Center stable manifolds for quasilinear parabolic pde and conditional stability of nonclassical viscous shock waves

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Abstract
Motivated by the study of conditional stability of traveling waves, we give an elementary $H^2$ center stable manifold construction for quasilinear parabolic PDE, sidestepping apparently delicate regularity issues by the combination of a carefully chosen implicit fixed-point scheme and straightforward time-weighted $H^s$ energy estimates. As an application, we show conditional stability of Lax- or undercompressive shock waves of general quasilinear parabolic systems of conservation laws by a pointwise stability analysis on the center stable manifold.

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1 Introduction

In this paper, extending our previous work in the semilinear case [Z5], we show by an elementary argument that an asymptotically constant stationary solution

\begin{equation}
  u(x, t) \equiv \bar{u}(x), \quad |\bar{u}(x) - u_\pm| \leq Ce^{-\theta|x|},
\end{equation}

\( \theta > 0 \), of a general quasilinear second-order parabolic system

\begin{equation}
  u_t = \mathcal{F}(u) := b(u)u_{xx} - h(u, u_x),
\end{equation}

\( x, t \in \mathbb{R}, \; u, h \in \mathbb{R}^n, \; b \in \mathbb{R}^{n \times n} \), possesses a local center stable manifold with respect to \( H^2 \). Combining this result with ideas from [ZH, HZ, RZ, TZ2, TZ3, Z5] we then establish conditional stability of nonclassical viscous shock solutions of strictly parabolic systems of conservation laws, similarly as was done in [Z5] for Lax shocks of semilinear parabolic systems.
1.1 Existence of center stable manifolds

We briefly describe our results. Following [Z5], assume:

- (A0) \( h \in C^{k+1}, k \geq 2 \).
- (A1) \( \Re \sigma(b) \geq \theta > 0 \).
- (A2) The linearized operator \( L = \frac{\partial F}{\partial u}(\bar{u}) \) about \( \bar{u} \) has \( p \) unstable (positive real part) eigenvalues, with the rest of its spectrum of nonpositive real part.
- (A3) \( |\partial_x^j \bar{u}(x)| \leq Ce^{-\theta|x|}, \theta > 0, \) for \( 1 \leq j \leq k + 2 \).

Then, we have the following basic version of the Center Stable Manifold Theorem.

**Proposition 1.1.** Under assumptions (A0)–(A2), there exists in an \( H^2 \) neighborhood of \( \bar{u} \) a Lipschitz (with respect to \( H^2 \)) center stable manifold \( \mathcal{M}_{cs} \), tangent to quadratic order at \( \bar{u} \) to the center stable subspace \( \Sigma_{cs} \) of \( L \) in the sense that

\[
|\Pi_u(u - \bar{u})|_{H^2} \leq C|\Pi_{cs}(u - \bar{u})|_{H^2}^2
\]

for \( u \in \mathcal{M}_{cs} \) where \( \Pi_{cs} \) and \( \Pi_u \) denote the center-stable and stable eigenprojections of \( L \), that is (locally) invariant under the forward time-evolution of (1.2) and contains all solutions that remain sufficiently close to \( \bar{u} \) in forward time. In general it is not unique.

Combining with ideas of [TZ1, Z5], we readily obtain by the same technique the following improved version respecting the underlying translation-invariance of (1.2), a property that is important in the applications [Z5].

**Theorem 1.2.** Under assumptions (A0)–(A2), there exists in an \( H^2 \) neighborhood of the set of translates of \( \bar{u} \) a translation invariant Lipschitz (with respect to \( H^2 \)) center stable manifold \( \mathcal{M}_{cs} \), tangent to quadratic order at \( \bar{u} \) to the center stable subspace \( \Sigma_{cs} \) of \( L \) in the sense of (1.3), that is (locally) invariant under the forward time-evolution of (1.2) and contains all solutions that remain bounded and sufficiently close to a translate of \( \bar{u} \) in forward time. In general it is not unique.

1.2 Conditional stability of nonclassical shocks

Now, specialize to the case

\[
h(u, u_x) = f(u)_x - (db(u)u_x)u_x
\]
that (1.2) corresponds to a parabolic system of conservation laws in standard form

\[(1.5) \quad u_t + f(u)_x = (b(u)u)_x,\]

\(u, f \in \mathbb{R}^n, b \in \mathbb{R}^{n \times n}, x, t \in \mathbb{R}.\) Following [ZH, HZ], assume:

\[(H0) \quad f, b \in C^{k+2}, k \geq 2.\]

\[(H1) \quad \Re \sigma(b) \geq \theta > 0.\]

\[(H2) \quad A_\pm := df(u_\pm) \text{ have simple, real, nonzero eigenvalues.}\]

\[(H3) \quad \Re \sigma(df i \xi - b|\xi|^2(u_\pm)) \leq -\theta|\xi|^2, \theta > 0, \text{ for all } \xi \in \mathbb{R}.\]

\[(H4) \quad \text{Nearby } \bar{u}, \text{ the set of all solutions (1.1) connecting the same values } u_\pm \text{ forms a smooth manifold } \{\bar{u}^\alpha\}, \alpha \in \mathcal{U} \subset \mathbb{R}^\ell, \bar{u}^0 = \bar{u}.\]

\[(H5) \quad \text{The dimensions of the unstable subspace of } df(u_-) \text{ and the stable subspace of } df(u_+) \text{ sum to either } n + \ell, \ell = 1 \text{ (pure Lax case), } n + \ell, \ell > 1 \text{ (pure overcompressive case), or } \leq n \text{ with } \ell = 1 \text{ (pure undercompressive case).}\]

Assume further the following spectral genericity conditions.

\[(D1) \quad L \text{ has no nonzero imaginary eigenvalues.}\]

\[(D2) \quad \text{The Evans function } D(\lambda) \text{ associated with } L \text{ vanishes at } \lambda = 0 \text{ to precisely order } \ell.\]

Here, the Evans function, as defined in [ZH, GZ] denotes a certain Wronskian associated with eigenvalue ODE \((L - \lambda)w = 0,\) whose zeros correspond in location and multiplicity with the eigenvalues of \(L.\) For history and basic properties of the Evans function, see, e.g. [AGJ, PW, GZ, MaZ1] and references therein.

As discussed in [ZH, MaZ1], (D2) corresponds in the absence of a spectral gap to a generalized notion of simplicity of the embedded eigenvalue \(\lambda = 0\) of \(L.\) Thus, (D1)–(D2) together correspond to the assumption that there are no additional (usual or generalized) eigenvalues on the imaginary axis other than the transational eigenvalue at \(\lambda = 0;\) that is, the shock is not in transition between different degrees of stability, but has stability properties that are insensitive to small variations in parameters.

With these assumptions, we obtain our remaining results characterizing the stability properties of \(\bar{u}.\) Denoting by

\[(1.6) \quad a_{\pm 1}^1 < a_{\pm 2}^1 < \cdots < a_{\pm n}^1\]

the eigenvalues of the limiting convection matrices \(A_\pm := df(u_\pm),\) define

\[(1.7) \quad \theta(x, t) := \sum_{a_j < 0} (1 + t)^{-1/2} e^{-|x - a_j t|^2/Mt} + \sum_{a_j > 0} (1 + t)^{-1/2} e^{-|x - a_j t|^2/Mt},\]

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\[ \psi_1(x, t) := \chi(x, t) \sum_{a_j < 0} (1 + |x| + t)^{-1/2} (1 + |x - a_j^{-} t|)^{-1/2} \\
+ \chi(x, t) \sum_{a_j > 0} (1 + |x| + t)^{-1/2} (1 + |x - a_j^{+} t|)^{-1/2} \]

and

\[ \psi_2(x, t) := (1 - \chi(x, t))(1 + |x - a_1^{-} t| + t^{1/2})^{-3/2} \\
+ (1 - \chi(x, t))(1 + |x - a_1^{+} t| + t^{1/2})^{-3/2}, \]

where \( \chi(x, t) = 1 \) for \( x \in [a_1^{-} t, a_1^{+} t] \) and zero otherwise, and \( M > 0 \) is a sufficiently large constant.

**Proposition 1.3.** Conditions (H0)–(H5) imply (A0)–(A3), so that there exists a translation-invariant center stable manifold \( \mathcal{M}_{cs} \) of \( \bar{u} \) and its translates.

**Theorem 1.4.** Under (H0)–(H5) and (D1)–(D2), \( \bar{u} \) is nonlinearly phase-asymptotically orbitally stable under sufficiently small perturbations \( v_0 \in H^4 \) lying on the codimension \( p \) center stable manifold \( \mathcal{M}_{cs} \) of \( \bar{u} \) and its translates with \( |(1 + |x|^2)^{3/4} v_0(x)|_{H^4} \leq E_0 \) sufficiently small, where \( p \) is the number of unstable eigenvalues of \( L \), in the sense that, for some \( \alpha(\cdot), \alpha_* \),

\[
\begin{align*}
|\partial_x^r (u(x, t) - \bar{u}^{\alpha_* + \alpha(t)}(x))| &\leq CE_0 (\theta + \psi_1 + \psi_2)(x, t), \\
|u(\cdot, t) - \bar{u}^{\alpha_* + \alpha(t)}|_{H^4} &\leq CE_0 (1 + t)^{-\frac{3}{4}}, \\
|\alpha_*| &\leq CE_0, \\
|\alpha(t)| &\leq CE_0 (1 + t)^{-1/2}, \\
|\dot{\alpha}(t)| &\leq CE_0 (1 + t)^{-1},
\end{align*}
\]  

(1.10)  

where \( u \) denotes the solution of (1.5) with initial data \( u_0 = \bar{u} + v_0 \) and \( \bar{u}^{\alpha} \) is as in (H4). Moreover, \( \bar{u} \) is orbitally unstable with respect to small \( H^2 \) perturbations not lying in \( \mathcal{M} \), in the sense that the corresponding solution leaves a fixed-radius neighborhood of the set of translates of \( \bar{u} \) in finite time.

**Remark 1.5.** Pointwise bound (1.10) yields as a corollary the sharp \( L^p \) decay rate

\[ |u(x, t) - \bar{u}^{\alpha_* + \alpha(t)}(x)|_{L^p} \leq CE_0 (1 + t)^{-\frac{1}{2}(1 - \frac{1}{p})}, \quad 1 \leq p \leq \infty \]

obtained by the \( L^p \) (rather than pointwise) analysis of [Z5]. However, we obtain here the additional information that phase \( \alpha_* + \alpha \) approaches a limit time-asymptotically at rate \( t^{-\frac{1}{2}} \).
Remark 1.6. Mixed over- or undercompressive type shocks are also possible [ZH, HZ], though we do not know of any physical examples. These can be treated with further effort as described in [HZ], Sections 5–6, or [RZ].

1.3 Discussion and open problems
As discussed in [GJLS], for semilinear problems like those considered in [GJLS, Z5], for which the nonlinear part of the associated evolution equations consists of a relatively compact perturbation of the linear part, construction of invariant manifolds reduces essentially to verification of a spectral mapping theorem for the linearized flow, after which the construction follows by already-well-developed theory. However, for quasilinear equations, the usual fixed-point construction of the standard theory does not close due to apparent loss of regularity. This appears to be a general difficulty in the construction of invariant manifolds for quasilinear systems; see [Li, LPS1, LPS2] for further discussion.

We overcome this in the present case in a very simple way, by (i) introducing a carefully chosen implicit fixed-point scheme for which the infinite-dimensional center stable part satisfies a standard Cauchy problem forced by the finite-dimensional (hence harmless in terms of regularity) unstable part, and (ii) making use of a straightforward time-weighted energy estimate for the Cauchy problem to control higher derivatives by lower ones, the latter of which may be estimated in standard linear fashion. For related arguments, see for example [MaZ2, MaZ3, Z2, Z3, RZ, TZ3].

Existence of invariant manifolds of quasilinear parabolic systems has been treated by quite different methods in [LPS1, LPS2] via a detailed study of the smoothing properties of the linearized flow. The advantage of the present method, besides its simplicity, is that carries over in straightforward fashion to the compressible Navier–Stokes equations of gas dynamics and MHD [Z6] for which the linearized flow, being of hyperbolic–parabolic type, is only partially smoothing. A disadvantage is that we do not see by this technique how to obtain smoothness of the center stable manifold, but only Lipschitz continuity; this seems to be the price of our simple energy-based approach.

