Abstract  The fact that one must evaluate the near-extremal and near-horizon limits of Kerr spacetime in a specific order, is shown to lead to discontinuity in the extremal limit, such that this limiting spacetime differs nontrivially from the precisely extremal spacetime. This is established by first showing a discontinuity in the extremal limit of the maximal analytic extension of the Kerr geometry, given by Carter. Next, we examine the ISCO of the exactly extremal Kerr geometry and show that on the event horizon of the extremal Kerr black hole, it coincides with the principal null geodesic generator of the horizon, having vanishing energy and angular momentum. We find that there is no such ISCO in the near-extremal geometry, thus garnering additional support for our primary contention. We relate this disparity between the two geometries to the lack of a trapping horizon in the extremal situation.

Keywords  ISCO; Extremal Kerr Blackhole; Trapped Surfaces.

1 Introduction

The extremal limit for four dimensional Reissner Nordstrom (RN) black holes has been extensively studied recently to probe whether the limit is continuous. The definition of the Entropy Function used to compare the
‘macroscopic’ entropy of a class of extremal black holes solutions of the supergravity limit of string theories, to the ‘microscopic’ entropy obtained from counting of string states \[5\] requires a bifurcate horizon -an attribute that extremal spacetimes do not possess. The trick of using a generic near-extremal black hole spacetime to evaluate the entropy function, though widely used in the literature, suffers from the pitfall that the extremal limit may have subtle discontinuities \[2\], \[3\], and may be different from the precisely extremal geometry. On our part, it has been shown \[1\] that there exists a class of stable circular orbits on the event horizon of the extremal Reissner Nordstrøm spacetime which coincides with the principal null geodesic generator of the horizon. The spacetime infinitesimally near the extremal does not admit this class of geodesics.

In this paper we examine similar features of the Kerr spacetime : whether in the extremal limit there exist geodesics on the horizon which can be characterized as ISCOs, as compared with the precisely extremal situation. Here, we employ Carter’s \[7,8\] maximal analytic extension instead to investigate the continuity of the extremal limit vis-a-vis the class of circular geodesics mentioned. While the use of conserved scalars like ‘energy’ and ‘angular momentum’ of test particles, both massive and massless, to study their geodesics, is usually deemed quite adequate in any coordinate system, here, there is a word of caution: since the geodesics in question lie on the horizon (and coincide with the null geodesic generator \[1\], \[6\]), the use of coordinate charts smooth on the horizon is certainly preferable to the use of charts which are not. Otherwise one may be led to conclusions which may have additional subtle pitfalls. The main result that emerges from our assay is what has been suspected earlier \[2\] and pointed out recently \[1\] for the Reissner Nordstrøm spacetime : all aspects of the extremal Kerr spacetime do not manifest themselves in the near-extremal limit. In particular, the class of geodesics close to the horizon in the extremal case does not overlap with the class in the case that we are infinitesimally close to the extremal.

The plan of the paper is as follows: in section \[2\], we motivate our paper by showing that the order in which the near-extremal and the near-horizon limits of the Kerr metric are taken, is important, following a similar demonstration made in \[1\] for the RN spacetime. The near horizon geometry of the extremal black hole is not the same as the extremal limit of the near-horizon geometry of the generic non-extremal black hole. In section \[3\] the Carter analytic extension of the Kerr spacetime is discussed, on the axis of symmetry, and the difficulties of extracting the analytic extension of the extremal geometry from the extremal limit of the analytic extension of the generic spacetime are elucidated. Equatorial circular orbits are next considered in detail in section \[4\] with regard to their stability, for timelike (outside the ergo-region) geodesics, and compared for the extremal and the near-extremal geometries. Section \[5\] gives a discussion on the non-existence of trapping surfaces in the precisely extremal situation and contrasts this with the near-extremal situation. We conclude our discussion in section \[7\]. The Carter maximal extensions off the symmetry axis for extremal spacetimes are discussed in an appendix.
2 Motivation

