A Free Boundary Minimal Surface via a 6-Sweepout

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Abstract
We prove that the Almgren–Pitts 6-width of the unit 3-ball is less than $2\pi$. We also prove that there exists a free boundary minimal surface in the unit 3-ball that has genus at most 1, index at most 5, area less than $2\pi$, and is not the equatorial disk or the critical catenoid.

Keywords
Minimal surfaces · Min–max theory · Free boundary · Geometric measure theory

Mathematics Subject Classification 53A10 · 58E12

1 Introduction

Given a compact Riemannian 3-manifold $M$, one can relate the topology of the space of all surfaces in $M$ to minimal surfaces in $M$ via Morse theory. One may even obtain information about the genus, Morse index, and area of the minimal surfaces. In this paper, we will illustrate this phenomenon by looking at surfaces with low genus and area in the compact Euclidean unit 3-ball $B^3$, via the Almgren–Pitts and the Simon–Smith min–max theory.

Let $\mathcal{E}$ denote the set of all surfaces, possibly with boundary, in $B^3$ that are smooth and properly embedded except possibly at finitely many points (see Sect. 2.1 for details). The reason for allowing singularities is that we want to study the space of all surfaces, regardless of their genus or number of connected components, as a whole. In fact, let us define on $\mathcal{E}$ the following topology inspired by the Simon–Smith min–max theory: For each finite set $P \subset B^3$, we define on the subset

$$\{ S \in \mathcal{E} : S \setminus P \text{ is smooth and properly embedded} \}$$

(1)
the topology induced by the graphical $C^\infty$-convergence within open sets $U \subset \subset B^3 \setminus P$ (meaning $\overline{U} \subset \subset B^3 \setminus P$). Now, we collect all open sets in (1) for all possible $P$ to form a base, thereby defining a topology on $\mathcal{E}$. Note that under this topology, one has continuous paths in $\mathcal{E}$ of surfaces with different genus or number of connected components via neck-pinching. Then our first main result is the following.

Let $\mathcal{E}_g \subset \mathcal{E}$ be the subset of smooth surfaces with genus $g$, and $\mathcal{E}^a \subset \mathcal{E}$ the subset of surfaces with area less than $a$ for $a \in (0, \infty]$. Note that $\mathcal{E}^\infty \neq \mathcal{E}$, since an element of $\mathcal{E}$ can have an infinite area, concentrated near a singularity. And as in the Simon–Smith min–max theory, the genus of a disconnected smooth surface is defined as the sum of the genus of each of its connected components.

**Theorem 1.1** The first to the sixth cohomology groups of

$$\mathcal{E}_0 \cup \mathcal{E}_1 \cap \mathcal{E}^{2\pi}$$

in $\mathbb{Z}_2$-coefficients are non-trivial: In fact, the cup-length of this space is at least 6. And the same is true for any subspace of $\mathcal{E}^\infty$ that contains $\mathcal{E}_0 \cup \mathcal{E}_1 \cap \mathcal{E}^{2\pi}$.

Note that $\overline{\mathcal{E}_0 \cup \mathcal{E}_1}$ denotes the closure of $\mathcal{E}_0 \cup \mathcal{E}_1$ in $\mathcal{E}$, and the cup-length of a space $X$ is defined as the maximum number of elements in the cohomology ring of $X$ with degree at least 1 such that their cup product is non-trivial. We remark that $2\pi$ is twice the area of the equatorial disk in $B^3$.

Let us mention the following results. In his celebrated work [22], Hatcher proved the Smale conjecture, implying that the space of smoothly embedded 2-spheres in the (round) 3-sphere deformation retracts to the subspace of great 2-spheres, which is homeomorphic to $\mathbb{RP}^3$ and thus has cup-length 3. Moreover, based on Marques–Neves’ ground-breaking resolution of the Willmore conjecture [35], Nurser showed that the space of flat 2-cycles in the unit round 3-sphere with area at most $2\pi^2$ (which is the area of the Clifford torus) has cup-length in $\mathbb{Z}_2$-coefficients at least 7 [39].

Theorem 1.1 follows immediately from the result below, of which the terminologies will be defined precisely in Sect. 2.

**Theorem 1.2** There exists in the Euclidean unit 3-ball a family $\Psi$ of surfaces such that

(A) $\Psi$ is a 6-sweepout in the sense of Almgren–Pitts min–max theory.

(B) $\Psi$ is a smooth family of surfaces with genus at most 1, in the sense of Simon–Smith min–max theory.

(C) The area of each element in $\Psi$ is less than $2\pi$.

Theorem 1.2 also gives the following result immediately.

**Corollary 1.3** The Almgren–Pitts 6-width of the Euclidean unit 3-ball is less than $2\pi$.

Currently, the Almgren–Pitts widths of $B^3$ are not well understood: While the first three widths are $\pi$ and are detected by the equatorial disk (since the collection of flat disks in $B^3$ is a 3-sweepout), the fourth already seems to be unknown. Regarding computations of Almgren–Pitts widths of other manifolds, see also [2, 5, 10, 14, 49].

Let us now turn to the other side of the story: Free boundary minimal surfaces in the $B^3$. In recent years, besides the two most basic examples, the equatorial disk
and the critical catenoid, an abundance of free boundary minimal surfaces in $\mathbb{B}^3$ were constructed. For example, by solving extremal eigenvalue problems, Fraser–Schoen constructed examples with genus 0 and arbitrary number of boundary components [20]. Using gluing techniques, Kapouleas–Li constructed embedded free boundary minimal surfaces of large genus that desingularize the union of the equatorial disk and the critical catenoid [28]. (See [8, 11, 17, 23–26, 29] for more examples.)

We will use min–max theory to produce a free boundary minimal surface. The advantage of this approach is that one can upper bound the Morse index of the minimal surface because of the work of Marques–Neves [36]. In general, Morse index is difficult to compute. For example, to our best knowledge, in $\mathbb{B}^3$ the only embedded free boundary minimal surfaces whose index are known are the equatorial disk and the critical catenoid: They have index 1 and 4, respectively [12, 45, 46]. In addition, from the recent resolution of the multiplicity one conjecture in the free boundary setting by Sun–Wang–Zhou [44] based on the work of Zhou [48], we know there exists a sequence $\{\Sigma_k\}$ of embedded free boundary minimal surfaces in $\mathbb{B}^3$ with area growth of order $k^{1/3}$ and index at most $k$. However, using the Almgren–Pitts min–max theory, one cannot control the genus of the surfaces. In this paper, we apply the Simon–Smith min–max theory to the family $\Psi_1$ in Theorem 1.2 to construct an example with index, genus, and area bound.

**Theorem 1.4** There exists in the Euclidean unit 3-ball an embedded free boundary minimal surface with genus 0 or 1, Morse index 4 or 5, and area in the range $(\pi, 2\pi)$, that is not the equatorial disk or the critical catenoid.

In fact, using the results of Sargent [41] and Ambrozio–Carlotto–Sharp [1] that lower bound the index of a free boundary minimal surface by its genus and number of boundary components, we know that the surface in Theorem 1.4 has at most 16 boundary components. But we believe this bound is far from optimal (see Sect. 1.1 below). We also note that, since we have to prove the index bound, we cannot use the equivariant min–max theory of Ketover [25].

**Remark 1.5** We remark that Carlotto–Franz–Schulz [8] showed, using equivariant min–max theory, there exists a free boundary minimal surface in $\mathbb{B}^3$ that has genus 1, area less than $3\pi$, a connected boundary, and symmetry group $D_2$, where $D_2 \subset SO(3)$ denotes the dihedral group with four elements (see the Geometric Analysis Gallery by Schulz [42]). In fact, as we will see in Sect. 3.5, Theorem 1.2 can reproduce their result and slightly improve the area bound from $3\pi$ to $2\pi$.

The family $\Psi$ in Theorem 1.2 can be modified to become a desirable 6-sweepout in $\mathbb{R}^3$ equipped with the Gaussian metric $\frac{1}{4\pi}e^{-|x|^2/4}g_0$, in which $g_0$ denotes the Euclidean metric, allowing one to construct a self-shrinker with genus, index, and Gaussian area control. However, the Gaussian metric has a singularity at infinity, which poses some challenges in carrying out the min–max theory. We plan to address this in our upcoming work.
1.1 Open Questions

Regarding Theorem 1.1, it would be interesting to change the genus 0 and 1 constraint, the area bound $2\pi$, or the ambient space $\mathbb{B}^3$ (to the round 3-sphere for example), and investigate the topology of the corresponding space of surfaces. It will be nice to have more examples of $k$-sweepouts for $k > 6$ that are smooth families. And for each $k$, among $k$-sweepouts $\Phi$ that are smooth families, is there a non-trivial lower bound for the maximum of the genus of elements in $\Phi$?

We conjecture that the free boundary minimal surface in Theorem 1.4, denoted $\Sigma$, has index 5. One can also ask if $\Sigma$ has the third lowest area among all free boundary minimal surfaces in $\mathbb{B}^3$, after the equatorial plane and the critical catenoid. Moreover, we speculate that $\Sigma$ is the same as the free boundary minimal surface constructed by Carlotto–Franz–Schulz [8] mentioned in Remark 1.5.

Concerning the Almgren–Pitts min–max theory in $\mathbb{B}^3$, we conjecture the 4-width is detected by the critical catenoid $K$. In particular, showing the 4-width is at least area($K$) seems challenging, as it may depend on the conjecture that the second least area of an immersed free boundary minimal surface in $\mathbb{B}^3$ is realized by the critical catenoid [31, Sect. 7]. As for the 5-width and the 6-width, it will be interesting to know if they are detected by the free boundary minimal surface of Theorem 1.4.

1.2 Overview of Proofs

Let us outline the construction of the smooth family $\Psi$ in Theorem 1.2. We first consider the saddle surface $\{x^2 - y^2 + z = 0\}$ in $\mathbb{R}^3$, and then translate, rescale, and rotate it arbitrarily: We even allow the scaling factor to be 0 or $\pm\infty$. Then we collect all such surfaces, and it turns out this collection can be parametrized by a 7-dimensional quotient space of some $D_2$-action on $\mathbb{R}P^4 \times SO(3)$. This is actually due to the $D_2$-symmetry of the saddle. However, this collection contains intersecting planes like $\{x^2 - y^2 = 0\}$, the blow down of the saddle, which has a singular line and thus is not allowed in the Simon–Smith setting. To resolve this, we desingularize the intersecting planes by adding a small $z^3$ term to their defining equations (e.g., see Fig. 1; Table 1), so that only isolated singularities appear. Finally, we intersect all surfaces with $\mathbb{B}^3$ to define $\Psi$.

Theorem 1.1 is an immediate consequence of Theorem 1.2: See Sect. 3.2.

Finally, for Theorem 1.4, we will use the smooth family $\Psi$ in Theorem 1.2 as follows. Let $\Psi^{(5)}$ denote the subfamily of $\Psi$ parametrized by a 5-skeleton of the parameter space of $\Psi$. By applying the Simon–Smith min–max theorem to $\Psi^{(5)}$, we obtain a free boundary minimal surface $\Gamma$ with genus at most 1, index at most 5, and area less than $2\pi$. Note that, although it is not known if the multiplicity one conjecture holds in the Simon–Smith setting, we can guarantee that $\Gamma$ has multiplicity one because area($\Gamma$) < $2\pi$ and the least possible area of a free boundary minimal surface in $\mathbb{B}^3$ is $\pi$. Then by the fact that $\Psi$ is a 6-sweepout and topological arguments of Lusternik–Schnirelmann, we show that the method above, with some modifications, produces a free boundary minimal surface with the desired properties that is not the equatorial disk or the critical catenoid.
Table 1 The surface \( \{x^2 - y^2 + s(b_3z + 0.1 + z^3) = 0\} \) for various \( b_3 \) and \( s \) is shown above

| \( s \) | \( b_3 \) | 0.1 | \(-3\sqrt[3]{\frac{1}{400}} \approx 0.407\) | 0.6 |
|---|---|---|---|---|
| 0.05 | | ![Diagram](image1.png) | ![Diagram](image2.png) | ![Diagram](image3.png) |
| 0.3 | | ![Diagram](image4.png) | ![Diagram](image5.png) | ![Diagram](image6.png) |

They illustrate the three cases where the polynomial \( a_3z + a_4 + a_5z^3 \) has 1, 2, and 3 roots, respectively.
1.3 Organization

We introduce some preliminaries in Sect. 2, and in Sect. 3 prove the results in Sect. 1. The proofs of some propositions used in Sect. 3 will be postponed to Sect. 4.

