The continuity of the map $\lim_T$ in Hausdorff spaces

J. E. Palomar Tarancón

Dep. Math. Inst. Jaume I
C./ Europa, 3-2E
112530-Burriana-(Castellón)-Spain

Email: jpalomar016n@cv.gva.es and jepalomar23@ono.com

Abstract

Consider a Hausdorff space $(X,T)$ and a set $C$ of converging nets in $X$. By virtue of the limit uniqueness, the relation $\text{Lim}$ which assigns each member $x$ of $X$ to every net $N$ lying in $C$ that converges to $x$ is a map. Of course, structuring $C$ with some topology $U$, $\text{Lim}$ can be a continuous map. If $T$ is a topology induced by a uniformity, and $F$ is a function space such that $X$ is the codomain of each $f$ in $F$, it is a well-known property the uniform limits of continuous functions to be continuous. In this paper, the author shows that the continuity of limits of continuous-map nets is implied by the continuity of the map $\text{Lim}$, whenever the involved topologies are large enough, being this result obtained without using neither the uniform convergence notion nor the uniformity concept.

AMS Classification numbers  Primary: 54C05

Secondary: 54B30, 54C35

Keywords: Continuity, limits, concrete categories, partial-morphisms, partial algebras, net-functor.
1 Introduction

Consider any non empty set $X$ and a topology $T$ for $X$. If $(X, T)$ is a Hausdorff space and $\mathcal{C}(T)$ stands for a collection of converging nets in $X$, then the relation $\lim_{T} \subseteq \mathcal{C}(T) \times X$ such that $(S, x) \in \lim_{T} S = x$ is a map. It is a natural question to consider some topology $\mathfrak{T}$ for $\mathcal{C}(T)$ such that $\lim_{T} : (\mathcal{C}(T), \mathfrak{T}) \rightarrow (X, T)$ is a continuous map. Another natural question consists of knowing under what conditions the limit $f$ of a net of continuous maps $S = (f_\delta)_{\delta \in D}$ is again a continuous one. The second question possesses a well-known answer for uniform spaces, namely, the uniform convergence. In this paper we shall prove (Theorem 2.9) that, under some circumstances, the first property implies the second one. This result has been performed without using neither the uniform convergence notion nor the uniformity concept.

Of course the continuity of a map $f : (X_1, T_1) \rightarrow (X_2, T_2)$ can be defined by means of an expression of the form $\lim_{T_2} f(x_\rho) = f(\lim_{T_1} x_\rho)$, or simply $\lim_{T_2} f = f(\lim_{T_1})$; accordingly, if $(f_\delta)_{\delta \in D}$ is a net of continuous maps converging to $f$, then the continuity of the limit $f$ can be stated by the relation $\lim_{T_2} f = f(\lim_{T_1})$, explicitly $\lim_{T_2} \lim_{T_1} f_\delta = \lim_{T_1} f_\delta(\lim_{T_2} x_\rho)$. Thus, since each of the $f_\delta$ is supposed to be continuous, the last equation implies that $\lim_{T_2} \lim_{T_1} f_\delta = \lim_{T_1} f_\delta(\lim_{T_2} x_\rho)$, and this equation can be obtained from $\lim_{T_2} \lim_{T_1} f_\delta = \lim_{T_2} f_\delta(\lim_{T_1} x_\rho)$ whenever the last one is true. Notice, that if $\lim_{T_2}$ is a map, then the last relation denotes $\lim_{T_2}$ to be continuous. This is why, at least in Hausdorff spaces, in order to state the continuity of the limits of converging continuous map nets, the continuity of the map $\lim_{T_2}$ is the natural condition, and such a requirement can be stated without using the uniform space concept.

Finally, in Theorem 2.10 it is shown that the continuity of the map $\lim_{T}$ implies some differential operators to be also continuous.

