Logarithmic Yangians in WZW models

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Abstract

A new action of the Yangians in the WZW models is displayed. Its structure is generic and level independent. This Yangian is the natural extension at the conformal point of the one unravelled in massive theories with current algebras. Expectingly, this new symmetry of WZW models will lead to a deeper understanding of the integrable structure of conformal field theories and their deformations.
1. Introduction

In recent years Yangian symmetry has been unraveled in various physical models. A particularly interesting manifestation of this symmetry is its occurrence in Haldane-Shastry [1] long-range interacting spin chains, which have been found to realize a discrete analogue of level-1 WZW theories [2]. This has led to a quasi-particle (spinon) description of these conformal models [3, 4, 5, 6, 7] expected to be tailor-made for the analysis of off-critical deformations. Although rather explicit, this construction has some intrinsic limitations: except for \( su(N) \)-WZW models at level 1, there are no explicit description of the Yangian currents and the \( su(N) \) results do not generalize directly to other algebras (see e.g., [6]).

This situation contrasts sharply with that pertaining to the integrable massive deformations of the CFT theories for which there is a Yangian symmetry for any affine Lie spectrum generating algebra at the fixed point, with a level-independent structure [8, 9, 10]. These Yangian generators are nothing but the direct quantum extensions of the classical nonlocal charges of the sigma model or current algebra models [11].

In the present work, we display a new Yangian symmetry in WZW models whose structure is generic and level independent. It provides a direct extension of the off-critical Yangian realization in massive theories. Unfortunately, what is gained in generality seems to be lost in explicitness.

2. Nonlocal currents in WZW models

The classical equations of motion for the WZW models at the conformally invariant points can be written as the conservation laws, \( \partial_\mu J_\mu^\pm = 0 \), for two chiral currents \( J_\mu^\pm \), respectively defined by

\[
J_\mu^+ = g^{-1} \partial_\mu g - \epsilon_{\mu\nu} g^{-1} \partial_\nu g,
J_\mu^- = \partial_\mu g g^{-1} + \epsilon_{\mu\nu} \partial_\nu g g^{-1}
\]  

As a consequence, there exists classically two nonlocal conserved currents defined by:

\[
\mathcal{J}_\mu^{\pm a} (x) = i f^{abc} \int_0^x dv^\sigma \epsilon^{\sigma\rho} J_\rho^{\pm b} (v) J_\mu^{\pm c} (x)
\]

Similarly, in most nonlinear sigma models, the equations of motion can be written as a conservation law for a curl-free current taking values in a finite dimensional Lie algebra. This ensures the existence of nonlocal currents [11] given by an expression similar to
(2.2). In favorable cases these currents persist in the quantum theory. They are then the generators of a Yangian algebra [9,8,10].

The structure of the nonlocal conserved currents in massive theories suggests the following form for the nonlocal density of the Yangian generator in the ultraviolet limit:

\[ Y^a(z; P) = \lim_{y \to z} \left\{ i f^{abc} \int_P^z dv J^b(v) J^c(y) + 2g J^a(y) \ln \left( \frac{z-y}{P-y} \right) \right\} \tag{2.3} \]

where \( J^a(z) \) are the generators of the spectrum generating current algebra,

\[ J^a(z) J^b(w) \sim \frac{if^{abc} J^c(w)}{z-w} + \frac{k \delta^{ab}}{(z-w)^2} \tag{2.4} \]

and \( f^{abc} \) are the completely antisymmetric structure constants, normalized according to

\[ f^{abc} f^{abd} = 2g \delta^{cd} \tag{2.5} \]

g stands for the dual Coxeter number and \( k \) is the level. The first integration contour in eq.(2.3), \( \int_P^z dv J^b(v) \), is the same as the contour used to define the second logarithmic term, \( \ln \left( \frac{z-y}{P-y} \right) = \int_P^z \frac{dv}{v-y} \). Note that (2.3) could equally well be written as

\[ Y^a(z; P) = \frac{1}{2\pi i} \oint \frac{dx}{x-z} \left\{ i f^{abc} \int_P^x dv J^b(v) J^c(z) + 2g J^a(z) \ln \left( \frac{x-z}{P-z} \right) \right\} \tag{2.6} \]

which is the usual way field products are regularized in conformal theories.

