Richard Thompson group $F$ is not amenable

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September 1, 2014

Abstract

Richard Thompson’s group $F$ is the group of piecewise linear homeomorphisms of the unit interval with a final number of break points, all at dyadic rational numbers (their denominators are powers of 2) and with slopes which are powers of 2. A discrete group $G$ is amenable if there exists a finitely additive probability measure on $G$ which is invariant under left translations and is defined on all subsets of $G$. The amenability question for $F$ is a well known open problem. In this paper we prove that group $F$ is not amenable.

1 Introduction

A discrete group $G$ is amenable if there exists a finitely additive probability measure on $G$ which is invariant under left translations and is defined on all subsets of $G$. If a group contains a free subgroup on two generators then it is not amenable. The amenability question is usually difficult for groups of exponential growth which do not contain a free group. One of such mysterious groups is the famous group $F$ defined by Richard Thompson. The amenability question for $F$ is a well known open problem. In this paper we prove that group $F$ is not amenable.

Richard Thompson’s group $F$ is the group of piecewise linear homeomorphisms of the unit interval with a final number of break points, all at dyadic rational numbers (their denominators are powers of 2) and with slopes which
are powers of 2. It can be presented by two generators with two relations. It can be also described as a diagram group [GS].

The conjecture that $F$ is not amenable was one of four conjectures about group $F$ stated by Ross Geoghegan in 1979 (page 549 of the book [GS]). The first three conjectures were affirmed in 1980’s. Brown and Geoghegan proved that $F$ has type $FP_{\infty}$ in [BG], the same authors proved that all homotopy groups of $F$ at infinity are trivial in [BGI] and finally Brin and Squier proved that $F$ has no non-abelian free subgroups in [BS]. Brin and Squier proved a more general fact that the whole group of orientation-preserving, piecewise linear homeomorphisms of the unit interval has no free subgroups of rank greater than one. Their work was connected with von Neumann conjecture that a discrete group is non-amenable if and only if it has a free subgroup of rank two.

This conjecture turned out to be false in 1980 when Alexander Ol’shanskii showed that the Tarski monster group, which does not have a free subgroup of rank 2, is not amenable [O]. Since first counterexamples are not finitely presented, the conjecture was considered for finitely presented groups. However, in 2003, Alexander Ol’shanskii and Mark Sapir in [OS] gave examples of finitely-presented groups which do not satisfy the conjecture. Another counterexample given in 2012 by Nicolas Monod [M] was a group of piecewise projective homeomorphisms of the line which is remarkably simple to understand. Being non-amenable, it has many properties of amenable groups.

Another problem concerning amenability was to find a group which is amenable but not elementary amenable. It was stated by Day [D] and solved in a class of finitely presented groups by Grigorchuk in [G]. It is known that group $F$ is not elemenary amenable [CFP]. It is surprising that $F$ has smaller presentation than Grigorchuk’s example. It has also smaller presentation than the mentioned example of Ol’shanski and Sapir.

Amenable groups are considered widely in functional analysis. The fact that $F$ is non-amenable and a result from a paper of Hadwin and Rosenberg [H] imply that $C^*$-algebra of $F$ is not quasidiagonal. This property with its relation to F is studied in [HP].

Damien Gaboriau in [Gab] considered amenable groups as well as group $F$ in relation to a so called cost of a group. He proved that amenable groups and Thompson group $F$ have the same properties - have cost equal to one and also have fixed price.

The paper is organized as follows.

In section 2 we recall the notion of Fölner sequence. The existence of a
Fölner sequence in $F$ is equivalent to the amenability of $F$. We consider the Cayley graph $\Gamma$ of $F$ corresponding to the presentation of $F$ with two generators $a, b$. The amenabilty of $F$ and the existence of a Fölner sequence would imply the existence of a finite subgraph $A$ of $\Gamma$ with a very large proportion of vertices of valence 4 in $A$. This in turn implies the existence of a relation in $F$ of a very special form (Theorem 3).

In section 3 we describe the interpretation of the group $F$ in terms of ordered, rooted binary tree diagrams and an equivalent interpretation in terms of diagrams over a presentation $P = < a \mid a = aa >$ of the trivial semigroup. Then we describe the action of the generators $a, b$ on a large diagram.

In section 4 we describe a notion of irregularity $\kappa$ of a word $W$ in generators $a, b$ which controls the properties of the relation described in Theorem 3. We prove, using $\kappa$, that the existence in $F$ of a relation predicted by Theorem 3 leads to a contradiction (Lemma 19).

Acknowledgements. We are grateful to Tadeusz Januszkiewicz for suggesting this problem to us and to Tomasz Luczak for turning us away from a wrong direction and to Mark Sapir for pointing out a mistake in the first version of the paper.

2 Amenable groups with two generators

In this section we shall establish some properties of a group $G$ generated by two elements which are related to the question whether $G$ is amenable.

A discrete group $G$ is amenable if there is a finite left-invariant finitely additive measure $\mu$ on $G$ defined on all subsets of $G$ (see [CFP]).

A Fölner sequence in $G$ is a sequence of finite subsets $A_n$ of $G$, such that for every $x \in G$

$$\frac{|xA_n \setminus A_n|}{|A_n|} \to 0.$$

Here $xA_n$ denotes the translation of the set $A_n$ by $x$, i.e. $xA_n = \{xc : c \in A_n\}$ and $|B|$ denotes the number of elements in the set $B$.

It is known (see [BO] Thm 2.6.8) that a discrete group $G$ is amenable if and only if it contains a Fölner sequence.

Clearly there exists a Fölner sequence $A_n$ for the left translations if and only if there exists a Fölner sequence $B_n$ for the right translations. It suffices to choose inverses of all the elements in $A_n$ to get $B_n$.

We prefer to consider the right action of a group on itself and to read words in the group from left to right so we shall adopt the definition of a
Félner sequence with the right translations.

Let \( G \) be a group with two generators \( a \) and \( b \). We denote by \( \overline{a} \) and \( \overline{b} \) the inverses of \( a \) and \( b \) respectively. A word \( W \) in letters \( a, a, b, b \) is called a relator in \( G \) if \( W \) represents the neutral element of \( G \).

Let \( A \) be a subset of \( G \). We consider \( A \) as a full subgraph of the Cayley graph \( \Gamma \) of \( G \). We connect vertices \( x \) and \( y \) of \( A \) with an edge oriented from \( x \) to \( y \) and labelled \( a \) (respectively \( b \)) if \( y = xa \) (respectively \( y = xb \)).

If there exists a Félner sequence then in particular for every \( \alpha > 0 \) there exists a finite subset \( A \subset G \) such that
\[
|Aa \setminus A| < \alpha|A|, \quad |A\overline{a} \setminus A| < \alpha|A|, \quad |Ab \setminus A| < \alpha|A|, \quad |A\overline{b} \setminus A| < \alpha|A|.
\]
(1)

For a finite subset \( A \) of \( G \) we denote by \( v_i(A) \) the set of vertices of the subgraph \( A \subset \Gamma \) of valence \( i \) in \( A \), for \( i=1,2,3,4 \).