2 The net-functor and related partial algebras

Let $\mathcal{D}$ be a small-class (set) of directed sets being stable under products, that is, $\mathcal{D}$ contains with every countable subset $\{(D_n, \leq_n) \in \mathcal{D} | n \in I \subseteq N\}$ the product directed set $\prod_{n \in I}(D_n, \leq_n)$, besides, for every $(D, \leq) \in \mathcal{D}$ and each map $f : D \rightarrow \mathcal{D}$ the product directed set $\prod_{d \in D} f(d)$ again belongs to $\mathcal{D}$. With these assumptions, denote by $\mathfrak{R}_D : \text{Set} \rightarrow \text{Set}$ the endofunctor in the category $\text{Set}$ of ordinary sets, the object-map $\mathfrak{R}_{DO}$ of which carries each set $X$ into the net set $\bigcup_{D \in \mathcal{D}} X^D$; and the arrow-map $\mathfrak{R}_{DA}$ carries each morphism $f : X_1 \rightarrow X_2$ into the map $\mathfrak{R}_D(f)(S)$ such that $(x_\delta)_{\delta \in D} \mapsto (f(x_\delta))_{\delta \in D} \in X_2^D$. 

2
Remark Henceforth, to improve the readability, whenever a morphism \( f \) occurs as the argument of a functor \( \mathfrak{N}_D \), and in the same expression occurs an element \( x \) as the argument for \( \mathfrak{N}_D(f)(x) \), that is, the symbol \( \langle \rangle \) in the expression \( \mathfrak{N}_D\langle \rangle \) indicates that \( \mathfrak{N}_D \) works as an arrow-map.

Definition 2.1 Given a partial morphism \( \mathfrak{N}_D(X) \xleftarrow{i_{\mathfrak{C}(T)}} \mathfrak{C}(T) \xrightarrow{\lambda_T} X \), where \( i_{\mathfrak{C}(T)} \) stands for the canonical inclusion and \( \mathfrak{C}(T) \) is any non-empty subset of \( \mathfrak{N}_D(X) \), say the pair \( (X, (i_{\mathfrak{C}(T)}, \lambda_T)) \) to be a \( \lambda \)-partial \( \mathfrak{N}_D \)-algebra, provided that the following condition is satisfied.

The set \( \{(S, x) \in \mathfrak{C}(T) \times X | x = \lambda(S)\} \) is a convergence-class such that the corresponding topology is separated.

Obviously, in the category \( \text{Set} \) of ordinary sets and maps, the canonical inclusion of a subset is a monomorphism, therefore so is \( i_{\mathfrak{C}(T)} : \mathfrak{C}(T) \to \mathfrak{N}_D(X) \), and for every map \( \lambda : \mathfrak{C}(T) \to Y \), the pair \( (i_{\mathfrak{C}(T)}, \lambda_T) \) is a partial map or partial morphism between sets.

Lemma 2.2 If \( (X, (i_{\mathfrak{C}(T)}, \lambda_T)) \) is a \( \lambda \)-partial \( \mathfrak{N}_D \)-algebra, then for every \( S \) in \( \mathfrak{C}(T) \) the relation \( \lambda(S) = \lim_T S \) holds.

Proof It is a straightforward consequence of the former definition. \( \square \)

Remark Although the considered directed set class \( D \) is a small one, that is, a set of directed sets, in general, it is sufficient. Convergence classes can be built over those directed set collections each member of which is isomorphic to a base for a neighbourhood system ordered by inclusion. For instance, if a topological space satisfies the first axiom of countability only countable directed sets are necessary.

From now on, a pair of the form \( (X, (i_{\mathfrak{C}(T)}, \lim_T)) \) will be interpreted implicitly as a \( \lambda \)-partial \( \mathfrak{N}_D \)-algebra, therefore it must be understood \( (X, T) \) to be a Hausdorff space, otherwise \( \lim_T \) need not be a map.

