The Uehling correction to the energy levels in a pionic atom

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Abstract: We consider a correction to energy levels in a pionic atom induced by the Uehling potential, i.e., by a free electron vacuum-polarization loop. The calculation is performed for circular states \((l = n - 1)\). The result is obtained in a closed analytic form as a function of \(Z\alpha\) and the pion-to-electron mass ratio. Certain asymptotics of the result are also presented.

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1. Introduction

The vacuum polarization effects play an important role in precision theory of energy levels and wave functions of atoms. While in conventional atoms such as hydrogen the effects of the self energy of an electron dominate over the vacuum polarization, in atoms with a heavier orbiting particle (such as a muon or pion) the vacuum polarization is responsible for the dominant QED effects.

In principle, studies of pionic and other exotic atoms involve effects of the strong interaction. There is a direct interaction of the pion and the nucleus. Additionally, the finite-nuclear-size effects are important. However, relations between values of various contributions depend on values of the nuclear charge \(Z\) and of the principal and orbital quantum numbers, \(n\) and \(l\), respectively. There are certain situations where the vacuum polarization dominates. In a number of other cases, it produces a correction, which is not dominant, but still important.

In this paper we focus on study of effects of the vacuum polarization within a relativistic framework. Their leading contribution to the energy levels is related to so-called Uehling potential, a correction to the Coulomb potential induced by a free electronic vacuum loop.

The non-relativistic results in muonic atoms (i.e., the results in the leading order in \(Z\alpha\)) have been known for a while [1] as well as various numerical relativistic calculations for the conventional (see, e.g., [2] and references therein) and muonic (see, e.g., [3] and references therein) atoms.

A relativistic calculation of vacuum polarization effects can be performed analytically. In particular, various effects induced by the Uehling potential were studied for the Dirac equation, i.e., for an orbiting particle with spin \(1/2\), in [4, 5, 6]. The results were obtained in a closed analytic form as a function of...
and a mass ratio of the orbiting particle and an electron. However, the result for pionic atoms was
not derived because it was unclear how to write relativistic perturbative series for particle with zero
spin.

Here, we extend our approach onto spinless orbiting particles, such as a pion. Relativistic descrip-
tion of such a particle is significantly different from that of an electron. A spin-1/2 particle is described
by an equation of the form
\[ H \Psi_i = E_i \Psi_i, \]
where the Hamiltonian \( H \) is a Hermitian operator and the wave function \( \Psi_i \) is its eigenfunction. The
correction to the energy due to a perturbation by an electrostatic potential \( \delta V \) is of an obvious form
\[ \delta E_i = \langle \Psi_i^{(0)} | \delta V | \Psi_i^{(0)} \rangle. \]  
(1)

In [4, 5, 6] various integrals, which could appear in calculation of such a martix element over the
Uehling potential in Coulomb bound system were studied.

The equation, that describes a pion, is Klein-Gordon-Fock equation, which for electrostatic poten-
tial is of the form
\[
[(E_i - V(r))^2 - \mathbf{p}^2 c^2 - m^2 c^4] \Phi_i(r) = 0.
\]

This equation cannot be considered as a problem for eigenfunctions and eigenvalues of a Hermitian
operator. The solutions of this equation, \( \Phi_i \), are not orthogonal and their normalization is not trivial.

While ‘exact’ solution of the Klein-Gordon-Fock equation with a non-Coulomb potential has been
possible (see, e.g., study of the problem of falling to the center in [8] and numerical calculations in
[9]), the perturbative theory with the Coulomb field has not been developed.

A simple perturbative series for the Klein-Gordon-Fock equation has not been known until now
and here we apply a recent result obtained in [7]. In particular, the analog of Eq. (1), convenient for
analytic calculations, is
\[
\delta E_i = \frac{\langle \Phi_i^{(0)} | (E_i^{(0)} - V^{(0)}) \delta V | \Phi_i^{(0)} \rangle}{\langle \Phi_i^{(0)} | (E_i^{(0)} - V^{(0)}) \Phi_i^{(0)} \rangle}. \]  
(2)

This equation is valid for any perturbation that can be described by a certain electrostatic potential and,
in particular, for the Uehling correction.

