ON $m$–FOLD HOLOMORPHIC DIFFERENTIALS AND MODULAR FORMS

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Abstract. Let $\Gamma$ be the Fuchsian group of the first kind. For an even integer $m \geq 4$, we study $m/2$–holomorphic differentials in terms of space of (holomorphic) cuspidal modular forms $S_m(\Gamma)$. We also give in depth study of Wronskians of cuspidal modular forms and their divisors.

1. Introduction

Let $\Gamma$ be the Fuchsian group of the first kind [7, Section 1.7, page 28]. Examples of such groups are the important modular groups such as $SL_2(\mathbb{Z})$ and its congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$ [7, Section 4.2]. Let $\mathbb{H}$ be the complex upper half-plane. The quotient $\Gamma \backslash \mathbb{H}$ can be compactified by adding a finite number of $\Gamma$-orbits of points in $\mathbb{R} \cup \{\infty\}$ called cusps of $\Gamma$ and we obtain a compact Riemann surface which will be denoted by $\mathcal{R}_\Gamma$. For $l \geq 1$, let $H^l(\mathcal{R}_\Gamma)$ be the space of all holomorphic differentials on $\mathcal{R}_\Gamma$ (see [5], or Section 3 in this paper).

Let $m \geq 2$ be an even integer. Let $S_m(\Gamma)$ be the space of (holomorphic) cusp forms of weight $m$ (see Section 2). It is well–known that $S_2(\Gamma)$ is naturally isomorphic to the vector space $H^1(\mathcal{R}_\Gamma)$ (see [7, Theorem 2.3.2]). This is employed on many instances in studying various properties of modular curves (see for example [19, Chapter 6]). In this paper we study the generalization of this concept to the holomorphic differentials of higher order. For an even integer $m \geq 4$, in general, the space $S_m(\Gamma)$ is too big to be isomorphic to $H^{m/2}(\mathcal{R}_\Gamma)$ due to presence of cusps and elliptic points. So, in general we define a subspace

$$S^H_m(\Gamma) = \{ f \in S_m(\Gamma); \ f = 0 \text{ or } f \text{ satisfies } (1-1) \},$$

where

$$c_f \geq \sum_{a \in \mathcal{R}_\Gamma, \text{elliptic}} \left[ \frac{m}{2} (1 - 1/e_a) \right] a + \left( \frac{m}{2} - 1 \right) \sum_{b \in \mathcal{R}_\Gamma, \text{cusp}} b.$$  

The integral divisor $c_f$ is defined in Lemma 2.2 while the multiplicities $e_a$ are defined in Section 2. Now, we have the following result (see Section 4):

Theorem 1.2. The usual map $f \mapsto \omega_f$ from the space of all cuspidal modular form into space of meromorphic differentials (see [7, Theorem 2.3.3]) induces the isomorphism of $S^H_m(\Gamma)$ onto $H^{m/2}(\mathcal{R}_\Gamma)$.

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We study the space $S_{m/2}^H(\Gamma)$ in detail in Section (see Section 4). The main results are contained in a very detailed Lemma 4-5 and Theorem 4-16. We recall (see [5, III.5.9] or Definition 3-6) that $a \in \mathcal{R}_\Gamma$ is a $m/2$-Weierstrass point if there exists a non–zero $\omega \in H^{m/2}(\mathcal{R}_\Gamma)$ such that

$$\nu_a(\omega) \geq \dim H^{m/2}(\mathcal{R}_\Gamma).$$

Equivalently [5, Proposition III.5.10], if

$$\nu_a(W(\omega_1, \ldots, \omega_t)) \geq 1,$$

where $W(\omega_1, \ldots, \omega_t)$ is the Wronskian of holomorphic differential forms $\omega_1, \ldots, \omega_t$ (see Section 3).

When $m = 2$ we speak about classical Weierstrass points. So, 1-Weierstrass points are simply Weierstrass points. Weierstrass points on modular curves are very-well studied (see for example [19, Chapter 6], [16], [17], [20], [21], [22]). Higher–order Weierstrass points has not been not studied much (see for example [16]).

The case $m \geq 4$ is more complex. We recall that $\mathcal{R}_\Gamma$ is hyperelliptic if $g(\Gamma) \geq 2$, and there is a degree two map onto $\mathbb{P}^1$. By general theory [9, Chapter VII, Proposition 1.10], if $g(\Gamma) = 2$, then $\mathcal{R}_\Gamma$ is hyperelliptic. If $\mathcal{R}_\Gamma$ is not hyperelliptic, then $\dim S_2(\Gamma) = g(\Gamma) \geq 3$, and the regular map $\mathcal{R}_\Gamma \rightarrow \mathbb{P}^{g(\Gamma) - 1}$ attached to a canonical divisor $K$ is an isomorphism onto its image [9, Chapter VII, Proposition 2.1].

Let $\Gamma = \Gamma_0(N), N \geq 1$. Put $X_0(N) = \mathcal{R}_{\Gamma_0(N)}$. We recall that $g(\Gamma_0(N)) \geq 2$ unless

$$\begin{cases} N \in \{1 - 10, 12, 13, 16, 18, 25\} & \text{when } g(\Gamma_0(N)) = 0, \\
N \in \{11, 14, 15, 17, 19 - 21, 24, 27, 32, 36, 49\} & \text{when } g(\Gamma_0(N)) = 1. \end{cases}$$

Let $g(\Gamma_0(N)) \geq 2$. Then, we remark that Ogg [17] has determined all $X_0(N)$ which are hyperelliptic curves. In view of Ogg’s paper, we see that $X_0(N)$ is not hyperelliptic for $N \in \{34, 38, 42, 43, 44, 45, 51 - 58, 60 - 70\}$ or $N \geq 72$. This implies $g(\Gamma_0(N)) \geq 3$.

We prove the following result (see Theorem 1-12).

**Theorem 1-3.** Let $m \geq 4$ be an even integer. Assume that $\mathcal{R}_\Gamma$ is not hyperelliptic. Then, we have

$$S_{m,2}^H(\Gamma) = S_m^H(\Gamma),$$

where we denote the subspace $S_{m,2}^H(\Gamma)$ of $S_m^H(\Gamma)$ spanned by all monomials

$$f_0^{\alpha_0}f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}, \quad \alpha_i \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^{g-1} \alpha_i = \frac{m}{2}.$$  

Here $f_0, \ldots, f_{g-1}, g = g(\Gamma)$, is a basis of $S_2(\Gamma)$.

The criterion is given by the following corollary (see Corollary 4-14):
**Corollary 1-4.** Let \( m \geq 4 \) be an even integer. Assume that \( \mathfrak{M}_\Gamma \) is not hyperelliptic. Assume that \( a_\infty \) is a cusp for \( \Gamma \). Let us select a basis \( f_0, \ldots, f_{g-1}, \) \( g = g(\Gamma) \), of \( S_2(\Gamma) \). Compute \( q \)-expansions of all monomials

\[
f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}, \quad \alpha_i \in \mathbb{Z}_{\geq 0}, \quad \sum_{i=0}^{g-1} \alpha_i = \frac{m}{2}.
\]

Then, \( a_\infty \) is **not** a \( \frac{m}{2} \)-Weierstrass point if and only if there exist a basis of the space of all such monomials, \( F_1, \ldots, F_t, \) \( t = \dim S_m^H(\Gamma) = (m-1)(g-1) \) (see Lemma 4-5 (v)), such that their \( q \)-expansions are of the form

\[
F_u = a_u q^{u+m/2-1} + \text{higher order terms in } q, \quad 1 \leq u \leq t,
\]

where

\[
a_u \in \mathbb{C}, \quad a_u \neq 0.
\]

This is useful for explicit computations in SAGE at least when \( \Gamma = \Gamma_0(N) \). We give examples in Section 5 (see Propositions 5-1 and 5-2). A different more theoretical criterion is contained in Theorem 4-16.

Various other aspects of modular curves has been studied in [1], [2], [3], [4], [10], [14], [15] and [24]. We continue the approach presented in [11], [12], and [13]. In the proof of Theorem 4-12 we give an explicit construction of a higher order canonical map i.e., a map attached to divisor \( \frac{m}{2} K \), where \( K \) is a canonical divisor of \( \mathfrak{M}_\Gamma \). The case \( m = 2 \) is studied in depth in many papers (see for example [3]).

In Section 6 we deal with a generalization of the usual notion of the Wronskian of cuspidal modular forms [22], (19), 6.3.1, ([10], the proof of Theorem 4-5), and ([12], Lemma 4-1). The main result of the section is Proposition 6-2 which in the most important case has the following form:

**Proposition 1-5.** Let \( m \geq 1 \). Then, for any sequence \( f_1, \ldots, f_k \in M_m(\Gamma) \), the Wronskian

\[
W(f_1, \ldots, f_k)(z) \overset{df}{=} \begin{vmatrix}
    f_1(z) & \cdots & f_k(z) \\
    \frac{df_1(z)}{dz} & \cdots & \frac{df_k(z)}{dz} \\
    \vdots & \ddots & \vdots \\
    \frac{d^{k-1}f_1(z)}{dz^{k-1}} & \cdots & \frac{d^{k-1}f_k(z)}{dz^{k-1}}
\end{vmatrix}
\]

is a cuspidal modular form in \( S_{k(m+k-1)}(\Gamma) \) if \( k \geq 2 \). If \( f_1, \ldots, f_k \) are linearly independent, then \( W(f_1, \ldots, f_k) \neq 0 \).

What is new and deep is the computation of the divisor of \( W(f_1, \ldots, f_k) \) (see Section 7). The main results are Proposition 7-2 and Theorem 7-3. A substantial example has been given in Section 8 in the case of \( \Gamma = SL_2(\mathbb{Z}) \) (see Proposition 8-4).

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2. Preliminaries

In this section we recall necessary facts about modular forms and their divisors [7]. We follow the exposition in ([12], Section 2).

Let \( \mathbb{H} \) be the upper half–plane. Then the group \( SL_2(\mathbb{R}) \) acts on \( \mathbb{H} \) as follows:
\[
g \cdot z = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).
\]
We let \( j(g, z) = \frac{cz + d}{y^2} \). The function \( j \) satisfies the cocycle identity:
\[
j(gg', z) = j(g, g'z)j(g', z).
\]

Next, \( SL_2(\mathbb{R}) \)–invariant measure on \( \mathbb{H} \) is defined by \( dxdy/y^2 \), where the coordinates on \( \mathbb{H} \) are written in a usual way \( z = x + \sqrt{-1}y, \ y > 0 \). A discrete subgroup \( \Gamma \subset SL_2(\mathbb{R}) \) is called a Fuchsian group of the first kind if
\[
\int \int_{\Gamma \setminus \mathbb{H}} \frac{dxdy}{y^2} < \infty.
\]
Then, adding a finite number of points in \( \mathbb{R} \cup \{\infty\} \) called cusps, \( \mathcal{F}_\Gamma \) can be compactified. In this way we obtain a compact Riemann surface \( \mathcal{R}_\Gamma \). One of the most important examples are the groups
\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \ c \equiv 0 \mod N \right\}, \quad N \geq 1.
\]

We write \( X_0(N) \) for \( \mathcal{R}_{\Gamma_0(N)} \).

