INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE 3RD DERIVATIVES BELONG TO \( Q(I) \)

M.EMIN OZDEMIR, MERVE AVCI ARDIC, AND MUSTAFA GURBUZ

Abstract. In this paper, we obtain some new inequalities of Hermite-Hadamard type and Simpson type for functions whose third derivatives belong to Godunova-Levin class.

1. INTRODUCTION

Following inequalities are well known in the literature as Hermite-Hadamard inequality and Simpson inequality respectively:

**Theorem 1.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). Then, the following double inequality holds

\[
\frac{f(a) + f(b)}{2} \leq \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

**Theorem 2.** Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping on \((a, b)\) and \( \|f^{(4)}\|_{\infty} = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \). Then, the following inequality holds:

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} \right] + 2f\left( \frac{a + b}{2} \right) \right| - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b - a)^4.
\]

In 1985, E. K. Godunova and V. I. Levin introduced the following class of functions (see [1]):

**Definition 1.** A map \( f : I \to \mathbb{R} \) is said to belong to the class \( Q(I) \) if it is nonnegative and for all \( x, y \in I \) and \( \lambda \in (0, 1) \), satisfies the inequality

\[
f(\lambda x + (1 - \lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda}.
\]

In [2], Moslehian and Kian obtained Hermite-Hadamard and Ostrowski type inequalities for functions whose first derivatives belong to \( Q(I) \).

To obtain our new results, it is necessary two lemmas.

---

2000 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.

*Key words and phrases.* Godunova-Levin Class Functions, Power-mean integral inequality, Hölder inequality, Hermite-Hadamard inequality, Simpson inequality.

\( *\)Corresponding Author.
Lemma 1. [3] Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a three times differentiable function on \( I^o \) with \( a, b \in I, \ a < b \). If \( f''' \in L[a,b] \), then
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \frac{b-a}{12} \left[ f(b) - f'(a) \right] = \frac{(b-a)^3}{12} \int_0^1 t(1-t)(2t-1) f'''(ta + (1-t)b) \, dt.
\]

Lemma 2. [4] Let \( f'' : I \subseteq \mathbb{R} \to \mathbb{R} \) be an absolutely continuous function on \( I^o \) such that \( f''' \in L[a,b] \), where \( a, b \in I, \ a < b \). Then
\[
\int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] = (b-a)^4 \int_0^1 p(t) f'''(ta + (1-t)b) \, dt,
\]
where
\[
p(t) = \begin{cases} \frac{1}{6} t^2 \left( t - \frac{1}{2} \right), & \text{if } t \in \left[ 0, \frac{1}{2} \right] \\ \frac{1}{6} (t-1)^2 \left( t - \frac{1}{2} \right), & \text{if } t \in \left( \frac{1}{2}, 1 \right]. \end{cases}
\]

In this paper, using Lemma 1 and Lemma 2, we obtain some new inequalities for functions whose third derivatives belong to \( Q(I) \).

2. Main Results

We obtain the following new inequalities via Lemma 1.

Theorem 3. Let \( f'' : I \subseteq \mathbb{R} \to \mathbb{R} \) be an absolutely continuous function on \( I^o \) such that \( f''' \in L[a,b] \), where \( a, b \in I, \ a < b \). If \( |f'''|^q \) belongs to \( Q(I) \), then the following inequality holds
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \frac{b-a}{12} \left[ f'(b) - f'(a) \right] \right| \leq \frac{(b-a)^3}{12} \left( \frac{1}{16} \right)^{\frac{1}{q}} \left[ \frac{|f'''(a)|^q + |f'''(b)|^q}{4} \right]^{\frac{1}{q}}
\]
for \( q \geq 1 \).

