NON-LIPSCHITZ FLOW OF THE NONLINEAR SCHröDINGER EQUATION ON SURFACES

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Abstract. We construct non-Lipschitz flow in $H^s$ for the cubic nonlinear Schrödinger equation on the 2-torus of revolution with a Lipschitz or smooth metric. The non-Lipschitz property holds for all $s < 2/3$ for Lipschitz metric and $s < 1/2$ for smooth metric. Both coincide with the Sobolev exponents for uniform local well-posedness.

1. Introduction

We consider the Cauchy problem for the cubic nonlinear Schrödinger equation on compact Riemannian surfaces without boundary:

$$
\begin{aligned}
&i \frac{\partial}{\partial t} u = -\Delta u + |u|^2 u, \\
&u(t = 0) = u_0,
\end{aligned}
$$

with either a smooth or Lipschitz metric $g$ and $\Delta$ is the corresponding Laplace-Beltrami operator. (Surfaces with smooth boundary are included in Lipschitz $g$.) We say that the Cauchy problem (1) is uniformly locally well-posed in $H^s$ if for all $R > 0$, there exists $T > 0$ and a Banach space $X_T$ included in $C^0((-T,T), H^s)$ such that for all $f \in H^s$ with $\|f\|_{H^s} \leq R$, (1.1) has a unique solution $u \in X_T$ and the flow map is uniformly continuous in $C^0$. The problem is globally well-posed if $T$ can be arbitrary large (without the uniformity requirement).

On the flat torus, using multiple space-time Fourier series, Bourgain [B] proved that (1) is locally well-posed in $H^s$ for all $s > 0$. Later Burq, Gerard and Tzvetkov [BGT1] proved local well-posedness in $H^s$ for $s > 1/2$ for smooth $g$; Blair, Smith and Sogge proved $s > 2/3$ for Lipschitz $g$ (cf. also [A]). The latter two use dispersive estimates valid on short time intervals. So (1.1) is energy subcritical and has global solutions in $H^s$ for $s \geq 1$. Recently, Hani [H] proved moreover that (1) has global solutions in $H^s$ for $s > 2/3$ and smooth $g$.

On the flat 2-torus, it is known [B, CCT] that $s > 0$ is optimal. On the 2-sphere, it is known [BGT2, 3] that $s > 1/4$ is optimal instead of $1/2$. The purpose of this note is to present a simple construction valid for smooth as well as Lipschitz $g$ to exhibit
non-Lipshitz flow for \( s < 1/2, 2/3 \) respectively. For Lipshitz \( g \), we assume that the singularities are of type \( |x| \).

Concretely, on the 2-torus of revolution we prove

**Proposition.** Let \( ds^2 = dx^2 + g(x)dy^2 \) with \( g \in C^3 \) or Lipshitz with isolated singularities and admitting a unique global maximum. Then there are initial data, which are eigenfunctions of the Laplacian, such that the flow map is not Lipshitz in \( H^s \) for \( s < 1/2 \) when \( g \) is \( C^3 \) and \( s < 2/3 \) when \( g \) is only Lipshitz.

**Remark.** We relate the time scale and the Sobolev scale when this non-Lipshitz behavior is manifest. The proposition uses data at single high frequency \( k \gg 1 \) and non-Lipshitz behavior is observed at time scale \( t \approx k^{1/2} \) for smooth \( g \) and \( t \approx k^{2/3} \) for Lipshitz \( g \). For \( t \) up to \( O(1) \), non-Lipshitz flow is observed for \( s < 1/4 \) and \( 1/3 \) respectively.

We note that this transition from \( 1/4 \) to \( 1/2 \) also occurs on the 2-sphere. So the Sobolev exponent \( 1/4 \) obtained in [BGT2, 3] is strictly local.

Previously, for smooth \( g \), non-Lipshitz flow is known [T] for \( s < 1/4 \) under the assumption of a stable non-degenerate periodic geodesic for \( t \to 0 \) as a negative power of \( k \) using semi-classical constructions. The fact that \( 1/2 \) and \( 2/3 \) are observed on the torus of revolution, but at a longer time scale essentially reflects the stability of high frequency data, cf. [H] in the present context and [W1, 2] (also the review paper [W3]) in the energy supercritical context.

### 2. Proof

Let the torus be the set \([-\pi, \pi)^2\] with the metric \( g \) and identify the end points. From separation of variables, the spectrum of the Laplacian decomposes into:

\[
\sigma(\Delta) = \bigoplus_k \sigma\left(-\frac{d^2}{dx^2} + k^2 g^{-1}\right),
\]

where \( k \in \mathbb{Z} \) is the Fourier dual of \( y \). We investigate, in the high frequency limit: \( |k| \gg 1 \), the ground state eigenfunction and the first two eigenvalues of the Schrödinger operator defined in the right side of (2). Below we assume \( k \) is positive, as negative \( k \) works likewise.

