Seiberg-Witten invariants of non-simple type and Einstein metrics

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Abstract
We construct examples of four dimensional manifolds with Spin' structures, whose moduli spaces of solutions to the Seiberg-Witten equations, represent a non-trivial bordism class of positive dimension, i.e. the Spin' structures are not induced by almost complex structures. As an application, we show the existence of infinitely many non-homeomorphic compact oriented 4-manifolds with free fundamental group and predetermined Euler characteristic and signature that do not carry Einstein metrics (see [10]).

1 Introduction
A smooth Riemannian manifold (M, g) is said to be Einstein if its Ricci curvature tensor r is a multiple of the metric i.e.
\[ r = \lambda g. \]

Not every smooth compact oriented 4-manifold admits such a metric. A well known obstruction is given by the following result due to N. Hitchin and J. Thorpe (see [2]). If M is a compact oriented 4-manifold and \( e(M) < \frac{3}{2} |\sigma(M)| \) then M does not admit an Einstein metric, where e and \( \sigma \) respectively denote the Euler characteristic and the signature. The Gauss-Bonnet-like formula
\[
2e(M) \pm 3\sigma(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} - \frac{|r_0|^2}{2} + 2|W_\pm|^2 \right) d\mu,
\]
implies Hitchin-Thorpe’s inequality because Einstein metrics are characterized by the vanishing of \( r_0 \), and this is the only negative term in the above integrand. Here \( s, r_0, W_+, W_- \) respectively denote the scalar, trace-free Ricci, self-dual Weyl, and anti-self-dual Weyl curvature tensors of a Riemannian metric.

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As C. LeBrun showed in [6] this result can be improved using careful estimates on the $L^2$-norm of the scalar curvature tensor $s$ and the $L^2$-norm of the self-dual part of the Weyl tensor $W_+$ arising from the Seiberg-Witten equations if, for example, the smooth 4-manifold $M$, admits a symplectic form. To obtain these estimates C. LeBrun used that such an $M$ admits irreducible solutions to the Seiberg-Witten equations for every metric $g$ rather than actually using the fact that $M$ has non-trivial Seiberg-Witten invariant. Our main result is

**Theorem A.** Let $(M, c)$ be a smooth compact Kähler surface with a Spin$^c$-structure $c$. There is a canonical Spin$^c$ structure in the connected sum manifold $M \# (S^1 \times S^3)$ which we will denote by $c_{0,1}$. Moreover $d(c_{0,1}) = d(c) + 1$. If $c$ is a non-trivial SW-class for $M$ then $c_{0,1}$ is a $B$-class for the connected sum $M \# (S^1 \times S^3)$.

The equality $d(c_{0,1}) = d(c) + 1$ implies that $c_{0,1}$ is not induced by an almost complex structure, and the statement $c_{0,1}$ is a $B$-class implies that there exist irreducible solutions to the Seiberg-Witten equations for every Riemannian metric. The technique that we have used to produce these Spin$^c$-structures does not rely on the well-known gluing-argument (compare with [9]).

The main application of our result is

**Theorem B.** For each admissible pair $(m,n)$ there exist an infinite number of non-homeomorphic compact oriented 4-manifolds which have Euler characteristic $m$, signature $n$, with free fundamental group and which do not admit an Einstein metric.

Similar examples but with very complicated fundamental group have been obtained by A. Sambusetti [10] using connected sums with real or complex hyperbolic 4-manifolds.

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## 2 SW-Moduli Space

**Definition 1.** Let $(M, c)$ be a smooth compact oriented 4-manifold with a Spin$^c$-structure $c$. Let $L_c = \det(c)$ be the determinant line bundle associated to $c$. Fix a Riemannian metric $g$ on $M$. The configuration space $C(c)$ consist of pairs $(A, \phi)$, where $A$ is an $U(1)$-connection on $L_c$ and $\phi \in C^\infty(S^+(c))$ is a self-dual spinor. We say that $(A, \phi)$ satisfy the Seiberg-Witten equations (SW-equations) if and only if

\[
D_A \phi = 0 \\
F_A^+ = q(\phi),
\]

where $q(\phi) = \phi \otimes \phi^* - \frac{|\phi|^2}{2} \text{Id.}$
Remark. $D_A$ is the associated Dirac operator of the Spin$^c$-bundle, and $F^+_A$ is the self-dual part of the curvature associated to the connection $A$, thought of as an endomorphism of the self-dual spinors.

**Definition 2.** We say that an element $(A, \phi)$ is irreducible if $\phi \not\equiv 0$, otherwise it is reducible. We denote by $C^*(\mathfrak{c})$ the open subset of irreducible configurations, by $\mathcal{G}(\mathfrak{c}) = \{ \sigma : M \to S^1 \}$ the gauge group, and by $\mathcal{B}^*(\mathfrak{c}) = C^*(\mathfrak{c})/\mathcal{G}(\mathfrak{c})$ the open subset of irreducible equivalence classes.

The naive definition of the Seiberg-Witten moduli space would be:

$$\mathcal{M}_g(\mathfrak{c}) = \{ (A, \phi) \in C(\mathfrak{c}) \mid D_A \phi = 0, \quad F_A^+ = q(\phi) \}/\mathcal{G}(\mathfrak{c}),$$

but in order to use the usual analytical tools, one has to extend the $C^\infty$ objects to appropriate Sobolev spaces. From now on we extend the configuration space $\mathcal{A}(\mathfrak{c})$ and the gauge group $\mathcal{G}(\mathfrak{c})$ by requiring $A$ and $\phi$ to be in $L^2$ and $\sigma$ to be in $L^2$. The SW-equations and the gauge actions make sense in this context also and we define:

**Definition 3.** The Seiberg-Witten moduli space is:

$$\mathcal{M}_g(\mathfrak{c}) = \{ (A, \phi) \in C(\mathfrak{c}) \mid D_A \phi = 0, \quad F_A^+ = q(\phi) \}/\mathcal{G}(\mathfrak{c}),$$

where $\mathcal{A}(\mathfrak{c})$ and $\mathcal{G}(\mathfrak{c})$ are the extended configuration space and gauge group.

The formal dimension (computed using the Atiyah-Singer index theorem) of this moduli space is

$$d(\mathfrak{c}) = \frac{c_1^2(\mathfrak{c}) - (2\varepsilon(M) + 3\sigma(M))}{4}.$$ 

In general there is no reason to expect that the moduli space form a smooth manifold. The best we can hope for is that generically it does. The next Theorem guarantees that this is the case. For the proof see [8].

**Theorem 1.** Suppose that $b_2^+ > 0$. Fix a metric $g$ on $M$. Then for a generic $C^\infty$ self-dual 2-form $h$ on $M$ the following holds. For any Spin$^c$-structure $\mathfrak{c}$ on $M$ the moduli space $\mathcal{M}_g(\mathfrak{c}, h) \subset \mathcal{B}(\mathfrak{c})$ of gauge equivalence classes of pairs $[A, \phi]$ which are solutions to the perturbed SW-equations

$$D_A \phi = 0$$

$$F_A^+ - q(\phi) = ih$$

form a smooth compact submanifold of $\mathcal{B}^*(\mathfrak{c})$ of dimension $d(\mathfrak{c})$.

Also in [8] it is shown that if $b_2^+ > 1$ then the bordism class of $\mathcal{M}_g(\mathfrak{c}, h)$ is an invariant of the smooth structure of $M$ and the Spin$^c$-structure $\mathfrak{c}$ on $M$. We will denote by $\mathcal{M}(\mathfrak{c})$ this bordism class.