Proof. From Lemma 1 and using the power mean inequality, we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \frac{b-a}{12} \left[ f'(b) - f'(a) \right] \right| \leq \frac{(b-a)^3}{12} \left( \int_0^1 t(1-t)(2t-1) |dt| \right)^{\frac{1}{q}} \left( \int_0^1 t(1-t)(2t-1) |f'''(ta + (1-t)b)|^q \, dt \right)^{\frac{1}{q}}.
\]
Then, since \( |f'''|^q \) belongs to \( Q(I) \), we can write for \( t \in (0,1) \)
\[
|f'''(ta + (1-t)b)|^q \leq \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1-t}.
\]
Hence,
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^3}{12} \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left( \int_0^1 t(1-t) |2t-1| \left[ \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1-t} \right] dt \right)^{\frac{1}{q}}
\]
\[
= \frac{(b-a)^3}{12} \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left( \int_0^1 [(1-t) |2t-1| f'''(a)|^q + t |2t-1| f'''(b)|^q \right] dt \right)^{\frac{1}{q}}
\]
\[
= \frac{(b-a)^3}{12} \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left[ \frac{|f'''(a)|^q + |f'''(b)|^q}{4} \right]^{\frac{1}{q}},
\]
where
\[
\int_0^1 t(1-t) |2t-1| dt = \frac{1}{16}
\]
and
\[
\int_0^1 t |2t-1| dt = \int_0^1 (1-t) |2t-1| dt = \frac{1}{4}.
\]
The proof is completed. \[ \square \]

**Corollary 1.** In Theorem 3 if we choose \( q = 1 \) we obtain the following inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^3}{48} \left[ |f'''(a)| + |f'''(b)| \right].
\]

**Theorem 4.** Let \( f'' : I \subseteq \mathbb{R} \to \mathbb{R} \) be an absolutely continuous function on \( I^o \) such that \( f''' \in L[a,b] \), where \( a, b \in I, a < b \). If \( |f'''|^q \) belongs to \( Q(I) \) and \( q > 1 \), then the following inequality holds
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^3}{12} \left( \frac{\beta(q,q+1)}{(p+1)^q} \right)^{\frac{1}{q}} \left[ |f'''(a)| + |f'''(b)| \right]^{\frac{1}{q}}
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( \beta(.,.) \) is Euler Beta function.
Proof. Since \( |f'''|^q \) belongs to \( Q(I) \), from Lemma 1 and using the Hölder inequality we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx - \frac{b - a}{12} [f'(b) - f'(a)] \right| \leq \frac{(b - a)^3}{12} \left( \int_0^1 |2t - 1|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 t^q (1 - t)^q |f'''(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}} \\
\leq \frac{(b - a)^3}{12} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left( \int_0^1 t^q (1 - t)^q \left[ \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1 - t} \right] dt \right)^{\frac{1}{q}} \\
= \frac{(b - a)^3}{12} \frac{(\beta (q, q + 1))^\frac{1}{q}}{(p + 1)^\frac{1}{p}} \left[ |f'''(a)|^q + |f'''(b)|^q \right]^{\frac{1}{q}},
\]
which completes the proof.

\[\Box\]

**Theorem 5.** Under the assumptions of Theorem 4, we have the following inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx - \frac{b - a}{12} [f'(b) - f'(a)] \right| \leq \frac{(b - a)^3}{24} \left( \frac{1}{(p + 1)(p + 3)} \right)^{\frac{1}{p}} \left[ |f'''(a)|^q + |f'''(b)|^q \right]^{\frac{1}{q}}
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. Since \( |f'''|^q \) belongs to \( Q(I) \), from Lemma 1 and using the Hölder inequality we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx - \frac{b - a}{12} [f'(b) - f'(a)] \right| \leq \frac{(b - a)^3}{12} \left( \int_0^1 t (1 - t) |2t - 1|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 t(1 - t) |f'''(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}} \\
\leq \frac{(b - a)^3}{12} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left( \int_0^1 t(1 - t) \left[ \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1 - t} \right] dt \right)^{\frac{1}{q}} \\
= \frac{(b - a)^3}{12} \frac{1}{\left(2 (p + 1)(p + 3)\right)^\frac{1}{p}} \left[ |f'''(a)|^q + |f'''(b)|^q \right]^{\frac{1}{q}},
\]
where we used
\[
\int_0^1 t (1 - t) |2t - 1|^p dt = \frac{1}{2 (p + 1)(p + 3)}.
\]
The proof is completed. \[\Box\]

Following result is obtained via Lemma 2.

