Product Decomposition of Periodic Functions in Quantum Signal Processing

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We consider an algorithm to approximate complex-valued periodic functions $f(e^{i\theta})$ as a matrix element of a product of $SU(2)$-valued functions, which underlies so-called quantum signal processing. We prove that the algorithm runs in time $\text{poly}(n \log(1/\epsilon))$ on a Turing machine where $n$ is the degree of the polynomial that approximates $f$ with accuracy $\epsilon$; previous efficiency claim assumed a strong arithmetic model of computation and lacked numerical stability analysis.

1 Introduction

Quantum signal processing [1–3] refers to a scheme to construct an operator $V$ from a more elementary unitary $W$ where $V = \sum_{\theta} f(e^{i\theta}) |\theta\rangle \langle \theta|$ and $W = e^{i\theta} |\theta\rangle \langle \theta|$ share the eigenvectors but the eigenvalues of $V$ are transformed by a function $f$ from those of $W$. The transformation requires only one ancilla qubit, and is achieved by implementing control-$W$ and control-$W^\dagger$, interspersed by single-qubit rotations on the control, and final post-selection on the control.\(^1\)

This technique produced gate-efficient quantum algorithms for, e.g., Hamiltonian simulations, which is asymptotically optimal when the Hamiltonian is sparse and given as a blackbox, or as a linear combination of oracular unitaries [4, 5]. Furthermore, this technique with rigorous error bounds appears to be useful and competitive even for explicitly described, rather than oracular, local Hamiltonian simulation problems [6, 7]. It is also promised to be useful in solving linear equations [3, 8, 9].

However, in quantum signal processing the classical preprocessing to find interspersing single-qubit rotations for a given transformation function $f$ has been so numerically unstable that it has been unclear whether it can be performed efficiently. In fact, Ref. [6, App. H.3] reports that the computation time is “prohibitive” to obtain sequences of length $\gtrsim 30$ of interspersing unitaries for Jactobi-Anger expansions that we explain in Section 5. The true usefulness of the quantum signal processing hinges upon the ability to compute long sequences of interspersing single-qubit rotations.

It has been asserted that there exists a polynomial time classical algorithm [2, 9], but this claim is based on an unusual computational model where any real arithmetic, including evaluation of radicals, trigonometric functions and polynomial root finding, can be performed.\(^1\)

\(^1\) Sometimes it is possible to avoid controlled version of $W$ [3, 9], but we contend ourselves with this implementation for its simplicity in presentation. The result of this paper is applicable for the ancilla-free variant; the only change one may have to do is to replace a variable $e^{i\theta/2}$ with $e^{i\theta}$. 
with arbitrary precision in unit time. Such a computational model is too powerful to be realistic. Allowing arbitrary integer arithmetic at unit cost, not only can one factor integers in time that is linear in the number of digits \([10]\), but also solve NP-hard problems in polynomial time \([11]\). In a seemingly mundane problem involving real numbers, the number of required bits during the computation can be a priori very large. For example, it is still an open problem whether one can decide the larger between \(\sum_{j=1}^{k} \sqrt{a_j}\) and \(\sum_{j=1}^{k} \sqrt{b_j}\), which are sums of square roots of positive integers \(a_j, b_j\) that are smaller than \(n\), in time \(\text{poly}(k \log n)\) on a Turing machine; see e.g. \([12, 13]\) for recent results.

The numerical instability of previous methods may be attributed to expansions of large degree polynomials that are found by roots of another polynomial. Crudely speaking, there are two problems in this approach. First, the polynomial expansions can be regarded as the computation of convolutions, which, when naively iterated, suffers from numerical instability. Second, although the root finding is a well-studied problem, to use the roots to construct another polynomial one has to understand the distribution of the roots to keep track of the loss of precision. These problems were not addressed previously.

Here, we refine a classical algorithm to find interspersing single-qubit rotations and bound the number of required bits in the computation for a desired accuracy to a final answer. We conclude that the classical preprocessing can indeed be done in polynomial time on a Turing machine. We generally adopt the methods of Ref. \([2]\), but make manipulations easier by avoiding trigonometric polynomials. Some generalizations are obtained with simpler calculations. For the numerical stability and analysis, our algorithm avoids too small numbers by sacrificing approximation quality in the initial stage, and replaces polynomial expansions by a Fourier transform. These modifications enable us to handle the problems that are mentioned above. However, it should be made clear that our refinement also requires high precision arithmetic. Specifically, we show that \(O(n \log(n/\epsilon))\) bits of precision during the execution of our algorithm is sufficient to produce a reliable final decomposition, where \(n\) is the degree of the polynomial that approximates a given transformation function \(f\) of eigenvalues up to additive error \(\epsilon\). Previously no such bound was known. On a sequential (rather than parallel) random access machine, our algorithm runs in time \(O(n^3 \text{polylog}(n/\epsilon))\).

We will start by reviewing quantum signal processing in the next section, and then develop an algorithm and analyze it. In two short sections later we provide self-contained treatment of polynomial approximations for Hamiltonian simulation and matrix inversion problems. Throughout the paper, we use \(U(1)\) to denote the group of all complex numbers of unit modulus. Sometimes we will refer to \(U(1)\) as the unit circle. As usual, \(i = \sqrt{-1}\), and \(X, Y, Z\) are Pauli matrices.

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\(^{2}\) The basic idea in Ref. \([11]\) is to relate the expansion of a Boolean formula in a conjunctive form into a disjunctive form with the expansion of a polynomial from its factors. Then, the satisfiability translates to the positiveness of certain coefficients in the expansion. The variable in the polynomial is set to a sufficiently large number, effectively encoding the entire polynomial in a single big number. An important elementary fact that underlies the power of arithmetic model is that we only need \(O(n)\) compositions of squaring operation \(x \mapsto x^2\) to reach a \(2^n\)-bit number.
2 Quantum Signal Processing

