Comment on ‘Time-energy uncertainty relations for neutrino oscillations and the Mössbauer neutrino experiment’

Evgeny Kh. Akhmedov\textsuperscript{a}, Joachim Kopp\textsuperscript{b}, Manfred Lindner\textsuperscript{c}

Max–Planck–Institut für Kernphysik,
Postfach 10 39 80, 69029 Heidelberg, Germany

October 3, 2008

Abstract

We discuss the implications of the time-energy uncertainty relation to recoillessly emitted and captured neutrinos (Mössbauer neutrinos) and show that it does not preclude oscillations of these neutrinos, contrary to a recent claim (J. Phys. G35 (2008) 095003, \texttt{arXiv:0803.0527}).

1 The time-energy uncertainty relation for Mössbauer neutrinos

In a recent interesting article \cite{Bilenky:2008}, Bilenky et al. considered implications of the time-energy uncertainty relation to neutrino oscillations. The authors applied their general results to recoillessly emitted and captured neutrinos (Mössbauer neutrinos, \textsuperscript{2,3}) and concluded that oscillations of such neutrinos would be in conflict with the time-energy uncertainty relation. They also suggested that a Mössbauer neutrino oscillation experiment could test whether the time-energy uncertainty relation is applicable to Mössbauer neutrinos.

We believe that the time-energy uncertainty relation, being based on fundamental principles of quantum theory, does apply to Mössbauer neutrinos. Therefore the conclusions of \cite{Bilenky:2008} are in conflict with the results of our recent detailed quantum field theoretical calculation \cite{Akhmedov:2008}, in which, with no a priori assumptions on the propagating neutrino and very well-established assumptions on the properties of the source and the detector, we have shown that oscillations of Mössbauer neutrinos do occur. We will show now that the contradiction between the conclusions of \cite{Bilenky:2008} and \cite{Akhmedov:2008} is due to an incorrect application of the general results of \cite{Bilenky:2008} to Mössbauer neutrinos.

The argument in \cite{Bilenky:2008} is based on the Mandelstam-Tamm relation

\[
\Delta E \Delta O \geq \frac{1}{2} \left| \frac{d}{dt} \mathcal{O}(t) \right|,
\]  

(1)

\textsuperscript{a}Also at: National Research Centre Kurchatov Institute, 123182 Moscow, Russia
\textsuperscript{b}Email: jkopp@mpi-hd.mpg.de
\textsuperscript{c}Email: lindner@mpi-hd.mpg.de
where $O$ is an arbitrary quantum mechanical operator in the Heisenberg representation, and $\overline{O}(t) = \langle \psi | O | \psi \rangle$ is its expectation value in a state $| \psi \rangle$. Choosing $O$ to be the projection operator onto the neutrino flavour $| \nu_l \rangle$, i.e. $O \equiv | \nu_l \rangle \langle \nu_l |$, one can derive the uncertainty relation

$$
\Delta E \geq \frac{1}{2} \frac{\left| \frac{d}{dt} P_{\nu_l \rightarrow \nu_l}(x,t) \right|}{\sqrt{P_{\nu_l \rightarrow \nu_l}(x,t) - P_{\nu_l \rightarrow \nu_l}^2(x,t)}}.
$$

(2)

Here $P_{\nu_l \rightarrow \nu_l}(x,t) = | \langle \nu_l | \Psi(x,t) \rangle |^2$, with $\Psi(x,t)$ being the neutrino wave function, is the probability for finding a neutrino of flavour $l$ at position $x$ and time $t$.

The authors of [1] have written $P_{\nu_l \rightarrow \nu_l}$ in Eq. (33) of their paper, which is their version of Eq. (2), as a function of only time. They do so because they seek to formulate their arguments not within quantum mechanics (QM), but within the more general framework of quantum field theory (QFT), where one often deals with $x$-independent asymptotic states. It is indeed possible to define a coordinate independent quantity $P(t) = \left| \langle \nu_l | \psi(t) \rangle \right|^2$ by interpreting $| \psi(t) \rangle$ not as a wave function, but as a quantum field theoretical state. However, such an $x$-independent quantity $P(t)$ has no physical meaning; in particular, it cannot be interpreted as an oscillation or survival probability unless the assumption

