Exact and paraxial Airy propagation of relativistic electron plasma waves

Maricarmen A. Winkler$^1$ and Felipe A. Asenjo$^2$

$^1$Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile.
$^2$Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Santiago 7491169, Chile.

It is shown that relativistic electron plasma waves can propagate in an Airy-like form. These solutions are possible even for relativistic temperatures. These waves can be constructed in an exact manner, showing that they propagate in an accelerated fashion along coordinates transverse to the thermal speed cone coordinate. Similarly, we show that Airy-like propagation is also a solution for relativistic electron plasma waves in the paraxial approximation. This regime is considered in time-domain, when the paraxial approximation is taken for frequency, and in space-domain when the paraxial approximation is taken for wavelength. In these two different cases, the waves remain structured in the transverse plane. Finally, we show that these solutions allow us to define general and arbitrary Airy-like wavepackets for electron plasma waves.

PACS numbers:

Keywords:

I. INTRODUCTION

Electron plasma waves are one of the simplest possible longitudinal modes of wave propagation in plasmas. These modes are ubiquitous at different energy and temperature scales [1], and they are usually solved in terms of plane waves, with a simple dispersion relation.

However, there are other possible propagation modes that have been quite unexplored so far. The purpose of this work is to show that other propagation modes exist in terms of Airy functions. These modes are of two different kinds. First, there exist exact three-dimensional spatial wavepacket solutions, that evolve in time and space and exhibit accelerated propagation. Besides, we show that there is a second kind of propagating solution, in the paraxial approximation, that can also be obtained in terms of Airy functions. Those wave solutions can also be constructed in two different manners, those that can accelerate along a longitudinal direction and others that have curved trajectories in space. Anew, these are solutions that are structured in the whole three-dimensional space.

Airy-like solutions are a general form for wave propagation that emerge in different natural phenomena, such as light [2, 12], fluids [13], sound [14, 15], gravitational waves [16], and heat diffusion [17], among many others. Therefore, it is expected to obtain this kind of propagation for plasma waves, and some simple Airy waves have been already explored as solutions [18, 19]. However, in this work, we do not only show that this wave solution exist in the relativistic regime (at kinetic and temperature values), but also prove that a general wavepacket Airy-like solution can be arbitrarily proposed in order to have structured and non-diffracting propagation of these relativistic waves. In this sense, we propose complete and new solutions for electron plasma waves, not considered previously.

We start with the equation for electron density perturbation $\tilde{n}$. This equation describes the wave propagation in a relativistic plasma fluid, under the presence of an electrostatic field, assuming a fixed ion fluid background with density $n_0$, and a mobile electron fluid (with charge $-e$ and mass $m$) that react to the small perturbations of the electrostatic field. By assuming a constant relativistic temperature for the electron fluid, then we get the equation for relativistic dynamics of electron plasma waves (see Appendix for derivation)

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{S_e^2}{f} \nabla^2 + \frac{\omega_p^2}{f} \right) \tilde{n} = 0, \tag{1}
\]

where $\omega_p = \sqrt{e^2 n_0 / m}$ is the electron plasma frequency, $f$ is the enthalpy per mass density of the electron fluid [20], and $S_e^2 = (1/m) \partial \tilde{p} / \partial \tilde{n}$ is the electron thermal speed, with perturbative pressure $\tilde{p}$. Usually, the thermal speed depends on temperature, $S_e = S_e(T)$. The function $f$ depends only on temperature $T$, and it can be calculated as [20]

\[
f(T) = \frac{K_3(m/T)}{K_2(m/T)}, \tag{2}
\]

where $K_3$ and $K_2$ are the modified Bessel functions of order 3 and 2, respectively. A more general and complete form of Eq. (1), for arbitrary background electron velocities, can be found in Ref. [20].

Relativistic effects appears through the $f$ function in Eq. (1). This function shows the role of thermal-inertial effects. Thus, it increases the inertia of the electron plasma ($f$ increases with temperature), producing a decreasing of the effective thermal speed and the effective plasma frequency. There is no kinetic relativistic effects in Eq. (1), as perturbatively, the electron fluid has Lorentz factor $\gamma = 1$ (see Appendix).

It is a common practice to solve Eq. (1) for electron plasma waves in terms of plane waves. Distinctively in
this work, we show that Airy wavepackets are also propagating solutions of Eq. (1) in an exact manner and in the paraxial wave approximation. These both solutions are different and they are analyzed below. We conclude by showing how these solutions can be used to construct electron plasma wavepackets with arbitrary transversal forms.

