ON THE HOLLMAN MCKENNA CONJECTURE: INTERIOR CONCENTRATION NEAR CURVES

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Abstract. Consider the problem
\[
\begin{cases}
-\varepsilon^2 \Delta u = |u|^p - \Phi_1 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
where \(\varepsilon > 0\) is a parameter, \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^2\) and \(p > 2\). Let \(\Gamma\) be a stationary non-degenerate closed curve relative to the weighted arc-length \(\int_{\Gamma} \Phi_1^{\frac{p+3}{2p}}\). We prove that for \(\varepsilon > 0\) sufficiently small, there exists a solution \(u_{\varepsilon}\) of the problem, which concentrates near the curve \(\Gamma\) whenever \(d(\Gamma, \partial \Omega) > c_0 > 0\). As a result, we prove the higher dimensional concentration for a Ambrosetti-Prodi problem, thereby proving an affirmative result to the conjecture by Hollman-McKenna [9] in two dimensions.

1. Introduction. In this paper, we are interested on the higher dimensional concentration of solutions for the elliptic problem of Ambrosetti-Prodi type. There has been a considerable interest in understanding the number of solutions of the elliptic problem
\[
\begin{cases}
-\Delta u = \zeta(u) - t \Phi_1 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  
(1.1)
where \(t\) is a positive parameter, \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\), \(\Phi_1\) is an eigenfunction of \(-\Delta\) with Dirichlet boundary condition corresponding to the first eigenvalue \(\lambda_1\), and \(\lim_{t \to +\infty} \frac{\zeta(t)}{t} = \mu > \lim_{t \to -\infty} \frac{\zeta(t)}{t} = \nu\), where \((\nu, \mu)\) contains some eigenvalues of \(-\Delta\) subject to Dirichlet boundary condition. Here, \(\mu = +\infty\) and \(\nu = -\infty\) are allowed.

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A problem of this kind was first studied by Ambrosetti and Prodi [1] and by many authors, especially in the 1980’s. We refer the readers to [2] and the references therein for a detailed bibliography on the topic. The main result is that if \( \zeta(t) \) grows sub-critically at \( \pm \infty \), then (1.1) has at least two solutions: one is a local minimum of the corresponding functional, the other being a solution of the mountain-pass type. If \( \zeta(\xi) = \xi^2 \) and \( \Omega \) is a unit square in \( \mathbb{R}^2 \), Brue– McKenna–Plum [2] showed that (1.1) admits at least four solutions using a computer assisted proof. Dancer–Yan [3], considered \( \zeta(\xi) = |\xi|^p \) where \( 1 < p < \frac{N+2}{N-2} \) and constructed solutions with sharp peaks near the local maximum points of \( \Phi_1 \) or near the boundary. Using this idea, one can produce solutions with arbitrarily many peaks, as \( t \to \infty \), thereby proving Lazer-Mckenna conjecture [10]. Moreover, in [3], the asymptotic behavior, as \( t \to +\infty \), of the mountain pass solution is studied, proving in particular that the single peak of this solution approaches the boundary of the domain. This type of concentration was already obtained by de Figueiredo-Santra-Srikanth [8] in the ball, proving that the mountain pass solution is non-radial. The paper in [8] was also motivated by [2] because the numerical approximation of the solutions helped them to guess the geometrical properties of the solutions and their Morse indices. Lazer- McKenna conjecture was later extended to different kind of nonlinearities by Dancer–Yan [4], Dancer–Santra [5] ; Li–Yan–Yang [11], [12] , Wei-Yan [18] for the critical case and del Pino-Munoz [6] for the exponential case and asymptotically linear case is due to Molle- Passaseo [17]. Moreover, there exists results due to del Pino-Kowalczyk-Wei [7] and Mahmoudi– Malchiodi–Montenegro [16] for the higher dimensional concentration for the nonlinear Schrödinger equation. For the Neumann case, see Malchiodi–Montenegro [14] [15] and [19].

Based on the numerical evidence, Hollman–McKenna [9] asked the following question whether there exist other types of concentrations for the superlinear Ambrosetti- Prodi problem. For example, is it possible to construct solutions which concentrates near an interior line or interior geodesic? Note all the results available for this problem are point concentrating solutions only. Numerical results performed in [9] strongly suggests that not only there are solutions which concentrate interior geodesic but also existence of layered solutions close to the boundary. The idea of our result relies on the fact that locally superlinear Ambrosetti-Prodi problem looks like a nonlinear Schrödinger equation.

In this paper, we discuss the conjecture related to the existence of interior concentrating solution of a superlinear Ambrosetti Prodi problem. Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain and consider the problem

\[
\begin{align*}
-\varepsilon^2 \Delta u &= |u|^p - \Phi_1 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]  
(1.2)

where \( p > 2 \). By [3], there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0] \) there exists a negative solution \( u_\varepsilon \) of (1.2) such that \( u_\varepsilon > -\Phi_1^{\frac{1}{p}} \) and the expansion

\[
u_\varepsilon(x) = -\Phi_1^{\frac{1}{p}}(x) - \varepsilon^2 \frac{\Delta \Phi_1^{1/p}(x)}{p\Phi_1^{-(p-1)/p}(x)} + o(\varepsilon^2) \quad (1.3)
\]

for every compact subsets of \( \Omega \) as \( \varepsilon \to 0 \). Furthermore, note that by the Hopf’s lemma, the function \( \Phi_1^{\frac{1}{p}} \) is continuous but not differentiable near the boundary as \( p > 1 \). As a result, near the boundary \( \partial \Omega \), solutions are constructed by the one
dimensional limiting problem
\[
\omega_{xx} - \omega^p + x = 0, x > 0, \omega(0) = 0, \omega - x^{\frac{1}{p}} \to 0 \text{ as } x \to \infty.
\]

Theorem 1.1. Let \( \Gamma \) be a stationary non-degenerate curve in \( \Omega \) with \( \text{dist}(\Gamma, \partial \Omega) > 0 \) for the weighted length described by the functional \( \int_{\Gamma} \Phi_{1}^{\frac{p-3}{p}} \). Then given \( c > 0 \), there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) satisfying the gap condition
\[
|\varepsilon^2 n^2 - \lambda_*| \geq c \varepsilon \quad \text{for all } n \in \mathbb{N}
\]
the problem (1.2) has a solution \( u_\varepsilon \) which concentrates near \( \Gamma \). Here \( \lambda_* > 0 \) is a number defined by (9.14).

The method is very constructive. We will extensively use the idea by del Pino-Kowalczyk-Wei [7] to prove the result. The advantage of the method is that we can not only prove the existence of solution for (1.2) but also obtain the profile of the solutions. Our method can be used in a symmetric domain to prove existence of solution which concentrate on interior spheres without any gap condition. The disadvantage of this method is that the estimates depend heavily on the exponential decay of the limiting problem and as a matter of fact the method cannot be extended to the exponential or jumping nonlinearities (see [5] and [6]).

Remark 1. The idea of the proof is the following. First we shall study the weighted functional \( \int_{\Gamma} \Phi_{1}^{\frac{p-3}{p}} \). We find a sufficient condition for a curve to be a non-degenerate stationary curve of this functional. In the next section, we shall find an approximate solution (locally on a strip) near a curve and the corresponding error term. In section 4, we introduce the gluing procedure to get the global approximation. This will lead to a nonlinear coupled system. Solving one equation of the system we reduce the problem on the strip. To solve the reduced problem first we check the invertibility of the corresponding linear operator involved in the reduced equation in a proper space in section 5. In section 6, using the invertibility of the linear operator we solve the linear projection problem. Then we adjust the parameters in such a way that the coefficients of the approximate solutions in the strip become identically zero which will actually imply the existence of a true solution of the problem. This has been done in last few sections.

2. Study of the functional \( \int_{\Gamma} \Phi_{1}^{\frac{p-3}{p}} \). In this section, we analyze the properties of the weighted functional \( W(\Gamma) = \int_{\Gamma} \Phi_{1}^{\frac{p-3}{p}} \).

Let \( l \) be the total length of the curve \( \Gamma \). Let \( \gamma(\theta) \) be a natural parametrization of \( \Gamma \) with a positive orientation, where \( \theta \) denotes the arc length parameter measured from a fixed point of \( \Gamma \). Let \( \nu(\theta) \) denote the outer unit normal to \( \Gamma \). Points \( y \) which are \( \delta \)-close to \( \Gamma \) can be represented in the form
\[
y = \gamma(\theta) + t\nu(\theta)
\]
whenever \(|t| < \delta \) and \( 0 \leq \theta \leq l \). Any curve sufficiently close to \( \Gamma \) can be parametrized as
\[
\gamma_g(\theta) = \gamma(\theta) + g(\theta)\nu(\theta)
\]
where \( g \) is a smooth, \( l \)-periodic function which is uniformly small. Let \( \Gamma_g \) be the curve defined in this way. Let us denote \( \Phi_{1}(t, \theta) \) by \( \Phi_{1}(y) \) where \( y \) is given in (2.1).
Here $y \to (t, \theta)$ is a local diffeomorphism. Then the weighted length of this curve is given by the functional $W(g)$ of $g$ as

$$W(g) = \int_0^t \Phi_1^{\frac{p+3}{2p}}(\gamma_g(\theta)) |\gamma'_g(\theta)| d\theta$$

$$= \int_0^t \Phi_1^{\frac{p+3}{2p}}(\gamma_g(\theta)) |\gamma' + g'\nu| d\theta.$$  

Since $|\gamma'| = 1$ and $\nu' = k(\theta)\gamma'$, where $k(\theta)$ denotes the curvature of $\Gamma$, we have the expression for $W(g)$ as

$$W(g) = \int_0^t V(g, \theta)[(1 + kg)^2 + (g')^2]^{\frac{1}{2}} d\theta.$$  \hspace{1cm} (2.3)$$

Where $V(g, \theta) = \Phi_1^{\frac{p+3}{2p}}(g(\theta), \theta)$. Differentiating we get

$$W'(f)[h] = \int_0^t \left( (1 + kf)^2 + (f')^2 \right)^{1/2} D_g V(f, \theta)[h] d\theta$$

$$+ \int_0^t V(f, \theta) \frac{(1 + kf)kh + f'h'}{\sqrt{(1 + kf)^2 + (f')^2}} d\theta,$$

and we have

$$W'(0)[h] = \int_0^t [D_g V(0, \theta)[h] + V(0, \theta)kh] d\theta.$$  

Hence $\Gamma$ is a stationary if and only if

$$\frac{p+3}{2p} \Phi_1, t(0, \theta) = -k(\theta)\Phi_1(0, \theta), \hspace{1cm} \forall \theta \in (0, l).$$  \hspace{1cm} (2.4)$$

Now let us calculate the second derivative $W''(0)[h, h]$.

$$W''(0)[h, h] = \lim_{t \to 0} \frac{1}{t} \left[ W'(th)[h] - W'(0)[h] \right]$$

$$= \lim_{t \to 0} \frac{1}{t} \left[ \int_0^t D_g V(th)[h] \left( (1 + kth)^2 + (th')^2 \right)^{1/2} d\theta$$

$$+ \int_0^t V(g, \theta) \frac{(1 + kth)kh + th'h'}{\sqrt{(1 + kth)^2 + (th')^2}} d\theta$$

$$- \int_0^t [D_g V(0, \theta)[h] + V(0, \theta)kh] d\theta \right]$$

$$= \frac{1}{2} \int_0^t D_g^2 V(0, \theta)[h, h] + 2kh D_g V(0, \theta)[h] + V(0, \theta)h'^2 d\theta.$$  

Assuming $\Gamma$ is stationary we obtain

$$W''(0)[h, h] = \frac{1}{2} \int_0^t \left( [\Phi_1^{\frac{p+3}{2p}}]_tt - 2\Phi_1^{\frac{p+3}{2p}} k^2|h|^2 + \Phi_1^{\frac{p+3}{2p}} |h'|^2 \right) d\theta$$

and $\Gamma$ is a non-degenerate curve for any $l$ periodic function $h \in H^1(0, l)$

$$[\Phi_1^{\frac{p+3}{2p}} h']' - [\Phi_1^{\frac{p+3}{2p}}]_tt - 2\Phi_1^{\frac{p+3}{2p}} k^2|h| = 0$$  \hspace{1cm} (2.5)$$
under the condition $h'(0) = h'(l)$ admits only trivial solutions. Hence we have

$$
\begin{align*}
  h'' + \frac{p + 3}{2p} \Phi_1^{-1} \Phi_{1,tt} - \left[ \frac{p + 3}{2p} \Phi_1^{-1} \Phi_{1,tt} - \left( \frac{3(p + 1)}{p + 3} \right) \right] h &= 0, \quad (0, l) \\
  h(0) = h(l); h'(0) = h'(l)
\end{align*}
$$

admits only trivial solution.

**Remark 2.** In an unit ball $B$, it is known that $\Phi_1(x) = \Phi_1(|x|) = \Phi_1(r)$. Define the radial function as $G(r) = \Phi_1^{\frac{p+3}{2p}}(r)$. Then $G(0) = 0$ and $G(1) = 0$ and $G$ is positive in $(0, 1)$. Hence there exists $\xi \in (0, 1)$ such that $G'(\xi) = 0$. Hence $\xi \Phi_1(r)(\xi) = -\frac{2p}{p+3} \Phi_1(\xi)$. Moreover,

$$
G''(\xi) = (\Phi_1^{\frac{p+3}{2p}})'(\xi) + \frac{p + 3}{p}(\Phi_1^{\frac{p+3}{2p}}(\xi))'.
$$

Furthermore, if $\xi$ is a local minimum point of $G$, then $G''(\xi) > 0$. This implies that the weighted functional is non-degenerate in the radial case. The case of an annulus is the same.

3. Set up for the approximation.

**Lemma 3.1.** There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a unique negative solution $u_\varepsilon$ of (1.2) such that $u_\varepsilon > -\Phi_1^{\frac{1}{p}}$, and $u_\varepsilon \to -\Phi_1^{\frac{1}{p}}$ in $C^2_{loc}(\Omega)$. Moreover,

$$
\begin{align*}
  u_\varepsilon(x) &= -\Phi_1^{\frac{1}{p}}(x) - \varepsilon^2 \frac{\Delta \Phi_1^{\frac{1}{p}}(x)}{p \Phi_1^{\frac{p+3}{2p}}(x)} + O(\varepsilon^4)
\end{align*}
$$

holds in $C^2_{loc}(\Omega)$.

