EXPLICIT CALCULATION OF THE MOD 4 GALOIS REPRESENTATION ASSOCIATED WITH THE FERMAT QUARTIC

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Abstract. We use explicit methods to study the 4-torsion points on the Jacobian variety of the Fermat quartic. With the aid of computer algebra systems, we explicitly give a basis of the group of 4-torsion points. We calculate the Galois action, and show that the image of the mod 4 Galois representation is isomorphic to the dihedral group of order 8. As applications, we calculate the Mordell-Weil group of the Jacobian variety of the Fermat quartic over each subfield of the 8-th cyclotomic field. We determine all of the points on the Fermat quartic defined over quadratic extensions of the 8-th cyclotomic field. Thus we complete Faddeev’s work in 1960.

1. Introduction

In this paper, we use explicit methods to study rational points and divisors on the Fermat quartic $F_4 \subset \mathbb{P}^2$ defined by the homogeneous equation

$$X^4 + Y^4 = Z^4.$$ 

We study the 4-torsion points $\text{Jac}(F_4)[4]$ on the Jacobian variety $\text{Jac}(F_4)$. Then we give applications to the Mordell-Weil group of $\text{Jac}(F_4)$ and the rational points on $F_4$.

In the 17th century, Fermat proved that all of the $\mathbb{Q}$-rational points on $F_4$ satisfy $XYZ = 0$. (The points on $F_4$ satisfying $XYZ = 0$ are called the cusps.) Since the work of Fermat, the arithmetic of $F_4$ has been studied by many mathematicians because, together with the Fermat curves of higher degree, it provides a nice testing ground of more general theories.

In 1960, Faddeev studied the arithmetic of $\text{Jac}(F_4)$ using methods from algebraic geometry [4]. He constructed an isogeny of degree 8 defined over $\mathbb{Q}$ from $\text{Jac}(F_4)$ to the product of three elliptic curves. Since the Mordell-Weil groups of these elliptic curves over $\mathbb{Q}$ are finite of 2-power order, it follows that the Mordell-Weil group $\text{Jac}(F_4)(\mathbb{Q})$ is also finite of 2-power order.

It is an interesting problem to study the 2-power torsion points on the Mordell-Weil group of $\text{Jac}(F_4)$. But the situation in the literature was not clear. In [4], Faddeev claimed (without proof) that $\text{Jac}(F_4)(\mathbb{Q})$ has order 32, and the Mordell-Weil group of $\text{Jac}(F_4)$ over the 8-th cyclotomic field $\mathbb{Q}(\zeta_8)$ is finite. (But he did not mention the precise structure of the Mordell-Weil groups of $\text{Jac}(F_4)$ over $\mathbb{Q}$ and over $\mathbb{Q}(\zeta_8)$.) He alluded that he could determine all of the points on $F_4$ defined over quadratic extensions of $\mathbb{Q}(\zeta_8)$; see the last paragraph in [4, p.1150]. However, as far as the authors of this paper know, the precise statements of Faddeev’s claims and proofs of them did not appear in the literature. On the other hand, Faddeev’s results and claims were cited several times. Kenku used them to study 2-power torsion points on elliptic...
curves over quadratic fields [8]. Klassen revisited Faddeev’s results in his thesis [9]. Schaefer-Klassen used the finiteness of $\text{Jac}(F_4)(\mathbb{Q}(\zeta_8))$ and Coleman’s theory of $p$-adic abelian integrals to study the torsion packet on $F_4$ [10, Section 6].

In this paper, we use explicit methods to study the 4-torsion points $\text{Jac}(F_4)[4]$ on $\text{Jac}(F_4)$. (It is a free $\mathbb{Z}/4\mathbb{Z}$-module of rank 6 because $\text{Jac}(F_4)$ is an abelian variety of dimension 3 over $\mathbb{Q}$.) We give an explicit description of the action of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\text{Jac}(F_4)[4]$. As applications of our results, we give precise statements alluded by Faddeev, and prove them. Thus we fill the gap in the literature.

Here is a summary of the results we obtain in this paper:

1. (Theorem 4.2) We give 6 divisors of degree 0 on $F_4$ which give a basis of $\text{Jac}(F_4)[4]$ as a $\mathbb{Z}/4\mathbb{Z}$-module. Five of them are supported on the cusps, but one of them is not.

2. (Theorem 5.1) We explicitly calculate the mod 4 Galois representation:

$$\rho_4 : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(\text{Jac}(F_4)[4]) \cong \text{GL}_6(\mathbb{Z}/4\mathbb{Z}).$$

We show that the image of $\rho_4$ is isomorphic to the dihedral group of order 8, and the kernel of $\rho_4$ corresponds to the number field $\mathbb{Q}(2^{1/4}, \zeta_8)$, which is a quadratic extension of $\mathbb{Q}(\zeta_8)$. (We also calculate the Weil pairing explicitly. We determine the image of $\rho_4$ inside the symplectic similitude group $\text{GSp}_6(\mathbb{Z}/4\mathbb{Z})$. See Theorem 5.1 and Corollary 5.3.)

3. (Theorem 6.3) We calculate the Mordell-Weil group $\text{Jac}(F_4)(\mathbb{Q}(\zeta_8))$. We shall show that $\text{Jac}(F_4)(\mathbb{Q}(\zeta_8))$ is isomorphic to $(\mathbb{Z}/4\mathbb{Z})^6 \oplus \mathbb{Z}/2\mathbb{Z}$ generated by divisors supported on the cusps. (We also calculate the Mordell-Weil group of $\text{Jac}(F_4)$ and its generators over each subfield of $\mathbb{Q}(\zeta_8)$ (i.e., over $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{-2})$). See Appendix A).

4. (Theorem 7.3) We determine all of the points on $F_4$ defined over quadratic extensions of $\mathbb{Q}(\zeta_8)$. We list all of them. It turns out that $F_4$ has 188 such points in total. Except for the 12 cusps, none of them is defined over $\mathbb{Q}(\zeta_8)$. There exist 48 points defined over $\mathbb{Q}(2^{1/4}, \zeta_8)$, 32 points defined over $\mathbb{Q}(\zeta_3, \zeta_8)$, and 96 points defined over $\mathbb{Q}(\sqrt{-7}, \zeta_8)$.

Note that we take advantage of using results and techniques which were not available in 1960’s. Especially, our proof depends on Rohrlich’s results on the subgroup of the Mordell-Weil group generated by divisors supported on the cusps [14]. We obtained and confirmed key computational results with the aid of computer algebra systems Maxima, Sage, and Singular [21, 18, 19, 20].

The organization of this paper is as follows. In Section 2 we introduce necessary notation on the Galois group and divisors on the Fermat quartic $F_4$. In Section 3, we recall Rohrlich’s results on linear equivalence relations between divisors supported on the cusps. In Section 4, we explicitly give a basis of the 4-torsion points $\text{Jac}(F_4)[4]$. In Section 5, we calculate the Galois action explicitly. Using these results, the calculation of the Mordell-Weil group of $\text{Jac}(F_4)$ over $\mathbb{Q}(\zeta_8)$ becomes an easy exercise in linear algebra. We calculate it in Section 6. In Section 7, we determine all of the points on $F_4$ defined over quadratic extensions of $\mathbb{Q}(\zeta_8)$. This paper has 5 appendices. In Appendix A, we calculate the Mordell-Weil group of $\text{Jac}(F_4)$ over each subfield of $\mathbb{Q}(\zeta_8)$. In Appendix B, we explicitly calculate the Weil pairing on $\text{Jac}(F_4)[4]$, and determine the image of $\rho_4$ inside the symplectic similitude group $\text{GSp}_6(\mathbb{Z}/4\mathbb{Z})$. In Appendix C, we explicitly calculate the action of the automorphism group of $F_4$ on the 4-torsion points.
Jac(F₄)[4]. In Appendix D we briefly explain the authors’ experimental methods to find a 4-torsion point which does not belong to the subgroup generated by divisors supported on the cusps. Finally, in Appendix E we give several remarks on the methods of calculation.

The initial motivation of this work was to understand Faddeev’s results and claims in [4], and to apply them to study linear and symmetric determinantal representations of the Fermat quartic F₄ over Q. Such applications will appear elsewhere; see [6] for the summary of our results.

2. Notation

In this section, we shall introduce necessary notation on the Galois group and divisors which will be used in the rest of this paper.

We fix an embedding of an algebraic closure \( \overline{\mathbb{Q}} \) of Q into the field \( \mathbb{C} \) of complex numbers. We also put \( \zeta_n := \exp(2\pi i/n) \).

Hence we have \( (\zeta_{mn})^m = \zeta_n \) for any \( n, m \geq 1 \). For a positive integer \( a \geq 1 \), we put \( \sqrt{-a} := \sqrt{a} \zeta_4 \).

The field \( \mathbb{Q}(\sqrt{2}, \zeta_8) \) generated by \( \sqrt{2} \) and \( \zeta_8 \) is a Galois extension of \( \mathbb{Q} \) of degree 8. The Galois group Gal(\( \mathbb{Q}(\sqrt{2}, \zeta_8)/\mathbb{Q} \)) is generated by the following two automorphisms \( \sigma, \tau \): \( \sigma \) is an element of order 4 satisfying \( \sigma(\zeta_8) = -\zeta_8 \) and \( \sigma(\sqrt{2}) = \sqrt{2} \zeta_4 \), and \( \tau \) is an element of order 2 satisfying \( \tau(\zeta_8) = \zeta_8^7 \) and \( \tau(\sqrt{2}) = \sqrt{2} \). These elements satisfy \( \tau\sigma\tau = \sigma^3 \). We see that Gal(\( \mathbb{Q}(\sqrt{2}, \zeta_8)/\mathbb{Q} \)) is isomorphic to the dihedral group with 8 elements.

The Fermat quartic is a smooth projective curve over \( \mathbb{Q} \) defined by

\[
F_4 := \{ [X : Y : Z] \in \mathbb{P}^2 \mid X^4 + Y^4 = Z^4 \}.
\]

Its Jacobian variety \( \text{Jac}(F_4) \) is an abelian variety of dimension 3 over \( \mathbb{Q} \). The group of 4-torsion points

\[
\text{Jac}(F_4)[4] := \{ \alpha \in \text{Jac}(F_4)(\overline{\mathbb{Q}}) \mid [4]\alpha = 0 \}
\]

is a free \( \mathbb{Z}/4\mathbb{Z} \)-module of rank 6 with an action of Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \)), where [4] denotes the multiplication-by-4 isogeny on \( \text{Jac}(F_4) \).

