Disjoint total dominating sets in near-triangulations

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Abstract
We show that every simple planar near-triangulation with minimum degree at least three contains two disjoint total dominating sets. The class includes all simple planar triangulations other than the triangle. This affirms a conjecture of Goddard and Henning.

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coupon coloring, planar near-triangulations, total domatic number, total dominating sets

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1 | INTRODUCTION

All graphs considered in this paper are finite, simple, and undirected. A near-triangulation is a simple planar graph embedded in the plane such that all its faces except possibly the outer one are bounded by three edges. The most general form of the result we establish in the paper is the following.

Theorem 1. Let $G$ be a near-triangulation and $V'$ be a subset of vertices of $G$ containing all the vertices of degree at least three and at most two vertices of degree two. Then, there exist two disjoint subsets $V_1$ and $V_2$ of $V(G)$ such that each vertex $v \in V'$ has at least one neighbor in each of $V_1$ and $V_2$.

The above result can be stated in the language of total domination. Let $G(V, E)$ be a graph. For $S \subseteq V$, the open neighborhood $N_G(S)$ of $S$ is the set of all vertices in $G$ which have at least one neighbor in $S$. The set $S$ is called dominating if $N_G(S) \cup S = V$ and total dominating if $N_G(S) = V$. The minimum size of a dominating (resp., total dominating) set is called the domination number $\gamma(G)$ (resp., total domination number $\chi(G)$) of $G$, while the maximum...
number of pairwise disjoint dominating (total dominating) sets in $G$ is called the *domatic number* $d(G)$ (total domatic number $d_t(G)$) of $G$. A triangulated disk (resp., triangulation) is a near-triangulation in which the outer face is bounded by a simple cycle (resp., triangle). The two corollaries below follow by restricting Theorem 1 to the respective graph classes.

**Corollary 2** (Goddard and Henning [9, Conjecture 30]). If $G$ is a planar triangulation of order at least four, then $d_t(G) \geq 2$.

**Corollary 3** (Speculated in Goddard and Henning [9]). If $G$ is a triangulated disk with minimum degree at least three, then $d_t(G) \geq 2$.

Both the above results were speculated by Goddard and Henning [9] in 2018. There are several results which established the conjecture on interesting classes of triangulations. This includes Hamiltonian triangulations [16], triangulations with all odd-degree vertices, triangulations with a Hamiltonian dual [9], triangulations with at most two vertices of degree at most four, and triangulations with a 2-factor in which no cycle has length congruent to 2 modulo 4 [3]. Some of these approaches also reformulated the conjecture in various equivalent and slightly stronger ways. The interested reader is invited to check [3] for a nice catalog.

Theorem 1 is tight in two senses. First, we cannot increase the number of degree two vertices in $V'$. The 3-sun, which is a graph obtained by adding a triangle of chords to a six-cycle, does not have two disjoint dominating sets [9]. Second, the result cannot be extended to general graphs, as shown by Zelinka [22]. He observed that for any positive integer $k$, the incidence graph of the complete $k$-uniform hypergraph $H$ on $n$ vertices with $n \geq 2k - 1$ does not have two disjoint total dominating sets even though the minimum degree is high ($k$). But we do not know whether Theorem 1 can be extended to planar graphs, which are not near-triangulations. Goddard and Henning [9] had also shown that $d_t(G) \leq 4$ for every planar graph $G$ and hence conditions which ensure $d_t(G) \geq 3$ are equally interesting. Goddard and Henning [9] conjecture that every triangulation $G$ with minimum degree four has $d_t(G) \geq 3$.

