The Feynman Path Integral:
An Historical Slice

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Abstract
Efforts to give an improved mathematical meaning to Feynman’s path integral formulation of quantum mechanics started soon after its introduction and continue to this day. In the present paper, one common thread of development is followed over many years, with contributions made by various authors. The present version of this line of development involves a continuous-time regularization for a general phase space path integral and provides, in the author’s opinion at least, perhaps the optimal formulation of the path integral.

The Feynman Path Integral, 1948

Much has already been written about Feynman path integrals, and, no doubt, much more will be written in the future. A comprehensive survey after more than fifty years since their introduction would be a major undertaking, and this paper is not such a survey. Rather, it is an attempt to follow one relatively narrow development regarding a special form of regularization used in the definition of path integrals. Since we deal with several different approaches, this paper does not go too deeply into any one of them; it is intended more as a conceptual overview rather than a detailed exposition.

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In order to set the stage, let us start our discussion with a review of the traditional approach to path integral construction.

We begin with the Schrödinger equation

\[ i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x) \psi(x, t) \]  

(1)

appropriate to a particle of mass \( m \) moving in a potential \( V(x) \), \( x \in \mathbb{R} \). A solution to this equation can be written as an integral,

\[ \psi(x'', t'') = \int K(x'', t''; x', t') \psi(x', t') \, dx' , \]  

(2)

which represents the wave function \( \psi(x'', t'') \) at time \( t'' \) as a linear superposition over the wave function \( \psi(x', t') \) at the initial time \( t' \), \( t' < t'' \). The integral kernel \( K(x'', t''; x', t') \) is known as the propagator, and according to Feynman [1] it may be given by

\[ K(x'', t''; x', t') = N \int e^{\frac{i}{\hbar} \int [(m/2)\dot{x}^2(t) - V(x(t))] \, dt} \, \mathcal{D}x, \]  

(3)

which is a formal expression symbolizing an integral over a suitable set of paths. This integral is supposed to run over all continuous paths \( x(t) \), \( t' \leq t \leq t'' \), where \( x(t'') = x'' \) and \( x(t') = x' \) are fixed end points for all paths. Note that the integrand involves the classical Lagrangian for the system.

Unfortunately, although highly suggestive, the preceding expression for the path integral is undefined as it stands: For example, the normalization constant \( N \) diverges, and the putative translation invariant measure \( \mathcal{D}x \) does not exist. To overcome these basic problems, Feynman adopted a lattice regularization as a procedure to yield well-defined integrals which was then followed by a limit as the lattice spacing goes to zero called the continuum limit. With \( \epsilon > 0 \) denoting the lattice spacing, the details regarding the lattice regularization procedure are given by

\[ K(x'', t''; x', t') = \lim_{\epsilon \to 0} \left( \frac{m}{2\pi i\hbar \epsilon} \right)^{(N+1)/2} \int \cdots \int \times \exp \left\{ \frac{i}{\hbar} \sum_{i=0}^{N} [(m/2\epsilon)(x_{i+1} - x_i)^2 - \epsilon V(x_i)] \right\} \Pi_{i=1}^{N} dx_i , \]  

(4)

where \( x_{N+1} = x'' \), \( x_0 = x' \), and \( \epsilon \equiv (t'' - t')/(N + 1) \), \( N \in \{1, 2, 3, \ldots \} \). In this version, at least, we have an expression that has a reasonable chance of
being well defined, provided, of course, that one interprets the conditionally convergent integrals involved in an appropriate manner. One common and fully acceptable interpretation adds a convergence factor to the exponent of the preceding integral in the form

\[-(\epsilon^2/2\hbar)\sum_{i=1}^{N} x_i^2, \tag{5}\]

which is a term that formally makes no contribution to the final result in the continuum limit save for ensuring that the integrals involved are now rendered absolutely convergent.

Accepting the fact that the integrals all converge ensures that a meaningful function of \(\epsilon\) emerges, but, by itself, that fact does not ensure convergence as \(\epsilon \to 0\) and, even if convergence holds, it furthermore does not guarantee that the result is correct! To ensure convergence to the correct result requires that some technical condition(s) must be imposed on the potential \(V(x)\). In this regard, we only observe that a correct result emerges whenever the potential has a lower bound, i.e., whenever \(V(x) \geq c\), for some \(c\), \(-\infty < c < \infty\), for all \(x\).

We recall that for the free particle of mass \(m\) the potential \(V(x) = 0\) for all \(x\). In that case, Eq. (4) reads

\[
K(x'', t''; x', t') = \lim_{\epsilon \to 0} \left(\frac{m}{2\pi i\hbar}\right)^{(N+1)/2} \int \cdots \int \\
\times \exp\left\{\frac{(m/2\epsilon)(x_{l+1} - x_l)^2}{\hbar}\right\} \Pi_{l=1}^{N} dx_l \\
= \frac{m}{2\pi i\hbar(t'' - t')} \exp\left(\frac{im(x'' - x')^2}{2\hbar(t'' - t')}\right), \tag{6}\]

which is the form of the quantum mechanical propagator for the free particle.

**Comments**

The procedure sketched above — whereby the action functional expressed as an integral over a continuous-time parameter is replaced by a natural Riemann sum approximation, which eventually is followed by a continuum limit \((\epsilon \to 0)\) as the final step — provides a satisfactory procedure for a suitable and large class of potentials to define the propagator \(K(x'', t''; x', t')\). However, it is important to stress that a lattice regularization followed by a continuum limit is by no means the only way to give a satisfactory definition
of a path integral, nor, in the author’s opinion does it even represent the most satisfactory definition, although admittedly this latter issue is in part subjective. What follows in this paper is a narrow review — an historical slice — of the development of various efforts to find a suitable continuous-time regularization procedure along with a subsequent limit to remove that regularization that ultimately should yield the correct propagator. While some of the work to be described did not follow directly from work that preceded it, all of this work does, with appropriate hindsight, seem to have a fairly natural progression that makes an interesting story.

