MINIMUM-DELAY DECODING OF TURBO CODES FOR UPPER-LAYER FEC

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ABSTRACT

In this paper we investigate the decoding of parallel turbo codes over the binary erasure channel suited for upper-layer error correction. The proposed algorithm performs “on-the-fly” decoding, i.e. it starts decoding as soon as the first symbols are received. This algorithm compares with the iterative decoding of codes defined on graphs, in that it propagates in the trellises of the turbo code by removing transitions in the same way edges are removed in a bipartite graph under message-passing decoding. Performance comparison with LDPC codes for different coding rates is shown.

1. INTRODUCTION

The binary erasure channel (BEC) introduced by Elias [1] is one of the simplest channel models: a symbol is either erased with probability \( p \), or exactly received with probability \( 1 - p \). The capacity of such a channel with a uniform source is given by:

\[
C = 1 - p
\]

Codes that achieve this capacity are called Maximum-Distance Separable (MDS) codes, and they can recover the \( K \) information symbols from any \( K \) of the \( N \) codeword symbols. An MDS code that is widely used over the BEC is the non-binary Reed-Solomon (RS) code, but its block length is limited by the Galois field cardinality that dramatically increases the decoding complexity. For large block lengths, low-density parity-check (LDPC) codes [2] [3] [4] [5] and repeat-accumulate (RA) [6] codes with message-passing decoding proved to perform very close to the channel capacity with reasonable complexity. Moreover, “rateless” codes [7] [3] that are capable of generating an infinite sequence of parity symbols were proposed for the BEC. Their main strength is their high performance together with linear time encoding and decoding. However, convolutional-based codes, that are widely used for Gaussian channels, are less investigated for the BEC. Among the few papers that treat convolutional and turbo codes [9] in this context are [10] [11] [12] [13] [14].

In practical systems, data packets received at the upper layers encounter erasures. In the Internet for instance, it is frequent to have datagrams that are discarded by the physical layer cyclic redundancy check (CRC) or forward error correction (FEC), or even by the transport level user datagram protocol (UDP) checksums. Another example would be the transmission links that exhibit deep fading of the signal (fades of 10dB or more) for short periods. This is the case of the satellite channel where weather conditions (especially rain) severely degrades the channel quality, or even the mobile transmissions due to terrain effect. In such situations, the physical layer FEC fails and we can either ask for re-transmission (only if a return channel exists, and penalizing in broadcast/multicast scenarios) or use upper layer (UL) FEC.

In this paper, we propose a minimum-delay decoding algorithm for turbo codes suited for UL-FEC, in the sense that the decoding starts since the reception of the first symbols where a symbol could be a bit or a packet. The paper is organized as follows: Section 2 gives the system model and a brief recall of the existing decoding algorithms. Section 3 explains the minimum-delay decoding algorithm. Simulation results and comparisons with LDPC codes are shown in Section 4 and Section 5 gives the concluding remarks.

2. SYSTEM MODEL AND NOTATIONS

We consider the transmission of a parallel turbo code [9] with rate \( R_c = K/N \) over the BEC. An information bit sequence of length \( K \) is fed to a recursive systematic convolutional (RSC) code with rate \( \rho = k/n \) to generate a first parity bit sequence. The same information sequence is scrambled via an interleaver \( \Pi \) to generate a second parity sequence. With half-rate RSC constituents, the resulting turbo code has rate 1/3. In order to raise the rate of the turbo code, parity bits are punctured. In this paper, we consider rate-1/3, punctured rate-1/2 and punctured rate-2/3 turbo codes. The decoding of turbo codes is performed iteratively using probabilities on information bits, which requires the reception of the entire codeword before the decoding process starts. For instance, the soft-input soft-output (SISO) “Forward-Backward” (FB) algorithm [13], optimal in terms of \( a posteriori \) probability (APP) on symbols, consists of one forward recursion and one backward recursion over the trellis of the two constituent codes. As turbo codes are classically used over Gaussian channels, a SISO algorithm (the FB or other sub-optimal decoding algorithms) are required to attain low error rates. Exchanging
hard information between the constituent codes using an algorithm such as the well-known Viterbi Algorithm (VA) \cite{16} (that is a Maximum-Likelihood Sequence Estimator (MLSE) for convolutional codes) is harshly penalizing. However, in the case of the BEC, a SISO decoding algorithm is not necessary. In fact, it has been shown in \cite{12} that the VA is optimal in terms of symbol (or bit) probability on the BEC, which means that one can achieve optimal decoding of turbo codes on the BEC without using soft information. In other words, if a bit is known to (or correctly decoded by) one trellis, its value cannot be modified by the other trellis. Motivated by this key property, we propose a decoding algorithm for turbo codes based on hard information exchange.