There are at least two other ways in which total domatic number is studied in literature. A $k$-coloring of the vertices of a graph $G$ is called a $k$-coupon coloring if every vertex sees at least one vertex of each color in its open neighborhood [4]. This is also known as a total domatic coloring [7, 8] and a thoroughly dispersed coloring [9]. The coupon chromatic number $\chi_c(G)$ of a graph $G$ is the maximum $k$ for which $G$ has a $k$-coupon coloring. It is easy to see that $d_t(G) = \chi_c(G)$ since every color class in a coupon coloring has to be a total dominating set. A hypergraph $H$ has a $k$-rainbow coloring if there is a $k$-coloring of the vertices of $H$ such that every hyperedge contains all the $k$ colors. Given a graph $G(V, E)$, its open neighborhood hypergraph is the hypergraph $H$ on the same vertex set $V$ in which the open-neighborhood of each vertex in $G$ is a hyperedge in $H$. Hence, a $k$-coupon coloring of $G$ corresponds to a $k$-rainbow coloring of the open-neighborhood hypergraph of $G$. In particular, Corollary 3 can be seen as a guarantee that the open-neighborhood hypergraph of a triangulated disk with minimum degree at least three has a 2-rainbow coloring.

There is considerable literature on total domination in graphs [12]. The concept of domatic number and total domatic number was introduced by Cockayne et al. [5] and Cockayne and Hedetniemi [6], respectively. In [21], Zelinka obtained the characterization of $r$-regular bipartite graphs with $d_t(G) = r$. Henning and Peterin [11] provided a constructive characterization of graphs that have two disjoint total dominating sets. Chen et al. [4] shown that every $r$-regular graph has $d_t(G) \geq (1 - o(1))r/\log r$ as $r \to \infty$, and the proportion of
r-regular graphs for which \( d_r(G) \leq (1 + o(1)) r / \log r \) tends to 1 as \( V(G) \to \infty \). In [14], Koivisto et al. showed that it is NP-complete to decide whether \( d_r(G) \geq 3 \) where \( G \) is a bipartite planar graph of bounded maximum degree. The first and the last authors of this paper have studied the domatic and total domatic number of Cartesian product graphs [7, 8]. In [15], Matheson and Tarjan showed that if \( G \) is a triangulated disk, then \( d(G) \geq 3 \) and conjectured that for large enough \( n \), every triangulation on \( n \) vertices has a dominating set of size at most \( n^{4/3} \). This conjecture is still open, but there has been a recent improvement by Špacapan [19] who showed that every plane triangulation on \( n > 6 \) vertices has a dominating set of size at most \( n^{1/75} \).

1.1 Terminology and notation

Let \( G \) be a graph. The vertex-set and the edge-set of \( G \) are denoted, respectively, by \( V(G) \) and \( E(G) \). The subgraph of \( G \) induced on a set \( S \subseteq V(G) \) is denoted by \( G[S] \). The open neighborhood of a vertex \( v \) in graph \( G \) is denoted by \( N(v) \). The degree \( d(v) \) of a vertex \( v \) in \( G \) is \( |N(v)| \). A vertex of degree exactly \( k \), at least \( k \) and at most \( k \) in \( G \) are, respectively, termed \( k \)-vertex, \( k^+ \)-vertex and \( k^- \)-vertex. For a vertex \( v \) (resp., edge \( e \)) of \( G \), the subgraph of \( G \) obtained by removing \( v \) (resp., \( e \)) from \( G \) is denoted by \( G - v \) (resp., \( G - e \)). A cut in \( G \) is a partition of \( V(G) \) into two disjoint subsets. An edge of \( G \) with one endpoint in each part of the cut is said to cross the cut.

Let \( K_n \) denote the complete graph on \( n \) vertices. A diamond is a \( K_4 \) with one edge removed. The edge between the two 3-vertices of a diamond is called its diagonal. A 3-sun is a six-cycle \( (v_1, v_2, v_3, v_4, v_5, v_6) \) together with three chords forming a triangle \( (v_1, v_3, v_5) \).

In a near-triangulation \( G \), we refer to the cycle bounding the outer face as the boundary \( B(G) \) of \( G \) and the vertices and edges in \( B(G) \) as boundary vertices and boundary edges of \( G \). The remaining vertices and edges are called internal. An internal edge between two boundary vertices is called a chord.

Given a (partial) \( k \)-coloring of the vertices of \( G \), a vertex \( v \) is said to be satisfied if \( N(v) \) contains at least one vertex of each color. Given a two-coloring of some set \( X \), we call the process of swapping the color of each vertex in \( X \) as flipping the colors.

