ON THE GAIN OF REGULARITY FOR THE
POSITIVE PART OF BOLTZMANN COLLISION OPERATOR
ASSOCIATED WITH SOFT-POTENTIALS

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Abstract. As for the positive part of Boltzmann’s collision operator associated with the collision kernel of soft-potential type, we evaluate its Fourier transform explicitly and prove a set of bilinear estimates for $L^p$ and Sobolev regularity.

1. Introduction. In the theory of Boltzmann equations, the collision operator arises as a mathematical model for describing binary and elastic collision dynamics. In a simplified situation, it may be defined as

$$Q(f, g)(v) = \int_{\mathbb{R}^d} \int_{S^{d-1}} B [f(v')g(v_s') - f(v)g(v_s)] d\sigma dv_s \quad (1.1)$$

for each $v \in \mathbb{R}^d$ with dimension $d \geq 3$, where

$$\begin{align*}
v' &= \frac{v + v_a}{2} + \frac{|v - v_a|}{2} \sigma, \\
v_s' &= \frac{v + v_a}{2} - \frac{|v - v_a|}{2} \sigma, \\
k &= \frac{v - v_a}{|v - v_a|}.
\end{align*} \quad (1.2)$$

$B = B(|v - v_a|, k \cdot \sigma)$ is a nonnegative function on $(0, \infty) \times [-1, 1]$ and $d\sigma$ stands for the area measure on the unit sphere $S^{d-1}$. Formally, it may be written as the difference of two parts $Q = Q^+ - Q^-$ where

$$Q^+(f, g)(v) = \int_{\mathbb{R}^d} \int_{S^{d-1}} B f(v')g(v_s') d\sigma dv_s, \quad (1.3)$$

$$Q^-(f, g)(v) = f(v) \int_{\mathbb{R}^d} \int_{S^{d-1}} B g(v_s) d\sigma dv_s, \quad (1.4)$$

often called the positive and negative collision operator, respectively.
Of principal interest in physics are the power-potential models in which the collision kernel $B$ takes the specific form

$$B(|v - v_*|, k \cdot \sigma) = |v - v_*|^{-\lambda} b(k \cdot \sigma) \quad (-1 \leq \lambda < d).$$

(1.5)

By the potential order $\lambda$, it is said to be hard-potential if $-1 \leq \lambda < 0$, Maxwellian if $\lambda = 0$ and soft-potential if $0 < \lambda < d$. In general, the angular part $b$ is assumed to be continuous everywhere except at a singularity which causes it to be non-integrable on the unit sphere $S^{d-1}$ as a function of $\sigma$. From a mathematical standpoint, however, it is common to assume $b \in L^1(S^{d-1})$, referred to as Grad’s angular cutoff assumption, by considering a monotone family of integrable kernels obtained from cutting off such a singularity. For the collision kernel of type (1.5), it can be shown that the decomposition $Q = Q^+ - Q^-$ is legitimate for a large class of functions $f,g$ under Grad’s angular cutoff assumption on $b$.

For $\alpha \geq 0$, $1 \leq p < \infty$, let $H^\alpha(R^d)$, $L^p_\alpha(R^d)$ denote the homogeneous Sobolev space of smoothing order $\alpha$ and $L^p$ space with weight $(1 + |v|)^\alpha$, respectively, with the corresponding norms

$$
\|f\|_{H^\alpha} = \left( \int_{R^d} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 d\xi \right)^{1/2},
$$

(1.6)

$$
\|f\|_{L^p_\alpha} = \left( \int_{R^d} |f(v)|^p (1 + |v|)^{\alpha p} dv \right)^{1/p}
$$

(1.7)

in which $\hat{f}$ stands for the Fourier transform of $f$. The inhomogeneous Sobolev space is defined as $H^\alpha(R^d) = L^2 \cap H^\alpha(R^d)$ with the usual norm.

In the present paper, we are interested in studying mapping properties of $Q^+$ with our focus on $L^p$ and Sobolev regularity. Regarding earlier results, let us point out the work of P.-L. Lions ([10]) on Sobolev regularity of $Q^+$ which sets the cornerstone in later research developments. In a simplified version, a theorem of Lions states that if $B$ is of class $C_0^\infty$ in each variable and if $f,g \in L^1 \cap H^\alpha(R^d)$ with any $\alpha \geq 0$, then

$$
\|Q^+(f,g)\|_{H^\alpha} \leq C \left( \|f\|_{H^\alpha} \|g\|_{L^1} + \|g\|_{H^\alpha} \|f\|_{L^1} \right)
$$

(1.8)

for a uniform constant $C$. In other words, it asserts that the bilinear operator $Q^+$ gains regularity of order $(d - 1)/2$ in the sense

$$
Q^+: [L^1 \cap H^\alpha(R^d)] \times [L^1 \cap H^\alpha(R^d)] \rightarrow H^{\alpha + \frac{d-1}{2}}(R^d)
$$

continuously for any $\alpha \geq 0$. The proof is based on the stationary phase method involving Fourier integral operators or generalized Radon transforms. A simplified proof is given by Wennberg ([15], [16]) and an extension to weighted Sobolev spaces is given by Mouhot & Villani ([13]).

Since Lions’ work, there have been a number of attempts by many authors to weaken the hypothesis on the kernel $B$ from practical points of view. In particular, Bouchut & Desvillettes ([5]) proved that if $B$ takes the form (1.5) with $-1 \leq \lambda \leq 0$ and $b \in L^2(S^{d-1})$, then

$$
\|Q^+(f,f)\|_{H^{\frac{d-1}{2}}} \leq C \|b\|_{L^2(S^{d-1})} \|f\|_{L^2_{\lambda}}^2
$$

(1.9)

for a uniform constant $C$. In other words, it states that the quadratic map $f \mapsto Q^+(f,f)$ is continuous from $L^2_{\lambda}(R^d)$ into $H^{\frac{d-1}{2}}(R^d)$. The proof is based on Bobylev’s identity for the Fourier transform of $Q^+$ after reducing the matters to the Maxwellian case. For its extension and further regularity results, we refer to [6],
[11] and [13] all of which deal with the case of cutoff hard-potentials, Maxwellian or its variants.

On the other hand, there are a great deal of literature concerning $L^p$ boundedness of $Q^+$. To mention some of results available for the collision kernel of type (1.5), we refer to Gustafsson ([8]), Mouhot & Villani ([13]) for the case of cutoff hard-potentials or Maxwellian and Alonso, Carneiro & Gamba ([1], [2]) for the case of cutoff soft-potentials.

Our main purpose here is to study $L^p$ and Sobolev regularity properties of $Q^+$ for the collision kernel of type (1.5) with cutoff soft-potentials. In particular, we aim to investigate if $Q^+$ has such regularity properties as those described above in the work of Lions or Bouchut & Desvillettes. As it will be explained later in detail, there is a $\dot{H}^{d/2}$ regularity theorem due to Bouchut & Desvillettes ([5]) when $0 < \lambda < d/2$, which is perhaps the only known regularity result available for the soft-potential case.

In order to accomplish our goals, we shall focus our study on the function space $L^1 \cap \dot{H}^\alpha(\mathbb{R}^d)$, $\alpha \geq 0$, chosen preferably for the following reasons:

- For the regularity problem, our principal target is to obtain results analogous with Lions’ theorem described as in (1.8).
- It is common to consider the space $L^1(\mathbb{R}^d)$ (or more generally $L^1_2(\mathbb{R}^d)$) as the basic solution space in the theory of Boltzmann equations.
- Our approach is based on an explicit formula for the Fourier transform of $Q^+(f, g)$ in terms of $\hat{f}, \hat{g}$ and the condition $f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d)$, with appropriate range of $\alpha$, ensures its validity.

It turns out that our results vary with the ranges of the potential order $\lambda$, the smoothing order $\alpha$ and the type of cutoff assumption on $b$. To give a brief summary, let us assume $b \in L^2(\mathbb{S}^{d-1})$ for simplicity.

(i) In the case $(d+1)/2 < \lambda < d$, $Q^+$ gains regularity of order $\beta \in [0, d - \lambda]$ in the sense

$$Q^+ : \left[ L^1 \cap \dot{H}^\alpha(\mathbb{R}^d) \right] \times \left[ L^1 \cap \dot{H}^\alpha(\mathbb{R}^d) \right] \to \dot{H}^{\alpha + \beta}(\mathbb{R}^d)$$

continuously for any $\alpha > \beta + \lambda - d/2$. In addition, $Q^+$ and its Fourier-transformed operator $\hat{Q}^+$ are bounded in $L^p$ for all $1 \leq p \leq \infty$. These results will be presented in Theorems 4.3 and 4.5.

