Overcharging magnetized black holes at linear order and the weak cosmic censorship conjecture

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Abstract

Evidences have been found that the weak cosmic censorship conjecture could be violated if test particles with charge and angular momentum are injected into a black hole. However, second-order corrections and fine-tunings on the particle’s parameters are required in previous studies, indicating that self-force and radiative effects must be taken into account. In this paper, we first consider a magnetically charged particle falling into an extremal Bardeen black hole, which is regular (with no singularity) and has a magnetic monopole at the center. We then investigate a general class of magnetic black holes with or without singularities. In all the cases, we show that the test particle with magnetic charge could overcharge the black hole, causing possible violation of the weak cosmic censorship conjecture. In contrast to previous arguments in the literature, second-order corrections are not necessary in our analysis and the results are not sensitive to the particle’s parameters. Our work indicates that the self-force effect, which is related to the second-order correction, may not help rescue the weak cosmic censorship conjecture in our examples.

1 Introduction

In general relativity, a singularity could form in gravitational collapse of massive stars. To avoid “naked singularities”, Penrose [1] introduced the weak cosmic censorship conjecture(WCCC) which states that singularities formed in gravitational collapse with physically reasonable matter must be covered by black hole horizons. The WCCC prevents singularities from being seen by distant observers. The general proof of WCCC is known to be extremely difficult. In a seminal work, Wald [2] proposed a gedanken experiment, showing that an extremal Kerr-Newman black hole cannot be destroyed by capturing a test particle. This work provided a strong support to WCCC. However, some possible counterexamples have been discussed in the past decades[3]-[18]. By carefully selecting the parameters of the particle, it was found that the black hole horizons could be destroyed in some cases. All the counterexamples require the second-order calculation in the particle’s charge and energy, while in the analysis by Wald, only linear terms are involved. Consequently, the allowed window of the particle’s parameters is very narrow and the self-force effect should be taken into account. Recently, Sorce and Wald [19] gave a complete analysis on the second-order corrections and found that the WCCC is still valid for Kerr-Newman black holes.

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Li an Bambi [20] discussed the possibility to destroy the horizon of a regular black hole, such as the rotating Bardeen black hole [21] and rotating Hayward black hole [22]. These black holes do not possess singularities. The source of the Bardeen black hole has been interpreted as a magnetic monopole [23]-[24], which can be derived from the nonlinear electrodynamics (NED). In [20], the test particle is neutral and then there is no electromagnetic interaction between the particle and the black hole. In this paper, we first send a test particle with magnetic charge into a Bardeen black hole to see if it can be overcharged. The first step is to derive the electromagnetic force exerted on the particle. Because the Bardeen black hole is surrounded by the nonlinear electromagnetic field, we need extend the well-known Lorentz force formula from Maxwell’s theory to NED. Starting from a general Lagrangian of nonlinear electrodynamics, we calculate the divergence of the stress-energy tensor and derive the Lorentz-like force exerted on the charged particle. Then we derive the conserved quantity along the charged particle moving in a stationary spacetime. Application of these formulas shows that the horizon of the Bardeen black hole could be destroyed by such a particle. Differing from previous literature, we do not need take higher-order terms into account and the required parameter window is not necessarily narrow. We further investigate a general class of black holes with magnetic charges. With similar methods, we show that the black hole horizons could be destroyed under certain conditions. Particularly, we prove that some magnetic black holes with singularities could be overcharged, leading to a possible violation of the WCCC. Again, no higher-order terms are necessary in our treatment. This is crucial because it implies that the self-force effect may not rescue the WCCC.

This paper is organized as follows. In section 2, we introduce the theory of NED and derive the extended Lorentz force exerted on a particle with magnetic charge. Then we derive the conserved quantity along the particle. In section 3, we send a particle with magnetic charge into the Bardeen black hole and demonstrate that the event horizon could be destroyed and the allowed window of the particle’s parameters is not necessarily narrow. In section 4, we investigate a general class of magnetized black holes with or without singularities and show that these black holes could be overcharged under certain conditions. Some concluding remarks are given in section 5.

2 Nonlinear electrodynamics in curved spacetime

In this section, we first introduce the field equations of NED in curved spacetime. Then we study the motion of a magnetically charged particle and derive the conserved quantities.