2 PROOF OF THEOREM 1

In this section, we prove the following theorem, which is a restatement of Theorem 1 in the language of vertex coloring. However, this is stronger than Corollary 3 since we handle all near-triangulations. This strengthening helps us to run a proof by induction, since near-triangulations is a family closed under deletion of vertices and edges from the boundary. On the other hand, if one restricts to triangulations (Corollary 2), then the initial observations in this section are unnecessary. We will say more about this simplification after Observation 11.

**Theorem 4.** Let \( G \) be a near-triangulation. Let \( T \) be the set of all \( 3^+ \)-vertices in \( G \) and let \( S \) be any subset of 2-vertices in \( G \) such that \( |S| \leq 2 \). There exists a two-coloring of \( V(G) \) such that each vertex \( v \in T \cup S \) sees both the colors in \( N_G(v) \).

Till we complete the proof of Theorem 4, we call near-triangulations which satisfy the conclusion of the theorem as good. Given a near-triangulation \( G \) and a subset \( S \) of at most two
2-vertices in $G$, a two-coloring which satisfies all the $3^+$-vertices and the vertices in $S$ is called a good coloring of $(G, S)$.

We begin the proof of Theorem 4 by considering a counterexample $(G, S)$ with the smallest $|V(G)| + |E(G)|$. The minimality in the choice of $G$ helps us to make the following observations.

**Observation 5.** $G$ is 2-connected.

**Proof.** If $G$ is disconnected, then each component of $G$ is smaller than $S$ and hence good. In this case, $G$ is easily seen to be good.

Suppose $G$ contains a bridge $e = uv$. Let $G_u$ (resp., $G_v$) be the component of $G\setminus e$ containing $u$ (resp., $v$). For $x \in \{u, v\}$, let $S_x = S \cap V(G_x)$. Without loss of generality we can assume that $|S_u| \leq 1$, and let $S'_v = S_v \cup \{v\}$, if $d_{G_u}(v) = 2$ and $S'_v = S_v$ otherwise. By the minimality of $G$, both $G_u$ and $G_v$ are good. If $d_G(u) > 3$ or $d_G(u) = 1$, then a good coloring of $(G_u, S_u)$ together with a good coloring of $(G_v, S'_v)$ will be a good coloring of $(G, S)$. If $d_G(u) \in \{2, 3\}$, then the above procedure will still give a good coloring of $(G, S)$ provided we flip the coloring of $G_v$ if $u$ does not see both colors in the first coloring.

Suppose $G$ contains a cut vertex $v$. We can consider $G$ as two smaller graphs $G_1$ and $G_2$ which share exactly one common vertex $v$. For $i \in \{1, 2\}$, let $S_i = S \cap V(G_i)$ and let $d_i = d_{G_i}(v)$. Since $G$ is bridgeless, $d_i \geq 2$. Without loss of generality, we can assume $|S_2| \leq 1$ and let $S'_2 = S_2 \cup \{v\}$ if $d_2 = 2$ and $S'_2 = S_2$ otherwise. A good coloring of $(G_1, S_1)$ can be combined with a good coloring of $(G_2, S'_2)$, flipping the coloring of $G_2$ if necessary to match the color of $v$ in both colorings, to obtain a good coloring of $(G, S)$. □

Since $G$ is a 2-connected near-triangulation, it is a triangulated disk. The only triangulated disks on at most 4-vertices are $K_3, K_4$ and the diamond. One can easily verify that all three of them are good. Henceforth, $G$ is a triangulated disk with at least 5-vertices.

**Observation 6.** There are no consecutive $4^+$-vertices on the boundary of $G$.

**Proof.** Let $e$ be the boundary edge between such a pair of consecutive vertices. A good coloring of $(G\setminus e, S)$, which exists by the minimality of $G$, will be also a good coloring of $(G, S)$. □

**Observation 7.** No 2-vertex in $G$ has a $3^-$-neighbor.

**Proof.** Let $v$ be a 2-vertex in $G$ with neighbors $u$ and $w$. Since $G$ is a simple triangulated disk on at least 5-vertices, all the three vertices $u$, $v$, and $w$ are on the boundary of $G$ and there exists an edge $uw$ which is internal in $G$. Hence $u$ and $w$ have a second common neighbor $x$ and $G[\{u, v, w, x\}]$ is a diamond. In particular, both $u$ and $w$ are $3^+$-vertices. Let one of them, say $w$, be a 3-vertex. Then the edge $wx$ is also a boundary edge of $G$ and hence $G\setminus \{v, w\}$ is also a near-triangulation. We can extend a good coloring of $(G\setminus \{v, w\}, S\setminus \{v\})$ to a good coloring of $(G, S)$ by giving $v$ the color different from that of $x$, and $w$ the color different from that of $u$. □