2 Preliminaries

In this section, we first discuss more about the space \( \mathcal{E} \) introduced in Sect. 1, and then state some preliminaries about min–max theory, in both the Almgren–Pitts setting [3, 4, 40] and the Simon–Smith setting [7, 43].

2.1 About the Space \( \mathcal{E} \)

A smooth embedded surface \( S \) in \( B^3 \) is said to be properly embedded if \( \partial S = S \cap \partial B^3 \) and \( S \) meets \( \partial B^3 \) transversely along \( \partial S \). By definition, \( \mathcal{E} \) consists of closed sets \( S \subset B^3 \) such that there exists a finite set \( P \) such that \( S \setminus P \) is a smooth and properly embedded surface. (Note that \( \partial(S \setminus P) \) does not include \( P \).) Now, for any open set \( U \subset B^3 \) \( \setminus P \), \( \epsilon > 0 \), and non-negative integer \( k \), denote by \( B_{P, U, \epsilon, k}(S) \subset \mathcal{E} \) the subset of all surface \( S' \in \mathcal{E} \) such that \( S' \setminus P \) is smooth and properly embedded and is \( \epsilon \)-close to \( S \) in the graphical \( C^k \)-distance within \( U \). Then the following proposition tells us that the topology on \( \mathcal{E} \) introduced in Sect. 1 is well defined.

**Proposition 2.1** The subsets \( B_{P, U, \epsilon, k}(S) \subset \mathcal{E} \) form a base.

**Proof** First, these subsets clearly cover \( \mathcal{E} \). So it suffices to show that if \( B_{P_1, U_1, \epsilon_1, k_1}(S_1) \cap B_{P_2, U_2, \epsilon_2, k_2}(S_2) \) contains some element \( S \), then it contains some subset \( B_{P, U, \epsilon, k}(S) \). Indeed, one can just take \( P := P_1 \cap P_2, U := U_1 \cup U_2, k := \max\{k_1, k_2\}, \) and \( \epsilon > 0 \) to be sufficiently small. \( \square \)

We will mention some mostly obvious remarks. First, \( \mathcal{E} \) contains disconnected surfaces, and also the empty surface \( \emptyset \) and any finite sets of points tautologically. Taking \( P = \emptyset \) in (1), we know \( \{\emptyset\} \) is an open subset of \( \mathcal{E} \). However, \( \emptyset \in \mathcal{E} \) is not an isolated point as for any \( p \in B^3 \), all open neighborhoods of \( \{p\} \) in \( \mathcal{E} \) has \( \emptyset \) as an element tautologically. Similarly, for any distinct points \( p_1, p_2 \in B^3 \), all open neighborhoods of \( \{p_1, p_2\} \) in \( \mathcal{E} \) has \( \{p_1\} \) as an element, but not vice versa. Moreover, for any \( p \in B^3 \), let \( B_r(p) \subset B^3 \) be the ball centered at \( p \) with radius \( r \). Then for \( n \geq 2, \partial B_{1/n}(p) \in \mathcal{E} \) and converge to \( \{p\} \) (not \( \emptyset \)) as \( n \to \infty \). Furthermore, the path \( r \mapsto \partial B_r(0) \) of spheres in \( \mathcal{E} \) for \( r \in (0, 2) \) is not well defined at \( r = 1 \), but by perturbing the spheres to ellipsoids, the path becomes well defined and continuous.

2.2 Simon–Smith Min–Max Theory

Let \( M \) be a compact oriented Riemannian 3-manifold with strictly mean convex boundary.
Definition 2.2 Let $X$ be a compact $k$-dimensional cubical complex, called the parameter space. Suppose we have a map $\Phi$ assigning to each $x \in X$ a closed subset $\Phi(x)$ of $M$ such that

1. There exists a dense subset $Y \subset X$ of parameters such that
   - For each $x \in Y$, $\Phi(x)$ is an oriented, smooth, and properly embedded surface with boundary.
   - For each $x \in X \setminus Y$, there exists a finite set $P(x)$ such that $\Phi(x) \setminus P(x)$ is a smooth and properly embedded surface with boundary.
   Moreover, we require that $|P(x)|$ is bounded independent of $x$. (We can say $P(x) = \emptyset$ for $x \in Y$ for convenience.)
2. $\Phi$ is continuous in the varifold topology.
3. For any $x_0 \in X$ and open set $U \subset M \setminus P(x_0)$ (i.e., $\overline{U} \subset M \setminus P(x_0)$), $\Phi(x) \rightarrow \Phi(x_0)$ in the graphical $C^\infty$-topology in $U$ whenever $x \rightarrow x_0$.
4. $\Phi(x)$ has genus at most $g$ for each $x \in Y$.

Then we call $\Phi$ a smooth family of surfaces with genus at most $g$, in brief, a genus $\leq g$ smooth family.

Note that when $\Phi(x)$ is disconnected, its genus is defined as the sum of the genus of each of its connected components. For (3), $\Phi(x_0)$ meets $\partial B^3$ transversely in $U$, thus the graphical convergence makes sense even near the boundary $\partial \Phi(x_0)$. Moreover, we required continuity in the varifold topology (see [8, 18]) instead of the Hausdorff topology because we want to allow a smooth family to contain empty sets: We will explain more about the minor variations between our definition of a smooth family and others’ later in the proof of Theorem 2.3.

Two smooth families $\Phi$ and $\Phi'$ parametrized by $X$ are said to be homotopic if there exists a map $\psi \in C^\infty(X \times M, M)$ such that $\psi(x, \cdot) \in \text{Diff}_0(M)$ for each $x$ (meaning each $\psi(x, \cdot)$ is homotopic via diffeomorphisms to the identity map), and $\psi(x, \Phi(x)) = \Phi'(x)$ for each $x$. Given a homotopy class $\Lambda$, its width is defined by

$$L_{SS}(\Lambda) := \inf_{\Phi \in \Lambda} \max_{x \in X} \text{area}(\Phi(x)).$$

A sequence $\{\Phi_i\}$ in $\Lambda$ is said to be minimizing if

$$\lim_{i \rightarrow \infty} \max_{x \in X} \text{area}(\Phi_i(x)) = L_{SS}(\Lambda).$$

If $\{\Phi_i\}$ is a minimizing sequence and we pick $x_i$ such that

$$\lim_{i \rightarrow \infty} \text{area}(\Phi_i(x_i)) = L_{SS}(\Lambda),$$

then $\{\Phi_i(x_i)\}$ is called a min–max sequence. Furthermore, a minimizing sequence is pulled-tight if all its min–max sequences approach the set of stationary varifolds in the varifold topology.
Theorem 2.3 Let $\Lambda$ be a homotopy class of genus $\leq g$ smooth families parametrized by $X$. Then there exists a pulled-tight minimizing sequence in $\Lambda$, which contains a min–max sequence converging in the varifold topology to some varifold $V = \sum_{i=1}^{N} n_i \Gamma^i$, in which $\Gamma^i$ are disjoint embedded free boundary minimal surfaces and $n_i$ are positive integers, such that

- $\|V\| = \text{LSS}(\Lambda)$.
- $\text{index}(\text{spt}(V)) \leq \dim(X)$.
- $\sum_{\Gamma^i \text{ orientable}} \text{genus}(\Gamma^i) + \frac{1}{2} \sum_{\Gamma^i \text{ non-orientable}} (\text{genus}(\Gamma^i) - 1) \leq g$.

**Proof** It suffices to prove the following statements:

1. There exists in $\Lambda$ a pulled-tight minimizing sequence $\{\Phi_n\}$.
2. There exists a function $r : M \to \mathbb{R}_{>0}$ and a min–max sequence $\{\Phi_n(x_n)\}$ of the minimizing sequence above such that
   - For every $p \in M$, in every annulus centered at $p$ with outer radius at most $r(p)$, $\Phi_n(x_n)$ is $1/n$-almost minimizing (see [7, Definition 3.2]) when $n$ is large enough.
   - In any such annulus, $\Phi_n(x_n)$ is smooth when $n$ is large enough.
   - $\Phi_n(x_n)$ converges to a stationary varifold $V$.
3. $V$ has the desired form $\sum_{i=1}^{N} n_i \Gamma^i$ mentioned above.
4. The index bound.
5. The genus bound.

Item (1) follows from the pull-tight procedure in [7, Sect. 4] of Colding–De Lellis. Item (2) follows from [7, Proposition 5.1] and its multi-parameter version [9, Appendix] by Colding–Gabai–Ketover. For the adaptation to the case of manifold with boundary, see [30] by Li and [18] by Franz. Note the following differences between our setting and the previous ones. First, our parameter space $X$ is a cubical complex instead of a cube, but we can embed it into some cube of high dimension so that the same proofs work. Second, even though unlike in [7] we are doing non-relative min–max theory, as we allow a homotopy to vary a smooth family on the boundary of its parameter space, the same argument of [7] is still applicable (in the Almgren–Pitts setting, the non-relative version was carried out in [37]). Third, in our definition of a smooth family $\Phi$, we allow the set $P(x)$ of singularities of $\Phi(x)$ to vary as $x$ varies. However, we can still ensure each $\Phi_n(x_n)$ to be smooth in any small annulus described in (2). This is because, by passing to a subsequence, we can assume that $P(x_n)$ converges as $n \to \infty$ to some finite set $P$ in the Hausdorff topology, so that our claim follows immediately by choosing $r(p)$ to be small enough (see the last paragraph of [7, Sect. 5]).

As for item (3), the regularity of $V$ is due to [7, Theorem 7.1] for the closed case and [30, Proposition 4.11] for the free boundary case: Notice that we have assumed $\partial M$ to be strictly mean convex, which via the maximum principle guarantees that the interior of $V$ does not touch $\partial M$. As for the index bound, it was first proven in the Almgren–Pitts setting, by Marques–Neves [36] in the closed case and Guang–Li–Wang–Zhou [21] in the free boundary case, and then adapted to the Simon–Smith
setting by Franz [18]. Finally, the genus bound is due to [30, Theorem 9.1] by Li, based on [13, Theorem 1.6] by De Lellis–Pellandini. We note that although the set $X \setminus Y$ of parameters that give non-smooth surfaces may not be finite, the proof of the genus bound (in [13, Section 2.3]) using Simon’s lifting lemma [13, Proposition 2.1] is still valid using the fact that the complement $(X \setminus Y)^c = Y$ is dense by assumption. (See also [26, 27] by Ketover, which provide a stronger genus bound for limits of min–max sequences of smooth surfaces.)

\[\square\]

2.3 Almgren–Pitts Min–Max Theory

Let $M$ be a compact $(n + 1)$-dimensional Riemannian manifold with boundary. Let $\mathcal{R}_k(M; \mathbb{Z}_2)$ (resp. $\mathcal{R}_k(\partial M; \mathbb{Z}_2)$) be the set of $k$-dimensional rectifiable currents in $M$ (resp. $\partial M$) with $\mathbb{Z}_2$-coefficients. For any $T \in \mathcal{R}_k(M; \mathbb{Z}_2)$ such that its support lies in $\partial M$, we define an equivalence relation by $T \sim S$ if $T - S \in \mathcal{R}_k(\partial M; \mathbb{Z}_2)$, and then denote by $\mathcal{Z}_k(M, \partial M; \mathbb{Z}_2)$ the set of such equivalence classes. The three common topologies on $\mathcal{Z}_k(M, \partial M; \mathbb{Z}_2)$ are given by the flat metric $F$, the $F$-metric, and the mass $M$, respectively: Since the definitions are standard, we refer the reader to, for example, [21, Section 3] for the details. Note that by [21, Section 3.3], under the metric $F$ or $M$, $\mathcal{Z}_k(M, \partial M; \mathbb{Z}_2)$ is homeomorphic and isometric to the space of relative $k$-cycles considered in [32, Section 2.2].