As usual, morphisms between algebras induced by endofunctors in \( \text{Set} \) are those maps with respect to which some diagrams commute. Thus, given two \( \lambda \)-partial \( \mathfrak{N}_D \)-algebras \( (X_2, (i_{\mathfrak{C}(T_2)}, \lim_{T_2})) \) and \( (X_2, (i_{\mathfrak{C}(T_2)}, \lim_{T_2})) \), a map \( f : X_1 \to X_2 \) is a morphism, provided that there is a unique map \( f^* \) such that the following
diagram commutes.

\[
\begin{array}{c}
\text{(1)}
\end{array}
\]

Of course, the map \( f^\ast \) is nothing but the restriction \( \mathcal{D}(f)\big|_{\mathcal{C}(T_1)} \) of \( \mathcal{D}(f) \) to \( \mathcal{C}(T_1) \); accordingly the former diagram can be also written as follows.

\[
\begin{array}{c}
\text{(2)}
\end{array}
\]

Lemma 2.3 Morphisms among \( \lambda \)-partial \( \mathcal{D} \)-algebras are, precisely, continuous maps.

Proof Taking into account (2), every morphism satisfies the relation

\[
\lim_{T_2} \mathcal{D}(f)(S) = f\left(\lim_{T_1} S\right)
\]

for every converging net \( S = (x_\delta)_{\delta \in D} \) in \( \mathcal{C}(T_1) \), and this expression being written in the usual notation is nothing but \( \lim_{T_2} f(x_\delta) = f\left(\lim_{T_1} x_\delta\right) \).

Remark For our purposes, it is a better notation \( \lim_{T_2} \mathcal{D}(f)(S) = f\left(\lim_{T_1} S\right) \) than \( \lim_{T_2} f(x_\delta) = f\left(\lim_{T_1} x_\delta\right) \), since in the first expression it is denoted explicitly \( \mathcal{D}(f) \) to have a net as argument.
Since in a Hausdorff space \((X, T)\), \(\lim_T : C(T) \to X\) is a map, it is a natural question to build some separated topology \(\mathfrak{T}\) for \(C(T)\) with respect to which \(\lim_T\) is a continuous map. According to (2), the continuity for \(\lim_T\) can be stated by means of the following diagram.

\[
\begin{array}{c}
\mathcal{N}_D(C(T)) \\
\mathcal{N}_D(\lim_T) \downarrow \downarrow \downarrow \downarrow \downarrow \\
\mathcal{N}_D(X) \\
\mathcal{N}_D(\lim_T) \downarrow \downarrow \downarrow \downarrow \downarrow \\
C(T) \\
\end{array}
\]

Thus, \(\lim_T : C(T) \to X\) is a continuous map if and only if the former diagram commutes, that is, the following relation holds.

\[
\lim_T \left( \lim_T \right) = \lim_T \left( \mathcal{N}_D \left( \lim_T \right) \right)
\]

2.1 Transposing a net of nets

When the underlying set of a \(\lambda\)-partial \(\mathcal{N}_D\)-algebra \((C(T), (i_{\mathcal{E}(T)}, \lim_{\mathcal{T}}))\) is a net set \(C(T)\) with a separate topology \(\mathcal{T}\), each net lying in \(C(\mathcal{T})\) is a net of nets. For example, given a directed set \((D, \leq) \in D\), let \(S = (S_\delta)_{\delta \in D}\) be a net lying in \(C(\mathcal{T})\). Of course, for each \(\delta \in D\), \(S_\delta = (x_{\delta, d})_{d \in D_\delta}\) is a net in \(C(T)\), where \((D_\delta, \leq_\delta)\) stands for a directed set in \(D\), for each \(\delta \in D\). Obviously, \(S\) can be regarded as a net of nets. If for every \(\delta \in D\), the relation \((D_\delta, \leq_\delta) = (D_0, \leq_0)\) holds, then the net \(S\) can be written in a matrix form, that is, \(S = (x_{\delta, d})_{(d, \delta) \in D_0 \times D}\), being the nets of \(S\) the \(S_\delta\). However, transposing the matrix we obtain the net of nets \(S^t = (x_{\delta, d})_{(\delta, d) \in D_0 \times D}\), which is formed by the nets \(S^t_d = (x_{\delta, d})_{\delta \in D}\), for each \(d \in D_0\).

