BOUNDARY VALUE PROBLEMS FOR SEMILINEAR SCHRÖDINGER EQUATIONS WITH SINGULAR POTENTIALS AND MEASURE DATA

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Abstract. We study boundary value problems with measure data in smooth bounded domains \( \Omega \), for semilinear equations. Specifically we consider problems of the form \(-L_V u + f(u) = \tau\) in \( \Omega \) and \( \text{tr}_V u = \nu \) on \( \partial \Omega \), where \( L_V = \Delta + V \), \( f \in C(\mathbb{R}) \) is monotone increasing with \( f(0) = 0 \) and \( \text{tr}_V u \) denotes the measure boundary trace of \( u \) associated with \( L_V \). The potential \( V \) is typically a Hölder continuous function in \( \Omega \) that blows up at a set \( F \subset \partial \Omega \) as \( \text{dist}(x, F) \to 0 \). In general the above boundary value problem may not have a solution. We are interested in questions related to the concept of ‘reduced measures’, introduced in [4] for \( V = 0 \). Our results extend results of [4] and [6] and apply to a larger class of nonlinear terms \( f \). In the case of signed measures, some of the present results are new even for \( V = 0 \).

Keywords: reduced measures, boundary trace, harmonic measures, Kato’s inequality.

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1. Introduction

Let $\Omega$ be a $C^2$ bounded domain in $\mathbb{R}^N$, $N \geq 3$, and let

$$L_V := \Delta + V$$

where $V \in C^\theta(\Omega)$, for some $\theta \in (0,1]$, satisfies the following conditions:

(A1) \quad \exists \bar{a} > 0 : \quad |V(x)| \leq \bar{a} \delta(x)^{-2} \quad \forall x \in \Omega,

$$\delta(x) := \text{dist}(x, \partial \Omega),$$

(A2) \quad \int_{\Omega} |

\nabla \phi |^2 dx \geq \int_{\Omega} \phi^2 V dx \quad \forall \phi \in H^1_0(\Omega).

These conditions imply the existence of a (minimal) Green function $G_V$ and of the Martin kernel $K_V$ for the operator $-L_V$. Related to this, the operator has a ground state that we denote by $\Phi_V$. In the present case $\Phi_V$ is a positive eigenfunction of $-L_V$ with eigenvalue $\lambda_V > 0$.

The function $\Phi_V$ and the Martin kernel $K_V$ are normalized at a reference point $x_0 \in \Omega$:

$$\Phi_V(x_0) = 1, \quad K_V(x_0, y) = 1 \quad \forall y \in \partial \Omega.$$ 

Notation. Denote

$$\mathbb{K}_V[\nu](x) := \int_{\partial \Omega} K_V(x, y) d\nu(y) \quad \forall \nu \in \mathcal{M}(\partial \Omega),$$

$$\mathbb{G}_V[\tau](x) := \int_{\Omega} G_V(x, y) d\tau(y) \quad \forall \tau \in \mathcal{M}(\Omega; \Phi_V).$$

Here $\mathcal{M}(\partial \Omega)$ denotes the space of finite Borel measures on $\partial \Omega$ and $\mathcal{M}(\Omega; \Phi_V)$ denotes the space of real Borel measures $\tau$ in $\Omega$ such that $\int_{\Omega} \Phi_V d|\tau| < \infty$. As usual $\mathcal{M}_+(\partial \Omega)$ and $\mathcal{M}_+(\Omega; \Phi_V)$ denote the positive cones of these spaces.

A function $u \in L^1_{\text{loc}}(\Omega)$ is $L_V$ harmonic (resp. subharmonic, superharmonic) in $\Omega$ if $-L_V u = 0$ (resp. $\leq$, $\geq$) in $\Omega$ in the distribution sense.

By the Martin representation theorem, for every positive $L_V$ harmonic function $u$ in $\Omega$ there exists $\nu \in \mathcal{M}_+(\partial \Omega)$ such that $u = \mathbb{K}_V[\nu]$.

By the Riesz decomposition lemma, a positive $L_V$ superharmonic $u$ can be represented in the form $u = p + h$ where $h$ is the largest
$L_V$ harmonic function dominated by $u$ and $p$ is an $L_V$ potential, i.e. a positive $L_V$ superharmonic function which does not dominate any positive $L_V$ harmonic.

A function $u$ is an $L_V$ potential if and only if there exists a positive measure $\tau \in \mathcal{M}(\Omega; \Phi_V)$ such that $u = \mathcal{G}_V[\tau]$. If $\tau$ is a positive Radon measure then either $\mathcal{G}_V[\tau]$ is finite everywhere in $\Omega$ or $\mathcal{G}_V[\tau] \equiv \infty$. Moreover, $\mathcal{G}_V[\tau] < \infty$ if and only if $\tau \in \mathcal{M}(\Omega; \Phi_V)$.

For these and other basic potential theory results we refer the reader to [1]. A brief survey can be found in [15].

In this paper we study boundary value problems of the form

\[ -L_V u + f(u) = \tau \quad \text{in } \Omega, \quad \text{tr}_V u = \nu \quad \text{on } \partial \Omega, \tag{1.1} \]

always assuming that

\[ f \in C(\mathbb{R}), \quad f \text{ is non-decreasing, } \quad f(0) = 0 \tag{1.2} \]

and

\[ \nu \in \mathcal{M}(\partial \Omega), \quad \tau \in \mathcal{M}(\Omega; \Phi_V). \]

Finally $\text{tr}_V u$, the $L_V$ boundary trace of $u$, is defined as follows.

**Definition 1.1.** A non-negative Borel function $u$ defined in $\Omega$ has an $L_V$ boundary trace $\nu \in \mathcal{M}(\partial \Omega)$ if

\[ \lim_{n \to \infty} \int_{\partial D_n} hu \, d\omega_{x_0,D_n}^{x_0,D_n} = \int_{\partial \Omega} h\nu \quad \forall h \in C(\Omega), \tag{1.3} \]

for every uniformly Lipschitz exhaustion $\{D_n\}$ of $\Omega$ such that $x_0 \in D_n$ for all $n$. Here $x_0$ is the reference point previously mentioned and $\omega_{x_0,D_n}^{x_0,D_n}$ denotes the harmonic measure for $L_V$ in $D_n$ relative to $x_0$. The $L_V$ boundary trace of $u$ is denoted by $\text{tr}_V u$.

A real Borel function $u$ defined in $\Omega$ has an $L_V$ boundary trace if

\[ \sup_n \int_{\partial D_n} |u| \, d\omega_{x_0,D_n}^{x_0,D_n} < +\infty \tag{1.4} \]

and (1.3) holds.

When $V = 0$ this definition reduces to the classical definition of measure boundary trace. Recall that

\[ d\omega_n^{x_0} = P_{V,n}(x_0, \cdot) dS \quad \text{on } \partial D_n, \]

where $P_{V,n}$ is the Poisson kernel of $-L_V$ in $D_n$. 

By [17, Lemma 2.3], if (A1), (A2) hold, the $L_V$ trace has the following properties:

$$\begin{align*}
    (i) & \quad \text{tr}_V(K_V[\nu]) = \nu \quad \forall \nu \in M(\partial \Omega), \\
    (ii) & \quad \text{tr}_V(G_V[\tau]) = 0 \quad \forall \tau \in M(\Omega; \Phi_V).
\end{align*}$$

**Notation.** (i) Let $(\lambda, \sigma)$ and $(\tau, \nu)$ be two couples of measures in $M(\Omega; \Phi_V) \times M(\partial \Omega)$. Then $(\lambda, \sigma) \prec (\tau, \nu)$ means $\lambda \leq \tau$ and $\sigma \leq \nu$.

(ii) For $\beta > 0$, denote

$$D_\beta := \{x \in \Omega : \delta(x) > \beta\}, \quad \Omega_\beta := \{x \in \Omega : \delta(x) < \beta\},$$

$$\Sigma_\beta := \{x \in \Omega : \delta(x) = \beta\}.$$  

Since $\Omega$ is a $C^2$ bounded domain, there exists $\beta_0 > 0$ such that for every $x \in \Omega_{\beta_0}$ there is a unique point $\sigma(x) \in \partial \Omega$ such that $|x - \sigma(x)| = \delta(x)$, and $x \mapsto \delta(x)$ is in $C^2(\Omega_{\beta_0})$ while $x \mapsto \sigma(x)$ is in $C^1(\Omega_{\beta_0})$.

In addition to (A1) and (A2), we assume that the ground state $\Phi_V$ satisfies the following condition:

There exist $a_0 \geq 1$ and $\alpha, \alpha^* > 0$ satisfying

$$0 \leq \alpha - \alpha^* < \frac{1}{2},$$

such that for every $a > a_0$ and every $x, z \in \Omega_{\beta_0}$ lying on a normal to $\partial \Omega$:

$$(C1) \quad a\delta(x) \leq \delta(z) \implies \frac{\Phi_V(x)}{\Phi_V(z)} \leq c(a)\frac{\delta(x)^{\alpha^*}}{\delta(z)^{\alpha}}.$$  

Conditions (A1), (A2) and (C1) are assumed, without further mention, throughout the paper.

**Definition 1.2.** Let $(\tau, \nu) \in M(\Omega; \Phi_V) \times M(\partial \Omega)$ and $u \in L^1_{\text{loc}}(\Omega)$.

(i) $u$ is a solution of (1.1) if $f(u) \in L^1(\Omega; \Phi_V)$, the equation holds in the distribution sense and $\text{tr}_V u = \nu$.

(ii) $u$ is a subsolution of (1.1) if $f(u) \in L^1(\Omega; \Phi_V)$, $-L_V u + f(u) \leq \tau$ in the distribution sense and $\text{tr}_V u \leq \nu$.

A supersolution is defined in the same way with the inverse inequalities.

With this definition, $u$ is a solution of (1.1) if and only if (see [17, Lemma 3.1])

$$u + G_V[f(u)] = G_V[\tau] + K_V[\nu] \quad \text{in } \Omega.$$  

If (1.1) has a solution we say that $(\tau, \nu)$ is a good couple.
If \((0, \nu)\) (respectively \((\tau, 0)\)) is a good couple we say that \(\nu\) (respectively \(\tau\)) is a good measure.

