Analysis of coupled micro rings resonators and coupled Fabry–Pérot resonators with a single physical view

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Keywords: micro ring resonators, Fabry–Pérot, transfer-matrix method

Abstract
Coupled micro ring resonators gained a lot of interest in recent years in the field of silicon photonics. Although it is well recognized that coupled ring resonators and coupled Fabry–Pérot resonators operate with similar physical principles, their analysis in the literature is artificially separated, often each resonator type is analysed with different physical arguments, techniques and nomenclature. As a result, coupled rings resonators and coupled Fabry–Pérot resonators appear as two different problems, and the similarity in the physical principles that govern their operation is blurred. What is more unfortunate is that the established physical intuition, familiar from Fabry–Pérot analysis, is lost in notation when dealing with coupled micro rings. Here, we argue how a lossless boundary between two dielectrics and a lossless waveguide-coupler can be viewed as a single entity: an abstract ‘boundary’ that transmits and reflects optical fields according to Stokes relations. Using this view, Fabry–Pérot and micro rings become a single type of resonator. Accordingly, we calculate effective reflection and transmission coefficients of several configurations of coupled Fabry–Pérot and rings resonators. We calculate these coefficients by the intuitive method of summing reflected fields and by the method of transfer matrix. We illustrate that the effective reflection and transmission coefficients of a coupled ring resonator is similar to 3 layers Fabry–Pérot resonator and discuss the subtle difference. It is hoped that the common nomenclature and analysis used for both types of coupled resonators will give the reader a clear and basic understanding of both types of resonators.

Introduction
Silicon photonics micro rings are becoming increasingly interesting for various application in bio-sensing, spectroscopy on chip, optical communication and optical computing \cite{1–5}. Clear analysis of coupled rings in the published literature is scarce. The existing analysis is confounding due to a conflict between nomenclature and physical intuition used by different sources \cite{6–11}. Although the reader senses that there is a strong physical analogy between the coupled ring resonator and the coupled Fabry–Pérot resonator, the different nomenclature and techniques used for analysis hinder a clear view of their common physics.

Here, we analyse the effective reflectance and transmittance of coupled Fabry–Pérot resonators and coupled rings resonators by stressing their resemblance. Since Fabry–Pérot resonators were studied much earlier than coupled micro ring resonators, and the nomenclature and analysis of Fabry–Pérot is well founded, we analyse both types of resonators using the nomenclature applied to Fabry–Pérot resonators.

First, we argue that it is advantageous to view the manner that a waveguide mode propagates into a coupler is identical to the way an incident plane wave is reflected and transmitted at a boundary between two dielectrics. It enables us to approach the problem of calculating the effective reflectance (transmittance) of Fabry–Pérot and ring resonator with a common intuition, nomenclature and method.
Second, we calculate the effective reflection and transmission coefficients of coupled Fabry–Pérot and coupled ring resonators by adhering to nomenclature and methods originated from Fabry–Pérot analysis. We start by using the intuitive approach of summing multi reflections of the incident field. A method familiar from Fabry–Pérot analysis in text books. Then we redo the same calculations using the transmission matrix method. The results show that apart from a phase factor, formally, a coupled ring resonator is identical to three layers Fabry–Pérot etalon.

It is hoped that this article will clarify the physical similarity, almost identity, between the two types of resonators and that it will be a reference for analysing complex structures of coupled resonators of each type.

**Background**

When a plane wave is incident on a boundary of two dielectrics, it is partially reflected into the first dielectric and it is partially transmitted into the second dielectric. The absolute value of the complex reflection/transmission coefficient describes how the amplitude of the reflected/transmitted wave will decrease while the argument of the complex number describes the phase shift with respect to the incident wave. These changes are calculated specifically from Fresnel’s relations.

In a similar manner, when a waveguide mode propagates into a coupler it is advantageous to view it as if there is an incident waveguide mode that is partially ‘reflected back’ into the same waveguide and it is partially ‘transmitted into’ the other waveguide. The coupler practically acts as a ‘boundary’. The specific changes this ‘boundary’ induce on the reflected and transmitted amplitudes of a waveguide mode are found from mode coupling theory (but not from Fresnel’s relations) [6, 11]. It is advantageous to ignore the physical structure of the coupler and treat it as an abstract ‘boundary’ that reflects and transmit waveguide modes.

