Neel order, quantum spin liquids and quantum criticality in two dimensions

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This paper is concerned with the possibility of a direct second order transition out of a collinear Neel phase to a paramagnetic spin liquid in two dimensional quantum antiferromagnets. Contrary to conventional wisdom, we show that such second order quantum transitions can potentially occur to certain spin liquid states popular in theories of the cuprates. We provide a theory of this transition and study its universal properties in an \( \epsilon \) expansion. The existence of such a transition has a number of interesting implications for spin liquid based approaches to the underdoped cuprates. In particular it considerably clarifies existing ideas for incorporating antiferromagnetic long range order into such a spin liquid based approach.

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I. INTRODUCTION

In recent years a number of old ideas on theories of the cuprate materials have been clarified and rejuvenated. A variety of experiments now support the notion that the underdoped cuprates are usefully regarded as doped Mott insulators - in other words proximity to the Mott insulator strongly influences the properties of the underdoped material. The undoped Mott insulating parent materials have long range antiferromagnetic order. However this order disappears rather quickly upon doping. Rather (at not too low temperatures) the pseudogap state that appears above the superconducting transition in the underdoped materials is probably best regarded as a doped version of a paramagnetic Mott insulator. Such an insulator has a 'built-in' spin (pseudo)gap. Vexing questions however remain on the precise theoretical connection between such a point of view and the actual occurrence of antiferromagnetic order in the undoped materials.

A clue to resolving this dilemma is provided by neutron scattering experiments that reveal the existence of a sharp magnetic resonance at \((\pi, \pi)\) in the doped superconductor. An appealing interpretation of this resonance is as a gapped version of the familiar magnon of a proximate antiferromagnetic state. Interestingly as the doping is reduced, the resonance frequency goes down proportionately to \(T^2\). This suggests that if the doped state is to be viewed as a doped paramagnet, then the latter may at least be connected to the Neel state by a second order transition (see Fig 1). Hence we are led to search for quantum paramagnetic states of spin-1/2 moments on a square lattice that are accessible from the collinear Neel state by a second order transition.

It is important at this point to also review another old idea in cuprate theory. Early work suggested that the undoped magnetically ordered Mott insulator is close to being disordered by quantum fluctuations into a paramagnetic featureless "spin liquid" phase. Such a spin liquid state was postulated to have neutral spin-1/2 spinon excitations and preserve all the symmetries of the underlying microscopic spin Hamiltonian (including spin rotation). Further it was argued that doping a spin liquid could possibly lead to high temperature superconductivity.

Despite its original appeal, this scenario was subsequently questioned by a number of significant theoretical developments. Calculations in a controlled large-N expansion of quantum Heisenberg spin models concluded that the natural result of destruction of collinear Neel order was not a featureless spin liquid but (for spin-1/2) a valence bond solid (VBS), which breaks various lattice symmetries. (Here natural refers to the possibility that the quantum paramagnet in question is potentially separated from the Neel state by a second order transition.) The VBS state also does not support fractionalized spinon excitations. This was supported by a number of other indirect arguments - for instance by studies of quantum dimer models on the square lattice. It was shown however that destruction of non-collinear Neel order could indeed lead to a fractionalized spin liquid state that preserves all lattice symmetries.

These calculations lead to the following folk wisdom (for a review see Ref. 15): "In two spatial dimensions collinear ordered magnets naturally give way to..."
confined VBS paramagnets when disordered by quantum fluctuations while non-collinear magnets naturally lead to spin liquids". As the magnetic ordering is undoubtedly collinear in the cuprates this folk wisdom apparently spells doom for the view of the cuprates as doped spin liquid paramagnetic Mott states.

The purpose of the present paper is to revisit these issues. We will first argue that, if at all a spin liquid based approach is to be pursued, experiments suggest a certain kind of paramagnetic spin liquid state as natural candidate ‘parent’ states of the doped cuprates. Next we argue that the existing theoretical work does not rule out a direct second order transition between the Neel state and this particular kind of spin liquid state. Finally we outline in some detail a theory for just such a direct second order transition. Thus our work calls into question the folklore described above and potentially frees the spin liquid based approach to the cuprates from one of its theoretical criticisms.

Based on these results we will develop a qualitative picture of the neutron resonance mode seen in experiments in the doped system and its relationship with other aspects of the observed spin physics. Our description will naturally unify two popular views of the resonance mode - one as a soft mode associated with antiferromagnetic long range orderRef.5,6,7, and the other as a spin exciton formed as a triplet particle-hole collective mode of fermionic quasiparticlesRef.10,17,18,19,20.

We begin with experiments. It is by now quite clearly established that the cuprate superconductors are d-wave paired and furthermore have nodal BCS-like quasiparticles. The existence of the nodal quasiparticles is theoretically significant. Indeed the possibility of d-wave paired superconductors without nodal quasiparticles has been much emphasized by Kivelson and coworkers (for a review see Ref.21). It is therefore of some interest to ask whether there exist paramagnetic Mott states that already have gapless nodal excitations. Such a state then builds-in enough of the spin physics seen in the experiments at finite doping that it would be an attractive ‘parent’ Mott insulator as a basis for a theory of the underdoped cuprates.

Remarkably such states are known to exist as stable quantum phasesRef.22,23 of quantum antiferromagnets magnets on a two dimensional square lattice, at least within an appropriate large-N expansion. In this paper we will focus on one such state that has played a central role in some previous theoretical workRef.4,5,14,15,27,28,29 on the cuprate problem. This state - dubbed the d-wave RVB or staggered flux (sF) spin liquid - is a quantum paramagnet that nevertheless has gapless spin carrying excitations. Recent theoretical workRef.22 has established the stability of such a state (in a suitable large-N expansion). A low energy description of the physics is usefully provided in terms of a theory of gapless nodal fermionic spinons coupled minimally to a fluctuating U(1) gauge field. Despite this however there really is no true quasiparticle description of the low energy spectrumRef.25,26.

In this paper we are interested in exploring the quantum phase transition between this algebraic spin liquid and the collinear Neel state. Previous theoretical work providing the basis for the folk wisdom mentioned above do not constrain the nature of this transition. First the large-N calculations of Ref.12 were based on a bosonic representation of the spins. This representation is not well-suited to access the algebraic spin liquid phase. Indeed it is tailor-made to access either VBS phases or gapped spin liquids with bosonic spinons. Arguments for the presence of VBS order in the paramagnet based on quantum dimer models on the square latticeRef.23 also do not help. Clearly the quantum dimer models are useful only for paramagnets with a full spin gap - and hence will not be able to access algebraic spin liquids. A more recent elegant argumentRef.30 attacks from the spin liquid side as follows. First it supposes that to access Neel ordered states from spin liquids, the latter must have bosonic spinons. Then the Neel state is reached simply by condensing the bosons. Examining the dispersion relation of a bosonic spinon in a spin liquid state reveals that it generically has a minimum at an incommensurate wavevector. Condensing such a spinon then naturally led to incommensurate spiral states and not to the simple collinear state. A loophole in this argument is the supposition that it is only a spin liquid state with bosonic spinons that can be proximate (i.e. separated by a second order transition) to a Neel state. Indeed we will show in this paper that the algebraic spin liquid state - which doesn’t have bosonic spinons - can be connected to the Neel state by a second order transition.

Our starting point is a mean field description of both the spin liquid and antiferromagnetic phases. This will be done in terms of a slave particle representation of the spin operator in terms of neutral fermionic spinons. At this mean field level the spin liquid state we study will have spinons that are paired into a d-wave state. The resulting spinon dispersion has four Fermi points in the Brillouin zone at which the spinons are gapless. Antiferromagnetism is obtained in this representation as a spin density wave transition of the spinons. This magnetic ordering produces a gap for the spinons. Such a mean field description of the antiferromagnetic state was first proposed by HsuRef.22. Similar ideas have subsequently been explored in a number of publications - see Refs.33,34,35,36. However it is important to realize that the resulting mean field state is not a conventional antiferromagnet. Though the spinons have acquired a gap they still have not disappeared from the spectrum. Thus the mean field description is apparently that of a fractionalized antiferromagnet. For the particular mean field state studied in this paper this problem is cured once fluctuations beyond the mean field are included. Indeed we will argue that these fluctuations confine the spinons in the magnetically ordered state.

Does the slave particle mean field description of the antiferromagnet still have any physical meaning if the spinons are confined? An answer to this question is pro-
vided by considering the magnetic state close to the transition to the spin liquid. The critical point we describe for this transition has some conceptual similarity with the deconfined quantum critical point studied recently. Specifically spinon variables will already prove useful in describing the critical point itself though they do not correspond to physical degrees of freedom deep in the Neel phase. Furthermore there are two diverging length/time scales - one of which diverges as a power of the other - as the transition is approached from the Neel side. The physics is that of the conventional Neel state only at the very longest scales. At length/time scales intermediate between the two diverging ones, the physics is correctly thought of in terms of the mean field spin density wave state formed out of fermionic Dirac-like spinons. This state (though ultimately unstable) describes an increasingly wider regime of intermediate scales close to the transition. Apart from the spin waves expected from broken spin symmetry it also contains gapped spin-1/2 spinons coupled minimally to a gapless spin-0 gauge boson (a "photon"). Thus in this large intermediate scale regime the spinon mean field description of the antiferromagnet is indeed the correct physical starting point. Note that accessing the true critical theory requires keeping fluctuations other than those of the Neel order parameter. This should hardly be a surprise given the non-trivial nature of the paramagnetic phase.