To understand how the eigenvalue transformation (signal processing) works, it is convenient to consider the action of control-\(W\) restricted to an arbitrary but fixed eigenstate \(|\theta\rangle\) of \(W\). The eigenvalue \(e^{i\theta}\) associated with \(|\theta\rangle\) is kicked back to the control qubit to induce a unitary \(|0\rangle\langle 0| + e^{i\theta}|1\rangle \langle 1|\) on the control qubit. Conjugating the control-\(W\) by a single qubit unitary on the control qubit, we see that \(|0\rangle\) and \(|1\rangle\) can be any orthonormal basis vectors \(|0''\rangle\), \(|1''\rangle\) of the control qubit. If we allow ourselves to implement the inverse of the control-\(W\), which is reasonable, we can also implement

\[
\left( |0'\rangle \langle 0'| + e^{i\theta} |1'\rangle \langle 1'| \right) \left( |0''\rangle \langle 0''| + e^{-i\theta} |1''\rangle \langle 1''| \right) = \left( e^{-i\theta/2} |0'\rangle \langle 0'| + e^{i\theta/2} |1'\rangle \langle 1'| \right) \left( e^{i\theta/2} |0''\rangle \langle 0''| + e^{-i\theta/2} |1''\rangle \langle 1''| \right)
\]

where \(|\{0', |1'\}\rangle\) and \(|\{0''\}, |1''\}\rangle\) are arbitrary orthonormal bases.\(^3\) When we alternate an even number of control-\(W\) and control-\(W^\dagger\), this trick allows us to assume that an implementable unitary on the control qubit is a product of \textbf{primitive matrices}

\[
E_F(t) = tP + t^{-1}Q
\]

where \(t = e^{i\theta/2}\) and \(P, Q\) are projectors of rank 1 that sum to the identity. Thus, an even number \(n\) of control-\(W\) and control-\(W^\dagger\) induces \(F(t) = E_{P_1}(t)E_{P_2}(t)\cdots E_{P_n}(t)\) on the control qubit. The product \(F(t)\) can be thought of as an \(SU(2)\)-valued function over the unit circle in the complex plane. By the same formulas, we define \(E_F(t)\) and \(F(t)\) on the entire complex plane except the origin \(t = 0\). Now if \(\langle +| F(t) |+\rangle\) is close to \(f(t^2 = e^{i\theta})\) for all \(t \in U(1)\), then post-selection on \(|+\rangle\) of the control qubit enacts \(V\). Here, the choice of \(|+\rangle\) is of course a convention. The success probability of the post-selection, of course, depends on the magnitude of \(f(e^{i\theta})\).

A natural question is then what \(F(t)\) is achievable in the form of

\[
F(t) = E_0E_{P_1}(t)E_{P_2}(t)\cdots E_{P_n}(t)
\]

where we inserted an additional single-qubit unitary \(E_0\). Since \(UE_F(t)U^\dagger = E_{UPU^\dagger}(t)\) for any unitary \(U\), the product \(F(t)\) as in Eq. (3) captures all possible actions on the control qubit. The answer to the achievability question turns out to be quite simple, as we show in the next section.

3 Polynomial functions \(U(1) \rightarrow SU(2)\)

\textbf{Definition 1.} For any integer \(n \geq 0\), let \(\mathcal{P}_n\) be the set of all Laurent polynomials \(F(t) = \sum_{j=-n}^{n} C_j t^j\) in \(t\) with coefficients \(C_j\) in 2-by-2 complex matrices, such that \(F(\eta) \in SU(2)\) for all complex numbers \(\eta\) of unit modulus. We say that \(F(t) \in \mathcal{P}_n\) has \textbf{degree} \(n\) if \(C_n \neq 0\) or \(C_{-n} \neq 0\). We define \(\mathcal{E}_n\) to be the subset of \(\mathcal{P}_n\) consisting of \(F(t)\) where the exponents of \(t\) in \(F(t)\) belong to \(\{-n, -n+2, -n+4, \ldots, n-2, n\}\). Note that \(\mathcal{P}_0 = \mathcal{E}_0 = SU(2)\), and for any

\(^3\) The square root function \(e^{i\theta} \rightarrow e^{i\theta/2}\) has a branch cut, but it hardly matters to us as long as we are consistent that \(e^{-i\theta/2}\) denotes the inverse of the square root.
orthogonal projector $P$ we have $E_P(t) \in \mathcal{E}_1$. We define $F^\dagger(t)$ to be $\sum_j C_j^\dagger t^j$. In a set-theoretic notation, the definitions are as the following.

$$
P_n = \left\{ F(t) = \sum_{j=-n}^n C_j t^j \in \text{Mat}(2,\mathbb{C})[t,t^{-1}] \mid \forall \eta \in \mathbb{C}, |\eta| = 1 \Rightarrow F(\eta) \in SU(2) \right\} 
$$

(4)

$$
\mathcal{E}_n = \left\{ F(t) = \sum_{j=-n}^n C_j t^j \in \mathcal{P}_n \mid \forall k \in 2\mathbb{Z} + 1, C_{-n+k} = 0 \right\}
$$

(5)

**Theorem 2.** Any n-fold product $E_{P_1}(t) \cdots E_{P_n}(t)$ belongs to $\mathcal{E}_n$. Conversely, every $F(t) \in \mathcal{E}_n$ of degree $n$ has a unique decomposition into primitive matrices and a unitary, as in Eq. (3).

This completely characterizes polynomial functions $U(1) \to SU(2)$ and covers all previous results on “achievable functions” in quantum signal processing [2, 9]. Our version is slightly more general since previous results implicitly assume that $\text{Tr}(P_j Z) = 0$.

**Proof.** The first statement is trivial by definition. The proof of the converse is by induction in $n$ where the base case $n = 0$ is trivial. The induction step is proved as follows. We are going to prove that for any $F(t) \in \mathcal{E}_n$ of degree $n > 0$ there exists an $E_P(t)$ such that $F(t)E_P(t) \in \mathcal{E}_{n-1}$.

Such $E_P(t)$ is unique since $F^\dagger(1/t)F(t) = I = E_P^\dagger(1/t)F^\dagger(1/t)F(t)E_P(t)$ for two possible $E_P(t)$ and $E'_P(t)$. Also, the product $F(t)E_P(t)$ must have degree $n - 1$; otherwise $F(t)$ would have degree less than $n$.

We now show the existence of the $E_P$. Consider $F(t) = \sum_{j=-n}^n C_j t^j$ as a 2-by-2 matrix of four Laurent polynomials. The defining property $\text{det} F(t) = 1$ holds for infinitely many values of $t$, and therefore it should hold as a polynomial equation. Taking the leading term, we have $t^{2n} \text{det} C_n + (\text{lower order terms}) = 1$. Similarly, taking the leading term in $t^{-1}$, we have $t^{-2n} \text{det} C_{-n} + (\text{higher order terms}) = 1$. Hence,

$$
\text{det} C_n = 0 = \text{det} C_{-n}.
$$

(6)

Similarly, from the equation $F^\dagger(1/t)F(t) = I = F(t)F^\dagger(1/t)$ we have $t^{2n} C_n^\dagger = C_{-n}^\dagger + \mathcal{O}(t^{2n-1}) = \mathcal{O}(t^{2n-1})$ and hence

$$C_n^\dagger = 0 = C_{-n}^\dagger.
$$

(7)

Suppose $C_n \neq 0$. Let $P$ be a rank-1 projector such that $C_n P = 0$, and let $Q = I - P$. Such $P$ is unique since $C_n$ is a two-by-two matrix of rank one. Then, we claim that $F(t)(tP + t^{-1}Q) \in \mathcal{E}_{n-1}$. Indeed, expanding the left-hand side we have

$$ t^{-n-1} C_{-n} Q + t^{-n+1}(C_{-n} P + C_{-n+2} Q) + \cdots + t^{n-1}(C_n Q + C_{n-2} P) + t^{n+1} C_n P. 
$$

(8)

(This is the only place we use $\mathcal{E}$ instead of $\mathcal{P}$.) If $C_{-n} = 0$, this implies the claim. If $C_{-n} \neq 0$, then $P \propto C_n^\dagger C_{-n}$ and therefore $C_{-n} Q = 0$, implying the claim. The case $C_{-n} \neq 0$ is completely parallel. (Actually, $C_{-n} \neq 0$ if $C_n \neq 0$ as we will see later.)

