Corrections to the Planck’s radiation law from loop quantum gravity

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Abstract

We study the dispersion relation obtained from the semiclassical loop quantum gravity. This dispersion relation is considered for a photon system at finite temperatures and the changes to the Planck’s radiation law, the Wien and Boltzmann laws are discussed. Corrections to the equation of state of the black body radiation are also obtained.

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1 Introduction

Recently it has been studied with great interest the possibility that particles undergo changes in their dispersion relation as a consequence of gravitational or high energy effects. Hawking’s spectrum for black holes, for instance, displays changes in the range of high frequencies \[1\]. An analogous effect appears in the inflationary cosmology for the inflaton spectrum \[2\]. Another example is found in the field theory of non-commutative spaces, where the dispersion relation for photons also changes \[3\]. In the semiclassical loop quantum gravity (SLQG) it is possible to find an expression for a field over the weak quantum gravitational background. For this field corrections to its dispersion relation are also obtained \[4, 5\]. For a review see Ref. \[6\]. The modified dispersion relation could give rise to several consequences as the breaking of the Lorentz symmetry, for example. As an alternative approach, modified dispersion relations have been used to provide an explanation of anomalies in ultrahigh energetic cosmic rays \[7\].

In general one would expect that the consequences of changes to a dispersion relation were observed only in highly energetic particles. However, this is not always the case. To see this let us consider the modified dispersion relation for photons,

\[p^2 = E^2 \left[1 + \xi E/E_p + \mathcal{O}\left((E/E_p)^2\right)\right], \quad (1)\]

where \(\xi \approx 1\) and \(E_p \approx 10^{18} \text{ GeV}\) is the Planck energy. In this case, a photon that travels a distance \(L\) will have a time delay

\[\Delta t \approx \xi \frac{L}{c} (E/E_p). \quad (2)\]

It can be seen from Eq. \(\[2\]\) that if the distance \(L\) is large enough, this time delay will be observed even if the energy of the particle is not large. In other words, this is a small microscopic effect that could be observed at a macroscopic scale provided the macroscopic variables are large enough. For the case of the SLQG, the corrections to the dispersion relation for photons are analogous to those described by Eq. \(\[1\]\).

A well known result from condensed matter is that small changes in the microscopic properties of a system can lead to important modifications in its
macroscopic behavior. For quantum liquids, for example, it can be seen that a change in the dispersion relation of the excitations can result in a completely different behavior. Liquid helium, $^4$He, provides a good example of a boson fluid with a small residual interaction. At low temperatures this interaction is the one responsible for the superfluid state [8]. The main excitations below $1K$ are just phonons with a dispersion relation $\omega \propto k$, but at higher temperatures roton excitations with $\omega \propto k^2$ must be included. Accounting for these, corrections to the thermodynamic properties can be observed [8, 9].

In this letter we study the changes to the macroscopic properties at finite temperatures of a photon system provided the dispersion relation proposed by the SLQG is assumed. To first order in the Planck length, the dispersion relation is linear in the polarization and so the thermodynamic properties remain unchanged. However, to second order, the Planck’s radiation law loses its universal character as it will be shown shortly. Consequences of losing this universal behavior, as it is a correction to the Wien’s displacement law, are also discussed. We finally find that the Boltzmann law for the radiated energy density and the radiation pressure change.

This work is organized as follows. In section 2 the dispersion relation obtained from the SLQG is presented. It is also shown here that the black body radiation law loses its universal behavior and that this leads to modifications to the Wien’s law. In section 3 the changes to the thermodynamic properties are obtained; and finally in section 4 our results are summarized.