The rest of the paper is organized as follows. In Section II we quickly outline the mean field theory of the spin liquid and the transition to the Neel state. Next in Section III we go beyond mean field, by including gauge fluctuation for spinons and dynamics for Neel field. We first briefly review the properties of the algebraic spin liquid phase with particular emphasis on precursor fluctuations of the magnetic ordering near the transition (Section IIIA). Then in subsection IIIB we discuss the magnetic state and show that it gets smoothly connected to the conventional Neel state once monopole fluctuations are included. We very briefly discuss the connection to projected wavefunction descriptions of the magnetism in undoped spin models in subsection IIIC. Next in Section IV we consider the phase transition. We show that within certain assumptions the transition has conceptually similar structure to the deconfined critical points of Ref. 37. We then present results for both the ASL phase and the transition to the Neel state within an appropriate expansion in Section V. In Section VI we discuss implications of our results for theories of cuprates before concluding in Section VII. Various appendices contain technical details of the calculations.

II. MEAN FIELD THEORY

Consider a generic $SU(2)$ symmetric spin-1/2 model on a square lattice predominantly antiferromagnetic short ranged interactions:

$$\mathcal{H} = J \sum_{\langle rr' \rangle} \vec{S}_r \cdot \vec{S}_{r'} + \cdots$$  \hspace{1cm} (1)$$

Here $J > 0$ (antiferromagnetic exchange), and the ellipses represent frustrating interactions that can be used to tune quantum phase transitions. We will require that the full Hamiltonian be invariant under $SU(2)$ spin rotations, time-reversal, and the full space group of the square lattice. It is well known that the nearest neighbour model has a Neel ordered ground state. Various paramagnetic ground states can be accessed (in principle) by appropriate frustrating interactions. As explained in the introduction, here we will focus on a particular paramagnetic state that is known as the dRVB algebraic spin liquid (also often referred to as the staggered flux spin liquid). A mean field theory for this state has been described several times in the literature and is well-known. First the spin is formally rewritten as a bilinear of fermionic "spinon" operators

$$\vec{S}_r = \frac{1}{2} f_{r \alpha}^{\dagger} \sigma_{\alpha \beta} f_{r \beta}.$$  \hspace{1cm} (2)$$

Here $\alpha = 1,2$, corresponding to spin up/spin down fermions. This is an exact rewriting when combined with the local constraint $f_{r \alpha}^{\dagger} f_{r \beta} = 1$. In the mean field approximation the exact Hamiltonian is replaced by one quadratic in the spinon operators but with self-consistently determined parameters. For the dRVB state, the mean field Hamiltonian takes the form

$$H_{SF} = - \sum_{\langle rr' \rangle} \left( (\chi_{rr'} + i \Delta_{rr'}) f_{r \alpha}^\dagger f_{r' \beta} + h.c \right)$$  \hspace{1cm} (3)$$

Here we take $r$ to belong to one sublattice of the square lattice. So that $r'$ belongs to the opposite sublattice. The constants $\Delta_{rr'} = +\Delta$ on horizontal bonds, and $-\Delta$ on vertical bonds while $\chi_{rr'} = t$ on all bonds. This describes fermionic spin-1/2 spinons on the square lattice with complex hopping amplitudes such that there is a non-zero flux that is staggered from plaquette to plaquette. Despite appearances, this saddle point possesses the full symmetry of the microscopic model including all lattice symmetries. (The apparent breaking of translational symmetry is a gauge artifact). Recent work has clarified the nature of fluctuations about this mean field state. But in this present section, we will stick to the

![Graphical representation of a quantum phase transition between Neel and dRVB algebraic spin liquid phases.](image)
mean field description and see how a transition to a Neel ordered state may be described.

To access a Neel state we modify the dRVB mean field Hamiltonian by adding a nearest neighbour antiferromagnetic interaction between the spinons. Such an interaction will anyway be induced once fluctuations beyond the mean field theory are considered. By including it explicitly, we can induce a spin density wave ordering beyond the mean field theory. We therefore consider it explicitly, we can induce a spin density wave ordering of the fermionic spinons. We therefore consider.

\[
H = -\sum_{\langle rr'\rangle} (T_{rr} f_{r}^\dagger f_{r'} + T_{rr'} f_{r'}^\dagger f_{r}) + \frac{N}{g} \sum_{r} \bar{S}_{r} \bar{S}_{r'}
\]

(4)

Here \(T_{rr'} = T = t + i\Delta\) on bonds as shown in Fig. 3 and equals \(T^* = t - i\Delta\) on other bonds as also shown in Fig. 3. We now treat the \(\frac{T}{g}\) term in a mean field approximation.

We look for a solution where \(\bar{S}_{r} = \epsilon_{r}N \hat{z}\) is non-zero (In mean field theory, \(\bar{N} = (\epsilon_{r} \bar{S}_{r})\) is constant. So we can choose it’s direction as \(z\) direction). Here \(\epsilon_{r} = (-1)^{x+y}\) is +1 on the A sublattice and -1 on the B sublattice. The mean field Hamiltonian reads

\[
H_{MF} = -\sum_{\langle rr'\rangle} (T f_{r}^\dagger f_{r'} + T^* f_{r'}^\dagger f_{r}) - \frac{N}{g} \sum_{r} \epsilon_{r} f_{r}^\dagger \sigma_{z} f_{r}
\]

(5)

The value of \(N\) is to be determined self-consistently. We can diagonalize this Hamiltonian using a two site unit cell as plotted in figure 3. This gives the following equation for energy eigenvalues and eigenstates:

\[
\begin{bmatrix}
-\frac{4N}{g} \bar{\sigma}_{z} & \epsilon(k) - i\Delta(k) \\
\epsilon(k) + i\Delta(k) & \frac{4N}{g} \bar{\sigma}_{z} \\
\end{bmatrix}
\begin{bmatrix}
f_{1} \\
f_{2}
\end{bmatrix}
= E_{k}\begin{bmatrix}
f_{1} \\
f_{2}
\end{bmatrix}
\]

(6)

Herex, \(f_{1}\) belongs to one sublattice and \(f_{2}\) to the other one. We have decomposed \(T\) as \(t - i\Delta\). \(\epsilon(k)\) and \(\Delta(k)\) are then defined as:

\[
\epsilon(k) = -2t(\cos(k_{y}) + \cos(k_{x}))
\]

\[
\Delta(k) = 2\Delta(\cos(k_{x}) - \cos(k_{y}))
\]

This gives spectrum of the two bands:

\[
E_{k}^{\pm} = \pm\sqrt{\left(\frac{2N}{g}\right)^{2} + \epsilon(k)^{2} + \Delta(k)^{2}}
\]

(7)

Now with this in hand we can derive the self-consistency equation for \(N\):

\[
\langle -\frac{\partial H_{MF}}{\partial (\frac{N}{g})} \rangle = \sum_{r} \langle \epsilon_{r} f_{r}^\dagger \sigma_{z} f_{r} \rangle = L^{2}N
\]

(8)

Where \(L\) is the linear system size. Using the spectral function of the lower band (we consider \(T = 0\)) this gives:

\[
gN = \frac{N}{L^{2}} \sum_{k} \frac{1}{\sqrt{\epsilon(k)^{2} + \Delta(k)^{2}}}
\]

(9)

This equation has a trivial solution \(N = 0\), but it is obvious from (7) that non-zero solution, if it exists, has lower energy. So the system has two phases, achieved by tuning the value of \(g\). The critical value, \(g_{c}\) is given by:

\[
g_{c} = \frac{1}{L^{2}} \sum_{k} \frac{1}{\sqrt{\epsilon(k)^{2} + \Delta(k)^{2}}}
\]

(10)

for \(g > g_{c}\) there is no non-zero solution and so \(N = 0\). But for \(g < g_{c}\) we have \(N \neq 0\). From (9) we can also derive the behavior of \(N\) at critical point within mean-field:

\[
N \propto \begin{cases} 
0 & \text{for } g > g_{c} \\
g - g_{c} & \text{for } g < g_{c}
\end{cases}
\]

(11)

Note that in the magnetic phase the non-zero \(N\) induces a gap to the spinons.

To study low energy properties, we will set up a continuum effective theory. In the next section we will include fluctuations in this continuum field theory. In the spin liquid state, the spectrum consists of two Fermi points, located at \(k_{x} = k_{y} = \frac{\pi}{N}\) and \(k_{x} = -k_{y} = -\frac{\pi}{N}\) (at these points \(\epsilon(k) = 0\)). In the reduced Brilliouin zone. There are gapless spinon excitations near these nodes with a Dirac-like linear dispersion. A low energy description of the spin liquid is then possible in terms of a continuum field theory of massless Dirac spinons (a brief review that helps fix notation is in appendix A). To study the magnetic transition described above within this continuum field theory, we need to introduce a ‘mean field’ that couples to the \((\pi,\pi)\) component of the physical spin density. The resulting action takes the form

\[
S_{m} = \int d^{2}x d\tau \bar{\psi}(\gamma^\mu \partial_{\mu} + i\lambda \mu^{z} \bar{\gamma} \frac{\sigma_{z}}{2})\psi
\]

(12)

In this representation, \(\psi\) consists of four two-component Dirac fields. The four Dirac fields arise from the presence of two physical spin species together with two pairs of nodes. The Pauli matrices \(\vec{\sigma}\) act on the spin index while the \(\vec{\mu}\) are Pauli matrices acting on the node index. It is readily seen that the combination \(\frac{1}{2} \bar{\psi} \mu^{z} \vec{\sigma} \psi\) is precisely the continuum form of the physical spin density near \((\pi,\pi)\). In the mean field theory \(N\) is to be determined self-consistently. The coupling \(\lambda\) is proportional.
to coupling \( \xi \) and from now on the momenta are considered with respect to the nodes. As expected a non-zero \( N \) gaps out the Dirac spinons. This gap vanishes upon approaching the phase transition to the spin liquid. The inverse of this gap determines a diverging length scale - within the mean field theory this length scale describes the decay of the connected part of the spin correlations near \((\pi, \pi)\). This may be seen by a direct calculation (described in appendix C) which gives:

\[
e^{i\tilde{Q}\cdot\tilde{r}}\langle \hat{S}_i(0)\hat{S}_j(r) \rangle_c \propto \frac{e^{-\frac{\pi}{r}}}{r^4} (1 + \frac{r}{\xi}) \delta_{ij} \quad (13)
\]

