2D Eigenvalue Problem II: Rayleigh Quotient Iteration and Applications

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Abstract

In Part I of this paper, we introduced a 2D eigenvalue problem (2DEVP) and presented theoretical results of the 2DEVP and its intrinsic connection with the eigenvalue optimizations. In this part, we devise a Rayleigh quotient iteration (RQI)-like algorithm, 2DRQI in short, for computing a 2D-eigentriplet of the 2DEVP. The 2DRQI performs 2× to 8× faster than the existing algorithms for large scale eigenvalue optimizations arising from the minmax of Rayleigh quotients and the distance to instability of a stable matrix.

Key words. eigenvalue problem; Rayleigh quotient; Rayleigh quotient iteration; distance to instability.

AMS subject classifications. 65F15, 65K10

1 Introduction

This is the second part of the paper in the sequel on the 2D eigenvalue problem (2DEVP), namely computing scalars $\mu, \lambda \in \mathbb{R}$ and nonzero vector $x \in \mathbb{C}^n$ such that

\[
\begin{cases}
(A - \mu C)x = \lambda x, & \quad (1.1a) \\
x^H C x = 0, & \quad (1.1b) \\
x^H x = 1, & \quad (1.1c)
\end{cases}
\]

where $A, C \in \mathbb{C}^{n \times n}$ are given Hermitian matrices and $C$ is indefinite. The pair $(\mu, \lambda)$ is called a 2D-eigenvalue, $x$ is called the corresponding 2D-eigenvector, and the triplet $(\mu, \lambda, x)$ is called a 2D-eigentriplet.

In Part I [28], we presented the theory of the 2DEVP (1.1), such as association with the parameter eigenvalue problem and existence and variational characterization of 2D-eigenvalues. We revealed that the 2DEVP has intrinsic relation with the problem of eigenvalue optimization. Specifically, the equation (1.1a) is a parameter eigenvalue problem of $H(\mu) = A - \mu C$. Since $A$ and $C$ are Hermitian, $H(\mu)$ has $n$ real eigenvalues $\lambda_1(\mu), \lambda_2(\mu), \ldots, \lambda_n(\mu)$ for any $\mu \in \mathbb{R}$. Suppose these eigenvalues are sorted such that $\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_n(\mu)$, then equation (1.1b) is a necessary condition for (local or global) maxima or minima of $\lambda_i(\mu)$.

In this paper, we focus on numerical algorithms for solving the 2DEVP (1.1). Rayleigh quotient iteration (RQI) is a classical and efficient algorithm for computing an eigenpair of an Hermitian
matrix, see [23, 29] and references therein. The RQI is locally cubically convergent, i.e., the number of correct digits triples at each iteration once the error is small enough and the eigenvalue is simple [5, Theorem 5.9]. In this paper, we devise an RQI-like algorithm called 2DRQI for solving the 2DEVP (1.1). One of main advantages of the 2DRQI is that the computational kernel of the 2DRQI is a linear systems of equation, similar to the classical RQI. Therefore, the 2DRQI is capable to solve large scale 2DEVP by exploiting the structure and sparsity of matrices $A$ and $C$.

As a part of main contributions of this part, the 2DRQI is further developed for applications in two eigenvalue optimization problems, namely finding the minmax of two Rayleigh quotients and computing the distance to instability (DTI) of a stable matrix. We will demonstrate the algorithmic advantages of treating these eigenvalue optimizations through the 2DEVP and the 2DRQI, such as introducing the notion of the backward error of a computed DTI for the first time and the significant reduction (2× to 8× speedups) in computing time comparing with the existing algorithms.

In the third part of this paper, we will provide a rigorous convergence analysis of the proposed 2DRQI, and prove that the 2DRQI is locally quadratically convergent under some mild assumptions.

The rest of this paper is organized as follows. In Section 2, we will introduce concept of 2D Rayleigh quotients (2DRQ) and Jacobian of the 2DEVP, and present the approximation properties of the 2DRQ. In Section 3, we derive a 2D Rayleigh quotient iteration (2DRQI). The backward error analysis of the 2DEVP for an approximate 2D-eigentriplet is in Section 4. Section 5 discusses the applications of the 2DRQI for finding the minmax of two Rayleigh quotients and computing the distance to instability (DTI) of a stable matrix. In Section 6, we present numerical examples to illustrate the algorithmic advantages of the 2DRQI and demonstrate its efficiency for the applications. Concluding remarks are in Section 7.

2 2D Rayleigh quotient

In this section, we first introduce the concepts of Rayleigh quotient and Ritz values for the 2DEVP (1.1), and then reveal their approximation property to 2D-eigentriplets.

Definition 2.1. Given an $n \times n$ Hermitian matrix pair $(A, C)$ and an $n \times p$ orthonormal matrix $V$, the $p \times p$ matrix pair $(V^H AV, V^H CV)$ is called a 2D Rayleigh quotient (2DRQ).

Let $(\nu, \theta, z)$ be a 2D-eigentriplet of the 2DRQ $(V^H AV, V^H CV)$ when $V^H CV$ is indefinite, i.e.,

\[
\begin{align*}
(V^H AV - \nu (V^H CV))z &= \theta z, \quad \text{(2.1a)} \\
z^H (V^H CV)z &= 0, \quad \text{(2.1b)} \\
z^H z &= 1, \quad \text{(2.1c)}
\end{align*}
\]

then $(\nu, \theta)$ is called a 2D Ritz value, $Vz$ a 2D Ritz vector, and $(\nu, \theta, Vz)$ a 2D Ritz triplet.

The pair $(V^H AV, V^H CV)$ is called a 2DRQ for two reasons. First, it is analogous to the definition [23, p.288] of the RQ for one matrix. Second, in Section 3.1 we will see that when $C = 0$, a Rayleigh quotient iteration (RQI) like method to solve the 2DEVP (1.1) degenerates to the well-known RQI for an eigenpair of a Hermitian matrix [23, Sec. 4.6] and [29].

The 2DEVP (1.1) can be formulated as the problem of finding the root of the following system of nonlinear equations

\[
F(\mu, \lambda, x) \equiv \begin{bmatrix}
Ax - \mu Cx - \lambda x \\
-x^H Cx/2 \\
-(x^H x - 1)/2
\end{bmatrix} = 0.
\]

When $\mu, \lambda$ and $x$ are real, the Jacobian of the function $F$ is well defined, see e.g. [13, p.65]. When $x$ is complex, the second and third elements of $F$ are not differentiable due to the violation of Cauchy-Riemann conditions [14]. In this case we have the following natural extension of the Jacobian of the nonlinear function $F$. 

\[
\text{when } x = \text{complex: } \frac{\partial F}{\partial x_i} = \left( \frac{\partial F}{\partial x_j} \right)^* 
\]
Definition 2.2. The Jacobian of \( F \) (and the 2DEVP) is defined as
\[
J(\mu, \lambda, x) = \begin{bmatrix}
A - \mu C - \lambda I & -Cx - x \\
-x^H C & 0 \\
-x^H & 0
\end{bmatrix}.
\] (2.2)

For an \( n \times 2 \) orthonormal matrix \( V \) of certain properties, the following theorem shows that if a 2D-eigenvector is near the subspace spanned by \( V \), then the 2D Ritz triplet induced by \( V \) will contain a good approximation to a 2D-eigentriplet. A proof of the theorem will be provided in [16].

Theorem 2.1. Let \((\mu_*, \lambda_*, x_*)\) be a 2D-eigentriplet of \((A, C)\). For any \( \gamma > 0 \), denote \( \mathcal{V}_\gamma \) as the set of \( n \times 2 \) orthonormal matrices \( V \) satisfying \( V^H CV \) is diagonal, \( \det(V^H CV) \leq -\gamma \), and \( |(V^H AV)_{12}| \geq \gamma \). Then there exists constants \( \alpha_1, \alpha_2, \alpha_3 \) only depending on \((A, C, \mu_*, \lambda_*, \gamma)\), such that for any \( V \in \mathcal{V}_\gamma \), let \( \epsilon = \text{dist}(x_*, \text{span}\{V\}) \equiv \min_{v \in \text{span}\{V\}} \|x - v\|_2 \) and assume \( \epsilon < 1 \), there exists a 2D Ritz triplet \((\nu, \theta, Vz)\) that satisfies
\[
|\nu - \mu_*| \leq \alpha_1 \epsilon, \quad |\theta - \lambda_*| \leq \alpha_2 \epsilon^2 \quad \text{and} \quad \|Vz - x_*\| \leq \alpha_3 \epsilon.
\] (2.3)

Theorem 2.1 indicates that to solve the 2DEVP, we should first search for a subspace \( V \) where a good approximation of a 2D-eigentriplet lies in. This is the essential idea guiding the derivation of a 2D Rayleigh quotient iteration in next section.

3 2D Rayleigh Quotient Iteration

The Rayleigh quotient iteration (RQI) is an efficient single-vector iterative method for solving the symmetric eigenvalue problem \([23]\) Section 4.6]. In this section, we derive a RQI-like method to solve the 2DEVP \([1, 1]\).

Theorem 2.1 indicates that the gist of a RQI-like algorithm is how to use the \( k \)-th approximation \((\mu_k, \lambda_k, x_k)\) of a 2D-eigentriplet \((\mu_*, \lambda_*, x_*)\) to obtain a projection subspace \( V_k \) closer to a 2D-eigenvector \( x_* \) and then define the \( k + 1 \)-st approximation \((\mu_{k+1}, \lambda_{k+1}, x_{k+1})\) using a 2D Ritz triplet.

To that end, assume the Jacobian \( J(\mu_k, \lambda_k, x_k) \) defined in (2.2) is nonsingular. Write
\[
\mu_* = \mu_k + \Delta \mu_k, \quad \lambda_* = \lambda_k + \Delta \lambda_k, \quad x_* = x_k + \Delta x_k,
\]
where \( |\Delta \mu_k| \leq \epsilon, |\Delta \lambda_k| \leq \epsilon \) and \( \|\Delta x_k\| \leq \epsilon \) for some small \( \epsilon > 0 \). Then by (1.1a), we have
\[
\begin{bmatrix}
A - \mu_k C - \lambda_k I & -Cx_k - x_k \\
-x_k^H C & 0 \\
-x_k^H & 0
\end{bmatrix}
\begin{bmatrix}
x_k \\
\Delta \mu_k \\
\Delta \lambda_k
\end{bmatrix}
\equiv \begin{bmatrix}
x_* \\
\Delta \mu_k \\
\Delta \lambda_k
\end{bmatrix}
= O(\epsilon^2),
\] (3.1)

This implies that up to the second-order approximation of \( \epsilon \), the vector \( \begin{bmatrix} x_* \\ \Delta \mu_k \\ \Delta \lambda_k \end{bmatrix} \) lies in the null subspace of \( \mathcal{J}_k \). Since the Jacobian \( J(\mu_k, \lambda_k, x_k) \) is assumed to be nonsingular, \( \mathcal{J}_k \) is of full rank and the dimension of the null subspace of \( \mathcal{J}_k \) is 2. Let \( \begin{bmatrix} \tilde{V}_k \\ R \end{bmatrix} \) be a basis matrix of the null subspace of \( \mathcal{J}_k \), where \( \tilde{V}_k \in \mathbb{C}^{n \times 2}, R \in \mathbb{C}^{2 \times 2} \). Then by (3.1), up to the second-order approximation of \( \epsilon \), \( x_* \) lies approximately in \( \text{span}\{\tilde{V}_k\} \). Thus a natural idea is to use the 2D-Ritz triplet based on the Rayleigh quotient induced by \( \text{span}\{\tilde{V}_k\} \) to define the next iterate \((\mu_{k+1}, \lambda_{k+1}, x_{k+1})\).