(ii) In the case $1/2 < \lambda \leq (d+1)/2$, $Q^+$ gains regularity of order $d/2 - \lambda$ in the sense

$$Q^+ : \left[ L^1 \cap \dot{H}^\alpha(\mathbb{R}^d) \right] \times \left[ L^1 \cap \dot{H}^\alpha(\mathbb{R}^d) \right] \to \dot{H}^{\alpha + d/2 - \lambda}(\mathbb{R}^d)$$

continuously for any $\alpha > \max(d/4, \lambda - d/2)$, which will be presented in Theorem 5.2.

(iii) In the case $0 < \lambda < d/2$, we revisit the work of Bouchut & Desvillettes to prove that $Q^+$ gains regularity of order $(d-1)/2$ in the sense

$$Q^+ : \left[ L^1_1 \cap \dot{H}^\alpha(\mathbb{R}^d) \right] \times \left[ L^1_1 \cap \dot{H}^\alpha(\mathbb{R}^d) \right] \to \dot{H}^{\alpha + \lambda/2}(\mathbb{R}^d)$$

continuously for any $\alpha > \lambda$, where $\dot{H}^\alpha_1(\mathbb{R}^d)$ denotes the weighted Sobolev space defined below in section 5. This result will be presented in Theorem 5.1.
In what follows, we shall say that $b \in L^s(S^{d-1})$ for $1 \leq s < \infty$ if
\[
\|b\|_{L^s(S^{d-1})} = \left( \int_{S^{d-1}} |b(k \cdot \sigma)|^s \, d\sigma \right)^{1/s} = \left( \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \int_{-1}^1 |b(t)|^s (1 - t^2)^{d-2} \, dt \right)^{1/s} < \infty
\]
and $b \in L^\infty(S^{d-1})$ as usual. For the Fourier transform of an integrable function on $\mathbb{R}^d$, we shall use the definition
\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot v} f(v) \, dv.
\]
We finally remark that we shall not care for the sharp constants in all of our estimates. For two mathematical quantities $X,Y$, we shall write $X \lesssim Y$ for indicating that the inequality $X \leq cY$ holds with an implicit constant $c$ free of functions involved, if any, in the definitions of $X,Y$.

2. Preliminaries.

2.1. Hausdorff-Young type inequalities. The classical Hausdorff-Young inequality states that if $f \in L^p(\mathbb{R}^d)$ for $1 \leq p \leq 2$, then its Fourier transform $\hat{f} \in L^{p'}(\mathbb{R}^d)$ with
\[
\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p} \quad \text{where} \quad \frac{1}{p'} + \frac{1}{p} = 1.
\]
We shall need an extension of Hausdorff-Young inequality as follows.

**Lemma 2.1.** For $1 \leq p \leq \infty$, let $\alpha > (1/p - 1/2)d$ when $1 \leq p < 2$ and $\alpha \geq 0$ when $2 \leq p \leq \infty$. If $f \in L^1 \cap H^\alpha(\mathbb{R}^d)$, then $\hat{f} \in L^p(\mathbb{R}^d)$ with
\[
\|\hat{f}\|_{L^p} \lesssim \|f\|_{L^1}^{1-\theta} \|f\|_{H^\alpha}^\theta \quad \text{where} \quad \theta = \frac{2d}{p(d + 2\alpha)} \in [0,1].
\]

**Proof.** Assume $1 \leq p < 2$. For any $\rho > 0$, Hölder’s inequality gives
\[
\int_{\mathbb{R}^d} |\hat{f}(\zeta)|^p \, d\zeta = \left( \int_{|\zeta| \leq \rho} + \int_{|\zeta| > \rho} \right) |\hat{f}(\zeta)|^p \, d\zeta \leq \|\hat{f}\|_{L^\infty}^p \int_{|\zeta| \leq \rho} d\zeta + \int_{|\zeta| > \rho} |\zeta|^{-\alpha p} |\zeta|^{\alpha p} |\hat{f}(\zeta)|^p \, d\zeta \lesssim \rho^d \|\hat{f}\|_{L^\infty}^p + \rho^{d-p(d+2\alpha)/2} \|f\|_{H^\alpha}^p.
\]
Optimizing in $\rho$, we obtain the stated inequality. For $p = 2$, we estimate
\[
\int_{\mathbb{R}^d} |\hat{f}(\zeta)|^2 \, d\zeta \lesssim \rho^d \|\hat{f}\|_{L^\infty}^2 + \rho^{-2\alpha} \|f\|_{H^\alpha}^2
\]
from which the inequality follows by optimization. For $2 < p \leq \infty$, it is a simple consequence of $L^2$ case and the obvious estimate
\[
\|\hat{f}\|_{L^p} \leq \|\hat{f}\|_{L^\infty}^{1-2/p} \|\hat{f}\|_{L^2}^{2/p}
\]
\hfill \Box

An application yields Sobolev type inequalities:
Lemma 2.2. For $1 \leq p < \infty$, let $\alpha > (1/2 - 1/p)d$ when $2 < p \leq \infty$ and $\alpha \geq 0$ when $1 \leq p \leq 2$. If $f \in L^1 \cap H^\alpha(\mathbb{R}^d)$, then $f \in L^p(\mathbb{R}^d)$ with
\[
\|f\|_{L^p} \lesssim \|f\|_{L^1}^{-\theta} \|f\|_{H^\alpha}^\theta \quad \text{where} \quad \theta = \frac{2d}{d + 2\alpha} \left(1 - \frac{1}{p}\right) \in [0, 1].
\] (2.4)

Proof. For $2 \leq p < \infty$, the inequalities (2.1), (2.2) show
\[
\|f\|_{L^p} \leq \|\hat{f}\|_{L^p} \lesssim \|f\|_{L^1}^{-\theta} \|f\|_{H^\alpha}^\theta
\]
with the same $\theta$ as in (2.4) provided $\alpha > (1/p' - 1/2)d = (1/2 - 1/p)d$. The case $p = 1$ is included in the assumption. For $1 < p < 2$, we use the elementary interpolation inequality
\[
\|f\|_{L^p} \leq \|f\|_{L^{2t}}^{1-t} \|f\|_{L^2}^t, \quad t = 2(1 - 1/p) \in (0, 1)
\]
and Plancherel’s theorem to obtain (2.4). \qed

2.2. Bilinear fractional integrals. As it will be shown, it turns out that the Fourier-transformed collision operator behaves essentially like a version of the bilinear fractional integrals acting on functions of $\mathbb{R}^d$ by
\[
I_\lambda(f, g)(x) = \int_{\mathbb{R}^d} |y|^{-d+\lambda} f(x+y)g(x-y) \, dy
\] (2.5)
for $0 < \lambda < d$, which reduces to the ordinary fractional integrals when $f \equiv 1$ or $g \equiv 1$. A theorem of Grafakos ([7]) and Kenig & Stein ([9]) states

Theorem 2.3. ([7], [9]) Assume that $p, q, r$ satisfy
\[0 < p < \infty, \quad 1 < q, r \leq \infty, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r} - \frac{\lambda}{d}.
\]
If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then $I_\lambda(f, g) \in L^r(\mathbb{R}^d)$ with
\[
\|I_\lambda(f, g)\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}. \quad (2.6)
\]

2.3. Surface integral estimates. In order to exploit fully the Fourier transform of $Q^+(f, g)$, to be evaluated in the next section, it will be essential to know the precise behaviors of the surface integrals described as follows.

Lemma 2.4. For $\xi, \eta \in \mathbb{R}^d$ with $d \geq 3$, if $a < d - 1$, then
\[
\Phi(a) = \int_{S^{d-1}} |\eta - |\xi|\sigma|^{-a} \, d\sigma \leq C(a, d) \left(|\xi| + |\eta|\right)^{-a} \quad (2.7)
\]
for some constant $C(a, d) > 0$ which depends only on $a, d$.