**Definition 2.4** If a net of nets \(S = (S_\delta)_{\delta \in D}\) belongs to \(C(\mathcal{T})\) for some \(\lambda\)-partial \(\mathcal{N}_D\)-algebra \((\mathcal{N}_D(X), (i_{\mathcal{E}(T)}, \lim_{\mathcal{T}}))\), then \(\mathcal{N}_D(\lim_T)(S)\) is the net \((x_\delta)_{\delta \in D} = (\lim_d x_{d, \delta})_{\delta \in D} = (\lim_T S_\delta)_{\delta \in D}\). Sometimes, the nets forming its transposed are also converging ones. In this case, there is also the net \(\mathcal{N}_D(\lim_T)(S^t)\) which is formed by the nets \((x_d)_{d \in D_0} = (\lim_\delta x_{\delta, d})_{d \in D_0} = \mathcal{N}_D(\lim_T)S^t\). From now on, say
a topology $\mathfrak{T}$ for a net set $\mathfrak{C}(T)$ to satisfy the transposition property, provided that for every $\mathfrak{T}$-converging net of the form $S = (x_{d,\delta})_{d \in D_0, \delta \in D}$ there exists $\mathfrak{N}_D(\lim T) S^t = (x_d)_{d \in D_0}$, besides, the following relation holds.

$$\mathfrak{N}_D(\lim T) S^t = \lim_{\mathfrak{T}} S$$

(6)

**Remark** If $S$ is a one–column or one–row net, for instance $S = (x_\delta)_{\delta \in D}$, both limits $\lim T S$ and $\lim T S^t$ are the same; consequently one can eliminate the transposition operator $()^t$.

**Remark** The continuity of the map $\lim T$, according to (5) is implied by the relation

$$\lim_{\mathfrak{T}} \mathfrak{N}_D(\lim T) S^t = \lim_{\mathfrak{T}} \lim_{\mathfrak{T}} S^t$$

(7)

However, if we consider the net of nets $S$ in a matrix form, that is, considering column-nets and row-nets, the former expression must be written as follows

$$\lim_{\mathfrak{T}} \mathfrak{N}_D(\lim T) S^t = \lim_{\mathfrak{T}} \lim_{\mathfrak{T}} S$$

(8)

because the net consisting of the limits of all rows is a column one, and vice versa.

### 2.2 Continuity of the map $\lim T$

The results in this section are implied by the continuity of the map $\lim T$ together with the transposition property of the involved topologies.

**Theorem 2.5** If $(X_1, (i_{\mathfrak{C}(T_1)}, \lim T_1))$ and $(X_1, (i_{\mathfrak{C}(T_1)}, \lim T_1))$ are two $\lambda$-partial algebras, and $\mathfrak{T}_1$ and $\mathfrak{T}_2$ two topologies for $\mathfrak{C}(T_1)$ and $\mathfrak{C}(T_2)$, respectively, each of which satisfies the transposition property, then for every continuous map $f : (X_1, T_1) \rightarrow (X_2, T_2)$ the image $\mathfrak{N}_D(f) : \mathfrak{N}_D(X_1, T_1) \rightarrow \mathfrak{N}_D(X_2, T_2)$ is again continuous.

**Proof** Let $S = (S_d)_{d \in D_1} \in \mathfrak{N}_D(\mathfrak{N}_D(X))$ be a converging net, such that for every $d \in D_1$, the net $S_d = (x_{d,\delta})_{\delta \in D_2} \in \mathfrak{N}_D(X)$ is also a converging one. Since $\lim T_2$ is a map, therefore a $\text{Set}$-morphism, by the morphism composition compatibility of any functor it follows, that

$$\mathfrak{N}_D(\lim_{T_2}) (\mathfrak{N}_D(\mathfrak{N}_D(f))(S))^t = \mathfrak{N}_D \left( \lim_{T_2} (\mathfrak{N}_D(f))(S)^t \right)$$