We are interested in questions related to the notion of ‘reduced measure’ introduced in [4] (for \(L_V = \Delta\)). In general, problem (1.1) is not solvable for every couple \((\tau, \nu)\).

A great deal of research has been devoted to a precise characterization of good measures or good couples in some specific cases. Most of this research dealt with the equation \(-\Delta u + f(u) = 0\) in \(\Omega\) and in particular with the case \(f(t) = |t|^p \text{sign} t, \ p > 1\) (see [9, 12, 13, 24] for \(1 < p \leq 2\) and [14, 22, 23] for every \(p > 1\) and the references therein). See also [2] where the problem was treated for a general class of nonlinearities \(f\) that satisfy the Keller - Osserman condition.

More recently the characterization of good measures was studied with respect to the equation \(-L_Vu + f(u) = 0\) in \(\Omega\), mainly when \(V\) is the Hardy potential and \(f(t) = |t|^p \text{sign} t\) (see, e.g. [7, 10, 18–20]). The question was also studied in the context of fractional Schrödinger equations (see, e.g. [11]).

The idea of ‘reduced measure’, introduced in [4], is to provide a reduction process that converges to the ‘good’ part of \(\tau\) and \(\nu\) when (1.1) has no solution.\

In [4], the authors study problem (1.1) for \(L_V = \Delta\), mainly in the case where \(\nu = 0, \tau \in \mathcal{M}(\Omega)\), assuming that \(f\) satisfies (1.2) and vanishes on \((-\infty, 0]\). A solution is a function \(u \in L^1(\Omega)\) such that \(f(u) \in L^1(\Omega)\) and \(u\) satisfies (1.1) in the weak sense. To determine the reduced measure the authors consider a sequence of problems

\[
(1.8) \quad -\Delta u + f_n(u) = \tau \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega,
\]

where \(f_n \in C(\mathbb{R})\) is a non-negative, nondecreasing function, \(f_n \uparrow f\) and (1.8) has a solution for every \(\tau \in \mathcal{M}(\Omega)\). (For instance, the functions \(f_n\) are bounded.) One of the main results states (see [4, Theorem 4.1]):

Let \(u_n\) be the unique solution of (1.8). Then the sequence \(\{u_n\}\) decreases and \(u^* := \lim u_n\) satisfies

\[
-\Delta u^* + f(u^*) = \tau^* \quad \text{in} \ \Omega, \quad u^* = 0 \quad \text{on} \ \partial \Omega,
\]

where \(\tau^*\) is the largest good measure dominated by \(\tau\).

In [6] a similar result is established in the case \((\tau, \nu) \in \mathcal{M}(\Omega) \times \mathcal{M}(\partial \Omega)\) (possibly signed measures) assuming as before that \(f = 0\) on \((-\infty, 0]\). It is also shown that \((\tau, \nu)\) is good if and only if, both \(\tau\) and \(\nu\) are good measures.

\(^1\)A related notion of ‘reduced limit’ was studied in [21, 3].
When \( \tau, \nu \) are signed measures and we drop the assumption ‘\( f = 0 \) on \((-\infty, 0)\)’, the situation is more complex even in the case \( V = 0 \).

The present definition of a solution of (1.1) is necessarily different from that of weak solution used in [4], [6]. But when \( V = 0 \) these are essentially equivalent.

In the case where \( \tau \) and \( \nu \) are positive our results are similar to those quoted above.

Let \( u_n \) denote the solution of the problem,

\[
(1.9) \quad -L_V u + f_n(u) = \tau \quad \text{in } \Omega, \quad \text{tr}_V u = \nu \quad \text{on } \partial \Omega
\]

where \( f_n \in C(\mathbb{R}) \) is non-decreasing, bounded and \( (f_n)_{\pm} \uparrow f_{\pm} \).

**Theorem 1.3.** Assume that \( f \) satisfies (1.2) and \( (\tau, \nu) \in \mathcal{M}_+(\Omega; \Phi_V) \times \mathcal{M}_+(\partial \Omega) \). For \( u_n \) as above: \( u_n \downarrow u^\# \) and

\[
(1.10) \quad -L_V u^\# + f(u^\#) = \tau^\# \quad \text{in } \Omega, \quad \text{tr}_V u^\# = \nu^\#
\]

where \( (0, 0) \prec (\tau^\#, \nu^\#) \prec (\tau, \nu) \). Moreover, \( u^\# \) is the largest subsolution of (1.1) and \( \tau^\#, \nu^\# \) are the largest good measures dominated by \( \tau \) and \( \nu \) respectively.

A corresponding result holds for couples of negative measures (see Remark 3.6).

In the case that \( \tau, \nu \) may be signed measures we prove:

**Theorem 1.4.** Assume that \( f \) satisfies (1.2) and \( (\tau, \nu) \in \mathcal{M}(\Omega; \Phi_V) \times \mathcal{M}(\partial \Omega) \). Let \( (\lambda, \sigma) \) be a couple of measures such that

\[
(1.11) \quad -(\tau_- \vee \nu_-) \prec (\lambda, \sigma) \prec (\tau_+ \vee \nu_+)
\]

and let \( u_n \) be the solution of (1.9) with \( (\tau, \nu) \) replaced by \( (\lambda, \sigma) \).

Every subsequence of \( \{u_n\} \) has a limit point with respect to a.e. convergence. If \( \tilde{u} \) is such a limit point then

\[
(1.12) \quad -L_V \tilde{u} + f(\tilde{u}) = \tilde{\lambda} \quad \text{in } \Omega, \quad \text{tr}_V \tilde{u} = \tilde{\sigma}
\]

and

\[
(1.13) \quad -(\tau_-, \nu_-)^\# \prec (\tilde{\lambda}, \tilde{\sigma}) \prec (\tau_+, \nu_+)^\#.
\]

Moreover, every couple \( (\lambda, \sigma) \) such that

\[
(1.14) \quad -(\tau_-, \nu_-)^\# \prec (\lambda, \sigma) \prec (\tau_+, \nu_+)^\#
\]

is a good couple.

This naturally leads to the following question: If \( (\lambda, \sigma) \) is a good couple in the interval (1.11) does it necessarily satisfy (1.14)?
As shown below, if \( f \) vanishes in \((-\infty, 0]\), the answer is positive. In the general case this is an open question.

**Theorem 1.5.** Assume that \( f \) satisfies (1.2) and that \( f(t) = 0 \) for \( t \leq 0 \). Let \((\tau, \nu) \in M(\Omega; \Phi_V) \times M(\partial\Omega)\). Then \((\tau, \nu)\) is a good couple if and only if
\[
(\tau, \nu) \prec (\tau^\#, \nu^\#)
\]
where \( \tau^\# \) and \( \nu^\# \) are the largest good measures dominated by \( \tau_+ \) and \( \nu_+ \) respectively.

Consequently, \((\tau, \nu)\) is a good couple with respect to (1.1) if and only if \( \tau \) and \( \nu \) separately are good measures.

This result extends [6, Theorem 6].

Our main tools include: two-sided estimates of \( G_V[\tau], \tau \in M_+(\Omega; \Phi_V) \), and \( K_V[\nu], \nu \in M_+(\partial\Omega) \) [16], the inverse maximum principle [8], Kato’s inequality and its extension due to [5] and a result of [4] regarding the diffuse part of reduced measures.

Here is the plan of the paper. In Section 2 we recall the estimates of [16] as well as properties of sub and supersolutions established in [17] that are frequently used in the present paper. In Section 3 we study the problem of reduced couple for (1.1) with positive measures. Theorem 1.3 is a consequence of Theorems 3.1 and 3.5. Section 4 is devoted to problem (1.1) with signed measures. Theorem 1.4 is a consequence of Theorem 4.1 and Proposition 4.2. The section is completed by the proof of Theorem 1.5.

2. Some results on sub and supersolutions.

In this section we gather several results from [16] and [17] that are frequently used in the sequel.

2.1. Estimates of \( L_V \) harmonic functions and \( L_V \) potentials.

The estimates stated below are derived in [16].

**Theorem 2.1.** ([16, Theorem 3.1]) Assume that (A1), (A2) and (C1) hold. Then for any \( \nu \in M_+(\partial\Omega) \),
\[
\frac{1}{C} \|\nu\|_{M(\partial\Omega)} \leq \int_{\Sigma_\beta} \frac{\Phi_V}{\delta} K_V[\nu] dS \leq C \|\nu\|_{M(\partial\Omega)} \quad \forall \beta \in (0, \beta_0),
\]
where the constant \( C \) depends on \( \bar{a}, \Omega \) and the constants in (C1).

\(^2\)Note that, when \( f = 0 \) on \((-\infty, 0]\), the couple \((-\tau, -\nu)\) is always good.
Next is an estimate of $L_V$ potentials.

**Theorem 2.2.** ([16, Theorems 3.3 and 3.4])

(i) Assume $(A1)$ and $(A2)$ hold. Then there exists a constant $c$ depending on $\bar{a}$, $\Omega$ such that, for every $\tau \in \mathcal{M}_+(\Omega; \Phi_V)$,

$$\frac{1}{c} \int_{\Omega} \Phi_V \, d\tau \leq \int_{\Omega} \frac{\Phi_V}{\delta} G_V[\tau] \, dx.$$

(ii) Assume $(A1)$, $(A2)$ and $(C1)$ hold. Then there exists $c' > 0$ depending on $\bar{a}$, $\Omega$ and the constants in $(C1)$ such that for every $\tau \in \mathcal{M}_+(\Omega; \Phi_V)$,

$$\int_{\Omega} \frac{\Phi_V}{\delta} G_V[\tau] \, dx \leq c' \int_{\Omega} \Phi_V \, d\tau.$$

2.2. **Remarks on subsolutions and supersolutions.** We list some properties of subsolutions and supersolutions from [17].