The main observation that needs to be done from the expression of this nonlocal current is the dependence upon a certain point \( P \). This is a novelty brought by the conformal invariance which prevents us to simply let \( P \to \infty \) from the onset. Much of the complications to follow are rooted in this simple fact. In the massive regime, \( P \) can be sent to \( \infty \) since correlation functions decrease exponentially; mathematically said, the points are no longer all conformally equivalent.

It should also be mentioned that this candidate nonlocal current is not unique. The following variant of \( Y^a \) is also regular and shares most of its properties:

\[ \tilde{Y}^a(z; P) = \lim_{y \to z} \left[ i f^{abc} \int_P^z dv J^b(v) J^c(y) + 2g J^a(y) \ln(z-y) \right] \tag{2.7} \]

In view of relating the charges associated to these currents with Yangian generators, we first need to evaluate the effect of inserting nonlocal currents in correlation functions, that is, to calculate \( \langle Y^a(z; P) \prod_j \phi_j(\zeta_j) \rangle \). Here \( \phi_j(\zeta_j) \) stands for a WZW primary field:

\[ J^a(z) \phi_j(w) \sim \frac{-t^a_j}{z-w} \phi(w) \tag{2.8} \]
where \(t^a_j\) is a matrix representation (specified by \(j\)) of the symmetry algebra: \([t^a_j, t^b_k] = i\delta_{jk} f^{abc} t^c_j\). A direct calculation yields

\[
\langle Y^a(z; P) \prod_j \phi_j(\zeta_j) \rangle = i f^{abc} \sum_{j,k} t^b_j t^c_k \ln \left( \frac{z - \zeta_j}{P - \zeta_j} \right) \langle \prod_j \phi_j(\zeta_j) \rangle
\]

(2.9)

The nonlocal structure of \(Y^a\) manifests itself through logarithmic terms. Once again, the logarithmic terms \(\ln \left( \frac{z - \zeta_j}{P - \zeta_j} \right)\) are defined by the integrals \(\int \frac{dv}{v - \zeta_j}\) where the integration contours are paths from \(P\) to \(z\). The nonlocality of \(Y(z; P)\) is encoded in the monodromy acquired by these integrals upon contour deformations.

3. Nonlocal charges in WZW models and comultiplication

The action of the charge \(Q^a_1\) – associated to the current \(Y^a(z; P)\) – on a field \(\phi_0(\zeta_0)\) is defined as follows [8]:

\[
\langle Q^a_1(\phi_0(\zeta_0)) \prod_{j \geq 1} \phi_j(\zeta_j) \rangle = \oint_{\gamma_0} dz \langle Y^a(z; P) \prod_{j \geq 0} \phi_j(\zeta_j) \rangle
\]

(3.1)

where the contour \(\gamma_0\) is defined as in Fig. (1a). Essentially, the contour circles around \(\zeta_0\) but excludes all other \(\zeta_i\)'s. However, the presence of the logarithmic cut calls for a more careful specification. The cut, which extends from \(\zeta_0\) to infinity is chosen to pass through the point \(P\) and the contour includes \(P\) although it remains slightly open due to the cut. The point \(z\) starts then just below (and at the left of) \(P\), hence just below the cut, follows the cut up to \(\zeta_0\) which is then encircled and returns to \(P\) above the cut (and again at the left of \(P\)). The phase of \((z - \zeta_0)\) above the cut differs by \(2\pi i\) from its value just below it.\(^4\)

In this way, making use of the first and third contour integrals of appendix A, we find the explicit expression for the action of the charge \(Q^a_1\) on a primary field:

\[
\langle Q^a_1(\phi_0(\zeta_0)) \prod_{j \geq 1} \phi_j(\zeta_j) \rangle = \left\{ -gi\pi t^a_0 + 2if^{abc} \sum_{k \neq 0} t^b_k t^c_0 \ln \left( \frac{\zeta_0 - \zeta_k}{P - \zeta_k} \right) \right\} \langle \phi_0(\zeta_0) \prod_{j \geq 1} \phi_j(\zeta_j) \rangle
\]

(3.2)

\(^4\) For subsequent applications of the charge \(Q^a_1\), we mention that, in Fig. (1a), the open contour around \(P\) has vanishing radius. Thus, for this part of \(\gamma_0\), the logarithmic terms do not contribute, while the pole contributions are picked up (which can arise for some of the terms in a correlation function), as if, in the absence of cuts, the contour was completely closed at the left of \(P\).
On the other hand, the action of the local charge $Q^a_0$ on $\phi_0$ is defined as usual in terms of a small contour around $\zeta_0$. This results in $Q^a_0(\phi_0) = -t^a_0\phi_0$. We recall that $Q^a_0$ acts additively on a product of fields.