Proof. We may assume that $a > 0$ as well as $\eta \neq 0$ for the estimate would be trivial in such instances. A parametrization of the unit sphere with parameter $\theta \in [0, \pi]$ defined through $\cos \theta = \eta \cdot \sigma/|\eta|$ yields
\[
\Phi(a) = |S^{d-2}| \int_0^\pi (|\eta|^2 + |\xi|^2 - 2|\eta||\xi|\cos \theta)^{-\frac{3}{2}} \sin^{d-2} \theta \, d\theta
\]
\[
= 2^{d-2} |S^{d-2}| \left(|\xi| + |\eta|\right)^{-a} \int_0^1 (1 - zt)^{-\frac{3}{2}} [t(1-t)]^{\frac{d-3}{2}} \, dt,
\] (2.8)
where the latter follows from changing variables $2t - 1 = \cos \theta$ and setting
\[
z = \frac{4|\xi||\eta|}{(|\xi| + |\eta|)^2} \in [0, 1].
\]
Using the elementary inequality $1 - zt \geq 1 - t$ for $t \in [0, 1]$, we find
\[ \Phi(a) \leq C(a, d) \left( |\xi| + |\eta| \right)^{-a}, \]
where the constant $C(a, d)$ stands for
\[ C(a, d) = 2^{d-2} \left| S^{d-2} \right| \int_0^1 (1 - t)^{-a+d-3} t^{-d-3} \, dt, \]
which is finite if and only if $a < d - 1$.

\[ \Phi(a) \approx \left( |\xi| + |\eta| \right)^{-d+1} \begin{cases} \frac{|\xi| - |\eta|}{|\xi| + |\eta|}^{-a+d-1} & \text{if } a > d - 1, \\ \log e \left( \frac{|\xi| + |\eta|}{|\xi| - |\eta|} \right) & \text{if } a = d - 1. \end{cases} \tag{2.9} \]
For elementary properties of hypergeometric functions, see [3].

3. Fourier transform of $Q^+$. In this section, we evaluate the Fourier transform of $Q^+(f, g)$ explicitly in terms of $\hat{f}, \hat{g}$ for the collision kernel of type (1.5) with soft-potentials and Grad’s cutoff assumption on $b$. We begin with a preliminary lemma which will be needed in determining an admissible condition on $f, g$.

**Lemma 3.1.** For $f \in L^1(\mathbb{R}^d)$, define
\[ J_\lambda(f) = \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\xi|^{-d+\lambda} |\hat{f}(\xi - \zeta)| \, d\zeta \quad (0 < \lambda < d). \tag{3.1} \]
If $f \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d)$ with $\alpha > \lambda - d/2$, then
\[ J_\lambda(f) \lesssim \|f\|_{L^1}^{1-\theta} \|f\|_{\dot{H}^\alpha}^\theta \quad \text{where} \quad \theta = \frac{2\lambda}{d+2\alpha} \in (0, 1). \tag{3.2} \]

**Proof.** Fix $\xi \in \mathbb{R}^d$ and let $\rho > 0$. Choosing $1 \leq p < d/\lambda$ momentarily, we apply Hölder’s inequality to estimate
\[ \int_{\mathbb{R}^d} |\xi|^{-d+\lambda} |\hat{f}(\xi - \zeta)| \, d\zeta \lesssim \rho^\lambda \|f\|_{L^1} + \rho^{\lambda-d/p} \|\hat{f}\|_{L^p}. \]
Optimizing in $\rho$, we obtain
\[ \int_{\mathbb{R}^d} |\xi|^{-d+\lambda} |\hat{f}(\xi - \zeta)| \, d\zeta \lesssim \|f\|_{L^1}^{1-p\lambda/d} \|\hat{f}\|_{L^p}^{p\lambda/d}. \]
An application of Hausdorff-Young type inequality (2.2) yields
\[ \|f\|_{L^1}^{1-p\lambda/d} \|\hat{f}\|_{L^p}^{p\lambda/d} \lesssim \|f\|_{L^{1,\theta}} \|f\|_{\dot{H}^\alpha}^{\theta}, \]
with the same $\theta$ defined as above and the proof is complete. \hfill \Box

Our main result is the following where
\[ \lambda_d = \pi \frac{d+\lambda}{2} \Gamma \left( \frac{d-\lambda}{2} \right) / \Gamma \left( \frac{\lambda}{2} \right) (0 < \lambda < d). \]

**Theorem 3.2.** Assume that $B = |v - v^*|^{-\lambda} b(\mathbf{k}, \sigma)$ where $b \in L^1(\mathbb{S}^{d-1})$ and $0 < \lambda < d$. Let $f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d)$ for some $\alpha \geq 0$.

\[ ^1 \text{In the case when } 0 < \lambda < d/2, \text{ it simply means that } \alpha \geq 0. \]
apply the Hardy-Littlewood-Sobolev theorem to deduce
\[ \|Q^+(f, g)\|_{L^1} \leq \|b\|_{L^1} \left( \|f\|_{L^1} \|g\|_{L^1} \right)^{1-\theta} \left( \|f\|_{H^\alpha} \|g\|_{H^{-\alpha}} \right)^\theta \]

where \( \theta = \frac{\lambda}{d+2\alpha} \in (0, 1) \).

(i) We change variables \((v, v_*, \sigma) \rightarrow (v', v'_*, k)\), which has unit Jacobian, and apply the Hardy-Littlewood-Sobolev theorem to deduce
\[
\int |Q^+(f, g)(v)| dv \leq \|b\|_{L^1} \int |v-v_*|^{-\lambda} |f(v)||g(v^*)| dv_* dv
\]
where \(1 < p, q < \infty\) and \(1/p + 1/q = 2 - \lambda/d\). With the choice of \(p = q\), the Sobolev type inequality (2.4) gives
\[
\|f\|_{L^p} \lesssim \|f\|_{L^1}^{\theta}, \quad \|g\|_{L^p} \lesssim \|g\|_{L^1}^{\theta},
\]
with the same \(\theta\) as stated, whence (3.3) follows at once.

(ii) If \(\alpha > \lambda - d/2\) in addition, then the Fourier transform of \(Q^+(f, g)\) is given explicitly by
\[
|Q^+(f, g)|^\gamma(2\xi) = \lambda_d \int_{\mathbb{R}^d} \lambda \int_{\mathbb{R}^{d-1}} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right)|\eta - |\sigma|^{-d+\lambda} \times \hat{f}(\xi + \eta) \hat{g}(\xi - \eta) d\sigma d\eta.
\]

Proof. (i) We change variables \((v, v_*, \sigma) \rightarrow (v', v'_*, k)\), which has unit Jacobian, and apply the Hardy-Littlewood-Sobolev theorem to deduce
\[
\int |Q^+(f, g)(v)| dv \leq \|b\|_{L^1} \int |v-v_*|^{-\lambda} |f(v)||g(v^*)| dv_* dv
\]
which is valid for any \(f \in L^1(\mathbb{R}^d)\) satisfying \(J_\lambda(f) < \infty\) and \(g \in L^1(\mathbb{R}^d)\). To verify, we recall the classical fact that the fractional integrals may be recovered from inverting its Fourier transforms in the manner
\[
\int_{\mathbb{R}^d} |v-v_*|^{-\lambda} \phi(v) dv = \lambda_d \int_{\mathbb{R}^d} e^{2\pi i v \cdot \zeta} |\zeta|^{-d+\lambda} \hat{\phi}(\zeta) d\zeta
\]
for an integrable function \(\phi\) provided that \(|\zeta|^{-d+\lambda} \hat{\phi}(\zeta)\) were also integrable (see [14] or Theorem 5.9 of [12] for an elementary proof). Applying this formula to the function \(\phi(v) = e^{-2\pi i \xi \cdot v} f(v)\), we obtain
\[
\int_{\mathbb{R}^d} e^{2\pi i v \cdot \zeta} |v-v_*|^{-\lambda} f(v) dv = \lambda_d \int_{\mathbb{R}^d} e^{2\pi i v_* \cdot \zeta} |\zeta|^{-d+\lambda} \hat{f}(\xi + \zeta) d\zeta,
\]
which is legitimate for
\[
\int_{\mathbb{R}^d} |\zeta|^{-d+\lambda} \hat{f}(\xi + \zeta) d\zeta \leq J_\lambda(f) < \infty.
\]
In view of Fubini’s theorem and the trivial evaluation
\[
\int_{\mathbb{R}^d} e^{-2\pi i \eta \cdot v} e^{2\pi i v_* \cdot \zeta} g(v_*) dv_* = \hat{g}(\eta - \zeta),
\]
the claimed identity (3.5) follows instantly.
Let us now fix \( f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d) \) with \( \alpha > \lambda - d/2 \). According to Lemma 3.1, \( J_\lambda(f) < \infty \) and hence (3.5) is valid. We change variables \((v, v_\times, \sigma) \rightarrow (v', v_\times', k)\) as before to write

\[
|Q^+(f, g)|^\times(\xi) = \int_{\mathbb{R}^d} |v - v_\times|^{-\lambda} f(v)g(v_\times)A_b(v, v_\times, \xi) \, dv_\times dv,
\]

\[
A_b(v, v_\times, \xi) = \int_{S^{d-1}} b(k \cdot \sigma) e^{-2\pi i \xi \cdot v'} \, d\sigma.
\]

As observed by Bobylev [4], it is simple to construct an isometry on \( S^{d-1} \) that exchanges the unit vectors \( k \) and \( \xi/|\xi| \) for nonzero \( \xi \). Due to the invariance of the area measure under such an isometry, we get

\[
A_b(v, v_\times, \xi) = \int_{S^{d-1}} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) e^{-2\pi i (\xi^+ \cdot v + \xi^- \cdot v_\times)} \, d\sigma,
\]

\[
\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2}.
\]

Interchanging the order of integrations, we obtain the representation formula (3.4) at once upon invoking (3.5) and changing variables suitably. \( \square \)

Remark 3.1. (a) The formula (3.4) remains valid under a weaker assumption on \( g \), namely, \( g \in L^1(\mathbb{R}^d) \). In fact, with the same type of \( f \), it can be shown that \( Q^+(f, g) \) gives rise to a tempered distribution on \( \mathbb{R}^d \) whose Fourier transform is given by (3.4) in the sense of distribution.