2.1 Source-free Nonlinear electrodynamics

It is well known that from the electromagnetic tensor $F_{ab}$ and its dual $*F_{ab}$, one can construct the two invariants

\[ F = F^{ab}F_{ab}, \]

\[ Y = \ast F^{ab}F_{ab}. \]  

In the absence of magnetic monopoles, $F_{ab}$ is a closed form and we can introduce the vector potential $A_a$ such that

\[ F_{ab} = \nabla_a A_b - \nabla_b A_a, \]  

which also gives

\[ \nabla_a \ast F^{ab} = 0. \]
The NED Lagrangian is a function of \( F \) and \( Y \), denoted by \( L_{NE}(F,Y) \). A general theory of sourceless nonlinear electromagnetic field in curved spacetime can be described by the action \[ S = \int d^4x \sqrt{-g} [R - L_{NE}(F,Y)], \]

where \( R \) is the scalar curvature. For later convenience, we define the following quantities

\[
\begin{align*}
    h &= \frac{\partial L_{NE}}{\partial F}, \\
    f &= \frac{\partial L_{NE}}{\partial Y}, \\
    G_{ab} &= h F_{ab}, \\
    H_{ab} &= f^* F_{ab}, \\
    L_{ab} &= G_{ab} + H_{ab}.
\end{align*}
\]

Varying the action (5) with respect to \( A_a \) and \( g_{ab} \) respectively yields the field equations

\[
\nabla_a L^{ab} = 0, \quad (11)
\]

and

\[
R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}, \quad (12)
\]

where the stress-energy tensor \( T_{ab} \) is given by

\[
T_{ab} = \frac{1}{4\pi} (L_a^c F_{bc} - \frac{1}{4} L_{NE} g_{ab}). \quad (13)
\]

### 2.2 Derivation of the generalized Lorentz force

To derive the electromagnetic force on a charged particle, we calculate

\[
\nabla^a T_{ab} = \frac{1}{4\pi} \nabla^a (L_a^c F_{bc} - \frac{1}{4} L_{NE} g_{ab})
\]

\[
= \frac{1}{4\pi} [F_{bc} \nabla_a L^{ac} + (G^{ac} + H^{ac}) \nabla_a F_{bc} - \frac{1}{4} \nabla_b L_{NE}]
\]

\[
= \frac{1}{4\pi} [F_{bc} \nabla_a L^{ac} + (h F^{ac} + f^* F_{ab}) \nabla_a F_{bc} - \frac{1}{4} (h \nabla_b F + f \nabla_b Y)]
\]

\[
= \frac{1}{4\pi} [F_{bc} \nabla_a L^{ac} + (h F^{ac} \nabla_a F_{bc} - \frac{1}{4} h \nabla_b F) + (f^* F_{ab} \nabla_a F_{bc} - \frac{1}{4} f \nabla_b Y)]
\]

\[
= \frac{1}{4\pi} F_{bc} \nabla_a L^{ac} - \frac{3}{8\pi} (h F^{ac} + f^* F^{ac}) \nabla_{[a} F_{cb]} \\
= \frac{1}{4\pi} F_{bc} \nabla_a L^{ac} - \frac{1}{4\pi} (h^* F_{bc} - f F_{bc}) \nabla_a F^{ac} \\
= \frac{1}{4\pi} F_{bc} \nabla_a L^{ac} - \frac{1}{4\pi} (G_{bc} + h F_{bc}) \nabla_a F^{ac} \\
= \frac{1}{4\pi} F_{bc} \nabla_a L^{ac} - \frac{1}{4\pi} L_{bc} \nabla_a F^{ac}. \quad (14)
\]

In the source-free case, Eq. (4) and Eq. (11) gives \( \nabla^a T_{ab} = 0 \), which is just the conservation law. However, if the particle carries electric and magnetic charge, we can define the 4-electric and 4-magnetic current densities by

\[
\begin{align*}
    -4\pi j^e &= \nabla_a L^{ac}, \\
    -4\pi j^m &= \nabla_a F^{ac}.
\end{align*}
\]

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Then (14) becomes
\[ \nabla a T_{ab} = - F_{bc} J^c + \ast L_{bc} J^c. \] (17)
According to Eqs. (8)-(10),(15) and (16), we have
\[ \nabla a J^a = 0, \nabla a \hat{J}^a = 0, \] (18)
which confirms the conservation of these currents.

