Bounds on the entropy generated when timing information is extracted from microscopic systems

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We consider Hamiltonian quantum systems with energy bandwidth $\Delta E$ and show that each measurement that determines the time up to an error $\Delta t$ generates at least the entropy $(\hbar/(\Delta t\Delta E))^2/2$. Our result describes quantitatively to what extent all timing information is quantum information in systems with limited energy. It provides a lower bound on the dissipated energy when timing information of microscopic systems is converted to classical information. This is relevant for low power computation since it shows the amount of heat generated whenever a band limited signal controls a classical bit switch.

Our result provides a general bound on the information-disturbance trade-off for von-Neumann measurements that distinguish states on the orbits of continuous unitary one-parameter groups with bounded spectrum. In contrast, information gain without disturbance is possible for some completely positive semi-groups. This shows that readout of timing information can be possible without entropy generation if the autonomous dynamical evolution of the “clock” is dissipative itself.

I. TIMING INFORMATION GAIN WITHOUT DISTURBANCE?

Listening to “folklore versions” of quantum mechanics one may consider it as a key statement of quantum theory that there is no measurement without disturbing the measured system. However, the fact that this is not true is well-known in modern quantum information theory and is, in some sense, the reason why classical information exists at all although our world is quantum. Consider a two-level system, i.e., a quantum system with Hilbert space $\mathbb{C}^2$ and denote its upper or lower state by $|1\rangle$ and $|0\rangle$, respectively. Assume that we know by prior information that the system is not in a quantum superposition but only in one of the two states $|j\rangle$. Then the measurement with projections $P_0 := |0\rangle\langle 0|$ and $P_1 := |1\rangle\langle 1|$ show which state is present without disturbing it at all. Here we have used the two-level system as classical bit.

The situation changes if the two-level system is used as quantum bit (“qubit”) and is prepared in a quantum superposition $|\psi\rangle := c_0|0\rangle + c_1|1\rangle$ where the complex coefficients $c_0$ and $c_1$ with $|c_0|^2 + |c_1|^2 = 1$ are unknown to the person who measures. Then any von Neumann measurement with projections $|\phi\rangle\langle \phi|$ and $|\phi^\perp\rangle\langle \phi^\perp|$ will on the one hand only provide some information about $|\psi\rangle$ and will on the other hand disturb the unknown state $|\psi\rangle$ since it “collapses” to the state $|\psi\rangle$ with probability $p := |\langle \psi|\phi\rangle|^2$ and to the orthogonal state $|\phi^\perp\rangle$ with probability $q := |\langle \psi|\phi^\perp\rangle|^2$. From the point of view of the person who has prepared the state $|\psi\rangle$ and does not notice the measured result (“non-selective operation”), the measurement process changes the density matrix of the system from $|\psi\rangle\langle \psi|$ to

$$p|\phi\rangle\langle \phi| + q|\phi^\perp\rangle\langle \phi^\perp|,$$

i.e., the measurement causes an entropy increase of $\Delta S = -p \ln p - q \ln q$.

The general condition under which information about unknown quantum states can be gained without disturbing them is well-known and reads as follows [1]:

Let $\rho$ be the unknown density matrix of a system. By prior information one knows that $\rho$ is an element of a set $\Gamma$ of possible states. Then one can get some information on $\rho$ if and only if there is a projection $Q$ commuting with all matrices in $\Gamma$ such that the value $tr(Q\rho)$ is not the same for all $\rho \in \Gamma$.

As noted in [2], this can never be the case if the set $\Gamma$ is the orbit $(\rho_t)_{t \in \mathbb{R}}$ of a Hamiltonian system evolving according to $\rho_t = \exp(-iHt)\rho\exp(iHt)$. This holds even if $\rho$ and $H$ act on an infinite dimensional Hilbert space. In this sense, timing information is always to some extent quantum information that cannot be read out without state disturbance. It can only become classical information if either (1) prior information tells us that the time $t$ is an element of some discrete set $\{t_1, t_2, \ldots\}$ (see [2]) or (2) in the limit of infinite system energy [3,2].

At first sight the statement that classical timing information can only exist in one of these two cases seems to be disproved by the following dissipative “quantum clock”:

Let $\rho_0 := |1\rangle\langle 1|$ be the upper state of a two-level system. Let the system’s time evolution for positive $t$ be described by the Bloch relaxation (see e.g. [4])

$$\rho_t := \exp(-\lambda t)|1\rangle\langle 1| + (1 - \exp(-\lambda t))|0\rangle\langle 0|.$$

Since all the states $\rho_t$ commute with the projections $P_0$ and $P_1$ one can certainly gain some information about

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t by the measurement with projections \((P_0, P_1)\). However, this situation is actually the infinite energy limit since semi-group dynamics of this form is generated by coupling the system to a heat bath of infinite size and infinite energy spectrum [5]. Of course the fact that well-known derivations of relaxation dynamics require heat baths with infinite energy spectrum does not prove our claim that this is necessarily the case. This claim is rather an implication of Theorem 1 in Section II.