**Lemma 1.** If \( A \) is a finite subset of \( G \) which satisfies (1) for \( \alpha < 1/4 \) then
\[
|v_4(A)| > (1 - 4\alpha)|A|.
\]

*Proof.* Let \( B = A \setminus v_4(A) \). If \( x \in B \) then one of the elements \( xa, x\overline{a}, xb, x\overline{b} \) of \( G \) does not belong to \( A \). For one of the generators \( a, b, \overline{a}, \overline{b} \), say for \( a \), the set \( \{ x \in B : xa \not\in A \} \) has at least \( \frac{1}{4}|B| \) elements. The corresponding \( xa \) belong to \( Aa \setminus A \), hence
\[
\frac{1}{4}|B| \leq |Aa \setminus A| < \alpha|A|.
\]

It follows that \( \frac{1}{4}(|A| - |v_4(A)|) < \alpha|A| \)
and \( |v_4(A)| > (1 - 4\alpha)|A| \). \( \square \)

We shall embed a finite subgraph \( A \) of the Cayley graph \( \Gamma \) of \( G \) in an orientable surface in a special way to make a better use of Lemma 1. To each vertex \( x \in A \) we assign an oriented square \( K_x \) (with the preferred side up) of the side length 1 with oriented labeled segments issued from the center of the square in a particular counterclockwise order as in Figure 1.

To each edge \((x, xa)\) (respectively \((x, xb)\)) in \( A \) we assign an oriented rectangle (with the preferred side up) \( R_{x,xa} \) (respectively \( R_{x,xb} \)) of width 1 and length 10 with an oriented segment labelled \( a \) (respectively \( b \)) connecting the mid-points of the shorter edges of the rectangle. Now we attach the shorter edges of rectangles to the edges of the suitable squares so that the arrows and the labels match (Figure 1). We get an oriented surface with boundary. In each square \( K_x \) if some segment issued from the center (see Figure 1) is not extended along a rectangle we erase the segment. We attach a disk to each component.
of the boundary and get an oriented surface $S$ with a labeled oriented graph isomorphic to $A$ embedded in the surface in such a way that each component of $S \setminus A$ is a disk.

We choose any component of $S \setminus A$ and run along its boundary on the prefered side of $S$ in the positive (counterclockwise) direction reading the labels of the edges along the way and writing the labels, from left to right, $a$ or $b$ if we run in the direction of the arrow and $\overline{a}$ or $\overline{b}$ if we run in the direction opposite to the arrow on the edge. We get a word in letters $a, b, \overline{a}, \overline{b}$ which is a relator in the group $G$ and has as many letters as there are vertices met on the way, say $n$. It is possible that the same vertex of $A$ is met more than once along the boundary. We count each vertex as many times as we pass it, except for the last vertex of the path, which is equal to the first vertex of the path and is not counted again.

To each vertex which we meet on the way there corresponds a pair of consecutive labels: one before the vertex and one after the vertex. It follows from our construction that if the vertex has valence 4 then the pair of the consecutive labels corresponding to the vertex must be $ab, b\overline{a}, \overline{a}b$ or $\overline{ba}$. Such pairs will be called regular pairs also if the corresponding vertex is not of valence 4. We consider the relation cyclically, so the last and the first letter of the word also form a pair, which may be regular or not. There are $n$ pairs of consecutive letters in the relation.

The following Lemma is the goal of the above construction.

**Lemma 2.** Let $A$ be a finite subset of $G$ and let $|v_4(A)| > \gamma |A|$. Then there is a relator in $G$ of some length $n$ such that the number of regular pairs of consecutive letters in the relator is at least $\gamma n$.

**Proof.** If $A$, as a subgraph of the Cayley graph of $F$, has several connected components $A_i$, then one of the components satisfies the inequality $|v_4(A_i)| > \gamma |A_i|$ so we may assume that $A$ is connected. Let us embed the graph $A$ in a surface $S$ as above. Consider all complementary regions $D_i$ in $S \setminus A$ and
their boundaries $\partial_i$ for $i = 1, 2, \ldots, m$. Every vertex of $A$ of valence $j$ appears exactly $j$ times altogether along the paths $\partial_i$. Along each boundary $\partial_i$ we have $p_i$ vertices of valence 4, $q_i$ vertices of valence 3, $r_i$ vertices of valence 2 and $s_i$ vertices of valence 1.

For each $i$ the number of the regular pairs along $\partial_i$ is at least $p_i$. It suffices to prove that for some $i$ we have

$$p_i \geq \gamma (p_i + q_i + r_i + s_i).$$

If it is not the case then

$$4|v_4(A)| = \Sigma p_i < \gamma (\Sigma p_i + \Sigma q_i + \Sigma r_i + \Sigma s_i) = \gamma (4|v_4(A)| + 3|v_3(A)| + 2|v_2(A)| + |v_1(A)|).$$

Therefore

$$4|v_4(A)| < \gamma (4|v_4(A)| + 3(|v_3(A)| + |v_2(A)| + |v_1(A)|) = \gamma (4|v_4(A)| + 3(|A| - |v_4(A)|)).$$

From this we get

$$|v_4(A)| < \frac{3\gamma |A|}{4 - \gamma}$$

and since $\gamma < 1$ we get a contradiction with the assumptions of the lemma.

As a corollary of the above Lemma we get the following

**Theorem 3.** Let $G$ be an amenable group generated by two elements $a$ and $b$. Then for every real number $\gamma \in (0, 1)$ there exists a relation in $G$ of the form

$$(*) \quad W \equiv c_1c_2 \ldots c_n = 1$$

where $c_i \in \{a, \bar{a}, b, \bar{b}\}$ for $i = 1, 2, \ldots, n$ and $c_ic_{i+1} \in \{ab, b\bar{a}, a\bar{b}, \bar{b}a\}$ for at least $\gamma n$ indices $i$ ($c_{n+1} = c_1$).

**Definition 4.** Let $W = c_1c_2 \ldots c_n$ be a word in a group with two generators $a$ and $b$. A pair of consecutive letters $c_ic_{i+1}$ in $W$ will be called regular if $c_ic_{i+1} \in \{ab, b\bar{a}, a\bar{b}, \bar{b}a\}$ and will be called irregular otherwise. The pair $c_nc_1$ is also considered and it may be regular or not. We fix a positive integer $R$. We consider subwords $c_ic_{i+1} \ldots c_j$ of $W$ (where $W$ is considered as a cyclic word so a subword may be of the form $c_ic_{i+1} \ldots c_nc_1 \ldots c_k$). A subword is regular if all pairs of consecutive letters contained in the subword are regular. A subword is long if its length is greater than $R$ (it contains at least $R$ pairs $c_kc_{k+1}$). A subword is short if its length is at most $R$. A regular pair $c_ic_{i+1}$ in $W$ is called isolated if it does not belong to a long regular subword of $W$.

The irregular pairs and the isolated regular pairs will play an essential role in the proof that $F$ is not amenable.
Lemma 5. Let $G$ be an amenable group generated by two elements $a$ and $b$. Let $R$ be a fixed positive integer. Then for every real number $\delta \in (0,1)$ there exists a relation in $G$ of the form $W \equiv c_1c_2\ldots c_n = 1$ such that the number of regular non-isolated pairs $c_ic_{i+1}$ in $W$ is greater than $\delta n$.

