ON TRANSVERSALS OF QUASIALGEBRAIC FAMILIES OF SETS

G.R. CHELNOKOV, V.L. DOL’NIKOV

Yaroslavl State University, Sovetskaya Str. 14, Yaroslavl, 150000, Russia

Abstract. The main results of this paper are generalizations some classical theorems about transversals for families of finite sets to some cases of families of infinite sets.

1. Statements of the problems and the results.

In this paper we consider Helly-Gallai numbers for families of sets that are similar to families of sets which are solutions for finite systems of equations.

Definition 1. A set $X$, $|X| \leq t$, is called a $t$-transversal of a family of sets $F$ if $A \cap X \neq \emptyset$ for every $A \in F$. By $\tau(F)$ denote the least positive integer $t$ such that there exists a $t$-transversal of the family $F$. This number $\tau(F)$ is called the transversal number (or piercing number) of $P$ (see [1,10]).

Definition 2. The Helly-Gallai number $HG(t, F)$ of a family of sets $F$ is called a minimal number $k$ such that if every subfamily $P \subseteq F$ with $|P| \leq k$ has a $t$-transversal, then the family $F$ has a $t$-transversal (see [1,2,10]).

For every family $F$, the existence of a 1-transversal is equivalent to the condition that the intersection of all sets of $F$ is nonempty. Therefore a number $HG(1,F)$ is called a Helly number $H(F)$ for a family $F$.

Remark. If $F$ is a family of intervals on the line, then $HG(t, F) = t + 1$. If $F$ is a family of a convex compact sets in $\mathbb{R}^d$, $d \geq 2$ and $t \geq 2$, then $HG(1,F) = d + 1$ and numbers $HG(t,F)$ for $t \geq 2$ don’t exist.

The Helly numbers for a family of algebraic varities were found by T.S. Motzkin [3].

Definition 3. Let $A^d_m$ be a family of sets of common zeroes in $\mathbb{R}^d$ for a finite collection of polynomials of $d$ variables and degree at most $m$.
Motzkin’s Theorem.

\[ H(A_m^d) = \binom{m + d}{d}. \]

The Helly–Gallai numbers for algebraic varieties \( A_m^d \) were determined by M. Deza and P. Frankl [4], and V.Dol’nikov [5]. They are given by the formula:

\[ HG(t, A_m^d) = \binom{(m + d) + t - 1}{t}. \]

In the papers [5,6] the Helly–Gallai numbers for families of sets of more general kind were considered. More precisely, families were the zero sets of linear finite-dimensional subspaces of functions from a ground set \( V \) to in a field \( \mathbb{F} \).

In particular, the Helly–Gallai numbers

\[ HG(t, S_{d-1}) = \binom{d + t + 1}{t} \]

for families of spheres \( S_{d-1} \) in \( \mathbb{R}^d \) were found. Independently Helly numbers \( H(S_{d-1}) \) were found by H. Maehara [7].

Now we give some bounds for the Helly–Gallai numbers of quasialgebraic families of sets.

**Definition 4.** Let \( F \) be a family of sets. Denote inductively \( F^0 = F \) and

\[ F^{k+1} = \{ B : B = A_i \cap A_j, \text{ where } A_i, A_j \in F^k \text{ and } A_i \neq A_j \}. \]

**Definition 5.** A family of sets \( F \) has the \( [d, m] \)-property if \( |A| \leq d \) for every \( A \in F^m \). If a family \( F \) has \( [d, m] \)-property, then such family is called a quasialgebraic family of a dimension \( m \) and a degree \( d \). Denote the class of such families by \( QA_m^d \).

**Examples.** A family of lines in \( \mathbb{R}^d \), where \( \mathbb{F} \) is a field, or a family of lines of a finite projective plain, a family of all edges of a simple graph \( G \) or a family of sets of all edges for a simple graph \( G \) that contain a given vertex are quasialgebraic families of a dimension 1 and a degree 1.

Such families \( F \in QA_1^1 \) are called linear families in a literature. Let us remark that linear families investigated in different aspects (for example see the well-known conjecture of Erdős, Faber and Lovasz [8] or [9], ch. 9).

The family of circles is a quasialgebraic family of dimension 1 and degree 2. The family of finite sets of cardinality \( d \) is a quasialgebraic family of dimension 0 and degree \( d \), but also it may be considered as a family of dimension \( k \) and degree \( d - k \) for \( k \geq 0 \).

More generally, the families of sets, which were considered in [5,6], are a quasialgebraic families too.
Definition 6. A family of sets $F$ has the $\{d, m\}$-property if $|\bigcap_{A \in G} A| \leq d$ for every $G \subset F, |G| = m$.

Remark. The family of hyperplanes in $\mathbb{F}^m$, where $\mathbb{F}$ is an arbitrary field, has the $[1, m]$-property. The family of hyperplanes in $\mathbb{F}^m$ in general position has the $\{1, m\}$-property.

Further just a few notes about $\{d, m\}$-property will be made and this notion will not be used for achieving main results.

H. Hadwiger and H. Debrunner introduced the following concepts [1], see also [10].

Definition 7. Let $p$ and $q$ be integers with $p \geq q \geq 2$. We say that a family of sets $F$ has the $(p, q)$-property (in this case we write $F \in \Pi_{p,q}$) provided $F$ has at least $p$ members and among every $p$ members of $F$ some $q$ of them have a common point.

By $M(p, q; m)$ we denote $\sup_{F} \tau(F)$ for all finite families of convex sets $F$ in $\mathbb{R}^m$ such that $F \in \Pi_{p,q}$. N. Alon and D. Kleitman [11] proved that if $m + 1 \leq q \leq p$, then $M(p, q; m) < \infty$.

By $P(p, q; m)$ we denote $\sup_{F} \tau(F)$ for all finite families of hyperplanes in $\mathbb{R}^m$ such that $F \in \Pi_{p,q}$.

