On the maximum induced density of directed stars and related problems

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Abstract

Let $k \geq 3$ be an integer, we prove that the maximum induced density of the $k$-vertex directed star in a directed graph is attained by an iterated blow-up construction. This confirms a conjecture by Falgas-Ravry and Vaughan, who proved this for $k = 3, 4$. This question provides the first known instance of density problem for which one can prove extremality of an iterated blow-up construction. We also study the inducibility of complete bipartite digraphs and discuss other related problems.

1 Introduction

In modern extremal combinatorics, a substantial number of problems study the asymptotic relations between densities of subgraphs, and can be formulated in the following language. Given a family $F$ of graphs and another graph $H$, define the Turán $H$-number of $F$ to be the maximum number of induced copies of $H$ in a $F$-free graph on $n$ vertices, and denote it by $ex_H(n, F)$. We also denote by $\pi_H(F)$ the limit of the maximum induced density of $H$ in a $F$-free graph when the number of vertices tends to infinity. Similar definitions can be as well made in the setting of $r$-uniform hypergraph, directed graph, and so forth. When $H$ is a single edge, $\pi_e(F)$ is just the classical Turán density. It has been a long-standing open problem in extremal combinatorics to understand these densities for families of hypergraphs and directed graphs. For results and techniques, we refer the readers to the survey [10].

On the other hand, when $F = \emptyset$, $\pi_H(\emptyset)$ studies the maximum induced density of $H$ in arbitrary graph, and is known as the inducibility of $H$. Although there are various works [1, 2, 3, 5, 9, 11] on the inducibility of graphs, there are relatively fewer results for directed graphs. Sperfeld [13] studied the inducibility of some digraphs on three vertices. Falgas-Ravry and Vaughan [6] determined $\pi_{\vec{S}_3}(\emptyset)$ and $\pi_{\vec{S}_k}(\emptyset)$ by flag algebra. Here $\vec{S}_k$ is the directed star on $k$ vertices, with one vertex being the center, and $k - 1$ edges oriented away from it. They further made the following conjecture that the extremal digraph having the maximum induced density of $\vec{S}_k$ is always an unbalanced blow-up of $\vec{S}_2$ iterated inside one part.

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Conjecture 1.1. For every integer $k \geq 3$,

$$\pi_{\vec{S}_k}(\emptyset) = \alpha_k = \max_{0 \leq x \leq 1} \frac{kx(1-x)^{k-1}}{1-x^k}.$$ 

Assume the maximum is attained by $x = x_k$, then the extremal configuration is constructed by starting with two parts $|A| = x_k n$ and $|B| = (1-x_k)n$, adding all the edges oriented from $A$ to $B$, and iterating this process inside $A$.

As we mentioned earlier, the proofs for cases $k = 3$ and $k = 4$ employ the method of flag algebra developed by Razborov [12] and are partly computer assisted. However since the search space and running time grow exponentially in $k$, a different approach may be needed for large $k$. It is also worth mentioning that for most Turán-type problems studying densities of subgraphs, if the conjectured extremal graph comes from such iterated construction instead of a simple blow-up of small graph, usually we do not know how to obtain an exact bound. For example, the Turán density of $K_4^-$ (the unique 3-graph on 4 vertices with 3 edges) was conjectured to be achieved by the iterated blow-up of certain 6-vertex 3-graph by Frankl and Füredi [7] and is still open. Another example is the special case of the well-known Caccetta-Häggkvist conjecture [4]: every $n$-vertex digraph with minimum outdegree at least $n/3$ contains a directed triangle. Its difficulty probably lies in the fact that the iterated blow-up of a directed 4-cycle is one of the conjectured extremal examples. To the best of our knowledge, the case $k = 3$ and 4 of Conjecture 1.1 are probably the only examples that the exact bound has been proved for an iterated construction, which leads us to believe there exists a simpler and more human-readable proof. Actually we are able to apply certain operations on digraphs, and reduce it to an optimization problem, and verify this conjecture for every directed star $\vec{S}_k$ for $k \geq 3$.

The rest of this short paper is organized as follows. In Section 2 we give a complete proof of Conjecture 1.1. Section 3 discusses the inducibility for complete bipartite digraphs. In the concluding remarks, we mention some related problems and possible future directions for research.