Consider the Kerr metric in Boyer-Lindquist coordinates,

$$ds^2 = -\frac{\Delta}{\rho^2} \left[ dt - a \sin^2 \theta d\phi \right]^2 + \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2) d\phi - adt \right]^2 + \rho^2 \left[ \frac{dr^2}{\Delta} + d\theta^2 \right].$$

where

$$a \equiv \frac{J}{M}, \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta$$

$$\Delta \equiv r^2 - 2Mr + a^2 \equiv (r - r_+)(r - r_-)$$

For simplicity, taking the metric along the axis of symmetry ($\theta = 0$), the Lorentzian 2-fold spanned by the coordinates $r, t$ has three possible geometries namely, near-horizon geometry, near extremal geometry and the precisely extremal geometry. They are defined in terms of two non-negative parameters $\epsilon \equiv (r - r_1)/r_1 << 1$ and $\delta \equiv (r_+ - r_-)/r_1 << 1$ where $r_1$ is the radius of the event horizon in the extremal case. In terms of these parameters,

$$ds^2 = -\epsilon(\epsilon + \delta) dt^2 + \frac{(1 + \epsilon)}{\epsilon(\epsilon + \delta)} dr^2.$$  

which gives two different limiting geometries depending on the order in which the limits $\delta \to 0$ and $\epsilon \to 0$ are taken. First taking the extremal limit $\delta \to 0$ and then the near-horizon limit, the local geometry is that of an $AdS_2$.

$$ds^2 \simeq -\epsilon^2 dt^2 + \frac{r_1^2}{\epsilon^2} d\epsilon^2.$$  

If, in contrast, the near horizon limit is taken before the extremal limit, one obtains

$$ds^2 \simeq -\epsilon \delta dt^2 + \frac{r_1^2}{\epsilon \delta} d\epsilon^2.$$  

which indicates that the local geometry is not an $AdS_2$ and the extremal limit is indeed now singular. What this establishes is the subtlety that the near-horizon geometry of the extremal spacetime is not the same as the extremal limit of the near-horizon geometry of the generic non-extremal spacetime, as advertised in the Introduction. Of course, for the latter case, the behavior of the spacetime away from a bifurcation surface is what is being considered.

3 Carter’s Maximal Analytical Extension of Kerr Spacetime Along the Symmetric Axis

The maximal analytic extension of the extremal Kerr spacetime along the axis of symmetry was first reported by Carter [7]. Since, there is no discussion in the literature about the discontinuity of this extension at the extremal limit $r_+ \to r_-$, we present here the complete maximal analytic extension following Carter, showing that in the extremal limit $r_+ \to r_-$ it is indeed discontinuous, necessitating a separate treatment.
3.1 Non-Extremal Case:

The tortoise coordinate $r^*$ is given by

\[
\frac{dr^*}{\Delta} = \frac{(r^2 + a^2)dr}{(r - r_-)(r - r_+)}.
\] (6)

Integrating this equation, we obtain

\[
F(r) = 2r^* = 2r + \kappa^{-1}_+ \ln |r - r_+| + \kappa^{-1}_- \ln |r - r_-| \quad (7)
\]

with $\kappa_{\pm}^{-1} \equiv \frac{2(r^2_\pm + a^2)}{(r_\pm - r_\mp)}$ and where as usual $r_{\pm} = M \pm \sqrt{M^2 - a^2}$. $\kappa_{\pm}$ is the surface gravity of the respective horizons. The outer horizon $r_+$ is an event horizon and the inner horizon $r_-$ is a Cauchy horizon. $\kappa_{\pm}$ are both positive. $F(r)$ is monotonic in each of the regions

\[
\text{Region I} : \quad (r_+ < r < \infty) \\
\text{Region II} : \quad (r_- < r < r_+) \\
\text{Region III} : \quad (0 < r < r_-) \quad (8)
\]

and blows up at the boundaries of the regions. Clearly, it is impossible to define a single coordinate patch which is regular (in terms of geodetic completeness) over the entire spacetime.

