many non-zero frieze patterns over $R$. Using this, we disprove a conjecture of Fontaine, by showing that for a complex $d$-th root of unity $\zeta_d$ there are only finitely many non-zero frieze patterns for a given height over $R = \mathbb{Z}[\zeta_d]$ if and only if $d \in \{1, 2, 3, 4, 6\}$.

1. Introduction

Frieze patterns have been introduced by Coxeter [4]. Shortly afterwards, Conway and Coxeter presented a beautiful theory for frieze patterns over natural numbers [3], among other things showing that there is a bijection between frieze patterns with entries in $\mathbb{N}$ and triangulations of polygons. This result has recently been generalized to $p$-angulations of polygons and certain frieze patterns over positive real numbers [9].

Since the invention of cluster algebras by Fomin and Zelevinsky around 2000, frieze patterns have attracted renewed interest because of a close connection to cluster algebras. In fact, if one allows Laurent polynomials as entries of a frieze pattern then starting with a set of indeterminates produces the cluster variables of the cluster algebra of Dynkin type $A$ as entries in the frieze pattern.

So it is very natural to consider frieze patterns over other sets of numbers than integers. In this paper we start out by considering frieze patterns with entries in subsets of the field of complex numbers. In general there are far too many frieze patterns, see for instance [5], and a general frieze pattern does not share any periodicity properties. Therefore, one usually restricts to studying tame frieze patterns [2]. This is a very large class of frieze patterns including basically all interesting classes studied so far (like friezes corresponding to cluster algebras, the Conway-Coxeter friezes etc.), sharing nice symmetry properties and allowing unified methods to be applied to study them. For more details on frieze patterns we refer the reader to a nice survey by Morier-Genoud [10].

However, it is still very subtle to describe all tame frieze patterns with entries from a given set $R$ of numbers. For $R = \mathbb{N}$ this is the classic Conway-Coxeter theory. Frieze patterns over $R = \mathbb{Z} \setminus \{0\}$ have been described by Fontaine [8]; here it turns out that only very few new frieze patterns appear in addition to the ones over $\mathbb{N}$. The situation changes drastically for tame frieze patterns over $\mathbb{Z}$, i.e. when zeroes are allowed as entries. Then plenty of new frieze

2010 Mathematics Subject Classification. 05E15, 05E99, 13F60, 51M20.

Key words and phrases. Frieze pattern, tame frieze pattern, quiddity cycle, cluster algebra, polygon, triangulation.
patterns appear. As one of the main results of this paper we show how every such frieze pattern can be obtained from a new combinatorial model. This combinatorial model generalizes the combinatorial model via triangulations from the Conway-Coxeter theory. See Section 6 for the description of frieze patterns over \( \mathbb{Z} \) and Section 7 for the corresponding combinatorial model.

As main tools for achieving this, we provide new general transformations on frieze patterns, or more precisely on quiddity cycles, which might turn out to be useful in other situations as well. See Section 4 for precise statements. These transformations substantially generalise the classic Conway-Coxeter theory, where for inductively proving the bijection between frieze patterns over \( \mathbb{N} \) and triangulations of polygons one only needs transformations which insert/remove an entry 1 in the quiddity cycle. Over other subsets \( R \subseteq \mathbb{C} \), quiddity cycles do not necessarily contain a 1. However, we can show in Section 3 that each quiddity cycle over the complex numbers contains an entry (even two entries) of absolute value less than 2; see Corollary 3.3.

The possible absence of 1’s in quiddity cycles is one of the reasons why combinatorially describing all frieze patterns over a given set \( R \) can be quite hard. It is even a non-trivial problem to decide whether for a given height there are finitely or infinitely many (non-zero) frieze patterns over \( R \).

As one main result of this paper we provide in Theorem 3.6 a criterion for having finitely many non-zero frieze patterns of any given height over certain subsets \( R \). As we show in Corollary 3.8 this criterion implies that for any discrete subset \( R \subseteq \mathbb{C} \) there are only finitely many non-zero frieze patterns over \( R \) for each height.

As an application we can disprove a conjecture by Fontaine [8, Conjecture 6.2] on frieze patterns over rings of the form \( R = \mathbb{Z}[\zeta_d] \) where \( \zeta_d \in \mathbb{C} \) is a primitive \( d \)-th root of unity. Namely, we show in Corollary 3.10 that there are only finitely many non-zero frieze patterns of height \( n \) over \( \mathbb{Z}[\zeta_d] \) if and only if \( d \in \{1, 2, 3, 4, 6\} \) (independent of the height of the frieze patterns).

Acknowledgement
We thank Sophie Morier-Genoud for answering questions about some aspects of this paper.

2.QUIDDITY CYCLES

In this section we collect some fundamental definitions and results which are later needed in the paper.

**Definition 2.1.** Let \( R \subseteq \mathbb{C} \) be a subset of the complex numbers.

1. A frieze pattern over \( R \) is an array \( F \) of the form

\[
\begin{array}{cccccccc}
... & & & & & & & \\
0 & 1 & c_{i-1,i+1} & c_{i-1,i+2} & \cdots & \cdots & c_{i-1,n+i} & 1 & 0 \\
0 & 1 & c_{i,i+2} & c_{i,i+3} & \cdots & \cdots & c_{i,n+i+1} & 1 & 0 \\
0 & 1 & c_{i+1,i+3} & c_{i+1,i+4} & \cdots & \cdots & c_{i+1,n+i+2} & 1 & 0 \\
... & & & & & & & \\
\end{array}
\]

where \( c_{i,j} \) are numbers in \( R \), and such that every (complete) adjacent \( 2 \times 2 \) submatrix has determinant 1. We call \( n \) the height of the frieze pattern \( F \). We say that the frieze pattern \( F \) is periodic with period \( m > 0 \) if \( c_{i,j} = c_{i+m,j+m} \) for all \( i,j \).

2. A frieze pattern is called tame if every adjacent \( 3 \times 3 \)-submatrix has determinant 0.

Frieze patterns have been introduced by Coxeter [4] and studied further by Conway and Coxeter [3]. More precisely, Conway and Coxeter studied frieze patterns over \( \mathbb{N} \), i.e. frieze patterns with positive integral entries. If one allows the entries in a frieze pattern to be rational functions over \( \mathbb{Q} \), still observing the local condition on \( 2 \times 2 \)-determinants, then starting with a set of indeterminates, one obtains the cluster variables of Fomin and Zelevinsky’s cluster algebras of Dynkin type \( A \) as entries in the frieze pattern. This is one instance, among others,
which shows that for applications of frieze patterns to other areas of mathematics (e.g. geometry, representation theory, integrable systems) it is useful to allow entries from various (semi-)rings.

**Example 2.2.**

1. **Conway-Coxeter frieze patterns** are exactly the frieze patterns over \( \mathbb{N} \). An intriguing feature of these frieze patterns is that there is a bijection between the frieze patterns of height \( n \) and the triangulations of a regular \((n+3)\)-gon. In particular, every frieze pattern over \( \mathbb{N} \) of height \( n \) is periodic with period \( n + 3 \). The following is an example: The numbers at the vertices of the hexagon are the numbers of triangles attached; and these numbers (in counterclockwise order) yield the first diagonal in the corresponding frieze pattern on the left.

\[
\begin{array}{ccccccc}
0 & 1 & 1 & 3 & 2 & 1 & 0 \\
0 & 1 & 4 & 3 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 2 & 3 & 4 & 1 & 0 \\
0 & 1 & 2 & 3 & 1 & 1 & 0 \\
0 & 1 & 2 & 1 & 2 & 1 & 0 \\
\end{array}
\]

2. The array

\[
\begin{array}{cccccccc}
0 & 1 & -i+1 & 1 & i+1 & 1 & 0 \\
0 & 1 & i+1 & 2i+1 & 2 & 1 & 0 \\
0 & 1 & -i+1 & 1 & i+1 & 1 & 0 \\
0 & 1 & i+1 & 2i+1 & 2 & 1 & 0 \\
0 & 1 & 2 & -2i+1 & -i+1 & 1 & 0 \\
\end{array}
\]

repeated infinitely many times to both sides, is a frieze pattern over the Gaussian integers \( \mathbb{Z}[i] \); it is periodic with period 6.

3. The fact that the frieze patterns in the above example are periodic, follows from some general results on frieze patterns. In fact, if all entries \( c_{i,j} \) in a frieze pattern of height \( n \) are non-zero, then the frieze pattern is periodic with period \( n + 3 \); see Proposition 2.4 below for details. A frieze pattern with zero entries might not be periodic at all. For instance, for every sequences \((a_i)_{i \in \mathbb{Z}}\) and \((b_i)_{i \in \mathbb{Z}}\) we have a frieze pattern of the form

\[
\begin{array}{ccccccc}
0 & 1 & a_1 & -1 & b_1 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & a_2 & -1 & b_2 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & a_3 & -1 & b_3 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
\end{array}
\]

It follows directly from the definition that every non-zero frieze pattern is uniquely determined by the entries \( c_{i,i+2} \) in the first diagonal of the frieze pattern (cf. Definition 2.1); in fact,
Proof. (1) Let \( i \) for all \( 2 \times 2 \)-submatrices \( M \)
for suitable \( a, b, c, d, e, f, s, t \). Now the fact that all adjacent \( 2 \times 2 \)-determinants are 1 implies
\[ 1 = b(sc + td) - d(sa + tb) = s(bc - ad) = -s, \]
so \( s = -1 \). Thus setting \( a = c_{i,j}, b = c_{i,j+1} \) we see that for fixed \( j \), there is a \( t_j \) such that
\[ (2.1) \quad \eta(t_j) \begin{pmatrix} c_{i,j+1} \\ c_{i,j} \end{pmatrix} = \begin{pmatrix} -c_{i,j} + t_j c_{i,j+1} \\ c_{i,j+1} \end{pmatrix} = \begin{pmatrix} c_{i,j+2} \\ c_{i,j+1} \end{pmatrix} \]
for all \( i \). If we extend the frieze by a row of 0’s and \(-1\)’s on both sides, then the \( \text{SL}_2 \)-condition is still satisfied and in each consecutive pair of rows we find \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) at the beginning and
\( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) at the end of the frieze pattern. Since next to the left matrix there is \( \begin{pmatrix} 1 & c_{i,j+2} \\ 0 & 1 \end{pmatrix} \),

if all entries are non-zero then one can use the condition for the \( 2 \times 2 \)-determinants to be 1
to compute successively the second diagonal, the third diagonal etc. (But note that the above
Example 2.2(3) shows that this is no longer true if zero entries appear.)

If \( F \) is a periodic frieze pattern with period \( m \) then the sequence \((c_{1,3}, c_{2,4}, \ldots, c_{m,m+2})\) of
entries in the first diagonal is called the quiddity cycle of the frieze pattern [3], cf. Definition
2.3 below. Note that a quiddity cycle of a frieze pattern is only unique up to rotation.

It is a priori not clear which sequences of numbers actually yield a frieze pattern. For getting
a criterion we need the following matrices which play a crucial role in the theory of frieze
patterns:

**Definition 2.3.** For \( c \in \mathbb{C} \), let \( \eta(c) = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} \).

The following result collects several fundamental properties of the above matrices and explains
their importance in the context of frieze patterns. These results are known but not well
documented in the literature, thus we give a precise statement and include a proof.