$$
x \simeq t
$$

(“space-to-time conversion”) is invoked. This seemingly innocent assumption, which is often made for relativistic neutrinos from conventional sources, is grossly invalid for Mössbauer neutrinos. Indeed, it is, strictly speaking, only correct for pointlike relativistic neutrinos or, more generally, in the case when the size of the neutrino wave packet is small compared to the distance $x$ traveled by neutrinos. This is not the case for Mössbauer neutrino experiments, for which the baselines of interest are of order of tens to hundreds of meters, whereas the lengths of the neutrino wave packets exceed 10 km because of near monochromaticity of Mössbauer neutrinos ($\Delta E \lesssim 10^{-11}$ eV [5]).

In [1], the authors obtain their main result by integrating their Eq. (33) (the coordinate independent version of our Eq. (2)) over time. Apart from the lack of physical meaning of the integrand, also the choice of the integration interval is problematic. In [1], the integral runs from 0 to $t_{\text{min}}$, where $t_{\text{min}} \equiv 2\pi E/\Delta m^2$ is supposed to be the time it takes the neutrino to travel to the first oscillation maximum (i.e. to the first minimum of the survival probability). Here $\Delta m^2$ is the neutrino mass squared difference, and following [1] we have adopted the two-flavour approximation for neutrino oscillations. However, from the fact that the size of the neutrino wave packet is much larger than the baseline it is clear that the arrival time is not well defined for Mössbauer neutrinos. It therefore makes no sense to integrate Eq. (2) over time, because the integration interval cannot be given a clear physical meaning. Instead, we will proceed by considering the unintegrated version of the time-energy uncertainty relation, i.e. Eq. (2) itself.\footnote{Note that the $P_{\nu_l \rightarrow \nu_l}$-dependent ratio on the right hand side of Eq. (2) can be considered as the reciprocal of the effective time scale $\Delta t$ over which the expectation value of $O \equiv | \nu_l \rangle \langle \nu_l |$ in the state $\Psi(x,t)$ varies significantly, so that Eq. (2) is equivalent to $\Delta E \Delta t \geq 1/2$.}
Following [6], we write the oscillation probability \( \nu_l \rightarrow \nu_l \) (i.e. the survival probability of the flavour eigenstate \( \nu_l \)) as

\[
P_{\nu_l \rightarrow \nu_l}(x, t) = \sum_{j,k} |U_{lj}|^2 |U_{lk}|^2 e^{-2i\phi(x,t)} g(x - v_j t) g(x - v_k t)^* ,
\]

(4)

Here \( g(x - v_j t) \) are the wave packet shape factors which depend on the group velocities \( v_j \) of the mass eigenstates and on the width and shape of the neutrino wave packets, and \( \phi \) is the oscillation phase, given by

\[
2\phi(x,t) = (E_j - E_k)t - (p_j - p_k)x .
\]

(5)

Eq. (4) is valid in the limit of no wave packet spreading, which is a very good approximation for neutrinos. The shape factors allow one to describe possible effects on oscillations of decoherence and of lack of localization of the neutrino emitter and absorber. As has been shown in [4], the coherence and localization conditions should be very well fulfilled in any realistic Mössbauer neutrino experiment, so \( g(x - v_j t) \) can be set equal to unity in the following. The probability \( P_{\nu_l \rightarrow \nu_l}(x, t) \) then takes the standard form

\[
P_{\nu_l \rightarrow \nu_l}(x, t) = 1 - \sin^2 2\theta \sin^2 \phi(x,t) ,
\]

(6)

where \( \theta \) is the two-flavour mixing angle. Substituting it into Eq. (2), one readily finds

\[
\Delta E \geq |E_1 - E_2| \frac{\sin 2\theta \cos \phi(x,t)}{\sqrt{1 - \sin^2 2\theta \sin^2 \phi(x,t)}} .
\]

(7)

It is sufficient to consider the case \( \sin^2 2\theta = 1 \), because the right hand side has a maximum as a function of \( \theta \) then. Phrased differently, (7) is certainly fulfilled if it is fulfilled for \( \sin^2 2\theta = 1 \). In this case the inequality (7) amounts to

\[
\Delta E \geq |E_1 - E_2| .
\]

(8)

Eq. (8) expresses the obvious requirement that the energy uncertainty of the neutrino state be larger than the difference of the energies of different mass eigenstates composing the given flavour state \( \nu_l \). It has to be fulfilled in any oscillation experiment, and will certainly be satisfied in Mössbauer neutrino experiments where, due to the large momentum uncertainty of the emitted neutrino state, the energy difference \( |E_1 - E_2| \) can be vanishingly small without violating the energy-momentum relation of relativistic neutrinos [4].