When such a ‘boundary’ is part of coupled rings, as in figure 1(a), we often need to calculate the total field in the ring. We can calculate the sum of all reflections of the incident wave, just as it is done in calculations of the total field in a Fabry–Pérot resonator, alternatively, we can calculate the total fields by the transfer matrix technique.

Figure 1(b) shows a general case of transmission and reflection from a boundary of two dielectrics. The plane wave amplitude $B_i$ is the sum of the immediate reflected field from $A_i$ plus the transmitted field from $B_j$. The plane wave amplitude $A_j$ is the sum of the transmitted field from $A_i$ and the reflected field from $B_j$. The transmission and reflection coefficients are $t_{ij}$ and $r_{ij}$, respectively. The first index $i$ refers to the dielectric where the field is incident and the second index $j$ refers to the dielectric that the fields arrives to. The absolute values of $t_{ij}$ and $r_{ij}$ gives the field’s decrease in amplitudes and their argument gives the field phase change. Equation (1) gives the relations between the amplitudes of the involved fields:

$$B_i = r_{12} A_1 + t_{21} B_2,$$
$$A_2 = t_{12} A_1 + r_{21} B_2.$$  \(1\)

Figure 1(a) shows the relation between the fields in a coupler between two rings. Once we view a coupler as a ‘boundary’, equation (1) describes the relations between the amplitudes just as well: the field $B_i$ is the sum of the reflected field from $A_i$ in ring 1 from the coupler back into ring 1, plus the transmitted field from $B_j$ through the coupler into ring 1. The field $A_2$ is the sum of the transmitted field from $A_i$ through the coupler into ring 2, plus the reflection of field from $B_j$ in ring 2 back into ring 2. As said, when we view a coupler as a ‘boundary’,
equation (1) holds also for the coupled rings (we keep the same notation for transmission and reflection coefficients $t_{ij}$ and $r_{ij}$, the index relates to ring number instead of dielectric number).

We express the amplitudes $A_{1}$, $B_{1}$ in terms of the amplitudes $A_{2}$, $B_{2}$:

$$A_{1} = \frac{n_{2}}{n_{1}}A_{2} - \frac{r_{2}}{t_{2}n_{2}}B_{2}$$

in matrix form

$$\begin{pmatrix} A_{1} \\ B_{1} \end{pmatrix} = \begin{pmatrix} n_{2}/n_{1} & -r_{2}/t_{2}n_{2} \\ 1 & t_{2}n_{2} - n_{2}r_{21} \end{pmatrix} \begin{pmatrix} A_{2} \\ B_{2} \end{pmatrix}$$

(2)

According to equation (1), the boundary between two dielectrics and the waveguide coupler is assumed lossless. I.e. energy flow is conserved and the reflection and transmission through a boundary or a waveguide coupler are time reversible processes. Therefore, we can require time reversibility on the flow of the field at a boundary between two dielectrics and also at a waveguide-coupler as is shown in figure 2.

As a consequence, from figure 2, we find that the same Stokes relations hold for a boundary between two dielectrics and also for a waveguide coupler:

$$0 = n_{2}r_{2}A_{1} + r_{21}t_{2}A_{1} \Rightarrow r_{2} = -r_{21},$$

$$A_{1} = n_{2}r_{2}A_{1} + r_{21}t_{2}A_{1} \Rightarrow n_{2}r_{2} + r_{21}t_{2} = 1.$$  

We substitute these relations into the matrix element of equation (2) and obtain:

$$\begin{pmatrix} A_{1} \\ B_{1} \end{pmatrix} = \frac{1}{n_{2}} \begin{pmatrix} 1 & -r_{21} \\ n_{2} & 1 \end{pmatrix} \begin{pmatrix} A_{2} \\ B_{2} \end{pmatrix}$$

(3)

which is applicable for a boundary between two dielectrics and also for a waveguide coupler.

In a boundary between two dielectrics with refractive indexes $n_{1}$, $n_{2}$, the transmission and reflection coefficients are found from Fresnel’s relations. The coefficients are given by:

$$t_{2} = \frac{2n_{1}}{n_{1} + n_{2}}, \quad r_{2} = \frac{n_{1} - n_{2}}{n_{1} + n_{2}}.$$  

From these relations, we can also confirm again by substitution that $t_{2}r_{2} - r_{2}t_{21} = 1$ and $n_{2}r_{2} = -r_{21}.$

In the literature, what we prefer to call the transmission coefficient, is referred to as the cross-coupling coefficient in coupled rings (sometimes it is called the self-coupling coefficient). The transmission coefficient is calculated using mode coupling theory. It is found to be real and equal for both directions $[6, 11] t_{2} = t_{21}$. Its value depends on the dielectric constant of the waveguides and the geometry of the coupler.