**Proposition 2.4.** Let \( R \subseteq \mathbb{C} \) be a subset.

(1) Let \( F \) be a frieze pattern over \( R \) of height \( n \), with entries denoted \( c_{i,j} \) as in Definition
2.1. For abbreviation we set \( c_k := c_{k,k+2} \) for \( k \in \mathbb{Z} \). If \( F \) is a tame frieze pattern then
\( F \) is periodic with period \( m = n + 3 \). Furthermore,
\[ \prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad c_{i,j+2} = (M_{i,j})_{1,1}, \quad \text{where} \quad M_{i,j} := \prod_{k=i}^{j} \eta(c_k). \]
(Here \((M_{i,j})_{1,1}\) is the entry of the matrix \( M_{i,j} \) in the first row and first column. Notice
further that we may assume \( j \geq i \) in the product defining \( M_{i,j} \) by possibly adding
multiples of \( 2m \) to \( j \).)

(2) Suppose that \((c_1, \ldots, c_m) \in R^m\) satisfies \( \prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \). We define \( c_k \) for
all \( k \in \mathbb{Z} \) by repeating the sequence \((c_1, \ldots, c_m)\) periodically and define the matrices
\( M_{i,j} := \prod_{k=i}^{j} \eta(c_k) \) as above. Then the array
\[ (a_{i,j+2})_{i,j} = ((M_{i,j})_{1,1})_{i,j} \]
(where \( i - 1 \leq j \leq m + i - 3 \)) defines a periodic frieze pattern over \( R \) with period \( m \)
and height \( m - 3 \). Moreover, this frieze pattern is tame.

**Proof.** (1) Let \( F = (c_{i,j}) \) be a frieze pattern over \( R \). Consider an adjacent \( 3 \times 3 \)-submatrix
\( M \) of \( F \). The first two columns of \( M \) cannot be linearly dependent because the upper left
\( 2 \times 2 \)-submatrix has determinant 1. But then since \( F \) is tame, the determinant of \( M \) is zero, so
\[ M = \begin{pmatrix} a & b & sa + tb \\ c & d & sc + td \\ e & f & se + tf \end{pmatrix} \]
for suitable \( a, b, c, d, e, f, s, t \). Now the fact that all adjacent \( 2 \times 2 \)-determinants are 1 implies
\[ 1 = b(sc + td) - d(sa + tb) = s(bc - ad) = -s, \]
so \( s = -1 \). Thus setting \( a = c_{i,j}, b = c_{i,j+1} \) we see that for fixed \( j \), there is a \( t_j \) such that
\[ (2.1) \quad \eta(t_j) \begin{pmatrix} c_{i,j+1} \\ c_{i,j} \end{pmatrix} = \begin{pmatrix} -c_{i,j} + t_j c_{i,j+1} \\ c_{i,j+1} \end{pmatrix} = \begin{pmatrix} c_{i,j+2} \\ c_{i,j+1} \end{pmatrix} \]
for all \( i \). If we extend the frieze by a row of 0’s and \(-1\)’s on both sides, then the \( \text{SL}_2 \)-condition is still satisfied and in each consecutive pair of rows we find \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) at the beginning and
\( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) at the end of the frieze pattern. Since next to the left matrix there is \( \begin{pmatrix} 1 & c_{i,j+2} \\ 0 & 1 \end{pmatrix} \),
Equation (2.1) implies $t_i = c_{i,i+2}$. Moreover, from this it is clear that the product of the matrices $\eta(c_{i,k})$, $k = i, \ldots, i + m - 1$ (from left to right) in each row $i$ is the negative of the identity matrix. Further, by Equation (2.1), the entries in the frieze pattern are just the top left entries in the intermediate products. For the periodicity notice that by the above consideration, comparing two consecutive rows gives sequences $t_1, \ldots, t_m$ and $t_2, \ldots, t_{m+1}$ such that

$$
\prod_{i=1}^{m} \eta(t_i) = \prod_{i=2}^{m+1} \eta(t_i) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},
$$

which implies $t_1 = t_{m+1}$ and hence $c_{i,i+2} = c_{i+m,i+m+2}$ for all $i$. Since all entries in $F$ are determined by these numbers, the whole frieze is periodic.

(2) There are several items to check, namely the determinants of the adjacent $2 \times 2$-submatrices, the tameness, the height (i.e. that the pattern ends with a diagonal of 1’s), and the periodicity.

We start by considering the determinants of adjacent $2 \times 2$-submatrices. Because of

$$
\eta(c_i) M_{i+1,j} = M_{i,j}
$$

we have $(M_{i+1,j})_{1,1} = (M_{i,j})_{2,1}$. Similarly, we obtain

$$(M_{i+1,j})_{1,1} = -(M_{i+1,j+1})_{1,2} = (M_{i,j})_{2,1} = -(M_{i+1,j})_{2,2}.$$

From this we get

$$
\det \begin{pmatrix} a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j+1} & a_{i+1,j+2} \end{pmatrix} = \det \begin{pmatrix} (M_{i,j+1})_{1,1} & (M_{i,j})_{1,1} \\ (M_{i+1,j})_{1,1} & (M_{i+1,j})_{1,1} \end{pmatrix}
= \det \begin{pmatrix} -(M_{i,j})_{1,2} & (M_{i,j})_{1,1} \\ -(M_{i+1,j})_{2,2} & (M_{i+1,j})_{2,1} \end{pmatrix} = \det M_{i,j} = 1.
$$

Thus, the proposed array satisfies the condition for the entries of a frieze pattern. Equation (2.2) and $M_{i,j+1} = M_{i,j} \eta(c_{j+1})$ also give

$$
\begin{pmatrix} a_{i,j+1} & a_{i,j+2} & a_{i,j+3} \\ a_{i+1,j} & a_{i+1,j+2} & a_{i+1,j+3} \end{pmatrix} = \begin{pmatrix} -(M_{i,j})_{1,2} & (M_{i,j})_{1,1} & (M_{i,j+1})_{1,1} \\ -(M_{i+1,j})_{2,2} & (M_{i+1,j})_{2,1} & (M_{i+1,j})_{2,1} + c_{i+1}(M_{i,j})_{1,1} \end{pmatrix}.
$$

This shows that any three consecutive columns in the array are linear dependent, hence the frieze pattern is tame.

For showing that the array is a frieze pattern of height $m - 3$, we have to show that the pattern has a bounding diagonal of 1’s. With the labelling as in Definition 2.4 this means that we have to show that $a_{i,i+1} = 1$ and $a_{i,i+m-1} = 1$. But $a_{i,i+1} = (M_{i,i+2m-1})_{1,1} = 1$ because this is the top left entry in $(\prod_{k=1}^{m} \eta(c_k))^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Moreover, $a_{i,i+m-1} = (M_{i,i+m-3})_{1,1} = 1$ because

$$
M_{i,i+m-3} = \prod_{k=i}^{i+m-3} \eta(c_k) = \prod_{k=i}^{i+m-1} \eta(c_k) \eta(c_{i+m-1})^{-1} \eta(c_{i+m-2})^{-1} = -\eta(c_{i+m-1})^{-1} \eta(c_{i+m-2})^{-1} = \begin{pmatrix} 1 & -c_{i+m-2} \\ c_{i+m-1} & 1 - cm c_{i+m-2} \end{pmatrix},
$$

so we get the desired diagonal of 1’s.

Finally, the periodicity follows immediately from the periodicity of the sequence $(c_k)_{k \in \mathbb{Z}}$. □

Proposition 2.4 motivates the following definition which generalizes the notion of a quiddity cycle of a frieze pattern informally already given above.

**Definition 2.5.** Let $R \subseteq \mathbb{C}$ be a subset. A quiddity cycle over $R$ is a sequence $(c_1, \ldots, c_m) \in R^m$ satisfying

$$
\prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
$$
Remark 2.6. Let \((c_1, \ldots, c_m) \in \mathbb{R}^m\) be a quiddity cycle.

(1) The rotated cycle \((c_m, c_1, \ldots, c_{m-1})\) and the reversed cycle \((c_m, c_{m-1}, \ldots, c_1)\) are again quiddity cycles.

In fact, the first assertion easily follows from the fact that the negative identity matrix commutes with every matrix. The second already appeared in [6, Proposition 5.3 (3)]; for completeness we include the argument. Let \(\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\); note that \(\tau^2\) is the identity matrix and that for every \(c \in \mathbb{R}\) we have \(\tau c \tau = \begin{pmatrix} 0 & 1 \\ -1 & c \end{pmatrix} = \eta(c)^{-1}\). It follows that

\[
\eta(c_m)\eta(c_{m-1}) \cdots \eta(c_1) = (\tau \eta(c_m)^{-1} \tau)(\tau \eta(c_{m-1})^{-1} \tau) \cdots (\tau \eta(c_1)^{-1} \tau) = \tau(\eta(c_1) \cdots \eta(c_{m-1}) \eta(c_m))^{-1} \tau = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(where in the last equation we have used that \((c_1, \ldots, c_m)\) is a quiddity cycle).

(2) We have \(\prod_{k=1}^m \eta(-c_k) = (-1)^{m+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\). In particular, if \(m\) is even, \((-c_1, \ldots, -c_m)\) is again a quiddity cycle.

In fact, set \(T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\), where \(i \in \mathbb{C}\) is the imaginary unit. A direct computation shows that \(T \eta(c) T = \eta(-c)\) for all \(c \in \mathbb{C}\). Moreover, since \(T^2\) is the negative of the identity matrix, we get

\[
\prod_{k=1}^m \eta(-c_k) = \prod_{k=1}^m T \eta(c_k) T = (-1)^{m-1} T (\prod_{k=1}^m \eta(c_k)) T = (-1)^m T^2 = (-1)^{m+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Example 2.7. From the definition of the matrices \(\eta(c)\) it is clear that there is no quiddity cycle of length \(m = 1\). A straightforward calculation yields

\[
\eta(c_1)\eta(c_2) = \begin{pmatrix} c_1c_2 - 1 & -c_1 \\ c_2 & -1 \end{pmatrix}
\]

from which it follows that the only quiddity cycle of length \(m = 2\) is \((0, 0)\). For \(m = 3\) we compute

\[
\eta(c_1)\eta(c_2)\eta(c_3) = \begin{pmatrix} c_1c_2c_3 - c_3 - c_1 & -c_1c_2 + 1 \\ c_2c_3 - 1 & -c_2 \end{pmatrix}
\]

and deduce that the only quiddity cycle of length \(m = 3\) is \((1, 1, 1)\). For \(m = 4\) we obtain

\[
\eta(c_1)\eta(c_2)\eta(c_3)\eta(c_4) = \begin{pmatrix} c_1c_2c_3c_4 - c_1c_4 - c_3c_4 + c_1c_2 + 1 & -c_1c_2c_3 + c_1 + c_3 \\ c_2c_3c_4 - c_4 - c_2 & -c_2c_3 + 1 \end{pmatrix}.
\]

For this to become the negative identity matrix we get \(c_2c_4 = 2\) (from the \((2,2)\)-entry) and then \(c_1 = c_3\) and \(c_2 = c_4\) from the off-diagonal entries; with these conditions the \((1,1)\)-entry becomes \(-1\). Hence the quiddity cycles of length \(m = 4\) are precisely \((c_1, 2c_1^{-1}, c_1, 2c_1^{-1})\) with \(c_1 \neq 0\).
Remark 2.8. For fixed \( n \in \mathbb{N} \), there may be different periodic frieze patterns with period \( m = n + 3 \) and of height \( n \) but with the same quiddity cycle: the array

\[
\begin{array}{ccccccc}
1 & a & -1 & d & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & b & -1 & e & 1 & 0 \\
0 & 1 & c & -1 & f & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
\vdots
\end{array}
\]

(repeated to both sides) is a frieze pattern for arbitrary \( a, b, c, d, e, f \), and it is periodic with period 6. On the other hand, a direct matrix calculation shows that

\[
\eta(a)\eta(0)\eta(b)\eta(0)\eta(c)\eta(0) = \begin{pmatrix} -1 & -a - b - c \\ 0 & -1 \end{pmatrix}.
\]

By Definition 2.5, the sequence \((a, 0, b, 0, c, 0)\) is a quiddity cycle if and only if \( a + b + c = 0 \). So for example if \( a = 1, b = 1, c = -2 \), then \((a, 0, b, 0, c, 0)\) is a quiddity cycle and the above array is a frieze pattern for arbitrary \( d, e, f \), but tame only if \( a = e, d = c, b = f \).

