WEAK DIAMOND, WEAK PROJECTIVITY, AND TRANSFINITE EXTENSIONS OF SIMPLE ARTINIAN RINGS

JAN TRLIFAJ

Abstract. We apply set-theoretic methods to study projective modules and their generalizations over transfinite extensions of simple artinian rings $R$. We prove that if $R$ is small, then the Weak Diamond implies that projectivity of an arbitrary module can be tested at the layer epimorphisms of $R$.

The classic Baer’s Criterion, saying that a module $M$ is injective, iff it is $R$-injective, is a basic tool of the structure theory of injective modules over an arbitrary ring $R$. However, unless $R$ is a perfect ring, there are no criteria available for the dual case, that is, for testing projectivity using a set of epimorphisms, [10].

For many non-perfect rings, it can be proved that there exist small (e.g., countably generated) non-projective $R$-projective modules (e.g., when $R$ is commutative noetherian of Krull dimension $\geq 1$). However, for each cardinal $\kappa$, there exists a non-right perfect ring $R_{\kappa}$ such that all $\leq \kappa$-generated $R_{\kappa}$-projective modules are projective, [12]. This is the best one can achieve in ZFC, because it is consistent with ZFC + GCH that if $R$ is not right perfect, then there always exist (large) $R$-projective modules that are not projective, cf. [1].

Consistency (and hence independence) of the coincidence of $R$-projectivity and projectivity for certain commutative non-noetherian rings was proved in [11]. This answered in the positive a question from [11, 2.8], and clarified the set-theoretic status of an old problem by Carl Faith [6, p.175]. The consistency result was extended in [12] to further classes of rings that are finite Loewy length extensions of simple artinian rings. The set theoretic tool used in [11] and [12] was Jensen’s Diamond.

The goal of the present paper is twofold: to enhance the algebraic tools to cover infinite Loewy length extensions of simple artinian rings, and to weaken the set-theoretic assumptions used in the proofs. In Theorem 3.2 below, we show that the Weak Diamond Principle $\Phi$ and CH are sufficient to prove coincidence of the classes of all weak $R$-projective, $R$-projective, and projective modules, in the case when $R$ has cardinality at most $\aleph_1$, its Loewy length is countable, and each proper layer of $R$ is countably generated. That is, in this case, $\Phi$ and CH imply that the projectivity of a module $M$ is equivalent to the factorization of all morphisms from $M$ with finitely generated images through the layer epimorphisms of $R$. The latter are just the canonical projections $\pi_{\alpha} : S_{\alpha+1} \to S_{\alpha+1}/S_{\alpha}$ ($\alpha < \sigma$), where $(S_{\alpha} \mid \alpha \leq \sigma + 1)$ is the socle sequence of $R$.

For basic notions and facts needed from ring and module theory, we refer to [2] and [8]; our references for set-theoretic homological algebra are [4] and [7].
1. Weak projectivity

We start by recalling the classic notion of relative projectivity from [2 §16]:

**Definition 1.1.** Let $R$ be a ring and $M, N$ be modules. Then $M$ is **$N$-projective** provided that for each submodule $P$ of $N$, each homomorphism $f : M \rightarrow N/P$ has a factorization through the canonical projection $N \rightarrow N/P$. That is, the functor $\text{Hom}_R(M, -)$ is exact on all short exact sequences whose middle term is $N$.

The following Lemma is well-known (see [2, 16.12 and 16.14]):

**Lemma 1.2.** The class of all modules $M$ such that $M$ is $N$-projective is closed under submodules, homomorphic images, and finite direct sums.

In particular, each finitely generated $R$-projective module is projective.

Let $R$ be a right semiartinian ring with the right socle sequence $(S_\alpha \mid \alpha \leq \sigma + 1)$. We will call $\sigma + 1$ the **Loewy length of $R$**. To avoid trivialities, we will tacitly assume that $\sigma > 0$, that is, that $R$ is not completely reducible.

For each $\alpha \leq \sigma$, we will call the completely reducible module $L_\alpha = S_{\alpha+1}/S_\alpha$ the $\alpha$th layer of $R$. The layers $L_\alpha$ for $0 < \alpha < \sigma$ are called proper. For each $\alpha \leq \sigma$, the canonical epimorphism $\pi_\alpha : S_{\alpha+1} \rightarrow L_\alpha$ is the $\alpha$th layer epimorphism of $R$.

In this setting, the following definition of weak $R$-projectivity was introduced in [2, 3.6]

**Definition 1.3.** A module $M$ is called **weakly $R$-projective** provided that for each $0 < \alpha \leq \sigma$, each $f \in \text{Hom}_R(M, L_\alpha)$ with a finitely generated image has a factorization through the $\alpha$th layer epimorphism $\pi_\alpha$.

Note that by Lemma [2] each $R$-projective module is weakly $R$-projective. The following easy observation [2, 2.3] shows that weak $R$-projectivity can equivalently be stated in a stronger form:

**Lemma 1.4.** Let $R$ be a right semiartinian ring with the right socle sequence $(S_\alpha \mid \alpha \leq \sigma + 1)$. Let $M$ be a module. Then the following are equivalent:

1. $M$ is weakly $R$-projective,
2. For each $\alpha \leq \sigma$, each submodule $K$ such that $S_\alpha \subseteq K \subseteq S_{\alpha+1}$ and each $f \in \text{Hom}_R(M, S_{\alpha+1}/K)$ with a finitely generated image, there exists $g \in \text{Hom}_R(M, S_{\alpha+1})$ such that $f = \pi_K g$, where $\pi_K : S_{\alpha+1} \rightarrow S_{\alpha+1}/K$ is the canonical projection.

