On an easy transition from operator dynamics to generating functionals by Clifford algebras

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Abstract

Clifford geometric algebras of multivectors are treated in detail. These algebras are build over a graded space and exhibit a grading or multivector structure. The careful study of the endomorphisms of this space makes it clear, that opposite Clifford algebras have to be used also. Based on this mathematics, we give a fully Clifford algebraic account on generating functionals, which is thereby geometric. The field operators are shown to be Clifford and opposite Clifford maps. This picture relying on geometry does not need positivity in principle. Furthermore, we propose a transition from operator dynamics to corresponding generating functionals, which is based on the algebraic techniques. As a calculational benefit, this transition is considerable short compared to standard ones. The transition is not injective (unique) and depends additionally on the choice of an ordering. We obtain a direct and constructive connection between orderings and the explicit form of the functional Hamiltonian. These orderings depend on the propagator of the theory and thus on the ground state. This is invisible in path integral formulations. The method is demonstrated within two examples, a non-linear spinor field theory and spinor QED. Antisymmetrized and normal-ordered functional equations are derived in both cases.

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1 Introduction

Modern quantum field theory is treated currently by means of path integrals [1, 2]. Path integrals provide formal solutions of functional differential equations. Such functional differential equations encode the matrix element hierarchies of Dyson–Schwinger–Freese [3, 4, 5, 6]. Generating functionals describe in a condensed form an infinite set of vacuum or transition matrix elements with a single mathematical entity. They prove to be useful in manipulations of the whole hierarchy.

Since path integrals are very compact in their formulation, one does not actually have an access to their meaning. Moreover, one is troubled with convergence...
problems, which can be treated up to now only by heuristic methods or in trivial cases, see chapter 6 of \[2\].

It seems therefore to be a good idea, to study generating functionals and functional differential equations at their own right. Furthermore, it is known from ordinary differential equations, that one can achieve informations about the solutions without being able to integrate the differential equations. We expect to be able to support the usage of path integrals, at least their algebraic properties, in studying operator dynamics with help of functional differential equations.

The paper is organized in three logical parts. First, in section 2, we establish the mathematical apparatus necessary to treat generating functionals algebraically correct. This will be done in the case of antisymmetry (fermions), but might be equally well established for symmetric algebras (bosons), see \[7\]. This part is mathematically rigorous.

In the second part, consisting of sections 3 and 4, the infinite dimensional case of QFT is treated formally, since the transition to infinite many generators causes serious convergence problems. We give an algebraic account on generating functionals based on the Clifford algebra of multivectors. Field operators are shown to be Clifford endomorphisms of the graded space underlying the Grassmann algebra build over the Schwinger sources. This opens the first time an algebraically motivated approach to quantum field theoretic functionals. In section 4 we use the Heisenberg equation to give an very simple transition from operator dynamics into the space of generating functionals. Once more, this can be achieved only if one admits the Clifford geometric point of view. A single operator dynamics might result in different functional equations. This peculiar observation is an essential insight to understand the computational power of functional differential equations and path integrals. The different functional equations obtained are subjected to different orderings and ultimately to different vacua. The correct choice of this ordering by means of normal-ordering with help of a specific propagator selects a unique vacuum encoded implicitly in the theory, as was shown in \[8\]. The Clifford geometric approach given in this paper makes it possible to investigate this connections and to have a constructive tool to implement inequivalent vacua in QFT.

The third part, sections 5 and 6 gives examples of the method. A non-linear spinor field theory, e.g. a Hubbard or Nambu–Jona-Lasinio type model, and spinor QED as an example for a boson-fermion coupling theory are treated. Both examples are dealt with in the antisymmetric and normal-ordered case. Explicit calculations of the functional equations shows furthermore the efficiency of our method. The usually necessary vertex-regularization is no longer apparent, since the algebraic treatment accounts for this transition correctly. This was know earlier on the operator level \[9\]. It is novel, that one can achieve the normal-ordered functional equation in one single step.

The conclusion summarizes the results and discusses the relevance to path integral calculations.

2 Clifford geometric algebra of multivectors

There are many possibilities to introduce Clifford algebras, each of them emphasize a different point of view. In our case, it is of utmost importance to have the Clifford algebra build over a graded linear space. This grading is obtained from the space underlying a Grassmann algebra. The Clifford algebra is then related to the endomorphism algebra of this space. This construction, the Chevalley deformation \[10\], was originally invented to be able to treat Clifford algebras over fields of char = 2; see appendix of \[11\] by Lounesto and \[12\]. However, we use this construction in an entire different context. With help of the construction of M. Riesz \[13\], one is
able to reconstruct the multivector structure and thereby a correspondence between the linear spaces underlying the Clifford algebra and the Grassmann algebra in use.

This reconstruction depends on an automorphism \( J \), which is arbitrary, see [13]. In fact this is just the reversed direction of our construction following Chevalley given below.

Let \( T(V) \) be the tensor algebra build over the \( \mathbb{K} \)-linear space \( V \). The field \( \mathbb{K} \) will be either \( \mathbb{R} \) or \( \mathbb{C} \). With \( V^0 \simeq \mathbb{K} \) we have

\[
T(V) = \mathbb{K} \oplus V \oplus V \otimes \mathbb{K} \oplus \ldots.
\]

The tensor algebra is associative and unital. In \( T(V) \) one has bilateral or two-sided ideals, which can be used to construct new algebras by factorization. As an example we define the Grassmann algebra in this way.

**Definition 1** The Grassmann algebra \( \bigwedge(V) \) is the factor algebra of the tensor algebra \( T(V) \) w.r.t. the bilateral ideal

\[
I_{Gr} = \{ y \mid y = a \otimes x \otimes x \otimes b, \quad a, b \in T(V), \ x \in V \}.
\]

\[
\bigwedge(V) = \pi(T(V)) = \frac{T(V)}{I_{Gr}} = \mathbb{K} \oplus V \oplus \bigwedge V \oplus \ldots.
\]

The canonical projection \( \pi : T(V) \mapsto \bigwedge(V) \) maps the tensor product \( \otimes \) onto the exterior or wedge product denoted by \( \wedge \).

One may note, that the factorization preserves the grading naturally inherited by the tensor algebra, since the ideal \( I_{Gr} \) is homogeneous. Defining homogeneous parts of \( T(V) \) and \( \bigwedge(V) \) by \( T^k(V) = V \otimes \ldots \otimes V \) and \( \bigwedge^k(V) = V \wedge \ldots \wedge V \) \( k \)-factors, we obtain \( \pi(T^k(V)) = \bigwedge^k(V) \).

Proceeding to Clifford algebras requires a further structure, the quadratic form.

**Definition 2** The map \( Q : V \mapsto \mathbb{K} \), satisfying

\[
i) \quad Q(\alpha x) = \alpha^2 Q(x), \quad \alpha \in \mathbb{K}, \ x \in V
\]

\[
ii) \quad B_p(x,y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)),
\]

where \( B_p(x,y) \) is a symmetric bilinear form is called a quadratic form.

It is tempting to introduce an ideal \( I_{CX} \)

\[
I_{CX} = \{ y \mid y = a \otimes (x \otimes x - Q(x)) \otimes b, \quad a, b \in T(V), \ x \in V \}
\]

to obtain the Clifford algebra by a factorization procedure. However, since we are interested in arbitrary bilinear forms underlying a Clifford algebra, we will take another approach, which is wide enough for such a structure. Furthermore, the Clifford algebra does not have an intrinsic multivector structure, but is only \( \mathbb{Z}_2 \)-graded, since the ideal \( I_{CX} \) is inhomogeneous but \( \mathbb{Z}_2 \)-graded.

Let \( V^* \) be the space of linear forms on \( V \), i.e. \( V^* \simeq \text{lin}[V, \mathbb{K}] \). Elements \( \omega \in V^* \) act on elements \( x \in V \), but there is no natural connection between \( V \) and \( V^* \). However, we can find a set of \( x_i \) which span \( V \) and dual elements \( \omega_k \) acting on the \( x_i \) in a canonical way

\[
\omega_k(x_i) = \delta_{ki}.
\]

This allows to introduce a map \( * : V \mapsto V^* \), \( x^*_i = \omega_i \) which may be called Euclidean dual isomorphism \([14]\). The pair \( (V^*, V) \) is connected by this duality which constitutes a pairing \( < \cdot | \cdot > : V^* \times V \mapsto \mathbb{K} \). Since \( V^* \) is isomorphic to \( V \) in finite
dimensions, it is natural to build a Grassmann algebra $\Lambda(V^*)$ over it. This is the algebra of Grassmann multiforms.

It is further a natural thing to extend the pairing of the grade-one space and its dual to the whole algebras $\Lambda(V)$ and $\Lambda(V^*)$, as can be seen by its frequent occurrence in literature \[12, 13, 15, 16, 17, 18\]. This can be done by the

**Definition 3** Let $\tau, \eta \in \Lambda(V^*)$, $\omega \in V^*$, $u, v \in \Lambda(V)$ and $x \in V$, then we can define a canonical action of $\Lambda(V^*)$ on $\Lambda(V)$ requiring

\[
\begin{align*}
\text{i) } & \quad \omega(x) = \langle \omega | x \rangle \\
\text{ii) } & \quad \omega(u \wedge v) = u \omega(v) + \hat{u} \wedge \omega(v) \\
\text{iii) } & \quad (\tau \wedge \eta)(u) = \tau(\eta(u))
\end{align*}
\]

where $\hat{u}$ is the main involution $\hat{V} = -V$ extended to $\Lambda(V)$.