To compute \( \tilde{V}_k \), one can apply the traditional methods for computing the null space of \( \mathcal{J}_k \), such as the rank revealing QR decomposition \([5\) p.107\]. However, for exploiting the underlying structure
and sparsity of \((A, C)\), we consider the following augmented linear equation of (3.1):

\[
J(\mu_k, \lambda_k, x_k) \begin{bmatrix} X_a \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.
\] (3.2)

By the first block row of (3.2), span\(\{X_a\} \subseteq \text{span}\{\tilde{V}_k\}\). Meanwhile, by the second and third block rows of (3.2), \(\dim(\text{span}\{X_a\}) = 2\). Since \(\dim(\text{span}\{V_k\}) \leq 2\), we have

\[
\text{span}\{X_a\} = \text{span}\{\tilde{V}_k\}.
\] (3.3)

Once \(X_a\) is computed, an orthonormal basis of \(\text{span}\{\tilde{V}_k\}\) is given by

\[V_k = \text{orth}(X_a),\] (3.4)

where \(\text{orth}(X)\) denotes an orthonormal basis for the range of the matrix \(X\). We note that since \(\tilde{J}_k\) is of full rank, \(V_k\) is well-defined (up to an orthogonal transformation) even when \(A - \mu_kC - \lambda_kI\) is singular. The approach described here for computing a basis of a null space of a matrix via an augmented system is inspired by [21, 22, 26] and can be traced back to [24].

After obtaining the orthonormal basis matrix \(V_k\) of the desired projection subspace, we can define the 2DRQ:

\[(A_k, C_k) \equiv (V_k^H AV_k, V_k^H CV_k),\] (3.5)

where for the sake of exposition, without loss of generality, we assume that \(V_k\) is up to another orthogonal transformation such that

\[C_k = V_k^H CV_k = \begin{bmatrix} c_{1,k} & \alpha C_k \\ c_{2,k} & \beta C_k \end{bmatrix} \quad \text{with} \quad c_{1,k} \geq c_{2,k}.\] (3.6)

When \(C_k\) is indefinite, we have the following \(2 \times 2\) 2DEVP of the 2DRQ (3.5):

\[
\begin{cases} 
(A_k - \nu C_k - \theta I)z = 0, \\
\alpha C_k z = 0, \\
\beta C_k z = 1.
\end{cases}
\] (3.7a, 3.7b, 3.7c)

By Section 3 of Part I [28], we know that for the \(2 \times 2\) 2DEVP (3.7), if \(a_{12,k} \neq 0\), where \(a_{ij,k}\) is the \((i, j)\) element of \(A_k\), then there are two distinct 2D-eigentriplets of (3.7)

\[
(\nu(\alpha_{k,i}), \theta(\alpha_{k,i}), \alpha) \quad \text{for} \quad i = 1, 2,
\] (3.8)

where \(\alpha_{k,i} = \pm |a_{12,k}|/a_{12,k}\), and

\[
\nu(\alpha) = \frac{z(\alpha)^H C_k A_k z(\alpha)}{\|C_k z(\alpha)\|^2}, \quad \theta(\alpha) = z(\alpha)^H A_k z(\alpha), \quad z(\alpha) = \begin{bmatrix} \frac{\sqrt{-c_{2,k}}}{c_{1,k} - c_{2,k}} \\ \frac{\sqrt{c_{1,k} - c_{2,k}}}{c_{1,k} - c_{2,k}} \end{bmatrix},
\]

Otherwise, if \(a_{12,k} = 0\), the 2D-eigentriplets of (3.7) are given by

\[
(\nu_1, \theta_1, \alpha) \equiv \begin{bmatrix} a_{11,k} - a_{22,k} \\ c_{1,k} - c_{2,k} \end{bmatrix} \quad \text{and} \quad z(\alpha) = \begin{bmatrix} a_{11,k} - a_{22,k} \\ c_{1,k} - c_{2,k} \end{bmatrix} \quad \text{for} \quad i = 1, 2.
\] (3.9)

From the 2D-eigentriplets (3.8) or (3.9) of \((A_k, C_k)\), we can use the following 2D Ritz triplet to define the \(k + 1\) iterate \((\mu_{k+1}, \lambda_{k+1}, x_{k+1})\):

\[
\mu_{k+1} = \nu(\alpha_{k,j}), \quad \lambda_{k+1} = \theta(\alpha_{k,j}) \quad \text{and} \quad x_{k+1} = V_k z(\alpha_{k,j}),
\] (3.10)
when \(a_{12,k} \neq 0\), where \(j\) is the index such that \(|\mu_k - \nu(\alpha_{k,j})| + |\lambda_k - \theta(\alpha_{k,j})|\) is smaller one for \(j = 1, 2\). Otherwise, when \(a_{12,k} = 0\), the \(k + 1\) iterate \((\mu_{k+1}, \lambda_{k+1}, x_{k+1})\) is given by

\[
\mu_{k+1} = \nu_1, \quad \lambda_{k+1} = \theta_1 \quad \text{and} \quad x_{k+1} = V_k z(1),
\]

(3.11)

where for the sake of convenience, we choose \(\alpha = 1\).

When \(C_k\) is not indefinite, as we may encounter at early stages of iterations, we propose the following strategy for determining the \(k + 1\) iterate \((\mu_{k+1}, \lambda_{k+1}, x_{k+1})\). First, since the exact 2D-eigenvector \(x_s\) satisfies \(x_s^H C x_s = 0\), we choose a unit vector \(x_{k+1}\) to minimize \(|x^H C x|\) for \(x \in \text{span}\{V_k\}\). Specifically, when \(c_{1,k} \neq c_{2,k}\), up to a scaling, \(x_{k+1}\) is uniquely determined by

\[
x_{k+1} = \begin{cases} V_k e_1, & |c_{1,k}| < |c_{2,k}|, \\ V_k e_2, & |c_{1,k}| > |c_{2,k}|. \end{cases}
\]

(3.12)

When \(c_{1,k} = c_{2,k}\), we use

\[
x_{k+1} = V_k w / \|V_k w\|,
\]

(3.13)

where \(w\) is a uniformly distributed random vector on \([-1, 1]\). Once \(x_{k+1}\) is determined by (3.12) or (3.13), \((\mu_{k+1}, \lambda_{k+1})\) is obtained by solving the following least squares problem:

\[
(\mu_{k+1}, \lambda_{k+1}) = \arg\min_{\nu, \theta \in \mathbb{R}} \|Ax_{k+1} - \nu C x_{k+1} - \theta x_{k+1}\|.
\]

(3.14)

### 3.1 Algorithm outline

Algorithm 1 summarizes the derivation in the previous section for an algorithm to compute a 2D-eigentriplet. It is called 2DRQI since the algorithm is an extension of the RQI for a Hermitian matrix \(A\). By (3.2) and (3.4), we see that when \(A - \mu_k C - \lambda_k I\) is nonsingular,

\[
\text{span}\{V_k\} = \text{span}\{(A - \mu_k C - \lambda_k I)^{-1} x_k, (A - \mu_k C - \lambda_k I)^{-1} C x_k\}.
\]

If \(C = 0\) and \(\lambda_k\) is the Rayleigh quotient of \(A\) and \(x_k\), then \(\text{span}\{V_k\} = \text{span}\{(A - \lambda_k I)^{-1} x_k\}\) is the one used in the classical RQI, see e.g. \([23\text{, Section 4.6}]\). A few remarks of Algorithm 1 are in order.

1. A proper initial \((\mu_0, \lambda_0, x_0)\) is critical for the rapid convergence of the algorithm. The initial pair \((\mu_0, \lambda_0)\) should be close to a 2D-eigenvalue of interest. For the initial vector \(x_0\), we first compute a 2D-Ritz triplet \((\nu, \theta, z)\) of 2DRQ \((X^H AX, X^H CX)\), where \(X\) consists of the two orthonormal eigenvectors corresponding to two eigenvalues of \(A - \mu_0 C\) closest to \(\lambda_0\), and then set \(x_0\) to be the 2D-Ritz vector \(X z\) associated with the 2D-Ritz value \((\nu, \theta)\) closest to \((\mu_0, \lambda_0)\).

2. To solve the linear system (3.2), we should exploit the structure and sparsity of matrices \(A\) and \(C\). See numerical examples in Section 6.

3. We use an estimate \(\eta_1\) of the backward error of approximate 2D-eigentriplet \((\mu_k, \lambda_k, x_k)\) as the stopping criterion, see Theorem 4.2 in Section 4.

In Section 6 we will provide examples to demonstrate that the 2DRQI is locally quadratically convergent. A formal convergence analysis of the 2DRQI will be presented in Part III of this paper [16].
If we prove by construction. We first find the desired perturbation matrix

\[ Q_e = \text{orth}(X_a) \]

Define \( \delta \) be used as the stopping criterion of the 2DRQI (Algorithm 1). In Section 5.2, the notion of the distance to instability in Section 5.2. We start with the following theorem.

\[ \text{backward error analysis of the 2DEVP will be extended to applications for the computation of the stopping criterion in an iterative algorithm. In this section, we provide a backward error analysis for an approximate 2D-eigentriplet of the 2DEVP (1.1). The resulting backward error estimate can be used as the stopping criterion of the 2DRQI (Algorithm 1). In Section 5.2 the notion of the backward error analysis of the 2DEVP will be extended to applications for the computation of the distance to instability in Section 5.2. We start with the following theorem.} \]

**Theorem 4.1.** Let \( (\hat{\mu}, \hat{\lambda}, \hat{x}) \) be an approximate 2D-eigentriplet of \((A, C)\) with \( \hat{\mu}, \hat{\lambda} \in \mathbb{R} \) and \( \|\hat{x}\| = 1 \). Then there exist Hermitian matrices \( \delta A \) and \( \delta C \) such that

\[ (i) \ C + \delta C \ is \ indefinite, \ and \]

\[ (ii) \ (\hat{\mu}, \hat{\lambda}, \hat{x}) \ is \ an \ exact \ 2D-eigentriplet \ of \ the \ perturbed \ matrix \ pair \ (A + \delta A, C + \delta C): \]

\[ \begin{cases} 
(A + \delta A - \hat{\mu}(C + \delta C)) \hat{x} = \hat{\lambda}\hat{x}, \\
\hat{x}^H(C + \delta C)\hat{x} = 0, \\
\hat{x}^H\hat{x} = 1. 
\end{cases} \]  

(4.1a) (4.1b) (4.1c)

**Proof.** We prove by construction. We first find the desired perturbation matrix \( \delta C \) to satisfy (4.1b). Define \( \delta C = -(\hat{x}^H C \hat{x}) I. \) Then

\[ \hat{x}^H(C + \delta C)\hat{x} = 0. \]

(4.2)

If \( C + \delta C \) is indefinite, then (4.1b) holds by taking \( \delta C = \delta C. \) If \( C + \delta C \) is not indefinite, then \( C + \delta C \) is positive or negative semi-definite. Equation (4.2) implies \( (C + \delta C)\hat{x} = 0. \) Let \( Q \) be an orthogonal matrix with \( Q e_1 = \hat{x}. \) Then we have \( Q^H(C + \delta C)Q e_1 = 0 \) and

\[ Q^H(C + \delta C) = \begin{bmatrix} 0 & 0 \\ 0 & \hat{C}_1 \end{bmatrix}, \]

4 Backward error analysis of 2DEVP

It is well-known that the backward error of an approximate solution is a reliable and effective stopping criterion in an iterative algorithm. In this section, we provide a backward error analysis for an approximate 2D-eigentriplet of the 2DEVP (1.1). The resulting backward error estimate can be used as the stopping criterion of the 2DRQI (Algorithm 1). In Section 5.2, the notion of the backward error analysis of the 2DEVP will be extended to applications for the computation of the distance to instability in Section 5.2. We start with the following theorem.