**Feynman-Kac Formula, 1951**

Through his own research, Mark Kac was fully aware of Wiener’s theory of Brownian motion and the associated diffusion equation that describes the corresponding distribution function. Therefore, it is not surprising that he was well prepared to give a path integral expression in the sense of Feynman for an equation similar to the time-dependent Schrödinger equation save for a rotation of the time variable by $-\pi/2$ in the complex plane, namely, by the change $t \to -it$. In particular, Kac [2] considered the equation

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{\nu}{2} \frac{\partial^2 \rho(x, t)}{\partial x^2} - V(x) \rho(x, t).$$

This equation is analogous to Schrödinger’s equation but of course differs from it in certain details. Besides certain constants which are different, and the change $t \to -it$, the nature of the dependent variable function $\rho(x, t)$ is quite different from the normal quantum mechanical wave function. For one thing, if the function $\rho$ is initially real it will remain real as time proceeds. Less obvious is the fact that if $\rho(x, t) \geq 0$ for all $x$ at some time $t$, then the function will continue to be nonnegative for all time $t$. Thus we can interpret $\rho(x, t)$ more like a probability density; in fact in the special case that $V(x) = 0$, then $\rho(x, t)$ is the probability density for a Brownian particle which underlies the Wiener measure. In this regard, $\nu$ is called the diffusion constant.

The fundamental solution of Eq. (7) with $V(x) = 0$ is readily given as

$$W(x, T; y, 0) = \frac{1}{\sqrt{2\pi \nu T}} \exp \left(-\frac{(x - y)^2}{2\nu T}\right),$$

""
which describes the solution to the diffusion equation subject to the initial condition

\[ \lim_{T \to 0^+} W(x, T; y, 0) = \delta(x - y) . \]  

Moreover, it follows that the solution of the diffusion equation for a general initial condition is given by

\[ \rho(x'', t'') = \int W(x'', t''; x', t') \rho(x', t') \, dx' . \]  

Iteration of this equation \( N \) times, with \( \epsilon = (t'' - t')/(N + 1) \), leads to the equation

\[ \rho(x'', t'') = N' \int \cdots \int e^{-\left(1/2\nu\right)\sum_{l=0}^{N}(x_{l+1} - x_l)^2} \prod_{l=1}^{N} dx_l \rho(x', t') \, dx' , \]  

where \( x_{N+1} \equiv x'' \) and \( x_0 \equiv x' \). This equation features the imaginary time propagator for a free particle of unit mass as given by

\[ W(x'', t''; x', t') = N' \int \cdots \int e^{-\left(1/2\nu\right)\sum_{l=0}^{N}(x_{l+1} - x_l)^2} \prod_{l=1}^{N} dx_l . \]  

Since this equation holds for all \( N > 0 \), we may assume that it also holds in the limit \( \epsilon \to 0 \), i.e., \( N \to \infty \), which we can write formally as

\[ W(x'', t''; x', t') = \mathcal{N} \int e^{-\left(1/2\nu\right)\int \dot{x}^2 \, dt} \, Dx , \]  

where \( \mathcal{N} \) denotes a formal normalization factor. (Symbols such as \( \mathcal{N} \) may stand for different factors in different expressions.)

The similarity of this expression with the Feynman path integral [for \( V(x) = 0 \)] is clear, but there is a profound difference between these equations. In the former (Feynman) case the underlying measure is only \textit{finitely additive}, while in the latter (Wiener) case the continuum limit actually defines a genuine measure, i.e., a \textit{countably additive measure} on paths, which is a version of the justly famous Wiener measure. In particular,

\[ W(x'', t''; x', t') = \int d\mu_W(x) , \]
where $\mu_W^\nu$ denotes a measure on continuous paths $x(t)$, $t' \leq t \leq t''$, for which $x(t'') \equiv x''$ and $x(t') \equiv x'$. Such a measure is said to be a pinned Wiener measure, since it specifies its path values at two time points, i.e., at $t = t'$ and at $t = t'' > t'$. (The traditional Wiener measure, which we shall not really deal with in this paper, specifies its values only at the initial time, and it corresponds to one additional integration over the final value $x''$ at the final time $t''$.)

We note without proof that Brownian motion paths have the property that with probability one they are concentrated on continuous paths. However, it is also true that the time derivative of a Brownian path is almost nowhere defined, which means that, with probability one, $\dot{x}(t) = \pm \infty$ for all $t$.

When the potential $V(x) \neq 0$ the propagator associated with (7) is formally given by

$$W(x'', t''; x', t') = \mathcal{N} \int e^{-\int \frac{1}{2} \dot{x}^2 dt - \int V(x) dt} \mathcal{D}x,$$

an expression which is well defined if $V(x) \geq c$, $-\infty < c < \infty$. A mathematically improved expression for (15) makes use of the Wiener measure and is given by

$$W(x'', t''; x', t') = \int e^{-\int V(x(t)) dt} d\mu_W^\nu(x).$$

This is an elegant relation in that it represents a solution to the differential equation (7) in the form of an integral over Brownian motion paths suitably weighted by the potential $V$. Incidentally, since the propagator is evidently a strictly positive function, it follows that the solution of the differential equation (7) is nonnegative for all time $t$ provided it is nonnegative for any particular time value.

**Gel’fand and Yaglom, 1956**

In an important review article [3], these two authors introduced the concept of a continuous-time regularization of Feynman path integrals. Although their procedure was later shown to be incorrect, their work may be said to have initiated an interesting line of development.
The idea of Gel’fand and Yaglom was relatively straightforward. Formally stated, their proposal was to define the propagator as

$$\lim_{\nu \to \infty} \mathcal{N}_\nu \int \exp\{(i/\hbar)\int [(m/2)\dot{x}^2 - V(x)] \, dt\} \exp\{- (1/2\nu) \int \dot{x}^2 \, dt\} \, Dx. \quad (17)$$

Formally, therefore, their proposal consisted of introducing an auxiliary factor into the integrand that is identical to the factor which leads to Wiener measure. The purpose of such a factor is to introduce a regularization into the original formal expression. Finally, the limit $\nu \to \infty$ is taken as a last step which amounts to formally removing the auxiliary factor and leaving behind the original integrand of the Feynman path integral.