**Proposition 2.** Consider a fixed $U(1)$-connection $A$ on $L_\mathfrak{c}$. Let $[A_i, \phi_i]$ be solutions to the SW-equations, and let $\{A_i, \phi_i\}$ be the unique representatives such that $A_i - A$ is co-closed (gauge fixing condition, see [8]), for $i = 1, 2$. If $\phi_1 = \phi_2$ then $A_1 = A_2$. 

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Proof. The first thing to notice is that \( A_2 = A_1 + \theta \), where \( \theta \) is a co-closed 1-form. Since \((A_1, \phi_1)\) and \((A_2, \phi_2)\) are solutions to the SW-equations we have

\[
F_{A_1}^+ = q(\phi_1) \quad = q(\phi_2) \quad = F_{A_2}^+.
\]

Therefore

\[
F_{A_2}^+ - F_{A_1}^+ = 0 \iff (d\theta)^+ = 0
\]
\[
\iff *d\theta = -d\theta
\]
\[
\Rightarrow d * d\theta = -dd\theta = 0
\]
\[
\iff *d * d\theta = 0.
\]

This last statement and the fact \( \delta \theta = 0 \) implies that

\[
\Delta \theta = d\delta\theta + \delta d\theta
\]
\[
= \delta d\theta
\]
\[
= 0.
\]

Since \((A_i, \phi_i)\) \( i = 1, 2 \) are solutions to the Seiberg-Witten equations we have

\[
0 = D_{A_2} \phi_2 = D_{A_1 + \theta} \phi_1 = D_{A_1} \phi_1 + \theta \cdot \phi_1 = \theta \cdot \phi_1,
\]

multipling by \( \theta \) both sides of the equality we get that \(|\theta|^2 \phi_1 = 0\). Taking the point-wise norm we will have \(|\theta|^2 |\phi_1| = 0\). If we denote by \( Z_{|\theta|^2} \) and \( Z_{|\phi_1|} \) the set of points where \(|\theta|^2 \) and \(|\phi_1| \) vanish respectively, and we denote by \( Z_{|\theta|^2}^c \) and \( Z_{|\phi_1|}^c \) their corresponding complements, we will have that \( Z_{|\phi_1|}^c \subset Z_{|\theta|^2} \), therefore if \([A_1, \phi_1]\) is not a reducible solution then \( Z_{|\phi_1|}^c \) is a non-empty open set. By a result of N. Aronszajn (see [1]) we will have that \( \theta = 0 \), since it vanishes in an open set.

Since \( C(c) \) is an affine space it is contractible. Also the space of reducible configurations \( A(c) \times \{0\} \) is contractible and has infinite codimension in \( C(c) \). Since \( C^*(c) \) is open in \( C(c) \) and it is the complement of \( A(c) \times \{0\} \) then it is contractible. \( B^*(c) = C^*(c)/G(c) \) is the classifying space of \( G(c) = Map(M, S^1) \) since \( G(c) \) acts freely on \( C^*(c) \).

Moreover,

\[
Map(M, S^1) \sim Map(M, S^1)_o \times \pi_0(Map(M, S^1)),
\]
where $\text{Map}(M, S^1)_o$ denotes homotopically constant maps. $\text{Map}(M, S^1)_o$ can be identified with $S^1$, therefore $\text{Map}(M, S^1) \sim S^1 \times H^1(M; \mathbb{Z})$, so the classifying space for $\text{Map}(M, S^1)$ is weakly homotopically equivalent to $\mathbb{CP}^\infty \times \frac{H^1(M; \mathbb{R})}{H^1(M; \mathbb{Z})}$, and

$$H^*(B^*(\mathfrak{c}); \mathbb{Z}) \cong \mathbb{Z}[U] \otimes \Omega^* H^1(M; \mathbb{Z}),$$

(1)

where $U$ is a generator for $H^*(\mathbb{CP}^\infty; \mathbb{Z})$.

**Definition 4.** The Seiberg-Witten invariant $SW(\mathfrak{c})$ for the Spin$^c$-structure $\mathfrak{c}$ is defined as follows

$$SW(\mathfrak{c}) = \begin{cases} (U^{d(\mathfrak{c})/2}, M(\mathfrak{c})|_{B^*(\mathfrak{c})}) & \text{if } d(\mathfrak{c}) \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that this invariant is a cobordism invariant of the moduli space $M(\mathfrak{c})$, therefore it does not depend on the metric we used to define the Dirac operator, it does define an invariant of the smooth manifold $M$.

From this definition it is easy to see that we are loosing information about the moduli space. For example if the moduli space is odd dimensional this invariant is zero, even though the moduli itself may not represent a trivial bordism class in $B^*(\mathfrak{c})$.

**Definition 5.** Let $(M, \mathfrak{c})$ be a smooth compact oriented 4-manifold with a Spin$^c$-structure $\mathfrak{c}$. We will say that $\mathfrak{c}$ is a $B$-class if for some (then for any) Riemannian metric $g$ on $M$, the moduli space $M_g(\mathfrak{c})$ of irreducible solutions to the SW-equations is a smooth manifold of dimension $d(\mathfrak{c}) \geq 0$ that represents a non-trivial bordism class in $B^*(\mathfrak{c})$, i.e. there exists $\eta \in H^*(B^*(\mathfrak{c}); \mathbb{Z})$ of degree $d(\mathfrak{c})$ such that

$$\langle \eta, M(\mathfrak{c})|_{B^*(\mathfrak{c})} \rangle \neq 0.$$
Proposition 3. Let $D_A$ and $D'_A$ be the Dirac operators (induced by the $U(1)$-connection $A$) defined over the conformally related Riemannian manifolds $(M,g)$ and $(M, gf)$ respectively. Then

$$\Psi_f \circ D_A = D'_A \circ \Psi_f$$

Corollary 4. There is bijection between ker $D_A$ and ker $D'_A$.

Let $(M, c)$ be a fixed smooth compact oriented 4-manifold with a fixed Spin$^c$-structure $c$. We want to relate the moduli spaces $\mathcal{M}(c)$ and $\mathcal{M}'(c)$ for two Riemannian metrics $g$ and $gf$ (respectively) in the same conformal class. It is well known (see [8]) that both moduli spaces represent the same bordism class (in $\mathcal{B}^*(c)$), but when one of the metrics is Kähler, both moduli spaces are diffeomorphic (see proposition 6).

Proposition 5. Let $(M, c)$ be a fixed smooth compact oriented 4-manifold with a fixed Spin$^c$-structure $c$. Let $g$ be a fixed Riemannian metric on $M$ and consider the conformal metric $g_f = e^{2f}g$. Solutions to the Seiberg-Witten equation for the metric $g_f$ are in one-to-one correspondence with solutions of the following pair of equations:

$$D_A \phi = 0$$

$$F_A^+ = e^{-f} q(\phi). \quad \text{(SW$_f$)}$$

The one-to-one correspondence is given by the map $(A, \phi) \mapsto (A, \Psi_f \phi)$.