The paper is organized as follows. In Section II we show quantitatively to what extent information on the (non-discrete) time is always quantum information as claimed that this is necessarily the case. This claim is rather an implication of Theorem 1 in Section II.

The following Lemma shows that the measurement can never decrease the entropy:

**Lemma 1** Let \(\gamma\) be an arbitrary density matrix on a Hilbert space and \((P_j)\) be a family of orthogonal projections with \(\oplus P_j = 1\). Set \(\tilde{\gamma} := \sum_j P_j \gamma P_j\). Then we have

\[
\Delta S := S(\gamma) - S(\tilde{\gamma}) = K(\gamma||\tilde{\gamma})
\]

where \(K(\gamma||\tilde{\gamma})\) is the Kullback-Leibler distance (or Relative Entropy) between \(\gamma\) and \(\tilde{\gamma}\). It is defined by [6]

\[
K(\gamma||\tilde{\gamma}) := tr(\gamma \ln \gamma) - tr(\gamma \ln \tilde{\gamma}).
\]

Proof: According to the definition of entropy it is sufficient to show the equation \(tr(\gamma \ln \tilde{\gamma}) = tr(\gamma \ln \gamma)\). The second term on the right in eq. (1) is equal to

\[
tr(\sum_i P_i \gamma \ln(\sum_j P_j \gamma P_j) P_i) = tr(\sum_i P_i \gamma \ln(\sum_j P_j \gamma P_j P_i) P_i)
\]

Note that \(\ln(\sum_j P_j \gamma P_j)\) commutes with each \(P_i\). Hence we get

\[
tr(\sum_i P_i \gamma \ln(\sum_j P_j \gamma P_j P_i)) =
\]

\[
tr(\gamma \ln \tilde{\gamma})
\]

This completes the proof. \(\square\)

Since the entropy increase is the Kullback-Leibler distance between the pre- and the post-measurement state we can obtain a lower bound on \(\Delta S\) in terms of the trace-norm distance between them:

**Lemma 2** For the entropy increase \(\Delta S\) we obtain the lower bound

\[
\Delta S \geq \frac{1}{2} ||\tilde{\gamma} - \gamma||_1^2
\]

where \(||a||_1 := tr(\sqrt{a^\dagger a})\) is the trace-norm of an arbitrary matrix \(a\).
The proof is immediate using
\[ K(\gamma|\tilde{\gamma}) \geq \frac{\|\tilde{\gamma} - \gamma\|^2}{2} \]
(see [6]).

Now we consider the states \( \rho_t \) on the orbit of the time evolution and show that the norm distance between \( \rho_t \) and \( \tilde{\rho}_t \) is large at every moment where the outcome probabilities of the measurement (\( P_j \)) change quickly, i.e., where the values \( tr(i[H,P_j] \rho_t) \) are large. This is the essential statement that is used to prove the information-disturbance trade-off relation. But first we have to introduce the energy bandwidth of a system. For Hamiltonians with discrete eigenvalues it is just the difference between the greatest and smallest eigenvalues of the energy states with non-zero occupation probability. The generalization to Hamiltonians with continuous and discrete parts in the spectrum reads:

**Definition 1** Let \( H \) be the Hamiltonian of a quantum system and \( (Q_E)_{E \in \mathbb{R}} \) be the spectral family corresponding to \( H \), i.e., \( Q_E \) projects onto the Hilbert space corresponding to energy values not greater than \( E \). Then \( f(E) := tr(Q_E \rho_t) \) is the distribution function of a (time-independent) probability measure \( \mu \) which is called the spectral measure of \( \rho_t \) corresponding to \( H \). Let \( [E_{\text{min}}, E_{\text{max}}] \) be the smallest interval supporting this spectral measure. Then \( \Delta E := E_{\text{max}} - E_{\text{min}} \) is the energy bandwidth of the system in the state \( \rho_t \).

By rescaling the Hamiltonian it is easy to see that the time evolution of the state \( \rho \) can equivalently be described by
\[ H' := (Q_{E_{\text{max}}} - Q_{E_{\text{min}}})H - \frac{1}{2}(E_{\text{max}} - E_{\text{min}})1, \]

since \( \exp(-iH't)\rho \exp(iH't) = \exp(-iH't)\rho \exp(iHt) \). Clearly, \( \|H'\| \leq \Delta E/2 \). The energy spread is decisive for our lower bound on the trace-norm distance between \( \rho_t \) and \( \tilde{\rho}_t \).