**Proof.** Since the negative solution is obtained by monotone iteration, using the maximum principle one can show that the negative solution is unique. But

$$
-\Delta u = (|u|^p - \Phi_1)\varepsilon^{-2}
$$

and hence the right hand side of (1.2) is equal to $f(x) + o(1)$ uniformly on compact subset (using (1.3)). By the $L^q$ estimate, $u_\varepsilon$ is bounded in $W^{2,q}_{loc}$ for any $q > 1$. This gives the $C^{1,\theta}_{loc}$ convergence for some $0 \leq \theta < 1$. Using the Schauder estimates, $u_\varepsilon$ is bounded in $C^2_{loc}(\Omega)$. By the Arzela–Ascoli theorem $u_\varepsilon \to -\Phi_1^{\frac{1}{p}}$ in $C^2_{loc}(\Omega)$ converges along a subsequence. But as $u_\varepsilon \to -\Phi_1^{\frac{1}{p}}$ converges in $C^0_{loc}(\Omega)$ by the uniqueness, $u_\varepsilon \to -\Phi_1^{\frac{1}{p}}$ in $C^2_{loc}(\Omega)$ as a whole sequence.

Let us write

$$
u_\varepsilon = -\Phi_1^{\frac{1}{p}} - \varepsilon^\alpha \psi_1.
$$

We want to determine $\alpha > 0$ and $\psi_1$ a $C^4$ function independent of $\alpha$. Hence we have

$$
\begin{align*}
-\varepsilon^2 \Delta (-\Phi_1^{\frac{1}{p}} - \varepsilon^\alpha \psi_1) &= |\Phi_1^{\frac{1}{p}} + \varepsilon^\alpha \psi_1|^p - \Phi_1 \\
&= |\Phi_1^{\frac{1}{p}} + \varepsilon^\alpha \psi_1|^p - \Phi_1 \\
&= p \varepsilon^\alpha \Phi_1^{\frac{p+1}{2p}} \psi_1 + p(p-1) \varepsilon^{2\alpha} \Phi_1^{\frac{p-2}{2p}} \psi_1^2 + o(\varepsilon^{2\alpha})
\end{align*}
$$
which implies that $\alpha = 2$ and $\psi_1 = \frac{\Delta \Phi_1^{1/p}(x)}{p\Phi_1^{(p-1)/p}(x)}$.

Hence we obtain (3.1). Moreover, it is easy to check that if $p > 3$

$$u_\varepsilon(x) = -\Phi_1^{\frac{1}{p}}(x) - \varepsilon^2 \psi_1 - \varepsilon^4 \psi_2 - \varepsilon^6 \psi_3 + O(\varepsilon^8) \quad (3.2)$$

where

$$\psi_1 = \frac{\Delta \Phi_1^{1/p}(x)}{p\Phi_1^{(p-1)/p}(x)}, \psi_2 = \frac{\Delta \psi_1 - p(p-1)\Phi_1^{(p-2)/p}\psi_1^2}{p\Phi_1^{(p-1)/p}} \quad (3.3)$$

and

$$\psi_3 = \frac{\Delta \psi_2 - 2p(p-1)\Phi_1^{(p-2)/p}\psi_1\psi_2 - p(p-1)(p-2)\Phi_1^{(p-3)/p}\psi_1^3}{p\Phi_1^{(p-1)/p}} \quad (3.4)$$

and the expansion (3.1) holds.

Let $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$. Then for $u_\varepsilon$ satisfying (1.2), $v_\varepsilon$ satisfies

$$\begin{cases} -\Delta v = |v|^p - \Phi_1(\varepsilon y) & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial \Omega_\varepsilon \end{cases} \quad (3.5)$$

where $\Omega_\varepsilon = \Omega / \varepsilon$.

Now given $u_\varepsilon$ a solution of (1.2), we look for a solution of (1.2) of the type

$$\tilde{v} = u + u_\varepsilon(x). \quad (3.6)$$

Then we can easily check that $u$ satisfies the equation

$$-\varepsilon^2 \Delta u = |u + u_\varepsilon(y)|^p - |u_\varepsilon(y)|^p, \ u \in H_0^1(\Omega). \quad (3.7)$$

Let us now calculate the Laplace operator in the new co-ordinate system: first note that

$$x_1 = \gamma_1(\theta) + \nu_1 \quad \text{and} \quad x_2 = \gamma_2(\theta) + \nu_2.$$

Hence

$$\begin{align*}
    dx_1 &= \gamma_1'(\theta)d\theta + \nu_1'd\theta + \nu_1dt = (1 + tk)\gamma_1'(\theta)d\theta + \nu_1dt, \\
    dx_2 &= \gamma_2'(\theta)d\theta + \nu_2'd\theta + \nu_2dt = (1 + tk)\gamma_2'(\theta)d\theta + \nu_2dt.
\end{align*}$$

The expression of the metric

$$g = (1 + tk)^2d\theta^2 + dt^2,$$

and the Laplace operator is given by

$$\Delta u = \frac{u_{\theta\theta}}{(1 + tk)^2} + u_{tt} + \frac{ku_t}{1 + tk} - \frac{tk'(\theta)u_\theta}{(1 + tk)^3}. \quad (3.8)$$

Also near the curve $\Gamma$ the equation (3.6) takes the form

$$-\varepsilon^2\left[\frac{u_{\theta\theta}}{(1 + tk)^2} + u_{tt} + \frac{ku_t}{1 + tk} - \frac{tk'(\theta)u_\theta}{(1 + tk)^3}\right] = |u + u_\varepsilon(y)|^p - |u_\varepsilon(y)|^p. \quad (3.9)$$

Now consider the change of variables

$$(s, z) = \varepsilon^{-1}(t, \theta),$$

the natural stretched coordinates associated to the curve $\Gamma$ in $\mathcal{S}$, where

$$\mathcal{S} := \{(s, z) \mid 0 \leq z \leq \frac{l}{\varepsilon} \text{ and } s \in (-\varepsilon^{-1}\delta_0, \varepsilon^{-1}\delta_0)\}. \quad (3.10)$$
Then in the new coordinates, the equation (3.6) reduces to

\[- u_{zz} - u_{ss} - |u - q(\varepsilon s, \varepsilon z)|^p + |q(\varepsilon s, \varepsilon z)|^p - B_1(u) = 0, \tag{3.8}\]

in the region (3.7), where

\[q(t, \theta) := -u_z(\gamma(\theta) + t\nu(\theta)). \tag{3.9}\]

\[B_1(u) = u_{zz} \left[ \frac{1}{(1 + \varepsilon k(\varepsilon z)s)^2} - 1 \right] + \frac{\varepsilon k(\varepsilon z)u_s}{1 + \varepsilon k(\varepsilon z)s} - \frac{\varepsilon^2 sk'(\varepsilon z)u_x}{(1 + \varepsilon k(\varepsilon z)s)^3}. \tag{3.10}\]

For further reference, it is convenient to expand this operator in the form

\[B_1(u) = (\varepsilon k - \varepsilon^2 sk^2)u_s + B_0(u), \tag{3.11}\]

where

\[B_0(u) = \varepsilon^2 sa_1 u_z + \varepsilon sa_2 u_{zz} + \varepsilon^3 s^2 a_3 u_s \tag{3.12}\]

and \(a_i\) are smooth functions in \((s, z)\) given by

\[a_1 = -\frac{k'(\varepsilon z)}{(1 + \varepsilon k(\varepsilon z)s)^3}, \quad a_2 = -\frac{2k(\varepsilon z) + \varepsilon^2 k'(\varepsilon z)s}{(1 + \varepsilon k(\varepsilon z)s)^2} \text{ and } a_3 = \frac{k^3(\varepsilon z)}{1 + \varepsilon k(\varepsilon z)s}. \]

We consider a further change of variable in (3.8) that replaces the main order of the potential \(q\) by 1. Let

\[\tilde{\alpha}(\theta) = q(0, \theta), \quad \tilde{\beta}(\theta) = |q(0, \theta)|^\frac{p-1}{2}, \tag{3.13}\]

\[\alpha(\theta) = q(0, \theta), \quad \beta(\theta) = q(0, \theta)^\frac{p-1}{2}, \tag{3.14}\]

where \(q(t, \theta) := \Phi_{1,2}^\frac{1}{2} (\gamma(\theta) + t\nu(\theta))\).

Let us now find the expression of \(\tilde{\alpha}(\theta), \tilde{\beta}(\theta)\) and its derivatives in terms of \(\alpha(\theta)\) and \(\beta(\theta)\) and its derivatives up to order \(\varepsilon^2\). From the expression (3.2) we get

\[\tilde{\alpha}(\theta) = q(0, \theta) = -u_z(\gamma(\theta)) \]

\[= \Phi_{1,2}^\frac{1}{2} (\gamma(\theta)) + \varepsilon^2 \psi_1(\gamma(\theta)) + O(\varepsilon^4) \]

\[= q(0, \theta) + \varepsilon^2 \psi_1(\gamma(\theta)) + O(\varepsilon^4) \]

\[= \alpha(\theta) + \varepsilon^2 \psi_1(\gamma(\theta)) + O(\varepsilon^4). \]

Now the convergence in (3.2) holds in \(C^2_{\text{loc}}\). Hence we have

\[\tilde{\alpha}'(\theta) = \frac{\partial}{\partial \theta} q(0, \theta) = \frac{\partial}{\partial \theta} \left( -u_z(\gamma(\theta)) \right) \]

\[= \frac{\partial}{\partial \theta} \left( \Phi_{1,2}^\frac{1}{2} (\gamma(\theta)) \right) + \varepsilon^2 \frac{\partial}{\partial \theta} \left( \psi_1(\gamma(\theta)) \right) + O(\varepsilon^4) \]

\[= \frac{\partial}{\partial \theta} \left( q(0, \theta) \right) + \varepsilon^2 \frac{\partial}{\partial \theta} \left( \psi_1(\gamma(\theta)) \right) + O(\varepsilon^4) \]

\[= \alpha'(\theta) + O(\varepsilon^2). \]

Similarly we can show

\[\tilde{\alpha}''(\theta) = \alpha''(\theta) + O(\varepsilon^2). \]
Now for the next term we shall again use the $C^2_{loc}$-convergence. First note that

$$
\tilde{\beta}(\theta) = |q(0, \theta)|^{\pm 1} = |\frac{\partial}{\partial \theta} \gamma(\theta)|^{\pm 1} 
= \left| \Phi^\beta (\gamma(\theta)) + \varepsilon^2 \psi_1(\gamma(\theta)) + O(\varepsilon^4) \right|^{\pm 1} 
= |\alpha(\theta) + \varepsilon^2 \psi_1(\gamma(\theta)) + O(\varepsilon^4)|^{\pm 1} 
= \alpha(\theta) \times (\pm 1) + \varepsilon^2 \frac{\partial}{\partial \theta} \psi_1(\gamma(\theta)) + O(\varepsilon^4) 
= q(0, \theta) \times (\pm 1) + O(\varepsilon^2) 
= \beta(\theta) + O(\varepsilon^2).
$$

Now for the first derivative we have

$$
\tilde{\beta}'(\theta) 
= \frac{\partial}{\partial \theta} |q(0, \theta)|^{\pm 1} = \frac{\partial}{\partial \theta} \left| -u_x(\gamma(\theta)) \right|^{\pm 1} 
= \frac{v_x}{\varepsilon^2} \left| -u_x(\gamma(\theta)) \right|^{\pm 2} \frac{\partial}{\partial \theta} (-u_x(\gamma(\theta))) 
= \frac{v_x}{\varepsilon^2} \left| \Phi^\beta (\gamma(\theta)) + \varepsilon^2 \psi_1(\gamma(\theta)) + O(\varepsilon^4) \right|^{\pm 3} \left( \frac{\partial}{\partial \theta} \Phi^\beta (\gamma(\theta)) + \varepsilon^2 \frac{\partial}{\partial \theta} \psi_1(\gamma(\theta)) + O(\varepsilon^4) \right) 
= \frac{v_x}{\varepsilon^2} \left| \Phi^\beta (\gamma(\theta)) \right|^{\pm 3} \frac{\partial}{\partial \theta} \Phi^\beta (\gamma(\theta)) + O(\varepsilon^2) 
= \frac{\partial}{\partial \theta} \alpha(\theta) \times (\pm 1) + O(\varepsilon^2) 
= \frac{\partial}{\partial \theta} q(0, \theta) \times (\pm 1) + O(\varepsilon^2) 
= \beta'(\theta) + O(\varepsilon^2).\n$$

Similarly we can show that

$$\tilde{\beta}''(\theta) = \beta''(\theta) + O(\varepsilon^2).$$

Fix a twice differentiable, $\ell$ periodic function $f(\theta + \ell) = f(\theta)$. We define $v(x, z)$ by

$$u(s, z) = \tilde{\alpha}(\varepsilon v(x, z), x = \tilde{\beta}(\varepsilon z)(s - f(\varepsilon z)), \quad (3.15)$$

and we obtain

$$u_s = \tilde{\alpha} \tilde{\beta} v_x, \quad u_{ss} = \tilde{\alpha} \tilde{\beta}^2 v_{xx},$$
$$u_z = \varepsilon \tilde{\alpha}' v + \tilde{\alpha} v_x + \tilde{\alpha} v_x (\tilde{\beta}(s - f)), z,$$
$$u_{zz} = \varepsilon^2 \tilde{\alpha}'' v + 2 \varepsilon \tilde{\alpha}' (v_x (\tilde{\beta}(s - f))_z + v_z)
+ \tilde{\alpha} [v_{xx} (\tilde{\beta}(s - f))_z^2 + 2 v_{xz} (\tilde{\beta}(s - f))_z + v_x (\tilde{\beta}(s - f))_{zz} + v_{zz}],$$

and

$$(\tilde{\beta}(s - f))_z = \varepsilon (\tilde{\beta}'(s - f) - \tilde{\beta}'')f', \quad (\tilde{\beta}(s - f))_{zz} = \varepsilon^2 (\tilde{\beta}''(s - f) - 2 \tilde{\beta}' f' - \tilde{\beta}''')f''.$$ Expanding

$$q(\varepsilon s, \varepsilon z) = q(0, \varepsilon z) + q_t(0, \varepsilon z)\varepsilon s + \frac{1}{2} q_{tt}(0, \varepsilon z)\varepsilon^2 s^2 + O(\varepsilon^3 s^3). \quad (3.16)$$
Then the left hand side of the equation (3.8) takes the form
\[
- \tilde{\alpha}^2 v_{xx} - \varepsilon^2 \tilde{\alpha}' v - 2\varepsilon\tilde{\alpha}'[v_x(\tilde{\beta}(s-f))_x + v_x] - \tilde{\alpha}[v_{xx}(\tilde{\beta}(s-f))_x^2 \\
+ 2v_x(\tilde{\beta}(s-f))_x + v_x(\tilde{\beta}(s-f))_x v_{zz} - |u - \tilde{\alpha} + \tilde{\alpha} - q|^p + |q|^p \\
- (\varepsilon k - \varepsilon^2 s k^2)\tilde{\alpha} v_x - B_0(u) \\
= - \tilde{\alpha}^2 v_{xx} + \tilde{\alpha}^2 v_{zz} + (\varepsilon k - \varepsilon^2 s k^2)\tilde{\alpha}^2 v_x + \varepsilon^2 \tilde{\alpha}^2 \tilde{\alpha}' v \\
+ \left(2\varepsilon \tilde{\alpha} \tilde{\alpha}'(\tilde{\beta}(s-f) + \varepsilon p(\varepsilon p(\tilde{\beta}(s-f))_x v_x \\
+ 2\varepsilon \tilde{\alpha} \tilde{\alpha}' v_x + \tilde{\alpha}^2(\tilde{\beta}(s-f))_x^2 v_{xx} + [2\tilde{\alpha}^2(\tilde{\beta}(s-f))_x v_{xz} \\
+ \tilde{\alpha}^2\left[|u - \tilde{\alpha} + \tilde{\alpha} - q|^p - |q|^p\right] + \tilde{\alpha}^2 B_0(u) \right).
\]

