LINEAR-QUADRATIC MEAN-FIELD TYPE STACKELBERG DIFFERENTIAL GAMES FOR STOCHASTIC JUMP-DIFFUSION SYSTEMS

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Abstract. In this paper, we consider linear-quadratic (LQ) leader-follower Stackelberg differential games for mean-field type stochastic systems with jump diffusions, where the system includes mean-field variables, i.e., the expected value of state and control variables. We first solve the LQ mean-field type control problem of the follower using the stochastic maximum principle and obtain the state-feedback representation of the open-loop optimal solution in terms of the coupled integro-Riccati differential equations (CIRDEs) via the Four-Step Scheme. Next, we solve the problem of the leader, which is the LQ control problem subject to the mean-field type forward-backward stochastic system with jump diffusions, where the constraint characterizes the rational behavior of the follower. Using the variational approach, we obtain the (mean-field type) stochastic maximum principle. However, to obtain the state-feedback representation of the open-loop optimal solution of the leader, there is a technical challenge due to the jump process. We consider two different cases, in which the state-feedback type control in terms of the CIRDEs can be characterized by generalizing the Four-Step Scheme. We finally show that the state-feedback type controls of the open-loop optimal solutions for the leader and the follower constitute the Stackelberg equilibrium.

1. Introduction. Let \( B \) be a standard Brownian motion and \( \tilde{N} \) a compensated Poisson process (see the notation at the end of this section and the precise problem formulation in Section 2). We consider the following stochastic differential equation (SDE) with jump diffusions:

\[
\begin{aligned}
dx(t) &= \left[ A(s)x(s-)+ \bar{A}(s)E[x(s-)] + B_1(s)u_1(s) + \bar{B}_1(s)E[u_1(s)] \\
&+ B_2(s)u_2(s) + \bar{B}_2(s)E[u_2(s)] \right] ds + \left[ C(s)x(s-)+ \bar{C}(s)E[x(s-)] \\
&+ D_1(s)u_1(s) + \bar{D}_1(s)E[u_1(s)] + D_2(s)u_2(s) \\
&+ D_2(s)E[u_2(s)] \right] dB(s) + \int_{\mathbb{R}} \left[ F(s,e)x(s-)+ \bar{F}(s,e)E[x(s-)] \\
&+ G_1(s,e)u_1(s) + \bar{G}_1(s,e)E[u_1(s)] + G_2(s,e)u_2(s) \\
&+ \bar{G}_2(s,e)E[u_2(s)] \right] \tilde{N}(de,ds) \\
&= a,
\end{aligned}
\]

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where $x$ is the state process with the initial condition $x(t) = a$, which is controlled by the leader $u_1$ and the follower $u_2$. The objective functional to be minimized by the leader is given by

$$J_1(a; u_1, u_2) = \mathbb{E} \left[ \int_t^T \left[ |x(s)|^2_{Q_1(s)} + |\mathbb{E}[x(s)]|^2_{R_1(s)} + |u_1(s)|^2_{M_1(s)} \right] ds + |x(T)|^2_{M_1} + |\mathbb{E}[x(T)]|^2_{M_1} \right],$$

and the objective functional to be minimized by the follower is as follows:

$$J_2(a; u_1, u_2) = \mathbb{E} \left[ \int_t^T \left[ |x(s)|^2_{Q_2(s)} + |\mathbb{E}[x(s)]|^2_{R_2(s)} + |u_2(s)|^2_{M_2(s)} \right] ds + |x(T)|^2_{M_2} + |\mathbb{E}[x(T)]|^2_{M_2} \right].$$

In the leader-follower Stackelberg game framework, the leader holds a dominating position in a decision-making process; the leader chooses and then announces her (or his) optimal strategy by considering the rational behavior of the follower [3, 31, 5]. We note that in (1.1)-(1.3), the expected value of state and control variables is included. In fact, (1.1) is known as a class of mean-field type SDEs (MF-SDEs), which can be applied to analyze macroscopic behavior of the interacting particle systems and to study mean-variance portfolio optimization problems. Therefore, the problem considered in this paper can be referred to as the linear-quadratic (LQ) leader-follower stochastic Stackelberg differential game for mean-field type SDEs with jump diffusions.

Various stochastic optimal control problems and differential games for MF-SDEs were studied extensively in the literature. Specifically, general stochastic maximum principles for MF-SDEs were established in [7, 14, 9]. LQ stochastic control problems for MF-SDEs were considered in [14, 24, 22, 30, 10, 23, 15, 20, 19, 18, 21]. Note that the references mentioned above considered MF-SDEs without jump diffusions. Recently, stochastic control problems and differential games for MF-SDEs with jump diffusions were studied in [2, 6, 27, 22, 12, 33, 17]. Specifically, stochastic maximum principles were established in [28, 27, 12, 33, 17] under different settings (including the delay case). LQ mean-field type differential games for jump-diffusion models were studied in [2]. As for applications, the mean-variance portfolio selection problem, mean cooperative dynamic model, and air conditioning control in building systems were considered as the MF-SDE framework [8, 29, 24, 27, 28].

We note that the aforementioned references correspond to a class of one-player mean-field type control problems and $N$-player noncooperative differential games. Within this formulation, hierarchical decision-making analysis between players cannot be considered. The class of leader-follower hierarchical decision-making differential games is also known as Stackelberg games, where the leader chooses his optimal decision by considering the rational behavior of the follower. Under this hierarchical setting, the leader’s optimal solution and the follower’s rational behavior constitute a Stackelberg equilibrium. Some earlier results on Stackelberg games their applications can be found in [3, 31, 5, 19, 25, 11, 4, 21] and the references therein. Recently, LQ Stackelberg differential games for MF-SDEs were studied in [16]. However, MF-SDEs in [16] do not have jump diffusions.
As mentioned above, we consider LQ leader-follower Stackelberg differential game for MF-SDEs with jump diffusions, captured in (1.1)-(1.3). Our problem and results can be viewed as extensions of [16] to the case of jump-diffusion models. Specifically, in our problem setup given in (1.1)-(1.3), the jump-diffusion term in (1.1) depends on control variables of the leader and the follower, which makes the problem challenging compared with the LQ mean-field type Stackelberg game without jumps in [16] and the classical LQ Stackelberg game without jumps in [31]. In addition, the cost parameters in (1.2)-(1.3) not being needed to be (positive) definite matrices leads to another challenge of this paper, as in the case of Stackelberg games without jumps in [16, 31]. Note also that our paper is different from mean-field type control problems and Nash differential games for jump-diffusion models in [2, 6, 28, 27, 12, 33, 17] in that they did not consider a hierarchical decision-making process modeled by the leader-follower framework. In fact, unlike our paper, [2, 6] considered the situation that players choose their optimal solutions noncooperatively and simultaneously to find a Nash equilibrium.

We first solve the LQ mean-field type control problem of the follower under an arbitrary control of the leader (minimizing (1.3) subject to (1.1) for an arbitrary $u_1$). In particular, by using the stochastic maximum principle (see Lemma 3.1), we obtain the open-loop type optimal solution of the follower in terms of the mean-field type forward-backward SDE (MF-FBSDE) with jump diffusions. Since the open-loop type solution is not implementable in practical situations, the state-feedback representation of the open-loop solution in terms of the coupled integro-Riccati differential equations (CIRDEs) is characterized via the Four-Step Scheme. We show that the corresponding state-feedback type control is the optimal solution for the follower’s problem via the completion of squares method (see Theorem 3.1).

We next address the problem of the leader, i.e., minimizing (1.2) subject to the optimal solution of the follower. This is the LQ stochastic control problem for the MF-FBSDE with jump diffusions, where the constraint is induced from the follower’s problem. In fact, the constraint in the leader’s problem (the MF-FBSDE with jump diffusions) characterizes the rational behavior of the follower. We obtain the necessary and sufficient condition of optimality for the leader’s problem in terms of the stochastic maximum principle via the variational approach (see Lemma 4.1). Then by the stochastic maximum principle, the open-loop optimal solution for the leader’s problem is obtained, which is expressed as the coupled MF-FBSDEs with jump diffusions.

The state-feedback representation of the open-loop optimal solution of the leader in terms of the CIRDEs is obtained by generalizing the Four-Step Scheme of [31, 16]. Unfortunately, there is a technical limitation when extending the Four-Step Scheme to the jump-diffusion model in (1.1). We provide a detailed discussion on the technical restriction in Section 5. Hence, we consider two different cases: (i) when the Poisson process $N$ has jumps of unit size ($E = \{1\}$ in (1.1)) and (ii) when the jump part in (1.1) does not depend on the follower’s control ($\hat{G}_2 = \hat{G}_2 = 0$). Note that the Four-Step Schemes in both cases are much more involved than that for Stackelberg games of SDEs without jumps studied in [31, 16, 21] due to the presence of the coupling terms by the Brownian motion and the Poisson process (see Theorems 4.1 and 4.2). Moreover, the CIRDEs obtained in the leader’s problem (see (4.18) and (4.27)) are integro-type, nonsymmetric and highly nonlinear, whereas the RDEs in [31, Theorem 3.3] and [16, Theorem 3.2] are local and/or symmetric. We also show that when the leader’s and follower’s problems are solvable, the corresponding...
open-loop optimal solutions constitute the Stackelberg equilibrium, and they admit
the state-feedback representation (see Corollaries 4.1 and 4.2).

This paper is organized as follows. The precise problem formulation is provided
in Section 2. We solve the mean-field type control problem of the follower and the
leader in Sections 3 and 4, respectively. The concluding remarks including several
potential future research problems are given in Section 5. The proofs of the main
results are provided in Appendices A-D.

Notation. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. For $x, y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of $x$, $\langle x, y \rangle$ is the inner product, and $|x| := \langle x, x \rangle^{1/2}$. Let $\mathbb{S}^n$ be the set of $n \times n$ symmetric matrices. Let $|x|^2_s := x^\top S x$ for $x \in \mathbb{R}^n$ and $S \in \mathbb{S}^n$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the natural filtration $\mathbb{F} := \{\mathcal{F}_s, 0 \leq s \leq t\}$ generated by the following two mutually independent stochastic processes and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$:

- an one dimensional standard Brownian motion $B$ defined on $[0, T]$;
- an $E$-marked right continuous Poisson random measure (process) $N$ defined on $E \times [0, T]$, where $E := E \setminus \{0\}$ with $E \subset \mathbb{R}$ being a Borel subset of $\mathbb{R}$ equipped with its Borel $\sigma$-field $\mathcal{B}(E)$. The intensity measure of $N$ is denoted by $\lambda'(de, dt) := \lambda(de)dt$, satisfying $\lambda(E) < \infty$, where $\{\tilde{N}(A, (0, t]) := (N - \lambda')(A, (0, t])\}_{t \in [0, T]}$ is an associated compensated $\mathcal{F}_t$-martingale random (Poisson) measure of $N$ for any $A \in \mathcal{B}(E)$. Here, $\lambda$ is an $\sigma$-finite Lévy measure on $(E, \mathcal{B}(E))$, which satisfies $\int_E (1 + e|e|^2)\lambda(de) < \infty$.

Remark 1. When $E = \{1\}$, i.e., the Poisson process $N$ has jumps of unit size, $\lambda'(dt) = \lambda dt$, where $\lambda > 0$ is the intensity of $N$, and $\{\tilde{N}((0, t]) := (N - \lambda')(0, t])\}_{t \in [0, T]}$ is the compensated Poisson process $[1, 26]$.

We introduce the following spaces $[1]$; for $t \in [0, T]$ with $s \in [t, T]$,

- $C^2_d(t, T; \mathbb{R}^n)$: the space of $\mathcal{F}_s$-adapted $\mathbb{R}^n$-valued stochastic processes, which is càdlàg and satisfies $\mathbb{E} [\sup_{s \in [t, T]} |x(s)|^2]^{1/2} < \infty$ for $x \in C^2_d(t, T; \mathbb{R}^n)$;
- $\mathcal{L}^2(t, T; \mathbb{R}^n)$: the space of $\mathcal{F}_s$-adapted $\mathbb{R}^n$-valued stochastic processes satisfying $\mathbb{E} \int_t^T |x(s)|^2 ds < \infty$ for $x \in \mathcal{L}^2(t, T; \mathbb{R}^n)$;
- $\mathcal{L}^2_{\mathbb{F}_\sigma}(t, T; \mathbb{R}^n)$: the space of $\mathcal{F}_s$-predictable $\mathbb{R}^n$-valued stochastic processes satisfying $\mathbb{E} \int_t^T |x(s)|^2 ds < \infty$ for $x \in \mathcal{L}^2_{\mathbb{F}_\sigma}(t, T; \mathbb{R}^n)$;
- $G^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^n)$: the space of square integrable functions such that for $k \in G^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^n)$, $k : E \to \mathbb{R}^n$ satisfies $(\int_E |k(e)|^2 \lambda(de))^{1/2} < \infty$, where $\lambda$ is an $\sigma$-finite Lévy measure on $(E, \mathcal{B}(E))$;
- $G^2_{\mathbb{F}_\sigma}(t, T, \lambda; \mathbb{R}^n)$: the space of stochastic processes such that $k : \Omega \times [t, T] \times E \to \mathbb{R}^n$, for $k \in G^2_{\mathbb{F}_\sigma}(t, T, \lambda; \mathbb{R}^n)$, is a $\mathcal{P} \times \mathcal{B}(E)$-measurable $\mathbb{R}^n$-valued $\mathcal{F}_s$-predictable process satisfying $\mathbb{E} \int_t^T \|k(s)\|_{\mathbb{S}^2} ds < \infty$, where $\mathcal{P}$ denotes the $\sigma$-algebra of $\mathcal{F}_s$-predictable subsets of $\Omega \times [t, T]$.

2. Problem formulation. In this section, we state the precise problem formulation of this paper given in (1.1)-(1.3). Below, we first recall (1.1)-(1.3) and provide their assumptions. The detailed problem statement is then followed.

Consider the following mean-field type SDE (MF-SDE) with jump diffusions on $[t, T]$ (recall (1.1)):

\[1\] The assumption of the one-dimensional $B$ and $\tilde{N}$ in (2.1) is only for notational convenience, and we can easily extend the results of this paper to the multi-dimensional case.
\[
\begin{align*}
dx(s) &= \left[ A(s)x(s-) + \bar{A}(s)\mathbb{E}[x(s-)] + B_1(s)u_1(s) + \bar{B}_1(s)\mathbb{E}[u_1(s)] \\
+ B_2(s)u_2(s) + \bar{B}_2(s)\mathbb{E}[u_2(s)] \right] ds + \left[ C(s)x(s-) + \bar{C}(s)\mathbb{E}[x(s-)] \\
+ D_1(s)u_1(s) + \bar{D}_1(s)\mathbb{E}[u_1(s)] + D_2(s)u_2(s) \\
+ \bar{D}_2(s)\mathbb{E}[u_2(s)] \right] dB(s) + \int_E \left[ F(s,e)x(s-) + \bar{F}(s,e)\mathbb{E}[x(s-)] \right] ds \\
+ G_1(s,e)u_1(s) + \bar{G}_1(s,e)\mathbb{E}[u_1(s)] + G_2(s,e)u_2(s) \\
+ \bar{G}_2(s,e)\mathbb{E}[u_2(s)] \right] \bar{N}(dc,ds)
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the controlled state process, \( u_1 \in \mathbb{R}^{m_1} \) is the control of the leader, and \( u_2 \in \mathbb{R}^{m_2} \) is the control of the follower. Let \( \mathcal{U}_1 := \mathcal{L}^2_{p,p}(t,T;\mathbb{R}^{m_1}) \) and \( \mathcal{U}_2 := \mathcal{L}^2_{p,p}(t,T;\mathbb{R}^{m_2}) \) be spaces of admissible controls for the leader and the follower, respectively.

As in (1.2), the objective functional to be minimized by the leader is given by

\[
J_1(a; u_1, u_2) = \mathbb{E} \left[ \int_t^T \left[ |x(s)|^2_{Q_1(s)} + |\mathbb{E}[x(s)]|^2_{Q_2(s)} + |u_1(s)|^2_{R_1(s)} + |u_2(s)|^2_{R_2(s)} \right] ds + |x(T)|^2_{M_1} + |\mathbb{E}[x(T)]|^2_{M_2} \right].
\]

Moreover, as given in (1.3), the objective functional that is minimized by the follower is as follows:

\[
J_2(a; u_1, u_2) = \mathbb{E} \left[ \int_t^T \left[ |x(s)|^2_{Q_1(s)} + |\mathbb{E}[x(s)]|^2_{Q_2(s)} + |u_1(s)|^2_{R_1(s)} + |u_2(s)|^2_{R_2(s)} \right] ds + |x(T)|^2_{M_1} + |\mathbb{E}[x(T)]|^2_{M_2} \right].
\]

Throughout this paper, the following assumption holds for the coefficients in (2.1)-(2.3):

**Assumption 1.** The coefficients \( A, \bar{A}, C, \bar{C} : [0,T] \rightarrow \mathbb{R}^{n \times n}, B_i, \bar{B}_i, D_i, \bar{D}_i : [0,T] \rightarrow \mathbb{R}^{n \times m_i}, i = 1, 2, F, \bar{F} : [0,T] \rightarrow G^2(E, \mathbb{B}(E), \lambda; \mathbb{R}^{n \times n}), G_i, \bar{G}_i : [0,T] \rightarrow G^2(E, \mathbb{B}(E), \lambda; \mathbb{R}^{n \times m_i}), i = 1, 2, Q_i, \bar{Q}_i : [0,T] \rightarrow \mathbb{S}^n \) and \( R_i, \bar{R}_i : [0,T] \rightarrow \mathbb{S}^{m_i}, i = 1, 2, \) are deterministic, which are continuous and uniformly bounded in \( t \in [0,T] \). Also, \( M_i, \bar{M}_i \in \mathbb{S}^n, i = 1, 2, \) are deterministic and bounded.

Note that for any \((u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2\), the MF-SDE in (2.1) admits a unique càdlàg solution in \( \mathcal{C}^2_p(t,T;\mathbb{R}^n) \) [28, Lemma 4.1] (see also the case of MF-SDEs with jump diffusions and delay in [27, Lemma 2.1] and [17, Theorem 3.2]). Moreover, \( Q_i, \bar{Q}_i, Q_i, \bar{R}_i, R_i, \bar{M}_i \) in (2.2)-(2.3) are not needed to be (positive) definite matrices.
Remark 2. When the Poisson process \( N \) has jumps of unit size, i.e., \( E = \{1\} \), the MF-SDE in (2.1) becomes (see Remark 1)

\[
\begin{align*}
    dx(t) &= \left[ A(x) dx(t) + \bar{A}(x) dt + B(x) dW(t) + \bar{B}(x) d\bar{W}(t) \right] dt + \left[ C(x) dx(t) + \bar{C}(x) dt + D(x) d\bar{W}(t) + \bar{D}(x) d\bar{W}(t) \right] dB(t) \\
    x(t) &= a.
\end{align*}
\]

This situation will be considered in Section 4.

