ON NEGAMI’S PLANAR COVER CONJECTURE

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Abstract. Given a finite cover \( f : \tilde{G} \to G \) and an embedding of \( \tilde{G} \) in the plane, Negami conjectures that \( G \) embeds in \( P^2 \). Negami proved this conjecture for regular covers. In this paper we define two properties (called Properties \( \mathcal{V} \) and \( \mathcal{E} \)), depending on the cover \( \tilde{G} \) and its embedding into \( S^2 \), and generalize Negami’s result by showing: (1) If Properties \( \mathcal{V} \) and \( \mathcal{E} \) are fulfilled then \( G \) embeds in \( P^2 \). (2) Regular covers always fulfill Properties \( \mathcal{V} \) and \( \mathcal{E} \). We give an example of an irregular cover fulfilling Properties \( \mathcal{V} \) and \( \mathcal{E} \). Covers not fulfilling Properties \( \mathcal{V} \) and \( \mathcal{E} \) are discussed as well.

1. Introduction

In [7] S. Negami proved that if a graph \( G \) has a finite, unbranched, regular, planar cover then \( G \) itself embeds in the projective plane \( P^2 \) (for definitions see Section 2). We call a graph that embeds in \( P^2 \) projective. Note that all planar graphs are projective. In the same paper, Negami conjectured that this holds in general:

Conjecture 1.1 (Negami’s Conjecture). If a graph \( G \) has a finite unbranched planar cover then \( G \) embeds in the projective plane \( P^2 \).

Negami’s result was extended by S. Kitakubo to branched regular covers in [6] (with the exception of Section 5, any cover considered in this paper may be branched). We note that “branched cover” is not a standard term for graphs; for a precise definition and discussion see Definitions 2.2 and Remark 2.3. The definition used by Kitakubo is what we call weak cover; it is immediate from the definitions that every branched cover is a weak cover. For regular covers the converse holds as well: every regular weak cover is a branched cover. We do not have a clear idea about Negami’s Conjecture for weak covers:

Question 1.2. Is Negami’s Conjecture true for weak covers?

Given a finite cover \( f : \tilde{G} \to G \) and an embedding of \( \tilde{G} \) into \( S^2 \) we define two properties called Property \( \mathcal{V} \) and Property \( \mathcal{E} \). These properties depend on the covering map \( f \) and the embedding of \( \tilde{G} \) into \( S^2 \). We prove Negami’s Conjecture for covers \( f : \tilde{G} \to G \) fulfilling Properties \( \mathcal{V} \) and \( \mathcal{E} \). We show that regular covers fulfill Properties \( \mathcal{V} \) and \( \mathcal{E} \) (perhaps after re-embedding \( \tilde{G} \) in \( S^2 \)). The converse does not...
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hold: in Example 6.1 we show an irregular cover fulfilling Properties $V$ and $E$, showing that the work here is more general than [7] and [6]. We now give a more detailed description of our results and the structure of this paper.

In Section 2 we give the necessary definition and background.

In Section 3 we define Properties $V$ and $E$ (Definitions 3.1 and 3.2) and show (Theorem 3.3 and Corollary 3.4) that if $f : \tilde{G} \rightarrow G$ is a finite planar cover fulfilling Properties $V$ and $E$ then $G$ is projective, thus proving Negami’s Conjecture in that case. In fact, we show a little more: the map $f$ can be extended to a map $f : S^2 \rightarrow F$, for some surface $F$, and this map is a branched cover. It is then easy to see (Lemma 2.8) that either $F \cong S^2$ or $F \cong P^2$.

In Section 4 we show that if $\tilde{G} \rightarrow G$ is a regular finite, planar cover then it fulfills Properties $V$ and $E$.

In [7] Negami mentions the following strategy for proving Conjecture 1.1: given a finite cover $f : \tilde{G} \rightarrow G$ it is a well-known application of group theory that there exists finite cover $\tilde{f} : \tilde{\tilde{G}} \rightarrow \tilde{G}$, so that the composition of the covers (i.e., $f \circ \tilde{f} : \tilde{\tilde{G}} \rightarrow G$) is a finite regular cover (we remark that the degree of $\tilde{\tilde{G}} \rightarrow \tilde{G}$ is usually quite high). However, Negami continues, even if we assume that $\tilde{G}$ is planar, it does not follow that $\tilde{\tilde{G}}$ is planar as well. In Section 5 we prove Theorem 5.1: let $\tilde{\tilde{G}} \rightarrow G$ be a finite unbranched cover and let $\tilde{f} : \tilde{\tilde{G}} \rightarrow \tilde{G}$ be a cover with $\tilde{G} \subset S^2$. If the cover $f \circ \tilde{f} : \tilde{\tilde{G}} \rightarrow G$ fulfills Properties $V$ and $E$, then either $\tilde{G}$ is planar and $f : \tilde{G} \rightarrow G$ fulfills Properties $V$ and $E$, or $\tilde{G}$ embeds in $P^2$; in that case, lifting $\tilde{G}$ to the universal cover of $P^2$ (i.e., to the double cover $S^2 \rightarrow P^2$) we obtain a finite planar cover of $G$ fulfilling Properties $V$ and $E$ (this is sharp—see Example 6.4). Thus, if we wish to pass to a cover of $\tilde{G}$ in order to prove that $G$ is planar, we need not look for further than double covers.

In Section 6 we give examples. The first is Example 6.1 which is an irregular cover fulfilling Properties $V$ and $E$. Next (Example 6.3) we give an example of two distinct double covers of the planar graph $K_4$, both fulfilling Properties $V$ and $E$. We use both covers to embed $K_4$. The first gives an embedding into $S^2$ and the second into $P^2$, demonstrating that the embedding depends on the cover in a non-trivial way. Next, we exemplify Theorem 5.1 by showing (Example 6.4) a planar cover $f : \tilde{K}_4 \rightarrow K_4$ that does not fulfill Properties $V$ and $E$ for any planar embedding of $\tilde{K}_4$. However, $\tilde{K}_4$ embeds in $P^2$ and the lift of that embedding to $S^2$ does fulfill Properties $V$ and $E$. Finally, we show a more disturbing phenomenon: in Conjecture 6.5 we show a planar finite cover $f : \tilde{K}_4 \rightarrow K_4$ and we conjecture that is has no finite cover $\tilde{f} : \tilde{\tilde{K}}_4 \rightarrow \tilde{K}_4$ so the composition $f \circ \tilde{f} : \tilde{\tilde{K}}_4 \rightarrow K_4$ fulfills Properties $V$ and $E$. What happens here is that the cover $\tilde{K}_4$ is simply the wrong cover, and this cannot be fixed by passing to a higher cover. This, perhaps, explains the difficulty in proving Negami’s Conjecture.
So is Negami’s Conjecture true? At the current time, the answer is not known. However, the work of many people (including D. Archdeacon, M. Fellows, P. Hliněný, Negami, R. Thomas) accumulated in proving that Negami’s Conjecture is equivalent to the statement: the graph $K_{1,2,2,2}$ has no finite planar cover. This seems to us very strong evidence supporting Conjecture 1.1. Assuming Conjecture 1.1 for a moment, in light of the results and examples in this paper we ask: suppose we are given a finite planar cover $f : \tilde{G} \to G$ not fulfilling Properties $\mathcal{V}$ and $\mathcal{E}$ for any embedding of $\tilde{G}$ (or a cover of $G$) into $S^2$ (i.e., a “wrong cover”), how can we correct that cover? We end Section 6 with two ways to “fix” the cover in Conjecture 6.5. The first is a cut-and-paste procedure that produces a cover fulfilling Properties $\mathcal{V}$ and $\mathcal{E}$; this changes the graph $\tilde{G}$. The second does not change $\tilde{G}$ but embeds it into some non-orientable surface (not unlike Theorem 5.1). This gives an embedding of $G$ into some surface, not necessarily $P^2$, with control over the Euler characteristic of that surface.