**Lemma 3** Let \( p_j(t) := tr(P_j \rho_t) \) be the probability of the measurement outcome \( j \) at time \( t \). Let \( \Delta E \) be the energy bandwidth of \( \rho_t \). Set
\[ \|\dot{\rho}(t)\|_1 := \sum_j \left| \frac{d}{dt} p_j(t) \right|. \]

Then we have
\[ \|\rho_t - \tilde{\rho}_t\|_1 \geq \frac{\|\dot{\rho}(t)\|_1}{\Delta E}. \]

**Proof:** For a specific moment \( t \) define the operator \( R(t) \) by
\[ R(t) := s_j(t)P_j \]
where \( s_j(t) = 1 \) if \( \dot{p}_j(t) \geq 0 \) and \( s_j(t) = -1 \) if \( \dot{p}_j(t) < 0 \). Note that we have
\[ \|\dot{\rho}(t)\|_1 = \sum_j s_j(t)\dot{p}_j(t) = tr(i[H,R(t)] \rho_t). \]

Furthermore easy computation shows that \( tr(i[H,R(t)] \rho_t) = 0 \). We conclude
\[ \|\dot{\rho}(t)\|_1 = tr(i[H,R(t)](\rho_t - \tilde{\rho}_t)) \leq 2\|H\| \rho_t - \tilde{\rho}_t\|_1, \]
since the operator norm of \( R(t) \) is 1. We assume \( \|H\| = \Delta E/2 \) without loss of generality and have
\[ \|\rho_t - \tilde{\rho}_t\|_1 \geq \frac{\|\dot{\rho}(t)\|_1}{\Delta E}. \]

\( \square \)

Now we are able to prove our main theorem.

**Theorem 1** Let the energy bandwidth \( \Delta E \) of a quantum system be less than \( \infty \). Assume that the true time \( t \) is in the interval \([0,T]\) for arbitrary \( T > 0 \). Let the prior probability for \( t \) be the uniform distribution on \([0,T]\). Let \( (P_j) \) be the family of projections corresponding to von-Neumann measurement. Assume that the measurement has the time resolution \( \Delta t \) in the following sense: For each \( t \in [0,T - \Delta t] \) there is a decision rule based on the measurement outcome that decides whether the state \( \rho_t \) or \( \rho_{t+\Delta t} \) is present with error probability at most \( 1/4 \).

Then the mean entropy increase \( \overline{\Delta S} \) (averaged over the interval \([0,T]\)) caused by the measurement is at least
\[ \frac{1}{2} \left( \frac{\hbar}{\Delta t \Delta E} \right)^2. \]

Note that our definition of time resolution is not the usual one since it does not require that the measurement distinguishes between \( \rho_t \) and \( \rho_{t+\Delta t} \), for instance. However, it is exactly the definition of time resolution that we need in the proof.

**Proof:** Each decision rule distinguishing between \( \rho_t \) and \( \rho_{t+\Delta t} \) that is based on the measurement outcome is of the following form: If the outcome is \( j \) it decides for \( t \) with probability \( q_j \). If the true state is \( \rho_t \) it decides for \( t \) with probability \( \sum_j q_j p_j(t) \) and if the true state is \( \rho_{t+\Delta t} \) it decides erroneously for \( t \) with probability
\[ \sum_j q_j p_j(t + \Delta t) = \sum_j q_j p_j(t) + \sum_j q_j \int_t^{t+\Delta t} \dot{p}_j(t')dt'. \]

The difference between the probability to decide correctly for \( t \) and to decide erroneously for \( t \) is at least \( 1/2 \) by assumption. Hence
\[ \left| \int_t^{t+\Delta t} \sum_j q_j p_j(t')dt' \right| \geq 1/2. \]
Note that $2q_j - 1$ is in the interval $[-1, 1]$ for all $j$. Therefore
\[
| \sum_j (2q_j - 1) \dot{p}_j(t) | \leq \sum_j | \dot{p}_j(t) |.
\]
Obviously we have $\sum_j \dot{p}_j(t) = 0$ since $\sum_j p_j(t) = 1$. We conclude
\[
\| \dot{p}(t) \|_1 \geq 2 \sum_j q_j \dot{p}_j(t),
\]
hence
\[
\int_t^{t+\Delta t} \| \dot{p}(t') \|_1 dt' \geq 1.
\]
Therefore we find that the average of $\| \dot{p}(t') \|_1$ over the interval $[t, \Delta t]$ is at least $1/\Delta t$. Since this holds for every $t \in [0, T - \Delta t]$ the average of $\| \dot{p}(t) \|_1$ over the whole interval [0, T] is at least $1/\Delta t$. We denote this average by
\[
\| \dot{p}(t) \|_1.
\]
For the average entropy generation we have
\[
\bar{\Delta S} = \frac{1}{T} \int_0^T S(\dot{p}_t) - S(p_t) dt \geq \frac{1}{2} \| \dot{p}_t - p_t \|_1^2
\]
by Lemma 2. Note that the average of the square is at least the square of the average. We conclude
\[
\bar{\Delta S} \geq \frac{1}{2} \| \dot{p}_t - p_t \|_1^2.
\]
Due to Lemma 3 we have
\[
\bar{\Delta S} \geq \frac{1}{2} \frac{1}{(\Delta t \Delta E)^2},
\]
as long as we measure the energy spread in units of $\hbar$. In physics, it is more common to use SI-units which lead to
\[
\bar{\Delta S} \approx \frac{1}{2} \frac{\hbar}{(\Delta t \Delta E)^2}.
\]
\[\square\]