So the connected correlation for \( r \ll \xi \) is a power-law decaying function with fourth power of \( r \). For \( r \gg \xi \) it is an exponential decaying function, with correlation length \( \xi \) and a pre-factor which decays as the third power of \( r \). The correlation length at critical point diverges as:

\[
\xi \propto \frac{1}{|\lambda - \lambda_c|} \sim \frac{1}{|g - g_c|} \quad (14)
\]

### III. BEYOND MEAN FIELD THEORY

In this Section we consider the effects of fluctuations beyond the mean field theory described above. In the spin liquid phase far from the magnetic transition the crucial fluctuations are those associated with the phase of the spinon hopping parameter. These are to be thought of as gauge fluctuations associated with a (compact) \( U(1) \) gauge field that is coupled minimally to the spinons. Recent work has shown that the dRVB spin liquid is stable to such gauge fluctuations\(^{23}\) (at least within a systematic large-\( N \) expansion where \( N \) is the number of Dirac spinons). Through out this paper we will assume that this stability persists to the physically relevant case \( N = 4 \). The low energy theory of the resulting phase is described by massless QED in three space-time dimensions:

\[
S = \int d^2x d\tau \left\{ -\bar{\psi}i\gamma^\mu(\partial_\mu + ie \ a_\mu)\psi + (\epsilon_{\mu\nu\kappa} \partial_\nu a_\kappa)^2 \right\}
\]

Here \( a_\mu \) is a fluctuating \( U(1) \) gauge field which may be taken to be non-compact at the low energies. This theory flows to a conformally invariant fixed point. Various physical quantities have non-trivial power law correlations at the resultant spin liquid fixed point\(^{49}\). In particular the \((\pi, \pi)\) spin correlator decays as a power law:

\[
e^{i\tilde{Q}\cdot\tilde{r}}\langle \hat{S}(0)\hat{S}(r) \rangle_c \sim \frac{1}{r^{2\Delta}} \quad (16)
\]

The exponent \( \Delta \) is not known - a rough estimate from projected wavefunctions\(^{40,41}\) gives \( \Delta \approx 0.75 \). The dynamical spin correlations at \((\pi, \pi)\) in the scaling limit follow straightforwardly from the relativistic invariance of the field theory above. For the full zero temperature dynamical spin susceptibility, we have

\[
\chi_{SL}^{\mu\nu}(q, \omega) \sim \frac{1}{\omega^{2-\eta}} F\left(\frac{\omega}{\eta q}\right) \quad (17)
\]

Here \( \eta \) is the deviation of the wavevector from \( \tilde{Q} = (\pi, \pi) \), \( F \) is a universal scaling function, and \( \epsilon \) is a non-universal spinon velocity associated with the nodal Dirac dispersion. The exponent \( \eta \) is the anomalous dimension of the staggered spin and is related to \( \Delta \) through \( 2\Delta = 1 + \eta \). Due to the power law spin correlations, this spin liquid phase has been dubbed as "algebraic spin liquid" (ASL)\(^{23}\). Note that the spinons are not good quasiparticles at low energies in the spin liquid phase. Indeed there presumably is no quasiparticle description of the spectrum (rather like at interacting quantum critical points). Nevertheless the field theory above in terms of spinons provides a useful description of the system.

A remarkable feature of the dRVB algebraic spin liquid phase is the emergence of a huge global symmetry group characterizing the low energy fixed point. The low energy theory has an \( SU(4) \) symmetry corresponding to free unitary rotations between the four Dirac species. In addition the irrelevance of space-time monopoles at low energies implies a non-trivial global \( U(1) \) symmetry associated physically with the conservation of internal magnetic flux. Ref.\(^{49}\) studied a number of consequences of the \( SU(4) \) symmetry. In particular it showed that several other competing order parameters had the same power law correlators as the Neel vector - these include the order parameter associated with the columnar/plaquette VBS orders.

Near the transition to the antiferromagnetic state, we must treat the Neel field introduced in the previous section as a fluctuating vector \( \vec{N} \). Further upon integrating out high energy spinons (i.e. ones far away from the nodes), this \( \vec{N} \) field will develop some dynamics of its own. The resulting action takes the form

\[
S = \int d^2x d\tau \left\{ -\bar{\psi}i\gamma^\mu(\partial_\mu + ie \ a_\mu)\psi + i\lambda\bar{\psi}\mu^2 \vec{N} \cdot \vec{\sigma} \psi + (\epsilon_{\mu\nu\kappa} \partial_\nu a_\kappa)^2 + \frac{(\partial_\mu \vec{N})^2}{2} + r(\vec{N})^2 + \frac{u}{4!}(\vec{N})^2 \right\}
\]

In writing this action we have ignored anisotropies in the spinon velocities at the Dirac node and any difference between the velocities of spinon and \( N \) fields. Later we will show that all of these velocity anisotropies are irrelevant (if small) at the critical fixed point between the dRVB ASL and Neel states (see\(^{30,45}\)).

In the presence of the coupling to the \( \vec{N} \) field, the action no longer has full \( SU(4) \) global symmetry. An \( SU(2) \) subgroup - corresponding to physical spin rotations - is still obviously a symmetry. This involves an \( SU(2) \) spin rotation of the \( \psi \) field together with an \( O(3) \) rotation of the \( \vec{N} \) vector. In addition, the global transformation

\[
\psi \rightarrow e^{i\theta\mu^\nu} \psi
\]
with $\theta$ a constant is also a symmetry. This is a $U(1)$ subgroup of the full $SU(4)$ symmetry. Thus the action has a global $SU(2) \times U(1)$ symmetry apart from the $U_{flux}(1)$ associated with the gauge flux conservation. The extra $U(1)$ symmetry that survives from the full $SU(4)$ has the consequence that fermion bilinears such as $\bar{\psi} \mu_x \psi, \bar{\psi} \mu_y \psi$ can be freely rotated into another. In the original spin model, these operators transform identically to the VBS order parameter. The $U(1)$ symmetry then implies that the columnar and plaquette order parameters can be rotated into one another, and hence will have identical correlations.

Let us now study some general aspects of the two phases, near the phase transition.

A. Precursor fluctuations in the spin liquid

We will first consider the precursor fluctuations of the magnetic ordering in the spin liquid side. First consider the limit $\lambda = 0$. Then the $\vec{N}$ field decouples from the spinon-gauge sector. It is instructive to think about the spectral function for the $(\pi, \pi)$ spin correlations in the spin liquid phase in this limit. It is simply a sum of two pieces as shown in Fig. 4- a diverging power law coming from the ASL, and a sharp delta function peak coming from the $N$ fluctuations. Now consider turning on a small non-zero $\lambda$. The low frequency divergence of the spin susceptibility will be unaffected by the coupling to the fluctuating $\vec{N}$ field (this follows from the assumed stability of the ASL fixed point). However the delta function peak coming from the $\vec{N}$ field will now be broadened due to decay into two spinons. The broadening may be described within a simple RPA approximation (for details see appendix B).

In Fig 5 we plot the dynamical spin susceptibility at $(\pi, \pi)$ as the phase transition is approached (by decreasing $r$). Note that as expected, the peak coming from the $\vec{N}$ fluctuations “softens” on approaching the transition.

B. Magnetic state

When the parameter $r$ is sufficiently negative the $\vec{N}$ field will condense leading to magnetic long range order.

In the continuum field theory this may be viewed as a spin density wave state that arises out of the dRVB ASL. In this subsection we will argue that contrary to naive expectations, it is in fact a different state from a conventional Neel state - rather it is a fractionalized antiferromagnet in the same spirit as that studied in Refs. 22,33. This apparent problem will be cured once we include the effects of monopole fluctuations (ignored so far). We will show that this fractionalized antiferromagnet evolves at long length/time scales into the conventional Neel antiferromagnet.

Consider first the description of the Neel ordered state within the mean field theory developed in Section II. The mean field spectrum consists of gapped spin-1/2 spinons. It is important to realize that the spinons are merely gapped - they however have not disappeared from the spectrum. Now consider including fluctuations.
as in Section III. The important fluctuations are those associated with slow rotation of the direction of the Neel order parameter \( \vec{N} \) (spin waves) and those associated with the phase of the fermion hopping \( T \) (gauge fluctuations). These are both conveniently discussed within the continuum theory in Eqn. 18 that obtains close to the critical point. Integrating out the gapped spinons, we may obtain an effective action for the spin waves and the gauge fluctuations. To quadratic order in both the transverse component of the Neel vector \( \vec{N}_\perp \) and the gauge field \( a \), we get

\[
S_{eff} = \int d^3 x \left( \partial_\mu \vec{N}_\perp \right)^2 + \frac{g}{2} (\epsilon_{\mu\nu\kappa} \partial_\mu a_\nu a_\kappa)^2 + \ldots \quad (20)
\]

where the ellipses refer to higher order terms that are unimportant at low energies. The first term describes the expected gapless spin wave excitations. The second term describes a gapless linear dispersing “photon” associated with the gauge fluctuations. This extra gapless mode provides a sharp low energy distinction between this Neel state (as described so far) and the conventional one. The gapless photon mode is minimally coupled to the gapped spinons - the presence of gapped spinons serves as another distinction with the conventional Neel state. Thus the antiferromagnet state is to be characterized as a “fractionalized antiferromagnet” with a \( U(1) \) gauge structure. Following the notation of Ref. 33, we will dub it \( U(1) AF^* \).