2. Preliminaries

For a graph $G$, we denote the vertices of $G$ by $V(G)$ and the edges of $G$ by $E(G)$. Naturally, any map between graphs is assumed to map vertices to vertices and edges to edges. We follow standard definitions and terminology used in topology: $N(\cdot)$ means closed regular (or normal) neighborhood, $\partial$ is read boundary, $\text{cl}$ is read closure, and int is read interior. A homeomorphism is a continuous bijection with a continuous inverse. $\chi(\cdot)$ stands for Euler characteristic. All surfaces are assumed to be connected.

As it is easy to reduce Negami’s Conjecture to graphs $G$ with no cycles of length one or two (that is, graphs $G$ in which every edge in $E(G)$ connects distinct vertices and distinct edges have at most one vertex in common). It is also easy to reduce Negami’s Conjecture to connected graphs $G$ and $\tilde{G}$. Therefore, throughout this paper we assume:

Assumptions 2.1. The graphs $G$ and $\tilde{G}$ are connected and $G$ has no cycles of length 2 or less.

We define branched covers in the two relevant situations, graphs and surfaces. In these definitions disks are modeled on $\{z \in \mathbb{C} : |z| < 1\}$, and a half disk is a set homeomorphic to (and modeled on) $\{z \in \mathbb{C} : |z| < 1 \text{ and } \Re(z) \geq 0\}$ (here, $\Re(z)$ is the imaginary part of $z$). The boundary of a half disk are the points corresponding to $\{z \in \mathbb{C} : |z| < 1 \text{ and } \Re(z) = 0\}$. The symbol $\sqcup$ is used for disjoint unions.

Definitions 2.2. In (1)–(3) below let $\tilde{G}$ and $G$ be finite graphs and $f : \tilde{G} \to G$ a map. In (4)–(5) below let $F_1$ and $F_2$ be compact surfaces and $f : F_1 \to F_2$ a proper map (that is, $f^{-1}(\partial F_1) = \partial F_2$).

1. $f : \tilde{G} \to G$ is called a unbranched cover if $f$ is onto and for any $\tilde{v} \in V(\tilde{G})$ $f$ maps the neighbors of $\tilde{v}$ bijectively onto the neighbors of $f(\tilde{v})$. 

(2) $f : \tilde{G} \to G$ is called a weak cover if $f$ is onto and for any $\tilde{v} \in V(\tilde{G})$ $f$ maps the neighbors of $\tilde{v}$ onto the neighbors of $f(\tilde{v})$.

(3) $f : \tilde{G} \to G$ is called a branched cover if $f$ is onto and for any $\tilde{v} \in V(\tilde{G})$ there is a positive integer $d = d(\tilde{v})$ so that every neighbor of $f(\tilde{v})$ has exactly $d$ preimages that are neighbors of $\tilde{v}$, i.e., the restriction of $f$ to the neighbors of $\tilde{v}$ is a map to the neighbors of $f(\tilde{v})$ onto and $d$-to-$1$.

We call $d$ the local degree of $f$ at $\tilde{v}$. If $d > 1$ then $\tilde{v}$ is called a singular point. The set of all singular points is called the singular set and the image of the singular set is called the branched set.

(4) map $f : F_1 \to F_2$ is called a branched cover if the following holds:

(a) Every point $p \in \partial F_2$ has a neighborhood $D \ni p$ so that $D$ is half a disk (with $p$ corresponding to 0) and $f^{-1}(D)$ is a disjoint collection of half disks $\sqcup_{i=1}^{n_i} D_i$ (for some $n_i$) with $\partial D_i \subset \partial F_1$, so that for every $i$, $f|_{D_i} : D_i \to D$ is a homeomorphism.

(b) Every point $p \in \text{int} F_2$ has a neighborhood $D \ni p$ (with $p$ corresponding to 0) so that $D$ is a disk and $f^{-1}(D)$ is a disjoint collection of disks $\sqcup_{i=1}^{n_i} D_i$ (for some $n_i$) so that for every $i$, $f|_{D_i} : D_i \to D$ is modeled on $z \mapsto z^d$ for some non-zero integer $d$.

If $f|_{D_i}$ is modeled on $z \mapsto z^d$ then $d$ is called the local degree at the center of $D_i$ (of course, for distinct values of $i$ we may have distinct local degrees). A point with local degree greater than one (in absolute value) is called a singular point, the union of the singular points is called the singular set, and the image of the singular set is called the branched set.

Note that the branch set is finite and contained in $\text{int} F_2$.

Thus we see that every unbranched cover of graphs is a branched cover of graphs with all local degrees one, and conversely a branched cover with all local degrees one is an unbranched cover. (Equivalently, unbranched covers are covers with empty branch set).

**Remark 2.3.** The definition of branched cover used in [6] is the definition of weak cover given above. However, Kitakubo only considered regular covers. It is left as an exercise to the reader to show that (under Assumptions 2.1) a weak regular cover is in fact a branched cover. We do not know if Negami’s Conjecture holds for weak covers, and it will probably be a nice project to the reader to find and classify the counterexamples. Weak covers will not appear in this paper again.

**Remark 2.4.** We identify $S^2$ with the Riemann sphere. Then any rational function $f$ gives a branched cover $f : S^2 \to S^2$. Conversely given a branched cover $f : S^2 \to S^2$ we can multiply $f$ by $p^{-1}$ (for an appropriately chosen polynomial $p$) so that no point in $\mathbb{C}$ is sent to $\infty$. It then follows from the Riemann Uniformization Theorem that (perhaps after conjugation) $f/p$ is a polynomial. Hence, after conjugating if necessary, $f$ is a rational function.
We conclude this section with a few well-known lemmas about branched covers; some of the proofs are sketched for the convenience of the reader. For the first lemma, the reader may consult, for example, [3] for the definition of cover from the circle $S^1$ to itself.

**Lemma 2.5.** Let $f : F_1 \to F_2$ be a branched cover between surfaces with non-empty boundary. Then the restriction $f|_{\partial F_1}$ an unbranched cover from $f|_{\partial F_1} : \partial F_1 \to \partial F_2$.

**Notation 2.6.** The **universal cover** of $P^2$ is the map $\pi : S^2 \to P^2$ given by identifying antipodal points. It is an unbranched double cover.

One of the basic facts about the universal cover of $P^2$ is:

**Lemma 2.7.** Let $f : S^2 \to P^2$ be a cover. Then $f$ factors through $\pi$, that is, there exists a cover $f' : S^2 \to S^2$ so that $\pi \circ f' = f$.

**Sketch of the proof.** Let $B \subset P^2$ be the branch set. Then $f|_{f^{-1}(P^2 \setminus B)} : f^{-1}(P^2 \setminus B) \to P^2 \setminus B$ is an unbranched cover. Since $P^2 \setminus B$ is non-orientable, it has an **orientation double cover** that is, an unbranched double cover $f_1 : F \to P^2 \setminus B$ from some orientable surface $F$. A basic property of the orientation double cover is that any cover from an orientable surface factors through it. Applying this in our setting, we see there a cover $f'|_{f^{-1}(P^2 \setminus B)} : f^{-1}(P^2 \setminus B) \to S^2$ so that $f'|_{f^{-1}(P^2 \setminus B)} \circ f_1 = f|_{f^{-1}(P^2 \setminus B)}$. The surfaces $P^2 \setminus B$, $F$ and $f^{-1}(P^2 \setminus B)$ are not compact. We compactify them by adding one point to each end. Denoting the compactification of $F_1$ by $F_2$, we get a cover $f_2 : F_2 \to P^2$ so that $f$ factors through $f_2$.

