All meager filters may be null

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Abstract

We show that it is consistent with ZFC that all filters which have the Baire property are Lebesgue measurable. We also show that the existence of a Sierpinski set implies that there exists a nonmeasurable filter which has the Baire property.

The goal of this paper is to show yet another example of nonduality between measure and category.

Suppose that $\mathcal{F}$ is a nonprincipal filter on $\omega$. Identify $\mathcal{F}$ with the set of characteristic functions of its elements. Under this convention $\mathcal{F}$ becomes a subset of $2^\omega$ and a question about its topological or measure-theoretical properties makes sense.

It has been proved by Sierpinski that every non-principal filter has either Lebesgue measure zero or is nonmeasurable. Similarly it is either meager or does not have the Baire property.

In [T] Talagrand proved that

**Theorem 0.1** There exists a measurable filter which does not have the Baire property.

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In fact we have an even stronger result. In [Ba] it is proved that

**Theorem 0.2** Every measurable filter can be extended to a measurable filter which does not have the Baire property. ■

We show that the dual result is false.

# 1 A model where all meager filters are null

In this section we prove the following theorem:

**Theorem 1.1** It is consistent with ZFC that every filter which has the Baire property is measurable.

**Proof** We will use the following more general result:

**Theorem 1.2** Let $V \models GCH$ and suppose that $V[G]$ is a generic extension of $V$ obtained adding $\omega_2$ Cohen reals. Then in $V[G]$ for any two sets $A, B \subseteq 2^\omega$ if $A + B = \{a + b : a \in A, b \in B\}$ is a meager set then either $A$ or $B$ has measure zero.

**Proof** Note that we apply this lemma only for the case $A = B$. Therefore to simplify the notation we assume that $A = B$. The proof of the general case is almost the same. We follow [Bu].

We will use the following notation. Let $F_n(X, 2) = \{s : \text{dom}(s) \subseteq [X]^{<\omega} \text{ and } \text{range}(s) \subseteq \{0, 1\}\}$ be the notion of forcing adding $|X|$-many Cohen reals. For $s \in F_n(X, 2)$ let $[s] = \{f \in 2^X : s \subseteq f\}$.

Let $V \models GCH$ be a model of ZFC and let $G_{\omega_2}$ be a $F_n(\omega_2, 2)$-generic filter over $V$. Clearly $c = \bigcup G_{\omega_2}$ is a generic sequence of $\omega_2$ Cohen reals and $V[c] = V[G_{\omega_2}]$.

Let $\{F_n : n \in \omega\}$ be a sequence of closed, nowhere dense sets such that $A + A \subseteq \bigcup_{n \in \omega} F_n$. Without loss of generality we can assume that $\{F_n : n \in \omega\} \in V$.

Let $\{a_\xi : \xi < \omega_2\}$ be an enumeration of all elements of $A$. For every $\xi < \omega_2$ let $\dot{a}_\xi$ be a name for $a_\xi$. In other words for every $\xi < \omega_2$ we have a countable set $I_\xi \subseteq \omega_2$ such that $\dot{a}_\xi$ is a Borel function from $2^{I_\xi}$ into $2^\omega$. Moreover $a_\xi$ is the value of of the function $\dot{a}_\xi$ on Cohen real i.e. $\dot{a}_\xi(c|I_\xi) = a_\xi$. In addition we can find a dense $G_\delta$ set $H_\xi \subseteq 2^{I_\xi}$ such that $\dot{a}_\xi|H_\xi$ is a continuous function.

For $\alpha, \xi, \eta < \omega_2$ define $\xi \simeq_\alpha \eta$ if

1. $I_\xi$ and $I_\eta$ are order isomorphic,

2. the order-isomorphism between $I_\xi$ and $I_\eta$ transfers $\dot{a}_\xi$ onto $\dot{a}_\eta$ and $H_\xi$ onto $H_\eta$.  

2
3. \( I_\xi \cap \alpha = I_\eta \cap \alpha \).

Notice that for every \( \alpha < \omega_2 \) the relation \( \simeq_\alpha \) is an equivalence relation with \( \omega_1 \) many equivalence classes.