In the proof of the next lemma, we will verify $N$-projectivity of a finitely generated module $M$ by a recursive procedure using a filtration of $N$.

Here, we call a chain $(N_\alpha \mid \alpha \leq \tau)$ of $\alpha$-submodules of a module $N$ a **filtration of $N$**, provided that $\tau$ is an ordinal, $N_0 = 0$, $N_\alpha \subseteq N_{\alpha+1}$ for each $\alpha < \tau$, $N_\alpha = \bigcup_{\beta < \alpha} N_\beta$ for each limit ordinal $\alpha \leq \tau$, and $N_\tau = N$.

**Lemma 1.5.** Let $R$ be a ring and $N$ be a module with a filtration $N = (N_\alpha \mid \alpha \leq \tau)$.

Let $M$ be a finitely generated module such that for each $\alpha < \tau$, each submodule $K$ such that $N_\alpha \subseteq K \subseteq N_{\alpha+1}$ and each $f \in \text{Hom}_R(M, N_{\alpha+1}/K)$, there exists $g \in \text{Hom}_R(M, N_{\alpha+1})$ such that $f = \pi_K g$, where $\pi_K : N_{\alpha+1} \rightarrow N_{\alpha+1}/K$ is the canonical projection.

Then $M$ is $N$-projective.

**Proof.** We claim that $M$ is $N_\alpha$-projective for each $\alpha \leq \tau$. If so, then for $\alpha = \tau$, we get that $M$ is $N$-projective.

The claim will be proved by induction on $\alpha$. There is nothing to prove for $\alpha = 0$.

Assume that the claim is true for some $\alpha < \tau$. Let $K$ be a submodule of $N_{\alpha+1}$. Let $\pi_K : N_{\alpha+1} \rightarrow N_{\alpha+1}/K$, $\rho : N_{\alpha+1} \rightarrow N_{\alpha+1}/(N_\alpha + K)$, $\eta : N_{\alpha+1}/K$ →
$N_{\alpha+1}/(N_{\alpha} + K)$, and $\theta : N_{\alpha} \to N_{\alpha}/(N_{\alpha} \cap K)$ denote the canonical projections. Also, let $\iota_{\alpha} : N_{\alpha}/(N_{\alpha} \cap K) \to (N_{\alpha} + K)/K$ be the canonical isomorphism (given by the assignment $\iota_{\alpha}(x + N_{\alpha} \cap K) = x + K$).

These homomorphisms fit in the following commutative diagram with exact rows (where, except for the zero maps, all the unnamed single arrows are inclusions):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_{\alpha} + K & \longrightarrow & N_{\alpha+1} & \longrightarrow & \frac{N_{\alpha+1}}{(N_{\alpha} + K)} & \longrightarrow & 0 \\
\end{array}
$$

Let $f \in \text{Hom}_R(M, N_{\alpha+1}/K)$. We have to show that $f$ factorizes through $\pi_K$.

By our assumption on $M$, there exists $g \in \text{Hom}_R(M, N_{\alpha+1})$ such that $\rho g = \eta f$. Since $\rho = \eta \pi_K$, $\eta(f - \pi_K g) = 0$. It follows that $\delta = f - \pi_K g$ maps $M$ into $\ker(\eta) = (N_{\alpha} + K)/K$, whence $\iota_{\alpha}^{-1}\delta \in \text{Hom}_R(M, N_{\alpha}/(N_{\alpha} \cap K))$. By the inductive premise, there exists $\epsilon : M \to N_{\alpha}$ such that $\iota_{\alpha}\epsilon = \delta$. As $\iota_{\alpha}\theta = \pi_K$, we conclude that $f = \pi_K g + \delta = \pi_K g + \pi_K \epsilon = \pi_K(g + \epsilon)$, which is the desired factorization of $f$ through $\pi_K$.

Let $\alpha \leq \tau$ be a limit ordinal. Let $K$ be a submodule of $N_{\alpha} = \bigcup_{\beta < \alpha} N_{\beta}$ and $f \in \text{Hom}_R(M, N_{\alpha}/K)$. Since $M$ is finitely generated, there exists $\beta < \alpha$ such that $f : M \to (N_{\beta} + K)/K$. Let $\iota_{\beta} : N_{\beta}/(N_{\beta} \cap K) \to (N_{\beta} + K)/K$ be the canonical isomorphism, and $\theta : N_{\beta} \to N_{\beta}/(N_{\beta} \cap K)$ the canonical projection. Then $\pi_K \mid (N_{\beta} + K) = \iota_{\beta}\theta$, and we have the following commutative diagram with exact rows (again, except for the zero maps, all the unnamed single arrows are inclusions):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & N_{\alpha} & \longrightarrow & \frac{N_{\alpha}}{K} & \longrightarrow & 0 \\
\end{array}
$$

By the inductive premise for $\beta$, there exists $g \in \text{Hom}_R(M, N_{\beta})$ such that $\theta g = \iota_{\beta}^{-1} f$. Then $f = \iota_{\beta}\theta g = \pi_K g$, which yields the desired factorization of $f$ through $\pi_K$.

The results above imply that weak $R$-projectivity is sufficient to guarantee projectivity for each finitely generated module:

**Corollary 1.6.** Let $R$ be any right semiartinian ring. Let $M$ be a finitely generated weakly $R$-projective module. Then $M$ is projective.

**Proof.** Since weak projectivity can be expressed in the stronger form of Lemma 1.4, we can apply Lemma 1.5 for $N = R$ and $N' = \text{the right socle sequence of } R$. Thus, $M$ is $R$-projective, and by Lemma 1.2, $M$ is projective. □
The question of the coincidence of weak $R$-projectivity, $R$-projectivity, and projectivity for infinitely generated modules over semiartinian rings, that is, whether projectivity can be tested at the layer epimorphisms of $R$, is much more delicate. We will see that for some semiartinian rings, it is actually independent of $\text{ZFC} + \text{GCH}$.