In fact we have given by definition 3 an isomorphism between the Grassmann algebra of multiforms $\Lambda(V^*)$ and the dual Grassmann algebra $[\Lambda(V)]^*$. This can be made much clearer in writing

\[
y_\cdot x = \omega_y(x) = \langle \omega_y | y \rangle = B(y, x),
\]

where we have used the canonical identification of $V$ and $V^*$ via the Euclidean dual isomorphism. One should be very careful in the distinction of $\Lambda(V^*)$ and $[\Lambda(V)]^*$, since they are isomorphic but not equivalent. Furthermore, we emphasize that in writing $y_\cdot$ we make explicitly use of a special dual isomorphism encoded in the contraction

\[
y_\cdot : V \mapsto V^* \\
y \mapsto y_\cdot = \omega_y.
\]

Since there is no natural, that is mathematically motivated or even better functorial relation between $V$ and $V^*$, we are called to seek for physically motivated reasons to select a dual isomorphism.

**Theorem 4** Let $(V, Q)$ be a pair of a $K$-linear space $V$ and $Q$ a quadratic form as in definition 2. There exists an injection $\gamma$ called Clifford map from $V$ into the associative unital algebra $C\ell(V, Q)$ which satisfies

\[
\gamma_x \gamma_x = Q(x) I.
\]

**Definition 5** The (smallest) algebra $C\ell(V, Q)$ generated by $I$ and $\gamma_x$ is called (the) Clifford algebra of $Q$ over $V$.

By polarization of this relation we get the usual commutation relations; $x, y \in V$

\[
\gamma_x \gamma_y + \gamma_y \gamma_x = 2B_p(x, y),
\]

where $B_p(x, y)$ is the symmetric polar form of $Q$ as defined in (6).

**Remarks:** i) We could have obtained this result directly factoring the tensor algebra with the ideal $I$. ii) There exists Clifford algebras which are universal, in this case it is convenient to speak from the Clifford algebra over $(V, Q)$. iii) If $V \simeq K^n \simeq C^n$ or $R^n$, we denote $CL(V, Q)$ also by $CL(C^n) \simeq C\ell_n$ or $C\ell(R_p,q)$ where the pair $p, q$ enumerate the number of positive and negative eigenvalues of $Q$. We can as well give the dimension $n$ and signature $s = p - q$ to classify all quadratic
forms over $\mathbb{R}$. In the case of the complex field, one remains with the dimension as can be seen e.g. from the Weyl unitary trick, letting $x_i \to ix_i$ which flips the sign. We do not use sesquilinear forms here, which could be included nevertheless.

We will now use Chevalley-deformations to construct the Clifford algebra of multivectors. The main idea is, that we can decompose the Clifford map as

$$\gamma_x = x \downarrow + x \wedge.$$  

We have thus a natural action of $\gamma_x$ on the space $F \wedge (V)$.

**Theorem 6 (Chevalley)** Let $F \wedge (V)$ be the space underlying the Grassmann algebra over $V$ and $\gamma_x : V \mapsto \operatorname{End}(F \wedge (V))$, $x \in V$ be defined as in (11), then $\gamma_x$ is a Clifford map.

We have shown that $\mathcal{C}$ is a subalgebra of the endomorphism algebra of $F \wedge (V)$,

$$\mathcal{C} \subseteq \operatorname{End}(F \wedge (V)).$$  

It is possible to interpret $x \downarrow$ and $x \wedge$ as annihilating and creation operators (on the space underlying the Grassmann algebra) [3].

With help of the relations (6) we can then lift this Clifford map to multivector actions. No symmetry requirement has to be made on the contraction. This leads to the

**Definition 7 (Clifford algebra of multivectors)** Let $B : V \times V \mapsto \mathbb{K}$ be an arbitrary bilinear form. The Clifford algebra $\mathcal{C}(V, B)$ obtained from lifting the Clifford map

$$\gamma_x = x \downarrow + x \wedge = < x | , > + x \wedge = B(x, .) + x \wedge$$  

(13) to $\operatorname{End}(F \wedge (V))$ using the relations (6) is called Clifford algebra of multivectors.

Note, that $B(x, .) = \omega_x$ is a map from $V \to V^*$ and incorporates a dual isomorphism. It is clear from the construction that $\mathcal{C}(V, B)$ has a multivector structure or say a $\mathbb{Z}_n$-grading inherited from the graded space $F \wedge (V)$.

$B$ admits a decomposition into symmetric and antisymmetric parts $B = G + F$. The symmetric part $G = B_p$ corresponds to a quadratic form $Q$, see (8).

**Theorem 8** The Clifford algebra $\mathcal{C}(V, Q) \simeq \mathcal{C}(V, G)$ is isomorphic as Clifford algebra to $\mathcal{C}(V, B)$, if $B$ admits a decomposition $B = G + F, G^T = G, F^T = -F$.

A proof can be found for low dimensions in [18] and in general in [14]. However, this result was implicitly known to physicists, see [3, 17, 20]. In fact, this is the old Wick rule of QFT. We will insist on the $\mathbb{Z}_n$-grading and therefore carefully distinguish Clifford algebras of multivectors with a common quadratic form $Q$ but different contractions $B$.

**Definition 9** The opposite algebra $A^{\text{op}}$ of an algebra $A$ with product $m(a, b) = ab$ is defined to be the same linear space $FA$ underlying $A$ endowed with the opposite or transposed product $m^{\text{op}}(a, b) = m(b, a) = ba$.

**Theorem 10 (Chevalley)** The opposite Clifford algebra $\mathcal{C}^{\text{op}}(V, Q)$ of $\mathcal{C}(V, Q)$ is isomorphic to $\mathcal{C}(V, -Q)$,

$$\mathcal{C}^{\text{op}}(V, Q) \simeq \mathcal{C}(V, -Q).$$  

(14)
We can generalize this theorem to Clifford algebras of multivectors as

**Theorem 11** For Clifford algebras of multivectors holds

\[
\mathcal{C}^{op}(V, B) \simeq \mathcal{C}(V, -B^T),
\]

where \(T\) denotes the transposition of the bilinear form \(B(x, y)^T = B(y, x)\).

The most general linear transformation on \(F\wedge(V)\) is achieved by left and right translations, we have

\[
\text{End}(F\wedge(V)) \simeq \wedge(V) \otimes [\wedge(V)]^* \simeq \mathcal{C}(V, B) \otimes \mathcal{C}^{op}(V, B) \simeq \mathcal{C}(V, -B^T) \simeq C\ell(V \oplus V, B \oplus -B^T),
\]

(16)

see e.g. [21] theorem 5.5 or [22]. This is the algebra which will be used in the subsequent sections.

The fact that in the above construction one has the graded tensor product is reflected by the following formulas. Let

\[
\gamma_x = x \mathbf{j}_B + x \wedge
\]

\[
\gamma_x^{op} = x \mathbf{j}_{B^T} + x \wedge
\]

(17)

be the Clifford maps into \(\mathcal{C}(V, B)\) and \(\mathcal{C}^{op}(V, B) \simeq \mathcal{C}(V, -B^T)\), then we obtain the following commutation relations

\[
\gamma_x \gamma_y + \gamma_y \gamma_x = +2G(x, y)
\]

\[
\gamma_x^{op} \gamma_y^{op} + \gamma_y^{op} \gamma_x^{op} = -2G(x, y)
\]

\[
\gamma_x \gamma_y^{op} + \gamma_y \gamma_x^{op} = 0.
\]

(18)

Remember the decompositions \(B = G + F\) and \(B^{op} = -B^T = -G + F\). If one wants to treat ordinary left and right translations

\[
L_a x = ax
\]

\[
R_a x = xa,
\]

(19)

one has to introduce a further involution which accounts for the grading in the above graded tensor product of \(\mathcal{C}\) and \(\mathcal{C}^{op}\). One defines

\[
L_x = \gamma_x
\]

\[
R_x = \gamma_x^{op}(\cdot),
\]

(20)

where \((\cdot)\) indicates the grade involution operator acting to the right. We obtain in this case the commutation relations as

\[
L_x L_y + L_y L_x = \gamma_x \gamma_y + \gamma_y \gamma_x = 2G(x, y)
\]

\[
R_x R_y + R_y R_x = \gamma_x^{op} \gamma_y^{op}(\cdot) + \gamma_y^{op} \gamma_x^{op}(\cdot) = 2G(x, y)(\cdot)
\]

\[
L_x R_y - R_y L_x = \gamma_x \gamma_y^{op}(\cdot) - \gamma_y^{op} \gamma_x^{op}(\cdot) = 0.
\]

(21)

We give some further notations. Let \(\{j_i\}\) be a set of elements spanning \(V \simeq <j_1, \ldots, j_n>\) and \(\{\partial_k\}\) be a set of dual elements. Building the Grassmann algebras \(\wedge(V), \wedge(V^*)\) and defining the action of the forms via \(\mathcal{B}\), one obtains the relations

\[
\text{i) } j_i \wedge j_i = 0 = \partial_i \wedge \partial_i
\]

\[
\text{ii) } \partial_i j_k + j_k \partial_i = B_{ik} + B_{ki} = 2G_{ik}.
\]