All that remains to show is that $f_2$ is in fact the universal cover $\pi$. We can easily see that $f_2$ is a double cover from a closed orientable surface $F_2$ to $P^2$. Since curves parallel to the punctures of $P^2 \setminus B$ are orientation preserving they lift to the double cover; hence this cover is not branched. Euler characteristic is multiplicative under unbranched cover, and so $\chi(F_2) = 2$. As $S^2$ is the only connected surface with Euler characteristic 2, we get that $F_2 \cong S^2$. By uniqueness of the universal cover, $f_2 = \pi$ (perhaps after conjugation).

The proof of the lemma below is an easy exercise in Euler characteristic:

**Lemma 2.8.** If $S^2$ branch covers $F$ then either $F \cong S^2$ or $F \cong P^2$.

The following lemma tells us when $F$ is $P^2$; (3) is particularly convenient since it requires only looking at one point:

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1. It is well-known that unbranched covers correspond to subgroups of the fundamental group. The orientation double cover correspond to the group consisting of all the orientation preserving loops in $P^2 \setminus B$.
2. A cover of $P^2 \setminus B$ is orientable if and only if the corresponding subgroup contains only orientation preserving loops. Any such subgroup is contained in the subgroup of all orientation preserving loops, and hence any such cover factors through the orientation double cover.
Lemma 2.9. Let $F_1$, $F_2$ be surfaces and suppose $F_1$ is orientable. Let $f : F_1 \to F_2$ be a cover. For $p \in F_2$, let $D_p$ be a open normal neighborhood of $p$ endowed with and orientation (since $D_p$ is a disk this is always possible). For $q \in F_1$ with $f(q) = p$ let $D_q$ be a disk so that $f|_{D_q} : D_q \to D_p$ is modeled on $z \to z^d$ (for some $d$). Note that the orientation on $D_p$ induces an orientation on $D_q$.

Then the following conditions are equivalent:

(1) $F_2$ is non-orientable.

(2) For any $p \in F_2$, there exist $q_1, q_2 \in F_1$ with $f(q_1) = f(q_2) = p$ so that the orientations induced orientations on $D_{q_1}$ and $D_{q_2}$ define opposite orientation on $F_1$.

(3) For some $p \in F_2$ there exist $q_1, q_2 \in F_1$ with $f(q_1) = f(q_2) = p$ so that the orientations induced orientations on $D_{q_1}$ and $D_{q_2}$ define opposite orientation on $F_1$.

Proof. (1) $\Rightarrow$ (2): let $p \in F_2$ be an arbitrary point, and let $\gamma \subset F_2$ be an orientation reversing loop (which exists by assumption) based at $p$. Since $F_1$ admits no orientation reversing loops, the lift of $\gamma$ is an arc connecting two points (say $q_1$ and $q_2$) that project to $p$. It is now easy to verify that the $D_{q_1}$ and $D_{q_2}$ induce opposite orientations on $F_1$ (in fact, this is equivalent to $\gamma$ being orientation reversing).

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Given $p, q_1, q_2$ as in the statement let $\alpha \subset F_1$ be any arc connecting $q_1$ and $q_2$. Then the image of $\alpha$ on $F_2$ is an orientation reversing loop, and hence $F_2$ is non-orientable.

$\square$

3. Property $\mathcal{V}$ and Property $\mathcal{E}$

Let $f : \tilde{G} \to G$ be a finite planar cover. The embedding of $\tilde{G}$ into $S^2$ induces a cyclic order around each vertex of $\tilde{G}$. Our first condition, Property $\mathcal{V}$, is a consistency condition requiring that this order induces a cyclic order around each vertex of $V(G)$. Fix $v \in G(V)$ and $\tilde{v} \in v(\tilde{G})$ in the preimage of $v$. There are two obstructions to inducing a cyclic order around $v$. The first obstruction is local: if $\tilde{v}$ is a singular vertex (say with local degree $d$), then every neighbor of $v$ has $d$ preimages around $\tilde{v}$; the ordering of the different preimages may contain inconsistencies. For example, if $v$ has three neighbors (say $v_1, v_2$, and $v_3$) and $d = 2$ then there are six lifts of $v_1, v_2$ and $v_3$ adjacent to $\tilde{v}$, say $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}'_1, \tilde{v}'_2, \tilde{v}'_3$ (with $f(\tilde{v}_i) = f(\tilde{v}'_i) = v_i, i = 1, 2, 3$). These vertices may be cyclically ordered as $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}'_1, \tilde{v}'_2, \tilde{v}'_3$. It is now not possible to induce a cyclic order around $v$. The second obstruction is global: it is possible that the order around each preimage induces an order around $v$ but different preimages induce distinct orders. We now define Property $\mathcal{V}$:
Property 3.1 (Property $\mathcal{V}$). Let $f : \tilde{G} \to G$ be a finite planar cover. We say that $f : \tilde{G} \to G$ fulfills Property $\mathcal{V}$ if for any $v \in V(G)$ the following two conditions are satisfied:

1. For any $\tilde{v} \in f^{-1}(v)$, the cyclic order of the neighbors of $\tilde{v}$ (induced by the embedding into $S^2$) induces an order around $v$. That is to say, denoting the neighbors of $v$ by $v_1, \ldots, v_n$, after reordering the indices of if necessary, projecting the neighbors of $\tilde{v}$ to $V(G)$ in order we get $v_1, v_2, \ldots, v_n, v_1, v_2, \ldots, v_n, \ldots, v_1, v_2, \ldots, v_n$. (This condition is vacuous if $\tilde{v}$ is not singular).

2. The order obtained is independent of choice of preimage.

Example: Suppose that $v \in V(G)$ has four neighbors. Figure 1 shows the two possibilities that a cover does not fulfill Property $\mathcal{V}$. The vertices $\tilde{v}, \tilde{v}'$ in that figure project to $v$; their neighbors are labeled using the labels $v_1, v_2, v_3, v_4$, according to which vertex of $G$ they project to. Using the same labeling scheme, in Figure 2 we show a little part of a cover that fulfills Property $\mathcal{V}$.

Let $f : \tilde{G} \to G$ be a finite planar cover fulfilling Property $\mathcal{V}$. Given $v \in V(G)$ and $\tilde{v} \in f^{-1}(v)$, moving around the neighbors of $\tilde{v}$ counterclockwise allows us to distinguish a particular cyclic order from its reverse order on the neighbors of $v$. We call this the counterclockwise cyclic order. We assign $\tilde{v}$ an arbitrary sign (plus or minus).\footnote{The freedom of choice will be useful in Section 5.} If $\tilde{v}'$ is another preimage of $v$ we assign $\tilde{v}'$ the same sign as $\tilde{v}$ if the counterclockwise cyclic order around $\tilde{v}'$ projects to the same order around $v$, and the
opposite sign otherwise. A sign assignment as above is called *valid*. We refer the reader to Section 6 for examples of valid sign assignments. Like Property V, Property E is a question of consistency:

**Property 3.2** (Property E). Let \( f : \tilde{G} \to G \) be a finite planar cover fulfilling Property V with valid signs on \( G(\tilde{V}) \). We say that \( f : \tilde{G} \to G \) fulfills Property E if for any edge \( e \in E(G) \) and any two edges \( \tilde{e}_1, \tilde{e}_2 \in E(\tilde{G}) \) that project to \( e \), we have that \( \tilde{e}_1 \) connects vertices of the same sign if and only if \( \tilde{e}_2 \) does.

We now state and prove Theorem 3.3. In that theorem, we consider a graph \( \tilde{G} \) embedded in \( S^2 \) (\( G \) embedded in some surface \( F \), resp.). The closure of the components of \( S^2 \setminus \tilde{G} \) (\( F \setminus G \), resp.) are called faces of \( S^2 \) (\( F \), resp.).