**Lemma 2.3.** Let $w \in L^1_{\text{loc}}(\Omega)$ be a non-negative $L_V$ subharmonic function. If $\text{tr}_V w = 0$ then $w \equiv 0$.

This is a consequence of [17, Corollary 2.6].

**Lemma 2.4.** ([17, Corollary 2.8]) Let $u \in L^1_{\text{loc}}(\Omega)$ and suppose that $-L_V u = \tau \in \mathcal{M}(\Omega; \Phi_V)$. In addition assume that for some smooth exhaustion $\{D_n\}$ of $\Omega$,

$$(2.1) \quad \sup_n \int_{\partial D_n} P_{V,n}(x_0, y) |u(y)| \, dS(y) < +\infty.$$

Then $\text{tr}_V u =:\nu$ exists and $u = G_V[\tau] + K_V[\nu]$ in $\Omega$.

**Remark.** If $\text{tr}_V u$ exists then, by definition, (2.1) holds.

If $u \geq 0$, condition (2.1) is not needed. In this case the result (stated below) is a consequence of the Riesz decomposition lemma.

**Lemma 2.5.** ([17, Lemma 2.11]) Let $u \in L^1_{\text{loc}}(\Omega)$ be a positive function such that $-L_V u = \tau \in \mathcal{M}(\Omega; \Phi_V)$. Then $u$ has an $L_V$ boundary trace, say $\nu$, and $u = G_V[\tau] + K_V[\nu]$ in $\Omega$.

**Notation.** A function $u$ is $L_V$ perfect if $u = G_V[\tau] + K_V[\nu]$ for some $\tau \in \mathcal{M}(\Omega; \Phi_V)$ and $\nu \in \mathcal{M}(\partial \Omega)$.

**Lemma 2.6.** ([17, Lemma 3.3]) Suppose that $u$ is an $L_V$ perfect function. If $\text{tr}_V u \leq 0$ then $\text{tr}_V u_+ = 0$.

Consider the equation

$$(2.2) \quad -L_V u + f(u) = \tau \quad \text{in } \Omega.$$
and the boundary value problem
(2.3) \[-L_V u + f(u) = \tau \text{ in } \Omega, \quad \text{tr}_V u = \nu,\]
where \(f\) satisfies (1.2), \(\tau \in \mathcal{M}(\Omega; \Phi_V)\) and \(\nu \in \mathcal{M}(\partial \Omega)\).

**Definition 2.7.** Let \(u \in L^1_{\text{loc}}(\Omega)\) be a function such that \(f(u) \in L^1_{\text{loc}}(\Omega)\).

The function \(u\) is a subsolution (supersolution) of (2.2) if
\[-L_V u + f(u) \leq (\geq) \tau \text{ in } \Omega \text{ in the distribution sense.}\]

The function \(u\) is a subsolution (supersolution) of (2.3) if it is a subsolution (supersolution) of (2.2), \(f(u) \in L^1(\Omega; \Phi_V)\) and \(u\) has an \(L_V\) boundary trace such that \(\text{tr}_V u \leq \nu\) (\(\text{tr}_V u \geq \nu\)).

**Lemma 2.8.** ([17, Lemma 3.1]) Assume that \(u \in L^1_{\text{loc}}(\Omega)\) and \(f(u) \in L^1(\Omega; \Phi_V)\).

If \(u\) is a subsolution (resp. supersolution) of problem (2.3) then \(u\) is \(L_V\) perfect. More precisely, there exist measures \(\lambda \in \mathcal{M}_+(\Omega; \Phi_V)\) and \(\sigma \in \mathcal{M}_+(\partial \Omega)\) such that:
(2.4) \[u = G_V[\tau - f(u) - \lambda] + K_V[\nu - \sigma] \]
\[u = G_V[\tau - f(u) + \lambda] + K_V[\nu + \sigma].\]

**Lemma 2.9.** ([17, Lemma 3.4]) (i) Let \(u_1\) (resp. \(u_2\)) be a supersolution (resp. subsolution) of (2.3). Then \(u_2 \leq u_1\).

(ii) Problem (2.3) has at most one solution.

**Lemma 2.10.** ([17, Corollary 3.7]) Suppose that \(u_1, u_2\) are respectively a supersolution and a subsolution of (2.2) such that \(f(u_i) \in L^1(\Omega; \Phi_V)\). If \(u_i\) has an \(L_V\) boundary trace, say \(\nu_i, i = 1, 2\), and \(\nu_2 \leq \nu_1\) then problem (2.3) has a (unique) solution for every measure \(\nu\) such that \(\nu_2 \leq \nu \leq \nu_1\).

3. The reduced measures for couples of positive measures.

**Notation.** If \(f\) and \(g\) are two non-negative functions on a set \(X\), we say that \(f\) and \(g\) are similar if there exists \(c > 0\) such that
\[\frac{1}{c}f(x) \leq g(x) \leq cf(x) \quad \forall x \in X.\]

This relation is denoted by \(f \sim g\).

**Theorem 3.1.** Let \((\tau, \nu) \in \mathcal{M}_+(\Omega; \Phi_V) \times \mathcal{M}_+(\partial \Omega)\). Let \(\{f_n\}\) be a sequence of continuous, bounded, non-decreasing functions on \(\mathbb{R}\) such that \(f_n(0) = 0\) and \((f_n)_\pm \uparrow f_\pm\).
Then the boundary value problem

\begin{equation}
- L_V u + f_n(u) = \tau \quad \text{in } \Omega, \quad \text{tr}_V u = \nu,
\end{equation}

has a unique solution \( u_n = u_n(\tau, \nu) \). The sequence \( \{u_n\} \) is a decreasing sequence of positive functions and its limit \( u^# = u^#(\tau, \nu) \) has the following properties.

(a) \( u^# \) satisfies

\begin{equation}
\|u^#\|_{L^1(\Omega; \Phi_V/\delta)} + \|f(u^#)\|_{L^1(\Omega; \Phi_V)} \leq C (\|\nu\|_{\mathcal{M}(\partial \Omega)} + \|\tau\|_{\mathcal{M}(\Omega; \Phi_V)})
\end{equation}

and

\begin{equation}
\|u^# + G_V[f(u^#)]\| \leq G_V[\tau] + K_V[\nu] =: \tilde{w} \quad \text{in } \Omega.
\end{equation}

(b) There exists a non-negative measure \( \tau^# \leq \tau \) such that

\begin{equation}
- L_V u^# + f(u^#) = \tau^# \quad \text{in } \Omega.
\end{equation}

(c) \( u^# \) has \( L_V \) boundary trace \( 0 \leq \nu^# \leq \nu \). Thus

\begin{equation}
\|u^# + G_V[f(u^#)]\| = G_V[\tau^#] + K_V[\nu^#] \quad \text{in } \Omega.
\end{equation}

(d) \( u^# \) is the largest subsolution of problem

\begin{equation}
- L_V u + f(u) = \tau \quad \text{in } \Omega, \quad \text{tr}_V u = \nu.
\end{equation}

In particular, if \( \text{(3.6)} \) has a solution \( u \) then \( u^# = u \), \( \tau^# = \tau \) and \( \nu^# = \nu \).

Proof. The function \( \tilde{w} := K_V[\nu] + G_V[\tau] \) is a supersolution of the equation

\begin{equation}
- L_V u + f_n(u) = \tau \quad \text{in } \Omega
\end{equation}

and \( \text{tr}_V \tilde{w} = \nu \). Obviously \( v \equiv 0 \) is a subsolution, \( v \leq \tilde{w} \) and \( f_n(\tilde{w}) \in L^1(\Omega; \Phi_V) \).

Therefore, by Lemma 2.10 there exists a unique solution \( u_n = u_n(\tau, \nu) \) of the boundary value problem \( \text{(3.1)} \) satisfying \( 0 \leq u_n \leq \tilde{w} \) in \( \Omega \). The solution \( u_n \) satisfies

\begin{equation}
u_n + G_V[f_n(u_n)] = G_V[\tau] + K_V[\nu] = \tilde{w} \quad \text{in } \Omega.
\end{equation}

Put \( w := u_{n+1} - u_n \). Then \( w \) is \( L_V \) perfect and \( \text{tr}_V w = 0 \). Consequently, by Lemma 2.6 \( \text{tr}_V w^+ = 0 \). Furthermore

\begin{equation}
- L_V w + f_{n+1}(u_{n+1}) - f_n(u_n) = 0 \quad \text{in } \Omega.
\end{equation}

By Kato’s inequality

\begin{equation}
- L_V w^+ + (f_{n+1}(u_{n+1}) - f_n(u_n)) \text{sign } w \leq 0 \quad \text{in } \Omega.
\end{equation}

In the set \( \{x \in \Omega : w^+ \geq 0\} \):

\begin{equation}
f_{n+1}(u_{n+1}) - f_n(u_n) \geq f_{n+1}(u_n) - f_n(u_n) \geq 0.
\end{equation}
Thus $w_+$ is $L_V$ subharmonic in $\Omega$. As $\text{tr}_V w_+ = 0$, Lemma 2.3 yields $w_+ \equiv 0$, i.e., $u_{n+1} \leq u_n$ in $\Omega$.

(a) Let $u^\# = \lim u_n$, then $0 \leq u^\# \leq \tilde{w}$ in $\Omega$. By Dini’s Lemma, $f_n(u_n) \to f(u^\#)$ a.e. in $\Omega$. Therefore, by (3.7) and Fatou’s lemma, we obtain (3.3).