There is one glaring difficulty with this proposal. The auxiliary factor which leads to Brownian motion paths with a diffusion constant $\nu < \infty$ has the property of ensuring that the paths are continuous, but they are nowhere differentiable. For such paths, the contribution of the potential is well defined, but the contribution of the kinetic energy is divergent for all paths. Gel’fand and Yaglom hoped to get around this difficulty by combining the kinetic energy with the corresponding factor in the auxiliary term giving rise to a Wiener measure with a complex diffusion coefficient $\sigma$, where

$$\frac{1}{\sigma} = \frac{1}{\nu} - \frac{im}{\hbar}. \quad (18)$$

Thus the strategy was to take the limit of a sequence of presumably well defined integrals over Wiener measure with a complex diffusion constant $\sigma$ as the real part of $\sigma$ vanished. Notice that this kind of regularization does not involve the introduction of a temporal lattice followed by a continuum limit but rather maintains a continuous time parameter throughout. It is for this reason that we refer to this kind of procedure as a continuous-time regularization scheme.

Cameron, 1960

Shortly after the appearance of the Gel’fand and Yaglom paper, Cameron [4] showed that the scheme was fatally flawed. In particular, Cameron showed that a Wiener measure defined with a complex diffusion constant $\sigma$ leads to a countably additive measure only when $\sigma \equiv \text{Re}\sigma > 0$, hence $\text{Im}\sigma \equiv 0$. It
is instructive to sketch the argument that leads to this conclusion. Consider the $N$-fold integral

$$\left(\frac{\lambda}{2\pi \epsilon}\right)^{(N+1)/2} \cdot \cdot \cdot \int \exp[-(\lambda/2\epsilon) \sum_{i=0}^{N} (x_{i+1} - x_i)^2] \prod_{i=1}^{N} dx_i$$

which represents one of the primary formulas involved in discussing Wiener measures. This formula holds for any complex $\lambda$ so long as $\text{Re}\lambda > 0$. If such a formula exists in the limit that $\epsilon \to 0$ and $N \to \infty$ such that $N\epsilon$ remains constant, then the desired path measure will also exist. Although that condition would appear to be evident, it is not true for general $\lambda$.

Integrals exist when they converge absolutely. Absolute convergence certainly holds for (19) for all $N < \infty$, but what about when $N \to \infty$? To test this issue, we consider the absolute integral associated with (19) given by

$$\left(\frac{|\lambda|}{2\pi \epsilon}\right)^{(N+1)/2} \cdot \cdot \cdot \int \exp[-(\text{Re}\lambda/2\epsilon) \sum_{i=0}^{N} (x_{i+1} - x_i)^2] \prod_{i=1}^{N} dx_i$$

Evidently, the final result exists as $N \to \infty$ only provided $|\lambda| \equiv \text{Re}\lambda > 0$, i.e., provided $\text{Im}\lambda \equiv 0$, as claimed. Consequently, the proposal of Gel’fand and Yaglom fails on the ground that whenever the diffusion constant for the Wiener measure is truly complex ($\text{Im}\sigma \neq 0$), then the so-defined Wiener measure is only finitely additive and not countably additive. In this case, finite additivity means that finite answers for general integrals involving this measure arise by delicate cancellation of both positive and negative divergent contributions. A regularization scheme that results in a finitely additive measure on paths is not qualitatively different from the original formal Feynman path integral, and therefore it does not constitute an acceptable continuous time regularization.

**Itô, 1962**

Soon after Cameron pointed out the lack of a countably additive measure for the Gel’fand-Yaglom procedure, Itô [5] proposed another version of a
continuous-time regularization that resolved some of the troublesome issues. In essence, the proposal of Itô takes the form given by

\[
\lim_{\nu \to \infty} N_\nu \int \exp\{(i/\hbar)\int [\frac{1}{2}m\dot{x}^2 - V(x)] dt\} \exp\{- (1/2\nu)\int [\dddot{x}^2 + \dddot{x}^2] dt\} \mathcal{D}x .
\]

(21)

Note well the alternative form of the auxiliary factor introduced as a regulator. The additional term \(\dddot{x}^2\), the square of the second derivative of \(x\), acts to smooth out the paths sufficiently well so that in the case of (21) both \(x(t)\) and \(\dot{x}(t)\) are continuous functions, leaving \(\dddot{x}(t)\) as the term which does not exist. However, since only \(x\) and \(\dot{x}\) appear in the rest of the integrand, the indicated path integral can be well defined; this is already a positive contribution all by itself!

To proceed further with (21) we need to decide on what values to fix at the initial and final times. As was the case with the procedure proposed by Gel’fand and Yaglom, let us first suppose that a composition law of the expression in (21) holds before the limit \(\nu \to \infty\) is taken. This requires that one fix not only \(x(t'') = x''\) and \(x(t') = x'\) but, in addition, that we fix \(\dot{x}(t'') = \dot{x}''\) and \(\dot{x}(t') = \dot{x}'\). Viewed as a solution of the Schrödinger equation, however, we are at a loss to understand the meaning of the values of \(x\) and \(\dot{x}\) to be held at any time. Thus, while this interpretation may yield a limiting function it is hard to see that such a result could be a solution to Schrödinger’s equation.

In point of fact, Itô did not choose the previous set of data to be fixed but he chose another set. Itô interpreted (21) as an integral in which \(x(t'') = x''\) and \(x(t') = x'\) are the only values held fixed. He therefore did not require any composition law on the expression (21) at finite \(\nu\), but only expected a composition law to hold for the expression that arises after \(\nu \to \infty\).