The next step is to define the action of $Q^a_1$ on two fields $\phi_1(\zeta_1)$ and $\phi_2(\zeta_2)$. This is still defined by a contour of integration:

$$\langle Q^a_1(\phi_1(\zeta_1)\phi_2(\zeta_2)) \prod_{j \geq 3} \phi_j(\zeta_j) \rangle = \oint_{\Gamma} dz \langle Y^a(z; P) \prod_{j \geq 0} \phi_j(\zeta_j) \rangle$$  \hspace{1cm} (3.3)

but with a different integration contour, which is illustrated in Fig. (1b). This contour can be deformed in two contours $\gamma_1$ and $\gamma_2$. However, after the first contour a shift of $2\pi i$ occurs, which is responsible for the following comultiplication rule

$$\Delta Q^a_1 = Q^a_1 \otimes 1 + 1 \otimes Q^a_1 - 2\pi f^{abc} Q^b_0 \otimes Q^c_0$$  \hspace{1cm} (3.4)

This is indeed the comultiplication of Yangians.

4. Yangian relations

We will now show that the charges $Q^a_0$ and $Q^a_1$ are the first two generators of a Yangian symmetry, ie. that they generate a representation of the Yangian algebra. This is our main result. It amounts to establishing the relations

$$Y(1) : [Q^a_0, Q^b_0] = if^{abc} Q^c_0$$
$$Y(2) : [Q^a_0, Q^a_1] = if^{abc} Q^c_1$$
$$Y(3) : [Q^a_1, [Q^b_1, Q^c_0]] - [Q^a_0, [Q^b_1, Q^c_1]] = C A^{abc:odef} \{Q^d_0, Q^e_0, Q^f_0\}_s$$  \hspace{1cm} (4.1)

where

$$A^{abc:odef} = f^{adp} f^{bce} f^{def} f^{pqr}$$  \hspace{1cm} (4.2)

${a, b, c}_s$ stand for the complete symmetrization of the three objects $a, b, c$ (which incorporates a factor 1/6) and $C$ is a constant fixed by the comultiplication. With our normalization for the comultiplication, we have

$$C = -4\pi^2$$  \hspace{1cm} (4.3)

A fourth relation, needed only for $su(2)$, will be ignored here.
We will prove the Yangian relations (4.1) by checking them for the action of the charges $Q_0^a$ and $Q_1^a$ on a set of primary fields. (Actually, this check provides only a partial proof of the Yangian relations since acting with the Yangian generators produces fields which are not generated by products of primary fields, i.e., nonlocal fields are produced.) As is well known, the action of $Q_0^a$ obtained previously implies $Y(1)$ directly. Having obtained the Yangian comultiplication, it is enough to verify the basic relation $Y(2)$ and the Serre relation $Y(3)$ on a single field, since this will imply that they are also verify on an arbitrary number of primary fields. Hence, it suffices to evaluate the following expectation values

$$\langle [Q_i^a(Q_1^b(\phi_0)) - Q_1^b(Q_i^a(\phi_0))] (\zeta_0) \prod_{j \geq 1} \phi_j(\zeta_j) \rangle$$

with $i = 0, 1$. Let us start with $Y(2)$:

$$\langle Y^a(z; P)J^b(w) \prod_j \phi_j(\zeta_j) \rangle =$$

$$= \left[ -f^{abc} f^{dbe} \ln \left( \frac{z - w}{P - w} \right) \left\{ \frac{k^{dce}}{(z - w)^2} + \sum_k \frac{-i f^{c p e t_k^p}}{(z - w)(w - \zeta_k)} + \sum_{k, \ell} \frac{t_k^c t_\ell^e}{(z - \zeta_k)(w - \zeta_\ell)} \right\} ight. - i f^{abc} \sum_n t_n^d \ln \left( \frac{z - \zeta_n}{P - \zeta_n} \right) \left\{ \frac{k^{dcb}}{(z - w)^2} + \sum_k \frac{-i f^{c p e t_k^p}}{(z - w)(w - \zeta_k)} + \sum_{k, \ell} \frac{t_k^c t_\ell^b}{(z - \zeta_k)(w - \zeta_\ell)} \right\} 