(9)
Now, by virtue of the transposition property the net of nets \( \mathcal{N}_D(\mathcal{N}_D(f))(S) \) satisfies (6), consequently

\[
\mathcal{N}_D(\lim_{T_2} (\mathcal{N}_D(\mathcal{N}_D(f))(S))^t = \mathcal{N}_D(\lim_{T_2} \mathcal{N}_D(\mathcal{N}_D(f)))(S) = \lim_{T_2} \mathcal{N}_D(\mathcal{N}_D(f))(S) \quad (10)
\]

hence, from (9) and (10) we obtain

\[
\mathcal{N}_D \left( \lim_{T_2} (\mathcal{N}_D(f))(S)^t \right) = \lim_{T_2} \mathcal{N}_D(\mathcal{N}_D(f))(S) \quad (11)
\]

and because \( f \) is assumed to be continuous, taking into account Lemma 2.3 and equation (3) one can obtain that

\[
\mathcal{N}_D \left( \lim_{T_1} (\mathcal{N}_D(f))(S)^t \right) = \mathcal{N}_D \left( f(\lim_{T_1} S)^t \right),
\]

and using this equality in (11),

\[
\mathcal{N}_D \left( f(\lim_{T_1} S)^t \right) = \mathcal{N}_D(f) \left( \mathcal{N}_D(\lim_{T_1} S)^t \right) = \lim_{T_2} \mathcal{N}_D(\mathcal{N}_D(f))(S) \quad (12)
\]

Finally, using (6) in the former equation, we have that

\[
\mathcal{N}_D(f) \left( \lim_{T_1} S \right) = \lim_{T_2} \mathcal{N}_D(\mathcal{N}_D(f))(S) \quad (13)
\]

and by virtue of (3), the former equation implies \( \mathcal{N}_D(f) = \mathcal{N}_D(f) \) to be continuous.

**Notation 2.6** Henceforth, to avoid any confusion in expressions containing iterated limits like the following one

\[
\lim_{T_1} \lim_{T_2} \lim (x_{\delta,d})_{(\delta,d) \in D_1 \times D_2}
\]

we shall write the generic member \( \delta \) of the corresponding directed set together with the symbol denoting the topology, being both symbols separated by a semicolon. Thus, the former expression will be written explicitly as follows.

\[
\lim_{T_1} \lim_{T_2} (x_{\delta,d})_{(\delta,d) \in D_1 \times D_2}
\]

**Lemma 2.7** Let \( (X_1, (i_{\in T_1}), \lim_{T_1}) \) and \( (X_2, (i_{\in T_2}), \lim_{T_2}) \) be two \( \lambda \)-partial \( \mathcal{N}_D \)-algebras and \( T \) a topology for \( F = \text{hom} ((X_1, T_1), (X_2, T_2)) \). If \( T \) is finer than or equivalent to the topology of the pointwise convergence, then the restriction of the arrow-map of \( \mathcal{N}_D \) to \( F \) is continuous.
Proof Let \( S = (f_\delta)_{\delta \in D_1} \) a net in \( \mathcal{F} \) converging to \( f \), so then for every net \( N = (x_\rho)_{\rho \in D_2} \) we have that
\[
\lim_T \mathcal{N}_D(S)(N) = \lim_T \mathcal{N}_D(f_\delta)_{\delta \in D_1}(N) = \lim_{T; \delta} \mathcal{N}_D(f_\delta)(x_\rho)_{\rho \in D_2}
\] (14)
and because \( T \) is assumed to be finer than or equivalent to the topology of pointwise convergence, for every \( \rho \in D_2 \) the relation \( \lim_{T; \delta} f_\delta(x_\rho) = \lim_{T_2; \delta} f_\delta(x_\rho) = f(x_\rho) \) holds, consequently
\[
\lim_{T; \delta} \mathcal{N}_D(f_\delta)(x_\rho)_{\rho \in D_2} = \lim_{T_2; \delta} \mathcal{N}_D(f_\delta)(x_\rho)_{\rho \in D_2} = \mathcal{N}_D(f)(x_\rho)_{\rho \in D_2}
\] that is to say, \( \lim_T \mathcal{N}_D(S)(N) = \mathcal{N}_D(f)(N) = \mathcal{N}_D(\lim S)(N) \), therefore
\[
\lim_T \mathcal{N}_D(S) = \mathcal{N}_D(\lim S)
\] (15)
\[\square\]