Let \( \Gamma \) be a Fuchsian group of the first kind. We consider the space \( M_m(\Gamma) \) (resp., \( S_m(\Gamma) \)) of all modular (resp., cuspidal) forms of weight \( m \); this is the space of all holomorphic functions \( f : \mathbb{H} \to \mathbb{C} \) such that \( f(\gamma \cdot z) = j(\gamma, z)^m f(z) \) \( (z \in \mathbb{H}, \ \gamma \in \Gamma) \) which are holomorphic (resp., holomorphic and vanish) at every cusp for \( \Gamma \). We also need the following obvious property: for \( f, g \in M_m(\Gamma, \chi), \ g \neq 0 \), the quotient \( f/g \) is a meromorphic function on \( \mathcal{R}_\Gamma \).

Next, we recall from ([7], 2.3) some notions related to the theory of divisors of modular forms of even weight \( m \geq 2 \) and state a preliminary result.

Let \( m \geq 2 \) be an even integer and \( f \in M_m(\Gamma) - \{0\} \). Then, \( \nu_{z - \xi}(f) \) denotes the order of the holomorphic function \( f \) at \( \xi \). For each \( \gamma \in \Gamma \), the functional equation \( f(\gamma \cdot z) = j(\gamma, z)^m f(z) \), \( z \in \mathbb{H} \), shows that \( \nu_{z - \xi}(f) = \nu_{z' - \xi'}(f) \) where \( \xi' = \gamma \cdot \xi \). Also, if we let
\[
eq = \#(\Gamma \xi / \Gamma \cap \{\pm 1\}),
\]
then \( \nu_{\xi} = \nu_{\xi'} \). The point \( \xi \in \mathbb{H} \) is elliptic if \( e_\xi > 1 \). Next, following ([7], 2.3), we define
\[
\nu_{\xi}(f) = \nu_{z - \xi}(f)/e_\xi.
\]
Clearly, \( \nu_\xi = \nu_{\xi'} \), and we may let
\[
\nu_{a_\xi}(f) = \nu_{\xi}(f),
\]
where
\[
a_\xi \in \mathcal{R}_\Gamma \text{ is the projection of } \xi \text{ to } \mathcal{R}_\Gamma,
\]
a notation we use throughout this paper. If \( x \in \mathbb{R} \cup \{ \infty \} \) is a cusp for \( \Gamma \), then we define \( \nu_x(f) \) as follows. Let \( \sigma \in SL_2(\mathbb{R}) \) such that \( \sigma.x = \infty \). We write

\[
\{ \pm 1 \} \sigma \Gamma_x \sigma^{-1} = \{ \pm 1 \} \left\{ \begin{pmatrix} 1 & lh' \\ 0 & 1 \end{pmatrix} ; \ l \in \mathbb{Z} \right\},
\]

where \( h' > 0 \). Then we write the Fourier expansion of \( f \) at \( x \) as follows:

\[
(f|_{m\sigma^{-1}})(\sigma.z) = \sum_{n=1}^{\infty} a_n e^{2\pi \sqrt{-1} \sigma.z/h'}.
\]

We let

\[
\nu_x(f) = l \geq 0,
\]

where \( l \) is defined by \( a_0 = a_1 = \cdots = a_{l-1} = 0, \ a_l \neq 0 \). One easily see that this definition does not depend on \( \sigma \). Also, if \( x' = \gamma.x \), then \( \nu_{x'}(f) = \nu_x(f) \). Hence, if \( b_x \in \mathcal{R}_\Gamma \) is a cusp corresponding to \( x \), then we may define

\[
\nu_{b_x} = \nu_x(f).
\]

Put

\[
div(f) = \sum_{a \in \mathcal{R}_\Gamma} \nu_a(f) a \in \mathbb{Q} \otimes \text{Div}(\mathcal{R}_\Gamma),
\]

where \( \text{Div}(\mathcal{R}_\Gamma) \) is the group of (integral) divisors on \( \mathcal{R}_\Gamma \).

Using ([7], 2.3), this sum is finite i.e., \( \nu_a(f) \neq 0 \) for only a finitely many points. We let

\[
\deg(div(f)) = \sum_{a \in \mathcal{R}_\Gamma} \nu_a(f).
\]

Let \( \mathfrak{d}_i \in \mathbb{Q} \otimes \text{Div}(\mathcal{R}_\Gamma), \ i = 1, 2 \). Then we say that \( \mathfrak{d}_1 \geq \mathfrak{d}_2 \) if their difference \( \mathfrak{d}_1 - \mathfrak{d}_2 \) belongs to \( \text{Div}(\mathcal{R}_\Gamma) \) and is non-negative in the usual sense.

**Lemma 2-2.** Assume that \( m \geq 2 \) is an even integer. Assume that \( f \in M_m(\Gamma) \), \( f \neq 0 \). Let \( t \) be the number of inequivalent cusps for \( \Gamma \). Then we have the following:

(i) For \( a \in \mathcal{R}_\Gamma \), we have \( \nu_a(f) \geq 0 \).

(ii) For a cusp \( a \in \mathcal{R}_\Gamma \), we have that \( \nu_a(f) \geq 0 \) is an integer.

(iii) If \( a \in \mathcal{R}_\Gamma \) is not an elliptic point or a cusp, then \( \nu_a(f) \geq 0 \) is an integer. If \( a \in \mathcal{R}_\Gamma \) is an elliptic point, then \( \nu_a(f) - \frac{m}{2}(1 - 1/e_a) \) is an integer.

(iv) Let \( g(\Gamma) \) be the genus of \( \mathcal{R}_\Gamma \). Then

\[
\deg(div(f)) = m(g(\Gamma) - 1) + \frac{m}{2} \left( t + \sum_{a \in \mathcal{R}_\Gamma, \ \text{elliptic}} (1 - 1/e_a) \right)
\]

\[
= \frac{m}{4\pi} \int_{\Gamma \setminus \Xi} dx dy / y^2.
\]

(v) Let \([x]\) denote the largest integer \( \leq x \) for \( x \in \mathbb{R} \). Then
\[ \dim S_m(\Gamma) = \begin{cases} (m-1)(g(\Gamma) - 1) + \left(\frac{m}{2} - 1\right)t + \sum_{a \in \mathcal{R}^r, \text{elliptic}} \left[\frac{m}{2}(1 - 1/e_a)\right], & \text{if } m \geq 4, \\ g(\Gamma), & \text{if } m = 2. \end{cases} \]

\[ \dim M_m(\Gamma) = \begin{cases} \dim S_m(\Gamma) + t, & \text{if } m \geq 4, \text{ or } m = 2 \text{ and } t = 0, \\ \dim S_m(\Gamma) + t - 1 = g(\Gamma) + t - 1, & \text{if } m = 2 \text{ and } t \geq 1. \end{cases} \]

(vi) There exists an integral divisor \( c'_f \geq 0 \) of degree
\[ \begin{cases} \dim M_m(\Gamma) + g(\Gamma) - 1, & \text{if } m \geq 4, \text{ or } m = 2 \text{ and } t \geq 1, \\ 2(g(\Gamma) - 1), & \text{if } m = 2 \text{ and } t = 0 \\ \end{cases} \]

such that
\[ \text{div}(f) = c'_f + \sum_{a \in \mathcal{R}^r, \text{elliptic}} \left(\frac{m}{2}(1 - 1/e_a) - \left[\frac{m}{2}(1 - 1/e_a)\right]\right) a. \]

(vii) Assume that \( f \in S_m(\Gamma) \). Then, the integral divisor defined by \( c_f \overset{\text{def}}{=} c'_f - \sum_{b \in \mathcal{R}^r, \text{cusp}} b \)
satisfies \( c_f \geq 0 \) and its degree is given by
\[ \begin{cases} \dim S_m(\Gamma) + g(\Gamma) - 1; & \text{if } m \geq 4, \\ 2(g(\Gamma) - 1); & \text{if } m = 2. \end{cases} \]

Proof. The claims (i)–(v) are standard ([7], 2.3, 2.4, 2.5). The claim (vi) follows from (iii), (iv), and (v) (see Lemma 4-1 in [10]). Finally, (vii) follows from (vi). \( \Box \)

3. Holomorphic Differentials and \( m \)-Weierstrass Points on \( \mathcal{R}_\Gamma \)

Let \( \Gamma \) be a Fuchsian group of the first kind. We let \( D^m(\mathcal{R}_\Gamma) \) (resp., \( H^m(\mathcal{R}_\Gamma) \)) be the space of meromorphic (resp., holomorphic) differential of degree \( m \) on \( \mathcal{R}_\Gamma \) for each \( m \in \mathbb{Z} \). We recall that \( D^0(\mathcal{R}_\Gamma) = \mathbb{C}(\mathcal{R}_\Gamma) \), and \( D^m(\mathcal{R}_\Gamma) \neq 0 \) for all other \( m \in \mathbb{Z} \). In fact, if we fix a non-zero \( \omega \in D^1(\mathcal{R}_\Gamma) \), then \( D^m(\mathcal{R}_\Gamma) = \mathbb{C}(\mathcal{R}_\Gamma) \omega^n \). We have the following:

\[ \deg(\text{div}(\omega)) = 2m(g(\Gamma) - 1), \; \omega \in D^m(\mathcal{R}_\Gamma), \; \omega \neq 0. \]

We shall be interested in the case \( m \geq 1 \), and in holomorphic differentials. We recall [5, Proposition III.5.2] that

\[ \dim H^m(\mathcal{R}_\Gamma) = \begin{cases} 0; & \text{if } m \geq 1, g(\Gamma) = 0; \\ g(\Gamma); & \text{if } m = 1, g(\Gamma) \geq 1; \\ g(\Gamma) / (2m - 1) (g(\mathcal{R}_\Gamma) - 1); & \text{if } m \geq 2, g(\Gamma) = 1; \\ (2m - 1) (g(\mathcal{R}_\Gamma) - 1); & \text{if } m \geq 2, g(\Gamma) \geq 2. \end{cases} \]

This follows easily from Riemann-Roch theorem. Recall that a canonical class \( K \) is simply a divisor on any non-zero meromorphic form \( \omega \) on \( \mathcal{R}_\Gamma \). Different choices of a \( \omega \) differ by a divisor of a non-zero function \( f \in \mathbb{C}(\mathcal{R}_\Gamma) \)
\[ \text{div}(f\omega) = \text{div}(f) + \text{div}(\omega). \]
Different choices of $\omega$ have the same degree since $\deg(\text{div}(f)) = 0$.

For a divisor $a$, we let

$$L(a) = \{ f \in \mathbb{C}(\mathcal{R}_F); \ f = 0 \text{ or } \text{div}(f) + a \geq 0 \}. $$

We have the following three facts:

1. for $a = 0$, we have $L(a) = \mathbb{C}$;
2. if $\deg(a) < 0$, then $L(a) = 0$;
3. the Riemann-Roch theorem: $\dim L(a) = \deg(a) - g(\Gamma) + 1 + \dim L(K - a)$.