Then by the Almgren isomorphism theorem [3] (see also [32, Section 2.5]), if $H_n(M, \partial M; \mathbb{Z}_2) = 0$, then when equipped with the flat topology, $\mathcal{Z}_n(M, \partial M; \mathbb{Z}_2)$ is connected and weakly homotopic equivalent to $\mathbb{RP}^\infty$. Thus we can denote its cohomology ring by $\mathbb{Z}[\bar{\lambda}]$. Then an $F$-continuous map $\Phi : X \to \mathcal{Z}_n(M, \partial M; \mathbb{Z}_2)$ is said to be a $k$-sweepout if there exists $\lambda \in H^1(X; \mathbb{Z}_2)$ such that

- $\lambda$ detects the 1-sweepouts, i.e., for any cycle $\gamma : S^1 \to X$, we have $\lambda(\gamma) \neq 0$ if and only if $\Phi \circ \gamma : S^1 \to \mathcal{Z}_n(M, \partial M; \mathbb{Z}_2)$ is a 1-sweepout.
- The cup product $\lambda^k \in H^k(X; \mathbb{Z}_2)$ is non-zero.

3 Proofs of Main Results

In this section, we prove the results stated in §1. From now on, we denote by $\mathcal{Z}$ the space $\mathcal{Z}_2(\mathbb{B}^3, \partial \mathbb{B}^3; \mathbb{Z}_2)$ with the flat topology.
3.1 Proof of Theorem 1.2

We will construct the desired family $\Psi$ that satisfies condition (A), (B), and (C) of Theorem 1.2 in two steps: In step 1, we construct a 6-sweepout (condition (A)). Then in step 2, we modify it such that it becomes, in addition, a genus $\leq 1$ smooth family (condition (B)) with maximal area less than $2\pi$ (condition (C)).

**Step 1.** We consider all scalings and translations of the saddle surface $\{x^2 - y^2 + z = 0\}$ in $\mathbb{R}^3$, and then intersect them with $\mathbb{B}^3$. Namely, we define a map $\Phi_4 : \mathbb{R}^4 \to \mathcal{Z}$ by assigning to each $a = [a_0 : a_1 : a_2 : a_3 : a_4] \in \mathbb{R}^4$ the zero set of the polynomial

$$p_{a_0, a_1, a_2, a_3, a_4}(x, y, z) := a_0(x^2 - y^2) + a_1 x + a_2 y + a_3 z + a_4$$

in $\mathbb{B}^3$. And then we add in rotations. Namely, we define $\Phi_7 : \mathbb{R}^4 \times SO(3) \to \mathcal{Z}$ by assigning each $(a, Q)$ to the surface “$\Phi_4(a)$ rotated by $Q^{-1}$,” i.e., the zero set of the polynomial $p_{a_0, a_1, a_2, a_3, a_4}(Q(x, y, z))$ in $\mathbb{B}^3$.

However, a loop in the $SO(3)$ factor does not produce a 1-sweepout (e.g., consider a disk rotating for $360^\circ$), and $\Phi_7$ is not yet a 6-sweepout. To get a 6-sweepout, one needs to take a quotient on the space $\mathbb{R}^4 \times SO(3)$ as follows:

We first observe $\{x^2 - y^2 + z = 0\}$ has a dihedral symmetry: Let

$$g_1 := \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right) \text{ and } g_2 := \left( \begin{array}{ccc} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right).$$

They are the $180^\circ$ rotation about the line $\{z = 0, x = y\}$ and $\{z = 0, x = -y\}$, respectively. Then $D_2 := \{\text{id}, g_1, g_2, g_1 g_2\}$ is a dihedral group, which preserves $\{x^2 - y^2 + z = 0\}$. Motivated by this, we define a $D_2$-action on $\mathbb{R}^4$ by

$$g_1[a_0 : a_1 : a_2 : a_3 : a_4] := [-a_0 : a_2 : a_1 : -a_3 : a_4],$$

$$g_2[a_0 : a_1 : a_2 : a_3 : a_4] := [-a_0 : -a_2 : -a_1 : -a_3 : a_4],$$

and then a $D_2$-action on $\mathbb{R}^4 \times SO(3)$ by

$$g_1(a, Q) := (g_1 a, g_1 Q),$$

$$g_2(a, Q) := (g_2 a, g_2 Q).$$

The whole reason we define the action by (2) is to ensure the following:

**Proposition 3.1** For each $g \in D_2$ and $a \in \mathbb{R}^4$, $g^{-1}(\Phi_4(g(a))) = \Phi_4(a)$.

**Proof** The proof is straightforward: Let $a = [a_0 : a_1 : a_2 : a_3 : a_4]$. Then

$$g_1^{-1}(\Phi_4(g_1(a))) = \{ -a_0(y^2 - x^2) + a_2 y + a_1 x - a_3(-z) + a_4 = 0 \} \cap \mathbb{B}^3,$$

$$g_2^{-1}(\Phi_4(g_2(a))) = \{ -a_0((-y)^2 - (-x)^2) - a_2(-y) - a_1(-x) - a_3(-z) + a_4 = 0 \} \cap \mathbb{B}^3,$$

which are both the same as $\Phi_4(a)$. \qed
Fig. 1 The surface \( \{ a_0(x^2 - y^2 + a_5z^3) + a_3z = 0 \} \) for various \( a_0 \) and \( a_3 \), with \( a_5 > 0 \) small and fixed

As an immediate result, \( \tilde{\Phi}_7(g(a, Q)) = \tilde{\Phi}_7(a, Q) \) for all \( g, a, \) and \( Q \), and hence one can pass to the quotient space to define a new collection \( \Phi_7 : \mathbb{RP}^4 \times SO(3) \rightarrow \mathcal{Z} \). Note that \( \Phi_7 \) is \( \mathcal{F} \)-continuous clearly because it parametrizes the collection of all scalings, translations, and rotations of the saddle surface \( \{ x^2 - y^2 + z = 0 \} \), intersected with \( \mathbb{B}^3 \). Then a crucial fact is

**Proposition 3.2** \( \Phi_7 \) is a 6-sweepout.

The proof of Proposition 3.2 is a lengthy calculation of algebraic topology. We postpone it to Sect. 4.1. Hence, \( \Phi_7 \) satisfies condition (A).

**Step 2.** However, \( \Phi_7 \) is not a smooth family, as it contains intersecting disks of the form \( \{(x - x_0)^2 - (y - y_0)^2 = 0\} \cap \mathbb{B}^3 \). We are going to desingularize them.

For each fixed number \( a_5 \geq 0 \), let us define a collection \( \Phi_{4s} : \mathbb{RP}^4 \rightarrow \mathcal{Z} \) by assigning \([a_0 : a_1 : a_2 : a_3 : a_4] \) to the zero set of the polynomial

\[
p_{a_0, a_1, a_2, a_3, a_4, a_5}(x, y, z) := a_0(x^2 - y^2 + a_5z^3) + a_1x + a_2y + a_3z + a_4 \quad (3)
\]

in \( \mathbb{B}^3 \). So for small \( a_5 > 0 \), \( \Phi_{4s} \) is slight modification of \( \Phi_4 \). In fact, as we will see, the effect of the \( z^3 \) term is three-fold: Desingularizing the intersecting disks, creating genus 1 surfaces (Fig. 1), and lowering the area to strictly below \( 2\pi \).

**Remark 3.3** Let us geometrically describe the family \( \Phi_{4s} \). We will focus on the cubic surfaces, so without loss of generality we put \( a_0 = 1 \). Since \( a_1 \) and \( a_2 \) just contribute to translation, let us assume they are both 0. Now, fix some \((a_3, a_4, a_5) \in \mathbb{R}^2 \times (0, \infty)\).
and let \( s > 0 \) varies. We claim that as \( s \) increases from 0, the surfaces

\[
\Phi_4^{(s)}([1 : 0 : 0 : a_3 s : a_4 s]) = \{ x^2 - y^2 + s(a_3 z + a_4 + a_5 z^3) = 0 \} \cap \mathbb{B}^3
\]  

(4)

desingularize the intersecting disks \( \{ x^2 - y^2 = 0 \} \cap \mathbb{B}^3 \) along the singular line. Indeed, we consider the three cases: \( a_3 z + a_4 + a_5 z^3 \) having 1, 2, or 3 roots; let \( z_i \)'s be the roots. Then in each case, as \( s \) increases from 0, the surfaces \( \Phi_4^{(s)}([1 : 0 : 0 : a_3 s : a_4 s]) \) stay fixed on the coordinate planes \( \{ z = z_i \} \), and “open up” smoothly above, below, and in between. This is because by (4), at each fixed height \( z \neq z_i \), the cross-section of the surface is a hyperbola, which dilates as \( s \) increases and has a distance of \( \sqrt{s} |a_3 z + a_4 + a_5 z^3| \) between the two branches. In Table 1, we show some examples.

Moreover, we can study for which \((a_3, a_4, a_5)\) the surface \( \Phi_4^{(s)}([1 : 0 : 0 : a_3 : a_4]) \) has singularities, and where they are: By solving

\[
\begin{cases}
p_{1,0,0,a_3,a_4,a_5} = x^2 - y^2 + a_3 z + a_4 + a_5 z^3 = 0 \\
\nabla p_{1,0,0,a_3,a_4,a_5} = (2x, -2y, a_3 + 3a_5 z^2) = (0, 0, 0)
\end{cases}
\]

we know when \( a_3 z + a_4 + a_5 z^3 \) has some double or triple root \( z_1 \), the surface has a singularity at \((0, 0, z_1)\).

But our goal is to modify \( \Phi_7 \), not just \( \Phi_4 \). To do that, first notice by a straightforward calculation that \( \Phi_4^{(s)} \) satisfies a property similar to Proposition 3.1, namely \( g^{-1}(\Phi_4^{(s)}(g(a))) = \Phi_4^{(s)}(a) \). The key idea behind is that the polynomial \( x^2 - y^2 + a_5 z^3 \) is invariant under the \( D_2 \)-action on \((x, y, z)\). As a result, we can construct a map

\[
\Phi_7^{(s)} : \frac{\mathbb{R}^4 \times SO(3)}{D_2} \rightarrow \mathbb{Z}
\]

by rotating all elements in \( \Phi_4^{(s)} \), just like how we constructed \( \Phi_7 \) from \( \Phi_4 \) in step 1. Hence we have obtained a modification \( \Phi_7^{(s)} \) of \( \Phi_7 \).

Now, from Remark 3.3, it follows easily that \( \Phi_7^{(s)} \) is \( \mathcal{F} \)-continuous and is homotopic in the \( \mathcal{F} \)-topology to \( \Phi_7 \), so that it is a \( 6 \)-sweepout (condition (A)). Moreover, we show that condition (B) can be satisfied:

**Proposition 3.4** For almost every \( a_5 \in [0, 1] \), \( \Phi_7^{(s)} \) is a genus \( \leq 1 \) smooth family.

**Proof** By Remark 3.3, the subset of parameters \( a \in \frac{\mathbb{R}^4 \times SO(3)}{D_2} \) such that \( \Phi_7^{(s)}(a) \) has singularities is 1-codimensional, and all such surfaces have at most one singularity. Moreover, it is straightforward to check that for all \( a \), \( \Phi_7^{(s)}(a) \) and \( \partial \mathbb{B}^3 \) touch (i.e., have coinciding tangent planes) at finitely many points. Also, by the transversality theorem and Remark 3.3, for a.e. \( a_5 \in [0, 1] \), the algebraic surfaces in \( \mathbb{R}^3 \) that define \( \Phi_7^{(s)}(a) \) (i.e., the rotations of the zero set of (3) in \( \mathbb{R}^3 \)) intersect \( \partial \mathbb{B}^3 \) transversely for a.e. \( a \in \frac{\mathbb{R}^4 \times SO(3)}{D_2} \). So for such \( a_5 \), \( \Phi_7^{(s)} \) satisfies Definition 2.2 (1).