Assume \( g \) is only Lipshitz, \( g \) smooth works similarly, cf. [C]. Since \( g \) has only isolated singularities and a unique global maximum and we are interested in the semi-classical limit \( k \gg 1 \), it suffices to take \( g = (|x| + 1)^{-1} \) with the unique singularity and global maximum at \( x = 0 \). Other cases can be reduced to this.
The Schrödinger operator is then
\[ H = -\frac{d^2}{dx^2} + k^2 |x| + k^2, \] (3)
on \(L^2[-\pi, \pi]\) with periodic boundary conditions. The reference operator is therefore
\[ A = -\frac{d^2}{dx^2} + k^2 |x| \] (4)
on \(L^2(\mathbb{R})\). It is classical that \(A\) has eigenvalues \(\alpha_n\), which are deduced from the zeroes of the Airy function or its derivative and that its eigenfunctions \(\psi_n\) are obtained from the Airy function for positive arguments so that the \(n\)th eigenfunction has parity \((-1)^n\), \(n = 0, 1\ldots\)

More precisely, \(\psi_n\) can be written as
\[
\psi_n(x) = C_n Ai(k^{2/3} |x| - a_n), \quad x \geq 0,
\]
\[
= (-1)^n C_n Ai(k^{2/3} |x| - a_n), \quad x < 0,
\]
where \(C_n\) is a normalizing constant and for \(n\) even, \(a_n\) is the \(n/2 + 1\) zero of the derivative of the Airy function \(Ai'(x)\) and for \(n\) odd, \(a_n\) is the \((n + 1)/2\) zero of the Airy function \(Ai(x)\).

Below \(k\) is large, \(k \gg 1\). For us, it suffices to know that
\[
\alpha_0, \text{ and } \alpha_1 - \alpha_0 = O(k^{4/3}),
\] (5)
\[
\|\psi_0\|_\infty = k^{1/3} \|\psi_0\|_2,
\] (6)
and
\[
|\psi_0(x)| \sim k^{1/6} e^{-k|x|^{3/2}} \frac{1}{|x|^{1/4}}, \quad |x| > k^{-2/3}.
\] (7)

Let \(\phi_0\) be the ground state eigenfunction, \(\lambda_0\) and \(\lambda_1\) the first two eigenvalues of \(H\). Using (5, 7) and standard perturbation theory, we then obtain
\[
\lambda_0 = k^2 + O(k^{4/3}) + O(e^{-k/2}) = k^2 + O(k^{4/3})
\] (8)
\[
\lambda_1 - \lambda_0 = O(k^{4/3}),
\] (9)
and
\[
\phi_0 = \psi_0 + O(e^{-k/2}),
\] (10)
where the \(O\) is in \(L^2\), which in turn gives
\[
\|\phi_0\|_\infty \sim k^{1/3} \|\phi_0\|_2
\] (11)
using (6) and Sobolev embedding in one dimension.

We now proceed to study the Cauchy problem (1) with the initial condition \( u_0 \):

\[
u_0(x, y) = ae^{iky}\phi_0(x),
\]

where \( \phi_0 \) is assumed to be normalized, \( \|\phi_0\|_2 = 1 \), \( a = O(k^{-s}) \), \( s \geq 0 \), so that \( \|u_0\|_{H^s} = O(1) \). The solution \( u \) can be written as

\[
u(x, y, t) = e^{iky}v(x, t)
\]

with \( v \) satisfying

\[
\begin{align*}
i \frac{\partial}{\partial t} v &= (-\frac{d^2}{dx^2} + k^2g^{-1})v + |v|^2v, \\
v(t = 0) &= a\phi_0.
\end{align*}
\]

We seek \( v \) in the form

\[
v(x, t) = a\gamma(t)e^{it(\lambda_0 + a^2\omega)}\phi_0(x) + \sum_{j=1}^{\infty} q_j(t)\phi_j(x),
\]

where \( \gamma(0) = 1 \), \( q_j(0) = 0 \),

\[
\omega = \frac{1}{2} \|\phi_0\|_4^4
\]

and \( \phi_j \) is the \( j \)th eigenfunction of the Schrödinger operator in (3). We note that \( \omega \) is the frequency modulation that is at the root of this non-Lipschitz flow.