As we are mainly interested in stability, it is not smoothness but quadratic-order tangency (1.3) that is our main concern. An observation of possibly general use is that this weaker property is satisfied whenever the underlying flow is $C^2$ at $\tilde{u}$, whereas global $C^2$ regularity of the center stable manifold would require global $C^{2+\alpha}$ regularity, $\alpha > 0$, of the flow. Moreover, it is easily
verified in the course of the standard construction of a Lipschitz invariant manifold, and so we obtain this key property with essentially no extra effort.

Regarding conditional stability, the main novelty of the present analysis is that we carry out a pointwise iteration scheme in order to treat nonclassical shock waves (for further discussion regarding the need for pointwise estimates, see [HZ]). It seems an observation (though elementary) of possibly wider use that our $H^2$ center stable manifold construction can be used to obtain pointwise control on the solution in this way, extending a bit further the basic ideas of [Z5]. As our study of conditional stability was partly motivated by discussions in [GZ, AMPZ, Z7] of certain unstable undercompressive shocks and their effect on solution structure through metastable behavior, it seems desirable to fit such nonclassical waves in the theory.

An interesting open problem is to investigate conditional stability of a planar standing shock $u(x,t) \equiv \bar{u}(x_1)$ of a multidimensional system of conservation laws

$$u_t + \sum_j f_j(u)_{x_j} = \sum_{jk} (b_{jk}(u)u_{x_k})_{x_j},$$

which likewise (by the multidimensional arguments of [Z1, Z2, Z3]) reduces to construction of a center stable manifold, in this case involving an infinite-dimensional unstable subspace corresponding to essential spectra of the linearized operator $L$ about the wave.

**Plan of the paper.** In Section 2, we establish existence of center stable manifolds for general quasilinear parabolic PDE. In Section 3, we establish conditional stability on the center stable manifold by a modification of the pointwise arguments of [HZ, RZ] in the stable ($p = 0$) case.

## 2 Existence of Center Stable Manifold

Defining the perturbation variable $v := u - \bar{u}$, we obtain after a brief computation the nonlinear perturbation equations

$$(2.1) \quad v_t - Lv = N(v),$$

where

$$(2.2) \quad Lv := b(\bar{u})v_{xx} + (db(\bar{u})v)\bar{u}_x - h_u(\bar{u}, \bar{u}_x)v - h_{ux}(\bar{u}, \bar{u}_x)v_x$$
denotes the linearized operator about the wave and

\[
N(v) := \left( b(\bar{u} + v)(\bar{u} + v)_{xx} - b(\bar{u})\bar{u}_{xx} - b(\bar{u})v_{xx} - (db(\bar{u})v)\bar{u}_{xx} \right) \\
- \left( h(\bar{u} + v, \bar{u}_x + v_x) - h(\bar{u}, \bar{u}_x) - h_u(\bar{u}, \bar{u}_x)v - h_{ux}(\bar{u}, \bar{u}_x)v_x \right)
\]  

(2.3)

is a quadratic order residual. We seek to construct a \( C^k(H^2) \) local center stable manifold about the equilibrium \( v \equiv 0 \), that is, a locally invariant \( C^k \) manifold tangent (in Frechet sense) with respect to \( H^2 \) to the center stable subspace \( \Sigma_{cs} \).

### 2.1 Preliminary estimates

Denote by \( \Sigma_u \) and \( \Sigma_{cs} \) the unstable and center stable subspaces of \( L \) and \( \Pi_u \) and \( \Pi_{cs} \) the associated spectral projections.

**Proposition 2.1 ([TZ1]).** Under assumptions (A0)–(A3), \( L \) generates an analytic semigroup \( e^{Lt} \) satisfying

\[
|e^{t\Pi_{cs}}|_{L^2 \to L^2} \leq C \omega e^{\omega t}, \\
|e^{-tL}\Pi_u|_{L^2 \to H^2} \leq C \omega e^{-\beta t},
\]

(2.4)

for some \( \beta > 0 \), and for all \( \omega > 0 \), for all \( t \geq 0 \).

**Proof.** Standard semigroup estimates for second-order elliptic operators; see [TZ1, Z5], or Appendix A.

Introducing a \( C^\infty \) cutoff function \( \rho(x) = \begin{cases} 1 & |x| \leq 1, \\ 0 & |x| \geq 2, \end{cases} \) let

\[
N^\delta(v) := \rho\left( \frac{|v|_{H^2}}{\delta} \right)N(v).
\]

**Lemma 2.2 ([TZ1]).** Assuming (A0)–(A3), the map \( N^\delta : H^2 \to L^2 \) is \( C^{k+1} \) and its Lipschitz norm with respect to \( v \) is \( O(\delta) \) as \( \delta \to 0 \). Moreover,

\[
|N^\delta(v)|_{H^2} \leq C |v|_{H^2}^2.
\]

(2.5)

**Proof.** See Appendix A. □
Corollary 2.3. Under assumptions (A0)—(A3),

\[ |e^{tL}\Pi_c N^\delta|_{H^2 \to L^2} \leq C_\omega e^{\omega t}, \]
\[ |e^{-tL}\Pi_u N^\delta|_{H^2 \to H^2} \leq C_\omega e^{-\beta t}, \]

for some \( \beta > 0 \), and for all \( \omega > 0 \), for all \( t \geq 0 \), with Lipschitz bounds

\[ |e^{tL}\Pi_c dN^\delta|_{H^2 \to L^2} \leq C_\omega \delta e^{\omega t}, \]
\[ |e^{-tL}\Pi_u dN^\delta|_{H^2 \to H^2} \leq C_\omega \delta e^{-\beta t}. \]

2.2 Fixed-point iteration scheme

Applying projections \( \Pi_j, j = cs, u \) to the truncated equation

\[ v_t - L v = N^\delta(v), \]

we obtain using the variation of constants formula equations

\[ \Pi_j v(t) = e^{L(t-t_0,j)}\Pi_j v(t_0,j) + \int_{t_0,j}^{t} e^{L(t-s)}\Pi_j N^\delta(v(s)) \, ds, \]

\( j = cs, u \), so long as the solution \( v \) exists, with \( t_{0,j} \) arbitrary. Assuming growth of at most \( |v(t)|_{H^2} \leq C e^{\beta t} \) in positive time, we find for \( j = u \) using bounds (2.4)(ii) and (2.7)(ii) that, as \( t_{0,u} \to +\infty \), the first term \( e^{L(t-t_{0,u})}\Pi_u v(t_{0,u}) \) converges to zero while the second, integral term converges to \( \int_{t}^{+\infty} e^{L(t-s)}\Pi_u N^\delta(v(s)) \, ds \), so that, denoting \( w := \Pi_c v, z := \Pi_u v \), we have

\[ z(t) = T(z, w)(t) := -\int_{t}^{+\infty} e^{L(t-s)}\Pi_u N^\delta((w + z)(s)) \, ds. \]

Likewise, choosing \( t_{0,cs} = 0 \), we have

\[ w(t) = e^{Lt}\Pi_c w_0 + \int_{0}^{t} e^{L(t-s)}\Pi_c N^\delta((w + z)(s)) \, ds, \]

\( w_0 := \Pi_c v(0) \). On the other hand, we find from the original differential equation projected onto the center stable component, after some rearrangement, that \( w \) satisfies the Cauchy problem

\[ w_t - b(\bar{u})w_{xx} = \Pi_c b(\bar{u})z_{xx} - \Pi_u b(\bar{u})w_{xx} + \Pi_c M(w + z) + \Pi_c N^\delta(w + z) \]
with initial data \( w_0 = \Pi_{cs}v(0) \) given at \( t = 0 \), where

\[(2.12) \quad M(v) := (db(\bar{u})v)\bar{u}_x - h_u(\bar{u}, \bar{u}_x)v - h_{ux}(\bar{u}, \bar{u}_x)v_x.\]

We shall use these two representations together to obtain optimal estimates, the first for decay, through standard linear semigroup estimates, and the second for regularity, through the nonlinear damping estimate (2.5) below.

Viewing (2.10), or alternatively (2.11), as determining \( w = \mathcal{W}(z, w_0) \) as a function of \( z \), we seek \( z \) as a solution of the fixed-point equation

\[(2.13) \quad z = \tilde{T}(z, w_0) := T(z, \mathcal{W}(z, w_0)).\]

As compared to the standard ODE construction of, e.g., [B, VI, TZ1, Z5], in which (2.9)–(2.10) together are considered as a fixed-point equation for the joint variable \((w, z)\), this amounts to treating \( w \) implicitly. This is a standard device in situations of limited regularity; see, e.g., [CP, GMWZ, RZ].

It remains to show, first, that \( \mathcal{W} \), hence \( \tilde{T} \), is well-defined on a space of slowly-exponentially-growing functions and, second, that \( \tilde{T} \) is contractive on that space, determining a \( C^k \) solution \( z = z(w_0) \) similarly as in the usual ODE construction. We carry out these steps in the following subsections.

### 2.3 Nonlinear energy estimates

**Lemma 2.4 ([Z5]).** Under assumptions (A0)–(A3), for all \( 1 \leq p \leq \infty \), \( 0 \leq r \leq 4 \),

\[(2.14) \quad |\Pi_u|_{L^p \to W^{r,p}}, |\Pi_{cs}|_{W^{r,p} \to W^{r,p}} \leq C.\]

**Proof.** See Appendix A \( \square \)

**Proposition 2.5.** Under assumptions (A0)–(A3), for \( \delta \) sufficiently small, if the solution of (2.11) exists on \( t \in [0, T] \), then, for some constants \( \theta_{1,2} > 0 \), and all \( 0 \leq t \leq T \),

\[(2.15) \quad |w(t)|_{H^2}^2 \leq Ce^{-\theta_1 t}|w_0|_{H^2}^2 + C \int_0^t e^{-\theta_2 (t-s)}(|w|^2_{L^2} + |z|^2_{L^2}) (s) \, ds.\]

**Proof.** Let us first consider the simpler case that \( b \) is uniformly elliptic,

\[\Re b := \frac{1}{2}(b + b^t) \geq \theta > 0.\]
Taking the $L^2$ inner product in $x$ of $\sum_{j=0}^2 \partial_x^j w$ against (2.11), integrating by parts and rearranging the resulting terms, we obtain
\[
\partial_t |w|^2_{H^2}(t) \leq - \langle \partial_x^3 w, b(\bar{u}) \partial_x^3 w \rangle + C \left( |w|^2_{H^2} + |z|^2_{H^4} \right),
\]
\[
\leq - \theta |\partial_x^3 w|^2_{L^2} + C \left( |w|^2_{H^2} + |z|^2_{H^4} \right),
\]
\[\theta > 0, \text{ for } C > 0 \text{ sufficiently large, where we have repeatedly used the bounds of Lemma 2.2 and of Lemma 2.4 with } p = 2 \text{ to absorb the error terms coming from the righthand side of (2.11), and have used Moser's inequality to bound } \|\partial_x N^\delta(v)\|_{L^2} \leq C \|v\|_{L^\infty} \|v\|_{H^3} \leq C \delta \|v\|_{H^3} \text{ whenever } N^\delta \text{ does not vanish, so that } |v|^2_{H^2} \leq 2\delta. \]

Using the Sobolev interpolation
\[
|w|^2_{H^2} \leq \tilde{C}^{-1} |\partial_x^3 w|^2_{L^2} + \tilde{C} |w|^2_{L^2},
\]
for $\tilde{C} > 0$ sufficiently large, and observing that
\[
(2.16) \quad |z|_{H^4} \leq C |z|_{L^2}
\]
by equivalence of norms on finite-dimensional spaces, we obtain
\[
\partial_t |w|^2_{H^2}(t) \leq - \tilde{\theta} |w|^2_{H^2} + C \left( |w|^2_{L^2} + |z|^2_{L^2} \right),
\]
from which (2.15) follows by Gronwall’s inequality.