Proof. Let $\gamma = \frac{R+\delta}{R+1}$. By Theorem 2 there exists a relator $W = c_1c_2\ldots c_n$ such that the number of regular pairs in $W$ is at least $\gamma n$. Therefore the number of the irregular pairs is at most equal to $(1-\gamma)n$. Consider the maximal sequences of consecutive regular pairs (maximal regular subwords of $W$). Two such sequences are separated by at least one irregular pair therefore the number of the maximal regular subwords is at most equal to $(1-\gamma)n$. The number of the short maximal regular subwords is at most equal to $(1-\gamma)n$, therefore the number of all isolated regular pairs is smaller than $(1-\gamma)nR$. It follows that the number of the non-isolated regular pairs is greater than $n - (1-\gamma)n - (1-\gamma)nR = n(1 - (1-\gamma)(R+1)) = \delta n$. 

\[\square\]

3 Richard Thompson group $F$

In this section we shall give some equivalent definitions of Richard Thompson group $F$ and we shall describe in a particular way the action of the generators of $F$ on ”large” ordered, rooted binary trees. In the next section we shall use this action to prove that the relation from Theorem 3 cannot exist in $F$, and therefore group $F$ is not amenable.

We follow [CFP] in the basic description of group $F$ and its properties.

Group $F$ is defined as a group of piecewise linear homeomorphisms $f$ of the interval $[0,1]$ onto itself. Each function $f$ has a finite number of ”break points”, where $f$ is not differentiable, each break point is a dyadic rational number (the denominator is a power of 2) and each slope is a power of 2. Composition of such functions is again in $F$.

3.1 Trees and tree diagrams

Group $F$ can be also described by tree diagrams. An ordered rooted binary tree has a root vertex $v_0$. If it has more than one vertex then the root has exactly two edges going out. Every other vertex has exactly one incoming edge, belonging to the shortest path connecting it with the root, and either no edges going out or exactly two edges going out. At each vertex with valence more than one the edges going out are ordered: there is the left edge and the
right edge. Vertices of valence 1 are called leaves. All leaves are ordered from left to right: leaf number 1, leaf number 2 and so on. From now on by a tree we shall mean an ordered rooted binary tree. By a subtree of a tree $T$ we shall mean an ordered rooted binary tree $T'$ which is a subtree of $T$, whose left edges are left edges of $T$, whose right edges are right edges of $T$, but whose root need not be the root of $T$.

A tree diagram is a pair $(R, S)$ of trees with the same number of leaves. We introduce an equivalence relation on tree diagrams. A caret is a tree with two edges. If $(R, S)$ is a tree diagram and we attach a caret at the leaf number $k$ in $R$ and at the leaf number $k$ in $S$ we get an equivalent tree diagram. Conversely if leaves number $k$ and $k + 1$ belong to a caret in $R$ and to a caret in $S$ we can cancel the carets. This induces an equivalence relation on tree diagrams. If no cancellation is possible we say that the tree diagram is reduced. Every tree diagram is equivalent to a unique reduced tree diagram.

There is a one to one correspondence between the elements of $F$ and the equivalence classes of tree diagrams. In this correspondence if $(Q, R)$ is a tree diagram corresponding to $f$ and $(R, S)$ is a tree diagram corresponding to $g$ then $(Q, S)$ is a tree diagram corresponding to the composition $gf$. The correspondence can be used to define a group structure on the diagrams which is isomorphic to the group $F$. The above property allows to define directly the composition law on the tree diagrams. The tree diagram inverse to $(R, S)$ is equal to $(S, R)$.

**Definition 6.** By the depth of a leaf in a tree $T$ we mean the distance of the leaf from the root. The depth of a tree is the maximal depth of its leaves. A tree is regular of depth $k$ if all its leaves have depth exactly $k$.

**Remark 7.** If each leaf of a tree $T$ has depth at least $k$ and if $S$ is a tree of depth $k$ then there is a unique sub-tree $S'$ of $T$ such that $S'$ is isomorphic to $S$ and the root of $S'$ coincides with the root of $T$. In particular, if $T$ is a regular tree of depth $k$ then it contains a ”copy” of every tree of depth $k$ and the root of the copy coincides with the root of $T$.

Now we can define explicitly the composition law in $F$. If $A = (R, S)$ and $B = (Q, R_1)$ are tree diagrams we can extend $R$ and $R_1$ to a regular tree $T$ of sufficient depth ($R$ and $R_1$ are subtrees of such a tree) and extend $S$ and $Q$ in a similar way to get tree diagrams $(T, S_1)$ and $(Q_1, T)$ equivalent to the initial diagrams $(R, S)$ and $(Q, R)$. Now the composition $AB$ of tree diagrams $(T, S_1)$ and $(Q_1, T)$ is equal $(Q_1, S_1)$. Of course we can extend $R$ and $R_1$ to any ”common multiple” $T$ which is not necessarily a regular tree.
Remark 8. The composition of tree diagrams \((R_1, Q)(S, R)\) is particularly easy if the tree \(R\) is isomorphic to a subtree \(R'\) of \(R_1\) such that the root of \(R'\) coincides with the root of \(R_1\). Then we simply replace \(R'\) by \(S\) in the tree \(R_1\) attaching the leaves of \(S\), in order, to the vertices of \(R_1 \setminus R'\) where the leaves of \(R'\) had been. We get a new tree \(R_2\) and a new tree diagram \((R_2, Q) = (R_1, Q)(S, R)\) and we reduce the tree diagram \((R_2, Q)\) if necessary. If no caret of the tree \(R'\) contains a leaf of \(R_1\) then the carets in \(R_2\) which contain leaves are the same as in \(R_1\) and if the diagram \((R_1, Q)\) was reduced then the diagram \((R_2, Q)\) is also reduced. It is true in particular if the depth of all leaves in \(R_1\) is greater than the depth of \(R\). Observe also that if the depth of the tree \(R\) is equal \(k\) then the depth of each leaf of \(R_2\) is at most by \(k\) less than the depth of the corresponding leaf of \(R_1\).

We know from [CFP] that the tree diagrams \(a\) and \(b\) in Figure 2 generate the group \(F\) and that \(F\) has a presentation

\[
 < a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] >.
\]

The order of leaves in the trees of the diagrams is also indicated in Figure 2.

### 3.2 From trees to planar diagrams

The action of a generator on a tree changes the tree considerably and it is difficult to keep track of the changes. We shall pass to a slightly different description of the group \(F\).

A tree corresponds to a plane diagram, a diagram over a presentation \(P = < a \mid a = aa >\) of the trivial semigroup (see [GS1] and [GS2], Example 2).

A diagram \(D\) is an embedded, oriented, finite planar graph of the following form.
All vertices of the graph lie on a horizontal line, called the base line, say $x$-axis. Suppose the graph has $M$ vertices altogether. They are numbered from left to right $v_1, v_2, \ldots, v_M$.

Every segment of the base line between the consecutive vertices is an edge of the graph. Every other edge of the graph is a semicircle in the upper half-plane based on the $x$-axis with the end-points at some vertices. The edges are oriented from left to right. There is an edge (called the top edge of the diagram) which connects the vertices $v_1$ and $v_M$. The boundary of each bounded complementary region of $D$ has exactly three edges, one top edge and two bottom edges. A complementary region is called a cell of the diagram. The vertex between the bottom edges of a cell is called the middle vertex of the cell.

Isotopic diagrams are considered equal if the isotopy preserves the orientation of the edges.