3. Bounds and finiteness

When considering frieze patterns over a subset \( R \subseteq \mathbb{C} \), one of the most fundamental questions is whether for a given \( n \in \mathbb{N} \) there are finitely or infinitely many frieze patterns over \( R \) of height \( n \). We have seen above (e.g. in Remark 2.8) that if zeroes are allowed as entries then usually one will get infinitely many frieze patterns and the situation is hard to control. Therefore, for the questions of finiteness addressed in this section we shall restrict to non-zero frieze patterns.

If for a certain subset \( R \) there are only finitely many non-zero frieze patterns over \( R \) of any height \( n \) then the additional question arises whether one can find a useful combinatorial model of these frieze patterns.

Both questions are very hard in general and so far have only received answers in very few cases. In fact, for \( R = \mathbb{N} \) the classic Conway-Coxeter results [3] yield a bijection between frieze patterns over \( \mathbb{N} \) of height \( n \) and triangulations of the regular \((n + 3)\)-gon. In particular, the frieze patterns over \( \mathbb{N} \) are counted by the famous Catalan numbers.

This result has been extended by Fontaine [8] to the case \( R = \mathbb{Z} \setminus \{0\} \), here only few new frieze patterns appear; more precisely, if \( n \) is even then every non-zero frieze pattern of height \( n \) over the integers is a Conway-Coxeter frieze pattern and if \( n \) is odd then there are twice as many non-zero frieze patterns of height \( n \) over the integers as Conway-Coxeter frieze patterns, and the new ones are obtained by multiplying in the Conway-Coxeter frieze patterns every second row by \(-1\).

In this section we will provide rather general criteria for when there are only finitely many non-zero frieze patterns over a subset \( R \subseteq \mathbb{C} \).

As mentioned above, a non-zero frieze pattern is uniquely determined by its quiddity cycle. Our approach here is to guarantee small entries and to give upper bounds for all entries in a quiddity cycle. For this we use the absolute value \(| \cdot |\) of complex numbers and its well-known properties, without further mentioning.

The following useful lemma is inspired by old results on continued fractions.

Lemma 3.1. Let \( c_1, \ldots, c_m, d, e \in \mathbb{C} \) with \( |c_m| \geq 1 \) and \( |c_1 e - d| > |e| \); moreover suppose that

\[
\prod_{j=1}^{m} \eta(c_j) = \begin{pmatrix} d & e * \\ e & * \end{pmatrix}.
\]
(Here the *’s denote arbitrary entries, not necessarily the same.) Then there exists an index \( j \in \{2, \ldots, m-1\} \) with \( |c_j| < 2 \).

Remark 3.2. One might wonder that the conclusion of the lemma only applies if \( m \geq 3 \). In fact, for \( m = 1 \) and \( m = 2 \) the assumptions of the lemma can not be satisfied. For \( m = 1 \) one would get \( e = 1 \) and \( d = c_1 \) and then \( |c_1 e - d| = 0 \). For \( m = 2 \) one has

\[
\eta(c_1)\eta(c_2) = \begin{pmatrix} c_1 c_2 - 1 & -c_1 \\ c_2 -1 \end{pmatrix},
\]

thus \( e = c_2 \) and \( d = c_1 c_2 - 1 \). But then the assumptions would yield \( 1 \leq |c_2| = |e| < |c_1 e - d| = 1 \), a contradiction.

Proof. Let \( a, b \in \mathbb{C} \) with \(|a| \geq |b| \) and \(|c| \geq 2 \). Then

\[ |ac - b| \geq |ac| - |b| = |a||(|c| - 1) + |a| - |b| \geq |a||(|c| - 1) \geq |a|. \]

From this inequality and

\[
\eta(c) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ac - b \\ e \end{pmatrix},
\]

we see that multiplying vectors in \( \mathbb{C}^2 \) from the left with \( \eta(c) \), where \(|c| \geq 2 \), preserves the property that the absolute value of the first entry is greater or equal to the absolute value of the second entry.

Now

\[
\eta(c_m) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_m & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_m \\ 1 \end{pmatrix}.
\]

From this and the assumption on the shape of \( \prod_{j=1}^{m-1} \eta(c_j) \) we get

\[
\left( \prod_{j=2}^{m-1} \eta(c_j) \right) \begin{pmatrix} c_m \\ 1 \end{pmatrix} = \eta(c_1)^{-1} \begin{pmatrix} 0 & 1 \\ -1 & c_1 \end{pmatrix} \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} e \\ c_1 e - d \end{pmatrix}.
\]

Note that by assumption we have \(|c_m| \geq 1 \) and \(|c| < |c_1 e - d| \). Thus \(|c_2|, \ldots, |c_{m-1}| \) all greater or equal to 2 would contradict the property stated after Equation (3.1).

In a special case we can draw a stronger conclusion which will turn out to be important for some later applications. Note that in particular the following corollary applies in the case of quiddity cycles (cf. Definition 2.3).

**Corollary 3.3.** Let \( (c_1, \ldots, c_m) \in \mathbb{C}^m \) such that \( \prod_{j=1}^{m} \eta(c_j) \) is a scalar multiple of the identity matrix. Then there are two different indices \( j, k \in \{1, \ldots, m\} \) with \(|c_j| < 2 \) and \(|c_k| < 2 \).

Remark 3.4. (i) Note that by Definition 2.3 the assumption of Corollary 3.3 can only be satisfied for \( m \geq 2 \). For \( m = 2 \), the only sequence \( (c_1, c_2) \) with \( \eta(c_1)\eta(c_2) = \begin{pmatrix} c_1 c_2 - 1 & -c_1 \\ c_2 -1 \end{pmatrix} \) a scalar multiple of the identity matrix is \( (c_1, c_2) = (0, 0) \).

(ii) Moreover, the product \( \prod_{j=1}^{m} \eta(c_j) \) is not the zero matrix, since the matrices \( \eta(c_j) \) have determinant 1.

Proof. The statement of the corollary clearly holds for the sequence (0, 0). So, according to Remark 3.4(i) we can assume that \( m \geq 3 \).

A crucial initial observation is that scalar multiples of the identity matrix commute with every matrix. This implies that if \( (c_1, \ldots, c_m) \in \mathbb{C}^m \) satisfies the assumption of the corollary, then also the rotated sequence \( (c_m, c_1, \ldots, c_{m-1}) \) does.

Suppose first that \(|c_m| < 1 \). If also \(|c_{m-1}| < 1 \), we are done. If \(|c_{m-1}| \geq 1 \) then we consider the rotated sequence \( (c_m, c_1, \ldots, c_{m-1}) \). By the initial observation, this satisfies the assumptions of Lemma 3.1 (with \( c = 0 \) and \( d \neq 0 \), cf. Remark 3.4(ii)), hence we obtain an index \( j \in \{1, \ldots, m-2\} \) with \(|c_j| < 2 \), and we are also done.
So suppose from now on that \( |c_m| \geq 1 \). Then we can, as before, apply Lemma 3.1 and obtain an index \( j \in \{ 2, \ldots, m-1 \} \) with \( |c_j| < 2 \). We then consider the rotated sequence \((c_j, c_{j+1}, \ldots, c_m, c_1, \ldots, c_{j-2}, c_{j-1})\). If \( |c_{j-1}| < 1 \) we are done (choose \( k = j - 1 \)). If \( |c_{j-1}| \geq 1 \) then we can again apply Lemma 3.1 (by iterating the initial observation, the sequence \((c_j, c_{j+1}, \ldots, c_m, c_1, \ldots, c_{j-2}, c_{j-1})\) satisfies the assumptions, with \( e = 0 \) and \( d \neq 0 \)). Thus we get an index \( k \in \{ j+1, \ldots, m, 1, \ldots, j-2 \} \) with \( |c_k| < 2 \). This finishes the proof. \( \square \)

Remark 3.5. Let us revisit again the classic case of Conway-Coxeter frieze patterns (or slightly more generally of frieze patterns over \( \mathbb{Z} \)). Then Corollary 3.3 states that in the quiddity cycle of every such frieze pattern there are two entries equal to 1 (or \( \pm 1 \) for frieze patterns over \( \mathbb{Z} \setminus \{ 0 \} \)). Actually this existence of 1’s in the quiddity cycle is the key for establishing in [3] an inductive argument for showing the Conway-Coxeter bijection between frieze patterns over \( \mathbb{N} \) and triangulations of regular polygons.

We now state our rather general criterion which gives, under certain conditions on the subset \( R \subseteq \mathbb{C} \), upper bounds for the absolute values of entries in the quiddity cycle of a frieze pattern over \( R \setminus \{ 0 \} \). The idea of the proof below is inspired by experiments in the Master thesis of D. Azadi [1].

Theorem 3.6. Let \( R \subseteq \mathbb{C} \) be a subset such that
\[
M := \inf\{|x| : x \in R \setminus \{ 0 \}\} > 0
\]
(i.e. there is a non-zero lower bound on the absolute values of non-zero elements in \( R \)). Let \( F \) be a frieze pattern over \( R \setminus \{ 0 \} \) with height \( n \in \mathbb{N} \). Then every entry in the quiddity cycle of \( F \) has absolute value at most \( \frac{(n-1)+2M}{M^2} \).

Proof. We set \( B := \frac{(n-1)+2M}{M^2} \) for abbreviation; note that \( B > 0 \) since \( M > 0 \) by assumption.