(22)
The space $\mathbf{V} = \mathbf{V} \oplus \mathbf{V}^T$ is thus spanned by (note the order of indices)
\[
\{e_1, \ldots, e_{2n}\} = \{j_1, \ldots, j_n, \partial_1, \ldots, \partial_n\}.
\] (23)

Usually, the choice of the contraction is taken to be the canonical one, with $G_{ik} = \delta_{ik}$. This leads to the so-called quantum algebra $[23]$. The resulting algebra is $\mathcal{O}(\mathbf{V}, B^{\text{can}})$, where
\[
[B^{\text{can}}(e_i, e_j)] = [B^{\text{can}}_{ij}] = \begin{pmatrix} 0_{n \times n} & \frac{1}{2} \mathbb{I}_{n \times n} \\ \frac{1}{2} \mathbb{I}_{n \times n} & 0_{n \times n} \end{pmatrix}
\] (24)

This leads to the far more restricted commutation relations of Sallers quantum algebra (which is in fact a CAR algebra)
\[
i) \quad j_i j_k + j_k j_i = 0 = \partial_i \partial_k + \partial_k \partial_i
\]
\[
ii) \quad \partial_i j_k + j_k \partial_i = \delta_{ik}.
\] (25)

However, see $[24]$ for the usefulness of a more general bilinear form $B$. We will employ the $j_i$ sources and $\partial_k$ duals to construct field operators. A more general bilinear form $B = G + F$ is then introduced explicitly via the quantization $G$ and the propagator $F$ in terms of $j$ sources and $\partial$ duals.

### 3 Generating functionals

Generating functionals originated from two major considerations. The first arose from studying covariant formulated perturbation theory $[3]$, see also $[25, 26]$, to be able to handle free-free transitions of particles via a scattering matrix. The main ingredients are vacuum expectation values of time-ordered products of field operators. Since in- and out-going states are assumed to be free, one can retain the Fock representation, which however is not valid in the interacting case. Path integrals originated out of this branch of physics $[27]$.

A second approach was developed to treat QFT in an axiomatic way $[28]$. The main tool are vacuum expectation values of ordinary operator products. But, beside some structural results, the most prominent of them the Wightman reconstruction theorem, the method has not lead to a computational useful tool $[29]$, apart from low dimensional models $[1]$.

Both of the constructions assume:

i) The existence of field operators, and operators build from them, the hermitean of which constitute the observable of the theory.

ii) The existence of a positive, linear, normalized functional, a state, which allows the calculation of vacuum expectation values and a statistical interpretation.

There are further requirements necessary, but even the given ones are questionable. It is well known from $C^*$-algebra theory, that representations exist in the thermodynamic limit, in which no field operators do exist, but only in some larger closure of the algebra.

We will remain with the first assumption, but tend to weaken the second one. It is mainly positivity which might be questioned in this context. QFT deals already with indefinite structures in describing interactions e.g. Faddeev-Popov ghosts. Additionally we know, that on orthogonal Clifford algebras, which are related to fermionic fields, there are no nonsingular forms in dimensions higher than two (generators) $[21, 27]$. 

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It is clear that a statistical interpretation breaks down without positivity. But such an interpretation is not called for in the interaction region. Only in- and outgoing states, which should have a particle interpretation, have to bear this feature. This idea was already proposed by Heisenberg.

However, we will go a step further and propose a geometric notation, which is valid in positive, indefinite and even singular cases. This makes it possible to decide afterwards in which situations positivity may be obtained and a statistical interpretation is meaningful, see [3]. The Dirac theory was already treated within this methods [31].

We will treat fermionic quantum fields, but give in section 6 also an example for a boson-fermion coupling theory, which can be developed straightforward.

Our starting point is the quantization rule imposed on field operators.

\[ \{ \psi_{I_1}, \psi_{I_2} \} = A_{I_1 I_2}, \]

where \( \psi_I \) is a ‘super field’ defined as

\[ \psi_I = \psi_{\Lambda K} = \begin{cases} \psi_K & \Lambda = 1 \\ \psi^{\Lambda}_K & \Lambda = 2. \end{cases} \]

The anti-involution \( \alpha \) is taken e.g. to be the hermitean or charge conjugation. \( A_{I_1 I_2} \) is a symmetric \( c \)-number function of space-time and algebraic freedoms which all together are summarized in the multiindex \( I = \{ \Lambda, \alpha, \ldots \} \).

The relation (26) can be seen as the defining relation of a Clifford algebra \( C\ell( <\psi>, A) \), when we identify the field operators as generators. To be able to give an algebraic basis of this Clifford algebra, we need an ordering. Let \( P \) denote such an ordering, which acts on the indices of the field operators. Clifford monomials in the field operators are then given by

\[ e_n := P(\psi_{I_1}, \ldots, \psi_{I_n}), \]

where \( P \) might be e.g. time-ordering, antisymmetry for equal times or normal-ordering w.r.t. a vacuum state. The linear space underlying \( C\ell( <\psi>, A) \) is thus the linear span of the \( e_n \) Clifford monomials

\[ FCL( <\psi>, A) \simeq <e_n>. \]

As we emphasized in section 2 it is of utmost importance to have a graded space underlying those algebras, which leads to the Clifford algebra of multivectors of definition 7. In quantum field theory this space is obtained from a Grassmann algebra by introducing the anticommuting Schwinger sources \( j \). Indeed, due to the relation \( j^2 = 0 \) this are Grassmann generators. Assuming that the indices are from the same set as in the case of the \( \psi_I \)’s, we build the Grassmann algebra over \( V \) with \( V = <j_I> \)

\[ \bigwedge( <j_I>) = \bigwedge( V). \]

Of course, we can identify the spaces underlying \( C\ell( <\psi>, A) \) and \( \bigwedge( <j_I>) \) by means of Chevalley’s identification

\[ FCL( <\psi>, A) \simeq F\bigwedge( <j_I>), \]

see section 2. This is the graded space physicists make actually use of in their calculations. This remains true for path integrals also, as can be seen by their functional derivatives and the explicit occurrence of Schwinger source terms \( \psi_I j_I \) in the exponent. It is therefore convenient to relate all entities to this space.
We know from Chevalley deformation, that the Clifford map $\psi_I$ constitutes an endomorphism of $\mathcal{F}\Lambda^j(\langle j_I \rangle)$ which does not respects the grading. $\psi_I$ may be decomposed in a grade lowering or annihilation operator $\partial_I$ and a grade rising or creation operator $1/2A_{I_1 I_2 J_1 J_2 \wedge}$,

$$\psi_I = \partial_I + \frac{1}{2} A_{I_1 I_2 J_1 J_2 \wedge}, \quad (32)$$

where we omitted some superfluous factors. The $\partial_I$'s are the canonical duals w.r.t. the Euclidean dual isomorphism. We have thus the commutation relations

$$\{j_I, j_K\}_+ = 0 = \{\partial_I, \partial_K\}_+$$

$$\{j_I, \partial_K\}_+ = \delta_{IK}, \quad (33)$$

which results with (32) in the quantization condition (26)

$$\{\psi_I, \psi_K\}_+ = A_{IK}. \quad (34)$$

The next step towards a QFT generating functional is to project the Clifford monomials $e_n = \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n})$ onto the graded space $\mathcal{F}\Lambda^j(\langle j_I \rangle)$. This is in fact a representation of the Clifford morphism obtained by lifting the $\psi_I$ action to the whole algebra, as was described in section 2. We write this projection as $\pi : \mathcal{FC}(V, A) \to \mathcal{F}\Lambda(V)$

$$\pi(e_n) = \pi(\mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n}))$$

$$= \langle \partial_{I_1} \wedge \ldots \wedge \partial_{I_n} \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n}) \rangle_{\pi} \langle j_{I_1} \wedge \ldots \wedge j_{I_n} \rangle$$

$$= \langle 0 \mid \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n}) \mid p \rangle_{j_{I_1} \wedge \ldots \wedge j_{I_n}}$$

$$= \rho_n(I_1, \ldots, I_n \mid p) j_{I_1} \wedge \ldots \wedge j_{I_n}. \quad (35)$$

The index $\pi$ parameterizes the linear normed functionals on $\mathcal{FC}(V, A)$, which is thereafter encoded in the bra $\langle 0 \rangle$ and ket $\mid p \rangle$ of the notation common to physicists. Observe, that

$$\langle \partial_{I_1} \wedge \ldots \wedge \partial_{I_n} \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n}) \rangle_{\pi} = \langle 0 \mid \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n}) \mid p \rangle$$

$$= \rho_n(I_1, \ldots, I_n \mid p) \quad (36)$$

is a c-number transition matrix element. If $\mid a \rangle$ is set equal to $\mid 0 \rangle$ we obtain vacuum expectation values which are known to be sufficient to describe the theory.