+ ik f^{abc} \left( \frac{1}{z - w} - \frac{1}{P - w} \right) \sum_k \frac{t_k^c}{z - \zeta_k} \langle \prod_j \phi_j(\zeta_j) \rangle \right]$$

The commutator $Y(2)$ is calculated from the difference between the two integration contours (see Fig. 2):

$$\oint_{\gamma_0} dz \oint_{\zeta_0} dw - \oint_{\gamma_0} dw \oint_{\gamma_0} dz$$

In the first step we integrate $w$ around $\zeta_0$ and then integrate $z$ along $\gamma_0$. The order of integration is reversed in the second step. Since there are logarithmic cuts, $z$ must still be integrated along $\gamma_0$. The second integration contour encloses the first one: $w$ follows then a contour similar to $\gamma_0$. However, we stress that in absence of cuts (and this indeed arises for some terms which display only a pole structure), the small contour around $P$ picks up pole contributions there. After performing all integrals, the resulting commutator is found to be equivalent to the action of $Q_1^a$ on the field $\phi_0$, i.e., $Y(2)$ is satisfied.\(^5\)

\(^5\) Note that this requires the cancellation of a term proportional to $\ln(1) - i\pi$, which means that a phase choice for $\ln(-1)$ is required.
The first step of the rather long calculation leading to $Y(3)$ is to obtain

$$\langle Y^a(z; P)Y^b(w; P)\prod_j \phi_j(\zeta_j) \rangle \quad (4.7)$$

whose appropriate integrations will produce the commutator $[Q^a_1, Q^b_1]$. An immediate difficulty must be bypassed which is that the resulting correlation function is singular simply because the same point $P$ appears in the two currents. This calls for a point-splitting regularisation:

$$\langle Y^a(z; P)Y^b(w; P)\prod_j \phi_j(\zeta_j) \rangle = \lim_{P' \to P} \langle Y^a(z; P')Y^b(w; P)\prod_j \phi_j(\zeta_j) \rangle \quad (4.8)$$

Once the above correlator is evaluated, the integrals are performed along the following contours (see Fig. 1):

$$\oint_{\gamma_0} dz \oint_{\gamma_0} dw - \oint_{\gamma_0} dw \oint_{\gamma_0} dz \quad (4.9)$$

where $\gamma_P$ is defined like $\gamma_0$. Some computational simplifying features should be noticed. First, symmetrizing with respect to $P$ and $P'$ in intermediate calculations does not affect the final result and provides substantial simplifications. Second, for the present purpose the commutator $[Q^a_1, Q^b_1]$ does not need to be evaluated in extenso because not all resulting terms contribute to $Y(3)$. For instance, it is simple to see that those terms proportional to $f^{abc}F^c$ for some functions $F^c$ do not contribute and can thus be ignored. Performing all the integrals (a collection of which is displayed in the Appendix) and substituting the results into

$$if^{bcd}[Q^a_1, Q^d_1] + if^{abd}[Q^c_1, Q^d_1] + if^{cad}[Q^b_1, Q^d_1] \quad (4.10)$$

(the left hand side of $Y(3)$), we indeed recover the right hand side of $Y(3)$, including the correct normalizing factor $C$. Thus $Q^a_0$ and $Q^a_1$ have been proved to be the first two Yangian generators.

A quartic relation, $Y(4)$, exists for all algebras; however it is a consequence of $Y(2)$ and $Y(3)$ for all algebras other than $su(2)$ [12]. For $su(2)$, the relation $Y(3)$ is trivial (both sides vanish) and the quartic relation has to be verified. We have not carried out this long calculation but we believe that $Y(4)$ should be satisfied in a manner similar to $Y(3)$.