The Fermat quartic \( F_4 \) has a \( \mathbb{Q} \)-rational point (such as \( [1 : 0 : 1] \)). Hence, for any extension \( k/\mathbb{Q} \), the degree 0 part of the Picard group of \( F_4 \otimes \mathbb{Q} k \) is canonically identified with the group of \( k \)-rational points on \( \text{Jac}(F_4) \):

\[
\text{Pic}^0(F_4 \otimes \mathbb{Q} k) \cong \text{Jac}(F_4)(k).
\]

For basic results on the Picard schemes and the Jacobian varieties, see [3, Chapter 8, Proposition 4]. (See also [13, Section 5.7.1].)

The Fermat quartic \( F_4 \) has 12 points satisfying \( XYZ = 0 \):

\[
A_i := [0 : \zeta_i^4 : 1], \quad B_i := [\zeta_i^4 : 0 : 1], \quad C_i := [\zeta_i 4\zeta_i^4 : 1 : 0]
\]

for \( 0 \leq i \leq 3 \). These points are called cusps. All of the cusps on \( F_4 \) are defined over \( \mathbb{Q}(\zeta_8) \).
We also introduce the following points defined over $\mathbb{Q}(2^{1/4}, \zeta_8)$:

$$P_1 := [2^{1/4}\zeta_4 : \zeta_8 : 1], \quad P_2 := [\zeta_8 : 2^{1/4}\zeta_4 : 1], \quad P_3 := [2^{-1/4} : 2^{-1/4} : 1].$$

For each $0 \leq i \leq 3$, we define $\alpha_i, \beta_i, \gamma_i \in \text{Jac}(F_4)(\bar{\mathbb{Q}})$ by

$$\alpha_i := [A_i - B_0], \quad \beta_i := [B_i - B_0], \quad \gamma_i := [C_i - B_0],$$

where the linear equivalence class of a divisor $D$ is denoted by $[D]$. We define $e_1, e_2, e_3, e_4, e_5 \in \text{Jac}(F_4)(\bar{\mathbb{Q}})$ by

$$e_1 := \alpha_1, \quad e_2 := \alpha_2, \quad e_3 := \beta_1,$$

$$e_4 := \beta_2, \quad e_5 := \gamma_1.$$

We also define $e_6, e_6' \in \text{Jac}(F_4)(\bar{\mathbb{Q}})$ by

$$e_6 := [A_1 + A_2 + B_1 + B_2 + C_1 + C_2 - 6B_0],$$

$$e_6' := [P_1 + P_2 + P_3 - 3B_0].$$

It turns out that the divisor $P_1 + P_2 + P_3$, defined over $\mathbb{Q}(2^{1/4}, \zeta_8)$, plays an important role in the arithmetic of the Fermat quartic $F_4$. This divisor does not seem to receive special attention before.

3. Rohrlich’s results on divisor classes of the cusps

The proof of our results heavily depends on Rohrlich’s results on the subgroup of the Mordell-Weil group generated by divisors supported on the cusps [14]. In this section, we briefly recall Rohrlich’s results. (For the notation on the cusps of the Fermat quartic $F_4$, see Section 2.)

Let

$$\mathcal{G} \subset \text{Jac}(F_4)(\bar{\mathbb{Q}})$$

be the subgroup generated by $\alpha_i, \beta_i, \gamma_i (0 \leq i \leq 3)$. (The group $\mathcal{G}$ is denoted by $\mathcal{G}_\infty / \mathcal{F}_\infty$ in [14].) Rohrlich determined all of the relations between $\alpha_i, \beta_i, \gamma_i$, and calculated the group $\mathcal{G}$.

Here is a summary of Rohrlich’s results:

**Proposition 3.1** (Rohrlich [14 p.117, Corollary 1]). The group $\mathcal{G}$ is isomorphic to the abelian group generated by $\alpha_i, \beta_i, \gamma_i (0 \leq i \leq 3)$ with the following relations:

$$0 = 4\alpha_i = 4\beta_i = 4\gamma_i,$$

$$0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3,$$

$$0 = \beta_1 + \beta_2 + \beta_3,$$

$$0 = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3,$$

$$0 = \alpha_1 + \beta_1 + 2(\alpha_2 + \beta_2) + 3(\alpha_3 + \beta_3),$$

$$0 = \beta_1 + \gamma_1 + 2(\beta_2 + \gamma_2) + 3(\beta_3 + \gamma_3),$$

$$0 = 2(\alpha_1 + \beta_1 + \gamma_1 + \alpha_2 + \beta_2 + \gamma_2).$$

From Proposition 3.1, it is straightforward to calculate $\mathcal{G}$ explicitly. (See also [14 Theorem 1, Theorem 2].)

**Corollary 3.2.** (1) $\mathcal{G}$ is a finite abelian group killed by 4.

(2) $e_6$ is killed by 2.
(3) The following homomorphism is an isomorphism:

\[
(Z/4Z)^{\oplus 5} \oplus (Z/2Z) \twoheadrightarrow \mathcal{C},
\]

\[
(c_1, c_2, c_3, c_4, c_5, c_6) \mapsto \sum_{i=1}^{5} c_i e_i + c_6 e_6.
\]

For later use, we note that the following equalities are satisfied:

\[
\begin{align*}
\alpha_0 &= 2e_1 + e_2 + 2e_3 + e_4, \\
\alpha_3 &= e_1 + 2e_2 + 2e_3 + 3e_4, \\
\beta_0 &= 0, \\
\beta_3 &= 3e_3 + 3e_4, \\
\gamma_0 &= 3e_1 + 3e_2 + e_3 + e_5 + e_6, \\
\gamma_2 &= 3e_1 + 3e_2 + 3e_3 + 3e_4 + 3e_5 + e_6, \\
\gamma_3 &= 2e_1 + 2e_2 + e_4 + 3e_5.
\end{align*}
\]

4. Explicit determination of the 4-torsion points

By Rohrlich’s results, we see that the subgroup \( \mathcal{C} \subset \text{Jac}(F_4)[4] \) has index 2; see Corollary 3.2. Hence we need only to find a 4-torsion point on \( \text{Jac}(F_4) \) which does not belong to \( \mathcal{C} \). Currently, there seems no practical algorithm to calculate such a point. Fortunately, after a number of trials and errors, the authors found that the divisor class \( e_6' \) does the job. (See also Appendix D.)

We have the following result.

**Proposition 4.1.** The element \( e_6' \) satisfies the following equality in \( \text{Jac}(F_4)(\overline{\mathbb{Q}}) \):

\[
2e_6' = 2e_2 + 2e_4 + e_6.
\]

**Proof.** It is enough to show that

\[
2P_1 + 2P_2 + 2P_3 - A_1 - 3A_2 + 4B_0 - B_1 - 3B_2 - C_1 - C_2
\]

is linearly equivalent to 0. We shall explicitly give a rational function \( f \) whose divisor coincides with the above divisor.

The field \( \mathbb{Q}(2^{1/4}, \zeta_8) \) is generated by an element \( \delta \) such that its minimal polynomial over \( \mathbb{Q} \) is

\[
X^8 - 4X^6 + 8X^4 - 4X^2 + 1,
\]

and the following equalities are satisfied:

\[
\begin{align*}
2 \cdot \delta^2 &= (2 - \sqrt{2})(1 + \zeta_4), \\
3 \cdot \zeta_8 &= 2\delta^6 - 7\delta^4 + 11\delta^2 - 1, \\
3 \cdot 2^{1/4} &= \delta^7 - 5\delta^5 + 10\delta^3 - 8\delta.
\end{align*}
\]
We define the elements \( c_1, c_2, c_3, c_4, c_5 \) by
\[
\begin{align*}
    c_1 & := \delta^6 - 2\delta^4 + \delta^2 + 7, \\
    c_2 & := 2\delta^6 - 10\delta^4 + 20\delta^2 - 13, \\
    c_3 & := 22\delta^7 + 9\delta^6 - 86\delta^5 - 36\delta^4 + 166\delta^3 + 63\delta^2 - 86\delta - 24, \\
    c_4 & := 10\delta^7 - 2\delta^6 - 26\delta^5 + 7\delta^4 + 22\delta^3 - 20\delta^2 + 46\delta - 14, \\
    c_5 & := 22\delta^6 - 77\delta^4 + 154\delta^2 - 44.
\end{align*}
\]

Then, it can be checked (with the aid of computer algebra systems) that the following rational function \( f \) satisfies the required conditions:
\[
\begin{align*}
    f & := \frac{g_2}{g_1}, \\
    g_1 & := 3(X^3 + Y^3 + Z^3) + c_1(X^2Y + X^2Z + XY^2 + XYZ + Y^2Z - Z^3) \\
    & \quad - c_2(XYZ + XZ^2 + YZ^2 + Z^3), \\
    g_2 & := 33(X^3 + XY^2 + XYZ - XZ^2 - Y^2Z - YZ^2) \\
    & \quad + c_3(-X^2Y + XYZ) + c_4(X^2Z + XYZ - XZ^2 - YZ^2) + c_5(XZ^2 - Z^3).
\end{align*}
\]

Hence the equality \( 2e'_6 = 2e_2 + 2e_4 + e_6 \) is satisfied. (The authors checked the calculations using Sage and Singular. See Appendix E for details.)

By Corollary 3.2, the element \( 2e_2 + 2e_4 + e_6 \) is killed by 2, but it is not divisible by 2 inside \( \mathcal{C} \). From these calculations, it is straightforward to give a basis of the \( \mathbb{Z}/4\mathbb{Z} \)-module \( \text{Jac}(F_4)[4] \).

**Theorem 4.2.** The group of 4-torsion points \( \text{Jac}(F_4)[4] \) is a free \( \mathbb{Z}/4\mathbb{Z} \)-module of rank 6 with basis \( e_1, e_2, e_3, e_4, e_5, e'_6 \). In other words, the following homomorphism is an isomorphism:
\[
\begin{align*}
    (\mathbb{Z}/4\mathbb{Z})^\oplus 6 & \xrightarrow{\cong} \text{Jac}(F_4)[4], \\
    (c_1, c_2, c_3, c_4, c_5, c'_6) & \mapsto \sum_{i=1}^{5} c_i e_i + c'_6 e'_6.
\end{align*}
\]

In particular, all of the 4-torsion points on \( \text{Jac}(F_4) \) are defined over \( \mathbb{Q}(2^{1/4}, \zeta_8) \).

**Remark 4.3.** The element \( e'_6 \) was found experimentally by the authors; see Appendix D. The authors were unaware of any theoretical meaning of this element. It would be interesting to look for a “natural” or “moduli theoretic” proof of Proposition 4.1. (Note that the Fermat quartic \( F_4 \) is isomorphic to the modular curve \( X_0(64) \) over \( \mathbb{Q} \); see [8, Proposition 2], [15, p.454], [16, p.107].)