III. ENTROPY GENERATED BY GENERALIZED MEASUREMENTS

The assumption that every measurement is described by projections $(P_j)$ and that the effect of the measurement is given by the projection postulate
\[
\rho \mapsto \sum_j P_j \rho P_j
\]
is not general enough. In general, a measurement can be described by a positive operator valued measure (POVM) (see [7] for the most general description of measurements). If the set of possible outcomes is finite, a POVM is defined by a family $(M_j)$ of positive operators with $\sum_j M_j = 1$ such that $tr(M_j \rho)$ is the probability for the outcome $j$. Furthermore the connection between the POVM and the effect on the state is given by the following consistency condition: There exist completely positive maps $G_j$ such that $tr(G_j(\rho)) = tr(M_j \rho)$ and that, given the outcome $j$, the post-measurement state is $G_j(\rho)/tr(M_j \rho)$. The unselected post-measurement state is hence given by
\[
\hat{\rho} := \sum_j G_j(\rho).
\]
Clearly, in this general setting it is even possible that the measurement can decrease the entropy of the clock since the maps $G_j$ may include processes that are cooling mechanisms for the clock and transport entropy to the environment.

However, we will show that each time measurement leads unavoidably to an entropy increase in the total system consisting of the measurement apparatus and the clock. Let $H_m$ be the Hilbert space of the measurement apparatus and be $H$ be the Hamiltonian its free time evolution as long as no measurement interaction is active. We assume the state $\gamma$ of the apparatus to be invariant under its free evolution. Otherwise the measurement apparatus would be a clock in its own since it contained some information about the time $t$. Then we switch on the measurement interaction which leads to a unitary transformation $u$ on the space $H \otimes H_m$ (the definition of the apparatus includes its environment such that the total evolution is unitary). This process is often called a \textit{pre-measurement}. The different outcomes $j$ correspond to mutual orthogonal subspaces of $H_m$ ("pointer states"). Let $(Q_j)$ be the projections onto these subspaces. Now we assume that decoherence on the measurement apparatus’ pointer takes place [8–10]. The process given by the unitary pre-measurement process $u$ and the decoherence is described by
\[
\rho \otimes \gamma \mapsto \sum_j (1 \otimes Q_j) u(\rho \otimes \gamma) u^\dagger (1 \otimes Q_j)
\]
and the entropy of this state is clearly the same as the entropy of
\[
\sum_j u^\dagger (1 \otimes Q_j) u(\rho \otimes \gamma) u^\dagger (1 \otimes Q_j) u.
\]
By defining $P_j := u^\dagger (1 \otimes Q_j) u$ we have reduced the problem to the problem in Section II with the only generalization that $(P_j)$ is not a family of projections acting on the Hilbert space $H$ of the clock but on the Hilbert space of clock $+ \text{ measurement apparatus}$ where the apparatus is at rest. Now we show that the stationarity of the apparatus state ensures that the bound in Theorem 1 holds for the considered kind of generalized measurement as well.

Formally, we claim:
Theorem 2 Let \( \rho \otimes \gamma \) be a density matrix on a Hilbert space \( \mathcal{H} \otimes \hat{\mathcal{H}} \). Let the time evolution of the composed system be given by the Hamiltonian \( \hat{H} \otimes 1 + 1 \otimes \hat{H} \) with \( [\gamma, \hat{H}] = 0 \). Let \( (P_j) \) be a von-Neumann measurement acting on the Hilbert space \( \mathcal{H} \otimes \hat{\mathcal{H}} \). Let \( \Delta E \) be the energy bandwidth of the left component of the composed system (given by the state \( \rho \) with Hamiltonian \( H \)). Let the measurement have the time resolution \( \Delta t \) in the sense of Theorem 1. Then we have the following lower bound on the average entropy increase \( \Delta S \) of the composed system caused by the measurement:

\[
\Delta S \geq \frac{1}{2} \left( \frac{\hbar}{\Delta t \Delta E} \right)^2.
\]

Proof: Clearly, the Hamiltonian \( \hat{H} \) is irrelevant for the evolution of the state \( \rho \otimes \gamma \) due to

\[
\exp(-iH,t)(\rho \otimes \gamma)\exp(iH,t) = \exp(-iHt)\rho \exp(iHt) \otimes \gamma.
\]

Hence we can treat the evolution as if it was implemented by the Hamiltonian \( H \otimes 1 \), which can assumed to be bounded as in Section II. \( \Box \).