Second author proposed a conjecture [12] (see also Oberwolfach sept.2011) that

$$P(p, q; m) = p - q + 1 \quad \text{if} \quad m + 1 \leq q \leq p.$$ 

The main purpose of this paper is to propose some general problems of quasialgebraic families. In the paper the following theorems are proved.

**Theorem 1.**

$$\binom{d + m + t}{d + m} \leq HG(t, QA^d_m) \leq \sum_{i=0}^{d+m} t^i.$$ 

**Corollary.** $H(QA^d_m) = d + m + 1$.

Indeed, if $t = 1$, then the upper and lower estimations of Theorem 1 coincide. Thus $H(QA^d_m) = d + m + 1$.

**Remark.** Note that the upper and lower bounds have the same multiplicity $t^{d+m}$ by $t$.

**Theorem 2.**

$$\binom{d + 3}{2} \leq HG(2, QA^d_1) \leq \max \left(2d^2 + 3, \binom{d + 3}{2} \right),$$

and

$$\binom{d + 3}{2} = HG(2, QA^d_1)$$

for $d = 1, 2, 3^1$.

$^1$First author has a proof of this equality for all $d \leq 7$. 

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Theorem 3. \[
\left( \frac{t+2}{2} \right) \leq HG(t, QA_1^1) \leq \max \left( t^2 - t + 3, \left( \frac{t+2}{2} \right) \right).
\]

Remark. If \( F \) is family of lines in \( \mathbb{F}^d \), where \( \mathbb{F} \) is a field, or a family of lines in a projective space over a field \( \mathbb{F} \), then
\[
HG(t, F) = \left( \frac{t+2}{2} \right) \quad [6].
\]
But if \( F \) is a family of lines of an arbitrary finite projective plain, then the authors don’t know uppers estimates better than in Theorem 3.

Theorem 4. If \( F \in QA_1^1, F \in \Pi_{p,q} \) and \( |F| \geq \left( \frac{p-q+3}{2} \right) \left( \left( \frac{p-q+2}{2} \right) - 1 \right) + p - q + 4 \), then
\[
\tau(F) \leq p - q + 1.
\]

2. Basic Definitions and Notations

Definition 8. A family of sets \( F \) has the \( t \)-property if a subfamily \( F \setminus \{A\} \) has a \( t \)-transversal \( X(A) \) for every \( A \in F \) such that \( X(A) \cap A = \emptyset \). Such \( t \)-transversal \( X(A) \) of \( F \setminus \{A\} \) is called a representation of the set \( A \).

Remark. Note that if \( A \neq B \in F \), then \( X(A) \neq X(B) \).

Definition 9. By \( a(d, m; t) \) denote the maximal \( HG(t, F) \) for a family \( F \) having the \([d, m]\)-property. By \( b(d, m; t) \) denote the maximal cardinality of a family \( F \) having the \([d, m]\)-property and the \( t \)-property. By \( c(d, m; t) \) denote the maximal cardinality of a family \( F \) having the \([d, m]\)-property and the \( t \)-property.

Remark. Later we’ll prove that \( a(d, m; t) \), \( b(d, m; t) \) and \( c(d, m; t) \) are finite for every triple \((d, m, t)\).

Proposition 1. \( b(d, m; t) \geq a(d, m; t) \).

Proof. Assume the converse \( a(d, m; t) > b(d, m; t) \). Then there exists a family \( F \) with the \((d, m)\)-property and the \( t \)-property such that every \( a(d, m; t) - 1 \) sets of \( F \) have a \( t \)-transversal, but \( F \) doesn’t. Consider a subfamily \( F' \subseteq F \) of the minimal cardinality that doesn’t have a \( t \)-transversal. Then \( |F'| > a(d, m; t) - 1 \geq b(d, m; t) \). Since \( F' \) is a minimal subfamily, it follows that every \( G \subset F' \) has a \( t \)-transversal. Thus \( F' \) has the \( t \)-property. Also, \( F' \) has the \([d, m]\)-property as a subfamily of \( F \). Hence the inequality \( |F'| > b(d, m; t) \) contradicts the definition of \( b(d, m; t) \).

Therefore the upper bounds for \( b(d, m; t) \) are also valid for \( a(d, m; t) \).
Proposition 2.

\[ b(d, m; t) \geq \binom{d + m + t}{t}. \]

Proof. Let \( F \) be a family of all \((d+m)\)-element subsets of a \((d+m+t)\)-element set. Obviously, the family \( F \) has the \([d, m]\)-property and the \( t \)-property. Clearly, \( F \setminus \{ A \} \) has a \( t \)-transversal for all \( A \in F \). Also, \( F \) does not have a \( t \)-transversal. It follows that

\[ a(d, m; t) \geq \binom{d + m + t}{t}. \]

Definition 10. Let \( F \) be a family of sets with the \( t \)-property. For a set \( W \) we denote

\[ G_W = \{ A : A \in F, \ W \subseteq A \} \quad \text{and} \quad G_W^k = \{ A : A \in F^k, \ W \subseteq A \}. \]

In particular, when \( W \) is a one element set \( W = \{ x \} \) we write \( G_x \) and \( G_x^k \) respectively. Also, we denote

\[ H_x = \{ A : A \in F, \ x \in X(A) \}. \]

Definition 11. A set \( B \) is called a proper set for the family \( F \) if there exists a non-empty subfamily \( H \subset F \) such that

\[ B = \bigcap_{A \in H} A \neq \emptyset. \]

Definition 12. Consider a set \( W \neq \emptyset \). Let \( B \) be a minimal proper set such that \( W \subset B \). By codim(\( W \)) denote the maximal number \( k \) such that \( B \in F^{k-1} \). If such \( B \) does not exists we say that codim(\( W \)) = 0. If \( F \) has the \([0, m]\)-property, then codim(\( W \)) < \( \infty \) of every nonempty set \( W \).

