THE GELFAND–PHILLIPS PROPERTY FOR LOCALL Y CONVEX SPACES

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Abstract. We extend the well-known Gelfand–Phillips property for Banach spaces to locally convex spaces, defining a locally convex space $E$ to be Gelfand–Phillips if every limited set in $E$ is precompact in the topology on $E$ defined by barrels. Several characterizations of Gelfand–Phillips spaces are given. The problem of preservation of the Gelfand-Phillips property by standard operations over locally convex spaces is considered. Also we explore the Gelfand–Phillips property in spaces $C(X)$ of continuous functions on a Tychonoff space $X$. If $\tau$ and $T$ are two locally convex topologies on $C(X)$ such that $T_p \subseteq \tau \subseteq T \subseteq T_k$, where $T_p$ is the topology of pointwise convergence and $T_k$ is the compact-open topology on $C(X)$, then the Gelfand–Phillips property of the function space $(C(X), \tau)$ implies the Gelfand–Phillips property of $(C(X), T)$. If additionally $X$ is metrizable, then the function space $(C(X), T)$ is Gelfand–Phillips.

1. Introduction

All locally convex spaces are assumed to be Hausdorff, infinite-dimensional and over the field $\mathbb{F}$ of real or complex numbers, and all topological spaces are assumed to be infinite and Tychonoff. We denote by $E'$ the topological dual of an lcs $E$. The dual space $E'$ of $E$ endowed with the weak* topology $\sigma(E', E)$ and the strong topology $\beta(E', E)$ is denoted by $E'_{w*}$ and $E'_{\beta}$, respectively. For a bounded subset $B \subseteq E$ and a functional $\chi \in E'$, we put

$$\|\chi\|_B := \sup \{ |\chi(x)| : x \in B \cup \{0\} \}.$$

Let $E$ be a Banach space. The closed unit ball of $E$ is denoted by $B_E$. A bounded subset $B$ of $E$ is called limited if each weak* null sequence $\{\chi_n\}_{n \in \omega}$ in $E'$ converges uniformly on $B$, that is $\lim_{n \to \infty} \|\chi_n\|_B = 0$. A Banach space $E$ is said to have the Gelfand–Phillips property ((GP) property for short) or is a Gelfand–Phillips space if every limited set in $E$ is precompact.

The classical result of Gelfand [16] states that every separable Banach space is Gelfand–Phillips. On the other hand, Phillips [19] showed that the non-separable Banach space $\ell_\infty = C(\beta\omega)$ is not Gelfand–Phillips, where $\beta\omega$ is the Stone–Čech compactification of the discrete space $\omega$ of nonnegative integers.

Following Bourgain and Diestel [5], a bounded linear operator $T$ from a Banach space $L$ into $E$ is called limited if $T(B_L)$ is a limited subset of $E$. In [12], Drewnowski noticed the next characterization of Banach spaces with the (GP) property.

**Theorem 1.1.** For a Banach space $E$ the following assertions are equivalent:

1. $E$ is Gelfand–Phillips;
2. every limited weakly null sequence in $E$ is norm null;
3. every limited operator with range in $E$ is compact.

This characterization of Gelfand–Phillips Banach spaces plays a crucial role in many arguments for establishing the (GP) property in Banach spaces. The Gelfand–Phillips property was intensively studied in particular in [4, 12, 13, 21]. It follows from results of Schlumprecht [20, 21] that the (GP) property is not a three space property (see also Theorem 6.8.h in [6]). In our recent paper [2], we give several new characterizations of Gelfand–Phillips Banach spaces.

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Another direction for studying the Gelfand–Phillips property is to characterize Gelfand–Phillips spaces that belong to some important classes of Banach spaces. In the next proposition, whose short proof is given in Corollary 2.2 of [2], we provide some of the most important and general results in this direction (for all relevant definitions see Section 2).

**Theorem 1.2.** A Banach space $E$ is Gelfand–Phillips if one of the following conditions holds:

(i) ([2] Cor. 2.2) the closed unit ball $B_{E'}$ of the dual space $E'$ endowed with the weak* topology is selectively sequentially pseudocompact;

(ii) ([16]) $E$ is separable;

(iii) ([7, Prop. 2]) $E$ is separably weak*-extensible ($\iff E$ has the separable $c_0$-extension property);

(iv) (cf. [12, Th. 2.2] and [21, Prop. 2]) the space $E_{w^*}$ is selectively sequentially pseudocompact at some $E$-norming set $S \subseteq E'$;

(v) ([13, Th. 4.1]) $E = C(K)$ for some compact selectively sequentially pseudocompact space $K$.

The aforementioned results and discussion suggest the problem of defining and studying the Gelfand–Phillips property in the class of all locally convex spaces which is the main purpose of the article. To this end, instead of the original topology on a locally convex space $E$ we consider the topology $\beta(E, E')$ on $E$ whose neighborhood base at zero consists of barrels (for more details about this topology, see § 8.4 of [18]). We shall say that a subset $A$ of $E$ is barrel-bounded or barrel-precompact if $A$ is bounded or, respectively, precompact in the space $(E, \beta(E, E'))$.

**Definition 1.3.** A barrel-bounded subset $A$ of a locally convex space $E$ is called limited if every weak* null sequence $\{\chi_n\}_{n \in \omega}$ in $E'$ converges uniformly on $A$, that is, $\lim_{n \to \infty} \|\chi_n\|_A = 0$.

It is clear that if $E$ is a Banach space, then a subset $A$ of $E$ is limited if and only if it is limited in the usual sense. Now Gelfand–Phillips spaces are defined in a very natural way as follows.

**Definition 1.4.** A locally convex space $E$ is said to have the Gelfand–Phillips property or else $E$ is a Gelfand–Phillips space if every limited subset of $E$ is barrel-precompact.

It is easy to see that a Banach space $E$ has the Gelfand–Phillips property if and only if it is Gelfand–Phillips in the sense of Definition 1.4.

Now we describe the content of the article. In Section 2 we give several characterizations of Gelfand–Phillips spaces generalizing and extending Theorem 1.1 and characterizations of Gelfand–Phillips Banach spaces obtained in [2], see Theorems 2.2 and 2.3. In Corollary 2.7 we generalize (i) and (ii) of Theorem 1.2 and Corollary 2.12 generalizes (iii) of Theorem 1.2. In Theorem 2.13 we give a sufficient condition on a locally convex space implying the failure of the (GP) property.

In Section 3 we study the Gelfand–Phillips property in function spaces $C(X)$ endowed with different locally convex topologies $T$, but mainly we consider the pointwise topology $T_p$ and the compact-open topology $T_k$. In Corollary 3.4 of Theorem 3.3 we complement (v) of Theorem 1.2 by showing that for every selectively sequentially pseudocompact space $K$, the function space $C_p(X)$ is Gelfand–Phillips. In Theorem 3.6 we show that for every Tychonoff space $X$, if $C_p(X)$ is Gelfand–Phillips then so is $C_k(X)$. We prove that for every metrizable space $X$, the function spaces $C_p(X)$ and $C_k(X)$ are Gelfand–Phillips, see Corollary 3.7.

2. A characterization of Gelfand–Phillips locally convex spaces

In this section we characterize locally convex spaces with the Gelfand–Phillips property. But first we recall some basic definitions.

Let $E$ be a locally convex space (lcs for short). We denote by $E_\beta$ the space $E$ endowed with the locally convex topology $\beta(E, E')$ whose neighborhood base at zero consists of barrels.