**Theorem 3.3.** Let \( \tilde{G} \) be a connected planar graph, and let \( f : \tilde{G} \to G \) be a finite cover. Then \( f : \tilde{G} \to G \) fulfills Properties V and E if and only if there exist a surface \( F \) containing \( G \) and a map \( f' : S^2 \to F \) with the following properties:

1. \( f' \) extends \( f \), that is, for every point \( p \in \tilde{G} \), \( f'(p) = f(p) \).
2. \( f' \) is a branched cover, and hence (by Lemma 2.8) \( f \cong P^2 \) or \( F \cong S^2 \).
3. The intersection of the branch points of \( f' \) with \( G \) is contained in \( V(G) \). More specifically, it is exactly the branch set of \( f \).
4. The intersection of the singular points of \( f' \) with \( \tilde{G} \) is contained in \( V(\tilde{G}) \). More specifically, it is exactly the set of singular points of \( f \) and with the same local degrees.
5. The faces of \( S^2 \) (\( F \), resp.) are all disks, and every such face contains at most one singular point (branch point, resp.).
6. For each \( v \in V(G) \), the cyclic order induced on the neighbors of \( v \) by \( f \) and the cyclic order given by the embedding of \( G \) into \( F \) are the same.

Negami’s Conjecture for finite planar covers fulfilling Properties V and E (Corollary 3.4 below) follows easily from points (1) and (2) above. However, since Negami’s Conjecture does not require (3)–(6), corollary 3.4 is not and “if and only if” statement.

**Corollary 3.4** (Negami’s Conjecture for covers fulfilling Properties V and E). Let \( \tilde{G} \) be a connected graph, \( \tilde{G} \subset S^2 \). Let \( f : \tilde{G} \to G \) be a cover fulfilling Properties V and E. Then \( G \) is projective.

**Proof of Theorem 3.3.** Suppose \( f : \tilde{G} \to G \) is a finite planar cover fulfilling Properties V and E. We construct the surface \( F \) and the map \( f' : S^2 \to F \):

**Step One: Vertices.** We first construct \( F \) near each vertex of \( G \) and extend the map \( f' \) from a neighborhood of \( V(\tilde{G}) \) to this neighborhood of \( V(G) \). For each \( \tilde{v} \in V(\tilde{G}) \) let \( D_{\tilde{v}} \) be a regular neighborhood of \( \tilde{v} \). By the Normal Neighborhood Theorem we may assume each \( D_{\tilde{v}} \) is a closed (sic.) disk and these disks are disjoint; moreover,
the edges of $\tilde{G}$ intersect $D_v$ in radial arcs, as in Figure 3. For $v \in V(G)$ we pick a small disk $D_v$ and embed a neighborhood of $v$ in $D_v$ as follows: $v$ is the center of $D_v$, and $E(G)$ intersects $D_v$ in radial arcs, each arc corresponding to the tip of an edge that is incident to $v$. This can be done in several different ways resulting in distinct cyclic orders around $v$. We embed $v$ in $D_v$ so that the cyclic order around $v$ agrees with the order induced by the neighbors of the preimages of $v$; this is well-defined since the cover fulfills Property $V$.

We assign $V(\tilde{G})$ a valid sign convention. Endowing $D_v$ with an orientation gives a specific counterclockwise cyclic order around $v$. We orient $D_v$ so that the counterclockwise cyclic order around $D_v$ agrees with the counterclockwise cyclic order along induced by the neighbors of a vertex of $f^{-1}(v)$ with plus sign and is opposite the counterclockwise cyclic order induced by vertices of negative sign. Since the sign assignment is valid, this is well defined.

The surface constructed so far, $\bigsqcup_{v \in V(G)} D_v$, is denoted $F_V$. Next we extend the map $f$ to $f'$: $\bigsqcup_{\tilde{v} \in V(\tilde{G})} D_{\tilde{v}} \to F_V$ by mapping $D_{\tilde{v}}$ to $D_{f(\tilde{v})}$ by a map modeled $z \mapsto z^d$ (where $d$ is the local degree at $\tilde{v}$).

We check each property listed in Theorem 3.3 in the same order:

1. Holds by construction: for any point $p \in \tilde{G} \cap (\bigsqcup_{\tilde{v} \in V(\tilde{G})} D_{\tilde{v}})$, $f'(p) = f(p)$.
2. Again by construction, $f': \bigsqcup_{\tilde{v} \in V(\tilde{G})} D_{\tilde{v}} \to F_V$ is a branched cover.
3. Holds by construction.
4. Holds by construction.
5. To be verified later.
6. Holds by construction. Since this is a local property and we will not modify $f'$ near $V(\tilde{G})$ any more, we will not need to check this property again.

**Remark 3.5.** $f': D_{\tilde{v}} \to D_v$ is orientation preserving if and only if the sign of $v$ is plus.

**Step Two: Edges.** Next, we extend the construction of $F_V$ to a neighborhood of $E(G)$ and extend the range of $f'$ from a neighborhood of $E(\tilde{G})$. The neighborhood of an edge $e \in E(G)$ (say $e = (v, u)$, for some $v, u \in V(G)$) is a closed (sic.) band that connects $D_v$ to $D_u$. We first glue the band to $D_v$ and extend the orientation of $D_v$.
along the band. We then glue the band to $D_u$ as follows: choose an edge $\tilde{e} \in f^{-1}(e)$. If $\tilde{e}$ connects vertices with the same sign, the orientation of the band agrees with the orientation of $D_u$ and if $\tilde{e}$ connects vertices with the opposite signs, the orientation of the band disagrees with the orientation of $D_u$.

Since $f$ fulfills Property $\mathcal{E}$ this construction is independent of choice. We denote the part of $F$ constructed so far by $F_E$.

Again, we check each property listed in Theorem 3.3 ignoring the properties we are done with.

(1) Holds by construction for any point of $\tilde{G}$. Since this is a local property and we will not modify $f'$ on $(f')^{-1}(F_E)$ any more, we will not need to check this property again.

(2) The map $f' : (f')^{-1}(F_E) \to F_E$ is a branched cover by construction.

(3) and (4) Since we did not introduce any new branch points or singular points, (3) and (4) still hold. We will not change $f'$ near $\tilde{G}$ so do not need to check these properties again.

(5) Note that so far, all singular (resp. branch) points are vertices of $\tilde{G}$ (resp. $G$). (5) will be verified later.

**Step Three: Closing $F_E$.** We have constructed a branched cover $f' : (f')^{-1}(F_E) \to F_E$, both compact surfaces with non-empty boundary. Let $\tilde{\gamma}$ denote a boundary component of $f^{-1}(F_E)$. By Lemma 2.5, $f|_{f^{-1}(F_E)}$ maps $\tilde{\gamma}$ to a boundary component of $F_E$, say $\gamma$, and the map $f|_{\tilde{\gamma}} : \tilde{\gamma} \to \gamma$ is a cover. Since $\tilde{\gamma}$ and $\gamma$ are both circle, $f|_{\tilde{\gamma}}$ is modeled on the restriction of $z \mapsto z^d$ to the unit circle (for some $0 \neq d \in \mathbb{Z}$, called the winding number of $f|_{\tilde{\gamma}}$). Since $\tilde{G}$ is connected and $f^{-1}(F_E)$ is a neighborhood of $\tilde{G}$, $f^{-1}(F_E)$ is connected as well; hence the components of $S^2 \setminus f^{-1}(F_E)$ are all disks. Let $D_{\tilde{\gamma}}$ be the closed disk bound by $\tilde{\gamma}$ disjoint from $\tilde{G}$, that is, the closure of the component of $S^2 \setminus f^{-1}(F_E)$ adjacent to $\tilde{\gamma}$. We attach a disk (say $D_{\gamma}$) to $\gamma$ and extend the map $f'$ by mapping $D_{\tilde{\gamma}}$ to $D_{\gamma}$ by coning. If the absolute value of the winding number $f|_{\tilde{\gamma}}$ is more than one we introduce exactly one singular point on $D_{\gamma}$ and one branch point on $D_{\gamma}$, otherwise no new singular or branch point is introduced. Continuing this way, we cap off every component of the boundary of $F_E$, finally constructing a closed surface $F$ containing $G$ and a branched cover $f : S^2 \to F$.