3. Proof of theorem 1

Lemma 1. \( b(0, 1; t) = t + 1 \).

Proof. Indeed, the \([0, 1]\)-property of a family \( F \) means that each two sets of \( F \) has an empty intersection. But \( F \setminus \{ A \} \) has a \( t \)-transversal, then it has at most \( t \) sets and \(|F| \leq t + 1\).

By proposition 2 \( b(0, 1; t) = t + 1 \).

Lemma 2. \( b(d, 1; 1) = d + 2 \).

Proof. By proposition 2 it is sufficient to prove \( b(d, 1; 1) \leq d + 2 \).

Suppose a family \( F \) has the \([d, 1]\)-property and \(|F| \geq d + 3 \). Consider different sets \( A_1, \ldots, A_{d+3} \in F \). Each representation \( X(A_i) \) consists of one element \( x_i \in \cap_{j \neq i} A_j \), and \( x_i \neq x_j \) if \( i \neq j \). Then

\[ A_{d+2} \cap A_{d+3} \supset \{ x_1, \ldots, x_{d+1} \}, \]

This contradiction completes the proof.
Lemma 3. Let $F$ be a family of sets, with the $(d, m)$-property and the $t$-property. Then $|G_x| \leq b(d-1, m; t)$ for all $x$. Also, $b(d, m; t) \leq tb(d-1, m; t) + 1$.

Proof. Consider the family of sets

$$E = \{A \setminus \{x\} : A \in G_x\}.$$  

Let $B \in E^m$. Obviously, $B \cup \{x\} \in G_x^m \subset F^m$. Then $|B \cup \{x\}| \leq d$, therefore $|B| \leq d - 1$. That means the family $E$ has the $[d, m]$-property.

Consider a set $A \in E$. Since $A \cup \{x\} \in F$, we may take $X(A \cup \{x\})$. Then we have that $x \notin X(A \cup \{x\})$ because $X(A \cup \{x\}) \cap (A \cup \{x\}) = \emptyset$. Thus the set $X(A \cup \{x\})$ is also a $t$-representation of the set $A$ in the family $E$. That means the family $E$ has the $t$-property.

Then $|G_x| = |E| \leq b(d, m; t)$.

Take $A \in F$ and $X(A) = \{x_1, \ldots, x_t\}$. Recall that

$$F = \{A\} \cup G_{x_1} \cup \cdots \cup G_{x_t},$$

then $|F| \leq 1 + tb(d, m; t)$.

Arguing as above, we see that.

Lemma 4. Let $F$ be a family of sets, with the $[d, m]$-property and the $t$-property. Then

$$|G| \leq b(d - | \bigcap_{A \in G} A|, m; t) \text{ for every } G \subset F.$$  

Lemma 5. Let $F$ be a family of sets, with the $[d, m]$-property and the $t$-property. Then

$$|G| \leq c(d - | \bigcap_{A \in G} A|, m; t) \text{ for every } G \subset F.$$  

Lemma 6. Let $B$ be a proper set. Then codim$(B)$ equals the maximal number $k$, for which there exist proper sets $B_1, \ldots, B_k$ such that $B = \bigcup B_i$.

The following inequality for $|G_x|$ is a core step in our proof of Theorem 1.

Lemma 7. Let a family $F$ has the $[0, m]$-property and the $t$-property. Suppose codim$(W) = k$ for a set $W$. Then

$$|G_W| \leq t^{k-1} + \cdots + t + 1.$$

In particular, 

$$|G_x| \leq t^{m-1} + t^{m-2} + \cdots + t + 1. \text{ Consequently } b(0, m; t) \leq t^m + t^{m-1} + \cdots + t + 1.$$
Proof. Induction on \(k\). Base for \(k = 1\) is obvious.

Suppose the statement is proved for all sets of the codimension \(\leq k - 1\). Consider a set \(W\) such that \(\operatorname{codim}(W) = k\). Assume the converse

\[
|G_W| \geq t^{k-1} + \cdots + t + 2.
\]

Let \(\{A_1, \ldots, A_s\} = G_W\). Since \(|G_x \setminus \{A_s\}| = s - 1\), it follows that there exists \(y \in X(A_s)\), which belongs to at least

\[
\left\lfloor \frac{s - 1}{t} \right\rfloor = t^{k-2} + \cdots + t + 2
\]

different sets of \(G_x\). Let \(\bigcap_{y \in A_i} A_i = U\). Obviously, \(W \subseteq U, y \in U\), but \(y \notin W\), thus

\[
\operatorname{codim}(U) \leq \operatorname{codim}(W) - 1 = k - 1.
\]

This contradiction concludes the proof of the first statement of Lemma 7.

Since \(\{x\}\) isn’t empty, \(\operatorname{codim}(\{x\}) \leq m\). Finally, we obtain

\[
|F| \leq t|G_x| + 1 \leq t^m + t^{m-1} + \cdots + t + 1.
\]

Lemma 8.

\[
b(d, m; t) \leq \sum_{i=0}^{d+m} t^i.
\]

Proof. By Lemmas 3 and 7

\[
b(0, m, t) \leq t|G_x| + 1 \leq t^m + t^{m-1} + \cdots + t + 1.
\]

Applying recursively the inequality \(b(d, m, t) \leq tb(d - 1, m, t) + 1\) from Lemma 3 we get

\[
b(d, m, t) \leq t^{m+d} + t^{m+d-1} + \cdots + t + 1 = \frac{t^{m+d+1} - 1}{t - 1}.
\]

Proof of Theorem 1. The first inequality follows from proposition 1 and Lemma 8. The second inequality is provided by the example in the proof of proposition 2.