A subset $A$ of an lcs $E$ is called precompact if for every neighborhood $U$ of zero there is a finite subset $F \subseteq E$ (one can take $F \subseteq A$) such that $A \subseteq F + U$. Therefore $A$ is barrel-precompact if it is a precompact subset of $E_\beta$. We say that a subset $B \subseteq E$ is barrel-bounded if it is a bounded subset
of $E$, that is, if for any barrel $U \subseteq E$ there is an $n \in \omega$ such that $B \subseteq nU$. A subset $A$ of $E$ is defined to be barrel-separated if there exists a barrel $B \subseteq E$ such that $A$ is $B$-separated in the sense that $a - a' \notin B$ for any distinct elements $a, a' \in A$. It is easy to see that a subset $A \subseteq B$ is not barrel-precompact if and only if it contains an infinite barrel-separated subset. The polar $D^\circ$ of a set $D \subseteq E$ is the set

$$D^\circ := \{ x \in E' : \| \chi \|_D \leq 1 \}.$$ 

To characterize locally convex spaces with the (GP) property, we need the following topological notions. Let $A$ be a subset of a topological space $X$. Then $A$ is defined to be relatively sequentially compact in $X$ if every sequence $\{ x_n \}_{n \in \omega} \subseteq A$ contains a subsequence $\{ x_{n_k} \}_{k \in \omega}$ that converges in $X$. Following [2], we define the space $X$ to be selectively sequentially pseudocompact at $A$ if for any open sets $U_n \subseteq X$, $n \in \omega$, intersecting the set $A$, there exists a sequence $\{ x_n \}_{n \in \omega} \subseteq \prod_{n \in \omega} U_n$ containing a subsequence $\{ x_{n_k} \}_{k \in \omega}$ that converges in $X$. It is clear that if $A$ is a relatively sequentially compact subset of $X$, then $X$ is selectively sequentially pseudocompact at $A$.

A topological vector space $X$ is defined to be selectively sequentially precompact at $A$ if for any open sets $U_n \subseteq X$, $n \in \omega$, intersecting the set $A$ there exists a sequence $\{ x_n \}_{n \in \omega} \subseteq \prod_{n \in \omega} U_n$ containing a Cauchy subsequence. We recall that a sequence $\{ x_n \}_{n \in \omega}$ in $X$ is Cauchy if for any neighborhood $U \subseteq X$ of zero there exists $n \in \omega$ such that $x_{m} - x_{n} \in U$ for every $m, k \geq n$. It is clear that if $X$ is selectively sequentially pseudocompact at $A$, the $X$ is also selectively sequentially precompact at $A$.

Analogously to limited and compact operators between Banach spaces we define

**Definition 2.1.** An operator $T : X \rightarrow Y$ between locally convex spaces is called

(i) limited if for every barrel-bounded subset of $X$, the image $T(B)$ is a limited subset of $Y$;

(ii) barrel-precompact if for every barrel-bounded subset $B$ of $X$, the image $T(B)$ is barrel-precompact;

(iii) $\beta$-to-$\beta$ precompact if the operator $T : X_\beta \rightarrow Y$ is barrel-precompact.

The following characterization of locally convex spaces with the (GP) property is the main result of this section.

**Theorem 2.2.** For a locally convex space $E$ the following assertions are equivalent:

(i) $E$ has the Gelfand–Phillips property;

(ii) For every barrel-bounded set $B \subseteq E$ which is not barrel-precompact, there is a weak* null sequence $\{ \chi_n \}_{n \in \omega}$ in $E'$ such that $\| \chi_n \|_B \not\rightarrow 0$.

(iii) For any infinite barrel-bounded, barrel-separated subset $D$ of $E$ and every $\delta > 0$ there exist a sequence $\{ x_n \}_{n \in \omega}$ in $D$ and a null sequence $\{ f_n \}_{n \in \omega}$ in $E'_\omega$ such that $|f_n(x_k)| < \delta$ and $|f_n(x_k)| > 1 + \delta$ for all natural numbers $k < n$.

(iv) For any infinite barrel-bounded barrel-separated set $D$ in $E$ there exists a continuous operator $T : E \rightarrow c^0(\omega)$ such that $T(D)$ is not precompact in the Banach space $c_0$.

(v) For each infinite barrel-bounded barrel-separated subset $D$ of $E$ there exist an infinite subset $X \subseteq D$ and a subset $S \subseteq E'_\omega$ such that $E'_\omega$ is selectively sequentially precompact at $S$ and $\sup_{s \in S} |s(x - x')| > 1$ for every distinct $x, x' \in X$.

(vi) Every limited operator $T : L \rightarrow E$ from an lcs space $L$ to $E$ is barrel-precompact.

(vii) Every limited operator $T : L \rightarrow E$ from a normed space $L$ to $E$ is barrel-precompact.

**Proof.** (i)$\Rightarrow$(ii) Fix any barrel-bounded set $B \subseteq E$ which is not barrel-precompact. By (i), $E$ is Gelfand–Phillips and hence $B$ is not limited. Therefore there exists a weak* null sequence $\{ \chi_n \}_{n \in \omega}$ in $E'$ such that $\| \chi_n \|_B \not\rightarrow 0$, as desired.

(ii)$\Rightarrow$(iii) Fix any $\delta > 0$ and any infinite barrel-bounded barrel-separated set $D$ in $E$. Find a barrel $B \subseteq E$ such that the set $D$ is $B$-separated. Observe that $D$ is not barrel-precompact because $D$ is $B$-separated and infinite. By (ii), there is a null sequence $\{ g_n \}_{n \in \omega}$ in $E'_\omega$ such that $\| g_n \|_D \not\rightarrow 0$. Passing to a subsequence and multiplying each functional $g_n$ by an appropriate constant if needed,
we can assume that

\[(2.1) \quad \|g_n\|_D = \sup_{d \in D} |g_n(d)| > 1 + \delta \quad \text{for every } n \in \omega.\]

Now, choose arbitrarily \(x_0 \in D\) such that \(|g_0(x_0)| > 1 + \delta\). Assume that, for \(k \in \omega\), we found \(x_0, \ldots, x_k \in D\) and a sequence \(0 = n_0 < n_1 < \cdots < n_k\) of natural numbers such that

\[|g_{n_i}(x_i)| < \delta \quad \text{and} \quad |g_{n_j}(x_j)| > 1 + \delta \quad \text{for every } 0 \leq i < j \leq k.\]

Since \(g_n \to 0\) in the topology \(\sigma(E', E)\), (2.1) implies that there are \(n_{k+1} > n_k\) and \(x_{k+1} \in D\setminus \{x_0, \ldots, x_k\}\) such that \(|g_{n_{k+1}}(x_{k+1})| > 1 + \delta\) and \(|g_{n_k}(x_i)| < \delta\) for every \(i \leq k\). For every \(k \in \omega\), put \(f_k := g_{n_k}\), and observe that \(f_k \to 0\) in \(E'_u\) and

\[|f_k(x_i)| = |g_{n_k}(x_i)| < \delta \quad \text{and} \quad |f_k(x_k)| = |g_{n_k}(x_k)| > 1 + \delta\]

for any numbers \(i < k\), as desired.

(iii) \(\Rightarrow\) (iv) Let \(D\) be any barrel-bounded barrel-separated set in \(E\). By (iii), for \(\delta = \frac{1}{2}\), there exist a sequence \(\{x_n\}_{n \in \omega}\) in \(D\) and a null sequence \(\{\chi_n\}_{n \in \omega}\) in \(E'_u\), such that \(|\chi_m(x_n)| < \frac{1}{2}\) and \(|\chi_m(x_n)| > \frac{3}{2}\) for all \(n < m\). Then \(|\chi_m(x_m) - \chi_m(x_n)| \geq |\chi_m(x_m) - |\chi_m(x_n)| > \frac{3}{2} - \frac{1}{2} = 1\) for every \(n < m\). It follows that the operator

\[T : E \to C^0_p(\omega), \quad T(x) := (\chi_n(x))_{n \in \omega},\]

is well-defined and continuous. Since \(\|T(x) - T(x')\|_\infty \geq |\chi_m(x_m) - \chi_m(x_n)| > 1\) for every \(n < m\), the set \(T(D) \supseteq \{T(x_n)\}_{n \in \omega}\) is not precompact in \(c_0\).