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For this calculation, there is no phase ambiguity since $\ln(-1)$ appears only squared.
5. Yangians densities as logarithmic operators

The OPE of the energy-momentum tensor and the nonlocal current is easily found to be:

$$T(w)Y^a(z; P) \sim \frac{Y^a(z; P)}{(w-z)^2} + \frac{\partial_z Y^a(z; P)}{w-z} - \frac{J^a(z)}{(w-z)^2} + \frac{J^a(z)}{(w-z)(w-P)} - \frac{G^a(z; P)}{w-P}$$  \hspace{1cm} (5.1)

where $G^a(z; P) = i f^{abc} J^b(P) J^c(z)$. Inserted in a correlation, this can be used to evaluate the commutator $[L_{-1}, Q^a_1]$ acting on the field $\phi_0(\zeta_0)$ using the contours (4.6) (see Fig. 2). The result is simply

$$\left[ \oint_{\gamma_0} dz \oint_{\zeta_0} dw - \oint_{\gamma_0} dw \oint_{\gamma_0} dz \right] \langle Y^a(z; P) T(w) \prod_j \phi_j(\zeta_j) \rangle = -2i f^{abc} \sum_{k \neq 0} \frac{t^b_j c_k}{P - \zeta_k} \langle \prod_j \phi_j(\zeta_j) \rangle$$  \hspace{1cm} (5.2)

which vanishes only when $P \to \infty$. Thus, if $Q^a_{1|\infty}$ refers to the charge associated to $Y^a(z; \infty)$ we have:

$$[L_{-1}, Q^a_1] |_{\infty} = 0$$  \hspace{1cm} (5.3)

This commutation relation has to be used carefully since it holds only in the limit $P \to \infty$ and with the limit taken after having performed the integration. The fact that the Yangian charges commute with the momentum operator $L_{-1}$ only once the limit $P \to \infty$ has been taken is the analogue of the well known fact that the quantum group symmetry of integrable lattice models only emerges once the infinite lattice limit has been taken.

When using this commutation relation, one has to remember that taking the limit $P \to \infty$ and integrating along the contour $\gamma_0$ are operations that do not commute. This noncommutativity of the limits prevents us from writing down directly the Ward identities for the charges $Q^a_{1|\infty}$. A similar phenomenon for nonlocal symmetry on finite lattices has been pointed out in [13].

If the point $P$ could be set to $\infty$, the OPE of $T$ with $Y^a(z; \infty)$ would simply be

$$T(w)Y^a(z; \infty) \sim \frac{Y^a(z; \infty)}{(w-z)^2} + \frac{\partial_z Y^a(z; \infty)}{w-z} - \frac{J^a(z)}{(w-z)^2}$$  \hspace{1cm} (5.4)

It reveals the standard Jordan cell structure observed in [14], which characterizes the presence of logarithmic operators. More precisely, it arises if at least two operators with the same dimension appear in the OPE of two other operators. Here the nonlocal operator $Y^a(z; \infty)$ would be the logarithmic partner of the operator $J^a(z)$. More generally by acting recursively on a local field of the WZW model with $Y^a(z; \infty)$ one will produce an infinite tower nonlocal fields [8, 15] which are recursively logarithmic partners and which form a Yangian representation. But once again one has to remember that the limit $P \to \infty$ does not commute with the contour integrals.
6. Conclusions

We have displayed a new Yangian symmetry in WZW models that extends to the massless regime the one already observed in massive theories with current algebras and in sigma models. It holds for any affine Lie spectrum generating algebra and any value of the level. It should be stressed that this new Yangian symmetry is rather different from the one associated to the spinon description [2,3,4,5,6,7]. Indeed, the spinon description provides a basis of the WZW model Hilbert space which diagonalizes an infinite set of commuting hamiltonians containing the boost operator $L_0$. The Yangian for which the $n$-spinon states form irreducible multiplets therefore commutes with these hamiltonians and thus with $L_0$, but not with the momentum operator $L_{-1}$. On the contrary the Yangian symmetry displayed here is more closely related to the $S$-matrix description of conformal field theory [16]. For the WZW models, the $S$-matrix is Yangian invariant. The Yangian multiplets are then formed by the $n$-asymptotic particle states which are eigenstates of an infinite set of hamiltonian containing the momentum operator $L_{-1}$. This provides an on-shell definition of the Yangian symmetry commuting with $L_{-1}$ but not with $L_0$. We have presented the off-shell construction of this Yangian symmetry. We have seen that the commutation relations with the boost and momentum operator are more subtle than for the on-shell Yangian generators. The Yangian action we described and the one associated to the spinon description are also distinguished by the way they act on product fields. The logarithmic Yangian acts by comultiplication, which is not invariant under field permutations, whereas the action of the spinon Yangian is invariant upon a permutation of the fields. However, both are compatible with the product operator expansions.