Near the event horizon $r = r_+$, the tortoise coordinates is given by

\[
r^* \approx \frac{1}{2\kappa_+} \ln |r - r_+| . \quad (9)
\]

Here $r^*$ has logarithm dependence and is singular at $r = r_+$. Therefore, introducing double null coordinates $u \equiv r^* + t, v \equiv r^* - t$, it is obvious that the event horizon $r = r_+$ occurs at $v + u = -\infty$. The analytic extension of the Kerr spacetime is obtained by choosing in the neighborhood of the event horizon the coordinate system

\[
U^+ = -\exp (-\kappa_+ u), \quad V^+ = \exp (\kappa_+ v) \quad (10)
\]

Therefore the metric becomes near $r = r_+$ along the axis of symmetry

\[
ds^2 \equiv \frac{r_+ r_- \exp (-2\kappa_+ r)}{\kappa_+^2} \left( \frac{r_-}{r - r_-} \right)^{\kappa_+ / \kappa_- - 1} dU^+ dV^+
\]

where

\[
U^+ V^+ = -\exp (2\kappa_+ r) \left( \frac{r - r_+}{r_+} \right) \left( \frac{r - r_-}{r_-} \right)^{\kappa_+} \quad (11)
\]

It is clear that $U^+ = V^+ = 0$ corresponds to the bifurcation 2-sphere for the generic Kerr spacetime. The coordinate patch used here is smooth in the
neighbourhood of the event horizon but not near the Cauchy horizon at \( r = r_- \). Therefore, when the two horizons merge in the extremal limit, this coordinate patch is invalid. This conundrum shows up in some metric coefficients in (11) blowing up in the extremal limit \( r_+ \rightarrow r_- \).

Similarly, a coordinate chart that is smooth across the Cauchy horizon can be constructed. The metric near \( r = r_- \) along the symmetry axis

\[
ds^2 = - \frac{r_+ r_-}{\kappa_-^2 (r^2 + a^2)} \left( \frac{r_+}{r - r_+} \right)^{\frac{\kappa_-}{\kappa_- + 1}} dU^- dV^-
\]

where

\[
U^- V^- = - \exp (2\kappa_- r) \left( \frac{r - r_-}{r_+} \right)^{\frac{\kappa_-}{\kappa_- + 1}} \left( \frac{r - r_+}{r_+} \right)^{\frac{\kappa_+}{\kappa_+ + 1}} \tag{12}
\]

Once again, this coordinate chart is not a valid one near the event horizon, and expectedly, the extended metric (12) in the neighborhood of the \( r_- \) blows up in the extremal limit \( r_+ \rightarrow r_- \).

### 3.2 Extremal Case:

Thus we cannot obtain the complete maximal analytic extension along the axis of symmetry of the extremal Kerr spacetime as a limiting case of the non-extremal Kerr spacetime; the extremal case needs to be treated separately [7].

The tortoise coordinate in this case is given by

\[
r^* = \int \frac{(r^2 + M^2)dr}{(r - M)^2} = r + 2M \left[ \ln |r - M| - \frac{M}{r - M} \right] \quad (13)
\]

Near the horizon \( r = M \) this has a leading pole-type singularity

\[
r^* \approx \frac{2M^2}{(r - M)} \quad \tag{14}
\]

instead of a logarithmic one. Defining the double null coordinates \( u \) and \( v \) as \( u \equiv r^* + t, \ v \equiv r^* - t \), the metric is given by

\[
ds^2 = \frac{(r - M)^2 \, dudv}{r^2 + M^2} \quad (15)
\]

To determine the position of event horizon at a finite region in the coordinate chart, we follow ref. [7] and introduce universal coordinates \( U, V \) such that

\[
u = \tan U, \ v = \cot V \quad (16)
\]

This implies that

\[
\tan U + \cot V = 2r^* (U, V) \quad (17)
\]
Therefore the extremal Kerr metric along the axis of symmetry in the Carter coordinate system is given by

$$ds^2 = -\frac{(r - M)^2}{r^2 + M^2} \sec^2 U \csc^2 V dUdV .$$ (18)