5. **Explicit Calculation of the Galois Action**

By Theorem 3.2, we see that the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on \( \text{Jac}(F_4)[4] \) factors through \( \text{Gal}(\mathbb{Q}(2^{1/4}, \zeta_8)/\mathbb{Q}) \). Hence we have the mod 4 Galois representation
\[
\rho_4 : \text{Gal}(\mathbb{Q}(2^{1/4}, \zeta_8)/\mathbb{Q}) \to \text{Aut}(\text{Jac}(F_4)[4]) \cong \text{GL}_6(\mathbb{Z}/4\mathbb{Z})
\]
with respect to the basis \( e_1, e_2, e_3, e_4, e_5, e'_6 \).

Our task is to calculate the matrices \( \rho_4(\sigma), \rho_4(\tau) \) explicitly.
Theorem 5.1. With respect to the basis $e_1, e_2, e_3, e_4, e_5, e_6^\prime \in \text{Jac}(F_4)[4]$, the actions of $\sigma, \tau$ on $\text{Jac}(F_4)[4]$ are represented by the following matrices:

$$
\rho_4(\sigma) = \begin{pmatrix}
1 & 0 & 0 & 0 & 2 & 1 \\
0 & 1 & 0 & 0 & 2 & 3 \\
0 & 0 & 1 & 0 & 0 & 3 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}, \quad \rho_4(\tau) = \begin{pmatrix}
1 & 0 & 0 & 0 & 3 & 0 \\
2 & 1 & 0 & 0 & 1 & 3 \\
2 & 0 & 3 & 0 & 3 & 0 \\
3 & 0 & 3 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 2 & 3
\end{pmatrix}.
$$

In particular, the mod 4 Galois representation $\rho_4$ is injective, and the image of $\rho_4$ is isomorphic to the dihedral group of order 8.

Proof. It is a straightforward exercise (with the aid of computer algebra systems). We briefly give a summary of our calculations.

The actions of $\sigma, \tau$ on the cusps $A_i, B_i, C_i$ ($0 \leq i \leq 3$) are calculated as follows:

$$
\begin{array}{cccccccc}
& A_0 & A_1 & A_2 & A_3 & B_0 & B_1 & B_2 & B_3 & C_0 & C_1 & C_2 & C_3 \\
\sigma & A_0 & A_1 & A_2 & A_3 & B_0 & B_1 & B_2 & B_3 & C_0 & C_1 & C_2 & C_3 \\
\tau & A_0 & A_3 & A_2 & A_1 & B_0 & B_3 & B_2 & B_1 & C_3 & C_2 & C_1 & C_0
\end{array}
$$

The actions of $\sigma, \tau$ on $P_1, P_2, P_3$ are calculated as follows:

| $P_1$ | $P_2$ | $P_3$ |
|-------|-------|-------|
| $-2^{1/4} : -\zeta_8 : 1$ | $-\zeta_8 : -2^{1/4} : 1$ | $[-2^{1/4} \zeta_4 : -2^{1/4} \zeta_4 : 1]$ |
| $-2^{1/4} \zeta_4 : \zeta_8 : 1$ | $[\zeta_8 : -2^{1/4} \zeta_4 : 1]$ | $P_3$ |

By Theorem 4.2, $\text{Jac}(F_4)[4]$ is a free $\mathbb{Z}/4\mathbb{Z}$-module with basis $e_1, e_2, e_3, e_4, e_5, e_6^\prime$. Hence we need to calculate the actions of $\sigma, \tau$ on these elements.

$$
\sigma e_1 = e_1, \quad \tau e_1 = e_2, \quad e_2 = e_2, \quad e_3 = e_3, \quad e_4 = e_4
$$

$$
\begin{array}{cccc}
& e_1 & e_2 & e_3 & e_4 \\
\sigma & e_1 & e_2 & e_3 & e_4 \\
\tau & [A_3 - B_0] & e_2 & [B_3 - B_0] & e_4
\end{array}
$$

$$
\begin{array}{cccc}
& e_5 & e_6^\prime \\
\sigma & 2e_1 + 2e_2 + e_4 + 3e_5 & e_1 + 3e_2 + 3e_3 + e_4 + 2e_5 + 3e_6^\prime \\
\tau & [C_3 - B_0] & e_5 + 3e_3 + e_4 + 2e_5 + 3e_6^\prime
\end{array}
$$

$$
\begin{array}{cccc}
& e_5 & e_6^\prime \\
\sigma & 2e_1 + 2e_2 + e_4 + 3e_5 & e_1 + 3e_2 + 3e_3 + e_4 + 2e_5 + 3e_6^\prime \\
\tau & [C_3 - B_0] & 3e_1 + e_2 + 3e_3 + e_4 + 3e_5 + 2e_6^\prime
\end{array}
$$

From these calculations, we see that the images of $\sigma, \tau$ under the homomorphism $\rho_4$ are given by the above matrices. \qed

6. Calculation of the Mordell-Weil Group over $\mathbb{Q}(\zeta_8)$

In this section, as an application of our results (Theorem 4.2, Theorem 5.1), we calculate the Mordell-Weil group of $\text{Jac}(F_4)$ over $\mathbb{Q}(\zeta_8)$.

We briefly recall the isogeny constructed by Faddeev [4]. Let $E_1$ (resp. $E_3$) be the smooth projective curve over $\mathbb{Q}$ birational to the affine curve $Y^2 = 1 - X^4$ (resp. $Y^2 = 1 + X^4$). Then, $E_1$ (resp. $E_3$) is isomorphic to the elliptic curve whose Weierstrass equation is $Y^2 = X^3 + 4X$ (resp. $Y^2 = X^3 - 4X$). (For any $\lambda$, the affine curve $C_\lambda: Y^2 = 1 - \lambda X^4$ is birational to the affine curve $C'_\lambda: Y^2 = X^3 + 4\lambda X$ via the rational map $C_\lambda \dashrightarrow C'_\lambda$, $(a, b) \mapsto (u, v) := (2\lambda a^2/(1 - b), 4\lambda a/(1 - b))$; the inverse
map is given by \( C'_\lambda \rightarrow C_\lambda, \ (u, v) \mapsto (2u/v, 1 - 8\lambda u/v^2). \) We put \( E_2 := E_1. \) Then we have morphisms \( f_i : F_4 \rightarrow E_i \) (1 \( \leq i \leq 3 \)) of degree 2 by

\[
\begin{align*}
    f_1([X : Y : Z]) &= ([Y/Z], (X/Z)^2), \\
    f_2([X : Y : Z]) &= ([X/Z], (Y/Z)^2), \\
    f_3([X : Y : Z]) &= ([X/Y], (Z/Y)^2).
\end{align*}
\]

These morphisms induce a homomorphism of abelian varieties

\[
(6.1) \quad \text{Jac}(F_4) \longrightarrow E_1 \times E_2 \times E_3
\]
defined over \( \mathbb{Q} \). It is an isogeny of degree 8.

The following result is presumably well-known.

**Lemma 6.1.** The Mordell-Weil group \( \text{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) is a finite abelian group whose order is a power of 2.

**Proof.** Since the degree of the isogeny \([6.1]\) is a power of 2, it is enough to show the same assertion for each \( E_i \) (1 \( \leq i \leq 3 \)). Over \( \mathbb{Q}(\zeta_8) \), the elliptic curves \( E_1, E_2, E_3 \) are isomorphic to each other. Hence it is enough to show the assertion for \( E_1 \).

It should be possible to calculate \( E_1(\mathbb{Q}(\zeta_8)) \) directly. But the calculation of the Mordell-Weil group, especially the determination of its rank, is a computationally hard problem over a number field of large degree.

Here is a proof of this lemma avoiding the calculation of the Mordell-Weil group over a number field other than \( \mathbb{Q} \). The Mordell-Weil group of \( E_1 \) over \( \mathbb{Q}(\zeta_8) \) is identified with the Mordell-Weil group of the Weil restriction \( \text{Res}_{\mathbb{Q}(\zeta_8)/\mathbb{Q}}(E_1) \), which is an abelian variety of dimension 4 over \( \mathbb{Q} \). Since \( \mathbb{Q}(\zeta_8) \) is a biquadratic field equal to the composite of \( \mathbb{Q}(\sqrt{-1}) \) and \( \mathbb{Q}(\sqrt{2}) \), there is an isogeny of 2-power degree from \( \text{Res}_{\mathbb{Q}(\zeta_8)/\mathbb{Q}}(E_1) \) to the product of the following 4 elliptic curves over \( \mathbb{Q} \):

\[
\begin{align*}
    Y^2 &= X^3 + 4X, & -Y^2 &= X^3 + 4X, \\
    2Y^2 &= X^3 + 4X, & -2Y^2 &= X^3 + 4X.
\end{align*}
\]

It is an easy exercise to check that the Mordell-Weil groups of these elliptic curves over \( \mathbb{Q} \) are finite abelian groups of 2-power order. \( \square \)

In the following, let \( \mathbb{F}_3 \) (resp. \( \mathbb{F}_9 \)) be the finite field with 3 (resp. 9) elements.

**Lemma 6.2.** Let \( \tilde{F}_4 \) be the smooth quartic over \( \mathbb{F}_3 \) defined by \( X^4 + Y^4 = Z^4 \). Let \( \text{Jac}(\tilde{F}_4) \) be the Jacobian variety of \( \tilde{F}_4 \).

1. For any prime number \( \ell \neq 3 \), every eigenvalue of the Frobenius morphism over \( \mathbb{F}_3 \) on the \( \ell \)-adic Tate module

\[
V_\ell \text{Jac}(\tilde{F}_4) := \left( \lim_{\to} \text{Jac}(\tilde{F}_4)[\ell^n] \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]

is a square root of \(-3\).

2. For any prime number \( \ell \neq 3 \), the Frobenius morphism over \( \mathbb{F}_9 \) acts on \( V_\ell \text{Jac}(\tilde{F}_4) \) via the scalar multiplication by \(-3\).

3. \( \text{Jac}(\tilde{F}_4)(\mathbb{F}_9) \) is isomorphic to \( (\mathbb{Z}/4\mathbb{Z})^\oplus 6 \). In particular, it is killed by 4.

**Proof.** (1) \( \text{Jac}(\tilde{F}_4) \) is isogenous to the product of three elliptic curves over \( \mathbb{F}_3 \) because the isogeny \([6.1]\) is also defined over \( \mathbb{F}_3 \). Each of them is isomorphic to either \( Y^2 = X^3 + 4X \) or \( Y^2 = X^3 - 4X \). The number of \( \mathbb{F}_3 \)-rational points (including the point
at infinity) of each of these elliptic curves is equal to 4. Hence every eigenvalue of
the Frobenius morphism over \( \mathbb{F}_3 \) on the \( \ell \)-adic Tate module of these elliptic curves is
a square root of \(-3\). Since the Frobenius eigenvalues are invariant under isogeny, the
same assertion holds for \( V_\ell \operatorname{Jac}(\widetilde{F}_4) \).

(2) This assertion follows from (1) and the semisimplicity of the action of the
Frobenius morphism on the \( \ell \)-adic Tate module.