If a general element of $\mathcal{F}\Lambda^j(\langle j_I \rangle)$ is written down, this reads

$$\mathcal{P}(j, p)_{\pi} = \sum_{i=0}^{n} \frac{i^n}{n!} \rho_n(I_1, \ldots, I_n \mid p) j_{I_1} \wedge \ldots \wedge j_{I_n} \mid 0 \rangle_{\pi}$$

$$\rho_n(I_1, \ldots, I_n \mid p) = \langle 0 \mid \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n}) \mid p \rangle \quad (37)$$

and is thus exactly the definition of a fermionic generating functional in quantum field theory [6, 32]. To emphasize the ‘vector’-state character, in fact the left linear structure, we have added a functional ket $\mid 0 \rangle_{\pi}$ which is usually subjected to the relations

$$\lambda < 0 \mid j_I = 0 = \partial_I \mid 0 \rangle_{\pi}. \quad (38)$$

This writings mimic in fact a linear form imposed on the algebra. We have thus given a Clifford algebraic account on quantum field theoretic generating functionals. The Clifford algebra involved was shown to be a Clifford algebra of multivectors build over a graded space $\mathcal{F}\Lambda^j(\langle j_I \rangle)$.

One may note, that due to the explicit introduction of a basis $\{j_I\}$ the functional is also basis dependent, especially the matrix elements $\rho_n(I_1, \ldots, I_n \mid p)$ are. However, from our general considerations it is clear, that the functional ‘state’ $\langle \mathcal{P}(j, p) \rangle_{\pi} \in \Lambda^j(\langle j_I \rangle) \rangle$ is an invariant object and basis independent.

9
4 Transition from operator dynamics to generating functionals

The most common case in quantum field theory is that an operator dynamics is given \textit{a priori}. Furthermore, quantization calls for the Hamiltonian picture which breaks explicitly covariance \cite{33}. This is not a problem as long as exact calculations are performed, since one is able to return to the manifest covariant theory.

Given a Hamiltonian, we can use the Heisenberg equation to write down the dynamics

\[i \dot{\psi}_I = [\psi, H]_-, \quad H = H[\psi]. \tag{39}\]

\(H\) is assumed to generate a one parameter family of automorphisms by integrating \cite{33} to

\[\psi_I(t) = e^{iHt}\psi_I(0)e^{-iHt}. \tag{40}\]

It can be shown under the restrictive conditions of \(C^*\)-theory, that this is always possible.

We know, that generating functionals are constructed by lifting the \(\psi_I\)-action to \(F \wedge \langle j_I \rangle\), so it is a natural question to ask for a Clifford morphism extension of \cite{33}. This will be the desired functional equation.

Since it is quite not clear, which \(H[\psi]\) integrate to a group action, we assume this explicitly. However, the space underlying a Clifford algebra can easily be made into a Lie-algebra by introducing the commutator product as a Lie-product. This has not to be confused with the Lie-algebras obtained as bi-vector sub-algebras under the commutator product.

A straightforward method to obtain derivatives of generating functionals is the Baker-Campbell-Hausdorff formula. If one writes the generating functional formally as

\[e^{\psi(t)} \simeq \mathcal{P}(j, p) \rangle_F \tag{41}\]

one has to calculate

\[i \partial_0 e^{\psi(t)} = \sum_{k=0}^{\infty} [\psi, H]_k \]

\[[\psi, H]_k := [[\psi, H]_{k-1}, H]_-, \quad k \geq 1\]

\[i \partial_0 \psi = [\psi, H]_0 = [\psi, H], \tag{42}\]

which is quite tedious. Furthermore, one has to apply the BCH formula a second time to eliminate the field operators, which prolongates the calculations enormous. The result is then the functional equation

\[i \partial_0 \mathcal{P}(j, p) \rangle_F = H[j, \partial]^P \mathcal{P}(j, p) \rangle_F \tag{43}\]

with the functional Hamiltonian \(H[j, \partial]^P\). This Hamiltonian depends in its explicit form on the ordering implicit in \(\mathcal{P}(j, p) \rangle_F\).

We will circumvent this lengthy calculations using the opposite algebra \(C^{op}\langle \psi_I \rangle, A\). To account for generating functionals, we reinterpret the field operators as Clifford endomorphisms on the space \(F \wedge \langle j_I \rangle\) spanned by the Grassmann sources. If we allow only a left action, we can not reach every endomorphism on this space. However, the algebra \(\wedge(\langle j_I \rangle) \otimes \wedge(\langle j_I \rangle^*)\) was shown to be isomorphic
to $\text{End}(F \wedge (< j_I >))$. We have therefore to construct the opposite Clifford algebra to handle right actions. In theorem 14 we obtained the isomorphism

$$C\ell^{op}(< \psi_I >, A) \simeq C\ell(< \psi_I >, -A).$$ (44)

The decomposition into creation and annihilation part reads then for the antisymmetric ordering

$$\psi_I = \partial_I + \frac{1}{2} A_{I'I} j_{I'} \wedge$$

$$\psi_{I'}^{op} = \partial_I - \frac{1}{2} A_{I'I} j_{I'} \wedge.$$ (45)

$\psi^{op}$ acts as an operator of right action from the left on the space $F \wedge (< j_I >)$. Indeed, one finds

$$\partial_I = \frac{1}{2}(\psi_I + \psi_{I'}^{op})$$

$$j_I = A_{I'I}(\psi_{I'} - \psi_{I'}^{op}),$$ (46)

where we assumed $AA = 1$, according to the Euclidean dual isomorphism. It is now an easy task to find the desired lifting of the Clifford morphism.

Let $g_n = \pi(e_n) \rho_n(I_1, \ldots, I_n), j_{I_1} \wedge \cdots \wedge j_{I_n}$ be the projected monomial. The action of $H[\psi]$ on this monomials $g_n$ may be defined as

$$i \dot{g}_n = [g_n, H[\psi]] = g_nH[\psi] - H[\psi]g_n$$

$$= H[\psi^{op}]g_n - H[\psi]g_n.$$ (47)

A detailed investigation of the grading properties should then follow. However, to reproduce the results in literature, the grade involution is suppressed. The action of the Hamiltonian on multivectors is thus defined to be

$$i \dot{g}_n = (H[\psi^{op}] - H[\psi])g_n.$$ (49)

Summing up the monomials yields the functional equation

$$i \dot{0} | P(j, a) >_F = (H[\psi^{op}] - H[\psi]) | P(j, a) >_F$$

$$= H[j, \partial]^P | P(j, a) >_F,$$ (50)

where $H[j, \partial]^P$ is the desired functional Hamiltonian acting on the whole Dyson-Schwinger-Freese hierarchy encoded in the generating functional.

Instead of tedious calculations by means of the BCH formula, we have simply to decompose $\psi$ and $\psi^{op}$ into $j$-$\partial$-parts and sum up the terms. In the above case of an antisymmetric ordering we have

$$H[j, \partial]^{as} = H[\psi^{op}] - H[\psi]$$

$$= H[\partial - 1/2 A j] - H[\partial + 1/2 A j].$$ (51)

The antisymmetric ordering was introduced in the second line due to the special form of the decomposition of $\psi_I$ into annihilating and creating parts. In the case of an arbitrary ordering, we have to use the Clifford algebra of multivectors with an arbitrary bilinear form $B = G + F = A + F$ ($G$ and $A$ are synonyms). The resulting decompositions are (without factors)

$$\psi = \partial + A j + F j$$

$$\psi^{op} = \partial - A j + F j,$$ (52)
due to theorem [13]. We obtain as $F$-ordered or say normal-ordered functional Hamiltonian

$$H[j, \partial]^{\mathcal{N}} = H[\psi^{op}] - H[\psi] = H[\partial - A_j + F_j] - H[\partial + A_j + F_j]$$

(53)

and the normal-ordered functional equation

$$i \partial_0 | \mathcal{N}(j,a) >_F = (H[\psi^{op}] - H[\psi]) | \mathcal{N}(j,a) >_F$$

$$= H[j, \partial]^{\mathcal{N}} | \mathcal{N}(j,a) >_F .$$

(54)

The usefulness of this approach will be shown in the following sections by explicit calculations. Bosons can be treated in an analogous way using symplectic Clifford algebras [2]. We have not accounted for all factors $i$ and 1/2 occurring for historical reasons in the definition of generating functionals, this will be done in the examples.

5 Non-linear spinor field model

It will be illuminating to treat every model in two versions. First the pure antisymmetric case which is the one-time limit of the time-ordered functionals and afterwards the more general normal-ordered case, where $F$ is either the correct propagator of the theory or taken as a parameter.

We use a compact notation, where spinor operators and charge conjugate spinors are united

$$\psi_I = \psi_{\Lambda K} = \begin{cases} \psi_K & \Lambda = 1, \\ \psi_{\bar{K}} & \Lambda = 2. \end{cases}$$

(55)

It is useful to work with charge conjugate field operators, since they allow a compact notation of the field equations. We define our model as (summation and integration over repeated indices implicitly assumed)

$$i \partial_0 \psi_I = D_{I I_1} \psi_{I_1} + g V^{(I_1 I_2 I_3)}_{(a)} \psi_{I_1} \psi_{I_2} \psi_{I_3}$$

$$\{\psi_I, \psi_{I_2}\}^+ = A_{I_1 I_2},$$

(56)

where $D_{I I_1}$ is a kinetic term and $V^{(I_1 I_2 I_3)}_{(a)}$ a constant vertex. The anticommutator is denoted by $A_{I_1 I_2}$ and we require the identity $A_{I_1 I_2} A_{I_2 I_3} = \delta_{I_1 I_3}$. For explicit expressions see e.g. [33]. But, it is our intention not to give explicit expressions for $D, V$ and $A$, since we want to demonstrate the model independency of our method.