With such an interpretation in mind we proceed to evaluate in a fairly heuristic manner the functional integral (21). For simplicity we shall discuss only the simple case where \(V(x) = 0\); moreover, we set \(t' = 0\), \(t'' = T\), and choose \(x(0) = 0\), \(x(T) = x\). Thus, prior to taking the limit \(\nu \to \infty\), let us focus on the formal integral

\[
N_\nu \int \exp\{(i/\hbar)\int \dot{x}g(t) dt\} \exp\{- (1/2\nu)\int [\dddot{x}^2 + a^2\dddot{x}^2] dt\} \delta(x(0)) \mathcal{D}x ,
\]

(22)

where \(a^2 \equiv 1 - im\nu/\hbar\), \(g(t)\) will be chosen below, and for the foregoing expression we assume there is no other pinning; the required second pinning
at $t'' = T$ will be introduced in a moment. Let us set

$$x(t) \equiv \int_0^t \xi(u) \, du,$$

for all $t$, so that the former integral can be replaced by

$$N_\nu \int \exp\{(i/\hbar) \int \xi g(t) \, dt\} \exp\{- (1/2\nu) \int [\dot{\xi}^2 + a^2\xi^2] \, dt\} \mathcal{D}\xi.$$

(24)

In this form, the integral involves an Ornstein-Uhlenbeck measure [6], and the answer is readily given by

$$\exp\{- (\nu/4a\hbar^2) \int g(t) g(u) e^{-a|t-u|} \, dt \, du\},$$

(25)

where we have chosen $N_\nu$ so that the answer is unity if $g(t) = 0$ for all $t$. As a next step we let $g(t) = \lambda$ for $0 \leq t \leq T$, and $g(t) = 0$ otherwise. Thus (25) becomes

$$\exp\{- (\nu\lambda^2/4a\hbar^2) \int_0^T \int_0^T e^{-a|t-u|} \, dt \, du\}. $$

(26)

Since

$$\int e^{(i/\hbar)\lambda} \int_0^T \xi(t) \, dt \, e^{-(i/\hbar)\lambda x} \, d\lambda/(2\pi\hbar)
= \delta(\int_0^T \xi(t) \, dt - x)
= \delta(x(T) - x),$$

(27)

it follows that

$$N_\nu \int \delta(x(T) - x) \, e^{-(1/2\nu) \int [\dot{\xi}^2 + a^2\xi^2] \, dt} \, D\xi
= \int e^{-\nu\lambda^2 F/4a\hbar^2} \, d\lambda/(2\pi\hbar)
= \sqrt{\frac{a}{\nu F\pi}} \exp\left(- \frac{ax^2}{\nu F}\right),$$

(28)

where

$$F \equiv \int_0^T \int_0^T e^{-a|t-u|} \, dt \, ds
= \frac{2T}{a} - \frac{2}{a^2} (1 - e^{-aT}).$$

(29)
As \( \nu \) becomes large, we may set \( a^2 \simeq -im\nu/\hbar \), in which case \( a/\nu F \) may be replaced by \( -im/2\hbar T \). Hence, we are led to the final result

\[
\sqrt{\frac{m}{2\pi i\hbar}} \exp \left( \frac{imx^2}{2\hbar T} \right),
\]

which is recognized as the quantum mechanical propagator \( K(x, T; 0, 0) \) for a free particle of mass \( m \) as given in (6).

Itô [5] made this story rigorous for constant, linear, and quadratic potentials, as well as those potentials of the form

\[
V(x) = \int e^{ixs} w(s) \, ds
\]

provided that \( \int |w(s)| \, ds < \infty \).

In summary, by introducing a regularization involving a higher derivative \((\dddot{x})^2\), Itô was able to soften the paths sufficiently to allow the kinetic energy to be well defined. In so doing he was able to give a satisfactory continuous-time regularization for (21) for a certain class of potentials \( V \).

**Feynman, 1951**

It is necessary to retrace history at this point to recall the introduction of the *phase space path integral* by Feynman [7]. In Appendix B to this article, Feynman introduced a formal expression for the configuration or \( q \)-space propagator given by

\[
K(q'', t''; q', t') = \mathcal{M} \int \exp \{ (i/\hbar) \int[p\dot{q} - H(p, q)] \, dt \} \, dp \, dq.
\]

In this equation one is instructed to integrate over all paths \( q(t), t' \leq t \leq t'' \), with \( q(t'') \equiv q'' \) and \( q(t') \equiv q' \) held fixed, as well as to integrate over all paths \( p(t), t' \leq t \leq t'' \), without restriction. As customary, this is a formal statement and in practice it needs to be given a precise meaning. The lattice prescription in which the continuous time parameter is replaced by a finite set of discrete points is the procedure that is typically followed. For completeness, we illustrate a common lattice space version of the formal phase space path integral expression, as given by

\[
K(q'', t''; q', t') = \lim_{\epsilon \to 0} \int \cdots \int \exp \{ (i/\hbar) \sum_{i=0}^{N} [\frac{1}{2}p_{i+1/2}(q_{i+1} - q_i) - \epsilon H(p_{i+1/2}, \frac{1}{2}(q_{i+1} + q_i))] \} \, \Pi_{i=0}^{N} dp_{i+1/2} / (2\pi \hbar) \, \Pi_{i=1}^{N} dq_i.
\]
In this expression, all \( p \) and \( q \) variables are integrated over except for \( q_{N+1} \equiv q'' \) and \( q_0 \equiv q' \), and, just as before, \( \epsilon = (t'' - t')/(N + 1) \). Since \( q_l \) implies a sharp \( q \) value at time \( t' + l\epsilon \), we have chosen to name the conjugate variable \( p_{l+1/2} \) to emphasize that a sharp \( p \) value must occur at a different time, here at \( t' + (l + 1/2)\epsilon \), since it is not possible to have sharp \( p \) and \( q \) values at the same time. Note that there is one more \( p \) integration than \( q \) integration in this formulation. This discrepancy becomes clear when one imposes the composition law which requires that

\[
K(q'', t''; q', t') = \int K(q''', t'''; q'', t'') \int K(p'', t'''; p', t') \, dq'' ,
\]

(34)
a relation which implies, just on dimensional grounds, that there must be one more \( p \) integration than \( q \) integration in the definition of each \( K \) expression.

Observe that (32) exhibits an apparent covariance under canonical coordinate transformations, but this is quite illusory. In fact, the lattice form (33) is correct only when the canonical variables are described by Cartesian coordinates, and this kind of limitation on straightforward lattice-space regularizations is quite general. In a later section of this paper we shall return to this point and explain why this limitation is necessary.