On the one hand, there is the following set-theoretic barrier for all non-right perfect rings $R$, given by Shelah’s Uniformization Principle (SUP). This principle is consistent with $\text{ZFC} + \text{GCH}$ (see [4, XIII.1.5] or [5, §2] for more details):

**Lemma 1.7.** Assume SUP. Let $R$ be a non-right perfect ring. Let $\kappa$ be a singular cardinal of cofinality $\omega$ such that $\text{card}(R) < \kappa$. Then there exists a $\kappa^+$ generated module $M$ of projective dimension equal to 1, such that $\text{Ext}^1_R(M,I) = 0$ for each right ideal $I$ of $R$. In particular, $M$ is $R$-projective, but not projective.

**Proof.** See [1, 2.4 and 2.5]. □

On the other hand, there are different extensions of $\text{ZFC} + \text{GCH}$ where weak $R$-projectivity, $R$-projectivity, and projectivity coincide for suitable semiartinian rings. These rings will be studied in the next section.

### 2. Transfinite extensions of simple artinian rings

Recall that a ring $R$ is *von Neumann regular* if for each $r \in R$ there exists $s \in R$ such that $rsr = r$. We refer to [8] for properties of von Neumann regular rings. A ring $R$ is said to have (right) *primitive factors artinian*, or (right) *pfa* for short, if $R/P$ is (right) artinian for each (right) primitive ideal of $R$.

If $R$ is von Neumann regular, then minimal right (left) ideals of $R$ correspond to primitive idempotents of $R$, so the right and left socle sequences of $R$ coincide. In particular, $R$ is right semiartinian, iff $R$ is left semiartinian. Similarly, by [8, 6.2], a von Neumann regular ring has right pfa, iff it has left pfa. The latter condition can equivalently be stated as a property of the module category $\text{Mod–}R$: each homogenous completely reducible module is injective, see [8, 6.28].

Let $R$ be a semiartinian von Neumann regular ring with pfa, and $(S_\alpha | \alpha \leq \sigma + 1)$ be its socle sequence. (Notice that $\sigma > 0$ implies that $R$ is not right perfect, because the Jacobson radical of any von Neumann regular ring is 0; thus, Lemma 1.7 applies here.)

By the following theorem from [9], semiartinian von Neumann regular rings with pfa can be viewed as transfinite extensions of full matrix rings over skew-fields (i.e., of simple artinian rings):

**Theorem 2.1.** Let $R$ be a right semiartinian ring and $(S_\alpha | \alpha \leq \sigma + 1)$ be the right socle sequence of $R$. The following conditions are equivalent:

1. $R$ is von Neumann regular with pfa.
2. For each $\alpha \leq \sigma$ there is a cardinal $\lambda_\alpha$, positive integers $n_{\alpha \beta}$ ($\beta < \lambda_\alpha$) and skew-fields $K_\alpha\beta$ ($\beta < \lambda_\alpha$), such that $L_\alpha \varphi_\alpha \cong \bigoplus_{\beta < \lambda_\alpha} M_{n_{\alpha \beta}}(K_\alpha\beta)$, as rings without unit. Moreover, $\lambda_\alpha$ is infinite iff $\alpha < \sigma$.

   The pre-image of $M_{n_{\alpha \beta}}(K_\alpha\beta)$ in the isomorphism $\varphi_\alpha$ coincides with the $\beta$th homogenous component of $\text{Soc}(R/S_\alpha)$, and it is a finitely generated as right $R/S_\alpha$-module for all $\beta < \lambda_\alpha$.

The structure of the rings characterized by Theorem 2.1 can be depicted as follows:
Here, \( \sigma + 1 \) is the Loewy length of \( R \). The rows in this picture represent the layers of \( R \); \( \lambda_\alpha \) is the number of homogenous components in the \( \alpha \)th layer \( L_\alpha = S_{\alpha + 1}/S_\alpha \) for each \( \alpha \leq \sigma \) (this number is infinite except for \( \alpha = \sigma \)).

\( n_{\alpha \beta} \) is the (finite) rank of the \( \beta \)th homogenous component of \( L_\alpha \), and \( K_{\alpha \beta} \) is the endomorphism ring (skew-field) of each simple module in the \( \beta \)th homogenous component of \( L_\alpha \) (for all \( \alpha \leq \sigma, \beta < \lambda_\alpha \)).

In particular, \( \sigma, \lambda_\alpha (\alpha \leq \sigma), n_{\alpha \beta}, \) and \( K_{\alpha \beta} (\alpha \leq \sigma, \beta < \lambda_\alpha) \) are invariants of the ring \( R \).

**Lemma 2.2.** Let \( R \) be a semiartinian von Neumann regular ring with pfa. Then the following hold:

1. The class of all weakly \( R \)-projective modules is closed under submodules.
2. All countably generated weakly \( R \)-projective modules are projective.

**Proof.** (1) This follows from the fact [8, 6.28] that all homogenous semisimple modules are injective, whence so are all finitely generated semisimple modules, and thus all finitely generated submodules of each of the layers \( L_\alpha (\alpha < \sigma) \).