Now, it is obvious that $f \omega^m \in H^m(\mathcal{R}_F)$ if and only if

$$\text{div}(f \omega^m) = \text{div}(f) + m \text{div}(\omega) = \text{div}(f) + mK \geq 0.$$ 

Equivalently, $f \in L(mK)$. Thus, we have that $\dim H^m(\mathcal{R}_F) = \dim L(mK)$. Finally, by the Riemann-Roch theorem, we have the following:

$$\dim L(mK) = \deg(mK) - g(\Gamma) + 1 + \dim L((1-m)K) = (2m-1)(g(\mathcal{R}_F) - 1) + \dim L((1-m)K).$$

Now, if $g(\Gamma) \geq 2$, then $\deg(K) = 2(g(\Gamma) - 1) > 0$, and the claim easily follows from (1) and (2) above. Next, assume that $g(\Gamma) = 1$. If $\omega \in \dim H^1(\mathcal{R}_F)$ is non-zero, then it has a degree zero. Thus, it has no zeroes. This means that $\omega H^{l-1}(\mathcal{R}_F) = H^l(\mathcal{R}_F)$ for all $l \in \mathbb{Z}$. But since obviously $H^0(\mathcal{R}_F)$ consists of constants only, we obtain the claim. Finally, the case $g(\Gamma) = 0$ is obvious from (2) since the degree of $mK$ is $2m(g(\Gamma) - 1) < 0$ for all $m \geq 1$.

Assume that $g(\Gamma) \geq 1$ and $m \geq 1$. Then, $\dim H^m(\mathcal{R}_F) \neq 0$. Let $t = \dim H^m(\mathcal{R}_F)$. We fix the basis $\omega_1, \ldots, \omega_t$ of $H^m(\mathcal{R}_F)$. Let $z$ be any local coordinate on $\mathcal{R}_F$. Then, locally there exists unique holomorphic functions $\varphi_1, \ldots, \varphi_t$ such that $\omega_i = \varphi_i (dz)^m$, for all $i$. Then, again locally, we can consider the Wronskian $W_z$ defined by

$$W_z (\omega_1, \ldots, \omega_t) \overset{\text{def}}{=} \begin{vmatrix} \varphi_1(z) & \cdots & \varphi_t(z) \\ \frac{d\varphi_1(z)}{dz} & \cdots & \frac{d\varphi_t(z)}{dz} \\ \vdots & \ddots & \vdots \\ \frac{d^{t-1}\varphi_1(z)}{dz^{t-1}} & \cdots & \frac{d^{t-1}\varphi_t(z)}{dz^{t-1}} \end{vmatrix}. $$

As proved in [5, Proposition III.5.10], collection of all

$$W_z (\omega_1, \ldots, \omega_t) (dz)^{t/2(2m-1+t)},$$

defines a non-zero holomorphic differential form

$$W (\omega_1, \ldots, \omega_t) \in H^{t/2(2m-1+t)}(\mathcal{R}_F).$$

We call this form the Wronskian of the basis $\omega_1, \ldots, \omega_t$. It is obvious that a different choice of a basis of $H^m(\mathcal{R}_F)$ results in a Wronskian which differ from $W (\omega_1, \ldots, \omega_t)$ by a multiplication by a non-zero complex number. Also, the degree is given by

$$\deg(\text{div}(W (\omega_1, \ldots, \omega_t))) = t(2m - 1 + t) (g(\mathcal{R}_F) - 1).$$

Following [5, III.5.9], we make the following definition:
**Definition 3-6.** Let $m \geq 1$ be an integer. We say that $a \in \mathcal{R}_\Gamma$ is a $m$-Weierstrass point if there exists a non–zero $\omega \in H^m(\mathcal{R}_\Gamma)$ such that

$$\nu_a(\omega) \geq \dim H^m(\mathcal{R}_\Gamma).$$

Equivalently [5, Proposition III.5.10], if

$$\nu_a(W(\omega_1, \ldots, \omega_t)) \geq 1.$$

When $m = 1$ we speak about classical Wierstrass points. So, 1-Weierstrass points are simply Weierstrass points.

### 4. Interpretation in Terms of Modular Forms

In this section we give interpretation of results of Section 3 in terms of modular forms. Again, $\Gamma$ stand for a Fuchsian group of the first kind. Let $m \geq 2$ be an even integer. We consider the space $A_m(\Gamma)$ be the space of all all meromorphic functions $f : \mathbb{H} \to \mathbb{C}$ such that $f(\gamma z) = j(\gamma, z)^m f(z)$ ($z \in \mathbb{H}$, $\gamma \in \Gamma$) which are meromorphic at every cusp for $\Gamma$. By [7, Theorem 2.3.1], there exists isomorphism of vector spaces $A_m(\Gamma) \to D^{m/2}(\mathcal{R}_\Gamma)$, denoted by $f \mapsto \omega_f$ such that the following holds (see Section 2 for notation, and [7, Theorem 2.3.3]):

$$\nu_{a_\xi}(f) = \nu_{a_\xi}(\omega_f) + \frac{m}{2} \left(1 - \frac{1}{e_{a_\xi}}\right) \quad \text{if } \xi \in \mathbb{H}$$

(4-1)

$$\nu_a(f) = \nu_a(\omega_f) + \frac{m}{2} \quad \text{for } \Gamma \text{-cusp } a.$$

$$\text{div}(f) = \text{div}(\omega_f) + \sum_{a \in \mathcal{R}_\Gamma, \text{elliptic}} \frac{m}{2} \left(1 - \frac{1}{e_a}\right) a,$$

where $1/e_a = 0$ if $a$ is a cusp. Let $f \in M_m(\Gamma)$. Then, combining Lemma 2-2 (vi) and (4-1), we obtain

(4-2) \hspace{1cm} \text{div}(\omega_f) = c'_f - \sum_{a \in \mathcal{R}_\Gamma, \text{elliptic}} \left[\frac{m}{2} (1 - 1/e_a)\right] a - \frac{m}{2} \sum_{b \in \mathcal{R}_\Gamma, \text{cusp}} b.$$

This shows that $\omega_f$ is holomorphic everywhere except maybe at cusps and elliptic points. Moreover, if $f \in S_m(\Gamma)$, then (see Lemma 2-2 (vii))

(4-3) \hspace{1cm} \text{div}(\omega_f) = c_f - \sum_{a \in \mathcal{R}_\Gamma, \text{elliptic}} \left[\frac{m}{2} (1 - 1/e_a)\right] a - \left(\frac{m}{2} - 1\right) \sum_{b \in \mathcal{R}_\Gamma, \text{cusp}} b.$$

Next, we determine all $f \in M_m(\Gamma)$ such that $\omega_f \in H^{m/2}(\mathcal{R}_\Gamma)$. From (4-2) we see that such $f$ must belong to $S_m(\Gamma)$, and from (4-3)

(4-4) \hspace{1cm} c_f \geq \sum_{a \in \mathcal{R}_\Gamma, \text{elliptic}} \left[\frac{m}{2} (1 - 1/e_a)\right] a + \left(\frac{m}{2} - 1\right) \sum_{b \in \mathcal{R}_\Gamma, \text{cusp}} b.$$

Now, we define the subspace of $S_m(\Gamma)$ by

$$S^H_m(\Gamma) = \{ f \in S_m(\Gamma); \ f = 0 \text{ or } f \text{ satisfies (4-4)} \}.$$
It is mapped via $f \mapsto \omega_f$ isomorphically onto $H^{m/2}(\mathcal{M}_\Gamma)$.

We remark that when $m = 2$, (4-4) and reduces to obvious $c_f \geq 0$. Hence, $S_2^H(\Gamma) = S_2(\Gamma)$ recovering the standard isomorphism of $S_2(\Gamma)$ and $H^1(\mathcal{M}_\Gamma)$ (see [7, Theorem 2.3.2]). We have the following result:

**Lemma 4-5.** Assume that $m, n \geq 2$ are even integers. Let $\Gamma$ be a Fuchsian group of the first kind. Then, we have the following:

(i) $S_2^H(\Gamma) = S_2(\Gamma)$.
(ii) $S_m^H(\Gamma)$ is isomorphic to $H^{m/2}(\mathcal{M}_\Gamma)$.
(iii) $S_m^H(\Gamma) = \{0\}$ if $g(\Gamma) = 0$.
(iv) Assume that $g(\Gamma) = 1$. Let us write $S_2(\Gamma) = \mathbb{C} \cdot f$, for some non–zero cuspidal form $f$. Then, we have $S_m^H(\Gamma) = \mathbb{C} \cdot f^{m/2}$.
(v) $\dim S_m^H(\Gamma) = (m - 1)(g(\Gamma) - 1)$ if $g(\Gamma) \geq 2$.
(vi) $S_m^H(\Gamma) \cdot S_n^H(\Gamma) \subset S_{m+n}^H(\Gamma)$.
(vii) There are no $m/2$–Weierstrass points on $\mathcal{M}_\Gamma$ for $g(\Gamma) \in \{0, 1\}$.
(viii) Assume that $g(\Gamma) \geq 2$, and $\alpha_\infty$ is a $\Gamma$-cusp. Then, $\alpha_\infty$ is a $m/2$–Weierstrass point if and only if there exists $f \in S_m^H(\Gamma)$, $f \neq 0$, such that

$$c'_f(\alpha_\infty) \geq \begin{cases} \frac{m}{2} + g(\Gamma) & \text{if } m = 2; \\ \frac{m}{2} + (m - 1)(g(\Gamma) - 1) & \text{if } m \geq 4. \end{cases}$$

(ix) Assume that $g(\Gamma) \geq 1$, and $\alpha_\infty$ is a $\Gamma$-cusp. Then, there exists a basis $f_1, \ldots, f_t$ of $S_m^H(\Gamma)$ such that their $q$–expansions are of the form

$$f_u = a_u q^{i_u} + \text{higher order terms in } q, \ 1 \leq u \leq t,$$

where

$$\frac{m}{2} \leq i_1 < i_2 < \cdots < i_t \leq \frac{m}{2} + m(g(\Gamma) - 1),$$

and

$$a_u \in \mathbb{C}, \ a_u \neq 0.$$

(x) Assume that $g(\Gamma) \geq 1$, and $\alpha_\infty$ is a $\Gamma$-cusp. Then, $\alpha_\infty$ is **not** a $m/2$–Weierstrass point if and only if there exists a basis $f_1, \ldots, f_t$ of $S_m^H(\Gamma)$ such that their $q$–expansions are of the form

$$f_u = a_u q^{u + m/2 - 1} + \text{higher order terms in } q, \ 1 \leq u \leq t,$$

where

$$a_u \in \mathbb{C}, \ a_u \neq 0.$$

**Proof.** (i) and (ii) follow from above discussion. Next, using above discussion and (3-2) we obtain

$$\dim S_m^H(\Gamma) = \dim H^{m/2}(\mathcal{M}_\Gamma) = \begin{cases} 0 & \text{if } m \geq 2, g(\Gamma) = 0; \\ g(\Gamma) & \text{if } m = 2, g(\Gamma) \geq 1; \\ g(\Gamma) & \text{if } m \geq 4, g(\Gamma) = 1; \\ (m - 1)(g(\Gamma) - 1) & \text{if } m \geq 4, g(\Gamma) \geq 2. \end{cases}$$
This immediately implies (iii) and (v). Next, assume that \( g(\Gamma) = 1 \). Then, we see that \( \dim S_m^H(\Gamma) \leq 1 \) for all even integers \( m \geq 4 \). It is well known that \( f^{m/2} \in S_m(\Gamma) \). Next, (4-4) for \( m = 2 \) implies \( \text{div}(\omega_f) = c_f \). Also, the degree of \( c_f \) is zero by Lemma 2-2 (vii). Hence,

\[
\text{div}(\omega_f) = c_f = 0.
\]