Note that each smooth surface of \( \Phi_4^{(s)} \) has genus 0 or 1 because they are obtained from opening up the intersecting disks \( \{ x^2 - y^2 = 0 \} \cap \mathbb{B}^3 \) above, below, and in between at most three horizontal planes, by Remark 3.3. So each smooth surface of \( \Phi_7^{(s)} \) has genus 0 or 1. Now, using Remark 3.3, one can show that Definition 2.2 (2) and (3) are also satisfied by \( \Phi_7^{(s)} \). So \( \Phi_7^{(s)} \) is a genus \( \leq 1 \) smooth family for a.e. \( a_5 \). \( \square \)
Next, we claim that for small $a_5 > 0$, $\Phi_7^{a_5}$ also satisfies condition (C), i.e., area $\circ \Phi_7^{a_5} < 2\pi$. Indeed, it suffices to show that area $\circ \Phi_4^{a_5} < 2\pi$ for small $a_5 > 0$, which follows straightforwardly from the following two propositions:

**Proposition 3.5** The area function area $\circ \Phi_4 : \mathbb{RP}^4 \to \mathbb{R}$ attains a strict global maximum at $[1 : 0 : 0 : 0 : 0]$.

Note that $\Phi_4([1 : 0 : 0 : 0 : 0]) = \{x^2 - y^2 = 0\} \cap \mathbb{B}^3$, which has area $2\pi$.

**Proposition 3.6** Define $\Phi_5 : \mathbb{RP}^4 \times [0, 1] \to Z$ by $\Phi_5(a, a_5) = \Phi_4^{a_5}(a)$. Then the area function area $\circ \Phi_5$ attains a strict local maximum at $([1 : 0 : 0 : 0 : 0], 0)$.

The proof of Proposition 3.5 is due to the MathOverflow user fedja [16]. We include it in Appendix 1. The proof of Proposition 3.6 is postponed to Sect. 4.2: It uses mainly calculus but is quite technical. However, intuitively Proposition 3.6 makes sense because Remark 3.3 tells us that $\Phi_5$ gives a desingularization of the intersecting equatorial disks, and desingularization should lower the area as the sharp bend along the singular line is smoothed.

Thus, for a.e. sufficiently small $a_5 > 0$, by Propositions 3.5 and 3.6, $\Phi_7^{a_5}$ satisfies condition (C) also. Defining $\Psi$ as one such $\Phi_7^{a_5}$, we finish the proof of Theorem 1.2.

### 3.2 Proof of Theorem 1.1

We will first do the case of $\overline{\mathcal{E}_0} \cup \overline{\mathcal{E}_1} \cap \mathcal{E}^{2\pi}$. By Theorem 1.2 (B) and (C), we can view the family $\Psi$ in Theorem 1.2 as the composition of a map $\Psi' : \mathbb{RP}^4 \times SO(3) \to \overline{\mathcal{E}_0} \cup \overline{\mathcal{E}_1} \cap \mathcal{E}^{2\pi}$ and the natural map $i$ from $\overline{\mathcal{E}_0} \cup \overline{\mathcal{E}_1} \cap \mathcal{E}^{2\pi}$ to the space of 2-cycles $Z_2(\mathbb{B}^3, \partial \mathbb{B}^3; Z_2)$ equipped with the flat topology (see Sect. 2.3). Note that $\Psi'$ is continuous (under the topology of $\mathcal{E}$ defined in Sect. 1) since $\Psi$ is a smooth family by Theorem 1.2 (B), and $i$ is continuous by the fact that the smooth convergence of surfaces is stronger than the flat convergence.

By Almgren isomorphism theorem (see Sect. 2.3), we denote the cohomology ring of $Z_2(\mathbb{B}^3, \partial \mathbb{B}^3; Z_2)$ in $Z_2$-coefficients as $Z_2[\lambda]$. To prove Theorem 1.1 for the space $\overline{\mathcal{E}_0} \cup \overline{\mathcal{E}_1} \cap \mathcal{E}^{2\pi}$, it suffices to show that $(i^\ast \lambda)^6 \neq 0$. Thus, it suffices to show $(i \circ \Psi')^\ast(\lambda^6) \neq 0$, i.e., $\Psi$ is a 6-sweepout, which is true by Theorem 1.2.

Reusing the argument above with $\overline{\mathcal{E}_0} \cup \overline{\mathcal{E}_1} \cap \mathcal{E}^{2\pi}$ replaced by any subspace of $\mathcal{E}^\infty$ that contains $\overline{\mathcal{E}_0} \cup \overline{\mathcal{E}_1} \cap \mathcal{E}^{2\pi}$, we finish the proof of Theorem 1.1. (Note that elements in $\mathcal{E}^\infty$ have finite area and thus belong to $Z_2(\mathbb{B}^3, \partial \mathbb{B}^3; Z_2)$.)

### 3.3 Proof of Corollary 1.3

Let $\Psi$ be the smooth family in Theorem 1.2. Since area $\circ \Psi < 2\pi$ and $\Psi$ is a 6-sweepout by Theorem 1.2, $\omega_6(\mathbb{B}^3) < 2\pi$.

### 3.4 Proof of Theorem 1.4

Let $\Psi$ be the family satisfying condition (A), (B), and (C) of Theorem 1.2, and $\Psi^{(5)}$ be the subfamily of $\Psi$ parametrized by a 5-skeleton of the parameter space of $\Psi$. Without
loss of generality, we can assume that $\Psi^{(5)}$ is also a smooth family. Now, since $\Psi$ is a 5-sweepout by (A), so is $\Psi^{(5)}$ (see the proof of [38, Proposition 7.1]). It follows that the width $L := L_{\text{SS}}(\Lambda(\Psi^{(5)}))$ is positive.

Now, we apply the Simon–Smith min–max theory to $\Psi^{(5)}$. Let $\{\Phi_i\}$ be a pulled-tight minimizing sequence of $\Lambda(\Psi^{(5)})$. Denote by $W$ the set of all stationary integral varifolds in $\mathbb{B}^3$ whose support is a smooth embedded free boundary minimal hypersurface, and by $C(\{\Phi_i\})$ the set of subsequential varifold limits of min–max sequences of $\{\Phi_i\}$. Then by Theorem 2.3, $C(\{\Phi_i\}) \cap W$ is non-empty. Now, there are three cases: $C(\{\Phi_i\}) \cap W$ contains (1) some element $\Gamma$ that is not the equatorial disk or the critical catenoid; (2) only critical catenoids; or (3) only equatorial disks (note that critical catenoids and equatorial disks cannot appear together in $C(\{\Phi_i\}) \cap W$ as they have different area). We will consider each case individually in the following.

**Case (1).** We will show that $\Gamma$ has the desired property stated in Theorem 1.4:

**Proposition 3.7** $\Gamma$ has multiplicity 1, genus 0 or 1, Morse index 4 or 5, and area in the range $(\pi, 2\pi)$.

**Proof** First, by Theorem 2.3, area($\Gamma$) when counted with possible multiplicities is equal to $L_{\text{SS}}(\Lambda(\Psi^{(5)}))$, which is less than $2\pi$ by (C). Then, since the least possible area of a free boundary minimal surface in $\mathbb{B}^3$ is $\pi$ by a result of Fraser–Schoen [19, Theorem 5.4], $\Gamma$ must have multiplicity 1. Moreover, by Theorem 2.3, (B) implies genus($\Gamma$) $\leq 1$, and index($\Gamma$) $\leq 5$ since the parameter space of $\Psi^{(5)}$ is 5-dimensional. Lastly, since $\Gamma$ is not the equatorial disk, we have area($\Gamma$) $> \pi$ again from [19] (and also [6] by Brendle), and index($\Gamma$) $\geq 4$ from [12, Section 5] by Devyver or [46, Section 3.1] by Tran.

Hence, case (1) is done.

**Case (2).** Now we turn to case (2). We will use a technique called *splitting of domains*. First, let $C$ denote the set of critical catenoids: Note that $C$ is homeomorphic to $\mathbb{R}P^2$. Fixing a small $\epsilon > 0$, and denoting by $W$ the parameter space of $\Psi^{(5)}$, we consider the set of parameters

$$\{x \in W : F(\Phi_i(x), C) \leq \epsilon\}.$$

This subset, after a slight thickening, can be assumed to be a cubical complex; we will denote it by $Z_i$, and then $W \setminus Z_i$ by $Y_i$. Now, as we mentioned $\Psi^{(5)}$ is a 5-sweepout, hence so is $\Phi_i$. Then using a topological argument by Lusternik–Schnirelmann [33] (see also [37, Claim 6.3]), we know that either $\Phi_i|Z_i$ is a 1-sweepout or $\Phi_i|Y_i$ is a 4-sweepout. Now note that, if $\epsilon$ is small enough, $\Phi_i|Z_i$ lies near $C$ and thus is homotopic in the $F$-topology to some $M$-continuous map into $C$ (using [39, Section 3.3.6] by Nurser, together with discretization and interpolation theorems in the free boundary setting by Li–Zhou [34, Section 4.2]). But no map into $C$ can be a 1-sweepout as $C$ can be contracted to just $\emptyset$, by shrinking each critical catenoid to its axis, which has no mass. Hence, $\Phi_i|Z_i$ cannot be a 1-sweepout, and so each $\Phi_i|Y_i$ must be a 4-sweepout.

Now, we claim that for some $i$, $\Phi_i|Y_i$ is homotopic (in the Simon–Smith setting) to another smooth family $\tilde{\Psi}$ with maximal area less than $L$. Indeed, if not, then by...
standard Simon–Smith min–max theory, there exists \( y_i \) such that \( \Phi_i|_{Y_i}(y_i) \) converges subsequentially to some smooth embedded free boundary minimal surface \( V \) with mass \( L \) (see [7, Section 5] and [9, Appendix]: Their arguments apply here because even though our parameter spaces \( Y_i \) depend on \( i \), they can all be embedded into some \( \mathbb{R}^N \) with \( N \) independent of \( i \)). Then note two facts: \( V \) cannot be the critical catenoid by the definition of \( Y_i \), and \( V \in C(\{\Phi_i\}) \cap \mathcal{W} \) clearly. However, these two facts are contradictory because we are in case (2). Thus, the desired smooth family \( \tilde{\Psi} \) exists.

We then apply the Theorem 2.3 to \( \Lambda(\tilde{\Psi}) \), and repeat the argument above. Namely, letting \( \{\tilde{\Phi}_i\} \) be a pulled-tight minimizing sequence of \( \Lambda(\tilde{\Psi}) \), there are two cases: \( C(\{\tilde{\Phi}_i\}) \cap \mathcal{W} \) either contains (2a) some element \( \tilde{\Gamma} \) that is not the equatorial disk or the critical catenoid, or (2b) only equatorial disks. The critical catenoid, having area \( L \), cannot appear because area \( \circ \tilde{\Psi} < L \).

If it is case (2a), one can reapply the proof of Proposition 3.7 to \( \tilde{\Gamma} \) to show that \( \tilde{\Gamma} \) has the desired properties in Theorem 1.4 (this time index \( (\tilde{\Phi}_i) \) is actually 4), and we are done. If it is case (2b), we split the domains again to arrive at a contradiction. This time the key ideas are as follows: There is no 3-sweepout near the set of equatorial disks, which is merely an \( \mathbb{R}P^2 \); and there is no 1-sweepout with maximal area less than \( \pi \), which is the least possible area for a free boundary minimal surface. Therefore case (2b) is impossible. Now case (2) is also done.