By Theorems 2.1 and 2.2, we have

$$\int_\Omega \frac{\Phi_V}{\delta} \tilde{w} \, dx = \int_\Omega \frac{\Phi_V}{\delta} (K_V[\nu] + G_V[\tau]) \, dx \sim \|\tau\|_{\gamma(\Omega; \Phi_V)} + \|\nu\|_{\gamma(\partial \Omega)}$$

and

$$\int_\Omega \frac{\Phi_V}{\delta} G_V[f_n(u_n)] \, dx \sim \int_\Omega f_n(u_n) \Phi_V \, dx.$$

Therefore, multiplying (3.7) by $\Phi_V/\delta$ and integrating over $\Omega$, we obtain the following similarity relations

$$\int_\Omega u_n \frac{\Phi_V}{\delta} \, dx + \int_\Omega f_n(u_n) \Phi_V \, dx$$

$$\sim \int_\Omega u_n \frac{\Phi_V}{\delta} \, dx + \int_\Omega \frac{\Phi_V}{\delta} G_V[f_n(u_n)] \, dx$$

$$= \int_\Omega \tilde{w} \frac{\Phi_V}{\delta} \, dx \sim \|\tau\|_{\gamma(\Omega; \Phi_V)} + \|\nu\|_{\gamma(\partial \Omega)}.$$ 

(b) Let $\zeta \in C_c^\infty(\Omega)$. Multiplying (3.7) by $-L_V \zeta$ and integrating over $\Omega$, we obtain

$$- \int_\Omega u_n L_V \zeta \, dx - \int_\Omega G_V[f_n(u_n)] L_V \zeta \, dx = - \int_\Omega G_V[\tau] L_V \zeta \, dx.$$

We used the fact that $K_V[\nu]$ is $L_V$ harmonic. Further, for every $\tau \in \mathcal{M}(\Omega; \Phi_V)$, $-L_V G_V[\tau] = \tau$, i.e.,

$$- \int_\Omega G_V[\tau] L_V \zeta \, dx = \int_\Omega \zeta \, d\tau.$$

Hence

$$- \int_\Omega u_n L_V \zeta \, dx + \int_\Omega f_n(u_n) \zeta \, dx = \int_\Omega \zeta \, d\tau.$$

Recall that $\{u_n\}$ converges to $u^\#$ and is dominated by $u_1$ in $L^1(\Omega; \Phi_V/\delta)$ and $\{f_n(u_n)\}$ converges to $f(u^\#)$ and – by (3.8) – is bounded in $L^1(\Omega; \Phi_V)$. Consequently, using Fatou’s lemma,

$$-C \int_\Omega \zeta \, dx \leq - \int_\Omega u^\# L_V \zeta \, dx + \int_\Omega f(u^\#) \zeta \, dx \leq \int_\Omega \zeta \, d\tau$$

for every $0 \leq \zeta \in C_c^\infty(\Omega)$ where $C \geq 0$ a constant independent of $\zeta$. Thus there exists a bounded measure $\tau^\# \leq \tau$ such that (3.4) holds.
Now, in Lemma 3.4 below, it is shown that $\tau^# \geq \tau_d$. This is based only on the assumptions of the present theorem, the definition of $u^#$ as the limit of the decreasing sequence $\{u_n\}$ and what is proved above. Therefore, as $\tau \geq 0$, we conclude that $\tau^# \geq 0$. This completes the proof of part (b).

(c) By (3.3), $u^# \leq \tilde{w}$. Since $\text{tr}_V \tilde{w}$ exists and $u^# \geq 0$, it follows that
\[
\sup_n \int_{\partial D_n} P_{V,n}(x_0, y)|u^#(y)|dS(y) \leq \sup_n \int_{\partial D_n} P_{V,n}(x_0, y)\tilde{w}(y)dS(y) < +\infty.
\]
By (3.4), as $f(u^#) \in L^1(\Omega; \Phi_V)$, the function $v := u^# + G_V[f(u^#)]$ satisfies $-L_V v = \tau$ and $v \geq 0$. Therefore, by Lemma 2.4 and (1.5), $\text{tr}_V v =: \nu^#$ exists, $\text{tr}_V u^# = \text{tr}_V v$ and (3.5) holds. As $0 \leq u^# \leq \tilde{w}$,
\[
0 \leq \nu^# \leq \text{tr}_V \tilde{w} = \nu.
\]

(d) Let $w$ be a positive subsolution of (3.6). Then
\[
-L_V w + f_n(w) \leq -L_V w + f(w) \leq \tau \quad \text{in } \Omega, \quad \text{tr}_V w \leq \nu.
\]
On the other hand, we have
\[
-L_V u_n + f_n(u_n) = \tau \quad \text{in } \Omega, \quad \text{tr}_V u_n = \nu.
\]
By Lemma 2.9, $w \leq u_n$ and thus $w \leq u^#$. This proves (d).

Obviously, if (3.6) has a solution $u$ then it is the largest subsolution of the problem so that $u^# = u$.

**Definition 3.2.** A measure $\tau \in \mathcal{M}_+(\Omega; \Phi_V)$ is a **good measure** with respect to $f$ if there exists a solution $u$ of equation (2.2) such that $f(u) \in L^1(\Omega; \Phi_V)$.

A couple of measures $(\tau, \nu) \in \mathcal{M}_+(\Omega; \Phi_V) \times \mathcal{M}_+(\partial \Omega)$ is a **good couple** with respect to $f$ if there exists a solution $u$ of problem (3.6).

The couple $(\tau^#, \nu^#)$ that satisfies (3.5) is called the **reduced couple** of $(\tau, \nu)$.

**Remark 3.3.** We note that as a consequence of Theorem 3.1 parts (b) and (c):

(i) For every $\nu \in \mathcal{M}_+(\partial \Omega)$ the reduced couple of $(0, \nu)$ is $(0, \nu^*)$ where $\nu^*$ is the largest good measure dominated by $\nu$.

(ii) For every $\tau \in \mathcal{M}_+(\Omega; \Phi_V)$ the reduced couple of $(\tau, 0)$ is $(\tau^*, 0)$ where $\tau^*$ is the largest good measure dominated by $\tau$. 

□
Notation. (a) Let $\tau \in \mathcal{M}_+(\Omega; \Phi_V)$. Denote by $\tau^\#(\nu)$ the measure $\tau^\#$ in Theorem 3.1. In particular $\tau^\#(0)$ is the reduced measure of problem

$$L_V u + f(u) = \tau \quad \text{in} \; \Omega, \quad \text{tr}_V u = 0.$$  

(b) Let $\lambda$ be a Borel measure in $\Omega$ such that $\lambda = \lambda_+ - \lambda_-$ where $\lambda_+$ are positive Radon measures. It is well-known (see e.g. [4]) that $\lambda$ has a unique representation of the form $\lambda = \lambda_c + \lambda_d$ where $\lambda_d$ vanishes on sets of (Newtonian) capacity zero while $\lambda_c$ is concentrated on a set of zero capacity. We say that $\lambda_d$ is the diffuse part of $\lambda$ while $\lambda_c$ is the concentrated part of $\lambda$. If $\lambda = \lambda_d$ we say that $\lambda$ is a diffuse measure. If $\lambda = \lambda_c$ we say that $\lambda$ is a concentrated measure.

For the proof of the next theorem we need a version of [4, Lemma 4.1] suitable for the present problem. The proof is essentially the same as in [4], but some slight modifications are needed. For the convenience of the reader we provide the proof below.

**Lemma 3.4.** Let $(\tau, \nu) \in \mathcal{M}_+(\Omega; \Phi_V) \times \mathcal{M}_+^c(\partial \Omega)$. Then under the assumptions and with the notation of Theorem 3.1, we have

$$\tau^\# \geq \tau_d \quad \text{and} \quad (\tau^\#)_d = \tau_d. \quad (3.9)$$

**Proof.** Let $u_n$ be the solution of $(3.1)$. Then $u_n \geq 0$, the sequence $\{u_n\}$ is decreasing and, as in Theorem 3.1, the function $u^# := \lim n u_n$ satisfies

$$-L_V u^# + f(u^#) = \tau^# \quad \text{in} \; \Omega, \quad (3.10)$$

where $\tau^#$ is a measure such that $\tau^# \leq \tau$.

Denote $T_k(s) := \min(k, s)$, $s \in \mathbb{R}$. By [5] and [4, Corollary 4.9],

$$\Delta T_k(u_n) \leq \chi_{[u_n \leq k]}(\Delta u_n)_d + ((\Delta u_n)_c)_+,$$

where $\chi_A$ denotes the characteristic function of $A \subset \mathbb{R}^N$. Since $u_n$ satisfies (3.1), we obtain

$$\begin{align*}
(\Delta u_n)_d &= -V u_n + f_n(u_n) - \tau_d, \\
(\Delta u_n)_c &= -\tau_c.
\end{align*}$$

As $\tau \geq 0$, these relations and the previous inequality yield

$$\Delta T_k(u_n) \leq \chi_{[u_n \leq k]}(-V u_n + f_n(u_n) - \tau_d) \quad \text{in} \; \Omega,$$

and

$$\begin{align*}
-L_V T_k(u_n) &\geq -V T_k(u_n) + \chi_{[u_n \leq k]}(V u_n - f_n(u_n) + \tau_d) \\
&= -\chi_{[u_n > k]} V k - \chi_{[u_n \leq k]}(f_n(u_n) - \tau_d).
\end{align*}$$

Since $\chi_{[u_n \leq k]} f_n(u_n) \leq f_n(T_k(u_n))$, it follows that

$$\begin{align*}
-L_V T_k(u_n) + f_n(T_k(u_n)) &\geq -\chi_{[u_n > k]} V |u_n| + \chi_{[u_n \leq k]} \tau_d \quad \text{in} \; \Omega.
\end{align*} \quad (3.11)$$
Let $\zeta \in C^\infty_c(\Omega)$ and $\zeta \geq 0$. Since $0 \leq u_n \leq u_1$, (3.11) yields
\[ -\int_\Omega T_k(u_n)L_V\zeta \, dx + \int_\Omega f_n(T_k(u_n))\zeta \, dx \geq -\int_\Omega \chi_{[u_1 > k]}|V|u_1\zeta \, dx + \int_\Omega \chi_{[u_1 \leq k]}\zeta \, d\tau_d. \]
By the dominated convergence theorem, letting $n \to \infty$, we obtain
\[ -\int_\Omega T_k(u^#)L_V\zeta \, dx + \int_\Omega f(T_k(u^#))\zeta \, dx \geq -\int_\Omega \chi_{[u_1 > k]}|V|u_1\zeta \, dx + \int_\Omega \chi_{[u_1 \leq k]}\zeta \, d\tau_d. \]
Finally, letting $k \to \infty$, we obtain
\[ -\int_\Omega u^#L_V\zeta \, dx + \int_\Omega f(u^#)\zeta \, dx \geq \int_\Omega \zeta \, d\tau_d. \]
In view of (3.10) this implies $\tau^# \geq \tau_d$ and therefore $(\tau^#)_d \geq \tau_d$. As $\tau^# \leq \tau$ we obtain $(\tau^#)_d = \tau_d$. This proves (3.9).