Suppose for a contradiction that there is an element \( x_1 \) in the quiddity cycle of \( F \) such that
\[
|x_1| > B.
\]
Then we consider two consecutive rows in the frieze pattern \( F \), as in the following figure.

\[
\begin{array}{cccccc}
0 & 1 & x_1 & \cdots & x_n & 1 & 0 \\
0 & 1 & y_1 & \cdots & y_n & 1 & 0
\end{array}
\]

We compare neighbouring entries and proceed inductively. By the defining rule for frieze patterns we have \( x_1y_1 - x_2 = 1 \), so \( |y_1| \leq \frac{1+|x_2|}{|x_1|} \) by the triangle inequality. By definition of \( M \) as infimum and by our assumption on \( x_1 \) (cf. Equation (3.2)) we can conclude that
\[
M \leq |y_1| \leq \frac{1+|x_2|}{|x_1|} < \frac{1+|x_2|}{B}.
\]
Since \( B > 0 \), this implies that
\[
|x_2| > MB - 1.
\]

Now we go one step further. Again, by the defining rule of frieze patterns, we have \( x_2y_2 - x_3y_1 = 1 \), so
\[
|y_2| \leq \frac{1+|x_3| \cdot |y_1|}{|x_2|}.
\]
Together with the definition of \( M \) and Equation (3.3) we obtain
\[
M \leq |y_2| \leq \frac{1+|x_3| \cdot |y_1|}{|x_2|} < \frac{B + |x_3| + |x_3| \cdot |x_2|}{B|x_2|} = \frac{1}{|x_2|} + \frac{|x_3|}{B} \left( \frac{1}{|x_2|} + 1 \right).
\]
Now we use Equation (3.4) and get
\[
M \leq |y_2| < \frac{1}{MB - 1} + \frac{|x_3|}{B} \cdot \frac{MB}{MB - 1} = \frac{1}{MB - 1} (1 + M|x_3|).
\]

\( ^{1} \)Frieze patterns with non-zero entries are always tame, due to Sylvester’s theorem, see for example [5].
Solving this inequality for $|x_3|$ yields

$$|x_3| > \frac{M(MB - 1) - 1}{M}. \quad (3.7)$$

(Note that $MB - 1 = \frac{n-1+2M}{M} - 1 = \frac{(n-1)+M}{M} > 0$ so the inequality sign is not reversed.)

Now we suppose inductively that we have already shown

$$|x_k| > \frac{M(MB - 1) - (k - 2)}{M} \text{ for } k = 2, \ldots, i + 1 \quad (3.8)$$

and

$$|y_k| < \frac{M}{M(MB - 1) - (k - 2)}(1 + M|x_{k+1}|) \text{ for } k = 2, \ldots, i. \quad (3.9)$$

Note that Equations (3.7) and (3.8) are the special case of Equation (3.9) for $k = 2$ and $k = 3$; moreover, Equation (3.6) shows that Equation (3.9) is valid for $k = 2$. This means that the induction base for $i = 2$ has been settled above.

For the induction step we now consider $y_{i+1}$ (where $i + 1 \leq n$); by the defining rule for frieze patterns this has the form $y_{i+1} = \frac{1+x_{i+2}}{x_{i+1}}$. Analogous to the arguments for the induction base we get the following series of inequalities by using the triangle inequality and the induction hypotheses for $|y_i|$ in Equation (3.9) and for $|x_{i+1}|$ in Equation (3.8), respectively.

\[
|y_{i+1}| \leq \frac{1 + |x_{i+2}| \cdot |y_i|}{|x_{i+1}|} < \frac{1}{|x_{i+1}|} + \frac{M|x_{i+2}|}{M(MB - 1) - (i - 2)} \left( \frac{1}{|x_{i+1}|} + M \right) < \frac{M}{M(MB - 1) - (i - 1)} + \frac{M|x_{i+2}|}{M(MB - 1) - (i - 2)} \left( \frac{M}{M(MB - 1) - (i - 1)} + M \right) = \frac{M}{M(MB - 1) - (i - 1)}(1 + M|x_{i+2}|).
\]

This shows the induction step for Equation (3.9). But using that $M \leq |y_{i+1}|$ and then solving the above inequality for $|x_{i+2}|$ yields

$$|x_{i+2}| > \frac{M(MB - 1) - i}{M}$$

and this gives the induction step for Equation (3.8) as well. (Note again that multiplication does not reverse the inequality sign since $M(MB - 1) - (i - 1) = n - 1 + M - (i - 1) = n - i + M > 0$ since $i \leq n - 1$.)

We are considering a frieze pattern $F$ of height $n$. Thus, $x_{n+1} = 1$ and eventually we get from Equation (3.9) that

$$|y_n| < \frac{M}{M(MB - 1) - (n - 2)}(1 + M). \quad (3.10)$$

Using the definition of $B$ we compute that

$$M(MB - 1) = M \left( \frac{(n-1)+2M}{M} - 1 \right) = n - 1 + M.$$  

Together with Equation (3.10) this yields $|y_n| < M$, contradicting the definition of $M$ as infimum and the fact that our frieze pattern $F$ has non-zero entries by assumption. \hfill \Box

**Remark 3.7.** For the classic case of Conway-Coxeter frieze patterns over $\mathbb{N}$ (or frieze patterns over $\mathbb{Z} \setminus \{0\}$) the above theorem yields that every entry in the quiddity cycle of such a frieze pattern of height $n$ has absolute value $\leq n + 1$. This is not hard to deduce from the bijection between frieze patterns over $\mathbb{N}$ and triangulations because the entries in the quiddity cycle are
given by the numbers of triangles attached to the vertices. However, the above proof does not refer to this bijection.

As one of our main applications of Theorem 3.6 we can deduce for a large class of subsets $R \subseteq \mathbb{C}$ that there are only finitely many frieze patterns over $R \setminus \{0\}$ for any given height.

**Corollary 3.8.** Let $R \subseteq \mathbb{C}$ be a discrete subset (i.e. $R$ has no accumulation point). Then for each $n \in \mathbb{N}$ there are only finitely many frieze patterns over $R \setminus \{0\}$ of height $n$.

**Proof.** The subset $R$ is discrete, in particular the origin is not an accumulation point for $R$, so $R$ satisfies the assumption of Theorem 3.6. Furthermore, again by discreteness, each closed disk of $\mathbb{C}$ contains only finitely many elements of $R$. Together with Theorem 3.6 this implies that for fixed height $n$ there are only finitely many possible elements of $R$ which can appear in a quiddity cycle of a frieze pattern of height $n$. But any non-zero frieze pattern is uniquely determined by its quiddity cycle. So the claim follows.

For dealing with a second main application of our finiteness criterion Theorem 3.6 we first state an observation which easily follows from a construction we will present in the next section.

**Proposition 3.9.** Let $R \subseteq \mathbb{C}$ be a subring containing infinitely many divisors of 2 (i.e. elements $t \in R$ s.t. $2/t \in R$). Then for each $n \in \mathbb{N}$ there are infinitely many frieze patterns of height $n$ over $R \setminus \{0\}$.

**Proof.** Let $t \in R$ be an element such that $2t^{-1} \in R$. Then the sequence $(t, 2t^{-1}, t, 2t^{-1})$ is the quiddity cycle of a frieze pattern of height 1 over $R \setminus \{0\}$. Using the rule from Corollary 4.2 produces infinitely many frieze patterns over $R \setminus \{0\}$ in every height greater than 1.

As a second main application of our results in this section we can provide an answer to a question posed by Fontaine in [5]. At the end of his paper, Fontaine considered frieze patterns over rings $\mathbb{Z}[\zeta]$ where $\zeta \in \mathbb{C}$ is a root of unity and he stated the conjecture that over each of these rings there are only finitely many non-zero frieze patterns of any given height (see [5] Conjecture 6.2).

This conjecture has a negative answer in general for any height $n$: only for few such rings one indeed gets finitely many frieze patterns. The following result gives a complete answer.

**Corollary 3.10.** For $d \in \mathbb{N}$, let $\zeta_d \in \mathbb{C}$ be a primitive $d$-th root of unity. The following statements are equivalent for every $n \in \mathbb{N}$:

(i) There are finitely many non-zero frieze patterns of height $n$ over the ring $\mathbb{Z}[\zeta_d]$.

(ii) $d \in \{1, 2, 3, 4, 6\}$.

**Proof.** Let us first suppose that (ii) holds. For the specific values given in (ii) we have $\mathbb{Z}[\zeta_1] = \mathbb{Z} = \mathbb{Z}[\zeta_2]$, the integers, $\mathbb{Z}[\zeta_3] = \mathbb{Z}[i]$, the Gaussian integers, and $\mathbb{Z}[\zeta_4] = \mathbb{Z}[1+i+\sqrt{-2}] = \mathbb{Z}[\zeta_6]$, the Eisenstein integers. These rings are easily seen to be discrete subsets of $\mathbb{C}$. So (i) follows directly from Corollary 3.8.

Conversely, suppose that (i) holds. Then it follows from Proposition 3.3 that the ring $\mathbb{Z}[\zeta_d]$ has only finitely many units. On the other hand, the ring $\mathbb{Z}[\zeta_d]$ is the ring of integers of the cyclotomic number field $\mathbb{Q}(\zeta_d)$ (see e.g. [12] Proposition 10.2). According to Dirichlet’s unit theorem, the rank of the group of units of $\mathbb{Z}[\zeta_d]$ has the form $r_1 + r_2 - 1$ where $r_1$ and $r_2$ are the numbers of real embeddings and pairs of complex embeddings of $\mathbb{Q}(\zeta_d)$, respectively. Moreover, $r_1 + 2r_2 = [\mathbb{Q}(\zeta_d) : \mathbb{Q}] = \varphi(d)$, where $\varphi$ denotes Euler’s totient function.

We have seen above that assumption (i) implies that the rank of the group of units must be 0, thus $r_1 + r_2 = 1$. The are only two possibilities: we can have $r_1 = 1$ and $r_2 = 0$, and then $\varphi(d) = r_1 + r_2 = 1$; or we have $r_1 = 0$ and $r_2 = 1$, and then $\varphi(d) = 2$. It is an easy exercise on Euler’s totient function to show that $\varphi(d) = 1$ if and only if $d \in \{1, 2\}$ and $\varphi(d) = 2$ if and only if $d \in \{3, 4, 6\}$. So statement (ii) holds.
Multiplication with $\eta$

Proof.

(a) Lemma 4.1, Equation (4.1) for $N$ of the Conway-Coxeter bijection between triangulations and frieze patterns over $\mathbb{Z}$ frieze patterns, there are plenty of new non-zero frieze patterns over the Gaussian integers $\mathbb{Z}[i]$ and the Eisenstein integers $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. A major complication compared to the classic Conway-Coxeter frieze patterns (or frieze patterns over $\mathbb{Z} \setminus \{0\}$) is that in the quiddity cycles one does not necessarily have an entry equal to 1 (or $\pm 1$). For instance, Example 2.2 (2) gives an example of a (non-zero) frieze pattern over the Gaussian integers without 1’s in the quiddity cycle. However, Corollary 3.3 yields that in each non-zero frieze pattern over $\mathbb{Z}[i]$ one has two entries of absolute value less than 2, i.e. entries in $\{\pm 1, \pm (1+i), \pm (1-i)\}$. For dealing with frieze patterns without 1’s in the quiddity cycle one would need new transformation rules to reduce such a frieze pattern to a smaller frieze pattern, generalizing the Conway-Coxeter approach to add/delete 1’s. Such rather general transformation rules will be presented in the next section. With these transformations we have obtained reduction rules for frieze patterns over the Gaussian integers which might be published elsewhere.

4. Transformations and rules

Similar to the classic Conway-Coxeter theory we would like to classify frieze patterns over some subset $R \subseteq \mathbb{C}$ via their quiddity cycles. More precisely, we want to prove that every quiddity cycle admits transformations leading to a shorter cycle. The following lemma contains some of the (apparently) most useful rules concerning products of matrices of the form $\eta(a)$, $a \in \mathbb{C}$. We will need them later in Section 6 for the main result about quiddity cycles over $\mathbb{Z}$.