Proof. This is a consequence of Proposition 3, the expression for $q$ (see Definition 1) and that $\star'|_{\Lambda^2} = \star|_{\Lambda^2}$, where $\star$ and $\star'$ are the Hodge operators of $g$ and $gf$, respectively. \hfill $\square$

Proposition 6. Let $(M, g)$ be a Kähler surface with Kähler metric $g$. Then for any smooth function $f : M \to \mathbb{R}$

- If the degree of $K_M$ is negative the only solutions to (SW$_f$) are reducible, i.e. $\mathcal{M}_{e^{2f}g}(c) = \emptyset$.
- Let $c$ be the Spin$^c$-structure determined by the complex structure. If the degree of $K_M$ is positive then $\# \mathcal{M}_{e^{2f}g}(c) = 1$.

Proof. The proof of this proposition can be carried out following the steps in the proof of Proposition 7.3.1 in [8] pg. 119, replacing the expression for $q$ with $e^{-f}q$. \hfill $\square$

Remark. Note that $\# \mathcal{M}_{e^{2f}g}(c) = 1$ is stronger than $\text{SW}_{e^{2f}g}(c) = 1$, which we already knew (see [8]).
4 SW-Moduli Space of a Manifold with a Cylindrical End

The last result shows that if \((M, g)\) is a Kähler surface with \(\text{deg}(K_M) > 0\) the Seiberg-Witten moduli space for any metric \(g_f = e^{2f}g\) in the same conformal class of \(g\) consists of a single point. In this Section we extend this result to a manifold with finitely many cylindrical ends.

**Definition 6.** We will say that \((M_\infty, g_\infty)\) is a manifold with a cylindrical end modeled on \(\mathbb{R}^+ \times S^3\), if \(M_\infty\) is diffeomorphic to \(M - \{p\}\) where \(M\) is a closed manifold, and \(F : U_p - \{p\} \rightarrow \mathbb{R}^+ \times S^3\) where \(F(x) = (\log(|x|^{-1}), x/|x|)\) is a diffeomorphism such that \((g_\infty)|_{U_p - \{p\}}\) is the \(F\)-pull-back of the standard product metric \(dt^2 + g_{S^3}\) on \(\mathbb{R}^+ \times S^3\) and \(U_p\) is a neighborhood of \(p\).

If \((M, g)\) is a Riemannian manifold such that \(g\) is flat in a \(\delta\)-neighborhood of \(p\), where \(\delta < \text{inj}(M, g)\), there is a canonical way to produce a manifold with a cylindrical end using the conformal class of \(g\). Here \(\text{inj}(M, g)\) denotes the injectivity radius of \((M, g)\). Choose a function \(\lambda_t : (0, 1] \rightarrow [1, \infty)\) which satisfies

\[
\lambda_t(r) = \begin{cases} 
1 & \text{if } 0 \leq r \leq e^{-t}\delta^3 \\
\delta^2/r & \text{if } e^{-t}\delta^2 \leq r \leq \delta^2 \\
1 & \text{if } r \geq \delta.
\end{cases}
\]

Consider the sequence of functions \(\{f_t\}\), where \(e^{f_t(x)} = \lambda_t(|x|)\) and the sequence of metrics \(g_t = e^{2f_t}g\). This sequence of metrics converges in the \(\mathcal{C}^0\)-topology on \(M - \{p\}\) to a metric \(g_\infty\). The pair \((M - \{p\}, g_\infty)\) is a manifold with a cylindrical end. We will denote by \(\Psi_t\) the associated conformal isomorphism defined above proposition 3.

The SW-equations make perfectly good sense on a manifold with a cylindrical end, but in order to use the usual analytical tools, one has to extend the \(\mathcal{C}^\infty\) objects to appropriate weighted Sobolev spaces (see [7]). From now on every time we work on a manifold with finitely many cylindrical ends we extend the configuration space \(\mathcal{A}(\epsilon)\) and the gauge group \(\mathcal{G}(\epsilon)\) by requiring \(A\) and \(\phi\) to be in \(L^2(x)(M_\infty, g_\infty)\) and \(\sigma\) to be in \(L^3 \sigma_x(M_\infty, g_\infty)\). The \(L^p,q_x(M_\infty, g_\infty)\) norm is defined as

\[
||h||_{p,q,\epsilon} = ||e^{\tilde{\epsilon}}h||_{p,q},
\]

where \(\tilde{\epsilon}\) is a smooth non-decreasing function with bounded derivatives, \(\tilde{\epsilon} : M \rightarrow [0, \epsilon]\), such that \(\tilde{\epsilon}(x) \equiv 0\) for \(x \notin \bar{B}_\delta(p)\) and \(\tilde{\epsilon}(x) \equiv \epsilon > 0\) for \(x \in B_{\delta/2}(p)\).

Here we choose the weight \(\epsilon < 1\) because we want to produce solutions on the manifold with cylindrical end from solutions on the manifold \((M, g)\) via the conformal process \((g_t \rightarrow g_\infty)\) using proposition 5.

**Proposition 7.** Let \((M, g)\) be any Riemannian 4-manifold, where \(g\) is flat in the neighborhood of some point \(p \in M\). If \((A, \phi)\) is a solution of \((\text{SW}_f)\) on
\((M,g)\) (where \(f = f_\infty\)) then \((A, \Psi_\infty \phi)\) is a solution of the SW-equations on \((M_\infty, g_\infty)\), such that \((A, \Psi_\infty \phi) \in L^2_t(M_\infty, g_\infty)\).

**Proof.** The fact that \((A, \Psi_\infty \phi)\) satisfies the SW-equations follows from proposition 5. We just need to show that \((A, \Psi_\infty \phi) \in L^2_t(M_\infty, g_\infty)\). In order to do this, we will use the metric \(g\) as the background metric.

\[
\begin{align*}
\|\Psi_\infty \phi\|_{2,1,\epsilon}^2 &= \|e^{\epsilon t} \Psi_\infty \phi\|_{2,1}^2 \\
&= \int_{M-B_\epsilon(p)} (|\phi|^2 + |\nabla \phi|^2) d\mu + \\
&\quad \int_{\mathbb{R}^+ \times S^3} (|e^{\epsilon t} \Psi_\infty \phi|^2 + e^{2t} \partial_t \nabla \Psi_\infty \phi |^2) dt d\mu_{S^3} \\
&= \int_{M-B_\epsilon(p)} (|\phi|^2 + |\nabla \phi|^2) d\mu + \\
&\quad \int_{B_\epsilon(p)-\{p\}} (|r^{n-1/2} \phi|^2 + |r^{n+3/2} \partial_r \nabla \phi|^2) \frac{1}{r} dr d\mu_{S^3} \\
&\quad \int_{B_\epsilon(p)-\{p\}} r^{-2} (|\phi|^2 + |r \partial_r \nabla \phi|^2) r^3 dr d\mu_{S^3} \\
&\leq C \|\phi\|_{2,1}^2.
\end{align*}
\]

To prove that \(A \in L^2_t(M_\infty, g_\infty)\) we need to recall that

\[\star_{g_f}|_p = e^{(n-2)p} f \star_g|_p\]

where \(g_f = e^{2f} g\). The computation is very similar to the one above.

\(\square\)

Our next task is to show that there is no loss of generality in assuming that a Kähler metric \(g\) is flat in a neighborhood of some point.

**Proposition 8.** Let \((M^{2n}, g)\) be a Kähler 2n-manifold with Kähler metric \(g\) and induced Kähler form \(\omega\). There is no local obstruction to finding a Kähler metric on \(M\), flat in a neighborhood of a point (a finite collection of points) without changing the Kähler class of \(\omega\).