Now we have
\[
|u - \tilde{\alpha} + \tilde{\alpha} - q|^p = |\tilde{\alpha}(1 - v) + (q - \tilde{\alpha})|^p \\
= \tilde{\alpha}^p|1 - v|^p + \tilde{\alpha}^{p-1}|1 - v|^{p-2}(1 - v)(q - \tilde{\alpha}) \\
+ \frac{p(p-1)}{2}\tilde{\alpha}^{p-2}|1 - v|^{p-2}(q - \tilde{\alpha})^2 + O(\varepsilon^3) \\
= \tilde{\alpha}^p\left[|1 - v|^p + p\tilde{\alpha}^{-1}|1 - v|^{p-2}(1 - v)(q\varepsilon s) + \frac{1}{2}q_t\varepsilon^2 s^2 \right] \\
+ \frac{p(p-1)}{2}\tilde{\alpha}^{-2}|1 - v|^{p-2}\varepsilon^2 s^2 q_t^2 + O(\varepsilon^3) \right] \\
= \tilde{\alpha}^p\left[|v - 1|^p + p\tilde{\alpha}^{-1}q_{tt}|v - 1|^{p-2}(v - 1)\varepsilon s - \frac{1}{2}p\tilde{\alpha}^{-1}q_{tt}|v - 1|^{p-2}(v - 1)\varepsilon^2 s^2 \\
+ \frac{p(p-1)}{2}\tilde{\alpha}^{-2}|v - 1|^{p-2}\varepsilon^2 s^2 q_t^2 + O(\varepsilon^3) \right].
\]

Also
\[
|q|^p = |\tilde{\alpha} + \varepsilon s q_t + \frac{\varepsilon^2 s^2}{2} q_{tt} + O(\varepsilon^3 s^3)|^p \\
= \tilde{\alpha}^p\left[1 + p\tilde{\alpha}^{-1}q_{tt}|v - 1|^{p-2}(v - 1)\varepsilon s \\
+ \frac{1}{2}p\tilde{\alpha}^{-1}q_{tt}|v - 1|^{p-2}(v - 1)\varepsilon^2 s^2 \\
+ \frac{p(p-1)}{2}\tilde{\alpha}^{-2}|v - 1|^{p-2}\varepsilon^2 s^2 q_t^2 + O(\varepsilon^3) \right].
\]

So we have
\[
|u - \tilde{\alpha} + \tilde{\alpha} - q|^p - |q|^p = \tilde{\alpha}^p\left[|v - 1|^p - 1 + p\tilde{\alpha}^{-1}q_{tt}|v - 1|^{p-2}(v - 1)\varepsilon s \\
+ \frac{1}{2}p\tilde{\alpha}^{-1}q_{tt}|v - 1|^{p-2}(v - 1)\varepsilon^2 s^2 \\
+ \frac{p(p-1)}{2}\tilde{\alpha}^{-2}|v - 1|^{p-2}\varepsilon^2 s^2 q_t^2 + O(\varepsilon^3) \right].
\]

Note that \( s - f = x/\tilde{\beta}, \ u - \tilde{\alpha} = \tilde{\alpha}(v - 1) \) and using the above expansions we have the equation satisfied by \( v \) in the new coordinates
\[
v_{xx} + \tilde{\alpha}^2 v_{zz} + |v - 1|^p - 1 + B_3(v) = 0, \tag{3.17}
\]

where
\[
B_3(v) = \tilde{\alpha}^2 (\varepsilon x - \tilde{\alpha}^2 f)^2 v_x \\
+ \varepsilon^2 \tilde{\alpha}^2 \tilde{\alpha}' v + \left(2\varepsilon \tilde{\alpha}^2 \tilde{\alpha}'(\tilde{\beta} f') + \tilde{\alpha}^2 \varepsilon^2 (\tilde{\beta} f')' \right) v_x
\]
\[ +2\varepsilon\tilde{\alpha}^{-p}\tilde{\alpha}'v_x + \varepsilon^2\tilde{\alpha}^{-1-p}\left(\frac{\beta'}{\beta}x - \tilde{\beta}f\right)^2v_{xx} + 2\varepsilon\tilde{\alpha}^{-1-p}\left(\frac{\beta'}{\beta}x - \tilde{\beta}f\right)v_{xz} \]
\[ +\tilde{\alpha}^{-1}p(|1 - v|^{p-2}(1 - v) - 1)q_t\varepsilon\left(\frac{x}{\beta} + f\right) \]
\[ +\frac{1}{2}(\tilde{\alpha}^{-1}p(|1 - v|^{p-2}(1 - v) - 1)q_{tt} \]
\[ +\tilde{\alpha}^{-2}p(p - 1)(|1 - v|^{p-2} - 1)q_t^2\varepsilon^2\left(\frac{x}{\beta} + f\right)^2 + B_2(v), \]
and
\[ B_2(v) = \tilde{\alpha}^{-p}B_0(u) + O(\varepsilon^3\left(\frac{x}{\beta} + f\right)^3). \] (3.18)

Hence \( u \) solves (3.8), if and only if \( v \) solves
\[ S(v) = v_{xx} + \tilde{\alpha}^{-1-p}v_{zz} + |v - 1|^{p\alpha} - 1 + B_2(v) = 0. \] (3.19)

Let \( w(x) \) denotes the unique positive solution of the following equation
\[ -w''(x) = |1-w|^{p-1}, \quad w(0) = \max_{x \in \mathbb{R}} w(x); \quad w \in H^1(\mathbb{R}). \] (3.20)

Then \( w \) decays exponentially at infinity as \( e^{-\sqrt{E}|x|} \). Taking \( w(x) \) as the first approximate solution the error becomes
\[ S(w) = B_3(w) \]
\[ = \tilde{\alpha}^{-1-p}\beta\left(\varepsilon k - \varepsilon^2\left(\frac{x}{\beta} + f\right)k^2\right)w_x \]
\[ +\left(2\varepsilon^2\tilde{\alpha}^{-1}\tilde{\alpha}'\left(\frac{\beta'}{\beta}x - \tilde{\beta}f\right) + \tilde{\alpha}^{-1-p}\varepsilon^2\left(\frac{\beta''}{\beta}x - 2\tilde{\beta}'f' - \tilde{\beta}f''\right)\right)w_x \]
\[ +\varepsilon^2\tilde{\alpha}^{-1-p}\left(\frac{\beta'}{\beta}x - \tilde{\beta}f\right)^2w_{xx} + \varepsilon^2\tilde{\alpha}^{-1-p}\tilde{\alpha}''w \]
\[ +\tilde{\alpha}^{-1}p(|1 - w|^{p-2}(1 - w) - 1)q_t\varepsilon\left(\frac{x}{\beta} + f\right) \]
\[ +\frac{1}{2}(\tilde{\alpha}^{-1}p(|1 - w|^{p-2}(1 - w) - 1)q_{tt} + \tilde{\alpha}^{-2}p(p - 1)(|1 - w|^{p-2} - 1)q_t^2)\varepsilon^2\left(\frac{x}{\beta} + f\right)^2 \]
\[ + B_2(w), \]
where \( B_2(w) \) turn out to be of \( O(\varepsilon^3) \). Gathering terms of order \( \varepsilon \) and \( \varepsilon^2 \), we get
\[ S(w) = \varepsilon\tilde{\alpha}^{-1}p(|1 - w|^{p-2}(1 - w) - 1)q_t f \]
\[ +\varepsilon\left[\tilde{\alpha}^{-1-p}\tilde{\alpha}'kw_x + \tilde{\alpha}^{-1}p(|1 - w|^{p-2}(1 - w) - 1)q_t\varepsilon\left(\frac{x}{\beta}^2\right)\right] \]
\[ +\varepsilon^2\left[ -\tilde{\alpha}^{-1-p}\tilde{\alpha}'\varepsilon^2k^2w_x + (2\tilde{\alpha}^{-p}\tilde{\alpha}'\frac{\beta}{\beta}x + \tilde{\alpha}^{-1-p}\frac{\beta''}{\beta}x)w_x \right. \]
\[ +\tilde{\alpha}^{-1-p}\frac{\beta'}{\beta^2}x^2 + \tilde{\beta}^2f'^2w_{xx} + \frac{1}{2}(\tilde{\alpha}^{-1}p(|1 - w|^{p-2}(1 - w) - 1)q_{tt} \]
\[ \left. +\tilde{\alpha}^{-2}p(p - 1)(|1 - w|^{p-2} - 1)q_t^2\right)\varepsilon^2\left(\frac{x^2}{\beta^2} + f^2\right) \]
Moreover, using the expansion (1.3) we have

\[ S(w) = \varepsilon \alpha^{-1} p((1 - w)|w|^{p-2} - 1)q_t f \\
+ \varepsilon \left[ \alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t \frac{x}{\beta} \right] \\
+ \varepsilon^2 \left[ - \alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t \left( \frac{x^2}{\beta} + f^2 \right) \right] \\
+ \frac{1}{2} \left( \alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t + \alpha^{-2} p(p - 1)((1 - w)|w|^{p-2} - 1)q_t^2 \right) \left( \frac{x^2}{\beta^2} + f^2 \right) \\
+ B_2(w) \\
= \varepsilon S_1 + \varepsilon S_2 + \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_2(w) + G. \]

where

\[ S_1 = \alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t f, \]
\[ S_2 = \alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t \frac{x}{\beta}, \]
\[ S_3 = -\alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t \frac{x}{\beta} \]
\[ + \alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t \]
\[ + \alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t \]
\[ + \alpha^{-2} p(p - 1)((1 - w)|w|^{p-2} - 1)q_t^2 \left( \frac{x^2}{\beta^2} + f^2 \right), \]
\[ S_4 = -\alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t \]
\[ - \alpha^{-1} p((1 - w)|w|^{p-2}(1 - w) - 1)q_t \]
\[ + \alpha^{-2} p(p - 1)((1 - w)|w|^{p-2} - 1)q_t^2 \frac{x^2}{\beta}. \]

Note that \( S_1, S_3 \) are even, and \( S_2, S_4 \) are odd functions of \( x \). Note that \( G \) is a function of order \( \varepsilon^3 \). We want to construct a further approximate solution which
eliminates the term of order $\varepsilon$ in the error. We obtain
\[
S(w + \varphi) = S(w) + \varphi_{xx} + \alpha^{1-p} \varphi_{zz} + |w + \varphi - 1|^p - |w - 1|^p + B_3(w + \varphi) - B_3(w),
\]
where
\[
L_0(\varphi) = \varphi_{xx} + \alpha^{1-p} \varphi_{zz} + p|w - 1|^{p-2}(w - 1)\varphi,
\]
\[
N_0(\varphi) = |w + \varphi - 1|^p - |w - 1|^p - p|w - 1|^{p-2}(w - 1)\varphi,
\]
and
\[
B_3(w, \varphi) = B_3(w + \varphi) - B_3(w).
\]
We choose $\varphi_1$ to be the solution of the following equation:
\[
\varphi_{1,xx} + p\varphi_{xx} + \alpha_1 = -\varepsilon(S_1 + S_2), \quad \varphi_1(\pm\infty) = 0.
\]
It is well known that it is uniquely solvable if
\[
\int_{-\infty}^{+\infty} (S_1 + S_2)w_x dx = 0,
\]
and $\int_{-\infty}^{+\infty} \varphi_1 w_x dx = 0$. Furthermore,
\[
\int_{-\infty}^{+\infty} (S_1 + S_2)w_x dx = \int_{-\infty}^{+\infty} S_2w_x dx
\]
\[
= k\beta^{1-p} \int_{-\infty}^{+\infty} w_x^2 dx - \frac{p\alpha^{1-q}q}{\beta} \int_{-\infty}^{+\infty} (|1 - w|^{p-2}(1 - w) - 1)w_x dx,
\]
since $w$ satisfies
\[
w'' + |1 - w|^{p-1} = 0,
\]
using integration by parts (multiply by $w$ and $xw_x$), we have
\[
\int w_x^2 dx = \int (|w - 1|^p - 1)wdx,
\]
\[
\frac{1}{2} \int w_x^2 dx = \frac{1}{p+1} \int \frac{\partial}{\partial x} (|w - 1|^p (1 - w) - 1)dx + \int wdx
\]
\[
= \frac{1}{p+1} \int (|w - 1|^p(1 - w) - 1)dx + \int wdx,
\]
we obtain
\[
\int wdx = \frac{p+3}{2p} \int w_x^2 dx.
\]
Since
\[
\int (|w - 1|^{p-2}(1 - w) - 1)w_x dx = \frac{1}{p} \int \frac{\partial}{\partial x} (|1 - w|^p - 1)x dx - \int xw_x dx
\]
\[
= \int wdx,
\]
we obtain
\[
\int S_2w_x dx = \left( k_{\beta}^{1-p} + \frac{p\alpha^{1-q}q}{\beta} \frac{p+3}{2p} \right) \int w_x^2 dx
\]
\[
= \left( k + \frac{p+3}{2p} \Phi_1^{-1}\Phi_1' \right) \frac{1}{2p} \int w_x^2 dx.
\]
Now the new error becomes
\[(3.28)\] and the assumption that \(\Gamma\) is stationary amounts to saying that \((3.28)\) is zero. Hence \((k + \frac{p^2 + p}{2p} \Phi_1 \Phi'_1) = 0\). Let the solution of \((3.24)\) has the form
\[
\varphi = \varphi_{11} + \varphi_{12},
\]
where
\[
\varphi_{11} = -\varepsilon p(\alpha^{-1} f\varphi_1 w_1), \varphi_{12} = -\varepsilon w_2.
\]
Moreover, \(w_1\) is the even function in \(x\):
\[
w_{1,xx} + p|1 - w|^{p-2}(w - 1)w_1 = |1 - w|^{p-2}(1 - w) - 1,
\]
and unique if \(\int R w_1 w_x = 0\). And \(w_2\) is an odd function in \(x\):
\[
w_{2,xx} + p|1 - w|^{p-2}(w - 1)w_2 = \alpha^{1-p} k \beta w_x + \alpha^{-1} p q t (|w - 1|^{p-2}(1 - w) - 1) \frac{\varepsilon}{\beta}.
\]
In fact
\[
w_1 = \frac{1 - p}{2p} x w_x - \frac{1}{p} w.
\]
Now the new error becomes
\[
S(w + \varphi_1) = \varepsilon^2 (S_3 + S_4) + B_2(w) + \alpha^{1-p} \varphi_{1,zz} + N_0(\varphi_1) + B_3(w, \varphi_1).
\]
Let \(Z(x)\) be the first eigenfunction of the problem
\[
Z'' + p|w - 1|^{p-2}(w - 1)Z = \lambda_0 Z; Z(\pm \infty) = 0.
\]
Then we find that
\[
0 = \int Z'' w_x + \int \frac{\partial}{\partial x} |w - 1|^p Z - \lambda_0 \int Z w_x dx
\]
\[
= - \int (w_{xx} + |w - 1|^p) Z' - \lambda_0 \int Z w_x dx
\]
\[
= - \int Z' - \lambda_0 \int Z w_x dx
\]
\[
= - \lambda_0 \int Z w_x dx.
\]
We now consider our basic approximation near the curve \(\Gamma_\varepsilon\) as
\[
w = w + \varphi_1 + \varepsilon(\varepsilon z) Z.
\]
The new error is
\[
E_1 = S(w)
\]
\[
= S(w + \varphi_1) + \varepsilon L_0(\varepsilon Z) + B_3(w + \varphi_1, \varepsilon\varepsilon Z) + p|w + \varphi_1 - 1|^{p-2}(w + \varphi_1 - 1) - |w - 1|^{p-2}(w - 1))\varepsilon z
\]
\[
+ \sum |w + \varphi_1 + \varepsilon\varepsilon Z - 1|^{p-2} - |w + \varphi_1 - 1|^{p-2} - p|w + \varphi_1 - 1|^{p-2}(w + \varphi_1 - 1)\varepsilon z|.
\]
and we have
\[
L_0(\varepsilon Z) = (\varepsilon Z)_{xx} + \alpha^{1-p}(\varepsilon Z)_{zz} + p|w - 1|^{p-2}(w - 1)\varepsilon Z
\]
\[
= (\lambda_0 + \varepsilon^2 \alpha^{1-p} e'' Z).
\]
Now we decompose \(E_1 = E_{11} + E_{12}\) where
\[
E_{11} = \varepsilon(\lambda_0 + \varepsilon^2 \alpha^{1-p} e'' Z)
\]
and
\[
E_{12} = E_1 - E_{11}.
\]
4. The gluing procedure. Let \( w(y) \) denote the first approximation constructed near the curve in the coordinate \( y \) in \( \mathbb{R}^2 \). Let \( \delta < \frac{\rho_0}{100} \) be a fixed number. We consider a smooth cut-off function \( \eta_\delta(t) \) where \( t \in \mathbb{R} \) such that \( \eta_\delta(t) = 1 \) if \( t < \delta \) and \( = 0 \) if \( t > 2\delta \). Denote as well \( \eta_\delta(s) = \eta_\delta(\varepsilon|s|) \), where \( s \) is the normal coordinate to \( \Gamma_\varepsilon \). We define our first global approximation to be simply