Notation 2.8 For every couple of sets \( X \) and \( Y \), denote by \( \text{ev} : Y^X \times X \rightarrow Y \) the general evaluation map, that is to say, \( \forall (f, x) \in Y^X \times Y, \text{ev}(f, x) = f(x) \). Thus, fixed a point \( x \in X \), \( \text{ev}_x(f) = \text{ev}(f, x) = f(x) \) is the ordinary evaluation map. By symmetry, fixed a map \( f \), denote \( \text{ev}_f : X \rightarrow Y \) by co-evaluation map, although it is nothing but the same \( f : X \rightarrow Y \).

Theorem 2.9 (Main theorem) Let \((X_1, (i_\epsilon(T_1), \lim T_1)), (X_2, (i_\epsilon(T_2), \lim T_2))\) and \((F, (i_\epsilon(T), \lim T))\) be three \( \lambda \)-partial \( \mathcal{N}_D \)-algebras, and \( \mathcal{T}_1, \mathcal{T}_2 \) and \( \mathcal{T} \) topologies for \( \mathcal{E}(T_1), \mathcal{E}(T_2) \) and \( \mathcal{E}(T) \) respectively. If \( F \) belongs to \( X_2^X \), \( F_0 \) stands for any non-empty subset of \( F \), and the following statements hold,
\begin{itemize}
  \item[a)] Each of the topologies \( \mathcal{T}_2, \mathcal{T}_2 \) and \( \mathcal{T} \) satisfies the transposition property.
  \item[b)] \( T \) is finer than or equivalent to the pointwise convergence topology.
  \item[c)] Every map in \( F_0 \) is continuous, and so is every evaluation map \( \text{ev}(f, x) \) for every \( x \in X_1 \).
  \item[d)] The map \( \lim_{T_2} : (\mathcal{E}(T_2), \mathcal{T}_2) \rightarrow (X_2, T_2) \) is continuous.
\end{itemize}
then, for every directed set \((D_1, \preceq)\) in \( D \), and each converging net \( S = (f_\delta)_{\delta \in D_1} \) in \( F_0 \) the limit \( f = \lim_T S = \lim_T (f_\delta)_{\delta \in D_1} \) is a continuous map.

Proof Let \( N = (x_\rho)_{\rho \in D_2} \) a converging net in \( X_1 \) and consider the net of nets
\[
S = \mathcal{N}_D(\mathcal{N}_D(\text{ev})) (\mathcal{N}_D(f_\delta)_{\delta \in D_1}, (x_\rho)_{\rho \in D_0})
\] (16)
Since, by hypothesis, the map \( \lim_{T_2} \) is continuous, from (8) it follows that
\[
\lim_{T_2} \mathcal{N}_D(\lim) S^T = \lim_{T_2} S \in \mathcal{T}_2
\] (17)
Now, since it is assumed that, for each \( f \in F_0 \) the map \( ev_f = f \) is continuous, and by hypothesis, so is each \( ev_x \) for every \( x \in X_1 \) together with their images under \( N_D \) and \( N_D \circ N_D \), as consequence of Theorem 2.5. Thus, by continuity, from the left hand of (17) we obtain that

\[
\lim_{T_2} N_D(\lim_{T_2} S) = \\
\lim_{T_2; \delta} N_D(\lim_{T_2; \rho} \left( N_D(\lim_{T_2; \delta} \left( N_D(f_\delta) \right)_{\delta \in D_1}, (x_\rho)_{\rho \in D_2} \right) \right)^t = \\
\lim_{T_2; \delta} \left( N_D(\lim_{T_2; \rho} f_\delta \left( \lim_{T_1} x_\rho \right) \right)^t = f(\lim_{T_1} x_\rho) \tag{18}
\]

and because, by assumption, \( T \) is finer than or equivalent to the pointwise convergence topology, the net \( (f_\delta)_{\delta \in D_1} \) pointwise converges to \( f \), therefore

\[
\lim_{T_2; \delta} f_\delta \left( \lim_{T_1} x_\rho \right) = f(\lim_{T_1} x_\rho) \tag{19}
\]

where the transposition is eliminated because of the expression contains only a one–row net.