**Proof.** Let \(p \in M\). The existence of such metric is equivalent to finding a neighborhood \(U\) of \(p\), and a Kähler form \(\omega'\) in the same Kähler class of \(\omega\), such that \(\omega'|_U = \omega_0 = \sum_{i=1}^n dz^i \wedge \bar{dz}^i\). It is well known that there exist an \(\epsilon\)-neighborhood \(U_\epsilon\) of \(p\) and a function \(f : U_\epsilon \rightarrow \mathbb{R}\) such that \(\omega|_{U_\epsilon} = i \partial \bar{\partial} (z\bar{z} + f(z)) > 0\), where \(|f(z)| \sim o(|z|^4)\) and \(|z|\) denotes the distance (using the Kähler metric \(g\)) on \(U_\epsilon\) to \(p\). Let \(K^\infty(f)\) be the space of smooth functions on \(M\) that satisfy

\[
K^\infty(f) = \{h_{s,t} \in C^\infty(M) | h(z) = -f(z) \text{ if } |z| < s, h(z) = 0 \text{ if } t < |z|\}
\]
where $0 < s < t \leq \epsilon$, depend on $h$. Observe that if $f$ is zero we do not have anything to prove, otherwise $0 \notin K^\infty(f)$, but $0 \in K^{3+\alpha}(f)$, where $K^{3+\alpha}(f)$ denotes the completion of $K^\infty(f)$ in the $C^{3+\alpha}$ topology. To see this consider the one-parameter family of functions $h_k(z) = -\rho(|z|) f(z)$, where $\rho$ is a smooth bump function such that

\[
\rho(r) = \begin{cases} 
1 & \text{if } 0 < r < 1/2 \\
0 & \text{if } 1/2 < r < 1.
\end{cases}
\]

All these functions are in $K^\infty(f)$ and satisfy

\[
|h_k(z)| \sim o(|z|^4) \\
|\nabla h_k(z)| \sim o(|z|^3) \\
|\nabla^2 h_k(z)| \sim o(|z|^2) \\
|\nabla^3 h_k(z)| \sim o(|z|) \\
|\nabla^4 h_k(z)| \sim o(1).
\]

It is not difficult to see that $h_k \to 0$ in the $C^{3+\alpha}$ topology. It is important to recall that the set $P(\omega)$ of smooth functions $h$ such that $\omega h = \omega + i\partial\bar\partial h > 0$, is open in the $C^\infty$ topology. This two facts allow us to find $h_{s,t} \in K^\infty(f) \cap P(\omega)$, $C^{3+\alpha}$ close to 0, such that

\[
\omega_{h_{s,t}} = \omega + i\partial\bar\partial h_{s,t} > 0 \\
= \omega_0 + i\partial\bar\partial(f + h_{s,t}),
\]

therefore we have

\[
\omega_{h_{s,t}}|_{B_s(p)} = \omega_0,
\]
where $B_s(p) = \{ z \in U_p | |z| < s \}$.

**Corollary 9.** For any compact oriented Kähler surface $(M,g)$ with canonical line bundle $K_M$ of positive degree, where $g$ is flat in a neighborhood of some point, the induced manifold with a cylindrical end $(M_\infty, g_\infty)$ admits solutions to the SW-equations.

In order to prove that the Seiberg-Witten moduli space of a manifold with a cylindrical end consists of only one point if $\deg(K_M) > 0$, we will need the following technical result.

**Proposition 10.** Let $(M_\infty, g_\infty)$ be a 4-manifold with a cylindrical end. If

\[
(A_\infty, \phi_\infty) \in C^\infty \cap L^2_{k,e}(M_\infty, g_\infty)
\]

is a solution of the SW-equations on the manifold with cylindrical end $(M_\infty, g_\infty)$, then $(A_\infty, \Psi^{-1}_\infty \phi_\infty)$ extends to a smooth solution of $(SW_f)$ on $(M, g)$, replacing the strictly positive function $e^{-f}$ by the non-negative function

\[
\lambda_\infty(x) = \begin{cases} 
|x|/\delta^2 & \text{if } |x| < \delta^2 \\
1 & \text{if } |x| > \delta
\end{cases}
\]
Proof. It is easy to see that \((A_\infty, \Psi_\infty^{-1}\phi_\infty) \in L^2(M,g)\), as it is to see that \((A_\infty, \Psi_\infty\phi_\infty)\) is a solution of \((SW_f)\) with function \(\lambda_\infty\) replacing \(e^{-f}\). The first equation in \((SW_f)\) tell us that \(\Psi_\infty^{-1}\phi_\infty\) is a holomorphic section on \(M - \{p\}\). Using Hartog’s Theorem we can extend this to a holomorphic section on \(M\).

All the analysis done in proving proposition 6 can be carry out if we replace the strictly positive function \(e^{-f}\) in \((SW_f)\) by a non-negative function \(\lambda_\infty\) whose zero set has measure zero.

Corollary 11. Let \((M,g)\) be a compact oriented Kähler surface with canonical line bundle \(K_M\) of positive degree, where \(g\) is flat in a neighborhood of some point. Then there exists a solution \((A_\infty, \phi_\infty)\) of the \(SW\)-equations on \((M_\infty, g_\infty)\). This solution is unique up to gauge equivalence.

Proof. Since all the analysis done in proving proposition 6 can be carry out if we replace the strictly positive function \(e^{-f}\) in \((SW_f)\) by a non-negative function \(\lambda_\infty\) whose zero set has measure zero, existence is a consequence of corollary 9 and uniqueness is obtained using proposition 10 and proposition 6.

5 Holonomy, Connected Sums with \(S^1 \times S^3\) and \(SW\)-Invariants

Consider the diffeomorphism

\[
F: \mathbb{R}^4 - \{0\} \to \mathbb{R} \times S^3, \quad F(x) = \left(\log|x|, \frac{x}{|x|}\right).
\]

It is easy to see that the pull-back of the standard product metric \(g\) on \(\mathbb{R} \times S^3\) under this diffeomorphism is given by

\[
F^*g(\xi, \eta) = \frac{1}{|x|^2} \langle \xi, \eta \rangle
\]

for \(|x| \leq 1\). Fix \(\delta > 0\) and choose a function \(\lambda_l : (0,1] \to [1,\infty)\) as in (2) and consider the metric

\[
g_l(\xi, \eta) = \lambda_l(|x|)^2 \langle \xi, \eta \rangle.
\]

Note that for \(e^{-l}\delta^2 \leq |x| \leq \delta^2\) this metric agrees with the above pull-back metric \(F^*g\).

It is convenient to think of the connected sum \(M \#(S^1 \times S^3)\) as follows. Let \(M\) be a smooth compact oriented 4-manifold. Fix two points \(p_1, p_2 \in M\), and choose a metric \(g\) on \(M\) which is flat in a \(\delta\)-neighborhood of \(p_i\). For every \(l \in \mathbb{N}\) consider the \(e^{-l-1}\delta^2\)-neighborhood of \(p_i\) (with respect to \(g\)) \(B_{p_i}(e^{-l-1}\delta^2)\), and denote by \(M_l\) the open subset of \(M\) given by the complement of \(B_{p_i}(e^{-l-1}\delta^2) \cup B_{p_2}(e^{-l-1}\delta^2)\). If we denote by \(T_i = T_i(e^{-l}\delta^2, e^{-l-1}\delta^2)\) the annulus centered at \(p_i\) with radii \(e^{-l-1}\delta^2\) and \(e^{-l}\delta^2\), it is easy to see that there exist a diffeomorphism
(orientation reversing) that takes $T_1$ into $T_2$ and if we define $g_t = \lambda_t^2 g$, such diffeomorphism becomes a $g_t$-isometry. Since we have observed that $T_1$ and $T_2$ are $g_t$-isometric we can identify $T_1$ with $T_2$, and call them $T_l$, to obtain a Riemannian manifold $(M\#_{l}(S^1 \times S^3), g_l)$. This manifold is simply the manifold $M$ with two cylindrical ends of length $l$ obtained by conformally rescaling the metric $g$ and identifying the annuli. It is easy to see that such manifold is diffeomorphic to the connected sum $M\#(S^1 \times S^3)$.