The open-loop type interaction between the leader and the follower is stated as follows. The leader first chooses her (or his) optimal solution by considering the rational reaction of the follower, and announces his (or her) the corresponding optimal solution. The follower then determines her (or his) optimal solution by responding to the optimal solution of the leader. The problem can then be referred to as the \( LQ \) stochastic Stackelberg differential game for mean-field type jump-diffusion systems.

Under this framework, the Stackelberg game of this paper can be solved in a reverse way [31, 5, 3]. Specifically, the main objective of the follower is

\[
(LQ-F) \quad J_2(a; u_1, \bar{u}_2[a, u_1]) = \inf_{u_2 \in U_2} J_2(a; u_1, u_2), \quad \forall u_1 \in U_1. \quad (2.4)
\]

From (2.4), \( \bar{u}_2 \) is an optimal strategy dependent on \( (a, u_1) \in \mathbb{R}^n \times U_1 \), i.e., \( \bar{u}_2 : \mathbb{R}^n \times U_1 \rightarrow U_2 \). We note that the state dynamics under the optimal solution of \((LQ-F)\) characterizes the rational behavior of the follower. Given the optimal solution of \((LQ-F)\), the problem of the leader is as follows:

\[
(LQ-L) \quad J_1(a; \bar{u}_1, \bar{u}_2[a, \bar{u}_1]) = \inf_{u_1 \in U_1} J_1(a; u_1, \bar{u}_2[a, u_1]), \quad (2.5)
\]

subject to the rational behavior of the follower induced from \((LQ-F)\). When the pair \((\bar{u}_1, \bar{u}_2[a, \bar{u}_1]) \in U_1 \times U_2 \) in (2.4) and (2.5) exists, we say that \((\bar{u}_1, \bar{u}_2[a, \bar{u}_1])\) constitutes an (adapted) open-loop type Stackelberg equilibrium [3, 31, 5, 21].

3. \( LQ \) mean-field type optimal control for the follower. This section solves \((LQ-F)\). We first state the necessary condition for optimality of \((LQ-F)\). Since we are not able to find an exact reference on the following lemma, we provide its proof in Appendix A.

Lemma 3.1. Suppose that \( \bar{u}_2 \) is the optimal solution of \((LQ-F)\), where \( \bar{x} \) is the corresponding optimal state process. Then there exists a unique \((p, q, r) \in C^0_t (t, T; \mathbb{R}^n) \times L^2_t \left( t, T; \mathbb{R}^n \right) \times G^2_{t, \lambda} (t, T; \mathbb{R}^n)\), which is the solution of the following mean-field type backward SDE (MF-BSDE) with jump diffusions:

\[
\begin{align*}
    dp(s) &= -\left[ A(s)^T p(s-) + \bar{A}(s)^T E[p(s-)] + C(s)^T q(s) \\
    &+ \bar{C}(s)^T E[q(s)] + Q_1(s) \bar{x}(s-) + \bar{Q}_1(s) E[\bar{x}(s-)] \\
    &+ \int_E F(s, e)^T r(s, e) \lambda(de) + \int_E \bar{F}(s, e)^T E[r(s, e)] \lambda(de) \right] ds \\
    &+ q(s) dB(s) + \int_E r(s, e) \bar{N}(de, ds) \\
    p(T) &= M_2 \bar{x}(T) + \bar{M}_2 E[\bar{x}(T)],
\end{align*}
\]

(3.1)
such that the following first-order optimality condition holds:

\begin{align}
R_2(s)\tilde{u}_2(s) + \bar{R}_2(s)\mathbb{E}[\tilde{u}_2(s)] &+ B_2(s)^T p(s) + \bar{B}_2(s)^T \mathbb{E}[p(s)] \\
+ D_2(s)^T q(s) + D_2(s)^T \mathbb{E}[q(s)] &+ \int_\mathcal{E} G_2(s,e)^T r(s,e)\lambda(de) \\
+ \int_\mathcal{E} \bar{G}_2(s,e)^T \mathbb{E}[r(s,e)]\lambda(de) &= 0, \text{ d}\mathbb{P}\text{-a.s., d}s\text{-a.e.}
\end{align}

Hence, from Lemma 3.1 (see (2.1) and (3.1)), we have the following mean-field type forward-backward SDE (MF-FBSDE) with jump diffusions:

\begin{equation}
\begin{aligned}
d\pi(s) &= \left[ A(s)\pi(s) + \bar{A}(s)\mathbb{E}[\pi(s)] + B_1(s)u_1(s) + \bar{B}_1(s)\mathbb{E}[u_1(s)] \\
+ B_2(s)\bar{\pi}_2(s) + B_2(s)\mathbb{E}[\bar{\pi}_2(s)] \right] ds + \left[ C(s)\pi(s) + \bar{C}(s)\mathbb{E}[\pi(s)] \\
+ D_1(s)u_1(s) + D_1(s)\mathbb{E}[u_1(s)] + D_2(s)\pi_2(s) \\
+ D_2(s)\mathbb{E}[\pi_2(s)] \right] dB(s) + \int_\mathcal{E} \left[ F(s,e)\pi(s) + \bar{F}(s,e)\mathbb{E}[\pi(s)] \\
+ G_1(s,e)u_1(s) + G_1(s,e)\mathbb{E}[u_1(s)] \\
+ G_2(s,e)\pi_2(s) + G_2(s,e)\mathbb{E}[\pi_2(s)] \right] d\lambda(de,ds) \\
p(s) &= -\left[ A(s)^T p(s) + \bar{A}(s)^T \mathbb{E}[p(s)] \\
+ C(s)^T q(s) + \bar{C}(s)^T \mathbb{E}[q(s)] + Q_1(s)\pi(s) + \bar{Q}_1(s)\mathbb{E}[\pi(s)] \\
+ Q_2(s)\pi_2(s) + \bar{Q}_2(s)\mathbb{E}[\pi_2(s)] \\
+ \int_\mathcal{E} F(s,e)^T r(s,e)\lambda(de) + \int_\mathcal{E} \bar{F}(s,e)^T \mathbb{E}[r(s,e)]\lambda(de) \right] ds \\
+ q(s)dB(s) + \int_\mathcal{E} r(s,e)d\lambda(de,ds) \\
x(t) &= a, \quad p(T) = M_2\pi(T) + M_2\mathbb{E}[\pi(T)],
\end{aligned}
\end{equation}

where \( \bar{\pi}_2 \in \mathcal{U}_2 \) holds the optimality condition in (3.2).

We take the following transformation in the Four-Step Scheme:

\begin{equation}
p(s) = P(s)(\pi(s) - \mathbb{E}[\pi(s)]) + Z(s)\mathbb{E}[\pi(s)] + \phi(s),
\end{equation}

where \( P \) and \( Z \) are \( \mathbb{S}^n \)-valued deterministic processes, which are assumed to satisfy the following form:

\begin{equation}
\begin{aligned}
\frac{dP(s)}{ds} &= \Lambda_1(s), \quad \frac{dZ(s)}{ds} = \Lambda_2(s) \\
P(T) &= M_2, \quad Z(T) = M_2 + M_2,
\end{aligned}
\end{equation}

and \( \phi \) is an \( \mathbb{R}^n \)-valued (linear) MF-BSDE with jump diffusions satisfying:

\begin{equation}
\begin{aligned}
d\phi(s) &= \Lambda_3(s)ds + \theta(s)dB(s) + \int_\mathcal{E} \psi(s,e)d\lambda(de,ds) \\
\phi(T) &= 0.
\end{aligned}
\end{equation}

Let \( \tilde{u}_1(s) := u_1(s) - \mathbb{E}[u_1(s)] \), \( \tilde{u}_2(s) := \bar{\pi}_2(s) - \mathbb{E}[\bar{\pi}_2(s)] \), and \( e(s) := \pi(s) - \mathbb{E}[\pi(s)] \), where note that \( e(t) = 0 \). We apply Itô’s formula to (3.4). Then with (3.5) and
\[ dp = - \left[ A^T p(s-) + \tilde{A}^T \mathbb{E}[p(s-)] + C^T q + \tilde{C}^T \mathbb{E}[q] + Q_2 \pi(s-) \right] \tag{3.7} \
+ Q_2 \mathbb{E}[\pi(s-)] + \int_E F(e)^T r(e) \lambda(de) + \int_E \tilde{F}(e)^T \mathbb{E}[r(e)] \lambda(de) \, ds \\
+ q(s) d\mathbb{B}(s) + \int_E r(s,e) \tilde{N}(de,ds) \]

Then we can see that (s is suppressed)

\[
\begin{cases} 
q(s) = P(s-) \left[ Ce(s-) + (C + \tilde{C}) \mathbb{E}[x(s-)] + D_1 \tilde{u}_1 \\
+ (D_1 + \tilde{D}_1) \mathbb{E}[u_1] + D_2 \tilde{u}_2 + (D_2 + \tilde{D}_2) \mathbb{E}[\pi_2] \right] + \theta(s) \\
r(s,e) = P(s-) \left[ Fe(s-) + (F + \tilde{F}) \mathbb{E}[x(s-)] + G_1 \tilde{u}_1 \\
+ (G_1 + \tilde{G}_1) \mathbb{E}[u_1] + G_2 \tilde{u}_2 + (G_2 + \tilde{G}_2) \mathbb{E}[\pi_2] \right] + \psi(s,e),
\end{cases}
\]

which implies

\[
\begin{cases} 
\mathbb{E}[q(s)] = P(s-) \left[ (C + \tilde{C}) \mathbb{E}[x(s-)] + (D_1 + \tilde{D}_1) \mathbb{E}[u_1] \\
+ (D_2 + \tilde{D}_2) \mathbb{E}[\pi_2] \right] + \mathbb{E}[\theta(s)] \\
\mathbb{E}[r(s,e)] = P(s-) \left[ (F + \tilde{F}) \mathbb{E}[x(s-)] \\
+ (G_1 + \tilde{G}_1) \mathbb{E}[u_1] + (G_2 + \tilde{G}_2) \mathbb{E}[\pi_2] \right] + \mathbb{E}[\psi(s,e)].
\end{cases}
\tag{3.8}
\]

and

\[
\begin{cases} 
q(s) - \mathbb{E}[q(s)] = P(s-) \left[ Ce(s-) + D_1 \tilde{u}_1 + D_2 \tilde{u}_2 \right] + \theta(s) - \mathbb{E}[\theta(s)] \\
r(s,e) - \mathbb{E}[r(s,e)] = P(s-) \left[ Fe(s-) + G_1 \tilde{u}_1 + G_2 \tilde{u}_2 \right] \\
+ \psi(s,e) - \mathbb{E}[\psi(s,e)].
\end{cases}
\tag{3.9}
\]

Note that the optimality condition in (3.2) is equivalent to (s is suppressed)

\[
\begin{cases} 
R_2 (\pi_2(s) - \mathbb{E}[\pi_2(s)]) + B_2^T (p(s-) - \mathbb{E}[p(s-)]) + D_2^T (q - \mathbb{E}[q]) \\
+ \int_E G_2(s,e)^T (r(s,e) - \mathbb{E}[r(s,e)]) \lambda(de) = 0 \\
(R_2 + B_2) \mathbb{E}[\pi_2(s)] + (B_2 + \tilde{B}_2)^T \mathbb{E}[p(s-)] + (D_2 + \tilde{D}_2)^T \mathbb{E}[q] \\
+ \int_E (G_2(s,e) + \tilde{G}_2(s,e))^T \mathbb{E}[r(s,e)] \lambda(de) = 0.
\end{cases}
\tag{3.10}
\]
Let \( \hat{\phi}(s) := \phi(s) - \mathbb{E}[\phi(s)] \), \( \hat{\theta}(s) := \theta(s) - \mathbb{E}[\theta(s)] \) and \( \hat{\psi}(s, e) := \psi(s, e) - \mathbb{E}[\psi(s, e)] \).

By substituting (3.4), (3.8) and (3.9) into (3.10), we have (s and \( e \) are suppressed)

\[
\begin{align*}
R_2 \hat{u}_2(a) + B_2^\top P(s) e(s) + B_2^\top \hat{\phi}(s) + D_2^\top P(s) \left[ C e(s) - \mathbb{E}[C(s)] \right] + D_1 \hat{u}_1(s) + D_2 \hat{\theta}(s) + \int_E G_2^\top P(s) \left[ F e(s) - \mathbb{E}[F(s)] \right] + G_1 \hat{u}_1(s) + G_2 \hat{\theta}(s) \lambda(de) + \int_E G_2^\top \hat{\psi}(s, e) \lambda(de) &= 0.
\end{align*}
\]

Then from (3.11), the state-feedback representation of \( \pi_2 \), which is the optimal solution for (LQ-F), can be written as

\[
\begin{align*}
\pi_2(s) - \mathbb{E} [\pi_2(s)] &= -\tilde{R}_2(s)^{-1} \tilde{S}_2(s)^\top [\pi(s) - \mathbb{E}[\pi(s)]] \\
&= -\tilde{R}_2(s)^{-1} B_2(s)^\top \phi(s) - \mathbb{E}[\phi(s)] \\
&= -\tilde{R}_2(s)^{-1} D_2(s)^\top \lambda \theta(s) - \mathbb{E}[\lambda \theta(s)] \\
&= -\tilde{R}_2(s)^{-1} \int_E G_2(s, e)^\top \psi(s, e) - \mathbb{E}[\psi(s, e)] \lambda(de) \\
&= -\tilde{R}_2(s)^{-1} \tilde{S}_1(s)(u_1(s) - \mathbb{E}[u_1(s)]) \\
\mathbb{E}[\pi_2(s)] &= -\tilde{R}_2(s)^{-1} \tilde{S}_2(s)^\top [\pi(s) - \mathbb{E}[\pi(s)]] \\
&= -\tilde{R}_2(s)^{-1} (B_2(s) + \tilde{B}_2(s))^\top \phi(s) \\
&= -\tilde{R}_2(s)^{-1} (D_2(s) + \tilde{D}_2(s))^\top \lambda \theta(s) \\
&= -\tilde{R}_2(s)^{-1} \int_E (G_2(s, e) + \tilde{G}_2(s, e))^\top \psi(s, e) \lambda(de) \\
&= -\tilde{R}_2(s)^{-1} \tilde{S}_1(s)[u_1(s)],
\end{align*}
\]

provided that \( \tilde{R}_2 \) and \( \tilde{S}_2 \) are invertible, where (s and \( e \) are suppressed)

\[
\begin{align*}
\tilde{R}_2 &:= R_2 + D_2 P(s) D_2 + \int_E G_2^\top P(s) G_2 \lambda(de) \\
\tilde{S}_2 &:= \left( B_2^\top P(s) + D_2^\top P(s) C + \int_E G_2^\top P(s) F \lambda(de) \right)^\top \\
\tilde{S}_1 &:= D_2^\top P(s) D_1 + \int_E G_2^\top P(s) G_1 \lambda(de) \\
\tilde{S}_2 &:= \left( (B_2 + \tilde{B}_2)^\top Z(s) + (D_2 + \tilde{D}_2)^\top P(s)(C + \tilde{C}) \right)^\top \\
&= \left( \int_E (G_2 + \tilde{G}_2)^\top P(s)(F + \tilde{F}) \lambda(de) \right)^\top \\
\tilde{S}_1 &:= (D_2 + \tilde{D}_2)^\top P(s)(D_1 + \tilde{D}_1) + \int_E (G_2 + \tilde{G}_2)^\top P(s)(G_1 + \tilde{G}_1) \lambda(de).
\end{align*}
\]

We can easily see that (3.12) depends on the arbitrary control of the leader \( u_1 \in U_1 \). Note that \( \pi_2 : \mathbb{R}^n \times U_1 \to U_2 \); hence, \( \pi_2 \in U_2 \) for a given \( u_1 \in U_1 \).
The state dynamics $\mathcal{F}$ in (3.3) under (3.12) can be written as

\[
\begin{align*}
\begin{aligned}
    \mathbf{d}\mathcal{F}(s) &= \left[ \hat{A}(s)\mathcal{F}(s) - \hat{A}(s)\mathbb{E}[\mathcal{F}(s)] + \hat{B}_2(s)\phi(s) + \hat{B}_2(s)\mathbb{E}[\phi(s)] \\
    &+ \hat{H}_2(s)\theta(s) + \hat{H}_2(s)\mathbb{E}[\theta(s)] + \int_E \hat{K}_2(s, e)\psi(s, e)\lambda(\text{d}e) \\
    &+ \int_E \hat{K}_2(s, e)\mathbb{E}[\psi(s, e)]\lambda(\text{d}e) + \hat{B}_1u_1(s) + \hat{B}_1\mathbb{E}[u_1(s)] \right] \text{d}s \\
    &\left[ \hat{C}(s)\mathcal{F}(s) + \hat{C}(s)\mathbb{E}[\mathcal{F}(s)] + \hat{H}_2(s)\mathbb{E}[\phi(s)] + \hat{H}_2(s)\mathbb{E}[\phi(s)] \\
    &+ \hat{H}_2(s)\theta(s) + \hat{H}_2(s)\mathbb{E}[\theta(s)] + \int_E \hat{K}_2(s, e)\psi(s, e)\lambda(\text{d}e) \\
    &+ \int_E \hat{K}_2(s, e)\mathbb{E}[\psi(s, e)]\lambda(\text{d}e) + \hat{D}_1u_1(s) + \hat{D}_1\mathbb{E}[u_1(s)] \right] \text{d}B(s) \\
    &\left[ \int_E \hat{F}(s, e)\mathcal{F}(s) + \hat{F}(s, e)\mathbb{E}[\mathcal{F}(s)] + \hat{K}_2(s, e)\theta(s) + \hat{K}_2(s, e)\mathbb{E}[\theta(s)] \\
    &+ \hat{K}_2(s, e)\mathbb{E}[\phi(s)] + \hat{K}_2(s, e)\mathbb{E}[\phi(s)] \\
    &+ \int_E \hat{K}_2(s, e, e')\psi(s, e')\lambda(\text{d}e') \\
    &+ \int_E \hat{K}_2(s, e, e')\mathbb{E}[\psi(s, e', e')]\lambda(\text{d}e') \\
    &+ \hat{G}_1(s, e)u_1(s) + \hat{G}_1(s, e)\mathbb{E}[u_1(s)] \right] \hat{N}(\text{d}e, \text{d}s)
\end{aligned}
\end{align*}
\]