To treat the general case, we note that $\Re \sigma(b) \geq \theta > 0$ by Lyapunov’s Lemma implies that there exists a positive definite matrix $P$ such that $\Re (Pb) \geq \frac{\theta}{2}$, whence, by a partition of unity argument, there exists a smooth positive definite matrix-valued function $P(u)$ such that $\Re (Pb) \geq \frac{\theta}{2}$. Taking the $L^2$ inner product in $x$ of $\sum_{j=0}^2 \partial_x^j P(\bar{u}) \partial_x^j w$ against (2.11), integrating by parts and rearranging the resulting terms, we obtain
\[
\partial_t \mathcal{E}(w) \leq - \langle \partial_x^3 w, Pb(\bar{u}) \partial_x^3 w \rangle + C \left( |w|^2_{H^2} + |z|^2_{H^4} \right),
\]
\[
\geq - \frac{\theta}{2} |\partial_x^3 w|^2_{L^2} + C \left( |w|^2_{H^2} + |z|^2_{H^4} \right),
\]
\[\theta > 0, \text{ where } \mathcal{E}(w) := \sum_{j=0}^2 \langle \partial_x^j w, Pb(\bar{u}) \partial_x^j w \rangle \text{ is equivalent to } |\cdot|^2_{H^2}. \]

By Sobolev interpolation and (2.16), we have therefore
\[
\partial_t \mathcal{E}(w) \leq - \tilde{\theta} \mathcal{E} + C \left( |w|^2_{L^2} + |z|^2_{L^4} \right),
\]
\[\tilde{\theta} > 0, \text{ from which (2.15) follows again by Gronwall’s inequality.} \]
Proposition 2.6. Under assumptions (A0)–(A3), if solutions $w_1, z_1$ and $w_2, z_2$ of (2.11) exist for all $t \geq 0$, then, for $\delta$ sufficiently small, any constant $\theta \geq 0$ and some $C = C(\theta)$, for all $t \geq 0$,

\begin{equation}
\int_0^t e^{-\theta s}|w_1 - w_2|^2_{H^2}(s) ds \leq C|w_{0,1} - w_{0,2}|^2_{H^1} \\
+ C \int_0^t e^{-\theta s}(|w_1 - w_2|^2_{L^2} + |z_1 - z_2|^2_{L^2})(s) ds, \tag{2.18}
\end{equation}

\begin{equation}
\int_t^{+\infty} e^{\theta(t-s)}|w_1 - w_2|^2_{H^2}(s) ds \leq C|w_1 - w_2|^2_{H^1}(t) \\
+ C \int_t^{+\infty} e^{\theta(t-s)}(|w_1 - w_2|^2_{L^2} + |z_1 - z_2|^2_{L^2})(s) ds \leq C e^{-\theta t}|w_{0,1} - w_{0,2}|^2_{H^1} \\
+ C e^{2\eta t}(|w_1 - w_2|^2_{L^2} + \| z_1 - z_2 \|^2_{L^2}). \tag{2.19}
\end{equation}

Proof. Subtracting the equations for $w_1, z_1$ and $w_2, z_2$, we obtain, denoting $\dot{w} := w_1 - w_2, \dot{z} := z_1 - z_2$, the equation

\begin{equation}
\dot{w}_t - b(\ddot{u})\dot{w}_{xx} = \Pi cs b(\ddot{u})\dot{z}_{xx} - \Pi u b(\ddot{u})\dot{\omega}_{xx} \\
+ \Pi cs M(\dot{w} + \dot{z}) + \Pi cs (N^\delta(w_1 + z_1) - N^\delta(w_2 + z_2)) \tag{2.20}
\end{equation}

with initial data $\dot{w}_0 = w_{0,1} - w_{0,2}$ at $t = 0$.

Performing an $H^1$ version of the energy estimate in the proof of Proposition 2.5– that is, taking the $L^2$ inner product in $x$ of $\sum_{j=0}^1 \partial_x^j \dot{P}(\ddot{u})\partial_x^j \dot{\omega}$ against (2.20), $P$ as in the proof of Proposition 2.5, integrating by parts, and rearranging the resulting terms– we obtain

\begin{equation}
\partial_x \mathcal{E}(\dot{w}) \leq -\frac{\alpha}{2} |\partial_x^2 \dot{w}|^2_{L^2} + C(\| \dot{w} \|^2_{H^1} + \| \dot{z} \|^2_{H^2}), \tag{2.21}
\end{equation}

$\alpha > 0$, where $\mathcal{E}(\dot{w}) := \sum_{j=0}^1 (\partial_x^j \dot{w}, P b(\ddot{u})\partial_x^j \dot{\omega})$ is equivalent to $|\dot{w}|^2_{H^1}$. Here, we have used in a key way the $H^2 \to L^2$ Lipschitz bound on $N^\delta$ to bound

\begin{equation}
|N^\delta(w_1 + z_1) - N^\delta(w_2 + z_2)|_{L^2} \leq C \delta(\| \dot{w} \|_{H^2} + \| \dot{z} \|_{H^2}), \quad \delta < < 1,
\end{equation}

and thus \( \langle \partial_x \dot{w}, P b(\ddot{u})\partial_x \dot{w} \rangle, N^\delta(w_1 + z_1) - N^\delta(w_2 + z_2) \rangle \leq C(\| \dot{w} \|^2_{H^2} + \| \dot{z} \|^2_{H^2}) \).
By Sobolev interpolation and (2.16), we have therefore

\begin{equation}
\partial_t E(\dot{w}) + |\dot{w}|^2_{H^2} \leq C(|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2}),
\end{equation}

whence (2.18) follows by Gronwall's inequality, \( E \geq 0 \), and \( |E(\dot{w}_0)| \leq C|\dot{w}_0|^2_{H^1} \).

The first line of (2.19) follows similarly, using \( |E(\dot{w}(t))| \leq C|\dot{w}|^2_{H^1}(t) \).

Noting that we could in place of (2.22) have rearranged (2.21) as

\begin{equation}
\partial_t E(\dot{w}) \leq -\frac{\alpha}{2} E(\dot{w}) + C(|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2}),
\end{equation}

and argued as in Proposition 2.5 to obtain

\begin{equation}
|\dot{w}(t)|^2_{H^1} \leq Ce^{-\theta t}|\dot{w}_0|^2_{H^1} + C \int_0^t e^{-\theta(t-s)}(|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2})(s) \, ds
\end{equation}

substituting in the first line of (2.19), and estimating similarly

\begin{equation}
\int_0^{+\infty} e^{\theta(t-s)}(|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2})(s) \, ds \leq Ce^{2\eta t}(||\dot{w}||_{L^2_{-\eta}} + ||\dot{z}||_{L^2_{-\eta}}),
\end{equation}

we obtain the second line of (2.19), completing the proof.

**Remark 2.7.** The absence of a uniform \( H^3 \to H^1 \) Lipschitz bound on \( N^\delta \) prevents us from obtaining a pointwise \( H^2 \) energy estimate on \( \dot{w} \) like the one obtained on \( w \) in Proposition 2.5.

### 2.4 Basic existence result

Define now the negatively-weighted sup norm

\[ ||f||_{-\eta} := \sup_{t \geq 0} e^{-\eta t} |f(t)|_{H^2}, \]

noting that \( |f(t)|_{H^2} \leq e^{\tilde{\eta} t} ||f||_{-\tilde{\eta}} \) for all \( t \geq 0 \), and denote by \( \mathcal{B}_{-\eta} \) the Banach space of functions bounded in \( || \cdot ||_{-\eta} \) norm. Define also the auxiliary norm

\[ ||f||_{L^2_{-\eta}} := \sup_{t \geq 0} e^{-\eta t} |f(t)|_{L^2}. \]
Lemma 2.8. Under assumptions (A0)–(A3), for $3\omega < \eta < \beta$ and $\delta > 0$ and $w_0 \in H^2$ sufficiently small, for each $z \in B_{-\eta}$, there exists a unique solution $w =: \mathcal{W}(z, w_0) \in B_{-\eta}$ of (2.10), (2.11), with

\begin{equation}
\|w\|_{-\eta} \leq C(|w_0|_{H^2} + \delta \|z\|_{-\eta}) \tag{2.24}
\end{equation}

and

\begin{equation}
\|\mathcal{W}(z_1, w_{0,1}) - \mathcal{W}(z_2, w_{0,2})\|_{L^2_{-\eta}} \leq C\delta\|z_1 - z_2\|_{-\eta} + C\|w_{0,1} - w_{0,2}\|_{-\eta} \tag{2.25}
\end{equation}

Proof. (i) ($H^2$ bound) By short-time $H^2$-existence theory, and the earlier-observed fact (2.16), an $H^2$ solution of (2.11) exists and remains bounded $H^2$ up to some time $T > 0$ provided that $w_0$ is bounded in $H^2$, whereupon (2.15) holds.

Using now the integral representation (2.10), and applying (2.4)(i), (2.7)(i) $\Rightarrow |N^t(v(t))| \leq \delta|v(t)|$, and $|w(t)|_{H^2} \leq e^{\eta|t|}\|w\|_{-\eta, T} := \sup_{0 \leq s \leq T} e^{-\eta s}|w(s)|_{H^2}$, we obtain for $0 \leq t \leq T$ that

$$|w(t)|_{L^2} \leq C e^{\omega|t|}|w_{cs}|_{L^2} + C\delta(\|w\|_{-\eta, T} + \|z\|_{-\eta}) \int_0^t e^{\omega|t-s|e^{\eta|s|}}ds,$$

hence, using $\omega \pm \eta > 0$ that $|w(t)|_{L^2} \leq C e^{\eta|t|}\left(|w_{cs}|_{L^2} + \delta(\|w\|_{-\eta, T} + \|z\|_{-\eta})\right)$.

Applying (2.15), we then obtain

$$|w(t)|_{H^2} \leq C e^{\eta|t|}\left(|w_{cs}|_{H^2} + \delta(\|w\|_{-\eta, T} + \|z\|_{-\eta})\right)$$

for all $0 \leq t \leq T$, and thus $\|w\|_{-\eta, T} \leq C\left(|w_{cs}|_{H^2} + \delta(\|w\|_{-\eta, T} + \|z\|_{-\eta})\right)$, whence, for $\delta$ sufficiently small, $\|w\|_{-\eta, T} \leq C(|w_{cs}|_{H^2} + \|z\|_{-\eta})$. Since this bound is independent of $T$, we obtain by continuation global existence of $w$ and, letting $T \to \infty$, (2.24) as claimed.