4. Proof of theorem 2

In following three parts we will consider quasialgebraic families for dimension 1 and degree 1 (lines); and for dimension 1, arbitrary degree and \(t = 2\).
Lemma 9. Let $F$ be a family with the $[d, 1]$-property and the 2-property. If

$$|F| \geq b(d - 1, 1, 2) + d + 3,$$

then

$$|H_x| \leq d \text{ and } |F \setminus G_x| - |H_x| \leq b(d - |H_x|, 1, 2)$$

for every $x$.

Proof. Let $H_x = \{A_1, \ldots, A_r\}$ and

$$A_1 = \{x, x_1\}, A_2 = \{x, x_2\}, \ldots, A_r = \{x, x_r\}.$$

If $|H_x| \geq d + 1$, then $(F \setminus G_x) \setminus \{A_1, \ldots, A_{d+1}\}$ contains at least two sets $B$ and $C$. Therefore

$$B \cap C \supset \{x_1, \ldots, x_{d+1}\}.$$

This contradiction proves that $|H_x| \leq d$.

Obviously

$$\{x_1, \ldots, x_r\} \subset A \text{ for every } A \in F \setminus (G_x \cup H_x).$$

Applying Lemma 4 we have

$$|F \setminus (G_x \cup H_x)| \leq b(d - \bigcap_{A \in F \setminus (G_x \cup H_x)} |A|, 1, 2) \leq b(d - r, 1, 2) = b(d - |H_x|, 1, 2).$$

Lemma 10. Let $F$ be a family of sets with the $[1, 1]$-property and the $t$-property. If $|G_x| = t + 1$ for some $x$, then $x \in X(A)$ for every $A \in F \setminus G_x$ and the family $F \setminus G_x$ has the $(t - 1)$-property. Therefore $|F| \leq t + 1 + b(1, 1, t - 1)$.

Proof. Assume the converse, i.e.

$$G_x = \{A_1, \ldots, A_{t+1}\},$$

but $x \notin X(B)$ for some $B \in F \setminus G_x$. Let $X(B) = \{x_1, \ldots, x_t\}$. Since each set $A_i \cap X(B) \neq \emptyset$ for $1 \leq i \leq t+1$, it follows that some $x_i$ belongs to at least two sets. Without loss of generality $x_1 \in A_1 \cap A_2$. Then $\{x, x_1\} \in A_1 \cap A_2$. This contradiction completes the proof.

Thus $x \in X(A)$ for every $A \in F \setminus G_x$. It is not hard to prove that $F \setminus G_x$ has the $(t - 1)$-property. Indeed, for every $A \in F \setminus G_x$ the family $F \setminus (G_x \cup |A|)$ has the $(t - 1)$-transversal $X(A) \setminus \{x\}$ and $A \cap (X(A) \setminus \{x\}) = \emptyset$. 
Lemma 11. \( b(1, 1, 2) = 6 \)

Proof. Let \( X(A) = \{x, y\} \) for some \( A \in F \). By Lemma 3 we have \( |G_x| \leq b(0, 1, 2) = 3 \). If \( |G_x| = 3 \) for some \( x \), then by Lemma 9

\[
|F| \leq 3 + b(1, 1, 1) = 6.
\]

Suppose \( |G_x| < 3 \) for every \( x \), then

\[
|F| \leq 1 + G_x + G_y \leq 5.
\]

Finally, \( b(1, 1, 2) \leq 6 \), hence by Proposition 2 \( b(1, 1, 2) = 6 \).

Lemma 12. Let \( F \) be a family of sets with the \([1, 1]\)-property and the \( t \)-property. Suppose \( |G_x| = |G_y| = |G_z| = t \) and \( G_x, G_y, G_z \) are mutually disjoint. Then each representation \( X(A) \), \( A \in F \), contains either all or none of elements \( x, y, z \) for every \( A \in F \).

Proof.

Consider \( x \in X(A) \) for some \( A \in F \) and \( X(A) = \{x, x_1, x_2, \ldots, x_{t-1}\} \). Suppose \( y, z \notin X(A) \). Since \( G_y \cap G_z = \emptyset \), it follows that \( A \notin G_y \) or \( A \notin G_z \). Without loss of generality \( A \notin G_y \). Using \( G_x \cap G_y = \emptyset \), we get \( x \notin B \) for every \( B \in G_y \). Then \( B \cap \{x, x_1, x_2, \ldots, x_{t-1}\} \neq \emptyset \) for each \( B \in G_y \). So there exist \( B_1, B_2 \in G_y \) such that \( B_1 \cap B_2 \cap \{x, x_1, x_2, \ldots, x_{t-1}\} \neq \emptyset \). Wlog \( B_1 \cap B_2 \cap \{x, x_1, x_2, \ldots, x_{t-1}\} = \{x_1\} \). Then \( x_1, y \in B_1 \cap B_2 \). This contradiction proves \( y \in X(A) \).

Let \( X(A) = \{x, y, x_1, \ldots, x_{t-2}\} \). Since \( |G_z \setminus \{A\}| \geq t - 1 \) and every set of \( G_z \setminus \{A\} \) contains some element of \( \{x_1, \ldots, x_{t-2}\} \), it follows that two sets of \( G_z \setminus \{A\} \) contains same element. Consequently \( z \in \{x_1, \ldots, x_{t-2}\} \).

Arguing as above we get the following statement.

Lemma 13. Let \( F \) be a family of sets with the \([1, 1]\)-property and the \( t \)-property. Suppose \( |G_x| = |G_y| = t \) and \( G_x \cap G_y = \emptyset \). Then each representation \( X(A) \), where \( A \in F \setminus (G_x \cup G_y) \), contains either all or none of elements \( x, y \).

Definition 13. Let \( B \neq C \in F \) and \( x \in B \cap C \). Triple \((B, C, x)\) corresponds to a representation \( X(A) \) if \( x \in X(A) \).