**Case (3).** Case (3) is entirely analogous to case (2b).

So we have finished the proof of Theorem 1.4.

### 3.5 Explanation of Remark 1.5

Letting \( a_5 > 0 \) be sufficiently small, we define the family \( \Phi_1: \mathbb{R}P^1 \to \mathcal{Z} \) of \( D_2 \)-symmetric surfaces:

\[
\Phi_1([a_0 : a_3]) := \Phi_4^{a_5}([a_0 : 0 : a_3 : 0]) = \{a_0(x^2 - y^2 + a_5z^3) + a_3z = 0\} \cap \mathbb{B}^3
\]

(see Fig. 1). Note that area \( \circ \Phi_1 < 2 \) as area \( \circ \Phi_4^{a_5} < 2 \) by Theorem 1.2 (C). Then applying the equivariant Simon–Smith min–max theorem to \( \Phi_1 \), we obtain a free boundary minimal surface. To show it has the desired properties mentioned in Remark 1.5, we just proceed in a way similar to [8, Section 4]. In particular, we can use the proof of [8, Lemma 4.1] to show that the number of boundary component of the minimal surface obtained is one. Indeed, we first note that for each surface \( \Phi_1(a) \), the complement of the three axes of rotations (the \( z \)-axis, \( \{z = 0, x = y\} \), and \( \{z = 0, x = -y\} \)) in \( \Phi_1(a) \) are topological disks. And this fact is what one need to carry out the proof of [8, Lemma 4.1].

### 4 Technical Ingredients

In this section, we prove Propositions 3.2 and 3.6.
4.1 Proof of Proposition 3.2

Throughout Sect. 4.1, we write \( X := \mathbb{RP}^4 \times SO(3) / D_2 \). The proof has four steps. In step 1, we compute \( H_1(X; \mathbb{Z}_2) \), and understand which first homology classes give 1-sweepouts under \( \Phi_7 \). In step 2, we find the cohomology class \( \lambda \in H^1(X; \mathbb{Z}_2) \) that detects the 1-sweepouts (as explained Remark 2.4), and understand its Poincaré dual. In step 3, we show that \( \lambda^6 \neq 0 \). And in step 4, we prove a technical lemma used in step 3. By Remark 2.4, we immediately obtain the desired claim that \( \Phi_7 \) is a 6-sweepout.

**Step 1** Let us first find \( \pi_1(X) \). Let \( Q_8 \) be the quaternion group \( \{ \pm 1, \pm i, \pm j, \pm k \} \), contained in the group \( S^3 \) of unit quaternions.

**Lemma 4.1** \( \pi_1(X) = \mathbb{Z}_2 \times Q_8 \).

**Proof** First, the universal cover of \( \mathbb{RP}^4 \times SO(3) \) is \( S^4 \times S^3 \); in fact, without loss of generality one may assume the double covering \( S^3 \to SO(3) \) maps \( \pm i \) to \( g_1 \) and \( \pm j \) to \( g_2 \), and thus \( Q_8 \) to \( D_2 \). Then to prove the lemma, it suffices to construct a \( \mathbb{Z}_2 \times Q_8 \)-action on \( S^4 \times S^3 \) that descends, under the projections \( S^4 \times S^3 \to \mathbb{RP}^4 \times SO(3) \) and \( \mathbb{Z}_2 \times Q_8 \to 1 \times D_2 \), to the \( D_2 \)-action on \( \mathbb{RP}^4 \times SO(3) \) defining \( X \).

First, we define a \( Q_8 \)-action on \( S^4 \times S^3 \) by

\[
(\pm i) \cdot ((a_0, a_1, a_2, a_3, a_4), q) := ((-a_0, a_1, -a_2, -a_3, a_4), \pm iq),
\]

\[
(\pm j) \cdot ((a_0, a_1, a_2, a_3, a_4), q) := ((-a_0, -a_2, a_1, -a_3, a_4), \pm jq).
\]

Then, let \( \mathbb{Z}_2 \) act on \( S^4 \times S^3 \) by acting antipodally on only the \( S^4 \) factor. After checking these two actions commute, we obtain a \( \mathbb{Z}_2 \times Q_8 \)-action on \( S^4 \times S^3 \), and it is straightforward to check that this action has the desired property. \( \square \)

Now, abelianizing \( \pi_1(X) = \mathbb{Z}_2 \times Q_8 \), we have \( H_1(X; \mathbb{Z}) = \mathbb{Z}_2 \times D_2 \), which then by the universal coefficient theorem gives \( H_1(X; \mathbb{Z}_2) = \mathbb{Z}_2 \times D_2 \). In fact, some first homology classes can be described explicitly as follows. Denote \( e_0 := (1, 0, 0, 0, 0) \), and let \( \tilde{x}_0 := (e_0, 1) \) be the base point in \( S^4 \times S^3 \). Then consider the path

\[
\{((a_0, 0, 0, \sqrt{1 - a_0^2}, 0), 1) : -1 \leq a_0 \leq 1\}
\]

in \( S^4 \times S^3 \) joining \( \tilde{x}_0 \) to \((e_0, 1)\), a path joining \( \tilde{x}_0 \) to \((e_0, i)\), and a path joining \( \tilde{x}_0 \) to \((e_0, j)\). Call the projection of these three paths onto \( X \), which are actually loops, \( c_1, c_2, \) and \( c_3 \), respectively. Then \( [c_1] = (1, \text{id}) \), \( [c_2] = (0, g_1) \), and \( [c_3] = (0, g_2) \) in \( H_1(X; \mathbb{Z}_2) = \mathbb{Z}_2 \times D_2 \), and hence they form a base.

**Lemma 4.2** \( (1, \text{id}), (0, g_1), (0, g_2), (1, g_1g_2) \) are exactly the homology classes that give 1-sweepouts under \( \Phi_7 \).

**Proof** It suffices to show that \( (1, \text{id}), (0, g_1), (0, g_2) \) give 1-sweepouts. To show that \( (1, \text{id}) \) gives a 1-sweepout, note that

\[
\Phi_7 \circ c_1 = \Phi_4([[a_0 : 0 : 0 : a_3 : 0] : a_0^2 + a_3^2 = 1])
\]
But in $\mathbb{RP}^4$ the loop $\{a_0^2 + a_3^2 = 1\}$ is homotopic to $\{a_3^2 + a_4^2 = 1\}$, which under $\Phi_4$ gives the collection of all horizontal planes together with the empty set, and that certainly is a 1-sweepout. So $\Phi_7 \circ c_1$ is a 1-sweepout.

To show that $(0, g_1)$ gives a 1-sweepout, note that $\Phi_7 \circ c_2$ gives the motion of rotating the intersecting planes $\{x^2 - y^2 = 0\}$ about the axis $\{z = 0, x = y\}$ by $180^\circ$. Let us call $\{y > |x|\}$ the inside region of $\{x^2 - y^2 = 0\}$, and $\{y < |x|\}$ the outside. Then the rotation switches the inside and the outside. So $\Phi_7 \circ c_2$ is a 1-sweepout.

The proof that $(0, g_2)$ gives a 1-sweepout is similar. \hfill $\Box$

**Step 2** By the universal coefficient theorem, $H^1(X; \mathbb{Z}_2) = \text{Hom}(H_1(X; \mathbb{Z}_2), \mathbb{Z}_2)$. Since $[c_1], [c_2], [c_3]$ form a base of $H_1(X; \mathbb{Z}_2)$, we can define, respectively, their Hom-duals $\lambda_i := [c_i]^* \in H^1(X; \mathbb{Z}_2)$ for $i = 1, 2, 3$. Let $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, then by Lemma 4.2, $\lambda$ is the cohomology class that detects exactly the 1-sweepouts. Hence, to prove Proposition 3.2, we need $\lambda^6 \neq 0$. We are going to prove this by considering the Poincaré dual of $\lambda^6$, so let us first understand the Poincaré dual $PD(\lambda_i) \in H_6(X; \mathbb{Z}_2)$ of $\lambda_i$.

In the remaining of Sect. 4.1, we will view $X$ as an $\mathbb{RP}^4$-bundle over the base $B := SO(3)/D_2$, and let $p : X \to B$ be the projection. Let $A_0$ be the 6-dimensional subbundle of $X$ over $B$ on which $a_0 = 0$. Note that $A_0$ is well defined because the subset $\{a_0 = 0\}$ of $\mathbb{RP}^4 \times SO(3)$ is $D_2$-invariant.

**Lemma 4.3** $PD(\lambda_1) = [A_0]$ in $H_6(X; \mathbb{Z}_2)$.

**Proof** This is because the loop $c_1$ intersects $A_0$ at only one point in $X$, $D_2 \cdot ([0 : 0 : 0 : 1 : 0], 1d)$. \hfill $\Box$

To construct $PD(\lambda_2 + \lambda_3)$, we will need to know the cohomology groups of $B$, which is $S^3/Q_8$. We quote the result [47, Theorem 2.2 (1)] of Tomoda–Zvengrowski:

**Proposition 4.4** The cohomology ring $H^*(S^3/Q_8; \mathbb{Z}_2)$ is given by

$$\mathbb{Z}_2[\alpha_1, \alpha'_1, \alpha_2, \alpha'_2, \alpha_3] / \sim,$$

in which the subscript of each generator denotes its degree, and $\sim$ denotes the following equivalence:

$$\alpha_1^2 = \alpha_2 + \alpha'_2, \quad \alpha_1\alpha'_1 = \alpha'_2, \quad (\alpha'_1)^2 = \alpha_2,$$

$$\alpha_1\alpha_2 = \alpha_1\alpha'_2 = \alpha'_1\alpha'_2 = \alpha_3, \quad \alpha'_1\alpha_2 = 0,$$

products of cohomology classes with total degree greater than 3 is 0.

Now, from the definition of $c_2$ and $c_3$, we know $p \circ c_2$ and $p \circ c_3$ form a base in $H_1(B; \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Remembering $*$ denotes the Hom-dual, we let $b_2, b_3$ be 2-dimensional submanifolds in $B$ such that

$$[b_2] = PD([p \circ c_2]^*), [b_3] = PD([p \circ c_3]^*) \in H_2(B; \mathbb{Z}_2).$$
We moreover assume $b_2$ and $b_3$ intersect transversely, and define $b := b_2 \cup b_3$. Let $X|_{b_2}$ be the restriction of the $\mathbb{R}P^d$-bundle $X$ over the base $b_2$, and similarly for $X|_{b_3}$ and $X|_{b}$. Note that they are 6-dimensional.

**Lemma 4.5** In $H_6(X; \mathbb{Z}_2)$, we have

1. $PD(\lambda_2) = [X|_{b_2}]$.
2. $PD(\lambda_3) = [X|_{b_3}]$.
3. $PD(\lambda_2 + \lambda_3) = [X|_{b}]$.

**Proof** Since $\lambda_2$ is the Hom-dual of $[c_2]$, to show that $[X|_{b_2}] = PD(\lambda_2)$, it suffices to show that the intersections number of $X|_{b_2}$ with $c_1$, $c_2$, and $c_3$, respectively, are 0, 1, and 0. Indeed, this is true because, respectively, $b_2$ can be perturbed to avoid the point $p \circ c_1$ (in $B$), the intersection number of $b_2$ with $p \circ c_2$ is 1, and the intersection number of $b_2$ with $p \circ c_3$ is 0.

Similarly, one can show $PD(\lambda_3) = [X|_{b_3}]$, and thus $PD(\lambda_2 + \lambda_3) = [X|_{b_2}] + [X|_{b_3}] = [X|_{b}]$. $\square$

**Step 3** Note that

$$\lambda^6 = \lambda_1^6 + \lambda_1^4(\lambda_2 + \lambda_3)^2 + \lambda_1^2(\lambda_2 + \lambda_3)^4 + (\lambda_2 + \lambda_3)^6.$$ 

To show $\lambda^6 \neq 0$, it suffices to show the following lemma.