\[
\tilde{w}(y) = \eta_{\delta\varepsilon}(t) w
\]

extended globally as 0 beyond the \( \frac{6\delta}{\varepsilon} \)-neighborhood of \( \Gamma_\varepsilon \). Denote

\[
S(u) = \Delta u + |u + u_\varepsilon(\varepsilon y)|^p - |u_\varepsilon(\varepsilon y)|^p
\]

for \( u = \tilde{w} + \tilde{\phi} \), where \( \tilde{\phi} \) is globally defined in \( \mathbb{R}^2 \). Then \( S(\tilde{w} + \tilde{\phi}) = 0 \) if and only if

\[
-\tilde{L}(\tilde{\phi}) = \tilde{E} + \tilde{N}(\tilde{\phi}),
\]

where

\[
\tilde{L}(\tilde{\phi}) = \Delta \tilde{\phi} + p|\tilde{w} - 1|^{p-2}(\tilde{w} - 1)\tilde{\phi},
\]

\[
\tilde{E} = S(\tilde{w})
\]

\[
\tilde{N}(\tilde{\phi}) = |\tilde{w} + \tilde{\phi} - q(\varepsilon s, \varepsilon z)|^p - |\tilde{w} - q(\varepsilon s, \varepsilon z)|^p - p|\tilde{w} - 1|^{p-2}(\tilde{w} - 1)\tilde{\phi}.
\]

We plan to decompose \( \tilde{\phi} \) in the following way

\[
\tilde{\phi} = \eta_{\delta\varepsilon}\phi + \psi
\]

where

\[
-\tilde{L}(\eta_{\delta\varepsilon}\phi) - \tilde{L}(\psi) = \tilde{E} + \tilde{N}(\eta_{\delta\varepsilon}\phi + \psi).
\]

That is

\[
\begin{align*}
&\left[-\eta_{\delta\varepsilon}\Delta \phi - \eta_{\delta\varepsilon}p|\tilde{w} - 1|^{p-2}(\tilde{w} - 1)\phi - \eta_{\delta\varepsilon}\tilde{N}(\eta_{\delta\varepsilon}\phi + \psi)\right]\\
&\quad - \eta_{\delta\varepsilon}^2 \tilde{E} - pm_{\varepsilon}^2 |\tilde{w} - 1|^{p-2}(\tilde{w} - 1)\psi\right] + \left[-\Delta \psi - p(1 - \eta_{\delta\varepsilon})|\tilde{w} - 1|^{p-2}(\tilde{w} - 1)\psi\right]\\
&\quad - (1 - \eta_{\delta\varepsilon})\tilde{N}(\eta_{\delta\varepsilon}\phi + \psi) - (1 - \eta_{\delta\varepsilon})\tilde{E} - 2\varepsilon \nabla \eta_{\delta\varepsilon} \nabla \phi - \varepsilon^2 \Delta \eta_{\delta\varepsilon} \phi = 0.
\end{align*}
\]
Hence we obtain a pair \((\psi, \phi)\) which satisfies the nonlinear coupled system:

\[-\dot{L}(\phi) = \eta_0^s \tilde{N}(\psi + \phi) + \eta_0^l \tilde{E} + p\eta_0^l |\tilde{w} - 1|^p - 2(\tilde{w} - 1)\psi \quad (4.6)\]

and

\[-\Delta \psi - p(1 - \eta_0^l) |\tilde{w} - 1|^p - 2(\tilde{w} - 1)\psi \]

\[= (1 - \eta_0^l) \tilde{E} + 2\varepsilon \nabla\eta_0^l \nabla \phi + \varepsilon^2 \Delta \eta_0^l \phi + (1 - \eta_0^l) \tilde{N}(\eta_0^l \phi + \psi). \quad (4.7)\]

Note that the operator \(\tilde{L}\) in the strip \(S\) may be taken as any compatible extension outside \(\frac{\delta}{\varepsilon}\) neighborhood of the curve.

Now we want to reduce the problem to a problem on the strip \(S\). To do this we solve, given a small \(\phi\) we solve for \(\psi\). Now \(\tilde{w}\) is exponentially decaying whenever \(|s| > \frac{\delta}{\varepsilon}\), where \(s\) is a normal coordinate of \(\Gamma_{\varepsilon}\), then the problem

\[\Delta \psi + p(1 - \eta_0^l) |\tilde{w} - 1|^p - 2(\tilde{w} - 1)\psi = h \quad (4.8)\]

has a unique bounded solution \(\psi\) whenever \(\|h\|_\infty < +\infty\). Moreover, by the Schauder estimates

\[\|\psi\|_\infty \leq C\|h\|_\infty.\]

Now we assume that \(\phi\) satisfies the estimate

\[|\nabla \phi(y)| + |\phi(y)| \leq e^{-p\frac{\gamma}{2}} \quad (4.9)\]

for \(|s| > \frac{\delta}{\varepsilon}\), for some constant \(\gamma > 0\). Since the remainder \(\tilde{N}\) has terms involving \(p > 1\), the direct application of fixed point theorem yields a unique solution

\[\|\psi(\phi)\|_\infty \leq C \varepsilon \left[\|\phi\|_{L^\infty(\Gamma_{\varepsilon} \cap |s| > \frac{\delta}{\varepsilon})} + \|\nabla \phi\|_{L^\infty(\Gamma_{\varepsilon} \cap |s| > \frac{\delta}{\varepsilon})}\right]\]

and the nonlinear operator satisfies the Lipschitz condition of the form

\[\|\psi(\phi_1) - \psi(\phi_2)\|_\infty \leq C \varepsilon \left[\|\phi_1 - \phi_2\|_{L^\infty(\Gamma_{\varepsilon} \cap |s| > \frac{\delta}{\varepsilon})} + \|\nabla \phi_1 - \nabla \phi_2\|_{L^\infty(\Gamma_{\varepsilon} \cap |s| > \frac{\delta}{\varepsilon})}\right].\]

Hence we are reduced to the problem on the strip \(S\)

\[L_2(\phi) = \eta_0^s \tilde{N}(\psi(\phi) + \phi) + \eta_0^l \tilde{E} + p\eta_0^l |\tilde{w} - 1|^p - 2(\tilde{w} - 1)\psi(\phi) \quad (4.10)\]

for a \(\phi \in H^2(S)\) satisfying (4.9). By \(L_2\) we mean the operator which coincides with \(\tilde{L}\) in the region \(|s| < \frac{10\delta}{\varepsilon}\).

We shall define this operator next. The operator \(\hat{L}\) for \(|s| < \frac{10\delta}{\varepsilon}\) is given in coordinates \((x, z)\) by formula (3.40). We extend it for functions \(\phi\) defined in the entire strip \(S\), in terms of \((x, z)\), as follows:

\[L_2(\phi) = L_1(\phi) + \chi(\varepsilon|x|)B_3(\phi) \quad (4.11)\]

where \(\chi(r)\) is a smooth function function with \(\chi(r) = 1\) if \(r < 10\delta\) and zero outside \(r > 20\delta\). Here \(L_1\) is the operator defined by (3.40).

Now we consider the projected problem in \(H^2(S)\) : given \(f\) and \(e\) satisfying \(\phi \in H^2(S)\) \(c, d \in L^2(S)\) such that

\[L_2(\phi) = \chi E_1 + N_2(\phi) + e(\varepsilon) \chi w_x + d(\varepsilon) \chi Z \quad (4.12)\]

\[\phi(x, 0) = \phi \left( x, \frac{1}{\varepsilon} \right); \phi_x(x, 0) = \phi_x \left( x, \frac{1}{\varepsilon} \right) \quad (4.13)\]

\[\int_{-\infty}^{\infty} \phi(x, z)w_x(x)dx = \int_{-\infty}^{\infty} \phi(x, z)Z(x)dx = 0 \quad (4.14)\]
for \(0 < z < \frac{l}{\varepsilon}\) and

\[
N_2(\phi) = \eta_0^2 \tilde{N}(\phi + \psi(\phi)) + p|w - 1|^{p-1}(w - 1)\psi(\phi).
\]  

(4.15)

We will prove that this problem has a unique solution whose norm is controlled by the \(L^2\)-norm, not of the whole \(E_1\) but rather that of \(E_{12}\). After this has been done, our task is to adjust the parameters \(f\) and \(e\) in such a way that \(c\) and \(d\) are identically zero. As we will see, this turns out to be equivalent to solving a nonlocal, nonlinear coupled second-order system of differential equations for the pair \((c, d)\) under periodic boundary conditions. As we will see, this system is solvable in a region where the bounds (3.42) and (3.43) hold. In order to solve (4.12)-(4.14), we need to investigate the invertibility of \(L_2\) in an \(L^2 - H^2\) setting under periodic boundary and orthogonality conditions.

5. Invertibility of the operator \(L_2\). We study the following problem

\[
L_2(\phi) = h + c(\varepsilon z)\chi w_x + d(\varepsilon z)\chi Z
\]  

(5.1)

\[
\phi(x,0) = \phi \left( x, \frac{l}{\varepsilon} \right); \quad \phi_x(x,0) = \phi_x \left( x, \frac{l}{\varepsilon} \right)
\]  

(5.2)

\[
\int_{-\infty}^{\infty} \phi(x,z)w_x(x)dx = \int_{-\infty}^{\infty} \phi(x,z)Z(x)dx = 0 \text{ for } 0 < z < \frac{l}{\varepsilon}.
\]  

(5.3)

**Proposition 1.** Let \(\delta\) be as in (4.11) is chosen sufficiently small, then there exists a constant \(C > 0\), independent of \(\varepsilon\), such that for all small \(\varepsilon\) problem (5.1)-(5.3) has a unique solution \(\phi = T(h)\) that satisfies the estimate

\[
||\phi||_{H^1(S)} \leq C||h||_{L^2(S)}.
\]  

(5.4)

First we consider a simpler problem

\[
L(\phi) = -\Delta \phi - p|w - 1|^{p-2}(w - 1)\phi = h
\]  

(5.5)

\[
\phi(x,0) = \phi \left( x, \frac{l}{\varepsilon} \right); \quad \phi_x(x,0) = \phi_x \left( x, \frac{l}{\varepsilon} \right)
\]  

(5.6)

\[
\int_{-\infty}^{\infty} \phi(x,z)w_x(x)dx = \int_{-\infty}^{\infty} \phi(x,z)Z(x)dx = 0 \text{ for } 0 < z < \frac{l}{\varepsilon}.
\]  

(5.7)

**Lemma 5.1.** Then there exists a constant \(C > 0\), independent of \(\varepsilon\), such that the problem (5.5)-(5.7) admits a solution \(\phi\) satisfying the estimate

\[
||\phi||_{H^1(S)} \leq C||h||_{L^2(S)}.
\]  

(5.8)

**Proof.** Let us consider the Fourier decomposition of \(h\) and \(\phi\) of the form

\[
\phi(x,z) = \sum_{k=0}^{\infty} \left[ \phi_{1k}(x) \cos \left( \frac{2\pi k}{l} \varepsilon z \right) + \phi_{2k}(x) \sin \left( \frac{2\pi k}{l} \varepsilon z \right) \right],
\]  

(5.9)

\[
h(x,z) = \sum_{k=0}^{\infty} \left[ h_{1k}(x) \cos \left( \frac{2\pi k}{l} \varepsilon z \right) + h_{2k}(x) \sin \left( \frac{2\pi k}{l} \varepsilon z \right) \right].
\]  