It is also instructive to consider an alternative phase space path integral for the momentum or \( p \)-space propagator formally given by

\[
K(p'', t''; p', t') = \mathcal{M} \int \exp\{-(i/\hbar)\int[-q\dot{p} - H(p, q)]dt\} \, dp \, dq ,
\]

(35)

which is obtained from (32) by Fourier transformation on both end variables, \( q'' \) and \( q' \). In this case a lattice space definition can be given by

\[
K(p'', t''; p', t') = \lim_{\epsilon \to 0} \int \cdots \int \exp\{(i/\hbar)\sum_{l=0}^{N}[\frac{-1}{2}q_{l+1/2}(p_{l+1} - p_l)
-\epsilon H(\frac{1}{2}(p_{l+1} + p_l), q_{l+1/2})] \} \prod_{l=1}^{N} dp_l \prod_{l=0}^{N} dq_{l+1/2} / (2\pi\hbar) ,
\]

(36)

and we see in this expression that there is one more \( q \) integration than \( p \) integration. Again this makes sense when one considers the composition law

\[
K(p'', t''; p', t') = \int K(p''', t'''; p'', t'') \int K(p'', t'''; p', t') \, dp'' .
\]

(37)
On the other hand, and following similar reasoning, one would be hard
pressed to give a satisfactory lattice space formulation of the putative formal
phase space path integral given by
\[ \mathcal{M} \int \exp\{ (i/\hbar) \int [\frac{1}{2}(p\dot{q} - q\dot{p}) - H(p,q)] \, dt \} \, \mathcal{D}p \mathcal{D}q . \] (38)
(In the light of the previous discussion, even the physical meaning of such an
expression seems uncertain, at least to the present author. On the other hand,
such an expression is easily understood in the coherent state representations
that follow.)

It is widely appreciated that the phase space path integral is more gen-
erally applicable than the original, Lagrangian, version of the path integral.
For instance, the original configuration space path integral is satisfactory for
Lagrangians of the general form
\[ L(x) = \frac{1}{2} m \dot{x}^2 + A(x) \dot{x} - V(x) , \] (39)
but it is unsuitable, for example, for the case of a relativistic particle with
the Lagrangian
\[ L(x) = -m \sqrt{1 - \dot{x}^2} \] (40)
expressed in units where the speed of light is unity. For such a system — as
well as many more general expressions — the phase space form of the path
integral is to be preferred. In particular, for the relativistic free particle, the
phase space path integral
\[ \mathcal{M} \int \exp\{ (i/\hbar) \int [p\dot{q} - \sqrt{p^2 + m^2}] \, dt \} \, \mathcal{D}p \mathcal{D}q , \] (41)
interpreted in the sense of (33), is readily evaluated and yields the correct
propagator.

Issues of proper coordinate choice also arise for the configuration space
path integral as well, and expressions such as (3) and (4) implicitly refer to
Cartesian coordinates.

**Coherent State Representations, 1960**

As a prelude to the following section it is pedagogically useful to recall some
basic properties of coherent states and the Hilbert space representations they
generate [8]. We focus on coherent states generated by Heisenberg canonical operators $P$ and $Q$ which obey the basic commutation relation $[Q, P] = i\hbar I$, where $I$ denotes the unit operator. More specifically, we shall assume the Weyl form of the commutation relation holds for self-adjoint momentum and position operators in the form

$$e^{-i\frac{p}{\hbar}Q} e^{i\frac{q}{\hbar}P} = e^{-i\frac{q}{\hbar}P} e^{i\frac{p}{\hbar}Q} \equiv U[p, q],$$

where $p$ and $q$ are arbitrary real variables. In an abstract Hilbert space, let us introduce a normalized fiducial vector $|\eta\rangle$, which is otherwise arbitrary, and define vectors of the form

$$|p, q\rangle \equiv U[p, q] |\eta\rangle$$

for all $(p, q) \in \mathbb{R}^2$. These states are the canonical coherent states, and they have a number of interesting properties. Since $U[p, q]$ is a unitary operator, it follows that each vector is normalized, i.e.,

$$\| |p, q\rangle \| \equiv \langle p, q|p,q\rangle^{1/2} = 1.$$  

Next, it should be noted that the coherent states are continuously parameterized; specifically, if $(p, q) \to (p', q')$ in the sense that $|p-p'|^2 + |q-q'|^2 \to 0$, then it follows for any choice of $|\eta\rangle$, that $\| |p, q\rangle - |p', q\rangle\| \to 0$, or as one says, the vectors are strongly continuous in the parameters $p$ and $q$. Finally, we want to stress the important property that these vectors admit a rather conventional looking resolution of unity as a superposition over one-dimensional projection operators onto the coherent states, and specifically that

$$\int |p, q\rangle \langle p, q| dpdq/(2\pi\hbar) = I$$

holds, where, as before, $I$ is the unit operator, and this relation holds for any choice of $|\eta\rangle$. It is easy to show that the resolution of unity holds weakly in the sense that, for arbitrary vectors $|\psi\rangle$ and $|\phi\rangle$,

$$\int \langle\phi|p, q\rangle \langle p, q|\psi\rangle dpdq/(2\pi\hbar) = \langle\phi|\psi\rangle.$$  

(It is less easy to show, but nevertheless true, that the resolution of unity also holds as a strong operator identity.)
The resolution of unity formula permits us to introduce novel representations of Hilbert space rather different from those customarily used, particularly in quantum mechanics. Let us introduce the functional representatives

$$\psi(p, q) \equiv \langle p, q|\psi \rangle,$$

which are defined for all $|\psi\rangle$ in the Hilbert space. As we have already done for the coherent states themselves, we suppress any dependence of the functions $\psi(p, q)$ on the state $|\eta\rangle$. Indeed, for any choice of $|\eta\rangle$, each such function is a bounded, continuous function of the variables $p$ and $q$. For each pair of such functions, such as $\psi(p, q) = \langle p, q|\psi \rangle$ and $\phi(p, q) = \langle p, q|\phi \rangle$, we assign the inner product given by

$$\langle \phi | \psi \rangle \equiv \int \phi(p, q)^* \psi(p, q) \, dpdq/(2\pi\hbar) = \langle \phi | \psi \rangle.$$  