A (rooted, ordered, binary) tree corresponds to such a diagram. Each edge of the diagram corresponds to a vertex of the tree. The top edge of the diagram corresponds to the root of the tree. The edges which are segments of the base line correspond to the leaves of the tree. Each cell of the diagram corresponds to a caret - the top edge of the boundary of the cell is connected to each of the bottom edges of the boundary, the left edge and the right edge. Conversely each rooted, ordered binary tree is isomorphic to a tree obtained from such a diagram.

Trees $R$ and $S$ have the same number of leaves if and only if the diagrams have the same number of vertices. Then the pair $(R, S)$ forms a tree diagram and the pair of the corresponding plane diagrams forms a double diagram. The reduction in the double diagrams and the product of the double diagrams is defined as in the tree diagrams.

We may represent a double diagram $(R, S)$ in a different way. We put diagram $S$ upside down reflecting it with respect to the $x$-axis. We denote the reflected diagram by $S'$ and we put it under the diagram $R$ with matching vertices. If a cell of $R$ and a cell of $S'$ have two common edges along the $x$ axis then the cells correspond to a pair of cancelling carets. We contract both cells to a segment of the base line removing the top edge of the cell above the base line and removing the bottom edge of the cell below the base line and cancelling the middle vertex of the cells.

Figure 3 shows the double diagrams representing the generators of $F$.

There is a straightforward correspondence between the trees and the diagrams. Trees are composed of carets and diagrams are composed of cells. All remarks and lemmas from the previous sub-section translate directly into the
language of diagrams. In particular Remark 8 describes the multiplication of a "big" diagram by a "small" diagram.

Before we proceed to explain the result of the multiplication of a diagram by a generator we add some more structure. Each diagram has a top cell the top edge of which coincides with the top edge of the diagram. The middle vertex of the top cell will be called the base point of the diagram. It is denoted by a larger dot in Figure 4. Each edge of the diagram is painted red or blue. An edge is painted blue if it is not contained in the base line and one of its end-points is equal to $v_1$ or to $v_M$, the left-most or the right-most vertex of $D$. The edge is painted red otherwise. Among the red edges there are maximal (or outside) red edges which are not covered from above by another red edge. Each vertex which is an end-point of a maximal red edge will be called an accessible vertex. It is not covered from above by a red edge. The base point is at one of the accessible vertices. The blue edges will be ignored. They are determined by the red edges and the base point of the diagram. Indeed if we know all the red edges and the base point then the blue edges with an end point at $v_1$ are exactly the semicircles which connect $v_1$ with the accessible vertices on the left side of the base point, including the base point, and the blue edges with an end-point at $v_M$ are the semicircles which connect $v_M$ with the accessible vertices on the right side of the base point, including the base point, and there is one more blue edge - the top edge of the diagram which connects $v_1$ and $v_M$. 

![Figure 3: Double diagrams a and b](image)
Figure 4: Multiplication by the generators
We consider diagrams instead of trees because when we remove the vertices of the tree corresponding to the blue edges we get a forest with an unfamiliar structure while when we remove the blue edges and the cells containing them we get again a diagram over the presentation $P = \langle a | a = aa \rangle$. Diagrams without the blue edges allow a simpler description of the action of the generators.

We now come back to the action of the generators on a diagram. Let $(D, E)$ be a reduced double diagram and assume that each leaf of the tree corresponding to $D$ has the depth at least 4. Since the trees which form the parts of the generators $a$ and $b$ of $F$ have the depth at most 3 the multiplication by a generator does not change part $E$ of the double diagram, by Remark 8, and it suffices to understand the change in the diagram $D$.

In order to multiply $(D, E)$ by a generator $a$ we seek in the diagram $D$ a sub-diagram $A$ isotopic to the right part of the double diagram $a$ from Figure 3 with the top edge of the diagram $A$ equal to the top edge of $D$, and we replace it by the left part of the diagram $a$ from Figure 3. We give a precise description. Each diagram in $a$ (Figure 3) is composed of two cells and its bottom path has three edges. When we remove from $D$ the two cells corresponding to the right diagram of $a$ the remaining diagram $\hat{D}$ has three edges in the top path. We attach the left diagram of $a$ above the diagram $\hat{D}$ matching the edges of the bottom path of the left diagram of $a$ to the edges of the top path of $\hat{D}$ and get a diagram $D_1$. We write $D_1 = Da$, the multiplication of $D$ by $a$. The subdiagrams of $D$ isotopic to right or left diagram in $a$ or $b$ are drawn in thicker lines in the left part of Figure 4 and their replacements by the other part of the generator are drawn in thicker lines in the right part of Figure 4. The multiplication by $a$ and $b$ replaces the right part of the generator by the left part and the multiplication by $\overline{a}$ and $\overline{b}$ replaces the left part of the generator by the right part. The results of the multiplication are shown in Figure 4.

We now remove the blue edges from the diagrams. The diagrams without the blue edges will be called the essential diagrams. The result of the multiplication by a generator restricted to an essential diagram is shown in Figure 5. Figure 5 shows only the maximal red edges and some interior red edges near the base point. We put the result in the form of a lemma.

**Lemma 9.** The multiplication by $a$ does not change the edges of the essential diagram and moves the base point to the next accessible vertex on the left.

The multiplication by $\overline{a}$ does not change the edges of the essential diagram and moves the base point to the next accessible vertex on the right.

The multiplication by $b$ removes the maximal edge with the left end at the
Figure 5: Multiplication by the generators, the essential diagrams.

The multiplication by $\bar{b}$ inserts a new maximal edge with the left end at the base point, which covers from above exactly two former maximal edges.

4 A proof of the main theorem

In this section we shall prove

Theorem 10. The Richard Thompson group $F$ is not amenable

We fix $R = 8$ till the end of the paper (see Definition 4). We shall prove that if $W = c_1c_2 \ldots c_n$ is a word in letters $a, b, \pi, \bar{b}$ and if the number of non-isolated regular pairs is greater than $0.999n$ then $W$ is not a relator in the group $F$. Therefore $F$ is not amenable by Lemma 5.

We shall say that a tree $T$ is sufficiently deep for a word $W$ of length $n$ if each leaf of $T$ has depth at least $4n$. Let $V = (T, S)$ be a reduced tree diagram with $T$ sufficiently deep for a word $W$. Then if we multiply $V$ on the right by the first letter of $W$ we get a new reduced tree diagram with the left tree
sufficiently deep for the remaining word, by Remark 8. In particular the tree $S$ of the tree diagram does not change after this action so we do not need to mention $S$.

**Remark 11.** An edge of a tree belongs to a caret. It corresponds to a pair of the boundary edges of a cell consisting of the top edge and of one of the bottom edges. It follows that the depth of the right-most leaf of a diagram (the right-most edge in the base line) is equal to the number of the blue edges with the end-point at $v_M$ and is equal to the number of the maximal red edges on the right side of the base-point. If the tree $T$ is sufficiently deep for the word $W$ with $n$ letters and we apply the consecutive letters of $W$ to $T$ we each time get a tree the leaves of which have depth at least $n$. In particular if we come to a letter $b$ in $W$ its action produces a new maximal red edge, not a blue edge (a blue edge would mean that there are only two maximal red edges to the right of the base point).

We assign a degree of irregularity to a word $W = c_1c_2 \ldots c_n$.