Even though the process above described can be realized on any smooth 4-manifold the following results are only valid when $M$ is a Kähler surface, because to prove them, we (strongly) use that on a given conformal class of metrics, the moduli spaces of solutions of the SW-equations for any two representatives are diffeomorphic, and this was proved for Kähler surfaces (see proposition 6).

Our next task is to explain how a Spin$^c$-structure on $M$ transforms into a Spin$^c$-structure on $M\#(S^1 \times S^3)$ under the process above described. The following Proposition will be very useful to explain it.

**Proposition 12.** There is a canonical projection map $\pi : M\#(S^1 \times S^3) \to M$. It has the following properties:

1. The induced maps in cohomology
   
   $\pi^* : H^i(M;\mathbb{F}) \to H^i(M\#(S^1 \times S^3);\mathbb{F})$

   are injective. Here $\mathbb{F} = \mathbb{Z}_2$ or $\mathbb{Z}$. Moreover for $i = 0, 2, 4$, $\pi^*$ is an isomorphism.

2. $\pi^*(w_2(M)) = w_2(M\#(S^1 \times S^3))$.

We will denote the Spin$^c$-structure obtained in the above proposition by $c_{0,1}$. It is not difficult to show that the formal dimension of the moduli space associated to $c_{0,1}$ is $d(c_{0,1}) = d(c) + 1$.

To explain the increment in the dimension above we need to recall the concept of holonomy. Let $P_G \to M$ be a principal $G$-bundle over $M$, with a connection $A$. Let $x \in M$ and denote by $C(x)$ the loop space at $x$. For each $\gamma \in C(x)$ the parallel displacement along $\gamma$ is an isomorphism of the fiber $\approx G$ onto itself and we will denote it by $\text{hol}_\gamma(A)$. The set of all such isomorphisms forms a group, the *holonomy group of $A$ with reference point $x$*.

Once and for all for each $l > 0$ we will choose $p_l \in T_1$, $q_l \in T_2$ and a path $\Gamma_l : I \to M$ from $p_l$ to $q_l$ such that after identifying $T_1$ with $T_2$ we obtain and embedding $\gamma_l : S^1 \to M\#_{l}(S^1 \times S^3)$. It is not difficult to observe that for all $l > 0$ $[\gamma_l] \approx 0 \in \pi_1(M\#(S^1 \times S^3))$, and in fact $\gamma_l$ represents the $S^1$ factor of the connected sum.

If $A$ is a $U(1)$-connection on the determinant line bundle $L_A$, we can trivialize $L_k$ along $\Gamma_l$ so that the parallel transport along $\Gamma_l$ induces the identity from the fiber at $p_l$ to the fiber at $q_l$. When we identify $T_1$ with $T_2$ we still have the extra degree of freedom of how to identify the fiber at $p_l$ with the fiber at $q_l$, and this is measured by $\text{hol}_\gamma(A)$, where $A$ is the glued connection. If we change of gauge, $\text{hol}_\gamma(A)$ remains unchanged because the structure group $U(1)$ is Abelian. In
this section we will prove that when $M$ is a Kähler surface then every solution to the Seiberg-Witten equations for a Spin$^c$-structure $\mathfrak{c}$, induces an $S^1$ family of solutions to the SW-equations for the Spin$^c$-structure $c_{0,1}$ on $M\#(S^1 \times S^3)$.

We can glue a solution $(A_\infty, \phi_\infty)$ of the SW-equations on $(M_\infty, g_\infty)$ to produce a solution $(A_l, \phi_l)$ of the following set of equations on $(M\#(S^1 \times S^3), g_l)$

\[
D_{A_l} \phi_l = \mu(A_l, \phi_l) = \mu_l \\
F_{A_l}^+ - q(\phi_l) = \nu(A_l, \phi_l) = \nu_l,
\]

where $(\mu_l, \nu_l) \in \mathcal{S}(c) \times \Omega^2_+(M\#(S^1 \times S^3); i\mathbb{R})$. It is not difficult to see that

\[
(\mu_l, \nu_l) \in L^1_2(M\#(S^1 \times S^3), g_l) \\
\lim_{l \to \infty} \| (\mu_l, \nu_l) \|_{2,1} = 0,
\]

**Definition 7.** We will denote by $\mathcal{M}_\theta(c_{0,1}) \subset \mathcal{M}(c_{0,1})$ the solution subspace of the SW-equations satisfying the extra condition

\[
\text{hol}_\gamma(A) = \theta,
\]

and by $SW_\theta(c_{0,1})$ the cobordism invariant associated to this moduli space (counting solutions with appropriate sign). Note that the condition $\text{hol}_\gamma(A) = \theta$ reduces the dimension of the moduli space by one.

**Proposition 13.** Let $(M\#(S^1 \times S^3), g_l)$ be the connected sum of $M$ with $S^1 \times S^3$ with a neck of length $l$. For every $\theta \in S^1$ and for every $l \gg 0$, there exists some generic perturbation $\eta_l \in \Omega^2_+(M\#(S^1 \times S^3); i\mathbb{R})$ with supp $\eta_l \subset T_l$ such that $SW^{-1}_{\theta,l}(0, \eta_l) \neq \emptyset$, where $SW_{\theta,l}(A, \phi) = (D_A \phi, F_A^+ - q(\phi))$ and $\text{hol}_\gamma(A) = \theta$.

**Proof.** Observe that the condition of $\eta_l$ having supp $\eta_l \subset T_l$ is not much of a restriction at all, because the space of such 2-forms is open and the set of generic perturbations is dense (see [8]).

Suppose otherwise, there exists some $\theta \in S^1$ such that for every $l \gg 0$ we have $SW^{-1}_{\theta,l}(0, \eta_l) = \emptyset$. This would imply that $SW^{-1}_{\theta}(0, 0) = \emptyset$ since we have seen (see Corollary 11) that $(M\#(S^1 \times S^3), g_l) \to (M_\infty, g_\infty)$, but this is a contradiction because we have proven (see Corollary 11), that $SW^{-1}_{\theta}(0, 0) \neq \emptyset$.

**Definition 8.** We will say that $(\tilde{A}_l, \tilde{\phi}_l)$ on $M_\infty$, $C^0$-extends a solution $(A_l, \phi_l)$ of $SW_{\theta,l}(A, \phi) = (0, \eta_l)$ on $M\#(S^1 \times S^3)$ if

\[
(\tilde{A}_l, \tilde{\phi}_l)|_{M_l} = (A_l, \phi_l) \text{ and } \quad (\tilde{A}_l(t, x), \tilde{\phi}_l(t, x)) = (A_l(x), e^{-2\pi t} \phi_l(x))
\]

for $(t, x) \in [l, \infty) \times S^3$.