For a point particle carrying electric charge \( q \) and magnetic charge \( g \), by integrating Eq. (17) over the worldline, one can obtain
\[ F_a = q F_{ab} U^b - g \ast L_{ab} U^b, \] (19)
where \( U^a \) is the four-velocity of the particle, i.e., the unit vector tangent to the worldline. We call this extended Lorentz force. If magnetic charges or monopoles are taken into account in Maxwell’s theory, the Lorentz force takes exactly the same form as Eq. (19) [26]. Authors [25] derived the force on a particle with electric charge (but without magnetic charge) in NED, which is also consistent with Eq. (19).

### 2.3 Conserved quantity of electric and magnetic charges

In rest of this paper, we only consider nonlinear electromagnetic field with \( Y = 0 \). So
\[ L_{ab} = G_{ab}. \] (20)
In the sourceless case, Eq. (11) indicates that \( \ast G_{ab} \) is closed and then we can introduce the potential \( \hat{A}_a \) such that
\[ \ast G_{ab} = \nabla a \hat{A}_b - \nabla b \hat{A}_a. \] (21)
According to Eq. (19), the equation of motion of a charged particle is determined by
\[ m U^b \nabla b (m \xi^a U_a) = q F_{ab} U^b - g \ast L_{ab} U^b. \] (22)
If the spacetime possesses a Killing vector field \( \xi^a \), we can find a conserved quantity along the worldline. For this purpose, we calculate
\[ U^b \nabla_b (m \xi^a U_a) = m \xi^a U^b \nabla_b U_a = q U^b [\xi^a \nabla_a A_b - \nabla_b (\xi^a A_a) + A_a \nabla_b (\xi^a) - g U^b [\xi^a \nabla_a \hat{A}_b - \nabla_b (\xi^a \hat{A}_a) + \hat{A}_a \nabla_b (\xi^a)], \] (23)
where in the first step, we have used the antisymmetry of \( \nabla_a \xi_b \) and in the second step, Eqs. (3), (21) and (22) have been used.

By rearranging Eq. (23), we find
\[ U^b \nabla_b (m \xi^a U_a + q \xi^a A_a - g \xi^a \hat{A}_a) = q U^b (\xi^a \nabla_a A_b + A_a \nabla_b \xi^a) - g U^b (\xi^a \nabla_a \hat{A}_b + \hat{A}_a \nabla_b \xi^a) = q U^b (\xi^a \nabla_a A_b - A_a \nabla^a \xi^b) - g U^b (\xi^a \nabla_a \hat{A}_b - \hat{A}_a \nabla^a \xi^b) = q U^b (L \xi^a A^b) - g U^b (L \xi^a \hat{A}^b) = 0. \] (24)
Therefore, the conserved quantity associated with \( \xi^a \) is
\[ P = m \xi^a U_a + q \xi^a A_a - g \xi^a \hat{A}_a. \] (25)
This quantity will play an important role in the following calculation.
3 Destroying the event horizon of a Bardeen black hole

In this section, we shall check whether a magnetically charged particle can destroy the event horizon of a Bardeen black hole.

The Bardeen model can be described by the line element

\[ ds^2 = -\left[1 - \frac{2M r^2}{(r^2 + Q^2)^{3/2}}\right] dt^2 + \left[1 - \frac{2M r^2}{(r^2 + Q^2)^{3/2}}\right]^{-1} dr^2 + r^2 d\Omega^2. \tag{26} \]

According to [24], the solution is generated from a nonlinear electrodynamic field with Lagrangian

\[ L_{NE}(F) = \frac{12M}{|Q|^3} \left( \frac{\sqrt{Q^2 F}}{\sqrt{2} + \sqrt{Q^2 F}} \right)^5, \tag{27} \]

where

\[ F_{ab} = \hat{Q}\sin\theta (d\theta_a d\phi_b - d\theta_b d\phi_a). \tag{28} \]

By virtue of Eqs. (15) and (16), we can obtain the total electric charge and magnetic charge of the black hole

\[ Q_e = \frac{1}{4\pi} \int_S *G_{ab}, \tag{29} \]

\[ Q_m = \frac{1}{4\pi} \int_S F_{ab}, \tag{30} \]

where S is any two-sphere surrounding the black hole.