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4A reader might find it unusual that the degree of a polynomial is decreasing under multiplication, but in the algebra of matrices two nonzero matrices may multiply to vanish.
3.1 Parity constraints

Recall that under the standard representation of Pauli matrices, $Z$ is diagonal, and $X, Y$ are off-diagonal. Suppose an $SU(2)$-valued function $\theta \mapsto F(e^{i\theta}) = a(e^{i\theta})I + b(e^{i\theta})iX + c(e^{i\theta})iY + d(e^{i\theta})iZ$ has even functions (reciprocal in $t = e^{i\theta}$) in the diagonal and odd (anti-reciprocal in $t = e^{i\theta}$) in the off-diagonal. We claim that if $F(t)$ is an element of $E_n$, then the primitive matrix $E_{P_n}(t)$ factored from $F(t)$ by Theorem 2 has a property that $\text{Tr}(ZP_n) = 0$. This implies $E_{P_n}(t)$ has reciprocal diagonal and anti-reciprocal off-diagonal. To prove the claim we observe that

$$Z = F(t)F^\dagger(1/t)Z$$
$$= F(t)(a(1/t) - b(1/t)iX - c(1/t)iY - d(1/t)iZ)Z$$
$$= F(t)(a(t) + b(1/t)iX + c(t)iY - d(t)iZ)Z$$
$$= F(t)Z(a(t) - b(t)iX - c(t)iY - d(t)iZ)$$
$$= F(t)ZF^\dagger(t).$$

It follows that $0 = \text{Tr}(Z) = \text{Tr}(F(t)ZF^\dagger(t)) = t^{2n} \text{Tr}(C_nZC_n^\dagger) + \cdots + t^{-2n} \text{Tr}(C_{-n}ZC_{-n}^\dagger)$ as a polynomial in $t$, and hence $\text{Tr}(C_n^\dagger C_n) = 0 = \text{Tr}(C_{-n}^\dagger C_{-n})$ when $n > 0$. The matrix $C_{\pm n}^\dagger C_{\pm n}$ is proportional to the projector $P_n$ up to the identity in the construction of the factor $E_{P_n}(t)$ in Theorem 2, and therefore we have $\text{Tr}(ZP_n) = 0$.

Moreover, it then follows that $F(t)E_{P_n}(1/t)$ has reciprocal diagonal and anti-reciprocal off-diagonals. Therefore, the unique projectors $P_1, \ldots, P_n$ in the decomposition, which defines the primitive matrices $E_{P_j}(t)$, have zero $Z$-component. This means that all projectors $P_j$ are of form

$$P_j = e^{iZ\phi_j/2} |+\rangle \langle +| e^{-iZ\phi_j/2}$$

where $\phi_j \in \mathbb{R}$ is some angle. In fact, Ref. [2] exclusively considered $E_P(t)$ of this form. This is contrasted to the general case where $P_j$ is identified with a point on the Bloch sphere. The constraint $\text{Tr}(ZP_j) = 0$ forces $P_j$ to lie on the equator of the Bloch sphere.

Note that $E_0$, the residual $SU(2)$ factor in the decomposition, has generally nonzero $Z$-component. Since $E_P(1) = I$ for any projector $P$, we know $E_0 = F(1) = a(1)I + b(1)iX + c(1)iY + d(1)iZ$, but $b(1) = c(1) = 0$ due to their anti-reciprocity. This implies that $E_0 = e^{iZ\phi_0/2}$ for some angle $\phi_0$. Hence, under the parity constraint of this subsection, $F(t) \in E_n$ is uniquely specified by $n + 1$ angles $\phi_0, \phi_1, \ldots, \phi_n$. Note that if $d(1)^2 = 1 - a(1)^2 = \mathcal{O}(\epsilon)$, then $d(1) = \mathcal{O}(\sqrt{\epsilon})$ and $\|E_0 - I\| = \mathcal{O}(\sqrt{\epsilon})$.

3.2 Complementing polynomials

Quantum signal processing does not use $F(t)$ itself, but rather a certain matrix element of it. Hence, it is important to know what a matrix element can be. Observe that any member of $SU(2)$ can be written as $aI + biX + ciY + diZ$ where the real numbers $a, b, c, d$ satisfy $a^2 + b^2 + c^2 + d^2 = 1$, and this decomposition is unique. (The group $SU(2)$ is identified with the group of all unit quaternions.) Thus, a member $F(z) \in \mathcal{P}_n$ can be written uniquely as $F(z) = a(z)iI + b(z)iX + c(z)iY + d(z)iZ$. Here, $a(z), b(z), c(z), d(z)$ are Laurent polynomials such that $a(z)^2 + b(z)^2 + c(z)^2 + d(z)^2 = 1$, and each takes real values on $U(1)$.
Definition 3. A (Laurent) polynomial with real $\mathbb{R}$ coefficients is called a real (Laurent) polynomial. The degree of a Laurent polynomial is the maximum absolute value of the exponent of the variable whose coefficient is nonzero. A Laurent polynomial $f(z)$ is reciprocal if $f(z) = f(1/z)$, or anti-reciprocal if $f(z) = -f(1/z)$. A Laurent polynomial function $f : \mathbb{C}^\times \rightarrow \mathbb{C}$ is real-on-circle if $f(z) \in \mathbb{R}$ for all $z \in \mathbb{C}$ of unit modulus. A real-on-circle Laurent polynomial $f(z)$ is pure if $f(z)$ is real reciprocal or $i f(z)$ is real anti-reciprocal.

The term “pure” is because a Laurent polynomial $f(z)$ with complex coefficients is real-on-circle if and only if both

$$f_+(z) := \frac{f(z) + f(1/z)}{2} \text{ and } f_-(z) := \frac{f(z) - f(1/z)}{2i}$$

are real Laurent polynomials. (Proof: Write $f(e^{i\theta}) = \sum_j a_j e^{ij\theta}$, with the complex conjugate being $\sum_j \overline{a}_j e^{-ij\theta} = \sum_j a_{-j} e^{ij\theta}$. Thus, $a_j = \overline{a}_{-j}$, and the claim follows.) This simply rephrases the fact that a real-valued function $\theta \mapsto f(e^{i\theta})$ has a trigonometric function series with real coefficients. Hence, for any real-on-circle Laurent polynomial $f(z)$, it is real if and only if it is reciprocal. In addition, a real and reciprocal Laurent polynomial is real-on-circle. That is, among the three properties, real, real-on-circle, and reciprocal, any two imply the third. Note that a real-on-circle Laurent polynomial is not necessarily real, and a real Laurent polynomial is not necessarily real-on-circle. Also, note that real-on-circle Laurent polynomials form an algebra over the real numbers.