**Theorem 3.5.** Under the assumptions and with the notation of Theorem 3.1, the following statements hold.

(i) For every $\tau$, $\nu^# = \nu^*$ with $\nu^*$ as in Remark 3.3. Thus $\nu^#$ does not depend on the data $\tau$.

(ii) For every $\nu$, $\tau^# = \tau^*$ with $\tau^*$ as in Remark 3.3. Thus $\tau^#$ does not depend on the boundary data $\nu$.

(iii) Let $(0,0) \prec (\lambda,\sigma) \prec (\tau,\nu)$. Then problem
\[ -L_Vu + f(u) = \lambda \quad \text{in } \Omega, \quad \text{tr}_V u = \sigma, \]
has a solution if and only if $(\lambda,\sigma) \prec (\tau^#,\nu^*)$.

**Proof.** (i) Let $u_n(\tau,\nu)$, $u^#(\tau,\nu)$ and $(\tau^#,\nu^#)$ be as in Theorem 3.1. For simplicity, we write $u^* = u^#(0,\nu)$ and $u^# = u^#(\tau,\nu)$. Since $0 \leq u_n(0,\nu) \leq u_n(\tau,\nu)$, it follows that $0 \leq u^* \leq u^#$ in $\Omega$. Therefore, using Theorem 3.1(c), we obtain
\[ \nu^* = \text{tr}_V u^* \leq \text{tr}_V u^# = \nu^# \leq \nu. \]
Consequently $u^*$ is a subsolution of problem
\[ -L_Vu + f(u) = 0 \quad \text{in } \Omega, \quad \text{tr}_V u = \nu^#. \]
Obviously $u^#$ is a supersolution of (3.13). Hence, by Lemma 2.10 there exists a unique solution $\bar{v}$ of problem (3.13) and $0 \leq u^* \leq \bar{v} \leq u^#$. By Theorem 3.1 $u^*$ is the largest solution of the equation $-L_Vu + f(u) = 0$ in $\Omega$ with $L_V$ boundary trace $\leq \nu$. Thus $\bar{v} \leq u^*$ and consequently $\nu^# = \text{tr}_V \bar{v} \leq \text{tr}_V u^* = \nu^*$. This inequality and (3.12) imply $\nu^# = \nu^*$. 


(ii) Here we denote by $R(\tau, \nu)$ the reduced couple of $(\tau, \nu)$. For given $\tau$ we denote by $\tau^#(\nu)$ the first component of $R(\tau, \nu)$.

In view of Theorem 3.1(d),

$$R(\tau, \nu^#) = R(\tau, \nu), \quad \tau^#(\nu) = \tau^#(\nu^#).$$

Hence, as $\nu^# = \nu^*$,

$$R(\tau, \nu) = R(\tau, \nu^*), \quad u^#(\tau, \nu) = u^#(\tau, \nu^*).$$

Therefore, in this part of the proof we may assume that $\nu = \nu^*$.

**Step I.** If $\nu_1, \nu_2 \in \mathcal{M}_+ (\partial \Omega)$ and $\nu_1 \leq \nu_2$ then

$$\tau^#(\nu_1) \geq \tau^#(\nu_2).$$

**Proof.** The assumption implies that $\nu_1^* \leq \nu_2^*$. Therefore by (3.14), we may assume that $\nu_1, \nu_2$ are good measures with respect to (3.6).

Let $\lambda$ be a measure such that $0 \leq \lambda \leq \tau$ and suppose that there exists a solution $u_2$ of

$$-Lu + f(u) = \lambda \quad \text{in } \Omega, \quad \text{tr}_V u = \nu_2.$$

Then $u_2$ is a supersolution of

$$-Lu + f(u) = \lambda \quad \text{in } \Omega, \quad \text{tr}_V u = \nu_1.$$

We also know that there exists $\lambda^# \leq \lambda$ such that the following problem has a solution $u_1$:

$$-Lu + f(u) = \lambda^# \quad \text{in } \Omega, \quad \text{tr}_V u = \nu_1.$$

The function $u_1$ is a subsolution of (3.16) and, by Lemma 2.9, $u_1 \leq u_2$. Hence there exists a solution $\bar{v}$ of (3.16). Clearly, $\bar{v}$ is a subsolution of

$$-L\bar{v} u + f(u) = \tau \quad \text{in } \Omega, \quad \text{tr}_V u = \nu_1$$

and therefore, by Theorem 3.1(d), $\lambda \leq \tau^#(\nu_1)$. As $\lambda = \tau^#(\nu_2)$ satisfies the conditions required above, this implies (3.15).

**Step II.** With $\nu^*$ as before we prove that

$$\tau^#(\nu^*) = \tau^#(0).$$

Let $u_0^*$ be the solution of

$$-L\nu u + f(u) = \tau^#(0) \quad \text{in } \Omega, \quad \text{tr}_V u = 0.$$

Then $u_0^*$ is a subsolution of (3.6). By Theorem 3.1(d) and (3.14),

$$u_0^# \leq u^#(\tau, \nu) = u^#(\tau, \nu^*).$$

Let $w := u^#(\tau, \nu^*) - u_0^#$ so that

$$-L\nu w + f(u^#(\tau, \nu^*)) - f(u_0^#) = \tau^#(\nu^*) - \tau^#(0) \quad \text{in } \Omega.$$
As $w \geq 0$, by the inverse maximum principle \[8\],

$$-(\Delta w)_c = (\tau^#(\nu^*) - \tau^#(0))_c \geq 0.$$  

Moreover, by Lemma \[3.4\]

$$\tau^#(\nu^*)_d = \tau^#(0)_d = \tau_d.$$  

Hence $\tau^#(\nu^*) \geq \tau^#(0)$. On the other hand, by step I, $\tau^#(\nu^*) \leq \tau^#(0)$.

This proves \[3.17\].

(iii) This is a simple consequence of statements (i) and (ii) and Theorem \[3.1(d)\].

**Remark 3.6.** Given a real function $h$ on $\mathbb{R}$, denote by $\hat{h}$ the function given by

$$\hat{h}(t) := -h(-t) \quad \forall t \in \mathbb{R}.$$  

Let $f_n, \tau, \nu$ be as in Theorem \[3.1\]. If $w_n$ is the solution of the boundary value problem

$$-L_V w + \hat{f}_n(w) = \tau \quad \text{in } \Omega, \quad \text{tr}_V w = \nu,$$

then $z_n = -w_n$ satisfies

$$-L_V z_n + f_n(z_n) = -\tau \quad \text{in } \Omega, \quad \text{tr}_V z_n = -\nu.$$  

Since $\hat{f}_n$ has the same properties as $f_n$, the sequence $\{w_n\}$ has the same properties as the sequence $\{u_n\}$ in Theorem \[3.1\].

Accordingly, for $(\tau, \nu) \in M_+(\Omega; \Phi_V) \times M_+(\partial \Omega)$ the reduced measures for $-\tau$ and $-\nu$ and the corresponding reduced couple are given by

$$(-\tau)^# := -(\tau^#_f), \quad (-\nu)^# := -(\nu^#_f),$$

$$(-\tau, -\nu)^# := -((\tau, \nu)^#_f) = ((-\tau)^#_f, (-\nu)^#_f).$$

(The subscript $\hat{f}$ above indicates that the reduced measure or couple is defined relative to this function.)

In this case $(-\tau)^#$ and $(-\nu)^#$ are the smallest good measures dominating $-\tau$ and $-\nu$ respectively. The relation between the solutions corresponding to the reduced couples is given below

$$u^#(-\tau, -\nu) = -u^#_f(\tau, \nu).$$

Here again the notation $u^#_f$ indicates that the reduced couple and the corresponding solution is defined relative to $\hat{f}$.

In view of this remark, the results of Theorems \[3.1\] and \[3.5\] with obvious modifications, also apply to couples of negative measures.
4. Signed measures

Theorem 4.1. Let $\tau \in \mathcal{M}(\Omega; \Phi_V)$ and $\nu \in \mathcal{M}(\partial \Omega)$. Let $\tau_1, \tau_2 \in \mathcal{M}_+(\Omega; \Phi_V)$ and $\nu_1, \nu_2 \in \mathcal{M}_+(\partial \Omega)$ be measures such that

\begin{equation}
-\tau_1 \leq \tau \leq \tau_2, \quad -\nu_1 \leq \nu \leq \nu_2.
\end{equation}

Let $\{f_n\}$ be a sequence of functions as in Theorem 3.1. Then the boundary value problem (3.1) has a unique solution $u_n = u_n(\tau, \nu)$.

The following statements hold.

(a) The sequence $\{u_n\}$ is bounded in $W^{1,p}_{\text{loc}}(\Omega)$ for every $p \in [1, N)$. Consequently every subsequence has a limit point in the sense of convergence in $L^p_{\text{loc}}(\Omega)$ and convergence a.e. in $\Omega$.