By employing Eqs. (29) and (30), we see that the Bardeen black hole possesses no electric charge and \( \hat{Q} \) is just its magnetic charge. So the source of the Bardeen solution is a magnetic monopole located at the center \( r = 0 \) [23, 24]. The black hole is regular because it has no singularity.

By definition, one can calculate

\[ *G_{ab} = h^* F_{ab} = \frac{15\hat{Q}Mr^4}{2(Q^2 + r^2)^{7/2}} (dt_a dr_b - dr_a dt_b), \tag{31} \]

and the corresponding potential

\[ \hat{A}_a = -\frac{3M}{2Q} \left( \frac{r^5}{(Q^2 + r^2)^{3/2}} - 1 \right) (dt)_a, \tag{32} \]

where the gauge has been fixed by the condition \( \hat{A}_a \to 0 \) at infinity.

From Eq. (26), we see that the horizon of the black hole can be found by solving

\[ 1 - \frac{2M r^2}{(r^2 + Q^2)^{3/2}} = 0. \tag{33} \]

This is a cubic equation of \( r^2 \). For

\[ \hat{Q} < \frac{16}{27} M^2, \tag{34} \]

the black hole possesses two horizons.

For

\[ \hat{Q} = \frac{16}{27} M^2, \tag{35} \]
there is only one horizon and the black hole is called \textit{extremal}.

For

\[ \dot{Q}^2 > \frac{16}{27} M^2, \]  

(36)

there is no horizon.

Consider a particle with mass \( m \) and magnetic charge \( g \) moving outside the extremal Bardeen black hole. If the particle can enter the black hole and the parameters of the resulting spacetime satisfy Eq. (36), the black hole is called \textit{overcharged} and its horizon could disappear. The conserved quantity associated with the Killing vector field \( \left( \frac{\partial}{\partial t} \right)^a \) is the energy, denoted by \( E \). By considering the radial motion, Eq. (25) yields

\[
E = - \left( mg_{\mu} \frac{dt}{d\tau} + qA_t - g\dot{A}_t \right) = - \frac{3Mg}{2Q} \left[ \frac{r^5}{(Q^2 + r^2)^{3/2}} - 1 \right] + m \frac{dt}{d\tau} \left[ 1 - \frac{2Mr^2}{(Q^2 + r^2)^{3/2}} \right].
\]  

(37)

The four-velocity takes the form

\[
U^a = \left( \frac{\partial}{\partial \tau} \right)^a i \left( \frac{\partial}{\partial t} \right)^a + \dot{r} \left( \frac{\partial}{\partial r} \right)^a,
\]  

(38)

where \( \tau \) is the proper time of the particle and the dot denotes derivative with respect to \( \tau \). Thus, the normalization condition is written as

\[
-1 = g_{ab} U^a U^b = - \left[ 1 - \frac{2Mr^2}{(r^2 + Q^2)^{3/2}} \right] \dot{t}^2 + \left[ 1 - \frac{2Mr^2}{(r^2 + Q^2)^{3/2}} \right]^{-1} \dot{r}^2,
\]  

(39)

which gives

\[
\dot{t} = \pm \left[ 1 - \frac{2Mr^2}{(r^2 + Q^2)^{3/2}} \right]^{-1/2} \sqrt{\dot{r}^2 + \left[ 1 - \frac{2Mr^2}{(r^2 + Q^2)^{3/2}} \right]}. \]  

(40)

Using the fact that \( U^a \) is future-directed, one can show that \( \dot{t} > 0 \). So we should take the positive sign in Eq. (40). Substituting Eq. (40) into Eq. (37), we have

\[
E = \frac{3gM}{2Q} \left[ 1 - \frac{r^5}{(Q^2 + r^2)^{3/2}} \right] + m \sqrt{\dot{r}^2 + \left[ 1 - \frac{2Mr^2}{(Q^2 + r^2)^{3/2}} \right]}. \]  