(5.10)

Then for \(i = 1, 2\) we obtain

\[
\frac{4\pi^2}{l^2} k^2 \varepsilon^2 \phi_{ik} + L_0(\phi_{ik}) = h_{ik}
\]  

(5.11)

where

\[
L_0(\phi_{ik}) = -\phi_{ik,xx} - p|w - 1|^{p-2}(w - 1)\phi_{ik}.
\]  

(5.12)
with the orthogonality condition as
\[ \int_{-\infty}^{\infty} \phi_{ik} w_x(x) dx = \int_{-\infty}^{\infty} \phi_{ik} Z(x) dx = 0. \tag{5.13} \]

Now we consider the bilinear form
\[ B(\phi, \psi) = \int_{-\infty}^{\infty} [(\phi')^2 + p|w - 1|^p - 2(1 - w)|\phi|^2] dx. \tag{5.14} \]

Moreover, as (5.13) holds we must have,
\[ C[\|\phi_{ik}\|_{L^2(\mathbb{R})}^2 + \|\phi_{ik}\|_{L^2(\mathbb{R})}^2] \leq B(\phi_{ik}, \phi_{ik}). \tag{5.15} \]
for some constant independent of \( i \) and \( k \). Furthermore, from (5.11)
\[ C\left[ 1 + \frac{16\pi^4 \varepsilon^4 k^4}{l^4} \right] \|\phi_{ik}\|_{L^2(\mathbb{R})}^2 + \|\phi_{ik}\|_{L^2(\mathbb{R})}^2 \leq \|\phi_{ik}\|_{L^2(\mathbb{R})}^2. \tag{5.16} \]
In particular, we have from (5.11), \( \phi_{ik} \) satisfies
\[ - \phi_{ik,xx} + \phi_{ik} = \tilde{h}_{ik} \tag{5.17} \]
where \( \|\tilde{h}_{ik}\|_{L^2(\mathbb{R})} \leq \|h_{ik}\|_{L^2(\mathbb{R})} \). Hence it follows that
\[ \|\phi_{ik}\|_{L^2(\mathbb{R})} \leq C\|h_{ik}\|_{L^2(\mathbb{R})}. \tag{5.18} \]
Hence we obtain,
\[ \|\nabla^2 \phi\|_{L^2(S)} + \|\nabla \phi\|_{L^2(S)} + \|\phi\|_{L^2(S)} \leq C\|h\|_{L^2(S)}. \tag{5.19} \]

Now we consider the following problem of \( \phi \in H^2(S) \)
\[ L(\phi) = h + c(\varepsilon z) w_x + d(\varepsilon z) Z \tag{5.20} \]
\[ \phi(x, 0) = \phi \left( x, \frac{l}{\varepsilon} \right); \phi_z(x, 0) = \phi_z \left( x, \frac{l}{\varepsilon} \right) \tag{5.21} \]
\[ \int_{-\infty}^{\infty} \phi(x, z) w_x(x) dx = \int_{-\infty}^{\infty} \phi(x, z) Z(x) dx = 0 \text{ for } 0 < z < \frac{l}{\varepsilon} \tag{5.22} \]
where \( h \in L^2(S) \) and \( c, d \in L^2(0, l) \).

**Lemma 5.2.** Then there exists a constant \( C > 0 \), independent of \( \varepsilon \), such that the problem (5.20)-(5.22) admits a unique solution \( \phi \) satisfying the estimate
\[ \|\phi\|_{H^2(S)} \leq C\|h\|_{L^2(S)}. \tag{5.23} \]

**Proof.** Let
\[ h(x, z) = \sum_{k=0}^{\infty} \left[ h_{1k}(x) \cos \left( \frac{2\pi k}{l} \varepsilon z \right) + h_{2k}(x) \sin \left( \frac{2\pi k}{l} \varepsilon z \right) \right]. \tag{5.24} \]

We consider the problem
\[ 4\pi^2 \varepsilon^2 k^2 \phi_{ik} + L_0(\phi_{ik}) = h_{ik} + c_{ik} w_x + d_{ik} Z \tag{5.25} \]
with (5.13). Then using the integration by parts, we obtain the problem is solvable if
\[ c_{ik} = -\frac{\int_{-\infty}^{+\infty} h_{ik} w_x dx}{\int_{-\infty}^{+\infty} w_x^2 dx}; \quad d_{ik} = -\frac{\int_{-\infty}^{+\infty} h_{ik} Z dx}{\int_{-\infty}^{+\infty} Z^2 dx}. \tag{5.26} \]
In particular, by the Parseval’s identity we have
\[ \sum_{k=0}^{\infty} |c_{ik}|^2 + |d_{ik}|^2 \leq C \varepsilon \|h\|_{L^2(S)}^2. \tag{5.27} \]

Now we define
\[ \phi(x, z) = \sum_{k=0}^{\infty} \left[ \phi_{1k}(x) \cos \left( \frac{2\pi k}{l} z \right) + \phi_{2k}(x) \sin \left( \frac{2\pi k}{l} z \right) \right] \tag{5.28} \]
and
\[ c(z) = \sum_{k=0}^{\infty} \left[ c_{1k} \cos \left( \frac{2\pi k}{l} z \right) + c_{2k} \sin \left( \frac{2\pi k}{l} z \right) \right], \tag{5.29} \]
\[ d(z) = \sum_{k=0}^{\infty} \left[ d_{1k} \cos \left( \frac{2\pi k}{l} z \right) + d_{2k} \sin \left( \frac{2\pi k}{l} z \right) \right]. \tag{5.30} \]

Estimate (5.27) gives that \( c(\varepsilon z)w_x \) and \( d(\varepsilon z)Z \) have their \( L^2(S) \) norms controlled by that of \( h \). Thus by the priori estimates of the previous lemma the series for \( \phi \) is convergent in \( H^2(S) \) and defines a unique solution for the problem with the desired bounds.

\[ \square \]

**Proof of Proposition 1.** We will reduce problem (5.1)–(5.3) to a small perturbation of a problem of the form (5.5)–(5.7) in which Lemma 5.1 is applicable. This will be done by introducing a change of variables that eliminates the weight \( \alpha^{1-\eta} \) in front of \( \phi_{zz} \). Let
\[ \phi(x, z) = \varphi(x, a(z)), a(z) = \int_0^{\varepsilon z} \alpha^{\frac{1-\eta}{2}}(r)dr. \tag{5.31} \]

The map \( a : [0, \frac{l}{\varepsilon}] \to [0, \frac{l}{\varepsilon}] \) is a diffeomorphism, where \( \tilde{l} = \int_0^{l} \beta(r)dr \) we denote then
\[ \bar{\phi}_x = \frac{1}{2} \bar{\phi}_x; \bar{\phi}_{zz} = \alpha^{1-\eta}(\varepsilon z)\varphi_{zz} + \frac{1-\eta}{2} \varepsilon \alpha^{-\frac{\eta+1}{2}} \alpha' (\varepsilon z) \varphi_{zz} \tag{5.32} \]
while differentiation in \( x \) does not change.

Then problem (4.12)–(4.14) reduces to
\[ \Delta \varphi + p|w - 1|^{p-2}(w - 1)\varphi + \frac{1-\eta}{2} \varepsilon \alpha^{-\frac{\eta+1}{2}} \alpha'(\varepsilon z) \varphi_{zz} \]
\[ - p(|w - 1|^{p-2}(w - 1) - |w - 1|^{p-2}(w - 1))\varphi = \hat{h} \]
\[ + \tilde{c}(\varepsilon z)\chi w_x + \tilde{d}(\varepsilon z)\chi Z \text{ in } S \]
\[ \varphi(x, 0) = \varphi \left( x, \frac{l}{\varepsilon} \right); \varphi_z(x, 0) = \varphi_z \left( x, \frac{l}{\varepsilon} \right) \tag{5.34} \]
\[ \int_{-\infty}^{\infty} \varphi(x, z')w_x(x)dx = \int_{-\infty}^{\infty} \varphi(x, z')Z(x)dx = 0 \tag{5.35} \]
for \( 0 < z < \frac{l}{\varepsilon} \).

Here \( \hat{h}(x, z) = h(x, a^{-1}(z)) \) and the operator \( \hat{B}_3 \) is defined by using the above formulas to replace the \( z \)-derivatives by \( z' \)-derivatives and the variable \( z \) by \( a^{-1}(z') \) in the operator \( B_3 \). The key point is the following: the operator
\[ B_4(\varphi) = \chi \hat{B}_3(\varphi) + \varepsilon \beta' \varphi_{zz} + p(|w - 1|^{p-2}(w - 1) - |w - 1|^{p-2}(w - 1))\varphi \tag{5.36} \]
is small in the sense that
\[ \|B_4(\varphi)\|_{L^2(S)} \leq C\delta \|\varphi\|_{L^2(S)}. \]
This last estimate is a rather straightforward consequence of the fact that \( |s| < \frac{20\delta}{\varepsilon} \) wherever the operator \( \tilde{B}_3 \) is supported, and the other terms are even smaller when \( \varepsilon \) is small. Thus by reducing \( \delta \) if necessary, we apply the invertibility result of Lemma 1. The result thus follows by transforming the estimate for \( \phi \) into a similar one for \( \varphi \) via a change of variables. This concludes the proof.

6. The nonlinear projected problem. In this section we will solve the problem (4.12)–(4.14)

\[
L_2(\phi) + B_3(\phi) = \chi E_1 + N_2(\phi) + c(\varepsilon z) \chi w_x + d(\varepsilon z) \chi Z
\]

under periodic boundary and orthogonality conditions in \( S \). Here

\[
N_2(\phi) = \chi N_1(\phi + \psi(\phi)) \quad (6.1)
\]

whenever this operator is well-defined, namely, for \( B \in L^2(\Omega) \). In fact, since (4.12)–(4.14), we have

\[
\int_{\Omega} L_n(\phi) \, dx \leq C \varepsilon^{2p-1} \|
\]

wherever the operator \( \hat{\phi} \) is small. Thus by reducing \( \delta \) if necessary, we apply the invertibility result of Lemma 1. The result thus follows by transforming the estimate for \( \phi \) into a similar one for \( \varphi \) via a change of variables. This concludes the proof.

\[
E_{11} = \varepsilon \left( e^{2\alpha - p} e'' + \lambda_0 \varepsilon \right) Z.
\]

Hence (4.12) reduces to

\[
L_2(\phi) + B_3(\phi) = \chi E_1 + N_2(\phi) + c(\varepsilon z) \chi w_x + d(\varepsilon z) \chi Z
\]

as we can absorb the term \( E_{11} \) into \( d(\varepsilon z) \chi Z \). Note that

\[
\|
\]

Furthermore, the Lipschitz dependence of the term of error \( E_{12} \) on the parameters \( f \) and \( e \) for the norms defined in (3.42)–(3.43). We have the validity of the estimate

\[
\|
\]

Let \( T \) be the operator defined by Proposition 5.1. Then the equation is equivalent to the fixed-point problem

\[
\phi = T(\chi E_1 + N_2(\phi)) = A(\phi). \quad (6.4)
\]

The operator \( T \) has the following useful property: Let \( h \) has support contained in \( |x| \leq \frac{20\delta}{\varepsilon} \). Then \( \phi = T(h) \) satisfies the estimate

\[
|\phi(x, z)| + |\nabla \phi(x, z)| \leq \|\phi\|_{\infty} e^{-\frac{4\lambda}{\varepsilon}} \text{ for } |x| > \frac{20\delta}{\varepsilon}. \quad (6.5)
\]

In fact, since \( B_3 \) is supported on \( |x| < \frac{20\delta}{\varepsilon} \) and so do the terms involving \( c \) and \( d \), then \( \phi \) satisfies for \( |x| \geq \frac{20\delta}{\varepsilon} \) an equation of the form

\[
\phi_{xx} + \alpha^{1-p} \phi_{zz} - (1 + o(1)) \phi = 0, \quad (6.6)
\]

where \( o(1) \to 0 \) as \( \varepsilon \to 0 \). For \( |x| \geq \frac{20\delta}{\varepsilon} \), we can use a barrier of the form \( \phi(x, z) = \|\phi\|_{\infty} e^{-\frac{4\lambda}{\varepsilon} (x - \frac{20\delta}{\varepsilon})} \) to conclude that for \( |x| > \frac{40\delta}{\varepsilon} \), we have

\[
\phi(x, z) \leq \|\phi\|_{\infty} e^{-\frac{10\lambda}{\varepsilon}}.
\]

The lower bound follows in a similar fashion. The bound for \( \nabla \phi \) follows using the standard elliptic estimates. The operator \( \psi(\phi) \) satisfies

\[
\|
\]

and by Lipschitz condition we have

\[
\|
\]

(6.7)
These facts will allow us to construct a region where the contraction mapping principle applies. As we have said,
\[ \| \chi_{E_{12}} \|_{L^2(S)} \leq C \varepsilon^{\frac{3}{2}} \]
for a certain constant \( C > 0 \). Now we consider a closed bounded set \( B \) by
\[ B = \{ \phi \in L^2(S) : \| \phi \|_{H^2(S)} \leq C \varepsilon^{\frac{3}{2}} ; \| \nabla \phi \| + \| \phi \|_{L^\infty(|x| \geq \frac{34}{1})} \leq \| \phi \|_{H^2(S)} e^{-\frac{1}{2}} \} \]
Then we claim for \( C > 0 \) large but fixed, the map \( A \) maps \( B \) into \( B \) is a contraction.

**Proposition 2.** There exists a constant \( C > 0 \) large but fixed such that for sufficiently small \( \varepsilon \) and all \((f,e)\) satisfying \((3.42)–(3.43)\), the problem \((4.12)–(4.14)\) has a unique solution \( \phi \) depending on \((f,e)\) satisfying
\[ \| \phi \|_{H^2(S)} \leq C \varepsilon^{\frac{3}{2}} ; \| \nabla \phi \| + \| \phi \|_{L^\infty(|x| \geq \frac{34}{1})} \leq \| \phi \|_{H^2(S)} e^{-\frac{1}{2}}. \]
Moreover, \( \phi \) is Lipschitz continuous with respect to the variables \( f \) and \( e \).

