A NON-LINEAR ROTH THEOREM FOR FRACTALS OF
SUFFICIENTLY LARGE DIMENSION

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Abstract. Suppose that $d \geq 2$, and that $A \subset [0, 1]$ has sufficiently large dimension,
$1 - \epsilon_d < \dim_H(A) < 1$. Then for any polynomial $P$ of degree $d$ with no constant
term, there exists $\{x, x - t, x - P(t)\} \subset A$ with $t \approx P 1$.

1. Introduction

In [3], the authors exhibit the existence of polynomial configurations in fractal
sets; a key assumption on these fractal sets is that they have sufficiently large Fourier
dimension, where we recall that the Fourier dimension of a set is given by

$\dim_F(E) := \sup \{\beta : E \text{ supports a probability measure, } \mu, \text{ so that } |\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\beta/2}\}$.

The purpose of this short note is to show that – in one dimension – this phenomenon
is independent on Fourier dimension of fractal sets, provided that $E$ has sufficiently
large Hausdorff dimension.

In particular, we have the following result.

Theorem 1.1. Suppose that $A \subset [0, 1]$ has sufficiently large dimension, $1 - \epsilon_d < \dim_H(A) < 1$, $d \geq 2$. Then for any polynomial $P$ of degree $d$ with no constant
term, there exists $\{x, x - t, x - P(t)\} \subset A$ with $t \approx P 1$.

1.1. Acknowledgement. The author would like to thank Alex Iosevich for his per-
spective on the interplay between Fourier and Hausdorff dimension in detecting point
configurations.

1.2. Notation. Here and throughout, $e(t) := e^{2\pi it}$. For real numbers $A$ (typically
taken to be dyadics), define $f_A$ to be the smooth Fourier restriction of $f$ to $|\xi| \approx A$,
similarly define $f_{\leq A}$, etc.

For bump functions $\phi$, we let $\phi_j(x) := 2^j \phi(2^j x)$.

We will make use of the modified Vinogradov notation. We use $X \lesssim Y$, or $Y \gtrsim X$,
to denote the estimate $X \leq CY$ for an absolute constant $C$. We use $X \approx Y$ as
shorthand for $Y \lesssim X \lesssim Y$. We also make use of big-O notation: we let $O(Y)$ denote
a quantity that is $\lesssim Y$. If we need $C$ to depend on a parameter, we shall indicate

Date: April 25, 2019.
this by subscripts, thus for instance $X \lesssim_p Y$ denotes the estimate $X \leq C_p Y$ for some $C_p$ depending on $p$. We analogously define $O_p(Y)$.

2. The Argument

2.1. Preliminaries. We need the following Lemmas; the first is essentially a consequence of the main result [1], or [2].

**Lemma 2.1.** Suppose that $0 \leq f_i = \phi_{t_i} * f$ for some bounded function $0 \leq f \leq 1_{[0,1]}$, $0 < t_i < \infty$ so that $\int f \geq \epsilon$. Then there exists $\delta = \delta(\epsilon) > 0$ so that

$$\int \int f_1(x)f_2(x-t)f_3(x-P(t)) \, dxdt \geq \delta.$$ 

The following refinement of [1, Lemma 5], due to [2], is our primary tool.

**Lemma 2.2** (Lemma 1.4 of [2]). Suppose that $\hat{f}_2$ is supported on $|\xi| \approx N$. Then there exists $\delta_0 > 0$ so that

$$\| \int f_1(x-t)f_2(x-P(t))\rho(t) \, dt \|_1 \lesssim N^{-\delta_0} \| f_1 \|_2 \| f_2 \|_2;$$

here $\rho = \rho_P$ is a bump function adapted to some scale bounded away from zero in terms of $P$.

Via the Fourier localization argument below, we see that this lemma essentially implies a Sobolev estimate; since sets of dimension $> d_E$ support a probability measure $\mu$ with

$$\| \phi_n * \mu \|_\infty \leq 2^n(1-d_E-\kappa),$$

for some $\kappa > 0$, we see that $\mu$ is in the (negative) Sobolev class

$$\mu \in H^{\kappa'-1/d_E}/2$$

for any $\kappa > \kappa' > 0$; this is essentially the key to the argument.