Using [7, Theorem 2.3.2], we obtain

\[
\omega_{f^{m/2}} = \omega_f^{m/2}.
\]

Hence

\[
\text{div}(\omega_{f^{m/2}}) = \frac{m}{2} \text{div}(\omega_f) = 0.
\]

Then, applying (4-3) with \( f^{m/2} \) in place of \( f \), we obtain

\[
\sum_{a \in \mathfrak{A}_\Gamma, \text{elliptic}} \left[ \frac{m}{2} (1 - 1/e_a) \right] a + \left( \frac{m}{2} - 1 \right) \sum_{b \in \mathfrak{A}_\Gamma, \text{cusp}} b.
\]

This shows that \( f^{m/2} \in S_m^H(\Gamma) \) proving (iv). Finally, (vi) follows from [7, Theorem 2.3.1]. We can also see that directly as follows. Let \( 0 \neq f \in S_m^H(\Gamma) \) and \( 0 \neq g \in S_n^H(\Gamma) \). Then, \( fg \in S_{m+n}(\Gamma) \) since \( f \in S_m^H(\Gamma) \subset S_m(\Gamma) \) and \( g \in S_n^H(\Gamma) \subset S_n(\Gamma) \). We have the following:

\[
\text{div}(f \cdot g) = \text{div}(f) + \text{div}(g).
\]

Using Lemma 2-2 (vi) we can rewrite this identity as follows:

\[
\sum_{a \in \mathfrak{A}_\Gamma, \text{elliptic}} \left[ \frac{m+n}{2} (1 - 1/e_a) \right] a =
\]

\[
\sum_{a \in \mathfrak{A}_\Gamma, \text{elliptic}} \left[ \frac{m}{2} (1 - 1/e_a) \right] a + \sum_{a \in \mathfrak{A}_\Gamma, \text{elliptic}} \left[ \frac{n}{2} (1 - 1/e_a) \right] a.
\]

By Lemma 2-2 (vii) we obtain:

\[
\sum_{a \in \mathfrak{A}_\Gamma, \text{elliptic}} \left[ \frac{m+n}{2} (1 - 1/e_a) \right] a - \left( \frac{m+n}{2} - 1 \right) \sum_{b \in \mathfrak{A}_\Gamma, \text{cusp}} b =
\]

\[
\left( \sum_{a \in \mathfrak{A}_\Gamma, \text{elliptic}} \left[ \frac{m}{2} (1 - 1/e_a) \right] a - \left( \frac{m}{2} - 1 \right) \sum_{b \in \mathfrak{A}_\Gamma, \text{cusp}} b \right) +
\]

\[
\left( \sum_{a \in \mathfrak{A}_\Gamma, \text{elliptic}} \left[ \frac{n}{2} (1 - 1/e_a) \right] a - \left( \frac{n}{2} - 1 \right) \sum_{b \in \mathfrak{A}_\Gamma, \text{cusp}} b \right).
\]

Finally, (vi) follows applying (4-4) since both terms on the right hand of equality are \( \geq 0 \).

Next, (vii) follows immediately from the discussion in Section 3 and it is well–known. (viii) is a reinterpretation of Definition 3-6. The details are left to the reader as an easy exercise.
Finally, (ix) and (x) in the case of \( g(\Gamma) = 1 \) are obvious since by Lemma 2-2 we have \( S_2(\Gamma) = \mathbb{C} \cdot f \) where

\[
c'_f = a_\infty + \sum_{b \in \mathcal{R}_\Gamma, \text{cusp}} b.
\]

We prove (ix) and (x) in the case of \( g(\Gamma) \geq 2 \). Let \( f \in S_H^m(\Gamma), f \neq 0 \). Then, by the definition of \( S_H^m(\Gamma) \), we obtain

\[
c'_f(a_\infty) = 1 + c_f(a_\infty) \geq 1 + \left( \frac{m}{2} - 1 \right) = \frac{m}{2}.
\]

On the other hand, again by the definition of \( S_H^m(\Gamma) \) (see (4-4)) and the fact that \( c'_f \geq 0 \), we obtain

\[
\deg(c'_f) = \sum_{a \in \mathcal{R}_\Gamma} c'_f(a) \geq 
\sum_{a \in \mathcal{R}_\Gamma, \text{elliptic}} c'_f(a) + \sum_{b \in \mathcal{R}_\Gamma, \text{cusp} b \neq a_\infty} c'_f(b) + c'_f(a_\infty) \geq 
\sum_{a \in \mathcal{R}_\Gamma, \text{elliptic}} \left[ \frac{m}{2}(1 - 1/e_a) \right] + \frac{m}{2}(t - 1) + c'_f(a_\infty)
\]

where \( t \) is the number of inequivalent \( \Gamma \)-cusps. The degree \( \deg(c'_f) \) is given by Lemma 2-2 (vi)

\[
\deg(c'_f) = \dim M_m(\Gamma) + g(\Gamma) - 1 
= \begin{cases} 
2(g(\Gamma) - 1) + t & \text{if } m = 2; \\
m(g(\Gamma) - 1) + \frac{m}{2}t + \sum_{a \in \mathcal{R}_\Gamma, \text{elliptic}} \left[ \frac{m}{2}(1 - 1/e_a) \right] & \text{if } m \geq 4.
\end{cases}
\]

Combining with the previous inequality, we obtain

\[
c'_f(a_\infty) \leq \frac{m}{2} + m(g(\Gamma) - 1) \quad \text{if } m \geq 2.
\]

Having in mind (4-6), the rest of (ix) has standard argument (see for example 10, Lemma 4.3]). Finally, (x) follows (viii) and (ix).

The criterion in Lemma 4-5 (x) is a quite good criterion to check whether or not \( a_\infty \) is a Weierstrass points (the case \( m = 2 \)) using computer systems such as SAGE since we need just to list the basis. This case is well-known (see 19, Definition 6.1]). This criterion has been used in practical computations in combination with SAGE in 13 for \( \Gamma = \Gamma_0(N) \).

But it is not good when \( m \geq 4 \), regarding the bound for \( S_H^m(\Gamma) \) given by Lemma 4-5 (ix), since then a basis of \( S_m(\Gamma) \) contains properly normalized cusp forms having leading terms \( q^{m/2}, \ldots, q^{m/2 + m(g(\Gamma) - 1)} \) at least when \( \Gamma \) has elliptic points for \( m \) large enough and we do not know which of them belong to \( S_H^m(\Gamma) \). We explain that in Corollary 4-10 below.
First, we recall the following result [12, Lemma 2.9] which is well-known in a slightly different notation ([20], [21]):

**Lemma 4-7.** Let \( m \geq 4 \) be an even integer such that \( \dim S_m(\Gamma) \geq g(\Gamma) + 1 \). Then, for all \( 1 \leq i \leq t_m - g \), there exists \( f_i \in S_m(\Gamma) \) such that \( c'_{f_i}(a_\infty) = i \).

**Lemma 4-8.** Assume that \( \Gamma \) has elliptic points. (For example, \( \Gamma = \Gamma_0(N) \).) Then, for a sufficiently large even integer \( m \), we have

\[
\frac{m}{2} + m(g(\Gamma) - 1) \leq \dim S_m(\Gamma) - g(\Gamma).
\]

**Proof.** Assume that \( m \geq 4 \) is an even integer. Then, by Lemma 2-2 (v), we obtain

\[
\dim S_m(\Gamma) - g(\Gamma) - \left( \frac{m}{2} + m(g(\Gamma) - 1) \right) = \left( \frac{m}{2} - 1 \right) t - \frac{m}{2} + \sum_{a \in R_{\Gamma, \text{elliptic}}} \left[ \frac{m}{2} \left( 1 - 1/e_a \right) \right] - 2g(\Gamma) + 1 \geq \sum_{a \in R_{\Gamma, \text{elliptic}}} \left[ \frac{m}{2} \left( 1 - 1/e_a \right) \right] - 2g(\Gamma) - t + 1.
\]

Since \( \Gamma \) has elliptic points, the last term is \( \geq 0 \) for \( m \) sufficiently large even integer. \( \square \)

**Corollary 4-10.** Assume that (4-9) holds. Then, given a basis \( f_1, \ldots, f_t \) of \( S^H_m(\Gamma) \) such that \( c'_{f_i}(a_\infty) = i_j \), \( 1 \leq j \leq t \), where

\[
\frac{m}{2} \leq i_1 < i_2 < \cdots < i_t \leq \frac{m}{2} + m(g(\Gamma) - 1)
\]

can be extended by additional \( g(\Gamma) \) cuspidal modular forms in \( S_m(\Gamma) \) to obtain the collection \( F_k, \frac{m}{2} \leq k \leq \frac{m}{2} + m(g(\Gamma) - 1) \) such that \( c'_{F_k}(a_\infty) = k \) for all \( k \).

**Proof.** This follows directly from Lemmas 4-7 and 4-8. \( \square \)

Now, explain the algorithm for testing that \( a_\infty \) is a \( \frac{m}{2} \) Weierstrass point for \( m \geq 6 \). It requires some geometry. We recall that \( \mathcal{R}_F \) is hyperelliptic if \( g(\Gamma) \geq 2 \), and there is a degree two map onto \( \mathbb{P}^1 \). By general theory [9, Chapter VII, Proposition 1.10], if \( g(\Gamma) = 2 \), then \( \mathcal{R}_F \) is hyperelliptic. If \( \mathcal{R}_F \) is not hyperelliptic, then \( \dim S_2(\Gamma) = g(\Gamma) \geq 3 \), and the regular map \( \mathcal{R}_F \to \mathbb{P}^{g(\Gamma)-1} \) attached to a canonical divisor \( K \) is an isomorphism onto its image [9, Chapter VII, Proposition 2.1].

Let \( \Gamma = \Gamma_0(N), N \geq 1 \). Put \( X_0(N) = \mathcal{R}_{\Gamma_0(N)} \). We recall that \( g(\Gamma_0(N)) \geq 2 \) unless

\[
\begin{aligned}
N &\in \{ 1 - 10, 12, 13, 16, 18, 25 \} \text{ when } g(\Gamma_0(N)) = 0, \text{ and } \\
n &\in \{ 11, 14, 15, 17, 19 - 21, 24, 27, 32, 36, 49 \} \text{ when } g(\Gamma_0(N)) = 1.
\end{aligned}
\]
Let \( g(\Gamma_0(N)) \geq 2 \). Then, we remark that Ogg [17] has determined all \( X_0(N) \) which are hyperelliptic curves. In view of Ogg’s paper, we see that \( X_0(N) \) is not hyperelliptic for \( N \in \{34, 38, 42, 43, 44, 45, 51 – 58, 60 – 70\} \) or \( N \geq 72 \). This implies \( g(\Gamma_0(N)) \geq 3 \).

Before we begin the study of spaces \( S^H_m(\Gamma) \) we give the following lemma.