To calculate the Uehling correction we consider the Coulomb problem as an unperturbed case. Its
eigenvalues and eigenfunctions are known (see, e.g., [10]):
\[
\Phi_{nl}^{(0)} = \frac{1}{r} R_{nl}(r) Y_{lm}(\theta, \phi),
\]
\[
R_{nl}(r) = N (\beta r)^{s+1} \exp\left(-\frac{\beta r}{2}\right) {}_1F_1(-n + l + 1, 2s + 2, \beta r),
\]
\[
E^{(0)} = \frac{mc^2}{\sqrt{1 + \left(\frac{Z\alpha}{n}\right)^2}}.
\]  
(3)

where \( Y_{lm}(\theta, \varphi) \) stands for spherical harmonics, \( {}_1F_1(a, b; z) \) is the confluent hypergeometric function,
\[
s = -\frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2} \approx l - \frac{(Z\alpha)^2}{2l + 1} + \ldots,
\]
\[
\eta = n - l + s \approx n - \frac{(Z\alpha)^2}{2l + 1} + \ldots,
\]
\[
\beta = \frac{2}{c \hbar} \frac{(Z\alpha) E^{(0)}}{\eta} \approx \frac{2(Z\alpha) mc}{n \hbar} \left(1 + \frac{2n - 2l - 1}{2n^2(2l + 1)}(Z\alpha)^2 + \ldots\right).
\]  

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and $m$ is the mass of the orbiting particle, presumably, a pion. Here, $N$ is a normalization factor, unspecified in [10], which is sufficient for our calculation, since the normalization is unimportant for any application of Eq. 2.

Our approach can be applied for an arbitrary state. In this paper we consider only circular states ($l = n - 1$), in which the wave function is of the simplest form

$$R_{nl}(r) = N (\beta r)^n \exp \left( -\frac{\beta r}{2} \right).$$

Such states in particular include the 1s and 2p states. Some of these states are also of experimental interest: the ground state (1s) is the easiest to create. Meantime, high-$l$ states are easier for theoretical interpretation because of a substantial reduction of the strong-interaction effects.

The Uehling potential, which is considered as a perturbation of the unperturbed problem with the Coulomb potential, is of the form [11]

$$\delta V = -\alpha(Z\alpha)^2 \frac{mc^2}{\sqrt{\eta^2 + (Z\alpha)^2}} \left[ \beta r \eta K_{2,2\eta}(\tilde{\kappa}_n) + \frac{2(Z\alpha)^2}{2\eta - 1} K_{2,2\eta-1}(\tilde{\kappa}_n) \right],$$

where $m_e$ is the electron mass.

The integrals needed for matrix element in (2) are very similar to the ones studied by us previously for a Dirac particle [4, 5, 6]. The result of calculations of the correction to the energy (2) with potential (5) over the wave functions from (3) and (4) is (cf., [4])

$$\delta E = -\frac{\alpha(Z\alpha)^2}{\eta} \int_0^1 dv \frac{v^2(1 - \frac{1}{2}v^2)}{1 - v^2} e^{-\lambda r},$$

where $\lambda = \frac{2}{\hbar} \frac{m_e c}{\sqrt{1 - v^2}}$.

The result for a spin-0 particle is similar to that a spin-1/2 particle [4]. We compare the related results for the 1s state in Fig. 1, where we present results for two values of the mass of the orbiting particle,

$$K_{abc}(\kappa) = \frac{1}{2} c^c B(a + 1/2, 1 - b/2 + c/2)$$

$$\times \ 3F_2(c/2, c/2 + 1/2, 1 - b/2 + c/2; 1/2, a + 3/2 - b/2 + c/2; \kappa^2)$$

$$- \frac{c}{2} c^{c+1} B(a + 1, 3/2 - b/2 + c/2)$$

$$\times \ 3F_2(c/2 + 1, c/2 + 1/2, 3/2 - b/2 + c/2; 3/2, a + 2 - b/2 + c/2; \kappa^2).$$

where $3F_2(a, b, c; d, e; z)$ stands for the generalized hypergeometric function and $B(a, b)$ is the beta function.

The result for a spin-0 particle is similar that a spin-1/2 particle [4]. We compare the related results for the 1s state in Fig. 1, where we present results for two values of the mass of the orbiting particle,
namely, for \( m = m_e \) and \( m = 250 \, m_e \) (for the latter we chose the \( m \) value to be somewhere in between \( m_{\mu} \) and \( m_{\pi} \)). The correction is presented in terms of a dimensionless function \( F(Z\alpha) \) defined as
\[
\delta E = \frac{\alpha}{\pi} \frac{(Z\alpha)^2 m e^2}{n^2} F(Z\alpha) .
\]
The similar plots for \( 2p \) and \( 5g \) states are presented in Figs. 2 and 3. We note that the results have different ranges of convergence, but for physical values \( Z\alpha < 1 \) the results for excited states are very close to each other.