With these lemmas in hand, we turn to the proof.

2.2. The Proof of Theorem 1.1.

**Proof.** Set $f = f_J := \mu * \phi_J$ for some sufficiently large $J$; it suffices to exhibit upper and lower bounds on

$$\int \int f(x)f(x-t)f(x-P(t))\rho(t) \, dt \, dx$$

independent of $J$, for $\rho$ an appropriate bump function. Split (2.3) into two terms:

$$\int \int f(x)f(x-t)f_{\leq B}(x-P(t))\rho(t) \, dt \, dx$$
and its complementary piece

\begin{equation}
(2.5) \quad \int \int f(x)f(x-t)f_{>B}(x-P(t)) \rho(t) \, dtdx
\end{equation}

where \( B \) is a large parameter to be determined later.

We begin with (2.4), which we write as

\begin{equation}
\int \int \hat{f}(-\xi - \eta)\hat{f}(\xi)\hat{f}_{\leq B}(\eta)m(\xi, \eta) \, d\xi d\eta,
\end{equation}

where

\[ m(\xi, \eta) := \int e(-\xi t - \eta P(t)) \rho(t) \, dt \]

\[ = m(\xi, \eta) \cdot 1_{|\xi| \leq \rho(\eta)} + 1_{|\xi| > \rho(\eta)} \cdot O_B((1 + |\xi|)^{-B}) \]

\[ =: m_1(\xi, \eta) + m_2(\xi, \eta) \]

by non-stationary phase considerations.

Now, with \( B \gtrsim A = A(B, P) \gg B \) a large threshold, decompose (2.4) as a sum of three terms:

\begin{equation}
(2.6) \quad (2.4) = \int \int f_{\leq A}(x)f_{\leq A}(x-t)f_{\leq B}(x-P(t)) \rho(t) \, dtdx
\end{equation}

\begin{equation}
(2.7) \quad + \int \int f_{>A}(x)f_{\leq A}(x-t)f_{\leq B}(x-P(t)) \rho(t) \, dtdx
\end{equation}

\begin{equation}
(2.8) \quad \int \int f(x)f_{>A}(x-t)f_{\leq B}(x-P(t)) \rho(t) \, dtdx.
\end{equation}

The first term is a main term; an upper bound is simply given by

\( (2.6) \lesssim \| f_{\leq A} \|_\infty \cdot \| f_{\leq A} \|_\infty \cdot \| f_{\leq B} \|_\infty \lesssim A^{3(1-d_E)}; \)

as we will see, (2.7) and (2.8) are lower order error terms, so this upper bound majorizes (2.4).

As for the lower bound, an application of Lemma 2.1 yields a lower bound of

\( (2.6) \gtrsim A^{d_E-1} \cdot \delta; \)

the loss of \( A^{d_E-1} \) comes from reproducing: we have

\[ \| f_{\leq A} \|_\infty \lesssim A^{1-d_E}. \]

The second term, (2.7), vanishes identically, since

\[ \hat{f}_{>A}(-\xi - \eta)\hat{f}_{\leq A}(\xi)\hat{f}_{\leq B}(\eta) = 0. \]
We express the third term using the Fourier transform:

\[(2.8) = \int \int \hat{f}(-\xi - \eta)\hat{f}_{> A}(\xi)\hat{f}_{\leq B}(\eta)m_1(\xi, \eta) \, d\xi d\eta + \int \int \hat{f}(-\xi - \eta)\hat{f}_{> A}(\xi)\hat{f}_{\leq B}(\eta)m_2(\xi, \eta) \, d\xi d\eta.\]