2 Main proof

In this section we will give a proof of Conjecture 1.1. Our proof is inspired by that of [3]. Assume $D_n$ is the extremal directed graph on $n$ vertices which has the maximum number of induced copies of of $\vec{S}_k$. We define an equivalence relation on its vertex set $V(D_n)$ as follows: $u \sim v$ iff they have the same in- and out-neighborhoods, i.e. $N^+(u) = N^+(v)$ and $N^-(u) = N^-(v)$. This equivalence relation naturally partitions the vertices of $D_n$ into the following equivalence classes: $V(D_n) = V_1 \cup \cdots \cup V_m$, where each $V_i$ induces an empty digraph. From the definition, between two classes $V_i$ and $V_j$ there are three possible scenarios: (i) all the edges are oriented from $V_i$ to $V_j$; (ii) all the edges are oriented from $V_j$ to $V_i$; (iii) there is no edge between $V_i$ and $V_j$. We claim that a sequence of operations can be applied on $D_n$ such that the number of induced copies of $\vec{S}_k$ does not decrease, and in the resulted digraph case (iii) never occurs.
Lemma 2.1. Given a directed graph $D_n$ with equivalence classes $V_1, \cdots, V_m$, and $1 \leq i \neq j \leq m$. If there is no edge between $V_i$ and $V_j$, we can merge $V_i$ and $V_j$ into one equivalence class without decreasing the number of induced copies of $\vec{S}_k$.

Proof. Assume $|V_i| = x$ and $|V_j| = y$. Denote by $D'_n$ the digraph formed by moving vertices between $V_i$ and $V_j$ and changing their neighborhoods accordingly, with $|V'_i| = z$, $|V'_j| = x + y - z$.

Let $N_{00}$ be the number of induced copies of $\vec{S}_k$ in $D$ not involving vertices in $V_i$ or $V_j$; $N_{10}$ be the number of induced copies of $\vec{S}_k$ using vertices from $V_i$ but not vertices from $V_j$; $N_{01}$ be the number of induced copies of $\vec{S}_k$ using vertices from $V_j$ but not vertices from $V_i$; and finally $N_{11}$ be the number of induced copies of $\vec{S}_k$ using vertices from both $V_i$ and $V_j$. Obviously the total number of induced copies of $\vec{S}_k$ is equal to $N_{00} + N_{10} + N_{11} + N_{11}$. Similarly we also define the parameters $N'_{00}, N'_{10}, N'_{01}, N'_{11}$ for $D'_n$.

Note that $N'_{00} = N_{00}$ since only the adjacencies involving vertices in $V_i \cup V_j$ might be changed. We also have $N_{11} = N'_{11}$. Consider an induced copy of $\vec{S}_k$ containing $v_i \in V_i$, and $v_j \in V_j$. Since there is no edge between the two parts $V_i$ and $V_j$, $v_i$ is not adjacent to $v_j$. Therefore both of them are the leaves of $\vec{S}_k$. Because all the leaves are equivalent in $\vec{S}_k$, moving vertices between $V_i$ and $V_j$ does not change the value of $N_{11}$.

Next we show that $z$ can be chosen such that $N'_{10} + N'_{01} \geq N_{10} + N_{01}$. Denote by $s_l$ the number of $(k-l)$-vertex sets $S$ in $[n] \setminus (V_i \cup V_j)$ such that $S$ together with any $l$ vertices in $V_i$ induce a copy of $\vec{S}_k$. Similarly let $t_l$ be the number of $(k-l)$-vertex sets $T$ such that $T$ together with any $l$ vertices in $V_j$ induce a copy of $\vec{S}_k$. Then by the definition of equivalence class, we have

$$N_{10} = \sum_{l=1}^{k} \binom{x}{l} s_l, \quad N_{01} = \sum_{l=1}^{k} \binom{y}{l} t_l,$$

$$N'_{01} = \sum_{l=1}^{k} \binom{z}{l} s_l, \quad N'_{10} = \sum_{l=1}^{k} \binom{x+y-z}{l} t_l.$$