**Theorem 4.1.** Let \( (\hat{\mu}, \hat{\lambda}, \hat{x}) \) be an approximate 2D-eigentriplet of \((A, C)\) with \( \hat{\mu}, \hat{\lambda} \in \mathbb{R} \) and \( \|\hat{x}\| = 1 \). Then there exist Hermitian matrices \( \delta A \) and \( \delta C \) such that

\[ (i) \ C + \delta C \ is \ indefinite, \ and \]

\[ (ii) \ (\hat{\mu}, \hat{\lambda}, \hat{x}) \ is \ an \ exact \ 2D-eigentriplet \ of \ the \ perturbed \ matrix \ pair \ (A + \delta A, C + \delta C): \]

\[ \begin{cases} 
(A + \delta A - \hat{\mu}(C + \delta C)) \hat{x} = \hat{\lambda}\hat{x}, \\
\hat{x}^H(C + \delta C)\hat{x} = 0, \\
\hat{x}^H\hat{x} = 1. 
\end{cases} \]  

(4.1a) (4.1b) (4.1c)

**Proof.** We prove by construction. We first find the desired perturbation matrix \( \delta C \) to satisfy (4.1b). Define \( \delta C = -(\hat{x}^H C \hat{x}) I. \) Then

\[ \hat{x}^H(C + \delta C)\hat{x} = 0. \]

(4.2)

If \( C + \delta C \) is indefinite, then (4.1b) holds by taking \( \delta C = \delta C. \) If \( C + \delta C \) is not indefinite, then \( C + \delta C \) is positive or negative semi-definite. Equation (4.2) implies \( (C + \delta C)\hat{x} = 0. \) Let \( Q \) be an orthogonal matrix with \( Q e_1 = \hat{x}. \) Then we have \( Q^H(C + \delta C)Q e_1 = 0 \) and

\[ Q^H(C + \delta C) = \begin{bmatrix} 0 & 0 \\ 0 & \hat{C}_1 \end{bmatrix}, \]
The following theorem provides a tight computable estimate of

\[ \text{Theorem 4.2.} \]

Let \( \eta \) and \((4.1b) \) holds. Then it is straightforward to verify that \((4.1a) \) holds with \( \eta \). This completes the proof.

By Theorem \([4.1]\) the backward error \( \eta \) of an approximate 2D-eigentriplet \((\hat{\mu}, \hat{\lambda}, \hat{x})\) of \((A, C)\) is defined as the infimum of normwise relative perturbation of \( A \) and \( C \) such that \((\hat{\mu}, \hat{\lambda}, \hat{x})\) is an exact 2D-eigentriplet of the perturbed 2DEVP \((4.1)\):

\[ \eta \equiv \inf \left\{ \epsilon \mid \exists \delta A, \delta C \text{ s.t. } ||\delta A|| \leq \epsilon ||A||, ||\delta C|| \leq \epsilon ||C||, \text{ and } (4.1) \text{ holds} \right\}. \]  

The following theorem provides a tight computable estimate of \( \eta \).

**Theorem 4.2.** Let \((\hat{\mu}, \hat{\lambda}, \hat{x})\) be an approximate 2D-eigentriplet of \((A, C)\) with \( \hat{\mu}, \hat{\lambda} \in \mathbb{R} \) and \( ||\hat{x}|| = 1 \), and

\[ \eta_1 = \max \left\{ \frac{||\gamma_A||}{||A||}, \frac{||\gamma_C||}{||C||}, \frac{||r||}{||A|| + ||\hat{\mu}|| ||C||} \right\}, \]  

where \( \gamma_A = \hat{x}^H A \hat{x} - \hat{\lambda}, \gamma_C = \hat{x}^H C \hat{x} \) and \( r = (A - \hat{\mu} C - \hat{\lambda} I) \hat{x} \). Then the backward error \( \eta \) of \((\hat{\mu}, \hat{\lambda}, \hat{x})\) defined in \((4.4)\) satisfies

\[ \eta_1 \leq \eta \leq \sqrt{2} \eta_1. \]  

**Proof.** We first prove the lower bound \( \eta \geq \eta_1 \). For any \( \delta A \) and \( \delta C \) satisfying the perturbed 2DEVP \((4.1)\), by \((4.1a)\) and \((4.1b)\), we have \( \hat{x}^H (A + \delta A) \hat{x} = \hat{\lambda} \). Hence by the definition \((4.4)\) of \( \eta \), we have

\[ \eta \geq \frac{||\delta C||}{||C||} \geq \frac{\hat{x}^H \delta C \hat{x}}{||C||} = \frac{||\hat{x}^H C \hat{x}||}{||C||} = \frac{||\gamma_C||}{||C||}. \]  

and

\[ \eta \geq \frac{||\delta A||}{||A||} \geq \frac{||\hat{x}^H \delta A \hat{x}||}{||A||} = \frac{||\hat{x}^H A \hat{x} - \hat{\lambda}||}{||A||} = \frac{||\gamma_A||}{||A||}. \]  

Now by \((4.1a)\), for the norm of the residual vector \( r \):

\[ ||r|| = ||(A - \hat{\mu} C - \hat{\lambda} I) \hat{x}|| = ||(\delta A - \hat{\mu} \delta C) \hat{x}|| \leq ||\delta A|| + ||\hat{\mu}|| ||\delta C|| \leq (||A|| + ||\hat{\mu}|| ||C||) \epsilon. \]  

Therefore, by the definition \((4.4)\) of \( \eta \), we have

\[ \eta \geq \frac{||r||}{||A|| + ||\hat{\mu}|| ||C||}. \]  

Combining \((4.7)\), \((4.8)\) and \((4.9)\), we have \( \eta \geq \eta_1 \).

The gist of finding the upper bound of \( \eta \), namely \( \eta \leq \sqrt{2} \eta_1 \), is to find two particular perturbation matrices \( \delta A \) and \( \delta C \) such that

\[ \begin{cases} \delta A \hat{x} - \hat{\mu} \delta C \hat{x} = -r, & (4.10a) \\ \hat{x}^H \delta C \hat{x} = -\gamma_C, & (4.10b) \end{cases} \]
and
\[ C + \delta C \text{ is indefinite,} \quad (4.11) \]
and then derive the upper bound of \( \eta \) from the upper bound of \( \max \left\{ \frac{\| \delta A \|}{\| A \|}, \frac{\| \delta C \|}{\| C \|} \right\} \). We first note that we can safely discard the condition (4.11). This is due to the fact that when (4.10) holds, using the same arguments as in the proof of Theorem 4.1 we can add infinitesimal perturbation to \( \delta A, \delta C \) to guarantee (4.10) and (4.11) hold. Since the backward error \( \eta \) takes the infimum, the quantity \( \max \left\{ \frac{\| \delta A \|}{\| A \|}, \frac{\| \delta C \|}{\| C \|} \right\} \) is still an upper bound.

To find \( \delta A \) and \( \delta C \) satisfying (4.10), let us define
\[ \tilde{a} \equiv - \frac{\| A \|}{\| A \| + |\tilde{\mu}| \| C \|} (I - \tilde{x}\tilde{x}^H) r, \quad \tilde{c} \equiv \frac{\text{sign}(\tilde{\mu}) \| C \|}{\| A \| + |\tilde{\mu}| \| C \|} (I - \tilde{x}\tilde{x}^H) r, \]
where \( \text{sign}(\tilde{\mu}) = \tilde{\mu} / |\tilde{\mu}| \). Then \( \tilde{a} \) and \( \tilde{c} \) are orthogonal to \( \tilde{x} \) and satisfy
\[ \tilde{a} - \tilde{\mu} \tilde{c} = -(I - \tilde{x}\tilde{x}^H) r. \]

Next, let us define \( a \equiv (-\tilde{x}^H r - \tilde{\mu}_C \tilde{x}^H) \tilde{x} + \tilde{a} \) and \( c \equiv -\gamma_C \tilde{x} + \tilde{c} \). Then it holds that
\[ \begin{cases} a - \tilde{\mu} c = (\tilde{a} - \tilde{\mu} \tilde{c}) - \tilde{x}^H r \tilde{x} = -(I - \tilde{x}\tilde{x}^H)r - \tilde{x}^H r \tilde{x} = -r, \\ \tilde{x}^H c = -\gamma_C. \end{cases} \]

From the vectors \( a \) and \( c \), we can construct Hermitian matrices \( \delta A \) and \( \delta C \), say real constant multiples of Householder reflections [11, Theorem 2.1.13] satisfying \( \delta A \tilde{x} = a, \delta C \tilde{x} = c \) and \( \| \delta A \| = \| a \| \) and \( \| \delta C \| = \| c \| \). Then \( \delta A \) and \( \delta C \) are desired matrices satisfying (4.10).

For \( \delta A \), by the definition of \( r \), we have \( -\tilde{x}^H r - \tilde{\mu}_C \tilde{x}^H = -\gamma_A \), and thus
\[ \frac{\| \delta A \|}{\| A \|} = \frac{\| a \|}{\| A \|} = \frac{-\| \gamma_A \tilde{x} + \tilde{a} \|}{\| A \|} = \frac{\sqrt{\gamma_A^2 + \| \tilde{a} \|^2}}{\| A \|} = \sqrt{\left( \frac{\| \gamma_A \|}{\| A \|} \right)^2 + \left( \frac{\| \tilde{a} \|}{\| A \|} \right)^2}, \]
\[ \leq \sqrt{\left( \frac{\| \gamma_A \|}{\| A \|} \right)^2 + \left( \frac{\| r \|}{\| A \| + |\tilde{\mu}| \| C \|} \right)^2} \leq \sqrt{2} \eta_1. \quad (4.12) \]
By an analogous derivation, for \( \delta C \), we have
\[ \frac{\| \delta C \|}{\| C \|} \leq \sqrt{2} \eta_1. \quad (4.13) \]
Combining the upper bounds (4.12) and (4.13), we have \( \eta \leq \sqrt{2} \eta_1 \). This completes the proof. \( \square \)

5 Applications

5.1 Minmax of Rayleigh Quotients

In Sections 2.1 and 7.1 of Part I [28], we discussed an application of the 2DEVP for the minmax of Rayleigh quotients (RQminmax) of \( n \times n \) Hermitian matrices \( A \) and \( B \):
\[ \min_{x \neq 0} \max \left\{ \frac{x^H Ax}{x^H x}, \frac{x^H Bx}{x^H x} \right\} \]
(5.1)
In [28, Theorem 2.1], we have shown that the RQminmax (5.1) can be divided into three cases. For Case-I and Case-II, one need to calculate the eigen-subspace corresponding to the minimum eigenvalues \( \lambda_A \) and \( \lambda_B \) of \( A \) and \( B \). This could be computational expensive when the multiplicity of \( \lambda_A \) or \( \lambda_B \) is larger than 1. In the following we derive a computational-friendly variant of [28, Theorem 2.1], which only need to calculate an eigenvector corresponding to \( \lambda_A \) and \( \lambda_B \), regardless of the multiplicities of \( \lambda_A \) and \( \lambda_B \).
Theorem 5.1. Let $\lambda_A$ be the minimum eigenvalue and $x_A$ be a corresponding eigenvector of $A$, $\lambda_B$ be the minimum eigenvalue and $x_B$ be a corresponding eigenvector of $B$, $\rho_A(x) = x^H A x / x^H x$ and $\rho_B(x) = x^H B x / x^H x$ be the Rayleigh quotients of $A$ and $B$, respectively.