**Observation 8.** $G$ has no 2-vertices outside $S$.
Proof. Suppose $G$ has a 2-vertex $v \not\in S$ and let $u$ and $w$ be its neighbors. By Observation 7, both $u$ and $w$ are $4^+$-vertices. Hence, any good coloring of $(G \setminus \{v\}, S)$ can be extended to a good coloring of $(G, S)$ by giving any color to $v$. Notice that we do not need to satisfy $v$, since it is a 2-vertex outside $S$. □

Due to Observation 8, we do not need to specify $S$ separately anymore: $S$ is the set of all 2-vertices in the triangulated disk $G$. Moreover, any good coloring of $(G, S)$ is an ordinary 2-coupon coloring of $G$, since it satisfies all the vertices of $G$.

Observation 9. There is no chord between two 3-vertices on the boundary of $G$.

Proof. Let $uv$ be a chord between two 3-vertices $u$ and $v$ on the boundary of $G$. Since $G$ is a triangulated disk, $uv$ is the diagonal of a diamond in $G$. Since $u$ and $v$ are 3-vertices, this diamond is the entire $G$. This contradicts the fact that $G$ has at least 5-vertices. □

We summarize the properties of $(G, S)$ seen so far. The counterexample $G$ is a triangulated disk on at least five vertices with no 2-vertices outside $S$, no 3-vertex adjacent to a 2-vertex, no chord between two 3-vertices, and no consecutive $4^+$-vertices on its boundary. We color $G$ in two stages. First, we select a special independent set $I$ which dominates all the 4-vertices of $G$ and color $V(G) \setminus I$. This coloring satisfies all the vertices in $V(G) \setminus N_G(I)$. In the second stage, we color the vertices of $I$ to satisfy the vertices which were unsatisfied in the first stage. Since our choice of $I$ is motivated by the second stage of coloring, we describe the second stage immediately after the construction of $I$.

Start with $I = \emptyset$. In Round 1, for each $4^+$-vertex on the boundary $B(G)$, we add its clockwise next boundary vertex to $I$. By Observations 6 and 9, $I$ is an independent set of 3-vertices after Round 1. In Round 2, we keep adding 3-vertices from $B(G)$ to $I$ as long as it does not violate the independence of $I$. In Round 3, we keep adding $4^+$-vertices from $G$ to $I$ as long as it does not violate the independence of $I$.

Observation 10. $I$ contains all the 2-vertices and none of the $4^+$-vertices from the boundary of $G$.

Proof. By Observation 7, all the 2-vertices in $B(G)$ are added to $I$ in Round 1. By Observation 6, no $4^+$-vertices in $B(G)$ are added to $I$ in Round 1 and since they are all dominated by $I$ after Round 1, they are not added to $I$ in the two subsequent rounds. □

Observation 11. If $v \in I$, then every pair of vertices in $N_G(v)$ has a second common neighbor in $G$.