(ii) ($H^2 \to L^2$ Lipschitz bounds) Now consider a pair of data $z_1, w_{0,1}$ and $z_2, w_{0,2}$, and compare the resulting solutions, denoting $(\hat{z}, \hat{w}, \hat{w}_0) := (z_1 - z_2, w_1 - w_2, w_{0,1} - w_{0,2})$. Using the integral representation (2.10), and applying (2.4)(i), (2.7)(i), and the definition of $\|\cdot\|_{-\eta}$, we obtain for all $t \geq 0$

\begin{equation}
|\hat{w}(t)|_{L^2} \leq C e^{\omega|t|}|\hat{w}_0|_{L^2} + C\delta \int_0^t e^{\omega|t-s|e^{\eta|s|}}ds \tag{2.26}
\end{equation}
Estimating as before

\[ C\delta \int_0^t e^{\omega(t-s)}|\dot{z}|_{H^2}(s) \, ds \leq C\delta \|\dot{z}\|_{-\eta} \int_0^t e^{\omega(t-s)} e^{\eta s} \, ds \leq C\delta \|\dot{z}\|_{-\eta} e^{\eta t} \]

and, by (2.18) with \( \theta = 3\omega \) together with the Cauchy–Schwarz inequality,

\[ C\delta \int_0^t e^{\omega(t-s)}|\dot{w}|_{H^2}(s) \, ds \leq C\delta \left( \int_0^t e^{-\omega(t-s)} \right)^{1/2} \left( \int_0^t e^{3\omega(t-s)}|\dot{w}|^2_{H^2}(s) \, ds \right)^{1/2} \]

\[ \leq C\delta \left( e^{3\omega t} |w_0|^2_{H^1} + \int_0^t e^{3\omega(t-s)}(|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2})(s) \, ds \right)^{1/2}, \]

we obtain, substituting in (2.26),

(2.27)

\[ |\dot{w}(t)|^2_{L^2} \leq C e^{6\omega t} |\dot{w}_0|^2_{H^1} + C\delta^2 \left( \|\dot{z}\|_{-\eta}^2 e^{2\eta t} + \int_0^t e^{3\omega(t-s)}(|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2})(s) \, ds \right) \]

\[ \leq e^{2\eta t} \left( C |\dot{w}_0|^2_{H^1} + C\delta^2 (\|\dot{w}\|_{L^2}^2 + \|\dot{z}\|_{L^2}^2) \right). \]

This yields

\[ \|\dot{w}(t)\|_{L^2_{-\eta}} \leq C|\dot{w}_0|_{-\eta} + C\delta (\|\dot{w}\|_{L^2_{-\eta}} + \|\dot{z}\|_{-\eta}), \]

from which (2.25) follows by smallness of \( \delta \).

\[ \square \]

**Proof of Proposition 1.1. (i) (Boundedness on a ball)** Again, working in \( B_{-\eta} \), recall that

\[ z(t) = \tilde{T}(z, w_0)(t) := -\int_t^{+\infty} e^{L(t-s)} \Pi_u N^\delta((w + z)(s)) \, ds, \]

where \( w = \mathcal{W}(z, w_0) \). Using (2.9), and applying (2.4)(ii), (2.7)(ii), \( |N^\delta(v(t))| \leq \delta|v(t)| \), and \( |z(t)|_{H^2} \leq e^{\eta t} \|z\|_{-\eta} \) and, by (2.24),

\[ |w(t)|_{H^2} \leq e^{\eta t} \|w\|_{-\eta} \leq C e^{\eta t} (|w_0|_{H^2} + \delta \|z\|_{-\eta}), \]

we obtain for \( 0 \leq t \) that

\[ |\tilde{T}(z, w_0)(t)| \leq C\delta (|w_0|_{H^2} + \delta \|z\|_{-\eta}) \int_t^{+\infty} e^{\beta(t-s)} e^{\eta s} \, ds, \]

hence, using \( \beta \geq \eta > 0 \) and taking \( \delta \) and \( |w_0|_{H^2} \) sufficiently small, that \( \tilde{T}(\cdot, w_0) \) maps the ball \( B(0, r) \) to itself, for \( r > 0 \) arbitrarily small but fixed.

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(i) (Lipschitz bounds) Similarly, we find using (2.25), (2.7)(ii), and Lemma 2.2 that
\begin{align}
\|\tilde{T}(z_1, w_{0,1}) - \tilde{T}(z_2, w_{0,2})\|_{-\eta} & \leq \sup_t C\delta e^{-rt} \int_t^{+\infty} e^{\beta(t-s)}(|\dot{w}|_{H^2} + |\dot{z}|_{H^2})(s) \, ds \\
& \leq \sup_t C_1 \delta \left(\|\dot{z}\|_{-\eta} + e^{-rt} \int_t^{+\infty} e^{\beta(t-s)}|\dot{w}|_{H^2}(s) \, ds\right).
\end{align}

Using the Cauchy–Schwarz inequality and (2.19) with \(\theta = \beta\) to estimate
\[
\int_t^{+\infty} e^{\beta(t-s)}|\dot{w}|_{H^2}(s) \, ds \leq \left(\int_t^{+\infty} e^{\beta(t-s)} ds\right)^{1/2} \left(\int_t^{+\infty} e^{\beta(t-s)}|\dot{w}|^2_{H^2}(s) \, ds\right)^{1/2} \\
\leq C_3 \left(|\dot{w}|^2_{H^1}(t) + \int_0^t e^{\beta(t-s)}(|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2})(s) \, ds\right)^{1/2} \\
\leq C_3 e^{\eta t} \left(|\dot{w}_0|_{H^1} + \|\dot{w}\|_{L^2_{-\eta}} + \|\dot{z}\|_{L^2_{-\eta}}\right)
\]
and applying (2.25), we obtain
\begin{align}
\|\tilde{T}(z_1, w_{0,1}) - \tilde{T}(z_2, w_{0,2})\|_{-\eta} & \leq C\delta(\|\dot{w}_0\|_{-\eta} + \|\dot{z}\|_{L^2_{-\eta}}).
\end{align}

This yields at once contractivity on \(B(0, r)\), hence existence of a unique fixed point \(z = Z(w_0)\), and Lipschitz continuity of \(Z\) from \(\Sigma_{cs}\) to \(\mathcal{B}_{-\eta}\), by the Banach Fixed-Point Theorem with Lipschitz dependence on parameter \(w_0\).

(ii) (Existence of a Lipschitz invariant manifold) Defining
\begin{align}
\Phi(w_0) := Z(w_0)|_{t=0} = -\int_0^{+\infty} e^{-Lt}\Pi_u N^\delta(v(s)) \, ds,
\end{align}
we obtain a Lipschitz function from \(\Sigma_{cs} \rightarrow \Sigma_u\), whose graph over \(B(0, r)\) is the invariant manifold of solutions of (2.8) growing at exponential rate \(|v(t)| \leq Ce^{\eta t}\) in forward time. From the latter characterization, we obtain evidently invariance in forward time. Since the truncated equations (2.8) agree with the original PDE so long as solutions remain small in \(H^2\), this gives local invariance with respect to (1.1) as well. By uniqueness of fixed point solutions, we have \(Z(0) = 0\) and thus \(\Phi_{cs}(0) = 0\), so that the invariant manifold passes through the origin. Likewise, any bounded, sufficiently small solution of (1.1) in \(H^2\) is a bounded small solution of (2.8) as well, so by uniqueness is contained in the center stable manifold.
(ii) (Quadratic-order tangency) By (2.30), (2.4), (2.5), and (2.24),

\[
|\Phi(w_{cs})|_{H^2} = \left| \int_{0}^{+\infty} e^{-Ls}\Pi uN^\delta(w + z)(s) \, ds \right|_{H^2} \\
\leq C \int_{0}^{+\infty} e^{-\beta s}(|W|_{H^2}^2 + |Z|_{H^2}^2)(s) \, ds \\
\leq C_1(\|W\|_{L^\infty}^2 + \|Z\|_{L^\infty}^2) \leq C_2(|w_{cs}|_{H^2}^2 + \|Z\|_{L^\infty}^2).
\]

By \(Z(0) = 0\) and Lipshitz continuity of \(Z\), we have \(\|Z\|_{L^\infty} \leq C\|w_{cs}\|_{H^2}\), whence

\[
(2.31) \quad |\Phi(w_{cs})|_{H^2} \leq C_3\|w_{cs}\|_{H^2}^2,
\]

verifying quadratic-order tangency at the origin.

### 2.5 Translation-invariance

We conclude by indicating briefly how to recover translation-invariance of the center stable manifold, following [Z5, TZ1]. Differentiating with respect to \(\alpha\) the relation \(0 = F(\bar{u}(x + \alpha))\), we recover the standard fact that \(\phi := \bar{u}_x\)

is an \(L^2\) zero eigenfunction of \(L\), by the assumed decay of \(\bar{u}_x\).

Define orthogonal projections

\[
(2.32) \quad \Pi_2 := \frac{\phi \langle \phi, \cdot \rangle}{|\phi|^2_{L^2}}, \quad \Pi_1 := \text{Id} - \Pi_2,
\]

onto the range of right zero-eigenfunction \(\phi := (\partial/\partial x)\bar{u}\) of \(L\) and its orthogonal complement \(\phi^\perp\) in \(L^2\), where \(\langle \cdot, \cdot \rangle\) denotes standard \(L^2\) inner product.

**Lemma 2.9.** Under the assumed regularity \(h \in C^{k+1}, \ k \geq 2, \ Pi_j, j = 1, 2\) are bounded as operators from \(H^s\) to itself for \(0 \leq s \leq k + 2\).

**Proof.** Immediate, by the assumed decay of \(\phi = \bar{u}_x\) and derivatives. \(\square\)

**Proof of Theorem 1.2.** Introducing the shifted perturbation variable

\[
(2.33) \quad v(x, t) := u(x + \alpha(t), t) - \bar{u}(x)
\]

we obtain the modified nonlinear perturbation equation

\[
(2.34) \quad \partial_t v = Lv + N(v) - \partial_t \alpha(\phi + \partial_x v),
\]
where \( L := \frac{\partial^2}{\partial u^2} (\bar{u}) \) and \( N \) as in (3.17) is a quadratic-order Taylor remainder.

Choosing \( \partial_\alpha \alpha \) so as to cancel \( \Pi_2 \) of the righthand side of (2.34), we obtain the reduced equations

\[
(2.35) \quad \partial_t v = \Pi_1 (Lv + N(v))
\]

and

\[
(2.36) \quad \partial_t \alpha = \frac{\pi_2 (Lv + N(v))}{1 + \pi_2 (\partial_x v)}
\]

\( v \in \phi^\perp, \pi_2 v := \langle \tilde{\phi}, v \rangle|\phi|^2_{L^2}, \) of the same regularity as the original equations.

Here, we have implicitly chosen \( \alpha(0) \) so that \( v(0) \in \phi^\perp, \) or

\[
\langle \phi, u_0(x+\alpha) - \bar{u}(x) \rangle = 0.
\]

Assuming that \( u_0 \) lies in a sufficiently small tube about the set of translates of \( \bar{u} \), or \( u_0(x) = (\bar{u} + w)(x - \beta) \) with \( |w|_{H^2} \) sufficiently small, this can be done in a unique way such that \( \tilde{\alpha} := \alpha - \beta \) is small, as determined implicitly by

\[
0 = \mathcal{G}(w, \tilde{\alpha}) := \langle \phi, \bar{u}(\cdot + \tilde{\alpha}) - \bar{u}(\cdot) \rangle + \langle \phi, w(\cdot + \alpha) \rangle,
\]

an application of the Implicit Function Theorem noting that

\[
\partial_{\tilde{\alpha}} \mathcal{G}(0, 0) = \langle \phi, \partial_x \bar{u}(\cdot) \rangle = |\phi|^2_{L^2} \neq 0.
\]

With this choice, translation invariance under our construction is clear, with translation corresponding to a constant shift in \( \alpha \) that is preserved for all time.

Clearly, (2.36) is well-defined so long as \( |\partial_x v|_{L^\infty} \leq C|v|_{H^2} \) remains small, hence we may solve the \( v \) equation independently of \( \alpha \), determining \( \alpha \)-behavior afterward to determine the full solution

\[
u(x,t) = \bar{u}(x - \alpha(t)) + v(x - \alpha(t),t).
\]

Moreover, it is easily seen (by the block-triangular structure of \( L \) with respect to this decomposition) that the linear part \( \Pi_1 L = \Pi_1 L \Pi_1 \) of the \( v \)-equation possesses all spectrum of \( L \) apart from the zero eigenvalue associated with eigenfunction \( \phi \). Thus, we have effectively projected out this zero-eigenfunction, and with it the group symmetry of translation.

We may therefore construct the center stable manifold for the reduced equation (2.35), automatically obtaining translation-invariance when we extend to the full evolution using (2.36). See [TZ1, Z5] for further details.
2.6 Application to viscous shock waves

Proof of Proposition 1.3. Clearly, (H0) implies (A0) by the form (1.4) of $h$, while (H1) and (A1) are identical. Plugging $u = \bar{u}(x)$ into (1.5), we obtain the standing-wave ODE $f(\bar{u})_x = (b(\bar{u})\bar{u}_x)_x$, or, integrating from $-\infty$ to $x$, the first-order system

$$b(\bar{u})\bar{u}_x = f(\bar{u}) - f(u_-).$$

Linearizing about the assumed critical points $u_\pm$ yields linearized systems $w_t = df(u_\pm)w$, from which we see that $u_\pm$ are nondegenerate rest points by (H2), as a consequence of which (A3) follows by standard ODE theory [Co].