I. If $\lambda_A \geq \rho_B(x_A)$, then $x_A$ is a solution of the RQminmax (5.1);

II. If $\lambda_B \geq \rho_A(x_B)$, then $x_B$ is a solution of the RQminmax (5.1);

III. Otherwise, namely $\lambda_A < \rho_B(x_A)$ and $\lambda_B < \rho_A(x_B)$, let $\mu_*$ be an optimizer of the eigenvalue optimization problem (EVopt):

$$\max_{\mu \in \mathbb{R}} \lambda_{\min}(A - \mu C),$$

and $V_{\mu_*}$ be the set of eigenvectors $x_*$ corresponding to $\lambda_{\min}(A - \mu_* C)$ and $x_*^H C x_* = 0$, where $C = A - B$, then (a) $\mu_*$ $\in$ $[0, 1]$, (b) $V_{\mu_*} \neq \emptyset$ and (c) any $x_* \in V_{\mu_*}$ is a solution of the RQminmax (5.1).

Proof. For Case-I, we note that for any $x \neq 0$,

$$\max\{\rho_A(x), \rho_B(x)\} \geq \rho_A(x) \geq \lambda_A.$$

On the other hand,

$$\max\{\rho_A(x_A), \rho_B(x_A)\} = \lambda_A.$$

Thus $x_* = x_A$ is a solution of the RQminmax (5.1).

Case-II can be proven by exchanging the roles of $A$ and $B$ in the proof of Case-I.

For Case-III we need to prove that under the conditions $\lambda_A < \rho_B(x_A)$ and $\lambda_B < \rho_A(x_B)$, we have the results (a), (b) and (c). To that end, let $S_A$ and $S_B$ be orthonormal bases of the eigenspaces of $\lambda_A$ and $\lambda_B$, respectively, and denote $\theta_A = \lambda_{\min}(S_B^H AS_B)$, $\theta_B = \lambda_{\min}(S_A^H BS_A)$. Let us divide Case-III into subcases based on the relation between $\lambda_A$ ($\lambda_B$) and $\theta_B$ ($\theta_A$). If $\lambda_A < \theta_B$ and $\lambda_B < \theta_A$, then it belongs to the case of Theorem 2.1(III) of Part I [28], and

$$\arg\min_{x \neq 0} \left(\max\{\rho_A(x), \rho_B(x)\}\right) = V_{\mu_*}.$$

The results (b) and (c) are immediately followed. The result (a) is contained in [28, Theorem 7.1], namely

$$\arg\max_{\mu \in \mathbb{R}} g(\mu) = \arg\max_{\mu \in (0, 1)} g(\mu),$$

where $g(\mu) = \lambda_{\min}(A - \mu C)$. Consequently, we only need to consider the subcase where the inequalities $\lambda_A < \theta_B$ and $\lambda_B < \theta_A$ do not hold simultaneously. This implies that at least one of the following conditions holds: (i) $\lambda_A \geq \theta_B$; (ii) $\lambda_B \geq \theta_A$. Let us consider (i) in the following. (ii) can be shown analogously.

By the condition under Case-III i.e., $\lambda_A < \rho_B(x_A)$ and $\lambda_B < \rho_A(x_B)$, we have

$$-x_A^H C x_A = -\lambda_A + \rho_B(x_A) > 0, \quad -x_B^H C x_B = -\rho_A(x_B) + \lambda_B < 0. \quad (5.3)$$

Note that $x_A$ and $x_B$ are also eigenvectors of $A - \cdot C = A$ and $A - \cdot C = B$, respectively. Thus by [28, Theorem 4.5], the inequalities in (5.3) imply that

$$g^{(-)}(0) = \lambda_{\max}(-S_B^H CS_A) \geq -x_A^H C x_A > 0, \quad g^{(+)}(1) = \lambda_{\min}(-S_B^H CS_B) \leq -x_B^H C x_B < 0. \quad (5.4)$$

Let $\mu_*$ be an optimizer of EVopt (5.2), then by (5.4) and the concavity of $g(\mu)$, we conclude that $\mu_* \in [0, 1]$. This completes the proof of the result (a).
For result (b), note that \((\mu_*, \lambda_{\min}(A - \mu_*C))\) is a 2D eigenvalue of \((A, C)\) according to \([28, \text{Theorem 5.1}]\). Then the associated 2D-eigenvectors belong to \(V_{\mu_*}\) and thus we obtain the result (b).

To prove the result (c), we first calculate the optimal value of the EVopt \([5, 2]\). Since \(\lambda_A \geq \theta_B\), by denoting \(z_B\) as the eigenvector of \(S_A^HBS_A\) corresponding to \(\theta_B\) and the definition of \(S_A\), we have

\[
\rho_A(S_Az_B) = \lambda_A \geq \theta_B = \rho_B(S_Az_B).
\]

Let \(\tilde{x}_A = S_Az_B\). Then \(-\tilde{x}_A^HCS_A \leq 0\) and thus by \([28, \text{Theorem 4.5}]\),

\[
g'_+(0) = \lambda_{\min}(-S_A^HCS_A) \leq -\tilde{x}_A^HCS_A \leq 0. \tag{5.5}
\]

According to \((5.4), (5.5)\) and the concavity of \(g(\mu) = \lambda_{\min}(A - \mu C)\) (see \([28, \text{Theorem 4.1}]\)), 0 is an optimizer of EVopt \((5.2)\) and thus

\[
\max_{\mu \in \mathbb{R}} \lambda_{\min}(A - \mu C) = \lambda_A. \tag{5.6}
\]

Now for any \(x_\star \in V_{\mu_*}\), we have

\[
\rho_B(x_\star) = \rho_A(x_\star) = \lambda_{\min}(A - \mu_*C) = \max_{\mu \in \mathbb{R}} \lambda_{\min}(A - \mu C) = \lambda_A = \min \{\rho_A(x), \rho_B(x)\},
\]

where the first equality results from \(x_\star^HCS_\star = 0\), the second equality results from the fact that \(\rho_A(x_\star) = \rho_A - \mu_*C(x_\star)\) and \(x_\star\) is an eigenvector corresponding to \(\lambda_{\min}(A - \mu_*C)\), the third equality comes from \(\mu_*\) is an optimizer, the fourth equality results from \((5.6)\) and the last equality holds according to Theorem 2.1(I) of Part I \([28]\) as \(\theta_B \leq \lambda_A\). Thus \(x_\star\) is the solution to the RQminmax \((5.1)\).

This completes the proof of the result (c).

By Definition 5.1 of Part I \([28]\), for Case III of Theorem 5.1 we know that \((\mu_*, \lambda_*)\) with \(\lambda_* = \lambda_{\min}(A - \mu_*C)\) is the minimum 2D-eigenvalue of \((A, C)\). On the other hand, by the definition of \(V_{\mu_*}\), up to a scaling, \(x_\star \in V_{\mu_*}\) if and only if \(x_\star\) is a 2D-eigenvector associated with \((\mu_*, \lambda_*)\). Thus the RQminmax \((5.1)\), excluding Cases I and II in Theorem 5.1, turns to calculating a minimum 2D-eigenvalue and the corresponding 2D-eigenvector of \((A, C)\).

Based on the fact that \(\mu_*\) of the minimum 2D-eigenvalue \((\mu_*, \lambda_*)\) must be in \([0, 1]\), we can derive a combination of the bisection search and the 2DRQI (Algorithm 1.1). Starting with the search interval \([a, b] = [0, 1]\) of the EVopt \((5.2)\), let

\[
\mu_0 = \frac{1}{2}(a + b) \quad \text{and} \quad \lambda_0 = \lambda_{\min}(A - \mu_0C) \tag{5.7}
\]

and \(x_0\) be the one as recommended for the 2DRQI (Algorithm 1.1). Then we can use the 2DRQI with the initial \((\mu_0, \lambda_0, x_0)\) to find a 2D-eigentriplet \((\mu, \lambda, \tilde{x})\) of \((A, C)\).

If \(\hat{\lambda} = \lambda_{\min}(A - \hat{\mu}C)\), then according to \([28, \text{Corollary 5.1}]\),

\[
\hat{\lambda} = \max_{\mu \in \mathbb{R}} \lambda_{\min}(A - \mu C).
\]

Thus \(\hat{\mu}\) is an optimizer of EVopt \((5.2)\) and \(\tilde{x}\) is the solution of RQminmax \((5.1)\).

If \(\hat{\lambda} \neq \lambda_{\min}(A - \hat{\mu}C)\), or the 2DRQI does not converge, then we can use the concavity of \(g(\mu) = \lambda_{\min}(A - \mu C)\) (see \([28, \text{Theorem 4.1}]\)) to bisect the interval \([a, b]\) and run the 2DRQI with a new initial \((\mu_0, \lambda_0, x_0)\). This bisection search strategy works due to the facts that

- if \((x^{(n)})^HCS(x^{(n)}) \leq 0\), where \(x^{(n)}\) is an eigenvector corresponding to \(\lambda_0 = \lambda_{\min}(A - \mu_0C)\), then by \([28, \text{Theorem 4.5}]\), \(g'_-(\mu_0) = \lambda_{\max}(-X_0(\mu_0)^HCSX_0(\mu_0)) \geq 0\), where \(X_0(\mu)\) is an orthonormal basis of the eigen-subspace of \(\lambda_{\min}(A - \mu C)\), and there is an optimizer \(\mu_*\) of the EVopt \((5.2)\) such that \(\mu_* \geq \mu_0\). Consequently, we set \(a = \mu_0\) to half the search interval.
• if \((x^{(n)})^H C x^{(n)} > 0\), then by [26] Theorem 4.5, \(g_+ (\mu_0) = \lambda_{\min}(-X_0(\mu_0)^HCX_0(\mu_0)) < 0\) and there is an optimizer \(\mu_\ast\) of the EVopt (5.2) such that \(\mu_\ast \leq \mu_0\). Consequently, we set \(b = \mu_0\) to half the search interval.

A combination of the 2DRQI (Algorithm 1) and the bisection search described above is summarized in Algorithm 2 for solving the RQminmax (5.1), where in line 9 we use whether

\[
|\hat{\lambda} - \lambda_{\min}(A - \hat{\mu}C)| = |\hat{\lambda} - \lambda_{\min}((1 - \hat{\mu})A + \hat{\mu}B)| < \text{reltol} \cdot (|1 - \hat{\mu}||A| + |\hat{\mu}||B|)
\]

to numerically check whether \(\hat{\lambda} = \lambda_{\min}(A - \hat{\mu}C)\).

**Algorithm 2 Minimax of two RQs**

**Input:** \(n\)-by-\(n\) Hermitian matrices \(A\) and \(B\), tolerance values \(\text{reltol}\) and \(\text{backtol}\).

**Output:** approximate solution \(\hat{x}\) and the optimal value \(\hat{\lambda}\) of RQminmax (5.1).

1: compute a minimum eigenpair \((\lambda_A, x_A)\) of \(A\). If \(\lambda_A \geq \rho_B(x_A)\), then return \((\hat{\lambda}, \hat{x}) = (\lambda_A, x_A)\).
2: compute a minimum eigenpair \((\lambda_B, x_B)\) of \(B\). If \(\lambda_B \geq \rho_A(x_B)\), then return \((\hat{\lambda}, \hat{x}) = (\lambda_B, x_B)\).
3: set \([a, b] = [0, 1]\);
4: for \(k = 0, 1, 2, \ldots\), do
5:   set \(\mu_0 = (a + b)/2;\)
6:   compute two smallest eigenpairs \((\lambda_n, x^{(n)}), (\lambda_{n-1}, x^{(n-1)})\) of \(A - \mu_0C\);
7:   compute the minimum 2D-Ritz triplet \((\nu, \theta, z)\) of \((Z^H A Z, Z^H C Z)\), where \(Z = [x^{(n-1)} x^{(n)}]^T\);
8:   apply the 2DRQI (Algorithm 1) with the initial \((\mu_0, \lambda_0 = \lambda_n, x_0 = Zz)\) and the backward error tolerance \(\text{backtol}\).
9:   if 2DRQI converges to \((\hat{\mu}, \hat{\lambda}, \hat{x})\) and \(|\hat{\lambda} - \lambda_{\min}(A - \hat{\mu}C)| < \text{reltol} \cdot (|1 - \hat{\mu}||A| + |\hat{\mu}||B|)\) then
10:      return \((\hat{\lambda}, \hat{x})\).
11:   else
12:      if \((x^{(n)})^H C x^{(n)} \leq 0\) then
13:         update \(a = \mu_0\).
14:      else
15:         update \(b = \mu_0\).
16:      end if
17:   end if
18: end for

For a robust implementation, we need to deal with the extreme case where the 2DRQI (Algorithm 1) does not converge to the correct 2D eigentriplet even when \(b - a\) is sufficiently small. Specifically, we terminate the outer iteration when \(b - a < \text{abstol}\), where \(\text{abstol}\) is a prescribed tolerance. According to our analysis, \(\mu_\ast \in [a, b]\) and thus \(\hat{\mu} = (a + b)/2, \hat{\lambda} = \lambda_{\min}(A - \hat{\mu}C)\) is already sufficiently close to the optimizer of the EVopt (5.2). The remaining issue is how to recover an approximation \(\hat{x}_\ast\) to the solution of the RQminmax (5.1), namely \(\hat{x}_\ast^H C \hat{x}_\ast \approx 0\).