(iv) \(\Rightarrow\) (i) Fix a barrel-bounded subset \(P \subseteq E\), which is not barrel-precompact. Then there exists a barrel \(B \subseteq E\) such that \(L \nsubseteq F + B\) for any finite subset \(F \subseteq E\). For every \(n \in \omega\), choose inductively a point \(z_n \in P\) so that \(z_n \notin \bigcup_{k < n}(z_k + B)\). Observe that the set \(D = \{z_n : n \in \omega\}\) is infinite, barrel-separated, and barrel-separated. By (iv), there exists a continuous operator \(T : E \to C^0_p(\omega)\) such that the set \(T(D)\) is not precompact in the Banach space \(c_0\). Since \(c_0\) is a Banach space, there exist a sequence \(\{a_n\}_{n \in \omega}\) in \(D\) and \(\delta > 0\) such that \(\|T(a_n) - T(a_m)\|_{c_0} \geq \delta\) for all distinct \(n, m \in \omega\). Observe that the sequence \(\{T(a_n)\}_{n \in \omega}\) is bounded in the Banach space \(c_0\). Therefore there are two sequences \(0 \leq n_0 < n_1 < \cdots\) and \(0 \leq m_0 < m_1 < \cdots\) of natural numbers such that

\[|c_{m_k}(T(a_n))| > \frac{\delta}{2} \quad \text{for every } k \in \omega,\]

where \(c_{m_k} : C^0_p(\omega) \to \mathbb{F}\) is the \(n\)th coordinate functional. For every \(k \in \omega\), set \(f_k := c_{m_k} \circ T\). It follows that \(\{f_k\}_{k \in \omega}\) is a null sequence in \(E'_u\), and

\[\|f_k\|_P = \sup_{x \in P} |f_k(x)| \geq |f_k(a_{n_k})| > \frac{\delta}{2}\]

for every \(k \in \omega\), witnessing that the set \(P\) is not limited.

(iii) \(\Rightarrow\) (v) By (iii), for any infinite barrel-bounded, barrel-separated set \(X \subseteq E\) there exists a null sequence \(S = \{s_n\}_{n \in \omega} \subseteq E'_u\) and a subsequence \(X_0 = \{x_n\}_{n \in \omega} \subseteq X\) such that \(|s_n(x_n)| > 2\) and \(|s_n(x_k)| < 1\) for all \(k < n\). Taking into account that \(S\) is a null sequence in \(E'_u\), we conclude that \(S\) is relatively sequentially compact in \(E'_u\), and hence \(E'_u\) is selectively sequentially precompact at \(S\). Finally, observe that for any \(k < n\)

\[\sup_{s \in S} |s(x_n - x_k)| \geq |s_n(x_n) - s_n(x_k)| > 2 - 1 = 1.\]

(v) \(\Rightarrow\) (ii) Fix any barrel-bounded subset \(P \subseteq E\), which is not barrel-precompact. Then there exists a barrel \(B \subseteq E\) such that \(P \nsubseteq F + B\) for any finite subset \(F \subseteq E\). For every \(n \in \omega\), choose inductively a point \(z_n \in P\) so that \(z_n \notin \bigcup_{k < n}(z_k + B)\). Observe that the set \(\{z_n : n \in \omega\}\) is infinite, barrel-bounded, and barrel-separated. By (v), there is an infinite countable subset \(X \subseteq \{z_n : n \in \omega\}\) and a subset \(S \subseteq E'_u\) such that \(E'_u\) is selectively sequentially precompact at \(S\) and

\[\sup_{s \in S} |s(x - x')| > 1\]

for every distinct \(x, x' \in X\).
Write the set $X$ as $\{x_n\}_{n \in \omega}$ for pairwise distinct points $x_n$. Then for every $n < m$, we can find a linear functional $f_{n,m} \in S$ such that $|f_{n,m}(x_m - x_n)| > 1$. It follows from the selective sequential precompactness of $E'_{u^*}$ at $S$ that the set $S$ is bounded in $E'_{u^*}$, see for example [15, Lemma 2.9]. Therefore, by Theorem 8.3.2 in [15], the polar $D := S^0$ is a barrel in $E$. Since the set $P$ is barrel-bounded, the real number $|P|_D = \inf\{r \geq 0 : P \subseteq r \cdot D\}$ is well-defined. It follows that $X = \{x_n : n \in \omega\} \subseteq P \subseteq |P|_D \cdot D$. Consider the disk $D = \{z \in \mathbb{F} : |z| \leq |P|_D\}$ in the field $\mathbb{F}$. Observe that for every $n < m$ the inclusion $f_{n,m} \in S \subseteq S^0 = D^0$ implies

$$f_{n,m}(X) \subseteq f_{n,m}(|P|_D \cdot D) = |P|_D \cdot f_{n,m}(D) \subseteq D,$$

which means that $f_{n,m}|_X \in \mathbb{D}^X$.

Using the sequential compactness of the compact metrizable space $\mathbb{D}^{X}$, we can construct a decreasing sequence $\{\Omega_n\}_{n \in \omega}$ of infinite sets in $\omega$ such that for every $n \in \omega$, the sequence $\{f_{n,m}|_X\}_{m \in \Omega_n}$ converges to some function $f_n \in \mathbb{D}^{X}$. Choose an infinite set $\Omega \subseteq \omega$ such that $\Omega \setminus \Omega_n$ is finite for every $n \in \omega$. Since the space $\mathbb{D}^X$ is sequentially compact, we can replace $\Omega$ by a smaller infinite set and additionally assume that the sequence $\{f_n\}_{n \in \Omega}$ converges to some element $f_\infty \in \mathbb{D}^X$. Since the set $\{f_\infty(x_n)\}_{n \in \Omega} \subseteq \mathbb{D}$ admits a finite cover by sets of diameter $< \frac{1}{4}$, we can replace $\Omega$ by a suitable infinite subset and additionally assume that the set $\{f_\infty(x_n)\}_{n \in \Omega}$ has diameter $< \frac{1}{4}$.

It follows that the function $f_\infty \in \mathbb{D}^X$ belongs to the closure of the set $\{f_{n,m}|_X : n, m \in \Omega, n < m\}$. Since the space $\mathbb{D}^X$ is first-countable, we can choose a sequence $\{(n_i, m_i)\}_{i \in \omega} \subseteq \{(n, m) \in \Omega \times \Omega : n < m\}$ such that the sequence $\{f_{n_i, m_i}|_X\}_{i \in \omega}$ converges to $f_\infty$. Since $\mathbb{F}^X$ is metrizable, for every $i \in \omega$ the element $f_{n_i, m_i}|_X$ of $\mathbb{D}^X \subseteq \mathbb{F}^X$ has an open neighborhood $V_i \subseteq \mathbb{F}^X$ such that the sequence $\{V_i\}_{i \in \omega}$ converges to $f_\infty$ in the sense that each neighborhood of $f_\infty$ in $\mathbb{F}^X$ contains all but finitely many sets $V_i$. Since $|f_{n_i, m_i}(x_n - x_m)| > 1$, we can replace each set $V_i$ by a smaller neighborhood of $f_{n_i, m_i}|_X$ and additionally assume that $|g(x_n) - g(x_m)| > 1$ for every $g \in V_i$. For every $i \in \omega$ consider the open neighborhood $W_i = \{f \in E'_{u^*} : f|_X \in V_i\}$ of the functional $f_{n_i, m_i}$ in the space $E'_{u^*}$.

Since $E'_{u^*}$ is selectively sequentially precompact at $S$, there exists a Cauchy sequence $\{g_k\}_{k \in \omega} \subseteq E'_{u^*}$ and an increasing number sequence $\{i(k)\}_{k \in \omega}$ such that $g_k \in W_{i(k)}(k)$ for every $k \in \omega$. Since the sequence $(g_k)_{k \in \omega}$ is Cauchy in $E'_{u^*} \subseteq \mathbb{F}^E$, it converges to some linear functional $g_\infty \in \mathbb{F}^E$ (which is not necessarily continuous on $E$). The continuity of the restriction operator $E'_{u^*} \to \mathbb{F}^X$, $f \mapsto f|_X$, and the choice of the open sets $V_i$, $i \in \omega$, guarantee that $g_\infty|_X = f_\infty$. Consequently, the sequence $(g_k|_X)_{k \in \omega}$ converges to $g_\infty|_X = f_\infty$ in $\mathbb{F}^X$.