With the exception of (2) and (5), we verified that the cover $f' : S^2 \to F$ fulfills all the conditions of Theorem 3.3. We note that $f'|_{f^{-1}(F_E)}$ is a branched cover, and

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4Since the disks $D_v$ are oriented, we can place them on a coffee table all facing up. For the edge $e = (u, v)$, the band $N(e)$ connects $D_u$ to $D_v$. If $\tilde{e}$ connects vertices of the same sign, this band is untwisted (i.e., lies flat on the table) and if $\tilde{e}$ connects vertices of the opposite sign, the band is twisted.

5Recall that we model both $D_{\delta}$ and $D_{\gamma}$ on the unit disk in $\mathbb{C}$. By coning we mean that $f'$ is modeled on $z \mapsto z^d$, extending $f'_{|_{\tilde{\gamma}}}$. 
by construction, $f'|_{\text{cl}(F \setminus F_e)}$ is a branched cover as well; this establishes (2). It is straightforward to see that the construction gives (5) as well.

Conversely, given $f : \tilde{G} \to G$ and a map $f'$ as in the statement of Theorem 3.3, the embedding of $G$ into $F$ induces a cyclic order around the vertices of $G$. Lifting these orders to $\tilde{G}$ we get the order around each vertex of $\tilde{G}$; this order coincides with the order given by the embedding of $\tilde{G}$ into $S^2$. It is now easy to see that Properties $V$ and $E$ follow.

This completes the proof of Theorem 3.3. □

We can know if $F \cong S^2$ or $F \cong P^2$ by looking at labels only:

**Proposition 3.6.** The following are equivalent:

1. The surface $F$ constructed in Theorem 3.3 is homeomorphic to $P^2$.
2. There exists a vertex $v \in V(G)$ with preimages of opposite signs.
3. Every vertex $v \in V(G)$ has preimages of opposite signs.

**Proof.** This is immediate from Remark 3.5 and Lemma 2.9. □

## 4. Regular covers

In this section we show that if $\tilde{G} \to G$ is a finite planar regular cover then it fulfills Properties $V$ and $E$. Some of the material in this section is from [7] and [6].

**Proposition 4.1.** Let $f : \tilde{G} \to G$ be a regular planar cover. Then (perhaps after re-embedding $\tilde{G}$) $f : \tilde{G} \to G$ fulfills Properties $V$ and $E$.

**Proof.** Recall that a cover $f : \tilde{G} \to G$ is called regular if there is a group $\Gamma$ acting on $\tilde{G}$ so that for any $\tilde{v}, \tilde{v}' \in V(\tilde{G})$ with $f(\tilde{v}) = f(\tilde{v}')$ there exist $\gamma \in \Gamma$ with $\gamma(\tilde{v}) = \tilde{v}'$, and for any $\tilde{e}, \tilde{e}' \in E(\tilde{G})$ with $f(\tilde{e}) = f(\tilde{e}')$ there exist $\gamma \in \Gamma$ with $\gamma(\tilde{e}) = \tilde{e}'$.

**Lemma 4.2.** Let $f : \tilde{G} \to G$ be a finite planar regular cover with group $\Gamma$. Then (perhaps after re-embedding $\tilde{G}$) the action of $\Gamma$ can be extended to an action on $S^2$.

**Proof of Lemma 4.2.** The proof is an induction on the number of vertices of $\tilde{G}$; when $\tilde{G}$ is not 3-connected we reduce this number. Hence the base case of the induction is:

**Base Case:** $\tilde{G}$ is 3-connected. This is a well-known theorem of Whitney [11].

**Inductive Step:** We assume $\tilde{G}$ is not 3-connected. The proofs for 2-connected and 1-connected graphs are similar, and we omit the easier case of 1-connected graphs. Let $\{\tilde{c}_1, \tilde{c}_2\}$ be a cut pair for $\tilde{G}$ and suppose removing $\tilde{c}_1, \tilde{c}_2$ from $\tilde{G}$ we obtain the graphs $\tilde{G}_0$ and $\tilde{G}_1$; we assume further that $\{\tilde{c}_1, \tilde{c}_2\}$ were chosen so that $\tilde{G}_0$ is minimal with respect to inclusion. Note that $V(\tilde{G}_0) \neq \emptyset$ and $V(\tilde{G}_1) \neq \emptyset$. For any $\gamma \in \Gamma$, 

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6This implies that $\tilde{G}_0$ is 2-connected, but $\tilde{G}_1$ need not be connected.
\{\gamma(\tilde{c}_1), \gamma(\tilde{c}_2)\} \text{ is a cut pair. Therefore minimality of } \tilde{G}_0 \text{ implies that } \gamma(\tilde{c}_i) \notin \tilde{G}_0 \quad (i = 1, 2). \text{ We use the notation } \text{ext}(\tilde{G}_0) \text{ (called the } extension \text{ of } \tilde{G}_0) \text{ for the subgraph spun by } V(\tilde{G}_0) \cup \{\tilde{c}_1, \tilde{c}_2\} \text{ and similarly } \text{ext}(\tilde{G}_1) \text{ is the graph spun by } V(\tilde{G}_1) \cup \{\tilde{c}_1, \tilde{c}_2\}. \text{ Since } \tilde{G} \text{ is 2-connected there is a pass (say } \alpha_1) \text{ in } \tilde{G} \setminus \tilde{G}_0 \text{ (equivalently, in } \text{ext}(\tilde{G}_1)) \text{ connecting } \tilde{c}_1 \text{ to } \tilde{c}_2. \text{ By choosing the shortest pass, we can guarantee that } \alpha_1 \text{ is embedded. Therefore, the graph obtained by adding an edge (say } \tilde{e}) \text{ that connects } \tilde{c}_1 \text{ to } \tilde{c}_2 \text{ to } \text{ext}(\tilde{G}_0) \text{ is planar. We denote this graph by } \text{ext}(\tilde{G}_0) \cup \{\tilde{e}\}.

Let } \Gamma_0 \text{ be the (possibly trivial) subgroup of } \Gamma \text{ that leaves } \text{ext}(\tilde{G}_0) \text{ invariant (equivalently, leaves } \tilde{G}_0 \text{ invariant). Then for any } \gamma \in \Gamma_0, \{\gamma(\tilde{c}_1), \gamma(\tilde{c}_2)\} \subset \text{ext}(\tilde{G}_0), \text{ and hence } \{\gamma(\tilde{c}_1), \gamma(\tilde{c}_2)\} = \{\tilde{c}_1, \tilde{c}_2\}. \text{ Therefore the action of } \Gamma_0 \text{ can be extended to an action on } \text{ext}(\tilde{G}_0) \cup \{\tilde{e}\}. \text{ Since } V(\tilde{G}_1) \neq \emptyset, \text{ } |V(\text{ext}(\tilde{G}_0) \cup \{\tilde{e}\})| < |V(\tilde{G})| \text{ (where } |V(\cdot)| \text{ denotes number of vertices and we may apply the inductive hypothesis to get a re-embedding of } \text{ext}(\tilde{G}_0) \cup \{\tilde{e}\} \text{ into } S^2 \text{ so that the action of } \Gamma_0 \text{ extends to } S^2. \text{ Removing } \tilde{e}, \text{ we obtain a re-embedding of } \text{ext}(\tilde{G}_0), \text{ denoted } \hat{G}_0.

**Remark 4.3.** It is not hard to argue that } \text{ext}(\tilde{G}_0) \cup \{\tilde{e}\} \text{ is 3-connected. Hence by Whitney \[\square\] (the base case of the induction) } \hat{G}_0 \text{ is the original embedding of } \text{ext}(\tilde{G}_0).