**Lemma 1.3** There exists \( \alpha^* < \omega_2 \) such that

\[
\forall \xi, \beta \exists \eta \ (\xi \simeq_\alpha \eta \land I_\eta \cap (\beta - \alpha^*) = \emptyset) .
\]

**Proof** For every \( \alpha < \omega_2 \) let \( E_\alpha \) be the set \( \{[\xi]_\alpha : \xi < \omega_2 \} \) of \( \simeq_\alpha \)-equivalence classes. Let

\[
E^0_\alpha = \{ E \in E_\alpha : \sup(\min(I_\eta - \alpha)) < \omega_2 \} \quad \text{and} \quad E^1_\alpha = E_\alpha - E^0_\alpha .
\]

Let

\[
\gamma(\alpha) = \sup_{E \in E^0_\alpha} (\sup(\min(I_\eta - \alpha))) .
\]

Note that \( \gamma(\alpha) < \omega_2 \) since \( |E_\alpha| < \aleph_1 \).

Find \( \alpha^* < \omega_2 \) such that \( \gamma(\alpha) < \alpha^* \) for all \( \alpha < \alpha^* \) and \( \text{cf}(\alpha^* ) = \omega_1 \). We claim that \( \alpha^* \) satisfies the statement of the lemma.

Take any \( \xi < \omega_2 \) and any \( \beta \). If \( \beta < \alpha^* \) or \( I_\xi \subseteq \alpha^* \), then we can choose \( \eta = \xi \).

So assume \( \beta > \alpha^* \) and \( I_\xi - \alpha^* \neq \emptyset \). There is \( \alpha < \alpha^* \) such that \( I_\xi \cap \alpha = I_\xi \cap \alpha^* \).

Let \( E = [\xi]_\alpha \).

**Case 1** \( E \in E^0_\alpha \). Then

\[
\sup_{\eta \in E}(\min(I_\eta - \alpha)) \leq \gamma(\alpha) < \alpha^*
\]

which is a contradiction since \( \min(I_\xi - \alpha) \geq \alpha^* \) and \( \xi \in E \).

**Case 2** \( E \notin E^0_\alpha \). So

\[
\sup_{\eta \in E}(\min(I_\eta - \alpha)) = \omega_2
\]

hence there is \( \eta \in E \) with \( \min(I_\eta - \alpha) \geq \beta \) i.e. \( I_\eta \cap (\beta - \alpha) = \emptyset \).

So \( I_\xi \cap \alpha^* = I_\xi \cap \alpha = I_\eta \cap \alpha = I_\eta \cap \alpha^* \), where the last equality holds because \( I_\eta \cap (\alpha^* - \alpha) \subseteq I_\eta \cap (\beta - \alpha) = \emptyset \). Also \( I_\eta \cap (\beta - \alpha^*) \subseteq I_\eta \cap (\beta - \alpha) = \emptyset \). \( \square \)

Let \( \alpha^* \) be the ordinal from the above lemma. Work in \( \mathbf{V}' = \mathbf{V}[\mathbf{c}|\alpha^*] \).

For every \( \xi < \omega_2 \) define

\[
D_\xi = \{ s \in Fn(\omega_2 - \alpha^*, 2) : \text{cl}(\hat{a}_\xi([s])) \text{ has measure zero} \} .
\]

**Lemma 1.4** \( D_\xi \) is dense in \( Fn(\omega_2 - \alpha^*, 2) \) for every \( \xi < \omega_2 \).
Proof Notice that it is enough to show that $D_\xi \cap Fn(I_\xi - \alpha^*, 2)$ is dense in $Fn(I_\xi - \alpha^*, 2)$ for $\xi < \omega_2$.

Suppose that this fails. Find $\xi < \omega_2$ and $s_0 \in Fn(I_\xi - \alpha^*, 2)$ such that for all $s \supseteq s_0$ the set $cl(\hat{a}_\xi([s]))$ has positive measure.

Using the lemma with $\beta > \sup(I_\xi)$ we can find $\eta < \omega_2$ such that $\xi \simeq \alpha^*, \eta$ and $(I_\xi - \alpha^*) \cap (I_\eta - \alpha^*) = \emptyset$. Notice that there exists $t_0 \in Fn(I_\eta - \alpha^*, 2)$ (the image of $s_0$ under the isomorphism between $I_\xi$ and $I_\eta$) such that for every $t \supseteq t_0$ the set $cl(\hat{a}_\eta([t]))$ has positive measure.