**Fig. 1.** The Uehling correction to the energy of the \( 1s \) state for spin-0 and spin-1/2 particles. The parameters are (a): \( m/m_e = 1 \) and (b): \( m/m_e = 250 \).

**Fig. 2.** The Uehling correction to the energy of the \( 2p \) states for spin-0 and spin-1/2 particles. The parameters are (a): \( m/m_e = 1 \) and (b): \( m/m_e = 250 \).

The most important difference is related to the range of values for the nuclear charge \( Z \) for which the results, derived here for a point-like nucleus, can be applied. In principle, it is not necessary that a correction is well defined for the same range of parameters as the leading term is. For instance, in a Dirac atom with a point-like nucleus the hyperfine interval (which can be considered as a perturbative...
correction to the electron energy levels due to an interaction with the nuclear spin) for the 1\textit{s} state is well defined for a narrower range \((Z\alpha < \sqrt{3}/2)\) than the leading term, namely, the Coulomb energy. Nevertheless, the Uehling correction for both spin-0 and spin-1/2 particles is well defined in the same \(Z\) range as the related Coulomb term. In particular, for the 1\textit{s} state of a Klein-Gordon particle the condition is \(Z\alpha < 1/2\), while for a Dirac particle it is \(Z\alpha < 1\). For the hyperfine splitting and the \(g\) factor of a Dirac particle the situation is similar — the Uehling correction is valid for the same range of \(Z\alpha\) as the Dirac term (for the reference to the Uehling correction in the 1\textit{s} state see \([4]\) for the Lamb shift, \([5]\) for the hyperfine splitting and \([6]\) for the \(g\) factor of a bound electron).

The result we obtained above (6) contains two free parameters: \(Z\alpha\) and

\[
\kappa_n = \frac{Z\alpha \, m}{n \, m_e},
\]

where

\[
\tilde{\kappa}_n = \frac{n}{\sqrt{\eta^2 + (Z\alpha)^2}} \kappa_n \simeq \kappa_n \times \left(1 + (Z\alpha)^2 \frac{2n-2l-1}{2n^2(2l+1)} + \ldots\right).
\]

In a pionic atom

\[
\kappa \simeq 2 \frac{Z}{n},
\]

while in other mesonic atoms this ratio can be even higher.

The general expression is rather complicated and we present here some useful asymptotics applying an expansion in either of these two parameters. First we consider a case of \(Z\alpha \ll 1\) which allows a non-relativistic expansion. The result is found to be

\[
\delta E = -\frac{\alpha (Z\alpha)^2 mc^2}{n^2} \left\{ K_{2,2n}(\kappa_n) + \frac{(Z\alpha)^2}{n(2n-1)} \left[ \frac{1}{\kappa_n} K_{3,2n+1}(\kappa_n) \right. \right.
\]

\[
+ \left. \left( \frac{3}{2n} - 1 \right) K_{2,2n}(\kappa_n) - 2n L_{2,2n}(\kappa_n) + 2K_{2,2n-1}(\kappa_n) \right\} + \ldots ,
\]

where

\[
K_{\ell,m}(\kappa_n) = \sum_{m=-\ell}^{\ell} \frac{\kappa_n^{m+\ell}}{m!} \left( \begin{array}{c} 2\ell+1 \\ m \end{array} \right) \left( \begin{array}{c} \ell+1 \\ m+\ell \end{array} \right) m^{\ell+1},
\]

and

\[
L_{\ell,m}(\kappa_n) = \sum_{m=-\ell}^{\ell} \frac{\kappa_n^{m+\ell}}{m!} \left( \begin{array}{c} 2\ell+1 \\ m \end{array} \right) \left( \begin{array}{c} \ell+1 \\ m+\ell \end{array} \right) m^{\ell+1}.
\]

\(\eta = \frac{1}{\sqrt{2}} \left( 1 - \sqrt{1 - \frac{4Z^2}{\sqrt{3}} \frac{1}{\kappa} \frac{m_e}{m} } \right)
\]

\(\kappa = \sqrt{3} \frac{m_e}{m}
\]
where

\[ L_{bc}(\kappa) = \frac{\partial}{\partial \kappa} K_{bc}(\kappa) \]

\[ = \int_0^1 dv \frac{v^2(1 - \frac{1}{2} v^2)}{(1 - v^2)^{3/2}} \left( \frac{\kappa \sqrt{1 - v^2}}{1 + \kappa \sqrt{1 - v^2}} \right)^c \ln \left( \frac{\kappa \sqrt{1 - v^2}}{1 + \kappa \sqrt{1 - v^2}} \right). \]

Various asymptotics of the integral above were studied in [5, 12] (see also below).