Since } \hat{G} \text{ is 2-connected there is a pass (say } \tilde{\alpha}_0) \text{ in } \text{ext}(\tilde{G}_0) \text{ connecting } \tilde{c}_1 \text{ to } \tilde{c}_2. \text{ By choosing the shortest pass, we can guarantee that } \tilde{\alpha}_0 \text{ is embedded. Similar to the process above, we replace } \tilde{G}_0 \text{ with an edge (say } \tilde{e}_0) \text{ connecting } \tilde{c}_1 \text{ and } \tilde{c}_2, \text{ obtaining the planar graph } (\hat{G} \setminus \hat{G}_0) \cup \tilde{e}. \text{ Of course, we cannot expect } \Gamma \text{ to act on this graph. To that end, we repeat this operation on the image of } \hat{G}_0 \text{ under } \Gamma \text{ and obtain the planar graph:}

\[(\hat{G} \setminus \cup_{\gamma \in \Gamma} \gamma(\tilde{G}_0)) \cup (\cup_{\gamma \in \Gamma} \gamma(\tilde{e}_0)).\]

Denote this graph by } \tilde{H}, \text{ and note that by construction } \Gamma \text{ acts on } \tilde{H}. \text{ Since } V(\tilde{G}_0) \neq \emptyset, \text{ } |V(\tilde{H})| < |V(\hat{G})| \text{ and we may apply the inductive hypothesis to get a re-embedding of } \tilde{H} \text{ into } S^2 \text{ so that the action of } \Gamma \text{ extends to } S^2. \text{ For every } \gamma \in \Gamma, \text{ We replace } \tilde{e}_0 \text{ by } \gamma(\tilde{G}_0). \text{ It is now easy to see that we obtain an embedding of } \hat{G} \text{ into } S^2 \text{ and the action of } \Gamma \text{ on this graph extends to } S^2, \text{ as desired.}

This completes the proof of Lemma 4.2. \[\square\]

After extending the action of } \Gamma \text{ to } S^2, \text{ we denote the group elements by } \phi : S^2 \rightarrow S^2.

**Lemma 4.4.** Let } \tilde{f} : \tilde{G} \rightarrow G \text{ be a finite planar regular cover with group } \Gamma. \text{ If any } \gamma \in \Gamma \text{ can be extended to a homeomorphism } \phi : S^2 \rightarrow S^2 \text{ then the cover fulfills Properties } \mathcal{V} \text{ and } \mathcal{E}.

**Proof of Lemma 4.4.** First, let } \tilde{v} \text{ be a singular vertex (say } \tilde{v} \text{ projects to } v). \text{ We need to show that the cyclic order of the neighbors of } \tilde{v} \text{ induces an order on the neighbors } v.
Claim 1. Let $\tilde{v}_1, \tilde{v}_2 \in V(\tilde{G})$ be consecutive vertices in the counterclockwise cyclic order around $\tilde{v}$ and let $\tilde{v}_1'$ be a neighbor of $\tilde{v}$ so that $f(\tilde{v}_1') = f(\tilde{v}_1)$.

Then the vertex that follows $\tilde{v}_1'$ in the counterclockwise cyclic order around $\tilde{v}$ projects to the same vertex as $\tilde{v}_2$.

Proof of Claim 1: Since $\tilde{v}_1$ and $\tilde{v}_1'$ project to the same vertex, there exists $\gamma \in \Gamma$ so that $\gamma(\tilde{v}_1) = \tilde{v}_1'$. Denote the edge $\{\tilde{v}, \tilde{v}_1'\}$ by $e_1$ and the image of $\{\tilde{v}_1, \tilde{v}\}$ under $\gamma$ by $e_2$ (that is, $e_2 = \{\tilde{v}_1', \gamma(\tilde{v})\}$). If $\tilde{v} \neq \gamma(\tilde{v})$ then $e_1 \cup e_2$ projects to a cycle of length 1 or 2, contradicting our assumption (recall Assumptions 2.1). We conclude that $\gamma$ fixes $\tilde{v}$. By assumption, there exists $\phi' : S^2 \to S^2$ a homeomorphism that extends $\gamma$; thus $\phi(\tilde{v}) = \tilde{v}$, $\phi(\tilde{v}_1) = \tilde{v}_1'$, and $\phi(\tilde{v}_2)$ is a vertex that projects to the same vertex as $\tilde{v}_2$, and follows $\tilde{v}_1'$ in the counterclockwise cyclic order around $\tilde{v}$. This proves Claim 1.

It follows immediately from Claim 1 that the counterclockwise cyclic order around $\tilde{v}$ induces a cyclic order around $v$.

Next, let $\tilde{v}, \tilde{v}' \in V(\tilde{G})$ be distinct vertices that project to the same vertex $v \in V(G)$. Then there exists $\gamma \in \Gamma$ so that $\gamma(\tilde{v}) = \tilde{v}'$, and by assumption there exists $\phi : S^2 \to S^2$ extending $\gamma$. It is easy to see that $\phi$ induces an order preserving bijection between the neighbors of $\tilde{v}$ and the image of $\tilde{v}'$. Therefore the neighbors of $\tilde{v}$ and the neighbors of $\tilde{v}'$ induce the same order on the neighbors of $v$; this establishes Property $\mathcal{V}$.

Since $f : \tilde{G} \to G$ fulfills Property $\mathcal{V}$, we can assign a valid sign assignment for $V(\tilde{G})$ (as described in Section 3). Let $\tilde{e}_1, \tilde{e}_2 \in E(\tilde{G})$ be edges that project to the same edge $e \in E(G)$; say $\tilde{e}_1 = \{\tilde{v}_1, \tilde{v}_1'\}$ and $\tilde{e}_2 = \{\tilde{v}_2, \tilde{v}_2'\}$. Then there exists $\gamma \in \Gamma$ so that $\gamma(\tilde{e}_1) = \tilde{e}_2$, equivalently $\{\gamma(\tilde{v}_1), \gamma(\tilde{v}_2)\} = \{\tilde{v}_1', \tilde{v}_2'\}$. If $\gamma$ is the same as the sign of $\gamma(\tilde{v}_1)$ and the sign of $\tilde{v}_2$ is the same as the sign of $\gamma(\tilde{v}_2)$; if $\gamma$ reverses the orientation of $S^2$ then the sign of $\tilde{v}_1$ is the opposite of the sign of $\gamma(\tilde{v}_1)$ and the sign of $\tilde{v}_2$ is opposite the sign of $\gamma(\tilde{v}_2)$. In both cases, $\tilde{e}_1$ connects vertices of the same sign if and only if $\tilde{e}_2$ does. This establishes Property $\mathcal{E}$ and completes the proof of Lemma 4.4.

Clearly, Proposition 4.1 follows from Lemmas 4.2 and 4.4.

5. Higher covers

In this section, we try to better our situation by passing to higher covers. Let $f : \tilde{G} \to G$ be a finite cover. When does there exist a finite planar cover $\tilde{f} : \tilde{G} \to \tilde{G}$ so that the composition $\tilde{f} \circ f : \tilde{G} \to G$ fulfills Properties $\mathcal{V}$ and $\mathcal{E}$? Assume such cover exists. To address this question, we wish to apply Theorem 3.3 to the cover $\tilde{f} : \tilde{G} \to \tilde{G}$. However, it does not follow that $\tilde{f} : \tilde{G} \to \tilde{G}$ fulfills Properties $\mathcal{V}$ and $\mathcal{E}$. To guarantee that, we need to add the assumption that the cover $f : \tilde{G} \to G$ is not branched.
Recall from Notation 2.6 that the map \( \pi: S^2 \to P^2 \) given by identifying antipodal points is an unbranched, double cover called the universal cover of \( P^2 \). Therefore, given a graph \( \tilde{G} \subset P^2, \pi^{-1}(\tilde{G}) \) is a graph that double covers \( \tilde{G} \) and naturally embeds in \( S^2 \).