II. EXACT PROPAGATION OF AIRY WAVEPACKET

Let us start by showing that Eq. (1) can be solved exactly in terms of an Airy-like propagating wave. This is a wavepacket that can be solved in terms of the electron thermal speed cone coordinates \( x \pm S_e t / \sqrt{f} \). In this way, this wavepacket accelerates in the transversal direction to such coordinates. In order to show this, let us consider the following form of the perturbed density

\[
\tilde{n}(t, \mathbf{x}) = \zeta(\eta, y, z) \exp \left( i \alpha \xi - i \frac{\omega^2}{4 \alpha S_e^2} \eta \right)
\]  

where \( \eta = x + S_e t / \sqrt{f} \), \( \xi = x - S_e t / \sqrt{f} \) [22], and \( \alpha \) is an arbitrary constant with inverse length unit dimensions. Using this in Eq. (1), we obtain that the function \( \zeta \) follows a Schrödinger-like equation

\[
\frac{1}{i} \frac{\partial \zeta}{\partial \eta} = - \frac{1}{4 \alpha} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \zeta,
\]

which can be solved by Airy functions \( \text{Ai} \). Following Lekner’s results [22], a normalizable wavepacket solution to the above equation can be written as \( \zeta(\eta, y, z) = \zeta_y(\eta, y) \zeta_z(\eta, z) \), where

\[
\zeta_{y,z} = \text{Ai} \left[ \left( 8 a_{y,z} \alpha^2 \right)^{1/3} \left( w_{y,z} - u_{y,z} \eta + iv_{y,z} \eta - a_{y,z} \frac{\eta^2}{2} \right) \right] \\
\times \exp \left[ 2 i \alpha a_{y,z} \eta \left( w_{y,z} - u_{y,z} \eta - a_{y,z} \frac{\eta^2}{3} \right) \right] \\
\times \exp \left[ 2 i \alpha v_{y,z} \left( w_{y,z} - u_{y,z} \eta + iv_{y,z} \eta - a_{y,z} \frac{\eta^2}{2} \right) \right] \\
\times \exp \left[ 2 i \alpha u_{y,z} \left( w_{y,z} - u_{y,z} \eta \right) \right] ,
\]

with \( w_y \equiv y, w_z \equiv z, u_{y,z} \) are arbitrary Galilean boost speeds in respective \( y \) and \( z \)-directions, and \( v_{y,z} > 0 \) are arbitrary factors that allow the solution to be integrable.

Besides, \( a_{y,z} \) are arbitrary accelerations in respective \( y \) and \( z \)-directions. In this way, the relativistic electron plasma wave packet can propagate in an exact manner in terms of Airy functions, showing that it has different high-intensity lobes present, independent accelerations in \( y \) and \( z \) directions with respect to the \( \eta = x + S_e t / \sqrt{f} \) direction. Of course, another straightforward solution exists that has acceleration in the transverse plane with respect to the \( x - S_e t / \sqrt{f} \) direction.

III. PARAXIAL PROPAGATION OF AIRY WAVEPACKETS

Differently to the above exact solution, we can also show that in the paraxial approximation, structured, localized and non-diffracting Airy-like propagation solutions exist. In fact, different behaviors are found depending on the kind of paraxial approximation used, say in time-domain or in space-domain. In the following, we discuss these two types of solutions, showing their different expected behaviors.

A. Accelerating relativistic wavepackets

In the paraxial approximation one can also obtain solutions that propagate in accelerated fashion in time as they move in space. In this case, the paraxial approximation should be performed in the time-domain of the solution.

In order to solve Eq. (1) under this assumption, let us consider a solution for the perturbed density \( \tilde{n} \) with slowly-varying amplitude \( \rho \) and a rapid-varying phase, with frequency \( \omega \), in the form

\[
\tilde{n}(t, \mathbf{x}) \propto \rho(t, \mathbf{x}) \exp(i \omega t).
\]

The paraxial approximation in time-domain can be taken when \( \omega \gg \partial_x^2 \rho / \partial t \rho \) is considered. In this way, Eq. (1) is written for \( \rho \) as

\[
\left( 2i \omega \frac{\partial}{\partial t} - \frac{S_e^2}{f} \nabla^2 + \frac{\omega^2}{f} - \omega^2 \right) \rho = 0.
\]