**Definition 12.** Let $W = c_1c_2 \ldots c_n$ be a word in $F$. We assign the weight to each pair $c_i c_{i+1}$ according to the following table.

- Pairs $ab$, $ba$, $\overline{ab}$, $\overline{ba}$ have the weight 200.
- Pairs $a\underline{a}$, $\underline{aa}$, $b\underline{b}$, $\underline{bb}$ have the weight 160.
- Pairs $aa$, $\underline{a}a$, $bb$, $\underline{b}b$ have the weight 120.
- An isolated regular pair has the weight 1.
- A non-isolated regular pair has the weight 0.

The sum of the weights of the irregular pairs in $W$ is denoted by $\sigma$.

The sum of the weights of the isolated regular pairs (the number the isolated regular pairs) is denoted by $\rho$.

The **irregularity** of $W$ is denoted by $\kappa$ and is equal to $\sigma + \rho$.

**Remark 13.** By Definition 4 if we move one letter from the end of $W$ to its beginning then all pairs of consecutive letters and all subwords in the new word are the same as in $W$ so the numbers $\sigma$, $\rho$ and $\kappa$ are the same for both words.

The irregularity of the word $W$ is at most 200 times the number of the irregular pairs and the isolated regular pairs together.

It follows that if there are at least $0.999n$ non-isolated regular pairs in $W$ then $\kappa < 200(n - 0.999n) = 0.2n < 0.25n$, hence $n > 4\kappa$.

Suppose there exists a relator $W = c_1c_2 \ldots c_n$ in $F$ which has at least $0.999n$ non-isolated regular pairs. In particular it satisfies $n > 4\kappa$. At this stage we do not assume that $W$ is a reduced word.
Remark 14. Observe that not all pairs $c_ic_{i+1}$ in a relator $W$ are regular. The defining relations of $F$ are commutators so the $a$-degree and the $b$ degree of each relator is equal to zero. If all pairs $c_ic_{i+1}$ are regular then $W = (aba)^k$ and it is easy to check, considering the action of $W$ on a sufficiently deep tree, that such a word is not a relator. In fact there are no elements of finite order in $F$. If $W$ is a relator which satisfies $n > 4\kappa$ then it cannot be very short, $n \geq 480$.

If a letter, say $x$, denotes one of the letters in $W$ then it may be any of the generators $a, \overline{a}, b, \overline{b}$ and $\overline{x}$ denotes the inverse of $x$.

We shall prove later that if $W$ is a relator in $F$ then there exist two disjoint subwords of $W$, each of the length at most 3, such that when we remove these subwords we get again a relator in $W$. We want to know how the irregularity $\kappa$ of the word $W$ changes when we remove a subword and in particular how the part $\rho$ of $\kappa$ changes (see Definition 4).

Lemma 15. Let $W = c_1c_2\ldots c_n$ be a word in $F$. Let $V$ be a short subword of $W$. Let $c_i$ be the left neighbor of $V$ and let $c_j$ be the right neighbor of $V$ in $W$. Let $P$ be the maximal regular subword of $W$ containing $c_i$ and let $Q$ be the maximal regular subword of $W$ containing $c_j$. (It may happen that $P = Q$.) Let $P_1$ be the initial part of $P$ which ends at $c_i$ and let $Q_1$ be the final part of $Q$ which begins at $c_j$. Let $W_1$ be the word which we obtain from $W$ when we remove $V$ and concatenate the remaining parts. Then all pairs $c_ic_{i+1}, c_{i+1}c_{i+2}, \ldots, c_{j-1}c_j$ vanish in $W_1$ and one new pair $c_ic_j$ appears in $W_1$. Let $\rho$ be the number of the isolated regular words in $W$ and let $\rho_1$ be the number of the isolated regular pairs in $W$. Let $m$ be the number of the regular pairs in $W$ which vanish in $W_1$.

If each of the words $P$ and $Q$ is either short or disjoint from $V$ then $\rho - \rho_1 = m$ if $c_ic_j$ is irregular and $\rho - \rho_1 \geq m - 1$ if $c_ic_j$ is regular.

If $P$ or $Q$ is a long word then $\rho_1 - \rho \leq 14$.

Proof. If $P$ and $Q$ are short words then all the regular pairs in $P$ and $Q$ and $V$ are isolated. Their number decreases by $m$ if $c_ic_j$ is irregular and by $m - 1$ if $c_ic_j$ is regular, but in the last case the pair $c_ic_j$ may form a long word with $P_1$ and $Q_1$ and additional pairs which are isolated in $W$ may become non-isolated in $W_1$.

If $P$ is disjoint from $V$ and is long then all of its pairs are non-isolated in $W$ and in $W_1$ so the count of the isolated regular pairs is the same as before.

If $P_1$ is a long word then all pairs in $P_1$ are non-isolated in $W$ and in $W_1$. There are no new isolated pairs on the left side of $c_ic_j$ in $W_1$. 


If $P$ is a long word and $P_1$ is a short word then the pairs in $P_1$ are non-isolated in $W$ but may become isolated in $W_1$. There are at most 7 such pairs. In a similar way if $Q$ is a long word and $Q_1$ is a short word then the pairs in $Q_1$ are non-isolated in $W$ but may become isolated in $W_1$. There are at most 7 such pairs. There is just one more new pair in $W_1$ which may be regular, the pair $c_ic_j$. If $c_ic_j$ is regular and there are 7 regular pairs in $P_1$ or in $Q_1$ then $c_ic_j$ together with $P_1$ and $Q_1$ form a long regular word and none of the pairs is isolated. All the pairs outside of $P$ and $Q$ and $V$ are not affected. Therefore $\rho_1 - \rho \leq 14$.

We now check the effect of reducing one cancelling pair in the case when $W$ is not reduced.

**Lemma 16.** Let $W$ be a word of length $n$ and of irregularity $\kappa$ satisfying $n > 4\kappa$ and suppose that $W$ contains a cancelling pair $xx^{-1}$. Let $W_1$ be the word obtained by cancelling the pair $xx^{-1}$ and let $n_1 = n - 2$ and $\kappa_1$ be respectively the length and the irregularity of $W_1$. Then $n_1 > 4\kappa_1$.

**Proof.** We want to prove that $\kappa_1 < \kappa$. Consider a longer subword $qx\bar{x}u$ of $W$ containing the cancelling pair $x\bar{x}$. It is replaced by $qu$. The sum of weights of two different irregular pairs is bigger at least by 40 than a weight of one irregular pair. By Lemma 15 the number of isolated regular pairs increases at most by 14. In our case three pairs: $qx$, $x\bar{x}$ and $\bar{x}u$ vanish and one new pair $qu$ appears. We only need to consider the case when both pairs $qx$ and $\bar{x}u$ are regular. But then, by the definition of the regular pairs, $q = u$. The pair $x\bar{x}$ is replaced by $qq$ and the weight goes down by 40. It follows that $\kappa \geq \kappa_1 + 1$. Since $n > 4\kappa$ we get $n_1 = n - 2 > 4\kappa - 2 \geq 4\kappa_1 + 2 > 4\kappa_1$ so the new word $W_1$ is shorter than $W$ and satisfies $n_1 > 4\kappa_1$. $\Box$

It follows that the relator $W$ of the minimal length which satisfy $n > 4\kappa$ is reduced and cyclically reduced.