**Theorem 5.1.** Let \( f: \tilde{G} \to G \) be an unbranched finite (not necessarily planar) cover and \( \tilde{f}: \tilde{G} \to \tilde{G} \) be a finite planar covers. Suppose that the composition \( \tilde{f} \circ f: \tilde{G} \to G \) fulfills Properties \( V \) and \( E \). Then one of the following holds:

1. \( \tilde{G} \) is planar and for some embedding of \( \tilde{G} \) into \( S^2 \), \( f: \tilde{G} \to G \) fulfills Properties \( V \) and \( E \).
2. \( \tilde{G} \) embeds in \( P^2 \) so that the cover \( f \circ \pi: \pi^{-1}(\tilde{G}) \to G \) fulfills Properties \( V \) and \( E \).

**Proof.** We begin by showing:

**Proposition 5.2.** Let \( f: \tilde{G} \to G \) be an unbranched finite planar cover and \( \tilde{f}: \tilde{G} \to \tilde{G} \) be a finite planar covers. Suppose that the composition \( \tilde{f} \circ f: \tilde{G} \to G \) fulfills Properties \( V \) and \( E \). Then \( \tilde{f}: \tilde{G} \to \tilde{G} \) fulfills Properties \( V \) and \( E \).

**Proof of Proposition 5.2.** Let \( \tilde{v}, \tilde{v}' \in V(\tilde{G}) \) be two vertices that project to the same vertex under \( \tilde{f} \) (say \( \tilde{v} \)) and denote \( f(\tilde{v}) \) by \( v \). By assumption, \( \tilde{f} \circ f \) fulfills Property \( V \). Therefore, the cyclic order on the neighbors \( (\tilde{f} \circ f)^{-1}(v) \) induced by \( f(\tilde{v}) \) is order preserving. Hence, the cyclic order around \( \tilde{f}^{-1}(\tilde{v}) \) (which is a subset of \( (\tilde{f} \circ f)^{-1}(v) \)) induces a cyclic order on the neighbors of \( \tilde{v} \). This establishes Property \( V \). By construction the bijection induced by \( f \) between the neighbors of \( \tilde{v} \) and the neighbors of \( f(\tilde{v}) \) is order preserving.

Before establishing Property \( E \) we must assign valid signs to the vertices of \( \tilde{G} \). Since \( \tilde{f} \circ f \) fulfills Property \( E \), some signs have been assigned already, and these signs are valid for \( f \circ \tilde{f} \). We show that the same signs are valid for \( \tilde{f} \). Let \( \tilde{v}, \tilde{v}' \in V(\tilde{G}) \) be vertices so that \( \tilde{f}(\tilde{v}) = \tilde{f}(\tilde{v}') \). Then \( f \circ \tilde{f}(\tilde{v}) = f \circ \tilde{f}(\tilde{v}') \) and therefore \( \tilde{v} \) and \( \tilde{v}' \) have the same (resp. opposite) sign if and only if the counterclockwise cyclic order around them is the same (resp. opposite) under \( f \circ \tilde{f} \). The order preserving bijection induced on the neighbors of \( \tilde{v} \) by \( f \) shows that the sign choice is valid for \( \tilde{f} \) as well.

Let \( \tilde{e}, \tilde{e}' \in E(\tilde{G}) \) be two edges that project to the same edge (say \( e \)) under \( \tilde{f} \). By assumption \( \tilde{f} \circ f \) fulfills Property \( E \). Since \( \tilde{e} \) and \( \tilde{e}' \) project to the same edge under \( f \circ \tilde{f} \), \( \tilde{e} \) connects vertices of the same sign if and only if \( \tilde{e}' \) does. Hence \( \tilde{f}: \tilde{G} \to \tilde{G} \) fulfills Property \( E \).

This completes the proof of Proposition 5.2.
By Proposition 5.2 we may apply Theorem 3.3 to \( \tilde{f} : \tilde{G} \to \tilde{G} \) and get an embedding of \( \tilde{G} \) into \( F \), where \( F \cong S^2 \) or \( F \cong P^2 \).

**Case One:** \( F \cong S^2 \): This case corresponds to Case (1) of Theorem 5.1. We need to show that \( f : \tilde{G} \to G \) fulfills Properties \( V \) and \( E \). By Theorem 3.3 (5) the embedding of \( \tilde{G} \) into \( S^2 \) and the map \( \tilde{f} \) induce the same order around the vertices of \( \tilde{G} \). Similarly, applying Theorem 3.3 to \( f \circ \tilde{f} \) we obtain an embedding of \( G \) into some surface \( F' \), and the embedding of \( G \) into \( F' \) and the map \( f \circ \tilde{f} \) induce the same order around the vertices of \( G \).

To verify Property \( V \), fix \( \tilde{v} \in V(\tilde{G}) \) and denote \( f(\tilde{v}) \) by \( v \). By assumption, the cover \( f \) is not branched. Therefore \( f \) induces a bijection between the neighbors of \( \tilde{v} \) and the neighbors of \( v \). Since the orders around \( \tilde{v} \) and around \( v \) are both induced from the order around their preimages in \( \tilde{G} \), this bijection is order preserving. Therefore, \( f \) fulfills Property \( V \).

Next we need a valid sign assignment on \( V(\tilde{G}) \). Fix \( \tilde{v} \in V(\tilde{G}) \). By assumption, \( F \cong S^2 \) and therefore by Proposition 3.6 all the preimages of \( \tilde{v} \) have the same sign. We assign \( \tilde{v} \) that sign. We need to verify that the assignment is valid. Fix \( \tilde{v}, \tilde{v}' \in V(\tilde{G}) \) that project to the same vertex, say \( v \). Let \( D_v \) be a small disk neighborhood of \( v \) oriented so that the restriction of \( f \circ \tilde{f} \) to a component of \( (f \circ \tilde{f})^{-1}(D_v) \) is orientation preserving if and only if the sign at the corresponding preimage of \( v \) is plus (this is possible since the sign assignment on \( V(\tilde{G}) \) is valid for \( f \circ \tilde{f} \)). We choose an orientation for \( F \cong S^2 \) so that \( \tilde{f} \) is orientation preserving. It is easy to see that given \( \tilde{v} \) with \( f(\tilde{v}) = v \), \( f|_{D_v} : D_v \to D_v \) is orientation preserving if and only if only if the sign at \( \tilde{v} \) is plus; it follows that the sign assignment is valid.

Finally we verify Property \( E \). Given \( e \in E(G) \) and \( \tilde{e}, \tilde{e}' \) that project to \( e \), the signs at the endpoints of \( \tilde{e} \) are the same as the signs at the endpoints of any edge that projects to \( \tilde{e} \) under \( \tilde{f} \), and similarly for \( \tilde{e}' \). Property \( E \) for \( f \) follows from Property \( E \) for \( \tilde{f} \) (that was established is Proposition 5.2).

**Case Two:** \( F \cong P^2 \): This case corresponds to Case (2) of Theorem 5.1. Let \( \tilde{G} \) be the lift of \( G \) to \( P^2 \), i.e., \( \tilde{G} = \pi^{-1}(G) \). (Recall the definition of the universal cover \( \pi : S^2 \to P^2 \) in Notation 2.6)

Although not essential to the proof, we show that \( \tilde{G} \) is connected. Every face of \( S^2 \) cut open along \( \tilde{G} \) is an unbranched cover of a face of \( \tilde{G} \), which by Theorem 3.3 (5) (applied to \( \tilde{f} \)) is a disk. Therefore the faces of \( \tilde{G} \) are disks as well and \( \tilde{G} \) is connected.

By Lemma 2.7 \( \tilde{f} \) factors through \( \pi \); that is, there exist a cover \( \tilde{f} : S^2 \to S^2 \) so that \( \tilde{f} = \pi \circ \tilde{f} \). Then \( \tilde{G} = \tilde{f}(\tilde{G}) \). Therefore the orders induced on the neighbors of every vertex of \( V(\tilde{G}) \) by the embedding into \( S^2 \) and by \( \tilde{f} \) are the same, and we conclude that \( f \circ \pi \) induces that same order on the neighbors of every vertex in \( V(G) \) as \( f \circ \tilde{f} \);
in particular, \( f \circ \pi \) induces some order on the neighbors of every vertex in \( V(G) \) and therefore fulfills Properties \( V \) and \( E \).

