GLUING DERIVED EQUIVALENCES TOGETHER

HIDETO ASASHIBA

Abstract. The Grothendieck construction of a diagram $X$ of categories can be seen as a process to construct a single category $\text{Gr}(X)$ by gluing categories in the diagram together. Here we formulate diagrams of categories as colax functors from a small category $I$ to the 2-category $\mathcal{k}\text{-Cat}$ of small $\mathcal{k}$-categories for a fixed commutative ring $\mathcal{k}$. In our previous paper we defined derived equivalences of those colax functors. Roughly speaking two colax functors $X, X' : I \to \mathcal{k}\text{-Cat}$ are derived equivalent if there is a derived equivalence from $X(i)$ to $X'(i)$ for all objects $i$ in $I$ satisfying some “$I$-equivariance” conditions. In this paper we glue the derived equivalences between $X(i)$ and $X'(i)$ together to obtain a derived equivalence between Grothendieck constructions $\text{Gr}(X)$ and $\text{Gr}(X')$, which shows that if colax functors are derived equivalent, then so are their Grothendieck constructions. This generalizes and well formulates the fact that if two $\mathcal{k}$-categories with a $G$-action for a group $G$ are “$G$-equivariantly” derived equivalent, then their orbit categories are derived equivalent. As an easy application we see by a unified proof that if two $\mathcal{k}$-algebras $A$ and $A'$ are derived equivalent, then so are the path categories $AQ$ and $A'Q$ for any quiver $Q$; so are the incidence categories $AS$ and $A'S$ for any poset $S$; and so are the monoid algebras $AG$ and $A'G$ for any monoid $G$. Also we will give examples of gluing of many smaller derived equivalences together to have a larger derived equivalence.

Keywords: Grothendieck constructions, 2-categories, lax functors, colax functors, pseudofunctors, derived equivalences

1. Introduction

Under the preparations in [6] we complete our project of the title in this paper. We fix a small category $I$ and a commutative ring $\mathcal{k}$ and denote by $\mathcal{k}\text{-Cat}$ (resp. $\mathcal{k}\text{-Ab}$, $\mathcal{k}\text{-Tri}$) the 2-category of small $\mathcal{k}$-categories (resp. abelian $\mathcal{k}$-categories, triangulated $\mathcal{k}$-categories). For a $\mathcal{k}$-category $\mathcal{C}$ a (right) $\mathcal{C}$-module is a contravariant functor from $\mathcal{C}$ to the category $\text{Mod}\mathcal{k}$ of $\mathcal{k}$-modules, and we denote by $\text{Mod}\mathcal{C}$ (resp. $\text{Prj}\mathcal{C}$, $\text{prj}\mathcal{C}$) the category of $\mathcal{C}$-modules (resp. projective $\mathcal{C}$-modules, finitely generated projective $\mathcal{C}$-modules). When we deal with derived equivalences, we usually assume that $\mathcal{k}$ is a field because Keller’s theorem in [11] or [12] on derived equivalences of categories requires that the $\mathcal{k}$-categories in consideration are $\mathcal{k}$-flat or $\mathcal{k}$-projective.

A $\mathcal{k}$-category $\mathcal{C}$ with an action of a group $G$ have been well investigated in connection with a so-called covering technique in representation theory of algebras (see e.g., [8]). The orbit category $\mathcal{C}/G$ and the canonical functor $\mathcal{C} \to \mathcal{C}/G$ are naturally constructed from these data, and one studied relationships between $\text{Mod}\mathcal{C}$ and $\text{Mod}\mathcal{C}/G$.

2000 Mathematics Subject Classification. 18D05, 16W22, 16W50.

This work is partially supported by Grant-in-Aid for Scientific Research (C) 21540036 from JSPS.
We brought this point of view to the derived equivalence classification problem of algebras in [1], and a main tool obtained there was fully used in the derived equivalence classifications in [2, 3]. The main tool was extended in [4] in the following form:

**Theorem 1.1.** Let $G$ be a group acting on categories $\mathcal{C}$ and $\mathcal{C}'$. Assume that $\mathcal{C}$ is $\mathbb{k}$-flat and that the following condition is satisfied:

$$(*) \text{ There exists a } G\text{-stable tilting subcategory } E \text{ of } \mathcal{K}^b(\text{prj}\mathcal{C}) \text{ such that there is a } G\text{-equivariant equivalence } \mathcal{C}' \rightarrow E.$$ 

Then the orbit categories $\mathcal{C}/G$ and $\mathcal{C}'/G$ are derived equivalent.

(In the above, $\mathcal{C}$ is called $\mathbb{k}$-flat if all morphism spaces are flat $\mathbb{k}$-modules, and $E$ is said to be $G$-stable if the set of objects in $E$ is stable under the $G$-action on $\mathcal{K}^b(\text{prj}\mathcal{C})$ induced from that on $\mathcal{C}$.) Observe that if we regard $G$ as a category with a single object $\ast$, then a $G$-action on a category $\mathcal{C}$ is nothing but a functor $X : G \rightarrow \mathbb{k}\text{-Cat}$ with $X(\ast) = \mathcal{C}$; and the orbit category $\mathcal{C}/G$ coincides with (the $\mathbb{k}$-linear version of) the Grothendieck construction $\text{Gr}(X)$ of $X$ defined in [10].

The purpose of this paper is to generalize this theorem to an arbitrary category $I$ and to any *colax functor* $\mathcal{X}, \mathcal{X}' : I \rightarrow \mathbb{k}\text{-Cat}$ (roughly speaking a colax functor $X$ is a family $(X(i))_{i \in I_0}$ of $\mathbb{k}$-categories indexed by the objects $i$ of $I$ with an action of $I$, the precise definition is given in Definition 2.1). Recall that if $\mathcal{C}$ is a category with an action of a group $G$, then the module category $\text{Mod}\mathcal{C}$ (resp. the derived category $\mathcal{D}(\text{Mod}\mathcal{C})$) has the induced $G$-action; thus both of them are again categories with $G$-actions. Hence for a colax functor $X$ the “module category” $\text{Mod}X$ (resp. the “derived category” $\mathcal{D}(\text{Mod}X)$) should again be a family of categories with an $I$-action, i.e., a colax functor from $I$ to $\mathbb{k}\text{-Ab}$ (resp. to $\mathbb{k}\text{-Tri}$). In addition, we need a notion of equivalences between colax functors for two purposes:

(a) to generalize the statement $(*)$; and
(b) to define a derived equivalence of colax functors $X, X'$ by an existence of an equivalence between colax functors $\mathcal{D}(\text{Mod}X)$ and $\mathcal{D}(\text{Mod}X')$.

To define equivalences of objects we need notions of 1-morphisms and 2-morphisms, thus we need a 2-categorical structure on the collection of colax functors, i.e., it is needed to define a 2-category $\mathcal{Colax}(I, \mathcal{C})$ of all colax functors from $I$ to a 2-category $\mathcal{C}$, which can be used for both (a) and (b) above. Having these things in mind we see that to generalize the theorem above we have to solve the following problems:

(1) Define the “module category” of a colax functor again as a colax functor.
(2) Define the “derived category” of a colax functor as a colax functor.
(3) Give a natural definition of an equivalence between colax functors using 2-morphisms of the 2-category of colax functors.
(4) Give a condition on a 1-morphism between colax functors to be an equivalence.
(5) Give a natural definition of a derived equivalence between colax functors by the equivalence (defined in (3)) of their “derived categories” defined in (2).

---

1In [6] we called them oplax functors. There are two versions of Grothendieck constructions: (1) for contravariant lax functors and (2) for covariant colax functors. Since skew group algebras are formulated as the second version we deal with colax functors here. See [4, Example 2.12].
(6) Characterize the existence of derived equivalences of colax functors by tilting subcategories, which turns out to be a generalization of Rickard’s Morita theorem for colax functors.

(7) Induce a derived equivalence of Grothendieck constructions of colax functors from the existence of tilting subcategories, which will be a generalization of the theorem above.

In our previous paper [6] we have solved the problems (1) – (6) and made clear the meaning of the condition (∗) in the setting of colax functors. In this paper we solve the problem (7), and in addition we give a unified way to solve (1) and (2) using the following general statement on compositions with pseudofunctors (cf. Gordon–Power–Street [9, Subsection 5.6]):

**Theorem** (Theorem 6.5). Let $B, C$ and $D$ be 2-categories and $V : C \to D$ a pseudofunctor. Then the obvious correspondence (see subsection 9.1 for details)

$\text{Colax}(B, V) : \text{Colax}(B, C) \to \text{Colax}(B, D)$

turns out to be a pseudofunctor.

The solutions of (1) and (2) use the correspondence on objects given by the pseudofunctor $\text{Colax}(B, V)$. The correspondence on 1-morphisms is needed also to solve (7). The following is our main result (see Definition 7.4 for definitions):

**Theorem** (Theorem 8.2). Let $X, X' \in \text{Colax}(I, \mathbb{k}\text{-Cat})$. Assume that $X$ is $\mathbb{k}$-flat and that there exists a tilting colax functor $\mathcal{T}$ for $X$ such that $\mathcal{T}$ and $X'$ are equivalent in $\text{Colax}(I, \mathbb{k}\text{-Cat})$. Then $\text{Gr}(X)$ and $\text{Gr}(X')$ are derived equivalent.

Note that there is an easier way (Lemma 7.1, a solution of (4)) to verify that $\mathcal{T}$ and $X'$ are equivalent in $\text{Colax}(I, \mathbb{k}\text{-Cat})$ in the above. As an easy application, the theorem above gives a unified proof of the following.

**Theorem** (Theorem 8.5). Assume that $\mathbb{k}$ is a field and that $\mathbb{k}$-algebras $A$ and $A'$ are derived equivalent. Then the following pairs are derived equivalent as well:

1. path-categories $AQ$ and $A'Q$ for any quiver $Q$;
2. incidence categories $AS$ and $A'S$ for any poset $S$; and
3. monoid algebras $AG$ and $A'G$ for any monoid $G$.

**Theorem** 8.2 can be used to glue many derived equivalences together as shown in Example 8.6.

The paper is organized as follows. In section 2 we recall the definition of the 2-category $\text{Colax}(I, \mathbb{C})$ of colax functors from a category $I$ to a 2-category $\mathbb{C}$. In section 3 we first define a diagonal 2-functor $\Delta : \mathbb{k}\text{-Cat} \to \text{Colax}(I, \mathbb{k}\text{-Cat})$ in an obvious way, and introduce a notion of $I$-coverings $(F, \psi) : X \to \Delta(\mathcal{C})$ for a colax functor $X \in \text{Colax}(I, \mathbb{k}\text{-Cat})_0$ and $\mathcal{C} \in \mathbb{k}\text{-Cat}_0$ (the subscript 0 stands for objects) as a generalization of $G$-coverings for a group $G$. In section 4 we define a $\mathbb{k}$-linear version of Grothendieck construction as a 2-functor $\text{Gr} : \text{Colax}(I, \mathbb{k}\text{-Cat}) \to \mathbb{k}\text{-Cat}$ and introduce the canonical morphism $(P, \phi) : X \to \Delta(\text{Gr}(X))$. In section 5 we will show that the
Grothendieck construction is a strict left adjoint to the diagonal 2-functor with a unit given by the family of canonical morphisms, in particular, this shows that the canonical morphism \((P, \phi): X \to \Delta(\text{Gr}(X))\) is an \(I\)-covering and any other \(I\)-covering \(X \to \Delta(C)\) is given as the composite of this followed by \(\Delta(H)\) for an equivalence \(H: \text{Gr}(X) \to C\). This will be used in the proof of the main result. In section 6 we redefine the module colax functor \(\text{Mod} X: I \to \mathbb{k}\text{-Ab}\) and its derived colax functor \(\mathcal{D}(\text{Mod} X): I \to \mathbb{k}\text{-Tri}\) for a colax functor \(X \in \mathcal{C}(\text{Colax}(I, \mathbb{k}\text{-Cat})_0\) by using Theorem 6.5. In addition, we also define \(\mathcal{K}^b(\text{prj} X)\) for \(X \in \mathcal{C}(\text{Colax}(I, \mathbb{k}\text{-Cat})_0\) and show that this construction preserves \(I\)-precoverings, which is also used in the proof of the main result. It is obvious that the definitions given here coincide with those given in our previous paper [6]. In section 7 we recall the definition of derived equivalences of colax functors in \(\mathcal{C}(\text{Colax}(I, \mathbb{k}\text{-Cat})\) and the theorem characterizing the derived equivalence by tilting colax functors (Theorem 7.5). In section 8 we give a proof of Theorem 8.2 and give some applications including an example of gluing of pieces of derived equivalences together to have a larger one. In the last section we give a proof of Theorem 6.5.