3. ON-THE-FLY DECODING OF TURBO CODES

The turbo code has two trellises that have $K$ steps each, and one codeword represents a path in the trellises. In a goal to minimize the decoding delay, we propose an algorithm that starts decoding directly after the reception of the first bits of the transmitted codeword. First, at every step of the trellises, if one of the $n$ bits of the binary labeling is received (i.e. is known), we remove the transitions that do not cover this bit. Similarly, if - at some step - there are no transitions arriving to a state $e_i$ on the left are removed, we then know that no transition arrives to this state at the previous step. Consequently, all the incoming transitions to state $e_i$ from the left are removed. Similarly, if - at some step - there are no transitions arriving to a state $e_j$ on the right, this means that we cannot leave state $e_j$ at the following step, and all the transitions outgoing from state $e_j$ are removed. This way the information propagates in the trellis and some bits can be determined without being received. This algorithm is inspired by the message-passing decoding of LDPC codes over the BEC, where transitions connected to a variable node are removed if this variable is received.

Now at some stage of the decoding process, if an information bit is determined in one trellis without being received, we set its interleaved (or de-interleaved) counterpart as known and the same propagation is triggered in the other trellis. The information exchange between the two trellises continues until propagation stops in both trellises. This way we can recover the whole transmitted information bits without receiving the whole transmitted codeword.

In the sequel, for the sake of clearness, we will only consider parallel turbo codes built from the concatenation of two RSC codes with generator polynomials $(7, 5)$ in octal (the polynomial $(7)_8$ being the feedback polynomial), constraint length $L = 3$, and coding rate $\rho = k/n = 1/2$, code that has a simple trellis structure with four states. The algorithm can be applied to any parallel turbo code built from other RSC constituents. The transitions of the RSC $(7, 5)_8$ code between two trellis steps are shown in Fig. 1. As the code is systematic, the bit $b_1$ represents the information bit, and the bit $b_2$ the parity bit. There are $2^k = 2$ transitions leaving and 2 transitions arriving to each state. The transitions between two steps of the trellis can be represented by a $2^{L-1} \times 2^{L-1}$ matrix (4 × 4 matrix in this case). For the $(7, 5)_8$ code for instance, the transition table is given by:

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|-------|-------|-------|-------|
| $e_1$ | 00  | X     | 11    | X     |
| $e_2$ | 11    | X     | 00    | X     |
| $e_3$ | X     | 10    | X     | 01    |
| $e_4$ | X     | 01    | X     | 10    |

where an $X$ means that the transition does not exist. For the need of the proposed algorithm, we will use the transition table of the code to build binary transition matrices $T_{xx}$, $T_{b_1 x}$, and $T_{b_2 x}$ with $b_1, b_2 \in \{0, 1\}$ that contain the allowed transitions depending on the known bits. These matrices will be stored at the decoder and used as look-up tables throughout the decoding process. For instance, if the two bits of the transition are unknown, we define the matrix:

$$T_{xx} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

where a one in position $(i, j)$ means that there is a transition between state $e_i$ and state $e_j$, and a zero means that no transition exists. However, if $b_1 = 0$ and $b_2$ is unknown, or if $b_1$ is unknown and $b_2 = 0$, we define the following matrices corresponding to the allowed transitions:

$$T_{b_1 x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad T_{x0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We build the other matrices similarly. Note that there are a total of $3^n$ matrices, each of size $2^{L-1} \times 2^{L-1}$.
On-the-fly decoding algorithm

1) Initialization step. We consider matrices $M_1(i)$ and $M_2(j)$ corresponding to transitions at steps $i$ and $j$ of the two trellises of the constituent codes. These matrices are initialized as follows:

$$M_{1,2}(0) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_{1,2}(1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{1,2}(K) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_{1,2}(K+1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_1(t) = M_2(t) = T_{xx}, \quad t = 2, \ldots, K - 1$$

The matrices at steps 0 and 1 (namely $M_1(0)$, $M_2(0)$, $M_1(1)$ and, $M_2(1)$) represent the fact that any codeword starts in the zero state. The matrices at steps $K$ and $K + 1$ represent the two steps required for trellis termination (i.e. ending in the zero state).