(41)

We shall be interested in the extremal case, where the charge and mass are related by Eq. (35). It is easy to find that the horizon of the black hole is located at

\[
r_h = \sqrt{\frac{32}{27} M}.
\]  

(42)

Substitution of Eqs. (35) and (42) into Eq. (41) yields

\[
E \geq \frac{3gM}{2Q} \left[ 1 - \frac{r^5}{(Q^2 + r_h^2)^{3/2}} \right] = \frac{9\sqrt{3} - 4\sqrt{2}}{8} \frac{\dot{Q}}{|\dot{Q}|} g.
\]  

(43)

This just gives the minimum energy that allows a particle to reach the horizon of the black hole.
On the other hand, Eq. (36) suggest that

$$ (Q + g)^2 > \frac{16}{27} (M + E)^2 \tag{44} $$

must hold if the particle can destroy the horizon. Since $g \ll \hat{Q}$ and $E \ll M$, by keeping the linear terms, we have

$$ \frac{\hat{Q}}{|\hat{Q}|} g > \frac{4}{\sqrt{27}} E. \tag{45} $$

Without loss of generality, we assume $\hat{Q} > 0$ and $g > 0$. Thus, Combining Eqs. (43) and (45), we obtain

$$ \frac{8}{9\sqrt{3} - 4\sqrt{2}} > \frac{g}{E} > \frac{4}{\sqrt{27}}, \tag{46} $$

or

$$ 0.806 > \frac{g}{E} > 0.770. \tag{47} $$

If the parameters of the particle fall into this interval, the particle could enter the horizon and destroy the horizon. Note that to obtain the above result, we only took the terms linear to $g$ and $E$. This is a crucial difference from previous examples.

Next, we check whether a particle released from infinity can overcharge the blackhole. For this purpose, we define the effective potential

$$ V_{eff} = -r^2. \tag{48} $$

By solving Eq. (41), we can write the effective potential of the extremal black hole as

$$ V_{eff} = 1 - \frac{3\sqrt{3}x^2}{2(1 + x^2)} - \frac{1}{m^2} \left\{ E - \frac{9\sqrt{3}g}{8} \left[ 1 - \left( \frac{1}{x^2} + 1 \right)^{-\frac{3}{2}} \right] \right\}^2, \tag{49} $$

where $x = \frac{\hat{Q}}{Q}$. Suppose that the particle stays still at infinity and then is released, which means $E = m$. Now Eq. (49) reduces to

$$ V_{eff} = 1 - \frac{3\sqrt{3}x^2}{2(1 + x^2)} - \left\{ 1 - \frac{9\sqrt{3}g}{8m} \left[ 1 - \left( \frac{1}{x^2} + 1 \right)^{-\frac{3}{2}} \right] \right\}^2. \tag{50} $$

Taking into account Eq. (46), it is easy to check $V_{eff}$ in Eq. (50) increases with $g/m$. So by substituting the upper bound in Eq. (46) into the right-hand side of Eq. (50), we have

$$ V_{eff} \leq 1 - \frac{3\sqrt{3}x^2}{2(1 + x^2)^{3/2}} - \left\{ 1 - \frac{27[1 - x^5(1 + x^2)^{-5/2}]}{27 - 4\sqrt{6}} \right\}^2. \tag{51} $$

One can check that the right-hand side of (51) is always negative outside the horizon ($x > \sqrt{2}$). This means that the particle released from infinity can go all the way to the horizon and then overcharge the Bardeen black hole.

4 General magnetized black holes and the WCCC

In this section, we will demonstrate the event horizons of a general class of magnetized black holes proposed by [27] can be destroyed by test particles. Particularly, some of them contain singularities, leading to possible violation of the WCCC.
4.1 Magnetized black holes

A static and spherically symmetric metric is described by the line element

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2. \]  

(52)

Introduce the mass function \( m(r) \) by

\[ f(r) = 1 - \frac{2m(r)}{r}. \]  

(53)

The black hole horizon is located at \( r = r_h \) satisfying

\[ r_h = 2m(r_h). \]  

(54)