(b) An important feature of the collision operator \( Q^+ \) and its Fourier transform is the dilation invariance property defined as

\[
Q^+(f_\delta, g_\delta)(v) = \delta^{d-\lambda} Q^+(f, g)(v/\delta),
\]

\[
[Q^+(f_\delta, g_\delta)]^\times(\xi) = \delta^{2d-\lambda} [Q^+(f, g)]^\times(\delta\xi)
\]

for any \( \delta > 0 \), where \( f_\delta(v) = f(v/\delta) \), \( g_\delta(v) = g(v/\delta) \), which may be used in determining necessary conditions for their mapping properties in the standard manner.

4. Bilinear estimates for \( L^p \) and Sobolev regularity: Main results. The purpose of this section is to study the problem of \( L^p \) and Sobolev regularity of \( Q^+ \) having collision kernel of type (1.5) with soft-potentials and cutoff \( b \) in which the potential order \( \lambda \) is relatively large. To simplify the notation in our calculations, we shall write

\[
\hat{Q^+}(\xi) = [Q^+(f, g)]^\times(2\xi) \quad \text{and} \quad \|b\|_s = \|b\|_{L^s(S^{d-1})}
\]

in the rest of this paper unless precise statements are necessary.

An application of Lemma 2.4 yields

Lemma 4.1. If \( b \in L^s(S^{d-1}) \) with \( 1 < s \leq \infty \) and \( 1 + (d - 1)/s < \lambda < d \), under the same settings as in Theorem 3.2, then

\[
\left|\hat{Q^+}(\xi)\right| \lesssim \|b\|_s \int_{\mathbb{R}^d} (|\xi| + |\eta|)^{-d+\lambda} |\hat{f}(\xi + \eta)| |\hat{g}(\xi - \eta)| \, d\eta.
\]

Proof. By Lemma 2.4 and Hölder’s inequality, note that

\[
\int_{S^{d-1}} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) |\eta - |\xi||\sigma|^{-d+\lambda} \, d\sigma \lesssim \|b\|_s (|\xi| + |\eta|)^{-d+\lambda}
\]

from which the claim follows immediately. \( \square \)
4.1. $L^p$ boundedness. We simply bound $(|\xi| + |\eta|)^{-d+\lambda} \leq |\eta|^{-d+\lambda}$ in the estimate (4.1) to see
\[ |\hat{Q^n}(\xi)| \lesssim \|b\|_s \int_{\mathbb{R}^d} |\eta|^{-d+\lambda} |\hat{f}(\xi + \eta)| |\hat{g}(\xi - \eta)| \, d\eta \]
\[ = \|b\|_s I_\lambda (|\hat{f}|, |\hat{g}|) (\xi), \tag{4.2} \]
where $I_\lambda$ denotes the bilinear fractional integral operator defined as in (2.5).

An application of Theorem 2.3 gives

Lemma 4.2. Assume that $B = |v - v_s|^{-\lambda} b(k \cdot \sigma)$ where $b \in L^s(S^{d-1})$ for some $1 < s \leq \infty$ and $1 + (d-1)/s < \lambda < d$. Let $p$, $q$, $r$ satisfy
\[ 0 < p < \infty, \quad 1 < q, r \leq \infty, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r} - \frac{\lambda}{d}. \]

Let $f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d)$ with $\alpha$ chosen so that the formula (3.4) is valid and assume further $\hat{f} \in L^q(\mathbb{R}^d)$, $\hat{g} \in L^r(\mathbb{R}^d)$. Then
\[ \| |Q^+(f,g)| |_{L^p} \|_{L^p} \lesssim \|b\|_{L^s(S^{d-1})} \|\hat{f}\|_{L^q} \|\hat{g}\|_{L^r}. \tag{4.3} \]

Combining with Hausdorff-Young type inequalities, we now prove that $Q^+(f,g)$ and its Fourier transform are bounded in $L^p$ for all $1 \leq p \leq \infty$ if $f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d)$ with suitably chosen $\alpha$.

Theorem 4.3. Assume that $B = |v - v_s|^{-\lambda} b(k \cdot \sigma)$ where $b \in L^s(S^{d-1})$ for some $1 < s \leq \infty$ and $1 + (d-1)/s < \lambda < d$. Let $f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d)$.

(i) For $1 \leq p \leq \infty$, if $\alpha$ satisfies
\[ \alpha > \max \left\{ \lambda - \frac{d}{2}, \frac{d}{2} \left( \frac{1}{p} + \frac{\lambda}{d} - 1 \right) \right\}, \]
then $|Q^+(f,g)| \in L^p(\mathbb{R}^d)$ and
\[ \| |Q^+(f,g)| |_{L^p} \|_{L^p} \lesssim \|b\|_{L^s(S^{d-1})} \left( \|f\|_{L^1} \|g\|_{L^1} \right)^{1-\theta} \left( \|f\|_{\dot{H}^\alpha} \|g\|_{\dot{H}^\alpha} \right)^\theta \]
where $\theta = \frac{d}{d + 2\alpha} \left( \frac{1}{p} + \frac{\lambda}{d} \right) \in (0, 1). \tag{4.4} \]

(ii) For $1 \leq p \leq \infty$, if $\alpha$ satisfies
\[ \alpha > \max \left\{ \lambda - \frac{d}{2}, \frac{d}{2} \left( \frac{\lambda}{d} - \frac{1}{p} \right) \right\}, \]
then $Q^+(f,g) \in L^p(\mathbb{R}^d)$ and
\[ \|Q^+(f,g)\|_{L^p} \lesssim \|b\|_{L^s(S^{d-1})} \left( \|f\|_{L^1} \|g\|_{L^1} \right)^{1-\mu} \left( \|f\|_{\dot{H}^\alpha} \|g\|_{\dot{H}^\alpha} \right)^\mu \]
where $\mu = \frac{d}{d + 2\alpha} \left( 1 - \frac{1}{p} + \frac{\lambda}{d} \right) \in (0, 1). \tag{4.5} \]

Proof. Owing to the $L^1$ estimate established in (3.3) of Theorem 3.2, we may assume that $1 \leq p < \infty$ in part (i) and $1 < p \leq \infty$ in part (ii). We also note that the Fourier transform formula (3.4) is valid for $\alpha > \lambda - d/2$.
(i) We fix $p$ and apply Lemma 4.2 with the choice of
$$\frac{1}{q} = \frac{1}{r} = \frac{1}{2} \left( \frac{1}{p} + \frac{\lambda}{d} \right) < 1$$
to deduce
$$\| [Q^+(f,g)]^- \|_{L^p} \lesssim \| b \|_{L^s(S^{d-1})} \| \hat{f} \|_{L^s} \| \hat{g} \|_{L^s}.$$
Since
$$\alpha > \left( \frac{1}{q} - \frac{1}{2} \right) d = \frac{1}{2} \left( \lambda - d + \frac{d}{p} \right),$$
the Hausdorff-Young type inequality (2.2) gives
$$\| \hat{f} \|_{L^q} \lesssim \| f \|_{L^1} \| f \|_{H^{\alpha}}, \quad \theta = \frac{2d}{q(d+2\alpha)} = \frac{d}{d+2\alpha} \left( \frac{1}{p} + \frac{\lambda}{d} \right)$$
and the same type of estimate for $g$, which yields the estimate (4.4).

(ii) We note that the case $2 \leq p \leq \infty$ follows from (i) via
$$\| Q^+(f,g) \|_{L^p} \leq \| Q^+(f,g) \|_{L^p}^\prime.$$
In the case $1 < p < 2$, we first use the interpolation inequality
$$\| Q^+(f,g) \|_{L^p} \leq \| Q^+(f,g) \|_{L^1}^{\frac{1}{t}} \| Q^+(f,g) \|_{L^2}^t, \quad t = 2(1 - 1/p)$$
with $0 < t < 1$ and then apply the $L^1$ estimate (3.3) together with the preceding $L^2$ estimate (via Plancherel’s theorem) to obtain (4.5).