Then for every $k \in \omega$, we can find a number $j_k > k$ such that

$$\max\{|f_\infty(x_{n(k)}) - g_{j_k}(x_{n(k)})|, |f_\infty(x_{m(k)}) - g_{j_k}(x_{m(k)})|\} < \frac{1}{5}.$$ 

For every $k \in \omega$, consider the functional $\mu_k := g_k - g_{j_k} \in E'$ and observe that the sequence $\{\mu_k\}_{k \in \omega}$ converges to zero in $E'_{u^*}$. On the other hand, for every $k \in \omega$, the choice of the sequence $(j_k)_{k \in \omega}$, the inequality $\text{diam}\{f_\infty(x_n)\}_{n \in \Omega} < \frac{1}{4}$, and the inclusion $g_k|_X \in V_{i(k)}$ imply

$$|g_{j_k}(x_{n(k)}) - g_{j_k}(x_{m(k)})| = |g_{j_k}(x_{n(k)}) - f_\infty(x_{n(k)}) + f_\infty(x_{m(k)}) - g_{j_k}(x_{m(k)}) + f_\infty(x_{n(k)}) - f_\infty(x_{m(k)})| \leq |g_{j_k}(x_{n(k)}) - f_\infty(x_{n(k)})| + |f_\infty(x_{m(k)}) - g_{j_k}(x_{m(k)})| + |f_\infty(x_{n(k)}) - f_\infty(x_{m(k)})| \leq \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2};$$

and

$$|\mu_k(x_{n(k)}) - \mu_k(x_{m(k)})| = |g_k(x_{n(k)}) - g_{j_k}(x_{n(k)}) - g_k(x_{m(k)}) + g_{j_k}(x_{m(k)})| \leq |g_k(x_{n(k)}) - g_k(x_{m(k)})| - |g_k(x_{n(k)}) - g_k(x_{m(k)})| > 1 - \frac{1}{2} = \frac{1}{2}.$$ 

Then

$$\sup_{x \in P} |\mu_k(x)| \geq \max\{|\mu_k(x_{n(k)})|, |\mu_k(x_{m(k)})|\} \geq \frac{1}{4},$$

and

$$\sup_{x \in P} |\mu_k(x)| \geq \frac{1}{4}.$$
witnessing that \( \| \mu_k \|_p \not\to 0 \).

(i)\(\Rightarrow\)(vi) Let \( T : L \to E \) be a limited operator from an lcs \( L \) to \( E \), and let \( B \) be a barrel-bounded subset of \( L \). Then the image \( T(B) \) is a limited subset of \( E \) and hence, by (i), \( T(B) \) is barrel-precompact. Thus \( T \) is a barrel-precompact operator.

(vi)\(\Rightarrow\)(vii) is trivial.

(vii)\(\Rightarrow\)(i) Fix a limited subset \( B \) of \( E \). It is clear that the closed absolutely convex hull \( D \) of the set \( B \) in \( E \) is also limited. Let \( L \) be the linear hull of \( D \). Since \( D \) is (barrel)-bounded in \( E \), the function \( \| \cdot \| : L \to [0, \infty) \), \( \| \cdot \| : x \mapsto \inf \{ r \geq 0 : x \in rD \} \) is a well-defined norm on the linear space \( L \) and the set \( D \) coincides with the closed unit ball \( B_L \) of the normed space \( (L, \| \cdot \|) \). Since the identity inclusion \( T : (L, \| \cdot \|) \to E \) is continuous and the set \( D = T(B_L) \) is limited in \( E \), the operator \( T \) is limited. By (vii), the set \( D = T(B_L) \) is barrel-precompact in \( E \) and so is the set \( B \subseteq D \). Thus \( E \) has the Gelfand–Phillips property.

The following theorem shows that the equivalence (i)\(\Leftrightarrow\)(ii) in Theorem 1.1 remains true for a wider class of locally convex spaces that includes all barrelled normed spaces, in particular, all Banach spaces.

**Theorem 2.3.** Let \( E \) be a locally convex space such that \( E \) has a bounded barrel and \( E' \) is dense in \( (E_{\beta})' \). Then the space \( E \) is Gelfand–Phillips if and only if every limited weakly null sequence in \( E \) converges to zero in \( E_{\beta} \).

**Proof.** To prove the “only if” part, assume that \( E \) is Gelfand–Phillips and take any limited weakly null sequence \( \{ x_n \}_{n \in \omega} \subseteq E \). Assuming that the sequence \( \{ x_n \}_{n \in \omega} \) does not converge to zero in the topology \( \beta(E, E') \), we can find a barrel \( B \subseteq E \) such that the set \( I = \{ n \in \omega : x_n \notin 3B \} \) is infinite. By the Gelfand–Phillips property of \( E \), the limited set \( \{ x_n \}_{n \in \omega} \) is barrel-precompact and hence \( \{ x_n \}_{n \in \omega} \subseteq F + B \) for some finite set \( F \). By the Pigeonhole Principle, there exists an element \( y \in F \) such that the set \( J = \{ n \in I : x_n \in y + B \} \) is infinite. Assuming that \( y \in 2B \), we would conclude that \( x_n \in y + B \subseteq 2B + B = 3B \) for every \( n \in J \subseteq I \), which contradicts the definition of the set \( I \). Therefore, \( 1/2y \notin B \) and, by the Hahn–Banach Theorem, there exists a functional \( \chi \in E' \) such that \( \sup \chi(B) < \frac{1}{2} \chi(y) = 1 \). Then

\[
\inf_{n \in J} \chi(x_n) \geq \inf \chi(y + B) \geq \chi(y) - \sup \chi(B) \geq 2 - 1 = 1,
\]

which means that the sequence \( \{ x_n \}_{n \in \omega} \) is not weakly null in \( E \). This contradiction shows that the sequence \( \{ x_n \}_{n \in \omega} \) converges to zero in the topology \( \beta(E, E') \).

The proof of the “if” part is more complicated. Assume that every limited weakly null sequence in \( E \) converges to zero in \( E_{\beta} \). Suppose for a contradiction that the space \( E \) is not Gelfand–Phillips. Then \( E \) contains a limited set \( L \) which is not barrel-precompact, and hence \( L \) contains a \( B \)-separated sequence \( \{ x_n \}_{n \in \omega} \) for some barrel \( B \). Since \( E \) contains a bounded barrel, we can assume that the barrel \( B \) is bounded. Then its gauge is a norm \( \| \cdot \| \) in the space \( E \) and the sequence \( \{ x_n \}_{n \in \omega} \) is 1-separated in this norm. Since the limited set \( L \) is barrel-bounded, the sequence \( \{ x_n \}_{n \in \omega} \) is bounded in the normed space \( (E, \| \cdot \|) \).

**Claim 2.4.** The sequence \( \{ x_n \}_{n \in \omega} \) contains no subsequences which are weakly Cauchy in the normed space \( (E, \| \cdot \|) \).

**Proof.** To derive a contradiction, assume that the sequence \( \{ x_n \}_{n \in \omega} \) does contain a subsequence \( \{ x_{n_k} \}_{k \in \omega} \) which is weakly Cauchy in the normed space \( (E, \| \cdot \|) \). Then the sequence \( \{ x_{n_{k+1}} - x_{n_k} \}_{k \in \omega} \) is weakly null in \( (E, \| \cdot \|) \) and hence weakly null in \( E \) by the continuity of the identity operator \( (E, \| \cdot \|) \to E \). Since the set \( \{ x_n \}_{n \in \omega} \subseteq L \) is limited, so is the set \( \{ x_{n_{k+1}} - x_{n_k} \}_{k \in \omega} \). By our assumption, the weakly null limited sequence \( \{ x_{n_{k+1}} - x_{n_k} \}_{k \in \omega} \) is null in \( E_{\beta} \), which is not possible as \( x_{n_{k+1}} - x_{n_k} \notin B \) for every \( k \in \omega \). \( \square \)
By Claim 2.4 and Rosenthal’s $\ell_1$-Theorem [10, p.201], $(x_n)_{n \in \omega}$ contains a subsequence $(x_{n_k})_{k \in \omega}$, equivalent to the standard basis in $\ell_1$. Replacing $(x_n)_{n \in \omega}$ by this subsequence, we can assume that $(x_n)_{n \in \omega}$ is equivalent to the standard basis in $\ell_1$. Then the closed linear hull $X$ of the set $\{x_n\}_{n \in \omega}$ in the completion $\hat{E}$ of the normed space $(E, \| \cdot \|)$ is isomorphic to the Banach space $\ell_1$. Observe that the identity inclusion $I : X \to \hat{E}$ is not compact. By [1], there exists a continuous operator $T : \hat{E} \to c_0$ such that $T|_X = T \circ I : X \to c_0$ is not compact. Let $D$ be the absolutely convex hull of the set $L_0 = \{x_n\}_{n \in \omega}$. Since $(x_n)_{n \in \omega}$ is equivalent to the standard basis in $\ell_1$, the set $D$ is a bounded neighborhood of zero in $X$. Then, by the non-compactness of the operator $T|_X$, the image $T(D)$ is not precompact in $c_0$. Since absolutely convex hulls of precompact sets are precompact and $T(D)$ is contained in the absolutely convex hull of the set $T(L_0)$ in $c_0$, the set $T(L_0)$ is not precompact in $c_0$ and hence $\|e'_n\|_{T(L_0)} \not\to 0$, where $(e'_n)_{n \in \omega}$ is the sequence of coordinate functionals in $c_0$. For every $n \in \omega$, consider the functional $e'_n \circ T|_E$ which is continuous in the norm $\| \cdot \|$ of the space $E$ and hence is a continuous functional on the space $E_\beta$. Since the barrel $B$ is bounded and $E'$ is dense in $(E_\beta)'$, there exists a functional $\chi_n \in E'$ such that $\|\chi_n - e'_n \circ T\|_B < \frac{1}{2^n}$. Then $(\chi_n)_{n \in \omega}$ is a weak* null sequence in $E'$ such that $\|\chi_n\|_{L_0} \not\to 0$, which implies that the set $L_0$ is not limited. But this contradicts the choice of the limited set $L \supset L_0$. 