**Lemma 4.6** In the cohomology ring $H^*(X; \mathbb{Z}_2)$, we have

1. $(\lambda_2 + \lambda_3)^3 = 0$.
2. $\lambda_1^2(\lambda_2 + \lambda_3)^2 = 1$.
3. $\lambda_1^3 = 0$.

**Proof** To prove (1), it suffices to perturb three copies of $X|_{b}$ and show that their intersection is empty. To achieve this, we can just perturb three copies of the base $b \subset B$. But their intersection number will be 0, because by Proposition 4.4 any element of $H^1(B; \mathbb{Z}_2)$ cubes to 0. Hence we have proven (1).

To prove (2) and (3), we need to find different representatives of $[A_0]$. Let $A_1, A_2, A_3, A_4$ be the subbundle of $X$ over $B$ on which $a_1 = a_2, a_1 = -a_2, a_3 = 0, and a_4 = 0$, respectively.

**Lemma 4.7** In $H_6(X; \mathbb{Z}_2)$, we have

1. $[A_1] = [A_0] + [X|_{b_3}]$.
2. $[A_2] = [A_0] + [X|_{b_2}]$.
3. $[A_3] = [A_0]$.
4. $[A_4] = [A_0] + [X|_{b}]$.

We postpone the proof of Lemma 4.7 to step 4.

To prove (2) of Lemma 4.6, first note that $A_0 \cap A_1 \cap A_2 \cap A_4$ is the $[0 : 0 : 0 : 1 : 0]$-bundle over $B$. Also, $b$ can be perturbed to some $\tilde{b}$ such that $b \cap \tilde{b}$ is non-trivial in $H_1(B; \mathbb{Z}_2)$, because Proposition 4.4 says the square of any element in $H^1(B; \mathbb{Z}_2)$ is
non-trivial. As a result, $A_0 \cap A_1 \cap A_2 \cap A_4 \cap X|_b \cap X|_g$ is non-trivial in $H_1(X; \mathbb{Z}_2)$. Hence, writing $\mu := \lambda_2 + \lambda_3$ for simplicity, we have

$$1 = PD(A_0)PD(A_1)PD(A_2)PD(A_4)PD(X|_b)^2$$

$$= \lambda_1(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_2)(\lambda_1 + \mu)^2$$

$$= \lambda_1^4\mu^2 + \lambda_1^3(\lambda_2 + \lambda_3 + \mu)^2 + \lambda_1^2(\lambda_2\lambda_3 + \lambda_2\mu + \lambda_3\mu)^2 + \lambda_1\lambda_2\lambda_3\mu^3$$

$$= \lambda_1^4\mu^2. \quad (5)$$

The first equality above is from Lemma 4.7. The last equality holds because $\lambda_2 + \lambda_3 + \mu, \lambda_2\lambda_3 + \lambda_2\mu + \lambda_3\mu$, and $\lambda_2\lambda_3\mu$ all are zero: This is straightforward to check by considering how the bases $b_2, b_3$, and $b$ intersect using Proposition 4.4. Hence, we have proven (2) of Lemma 4.6.

To prove (3) of Lemma 4.6, one note that $A_0 \cap A_1 \cap A_2 \cap A_3 \cap A_4$ is empty, and then $\lambda_1^5 = 0$ would follow from a calculation analogous to (5).

Lemma 4.6 implies $\lambda^6 \neq 0$, finishing the proof of Proposition 3.2.

**Step 4.** Finally, let us prove Lemma 4.7.

**Proof of Lemma 4.7** For Lemma 4.7 (3): $[A_3] = [A_0]$ because we can homotope $A_0$ to $A_3$ using the $((1 - s)a_0 = sa_3)$-bundles over $B$, for $0 \leq s \leq 1$. (Note that we cannot prove, say, $[A_2] = [A_0]$ this way because the $((1 - s)a_0 = sa_2)$-bundles over $B$ is not well defined.)

We now prove Lemma 4.7 (1), by acting on the basic elements $[c_1], [c_2], \text{ and } [c_3]$ of $H_1(X; \mathbb{Z}_2)$. More precisely, recall that $\lambda_i$ is by definition the Hom-dual of $c_i$. So by Lemmas 4.3 and 4.5, $PD([A_0] + [X|_b])$ acts on $[c_1], [c_2], \text{ and } [c_3]$ to give $1+0, 0+0, \text{ and } 0+1$, respectively. Therefore, to prove (1) it suffices to show that

- $PD([A_1])[c_1] = 1$.
- $PD([A_1])[c_2] = 0$.
- $PD([A_1])[c_3] = 1$.

To show that $PD([A_1])[c_1] = 1$, just observe that we can homotope $c_1$ to the loop $[(0 : a_1 : 0 : 0 : a_4) : a_1^2 + a_4^2 = 1]$ within the same fiber $X|_{D_2;id}$ as $\pi_1(\mathbb{R}P^4) = \mathbb{Z}_2$, and this loop intersects $A_1$ only once.

To show that $PD([A_1])[c_2] = 0$, we will perturb $c_2$ to another loop $\tilde{c}_2$ as follows. Let $d_2 : [0, 1] \to SO(3)$ be the path that lifts $p \circ c_2 \subset B = SO(3)/D_2$, starts at id, and ends at $g_1$. Fix a small constant $\epsilon_0 > 0$. We define $\tilde{c}_2 : [0, 1] \to X$ to be such that it is over the same base $p \circ c_2$ as $c_2$, but has different fibers:

$$\tilde{c}_2(s) := D_2 \cdot [(1 : \epsilon_0 : -\epsilon_0 : 0 : 0), d_2(s)] \quad (6)$$

Then one can check that $\tilde{c}_2(0) = \tilde{c}_2(1)$ so that $\tilde{c}_2$ is a loop, and $\tilde{c}_2$ does not intersect $A_1$. Thus $PD([A_1])[c_2] = 0$.

To show that $PD([A_1])[c_3] = 1$, we will perturb $c_3$ to $\tilde{c}_3$ as follows. Let $d_3 : [0, 1] \to SO(3)$ be the path that lifts $p \circ c_3$, starts at id, and ends at $g_2$. This time, we let $\epsilon$ be a function from $[0, 1]$ to $\mathbb{R}$ that strictly decreases from $\epsilon_0$ to $-\epsilon_0$, for some
fixed small $\epsilon_0 > 0$. We define $\tilde{c}_3 : [0, 1] \to X$ by

$$\tilde{c}_3(s) := D_2 \cdot ([1 : \epsilon(s) : -\epsilon(s) : 0 : 0], d_3(s)) \quad (7)$$

One can again check $\tilde{c}_3$ is indeed a loop, but $\tilde{c}_3$ intersects $A_1$ at one point, where $\epsilon(s) = 0$. Thus $PD([A_1])[c_3] = 1$. This finishes the proof of Lemma 4.7 (1).

The proof of Lemma 4.7 (2) is similar. We only state the modifications needed: To prove $PD([A_2])[c_2] = 1$ and $PD([A_2])[c_3] = 0$, instead of (6) and (7), respectively, we use

$$D_2 \cdot ([1 : \epsilon(s) : \epsilon(s) : 0 : 0], d_2(s)) \text{ and } D_2 \cdot ([1 : \epsilon_0 : \epsilon_0 : 0 : 0], d_3(s)).$$

The proof of Lemma 4.7 (4) is also similar. We only state the modifications needed: To prove $PD([A_4])[c_2] = 1$ and $PD([A_4])[c_3] = 1$, instead of (6) and (7), respectively, we use

$$D_2 \cdot ([1 : 0 : 0 : 0 : \epsilon(s)], d_2(s)) \text{ and } D_2 \cdot ([1 : 0 : 0 : 0 : \epsilon(s)], d_3(s)).$$

### 4.2 Proof of Proposition 3.6

**Step 1** For convenience, let us reparametrize the family $\Phi_5 : \mathbb{R}^4 \times [0, 1] \to Z$ as follows. First, we write $\mathbb{R}^4$ as $\mathbb{R}^4 \sqcup \mathbb{R}^3$ in which $\mathbb{R}^3$ is where $a_0 = 0$. Then on $\mathbb{R}^4 \times [0, 1]$, we reparametrize the family $\Phi_5$ by

$$\Phi_5(b_1, b_2, b_3, b_4, b_5) := \{(x - b_1)^2 - (y - b_2)^2 + b_3z + b_4 + b_5z^3 = 0\} \cap \mathbb{R}^3.$$

Throughout this section we will adopt this new parametrization. And then our goal is to show that area $\circ \Phi_5$ has a strict local maximum at $(0, 0, 0, 0, 0)$. In fact, it suffices to prove the following:

**Proposition 4.8** There exists $\epsilon_1, \epsilon_2 > 0$ such that for any $(b_1, b_2) \in (-\epsilon_1, \epsilon_1)^2$, $(b_3, b_4, b_5) \in \mathbb{R}^2 \times [0, 1]$ such that $b_2^2 + b_4^2 + b_5^2 = 1$, and $t \in (0, \epsilon_2)$,

$$\text{area}(\Phi_5(b_1, b_2, b_3t, b_4t, b_5t)) < \text{area}(\Phi_5(0, 0, 0, 0, 0)). \quad (8)$$

Geometrically, $t$ governs how much the surface opens up (see Remark 3.3). Namely, it is elementary to show that the width of the hole opened up in $\Sigma_t$ is at most $\sqrt{3}t$.

**Step 2** We begin to prove Proposition 4.8. Let $b_1, b_2, b_3, b_4, b_5$ satisfy the assumptions—we will explain how small $\epsilon_1, \epsilon_2$ need to be later. For each $t \in (0, \epsilon_2)$, denote $\tilde{\Sigma}_t := \Phi_5(b_1, b_2, b_3t, b_4t, b_5t)$. Then by Lemma B.1,

$$\frac{d}{dt} \text{area}(\tilde{\Sigma}_t) = -\int_{\tilde{\Sigma}_t} H \cdot V - \int_{\partial\tilde{\Sigma}_t} \frac{n \cdot w}{\nu \cdot w} V \cdot n,$$

which one would hope to show to be negative in order to prove Proposition 3.6. However, the second integral is difficult to bound, since $\nu \cdot w$ can be zero on $\partial\mathbb{R}^3$. To
prevent \( v \cdot w = 0 \) on \( \partial B^3 \), we will slightly enlarge \( B^3 \) to some domain \( \Omega \), which we will soon define. Then for each \( s \in (0, t] \), we let

\[
\Sigma_s := \{(x - b_1)^2 - (y - b_2)^2 + s(b_3z + b_4 + b_5z^3) = 0\} \cap \Omega. \tag{9}
\]

(Here, \( \Sigma_s \) has a boundary, so the notation is different from Lemma B.1.) Then

\[
\text{area}(\Phi_5(b_1, b_2, b_3t, b_4t, b_5t)) - \text{area}(\Phi_5(0, 0, 0, 0, 0)) \\
< \text{area}(\Sigma_t) - \text{area}(\Phi_5(0, 0, 0, 0, 0)) \\
= (\text{area}(\Sigma_0) - \text{area}(\Phi_5(0, 0, 0, 0, 0))) + \int_0^t \frac{d}{ds} \text{area}(\Sigma_s) ds \tag{10}
\]

Therefore to prove Proposition 4.8, it suffices to show that expression (10) is negative. We will achieve this by showing the initial area added by enlarging \( B^3 \) to \( \Omega \), which is the first term of (10), is dominated by the area decrease as \( s \) increases from 0 to \( t \), which is the second term of (10).