**Remark.** Note that $(\tilde{A}_l, \tilde{\phi}_l) \in L^2_{\text{hol}}(M_\infty, g_\infty)$. From now on we will fix a $U(1)$-connection $A$ on $L_c$. 

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Lemma 14. If for every $l \gg 0$ there exist two different irreducible solutions $[A_1^i, \phi_1^i]$ and $[A_2^i, \phi_2^i]$ of

\[
D_A \phi = 0 \\
F_A^+ - q(\phi) = \eta_l
\]
on $M \#_i(S^1 \times S^3)$ for some generic perturbations $\eta_l$, then

\[
(C_l, \psi_l) = (\|\tilde{\phi}_1^i - \tilde{\phi}_2^i\|_{2,0,\epsilon}(\tilde{A}_1^i - \tilde{A}_2^i), \frac{1}{\|\tilde{\phi}_1^i - \tilde{\phi}_2^i\|_{2,0,\epsilon}}(\tilde{\phi}_1^i - \tilde{\phi}_2^i))
\]
satisfies

\[
(C_l, \psi_l) \to (C, \psi) \in L^2_{1,1}(M_{\infty}, g_{\infty}) \\
\|\psi\|_{2,0,\epsilon} = 1,
\]

where $(\tilde{A}_1^i, \tilde{\phi}_1^i) \mathcal{C}^0$-extends $(A_1^i, \phi_1^i)$ to $(M_{\infty}, g_{\infty})$ for $i = 1, 2$, and $(A_1^i, \phi_1^i)$ are the unique representatives obtained by the gauge fixing condition $\delta(A_1^i - A) = 0$.

Lemma 15. The same hypothesis as before. If $(\tilde{A}_1^i, \tilde{\phi}_1^i) \to (A_\infty, \phi_\infty)$ in the $L^2_{1,1}(M_{\infty}, g_{\infty})$ topology, then we have

\[
\lim_{l \to \infty} \|D(SW_{\theta,1})(\tilde{A}_1^i, \tilde{\phi}_1^i)(C_l, \psi_l)\|_{2,0,\epsilon} = 0
\]

Remark. The proof of the previous two lemmas is not difficult but technical (a straightforward computation) so we will omit the details.

Proposition 16. For every $\theta \in S^1$, $SW_{\theta}(c_{0,1}) = 1$.

Proof. Assume that $SW_{\theta}(c_{0,1}) \neq \pm 1$. Proposition 13 implies, for $l \gg 0$ there exist (at least) two different irreducible solutions $(A_1^i, \phi_1^i), i = 1, 2$ on $(M \#_i(S^1 \times S^3), g_l)$. By lemma 14 and lemma 15 we would have an element of $\ker DSW_{\infty}$ at $(A_\infty, \phi_\infty)$ the unique solution on $(M_{\infty}, g_{\infty})$, obtained in corollary 11. But this is a contradiction since $(A_\infty, \phi_\infty)$ is a smooth point. The same kind of argument shows that $SW_{\theta}(c_{0,1}) = 1$ since $SW_{\infty}(c) = 1$. 

6 Cohomology of $B^*(c)$

In this section we will build cohomology classes for $B^*(c)$ in order to detect $B$-classes (see Definition 5). To describe the cohomology of $B^*(c)$ we have to introduce the concept of universal family of SW-connections associated to a Spin$^c$ structure $c$, parameterized by $B^*(c)$. A SW-connection is simply a pair $(A, \phi)$, where $A$ is a $U(1)$-connection on $L_c$ and $0 \neq \phi \in S^+(c)$.

A cohomology class $\beta \in H^i(B^*(c); \mathbb{Z})$ can be thought of as a homomorphism $\beta : H_i(B^*(c); \mathbb{Z}) \to \mathbb{Z}$, and the elements of $H_i(B^*(c); \mathbb{Z})$ can be thought of as homotopic classes of maps $f : T \to B^*(c)$, where $T$ is a compact space. The maps $f : T \to B^*(c)$ are naturally interpreted in terms of families of SW-connections.
Definition 9. A family of SW-connections in a bundle $L_c \to M$ parametrized by a space $T$ is a bundle $L \to T \times M$ with the property that each slice $L_t = L|_{(t) \times M}$ is isomorphic to $L_c$, together with a SW-connection $(A_\phi)_t = (A_t, \phi_t)$ in $L_t$, forming a family $A_\phi = \{(A_\phi)_t\}$.

Let $p_2 : C^*(\epsilon) \times M \to M$ be the projection onto the second factor and let $L_c \to C^*(\epsilon) \times M$ be the pull-back line bundle, $p_2^*L_c$. Then $L_c$ carries a tautological family of SW-connections $A_\phi$, in which the SW-connection on the slice $L_c|_{\{(A, \phi)\}}$ over $\{(A, \phi)\} \times M$ is $(p_2^*(A), p_2^*(\phi))$. The group $G(\epsilon)$ acts freely on $C^*(\epsilon) \times M$ as well as on $L_c = C^*(\epsilon) \times L_c$, and there is therefore a quotient bundle

$$\mathbb{L}_c \to B^*(\epsilon) \times M$$
$$\mathbb{L}_c = L_c/G(\epsilon).$$

The family of SW-connections $A_\phi$ is preserved by $G(\epsilon)$, so $\mathbb{L}_c$ carries an inherited family of SW-connections $\mathbb{K}_\phi$. This is the universal family of SW-connections in $L_c \to M$ parameterized by $B^*(\epsilon)$.

If a family of SW-connections is parameterized by a space $T$ and carried by a bundle $L \to T \times M$, there is an associated map $f : T \to B^*(\epsilon)$ given by

$$f(t) = [A_t, \phi_t].$$

Conversely, given $f : T \to B^*(\epsilon)$ there is a corresponding pull-back family of connections carried by $(f \times I)^*\mathbb{L}_c$. These two constructions are inverses of one another: if $f$ is determined by the above equation, then for each $t$ there is a unique isomorphism $\psi_t$ between the SW-connections in $L_t$ and $(f \times I)^*(\mathbb{L}_c)_t$, and as $t$ varies these fit together to form an isomorphism $\psi : L \to (f \times I)^*\mathbb{L}_c$ between these two families. (The uniqueness of $\psi_t$ results from the fact that $G(\epsilon)$ acts freely on $C^*(\epsilon)$). Thus:

Lemma 17. The maps $f : T \to B^*(\epsilon)$ are in one-to-one correspondence with families of SW-connections on $M$ parameterized by $T$, and this correspondence is obtained by pulling back from the universal family $(\mathbb{L}_c, \mathbb{K}_\phi)$.

Remark. Let $\{\gamma_i\}$ be fixed representatives for the generators of the free part of $H_1(M; \mathbb{Z})$. If $f_1, f_2 : T \to B^*(\epsilon)$ are homotopic, the corresponding bundles $L_1$ and $L_2$ are isomorphic, and the corresponding holonomy maps $h_1 : T \to (S^1)^{b_1}$ and $h_2 : T \to (S^1)^{b_1}$ are homotopic, where the holonomy map is defined as $h_i(t) = (\text{hol}_{\gamma_1}(f_i(t)), \ldots, \text{hol}_{\gamma_{n_1}}(f_i(t)))$.