**Remark 4.1.** In their recent work [1], [2], Alonso, Carneiro & Gamba established the following $L^p$ boundedness of $Q^+$ for the same type of kernel $B$ as above:

Suppose that
$$\| b \|_{a,\lambda} = \int_{S^{d-1}} b(k \cdot \sigma)(1 - k \cdot \sigma)^{-\left(\frac{d-\lambda}{d}\right)} d\sigma < \infty \quad (4.6)$$
for some $a > 1$ and $0 < \lambda < d - 1$. Then
$$\| Q^+(f,g) \|_{L^p} \lesssim \| b \|_{a,\lambda} \| f \|_{L^s} \| g \|_{L^r} \quad (4.7)$$
for $1 < p, q, r < \infty$ satisfying $1/p = 1/q + 1/r + \lambda/d - 1$ and
$$\max \left\{ \frac{1}{p'}, 1 - \frac{1}{q(1 - \lambda/d)} \right\} < \frac{1}{a} < \min \left\{ \frac{1}{p'(1 - \lambda/d)}, \frac{1}{q'} \right\} \quad (4.8)$$
where $p', q'$ denote the Hölder exponents of $p, q$, respectively.

Let us point out the following features and consequences:

(a) The condition (4.6) is stronger than the condition $\| b \|_{L^1(S^{d-1})} < \infty$ for any $a > 1$ but it is weaker than the one $\| b \|_{L^s(S^{d-1})} < \infty$ for any $a > 1, s > 1$ satisfying
$$a > \frac{s(d - \lambda)}{(s - 1)(d - 1)}$$
in view of Hölder’s inequality. Otherwise, they are independent.

(b) Setting $\lambda_s = 1 - \lambda/d$, the restriction (4.8) is equivalent to
$$\frac{a}{a - \lambda_s} < p < \frac{a}{a - 1}, \quad q < \frac{a}{\lambda_s(a - 1)}$$
and $p, q, r$ must satisfy $1/p = 1/q + 1/r - \lambda_s, 1/r > \lambda_s.$
(c) One may obtain $L^p$ boundedness of $Q^+$ of kind (4.5) directly from the result (4.7). As a matter of fact, for a fixed $a > 1$, if we take $q = r$ so that $1/p = 2/q - \lambda_s$, then the Sobolev type inequality (2.4) gives
\[
\|f\|_{L^p} \lesssim \|f\|_{L^2}^{1-\mu} \|f\|^{\mu}_{\dot{H}^{\alpha}}
\]
with the same $\mu$ defined as in (4.5). To find out the required range of $\alpha$ and $p$, we look at the index-diagram in the unit square
\[
D = \left\{ \left( \frac{1}{q}, \frac{1}{p} \right) : \lambda_s < \frac{1}{q} < \frac{a-1}{a} < \frac{1}{p} < \frac{a-\lambda_s}{a}, \quad \frac{1}{p} = \frac{2}{q} - \lambda_s \right\}.
\]
Note that $D$ is not empty only if $\lambda_s < (a-1)/a$ or $\lambda > d/a$. Since $\lambda < d-1$, we must have $a > d/(d-1)$. A careful analysis on $D$ reveals the following regularity result under the assumption (4.6):

**Theorem 4.4.** Assume $B = |v - v_*|^{-\lambda} b(k \cdot \sigma)$ where $b$ satisfies (4.6) for some $a > d/(d-1)$ and $d/a < \lambda < d-1$. Let
\[
\frac{a}{a-\lambda_s} < p < \frac{a}{a-1}, \quad \lambda_s = 1 - \lambda/d.
\]
Choose $\alpha$ according to the following rule:

(i) $\alpha \geq 0$ if $a > 2$ and $d/a < \lambda < d/2$.

(ii) $\alpha > (\lambda - d/p)/2$ if $a > 2$ and $d/2 \leq \lambda < d-1$ or $a \leq 2$.

If $f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d)$, then $Q^+(f, g) \in L^p(\mathbb{R}^d)$ with
\[
\|Q^+(f, g)\|_{L^p} \lesssim \|b\|_{\dot{H}^\alpha} \left( \|f\|_{L^1} \|g\|_{L^1} \right)^{1-\mu} \left( \|f\|_{\dot{H}^\alpha} \|g\|_{\dot{H}^\alpha} \right)^{\mu}
\]
where $\mu = \frac{d}{d+2\alpha} \left( \frac{1}{p} + \frac{\lambda}{d} \right) \in (0, 1)$.

**4.2. Sobolev regularity.** Our principal result is the following:

**Theorem 4.5.** Assume that $B = |v - v_*|^{-\lambda} b(k \cdot \sigma)$ where $b \in L^s(\mathbb{S}^{d-1})$ for some $1 < s \leq \infty$ and $1 + (d-1)/s < \lambda < d$. For $0 \leq \beta \leq d - \lambda$, if $f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d)$ with $\alpha > \beta + \lambda - d/2$, then
\[
\|Q^+(f, g)\|_{\dot{H}^{\alpha + \beta}} \lesssim \|b\|_{L^s(\mathbb{S}^{d-1})} \left\{ \|f\|_{\dot{H}^\alpha} \|g\|_{L^1}^{1-\theta} \|g\|_{\dot{H}^{\alpha+\beta}}^{\theta} + \|g\|_{\dot{H}^\alpha} \|f\|_{L^1}^{1-\theta} \|f\|_{\dot{H}^{\alpha+\beta}}^{\theta} \right\}
\]
where $\theta = \frac{2(\beta + \lambda)}{d+2\alpha} \in (0, 1)$.

**Proof.** As before, we note that the Fourier transform formula (3.4) is valid in the present case. Assume first $0 \leq \beta < d - \lambda$. In the estimate (4.1) of Lemma 4.1, we use the elementary inequality
\[
(|\xi| + |\eta|)^{-d+\lambda} \leq C_\alpha |\xi|^{-\alpha} |\eta|^{-d+\beta + \lambda} (|\xi + \eta|^\alpha + |\xi - \eta|^\alpha)
\]
with $C_\alpha = \max \{1, 2^{\alpha/2-1}\}$, for instance, to bound
\[
|\xi|^{\alpha + \beta} \|Q^+(\xi)\| \lesssim \|b\| \left\{ |\hat{f}_{\beta + \lambda} (|\hat{\xi}_\alpha|, |\hat{g}|) (\xi) + I_{\beta + \lambda} \left( |\hat{f}|, |\hat{g}| \right) (\xi) \right\},
\]
where we put $\hat{f}_\alpha(n) = |\xi|^\alpha \hat{f}(\xi), \hat{g}_\alpha(n) = |\xi|^\alpha \hat{g}(\xi).$ By Lemma 4.2,
\[
\|I_{\beta + \lambda} \left( |\hat{f}_\alpha|, |\hat{g}| \right) \|_{L^2} \lesssim \|\hat{f}_\alpha\|_{L^2} \|\hat{g}\|_{L^2(\beta + \lambda)} \lesssim \|f\|_{\dot{H}^\alpha} \|g\|_{L^1}^{1-\theta} \|g\|_{\dot{H}^{\alpha+\beta}}^{\theta}
\]
with the same \( \theta \) as in the above statement, where we have invoked the Hausdorff-Young type inequality (2.2) for the last inequality. Since the same type estimate holds for \( L^{p+\lambda} \), the result (4.11) follows by taking \( L^2 \) norms on both sides of (4.12).

In the case \( \beta = d - \lambda \), we bound with appropriate change of variables

\[
|\xi|^{\alpha + \beta}|\hat{Q}^+(\xi)| \lesssim \|b\|_s \left| \hat{Q}_1(\xi) + \hat{Q}_2(\xi) \right|,
\]

where

\[
\hat{Q}_1(\xi) = \int_{\mathbb{R}^d} |\hat{f}_\alpha(2\xi - \zeta)| |\hat{g}(\zeta)| \, d\zeta,
\]

\[
\hat{Q}_2(\xi) = \int_{\mathbb{R}^d} |\hat{f}(\zeta)| |\hat{g}_\alpha(2\xi - \zeta)| \, d\zeta.
\]

An application of Minkowski’s integral inequality gives

\[
\left\| \hat{Q}_1 \right\|_{L^2} \leq \left\| \hat{f}_\alpha \right\|_{L^2} \left\| \hat{g} \right\|_{L^1} \lesssim \|f\|_{H^\lambda}\|g\|_{L^{1-\theta}}\|g\|_{H^\alpha},
\]

with \( \theta = 2d/(d + 2\alpha) \), where we have invoked the Hausdorff-Young type inequality (2.2) for \( \|\hat{g}\|_{L^1} \) once again. Since the same type estimate holds for \( \hat{Q}_2 \), we obtain the result (4.11) as in the preceding manner.