We use Theorem 2.2 to obtain some hereditary properties of the class of Gelfand–Phillips spaces. We define a linear subspace $X$ of a locally convex space $E$ to be $\beta$-embedded if the identity inclusion $X_\beta \to E_\beta$ is a topological embedding. It is easy to see that $X$ is $\beta$-embedded in $E$ if and only if for any barrel $B \subseteq X$ there exists a barrel $D \subseteq E$ such that $D \cap X \subseteq B$.

**Proposition 2.5.** A subspace $X$ of a locally convex space $E$ is $\beta$-embedded if one of the following conditions is satisfied:

(i) $X$ is complemented in $E$;

(ii) $X$ is barrelled;

(iii) $X$ and $E$ are $\beta$-Banach and $X_\beta$ is closed in $E_\beta$.

**Proof.** Given a barrel $B \subseteq X$, we should find a barrel $D \subseteq E$ such that $D \cap X \subseteq B$.

(i) If $X$ is complemented in $E$, then there exists a linear continuous operator $R : E \to X$ such that $R(x) = x$ for all $x \in X$. In this case the set $D = R^{-1}(B)$ is a barrel in $E$ with $D \cap X = B$.

(ii) If $X$ is barrelled, then the barrel $B$ is a neighborhood of zero. Since $X$ is a subspace of $E$, there exists a barrel neighborhood $D \subseteq E$ of zero such that $D \cap X \subseteq B$.

(iii) Assume that the spaces $X$ and $E$ are $\beta$-Banach and $X_\beta$ is closed in $E_\beta$. Then the identity inclusion $I : X_\beta \to E_\beta$ is a continuous injective operator between Banach spaces such that the image $I(X_\beta)$ is closed in $E_\beta$. By the Banach Open Mapping Principle, the operator $I : X_\beta \to E_\beta$ is a topological embedding.

In the next corollary we give some sufficient conditions on a subspace of a Gelfand–Phillips space to have the Gelfand–Phillips property.

**Corollary 2.6.** Assume that a locally convex space $E$ is Gelfand–Phillips. Then:

(i) Every $\beta$-embedded subspace of $E$ is Gelfand–Phillips.

(ii) Every barrelled subspace of $E$ is Gelfand–Phillips.

(iii) If $E_\beta$ is barrelled (for example, $E$ is $\beta$-Banach), then $E_\beta$ is Gelfand–Phillips.

**Proof.** (i) Let $X$ be a $\beta$-embedded subspace of $E$ and $D$ be an infinite barrel-bounded barrel-separated subset of $X$. Take any barrel $B \subseteq X$ such that $D$ is $B$-separated. Since $X$ is $\beta$-embedded, there exists a barrel $B' \subseteq E$ such that $B' \cap X \subseteq B$. Observe that $D$ is also $B'$-separated in $E$. Since $E$ is Gelfand–Phillips, by Theorem 2.2 there exist a continuous operator $T : E \to c_0((\omega))$ such that $T(D)$ is not precompact in $c_0$. Then the restriction $T|_X$ of $T$ onto $X$ is continuous and $T|_X(D) = T(D)$ is not precompact in $c_0$. Thus, by Theorem 2.2 the lcs $X$ is Gelfand–Phillips.
(ii) follows from (i) and Proposition 2.5(ii).

(iii) Assume that the space $E_β$ is barrelled and take any infinite barrel-bounded barrel-separated subset $D ⊆ E_β$. Find a barrel $B ⊆ E_β$ such that $D$ is $B$-separated. Since $E_β$ is barrelled, $B$ is a neighborhood of zero in $E_β$. By the definition of $E_β$, $B$ is a barrel in $E$. Therefore, by Theorem 2.2, there exist a continuous operator $T : E → \mathcal{C}_p^0(\omega)$ such that $T(D)$ is not precompact in $c_0$. It is clear that for the identity inclusion $i : E_β → E$ and the operator $T \circ i : E_β → \mathcal{C}_p^0(\omega)$, the image $T \circ i(D) = T(D)$ is not precompact in $c_0$. Thus, by Theorem 2.2, the lcs $E_β$ is Gelfand–Phillips. □

Theorem 2.2 suggests also to study topological spaces which are selectively sequentially pseudo-compact at itself. Following [11], we call a topological space $X$ selectively sequentially pseudocompact if $X$ is selectively sequentially pseudocompact at $X$. Clearly, every sequentially compact space is selectively sequentially pseudocompact, and every selectively sequentially pseudocompact space is pseudocompact. It is easy to see that a topological space $X$ is selectively sequentially pseudocompact if and only if $X$ is selectively sequentially pseudocompact at some dense set $A ⊆ X$. Compact selectively sequentially pseudocompact spaces form the class $\mathcal{K}''$ introduced by Drewnowski and Emmanuele [13].

**Corollary 2.7.** A locally convex space $E$ is Gelfand–Phillips if one of the following conditions holds:

(i) for every barrel $B ⊆ E$ there exists a barrel $D ⊆ B$ such that $E_{w^*}'$ is selectively sequentially precompact at the polar $D^o ⊆ E_{w^*}''$;

(ii) for every barrel $B ⊆ E$ there exists a barrel $D ⊆ B$ such that $E_{w^*}'$ is selectively sequentially pseudocompact at the polar $D^o ⊆ E_{w^*}''$;

(iii) for any barrel $B ⊆ E$ there exists a barrel $D ⊆ B$ whose polar $D^o$ endowed with the weak$^*$-topology is selectively sequentially pseudocompact;

(iv) $E$ is separable and barrelled.

**Proof.** (i) In order to apply Theorem 2.2(v), fix any infinite barrel-bounded barrel-separated subset $X$ of $E$. Choose a barrel $B ⊆ E$ such that $X$ is $B$-separated. By (i), there exists a barrel $D ⊆ B$ such that the space $E_{w^*}'$ is selectively sequentially precompact at the set $D^o$. By the Hahn–Banach Separation Theorem [18 7.3.5], $\sup_{s \in D^o} |s(x)| > 1$ for any $x \in E \setminus D$. In particular, $\sup_{s \in D^o} |s(x - x')| > 1$ for any distinct points $x, x' \in X$. Now it is legal to apply the implication (v)⇒(i) in Theorem 2.2 and conclude that the locally convex space $E$ is Gelfand–Phillips.

(ii) follows from (i) since if $E_{w^*}'$ is selectively sequentially pseudocompact at $D^o$, then $E_{w^*}'$ is also selectively sequentially precompact at $D^o$.