Now, let us consider an electron plasma wavepacket propagating in a longitudinal direction, say \( z \), where in the transverse direction it remains with a structure described by Bessel functions \( J_n \). In that case,

\[
\rho(t, \mathbf{x}) = \rho_z(t, z) J_n(\alpha r) \exp \left( i \phi - \frac{f \omega^2 - S_e^2 \alpha^2 - \omega_f^2}{2 f \omega} \right),
\]

where \( r = \sqrt{x^2 + y^2} \), and \( \phi = \arctan(y/x) \). Using this ansatz in Eq. (4), we can obtain an expression for the dynamics of \( \rho_z \), which becomes simply

\[
\frac{i}{\partial z} \frac{\partial \rho_z}{\partial z} = \frac{S_e^2}{2 f \omega} \frac{\partial^2 \rho_z}{\partial z^2}.
\]

This last equation can again be solved in terms of Airy functions [22, 23] to have the normalizable solution

\[
\rho_z(t, z) = \text{Ai} \left[ \left( 2 \frac{f \omega^2}{S_e} \right)^{1/3} \left( z - ut + ivt - \frac{a t^2}{2} \right) \right] \\
\exp \left( \frac{i f \omega}{S_e} ut \left( z - ut - \frac{a t^2}{3} \right) \right) \\
\exp \left( \frac{i f \omega}{S_e} vt \left( z - ut + iv \frac{t}{2} - at^2 \right) \right) \\
\exp \left( \frac{i f \omega}{S_e} u \left( z - w \frac{t}{2} \right) \right) ,
\]
where \( u \) is a Galilean boost speed \([22]\), \( v \) is an arbitrary factor that enables the normalization, and \( a \) is the arbitrary acceleration of the different lobes of the solution in the \( z-t \) plane.

The above solution \( \upsilon \) can be used to construct the most general solution for this kind of wavepackets. A general accelerating electron wavepacket in the paraxial approximation in time-domain, can be written as \([24]\)

\[
\tilde{n}(t, x) = \int d\alpha \ g(\alpha) \rho_x(t, z) J_1(\alpha r) \times 
\exp \left( il\phi + \frac{f \omega^2 + S^2 \alpha^2 + \omega_p^2}{2f \omega} \right), \quad (11)
\]

where \( g(\alpha) \) is an arbitrary spectral function, that can be chosen at will to define different wavepacket structures.

Thus, relativistic electron plasma wavepacket \( \upsilon \) present acceleration in a longitudinal direction, while it remains arbitrary structured in the transversal directions.

**B. Curved trajectories relativistic wavepackets**

We can also perform a different paraxial approximation, now in space-domain. This solution consists in wavepackets following curved parabolic trajectories in space, as we show in the following.

Let us consider that the solution of Eq. \( \upsilon \) can be put in the form

\[
\tilde{n}(t, x) \propto \rho(x, y, z) \exp(i\omega t + ikz).
\]

where \( \omega \) is a constant frequency, and the paraxial approximation in space-domain is obtained when \( k \gg \nabla^2 \rho / \nabla \rho \).

In this case, we arrive to the equation

\[
i \frac{\partial \rho}{\partial z} = -\frac{1}{2k} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \rho + \frac{\beta^2}{2k} \rho,
\]

where we have defined the constant \( \beta^2 = k^2 + \omega_p^2/S^2 - \omega^2 f/S^2 \). Notice that this constant is arbitrary and it defines a dispersion relation for the frequency. In principle, it can be chosen to be 0. However, as we will show below, a more general wavepacket solution can be constructed from its arbitrary value.

Equation \( \upsilon \) has wavepacket solutions that show arbitrary parabolic trajectories in the \( x-z \) and \( y-z \) planes \([22, 23]\). The solution is given in terms of \( \rho(x, y, z, \beta) = \rho_x(x, z) \rho_y(y, z) \exp(-i\beta^2 z / 2k) \), with

\[
\rho_{x,y} = \mathrm{Ai} \left( 2a_{x,y} k^3 \right)^{1/3} \left( w_{x,y} - u_{x,y} z + i v_{x,y} z^2 - a_{x,y} \frac{z^2}{2} \right) \times 
\exp \left( i k a_{x,y} \frac{w_{x,y} - u_{x,y} z - a_{x,y} \frac{z^2}{3}}{2} \right) \times 
\exp \left( k v_{x,y} \left( w_{x,y} - u_{x,y} z + i v_{x,y} \frac{z^2}{2} - a_{x,y} \frac{z^2}{2} \right) \right) \times 
\exp \left( i k u_{x,y} \left( w_{x,y} - u_{x,y} \frac{z}{2} \right) \right),
\]

satisfies all the conditions of the paraxial approximation in space-domain. Here, \( g(\beta) \) is the arbitrary spectral function \([24]\) that allows the construction of different relativistic electron plasma wavepackets.