All results in this section hold for matrices over an arbitrary commutative ring $R$, so we take a slightly more general viewpoint here, setting $\eta(c) = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}$ for $c \in R$ as in Definition 2.3.

Lemma 4.1. Let $a, u, v, b, \lambda \in R$ and assume that $uv - 1$ and $\lambda$ are invertible in $R$. Then we have:

\begin{align*}
(4.1) \quad \eta(a)\eta(u)\eta(v)\eta(b) &= \eta \left( a + \frac{1-v}{uv-1} \right) \eta(uv - 1) \eta \left( b + \frac{1-u}{uv-1} \right) \\
(4.2) \quad \eta(a)\eta(u)\eta(v)\eta(b) &= \eta \left( a + \frac{(1-v)u}{uv-1} \right) \eta(\lambda u)\eta \left( \frac{v}{\lambda} \right) \eta \left( b + \frac{(\lambda-1)u}{uv-1} \right) \\
(4.3) \quad \eta(a)\eta(0)\eta(b) &= \eta(a+u)\eta(0)\eta(b-u) = -\eta(a+b)
\end{align*}

Proof. Each formula can be verified by straightforward (though slightly tedious) matrix computations. We leave the details to the reader. \qed

We also note some important special cases. Actually, the first formula below lies at the heart of the Conway-Coxeter bijection between triangulations and frieze patterns over $\mathbb{N}$; namely it allows to insert/remove 1’s from the quiddity cycle of such a frieze pattern. The formulas in the above lemma can thus be seen as more general tools for dealing with frieze patterns over other subsets of the complex numbers.

Corollary 4.2 (See also [8] Lemma 5.2]). For all $a, b \in R$ we have

(a) $\eta(a)\eta(b) = \eta(a+1)\eta(1)\eta(b+1)$.

(b) $\eta(a)\eta(b) = -\eta(a-1)\eta(-1)\eta(b-1)$.

Proof. (a) Lemma 4.1, Equation (4.1) for $u = 1$ reads

$$\eta(a)\eta(1)\eta(v)\eta(b) = \eta(a-1)\eta(v-1)\eta(b).$$

Multiplication with $\eta(b)^{-1}$ and an obvious transformation of variables yields the claim.
(b) We use the formula $T\eta(c)T = \eta(-c)$ for all $c \in \mathbb{C}$ from Remark 2.6 (2) where $T^2$ is minus the identity matrix. Then using part (a) we get

$$\eta(a-1)\eta(-1)\eta(b-1) = (-1)^2 T\eta(-a+1)\eta(1)\eta(-b+1)T = T\eta(-a)\eta(-b)T = -\eta(a)\eta(b).$$

\[ \square \]

Finally, the following transformation is the key to reductions for quiddity cycles over several subsets of $\mathbb{C}$, in particular for the Gaussian integers.

**Lemma 4.3.** Let $a, b \in R$ and assume that $u, z$ are invertible in $R$. Then

$$\begin{pmatrix} \frac{z}{u} & 0 \\ 0 & z \end{pmatrix} \eta(a)\eta(u)\eta(b) \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} = \eta \left( \frac{a}{z^2} - \frac{1}{u} \right) \eta(u) \eta \left( z^2b - \frac{z^2-1}{u} \right).$$

**Proof.** The formula can easily be verified by direct computation. \[ \square \]

### 5. Frieze Patterns as Specializations of Cluster Algebras

In this section we consider the close connection between frieze patterns and Fomin and Zelevinsky’s cluster algebras in Dynkin type $A$. If you place indeterminates $x_1, \ldots, x_n$ on a row of a frieze pattern of height $n$ (say on the positions labelled $c_0, 2, c_0, 3, \ldots, c_0, n+1$ in Definition 2.1) and still follow the local condition $ad - bc = 1$, then one obtains a frieze pattern over the rational function field $\mathbb{Q}(x_1, \ldots, x_n)$ with entries being the cluster variables of a cluster algebra of Dynkin type $A_n$. See Figure 1 for the case $n = 2$.

These frieze patterns corresponding to cluster algebras of Dynkin type $A_n$ are not only periodic of period $n + 3$ (cf. Proposition 2.4), but they have an additional glide symmetry. In other words, a fundamental domain for the entries in these frieze patterns (containing each cluster variable once) is given by a triangular shape.

Note that the pairs of indices of the elements in this fundamental region are in bijection with pairs of different numbers from $\{0, \ldots, n + 2\}$, and these are in bijection with the edges and the diagonals of a regular $(n + 3)$-gon $P$ (with vertices labelled consecutively).

It is well known that there is an alternative description of frieze patterns with such a glide symmetry. Namely, such a frieze pattern can be seen as an assignment of numbers, called labels, to the edges and the diagonals of $P$ such that the labels of edges are 1 and such that for each pair of crossing diagonals $(i, j)$ and $(k, \ell)$ the Ptolemy condition

$$c_{i,j}c_{k,\ell} = c_{i,k}c_{j,\ell} + c_{i,\ell}c_{j,k}$$

is satisfied; see Figure 2.

Any maximal set of pairwise non-crossing diagonals of $P$ is called a cluster. Note that the diagonals of a cluster form a triangulation of the polygon $P$. 

\[ \text{Figure 1. The frieze pattern corresponding to the cluster algebra of Dynkin type } A_2. \]
This notion is perfectly in line with the fundamental notion of cluster in the cluster algebras of Dynkin type $A$. In fact, a subset of cluster variables forms a cluster of the cluster algebra precisely when the positions in the frieze pattern yield a triangulation, i.e. a cluster of $\mathcal{P}$ (and mutation of clusters corresponds to flipping diagonals in the triangulation).

A deep theorem on cluster algebras, the Laurent phenomenon, states that if you start with placing indeterminates $x_1, \ldots, x_n$ on the positions of a cluster then all cluster variables are rational functions with denominator a monomial $x_1^{a_1} \cdots x_n^{a_n}$; see for instance [7, Theorem 3.3.1].

We want to study what happens when the indeterminates, placed on the positions of a cluster, are specialized to values of $\mathbb{C}$. From the above mentioned Laurent phenomenon it is clear that one can not specialize an indeterminate to zero (because this resulted in undefined denominators).

We call a frieze pattern over $\mathbb{C}$ of height $n$ a specialization of the cluster algebra of Dynkin type $A_n$ if the frieze pattern contains a cluster with only non-zero entries.

The main result of this section will describe the frieze patterns over $\mathbb{C}$ which are specializations of the cluster algebras of Dynkin type $A_n$. For proving this result we need some preparation.

Proposition 5.1. Suppose $m \in \mathbb{N}$ is even. Let $c = (c_1, \ldots, c_m) \in \mathbb{C}^m$ be a quiddity cycle, and $0 \neq t \in \mathbb{C}$. Then the following hold:

(a) $c' = (c'_1, \ldots, c'_m) = (tc_1, t^{-1}c_2, tc_3, \ldots, t^{-1}c_m)$ is a quiddity cycle as well.

(b) The frieze patterns $F$ and $F'$ corresponding to $c$ and $c'$ have their zero entries at exactly the same positions.

Proof. (a) For $t \in \mathbb{C} \setminus \{0\}$ we consider the complex matrix

$$T := \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}.$$ 

Then a simple computation yields that for every $c \in \mathbb{C}$ we have

$$\eta(tc) = T\eta(c)T, \quad \eta(t^{-1}c) = T^{-1}\eta(c)T^{-1}.$$ 

This implies that

$$\eta(tc_1)\eta(t^{-1}c_2)\cdots\eta(tc_{m-1})\eta(t^{-1}c_m) \eta(t^{-1}c_m)T^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

where for the last equality we have used the assumption that $c$ is a quiddity cycle (cf. Definition 2.5). Thus, $c'$ is again a quiddity cycle.

(b) We denote the entries in the frieze patterns $F$ and $F'$ by $c_{i,j}$ and $c'_{i,j}$, respectively. According to Proposition 2.4 the entries in the frieze patterns are given by the $(1,1)$-entry in some product.
of $\eta$-matrices. More precisely, we get by using Equation \[\text{Proposition 5.1}\]
that for all $i \leq j - 2$:
\[c'_{i,j} = \left(\prod_{k=i}^{j-2} \eta(c_k')\right)_{1,1} = (T^{\delta} \left(\prod_{k=i}^{j-2} \eta(c_k)\right)T^{\epsilon})_{1,1}\]
with some $\delta, \epsilon \in \{1, -1\}$. But it is straightforward to check that for every complex $2 \times 2$-matrix $A$ we have
\[(TAT)_{1,1} = tA_{1,1}, \quad (TAT^{-1})_{1,1} = A_{1,1}, \quad (T^{-1}AT)_{1,1} = A_{1,1}, \quad (T^{-1}AT^{-1})_{1,1} = t^{-1}A_{1,1}.
So the entries in the frieze patterns corresponding to $c$ and $c'$ only differ up to (possibly) multiplication with $t$ or $t^{-1}$. In particular, the non-zero entries appear at the same positions. □

**Lemma 5.2.** Suppose that $m \in \mathbb{N}$ is odd. Let $c = (c_1, \ldots, c_m) \in \mathbb{C}^m$ be a quiddity cycle, and assume that for all $i = 1, \ldots, m$ we have $c_ic_{i+1} = 1$ or $c_i = c_{i+1} = 0$. Then $c \in \{(0, \ldots, 0), (1, \ldots, 1)\}$.

**Proof.** If $c_1 = 0$ then it follows directly from the assumption that $c_1 = c_2 = \ldots = c_m = 0$, i.e. $c = (0, \ldots, 0)$.

So we assume now that $c_1 \neq 0$. But then $c_2 = c_1^{-1}$ by the assumption, and inductively $c = (c_1, c_1^{-1}, c_1, \ldots, c_1^{-1}, c_1)$ (use that $m$ is odd). By assumption, $c$ is a quiddity cycle, i.e. $\prod_{k=1}^{m} \eta(c_k)$ is minus the identity matrix (cf. Definition 2.5). We claim that this can only happen if $m \equiv 3 \pmod{6}$ and that in this case $c_1 = 1$, i.e. $c \in \{(1, \ldots, 1)\}$. In fact, one easily computes that $\eta(c_1)\eta(c_1^{-1})\eta(c_1)\eta(c_1^{-1})\eta(c_1)\eta(c_1^{-1})$ equals the identity matrix. So it suffices to consider the cases $m \in \{1, 3, 5\}$. Clearly, this can not happen for $m = 1$. For $m = 3$ we have
\[\eta(c_1)\eta(c_1^{-1})\eta(c_1) = \begin{pmatrix} -c_1 & 0 \\ 0 & -c_1^{-1} \end{pmatrix}\]
and this implies $c_1 = 1$. For $m = 5$ we compute
\[\eta(c_1)\eta(c_1^{-1})^2\eta(c_1) = \begin{pmatrix} 0 & 1 \\ -1 & c_1^{-1} \end{pmatrix}\]
which is clearly not a scalar multiple of the identity matrix. □

**Proposition 5.3.** Let $c = (1, \ldots, 1) \in \mathbb{C}^m$ be a quiddity cycle and let $\mathcal{F}$ be the corresponding frieze pattern. Then there exists a cluster without zero entry.
Proof. We have seen in the proof of Lemma 5.2 that if \((1, \ldots, 1)\) is a quiddity cycle then \(m \equiv 3(\text{mod}\ 6)\). Let \(c_{i,j}\) denote the entries in the corresponding frieze pattern. From Proposition 2.4 we know that these entries appear as the top left entry in matrices of the form \(\eta(1)^k\) for \(1 \leq k \leq 6\) (note that since \(\eta(1)^k\) is the identity matrix, the exponents can be reduced modulo 6). Then an easy computation yields that for every \(1 \leq i < j \leq m\) we have

\[
c_{i,j} = \begin{cases} 
1 & \text{if } j - i \equiv 1 \text{ (mod } 6) \text{ or } j - i \equiv 2 \text{ (mod } 6), \\
0 & \text{if } j - i \equiv 0 \text{ (mod } 6) \text{ or } j - i \equiv 3 \text{ (mod } 6), \\
-1 & \text{if } j - i \equiv 4 \text{ (mod } 6) \text{ or } j - i \equiv 5 \text{ (mod } 6). 
\end{cases}
\]

Thus it suffices to find a triangulation \(C\) of the regular \(m\)-gon such that \(j - i \neq 0 \text{ (mod } 3)\) for all diagonals \((i, j) \in C\). For example, writing \(m = 6\ell + 3\) we can take

\[
C = \{(1, 3)\} \cup \{(1, 3k - 1), (1, 3k) \mid 2 \leq k \leq 2\ell\} \cup \{(3k, 3k + 2) \mid 1 \leq k \leq 2\ell\} \cup \{(1, 6\ell + 2)\},
\]

see also Figure 3.