Finally, linearizing PDE (1.2) about the constant solutions $u \equiv u_\pm$, we obtain $w_t = L_{\pm}w := -df(u_\pm)w_x - b(u_\pm)w_{xx}$. By Fourier transform, the limiting operators $L_{\pm}$ have spectra $\lambda_{\pm}^j(k) = \sigma(-ikA_{\pm} - b_{\pm}k^2)$, where the Fourier wave-number $k$ runs over all of $\mathbb{R}$; in particular, $L_{\pm}$ have spectra of nonpositive real part by (H3). By a standard result of Henry [He], the essential spectrum of $L$ lies to the left of the rightmost boundary of the spectra of $L_{\pm}$, hence we may conclude that the essential spectrum of $L$ is entirely nonpositive. As the spectra of $L$ to the right of the essential spectrum by sectoriality of $L$, consists of finitely many discrete eigenvalues, this means that the spectra of $L$ with positive real part consists of $p$ unstable eigenvalues, for some $p$, verifying (A2).

3 Conditional stability analysis

From now on we specialize to the case (1.4), (1.5) of a system of viscous conservation laws $u_t = \mathcal{F}(u) = (b(u)u_x)_x - f(u)x$, with associated linearized operator $L = \partial\mathcal{F}u(\bar{u})$.

3.1 Projector bounds

Let $\Pi_u$ denote the eigenprojection of $L$ onto its unstable subspace $\Sigma_u$, and $\Pi_{cs} = \text{Id} - \Pi_u$ the eigenprojection onto its center stable subspace $\Sigma_{cs}$.

Lemma 3.1 ([Z5]). Assuming (H0)–(H1),

$$\Pi_j \partial_x = \partial_x \Pi_j \tag{3.1}$$

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for \( j \) = 1, \( cs \) and, for all \( 1 \leq p \leq \infty \), \( 0 \leq r \leq 4 \),

\[
\|\Pi_u\|_{L^p \to W^{r,p}} , \|\tilde{\Pi}_u\|_{L^p \to W^{r,p}} \leq C ,
\]

and

\[
\|\Pi_{cs}\|_{W^{r,p} \to W^{r,p}} , \|\tilde{\Pi}_{cs}\|_{W^{r,p} \to W^{r,p}} \leq C ,
\]

\[
|(1 + |x|^2)^{3/4} \Pi_u f|^2_{H^4} \leq |f|^2_{L^p} , \\
|(1 + |x|^2)^{3/4} \tilde{\Pi}_u f|^2_{H^4} \leq |(1 + |x|^2)^{3/4} f|^2_{H^4} .
\]

Moreover,

\[
|\Pi_{cs} f(x)| \leq Ce^{-\theta |x|} \sup_y |e^{-\theta |y|} f(y)| .
\]

\textbf{Proof.} Recalling that \( L \) has at most finitely many unstable eigenvalues, we find that \( \Pi_u \) may be expressed as

\[
\Pi_u f = \sum_{j=1}^p \phi_j(x) \langle \tilde{\phi}_j , f \rangle ,
\]

where \( \phi_j \), \( j = 1, \ldots, p \) are generalized right eigenfunctions of \( L \) associated with unstable eigenvalues \( \lambda_j \), satisfying the generalized eigenvalue equation \( (L - \lambda_j)^r \phi_j = 0 \), \( r_j \geq 1 \), and \( \tilde{\phi}_j \) are generalized left eigenfunctions. Noting that \( L \) is divergence form, and that \( \lambda_j \neq 0 \), we may integrate \( (L - \lambda_j)^r \phi_j = 0 \) over \( \mathbb{R} \) to obtain \( \lambda_j^{r_j} \int \phi_j dx = 0 \) and thus \( \int \phi_j dx = 0 \). Noting that \( \phi_j \), \( \tilde{\phi}_j \) and derivatives decay exponentially by standard theory [He, ZH, MaZ1], we find that

\[
\phi_j = \partial_x \Phi_j
\]

with \( \Phi_j \) and derivatives exponentially decaying, hence

\[
\tilde{\Pi}_u f = \sum_j \Phi_j \langle \partial_x \tilde{\phi}_j , f \rangle .
\]

Estimating

\[
|\partial_x^2 \Pi_u f|_{L^p} = \left| \sum_j \partial_x^2 \phi_j \langle \tilde{\phi}_j f \rangle \right|_{L^p} \leq \sum_j |\partial_x^2 \phi_j|_{L^p} |\tilde{\phi}_j|_{L^q} |f|_{L^p} \leq C |f|_{L^p}
\]

for \( 1/p + 1/q = 1 \) and similarly for \( \partial_x^r \tilde{\Pi}_u f \), we obtain the claimed bounds on \( \Pi_u \) and \( \tilde{\Pi}_u \), from which the bounds on \( \Pi_{cs} = \text{Id} - \Pi_u \) and \( \tilde{\Pi}_{cs} = \text{Id} - \tilde{\Pi}_u \) follow immediately. Bounds (3.3) and (3.4) follow similarly. \( \square \)
3.2 Linear estimates

Let $G_{cs}(x, t; y) := \Pi_{cs} e^{Lt} \delta_y(x)$ denote the Green kernel of the linearized solution operator on the center stable subspace $\Sigma_{cs}$. Then, we have the following detailed pointwise bounds established in [TZ2, MaZ1].

**Proposition 3.2** ([TZ2, MaZ1]). Assuming $(H0)$–$(H5)$, $(D1)$–$(D2)$, the center stable Green function may be decomposed as $G_{cs} = E + \tilde{G}$, where

$$E(x, t; y) = \sum_{j=1}^{\ell} \frac{\partial^{\alpha_j} u}{\partial \alpha_j} |_{\alpha = \alpha_*} e_j(y, t),$$

$$e_j(y, t) = \sum_{a_k > 0} \left( \text{erf}_n \left( \frac{y + a_k^{-1} (t + 1)}{\sqrt{4\beta_k} (t + 1)} \right) - \text{erf}_n \left( \frac{y - a_k^{-1} t}{\sqrt{4\beta_k} (t + 1)} \right) \right) l_{jk}(y),$$

for $y \leq 0$ and symmetrically for $y \geq 0$, with

$$|l_{jk}^\pm| \leq C, \quad |(\partial/\partial y) l_{jk}^\pm| \leq C e^{-\eta|y|},$$

and

$$|\partial^s x \tilde{G}(x, t; y)| \leq C e^{-\eta(|x-y|+t)}$$

$$+ C (t^{-\frac{s}{2}} + e^{-\theta|x|}) \left( \sum_{k=1}^{n} t^{-\frac{1}{2}} e^{-(x-y-a_k^{-1} t)^2/M t e^{-\eta x^+}} \right)$$

$$+ \sum_{a_k > 0, a_j < 0} \chi_{\{|a_k^{-1} t| \geq |y|\}} t^{-1/2} e^{-(x-a_j^{-1} (t-|y/a_k|))^{2/M t e^{-\eta x^+}}}$$

$$+ \sum_{a_k > 0, a_j > 0} \chi_{\{|a_k^{-1} t| \geq |y|\}} t^{-1/2} e^{-(x-a_j^{-1} (t-|y/a_k|))^{2/M t e^{-\eta x^-}}}.$$

$$|\partial^s \partial_y \tilde{G}(x, t; y)| \leq C e^{-\eta(|x-y|+t)}$$

$$+ C t^{-\frac{s}{2}} (t^{-\frac{s}{2}} + e^{-\theta|x|} + \gamma e^{-\theta|y|}) \left( \sum_{k=1}^{n} t^{-1/2} e^{-(x-y-a_k^{-1} t)^2/M t e^{-\eta x^+}} \right)$$

$$+ \sum_{a_k > 0, a_j < 0} \chi_{\{|a_k^{-1} t| \geq |y|\}} t^{-1/2} e^{-(x-a_j^{-1} (t-|y/a_k|))^{2/M t e^{-\eta x^+}}}$$

$$+ \sum_{a_k > 0, a_j > 0} \chi_{\{|a_k^{-1} t| \geq |y|\}} t^{-1/2} e^{-(x-a_j^{-1} (t-|y/a_k|))^{2/M t e^{-\eta x^-}}}.$$
for $0 \leq s \leq 1$, for $y \leq 0$ and symmetrically for $y \geq 0$, for some $\eta$, $C$, $M > 0$, where $a_{j}^{\pm}$ are as in (1.6), $\beta_{k}^{\mp} > 0$, $x^{\pm}$ denotes the positive/negative part of $x$, indicator function $\chi_{\{|a_{k}^{-}t| \geq |y|\}}$ is $1$ for $|a_{k}^{-}t| \geq |y|$ and $0$ otherwise, and $\gamma = 1$ in the mixed or undercompressive case and $0$ in the pure Lax or overcompressive case.

**Proof.** This follows from the observation [TZ2] that the contour integral (inverse Laplace Transform) representation of $G_{cs}$ is exactly that for the full Green kernel in the stable case $p = 0$, and that the resolvent kernel satisfies the same bounds. Thus, we obtain the stated bounds by the same argument used in [MaZ1] to bound the full Green kernel in the stable case. See Appendix A for further discussion.

**Corollary 3.3 ([HZ]).** Assuming (H0)–(H5), (D1)–(D2), (3.10)

$$|e_{j}(y, t)| \leq C \sum_{a_{k}^{+} > 0} \left( \text{erf} \left( \frac{y + a_{k}^{-}t}{\sqrt{4\beta_{k}^{-}(t+1)}} \right) - \text{erf} \left( \frac{y - a_{k}^{-}t}{\sqrt{4\beta_{k}^{-}(t+1)}} \right) \right),$$

$$|e_{j}(y, t) - e_{j}(y, +\infty)| \leq C \text{erf} \left( |y| - at \frac{\sqrt{t}}{\sqrt{\eta \gamma}} \right),$$

$$|\partial_{t} e_{j}(y, t)| \leq C t^{-1/2} \sum_{a_{k}^{+} > 0} e^{-|y + a_{k}^{-}t|^{2}/Mt},$$

$$|\partial_{y} e_{j}(y, t)| \leq C t^{-1/2} \sum_{a_{k}^{+} > 0} e^{-|y + a_{k}^{-}t|^{2}/Mt} + C \gamma e^{-\eta |y|} \times \left( \text{erf} \left( \frac{y + a_{k}^{-}t}{\sqrt{4\beta_{k}^{-}(t+1)}} \right) - \text{erf} \left( \frac{y - a_{k}^{-}t}{\sqrt{4\beta_{k}^{-}(t+1)}} \right) \right),$$

$$|\partial_{y} e_{j}(y, t) - \partial_{y} e_{j}(y, +\infty)| \leq C t^{-1/2} \sum_{a_{k}^{+} > 0} e^{-|y + a_{k}^{-}t|^{2}/Mt}$$

$$|\partial_{tt} e_{j}(y, t)| \leq C (t^{-1} + \gamma t^{-1/2} e^{-\eta |y|}) \sum_{a_{k}^{+} > 0} e^{-|y + a_{k}^{-}t|^{2}/Mt},$$

for $y \leq 0$, and symmetrically for $y \geq 0$, where $\gamma$ as above is one for undercompressive profiles and zero otherwise.

**Proof.** Direct computation using definition (3.6); see [MaZ1].
3.3 Convolution bounds

From the above pointwise bounds, there follow by direct computation the following convolution estimates established in [HZ], here stated without proof.

Lemma 3.4 (Linear estimates [HZ]). Assuming (H0)–(H5), (D1)–(D2),

\[ \int_{-\infty}^{+\infty} |\tilde{G}(x,t;y)|(1 + |y|)^{-3/2} dy \leq C(\theta + \psi_1 + \psi_2)(x,t), \]
\[ \int_{-\infty}^{+\infty} |\tilde{G}_x(x,t;y)|(1 + |y|)^{-3/2} dy \leq C(t^{-\frac{1}{2}} + 1)(\theta + \psi_1 + \psi_2)(x,t), \]
\[ \int_{-\infty}^{+\infty} |\tilde{e}_t(y,t)|(1 + |y|)^{-3/2} dy \leq C(1 + t)^{-3/2}, \]
\[ \int_{-\infty}^{+\infty} |\tilde{e}(y,t)|(1 + |y|)^{-3/2} dy \leq C, \]
\[ \int_{-\infty}^{+\infty} |e(y,t) - e(y, +\infty)|(1 + |y|)^{-3/2} dy \leq C(1 + t)^{-1/2}, \]

for \(0 \leq t \leq +\infty, C > 0\), where \(\tilde{G}\) and \(e\) are defined as in Proposition 3.2.