Let us now include monopole fluctuations. In this magnetically ordered phase, the low energy gauge action is that of free Maxwell theory in \( 2 + 1 \) dimensions. Then standard arguments show that the monopoles are strongly relevant. Thus the \( U(1) AF^* \) state (in two dimensions) is ultimately unstable to monopole proliferation. The result is to gap out the photon mode and cause confinement of all objects that carry non-zero gauge charge. In particular it implies that the spinons (which survived as gapped excitations when monopoles were ignored) will now be confined and disappear from the spectrum. The resulting state is thus simply smoothly connected to the conventional Neel state. Thus including monopole fluctuations cause an instability of the unconventional \( U(1) AF^* \) state toward the conventional Neel state.

C. Projected wavefunctions

Before continuing we digress briefly to make contact with the large body of work on Gutzwiller projected superconducting wavefunctions (see Ref. 11 and references therein), as a route to implementing RVB ideas. Of interest to us, will be studies on projected \( d \)-wave BCS states and their variants. In the slave particle description, a useful guess for a prototypical wavefunction for a state is obtained by taking the mean field state and projecting it onto the space of physical states. At half-filling this is equivalent to doing a Gutzwiller projection on the mean field state. According to this prescription, a guess for the wavefunction of the dRVB algebraic spin liquid will simply be

\[
|dRVB >= P_G |dBCS >
\]

where \( |dBCS > \) is the mean field ground state of a \( d \)-wave superconductor at half-filling with just nearest neighbour hopping and pairing on the square lattice. Correspondingly, a guess for the wavefunction of the magnetic state, as we have obtained it, would simply be

\[
|AF >= P_G |dBCS + SDW >
\]

The preprojected state on the right simply has spin density wave order at \((\pi, \pi)\) coexisting with the \( d \)-wave superconductivity. Such wavefunctions have been studied numerically and are known to have excellent energy for the nearest neighbour Heisenberg model. From our considerations in the previous section, we would expect that this wavefunction is a prototype for a confined antiferromagnet with no finite energy spinons. Some support for this expectation comes from the work of Ref. 43 which studied the properties of a single hole in that state. The quasiparticle residue was found to be non-zero consistent with that expected in a confined antiferromagnet.

IV. PHASE TRANSITION: GENERALITIES

Let us now consider the phase transition between the dRVB ASL and the Neel state. In the limit \( \lambda = 0 \), the \( \vec{N} \) vector fluctuations are decoupled from the ASL and the magnetic transition is simply in the universality class of the usual \( O(3) \) fixed point in \( D = 3 \) space-time dimensions. Note that in this limit mean field theory predicts that the Neel order parameter vanishes with exponent \( \beta = 1/2 \) on approaching the transition. What is the effect of turning on a weak \( \lambda \) at this decoupled transition? First note that in the mean field theory of Section III we found that the Neel order parameter vanished with exponent \( \beta = 1 \) clearly different from the \( \lambda = 0 \) limit. Thus a non-zero \( \lambda \) already changes the answers within mean field theory. More generally the effects of a weak \( \lambda \) may be assessed by considering the renormalization group flow of \( \lambda \) at the decoupled fixed point. We have

\[
\frac{d\lambda}{dl} = (D - \Delta_N - \Delta) \lambda
\]

where \( D = 3 \) is the space-time dimension, \( \Delta_N \) is the scaling dimension of the \( N \) field at the \( D = 3 \) \( O(3) \) fixed point, and \( \Delta \) is the scaling dimension of the spin operator near \((\pi, \pi)\) at the dRVB ASL fixed point. Here \( l \) is the usual logarithmic renormalization scale. We have \( \Delta_N = \frac{1 + \eta}{2} \approx \frac{1}{2} \) and \( \Delta = \frac{1 + \eta}{4} \). Thus \( D - \Delta_N - \Delta \approx 2 - \frac{\eta}{2} \). With the rough estimate \( \eta \approx 0.5 \), we find that \( \lambda \) is strongly relevant at the decoupled fixed point. Thus the true critical behavior will involve strong coupling between the \( \vec{N} \) field and the spinons of the ASL. In Section.
we will study this critical behavior in a controlled $3 - \epsilon$ dimension. A very similar field theory where a fluctuating $O(3)$ vector field was coupled to massless Dirac fermions was studied many years ago by Balents et al in a different physical context. The main difference between the theory of Balents et al and the action in Eqn. 18 is the presence of the gapless gauge fields in the latter. We will see that this modifies the universality class of the transition from that in Ref. 44.

What about monopole fluctuations at this critical point? Let us first review the situation in the paramagnetic algebraic spin liquid state. Recent work has argued that when the number of Dirac species $N$ is sufficiently large (i.e. bigger than some $N_c$) the monopoles are formally irrelevant at the ASL fixed point. In the large-$N$ expansion, the monopole scaling dimension will be $o(N)$ both at the ASL fixed point and at the critical fixed point - thus at least for large enough $N$ the monopoles are irrelevant at the critical fixed point as well. In this paper we will make the crucial assumption that this irrelevance continues to hold at $N = 4$, i.e. for $SU(2)$ spin models (see figure 6).

\[ \text{FIG. 6: Renormalization flow diagram near the critical fixed point. The vertical axis is the monopole fugacity; the horizontal axis is a coupling $g$ which describes the strength of the short range part of the spinon interaction.} \]

With this assumption the monopole fugacity is irrelevant at the critical fixed point (and the paramagnetic ASL fixed point) but relevant at the ordered fixed point of the continuum field theory in Eqn. 18. In renormalization group language, the monopole fugacity is a dangerously irrelevant coupling. The length scale $\xi_\text{m}$ at which the photon gets gapped (which may loosely be dubbed the “confinement scale”) in the magnetic side may be estimated as follows. Let the monopole scaling dimension at the critical point be $\Delta_\text{m} > 3$. Upon scaling out of the critical region in the ordered state to the correlation length scale $\xi$, the monopole fugacity will renormalize to $\xi \sim \xi^{3 - \Delta_\text{m}}$. It is beyond this scale $\xi$ that the action in Eqn. 20 starts applying. In the free Maxwell theory that obtains beyond $\xi$, the monopole fugacity grows. A standard matching argument now gives $\xi_\text{m} \sim \xi^{(\Delta_\text{m} - 1)/2}$. Thus $\xi_\text{m}$ diverges faster than $\xi$. The physics on scales $\xi \ll L \ll \xi_\text{m}$ is that of the fractionalized antiferromagnet $U(1) AF^\ast$. It is only at the longest scales $L \gg \xi_\text{m}$ that the conventional Neel behavior is obtained (fig. 7).

\[ \text{FIG. 7: Crossover length scales in the magnetic state close to the transition to the spin liquid. The shorter length scale $\xi$ describes the crossover from the critical state to the fractionalized antiferromagnet. The longer scale $\xi_\text{m}$ is where this exotic antiferromagnet crosses over to the conventional Neel state through confinement. Both scales diverge near the critical point but $\xi_\text{m}$ diverges faster than $\xi$.} \]

V. $\epsilon$ Expansion for Critical Properties

In this section we will show how the structure of the critical fixed point can be studied in a formal expansion near three space dimensions. Some care is necessary in dealing with the Dirac matrices in arbitrary dimension. But here following Ref. 14 we will sidestep this issue. We will perform calculations in a perturbative expansion of the coupling constants in $d = 2$, take the traces over the Dirac matrices and then finally, in evaluating momentum integrals, set $d = 3 - \epsilon$. As a warm-up we first describe the ASL fixed point within this $\epsilon$ expansion.

A. ASL in the $\epsilon$ Expansion

Before starting the $\epsilon$ expansion studies for the full theory, we study ASL phase with this approach. To do that we consider action 18 and assume $N = 0$. Then it reduces to the usual $QED_3$ action. The flow equation for this theory is well known (and also calculated in appendix C):

\[ \beta_{\epsilon_2} = \epsilon e^2 - \frac{16}{3} \frac{\epsilon^4}{(4\pi)^2} \]

(24)

Here $\epsilon = 3 - d$. This flow equation indicates the presence of nontrivial fixed point of order $\epsilon$ at:

\[ e^{*2} = 3\pi^2 \epsilon \]

(25)

This is the pure ASL fixed point. The microscopic derivation of the continuum field theory for the spin liquid allows for a velocity anisotropy between the two spinon nodes. This anisotropy was found to be irrelevant in the large-$N$ limit of $QED_3$. Here we examine its fate within the $\epsilon$ expansion. Direct calculation (appendix C.5) shows that $\beta_{\epsilon} = -\delta \frac{14}{3(4\pi)^2} e^{*2} = -\frac{7}{8} \delta \epsilon$, where $\delta$ measures the velocity anisotropy (i.e. $\delta = 0$ corresponds to isotropic $QED_3$). So that it is irrelevant at $o(\epsilon)$. Combined with the large-$N$ result, this is strong evidence for its irrelevance at the physically relevant ASL fixed point for $N = 4$ in two space dimensions. As noted in previous papers, this irrelevance implies that the $N = 4$ ASL fixed point has global SU(4) symmetry corresponding to free unitary rotation between the four Dirac species.
Finally we can examine the scaling of gauge neutral fermion bilinears (such as the Neel vector) in the $\epsilon$-expansion. This is conveniently done by adding a source term that couples to such a bilinear, and calculating the one loop correction to the corresponding vertex (see appendix C). We find that the $\pi, \pi$ component of the spin has scaling dimension $\Delta = 3 - 1.94\epsilon$. Setting $\epsilon = 1$ gives the estimate $\Delta \approx 1.06$. This implies extremely slow decay of the corresponding correlator. This estimate may be compared with that from the $1/N$ expansion directly in $d = 2$ which gives $\Delta \approx 1.54$. Thus both expansions give slow decay for the Neel correlations that are strongly enhanced compared to the mean field results.

