Catalan-like numbers and Stieltjes moment sequences

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Abstract

We provide sufficient conditions under which the Catalan-like numbers are Stieltjes moment sequences. As applications, we show that many well-known counting coefficients, including the Bell numbers, the Catalan numbers, the central binomial coefficients, the central Delannoy numbers, the factorial numbers, the large and little Schröder numbers, are Stieltjes moment sequences in a unified approach.

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1 Introduction

A sequence \((m_n)_{n \geq 0}\) of numbers is said to be a Stieltjes moment sequence if it has the form

\[
m_n = \int_0^{+\infty} x^n d\mu(x),
\]

where \(\mu\) is a non-negative measure on \([0, +\infty)\). It is well known that \((m_n)_{n \geq 0}\) is a Stieltjes moment sequence if and only if \(\det[m_{i+j}]_{0 \leq i,j \leq n} \geq 0\) and \(\det[m_{i+j+1}]_{0 \leq i,j \leq n} \geq 0\) for all \(n \geq 0\) [9, Theorem 1.3]. Another characterization for Stieltjes moment sequences comes from the theory of total positivity.

Let \(A = [a_{n,k}]_{n,k \geq 0}\) be a finite or an infinite matrix. It is totally positive (TP for short), if its minors of all orders are nonnegative. Let \(\alpha = (a_n)_{n \geq 0}\) be an infinite sequence of nonnegative numbers. Define the Hankel matrix \(H(\alpha)\) of the sequence \(\alpha\) by

\[
H(\alpha) = [a_{i+j}]_{i,j \geq 0} =
\begin{bmatrix}
    a_0 & a_1 & a_2 & a_3 & \cdots \\
    a_1 & a_2 & a_3 & a_4 & \cdots \\
    a_2 & a_3 & a_4 & a_5 & \cdots \\
    a_3 & a_4 & a_5 & a_6 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

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Then $\alpha$ is a Stieltjes moment sequence if and only if $H(\alpha)$ is totally positive (see [7, Theorem 4.4] for instance).

Many counting coefficients are Stieltjes moment sequences. For example, the factorial numbers $n!$ form a Stieltjes moment sequence since

$$n! = \int_0^\infty x^n e^{-x} dx. \quad (1.2)$$

The Bell numbers $B_n$ form a Stieltjes moment sequence since $B_n$ can be interpreted as the $n$th moment of a Poisson distribution with expected value 1 by Dobinski’s formula

$$B_n = \frac{1}{e} \sum_{k \geq 0} \frac{k^n}{k!}.$$  

The Catalan numbers $C_n = \binom{2n}{n}/(n+1)$ form a Stieltjes moment sequence since

$$\det[C_{i+j}]_{0 \leq i,j \leq n} = \det[C_{i+j+1}]_{0 \leq i,j \leq n} = 1, \quad n = 0, 1, 2, \ldots$$

(see Aigner [1] for instance). Bennett [3] showed that the central Delannoy numbers $D_n$ and the little Schröder numbers $S_n$ form Stieltjes moment sequences by means of their generating functions (see Remark 2.11 and Example 2.12). All these counting coefficients are the so-called Catalan-like numbers. In this note we provide sufficient conditions such that the Catalan-like numbers are Stieltjes moment sequences by the total positivity of the associated Hankel matrices. As applications, we show that the Bell numbers, the Catalan numbers, the central binomial coefficients, the central Delannoy numbers, the factorial numbers, the large and little Schröder numbers are Stieltjes moment sequences in a unified approach.

### 2 Main results and applications

Let $\sigma = (s_k)_{k \geq 0}$ and $\tau = (t_k)_{k \geq 1}$ be two sequences of nonnegative numbers and define an infinite lower triangular matrix $R := R^{\sigma,\tau} = [r_{n,k}]_{n,k \geq 0}$ by the recurrence

$$r_{0,0} = 1, \quad r_{n+1,k} = r_{n,k-1} + s_k r_{n,k} + t_{k+1} r_{n,k+1}, \quad (2.1)$$

where $r_{n,k} = 0$ unless $n \geq k \geq 0$. Following Aigner [2], we say that $R^{\sigma,\tau}$ is the recursive matrix and $r_{n,0}$ are the Catalan-like numbers corresponding to $(\sigma, \tau)$.