To that end, we compute eigenvectors \(x_a, x_b\) associated with the minimum eigenvalue of \(A - aC\) and \(A - bC\). After proper scaling we can assume \(\|x_a\| = \|x_b\| = 1\), and \(x_a^H x_b\) is real and non-negative. If \(x_a^H C x_a = 0\) or \(x_b^H C x_b = 0\) or \(x_a \parallel x_b\), i.e., \(x_a = \alpha x_b\) for a constant \(\alpha\), we set \(\hat{x} = x_a\) or \(\hat{x} = x_b\) or \(\hat{x} = x_a\). Otherwise, we denote \(q_a = x_a, q_b = \frac{(I - x_a x_a^H) x_b}{\|I - x_a x_a^H\| x_b\|} = \frac{(I - x_a x_a^H) x_b}{\sqrt{1 - \|x_a^H x_b\|^2}}\) and \(u(\theta) = \cos \theta q_a + \sin \theta q_b\). We find \(\theta\) that minimizes \(|u(\theta)^H C u(\theta)|\) among \([0, \arccos(x_a^H x_b)]\). We then set \(\hat{x} = u(\theta)\) and return \((\hat{\lambda}, \hat{x})\) as the solution of the RQminmax (5.1). The following proposition shows that this strategy is valid under mild conditions.

**Proposition 5.1.** If \(\lambda_{\min}(A - aC)\) and \(\lambda_{\min}(A - bC)\) are simple eigenvalues, then

(a) \(\hat{x}^H A \hat{x} = \hat{x}^H B \hat{x}\).
(b) \( \| (A - \mu_s C) \widehat{x} - \lambda_s \widehat{x} \| \leq 6(b - a)\| C \|. \)

(c) \( | \widehat{x}^H A \widehat{x} - \lambda_s | \leq 6(b - a)\| C \|. \)

Proof. First consider the case \( x_a^H C x_a = 0 \). Let \( \widehat{x} = x_a \), then the result (a) holds. The result (c) holds due to the fact that

\[
| \widehat{x}^H A \widehat{x} - \lambda_s | = | \lambda_{\min}(A - aC) - \lambda_{\min}(A - \mu_s C) | \leq (b - a)\| C \|,
\]

where the last inequality comes from Weyl theorem [8, p. 203, Corollary 4.10]. The result (b) holds since

\[
(A - \mu_s C)x_a - \lambda_s x_a = (A - aC)x_a - \lambda_{\min}(A - aC)x_a - (\mu_s - a)C x_a - (\lambda_s - \lambda_{\min}(A - aC)) x_a
\]

\[= -(\mu_s - a)C x_a - (\lambda_s - \lambda_{\min}(A - aC)) x_a
\]

and

\[
\| (A - \mu_s C)x_a - \lambda_s x_a \| \leq 2(b - a)\| C \|.
\]

The argument for the case \( x_b^H C x_b = 0 \) is similar. The remaining is the case where both \( x_a^H C x_a \) and \( x_b^H C x_b \) are nonzero.

According to the concavity of \( g(\mu) = \lambda_{\min}(A - \mu C) \) and \( a \leq \mu_s \leq b, \ g'(a) \geq 0 \) \( \text{and} \ g'(b) \leq 0 \). This implies \( x_a \parallel x_b \), and thus \( q_a, q_b \) are well-defined.

Note that \( q_a^H q_b = 0 \) and \( \| q_a \| = \| q_b \| = 1 \). Then \( \| u(\theta) \| = 1 \) \( \forall \theta \). Furthermore, let \( \theta_a = 0, \theta_b = \arccos(x_a^H x_b) \). Straight calculation shows that \( u(\theta_a) = x_a \) \( \text{and} \ u(\theta_b) = x_b \).

Define function \( h(\theta) = u(\theta)^H C u(\theta) \). Then \( h(\theta_a) = x_a^H C x_a < 0 \) \( \text{and} \ h(\theta_b) = x_b^H C x_b > 0 \). By continuity, there exists \( \theta \in (\theta_a, \theta_b) \) such that \( h(\theta) = 0 \). Thus \( \min_{\theta \in [\theta_a, \theta_b]} | u(\theta)^H C u(\theta) | = 0 \) \( \text{and we have} \)

\( \hat{x}^H C \hat{x} = 0 \). The result (a) is obtained.

We next show \( \hat{x} \) lies approximately in the eigen-subspace of \( A - \mu_s C \) in the backward sense. Denote \( r_a, r_b \) such that \( (A - \mu_s C)x_a = \lambda_s x_a + r_a, (A - \mu_s C)x_b = \lambda_s x_b + r_b \). We have

\[
(A - \mu_s C)\hat{x} = (A - \mu_s C) \left( \cos \hat{\theta} x_a + \sin \hat{\theta} \frac{(I - x_a x_a^H)x_b}{\sqrt{1 - (x_a^H x_b)^2}} \right)
\]

\[= \cos \hat{\theta} \lambda_s x_a + r_a + \sin \hat{\theta} \frac{\lambda_s x_b + r_b - x_a^H x_b (\lambda_s x_a + r_a)}{\sqrt{1 - (x_a^H x_b)^2}}
\]

\[= \lambda_s \hat{x} + \cos \hat{\theta} r_a + \sin \hat{\theta} \frac{r_b - r_a x_a^H x_b}{\sqrt{1 - (x_a^H x_b)^2}}
\]

\[\equiv \lambda_s \hat{x} + \hat{r},
\]

with

\[
\| \hat{r} \| \leq \| r_a \| + \sin \theta_b \frac{\| r_b - r_a x_a^H x_b \|}{\sqrt{1 - (x_a^H x_b)^2}}
\]

\[= \| r_a \| + \| r_b - r_a x_a^H x_b \|
\]

\[\leq 2\| r_a \| + \| r_b \|.
\]

Note that using the same argument in \((5.8)\) \(\text{and} \ (5.9)\), we can obtain \( \| r_a \| \leq 2(b - a)\| C \| \) \(\text{and} \ \| r_b \| \leq 2(b - a)\| C \| \). Thus we have

\[
\| \hat{r} \| \leq 6(b - a)\| C \|
\]

and we reach the result (b). Multiplying \( \hat{x}^H \) on the left of \((5.10)\) leads to the result (c).

Numerical examples for large scale RQminmax \((5.1)\) arising from signal processing will be presented in Section \(6\).
5.2 The distance to instability

As discussed in Section 2.2 of Part I [28], the distance to instability (DTI) of a stable matrix \( \hat{A} \in \mathbb{C}^{n \times n} \) can be recast as the eigenvalue optimization as follows:

\[
\beta(\hat{A}) \equiv \min \left\{ \| E \| \mid \hat{A} + E \text{ is unstable} \right\} = \min_{\mu \in \mathbb{R}} \lambda_n(\mu), \quad (5.12)
\]

where \( \lambda_n(\mu) \) is the smallest positive eigenvalue of \( A - \mu C \) with

\[
A = \begin{bmatrix} \hat{A} & \hat{A}^H \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -jI & jI \end{bmatrix}. \quad (5.13)
\]

Furthermore, in Section 7.2 of Part I [28], we know that if \( \mu^* \) is an optimizer of (5.12), then \( (\mu^*, \lambda_n(\mu^*)) = (\beta(\hat{A})) \) is a 2D-eigenvalue of \((A, C)\) and \( \mu^* \in [-\|A\|, \|A\|] \), and

\[
\beta(\hat{A}) = \min \{|\lambda| \mid (\mu, \lambda) \text{ is a 2D-eigenvalue of } (A, C) \text{ and } \lambda > 0\} \quad (5.14a)
\]

\[
= -\max \{|\lambda| \mid (\mu, \lambda) \text{ is a 2D-eigenvalue of } (A, C) \text{ and } \lambda < 0\} \quad (5.14b)
\]

\[
= \min \{|\lambda| \mid (\mu, \lambda) \text{ is a 2D-eigenvalue of } (A, C)\}. \quad (5.14c)
\]

In addition, we note that by the structure of \( A \) and \( C \) in (5.13) and equations (1.1b) and (1.1c) of the 2DEVP (1.1), the corresponding 2D-eigenvector \( x^* = [x_1^* \ x_2^*] \) of \((\mu^*, \lambda_n(\mu^*))\) must obey

\[
\text{Imag}(x_1^H x_2) = 0 \quad \text{and} \quad x_1^H x_1 + x_2^H x_2 = 1. \quad (5.15)
\]

Meanwhile, from equation (1.1a),

\[
\hat{A} x_2 = \mu^* j x_2 + \beta(\hat{A}) x_1, \quad \hat{A}^H x_1 = -\mu^* j x_1 + \beta(\hat{A}) x_2.
\]

Since \( x_1^H \hat{A} x_2 = x_2^H \hat{A}^H x_1 \), we have

\[
\mu^* j x_1^H x_2 + \beta(\hat{A}) x_1^H x_1 = -\mu^* j x_2^H x_1 + \beta(\hat{A}) x_2^H x_2,
\]

which, by noting \( \beta(\hat{A}) \neq 0 \), is equivalent to

\[
x_1^H x_1 = x_2^H x_2. \quad (5.16)
\]

Hence the 2D-eigenvector \( x^* \) must satisfy the relations (5.15) and (5.16).

**Algorithm 3 DTI by 2DRQI**

**Input:** \( m \times m \) stable matrix \( \hat{A} \), reltol, tol.

**Output:** 2D-eigentriplet \((\hat{\mu}, \hat{\lambda}, \hat{x})\), where \( \hat{\lambda} \) is an estimate of the DTI \( \beta(\hat{A}) \), and a backward error estimate \( \eta_2 \).

1: set \( \mu_0 \) as the imaginary part of the rightmost eigenvalue of \( \hat{A} \).
2: compute the singular triplet \((u, \lambda_0, v)\) corresponding to the smallest singular value of \( \hat{A} - \mu_0 i I \).
3: apply the 2DRQI (Algorithm 1) with initial \((\mu_0, \lambda_0, x_0 = \sqrt{2/2} [u] \) and stopping tolerance \( \text{tol} \) to compute an approximate 2D-eigentriplet \((\hat{\mu}, \hat{\lambda}, \hat{x})\) of \((A, C)\) and the corresponding backward error estimate \( \eta_2 \).
4: validate the computed DTI \( \hat{\lambda} \) with reltol (optional).

Algorithm 3 is an outline of a 2DRQI-based algorithms for computing \( \beta(\hat{A}) \). Two remarks are in order.
1. The initial \((\mu_0, \lambda_0, x_0)\) (lines 1 and 2) follows the recommendation in [7], and is critical for the success of the computation.