In the literature, what we prefer to call the reflection coefficient, is referred to as the straight-through coupling coefficients in coupled rings. The general Stokes relations tell us that $r_{2} = -r_{21}$. This equation is
Table 1. The transmission matrix for a boundary between two dielectrics and for a waveguide coupler. The reflection coefficient is calculated specifically by Fresnel’s relation in the case of a boundary between two dielectrics.

|                  | Dielectric boundary | Waveguide coupler |
|------------------|---------------------|-------------------|
| Transmission matrix | $\begin{bmatrix} 1 & 1 - r_2 \\ r_2 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 - r_2 \\ r_2 & 1 \end{bmatrix}$ |
| Propagation matrix | $e^{i\varphi}$ $1$ | $e^{i\varphi}$ $1$ |
|                  | $1$ $e^{-i\varphi}$ | $1$ $e^{-i\varphi}$ |

sufficient for calculating effective reflectance and transmittance, up to a phase factor in the expression for reflectance (transmittance).

To complete the preparation for transmission matrix analysis of coupled Fabry–Pérot resonators and coupled rings resonators, we need also to consider the phase shift between boundaries (waveguide-couplers) due to a propagation distance $l$. In a dielectric, when the plane wave with amplitude $A_i$ propagates in the positive $z$ direction, it experiences a phase shift of $e^{-i\beta \ell}$, and the wave that propagates in the negative $z$ direction has a phase shift of $e^{i\beta \ell}$. In coupled rings there is a similar phase shift of $e^{-i\beta \ell}$, where $\beta$ is the propagation vector of the waveguide mode. This phase shift due to propagation is considered via a propagation matrix. Table 1 summarizes the required transmission and propagation matrices. We include in the propagation matrices also the effect of loss.

When a plane wave is incident on a stack of dielectric layers, i.e. the plane wave propagates through several dielectric boundaries, it is easy to convince ourselves that the whole structure of several boundaries can be collapsed into one effective boundary with effective transmission, reflection and phase shift. In analogy, the same can be thought of with respect to a ‘stack’ of couplers. The stack of couplers can be viewed as one coupler with an effective transmission, reflection and phase shift.

The transmission matrix technique is very useful for calculating a stack of dielectric layers, or coupled rings. Each layer/coupler is represented by a transfer matrix and the optical path between two boundaries/couplers is represented by a propagation matrix. A chain of multiplied transmission matrices and propagation matrices represents the chain of layers/couplers and the distances between them. The obtained product matrix can be used to extract the effective reflection and transmission coefficients of the stack.

**Fabry–Pérot etalon and interferometer**

An incident, right angle, plane wave with amplitude $A_i$ arrives at two boundaries as shown in the top of figure 3.

A plane wave with amplitude $B_i$ is reflected from the first boundary and a plane wave with amplitude $A_i$ is transmitted through both boundaries. $A_{fl}$ and $B_{fl}$ are the effective reflection and transmission coefficients from the two boundaries, respectively. The total transmitted field is given as the superposition of all waves propagating to the right hand side:

$$A_3 = h_2t_2A_i e^{-i\varphi} + h_2t_2A_i e^{i\varphi} r_21 r_23 e^{-i2\varphi} + \ldots + h_2t_2A_i e^{-i\varphi} (r_21 r_23 e^{-i2\varphi})^n. $$

The first term is the immediate transmitted wave. The rest of the terms result from multiple reflections that end up transmitted through the right hand side boundary.

The total reflected field is given as a superposition of all reflected waves to the left-hand side:

$$B_1 = r_2A_i + h_2t_2A_i r_23 e^{-i2\varphi} + h_2t_2A_i r_23 (r_21 r_23 e^{-i2\varphi})^2 + \ldots + h_2t_2A_i r_23 (r_21 r_23 e^{-i2\varphi})^n. $$

The first term is the immediate reflected wave. The rest of the terms are from multi reflections that end up moving towards the left-hand boundary.