The Catalan-like numbers unify many well-known counting coefficients, such as

1. the Catalan numbers $C_n$ if $\sigma = (1, 2, 2, \ldots)$ and $\tau = (1, 1, 1, \ldots)$;
2. the central binomial coefficients $\binom{2n}{n}$ if $\sigma = (2, 2, 2, \ldots)$ and $\tau = (2, 1, 1, \ldots)$;
3. the central Delannoy numbers $D_n$ if $\sigma = (3, 3, 3, \ldots)$ and $\tau = (4, 2, 2, \ldots)$;
4. the large Schröder numbers $r_n$ if $\sigma = (2, 3, 3, \ldots)$ and $\tau = (2, 2, 2, \ldots)$;
(5) the little Schröder numbers \(S_n\) if \(\sigma = (1, 3, 3, \ldots)\) and \(\tau = (2, 2, \ldots)\);

(6) the (restricted) hexagonal numbers \(h_n\) if \(\sigma = (3, 3, 3, \ldots)\) and \(\tau = (1, 1, 1, \ldots)\);

(7) the Bell numbers \(B_n\) if \(\sigma = \tau = (1, 2, 3, 4, \ldots)\);

(8) the factorial numbers \(n!\) if \(\sigma = (1, 3, 5, 7, \ldots)\) and \(\tau = (1, 4, 9, 16, \ldots)\).

Rewrite the recursive relation (2.1) as

\[
\begin{bmatrix}
    r_{1,0} & r_{1,1} \\
    r_{2,0} & r_{2,1} & r_{2,2} \\
    r_{3,0} & r_{3,1} & r_{3,2} & r_{3,3} \\
    \vdots & & \ddots & \\
\end{bmatrix}
= 
\begin{bmatrix}
    r_{0,0} & r_{1,1} \\
    r_{2,0} & r_{2,1} & r_{2,2} \\
    \vdots & & \ddots & \\
\end{bmatrix}
\begin{bmatrix}
    s_0 & 1 \\
    t_1 & s_1 & 1 \\
    t_2 & s_2 & \ddots & \\
    \vdots & & \ddots & \\
\end{bmatrix},
\]

or briefly,

\[
\begin{array}{c}
R \\
\end{array} = RJ
\]

where \(\overline{R}\) is obtained from \(R\) by deleting the 0th row and \(J\) is the tridiagonal matrix

\[
J := J^{\sigma, \tau} = 
\begin{bmatrix}
    s_0 & 1 \\
    t_1 & s_1 & 1 \\
    t_2 & s_2 & \ddots & \\
    \vdots & & \ddots & \\
\end{bmatrix}.
\]

Clearly, the recursive relation (2.1) is decided completely by the tridiagonal matrix \(J\). Call \(J\) the coefficient matrix of the recursive relation (2.1).

**Theorem 2.1.** If the coefficient matrix is totally positive, then the corresponding Catalan-like numbers form a Stieltjes moment sequence.

**Proof.** Let \(H = [r_{n+k,0}]_{n,k \geq 0}\) be the Hankel matrix of the Catalan-like numbers \((r_{n,0})_{n \geq 0}\). We need to show that \(H\) is totally positive. We do this by two steps. We first show the total positivity of the coefficient matrix \(J\) implies that of the recursive matrix \(R\). Let \(R_n = [r_{i,j}]_{0 \leq i,j \leq n}\) be the \(n\)th leading principal submatrix of \(R\). Clearly, to show that \(R\) is TP, it suffices to show that \(R_n\) are TP for \(n \geq 0\). We do this by induction on \(n\). Obviously, \(R_0\) is TP. Assume that \(R_n\) is TP. Then by (2.1), we have

\[
R_{n+1} = 
\begin{bmatrix}
    1 & O \\
    O & R_n
\end{bmatrix} L_n,
\]

where

\[
L_n = 
\begin{bmatrix}
    1 & & & \\
    s_0 & 1 & & \\
    t_1 & s_1 & 1 & \\
    t_2 & s_2 & \ddots & \\
    \vdots & \ddots & \ddots & 1 & \\
    t_{n-1} & s_{n-1} & 1 & & \\
    t_n & s_n & 1 & & \\
\end{bmatrix}.
\]
Clearly, the total positivity of $R_n$ implies that of $\begin{bmatrix} 1 & O \\ O & R_n \end{bmatrix}$. On the other hand, $J$ is TP, so is its submatrix $J_n$, as well as the matrix $L_n$. Thus the product matrix $R_{n+1}$ is TP, and $R$ is therefore TP by induction.

Secondly we show the total positivity of $R$ implies that of $H$. Let $T_0 = 1, T_k = t_1 \cdots t_k$ and

$$T = \begin{bmatrix} T_0 & T_1 & \cdots \\ & T_2 \\ & & \ddots \end{bmatrix}.$$

Then it is not difficult to verify that $H = RT^t R^t$ (see [2, (2.5)]). Now $R$ is TP, and so is its transpose $R^t$. Clearly, $T$ is TP. Thus the product $H$ is also TP. This completes the proof.