**Proof.** Let us analyze the Lipschitz character of the nonlinear operator involved in \( A \) for functions in \( B \)
\[ N_2(\phi) = \chi N_1(\phi + \psi(\phi)). \]
where \( N_1(\phi) = p[(w - 1 + t\phi)^{p-1} - (w - 1)^{p-1}] \phi \)
for \( t \in (0,1) \). Hence it follows that
\[ |N_1(\phi)| \leq C|\phi|^p + |\phi|^2, \]
so that denoting \( S_\delta = S \cap \{|x| < \\frac{104 \delta}{\varepsilon}\} \), we have that for \( \phi \in B \),
\[ \|N_1(\phi)\|_{L^2(S)} \leq C[\|\phi\|_{L^2(S)}^p + \|\phi\|_{L^4(S)}^2 + \|\phi\|_{L^2(S)}^p + \|\phi\|_{L^4(S)}^2] \]
By the Sobolev embedding theorem we obtain
\[ \|\phi\|_{L^2(S)}^p + \|\phi\|_{L^4(S)}^2 \leq C[\|\phi\|_{H^2(S)}^p + \|\phi\|_{H^2(S)}^2] \]
while using estimate \( \phi \in B \), \((6.4)\), \((6.7)\) and the fact that the area of \( S_\delta \) is of order \( O(\frac{1}{\varepsilon^3}) \), and the Sobolev embedding, we get
\[ \|\psi(\phi)\|^p_{L^2(S)} + \|\psi(\phi)\|^2_{L^4(S)} \leq C[1 + \|\phi\|_{H^2(S)}^p + \|\phi\|_{H^2(S)}^2] e^{-\frac{1}{2}}. \]
As a result we obtain
\[ \|N_2(\phi)\|_{L^2(S)} \leq C(\varepsilon^{\frac{3}{2}} + \varepsilon^3). \]
Furthermore, by the Lipschitz property we have
\[ \|N_1(\phi_1) - N_1(\phi_2)\|_{L^2(S)} \leq C[\|\phi_1\|^p_{L^2(S)} + \|\phi_1\|^2_{L^4(S)} + \|\phi_2\|^p_{L^2(S)} + \|\phi_2\|^2_{L^4(S)}] \times \|\phi_1 - \phi_2\|_{L^4(S)}. \]
Moreover,
\[ \|N_2(\phi_1) - N_2(\phi_2)\|_{L^2(S)} \leq \|N_1(\phi_1 + \psi(\phi_1)) - N_2(\phi_2 + \psi(\phi_2))\|_{L^2(S)} \]
\[ \leq C[\|\phi_1\|^{p-1}_{L^2(S)} + \|\phi_1\|^2_{L^4(S)} + \|\phi_2\|^{p-1}_{L^2(S)} + \|\phi_2\|^2_{L^4(S)}] \times \|\phi_1 - \phi_2\|_{L^4(S)}. \]
where \( A = A_1 + A_2 \) with
\[ A_i = \|\phi_i\|^{p-1}_{L^2(S_i)} + \|\psi(\phi_i)\|^2_{L^4(S_i)}. \]
Hence we conclude that
\[ \|N_2(\phi_1) - N_2(\phi_2)\|_{L^2(S)} \leq C(\varepsilon^{\frac{3(p-1)}{2}} + \varepsilon^3)\|\phi_1 - \phi_2\|_{H^2(S)}. \]
If $\phi \in \mathcal{B}$, then $\psi = A(\phi)$ satisfies
$$\|\psi\|_{H^2(S)} \leq \|T\|(C\varepsilon^2 + CD\varepsilon^2).$$
Hence we obtain
$$\|\psi\|_{H^2(S)} \leq D\varepsilon^3.$$
For the second estimate in (6.9) we use the fact that
$$\|\psi\|_{L^\infty(S)} \leq C\|\psi\|_{H^2(S)}.$$
But $\psi$ satisfies an equation of the form $L_2^2(\psi) = h$ with $h$ compactly supported. Hence $\psi$ belongs to $\mathcal{B}$. As a result, $A$ is a contraction mapping thanks to (6.12). We conclude that map $A$ has a unique fixed point in $\mathcal{B}$.

The error $E_2$ and the operator $T$ itself carry the functions $f$ and $e$ as parameters. A tedious but straightforward analysis of all terms involved in the differential operator and in the error yield that this dependence is indeed Lipschitz with respect to the $H^2$-norm (for each fixed $\varepsilon$).

In the operator, consider, for instance, the following only term involving $f$:
$$B_f(\phi) = \varepsilon^2 f''(\varepsilon z)\phi_x. \quad (6.13)$$
Then we have
$$\|B_f(\phi)\|_{L^2(S)} \leq \varepsilon^3 \int_0^1 |f''(\theta)|^2 d\theta \left( \sup_{z} \int_{-\infty}^{\infty} |\phi_x(x,z)|^2 dx \right).$$
Let $b(z) = \int_{-\infty}^{\infty} |\phi_x(x,z)|^2 dx$. Then using the Young’s and interpolation inequality
$$\sup_z b(z) \leq \varepsilon \int_{\mathcal{S}} \phi_x^2 + 2 \int_{\mathcal{S}} |\phi_x| |\phi_{xx}|$$
$$ \leq \frac{3\varepsilon}{2} \sup_z b(z) + \frac{4}{\varepsilon} \int_{\mathcal{S}} |\phi_{xx}|^2$$
which in fact implies that
$$b(z) \leq C\varepsilon^{-1}\|\phi\|_{H^2(S)}^2 \quad (6.14)$$
and as a result we obtain
$$\|B_f(\phi)\|_{L^2(S)} \leq \varepsilon\|f\|_a.$$ 
For the other terms the analysis follows in a simpler way. Emphasizing the dependence on $f$, we obtain that the linear operator $T$ satisfies
$$\|T_{f_1} - T_{f_2}\| \leq C\varepsilon\|f_1 - f_2\|_a.$$
We recall that we have the Lipschitz dependence (6.3). Moreover, the operator $N$ also has Lipschitz dependence on $(f,e)$. It can be easily checked that for $\phi \in \mathcal{B}$ we have,
$$\|N_{f_1,e_1} - N_{f_2,e_2}\| \leq C\varepsilon^2\|f_1 - f_2\|_a + \|e_1 - e_2\|_b.$$
Hence from the fixed point theory we obtain
$$\|\phi_{f_1,e_1} - \phi_{f_2,e_2}\| \leq C\varepsilon^2\|f_1 - f_2\|_a + \|e_1 - e_2\|_b. \quad (6.15)$$
7. Error estimates. In this section, we set up the equations for \( f \) and \( e \) that are equivalent to making \( c \) and \( d \) identically zero. These equations are obtained by simply integrating the equation (only in \( x \)) against \( w_x \) and \( Z \), respectively. It is therefore of crucial importance to carry out computations of the terms \( \int_{\mathbb{R}} E_1 w_x \, dx \) and \( \int_{\mathbb{R}} E_1 Z \, dx \). Using the fact that \( \int Z w_x \, dx = 0 \) we have (\( \varphi_1 \) solving (3.24))

\[
\int_{\mathbb{R}} E_1 w_x \, dx = \int_{\mathbb{R}} S(w + \varphi_1)w_x + B_3(w + \varphi_1, \varepsilon e Z)w_x.
\]

Let us first consider \( \int S(w + \varphi_1)w_x \, dx \): we shall use the expression (3.34) and the fact that \( S_3 \) is even and \( B_2(w) = O(\varepsilon^3) \) to get

\[
\int S(w + \varphi_1)w_x \, dx = \varepsilon^2 \int S_4 w_x \, dx + \int B_3(w, \varphi_1)w_x \, dx + \int N_0(\varphi_1)w_x \, dx + O(\varepsilon^3),
\]

\[
\int S_4 w_x \, dx = \int \left[ -\alpha^{-p} \beta f k^2 w_x + (-2\alpha^{-p} \alpha' \beta f' + \alpha^{-p}(-2\beta' f' - \beta f''))w_x \\
- 2\alpha^{-p} \beta f' x w_x + \frac{1}{2} (\alpha^{-1}p(1 - w)|^p - 2p(1 - w) - 1) q_{tt} \\
+ \alpha^{-2} p(p - 1)(1 - w)|^p - 2p - 1) q_t^2 \frac{2p \beta}{\beta} w_x \, dx \\
= -\alpha^{-p} \beta f'' \int w_x^2 \, dx - (\alpha^{-p} \beta' + 2\alpha^{-p} \beta' \beta') f' \int w_x^2 \, dx \\
+ f \left[ -\alpha^{-p} \beta k^2 \int w_x^2 \, dx + \frac{p + 3}{2\beta} \alpha^{-1} q_{tt} \int w_x^2 \, dx \right. \text{ (using (3.26), (3.27))} \\
\left. + \frac{p(p - 1)}{\beta} \alpha^{-2} q_t^2 \int (|w - 1|^p - 2 - 1) x w_x \, dx \right].
\]

\[
\int N_0(\varphi_1)w_x \, dx = \int (|w + \varphi_1 - 1|^p - |w - 1|^p) \int B_3(\varphi_1)w_x \, dx \\
= \frac{p(p - 1)}{2} \int |w - 1|^p - 2 \varphi_1^2 w_x \, dx + O(\varepsilon^3) \\
= p(p - 1) \int |w - 1|^p - 2 \varphi_1 \varphi_2 w_x \, dx + O(\varepsilon^3) \\
= \varepsilon^2 p^2 (p - 1) \alpha^{-1} f_{tt} \int |w - 1|^p - 2 w_x w_x \, dx + O(\varepsilon^3),
\]

Differentiating the equation (3.31) we get

\[
w_{1xxx} + p|1 - w|^p - 2(w - 1)w_{1x} - p(p - 1)|1 - w|^p - 2w_{1x} = -(p - 1)|1 - w|^p - 2 w_{1x} \quad (7.1)
\]

and multiplying the equation (3.31) by \( w_{1x} \) we get

\[
\int w_{1x} w_{2xx} + p|1 - w|^p - 2(w - 1)w_{1x} \, dx = \int S_2 w_{1x} \, dx.
\]

Integrating by parts we have

\[
\int S_2 w_{1x} \, dx = -p(p - 1) \int |1 - w|^p - 2 w_{1x} w_x \, dx \\
p(p - 1) \int |1 - w|^p - 2 w_{1x} w_x \, dx = -\int S_2 w_{1x} \, dx - (p - 1) \int |1 - w|^p - 2 w_x \, dx.
\]
and using equation (3.32), we obtain
\[ \int N_0(\varphi_1)w_xdx \]
\[ = -\varepsilon^2 pa^{-1}q_t \int S_2 w'_1 dx - \varepsilon^2 p(p - 1)\alpha^{-1} q_t \int |w - 1|^{p-2}w_xw_2dx + O(\varepsilon^3), \]
\[ = -\varepsilon^2 pa^{-1}q_t \int (\alpha^{-1} p\beta k w_x + \alpha^{-1} p[1 - w]^{p-2}(1 - w) - 1)q_t \frac{x}{\beta}w'_1 dx \]
\[ - \varepsilon^2 p(p - 1)\alpha^{-1} q_t \int |w - 1|^{p-2}w_xw_2dx + O(\varepsilon^3), \]
\[ = -\varepsilon^2 pa^{-1}q_t \int (\alpha^{-1} p\beta k w_x w_1' dx - \varepsilon^2 p^2\beta^2 q_t^2 \int ((1 - w)^{p-2}(1 - w) - 1)x w_1' dx \]
\[ - \varepsilon^2 p(p - 1)\alpha^{-1} f q_t \int |w - 1|^{p-2}w_xw_2dx + O(\varepsilon^3), \]
and
\[ \int B_3(w, \varphi_1)w_xdx \]
\[ = \varepsilon \int \left[ \alpha^{-1} p\beta k \varphi_{1,x} - p(p - 1)\alpha^{-1} q_t \frac{x}{\beta} + f \right] |w - 1|^{p-2}\varphi_1 w_x dx + O(\varepsilon^3) \]
\[ = \varepsilon \left\{ \alpha^{-1} p\beta k \int \varphi_{1,x} w_x dx - \frac{\alpha^{-1} q_t}{\beta} p(p - 1) \int |w - 1|^{p-2}\varphi_{1,x} w_x dx \right\} \]
\[ - \alpha^{-1} q_t f p(p - 1) \int |w - 1|^{p-2}\varphi_1 w_x dx + O(\varepsilon^3) \}
\[ = -\varepsilon^2 pa^{-1}q_t \int (\alpha^{-1} p\beta k w_x w_1' dx + \varepsilon^2 p^2\beta^2 q_t^2 \int f(p - 1)|1 - w|^{p-2}w_xw_1 \frac{x}{\beta} dx \]
\[ + \varepsilon^2 p(p - 1)\alpha^{-1} f q_t \int |w - 1|^{p-2}w_xw_2dx + O(\varepsilon^3). \]

Note that \( w_1 = \frac{1 - x}{2p}w_x - \frac{1}{p}w \), to obtain
\[ \int (N_0(\varphi_1) + B_3(w, \varphi_1))w_x dx \]
\[ = f \left[ -2\varepsilon^2 pa^{-1} q_t \int (\alpha^{-1} p\beta k w_x + \alpha^{-1} p[1 - w]^{p-2}(1 - w) - 1)w'_1 dx \right] \]
\[ + \varepsilon^2 p^2 (p - 1) \frac{\alpha^{-2} q_t^2}{\beta} \int |w - 1|^{p-2}w_xw_2 dx \]
\[ = -\varepsilon^2 f \left[ 2\alpha^{-1} p\beta k \int w_x w'_1 dx - p^2 \frac{\alpha^{-2} q_t^2}{\beta} \int (|w - 1|^{p-2}(w - 1) + 1)x w'_1 dx \right] \]
\[ - p^2 \frac{\alpha^{-2} q_t^2}{\beta} \int (p - 1)|w - 1|^{p-2}w_xw_1 dx \].