(2) Let \( C \) be a countably generated weakly \( R \)-projective module. So \( C = \bigcup_{n < \omega} F_n \) where \( (F_n \mid n < \omega) \) is a chain of finitely generated submodules of \( C \). By part (1), each \( F_n \) is weakly \( R \)-projective, hence projective by Corollary [1.6]

Since \( R \) is von Neumann regular, \( F_n \) is a direct summand in \( F_{n+1} \) for each \( n < \omega \), whence \( C \) is projective. \( \square \)

From Lemma 2.2(1), we see that if the classes of all weak \( R \)-projective and projective modules coincide, then \( R \) is a right hereditary ring. What that means in out setting is partially clarified in the next lemma:

**Lemma 2.3.** Let \( R \) be a semiartinian von Neumann regular ring with pfa. Consider the following two conditions:

1. \( R \) is (left and right) hereditary.
2. \( \sigma \) is a countable ordinal, and all proper layers of \( R \) are countably generated

(i.e., \( \lambda_\alpha \) is countable for each \( 0 < \alpha < \sigma \)).

Then (2) implies (1). Moreover, if \( \lambda_0 \) is countable, then (1) and (2) are equivalent.

**Proof.** That (2) implies (1) was proved in [12, 3.10].

Assume \( \lambda_0 \) is countable and \( R \) is right hereditary. Since all the homogenous components of \( \text{Soc}(R) \) are injective, and hence finitely generated, \( \text{Soc}(R) \) is countably generated.

Assume (2) fails. If \( \sigma \) is uncountable, then \( \sigma = \tau + n \) for an uncountable limit ordinal \( \tau \) and some \( n < \omega \), whence \( I = S_\tau \) is not countably generated. If \( \lambda_\alpha \) is uncountable for some ordinal \( 0 < \alpha < \sigma \), then the \( \alpha \)th layer \( L_\alpha \) is not countably generated, and the same is true of \( I = S_{\alpha + 1} \).

Since projective modules over von Neumann regular rings are isomorphic to direct sums of cyclic modules generated by idempotents of \( R \), in either case there is a cardinal \( \kappa > \lambda_0 \) such that \( I \cong \bigoplus_{\gamma < \kappa} e_\gamma R \) where \( e_\gamma^2 = e_\gamma \in R \) for each \( \gamma < \kappa \). Since \( R \) is semiartinian, \( \text{Soc}(I) \cong \bigoplus_{\gamma < \kappa} \text{Soc}(e_\gamma R) \) is a direct sum of uncountably many non-zero completely reducible modules, in contradiction with \( \text{Soc}(I) \) being a direct summand in the countably generated module \( \text{Soc}(R) \). \( \square \)
Notice that Lemma 2.3 implies that all semiartinian von Neumann regular rings with pfa of Loewy length 2 are hereditary. The simplest such example is the $K$-subalgebra $R_1$ of all eventually constant sequences in $K^\omega$ for a field $K$, studied in [11]. The corresponding picture for $R_1$ is a follows:

\[
\begin{array}{c}
L_1 = K \\
L_0 = K(\omega) \\
K \oplus \ldots \oplus K \oplus \ldots
\end{array}
\]

However, semiartinian von Neumann regular rings with pfa of Loewy length 3 need not be hereditary: for an example, take a set $S$ of cardinality $\omega_1$ consisting of almost disjoint infinite subsets of $\omega$, and consider the $K$-subalgebra $R'$ of $K^\omega$ generated (as a $K$-linear space) by $B \cup C \cup \{1\}$ where 1 is the unit of $K^\omega$, $B$ is the canonical basis of $K^{(\omega)}$ and $C$ is the set of characteristic functions of all the sets in $S$. By Lemma 2.3, $R'$ is not hereditary. The corresponding picture for $R'$ is

\[
\begin{array}{c}
L_2 = K \\
L_1 = K(\omega_1) \\
L_0 = K(\omega) \\
K \oplus \ldots \oplus K \oplus \ldots
\end{array}
\]

The following recursive construction shows that hereditary semiartinian von Neumann regular rings with pfa of Loewy length $\alpha + 1$ do exist for each countable ordinal $\alpha > 0$. The induction step makes use of a construction of semiartinian von Neumann regular rings from [3, 2.4]:

**Example 2.4.** Let $K$ be a field. By induction on $0 < \alpha < \omega_1$, we will construct semiartinian von Neumann regular $K$-algebras with pfa, $R_\alpha$, of Loewy length $\alpha + 1$ such that $R_\alpha$ has countably generated layers, together with $K$-algebra embeddings $\nu_\alpha : R_\alpha \to K^\omega$, and for each $0 < \beta < \alpha$, non-unital $K$-algebra monomorphisms $f_{\alpha\beta} : R_\beta \to R_\alpha$ and $g_{\alpha\beta} : K^\omega \to K^\omega$ such that the squares

\[
\begin{array}{c}
R_\beta \xrightarrow{f_{\alpha\beta}} R_\alpha \\
\nu_\alpha \downarrow \quad \downarrow \nu_\alpha \\
K^\omega \xrightarrow{g_{\alpha\beta}} K^\omega
\end{array}
\]

are commutative, $g_{\alpha\beta}$ splits, and the complement of $\text{Im}(g_{\alpha\beta})$ in $K^\omega$ is isomorphic to $K^\omega$.

For $\alpha = 1$, we let $R_1$ be the $K$-algebra of all eventually constant sequences of elements of $K$ mentioned above. In particular, $R_1$ is a $K$-subalgebra of $K^\omega$ of Loewy length 2, and $\nu_1$ is defined as the inclusion of $R_1$ into $K^\omega$.