This completes the proof of Theorem 5.1. \( \square \)

6. Examples

Our first example is very simple. It shows a cover of a graph with one vertex and two edges (the bouquet of two circles). The cover is given in Figure 6 and is an irregular triple cover that fulfills Properties \( V \) and \( E \). (We note that any double cover is regular.) In that figure, we use the following labels \( a \) and \( b \), where the three edges labeled \( a \) project to the same edge and the three edges labeled \( b \) project to the other edge. The the bouquet of two circles is not shown in the figure.

**Example 6.1.** Let \( G \) be the bouquet of two circles and \( \tilde{G} \) the triple cover given in figure 6; the projection \( f \) is indicated by labels and arrows. Then \( f : \tilde{G} \to G \) is an irregular cover fulfilling Properties \( V \) and \( E \).

Of course, the bouquet of two circles cannot be regarded as an “interesting” graph. The reader can soup this example up by replacing the edges of the bouquet of two circles by planar graphs. However, the resulting cover is not 3-connected. We ask:

**Question 6.2.** Let \( f : \tilde{G} \to G \) be a finite planar graph fulfilling Properties \( V \) and \( E \). Suppose \( G \) has no 1 or 2-cycles and \( \tilde{G} \) is 3-connected, is the cover regular?

All the remaining examples in this section are covers of \( K_4 \). The labels used are as follows: the vertices of \( K_4 \) are labeled \( a, b, c, \) and \( d \) and vertices of the covers are labeled by the vertex they project to; edges are not labeled.

**Example 6.3.** Figures 5 and 6 give two regular, unbranched, planar double covers of \( K_4 \). The first cover yields an embedding of \( K_4 \) into \( S^2 \) while the second cover yields an embedding of \( K_4 \) into \( P^2 \).

We see by inspection that both covers fulfill Properties \( V \) and \( E \). It follows from Proposition 3.6 the first cover gives an embedding of \( K_4 \) into \( S^2 \) while the second embeds \( K_4 \) in \( P^2 \). Alternatively, we can see that the second cover yields an embedding into \( P^2 \) by observing that it has two cycles of length four that bound faces. These
cycles project to two cycles of length two or a single cycle of length four in \( K_4 \) (in fact, they project to a single cycle of length four). By Theorem 5.1 (5) this cycle bounds a face. But the unique embedding of \( K_4 \) into \( S^2 \) has only triangular faces.

The next example, Example 6.4, is the triple cover \( f : \tilde{K}_4 \rightarrow K_4 \) given in Figure 7. We observe that with the given embedding the cover does not fulfill Property \( E \). This cover is not 3-connected so the embedding into \( S^2 \) is not unique. However, the only other embedding is given by “flipping” the disk contained in the dashed circle. It is easy to see that it too does not fulfill Property \( E \).

We seek a planar cover \( \tilde{\tilde{K}}_4 \rightarrow \tilde{K}_4 \) so that the composition \( \tilde{\tilde{K}}_4 \rightarrow K_4 \) fulfills Properties \( V \) and \( E \). Although covers of this type may have arbitrarily high degree, by Theorem 5.1 if such cover exists, then a double cover with this property exists as well. Moreover, Theorem 5.1 tells us how to find this cover: let \( \pi : S^2 \rightarrow P^2 \) be the universal cover; embed \( \tilde{K}_4 \) in \( P^2 \) and lift to \( S^2 \); this is the cover we need.

**Example 6.4.** The cover given in Figure 7 does not fulfill Properties \( V \) and \( E \) for any embedding of \( \tilde{G} \) into \( S^2 \). However, the embedding of \( \tilde{K}_4 \) into \( P^2 \) given in Figure 8 (where we view \( P^2 \) as a disk union a Möbius band) has a double cover (given in Figure 10) that embeds in \( S^2 \) and fulfills Properties \( V \) and \( E \).

Figure 10 was constructed by taking two copies of the disk in Figure 8 and an annulus that double covers the Möbius band in Figure 8 see Figure 9. Pasting the disks to the annulus in Figure 9 gives Figure 10.

Our final example is far more disturbing in nature. We give a cover \( f : \tilde{K}_4 \rightarrow K_4 \) that does not fulfill Properties \( V \) and \( E \) (quite similar to Example 6.4) and conjecture
that for any finite planar cover $\tilde{K}_4 \to \tilde{K}_4$, the composition $\tilde{K}_4 \to K_4$ does not fulfilling Properties $\mathcal{V}$ and $\mathcal{E}$.

**Conjecture 6.5.** Consider the cover $\tilde{K}_4$ given in Figure 11. Let $\tilde{K}_4 \to \tilde{K}_4$ be a finite planar cover. Then the composition $\tilde{K}_4$ does not fulfill Properties $\mathcal{V}$ and $\mathcal{E}$. 
We end this paper by describing two ways of dealing with the cover $\tilde{K}_4$ given in Figure 11. First, note that if the four disks enclosed in dashed circles in Figure 11 are replaced by the disks given in Figure 12 (and some of the “mushrooms” are reflected), we obtain the cover shown in Figure 13 that does fulfill Properties $V$ and $E$. The work in this paper suggests that topological “cut-and-paste” techniques such as this could be very useful.

Next, recall Figures 7 and 8 where we replaced a disk with a Möbius band. Similarly, we can replace the four disks in Figure 11 with four Möbius bands, obtaining an embedding of $\tilde{K}_4$ into a non-orientable surface $S$; since replacing a disk with a Möbius band lowers the Euler characteristic by 1, $\chi(S) = -2$. Just like $P^2$, $S$ has an oriented double cover (say $\tilde{S}$). The Euler characteristic is multiplicative under unbranched covers, and we conclude that $\tilde{S}$ is the surface of genus 3. (In general, if we replace $n > 0$ disks with Möbius bands the oriented double cover will have genus $n - 1$.) The reader can verify that the proof of Theorem 3.3 is valid, and gives a surface $F$, an embedding $G \subset F$, and a cover $f' : \tilde{S} \to F$ that extends $f$. As in Lemma 2.8, $\chi(F) \geq -2$. (In general, $\chi(F) \geq 2 - n$.) This does not give an
embedding of $G$ into $P^2$, but does give an embedding of $G$ into a surface with some control over its Euler characteristic.

References

[1] Dan Archdeacon. Two graphs without planar covers. *J. Graph Theory*, 41(4):318–326, 2002.
[2] Dan Archdeacon and R. Bruce Richter. On the parity of planar covers. *J. Graph Theory*, 14(2):199–204, 1990.
[3] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
[4] Petr Hliněný. Another two graphs with no planar covers. *J. Graph Theory*, 37(4):227–242, 2001.
[5] Petr Hliněný and Robin Thomas. On possible counterexamples to Negami’s planar cover conjecture. *J. Graph Theory*, 46(3):183–206, 2004.
[6] Shigeru Kitakubo. Planar branched coverings of graphs. *Yokohama Math. J.*, 38(2):113–120, 1991.
[7] Seiya Negami. The spherical genus and virtually planar graphs. *Discrete Math.*, 70(2):159–168, 1988.
[8] Seiya Negami. Composite planar coverings of graphs. *Discrete Math.*, 268(1-3):207–216, 2003.
[9] Seiya Negami. Projective-planar double coverings of graphs. *European J. Combin.*, 26(3-4):325–338, 2005.
[10] Yusuke Suzuki and Seiya Negami. Projective-planar double coverings of 3-connected graphs. *Yokohama Math. J.*, 50(1-2):87–95, 2003.
[11] Hassler Whitney. Congruent Graphs and the Connectivity of Graphs. *Amer. J. Math.*, 54(1):150–168, 1932.