Since
\[ \int w_x w'_1 dx = \int w_x(-\frac{1}{p}w_x + \frac{1 - p}{2p}w - \frac{1 - p}{2p}w_xw_{xx})dx = -\frac{p + 3}{4p} \int w_x^2 dx \]
we have
\[ \int (|w - 1|^{p-2}(w - 1) + 1)x w'_1 dx + \int (p - 1)|w - 1|^{p-2}w_xw_1 dx \]
hence we obtain

\[
\int (N_0(\varphi_1) + B_3(w, \varphi_1))w_x\,dx
\]

\[
= -\varepsilon^2 f \left[ -\frac{p}{2} + 3 \beta \alpha^{-p} \int w_x^2\,dx - \frac{p(1-p)}{2\beta} \alpha^{-2} q_t^2 \int wdx \right.
\]

\[
- \frac{p}{\beta} \alpha^{-2} q_t^2 \int (|w - 1|^{p-2}(w - 1) + 1)\,wdx \right].
\]

As a result using (2.4), we obtain the major terms as

\[
\int S(w + \varphi_1)w_x\,dx = \varepsilon^2 \left[ -\alpha^{-p} \beta f'' \int w_x^2\,dx + (\alpha^{1-p} \beta' + 2\alpha^{-p} \alpha' \beta) f' \int w_x^2\,dx \right.
\]

\[
+ \varepsilon^2 \frac{p}{2\beta} \alpha^{-1} q_t \int w_x^2\,dx \right]
\]

\[
+ \varepsilon^2 \frac{p(p-1)}{2\beta} \alpha^{-2} q_t^2 \int (|w - 1|^{p-2} - 1)w_x\,dx + \frac{p(1-p)}{2\beta} \alpha^{-p} q_t \int w_x^2\,dx \right].
\]

Using \( q_t = -\frac{2}{p+3} \Phi_1^{-\frac{1}{2}} k = -\frac{2}{p+3} \alpha k \) we have

\[
\int S(w + \varphi_1)w_x\,dx
\]

\[
= -\varepsilon^2 \left[ \frac{p+3}{2p} \Phi_1^{\frac{1}{2}} f'' \int w_x^2\,dx + \left( \frac{p+3}{2p} \frac{\Phi_1^{\frac{1}{2}}}{\Phi_1^{\frac{1}{2}}} \Phi_1^{-1} \Phi_1' \right) f' \int w_x^2\,dx \right.
\]

\[
+ \varepsilon^2 \frac{p}{2p} \beta \alpha^{-1} \left( \frac{1}{p} \Phi_1^{-\frac{1}{2}} \Phi_1'' + \frac{4(1-p)}{(p+3)^2} \Phi_1^{\frac{1}{2}} k^2 \right) \int w_x^2\,dx - 2\alpha^{-p} \beta k^2 \int w_x^2\,dx \right.
\]

\[
+ \varepsilon^2 \frac{p(p-1)}{2\beta} \alpha^{-2} q_t^2 \int (|w - 1|^{p-2} - 1)w_x\,dx \right]
\]

\[
+ \frac{p(1-p)}{2\beta} \alpha^{-2} q_t^2 \int wdx + \frac{p}{\beta} \alpha^{-2} q_t^2 \int (|w - 1|^{p-2}(w - 1) + 1)\,wdx \right]
\]

\[
= -\varepsilon^2 \left( \Phi_1^{\frac{1}{2}} f'' + \left( \frac{p+3}{2p} \Phi_1^{\frac{1}{2}} \Phi_1^{-1} \Phi_1' \right) f' - \frac{p+3}{2p} \Phi_1^{-1} \Phi_1'' \right) \int w_x^2\,dx
\]

\[
+ \varepsilon^2 \left[ -2\Phi_1^{\frac{1}{2}} k^2 \int w_x^2\,dx + \frac{2(1-p)}{p+3} \Phi_1^{\frac{1}{2}} k^2 \int w_x^2\,dx \right.
\]

\[
+ \frac{4p}{(p+3)^2} \Phi_1^{\frac{1}{2}} k^2 \int (p-1)(|w - 1|^{p-2} - 1)w_x\,dx \right].
\]
Summarizing, we obtain
\[\int w_x^2 dx + \frac{4p}{(p+3)^2} \Phi^{\frac{p+3}{p}} k^2 \int (|w - 1|^{p-2}(w - 1) + 1)wdx\]
and finally we have
\[\int S(w + \varphi_1)w_x dx\]
Using the Taylor’s theorem and (3.30)
\[\int p|w + \varphi_1 - 1|^{p-2}(w + \varphi_1 - 1) - |w - 1|^{p-2}(w - 1)\varepsilon eZw_x dx = \varepsilon^2 p(p-1)e \int |w - 1|^{p-2}w_x^2 Zw_x dx + O(\varepsilon^3),\]
and
\[\int |w + \varphi_1 + \varepsilon eZ - 1|^p - |w + \varphi_1 - 1|^p - p|w + \varphi_1 - 1|^{p-2}(w + \varphi_1 - 1)\varepsilon eZw_x dx = \frac{p(p-1)}{2} \varepsilon^2 \int |w - 1|^{p-2}e^2 Z^2 w_x dx + O(\varepsilon^3)\]
Furthermore,
\[\int B_3(w + \varphi_1, \varepsilon eZ)w_x dx = \varepsilon^2 \left( \int \alpha^{-p} \beta k Zxw_x dx + p(p-1)\frac{\alpha^{-1}}{\beta} q_t \int |w - 1|^{p-2}Zxw_x dx \right) - 2\varepsilon f' \int \alpha^{-p} \beta Zxw_x dx + O(\varepsilon^3),\]
and
\[\int L_0(eZ)w_x dx = 0.
\]
Summarizing, we obtain
\[\int E_1w_x dx\]
\[= -\varepsilon^2 \left( f'' + \Phi_{\frac{1}{p}} \Phi_{\frac{1}{p}} f' - \left( \frac{p + 3}{2p} \Phi_{\frac{1}{p}} \Phi_{\frac{1}{p}} f' - \frac{3p + 3}{p + 3} k^2 \right) f \right) \int w_x^2 dx \]
+ \varepsilon^2 p(p-1)e \int |w - 1|^{p-2}w_x Zw_x dx
+ \varepsilon^2 e \left( \int \alpha^{-p} \beta k Zxw_x dx + p(p-1)\frac{\alpha^{-1}}{\beta} q_t \int |w - 1|^{p-2}Zxw_x dx \right)
- 2\varepsilon^2 f' \int \alpha^{-p} \beta Zxw_x dx + O(\varepsilon^3).
\]
The next computation correspond to the projection onto $Z$ of the error.
\[\int L_0(eZ)Z dx = \varepsilon(\varepsilon^2 \alpha^{-p} e'' + \lambda_1 \varepsilon) \int Z^2 dx,\]
\[ \int S(w + \varphi_1)Zdx = \int \left[ \varepsilon^2 S_3 + L_0(\phi_1) + N_0(\phi_1) + B_3(w, \phi_1) + B_2(w) \right]Zdx \]

where

\[ \int S_3Zdx = \int \left[ - \alpha^{-p} \beta^2 \frac{x^2}{\beta} k^2 w_x \right. \]
\[ + (2\alpha^{-p} \alpha' \frac{1}{\beta} x + \alpha^{-p} \beta^2 \frac{x}{\beta} x)w_x + \alpha^{-p} \alpha'' w - \alpha^{-p} \frac{(\beta' x^2 + \beta^2 f^2) w_{xx} - \frac{1}{2} (\alpha^{-1} p(|1 - w|^{p-2}(1 - w) - 1)q_u - \alpha^{-p} p(p - 1)(1 - w)^{p-2}(1 - w) - 1) x^2 Zdx \]
\[ - \alpha^{-p} p(p - 1)(1 - w)^{p-2} x^2 Zdx \]
\[ + \frac{1}{2} p(p - 1) \alpha^2 \frac{q_1}{\beta} \int (|w - 1|^{p-2} - 1) x^2 Zdx \]
\[ + \varepsilon^2 O(f^2 + f'^2) + O(\varepsilon^3), \]

and

\[ \int N_0(\varphi_1)Zdx = \frac{1}{2} p(p - 1) \int |w - 1|^{p-2} \phi^2_1 Zdx \]

Similarly, we obtain

\[ \int B_3(w, \varphi_1)Zdx \]
\[ = \varepsilon \int \alpha^{-p} \beta k \phi_{1,x} Z + \varepsilon \int \alpha^{-1} p(p - 1) q_1 |w - 1|^{p-2} \left( \frac{x}{\beta} + f \right) \phi_1 Zdx \]
\[ = -\varepsilon^2 \alpha^{-p} \beta k \int w_{2,x} Zdx - \varepsilon \frac{\alpha^{-1} q_1 p(p - 1)}{\beta} \int |w - 1|^{p-2} w_{2,x} Zdx \]
\[ + \varepsilon^2 O(f^2) + O(\varepsilon^3), \]

and

\[ \int p||w + \varphi_1 - 1|^{p-2}(w + \varphi_1 - 1) - |w - 1|^{p-2}(w - 1)|\varepsilon e Z^2 dx \]
\[ + \int ||w + \varphi_1 + \varepsilon e Z - 1|^{p} - |w + \varphi_1 - 1|^{p} - p|w + \phi_1 - 1|^{p-2}(w + \varphi_1 - 1)\varepsilon e Z| Zdx \]
\[ = -\varepsilon^2 p(p - 1) a_{11} \int |w - 1|^{p-2} w_{1} Z^2 dx + \varepsilon^2 \varepsilon^2 \frac{p(p - 1)}{2} \int |w - 1|^{p-2} Z^3 dx \]
\[ + O(\varepsilon^3) \]
\[ = O(\varepsilon^2)(ef + e^2) + O(\varepsilon^3), \]
\[
\int B_3(w + \varphi_1, \varepsilon eZ)Zdx
\]
\[
= \int [\varepsilon^2 \alpha_1^{-p} \beta kew + 2\varepsilon^3 \alpha^{-p} \alpha' e' Z + 2\varepsilon^3 \alpha_1^{-p}(\frac{\beta'}{\beta} x - \beta f')e'Z
\]
\[
+ \varepsilon^2 \alpha^{-1} q_1 p(p - 1)(\frac{x}{\beta} + f)|w - 1|^{p-2} e eZ]Zdx
\]
\[
= 2\varepsilon^3 e'(\alpha^{-p} \alpha') \int Z^2dx + \frac{\beta'}{\beta} \alpha_1^{-p} \int xZ_x Zdx
\]
\[
+ \varepsilon^2 e(\alpha^{-1} q_1 p(p - 1)f \int |w - 1|^{p-2} Z^2dx)
\]
\[
= 2\varepsilon^3 e'(\alpha^{-p} \alpha') \int Z^2dx + \frac{\beta'}{\beta} \alpha_1^{-p} \int xZ_x Zdx
\]
\[
+ O(\varepsilon^2)e f + O(\varepsilon^3).
\]

Summing them up, we obtain
\[
\int E_1 Zdx = \varepsilon(\varepsilon^2 \alpha_1^{-p} e'' + \lambda_0 e) \int Z^2dx
\]
\[
+ 2\varepsilon^3 e'(\alpha^{-p} \alpha' - \frac{\beta'}{\beta} \alpha_1^{-p}) \int Z^2dx + \varepsilon^2 \left[- \alpha_1^{-p} k^2 \int xw_x Zdx
\right.
\]
\[
+ \frac{\alpha_1^{-p} \beta^2 k^2}{\beta} \int x^2 w_x^2 Zdx - \frac{1}{2 \beta^2} p \alpha^{-1} q t \int (|w - 1|^{p-2}(1 - w) - 1)x^2 Zdx
\]
\[
+ \frac{1}{2 \beta^2} p(p - 1) \alpha^{-2} q^2 t \int (|w - 1|^{p-2} - 1)x^2 Zdx + \frac{1}{2} p(p - 1) \int |w - 1|^{p-2} w_x^2 Zdx
\]
\[
- \alpha_1^{-p} \beta k \int w_x Zdx - \varepsilon^2 \alpha^{-1} q_1 p(p - 1) \int |w - 1|^{p-2} w_x Zdx
\]
\[
+ \varepsilon^2 O(f^2 + f'^2 + ef + e^2) + O(\varepsilon^3).
\]

8. Projection of the term involving $\phi$. Now we estimate involving $\phi$ in (4.12)–(4.14) integrated against $w$ and $Z$. Note that the terms associated with $\phi$ are
\[
\Delta \phi + p|w - 1|^{p-2}(w - 1)\phi + \chi(|x|)B_3(\phi),
\]
$N_2(\phi)$ and $(|w - 1|^{p-2}(w - 1) - |w - 1|^{p-2}(w - 1))\phi$. So in this section we are going to estimate these quantities and their Lipschitz continuity.

First note that $w_x$ is exponentially decaying function and hence
\[
\left\| \int_\mathbb{R} B_3(\phi)w_x \right\|_{L^2(0,1)} = O(\varepsilon^3).
\]

Now we consider the worst terms $g_{11}$ and $g_{12}$ defined by
\[
g_{11} = \varepsilon^2 f'' \int \phi_x Z \text{ and } g_{12} = \int \phi_{xx} Z
\]
when integrated against $Z$. Then (6.15)
\[
g_{11} = -\varepsilon^3 f'' \int \phi Z_x.
\]

Then using the Lipschitz property of $\phi$
\[
\|g_{11}(f_1, \varepsilon_1) - g_{11}(f_2, \varepsilon_2)\|_{L^2(0,1)} \leq C\varepsilon^{3+\frac{1}{2}}[\|f_1 - f_2\|_a + \|\varepsilon_1 - \varepsilon_2\|_b].
\]
Similarly, using (6.15) we obtain
\[\|g_1(f_1, e_1) - g_1(f_2, e_2)\|_{L^2(0, t)} \leq C\varepsilon^3 [\|f_1 - f_2\|_a + \|e_1 - e_2\|_a]. \tag{8.3}\]
Moreover, if
\[g_2(\varepsilon z) = \int_R \tilde{N}(\phi)w_x \leq C \int_R |\phi|^2 w_x \]
which implies that
\[\int_0^1 g_2^2(\varepsilon z) \leq C \int_0^1 (\int_R |\phi|^2 w_x)^2 \]
using the fact that \(w_x\) is exponentially decreasing. Similarly, if
\[g_3(\varepsilon z) = \int_R (|w - 1|^{p-2}(w - 1) - |w - 1|^{p-2}(w - 1))\phi w_x.\]
Using the fact that \(w = w + \varphi_1 + \varepsilon e(z)Z\) and \(\varphi_1\) can be estimated as
\[\varepsilon |eZ| + |\varphi_1(x, z)| \leq C\varepsilon (|x|^2 + 1)e^{-|x|}\]
and hence
\[\|(w - 1|^{p-2}(w - 1) - |w - 1|^{p-2}(w - 1))w_x| \leq Ce^{-\sigma_0|x|}\]
for some \(\sigma_0 > 0\). Hence we obtain
\[\|g_3\|_{L^2(0, t)} = O(\varepsilon^3).\]
The estimate is almost the same when integrated against \(Z\). These crucial estimates will be used in the next section to show that there contribution is very small and of the order \(O(\varepsilon^3)\).