(3) By (2), the order of \( \operatorname{Jac}(\widetilde{F}_4)(\mathbb{F}_9) \) is equal to \((1 - (-3))^6 = 4096\); see [12, Chapter
IV, Section 21, Theorem 4]. For any \( \ell \neq 3 \), a torsion point \( x \in \operatorname{Jac}(\widetilde{F}_4)[\ell^\infty] \) of \( \ell \)-
power order is defined over \( \mathbb{F}_9 \) if and only if \([-3]x = x \). Hence we have \( \operatorname{Jac}(\widetilde{F}_4)(\mathbb{F}_9) = \operatorname{Jac}(\widetilde{F}_4)[4] \). This group is isomorphic to \((\mathbb{Z}/4\mathbb{Z})^\oplus 6\).

\( \square \)

Remark 6.3. For an abelian variety \( A \) of dimension \( g \) over the finite field \( \mathbb{F}_q \) with \( q \) elements, the Riemann Hypothesis (also called the Weil conjecture, proved by Weil
himself) for \( A \) implies the number of \( \mathbb{F}_q \)-rational points on \( A \) is less than or equal to
\((1 + q^{1/2})^{2g}\); see [12, Chapter IV, Section 21, Theorem 4]. Lemma 6.2 (3) shows that
the maximal number of elements is realized by \( \operatorname{Jac}(\widetilde{F}_4) \) over \( \mathbb{F}_9 \).

We shall show the Mordell-Weil group \( \operatorname{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) is killed by 4. This result is
presumably well-known. One possible approach is to use the isogeny \([6.1]\). But it is
a non-trivial task to eliminate the possibility that \( \operatorname{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) might have a non-
trivial 8-torsion point. Instead of using \([6.1]\), in the following, we shall give a proof
using the reduction modulo 3 of \( \operatorname{Jac}(F_4) \).

Proposition 6.4. \( \operatorname{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) is killed by 4.

Proof. Let \( v \) be a finite place of \( \mathbb{Q}(\zeta_8) \) above 3. The residue field \( \kappa(v) \) at \( v \) is \( \mathbb{F}_9 \).
The Fermat quartic \( F_4 \) naturally extends to a smooth projective curve over the ring of
integers of the completion of \( \mathbb{Q}(\zeta_8) \) at \( v \). Its special fiber is the quartic \( \widetilde{F}_4 \) in Lemma
6.2. By Lemma 6.1, every element of \( \operatorname{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) is of finite order. Its order is a
power of 2. (In particular, its order is prime to 3.) Hence the reduction modulo \( v \) homomorphism
\[
\operatorname{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \longrightarrow \operatorname{Jac}(F_4,v)(\mathbb{F}_9)
\]
is injective. The group \( \operatorname{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) is killed by 4 by Lemma 6.2 (3). \( \square \)

By Proposition 6.4, we have only to calculate the subgroup of \( \operatorname{Jac}(F_4)[4] \) fixed by
\( \operatorname{Gal}(\mathbb{Q}(2^{1/4},\zeta_8)/\mathbb{Q}(\zeta_8)) \). By Theorem 5.1, it is solved by a standard method in linear
algebra.

Theorem 6.5. The Mordell-Weil group \( \operatorname{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) coincides with the group \( C \) generated by divisors supported on the cusps. In particular, the following homomor-
phism is an isomorphism:
\[
(\mathbb{Z}/4\mathbb{Z})^\oplus 5 \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow \operatorname{Jac}(F_4)(\mathbb{Q}(\zeta_8)),
\]
\[
(c_1, c_2, c_3, c_4, c_5, c_6) \mapsto \sum_{i=1}^5 c_i e_i + c_6 e_6.
\]
By the Galois action on \( \text{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) defined over \( \mathbb{K} \), the matrix \( \rho \) is satisfied. Moreover, if the equality \( \rho(\sigma)^2 = v \) holds, then the inequality \( \text{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \cong \{ v \in (\mathbb{Z}/4\mathbb{Z})^6 \mid \rho(\sigma)^2v = v \} \).

By Theorem 5.1, the matrix \( \rho_M^2 \) is calculated as

\[
\rho_M^2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

From this, it is easy to see that \( \text{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) is generated by \( e_1, e_2, e_3, e_4, e_5, e_6 \).

Hence it coincides with the group \( \mathcal{C} \) generated by divisors supported on the cusps; see Corollary 3.2.

\[\square\]

7. Points over Quadratic Extensions of \( \mathbb{Q}(\zeta_8) \)

In this section, we shall determine all of the points on the Fermat quartic \( F_4 \) defined over quadratic extensions of \( \mathbb{Q}(\zeta_8) \).

We shall use the following notation: a pair of \( \overline{Q} \)-rational points \( P, Q \in C(\mathbb{Q}) \) on a curve \( C \) over a number field \( K \) is called a conjugate pair over \( K \) if none of \( P, Q \) is defined over \( K \), and there exists a quadratic extension \( L/K \) such that both of \( P, Q \) are defined over \( L \) and \( P, Q \) are interchanged by the action of \( \text{Gal}(L/K) \). If a pair \( P, Q \) is a conjugate pair over \( K \), then the divisor \( P + Q \) is defined over \( K \).

The following result is essentially due to Faddeev.

**Lemma 7.1** (Faddeev [4, p.1150, Section 3]). Let \( K \) be a number field. Let \( C \subset \mathbb{P}^2 \) be a smooth plane quartic defined over \( K \). Assume that the following conditions are satisfied:

1. \( C \) has at least \( N_1 \) points \( P_1, \ldots, P_{N_1} \) defined over \( K \).
2. \( C \) has at least \( N_2 \) conjugate pairs \( Q_1, Q_1, \ldots, Q_{N_2}, Q_{N_2} \) over \( K \) (i.e. for each \( 1 \leq i \leq N_2 \), the pair \( Q_i, Q_i \) is a conjugate pair over \( K \));
3. the number of linear equivalence classes of effective divisors of degree 2 on \( C \) defined over \( K \) is equal to \( M \).

Then the inequality

\[
N_1(N_1 + 1)/2 + N_2 \leq M
\]

is satisfied. Moreover, if the equality

\[
N_1(N_1 + 1)/2 + N_2 = M
\]

is satisfied, the points \( P_1, \ldots, P_{N_1}, Q_1, Q_1, \ldots, Q_{N_2}, Q_{N_2} \) are the only points on \( C \) defined over quadratic extensions of \( K \).

**Proof.** We shall observe that if \( D = P + Q \) is an effective divisor of degree 2 defined over \( K \), then either both of \( P, Q \) are \( K \)-rational points, or \( P, Q \) is a conjugate pair over \( K \). In the setting of this lemma, \( C \) has the following effective divisors of degree 2 defined over \( K \):

\[
P_i + P_j \quad (1 \leq i \leq j \leq N_1), \quad Q_k + \overline{Q}_k \quad (1 \leq k \leq N_2).
\]
In total, we have \( N_1(N_1 + 1)/2 + N_2 \) effective divisors of degree 2 defined over \( K \). Since \( C \) is non-hyperelliptic, any two of them are not linearly equivalent to each other. Hence the inequality \( N_1(N_1 + 1)/2 + N_2 \leq M \) is satisfied.

Moreover, if the equality \( N_1(N_1 + 1)/2 + N_2 = M \) is satisfied, the above divisors represent all of the linear equivalence classes of effective divisors of degree 2 on \( C \) defined over \( K \). Therefore, there does not exist any \( K \)-rational point other than \( P_1, \ldots, P_M \), and there does not exist any conjugate pair over \( K \) other than \( Q_1, \overline{Q}_1, \ldots, Q_{N_2}, \overline{Q}_{N_2} \).

Here are some points on \( F_4 \) defined over quadratic extensions of \( \mathbb{Q}(\zeta_8) \).

1. There are 12 cusps \( A_i, B_i, C_i \) (0 \( \leq i \leq 3 \)). All of them are defined over \( \mathbb{Q}(\zeta_8) \).
2. For any \( i, j \) with \( 0 \leq i, j \leq 3 \), the following points
   \[
   [21/4 \zeta_8^i : \zeta_8^{1+2j} : 1], \quad [\zeta_8^{1+2j} : 2^{1/4} \zeta_8^i : 1], \quad [2^{-1/4} \zeta_8^i : 2^{-1/4} \zeta_8^j : 1]
   \]
   are defined over \( \mathbb{Q}(2^{1/4}, \zeta_8) \). In total, we have 48 points defined over \( \mathbb{Q}(2^{1/4}, \zeta_8) \). Since none of them is defined over \( \mathbb{Q}(\zeta_8) \), we have 24 conjugate pairs over \( \mathbb{Q}(\zeta_8) \).
3. For any \( i, j \) with \( 0 \leq i, j \leq 3 \), the following points
   \[
   [\zeta_8^{1+2i} : \zeta_8^{2+1+2j} : 1], \quad [\zeta_8^{2+1+2i} : \zeta_8 \zeta_8^{1+2j} : 1]
   \]
   are defined over \( \mathbb{Q}(\zeta_3, \zeta_8) \). In total, we have 32 points defined over \( \mathbb{Q}(\zeta_3, \zeta_8) \). Since none of them is defined over \( \mathbb{Q}(\zeta_8) \), we have 16 conjugate pairs over \( \mathbb{Q}(\zeta_8) \).
4. We put \( \alpha := (1 + \sqrt{-7})/2 \) and \( \overline{\alpha} := (1 - \sqrt{-7})/2 \). For any \( i, j \) with \( 0 \leq i, j \leq 3 \), the following points
   \[
   [\alpha \zeta_8^i : \alpha \zeta_8^{1+2j} : 1], \quad [\overline{\alpha} \zeta_8^i : \alpha \zeta_8^{1+2j} : 1]
   \]
   are defined over \( \mathbb{Q}((\sqrt{-7}), \zeta_8) \). Note that the points in the second (resp. the third) row are obtained from the points in the first row by applying the automorphism \( \theta_1 \) (resp. \( \theta_2^2 \)); see Appendix \[11\]. In total, we have 96 points defined over \( \mathbb{Q}((\sqrt{-7}), \zeta_8) \). Since none of them is defined over \( \mathbb{Q}(\zeta_8) \), we have 48 conjugate pairs over \( \mathbb{Q}(\zeta_8) \).

**Remark 7.2.** The 4 points \([\pm \alpha : \pm \overline{\alpha} : 1], [\pm \overline{\alpha} : \pm \alpha : 1] \) are defined over \( \mathbb{Q}(\sqrt{-7}) \). These points were given by Faddeev in [11] p.1150, Section 3. Note that these points were already known by Aigner in 1934 [11]. (See also Mordell’s paper [11].)