Note that our arguments do not require that the time evolution \( u \) takes place on a smaller time scale than \( \Delta t \). The reason is that we have argued that \( P_j := u^\dagger (1 \otimes Q_j) u \) may formally be considered as projections of a measurement performed before the interaction was switched on.

Our results in Section II and III may be given an additional interpretation as information-disturbance trade-off relation. Lemma 1 shows that the entropy increase caused by the measurement is the Kullback-Leibler distance between pre- and post-measurement state. Information-disturbance trade-off relations are an important part of quantum information theory [11]. Unfortunately, Theorem 1 is restricted to von-Neumann measurements. Theorem 2 extends the statement to general measurements as far as the disturbance of the total state \( \rho_t \otimes \gamma \) of the “clock” and the ancilla system is considered. However, this is interesting from the thermodynamical point of view taken in this article but not in the setting of information-disturbance trade-off. In the latter setting, the disturbance of the ancilla state that is used to implement a non-von-Neumann measurement is not of interest. Therefore we admit, that our bound on the state disturbance does only hold for von-Neumann measurements.

IV. HOW TIGHT IS THE BOUND?

The entropy increase predicted by our results can never be greater than 1/2. This can be seen by the following argument. Using the definitions of the proof of Lemma 3 we have

\[
\|\dot{\rho}(t)\|_1 = tr(i[H,R(t)]\rho_t) \leq 2\|H\| \|R(t)\| = \Delta E \|R(t)\| = \Delta E.
\]

Note also the connection to the Heisenberg uncertainty principle

\[
\Delta' E \Delta' t \geq \hbar/2.
\]

where \( \Delta' E \) is the energy spread of the system, i.e., the standard deviation of the energy values and \( \Delta' t \) the standard deviation of the time estimation [12] based on any measurement. However, using the symbols \( \Delta' \) instead of \( \Delta \), we emphasize that these definitions do not agree with ours. Note that the energy spread \( \Delta' E \) is at most the energy bandwidth \( \Delta E \). But \( \Delta' t \) can exceed our time resolution \( \Delta t \) by an arbitrary large value. This shows the following example: assume the clock to be a two-level system with period \( T \). Set \( \rho_t := |\psi_t\rangle \langle \psi_t| \) with

\[
|\psi_t\rangle := \frac{1}{\sqrt{2}}(|0\rangle + \exp(i2\pi/\tilde{T})|1\rangle).
\]

Given the prior information that the time is in an interval \( [0,T] \) with \( T \gg \tilde{T} \) we can only estimate the time up to multiples of \( \tilde{T} \) and obtain an error in the order \( T \). In contrast, the time resolution \( \Delta t \) in our sense is \( T/2 \) since we can certainly distinguish between the state at the time \( t \) and at \( t + \tilde{T}/2 \). However, we conjecture that similar bounds as in Theorem 1 on the entropy generation can be given for systems where the energy spectrum is essentially supported by an interval \( [E_{\min}, E_{\max}] \). Therefore our results suggest the following interpretation: every measurement that allows to estimate the time up to an error that is not far away from the Heisenberg limit produces a non-negligible amount of entropy. The following example suggests that the necessary entropy generation may even be much above our lower bound if “time measurements” are used that are extremely close to the Heisenberg limit. Consider the wave function of a free Schrödinger particle moving on the real line. A natural way to measure the time would be to measure its position. We may use, for instance, von-Neumann measurements that correspond to a partition of the real line into intervals of length \( \Delta x \). Assume one wants to improve the time accuracy by decreasing \( \Delta x \) arbitrarily. The advantage for the time estimation is small if \( \Delta x \) is smaller than the actual position uncertainty of the particle. However, if the state is pure it is easy to see that the generated entropy goes to infinity for \( \Delta x \to 0 \).

The following example shows that the entropy generation in a time measurement can really go to zero when \( \Delta E \) goes to infinity. Consider the Hilbert space \( \mathcal{H} := l^2(\mathbb{Z}) \) of square summable functions over the set of integers. Let \( |j\rangle \) with \( j \in \mathbb{Z} \) be the canonical basis vectors. Let the Hamiltonian be the diagonal operator

\[
H|j\rangle := j|j\rangle.
\]

Consider the state
We assume to know that the true time is in the inter-
decide for each \( t \) whether the true time is \( t \) or \( t + \pi \) with confidence that is increasing with \( k \). We use a measurement with projections \( P_1, P_2, P_3, P_4 \) projecting on the four sectors of the unit circle. If the angle uncertainty is considerably smaller than \( \pi/2 \) this measurement can clearly distinguish between \( t \) and \( t + \pi \) with high confidence. Hence, for \( k \) large enough, we get \( \Delta t = \pi \) as time resolution of the measurement. Note that we have only non-negligible entropy generation at that moments where the main part of the wave packet crosses the border between the sectors, i.e., when there are two sectors containing a non-negligible part of the wave function. If, for instance, each sector contains about one half of the probability, we generate the entropy one bit, i.e., the entropy \( \Delta S = \ln 2 \) in natural units.