We are now ready to state a sufficient condition under which a complex polynomial qualifies to be a matrix element of some $F(t) \in \mathcal{P}_n$. We think of $a(z)$ and $b(z)$ below as the real and imaginary part of a complex function, respectively.

Lemma 4. Let $a(z)$ and $b(z)$ be real-on-circle Laurent polynomials of degree at most $n$ such that $a(\eta)^2 + b(\eta)^2 < 1$ for all $\eta \in U(1)$. If $a(z)^2 + b(z)^2$ is reciprocal (e.g., $a(z)$ and $b(z)$ are pure), then there exist pure real-on-circle Laurent polynomials $c(z) = c_+(z)$ and $d(z) = id_-(z)$ of degree at most $n$ such that $a(z)^2 + b(z)^2 + c(z)^2 + d(z)^2 = 1$.

Note that if $a(z)$ is pure and $a(z)^2 + b(z)^2$ is reciprocal, then $b(z)$ is also pure. (Proof: $b(z)^2 = (b_+(z) + ib_-(z))^2 = b_+(z)^2 - b_-(z)^2 + 2ib_+(z)b_-(z)$ has to be real.)

Proof. The Laurent polynomial $1 - a(z)^2 - b(z)^2$ is reciprocal real of degree $n'$ that is at most $2n$; the leading terms of $a(z)^2$ and $b(z)^2$ might cancel each other so that $n' < 2n$. Due to the reciprocity, any real root $r$ comes in a pair $(r, 1/r)$ where $r \neq \pm 1$ since no root exists on $U(1)$. For the same reason, any nonreal complex root $z$ comes in a quadruple $(z, \bar{z}, 1/z, 1/\bar{z})$, and there is no degenerate quadruple. Enumerate representatives of the pairs and quadruples of the roots:

$$\mathcal{C} = \left\{ r \in \mathbb{C} : 1 - a(r)^2 - b(r)^2 = 0, \ 3\text{m}(r) > 0, \ |r| < 1 \right\},$$
$$\mathcal{R} = \left\{ r \in \mathbb{R} : 1 - a(r)^2 - b(r)^2 = 0, \ |r| < 1 \right\}.$$

These are lists rather than sets, as we take the multiplicities into account. So, $4|\mathcal{C}| + 2|\mathcal{R}| = 2n'$. (A reciprocal Laurent polynomial of degree $n'$ has $2n'$ roots.) If $n'$ is even, let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$
be a partition into sublists of equal lengths; if \( n' \) is odd, let \( \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \) be a partition into sublists of lengths differing by one. Consider factors of \( 1 - a(z)^2 - b(z)^2 \):

\[
e(z) = \prod_{r \in \mathbb{C}} \left(z - 2\Re(r) + \frac{|r|^2}{z}\right) \cdot \prod_{r_1 \in \mathcal{R}_1} (z - r_1) \cdot \prod_{r_2 \in \mathcal{R}_2} \left(\frac{1}{z} - r_2\right). \tag{13}\]

The factor \( e(z) \) is a real Laurent polynomial of degree \( \lfloor n'/2 \rfloor \). Then, the product \( e(z)e(1/z) \) is real reciprocal and has degree \( 2|C| + |\mathcal{R}_1| + |\mathcal{R}_2| = n^5 \). Now the two Laurent polynomials \( e(z)e(1/z) \) and \( 1 - a(z)^2 - b(z)^2 \) are reciprocal real, and have the same roots, and therefore they differ by a constant factor \( \alpha = \frac{1-a(z)^2-b(z)^2}{e(z)e(1/z)} \in \mathbb{R} \). Evaluating this expression at \( z = 1 \), we see that \( \alpha \) is positive. Thus, we finish the proof by observing

\[
1 - a(z)^2 - b(z)^2 = \alpha e(z)e(1/z) = \left(\frac{e(z) + e(1/z)}{2}\sqrt{\alpha}\right)^2 + \left(\frac{e(z) - e(1/z)}{2i}\sqrt{\alpha}\right)^2. \tag{14}\]

Both the reciprocal \( (c(z)) \) and anti-reciprocal \( (d(z)) \) combinations have degree \( \leq \lfloor n'/2 \rfloor \leq n \).

As one can see in the proof, given \( a(z) \) and \( b(z) \), the “remainder” Laurent polynomials \( c(z), d(z) \) are not unique. At least, the partition \( \mathcal{R}_1 \cup \mathcal{R}_2 \) is arbitrary.

### 4 Efficient implementation with bounded precision

In this section we consider an algorithm to find interspersing single-qubit unitaries given a complex function \( A(\varphi) + iB(\varphi) \). The algorithm consists of two main parts: first, we have to find a \( SU(2) \)-valued function of \( \varphi \) such that a particular matrix element is the input function. It suffices to find a good approximation. Second, we have to decompose the \( SU(2) \)-valued function into a product of primitive matrices. We have already given constructive proofs for both the steps, but we tailor the construction so that numerical error is reduced and traceable. We will outline our algorithm first, deferring certain details to the next subsection. The computational complexity will be analyzed subsequently.

#### 4.1 Algorithm

0. An input is a complex-valued periodic function \( A(\varphi) + iB(\varphi) \) with period \( 2\pi \) where \( A \) and \( B \) are real-valued, such that \( A(\varphi)^2 + B(\varphi)^2 \) is an even function of \( \varphi \) and is \( \leq 1 \) for any real \( \varphi \). Given \( \epsilon > 0 \), the order of error we tolerate, analyze \( A(\varphi), B(\varphi) \) to obtain \( \epsilon \)-close real-on-circle Laurent polynomials \( \tilde{A}(z), \tilde{B}(z) \) for \( z = e^{i\varphi} \).

1. Compute rational Laurent polynomials \( a(z) \) and \( b(z) \) of degree \( n \) such that (i) \( a(z)^2 + b(z)^2 \) is real reciprocal, (ii) they are \( \epsilon \)-close to the true target functions on \( U(1) \), (iii) \( a(z)^2 + b(z)^2 \leq 1 - \epsilon \) for \( z \in U(1) \), and (iv) every coefficient in \( a(z) \) or \( b(z) \) has magnitude \( \geq \epsilon/n \).