Definition 14. Consider a representation \( X(A) \). By the price \( P(X(A)) \) we denote the following sum

\[
P(X(A)) = \sum_{x \in X(A)} \left( \frac{|G_x|}{2} \right) / |H_x|.
\]

Next two statements are obvious corollaries of definitions.
Lemma 14. The sum \( \sum_{A \in F} P(X(A)) \) equals to the number of triples, corresponding to at least one representation.

Lemma 15. The number of all triples is \( \sum_{B \neq C \in F} |B \cap C| \).

Lemma 16. \( b(d, 1, 2) \leq 2d^2 + 3 \) for \( d \geq 2 \).

Proof. Assume the converse. Let \( d \) be a minimal number such that \( b(d, 1, 2) > 2d^2 + 3 \) and \( d \geq 2 \). Consider the following cases.

First case
\[
b(d, 1, 2) \leq b(d - 1, 1, 2) + d + 2.
\]
Furthermore, if \( d > 2 \), then
\[
b(d - 1, 1, 2) \leq 2(d - 1)^2 + 3,
\]
thus
\[
b(d, 1, 2) \leq 2(d - 1)^2 + 3 + d + 2 < 2d^2 + 3.
\]
If \( d = 2 \), using \( b(1, 1, 2) = 6 \) from Lemma 11, we get \( b(2, 1, 2) \leq 6 + 4 < 11 \). The case is proven.

Assume the opposite case \( b(d, 1, 2) \geq b(d - 1, 1, 2) + d + 3 \).

Consider a family \( F \) with the \([d, 1]\)-property and the 2-property of maximal cardinality. Denote \( |F| = b(d, 1, 2) = b \). Since
\[
|B \cap C| \leq d \quad \text{for every} \quad B, C \in F, B \neq C,
\]
it follows that
\[
\sum_{B, C \in F, B \neq C} |B \cap C| \leq \frac{d(b - 1)}{2}.
\]

By the other hand. Suppose \( X(A) = \{x, y\} \) for some \( A \in F \). From
\[
G_x \bigcup G_y = F \setminus \{A\} \quad \text{deduce} \quad |G_x| + |G_y| \geq b - 1.
\]

Also, by Lemma 9 \( |H_x| \leq d, |H_y| \leq d \). Consequently
\[
P(X(A)) = \left( \frac{|G_x|}{2} \right)/|H_x| + \left( \frac{|G_y|}{2} \right)/|H_y| \geq \left( \left( \frac{|G_x|}{2} \right) + \left( \frac{|G_y|}{2} \right) \right)/d.
\]

By the convexity of the function \( \left( \frac{x}{2} \right) \)
\[
\left( \left( \frac{|G_x|}{2} \right) + \left( \frac{|G_y|}{2} \right) \right)/d \geq 2 \left( \frac{|G_x| + |G_y|}{2} \right)/d \geq \frac{(b - 1)(b - 3)}{4d}.
\]
Finally, by Lemma 14 the number of all triples, corresponding to some representation is at least \( \frac{b(b-1)(b-3)}{4d} \), and by Lemma 15 the number of all triples is at most \( \frac{b(b-1)}{2}d \). Thus we get
\[
\frac{b(b-1)(b-3)}{4d} \leq \frac{b(b-1)}{2}d,
\]
that is \( b \leq 2d^2 + 3 \).

The first part of Theorem 2 is provided by Lemma 16 and Proposition 2. For the proof of the second part we need the following lemmas.

**Lemma 17.** \( b(2,1,2) = 10 \).

*Proof.* By Proposition 2 we need to prove \( b(2,1,2) \leq 10 \). Consider a family \( F \) with the \([2,1]\)-property and the 2-property. Suppose \( |F| = 11 \).

By Lemmas 3 and 11 \( |G_x| \leq 6 \) for every \( x \). If \( |G_x| = 6 \), then \( |H_x| \leq 2 \) by Lemma 9. If \( |G_x| < 6 \), then \( |H_x| \leq 1 \) by Lemma 9.

It isn’t hard to see that \( P(X(A)) \geq 27/2 \) for every \( A \in F \). Indeed,

\[
\text{if } |G_x| = 6, \text{ then } \left( \frac{|G_x|}{2} \right)/|H_x| = 15/2, \text{ if } 4 \leq |G_y| \leq 6, \text{ then } \left( \frac{|G_y|}{2} \right)/|H_x| \geq 6.
\]

Consider a representation \( X(A) = \{ x, y \} \). If \( |G_x| = 6 \), then
\[
P(X(A)) = \left( \frac{|G_x|}{2} \right)/|H_x| + \left( \frac{|G_y|}{2} \right)/|H_y| \geq 15/2 + 6 = 27/2.
\]

If both \( |G_x| < 6 \) and \( |G_y| < 6 \), then \( |G_x| = |G_y| = 5 \) since \( |G_x| + |G_y| \geq 10 \). In this case
\[
\left( \frac{|G_x|}{2} \right)/|H_x| + \left( \frac{|G_y|}{2} \right)/|H_y| = 2 \left( \frac{5}{2} \right) = 20.
\]

So \( 11 \times 27/2 > 2 \frac{11 \times 10}{2} \), we got a contradiction with Lemmas 14 and 15.

Arguing as above, we get the following statement.

**Lemma 18.** \( b(3,1,2) = 15 \)

*Proof.* By Proposition 2 we need to prove \( b(3,1,2) \leq 15 \). Consider a family \( F \) with the \([3,1]\)-property and the 2-property. Suppose \( |F| = 16 \).

By Lemmas 3 and 17 \( |G_x| \leq 10 \) for every \( x \). If \( |G_x| = 10 \), then \( |H_x| \leq 3 \) by Lemma 9. If \( |G_x| < 10 \), then \( |H_x| \leq 2 \) and if \( |G_x| < 8 \), then \( |H_x| \leq 1 \) by Lemma 9. That is for values \( |G_x| = 5, 6, 7, 8, 9, 10 \) we obtain that the values
\[
\frac{|G_x|(|G_x| - 1)}{2|H_x|} \geq 10, 15, 21, 14, 18, 15
\]
respectively. For any representation \(\{x, y\}\) we have \(|G_x| + |G_y| \geq 15\).