The antisymmetry of $V^{(I_1 I_2 I_3)}_{(a)}$ can be pushed over to the field operator products. In this case one has to be careful since there are relations of the type

$$: \psi_{I_2} \wedge \psi_{I_3} \wedge \psi_{I_4} : = \psi_{I_2} \wedge \psi_{I_3} \wedge \psi_{I_4}$$

$$= \psi_{I_2} \wedge \psi_{I_3} \wedge \psi_{I_4} + F_{I_2 I_3} \psi_{I_4} - F_{I_2 I_4} \psi_{I_3} + F_{I_3 I_4} \psi_{I_2} ,$$

(57)

which result from the normal-ordering [3]. We have thus carefully to distinguish the different wedge products, which ultimately lead to different vacua [8]. Literally this is done by the $: \ldots :$ notations used by physicists, but there without an algebraic theory behind. From the dynamics (56) we obtain the Hamilton operators

$$H[\psi] = \frac{1}{2} A_{I_1 I_3} D_{I_1 I_2} \psi_{I_1} \wedge \psi_{I_2} + \frac{g}{4} A_{I_1 I_3} V^{I_2 I_3 I_4}_{I_1} \psi_{I_1} \wedge \psi_{I_2} \wedge \psi_{I_3} \wedge \psi_{I_4}$$

(58)

and

$$H[\psi] = \frac{1}{2} A_{I_1 I_3} D_{I_1 I_2} \psi_{I_1} \wedge \psi_{I_2} + \frac{g}{4} A_{I_1 I_3} V^{I_2 I_3 I_4}_{I_1} \psi_{I_1} \wedge \psi_{I_2} \wedge \psi_{I_3} \wedge \psi_{I_4} ,$$

(59)

where the superscript wedges indicate the usage of the wedge or dotted wedge products.
5.1 Antisymmetric case

The antisymmetric case is connected with the normal wedge. Furthermore, the field operators corresponding to this ordering are given as

\[ \psi = \frac{1}{i} \partial_j - i \frac{1}{2} A_{II_1 j_1} = \partial - A_j \]

\[ \psi^{op} = \frac{1}{i} \partial_j + i \frac{1}{2} A_{II_1 j_1} = \partial + A_j. \]  

(60)

Using the functional Heisenberg equation (50), we have to calculate

\[ i\partial_0 |A(j, a) >_F = E_{0a} |A(j, a) >_F \]

\[ = (H[\partial - A j] - H[\partial + A j]) |A(j, a) >_F, \]  

(61)

where \( |a > \) is assumed to be an energy eigen-state here and in the sequel. Hence, we have to decompose the Clifford map \( \psi \) and \( \psi^{op} \) into creation and annihilation parts w.r.t. the space \( F \backslash (j i) >. \) The first term of (68) is given as

\[ T_1[\psi]^\wedge = \frac{1}{2} A_{I_1 I_3} D_{I_3 I_2} \psi_{I_1} \wedge \psi_{I_2}, \]  

(62)

which results in functional form as

\[ T_1[j, \partial]^\wedge = T_1[\partial + A j] - T_1[\partial - A j] \]

\[ = \frac{1}{2} A_{I_1 I_3} D_{I_3 I_2} \left( \frac{1}{i} \partial_{I_3} + i \frac{1}{2} A_{I_2 I_4 j_1} \left( \frac{1}{i} \partial_{I_2} + \frac{i}{2} A_{I_1 I_3 j_1} \right) \right. \]

\[ - \left. \left( \frac{1}{i} \partial_{I_2} - i \frac{1}{2} A_{I_2 I_4 j_1} \left( \frac{1}{i} \partial_{I_2} - \frac{i}{2} A_{I_1 I_3 j_1} \right) \right) \right) \]

\[ = \frac{1}{2} A_{I_1 I_3} D_{I_3 I_2} \left( \frac{1}{i} \partial_{I_2} + \frac{i}{2} A_{I_2 I_4 j_1} \left( \frac{1}{i} \partial_{I_2} + \frac{i}{2} A_{I_1 I_3 j_1} \right) \right) \]

\[ + \frac{1}{4} \{ \partial_{I_2}, \partial_{I_1} \} + \frac{i^2}{4} A_{I_2 I_4} A_{I_1 I_3} \{ j_{I_2}, j_{I_1} \} + \]  

\[ + \frac{4 i^2}{2} A_{I_2 I_4 j_{I_2} \partial_{I_1}}. \]  

(63)

Using the commutation relations of the \( j, \partial \) generators and the antisymmetry of \( AD, \) we obtain

\[ T_1[j, \partial]^\wedge = - \frac{1}{2} A_{I_2 I_3} D_{I_3 I_2} 2 A_{I_2 I_4 j_{I_1}} \partial_{I_1} \]

\[ = - D_{I_2 I_3} j_{I_1} \partial_{I_1}, \]

\[ = - D_{I_2 I_3} j_{I_1} \partial_{I_2}. \]  

(64)

The interaction term

\[ T_2[\psi]^\wedge = \frac{9}{4} A_{I_1 I_3} V_{I_3} \psi_{I_1} \wedge \psi_{I_2} \wedge \psi_{I_3} \wedge \psi_{I_4} \]  

(65)

is decomposed as

\[ T_2[j, \partial]^\wedge = T_2[\partial + A j] - T_2[\partial - A j] \]

\[ = \frac{9}{4} A_{I_1 I_3} V_{I_3} \left( \times \right. \]

\[ \times \left( \frac{1}{i} \partial_{I_4} + \frac{i}{2} A_{I_3 I_4 j_{I_4}} \left( \frac{1}{i} \partial_{I_3} + \frac{i}{2} A_{I_3 I_4 j_{I_4}} \right) \right. \times \]  

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The functional Hamiltonian

\begin{align*}
\frac{1}{i} \partial_t - \frac{i}{2} A_{I_1, I_2} j_{I_2}' \left( \frac{1}{i} \partial_t - \frac{i}{2} A_{I_1, I_2} j_{I_2}' \right) \\
\frac{1}{i} \partial_t + \frac{i}{2} A_{I_1, I_2} j_{I_2}' \left( \frac{1}{i} \partial_t - \frac{i}{2} A_{I_1, I_2} j_{I_2}' \right)
\end{align*}

Since \(T_2[j, \partial]^\wedge\) is an even function of the generators, we remain with terms of an odd number of \(j\) and \(\partial\) generators. Furthermore we use the binomial theorem. This can be done, since new commutator terms does not arise due to antisymmetry of \(AV\) in \(I_1, \ldots, I_4\). Accounting finally for a factor 2 from the two commutator terms, the result is

\begin{align}
T_2[j, \partial]^\wedge &= g A_{I_1, I_2} V_{I_1}^{I_2} A_{I_1, I_2} j_{I_2}' \partial_{I_3} \partial_{I_2} \\
&+ \frac{g}{4} A_{I_1, I_2} V_{I_1}^{I_2} A_{I_1, I_2} j_{I_2}' A_{I_1, I_2} j_{I_2}' A_{I_1, I_2} j_{I_2}' \partial_{I_1} \\
&= g j_{I_1} V_{I_1}^{I_2} A_{I_1, I_2} j_{I_2}' \partial_{I_3} \partial_{I_2} + \frac{1}{4} A_{I_1, I_2} A_{I_1, I_2} j_{I_2}' \partial_{I_1}. \tag{66}
\end{align}

The functional energy equation reads

\begin{align*}
E_{\alpha 0}|A(j, a)>_F &= \left\{ D_{I_1, I_2} j_{I_2} \partial_{I_2} \\
&+ g j_{I_1} V_{I_1}^{I_2} A_{I_1, I_2} j_{I_2}' \partial_{I_3} \partial_{I_2} + \frac{1}{4} A_{I_1, I_2} A_{I_1, I_2} j_{I_2}' \partial_{I_1} \right\}|A(j, a)>_F \tag{68}
\end{align*}

in accordance with formula (3.98) in [20].

### 5.2 Normal-ordered case

The normal-ordered functional equation is related to the dotted wedge. It contains explicitly the propagator. This results in another functional Hamiltonian \(H[j, \partial]^\wedge\) .