**Theorem 7.3.** The above points on the Fermat quartic \( F_4 \) are the only points defined over quadratic extensions of \( \mathbb{Q}(\zeta_8) \). In particular, the number of points on \( F_4 \) defined over quadratic extensions of \( \mathbb{Q}(\zeta_8) \) is
\[
12 + 48 + 32 + 96 = 188.
\]

**Proof.** We have already seen that \( F_4 \) has at least 12 points defined over \( \mathbb{Q}(\zeta_8) \). Also, \( F_4 \) has at least
\[
88 = (48 + 32 + 96)/2
\]
conjugate pairs over \( \mathbb{Q}(\zeta_8) \). Since
\[
12(12 + 1)/2 + 88 = 166,
\]
it is enough to show that the number of linear equivalence classes of effective divisors of degree 2 on \( C \) defined over \( \mathbb{Q}(\zeta_8) \) is equal to 166.

This can be shown as follows. Since \( F_4 \) has a \( \mathbb{Q}(\zeta_8) \)-rational point (such as \( B_0 = [1 : 0 : 1] \)), the following map is bijective:

\[
\text{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \cong \text{Pic}^2(F_4 \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_8)),
\]

\[
[D] \mapsto [D + 2B_0].
\]

By Theorem 6.5, we know that \( \text{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) is an abelian group of order 2048.

We also know generators of this group explicitly. Therefore, we can make a list of divisors representing all of the 2048 elements of \( \text{Pic}^2(F_4 \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_8)) \). For each divisor of degree 2, there is an algorithm using Gröbner basis which calculates the dimension of the global sections of the line bundle associated to it. It is implemented in standard computer algebra systems. Hence we can count the number of linear equivalence classes of effective divisors of degree 2 on \( F_4 \) defined over \( \mathbb{Q}(\zeta_8) \). It turns out that this number is equal to 166. (The authors calculated it using Singular. See Appendix E for the methods of calculation.)

The following result is considered as an analogue of Fermat’s Last Theorem for the quartic equation \( X^4 + Y^4 = Z^4 \) over quadratic extensions of \( \mathbb{Q}(\zeta_8) \).

**Corollary 7.4.** Let \( K \) be a quadratic extension of \( \mathbb{Q}(\zeta_8) \) which does not contain any of \( 2^{1/4}, \zeta_3, \sqrt{-7} \). Then there do not exist any \( K \)-rational point on the Fermat quartic \( F_4 \) other than the 12 cusps \( A_i, B_i, C_i \) (\( 0 \leq i \leq 3 \)).

**Remark 7.5.** The non-existence of non-cuspidal \( \mathbb{Q}(\zeta_8) \)-rational points on the Fermat quartic \( F_4 \) was proved by Klassen-Schaefer [10, Proposition 6.1]. They used the finiteness of \( \text{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \) and Coleman’s theory of \( p \)-adic abelian integrals (for \( p = 5 \)). Their proof and our proof are completely different.

**Appendix A. The Mordell-Weil group over subfields of \( \mathbb{Q}(\zeta_8) \)**

The 8-th cyclotomic field \( \mathbb{Q}(\zeta_8) \) contains the following 4 proper subfields: \( \mathbb{Q} \), \( \mathbb{Q}(\sqrt{-1}) \), \( \mathbb{Q}(\sqrt{2}) \), \( \mathbb{Q}(\sqrt{-2}) \). In this appendix, we shall calculate the Mordell-Weil group of \( \text{Jac}(F_4) \) over each subfield of \( \mathbb{Q}(\zeta_8) \).

**A.1. The Mordell-Weil group over \( \mathbb{Q} \).** In order to illustrate our methods, we shall explain how to calculate \( \text{Jac}(F_4)(\mathbb{Q}) \) using our results in this paper.

The Galois group \( \text{Gal}(\mathbb{Q}(2^{1/4}, \zeta_8)/\mathbb{Q}) \) is generated by \( \sigma \) and \( \tau \). Hence we have

\[
\text{Jac}(F_4)(\mathbb{Q}) \cong \{ v \in (\mathbb{Z}/4\mathbb{Z})^\oplus 6 \mid (\rho_4(\sigma) - I_6)v = (\rho_4(\tau) - I_6)v = 0 \},
\]

where \( I_6 \) denotes the identity matrix of size 6.

The calculation of the right hand side is an easy exercise in linear algebra. We shall consider the following invertible matrix mod 4:

\[
P := 
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 3 & 2 \\
0 & 0 & 2 & 0 & 3 & 3 \\
\end{pmatrix}
\]
(This matrix \( P \) was found by elementary column operations applied to both matrices \( \rho_4(\sigma) - I_6 \) and \( \rho_4(\tau) - I_6 \) simultaneously.)

Then we have

\[
(\rho_4(\sigma) - I_6)P = \begin{pmatrix}
0 & 0 & 2 & 0 & 1 & 3 \\
0 & 0 & 2 & 0 & 3 & 1 \\
0 & 0 & 2 & 0 & 1 & 1 \\
2 & 0 & 2 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 2 \\
\end{pmatrix},
\]

\[
(\rho_4(\tau) - I_6)P = \begin{pmatrix}
2 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 \\
\end{pmatrix}.
\]

We put

\[
P^{-1}v = (x_1, x_2, x_3, x_4, x_5, x_6)^T \quad \text{for some } x_1, \ldots, x_7 \in \mathbb{Z}/4\mathbb{Z},
\]

where the superscript \( T \) denotes the transpose.

Assume that

\[
(\rho_4(\sigma) - I_6)v = (\rho_4(\tau) - I_6)v = 0.
\]

Then, from the fourth row of \( (\rho_4(\tau) - I_6)P \), we have \( x_1 = 0 \). Similarly, from the third and the second rows of \( (\rho_4(\tau) - I_6)P \), we have \( x_5 = x_6 = 0 \). Then, from the first row of \( (\rho_4(\sigma) - I_6)P \), we have \( 2x_3 = 0 \). Hence we have \( x_3 \in 2\mathbb{Z}/4\mathbb{Z} \). The elements \( x_2, x_4 \) are arbitrary because all of the entries in the second and the fourth columns of \( (\rho_4(\sigma) - I_6)P \), \( (\rho_4(\tau) - I_6)P \) are zero.

From these calculations, we see that \( \text{Jac}(F_4)(\mathbb{Q}) \) is an abelian group of order 32 isomorphic to \((\mathbb{Z}/4\mathbb{Z})^6 \oplus \mathbb{Z}/2\mathbb{Z}\). Moreover, looking at the second, third, and the fourth columns of \( P \), we see that the following homomorphism is an isomorphism:

\[
(\mathbb{Z}/4\mathbb{Z})^6 \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \text{Jac}(F_4)(\mathbb{Q}),
\]

\[
(c_1, c_2, c_3) \mapsto c_1e_2 + c_2e_4 + c_3(2e_1 + 2e_3).
\]

Since \( F_4 \) has a \( \mathbb{Q} \)-rational point (such as \( B_0 = [1 : 0 : 1] \)), every element of \( \text{Jac}(F_4)(\mathbb{Q}) \) is represented by a divisor defined over \( \mathbb{Q} \); see [3, Chapter 8, Proposition 4], [13, Section 5.7.1]. Let us confirm it for the basis \( e_2, e_4, 2e_1 + 2e_3 \) of \( \text{Jac}(F_4)(\mathbb{Q}) \). By definition, \( e_2, e_4 \) are represented by divisors defined over \( \mathbb{Q} \). It is a non-trivial fact that the divisor class

\[
2e_1 + 2e_3 = [2A_1 + 2B_1 - 4B_0]
\]

is represented by a divisor defined over \( \mathbb{Q} \). (The points \( A_1, B_1 \) are not \( \mathbb{Q} \)-rational points.) To see this, we note that the divisor \( 2A_1 + 2B_1 - 4B_0 \) is linearly equivalent to \( A_0 + B_0 - A_2 - B_2 \). (This can be checked using Rohrlich’s results; see Section [3]) Hence we have

\[
2e_1 + 2e_3 = [A_0 + B_0 - A_2 - B_2].
\]

Since \( A_0, A_2, B_0, B_2 \) are \( \mathbb{Q} \)-rational points, we see that \( 2e_1 + 2e_3 \) is represented by a divisor defined over \( \mathbb{Q} \).
Remark A.1. The structure of $\text{Jac}(F_4)(\mathbb{Q})$ is presumably well-known. In [4], Faddeev claimed (without proof) that $\text{Jac}(F_4)(\mathbb{Q})$ has order 32. In [9, p.19, Proposition 2], Klassen describes an isomorphism $\text{Jac}(F_4)(\mathbb{Q}) \cong (\mathbb{Z}/4\mathbb{Z})^6 \oplus \mathbb{Z}/2\mathbb{Z}$, which is essentially the same as above. Klassen also gave a sketch of the proof in [9, p.18-20], but it does not seem complete. The authors of this paper could not find a complete proof of this isomorphism in the literature.

A.2. The Mordell-Weil group over $\mathbb{Q}(\sqrt{-1})$. The method of the calculation is basically the same as in the case of $\mathbb{Q}$.

The Galois group $\text{Gal}(\mathbb{Q}(2^{1/4}, \zeta_8)/\mathbb{Q}(\sqrt{-1}))$ is a cyclic group of order 4 generated by $\sigma$. Hence we have

$$\text{Jac}(F_4)(\mathbb{Q}(\sqrt{-1})) \cong \{ v \in (\mathbb{Z}/4\mathbb{Z})^6 \mid (\rho_4(\sigma) - I_6)v = 0 \}.$$ 

For the square matrix

$$P := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 3 \\
\end{pmatrix},$$

we have

$$(\rho_4(\sigma) - I_6)P = \begin{pmatrix}
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.$$}

Therefore, $\text{Jac}(F_4)(\mathbb{Q}(\sqrt{-1}))$ is isomorphic to $(\mathbb{Z}/4\mathbb{Z})^6$. The following homomorphism is an isomorphism:

$$(\mathbb{Z}/4\mathbb{Z})^6 \xrightarrow{\cong} \text{Jac}(F_4)(\mathbb{Q}(\sqrt{-1})),
\begin{pmatrix}
c_1, c_2, c_3, c_4
\end{pmatrix} \mapsto c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4.$$ 

From the definition of the divisor classes $e_1, e_2, e_3, e_4$, we see that they are represented by divisors defined over $\mathbb{Q}(\sqrt{-1})$.

A.3. The Mordell-Weil group over $\mathbb{Q}(\sqrt{2})$. It is easy to see that the Galois group $\text{Gal}(\mathbb{Q}(2^{1/4}, \zeta_8)/\mathbb{Q}(\sqrt{2}))$ is generated by $\sigma^2$ and $\tau$.