(iii) Observing that for any selectively sequentially pseudocompact subspace $S ⊆ E_{w^*}'$, the space $E_{w^*}'$ is selectively sequentially pseudocompact at $S$, we conclude that (iii) follows from (ii).

(iv) Assume that $E$ is barrelled and separable. Since $E$ is barrelled, each barrel $B$ in $E$ is a neighborhood of zero. By the Alaoglu–Bourbaki Theorem [18 8.5.2], the polar $B^o$ is compact in $E_{w^*}'$. Since $E$ is separable, the compact space $B^o$ is metrizable according to [18 8.5.3]. Being metrizable and compact, the space $B^o$ is selectively sequentially pseudocompact. Applying (iii), we conclude that $E$ has the Gelfand–Phillips property. □

In the next proposition we use the following simple lemma.

**Lemma 2.8.** Each continuous operator $T : L → \mathcal{C}_p^0(\omega)$ from a barrelled space $L$ remains continuous as an operator from $L$ to $c_0$.

**Proof.** Let $\{e'_n\}_{n \in \omega}$ be the sequence of coordinate functionals on the Banach space $c_0$. The definition of the topology of the space $\mathcal{C}_p^0(\omega)$ ensures that each functional $e'_n$ remains continuous on the locally convex space $\mathcal{C}_p^0(\omega)$. Observe that the intersection $B := \bigcap_{n \in \omega} \{x \in \mathcal{C}_p^0(\omega) : |e'_n(x)| ≤ 1\}$ coincides with the closed unit ball of the Banach space $c_0$. Since $B$ is a barrel also in $\mathcal{C}_p^0(\omega)$, the continuity of the operator $T$ implies that the set $T^{-1}(B)$ is a barrel in $L$. Since $L$ is barrelled, $T^{-1}(B)$ is a neighborhood of zero, which means that the operator $T : L → c_0$ is continuous. □
Let $Y$ be a locally convex space. A locally convex space $E$ is defined to have the separable $Y$-extension property if every separable subspace of $E$ is contained in a barrelled separable linear subspace $X \subseteq E$ such that each continuous operator $T : X \to Y$ can be extended to a continuous operator $\bar{T} : E \to Y$. This definition implies that each separable barrelled locally convex space has the separable $Y$-extension property for any locally convex space $Y$.

**Proposition 2.9.** A (barrelled) locally convex space $E$ has the separable $C^0_p(\omega)$-extension property if (and only if) $E$ has the separable $c_0$-extension property.

**Proof.** Assume that $E$ has the separable $c_0$-extension property, and let $H$ be a separable subspace of $E$. By the separable $c_0$-extension property, $H$ is contained in a barrelled separable linear subspace $L \subseteq E$ such that each continuous operator $T : L \to c_0$ can be extended to a continuous operator $\bar{T} : E \to c_0$. Let $T : L \to C^0_p(\omega)$ be a continuous operator. Since $L$ is barrelled, Lemma 2.8 implies that $T$ is also continuous as an operator from $L$ to the Banach space $c_0$. Therefore $T$ can be extended to a continuous operator $\bar{T} : E \to c_0$. Clearly, $\bar{T}$ is also continuous as an operator from $E$ to $C^0_p(\omega)$. Thus $E$ has the separable $C^0_p(\omega)$-extension property.

If $E$ is barrelled and has the separable $C^0_p(\omega)$-extension property, then an analogous argument shows that $E$ has the separable $c_0$-extension property. □

Using the classical Sobczyk Theorem [10, p.72] (which states that if $H$ is a linear subspace of a separable Banach space $E$ and $T : H \to c_0$ is a bounded operator, then there is a bounded operator $S : E \to c_0$ extending $T$ to the whole $E$), one can prove the following characterization showing that for Banach spaces our definition of separable $c_0$-extension property is equivalent to that introduced by Correa and Tausk [8, 9].

**Proposition 2.10.** A Banach space $E$ has the separable $c_0$-extension property if and only if every continuous operator $T : X \to c_0$ defined on a separable subspace $X \subseteq E$ can be extended to a continuous operator $\bar{T} : E \to c_0$.

By [8, 9] the class of Banach spaces with the separable $c_0$-extension property includes all weakly compactly generated Banach spaces, all Banach spaces with the separable complementation property (=every separable subspace is contained in a separable complemented subspace), and all Banach spaces $C(K)$ over $K_0$-monolithic compact lines $K$. Let us recall that a topological space $X$ is $K_0$-monolithic if each separable subspace of $X$ has a countable network.

**Theorem 2.11.** Every locally convex space $E$ with the separable $C^0_p(\omega)$-extension property is Gelfand–Phillips.

**Proof.** In order to apply Theorem 2.2(iv), fix an infinite barrel-bounded, barrel-separated set $D \subseteq E$. Choose any countable infinite subset $I \subseteq D$. By the separable $C^0_p(\omega)$-extension property, $I$ is contained in a barrelled separable linear subspace $X \subseteq E$ such that every continuous operator $T : X \to C^0_p(\omega)$ can be extended to a continuous operator $\bar{T} : E \to C^0_p(\omega)$.

By Corollary 2.7(iv), the barrelled separable lcs $X$ is Gelfand–Phillips. By Theorem 2.2, there exists a continuous operator $T : X \to C^0_p(\omega)$ such that the set $T(I)$ is not precompact in the Banach space $c_0$. By the choice of $X$, the operator $T$ can be extended to a continuous operator $\bar{T} : E \to C^0_p(\omega)$. Observing that the set $\bar{T}(D) \supseteq T(I)$ is not precompact in $c_0$, we can apply Theorem 2.2 and conclude that $E$ is Gelfand–Phillips. □

Proposition 2.9 and Theorem 2.11 imply

**Corollary 2.12.** A barrelled space with the separable $c_0$-extension property is Gelfand–Phillips.

We finish this section with some conditions on a locally convex space implying the failure of the Gelfand–Phillips property.
Theorem 2.13. A locally convex space $E$ is not Gelfand–Phillips if the identity map $E'_{w^*} \to (E'_p)_w$ is sequentially continuous and $E$ admits a continuous operator $T : c_0 \to E$ which is not $\beta$-to-$\beta$ precompact.

Proof. Since the operator $T : c_0 \to E$ is not $\beta$-to-$\beta$ precompact, the image $T(B)$ of the closed unit ball $B = \{ x \in c_0 : \|x\|_0 \leq 1 \}$ of $c_0$ is not barrel-precompact in $E$. Assuming that the space $E$ is Gelfand–Phillips, we can find a null sequence $\{ \mu_n \}_{n \in \omega} \subseteq E'_{w^*}$ such that $\|\mu_n\|_{T(B)} \not\to 0$. By our assumption, the identity map $E'_{w^*} \to (E'_p)_w$ is sequentially continuous, which implies that the sequence $\{ \mu_n \}_{n \in \omega}$ converges to zero in the weak topology of the strong dual space $E'_p$. Then for the dual operator $T^* : (E'_p)_w \to (c_0)'_w = (\ell_1)_w$, the sequence $\{ T^* \mu_n \}_{n \in \omega}$ converges to zero in the weak topology of the Banach space $\ell_1$. By the Schur Theorem [10, VII], this sequence converges to zero in norm. Observe that for every $n \in \omega$ and each $x \in B$, we have

$$\|\mu_n\|_{T(B)} = \sup_{x \in B} |\mu_n(T(x))| = \sup_{x \in B} |(T^* \mu_n)(x)| = \|T^* \mu_n\| \to 0,$$

which contradicts the choice of the sequence $\{ \mu_n \}_{n \in \omega}$. \qed

3. The Gelfand–Phillips property in function spaces

In this section we apply the results from the preceding sections to study the Gelfand–Phillips property in function spaces.

We shall use repeatedly the following assertion whose proof can be found in [4].

Proposition 3.1. Let $X$ be a Tychonoff space, and let $T$ be a locally convex topology on $C(X)$ such that $\mathcal{T}_p \subseteq \mathcal{T} \subseteq \mathcal{T}_k$. Then:

(i) for every barrel $D$ in $C_T(X)$, there are a functionally bounded subset $A$ of $X$ and $\varepsilon > 0$ such that $[A; \varepsilon] \subseteq D$.