B. Critical fixed point

In this section we study the critical point within the $\epsilon$ expansion by including the fluctuating $\vec{N}$ field. From the action (18) we have three types of vertices and eleven different one loop diagrams. At one loop level, as derived in appendix C we get the following set of flow equations:

\[
\begin{align*}
\beta_{e^2} & = e\epsilon^2 - \frac{16}{3} \frac{e^4}{(4\pi)^2} \\
\beta_{\lambda^2} & = \epsilon \lambda^2 - \frac{5}{12} \frac{\lambda^4}{(4\pi)^2} + \frac{10}{(4\pi)^2} \epsilon^2 \lambda^2 \\
\beta_{u} & = \epsilon u - \frac{5}{3} \frac{u^2}{(4\pi)^2} + 16 \frac{\lambda^2 u}{(4\pi)^2} + 96 \frac{\lambda^4}{(4\pi)^2} \\
\beta_{v} & = (2 - \frac{5}{3} \frac{u}{(4\pi)^2} - \frac{\lambda^2}{(4\pi)^2}) v
\end{align*}
\]

Note that flow equation for electric charge is the same as usual quantum electrodynamics. In fact, gauge invariance dictates this form (25,26) (to keep the form $\epsilon a_{\mu}$ invariant under RG flow we need to have $Z_a/\sqrt{Z_a} = 1$). From these we can get the following fixed points:

\[
\begin{align*}
\epsilon^{*2} & = 0 \\
\lambda^{*2} & = \frac{8\pi^2}{5} \epsilon \\
u^{*} & = \frac{384\pi^4}{55} \epsilon
\end{align*}
\]

This fixed point describes the transition in the absence of the gauge field and was first discussed by Balents et al.24. Our calculations at this fixed point matches this previous work which therefore provides a useful check. As another check at this fixed point, if we consider our theory, with the three component $\vec{N}$ field replaced by a scalar field $\phi$, it represent a Yukawa like theory, which has been studied in Ref. 13. Our results then can be partially checked against these previous calculations. The full flow equations admit another fixed point located at:

\[
\begin{align*}
\epsilon^{*2} & = 3\pi^2 \epsilon \\
\lambda^{*2} & = \frac{23\pi^2}{5} \epsilon \\
u^{*} & = \frac{12\pi^2}{55} (-36 + \sqrt{12934}) \epsilon
\end{align*}
\]

It is readily checked that the $\epsilon^2 = 0$ fixed point is unstable towards this one. Thus the presence of the gauge field has changed the universality class of the transition. Here we should also note that the at the one loop level, calculation in appendix C shows that at this fixed point, the velocity anisotropy is irrelevant at $o(\epsilon)$. So this fixed point is also stable against small velocity anisotropy. Now using the flow equation for $r$ (Eqn. 29), it is easy to extract exponent $\nu$ for this fixed point:

\[
\frac{1}{\nu} = 2 - 4.07 \epsilon
\]

Note that simply setting $\epsilon = 1$ gives an unphysical answer. This is a signal that the leading order $\epsilon$ expansion is not quantitatively very accurate in estimating scaling dimensions in two space dimensions. Despite this the $\epsilon$ expansion is useful to describe the structure of the fixed points and the trends of the various exponents.

It is very interesting to ask about the behavior of the staggered spin correlations (i.e. near $(\pi, \pi)$) at this critical point. Naively there are two different physical operators that have the same symmetries as the staggered spin: the vector $\vec{N}$ and the fermion bilinear $\vec{N}_A \equiv \bar{\psi} \mu \vec{\sigma} \psi$. Thus in writing down an expression for the staggered spin in terms of the fields of the continuum theory, we must include both contributions:

\[e^{i\vec{Q}.\vec{A}} \tilde{S}(\vec{x}) \sim c_1 \vec{N} + c_2 \vec{N}_A + ...\]

The ellipses refer to other operators with larger scaling dimension that also have the same symmetries as the staggered spin. The coefficients $c_{1,2}$ are non-universal. Now consider the scaling of the staggered spin. If $\lambda = 0$, then the fermion bilinear and $\vec{N}$ scale independently. Near $d = 3$, and with the available estimate of the scaling dimension of the fermion bilinear at the ASL, it is readily checked that $\vec{N}$ has the lower scaling dimension. Hence the long distance decay of the staggered spin correlations will be determined by $\vec{N}$ in the $\lambda = 0$ limit. What happens when $\lambda$ is non-zero as at the non-trivial fixed point above in the $\epsilon$ expansion? It is expected that the true scaling fields will be some linear combinations of $\vec{N}$ and $\vec{N}_A$ which will have the form

\[
\begin{align*}
\bar{\phi}_1 & \sim A_1 \vec{N} + A_2 \vec{N}_A \\
\bar{\phi}_2 & \sim B_1 \vec{N} + B_2 \vec{N}_A
\end{align*}
\]
be dominated by $\bar{\phi}_1$. The coefficients $A_{1,2}$ and $B_{1,2}$ will be determined by the fixed point theory. At $o(\epsilon)$ we expect $A_1 \sim o(1), A_2 \sim o(\epsilon), B_1 \sim o(\epsilon), B_2 \sim o(1)$. Thus to obtain the scaling dimensions $\Delta_{1,2}$ to $o(\epsilon)$ we can ignore the ‘mixing’ terms $A_3, B_3$, and simply calculate the anomalous dimension of $\vec{N}$ and $N_A^z$.

The exponent $\eta$ for $N$ field (which determines the scaling dimension) is easily calculated from the field renormalization coefficient $Z_N$:

$$Z_N = 1 - \frac{8}{(4\pi)^2} \frac{\lambda^2}{\epsilon}$$

(40)

$\eta$ is then given by coefficient of $\frac{1}{\epsilon}$ in $Z_N$:

$$\eta = 2.3\epsilon$$

(41)

so that $\Delta_1 = 1 + 0.65\epsilon$. The dimension $\Delta_2$ is readily calculated as in our discussion of the ASL above. We find $\Delta_2 = 3 - 1.65\epsilon$.

It is straightforward to determine the scaling dimension of all the fermion bilinears related to $\vec{N}_A^z$ by $SU(4)$ rotations. These are listed in Table I the corresponding Feynman diagrams are in Appendix C. The absence of $SU(4)$ symmetry at the critical fixed point implies that these bilinears mostly all have different scaling dimensions. Some weak constraints follows from the $U(1)$ subgroup of the $SU(4)$ that remains unbroken. For instance it implies that $N_A^z, N_A^y$ have the same scaling dimension. As emphasized previously physically this implies identical scaling of the plaquette and columnar VBS order parameters at this critical point.

| Field theory | Spin Model | Scaling dimension |
|--------------|------------|------------------|
| $N_A^x \cdot N_A^y$ | $(\pm1)^{v+y+1} \bar{S}_r \times \bar{S}_{r+g} \cdot (\pm1)^{v} \bar{S}_x \times \bar{S}_{x+\bar{x}}$ | $3-0.5\epsilon$ |
| $N_A^z$ | $(-1)^{v+y+x} \bar{S}_r$ | $3-1.65\epsilon$ |
| $N_B$ | $(-1)^{v+y+x} [(\bar{S}_1 + \bar{S}_2)(\bar{S}_y \cdot \bar{S}_z)] + (\bar{S}_2 + \bar{S}_3)(\bar{S}_1 \cdot \bar{S}_z)$ | $3-1.65\epsilon$ |
| $N_C^y \cdot N_C^z$ | $(-1)^{v+y} \bar{S}_r \times \bar{S}_{r+y} \cdot (-1)^{v} \bar{S}_x \times \bar{S}_{x+y}$ | $3-2.8\epsilon$ |
| $N_C^x$ | $[\bar{S}_1 \cdot (\bar{S}_2 \times \bar{S}_1) - \bar{S}_2 \cdot (\bar{S}_1 \times \bar{S}_1) + \bar{S}_3 \cdot (\bar{S}_4 \times \bar{S}_3) - \bar{S}_4 \cdot (\bar{S}_3 \times \bar{S}_3)]$ | $3+0.65\epsilon$ |
| $M$ | $[\bar{S}_1 \cdot (\bar{S}_2 \times \bar{S}_1) + \bar{S}_2 \cdot (\bar{S}_3 \times \bar{S}_1) + \bar{S}_3 \cdot (\bar{S}_4 \times \bar{S}_2) + \bar{S}_4 \cdot (\bar{S}_1 \times \bar{S}_3)]$ | $3+0.65\epsilon$ |

TABLE I: List of observable in the spin model that are symmetry-equivalent to the $N^x$ and $M$ fermion bilinears. For some of these we label the sites around the plaquette with lower-left corner at $r$ by the numbers $1, \ldots, 4$. Precisely, $\bar{S}_1 = \bar{S}_r, \bar{S}_2 = \bar{S}_{r+x}, \bar{S}_3 = \bar{S}_{r+x+y}$ and $\bar{S}_4 = \bar{S}_{r+y}$.

Here we have defined the observable as:

$$N_A^c = -i\bar{\psi} \mu^\dagger \sigma^c \psi$$

(42)

$$N_B^c = -i\bar{\psi} \sigma^c \psi$$

(43)

$$N_C^c = -i\bar{\psi}^\dagger \mu^c \psi$$

(44)

$$M = -i\bar{\psi}^\dagger \psi$$

(45)

We see that $N_A^c$ corresponds to Neel vector and $iN_A^x + N_A^y$ corresponds to VBS order parameter.