Acknowledgements

Most part of this work was done during my stay in Bielefeld in February and September, 2010; and a final part (Theorem 6.4) in September, 2011. I would like to thank Claus M. Ringel and Henning Krause for their hospitality and nice discussions. The results were announced at the seminars in the Universities of Bielefeld, Bonn, Paris 7, and in Beijing Normal University. I would like to thank Jan Schröer, Bernhard Keller and Changchang Xi for their kind invitations. The results were also announced at conferences: ICRA XIV held in August 2010 in Tokyo (functor version), the 6-th China-Japan-Korea International Conference on Ring and Module Theory held in June 2011 at Kyung Hee University at Suwon, and Shanghai International Conference on Representation Theory of Algebras held in October 2011 at Shanghai Jiao Tong University. I would like to thank the organizers for their kind invitations and hospitality. Finally, I would like to thank D. Tamaki for useful discussions with him on Grothendieck constructions and for his expositions on 2-categorical notions through his preprints [15, 16] that aimed at a generalization of [5]. In addition I would also like to thank the referee for his/her careful reading, suggestions and questions, by which the paper became easier to read and I could notice that I forgot to consider the naturality property (0) of 1-morphisms in Definition 6.1 and I could add the verification of this property in the proof of Lemma 9.3 also I changed the terminology “oplax” to “colax”.

2. Preliminaries

In this section we recall the definition of the 2-category of colax functors from \(I\) to a 2-category from [6] (see also Tamaki [15]).

**Definition 2.1.** Let \(C\) be a 2-category. A *colax functor* (or an *oplax functor*) from \(I\) to \(C\) is a triple \((X, \eta, \theta)\) of data:

- a quiver morphism \(X: I \to C\), where \(I\) and \(C\) are regarded as quivers by forgetting additional data such as 2-morphisms or compositions;
• a family \( \eta := (\eta_i)_{i \in I_0} \) of 2-morphisms \( \eta_i : X(i) \Rightarrow 1_{X(i)} \) in \( C \) indexed by \( i \in I_0 \); and
• a family \( \theta := (\theta_{b,a})_{(b,a)} \) of 2-morphisms \( \theta_{b,a} : X(ba) \Rightarrow X(b)X(a) \) in \( C \) indexed by \( (b,a) \in \text{com}(I) := \{(b,a) \in I_1 \times I_1 \mid ba \text{ is defined}\} \)

satisfying the axioms:

(a) For each \( a : i \to j \) in \( I \) the following are commutative:

\[
\begin{align*}
X(a1l_i) & \xrightarrow{\theta_{a,l_i}} X(a)X(l_i) \\
& \xrightarrow{X(\eta_i)} X(a)1_{X(i)} \\
X(1_{X(i)}l_i) & \xrightarrow{\theta_{1_{X(i)},l_i}} X(1_{X(i)})X(a) \\
& \xrightarrow{1_{X(j)}X(\eta_i)} X(j)X(a);
\end{align*}
\]

(b) For each \( i \to j \to k \to l \) in \( I \) the following is commutative:

\[
\begin{align*}
X(cba) & \xrightarrow{\theta_{c,b,a}} X(c)X(ba) \\
& \xrightarrow{\theta_{c,b,a}} X(c)X(b)X(a) \\
X(cb)X(a) & \xrightarrow{\theta_{c,b,X(a)}} X(c)X(b)X(a);
\end{align*}
\]

**Definition 2.2.** Let \( C \) be a 2-category and \( X = (X, \eta, \theta) \), \( X' = (X', \eta', \theta') \) be colax functors from \( I \) to \( C \). A 1-morphism (called a left transformation) from \( X \) to \( X' \) is a pair \((F, \psi)\) of data

• a family \( F := (F(i))_{i \in I_0} \) of 1-morphisms \( F(i) : X(i) \to X'(i) \) in \( C \); and
• a family \( \psi := (\psi(a))_{a \in I_1} \) of 2-morphisms \( \psi(a) : X'(a)F(i) \Rightarrow F(j)X(a) \)

in \( C \) indexed by \( a : i \to j \) in \( I_1 \)

satisfying the axioms

(a) For each \( i \in I_0 \) the following is commutative:

\[
\begin{align*}
X'(1_i)F(i) & \xrightarrow{\psi(1_i)} F(i)X(1_i) \\
& \xrightarrow{\eta'_iF(i)} F(i)X(\eta_i) ;
\end{align*}
\]
(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in $I$ the following is commutative:

$$
\begin{array}{ccc}
X'(ba)F(i) & \xrightarrow{\theta_{a,b}F(i)} & X'(b)X'(a)F(i) & \xrightarrow{X'(b)\psi(a)} & X'(b)F(j)X(a) \\
\downarrow{\psi(ba)} & & \downarrow{\psi(b)X(a)} & & \downarrow{\psi(b)X(a)} \\
F(k)X(ba) & \xrightarrow{F(k)\theta_{b,a}} & F(k)X(b)X(a) & & \\
\end{array}
$$

A 1-morphism $(F, \psi): X \to X'$ is said to be $I$-equivariant if $\psi(a)$ is a 2-isomorphism in $C$ for all $a \in I_1$.

**Definition 2.3.** Let $C$ be a 2-category, $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from $I$ to $C$, and $(F, \psi)$, $(F', \psi')$ 1-morphisms from $X$ to $X'$. A 2-morphism from $(F, \psi)$ to $(F', \psi')$ is a family $\zeta = (\zeta(i))_{i \in I_0}$ of 2-morphisms $\zeta(i): F(i) \Rightarrow F'(i)$ in $C$ indexed by $i \in I_0$ such that the following is commutative for all $a: i \to j$ in $I$:

$$
\begin{array}{ccc}
X'(a)F(i) & \xrightarrow{X'(a)\zeta(i)} & X'(a)F'(i) \\
\downarrow{\psi(a)} & & \downarrow{\psi(a)} \\
F(j)X(a) & \xrightarrow{\zeta(j)X(a)} & F'(j)X(a). \\
\end{array}
$$

**Definition 2.4.** Let $C$ be a 2-category, $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ and $X'' = (X'', \eta'', \theta'')$ be colax functors from $I$ to $C$, and let $(F, \psi): X \to X'$, $(F', \psi'): X' \to X''$ be 1-morphisms. Then the composite $(F', \psi')(F, \psi)$ of $(F, \psi)$ and $(F', \psi')$ is a 1-morphism from $X$ to $X''$ defined by

$$(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),$$

where $F'F := ((F'(i)F(i))_{i \in I_0}$ and for each $a: i \to j$ in $I$, $(\psi' \circ \psi)(a) := F'(j)\psi(a) \circ \psi'(a)F(i)$ is the pasting of the diagram

$$
\begin{array}{ccc}
X(i) & \xrightarrow{F(i)} & X'(i) & \xrightarrow{F'(i)} & X''(i) \\
\downarrow{\psi(a)} & & \downarrow{\psi'(a)} & & \downarrow{\psi'_aX''_a} \\
X(j) & \xrightarrow{F(j)} & X'(j) & \xrightarrow{F'(j)} & X''(j). \\
\end{array}
$$

The following is straightforward to verify.

**Proposition 2.5.** Let $C$ be a 2-category. Then colax functors $I \to C$, 1-morphisms between them, and 2-morphisms between 1-morphisms (defined above) define a 2-category, which we denote by $\text{Colax}(I, C)$.

**Notation 2.6.** Let $C$ be a 2-category. Then we denote by $C^{\text{op}}$ (resp. $C^{\text{co}}$) the 2-category obtained from $C$ by reversing the 1-morphisms (resp. the 2-morphisms), and we set $C^{\text{coop}} := (C^{\text{co}})^{\text{op}} = (C^{\text{op}})^{\text{co}}$. 
3. $I$-COVERINGS

In this section we introduce the notion of $I$-coverings that is a generalization of that of $G$-coverings for a group $G$ introduced in [4], which was obtained by generalizing the notion of Galois coverings introduced by Gabriel in [8]. This will be used in the proof of our main theorem.

**Definition 3.1.** We define a 2-functor $\Delta : \text{k-Cat} \rightarrow \text{Colax}(I, \text{k-Cat})$ as follows, which is called the *diagonal* 2-functor:

- Let $\mathcal{C} \in \text{k-Cat}$. Then $\Delta(\mathcal{C})$ is defined to be a functor sending each morphism $a : i \rightarrow j$ in $I$ to $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$.

- Let $E : \mathcal{C} \rightarrow \mathcal{C}'$ be a 1-morphism in $\text{k-Cat}$. Then $\Delta(E) : \Delta(\mathcal{C}) \rightarrow \Delta(\mathcal{C}')$ is a 1-morphism $(F, \psi)$ in $\text{Colax}(I, \text{k-Cat})$ defined by $F(i) := E$ and $\psi(a) := 1_{E}$ for all $i \in I_{0}$ and all $a \in I_{1}$:

- Let $E, E' : \mathcal{C} \rightarrow \mathcal{C}'$ be 1-morphisms in $\text{k-Cat}$, and $\alpha : E \Rightarrow E'$ a 2-morphism in $\text{k-Cat}$. Then $\Delta(\alpha) : \Delta(E) \Rightarrow \Delta(E')$ is a 2-morphism in $\text{Colax}(I, \text{k-Cat})$ defined by $\Delta(\alpha) := (\alpha)_{i \in I_{0}}$.

**Remark 3.2.** Let $\mathcal{C}$ be a 2-category, $X = (X, \eta, \theta) \in \text{Colax}(I, \mathcal{C})$, and $C \in \mathcal{C}_{0}$. Further let

- $F$ be a family of 1-morphisms $F(i) : X(i) \rightarrow C$ in $\mathcal{C}$ indexed by $i \in I_{0}$; and

- $\psi$ be a family 2-morphisms $\psi(a) : F(i) \Rightarrow F(j)X(a)$ indexed by $a : i \rightarrow j$ in $I$:

Then $(F, \psi)$ is in $\text{Colax}(I, \mathcal{C})(X, \Delta(C))$ if and only if the following hold.

(a) For each $i \in I_{0}$ the following is commutative:

$$
\begin{array}{c}
F(i) \xrightarrow{\psi(1_{i})} F(i)X(1_{i}) \\
\downarrow \quad \downarrow \quad \downarrow \\
F(i)1_{X(i)}
\end{array}
$$
Definition 3.3. Let \( \mathcal{C} \in \kab - \text{Cat} \) and \((F, \psi): X \to \Delta(\mathcal{C})\) be in \(\text{Colax}(I, \kab - \text{Cat})\). Then

(1) \((F, \psi)\) is called an \(I\)-precovering (of \(\mathcal{C}\)) if the homomorphism
\[
(F, \psi)^{(1)}: \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \mathcal{C}(F(i)x, F(j)y)
\]
of \(\mathbb{k}\)-modules defined by \(\left(f_a : X(a)x \to y\right)_{a \in I(i,j)} \mapsto \sum_{a \in I(i,j)} F(j)(f_a) \circ \psi(a)(x)\) is an isomorphism for all \(i, j \in I_0\) and all \(x \in X(i)_0, y \in X(j)_0\).

(2) \((F, \psi)\) is called an \(I\)-covering if it is an \(I\)-precovering and is dense, i.e., for each \(c \in C_0\) there exists an \(i \in I_0\) and \(x \in X(i)_0\) such that \(F(i)(x)\) is isomorphic to \(c\) in \(\mathcal{C}\).

4. GROTHENDIECK CONSTRUCTIONS

In this section we define a 2-functor \(\text{Gr}: \text{Colax}(I, \kab - \text{Cat}) \to \kab - \text{Cat}\) whose correspondence on objects is a \(\mathbb{k}\)-linear version of (the opposite version of) the original Grothendieck construction (cf. [15]).

Definition 4.1. We define a 2-functor \(\text{Gr}: \text{Colax}(I, \kab - \text{Cat}) \to \kab - \text{Cat}\), which is called the Grothendieck construction.

On objects. Let \(X = (X, \eta, \theta) \in \text{Colax}(I, \kab - \text{Cat})_0\). Then \(\text{Gr}(X) \in \kab - \text{Cat}_0\) is defined as follows.

- \(\text{Gr}(X)_0 := \bigcup_{i \in I_0} \{i\} \times X(i)_0 = \{i_x := (i, x) \mid i \in I_0, x \in X(i)_0\}\).
- For each \(i_x, j_y \in \text{Gr}(X)_0\), we set
  \[
  \text{Gr}(X)(i_x, j_y) := \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y).
  \]
- For each \(i_x, j_y, k_z \in \text{Gr}(X)_0\) and each \(f = (f_a)_{a \in I(i,j)} \in \text{Gr}(X)(i_x, j_y), g = (g_b)_{b \in I(j,k)} \in \text{Gr}(X)(j_y, k_z)\), we set
  \[
  g \circ f := \left( \sum_{a \in I(i,j)} g_b \circ X(b)f_a \circ \theta_{b,a}x \right)_{c \in I(i,k)}
  \]
  where each summand is the composite of
  \[
  X(ba)x \xrightarrow{\theta_{b,a}x} X(b)X(a)x \xrightarrow{X(b)f_a} X(b)y \xrightarrow{g_b} z.
  \]
• For each \( i \in \text{Gr}(X)_0 \) the identity \( 1_{ix} \) is given by

\[
1_{ix} = (\delta_{a,1}, \eta_i(x))_{a \in I(i,i)} \in \bigoplus_{a \in I(i,i)} X(i)(X(a)x, x),
\]

where \( \delta \) is the Kronecker delta\(^3\).

### On 1-morphisms.

Let \( X = (X, \eta, \theta), X' = (X', \eta', \theta') \) be objects of \( \text{Colax}(I, \mathbb{k}\text{-Cat}) \)
and \( (F, \psi) : X \to X' \) a 1-morphism in \( \text{Colax}(I, \mathbb{k}\text{-Cat}) \). Then a 1-morphism

\[
\text{Gr}(F, \psi) : \text{Gr}(X) \to \text{Gr}(X')
\]
in \( \mathbb{k}\text{-Cat} \) is defined as follows.