5. Further regularity results. The preceding methods of obtaining \( L^p \) and Sobolev regularity results are all based on the idea of dominating the Fourier transform of \( Q^+(f, g) \) by the bilinear integral operator defined as in (4.1) of Lemma 4.1, which is in turn made possible due to the surface integral estimate of Lemma 2.4.

Under the cutoff assumption \( b \in L^s(\mathbb{R}^d) \) with \( 1 < s \leq \infty \), if the potential order \( \lambda \) is in the range \( 0 < \lambda \leq 1 + (d - 1)/s \), such an idea breaks down owing to the singular behavior of the surface integral depicted as in (2.8) of Remark 3.1. On the other hand, in the case when \( b \) is merely integrable on \( S^{d-1} \), the preceding methods may not be applicable either in view of the structure of the Fourier transform of \( Q^+ \).

The purpose of this section is to obtain an alternative or extension of Theorem 4.5 in such a situation described as in the above by employing different methods. To simplify the matters, we shall restrict our attention to the case of \( s = 2 \) and present three types of regularity results.

5.1. Gain of \( \frac{d-1}{2} \) regularity. In the first place, we revisit the work [5] of Bouchut & Desvillettes and reformulate their theorem of \( \dot{H}^{\frac{d-1}{2}} \) regularity for \( Q^+ \) in the present framework. The original theorem of Bouchut & Desvillettes states

\[
\|Q^+(f, f)\|_{\dot{H}^{\frac{d-1}{2}}} \lesssim \|b\|_{L^2(\mathbb{R}^{d-1})} \|f\|_{L^2}^2,
\]

where \( p_\lambda = 2d/(d - \lambda) \) and \( 0 < \lambda < d/2 \).

We observe that \( f \in L^1(\mathbb{R}^d) \), the weighted \( L^1 \) space, if and only if \( f \) is integrable on \( \mathbb{R}^d \) and \( \forall f \in L^1(\mathbb{R}^d) \), that is, \( f \) has finite moments of first order. In an analogous manner, we shall say that \( f \in \dot{H}^p(\mathbb{R}^d) \), the weighted Sobolev space, if and only if \( f, \forall f \in \dot{H}^p(\mathbb{R}^d) \) and define

\[
\|f\|_{\dot{H}^1} = \left( \int_{\mathbb{R}^d} |\xi|^{2\alpha} \left( |\hat{f}(\xi)|^2 + |\nabla \hat{f}(\xi)|^2 \right) \, d\xi \right)^{1/2},
\]

\[
\approx \left( \int_{\mathbb{R}^d} |\xi|^{2\alpha} \left( |\hat{f}(\xi)|^2 + |\nabla \hat{f}(\xi)|^2 \right) \, d\xi \right)^{1/2},
\]

(5.2)
which clearly makes sense for any \( f \in L^1_1(\mathbb{R}^d) \).

Our reformulation of (5.1) in the space \( L^1_1 \cap \dot{H}^\alpha_1(\mathbb{R}^d) \) reads as follows.

**Theorem 5.1.** Assume that \( B = |v - v_*|^{-\lambda} b(k \cdot \sigma) \) where \( b \in L^2(S^{d-1}) \) and \( 0 < \lambda < d/2 \). If \( f, g \in L^1_1 \cap \dot{H}^\alpha_1(\mathbb{R}^d) \) with \( \alpha > \lambda \), then

\[
\|Q^+ (f,g)\|_{H^{\alpha+2-\lambda}} \lesssim \|b\|_{L^2} \left\{ \|f\|_{H^\alpha_1} \|g\|_{H^\alpha_1}^{1-\theta} + \|g\|_{H^\alpha_1} \|f\|_{H^\alpha_1}^{1-\theta} \right\}
\]

where \( \theta = \frac{d + 2\lambda}{d + 2\alpha} \in (0,1) \).

**Proof.** We write the Fourier transform formula (3.4) in the form

\[
\hat{Q}^+ (\xi) = \int_{S^{d-1}} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \hat{F}(\xi + |\xi| \sigma, \xi - |\xi| \sigma) d\sigma,
\]

(5.5)

\[
\hat{F}(\xi, \eta) = \lambda_d \int_{\mathbb{R}^d} \frac{\xi - d+\lambda \eta}{d+\lambda} \hat{f}(\xi + \zeta) \hat{g}(\eta - \zeta) d\zeta.
\]

(5.6)

Making use of the identity

\[
\int_{S^{d-1}} \left| \hat{F}(\xi + |\xi| \sigma, \xi - |\xi| \sigma) \right|^2 d\sigma = - \int_{S^{d-1}} \int_{|\eta| = |\xi|} \frac{\partial}{\partial r} \left| \hat{F}(\xi + r\sigma, \xi - r\sigma) \right|^2 r dr d\sigma
\]

and the readily-verified estimate

\[
\left| \frac{\partial}{\partial r} \left| \hat{F}(\xi + r\sigma, \xi - r\sigma) \right|^2 \right| \leq 2 \left| \hat{F}(\xi + r\sigma, \xi - r\sigma) \right| \left| \left( \nabla_2 \hat{F} - \nabla_1 \hat{F} \right) (\xi + r\sigma, \xi - r\sigma) \right|
\]

where \( \nabla_2, \nabla_1 \) denote gradients with respect to the first and the second \( d \)-components, respectively, we apply the Cauchy-Schwarz inequality and the formula for integrating in polar coordinates to deduce from (5.5)

\[
\left| \hat{Q}^+ (\xi) \right|^2 \lesssim \|b\|_{L^2} \|\xi\|^{-d+1} \int_{|\zeta| < |\xi|} \left| \hat{F}(\xi + \zeta, \xi - \zeta) \right| \left| \left( \nabla_2 \hat{F} - \nabla_1 \hat{F} \right) (\xi + \zeta, \xi - \zeta) \right| d\zeta
\]

\[
\lesssim \|b\|_{L^2} \|\xi\|^{-d+1-2\alpha} \int \|\xi\|^{2\alpha} \left| \hat{F}(\xi + \zeta, \xi - \zeta) \right| \left| \left( \nabla_2 \hat{F} - \nabla_1 \hat{F} \right) (\xi + \zeta, \xi - \zeta) \right| d\zeta
\]

for any \( \alpha \geq 0 \). Multiplying both sides by \( |\xi|^{d+1+2\alpha} \) and integrating about \( d\xi \) with the change of variables \( \xi + \zeta \to u \), \( \xi - \zeta \to v \), we are led to

\[
\int_{\mathbb{R}^d} |\xi|^{d+1+2\alpha} \left| \hat{Q}^+ (\xi) \right|^2 d\xi \lesssim \|b\|_{L^2} \left( \int |u + v|^{2\alpha} \left| \hat{F}(u,v) \right| \left| \left( \nabla_2 \hat{F} - \nabla_1 \hat{F} \right) (u,v) \right| dudv \right) \]

\[
\lesssim \|b\|_{L^2} \left( \left\| u + v \right\|_{L^2}^{\alpha} \left( \left\| \nabla_1 \hat{F} \right\|_{L^2} + \left\| u + v \right\|_{L^2}^{\alpha} \right) \right)
\]

(5.7)

for which \( L^2 \) norms should be taken on the product space \( \mathbb{R}^d \times \mathbb{R}^d \).
Setting \( \hat{f}_\alpha(u) = |u|^\alpha \hat{f}(u) \), \( \hat{g}_\alpha(u) = |u|^\alpha \hat{g}(u) \) as before, let us put

\[
\tilde{M}_\alpha (u, v) = \int |\zeta|^{-d+\lambda} |\hat{f}_\alpha(u + \zeta)| \, |\hat{g}(v - \zeta)| \, d\zeta, \\
\tilde{N}_\alpha (u, v) = \int |\zeta|^{-d+\lambda} |\hat{f}(u + \zeta)| \, |\hat{g}_\alpha(v - \zeta)| \, d\zeta.
\]