2. To satisfy the conditions (5.15) for the approximate 2D-eigenvector \(x_k = \begin{bmatrix} x_{k,1} \\ x_{k,2} \end{bmatrix}\), we should add the following steps after line (13) in the 2DRQI (Algorithm 1):

1: \(x_{k+1,1} = \sqrt{\frac{\beta}{2}} x_{k+1,1} / \|x_{k+1,1}\|\).
2: \(x_{k+1,2} = \sqrt{\frac{\beta}{2}} x_{k+1,2} / \|x_{k+1,2}\|\).

For the stopping criterion of the 2DRQI, we use a backward error estimate of the computed DTI. It has been a challenge to properly define the stopping criterion of iterative methods for computing DTI [7, 10, 12, 30]. A main reason is that it is meaningless to define the backward error for a estimated DTI \(\hat{\beta}\) only. Specifically, if a backward error \(\tilde{\eta}\) of \(\hat{\beta}\) is defined as

\[\tilde{\eta} = \inf \left\{ \epsilon \mid \exists \hat{\Delta} \text{ such that } \|\hat{\Delta}\| \leq \epsilon \|\hat{\Delta}\| \text{ and } \beta(\hat{\Delta} + \delta \hat{\Delta}) = \hat{\beta} \right\}, \tag{5.17}\]

then the following proposition shows that the calculation of the backward error \(\tilde{\eta}\) is as hard as the calculation of the original \(\beta(\hat{\Delta})\).

**Proposition 5.2.** If \(\beta(\hat{\Delta}) > \hat{\beta}\), then \(\tilde{\eta} = \frac{\beta(\hat{\Delta}) - \hat{\beta}}{\|\hat{\Delta}\|}\).

**Proof.** We first prove the inequality \(\tilde{\eta} \geq \frac{\beta(\hat{\Delta}) - \hat{\beta}}{\|\hat{\Delta}\|}\). By the definition of \(\tilde{\eta}\), for any \(t > 0\), there exists a matrix \(\delta \hat{\Delta}\) such that \(\beta(\hat{\Delta} + \delta \hat{\Delta}) = \hat{\beta}\) and \(\|\delta \hat{\Delta}\| \leq (\tilde{\eta} + t)\|\hat{\Delta}\|\). By (5.12), there exists \(E_{\hat{\beta}}\) such that \(\|E_{\hat{\beta}}\| = \tilde{\beta}\) and \((\hat{\Delta} + \delta \hat{\Delta}) + E_{\hat{\beta}}\) is unstable. Thus \(\hat{\Delta}\) is unstable under the perturbation \(\delta \hat{\Delta} + E_{\hat{\beta}}\). By the definition of \(\beta(\hat{\Delta})\), this implies

\[\|\delta \hat{\Delta} + E_{\hat{\beta}}\| \geq \beta(\hat{\Delta}).\]

Thus

\[\beta(\hat{\Delta}) \leq \|\delta \hat{\Delta} + E_{\hat{\beta}}\| \leq \|\delta \hat{\Delta}\| + \|E_{\hat{\beta}}\| \leq (\tilde{\eta} + t)\|\hat{\Delta}\| + \tilde{\beta}\]

holds for any \(t > 0\). Let \(t \to 0\), then we have

\[\tilde{\eta} \geq \frac{\beta(\hat{\Delta}) - \tilde{\beta}}{\|\hat{\Delta}\|}. \tag{5.18}\]

We next prove the inequality \(\tilde{\eta} \leq \frac{\beta(\hat{\Delta}) - \tilde{\beta}}{\|\hat{\Delta}\|}\). By the definition of \(\beta(\hat{\Delta})\), there exists a matrix \(E_{\beta}\) such that \(\|E_{\beta}\| = \beta(\hat{\Delta})\) and \(\hat{\Delta} + E_{\beta}\) is unstable. Let \(F = \frac{\beta(\hat{\Delta}) - \tilde{\beta}}{\beta(\hat{\Delta})} E_{\beta}\). Then \(\|F\| = \beta(\hat{\Delta}) - \tilde{\beta} < \beta(\hat{\Delta})\) and thus \(\hat{\Delta} + F\) must be stable.

Consider \(\beta(\hat{\Delta} + F)\). Since \(\hat{\Delta} + F + (E_{\beta} - F) = \hat{\Delta} + E_{\beta}\) is unstable, we have

\[\beta(\hat{\Delta} + F) \leq \|E_{\beta} - F\| = \tilde{\beta}. \tag{5.19}\]

On the other hand, assume there is a matrix \(G\) satisfies \(\hat{\Delta} + F + G\) is unstable, then by the definition of \(\beta(\hat{\Delta})\),

\[\beta(\hat{\Delta}) \leq \|F + G\| \leq \beta(\hat{\Delta}) - \tilde{\beta} + \|G\|.

Thus \(\|G\| \geq \tilde{\beta}\), which implies

\[\beta(\hat{\Delta} + F) \geq \tilde{\beta}. \tag{5.20}\]

By (5.19) and (5.20), we have

\[\beta(\hat{\Delta} + F) = \tilde{\beta}. \tag{5.21}\]
By \((5.24)\) and the definition of \(\bar{\eta}\),
\[
\| \hat{A} \| \bar{\eta} \leq \| F \| = \beta(\hat{A}) - \hat{\beta}.
\]

Then we have the inequality
\[
\bar{\eta} \leq \frac{\beta(\hat{A}) - \hat{\beta}}{\| \hat{A} \|}.
\] (5.22)

The proposition is then proven by \((5.18)\) and \((5.22)\).

Proposition 5.2 implies the exact calculation of the backward error \(\bar{\eta}\) could be as hard as the calculation of the original \(\beta(\hat{A})\). This is analogous to the fact that for eigenvalue problems we do not define the backward error of an approximate eigenvalue only. We consider the backward error of an approximate eigenpairs, see e.g. [27, Thm.1.3]. As an advantage of treating the DTI via the 2DEVP, we can establish the notion of the backward error for a computed DTI via an approximate 2D-eigentriplet. The resulting backward error estimation naturally leads to a reliable stopping criterion for an iterative DTI algorithm.

To that end, let the approximate 2D eigentriplet \((\hat{\mu}, \hat{\lambda}, \hat{x})\) of \((A, C)\) be an exact 2D-eigentriplet of structurely-perturbed 2DEVP
\[
\begin{align}
\hat{A}^H + \delta \hat{A} &\ 
\begin{bmatrix}
0 & \hat{A}^H + \delta \hat{A} \\
\hat{A}^H + \delta \hat{A} & 0
\end{bmatrix} \begin{bmatrix}
\hat{x} \\
\hat{x}
\end{bmatrix} = \hat{\mu} \hat{C} \begin{bmatrix}
\hat{x} \\
\hat{x}
\end{bmatrix}, \\
\hat{x}^H \hat{C} \hat{x} &= 0, \\
\hat{x}^H \hat{x} &= 1.
\end{align}
\] (5.23a)

for some \(\delta \hat{A}\). Then we can define a structure-preserving backward error of the 2DEVP of the DTI problem as follows:
\[
\hat{\eta}_\beta(\hat{\mu}, \hat{\lambda}, \hat{x}) = \inf \{ \epsilon \mid \exists \delta \hat{A} \text{ such that } \| \delta \hat{A} \| \leq \epsilon \| \hat{A} \| \text{ and } (5.23) \text{ holds} \}.
\] (5.24)

We first note that the set in \((5.21)\) is nonempty when the approximate 2D-eigenvector \(\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}\) satisfies the conditions \((5.15)\). In fact, denote \(r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\), where \(r_1 = \hat{A} \hat{x}_2 - \hat{\mu} \hat{x}_2 - \hat{\lambda} \hat{x}_1\) and \(r_2 = \hat{A}^H \hat{x}_1 + \hat{\mu} \hat{x}_1 - \hat{\lambda} \hat{x}_2\). Then it can be shown that the matrix
\[
\delta \hat{A} = \delta \hat{A}_1 + \delta \hat{A}_2 \quad \text{with} \quad \delta \hat{A}_1 = - \left( I - \frac{\hat{x}_1 \hat{x}_1^H}{\hat{x}_1^H \hat{x}_1} \right) \frac{r_1 \hat{x}_2^H}{\hat{x}_2^H \hat{x}_2} \quad \text{and} \quad \delta \hat{A}_2 = - \frac{\hat{x}_1 r_2^H}{\hat{x}_1^H \hat{x}_1},
\]
is in the set \((5.24)\). Meanwhile, we have
\[
\| \delta \hat{A} \| = \max_{\| z \| = 1} \| (\delta \hat{A}_1 + \delta \hat{A}_2) z \| = \max_{\| z \| = 1} \sqrt{\| \delta \hat{A}_1 z \|^2 + \| \delta \hat{A}_2 z \|^2} \\
\leq \sqrt{\| \delta \hat{A}_1 \|^2 + \| \delta \hat{A}_2 \|^2} \leq \sqrt{2 \| r_1 \|^2 + 2 \| r_2 \|^2} = \sqrt{2 \| r \|^2},
\] (5.25)

where the second equality lies in the fact that \(\delta \hat{A}_1 z\) is orthogonal to \(\delta \hat{A}_2 z\).

Next we provide an estimate of \(\hat{\eta}_\beta\). Since \(\hat{\eta}_\beta\) is the backward error of the structured 2DEVP \((5.23)\), the backward error \(\eta\) in \((4.4)\) of a generic (unstructured) 2DEVP is the lower bound of \(\hat{\eta}_\beta\):
\[
\hat{\eta}_\beta \geq \eta \geq \eta_1.
\] (5.26)

where \(\eta_1\) is defined in \((4.5)\). On the other hand, by the definition of \(\hat{\eta}_\beta\) and \((5.25)\), we have an upper bound of \(\hat{\eta}_\beta\):
\[
\hat{\eta}_\beta \leq \eta_2 \equiv \sqrt{2 \| r \| \| A \|}.
\] (5.27)
By the facts that \( \| \hat{A} \| = \| A \| \) and \( \| C \| = 1 \), we have
\[
\frac{\eta_2}{\eta_1} \leq \sqrt{\frac{2 \| r \|}{\| A \| (\| r \| + \| \hat{A} \|)}} = \sqrt{2} \left( 1 + \frac{\| \hat{\mu} \|}{\| A \|} \right).
\] (5.28)
Combining (5.26), (5.27), and (5.28), we have
\[
\frac{1}{\sqrt{2} \left( 1 + \frac{\| \hat{\mu} \|}{\| A \|} \right)} \eta_2 \leq \hat{\eta} \leq \eta_2.
\] (5.29)
Therefore \( \eta_2 \) defined in (5.27) can be used as an estimate of \( \hat{\eta} \). Consequently, the stopping criteria (line 15) of the 2DRQI (Algorithm 1) should be
\[
| \text{Im} (x_{k,1}^H x_{k,2}) | \leq \text{tol} \quad \text{and} \quad \eta_2(\mu_k, \lambda_k, x_k) \leq \text{tol},
\] (5.30)
where \( \text{tol} \) is a prescribed tolerance value. In addition, to handle the possible stagnation of the 2DRQI, we can also include the following test for possible stagnation:
\[
\eta_2(\mu_k, \lambda_k, x_k) \geq \frac{1}{2} \left( \eta_2(\mu_{k-2}, \lambda_{k-2}, x_{k-2}) + \eta_2(\mu_{k-1}, \lambda_{k-1}, x_{k-1}) \right).
\] (5.31)
For the optional validation step of Algorithm 3, we know that if the computed \( \hat{\lambda} \) is an acceptable estimate of DTI \( \beta(\hat{A}) \), it should satisfy
\[
(1 - \text{reltol}) \hat{\lambda} \leq \beta(\hat{A}) \leq \hat{\lambda}
\] (5.32)
for a small \( \text{reltol} \), where without loss of generality, we assume \( \hat{\lambda} > 0 \). Otherwise, according to the symmetric properties of 2D eigenvalues in DTI, we can use \( -\hat{\lambda} \) as an estimate of the DTI \( \beta(\hat{A}) \).
The upper bound of (5.32) naturally holds according to (5.14) and \( (\hat{\mu}, \hat{\lambda}) \) is a 2D-eigenvalue. For the lower bound of (5.32), we just need to verify that \( H((1 - \text{reltol}) \hat{\lambda}) \) has no imaginary eigenvalue. This is based on the following lemma.