Hence we have

$$\text{Jac}(F_4)(\mathbb{Q}(\sqrt{2})) \cong \{ v \in (\mathbb{Z}/4\mathbb{Z})^6 \mid (\rho_4(\sigma)^2 - I_6)v = (\rho_4(\tau) - I_6)v = 0 \}.$$ 

For the square matrix

$$P := \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 3 & 2 \\
0 & 0 & 2 & 0 & 3 & 3 \\
\end{pmatrix},$$
we have

\[(\rho_4(\sigma)^2 - I_6)P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix},\]

\[(\rho_4(\tau) - I_6)P = \begin{pmatrix}
2 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 
\end{pmatrix}.\]

Therefore, \(\text{Jac}(F_4)(\mathbb{Q}(\sqrt{2}))\) is isomorphic to \((\mathbb{Z}/4\mathbb{Z})^{\oplus 3}\). The following homomorphism is an isomorphism:

\[(\mathbb{Z}/4\mathbb{Z})^{\oplus 3} \xrightarrow{\cong} \text{Jac}(F_4)(\mathbb{Q}(\sqrt{2})),\] 

\[(c_1, c_2, c_3) \mapsto c_1 e_2 + c_2 (e_1 + e_3 + 2e'_6) + c_3 e_4.\]

Let us give divisors defined over \(\mathbb{Q}(\sqrt{2})\) which give a basis of \(\text{Jac}(F_4)(\mathbb{Q}(\sqrt{2}))\). The divisor classes \(e_2, e_4\) are defined over \(\mathbb{Q}\). Hence they are defined over \(\mathbb{Q}(\sqrt{2})\). It can be checked (with the aid of computer algebra systems) that the divisor class \(e_1 + e_3 + 2e'_6\) is represented by the divisor

\([-2^{-1/4} : 2^{-1/4} : 1] + [2^{-1/4} : -2^{-1/4} : 1] + A_0 - 3B_0.\]

This divisor is defined over \(\mathbb{Q}(\sqrt{2})\).

A.4. The Mordell-Weil group over \(\mathbb{Q}(\sqrt{-2})\). It is easy to see that the Galois group \(\text{Gal}(\mathbb{Q}(2^{1/4}, \zeta_8)/\mathbb{Q}(\sqrt{-2}))\) is generated by \(\sigma^2\) and \(\tau \sigma\).

Hence we have

\[\text{Jac}(F_4)(\mathbb{Q}(\sqrt{-2})) \cong \{ v \in (\mathbb{Z}/4\mathbb{Z})^{\oplus 6} \mid (\rho_4(\sigma)^2 - I_6)v = (\rho_4(\tau \sigma) - I_6)v = 0 \},\]

For the square matrix

\[P := \begin{pmatrix}
3 & 0 & 3 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 2 & 0 & 1 & 3 
\end{pmatrix},\]
we have

\[ (\rho_4(\sigma)^2 - I_6)P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \]

\[ (\rho_4(\tau\sigma) - I_6)P = \begin{pmatrix}
2 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \]

Therefore, \( \text{Jac}(F_4)(\mathbb{Q}(\sqrt{-2})) \) is isomorphic to \((\mathbb{Z}/4\mathbb{Z})^{\oplus 3} \). The following homomorphism is an isomorphism:

\[ (\mathbb{Z}/4\mathbb{Z})^{\oplus 3} \overset{\cong}{\to} \text{Jac}(F_4)(\mathbb{Q}(\sqrt{-2})), \]

\[ (c_1, c_2, c_3) \mapsto c_1e_2 + c_2(3c_1 + e_3 + 2c_5 + 2e_6') + c_3e_4. \]

Let us give divisors defined over \( \mathbb{Q}(\sqrt{-2}) \) which give a basis of \( \text{Jac}(F_4)(\mathbb{Q}(\sqrt{-2})) \). The divisor classes \( e_2, e_4 \) are defined over \( \mathbb{Q} \). Hence they are defined over \( \mathbb{Q}(\sqrt{-2}) \).

It can be checked (with the aid of computer algebra systems) that the divisor class \( 3c_1 + e_3 + 2c_5 + 2e_6' \) is represented by the divisor

\[ B_2 + C_2 + C_3 - 3A_0 = [-1 : 0 : 1] + [\zeta_6^3 : 1 : 0] + [\zeta_6^7 : 1 : 0] - 3[0 : 1 : 1]. \]

This divisor is defined over \( \mathbb{Q}(\sqrt{-2}) \) since \( \zeta_6^3, \zeta_6^7 \) are the roots of the quadratic polynomial \( X^2 + \sqrt{-2}X - 1 \) over \( \mathbb{Q}(\sqrt{-2}) \).

**Appendix B. Explicit calculation of the Weil pairing**

Recall that we have a non-degenerate alternating bilinear form

\[ \langle, \rangle : \text{Jac}(F_4)[4] \times \text{Jac}(F_4)[4] \longrightarrow \mu_4 \cong \mathbb{Z}/4\mathbb{Z}, \]

called the Weil pairing, where \( \mu_4 := \{ \pm 1, \pm \zeta_4 \} \) is the group of fourth roots of unity.

The isomorphism \( \mu_4 \cong \mathbb{Z}/4\mathbb{Z} \) is given by \( \zeta_4 \mapsto 1 \).

Since the Weil pairing \( \langle, \rangle \) is Galois equivariant, the image of the mod 4 Galois representation \( \rho_4 \) sits in the symplectic similitude group

\[ \text{GSp}(\text{Jac}(F_4)[4], \langle, \rangle) \]

\[ := \left\{ (g, c) \in \text{Aut}(\text{Jac}(F_4)[4]) \times (\mathbb{Z}/4\mathbb{Z})^\times \mid \langle g\alpha, g\beta \rangle = c\langle \alpha, \beta \rangle \text{ for all } \alpha, \beta \in \text{Jac}(F_4)[4] \right\}. \]

In order to calculate the image of \( \rho_4 \) inside \( \text{GSp}(\text{Jac}(F_4)[4], \langle, \rangle) \), we need to calculate the matrix

\[
\begin{align*}
\begin{pmatrix}
e_1, e_1 & e_1, e_2 & e_1, e_3 & e_1, e_4 & e_1, e_5 & e_1, e_6 \\
e_2, e_1 & e_2, e_2 & e_2, e_3 & e_2, e_4 & e_2, e_5 & e_2, e_6 \\
e_3, e_1 & e_3, e_2 & e_3, e_3 & e_3, e_4 & e_3, e_5 & e_3, e_6 \\
e_4, e_1 & e_4, e_2 & e_4, e_3 & e_4, e_4 & e_4, e_5 & e_4, e_6 \\
e_5, e_1 & e_5, e_2 & e_5, e_3 & e_5, e_4 & e_5, e_5 & e_5, e_6 \\
e_6, e_1 & e_6, e_2 & e_6, e_3 & e_6, e_4 & e_6, e_5 & e_6, e_6
\end{pmatrix}
\end{align*}
\]
representing the Weil pairing with respect to the basis $e_1, e_2, e_3, e_4, e_5, e'_6$.

Here is the summary of our results on the explicit calculation of the Weil pairing.

**Theorem B.1.** With respect to the basis $e_1, e_2, e_3, e_4, e_5, e'_6$ of $\text{Jac}(F_4)[4]$, the matrix representing the Weil pairing $\langle \cdot, \cdot \rangle$ is given by

$$
\begin{pmatrix}
0 & 1 & 3 & 0 & 3 & 1 \\
3 & 0 & 3 & 0 & 1 & 1 \\
1 & 1 & 0 & 2 & 3 & 0 \\
0 & 0 & 2 & 0 & 0 & 1 \\
1 & 3 & 1 & 0 & 0 & 1 \\
3 & 3 & 0 & 3 & 3 & 0
\end{pmatrix}.
$$

**Proof.** First, we recall how to calculate the Weil pairing on a smooth projective curve $C$ over a field $k$. Let $n \geq 1$ be a positive integer invertible in $k$, and $D, E$ divisors of degree 0 on $C$ defined over $\overline{k}$ representing $n$-torsion points on $\text{Jac}(F_4)$. Take non-zero rational functions $f_D, f_E$ satisfying $\text{div } f_D = nD$ and $\text{div } f_E = nE$. Then the Weil pairing is calculated by the following formula:

$$
\langle [D], [E] \rangle = \prod_{P \in C(\overline{k})} (-1)^{n(\text{ord}_P D)(\text{ord}_P E)} \frac{f^\text{ord}_P D}{f^\text{ord}_P E}(P).
$$

(See [5] Theorem 1] and references therein.)

We shall apply the above formula for the Fermat quartic $F_4$ and $n = 4$. Let $\delta$ be the element as in the proof of Proposition 4.1. We define $c'_1, c'_2, c'_3, c'_4, c'_5$ by

- $c'_1 := -52\delta^7 + 9\delta^6 + 224\delta^5 - 36\delta^4 - 496\delta^3 + 63\delta^2 + 452\delta + 93$,
- $c'_2 := 148\delta^7 + 80\delta^6 - 476\delta^5 - 280\delta^4 + 736\delta^3 + 449\delta^2 + 88\delta - 58$,
- $c'_3 := 216\delta^7 + 716\delta^6 - 780\delta^5 - 253\delta^4 + 1476\delta^3 + 515\delta^2 - 312\delta - 160$,
- $c'_4 := 736\delta^7 - 365\delta^6 + 730\delta^5 - 365$,
- $c'_5 := -80\delta^7 + 1586\delta^6 + 352\delta^5 - 553\delta^4 - 824\delta^3 + 728\delta^2 + 568\delta + 203$.

We define linear polynomials $f_1, f_2, f_3, f_4, f_5, g_1$ by

- $f_1 := Y - \zeta_4 Z$,
- $f_2 := Y + Z$,
- $f_3 := X - \zeta_4 Z$,
- $f_4 := X + Z$,
- $f_5 := X - \zeta_8^2 Y$,
- $g_1 := X - Z$.

Moreover, we define a cubic polynomial $f'_6$ by

$$
f'_6 := 219(X^3 + Y^3) + c'_1(X^2Y + XY^2) + c'_2(X^2Z + Y^2Z) + c'_3(XZ^2 + YZ^2) + c'_4Z^3 + c'_5XY Z.
$$

Then, it can be checked (with the aid of computer algebra systems) that the divisors associated with the rational functions $f_1/g_1, f_2/g_1, f_3/g_1, f_4/g_1, f_5/g_1, f'_6/g'_1$ are

- $\text{div } (f_1/g_1) = 4(A_1 - B_0)$,
- $\text{div } (f_2/g_1) = 4(A_2 - B_0)$,
- $\text{div } (f_3/g_1) = 4(B_1 - B_0)$,
- $\text{div } (f_4/g_1) = 4(B_2 - B_0)$,
- $\text{div } (f_5/g_1) = 4(C_1 - B_0)$,
- $\text{div } (f'_6/g'_1) = 4(P_1 + P_2 + P_3 - 3B_0)$.

Therefore, we can use these rational functions to calculate the Weil pairings explicitly. (The authors calculated them using Singular.)
Remark B.2. There is a sign issue on the Weil pairing. In some literature, some authors seem to use the Weil pairing with opposite signs. In Theorem B.1, we used the formula of the Weil pairing as presented in [5, Theorem 1]. If we calculate the Weil pairing with opposite sign convention, the matrix in Theorem B.1 is replaced by the transpose of it.