2 Evolution in time vs. evolution in space and time

In the literature, there exist different approaches (or schemes) for describing neutrino oscillations (“oscillations in time”, “oscillations in space”, “oscillations in space and time”). The authors of [1] assert that only the experiment can decide which scheme is the correct one, and argue that in fact only the Mössbauer neutrino experiments can do the job. While we do not consider the theory of neutrino oscillations to be finished or closed, we believe that the standard QFT cannot yield different predictions for the same process. In our opinion,
there exist different approximations (not mechanisms or schemes), and to find out which of them are justified, one does not need to perform an experiment: it is sufficient to carefully examine the validity of the invoked assumptions in each particular case.

In [1], the approximation of “oscillations in time” is advocated. We consider this approximation to be invalid for Mössbauer neutrinos for several reasons. Firstly, it is not possible to define a “time of flight” for these neutrinos because they are produced and absorbed recoillessly, and with no accompanying charged leptons being emitted from the atom. Thus, detection of the nuclear recoil or of accompanying charged leptons cannot be used for a precise determination of the neutrino emission or absorption time. It is easy to see that a detection of the recoil of the crystal as a whole cannot be used for this purpose either. Indeed, it would require very long times because one would have to detect a microscopic momentum transfer to a macroscopic body (the time necessary for the crystal to be displaced by an interatomic distance is $\gtrsim 10^{10}$ s). Taking into account also the fact that the length of the Mössbauer neutrino wave packets is about 10 km, we see that the uncertainty of the neutrino emission and absorption times greatly exceeds the time it would take a classical relativistic pointlike particle to travel from the source to the detector.

As another argument against the “evolution in time” picture, note that, if this picture were true at a fundamental level, i.e. without any “space-to-time conversion”, no far detectors would be required in oscillation experiments because the oscillation probability would depend only on $t$, not on $x$. It is only through the assumption $x \simeq t$, Eq. (3), that the standard oscillation phenomenology is recovered. However, as we have shown above, Eq. (3) does not hold for Mössbauer neutrinos.

One possible argument for “evolution only in time” is that such a description is usually employed (and is known to work well) for oscillations of neutral $K$ and $B$ mesons. This actually corresponds to going into the rest frame of the mesons and considering their evolution with proper time. While this approach is justified for $K$ and $B$ mesons, which are extremely degenerate in mass, it is not necessarily applicable to neutrinos, for which the rest frame of flavour states may simply not exist. Indeed, if neutrino masses are hierarchical, in the reference frame where one of the mass eigenstates composing a given flavour state is at rest, the others will be relativistic. More importantly, neutral $K$ and $B$ mesons are not even nearly as monochromatic as Mössbauer neutrinos, so that their wave packets are of microscopic size, and for all practical purposes they can be considered pointlike. As we discussed above, this is not the case for Mössbauer neutrinos, for which the coordinate dependence cannot be ignored even in their rest frame (if it exists), simply because their wave packets are of macroscopic size.

The authors of [1] have correctly pointed out that in QFT the evolution of the states is described by the Schrödinger equation. This does not, however, mean that the evolution in QFT occurs only in time: In fact, the Schrödinger equation of QFT results in just the standard Feynman rules, which can also be obtained from the covariant Lagrangian formalism and which describe the space-time development of the processes. Since we have used the standard Feynman rules in our calculations in [4], the approach based on using the Schrödinger equation must yield results identical to ours. One may then wonder why the approach of Bilenky et al. actually gives different results. In our opinion, this is related to their complete disregard of the spatial evolution of flavour-eigenstate neutrinos. Unlike in QM, in QFT the production and detection processes for mixed states have to be included
into the consideration. This brings in the necessary dependence of the transition probability on the coordinate (through the coordinates of the neutrino source and detector). Moreover, this ensures that the asymptotic (i.e. \textit{in-} and \textit{out-}) states are mass eigenstates, as they have to be in the standard QFT.