2. Find all roots of \( 1 - a(z)^2 - b(z)^2 \) to accuracy \( 2^{-R} \) where \( R = \Omega(n \log(n/\epsilon)) \).

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\(^5\) The degree of a Laurent polynomial in our definition is only subadditive under multiplication of two Laurent polynomials. For example, the product \( (z-1)(z^{-1} - 2) \) has degree one.
3. Evaluate the complementary polynomials computed from the roots of Step 2 according to Lemma 4 at points of \(T\) where

\[
T := \{ e^{2\pi ik/D} \mid k = 1, \ldots, D \}
\]  
and \(D\) is a power of 2 that is larger than \(2n + 1\). One should not expand \(c(z), d(z)\) before evaluation, but should substitute numerical values for \(z\) with accuracy \(2^{-R}\) in the factorized form of \(e(z)\), and then read off the real \((c(z))\) and imaginary \((d(z))\) parts.

4. Compute the fast Fourier transform of the function value list to obtain

\[
C_{2j}^{(2n)} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-ij\theta} F(e^{i\theta})
\]

for \(j = -n, -n+1, \ldots, n-1, n\). (In exact arithmetic, we would have \(F(z) = \sum_{j=-n}^n C_{2j}^{(2n)} z^j\).)

5. For \(m = 2n, 2n-1, \ldots, 2, 1\) sequentially in decreasing order, (i) obtain a primitive matrix \(E^{(m)}(t)\) by

\[
E^{(m)}(t) = t(I - Q^{(m)}) + t^{-1}Q^{(m)} = tP^{(m)} + t^{-1}(I - P^{(m)})
\]

where \(Q^{(m)} = \frac{C_m^{(m)}\dagger C_m^{(m)}}{\|C_m^{(m)}\|^2}, \quad P^{(m)} = \frac{C_{-m}^{(m)}\dagger C_{-m}^{(m)}}{\|C_{-m}^{(m)}\|^2}\),

and (ii) compute the coefficient list of \(F^{(m-1)}(t) = E^{(m)}(t)E^{(m)}(t)\) by

\[
C_k^{(m-1)} = C_k^{(m)}P^{(m)} + C_{k+1}^{(m)}Q^{(m)}
\]

where \(k = -m + 1, -m + 3, \ldots, m - 3, m - 1\).

6. Output \(E^{(2n)}(t), \ldots, E^{(1)}(t)\) and \(E^{(0)} = C_0^{(0)}\). Then,

\[
\langle + | E^{(0)}E^{(1)\dagger}(1/t) \cdots E^{(2n)\dagger}(1/t) | + \rangle
\]

is \(O(\epsilon)\)-close to \(A(t^2) + iB(t^2)\) for all \(t \in U(1)\).

### 4.2 Further details and Analysis

**Step 0.** The complexity of this Step depends on the nature of the true target functions \(A(z)\) and \(B(z)\); the smoother the functions are, the lower the complexity is. An obvious approach is to expand the given function in Fourier series and truncate it at a certain degree to desired accuracy. See Sections 5 and 6 for examples.

**Step 1.** Put \(\tilde{a}(z) = (1 - 2\epsilon)\tilde{A}(z) = \sum_{k \in \mathbb{Z}} \tilde{a}_k z^k\) and \(\tilde{b}(z) = (1 - 2\epsilon)\tilde{B}(z) = \sum_{k \in \mathbb{Z}} \tilde{b}_k z^k\), where only finitely many \(\tilde{a}_k, \tilde{b}_k\) are nonzero. Let \(n\) be the largest positive integer such that at least one of \(|\tilde{a}_n|, |\tilde{a}_{-n}|, |\tilde{b}_n|, |\tilde{b}_{-n}|\) is \(\geq \epsilon/n\).

Approximate every coefficient of \(\tilde{a}(z)\) and \(\tilde{b}(z)\) by rational numbers up to error \(\epsilon/n\) to obtain \(a(z)\) and \(b(z)\); this can be done by continued fractions. The rational approximation should preserve reality and (anti-)reciprocity if exists. Any coefficient that are smaller than
$\epsilon/n$ is made zero by the rational approximation. By the choice of $n$, the degrees of $a(z)$ and $b(z)$ are at most $n$, and one of them is indeed $n$. Then, one can easily show that $a(z)$ and $b(z)$ satisfies all four conditions. The number of bits to represent all the rational coefficients is $O(n \log(n/\epsilon))$.

The reason to use rational Laurent polynomials is mainly for the convenience of analysis, as its evaluation can be made arbitrarily accurate and we do not worry about the precision of the coefficients of $a(z)$ and $b(z)$.

**Step 2.** There exists a root-finding algorithm with computational complexity $\tilde{O}(n^3 + n^2 R)$ under the assumption that all the roots have modulus at most 1 [14]. In our case, the rational Laurent polynomial $p(z) = 1 - a(z)^2 - b(z)^2$ does not satisfy the modulus condition; however, this is a minor problem. Every coefficient of $p(z)$ is the Fourier coefficient of the periodic function $p(e^{i\theta}) < 1$, and hence is bounded by 1. By the condition (iv) of Step 1, the leading coefficient of $p(z)$ is at least $(\epsilon/n)^2$ in magnitude. (The leading coefficients of $a(z)$ and $b(z)$ may cancel each other in $p(z)$, yielding a much smaller leading coefficient in $p(z)$, but some arbitrary perturbation of magnitude $\epsilon/n$ in the rational approximation can easily remove such fine-tuned cancellation. Consequently, we may assume that $p(z)$ has degree $2n$.) Converting $p(z)$ into a monic polynomial $q(z)$ (after multiplying $z^{2n}$), we have $q(z) = z^{4n} + \sum_{j=0}^{4n-1} q_j z^j$ with $|q_j| \leq (n/\epsilon)^2$. If $q(z_0) = 0$ with $|z_0| > 1$, then

$$|z_0|^{4n} \leq \sum_{j=0}^{4n-1} |q_j||z_0|^j \leq O(n^3|z_0|^{4n-1}/\epsilon^2) \quad (20)$$

implying $|z_0| \leq O(n^3/\epsilon^2)$. (If $|z_0| \leq 1$, this is trivial.) We can use the algorithm of Ref. [14] after rescaling $z$ by a known factor. The loss of accuracy due to the rescaling is negligible since $R = \Omega(n \log(n/\epsilon))$.

**Step 3.** We need to evaluate $e(z)$ that is defined by roots found in Step 2. For $z \in U(1)$, the following Lemma 5 guarantees that any root of $1 - a(z)^2 - b(z)^2$ is $\Omega(\epsilon/n)$-away (or $\Omega(1/n)$-away if $|1 - A(z)^2 - B(z)^2| = O(\epsilon)$) from the unit circle. The number of significant bits that are lost is at most $O(\log(n/\epsilon))$, which is negligible compared to $R$.

**Lemma 5.** If a Laurent polynomial $f(z)$ of degree $d$ satisfies $0 < m \leq f(z) \leq M$ for all $z \in U(1)$, then every zero of $f$ is at least $m/(6Md)$-away from $U(1)$.