2 The dispersion relation from the LQG

The dispersion relation obtained from the SLQG for photons is given by [1, 9]:

$$\omega = ck + \lambda a_1 k^2 - a_2 k^3,$$

where $a_1$ is proportional to the Planck length $l_p$, $a_2$ is proportional to $l_p^2$ and $\lambda$ is the polarization of the photons that can take values $\lambda = \pm 1$. The coefficients $a_1$ and $a_2$ depend on the semiclassical state chosen in the LQG formalism.
It is worth mentioning that Eq. (3) has been obtained under the hypothesis that $kl_p \ll 1$; which forbids the solution $k_c \neq 0$ for $\omega = 0$. Notice that such a $k_c$ is of the order of
\[ k_c \propto \frac{1}{l_p}, \tag{4} \]
which is very large and does not satisfy $kl_p \ll 1$. Therefore, moments $k \geq k_c$ can be neglected. It can also be observed that there is a value $k_{max}$ for which \(\omega\) from Eq. (3) has its maximum. This $k_{max}$ is of the same order of magnitude as $k_c$.

Let us now proceed to calculate the modified Planck’s radiation law. The number of particles inside a volume element in phase space is
\[ dN_\omega = \rho(\omega) \frac{4\pi V}{(2\pi)^3} k^2 dk, \tag{5} \]
where $V$ is the volume and $\rho(\omega)$ is given by
\[ \rho(\omega) = \frac{1}{e^{\beta \hbar \omega} - 1}, \tag{6} \]
with $\beta = (k_B T)^{-1}$, $k_B$ being the Boltzmann constant and $T$ the temperature. The energy associated with the above volume element is
\[ dU_\omega = \hbar \omega \rho(\omega) \frac{V}{2\pi^2} k^2 dk. \tag{7} \]
Using the dispersion relation in Eq. (3) to first order in $l_p$ we obtain for the momentum $k$
\[ k = \frac{\omega}{c} \left[ 1 - \frac{\lambda a_1}{c^2} \omega \right]. \tag{8} \]
However, if considered to second order, we get
\[ k = \frac{\omega}{c} \left[ 1 - \frac{\lambda a_1}{c^2} \omega + \left( \frac{2a_1^2}{c^4} + \frac{a_2}{c^3} \right) \omega^2 \right], \tag{9} \]
so that, to this order
\[ k^2 dk = \frac{\omega^2}{c^3} [1 - \lambda C_1 \omega + C_2 \omega^2] d\omega, \tag{10} \]
with
\[
C_1 = \frac{4a_1}{c^2} \quad \text{and} \quad C_2 = \frac{15a_1^2}{c^4} + \frac{5a_2}{c^3}.
\] (11)

Therefore, the energy in Eq. (7) becomes
\[
dU_\omega = \frac{\hbar V}{2\pi^2 c^3} \rho(\omega) [1 - \lambda C_1 \omega + C_2 \omega^2] d\omega.
\] (12)

Notice, in this equation, that after summing over all possible \(\lambda\) values, the corrections to first order in \(l_p\) get canceled. Therefore the first corrections to the thermodynamic quantities appear to second order. That is
\[
dU_\omega = \frac{V \hbar}{\pi^2 c^3} \rho(\omega)^3 \omega [1 + C_2 \omega^2] d\omega = \frac{V \hbar}{\pi^2 c^3 (\hbar \beta)^4} x^3 e^x - 1 \left[1 + \frac{C_2}{(\hbar \beta)^2} x^2\right] dx,
\] (13)

with \(x = \hbar \omega \beta\). To order zero Eq. (13) yields the usual Planck distribution,
\[
P_0(x) = \frac{x^3}{e^x - 1},
\] (14)

which is valid for the radiation of a black body at any given temperature and therefore it is universal. However, the first corrections to Eq. (13) yield the modified distribution
\[
P(x) = \frac{x^3}{e^x - 1} (1 + \alpha x^2), \quad \alpha = \frac{C_2 k_B^2}{\hbar^2 T^2},
\] (15)

that does depend on temperature and thus it loses the universal property observed in \(P_0(x)\). Figure 1 shows the behavior of the modified Planck distribution \(P(x)\) from Eq. (13) compared to the usual Planck distribution \(P_0(x)\) for different temperatures.