The probability that the measurement is performed at a time in which non-negligible entropy generation takes place is about \( 2\Delta\phi/\pi \). The average entropy generation decreases therefore for increasing \( k \), i.e., for increasing \( \Delta E = k - 1 \). Let \( p_j^k(t) \) for \( j = 1, \ldots, 4 \) be the probabilities for the 4 possible outcomes when the wave packet with energy spread \( k - 1 \) is measured at the time instant \( t \). There are four moments where the maximum of the wave packet is exactly on the border of the sectors. The probability to meet these times is zero. For all the other times \( t \) there is one \( j \) such that \( 1 - p_j^k(t) \) tends to zero with \( O(1/k^2) \) by standard Fourier analysis arguments. Note that a measurement at time \( t \) produces the entropy

\[
\Delta S = - \sum_j p_j^k(t) \ln p_j^k(t).
\]

We conclude by elementary analysis that \( \Delta S \) tends to zero with \( O((\ln k)/k^2) \) for all times \( t \) except from 4 irrelevant values \( t \) and therefore the average entropy generation tends to zero with \( O((\ln k)/k^2) \). Note that the decrease of the lower bound of Theorem 1 is asymptotically a little bit faster since it is \( O(1/k^2) \) (due to \( \Delta E = k - 1 \)).

V. CONTROLLING A CLASSICAL BIT SWITCH BY A MICROSCOPIC CLOCK

Now we consider the system consisting of the clock, its environment and the classical bit. For the moment we ignore the fact that the bit is quantum and claim that the two logical states 0 and 1 correspond to an orthogonal decomposition

\[
\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1
\]

de of subspaces of the Hilbert space of the composed system. If \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) are non-isomorphic as Hilbert spaces we extend the smaller space such that they are isomorphic. Then we can assume without loss of generality \( \mathcal{H} \) to be of the form \( \mathcal{H}_0 \otimes \mathbb{C}^2 \) without assuming that the bit is physically realized by a two-level quantum system. The fact that the bit is classical will now be described by the fact that it is subjected to decoherence, i.e., that all superpositions between \( |0\rangle \) and \( |1\rangle \) are destroyed and changed to mixtures on a time scale that is not larger than the switching time. Decoherence keeps the diagonal values of the density matrix whereas its non-diagonal entries decay. If the process is a uniform time evolution, i.e., given by a semi-group dynamics, decoherence of the two-level system is given by an exponential decay of both non-diagonal entries:

\[
\gamma_t := \gamma_{00}|0\rangle\langle 0| + \exp(-\lambda t)\gamma_{01}|0\rangle\langle 1| + \exp(-\lambda t)\gamma_{10}|1\rangle\langle 0| + \gamma_{11}|1\rangle\langle 1|,
\]

where \( \gamma_{ij} \) are the coefficients of the density matrix. Note that it is assumed that the effect of the environment does not cause any bit-flips in the system but only destroys coherence. The parameter \( \lambda \geq 0 \) defines the decoherence rate. Let \( P_0 \) and \( P_1 \) be the projections onto the states \( |0\rangle \) and \( |1\rangle \), respectively. The decoherence process can be simulated by measurement processes that are performed at randomly chosen time instants. This analogy is explicitly given as follows: Let

\[
\tilde{G} = P_0 \gamma P_0 + P_1 \gamma P_1
\]

be the effect of the measurement that distinguishes the two logical states. Then we have

\[
\gamma_t = \lim_{n \to \infty} \left((1-\lambda/n)\text{id} + \lambda \tilde{G}/n\right)^n(\gamma) = \exp(\lambda t(\tilde{G} - \text{id})) (\gamma).
\]

The second expression provides the following intuitive approximation of the process: In each small time interval of length \( 1/n \) a measurement is performed with probability \( \lambda/n \). Let \( \tilde{G} := 1 \otimes \tilde{G} \) be the extension of \( \tilde{G} \) to the total system. We assume that the dynamical evolution of the total system is generated by

\[
F := i[., H] + (G - \text{id}),
\]