Proof. Let $v$ be a boundary vertex in $I$. By Observation 10, $v$ is 3-vertex. Suppose $v$ is a 2-vertex with neighbors $u$ and $w$. Since $G$ is a triangulated disk with at least 5-vertices, $uw$ is a chord of $G$ and hence $u$ and $w$ have a second common neighbor. Suppose $v$ is a 3-vertex in $I$ with neighbors $u$, $v'$, and $w$, where $u$ and $w$ are in $B(G)$. Since $v$ has degree exactly three, $v'$ is a common neighbor of $u$ and $w$. By Observation 7, $u$ and $w$ are 3-vertices. Hence $uv'$ and $v'w$ are internal edges of $G$ and hence $u$ and $v'$ as well as $v'$ and $w$ have common neighbors other than $v$. □
Let $v$ be an internal vertex in $I$. By construction, $v$ is a 4-vertex. Let $u$ and $w$ be two distinct vertices in $N_G(v)$. If $u$ and $w$ are nonadjacent in $G$, then $v$ is a 4-vertex, and hence both the remaining vertices in $N_G(v)$ are common neighbors of $u$ and $w$. If $uw$ is an internal edge of $G$, then since $G$ is a near-triangulation, $u$ and $w$ have a common neighbor other than $v$. Suppose $uw$ is a boundary edge of $G$, then by Observation 6, either $u$ or $w$ is a 3-vertex, say $w$. Then the neighbor of $w$ in $B(G)\{u\}$, say $x$, is also a neighbor of $v$. At least one vertex in $\{u, w, x\}$ will be included in $I$ by the end of Round 2. Hence, $v$ would not have been added to $I$.

\[\square\]

**Remark.** If $G$ is a triangulation of order at least 4, Observation 11 can be directly argued for any 4-vertex $v$, since every edge is part of two triangles. In that case, we can pick $I$ to be any maximal independent set of 4-vertices in $G$ and skip all previous observations made in this section. This would have sufficed if our aim was limited to proving Corollary 2.

Observation 11 leads us to a simple idea which helped us unlock this problem.

**Observation 12 (Key observation).** If there exists a two-coloring $f$ of $V(G)\setminus I$ such that every vertex in $V(G)\setminus N(I)$ is satisfied, then $f$ can be extended to a two-coloring of $V(G)$ which satisfies every vertex in $G$.

**Proof.** We extend $f$ to a two-coloring of $V(G)$ by coloring the vertices in $I$ arbitrarily. Suppose there is a vertex $v \in I$ such that a vertex in $N_G(v)$ is unsatisfied. Let $G_v = G\setminus \{v\}$ and $f_v$ be $f$ restricted to $V(G_v)$. Let $U \subseteq N_G(v)$ be the set of unsatisfied vertices in $N_G(v)$ under $f_v$. Note that this may contain vertices which were satisfied under $f$. If $|U| \geq 2$, by Observation 11, each pair of vertices in $U$ have a common neighbor in $G_v$ and hence miss the same color under $f_v$ (the color different from that of the common neighbor). Since this is true for every pair in $U$, all the vertices in $U$ miss the same color, say $c$, under $f_v$. If $|U| = 1$, we choose $c$ to be the color missing for the single vertex $u \in U$ under $f_v$.

Recoloring $v$ to $c$ in $f$ gives a new two-coloring $f'$ of $V(G)$, which satisfies all the vertices in $U$. The vertices in $N_G(v) \setminus U$ were satisfied under $f_v$ and hence remains satisfied in $f'$. The vertices in $V(G)\setminus N(v)$ see no change in their neighborhood. Hence, the total number of satisfied vertices goes up at least by one. We can repeat this procedure till every vertex in $V(G)$ is satisfied to get a 2-coupon coloring of $G$.

In view of Observation 12, we can focus on finding a two-coloring of $G\setminus I$ such that every vertex in $V(G)\setminus N(I)$ is satisfied. We do this by finding a special four-coloring of a supergraph $G'$ of $G\setminus I$ which properly colors a selected subset $P' \subseteq E(G')$.

Let $P$ be the set of edges in $G$ between two vertices in $N_G(v)$ for each $v \in I$. Let $G'$ be a graph obtained from $G\setminus I$ by adding any missing edge between two vertices in $N_G(v)$ for each $v \in I$ without violating planarity. Let $P'$ be the union of $P$ and the set of all newly added edges. We call the edges in $P'$ and their endpoints protected and the remaining vertices unprotected.

**Observation 13.** Every unprotected vertex in $G'$ is a 5-vertex.

**Proof.** Since $I$ is a maximal independent set of 4-vertices, every 4-vertex $v$ is in $I \cup N(I)$. If $v \in I$, then it gets deleted; and if $v \in N(I)$, then it gets protected. Hence,
every unprotected vertex $u$ in $G'$ is a $5^+$-vertex in $G$. Since $u \notin N(I)$, it remains a $5^+$-vertex in $G' \setminus I$ and in $G'$, which is a supergraph of $G \setminus I$.