From energy conservation, we have

\[
a^2\lambda_0 + \frac{1}{2}a^4\|\phi_0\|_4^4 = a^2\gamma^2(t)\lambda_0 + \sum_{j=1}^{\infty} \lambda_j|q_j(t)|^2 + \frac{1}{2}\|v\|_4^4,
\]

since \( \|\phi_0\|_2 = 1 \). Using \( L^2 \) conservation:

\[
a^2\gamma^2 + \sum |q_j|^2 := a^2\gamma^2 + \|q\|_2^2 = a^2,
\]

we then obtain from (16)

\[
\frac{1}{2}a^4\|\phi_0\|_4^4 = \sum (\lambda_j - \lambda_0)|q_j|^2 + \frac{1}{2}\|v\|_4^4
\]

\[
\geq k^{4/3}\|q\|_2^2 + \frac{1}{2}\|v\|_4^4,
\]
where we used (9) to reach the last estimate.

(18) gives the following estimates valid for all $t$:

\[ \|q\|_4 \leq \|v\|_4 + a\|\phi_0\|_4 \leq 2a\|\phi_0\|_4, \]  

(19) \[ \|q\|_2 \leq k^{-2/3}a^2\|\phi_0\|_4^2. \]  

(17, 20) then give for all $t$:

\[ (1 - \gamma^2(t)) \leq k^{-4/3}a^2\|\phi_0\|_4^4, \]  

(21) Further, the first line of (18) gives

\[ \frac{1}{2}a^4\|\phi_0\|_4^4 \geq k^{4/3(1-s)}\sum(\lambda_j - \lambda_0)^s|q_j|^2 \geq k^{4/3(1-s)}(\|q\|_{H^s}^2 - \lambda_0\|q\|_2^2), \quad s < 1, \]

which leads to

\[ \|q\|_{H^s} \leq a^2\|\phi_0\|_4^2k^{s-2/3}, \quad s < 1, \]  

(22) where we also used (20).

To solve for $\gamma$, we project $v$ onto $\phi_0$ and obtain

\[ \gamma(t) = a^{-1}e^{it(\lambda_0 + a^2\omega)}(v, \phi_0). \]

So time derivative satisfies

\[ i\dot{\gamma}(t) = -a^{-1}(\lambda_0 + a^2\omega)e^{it(\lambda_0 + a^2\omega)}(v, \phi_0) + a^{-1}e^{-it(\lambda_0 + a^2\omega)}([(-\frac{d^2}{dx^2} + k^2g^{-1})v + |v|^2v], \phi_0), \]

where we used (14). The choice of $\omega$ cancels the leading order nonlinear term and we obtain

\[ |\dot{\gamma}| \leq a^2\omega(1 - |\gamma|^2)|\gamma| + a^2\mathcal{O}(\|q\|_2\|\phi_0\|_6^3 + \|\phi_0\|_4^2\|q\|_2^2|\gamma| + \|\phi_0\|_\infty\|q\|_4^2\|q\|_2). \]

Using

\[ a = \mathcal{O}(k^{-s}), \quad \|\phi_0\|_4 \asymp k^{1/6}, \quad \|\phi_0\|_6 \asymp k^{2/9} \text{ and } \|\phi_0\|_\infty \asymp k^{1/3}, \]

we obtain

\[ \dot{\gamma} = \mathcal{O}(k^{-4s} + k^{-1/3}s^{-4s} + k^{-6s} + k^{-6s+1/3}), \quad s < 1. \]

We note that the precise value of the right side is not important as long as $\dot{\gamma}t$ is sufficiently small.
Let $S_t$ be the flow at $t$ (if it exists). We note that for $u_0$ of the form (12), the solution $u_t$ exists globally in $H^1$. Let $s \sim 2/3^- := 2/3 - \delta$ for arbitrarily small $\delta > 0$. Choose two initial data as in (12) with $a_1 = k^{-2/3+\delta}$, $a_2 = a_1 + \epsilon$ with $\epsilon = k^{-2/3-2\delta}$ and $t \asymp k^{2/3}$.

Let $a = a_1$, $a_2$ and $q = q_1$, $q_2$ be the remainders, then using also (15, 22) we have

$$
\|q\|_{H^s} \leq k^{-1+\delta} \ll a \omega t \asymp k^{-\delta} \ll 1,
$$

$$
a^2 \omega t \asymp k^{2\delta} \gg 1.
$$

So

$$
\|S_t\|_{\text{Lip}} \geq \frac{\|u_t^{(2)} - u_t^{(1)}\|_{H^s}}{\|u_0^{(2)} - u_0^{(1)}\|_{H^s}} \asymp a^2 \omega t \asymp k^{2\delta} \gg 1
$$

for $t \asymp k^{2/3}$, $k \gg 1$.

Using exactly the same argument for smooth $g$ but Hermite instead of Airy function gives

$$
\|S_t\|_{\text{Lip}} \geq k^{0^+}
$$

for $t \asymp k^{1/2}$ in $H^{1/2-}$, cf. [C]. □

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