We will prove

\[
\frac{|G_x|(|G_x| - 1)}{2|H_x|} + \frac{|G_y|(|G_y| - 1)}{2|H_y|} \geq 25.
\]

Assume the converse

\[
\frac{|G_x|(|G_x| - 1)}{2|H_x|} + \frac{|G_y|(|G_y| - 1)}{2|H_y|} < 25,
\]

also w.l.o.g.

\[
\frac{|G_x|(|G_x| - 1)}{2|H_x|} \geq \frac{|G_y|(|G_y| - 1)}{2|H_y|}.
\]

This implies

\[
\frac{|G_y|(|G_y| - 1)}{2|H_y|} < \frac{25}{2},
\]

which is possible only if \(|G_y| = 5\). In this case \(|G_x| = 10\), so

\[
\frac{|G_x|(|G_x| - 1)}{2|H_x|} + \frac{|G_y|(|G_y| - 1)}{2|H_y|} \geq 15 + 10 = 25.
\]

Then for every \(A \in F\)

\[
P(X(A)) \geq \frac{|G_x|(|G_x| - 1)}{2|H_x|} + \frac{|G_y|(|G_y| - 1)}{2|H_y|} \geq 25,
\]

which contradicts Lemmas 14 and 15.

Proposition 2 and Lemmas 11, 17 and 18 proves the second part of Theorem 2.

5. PROOF OF THEOREM 3

Lemma 19. \(b(1, 1, 3) = 10\)

The proof is word by word as the proof of Lemma 11.

Lemma 20. \(b(1, 1, 4) = 15\).

Proof. By Lemma 3 \(|G_x| \leq 5\). If \(|G_x| = 5\) for some \(x\), then by Lemma 10 we have

\[
|F| \leq |G_x| + b(1, 1, 3) = 15.
\]

If \(|G_x| < 5\) for every \(x\), then suppose w.l.o.g. \(|F| = 16\). Denote \(X(A) = \{x, y, z, t\}\) for some \(A \in F\). Since

\[
F \setminus \{A\} = G_x \cup G_y \cup G_z \cup G_t \quad \text{and all} \quad |G_x|, |G_y|, |G_z|, |G_t| \leq 4,
\]
it follows that some three of them are mutually disjoint and have exactly 4 elements each. Indeed, at least one of $G_x, G_y, G_z, G_t$ have 4 elements, w.l.o.g. $G_x$. Using

$$|(G_y \setminus G_x) \cup (G_z \setminus G_x) \cup (G_t \setminus G_x)| = 11,$$

we get that at least one of sets

$$G_y \setminus G_x, G_z \setminus G_x, G_t \setminus G_x$$

have 4 elements, w.l.o.g. $G_y \setminus G_x$. Then

$$|G_y| = 4 \quad \text{and} \quad G_y \cap G_x = \emptyset.$$

Arguing analogously we get

$$|G_z| = 4 \quad \text{and} \quad G_z \cap G_x = G_x \cap G_y = \emptyset.$$

By Lemma 12 the representation of each set of $F$ contains either all or none of elements $x, y, z$. Suppose there exists $B \neq A$ such that $x, y, z \in X(B)$. Denote

$$X(A) = \{x, y, z, t'\}, \quad \{C, D\} = |F \setminus (\{A, B\} \cup G_x \cup G_y \cup G_z \cup G_t)| = 2.$$

Since $\{t, t'\} \subset C \cap D$, we got a contradiction. Therefore we proved that $x, y, z$ belongs to the representation of only one set of $F$. Then the price

$$P(X(A)) = \frac{|G_x|(|G_x - 1|)}{2|H_x|} + \frac{|G_y|(|G_y - 1|)}{2|H_y|} + \frac{|G_z|(|G_z - 1|)}{2|H_z|} + \frac{|G_t|(|G_t - 1|)}{2|H_t|} =$$

$$= 6 + 6 + 6 + \frac{|G_t|(|G_t - 1|)}{2|H_t|} > 18$$

for each $A \in F$. Since $18 \times 16 > \frac{16 \times 15}{2}$, we got a contradiction with Lemmas 14 and 15.

Further we use the following notation. Let the family $F$ has the $t$-property and $A \in F$. By $x_1$ we denote $x \in X(A)$ such that $|G_x|$ is maximal for all $x \in X(A)$. Denote by $x_i$ an element $x \in X(A) \setminus \{x_1, \ldots, x_{i-1}\}$ such that the number

$$|G_x \setminus \left( \bigcup_{1 \leq j < i} G_{x_j} \right)|$$

is maximal for all $x \in X(A) \setminus \{x_1, \ldots, x_{i-1}\}$. Denote

$$K^A_{x_i} = G_{x_i} \setminus (G_{x_1} \cup \cdots \cup G_{x_{i-1}}) \quad \text{and} \quad k_i = |K^A_{x_i}|.$$

Remark. Obviously, $k_1 \geq k_2 \geq \cdots \geq k_t$. Also if $k_i = k_j$, then $G_{x_i} \cap G_{x_j} = \emptyset$.

By $s$ denote the maximal index such that $x_s = t$.

We need the following lemmas to prove Theorem 3.
Lemma 21. Let a family $F$ has the $[1,1]$-property and the $t$-property, $|F| \geq t^2 - t + 4$ and $|G_x| \leq t$ for every $x$. Then

$$\sum_{i=1}^{t} k_i = |F| - 1, \quad s \geq 3 \quad \text{and} \quad k_i \geq t - s + 2 \quad \text{for each} \ i.$$ 

Also,

$$\sum_{i=s}^{t} t - k_i \leq t - 3.$$ 

Proof. The first statement is an obvious corollary of the definition of the numbers $k_1, \ldots, k_t$.