Lemma 3.5 (Nonlinear estimates [HZ]). Under (H0)–(H5), (D1)–(D2),

\[ \int_{0}^{t} \int_{-\infty}^{+\infty} |\tilde{G}_y(x,t-s;y)|\Psi(y,s) dy ds \leq C(\theta + \psi_1 + \psi_2)(x,t), \]
\[ \int_{0}^{t-1} \int_{-\infty}^{+\infty} |\tilde{G}_{yx}(x,t-s;y)|\Psi(y,s) dy ds \leq C(\theta + \psi_1 + \psi_2)(x,t), \]
\[ \int_{t-1}^{t} \int_{-\infty}^{+\infty} |\tilde{G}_x(x,t-s;y)|(\theta + \psi_1 + \psi_2)(y,s) dy ds \leq C(\theta + \psi_1 + \psi_2)(x,t), \]
\[ \int_{0}^{t} \int_{-\infty}^{+\infty} |e_{yt}(y,t-s)|\Psi(y,s) dy ds \leq C(1 + t)^{-1}, \]
\[ \int_{t}^{+\infty} \int_{-\infty}^{+\infty} |e_y(y, +\infty)|\Psi(y,s) dy \leq C\gamma(1 + t)^{-1/2}, \]
\[ \int_{0}^{t} \int_{-\infty}^{+\infty} |e_y(y, t-s) - e_y(y, +\infty)|\Psi(y,s) dy ds \leq C(1 + t)^{-1/2}, \]
for $0 \leq t \leq +\infty$, $C > 0$, where $\tilde{G}$ and $e$ are as in Proposition 3.2 and
\begin{equation}
\Psi(y,s) := (1 + s)^{1/2} s^{-1/2} (\theta + \psi_1 + \psi_2)^2 (y,s) + (1 + s)^{-1} (\theta + \psi_1 + \psi_2) (y,s).
\end{equation}

We have by standard short-time theory the following additional bound.

**Lemma 3.6 (Commutator bound).** Assuming (H0)–(H5), (D1)–(D2),
\begin{equation}
\int_{-\infty}^{+\infty} |(\tilde{G}_x + \tilde{G}_y)(x,t;y)| (1 + |y|)^{-3/2} dy \leq C (\theta + \psi_1 + \psi_2)(x,t),
\end{equation}
for $0 \leq t \leq 1$, $C > 0$, where $\tilde{G}$ is as in Proposition 3.2.

**Proof.** By standard short-time parametrix bounds [Fr] on the entire Green function, we have for $0 \leq t \leq 1$ that $|G_x + G_y| \leq C t^{-1} e^{-\frac{|x-y|^2}{4t}}$ is of the same order order as $\tilde{G}$, whence
\begin{equation}
\int_{-\infty}^{+\infty} |(G_x + G_y)(x,t;y)| (1 + |y|)^{-3/2} dy \leq C (1 + |x|)^{-3/2}.
\end{equation}

By direct computation, integration against $|\partial_x G_u|$, $|\partial_y G_u|$, $|\partial_x E|$, or $|\partial_y E|$ gives a contribution that is also bounded by $C (1 + |x|)^{-3/2}$, yielding the result. 

### 3.4 Reduced equations II

We now restrict to the pure Lax or undercompressive case $\ell = 1$, following the simple stability argument of Section 3, [HZ]. The pure Lax or overcompressive case may be carried out following the similar but slightly more complicated argument of Section 4, [HZ].

Define similarly as in Section 2.5 the perturbation variable
\begin{equation}
v(x,t) := u(x + \alpha(t), t) - \bar{u}(x)
\end{equation}
for $u$ a solution of (1.5), where $\alpha$ is to be specified later in a way appropriate for the task at hand. Subtracting the equations for $u(x + \alpha(t), t)$ and $\bar{u}(x)$, we obtain the nonlinear perturbation equation
\begin{equation}
v_t - Lv = N(v)_x - \partial_t \alpha (\phi + \partial_x v),
\end{equation}
where $L$ as in (2.2) denotes the linearized operator about $\bar{u}$, $\phi = \bar{u}_x$, and

(3.17) \quad N(v) := -(f(\bar{u} + v) - f(\bar{u}) - df(\bar{u})v)

where, so long as $|v|_{H^1}$ (hence $|v|_{L^\infty}$ and $|u|_{L^\infty}$) remains bounded,

(3.18)

\[
N(v) = O(|v| |v_x|), \quad \partial_x N(v) = O(|v| |v_{xx}| + |v_x|^2).
\]

Recalling that $\partial_x \bar{u}$ is a stationary solution of the linearized equations $u_t = Lu$, so that $L \partial_x \bar{u} = 0$, or

\[
\int_{-\infty}^{\infty} G(x, t; y) \bar{u}_x(y) dy = e^{Lt} \bar{u}_x(x) = \partial_x \bar{u}(x),
\]

we have, applying Duhamel’s principle to (3.16),

\[
v(x, t) = \int_{-\infty}^{\infty} G(x, t; y)v_0(y) dy - \int_0^t \int_{-\infty}^{\infty} G_y(x, t - s; y)(N(v) + \dot{\alpha} v)(y, s) dy ds + \alpha(t) \partial_x \bar{u}(x).
\]

Defining

\[
\alpha(t) = -\int_{-\infty}^{\infty} e(y, t)v_0(y) dy + \int_0^t \int_{-\infty}^{\infty} e_y(y, t - s)(N(v) + \dot{\alpha} v)(y, s) dy ds,
\]

(3.19)

following [ZH, Z4, MaZ2, MaZ3], where $e$ is defined as in (3.6), and recalling the decomposition $G = E + G_u + \tilde{G}$ of (A.2), we obtain the reduced equations

(3.20)

\[
v(x, t) = \int_{-\infty}^{\infty} (G_u + \tilde{G}_y)(x, t; y)v_0(y) dy - \int_0^t \int_{-\infty}^{\infty} (G_u + \tilde{G})_y(x, t - s; y)(N(v) + \dot{\alpha} v)(y, s) dy ds,
\]

and, differentiating (3.19) with respect to $t$, and observing that $e_y(y, s) \to 0$ as $s \to 0$, as the difference of approaching heat kernels,

(3.21)

\[
\dot{\alpha}(t) = -\int_{-\infty}^{\infty} e_t(y, t)v_0(y) dy + \int_0^t \int_{-\infty}^{\infty} e_{yt}(y, t - s)(N(v) + \dot{\alpha} v)(y, s) dy ds.
\]
We emphasize that this (nonlocal in time) choice of $\alpha$ and the resulting reduced equations are different from those of Section 2.5. As discussed further in [Go, Z4, MaZ2, MaZ3, Z2], $\alpha$ may be considered in the present context as defining a notion of approximate shock location.

3.5 Nonlinear damping estimate

**Proposition 3.7 ([MaZ3]).** Assuming (H0)-(H5), let $v_0 \in H^4$, and suppose that for $0 \leq t \leq T$, the $H^4$ norm of $v$ remains bounded by a sufficiently small constant, for $v$ as in (3.15) and $u$ a solution of (1.5). Then, for some constants $\theta_{1,2} > 0$, for all $0 \leq t \leq T$,

$$ (3.22) \quad |v(t)|^2_{H^4} \leq C e^{-\theta_1 t} |v(0)|^2_{H^4} + C \int_0^t e^{-\theta_2 (t-s)} (|v|^2_{L^2} + |\dot{\alpha}|^2(s)) \, ds. $$

**Proof.** Energy estimates essentially identical to those in the proof of Proposition 2.5, but involving the new forcing term $\dot{\alpha}(t)\partial_x \bar{u}(x)$. Observing that $\partial_x^j (\partial_x \bar{u})(x) = O(e^{-\eta|x|})$ is bounded in $L^1$ norm for $j \leq 4$, we obtain (in the simpler, uniformly elliptic case), taking account of this new contribution, the inequality

$$ \partial_t |v|^2_{H^4} \leq -\theta |\partial_x^5 v|^2_{L^2} + C \left( |v|^2_{H^4} + |\dot{\alpha}(t)|^2 \right), $$

$\theta > 0$, for $C > 0$ sufficiently large, so long as $|v|^2_{H^4}$ remains sufficiently small, yielding (3.22) as before by Sobolev interpolation followed by Gronwall’s inequality. See also [MaZ3, RZ].

3.6 Short time existence theory

**Lemma 3.8 ([RZ]).** Assuming (H0)-(H5), let $M_0 := |(1 + |x|^2)^{3/4} v_0(x)|_{H^4} < \infty$, and suppose that, for $0 \leq t \leq T$, the supremum of $|\dot{\alpha}|$, and the $H^4$ norm of $v$, determined by (3.16), each remain bounded by some constant $C > 0$. Then there exists some $M = M(C) > 0$ such that, for all $0 \leq t \leq T$,

$$ (3.23) \quad |(1 + |x|^2)^{3/4} v(x,t)|^2_{H^4} \leq M e^{M t} (M_0 + \int_0^t |\dot{\alpha}|^2(\tau) \, d\tau). $$

**Proof.** This follows by standard Friedrichs symmetrizer estimates carried out in the weighted $H^4$ norm. Specifically, making the coordinate change

$$ v = (1 + |x|^2)^{3/4} w, $$

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we obtain from (3.16) a similar equation plus lower-order commutator terms, and similarly in the equations for $\partial_j^x w$ for $j = 1, \ldots, 4$. Performing the same energy estimates as carried out on (3.16) in the proof of Lemma 3.7, we readily obtain the result by Gronwall’s inequality. We refer to [RZ, Lemma 5.2] for further details in the general partially parabolic case.

**Remark 3.9.** Using Sobolev embeddings and equation (3.16), we see that Lemma 3.8 immediately implies that, if $|(1 + |x|^2)^{3/4} v_0(x)|_{H^4} < \infty$ and if $|v(\cdot, t)|_{H^4}$, $|\dot{\alpha}(t)|$ are uniformly bounded on $0 \leq t \leq T$, then

$$|(1 + |x|^2)^{3/4} v(x, t)| \quad \text{and} \quad |(1 + |x|^2)^{3/4} v_t(x, t)|$$

are uniformly bounded on $0 \leq t \leq T$ as well.

### 3.7 Proof of nonlinear stability

Decompose now the nonlinear perturbation $v$ as

$$v(x, t) = w(x, t) + z(x, t), \quad (3.24)$$

where

$$w := \Pi_{cs} v, \quad z := \Pi_u v. \quad (3.25)$$

Applying $\Pi_{cs}$ to (3.20) and recalling commutator relation (3.1), we obtain an equation

$$w(x, t) = \int_{-\infty}^{\infty} \tilde{G}(x, t; y) w_0(y) dy$$

$$- \int_0^t \int_{-\infty}^{\infty} \tilde{G}_y(x, t-s; y) \Pi_{cs} (N(v) + \dot{\alpha} v)(y, s) dy ds$$

(3.26)

for the flow along the center stable manifold, parametrized by $w \in \Sigma_{cs}$.

**Lemma 3.10.** Assuming (H0)–(H1), for $v$ lying initially on the center stable manifold $\mathcal{M}_{cs}$,

$$|\partial^r_x z(x, t)| \leq Ce^{-\theta|x||w|_{H^2}^2} \quad (3.27)$$

for all $0 \leq r \leq 4$, some $C > 0$, so long as $|w|_{H^2}$ remains sufficiently small.
Proof. By quadratic-order tangency at $\bar{u}$ of the center stable manifold to $\Sigma_{cs}$, estimate (1.3), we have immediately $|z|_{H^2} \leq C|w|^2_{H^2}$, whence (3.27) follows by equivalence of norms for finite-dimensional vector spaces, applied to the $p$-dimensional subspace $\Sigma_u$. (Alternatively, we may see this by direct computation using the explicit description of $\Pi_u v$ afforded by Lemma 3.1.)