The last part of this relation shows that the functional inner product equals the inner product in the abstract Hilbert space. The so-defined functional representation is complete simply because every element of the abstract Hilbert space is imaged as a function of the form $\psi(p, q)$. It should be noticed that the space of functions defined in this way is a complete and proper subspace of the space of square integrable functions over two variables. The space of functions $\{\psi(p, q) \equiv \langle p, q|\psi \rangle\}$ defined for all $|\psi\rangle$ and all $(p, q)$ constitutes the coherent state functional representation of a one-particle Hilbert space. In this representation, for example, the basic Heisenberg operators are given by

$$P \rightarrow -i\hbar \frac{\partial}{\partial q}, \quad Q \rightarrow q + i\hbar \frac{\partial}{\partial p},$$

and thus the coherent state representation of the Schrödinger equation is given by

$$i\hbar \frac{\partial \psi(p, q, t)}{\partial t} = \mathcal{H}(\psi(p, q) = \mathcal{H}(-i\hbar \partial/\partial q, q + i\hbar \partial/\partial p) \psi(p, q, t).$$

What makes this a one-particle problem (rather than a special kind of two-particle problem) is the proper choice of acceptable initial conditions for this equation. In particular, it suffices to take as an initial condition the coherent state overlap function $\langle p, q|p', q' \rangle$, for arbitrary $p'$ and $q'$. 

15
The solution to this form of Schrödinger’s equation can be expressed in the form
\[ \psi(p'', q'', t'') = \int K(p'', q'', t''; p', q', t') \psi(p', q', t') \, dp' \, dq' / (2\pi \hbar), \] (51)
where \( K(p'', q'', t''; p', q', t') \) denotes the propagator in the coherent state representation.

The coherent state representation differs from the usual Schrödinger representation given in (1), and it also has a different physical interpretation as well. For one thing, the meaning of \( |\psi(x)|^2 \) is the probability density to find the particle at position \( x \). The meaning of \( |\psi(p, q)|^2 \), however, is rather different; this quantity is simply the probability that the state \( |\psi\rangle \) can be found in the state \( |p, q\rangle \). The physical meaning of the variable \( x \) in the Schrödinger representation is that of a sharp position. Correspondingly, we may ask what is the physical meaning of the parameters \( p \) and \( q \)? To answer that question it is convenient to restrict the choice of the fiducial vector very slightly by imposing the conditions that
\[ \langle \eta | P | \eta \rangle = \langle \eta | Q | \eta \rangle = 0. \] (52)

implying that unlike \( x \) in the Schrödinger representation, the variables \( p \) and \( q \) represent mean values in the coherent states. It is for this reason that we can specify both values simultaneously at the same time, something that could not be done if instead they both had represented sharp eigenvalues.

The propagator for the coherent state representation of Schrödinger’s equation can also be given a formal phase space path integral form, namely
\[ K(p'', q'', t''; p', q', t') = \mathcal{M} \int \exp\{(i/\hbar) \int [p \dot{q} - H(p, q)] \, dt\} \, Dp \, Dq, \] (53)
which superficially is just the same expression as (32)! What makes these expressions different is what values are pinned and more explicitly what are the respective lattice space formulations. In the coherent state case, we have
\[ K(p'', q'', t''; p', q', t') = \lim_{\epsilon \to 0} \int \cdots \int \exp\{(i/\hbar) \sum_{i=0}^{N} \left[ \frac{1}{2} (p_{i+1} + p_{i})(q_{i+1} - q_{i}) - i \epsilon \left( p_{i+1} + p_{i} \right) + i \frac{\epsilon}{2} (q_{i+1} - q_{i}) + \frac{1}{2} (p_{i+1} - p_{i}) \right] \} \times \exp\{- (1/4\hbar) \sum_{i=0}^{N} (p_{i+1} - p_{i})^2 + (q_{i+1} - q_{i})^2 \} \right\} \prod_{i=1}^{N} dp_{i} \, dq_{i} / (2\pi \hbar). \] (54)
Observe that there are the same number of $p$ and $q$ integrations in this expression. Such a conclusion is fully in accord with the combination law as expressed in the coherent state representation, namely

$$K(p'''', q'''', t'''; p', q', t') = \int K(p''', q'''', t'''; p'', q'', t'') K(p'', q'', t''; p', q', t') \frac{dp'' dq''}{(2\pi \hbar)} . \quad (55)$$

Daubechies and Klauder, 1985

By any measure, phase space is the natural arena for both classical and quantum mechanics. In classical physics, dynamics evolves in phase space in a highly symmetric fashion for general Hamiltonians, while in quantum physics both $P$ and $Q$ appear in remarkably similar roles within the general formalism. Moreover, as stressed previously, the phase space path integral is more widely applicable than the original configuration space path integral.

In 1985, Daubechies and Klauder [9] examined the phase space path integral in a new light. In particular, they were led to consider the expression

$$\lim_{\nu \to \infty} \mathcal{M}_{\nu} \int \exp\{\frac{i}{\hbar} \int [p\dot{q} - H(p, q)] \, dt\} \exp\{-\frac{1}{2\nu} \int [\dot{p}^2 + \dot{q}^2] \, dt\} \, Dp \, Dq, \quad (56)$$

which extends the continuous-time regularization procedure to phase space path integrals. Let us initially look at this expression from a general viewpoint. Evidently we can interpret the regularization as two independent Brownian motion regularizations in the limit as the diffusion constant diverges, which is rather like the original Gel’fand-Yaglom procedure. Alternatively, and since from a classical point of view $p$ is often rather like $\dot{q}$ (depending on $H$, of course), this procedure has some features superficially in common with that of Itô. However, there are also important differences with those previous procedures as well. The chosen regularization ensures that both $p$ and $q$ are continuous functions but that $\dot{p}$ and $\dot{q}$ are almost nowhere defined. The derivatives enter the integrand only in the form $\int p\dot{q} dt$. However, this term can be interpreted as $\int p dq$, which for Brownian motion paths is a standard stochastic integral that has been well studied in both its Itô and Stratonovich versions; for reasons to be given below, when it becomes important to choose a rule we shall choose the Stratonovich rule. We interpret the integration in (56) to be pinned for both $p$ and $q$ at both end points.
specifically, \((p(t''), q(t'')) = (p'', q'')\) and \((p(t'), q(t')) = (p', q')\). In this case it is clear that the result of (56) yields a function of the form

\[ K(p'', q'', t''; p', q', t') . \]  

(57)

By the time of 1985, representations of propagators for one-particle quantum systems in the form of coherent state representations, such as those described in an earlier section, were well known to many researchers. If, indeed, (56) turned out to yield coherent state representations for solutions to the Schrödinger equation — as opposed, say, to either the \(q\)-space or \(p\)-space representations — there would be no reason for concern. In fact, that is exactly what happens!