**Lemma 4.11.** Let \( m \geq 4 \) be an even integer. Let us select a basis \( f_0, \ldots, f_{g-1}, g = g(\Gamma), \) of \( S^2(\Gamma) \). Then, all of \( (g + m^2 - 1) \) monomials \( f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}, \alpha_i \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^{g-1} \alpha_i = \frac{m^2}{2} \), belong to \( S^H_m(\Gamma) \). We denote this subspace of \( S^H_m(\Gamma) \) by \( S^H_{m,2}(\Gamma) \).

**Proof.** This follows from Lemma 4-5 (vi) since \( S^2(\Gamma) = S^H_{2,2}(\Gamma) \) (see Lemma 4-5 (i)). \( \square \)

**Theorem 4.12.** Let \( m \geq 4 \) be an even integer. Assume that \( \mathfrak{A}_\Gamma \) is not hyperelliptic. Then, we have

\[
S^H_{m,2}(\Gamma) = S^H_m(\Gamma).
\]

**Proof.** We use notation of Section 3 freely. The reader should review Lemma 4-5. Let \( F \in S^2(\Gamma), F \neq 0 \). We define a holomorphic differential form \( \omega \in H(\mathfrak{A}_\Gamma) \) by \( \omega = \omega_F \). Define a canonical class \( K \) by \( K = \text{div}(\omega) \). We prove the following:

\[
L\left(\frac{m}{2}K\right) = \left\{ \frac{f}{F^{m/2}}; \ f \in S^H_m(\Gamma) \right\}.
\]

The case \( m = 2 \) is of course well–known. By the Riemann-Roch theorem and standard results recalled in Section 3 we have

\[
\dim L\left(\frac{m}{2}K\right) = \deg\left(\frac{m}{2}K\right) - g(\Gamma) + 1 + \dim L\left(\left(1 - \frac{m}{2}\right)K\right) = (m - 1)(g(\Gamma) - 1) + \begin{cases} 1 & \text{if } m = 2; \\ 0 & \text{if } m \geq 4. \end{cases}
\]

Next, we recall that \( S^2(\Gamma) = S^H_2(\Gamma) \) (see Lemma 4-5 (i)). Then, Lemma 4-5 (vi) we obtain \( F^{m/2} \in S^H_m(\Gamma) \). Therefore, \( f/F^{m/2} \in \mathbb{C}(\mathfrak{A}_\Gamma) \) for all \( f \in S^H_m(\Gamma) \).

By the correspondence described in (4-1) we have

\[
\text{div}(F) = \text{div}(\omega_F) + \sum_{a \in \mathfrak{A}_\Gamma} \left(1 - \frac{1}{e_a}\right)a = K + \sum_{a \in \mathfrak{A}_\Gamma} \left(1 - \frac{1}{e_a}\right)a = K + \sum_{a \in \mathfrak{A}_\Gamma, \text{elliptic}} (1 - 1/e_a)a + \sum_{b \in \mathfrak{A}_\Gamma, \text{cusp}} b.
\]
Thus, for \( f \in S_m^H(\Gamma) \), we have the following:

\[
\text{div} \left( \frac{f}{F^{m_2}} \right) + \frac{m}{2} K = \text{div}(f) - \frac{m}{2} \text{div}(F) + \frac{m}{2} K
\]

\[
= \text{div}(f) - \frac{m}{2} \sum_{a \in \mathfrak{A}, \text{elliptic}} (1 - 1/e_a) a - \frac{m}{2} \sum_{b \in \mathfrak{A}, \text{cusp}} b
\]

Next, using Lemma 2-2 (vi), the right-hand side becomes

\[
c'_f - \sum_{a \in \mathfrak{A}, \text{elliptic}} \left[ \frac{m}{2} (1 - 1/e_a) \right] a - \frac{m}{2} \sum_{b \in \mathfrak{A}, \text{cusp}} b \geq 0
\]

by the definition of \( S_m^H(\Gamma) \). Hence, \( f/F^{m_2} \in L \left( \frac{m}{2} K \right) \). Now, comparing the dimensions of the right-hand and left-hand side in (4-13), we obtain their equality. This proves (4-13).

Let \( W \) be any finite dimensional \( \mathbb{C} \)-vector space. Let \( \text{Symm}^k(W) \) be symmetric tensors of degree \( k \geq 1 \). Then, by Max Noether theorem ([9], Chapter VII, Corollary 3.27) the multiplication induces surjective map

\[
\text{Symm}^k(L(K)) \twoheadrightarrow L \left( \frac{m}{2} K \right).
\]

The theorem follows. \( \square \)

Now, we combine Theorem 4-12 with Lemma 4-5 (x) to obtain a good criterion in the case \( m \geq 4 \) for testing that \( a_\infty \) is a \( \frac{m}{2} \)-Weierstrass point. We give examples in Section 5 (see Propositions 5-1 and 5-2).

**Corollary 4-14.** Let \( m \geq 4 \) be an even integer. Assume that \( \mathfrak{A}_\Gamma \) is not hyperelliptic. Assume that \( a_\infty \) is a cusp for \( \Gamma \). Let us select a basis \( f_0, \ldots, f_{g-1} \), \( g = g(\Gamma) \), of \( S_2(\Gamma) \). Compute \( q \)-expansions of all monomials

\[
f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}, \quad \alpha_i \in \mathbb{Z}_{\geq 0}, \quad \sum_{i=0}^{g-1} \alpha_i = \frac{m}{2}.
\]

Then, \( a_\infty \) is **not** a \( \frac{m}{2} \)-Weierstrass point if and only if there exist a basis of the space of all such monomials, \( F_1, \ldots, F_t \), \( t = \dim S_m^H(\Gamma) = (m - 1)(g - 1) \) (see Lemma 4-3 (v)), such that their \( q \)-expansions are of the form

\[
F_u = a_u q^{u+m/2-1} + \text{higher order terms in } q, \quad 1 \leq u \leq t,
\]

where

\[
a_u \in \mathbb{C}, \quad a_u \neq 0.
\]

When \( \mathfrak{A}_\Gamma \) is hyperelliptic, for example if \( g(\Gamma) = 2 \), the space \( S_m^H(\Gamma) \) could be a proper subspace of \( S_m^H(\Gamma) \). For example, if \( N = 35 \), then \( g(\Gamma_0(N)) = 3 \) and \( X_0(N) \) is hyperelliptic.
For $m = 4, 6, 8, 10, 12, 14$ we checked that $\dim S^H_{m,2}(\Gamma) = m + 1$ while by general theory $\dim S^H_m(\Gamma) = (m - 1)(g(\Gamma_0(N)) - 1) = 2(m - 1)$. We see that

$$\dim S^H_m(\Gamma) - \dim S^H_{m,2}(\Gamma) = m - 3 \geq 1, \quad \text{for } m = 4, 6, 8, 10, 12, 14.$$ 

In fact, the case of $g(\Gamma) = 2$ could be covered in full generality. We leave easy proof of the following proposition to the reader.

**Proposition 4-15.** Assume that $g(\Gamma) = 2$. Let $f_0, f_1$ be a basis of $S_2(\Gamma)$. Then, for any even integer $m \geq 4$, $f_0^u f_1^{\frac{m}{2} - u}$, $0 \leq u \leq m$ is a basis of $S^H_{m,2}(\Gamma)$. Therefore,

$$\dim S^H_m(\Gamma) = (m - 1)(g(\Gamma) - 1) = m - 1 > \frac{m}{2} + 1, \quad \text{for } m \geq 6,$$

and $S^H_{4,2}(\Gamma) = S^H_4(\Gamma)$.

We end this section with the standard yoga.

**Theorem 4-16.** Let $m \geq 2$ be an even integer. Let $\Gamma$ be a Fuchsian group of the first kind such that $g(\Gamma) \geq 1$. Let $t = \dim S^H_m(\Gamma) = \dim H^{m/2}(\mathfrak{R}_\Gamma)$. Let us fix a basis $f_1, \ldots, f_t$ of $S^H_m(\Gamma)$, and let $\omega_1, \ldots, \omega_t$ be the corresponding basis of $H^{m/2}(\mathfrak{R}_\Gamma)$. As above, we construct holomorphic differential $W(\omega_1, \ldots, \omega_t) \in H^1_{\Gamma_0} H^{m/2}(\mathfrak{R}_\Gamma)$. We also construct the Wronskian $W(f_1, \ldots, f_t) \in S_{(m+t-1)}(\Gamma)$ (see Proposition 6-11). Then, we have the following equality:

$$\omega_{W(f_1,\ldots,f_t)} = W(\omega_1, \ldots, \omega_t).$$

In particular, we obtain the following:

$$W(f_1, \ldots, f_t) \in S^H_{(m+t-1)}(\Gamma).$$

Moreover, assume that $a_{\infty}$ is a $\Gamma$-cusp. Then, $a_{\infty}$ is a $\frac{m}{2}$-Weierstrass point if and only if

$$c_{W(f_1,\ldots,f_t)}(a_{\infty}) \geq 1 + \frac{t}{2}(m - 1 + t) \quad \text{i.e.,} \quad c_{W(f_1,\ldots,f_t)}(a_{\infty}) \geq \frac{t}{2}(m - 1 + t)$$

(See also Lemma 4-17 (x) for more effective formulation of the criterion.)

**Proof.** Since this is a equality of two meromorphic differentials, it is enough to check the identity locally. Let $z \in \mathbb{H}$ be a non–elliptic point, and $z \in U \subset \mathbb{H}$ a chart of $a_z$ such that $U$ does not contain any elliptic point. Then, one can use [7] Section 2.3 to check the equality directly. Indeed, we have the following argument.

Let $z_0 \in \mathbb{H}$ be a non–elliptic point, and $z \in U \subset \mathbb{H}$ a chart of $a_{z_0}$ such that $U$ does not contain any elliptic point.

Let $t_a$ be a local coordinate on a neighborhood $V_a = \pi(U)$. By the [7] Section 1.8 if $U$ is small enough, projection $\pi$ gives homeomorphism of $U$ to $V_a$ such that

$$t_a \circ \pi(z) = z \quad \text{for } z \in U.$$  

Let $f \in \mathcal{A}_m(\Gamma)$ and let $\omega_f$ be the corresponding differential. Locally there exist unique meromorphic function $\varphi$ such that $\omega_f = \varphi (dz)^{m/2}$. 