Proof of Theorem 1.4. Recalling by Theorem 1.2 that solutions remaining for all time in a sufficiently small radius neighborhood $N$ of the set of translates of $\bar{u}$ lie in the center stable manifold $M_{cs}$, we obtain trivially that solutions not originating in $M_{cs}$ must exit $N$ in finite time, verifying the final assertion of orbital instability with respect to perturbations not in $M_{cs}$.

Consider now a solution $v \in M_{cs}$, or, equivalently, a solution $w \in \Sigma_{cs}$ of (3.26) with $z = \Phi_{cs}(w) \in \Sigma_u$. Define

$$\zeta(t) := \sup_{y, 0 \leq s \leq t} \left( |w| + |w_x|(\theta + \psi_1 + \psi_2)^{-1}(y, t) + |w|_{H^4}(1 + s)^{1/4} + |\dot{\alpha}(s)|(1 + s) \right).$$

(3.28)

We shall establish:

Claim. For all $t \geq 0$ for which a solution exists with $\zeta$ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \leq C_2(E_0 + \zeta(t)^2).$$

(3.29)

From this result, provided $E_0 < 1/4C_2^2$, we have that $\zeta(t) \leq 2C_2E_0$ implies $\zeta(t) < 2C_2E_0$, and so we may conclude by continuous induction that

$$\zeta(t) < 2C_2E_0$$

(3.30)

for all $t \geq 0$, from which we readily obtain the stated bounds. (By standard short-time $H^s$ existence theory together with Remark 3.9, $v \in H^4$ exists and $\zeta$ remains continuous so long as $\zeta$ remains bounded by some uniform constant, hence (3.30) is an open condition.)

Proof of Claim. By (3.3), $|(1 + |x|^2)^{3/4}w_0|_{H^4} = |(1 + |x|^2)^{3/4}(1)_{H^4} \leq CE_0$. Conversely, by Lemma 3.10, (3.28), (3.18), and (3.4), for $0 \leq s \leq t$,

$$|\bar{\Pi}_{cs}(N(v) + \dot{\alpha}v)(y, s)| \leq C\zeta(t)^2\Psi(y, s),$$

$$|(N(v) + \dot{\alpha}v)(y, s)| \leq C\zeta(t)^2\Psi(y, s),$$

(3.31)
$\Psi$ as in (3.13), while

$$|\Pi_{cs} \partial_y (N(v) + \dot{\alpha}v)(y, s)| \leq C \zeta(t)^2 C(1 + s)^{-\frac{3}{2}}(\theta + \psi_1 + \psi_2)(x, t).$$

Combining (3.31) and (3.32) with representations (3.26) and (3.21) and applying Lemmas 3.4 and 3.5, we obtain

$$|w(x, t)| \leq \int_{-\infty}^{\infty} |\tilde{G}(x, t; y)||w_0(y)| dy$$
$$+ \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x, t - s; y)||\Pi_{cs}(N(v) + \dot{\alpha}v)(y, s)| dy ds$$

(3.33)
$$\leq E_0 \int_{-\infty}^{\infty} |\tilde{G}(x, t; y)|(1 + |y|)^{-3/2} dy$$
$$+ C \zeta(t)^2 \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x, t - s; y)||\Psi(y, s)| dy ds$$
$$\leq C(E_0 + \zeta(t)^2)(\theta + \psi_1 + \psi_2)(x, t)$$

and, similarly,

$$|\dot{\alpha}(t)| \leq \int_{-\infty}^{\infty} |e_t(y, t)||u_0(y)| dy$$
$$+ \int_0^t \int_{-\infty}^{+\infty} |e_{yt}(y, t - s)||\Pi_{cs}(N(v) + \dot{\alpha}v)(y, s)| dy ds$$

(3.34)
$$\leq \int_{-\infty}^{\infty} E_0|e_t(y, t)|(1 + |y|)^{-3/2} dy$$
$$+ \int_0^t \int_{-\infty}^{+\infty} C \zeta(t)^2|e_{yt}(y, t - s)||\Psi(y, s)| dy ds$$
$$\leq C(E_0 + \zeta(t)^2)(1 + t)^{-1}.$$
Likewise, we may estimate for \( t \geq 1 \)

\[
\int_{-\infty}^{\infty} |\tilde{G}_x(x, t; y)||w_0(y)| \, dy \\
+ \int_0^{t} \int_{-\infty}^{\infty} |\tilde{G}_{xy}(x, t - s; y)||\tilde{\Pi}_{cs}(N(v) + \dot{\alpha}v)(y, s)| \, dy \, ds \\
+ \int_t^{\infty} \int_{-\infty}^{\infty} |\tilde{G}_x(x, t - s; y)||\Pi_{cs}\partial_y(N(v) + \dot{\alpha}v)(y, s)| \, dy \, ds \\
\leq E_0 \int_{-\infty}^{\infty} |\tilde{G}(x, t; y)|(1 + |y|)^{-3/2} \, dy \\
+ C\zeta(t)^2 \int_0^{t-1} \int_{-\infty}^{\infty} |\tilde{G}_{xy}(x, t - s; y)||\Psi(y, s)| \, dy \, ds \\
+ C\zeta(t)^2 \int_t^{\infty} \int_{-\infty}^{\infty} |\tilde{G}_x(x, t - s; y)|(\theta + \psi_1 + \psi_2)(y, s) \, dy \, ds \\
\leq C(E_0 + \zeta(t)^2)(\theta + \psi_1 + \psi_2)(x, t),
\]

and for \( 0 \leq t \leq 1 \), substitute for the first term on the righthand side instead

\[
\int_{-\infty}^{\infty} |(\tilde{G}_x + \tilde{G}_y)(x, t; y)||w_0(y)| \, dy + \int_{-\infty}^{\infty} |\tilde{G}(x, t; y)||\partial_yw_0(y)| \, dy
\]

to obtain the same bound with the aid of (3.14).

By Lemma 3.10,

\[
|z|_{H^4}(t) \leq C|w|_{H^2}^2(t) \leq C\zeta(t)^2.
\]

In particular, \(|z|_{L^2}(t) \leq C\zeta(t)^2(1 + t)^{-\frac{1}{2}}\). Applying Proposition 3.7 and using (3.33) and (3.34), we thus obtain

\[
|w|_{H^4}(t) \leq C(E_0 + \zeta(t)^2)(1 + t)^{-\frac{1}{4}}.
\]

Combining (3.33), (3.34), (3.35), and (3.37), we obtain (3.29) as claimed.

As discussed earlier, from (3.29), we obtain by continuous induction (3.30), or \( \zeta \leq 2C_2|v_0|_{L^1 \cap H^4} \), whereupon the claimed bounds on \(|v|, |v_x|\) and \(|v|_{H^4}\) follow by (3.33), (3.35) and (3.37) together with (3.27), and the bounds on \(|\dot{\alpha}|\) by (3.34). It remains to establish the bound on \(|\alpha|\), expressing convergence of phase \( \alpha \) to a limiting value \( \alpha(\infty) \).
By Lemmas 3.4–3.5 together with the previously obtained bounds (3.31) and \( \zeta \leq CE_0 \), and the definition (3.28) of \( \zeta \), the formal limit

\[
\alpha(\pm \infty) := \int_{-\infty}^{\infty} e(y, \pm \infty) u_0(y) \, dy
\]

\[
+ \int_{0}^{+\infty} \int_{-\infty}^{+\infty} e_y(y, \pm \infty)(N(v) + \dot{\alpha}v)(y) \, dy \, ds
\]

\[
\leq \int_{-\infty}^{\infty} E_0 |e(y, \pm \infty)| (1 + |y|)^{-3/2} \, dy
\]

\[
+ \int_{0}^{+\infty} \int_{-\infty}^{+\infty} CE_0 |e_y(y, \pm \infty)| \Psi(y, s) \, dy \, ds
\]

\[
\leq CE_0
\]

is well-defined, as the sum of absolutely convergent integrals.

Applying Lemmas 3.4–3.5 a final time, we obtain

\[
|\alpha(t) - \alpha(\pm \infty)| \leq \int_{-\infty}^{\infty} |e(y, t) - e(y, \pm \infty)| |v_0(y)| \, dy
\]

\[
+ \int_{0}^{t} \int_{-\infty}^{+\infty} |e_y(y, t-s) - e_y(y, \pm \infty)|
\]

\[
\times |(N(v) + \dot{\alpha}v)(y, s)| \, dy \, ds
\]

\[
+ \int_{t}^{+\infty} \int_{-\infty}^{+\infty} |e_y(y, \pm \infty)||Q(v) + \dot{\alpha}v)(y, s)| \, dy \, ds
\]

\[
\leq \int_{-\infty}^{\infty} E_0 |e(y, t) - e(y, \pm \infty)| (1 + |y|)^{-3/2} \, dy
\]

\[
+ \int_{0}^{t} \int_{-\infty}^{+\infty} |e_y(y, t-s) - e_y(y, \pm \infty)| CE_0 \Psi(y, s) \, dy \, ds
\]

\[
+ \int_{t}^{+\infty} \int_{-\infty}^{+\infty} |e_y(y, \pm \infty)| CE_0 \Psi(y, s) \, dy \, ds
\]

\[
\leq CE_0 (1 + t)^{-1/2},
\]

establishing the remaining bound and completing the proof.

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A Proofs of miscellaneous lemmas

We include for completeness the proofs of earlier cited lemmas that were not proved in the main body of the text.

Proof of Proposition 2.1. By sectoriality of $L$, we have the inverse Laplace transform representations

$$e^{tL}\Pi_u := \int_{\Gamma_u} e^{\lambda t}(\lambda - L)^{-1} d\lambda,$$

(A.1)

$$e^{tL}\Pi_{cs} := \int_{\Gamma_{cs}} e^{\lambda t}(\lambda - L)^{-1} d\lambda,$$

where $\Gamma_{cs}$ denotes a sectorial contour bounding the center and stable spectrum to the right [Pa], which by (A2) may be taken so that $\Re\Gamma_s \leq \omega$, and $\Gamma_u$ denotes a closed curve enclosing the unstable spectrum of $L$, with $\Re\Gamma_s \geq \beta > 0$. Estimating $|e^{tL}\Pi_j|_{L^2} \leq \int_{\Gamma_j} |e^{\lambda t}||\lambda - L||^{-1}|_{L^2}|d\lambda|$ using sectorial resolvent estimates, we obtain the stated bounds from $L^2 \to L^2$; see [He].

Applying the resolvent formula $L(\lambda - L)^{-1} = \lambda(\lambda - L)^{-1} - Id$, we obtain

$$Le^{tL}\Pi_j := \int_{\Gamma_j} e^{\lambda t}(\lambda - L)^{-1} d\lambda,$$

from which we obtain $|Le^{tL}\Pi_u|_{L^2} \leq Ce^{-\beta t}$ for $t \leq 0$ by boundedness of $\Gamma_u$, yielding the stated bound from $L^2 \to H^2$.

Proof of Lemma 2.4. Recalling that $L$ has at most finitely many unstable eigenvalues, we find that $\Pi_u$ may be expressed as

$$\Pi_u f = \sum_{j=1}^{p} \phi_j(x)\langle \tilde{\phi}_j, f \rangle,$$

where $\phi_j$, $j = 1, \ldots, p$ are generalized right eigenfunctions of $L$ associated with unstable eigenvalues $\lambda_j$, satisfying the generalized eigenvalue equation $(L - \lambda_j)^{r_j}\phi_j = 0$, $r_j \geq 1$, and $\tilde{\phi}_j$ are generalized left eigenfunctions. Noting that $\phi_j$, $\tilde{\phi}_j$ and derivatives decay exponentially by standard theory [He, ZH, MaZ1], and estimating

$$|\partial^2_x\Pi_u f|_{L^p} = |\sum_j \partial^2_x\phi_j \langle \tilde{\phi}_j, f \rangle|_{L^p} \leq \sum_j |\partial^2_x\phi_j|_{L^p}|\tilde{\phi}_j|_{L^2}|f|_{L^p} \leq C|f|_{L^p}$$
for $1/p + 1/q = 1$, we obtain the claimed bounds on $\Pi_u$, from which the bounds on $\Pi_{cs} = \text{Id} - \Pi_u$ follow immediately.