By Theorem B.1, it is an easy exercise in linear algebra to determine the image of the mod 4 Galois representation \( \rho_4 \) inside the symplectic similitude group \( \text{GSp}_6(\mathbb{Z}/4\mathbb{Z}) \).

Here \( \text{GSp}_6(\mathbb{Z}/4\mathbb{Z}) \) is defined by

\[
\text{GSp}_6(\mathbb{Z}/4\mathbb{Z}) := \left\{ (g, c) \in \text{GL}_6(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})^\times \mid g^T J g = c J \right\},
\]

where \( g^T \) denotes the transpose of \( g \). The matrix \( J \in \text{GL}_6(\mathbb{Z}/4\mathbb{Z}) \) is defined by

\[
J := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

(Note that \( 3 \equiv -1 \pmod{4} \).)

We define \( e''_1, e''_2, e''_3, e''_4, e''_5, e''_6 \in \text{Jac}(F_4)[4] \) by

\[
e''_1 := e_1,
e''_2 := e_2 + 3e'_6,
e''_3 := e_3 + e'_6,
e''_4 := 3e_4 + 3e_5 + 3e'_6,
e''_5 := e_3 + 2e_4 + e_5 + 2e'_6,
e''_6 := 3e_2 + e_3 + 2e_4 + 3e_5 + 2e'_6.
\]

Then we have the following corollary.

**Corollary B.3.** The elements \( e''_1, e''_2, e''_3, e''_4, e''_5, e''_6 \) form a basis of \( \text{Jac}(F_4)[4] \) such that the Weil pairing \( \langle , \rangle \) is represented by \( J \). Moreover, the images of \( \sigma, \tau \in \text{Gal}(\mathbb{Q}(2^{1/4}, \zeta_8)/\mathbb{Q}) \) by the mod 4 Galois representation

\[
\text{Gal}(\mathbb{Q}(2^{1/4}, \zeta_8)/\mathbb{Q}) \to \text{GSp}_6(\mathbb{Z}/4\mathbb{Z})
\]

with respect to the basis \( e''_1, e''_2, e''_3, e''_4, e''_5, e''_6 \) are given by the following matrices:

\[
\begin{pmatrix}
1 & 3 & 1 & 1 & 0 & 0 \\
0 & 2 & 3 & 3 & 0 & 0 \\
0 & 0 & 1 & 3 & 1 & 3 \\
0 & 3 & 1 & 3 & 3 & 1 \\
0 & 1 & 3 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 1 & 3 & 1 \\
2 & 1 & 2 & 1 & 1 & 3 \\
3 & 1 & 3 & 1 & 1 & 1 \\
3 & 1 & 2 & 1 & 2 & 0 \\
3 & 0 & 1 & 3 & 3 & 0 \\
0 & 3 & 3 & 1 & 2 & 3
\end{pmatrix}.
\]

**Proof.** The assertions immediately follow from Theorem 5.1 and Theorem B.1. \( \square \)
Appendix C. The action of automorphisms on $\text{Jac}(F_4)[4]$

We shall recall the structure of the automorphism group of the Fermat quartic $F_4$. It is classically known that the automorphism group of the Fermat quartic $F_4 \otimes \mathbb{Q} \mathbb{C}$ over $\mathbb{C}$ is a non-abelian group of order 96. All of the automorphisms of $F_4 \otimes \mathbb{Q} \mathbb{C}$ are defined over $\mathbb{Q}(\zeta_8)$. The automorphism group $\text{Aut}(F_4 \otimes \mathbb{Q}(\zeta_8))$ is generated by the following elements (see [17, p.173, Theorem]):

- $\theta_1: [X : Y : Z] \mapsto [\zeta_4 X : Y : Z]$,
- $\theta_2: [X : Y : Z] \mapsto [Y : X : Z]$,
- $\theta_3: [X : Y : Z] \mapsto [\zeta_7^3 Y : \zeta_4^3 Z : X]$.

Remark C.1. The following explanation might be useful to understand the nature the automorphism group of $F_4$. Over the 8-th cyclotomic field $\mathbb{Q}(\zeta_8)$, the Fermat quartic $F_4$ is isomorphic to another quartic $F'_4 := \{ [X : Y : Z] \in \mathbb{P}^2 \mid X^4 + Y^4 + Z^4 = 0 \}$ via the isomorphism:

$$\psi: F_4 \otimes \mathbb{Q}(\zeta_8) \cong F'_4 \otimes \mathbb{Q}(\zeta_8)$$

$$[X : Y : Z] \mapsto [X : Y : \zeta_8^7 Z].$$

Under this isomorphism $\psi$, the automorphism $\theta_3$ is translated into the cyclic permutation of the coordinates $X, Y, Z$. Precisely, we have

$$\psi^{-1}\theta_3\psi: [X : Y : Z] \mapsto [Y : Z : X].$$

It is easy to see that $\theta_2, \theta_3$ generate a subgroup of $\text{Aut}(F_4 \otimes \mathbb{Q}(\zeta_8))$ isomorphic to the symmetric group $S_3$ of order 6. Since $\theta_2^2 = \text{id}$, we see that $\theta_2\theta_1\theta_2$ is conjugate to $\theta_1$. We have

$$\theta_2\theta_1\theta_2: [X : Y : Z] \mapsto [X : \zeta_4 Y : Z].$$

From this, we see that $\theta_1$ commutes with $\theta_2\theta_1\theta_2$. Hence $\theta_1$ and $\theta_2\theta_1\theta_2$ generate an abelian subgroup of $\text{Aut}(F_4 \otimes \mathbb{Q}(\zeta_8))$ of order 16, which is isomorphic to $(\mathbb{Z}/4\mathbb{Z})^\oplus 2$. Therefore, $\text{Aut}(F_4 \otimes \mathbb{Q}(\zeta_8))$ is isomorphic to the semi-direct product $(\mathbb{Z}/4\mathbb{Z})^\oplus 2 \rtimes S_3$.

The action of the automorphisms on the 4-torsion points gives a homomorphism

$$\varphi_4: \text{Aut}(F_4 \otimes \mathbb{Q}(\zeta_8)) \to \text{Aut}(\text{Jac}(F_4)[4]) \cong \text{GL}_6(\mathbb{Z}/4\mathbb{Z}).$$

We shall calculate this homomorphism explicitly as follows.
Theorem C.2. With respect to the basis $e_1, e_2, e_3, e_4, e_5, e_6' \in \text{Jac}(F_4)[4]$, the actions of the automorphisms $\theta_1, \theta_2, \theta_3$ are represented by the following matrices:

\[
\varphi_4(\theta_1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 3 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
3 & 3 & 3 & 2 & 2 & 1 \\
0 & 0 & 1 & 3 & 1 & 2 \\
0 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 2 & 3 \\
\end{pmatrix}, \quad \varphi_4(\theta_2) = \begin{pmatrix}
2 & 2 & 3 & 1 & 2 & 1 \\
3 & 3 & 3 & 0 & 0 & 1 \\
3 & 2 & 2 & 2 & 1 & 2 \\
3 & 0 & 3 & 3 & 0 & 1 \\
0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 \\
\end{pmatrix}, \\
\varphi_4(\theta_3) = \begin{pmatrix}
3 & 2 & 3 & 0 & 3 & 1 \\
2 & 3 & 3 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 & 2 & 3 \\
1 & 2 & 1 & 1 & 2 & 3 \\
1 & 3 & 0 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & 0 & 3 \\
\end{pmatrix}.
\]

Proof. The actions of $\theta_1, \theta_2, \theta_3$ on the cusps $A_i, B_i, C_i \ (0 \leq i \leq 3)$ are calculated as follows:

\[
\begin{array}{c|cccccccc}
& A_0 & A_1 & A_2 & A_3 & B_0 & B_1 & B_2 & B_3 & C_0 & C_1 & C_2 & C_3 & C_4 \\
\hline
\theta_1 & A_0 & A_1 & A_2 & A_3 & B_0 & B_1 & B_2 & B_3 & C_0 & C_1 & C_2 & C_3 & C_4 \\
\theta_2 & B_0 & B_1 & B_2 & B_3 & A_0 & A_1 & A_2 & A_3 & C_0 & C_1 & C_2 & C_3 & C_4 \\
\theta_3 & C_0 & C_1 & C_2 & C_3 & A_3 & A_2 & A_1 & A_0 & B_3 & B_2 & B_1 & B_0 & \\
\end{array}
\]

The actions of $\theta_1, \theta_2, \theta_3$ on the points $P_1, P_2, P_3$ are calculated as follows:

\[
\begin{array}{c|c|c}
P_1 & P_2 & P_3 \\
\hline
\theta_1 & [-2^{1/4} : \zeta_8 : 1] & [\zeta_8^3 : 2^{1/4} \zeta_4 : 1] & [-2^{-1/4} \zeta_4 : -2^{-1/4} : 1] \\
\theta_2 & P_2 & P_1 & P_3 \\
\theta_3 & [2^{-1/4} \zeta_4^3 : -2^{-1/4} : 1] & [2^{1/4} : \zeta_8^3 : 1] & [\zeta_8^3 : 2^{1/4} \zeta_4^3 : 1] \\
\end{array}
\]

Therefore, with the aid of computer algebra systems, the actions of $\theta_1, \theta_2, \theta_3$ on the elements $e_1, e_2, e_3, e_4, e_5, e_6'$ are calculated as follows:

\[
\begin{array}{c|c|c}
\theta_1 & e_1 & e_2 \\
\hline
\theta_1 & [A_1 - B_1] = e_1 + 3e_3 & [A_2 - B_1] = e_2 + 3e_3 \\
\theta_2 & [B_1 - A_0] = 2e_1 + 3e_2 + 3e_3 + 3e_4 & [B_2 - A_0] = 2e_1 + 3e_2 + 2e_3 \\
\theta_3 & [C_1 - A_3] = 3e_1 + 2e_2 + 2e_3 + e_4 + e_5 & [C_2 - A_3] = 2e_1 + 3e_2 + 3e_3 + 2e_4 + 3e_5 + 2e_6' \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\theta_1 & e_3 & e_4 \\
\hline
\theta_1 & [B_2 - B_1] = 3e_3 + e_4 & [B_3 - B_1] = 2e_3 + 3e_4 \\
\theta_2 & [A_1 - A_0] = 3e_1 + 3e_2 + 2e_3 + 3e_4 & [A_2 - A_0] = 2e_1 + 2e_3 + 3e_4 \\
\theta_3 & [A_2 - A_3] = 3e_1 + 3e_2 + 2e_3 + e_4 & [A_1 - A_3] = 2e_2 + 2e_3 + e_4 \\
\end{array}
\]

\[
\begin{array}{c|c}
\theta_1 & e_5 \\
\hline
\theta_1 & [C_2 - B_1] = 3e_1 + e_2 + 2e_3 + e_4 + 3e_5 + 2e_6' \\
\theta_2 & [C_2 - A_0] = e_1 + 3e_2 + 3e_5 + 2e_6' \\
\theta_3 & [B_2 - A_0] = e_1 + 3e_2 + 2e_3 + 2e_4 \\
\end{array}
\]
The 4-torsion point \( e'_6 \) (defined in Section 2) plays an important role in this paper. The authors found it experimentally. Here are the methods of the authors to find this element.