Another way to express the universal behavior of \(P_0(x)\) can be obtained by considering that it is invariant under the transformation
\[
\omega \rightarrow \gamma \omega, \quad T \rightarrow \gamma T.
\] (16)

If Eq. (16) is assumed, to zero order in \(l_p\), the energy \(dU_\omega\) scales with \(\gamma^4\). That is, it preserves form but presents changes in its global scale. Including
the quadratic corrections, \( dU_\omega \) changes both its form and scale.

A consequence of the breaking of universality is that the maximum of \( P(x) \) depends on temperature. For the usual Planck distribution, \( P_0(x) \) the maximum occurs when

\[
x_{0\text{max}} = \frac{\hbar \omega_{\text{max}}}{k_B T} \approx 2.822,
\]

which is invariant under the transformation in Eq. (16). The expression for \( x_{0\text{max}} \) in Eq. (17) is the well known Wien’s displacement law: the frequency for the maximum in the Planck distribution variates linearly with temperature. However, for \( P(x) \) the maximum occurs when \( x \) satisfies the condition

\[
(3 - x)e^x - 3 + \alpha x^2(5e^x - xe^x - 5) = 0.
\]
To first order in $\alpha$, the solution is given by

$$x_{\text{max}} \simeq x_{0\text{max}} + (18.218)\alpha.$$  \hfill (19)

Notice that $x_{\text{max}}$ does not have the symmetry of Eq. (16) because of its temperature dependence. From Eq. (19) we get

$$\omega_{\text{max}} = \left(\frac{2.822k_B}{\hbar}\right)T + \left(\frac{18.218C_2k_B^3}{\hbar^3}\right)T^3,$$  \hfill (20)

with $C_2$ defined in Eq. (11). Therefore, the Wien’s displacement law gets a correction proportional to $T^3$.

It is clear from Wien’s law, Eq. (17), that if an astrophysical object radiates with a well known frequency, then it will be possible to know its temperature. However, from the modified Wien’s law, Eq. (20), we notice that the expression for $\omega_{\text{max}}$ is cubic in $T$ and no longer linear. Nevertheless, these cubic corrections are usually very small so they can only be observed at high temperatures.

Now, we investigate the cosmological consequences of losing universality in Eq. (13). If the Robertson-Walker metric is considered, the frequency and temperature for the scaling are both proportional to the inverse radius of the universe $R$

$$\omega \propto \frac{1}{R}, \quad T \propto \frac{1}{R}.$$  \hfill (21)

By assuming this scaling transformation it can be seen that the energy $dU_\omega$ from Eq. (13) to order zero preserves form but changes scale with expansion of the universe. However, once the corrections are included, it displays modifications in both form and scale.

The dependence on $R$ of the temperature is derived from the Boltzmann law. We will see in the next section that this law is modified. This last result changes the dependence on $R$ of $T$, but does not solve the problem of losing the universal character of $P(x)$.

There are theories from which one can also get corrections to the Planck distribution [11, 12]. Some of these do not actually break universality of
the distribution. As an example, in quantum groups theory, the Planck distribution obtained has a complex structure but the universal character remains [11]. However, if a quantum mechanics consistent with the existence of a minimal length is considered, the Planck distribution loses universality [12].

3 Changes in the thermodynamic properties

The energy of the photon system can be obtained by integrating out Eq. (13) and evaluating the upper limit at the momentum $k_{\text{max}}$ from section 2. As the momentum $k_{\text{max}}$ is very large and the energy decreases rapidly for $\omega > \omega_{\text{max}}$, we can approximate the value of the energy by the integral over all possible values of $k$. Considering these simplifications we find

$$U = \frac{4V}{c} T^4 \sigma(T),$$

with $\sigma(T)$ given by

$$\sigma(T) = \sigma_0 \left[ 1 + \left( \frac{40C_2k_B^2\pi^2}{21\hbar^2} \right) T^2 \right], \quad \sigma_0 = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}. \quad (23)$$

Equation (22) represents the modified Boltzmann law. The correction obtained appears in the Stephan-Boltzmann constant, Eq. (23), which now displays a quadratic dependence on temperature.