The geodesic orbits of the test particle on the 2-fold can be obtained by considering the spacetime on the symmetry axis. Conserved quantities like test particle energy and angular momentum (both per unit mass) are defined in terms of Killing vector fields corresponding to stationarity and axisymmetry of the Kerr spacetime. E.g., the energy is defined as $\xi \cdot u \equiv -E$, where $u$ is the four velocity of the test particle and $\xi$ is the energy per unit mass of the test particle. The timelike Killing vector denoted as $\xi \equiv \partial_t$ and its components are $\xi^U = \cos^2 U , \xi^V = \sin^2 V$. In the next section we will be describe it in more detail. Now using the normalization condition of four velocity and for circular orbit we get the part of the squared potential along the symmetry axis

$$\mathcal{E}^2 = V_{\text{squared}} = -\frac{(r - M)^2}{r^2 + M^2} \kappa,$$ (19)

where $\kappa = -1$ for timelike geodesics and $\kappa = 0$ for null geodesics. In other words, the ISCO turns into a null geodesic on the horizon $r(U, V) = M$ with the energy vanishing as well. Thus, on the horizon, the only class of geodesics that survives must be null with vanishing energy. Hence as $r(U, V) \to M$

$$\mathcal{E}^2 = V_{\text{squared}} = U_{\text{squared}} = 0$$ (20)

where $V_{\text{squared}}$ and $U_{\text{squared}}$ are the squared potentials corresponding to the timelike and null cases respectively. This implies that in the extremal geometry, the geodesic on the horizon corresponds to the null geodesic generator.

4 Equatorial Circular Orbits in Kerr spacetime: Doran/Ingoing Kerr Coordinates

To compute the effective potential for circular orbits we choose nonsingular coordinates like ingoing Kerr coordinates rather than the Boyer Lindquist coordinates. In this coordinate system the metric[7] can be written as

$$ds^2 = -(1 - \frac{2Mr}{\rho^2})dv^2 + 2dvd\rho + \rho^2 d\theta^2 - 2a \sin^2 \theta d\rho d\tilde{\phi} + \frac{1}{\rho^2} [(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta] \sin^2 \theta d\tilde{\phi}^2 - \frac{4aMr}{\rho^2} \sin^2 \theta d\tilde{\phi} dv .$$ (21)

Taking the transformations

$$dv = dt + \frac{dr}{1 + \sqrt{2Mr(r^2 + a^2)}}$$ (22)

$$d\tilde{\phi} = d\phi + \frac{a}{r^2 + a^2 + \sqrt{2Mr(r^2 + a^2)}} dr .$$ (23)
we obtain the Doran form of Kerr metric\[11\] which is as follows:

\[
    ds^2 = -dt^2 + \left[ \frac{2Mr}{\rho^2}(dt - a\sin^2\theta d\phi) + \sqrt{\frac{\rho^2}{r^2 + a^2}} \right]^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2.
\]  

(24)

4.1 Circular Geodesics in the Precisely Extremal Case

On the equatorial plane \(\theta = \pi/2\), the extremal Kerr metric in Doran coordinate is given by

\[
    ds^2 = -(1 - 2\frac{M}{r})dt^2 - \frac{4M^2}{r}dtd\phi + 2\sqrt{\frac{2Mr}{r^2 + M^2}}dtdr - 2M\sqrt{\frac{2Mr}{r^2 + M^2}}drd\phi + \frac{r^2}{r^2 + M^2}dr^2
\]

\[+ \left[ r^2 + M^2 + \frac{2M^3}{r} \right]d\phi^2. \tag{25} \]