It follows from the formula (5.6) that \( |u + v|^\alpha |\hat{F}(u, v)| \) is bounded by

\[
\int |\zeta|^{-d+\lambda} |u + v|^\alpha |\hat{f}(u + \zeta)| \, |\hat{g}(v - \zeta)| \, d\zeta \\
\leq \int |\zeta|^{-d+\lambda} (|u + \zeta|^\alpha + |v - \zeta|^\alpha) \, |\hat{f}(u + \zeta)| \, |\hat{g}(v - \zeta)| \, d\zeta \\
= \tilde{M}_\alpha (u, v) + \tilde{N}_\alpha (u, v),
\]
which yields

\[
\|u + v|^\alpha \hat{F}\|_{L^2} \lesssim \|\tilde{M}_\alpha\|_{L^2} + \|\tilde{N}_\alpha\|_{L^2}. \tag{5.8}
\]

We apply Minkowski’s integral inequality and the ordinary fractional integration theorem of Hardy, Littlewood and Paley to estimate

\[
\|\tilde{M}_\alpha\|_{L^2} = \left( \iint \left| \int |\zeta|^{-d+\lambda} |\hat{f}_\alpha(u + \zeta)| \, |\hat{g}(v - \zeta)| \, d\zeta \right|^2 \, du \, dv \right)^{1/2} \\
\leq \|\hat{f}_\alpha\|_{L^2} \left( \int \left[ \int |\zeta|^{-d+\lambda} \, |\hat{g}(v - \zeta)| \, d\zeta \right]^2 \, dv \right)^{1/2} \\
\leq \|\hat{f}_\alpha\|_{L^2} \|\hat{g}\|_{L^q} \quad \text{where} \quad \frac{1}{q} = \frac{1}{2} + \frac{\lambda}{d}.
\]

Since \( 1 < q < 2 \) in the present case of \( 0 < \lambda < d/2 \), the Hausdorff-Young type inequality (2.2) gives

\[
\|\tilde{M}_\alpha\|_{L^2} \lesssim \|f\|_{\dot{H}^\alpha} \|g\|_{L^1}^{1-\theta} \|g\|_{H^\alpha}^\theta \quad \text{where} \quad \theta = \frac{d + 2\lambda}{d + 2\alpha},
\]
provided \( \alpha > \lambda \). Interchanging the roles of \( f, g \), we also get

\[
\|\tilde{N}_\alpha\|_{L^2} \lesssim \|g\|_{\dot{H}^\alpha} \|f\|_{L^1}^{1-\theta} \|f\|_{H^\alpha}^\theta.
\]

It follows from (5.8) that

\[
\|u + v|^\alpha \hat{F}\|_{L^2} \lesssim \|f\|_{\dot{H}^\alpha} \|g\|_{L^1}^{1-\theta} \|g\|_{H^\alpha}^\theta + \|g\|_{\dot{H}^\alpha} \|f\|_{L^1}^{1-\theta} \|f\|_{H^\alpha}^\theta. \tag{5.9}
\]

Differentiating under the integral sign and proceeding in the same way as above, it is straightforward to obtain

\[
\|u + v|^\alpha \nabla_1 \hat{F}\|_{L^2} \lesssim \|v^\alpha f\|_{\dot{H}^\alpha} \|g\|_{L^1}^{1-\theta} \|g\|_{H^\alpha}^\theta + \|g\|_{\dot{H}^\alpha} \|v^\alpha f\|_{L^1}^{1-\theta} \|v f\|_{H^\alpha}^\theta,
\]
\[
\|u + v|^\alpha \nabla_2 \hat{F}\|_{L^2} \lesssim \|f\|_{\dot{H}^\alpha} \|v g\|_{L^1}^{1-\theta} \|v g\|_{H^\alpha}^\theta + \|v g\|_{\dot{H}^\alpha} \|f\|_{L^1}^{1-\theta} \|f\|_{H^\alpha}^\theta. \tag{5.10}
\]

Since \( L^1, \dot{H}^\alpha \) norms are dominated by \( L^1, H^\alpha \) norms, respectively, in all of the estimates (5.9)-(5.10), we obtain the desired conclusion at once upon inserting those estimates into (5.7) and taking square roots on both sides.
Remark 5.1. (a) As the proof clearly indicates, the explicit formula (5.6) plays a key role in proving Theorem 5.1. On the contrary, we remark that the result (5.1) of Bouchut & Desvillettes was obtained without evaluating \( \hat{F} \) explicitly. As a matter of fact, the identity (3.5) shows that \( \hat{F} \) is the Fourier transform of \( F(v, v_s) = |v - v_s|^{-\lambda} f(v)g(v_s) \) with phase variables \((\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d\). One may apply Plancherel’s theorem and the fractional integration theorem to deduce
\[
\|\hat{F}\|_{L^2} = \|F\|_{L^2} \lesssim \|f\|_{L^{p_\lambda}} \|g\|_{L^{p_\lambda}}.
\]
Combining with similar estimates for the gradient terms, the estimate (5.7) with \( \alpha = 0 \) yields a bilinear version of (5.1) in the form
\[
\|Q^+(f, g)\|_{H^{\alpha+, \beta+}} \lesssim \|b\|_2 \left(\|f\|_{L^{p_\lambda}} \|g\|_{L^{p_\lambda}} + \|f\|_{L^{p_\lambda}} \|g\|_{L^{p_\lambda}}\right). \tag{5.11}
\]
(b) The problem of \( L^p \) boundedness of \( Q^+ \) is not pursued here for avoiding unnecessarily long presentations. Incidentally, that \( Q^+(f, g) \) is bounded in \( L^p \) for all \( 1 \leq p \leq 2 \) under the same assumptions on \( f, g \) as above, is an immediate consequence of Sobolev type inequality (2.4) in view of Theorem 5.1 and the \( L^1 \) estimate (3.3).
(c) In Theorem 5.1, the minimally required regularity \( \alpha \) is smaller than \( (d-1)/2 \) only when \( 0 < \lambda < (d-1)/2 \).

5.2. An extension of Theorem 4.5. Under the assumption \( b \in L^2(S^{d-1}) \), the regularity results in Theorem 4.5 are valid only for the potential order \( \lambda \) with \( (d+1)/2 < \lambda < d \). We now extend the range to \( \lambda > 1/2 \) at the expense of gaining less regularity.

Theorem 5.2. Assume that \( B = |v - v_s|^{-\lambda} b(k \cdot \sigma) \) where \( b \in L^2(S^{d-1}) \) and \( 1/2 < \lambda < d \). If \( f, g \in L^1 \cap H^\alpha(\mathbb{R}^d) \) with \( \alpha > \max(d/4, \lambda - d/2) \), then
\[
\|Q^+(f, g)\|_{H^{\alpha+, \beta+}} \lesssim \|b\|_{L^2(S^{d-1})} \left\{\|f\|_{H^\alpha} \|g\|_{L^1}^{\theta} \|g\|_{H^\alpha}^{\bar{\theta}} + \|g\|_{H^\alpha} \|f\|_{L^1}^{\bar{\theta}} \|f\|_{H^\alpha}^{\theta}\right\}
\]
where \( \theta = \frac{d}{d+2\alpha} \in (0, 1) \). \tag{5.12}

Proof. A well-known identity for the beta integral states that
\[
\int_{\mathbb{R}^d} |x - y|^{-\alpha+\beta} \, dy = C(\alpha, \beta, d) |x|^{-d+\alpha+\beta} \tag{5.13}
\]
for \( 0 < \alpha, \beta < d \) with \( \alpha + \beta < d \), where
\[
C(\alpha, \beta, d) = \pi^{d/2} \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\theta}{2}\right)}{\Gamma\left(\frac{\alpha+\beta}{2}\right)} \cdot \frac{\Gamma\left(\frac{d-\alpha-\beta}{2}\right)}{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d-\beta}{2}\right)}
\]
(see [12] for instance).