Since $s_0$ and $t_0$ have disjoint domains, $s_0 \cup t_0 \in Fn(\omega_2 - \alpha^*, 2)$. Find $n \in \omega$ and a condition $u \in Fn(\omega_2 - \alpha^*, 2)$ extending $s_0 \cup t_0$ such that $u \vdash \hat{a}_\xi(\delta) + \hat{a}_\eta(\delta) \in F_n$. $u$ can be written as $u_1 \cup u_2 \cup u_3$ where $s_0 \subseteq u_1 \in Fn(I_\xi - \alpha^*, 2)$, $t_0 \subseteq u_2 \in Fn(I_\eta - \alpha^*, 2)$ and $u_3 \in Fn(\omega_2 - (I_\xi \cup I_\eta \cup \alpha^*), 2)$. By the assumption the sets $cl(\hat{a}_\xi([u_1])), cl(\hat{a}_\eta([u_2]))$ have positive measure. By well-known theorem of Steinhaus the set $cl(\hat{a}_\xi([u_1])) + cl(\hat{a}_\eta([u_2]))$ contains an open set (hence also $(cl(\hat{a}_\xi([u_1])) + cl(\hat{a}_\eta([u_2])) - F_n$ contains an open set). Using the fact that $\hat{a}_\xi$ and $\hat{a}_\eta$ are continuous functions we can find $u_1 \subseteq s_1 \in Fn(I_\xi - \alpha^*, 2)$ and $u_2 \subseteq t_1 \in Fn(I_\eta - \alpha^*, 2)$ such that $(cl(\hat{a}_\xi([s_1])) + cl(\hat{a}_\eta([t_1]))) \cap F_n = \emptyset$. But this is a contradiction since

$$s_1 \cup t_1 \cup u_3 \vdash \hat{a}_\xi(\delta) + \hat{a}_\eta(\delta) \not\in F_n \, \blacksquare$$

Notice that for $\xi < \omega_2$

$$D_\xi = \{s \in Fn(I_\xi) : \text{there exists a closed measure zero set } F \in V' \text{ such that } s \vdash \hat{a}_\xi(\delta) \in F \} .$$

Therefore by the above lemma

$$A \subseteq \bigcup \{ F : F \text{ is a closed measure zero set coded in } V' \} .$$

Since $V$ contains Cohen reals over $V'$, the union of all closed measure zero sets coded in $V'$ has measure zero in $V$. We conclude that $A$ has measure zero. \[ \blacksquare \]

Let $\mathcal{F}$ be a non-principal filter. Denote by $\mathcal{F}^c = \{ X \subseteq \omega : \omega - X \in \mathcal{F} \}$. $\mathcal{F}^c$ is an ideal and it is very easy to see that $\mathcal{F}$ is measurable (has the Baire property) iff $\mathcal{F}^c$ is measurable (has the Baire property).

**Lemma 1.5** $\mathcal{F} + \mathcal{F} = \mathcal{F}^c$.

**Proof** Suppose that $X, Y \in \mathcal{F}$. Then $\{ n : X(n) + Y(n) = 0 \} \supseteq X^{-1}(1) \cap Y^{-1}(1) \in \mathcal{F}$. In general $\mathcal{F} + \cdots + \mathcal{F}$ is equal to $\mathcal{F}$ or $\mathcal{F}^c$ depending whether there is an even or odd number of $\mathcal{F}$’s.

Let $V \models GCH$ and suppose that $V[G]$ is a generic extension of $V$ obtained by adding $\omega_2$ Cohen reals. By the above lemma if $\mathcal{F}$ is a meager filter then $\mathcal{F}^c = \mathcal{F} + \mathcal{F}$ is meager. So by $\blacksquare$ $\mathcal{F}$ has measure zero. \[ \blacksquare \]
2 Filters which are meager and nonmeasurable

Theorem 1.1 shows that in order to construct a filter which is meager and nonmeasurable we need some extra assumptions.

In [T] Talagrand showed that

Theorem 2.1 Suppose that the real line is not the union of \(<2^{\aleph_0}\) many measure zero sets. Then there exists a nonmeasurable filter which is meager. □

Let \(\kappa\) be a regular uncountable cardinal. Recall that \(S\) is a generalized Sierpinski set of size \(\kappa\) if \(|S \cap H| < \kappa\) for every null set \(H\). It is clear that all \(S' \subseteq S\) of size \(\kappa\) are also nonmeasurable.