Consider now such a relator $W$ and its action on an element $V$ of $F$ represented by a tree diagram $(T, S)$ with $T$ sufficiently deep for $W$.

It suffices to consider the action on the tree $T$ and we may represent $T$ by an essential diagram. There is a base-point $v_0$ of the diagram. When we apply the consecutive letters of $W$ the base line of the diagram with its edges and vertices does not change, by Remark 8 but the other edges may change producing new diagrams. At each step the base-point of the diagram either stays put or moves to the next accessible vertex to the left or to the right and travels along a closed loop.
Lemma 17. Suppose $U$ is a word in $F$ and $T$ is a tree sufficiently deep for $U$. Consider the essential diagram of $T$ and consider the action of $U$ on $T$. Let $v_1$ be the base point of the diagram. Suppose that $U$ begins with the letter $a$ and when we apply the consecutive letters of $U$ the base point never comes to $v_1$ or to a vertex on the right side of $v_1$ and only at the end, after we apply the last letter of $U$, the base point comes back to $v_1$. Moreover assume that $v_1$ is always accessible, that it is not covered from above by a red edge for any essential diagram of the orbit of $U$. Then $U$ commutes with $b$.

Proof. Suppose the lemma is not true. Let $U$ be the shortest word for which there exists a sufficiently deep tree such that the assumptions of the lemma are satisfied and $U$ does not commute with $b$. If $U$ is not reduced then reducing one cancelling pair will produce a shorter word with the same properties, so $U$ must be reduced.

Consider the word $bUb$. When we apply $b$ a new edge appears. Its left end is at $v_1$ and it covers two former consecutive maximal edges on the right side of $v_1$. We now apply the consecutive letters of $U$. Their action is restricted to the left side of $v_1$. The base point never comes to $v_1$ except at the last moment and the edges produced or removed by the action of $b$ or the action of $b$ lie entirely on the left side of $v_1$. At the end, when the base point returned to $v_1$, we apply $b$ and this removes the edge which appeared in the first step.

If we omit the first step and start with the first letter of $U$ and continue we pass through the same moves except for the new edge on the right side of $v_1$, which does not exist now. When we come to the last letter of $U$ we have the same situation as after the action of $bUb$. Therefore $U$ commutes with $b$ which contradicts the assumptions that there may exist such $U$ which does not commute with $b$. \[\square\]

Lemma 18. Suppose that the word $W = c_1c_2\ldots c_n$ is a relator in $F$ which is reduced and cyclically reduced. Then there exists a subword $U_1$ of $W$ (where $W$ is considered as a cyclic word) of the form

$$U_1 = xpb^k atVyb^s qz$$

such that

- $U = atVyb$ commutes with $b$,
- $k, s$ are positive integers,
- $p$ and $q$ belong to $\{a, \bar{a}\}$,
- $y \neq b$ and if $p = \bar{a}$ then $x \neq b$.

Moreover
if $s = 1$, $k = 1$, $x = \overline{a}$, $p = \overline{a}$, $t = b$ and $q = \overline{a}$ then $q$ does not belong to a long regular subword of $W$ and $t$ does not belong to a long regular subword of $W$.

Proof. We fix a tree $T$ regular of depth $8n$. In particular $T$ is sufficiently deep for $W$. We consider the corresponding essential diagram. We apply the consecutive letters of $W$ to the diagram and watch the consecutive positions of the base point. Let $U_1$ be any initial subword of $W$ and let $U_2$ be the complementary subword, so that $W = U_1U_2$. Let $T_1$ be the essential diagram obtained after the application of $U_1$ to $T$ and let $v_1$ be the base-point of $T_1$. Then $W_1 = U_2U_1$ is also a relator in $F$ and if we start with the diagram $T_1$ and apply the consecutive letters of $W_1$ we obtain the same diagrams (the same orbit) as in the action of $W$ on $T$, in the same cyclic order, but we start in a different place. The tree $T_1$ is sufficiently deep for $W_1$. So we may start our journey at any point of the orbit of the word $W$.

We shall prove that there exists a point (an essential diagram) of the orbit of $W$ with the base point $v_1$ such that if we start at this point along the orbit and apply the consecutive letters of the corresponding word $W_1$ then there is an initial subword $U$ of $W_1$ which satisfies the assumptions of Lemma 17, and if we choose it carefully the claim of the Lemma will be satisfied.

Let $v_0$ be the rightmost vertex which becomes the base-point at some point of the orbit of $W$. By the previous remark we may start the journey along the orbit of $W$ one step before $v_0$, so that the first letter of $W$ is $\overline{a}$ and the application of $\overline{a}$ brings the base point to $v_0$. Suppose that at some point of the orbit the vertex $v_0$ is covered by a new red edge. (The other case will be treated at the end of the proof.) Among all edges which are produced along the orbit of $W$ and cover $v_0$ we choose the edges with the right end at the vertex with the largest number (the right-most vertex), say at vertex $v_2$. We choose the last edge (the edge which appears last along the orbit), call it $\alpha$, which covers $v_0$ and has the right end at $v_2$. Let $v_1$ be the left end of $\alpha$. We arrive at $v_1$ from the left or from the right, applying the letter $p$. The previous letter of $W$ is $x$. Next comes some positive power of $\overline{b}$ and the last letter $\overline{b}$ of this power produces the edge $\alpha$. Since $\alpha$ covers $v_0$ the next letter of $W$ must be $a$, because the letter $\overline{a}$ would put the base point on the right side of $v_0$. Now the base point is on the left side of $v_1$. Eventually the edge $\alpha$ must be removed because $v_0$ is accessible in the initial diagram of the orbit of $W$. In order to do it the base point must come back to $v_1$. Before it happens the base point cannot pass to the right of $v_1$, because the next accessible point on the right side of $v_1$ is on the right side of $v_0$, and no edge produced along the orbit
may cover \( v_1 \), because then it would cover \( v_0 \) and would end at \( v_2 \) (or further to the right) and \( \alpha \) was the last edge with this property. The base point comes to \( v_1 \) from the left as a result of the application of \( \overline{\pi} \) and the previous letter of \( W \) is \( y \), so we get a word \( U = a t V y \overline{\pi} \) which satisfies Lemma [17] and commutes with \( b \). The next letter of \( W \) must be \( b \) because \( \overline{\alpha} \) would take the base point to the right side of \( v_0 \) (the edge \( \alpha \) still exists) and \( \overline{b} \) would produce a new edge covering \( v_0 \) against the definition of \( \alpha \). So \( \overline{\alpha} \) is followed by a positive power of \( b \) and then by letters \( q \) and \( z \).

We now check the next claim.

Two steps before the end of \( U \), before applying \( y \), the base point was at \( v_3 \) on the left side of \( v_1 \). If \( y = b \) then a maximal edge between \( v_3 \) and \( v_1 \) is removed by \( b \) and now there is at least one more accessible vertex between \( v_3 \) and \( v_1 \) so \( \overline{\alpha} \) does not bring the base point to \( v_1 \) (and we know that it does). So \( y \neq b \).