Let us fix momentarily a number \( \beta \) satisfying
\[
\max\left(\frac{d}{2}, \lambda\right) < \beta < \min\left(\lambda + \frac{d-1}{2}, d\right). \tag{5.14}
\]
An application of the identity (5.13) yields
\[
\left(\int_{\mathbb{R}^d} |\eta - |\xi|\sigma|^{-2(d-\beta)} |\eta|^{-\beta} \, d\eta\right)^{1/2} \lesssim |\xi|^{-\frac{(\alpha+\beta)}{2}}.
\]
It follows from the Cauchy-Schwarz inequality that
\[ \int_{\mathbb{R}^d} |\eta - |\xi||^{-d+\lambda} |f(\xi + \eta)\hat{g}(\xi - \eta)| \, d\eta \]
\[ \lesssim |\xi|^{-\frac{d-2}{2}} \left( \int_{\mathbb{R}^d} |\eta - |\xi||^{2(\lambda - \beta)} |\eta|^\beta |f(\xi + \eta)\hat{g}(\xi - \eta)|^2 \, d\eta \right)^{1/2}. \]

Since \(2(\beta - \lambda) < d - 1\) for \(\beta < \lambda + (d - 1)/2\), which is justified by the condition (5.14), Lemma 2.4 is applicable and gives
\[ \int_{\mathbb{R}^d} |\eta - |\xi||^{2(\lambda - \beta)} \, d\sigma \lesssim (|\xi| + |\eta|)^{2(\lambda - \beta)}. \]

Using these estimates, we apply the Cauchy-Schwarz inequality to bound
\[ \left| \hat{Q}^+(\xi) \right| \]
\[ \leq \int_{\mathbb{R}^d} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( \int_{\mathbb{R}^d} |\eta - |\xi||^{-d+\lambda} |f(\xi + \eta)\hat{g}(\xi - \eta)| \, d\eta \right) \, d\sigma \]
\[ \lesssim \|b\|_2 \left| \xi \right|^{-\frac{d-4}{2}} \left( \int_{\mathbb{R}^d} (|\xi| + |\eta|)^{2(\lambda - \beta)} |\eta|^\beta |f(\xi + \eta)\hat{g}(\xi - \eta)|^2 \, d\eta \right)^{1/2}. \]

For any \(\delta > 0\), note that
\[ (|\xi| + |\eta|)^{2(\lambda - \beta)} |\eta|^\beta \leq |\xi|^{2(\lambda - \beta)} |\eta|^\beta \]
\[ \leq |\xi|^{2(\lambda - \beta - \delta)} \left| \xi \right|^\delta |\eta|^\beta \]
\[ \lesssim |\xi|^{2(\lambda - \beta - \delta)} (|\xi + \eta|^{\beta + \delta} + |\xi - \eta|^{\beta + \delta}). \]

Making use of this elementary inequality, we deduce
\[ \int_{\mathbb{R}^d} |\xi|^{d+\beta+\delta-2\lambda} \left| \hat{Q}^+(\xi) \right|^2 \, d\xi \]
\[ \lesssim \|b\|_2^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|\xi + \eta|^{\beta + \delta} + |\xi - \eta|^{\beta + \delta}) |f(\xi + \eta)\hat{g}(\xi - \eta)|^2 \, d\eta d\xi \]
\[ \lesssim \|b\|_2^2 \left( \|f\|_{\dot{H}^{\frac{\alpha + \delta}{2}}}^2 \|\hat{g}\|_{L^2}^2 + \|g\|_{\dot{H}^{\frac{\alpha + \delta}{2}}}^2 \|\hat{f}\|_{L^2}^2 \right) \]
where the letter follows from changing variables \(\xi + \eta \to u, \xi - \eta \to v\). In summary, we have proved
\[ \|Q^+(f, g)\|_{\dot{H}^{\frac{\alpha + \delta}{2} - \lambda}} \lesssim \|b\|_2 \left( \|f\|_{\dot{H}^{\frac{\alpha + \delta}{2}}} \|\hat{g}\|_{L^2} + \|g\|_{\dot{H}^{\frac{\alpha + \delta}{2}}} \|\hat{f}\|_{L^2} \right). \quad (5.15) \]

Setting \(\alpha = (\beta + \delta)/2\) and applying the Hausdorff-Young type inequality (2.2) for \(L^2\) norms, we obtain the desired estimate (5.12).

\textbf{Remark 5.2.} In Theorem 5.2, the collision operator \(Q^+\) gains regularity only when \(1/2 < \lambda < d/2\). It is obvious that Theorem 4.5 gives better regularity results than Theorem 5.2 in the case of \(\lambda > (d + 1)/2\) and hence we may restrict our attention to the case of \(1/2 < \lambda \leq (d + 1)/2\) in the statement of Theorem 5.2.
5.3. Regularity under \( b \in L^1(\mathbb{S}^{d-1}) \). While all of our preceding results are established under \( b \in L^s(\mathbb{S}^{d-1}) \) for some \( s > 1 \), it is possible to extend Theorem 5.2 under the original Grad’s cutoff condition \( b \in L^1(\mathbb{S}^{d-1}) \) for certain range of \( \lambda \). Note that the condition \( b \in L^s(\mathbb{S}^{d-1}) \) is stronger than the condition \( b \in L^1(\mathbb{S}^{d-1}) \) for any \( s > 1 \) in view of Hölder’s inequality.

**Theorem 5.3.** Assume that \( B = |v - v_*|^{-\lambda} b(k \cdot \sigma) \) where \( b \in L^1(\mathbb{S}^{d-1}) \) and \( d/2 < \lambda < d \). If \( f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d) \) with \( \alpha > \lambda/2 \), then
\[
\| Q^+(f, g) \|_{\dot{H}^{\alpha+}(\frac{d}{2})} \lesssim \| b \|_{L^1(\mathbb{S}^{d-1})} \left\{ \| f \|_{\dot{H}^{\alpha}} \| g \|_{\dot{H}^{\alpha}}^{1-\theta} + \| g \|_{\dot{H}^{\alpha}} \| f \|_{\dot{H}^{\alpha}}^{1-\theta} \right\}
\]
where \( \theta = \frac{d}{d + 2\alpha} \in (0, 1) \).

**Proof.** In the proof of Theorem 5.2, if we choose \( \beta = \lambda \) and proceed in the same way, then we get
\[
\left| \hat{Q^+}(\xi) \right| \lesssim \| b \|_1 \| \xi \|^{-\left(\frac{d-\lambda}{2}\right)} \left( \int_{\mathbb{R}^d} |\eta \lambda|^2 |\hat{f}(\xi + \eta)\hat{g}(\xi - \eta)|^2 \, d\eta \right)^{1/2}.
\]
Estimating \(|\eta \lambda|^2\) in the form
\[
|\eta \lambda|^2 \lesssim |\xi|^{-\delta} (|\xi + \eta \lambda|^{\lambda+\delta} + |\xi - \eta \lambda|^{\lambda+\delta})
\]
for any \( \delta \geq 0 \), it is straightforward to deduce
\[
\| Q^+(f, g) \|_{\dot{H}^{\alpha+}(\frac{d}{2})} \lesssim \| b \|_1 \left( \| f \|_{\dot{H}^{\alpha}} \| g \|_{\dot{H}^{\alpha}} + \| g \|_{\dot{H}^{\alpha}} \| f \|_{\dot{H}^{\alpha}} \right) \cdot
\]
Upon setting \( \alpha = (\lambda + \delta)/2 \), we obtain the desired result. \( \square \)

**Corollary 5.1.** Assume that \( B = |v - v_*|^{-\lambda} b(k \cdot \sigma) \) where
\[
\| b \|_* = \int_{\mathbb{S}^{d-1}} b(k \cdot \sigma) (1 - k \cdot \sigma)^{-\left(\frac{d-\lambda}{2}\right)} \, d\sigma < \infty
\]
and \( d/2 < \lambda < d \). If \( f, g \in L^1 \cap \dot{H}^\alpha(\mathbb{R}^d) \) with \( \alpha > \lambda/2 \), then
\[
\| Q^+(f, g) \|_{\dot{H}^{\alpha+}(\frac{d}{2})} \lesssim \| b \|_* \left\{ \| f \|_{\dot{H}^{\alpha}} \| g \|_{\dot{H}^{\alpha}}^{1-\theta} + \| g \|_{\dot{H}^{\alpha}} \| f \|_{\dot{H}^{\alpha}}^{1-\theta} \right\}
\]
where \( \theta = \frac{d}{d + 2\alpha} \in (0, 1) \).

**Proof.** Use the beta integral estimate
\[
\left( \int_{\mathbb{R}^d} |\eta |^{\lambda} |\xi |^{2(d-\lambda)} |\xi - \eta |^{-\lambda} \eta \right)^{1/2}
\]
\[
\lesssim |\xi - \xi |^{\left(\frac{d-\lambda}{2}\right)} \approx |\xi|^{-\left(\frac{d-\lambda}{2}\right)} \left( 1 - \frac{\xi}{|\xi|} \cdot \sigma \right)^{-\left(\frac{d-\lambda}{2}\right)}
\]
to bound the Fourier transform
\[
\left| \hat{Q^+}(\xi) \right| \lesssim \| b \|_* \| \xi \|^{-\left(\frac{d-\lambda}{2}\right)} \left( \int_{\mathbb{R}^d} |\xi - \eta |^{\lambda} |\hat{f}(\xi + \eta)\hat{g}(\xi - \eta)|^2 \, d\eta \right)^{1/2},
\]
and proceed as in the proofs of Theorems 5.2–5.3. \( \square \)
Remark 5.3. The condition (5.18) is stronger than $b \in L^1(S^{d-1})$ but weaker than the condition (4.6) for any $a > 1$ which appears in the work of Alonso, Carneiro & Gamba.

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