It is not difficult to verify that $\binom{z}{l}$ and $\binom{x+y-z}{l}$ are both convex functions in the variable $z$. Therefore one could merge these two equivalence classes $V_i$ and $V_j$ by taking either $z = 0$ or $z = x+y$ in the new digraph $D'_n$, such that $N'_{01} + N'_{10} \geq N_{01} + N_{10}$. \hfill \Box

Note that after merging vertices in Lemma 2.1, the number of equivalence classes decreases, so this process stops after a finite number of steps. We may assume that in the extremal digraph $D_n$ with equivalent classes $\{V_1, \cdots, V_m\}$, and any $i \neq j$, either there is a complete bipartite digraph with every edge oriented from $V_i$ to $V_j$, or from $V_j$ to $V_i$. We denote them by $V_i \rightarrow V_j$ and $V_j \rightarrow V_i$ respectively. Assume $|V_i| = w_i n$ where $\sum_{i=1}^{m} w_i = 1$. The induced density of $\vec{S}_k$ in $D_n$ is equal to

$$\frac{1}{\binom{m}{k}} \sum_{i=1}^{m} w_i n \binom{w_i n}{k-1} \sum_{i=1}^{m} k w_i w_j^{k-1} + o(1).$$

Since $\pi_{\vec{S}_k}(\emptyset)$ is the limit of the maximum induced density when $n$ tends to infinity, we can neglect the $o(1)$ term here. Without loss of generality, we may assume that $w_1 \geq w_2 \geq \cdots \geq w_m$ by
Therefore which implies that \( \alpha \) converges to a limit, denoted by \( \beta \), and bounded on the compact set \([0, 1]\). Reordering \( \{w_i\} \), the induced density increases by \( w_jw_i^{k-1} - w_iw_j^{k-1} \geq 0 \) Therefore we can assume \( V_j \rightarrow V_i \) for any \( i < j \). Basically speaking we obtain \( D_n \) to be the unbalanced blow-up of a transitive tournament, and the induced density of \( \bar{S}_k \) in \( D_n \) is now equal to

\[
 f_m(w_1, \ldots, w_m) = k \sum_{1 \leq i < j \leq m} w_i^{k-1}w_j = k \cdot \left( w_1^{k-1}(w_2 + \cdots + w_m) + w_2^{k-1}(w_3 + \cdots + w_m) + \cdots + w_m^{k-1}w_m \right)
\]

Let \( F_m(x) = \max f_m(w_1, \ldots, w_m) \) subject to \( \sum_i w_i = x \) and \( w_i \geq 0 \), then \( \pi_{\bar{S}_k}(\emptyset) = \lim \sup_{m \to \infty} F_m(1) \). Because \( f_m \) is a homogeneous polynomial of degree \( k \), we have \( F_m(x) = F_m(1)x^k \) and thus

\[
 F_m(1) = \max_{0 \leq w_1 \leq 1} kw_1^{k-1}(1 - w_1) + F_{m-1}(1 - w_1) = \max_{0 \leq w_1 \leq 1} kw_1^{k-1}(1 - w_1) + (1 - w_1)^k F_{m-1}(1).
\]  

(1)

Taking \( w_1 = 0 \) in (1) shows that \( F_m(1) \geq F_{m-1}(1) \). Due to the fact that the induced density can never be greater than 1, \( \{F_m(1)\} \) is a bounded monotone non-decreasing sequence and thus converges to a limit, denoted by \( \alpha_k \). Let \( m \to \infty \) in (1), we have

\[
 \alpha_k = \max_{0 \leq x \leq 1} kx^{k-1}(1 - x) + (1 - x)^k \alpha_k
\]