If \( p = \overline{\alpha} \) the situation is similar. Letter \( p = \overline{\pi} \) takes the base point to the right from \( v_4 \) to \( v_1 \). In particular \( v_1 \) is accessible. If the previous letter is \( b \) then in the previous step the base point is at \( v_4 \) on the left side of \( v_1 \) and when we apply \( b \) a maximal edge between \( v_4 \) and \( v_1 \) is removed and now there is at least one more accessible vertex between \( v_4 \) and \( v_1 \) so \( \overline{\alpha} \) does not bring the base point to \( v_1 \) (and we know that it does). So \( x \neq b \).

Suppose now that \( s = 1 \) and \( q = \overline{\alpha} \) so the end of the word \( U_1 \) has the form \( y \overline{\alpha} \overline{\pi} \). If \( q \) (which is equal to the last \( \overline{\alpha} \)) belongs to a long regular subword of \( W \) (of the length at least 9) then the subword begins with the letter \( b \) after \( y \overline{\alpha} \) because the pair \( \overline{\alpha} \overline{b} \) is not regular. \( U_1 \) extends to the right in the following way

\[ y \overline{\alpha} \overline{\pi} \overline{\alpha} \overline{b} \overline{\pi} \ldots \]

The first move \( b \) removes the arc \( \alpha \) which connects vertices \( v_1 \) and \( v_2 \). By the structure of an essential diagram there is now exactly one accessible vertex \( v_3 \) between \( v_1 \) and \( v_2 \). Move \( \overline{\alpha} \) takes the base point to \( v_3 \). Next we apply \( \overline{b} \) which introduces a new edge which connects \( v_3 \) to a point \( v_4 \) on the right side of \( v_2 \). Since \( v_3 \) is a base point the vertex \( v_0 \) cannot lie on the left side of \( v_3 \) by the definition of \( v_0 \). It also cannot lie on the right side of \( v_3 \) because the new edge reaches beyond \( v_2 \) and if it covers \( v_0 \) we get a contradiction with the definition of \( v_2 \) and of \( \alpha \). There remains the possibility that \( v_3 = v_0 \). We continue our journey. Next move \( a \) brings the base point to \( v_1 \) again. Next move \( b \) removes the maximal arc connecting \( v_1 \) to \( v_0 \) and uncovers a new unique accessible vertex \( v_5 \) between \( v_1 \) and \( v_0 \). Next \( \overline{\alpha} \) takes the base point from \( v_1 \) to \( v_5 \). Next \( \overline{b} \) introduces a new edge which covers \( v_0 \) and connects \( v_5 \) to \( v_4 \) on the right side of \( v_2 \) and we get a contradiction with the definition of \( v_2 \). Therefore \( q \) does not
belong to a long regular subword of $W$ in this case.

Suppose next that $k = 1$ and $p = \overline{a}$ and $x = \overline{a}$. Then the beginning of $U_1$ has the form $\overline{a}a\overline{a}ab$. If the letter $t$ (which is equal to the last $b$) belongs to a long regular subword of $W$ then the subword begins at the second letter \overline{a} because the pair $\overline{a}a$ is not regular. Therefore the beginning of $U_1$ must be $\overline{a}a\overline{a}ab\ldots$

Thus, after constructing $\alpha$ with the move $b$ we move the base point to the left to the next accessible vertex, say $v_3$, with the move $a$. Next we remove the maximal arc connecting $v_3$ and $v_1$ with the move $b$ and we introduce a new accessible vertex $v_4$ between $v_3$ and $v_1$. Next we move the base point to the right to vertex $v_4$ with the move $\overline{a}$. Next we apply $\overline{b}$ and it introduces a new maximal edge which covers the edge connecting $v_4$ to $v_1$ and the next edge, the edge $\alpha$. Thus the new edge reaches from $v_4$ to $v_2$. This is forbidden, the edge $\alpha$ was the last edge which covers $v_0$ and reaches to $v_2$. Therefore $t$ does not belong to a long regular subword of $W$ in this case.

Let us consider the case when $v_0$ is always accessible along the orbit of $W$. Then we let $v_1 = v_0$. The base point arrives at $v_1$ several times, each time from the left side after application of $\overline{a}$. When it arrives at $v_1$ the next letter must be $b$ which removes an edge on the right side of $v_1$, or $\overline{b}$ which introduces a new edge on the right side of $v_1$. After the power of $b$ comes $a$ because $\overline{a}$ would bring the base point to the right side of $v_1$. We must have a pair of consecutive arrivals of the base point at $v_1$ such that the first arrival is followed by a negative power of $b$ and the second arrival is followed by a positive power of $b$. The word $U$ in-between satisfies the assumptions of Lemma 17 so we get a subword $U = atVy\overline{a}$ of $W$ commuting with $b$. It is preceded by $x\overline{a}^k$ and followed by $b^*az$ so we have a subword $x\overline{a}^k atVy\overline{a}^*az$ of $W$. In particular the letters $p$ and $q$ from the claim of the lemma satisfy $p = \overline{a}$ and $q = a$ in this case. We repeat the proof of the properties of $x, y$ in this case.

If $x = b$ then $\overline{a}$ which follows it does not bring the base point to $v_1$.

If $y = b$ then $\overline{a}$ which follows it does not bring the base point to $v_1$.

Since in this case $q = a$ we do not need to check the last claim of the lemma.

\begin{lemma}
Suppose that the word $W$ is a relator in $F$ which is reduced and cyclically reduced, has the length $n$ and the irregularity $\kappa$ and satisfies $n > 4\kappa$.
Then there exists a shorter relator $W_1$ of length $n_1$ and irregularity $\kappa_1$ which satisfies $n_1 > 4\kappa_1$.
\end{lemma}

\textbf{Proof.} We fix a tree $T$ regular of depth $8n$. By Remark 13 we may change $W$
cyclically and \( \kappa \) will remain the same. We choose a subword

\[
U_1 = xpb^k atV y \overline{ab}^s qz
\]

of \( W \) as in Lemma \( \text{L18} \). In particular \( U = atV y \overline{a} \) commutes with \( b \) so we may cancell one \( b \) from \( b^s \) with one \( \overline{b} \) from \( \overline{b}^k \). We get a new shorter relator. We check what happens with \( \kappa \) and \( n \) when we do it.

We now consider separately what happens with \( \kappa \) when we replace \( b^s \) by \( b^{s-1} \) and what happens with \( \kappa \) when we replace \( \overline{b}^k \) by \( \overline{b}^{k-1} \). We want to prove that \( \kappa \) goes down.

We check first what happens with \( \sigma \), the part of \( \kappa \) related to the irregular pairs.

We consider the different cases.

1. \( s > 1 \). The subword \( b^s \) is replaced by \( b^{s-1} \). One pair \( bb \) vanishes and no new pair appears and \( \sigma \) goes down by 120.

2. \( s = 1 \) and \( q = a \). The word \( \overline{a}ba \) is replaced by \( \overline{a}a \). Irregular pairs \( \overline{ab} \) and \( ba \) are replaced by \( \overline{a}a \) and \( \sigma \) goes down by 240.

3. \( s = 1, q = \overline{a} \). The subword \( \overline{ab}a \) is replaced by \( \overline{a}a \). Irregular pair \( \overline{ab} \) vanishes and \( \overline{a}a \) appears and \( \sigma \) goes down by 80.

4. \( k > 1 \). The subword \( \overline{b}^k \) is replaced by \( \overline{b}^{k-1} \). One pair \( \overline{bb} \) vanishes and no new pair appears. \( \sigma \) goes down by 120.