• For each \( i \in \text{Gr}(X)_0 \), \( \text{Gr}(F, \psi)(i, x) := (F(i)x) \).

• For each \( i, j \in \text{Gr}(X)_0 \) and each \( f = (f_a)_{a \in I(i,j)} \in \text{Gr}(X)(i, j, x, y) \), we set

\[
\text{Gr}(F, \psi)(f) := (F(j)f_a \circ \psi(a)x)_{a \in I(i,j)},
\]

where each entry is the composite of

\[
X'(a)F(i)x \xrightarrow{\psi(a)x} F(j)X(a)x \xrightarrow{F(j)f_a} F(j)y.
\]

### On 2-morphisms.

Let \( X = (X, \eta, \theta), X' = (X', \eta', \theta') \) be objects of \( \text{Colax}(I, \mathbb{k}\text{-Cat}) \)
and \( (F, \psi), (F', \psi') : X \to X' \) 1-morphisms in \( \text{Colax}(I, \mathbb{k}\text{-Cat}) \), and let \( \zeta : (F, \psi) \Rightarrow (F', \psi') \) be a 2-morphism in \( \text{Colax}(I, \mathbb{k}\text{-Cat}) \). Then a 2-morphism

\[
\text{Gr}(\zeta) : \text{Gr}(F, \psi) \Rightarrow \text{Gr}(F', \psi')
\]
in \( \mathbb{k}\text{-Cat} \) is defined by

\[
\text{Gr}(\zeta)_i x := (\delta_{a,1}, \zeta(i)(x))_{a \in I(i,i)} : i(F(i)x) \to i(F'(i)x)
\]
in \( \text{Gr}(X') \) for each \( i \in \text{Gr}(X)_0 \).

### Example 4.2.

Let \( A \) be a \( \mathbb{k} \)-algebra regarded as a \( \mathbb{k} \)-category with a single object. Then \( A \in \mathbb{k}\text{-Cat}_0 \). Consider the functor \( X := \Delta(A) : I \to \mathbb{k}\text{-Cat} \). Then it is straightforward to verify the following.

1. If \( I \) is a free category defined by the quiver \( 1 \to 2 \), then \( \text{Gr}(X) \) is isomorphic to the triangular algebra \( \begin{bmatrix} A & 0 \\ A & A \end{bmatrix} \).

2. If \( I \) is a free category defined by a quiver \( Q \), then \( \text{Gr}(X) \) is isomorphic to the path-category \( AQ \) of \( Q \) over \( A \).

3. If \( I \) is a poset \( S \), then \( \text{Gr}(X) \) is isomorphic to the incidence category \( AS \) of \( S \) over \( A \).

4. If \( I \) is a monoid \( G \), then \( \text{Gr}(X) \) is isomorphic to the monoid algebra \( AG \) of \( G \) over \( A \).

In (3) above, \( AS \) is defined to be the factor category of the path-category \( AQ \) modulo the ideal generated by the full commutativity relations in \( Q \), where \( Q \) is the Hasse diagram of \( S \) regarded as a quiver by drawing an arrow \( x \to y \) if \( x \leq y \) in \( Q \). If \( S \) is a finite poset, then \( AS \) is identified with the usual incidence algebra.

---

\(^3\)This is used to mean that the \( a \)-th component is \( \eta_i x \) if \( a = 1_i \), and 0 otherwise.

\(^4\)Since \( AG \) has the identity \( 1_A 1_G \), this is regarded as a category with a single object.
See [7] for further examples of the Grothendieck constructions of functors, in which the examples (2) and (3) above are unified and generalized.

**Definition 4.3.** Let $X \in \mathcal{Colax}(I, \mathbb{k}\text{-Cat})$. We define a left transformation $(P_X, \phi_X) := (P, \phi): X \to \Delta(\text{Gr}(X))$ (called the canonical morphism) as follows.

- For each $i \in I_0$, the functor $P(i): X(i) \to \text{Gr}(X)$ is defined by
  \[
  P(i)x := x, \quad P(i)f := (\delta_{a,1}f \circ (\eta_i x))_{a \in I(i,i)}: x \to y \text{ in } \text{Gr}(X)
  \]
  for all $f: x \to y$ in $X(i)$.
- For each $a: i \to j$ in $I$, the natural transformation $\phi(a): P(i) \Rightarrow P(j)X(a)$
  \[
  \begin{array}{ccc}
  X(i) & \xrightarrow{P(i)} & \text{Gr}(X) \\
  X(a) \downarrow \nearrow & & \downarrow \phi(a) \\
  X(j) & \xrightarrow{P(j)} & \text{Gr}(X)
  \end{array}
  \]
  is defined by $\phi(a)x := (\delta_{b,a}1_{X(a)x})_{b \in I(i,j)}$ for all $x \in X(i)_0$.

**Lemma 4.4.** The $(P, \phi)$ defined above is a 1-morphism in $\mathcal{Colax}(I, \mathbb{k}\text{-Cat})$.

**Proof.** This is straightforward by using Remark 3.2. 

**Proposition 4.5.** Let $X \in \mathcal{Colax}(I, \mathbb{k}\text{-Cat})$. Then the canonical morphism $(P, \phi): X \to \Delta(\text{Gr}(X))$ is an $I$-covering. More precisely, the morphism

\[
(P, \phi)^{(1)}_{x,y}: \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \text{Gr}(X)(P(i)x, P(j)y)
\]

is the identity for all $i, j \in I_0$ and all $x \in X(i)_0$, $y \in X(j)_0$.

**Proof.** By the definitions of Gr$(X)_0$ and of $P$ it is obvious that $(P, \phi)$ is dense. Let $i, j \in I_0$ and $x \in X(i)$, $y \in X(j)$. We only have to show that

\[
(P, \phi)^{(1)}_{x,y}: \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \text{Gr}(X)(P(i)x, P(j)y)
\]
is the identity. Let \( f = (f_a)_{a \in I(i,j)} \in \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \). Then
\[
(P, \phi)_{x,y}^{i,j}(f) = \sum_{a \in I(i,j)} P(j)(f_a) \circ \phi(a)x
\]
\[
= \sum_{a \in I(i,j)} (\delta_{i,j} f_a \circ (\eta_j x))_{b \in I(j,i,j)} \circ (\delta_{c,a} I_{X(a)x} \circ \theta_{b,c} x)_{c \in I(i,j)}
\]
\[
= \sum_{a \in I(i,j)} (\delta_{d,a} f_a \circ (\eta_j x) \circ I_{X(1)X(a)x} \circ \theta_{1,a} x)_{d \in I(i,j)}
\]
\[
= (f_a \circ (\eta_j x) \circ \theta_{1,a} x)_{a \in I(i,j)} = (f_a)_{a \in I(i,j)}
\]
as required.

\[\square\]

**Lemma 4.6.** Let \( X \in \text{Colax}(I, k\text{-Cat})_0 \) and \( H : \text{Gr}(X) \to C \) be in \( k\text{-Cat} \) and consider the composite 1-morphism \((F, \psi) : X \xrightarrow{(P, \phi)} \Delta(\text{Gr}(X)) \xrightarrow{\Delta(H)} \Delta(C)\). Then \((F, \psi)\) is an \( I\)-covering if and only if \( H \) is an equivalence.

**Proof.** Obviously \((F, \psi)\) is dense if and only if so is \( H \). Further for each \( i, j \in I_0 \), \( x \in X(i) \) and \( y \in X(j) \), \((F, \psi)^{(1)}_{x,y}\) is an isomorphism if and only if so is \( H_{x,y} \) because we have a commutative diagram

\[
\begin{align*}
\bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) &\xrightarrow{(F, \psi)^{(1)}_{x,y}} C(F(i)x, F(j)y) \\
\text{Gr}(X)(ix, jy) &\xrightarrow{H_{ix, jy}}
\end{align*}
\]

by Proposition 4.5. \[\square\]

5. **Adjoints**

In this section we will show that the Grothendieck construction is a strict left adjoint to the diagonal 2-functor, and that \( I\)-coverings are essentially given by the unit of the adjunction.

**Definition 5.1.** Let \( C \in \text{k-Cat} \). We define a functor \( Q_C : \text{Gr}(\Delta(C)) \to C \) by

- \( Q_C(ix) := x \) for all \( ix \in \text{Gr}(\Delta(C))_0 \); and
- \( Q_C((f_a)_{a \in I(i,j)}) := \sum_{a \in I(i,j)} f_a \) for all \((f_a)_{a \in I(i,j)} \in \text{Gr}(\Delta(C))(ix, jy)\) and for all \( ix, jy \in \text{Gr}(\Delta(C))_0 \).

It is easy to verify that \( Q_C \) is a \( \text{k}\)-functor.
Theorem 5.2. The 2-functor \( \text{Gr} \colon \text{Colax}(I, \mathbf{k}\text{-Cat}) \to \mathbf{k}\text{-Cat} \) is a strict left 2-adjoint to the 2-functor \( \Delta \colon \mathbf{k}\text{-Cat} \to \text{Colax}(I, \mathbf{k}\text{-Cat}) \). The unit is given by the family of canonical morphisms \((P_X, \phi_X) : X \to \Delta(\text{Gr}(X))\) indexed by \(X \in \text{Colax}(I, \mathbf{k}\text{-Cat})\), and the counit is given by the family of \(Q_C : \text{Gr}(\Delta(C)) \to C\) defined as above indexed by \(C \in \mathbf{k}\text{-Cat}\).

In particular, \((P_X, \phi_X)\) has a strict universality in the comma category \((X \downarrow \Delta)\), i.e., for each \((F, \psi) : X \to \Delta(C)\) in \(\text{Colax}(I, \mathbf{k}\text{-Cat})\) with \(C \in \mathbf{k}\text{-Cat}\), there exists a unique \(H : \text{Gr}(X) \to C\) in \(\mathbf{k}\text{-Cat}\) such that the following is a commutative diagram in \(\text{Colax}(I, \mathbf{k}\text{-Cat})\):

\[
\begin{array}{ccc}
X & \xrightarrow{(F, \psi)} & \Delta(C) \\
(P_X, \phi_X) \downarrow & & \downarrow (\Delta(H)) \\
\Delta(\text{Gr}(X)) & &
\end{array}
\]

Proof. For simplicity set \(\eta := ((P_X, \phi_X))_{X \in \text{Colax}(I, \mathbf{k}\text{-Cat})_0}\) and \(\varepsilon := (Q_C)_{C \in \mathbf{k}\text{-Cat}_0}\).

Claim 1. \(\Delta\varepsilon \cdot \eta\Delta = \text{id}_\Delta\).

Indeed, Let \(C \in \mathbf{k}\text{-Cat}\). It is enough to show that \(\Delta(Q_C) \cdot (P_{\Delta(C)}, \phi_{\Delta(C)}) = \text{id}_{\Delta(C)}\).

Now
\[
\text{LHS} = \left((Q_C P_{\Delta(C)}(i))_{i \in I_0}, (Q_C \phi_{\Delta(C)}(a))_{a \in I_1}\right),
\]
and
\[
\text{RHS} = \left((\text{id}_C)_{i \in I_0}, (\text{id}_C)_{a \in I_1}\right).
\]

First entry: Let \(i \in I\). Then \(Q_C P_{\Delta(C)}(i) = \text{id}_C\) because for each \(x, y \in C_0\) and each \(f \in C(x, y)\) we have \((Q_C P_{\Delta(C)}(i))(x) = Q_C(x, x) = x\); and \((Q_C P_{\Delta(C)}(i))(f) = (\delta_{a, 1}, f \cdot ((\eta_{\Delta(C)}))_{a \in I_1})\).

Second entry: Let \(a \colon i \to j \in I\). Then \(Q_C \phi_{\Delta(C)}(a) = \text{id}_C\) because for each \(x \in C_0\) we have \(Q_C \phi_{\Delta(C)}(a) x = Q_C ((\delta_{b, a} \text{id}_{\Delta(C)}(a))_{b \in I_1}) = \sum_{b \in I_1} \delta_{b, a} \text{id}_x = \text{id}_x = \text{id}_{\Delta(C) x}\). This shows that \(\text{LHS} = \text{RHS}\).

Claim 2. \(\varepsilon \cdot \text{Gr} \cdot \text{Gr} \eta = \text{id}_{\text{Gr}}\).

Indeed, let \(X \in \text{Colax}(I, \mathbf{k}\text{-Cat})\). It is enough to show that \(Q_{\text{Gr}(X)} \cdot \text{Gr}(P_X, \phi_X) = \text{id}_{\text{Gr}(X)}\).

On objects: Let \(i x \in \text{Gr}(X)_0\). Then \(Q_{\text{Gr}(X)}(\text{Gr}(P_X, \phi_X)(x)) = Q_{\text{Gr}(X)}(P_X(i)(x)) = i x\).

On morphisms: Let \(f = (f_a)_{a \in I_{ij}} : i x \to j y\) be in \(\text{Gr}(X)\). Then \(Q_{\text{Gr}(X)} \cdot \text{Gr}(P_X, \phi_X)(f) = Q_{\text{Gr}(X)}((P_X(j)(f_a) \circ (\phi_X(a))_{a \in I_{ij}}) = \sum_{a \in I_{ij}} P_X(j)(f_a) \circ (\phi_X(a)) x = (P_X, \phi_X)^{1}(f) = f\). Thus the claim holds.