(3.13)

where $(s$ and $e$ are suppressed)

\[
\begin{align*}
\hat{A} &:= A - B_2R_2^{-1}\bar{S}_2^T, \quad \hat{A} := (A + A) - (B_2 + B_2)R_2^{-1}\bar{S}_2^T - \hat{A} \\
\hat{B}_2 &:= -B_2R_2^{-1}B_2^T, \quad \hat{B}_2 := -(B_2 + B_2)R_2^{-1}(B_2 + B_2)^T - \hat{B}_2 \\
\hat{H}_2 &:= -B_2R_2^{-1}D_2^T, \quad \hat{H}_2 := -(B_2 + B_2)R_2^{-1}(D_2 + D_2)^T - \hat{H}_2 \\
\hat{K}_2(s, e) &:= -(B_2 + B_2)R_2^{-1}G_2(s, e) - \hat{K}_2(s, e) \\
\hat{K}_2(s, e) &:= -B_2R_2^{-1}G_2(s, e) - \hat{K}_2(s, e) \\
\hat{B}_1 &:= B_1 - B_2R_2^{-1}\bar{S}_1, \quad \hat{B}_1 := (B_1 + B_1) - (B_2 + B_2)R_2^{-1}\bar{S}_1 - \hat{B}_1 \\
\hat{C} &:= C - D_2R_2^{-1}\bar{S}_2^T, \quad \hat{C} := (C + C) - (D_2 + D_2)^T R_2^{-1}\bar{S}_2^T - \hat{C} \\
\hat{H}_2 &:= -D_2R_2^{-1}D_2^T, \quad \hat{H}_2 := -(D_2 + D_2)R_2^{-1}(D_2 + D_2)^T - \hat{H}_2 \\
\hat{K}_2(s, e) &:= -D_2R_2^{-1}G_2(s, e)^T \hat{K}_2(s, e) \\
\hat{K}_2(s, e) &:= -(D_2 + D_2)R_2^{-1}(G_2(s, e) + G_2(s, e)) - \hat{K}_2(s, e) \\
\hat{D}_1 &:= D_1 - D_2R_2^{-1}\bar{S}_1, \quad \hat{D}_1 := (D_1 + D_1) - (D_2 + D_2)R_2^{-1}\bar{S}_1 - \hat{D}_1 \\
\hat{F}(s, e) &:= F(s, e) - G_2(s, e)R_2^{-1}\bar{S}_2^T \\
\hat{F}(s, e) &:= (F(s, e) + F(s, e)) - (G_2(s, e) + G_2(s, e))R_2^{-1}\bar{S}_2^T - \hat{F}(s, e) \\
\hat{K}_2(s, e, e') &:= -G_2(s, e)R_2(s)^{-1}G_2(s, e') \\
\hat{K}_2(s, e, e') &:= -(G_2(s, e) + G_2(s, e'))R_2(s)^{-1}(G_2(s, e) + G_2(s, e')) - \hat{K}(s, e, e') \\
\hat{G}_1(s, e) &:= G_1(s, e) - G_2(s, e)R_2(s)^{-1}\bar{S}_1 \\
\hat{G}_1(s, e) &:= (G_1(s, e) + G_1(s, e)) - G_2(s, e)R_2(s)^{-1}\bar{S}_1 - \hat{G}_1(s, e).
\end{align*}
\]

(3.14)
Note that (3.13) characterizes the rational behavior of the follower under the state-feedback type optimal solution given in (3.12).

By substituting (3.9) and (3.12) into (3.7), we can show that $P$ and $Z$ in (3.5) satisfy the following coupled integro-Riccati differential equations (CIRDEs):

$$
\begin{align*}
-\frac{dP(s)}{ds} &= A(s)^T P(s-) + P(s-)A(s) + Q_2(s) + C(s)^T P(s-)C(s) \nonumber \\
&\quad + \int_E F(s,e)^T P(s-)F(s,e)\lambda(de) - \tilde{S}_2(s) \tilde{R}_2(s)^{-1} \tilde{S}_2(s)^T \\
-\frac{dZ(s)}{ds} &= (A(s) + A(s))^T Z(s-) + Z(s-)(A(s) + \tilde{A}(s)) \\
&\quad + \int_E (F(s,e) + \tilde{F}(s,e))^T P(s-)(F(s,e) + \tilde{F}(s,e))\lambda(de) \\
&\quad - \tilde{S}_2(s) \tilde{R}_2(s)^{-1} \tilde{S}_2(s)^T \\
&\quad + \int_E F(s,e)^T P(s-)F(s,e)\lambda(de) \\
&\quad - \tilde{S}_2(s) \tilde{R}_2(s)^{-1} \tilde{S}_2(s)^T \\
P(T) &= M_2, \quad Z(T) = M_2 + \hat{M}_2, \\
\det(\tilde{R}_2(s)) &\neq 0, \quad \det(\tilde{R}_2(s)) &\neq 0, \quad \forall s \in [t, T],
\end{align*}
$$

(3.15)

and $(\phi, \theta, \psi)$ in (3.6) satisfies the following MF-BSDE with jump diffusions:

$$
\begin{align*}
\frac{d\phi(s)}{ds} &= -[\hat{A}(s)^T \phi(s-) + \hat{A}(s)^T \mathbb{E}[\phi(s-)] + \hat{C}(s)^T \theta(s)] \\
&\quad + \hat{C}(s)^T \mathbb{E}[\theta(s)] + \int_E \hat{F}(s,e)^T \psi(s,e)\lambda(de) \\
&\quad + \int_E \hat{F}(s,e)^T \mathbb{E}[\psi(s,e)]\lambda(de) + \hat{H}_1(s)^T u_1(s) + \hat{H}_1(s)^T \mathbb{E}[u_1(s)] \\
&\quad + \int_E \hat{K}_1(s,e)^T u_1(s)\lambda(de) + \int_E \hat{K}_1(s,e)^T \mathbb{E}[u_1(s)]\lambda(de) ds \\
&\quad + \theta(s)dB(s) + \int_E \psi(s,e)\mathbb{N}(de, ds) \\
\phi(T) &= 0,
\end{align*}
$$

(3.16)

where $(s$ is suppressed)

$$
\begin{align*}
\hat{H}_1 &= (C^T P(s-)D_1 + P(s-)B_1 - \tilde{S}_2 \tilde{R}_2^{-1} \tilde{S}_1)^T \\
\hat{H}_1 &= (Z(s-)B_1 + \tilde{B}_1^T + (C + \hat{C})^T P(s-)(D_1 + \tilde{D}_1) - \tilde{S}_1 \tilde{R}_2^{-1} \tilde{S}_1)^T - \hat{H}_1 \\
\hat{K}_1(s,e) &= (F(s,e) + \hat{F}(s,e))^T P(s-)(G_1(s,e) + \hat{G}_1(s,e))^T - \hat{K}_1(s,e).
\end{align*}
$$

To summarize the above analysis, we have the following result. The proof is provided in Appendix B.

**Theorem 3.1.** Suppose that Assumption 1 holds. Assume that the CIRDEs in (3.15) admit unique solutions $(P, Z)$, where $\tilde{R}_2$ and $\hat{R}_2$ are uniformly positive definite for all $s \in [t, T]$. Then the MF-BSDE with jump diffusions given in (3.16) admits a unique solution of $(\phi, \theta, \psi) \in \mathcal{C}^2 (\mathbb{T}_2(t, T; \mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F}_T}(t, T; \mathbb{R}^n) \times \mathcal{G}^2_{\mathcal{F}_T}(t, T; \mathbb{R}^n)$.

Also, the state-feedback representation of the optimal control for (LQ-F) is given by (note that (3.17) below has an argument of the form s — to preserve predictability of the control process.)

$$
\begin{align*}
\mathbb{E}[\pi_2(s)] &= -\tilde{R}_2(s)^{-1} \tilde{S}_2(s)^T ( x(s-) - \mathbb{E}[x(s-)] ) - \tilde{R}_2(s)^{-1} \tilde{f}(s) \\
\mathbb{E}[\pi_2(s)] &= -\tilde{R}_2(s)^{-1} \tilde{S}_2(s)^T \mathbb{E}[x(s-)] - \tilde{R}_2(s)^{-1} \tilde{f}(s),
\end{align*}
$$

(3.17)
where

\[
\begin{align*}
\tilde{f}(s) & := B_2(s)\top (\phi(s) - E[\phi(s)]) + D_2(s)\top (\theta(s) - E[\theta(s)]) \\
& \quad + \int_E G_2(s, e)\top (\psi(s, e) - E[\psi(s, e)]) \lambda(de) + \bar{S}_1(s) (u_1(s) - E[u_1(s)]) \\
\hat{f}(s) & := (B_2(s) + \bar{B}_2(s))\top E[\phi(s)] + (D_2(s) + \bar{D}_2(s))\top E[\theta(s)] \\
& \quad + \int_E (G_2(s, e) + \bar{G}_2(s, e))\top E[\psi(s, e)] \lambda(de) + \bar{S}_1(s) E[u_1(s)].
\end{align*}
\]

The corresponding optimal cost under (3.17) can be written as (s and e are suppressed)

\[
J_2(\alpha; u_1, \bar{u}_2) = E \left[ |a|_Z^2(0) + 2\langle a, \phi(0) \rangle + \int_t^T \left( -|\tilde{f}(s)|^2_{\bar{R}_Z^{-1}} - |\hat{f}(s)|^2_{\bar{R}_Z^{-1}} + |\bar{u}_1(s)|^2_{\bar{D}_1^{-1}} P_{D_1} \right. \\
+ \int_E |\bar{u}_1(s)|^2_{G_1^{-1} P_{G_1}} \lambda(de) + 2\langle \bar{u}_1(s), B_1\top \hat{\phi}(s) + D_1 \hat{\theta}(s) \rangle + \int_E G_1\top \psi(s, e) \lambda(de) \\
+ E[u_1(s)]\| (D_1 + \bar{D}_1)\top P(D_1 + \bar{D}_1) + (B_1 + \bar{B}_1)\top Z(B_1 + \bar{B}_1) \\
+ \int_E E[u_1(s)]\| (F_1 + \bar{F}_1)\top P(F_1 + \bar{F}_1) \lambda(de) + 2E[u_1(s)], (B_1 + \bar{B}_1)\top E[\phi(s)] \\
+ \left( D_1 + \bar{D}_1 \right)\top E[\theta(s)] + \int_E (G_1 + \bar{G}_1)\top E[\psi(s, e)] \lambda(de) \right] ds.
\]

Remark 3. Recently, it was shown in [32, Theorem 4.1] (see also the LQ mean-field type control without jumps in [30, Theorem 4.1]) that \( P \) of the CIRDEs in (3.15) admits a unique positive solution when \( R_2, R_2 + R_2, Q_2, Q_2 + \bar{Q}_2, M_2, M_2 + \bar{M}_2 \) are uniformly positive definite for all \( s \in [0, T] \). Hence, under this condition, \( \bar{Z} \) in (3.15) also admits a unique positive solution.

4. LQ mean-field type optimal control for the leader. In this section, we solve (LQ-L) in (2.5). Specifically, based on Theorem 3.1 of (LQ-F) in Section 3, (LQ-L) can be reformulated as follows:

\[
(LQ-L) \quad J_1(\alpha; \bar{u}_1, \bar{u}_2[a, \bar{u}_1]) = \inf_{u_1 \in U_1} J_1(\alpha; u_1, \bar{u}_2[a, u_1]), \text{ subject to (3.13) and (3.16)}.
\]

Note that from Theorem 3.1, (3.13) and (3.16) correspond to the MF-FBSDE with jump diffusions, which characterize the rational behavior of the follower under the state-feedback type optimal solution in (3.17). As can be seen, (LQ-L) is the LQ stochastic optimal control for the MF-FBSDE with jump diffusions.

The following lemma states the necessary and sufficient condition for optimality of (LQ-L). The proof can be found in Appendix C.

Lemma 4.1. Assume that Assumption 1 holds. Suppose that \( \bar{u}_1 \in U_1 \), where \( \bar{u} \) is the corresponding state trajectory. Suppose that \( (\bar{\tau}, \beta) \in C_2^2(t, T; \mathbb{R}^n \times \mathbb{R}^n) \), \( (\phi, \theta, \psi) \in C_2^2(t, T; \mathbb{R}^n) \times C_2^2(t, T; \mathbb{R}^n) \times \mathbb{G}_2^2(t, T; \mathbb{R}^n) \) and \( (\alpha, \eta, \gamma) \in C_2^2(t, T; \mathbb{R}^n) \times C_2^2(t, T; \mathbb{R}^n) \times \mathbb{G}_2^2(t, T; \mathbb{R}^n) \) are the solutions of the following coupled
MF-FBSDEs with jump diffusions (s is suppressed):

\[
\begin{align*}
\mathrm{d}\pi(s) &= \left[ \hat{A}\pi(s) + \hat{A}\mathbb{E}[\pi(s)] + \hat{B}_2\phi(s) + \hat{B}_2\mathbb{E}[\phi(s)] \right]
\quad + \hat{H}_2\theta(s) + \hat{H}_2\mathbb{E}[\theta(s)] + \int_E \hat{K}_2(s, e)\psi(s, e)\lambda(de)
\quad + \int_E \hat{K}_2(s, e)\mathbb{E}[\psi(s, e)]\lambda(de) + \hat{B}_1\pi_1(s) + \hat{B}_1\mathbb{E}[\pi_1(s)] ds \\
+ \left[ \hat{C}\pi(s) + \hat{C}\mathbb{E}[\pi(s)] + \hat{H}_2^\top\phi(s) + \hat{H}_2^\top\mathbb{E}[\phi(s)] \right]
\quad + \hat{H}_2\theta(s) + \hat{H}_2\mathbb{E}[\theta(s)] + \int_E \hat{K}_2(s, e)\psi(s, e)\lambda(de)
\quad + \int_E \hat{K}_2(s, e)\mathbb{E}[\psi(s, e)]\lambda(de) + \hat{D}_1\pi_1(s) + \hat{D}_1\mathbb{E}[\pi_1(s)] dB(s) \\
+ \int_E \hat{F}(s, e)\mathbb{E}[\pi(s)] + \hat{F}(s, e)\mathbb{E}[\pi(s)] + \hat{K}_2(s, e)^\top\phi(s)
\quad + \hat{K}_2(s, e)^\top\mathbb{E}[\phi(s)] + \hat{K}_2(s, e)^\top\theta(s) + \hat{K}_2(s, e)^\top\mathbb{E}[\theta(s)]
\quad + \int_E \hat{K}_2(s, e, e')\psi(s, e')\lambda(de') + \int_E \hat{K}_2(s, e, e')\mathbb{E}[\psi(s, e, e')]\lambda(de')
\quad + \hat{G}_1(s, e)\pi_1(s) + \hat{G}_1(s, e)\mathbb{E}[\pi_1(s)] \hat{N}(de, ds) \\
\mathrm{d}\phi(s) &= - \left[ \hat{A}^\top\phi(s) + \hat{A}^\top\mathbb{E}[^\phi(s)] + \hat{C}^\top\theta(s) + \hat{C}^\top\mathbb{E}[\theta(s)] \\
+ \int_E \hat{F}(s, e)^\top\psi(s, e)\lambda(de) + \int_E \hat{F}(s, e)^\top\mathbb{E}[\psi(s, e)]\lambda(de) \\
+ \hat{H}_1\pi_1(s) + \hat{H}_1\mathbb{E}[\pi_1(s)] + \int_E \hat{K}_1(s, e)^\top u_1(s)\lambda(de) \\
+ \int_E \hat{K}_1(s, e)^\top\mathbb{E}[u_1(s)]\lambda(de) \right] ds \\
+ \theta(s)dB(s) + \int_E \psi(s, e)\hat{N}(de, ds)
\end{align*}
\]

(4.1)

\[
\begin{align*}
\mathrm{d}\alpha(s) &= \left[ \hat{A}\alpha(s) + \hat{A}\mathbb{E}[\alpha(s)] + \hat{C}^\top\eta(s) + \hat{C}^\top\mathbb{E}[\eta(s)] \\
+ \int_E \hat{F}(s, e)^\top\gamma(s, e)\lambda(de) + \int_E \hat{F}(s, e)^\top\mathbb{E}[\gamma(s, e)]\lambda(de) \\
+ \hat{Q}_1\pi(s) + \hat{Q}_1\mathbb{E}[\pi(s)] \right] ds + \eta(s)dB(s) + \int_E \gamma(s, e)\hat{N}(de, ds) \\
\mathrm{d}\beta(s) &= \left[ \hat{A}\beta(s) + \hat{A}\mathbb{E}[\beta(s)] + \hat{B}_2\alpha(s) + \hat{B}_2\mathbb{E}[\alpha(s)] \\
+ \hat{H}_2\eta(s) + \hat{H}_2\mathbb{E}[\eta(s)] + \int_E \hat{K}_2(s, e)\gamma(s, e)\lambda(de) \\
+ \int_E \hat{K}_2(s, e)\mathbb{E}[\gamma(s, e)]\lambda(de) \right] ds \\
+ \left[ \hat{C}\beta(s) + \hat{C}\mathbb{E}[\beta(s)] + \hat{H}_2^\top\alpha(s) + \hat{H}_2^\top\mathbb{E}[\alpha(s)] \\
+ \hat{H}_2\eta(s) + \hat{H}_2\mathbb{E}[\eta(s)] + \int_E \hat{K}_2(s, e)\gamma(s, e)\lambda(de) \\
+ \int_E \hat{K}_2(s, e)\mathbb{E}[\gamma(s, e)]\lambda(de) \right] dB(s) \\
+ \int_E \hat{F}(s, e)\beta(s) + \hat{F}(s, e)\mathbb{E}[\beta(s)] + \hat{K}_2^\top\alpha(s) \\
+ \hat{K}_2^\top\mathbb{E}[\alpha(s)] + \hat{K}_2^s\gamma(s) + \hat{K}_2^s\mathbb{E}[\gamma(s)] \\
+ \int_E \hat{K}(s, e)^\top\mathbb{E}[\alpha(s)] + \hat{K}(s, e)^\top\mathbb{E}[\gamma(s)] + \int_E \hat{K}(s, e)^\top\mathbb{E}[\eta(s)] \\
+ \hat{M}_1\pi(T) + \hat{M}_1\mathbb{E}[\pi(T)] \\
\end{align*}
\]

\[
\begin{align*}
x(t) = \alpha, \quad \beta(t) = 0, \quad \phi(T) = 0, \quad \alpha(T) = M_1\pi(T) + \hat{M}_1\mathbb{E}[\pi(T)].
\end{align*}
\]