The induction step is modeled on [3, 2.4]: Assume the construction is done up to some $0 < \alpha < \aleph_1$. Let $I = \bigoplus_{i<\omega} R_\alpha$. Then $\iota_\alpha = \bigoplus_{i<\omega} \nu_\alpha$ embeds $I$ into $D = \bigoplus_{i<\omega} K^\omega \subseteq K^{\omega^\omega}$. As in [3], $\omega^\omega$ denotes the ordinal exponentiation, so $\omega^\omega = \sup_{i<\omega} \omega^i$ and $\omega^{i+1} = \omega^i \times \omega$ for each $0 < i < \omega$. Note that $\iota_\alpha(I)$ is a non-unital $K$-subalgebra of $K^{\omega^\omega}$.

Since the ordinal $\omega^\omega$ is countable, there is a $K$-algebra isomorphism $\psi : K^{\omega^\omega} \to K^\omega$. Let $R_{\alpha+1} = \psi(S_{\alpha+1})$, and let $\nu_{\alpha+1}$ denote the inclusion of $R_{\alpha+1}$ into $K^\omega$.

Let $\mu_\alpha$ be the embedding of the first copy of $K^\omega$ in $D$ composed with the inclusion $D \subseteq K^{\omega^\omega}$. Notice that $\mu_\alpha$ is a split non-unital embedding of $K$-algebras, and so is $\psi \mu_\alpha$. In fact, $\psi \mu_\alpha(K^\omega)$ is a direct summand in $K^\omega$ with a complement isomorphic to $K^\omega$. Moreover, $\mu_\alpha \mid R_\alpha$ is a non-unital $K$-algebra embedding of $R_\alpha$ into $S_{\alpha+1}$.
We have the following commutative diagram, where the vertical maps are $K$-algebra embeddings, and the horizontal ones are non-unital $K$-algebra embeddings:

We let $f_{\alpha+1,0} = \psi_{0,\alpha} \mid R_\alpha$, $g_{0,0,0} = \psi_{0,0}$, and for each $0 < \beta < \alpha$, $f_{\alpha+1,\beta} = f_{\alpha+1,0} \cdot f_{0,\beta}$ and $g_{0,\alpha,0} = g_{0,0,0} \cdot g_{0,\beta}$.

By Lemma 2.3, the Loewy length of $R_{\alpha+1}$ is $\alpha + 2$. Moreover, for each $\beta \leq \alpha$, the $\beta$th layer of $R_{\alpha+1}$ is countably generated, while its $(\alpha + 1)$th layer, $R_{\alpha+1}/\psi_{\alpha}(I)$, is isomorphic to $K$.

If $\alpha < \omega_1$ is a limit ordinal, then $\alpha = \sup_{n<\omega} \beta_n$ for a strictly increasing chain of countable ordinals $(\beta_n \mid n < \omega)$. We fix one such chain with $\beta_0 > 0$. By the induction hypothesis, we have the following commutative diagram, where $\nu_{\beta_n}$ ($n < \omega$) are $K$-algebra embeddings, $f_{\beta_{n+1},0} : g_{\beta_{n+1},0} (n < \omega)$ are non-unital $K$-algebra embeddings, $g_{\beta_{n+1},\beta_n}$ splits, and the complement of $\text{Im}(g_n)$ in $K^\omega$ is isomorphic to $K^\omega (n < \omega)$:

Let $T_\alpha = \lim_{n<\omega} R_{\beta_n}$. Since the direct limit of the split embeddings in the bottom row is isomorphic to $\bigoplus_{n<\omega} K^\omega$, we obtain a non-unital $K$-algebra embedding $\lim_{n<\omega} \nu_{\beta_n} : T_\alpha \to \bigoplus_{n<\omega} K^\omega$ which can be extended to a $K$-algebra homomorphism $\nu_\alpha : R_\alpha \to K^\omega$ by the same procedure as in the induction step above.

The direct limit of the non-unital $K$-algebra embeddings in the top row yields $f_{\alpha,0} : R_{\beta_n} \to T_\alpha \to R_\alpha$ for each $n < \omega$. Moreover, we have the split non-unital $K$-algebra embeddings $g_{\alpha,0} : K^\omega \to K^\omega$ that make the corresponding squares for $\beta_n$ and $\alpha$ commute.

Since for each $\beta < \alpha$, there exists $n < \omega$ such that $\beta < \beta_n$, we can define $f_{\alpha,\beta} = f_{\alpha,\beta_n} \cdot f_{\beta_n,\beta}$ and $g_{\alpha,\beta} = g_{\alpha,\beta_n} \cdot g_{\beta_n,\beta}$. Then also the squares for $\beta$ and $\alpha$ commute, by the induction premise. Finally, $R_\alpha$ is a semiarithmetic von Neumann regular ring with pfa of Loewy length $\alpha + 1$, all of whose layers are countably generated.

By Lemma 2.3, $R_\alpha$ is hereditary for each $0 < \alpha < \omega_1$. The corresponding picture for $R_\alpha$ is as follows:

```
L_0 = K  K
\ldots
L_\beta = K(\omega)
\ldots
L_1 = K(\omega)
L_0 = K(\omega)
```
Finally, we recall that in the hereditary setting of Lemma 2.3, there is a short exact sequence that tests for weak \( R \)-projectivity of any module \( M \):

**Lemma 2.5.** Let \( R \) be a semiartinian von Neumann regular ring with pfa. Assume \( \sigma \) is a countable ordinal and all proper layers of \( R \) are countably generated.

Then there exist a module \( B \) which is a countable direct product of certain ideals of \( R \), an injective module \( N \), and an epimorphism \( \pi : B \to N \), such that the following are equivalent for a module \( M \):

1. \( M \) is weakly \( R \)-projective.
2. The homomorphism \( \text{Hom}_R(M, \pi) : \text{Hom}_R(M, B) \to \text{Hom}_R(M, N) \) is surjective.