If the black hole is coupled to nonlinear electrodynamics, we can substitute Eqs. (13) and (20) into Einstein’s equation (12) and find the nonzero components of \( F_{ab} \):

\[ F_{tr} = -F_{rt} = \pm \sqrt{\frac{1}{2\hbar} \left[ \frac{2m'(r)}{r^2} - \frac{1}{2} L_{NE}(F) \right]}, \]  

(55)

\[ F_{\theta\phi} = -F_{\phi\theta} = \pm r^2 \sin \theta \sqrt{\frac{1}{2\hbar} \left( \frac{1}{2} L_{NE}(F) - \frac{m''(r)}{r} \right)}. \]  

(56)

By performing the integrations in Eqs. (29) and (30) over a two-sphere \( S \), we obtain the electric charge and magnetic charge of the black hole

\[ Q = \frac{1}{8\pi} \int_S c^{cd} \epsilon_{cdab} = -\frac{r^2}{4\hbar} \int_S hF_{tr} \sin \theta d\theta d\phi = -r^2 hF_{tr}, \]  

(57)

\[ \dot{Q} = \frac{1}{4\pi} \int_S F_{\theta\phi} d\theta d\phi = \frac{r^2 \sin \theta}{|\dot{F}_{\phi\phi}|}. \]  

(58)

Therefore,

\[ F_{ab} = -\frac{Q}{r^2 h}(dt_a dr_b - dr_a dt_b) + \dot{Q} \sin \theta (d\theta_a d\phi_b - d\theta_b d\phi_a), \]  

(59)

and

\[ ^*G_{ab} = \frac{h\dot{Q}}{r^2}(dt_a dr_b - dr_a dt_b) + Q \sin \theta (d\theta_a d\phi_b - d\theta_b d\phi_a). \]  

(60)

Combining (3)(21) and (59)(60), we get the potentials

\[ A_a = Q \int \frac{dr}{r^2 h'(F)} (dt)_a - \dot{Q} \cos \theta (d\phi)_a, \]  

(61)

\[ \dot{A}_a = -\dot{Q} \int \frac{h dr}{r^2} (dt)_a - Q \cos \theta (d\phi)_a. \]  

(62)

From now on, we only consider magnetized black holes, i.e.,

\[ Q = 0, \]  

(63)

and

\[ F_{tr} = 0. \]  

(64)
Thus,

\[ F = 2F_{\theta\phi}F^{\theta\phi} = \frac{\hat{Q}^2}{r^4}. \]  

(65)

Combining Eqs. (55) and (64), we have

\[ L_{NE}(F) = 4 m'(r) \frac{r}{r^2}. \]  

(66)

Substituting Eqs. (63)(65)(66) into (61)(62)

\[ A_a = -\hat{Q}\cos\theta (d\phi)_a, \]  

(67)

\[ \dot{A}_a = -\hat{Q} \int \frac{1}{r^2} \frac{dL_{NE}}{dF} dr (dt)_a 
= -\hat{Q} \int \frac{1}{r^2} \frac{dr}{dF} dL_{NE} (dt)_a 
= \frac{1}{8Q} \int r^3 dL_{NE} (dt)_a 
= \frac{1}{2Q} \int (rm'' - 2m') dr (dt)_a 
= \frac{1}{2Q} [-3m(r) + rm'(r) + 3m(r)|_{r=\infty}] (dt)_a. \]  

(68)

In the last step, we have assumed that \( rm'(r) \) vanishes at infinity and the integration constant has been chosen such that \( \dot{A}_a = 0 \) as \( r \to \infty \).

4.2 Energy of a test particle

We only consider radial movement of a test particle with mass \( m \) and magnetic charge \( g \). Substituting Eq. (67) and (68) into (25) and taking into account Eq. (52), we obtain the conserved energy

\[ E = -\left( mg \dot{t} - g \dot{A}_t \right) 
= mf(r) \dot{t} + \frac{g}{2Q} [-3m(r) + rm'(r) + 3m(r)|_{r=\infty}]. \]  

(69)

Again, the normalization condition yields

\[ -1 = g_{ab} U^a U^b = -f(r) \dot{r}^2 + f(r)^{-1} \dot{\tau}^2. \]  

(70)