(ii) a subset $\mathcal{F} \subseteq C_T(X)$ is barrel-bounded if and only if for any functionally bounded set $A \subseteq X$, the set $\mathcal{F}(A) := \bigcup_{f \in \mathcal{F}} f(A)$ is bounded in $\mathcal{F}$.

(iii) $(C_T(X))_\beta = C_b(X)$.

Below we give examples of locally convex spaces without the (GP) property. Recall that a Tychonoff space $X$ is called an $F$-space if every functionally open set $A$ in $X$ is $C^*$-embedded in the sense that every bounded continuous function $f : A \to \mathbb{R}$ has a continuous extension $\tilde{f} : X \to \mathbb{R}$. For various equivalent conditions for a Tychonoff space $X$ being an $F$-space, see [17, 14.25]. In particular, the Stone–Čech compactification $\beta X$ of a discrete space $X$ is compact.

Example 3.2. For any infinite compact $F$-space $K$, the spaces $C(K)$ and $C_p(K)$ are not Gelfand–Phillips.

Proof. We proved in [2] that for any compact $F$-space $K$ the Banach space $C(K)$ is not Gelfand–Phillips. By Proposition 3.1 $(C_p(K))_\beta = C(K)$. By (iii) of Corollary 2.6 the function space $C_p(K)$ is not Gelfand–Phillips. \qed

Below we provide a sufficient condition on the space $X$ for which $C_p(X)$ is Gelfand–Phillips.

Theorem 3.3. Let $X$ be a Tychonoff space such that every functionally bounded subset $K \subseteq X$ is contained in a subset $K'$ such that $X$ is selectively sequentially pseudocompact at $K'$. Then the function space $C_p(X)$ is Gelfand–Phillips.

Proof. In order to apply Theorem 2.2 to the locally convex space $E = C_p(X)$, take any infinite barrel-bounded barrel-separated subset $A$ of $E$. Find a barrel $B \subseteq E$ such that $A$ is $B$-separated. Choose an arbitrary sequence of pairwise distinct elements $\{ a_n \}_{n \in \omega}$ of $A$. Then the set $\{ a_n - a_m : n, m \in \omega, n \neq m \}$ is contained in $E \setminus B$. By (i) of Proposition 3.1 there exists a functionally bounded closed set $K \subseteq X$ and a real number $\varepsilon > 0$ such that the barrel $D := \{ f \in C_p(X) : \|f\|_K \leq \varepsilon \}$ is contained
in $B$. By our assumption, $K$ is contained in a subset $K' \subseteq X$ such that $X$ is selectively sequentially pseudocompact at $K'$. Then for any element $a_n - a_m \in E \setminus B \subseteq E \setminus D$ we have

$$\sup_{x \in K'} |a_n(x) - a_m(x)| \geq \sup_{x \in K} |a_n(x) - a_m(x)| > \varepsilon.$$

Let $\delta : X \to E_{w^*}$ be the continuous function assigning to each point $x \in X$ the evaluation functional $\delta_x \in E'$ at $x$ (so $\delta_x(f) = f(x)$ for every $f \in C_p(X)$). Consider the set $S := \{ \frac{1}{\varepsilon} \delta_x : x \in K' \} \subseteq E_{w^*}$. It is well known that $\delta$ is a homeomorphic embedding with the closed image. This fact and the selective sequential pseudocompactness of $X$ at $K'$ imply that the space $E_{w^*}$ is selectively sequentially pseudocompact at $\delta(K')$ and also at its homothetic copy $S = \frac{1}{\varepsilon} \delta(K')$. Observe that for every positive integers $n \neq m$, we have

$$\sup_{s \in S} |s(a_n - a_m)| = \frac{1}{\varepsilon} \sup_{x \in K'} |a_n(x) - a_m(x)| > \frac{1}{\varepsilon} \cdot \varepsilon = 1.$$

Applying Theorem 2.2 we conclude that the locally convex space $E = C_p(X)$ is Gelfand–Phillips. □

Corollary 2.6 and Theorem 3.3 imply

**Corollary 3.4.** For every selectively sequentially pseudocompact space $K$, the function spaces $C_p(K)$ and $C(K)$ are Gelfand–Phillips.

Since the spaces $[0, \omega_1]$ and $[0, \omega_1)$ are sequentially compact, Corollary 3.4 implies

**Example 3.5.** The spaces $C_p[0, \omega_1]$ and $C_p[0, \omega_1)$ are Gelfand–Phillips.

In the next two theorems we show that the Gelfand–Phillips property for function spaces satisfies some “hereditary property” in the sense that if $C(X)$ has the Gelfand–Phillips property for a topology $T$ then the space $C(X)$ has this property also for a finer topology. For a locally convex topology $T$ on $C(X)$ we will denote the locally convex space $(C(X), T)$ by $C_T(X)$.

**Theorem 3.6.** Let $X$ be a Tychonoff space, and let $\tau$ and $T$ be locally convex topologies on $C(X)$ such that $T_p \subseteq \tau \subseteq T \subseteq T_0$. If the space $C_\tau(X)$ is Gelfand–Phillips, then so is $C_T(X)$. In particular, if $C_p(X)$ is Gelfand–Phillips, then so is $C_b(X)$.

**Proof.** Fix any barrel-bounded set $B \subseteq C_T(X)$, which is not barrel-precompact in $C_T(X)$. The continuity of the identity operator $C_T(X) \to C_\tau(X)$ implies that $B$ is barrelbounded in $C_\tau(X)$.

We claim that $B$ is not barrel-precompact in $C_\tau(X)$. Indeed, since $B$ is not barrel-precompact in $C_T(X)$, there exists a barrel $D$ in $C_T(X)$ such that $B \nsubseteq F + D$ for every finite set $F \subseteq B$. By (i) of Proposition 3.3, there are a functionally bounded subset $A$ of $X$ and $\varepsilon > 0$ such that $[A; \varepsilon] \subseteq D$. It is clear that $[A; \varepsilon]$ is a barrel in $C_p(X)$ and hence also in $C_\tau(X)$. Since $[A; \varepsilon] \subseteq D$ it follows that $B \nsubseteq F + [A; \varepsilon]$ for every finite set $F \subseteq B$, which means that $B$ is not barrel-precompact in $C_\tau(X)$.

Since the lcs $C_\tau(X)$ is Gelfand–Phillips, there exists a weak* null sequence $\{ \mu_n \}_{n \in \omega}$ in $C_\tau(X)'$ such that $\| \mu_n \|_B \not\to 0$. The continuity of the identity operator $C_T(X) \to C_\tau(X)$ ensures that the sequence $\{ \mu_n \}_{n \in \omega}$ remains weak* null in $C_T(X)'$ and hence $C_T(X)$ is Gelfand–Phillips. □

Theorems 3.3 and 3.6 immediately imply the next assertion.

**Corollary 3.7.** Let $X$ be a $\mu$-space whose compact subsets are selectively sequentially pseudocompact (for example $X$ is metrizable). Then the function spaces $C_p(X)$ and $C_k(X)$ are Gelfand–Phillips.

Under an additional assumption that the space $C_b(X)$ is barrelled we can strengthen Theorem 3.6 as follows.

**Theorem 3.8.** Let $X$ be a Tychonoff space such that the space $C_b(X)$ is barrelled, and let $\tau$ and $T$ be locally convex topologies on $C(X)$ such that $T_p \subseteq \tau \subseteq T \subseteq T_0$. If the space $C_\tau(X)$ is Gelfand–Phillips, then so is $C_T(X)$; in particular, $C_b(X)$ is Gelfand–Phillips.
Problem 3.9. Let $X$ be a Tychonoff space such that the space $C_k(X)$ is Gelfand–Phillips. Are the spaces $C_k(νX)$, $C_k(μX)$ or $C_b(X)$ Gelfand–Phillips?

The aforementioned results suggest the following intriguing problem.

Problem 3.10. (i) Is there a compact space $K$ such that the Banach space $C(K)$ is Gelfand–Phillips but the function space $C_p(K)$ is not Gelfand–Phillips?

(ii) Is there a pseudocompact space $K$ such that the Banach space $C(K)$ is Gelfand–Phillips but the function space $C_b(K)$ is not Gelfand–Phillips?