Theorem 2.2 Assume that there exists a generalized Sierpinski set. Then there exists a nonmeasurable meager filter.

Proof Let \(S\) be a generalized Sierpinski set of size \(\kappa\). Build a sequence \(\{x_\xi : \xi < \alpha\} \subseteq S\) and an elementary chain of models \(\{M_\xi : \xi < \kappa\}\) of size \(\kappa\) such that

1. \(\{x_\xi : \xi < \alpha\} \subseteq M_\alpha\) for \(\alpha < \kappa\),
2. \(x_\beta\) is a random real over \(M_\alpha\) for \(\beta > \alpha\).

Suppose that \(M_\beta, x_\beta\) are already constructed for \(\beta < \alpha\). Since \(S\) is a Sierpinski set

\[\bigcup\{S \cap H : H\text{ is a null set coded in } M_\beta\} \text{ for } \beta < \alpha\]

has size \(< \kappa\). Let \(x_\alpha\) be any element of \(S\) avoiding this set.

Let \(X_\xi = x_\xi^{-1}(1)\) for \(\xi < \kappa\). Let \(\mathcal{F}\) be the filter generated by the family \(\{X_\xi : \xi < \kappa\}\). We will show that \(\mathcal{F}\) has the required properties.

For \(X \subseteq \omega\) let

\[d(X) = \lim_{n \to \infty} \frac{|X \cap n|}{n}\]

if the above limit exists.

By easy induction we show that for \(\xi_1, \ldots, \xi_n < \kappa\) we have \(d(X_{\xi_1} \cap \cdots \cap X_{\xi_n}) = 2^{-n}\). This shows that

\[\mathcal{F} \subseteq \{X \subseteq \omega : \liminf_{n \to \infty} \frac{|X \cap n|}{n} > 0\}\]

which is a meager set. To check that \(\mathcal{F}\) is nonmeasurable notice that \(\mathcal{F}\) contains the nonmeasurable set \(\{x_\xi : \xi < \kappa\}\) □

It is an open problem whether one can construct a meager nonmeasurable filter assuming the existence of a nonmeasurable set of size \(\aleph_1\). We only have some partial results.

Let \(b\) be the size of the smallest unbounded family in \(\omega^\omega\) and let \(\text{unif}\) be the size of the smallest nonmeasurable set.

For \(X \subseteq \omega\) let \(f_X \in \omega^\omega\) be an increasing function enumerating \(X\). For a filter \(\mathcal{F}\) let \(\mathcal{F}^* = \{f_X : X \in \mathcal{F}\}\). In [J] it is proved that
Theorem 2.3  For every filter $\mathcal{F}$, $\mathcal{F}$ has the Baire property iff $\mathcal{F}^\star$ is bounded.

Theorem 2.4  Suppose that $\text{unif} < b$. Then there exists a nonmeasurable filter which is meager.

Proof Let $X \subseteq 2^\omega$ be a nonmeasurable set of size $\text{unif}$. Let $M$ be a model of the same size containing $X$ as a subset. Then $M \cap 2^\omega$ does not have measure zero, so it is nonmeasurable. Consider any filter $\mathcal{F}$ such that $M \models \mathcal{F}$ is an ultrafilter. $\mathcal{F}$ generates a filter in $V$ and this filter is meager by 2.3 and the fact that it is generated by $\text{unif} < b$ many elements. On the other hand $M \models 2^\omega = \mathcal{F} \cup \mathcal{F}^c$ and we know that $M \cap 2^\omega$ is a nonmeasurable set. Hence $\mathcal{F}$ is nonmeasurable.

The previous theorem depended on the implication:

If $\mathcal{F}$ has measure zero then $M \cap 2^\omega$ has measure zero.

This implication is not true in general for any set $X \in M$ having outer measure 1 in $M$ as is showed by the following example.