By [7, Section 2.3], local correspondence $f \mapsto \omega_f$ is given by
\[ \varphi(t_a \circ \pi(z)) = f(z) \left( d(t_a \circ \pi)/dz \right)^{-m/2}, \]
which by the choice of local chart [4-17] become
\[ \varphi(z) = f(z) \quad \text{for} \quad z \in U. \]
So, in the neighborhood of non-elliptic point $z \in \mathbb{H}$ we have
\[ \omega_f = f \left( dz \right)^{m/2}. \]
This gives us local identity
\[ W_z (\omega_1, \ldots, \omega_t) = W(f_1, \ldots, f_t). \]
Since above is valid for any even $m \geq 2$, we get local identity of two meromorphic differentials
\[ \omega_{W(f_1, \ldots, f_t)} = W(f_1, \ldots, f_t) \left( dz \right)^{\frac{t}{2}(m-1+t)} = W_z (\omega_1, \ldots, \omega_t) \left( dz \right)^{\frac{t}{2}(m-1+t)} = W (\omega_1, \ldots, \omega_t) \]
Now, assume that $a_\infty$ is a $\Gamma$-cusp. Then, $a_\infty$ is a $\frac{m}{2}$-Weierstrass point if and only if
\[ \nu_{a_\infty} (W (\omega_1, \ldots, \omega_t)) \geq 1 \]
i.e.,
\[ \nu_{a_\infty} (\omega_{W(f_1, \ldots, f_t)}) \geq 1 \]
by the first part of the proof. Finally, by [4-1], this is equivalent to
\[ c'_{W(f_1, \ldots, f_t)}(a_\infty) = \nu_{a_\infty} (\omega_{W(f_1, \ldots, f_t)}) + \frac{t}{2} (m - 1 + t) \geq 1 + \frac{t}{2} (m - 1 + t). \]
This completes the proof of the theorem. $\square$

5. Explicit Computations Based on Corollary 4-14 for $\Gamma = \Gamma_0(N)$

In this section we apply the algorithm in Corollary [4-14] combined with SAGE. The method is the following. We take $q$-expansions of the base elements of $S_2(\Gamma_0(N))$:
\[ f_0, \ldots, f_{g-1}, \]
where $g = g(\Gamma_0(N))$. For even $m \geq 4$, we compute $q$-expansions of all monomials of degree $m/2$:
\[ f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}, \quad \alpha_i \in \mathbb{Z}_{\geq 0}, \quad \sum_{i=0}^{g-1} \alpha_i = \frac{m}{2}. \]
The number of monomials is
\[ \binom{g + m/2 - 1}{m/2} \]
By selecting first \( m/2 + m \cdot (g - 1) \) terms from \( q \)-expansions of the monomials (see Lemma 4.5 (ix)), we can create a matrix of size

\[
\left( \frac{g + m/2 - 1}{m/2} \right) \times \left( \frac{m}{2} + m \cdot (g - 1) \right).
\]

Then, we perform suitable integral Gaussian elimination method to transform the matrix into row echelon form. The procedure is as follows. We successively sort and transform the row matrices to cancel the leading row coefficients with the same number of leading zeros as their predecessor. We use the Quicksort algorithm for sorting. We obtain the transformed matrix and the transformation matrix. The non-null rows of the transformed matrix give the \( q \)-expansions of the basis elements, and the corresponding rows of the transformation matrix give the corresponding linear combinations of monomials.

Using above described method we perform various computations mentioned below. For example, we can easily verify particular cases Theorem 4.12.

**Proposition 5.1.** For \( m = 4, 6, 8, 10, 12 \) and \( N = 34, 38, 44, 55 \), and for \( m = 4, 6, 8, 10 \) and \( N = 54, 60 \), we have \( S_{m}^{H} (\Gamma_0(N)) = S_{m^2}^{H} (\Gamma_0(N)) \). (We remark that all curves \( X_0(N), \ N \in \{34, 38, 44, 54, 55, 60\} \) are not hyperelliptic (see the paragraph after Corollary 4.10).

We can also deal with generalized Weierstrass points. For example, we can check the following result:

**Proposition 5.2.** For \( m = 2, 4, 6, 8, 10 \), \( a_{\infty} \) is not \( \frac{m}{2} \)-Weierstrass point for \( X_0(34) \). Next, \( a_{\infty} \) is not \((1-)\)Weierstrass point for \( X_0(55) \), but it is \( \frac{m}{2} \)-Weierstrass point for \( X_0(55) \) and \( m = 4, 6, 8, 10 \).

For example, let \( m = 4 \). Then, for \( X_0(34) \) the monomials are

\[
\begin{align*}
f_0^2 &= q^2 - 4q^5 - 4q^6 + 12q^8 + 12q^9 - 2q^{10} \\
f_0f_1 &= q^3 - q^5 - 2q^6 - 2q^7 + 2q^8 + 5q^9 + 2q^{10} \\
f_0f_2 &= q^4 - 2q^5 - q^6 - q^7 + 6q^8 + 6q^9 + 2q^{10} \\
-f_1^2 + f_0f_2 &= -2q^5 + q^6 - q^7 + 5q^8 + 6q^9 + 4q^{10} \\
-f_1^2 + f_0f_2 + 2f_1f_2 &= -3q^6 - 5q^7 + 11q^8 + 16q^9 + 2q^{10} \\
-f_1^2 + f_0f_2 + 2f_1f_2 + 3f_2^2 &= -17q^7 + 17q^8 + 34q^9 + 17q^{10}
\end{align*}
\]

Their first exponents are \( \frac{m}{2} = 4, 5, 6, \frac{m}{2} + (m - 1)(g - 1) - 1 = 7 \) which shows that \( a_{\infty} \) is not 2–Weierstrass point for \( X_0(34) \).
For \(X_0(55)\) the monomials are

\[
\begin{align*}
&f_0^2 \\
&f_0f_1 \\
&f_0f_2 \\
&f_0f_3 \\
&f_0f_4 \\
&- f_1f_2 + f_0f_3 \\
&- f_1f_2 + f_0f_3 + 2f_2f_3 \\
&- f_1f_2 + f_0f_3 + 2f_2f_3 - f_3^2 \\
&- f_1f_2 + f_0f_3 + 2f_2f_3 - f_3^2 - 2f_3f_4 \\
&- f_1f_2 + f_0f_3 + 2f_2f_3 - f_3^2 - 2f_3f_4 + f_4^2 \\
&- f_1f_2 - f_2^2 + f_0f_3 + 2f_2f_3 - f_3^2 + f_0f_4 - 6f_3f_4 - f_4^2 \\
&- f_2^2 + f_3^2 + f_0f_4 - f_2f_4 - 4f_3f_4 + 2f_4^2.
\end{align*}
\]

Their \(q\)-expansions are given by the following expressions:

\[
\begin{align*}
q^2 - 2q^8 - 2q^9 - 2q^{10} + 2q^{11} - 4q^{13} + 3q^{14} + 4q^{15} + 3q^{16} - 2q^{17} + 5q^{18} \\
q^3 - 2q^7 + q^{10} - 2q^{11} + q^{12} - 2q^{14} - 4q^{16} + 5q^{18} \\
q^4 - 2q^7 - q^8 + 3q^9 + 4q^{10} - 4q^{11} - q^{13} - 2q^{14} - 3q^{15} - 10q^{16} - 2q^{17} + 3q^{18} \\
q^5 - 2q^7 - q^8 + 3q^9 + 4q^{10} - 4q^{11} - 3q^{13} + q^{14} - q^{15} - 11q^{16} - 2q^{17} + 5q^{18} \\
q^6 - 2q^{11} - q^{12} - q^{13} - q^{14} + q^{15} - q^{16} + 3q^{18} \\
- 2q^7 + q^8 + 6q^9 + q^{10} - 10q^{11} - 3q^{12} - 5q^{13} + 13q^{14} + 21q^{15} - 17q^{16} - 8q^{17} - 14q^{18} \\
q^8 + 2q^9 - 5q^{10} - 6q^{11} + 19q^{12} + 7q^{13} - 13q^{14} - 33q^{15} - 7q^{16} + 38q^{17} + 14q^{18} \\
2q^9 - q^{10} - 4q^{11} + 9q^{12} - 5q^{13} + 4q^{14} - 13q^{15} - 12q^{16} + 18q^{17} + 4q^{18} \\
- q^{10} + 11q^{12} - 11q^{13} - 7q^{15} - 22q^{16} + 22q^{17} + 22q^{18} \\
11q^{12} - 11q^{13} - 11q^{15} - 22q^{16} + 22q^{17} + 22q^{18} \\
- 22q^{13} + 44q^{15} - 44q^{16} + 44q^{18} \\
- 22q^{14} + 22q^{15} - 22q^{16} + 44q^{18}
\end{align*}
\]

The last exponent is \(14 > \frac{m}{2} + (m - 1)(g - 1) - 1 = 13\). So, \(a_\infty\) is not 2–Weierstrass point for \(X_0(55)\).
6. Wronskians of Modular Forms

In this section we deal with a generalization of the usual notion of the Wronskian of cuspidal modular forms [22], ([19], 6.3.1), ([10], the proof of Theorem 4-5), and ([12], Lemma 4-1).

Lemma 6-1. Let \( f \in M_m(\Gamma, \chi) \). Let \( \gamma \in \Gamma \). Then, for \( k \geq 0 \), \( k \)-th derivative of the function \( f(\sigma.z) \) is given by

\[
\frac{d^k}{dz^k} f(\gamma.z) = \chi(\gamma) j(\gamma, z)^{m+2k} \sum_{i=0}^{k-1} D_{ik} \cdot j(\gamma, z)^{m+k+i} \cdot \frac{d^i f(z)}{dz^i},
\]

where \( D_{ik} \) are some constants depending on \( m, k \), and \( \gamma \). If \( \Gamma \subset SL_2(\mathbb{Z}) \), then the constants can be taken to be from \( \mathbb{Z} \).

Proof. This follows by an easy induction on \( k \) using the fact that \( \frac{d}{dz} j(\gamma, z) = j(\gamma, z)^{-2} \).

See also the proof of Theorem 4-5 in [10], the text between the lines (4-6) and (4-8). \( \square \)

The following proposition is the main result of the present section:

Proposition 6-2. Let \( m \geq 1 \). Then, for any sequence \( f_1, \ldots, f_k \in M_m(\Gamma, \chi) \), the Wronskian

\[
W (f_1, \ldots, f_k) (z) \overset{\text{def}}{=} \begin{vmatrix} f_1(z) & \cdots & f_k(z) \\ \frac{df_1(z)}{dz} & \cdots & \frac{df_k(z)}{dz} \\ \vdots & \ddots & \vdots \\ \frac{d^{k-1}f_1(z)}{dz^{k-1}} & \cdots & \frac{d^{k-1}f_k(z)}{dz^{k-1}} \end{vmatrix}
\]

is a cuspidal modular form in \( S_{k(m+k-1)}(\Gamma, \chi^k) \) if \( k \geq 2 \). If \( f_1, \ldots, f_k \) are linearly independent, then \( W (f_1, \ldots, f_k) \neq 0 \).