For \(u_i^* \in U_i\), let \((x', \alpha') \in C_b^2(t, T; \mathbb{R}^{n+}), (\phi', \theta', \psi') \in C_b^2(t, T; \mathbb{R}^{n+}) \times \mathcal{L}^2_{p,t}(t, T; \mathbb{R}^{n+}) \times \mathcal{G}^2_{p,T}(t, T; \mathbb{R}^{n+})\) and \((\alpha', \eta', \gamma') \in C_b^2(t, T; \mathbb{R}^{n+}) \times \mathcal{L}^2_{p,t}(t, T; \mathbb{R}^{n+}) \times \mathcal{G}^2_{p,T}(t, T, \lambda; \mathbb{R}^{n+})\) be the solution of the coupled MF-FBSDEs with jump diffusions given in (4.1) with
(x′(t), β′(t), φ′(T), α′(T)) = (0, 0, 0, M_1 x′(T) + M_1 E[x′(T)]). Assume that the following condition holds:

\[
\begin{align*}
\mathbb{E} \left[ \int_t^T \left( u_1'(s) R_1(s) u_1'(s) + \bar{R}_1(s) E[u_1'(s)] + \bar{B}_1(s)^T \alpha'(s-) \right. \\
+ \bar{B}_1(s)^T E[\alpha'(s-)] + \bar{D}_1(s)^T \eta'(s) + \bar{D}_1(s)^T E[\eta'(s)] \\
+ \int_E \bar{G}_1(s,e)^T \gamma'(s,e) \lambda(de) + \int_E \bar{G}_1(s,e)^T E[\gamma'(s,e)] \lambda(de) \\
+ \bar{H}_1(s) \beta'(s-) + \bar{H}_1(s) E[\beta'(s-)] + \int_E \bar{K}_1(s,e) \beta'(s-) \lambda(de) \\
+ \int_E \bar{K}_1(s,e) E[\beta'(s-) \lambda(de)] ds \right] \geq 0.
\end{align*}
\]

Then \( u_1 \in U_t \) is the optimal control for (LQ-L) if and only if the following first-order optimality condition holds:

\[
\begin{align*}
R_1(s) \bar{u}_1(s) + \bar{R}_1(s) E[\bar{u}_1(s)] + \bar{B}_1(s)^T \alpha(s-) + \bar{B}_1(s)^T E[\alpha(s-)] \\
+ \bar{D}_1(s)^T \eta(s) + \bar{D}_1(s)^T E[\eta(s)] + \int_E \bar{G}_1(s,e)^T \gamma(s,e) \lambda(de) \\
+ \int_E \bar{G}_1(s,e)^T E[\gamma(s,e)] \lambda(de) + \bar{H}_1(s) \beta(s-) + \bar{H}_1(s) E[\beta(s-)] \\
+ \int_E \bar{K}_1(s,e) \beta(s-) \lambda(de) + \int_E \bar{K}_1(s,e) E[\beta(s-) \lambda(de)] = 0, \text{ } d\mathbb{P}-\text{a.s., } ds-\text{a.e.}
\end{align*}
\]

**Remark 4.** Note that

\[
J_1(0, u_1', \bar{u}_2) = J_{11}(0, u_1', \bar{u}_2) + J_{12}(0, u_1', \bar{u}_2),
\]

where

\[
\begin{align*}
J_{11}(0; u_1', \bar{u}_2) &= \mathbb{E} \left[ \int_t^T \left( |x'(s) - E[x'(s)]|_{Q_1(s)}^2 + |u_1'(s) - E[u_1'(s)]|_{R_1(s)}^2 \right) ds \\
&+ |x'(T) - E[x'(T)]|_{\tilde{M}_1}^2 \right] \\
J_{12}(0; u_1', \bar{u}_2) &= \mathbb{E} \left[ \int_t^T \left( |E[x'(s)]|_{Q_1(s)}^2 + |E[u_1'(s)]|_{R_1(s) + \tilde{R}_1(s)}^2 \right) ds \\
&+ |E[x'(T)]|_{\tilde{M}_1 + \tilde{N}_1}^2 \right].
\end{align*}
\]

Hence, from (C.2) in Appendix C, (4.2) holds when \( Q_1 + \tilde{Q}_1, M_1, M_1 + \tilde{M}_1, R_1 \) and \( R_1 + \tilde{R}_1 \) are (uniformly) positive (semi)definite for all \( s \in [t, T] \).

We now obtain the state-feedback representation of the optimal solution of (LQ-L). Let

\[
\mathcal{X}(s) := \begin{bmatrix} \pi(s) \\ \beta(s) \end{bmatrix}, \quad \mathcal{Y}(s) := \begin{bmatrix} \alpha(s) \\ \phi(s) \end{bmatrix}, \quad \mathcal{Z}(s) := \begin{bmatrix} \eta(s) \\ \theta(s) \end{bmatrix}, \quad \mathcal{K}(s, e) := \begin{bmatrix} \gamma(s,e) \\ \psi(s,e) \end{bmatrix},
\]

where \( \mathcal{X} := \mathcal{X}(t) = \begin{bmatrix} a \\ 0 \end{bmatrix} \) is the initial condition.
Define \((s \text{ is suppressed})\):

\[
\mathbf{A} := \begin{bmatrix} \hat{A} & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad \mathbf{B}_2 := \begin{bmatrix} 0 & \hat{B}_2 \\ \hat{B}_2 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}}_2 := \begin{bmatrix} 0 & \tilde{B}_2 \\ \tilde{B}_2 & 0 \end{bmatrix}
\]

\[
\mathbf{N} := \begin{bmatrix} \hat{N} & 0 \\ 0 & \hat{N} \end{bmatrix}, \quad \mathbf{P} := \begin{bmatrix} 0 & \hat{P} \\ \hat{P} & 0 \end{bmatrix}, \quad \tilde{\mathbf{P}} := \begin{bmatrix} 0 & \tilde{P} \\ \tilde{P} & 0 \end{bmatrix}
\]

\[
\mathbf{Q} := \begin{bmatrix} \hat{Q} & 0 \\ 0 & \hat{Q} \end{bmatrix}, \quad \mathbf{R} := \begin{bmatrix} 0 & \hat{R} \\ \hat{R} & 0 \end{bmatrix}, \quad \tilde{\mathbf{R}} := \begin{bmatrix} 0 & \tilde{R} \\ \tilde{R} & 0 \end{bmatrix}
\]

\[
\mathbf{S} := \begin{bmatrix} \hat{S} & 0 \\ 0 & \hat{S} \end{bmatrix}, \quad \mathbf{H} := \begin{bmatrix} 0 & \hat{H} \\ \hat{H} & 0 \end{bmatrix}, \quad \tilde{\mathbf{H}} := \begin{bmatrix} 0 & \tilde{H} \\ \tilde{H} & 0 \end{bmatrix}
\]

Then the coupled MF-FBSDEs with jump diffusions in (4.1) can be written into the following compact form \((s \text{ is suppressed})\):

\[
\begin{align*}
\mathrm{d}\mathbf{X}(s) &= \mathbf{A}\mathbf{X}(s) + \hat{\mathbf{A}}\mathbb{E}[\mathbf{X}(s)] + \mathbf{B}_2\mathbb{Y}(s) + \tilde{\mathbf{B}}_2\mathbb{E}[\mathbf{Y}(s)] \\
&\quad + \mathbf{N}\mathbb{Z}(s) + \hat{\mathbf{N}}\mathbb{E}[\mathbb{Z}(s)] + \int_{s}^{t} \hat{\mathbf{K}}(s,e)\mathbb{K}(s,e)\lambda(\mathrm{de}) \\
&\quad + \int_{s}^{t} \hat{\mathbf{K}}(s,e)\mathbb{E}[\mathbb{K}(s,e)]\lambda(\mathrm{de}) + \hat{\mathbb{B}}_1\mathbb{\tilde{\pi}}(s) + \tilde{\mathbb{B}}_1\mathbb{\mathbb{E}[\tilde{\pi}(s)]} \mathrm{ds} \\
&\quad + \mathbf{C}\mathbb{X}(s) + \hat{\mathbf{C}}\mathbb{E}[\mathbb{X}(s)] + \mathbf{H}\mathbb{Y}(s) + \tilde{\mathbf{H}}\mathbb{E}[\mathbb{Y}(s)] \\
&\quad + \mathbf{F}(s,e)\mathbb{X}(s) + \hat{\mathbf{F}}(s,e)\mathbb{E}[\mathbb{X}(s)] + \hat{\mathbb{F}}(s,e)\mathbb{\mathbb{E}[\hat{\mathbb{F}}(s,e)]} \mathrm{ds} \\
&\quad + \int_{s}^{t} \hat{\mathbf{G}}(s,e)\mathbb{\mathbb{E}[\mathbb{G}(s,e)]} \mathrm{dB}(s) \\
\mathrm{d}\mathbf{Y}(s) &= -[\mathbf{A}^\top\mathbb{Y}(s) + \hat{\mathbf{A}}^\top\mathbb{E}[\mathbb{Y}(s)] + \mathbf{Q}\mathbb{X}(s) + \mathbf{Q}\mathbb{E}[\mathbb{X}(s)]] \\
&\quad + \mathbf{C}^\top\mathbb{Z}(s) + \hat{\mathbf{C}}^\top\mathbb{E}[\mathbb{Z}(s)] + \int_{s}^{t} \hat{\mathbf{F}}(s,e)^\top\mathbb{K}(s,e)\lambda(\mathrm{de}) \\
&\quad + \int_{s}^{t} \hat{\mathbf{F}}(s,e)^\top\mathbb{E}[\mathbb{K}(s,e)]\lambda(\mathrm{de}) + \hat{\mathbb{H}}_1\mathbb{\tilde{\pi}}(s) + \tilde{\mathbb{H}}_1\mathbb{\mathbb{E}[\tilde{\pi}(s)]} \mathrm{ds} \\
&\quad + \mathbf{Z}(s)\mathrm{d}B(s) + \int_{s}^{t} \mathbf{K}(s,e)\tilde{\mathbb{N}}(s) \mathrm{ds}, \quad s \in [t, T)
\end{align*}
\]
where the optimality condition in (4.3) is equivalent to (dP-a.s., ds-a.e.)

\[
\begin{align*}
R_1(s)[\pi_1(s) - \mathbb{E}[\pi_1(s)]] + \mathbb{E}_1(s)^\top (\mathcal{Y}(s) - \mathbb{E}[\mathcal{Y}(s)]) \\
+ D_1(s)^\top (\mathcal{Z}(s) - \mathbb{E}[\mathcal{Z}(s)]) + H_1(s)(\mathcal{X}(s) - \mathbb{E}[\mathcal{X}(s)]) \\
+ \int_E G_1(s, e)^\top (K(s, e) - \mathbb{E}[K(s, e)])\lambda(de) \\
+ \int_E \mathbb{E}_1(s, e)(\mathcal{X}(s) - \mathbb{E}[\mathcal{X}(s)])\lambda(de) = 0
\end{align*}
\]

\[\tag{4.6}\]

We introduce the following transformation in the Four-Step Scheme:

\[\mathcal{Y}(s) = \mathcal{P}(s)(\mathcal{X}(s) - \mathbb{E}[\mathcal{X}(s)]) + \mathcal{W}(s)\mathbb{E}[\mathcal{X}(s)], \tag{4.7}\]

where \(\mathcal{P} \) and \(\mathcal{W}\) are \(\mathbb{R}^{2n \times 2n}\)-valued deterministic processes, which satisfy the following form:

\[
\begin{align*}
\frac{d\mathcal{P}(s)}{ds} &= \Lambda_3(s), \quad \frac{d\mathcal{W}(s)}{ds} = \Lambda_4(s) \\
\mathcal{P}(T) &= M_1, \quad \mathcal{W}(T) = M_1 + M_1, \tag{4.8}
\end{align*}
\]

Remark 5. Note that (4.7) is defined to have an argument of the form \(s-\) for \(\alpha\) and \(\beta\) to preserve predictability of the control process.

Let \(\mathcal{K}(s) := \mathcal{X}(s) - \mathbb{E}[\mathcal{X}(s)], \mathcal{Y}(s) := \mathcal{Y}(s) - \mathbb{E}[\mathcal{Y}(s)], \mathcal{Z}(s) := \mathcal{Z}(s) - \mathbb{E}[\mathcal{Z}(s)]\) and \(\mathcal{K}(s, e) := \mathcal{K}(s, e) - \mathbb{E}[\mathcal{K}(s, e)]\). Then using (4.8) and applying Itô’s formula to (4.7) yield \((s\text{ is suppressed})\)

\[
d\mathcal{Y}(s) = \Lambda_3(s)ds + \Lambda_4(s)ds \\
+ \mathcal{P}(s-)[A\mathcal{X}(s-)+B_2\mathcal{P}(s-)\mathcal{X}(s-)+H\mathcal{Z}(s)+\int_E \mathcal{K}(s, e)\mathcal{K}(s, e)\lambda(de)] \\
+ B_1\mathfrak{u}_1(s)ds + \mathcal{P}(s-)[C\mathcal{X}(s-)+(C+\mathcal{C})\mathbb{E}[\mathcal{X}(s-)] \\
+ \mathfrak{H}_1^\top \mathcal{P}(s-)\mathcal{X}(s-)+(\mathfrak{H}_1+\mathfrak{H}_2)^\top \mathcal{W}(s-)\mathbb{E}[\mathcal{X}(s-)] + \mathcal{H}^\top \mathcal{Z}(s) + (\mathfrak{H}+\mathfrak{H})\mathbb{E}[\mathcal{Z}(s)] \\
+ \int_E \mathcal{K}(s, e)\mathcal{K}(s, e)\lambda(de) + \int_E (\mathcal{K}(s, e)+\mathfrak{K}(s, e))\mathbb{E}[\mathcal{K}(s, e)]\lambda(de) \\
+ D_1\mathfrak{u}_1(s)+D_1\mathfrak{u}_1(s) + (D_1+D_1)\mathbb{E}[\pi_1(s)]dB(s) \\
+ \int_E \mathcal{P}(s-)[\mathcal{F}(s, e)\mathcal{X}(s-)+(\mathcal{F}(s, e)+\mathfrak{F}(s, e))\mathbb{E}[\mathcal{X}(s-)] \\
+ \mathfrak{K}(s, e)^\top \mathcal{P}(s-)\mathcal{X}(s-)+(\mathfrak{K}(s, e)+\mathfrak{K}(s, e))^\top \mathcal{W}(s-)\mathbb{E}[\mathcal{X}(s-)] \\
+ \mathfrak{K}(s, e)^\top \mathcal{Z}(s)+(\mathfrak{K}(s, e)+\mathfrak{K}(s, e))^\top \mathbb{E}[\mathcal{Z}(s)] + \int_E \mathfrak{K}(s, e, e')\mathfrak{K}(s, e')\lambda(de') \\
+ \int_E (\mathfrak{K}(s, e, e') + \mathfrak{K}(s, e, e'))\mathbb{E}[\mathcal{K}(s, e')]\lambda(de')
\]

\[\tag{4.9}\]
under the following two different cases:

4.1. Remark 6. \( \text{is suppressed} \)

\[ \lambda + (\lambda + \tilde{\lambda})\mathcal{W}(s-)E[X(s-)] + Q\tilde{X}(s-) \]

\[ + \int_E (\tilde{K}(s, e) + \bar{K}(s, e))E[K(s, e)]\lambda(\text{de}) \]

\[ + \int_E \tilde{u}_1(s) + (\tilde{H}_1 + \bar{H}_1)^\top E[\pi_1(s)] + \int_E K_1(s, e)^\top \tilde{u}_1(s)\lambda(\text{de}) \]

\[ + \int_E (K_1(s, e) + \bar{K}_1(s, e))^\top E[\pi_1(s)]\lambda(\text{de}) \]

\[ + \mathcal{Z}(s)d\mathcal{B}(s) + \int_E \mathcal{K}(s, e)\tilde{N}(\text{de}, ds). \]

We now obtain the state-feedback representation of the optimal solution for (LQ-L) under the following two different cases:

Case I: the Poisson process \( N \) has the unit jump \( (E = \{1\}) \);

Case II: the follower is not included in the jump part of (2.1) \( (G_2 = \bar{G}_2 = 0) \).

**Remark 6.** We provide a detailed discussion on these two assumptions in Section 5.

4.1. Case I: \( N \) has the unit jump. We first consider Case I, where

**Assumption 2.** The Poisson process \( N \) has the unit jump, i.e., \( E = \{1\} \).

Based on Assumption 2 and (4.7) (see also Remark 2), (4.9) can be written as\(^2\) (\( s \) is suppressed)

\[ d\mathcal{Y}(s) = \Lambda_3(s)ds + \Lambda_4(s)ds \]

\[ + \mathcal{P}(s-)\left[ A\tilde{X}(s-)^\top + B_2P(s-)\tilde{X}(s-) + \tilde{H}_1\mathcal{Z}(s) + \lambda\tilde{K}\mathcal{K}(s) + B_1\tilde{u}_1(s) \right] ds \]

\[ + \mathcal{P}(s-)\left[ \mathcal{F}(s) + (\mathcal{C} + \mathcal{C})E[\mathcal{X}(s-)] + \tilde{K}^\top \mathcal{P}(s-) \tilde{X}(s-) \right] \]

\[ + \int_E \mathcal{F}(s, e)^\top \tilde{K}(s, e)\lambda(\text{de}) + \int_E (\tilde{F}(s, e) + \bar{F}(s, e))^\top E[K(s, e)]\lambda(\text{de}) \]

\[ + \int_E (\tilde{K}_1(s, e) + \bar{K}_1(s, e))^\top E[\pi_1(s)]\lambda(\text{de}) \]

\[ + \mathcal{Z}(s)d\mathcal{B}(s) + \int_E \mathcal{K}(s, e)\tilde{N}(\text{de}, ds). \]

\(^2\)Note that under Assumption 2, \( \int_E g(s, e)\lambda(\text{de})ds = g(s)\lambda ds \) and \( \int_E E[g(s, e)]\lambda(\text{de})ds = E[g(s)]\lambda ds \) for \( g \in L^2_{\mathbb{F}}(t, T; \lambda; \mathbb{R}^n) \), where \( \lambda > 0 \) is the intensity of \( N \) \( [1, 20] \).
+ \mathcal{W}(s-)[(A + \tilde{A})E[\mathcal{X}(s-)] + (B_2 + \tilde{B}_2)W(s-)E[\mathcal{X}(s-)]]
+ \left[(\bar{H} + \tilde{H})E[\mathcal{Z}(s)] + \lambda(\bar{K} + \tilde{K})E[\mathcal{K}(s)] + (B_1 + \tilde{B}_1)E[\pi_1(s)]\right]ds
\]

= -\left[A^T \mathcal{P}(s-)\hat{\mathcal{X}}(s-) + (A + \tilde{A})^T \mathcal{W}(s-)E[\mathcal{X}(s-)] + Q\hat{\mathcal{X}}(s-)
+ (Q + \tilde{Q})E[\mathcal{X}(s-)] + C^T \tilde{Z}(s) + (C + \tilde{C})^T E[\mathcal{Z}(s)]
+ \lambda F^T \bar{K}(s) + \lambda(F + \tilde{F})^T E[\mathcal{K}(s)] + H_1^T \bar{u}_1(s) + (H_1 + \tilde{H}_1)^T E[\pi_1(s)]
+ \lambda \bar{K}_1^T \bar{u}_1(s) + \lambda(\bar{K}_1 + \tilde{K}_1)^T E[\pi_1(s)]\right]ds
+ \mathcal{Z}(s)dB(s) + \mathcal{K}(s)d\tilde{\mathcal{N}}(s).