**Proof of Lemma 2.2.** The norm in $H^2$ is a quadratic form, hence the map

$$v \in H^2 \mapsto \rho\left(\frac{|v|_{H^2}}{\delta}\right) \in \mathbb{R}_+,$$

is smooth, and $N^\delta$ is as regular as $N$. The Lipschitz bound follows by

$$|N^\delta(v_1) - N^\delta(v_2)|_{L^2} \leq |\rho\left(\frac{|v_1|_{H^2}}{\delta}\right) - \rho\left(\frac{|v_2|_{H^2}}{\delta}\right)|_{L^\infty} |N(v_1)|_{L^2}$$

$$+ |\rho\left(\frac{|v_2|_{H^2}}{\delta}\right)|_{L^\infty} |N(v_1) - N(v_2)|_{L^2}$$

$$\leq 3|v_1 - v_2|_{H^2} \left( \sup_{|v|_{H^2} < \delta} \frac{|N(v)|_{L^2}}{\delta} + \sup_{|v|_{H^2} < \delta} |dN(v)|_{L^2} \right),$$

so that $\sup_{|v|_{H^2} < \delta} |N(v)|_{L^2} = O(\delta^2)$, $\sup_{|v|_{H^2} < \delta} |dN(v)|_{L^2} = O(\delta)$. Finally, $|N^\delta(v)|_{L^2} \leq |N(v)|_{L^2} \leq C |v|_{H^2}^2$ for $|v|_{H^2} \leq 2\delta$ by Moser’s inequality, while $N^\delta(v) \equiv 0$ for $|v|_{H^2} \geq 2\delta$, yielding (2.5).

**Proof of Proposition 3.2.** As observed in [TZ2], it is equivalent to establish decomposition

\[(A.2) \quad G = G_u + E + \tilde{G}\]

for the full Green function $G(x, t; y) := e^{Lt} \delta_y(x)$, where

$$G_u(x, t; y) := \Pi_u e^{Lt} \delta_y(x) = e^{\gamma t} \sum_{j=1}^{p} \phi_j(x) \tilde{\phi}_j(y)^t$$

for some constant matrix $M \in \mathbb{C}^{p \times p}$ denotes the Green kernel of the linearized solution operator on $\Sigma_u$, $\phi_j$ and $\tilde{\phi}_j$ right and left generalized eigenfunctions associated with unstable eigenvalues $\lambda_j$, $j = 1, \ldots, p$.

The problem of describing the full Green function has been treated in [ZH, MaZ3], starting with the Inverse Laplace Transform representation

\[(A.3) \quad G(x, t; y) = e^{Lt} \delta_y(x) = \int_{\Gamma} e^{\lambda t} (\lambda - L(\varepsilon))^{-1} \delta_y(x) d\lambda ,\]

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where
\[ \Gamma := \partial \{ \lambda : \Re \lambda \leq \eta_1 - \eta_2 |\Im \lambda| \} \]
is an appropriate sectorial contour, \( \eta_1, \eta_2 > 0 \); estimating the resolvent kernel
\[ G_\varepsilon(x,y) := (\lambda - L(\varepsilon))^{-1}\delta_y(x) \]
using Taylor expansion in \( \lambda \), asymptotic ODE techniques in \( x, y \), and judicious decomposition into various scattering, excited, and residual modes; then, finally, estimating the contribution of various modes to (A.3) by Riemann saddlepoint (Stationary Phase) method, moving contour \( \Gamma \) to an optimal, “minimax” positions for each mode, depending on the values of \( (x, y, t) \).

In the present case, we may first move \( \Gamma \) to a contour \( \Gamma' \) enclosing (to the left) all spectra of \( L \) except for the \( p \) unstable eigenvalues \( \lambda_j, j = 1, \ldots, p \), to obtain
\[ G(x; t; y) = \oint_{\Gamma'} e^{\lambda t}(\lambda - L)^{-1}d\lambda + \sum_{j=\pm} \text{Residue}_{\lambda_j(\varepsilon)}(e^{\lambda t}(\lambda - L)^{-1}\delta_y(x)) ; \]
where \( \text{Residue}_{\lambda_j(\varepsilon)}(e^{\lambda t}(\lambda - L)^{-1}\delta_y(x)) = G_u(x; t; y) \), then estimate the remaining term \( \oint_{\Gamma'} e^{\lambda t}(\lambda - L)^{-1}d\lambda \) on minimax contours as just described. See the proof of Proposition 7.1, [MaZ3], for a detailed discussion of minimax estimates \( E + G \) and of Proposition 7.7, [MaZ3] for a complementary discussion of residues incurred at eigenvalues in \( \{ \Re \lambda \geq 0 \} \setminus \{0\} \). See also [TZ2].  

References

[AGJ] J. Alexander, R. Gardner and C.K.R.T. Jones, *A topological invariant arising in the analysis of traveling waves*, J. Reine Angew. Math. 410 (1990) 167–212.

[AMPZ] A. Azevedo-D. Marchesin-B. Plohr-K. Zumbrun, *Nonuniqueness of solutions of Riemann problems*. Z. Angew. Math. Phys. 47 (1996), 977–998.

[B] A. Bressan, *A tutorial on the center manifold theorem*, Appendix A, “Hyperbolic Systems of Balance Laws,” Lecture Notes in Mathematics 1911, Springer-Verlag, 2007.

[BeSZ] M. Beck, B. Sandstede, and K. Zumbrun, *Nonlinear stability of time-periodic shock waves*, preprint (2008).

[CP] J. Chazarain-A. Piriou, *Introduction to the theory of linear partial differential equations*, Translated from the French. Studies in Mathematics and its Applications, 14. North-Holland Publishing Co., Amsterdam-New York, 1982. xiv+559 pp. ISBN: 0-444-86452-0.

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W.A. Coppel, *Stability and asymptotic behavior of differential equations*, D.C. Heath and Co., Boston, MA (1965) viii+166 pp.

A. Friedman, *Partial differential equations of parabolic type*. Prentice-Hall, 1964.

Gues, O., Metivier, G., Williams, M., and Zumbrun, K., *Paper 4, Navier-Stokes regularization of multidimensional Euler shocks*, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 1, 75–175.

J. Goodman, *Remarks on the stability of viscous shock waves*, in: Viscous profiles and numerical methods for shock waves (Raleigh, NC, 1990), 66–72, SIAM, Philadelphia, PA, (1991).

R. Gardner and K. Zumbrun, *The Gap Lemma and geometric criteria for instability of viscous shock profiles*.

D. Henry, *Geometric theory of semilinear parabolic equations*. Lecture Notes in Mathematics, Springer–Verlag, Berlin (1981), iv + 348 pp.

P. Howard and K. Zumbrun, *Stability of undercompressive viscous shock waves*, in press, J. Differential Equations 225 (2006), no. 1, 308–360.

F. Gesztesy, C.K.R.T. Jones, Y. Latushkin, and M. Stanislavova, M., *A spectral mapping theorem and invariant manifolds for nonlinear Schrödinger equations*, Indiana Univ. Math. J. 49 (2000), no. 1, 221–243.

B. Kwon and K. Zumbrun, *Asymptotic Behavior of Multidimensional scalar Relaxation Shocks*, to appear, JHDE.

Y. Latushkin, J. Prüss, Jan, and R. Schnaubelt, *Stable and unstable manifolds for quasilinear parabolic systems with fully nonlinear boundary conditions*, J. Evol. Equ. 6 (2006), no. 4, 537–576.

Y. Latushkin, J. Prüss, Jan, and R. Schnaubelt, *Center manifolds and dynamics near equilibria of quasilinear parabolic systems with fully nonlinear boundary conditions*, Discrete Contin. Dyn. Syst. Ser. B 9 (2008), no. 3-4, 595–633.

C. Li, *On the Dynamics of Navier-Stokes and Euler Equations*, Preprint, (2006).

C. Mascia and K. Zumbrun, *Pointwise Green function bounds for shock profiles of systems with real viscosity*, Arch. Ration. Mech. Anal. 169 (2003), no. 3, 177–263.
[MaZ2] C. Mascia and K. Zumbrun, *Stability of small-amplitude shock profiles of symmetric hyperbolic-parabolic systems*, Comm. Pure Appl. Math. 57 (2004), no. 7, 841–876.

[MaZ3] C. Mascia and K. Zumbrun, *Stability of large-amplitude viscous shock profiles of hyperbolic-parabolic systems*. Arch. Ration. Mech. Anal. 172 (2004), no. 1, 93–131.

[MaZ4] C. Mascia and K. Zumbrun, *Pointwise Green’s function bounds and stability of relaxation shocks*. Indiana Univ. Math. J. 51 (2002), no. 4, 773–904.

[Pa] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*. Applied Mathematical Sciences, 44, Springer-Verlag, New York-Berlin, (1983) viii+279 pp. ISBN: 0-387-90845-5.

[PW] R. L. Pego-M.I. Weinstein, *Asymptotic stability of solitary waves*, Comm. Math. Phys. 164 (1994), no. 2, 305–349.

[RZ] M. Raoofi and K. Zumbrun, *Stability of undercompressive viscous shock profiles of hyperbolic–parabolic systems*, Preprint (2007).

[SS] B. Sandstede and A. Scheel, *Hopf bifurcation from viscous shock waves*, SIAM J. Math. Anal. 39 (2008) 2033–2052.

[S] D. Sattinger, *On the stability of waves of nonlinear parabolic systems*. Adv. Math. 22 (1976) 312–355.

[TZ1] B. Texier and K. Zumbrun, *Relative Poincaré–Hopf bifurcation and galloping instability of traveling waves*, Methods Anal. and Appl. 12 (2005), no. 4, 349–380.

[TZ2] B. Texier and K. Zumbrun, *Galloping instability of viscous shock waves*, to appear, Physica D (2008).

[TZ3] B. Texier and K. Zumbrun, *Hopf bifurcation of viscous shock waves in gas dynamics and MHD*, to appear, Archive for Rat. Mech. Anal. (2008).

[TZ4] B. Texier and K. Zumbrun, *Transition to longitudinal instability of detonation waves is generically associated with Hopf bifurcation to time-periodic galloping solutions*, preprint (2008).

[VI] A. Vanderbauwhede and G. Iooss, *Center manifold theory in infinite dimensions*, Dynamics reported: expositions in dynamical systems, 125–163, Dynam. Report. Expositions Dynam. Systems (N.S.), 1, Springer, Berlin, 1992.

[Z1] K. Zumbrun, *Multidimensional stability of planar viscous shock waves*, Advances in the theory of shock waves, 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.
[Z2] K. Zumbrun, *Stability of large-amplitude shock waves of compressible Navier–Stokes equations*, in Handbook of Mathematical Fluid Dynamics, Elsevier (2004).

[Z3] K. Zumbrun, *Planar stability criteria for viscous shock waves of systems with real viscosity*, in Hyperbolic Systems of Balance Laws, CIME School lectures notes, P. Marcati ed., Lecture Note in Mathematics 1911, Springer (2004).

[Z4] K. Zumbrun, *Refined Wave–tracking and Nonlinear Stability of Viscous Lax Shocks*. Methods Appl. Anal. 7 (2000) 747–768.

[Z5] K. Zumbrun, *Conditional stability of unstable viscous shocks*, preprint (2008).

[Z6] K. Zumbrun, *Conditional stability of unstable viscous shock waves in compressible gas dynamics and MHD*, preprint (2008).

[Z7] K. Zumbrun, *Dynamical stability of phase transitions in the p-system with viscosity-capillarity*, SIAM J. Appl. Math. 60 (2000), no. 6, 1913–1924.

[ZH] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves*. Indiana Univ. Math. J. 47 (1998), 741–871; Errata, Indiana Univ. Math. J. 51 (2002), no. 4, 1017–1021.

[ZS] K. Zumbrun and D. Serre, *Viscous and inviscid stability of multidimensional planar shock fronts*, Indiana Univ. Math. J. 48 (1999) 937–992.