**Observation 14.** If $v \in I$ is a $3^+$-vertex in $G$, then $G'[N_G(v)]$ contains a triangle, all whose edges are in $P'$. If $v \in I$ is a $2$-vertex in $G$, then the edge $G'[N_G(v)]$ is in $P'$.

**Proof.**

For every vertex $v \in I$, $N_G(v) \subseteq V(G')$. If $v$ is an internal $3$-vertex of $G$, then $N_G(v)$ induces a triangle in $G$ and hence in $G'$. If $v$ is a boundary $3$-vertex of $G$ and $N_G(v)$ induced a $2$-length path, it is completed to a triangle in $G'$ by connecting the two boundary neighbors of $v$ through the outer face of $G$. Finally, if $v$ is an internal $4$-vertex, then $N_G(v)$ contains at least one pair of nonadjacent vertices in $G$ (since $G$ is $K_5$-free). They are connected in $G'$ by an edge through the $4$-face created by deleting $v$ from $G$, resulting in at least two triangles in $G'[N_G(v)]$. The required memberships in $P'$ follow from the construction.

**Definition 15** (Weak $4$-coupon coloring). Given a graph $G$ and $P \subseteq E(G)$, a four-coloring of $V(G)$ is called a weak $4$-coupon coloring of $(G, P)$ if the endpoints of every edge in $P$ gets different colors and every vertex $v$ not incident to an edge in $P$ sees at least three colors in $N_G(v)$.

**Lemma 16.** Let $\Gamma$ be a planar graph and let $P \subseteq E(\Gamma)$ such that every $4^+$-vertex in $\Gamma$ is incident to at least one edge in $P$. Then $(\Gamma, P)$ has a weak $4$-coupon coloring.

**Proof.**

We call the edges in $P$ and their endpoints protected and the remaining vertices unprotected. Consider a cut $(A, B)$ in $\Gamma$ with a maximum number of edges crossing the parts subject to the constraint that no protected edge crosses the cut. By the maximality of the cut, every unprotected vertex $v$ has at least half of its neighbors in the opposite part. Otherwise, we can shift $v$ to the other part to get a larger cut without violating the constraint. Since all unprotected vertices in $G$ are $5^+$-vertices, they have at least three neighbors on the other side.

We color $V(\Gamma)$ by coloring $A$ and $B$ independently, starting with $A$. Pick a plane drawing of $\Gamma$. As long as there is an unprotected $B$-vertex $v$ in the drawing, remove $v$ and choose any three neighbors of $v$ from $A$ and add any missing edges between them to create a triangle in $N_{\Gamma}(v) \cap A$. This can be done without violating planarity since these edges can be added inside the new face that is created by removing $v$. Finally, remove the remaining $B$-vertices to obtain a planar graph $H_A$ on $A$. By the four-color theorem [1, 2], there exists a proper four-coloring $f_A$ of $H_A$. Repeat the same procedure to find a coloring $f_B$ of the vertices in $B$ and combine them to get a coloring $f$ of $\Gamma$. For every unprotected vertex $v \in B$ (resp., $v \in A$), there is a triangle induced in $H_A$ (resp., $H_B$) among the neighbors of $v$ in $\Gamma$. Hence, $N_{\Gamma}(v)$ sees three different colors in $f$. None of the edges in $P$ crosses the cut, and all of them are present in either $H_A$ or $H_B$. Hence, the endpoints of every protected edge get different colors. Therefore, $f$ is a weak $4$-coupon coloring of $(\Gamma, P)$.

**Remark.** The idea of using a max-cut (without any constraint) was used by Bérczi and Gábor [3] to prove that triangulations with at most two vertices of degree at most four have a 2-coupon coloring.
By Observation 13 and Lemma 16, the pair \((G', P')\) has a weak 4-coupon coloring \(f\). We construct a two-coloring of \(G\) from \(f\) as follows.