By comparing the diffusion terms in (4.10), (s is suppressed)

\begin{align*}
\mathcal{Z}(s) &= \mathcal{P}(s-)[(C + \tilde{C})E[\mathcal{X}(s-)] + (\bar{H} + \tilde{H})^T \mathcal{W}(s-)E[\mathcal{X}(s-)]]
+ \lambda \bar{K}\bar{K}(s) + \lambda(\bar{K} + \tilde{K})E[\mathcal{K}(s)] + D_1 \bar{u}_1(s) + (D_1 + \tilde{D}_1)E[\pi_1(s)]
\end{align*}

\begin{align*}
\mathcal{K}(s) &= \mathcal{P}(s-)[(F + \tilde{F})E[\mathcal{X}(s-)] + (\bar{K} + \tilde{K})^T \mathcal{W}(s-)E[\mathcal{X}(s-)]]
+ \lambda \bar{K}\bar{K}(s) + \lambda(\bar{K} + \tilde{K})E[\mathcal{K}(s)] + G_1 \bar{u}_1(s) + (G_1 + \tilde{G}_1)E[\pi_1(s)]
\end{align*}

which implies

\begin{align*}
\mathcal{E}[\mathcal{Z}(s)] &= \mathcal{P}(s-)[(C + \tilde{C})E[\mathcal{X}(s-)] + (\bar{H} + \tilde{H})^T \mathcal{W}(s-)E[\mathcal{X}(s-)]]
+ \lambda \bar{K}\bar{K}(s) + \lambda(\bar{K} + \tilde{K})E[\mathcal{K}(s)] + (D_1 + \tilde{D}_1)E[\pi_1(s)]
\end{align*}

\begin{align*}
\mathcal{E}[\mathcal{K}(s)] &= \mathcal{P}(s-)[(F + \tilde{F})E[\mathcal{X}(s-)] + (\bar{K} + \tilde{K})^T \mathcal{W}(s-)E[\mathcal{X}(s-)]]
+ \lambda \bar{K}\bar{K}(s) + \lambda(\bar{K} + \tilde{K})E[\mathcal{K}(s)] + (G_1 + \tilde{G}_1)E[\pi_1(s)]
\end{align*}

and

\begin{align*}
\hat{\mathcal{Z}}(s) &= \mathcal{P}(s-)[(C + \tilde{C})\hat{\mathcal{X}}(s-)] + (\bar{H} + \tilde{H})^T \mathcal{W}(s-)E[\mathcal{X}(s-)]]
+ \lambda \bar{K}\bar{K}(s) + \lambda(\bar{K} + \tilde{K})E[\mathcal{K}(s)] + D_1 \bar{u}_1(s)
\end{align*}

\begin{align*}
\hat{\mathcal{K}}(s) &= \mathcal{P}(s-)[(F + \tilde{F})\hat{\mathcal{X}}(s-)] + (\bar{K} + \tilde{K})^T \mathcal{W}(s-)E[\mathcal{X}(s-)]]
+ \lambda \bar{K}\bar{K}(s) + \lambda(\bar{K} + \tilde{K})E[\mathcal{K}(s)] + G_1 \bar{u}_1(s)
\end{align*}

Then from (4.11) and (4.12), we have (s is suppressed)

\begin{align*}
\Phi\begin{bmatrix}
\mathcal{Z}(s) \\
\hat{\mathcal{K}}(s)
\end{bmatrix} &= \begin{bmatrix}
\Phi_{11} & \Phi_{12} \\
\Phi_{21} & \Phi_{22}
\end{bmatrix}\begin{bmatrix}
\mathcal{Z}(s) \\
\hat{\mathcal{K}}(s)
\end{bmatrix} = \begin{bmatrix}
\Delta_{11} & \Delta_{12} \\
\Delta_{21} & \Delta_{22}
\end{bmatrix}\begin{bmatrix}
\mathcal{X}(s) \\
\hat{\mathcal{K}}(s)
\end{bmatrix} + \begin{bmatrix}
\Delta_{11} & \Delta_{12} \\
\Delta_{21} & \Delta_{22}
\end{bmatrix}\begin{bmatrix}
\mathcal{X}(s) \\
\hat{\mathcal{K}}(s)
\end{bmatrix} \hat{u}_1(s)
\end{align*}

\begin{align*}
\Phi\begin{bmatrix}
\mathcal{E}[\mathcal{Z}(s)] \\
\mathcal{E}[\mathcal{K}(s)]
\end{bmatrix} &= \begin{bmatrix}
\Phi_{11} & \Phi_{12} \\
\Phi_{21} & \Phi_{22}
\end{bmatrix}\begin{bmatrix}
\mathcal{E}[\mathcal{Z}(s)] \\
\mathcal{E}[\mathcal{K}(s)]
\end{bmatrix} = \begin{bmatrix}
\Delta_{11} & \Delta_{12} \\
\Delta_{21} & \Delta_{22}
\end{bmatrix}\begin{bmatrix}
\mathcal{E}[\mathcal{X}(s)] \\
\mathcal{E}[\mathcal{K}(s)]
\end{bmatrix} + \begin{bmatrix}
\Delta_{11} & \Delta_{12} \\
\Delta_{21} & \Delta_{22}
\end{bmatrix}\begin{bmatrix}
\mathcal{E}[\mathcal{X}(s)] \\
\mathcal{E}[\mathcal{K}(s)]
\end{bmatrix} \hat{u}_1(s),
\end{align*}

(4.13)
where

\[
\begin{align*}
\Phi(s) &= \begin{bmatrix}
I - \mathcal{P}(s-)\widetilde{H} & -\lambda\mathcal{P}(s-)\widetilde{K} \\
-\mathcal{P}(s-)(\widetilde{K} + \tilde{K}) & I - \lambda\mathcal{P}(s-)(\widetilde{K} + \tilde{K})
\end{bmatrix} \\
\Phi(s) &= \begin{bmatrix}
I - \mathcal{P}(s-)(\widetilde{H} + \tilde{H}) & -\lambda\mathcal{P}(s-)(\tilde{K} + \widetilde{K}) \\
-\mathcal{P}(s-)(\tilde{K} + \widetilde{K}) & I - \lambda\mathcal{P}(s-)(\tilde{K} + \widetilde{K})
\end{bmatrix}
\end{align*}
\]

\[
\Delta_{11} := \mathcal{P}(s-)C + \mathcal{P}(s-)(\widetilde{H} + \tilde{H})^\top\mathcal{P}(s-), \Delta_{12} := \mathcal{P}(s-)D_1 \\
\Delta_{21} := \mathcal{P}(s-)F + \mathcal{P}(s-)(\tilde{K} + \widetilde{K})^\top\mathcal{W}(s-), \Delta_{22} := \mathcal{P}(s-)G_1 \\
\Delta_{11} := \mathcal{P}(s-)(C + \tilde{C}) + \mathcal{P}(s-)(\widetilde{H} + \tilde{H})^\top\mathcal{W}(s-), \Delta_{12} := \mathcal{P}(s-)(D_1 + \tilde{D}_1), \Delta_{22} := \mathcal{P}(s-)(G_1 + \tilde{G}_1).
\]

Under the invertibility of \(\Psi := \Phi^{-1}\) and \(\overline{\Phi} := \Phi^{-1}\) and using the matrix inversion lemma [13, page 18], (4.13) is expressed as follows (\(s\) is suppressed)

\[
\begin{align*}
\begin{bmatrix}
\dot{\overline{Z}}(s) \\
\dot{\overline{K}}(s)
\end{bmatrix} &= 
\begin{bmatrix}
\Psi_{11}\Delta_{11} + \Psi_{12}\Delta_{21} \\
\Psi_{21}\Delta_{11} + \Psi_{22}\Delta_{21}
\end{bmatrix} \dot{\mathcal{X}}(s) \\
\begin{bmatrix}
\dot{E}[Z(s)] \\
\dot{E}[K(s)]
\end{bmatrix} &= 
\begin{bmatrix}
\Psi_{11}\Delta_{11} + \Psi_{12}\Delta_{21} \\
\Psi_{21}\Delta_{11} + \Psi_{22}\Delta_{21}
\end{bmatrix} \dot{E}[\mathcal{X}(s)] + 
\begin{bmatrix}
\Psi_{11}\Delta_{12} + \Psi_{12}\Delta_{22} \\
\Psi_{21}\Delta_{12} + \Psi_{22}\Delta_{22}
\end{bmatrix} \dot{E}[\mathcal{Y}(s)],
\end{align*}
\]

(4.14)

where

\[
\begin{align*}
\Psi_{11} &= (\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21})^{-1}, \quad \Psi_{12} := \Phi_{11}^{-1}\Phi_{12}(\Phi_{21}\Phi_{11}^{-1}\Phi_{12} - \Phi_{22})^{-1} \\
\Psi_{21} &= \Phi_{22}^{-1}\Phi_{21}(\Phi_{12}\Phi_{22}^{-1}\Phi_{21} - \Phi_{11})^{-1}, \quad \Psi_{22} := (\Phi_{22} - \Phi_{21}\Phi_{11}^{-1}\Phi_{12})^{-1} \\
\widetilde{\Psi}_{11} &= (\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21})^{-1}, \quad \widetilde{\Psi}_{12} := \Phi_{11}^{-1}\Phi_{12}(\Phi_{21}\Phi_{11}^{-1}\Phi_{12} - \Phi_{22})^{-1} \\
\widetilde{\Psi}_{21} &= \Phi_{22}^{-1}\Phi_{21}(\Phi_{12}\Phi_{22}^{-1}\Phi_{21} - \Phi_{11})^{-1}, \quad \widetilde{\Psi}_{22} := (\Phi_{22} - \Phi_{21}\Phi_{11}^{-1}\Phi_{12})^{-1}.
\end{align*}
\]

From Assumption 2 and (4.7), the optimality condition in (4.6) is equivalent to (\(s\) is suppressed)

\[
\begin{align*}
\begin{bmatrix}
R_1\tilde{u}_1(s) + \mathbb{E}[\mathcal{P}(s-)\mathcal{X}(s-)] + \mathbb{E}[\mathcal{Z}(s)] + \lambda\mathbb{E}[\mathcal{K}(s)] \\
+ \mathbb{E}[\mathcal{H}(s-)] + \lambda\mathbb{E}[\mathcal{Y}(s-)]
\end{bmatrix} &= \mathbb{E}[\mathcal{H}(s-)] \\
R_1 + \mathbb{E}[\mathcal{P}(s-)\mathcal{X}(s-)] + \mathbb{E}[\mathcal{Z}(s-)] + \lambda\mathbb{E}[\mathcal{K}(s)] \\
+ \mathbb{E}[\mathcal{H}(s-)] + \lambda\mathbb{E}[\mathcal{Y}(s-)] &= 0,
\end{align*}
\]

(4.15)

and by substituting (4.14) into (4.15), the explicit expressions of \(\tilde{u}_1\) and \(\mathbb{E}[\mathcal{Y}]\) can be obtained by

\[
\begin{align*}
\begin{bmatrix}
\mathcal{H}(s-)^{-1}\mathcal{H}(s) - \mathbb{E}[\mathcal{H}(s-)] \\
\mathbb{E}[\mathcal{H}(s-)] - \mathcal{H}(s-)^{-1}\mathcal{H}(s)
\end{bmatrix} &= 0.
\end{align*}
\]

(4.16)
provided that $R_1$ and $\widetilde{R}_1$ are invertible, where $(s$ is suppressed)

$$
\begin{align*}
\mathcal{R}_1 & := R_1 + \left( P_1^T (\Psi_{11} \Delta_{12} + \Psi_{12} \Delta_{22}) + \lambda \mathcal{G}_{11}^T (\Psi_{21} \Delta_{12} + \Psi_{22} \Delta_{22}) \right) \\
\widetilde{\mathcal{R}}_1 & := \tilde{R}_1 + \tilde{R}_1 + \left( (\mathcal{D}_1 + \mathcal{D}_1) \mathcal{B}_1 + \mathcal{D}_1 + \mathcal{D}_1 \right) \left( \tilde{\Psi}_{11} \tilde{\Delta}_{12} + \tilde{\Psi}_{12} \tilde{\Delta}_{22} \right) + \lambda \left( \mathcal{G}_1 + \mathcal{G}_1 \right) \left( \tilde{\Psi}_{21} \tilde{\Delta}_{12} + \tilde{\Psi}_{22} \tilde{\Delta}_{22} \right) \right) E[\mathcal{X}(s-)]
\end{align*}
$$

$$
\mathcal{H}_1 := \left( \mathcal{B}_1^T \mathcal{P}(s-) + \mathcal{H}_1 + \lambda \mathcal{K}_1 \right) + \left( \mathcal{D}_1 \mathcal{B}_1 \right) \left( \Psi_{11} \Delta_{11} + \Psi_{12} \Delta_{21} \right) + \lambda \left( \mathcal{G}_1 + \mathcal{G}_1 \right) \left( \Psi_{21} \Delta_{11} + \Psi_{22} \Delta_{21} \right)
$$

Note that (4.16) is the state-feedback type optimal solution of (LQ-L). We observe that (4.16) depends only on the state process.

By substituting (4.16) into (4.14), it follows that $(s$ is suppressed)

$$
\begin{align*}
\begin{bmatrix}
\mathcal{Z}(s) \\
\mathcal{K}(s)
\end{bmatrix} &= \begin{bmatrix}
\mathcal{X}_1 - \mathcal{X}_2 \mathcal{R}_1^{-1} \mathcal{H}_1 \\
\mathcal{X}_2 - \mathcal{X}_2 \mathcal{R}_1^{-1} \mathcal{H}_1
\end{bmatrix} \mathcal{X}(s) - E[\mathcal{X}(s-)]
\end{align*}
$$

where

$$
\begin{align*}
\mathcal{X}_1 & := \Psi_{11} \Delta_{11} + \Psi_{12} \Delta_{21}, & \mathcal{X}_2 & := \Psi_{11} \Delta_{12} + \Psi_{12} \Delta_{22} \\
\mathcal{X}_2 & := \Psi_{21} \Delta_{11} + \Psi_{22} \Delta_{21}, & \mathcal{X}_2 & := \Psi_{21} \Delta_{12} + \Psi_{22} \Delta_{22} \\
\mathcal{X}_1 & := \Psi_{11} \Delta_{11} + \Psi_{12} \Delta_{21}, & \mathcal{X}_2 & := \Psi_{11} \Delta_{12} + \Psi_{12} \Delta_{22} \\
\mathcal{X}_2 & := \Psi_{21} \Delta_{11} + \Psi_{22} \Delta_{21}, & \mathcal{X}_2 & := \Psi_{21} \Delta_{12} + \Psi_{22} \Delta_{22}.
\end{align*}
$$

We substitute (4.16) and (4.17) into (4.10). Then by comparing its expression with (4.8), we can show that $\mathcal{P}$ and $\mathcal{W}$ in (4.8) satisfy the following coupled integro-RDEs (CIRDEs) $(s$ is suppressed):

$$
\begin{align*}
\begin{bmatrix}
-\frac{d\mathcal{P}(s)}{ds} \\
-\frac{d\mathcal{W}(s)}{ds}
\end{bmatrix} &= \begin{bmatrix}
\mathcal{A}_1^T \mathcal{P}(s-) + \mathcal{P}(s-)[\mathcal{A}_1 + \mathcal{Q} + \mathcal{P}(s-) \mathcal{B}_2 \mathcal{P}(s-)] \\
(\mathcal{C}_1^T + \mathcal{P}(s-) \mathcal{H}_1)(\mathcal{A}_1 - \mathcal{A}_2 \mathcal{R}_1^{-1} \mathcal{H}_1)
\end{bmatrix} \\
&+ (\mathcal{A}_2^T + \lambda \mathcal{P}(s-) \mathcal{R}_1^{-1} \mathcal{H}_1)(\mathcal{A}_2 - \mathcal{A}_2 \mathcal{R}_1^{-1} \mathcal{H}_1) \\
&+ \left( \mathcal{A}^T + \lambda \mathcal{P}(s-) \mathcal{R}_1^{-1} \mathcal{H}_1 \right)(\mathcal{A} + \mathcal{A}) + (\mathcal{Q} + \mathcal{Q}) \\
&+ \mathcal{W}(s-) \mathcal{B}_2 \mathcal{P}(s-) \\
&+ \lambda \mathcal{W}(s-) \mathcal{B}_2 \mathcal{P}(s-) \\
&+ \lambda \mathcal{W}(s-) \mathcal{B}_2 \mathcal{P}(s-) \\
&+ \lambda \mathcal{W}(s-) \mathcal{B}_2 \mathcal{P}(s-) \\
&+ \lambda \mathcal{W}(s-) \mathcal{B}_2 \mathcal{P}(s-) \\
&+ \lambda \mathcal{W}(s-) \mathcal{B}_2 \mathcal{P}(s-) \\
&+ \lambda \mathcal{W}(s-) \mathcal{B}_2 \mathcal{P}(s-) \\
&+ \lambda \mathcal{W}(s-) \mathcal{B}_2 \mathcal{P}(s-) \\
&+ \lambda \mathcal{W}(s-) \mathcal{B}_2 \mathcal{P}(s-)
\end{align*}
$$

\text{for } T = \mathcal{M}_1, \mathcal{W}(T) = \mathcal{M}_1 + \mathcal{M}_1
$$

$$
\begin{align*}
\det(\mathcal{R}_1(s)) & \neq 0, \det(\tilde{\mathcal{R}}_1(s)) \neq 0, \forall s \in [t, T] \\
\det(\mathcal{P}(s)) & \neq 0, \det(\tilde{\mathcal{R}}_1(s)) \neq 0, \forall s \in [t, T] \\
\det(\mathcal{P}(s) - \mathcal{P}(s) \mathcal{P}(s^{-1}) \mathcal{P}(s) \mathcal{P}(s^{-1}) \mathcal{P}(s)) & \neq 0, \forall s \in [t, T] \\
\det(\mathcal{P}(s) - \mathcal{P}(s) \mathcal{P}(s^{-1}) \mathcal{P}(s) \mathcal{P}(s^{-1}) \mathcal{P}(s)) & \neq 0, \forall s \in [t, T].
\end{align*}
$$
We note that among the four determinant conditions given in (4.18), the first line \( (\det(\mathcal{R}_1(s)) \neq 0 \text{ and } \det(\tilde{\mathcal{R}}_1(s)) \neq 0 \text{ for all } s \in [t, T]) \) is induced due to the invertibility in (4.16). The other three determinant conditions are related to the block matrix inversion lemma in (4.14) and (4.17) (see the definition of \( \Psi_{ij} \) and \( \tilde{\Psi}_{ij}, i, j = 1, 2 \)) [13, page 18].