C. Discussion

Now let us examine the trends shown by the exponents calculated in Table I. Note that the scaling dimension of these gauge invariant bilinears are the same in ASL phase due to the $SU(4)$ symmetry. So for all of them we have $\Delta = 3 - 1.94\epsilon$. Remarkably the scaling dimension of the VBS order parameter $(N_A^x, N_A^y)$ is smaller at the critical point than it is in the ASL phase. Thus the VBS fluctuations are enhanced by the critical $\vec{N}$ vector fluctuations. All other fermion bilinears decay faster at the critical point as their scaling dimension is increased.

This includes the vector $N_A^z$ which is the contribution from the gapless fermions to the Neel vector. It is at present not clear whether in $d = 2$ the susceptibilities of these other operators (such as $N_A^z$, etc) will diverge at the critical point (though they apparently do in the ASL phase). In contrast the VBS susceptibility will presumably diverge at the critical point. The divergence will be faster at the critical point (as say a function of temperature) than in the ASL phase. On the magnetic side the VBS susceptibility will of course be finite in the ground state. However it will diverge as the transition is approached and will thus be large if the antiferromagnet is to be regarded as being close to this critical point. Thus a qualitative conclusion from our calculations is that in the limit that the Neel state can be usefully regarded as being born out of the dRVB spin liquid, it will also have enhanced VBS susceptibility.

The diverging VBS susceptibility also provides an interesting way to define the correlation length $\xi$ in terms of directly measurable quantities. Consider the VBS correlations in the magnetic side as a function of length scale. At scales smaller than $\xi$ they will decay as a power law. However at scales larger a length set by $\xi$, they will decay exponentially. Thus $\xi$ may be usefully defined as the correlation length for VBS fluctuations in the ordered antiferromagnet.

VI. IMPLICATIONS FOR CUPRATE THEORY

We now discuss some of the implications of our results for theories of the cuprates. As we emphasized in the introduction, a second order transition between a collinear Neel state and a gapless spin liquid is attractive for a number of reasons. Here we explore this in some greater detail. Our thinking on the cuprates is guided by Fig I. We suppose that increasing magnetic frustration (the parameter $g$) at zero doping can induce a transition out of the collinear Neel state to a spin liquid. Theoretically the spin liquid is expected to evolve rather naturally into a superconductor when it is doped. The real material starts off in the antiferromagnetic state at zero doping.
The idea is that doping (apart from introducing holes) also has the effect of increasing $g$. Then we can hope that the intermediate and long scale physics of the resulting doped superconductor may be fruitfully described as a doped spin liquid. This is the rationale behind the spin liquid based approach to the cuprates.

With this point of view in mind let us consider the effects of doping the dRVB algebraic spin liquid phase discussed in this paper. Previous papers (for a review see Ref. [41]) have shown how a $d$-wave superconductor with gapless nodal quasiparticles emerges quite naturally upon doping this spin liquid. Now consider reducing the doping in the real material. According to Fig. 1 this also has the effect of reducing the magnetic frustration $g$. This pushes the “parent” spin liquid state closer to the transition to antiferromagnetism. The magnetic response of the parent spin liquid at wave vector $(\pi, \pi)$ then evolves in the manner shown in Fig. 1. Note in particular that if the transition to the magnetic state is second order then the “resonance” due to the $\vec{N}$ fluctuations softens. How does this impact the magnetic response in the doped superconductor?

The doping of the spin liquid is incorporated theoretically by the introduction of two species of charged spinless bosons $b_1$ and $b_2$. These bosons also carry gauge charges $+1$ and $-1$ respectively. Superconductivity is achieved when both $b_1$ and $b_2$ condense with equal amplitude. This route from the dRVB spin liquid to the dSC has two important features. First the gauge charge carried by the bosons implies that the gapless gauge fluctuations of the spin liquid are quenched in the superconducting state (by the Anderson Higgs mechanism). The spinons evolve naturally into the fermionic quasiparticle excitations of the dSC. The nodal structure of the spinons is retained - however coupling between the $b$ and $f$ fields moves the nodes of the quasiparticles away from $(\pi/2, \pi/2)$ by an amount proportional to the doping $x$.

For the magnetic response this has some crucial implications. First when compared with Fig. 1 the diverging low frequency response is killed as it comes entirely due to the gapless gauge fluctuations of the spin liquid. The resonance due to the triplon $\vec{N}$ mode then becomes the most prominent feature in the $(\pi, \pi)$ response. Further its frequency will soften as the doping is reduced. Second as the fermionic quasiparticles no longer have nodes at $(\pi, \pi)$ they only weakly damp out this resonance. Finally the fermionic quasiparticles will still contribute some background magnetic response which can now be usefully addressed in a standard RPA calculation. Such calculations have been reported before in the literature, and give rise to incommensurate continuum scattering at frequencies below the resonance that appear to be consistent with experiment.

Since the original discovery of the neutron resonance peak, there have been two more or less independent interpretations. One view is to describe it as a soft mode associated with the magnetism of the undoped Mott insulator. This view has the advantage that it provides a natural explanation of the softening of the resonance frequency with underdoping. The other view has been to simply regard it as a $S = 1$ collective mode of weakly correlated fermionic quasiparticles in the superconducting state. The description given above unifies these two different interpretations. Indeed in the parent spin liquid the triplon $\vec{N}$ mode may be viewed as a particle-hole triplet exciton made out of spinons - rather than electrons. This mode appears naturally as a recognizable peak in the magnetic response upon approaching the AF state. Doping this spin liquid then leads to a superconductor with gapless fermionic quasiparticles and a sharp gapped $S = 1$ triplon.

VII. CONCLUSION

In this paper we have revisited the issue of possible second order phase transitions out of the collinear Neel state into paramagnetic spin liquid states in two dimensional quantum antiferromagnets. The particular spin liquid we considered is a dRVB state which has gapless spin excitations. Correspondingly there are non-trivial power law corelations in the spin and other quantities. A useful description is provided in terms of gapless Dirac-like spinons that are coupled to a fluctuating $U(1)$ gauge field. However there is possibly no true quasiparticle description of the spectrum. Indeed this state is in a critical phase that is the two dimensional analog of the one dimensional spin-1/2 chain. In contrast to other simpler spin liquids which have a spin gap, a direct second order transition to the collinear Neel state appears to be possible for such a two dimensional algebraic spin liquid. We developed in some detail a theory for such a transition. Magnetic long range order was obtained as a spin density wave transition of spinons. We argued that gauge fluctuations convert the resulting magnetic state into a conventional one that is smoothly connected to the usual Neel state. Thus the spinons disappear from the spectrum in the magnetic state. The theory for the transition shares a number of similarities with the deconfined critical points studied recently. Most importantly, there are two diverging length/time scales as the transition is approached from the magnetic side. The shorter of the two scales is associated with the onset of magnetic order from a critical soup of spinons. The second longer scale is associated with confinement of the spinons. It is in the intermediate length/time scale regime (i.e. between the two diverging lengths) that the magnetic ordering is correctly described as a spin density wave formed out of spinons. This intermediate scale regime may also be characterized as a fractionalized antiferromagnet.

We noted several implications of our results for theories of the cuprates that regard them as doped spin liquids. First it allows us to develop a qualitative picture of the resonance mode seen in neutron experiments. Our picture unifies the existing descriptions as a soft mode associated with the magnetic ordering in the insulator and...
as a triplet excitation formed from a particle-hole pair of fermionic BCS quasiparticles. Indeed in our description the resonance is a soft mode of the magnetic ordering that is formed as a particle-hole triplet exciton of spinons. This picture is closest to that in Ref. 20.

We have shown how magnetism may be incorporated into the spin liquid based approach to the cuprates. Central to this is the description of magnetism as a spin density wave ordering of spinons. Such a description has been explored before in a number of publications. As summarized in the first paragraph of this section, our work clarifies the range of validity of such a description. Indeed should experiments reveal clear signatures for a fermionic spinon description of the intermediate scale spin physics of the undoped cuprates then we could take that to be a signature of proximity to the quantum transition to the dRVB algebraic spin liquid.

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APPENDIX A: DIRAC ACTION

Here we show that low energy effective action is described by a continuum theory of fermionic spinons with Dirac dispersion. As mentioned before, we need to expand the Hamiltonian close to nodes at \((\pi,\pi)\) and \((-\pi,\pi)\). From now on \((k_x, k_y)\) refer to deviation from \((\pi,\pi)\) or \((-\pi,\pi)\) points. Here we explicitly derive the Hamiltonian near \((\pi,\pi)\). The other node is similar:

\[
H = \begin{bmatrix}
-\frac{\alpha}{2} \vec{N} \cdot \vec{\sigma} & 2t k_+ - 2i\Delta k_- \\
2t k_+ + 2i\Delta k_- & \frac{\alpha}{2} \vec{N} \cdot \vec{\sigma}
\end{bmatrix}
\]

(A1)

where \(k_+ = k_x + k_y\) and \(k_- = k_x - k_y\). This Hamiltonian could be written in terms of two by two Pauli matrices which are defined as:

\[
\begin{align*}
\tau^x &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \tau^y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & \tau^z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{align*}
\]

Using this notation and also adding the contribution from \((-\pi,\pi)\) node we get the following form for low energy Hamiltonian:

\[
H = \sum_{k_+, k_-} c_1^\dagger \left( 2t k_+ \tau^x + 2\Delta k_- \tau^y - \frac{4}{g} \tau^z \vec{N} \cdot \vec{\sigma} \right) c_1
- c_2^\dagger \left( 2t k_- \tau^x + 2\Delta k_+ \tau^y + \frac{4}{g} \tau^z \vec{N} \cdot \vec{\sigma} \right) c_2
\]