9. Further study of the system for \((f, e)\). In this section we set up the equation relating \(f\) and \(e\) in such a way that the solution \(\phi\) of (4.12)–(4.14) predicted by Proposition 2, one has that the coefficient \(c(\varepsilon z)\) is identically zero. To achieve this, we first multiply the equation against \(w_x\) and integrate only in \(x\) variable. The equation \(c = 0\) is then equivalent to the relation
\[\int_R E_1 w_x + \int_R (N_2(\phi) + B_3(\phi)) - p(|w - 1|^{p-2}(w - 1) - |w - 1|^{p-2}(w - 1))\phi w_x = 0\]
and the equation for \(d = 0\) can be written as
\[\int_R E_1 Z + \int_R (N_2(\phi) + B_3(\phi)) - p(|w - 1|^{p-2}(w - 1) - |w - 1|^{p-2}(w - 1))\phi Z = 0.\]
Using the previous section, we obtain a system of nonlinear second order differential equation for \((f, e)\) given by
\[
\left( f'' + \Phi_1^{-1} \Phi_1 f' - \left( \frac{p + 3}{2p} - 3 - \frac{3p + 3}{p + 3} \right) f \right) \int w_x^2 dx = p(p - 1) \varepsilon \int |w - 1|^{p-2} w_x^2 Z w_x dx
\] 
\[+ \varepsilon \left( \int \alpha^{-p} \beta k Z w_x dx + p(p - 1) \frac{\alpha^{-1}}{\beta} \varepsilon \int |w - 1|^{p-2} Z w_x dx \right) - 2\varepsilon f' \int \alpha^{-p} \beta Z w_x dx + O(\varepsilon). \tag{9.1}\]
and

\[ \varepsilon^2 \Phi_{\frac{1+\varepsilon}{p}} e'' - 2\varepsilon \left( \frac{p^2 - p - 4}{4p} \right) \Phi_{\frac{1+\varepsilon}{p}}' e' + \lambda_0 e \right) \int Z^2 dx \\
= -\varepsilon \left[ -\alpha^{-1} p \Phi_{\frac{1+\varepsilon}{p}}' e' + \alpha^1 \frac{\beta^2}{\beta^2} \right] \int x w_x Z dx + \alpha^1 \frac{\beta^2}{\beta^2} \int x^2 w_x^2 Z dx \\
+ \frac{1}{2\beta^2} p \alpha^{-1} q_{\omega} \int (|w-1|^{p-2} (1-w) - 1)x^2 Z dx \\
- \frac{1}{2\beta^2} p(p-1) \alpha^{-2} q_{\omega}^2 \int (|w-1|^{p-2} - 1)x^2 Z dx \\
- \frac{1}{2} p(p-1) \int |w-1|^{p-2} w_x^2 Z dx + \alpha^1 \frac{\beta^2}{\beta^2} \int w_x Z dx \\
- \frac{\alpha^{-1} q_{\omega}^2 (p-1)}{\beta} \int |w-1|^{p-2} w_x Z dx \\
+ \varepsilon O(f^2 + f'^2 + ef + e^2) + O(\varepsilon^2). \]

(9.2)

Now we are going to solve the above system. Using the non-degeneracy condition of the weighted geodesic, we observe that whenever we are away from the boundary of \( \Omega \), the operator on the left hand side of (9.1) is invertible under periodic boundary conditions. Hence there exist a unique solution \( f \in H^2(0, l) \) such that

\[ \mathcal{H}_1(f) = \left( f'' + \Phi_{\frac{1}{p}}^{-1} \Phi_{\frac{1}{p}}' f' - \left( \frac{p + 3}{2p} \Phi_{\frac{1}{p}}^{-1} \Phi_{\frac{1}{p}}' - \frac{3p + 3}{p + 3} k^2 \right) f \right) = g \]

(9.3)

which in other words implies that

\[ \|f\|_{H^2(0, l)} \leq C \|g\|_{L^2(0, l)}. \]

Now we are going to study the study the operator

\[ \mathcal{H}_2(e) = \left( \varepsilon^2 \Phi_{\frac{1+\varepsilon}{p}} e'' - 2\varepsilon \left( \frac{p^2 - p - 4}{4p} \right) \Phi_{\frac{1+\varepsilon}{p}}' e' + \lambda_0 e \right) \]

(9.4)

under the gap condition introduced in Theorem 1.1.

**Lemma 9.1.** Assume that condition (1.4) holds. If \( d \in L^2(0, l) \), then there exists unique solution \( e \in H^2(0, l) \) of \( \mathcal{H}_2(e) = d \) with

\[ \varepsilon^2 \|e\|_{L^2(0, l)} + \varepsilon \|e'\|_{L^2(0, l)} + \|e\|_{L^\infty(0, l)} \leq C \varepsilon^{-1} \|d\|_{L^2(0, l)}. \]

(9.5)

Moreover, if \( d \in H^2(0, l) \), then

\[ \varepsilon^2 \|e\|_{L^2(0, l)} + \varepsilon \|e'\|_{L^2(0, l)} + \|e\|_{L^\infty(0, l)} \leq C \|d\|_{H^2(0, l)}. \]

(9.6)

We consider

\[ \mathcal{H}_2(e_1) = \varepsilon B \]
Then (9.12) is transformed into

\[ B = \left[ -\alpha^{1-p} k^2 \int xw_x Z dx + \alpha^{1-p} \frac{\beta^2}{\beta} \int x^2 w_x Z dx \right. \\
+ \left. \frac{1}{2\beta^2} p\alpha^{-1} q \alpha \int (|w-1|^{p-2}(1-w)-1) x^2 Z dx \right. \\
- \left. \frac{1}{2\beta^2} p(p-1) \alpha^{-2} q \alpha \int (|w-1|^{p-2}-1) x^2 Z dx \right. \\
- \left. \frac{1}{2} p(p-1) \int |w-1|^p w_x^2 Z dx + \alpha^{1-p} \beta k \int w_{2x} Z dx \right. \\
\left. - \frac{\alpha^{-1}}{\beta} q \alpha p(p-1) \int |w-1|^p w_{2x} Z dx \right] \\
+ O(f^2 + f'^2 + ef + e'^2). \quad (9.7) \]

We consider \( e = e_1 + e_2 \). Then by lemma 9.1 we obtain

\[ \varepsilon^2 \| e_1 \|_{L^2(0,l)} + \| e \|_{L^\infty(0,l)} \leq C \varepsilon. \quad (9.8) \]

Hence the system arising has the same form (3.42)–(3.43) except the fact that now the term \( \varepsilon B \) disappears. As a result the linear operator

\[ H(f,e) = (H_1(f) - \gamma_3 e - \varepsilon^2 \gamma_4 e'', H_2(e)) \quad (9.9) \]

is invertible with bounds \( \| H(f,e) = (g,d) \| \) given by

\[ \| f \|_a + \| e \|_6 \leq C \| g \|_2 + \varepsilon^{-1} \| d \|_2. \quad (9.10) \]

Hence by the contraction mapping theorem, the problem

\[ [H + (\varepsilon A_{1,\varepsilon}, \varepsilon^2 A_{2,\varepsilon})](f,e) = (g,d) \quad (9.11) \]

is uniquely solvable for \( (f,e) \) satisfying (3.42)–(3.43) if \( \| g \|_{L^2(0,l)} < \varepsilon^{1+\sigma} \) and \( \| d \|_{L^2(0,l)} < \varepsilon^{2+\sigma} \) for some \( \sigma > 0 \). The desired result for the full problem (9.1)–(9.2) follows directly from the Schauder’s fixed-point theorem. In fact, refining the fixed-point region, one obtains \( \| f \|_a + \| e \|_6 = O(\varepsilon) \) for the solution.

Now we are in a position to prove Lemma 9.1.

**Proof of Lemma 9.1.** We consider the problem

\[ H_2(e) = d \quad (9.12) \]

subject to the condition \( e(0) = e(l); e'(0) = e'(l) \). Define

\[ l_0 = \int_0^l \Phi_1^{\frac{p-1}{p}}(\theta), t(\theta) = \int_0^\theta \Phi_1^{\frac{p-1}{p}}(s) \frac{d\theta}{\theta}, \quad \lambda_* = \frac{l_0^2}{\pi^2} \lambda_0; \Psi(\theta) = \Phi_1^{\frac{p-1}{p}}(\theta) \exp \left(-\frac{1}{2} \Phi_1^{\frac{p-1}{p}}(\theta) d\theta \right), \quad (9.13) \]

\[ m(t) = \Psi^{-1}(\theta) e(\theta); q(t) = \frac{l_0^2 \Psi'}{\pi^2 \beta^2 \Psi}. \quad (9.14) \]

Then it is easy to check that \( t(0) = 0 \) and \( t(\pi) = \pi \). Moreover, \( \Psi \) is periodic in \( \theta \). Then (9.12) is transformed into

\[ H_2(m) = \varepsilon^2 (m'' + q(t) m') + \lambda_* m(\theta) = \bar{d}, m(0) = m(\pi), m'(0) = m' (\pi) \quad (9.15) \]
and it then suffices to establish the estimates in Lemma 9.1 for the solution of this problem in terms of the corresponding norms of $d$. Consider the eigenvalue problem

$$m'' + q(t)m + \lambda m = 0, \quad m(0) = m(\pi), \quad m'(0) = m'(_{\pi}).$$  \hspace{1cm} (9.17)

Then (9.17) has a sequence of eigenvalues $\lambda_k$, $k \geq 0$, with an associated orthonormal basis in $L^2(0, \pi)$ constituted by $m_k$. Moreover, from [13] as $k \to +\infty$ the asymptotic expansion of the eigenvalues are related to the eigenvalues of the problem

$$m'' + \lambda m = 0, \quad m(0) = m(\pi), \quad m'(0) = m'(_{\pi})$$

by

$$\sqrt{\lambda_k} = 2k + O\left(\frac{1}{k^3}\right).$$  \hspace{1cm} (9.18)

Hence (9.16) is solvable if and only if $\lambda_k \varepsilon^2 \neq \lambda_*$ for all $k$ and the solution (9.16) is given by

$$m(t) = \sum_{k=0}^{\infty} \frac{\tilde{d}_k}{\lambda_* - \lambda_k \varepsilon^2} m_k(t)$$  \hspace{1cm} (9.19)

in $L^2$. Hence

$$\|m\|_{L^2(0, t)}^2 = \sum_{k=0}^{\infty} \frac{|\tilde{d}_k|^2}{(\lambda_* - \lambda_k \varepsilon^2)^2}.$$  \hspace{1cm} (9.20)

If we choose $\varepsilon > 0$ such that

$$|4k^2 \varepsilon^2 - \lambda_*| \geq c \varepsilon$$  \hspace{1cm} (9.21)

for all $k$, where $c$ is small. From (9.16) we obtain $|\tilde{\lambda}_0 - \lambda_k \varepsilon^2| \geq \frac{\varepsilon}{2}$. Hence we have $\|m\|_{L^2(0, t)} \leq C \varepsilon^{-1} \|d\|_{L^2(0, t)}$. Furthermore, we have

$$\|m'\|_{L^2(0, t)}^2 \leq C \sum_{k=0}^{\infty} \frac{|\tilde{d}_k|^2(1 + |\lambda_k|)}{(\lambda_* - \lambda_k \varepsilon^2)^2} \leq \frac{1}{\varepsilon^2} \sum_{k=0}^{\infty} (1 + k^2)|\tilde{d}_k|^2;$$

and hence

$$\varepsilon \|m'\|_{L^2(0, t)} + \|m\|_{L^\infty(0, t)} \leq C \varepsilon^{-1} \|\tilde{d}\|_{L^2(0, t)}.$$  \hspace{1cm} (9.22)

Furthermore, if $d \in H^2(0, t)$ we obtain

$$\varepsilon^2 \|m''\|_{L^2(0, t)} + \|m'\|_{L^2(0, t)} + \|m\|_{L^\infty(0, t)} \leq C \|\tilde{d}\|_{H^2(0, t)}.$$  \hspace{1cm}

This finishes of the proof. \hfill $\square$

**Remark.** In a forthcoming paper we show the existence of higher dimensional spherical concentrating solutions for this problem in an annular domain.

**REFERENCES**

[1] A. Ambrosetti and G. Prodi, On the inversion of some differentiable mappings with singularities between Banach spaces, Ann. Mat. Pura Appl., (4) 93 (1972), 231–246.

[2] B. Breuer, P. J. McKenna and M. Plum, Multiple solutions for a semilinear boundary value problem: A computational multiplicity proof, J. Differential Equations, 195 (2003), 243–269.

[3] E. N. Dancer and S. Yan, On the superlinear Lazer-McKenna conjecture, J. Differential Equations, 210 (2005), 317–351.

[4] E. N. Dancer and S. Yan, On the superlinear Lazer-McKenna conjecture: Part II, Comm. in Partial Differential Equations, 30 (2005), 1331–1358.

[5] E. N. Dancer and S. Santra, On the superlinear Lazer-McKenna conjecture: The nonhomogeneous case, Adv. Differential Equations, 12 (2007), 961–993.

[6] M. del Pino and C. Munoz, The two-dimensional Lazer-McKenna conjecture for an exponential nonlinearity, J. Differential Equations, 231 (2006), 108–134.
[7] M. Del Pino, M. Kowalczyk and J. Wei, Concentration on curves for nonlinear Schrödinger equations, *Comm. Pure Appl. Math.*, 60 (2007), 113–146.

[8] de Djairo G. Figueiredo, S. Santra and P. Srikanth, Non-radially symmetric solutions for a superlinear Ambrosetti-Prodi type problem in a ball, *Commun. Contemp. Math.*, 7 (2005), 849–866.

[9] L. Hollman and P. J. McKenna, A conjecture on multiple solutions of a nonlinear elliptic boundary value problem: Some numerical evidence, *Commun. Pure Appl. Anal.*, 10 (2011), 785–802.

[10] A. Lazer and P. J. McKenna, On the number of solutions of a nonlinear Dirichlet problem, *J. Math. Anal. Appl.*, 84 (1981), 282–294.

[11] G. Li, S. Yan and J. Yang, The Lazer-McKenna conjecture for an elliptic problem with critical growth, *Calc. Var PDE.*, 28 (2007), 471–508.

[12] G. Li, S. Yan and J. Yang, The Lazer-McKenna conjecture for an elliptic problem with critical growth: Part 2, *J. Differential Equations*, 227 (2006), 301–332.

[13] B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac Operators*, Mathematics and its Applications (Soviet Series), 59. Kluwer Academic Publishers Group, Dordrecht, 1991.

[14] A. Malchiodi and M. Montenegro, Boundary Concentration Phenomena for a singularly perturbed elliptic problem, *Comm. Pure Appl. Math.*, 55 (2002), 1507–1568.

[15] A. Malchiodi and M. Montenegro, Multidimensional boundary layers for a singularly perturbed Neumann problem, *Duke Math. J.*, 124 (2004), 105–143.

[16] F. Mahmoudi, A. Malchiodi and M. Montenegro, Solutions to the nonlinear Schrödinger equation carrying momentum along a curve, *Comm. Pure Appl. Math.*, 62 (2009), 1155–1264.

[17] R. Molle and D. Passaseo, Existence and multiplicity of solutions for elliptic equations with jumping nonlinearities, *J. Funct. Anal.*, 259 (2010), 2253–2295.

[18] J. Wei and S. Yan, Lazer-McKenna conjecture: The critical case, *J. Funct. Anal.*, 244 (2007), 639–667.

[19] J. Wei and J. Yang, Concentration on lines for a singularly perturbed Neumann problem in two-dimensional domains, *Indiana Univ. Math. J.*, 56 (2007), 3025–3073.

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