If $\{u_{nk}\}$ is a subsequence of $\{u_n\}$ converging to $\tilde{u}$ a.e. then:

\begin{itemize}
  \item[(i)] $u_{nk} \to \tilde{u}$ in $L^1(\Omega; \Phi_V/\delta)$,
  \item[(ii)] $f_n(u_{nk}) \to f(\tilde{u})$ a.e. in $\Omega$.
\end{itemize}

(b) The following inequality holds

\begin{equation}
\|\tilde{u}\|_{L^1(\Omega; \Phi_V/\delta)} + \|f(\tilde{u})\|_{L^1(\Omega; \Phi_V)} \leq C \sum_{i=1,2} (\|\tau_i\|_{\mathcal{M}(\Omega; \Phi_V)} + \|\nu_i\|_{\mathcal{M}(\partial \Omega)}).
\end{equation}

(c) There exist $\tilde{\tau} \in \mathcal{M}(\Omega; \Phi_V)$ and $\tilde{\nu} \in \mathcal{M}(\partial \Omega)$ such that

\begin{equation}
(-\tau_1)^\# \leq \tilde{\tau} \leq (-\tau_2)^\#, \quad (-\nu_1)^\# \leq \tilde{\nu} \leq (-\nu_2)^\#
\end{equation}

and

$$\tilde{u} + G_V[f(\tilde{u})] = G_V[\tilde{\tau}] + K_V[\tilde{\nu}] \quad \text{in } \Omega.$$ 

Thus $\tilde{u}$ is the solution of the boundary value problem

\begin{equation}
-L_V \tilde{u} + f(\tilde{u}) = \tilde{\tau} \quad \text{in } \Omega, \quad \tr_V \tilde{u} = \tilde{\nu}.
\end{equation}

Proof. Let $v_{2,n}$ denote the solution of the boundary value problem

\begin{equation}
-L_V v + f_n(v) = \tau_2 \quad \text{in } \Omega, \quad \tr_V v = \nu_2,
\end{equation}

and $v_{1,n}$ denote the solution of the boundary value problem

\begin{equation}
-L_V v + f_n(v) = -\tau_1 \quad \text{in } \Omega, \quad \tr_V v = -\nu_1.
\end{equation}

By Theorem 3.1 and Remark 3.6, $v_{2,n}$ and $-v_{1,n}$ are positive and the following inequalities hold

\begin{equation}
\|v_{i,n}\|_{L^1(\Omega; \Phi_V/\delta)} + \|f_n(v_{i,n})\|_{L^1(\Omega; \Phi_V)} \leq C(\|\tau_i\|_{\mathcal{M}(\Omega; \Phi_V)} + \|\nu_i\|_{\mathcal{M}(\partial \Omega)}), \quad i = 1, 2.
\end{equation}
Moreover,
\[ v_{1,n} \uparrow v_1^\#(-\tau, -\nu) =: w_1, \quad v_{2,n} \downarrow v_2^\#(\tau, \nu) =: w_2 \quad \text{a.e. in } \Omega, \]
\[ f(v_i^\#) \in L^1(\Omega; \Phi_V), \quad i = 1, 2, \]
and there exist
\[ \tau_2^\#, (-\tau_1)^\# \in \mathcal{M}(\Omega; \Phi_V), \quad \nu_2^\#, (-\nu_1)^\# \in \mathcal{M}(\partial \Omega) \]
such that
\[ -LVw_2 + f(w_2) = \tau_2^\# \quad \text{in } \Omega, \quad \text{tr}_Vw_2 = \nu_2^#, \]
\[ -LVw_1 + f(w_1) = (-\tau_1)^\# \quad \text{in } \Omega, \quad \text{tr}_Vw_1 = (-\nu_1)^#. \]

The monotone convergence of the sequences \( \{v_{i,n}\} \) and (4.8) imply
\[ v_{i,n} \to w_i \quad \text{in } L^1(\Omega; \Phi_V/\delta) \]
and, by Dini’s lemma,
\[ f_n(v_{i,n}) \to f(w_i) \quad \text{a.e. in } \Omega, \quad i = 1, 2. \]

(a) By (3.1) and (4.6), we have
\[ -LV(u_n - v_{2,n}) + f_n(u_n) - f_n(v_{2,n}) = \tau - \tau_2 \quad \text{in } \Omega \]
\[ \text{tr}_V(u_n - v_{2,n}) = \nu - \nu_2. \]
By (4.1), \( \tau - \tau_2 \leq 0 \) and \( \nu - \nu_2 \leq 0 \). Therefore, using Kato’s inequality, we obtain
\[ -LV(u_n - v_{2,n})_+ \leq 0 \quad \text{and, by Lemma 2.6, } \text{tr}_V(u_n - v_{2,n})_+ = 0. \]
By Lemma 2.3, \( (u_n - v_{2,n})_+ = 0 \), which implies
\[ u_n \leq v_{2,n} \quad \text{in } \Omega. \]

Similarly, by (3.1) and (4.7), we obtain
\[ v_{1,n} \leq u_n \quad \text{in } \Omega. \]
(Recall that \( v_{1,n} \leq 0 \).) Thus
\[ |u_n| \leq \max\{-v_{1,n}, v_{2,n}\} \quad \text{in } \Omega. \]
This inequality and the monotonicity of \( f_n \) imply that
\[ |f_n(u_n)| \leq \max\{f_n(-v_{1,n}), f_n(v_{2,n})\} \quad \text{in } \Omega. \]

Inequalities (4.15), (4.16) and (4.8) together with the fact that \( V \) is locally bounded in \( \Omega \) imply that \( \{|V u_n + f_n(u_n)|\} \) is bounded in \( L^1_{\text{loc}}(D) \). Consequently \( \{u_n\} \) is bounded in \( W^{1,p}_{\text{loc}}(\Omega) \) for every \( p \in [1, \frac{N}{N-1}) \). Therefore \( \{u_n\} \) is precompact in \( L^p_{\text{loc}}(\Omega) \). Hence there exists a subsequence \( \{u_{n_k}\} \) that converges to a function \( \tilde{u} \) in \( L^p_{\text{loc}}(\Omega) \) and a.e. in \( \Omega. \)
By the generalized dominated convergence theorem, the convergence of \( \{u_{n_k}\} \) to \( \tilde{u} \) a.e. in \( \Omega \), (4.15) and (4.10) imply (4.2) (i).

The assumption \( f_n^\pm \uparrow f^\pm \) and Dini’s lemma imply (4.2) (ii).

To simplify the presentation, in the remainder of the proof we assume that \( \{u_n\} \) is a sequence converging a.e. to \( \tilde{u} \).

(b) By (4.15), (4.8) and (4.2)(i), we obtain
\[
\|\tilde{u}\|_{L^1(\Omega;\Phi_V/\delta)} = \lim_{n \to \infty} \|u_n\|_{L^1(\Omega;\Phi_V/\delta)} \\
\leq C \sum_{i=1,2} (\|\tau_i\|_{\mathcal{M}(\Omega;\Phi_V)} + \|\nu_i\|_{\mathcal{M}(\partial\Omega)}).
\]

By Fatou’s lemma, (4.16), (4.8) and (4.2) yield
\[
\|f(\tilde{u})\|_{L^1(\Omega;\Phi_V)} \leq \lim_{n \to \infty} \|f_n(u_n)\|_{L^1(\Omega;\Phi_V)} \\
\leq \limsup_{n \to \infty} \|\max\{f_n(-v_{1,n}), f_n(v_{2,n})\}\|_{L^1(\Omega;\Phi_V)} \\
\leq C \sum_{i=1,2} (\|\tau_i\|_{\mathcal{M}(\Omega;\Phi_V)} + \|\nu_i\|_{\mathcal{M}(\partial\Omega)}).
\]

This proves (4.3).

(c) Inequalities (4.13) and (4.14) imply \( w_1 \leq \tilde{u} \leq w_2 \) a.e. in \( \Omega \).

Let \( \zeta \in C_\infty_c(\Omega) \) and \( \zeta \geq 0 \). By (4.12),
\[
(4.17) \quad -\int_\Omega (v_{2,n} - u_n)L_V \zeta \, dx + \int_\Omega (f_n(v_{2,n}) - f_n(u_n)) \zeta \, dx = \int_\Omega \zeta \, d(\tau_2 - \tau).
\]

By (4.13), (4.11) and Fatou’s lemma,
\[
\int_\Omega (f(w_2) - f(\tilde{u})) \zeta \, dx \leq \liminf_{n \to \infty} \int_\Omega (f_n(v_{2,n}) - f_n(u_n)) \zeta \, dx.
\]

Therefore, by (4.17) and (4.10),
\[
(4.18) \quad -\int_\Omega (w_2 - \tilde{u})L_V \zeta \, dx + \int_\Omega (f(w_2) - f(\tilde{u})) \zeta \, dx \leq \int_\Omega \zeta \, d(\tau_2 - \tau).
\]