Using Eq. (22) we obtain for the specific heat at constant volume,

$$C_V = \frac{16VT^3}{c} \sigma_0 \left[ 1 + \left( \frac{60C_2k_B^2\pi^2}{21\hbar^2} \right) T^2 \right]. \quad (24)$$

To get other thermodynamic quantities it is necessary to know not only the energy $U$ but also the partition function. In the grand canonical ensemble, the grand potential for bosons is given by [13]

$$\Omega = k_B T \sum_{l>0} \sum_{\lambda} \ln(1 - e^{-\beta(\epsilon_{l\lambda}-\mu)}), \quad (25)$$

where $l$ corresponds to the $l$-th momentum and $\lambda$ is the polarization. Additionally, for the case of photons $\mu = 0$. The sum above can be replaced by
an integral with aid of the rule

$$\sum_l \to \frac{V}{(2\pi)^3} \int dk^3 = \frac{V4\pi}{(2\pi)^3} \int k^2 dk.$$ \hspace{1cm} (26)

Performing these substitutions we obtain

$$\Omega = \frac{\kappa TV}{2\pi^2 c^3} \sum_{\lambda} \int_0^\infty \ln(1 - e^{-\beta h\omega}) \omega_\lambda^2 [1 - \lambda C_1 \omega_\lambda + C_2 \omega_\lambda^2] d\omega_\lambda$$

$$= \frac{\kappa TV}{\pi^2 c^3} \int_0^\infty \ln(1 - e^{-\beta h\omega}) \omega^2 [1 + C_2 \omega^2] d\omega$$

$$= -\left(\frac{V\kappa^4\pi^2}{45\hbar^3 c^3}\right) T^4 \left[1 + \left(\frac{8C_2\kappa^2\pi^2}{7\hbar^2}\right) T^2\right], \hspace{1cm} (27)$$

so that the entropy is

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_V = \left(\frac{4\kappa^4\pi^2}{45\hbar^3 c^3}\right) T^3 \left[1 + \left(\frac{12C_2\kappa^2\pi^2}{7\hbar^2}\right) T^2\right]. \hspace{1cm} (28)$$

With this result, considering Eq. \((27)\), and using the thermodynamic relation

$$U = \Omega + TS, \hspace{1cm} (29)$$

we recover Eq. \((22)\) for the energy \(U\). In addition, for the radiation pressure we obtain

$$P = \left(\frac{\kappa^4\pi^2}{45\hbar^3 c^3}\right) T^4 \left[1 + \left(\frac{8C_2\kappa^2\pi^2}{7\hbar^2}\right) T^2\right]. \hspace{1cm} (30)$$

From this it can be observed that the radiation pressure has a small increase respect to the standard result appreciable only at high temperatures.

The equation of state for the radiation pressure and energy density is usually expressed in the form

$$P = \bar{\omega} \frac{U}{V}. \hspace{1cm} (31)$$

In general, \(\bar{\omega}\) is constant, but in this case it has a temperature dependence of the form

$$\bar{\omega} = \frac{1}{3} \left[1 - \left(\frac{16C_2\kappa^2\pi^2}{21\hbar^2}\right) T^2\right]. \hspace{1cm} (32)$$

Notice that the correction to the usual equation of state is negative, which is analogous to the equation of state for a space with cosmological constant \(\Omega\).
4 Summary

In this work we consider the dispersion relation for photons obtained from the SLQG and study the thermodynamic changes. One of these changes is that the modified Planck distribution does not preserve universality. This is because the corrections depend on frequency but not on temperature and therefore it is not possible to express the modified distribution in terms of a dimensionless variable. Some consequences of this lack of universality were reviewed. One of these is the corrections to the Wien’s displacement law. Here, there is a small shift of the maximum frequency of the Planck distribution which depends on the cubic temperature. Boltzmann’s law and the radiation pressure also get small corrections. The equation of state for the energy density and pressure also gets modified. The constant relating pressure and energy density now has negative corrections that depend quadratically on temperature.

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