Assume $s \leq 2$. Then $k_1 \leq t$, $k_2 \leq t$, $k_i \leq t - 1$ for $3 \leq i \leq t$. Thus

$$t^2 - t + 3 \leq |F| - 1 = \sum_{i=1}^{t} k_i \leq t^2 - t + 2.$$ 

This contradiction proves that $s \geq 3$.

Suppose $k_i \leq t - s + 1$ for some $i$. Then $k_t \leq t - s + 1$. Recall that $k_j \leq t - 1$ for $s < j < t$. Thus

$$t^2 - t + 3 \leq |F| - 1 = \sum_{i=1}^{t} k_i \leq ts + (t - 1)(t - s - 1) + t - s + 1 \leq t^2 - t + 2.$$ 

This contradiction proves that $k_i \geq t - s$ for each $i$.

Finally, we have

$$\sum_{i=s}^{t} t - k_i = \sum_{i=1}^{t} t - k_i = t^2 - |F| + 1 \leq t - 3.$$ 

Lemma 22. Under the conditions of lemma 21, if $X(A) \cap X(B) \neq \emptyset$ for some $B \in F$, then

$$\{x_1, \ldots, x_s\} \cap X(B) \neq \emptyset.$$ 

Proof. Suppose $x_i \in X(A) \cap X(B)$. Since $k_i \geq t - s + 2$ by Lemma 21, we have

$$| (G_{x_i} \setminus K^A_{x_i}) \cup \{B\} | \leq s - 1.$$ 

Note that $K^A_{x_1}, \ldots, K^A_{x_s}$ are mutually disjoint, then among the subfamilies $K^A_{x_1}, \ldots, K^A_{x_s}$ there exists subfamily $K^A_{x_i}$ such that

$$K^A_{x_i} \cap ((G_{x_i} \setminus K^A_{x_i}) \cup \{B\}) = \emptyset.$$ 

Assume that $x_i \notin X(B)$. Then each set of $K^A_{x_i}$ contains one of the elements of $X(B)$, since $B \notin K^A_{x_i}$. Each element of $X(B)$ belongs to at most one of the sets of $K^A_{x_i}$ and $x_i$ belongs to none of them. This contradicts the assumption. Lemma 22 is proved.
Lemma 23. Under the conditions of Lemma 21, if \(X(A) \cap X(B) \neq \emptyset\) for some \(B \in F\), then \(\{x_1, \ldots, x_s\} \subseteq X(B)\).

Proof. Suppose \(x_i \in X(A) \cap X(B)\). By Lemma 22 there exists some \(l \leq s\) and \(x_l \in X(B)\). By Lemma 21 \(s \geq 3\), then Lemma 12 implies that \(\{x_1, \ldots, x_s\} \subseteq X(B)\).

Lemma 24. Under the conditions of Lemma 21,

\[X(A) \cap X(B) = \emptyset \text{ for every } B \in F, B \neq A.\]

Proof. Assume the converse. Let there exist some \(i\) such that \(x_i \in X(B)\). We will prove that \(x_j \in X(B)\) for all \(j, \ 1 \leq j \leq t\). It means \(X(A) = X(B)\), thus \(A = B\).

The proof is by induction on \(t - k_j\). The base claims that if there is some \(x_i \in X(B)\), then \(\{x_1, \ldots, x_s\} \subseteq X(B)\). Thus the base is provided by Lemma 23.

The induction step. Consider it is proved that \(x_i \in X(B)\) for all \(i\) such that \(k_i \geq t - n\) for some integer nonnegative \(n\). By \(q\) denote maximal \(i\) such that \(k_i \geq t - n\). Take some index \(j\) such that \(k_j < t - n\) and \(k_j\) is a maximal among all \(k_l, k_l < t - n\). Clearly, \(k_j = k_{q+1}\). We have to prove \(x_j \in X(B)\).

Since by Lemma 21 \(\sum_{l>q} t - k_l \leq t - 3\), it follows that

\[t - k_j = t - k_{q+1} = \sum_{l>q} t - k_l - \sum_{l>q+1} t - k_l \leq \sum_{l>q} t - k_l - \sum_{l>q+1} 1 \leq (t - 3) - (t - q - 1) = q - 2.\]

Obviously, the first inequality is sharp only in two cases: \(k_l = t - 1\) for \(l > q\) or \(q = t - 1\). Thus \(k_j \geq t - q + 2\), consequently

\[|K^A_{x_j} \setminus \{B\}| \geq t - q + 1.\]

If \(x_j \notin X(B)\), then each set of \(K^A_{x_j} \setminus \{B\}\) contains unique element of \(X(B)\). Recall that \(K^A_{x_i}\) are mutually disjoint, then none of sets of \(K^A_{x_j} \setminus \{B\}\) contains \(x_1, \ldots, x_q\). Since there are only \(t - q\) elements in \(X(B) \setminus \{x_1, \ldots, x_q\}\), we obtain the contradiction.

Lemma 25.

\[b(1, 1, t) \leq \max \left( t^2 - t + 3, \binom{t + 2}{2} \right).\]

Proof. We’ll prove this by induction on \(t\). The base for \(t = 1, 2, 3, 4\) is provided by Lemmas 2, 11, 19 and 20.
The step. Consider \( t > 4 \) and suppose the statement is proved for all \( t' \) such that \( 4 \leq t' < t \). Note that
\[
t^2 - t + 3 \geq \left( \frac{t + 2}{2} \right) \quad \text{for} \quad t \geq 4.
\]

Assume that there exists a family \( F \) with the \([1,1]\)-property and the \( t \)-property such that \( |F| \geq t^2 - t + 4 \).