The functional energy equation reads

\begin{align*}
\frac{i}{\hbar} \partial_j |N(j, a)>_F &= E_{\alpha 0} |N(j, a)>_F \\
&= \left( H[\psi^{op}] - H[\psi] \right) |N(j, a)>_F, \tag{69}
\end{align*}

where

\begin{align*}
\psi^{op} &= \frac{1}{i} \partial + \frac{i}{2} A j \wedge = \frac{1}{i} \partial + \frac{i}{2} A j \wedge + iF j \wedge \simeq \partial + A j + F j \\
\psi &= \frac{1}{i} \partial - \frac{i}{2} A j \wedge = \frac{1}{i} \partial - \frac{i}{2} A j \wedge + iF j \wedge \simeq \partial - A j + F j \tag{70}
\end{align*}

are the Clifford maps related to the dotted wedge, written in terms of the undoted ones. This allows us to use the same \(j\) and \(\partial\) generators. Since the operator dynamics is formally the same, we have once more to calculate \(T_1\) and \(T_2\). Now

\begin{align}
T_1[j, \partial]^\wedge &= T_1[\partial + A j + F j] - T_1[\partial - A j + F j] \\
&= \frac{1}{2} A_{I_1, I_2} D_{I_1, I_2} \left\{ \right.
\frac{1}{i} \partial_{I_3} + \frac{i}{2} A_{I_3, I_4} j_{I_4}' + iF_{I_3, I_4} j_{I_4}' \left( \frac{1}{i} \partial_{I_3} + \frac{i}{2} A_{I_3, I_4} j_{I_4}' + iF_{I_3, I_4} j_{I_4}' \right) \\
&- \frac{1}{i} \partial_{I_3} - \frac{i}{2} A_{I_3, I_4} j_{I_4}' + iF_{I_3, I_4} j_{I_4}' \left( \frac{1}{i} \partial_{I_3} - \frac{i}{2} A_{I_3, I_4} j_{I_4}' + iF_{I_3, I_4} j_{I_4}' \right) \right\} \tag{71}
\end{align}
is no longer odd or even in $j, \partial$ and we can only use antisymmetry arguments. Inspection of the equations shows, that only odd numbers of $A$'s can occur. This are

$$A_j \partial : \quad 2 \frac{i}{2!} A_{i_1 i_2} j_{i_1} j_{i_2} \partial_{i_1},$$

$$A F j j : \quad 2 \frac{i^2}{2} A_{i_1 i_2} F_{i_1 i_2} j_{i_1} j_{i_2}. \quad (72)$$

With the same considerations as above, we obtain

$$T_1[j, \partial]^{\lambda} = -D_{i_1 i_2} j_{i_1} \partial_{i_2} + D_{i_1 i_2} F_{i_3 i_4} j_{i_3} j_{i_4}.$$ \quad (73)

The interaction term $T_2$ is treated along the same lines. We have to use the polynomial theorem and the polynomial coefficients

$$P_{k_1 \ldots k_n} := \frac{k!}{k_1! \ldots k_n!}, \quad k = \sum k_n.$$ \quad (74)

The resulting 6 terms occurring twice are given in table 1.

| (#$\partial$, #$A_j$, #$F_{j j}$) | $P_{k_1 \ldots k_n}$ | factor | resulting term |
|---------------------------------|-------------------|--------|----------------|
| (3,1,0)                         | $\frac{4!}{3!} = 4$ | $\frac{4}{2!} = -2$ | $-2 A_{i_1 i_2} j_{i_1} \partial_{i_2} \partial_{i_1}$ |
| (2,1,1)                         | $\frac{9!}{3!} = 12$ | $\frac{12}{2} = 6$ | $6 A_{i_1 i_2} F_{i_3 i_4} j_{i_1} j_{i_2} \partial_{i_1}$ |
| (1,3,0)                         | $\frac{4!}{3!} = 4$ | $\frac{4}{3} = -\frac{1}{3}$ | $-\frac{1}{3} A_{i_1 i_2} A_{i_3 i_4} j_{i_1} j_{i_2} j_{i_3} j_{i_4} \partial_{i_1}$ |
| (1,1,2)                         | $\frac{9!}{3!} = 12$ | $\frac{12}{2} = 6$ | $-6 A_{i_1 i_2} F_{i_3 i_4} F_{i_5 i_6} j_{i_1} j_{i_2} j_{i_3} j_{i_4} \partial_{i_1}$ |
| (2,1,1)                         | $\frac{9!}{3!} = 12$ | $\frac{12}{2} = 6$ | $-6 A_{i_1 i_2} F_{i_3 i_4} F_{i_5 i_6} j_{i_1} j_{i_2} j_{i_3} j_{i_4} \partial_{i_1}$ |
| (0,3,1)                         | $\frac{4!}{3!} = 4$ | $\frac{4}{3} = \frac{1}{3}$ | $\frac{1}{3} A_{i_1 i_2} A_{i_3 i_4} F_{i_5 i_6} j_{i_1} j_{i_2} j_{i_3} j_{i_4} j_{i_5} j_{i_6}$ |
| (0,1,3)                         | $\frac{4!}{3!} = 4$ | $\frac{4}{3} = \frac{1}{3}$ | $\frac{1}{3} A_{i_1 i_2} A_{i_3 i_4} F_{i_5 i_6} j_{i_1} j_{i_2} j_{i_3} j_{i_4} j_{i_5} j_{i_6}$ |

After juggling the indices and summarizing we obtain

$$T_2[j, \partial]^{\lambda} = g V^2 F_{i_5 i_6} \left[ -j_{i_1} \partial_{i_1} \partial_{i_2} \partial_{i_3} - 3 F_{i_5 i_6} j_{i_1} \partial_{i_1} \partial_{i_2} \partial_{i_3} + \frac{1}{4} A_{i_1 i_2} A_{i_3 i_4} j_{i_1} j_{i_2} j_{i_3} j_{i_4} \partial_{i_1} \partial_{i_2} \partial_{i_3} - \frac{1}{4} A_{i_1 i_2} A_{i_3 i_4} F_{i_5 i_6} j_{i_1} j_{i_2} j_{i_3} j_{i_4} j_{i_5} j_{i_6} \partial_{i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4}ight]. \quad (75)$$

The normal-ordered functional energy equation obtained directly from the operator dynamics in one single step reads then

$$E_{00} |N(j, a) >_F = H[j, \partial]^{\lambda} |N(j, a) >_F = \left\{ D_{i_1 i_2} j_{i_1} \partial_{i_2} - D_{i_1 i_2} F_{i_3 i_4} j_{i_1} j_{i_2} + g V^2 F_{i_5 i_6} \left[ j_{i_1} \partial_{i_1} \partial_{i_2} \partial_{i_3} - 3 F_{i_5 i_6} j_{i_1} \partial_{i_1} \partial_{i_2} \partial_{i_3} + \frac{1}{4} A_{i_1 i_2} A_{i_3 i_4} j_{i_1} j_{i_2} j_{i_3} j_{i_4} \partial_{i_1} \partial_{i_2} \partial_{i_3} - \frac{1}{4} A_{i_1 i_2} A_{i_3 i_4} F_{i_5 i_6} j_{i_1} j_{i_2} j_{i_3} j_{i_4} j_{i_5} j_{i_6} \partial_{i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \right] j_{i_5} j_{i_6} \partial_{i_5} \partial_{i_6} \right\} |N(j, a) >_F \quad (76) \right.$$  

$H[j, \partial]^{\lambda}$ is the normal-ordered functional Hamiltonian. The result agrees with (3.106) in [20].

**Remark:** In QFT the normal-ordered functional equation is obtained in two steps. Only after deriving the antisymmetric functional equation normal-ordering is performed explicitly via the non-perturbative Wick transformation

$$|A(j, a) >_F = e^{-1/2 j F} |N(j, a) >_F, \quad H[j, \partial]^{\lambda} = e^{-1/2 j F} H[j, \partial]^{\lambda} e^{1/2 j F}. \quad (77)$$
Observe, that $j$ commutes with $exp(-1/2 jFj)$, but not $\partial$, so that the transition can be given in terms of generators also

$$\partial e^{-1/2 jFj} = e^{-1/2 jFj}(\partial - Fj)$$
$$d := \partial - Fj$$
$$H[j, \partial] = H[j, d],$$

(compare [31]). This method requires explicit vertex regularization, if the products $\wedge$ and $\dot{\wedge}$ are not explicitly treated as distinct $j_1, \dot{j}_1, j_2, \dot{j}_2 = j_1 \wedge j_2 + F_{1, 2}$, as was shown in [9].

6 Spinor quantum electrodynamics

We give spinor QED as an example for a boson-fermion coupling theory. We have to deal with a constrained theory, since spinor QED is a gauge theory. Especially on the quantum level this is a difficult task [34]. Hence in a first step we will eliminate the gauge freedom classically and perform the quantization afterwards. This is only possible in QED, because of the simple Abelian gauge coupling. In non abelian Yang–Mills Theories as QCD one has a nonlinear term in the Gauss’ law, and a classical elimination leads to difficulties. However, this can be done using functional techniques [34, 35].

We start with the equations of classical spinor ED

$$i \partial_\nu F_{\mu \nu} + \frac{ie_0}{2} \Psi C\gamma^\mu \sigma^2 \Psi = 0$$
$$ii) \quad (i\gamma^\mu \partial_\mu - m_0)\Psi + e_0 A_\mu \gamma^\mu \sigma^3 \Psi = 0$$
$$iii) \quad F_{\mu \nu} := \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where we have used the definition

$$\Psi = \Psi_{\alpha \Lambda} = \left\{ \begin{array}{ll}
\Psi_{\alpha} & \Lambda = 1 \\
C_{\alpha \beta} \bar{\Psi}_{\beta} & \Lambda = 2
\end{array} \right. (80)$$

and omit the matrix indices. By definition the sigma matrices act on the super index $\Lambda$ distinguishing the field and its charge conjugated field, which has to be treated as an independent quantity. The gamma matrices and the charge conjugation matrix $C$ act on the spinor index $\alpha$ of the $\Psi$ field.