Combining (69) and (70), we have

\[ \dot{r}^2 = \frac{1}{m^2} \left\{ E - \frac{g}{2Q} [-3m(r) + rm'(r) + 3m(r)|_{r=\infty}] \right\}^2 - 1 + \frac{2m(r)}{r}. \]  

(71)

By solving Eq. (71), we have

\[ E = \frac{g}{2Q} [-3m(r) + rm'(r) + 3m(r)|_{r=\infty}] + m \sqrt{\left( \frac{dr}{d\tau} \right)^2 + \left[ 1 - \frac{2m(r)}{r} \right]}. \]  

(72)

At the horizon \( r = r_h \), an extremal black hole satisfies

\[ f(r_h) = 0, \]  

(73)

\[ f'(r_h) = 0. \]  

(74)
Therefore, Eq. (53) gives

\[2m(r_h) = r_h, \quad (75)\]
\[m'(r_h) = \frac{1}{2}. \quad (76)\]

Substituting Eq. (75) and (76) into Eq. (72), we find the constraint for the particle to reach the horizon:

\[E \geq \frac{g}{2Q} [-r_h + 3m(r)|_{r=\infty}]. \quad (77)\]

### 4.3 Overcharging some general magnetized black holes

A generic class of magnetized black hole solutions have been proposed in \[27\]-[28] by taking the mass function in the form

\[m(r) = \frac{Mr^\mu}{(r^\nu + \hat{Q}^\nu)^{\frac{\mu}{\nu}}}. \quad (78)\]

The corresponding Lagrangian is

\[L_{NE}(F) = \frac{4\mu}{\alpha} \frac{(\alpha F)^{\frac{\nu+2}{4}}}{[1 + (\alpha F)^{\nu/4}]^{\frac{\nu}{2}}} \cdot (79)\]

Here \(\mu > 0\) and \(\nu > 0\) are dimensionless constants and the value of \(\nu\) characterizes the strength of the nonlinear electromagnetic field in the weak field limit. Eq. (78) covers several well-known black hole solutions. For example, \(\nu = 1\), \(\nu = 2\) and \(\nu = 3\) correspond to the Maxwellian solution in the weak field regime, Bardeen-like solutions, and Hayward-like solutions, respectively. For \(\mu \geq 3\), the black hole is regular. For \(0 < \mu < 3\), the black hole possesses a singularity. More properties have been studied in \[27\],\[29\]-[30].

One can show that \(M\) is the gravitational mass and \(\hat{Q}\) is the magnetic charge \[28, 30\]. We still consider an extremal black hole and assume \(\mu > 1\). From Eqs. (75),(76) and (78), it is not difficult to find

\[\hat{Q}^\nu = (2M)^\nu \frac{(\mu - 1)^{\mu - 1}}{\mu}, \quad (80)\]
\[r_h = 2M \left( 1 - \frac{1}{\mu} \right)^{\frac{\nu}{\mu}}, \quad (81)\]
\[M = m(r)|_{r=\infty}. \quad (82)\]

Without loss of generality, we assume \(\hat{Q} > 0\). Consider a test particle with energy \(E\) and magnetic charge \(g > 0\). Substituting (80)(82) into (77), we find the condition for the particle to reach the horizon:

\[E > \frac{g}{4} \left[ \frac{\mu^\mu}{(\mu - 1)^{\mu - 1}} \right]^{\frac{1}{\nu}} \left[ 3 - 2 \left( 1 - \frac{1}{\mu} \right)^{\frac{\nu}{\mu}} \right]. \quad (83)\]

On the other hand, in order to overcharge the black hole, the following inequality must be satisfied

\[(\hat{Q} + g)^\nu > (2M + 2E)^\nu \frac{(\mu - 1)^{\mu - 1}}{\mu^\mu}. \quad (84)\]

where Eq. (80) has been used. Since \(g \ll \hat{Q}\) and \(E \ll M\), by taking the linear orders, Eq. (84) becomes

\[E < \frac{g}{2} \left[ \frac{\mu^\mu}{(\mu - 1)^{\mu - 1}} \right]^{\frac{1}{\nu}}. \quad (85)\]
Putting Eq. (83) and (85) together, we obtain
\[ \frac{g}{4} \left[ \frac{\mu^2}{(\mu - 1)^{\mu - 1}} \right] ^{\frac{1}{\nu}} \left[ 3 - 2 \left( 1 - \frac{1}{\mu} \right) ^{\frac{1}{\nu}} \right] < E < \frac{g}{2} \left[ \frac{\mu^2}{(\mu - 1)^{\mu - 1}} \right] ^{\frac{1}{\nu}} . \] (86)