By Lemmas 3 and 1 \( |G_x| \leq b(0,1,t) = t + 1 \) for all \( x \).

Assume there exists \( x \) such that \( G_x = t + 1 \). Then by Lemma 10
\[
|F| \leq t + 1 + b(1,1,t-1) \leq t + 1 + (t - 1)^2 - (t - 1) + 3 < t^2 - t + 3.
\]

Consider the opposite case. Let \( |G_x| \leq t \) for every \( x \). Then by Lemma 24
\[
X(A) \cap X(B) = \emptyset,
\]
that means \( |H_x| \leq 1 \) for every \( x \). Thus the price of any representation
\[
P(X(A)) = \sum_{i=1}^{t} \frac{|G_{x_i}|(|G_{x_i} - 1|)}{2|H_{x_i}|} = \sum_{i=1}^{t} \frac{|G_{x_i}|(|G_{x_i} - 1|)}{2} \geq \sum_{i=1}^{t} \frac{k_i(k_i - 1)}{2} \geq t \frac{1}{2} \sum_{i=1}^{t} k_i \left( \frac{\sum_{i=1}^{t} k_i}{t} - 1 \right) = t \frac{1}{2} \left( \frac{|F| - 1}{t} \right) \left( \frac{|F| - 1}{t} - 1 \right).
\]

Thus by Lemma 14 the number of triples, corresponding to some representation is at least
\[
|F| \frac{1}{2} \left( \frac{|F| - 1}{t} \right) \left( \frac{|F| - 1}{t} - 1 \right),
\]
and by Lemma 15 the number of all triples is at most
\[
\frac{1}{2} |F|(|F| - 1).
\]

Then
\[
|F| \frac{1}{2} \left( \frac{|F| - 1}{t} \right) \left( \frac{|F| - 1}{t} - 1 \right) \leq \frac{1}{2} |F|(|F| - 1),
\]
i.e. \( |F| \leq 2t + 1 \). But the inequality
\[
t^2 - t + 4 \leq 2t + 1
\]
is impossible.

Now Theorem 3 follows from Propositions 1 and 2 and Lemma 25.
5. Proof of Theorem 4

Evidently if a family $F$ has the $(p, q)$-property, then $F$ has the $(p - 1, q - 1)$-property. Consequently a family $F$ has the $(p - q + 3, 3)$-property. Thus it is sufficient to prove Theorem 4 when a family $F$ has the $(p, 3)$-property.

The proof will use the induction on $p$. The base for $p = 3, 4$.

If $p = 3$ the statement is obvious. If $p = 4$, then there exists $A, B, C \in F$ such that $A \cap B \cap C = \{x\}$. We will prove that there exists at most one $D \in F$ such that $x \notin D$. Assume the converse. Let there exist $D \notin E \in F$ such that $x \notin D$ and $x \notin E$. By $x_A, x_B, x_C$ denote the elements $A \cap D, B \cap D, C \cap D$ respectively, if these elements exist.

Consider sets $A, B, D, E$. Then either $A \cap D \cap E \neq \emptyset$ or $A \cap D \cap E \neq \emptyset$ since

$$A \cap B \cap D = A \cap B \cap E = \emptyset.$$ 

Then either $x_A \in E$ or $x_B \in E$. Analogously we get that either $x_B \in E$ or $x_C \in E$ and either $x_A \in E$ or $x_C \in E$.

Thus $|D \cap E| \geq 2$. This contradiction completes the proof.

The base for the induction. Suppose $p \geq 4$. Consider a family $F$ having the $(p + 1, 3)$-property and

$$|F| \geq \left(\frac{p}{2}\right) \left(\left(\frac{p - 1}{2}\right) - 1\right) + p + 1.$$ 

If the family $F$ has the $(p, 3)$-property, then by the inductive assumption $\tau(F) \leq p - 2 < p - 1$.

Consider $Q \subset F$ such that $|Q| = p$ and $A \cap B \cap C = \emptyset$ for every $A, B, C \in Q$. Denote

$$W(Q) = \{x : A \cap B = \{x\} \text{ for } A, B \in Q\}.$$ 

Then $|W(Q)| \leq \binom{p}{2}$. Let $P \subset F$ and

$$|P| = \left(\frac{p}{2}\right) \left(\left(\frac{p - 1}{2}\right) - 1\right) + 1.$$ 

Consider $A \in P$. Since the family $F$ have the $(p + 1, 3)$-property, we have $A \cap W(Q) \neq \emptyset$. Then there exists $x \in W(Q)$ such that

$$|G_x| \geq 2 + \frac{|F| - p}{|W(Q)|} \geq \left(\frac{p - 1}{2}\right) + 2.$$ 

Take subfamily $F \setminus G_x$. Two cases are possible: $|F \setminus G_x| \leq p - 2$ and $|F \setminus G_x| \geq p - 1$.

In the first case evidently, $\tau(F) \leq p - 1$.

In the second case we prove that the family $F \setminus G_x$ has the $(p - 1, 3)$-property. Assume the converse. Then there exists $Q' \subset F \setminus G_x$ such that $|Q'| = p - 1$ and $A \cap B \cap C = \emptyset$ for every $A, B, C \in Q'$.

Obviously $|W(Q')| \leq \binom{p - 1}{2}$. Then there exist $A, B \in G_x$ such that $A \cap W(Q') = \emptyset$ and $B \cap W(Q') = \emptyset$. Consider the family $R = \{A, B\} \cup Q'$. Then $|R| = p + 1$ and $A \cap B \cap C = \emptyset$ for every $A, B, C \in R$. This contradiction proves that $F \setminus G_x$ has the $(p - 1, 3)$-property.

By the inductive assumption, $\tau(F \setminus G_x) \leq p - 3$, then $\tau(F) \leq p - 3 + 1 = p - 2 < p - 1$. 


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