An alternative case with \( \kappa_n \gg 1 \) is realized for low-lying states in medium-Z and heavy pionic atoms. The result is of the form

\[ \delta E = \frac{\alpha}{\pi} \frac{(Z\alpha)^2 m c^2}{v^2 + (Z\alpha)^2} \left[ \left( \eta + \frac{2(Z\alpha)^2}{2\eta - 1} \right) \left( \frac{2}{3} \ln(2\kappa_n) + \frac{2}{3} \psi(1) - \frac{5}{9} \right) \right. \]

\[ - \frac{2}{3} \eta \psi(2\eta) - \frac{4}{3} \frac{(Z\alpha)^2}{2\eta - 1} \psi(2\eta - 1) + \frac{\pi}{2} \eta^2 + (Z\alpha)^2 \]

\[ \left. - \frac{\eta}{2} \frac{\eta(2\eta + 1) + 2(Z\alpha)^2}{\kappa_n^2} + \ldots \right], \quad (9) \]

where \( \psi(x) = \frac{\partial}{\partial x} \ln(\Gamma(x)) \) is the logarithmic derivative of the Euler gamma function.

To check our calculations we have examined a double asymptotic behavior for the case \( Z\alpha \ll 1 \) and \( \kappa_n \gg 1 \). The results derived from (8) and (9) agreed. The result for the double asymptotics\(^2\)

\[ \delta E = -\frac{\alpha}{\pi} \frac{(Z\alpha)^2 m c^2}{v^2 + (Z\alpha)^2} \left[ \left( \frac{2}{3} \ln(2\kappa_n) + \frac{2}{3} (\psi(1) - \psi(2n)) - \frac{5}{9} + \frac{\pi n}{2\kappa_n} - \frac{n(2n + 1)}{2\kappa_n^2} \right) + \ldots \right] \]

\[ + \frac{(Z\alpha)^2}{n(2n - 1)} \left[ \left( \frac{3}{2n - 1} + 1 \right) \left( \frac{2}{3} \ln(2\kappa_n) + \frac{2}{3} \psi(1) - \frac{5}{9} \right) + \frac{1}{3n} \right. \]

\[ - \frac{2}{3} \left[ \left( \frac{3}{2n} - 1 \right) \psi(2n) + 2 \psi(2n - 1) - 2n \psi'(2n) \right] + \frac{4n^2 + 6n - 1}{4\kappa_n^2} \right] + \ldots \right] \]

is more transparent than the single-parameter expansions (8) and (9).

Other asymptotic cases of interest are related to \( \kappa_n \ll 1 \). These limit is approached at a pionic atom for high \( n \). For arbitrary \( Z\alpha \) the expansion is

\[ \delta E = -\frac{\alpha}{\pi} \frac{(Z\alpha)^2 m c^2}{3(\eta^2 + (Z\alpha)^2)^{3/2}} \left[ \frac{2n}{2n - 1} \frac{1}{(Z\alpha)^2} B(5/2, \eta - 1/2) \kappa_n^{2n - 1} \right. \]

\[ + \left( \eta + 1 \right) \left( \eta - 2(Z\alpha)^2 \right) B(5/2, \eta) \kappa_n^{2n} + \ldots \] \quad (10)

while for small \( Z\alpha \) we arrive at

\[ \delta E = -\frac{\alpha}{\pi} \frac{(Z\alpha)^2 m c^2}{3n^2} \left\{ (n + 1)B(5/2, n)\kappa_n^{2n} - n(2n + 3)B(5/2, n + 1/2)\kappa_n^{2n + 1} + \ldots \right\} \]

\[ + \frac{(Z\alpha)^2}{n(2n - 1)} \left[ (2n + 1)B(5/2, n - 1/2)\kappa_n^{2n - 1} \right. \] \quad (11)
\( - \frac{n(n+1)}{(n+1)(n+3)} \left\{ 2 \ln(\kappa_n) + \psi(n) - \psi(n+3/2) + \frac{8n^2 - 4n - 3}{2n^2} \right\} \) \( B(5/2, n) \kappa_n^{2n} + \ldots \} \).

Above we considered the Uehling correction to the energy of the circular states in a pionic atom and found the general expression and its various asymptotics. The same approach can be applied for any state. The result for low \( Z \), where the non-relativistic effects dominate, is similar to that for a bound Dirac particle. At higher \( Z \) the correction is different and, in particular, its has a different range of applicability (see, e.g., Figs. 1–3). More detail will be presented elsewhere.

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