**Theorem 6.** Let \( f'' : I \subseteq \mathbb{R} \to \mathbb{R} \) be an absolutely continuous function on \( I \) such that \( f''' \in L[a, b] \), where \( a, b \in I \), \( a < b \). If \( |f'''|^q \) belongs to \( Q(I) \), then the following
inequality holds

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] \right| \\
\leq \frac{(b-a)^4}{6} \left( \frac{1}{192} \right)^{1-\frac{1}{q}} \left\{ \left( \left( \frac{|f''(a)|^q}{48} + \left( \frac{17}{48} - \frac{1}{2} \ln 2 \right) |f'''(b)|^q \right)^{\frac{1}{q}} \right) \\
+ \left( \left( \frac{17}{48} - \frac{1}{2} \ln 2 \right) |f''(a)|^q + \frac{|f'''(b)|^q}{48} \right)^{\frac{1}{q}} \right\}
\]

for \( q \geq 1 \).

**Proof.** Since \( |f'''| \) belongs to \( Q(I) \), from Lemma 2 and using the power mean inequality we have

\[
(2.1) \quad \left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] \right| \\
\leq \frac{(b-a)^4}{6} \left( \frac{1}{192} \right)^{1-\frac{1}{q}} \left\{ \left( \int_0^{1/2} t^2 \left( \frac{1}{2} - t \right) dt \right)^{1-\frac{1}{q}} \left( \int_0^{1/2} t^2 \left( \frac{1}{2} - t \right) |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
+ \left( \int_0^{1/2} t^2 \left( \frac{1}{2} - t \right) \left( \frac{|f''(a)|^q}{t} + \frac{|f'''(b)|^q}{1-t} \right) dt \right) \right\}^{\frac{1}{q}}
\]

If we use the inequalities below in (2.1), we get the desired result:

\[
\int_0^{1/2} t^2 \left( \frac{1}{2} - t \right) dt = \int_0^{1/2} (t-1)^2 \left( t - \frac{1}{2} \right) dt = \frac{1}{192}
\]

and

\[
\int_0^{1/2} t^2 \left( \frac{1}{2} - t \right) \frac{1}{1-t} dt = \int_0^{1/2} \frac{1}{t} (t-1)^2 \left( t - \frac{1}{2} \right) dt \\
= \frac{17}{48} - \frac{1}{2} \ln 2.
\]

\[ \Box \]

**Corollary 2.** In Theorem 2, if we choose \( q = 1 \) we obtain the following inequality

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] \right| \\
\leq \frac{(b-a)^4}{6} \left( \frac{3}{8} - \frac{1}{2} \ln 2 \right) \left[ |f''(a)| + |f'''(b)| \right] .
\]
References

[1] E. K. Godunova and V. I. Levin, ‘Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions’ in: Numerical Mathematics and Mathematical Physics (Moskov. Gos. Ped. Inst, Moscow, 1985), pp. 138–142, 166 (in Russian).

[2] M. S. Moslehian and M. Kian, Jensen type inequalities for $Q$–class functions, Bull. Aust. Math. Soc. 85 (2012), 128–142, doi:10.1017/S0004972711002863.

[3] L. Chun and F. Qi, Integral inequalities for Hermite-Hadamard type for functions whose 3rd derivatives are $s$–convex, Applied Mathematics, 3 (2012), 1680-1885.

[4] M. Alomari and S. Hussain, Two inequalities of Simpson type for quasi–convex functions and applications, Applied Mathematics E-Notes, 11 (2011), 110-117.

♦ ATATURK UNIVERSITY, K.K. EDUCATION FACULTY, DEPARTMENT OF MATHEMATICS, 25240 CAMPUS, ERZURUM, TURKEY
E-mail address: emos@atauni.edu.tr

* ADIYAMAN UNIVERSITY, FACULTY OF SCIENCE AND ART, DEPARTMENT OF MATHEMATICS, 02040, ADIYAMAN, TURKEY
E-mail address: mavci@posta.adiyama.edu.tr

▲ Ağrı İbrahim Çeçen University, Faculty of Education, Department of Mathematics, 04100, Ağrı, Turkey
E-mail address: mgurbuz@agri.edu.tr