We now turn to the total positivity of tridiagonal matrices. Such a matrix is often called a Jacobi matrix. There are many well-known results about the total positivity of tridiagonal matrices. For example, a finite nonnegative tridiagonal matrix is totally positive if and only if all its principal minors containing consecutive rows and columns are nonnegative [7, Theorem 4.3]; and in particular, an irreducible nonnegative tridiagonal matrix is totally positive if and only if all its leading principal minors are positive [6, Example 2.2, p. 149]. Clearly, $J_{\sigma,\tau}$ is irreducible. So, to show the total positivity of $J_{\sigma,\tau}$, it suffices to show that all its leading principal minors are positive.

Example 2.2. For the Catalan-like numbers $n!$, we have $s_k = 2k + 1$ and $t_k = k^2$. It is not difficult to show that the $n$th leading principal minor of $J_{\sigma,\tau}$ is equal to $n!$. Thus $J_{\sigma,\tau}$ is totally positive, and so the factorial numbers $n!$ form a Stieltjes moment sequence, a well-known result.

Lemma 2.3. If $s_0 \geq 1$ and $s_k \geq t_k + 1$ for $k \geq 1$, then the tridiagonal matrix $J_{\sigma,\tau}$ is totally positive.

Proof. Let $D_n$ be the $n$th leading principal minor of $J_{\sigma,\tau}$. It suffices to show that all $D_n$ are nonnegative. We do this by showing the following stronger result:

$$1 \leq D_0 \leq D_1 \leq D_{n-2} \leq D_{n-1} \leq D_n \leq \cdots.$$

Obviously, $D_0 = s_0 \geq 1$ and $D_1 = s_0 s_1 - t_1 \geq s_0 = D_0$ since $s_1 \geq t_1 + 1$. Assume now that $D_{n-1} \geq D_{n-2} \geq 1$ for $n \geq 2$. Then by expanding the determinant $D_n$ along the last row or column, we obtain

$$D_n = s_n D_{n-1} - t_n D_{n-2} \geq (s_n - t_n)D_{n-1} \geq D_{n-1} \geq 1$$

by $s_n \geq t_n + 1$ and the induction hypothesis, as required. The proof is complete.

Combining Theorem 2.1 and Lemma 2.3 we obtain the following.

Corollary 2.4. If $s_0 \geq 1$ and $s_k \geq t_k + 1$ for $k \geq 1$, then the Catalan-like numbers corresponding to $(\sigma, \tau)$ form a Stieltjes moment sequence.
Example 2.5. The Bell numbers $B_n$, the Catalan numbers $C_n$, the central binomial coefficients $\binom{2n}{n}$, the (restricted) hexagonal numbers $H_n$, and the large Schröder numbers $r_n$ form a Stieltjes moment sequence respectively.

In what follows we apply Theorem 2.1 to the recursive matrix $R(p, q; s, t) = [r_{n,k}]_{n,k \geq 0}$ defined by

$$
\begin{align*}
    r_{0,0} &= 1, & r_{n+1,0} &= pr_{n,0} + qr_{n,1}, \\
    r_{n+1,k+1} &= r_{n,k} + sr_{n,k+1} + tr_{n,k+2}. \\
\end{align*}
$$

(2.2)

The coefficient matrix of (2.2) is

$$
\begin{bmatrix}
p & 1 \\
q & s & 1 \\
t & s & 1 & \ddots \\
&s & \ddots & \ddots \\
\end{bmatrix}
$$

(2.3)

The following result is a special case of [4, Proposition 2.5].

Lemma 2.6. The Jacobi matrix $J(p, q; s, t)$ is totally positive if and only if $s^2 \geq 4t$ and $p(s + \sqrt{s^2 - 4t})/2 \geq q$.

On the other hand, $R(p, q; s, t)$ is also a Riordan array. A Riordan array, denoted by $(d(x), h(x))$, is an infinite lower triangular matrix whose generating function of the $k$th column is $x^k h^k(x)d(x)$ for $k = 0, 1, 2, \ldots$, where $d(0) = 1$ and $h(0) \neq 0$ [8]. A Riordan array $R = [r_{n,k}]_{n,k \geq 0}$ can be characterized by two sequences $(a_n)_{n \geq 0}$ and $(z_n)_{n \geq 0}$ such that

$$
\begin{align*}
    r_{0,0} &= 1, & r_{n+1,0} &= \sum_{j \geq 0} z_j r_{n,j}, & r_{n+1,k+1} &= \sum_{j \geq 0} a_j r_{n,k+j}. \\
\end{align*}
$$

(2.4)

for $n, k \geq 0$ (see [3] for instance). Let $Z(x) = \sum_{n \geq 0} z_n x^n$ and $A(x) = \sum_{n \geq 0} a_n x^n$. Then it follows from (2.4) that

$$
d(x) = \frac{1}{1 - xZ(xh(x))}, \quad h(x) = A(xh(x)).
$$

(2.5)