The following theorem is the main result of this section and explains the connection between tame frieze patterns and specializations of cluster variables of Dynkin type \(A\).

**Theorem 5.4.** For \(m \geq 4\), let \(c = (c_1, \ldots, c_m) \in \mathbb{C}^m\) be a quiddity cycle with at least one non-zero entry and let \(\mathcal{F}\) be the corresponding frieze pattern. Then there exists a cluster of \(\mathcal{F}\) without zero entry. In particular, every such quiddity cycle defines a specialization of the variables of the cluster algebra of Dynkin type \(A\) to complex numbers.

**Proof.** We proceed by induction over the length \(m\) of the quiddity cycle.

For \(m = 4\) every quiddity cycle has the form \((c_1, \frac{c_2}{c_1}, c_1, \frac{c_3}{c_1})\) with \(c_1 \neq 0\) (cf. Example 2.7). The clusters of the corresponding quadrangle consist of only one diagonal, and this is labeled by \(c_1\) or by \(\frac{c_2}{c_1}\); since both are non-zero, the claim holds for \(m = 4\).

Suppose now that \(m > 4\).

Case 1: Assume first that there is a \(k\) with \(c_k = 1\). By Proposition 5.3 we can assume that \(c\) contains an entry not equal to 1. So we can choose \(k\) maximal such that \(c_k = 1\) and \(c_{k+1} \neq 1\). From Corollary 4.2(a) we know that \(c' = (c_1, \ldots, c_{k-2}, c_{k-2} - 1, c_{k-1} - 1, c_{k+2}, \ldots, c_m)\) is again a quiddity cycle. By maximality of \(k\), it has a non-zero entry, i.e. it satisfies the assumption of the theorem. Now by induction hypothesis we find a cluster without zero for the frieze pattern \(\mathcal{F}'\) corresponding to the quiddity cycle \(c'\) of length \(m - 1\). In other words, there is a triangulation of the \((m - 1)\)-gon obtained from the \(m\)-gon by removing the vertex \(k\) such that each diagonal has a non-zero label. The diagonal \((k - 1, k + 1)\) in the \(m\)-gon has label \(c_k = 1 \neq 0\). So the cluster for \(\mathcal{F}'\) may be extended to a cluster without zero of the original frieze pattern \(\mathcal{F}\) by adding the diagonal \((k - 1, k + 1)\).

Case 2: Suppose that there is no 1 in the quiddity cycle and that \(m\) is even. Then we choose any \(k\) with \(c_k \neq 0\). Applying Proposition 5.1 with \(t = c_k\) (if \(k\) is even) or \(t = c_k^{-1}\) (if \(k\) is odd) we obtain a quiddity cycle of the same length, with the same clusters without zero and with a 1 at position \(k\). We are thus in Case 1 and hence the claim follows.

Case 3: Finally, suppose now that there is no 1 in the quiddity cycle and that \(m\) is odd. Moreover, we may assume that there is a \(k\) such that \(c_k \neq 0\) or \(c_{k+1} \neq 0\) and such that \(c_k c_{k+1} - 1 \neq 0\) (otherwise we are finished by Lemma 5.2 and Proposition 5.3). We consider the quiddity cycle (see Lemma 4.1, Equation (4.1))

\[
c'' = (c_1, \ldots, c_{k-2}, c_{k-1} + \frac{1 - c_{k+1}}{c_k c_{k+1} - 1}, c_k c_{k+1} - 1, c_{k+2} + \frac{1 - c_k}{c_k c_{k+1} - 1}, c_{k+3}, \ldots, c_m).
\]

Using Proposition 5.1 (notice that \(c''\) has even length) we get a quiddity cycle with a 1 instead of the entry \(c_k c_{k+1} - 1\). It has the same length, with the same clusters without zero; we are thus in Case 1 and find a suitable cluster for \(c''\). In particular, this cluster contains the diagonal corresponding to the entry \(c_k c_{k+1} - 1\). Extending this triangulation by the diagonal \((k, k + 2)\)
if \(c_k \neq 0\) or \((k + 1, k + 3)\) if \(c_{k+1} \neq 0\) yields a cluster for the original quiddity cycle of length \(m\), since the diagonal corresponding to the entry \(c_k c_{k+1} - 1\) is now \((k, k + 3)\).

\[\square\]

**Example 5.5.** The following frieze pattern is tame but every cluster contains a zero entry:

\[
\begin{array}{cccccccc}
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
\end{array}
\]

By Theorem 5.4 this is (up to repeating this picture periodically) the only such example.

6. Frieze patterns over \(\mathbb{Z}\)

In this section we describe the quiddity cycles (and hence tame frieze patterns, cf. Proposition 2.4) over the integers \(\mathbb{Z}\). As a special case this includes the classic Conway-Coxeter frieze patterns over \(\mathbb{N}\) and also more recent work of Fontaine \[8\] on frieze patterns over \(\mathbb{Z} \setminus \{0\}\).

It turned out that over \(\mathbb{Z} \setminus \{0\}\) only very few new frieze patterns appear in addition to the Conway-Coxeter ones and the new ones are closely linked to the old ones. As we will show in this section the situation changes drastically when zeroes are allowed, i.e. when considering quiddity cycles and frieze patterns over \(\mathbb{Z}\). Then a plethora of new frieze patterns emerges.

Still, we will provide in the next section a nice combinatorial model for obtaining all quiddity cycles over \(\mathbb{Z}\) from certain labelled triangulations.

Before we prove the (more technical) theorem (Thm. 6.4) giving reductions of quiddity cycles, we consider a bigger class of cycles which contains our quiddity cycles.

**Definition 6.1.** Let \(\varepsilon \in \{\pm 1\}\). An \(\varepsilon\)-cycle is a sequence \((c_1, \ldots, c_m) \in \mathbb{Z}^m\) satisfying

\[
\prod_{k=1}^m \eta(c_k) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}.
\]

Such cycles have recently also been considered by Ovsienko \[13\].

**Theorem 6.2.** Let \((c_1, \ldots, c_m) \in \mathbb{Z}^m\) be an \(\varepsilon\)-cycle, \(\varepsilon \in \{\pm 1\}\). Then we have at least one of the following cases:

1. \(m = 2\) and \((c_1, \ldots, c_m) = (0, 0)\).
2. There exists an index \(k\) with \(c_k = 1\) and
   \[
   (c_1, \ldots, c_{k-2}, c_{k-1} - 1, c_{k+1} - 1, c_{k+2} \ldots, c_m)
   \]
   is an \(\varepsilon\)-cycle (of length \(m - 1\)).
3. There exists an index \(k\) with \(c_k = 0\) and
   \[
   (c_1, \ldots, c_{k-2}, c_{k-1} + c_{k+1}, c_{k+2} \ldots, c_m)
   \]
   is an \(-\varepsilon\)-cycle (of length \(m - 2\)).
4. There exists an index \(k\) with \(c_k = -1\) and
   \[
   (c_1, \ldots, c_{k-2}, c_{k-1} + 1, c_{k+1} + 1, c_{k+2} \ldots, c_m)
   \]
   is an \(-\varepsilon\)-cycle (of length \(m - 1\)).

**Proof.** By Corollary 3.3 there exists a \(k\) with \(|c_k| < 2\). But \(c_k \in \mathbb{Z}\), thus \(c_k \in \{-1, 0, 1\}\). If \(m = 2\) then the cycle is \((0, 0)\) (see for example the proof of Cor. 3.3). Otherwise, the rules in Equation (4.3) and Corollary 4.2 always apply; when \(c_k \in \{-1, 0\}\) then a sign appears and an \(\varepsilon\)-cycle becomes a \(-\varepsilon\)-cycle. \(\square\)

To show that there are even reductions producing cycles of the same type \((\varepsilon = -1\), i.e. within the class of quiddity cycles\), we first need the following refinement of Corollary 3.3 in the special case \(R = \mathbb{Z}\):
Corollary 6.3. Let \((c_1, \ldots, c_m) \in \mathbb{Z}^m\) be a quiddity cycle with \(m > 3\). Then there are two indices \(j, k \in \{1, \ldots, m\}\) with \(|j - k| > 1\) and \((j, k) \neq (1, m)\) such that \(|c_j| < 2\) and \(|c_k| < 2\).

Proof. Before entering the general argument we deal with the case \(m = 4\) separately. By Example 2.7 the quiddity cycles of length 4 have the form \((c_1, c_2, c_3, \pm 1)\) with \(c_1 \neq 0\). Over the integers \(\mathbb{Z}\), the only possibilities are \(c_1 \in \{ \pm 1, \pm 2\}\). It is easy to see that in each of these possible cases the assertion holds.

So from now we assume that \(m \geq 5\). Corollary 3.3 already gives us two indices \(i \neq j\) with \(|c_i| < 2\) and \(|c_j| < 2\). If \(|i - j| > 1\) and \((i, j) \neq (1, m)\) then we are done. Otherwise, by rotating the cycle (cf. Remark 2.6), we may assume without loss of generality that \(i = 1\) and \(j = m\). We distinguish two cases.

First, if \(c_1 = c_m = 0\), then by the definition of a quiddity cycle we get

\[
\prod_{\ell = 2}^{m-1} \eta(c_\ell) = \eta(0)^{-1} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \eta(0)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

If \(c_{m-1} = 0\), we are done. Otherwise, we can apply Lemma 3.1 to the product in equation (6.2); this yields an index \(k \in \{3, \ldots, m-2\}\) with \(c_k < 2\) (note that we need our assumption \(m \geq 5\) for this \(k\) to exist). Then clearly either \((1, k)\) or \((k, m)\) is a pair of indices as required.

Secondly, otherwise one of \(|c_1|\) or \(|c_m|\) is 1. Reversing the quiddity cycle if required (cf. Remark 2.6), we may assume that \(|c_m| = 1\), hence Lemma 3.1 yields a \(k \in \{2, \ldots, m-1\}\) with \(|c_k| < 2\) and again we are finished because \(m \geq 3\).