The two claims above prove the assertion. \(\square\)

Corollary 5.3. Let \((F, \psi) : X \to \Delta(C)\) be in \(\text{Colax}(I, \mathbf{k}\text{-Cat})\). Then the following are equivalent.

1. \((F, \psi)\) is an \(I\)-covering;
(2) There exists an equivalence $H : \text{Gr}(X) \to \mathcal{C}$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{(F, \phi)} & \Delta(\mathcal{C}) \\
\downarrow{(P_X, \phi_X)} & & \downarrow{\Delta(H)} \\
\Delta(\text{Gr}(X)) & \text{is strictly commutative.}
\end{array}
$$

Proof. This immediately follows by Theorem 5.2 and Lemma 4.6 \qed

6. The Module Colax Functor

Let $X : I \to \mathbf{k}\text{-Cat}$ be a colax functor. In this section we simplify the definition of the “module category” $\text{Mod} X$ of $X$ as a colax functor $I \to \mathbf{k}\text{-Cat}$ given in our previous paper [6]. Recall that the module category $\text{Mod} C$ of a category $C \in \mathbf{k}\text{-Cat}$ is defined to be the functor category $\mathbf{k}\text{-Cat}(C^{\text{op}}, \text{Mod} \mathbf{k})$, where $\text{Mod} \mathbf{k}$ denotes the category of $\mathbf{k}$-modules. Since $\mathbf{k}\text{-Cat}$ is a 2-category, this is extended to a representable 2-functor

$$
\text{Mod}' := \mathbf{k}\text{-Cat}((-)^{\text{op}}, \text{Mod} \mathbf{k}) : \mathbf{k}\text{-Cat} \to \mathbf{k}\text{-Ab}^{\text{coop}}
$$

(see Notation 2.6). As is easily seen the composite $\text{Mod}' \circ X$ turns out to be a colax functor $I \to \mathbf{k}\text{-Ab}^{\text{coop}}$, i.e., a contravariant lax functor $I \to \mathbf{k}\text{-Ab}$. When $X$ is a group action, namely when $I$ is a group $G$ and $X : G \to \mathbf{k}\text{-Cat}$ is a functor, the usual module category $\text{Mod} X$ with a $G$-action of $X$ was defined to be the composite functor $\text{Mod} X := \text{Mod}' \circ X \circ i$, where $i : G \to G$ is the group anti-isomorphism defined by $x \mapsto x^{-1}$ for all $x \in G$. In this way we can change $\text{Mod}' \circ X$ to a covariant one. But in general we cannot assume the existence of such an isomorphism $i$. Instead in this paper we will use a covariant “pseudofunctor” $\text{Mod} : \mathbf{k}\text{-Cat} \to \mathbf{k}\text{-Ab}$ defined below and will define $\text{Mod} X$ as the composite $\text{Mod} \circ X$, which can be seen as a “lax” extended version of the module category construction of a category with a $G$-action stated above. We start with a notion of colax functors from a 2-category to a 2-category. Compare our definitions of colax functors, left transformations (1-morphisms) and 2-morphisms in the setting of 2-categories given below with definitions of morphisms, transformations and modifications in the setting of bicategories (see Leinster [13] for instance).

Definition 6.1. Let $\mathbf{B}$ and $\mathbf{C}$ be 2-categories.

(1) A colax functor from $\mathbf{B}$ to $\mathbf{C}$ is a triple $(X, \eta, \theta)$ of data:

- a triple $X = (X_0, X_1, X_2)$ of maps $X_i : \mathbf{B}_i \to \mathbf{C}_i$ ($\mathbf{B}_i$ denotes the collection of $i$-morphisms of $\mathbf{B}$ for each $i = 0, 1, 2$) preserving domains and codomains of all 1-morphisms and 2-morphisms (i.e. $X_1(B_1(i, j)) \subseteq C_1(X_0i, X_0j)$ for all $i, j \in \mathbf{B}_0$ and $X_2(B_2(a, b)) \subseteq C_2(X_1a, X_1b)$ for all $a, b \in \mathbf{B}_1$ (we omit the subscripts of $X$ below));

- a family $\eta := (\eta_i)_{i \in \mathbf{B}_0}$ of 2-morphisms $\eta_i : X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in \mathbf{B}_0$; and

- a family $\theta := (\theta_{b,a})_{(b,a)}$ of 2-morphisms $\theta_{b,a} : X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b, a) \in \text{com}(\mathbf{B}) := \{(b, a) \in \mathbf{B}_1 \times \mathbf{B}_1 \mid ba \text{ is defined}\}$

satisfying the axioms:
(i) $(X_1, X_2): \mathcal{B}(i, j) \to \mathcal{C}(X_{0i}, X_{0j})$ is a functor for all $i, j \in \mathcal{B}_0$;
(ii) For each $a: i \to j$ in $\mathcal{B}_1$ the following are commutative:
\[
\begin{array}{ccc}
X(a1) & \xrightarrow{\theta_{a,b}} & X(a)X(1) \\
\downarrow X(1) & & \downarrow X(1)
\end{array}
\quad \quad
\begin{array}{ccc}
X(1) & \xrightarrow{\theta_{a,b}} & X(1)X(a) \\
\downarrow X(1) & & \downarrow X(1)
\end{array}
\]

(iii) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in $\mathcal{B}_1$ the following is commutative:
\[
\begin{array}{ccc}
X(cba) & \xrightarrow{\theta_{c,b,a}} & X(c)X(ba) \\
\downarrow X(c) & & \downarrow X(c)
\end{array}
\quad \quad
\begin{array}{ccc}
X(cb) & \xrightarrow{\theta_{c,b,a}} & X(c)X(b)X(a) \\
\downarrow X(c) & & \downarrow X(c)
\end{array}
\]

(iv) For each $a, a': i \to j$ and $b, b': j \to k$ in $\mathcal{B}_1$ and each $\alpha: a \to a'$, $\beta: b \to b'$ in $\mathcal{B}_2$ the following is commutative:
\[
\begin{array}{ccc}
X(ba) & \xrightarrow{\theta_{b,a}} & X(b)X(a) \\
\downarrow X(b) & & \downarrow X(b)
\end{array}
\quad \quad
\begin{array}{ccc}
X(b'a') & \xrightarrow{\theta_{b,a'}} & X(b')X(a') \\
\downarrow X(b') & & \downarrow X(b')
\end{array}
\]

(2) A lax functor from $\mathcal{B}$ to $\mathcal{C}$ is a colax functor from $\mathcal{B}$ to $\mathcal{C}^{\text{co}}$ (see Notation 2.6).
(3) A pseudofunctor from $\mathcal{B}$ to $\mathcal{C}$ is a colax functor with all $\eta_i$ and $\theta_{b,a}$ 2-isomorphisms.
(4) We define a 2-category $\text{Colax}(\mathcal{B}, \mathcal{C})$ having all the colax functors $\mathcal{B} \to \mathcal{C}$ as the objects as follows.

1-morphisms. Let $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be colax functors from $\mathcal{B}$ to $\mathcal{C}$. A 1-morphism (called a left transformation) from $X$ to $X'$ is a pair $(F, \psi)$ of data

- a family $F := (F(i))_{i \in \mathcal{B}_0}$ of 1-morphisms $F(i): X(i) \to X'(i)$ in $\mathcal{C}$; and
- a family $\psi := (\psi(a))_{a \in \mathcal{B}_1}$ of 2-morphisms $\psi(a): X'(a)F(i) \Rightarrow F(j)X(a)$

in $\mathcal{C}$ indexed by $a: i \to j$ in $\mathcal{B}_1$ with the property that

(0) for each $\alpha: a \Rightarrow b$ in $\mathcal{B}(i, j)$ the following is commutative:
\[
\begin{array}{ccc}
X'(a)F(i) & \xrightarrow{X'(a)\psi(i)} & X'(b)F(i) \\
\downarrow \psi(a) & & \downarrow \psi(b)
\end{array}
\quad \quad
\begin{array}{ccc}
F(j)X(a) & \xrightarrow{F(j)\psi(a)} & F(j)X(b)
\end{array}
\]
thus a family of natural transformations of functors

\[
\begin{array}{ccc}
\mathcal{B}(i, j) & \xrightarrow{\psi_{ij}} & \mathcal{C}(X'(i), X'(j)) \\
\downarrow X & & \downarrow \mathcal{C}(F(i), X'(j)) \\
\mathcal{C}(X(i), X(j)) & \xrightarrow{\mathcal{C}(X(i), F(j))} & \mathcal{C}(X(i), X'(j))
\end{array}
\]

satisfying the axioms

(a) For each \( i \in \mathcal{B}_0 \) the following is commutative:

\[
\begin{align*}
X'(1_i)F(i) & \xrightarrow{\psi(1_i)} F(i)X(1_i) \\
\eta_i F(i) & \xrightarrow{\psi(i)} F(i)\eta_i \\
1_{X'(i)}F(i) & \xrightarrow{\psi(i)} F(i)1_{X(i)}
\end{align*}
\]

(b) For each \( i \xrightarrow{a} j \xrightarrow{b} k \) in \( \mathcal{B}_1 \) the following is commutative:

\[
\begin{array}{ccc}
X'(ba)F(i) & \xrightarrow{\theta_{b,a} F(i)} & X'(b)X'(a)F(i) \\
\psi(ba) & & \xrightarrow{X'(b)\psi(a)} X'(b)F(j)X(a) \\
F(k)X(ba) & \xrightarrow{F(k)\theta_{b,a}} & F(k)X(b)X(a).
\end{array}
\]

2-morphisms. Let \( X = (X, \eta, \theta) \), \( X' = (X', \eta', \theta') \) be colax functors from \( \mathcal{B} \) to \( \mathcal{C} \), and \( (F, \psi), (F', \psi') \) 1-morphisms from \( X \) to \( X' \). A 2-morphism from \( (F, \psi) \) to \( (F', \psi') \) is a family \( \zeta = (\zeta(i))_{i \in \mathcal{B}_0} \) of 2-morphisms \( \zeta(i) : F(i) \Rightarrow F'(i) \) in \( \mathcal{C} \) indexed by \( i \in \mathcal{B}_0 \) such that the following is commutative for all \( a : i \to j \) in \( \mathcal{B}_1 \):

\[
\begin{array}{ccc}
X'(a)F(i) & \xrightarrow{X'(a)\zeta(i)} & X'(a)F'(i) \\
\psi(a) & & \xrightarrow{\psi(a)} \psi(a) \\
F(j)X(a) & \xrightarrow{\zeta(j)X(a)} & F'(j)X(a).
\end{array}
\]

Composition of 1-morphisms. Let \( X = (X, \eta, \theta) \), \( X' = (X', \eta', \theta') \) and \( X'' = (X'', \eta'', \theta'') \) be colax functors from \( \mathcal{B} \) to \( \mathcal{C} \), and let \( (F, \psi) : X \to X' \), \( (F', \psi') : X' \to X'' \) be 1-morphisms. Then the composite \( (F', \psi')(F, \psi) \) of \( (F, \psi) \) and \( (F', \psi') \) is a 1-morphism from \( X \) to \( X'' \) defined by

\[
(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),
\]

where \( F'F := ((F'(i)F(i)))_{i \in \mathcal{B}_0} \) and for each \( a : i \to j \) in \( \mathcal{B} \), \( (\psi' \circ \psi)(a) := F'(j)\psi(a) \circ \psi'(a)F(i) \) is the pasting of the diagram

\[
\begin{array}{ccc}
X(i) & \xrightarrow{F(i)} & X'(i) \\
\downarrow X(a) & & \downarrow F'(i) \\
X(j) & \xrightarrow{F(j)} & X'(j)
\end{array}
\]

\[
\begin{array}{ccc}
X'(i) & \xrightarrow{\psi'(a)} & \psi'(a) \\
\downarrow X'(a) & & \downarrow \psi'(a) \\
X''(i) & \xrightarrow{\psi'(a)} & \psi'(a)
\end{array}
\]

\[
\begin{array}{ccc}
X'(j) & \xrightarrow{\psi'(a)} & \psi'(a) \\
\downarrow X'(a) & & \downarrow \psi'(a) \\
X''(j) & \xrightarrow{\psi'(a)} & \psi'(a)
\end{array}
\]

\[
\begin{array}{ccc}
X(i) & \xrightarrow{F(i)} & X'(i) \\
\downarrow X(a) & & \downarrow F'(i) \\
X''(i) & \xrightarrow{\psi'(a)} & \psi'(a)
\end{array}
\]

\[
\begin{array}{ccc}
X'(j) & \xrightarrow{\psi'(a)} & \psi'(a) \\
\downarrow X'(a) & & \downarrow \psi'(a) \\
X''(j) & \xrightarrow{\psi'(a)} & \psi'(a)
\end{array}
\]
Remark 6.2. (1) Note that a (strict) 2-functor from $B$ to $C$ is a pseudofunctor with all $\eta_i$ and $\theta_{b,a}$ identities.

(2) By regarding the category $I$ as a 2-category with all 2-morphisms identities, the definition (1) of colax functors above coincides with Definition 2.1.

(3) When $B = I$, the definition (4) of $\text{Colax}(B, C)$ above coincides with that of $\text{Colax}(I, C)$ given before.