Using the sum formula for infinite geometric series, the sum of each of the fields becomes:

$$A_3 = h_2t_2A_i e^{-i\varphi}(1 + (r_21 r_23 e^{-i2\varphi}) + \ldots + (r_21 r_23 e^{-i2\varphi})^n) \ldots) = \frac{h_2t_2A_i e^{-i\varphi}}{1 - r_21 r_23 e^{-i2\varphi}}, $$

$$B_1 = r_2A_i + h_2t_2A_i r_23 e^{-i2\varphi}(1 + (r_21 r_23 e^{-i2\varphi}) + \ldots + (r_21 r_23 e^{-i2\varphi})^n) \ldots) $$

$$ = r_2A_i + \frac{h_2t_2A_i r_23 e^{-i2\varphi}}{1 - r_21 r_23 e^{-i2\varphi}} = \frac{r_2A_i + \hat{A} r_23 e^{-i2\varphi}}{1 - r_21 r_23 e^{-i2\varphi}}. $$
The effective reflection and transmission coefficients are given by:

\[
\frac{R}{A_1} = r = \frac{n_2 + r_2 e^{-2\alpha l}}{1 - r_2 n_2 e^{-2\alpha l}}, \quad \frac{A_1}{A_1} = t = \frac{r_2 n_2 e^{-\alpha l}}{1 - r_2 n_2 e^{-2\alpha l}}. \tag{4a}
\]

The same results are obtained by using the transmission matrix of two dielectric boundaries as shown at the bottom of figure 3. We consider the case when there is only one input wave \( A_1 \) and no backward input wave, i.e. \( B_0 = 0 \). It is straightforward to calculate that when \( B_3 = 0 \), \( r = \frac{T_{12}}{T_{11}} \) and \( t = \frac{T_{13}}{T_{11}} \). A direct substitution of matrix elements gives:

\[
r = \frac{n_2 + r_2 e^{-2\alpha l}}{1 - r_2 n_2 e^{-2\alpha l}}, \quad t = \frac{r_2 n_2 e^{-\alpha l}}{1 - r_2 n_2 e^{-2\alpha l}}. \tag{4b}
\]

In agreement with the method of superposition.

We take \( n_2 = |n_{12}| e^{-\frac{i}{2} \varphi_{12}}, r_{21} = |r_{21}| e^{-\frac{i}{2} \varphi_{21}}, r_{23} = |r_{23}| e^{-\frac{i}{2} \varphi_{23}} \) and compute the power reflectance and transmittance:

\[
R = |r|^2 = \left( \frac{n_2 + r_2 e^{-2\alpha l}}{1 - r_2 n_2 e^{-2\alpha l}} \right)^2, \quad T = \left( \frac{r_2 n_2 e^{-\alpha l}}{1 - r_2 n_2 e^{-2\alpha l}} \right)^2.
\]

In the case where \( n_1 > n_2 = n_3 \), according to Fresnel relations \( \varphi_{12} = \pi, \varphi_{21} = 0, \varphi_{23} = 0 \). Also, at resonance, when the distance \( l \) between the two boundaries equals a natural number \( m \) times half wavelength:

\[
l = \frac{m \lambda}{2}. \quad \text{The optical path length} \quad 2kl = \frac{2\pi}{\lambda} m^2 \pi = m_2 \pi \quad \text{and the argument of cosine at the nominator equals} \quad \pi.
\]

Hence, at resonance, the cosine at the nominator equals \(-1\) and the power reflectance and transmittance will be:

\[
R = \frac{1 - n_2 |r_{21}|^2 e^{-2\alpha l}}{1 - n_2 |r_{21}|^2 e^{-\alpha l}}, \quad T = \frac{|r_{21}|^2 r_{23}^2 e^{-\alpha l}}{1 - n_2 |r_{21}|^2 e^{-\alpha l}}. \tag{5}
\]

If, instead of the boundaries between the dielectrics \( n_1, n_2 \) and \( n_2, n_3 \), we put two dielectric mirrors \( M_1 \) and \( M_2 \) as shown in figure 4 we are dealing with a realistic Fabry–Pérot interferometer.

We recall that a dielectric mirror is made of multi-layers of dielectrics. In fact, the three dielectric layers in figure 3 can be seen as an example for a dielectric mirror. The three layers have an effective reflection and transmission coefficients as we calculated in equations (4a) and (4b). Practical dielectric mirrors may have dozens of thin dielectric layers. This similarity means that each mirror can be collapsed to a boundary \( M_1 \) and \( M_2 \) that is characterized with effective reflectivity, transmission and phase shift coefficients: \( n_{12}, \varphi_{12} \text{ and } r_{23}, l_{23} \). 
respectively. (In fact, multi-layers dielectric mirrors are designed by using the same transfer matrix technique to obtain a desired effective $\text{r}_{12}, \text{r}_{21}, \varphi_{12}$ for a given frequency band.)