We can now state the main result of this section, guaranteeing adequate occurrences of entries 0, 1 or \(-1\) in quiddity cycles over \(\mathbb{Z}\) and providing reductions to shorter quiddity cycles. This theorem will be the crucial input for the combinatorial model to be presented in the next section.

Theorem 6.4. Let \((c_1, \ldots, c_m) \in \mathbb{Z}^m\) be a quiddity cycle. Then we have at least one of the following cases:

1. \(m < 4\) and \((c_1, \ldots, c_m) \in \{(0, 0), (1, 1, 1)\}\).
2. There exists an index \(k\) with \(c_k = 1\) and
   \[(c_1, \ldots, c_{k-2}, c_{k-1} - 1, c_k + 1 - 1, c_{k+2}, \ldots, c_m)\]
   is a quiddity cycle (of length \(m - 1\)).
3. There exists an index \(k\) with \(c_k = 0\), \(m\) is odd, and
   \[(-c_1, \ldots, -c_{k-2}, -c_{k-1} - c_{k+1}, -c_{k+2}, \ldots, -c_m)\]
   is a quiddity cycle (of length \(m - 2\)).
4. There exists an index \(k\) with \(c_k = -1\), \(m\) is even, and
   \[(-c_1, \ldots, -c_{k-2}, -c_{k-1} - 1, -c_{k+1} - 1, -c_{k+2}, \ldots, -c_m)\]
   is a quiddity cycle (of length \(m - 1\)).
5. There exist \(j, k\) with \(|j - k| > 1\), \(c_j = c_k = 0\) and
   \[(c_1, \ldots, c_{j-2}, c_{j-1} + c_{j+1}, c_{j+2}, \ldots, c_{k-2}, c_{k-1} + c_{k+1}, c_{k+2}, \ldots, c_m)\]
   is a quiddity cycle if \(|j - k| > 2\), or
   \[(c_1, \ldots, c_{j-2}, c_{j-1} + c_{j+1}, c_{j+2}, \ldots, c_{k-2}, c_{k-1} + c_{k+1}, c_{k+2}, \ldots, c_m)\]
   is a quiddity cycle if \((w.l.o.g.)\) \(j + 1 = k - 1\) (in both cases of length \(m - 4\)).
6. There exist \(j, k\) with \(|j - k| > 1\), \(c_j = c_k = -1\) and
   \[(c_1, \ldots, c_{j-2}, c_{j-1} + 1, c_{j+1} + 1, c_{j+2}, \ldots, c_{k-2}, c_{k-1} + 1, c_{k+1} + 1, c_{k+2}, \ldots, c_m)\]
   is a quiddity cycle if \(|j - k| > 2\) or
   \[(c_1, \ldots, c_{j-2}, c_{j-1} + 1, c_{j+1} + 2, c_{k+1} + 1, c_{k+2}, \ldots, c_m)\]
   is a quiddity cycle if \((w.l.o.g.)\) \(j + 1 = k - 1\) (in both cases of length \(m - 2\)).
Proof. If \( m < 4 \), then the only quiddity cycles are \((0,0)\) or \((1,1,1)\) (cf. Example 2.7) and we are in case \([1]\). Otherwise, \( m > 3 \) and by Corollary 6.3 there are two entries equal to \(-1,0,1\), say \( c_j \) and \( c_k \), with \(|j-k| > 1 \) and \((j,k) \neq \{1,m\} \). If one of them is \( 1 \), then depending on whether \( m \) is odd or even, we are in case \([2]\) or case \([3]\), respectively. Otherwise, \( c_j \neq 1 \neq c_k \), \( c_j = c_k \), and one of \([4]\) or \([5]\) is the case.

It remains to prove that the reduced sequences given in the respective cases are indeed quiddity cycles.

For case \([1]\) this follows immediately from Corollary 4.2.

For case \([2]\), we have a quiddity cycle \((c_1,\ldots,c_{k-1},0,c_{k+1},\ldots,c_m)\). Using Remark 2.6\([2]\), the assumption \( m \) odd and equation \([4.3]\) we deduce that

\[
\left( \begin{array}{cc}
-1 & 0 \\
0 & -1 \\
\end{array} \right) = \eta(-c_1)\eta(-c_{k-1})\eta(0)\eta(-c_{k+1})\ldots\eta(-c_m)
\]

thus \((-c_1,\ldots,-c_{k-2},-c_{k-1}-c_{k+1},-c_{k+2},\ldots,-c_m)\) is a quiddity cycle.

For case \([3]\), the quiddity cycle has the form \((c_1,\ldots,c_{k-1},-1,c_{k+1},\ldots,c_m)\). Similar to the previous case, we use Remark 2.6\([2]\), the assumption \( m \) even and Corollary 4.2 to obtain

\[
\left( \begin{array}{cc}
-1 & 0 \\
0 & -1 \\
\end{array} \right) = \eta(-c_1)\eta(-c_{k-2})\eta(-c_{k-1})\eta(1)\eta(-c_{k+1})\eta(-c_{k+2})\ldots\eta(-c_m)
\]

thus \((-c_1,\ldots,-c_{k-2},-c_{k-1}-1,-c_{k+1},-c_{k+2},\ldots,-c_m)\) is a quiddity cycle.

For case \([4]\), we can apply equation \([4.3]\) twice and the claims follow directly in either case (note that the two minus signs from equation \([4.3]\) cancel).

Finally, for case \([5]\) we apply Corollary 1.2\([b]\) twice and the claims about the quiddity cycles follow.

Example 6.5. The following examples illustrate some of the cases of Theorem 6.4.

(1) \( m \) even, no \( 1 \) and no \(-1 \) in the quiddity cycle:

\[
\begin{array}{cccccccc}
0 & 1 & 0 & -1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & -5 & -2 & 1 & 0 & 0 \\
0 & 1 & -2 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & -5 & -2 & 1 & 0 & 0 \\
0 & 1 & -2 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

(2) \( m \) odd, no \( 1 \) in the quiddity cycle:

\[
\begin{array}{cccccccc}
0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

(3) \( m \) odd, no \( 1 \) and no \(-1 \) in the quiddity cycle:

\[
\begin{array}{cccccccc}
0 & 1 & 0 & -1 & 5 & 1 & -1 & -3 & 1 & 0 \\
0 & 1 & -4 & 19 & 4 & -3 & -10 & 3 & 1 & 0 \\
0 & 1 & -5 & -1 & 1 & 3 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & -2 & 1 & -1 & -4 & 1 & 0 \\
0 & 1 & 4 & 7 & -4 & 5 & 19 & -5 & 1 & 0 \\
0 & 1 & 2 & -1 & 1 & 4 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & -3 & 1 & -1 & 4 & 1 & 0 \\
0 & 1 & -3 & -10 & 3 & -2 & 7 & 2 & 1 & 0 \\
0 & 1 & 3 & -1 & 1 & -4 & -1 & 0 & 1 & 0 \\
\end{array}
\]
(4) $m$ even, no 1 and no 0 in the quiddity cycle:

\[
\begin{align*}
0 & 1 -1 -3 10 -7 -3 1 0 \\
0 & 1 2 -7 5 2 -1 1 0 \\
0 & 1 -3 2 1 0 -1 1 0 \\
0 & 1 -1 0 1 -3 2 1 0 \\
0 & 1 -1 -3 10 -7 -3 1 0 \\
0 & 1 2 -7 5 2 -1 1 0 \\
0 & 1 -3 2 1 0 -1 1 0 \\
0 & 1 -1 0 1 -3 2 1 0 
\end{align*}
\]

7. Combinatorial model

**Definition 7.1.** For $m \in \mathbb{N}_{\geq 2}$, let $T$ be a triangulation of a regular $m$-gon. A *labelling* of $T$ is an assignment of integers $a_t$, called labels, to the triangles $t$ of $T$. Let $d$ be the sum of the number of negative labels and half the number of labels 0. We call $(-1)^d$ the *sign* of the labelling if $d$ is an integer. A labelling is called *admissible* if the following conditions are satisfied:

(i) The set of triangles $t$ with $a_t \notin \{1, -1\}$ can be written as a disjoint union of two-element subsets $\{t_1, t_2\}$ (called *squares*) such that $t_1$ and $t_2$ have a common edge (i.e. are neighbouring triangles) and $a_{t_1} = -a_{t_2}$.

(ii) The sign is 1, i.e. the sum of the number of negative labels and half the number of labels 0 is even.

Note that by condition (i), the sign is defined for any admissible labelling.

**Example 7.2.** Figure 4 shows two examples of admissible labellings of a triangulation. The corresponding friezes defined by the quiddity cycles as given in Theorem 7.3 are:

\[
\begin{align*}
0 & 1 1 2 3 4 5 1 0 \\
0 & 1 3 5 7 9 2 1 0 \\
0 & 1 2 3 4 1 1 1 0 \\
0 & 1 2 3 1 2 3 1 0 \\
0 & 1 2 1 3 5 2 1 0 \\
0 & 1 1 4 7 3 2 1 0 \\
0 & 1 5 9 4 3 2 1 0 \\
0 & 1 2 1 1 1 1 1 0 
\end{align*}
\]
Theorem 7.3.  
(a) Let $\mathcal{T}$ be a triangulation of a regular $m$-gon with vertices denoted (in counterclockwise order) $1, 2, \ldots, m$, and assume that we have an admissible labelling of $\mathcal{T}$. For each vertex $i$ let $c_i$ be the sum of the labels of the triangles attached at the vertex $i$. Then $(c_1, \ldots, c_m)$ is a quiddity cycle over $\mathbb{Z}$ (in the sense of Definition 2.5).

(b) Every quiddity cycle over $\mathbb{Z}$ can be obtained as in (a) from an admissible labelling.

Proof. (b) By Theorem 6.4, there is a (not necessarily unique) sequence of transformations which reduces a quiddity cycle over $\mathbb{Z}$ to $(0, 0)$ or $(1, 1, 1)$. Actually, $(0, 0)$ suffices since $(1, 1, 1)$ can be reduced to $(0, 0)$ by Corollary 4.2(a). We translate each of these transformations into combinatorial pictures and obtain the figures 5, 6, 7, 8, and 9. (We write “$(-1)$” into the polygon to indicate that all entries are multiplied by $-1$.) To obtain an admissible labelling for a given quiddity cycle, just apply such a sequence of transformations in the reverse ordering to the 2-gon.