There is a general construction which produces cohomology classes in $B^*(\epsilon)$, using the slant-product pairing

$$/ : H^{d-i}(B^*(\epsilon); \mathbb{Z}) \times H_i(M; \mathbb{Z}) \to H^i(B^*(\epsilon); \mathbb{Z}).$$

We have built over $B^*(\epsilon) \times M$ a line bundle $\mathbb{L}_c$, so we can define a map

$$\mu : H_2(X; \mathbb{Z}) \to H^{2-i}(B^*(\epsilon); \mathbb{Z})$$

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by

\[ \mu(\alpha) = c_1([L_\epsilon])/\alpha. \]

If \( T \) is any \((2-i)\)-cycle in \( B^*(c) \), the class \( \mu(\alpha) \) can be evaluated on \( T \) using the formula

\[ \langle \mu(\alpha), T \rangle_{B^*(c)} = \langle c_1([L_\epsilon]), T \times \alpha \rangle_{B^*(c) \times M}, \]

which expresses the fact that the slant product is the adjoint of the cross-product homomorphism. Next we will describe another way to build cohomology classes.

**Definition 10.** A closed curve \( \gamma : S^1 \to M \) induces a holonomy map

\[ \text{hol}_\gamma : B^*(c) \to S^1 \]

defined as the holonomy of the SW-connections \( A_\phi \) along \( \gamma \). The pull-back of the canonical class \( d\theta \) of \( S^1 \) defines a cohomology class on \( H^1(B^*(c);\mathbb{Z}) \) which we will call the holonomy class along \( \gamma \).

**Proposition 18.** The cohomology groups of \( B^*(c) \) are generated by the image of the map \( \mu : H_i(X;\mathbb{Z}) \to H^{2-i}(B^*(c);\mathbb{Z}) \). Moreover, given \( \gamma \in H_1(M;\mathbb{Z}) \), \( \mu(\gamma) \) is the holonomy class along \( \gamma \), \( \text{hol}_\gamma(d\theta) \).

**Proof.** First we will prove that if \( \{\gamma_i\} \) are fixed representatives for the generators for the free part of \( H_1(M;\mathbb{Z}) \) then \( \{\mu(\gamma_i)\} \) generates \( H^1(B^*(c);\mathbb{Z}) \). It is enough to prove that for every \( i \) we can find \( \beta_i : S^1 \to B^*(c) \) such that \( \langle \mu(\gamma_i), \beta_i \rangle_{B^*(c)} = 1 \). Consider the line bundle \( \gamma_i^*L_\epsilon \to S^1 \), and observe that there is no obstruction to extend it to a line bundle \( L \to S^1 \times S^1 \) such that \( \deg L = \langle c_1(L), S^1 \times S^1 \rangle = 1 \).

Let \( A_i \) be a \( U(1) \)-connection on \( L \) and consider the map

\[ \text{hol}_{\bullet \times S^1}(A_i) : S^1 \to S^1. \]

It is not difficult to see that \( \deg L = \deg(\text{hol}_{\bullet \times S^1}(A_i)) \). After extending \( A_i(t,\gamma_i) \) to a \( U(1) \)-connection on \( L_\epsilon \to M \) for each \( t \), we obtain (see remark below lemma 17) our desired maps \( \beta_i : S^1 \to B^*(c) \).

To prove the last statement we proceed as follows: let \( \alpha : S^1 \to B^*(c) \),

\[ \langle \mu(\gamma_i), \alpha \rangle_{B^*(c)} = \langle c_1([L_\epsilon]), \alpha \times \gamma_i \rangle_{B^*(c) \times M} = \langle c_1((\alpha \times \gamma_i)^*([L_\epsilon])), S^1 \times S^1 \rangle = \deg(\text{hol}_{\bullet \times S^1}(A_i) : S^1 \to S^1) = \deg(\text{hol}_{\gamma_i} \alpha : S^1 \to S^1) = \langle \deg_{\beta_i}^*(d\theta), \alpha \rangle_{B^*(c)}. \]

Finally we have to show that if \( x \in M \) then \( \mu(x) \) generates the cohomology of the \( CP^\infty \) factor. Since \( Map(M,S^1)_0 \) acts freely on \( C^*(c) \), then it is easy to show that \( [L_\epsilon]_{B^*(c)} \approx C^*(c)/\mathcal{G}(c) \), where \( \mathcal{G}(c) \) is the kernel of the homomorphism \( \mathcal{G}(c) \to S^1 \) given by evaluating on the fiber over \( x \). \( \square \)
7 Applications

C. LeBrun [6] showed that under some mild conditions on \( M \), \( M \# k \mathbb{C}\mathbb{P}^2 \) does not admit Einstein metrics. The precise statement is the following:

**Theorem (C. LeBrun).** Let \( M \) be a smooth compact oriented 4-manifold with \( 2e + 3\sigma > 0 \). Assume, moreover, that \( M \) has a non-trivial Seiberg-Witten invariant. If \( k \geq \frac{2e}{3}\sigma \) then \( M \# k \mathbb{C}\mathbb{P}^2 \) does not admit an Einstein metric.

**Remark.** The proof of this theorem only requires that \( M \) has a \( \text{Spin}^c \)-structure \( c \) that is a \( B \)-class.

**Theorem A.** Let \((M, c)\) be a smooth compact Kähler surface with a \( \text{Spin}^c \)-structure \( c \). There is a canonical \( \text{Spin}^c \) structure in the connected sum manifold \( M \# (S^1 \times S^3) \) which we will denote by \( c_{0,1} \). Moreover \( d(c_{0,1}) = d(c) + 1 \). If \( c \) is a non-trivial SW-class for \( M \) then \( c_{0,1} \) is a \( B \)-class for the connected sum \( M \# (S^1 \times S^3) \).

**Proof.** \( SW_\theta(c_{0,1}) \) is a cobordism invariant for every \( \theta \in S^1 \). Consider the smooth cobordism induced by the family of metrics \( g_l \) on \( M \# (S^1 \times S^3) \) as \( l \to \infty \) and observe (corollary 11) that \( SW_\infty(c) = 1 \). This shows that

\[
\langle \text{hol}^*_\gamma(d\theta), M(c_{0,1}) \rangle |_{B^*(c_{0,1})} = 1,
\]

where \( \gamma \) is a representative for the \( S^1 \) factor of the connected sum. This, the definition of a \( B \)-class and Proposition 18 complete the proof.

**Corollary 19.** Let \((M, c)\) be a smooth compact oriented Kähler surface with a \( \text{Spin}^c \)-structure \( c \). There is a canonical \( \text{Spin}^c \) structure in the connected sum \( M \# 2(S^1 \times S^3) \) which we will denote by \( c_{0,2} \). Moreover \( d(c_{0,2}) = d(c) + 2 \). If \( c \) is a non-trivial SW-class then \( c_{0,2} \) is a \( B \)-class but has trivial Seiberg-Witten invariant.

**Proof.** Theorem A shows that every time that we perform a connected sum with \( S^1 \times S^3 \) we add a cycle to the moduli space, that lies entirely in the \( H^1(M \# (S^1 \times S^3); \mathbb{R})/H^1(M \# (S^1 \times S^3); \mathbb{Z}) \) part of \( B^*(c_{0,1}) \).