2) Reception step. Each time a bit $r \in \{0, 1\}$ is received:

- If $r$ is an information bit, it is placed in appropriate positions in both trellises as $r_{2i} = r_{2j} = r$, where $j = \Pi(i)$. We then compute:

$$M_1(i) = M_1(i) \land T_{rx} \quad \text{and} \quad M_2(j) = M_2(j) \land T_{rx}$$

where the $\land$ operator is a logical AND between corresponding entries of the two matrices. In other words, we only keep the transitions in $M_1(i)$ with $b_1 = r$.

- If $r$ is a parity bit, we set $r_{2i+1} = r$ if $r$ belongs to the first trellis, or $r_{2j+1} = r$ if $r$ belongs to the second one. We then compute:

$$M_1(i) = M_1(i) \land T_{xr} \quad \text{or} \quad M_2(j) = M_2(j) \land T_{xr}$$

3) Propagation step. If either $M_1(i)$ or $M_2(j)$ has at least one all-zero column or one all-zero row, the algorithm is able to propagate in either direction in either trellis using the following rule:

- Let $d \in \{1, 2\}$ represent the trellis indices and initialize a counter $t \in \{i, j\}$ representing the step index through each trellis.

- Left propagation: an all-zero row with index $u$ in $M_d(t)$ generates an all-zero column with index $u$ in $M_d(t-1)$.

- Right propagation: an all-zero column with index $v$ in $M_d(t)$ generates an all-zero row with index $v$ in $M_d(t+1)$.

If we get new all-zero columns or new all-zero rows at steps $t \pm 1$, we set $t \leftarrow t \pm 1$ and continue the propagation (Step 3).

4) Duplication step. If during the propagation we get some $M_{d}(t) \subseteq T_{bx}$ (i.e. the value of the information bit of the $t$th transition in the $d$th trellis is equal to $b$), we proceed as follows:

- If $M_1(t) \subseteq T_{bx}$, we compute:

$$M_2 \left(\Pi(t)\right) = M_2 \left(\Pi(t)\right) \land T_{bx}$$

and then we propagate from $\Pi(t)$ in the second trellis (Step 3).

- If $M_2(t) \subseteq T_{bx}$, we compute:

$$M_1 \left(\Pi^{-1}(t)\right) = M_1 \left(\Pi^{-1}(t)\right) \land T_{bx}$$

and then we propagate from $\Pi^{-1}(t)$ in the first trellis (Step 3).

5) New reception step. If the propagation in both trellises stops, we go back to step 2.

6) Decoding stop. The decoding is successful if $M_1(i) \subseteq T_{bx}$ for all $i \in \{0, \ldots, K-1\}$. We then define the inefficiency ratio $\mu$ as follows:

$$\mu = \frac{r_{\text{stop}}}{K}$$

where $r_{\text{stop}} \geq K$ is the number of bits received at the moment when the decoding stops. An illustration of the proposed algorithm is shown in Fig. 2. First, at the reception of an information bit $b_1 = 0$, we remove the transitions in the corresponding step in the trellis where $b_1 = 1$. Note that this step is done in interleaved positions in both trellises at the reception of an information bit. At this stage, no propagation in the trellis is possible as all the states are still connected. Next we receive a parity bit $b_2 = 1$; the remaining transitions corresponding to $b_2 = 0$ are removed. At that point, we notice that state $e_1$ and $e_2$ on the left are not connected. This means that the transitions arriving from the left to these states are not allowed anymore, thus they are removed. Similarly, we remove the transitions leaving the states $e_1$ and $e_3$ on the right.