Now $R(p, q; s, t)$ is a Riordan array with $Z(x) = p + qx$ and $A(x) = 1 + sx + tx^2$. Let $R(p, q; s, t) = (d(x), h(x))$. Then by (2.5), we have

$$
d(x) = \frac{1}{1 - x(p + q x h(x))}, \quad h(x) = 1 + sxh(x) + tx^2h^2(x).
$$

It follows that

$$
h(x) = \frac{1 - sx - \sqrt{1 - 2sx + (s^2 - 4t)x^2}}{2tx^2}
$$

and

$$
d(x) = \frac{2t}{2t - q + (qs - 2pt)x + q\sqrt{1 - 2sx + (s^2 - 4t)x^2}}
$$

(see [10] for details).

Combining Theorem 2.1 and Lemma 2.6 we have the following.
Theorem 2.7. Let $p, q, s, t$ be all nonnegative and
\[
\sum_{n \geq 0} d_n x^n = \frac{2t}{2t - q + (qs - 2pt)x + q\sqrt{1 - 2sx + (s^2 - 4t)x^2}}.
\]
If $s^2 \geq 4t$ and $p(s + \sqrt{s^2 - 4t})/2 \geq q$, then $(d_n)_{n \geq 0}$ is a Stieltjes moment sequence.

Setting $q = t$ in Theorem 2.7, we obtain

Corollary 2.8. Let $p, s, t$ be all nonnegative and
\[
\sum_{n \geq 0} d_n x^n = \frac{2}{1 + (s - 2p)x + \sqrt{1 - 2sx + (s^2 - 4t)x^2}}.
\]
If $s^2 \geq 4t$ and $p(s + \sqrt{s^2 - 4t})/2 \geq t$, then $(d_n)_{n \geq 0}$ is a Stieltjes moment sequence.

In particular, taking $p = s$ in Corollary 2.8 and noting $s^2 \geq 4t$ implies that $s(s + \sqrt{s^2 - 4t})/2 \geq t$, we have

Corollary 2.9. Let $s, t$ be nonnegative and
\[
\sum_{n \geq 0} d_n x^n = \frac{2}{1 - sx + \sqrt{1 - 2sx + (s^2 - 4t)x^2}}.
\]
If $s^2 \geq 4t$, then $(d_n)_{n \geq 0}$ is a Stieltjes moment sequence.

On the other hand, taking $p = s$ and $q = 2t$ in Theorem 2.7, we obtain

Corollary 2.10. Let $s, t$ be nonnegative and
\[
\sum_{n \geq 0} d_n x^n = \frac{1}{\sqrt{1 - 2sx + (s^2 - 4t)x^2}}.
\]
If $s^2 \geq 4t$, then $(d_n)_{n \geq 0}$ is a Stieltjes moment sequence.

Remark 2.11. Corollaries 2.9 and 2.10 have occurred in Bennett [3, §9].

Example 2.12. The Catalan numbers $C_n$, the central binomial coefficients $\binom{2n}{n}$, the central Delannoy numbers $D_n$, the large Schröder numbers $r_n$, the little Schröder numbers $S_n$ have generating functions
\[
\sum_{n \geq 0} C_n x^n = \frac{2}{1 + \sqrt{1 - 4x}},
\]
\[
\sum_{n \geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}},
\]
\[
\sum_{n \geq 0} D_n x^n = \frac{1}{\sqrt{1 - 6x + x^2}},
\]
\[
\sum_{n \geq 0} r_n x^n = \frac{2}{1 - x + \sqrt{1 - 6x + x^2}},
\]
\[
\sum_{n \geq 0} S_n x^n = \frac{2}{1 + x + \sqrt{1 - 6x + x^2}}
\]
respectively. Again we see that these numbers are all Stieltjes moment sequences.
3 Remarks

A Stieltjes moment sequence \((m_n)\) is called *determinate*, if there is a unique measure \(\mu\) on \([0, +\infty)\) such that (1.1) holds; otherwise it is called *indeterminate*. For example, \((n!)\) is a Stieltjes moment sequence of the exponential distribution by (1.2) and determinate by Stirling’s approximation and Carleman’s criterion which states that the divergence of the series

\[
\sum_{n \geq 0} \frac{1}{x^{2m_n}}
\]

implies the determinacy of the moment sequence \((m_n)\) (see [9, Theorem 1.11] for instance). We have shown that many well-known Catalan-like numbers are Stieltjes moment sequences. However, we do not know how to obtain the associated measures in general and whether these moment sequences are determinate.

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