According to the reflection and transmission coefficients, the power reflectance and transmittance of a Fabry–Pérot with two dielectric mirrors $M_1$ and $M_2$ are the same as given by equation (5). The effective phase shift introduced by each dielectric mirror will depend on the design of the multi-layer coatings. If the coating design was optimal so that when the optical path $p=2kl = 2\frac{\lambda}{2} m \frac{\lambda}{2} = m2\pi$ and the sum and difference of the effective phase shifts $\varphi_{12} - \varphi_{23} = \varphi_{23} + \varphi_{23} = \pi$. Then reflectance at resonance of the Fabry–Pérot interferometer will be given by equation (6).

We note that when $|\text{r}_{12}| = |\text{r}_{23}| e^{-2i\theta}$ the reflectance is zero. It is called impedance matching (also critical coupling) in a Fabry–Pérot resonator. In this situation, the field that is immediately reflected is equal, but it is in opposite phase to the total field that is transmitted from the cavity back to the direction of the immediately reflected field. Hence, the sum is zero, there is no observed reflection.

**Waveguide coupled to a ring resonator**

An ‘incident’ waveguide mode with amplitude $A_i$ arrives at the coupler as shown at the top of figure 5.

A waveguide mode with amplitude $B_i$ is reflected from the coupler back into the waveguide. $\frac{A_i}{A_i}$ is the effective reflection. We calculate the effective reflectivity by summing all reflections and transmissions in the direction of $B_i$.

$$B_i = \text{r}_{12} A_i + \text{r}_{12} \text{r}_{21} A_i e^{-i\delta} + \text{r}_{12} \text{r}_{21} A_i (r_{21} e^{-i\delta})^2 + \ldots + \text{r}_{12} \text{r}_{21} A_i (r_{21} e^{-i\delta})^n.$$  

The first term is the immediate reflection. The rest of the terms arrive from multiple reflections in the same direction as the immediate reflection.
on the wavelength \( \lambda \) is a periodic function of the argument of the cosine, we can set \( A_1 \) to be: \( A_n = e^{-i n \phi} \).

The reflectivity coefficient of a ring is like the reflectivity coefficient in equations (4a) and (4b) of a Fabry–Pérot etalon with \( |a_1| = 1 \).

Figure 5 at the bottom illustrates how the transmission matrix is applied to a coupled ring using the same view as for analysing a Fabry–Pérot. The input amplitude \( A_1 \) is partially reflected into the waveguide from the coupler and is partially transmitted into the ring through the coupler. The field \( B_1 \) that is built in the ring, is partially reflected from the coupler back into the ring, and it is also partially transmitted into the waveguide.

We note that in this configuration \( B_2 = A_2 e^{-i \phi} \). We plug it into the transmission matrix and calculate:

\[
B_1 = n_2 A_1 + \frac{h_1 t_2 A_1 e^{-i \phi}}{1 - n_2 e^{-i \phi}}(1 + r_{21} e^{-i \phi} + (r_{21} e^{-i \phi})^2 + \ldots + (r_{21} e^{-i \phi})^m)
\]

\[
B_1 = n_2 A_1 + \frac{h_1 t_2 A_1 e^{-i \phi}}{1 - n_2 e^{-i \phi}} \Rightarrow \frac{B_1}{A_1} = r = \frac{n_2 + e^{-i \phi}}{1 - n_2 e^{-i \phi}} \quad \text{(we used Stokes' relations)}.
\]

The specific phase argument of the reflection coefficients \( r_{12}, r_{21} \) will come from a detailed modelling of the coupler. However we know from Stokes relations that \( r_{12} = -r_{21} \) must hold. Since the reflectance \( R \) is a periodic function of the argument of the cosine, we can set \( r_{12} = |r_{21}| e^{i \phi} \) without spoiling the functional dependence of \( R \) on the wavelength (apart from a small shift). At resonance, the length \( l \) of the ring equals a natural number times the wavelength \( = m \lambda \). The argument of the cosine at the nominator is given by:

\[
\beta + \pi = \frac{2 \pi}{\lambda} m \lambda + \pi = m 2 \pi + \pi.
\]

Hence the cosine at the nominator equals \(-1\) and the reflectance of the ring is:

\[
R = \frac{|r_{12}|e^{-i \phi} e^{-i \phi}}{(1 - |r_{21}| e^{-i \phi})}.
\]

When \( |r_{12}| = e^{-i \phi} \) the reflectance is zero. It is similar to the case of impedance matching in the Fabry–Pérot etalon resonator with \( |a_1| = 1 \). Also, here, at impedance matching (critical coupling), the field that is immediately reflected is equal, but is in opposite phase to the resonating field that is transmitted from the ring back into the waveguide. Hence, the sum is zero.