Let us finally comment briefly on the possibly counterintuitive result, used in several places throughout this article, that Mössbauer neutrino wave packets have macroscopic spatial and temporal extents \( \sigma_x \simeq \sigma_t \sim 10 \text{ km} \). In a QM approach, this follows immediately from the time-energy uncertainty relation applied to the production process, which tells us that \( \sigma_x \simeq \sigma_t \sim 1/\Delta E \), where \( \Delta E \) is the energy uncertainty associated with the emission process. This relation was confirmed in \cite{4} by direct calculations performed within QFT. For all the regimes we have considered there (inhomogeneous broadening as well as homogeneous broadening, including the case of the natural linewidth dominance) we invariably found that the coherence length for Mössbauer neutrinos was given by \( L_{\text{coh}} \sim 1/\Delta E \Delta v_g \), with \( \Delta E \) the corresponding neutrino linewidth and \( \Delta v_g \) the difference of the group velocities of the wave packets corresponding to different mass eigenstates. Comparing this to the standard expression \( L_{\text{coh}} \sim \sigma_x/\Delta v_g \), we find \( \sigma_x \sim 1/\Delta E \). Thus, our conclusion that the lengths of Mössbauer neutrino wave packets greatly exceed the source–detector distance holds both within QM and QFT.

3 Conclusions

We conclude that, while the general results of Bilenky et al. \cite{1} on implications of the time-energy uncertainty relation to neutrino oscillations are mostly correct, their application of these results to Mössbauer neutrinos was flawed. A proper interpretation of the time-energy uncertainty relation is fully consistent with oscillations of Mössbauer neutrinos.

The main reason for the incorrect conclusion of Bilenky et al. regarding Mössbauer neutrinos was their improper treatment of the evolution of the neutrino state in QFT. In our opinion, the only meaningful way to treat neutrino oscillations in QFT is to explicitly include the production and detection processes, so that the neutrino appears only as an intermediate state. Since the source and the detector are spatially localized, the oscillation probability in this approach exhibits the proper coordinate dependence, i.e. it describes the evolution in space and time. The “evolution in time” approximation could only be justified by using the assumption \( x \simeq t \), which implies the equivalence of “evolution in space” and “evolution in time”. Such an equivalence indeed holds for neutrinos from conventional sources, but not for Mössbauer neutrinos, for which the distance traveled is well defined by that between the source and the detector, while evolution in time has no clear physical meaning because of the very large lengths of the neutrino wave packets.

We are grateful to S. Bilenky, F. von Feilitzsch and W. Potzel for many useful discussions clarifying their point of view.

\textit{Note added.} After the first version of this comment (arXiv:0803.1424v1) was submitted to the archive, the paper \cite{7} has appeared, in which the authors rejected our criticism. In

\footnote{This expression can be easily understood if one notes that decoherence occurs after the wave packets corresponding to different mass eigenstates have separated in coordinate space. This happens after a distance \( \sigma_x/\Delta v_g \).}
the present version of our paper we both comment on \cite{1} and answer the criticism presented in \cite{7}.

\begin{thebibliography}{9}
\bibitem{1} S. M. Bilenky, F. von Feilitzsch, and W. Potzel (2008), \texttt{arXiv:0803.0527[hep-ph]}
\bibitem{2} R. S. Raghavan (2005), \texttt{hep-ph/0511191}
\bibitem{3} R. S. Raghavan (2006), \texttt{hep-ph/0601079}
\bibitem{4} E. K. Akhmedov, J. Kopp, and M. Lindner, JHEP \textbf{05}, 005 (2008), \texttt{0802.2513}
\bibitem{5} W. Potzel, Phys. Scripta \textbf{T127}, 85 (2006).
\bibitem{6} C. Giunti and C. W. Kim, Phys. Rev. \textbf{D58}, 017301 (1998), \texttt{hep-ph/9711363}
\bibitem{7} S. M. Bilenky, F. von Feilitzsch, and W. Potzel (2008), \texttt{arXiv:0804.3409[hep-ph]}
\end{thebibliography}