In [9] it was rigorously established that

\[
\lim_{\nu \to \infty} \mathcal{M}_\nu \int e^{(i/\hbar)\int [p\dot{q} - H(p,q)] dt} e^{-(1/2\nu)\int [\dot{p}^2 + \dot{q}^2] dt} \mathcal{D}p \mathcal{D}q
= \lim_{\nu \to \infty} 2\pi \hbar e^{\nu T/2} \int e^{(i/\hbar)\int [p\dot{q} - H(p,q)] dt} d\mu_\nu(p,q)
\equiv \langle p'', q'' | e^{-i(t'' - t')\mathcal{H}/\hbar} | p', q' \rangle
\equiv K(p'', q'', t''; p', q', t'),
\]  

(58)

where the second line of (58) is a mathematically rigorous formulation of the heuristic and formal first line, and \(\mu_\nu\) denotes pinned Wiener measure. In fact, the connection indicated here is far reaching in that the choice of regularization also determines that

\[
|p, q\rangle \equiv e^{-i\frac{pQ}{\hbar}} e^{i\frac{pQ}{\hbar}} |0\rangle , \quad [Q, P] = i\hbar I , \quad (Q + iP) |0\rangle = 0 ,
\]  

(59) 

\[
\mathcal{H} = \int H(p,q) |p,q\rangle \langle p,q| \ dp \ dq/(2\pi \hbar) .
\]  

(60)

A sufficient set of technical assumptions ensuring the validity of this representation is given by

\begin{itemize}
  \item[a)] \( \int H(p, q)^2 e^{-\alpha(p^2 + q^2)} \ dp \ dq < \infty , \) for all \( \alpha > 0 \),
  \item[b)] \( \int H(p, q)^4 e^{-\beta(p^2 + q^2)} \ dp \ dq < \infty , \) for some \( \beta < 1/2\hbar \),
  \item[c)] \( \mathcal{H} \) is essentially self adjoint on the span of finitely many number eigenstates .
\end{itemize}

(62) 

(63) 

(64)
A wide class of Hamiltonians — but by no means all acceptable such operators — consists of all semibounded, symmetric (Hermitian) polynomials of $P$ and $Q$. We also note that the connection between the operator $H$ and its symbol $H(p, q)$ given in (61) is that of anti-normal ordering. In particular, this means that

$$e^{-(r-is)(Q+iP)/2\hbar}e^{(r+is)(Q-iP)/2\hbar} = \int e^{i(sq-rp)/\hbar} |p, q\rangle \langle p, q| dpdq/(2\pi\hbar),$$

(65)

and any other connection follows by suitable linear combinations. As an alternative association we also note that

$$H(p, q) = e^{-(h/4)(\partial_p^2 + \partial_q^2)} H_W(p, q),$$

(66)

where $H_W(p, q)$ denotes the well-known Weyl symbol of the operator $H$. [In most of the author’s earlier work, the particular symbol $H(p, q)$ used here has been denoted by the lower case symbol $h(p, q)$.

Observe that the auxiliary factor in (56) that provides the regularization involves a metric, specifically, $d\sigma = d\sigma(p, q) = dp^2 + dq^2$, on classical phase space. Such a metric characterizes a flat, two-dimensional phase space expressed in Cartesian coordinates.

It is interesting to consider the transformation of (58) under a canonical change of phase space coordinates. To this end we introduce $\overline{p} = \overline{p}(p, q)$ and $\overline{q} = \overline{q}(p, q)$ as two new classical canonical coordinates that are related to the original canonical coordinates by the relation

$$\overline{p}d\overline{q} = pdq + dF(\overline{q}, q),$$

(67)

where $F$ is known as the generator of the transformation. This is a standard form for this kind of relation in which $\overline{q}$ and $q$ are regarded as the independent variables; another version that we will find more useful is given by

$$\overline{p}d\overline{q} + dG(\overline{p}, \overline{q}) = pdq.$$  

(68)

Under such a canonical coordinate transformation, the metric $d\sigma^2 = dp^2 + dq^2$ assumes the form

$$d\sigma^2 = d\sigma(\overline{p}, \overline{q})^2 = A(\overline{p}, \overline{q}) d\overline{p}^2 + 2B(\overline{p}, \overline{q}) d\overline{p} d\overline{q} + C(\overline{p}, \overline{q}) d\overline{q}^2,$$ 

(69)
for suitable \( A, B, \) and \( C \), but, nevertheless, the metric still characterizes
the same flat, two-dimensional phase space even though it is now expressed,
generally speaking, in curvilinear canonical coordinates.

In the original Cartesian coordinates the stochastic integral \( \int p\,dq \) evaluated
by the Stratonovich rule is identical to its evaluation using the Itô rule. Under coordinate transformations, we adopt the Stratonovich definition
[mid-point rule, cf.(54)] since for this rule the differential and integral
laws of ordinary calculus still apply. Consequently, the differential rules (67)
and (68) which apply for classical functions (say \( C^2 \)), apply equally well
to Brownian paths (which are only \( C^0 \)). Recognizing that the Hamiltonian
function transforms as a scalar, we see that

\[
H(p, q) \equiv H(p(\bar{p}, \bar{q}), q(\bar{p}, \bar{q})) = H(p, q) .
\]  

\( (70) \)