**Ring coupled on both sides to a waveguide**

The top of figure 6 shows an ‘incident’ waveguide mode with amplitude \( A_1 \) arriving at the first coupler.

A waveguide mode with amplitude \( A_3 \) is reflected from the coupler. \( \frac{A_3}{A_1} \) is the effective reflection. A waveguide mode with amplitude \( A_3 \) is transmitted through the second coupler. \( \frac{A_3}{A_1} \) is the effective transmission. The field that is transmitted into the waveguide at the opposite side is given by:

\[
A_3 = t_3^2 A_3 e^{-i \phi} + t_3^2 A_3 e^{i \phi} + t_3^2 A_3 e^{-i \phi} (r_{23} e^{-i \phi})^1 + \ldots + t_3^2 A_3 e^{i \phi} (r_{23} e^{i \phi})^n.
\]

The first term is the immediately transmitted amplitudes, and all the other terms are the transmitted reflections. The field that is reflected to the left is given by:

\[
B_1 = n_2 A_1 + t_2 A_1 e^{-i \phi} + t_2 A_1 e^{i \phi} (r_{23} e^{-i \phi})^1 + \ldots + t_2 A_1 e^{i \phi} (r_{23} e^{i \phi})^n.
\]

The first term is the immediately reflected field, and all the other terms are the transmitted reflections.

\[
A_3 = t_3^2 A_3 e^{-i \phi} (1 + (r_{23} e^{-i \phi})^1 + \ldots + (r_{23} e^{-i \phi})^m) \Rightarrow \frac{A_3}{A_1} = \frac{t_3^2 A_3 e^{-i \phi}}{1 - r_{23} e^{-i \phi}},
\]

\[
B_1 = n_2 A_1 + t_2 A_1 e^{-i \phi} (1 + (r_{23} e^{-i \phi})^1 + \ldots + (r_{23} e^{-i \phi})^m) \Rightarrow \frac{B_1}{A_1} = \frac{t_2 A_1 + A_3 e^{-i \phi}}{1 - r_{23} e^{-i \phi}}.
\]

We can write the transmission and reflection coefficients as:

\[
t = \frac{A_3}{A_1} \quad \frac{B_1}{A_1} = \frac{t_2 A_1 + A_3 e^{-i \phi}}{1 - r_{23} e^{-i \phi}}.
\]

These transmission and reflection coefficients are formally similar to those of three layers Fabry–Pérot etalon, equations (4a) and (4b). They are identical if we regard the circumference of a ring as half the length of the resonator.

The bottom of figure 6 illustrates how the transmission matrix is applied to a doubly coupled ring. The input amplitude beam is \( A_1 \) and the output field \( B_3 \) are zero. The transmission matrix of this coupled ring resonator is
identical to that of a stack of three dielectric layers. Hence the reflection and transmission coefficients at the photodiode must be:

\[
\begin{pmatrix}
A_1 \\
B_1
\end{pmatrix} = \frac{1}{t_{12}} \begin{pmatrix} 1 & -r_{21} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & r_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}
\]

Figure 6. Top: the incident input field \(A_1\) is partially reflected towards \(B_1\) and partially transmitted into the ring and into the opposite waveguide. The reflected beam inside the ring is partially transmitted back into both waveguides. Bottom: the transmission matrix of the coupled ring.