**Remark 7.** We can observe that unlike the CIRDEs of \( (\text{LQ-F}) \) in (3.15), the CIRDEs of \( (\text{LQ-L}) \) in (4.18) are nonsymmetric. The solvability of (4.18) is an interesting problem, which will be studied in the future research topic.

We substitute (4.16), (4.7) and (4.17) into the dynamics \( \mathcal{X} \) in (4.5). Then (s is suppressed)

\[
\begin{align*}
\mathcal{X}(t) &= \tilde{\mathcal{X}}, \\
\frac{d\mathcal{X}}{ds} &= \begin{bmatrix}
\hat{A} \mathcal{X}(s) + \tilde{A} \mathcal{E}[\mathcal{X}(s-)] \\
\hat{B} \mathcal{X}(s) + \tilde{B} \mathcal{E}[\mathcal{X}(s-)] \\
\end{bmatrix} dB(s) + \begin{bmatrix}
\hat{C} \mathcal{X}(s) + \tilde{C} \mathcal{E}[\mathcal{X}(s-)] \\
\hat{D} \mathcal{X}(s) + \tilde{D} \mathcal{E}[\mathcal{X}(s-)] \\
\end{bmatrix} d\tilde{N}(s)
\end{align*}
\]

where

\[
\begin{align*}
\hat{A} &= \mathcal{A} + \mathcal{E} \mathcal{F} \mathcal{P}(s-) + \mathcal{E} \mathcal{C} \\
\hat{B} &= \mathcal{A} + \mathcal{E} \mathcal{G} \mathcal{P}(s-) + \mathcal{E} \mathcal{D} \\
\hat{C} &= \mathcal{C} + \mathcal{E} \mathcal{H} \mathcal{P}(s-) + \mathcal{E} \mathcal{F} \mathcal{G} \mathcal{D} \mathcal{P}(s-) + \mathcal{E} \mathcal{D} \mathcal{G} \mathcal{D} \mathcal{P}(s-) \\
\hat{D} &= \mathcal{E} \mathcal{H} \mathcal{P}(s-) + \mathcal{E} \mathcal{F} \mathcal{G} \mathcal{D} \mathcal{P}(s-) + \mathcal{E} \mathcal{D} \mathcal{G} \mathcal{D} \mathcal{P}(s-) \\
\end{align*}
\]

To summarize the above analysis, we state the following result, where its proof is given in Appendix D.

**Theorem 4.1.** Suppose that Assumptions 1 and 2 hold. Assume that the pair \( (\mathcal{P}, \mathcal{W}) \) is the solution of the CIRDEs in (4.18), and \( \mathcal{X} \) is the solution of (4.19). Define the transformations in (4.7) and (4.17), and consider the control in (4.16). Then (4.5) and (4.6) hold. In addition, suppose that (4.2) holds. Then the state-feedback type control in (4.16) is the optimal solution for \( (\text{LQ-L}) \), and the associated optimal cost is given by

\[
J_1(a; \mathcal{W}_1, \mathcal{W}_2) = \inf_{u_1 \in \mathcal{U}_1} J_1(a; u_1, \mathcal{W}_2) = \langle a, \mathcal{W}_1(t) a \rangle,
\]

where \( \mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} \) with \( \mathcal{W}_{11} \) being an \( \mathbb{R}^{n \times n} \)-valued process.
Finally, from (3.17) and (4.15), consider

\[
\begin{align*}
\begin{cases}
\overline{\pi}_1(s) - E[\overline{\pi}_1(s)] &= -\mathcal{R}_1(s)^{-1}\mathcal{H}_1(s) \left[ x(s-), \beta(s-) \right] - E\left[ x(s-), \beta(s-) \right], \\
E[\overline{\pi}_1(s)] &= -\mathcal{R}_1(s)^{-1}\mathcal{H}_1(s) E\left[ x(s-), \beta(s-) \right],
\end{cases}
\end{align*}
\tag{4.21}
\]

and

\[
\begin{align*}
\begin{cases}
\overline{\pi}_2(s) - E[\overline{\pi}_2(s)] &= -\mathcal{R}_2(s)^{-1}\mathcal{S}_2(s)^\top (x(s-) - E[x(s-)]) \\
-\mathcal{R}_2(s)^{-1}B_2(s)^\top (\phi(s-) - E[\phi(s-)]) \\
-\mathcal{R}_2(s)^{-1}D_2(s)^\top (\theta(s) - E[\theta(s)]) \\
-\mathcal{R}_2(s)^{-1}\int_E G_2(s,e)^\top (\psi(s,e) - E[\psi(s,e)]) \lambda(de) \\
-\mathcal{R}_2(s)^{-1}\mathcal{S}_1(s)(\overline{\pi}_1(s) - E[\overline{\pi}_1(s)]) \\
E[\overline{\pi}_2(s)] &= -\mathcal{R}_2(s)^{-1}\mathcal{S}_2(s)^\top E[x(s-)] \\
-\mathcal{R}_2(s)^{-1}(B_2(s) + B_2(s))^\top E[\phi(s-)] \\
-\mathcal{R}_2(s)^{-1}(D_2(s) + D_2(s))^\top E[\theta(s)] \\
-\mathcal{R}_2(s)^{-1}\int_E (G_2(s,e) + G_2(s,e))^\top E[\psi(s,e)] \lambda(de) \\
-\mathcal{R}_2(s)^{-1}\mathcal{S}_1(s)E[\overline{\pi}_1(s)].
\end{cases}
\end{align*}
\tag{4.22}
\]

Note that (4.22) is the state-feedback type optimal control of the follower in Theorem 3.1 when $u_1 \equiv \overline{\pi}_1$. This describes the situation when the leader announces $\overline{\pi}_1$ in (4.21) to the follower in the Stackelberg game.

We state the following result:

**Corollary 4.1.** Suppose that the assumptions of Theorems 3.1 and 4.1 hold. Then the pair $(\overline{\pi}_1, \overline{\pi}_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ given in (4.21) and (4.22) constitutes the state-feedback representation for the open-loop Stackelberg game of the leader and the follower.

4.2. **Case II:** The jump part in (2.1) does not depend on the control of the follower. Let us assume that

**Assumption 3.** The control of the follower is not included in the jump part of (2.1), i.e., $G_2 = \tilde{G}_2 = 0$.

**Remark 8.** From (3.14) and (4.4), Assumption 3 implies that $\tilde{K}_2 = \tilde{K}_2 = \tilde{K}_2 = \tilde{K}_2 = \tilde{K}_2 = \tilde{K}_2 = \tilde{K}_2 = 0$ and $\tilde{K} = \tilde{K} = \tilde{K} = \tilde{K} = \tilde{K} = 0$.

Under Assumption 3, (4.9) becomes (s is suppressed)

\[
d\mathcal{Y}(s) = \Lambda_3(s) ds + \Lambda_4(s) ds
\tag{4.23}
\]

\[
+ \mathcal{P}(s-) \left[ A\mathcal{X}(s-) + B_2 \mathcal{P}(s-), E[\mathcal{X}(s-)] + \tilde{H}\mathcal{Z}(s) + \beta_1 \overline{\pi}_1(s) \right] ds
\]

\[
+ \mathcal{P}(s-) \left[ C\mathcal{X}(s-) + (C + \overline{\mathcal{C}})E[\mathcal{X}(s-)] + \tilde{H}\mathcal{P}(s-), \mathcal{X}(s-) \right.
\]

\[
+ \tilde{H} \mathcal{W}(s-) E[\mathcal{X}(s-)] + \tilde{H}\mathcal{Z}(s) + \tilde{H}\mathcal{E}[\mathcal{Z}(s)]
\]

\[
+ \beta_1 \overline{\pi}_1(s) + (\beta_1 + \beta_1) E[\overline{\pi}_1(s)] \right]\ dB(s)
\]

\[
+ \int_E \mathcal{P}(s-) \left[ F(s,e), \mathcal{X}(s-) + (F(s,e) + \bar{F}(s,e)) E[\mathcal{X}(s-)] \right.
\]

\[
+ G_1(s,e) \overline{\pi}_1(s) + (G_1(s,e) + \bar{G}_1(s,e)) E[\overline{\pi}_1(s)] \right] \bar{N}(de, ds)
\]
By substituting (4.7) and (4.24) into the optimality condition in (4.6), we have

\[ + \mathcal{W}(s-) \left[ (A + \hat{A}) \mathbb{E}[\mathcal{X}(s-)] + (B_2 + \hat{B}_2) \mathcal{W}(s-) \mathbb{E}[\mathcal{X}(s-)] \right] \\
+ (B_1 + \hat{B}_1) \mathbb{E}[\pi_1(s)] + (\hat{H} + \hat{H}) \mathbb{E}[Z(s)] \] \\
= - \left[ A^\top \mathcal{P}(s-) \mathcal{X}(s-) + (A + \hat{A})^\top \mathcal{W}(s-) \mathbb{E}[\mathcal{X}(s-)] + Q \mathcal{X}(s-) \right] \\
+ (Q + \hat{Q}) \mathbb{E}[\mathcal{X}(s-)] + C^\top \tilde{Z}(s) + (C + \hat{C})^\top \mathbb{E}[Z(s)] \\
+ \int_E F(s, e)^\top \tilde{K}(s, e) \lambda(de) + \int_E (F(s, e) + \bar{F}(s, e))^\top \mathbb{E}[K(s, e)] \lambda(de) \\
+ \mathbb{H}_1^\top \tilde{u}_1(s) + (\mathbb{H}_1 + \hat{\mathbb{H}}_1)^\top \mathbb{E}[\pi_1(s)] \\
+ \int_E (K_1(s, e) + \bar{K}_1(s, e))^\top \mathbb{E}[\pi_1(s)] \lambda(de) \right] ds \\
+ \mathbb{Z}(s) dB(s) + \int_E K(s, e) \tilde{\mathcal{N}}(de, ds).

Then using the invertibility assumption, the diffusion terms in (4.23) become (s is suppressed) (note that \( \tilde{Z}(s) = Z(s) - \mathbb{E}[Z(s)] \) and \( \tilde{K}(s, e) = K(s, e) - \mathbb{E}[K(s, e)] \))

\[
\left\{ \begin{align*}
\tilde{Z}(s) &= (I - \mathcal{P}(s-)(\hat{H} + \hat{\mathbb{H}}))^{-1}\left[ \mathcal{P}(s-)(C + \hat{H}^\top \mathcal{P}(s-))\tilde{\mathcal{X}}(s-) + \mathcal{P}(s-) \mathbb{D}_1 \tilde{u}_1(s) \right] \\
\tilde{K}(s, e) &= \mathcal{P}(s-)(F(s, e) \tilde{\mathcal{X}}(s-) + \mathcal{P}(s-) \mathbb{G}_1(s, e) \tilde{u}_1(s) \\
\mathbb{E}[Z(s)] &= (I - \mathcal{P}(s-)(\hat{H} + \hat{\mathbb{H}}))^{-1}\left[ \mathcal{P}(s-)(C + \hat{C}) \right] \\
+ \mathbb{H}_1^\top \tilde{u}_1(s) + (\mathbb{H}_1 + \hat{\mathbb{H}}_1)^\top \mathbb{E}[\pi_1(s)] \\
\mathbb{E}[K(s, e)] &= \mathcal{P}(s-)(F(s, e) + \bar{F}(s, e)) \mathbb{E}[\mathcal{X}(s-)] \\
+ \mathcal{P}(s-)(\mathbb{G}_1(s, e) + \bar{G}_1(s, e)) \mathbb{E}[\pi_1(s)].
\end{align*} \right. 
\]

(4.24)

By substituting (4.7) and (4.24) into the optimality condition in (4.6), we have

\[
\left\{ \begin{align*}
\pi_1(s) - \mathbb{E}[\pi_1(s)] &= -\tilde{\mathcal{R}}_1(s) \hat{B}_1(s) (\mathcal{X}(s-) - \mathbb{E}[\mathcal{X}(s-)]) \\
\mathbb{E}[\pi_1(s)] &= -\tilde{\mathcal{R}}_1(s) \hat{B}_1(s) \mathbb{E}[\mathcal{X}(s-)],
\end{align*} \right. 
\]

(4.25)

provided that \( \tilde{\mathcal{R}}_1 \) and \( \hat{B}_1 \) are invertible, where (s and e are suppressed)

\[
\left\{ \begin{align*}
\tilde{\mathcal{R}}_1 &= R_1 + \mathbb{D}_1^\top (I - \mathcal{P}(\hat{H}))^{-1} \mathcal{P} \mathbb{D}_1 + \int_E \mathbb{G}_1^\top \mathcal{P} \mathbb{G}_1 \lambda(de) \\
\hat{B}_1 &= \mathbb{B}_1^\top \mathcal{P} + \mathbb{H}_1 + \int_E K_1(s, e) \lambda(de) + \int_E \mathbb{G}_1^\top \mathcal{P} \mathbb{F} \lambda(de) \\
&\quad + \mathbb{D}_1^\top (I - \mathcal{P}(\hat{H}))^{-1} \mathcal{P}(C + \hat{C}) \mathbb{D}_1 \\
\tilde{\mathcal{R}}_1 &= (R_1 + \hat{R}_1) + (\mathbb{D}_1 + \hat{\mathbb{D}}_1)^\top (I - \mathcal{P}(\hat{H}))^{-1} \mathcal{P}(\mathbb{D}_1 + \hat{\mathbb{D}}_1) \\
&\quad + \int_E (\mathbb{G}_1 + \hat{\mathbb{G}}_1)^\top \mathcal{P} (\mathbb{G}_1 + \hat{\mathbb{G}}_1) \lambda(de) \\
\hat{B}_1 &= (\mathbb{B}_1 + \hat{\mathbb{B}}_1)^\top \mathcal{W} + (\mathbb{H}_1 + \hat{\mathbb{H}}_1) + \int_E (\mathbb{K} + \hat{\mathbb{K}}_1) \lambda(de) \\
&\quad + \int_E (\mathbb{G}_1 + \hat{\mathbb{G}}_1)^\top \mathcal{P} (\mathbb{F} + \hat{\mathbb{F}}) \lambda(de) + (\mathbb{D}_1 + \hat{\mathbb{D}}_1)^\top \mathcal{W}(s-) \\
&\quad \times (I - \mathcal{P}(s-)(\hat{H} + \hat{\mathbb{H}}))^{-1} \mathcal{P}(s-)((C + \hat{C}) + (\hat{H} + \hat{\mathbb{H}})^\top \mathcal{W}(s-)).
\end{align*} \right. 
\]
In (4.27), the conditions that detects solvability will be studied in the future research topic.

Then from (4.25), (4.24) becomes

\[
\begin{aligned}
\dot{Z}(s) &= \mathcal{F}(s)\dot{X}(s) - \mathcal{K}(s)\mathcal{Y}(s) - \mathcal{W}(s) + \mathcal{R}(s) + \lambda(s)\mathcal{E}(s), \\
\dot{K}(s,e) &= \mathcal{G}(s,e)\mathcal{Y}(s) - \mathcal{R}(s)\mathcal{E}(s),
\end{aligned}
\tag{4.26}
\]

where (s and e are suppressed)

\[
\begin{aligned}
\mathcal{F} &:= (I-P\hat{\mathcal{H}})^{-1}(P(C + \hat{\mathcal{H}}^\top P) - P\hat{\mathcal{R}}_1\hat{\mathcal{B}}_1) \\
\mathcal{G} &:= P\mathcal{F} - P\mathcal{G}_1\hat{\mathcal{R}}_1\hat{\mathcal{B}}_1 \\
\mathcal{F} &:= (I-P(\hat{\mathcal{H}} + \hat{\mathcal{H}}))^{-1}(P(C + \hat{\mathcal{C}} + (\hat{\mathcal{H}} + \hat{\mathcal{H}})^\top \mathcal{W}) - P(D_1 + \hat{\mathcal{D}}_1)\hat{\mathcal{R}}_1\hat{\mathcal{B}}_1) \\
\mathcal{G} &:= P(\hat{\mathcal{F}} + \hat{\mathcal{G}}) - P(\mathcal{G}_1 + \hat{\mathcal{G}}_1)\hat{\mathcal{R}}_1\hat{\mathcal{B}}_1.
\end{aligned}
\]

By substituting (4.25) and (4.26) into (4.23) and comparing its coefficients with (4.8), we can show that \(\mathcal{P}\) and \(\mathcal{W}\) in (4.8) satisfy the following coupled integro-RDEs (CIRDEs) (s is suppressed):

\[
\begin{aligned}
-\frac{d\mathcal{P}(s)}{ds} &= A^\top \mathcal{P}(s) + \mathcal{P}(s)A + Q + \mathcal{P}(s)\mathcal{E}_2\mathcal{P}(s) - \mathcal{P}(s)\mathcal{E}_1\mathcal{P}(s) + ((C + \hat{\mathcal{C}})^\top + \mathcal{W}(s)(\hat{\mathcal{H}} + \hat{\mathcal{H}})^\top)\mathcal{F} + \int_E\mathcal{F}(s,e)^\top \mathcal{G}(s,e)\lambda(de) \\
-\frac{d\mathcal{W}(s)}{ds} &= (A + \hat{\mathcal{A}})^\top \mathcal{W}(s) + \mathcal{W}(s)(\hat{\mathcal{A}} + \hat{\mathcal{A}}) + (Q + \hat{\mathcal{Q}}) + \mathcal{W}(s)(\hat{\mathcal{B}}_2 + \hat{\mathcal{B}}_2)\mathcal{W}(s) - \mathcal{W}(s)(\hat{\mathcal{B}}_1 + \hat{\mathcal{B}}_1)^\top \mathcal{F} + \int_E\mathcal{F}(s,e)\mathcal{G}(s,e)\lambda(de) \\
\mathcal{P}(T) &= M_1, \quad \mathcal{W}(T) = M_1 + \hat{\mathcal{M}}_1 \\
det(\hat{\mathcal{R}}_1(s)) &\neq 0, \quad \det(\hat{\mathcal{R}}_1(s)) \neq 0, \quad \forall s \in [t,T] \\
det(I-P(s)\hat{\mathcal{H}}(s)) &\neq 0, \quad \forall s \in [t,T] \\
det(I-P(s)(\hat{\mathcal{H}}(s) + \hat{\mathcal{H}}(s))) &\neq 0, \quad \forall s \in [t,T].
\end{aligned}
\tag{4.27}
\]

In (4.27), the conditions that \(\det(\hat{\mathcal{R}}_1(s)) \neq 0\) and \(\det(\hat{\mathcal{R}}_1(s)) \neq 0\) for all \(s \in [t,T]\) follow from (4.25), while the other two determinant conditions are due to (4.26).