Example 6.3. (1) Since $k\text{-Cat}$ is a 2-category, $\text{Mod}' := k\text{-Cat}((-)^{op}, \text{Mod} k) : k\text{-Cat} \to k\text{-Ab}^{op}$ is a 2-functor, which we can regard as a contravariant lax functor

$$\text{Mod}' := k\text{-Cat}((-)^{op}, \text{Mod} k) : k\text{-Cat} \to k\text{-Ab}.$$ 

(2) We define a pseudofunctor $\text{Mod} : k\text{-Cat} \to k\text{-Ab}$ as follows.

- For each $C \in k\text{-Cat}_0$ we set $\text{Mod} C := \text{Mod}' C$.
- For each $F : C \to C'$ in $k\text{-Cat}_1$ we set $\text{Mod} F := - \otimes_C F : \text{Mod} C \to \text{Mod} C'$, where $F$ is the $C$-$C'$-bimodule defined by $F(x, y) := C'(y, F(x))$ for all $x \in C_0$, $y \in C'_0$, which we sometimes write as $F := C'(? , F(-))$.
- For each $\alpha : F \Rightarrow G$ in $k\text{-Cat}_2$ (with $F, G : C \to C'$ in $k\text{-Cat}_1$) we define $\text{Mod} \alpha : \text{Mod} F \Rightarrow \text{Mod} G$ by setting $(\text{Mod} \alpha)x := (\text{Mod} C)(?, \alpha x) : \text{Mod} C(? , Fx) \Rightarrow \text{Mod} C(? , Gx)$ for all $x \in C_0$.
- For each $C \in k\text{-Cat}$ we define $\eta_C : \text{Mod} \mathbb{1}_C \Rightarrow \mathbb{1}_{\text{Mod} C}$ by setting $\eta_C M : M \otimes_C C(?, ?) \to M$ to be the canonical isomorphism for all $M \in \text{Mod} C$.
- For each pair of functors $C \xrightarrow{F} C' \xrightarrow{\theta} C''$ in $k\text{-Cat}$ we define $\theta_{C,F} : \text{Mod} GF \Rightarrow \text{Mod} G \circ \text{Mod} F$ as the inverse of the canonical isomorphism

$$- \otimes_C C'(?, F(-)) \otimes_C C''(?, G(-)) \Rightarrow - \otimes_C C''(?, GF(-)).$$

It is straightforward to check that this defines a pseudofunctor.

(3) Denote by $k\text{-ModCat}$ the 2-subcategory of $k\text{-Ab}$ consisting of the following:

- objects: $\text{Mod} C$ with $C \in k\text{-Cat}_0$,
- 1-morphisms: functors between objects having exact right adjoints, and
- 2-morphisms: all natural transformations between 1-morphisms.

Then note that the pseudofunctor $\text{Mod} : k\text{-Cat} \to k\text{-Ab}$ defined above can be seen as a pseudofunctor $k\text{-ModCat} \to k\text{-ModCat}$. For each $\text{Mod} C$ with $C \in k\text{-Cat}_0$ we denote by $k\text{-ModCat}_0$ the full subcategory of the homotopy category $k\text{-Cat}$ of $\text{Mod} C$ consisting of homotopically projective objects $M$, i.e., objects $M$ such that $k\text{-Cat}(M, A) = 0$ for all acyclic objects $A$. Recall that there is a natural embedding $\text{id}_C : k\text{-ModCat}_0 \to \text{Mod} C$ having a left adjoint $\text{id}^{-1}_C$ such that there exists a quasi-isomorphism $\eta_C M : \text{id}_C \text{id}^{-1}_C M \Rightarrow M$ for each $M \in \text{Mod} C$ and that $\text{id}^{-1}_C \text{id}_C = \text{id}_k\text{-ModCat}_0$. Then we can define a pseudofunctor $\text{Mod} : k\text{-ModCat} \to k\text{-Tri}$ as follows.

- For each $\text{Mod} C$ in $k\text{-ModCat}_0$ with $C \in k\text{-Cat}$ we set $\text{Mod} C$ to be the derived category of $\text{Mod} C$.
- For each $F : \text{Mod} C \to \text{Mod} C'$ in $k\text{-ModCat}_1$, $F$ naturally induces a functor $\text{id}_C \text{id}_C F : \text{id}_C \text{id}_C \text{Mod} C \to \text{id}_C \text{id}_C \text{Mod} C'$, which restricts to a functor $\text{id}_C \text{id}_C F : \text{id}_C \text{id}_C \text{Mod} C \to \text{id}_C \text{id}_C \text{Mod} C$ because $F$ has an exact right adjoint. Then we set $\text{id}_C \text{id}_C F$ to be the
left derived functor $LF: D(\text{Mod} C) \to D(\text{Mod} C')$ of $F$, which is defined as the composite $LF := j_c'(K_p F)p_c$.

- For each $\alpha: F \Rightarrow F'$ in $k\text{-ModCat}_2$ with $F, F': \text{Mod} C \to \text{Mod} C'$ in $k\text{-ModCat}_1$, $\alpha$ naturally induces a natural transformation $K_p\alpha: K_p F \Rightarrow K_p F'$. Then we define $D\alpha := j_c'(K_p\alpha)p_c$.
- We define $\eta_{\text{Mod} C}: D(\text{Id}(\text{Mod} C)) = (jc p_c) \Rightarrow \text{I}D(\text{Mod} C)$ by $\eta_{\text{Mod} C} := (\eta_M M)_{M \in D(\text{Mod} C)}$.
- Note that we can define two 2-functors $\text{ModCat} \to \text{Cat}$ by taking formal additive hulls (see e.g., [2, Subsection 4.1]) and by taking split idempotent completions (see e.g., [4, Definition 3.1]), respectively. Then the Yoneda embeddings $Y_C: \text{C} \to \text{K}\text{-Cat}, x \mapsto \text{C}(\cdot, x) (C \in k\text{-Cat}_0)$ induce a natural 2-isomorphism $Y: \text{sic} \circ \oplus \Rightarrow \text{prj}$:

$$\text{k-Cat} \xrightarrow{\oplus} \text{k-add} \xrightarrow{\text{prj}} \text{k-add} \xrightarrow{\text{sic}} \text{k-Cat}$$

(2) A 2-functor $K^b: \text{k-add} \to \text{k-Tri}$ is canonically defined by setting $K^b(C)$ to be the homotopy category of bounded complexes in $C$ for all $C \in \text{k-add}$. Then the composite pseudofunctor $K^b \circ \text{prj}: \text{k-Cat} \to \text{k-Tri}$ turns out to be a subpseudo-functor of $D \circ \text{Mod}: \text{k-Cat} \to \text{k-Tri}$.

The following is a useful tool to define new colax functors from an old one by composing with pseudofunctors. The proof will be given in the last section.

**Theorem 6.5.** Let $B, C$ and $D$ be 2-categories and $V: C \to D$ a pseudofunctor. Then the obvious correspondence (see subsection [7.7] for details)

$$\hat{\text{Colax}}(B, V): \hat{\text{Colax}}(B, C) \to \hat{\text{Colax}}(B, D)$$

turns out to be a pseudofunctor.

**Definition 6.6.** Let $X = (X, \eta, \theta) \in \hat{\text{Colax}}(I, \text{k-Cat})$.

1. We define the *module colax functor* $\text{Mod} X = (\text{Mod} X, \text{Mod} \eta, \text{Mod} \theta): I \to k\text{-ModCat}$ of $X$ as the composite $\text{Mod} X := \text{Mod} \circ X = \hat{\text{Colax}}(I, \text{Mod})(X): I \xrightarrow{X} k\text{-Cat} \xrightarrow{\text{Mod}} k\text{-ModCat}$. By applying Theorem [6.5] to $B := I, C := k\text{-Cat}, D :=$
Then we have

- for each $i \in I_0$, $(\text{Mod} \ X)(i) = \text{Mod}(X(i))$; and
- for each $a : i \to j$ in $I$ the functor $(\text{Mod} \ X)(a) : (\text{Mod} \ X)(i) \to (\text{Mod} \ X)(j)$ is given by $(\text{Mod} \ X)(a) = - \otimes_{X(i)} X(a)$, where $X(a)$ is an $X(i)$-$X(j)$-bimodule defined by
  \[ X(a)(x, y) := X(j)(y, X(a)(x)) \]
  for all $x \in X(i)_0$ and $y \in X(j)_0$.

(2) By Theorem 6.5 and Example 6.3 we can define a colax functor $\mathcal{D}(\text{Mod} \ X) \in \text{Colax}(I, \mathbb{k} \text{-ModCat})$ as the composite $\mathcal{D}(\text{Mod} \ X) := \mathcal{D} \circ \text{Mod} \ X$, which we call the derived module colax functor of $X$. Then for each $a : i \to j$ in $I$, $\mathcal{D}(\text{Mod} \ X)(i) \xrightarrow{\mathcal{D}(\text{Mod} \ X)(a)} \mathcal{D}(\text{Mod} \ X)(j)$ is equal to
  \[ \mathcal{D}(\text{Mod} \ X(i)) \xrightarrow{\mathcal{D}_X(i, X(a))} \mathcal{D}(\text{Mod} \ X(j)). \]

(3) By Theorem 6.5 and Example 6.4 we can define a pseudofunctor $\text{Colax}(I, \mathbb{k} \text{-Cat}) \to \text{Colax}(I, \mathbb{k} \text{-Tri})$ sending each $X \in \text{Colax}(I, \mathbb{k} \text{-Cat})$ to $\mathbb{k} \text{-bimodule} \mathbb{k} \text{b}(\text{prj} \ X)$. By the remark in Example 6.4(2) $\mathbb{k} \text{b}(\text{prj} \ X)$ is a colax subfunctor of $\mathcal{D}(\text{Mod} \ X)$.

Remark 6.7. Let $\mathcal{C} \in \mathbb{k} \text{-Cat}_0$. Then it is obvious by definitions that
  \[ \Delta(\mathbb{k} \text{b}(\text{prj} \ \mathcal{C})) = \mathbb{k} \text{b}(\text{prj} \ \Delta(\mathcal{C})). \]

Proposition 6.8. The pseudofunctor $\mathbb{k} \text{b} \circ \text{prj}$ preserves $I$-precoverings, that is, if $(F, \psi) : X \to \Delta(\mathcal{C})$ is an $I$-precovering in $\text{Colax}(I, \mathbb{k} \text{-Cat})$ with $\mathcal{C} \in \mathbb{k} \text{-Cat}_0$, then so is $\mathbb{k} \text{b}(\text{prj}(F, \psi)) : \mathbb{k} \text{b}(\text{prj} \ X) \to \Delta(\mathbb{k} \text{b}(\text{prj} \ \mathcal{C}))$ in $\text{Colax}(I, \mathbb{k} \text{-Tri})$.

Proof. It is straightforward to verify that the 2-functors $\oplus$, $\text{sic}$ and $\mathbb{k} \text{b}$ defined in Example 6.4(2) preserve $I$-precoverings. Then the assertion follows from the natural 2-isomorphism $Y : \text{sic} \circ \oplus \Rightarrow \text{prj}$. \hfill \Box

7. Derived equivalences of colax functors

In this section we recall necessary terminologies and the main theorem in our previous paper [6]. First we cite the following. See [6] for the proof.

Lemma 7.1. Let $\mathcal{C}$ be a 2-category and $(F, \psi) : X \to X'$ a 1-morphism in the 2-category $\text{Colax}(I, \mathcal{C})$. Then $(F, \psi)$ is an equivalence in $\text{Colax}(I, \mathcal{C})$ if and only if

1. For each $i \in I_0$, $F(i)$ is an equivalence in $\mathcal{C}$; and
2. For each $a \in I_1$, $\psi(a)$ is a 2-isomorphism in $\mathcal{C}$ (namely, $(F, \psi)$ is $I$-equivariant).

Definition 7.2. Let $X, X' \in \text{Colax}(I, \mathbb{k} \text{-Cat})$. Then $X$ and $X'$ are said to be derived equivalent if $\mathcal{D}(\text{Mod} \ X)$ and $\mathcal{D}(\text{Mod} \ X')$ are equivalent in the 2-category $\text{Colax}(I, \mathbb{k} \text{-Tri})$.

By Lemma 7.1 we obtain the following.
Proposition 7.3. Let \( X, X' \in \tilde{\text{Colax}}(I, \text{k-Cat}) \). Then \( X \) and \( X' \) are derived equivalent if and only if there exists a 1-morphism \((F, \psi) : D(\text{Mod} X) \to D(\text{Mod} X')\) in \( \tilde{\text{Colax}}(I, \text{k-Tri}) \) such that

1. For each \( i \in I_0 \), \( F(i) \) is a triangle equivalence; and
2. For each \( a \in I_1 \), \( \psi(a) \) is a natural isomorphism (i.e., \((F, \psi)\) is \( I \)-equivariant).

A \( \text{k} \)-category \( A \) is called \( \text{k-projective} \) (resp. \( \text{k-flat} \)) if \( A(x, y) \) are projective (resp. flat) \( \text{k} \)-modules for all \( x, y \in A_0 \).

Definition 7.4. Let \( X : I \to \text{\text{k-Cat}} \) be a colax functor.