In this form, solution \( \upsilon \) represents the most general form of propagation that shows parabolic trajectories in space. This Airy-like wavepacket propagating solution is a generalization of the one presented in Ref. \([18]\). It is not just only a relativistic version of the solution found in Ref. \([18]\), but our solution \( \upsilon \) allows to have a whole broad family of different structured wavepackets, as we demand that \( \beta \neq 0 \). By restricting ourselves to the \( \beta = 0 \) choice, we recover the solution found in Ref. \([18]\), but we rule out the possibility to define general wavepackets.

**IV. CONCLUSIONS**

It has been established in this work that other wave propagation modes exist in a relativistic electron plasma, very different from the usual plane wave solution. These modes correspond to Airy-like wavepackets, that represent structured and non-spreading propagation of these relativistic waves.

All of the previously shown Airy-like forms of propagation for relativistic electron plasma waves, such as the exact form \( \upsilon \) and, paraxial approximated forms \( \upsilon \) and \( \upsilon \), are the most general solutions of this kind for the perturbed density following Eq. \( \upsilon \); all in different regimes, where relativistic effects decrease the effective value of thermal speed.

The exact solution \( \upsilon \) shows that this mode has different high-intensity lobes and independent accelerations in the \( y \) and \( z \) directions, with respect to \( \eta \) and \( \xi \) directions.

The paraxial propagation of Airy wavepackets can be studied from two different approximations: time-domain \( \upsilon \) and space-domain \( \upsilon \). In the first case, the solution accelerates in time as it moves in space. This acceleration is present in a longitudinal direction, while it remains arbitrarily structured in the transversal direction. In the latter, wavepackets follow curved parabolic trajectories in space, that accelerate in the transverse direction.
These Airy-like solutions can be used to construct general classes of arbitrary wavepackets. They will be dependent on the spectral functions, and can be used to produce new kinds of three-dimensional structured of electron plasma propagation. This is currently under investigation.

Finally, the same kind of procedure developed in this work, can be used to study similar Airy-like propagation for other plasma waves, such as electromagnetic plasma waves. In those cases, it is also expected that general wavepacket solutions can be constructed.

Data sharing is not applicable to this article as no new data were created or analyzed in this study. The authors have no conflicts to disclose.

Appendix A: Relativistic electron plasma fluid

Let us consider a relativistic plasma fluid, under the presence of an electrostatic field. Let us assume a fixed ion fluid background, with a mobile electron fluid with charge $-e$ and mass $m$. The plasma is immersed in a perturbed electrostatic field $\tilde{E}$. In such case, the Gauss’ Law reads

$$\nabla \cdot \tilde{E} = -e \tilde{n}, \quad (A1)$$

where $\tilde{n}$ is the electron density perturbation. We have assumed that the electron fluid is at rest in the background, with density $n_0$ (equal to density ion background), and it moves in a perurbative manner with velocity $\tilde{v}$, and relativistic Lorentz factor $\hat{\gamma} = (1 - \tilde{v} \cdot \tilde{v})^{-1/2} \approx 1$. The electron fluid fulfill the conservation law $^{[20]}$

$$\frac{\partial \tilde{n}}{\partial t} + n_0 \nabla \cdot \tilde{v} = 0. \quad (A2)$$

Lastly, the electron fluid momentum equation is simply $^{[20]}$

$$mf \frac{\partial \tilde{v}}{\partial t} = -e \tilde{E} - \frac{mS^2}{n_0} \nabla \tilde{n}, \quad (A3)$$

where $f$ is the enthalpy per mass density $^{[20]}$ of the electron fluid, and $S^2 = (1/m)\partial \tilde{p}/\partial \tilde{n}$ is the electron thermal speed, with perturbative pressure $\tilde{p}$.

Using Eqs. $^{[A1]}-^{[A3]}$, we are able to obtain the dynamical Eq. $^{[1]}$.

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