In Doran coordinates, the spacetime allows the timelike Killing vector \(\xi \equiv \partial_t\), whose projection along the 4-velocity \(u^t = -1\) for timelike and \(u^t = 0\) for null) of geodesics: \(\xi \cdot u = -E\), is conserved along such geodesics. There is also the ‘angular momentum ’ \(L \equiv \zeta \cdot u\) (where \(\zeta \equiv \partial_\phi\)) which is similarly conserved. Thus, in this coordinate chart, \(E\) and \(L\) can be expressed as

\[
    E = (1 - 2\frac{M}{r})u^t - \sqrt{\frac{2Mr}{r^2 + M^2}}u^r + \frac{2M^2}{r}u^\phi \tag{26}
\]

\[
    L = (r^2 + M^2 + \frac{2M^3}{r})u^\phi - M\sqrt{\frac{2Mr}{r^2 + M^2}}u^r - \frac{2M^2}{r}u^t. \tag{27}
\]

We can not solve \(u^t\) and \(u^\phi\) in terms of \(E\) and \(L\) on the horizon because in that situation \(L = 2M\) with

\[
    E = -u^t + 2Mu^\phi \tag{28}
\]

where, on the null surface \(r = M\) , \(u^r = 0\). Because of this degeneracy, a nontrivial solution for \(u^t\), \(u^\phi\) ensues if and only if \(E = 0 = L\). The norm of the four velocity at \(r = M\) is thus

\[
    u \cdot u = (u^t)^2 + 4M^2(u^\phi)^2 - 4Mu^\phi u^t = E^2 \tag{29}
\]

implying that this is a null geodesic generator of the horizon. This agrees with the foregoing analysis in the Carter frame in the previous section. Precisely on the event horizon in the exactly extremal geometry, there is no geodesic with non-vanishing energy or angular momentum (per unit mass).
4.2 Near Extremal Kerr Spacetime:

Proceeding similarly for non-extremal Kerr spacetime in Doran coordinates, the energy and angular momentum per unit mass of the test particle are

\[ E = \left(1 - \frac{2M}{r}\right)u^t - \sqrt{\frac{2Mr}{r^2 + a^2}} u^r + \frac{2aM}{r} u^\phi. \]  

(30)

\[ L = (r^2 + a^2 + \frac{2Ma^2}{r})u^\phi - \frac{a}{\sqrt{\frac{2Mr}{r^2 + a^2}}} u^r - \frac{2aM}{r} u^t. \]  

(31)

For circular orbits at \( r = r_0 \), \( u^r = 0 \), so that

\[ u^t = \frac{1}{\Delta} \left[ (r_0^2 + a^2 + \frac{2Ma^2}{r})E - \frac{2aM}{r_0} L \right]. \]  

(32)

\[ u^\phi = \frac{1}{\Delta} \left[ (1 - \frac{2M}{r_0})L + \frac{2aM}{r_0} E \right]. \]  

(33)

The squared norm of the four velocity

\[ u \cdot u = -(1 - \frac{2M}{r_0})(u^t)^2 - \frac{4aM}{r_0} u^\phi u^t + (r_0^2 + a^2 + \frac{2Ma^2}{r_0})(u^\phi)^2 \]  

(34)

which reduces to

\[ u \cdot u = -E u^t + L u^\phi \]  

(35)

More explicitly

\[ u \cdot u = \frac{1}{\Delta} \left[ (L^2 - a^2L^2) - r_0^2 E^2 - \frac{2M}{r_0} (L - aE)^2 \right]. \]  

(36)

Now if we take the extremal limit \( a \to M \) and the near-horizon limit \( \Delta \equiv (r_0 - M)^2 \to 0 \), one obtains,

\[ u \cdot u = -\frac{(L - 2ME)^2}{(r_0 - M)^2}. \]  

(37)