By (4.17), we have
\[
\left|\int_\Omega (v_{2,n} - u_n)L_V \zeta \, dx\right| \leq \int_\Omega (f_n(v_{2,n}) - f_n(u_n)) \zeta \, dx + \int_\Omega \zeta \, d(\tau_2 - \tau).
\]
By (4.8), (4.15) and (4.16),
\[
0 \leq \int_\Omega (f_n(v_{2,n}) - f_n(u_n)) \zeta \, dx + \int_\Omega \zeta \, d(\tau_2 - \tau) \\
\leq c \sup(\zeta/\Phi_V) \left( \sum_{i=1,2} (\|\tau_i\|_{\mathbb{M}(\Omega;\Phi_V)} + \|\nu_i\|_{\mathbb{M}(\partial\Omega)} + \|\tau\|_{\mathbb{M}(\Omega;\Phi_V)}) \right) \\
\leq C \sup(\zeta/\Phi_V).
\]
Since
\[
\lim_{n \to \infty} \int_\Omega (v_{2,n} - u_n) L_V \zeta \, dx = \int_\Omega (w_2 - \tilde{u}) L_V \zeta \, dx,
\]
it follows that
\[
\left| \int_\Omega (w_2 - \tilde{u}) L_V \zeta \, dx \right| \leq C \sup(\zeta/\Phi_V).
\]
Hence, by (4.18), there exists a measure \( \lambda \leq \tau_2 - \tau \) such that \( \lambda \in \mathcal{M}(\Omega; \Phi_V) \) and
\[
-\int_\Omega (w_2 - \tilde{u}) L_V \zeta \, dx + \int_\Omega (f(w_2) - f(\tilde{u})) \zeta \, dx = \int_\Omega \zeta \, d\lambda \quad \forall \zeta \in C_\infty^*(\Omega)
\]
or equivalently,
\[
(4.19) \quad -L_V(w_2 - \tilde{u}) + f(w_2) - f(\tilde{u}) = \lambda \quad \text{in } \Omega.
\]
Consequently, by (4.9),
\[
(4.20) \quad -L_V \tilde{u} + f(\tilde{u}) = \tilde{\tau} \quad \text{in } \Omega \quad \text{where } \tilde{\tau} := \tau_2^# - \lambda \geq \tau_2^# - \tau_2 + \tau.
\]
Next, by (3.1) and (4.7), for \( \zeta \in C_\infty^*(\Omega) \),
\[
-\int_\Omega (u_n - v_{1,n}) L_V \zeta \, dx + \int_\Omega (f_n(u_n) - f_n(v_{1,n})) \zeta \, dx = \int_\Omega \zeta \, d(\tau_1 + \tau).
\]
By the same argument as above it follows that there exists a measure \( \lambda' \leq \tau_1 + \tau \) such that \( \lambda' \in \mathcal{M}(\Omega; \Phi_V) \) and
\[
(4.21) \quad -L_V(\tilde{u} - w_1) + f(\tilde{u}) - f(w_1) = \lambda' \quad \text{in } \Omega.
\]
Consequently, by (4.9),
\[
(4.22) \quad \tilde{\tau} = \lambda' + (-\tau_1)^# \leq \tau_1 + \tau + (-\tau_1)^#.
\]
Next we show that \( \tilde{\tau} \) satisfies (4.4). By Lemma 3.4
\[
(4.23) \quad (\tau_2^#)_d = (\tau_2)_d, \quad (-(\tau_1)^#)_d = -(\tau_1)_d.
\]
Therefore, by (4.20), \( \tilde{\tau}_d \geq \tau_d \) and by (4.22), \( \tilde{\tau}_d \leq \tau_d \). Thus
\[
(4.24) \quad \tau_d = \tilde{\tau}_d.
\]
By (4.19) and (4.20),

\[-L_V(w_2 - \tilde{u}) + f(w_2) - f(\tilde{u}) = \tau_2^\# - \tilde{\tau} \text{ in } \Omega.\]

Since \(\tilde{u}\) and \(w_2\) are diffuse and \(f(0) = 0\), it follows that

\[(4.25) \quad (-\Delta(w_2 - \tilde{u})).c = (\tau_2^\# - \tilde{\tau}).c.\]

As \(w_2 - \tilde{u} \geq 0\), by the inverse maximum principle \([8]\), \((-\Delta(w_2 - \tilde{u})).c \geq 0\).

Consequently,

\[(\tau_2^\# - \tilde{\tau}).c \geq 0.\]

As \(\tau \leq \tau_2\), (4.23), (4.24) yield,

\[(\tau_2^\#)_d = (\tau_2)_d \geq \tau_d = \tilde{\tau}_d.\]

This inequality and (4.25) imply

\[(4.26) \quad \tilde{\tau} \leq \tau_2^\#.\]

Similarly by (4.21) and (4.22),

\[-L_V(\tilde{u} - w_1) + f(\tilde{u}) - f(w_1) = \tilde{\tau} - (\tau_1^\#) \text{ in } \Omega.\]

Since \(\tilde{u} - w_1 \geq 0\), another application of the inverse maximum principle yields \((-\Delta(\tilde{u} - w_1)).c \geq 0\) and consequently

\[(4.27) \quad (\tilde{\tau} - (\tau_1^\#)).c \geq 0.\]

As \(-\tau_1 \leq \tau\), (4.23), (4.24) imply

\[\tilde{\tau}_d = \tau_d \geq (\tau_1)_d = ((\tau_1^\#))_d.\]

This and (4.27) yield

\[\tilde{\tau} \geq (\tau_1^\#).\]

Finally, this and (4.26) imply (4.3) with respect to \(\tilde{\tau}\).

It remains to show that \(\tilde{u}\) has an \(L_V\) boundary trace and that the second inequality in (4.4) holds.

By (4.19) \(-L_V(w_2 - \tilde{u}) = \mu\) where \(\mu \in \mathfrak{M}(\Omega; \Phi_V)\). Since \(w_2 - \tilde{u} \geq 0\), by Lemma 2.5, \(w_2 - \tilde{u}\) has an \(L_V\) boundary trace, say \(\sigma\), and \(w_2 - \tilde{u} = G_V[\mu] + K_V[\sigma]\). Obviously \(\sigma \geq 0\). Therefore

\[\text{tr}_V \tilde{u} = \text{tr}_V w_2 - \sigma \leq \text{tr}_V w_2 = \nu_2^\#.\]

Similarly, starting with (4.21) we conclude that there exists \(\sigma' \in \mathfrak{M}(\partial\Omega)\) such that \(\text{tr}_V (\tilde{u} - w_1) = \sigma' \geq 0\). Therefore

\[\text{tr}_V \tilde{u} = \sigma' + \text{tr}_V w_1 \geq \text{tr}_V w_1 = (\nu_1)^\#.\]

This completes the proof. \(\square\)

The theorem is complemented by the following consequence of [17, Corollary 3.7] (see Lemma 2.10).
Proposition 4.2. Let \((\lambda_i, \sigma_i) \in \mathcal{M}(\Omega; \Phi_V) \times \mathcal{M}(\partial\Omega), i = 1, 2\). Suppose that these are good couples with respect to (1.1) and that \((\lambda_1, \sigma_1) \prec (\lambda_2, \sigma_2)\). Then, every couple \((\lambda, \sigma)\) such that
\[
(\lambda_1, \sigma_1) \prec (\lambda, \sigma) \prec (\lambda_2, \sigma_2)
\]
is a good couple.

Proof. Let \(v_i\) be the solution corresponding to the couple \((\lambda_i, \sigma_i), i = 1, 2\) and let \((\lambda, \sigma)\) be as in (4.28). Then \(v_2\) is a supersolution and \(v_1\) a subsolution of equation 
\[
-L_V u + f(u) = \lambda \quad \text{and} \quad \operatorname{tr} V u_1 \leq \sigma \leq \operatorname{tr} V v_2.
\]
Therefore the stated result is a consequence of Lemma 2.10). □

Remark. As mentioned before, Theorem 4.1 and Proposition 4.2 imply Theorem 1.4. However we emphasize that, in contrast to Theorem 4.1 in Proposition 4.2 \((\lambda_i, \sigma_i)\) may be couples of signed measures.

Proposition 4.3. In addition to the assumptions of Theorem 4.1, assume that \((\tau_2, \nu_2)\) and \((-\tau_1, -\nu_1)\) are good couples.

Suppose that \((-\tau_1, -\nu_1) \prec (\tau, \nu) \prec (\tau_2, \nu_2)\). (By the previous result, \((\tau, \nu)\) is a good couple.) Let \(u\) be the solution of problem (2.3) and let \(u_n\) denote the solution of the ‘approximating’ problem
\[
-L_V u + f_n(u) = \tau \quad \text{in} \ \Omega, \quad \operatorname{tr} V u = \nu.
\]

If \(\Phi_V\) satisfies the additional condition
\[
(4.29) \quad \int_{\Sigma_\beta} \Phi^\beta_V / \delta \, dS \to 0 \quad \text{as} \ \beta \to 0
\]
then \(u_n \to u\), i.e. \(\tilde{u} = u\).

Proof. We use the notation in the proof of Theorem 4.1.

Let \(v_{1,n}\) and \(v_{2,n}\) be the solutions of (4.7) and (4.6). Then
\[
(4.30) \quad v_{1,n} \leq u_n \leq v_{2,n} \quad \text{in} \ \Omega.
\]
The sequences \(\{v_{i,n}\}, i = 1, 2\) satisfy (4.10) and (4.11). In addition, by Proposition A.1 (see Appendix),
\[
(4.31) \quad f_n(v_{i,n}) \to f(w_i) \quad \text{in} \ L^1(\Omega; \Phi_V).
\]

By (4.30) and the monotonicity of \(f_n\),
\[
f_n(v_{1,n}) \leq f_n(u_n) \leq f_n(v_{2,n}) \quad \text{in} \ \Omega.
\]
Therefore, taking a subsequence for which (4.2) holds, the (generalized) dominated convergence theorem implies
\[ f_n(u_n) \to f(\tilde{u}) \text{ in } L^1(\Omega; \Phi_V) \text{ as } n \to \infty. \]
Hence, by Theorem 2.2,
\[ \mathbb{G}_V[f_n(u_n)] \to \mathbb{G}_V[f(\tilde{u})] \text{ in } L^1(\Omega; \Phi_V/\delta). \]
By (4.2),
\[ u_n \to \tilde{u} \text{ in } L^1(\Omega; \Phi_V/\delta). \]
As
\[ u_n + \mathbb{G}_V[f_n(u_n)] = \mathbb{G}_V[\tau] + \mathbb{K}_V[\nu] \text{ in } \Omega \]
we conclude that
\[ \tilde{u} + \mathbb{G}_V[f(\tilde{u})] = \mathbb{G}_V[\tau] + \mathbb{K}_V[\nu] \text{ in } \Omega. \]
Thus \( \tilde{u} \) is a solution of (2.3). By uniqueness (Lemma 2.9), \( \tilde{u} = u \). □

**Proof of Theorem 1.5** Since \( f \) vanishes on \((-\infty, 0] \), if \( w \) is a real function on \( \Omega \) then
\[ f(w) = f(w_+) + f(-w_-) = f(w_+). \]
Suppose that \((\tau, \nu)\) is a good couple, i.e. (3.6) has a solution \( u \) such that \( f(u) \in L^1(\Omega; \Phi_V) \).

Let \( u_n \) and \( \tilde{u} \) be as in Theorem 4.1. In view of (4.32), \( \{u_n\} \) is decreasing (by Lemma 2.9) and \( \tilde{u} = \lim u_n \).