Likewise, taking into account that \( T_2 \) satisfies the transposition property, using (6) in (17), from the right hand of (17) we obtain that

\[
\lim_{T_2} \lim_{T_2} S = \lim_{T_2; \rho} \lim_{T_2; \delta} N_D(\lim_{T_2; \delta} N_D(\lim_{T_2; \rho} f_\delta) \left( (N_D(f_\delta))_{\delta \in D_1}, (x_\rho)_{\rho \in D_2} \right)) = \\
\lim_{T_2; \rho} N_D(\lim_{T_2; \delta} \left( N_D(\lim_{T_2; \rho} f_\delta) \left( N_D(f_\delta) \right)_{\delta \in D_1}, (x_\rho)_{\rho \in D_2} \right)) \tag{20}
\]

Now, by continuity it follows that

\[
\lim_{T_2; \rho} \left( N_D(\lim_{T_2; \delta} \left( N_D(\lim_{T_2; \rho} f_\delta) \right)_{\delta \in D_1}, (x_\rho)_{\rho \in D_2} \right) \right)^t = \\
\lim_{T_2; \rho} \left( N_D(\lim_{T_2; \delta} \left( N_D(\lim_{T_2; \rho} f_\delta) \right)_{\delta \in D_1}, (x_\rho)_{\rho \in D_2} \right) \right)^t = \\
\lim_{T_2; \rho} \left( N_D(\lim_{T_2; \delta} f_\delta) \left( N_D(f_\delta) \right)_{\delta \in D_1}, (x_\rho)_{\rho \in D_2} \right) \right)^t \tag{21}
\]

and because \( T \) is assumed to be finer than or equivalent to the topology of
pointwise convergence,
\[
\lim_{T_2;\rho} \left( \mathcal{D}(\lim f_\delta)_{\delta \in D_1}, (x_\rho)_{\rho \in D_2} \right) = \\
\lim_{T_2;\rho} \left( \mathcal{D}(\lim f_\delta)_{\delta \in D_1}, (x_\rho)_{\rho \in D_2} \right) = \\
\lim_{T_2;\rho} \left( \mathcal{D}(f), (x_\rho)_{\rho \in D_2} \right) = \\
\lim_{T_2;\rho} \left( \mathcal{D}(f), (x_\rho)_{\rho \in D_2} \right) = \lim_{T_2;\rho} \mathcal{D}(f) \left( (x_\rho)_{\rho \in D_2} \right) = f(\lim_{T_1} x_\rho) \quad (22)
\]

Finally, by virtue of equation (17), from (19) and (22) it follows immediately,
\[
\lim_{T_2} \mathcal{D}(f) \left( (x_\rho)_{\rho \in D_2} \right) = f(\lim_{T_1} x_\rho) \quad (23)
\]
and because the net \((x_\rho)_{\rho \in D_2}\) is arbitrary, \(f\) is a continuous map. □