**Proof.** Pick any root $z_0$ and choose the closest point $u \in U(1)$ so that $f(z_0) = u + \eta = 0$. We will lower bound the magnitude of $\eta$. Assume $|\eta| < \frac{m}{6Md}$ on the contrary to the claim; in particular, $|\eta| < 1/(2d)$. Since $f$ is analytic except at $z = 0$, the Taylor series of $f$ at $u$ converges at $z = u + \eta$.

$$0 = f(u + \eta) = \sum_{k \geq 0} \frac{f^{(k)}(u)}{k!} \eta^k. \quad (21)$$

Let us estimate the magnitude of derivatives at $u$. Observe that $f^{(k)}(u) = (e^{-i\theta} \partial_\theta)^k f(e^{i\theta})$. Also, observe that $f(e^{i\theta}) = \sum_{j=-d}^{d} a_j e^{ij\theta}$ where $a_j = (2\pi)^{-1} \int_0^{2\pi} d\theta \ f(e^{i\theta}) e^{-ij\theta}$. Hence, $|a_j| \leq M$ and thus $|f^{(k)}(u)| \leq 2d \cdot M \cdot d(d+1) \cdots (d+k-1)$; there are $2d$ terms and the maximum
absolute value of the exponent increases by at most 1 every time we differentiate due to the negative degree term. Therefore,

\[ m \leq |f(u)| \leq \sum_{k \geq 1} 2dM \left( \frac{d+k-1}{k} \right) |\eta|^k = 2dM \left( \frac{1}{(1-|\eta|)^d} - 1 \right) \leq 2dM \cdot 3|\eta| \quad (22) \]

where in the last inequality we use the hypothesis that $|\eta| \leq 1/(2d)$.

**Step 4.** This is essentially expanding the polynomials $c(z)$ and $d(z)$ found in the root finding step, but we use the fast Fourier transform (FFT) for its better accuracy. It has been shown [15] that the FFT on an $N$-component input $F$ where each component is accurate to (relative) error $\delta$ gives Fourier coefficients $\tilde{F}_\omega$ with error

\[ \max_\omega |\tilde{F}_\omega - \hat{F}_\omega| \leq \mathcal{O}(\sqrt{N} \log N) \cdot \delta \sqrt{\frac{1}{N} \sum_\omega |\hat{F}_\omega|^2} \quad (23) \]

where $\hat{F}$ is the true Fourier spectrum. Since the input “vector” $F$ in this Step is a list of unitary matrices, the root-mean-square factor is $O(1)$, and the distinction between relative and absolute error is immaterial. By the analysis of Step 3 above, $\delta$ is $2^{-R+\mathcal{O}(\log(n/\epsilon))}$. Thus, the error in any Fourier coefficient $C_{2k}^{(2n)}$ is $\delta_{2n} = \mathcal{O}(\delta \sqrt{n} \log n)$.

**Step 5.** This is an implementation of Theorem 2. We have to bound the error propagated through the loop over $m$. The error in $E^{(m)}(t)$ is of the same order as the error of $Q^{(m)}$ or $P^{(m)}$. Thanks to the condition (iv) of Step 1,

\[ \|C^{(m)}_{\pm m}\| \geq \|C_{\pm 2n}^{(2n)}\| \geq \epsilon/n, \quad (24) \]

(where the first inequality is because $C^{(m)}_{\pm m}$ is a product of one unitary and $m$ projectors for any $m \geq 0$) and the error in $Q^{(m)}$ or $P^{(m)}$ is at most $\mathcal{O}(\delta_m n/\epsilon)$. Since the projectors $P^{(m)}$, $Q^{(m)}$ or the Fourier coefficients $C_k^{(m)}$ of a unitary-valued function has norm at most 1, we see that

\[ \delta_{m-1} \leq \mathcal{O}(\delta_m n/\epsilon). \quad (25) \]

Summarizing,

\[ \log_2(1/\delta_{2n}) = R - \mathcal{O}(\log(n/\epsilon)), \]
\[ \log_2(1/\delta_{m-1}) \geq \log_2(1/\delta_m) - \mathcal{O}(\log(n/\epsilon)), \]
\[ \log_2(1/\delta_0) \geq R - \mathcal{O}(n \log(n/\epsilon)). \quad (26) \]

Since we want $\delta_0$ to be sufficiently smaller than $\epsilon/n$, it suffices to have $R = \Omega(n \log(n/\epsilon))$.

**Step 6.** The correctness of the algorithm is clear from the construction and error analysis above.
4.3 Computational complexity

All arithmetic in the algorithm above operates with at most \( R \)-bit numbers, where \( R \) is chosen to be \( \Theta(n \log(n/e)) \). The number of elementary bit operations (\( AND, OR, NOT \)) to perform one basic arithmetic operation (\( +, -, \times, / \)) on \( u \)-bit numbers is upper bounded by \( O(u \log(u)) \) [16]. Let us count the number of arithmetic operations. Assuming that the coefficients of the true function \( A(z) \) and \( B(z) \) are given to accuracy \( O(\epsilon/n) \), it takes \( O(n \log(n/e)) \) arithmetic operations to find rational approximations, since continued fraction converges exponentially. The root finding takes time \( O(n^3 + n^2 R) = O(n^3 \log(n/e)) \) [14]; this count includes all the cost of bit operations for high precision arithmetic. The polynomial function evaluation involves \( O(n) \) arithmetic operations. The FFT requires \( O(n \log n) \) operations given \( O(\log n) \) trigonometric function values, which can be computed by Taylor expansions of order \( R \), invoking \( O(R \log n) \) arithmetic operations. These result in arithmetic complexity \( O(n \log(n) \log(n/e)) \) for the FFT. Updating the Fourier coefficient list \( C^{(m)}_{k} \) involves \( O(n) \) arithmetic operations, which we do for \( O(n) \) times, resulting in \( O(n^2) \) arithmetic operations. Overall, the computational complexity for any constant \( \epsilon \) is \( O(n^3 \log(n)) \), under the random-access memory model of computation.

5 Application to Hamiltonian simulation

Suppose we are given with a unitary \( W \) whose eigenvalues \( \mp e^{\pm i\theta} \) are associated with those \( \lambda \) of a Hermitian matrix \( H \) with \( \|H\| \leq 1 \) as

\[
\sin \theta = \lambda.
\]

The correspondence between \( W \) and \( H \) might seem contrived at this stage, but when \( H \) is represented as a linear combination of unitaries [4, 5], it is possible to construct such \( W \) as a quantum circuit [3]. The relation of Eq. (27) is in fact common whenever quantum walk is used [17, 18]. (See Appendix A for some detail.) So, the desired transformation is

\[
e^{i\theta} \mapsto e^{-\tau \sin \theta} = \langle + | F(e^{i\theta/2}) | + \rangle
\]

where \( F(t) \) should be constructed by quantum signal processing. Since the product of \( n \) primitive matrices yields a Fourier component of frequency at most \( n/2 \), \( F(t) \) must consist of at least \( 2\tau \) factors. (The factor of 2 is due to the half-angle in the argument of \( F \).) Note that the success probability of the post-selection is close to 1 since \( |f(e^{i\theta})| = 1 \).