Let \( w \) be a subsolution of (3.6) such that \( f(w) \in L^1(\Omega; \Phi_V) \). Then
\[ -L_V w + f_n(w) \leq -L_V w + f(w_+) \leq \tau \text{ in } \Omega, \quad \text{tr}_V w \leq \nu. \]
As \( u_n \) satisfies (3.1), Lemma 2.9 implies that \( w \leq u_n \). Thus \( w \leq \tilde{u} \) and, in particular,
\[ u \leq \tilde{u} \text{ in } \Omega. \]

Let \( \nu := \tilde{u} - u \). Then \( \nu \geq 0 \), \( \Delta \nu \) is a measure and therefore, by the inverse maximum principle,
\[ (-\Delta \nu)_c = (\tilde{\tau} - \tau)_c \geq 0. \]
As in the proof of Theorem 4.1 (see (4.24)) \( \tau_d = \tilde{\tau}_d \). Therefore
\[ \tau \leq \tilde{\tau}. \]
In addition, by (4.33),
\[ \nu = \text{tr}_V u \leq \text{tr}_V \tilde{u} = \tilde{\nu}. \]
Thus
\[ (\tau, \nu) \prec (\tilde{\tau}, \tilde{\nu}). \]

If \((\tau_1, \nu_1) := (\tau_-, \nu_-) \) and \((\tau_2, \nu_2) := (\tau_+, \nu_+) \) then \( \tau, \nu \) satisfy (4.1) and by Theorem 4.1(c),
\[ (\tilde{\tau}, \tilde{\nu}) \prec (\tau_+, \nu_+)^\#. \]
Hence, by (4.34), we obtain (1.15).

Conversely, assume that \((\tau, \nu)\) satisfies (1.15). Recall that every couple of negative measures is good relative to \(f\). Therefore, by Proposition 4.2, the relation

\[-(\tau_-, \nu_-) \prec (\tau, \nu) \prec (\tau_+, \nu_+)^\#

implies that \((\tau, \nu)\) is a good couple.

The last assertion of the theorem is obvious. □

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Appendix A.

We prove an auxiliary result that is used in the proof of Proposition 4.3.

**Proposition A.1.** Assume (A1), (A2), (C1) and (4.29) hold. Suppose that \((\tau, \nu) \in \mathcal{M}_+^+(\Omega; \Phi_V) \times \mathcal{M}_+^+(\partial \Omega)\) is a good couple of measures. Let \(u\) be the corresponding solution of problem (3.6) and \(u_n\) be the solution of (3.1). Then

\[u_n \to u \quad \text{in} \quad L^1(\Omega; \Phi_V/\delta), \quad f_n(u_n) \to f(u) \quad \text{in} \quad L^1(\Omega; \Phi_V).\]

The proof is based on the following lemma that was established in [20] for a more restricted class of potentials.

**Lemma A.2.** Assume (A1), (A2), (C1) and (4.29) hold. Let \(\tau \in \mathcal{M}(\Omega; \Phi_V)\) and let \(\lambda_V\) be the eigenvalue of \(-L_V\) corresponding to \(\Phi_V\). Then

\[
\lambda_V \int_\Omega G_V[\tau] \Phi_V \, dx = -\int_\Omega G_V[\tau] L_V \Phi_V \, dx = \int_\Omega \Phi_V \, d\tau.
\]

**Proof.** By linearity, we may assume that \(\tau \geq 0\). For \(\beta > 0\), put

\[I_\beta(\tau) := -\lambda_V \int_{D_\beta} G_V[\tau] \Phi_V \, dx + \int_{D_\beta} \Phi_V \, d\tau,
\]

where \(D_\beta = \{ x \in \Omega : \delta(x) > \beta \}\). To prove (a.1) we show that

\[
\lim_{{\beta \to 0}} I_\beta(\tau) = 0.
\]
By Theorem 2.2 (ii),

\[ I_0(\tau) \leq C \int_\Omega \Phi_V d\tau. \]

Given \( \epsilon > 0 \) we choose \( 0 < \gamma \) sufficiently small so that, for \( \tau_\gamma := \tau 1_{\Omega\gamma} \),

\[ I_0(\tau_\gamma) < \epsilon. \]  

(a.3)

Therefore it is sufficient to prove

\[ \lim_{\beta \to 0} I_\beta(\tau - \tau_\gamma) = 0. \]

Thus it is sufficient to prove (a.2) when \( \tau \) has compact support in \( \Omega \).

Let \( \beta_\tau = \frac{1}{2} \text{dist} (\text{supp } \tau, \partial \Omega) \) and let \( \beta \in (0, \beta_\tau) \). Applying Green’s theorem in \( D_\beta \), we obtain

\[
\lambda_V \int_{D_\beta} G_V[\tau] \Phi_V \, dx = -\int_{D_\beta} G_V[\tau] L_V \Phi_V \, dx \\
= \int_{D_\beta} \Phi_V \, d\tau + \int_{\Sigma_\beta} \frac{\partial G_V[\tau]}{\partial \mathbf{n}} \Phi_V \, dS(x) - \int_{\Sigma_\beta} \frac{\partial \Phi_V}{\partial \mathbf{n}} G_V[\tau] \, dS(x).
\]

Thus

(a.4)

\[ I(\beta) = -\int_{\Sigma_\beta} \frac{\partial G_V[\tau]}{\partial \mathbf{n}} \Phi_V \, dS(x) + \int_{\Sigma_\beta} \frac{\partial \Phi_V}{\partial \mathbf{n}} G_V[\tau] \, dS(x). \]

Note that

\[ G_V(x, y) \sim \Phi_V(x) \quad \forall (x, y) \in \Omega_{\beta_\tau} \times \text{supp } \tau. \]

Therefore

(a.5)

\[ G_V[\tau](x) = \int_\Omega G_V(x, y) \, d\tau(y) \sim \Phi_V(x), \quad \forall x \in \Sigma_\beta. \]

By interior elliptic estimates, for every \( x \in \Sigma_\beta \),

\[ \left| \frac{\partial \Phi_V}{\partial \mathbf{n}}(x) \right| \leq C \sup_{|\xi - x| < \beta/4} \Phi_V(\xi) \beta^{-1}. \]

Therefore by Harnack’s inequality, we deduce

\[ \left| \frac{\partial \Phi_V}{\partial \mathbf{n}}(x) \right| \leq C \Phi_V(x) \beta^{-1} \quad \forall x \in \Sigma_\beta. \]

Hence, by (a.5) and assumption (4.29),

\[ \lim_{\beta \to 0} \int_{\Sigma_\beta} \frac{\partial \Phi_V}{\partial \mathbf{n}} G_V[\tau] \, dS(x) = 0. \]
In $D_{\beta}$: $G_v[\tau]$ is $L_V$ harmonic and $G_v[\tau] \sim \Phi_V$. Therefore the same argument as above yields,

$$\lim_{\beta \to 0} \int_{\Sigma_\beta} \frac{\partial G_v[\tau]}{\partial n} \Phi_V \, dS(x) = 0.$$  

Combining (a.4) – (a.6), we obtain (a.2) for measures $\tau$ with compact support. In view of previous remarks, this implies (a.2) for any measure $\tau \in \mathcal{M}(\Omega; \Phi_V)$. This in turn implies (a.1).

**Proof of Proposition A.1**  
By Theorem 3.1(d), $u = u^\#$. From the proof of Theorem 3.1, $u_n \geq 0$ satisfies (3.7), $u_n \downarrow u^\# = u$ and $f_n(u_n) \to f(u)$ a.e. in $\Omega$. By (3.7), $u_n \leq G_v[\tau] + K_v[\nu] \in L^1(\Omega; \Phi_V/\delta)$. Therefore, by the dominated convergence theorem,

$$u_n \to u \quad \text{in} \quad L^1(\Omega; \Phi_V/\delta).$$

By Lemma A.2 with $\tau$ replaced by $f_n(u_n)$ (recall that $f_n$ is a bounded function) we have,

$$\int_\Omega f_n(u_n) \Phi_V \, dx = \lambda_V \int_\Omega G_v[f_n(u_n)] \Phi_V \, dx. \quad \text{(a.7)}$$

Since $u_n$ is the solution of (3.1) it satisfies

$$u_n + G_v[f_n(u_n)] = G_v[\tau] + K_v[\nu].$$

Multiplying this equality by $\lambda_V \Phi_V$ and using (a.7) we obtain

$$\int_\Omega f_n(u_n) \Phi_V \, dx = \lambda_V \int_\Omega G_v[f_n(u_n)] \Phi_V \, dx$$

$$= -\lambda_V \int_\Omega u_n \Phi_V \, dx + \lambda_V \int_\Omega G_v[\tau] \Phi_V \, dx + \lambda_V \int_\Omega K_v[\nu] \Phi_V \, dx.$$

Hence,

$$\lim_{n \to \infty} \int_\Omega f_n(u_n) \Phi_V \, dx =$$

$$\lambda_V \int_\Omega u \Phi_V \, dx + \lambda_V \int_\Omega G_v[\tau] \Phi_V \, dx + \lambda_V \int_\Omega K_v[\nu] \Phi_V \, dx. \quad \text{(a.8)}$$

Since $u$ is the solution of (3.6), $f(u) \in L^1(\Omega; \Phi_V)$ and $u + G_v[f(u)] = G_v[\tau] + K_v[\nu]$. In addition, by Lemma A.2

$$\int_\Omega f(u) \Phi_V \, dx = \lambda_V \int_\Omega G_v[f(u)] \Phi_V \, dx.$$
Therefore, as before,
\[
\int_{\Omega} f(u) \Phi_V \, dx
\]
\[
= -\lambda_V \int_{\Omega} u \Phi_V \, dx + \lambda_V \int_{\Omega} G_V \tau \Phi_V \, dx + \lambda_V \int_{\Omega} K_V \nu \Phi_V \, dx.
\]

By (a.8) and (a.9), \( \| f_n(u_n) \|_{L^1(\Omega; \Phi_V)} \to \| f(u) \|_{L^1(\Omega; \Phi_V)} \). As \( f_n(u_n) \to f(u) \) a.e. in \( \Omega \), it follows that
\[
f_n(u_n) \to f(u) \quad \text{in} \quad L^1(\Omega; \Phi_V).
\]
The proof is complete. \(\square\)

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