2.3 The continuity of differential operators

Let \(\mathfrak{A}_1 = (F_1, (i_\mathfrak{A}(T_1), \lim_{T_1}))\) be a \(\lambda\)-partial \(\mathfrak{A}_D\)-algebra each member \(f\) of which is a differentiable being its domain dense in a fixed closed subset \(U\) of \(\mathbb{R}\) with non-empty interior, and the codomain \(\mathbb{R}^n\). Assume \((F_1, T_1)\) to be a Hausdorff space. Of course, although the domains of members of \(F_1\) need not be the same, \((F_1, T_1)\) can be a topological vector space, for there are separated topologies and algebraic structures for such a kind of function sets, for instance, see [3]. Let \(\mathfrak{A}_2 = (F_2, (i_\mathfrak{A}(T_2), \lim_{T_2}))\) another \(\lambda\)-partial \(\mathfrak{A}_D\)-algebra such that the associated topological vector space \((F_2, T_2)\) contains the derivative of every member of \((F_1, T_1)\). Obviously, the differential operator \(\frac{d}{dt} : (F_1, T_1) \to (F_2, T_2)\) is continuous, provided that for each directed set \((D, \leq)\) in \(\mathcal{D}\) and every converging net \(S = (f_\delta)_{\delta \in D}\) the following relation holds.
\[
\lim_{T_2} \mathcal{D}(\frac{d}{dt})(S) = \frac{d}{dt} \lim_{T_1} S \quad (24)
\]

**Theorem 2.10** Let \((X \subseteq \mathbb{R}, (i_\mathfrak{A}(T), \lim_{T}))\) and \((\mathbb{R}^n, (i_\mathfrak{A}(T_u), \lim_{T_u}))\) two \(\lambda\)-partial \(\mathfrak{A}_D\)-algebras, where \(T\) and \(T_u\) are the standard topologies inducing the ordinary smooth structure for \(\mathbb{R}\) and \(\mathbb{R}^n\) respectively. Let \(F_1 \subseteq F_2\) be two subsets of \((\mathbb{R}^n)^X\) such that each member of \(F_1\) is defined and differentiable in a dense subset of \(X\) and its derivative belongs to \(F_2\). Let \(T\) stand for a separated topology for \(F_2\) and \(T_0\) the relative one for \(F_1\). Let \((F_1, (i_\mathfrak{A}(T_0), \lim_{T_0}))\) and \((F_2, (i_\mathfrak{A}(T), \lim_{T}))\) two \(\lambda\)-partial \(\mathfrak{A}_D\)-algebras. If the following statements hold,
a) $\mathcal{T}$ is finer than or equivalent to the pointwise convergence topology.

b) The map $\lim_{T_u} \subseteq (F_1, T_1) \rightarrow (F_2, T_2)$ is continuous.

\[ \frac{d}{dt} \mid_{T_0} (f_\delta)_{\delta \in D} = \lim_{T_u} \mathcal{H}_D (\lim_{T_0} g_{\delta, p})(N) = \lim_{T_u} \mathcal{H}_D (\mathcal{H}_D (g_{\delta, p})_{\delta \in D}) (N) \quad (25) \]

Since $\lim_{T_u}$ by assumption is continuous, and taking into account statement a),

\[ \lim_{T_u} \mathcal{H}_D (g_{\delta, p})_{\delta \in D} (N) = \mathcal{H}_D (\lim_{T_u} g_{\delta, p})_{\delta \in D} (N) = \lim_{T} \left( \mathcal{H}_D (\lim_{T_u} g_{\delta, p}) \right)_{\delta \in D} (N) = \lim_{T} \mathcal{H}_D \left( \frac{d}{dt} \right) (f_\delta)_{\delta \in D} \mid_{p} \quad (26) \]

and because $p$ is arbitrary, from (25) and (26) we obtain that

\[ \frac{d}{dt} \mid_{T_0} (f_\delta)_{\delta \in D} = \lim_{T} \mathcal{H}_D \left( \frac{d}{dt} \right) (f_\delta)_{\delta \in D} \quad (27) \]

\[ \square \]

References

[1] Jir Adámek, Horst Herrlich, George E Strecker, Abstract and concrete categories. The joy of cats., Wiley-Interscience Publication. New York etc.: John Wiley & Sons, Inc. xii, 482 p. (1990)

[2] John L Kelley, General topology, Springer-Verlag, New York (1975)

[3] J E Palomar Tarancón, Vector spaces over function fields. Vector spaces over analytic function fields being associated to ordinary differential equations, Southwest J. Pure Appl. Math. (2000) 60–87 (electronic)

[4] Giuseppe Rosolini, Domains and dominical categories, Riv. Mat. Univ. Parma (4) 11 (1985) 387–397