(A2)

Here \(c_1\) and \(c_2\) refer to \((\pi,\pi)\) and \((-\pi,\pi)\) nodes, respectively. They are two component fermionic operators, each component representing one of the sites in the unit cell. The \(\tau^\dagger\) matrices operate in the space of these two components. Each component has a \(SU(2)\) spin index, where \(\sigma\) matrices operates. Now assume \(t = \Delta\) (i.e. ignore the velocity un-isotropy which is irrelevant in renormalization group language) and we rename \(k_+\) as \(k_y\) and \(k_-\) as \(k_x\). Then introduce the new fermionic operators:

\[
\psi_1 = -i\tau^x c_1 \\
\psi_2 = e^{i\pi/2} \tau^x c_2
\]

(A3)

(A4)

and subsequently:

\[
\tilde{\psi}_{1,2} = \psi_{1,2}^\dagger (i\tau^z)
\]

(A5)

With these new variables, the Hamiltonian takes the following simple form:

\[
H = \sum_{k_x, k_y} \bar{\psi}_1 (k_x \tau^x + k_y \tau^y + J\vec{N} \cdot \vec{\sigma} \pi/2) \psi_1
+ \bar{\psi}_2 (k_x \tau^x + k_y \tau^y - J\vec{N} \cdot \vec{\sigma} \pi/2) \psi_2
\]

(A6)

Then using \(\mu\) matrices that operates in the space, made by presence of two different nodes, we can put but nodes contribution \((\psi_1 \text{ and } \psi_2)\) as a single vector, \(\psi\). Then the Hamiltonian takes the form:

\[
H = \sum_{k_x, k_y} \bar{\psi}_1 (k_x \tau^x) + k_y \tau^y + iJ\mu \cdot \vec{N} \cdot \vec{\sigma} \pi/2) \psi
\]

(A7)

This form in continuum limit and in real space leads to action given in 12.

APPENDIX B: RANDOM PHASE CALCULATION

We start with the partition function with an external magnetic field \(h\) that couples to the Neel vector:

\[
Z = \int [D\psi] [D\bar{\psi}] [Da][DN] e^{-S}
\]

(B1)

\[
S = S_N + S_{\psi,a} + S_{\text{mixing}} + S_h
\]

(B2)

\[
S_N = \int d^2 x d\tau \left( \frac{1}{2} (\partial_\mu N)^2 + \frac{r}{2} N^2 \right)
\]

(B3)

\[
S_{\psi,a} = \int d^2 x d\tau \bar{\psi}_1 \left[ -i \gamma^\mu (\partial_\mu + i e a_\mu) \right] \psi_1
\]

(B4)

\[
S_{\text{mixing}} = \int d^2 x d\tau \left( i \bar{\psi}_1 \sigma^{\mu} \psi_1 (\lambda \vec{N}) \right)
\]

(B5)

\[
S_h = \int d^2 x d\tau \bar{\psi}_1 \left( a \psi_1 \sigma^{\mu} \psi_1 + b \vec{N} \right)
\]

(B6)

Here \(a\) and \(b\) are non-universal constants that depend on details of the microscopic physics.
Now by expanding the the terms which contain $i\bar{\psi}\vec{\mu}\vec{\sigma}\psi$ (one coming from $S_{mixing}$ and one from coupling to $\vec{h}$ field, up to quadratic order, and performing the integral over $\psi$ and $a$ fields we get the following form for the partition function:

$$Z = \int [DN] e^{-S_N - b \int d^2x \bar{\psi} \vec{N} \psi} \int d^2q d\omega \chi(q,\omega)|\vec{h} + \lambda \vec{N}|^2$$

(B7)

Where $\chi(q,\omega)$ is the susceptibility of $i\bar{\psi}\vec{\sigma}\vec{\mu}\psi$ operator calculated using the action $S_{\psi,a}$. We can rewrite the $\chi(q,\omega)|\vec{h} + \lambda \vec{N}|^2$ up to quadratic order in $N$ as an exponential. Putting this form in (B7) we get the following for the partition function:

$$Z = \int [DN] e^{-S^{ff}_N}$$

(B8)

$$S^{ff}_N = S_N + \int d^2q d\omega \left\{ b \bar{h}_q \vec{N} \vec{N} \psi - \frac{\chi(q,\omega)}{2} [a \bar{h}_q \lambda \vec{N} \vec{N} \psi] \right\}$$

$$= \int d^2q d\omega \left\{ \frac{\chi_N(q,\omega)}{2} |\bar{h}_q \vec{N} \vec{N} \psi|^2 + \lambda |\bar{h}_q \vec{N} \vec{N} \psi|^2 - \frac{\chi(q,\omega)}{2} |\vec{h} + \lambda \vec{N}|^2 \right\}$$

(B9)

Now to get the quadratic action for $\vec{h}$ field we should integrate out $\vec{N}$ field, which is a simple Gaussian integral now. This leads to the following value for effective susceptibility:

$$\chi_{eff} = \frac{a^2 \chi + b^2 \chi_N - 2ab \chi \chi_N}{1 - \lambda^2 \chi \chi_N}$$

(B10)

You can see that as one expects, for $\lambda = 0$, this reduces to some of susceptibilities for $\vec{N}$ and $i\bar{\psi}\vec{\mu}\vec{\sigma}\psi$ fields. Also note that, this is valid only for the frequencies where $\lambda^2 \chi \chi_N$ is not of order one (i.e. it is not valid for small frequencies). Now by analytic continuation to real frequencies and taking the imaginary part, we get the spectral functions plotted in section IIIA.

APPENDIX C: FEYNMAN DIAGRAMS

1. Spin-Spin correlation in mean-field theory

For spin-spin correlation we need to study the action 12:

$$\langle S_i S_j \rangle = 2 \frac{\lambda N r}{r^4} (1 + 2 \lambda N r)$$

(C4)

We already saw that $N$ goes to zero as $g$ approaches $g_c$ like $g - g_c$. So if we define correlation length as $\xi = \frac{r}{2\lambda N r}$ we get the equations 113 and 114.

2. Beyond mean field: $\epsilon$-expansion

Now to do the $\epsilon$-expansion we need to study the action given in 13. We can see that there are three different types of vertices present in the theory and eleven different one loop diagrams which cause field renormalizations, vertex corrections and mass renormalization. We use the following diagrammatic representation for the propagators of the fields, in the theory:

$$\frac{a}{p} = \frac{\delta^{ab}}{p^2 + r}$$

$$\frac{b}{p} = \frac{1}{p}$$

$$\frac{\mu}{p} = \frac{\delta^{\mu\nu}}{p^2}$$

(C5)

Here we have used Feynman gauge for the gauge field propagator, and Euclidian metric. Now we start to calculate the one loop diagrams.
FIG. 9: fermion self energy

(a)

(b)

There are two, one loop diagrams which will generated fermions self energy. Diagram 9(a) represents $N$ field contribution to fermion self energy:

$$\Sigma_{\psi} \delta \psi = (i\lambda)^2 \int \frac{d^4k}{(2\pi)^d} \frac{\delta^i \delta^j}{k^2(k+p)^2}$$

(C6)

Diagram 9(b) represents gauge field contribution to fermion self energy:

$$\Sigma_{\psi} \delta \psi = e^2 \int \frac{d^4k}{(2\pi)^d} \frac{\gamma^\mu \delta^\nu}{k^2(k+p)^2}$$

(C7)

Diagram 9(b) represents gauge field contribution to fermion self energy:

$$\Sigma_{\psi} \delta \psi = e^2 \int \frac{d^4k}{(2\pi)^d} \frac{\gamma^\mu \delta^\nu}{k^2(k+p)^2}$$

(b. N Field Self Energy)

Since $N$ field is coupled to fermion field only, there is just one diagram contributing to its self energy.

FIG. 10: N field self energy

This has the following contribution:

$$\Sigma_N^\psi \delta \psi = (-1)(i\lambda)^2 \int \frac{d^4k}{(2\pi)^d} \frac{tr(\gamma^\mu \rho \gamma^\nu)(k+p)}{k^2(k+p)^2}$$

(C8)

Here the minus sign appears because of presence of a fermionic loop in the diagram.

c. Gauge field self energy

Gauge field is only coupled to fermion field as well, so there is only one diagram generates gauge field self energy:

FIG. 11: Gauge Field Self Energy

This diagram has the following contribution:

$$\Sigma_N^\psi \delta \psi = (-1)e^2 \int \frac{d^4k}{(2\pi)^d} \frac{tr(k^\mu \rho \gamma^\nu)(k+p)}{k^2(k+p)^2}$$

(C9)

Note that in this relation the term proportional to $p^\mu p^\nu$ does not contribute to physical observables (S-matrix elements). This is guarantied by Ward identity.

d. u($\vec{N}$)$^2$ vertex correction

There are two diagrams contributing to this vertex renormalization at one loop level. Note that up to one loop order, it is enough to calculate vertex corrections with zero external momentum. Diagram 12(a) gives:

FIG. 12: u renormalization

$$\Delta_N^u \delta \psi \delta = \frac{u^2}{6} \int \frac{d^4k}{(2\pi)^d} \frac{\delta^i \delta^j \delta^k \delta^l}{k^2(k+p)^2}$$

(C10)
The contribution of diagram 13(b) is similarly calculated:

\[
\Delta_{\psi}^\alpha (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})
\]
\[
= -3(-1)(i\lambda)^4 \int \frac{d^d k}{(2\pi)^d} \frac{tr(k^4)}{k^8} tr(\sigma^i \sigma^j \sigma^k \sigma^l)
\]
\[
= \frac{96}{(4\pi)^2} \frac{\lambda^4}{\epsilon} \left( \delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right)
\]  