To understand better the representations associated to the action of the Yangian charges on the fields clearly remains an open problem. Since the logarithmic Yangian acts on-shell on the $n$-particle states, to introduce these representations will be necessary if one wants to preserve the particle-field duality for massless theories. It would be interesting to see whether this action could be intertwined by transfer matrices associated to inhomogeneous spin chains. Another problem is to write the Yangian Ward identities in a compact form, the difficulty being related to contour deformations in presence of logarithmic cuts. Note finally that in view of the $S$-matrices of the massive (current-current perturbations of WZW models) theories, which are products of Yangian matrices and R-matrices of the

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7 That the Yangian does not act by comultiplication on a $n$-spinon state can be seen explicitly from eq.(4.6) in [3]; it follows from the presence of the $\theta_{ij}$ factor in the action of $Q_1^a$. 

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RSOS type, the logarithmic Yangian is expected to commute with the generators of the (restricted) affine quantum group symmetries $U_q(\mathcal{G})$ which are associated to the RSOS factor [17].

Appendix A. Useful integrals and formulas

Here we give a short list of useful integrals involving logs – $f(z)$ stands for a function which is analytic everywhere on and inside the contour $\gamma$:

$$
\frac{1}{2\pi i} \oint_{\gamma \zeta} dw f(w) \ln \left( \frac{w - \zeta}{P - \zeta} \right) = \int_\zeta^P dw f(w) \\
\frac{1}{2\pi i} \oint_{\gamma \zeta} dw f(w) \ln^2 \left( \frac{w - \zeta}{P - \zeta} \right) = 2 \int_\zeta^P dw f(w) \left[ i\pi + \ln \left( \frac{w - \zeta}{P - \zeta} \right) \right] \\
\frac{1}{2\pi i} \oint_{\gamma \zeta} dw \frac{f(w)}{w - \zeta} \ln \left( \frac{w - \zeta}{P - \zeta} \right) = i\pi f(\zeta) + \int_\zeta^P dw \frac{f(w) - f(\zeta)}{w - \zeta} \tag{A.1}
$$

To this list we add the defining relation of the dilogarithm function $Li_2(x)$:

$$
Li_2(x) = -\int_0^x dx \frac{\ln(1-x)}{x} \tag{A.2}
$$

which allows us to express the remaining log integrals, i.e.,

$$
\int_\zeta^{P'} dw \frac{\ln \left( \frac{P - w}{P - \zeta} \right)}{w - \zeta} = -Li_2 \left( \frac{P' - \zeta}{P - \zeta} \right) \tag{A.3}
$$

When integrating a dilogarithm function, we need to be careful about the cuts; this function has a cut from 1 to $\infty$. To avoid the two troublesome points, we can always transform the argument by means of one of the following two relations [18]:

$$
Li_2(-1/x) + Li_2(-x) = -\frac{\pi^2}{6} - \frac{1}{2} \ln^2 x \\
Li_2(x) + Li_2(1 - x) = \frac{\pi^2}{6} - \ln x \ln(1 - x) \tag{A.4}
$$
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Fig. 1: a) The contour $\gamma_0$ consists of an open contour of vanishing radius, starting below and to the left of $P$, and of a contour encircling $\zeta_0$ (dashed line). The second contour can be deformed into the contour drawn with a solid line. b) The contour (broken line) circling around $\zeta_1$ and $\zeta_2$ is deformed into the contours drawn with solid lines.

Fig. 2: For the first term, we carry out the $w$-integration on the inner circle first and then the $z$-integration on the outer contour. For the second term the $z$-integration is carried out on the inner contour and the $w$-integration on the outer contour.