Proof. This is a standard fact. We apply Lemma 6-1 to conclude

\[
W (f_1, \ldots, f_k) (\gamma.z) = \chi^{k} \gamma(j(\gamma, z)^{k(m+k-1)}W (f_1, \ldots, f_k) (z), \ \gamma \in \Gamma, \ z \in \mathbb{H}.
\]

Let \( x \in \mathbb{R} \cup \{\infty\} \) be a cusp for \( \Gamma \). Let \( \sigma \in SL_2(\mathbb{R}) \) such that \( \sigma.x = \infty \). We write

\[
\{\pm 1\} \sigma \Gamma_x \sigma^{-1} = \{\pm 1\} \begin{cases} 1 & \text{if } l = 0 \text{ or } l = h' \\ 1 & \text{otherwise} \end{cases}.
\]

where \( h' > 0 \) is the width of the cusp. Then we write the Fourier expansion of each \( f_i \) at \( x \) as follows:

\[
(f_i|_m \sigma^{-1})(\sigma.z) = \sum_{n=0}^{\infty} a_{n,i} \exp \frac{2\pi \sqrt{-1} n \sigma.z}{h'}.
\]

Using the cocycle identity

\[
1 = j(\sigma^{-1}, z) = j(\sigma^{-1}, \sigma.z) j(\sigma, z),
\]

we conclude the proof. \( \square \)
this implies the following:

\[ j(\sigma, z)^m \cdot f_i(z) = \sum_{n=0}^{\infty} a_{n,i} \exp \frac{2\pi \sqrt{-1} n \sigma \cdot z}{h'}. \]

By induction on \( t \geq 0 \), using

\[ \frac{d}{dz} \sigma \cdot z = j(\sigma, z)^{-2}, \]

we have the following:

\[ j(\sigma, z)^{m+2t} \frac{d^t f_i(z)}{dz^t} + \sum_{u=0}^{t-1} D_{i,\ell} j(\sigma, z)^{m+t+u} \frac{d^u f_i(z)}{dz^u} = \sum_{n=0}^{\infty} a_{n,i,t} \exp \frac{2\pi \sqrt{-1} n \sigma \cdot z}{h'}. \]

for some complex numbers \( D_{i,\ell} \) and \( a_{n,i,t} \), where

\[ a_{0,i,t} = 0, \ t \geq 1. \]

Now, by above considerations, using (6-3), we have

\[ (W(f_1, \ldots, f_k)|_{k(m+k-1)\sigma^{-1}}(\sigma, z)) = j(\sigma, z)^{(m+k-1)} W(f_1, \ldots, f_k)(z) \]

\[ = \left| \begin{array}{ccc} j(\sigma, z)^m f_1(z) & \cdots & j(\sigma, z)^m f_k(z) \\ j(\sigma, z)^{m+2} \frac{df_1(z)}{dz} & \cdots & j(\sigma, z)^{m+2} \frac{df_k(z)}{dz} \\ \vdots & \ddots & \vdots \\ j(\sigma, z)^{m+2(k-1)} \frac{d^{k-1} f_1(z)}{dz^{k-1}} & \cdots & j(\sigma, z)^{m+2(k-1)} \frac{d^{k-1} f_k(z)}{dz^{k-1}} \end{array} \right| = \det \left( \sum_{n=0}^{\infty} a_{n,i,t+1} \exp \frac{2\pi \sqrt{-1} n \sigma \cdot z}{h'} \right)_{0 \leq i, t \leq k-1} \]

Now, we see that the Wronskian is holomorphic at each cup of \( \Gamma \) and vanishes at the order at least \( k - 1 \). In particular, it belongs to \( S_{k(m+k-1)}(\Gamma, \chi^k) \) if \( k \geq 2 \).

The claim that linear independence is equivalent to the fact that Wronskian is not identically zero is standard ([9], Chapter VII, Lemma 4.4). \( \square \)

We end this section with an elementary remark regarding Wronskians. In the case when \( \Gamma \) has a cusp at the infinity \( a_\infty \), it is more convenient to use the derivative with respect to

\[ q = \exp \frac{2\pi \sqrt{-1} z}{h}, \]

where \( h > 0 \) is the width of the cusp since all modular forms have \( q \)-expansions. Using the notation from Proposition 6-2. It is easy to see

\[ \frac{d}{dz} = \frac{2\pi \sqrt{-1}}{h} \cdot q \frac{d}{dq}. \]

This implies that
\[
\frac{d^k}{dz^k} = \left( \frac{2\pi \sqrt{-1}}{h} \right)^k \cdot \left( q \frac{d}{dq} \right)^k, \quad k \geq 0.
\]

Thus, we may define the \(q\)-Wronskian as follows:

\[
W_q(f_1, \ldots, f_k) \overset{\text{def}}{=} \begin{vmatrix}
  f_1 & \cdots & f_k \\
  q \frac{d}{dq} f_1 & \cdots & q \frac{d}{dq} f_k \\
  \vdots & \ddots & \vdots \\
  \left( q \frac{d}{dq} \right)^{k-1} f_1 & \cdots & \left( q \frac{d}{dq} \right)^{k-1} f_k
\end{vmatrix},
\]

considering \(q\)-expansions of \(f_1, \ldots, f_k\).

We obtain

\[
W(f_1, \ldots, f_k) = \left( \frac{2\pi \sqrt{-1}}{h} \right)^{k(k-1)/2} W_q(f_1, \ldots, f_k).
\]

### 7. ON A DIVISOR OF A WRONSKIAN

In this section we discuss the divisor of cuspidal modular forms constructed via Wronskians (see Proposition 6-2). We start with necessary preliminary results.

**Lemma 7-1.** Let \(\varphi_1, \cdots, \varphi_l\) be a sequence of linearly independent meromorphic functions on some open set \(U \subset \mathbb{C}\). We define their Wronskian as usual \(W(\varphi_1, \cdots, \varphi_k) = \det \left( \frac{d}{dz} \varphi \right)_{i,j=1,\ldots,k}\).

Then, we have the following:

(i) The Wronskian \(W(\varphi_1, \cdots, \varphi_k)\) is a non–zero meromorphic function on \(U\).
(ii) We have \(W(\varphi_1, \cdots, \varphi_k) = \varphi^k W(\varphi/\varphi_1, \cdots, \varphi/\varphi_k)\) for all non–zero meromorphic functions \(\varphi\) on \(U\).
(iii) Let \(\xi \in U\) be such that all \(\varphi_i\) are holomorphic. Let \(A\) be the \(\mathbb{C}\)–span of all \(\varphi_i\). Then, all \(\varphi \in A\) are holomorphic at \(\xi\), and the set \(\{\nu_{z-\xi}(\varphi); \ \varphi \in A, \ \varphi \neq 0\}\) has exactly \(k = \dim A\) different elements (Here as in Section 2, \(\nu_{z-\xi}\) stands for the order at \(\xi\)). Let \(\nu_{z-\xi}(\varphi), \ldots, \varphi_k\) be the sum of all \(\dim A\)–values of that set. Then, \(W(\varphi_1, \cdots, \varphi_k)\) is holomorphic at \(\xi\), and the corresponding order is

\[
\nu_{z-\xi}(\varphi_1, \ldots, \varphi_k) = \frac{k(k-1)}{2}.
\]

**Proof.**
(i) is well–known. See for example \([9,\text{Chapter VII, Lemma 4.4}]\) or it is a consequence of \([5,\text{Proposition III.5.8}]\). (ii) is a consequence of the proof of \([5,\text{Proposition III.5.8}]\) (see formula (5.8.4)). Finally, we prove (iii). Then, by the text before the statement of \([5,\text{Proposition III.5.8}]\), we see that we can select another basis \(\psi_1, \ldots, \psi_k\) of \(A\) such that

\[
\nu_{z-\xi}(\psi_1) < \nu_{z-\xi}(\psi_2) < \ldots < \nu_{z-\xi}(\psi_k).
\]
Then, by [5] Proposition III.5.8, we have that the order of $W(\psi_1, \ldots, \psi_k)$ at $z$ is equal to

$$\sum_{i=1}^{k} (\nu_{z-\xi}(\psi_i) - i + 1) = \nu_{z-\xi}(\varphi_1, \ldots, \varphi_k) - \frac{k(k - 1)}{2}.$$ 

But $\varphi_1, \ldots, \varphi_k$ is also a basis of $A$. Thus, we see that we can write

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_k \end{pmatrix} = A \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_k \end{pmatrix}$$

for some $A \in GL_k(\mathbb{C})$. This implies

$$\begin{pmatrix} \varphi_1 & d\varphi_1/dz & \cdots & d^{k-1}\varphi_1/dz^{k-1} \\ \varphi_2 & d\varphi_2/dz & \cdots & d^{k-1}\varphi_2/dz^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_k & d\varphi_k/dz & \cdots & d^{k-1}\varphi_k/dz^{k-1} \end{pmatrix} = A \cdot \begin{pmatrix} \psi_1 & d\psi_1/dz & \cdots & d^{k-1}\psi_1/dz^{k-1} \\ \psi_2 & d\psi_2/dz & \cdots & d^{k-1}\psi_2/dz^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_k & d\psi_k/dz & \cdots & d^{k-1}\psi_k/dz^{k-1} \end{pmatrix}.$$ 

Hence

$$W(\varphi_1, \ldots, \varphi_k) = \det A \cdot W(\psi_1, \ldots, \psi_k)$$

has the same order at $z$ as $W(\psi_1, \ldots, \psi_k)$. $\square$

As a direct consequence of Lemma 7-1, we obtain the following result. At this point the reader should review the text in Section 2 before the statement of Lemma 2-2 as well as Proposition 6-2. The proof is left to the reader as an exercise.

**Proposition 7-2.** Assume that $m \geq 2$ is even. Let $f_1, \ldots, f_k \in M_m(\Gamma)$ be a sequence of linearly independent modular forms. Let $\xi \in \mathbb{H}$. Then, we have the following:

$$\nu_{a_\xi}(W(f_1, \ldots, f_k)) = \frac{1}{e_\xi} \left( \nu_{z-\xi}(\varphi_1, \ldots, \varphi_k) - \frac{k(k - 1)}{2} \right).$$

The case of a cusp requires a different technique but final result is similar:

**Theorem 7-3.** Assume that $m \geq 2$ is even. Suppose that $a_\infty$ is a cusp for $\Gamma$. Let $f_1, \ldots, f_k \in M_m(\Gamma)$ be a sequence of linearly independent modular forms. Let $i \in \{1, \ldots, k\}$. Consider $f_1, \ldots, f_k$ as meromorphic functions in a variable $q$ in a neighborhood of $q = 0$, and define $\nu_{q-0}(f_1, \ldots, f_k)$ as in Lemma 7-1 (iii). Then, we have the following identity:

$$\nu_{a_\infty}(W(f_1, \ldots, f_k)) = \nu_{q-0}(f_1, \ldots, f_k).$$

**Proof.** By Lemma 7-1 (ii), we can write

$$(7-4) \quad W(f_1, \ldots, f_k) = f_1^k \cdot W(1, f_2/f_1, \ldots, f_k/f_1)$$

as meromorphic functions on $\mathbb{H}$. But the key fact that $1, f_2/f_1, \ldots, f_k/f_1$ can be regarded as meromorphic (rational) functions on $\mathfrak{H}$ i.e., they are elements of $\mathbb{C}(\mathfrak{H})$. 

The key point now is that these meromorphic functions, and their Wronskian define meromorphic $k(k-1)/2$–differential form, denoted by $W_\Gamma$. Details are contained in [9, Section 4, Lemma 4.9]. We recall the following.