5. \( k = 1, p = \overline{a} \). Subword \( x\overline{ab}at \) is replaced by \( x\overline{a}at \) and we reduce it further to \( xt \). Since \( x \neq b \) the pair \( x\overline{a} \) is irregular. Its weight is equal to 120 at least. It is removed and one new pair \( xt \) appears. Its weight is equal to 200 at most. Therefore \( \sigma \) is increased by 80 at most. This happens when \( x\overline{a} \) has weight 120, and thus \( x = \overline{a} \) and \( xt \) has weight 200 and thus \( t = b \). In the other cases \( \sigma \) increases by 40 at most.

6. \( k = 1, p = a \). Subword \( \overline{aba} \) is replaced by \( aa \). The pair \( \overline{ab} \) of weight 200 vanishes and the pair \( aa \) of weight 120 appears. \( \sigma \) goes down by 80.

Each of the cases 1,2,3 has to be matched with each of the cases 4,5,6. For each pair of cases we remove two disjoint subwords from the word \( W \). By Lemma \( \text{L15} \) \( \rho \) goes up by at most 28. By the above discussion for each pair of cases the number \( \sigma \) goes down by 40 at least, and then \( \kappa \) goes down, with the exception of the case \( s = 1, q = \overline{a}, k = 1, p = \overline{a}, x = \overline{a}, t = b \) and then \( \sigma \).
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does not change. This case matches exactly the assumptions of the last claim of Lemma \[18\]

It remains to consider carefully the last case and check what happens with \( \rho \).

The subword \( x \rho \overline{b}^k U \rho^* q \rho z \) of \( W \) has now the form

\[
U_1 = \overline{aabab} V y \overline{a} \overline{a} z
\]

and is replaced by

\[
\overline{a} b V y \overline{a} \overline{a} z.
\]

We have to compute the change in \( \rho \). Consider Lemma \[15\] Disjoint subwords \( \overline{a} \overline{a} b \) and \( b \) are removed. We shall prove that for each of them the corresponding subwords \( P \) and \( Q \) are short or disjoint from the vanishing part.

1. The first letter of \( U_1 \) (the left neighbor of the subword \( \overline{a} \overline{a} b \) which vanishes in \( W_1 \)) belongs to a maximal regular subword of \( W \) disjoint from the second letter because the pair \( \overline{a} \overline{a} \) is not regular.

2. The fifth letter of \( U_1 \) (letter \( t = b \), the right neighbor of \( \overline{a} \overline{a} b \)) does not belong to a long regular subword of \( W \), by Lemma \[18\]

3. The fourth letter from the end of \( U_1 \) (letter \( \overline{a} \), the left neighbor of the vanishing \( b \)) does not belong to a long regular subword of \( W \) because it forms an irregular pairs with its left and right neighbor (\( y \neq b \)).

4. The second letter from the end of \( U_1 \) (letter \( q = \overline{a} \), the right neighbor of the vanishing \( b \)) does not belong to a long regular subword of \( W \) by Lemma \[18\]

We now use Lemma \[15\]. We are in the first case of the lemma so \( \rho \) decreases by the number of regular pairs in \( W \) which vanish in \( W_1 \) minus possibly the two new pairs in \( W_1 \) if they are regular. On the left side of \( U_1 \) there are three isolated regular pairs \( \overline{a} \overline{b} \) and \( \overline{b} a \) and \( a b \) which vanish and the new pair \( \overline{a} b \) is not regular. On the right side of \( U_1 \) we have one isolated regular pair \( b \overline{a} \) which vanishes and the new pair \( \overline{a} \overline{a} \) is not regular. Therefore \( \rho \) goes down by 4 and the whole \( \kappa \) goes down.

If we denote by \( n_1 \) and \( \kappa_1 \) the length and the irregularity of the new word \( W_1 \) then \( n_1 \geq n - 4 \) (we reduce \( b \) and \( \overline{b} \) and in case 5 also \( a \) and \( \overline{a} \)) and \( \kappa_1 \leq \kappa - 1 \), therefore

\[
n_1 \geq n - 4 > 4 \kappa - 4 \geq 4(\kappa_1 + 1) - 4 = 4 \kappa_1.
\]

\( \square \)
This concludes the proof of Theorem 10. If $F$ is amenable then by Lemma 5 there exists a relator $W = c_1c_2\ldots c_n$ in $F$ such that the number of non-isolated regular pairs in $W$ is greater than $0.999n$. Then $W$ also satisfies the condition $n > 4\kappa$, by Remark 13. We may assume that $W$ is the shortest such relator. By Lemma 19 there exists a shorter relator which also satisfies $n \geq 4\kappa$. This is a contradiction which shows that group $F$ is not amenable.

References

[BG] Kenneth S. Brown; R. Geoghegan, *An infinite-dimensional torsion-free $FP_\infty$ group*, Invent. Math. 77 (1984) 367-382

[BG1] Kenneth S. Brown; R. Geoghegan, *Cohomology with free coefficients of the fundamental group of a graph of groups*, Comment. Math. Helv. 60 (1985) 31-45

[BO] N. P. Brown, N. Osawa *$C^*$-Algebras and Finite-Dimensional Approximation*, GraduateStudies in Mathematics, 88 AMS, Providence, RI, 2008.

[BS] Matthew G. Brin, Craig C. Squier, *Groups of piecewise linear homeomorphisms of the real line*, Invent. Math. 79 (1985) 485-498

[CFP] J. W. Cannon, W. J. Floyd, and W. R. Parry *Introductory notes on Richard Thompson's groups*, L'Enseignement Mathematique, 42 (1996), 215-256.

[D] Day, Mahlon M. *Amenable semigroups*, Illinois Journal of Mathematics 1 (1957), no. 4, 509-544.

[G] R. I. Grigorchuk, *An example of a finitely presented amenable group not belonging to the class $EG$*, Sb. Math. 189 (1998) 79-100.

[GS1] V. Guba, M. Sapir, *Diagram groups*, Mem. Amer. Math. Soc. 130 (1997).

[GS2] Guba, V. S.; Sapir, M. V. *On subgroups of the R. Thompson group $F$ and other diagram groups*, (Russian) Mat. Sb. 190 (1999), no. 8, 3–60; translation in Sb. Math. 190

[Gab] D. Gaboriau *Coût des relations d’équivalence et des groupes*, Invent. Math. 139 (2000) 41-98.

[GS] S. M. Gersten and John R. Stallings, *Combinatorial group theory and topology*, Princeton University Press, 1987.

[HP] U. Haagerup, G. Picioroaga, *New presentations of Thompson’s groups and applications*, J. Operator Theory 66 (2011) 217-232.

[H] D. Hadwin, *Strongly quasidiagonal $C^*$-algebras (with an Appendix by Jonathan Rosenberg)*, J. Operator Theory 18 (1987) 3-18.

[M] Monod, N, *Groups of piecewise projective homeomorphisms*, Proceedings of the National Academy of Sciences of the United States of America 110 (2013) 4524-4527.

[O] Ol’shanskii, A., *On the question of the existence of an invariant mean on a group*, Uspekhi Mat. Nauk (in Russian) 35 (1980) 199-200

[OS] Ol’shanskii, A.; Sapir, M., *Non-amenable finitely presented torsion-by-cyclic groups*, Publications Mathématiques de l’IHÉS 96, (2003)