Since every divisor of degree 3 on \( F_4 \) is linearly equivalent to an effective divisor, the authors tried to find a divisor \( D \) of degree 3 such that \([D - 3B_0]\) gives a 4-torsion point on \( \text{Jac}(F_4) \) which does not belong to the subgroup \( \mathcal{C} \) generated by divisors supported on the cusps.

The authors first studied possible candidates of the field of definition of such a divisor \( D \). (In fact, the following strategy is inspired by recent developments on Iwasawa theory and extensions between mod \( p \) Galois representations.) Let \( e'_6 \) be a 4-torsion point on \( \text{Jac}(F_4) \) which does not belong to \( \mathcal{C} \). Let \( K \) be the smallest extension of \( \mathbb{Q}(\zeta_8) \) where \( e'_6 \) is defined. Since the 4-torsion points \( \text{Jac}(F_4)[4] \) are generated by \( \mathcal{C} \) and \( e'_6 \), the extension \( K/\mathbb{Q} \) corresponds to the kernel of the mod 4 Galois representation

\[
\rho_4: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(\text{Jac}(F_4)[4]) \cong \text{GL}_6(\mathbb{Z}/4\mathbb{Z}).
\]

In particular, we see that

- \( K/\mathbb{Q} \) is a Galois extension.

Since all of the 2-torsion points on \( \text{Jac}(F_4) \) are defined over \( \mathbb{Q}(\zeta_8) \), for each \( s \in \text{Gal}(K/\mathbb{Q}(\zeta_8)) \), the difference \( s(e'_6) - e'_6 \) is killed by 2. Hence we have an injective homomorphism

\[
\psi: \text{Gal}(K/\mathbb{Q}(\zeta_8)) \to \text{Jac}(F_4)[2], \quad s \mapsto s(e'_6) - e'_6.
\]

We see that

- the Galois group \( \text{Gal}(K/\mathbb{Q}(\zeta_8)) \) is isomorphic to an elementary abelian group of type \((2, \ldots, 2)\).

Since \( \text{Jac}(F_4) \) has good reduction outside 2, the extension \( K/\mathbb{Q} \) satisfies the following condition:

- \( K/\mathbb{Q} \) is unramified outside 2.

Moreover, \( K \) must satisfy the following condition:

- Every finite place of \( \mathbb{Q}(\zeta_8) \) above 3 splits in \( K/\mathbb{Q}(\zeta_8) \).

(Here is a proof of this claim. Let \( v \) be a finite place of \( \mathbb{Q}(\zeta_8) \) above 3, and take a finite place \( w \) of \( K \) above \( v \). Since \( \psi \) is injective, the Frobenius element \( \text{Frob}_w \) at \( w \) has non-zero image under \( \psi \). Hence \( \text{Frob}_w \) acts non-trivially on the reduction of \( e'_6 \) modulo \( w \). It contradicts the fact that the residue field at \( v \) is \( \mathbb{F}_9 \), and all of the 4-torsion points on the reduction modulo 3 of \( \text{Jac}(F_4) \) are defined over \( \mathbb{F}_9 \); see Lemma 6.2 (3).)
The authors guessed $K/\mathbb{Q}(\zeta_8)$ is a quadratic extension, and tried to find candidates of $K$. Using Sage, it is not difficult to check that $\mathbb{Q}(\zeta_8)$ has exactly 7 quadratic extensions which are unramified outside 2. Among them, exactly 3 quadratic extensions are Galois over $\mathbb{Q}$. Looking at the finite places above 3, it turned out that $\mathbb{Q}(2^{1/4}, \zeta_8)$ is the only quadratic extension of $\mathbb{Q}(\zeta_8)$ satisfying all of the above conditions.

Then, the authors guessed $K$ is $\mathbb{Q}(2^{1/4}, \zeta_8)$, and tried to find a divisor of degree 3 defined over $\mathbb{Q}(2^{1/4}, \zeta_8)$. It was not difficult to find the $\mathbb{Q}(2^{1/4}, \zeta_8)$-rational points $P_1, P_2, P_3$. (See Section 2 for the definition of these points.) Calculating the divisor class of $P_1 + P_2 + P_3 - 3B_0$ using Singular, the authors finally found the divisor class $e'_6 := [P_1 + P_2 + P_3 - 3B_0]$ is what they were looking for.

**APPENDIX E. METHODS OF CALCULATION**

In this appendix, we give some remarks on the methods of calculation.

The key computational results in this paper are the identity $2e'_6 = 2e_2 + 2e_4 + e_6$ in Proposition 4.1 and the calculation of the number of effective divisor classes in $\text{Pic}^2(F_4 \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_8))$ in the proof of Theorem 7.3.

Here is a sample source code for Singular which performs necessary calculation.

```singular
LIB "divisors.lib";
ring r=(0,d),(x,y,z),dp; minpoly = d^8 - 4*d^6 + 8*d^4 - 4*d^2 + 1;
ideal I = x^4 + y^4 - z^4; qring Q = std(I);
number z8 = (2*d^6 - 7*d^4 + 11*d^2 - 1)/3; number z4 = z8^2;
number a = (d^7 - 5*d^5 + 10*d^3 - 8*d)/3;
divisor A0 = makeDivisor(ideal(x, y-z), ideal(1));
divisor A1 = makeDivisor(ideal(x, y-z4*z), ideal(1));
divisor A2 = makeDivisor(ideal(x, y+z), ideal(1));
divisor A3 = makeDivisor(ideal(x, y+z4*z), ideal(1));
divisor B0 = makeDivisor(ideal(x-z, y), ideal(1));
divisor B1 = makeDivisor(ideal(x-z4*z, y), ideal(1));
divisor B2 = makeDivisor(ideal(x+z, y), ideal(1));
divisor B3 = makeDivisor(ideal(x+z4*z, y), ideal(1));
divisor C0 = makeDivisor(ideal(x-z8*y, z), ideal(1));
divisor C1 = makeDivisor(ideal(x-z8*z4*y, z), ideal(1));
divisor C2 = makeDivisor(ideal(x+z8*y, z), ideal(1));
divisor C3 = makeDivisor(ideal(x+z8*z4*y, z), ideal(1));
divisor E1 = A1 + multdivisor(-1, B0);
divisor E2 = A2 + multdivisor(-1, B0);
divisor E3 = B1 + multdivisor(-1, B0);
divisor E4 = B2 + multdivisor(-1, B0);
divisor E5 = C1 + multdivisor(-1, B0);
divisor E6 = A1 + B1 + C1 + A2 + B2 + C2 + multdivisor(-6, B0);
divisor P1 = makeDivisor(ideal(x-a*z4*z, y-z8*z), ideal(1));
divisor P2 = makeDivisor(ideal(x-z8*z, y-a*z4*z), ideal(1));
divisor P3 = makeDivisor(ideal(x-a*(-1)*z, y-a*(-1)*z), ideal(1));
divisor EE6 = P1 + P2 + P3 + multdivisor(-3, B0);
print("Is 2e'_6 equal to 2e_2 + 2e_4 + e_6?");
if(linearlyEquivalent(multdivisor(2, EE6), multdivisor(2, E2) + multdivisor(2, E4) + E6)[1] != 0) {
  print("Yes"); }else{ print("No"); }
```
divisor D1 = multdivisor(2, P1 + P2 + P3) + multdivisor(-1, A1) + multdivisor(-3, A2) + multdivisor(4, B0) + multdivisor(-1, B1) + multdivisor(-3, B2) + multdivisor(-1, C1 + C2);
number c1 = d^6 - 2*d^4 + d^2 + 7;
number c2 = 2*d^6 - 10*d^4 + 20*d^2 - 13;
number c3 = 22*d^7 + 9*d^6 - 86*d^5 - 36*d^4 + 166*d^3 + 63*d^2 - 86*d - 24;
number c4 = 10*d^7 - 2*d^6 - 26*d^5 + 7*d^4 + 22*d^3 - 20*d^2 + 46*d - 14;
number c5 = 22*d^6 - 77*d^4 + 154*d^2 - 44;
divisor D2 = makeDivisor(
    ideal(33*(x^3 + x*y^2 + x*y*z - x*z^2 - y^2*z - y*z^2) + c3*(-x^2*y + x*y*z) + c4*(x^2*z + x*y*z - x*z^2 - y*z^2) + c5*(x*z^2 - z^3)),
    ideal(3*(x^3 + y^3 + z^3) + c1*(x^2*y + x^2*z + x*y^2 + x*y*z + y^2*z - z^3) - c2*(x*y*z + x*z^2 + y*z^2 + z^3)));
print("Proof of Proposition 4.1: calculation of div(f)");
if(isEqualDivisor(D1, D2) == 1){ print("OK"); }else{ print("Not OK"); }
print("The number of effective elements in Pic^2(F_4)(Q(zeta_8))");
int c, p1, p2, p3, p4, p5, p6; c = 0;
for(p1=0; p1<=3; p1++){ for(p2=0; p2<=3; p2++) { for(p3=0; p3<=3; p3++) { for(p4=0; p4<=3; p4++) { for(p5=0; p5<=3; p5++) { for(p6=0; p6<=1; p6++) {
    if(size(globalSections(multdivisor(p1, E1) + multdivisor(p2, E2) + multdivisor(p3, E3) + multdivisor(p4, E4) + multdivisor(p5, E5) + multdivisor(p6, E6) + multdivisor(2, B0))[1]) == 1) { c = c+1; }}}}}}}};
print(c);

Generally speaking, the calculation of divisor classes over a number field of large degree (such as Gal(Q(2^{1/4}, ζ_8))) is very slow. Here is a simple trick to reduce the amount of computer calculation: in order to confirm identities between 4-torsion points on Jac(F_4), it is enough to confirm them over any finite field of characteristic different from 2. In fact, the authors took the prime number 73, which is the smallest prime number which splits completely in Q(2^{1/4}, ζ_8). They confirmed identities in this paper between the 4-torsion points by calculating them over F_{73}. The calculation over F_{73} is much faster because F_{73} is a prime field. The authors chose 10 (mod 73) as a primitive 8-th root of unity in F_{73}, and 18 (mod 73) as a fourth root of 2 in F_{73} because they satisfy the relation 10 + 10^7 ≡ 18^2 (mod 73).

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