But let us first define \( \Omega \). We now fix \( t \in (0, \epsilon_2) \) also. The new region \( \Omega \) will depend on \( t \) as follows. Let

\[
R := 20\sqrt{t}. \tag{11}
\]

Let \( S_1, S_2, S_3 \) be the solid cylinders in \( \mathbb{R}^3 \) with axis \( \{x = b_1, y = b_2\} \) and radius \( R, 2R, \frac{1}{4} \), respectively. By letting \( \epsilon_2 \) be small we can assume \( S_2 \subset S_3 \). Let \( \Omega \) be the unit 3-ball with a bump within \( S_1 \) (see Fig. 2). Moreover, let us view \( \partial B^3 \cap S_2 \) (resp. \( \partial \Omega \cap S_2 \)) as the 2-sheeted graph of some function \( \pm f \) (resp. \( \pm g \)) over a disk \( D \) on the xy-plane. Then since \( \text{dist}(p, (0, 0)) < 2\epsilon_1 + 40\sqrt{\epsilon_2} \) for all \( p \in D \), we know that \( |\nabla f| \lesssim (2\epsilon_1 + 40\sqrt{\epsilon_2})^2 \) (here \( \lesssim \) means the inequality holds up to a multiplicative constant that is universal), thus we can also assume \( |\nabla g| \lesssim (2\epsilon_1 + 40\sqrt{\epsilon_2})^2 \). As a result, if we write the outward unit normal \( w \) of \( \Omega \) as \( (w_1, w_2, w_3) \), then for \( \epsilon_1, \epsilon_2 \) sufficiently small, we have in \( S_2 \)

\[
|\nabla g|, |w_1|, |w_2| < C''(\epsilon_1 + \epsilon_2) \tag{12}
\]

for some universal constant \( C'' \).

Now we have defined \( \Omega \), it suffices to show that expression (10) is negative. The second term of (10) can be computed using the first variation formula in Lemma B.1. In order to estimate, let us derive some preliminary results in the next step.

**Step 3** We are interested in surfaces \( \Sigma_s \) defined in (9), which is the zero set in \( \Omega \) of the polynomial

\[
p(x, y, z) := (x - b_1)^2 - (y - b_2)^2 + b_3sz + b_4s + b_5sz^3,
\]

for \( s \) increases from 0 to \( t \), where \( t \in (0, \epsilon_2) \) is fixed. Note that

\[
\partial_s p = b_3z + b_4 + b_5z^3, \quad \nabla p = (2(x - b_1), -2(y - b_2), b_3 + 3b_5sz^2),
\]
Fig. 2 A schematic picture of $\Omega$

\[
\text{Hess}(p) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6b_5sz \end{pmatrix}, \quad \Delta p = 6b_5sz.
\]

Moreover, denoting $H = Hn$,

\[
H = \frac{\nabla p \text{Hess}(p)\nabla p^T - |\nabla p|^2 \Delta p}{|\nabla p|^3}
\]

\[
= \frac{1}{|\nabla p|^3} \left[ 18(x - b_1)^2 - 8(y - b_2)^2 + 6b_5sz(b_3s + 3b_5sz^2)^2 \\
- (4(x - b_1)^2 + 4(y - b_2)^2 + (b_3s + 3b_5sz^2)^2)(6b_5sz) \right]
\]

\[
= \frac{1}{|\nabla p|^3} \left[ -8s \partial_s p - 24b_5sz((x - b_1)^2 + (y - b_2)^2) \right], \tag{13}
\]

in which in the third equality we used $p = 0$ on $\Sigma_s$. We can choose $\frac{\nabla p}{|\nabla p|}$ as the normal vector field $n$, and $V := -\frac{\partial_s p}{|\nabla p|^2} \nabla p$ as the deformation vector field of $\Sigma_s$ (because by differentiating $p(s, x(s)) = 0$ with respect to $s$, one has $\partial_s p + \nabla p \cdot x' = 0$). As a result, by (13) and Lemma B.1,

\[
\frac{d}{ds} \text{area}(\Sigma_s) = -\int_{\Sigma_s} H \cdot V - \int_{\partial \Sigma_s} \frac{n \cdot w}{v \cdot w} V \cdot n
\]
Lemma 4.9 There exist some large universal constant $C > 0$ and small $\epsilon_1, \epsilon_2 > 0$ such that the following is true. For any $(b_1, b_2) \in (-\epsilon_1, \epsilon_1)^2$, $(b_3, b_4, b_5) \in \mathbb{R}^2 \times [0, 1]$ such that $b_2^2 + b_4^2 + b_5^3 = 1$, and $0 < s < t < \epsilon_2$, we have

- $I_1 < -\frac{1}{C\sqrt{s}} < 0$.
- $|I_2|, |I_3|, |I_4|, |I_6| < C$.
- $|I_5| < -C \log(400s)$.
- area$(\Sigma_0) - \text{area}(\Phi_5(0, 0, 0, 0, 0)) < Ct$.

Indeed, from this lemma it follows that in (10), when $\epsilon_1, \epsilon_2$ are small, the dominating term is $\int_0^t I_1 ds$, which is of order $\sqrt{t}$ and is negative. Thus the expression (10) is negative, as desired.

Step 4 We now begin to prove Lemma 4.9, by bounding the seven quantities listed one by one.

First, $I_1 < 0$ is clear, so we just need to lower bound $|I_1|$. By Lemma C.1, there exists a universal constant $h > 0$ and an interval $[z_0, z_0 + \frac{1}{8}] \subset [-\frac{1}{2}, \frac{1}{2}]$ on which $\partial_s p = b_3z + b_4 + b_5z^3 > h$ or $\partial_s p < -h$.

Let us first tackle the case $\partial_s p > h$. Namely, we have

$$|I_1| = \int_{\Sigma_z} \frac{8s(\partial_s p)^2}{|\nabla p|^4} \geq \int_{\partial \Sigma_z |z_0 < z < z_0 + \frac{1}{8}} \frac{8sh^2}{(x - b_1)^2 + 4s\partial_s p + (b_3 + 3b_5z^2)^2s^2}$$

$$\geq \int_{\Sigma_z |z_0 < z < z_0 + \frac{1}{8}} \frac{8sh^2}{(x - b_1)^2 + 13s^2}.$$  

Note that in the first inequality we rewrote $(y - b_2)^2$ in $|\nabla p|^4$ using $p = 0$, and used that $\partial_s p > h$ for $z \in [z_0, z_0 + \frac{1}{8}]$, while the second inequality holds because $|\partial_s p| \leq 3$ and $(b_3 + 3b_5z^2)^2s^2 < s$ if $\epsilon_2$ and thus $s$ is small. Moreover, note that the domain of the last integral is a two-sheeted graph over the rectangle $[-\frac{1}{2}, \frac{1}{2}] \times [z_0, z_0 + \frac{1}{8}]$ on the $xz$-plane, and clearly the graph has a larger area than the rectangle (see Fig. 3). As a result,

$$|I_1| > 2 \cdot \frac{1}{8} \int_{-1/2}^{1/2} \frac{8sh^2}{(x - b_1)^2 + 13s^2}dx \geq \frac{1}{\sqrt{s}}.$$  

This finishes proving $I_1 < -\frac{1}{C\sqrt{s}}$ for the case $\partial_s p > h$. The second case $\partial_s p < -h$ is similar: One would integrate with respect to $y$ instead of $x$ in the last step.
To bound $I_2$, we observe that

$$|\nabla p|^2 \geq 4(x - b_1)^2 + 4(y - b_2)^2 \geq 4|(x - b_1)^2 - (y - b_2)^2| = 4s|\partial_s p|. \quad (15)$$

So

$$|I_2| \leq \int_{\Sigma_s} 24b_5z^2 |\frac{s\partial_s p}{|\nabla p|^2} (x - b_1)^2 + (y - b_2)^2}{|\nabla p|^2} \leq \int_{\Sigma_s} 24 \cdot 1 \cdot 1.$$

Now, since when $\epsilon_1, \epsilon_2$ are small enough, $\Sigma_s$ is close to $\Sigma_0$, which is two disks, we can assume area$(\Sigma_s) < 3\pi$. Hence, $|I_2| < 24 \cdot 3\pi$.

To bound $I_3$, note that

$$\frac{(n \cdot w)(V \cdot n)}{v \cdot w} = \frac{(n \cdot w)(V \cdot n)}{\sqrt{1 - (n \cdot w)^2}} = \frac{(n \cdot w)(-\partial_s p)}{|\nabla p|\sqrt{1 - (n \cdot w)^2}} = \frac{(\nabla p \cdot w)(-\partial_s p)}{|\nabla p|\sqrt{|\nabla p|^2 - (\nabla p \cdot w)^2}}. \quad (16)$$

Now, inside $S_1$, $\partial \Omega$ is horizontal by definition and so $w = \pm e_3$. Thus

$$\left|\frac{(n \cdot w)(V \cdot n)}{v \cdot w}\right| \leq \frac{|\nabla p \cdot w||\partial_s p|}{|\nabla p|^2 - (\nabla p \cdot w)^2} \leq \frac{|s(b_3 + 3b_5z^2)||\partial_s p|}{4(x - b_1)^2 + 4(y - b_2)^2} \leq \frac{4s|\partial_s p|}{4s|\partial_s p|} \leq 1.$$

Fig. 3 A figure illustrating $\Sigma_s$
in which the third inequality used (15). Now by Remark 3.3, \( \partial \Sigma_s \cap \Sigma_1 \) is two hyperbolas near, respectively, the north and the south pole (see Fig. 4). Since the radius of \( S_1 \) is \( R \), it is elementary to show that length(\( \partial \Sigma_s \cap \Sigma_1 \)) \( < C''R \) for some universal constant \( C'' \). Hence, using (11), \(|I_3| < C'R < 20C'\sqrt{\epsilon_2} \), which is less than some universal constant, assuming, say, \( \epsilon_2 < 1 \).

To bound \( I_4 \), using (12), we have

\[
|\mathbf{n} \cdot \mathbf{w}| \leq \frac{|2(x - b_1)|}{|\nabla p|} |w_1| + \frac{|2(y - b_2)|}{|\nabla p|} |w_2| + \frac{|s(b_3 + 3b_5z^2)|}{|\nabla p|} |w_3| \\
\leq 1 \cdot C''(\epsilon_1 + \epsilon_2) + 1 \cdot C''(\epsilon_1 + \epsilon_2) + \frac{4s}{\sqrt{4(x - b_1)^2 + 4(y - b_2)^2}} \cdot 1 \\
\leq 2C''(\epsilon_1 + \epsilon_2) + \frac{4(R/20)^2}{2R} < \frac{1}{10}. \tag{17}
\]

Note that in the second inequality we used that \( s \leq t = (R/20)^2 \) by (11), and in the last inequality assumed \( \epsilon_1, \epsilon_2 \) are small.