**Lemma 5.1** ([2]). For any \( \lambda > 0 \), \( \lambda < \beta(\hat{A}) \) if and only if \( G(\lambda) \) has no pure imaginary eigenvalue, where \( G(\lambda) \) is an Hamiltonian matrix of the form
\[
G(\lambda) = \begin{bmatrix}
\hat{A} & -\lambda I \\
\lambda I & -\hat{A}^H
\end{bmatrix}.
\] (5.33)
This validation procedure is the one proposed in [2]. However, it should be noted that checking whether \( G((1 - \text{reltol}) \hat{\lambda}) \) has no imaginary eigenvalue could be prohibitively expensive for large scale problems. Therefore, the validation step is optional in all existing algorithms for computing DTI [7] [10] [12]. In Section 6, we will provide a numerical example to show that the 2DRQI outperforms a recently proposed subspace method for the DTI computation.

### 6 Numerical examples

In this section, we first present a numerical example to illustrate the convergence behavior of the 2DRQI (Algorithm 1), and then present two examples for finding the minmax of two Rayleigh quotients (Algorithm 2) and for computing the DTI (Algorithm 3). All algorithms are implemented in MATLAB. Numerical experiments are performed on a HP computer with an Intel(R) Core(TM) 2.60GHz i7-6700HQ CPU and 8GB RAM.
Example 1. This example illustrates convergence behaviors of the 2DRQI (Algorithm 1). Let us consider the 2DEVP (1.1) of the matrices

\[
A = \begin{bmatrix}
-0.7 & 0.01 & 0.2 \\
0.01 & 0 & 0 \\
0.2 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
C = \begin{bmatrix}
0.3 & 0.01 & 0.2 \\
0.01 & 1 & 0 \\
0.2 & 0 & -1
\end{bmatrix}.
\]

It can be verified that \((\mu_1, \lambda_1, x_1) = (1, 1, \sqrt{2})\) is a 2D-eigentriplet and \(\lambda_1 = 1\) is an eigenvalue of \(A - \mu_1 C\) with multiplicity 2. In addition, by a brute-force bisection search following the sorted eigencurves \(\lambda_j(\mu)\) for \(j = 1, 2, 3\), we found additional two 2D-eigenvalues to the machine precision:

\[
(\mu_2, \lambda_2) = (-0.665101440190437, -0.239801782612878) \\
(\mu_3, \lambda_3) = (-0.145810069397438, -0.744080780565709).
\]

Moreover, \(\lambda_2\) and \(\lambda_3\) are the simple eigenvalue of \(A - \mu_2 C\) and \(A - \mu_3 C\), respectively. The left plot of Figure 6.1 are the sorted eigencurves \(\lambda_j(\mu)\) for \(j = 1, 2, 3\). The maximum 2D-eigenvalue \((\mu_1, \lambda_1) = (1, 1)\) is marked in red. The 2D-eigenvalue \((\mu_2, \lambda_2)\) is blue. The minimum 2D-eigenvalue \((\mu_3, \lambda_3)\) is green.

Figure 6.1: Left: sorted eigencurves and corresponding 2D-eigenvalues of \((A, C)\) in Example 1. Right: Computed 2D-eigenvalues with different initials.

We use each grid point on the 100 \(\times\) 100 mesh of the domain \((\mu, \lambda) = [-1.5, 1.5] \times [-2, 2]\) as an initial \((\mu_0, \lambda_0)\) and the vector \(x_0\) is generated based on the recommendation of Algorithm 1. If the 2DRQI with the initial \((\mu_0, \lambda_0, x_0)\) and \(\text{tol} = n \cdot \text{macheps}\) and \(\text{maxit} = 15\), converges to the \(i\)-th 2D-eigenvalue \((\mu_i, \lambda_i)\), then we use the same color for the initial \((\mu_0, \lambda_0)\) and \((\mu_i, \lambda_i)\). The right plot of Figure 6.1 shows that the 2DRQI converges to a 2D-eigentriplet for all 10,000 initials \((\mu_0, \lambda_0, x_0)\).

Table 6.1 records the convergence history of a sequence \(\{(\mu_{3,k}, \lambda_{3,k}, x_{3,k})\}\) to the minimum 2D-eigenvalue \((\mu_3, \lambda_3)\), marked in green in Figure 6.1. We observe that the sequence \(\{(\mu_{3,k}, \lambda_{3,k})\}\) converges quadratically, the matrix \(C_k\) of the 2DRQ \((A_k, C_k)\) remains to be indefinite and \(a_{12,k} \neq 0\).
Table 6.1: Convergence history of \( \{ (\mu_{3,k}, \lambda_3, x_3) \} \) to \( (\mu_3, \lambda_3, x_3) \)

| \( k \) | \( |\mu_{3,k} - \mu_3| \) | \( |\lambda_3 - \lambda_{3,k}| \) | \( \eta_{1}(\mu_{3,k}, \lambda_{3,k}, x_{3,k}) \) | \( (c_{1,k}, c_{2,k}) \) | \( |a_{12,k}| \) |
|------|------------------|------------------|-------------------------|-----------------|------------------|
| 0    | 1.0e0            | 8.9e-1           | 4.1e-1                  | (-1.0e0, 3.3e-1)| 2.9e-1           |
| 1    | 2.6e-3           | 8.4e-3           | 7.1e-2                  | (-1.0e0, 3.3e-1)| 2.9e-1           |
| 2    | 2.2e-5           | 1.2e-7           | 2.7e-4                  | (-1.0e0, 3.3e-1)| 2.9e-1           |
| 3    | 6.5e-13          | 1.1e-16          | 2.1e-9                  | (-1.0e0, 3.3e-1)| 2.9e-1           |
| 4    | 3.4e-16          | 2.6e-16          | 1.1e-16                 | (-1.0e0, 3.3e-1)| 2.9e-1           |

Table 6.2 shows the convergence history of a sequence \( \{ (\mu_{1,k}, \lambda_{1,k}, x_{1,k}) \} \) to the maximum 2D-eigenvalue \( (\mu_1, \lambda_1) \), marked in red in Figure 6.1. Note that \( \lambda_1 \) is an eigenvalue of \( A - \mu_1 C \) with multiplicity 2. We observe that the sequence \( \{ \mu_{1,k}, \lambda_{1,k} \} \) converges quadratically and the matrix \( C_k \) of the 2DRQ \( (A_k, C_k) \) remains to be indefinite. However, \( a_{12,k} \) approaches to 0.

Table 6.2: Convergence history for \( \{ (\mu_{1,k}, \lambda_{1,k}, x_{1,k}) \} \) to \( (\mu_1^*, \lambda_1^*, x_1^*) \).

| \( k \) | \( |\mu_{1,k} - \mu_1| \) | \( |\lambda_{1,k} - \lambda_1| \) | \( \eta_{1}(\mu_{1,k}, \lambda_{1,k}, x_{1,k}) \) | \( (c_{1,k}, c_{2,k}) \) | \( |a_{12,k}| \) |
|------|------------------|------------------|-------------------------|-----------------|------------------|
| 0    | 1.0e0            | 1.0e0            | 3.2e-1                  | (-1.0e0, 7.9e-1)| 9.3e-2           |
| 1    | 3.3e-1           | 4.8e-1           | 3.1e-1                  | (-9.6e-1, 9.6e-1)| 3.2e-2           |
| 2    | 5.0e-2           | 9.0e-2           | 1.3e-1                  | (-1.0e0, 1.0e0  )| 2.4e-4           |
| 3    | 5.2e-4           | 3.3e-4           | 8.1e-3                  | (-1.0e0, 1.0e0  )| 3.2e-9           |
| 4    | 3.8e-10          | 2.2e-11          | 2.1e-6                  | (-1.0e0, 1.0e0  )| 8.0e-16          |
| 5    | 4.2e-16          | 2.2e-16          | 2.5e-16                 | (-1.0e0, 1.0e0  )| 4.6e-16          |

In [16], we will prove that the 2DRQI locally quadratically converges to a 2D-eigentriplet \( (\mu_*, \lambda_*, x_*) \) We will see that though the algorithm and local quadratic convergence rate are the same regardless the multiplicity of the eigenvalue \( \lambda_* \) of \( A - \mu_* C \), convergence analysis needs to be treated differently as indicated by whether \( |a_{12,k}| \) approaches to 0.

Example 2. We use Algorithm 2 to solve the RQminmax [5,1] arising from a MIMO relay precoder design problem in signal communication, and compare with an algorithm proposed in [9].

The MIMO relay precoder design problem is to minimize the total relay power subject to SINR constraints at the receivers [3]. Consider the multi-point to multi-point communication with two sources. The signals \( r_o \) after MIMO relay processing and signals \( y \) received by destinations are

\[
  r_o = ZH_{up}s + Zn_r \quad \text{and} \quad y = H_{dl}^HZH_{up}x + H_{dl}^HZn_r + n_d,
\]

where \( s \) is the transmit signals of the sources, \( n_r \) and \( n_d \) are zero-mean circularly symmetric complex Gaussian random variables with variance \( \sigma_r^2 \) and \( \sigma_d^2 \). \( H_{up} = [h_1, h_2] \in \mathbb{C}^{m \times 2} \) denotes channels between two sources and antennas, \( H_{dl} = [g_1, g_2] \in \mathbb{C}^{m \times 2} \) denotes channels between antennas and two destinations, \( m \) is the number of antennas at the relay. \( Z \in \mathbb{C}^{m \times m} \) is the MIMO relay processing matrix to be designed. Under the assumption that the source transmit signals \( s \) are zero-mean, statistically independent with the unit power, the goal of the MIMO relay precoder design is to minimize the relay power while maintaining SINR no less than a prescribed threshold \( \gamma_{th} \).

After some algebraic manipulations, the MIMO precoder relay design problem becomes solving the following homogeneous quadratic constrained programming (HQCQP):

\[
  \min_u u^HTu \quad \text{s.t.} \quad u^HP_iu + 1 \leq 0 \quad \text{for} \quad i = 1, 2, \quad (6.1)
\]

where \( u = \text{vec}(Z) \) is a vector of length \( n = m^2 \), \( T = \hat{F}_0 \otimes I \), \( P_1 = \hat{F}_1 \otimes g_1^H \) and \( P_2 = \hat{F}_2 \otimes g_2^H \).
are of dimension \( n = m^2 \), with
\[
\begin{align*}
\hat{F}_0 &= \overline{r}_1 h_1^T + \overline{r}_2 h_2^T + \sigma_r^2 I, \\
\hat{F}_1 &= \frac{1}{\gamma_{th} \sigma_d^2} (\gamma_{th} \overline{r}_2 h_2^T + \gamma_{th} \sigma_r^2 I - \overline{r}_1 h_1^T), \\
\hat{F}_2 &= \frac{1}{\gamma_{th} \sigma_d^2} (\gamma_{th} \overline{r}_1 h_1^T + \gamma_{th} \sigma_r^2 I - \overline{r}_2 h_2^T).
\end{align*}
\]

Note that \( \hat{F}_0 \) and \( \hat{F}_i \) are \( m \times m \) Hermitian matrices with \( \hat{F}_0 \) positive definite. Gaurav and Hari [9] show that the HQCQP (6.1) is equivalent to the RQminmax (5.1) of the matrices
\[
A = S^H P_1 S = F_1 \otimes g_1 g_1^H, \quad B = S^H P_1 S = F_2 \otimes g_2 g_2^H,
\]
where \( S = T^{-\frac{1}{2}} \) is the square root of \( T^{-1} \), \( F_1 = \hat{F}_0^{-\frac{1}{2}} \hat{F}_1 \hat{F}_0^{-\frac{1}{2}} \) and \( F_2 = \hat{F}_0^{-\frac{1}{2}} \hat{F}_2 \hat{F}_0^{-\frac{1}{2}} \). We note that by exploiting the structure of \( A \) and \( B \), the matrix-vector multiplications \( Ax \) and \( Bx \) can be performed efficiently.