1. \( X \) is called \( \text{k-projective} \) (resp. \( \text{k-flat} \)) if \( X(i) \) are \( \text{k-projective} \) (resp. \( \text{k-flat} \)) for all \( i \in I_0 \).
2. A colax subfunctor \( T \) of \( \mathcal{K} \text{b}(\text{prj} X) \) is called \textit{tilting} if for each \( i \in I_0 \), \( T(i) \) is a tilting subcategory of \( \mathcal{K} \text{b}(\text{prj} X(i)) \), namely,
   - \( \mathcal{K} \text{b}(\text{prj} X(i))(U, V[n]) = 0 \) for all \( U, V \in T(i)_0 \) and \( 0 \neq n \in \mathbb{Z} \); and
   - the smallest thick subcategory of \( \mathcal{K} \text{b}(\text{prj} X(i)) \) containing \( T(i) \) is equal to \( \mathcal{K} \text{b}(\text{prj} X(i)) \).
3. A tilting colax subfunctor \( T \) of \( \mathcal{K} \text{b}(\text{prj} X) \) with an \( I \)-equivariant inclusion \((\sigma, \rho) : T \hookrightarrow \mathcal{K} \text{b}(\text{prj} X)\) is called a \textit{tilting colax functor} for \( X \).

The following was our main theorem in [6] that gives a generalization of the Morita type theorem characterizing derived equivalences of categories by Rickard [14] and Keller [11] in our setting.

Theorem 7.5. Let \( X, X' \in \tilde{\text{Colax}}(I, \text{k-Cat}) \). Consider the following conditions.

1. \( X \) and \( X' \) are derived equivalent.
2. \( \mathcal{K} \text{b}(\text{prj} X) \) and \( \mathcal{K} \text{b}(\text{prj} X') \) are equivalent in \( \tilde{\text{Colax}}(I, \text{k-Tri}) \).
3. There exists a tilting colax functor \( T \) for \( X \) such that \( T \) and \( X' \) are equivalent in \( \tilde{\text{Colax}}(I, \text{k-Cat}) \).

Then

(a) (1) implies (2).
(b) (2) implies (3).
(c) If \( X' \) is \( \text{k-projective} \), then (3) implies (1).

8. Derived equivalences of Grothendieck constructions

First we cite the statement [11, Corollary 9.2] in the \( \text{k} \)-category case.

Theorem 8.1 (Keller). Let \( A \) and \( B \) be \( \text{k} \)-categories and assume that \( A \) is \( \text{k-flat} \). Then the following are equivalent:

1. \( A \) and \( B \) are derived equivalent.
2. \( B \) is equivalent to a tilting subcategory of \( \mathcal{K} \text{b}(\text{prj} A) \).

The following is our main result in this paper.
Theorem 8.2. Let $X, X' \in \text{Colax}(I, \kcat)$. Assume that $X$ is $\k$-flat and that there exists a tilting colax functor $T$ for $X$ such that $T$ and $X'$ are equivalent in $\text{Colax}(I, \kcat)$ (the condition (3) in Theorem 7.5). Then $\text{Gr}(X)$ and $\text{Gr}(X')$ are derived equivalent.

Proof. Note that $\text{Gr}(X)$ is also $\k$-flat by definition of $\text{Gr}(X)$. Let $T$ be a tilting colax subfunctor of $\mathcal{K}^b(\text{prj } X)$ with an $I$-equivariant inclusion $(\sigma, \rho): T \hookrightarrow \mathcal{K}^b(\text{prj } X)$. Put $(P, \phi) := (P_X, \phi_X)$ for short. Let $T'$ be the full subcategory of $\mathcal{K}^b(\text{prj } \text{Gr}(X))$ consisting of the objects $\mathcal{K}^b(\text{prj } P(i))(U)$ with $i \in I_0$ and $U \in T(i)_0$. Then $T'$ is a tilting subcategory of $\mathcal{K}^b(\text{prj } \text{Gr}(X))$. Indeed, let $L, M \in T'_0$ and $0 \neq p \in \mathbb{Z}$. Then $L = \mathcal{K}^b(\text{prj } P(i))(U)$ and $M = \mathcal{K}^b(\text{prj } P(j))(V)$ for some $i, j \in I_0$ and some $U \in T(i)_0$, $V \in T(j)_0$. Since

$$
\mathcal{K}^b(\text{prj } P, \phi): \mathcal{K}^b(\text{prj } X) \rightarrow \Delta(\mathcal{K}^b(\text{prj } \text{Gr}(X)))
$$

is an $I$-precovering by Proposition 6.8, we have

$$
\mathcal{K}^b(\text{prj } \text{Gr}(X))(L, M[p]) \cong \mathcal{K}^b(\text{prj } \text{Gr}(X))(\mathcal{K}^b(\text{prj } P, \phi)(U), \mathcal{K}^b(\text{prj } P, \phi)(V)[p])
$$

$$
\cong \bigoplus_{a \in I(i,j)} \mathcal{K}^b(\text{prj } X(j))(\mathcal{K}^b(\text{prj } X)(a)(U), V[p])
\cong \bigoplus_{a \in I(i,j)} \mathcal{K}^b(\text{prj } X(j))(T(a)U, V[p]) \overset{(b)}{=} 0,
$$

where the isomorphism (a) follows using the natural isomorphism $\rho(a)$:

$$
\begin{array}{ccc}
\mathcal{K}^b(\text{prj } X(i)) & \overset{\rho(a)}{\cong} & \mathcal{K}^b(\text{prj } X(j)) \\
T(a) & \downarrow & T(i) \\
\mathcal{K}^b(\text{prj } X(i)) & \overset{\rho(a)}{\cong} & \mathcal{K}^b(\text{prj } X(j))
\end{array}
$$

and the equality (b) follows by assumption from the fact that $T(a)U, V \in T(j)$. Now for a triangulated category $U$ and a class of objects $V$ in $U$ we denote by thick $V$ the smallest thick subcategory of $U$ containing $V$. Then for each $i \in I_0$ and $x \in X(i)$ we have $\mathcal{K}^b(\text{prj } P(i))(X(i)(-, x)) \cong (\text{prj } P(i))(X(i)(-, x)) \cong \text{Gr}(X)(-, P(i)(x)) = \text{Gr}(X)(-, x)$ by the formula 622, and hence

$$
\text{Gr}(X)(-, x) \cong \mathcal{K}^b(\text{prj } P(i))(X(i)(-, x))
\in \mathcal{K}^b(\text{prj } P(i))(\text{thick } T(i))
\subseteq \text{thick} \{\mathcal{K}^b(\text{prj } P(i))(U) \mid U \in T(i)\}
\subseteq \text{thick } T'.
$$

Therefore thick $T' = \mathcal{K}^b(\text{prj } \text{Gr}(X))$, and hence $T'$ is a tilting subcategory of $\mathcal{K}^b(\text{prj } \text{Gr}(X))$, as desired. Hence $\text{Gr}(X)$ and $T'$ are derived equivalent because $\text{Gr}(X)$ is $\k$-flat. Let $(F, \psi)$ be the restriction of $\mathcal{K}^b(\text{prj } P, \phi)$ to $T$. Then $(F, \psi): T \rightarrow \Delta(T')$ is a dense functor and an $I$-precovering, thus it is an $I$-covering, which shows that $T' \simeq \text{Gr}(T)$ by Corollary 533. Since $T$ and $X'$ are equivalent in $\text{Colax}(I, \kcat)$,
we have $\text{Gr}(\mathcal{T}) \simeq \text{Gr}(X')$. As a consequence, $\text{Gr}(X)$ and $\text{Gr}(X')$ are derived equivalent. □

**Corollary 8.3.** Let $X, X' \in \mathcal{Colax}(I, k\text{-Cat})$. If $X$ and $X'$ are derived equivalent, then so are $\text{Gr}(X)$ and $\text{Gr}(X')$.

**Proof.** Assume that $X$ and $X'$ are derived equivalent, namely that the condition (1) in Theorem 7.5 is satisfied. Then the condition (3) in Theorem 7.5 holds by Theorem 7.5 (a) and (b). Hence $\text{Gr}(X)$ and $\text{Gr}(X')$ are derived equivalent by the theorem above. □

The following is easy to verify.

**Lemma 8.4.** Let $C, C' \in k\text{-Cat}$. If $C$ and $C'$ are derived equivalent, then so are $\Delta(C)$ and $\Delta(C')$. □

**Corollary 8.3** together with the lemma above and Example 4.2 gives us a unified proof of the following fact.

**Theorem 8.5.** Assume that $k$ is a field and that $k$-algebras $A$ and $A'$ are derived equivalent. Then the following pairs are derived equivalent as well:

1. path-categories $AQ$ and $A'Q$ for any quiver $Q$;
2. incidence categories $AS$ and $A'S$ for any poset $S$; and
3. monoid algebras $AG$ and $A'G$ for any monoid $G$.

□

**Example 8.6.** Assume that $k$ is a field. Let $n$ be a natural number $\geq 3$, and $I$ the free category defined by the quiver $Q$: $2 \overset{\alpha_2}{\rightarrow} 3 \overset{\alpha_3}{\rightarrow} \cdots \overset{\alpha_{n-1}}{\rightarrow} n$. Define functors $X, X': I \rightarrow k\text{-Cat}$ as follows.

For each $i \in I_0 = \{2, \ldots, n\}$ let $X(i)$ be the $k$-category defined by the quiver

$$
\begin{array}{c}
1 \\
\alpha_1 \\
\beta_1 \\
2 \\
\alpha_2 \\
\beta_2 \\
3 \\
\alpha_3 \\
\beta_3 \\
\ldots \\
\alpha_{i-1} \\
\beta_{i-1} \\
i
\end{array}
$$

with relations $\alpha_j \alpha_j = 0$, $\beta_j \beta_j = 0$, $\alpha_j \beta_j = \beta_{j+1} \alpha_{j+1}$ for all $j = 1, \ldots, i - 1$ and $\alpha_1 \beta_1 \alpha_1 = 0$, $\beta_{i-1} \alpha_{i-1} \beta_{i-1} = 0$. For each $a_i: i \rightarrow i + 1$ in $I_1$ let $X(a_i): X(i) \rightarrow X(i + 1)$ be the inclusion functor. This defines a functor $X: I \rightarrow k\text{-Cat}$.

For each $i \in I_0 = \{2, \ldots, n\}$ let $X'(i)$ be the $k$-category defined by the quiver

$$
\begin{array}{c}
1 \\
\gamma_1 \\
2 \\
\gamma_2 \\
3 \\
\gamma_3 \\
\ldots \\
\gamma_i \\
i
\end{array}
$$

with relations $\gamma_j \gamma_j = 0$ for all $j \in \mathbb{Z}/i\mathbb{Z}$. For each $a_i: i \rightarrow i + 1$ in $I_1$ let $X(a_i): X(i) \rightarrow X(i + 1)$ be the functor defined by the correspondence $1 \mapsto 1$, $j \mapsto j + 1$ and $\alpha_1 \mapsto \alpha_2 \alpha_1$, $\alpha_j \mapsto \alpha_j + 1$ for all $j = 2, \ldots, i$. This defines a functor $X': I \rightarrow k\text{-Cat}$.
As is explained in [1] we have a tilting spectroid \( \mathcal{T}(i) \) for \( X(i) \) that is a full subcategory of \( \mathcal{K}^b(\text{prj} \ X(i)) \) consisting of the following \( i \) objects

\[
\begin{align*}
T(i)_1 := (P_1), \\
T(i)_2 := (P_2 \xrightarrow{P(\alpha_2)} P_3 \xrightarrow{P(\alpha_3)} \cdots \xrightarrow{P(\alpha_{i-1})} P_i), \\
T(i)_3 := (P_2 \xrightarrow{P(\alpha_2)} P_3 \xrightarrow{P(\alpha_3)} \cdots \xrightarrow{P(\alpha_{i-2})} P_{i-1}), \\
\vdots \\
T(i)_i := (P_2),
\end{align*}
\]

where \( P_j := X(i)(\cdot, j) \in \text{prj} \ X(i) \) for all \( j \in X(i)_0 \), \( P(\alpha) := X(i)(\cdot, \alpha) \) for all \( \alpha \in X(i)_1 \) and the underline indicates the place of degree zero. Again by [1], \( \mathcal{T}(i) \) is presented by the same quiver with relations as \( \mathcal{X}'(i) \) and we have an isomorphism \( F(i) : \mathcal{X}'(i) \rightarrow \mathcal{T}(i) \) sending \( j \) to \( T(i)_j \) for all \( j = 1, \ldots, i \) and \( \gamma_j \) to a morphism \( \delta(j) : T(i)_j \rightarrow T(i)_{j+1} \) for all \( j \in \mathbb{Z}/\mathbb{Z} \), where \( \delta(i)_1 := (P(\alpha_1)), \delta(i)_j := (1_{P_2}, \ldots, 1_{P_{j-1}}, 0) \) for all \( j = 2, \ldots, i-1 \) and \( \delta(i)_i := (P(\beta_i)) \). Thus \( \mathcal{T}(i) \) gives a derived equivalence between \( X(i) \) and \( \mathcal{X}'(i) \).