We use the Coulomb gauge and set

$$\partial_k A^k = 0. (81)$$

Furthermore we split the ‘bosonic’ fields $A$ and $E$ into longitudinal and transverse parts.

$$A_k = A^{tr}_k,$$  because of  $$\partial_k A^k = 0$$
$$E_k = E^{tr}_k + E^j_k, \quad E_k := -F^{0k} = F_{0k}. (82)$$

Furthermore we decompose the field equations into dynamic ones and the remaining constraint, the Gauss’ law. Looking at (79-i) yields for

$$\nu = 0: \quad i\partial_0 E_k = -\frac{1}{2} e_0 \Psi C\gamma^k \sigma^2 \Psi + i(\partial_j \partial^k A^j - \partial_j \partial^j A^k)$$
$$\nu = k: \quad \partial^k E_k = -\frac{i e_0}{2} \Psi C\gamma_0 \sigma^2 \Psi$$

(83)
and from the definition of $E^k$

$$i\partial_\alpha A^{tr,k} = iE^k - i\partial_k A^0.$$ (84)

With help of the current conservation the Gauss’ law \( [\#] \mu = k \) can now be used to yield an expression for the $A_0$ field in terms of spinors

$$A_0 = -i\frac{e_0}{2}\Delta^{-1}\Psi C\gamma_0\sigma^2\Psi.$$ (85)

The $\Delta^{-1}$ symbol is used for the integral kernel of the inverse of the Laplacian. For a further compactification of the notation we introduce a ‘super field’ $B_\eta$ in the ‘bosonic’ fields also,

$$B^\eta_k := \begin{cases} A^{tr}_k & \eta = 1 \\ E^{tr}_k & \eta = 2 \end{cases}.$$ (86)

The equations of motion reads then

$$i\partial_\alpha \Psi_{I_1} = D_{I_1 I_2} \Psi_{I_2} + W^{K}_{I_1 I_2} B_{K} \Psi_{I_2} + U^{I_1 I_2 I_3 I_4}_{I_1} \Psi_{I_2} \Psi_{I_3} \Psi_{I_4}$$

$$i\partial B_{K_1} = L_{K_1 K_2} B_{K_2} + J^{I_1 I_2}_{K_1} \Psi_{I_1} \Psi_{I_2}.$$ (87)

with the definitions

$I := \{\alpha, \Lambda, r\}$

$K := \{\eta, \zeta\}$

$P^{tr} := 1 - \Delta^{-1} \nabla \otimes \nabla$

$D_{I_1 I_2} := -(i\gamma_0 \gamma^k \partial_k - \gamma_0 m)_{\alpha_1 \alpha_2} \delta_{\Lambda_1 \Lambda_2} \delta(r_1 - r_2)$

$W^{K}_{I_1 I_2} := e_0 (\gamma^k)_{\alpha_1 \alpha_2} \delta(r_1 - r_2) \delta(r_1 - z) \delta_{\eta_1 \eta_2} \sigma^2_{\Lambda_1 \Lambda_2}$

$U^{I_1 I_2 I_3 I_4}_{I_1} := -i\frac{\sqrt{2}}{8\pi e_0} \left[ (\Delta \gamma_0)_{\alpha_2 \alpha_3} \delta_{\alpha_1 \alpha_4} \sigma^2_{\Lambda_2 \Lambda_3} \sigma^2_{\Lambda_1 \Lambda_4} \delta(r_2 - r_3) \delta(r_1 - r_4) \right]_{as(I_1 I_2 I_3 I_4)}$

$L_{K_1 K_2} := i\delta(z_2 - z_1) \delta_{k_1 k_2} \delta_{\eta_1 \eta_2} + i\Delta(z_1) \delta(z_1 - z_2) \delta_{k_1 k_2} \delta_{\eta_1 \eta_2}$

$J^{I_1 I_2}_{K} := -i\frac{\sqrt{2} e_0 P^{tr}(z - r_1) \delta(r_1 - r_2)}{(\Delta \gamma_0)_{\alpha_1 \alpha_2} \delta_{\eta_1 \eta_2} \sigma^2_{\Lambda_1 \Lambda_2}}$. (88)

We impose commutation relations on the $\Psi$ and $B$ fields which become thereby operators acting on a suitable state space. These states should belong to an Hilbert space, if the theory would be renormalized. The renormalization was discussed in \([36]\) in a perturbative manner. As a postulate we introduce the equal time commutation relations

$$i) \{\Psi_{I_1}, \Psi_{I_2}\}^\dagger := A_{I_1 I_2} = C\gamma_0 \sigma^1 \delta(r_1 - r_2)$$

$$ii) [B_{K_1}, \Psi_{I}]^\dagger := 0$$

$$iii) [B_{K_1}, B_{K_2}]^\dagger := C_{K_1 K_2}.$$ (89)

Equation (84-i) is the canonical commutation relation for $\Psi_\alpha$ and $\bar{\Psi}_\alpha$, the Dirac adjoint field. In our notation with charge conjugate spinor operators, the somehow unusual $A_{I_1 I_2}$ occurs. Because of the super field notation we have all four commutators belonging to the $\Psi_\alpha$ and $\bar{\Psi}_\alpha$ integrated in this relation.

Equation (84-ii) states, that the boson fields are looked at as elementary fields. $B_{K}$ is not a function of the $\Psi_I$ fields. Equation (84-ii) is usually postulated in QED.

Relation (84-iii) specifies the boson commutator. The special form of the c-number function $C_{K_1 K_2}$ can be calculated as a consequence of requiring a consistent quantum field theory \([36]\), see below \([61]\).
The set of equations (87) and (89) are the defining relations of Coulomb gauged quantized spinor ED. All previous steps are classical and only for convenience to state a somehow selfconsistent QFT. If one would quantize first and then try to eliminate the Gauss’ law, one would yield an other QFT! The functional Hamiltonians (see below) would then have infinite many terms [37].

We obtain the Hamiltonian corresponding to the equations (87) as

\[ H(\Psi, B) = \frac{1}{2} A_{I_1 J_1} D_{I_1 J_1} \Psi_{I_1} \Psi_{J_1} + \frac{1}{2} A_{I_1 J_1} W_{I_1 J_1}^K B_K \Psi_{I_1} \Psi_{J_1} \]
\[ + \frac{1}{4} A_{I_1 J_1} U_{I_1 J_1}^L \Psi_{I_1} \Psi_{J_1} \Psi_{I_5} \Psi_{J_5} \]
\[ + \frac{1}{2} C_{K_1 K_2} L_{K_3 K_2} B_{K_1} B_{K_2}. \]  

(90)

There is no term \( J_{I_1 J_2} \) in the Hamiltonian. This term was eliminated with help of the commutator (89–ii). Taking the time derivative and using the dynamics, one gets

\[ C_{K_1 K_2} W_{I_1 J_2}^K \Psi_{J_2} = 2 A_{I_1 J_2} J_{K_2}^I \Psi_{I_2}. \]  

(91)

This sort of consistency relation may be motivated by the physical assumption, that there is no asymmetry in the boson–fermion and fermion–boson interaction (actio equals reactio, Newton). Furthermore a consistency consideration leads to the above relation [36].

6.1 Antisymmetric case

We give the definitions of generating functionals for spinor QED. We introduce \( B \) sources and \( \partial^b \) duals

\[ [b_{K_1}, b_{K_2}] = 0 = [\partial^b_{K_1}, \partial^b_{K_2}] \]
\[ [\partial^b_{K_1}, b_{K_2}] = \delta_{K_1 K_2}. \]  

(92)

analogous to the Schwinger sources. Therewith we define

\[ |A(a, j, b)\rangle_F = \sum_{n,m=0}^{\infty} \frac{i^n}{n! m!} \tau_{nm}(I_1, \ldots, I_n, K_1, \ldots, K_m | a) \]
\[ j_{I_1} \wedge \ldots \wedge j_{I_n} b_{K_1} \circ \ldots \circ b_{K_m} | 0 >_F \]
\[ \tau_{nm}(I_1, \ldots, I_n, K_1, \ldots, K_m | a) := <0 | A(I_1, \ldots, I_n) S(B_{K_1}, \ldots, B_{K_m}) | a >_F \]  

(93)

and

\[ |N(a, j, b)\rangle_F = \sum_{n,m=0}^{\infty} \frac{i^n}{n! m!} \phi_{nm}(I_1, \ldots, I_n, K_1, \ldots, K_m | a) \]
\[ j_{I_1} \wedge \ldots \wedge j_{I_n} b_{K_1} \circ \ldots \circ b_{K_m} | 0 >_F \]
\[ \phi_{nm}(I_1, \ldots, I_n, K_1, \ldots, K_m | a) := <0 | N(I_1, \ldots, I_n) S(B_{K_1}, \ldots, B_{K_m}) | a >_F \]  

(94)

where \( \circ \) denotes the symmetric product of the \( b_K \)'s, \( A \) the antisymmetrizer, \( S \) the symmetrizer and \( N \) the normal-ordering and \( |a> \) an energy eigen-’state’. We assume furthermore, that the \( j, \partial \) and \( b, \partial^b \) generators commute. We do not consider bosonic normal-ordering. The field operators in the antisymmetric case are given as

\[ \psi_I = \frac{1}{i} \partial_I - \frac{i}{2} A_{IJ} j^I j^J. \]  

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Inserting these operators into (50) with the Hamiltonian (90) yields

while we have for the normal ordered fermions

\[ \psi_{op}^I = \frac{1}{i} \partial_I + \frac{i}{2} A_{I'I'J'} \]

\[ B_K = \partial_K^b - \frac{1}{2} C_{KK'} b_{K'} \]

\[ B_{op}^K = \partial_K^b + \frac{1}{2} C_{KK'} b_{K'} \, . \quad (95) \]

Inserting these operators into (50) with the Hamiltonian (90) yields

\[ E_{0a} | A(a, j, b) > = \left\{ D_{I_1 I_2 J_1 I_2} \partial_{I_1} + W_{I_1 I_2 J_1 I_2} \partial_{I_2} + L_{K_1 K_2} \partial_{K_1} + \frac{1}{4} A_{I_1 I_2} A_{I_1 I_2} \right\} \]

\[ + J_{K_1 K_2} (\partial_{I_1} \partial_{I_2} - \frac{1}{2} A_{I_1 I_2} A_{I_1 I_2} \psi_{K_1} \psi_{K_2}) \]

\[ + J_{K_1 K_2} (\partial_{I_1} \partial_{I_2} + \frac{1}{4} A_{I_1 I_2} A_{I_1 I_2} \psi_{K_1} \psi_{K_2}) \} | A(a, j, b) > , \quad (96) \]

where the \( A_{I_1 I_2} \) terms stem from the field quantization of the fermions. Boson quantization terms would again occur, if the \( J_{K_1 K_2} \) would be eliminated in the above mentioned way. This result can be obtained from the normal-ordered equation by setting \( F = 0 \), so we treat that case in more detail.