**Proof.** This follows by [12, 4.2]. □

**Remark 2.6.** By [12, 4.1], the epimorphism \( \pi \) is the product of restrictions of the layer epimorphisms \( \pi_\alpha \ (\alpha \leq \sigma) \) to the right ideals \( N_{\alpha, F} \), where \( F \) runs over all finite subsets of \( \Lambda \), \( S_\alpha \subseteq N_{\alpha, F} \subseteq S_{\alpha+1} \), and \( N_{\alpha, F}/S_\alpha \cong \bigoplus_{g \in F} M_{\alpha, g} \) is an injective module, cf. Theorem 2.1.

In particular, [12, 4.2] implies that if \( M \) is not weakly \( R \)-projective, then there exist an \( \alpha \leq \sigma \) and a finite subset \( F \) of \( \Lambda \) such that the homomorphism \( \text{Hom}_R(M, \pi_\alpha \mid N_{\alpha, F}) : \text{Hom}_R(M, N_{\alpha, F}) \to \text{Hom}_R(M, N_{\alpha, F}/S_\alpha) \) is not surjective.

3. **Weak diamond and weak projectivity**

By Lemma 2.3, if \( R \) is a semiartinian von Neumann regular ring with pfa, then the notions of a weak \( R \)-projective, \( R \)-projective, and projective module coincide for any countably generated module. In contrast with Lemma 1.7, we will show that in the extension of ZFC + CH where the prediction principle \( \Phi \) (Weak Diamond) holds, these notions coincide for arbitrary modules, provided that condition (2) of Lemma 2.3 holds (whence \( R \) is hereditary) and \( \text{card}(R) \leq \aleph_1 \).

To simplify our notation, we introduce the following definition:

**Definition 3.1.** Let \( R \) be a semiartinian von Neumann regular ring with pfa. Then \( R \) is small, if \( \text{card}(R) \leq \aleph_1 \), \( \sigma \) is a countable ordinal, and all proper layers of \( R \) are countably generated.

Notice that our notion of smallness is more general than the one in [12, Definition 4.3] which required the ordinal \( \sigma \) to be finite rather than countable.

Before introducing the Weak Diamond Principle, we need to recall several basic set-theoretic notions:

Let \( \kappa \) be a regular uncountable cardinal. Let \( A \) be a set of cardinality \( \leq \kappa \). An increasing continuous chain, \( (A_\gamma \mid \gamma < \kappa) \), consisting of subsets of \( A \) of cardinality \( < \kappa \), is a \( \kappa \)-filtration of the set \( A \) in case its union is \( A \). If \( A \) is moreover a module, we will also use the term \( \kappa \)-filtration of the module \( A \), which has the additional assumptions that all the \( A_\gamma \ (\gamma < \kappa) \) are submodules of \( A \), and \( A_0 = 0 \).

A subset \( C \subseteq \kappa \) is a club in \( \kappa \), if \( C \) is unbounded (i.e., \( \sup C = \kappa \)) and closed (i.e. for each \( D \subseteq C \), if \( s = \sup D < \kappa \) then \( s \in C \)). A subset \( E \subseteq \kappa \) is stationary in \( \kappa \), if \( E \cap C \neq \emptyset \) for each club \( C \) in \( \kappa \).

Now, we can introduce the Weak Diamond Principle, \( \Phi \). We will use it in the following form presented in [11, Lemma VI.1.7] and [7, Theorem 18.12]:

\begin{enumerate}
\item[(\( \Phi \))] Let \( \kappa \) be a regular uncountable cardinal and \( E \) a stationary subset in \( \kappa \). Let \( A \) and \( B \) be sets of cardinality \( \leq \kappa \). Let \( (A_\gamma \mid \gamma < \kappa) \) be a \( \kappa \)-filtration of \( A \), and \( (B_\gamma \mid \gamma < \kappa) \) a \( \kappa \)-filtration of \( B \). For each \( \gamma \in E \), let \( c_\gamma : A_\gamma B_\gamma \to 2 \).
Then there exists a function \( c : E \to 2 \), such that for each \( x \in A B \), the set \( E(x) = \{ \gamma \in E \mid x \in A_\gamma B_\gamma \text{ and } c(\gamma) = c_\gamma(x \mid A_\gamma) \} \) is stationary in \( \kappa \).
\end{enumerate}
The Weak Diamond $\Phi$ is easily seen to be a consequence the better known (and stronger) Jensen’s Diamond $\Diamond$, which in turn is a consequence of Gödel’s Axiom of Constructibility, and hence is consistent with $\text{ZFC} + \text{GCH}$. We refer to [4] §VI.1 and [7] §18 for more details.

We arrive at the promised generalization of [12] 4.4 requiring only the Weak Diamond (rather than the stronger Jensen’s Diamond), and allowing for arbitrary countable Loevy length of $R$:

**Theorem 3.2.** Assume $\text{CH} + \Phi$. Let $R$ be small. Then all weakly $R$-projective modules, and hence all $R$-projective modules, are projective.

**Proof.** Let $M$ be a weakly $R$-projective module. By induction on the minimal number of generators, $\kappa$, of $M$, we will prove that $M$ is projective. For $\kappa \leq \aleph_0$, the result follows by Lemma 2.2. If $\kappa$ is a singular cardinal, then we use the fact that the class of all weakly $R$-projective modules is closed under submodules (Lemma 2.2) and apply Shelah’s Singular Compactness Theorem (e.g., in the version from [7] 7.9, see also [4] XII.1.14).