**Lemma 20.** Let \((M, c)\) be a smooth compact oriented Kähler surface with a \( \text{Spin}^c \)-structure \( c \) and \( 2e + 3\sigma > 0 \). Assume that \( c \) is a non-trivial SW-class. Let \( k, l \) be any two natural numbers. Then there is a \( B \)-class \( c_{k,l} \) on \( M_{k,l} = M \# k \mathbb{C}\mathbb{P}^2 \# l(S^1 \times S^3) \) such that

\[
(c^+_k(c_{k,l}))^2 \geq (2e + 3\sigma)(M).
\]

**Proof.** First observe that \( M_{k,l} = (M \# k \mathbb{C}\mathbb{P}^2)_0 \). Since \( M \) is a Kähler surface, we know that \( M \# k \mathbb{C}\mathbb{P}^2 \) is also a Kähler surface, and its associated \( \text{Spin}^c \) structure...
$c_{k,0}$ satisfies $c_1(c_{k,0}) = c_1(c) + \sum_{j=1}^{k} E_j$, where $E_1, \ldots, E_k$ are generators for the pull-backs to $M \# k \mathbb{CP}^2$ of the $k$ copies of $H^2(\mathbb{CP}^2, \mathbb{Z})$ so that

$$c_1^+(c) \cdot E_j \geq 0, \quad j = 1, \ldots, k.$$ 

Let $c_1(c_{k,l})$ be the first Chern class of $(c_{k,0})_{0,l}$ which is a $B$-class by theorem A, and notice that $c_1(c_{k,l}) = c_1(c_{k,0})$. One then has

$$(c_1^+(c_{k,l}))^2 = (c_1^+(c_{k,0}))^2$$

$$= \left( c_1^+(c) + \sum_{j=1}^{k} E_j^+ \right)^2$$

$$= (c_1^+(c))^2 + 2 \sum_{j=1}^{k} c_1^+(c_{0,l}) \cdot E_j^+ + (\sum_{j=1}^{k} E_j^+)^2$$

$$\geq (c_1^+(c))^2$$

$$\geq (c_1(c))^2$$

$$= (2e + 3\sigma)(M).$$

LeBrun’s theorem can be generalized in the following way:

**Theorem 21.** Let $(M, c)$ be a smooth compact oriented Kähler surface with a Spin$^c$-structure $c$ and $2e + 3\sigma > 0$. Assume that $c$ is a $B$-class. If $k + 4l \geq \frac{25}{2}(2e + 3\sigma)$ then $M_{k,l} = M \# k \mathbb{CP}^2 \# l(S^1 \times S^3)$ does not admit an Einstein metric.

**Proof.** The proof is the same as the one given by C. LeBrun [6].

There exists two well known topological obstructions to the existence of Einstein metrics on a differentiable compact oriented 4-manifold $M$.

The first one is Thorpe’s inequality (see [2]), that comes from the Gauss-Bonnet-Chern formula for the Euler characteristic $e(M)$ of $M$ and from the Hirzebruch formula for the signature $\sigma(M)$ of $M$, which allow us to express these two topological invariants in terms of the irreducible components of the curvature under the action of $SO(4)$. It can be stated in the following way

**Theorem (N. Hitchin, J. Thorpe).** Let $M$ be a compact oriented manifold of dimension 4. If $e(M) < \frac{3}{2} |\sigma(M)|$ then $M$ does not admit any Einstein metric. Moreover, if $e(M) = \frac{4}{2}|\sigma(M)|$ then $M$ admits no Einstein metric unless it is either flat, or a K3 surface, or an Enriques surface, or the quotient of an Enriques surface by a free antiholomorphic involution.

This theorem implies a previous result of M. Berger who proved that there exists no compact Einstein 4-manifold with a negative Euler characteristic. On the other hand, combining the Gauss-Bonnet-Chern formula for the Euler characteristic with Gromov’s estimation of simplicial volume $\|M\|$ of a Riemannian manifold $M$ (see [4]), M. Gromov obtained the following obstruction

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Theorem (M. Gromov). Let $M$ be a compact manifold of dimension 4. If 
\[ e(M) < \frac{1}{2592\pi} \|M\| \] then $M$ does not admit any Einstein metric.

A. Sambusetti (see [10]) found a topological obstruction to the existence of Einstein metrics on compact 4-manifolds which admit a non-zero degree map onto some compact real or complex hyperbolic 4-manifold. As a consequence, by connected sums, he produces infinitely many non-homeomorphic 4-manifolds which admit no Einstein metrics. This fact is not a consequence of Hitchin-Thorpe’s or Gromov’s obstruction theorems. A. Sambusetti also proves that any Euler characteristic and signature can be simultaneously realized by these non-homeomorphic manifolds admitting no Einstein metrics.

**Definition 11.** We say that a pair $(m,n) \in \mathbb{Z}^2$ is admissible if there exists a smooth compact oriented 4-manifold with Euler characteristic $m$ and signature $n$. In fact a necessary and sufficient condition for $(m,n) \in \mathbb{Z}^2$ to be an admissible pair is that $m \equiv n \mod 2$.

To prove our last result we need the following theorem by Z. Chen (see [3]).

**Theorem (Z. Chen).** Let $x, y$ be integers satisfying
\[
\frac{352}{89} x + 140.2 x^{2/3} < y < \frac{18644}{2129} x - 365.7 x^{2/3},
\]
x > C, where $C$ is a large constant. There exists a simply connected minimal surface $M$ of general type with $c_1^2(M) = y$, $\chi(M) = x$. Furthermore, $M$ can be represented by a surface admitting a hyperelliptic fibration.

**Remark.** Recall that $\chi(M)$ denotes the Euler-Poincaré characteristic of the invertible sheaf $\mathcal{O}_M$. Using Noether’s formula we have that
\[
\chi(M) = \frac{c_1^2(M) + e(M)}{12} = \frac{e(M) + \sigma(M)}{4}.
\]

If $M$ is not a complex surface $e(M) + \sigma(M)$ is not necessarily a multiple of 4 but it is always an even number.

**Theorem B.** For each admissible pair $(m,n)$ there exist an infinite number of non-homeomorphic compact oriented 4-manifolds which have Euler characteristic $m$ and signature $n$, with free fundamental group and which do not admit Einstein metric.

**Proof.** Let $(m_0,n_0)$ be an admissible pair and consider the pair of integers $(x'_0, y_0) = \left(\frac{m_0 + n_0}{2}, 2m_0 + 3n_0\right)$. It is always possible to find (infinitely many) positive integers $k$ and $l$ such that
\[
(x, y) = \left(\frac{x'_0 + l}{2}, y_0 + k\right) \in \mathbb{Z}
\]
\[
4l + \frac{32}{54} k \geq \frac{25}{54} y_0
\]
where $Z$ denotes the set of $(x, y) \in Z^2$ that satisfy the conditions of Chen’s theorem. The reason for this last statement is that the region $Z$ determine by $(x, y) \in \mathbb{R}^2$ such that
\[
\frac{352}{89} x + 140.2x^{2/3} < y < \frac{18644}{2129} x - 365.7x^{2/3},
\]
\[x > C,
\]
is open, connected and not bounded, where $C$ is the same constant as in Chen’s theorem.

If we denote by $M$ the simply connected Kähler surface with $c_1^2 = y$ and $\chi = x$, then $M_{k,l} = M \# k \mathbb{CP}^2 \# l(S^1 \times S^3)$, is a manifold that realizes the pair $(m_0, n_0)$ and does not admit any Einstein metric. This last statement is a consequence of theorem 21.

\[\square\]

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