Then using (16),

\[
\left| \frac{(\mathbf{n} \cdot \mathbf{w})(V \cdot \mathbf{n})}{\nu \cdot \mathbf{w}} \right| = \frac{|\mathbf{n} \cdot \mathbf{w}||\partial_s p|}{|\nabla p|\sqrt{1 - (\mathbf{n} \cdot \mathbf{w})^2}} \leq \frac{(1/10) \cdot 4}{2R\sqrt{1 - (1/10)^2}} < \frac{1}{R}.
\]

Again, it is elementary to show that the length of \( \partial \Sigma_s \cap (S_2 \setminus S_1) \) is less than \( CR \) for some universal constant \( C \). As a result, \(|I_4| \) is less than \( CR \cdot \frac{1}{R} = C \).
To bound $I_5$, using the fact that $p = 0$ we can rewrite
\[
\nabla p \cdot w = 2(x - b_1)x - 2(y - b_2)y + s(b_3 + 3b_5z^2)z \\
= 2(x - b_1)b_1 - 2(y - b_2)b_2 + s(-b_3z - 2b_4 + b_5z^3).
\]
Therefore, using (11) and that $|\nabla p| > 2R$, and assuming $\epsilon_1, \epsilon_2$ to be small,
\[
|n \cdot w| \leq \frac{|2(x - b_1)|}{|\nabla p|} |b_1| + \frac{|2(y - b_2)|}{|\nabla p|} |b_2| + \frac{|s(-b_3z - 2b_4 + b_5z^3)|}{|\nabla p|} \\
\leq 1 \cdot \epsilon_1 + 1 \cdot \epsilon_1 + \frac{4s}{2R} \leq 2\epsilon_1 + \frac{4(R/20)^2}{2R} < \frac{1}{10}. \tag{18}
\]
Then
\[
\left| \frac{(n \cdot w)(V \cdot n)}{v \cdot w} \right| = \left| \frac{n \cdot w}{|\nabla p|} \cdot |\partial_s p| \right| \leq \frac{1}{|\nabla p|} \cdot \frac{(1/10) \cdot 4}{\sqrt{4(x - b_1)^2 \cdot 1 - (1/10)^2}} \\
< \frac{1}{|x - b_1|}. \tag{19}
\]

Now, on $\Sigma_3 \cap (S_3 \setminus S_2)$, it follows from the definition that $|x - b_1|, |y - b_2| > R$. From this, it is elementary to estimate $n$ and show that in $S_3 \setminus S_2$ the tangent planes of $\Sigma_3$ are close to that of $\{(x - b_1)^2 - (y - b_2)^2 = 0\}$ (see Fig. 4). In this step, the dependence of $R$ on $t$, namely $R = 20\sqrt{t}$, is crucial: We choose $20\sqrt{t}$ because the width of the hole opened up in $\Sigma_3$ is at most $\sqrt{3}s$, which we want to be small compared to $R$. As a result, $\partial \Sigma_3 \cap (S_3 \setminus S_2)$ consists of eight arcs such that if we let $\rho$ be the orthogonal projection map from $\partial \Sigma_3 \cap (S_3 \setminus S_2)$ to the $x$-axis, then the norm of the derivative $D \rho$ is lower bounded by some universal constant $C' > 0$. In addition, the image $J$ of $\rho$ is contained in $[b_1 - \frac{1}{4}, b_1 - R] \cup [b_1 + R, b_1 + \frac{1}{4}]$, and the preimage of each $x \in J$ has at most 4 points. Therefore, by (19), for $\epsilon_2$ small,
\[
|I_5| \leq 4 \int_{[b_1 - \frac{1}{4}, b_1 - R] \cup [b_1 + R, b_1 + \frac{1}{4}]} \frac{1}{C'|x - b_1|} dx \lesssim - \log(R) \lesssim - \log(400s).
\]

To bound $I_6$, note that $|\nabla p| > \frac{1}{2}$ outside $S_3$. Then as in (18), we have $n \cdot w < \frac{1}{10}$. It follows easily that the expression (16) is bounded by some universal constant. Then since length($\partial \Sigma_3 \cap (\mathbb{R}^3 \setminus S_3)$) bounded, so is $|I_6|$.

Finally, we prove the last item of Lemma 4.9. Note that the difference between $\Sigma_0$ and $\Phi_5(0, 0, 0, 0, 0)$ is $\{x^2 - y^2 = 0\} \cap (\Omega \setminus \mathbb{R}^3)$, which is four small planar pieces (two near the north pole and two near the south). Each piece can be contained in a rectangle of width $4R$ and, by (12), height $C''(\epsilon_1 + \epsilon_2)(4R)$. As a result, using (11),
\[
\text{area}(\Sigma_0) - \text{area}(\Phi_5(0, 0, 0, 0, 0)) \leq 4(4R) \cdot C''(\epsilon_1 + \epsilon_2)(4R) \lesssim t.
\]

This finishes the proof of Lemma 4.9. Hence, we have proven Proposition 4.8, and thus Proposition 3.6.
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Appendix A: Proof of Proposition 3.5

The following proof is due to the MathOverflow user fedja [16]. Let \( M \) denote the saddle in \( \mathbb{R}^3 \) given by \( x^2 - y^2 + z = 0 \). Then to prove Proposition 3.5, it suffices, by rescaling, to show that for any ball \( B \) with center \((x_0, y_0, z_0)\) and radius \( R > 0 \), the area of \( M \cap B \) is less than \( 2\pi R^2 \).

Recall that \( M \) is foliated by straight lines: It can be parametrized by \( x(s, t) = (s + t, s - t, 4st) \). Then the Jacobian of \( x \) is \( 2\sqrt{1 + 8s^2 + 8t^2} \). Thus we have

\[
\text{area}(M \cap B) < \int\int_{\{(s,t):x(s,t)\in B\}} (2\sqrt{1 + 8s^2 + 2\sqrt{1 + 8t^2}})dsdt. \quad (20)
\]

Now, for each fixed \( s \), let \( L_s \) be the corresponding coordinate line segment in \( M \cap B \). Letting \( d \) be the distance between \( L_s \) and the center of \( B \), we have

\[
d^2 \geq \min_{t \in \mathbb{R}} [(s + t - x_0)^2 + (s - t - y_0)^2] = 2\left(s - \frac{x_0 + y_0}{2}\right)^2. \quad (21)
\]

Note that \( L_s \) is parameterized by a time interval of length

\[
\frac{\text{length}(L_s)}{\|\partial_t x\|} = \frac{2\sqrt{(R^2 - d^2)^+}}{2\sqrt{1 + 8s^2}}, \quad (22)
\]

where \(^+\) denotes the positive part. It follows that, using (21) and (22),

\[
\int\int_{\{(s,t):x(s,t)\in B\}} 2\sqrt{1 + 8s^2}dtds \leq \int_{s \in \mathbb{R}} 2\sqrt{2} \sqrt{R^2 - 2\left(s - \frac{x_0 + y_0}{2}\right)^2}^+ ds = \pi R^2.
\]

The second integral in (20) can be similarly bounded, by integrating with respect to \( s \) first. So area\((M \cap B) < 2\pi R^2\), finishing the proof of Proposition 3.5.

Appendix B: First Variation Formula

Lemma B.1 Let \( \Omega \) be a compact \((n + 1)\)-dimensional region with smooth boundary in \( \mathbb{R}^{n+1} \), \( \{\Sigma_s\} \) a 1-parameter family of hypersurfaces without boundary in \( \mathbb{R}^3 \), and \( \nabla \) a
deformation vector field of \( \{ \Sigma_s \} \). Then

\[
\frac{d}{ds} \text{area}(\Sigma_s \cap \Omega) = - \int_{\Sigma_s \cap \Omega} H \cdot V - \int_{\partial(\Sigma_s \cap \Omega)} \frac{\mathbf{n} \cdot \mathbf{w}}{\nu} V \cdot \mathbf{n},
\]

where \( \mathbf{n} \) is a chosen unit normal vector field of \( \Sigma_s \), \( \mathbf{w} \) the outward unit normal of \( \partial \Omega \), and \( \nu \) the outward unit conormal of \( \Sigma_s \) on \( \partial \Omega \).

**Proof** We first smoothly extend \( \mathbf{w} \) to a unit vector field on a neighborhood of \( \partial \Omega \) in \( \Omega \), and \( \nu \) to a unit tangent vector field on a neighborhood of \( \partial \Sigma_s \) in \( \Sigma_s \). Let \( \epsilon > 0 \), and \( \Omega_\epsilon \subset \Omega \) be where the distance from \( \partial \Omega \) is at least \( \epsilon \). Then by using the function \( \text{dist}(., \partial \Omega) \) on \( \Omega \), with suitable smoothening, we can approximate the indicator function \( \chi_\Omega \) by a smooth function \( \chi_\epsilon \) that is 0 outside \( \Omega \) and 1 on \( \Omega_\epsilon \), with \( \nabla \chi_\epsilon = -|\nabla \chi_\epsilon| \mathbf{w} \) in between.

Now, using the first variation formula (4.2) in [15, p.49],

\[
\frac{d}{ds} \int_{\Sigma} \chi_\epsilon = \int_{\Sigma} -\chi_\epsilon \nabla \chi_\epsilon \cdot \mathbf{V} + \nabla \chi_\epsilon \cdot \mathbf{n} \mathbf{V} \cdot \mathbf{n}.
\]

Note that

\[
\nabla \chi_\epsilon \cdot \mathbf{n} \mathbf{V} \cdot \mathbf{n} = -|\nabla \chi_\epsilon| \mathbf{w} \cdot \mathbf{n} \mathbf{V} \cdot \mathbf{n} = \nabla \chi_\epsilon \cdot \mathbf{v} \frac{\mathbf{w} \cdot \mathbf{n}}{\mathbf{w} \cdot \nu} \mathbf{V} \cdot \mathbf{n}.
\]

Denoting \( g := \frac{\mathbf{w} \cdot \mathbf{n}}{\mathbf{w} \cdot \nu} V \cdot \mathbf{n} \), we then have

\[
\int_{\Sigma} \nabla \chi_\epsilon \cdot \mathbf{n} \mathbf{V} \cdot \mathbf{n} = \int_{\Sigma} \nabla \chi_\epsilon \cdot g \nu
\]

\[
= - \int_{\Sigma} \chi_\epsilon \text{div}(g \nu) \xrightarrow{\epsilon \to 0} - \int_{\Sigma \cap \Omega} \text{div}(g \nu)
\]

\[
= - \int_{\partial(\Sigma \cap \Omega)} g,
\]

in which the second and the third equality are due to divergence theorem. Hence,

\[
\frac{d}{ds} \text{area}(\Sigma_s \cap \Omega) \text{ is equal to }
\]

\[
\lim_{\epsilon \to 0} \frac{d}{ds} \int_{\Sigma} \chi_\epsilon = \lim_{\epsilon \to 0} \int_{\Sigma} -\chi_\epsilon H \cdot V + \nabla \chi_\epsilon \cdot \mathbf{n} \mathbf{V} \cdot \mathbf{n}
\]

\[
= - \int_{\Sigma \cap \Omega} H \cdot V - \int_{\partial(\Sigma \cap \Omega)} g.
\]
Appendix C: A Lemma About Cubic Polynomials

**Lemma C.1** There exists $h > 0$ such that the following is true. For any $a, b, c$ such that $a^2 + b^2 + c^2 = 1$, define $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ by $f(x) = ax^3 + bx + c$. Then there exists some interval of length $\frac{1}{8}$ in $[-\frac{1}{2}, \frac{1}{2}]$ on which $|f| > h$.

**Proof** Assume, by contradiction, that for each positive integer $n$ there exists a cubic function $f_n(x) = a_n x^3 + b_n x + c_n$, with $a_n^2 + b_n^2 + c_n^2 = 1$ such that there is no interval of length $\frac{1}{n}$ in $[-\frac{1}{2}, \frac{1}{2}]$ on which $|f| > \frac{1}{n}$. For each $n$, let $x_i$, for $i$ runs from 1 to at most 3, be the roots of $f_n(x) = 0$, and $I_i \subset [-\frac{1}{2}, \frac{1}{2}]$ be the maximal interval containing $x_i$ on which $|f_n| < \frac{1}{n}$. Then $[-\frac{1}{2}, \frac{1}{2}] \setminus (I_1 \cup I_2 \cup I_3)$ is a union of at most 4 intervals, each of which has length at most $\frac{1}{8} n$. Thus, $I_1 \cup I_2 \cup I_3$ has length at least $1 - 4 \cdot \frac{1}{8} = \frac{1}{2}$, and on $I_1 \cup I_2 \cup I_3$ it follows easily that $\sup_{x \in [-\frac{1}{2}, \frac{1}{2}]} |f_n(x)| \to 0$ as $n \to \infty$. Since $f_n'(x) = 3a_n x^2 + b_n$, we must have $a_n \to 0$ and $b_n \to 0$ too, which forces $c_n \to 1$ since $a_n^2 + b_n^2 + c_n^2 = 1$. But then $f_n$ is very close to 1 on $[-\frac{1}{2}, \frac{1}{2}]$, contradicting that $|f_n| < \frac{1}{n}$ on a set of length at least $\frac{1}{2}$.

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