(iii) Is there a Tychonoff space $X$ such that $C_k(X)$ is Gelfand–Phillips but the function space $C_p(K)$ is not Gelfand–Phillips?

Proposition 3.11. Let $f : L → K$ be a surjective map between two pseudocompact spaces. If the function space $C_p(L)$ (resp. $C(L)$) is Gelfand–Phillips, then also the space $C_p(K)$ (resp. $C(K)$) is Gelfand–Phillips.

Proof. It follows that the dual operators $T_p : C_p(K) → C_p(L)$ and $T : C(K) → C(L)$ assigning to each continuous function $φ : K → F$ the composition $φ ∘ f : L → F$ are isomorphic topological embeddings. The Banach subspace $T(C(K))$ of $C(L)$ is $β$-embedded into $C(L)$ by Proposition 2.20(ii). If the Banach space $C(L)$ is Gelfand–Phillips, then, by Corollary 2.6(i), the space $T(C(K))$ is Gelfand–Phillips and so is its isomorphic copy $C(K)$.

Now assume that the function space $C_p(L)$ is Gelfand–Phillips. By (iii) of Proposition 3.1 (C_p(K))_β = C(K) and (C_p(L))_β = C(L), which implies that the subspace $T_p(C_p(K))$ is $β$-embedded into $C_p(L)$. By Corollary 2.16 the space $T_p(C_p(K))$ is Gelfand–Phillips and so is its isomorphic copy $C_p(K)$. □

In fact, the selective sequential pseudocompactness of the compact space $K$ in Corollary 3.8 can be replaced by the selective sequential precompactness of the space $P_ω(K)$ of finitely supported probability measures on $K$ in $C(K)'_{w^∗}$.

For a Tychonoff space $X$ by a finitely supported probability measure on $X$ we understand an element of the convex hull $P_ω(X)$ of the set $\{δ_z : z ∈ X\}$ of Dirac measures in the dual space $C_p(X)'_{w^∗}$. The Dirac measure $δ_x$ concentrated at a point $x ∈ X$ is the linear continuous functional $δ_x : C_p(X) → F$, for
\( \delta_x : f \mapsto f(x) \). For a subset \( K \subseteq X \), let \( P_\omega(K) \) be the convex hull of the set \( \{ \delta_x : x \in K \} \subseteq C_p(X) \), of Dirac measures concentrated at points of the set \( K \).

**Proposition 3.12.** For a Tychonoff space \( X \), the locally convex space \( E = C_p(X) \) is Gelfand–Phillips whenever for every closed functionally bounded set \( K \) in \( X \) the space \( E_\omega^* \) is selectively sequentially precompact at \( P_\omega(K) \).

**Proof.** In order to apply Theorem 2.2 to the locally convex space \( E = C_p(X) \), take any infinite barrel-bounded, barrel-separated subset \( A \) of \( E \). Find a barrel \( B \subseteq E \) such that \( A \) is \( B \)-separated. By (i) of Proposition 3.11, there exist a closed functionally bounded set \( K \subseteq X \) and a real number \( \varepsilon > 0 \) such that \( \{ f \in C_p(X) : \| f \|_K \leq \varepsilon \} \subseteq B \). Assuming that \( E_\omega^* \) is selectively sequentially precompact at the set \( P_\omega(K) \), we conclude that \( E_\omega^* \) is also selectively sequentially precompact at the set \( S = \frac{1}{\varepsilon}P_\omega(K) \).

Since \( A \) is \( B \)-separated, it follows that for any distinct points \( a, b \in A \), we have \( a - b \notin B \) and hence \( \| a - b \|_K > \varepsilon \), which implies

\[
\sup_{s \in S} |s(a - b)| = \frac{1}{\varepsilon} \sup_{\mu \in P_\omega(K)} |\mu(a - b)| \geq \frac{1}{\varepsilon} \sup_{x \in K} |a(x) - b(x)| = \frac{1}{\varepsilon} \| a - b \|_K > 1.
\]

By Theorem 2.2 the locally convex space \( C_p(X) \) is Gelfand–Phillips. \( \square \)

For a compact space \( K \), the space

\[
P(K) := \{ \mu \in C(X)' : \| \mu \| = \mu(1_X) = 1 \}
\]

endowed with the weak* topology, is called the space of probability measures on \( K \). By the Riesz Representation Theorem [18, 7.6.1], each functional \( \mu \in P(K) \) can be identified with a regular probability Borel measure on \( K \). Each point \( x \in K \) can be identified with the Dirac measure \( \delta_x \in C(K)' \), \( \delta_x : f \mapsto f(x) \). It is well known that the map \( \delta : K \to P(K) \), \( \delta : x \mapsto \delta_x \), is a topological embedding and the convex hull \( P_\omega(K) \) of the set \( \delta(K) \subseteq P(K) \) is dense in \( P(K) \).

**Corollary 3.13.** Let \( K \) be an infinite compact space. Then:

(i) if \( P(K) \) is selectively sequentially pseudocompact, then \( C_p(P(K)) \), \( C(P(K)) \) and \( C(K) \) are Gelfand–Phillips;

(ii) if \( P(K) \) is selectively sequentially pseudocompact at \( P_\omega(K) \), then the spaces \( C_p(K) \) and \( C(K) \) are Gelfand–Phillips.

**Proof.** (i) By Corollary 3.4, the spaces \( C_p(P(K)) \) and \( C(P(K)) \) are Gelfand–Phillips. The space \( C(K) \) is Gelfand–Phillips by Corollary 2.2(vi) of [2].

(ii) If \( P(K) \) is selectively sequentially pseudocompact at \( P_\omega(K) \), then also \( E_\omega^* \) is selectively sequentially precompact at \( P_\omega(K) \). Therefore, by Proposition 3.12 the space \( C_p(K) \) Gelfand–Phillips and so is the Banach space \( C(K) = C_k(K) \), see Theorem 3.6. \( \square \)

The path-connected space \( P(K) \) can fail to be selectively sequentially pseudocompact as the following example shows.

**Example 3.14.** Let \( K \) be an infinite compact \( F \)-space. Then the function spaces \( C_p(P(K)) \) and \( C(P(K)) \) are not Gelfand–Phillips. Consequently, the compact space \( P(K) \) is not selectively sequentially pseudocompact.

**Proof.** Observe that the operator \( T : C(K) \to C(P(K)) \) assigning to each function \( f \in C(K) \) the function \( Tf : P(K) \to \mathbb{F}, Tf : \mu \mapsto \mu(f) \), is an isomorphic embedding of the Banach space \( C(K) \) into the Banach space \( C(P(K)) \). Assuming that the Banach space \( C(P(K)) \) is Gelfand–Phillips, we can apply Corollary 2.7(ii) and conclude that also the Banach space \( C(K) \) is Gelfand–Phillips. But this contradicts Example 3.2. This contradiction shows that the Banach space \( C(P(K)) \) is not Gelfand–Phillips.

Now Theorem 3.6 implies the function space \( C_p(P(K)) \) is not Gelfand–Phillips, and hence, by Corollary 3.4, the compact space \( P(K) \) is not selectively sequentially pseudocompact. \( \square \)
By Corollary 3.4 if a compact space $K$ is selectively sequentially pseudocompact, then the space $C_p(K)$ is Gelfand–Phillips. So it is natural to ask whether the converse is true, namely, let $K$ be a compact space such that the space $C_p(K)$ is Gelfand–Phillips. Is then $K$ selectively sequentially pseudocompact? Since every selectively sequentially pseudocompact space contains non-trivial convergent sequences, one can also ask a much weaker question: Does $K$ contain a non-trivial convergent sequence? Under an additional Set-Theoretic assumption (namely, the Jensen Diamond Principle ♦) we answer this question in the negative. Recall that a compact space $X$ is called Efimov if $X$ contains neither a non-trivial convergent sequence nor topological copy of $\beta\omega$.

Example 3.15. Under ♦, there exists a Efimov space $X$ whose function space $C_p(X)$ is Gelfand–Phillips.

Proof. Under ♦, we constructed in [3] a simple Efimov space $X$ such that $P(X)$ is selectively sequentially pseudocompact. By Corollary 3.13 the function space $C_p(X)$ is Gelfand–Phillips. □

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