**Remark 9.** As stated in Remark 7, the CIRDEs in (4.27) are nonsymmetric, and its solvability will be studied in the future research topic.

Finally, we substitute (4.25), (4.7) and (4.26) into the dynamics \(\mathcal{X}\) in (4.5). Then (s is suppressed)

\[
\begin{aligned}
\text{d}\mathcal{X} &= \left[\hat{\mathcal{A}}\mathcal{X}(s) + \hat{\mathcal{A}}\mathcal{E}[\mathcal{X}(s)]\right]ds + \left[\hat{\mathcal{C}}\mathcal{X}(s) + \hat{\mathcal{C}}\mathcal{E}[\mathcal{X}(s)]\right]dB(s) \\
&\quad + \int_E\left[\hat{\mathcal{F}}(s,e)\mathcal{X}(s) + \hat{\mathcal{F}}(s,e)\mathcal{E}[\mathcal{X}(s)]\right]d\hat{\mathcal{N}}(de,ds) \\
\mathcal{X}(t) &= \mathcal{X},
\end{aligned}
\tag{4.28}
\]
where

\[
\begin{align*}
\hat{A} &:= A + B_2 P(s-) + \hat{H} \hat{F} - B_1 \hat{R}_1^{-1} \hat{H}_1 \\
\hat{C} &:= C + \hat{H} \hat{F} - D_1 \hat{R}_1^{-1} \hat{H}_1 \\
\tilde{C} &:= (C + C) + (\hat{H} + \hat{H})^T W(s-) + (\hat{H} + \hat{H}) \hat{F} - (D_1 + D_1) \hat{R}_1^{-1} \hat{H}_1 - \hat{C} \\
\tilde{F} &:= F - \hat{G}_1 \hat{R}_1^{-1} \hat{H}_1, \quad \hat{F} := (\hat{F} + \tilde{F}) - (\hat{G}_1 + \hat{G}_1) \hat{H}_1 - \hat{F}.
\end{align*}
\]

As in Theorem 4.1, we state the following result. Its proof is omitted, since it resembles that of Theorem 4.1.

**Theorem 4.2.** Suppose that Assumptions 1 and 3 hold. Assume that the pair \((P, W)\) is the solution of the CIRDEs in (4.27), and \(X\) is the solution of (4.28). Define the transformations in (4.7) and (4.26), and consider the control in (4.25). Then (4.5) and (4.6) hold. In addition, suppose that (4.2) holds. Then the state-feedback type control in (4.25) is the optimal solution for \((LQ-L)\), and the associated optimal cost is given by

\[
J_1(a; \pi_1, \pi_2) = \inf_{u_1 \in U_1} J_1(a; u_1, \pi_2) = (a, W_{11}(t)a), \tag{4.29}
\]

where \(W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}\) with \(W_{11}\) being an \(\mathbb{R}^{n \times n}\)-valued process.

We consider

\[
\begin{align*}
\pi_1(s) - \mathbb{E}[\pi_1(s)] &= -\hat{R}_1(s) \hat{B}_1(s) \begin{bmatrix} x(s-) \\ \beta(s-) \end{bmatrix} - \mathbb{E} \begin{bmatrix} x(s-) \\ \beta(s-) \end{bmatrix} \\
\mathbb{E}[\pi_1(s)] &= -\tilde{R}_1(s) \tilde{B}_1(s) \mathbb{E} \begin{bmatrix} x(s-) \\ \beta(s-) \end{bmatrix}, \tag{4.30}
\end{align*}
\]

and

\[
\begin{align*}
\pi_2(s) - \mathbb{E}[\pi_2(s)] &= -\tilde{R}_2(s)^{-1} \tilde{S}_2(s)^\top (x(s-) - \mathbb{E}[x(s-))] \\
-\tilde{R}_2(s)^{-1} B_2(s)\top (\phi(s-) - \mathbb{E}[\phi(s-)]) \\
-\tilde{R}_2(s)^{-1} D_2(s)^\top (\theta(s) - \mathbb{E}[\theta(s)]) \\
-\tilde{R}_2(s)^{-1} \tilde{S}_1(s)(\pi_1(s) - \mathbb{E}[\pi_1(s)]) \\
\mathbb{E}[\pi_2(s)] &= -\hat{R}_2(s)^{-1} \hat{S}_2(s)^\top \mathbb{E}[x(s-)] \\
-\hat{R}_2(s)^{-1} (B_2(s) + \hat{B}_2(s))^\top \mathbb{E}[\phi(s-)] \\
-\hat{R}_2(s)^{-1} (D_2(s) + \hat{D}_2(s))^\top \mathbb{E}[\theta(s)] - \hat{R}_2(s)^{-1} \hat{S}_1(s) \mathbb{E}[\pi_1(s)]. \tag{4.31}
\end{align*}
\]

Then analogous to Corollary 4.1, we state the following result:

**Corollary 4.2.** Suppose that the assumptions of Theorems 3.1 and 4.2 hold. Then the pair \((\pi_1, \pi_2) \in U_1 \times U_2\) given in (4.30) and (4.31) constitutes the state-feedback representation for the open-loop Stackelberg game of the leader and the follower.

5. **Concluding remarks.** In this paper, we have considered the linear-quadratic mean-field type Stackelberg differential games for stochastic (mean-field type) jump-diffusion systems. By generalizing the Four-Step Scheme, we have obtained the explicit feedback-type optimal solutions for the leader and the follower in terms of the coupled integro-Riccati differential equations, which constitute the Stackelberg equilibrium. Note that the LQ mean-field type problem of the leader is new and
nontrivial due to the coupled MF-FBSDEs constraint induced by the rational behavior of the follower. We have obtained the mean-field type stochastic maximum principle for this problem via the variational approach.

As mentioned in Section 1, when the MF-SDE in (2.1) has no jumps, i.e., \( F = \tilde{F} = G_1 = G_1 = G_2 = G_2 = 0 \), Theorems 3.1-4.2 (and Corollaries 4.1 and 4.2) of this paper are reduced to the case of the LQ mean-field type Stackelberg game in a Brownian setting without jumps studied in [16, Theorems 3.1-3.3]. In addition, when (2.1) does not have the mean-field term, i.e., \( \bar{A} = B_1 = B_2 = \bar{C} = \bar{D}_1 = \bar{D}_2 = \tilde{F} = \bar{G}_1 = \bar{G}_2 = 0 \), Theorems 3.1-4.2 (and Corollaries 4.1 and 4.2) become equivalent to the results of (classical) LQ Stackelberg games in [31, Theorems 2.3 and 3.3, Section 5].

In Section 4, we have considered two different cases to obtain the feedback-type optimal solution of the leader. Unfortunately, it is hard to consider the general case without Assumption 2 or Assumption 3. Specifically, without Assumption 2 or Assumption 3, we need to consider (4.9) directly. This implies that the diffusion terms in (4.9) become \((s \text{ and } e \text{ are suppressed})\)

\[
\begin{align*}
Z &= \mathcal{P}\left[ C\tilde{\mathcal{X}}(s-) + (C + \bar{C})\mathbb{E}[\mathcal{X}] + \tilde{H}^\top \mathcal{P}\tilde{\mathcal{X}} + (\tilde{H} + \bar{H})^\top \mathbb{E}[\mathcal{X}] + \int_{\mathcal{E}} \tilde{K} \tilde{\mathcal{X}} \lambda(\text{d}e) + \int_{\mathcal{E}} (\bar{K} + \tilde{K}) \mathbb{E}[\mathcal{K}] \lambda(\text{d}e) \right. \\
&\left. + D_1 \tilde{u}_1 + (D_1 + \bar{D}_1) \mathbb{E}[\pi_1] \right] \\
\mathcal{K} &= \mathcal{P}\left[ F\tilde{\mathcal{X}} + (F + \bar{F}) \mathbb{E}[\mathcal{X}] + \tilde{K}^\top \mathcal{P}\tilde{\mathcal{X}} + (\tilde{K} + \bar{K})^\top \mathbb{E}[\mathcal{X}] + \int_{\mathcal{E}} \tilde{K} e \lambda(\text{d}e') + \int_{\mathcal{E}} (\bar{K} + \tilde{K}) \mathbb{E}[\mathcal{K}] \lambda(\text{d}e') \right. \\
&\left. + \int_{\mathcal{E}} (\mathbb{E}[\mathcal{K}] + \tilde{K} e) \lambda(\text{d}e') + G_1 \tilde{u}_1 + (G_1 + \bar{G}_1) \mathbb{E}[\pi_1] \right].
\end{align*}
\tag{5.1}
\]

Then it is necessary to obtain the explicit expressions of \( Z \) and \( \mathcal{K} \) from (5.1). However, due to the integral terms \( \int_{\mathcal{E}} \tilde{K} \tilde{\mathcal{X}} \lambda(\text{d}e) \), \( \int_{\mathcal{E}} (\tilde{K} + \bar{K}) \mathbb{E}[\mathcal{K}] \lambda(\text{d}e) \), \( \int_{\mathcal{E}} \tilde{K} \lambda(\text{d}e') \), and \( \int_{\mathcal{E}} (\tilde{K} + \bar{K}) \mathbb{E}[\mathcal{K}] \lambda(\text{d}e') \), there is a technical challenge to find the explicit expressions \( Z \) and \( \mathcal{K} \) (or \( \tilde{\mathcal{Z}} \), \( \tilde{\mathcal{K}} \), \( \mathbb{E}[\mathcal{Z}] \), and \( \mathbb{E}[\mathcal{K}] \)). We can see that Assumption 2 implies \( \int_{\mathcal{E}} \tilde{K} \tilde{\mathcal{X}} \lambda(\text{d}e) = \lambda \tilde{K} \tilde{\mathcal{X}} \), \( \int_{\mathcal{E}} (\tilde{K} + \bar{K}) \mathbb{E}[\mathcal{K}] \lambda(\text{d}e) = \lambda (\tilde{K} + \bar{K}) \mathbb{E}[\mathcal{K}] \), \( \int_{\mathcal{E}} \tilde{K} \lambda(\text{d}e') = \lambda \tilde{K} \), and \( \int_{\mathcal{E}} (\tilde{K} + \bar{K}) \mathbb{E}[\mathcal{K}] \lambda(\text{d}e') = \lambda (\tilde{K} + \bar{K}) \mathbb{E}[\mathcal{K}] \), and Assumption 3 leads to \( \int_{\mathcal{E}} \tilde{K} \tilde{\mathcal{X}} \lambda(\text{d}e) = \int_{\mathcal{E}} (\tilde{K} + \bar{K}) \mathbb{E}[\mathcal{K}] \lambda(\text{d}e) = \int_{\mathcal{E}} \tilde{K} \lambda(\text{d}e') = \int_{\mathcal{E}} (\tilde{K} + \bar{K}) \mathbb{E}[\mathcal{K}] \lambda(\text{d}e') = 0 \). Therefore, in both cases, we are able to find the explicit expressions of \( Z \) and \( \mathcal{K} \), which are given in (4.17) and (4.26).

We now state several potential future research problems. First, the solvability of the CIRDEs in (4.18) and (4.27) has to be studied to show the existence of the Stackelberg equilibrium in view of Corollaries 4.1 and 4.2. Second, unlike the hierarchical interaction between the leader and the follower, we may consider the symmetric interaction between players, which can be formulated as mean-field type stochastic zero-sum differential games. This can be viewed as an extension of [18] to the case of jump-diffusion models. Finally, it is interesting to study LQ mean-field type Stackelberg games for Markov regime-switching jump-diffusion models, in which the MF-SDE in (2.1) has an additional Markov jump parameters. In this problem, we need to generalize the stochastic maximum principle in [33] to the case of MF-FBSDEs with Markov regime-switching jump-diffusions.
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Appendix A. Proof of Lemma 3.1.

Proof of Lemma 3.1. Since $u_1$ is not involved in (LQ-F) and the SDE is linear, without loss of generality, we may assume $u_1 \equiv 0$. Moreover, note that since (3.1) is a linear MF-BSDE, in view of [28, Theorem 3.1], (3.1) admits a unique solution of

$$J_2(a, 0, \overline{u}_2 + \kappa u'_2) - J_2(a, 0, \overline{u}_2)$$

$$= 2\kappa \mathbb{E} \left[ \int_0^T \left( (Q_2(s)\bar{\pi}(s), x'(s)) + (Q_2(s)\mathbb{E} \bar{\pi}(s), \mathbb{E} x'(s)) \right) + (R_2(s)\overline{u}_2(s), u'_2(s)) + (\bar{R}_2(s)\mathbb{E} \overline{u}_2(s), \mathbb{E} u'_2(s)) \right] ds$$

$$+ (M_2\bar{\pi}(T), x'(T)) + \langle \bar{M}_2\mathbb{E} \bar{\pi}(T), \mathbb{E} x'(T) \rangle + \kappa^2 J_2(0, 0, u'_2)$$

$$= 2\kappa \mathbb{E} \left[ \int_0^T \left( (Q_2(s)\bar{\pi}(s) + \bar{Q}_2(s)\mathbb{E} \bar{\pi}(s), x'(s)) \right) + (R_2(s)\overline{u}_2(s) + \bar{R}_2(s)\mathbb{E} \overline{u}_2(s), u'_2(s)) \right] ds$$

$$+ \langle M_2\bar{\pi}(T) + \bar{M}_2\mathbb{E} \bar{\pi}(T), x'(T) \rangle \right] + \kappa^2 J_2(0, 0, u'_2) \geq 0.$$
We apply Itô’s formula to \((forward)\) MF-SDEs, from [28, Lemma 4.1], (3.13) admits a unique solution in Proof of Theorem 3.1.

Let \(J_2(a, 0, \tau_2 + \kappa u_2') = J_2(a, 0, \tau_2)
\]
\[
\kappa \mathbb{E} \left[ \int_t^T \langle R_2(s) \tau_2(s) + \tilde{R}_2(s) \mathbb{E}[\tau_2(s)] + B_2(s) \tau_2(s) + \tilde{B}_2(s) \mathbb{E}[\tau_2(s)] + D_2(s) \tau_2(s) + \tilde{D}_2(s) \mathbb{E}[\tau_2(s)] \rangle \mathbb{E}[r(s, e)] \lambda(de) + \int_E \tilde{G}_2(s, e) \tau_2(s) \mathbb{E}[r(s, e)] \lambda(de), u_2'(s) \rangle ds \right] + \kappa^2 J_2(0, 0, u_2') \geq 0.
\]

Hence, we have
\[
J_2(a, 0, \tau_2 + \kappa u_2') - J_2(a, 0, \tau_2)
\]
\[
= \kappa \mathbb{E} \left[ \int_t^T \langle R_2(s) \tau_2(s) + \tilde{R}_2(s) \mathbb{E}[\tau_2(s)] + B_2(s) \tau_2(s) + \tilde{B}_2(s) \mathbb{E}[\tau_2(s)] + D_2(s) \tau_2(s) + \tilde{D}_2(s) \mathbb{E}[\tau_2(s)] \rangle \mathbb{E}[r(s, e)] \lambda(de) + \int_E \tilde{G}_2(s, e) \tau_2(s) \mathbb{E}[r(s, e)] \lambda(de), u_2'(s) \rangle ds \right] + \kappa^2 J_2(0, 0, u_2') \geq 0.
\]

This implies that \(\tau_2 \in \mathcal{U}_2\) holds the first-order necessary condition in (3.2). This completes the proof. \(\square\)

**Appendix B. Proof of Theorem 3.1.**

**Proof of Theorem 3.1.** We note that (3.16) is a class of linear MF-BSDEs with jump diffusions, which holds [28, (A.3) and (A.4)]. Then in view of [28, Theorem 3.1], (3.16) admits a unique solution of \((\phi, \theta, \psi) \in \mathcal{C}^2_p(t, T; \mathbb{R}^n) \times \mathcal{L}^2_p(t, T; \mathbb{R}^n) \times \mathcal{G}^2_{\alpha, \beta}(t, T; \mathbb{R}^n)\) (see also [27, Lemma 2.2]). Moreover, note that \(\tau_2 : \mathbb{R}^n \times \mathcal{U}_1 \rightarrow \mathcal{U}_2\), which implies \(\tau_2 \in \mathcal{U}_2\) for a given \(u_1 \in \mathcal{U}_1\). Then since (3.13) is a class of linear (forward) MF-SDEs, from [28, Lemma 4.1], (3.13) admits a unique solution in \(\mathcal{C}^2_p(t, T; \mathbb{R}^n)\) (see also the case of MF-SDEs with jump diffusions and delay in [27, Lemma 2.1] and [17, Theorem 3.2]).