Assume that $\kappa$ is a regular uncountable cardinal. Let $A = \{m_\gamma : \gamma < \kappa\}$ be a minimal set of $R$-generators of $M$. For each $\gamma < \kappa$, let $A_\gamma = \{m_\delta : \delta < \gamma\}$. Let $M_\gamma$ be the submodule of $M$ generated by $A_\gamma$. By the inductive premise, $M_\gamma$ is projective, and $M = (M_\gamma : \gamma < \kappa)$ is a $\kappa$-filtration of the module $M$. Possibly skipping some terms of $M$, we can w.l.o.g. assume that $M$ has the following property for each $\gamma < \kappa$: If $M_\delta/M_\gamma$ is not weakly $R$-projective for some $\gamma < \delta < \kappa$, then also $M_{\delta+1}/M_\gamma$ is not weakly $R$-projective.

$(\dagger)$ Let $E$ be the set of all $\gamma < \kappa$ such that $M_{\gamma+1}/M_\gamma$ is not weakly $R$-projective. Also, let $\pi : B \to N$ be the epimorphism from Lemma 2.5. By that Lemma, for each $\gamma \in E$, we can choose an $h_\gamma \in \text{Hom}_R(M_{\gamma+1}/M_\gamma, N)$ such that $h_\gamma$ does not factorize through $\pi$.

We claim that $E$ is not stationary in $\kappa$. Assume this claim is not true. Note that CH implies $\text{card}(B) \leq \aleph_1 \leq \kappa$, so we can fix a $\kappa$-filtration of the set $B$, $(B_\gamma : \gamma < \kappa)$. Let $\gamma < \kappa$. For each $g \in \text{Hom}_R(M_\gamma, N)$, we choose $g^+ \in \text{Hom}_R(M_{\gamma+1}, N)$ such that $g^+ |_{M_\gamma} = g$. This is possible because $N$ is an injective module. Further, for each $f \in \text{Hom}_R(M_\gamma, B)$, we choose $f^+ \in \text{Hom}_R(M_{\gamma+1}, B)$ such that $\pi f^+ = (\pi f)^+$. This is possible since $M_{\gamma+1}$ is projective. Notice that $\pi f^+ |_{M_\gamma} = (\pi f)^+ |_{M_\gamma} = \pi f$, so $\delta f := f^+ |_{M_\gamma} - f \in \text{Hom}_R(M_\gamma, K)$ where $K = \text{Ker}(\pi)$.

For each $\gamma \in E$, we define $c_\gamma : A^\gamma B_\gamma \to 2$ as follows: If $x : A_\gamma \to B_\gamma$ is a restriction of a (necessarily unique) morphism $f \in \text{Hom}_R(M_\gamma, B)$ such that the morphism $\delta f = f^+ |_{M_\gamma} - f$ can be extended to a morphism from $\text{Hom}_R(M_{\gamma+1}, K)$, then we put $c_\gamma(x) = 1$. Otherwise, we let $c_\gamma(x) = 0$.

In this setting, $\Phi$ yields a function $c : E \to 2$ such that for each $x \in A^\gamma$, the set $E(x) = \{ \gamma \in E : x | A_\gamma \in A^\gamma B_\gamma \}$ and $c(\gamma) = c_\gamma(x | A_\gamma)$ is stationary in $\kappa$. We will use $c$ to define a morphism $g \in \text{Hom}_R(M, N)$ as follows:

By induction on $\gamma < \kappa$, we define a sequence $(g_\gamma : \gamma < \kappa)$ such that $g_\gamma \in \text{Hom}_R(M_\gamma, N)$. First, $g_0 = 0$. If $\gamma < \kappa$, and $g_\gamma$ is already defined, we distinguish two cases:

(I) $\gamma \notin E$ or $c(\gamma) = 0$. In this case, we put $g_{\gamma+1} = (g_\gamma)^+$. 

(II) $\gamma \in E$ and $c(\gamma) = 1$. In this case, we let $g_{\gamma+1} = (g_\gamma)^+ + h_\gamma \rho_\gamma$, where $\rho_\gamma : M_{\gamma+1} \to M_{\gamma+1}/M_\gamma$ is the canonical projection modulo. $M_\gamma$.

Notice that in both cases $g_{\gamma+1} |_{M_\gamma} = g_\gamma$. We let $g_\gamma = \bigcup_{\delta < \gamma} g_\delta$ in case $\gamma < \kappa$ is a limit ordinal. Then $g = \bigcup_{\gamma < \kappa} g_\gamma \in \text{Hom}_R(M, N)$. 


Since $M$ is weakly $R$-projective, there exists $f \in \text{Hom}_R(M, B)$ such that $g = \pi f$.
By $\Phi$, the set $E(f \upharpoonright A) = \{ \gamma \in E \mid f \upharpoonright A_{\gamma} \in \mathcal{A}_{\gamma} B_{\gamma} \}$ and $c(\gamma) = c(f \upharpoonright A_{\gamma})$ is stationary in $\kappa$. Let $\gamma \in E(f \upharpoonright A)$.
Assume that $c(\gamma) = 0$. Then we are in case (I), so $\pi(f \upharpoonright M_{\gamma+1}) = g_{\gamma+1} = (g_{\gamma})^+ = (\pi f \upharpoonright M_{\gamma})^\kappa = \pi(f \upharpoonright M_{\gamma})^\kappa$. Then the morphism $(f \upharpoonright M_{\gamma})^\kappa \to f \upharpoonright M_{\gamma+1} \in \text{Hom}_R(M_{\gamma+1}, K)$ is an extension of $\delta_{f|M_{\gamma}} = (f \upharpoonright M_{\gamma})^\kappa \upharpoonright M_{\gamma} = f \upharpoonright M_{\gamma} \upharpoonright M_{\gamma+1}$, in contradiction with $c_\gamma(f \upharpoonright A_{\gamma}) = c(\gamma) = 0$.