For each \( a_i : i \rightarrow i+1 \) in \( I_1 \) define a functor \( \mathcal{T}(a_i) : \mathcal{T}(i) \rightarrow \mathcal{T}(i+1) \) by the correspondence \( T(i)_1 \mapsto T(i+1)_1, T(i)_j \mapsto T(i+1)_{j+1} \) and \( \delta(i)_1 \mapsto \delta(i+1)_2 \delta(i+1)_1, \delta(i)_j \mapsto \delta(i+1)_{j+1} \) for all \( j = 2, \ldots, i \). This defines a functor \( \mathcal{T} : I \rightarrow \mathbb{k}\text{-Cat} \). Then we have a strict commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}'(i) & \xrightarrow{F(i)} & \mathcal{T}(i) \\
\mathcal{X}'(a_i) \downarrow & & \downarrow \mathcal{T}(a_i) \\
\mathcal{X}'(i+1) & \xrightarrow{F(i+1)} & \mathcal{T}(i+1)
\end{array}
\]

in \( \mathbb{k}\text{-Cat} \) for all \( i \in I_0 \), which shows that \( \mathcal{X}' \) and \( \mathcal{T} \) are equivalent in \( \text{Colax}(I, \mathbb{k}\text{-Cat}) \). Finally by definition of \( \mathcal{T}(a_i)'s \) it is easy to see that we have an \( I \)-equivariant inclusion \( (\sigma, \rho) : \mathcal{T} \hookrightarrow \mathcal{K}^b(\text{prj} \ X) \):

\[
\begin{array}{ccc}
\mathcal{T}(i) & \xrightarrow{\sigma(i)} & \mathcal{K}^b(\text{prj} \ X(i)) \\
\mathcal{T}(a_i) \downarrow & \downarrow \rho(a_i) & \downarrow \mathcal{K}^b(\text{prj} \ X(a_i)) \\
\mathcal{T}(i+1) & \xrightarrow{\sigma(i+1)} & \mathcal{K}^b(\text{prj} \ X(i+1)).
\end{array}
\]

Hence by Theorem 8.2 we can glue derived equivalences between \( X(i)'s \) and \( X'(i)'s \) together to have a derived equivalence between \( \text{Gr}(X) \) and \( \text{Gr}(X') \). For example when
$n = 5$, these are presented by the following quivers

$$\text{Gr}(X) = \begin{array}{ccc}
1 & \overset{\alpha_1}{\rightarrow} & 2 \\
\downarrow & & \downarrow \\
1 & \overset{\beta_1}{\rightarrow} 2 & \overset{\alpha_2}{\rightarrow} 3 \\
\downarrow & & \downarrow \\
1 & \overset{\beta_1}{\rightarrow} 2 & \overset{\beta_2}{\rightarrow} 3 & \overset{\alpha_3}{\rightarrow} 4 \\
\downarrow & & & \downarrow \\
1 & \overset{\beta_1}{\rightarrow} 2 & \overset{\beta_2}{\rightarrow} 3 & \overset{\beta_3}{\rightarrow} 4 & \overset{\alpha_4}{\rightarrow} 5 \\
\end{array} \quad \text{,} \quad \text{Gr}(X') = \begin{array}{ccc}
1 & \overset{\gamma_1}{\rightarrow} & 2 \\
\downarrow & & \downarrow \\
1 & \overset{\gamma_1}{\rightarrow} 2 & \overset{\gamma_2}{\rightarrow} 3 & \overset{\gamma_3}{\rightarrow} 4 & \overset{\gamma_4}{\rightarrow} 5 \\
\end{array}$$

with suitable relations as calculated in [7]. Note that if we start with $I$ presented by the same quiver $Q$ as above with relations $a_{i+1}a_i = 0$ for all $i = 2, \ldots, n-2$, then both $\text{Gr}(X)$ and $\text{Gr}(X')$ are presented by the same quivers with relations consisting of the same relations as before together with the additional relations that the vertical paths of length 2 are zero, respectively.

9. The composite of colax functors and pseudofunctors

In this section we prove Theorem 6.5. Throughout this section $B, C$ and $D$ are 2-categories.

**Notation 9.1.** When we denote a colax functor by a letter $X$ the 1-st (resp. 2-nd and 3-rd) entry of $X$ is denoted by $X_{012} := (X_0, X_1, X_2)$ (resp. $\eta^X$ and $\theta^X$), thus we set $X = (X_{012}, \eta^X, \theta^X)$, and sometimes we simply write $X$ for $X_d$ for all $d = 0, 1, 2$ if this seems to make no confusion.

9.1. Correspondences on cells.

**Lemma 9.2.** Let $X: B \to C$ and $V: C \to D$ be colax functors. We define the composite $VX: B \to D$ as follows.

- $(VX)_d := V_d X_d: B_d \xrightarrow{X_d} C_d \xrightarrow{V_d} D_d$ for all $d = 0, 1, 2$.
- $\eta^{VX}_i := \eta^{V}_{X(i)} \circ V \eta^X_i: VX(\mathbf{1}_i) \xrightarrow{V \eta^X_i} V(\mathbf{1}_{X(i)}) \xrightarrow{\eta^X_{(i)}} \mathbf{1}_{(VX)(i)}$ for all $i \in B_0$.
- $\theta^{VX}_{b,a} := \theta^{V}_{X(b), X(a)} \circ V \theta^X_{b,a}: VX(ba) \xrightarrow{V \theta^X_{b,a}} V(X(b) \circ X(a)) \xrightarrow{\theta^X_{X(b), X(a)}} VX(b) \circ VX(a)$ for all $(b, a) \in \text{com}(B)$.

Namely, $VX := ((V_0 X_0, V_1 X_1, V_2 X_2), (\eta^{VX}_{X(i)} \circ V \eta^X_i)_{i \in B_0}, (\theta^{VX}_{X(b), X(a)} \circ V \theta^X_{b,a})_{(b,a) \in \text{com}(B)})$.

Then the composite $\mathcal{C}_{\text{Colax}}(B, V)(X) := VX: B \to D$ is again a colax functor.

**Proof.** It is enough to verify the axioms (i) – (iv) in Definition 6.1.

(i) $(VX)_1, (VX)_2): B(i, j) \xrightarrow{(X_1, X_2)} C(X(i), X(j)) \xrightarrow{(V_1, V_2)} D(VX(i), VX(j))$ is a functor for all $i, j \in B_0$ as a composite of the functors $(X_1, X_2)$ and $(V_1, V_2)$. 
(ii) For each \( a: i \to j \) in \( B \) we have the following commutative diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
V X(a) \mathbb{1}_{V X(i)} & \xrightarrow{V X(a)\eta_X^i} & V X(a) V(\mathbb{1}_{V X(i)}) & \xleftarrow{V X(a)\theta_X^i} & V X(a) V(\mathbb{1}_i) \\
& \downarrow\theta^V_{X(a),1X(i)} & \downarrow\theta^V_{X(a),X(1)} & & \downarrow\theta^V_{X(a),X(1_i)} \\
V(X(a) \mathbb{1}_{X(i)}) & \xrightarrow{V(X(a)\eta_X^i)} & V(X(a) V(\mathbb{1}_{X(i)})) & \xleftarrow{V(X(a)\theta_X^i)} & V(X(a) V(\mathbb{1}_i)) \\
& & & \downarrow V(\theta^V_{a,1}) & \\
& & & V(X(a) \mathbb{1}_i) & \\
\end{array}
\end{array}
\]

The commutativity of the square follows from the axiom (iv) for \( \theta^V \). The remaining commutative diagram is obtained similarly. These two commutative diagrams verify the axiom (ii) of colax functors.

(iii) For each \( i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l \) in \( B \) we have the following commutative diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
V X(cb) & \xrightarrow{V \theta^X_{c,b}} & V X(c) X(ba) & \xrightarrow{\theta^V_{X(c), X(ba)}} & V X(c) \cdot V X(ba) \\
& & \downarrow V(1_X(c, \theta^X_{b,a}) & & \downarrow V X(c) \cdot V \theta^X_{b,a} \\
V(X(cb) X(a)) & \xrightarrow{V(\theta^X_{c,b} \mathbb{1}_{X(a)})} & V(X(c) X(ba) X(a)) & \xrightarrow{\theta^V_{X(c), X(ba) X(a)}} & V X(c) V X(ba) X(a) \\
\downarrow \theta^V_{X(c), X(ba) X(a)} & & \downarrow \theta^V_{X(c), X(ba) X(a)} & & \downarrow \theta^V_{X(c), X(ba) X(a)} \\
V X(cb) \cdot V X(a) & \xrightarrow{V(\theta^X_{c,b} \mathbb{1}_{V X(a)})} & V X(c) X(ba) V X(a) & \xrightarrow{\theta^V_{X(c), X(ba) V X(a)}} & V X(c) \cdot V X(ba) \cdot V X(a), \\
\end{array}
\end{array}
\]

which verifies the axiom (iii) of colax functors.

(iv) Let \( a, a': i \to j; b, b': j \to k; \alpha: a \Rightarrow a' \) and \( \beta: b \Rightarrow b' \) be in \( B \). Then we have the following commutative diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
V X(ba) & \xrightarrow{V(\theta^X_{b,a})} & V X(ba) X(a) & \xrightarrow{\theta^V_{X(ba) X(a)}} & V X(ba) \cdot V X(a) \\
& \downarrow V X(\beta \cdot X(a)) & \downarrow V X(\beta \cdot X(a)) & \downarrow V X(\beta \cdot X(a)) & \\
V X(ba) & \xrightarrow{V(\theta^X_{b,a})} & V X(ba) X(a) & \xrightarrow{\theta^V_{X(ba) X(a)}} & V X(ba) \cdot V X(a), \\
\end{array}
\end{array}
\]

which verifies the axiom (iv) of colax functors. \qed
Lemma 9.3. Let $X, X': B \to C$ and $V: C \to D$ be colax functors and $(F, \psi): X \to X'$ a 1-morphism in $\text{Colax}(B, C)$, and consider the diagram

\[
\begin{array}{c}
\xymatrix{ 
 VX(i) \ar@{..}[rr]_{\theta_{X(i), F(i)}^V} & & VX'(i) \\
 VX(a) \ar[d]_{\theta_{F(j), X(a)}^V} & & VX'(a) \ar[d]^{\theta_{F(j), X(a)}^V} \\
 VX(j) \ar[r]_{VF(j)} & VX'(j). & \\
}
\end{array}
\]

(9.3)

Assume that $\theta_{d,e}^V$ are isomorphisms for all $(d, e) \in \text{com}(C)$ (e.g., that $V$ is a pseudofunctor). Then we can define a 1-morphism $\Gamma_{\text{Colax}(B, C)}(V, \psi) := V(F, \psi): VX \to VX'$ in $\text{Colax}(B, D)$ by

\[
V(F, \psi) := ((V(F(i)))_{i \in B_0}, (\psi_V(a))_{a \in B_1}), \text{ where for } a : i \to j
\]

\[
\psi_V(a) := \theta_{F(j), X(a)}^V \cdot V(\psi(a)) \cdot \theta_{X(a), F(i)}^{-1}.
\]

Proof. We set $X = (X, \eta, \theta)$ and $X' = (X', \eta', \theta')$ for short.

First, the functor $V_{12}: C(X(i), X'(j)) \to D(VX(i), VX'(j))$ sends the commutative square (6.1) to the commutative square (*) below

\[
\begin{array}{c}
\xymatrix{ 
 VX'(a) \cdot VF(i) \ar[d]_{\psi_V(a)} \ar[rr]^{\theta_{X'(a), F(i)}^V} & & VX'(b) \cdot VF(i) \ar[d]^{\psi_V(b)} \\
 V(X'(a)F(i)) \ar[r]_{\theta_{X'(a)F(i), X'(a)}} & V(X'(b)F(i)) & \\
 VF(j) \cdot VX(a) \ar[u]_{V(\psi(a))} \ar[rr]_{\theta_{F(j), X(a)}^V} & & VF(j) \cdot VX(b), & \\
}
\end{array}
\]

which is completed to the commutative diagram above. Hence the family $(\psi_V(a))_{a \in B_1}$ has the property (0) of 1-morphisms in $\text{Colax}(B, D)$ (Definition 4).

(a) For each $i \in B_0$ we have the following commutative diagram:

\[
\begin{array}{c}
\xymatrix{ 
 VX'(i) \cdot VF(i) \ar[r]_{\theta_{VX'(i), F(i)}^V} & V(VX(i) \cdot F(i)) \ar[r]_{V(\psi(i))} & V(F(i) \cdot VX(i)) \ar[r]_{\theta_{F(i), VX(i)}^V} & VF(i) \cdot VX(i) \\
 V(X'(i)F(i)) \ar[r]_{\theta_{X'(i), F(i)}^V} & V(VX(i) \cdot F(i)) \ar[r]_{V(\psi(i))} & V(F(i) \cdot VX(i)) \ar[r]_{\theta_{F(i), VX(i)}^V} & VF(i) \cdot VX(i) \\
 \end{array}
\]

\[
\begin{array}{c}
\xymatrix{ 
 V(1X'(i)) \cdot VF(i) \ar[r]_{\theta_{1X'(i), F(i)}^V} & V(V1X'(i) \cdot F(i)) \ar[r]_{V(\psi(i))} & V(F(i) \cdot 1X(i)) \ar[r]_{\theta_{F(i), 1X(i)}^V} & VF(i) \cdot 1X(i) \\
 \end{array}
\]

\[
\xymatrix{ 
 1VX'(i) \cdot VF(i) \ar[r]_{\theta_{1VX'(i), F(i)}^V} & V(V1X'(i) \cdot F(i)) \ar[r]_{V(\psi(i))} & V(F(i) \cdot 1X(i)) \ar[r]_{\theta_{F(i), 1X(i)}^V} & VF(i) \cdot 1X(i), \\
}
\]
which verifies the axiom (a) of 1-morphisms.