Let $w \in \mathbb{H}$ be a non–elliptic point for $\Gamma$, such that $f_i \neq 0$, and $U \subset \mathbb{H}$ small neighborhood of $w$ giving a chart of $\mathcal{A}_w$ on the curve $\mathcal{A}_\Gamma$. Then, in the chart $U$ we have:

$$W_\Gamma = W(1, f_2(z)/f_1(z), \ldots, f_k(z)/f_1(z)) (dz)^{k(k-1)/2}. $$

On the other hand, in a chart of $\mathcal{A}_\infty$, $W_\Gamma$ is given by the usual Wronskian, denoted by $W_{\Gamma,q}(1, f_2/f_1, \ldots, f_k/f_1)$, of $1, f_2/f_1, \ldots, f_k/f_1$ presented by $q$–expansions with respect to the derivatives $d^k/dq^i$, $0 \leq i \leq k-1$, multiplied by $(dq)^{k(k-1)/2}$ i.e.,

$$W_\Gamma = W(1, f_2/f_1, \ldots, f_k/f_1) (dq)^{k(k-1)/2}. $$

Next, we insert $q$–expansions of $f_1, \ldots, f_k$ into $W(1, f_2/f_1, \ldots, f_k/f_1)$. So, we can express

$$W(1, f_2/f_1, \ldots, f_k/f_1) = c_m q^m + c_{m+1} q^{m+1} + \cdots,$$

where $c_m \neq 0$, $c_{m+1}, c_{m+2}, \ldots$ are complex numbers. Hence,

$$\nu_{\mathcal{A}_\infty}(W(1, f_2/f_1, \ldots, f_k/f_1)) = \nu_{q=0}(W(1, f_2/f_1, \ldots, f_k/f_1)) = m.$$  

Let us fix a neighborhood $U$ of $\infty$ such that it is a chart for $\mathcal{A}_\infty$, and there is no elliptic points in it. Then, we fix $w \in U$, $w \neq \infty$, and a chart $V$ of $w$ such that $V \subset \mathbb{H} \cap U$. Now, on $V$, we have the following expression for $W_\Gamma$:

$$W(1, f_2/f_1, \ldots, f_k/f_1) (dz)^{k(k-1)/2} = (c_m q^m + c_{m+1} q^{m+1} + \cdots) (dz)^{k(k-1)/2}, \ q = \exp \frac{2\pi \sqrt{1-z}}{h}. $$

On the other hand, on $U$, we must have the expression for $W_\Gamma$ of the form

$$W_{\Gamma,q}(1, f_2/f_1, \ldots, f_k/f_1) (dq)^{k(k-1)/2} = (d_n q^n + d_{n+1} q^{n+1} + \cdots) (dq)^{k(k-1)/2}, $$

where $d_n \neq 0$, $d_{n+1}, d_{n+2}, \ldots$ are complex numbers. We have

$$\nu_{\mathcal{A}_\infty}(W_\Gamma) = \nu_{q=0}(W_{\Gamma,q}(1, f_2/f_1, \ldots, f_k/f_1)) = n.$$  

By definition of meromorphic $k(k-1)/2$–differential, on $V$ these expressions must be related by

$$c_m q^m + c_{m+1} q^{m+1} + \cdots = (d_n q^n + d_{n+1} q^{n+1} + \cdots) \left(\frac{dq}{dz}\right)^{k(k-1)/2}.$$ 

Hence, we obtain

$$n = m - k(k-1)/2.$$ 

Using (7-5) and (7-6) this can be written as follows:

$$\nu_{q=0}(W_{\Gamma,q}(1, f_2/f_1, \ldots, f_k/f_1)) = \nu_{q=0}(W(1, f_2/f_1, \ldots, f_k/f_1)) - k(k-1)/2.$$  

Consider again $f_1, \ldots, f_k$ as meromorphic functions in a variable $q$ in a neighborhood of $q = 0$, and define the Wronskian $W_{\Gamma,q}(f_1, \ldots, f_k)$ using derivatives with respect to $q$. Then,
Lemma 7.1 (ii) implies
\[ \nu_{a,\infty} (W(f_1, \ldots, f_k)) = \nu_{a,\infty}(f_1^k) + \nu_{a,\infty}(W(1, f_2/f_1, \ldots, f_k/f_1)) \]
\[ = \nu_{q-0}(f_1^k) + \nu_{q-0}(W(1, f_2/f_1, \ldots, f_k/f_1)) \]
\[ = \nu_{q-0}(f_1^k) + \nu_{q-0}(W_{\Gamma,q}(1, f_2/f_1, \ldots, f_k/f_1)) + (k-1)/2 \]
\[ = \nu_{q-0}(W_{\Gamma,q}(f_1, f_2, \ldots, f_k)) + (k-1)/2. \]

Finally, we apply Lemma 7.1 (iii). \( \square \)

8. Computation of Wronskians for \( \Gamma = SL_2(\mathbb{Z}) \)

Assume that \( m \geq 4 \) is an even integer. Let \( M_m \) be the space of all modular forms of weight \( m \) for \( SL_2(\mathbb{Z}) \). We introduce the two Eisenstein series
\[ E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \]
\[ E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \]
of weight 4 and 6, where \( q = \exp(2\pi i z) \). Then, for any even integer \( m \geq 4 \), we have
\[ M_m = \bigoplus_{\alpha,\beta \geq 0} \mathbb{C} E_4^{\alpha} E_6^{\beta}. \]

We have
\[ k = k_m \text{ def } \dim M_m = \begin{cases} [m/12] + 1, & m \not\equiv 2(\text{mod } 12); \\ [m/12], & m \equiv 2(\text{mod } 12). \end{cases} \]

We let
\[ \Delta(z) = q + \sum_{n=2}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 + \cdots = \frac{E_4^3(z) - E_6^2(z)}{1728}. \]
be the Ramanujan delta function.

It is well–known that the map \( f \mapsto f \cdot \Delta \) is an isomorphism between the vector space of modular form \( M_m \) and the space of all cuspidal modular forms \( S_{m+12} \) inside \( M_{m+12} \). In general, we have the following:
\[ \dim S_m = \dim M_m - 1, \]
for all even integers \( m \geq 4 \).

Now, we are ready to compute our first Wronskian (see (6.1) for notation).
**Proposition 8-3.** We have the following:

(i) \( W_q(E^3_4, E^2_6) = -1728 \cdot \Delta \cdot E_4^2 E_6. \)

(ii) \( 2E_4 \frac{d}{dq} E_6 - 3E_6 \frac{d}{dq} E_4 = -1728 \cdot \Delta \cdot q^{-1}. \)

**Proof.** We compute

\[
W_q(E_4, E_6) = \begin{vmatrix}
E^3_4 & E^2_6 \\
q \frac{d}{dq} E^3_4 & q \frac{d}{dq} E^2_6
\end{vmatrix}
\]

\[
= 2E_4^3 E_6 \cdot q \frac{d}{dq} E_6 - 3E_4^2 E_6^2 \cdot q \frac{d}{dq} E_4
\]

\[
= E_4^3 E_6 \cdot q \cdot \left( 2E_4 \frac{d}{dq} E_6 - 3E_6 \frac{d}{dq} E_4 \right).
\]

But we know that \( W_q(E_4, E_6) \) is a cusp form of weight \( 2 \cdot (12 + 2 - 1) = 26 \). Thus, we must have that is equal to

\[
W_q(E_4, E_6) = \lambda \cdot \Delta \cdot E_4^2 E_6,
\]

for some non-zero constant \( \lambda \). This implies that

\[
2E_4 \frac{d}{dq} E_6 - 3E_6 \frac{d}{dq} E_4 = \lambda \cdot \Delta \cdot q^{-1}
\]

Considering explicit \( q \)-expansions, we find that

\[
\lambda = -1728.
\]

This proves both (i) and (ii). \( \square \)

The general case requires a different proof based on results of Section 7.

**Proposition 8-4.** Assume that \( m = 12t \) for some \( t \geq 1 \). Then, we write the basis of \( M_m \) as follows: \( (E^3_4)^u (E^2_6)^{t-u}, \ 0 \leq u \leq t. \) Then, we have the following

\[
W_q \left( (E^3_4)^u (E^2_6)^{t-u}, \ 0 \leq u \leq t \right) = \lambda \cdot \Delta \frac{t+1}{2} E_4^{(t+1)} E_6^{\frac{t+1}{2}},
\]

for some non-zero constant \( \lambda \).

**Proof.** We can select another basis \( f_0, \ldots, f_t \) of \( M_m \) such that \( f_i = c_i q^i + d_i q^{i+1} + \cdots \), \( 0 \leq i \leq t \), where \( c_i \neq 0, d_i, \ldots \) are some complex constants. An easy application of Theorem 7-3 gives

\[
\nu_{a_{\infty}} \left( W_q \left( (E^3_4)^u (E^2_6)^{t-u}, \ 0 \leq u \leq t \right) \right) = \frac{t(t+1)}{2}.
\]

But since \( \text{div}(\Delta) = a_{\infty} \), we obtain that

\[
f \stackrel{\text{def}}{=} W_q \left( (E^3_4)^u (E^2_6)^{t-u}, \ 0 \leq u \leq t \right) / \Delta \frac{t+1}{2}
\]

is a non-cuspidal modular form of weight

\[
l = k \cdot (m + k - 1) - 12 \frac{t(t+1)}{2} = (t+1)(12t + t) - 12t = 7t(t+1).
\]
It remains to determine \( f \). In order to do that, we use Proposition 7-2 and consider the order of vanishing of \( W_q \left( (E_4^3)^u (E_6^2)^{t-u}, \ 0 \leq u \leq t \right) \) at elliptic points \( i \) and \( e^{\pi i/3} = (1 + i\sqrt{3})/2 \), of order 2 and 3, respectively. We recall (see [12], Lemma 4-1) that 
\[
\text{div}(E_4) = \frac{1}{3}a_{(1+i\sqrt{3})/2}.
\]
Similarly we show that 
\[
\text{div}(E_6) = \frac{1}{2}a_i.
\]
This implies that \( (E_4^3)^u (E_6^2)^{t-u} \) has order 3\( u \) and 2\( (t-u) \) at \( (1 + i\sqrt{3})/2 \) and \( i \), respectively. Hence, \( W_q \left( (E_4^3)^u (E_6^2)^{t-u}, \ 0 \leq u \leq t \right) \) has orders 
\[
\nu_{a_{(1+i\sqrt{3})/2}} \left( W_q \left( (E_4^3)^u (E_6^2)^{t-u}, \ 0 \leq u \leq t \right) \right) = \frac{1}{3}t(t+1),
\]
and 
\[
\nu_{a_{(1+i\sqrt{3})/2}} \left( W_q \left( (E_4^3)^u (E_6^2)^{t-u}, \ 0 \leq u \leq t \right) \right) = \frac{1}{4}t(t+1).
\]
This implies the following: 
\[
\nu_{a_{(1+i\sqrt{3})/2}} (f) = \frac{1}{3} \cdot t(t+1),
\]
and 
\[
\nu_{a_i} (f) = \frac{1}{4} \cdot t(t+1),
\]
Since, \( f \in M_{7t(t+1)} \), comparing divisors as before, we conclude that 
\[
f = \lambda \cdot E_4^{t(t+1)} E_6^{(t+1)/2},
\]
for some non–zero constant \( \lambda \).

We are not able to determine constant \( \lambda \) in Proposition 8-4 for all \( t \geq 1 \). It should come out of comparison of \( q \)-expansions of left and right sides of the identity in Proposition 8-4. For \( t = 1 \), Proposition 8-3 implies that \( \lambda = -1728 \). Experiments in SAGE shows that \( \lambda = -2 \cdot 1728^3 \) for \( t = 2 \), and \( \lambda = 12 \cdot 1728^6 \) for \( t = 3 \).

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