i.e., the decoherence of the bit is the only contact of the system to its environment. Define the switching time as the length of the time interval \([0, \Delta t]\) where the probability of one of the logical states changes from \( 1/4 \) to \( 3/4 \). The reason that we do not assume it to switch between 0 and 1 is that this is impossible within a finite time interval with a Hamiltonian that has limited spectrum. This can be seen by the fact that all expectation values with
Hamiltonians of lower (!) bounded spectrum are analytical functions [13]. Consider the case that the decoherence time is small compared to the switching time \( \Delta t \), i.e., \( \lambda \Delta t \ll 1 \). Then the probability to have more than one measurement during the switching process is small (of second order in \( \lambda \)) and the probability that one measurement occurs is about \( \lambda \Delta t \). Given that it occurred, the probabilities of its time of occurrence is equally distributed in \( [0, \Delta t] \). Since entropy is convex the entropy generated by the switching process is at least \( \lambda \Delta t \) times the entropy that is generated if a measurement has occurred during the switching process. Also by convexity arguments, we conclude that the entropy generated by the process “perform a measurement at a randomly chosen (unknown) time instant in \( [0, \Delta t] \)” is at least the average entropy generation if the time instant is known. The latter situation meets exactly the assumptions of Theorem 1 with the following parameters: Set \( T := \Delta t \), i.e., assume that the prior information is that the true time is in \( [0, \Delta t] \). Furthermore the switching time \( \Delta t \) coincides with the time resolution \( \Delta t \) of Theorem 1 since reading out the logical state can distinguish between the times 0 and \( \Delta t \) with error probability at most 1/4. Taking into account that a measurement occurs only with probability \( \lambda \Delta t \) we multiply the bound of Theorem 1 by this factor and obtain:

\[
\Delta S \geq \frac{\lambda \Delta t}{2} \left( \frac{\hbar}{\Delta t \Delta E} \right)^2 = \frac{\lambda \hbar^2}{2 \Delta t (\Delta E)^2} \tag{3}
\]

Note that we do not speak about average entropy generation since we consider (in contrast to Theorem 1) the entropy of a single density matrix which is already an average over a set of density matrices which are obtained when the measurement occurs at different times.

Consider now the case that the decoherence time is so small that more than one measurement during the switching process is likely. Then we have to worry about the post-measurement state and whether its energy spectrum is still bounded. If not, we cannot apply our arguments. If it is, we can use the bound (3) nevertheless. This is less obvious than it may seem at first sight. The total dynamics is assumed to be the semi-group evolution

\[
\rho_t := \exp(Ft)\rho
\]

with \( F \) as in eq. (2). Now we investigate the amount of entropy generated by a measurement at time \( t \). Note that we do not have a lower bound by Theorem 1 since the dynamical evolution leading to the state \( \rho_t \) was not Hamiltonian. However, the bound

\[
\Delta S \geq \left( \frac{\| \dot{\rho}(t) \|_1}{\Delta E} \right)^2
\]

holds nevertheless. This is seen by the observation that the generator \( G \) is irrelevant if the proof of Lemma 3 should be converted to the situation here. With \( \tilde{\rho}_t := \sum_j P_j \rho_t P_j \) we have

\[
\dot{\rho}_j(t) = tr(P_j F(\rho_t - \tilde{\rho}_t)) = tr(P_j i[\rho_t - \tilde{\rho}_t, H])
\]

due to \( tr(P_j (G - id)\gamma)) = 0 \) for every state \( \gamma \). It should be emphasized that this argument would fail if the two-level system was not only subjected to decoherence but also to relaxation. If \( L \) is the generator of a relaxation process that causes directly transitions from the state \( |1 \rangle \) to the state \( |0 \rangle \) the equation \( tr(P_j L\gamma) = 0 \) does not hold, i.e., \( L \) would not be irrelevant for our proof. This is consistent with the observation in Section I that the “relaxation clock” can be read out without generating entropy by the measurement. We conclude that the bound (3) holds for arbitrarily high decoherence rate. Formally, the bound predicts infinite entropy production if \( \Delta t \) and \( \Delta E \) are constant and \( \lambda \to \infty \). However, this asymptotic is meaningless due to the quantum zeno effect (see e.g. [14]). Infinite decoherence would stop the switching process completely. Consider a short time \( t \) after a measurement has been performed on the state. Then the derivative of the probabilities \( p_j \) corresponding to a second measurement is only of the order \( \Delta E t \). The essential consequence is that in particular for small \( \Delta E \) when our bound would predict large entropy generation fast switching processes are impossible.

In order to show how to apply our bound to realistic situations consider the following example: A light signal (guided by an optical fiber, for instance) is sent to an apparatus that contains a two-level system. The incoming signal triggers the transition \( |0 \rangle \to |1 \rangle \) from the lower to the upper level (see Fig.1).

FIG. 1. Classical bit switch controlled by a light pulse. For time \( t \to -\infty \) the time evolution is approximatively the free evolution of the light field without any interaction with the apparatus.