The existence of \( E \) in Eq. (86) requires
\[ 2^{\nu} > \left( \frac{\mu}{\mu - 1} \right)^{\mu} . \] (87)

As shown in Fig. 1, Eq. (87) is satisfied in the region above the solid line.

![Figure 1: The plot of \( \mu - \nu \). Above the solid line, the horizon could be destroyed.](image)

For \( \nu = 1 \) and in the weak field limit \( F \ll 1 \), the solution reduces to the Maxwellian solution, which is obviously below the solid line. This is consistent with the previous conclusion that black holes associated with Maxwell’s theory cannot be overcharged by test particles.

When \( \mu = 3 \) and \( \nu = 2 \), the solution is just the Bardeen black hole, as we have discussed.

When \( \mu = 3 \) and \( \nu = 3 \), we recover the Hayward black hole [22], which is also regular. Eq. (86) becomes
\[ \frac{5}{4 \sqrt{4}} g < E < \frac{3}{2 \sqrt{4}} g , \] (88)

or
\[ 0.79g < E < 0.94g \] (89)

It is important to consider the case \( \mu = 2 \) and \( \nu = 3 \) because the solution describes a black hole with singularity[27]. It follows from Eq. (86) that
\[ 0.69g \leq E < 0.79g . \] (90)

Therefore, the black hole could be destroyed and the singularity becomes naked. One can check that this spacetime satisfies the weak energy condition. Furthermore, from Eq. (90) we calculate
\[ \frac{\Delta E}{E} = \frac{0.79g - 0.69g}{(0.79g + 0.69g)/2} = 0.14 . \] (91)

This value is much larger than that in other cases. For example, the corresponding value for an extremal Kerr-Newman black hole is \( \frac{\Delta E}{E} \approx 10^{-3} \) [13]. This is because higher order terms considered in previous literature are much smaller than the linear order terms in our analysis.
Similar to the treatment at the end of section 3, we find that the effective potential satisfies

$$V_{\text{eff}} < -\left(1 - \left[3 - 2 \left(1 - \frac{1}{\mu}\right)^{\frac{3}{2}}\right]^{-1} \left[3 + \frac{x^\mu(-3x^\nu + \mu - 3)}{(1 + x^\nu)^{1+\mu/\nu}}\right]\right) + 1 - \left[\frac{\mu^\mu}{(\mu - 1)^{\mu - 1}}\right]^{\frac{1}{\nu}} \frac{x^{\mu - 1}}{(x^\nu + 1)^{\mu/\nu}} ,$$

(92)

where \(x = r/\hat{Q}\). For \(\mu = 2\) and \(\nu = 3\), we have

$$V_{\text{eff}} < 1 - \frac{\sqrt{4x}}{(1 + x^3)^{2/3}} - \left\{ -1 + \frac{1}{3 - \sqrt{2}}[3 - x^2(1 + 3x^3)(1 + x^3)^{-5/3}] \right\} .$$

(93)

It’s easy to check that this formula is always negative outside the horizon \((x > 1)\). So the test particle can be released from infinity and overcharge this extremal black hole.

5 Conclusions

In this paper, we first derived the Lorentz-like force on a magnetically charged particle in the NED theory. We then show that such a particle could enter the Bardeen black hole and make its horizon disappear. Furthermore, we explored a general class of magnetically charged black holes. In particular, we have shown that a particle with magnetic charge could overcharge a black hole with singularity, leading to a possible violation of the WCCC. In contrast to previous gedanken experiments, our results do not require higher order terms and fine-tunings on the particle’s parameters. This indicates that the second-order effects, like the self-force, may not rescue the WCCC in this case. Although no experimental evidence for the existence of magnetic charges or monopoles has been found yet, our work could shed some new light on the research of WCCC.

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