In fact, the average decoding inefficiency $\mu_{av}$ of the code relates to its erasure recovery capacity as follows: suppose that, on average, the proposed decoding algorithm requires $K' \geq K$ symbols to be able to recover the $K$ information symbols. We can write the following:

$$\mu_{av} = \frac{K'}{K} = \frac{(1 - p_{th}) N}{K} = \frac{1 - p_{th}}{R_c}$$

where the threshold probability $p_{th}$ corresponds to the average fraction of erasures the decoder can recover. We can then write $p_{th}$ as:

$$p_{th} = 1 - \mu_{av} R_c$$
In this section, the performance of the proposed algorithm with parallel turbo codes is shown. The coding rate of the turbo code using half-rate constituent codes is $R_c = 1/3$. However, we also consider turbo codes with $R_c = 1/2$ and $R_c = 2/3$ obtained by puncturing the $R_c = 1/3$ turbo code. We use two types of interleavers: 1) Pseudo-random (PR) interleavers (not optimized) and 2) Quasi-cyclic (QC) bi-dimensional interleavers [17] that are the best known interleavers in the literature: in fact, it was shown in [18] that the minimum distance $d_{min}$ of a turbo code is upper-bounded by a quantity that grows logarithmically with the interleaver size $K$, and the QC interleavers always achieve this bound.

The comparison is made with regular and irregular staircase LDPC codes. An LDPC code is said to be staircase if the right hand side of its parity check matrix consists of a double diagonal. The advantage of a staircase LDPC code is that the encoding can be performed in linear time using the parity check matrix, therefore there is no need for the generator matrix, which generally is not low density. A staircase LDPC code is said to be regular if the left hand side of the parity check matrix is regular, i.e. the number of 1’s per column is constant. Otherwise it is said to be irregular. In this section, we consider regular staircase LDPC codes with four 1’s per each left hand side column. Irregular staircase LDPC codes are optimized for the BEC channel by density evolution. In Fig. 3, we compare the performance of turbo codes and LDPC codes for $R_c = 1/3$. Turbo codes with RSC $(7,5)_8$ and PR interleaving achieve an average inefficiency $\mu_{av}$ of about 1.09, which means they require $K' = 1.09K$ received bits (or 9% overhead) to be able to recover the $K$ information bits. However, using a QC interleaver, the overhead with the same turbo code is of about 7.6%, which is very

\[
\Delta_p = p - p_{th} \simeq 0.025
\]

With codes such as LDPC or turbo codes, it is possible to achieve near-capacity performance with iterative decoding, with $\mu_{av} \simeq 1$. Ideally, an MDS code (that achieves capacity) has $\mu_{av} = 1$, i.e. it is capable of recovering the $K$ information symbols from any $K$ received symbols out of the $N$ codeword symbols.

Finally, it is important to note that the algorithm proposed in this section is linear in the interleaver size $K$. In fact, an RSC code with $2^{L-1}$ states and $2^k$ transitions leaving each state has $2^{L-1} \times 2^k = 2^{k+L-1}$ transitions between two trellis steps. This means that the turbo code has a total of approximately $2 \times K \times 2^{k+L-1}$ transitions. Even if the decoding is exponential in $k$ and $L$, it is linear in $K$. As we can obtain very powerful turbo codes with relatively small $k$ and $L$, we can say that a turbo code with the proposed algorithm has linear time encoding and decoding, and thus it is suited for applications were low-complexity “on-the-fly” encoding/decoding are required (as with the “Raptor codes” [8] for instance).

**4. SIMULATION RESULTS**

![Fig. 3. Average inefficiency ($\mu_{av}$) with respect to interleaver size $K$ over the BEC. Turbo code with half-rate RSC constituents versus LDPC codes, $R_c = 1/3$.](image)
In this paper we proposed a novel decoding algorithm for turbo codes over the BEC. This algorithm, characterized by “on-the-fly” propagation in the trellises and hard information exchange between the two codes, is appropriate for UL-FEC. Performance results with very small overhead were shown for different interleaver sizes and coding rates. Although the turbo codes presented in this paper were not optimized for the BEC, the results are very promising. Further improvements can be done by optimizing turbo codes for this channel.

5. CONCLUSION

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