Consider \(f\) as a partial four-coloring of \(G\). Note that for each 2-vertex \(v \in I\), \(N_G(v)\) contains a protected edge (Observation 14) and hence \(v\) sees two colors in its neighborhood under \(f\). Group the four colors into two pairs so that for each 2-vertex \(v \in I\), the two colors on the neighbors of \(v\) go to different pairs. This is indeed possible since we have at most two 2-vertices in \(G\). Now merge the two colors in each pair into a single color. This gives a two-coloring \(f_2\) of \(G\setminus I\).

**Observation 17.** In the two-coloring \(f_2\), every vertex in \(V(G)\setminus N(I)\) is satisfied.

**Proof.** Recall that \(V(G)\setminus N(I)\) consists of two types of vertices—the unprotected vertices in \(G'\) and the vertices in \(I\). If \(v\) is an unprotected vertex, since \(f\) is a weak 4-coupon coloring, \(N_G(v)\) contains vertices of at least three colors under \(f\) and hence two colors under \(f_2\). If \(v\) is a 2-vertex in \(I\), then it is satisfied due to the selection of pairs of colors that are merged. If \(v\) is a 3\(^\pm\)-vertex in \(I\), then \(N_G(v)\) in \(G'\) contains a triangle, all of whose edges are protected (Observation 14). Hence, \(v\) sees at least three colors in its neighborhood under \(f\) and hence two colors under \(f_2\). \(\square\)

Observation 17 says that the \(f_2\) satisfies the premise of Observation 12 and hence we can conclude that \(f_2\) can be extended to a two-coloring which satisfies every vertex in \(G\). Hence, \(G\) is good. This completes the proof of Theorem 4 and equivalently Theorem 1.

**Remark.** It should be clear by now that the key role played by the proper four-coloring of \(H_A\) and \(H_B\) is in ensuring that no triangle is monochromatic after the merger into a two-coloring. The existence of such two-colorings can be proved without resorting to the four-color theorem (cf. [13, 20]). Perhaps the easiest way (due to Barnette) is to use a stronger version of Petersen’s theorem, which asserts that every edge of a bridgeless cubic multigraph is contained in a 1-factor [18]. Hence, we could have bypassed the use of four-color theorem if we did not have to handle the two 2-vertices. In particular, one can prove Corollary 2 without using the four-color theorem.

### 3 CONCLUDING REMARKS

Our proof of Theorem 4 lends itself to a polynomial-time algorithm. Notice that even though finding a max-cut is NP-hard for general graphs, it is polynomial-time solvable for planar graphs [10]. Four coloring of a planar graph can be done in quadratic time [17].

While we were able to affirm two of the conjectures in [9], we could not solve a tantalizing strengthening which states that the vertex set of a triangulation \(G\) with at least four vertices can be partitioned into two total dominating sets; both of which induce a bipartite subgraph of \(G\). Equivalently, there exists a proper four-coloring with color classes \(\{V_1, V_2, V_3, V_4\}\) such that both \(V_1 \cup V_2\) and \(V_3 \cup V_4\) are total dominating sets ([9, Conjecture 32]). Our method seems to be limited in power when we need a proper coloring.

Since we have settled two conjectures in this paper, we wish to restore the balance by posing two of our own. The first one stems out of the key technique in our proof and the second one comes out of our attempts to refute the original conjecture which we ended up proving.
Conjecture 1. Every near-triangulation $G$ has a four-coloring of its vertices, such that every vertex $v$ sees at least $\min\{d_G(v), 3\}$ different colors in $N_G(v)$.

Conjecture 1 will immediately give Theorem 4 via the color merger argument. Some of the earlier attempts to prove Corollary 2 on certain classes of triangulations can be modified to affirm Conjecture 1 for those classes. For example, one can see that triangulations with acyclic chromatic number at most four (this includes triangulations with all vertex degrees odd) satisfy Conjecture 1.

Conjecture 2. If $G$ is a planar graph with minimum degree at least three, then $d_t(G) \geq 2$.

A look into the coloring part in the proof of Lemma 16 will show that if $G$ is a planar graph which has a cut such that every vertex $v$ has at least three neighbors in the opposite part, then $(G, \emptyset)$ has a weak 4-coupon coloring and hence $G$ has a 2-coupon coloring. This suffices to confirm Conjecture 2 for all planar graphs with minimum degree at least five and all bipartite planar graphs.

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