Let \(e := x - \mathbb{E}[x], \bar{u}_i := u_i - \mathbb{E}[u_i], \tilde{\phi} := \phi - \mathbb{E}[\phi], \tilde{\theta} := \theta - \mathbb{E}[\theta], \) and \(\tilde{\psi} := \psi - \mathbb{E}[\psi].\) We apply Itô’s formula to \(d(e(s), P(e(s)) + d(\mathbb{E}[x(s)], Z(s) \mathbb{E}[x(s)]) + 2d(x(s), \phi(s)).\) Then by integrating it from 0 to \(T, (s \in \mathbb{R}^n)\) is suppressed
\[
J(a; u_1, u_2) = \mathbb{E} \left[ |a|^2 \mathbb{E}[x(0)] + 2(a, \phi(0)) + \int_t^T \left[ \tilde{R}_2^{-1} \tilde{S}_2 \mathbb{E}[e(s) - \hat{f}(s)]^2 \right] ds \right.
\]
\[
+ |\bar{u}_2(s)|^2 \mathbb{E}[x(0)] + 2(\bar{u}_2(s), \tilde{S}_2 \mathbb{E}[e(s)] + B_2 \tilde{\phi}(s) + D_2 \tilde{\theta}(s)
\]
\[
+ \int_E G_2 \tilde{\psi}(s, e) \lambda(de) + \tilde{S}_1 \tilde{\mu}_1(s)) + 2(\tilde{S}_2 e(s), \tilde{R}_2^{-1} (B_2(s)) \tilde{\phi}(s)
\]
\[
+ D_2 \tilde{\theta}(s) + \int_E G_2 \tilde{\psi}(s, e) \lambda(de) + \tilde{S}_1 \tilde{\mu}_1(s))
\]
\[
+ 2(B_2 \tilde{\psi}(s), \tilde{R}_2^{-1} (D_2 \tilde{\theta}(s) + \int_E G_2 \tilde{\psi}(s, e) \lambda(de) + \tilde{S}_1 \tilde{\mu}_1(s)))
\]
\[
+ 2(D_2 \tilde{\theta}(s), \tilde{R}_2^{-1} (\int_E G_2 \tilde{\psi}(s, e) \lambda(de) + \tilde{S}_1 \tilde{\mu}_1(s)))
\]
\[
+ 2(\int_E G_2 \tilde{\psi}(s, e) \lambda(de), \tilde{R}_2^{-1} \tilde{S}_1 \tilde{\mu}_1(s)) + |\tilde{f}(s)|^2 \mathbb{E}[x(0)] - |\tilde{f}(s)|^2 \mathbb{E}[x(0)].
Completing the integrand terms above with respect to \( \tilde{u}_2 \) and \( \mathbb{E}[u_2] \) yields

\[
J_2(a; u_1, u_2) = \mathbb{E} \left[ a \left( \tilde{u}_2^2(0) + 2(a, \phi(0)) + \int_t^T |\tilde{u}_2(s) + \tilde{R}_2(s)^{-1} \tilde{S}_2(s)\lambda(es)\right) ds \right] 
\]

\[
+ \tilde{R}_2(s)^{-1} B_2(s)^{\top} \phi(es) + \tilde{R}_2(s)^{-1} D_2(s)^{\top} \bar{\theta}(s) 
\]

\[
+ \tilde{R}_2(s)^{-1} \int_E G_2(s, e)^{\top} \psi(es) \lambda(de) + \tilde{R}_2(s)^{-1} \tilde{S}_1(s) \tilde{u}_1(s)^2 R_2(s) ds 
\]

\[
+ \int_t^T \mathbb{E}[u_2(s)] + \tilde{R}_2(s)^{-1} \tilde{S}_2(s)^{\top} \mathbb{E}[x(s-)] + \tilde{R}_2(s)^{-1} (B_2(s) + \tilde{B}_2(s) + \bar{B}_2(s)) \mathbb{E}[\theta(s)] \bigg\| \mathbb{E}[u_1(s)]^{2} R_2(s) ds 
\]

\[
\times \mathbb{E}[\phi(es-)] + \tilde{R}_2(s)^{-1} (D_2(s) + \tilde{D}_2(s))^{\top} \mathbb{E}[\theta(s)] + \tilde{R}_2(s)^{-1} \int_E (G_2(s, e) \mathbb{E}[\psi(es)]^{2} R_2(s) ds 
\]

\[
+ G_2(s, e)^{\top} \mathbb{E}[\psi(es)] \lambda(de) + \tilde{R}_2(s)^{-1} \tilde{S}_1(s) \mathbb{E}[u_1(s)]^{2} R_2(s) ds 
\]

\[
+ \int_t^T \left[ -|\tilde{f}(s)|^2_{R_2^{-1}} - |\tilde{f}(s)|^2_{R_2^{-1}} + |\tilde{u}_1(s)|^2_{\tilde{G}_1^{\top} \tilde{P} \tilde{D} \tilde{D}^{\top} \tilde{P} \tilde{G} \tilde{I} \lambda(de)} + 2(\tilde{u}_1(s), B_1^{\top} \phi(es) + D_1 \bar{\theta}(s) + \int_E G_1^{\top} \tilde{\psi}(es) \lambda(de)) 
\]

\[
+ |\mathbb{E}[u_1(s)]^{2} (D_1 + \tilde{D}_1)^{\top} P (D_1 + \tilde{D}_1) (B_1 + \tilde{B}_1)^{\top} Z (B_1 + \tilde{B}_1) \right] ds 
\]
\[ + \int_E \left| E[u_1(s)] \right|^2 \left( B_1 + \tilde{B}_1 \right)^\top E[\phi(s-)] + \int_E (D_1 + \tilde{D}_1)^\top E[\theta(s)] \right) \right) ds \] 

Since \( \tilde{R}_2 \) and \( \tilde{R}_2 \) are uniformly positive definite for all \( s \in [t, T] \),

\[ J_2(u; u_1, u_2) - J_2(u; u_1, \bar{u}_2) \]

\[ = \mathbb{E} \left[ \int_t^T \left| \tilde{u}_2(s) + \tilde{R}_2(s)^{-1} \tilde{S}_2(s)^\top \epsilon(s-) + \tilde{R}_2(s)^{-1} \tilde{f}(s) \right|^2 \tilde{R}_2(s) ds \right. 
\ \left. + \int_t^T \left| E[u_2(s)] + \tilde{R}_2(s)^{-1} \tilde{S}_2(s)^\top E[x(s-)] + \tilde{R}_2(s)^{-1} \tilde{f}(s) \right|^2 \tilde{R}_2(s) ds \right] \geq 0, \]

which shows that (3.17) is the (state-feedback type) optimal control for (LQ-F).

We can observe from (B.2) that (3.17) has an argument of the form \( s^- \) to preserve predictability of the control process. The optimal cost under (3.17) can be deduced easily from (B.1) and (B.2). This completes the proof of the theorem. \( \square \)

Appendix C. Proof of Lemma 4.1.

\textbf{Proof of Lemma 4.1.} First, from Theorem 3.1, we have \((\pi, \phi, \theta, \psi) \in C^{4\beta}(t, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times G^{2}_{\mathcal{F},p}(t, T, \lambda; \mathbb{R}^n) \) in (4.1). Then \( \beta \) is a linear (forward) MF-SDE with jump diffusions; hence, as in (2.1), it has a unique solution in \( C^2(t, T; \mathbb{R}^n) \). Furthermore, \((\alpha, \eta, \gamma) \) is a (linear) MF-BSDE with jump diffusions, which admits a unique solution in \( C^2(t, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times G^{2}_{\mathcal{F},p}(t, T, \lambda; \mathbb{R}^n) \); see [28, Theorem 3.1].

Applying Itô's formula to \( d(x'(s), \alpha(s)) \) yields \((s \text{ is suppressed})\)

\[ d(x'(s), \alpha(s)) = \left[ \hat{A}x'(s-) + \hat{B}_2 \phi(s-) + \hat{B}_2 E[\phi(s-)] + \hat{H}_2 \theta(s) + \hat{H}_2 E[\theta(s)] \right] + \int_E \left( \hat{K}_2(s, e) \psi(s, e) \lambda(\text{de}) + \hat{B}_1 u'_1(s) + \hat{B}_1 E[u'_1(s)] \right) \alpha(s-) ds \]

\[ - x'(s-) \left[ \hat{A}^{\top} \alpha(s-) + \hat{L}^{\top} E[\alpha(s-)] + \hat{C}^{\top} \eta(s) + \hat{C}^{\top} E[\eta(s)] \right] ds \]

\[ + \int_E \left( \hat{F}(s, e) \gamma(s, e) \lambda(\text{de}) + \hat{K}_2(s, e) E[\gamma(s, e)] \lambda(\text{de}) + \hat{Q}_1 x(s-) + \hat{Q}_1 E[x(s-)] \right) \]

\[ + \left[ \hat{C} x'(s-) + \hat{C} E[x'(s-)] + \hat{H}_2^{\top} \phi'(s-) + \hat{H}_2 E[\phi'(s-)] + \hat{F}_3(s, e) \lambda(\text{de}) + \hat{B}_1 u'_1(s) + \hat{B}_1 E[u'_1(s)] \right] \eta(s) ds \]

\[ + \int_E \left( \hat{F}(s, e) x'(s-) + \hat{K}_2(s, e) E[x'(s-)] + \hat{K}_2(s, e) \phi'(s-) + \hat{K}_2(s, e) \psi'(s, e') \lambda(\text{de'}) \right] \]

\[ + \hat{K}_2(s, e) \theta'(s) + \hat{K}_2(s, e) \gamma(s, e') \lambda(\text{de'}) + \hat{G}_1(s, e) u'_1(s) + \hat{G}_1(s, e) E[u'_1(s)] \right) \gamma(s, e) \lambda(\text{de}) + \cdots \]
Moreover, we have \((s \text{ is suppressed})\)
\[
d\langle \phi'(s), \beta(s) \rangle = \left[ -\tilde{A}^T \phi'(s-) + \tilde{A}^T E[\phi'(s-)] + \tilde{C}^T \theta'(s) + \tilde{C}^T E[\theta'(s)] \right. \\
+ \int_E \tilde{F}(s,e)^T \psi'(s,e) \lambda(de) + \int_E \tilde{F}(s,e)^T E[\psi'(s,e)] \lambda(de) + \tilde{H}_1 \tilde{u}_1(s) + \tilde{H}_1^T E[u'_1(s)] \\
+ \int_E \tilde{K}_1(s,e)^T \tilde{u}_1'(s) \lambda(de) + \int_E \tilde{K}_1(s,e)^T E[u'_1(s)] \lambda(de) \bigg] \tau(s-) ds \\
+ \phi'(s-) [\tilde{A} \beta(s-) + \tilde{A} E[\beta(s-)] + \tilde{B}_2 \alpha(s-) + \tilde{B}_2 E[\alpha(s-) - \tilde{H}_2 \eta(s) + \tilde{H}_2 E[\eta(s)] \\
+ \int_E \tilde{K}_2(s,e) \gamma(s,e) \lambda(de) + \int_E \tilde{K}_2(s,e) E[\gamma(s,e)] \lambda(de) \bigg] ds \\
+ \theta'(s-) [\tilde{C} \beta(s-) + \tilde{C} E[\beta(s-)] + \tilde{H}_2 \alpha(s-) + \tilde{H}_2 E[\alpha(s-) - \tilde{H}_2 \eta(s) + \tilde{H}_2 E[\eta(s)] \\
+ \int_E \tilde{K}_2(s,e) \gamma(s,e) \lambda(de) + \int_E \tilde{K}_2(s,e) E[\gamma(s,e)] \lambda(de) \bigg] ds \\
+ \int_E \psi'(s,e)^T \left[ \tilde{F}(s,e) \beta(s-) + \tilde{F}(s,e) E[\beta(s-)] + \tilde{K}_2 \alpha(s-) + \tilde{K}_2(s,e)^T E[\alpha(s-) \right] \\
+ \tilde{K}_2(s,e)^T \eta(s) + \tilde{K}_2(s,e)^T E[\eta(s)] + \int_E \tilde{K}(s,e,e') \gamma(s,e') \lambda(de') \\
+ \int \tilde{K}(s,e,e') E[\gamma(s,e')] \lambda(de') \bigg] \lambda(de) + \left[ \cdots \right] dB(s) + \int \left[ \cdots \right] \tilde{N}(de, ds).
\]

Then we have (note that \(\tilde{B}_2, \tilde{B}_2, \tilde{H}_2, \tilde{H}_2, \tilde{K}_2 \) and \(\tilde{K}_2 \) are symmetric)
\[
E[\langle x'(T), M_1 \bar{I}(T) + \langle x'(T), M_1 \bar{I}(T) \rangle \rangle] = \mathbb{E} \left[ \langle x'(T), \alpha(T) \rangle - \langle x'(t), \alpha(t) \rangle - \langle \phi'(T), \beta(T) \rangle + \langle \phi'(t), \beta(t) \rangle \right] \\
= \mathbb{E} \left[ \int_0^T [ -\langle x'(s), Q_1(s) \bar{I}(s) + \bar{Q}_1(s) E[\bar{I}(s)] \rangle \right. \\
+ \langle \tilde{u}_1'(s), \tilde{B}_1(s)^T \alpha(s-) + \tilde{B}_1(s)^T E[\alpha(s-)] + \tilde{D}_1(s)^T \eta(s) \\
+ \tilde{D}_1(s)^T E[\eta(s)] + \int \tilde{G}_1(s)^T \gamma(s,e) \lambda(de) + \int \tilde{G}_1(s)^T E[\gamma(s,e)] \lambda(de) + \tilde{H}_1(s) \beta(s-) \\
+ \tilde{H}_1(s) E[\beta(s-)] + \int \tilde{K}_1(s,e) \beta(s-) \lambda(de) + \int \tilde{K}_1(s,e) E[\beta(s-)] \lambda(de) \bigg] ds \bigg].
\]

Similarly,
\[
J_1(0; u'_1, \bar{u}_2) = \mathbb{E} \left[ \int_0^T \langle \tilde{u}_1'(s), R_1(s) u'_1(s) + \bar{R}_1(s) E[u'_1(s)] + \tilde{B}_1(s)^T \alpha'(s-) \\
+ \tilde{B}_1(s)^T E[\alpha'(s-)] + \tilde{D}_1(s)^T \eta'(s) + \tilde{D}_1(s)^T E[\eta'(s)] \\
+ \int \tilde{G}_1(s)^T \gamma'(s,e) \lambda(de) + \int \tilde{G}_1(s)^T E[\gamma'(s,e)] \lambda(de) + \tilde{H}_1(s) \beta'(s-) \\
+ \tilde{H}_1(s) E[\beta'(s-) + \int \tilde{K}_1(s,e) \beta'(s-) \lambda(de)] + \int \tilde{K}_1(s,e) E[\beta'(s-)] \lambda(de) \bigg] ds \bigg],
\]
Proof of Theorem 4.1.

For $J_1(a, π_1 + κu_1', π_2) = J_1(a, π_1, π_2)$, we have

$$J_1(a, π_1 + κu_1', π_2) = 2κE\left[\int_t^T [x'(s), Q_1(s)π(s)] + (E[x'(s)], Q_1(s)E[π(s)]) + (u_1'(s), R_1(s)π_1(s))\right]$$

$$+ \kappa^2 J_1(0, u_1', π_2)$$

From (C.1) and (C.2), it follows that

$$J_1(a, π_1 + κu_1', π_2) − J_1(a, π_1, π_2)$$

$$= 2κE\left[\int_t^T \langle u_1'(s), R_1(s)π_1(s) + \hat{R}_1(s)E[π_1(s)] + \hat{B}_1(s)^T α(s) + \hat{B}_1(s)^T E[α(s)]\rangle$$

$$+ \hat{D}_1(s)^T η(s) + \hat{D}_1(s)^T E[η(s)] + \int_E \hat{G}_1(s)^T γ(s,e)λ(de) + \int_E \hat{G}_1(s)^T E[γ(s,e)]λ(de)$$

$$+ \hat{H}_1(s)β(s) + \hat{H}_1(s)E[β(s)] + \int_E \hat{K}_1(s,e)β(s)λ(de) + \int_E \hat{K}_1(s,e)E[β(s)]λ(de)\right]$$

$$+ \kappa^2 E\left[\int_t^T \langle u_1'(s), R_1(s)u_1'(s) + \hat{R}_1(s)E[u_1'(s)] + \hat{B}_1(s)^T α'(s) + \hat{B}_1(s)^T E[α'(s)]\rangle$$

$$+ \hat{D}_1(s)^T η'(s) + \hat{D}_1(s)^T E[η'(s)] + \int_E \hat{G}_1(s)^T γ'(s,e)λ(de) + \int_E \hat{G}_1(s)^T E[γ'(s,e)]λ(de)$$

$$+ \hat{H}_1(s)β'(s) + \hat{H}_1(s)E[β'(s)] + \int_E \hat{K}_1(s,e)β'(s)λ(de) + \int_E \hat{K}_1(s,e)E[β'(s)]λ(de)\right]$$

$$+ \int_E \hat{K}_1(s,e)E[β'(s)]λ(de)ds \geq 0,$$
We now prove (4.29). From (C.3) and (4.7), it follows that (s is suppressed) 
\[ J_1(a; \overline{\pi}_1, \overline{\pi}_2) \]
\[ = \mathbb{E} \left[ \langle \mathcal{X}(t), \mathcal{P}(t) (\mathcal{X}(t) - \mathbb{E}[\mathcal{X}(t)]) \rangle + \langle \mathcal{X}(t), \mathcal{W}(t) \mathbb{E}[\mathcal{X}(t)] \rangle \right] \]
\[ + \int_t^T \left( \frac{\overline{\pi}_1(s)}{\bar{\mathbb{E}}_1} + \bar{R}_1 \mathbb{E}[\overline{\pi}_1(s)] + \mathbb{E}_1 \overline{\mathcal{Y}}(s) - \mathbb{E}_1 \mathbb{E}[\overline{\mathcal{Y}}(s)] + D_1 \bar{Z}(s) \right) \]
\[ + \bar{H}_1 \mathbb{E}[\overline{\mathcal{Y}}(s)] + \lambda \bar{K}_1(\mathcal{e}) \mathcal{X}(s) - \lambda \mathbb{E} [\bar{K}_1(\mathcal{e}) \mathcal{X}(s)] \right) ds \]
\[ = \langle a, \mathcal{W}_{11}(t) a \rangle, \]
which shows the desired result. We complete the proof. 

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