The first term vanishes identically since \(|\xi|\) is so much larger than \(|\eta|\), for an appropriate choice of \(A\). As for the error term, we estimate:

\[
\left| \int \int \hat{f}(-\xi - \eta)\hat{f}_{> A}(\xi)\hat{f}_{\leq B}(\eta)m_2(\xi, \eta) \, d\xi d\eta \right|
\leq \sum_{N > A} \int \int |\hat{f}_N(\xi)| \cdot |\hat{f}_{\leq B}(\eta)| \cdot |m_2(\xi, \eta)| \, d\xi d\eta
\leq B^{1-dE/2} \cdot A^{1-dE/2-B} \lesssim A^{2-dE-B} =: A^{-C};
\]

note the use of the trivial estimate \(\|\hat{f}\|_\infty \leq \|f\|_1\) in passing to the second line.

In particular, we have exhibited

\[(2.9) \quad (2.4) \gtrsim A^{dE-1}\delta - A^{-C} \gtrsim A^{dE-1}\delta.\]

We next term to \((2.5)\), which we decompose as a sum of \(N > B\):

\[(2.4) = \int \int f(x)f(x - t)f_{> B}(x - P(t))\rho(t) \, dt dx
\]

\[(2.10) \quad = \sum_{N > B} \int \int \hat{f}_{\leq N}(-\xi - \eta)\hat{f}_{\leq N}(\xi)\hat{f}_{N}(\eta)m_1(\xi, \eta) \, d\xi d\eta
\]

\[(2.11) \quad + \sum_{N > B} \int \int \hat{f}_{> N}(-\xi - \eta)\hat{f}_{> N}(\xi)\hat{f}_{N}(\eta)m_2(\xi, \eta) \, d\xi d\eta
\]

We begin with \((2.11)\); by arguing as previously, the \(N\)th term admits an upper bound of \(N^{-C}\) for a very large \(C = C(B)\), which leads to the estimate

\[
|(2.11)| \lesssim B^{-C}.
\]
It remains to consider (2.10); we extract the $N$th term once again,
\[
\left| \int \int \hat{f}_{\leq N}(-\xi - \eta) \hat{f}_{\leq N}(\xi) \hat{f}_{\leq N}(\eta) m_1(\xi, \eta) \, d\xi d\eta \right|
\]
\[
= \left| \int \int \hat{f}_{\leq N}(-\xi - \eta) \hat{f}_{\leq N}(\xi) \hat{f}_{\leq N}(\eta) m(\xi, \eta) \, d\xi d\eta \right|
\]
\[
\leq \|f_{\leq N}\|_{\infty} \cdot \left\| \int f_{\leq N}(x - t) f_N(x - P(t)) \rho(t) \, dt \right\|_{L^1}
\]
\[
\lesssim N^{1-d_E-\kappa} \cdot N^{-\delta_0} \cdot \|f_{\leq N}\|_2^2
\]
\[
\lesssim N^{2-2d_E-\delta_0-\kappa}.
\]
In passing from the first line to the second, we have (possibly) discarded $O(1)$ terms of the form
\[
\int \int |\hat{f}_{\leq C N}(-\xi - \eta)| \cdot |\hat{f}_{\leq C' N}(\xi)| \cdot |\hat{f}_{\leq C N}(\eta)| \cdot |m_1(\xi, \eta)| \, d\xi d\eta
\]
for some large $1 \ll C, C' \lesssim 1$ as we drop the Fourier restriction in the definition of $m_1$; but, on this domain, we retain the pointwise bound $|m_1(\xi, \eta)| \lesssim (1 + |\xi|)^{-B}$, so we may handle these error terms as above.

In particular, provided that $1 - \delta_0/2 < d_E$, we have exhibited a upper bound
\[
(2.12) \quad (2.5) \lesssim B^{-\delta_0-2d_E+2-\kappa}
\]
Combining (2.9) and (2.12), we see that we may estimate from below
\[
(2.3) \geq A^{d_E-1} \delta - CB^{-\delta_0-2d_E+2-\kappa}.
\]
which remains bounded away from zero for sufficiently large $d_E < 1$, since $A \approx_P B$.

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