(C11)

Again one minus sign appears because of the fermionic loop.

e. N-\psi vertex correction

Again, there are two different diagrams associated with this correction at one-loop level:

As before we set the external momentums to zero. Then the contribution of diagram 13(a) is:

\[
i \Delta_{N}^\lambda \sigma^k = i(\lambda)^3 \int \frac{d^d k}{(2\pi)^d} \frac{k^k \sigma^i \sigma^k \sigma^j \sigma^l \gamma^j k^l}{k^2 + r}
\]  

(C12)

and diagram 13(b) gives:

\[
i \Delta_{N}^\lambda = i \lambda \epsilon^2 \int \frac{d^d k}{(2\pi)^d} \frac{k^k \gamma^\mu \rho_{\mu\nu} k^\nu \gamma^\nu k^l}{k^2 + r}
\]  

(C13)

f. a-\psi vertex correction

Similar to C2e, there are two diagrams:

The first diagram 14(a) gives:

\[
\Delta_{N}^\epsilon \gamma^\alpha = \epsilon^3 \int \frac{d^d k}{(2\pi)^d} \gamma^\mu \gamma^\nu \frac{k^k \gamma^\mu \rho_{\mu\nu} k^\nu \gamma^\nu k^l}{k^2 + r}
\]  

(C14)

FIG. 13: \( \lambda \) renormalization

and 14(b) gives:

\[
\Delta_{N}^\epsilon \gamma^\alpha = \epsilon^3 \int \frac{d^d k}{(2\pi)^d} \frac{k^k \gamma^\alpha \delta^{ij} k^l}{k^2 + r}
\]

\[
= \frac{3}{(4\pi)^2} \epsilon \gamma^\alpha
\]  

(C15)

Note that C14 is minus the contribution of C7 and C15 has the minus contribution of C6. This, in fact, should be the case to keep gauge invariance of the theory.

g. Mass renormalization

The last one loop diagram, we consider, generates mass renormalization:

\[
\Delta_{\epsilon} \sigma^k = \frac{u}{6} (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) \int \frac{d^d k}{(2\pi)^d} \frac{\delta^{lk}}{k^2 + r}
\]

\[
= -\frac{5}{(4\pi)^2} \frac{u}{3} \delta^{ij}
\]

(C16)

3. Renormalization conditions

Now with these in hand, we can proceed using minimal subtraction scheme. Introducing a mass scale \( m \propto \sqrt{r} \) we can write the following set of renormalization conditions:

\[
p \left( Z_\psi - (\Sigma^\psi + \Sigma^\psi) \right) = finite \ O(2 \ loops)
\]
\[
m^{-\frac{1}{2}} e_0 Z_\psi \sqrt{Z_a - (\Delta^a_\psi + \Delta^a_\psi)} = finite \ O(2 \ loops)
\]
\[
Z_N (p^2 + r) + (r \Delta_N - \Sigma^N_p) = finite \ O(2 \ loops)
\]
\[
m^{-\frac{1}{2}} \lambda_0 Z_N \sqrt{Z_N - (\Delta^N_N + \Delta^N_N)} = finite \ O(2 \ loops)
\]
\[
p^2 (Z_a - \Sigma_a) = finite \ O(2 \ loops)
\]
\[
m^{-\frac{1}{2}} u_0 Z^2_N + (\Delta^u_N + \Delta^u_N) = finite \ O(2 \ loops)
\]
Here $c_0$, $\lambda_0$ and $u_0$ are bare coupling constants and so does not flow with mass scale. These relations give field renormalization coefficients directly, since the divergence part of self energy diagrams should cancel out with these field renormalization coefficients.

\[
Z_\phi = 1 + \Sigma_\psi^N + \Sigma_\psi^\alpha \quad (C17)
\]

\[
Z_N = 1 + \Sigma_N^\psi \quad (C18)
\]

\[
Z_a = 1 + \Sigma_a^\psi \quad (C19)
\]

Putting them back in renormalization condition equations, and letting the mass scale flow, we get the equations given in section V.

4. Renormalization of bilinear operators

Here we introduce a new term in the Lagrangian with the general form:

\[
v \bar{\psi} \mathcal{O} \psi, \quad (C20)
\]

where $\mathcal{O}$ is the combination of $\mu$ and $\sigma$ matrices which are given in table I. There are two new diagrams corresponding to this new vertex [15(a), 15(b)] Diagram 15(a)

\[
\Delta_N^\mathcal{O} = (i\lambda)^2 \int \frac{d^4k}{(2\pi)^2} \sigma^j \frac{\hat{\gamma}^\mu}{k^2} \frac{\hat{\gamma}^\nu}{k^2} \frac{\delta^{ij}}{k^2} = -\frac{\lambda^2}{(4\pi)^2} \mathcal{O}, \quad \mathcal{A}_\mathcal{O} = \sigma^j \mu^\nu \mathcal{O} \hat{\gamma}^\mu \hat{\gamma}^\nu \quad (C21)
\]

and diagram 15(b) gives:

\[
\Delta_a^\mathcal{O} = e^2 \int \frac{d^4k}{(2\pi)^2} \mathcal{O} \hat{\gamma}^\mu \frac{\delta^{\mu\nu}}{k^2} \frac{\hat{\gamma}^\nu}{k^2} = e^2 \frac{6}{(4\pi)^2} \mathcal{O} \quad (C22)
\]

Whit these in hand we can get the scaling dimension of $v$:

\[
\Delta_v = 1 + \delta_v = [1 + (\Delta_N^\mathcal{O} + \Sigma_\psi^N) \lambda^2 + (\Delta_a^\mathcal{O} + \Sigma_\psi^\alpha) e^2] v
\]

\[
= [1 + (-2\Delta_\mathcal{O} - 3) \frac{\lambda^2}{(4\pi)^2} + 5 \frac{e^2}{(4\pi)^2}] v \quad (C23)
\]

This gives the scaling dimension of $v$ ($1 + \delta_v$). Now to get the scaling dimension of $\mathcal{O}$ ($\Delta_\mathcal{O}$) note:

\[
\Delta_\mathcal{O} = D - (1 + \delta_v) = 3 - \epsilon - \delta_v \quad (C24)
\]

Using the fixed point values for $\lambda$ and $e$, we get the scaling dimensions mentioned in table I.

5. Velocity anisotropy

In this section we assume a small velocity anisotropy and treat it as a perturbation to our $QED_3$ theory. Following the notation used in Ref. 30, this anisotropy is presented by:

\[
K_a = -i\delta \bar{\psi} \hat{\gamma}^\mu (\partial_\mu + ie a_\mu) \psi \quad (C25)
\]

\[
\text{FIG. 16: Velocity anisotropy vertexes}
\]

Here, $\delta$ is the small perturbation parameter (we are essential interested in its behavior under renormalization) and $\hat{\gamma}^\mu = \gamma^x \delta_{x,\mu} - \gamma^y \delta_{y,\mu}$. There are two vertexes associated with this perturbation. One is correction in fermionic kinetic energy presented in figure 16(a) and correction to the $a - \psi$ vertex presented in figure 16(b)

\[
\delta_{\mu} \hat{k} \quad (C26)
\]

\[
-\delta_{\mu} \hat{k} \quad (C27)
\]

So there are three different type of one-loop diagrams, in addition to field renormalization factors calculated before contributing renormalization of the anisotropy term in fermionic kinetic energy (i.e. $\delta$). Diagram 17(a) gives:

\[
\text{FIG. 17: Velocity anisotropy one-loop renormalization}
\]

[... ]
\[-\delta \Sigma^N_\delta \hat{\phi} = -\delta (i\lambda)^2 \int \frac{d^dk}{(2\pi)^d} \lambda^2 \frac{1}{k} \frac{1}{k} \frac{1}{k} \delta^{ij} (p-k)^2 + r \]

\[= -\frac{\delta}{(4\pi)^2} \lambda^2 \epsilon \hat{\phi} \]

(Diagram 17(b)) similarly gives:

\[-\delta \Sigma^3_\delta \hat{\phi} = -\delta \epsilon^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu \frac{1}{k} \frac{1}{k} \frac{1}{k} \delta^{\mu\nu} \gamma^\nu \]

\[= -\frac{\delta}{3(4\pi)^2} \frac{\epsilon^2}{3} \hat{\phi} \]

(Diagram 17(c) and the similar one with the correction on the right vertex give:

\[-\delta \Delta^a_\delta \hat{\phi} = \delta \epsilon^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu \frac{1}{k} \frac{1}{k} \frac{1}{k} \delta^{\mu\nu} \gamma^\nu \]

\[= \frac{\delta}{(4\pi)^2} \frac{2\epsilon^2}{\epsilon} \hat{\phi} \]

This proves that the small velocity anisotropy is irrelevant.

Now with these in and also the field renormalization coefficients calculated perviously we can get the RG flow for \(\delta\):

\[\beta_\delta = (2 \Delta^a_\delta + \Sigma^N_\delta + \Sigma^a_\delta + \Sigma^\gamma_\delta) \]

\[= -\delta \left( \frac{14}{3(4\pi)^2} \epsilon^2 + \frac{2}{(4\pi)^2} \lambda^2 \right) \]

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51 Note that if $\vec{N} = N_0 \hat{z}$, then $S^z$ is a good quantum number and can be used to label the states. The spinons have $S^z = \pm 1/2$.
52 It is at present not known what the value of $N_c$ is though existing numerical work suggests the bound $N_c < 8$. For the SU(2) magnet studied in this paper we have $N = 4$ and have simply assumed that $N_c < 4$. 