(a) Let $(a_t)_{t \in \mathcal{T}}$ be an admissible labelling of $\mathcal{T}$. This may be constructed inductively (although not uniquely) by successively gluing building blocks, i.e. triangles with entries 1 or $-1$ and squares containing $c$, $-c$ for some values $c$. We start with the “triangulation” of the 2-gon and labels $0, 0$ at the vertices; this is admissible and gives the quiddity cycle $(0, 0)$. In each step, we glue a building block at an edge $(i, i+1)$ of a triangulation with vertices labelled as above (sum of the labels of the triangles), say $a = c_i$, $b = c_{i+1}$. For each type of block, this produces a new sequence of labels at the vertices. However, if the sign of the obtained labelling is $\varepsilon$, then the product of matrices of the form $\eta(\cdot)$ of the labels at the vertices is $\varepsilon$ times the identity matrix, thus the labels are a quiddity cycle only if $\varepsilon = 1$:

- If we glue a triangle with a 1: $(\ldots, a, b, \ldots)$ becomes $(\ldots, a + 1, 1, b + 1, \ldots)$ which is compatible with $\eta(a)\eta(b) = \eta(a + 1)\eta(1)\eta(b + 1)$ (Corollary 4.2), the sign remains the same.
- If we glue a triangle with a $-1$: $(\ldots, a, b, \ldots)$ becomes $(\ldots, a - 1, -1, b - 1, \ldots)$. Here the sign changes: Corollary 4.2 tells us $\eta(a)\eta(b) = -\eta(a - 1)\eta(-1)\eta(b - 1)$.
- If we glue a square containing $c$, $-c$: $(\ldots, a, b, \ldots)$ becomes $(\ldots, a, c, 0, b - c, \ldots)$ or $(\ldots, a - c, 0, c, b, \ldots)$. Here the sign changes as well: Equation 4.3 tells us $\eta(a)\eta(c)\eta(0)\eta(b - c) = -\eta(a)\eta(b) = \eta(a - c)\eta(0)\eta(c)\eta(b)$.

In any case, the sign of the labelling is 1 if and only if the obtained sequence is a quiddity cycle. \qed
Figure 6. Theorem 6.4 case (2) \((m \text{ odd})\)

Figure 7. Theorem 6.4 case (3) \((m \text{ even})\)

Figure 8. Theorem 6.4 case (4)

Figure 9. Theorem 6.4 case (5) \((m \text{ odd})\)
Remark 7.4. (1) The construction described above directly generalizes the classic case of Conway-Coxeter frieze patterns \[3\]. In fact, in the classic case, the quiddity cycle of a frieze pattern over \( \mathbb{N} \) is determined from the corresponding triangulation by counting the number of triangles attached to each vertex. In other words, in the classic case one only considers admissible labellings where each triangle is labelled by 1.

(2) Fontaine \[8\] has described frieze patterns over \( \mathbb{Z} \setminus \{0\} \). His main theorem in this direction states that for even height \( n \) each such frieze pattern is a Conway-Coxeter frieze pattern, and for \( n \) odd there are twice as many frieze patterns, obtained from the Conway-Coxeter frieze patterns by multiplying each second row by \(-1\). In particular, in the latter case, the quiddity cycle gets multiplied by \(-1\). So in the above language of labellings this means that for \( n \) odd the labellings for the new frieze patterns are constantly \(-1\) on each triangle. (Note that for \( n \) odd, the corresponding triangulation of the \((n+3)\)-gon has \( n+1 \) triangles; this is an even number, so condition (ii) of Definition 7.1 is satisfied and this labelling is admissible.)

(3) Theorem 7.3 shows that in addition to the friezes considered by Conway-Coxeter and Fontaine there are plenty of other frieze patterns over \( \mathbb{Z} \) (but these would all have at least one zero entry).

(4) Theorem 7.3 provides a combinatorial model for producing all quiddity cycles over \( \mathbb{Z} \). Unfortunately, in this way one does not get a bijection between quiddity cycles and admissible labellings of triangulations. There exist different admissible labellings giving the same quiddity cycle, as the following hexagonal example shows; for any \( a \in \mathbb{Z} \), both admissible labellings yield the quiddity cycle \((1, 1, -a, -1, -1, a)\) (starting from the top left vertex).

\[
\begin{array}{cccccc}
1 & a & -a & -1 \\
1 & a-1 & -a+1 & -1
\end{array}
\]

It seems to be tricky to describe combinatorially precisely when two admissible labellings give the same quiddity cycle.

We now prove an analogue to Theorem 6.4 for the combinatorial model. The following lemma is the key to obtain all required reductions.

Lemma 7.5. Let \( \mathcal{C} = (c_1, ..., c_m) \), \( m > 3 \) be a quiddity cycle of a triangulation, i.e. the quiddity cycle of the Conway-Coxeter frieze pattern of this triangulation. Then \( \mathcal{C} \) contains two disjoint subsequences \((1, 2)\) or \((2, 1)\), or it contains the subsequence \((1, 3, 1)\).

Proof. Assume that \((1, 3, 1)\) is not a subsequence of \( \mathcal{C} \). Now consider all subsequences in \( \mathcal{C} \) of the form \((a, 1, b)\) with \( a, b > 2 \). If \((a, 1, b, 1, d)\) with \( a, b, d > 2 \) is a subsequence, then \( b > 3 \) by assumption. Thus we may apply Corollary 4.2 (a) to all these subsequences simultaneously and obtain a new quiddity cycle \( \mathcal{C}' \) in which \((a, 1, b)\) with \( a, b > 2 \) is never a subsequence; this \( \mathcal{C}' \) has length at least 4 because it contains at least one entry bigger than 1. In particular, \((1, 1)\) is not a subsequence of \( \mathcal{C}' \).

Hence every 1 in \( \mathcal{C}' \) has a neighbouring 2, and we have at least two 1’s since the length of \( \mathcal{C}' \) is greater than 3. If the 1’s are at positions \( k \) and \( k+2 \) and the neighbouring 2’s are both at \( k+1 \), i.e. \((1, 2, 1)\) is a subsequence, then \( \mathcal{C}' = (1, 2, 1, 2) \). Otherwise we find two disjoint subsequences in \( \mathcal{C}' \). Now we go back to \( \mathcal{C} \): including the 1’s we have removed, the disjoint subsequences found in \( \mathcal{C}' \) remain, since \((1, 3, 1)\) is not a subsequence of \( \mathcal{C} \). \(\square\)
Given a triangulation of a polygon, a vertex of the polygon is called an ear; if it is attached to only one triangle.

The following theorem is a combinatorial reformulation of Theorem 6.4.

**Theorem 7.6.** Let $T$ be a triangulation of a regular $m$-gon with an admissible labelling. Then we have at least one of the following cases:

1. There exists an index $k$ such that $(k-1,k,k+1)$ is a triangle labelled 1. Removing this triangle yields an admissible labelling of a triangulation of an $(m-1)$-gon.
2. There exists an index $k$ such that $(k-1,k+1,k+2)$ is a square labelled $(c,-c)$ and $m$ is even. Removing this square and multiplying each entry by $-1$ yields an admissible labelling of a triangulation of an $(m-2)$-gon.
3. There exists an index $k$ such that $(k-1,k,k+1)$ is a triangle labelled $-1$ and $m$ is even. Removing this triangle and multiplying each entry by $-1$ yields an admissible labelling of a triangulation of an $(m-1)$-gon.
4. There exist $j,k$ with $|j-k|>1$ and $(j-1,j,j+1,j+2)$, $(k-1,k,k+1,k+2)$ are squares labelled $(c,-c)$, $(d,-d)$. Removing these two squares yields an admissible labelling of a triangulation of an $(m-4)$-gon.
5. There exist $j,k$ with $|j-k|>1$ and $(j-1,j,j+1)$, $(k-1,k,k+1)$ are triangles labelled $-1$. Removing these two triangles yields an admissible labelling of a triangulation of an $(m-2)$-gon.

**Proof.** If $m<4$ then we are in case (0). Thus assume that $m>3$. The triangulation $T$ has an ear, thus if it is labelled 1, then we have case (1). Thus assume that no ear is labelled 1. By Lemma 7.5 we have the following cases for the quiddity cycle $q$ of the Conway-Coxeter frieze pattern of the triangulation $T$:

1. The cycle $q$ contains $(1,3,1)$, say at position $k,k+1,k+2$. If the pentagon $(k-1,k,k+1,k+2,k+3)$ has no triangle with label 1, then either we have two ears labelled $-1$ and we are in case (1), or the pentagon consists of a triangle labelled $-1$ and a square labelled $(c,-c)$, we are then in case (2) or (3) depending on whether $m$ is odd or even: If $m$ is odd, remove the triangle labelled $-1$; the sign of the smaller labelling is different, but since $m$ is odd, multiplying each entry by $-1$ remedies this. If $m$ is even, remove the square; again the sign changes and is remedied by multiplying each entry by $-1$.
2. The cycle $q$ is equal to $(1,2,1,2)$. Then both triangles are labelled $-1$ because this is the only admissible labelling without a 1, and we are in case (3).
3. The cycle $q$ contains two disjoint subsequences $(1,2)$ or $(2,1)$ and $m>4$. If both ears are labelled $-1$ then we are in case (4). If both subsequences belong to squares labelled $(c,-c)$, $(d,-d)$, then we are in case (4). Otherwise, we are in case (2) or (3) depending on whether $m$ is odd or even (same argument as in case 1 to see that the new labelling is admissible).

**References**

[1] D. Azadi, *Friesmuster und Triangulierungen*, Master thesis, Leibniz Universität Hannover (2016), 62 pp.
[2] F. Bergeron, C. Reutenauer, *SL₃-tilings of the plane*, Illinois J. Math. 54 (2010), no. 1, 283-300.
[3] J. H. Conway, H. S. M. Coxeter, *Triangulated polygons and frieze patterns*, Math. Gaz. 57 (1973), no. 400, 87-94 and no. 401, 175-183.
[4] H. S. M. Coxeter, *Fries pattern*, Acta Arith. 18 (1971), 297-310.
[5] M. Cuntz, *On wild frieze patterns*, Exp. Math. 26,3 (2017), 342-348.
[6] M. Cuntz, I. Heckenberger, *Weyl groupoids of rank two and continued fractions*, Algebra Number Theory 3 (2009), 317-340.
[7] S. Fomin, L. Williams, A. Zelevinsky, *Introduction to Cluster Algebras. Chapters 1-3*, Preprint (2016), arXiv:1608.05735.
[8] B. Fontaine, *Non-zero integral friezes*, Preprint (2014), arXiv:1409.6926.
[9] T. Holm, P. Jørgensen, *A p-angulated generalisation of Conway and Coxeter’s theorem on frieze patterns*, Preprint (2017), arXiv:1709.09861.
[10] S. Morier-Genoud, *Coxeter’s frieze patterns at the crossroads of algebra, geometry and combinatorics*, Bull. Lond. Math. Soc. 47 (2015), no. 6, 895-938.
[11] S. Morier-Genoud, V. Ovsienko, R. E. Schwartz, S. Tabachnikov, *Linear difference equations, frieze patterns, and the combinatorial Gale transform*, Forum Math. Sigma 2 (2014), e22, 45.

[12] J. Neukirch, *Algebraic number theory*. Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by G. Harder. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 322. Springer-Verlag, Berlin, 1999. xviii+571 pp.

[13] V. Ovsienko, *Partitions of unity in \(SL(2,\mathbb{Z})\), negative continued fractions, and dissections of polygons*, Preprint (2017), arXiv:1710.02996.

MICHAEL CUNTZ, LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISCRETE MATHEMATIK, FAKULTÄT FÜR MATHEMATIK UND PHYSIK, WELFENGARTEN 1, D-30167 HANNOVER, GERMANY  
*E-mail address*: cuntz@math.uni-hannover.de  
*URL*: https://www.iazd.uni-hannover.de/cuntz.html

THORSTEN HOLM, LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISCRETE MATHEMATIK, FAKULTÄT FÜR MATHEMATIK UND PHYSIK, WELFENGARTEN 1, D-30167 HANNOVER, GERMANY  
*E-mail address*: holm@math.uni-hannover.de  
*URL*: http://www2.iazd.uni-hannover.de/~holm