The apparatus may be a huge physical system and we are only interested in the fact that the transition time is given by the incoming light signal, the energy may be provided by the apparatus itself. Assume the light signal consists of photons with frequency bandwidth \( \Delta \omega \). The signal is assumed to contain at most \( k \) photons. Then the energy bandwidth of the signal is at most \( \Delta \omega k \). For \( t \to -\infty \) the time evolution of the total system is evolving approximatively as

\[
\rho_t \otimes \gamma
\]

where \( \gamma \) is the (stationary) state of the apparatus. When the signal meets the apparatus the time evolution is given
by an unknown Hamiltonian $H_c$ of the total system combined with the decoherence of the two-level system. The energy bandwidth of the total system according to $H_c$ is unknown. The “measurement” on the two-level system takes definitely place at a moment where the total evolution of the system is generated by the unknown Hamiltonian $H_c$. Nevertheless we can apply the bound of Theorem 2. In Section III we have argued that Theorem 2 does not assume that the measurement interaction was only switched on during a time interval that is small compared to the time resolution $\Delta t$. We observed that the measurement can formally be considered as a measurement that had been performed before the interaction had been switched on. Analogously, we argue that the unknown interaction between light field and apparatus is irrelevant: Let $\sigma_t$ be the state of the total system at the time $t$. Let the switching process happen during the interval $[0, \Delta t]$. Let $(u_t)_{t \in \mathbb{R}}$ be the time evolution of the total system implemented by a possibly unbounded Hamiltonian. The entropy generated by “measurements” $P_j$ performed on the state $\sigma_t$ during this period due to the decoherence is the same as produced by $u_t^* P_j u_t$ performed on the state $u_t \sigma_t u_t^\dagger$. For $s \to -\infty$ the family of states $(u_t \sigma_t u_t^\dagger)_{t \in [0, \Delta t]}$ evolve approximatively like the free evolution $\rho_\text{free} \otimes \gamma$ of the light field. We conclude that the bound in eq. (3) can be applied and the relevant energy bandwidth is the energy bandwidth of the free light field. The entropy generation is at least

$$\frac{\lambda}{2 \Delta t (\Delta \omega)^2}.$$ 

Statements of this kind may be relevant in future computer technology when miniaturization reduces signal energies on the one hand and requires on the other hand reduction of power consumption. Our results show that reduction of signal energy down to the limit of the energy-time uncertainty principle leads unavoidably to heat generation as long as the signal control classical bits. The entropy generation $\Delta S$ leads to an energy loss of $\Delta S k T$ (where $k$ is Boltzmann’s constant and $T$ is the absolute temperature) due to the second law of thermodynamics.

**VI. GENERALIZATION TO OTHER ONE-PARAMETER GROUPS**

Obviously, our results generalize to other one-parameter groups since the proofs do not rely on the interpretation of the unitary group $\exp(-iHT)$ as the system’s time evolution.

Consider for instance the case that the momentum of a Schrödinger wave package is restricted to the interval $[p, p + \Delta p]$. Let $\rho$ be the particle’s density matrix and $\rho_x$ the state translated by $x \in \mathbb{R}$. Then we conclude that every measurement that is suitable to distinguish between $\rho_x$ and $\rho_{x+\Delta x}$ in the sense of Theorem 1 produces at least the entropy $(\hbar/(\Delta x \Delta p))^2/2$. Note that the measurement is not necessarily a position measurement, it has not even to be compatible with the position operator. In this sense, our results may be interpreted as a kind of “generalized uncertainty relation”.

Another very natural application is to consider the group of rotations around a specific axis. Consider a spin-$k/2$ particle and the group of rotations

$$\exp(iL_\alpha/\hbar)_{\alpha \in \mathbb{R}}$$

on its Hilbert space $\mathcal{C}^{k+1}$, where $L_\alpha$ is the operator of its angular momentum in $z$-direction. We have $\Delta L_\alpha = k\hbar$ and conclude the following: Each measurement that distinguishes between $\rho_\alpha$ and $\rho_{\alpha+\Delta\alpha}$ with error probability at most 1/4 produces at least the entropy $1/(2(k \Delta \alpha)^2)$.

**VII. CONCLUSIONS**

Our bound on the entropy that is generated when information about the actual time is extracted from quantum system holds only for Hamiltonian time evolution. A simple counterexample in Section I has shown that time readout without state disturbance is possible for some dissipative semi-group dynamics. This leads to an interesting question: In physical systems, each dissipative semi-group dynamics that is induced by weak coupling to a reservoir in thermal equilibrium is unavoidably accompanied by some loss of free energy. This shows that the loss of free energy caused by the time measurement can only be avoided by systems that loose energy during its autonomous evolution. It would be desirable to know whether there is a general lower bound (including dissipative dynamics) on the total amount of free energy that is lost as soon as timing information is converted to classical information.

**ACKNOWLEDGMENTS**

Thanks to Thomas Decker for helpful comments. This work has been supported by grants of the DFG project “Komplexität und Energie” of the “Schwerpunktprogramm verlustarme Informationsverarbeitung” Be 887/12.
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