With \( e^{i\varphi} = z \), we write

\[
\exp(i\tau \sin \varphi) = \exp \left( \tau \frac{z - z^{-1}}{2} \right) = \sum_{k \in \mathbb{Z}} J_k(\tau) z^k
\]

where \( J_k \) are the Bessel functions of the first kind. This is the Fourier series of \( \exp(i\tau \sin \varphi) \), and we see that \( J_k \) is an even function if \( k \) is even, and odd if \( k \) is odd. Also, \( J_{-k} = (-1)^k J_k \). One can take Eq. (29) as a definition of the Bessel functions. We separate the reciprocal and anti-reciprocal parts of the expansion as

\[
\exp \left( \tau \frac{z - z^{-1}}{2} \right) = \sum_{k \in \mathbb{Z}} J_k(\tau) \frac{z^k + z^{-k}}{2} + \left( A(z) \right) \sum_{k \in \mathbb{Z}+1} J_k(\tau) \frac{z^k - z^{-k}}{2i} \left( B(z) \right).
\]
This expansion, called the Jacobi-Anger expansion, converges absolutely at a superexponential rate, as shown below by a steepest descent method [19].

**Lemma 6.** For any real number \(\tau\) and any integer \(k \neq 0\),

\[
|J_k(\tau)| \leq \frac{e}{2} \left( \frac{|\tau|}{2(|k| + 1)} \right)^{|k|}. \tag{31}
\]

**Proof.** It suffices to consider the case where \(\tau \geq 0\) and \(k \geq 1\).

\[
J_k(\tau) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \ e^{i\tau \sin \varphi} e^{-ik\varphi} = \frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{z} e^{(\tau/2)(z - z^{-1})} z^{-k} \tag{32}
\]

where the contour \(\Gamma\) is the unit circle. Since the integrand has a pole only at \(z = 0\) and it vanishes on large \(z\) with \(\Re(z) < r\) for any fixed \(r < \infty\), the contour may be replaced with a semi-circle with an infinite negative real together with a line \(z(y) = (2(k + 1)/\tau) + iy\) where \(y \in \mathbb{R}\). The contribution from the semi-circle at infinity is zero. Therefore,

\[
|J_k(\tau)| < \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \ \frac{e^{k+1}}{|(2(k + 1)/\tau) + iy|^{k+1}}
\]

\[
= \frac{e}{2\pi} \left( \frac{e\tau}{2(k + 1)} \right)^k \int_{-\infty}^{\infty} \frac{dy}{(1 + y^2)^{(k+1)/2}}
\]

\[
\leq \frac{e}{2\pi} \left( \frac{e\tau}{2(k + 1)} \right)^k \int_{-\infty}^{\infty} \frac{dy}{1 + y^2}
\]

\[
= \frac{e}{2} \left( \frac{e\tau}{2(k + 1)} \right)^k. \tag{33}
\]

One can improve the second inequality by evaluating the integral explicitly, to give an extra factor of \(O(k^{-1/2})\) in the bound.

A partial sum of the Jacobi-Anger expansion can be written as

\[
\sum_{k:|k| \leq n} J_k(\tau) z^k = \underbrace{\sum_{\text{even } k:0 \leq k \leq n} J_k(\tau) \frac{z^k + z^{-k}}{2}}_{\tilde{A}(z)} + i \underbrace{\sum_{\text{odd } k:0 \leq k \leq n} J_k(\tau) \frac{z^k - z^{-k}}{2i}}_{\tilde{B}(z)}. \tag{34}
\]

This is \(\epsilon\)-close to the full expansion if \(n = \Omega(\tau + \log(1/\epsilon))\) by Lemma 6 for any \(z \in U(1)\). Numerical experiments suggest that the bound is quite tight and it suffices to choose

\[
n \approx \frac{e}{2} \tau + \ln(1/\epsilon) \approx 1.36\tau + 2.30 \log_{10}(1/\epsilon). \tag{35}
\]

The Laurent polynomials \(\tilde{A}(z)\) and \(\tilde{B}(z)\) are pure real-on-circle. Applying our algorithm, we obtain real-on-circle Laurent polynomials \(a(z) = a_+(z), b(z) = ib_-(z), c(z) = ic_-(z), d(z) = d_+(z)\). The pure Laurent polynomials \(c(z), d(z)\) are calculated by Lemma 4 where we choose \(c(z)\) to be anti-reciprocal and \(d(z)\) reciprocal. (This choice is to have our results in the same convention as those of Ref. [2].) Note that every exponent of \(z\) of the
polynomial $1 - a(z)^2 - b(z)^2$ here, whose roots must be computed, is even since $a(z)$ has only even exponents and $b(z)$ has only odd exponents, so it is always better to feed a Laurent polynomial $g$ of degree $n$, instead of $2n$, where

$$g(z^2) = 1 - a(z)^2 - b(z)^2 \quad (36)$$

into the root finding routine. Given the expanded form of $1 - a(z)^2 - b(z)^2$, it takes no effort to find $g$. In this case, the intermediate polynomial $e(z)$ of Lemma 4 is

$$e(z) = \prod_{r \in \mathbb{R} : g(r) = 0, |r| > 1} \left( z - \frac{r}{2} \right) \prod_{c \in \mathbb{C} : g(c) = 0, |c| > 1, m(c) > 0} \left( z^2 - (c + c) + \frac{c\bar{c}}{z^2} \right). \quad (37)$$

6 Application to Matrix inversion

While there are slightly more efficient implementations of matrix inversion problems [8] using quantum signal processing [9], here we contend ourselves with an eigenvalue transformation perspective. The techniques of Refs. [9] reduces the number of ancilla qubits by one or two, and hence relieves some burden of implementing controlled unitary, but the underlying mathematics, regarding polynomial approximations and finding interspersing single-qubit unitaries, is unchanged.

Suppose a hermitian matrix $H$ of norm at most 1 that we wish to invert is block-encoded in a unitary $W$ so that $W$ has eigenvalues $\mp e^{\pm i\theta}$ associated with an eigenvalue $\lambda$ of $H$. This encoding is the same as in the Hamiltonian simulation above. The condition for $H$ being hermitian is not too restrictive since, for any matrix $M$, an enlarged matrix $|0\rangle \langle 1| \otimes M + |1\rangle \langle 0| \otimes M^\dagger$ is always hermitian. Then, we want eigenvalue transformation $\mp e^{\pm i\theta} \mapsto 1/\sin \theta \lambda$.