### 6.2 Normal-ordered case

The (fermion) normal-ordered equation is obtained by using the normal-ordered fermions

\[ \psi_I = \frac{1}{i} \partial_I - \frac{i}{2} A_{I'I'J'} + i F_{I'I'J'} \]

\[ \psi_{op}^I = \frac{1}{i} \partial_I + \frac{i}{2} A_{I'I'J'} + i F_{I'I'J'} \, , \quad (97) \]

the boson operators remain unchanged. Before we proceed, we give the commutators of the fields, not displayed commutators vanish. The bosons obey

\[ [\partial_K^b + \frac{1}{2} C_{K_1 K_1'} b_{K_1'} , \partial_K^b + \frac{1}{2} C_{K_2 K_2'} b_{K_2'} ] = 0 \, . \quad (98) \]

while we have for the normal ordered fermions

\[ \left\{ \frac{1}{i} \partial_I \pm i \frac{1}{2} A_{I_1 I_2} j_{I_1} j_{I_2} \right\} \]

\[ \left\{ \frac{1}{i} \partial_I \pm i \frac{1}{2} A_{I_1 I_2} j_{I_1} j_{I_2} \right\} = \pm A_{I_1 I_2} \]

\[ = 0 \, . \quad (99) \]

We calculate directly the (fermion) normal-ordered functional Hamiltonian from \( H[\Psi, B] \). Therefore we have to build

\[ H[j, b, \partial, \partial] = H[ \frac{1}{i} \partial_I + \frac{i}{2} A_{I'I'J'} + i F_{I'I'J'} , \partial_K + \frac{1}{2} C_{KK'} b_{K'} ] \]

\[ - H[ \frac{1}{i} \partial_I - \frac{i}{2} A_{I'I'J'} + i F_{I'I'J'} , \partial_K - \frac{1}{2} C_{KK'} b_{K'} ] \, . \quad (100) \]

We have according to (90) four terms \( T_i \). The \( T_1 \) term is of the same form as in the non-linear spinor field model

\[ T_1[j, \partial] = D_{I_1 I_2} J_{I_1 I_2} \partial_I - D_{I_1 I_2} F_{I_1 I_2} J_{I_1 I_2} \, , \quad (101) \]

19
where the renormalization was discussed in [30]. The term $T_2^\wedge$ is given as

$$T_2[j, \partial]^\wedge = \frac{1}{2} A_{t_1} A_{t_2} W^K_{t_1 t_2} \left[ (\partial_K^b + \frac{1}{2} C_{KK'} b_{K'}) (\ldots) (\ldots) \right. $$

$$\left. - (\partial_K^b - \frac{1}{2} C_{KK'} b_{K'}) (\ldots) (\ldots) \right]$$

$$= \frac{1}{2} A_{t_1} A_{t_2} W^K_{t_1 t_2} \left[ (\partial_K^b \{ (\ldots) (\ldots) -(\ldots) (\ldots) \} \right. $$

$$\left. + \frac{1}{2} C_{KK'} b_{K'} \{ (\ldots) (\ldots) + (\ldots) (\ldots) \} \right]. \quad (102)$$

The first term is just the same as before, while the second yields

$$2. \{ \} = -2 \partial_{t_1} \partial_{t_2} - 2 \frac{1}{4} A_{t_1} A_{t_3} A_{t_2} J_{t_1 t_2} j_{t_2} $$

$$+ 4 F_{t_1} F_{t_2} j_{t_1} \partial_{t_2} - 2 F_{t_1} F_{t_2} j_{t_1} \partial_{t_2}. \quad (103)$$

We may use (11) to obtain

$$\frac{1}{4} A_{t_1} W^K_{t_1 t_2} C_{KK'} b_{K'} = - \frac{1}{4} A_{t_1} A_{t_3} C_{KK'} b_{K'} W^K_{t_1 t_2} \quad (104)$$

$$= - \frac{1}{2} A_{t_1} A_{t_3} J_{t_1 t_2} b_{K'} $$

$$= - \frac{1}{2} J_{t_1 t_2} b_{K} \quad (105)$$

and finally the result

$$T_2[j, \partial]^\wedge = W^K_{t_1 t_2} [j_{t_1} \partial_{t_2} - F_{t_2} j_{t_1} \partial_{t_2}] \partial_K^b $$

$$+ J_{t_1 t_2} b_{K} [\partial_{t_1} \partial_{t_2} - 2 F_{t_1} j_{t_1} \partial_{t_2} $$

$$+ (F_{t_1} F_{t_2} + \frac{1}{4} A_{t_3} A_{t_2} j_{t_1} j_{t_2}] \cdot \quad (106)$$

The third term will be omitted, as it has the same structure as $T_2$ in the non-linear spinor model. The last term runs along the lines of $T_1$ and has not changed at all from [30]. Our result is then

$$E_{0a} |N(a, j, b) > = \left\{ D_{t_1} j_{t_2} \partial_{t_2} - Z_{t_1} j_{t_2} F_{t_1} j_{t_1} j_{t_2} $$

$$+ W^K_{t_1 t_2} [j_{t_1} \partial_{t_2} - F_{t_2} j_{t_1} \partial_{t_2}] \partial_K^b $$

$$+ J_{t_1} b_{K} [\partial_{t_1} \partial_{t_2} - 2 F_{t_1} j_{t_1} \partial_{t_2} $$

$$+ (F_{t_1} F_{t_2} + \frac{1}{4} A_{t_3} A_{t_2} j_{t_1} j_{t_2}] $$

$$+ U_{t_1 t_2} j_{t_1} j_{t_2} [\partial_{t_1} \partial_{t_2} \partial_{t_3} - 3 F_{t_1} j_{t_1} j_{t_2} \partial_{t_2}$$

$$+(3 F_{t_1} F_{t_2} + \frac{1}{4} A_{t_3} A_{t_2} j_{t_1} j_{t_2} \partial_{t_1} $$

$$- (F_{t_1} F_{t_2} F_{t_1} j_{t_2} + \frac{1}{4} A_{t_3} A_{t_2} A_{t_1} j_{t_1} j_{t_2} j_{t_2} $$

$$+ L_{K_1 K_2} b_{K_1} \partial_{K_2} ] |N(a, j, b) >. \quad (107)$$

which coincides with (8.59) in [20] and (63) in [30]. We have succeeded in deriving the normal-ordered functional equation in one single step by Clifford algebraic considerations. No normal-ordering problems, as a vertex normal ordering, was necessary. No singular intermediate equation has occurred. And as a benefit, the calculation was considerable short.
7 Conclusion

We have identified Clifford geometric algebras of multivectors as the appropriate mathematical tool in modeling quantum field theoretic generating functions. These algebras were studied in section 2. Great emphasis was made on the connection of the spaces underlying the Clifford and Grassmann algebras, which induce the multivector structure in a Clifford algebra. This structure is ubiquitously used in physics, but not inherent in an ordinary Clifford algebra. However, if a Clifford algebra is constructed from generators and relations, one has implicitly imposed an ordering, which results in the explicit multivector structure. Only the Clifford algebra of multivectors can systematically account for this situations and uncover the role played by the propagator. The propagator or say antisymmetric part of the contraction directly parameterizes the vacua underlying the theory as was shown in 8.

The 3rd section develops an complete algebraic approach to quantum field theoretic generating functionals by means of Clifford algebras of multivectors. It was shown, that the field operators can be seen as Clifford maps acting on the Grassmann algebra build from the Schwinger source terms. The vacuum was shown to be involved in the definition of matrix elements. This result remains true for path integrals, which rely on the same generating functionals. However, in the very compact notation of path integrals no direct access to this structures is possible.

In section 4 we gave the transition from operator dynamics to generating functionals. This transition was motivated by the Heisenberg equation. However, there is no unique lifting of such an expression. To be able to reproduce the results in literature, we chose the opposite action, not the right action. Further investigations on that topic seem to be necessary. The last sections 5 and 6 gave examples, which showed the easiness and computational advantages of this method. A non-linear spinor field was treated in the antisymmetric and normal-ordered case. The same was done for spinor QED as an example for a boson-fermion coupling theory. The most remarkable result was, that it was possible to derive normal-ordered generating functionals directly, that is in one single step, from the operator dynamics. Furthermore, this showed that the algebraic theory has full control over the involved orderings and thereby over the involved vacuum states. This was investigated in detail in 8. The method was already applied to Dirac theory with some success 31.

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