Let \( \beta_k = \max_{0 \leq x \leq 1} \frac{kx^{k-1}(1 - x)}{1 - (1 - x)^k} \), we now prove that \( \alpha_k = \beta_k \). Since \( \frac{kx^{k-1}(1 - x)}{1 - (1 - x)^k} \) is continuous and bounded on the compact set \([0, 1]\), \( \beta_k = \frac{ky^{k-1}(1 - y)}{1 - (1 - y)^k} \) for some \( y \in [0, 1] \) and thus

\[
 \alpha_k \geq ky^{k-1}(1 - y) + (1 - y)^k \alpha_k = (1 - (1 - y)^k)\beta_k + (1 - y)^k \alpha_k,
\]

which implies that \( \alpha_k \geq \beta_k \). On the other hand, suppose \( z \in [0, 1] \) maximizes \( kx^{k-1}(1 - x) + (1 - x)^k \alpha_k \), then \( \alpha_k = k\beta - (1 - z)^k\alpha_k \), and

\[
 \beta_k \geq \frac{k\beta - (1 - z)^k}{1 - (1 - z)^k} = \alpha_k.
\]

Therefore

\[
 \pi_{\bar{S}_k}(\emptyset) = \alpha_k = \beta_k = \max_{0 \leq x \leq 1} \frac{kx^{k-1}(1 - x)}{1 - (1 - x)^k} = \max_{0 \leq x \leq 1} \frac{kx(1 - x)^{k-1}}{1 - x^k}.
\]

Suppose \( x_k \in [0, 1] \) maximizes \( kx(1 - x)^{k-1}/(1 - x^k) \), from the above proof, we can see that the bound is obtained uniquely by the infinite sequence \( w_i = x_i^{k-1}(1 - x_k), i = 1, 2, \ldots \), which corresponds to the iterated blow-up of \( \bar{S}_2 \) and concludes the proof of Conjecture 1.1.
Theorem 3.1. For integers $m \geq 0$, let $w_i$ be non-negative reals. Then the coefficient of the term $w_i w_j$ in the expansion of $(w_1 + \cdots + w_m)^s$, when $s \geq 2$, is greater than or equal to $s$, which is the coefficient of $w_1 w_j$ in the expansion of $w^s_j$. The second inequality follows from the fact that the coefficient of $w_i w_j^{s-1}$ in the expansion of $(w_1 + \cdots + w_m)^s$ is equal to $s$, which is greater than the coefficient 1 of the corresponding term in the left hand side, whenever $s \geq 2$. It follows from inequality (2) that $F_m \leq F_2$. By elementary calculus, one can easily show that $w_1^t w_2^s$ is maximized when $w_1 = \frac{t}{s+t}$ and $w_2 = \frac{s}{s+t}$, which finishes the proof of Theorem 3.1.
4 Concluding remarks

• In [6], the authors mention that for any given digraph $D$, an auxiliary 3-uniform hypergraph $G(D)$ can be defined by setting $xyz$ to be a 3-edge whenever $\{x, y, z\}$ induces a copy of $\vec{S}_3$ in $D$. It is not hard to check that $G(D)$ is always a $C_5$-free 3-graph. Here $C_5$ refers to the tight cycle on 5 vertices, whose edges are (123), (234), (345), (451), and (512). Mubayi and Rödl conjectured that the Turán number $\pi(C_5)$ is equal to $2\sqrt{3} - 3$, with exactly the same iterated construction in the $\vec{S}_3$ problem. The result $\pi_{\vec{S}_3}(\emptyset) = \max_{0 \leq x \leq 1} 3x(1-x)^2/(1-x^3) = 2\sqrt{3} - 3$ settles the special case when the 3-graph has the form $G(D)$ from a digraph $D$.

• Sperfeld [13] studies the maximum induced density of some small digraphs, and in particular he proved that $\pi_{\vec{C}_3}(\emptyset) = 1/4$ with the extremal example including the random tournament and the iterated blow-up of $\vec{C}_3$, and conjecture that $\pi_{\vec{C}_4}(\emptyset)$ is achieved by the iterated blow-up of $\vec{C}_4$. It would be of great interest to develop new techniques to attack this problem, since the solution of this problem might as well provide insights into solving the Caccetta-Häggkvist conjecture.

• Although obtaining a general solution to the graph or digraph inducibility problem seems to be difficult, Hatami, Hirst and Norine [8] showed that for a given graph $H$, the $n$-vertex graph $G$ containing the most number of induced copies of sufficiently large balanced blow-up of $H$, is itself essentially a blow-up of $H$. It would be interesting if similar results can be proved for the inducibility of directed graphs, which may also involve some iterated blow-ups.

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