So necessarily $c(\gamma) = 1$, and we are in case (II), whence $g_{\gamma+1} = (g_{\gamma})^+ + h_{\gamma}\rho_{\gamma}$.
As $c_\gamma(f \upharpoonright A_{\gamma}) = c(\gamma) = 1$, the morphism $\delta_{f|M_{\gamma}} = (f \upharpoonright M_{\gamma})^\kappa \upharpoonright M_{\gamma} = f \upharpoonright M_{\gamma} \upharpoonright M_{\gamma+1}$ can be extended to a morphism $\Delta_{f|M_{\gamma}} \in \text{Hom}_R(M_{\gamma+1}, K)$.

Again, $g_{\gamma+1} = \pi(f \upharpoonright M_{\gamma+1})$ and $(g_{\gamma})^+ = \pi(f \upharpoonright M_{\gamma})^\kappa$, so
\[ h_{\gamma}\rho_{\gamma} = \pi(f \upharpoonright M_{\gamma+1} - (f \upharpoonright M_{\gamma})^\kappa) = \pi u_{\gamma}, \]
where $u_{\gamma} = f \upharpoonright M_{\gamma+1} - (f \upharpoonright M_{\gamma})^\kappa + \Delta_{f|M_{\gamma}} \in \text{Hom}_R(M_{\gamma+1}, B)$. However, $u_{\gamma} \upharpoonright M_{\gamma} = f \upharpoonright M_{\gamma} - (f \upharpoonright M_{\gamma})^\kappa \upharpoonright M_{\gamma} + \delta_{f|M_{\gamma}} = 0$. Hence $u_{\gamma}$ factorizes through $\rho_{\gamma}$ by some $v_{\gamma} \in \text{Hom}_R(M_{\gamma+1}/M_{\gamma}, B)$, that is, $u_{\gamma} = v_{\gamma}\rho_{\gamma}$.

Thus $h_{\gamma}\rho_{\gamma} = \pi v_{\gamma}\rho_{\gamma}$. Since $\rho_{\gamma}$ is surjective, $h_{\gamma} = \pi v_{\gamma}$, in contradiction with our choice of $h_{\gamma}$.

This proves our claim about the set $E$. So there is a club $C$ in $\kappa$ such that $C \cap E = \emptyset$. Let $\gamma : \kappa \to \kappa$ be a strictly increasing continuous function whose image is $C$. For each $\gamma < \kappa$, let $N_{\gamma} = M_{\gamma(\gamma)}$. Then $(N_{\gamma} \mid \gamma < \kappa)$ is a $\kappa$-filtration of the module $M$ such that $N_{\gamma+1}/N_{\gamma}$ is weakly $R$-projective for each $\gamma < \kappa$. By the inductive premise, $N_{\gamma+1}/N_{\gamma}$ is projective, hence $N_{\gamma+1} = N_{\gamma} \oplus P_{\gamma}$ for a projective module $P_{\gamma}$, for each $\gamma < \kappa$. We conclude that $M = N_0 \oplus \bigoplus_{\gamma < \kappa} P_{\gamma}$ is projective. This finishes the inductive step for the case when $\kappa$ is a regular uncountable cardinal. □

Remark 3.3. As observed by Jan Šaroch, Remark 2.6 makes it possible to prove Theorem 3.2 without the assumption of CH, i.e., assuming only $\Phi$. (That is indeed a weaker assumption, since unlike $\Diamond$, the $\Phi$ does not imply CH.) The only modification needed concerns the set $E$ of all $\gamma < \kappa$ such that $M_{\gamma+1}/M_{\gamma}$ is not weakly $R$-projective, and the morphisms $h_{\gamma} (\gamma \in E)$ defined in part (i) of the proof of Theorem 3.2.

For each $\gamma \in E$, Remark 2.6 yields an $\alpha_{\gamma} \leq \sigma$, a finite subset $F_{\gamma}$ of $\lambda_{\alpha_{\gamma}}$, and an $h_{\gamma} \in \text{Hom}_R(M_{\gamma+1}/M_{\gamma}, N_{\alpha_{\gamma}}/S_{\alpha_{\gamma}})$ which does not factorize through $\pi_{\alpha_{\gamma}} \upharpoonright N_{\alpha_{\gamma}} F_{\gamma}$. For each $\alpha \leq \sigma$ and each finite subset $F$ of $\lambda_{\alpha}$, let $E_{\alpha, F} = \{ \gamma \in E \mid \alpha = \alpha \land F_{\gamma} = F \}$. Then $E = \bigcup_{\alpha, F} E_{\alpha, F}$. Notice that since $R$ is small, the set of all such pairs $(\alpha, F)$ is countable.

Thus, if $E$ is stationary, then so is one of the $E_{\alpha, F}$ (see e.g. [4, II.4.3]), say $E_{\alpha', F'}$. The proof of a contradiction with the stationarity of $E$ then proceeds as that of Theorem 3.2 in the parts following (i), but for $B = N_{\alpha', F'}$, $N = N_{\alpha', F'}/S_{\alpha'}$, $\pi = \pi_{\alpha'} \upharpoonright N_{\alpha', F'}$, and $E = E_{\alpha', F'}$. The point is that in this setting, the cardinality of $B$ is $\leq \text{card}(R) \leq \aleph_0$ even without the assumption of CH.

Combining Lemma 1.7 and Theorem 3.2 we obtain

Corollary 3.4. Let $R$ be small. Then the assertion ‘All weakly $R$-projective modules are projective’ is independent of ZFC + GCH.

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Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 75 Prague 8, Czech Republic

Email address: trlifaj@karlin.mff.cuni.cz