Example It is consistent with ZFC that there are models $M \subset V$ such that only some sets which have outer measure 1 in $M$ have measure 0 in $V$. Let $V = L[c]|(r_\xi : \xi < \omega_1)$ where $c$ is a Cohen real over $L$ and $(r_\xi : \xi < \omega_1)$ is a sequence of random reals over $L[c]$ (added side by side). Let $M = L[(r_\xi : \xi < \omega_1)]$. Consider the set $X = L \cap 2^\omega$. It is known that $X$ is a nonmeasurable set in $M$ but $X$ has measure 0 in $V$. On the other hand the set $\{r_\xi : \xi < \omega_1\}$ is nonmeasurable in $V$.

We conclude the paper with a canonical example of a filter which does not generate an ultrafilter. In other words we have the following:

Theorem 2.5  Let $M$ be a model for ZFC and let $r$ be a real which does not belong to $M$. Then there exists a filter $\mathcal{F}$ such that $M \models \mathcal{F}$ is an ultrafilter but

$$M[r] \models \{X \subseteq \omega : \exists Y \in \mathcal{F} \ Y \subseteq X\} \text{ is not an ultrafilter}.$$ 

Proof Let $\{k_n : n \in \omega\}$ be a fast increasing sequence of natural numbers. Let $T$ be a tree on $2^{<\omega}$ such that:

1. For $s \in T$ we have $|s| = k_n$ iff $s \downarrow 0 \in T$ and $s \downarrow 1 \in T$,
2. let $\{s_1, \ldots, s_{2^n}\}$ be the list of $T \cap 2^{k_n}$ in lexicographical order. Then for every $w \subseteq \mathcal{P}(2^n) - \{\emptyset, 2^n\}$ there exists $m \in [k_n + 1, k_{n+1}]$ such that $s_i(m) = 0$ iff $l \in w$,
3. there is no $m \in \omega$ such that for all $s \in T \cap 2^{m+1}$ we have $s(m) = 0$ or for all $s \in T \cap 2^{m+1}$ we have $s(m) = 1$. 


Let $S \subseteq T$ be a subtree of $T$. Define
\[ A^0_S = \{ m : \forall s \in S \cap 2^{m+1} \ s(m) = 0 \} \quad \text{and} \quad A^1_S = \{ m : \forall s \in S \cap 2^{m+1} \ s(m) = 1 \}. \]

Let $J$ be the ideal generated by sets $\{ A^0_S, A^1_S : S \text{ is a perfect subtree of } T \}$.

One can easily verify that all finite subsets of $\omega$ belong to $J$.

**Lemma 2.6** $J$ is a proper ideal.

**Proof** Let $S_1, \ldots, S_m$ be perfect subtrees of $T$. Find $n$ sufficiently big so that $|S_j \cap 2^{k_n}| > m$ for $j \leq m$. Let $s_1, \ldots, s_m$ be the list of $T \cap 2^{k_n}$ in lexicographical ordering. Let $w_1, \ldots, w_m$ be such that $S_j \cap 2^{k_n} = \{ s_i : i \in w_j \}$ for $j \leq m$. Let $w = \{ \min(w_1), \ldots, \min(w_m) \}$. Then for all $j$, $w_j \nsubseteq w$ and $w_j \cap w \neq \emptyset$. By the definition of $T$ there is $k < k_n$ such that $w = \{ l : s_l(k) = 0 \}$. By the property of $w$ for every $j \leq m$ there exist $s^0, s^1 \in S_j \cap 2^{k_n}$ such that $s^0(k) = 0$ and $s^1(k) = 1$. Therefore $k \notin A^0_{S_j} \cup A^1_{S_j} \cup \cdots \cup A^0_{S_m} \cup A^1_{S_m}$.

Let $F$ be any ultrafilter in $M$ extending the filter $\{ \omega - X : X \in J \}$. Let $r$ be a real which does not belong to $M$. Without loss of generality we can assume that $r$ is a branch through $T$.

Assume that $F$ generates an ultrafilter and let $X_r = \{ n : r(n) = 1 \}$. We can assume that there exists an element $X \in F$ such that $X \subseteq X_r$. Let $S = \{ s \in T : \forall k \in X \ (|s| > k \rightarrow s(k) = 1) \}$. Clearly $r$ is a branch through $S$. But in that case $S$ contains a perfect subtree $S_1 \subseteq S$ (since it contains a new branch). Therefore $X \subseteq A^1_{S_1} \in J$. Contradiction.

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