Algorithm 2 first checks the Cases-I and II of the RQminmax (5.1) described in Theorem 5.1 for possible early exit. Then it uses a combination of the 2DRQI and the bisection search to find an optimizer \( \mu^{(RQI)}_* \) of the EVopt (5.2) for the general Case-III.

A dichotomous method is proposed in [9] for solving the EVopt (5.2). Starting from a search interval \([a, b]\) containing the global maximum of the concave function \( g(\mu) = \lambda_{\min}(A - \mu C) \), where \( C = A - B \), the dichotomous method compares \( g(a) \), \( g(b) \), \( g(\frac{a+b}{2} - \epsilon_r) \) and \( g(\frac{a+b}{2} + \epsilon_r) \) for a small scalar \( \epsilon_r \), and then by using the concavity of \( g(\mu) \), replaces \( a \) with \( \frac{a+b}{2} - \epsilon_r \), or \( b \) with \( \frac{a+b}{2} + \epsilon_r \) for the next iteration. When the search interval width \( b - a \) is less than a prescribed tolerance \( tol \), it returns an approximate optimal value \( \mu^{(Dich)}_* \).

For numerical experiments described in [9], \( H_{up} \) and \( H_{dl} \) are complex Gaussian random matrices. The SINR is set to 3dB and noise variances are set to -10dB, i.e., \( \gamma_{th} = 10^{-3} \), and \( \sigma_d^2 = \sigma_r^2 = 10^{-1} \).

We observed that the optimizers of the EVopt (5.2) on the interval \([0, 1]\) computed by the dichotomous method with \( tol = 1e-8 \) and Algorithm 2 with \( backtol = n \epsilon \) for the 2DRQI and \( reltol = 1e-8 \) agree up to 8 significant digits: \( |\mu^{(RQI)}_* / \mu^{(Dich)}_*| \leq 1e-8 \) for 20 runs of each of dimensions \( n = 10^2, 100^2, 200^2, 400^2 \).

The third column of Table 6.3 reports the average runtime (in seconds) of 100 runs of the dichotomous method for finding the optimizer of the EVopt (5.2) on the interval \([0, 1]\) with the accuracy \( tol = 1e-4 \). The fifth column of Table 6.3 reports the average runtime of 100 runs of Algorithm 2 with \( reltol = 1e-8 \) and \( backtol = n \epsilon \), excluding the lines 1 and 2 of Algorithm 2 for checking the Cases-I and II.

| \( n = m^2 \) | Dichotomous method | Algorithm 2 |
|---|---|---|
| 10^2 | 15 | 0.11 | 3.1 | 0.026 |
| 100^2 | 15 | 1.2 | 2.6 | 0.19 |
| 200^2 | 15 | 4.6 | 2.4 | 0.57 |
| 400^2 | 15 | 29 | 2.1 | 3.6 |

The significant performance gain of Algorithm 2 in speed is due to the reduction of the number of iterations shown in the “niter” columns of Table 6.3, and the fact that each iteration of the dichotomous method needs to solve two eigenvalue problems of \( A - \mu_i C \) for computing \( g(\mu_i) = \lambda_{\min}(A - \mu_i C) \), where we use the sparse eigensolver \texttt{eigs}. In contrast, each iteration of Algorithm 2
calls the 2DRQI (Algorithm 1) once, which in turn only needs to solve the linear system \((6.2)\), where we use the linear solver \texttt{gmres}.

**Example 3.** The purpose of this example is to show that Algorithm 3 is more efficient than recently proposed subspace method [12] for large scale DTI computation.

An \(n \times n\) Orr-Sommerfeld matrix from finite difference discretization of the Orr-Sommerfeld operator for planar Poiseuille flow is of the form

\[
\hat{A}_n = L_n^{-1}B_n,
\]

where \(L_n = (1/h^2)\text{tridiag}(1, -(2+h^2), 1)\), \(B_n = \frac{1}{\mathcal{R}_e} L_n^2 - 1(U_n L_n + 2I)\) and \(U_n = \text{diag}(1-u_1^2, \ldots, 1-u_n^2)\). \(h = 2/(n + 1)\) is the stepsize of discretization, \(u_k = -1 + kh\), \(\mathcal{R}_e\) is the Reynolds number (\(\mathcal{R}_e = 1000\) in numerical experiments) and \(i = \sqrt{-1}\). The stability of the Orr-Sommerfeld matrices has been extensively studied [6, 17, 25]. It is known that the eigenvalues of Orr-Sommerfeld matrices are highly sensitive to perturbations. The DTI is an important measure of the stability under perturbation [7, 10, 12].

To apply Algorithm 3 for computing the DTI of \(\hat{A}_n\), we need to solve the linear equation \((3.2)\) in the 2DRQI. For computational efficiency, we first transform the Jacobian \(J(\mu_k, \lambda_k, x_k)\) into a banded arrow matrix [4, p. 86] through a permutation, and then apply a Schur complement technique [20, p. 406].

For the initial \((\mu_0, \lambda_0, x_0)\) of the 2DRQI, we apply the Cayley-Arnoldi algorithm with complex shift for computing \(\mu_0\) [18], and then use MATLAB’s \texttt{svds} to compute the smallest singular triplet of the matrix \(\hat{A} - \mu_0 I\). We set \(\texttt{reitol} = 1e-9\) and \(\texttt{tol} = n\epsilon\).

A subspace method [12] for eigenvalue optimization is recently applied for computing DTI \(\beta(\hat{A}_n)\) based on the singular value minimization:

\[
\beta(\hat{A}_n) = \min_{\mu \in \mathbb{R}} \sigma_{\min}(\hat{A}_n - \mu I). \tag{6.3}
\]

With a prescribed search interval \([a, b]\) and an initial \(\mu_0 \in [a, b]\), the subspace method first computes \(\sigma_{\min}(\hat{A}_n - \mu_0 I)\) and the corresponding right singular vector \(v_0\) and then sets the initial projection subspace \(V_0 = v_0\). At the \(k\)-th iteration for \(k \geq 1\), the subspace method projects the minimization \((6.3)\) onto the subspace \(V_{k-1}\) and solves the reduced problem:

\[
\sigma_{\min}^{(k)} = \min_{\mu \in [a, b]} \sigma_{\min}(\hat{A}_n V_{k-1} - \mu I V_{k-1}). \tag{6.4}
\]

With a minimizer \(\mu_k\) of the reduced problem \((6.4)\), the subspace method computes \(\sigma_{\min}(\hat{A}_n - \mu_k I)\) and the corresponding right singular vector \(v_k\), and then updates the projection subspace \(V_k = \text{Orth}(v_{k-1}, v_k)\). The iteration terminates when \(\sigma_{\min}^{(k-1)} - \sigma_{\min}^{(k)} < \texttt{tol}\) for a prescribed \(\texttt{tol}\), or the number of iterations exceeds \(\sqrt{n}\).

\texttt{leigopt} is an implementation of the subspace method in MATLAB [12]. To improve computational efficiency, the following minor modifications are made in \texttt{leigopt}. (1) We set the dimension of the projection subspace \texttt{opts.p} = 20 in \texttt{eigs} or \texttt{svds}, instead of \texttt{round(sqrt(n))} used in \texttt{leigopt}. It is observed significant reduction in computational cost. (2) \texttt{leigopt} uses \texttt{eigopt}, a quadratic supporting functions based method [19], to solve the reduced problem \((6.4)\). For the Orr-Sommerfeld matrices, \texttt{eigopt} is too time consuming. Instead, we use a modified Boyd-Balakrishnan method [11]. For numerical experiments, the search interval of \texttt{leigopt} is set to \([a, b] = [-60, 60]\), the initial \(\mu_0 = 0\) and the tolerance \(\texttt{tol} = 1e-12\).

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[1] The formulation in [10, 12] has some typos.

[2] \url{http://home.ku.edu.tr/~emengi/software/leigopt}, downloaded on October 2, 2021.

[3] This strategy is also recommended by Mengi, one of the authors of \texttt{leigopt} in a private communication.
Table 6.4 shows the performance of Algorithm 3 and the subspace method. The runtime of Algorithm 3 is written as $t_1 + t_2$ with $t_1$ for calculating the rightmost eigenvalue of $\hat{A}_n$ and the singular triplet of $\hat{A}_n - \mu_k I$ (i.e., lines 1 and 2 of Algorithm 3), and $t_2$ for the rest of calculation. The runtime of the subspace method is written as $t_{all}(t_{sub})$ with $t_{all}$ for the total time and $t_{sub}$ for solving the subproblems.

We observe that the computed $\hat{\beta}(\hat{A}_n)$ by two algorithms agrees from 4 to 8 significant digits. However, Algorithm 3 uses no more than half of the runtime of the subspace method. The speedup of Algorithm 3 comes from two-fold. Algorithm 3 uses less iterative steps. The major cost of the subspace method is on computing the right singular vector $v_k$ corresponding to $\sigma_{\text{min}}(\hat{A} - \mu_k I)$. In contrast, in Algorithm 3 we only need to solve a linear equation of the form (3.2) in each iteration of 2DRI (Algorithm 1).

Table 6.4: DTI computation by the subspace method and Algorithm 3

| $n$   | The subspace method | Algorithm 3 |
|-------|---------------------|-------------|
|       | $n_{iter}$ | runtime | $\hat{\beta}(\hat{A}_n)$ | $n_{iter}$ | runtime | $\hat{\beta}(\hat{A}_n)$ |
| 1000  | 9.7          | 0.16(0.013) | 1.97789572460e-3 | 5.8          | 0.025 + 0.032  | 1.9778957275e-3 |
| 4000  | 9.7          | 0.44(0.017)  | 1.97809438632e-3 | 4.9          | 0.062 + 0.095  | 1.9780964583e-3 |
| 16000 | 8.9          | 1.53(0.035)  | 1.9376364536e-3  | 4.8          | 0.25 + 0.38    | 1.9379706543e-3 |

We note that the validation step for computed $\hat{\beta}(\hat{A}_n)$ by Algorithm 3 and the subspace method is not reported in Table 6.4. For the matrix size $n = 1000$, it is verified that both algorithms pass the validation procedure described in Section 5.2. Although there exists an algorithm [15] for checking whether $G(\lambda)$ defined in Lemma 5.1 has pure imaginary eigenvalues, it would be too expensive for large matrix sizes. As a common practice of existing algorithms [7, 10, 12, 30], there is no validation procedure for large scale DTI calculation.

7 Concluding remarks

Based on the theoretical results presented in Part I of this paper [28], we devised an RQI-like algorithm, 2DRQI in short, for solving the 2DEVP (1.1). The computational kernel of the 2DRQI is on solving a linear systems of equation. The efficiency of the 2DRQI is demonstrated for solving large scale 2DIEVP arising from the minmax problem of two Rayleigh quotients and the computation of the distance to instability of a stable matrix. A rigorous convergence analysis of the proposed 2DRQI will be presented in the third part of this paper.

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