(b) For each $i \to j \to k$ in $B$ we have the following commutative diagrams:

\[
\begin{array}{c}
\theta^V_{X'(ba), F(i)} & \theta^V_{X'(ba), X'(a), F(i)} & \theta^V_{X'(ba), X'(a), X'(a)} \cdot VF(i) \\
\theta^V_{X'(ba), F(i)} & \theta^V_{X'(ba), X'(a), F(i)} & \theta^V_{X'(ba), X'(a), X'(a)} \cdot VF(i) \\
V(X'(ba) \cdot VF(i)) & V(X'(ba)X'(a)) \cdot VF(i) & V(X'(ba)X'(a)X'(a)) \cdot VF(i)
\end{array}
\]

\[
\begin{array}{c}
\theta^V_{X'(ba), P(i)} & \theta^V_{X'(ba), X'(a), P(i)} & \theta^V_{X'(ba), X'(a), X'(a)} \cdot VF(i) \\
\theta^V_{X'(ba), P(i)} & \theta^V_{X'(ba), X'(a), P(i)} & \theta^V_{X'(ba), X'(a), X'(a)} \cdot VF(i) \\
V(X'(ba)F(i)) & V(X'(ba)X'(a)F(i)) & V(X'(ba)X'(a)X'(a)F(i))
\end{array}
\]

and

\[
\begin{array}{c}
\theta^V_{F(k), X(ba)} & \theta^V_{F(k), X(ba)} & \theta^V_{F(k), X(ba)} \cdot VF(i) \\
\theta^V_{F(k), X(ba)} & \theta^V_{F(k), X(ba)} & \theta^V_{F(k), X(ba)} \cdot VF(i) \\
V(F(k) \cdot V(ba)) & V(F(k) \cdot V(ba)) & V(F(k) \cdot V(ba))
\end{array}
\]

\[
\begin{array}{c}
\theta^V_{X'(ba), F(i)} & \theta^V_{X'(ba), X'(a), F(i)} & \theta^V_{X'(ba), X'(a), X'(a)} \cdot VF(i) \\
\theta^V_{X'(ba), F(i)} & \theta^V_{X'(ba), X'(a), F(i)} & \theta^V_{X'(ba), X'(a), X'(a)} \cdot VF(i) \\
V(X'(ba)F(i)) & V(X'(ba)X'(a)F(i)) & V(X'(ba)X'(a)X'(a)F(i))
\end{array}
\]

\[
\begin{array}{c}
\theta^V_{X'(ba), F(i)} & \theta^V_{X'(ba), X'(a), F(i)} & \theta^V_{X'(ba), X'(a), X'(a)} \cdot VF(i) \\
\theta^V_{X'(ba), F(i)} & \theta^V_{X'(ba), X'(a), F(i)} & \theta^V_{X'(ba), X'(a), X'(a)} \cdot VF(i) \\
V(F(k) \cdot V(ba)) & V(F(k) \cdot V(ba)) & V(F(k) \cdot V(ba))
\end{array}
\]

Glue these two diagrams together along the common row to get a large diagram, which verifies the axiom (b) of 1-morphisms. 

---

**Lemma 9.4.** Let $X, X': B \to C$ and $V: C \to D$ be colax functors, $(F, \psi), (F', \psi'): X \to X'$ 1-morphisms, and $\alpha: (F, \psi) \Rightarrow (F', \psi')$ a 2-morphism in $\text{Colax}(B, C)$. Assume that all $\theta^V_{d,c}$ are isomorphisms (e.g., that $V$ is a pseudofunctor). Then we can define a 2-morphism $\text{Colax}(B, V)(\alpha) := V\alpha: V(F, \psi) \Rightarrow V(F', \psi')$ in $\text{Colax}(B, D)$ by

\[
V\alpha := (V\alpha_i)_{i \in B_b}.
\]

**Proof.** Let $a: i \to j$ be in $B$. It is enough to show the commutativity of the following diagram:

\[
\begin{array}{c}
VX'(a) \cdot VF(i) & \theta^V_{X'(a), F(i)} & V(X'(a)F(i)) & \theta^V_{X'(a), F(i)} & V(X'(a)F(i)) \cdot VF(i) \\
VX'(a) \cdot V\alpha_i & \theta^V_{X'(a), F(i)} & V(X'(a)F(i)) & \theta^V_{X'(a), F(i)} & V(X'(a)F(i)) \cdot V\alpha_i \\
VX'(a) \cdot VF'(i) & \theta^V_{X'(a), F'(i)} & V(X'(a)F'(i)) & \theta^V_{X'(a), F'(i)} & V(X'(a)F'(i)) \cdot VF'(i)
\end{array}
\]

\[
\begin{array}{c}
VX'(a) \cdot VF(i) & \theta^V_{X'(a), F(i)} & V(X'(a)F(i)) & \theta^V_{X'(a), F(i)} & V(X'(a)F(i)) \cdot VF(i) \\
VX'(a) \cdot V\alpha_i & \theta^V_{X'(a), F(i)} & V(X'(a)F(i)) & \theta^V_{X'(a), F(i)} & V(X'(a)F(i)) \cdot V\alpha_i \\
VX'(a) \cdot VF'(i) & \theta^V_{X'(a), F'(i)} & V(X'(a)F'(i)) & \theta^V_{X'(a), F'(i)} & V(X'(a)F'(i)) \cdot VF'(i)
\end{array}
\]
Since $\alpha = (\alpha_i : F(i) \Rightarrow F'(i))_{i \in B_0}$ is a 2-morphism in $\overline{\text{Colax}}(B, C)$, we have the commutative diagram
\[
\begin{array}{ccc}
X'(a)F(i) & \xrightarrow{\psi(a)} & F(j)X(a) \\
\downarrow \alpha_i(a) & & \downarrow \alpha_jX(a) \\
X'(i)F'(i) & \xrightarrow{\psi'(a)} & F'(j)X(a).
\end{array}
\]

This gives the commutativity of the central square of the diagram above by applying the proof of Theorem 6.5.

\[\square\]

9.2. **Proof of Theorem 6.5.** By the three lemmas above we can define a correspondence
\[
\overline{\text{Colax}}(B, V)^{012} : \overline{\text{Colax}}(B, C) \rightarrow \overline{\text{Colax}}(B, D)
\]
sending $i$-cells to $i$-cells for all $i = 0, 1, 2$ preserving domains and codomains. It remains to define families $H = (H_X)_{X \in \overline{\text{Colax}}(B, C)_0}$ and $\Theta = (\Theta_{F', F})_{(F', F) \in \text{com}(\overline{\text{Colax}}(B, C))}$ and to show that $\overline{\text{Colax}}(B, V) := (\overline{\text{Colax}}(B, V)^{012}, H, \Theta)$ becomes a pseudofunctor $\overline{\text{Colax}}(B, C) \rightarrow \overline{\text{Colax}}(B, D)$.

For each $X \in \overline{\text{Colax}}(B, C)_0$ we define $H_X : V(1_X) \Rightarrow 1_{VX}$ by setting
\[
H_X := (\eta^V_{X(i)} : V(1_{X(i)}) \Rightarrow 1_{VX(i)})_{i \in B_0}.
\]

Then $H_X$ turns out to be a 2-morphism because by definitions of $\theta^V$ and $\eta^V$ we have a commutative diagram
\[
\begin{array}{ccc}
VX(a) \cdot V(1_{X(i)}) & \xrightarrow{\left(\theta^V_{X(a), 1_{X(i)}}\right)^{-1}} & V(1_{X(j)}) \cdot VX(a) \\
\downarrow \psi^V_{X(a), VX(i)} & & \downarrow \eta^V_{X(j), VX(a)} \\
VX(a) \cdot 1_{VX(i)} & \xrightarrow{\eta^V_{X(i), VX(a)}} & 1_{VX(j)}VX(a)
\end{array}
\]

for all $a : i \Rightarrow j$ in $B$. Note that $H_X$ are isomorphisms because $\eta^V_k$ are for all $k \in C_0$.

For each $(F', F) \in \text{com}(\overline{\text{Colax}}(B, C))$, say $F : X \Rightarrow X'$ and $F' : X' \Rightarrow X''$, we define $\Theta_{F', F} : V(F''F) \Rightarrow VF' \circ VF$ by setting
\[
\Theta_{F', F} := (\theta^V_{F'(i), F(i)} : V(F'(i)F(i)) \Rightarrow VF'(i) \cdot VF(i))_{i \in B_0}.
\]

Then $\Theta_{F', F}$ turns out to be a 2-morphism. Indeed, it is enough to show the commutativity of the diagram
\[
\begin{array}{ccc}
VX''(a) \cdot V(F'(i)F(i)) & \xrightarrow{\psi(a)} & V(F'(j)F(j)) \cdot VX(a) \\
\downarrow \psi^V_{X''(a), F'(i), F(i)} & & \downarrow \psi^V_{F'(j), F(j), VX(a)} \\
VX''(a) \cdot VF'(i) \cdot VF(i) & \xrightarrow{\theta_{F'(i), F(i), VX(a)}} & VF'(j)VF(j)VX(a)
\end{array}
\]
for all \(a: i \to j\) in \(B\), where we set \(V(F'F) = ((VF(i))F(i))_{i \in B_0}, (\Psi(a))_{a \in B_1}\) and \(VF' \cdot VF = ((VF(i)) \cdot VF(i))_{i \in B_0}, (\Psi'(a))_{a \in B_1}\), namely
\[
\Psi(a) = \theta_{F(j),F(i),X(a)}' \cdot V((F'(j) \cdot \psi(a)) \cdot V(\psi'(a) \cdot F(i)) \cdot (\theta_{X'(a),F(i),F(j)})^{-1}
\]
\[
\Psi'(a) = (VF'(j) \cdot (\theta_{F(j),X(a)}) \cdot V(\psi'(a)) \cdot (\theta_{X'(a),F(j),F(i)})^{-1} \circ (\theta_{F(j),X'(a)} \cdot V(\psi'(a)) \cdot (\theta_{X'(a),F(j),F(i)})^{-1} \cdot VF(i))
\]
for all \(a: i \to j\) in \(B\). This follows from the coassociativity of \(V\) and the naturality of \(\theta^V\). Note that \(\Theta_{F',F}\) are isomorphisms because \(\theta^V_{b,a}\) are for all \(a, b \in C_0\).

Now the defining conditions of \(\theta^V\) and \(\eta^V\) directly show that \((\text{Colax}(B,V)_{012}, H, \Theta)\) is a colax functor, hence a pseudofunctor because all \(H_X\) and \(\Theta_{F',F}\) are isomorphisms.

\section*{References}

[1] Asashiba, H.: \textit{A covering technique for derived equivalence}, J. Algebra., 191 (1997), 382–415.
[2] Asashiba, H.: \textit{The derived equivalence classification of representation-finite selfinjective algebras}, J. Algebra, 214 (1999), 182–221.
[3] Asashiba, H.: \textit{Derived and stable equivalence classification of twisted multifold extensions of piecewise hereditary algebras of tree type}, J. Algebra 249 (2002), 345–376.
[4] Asashiba, H.: \textit{A generalization of Gabriel’s Galois covering functors and derived equivalences}, J. Algebra 334 (2011), 109–149.
[5] Asashiba, Hideto: \textit{A generalization of Gabriel’s Galois covering functors II: 2-categorical Cohen-Montgomery duality}, preprint arXiv: 0905.3884.
[6] Asashiba, H.: \textit{Derived equivalences of actions of a category}, Appl. Categor. Struct. DOI 10.1007/s10485-012-9284-5. (arXiv:1111:2239).
[7] Asashiba, H. and Kimura, M.: \textit{Presentations of Grothendieck constructions}, to appear in Comm. in Alg., (arXiv:1111:3845).
[8] Gabriel, P.: \textit{The universal cover of a representation-finite algebra}, In: Lecture Notes in Math., vol. 903, Springer-Verlag, Berlin/New York, 1981, pp. 68–105.
[9] Gordon, R., Power, A. J. and Street, R.: \textit{Coherence for tricategories}. Mem. Amer. Math. Soc., 117 (558):vi+81, 1995.
[10] Grothendieck, A.: Revêtements étals et groupe fondamental, Springer-Verlag, Berlin, 1971.
[11] Keller, B.: \textit{Deriving DG categories}, Ann. scient. Éc. Norm. Sup., 4e série, t. 27, 1994, 63–102.
[12] \textit{Bimodule complexes via strong homotopy actions}, Algebras and Representation theory, Vol. 3, 2000, 357–376. Special issue dedicated to Klaus Roggenkamp on the occasion of his 60th birthday.
[13] Leinster, T.: \textit{Basic Bicategories}, arXiv:math.CT/9810017
[14] Rickard, J.: \textit{Morita theory for derived categories}, J. London Math. Soc., 39 1989, 436–456.
[15] Tamaki, D.: \textit{The Grothendieck construction and gradings for enriched categories}, preprint, arXiv:0907.0061.
[16] Tamaki, D.: \textit{Draft 6 on Grothendieck constructions and smash product constructions}, draft of a preprint.

\textit{E-mail address:} shasash@ipc.shizuoka.ac.jp

\textsc{Department of Mathematics, Faculty of Science, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka, 422-8529, Japan.}