Table 2. \(l\) is the length of a Fabry–Pérot resonator or the circumference of a ring resonator.

|                      | Fabry–Pérot interferometer | Coupled ring | Input–output coupled ring (\(l\) is the circumference) |
|----------------------|---------------------------|--------------|------------------------------------------------------|
| Effective reflectance at resonance | \(R = \frac{\abs{n_2} - \abs{r_{21}} e^{-2\alpha l}}{1 - \abs{n_2} \abs{r_{21}} e^{-2\alpha l}}\) | \(R = \frac{\abs{n_2} - \abs{r_{21}} e^{-\alpha l}}{1 - \abs{n_2} \abs{r_{21}} e^{-\alpha l}}\) | \(R = \frac{\abs{n_2} - \abs{r_{21}} e^{-\alpha l}}{1 - \abs{n_2} \abs{r_{21}} e^{-\alpha l}}\) |
| Effective transmittance at resonance | \(T = \frac{\abs{n_2} \abs{r_{21}} e^{-\alpha l}}{1 - \abs{n_2} \abs{r_{21}} e^{-\alpha l}}\) | \(T = \frac{\abs{n_2} \abs{r_{21}} e^{-\alpha l}}{1 - \abs{n_2} \abs{r_{21}} e^{-\alpha l}}\) | \(T = \frac{\abs{n_2} \abs{r_{21}} e^{-\alpha l}}{1 - \abs{n_2} \abs{r_{21}} e^{-\alpha l}}\) |
| Impedance matching condition | \(\abs{n_2} = \abs{r_{21}} e^{-2\alpha l}\) | \(\abs{n_2} = e^{-\alpha l}\) | \(\abs{n_2} = \abs{r_{21}} e^{-\alpha l}\) |

In summary, A boundary between two dielectrics induces amplitude and phase changes on an incident wave. The specific phase and amplitude changes are given from the simple Fresnel’s relations.
A waveguide coupler also induces amplitude and phase changes on an incident waveguide mode. The specific phase and amplitude changes are given specifically from mode coupling theory and more generally from Stokes relations.

It is practically advantageous to view the boundary between two dielectrics and the waveguide coupler as a single abstract element, a ‘boundary’, characterized by a single nomenclature ‘reflection and transmission coefficients’.

With this view, analysis is intuitive and simple: we applied the method of summing multiple reflections to calculate the total fields and the effective reflection and transmission coefficients in ring and Fabry–Pérot resonators. We found the same coefficients by applying the method of transfer matrix. The Stokes relations between the reflection and transmission coefficients simplified the transfer matrix for of the coupler and dielectric boundary. In fact, the coupler and the boundary between two dielectric are represented by the same transfer matrix. The subtle difference is in the phase factor of the reflection coefficient. In a boundary between two dielectrics, there is a π phase shift when the plane wave is reflected from the dense media. In the coupler, the phase shift is arbitrary, depending on the effective length of the coupler. However Stokes relation must hold. It results in a subtle difference between the calculation of effective reflection (transmission) coefficients for a coupled ring and a coupled Fabry–Pérot resonator.

The presented examples showed that when we perceive a coupler as an abstract ‘boundary’ there is a close analogy between Fabry–Pérot and ring resonators. An extension into examples that illustrate this analogy between coupled multi-ring resonators and coupled multi-Fabry–Pérot resonators is straightforward:

When a plane wave is incident on a stack of dielectric layers, the plane wave is propagating through several dielectric boundaries. This is a case of coupled multi-resonators of Fabry–Pérot. The effective transmittance and reflectance may be calculated by summing all reflections terms and all transmitted terms in each direction of propagation as was done in the example for a plain Fabry–Pérot resonator. This approach is intuitive but it becomes cumbersome with increasing coupled resonators due to large book keeping of reflection terms from each dielectric boundary. Preferably, when one analyses multi-resonator coupling of Fabry–Pérot resonators, it is advantageous to revert to transmission matrix technique. This is done routinely in designing multi-layers dielectric coating for dielectric mirrors. Such analysis is rooted in the practice and therefore it is considered intuitive.

Viewing the coupler between neighbouring rings as an abstract ‘boundary’ facilitates straightforward analysis of coupled multi-resonators of the ring type. When a waveguide mode is incident on a stack of coupled rings, one can approach the analysis with the same intuition gained from analysis of coupled multi-resonators of Fabry–Pérot. The waveguide mode is reflected or transmitted at each coupler (a ‘boundary’). Hence, the effective transmission can be calculated by summing the reflected terms to each direction of propagation, just as was done with the simple ring with one coupler, but more terms appear from each additional coupler (‘boundary’). The transmission matrix technique becomes handy also in this case. By using the appropriate matrices as is summarized in table 1, the analysis is straightforward. Figure 7 illustrates coupled multi-resonator of both types and illustrates the analogy.
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