Assembling the separate parts, we learn that under a canonical coordinate
transformation, (58) becomes

\[
\lim_{\nu \to \infty} M_\nu \int \frac{e^{(i/h)\int[\overline{\mathcal{G}(\bar{\mathbf{p}}, \bar{\mathbf{q}})] - \overline{\mathcal{H}(\bar{\mathbf{p}}, \bar{\mathbf{q}})]}} dt}{\nu^{(1/2\nu)}\int[ds(\bar{\mathbf{p}}, \bar{\mathbf{q}})]^2 / d\nu^2} \, D\bar{\mathbf{p}} D\bar{\mathbf{q}}
= \lim_{\nu \to \infty} 2\pi h e^{e^{T/2h}} \int \frac{e^{(i/h)\int[\overline{\mathcal{G}(\bar{\mathbf{p}}, \bar{\mathbf{q}})] - \overline{\mathcal{H}(\bar{\mathbf{p}}, \bar{\mathbf{q}})]}} dt}{\nu^{(1/2\nu)}\int[ds(\bar{\mathbf{p}}, \bar{\mathbf{q}})]^2 / d\nu^2} \, d\mathbf{p}_W(\bar{\mathbf{p}}, \bar{\mathbf{q}})
\equiv \langle \bar{\mathbf{p}}', \bar{\mathbf{q}}' | e^{-i(t''-t')H/h} | \bar{\mathbf{p}}, \bar{\mathbf{q}} \rangle
\equiv \overline{K}(\bar{\mathbf{p}}', \bar{\mathbf{q}}', t'', \bar{\mathbf{p}}, \bar{\mathbf{q}}, t') ,
\]  

\( (71) \)

where \( \mathbf{p}_W \) denotes pinned Wiener measure on a two-dimensional flat phase
space expressed in curvilinear coordinates. The term

\[
\int d\mathcal{G}(\bar{\mathbf{p}}, \bar{\mathbf{q}}) = \overline{\mathcal{G}(\bar{\mathbf{p}}', \bar{\mathbf{q}}')} - \overline{\mathcal{G}(\bar{\mathbf{p}}, \bar{\mathbf{q}})} ,
\]  

\( (72) \)

and thus its presence amounts to no more than a phase change of the corre-
sponding coherent states. Specifically,

\[
|\bar{\mathbf{p}}, \bar{\mathbf{q}}\rangle \equiv e^{-\overline{\mathcal{G}(\bar{\mathbf{p}}, \bar{\mathbf{q}})/h}} e^{-ip\bar{\mathbf{p}} - iq\bar{\mathbf{q}}/h} e^{ip\bar{\mathbf{p}} - iq\bar{\mathbf{q}}/h} |0\rangle = e^{-\overline{\mathcal{G}(\bar{\mathbf{p}}, \bar{\mathbf{q}})/h}} |p, q\rangle ,
\]  

\( (73) \)

\[
[Q, P] = ihI , \quad (Q + iP) |0\rangle = 0 ,
\]  

\( (74) \)

\[
\mathcal{H} = \int \overline{\mathcal{H}(\bar{\mathbf{p}}, \bar{\mathbf{q}})} |\bar{\mathbf{p}}, \bar{\mathbf{q}}\rangle \langle \bar{\mathbf{p}}, \bar{\mathbf{q}}| \, d\bar{\mathbf{p}} d\bar{\mathbf{q}}/(2\pi h) .
\]  

\( (75) \)

Under a canonical coordinate transformation, therefore, and apart from a possible phase change, observe that the coherent state vectors are unchanged;
it is only their *labels* that have changed. Observe further that the Hamiltonian operator $\mathcal{H}$ is completely unchanged; only the details of its representation are different in the new coordinate system.

In summary, we see that a continuous-time, Brownian motion regularization of the phase space path integral can be rigorously established. It applies to a wide class of Hamiltonians, and the formulation is fully covariant under general canonical coordinate transformations. The only additional ingredient to attain this covariance is the introduction of a *metric*, $d\sigma^2$, on classical phase space which is ultimately used to support the Wiener measure regularization. As a general rule, it is noteworthy that quantization schemes that agree with orthodox quantum mechanics — and therefore agree with a vast number of experiments — do *not* exhibit manifest covariance under canonical coordinate transformations. Hence, the fact that the present procedure exhibits covariance under canonical coordinate transformations is especially to be welcomed.

As a final comment, we remind the reader of the general requirement that canonical quantization, i.e., the choice of certain phase space variables to be “promoted” to operators, should be carried out in Cartesian coordinates [10]. This rule has caused much concern over the years and its elimination has been, partially at least, a motivating factor in certain alternative quantization schemes. Instead of seeking to eliminate the Cartesian coordinates, we fully accept them, and, indeed, we understand the need for them as follows [11]: With only a symplectic form on phase space — and in the absence of any phase space metric — there is no self-referential mechanism to associate the correct *physical meaning* of any given *mathematical expression*, say, for the Hamiltonian $H(p,q)$. In particular, if $H(p,q) = p^2/2$, and in the absence of any further information, how is one to determine that this particular expression is the Hamiltonian for a free particle, or for an harmonic oscillator, or for some particular anharmonic oscillator, etc., any one of which can be brought to that mathematical form by a suitable canonical coordinate transformation. It is with the addition of a flat-space phase-space metric, and with due regard to its coordinatized expression, that the physical significance of a given mathematical expression can be determined, and with that determination, the quantization procedure can lead to the correct energy spectrum for the physical system under consideration. The phase space continuous-time regularization scheme discussed in this section offers the great advantage that the introduction of the metric into the formal phase space path integral si-
multaneously gives mathematical and physical meaning to the path integral by assuring that the proper interpretation of the Hamiltonian is maintained throughout the quantization procedure. In point of fact, one may say that the procedure we have illustrated offers a true geometric quantization procedure, and consequently, the entire procedure can therefore be expressed in a coordinate free form. That is indeed the case, and the interested reader is referred to [12] for further discussion of this point.

Final remarks

A number of further developments and applications regarding continuous-time regularizations have appeared in recent years. We do not enter into any details of these works here, but rather list just a few additional papers for further study: [13, 14, 15].

Dedication

It is a distinct pleasure to dedicate this paper to my long time friend and colleague, Hiroshi Ezawa. I have known Hiroshi for approximately forty years and I have profited greatly from the many times we have been together and during which we have often worked closely on a variety of topics of mutual interest. I wish him many more years of good health and productive research.

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