As we should not invert a singular matrix, we assume that eigenvalues of $H$ are bounded away from zero by $1/\kappa$ where $\kappa \geq 1$ is the condition number of $H$. (Strictly zero eigenvalues are fine if we are interested in a pseudo-inverse.) Thanks to the condition number assumption, we need to find a polynomial approximation to the function $\mp e^{\pm i\theta} \mapsto 1/\sin \theta \lambda$ that is good for values $\sin \theta \lambda$ away from zero by $1/\kappa$. For this purpose, there is a useful polynomial [20]:

**Lemma 7.** Let $b \geq b' \geq 1$ be any integers, and let $z = e^{i\varphi} \in U(1)$ be any complex number. Then,

$$\left| \left( \frac{z + z^{-1}}{2} \right)^{2b} - \frac{1}{2^{2b}} \sum_{k=-b'}^{b'} \binom{2b}{b+k} z^{2k} \right| \leq 2e^{-b'/b}. \quad (38)$$

If $\left| \frac{z - z^{-1}}{2i} \right| = |\sin \varphi| \geq 1/\kappa$, then

$$\left| \left( \frac{z + z^{-1}}{2} \right)^{2b} \right| \leq e^{-b/\kappa^2}. \quad (39)$$

**Proof.** The first claim is proved as $2^{-2b} \sum_{k : |k| > b'} \binom{2b}{b+k} z^{2k} \leq 2^{-2b} \sum_{k : |k| > b'} \binom{2b}{b+k}$ and then using Hoeffding’s inequality on the tail of binomial probability distributions. The second claim is proved as $(1 - \sin^2 \varphi)^b \leq e^{-b/\kappa^2}$ whenever $|\sin \varphi| \geq 1/\kappa$. \qed
Thus, for large $b$ we see that $1 - ((z + z^{-1})/2)^{2b}$ vanishes when $\sin \varphi = 0$ (i.e., $z = \pm 1$), but is close to 1 for $|\sin \varphi| \geq 1/\kappa$. By the first statement of the lemma this function can be replaced with a lower degree polynomial function. Quantitatively, with $z = e^{i\varphi}$ we have an $\epsilon$-approximation of the desired inversion function by Laurent polynomial of degree $2b' - 1$ for $|\sin \varphi| \geq 1/\kappa$:

$$
\frac{2i}{2^{2b}\kappa(z - z^{-1})} \sum_{k=-b'}^{b'} \binom{2b}{b+k} (1 - z^{2k}) \approx \frac{2i}{\kappa(z - z^{-1})}
$$

(40)

where $b = \kappa^2 \log(2/\epsilon)$ and $b' = \sqrt{b \log(4/\epsilon)} = \mathcal{O}(\kappa \log(1/\epsilon))$. The left-hand side is a genuine Laurent polynomial since the sum vanishes at $z = \pm 1$. Feeding this polynomial, which is real-on-circle and anti-reciprocal, into our algorithm, we obtain a desired eigenvalue inversion quantum algorithm. The success probability can be as small as $\Omega(1/\kappa^2)$, and hence we had better amplify the amplitude for post-selection, enlarging the quantum gate complexity by a factor of $\mathcal{O}(\kappa)$.

7 Discussion

We have determined the scope of $SU(2)$-valued periodic polynomial functions and their decomposition, and analyzed the algorithmic aspects. Our algorithm for the decomposition is not numerically stable in a usual sense — a numerically stable algorithm should only require polylog($n/\epsilon$) bits of precision, rather than poly($n \log(1/\epsilon)$). The instability appears to be generally unavoidable in any method that reduces polynomial degree iteratively as it arises due to the small norm of leading coefficients, but the small leading coefficients are necessary if a nonpolynomial function admits converging polynomial approximations. On the contrary, obviously, if leading coefficients are large in norm, then there is no issue of numerical instability.

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A Jordan’s Lemma and block encoding of Hamiltonians

The following is a well-known fact, but we include it here for completeness.

**Lemma 8.** Let $P$ and $Q$ be arbitrary self-adjoint projectors on a finite dimensional complex vector space $V$. Then, $V$ decomposes into orthogonal subspaces $V_j$ invariant under $P$ and $Q$, where each $V_j$ has dimension 1 or 2.

For any hermitian operator $H$, let $\text{supp}(H)$ denote the subspace support of $H$, i.e., the orthogonal complement of the kernel of $H$. Clearly, $\text{supp}(H)$ is an invariant subspace of $H$, and $H$ is invertible within $\text{supp}(H)$. 
Proof. We will find a subspace $W$ of dimension at most 2 that is invariant under both $P$ and $Q$. This is sufficient since the orthogonal complement of $W$ is also invariant and the proof will be completed by induction in the dimension of $V$.

Put $P' = I - P$ and $Q' = I - Q$, and consider the identities

$$PQP + PQ'P + P'Q'P' = I, \quad \text{supp}(PQP) + \text{supp}(PQ'P) + \text{supp}(P'Q'P') = V,$$

where the second equality is because the intersection of the orthogonal complements of the four supports is zero. Therefore, at least one of four supports is nonzero, and without loss of generality assume $S = \text{supp}(PQP) \neq 0$. Let $|\psi\rangle \in S$ be an eigenvector of $PQP$; $PQP |\psi\rangle = a |\psi\rangle$. The associated eigenvalue $a$ is nonzero by definition of $S$. Now consider $W = \text{span}\{|\psi\rangle, Q |\psi\rangle\}$. Observe that $a |\psi\rangle = PQP |\psi\rangle = PPQP |\psi\rangle = aP |\psi\rangle$, and hence $P |\psi\rangle = |\psi\rangle$. Moreover, $PQ |\psi\rangle = PQP |\psi\rangle = a |\psi\rangle$. Therefore, $W$ is a nonzero invariant subspace under both $P$ and $Q$. □

This Jordan’s lemma can be applied to two hermitian unitaries (reflections) $U_1, U_2$ as any hermitian unitary is $2P - I$ for some projector $P$. It immediately follows that there is a basis where $U_1$ is diagonal and $U_1U_2$ is block-diagonal with at most two-dimensional blocks and each block belongs to $U(1)$ or $U(2)$. In any such two-dimensional block, we must have

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} \frac{\lambda}{\sqrt{1 - \lambda^2}} e^{i\phi} & e^{i\phi} \sqrt{1 - \lambda^2} \\ e^{-i\phi} \sqrt{1 - \lambda^2} & -\lambda \end{pmatrix}$$

for some real number $\lambda \in [-1, 1]$ and an angle $\phi \in \mathbb{R}$, up to a permutation of rows and columns. Therefore, the product $W = -iU_1U_2$ is a rotation in a two-dimensional subspace that appears in Grover search algorithm [21], and has eigenvalues $\pm e^{\pm i\theta}$ where $\sin \theta = \lambda$. This is relevant in a Hamiltonian simulation problem where $U_1 = 2 |G\rangle \langle G| - 1$ and the Hamiltonian is “block-encoded” as $H = \langle G| U_2 |G\rangle$, which is the case if $H$ is represented as, e.g., a linear combination of Pauli operators.

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