The proof that in the extremal limit the circular ISCO turns into a null geodesic generator must first demonstrate that the 4-velocity of the geodesic has vanishing norm on the horizon. However, it is not clear that eq. (37) does lead to that, except for a specific order of limits mentioned above. Thus, in the extremal limit, we do have \( L = 2ME \), but to get \( u \cdot u = 0 \), we must move slightly away from the horizon \( r_0 = M \); but this new hypersurface is not necessarily null, so the geodesic on it need not be null. On the other hand, if we move away from extremality \( L = 2ME \), then the near-horizon value of \( u \cdot u \) diverges, and one can scarcely claim the existence of a null geodesic generator. There may be a way to evaluate the limits simultaneously, but it is not clear how that establishes that the geodesic on the horizon at extremality is null. This conundrum is related to the issues mentioned in the Introduction,
which motivate our primary contention that the extremal limit and the exactly extremal spacetimes have subtle disparities. It may be noted that our computational results in this subsection are consistent with those in ref. [6], as they are with those of the earlier paper [9]. However, rather than focus on the proper distance between the bifurcation sphere (nonexistent in the precisely extremal geometry) and the geodesic in question, we have chosen here to focus on the squared norm of the geodesic on the horizon in the extremal limit, and compare it with the behaviour in the exactly extremal situation. It is not enough to merely argue that the extremal limit exists, it is important to establish that an independent treatment of the exactly extremal case yields the same results as in the limiting situation. Since the precisely extremal situation has not been analyzed in [6], the comparison of the results there with those in the exactly extremal geometry cannot be made. In contrast, in this paper, we have already discussed the exactly extremal Kerr geometry in the last subsection, where ambiguities in evaluating limits seen in this subsection are not present. This is yet another instance of the subtle disparity between the exactly extremal and the extremal limit of a non-extremal Kerr spacetime, as we have contended, now observed within the Doran frame. In the next section, we reanalyze the extremal situation using Carter’s maximal analytic extension of the extremal Kerr geometry.

5 Carter’s Maximal Analytic Extension of Extremal Kerr Spacetime:

In section 4.1 we observed that both Doran and ingoing Kerr coordinates are better behaved at \( r = M \) than the Boyer-Lindquist coordinates; but they are not fully well behaved because “the outgoing coordinates describe in a non-pathological manner the ejection of particles outward from \( r = 0 \) through \( r = 2M \); but their descriptions of in fall through \( r = 2M \) has the same pathology as the description given by Schwarzschild coordinates. Similarly, the ingoing coordinates describe well the in fall of a particle through \( r = 2M \), but they give a pathological description of outgoing trajectories…” [Page-831]. This is exactly the same pathology of Boyer-Lindquist coordinates for the descriptions of Kerr spacetimes also. So here we will use Carter’s coordinates/Universal like coordinates for the particular \( r = M \) geodesics on the horizon. Carter’s coordinates are explicitly derived in appendix 8. Without loss of generality we set \( \theta = 0 \) and \( \theta = \text{constant} = \frac{\pi}{2} \) for the equatorial plane. Therefore from (77) the extremal Kerr metric on the equatorial plane can be written as

\[
\begin{align*}
\frac{\text{ds}^2}{\text{A}} &= \frac{1}{8} (1 - \frac{M}{r})^2 \left( \frac{r^2}{r^2 + M^2} + \frac{1}{2} \right) \left( \frac{r^2 - M^2}{r^2 + M^2} \right) + \frac{(M^2 - r^2)^2}{16M^2r^2}.
\end{align*}
\]
\[ B = \frac{(M^2 - r^2)^2}{8Mr^2} - \frac{1}{2} \left(1 - \frac{M}{r}\right)^2 \left[ \frac{r^4}{(r^2 + M^2)^2} + \frac{1}{4} \right]. \tag{40} \]

\[ C = \frac{(r^2 + M^2)^2}{2r^2} - M^2 \left(1 - \frac{M}{r}\right)^2, \quad D = \frac{1}{2} M \left(1 - \frac{M}{r}\right)^2 - \frac{(M^4 - r^4)}{2Mr^2}. \tag{41} \]

For our simplicity we take \( \phi \) instead of \( \phi^* \) in the subsequent analysis. The spacetime metric (38) has a timelike isometry. The generator of this isometry is the Killing vector field \( \xi \) whose projection along the 4-velocity \( u \) of timelike geodesics: \( \xi \cdot u = -E \), is conserved along such geodesics. Now, \( \xi \) has non-vanishing components \( \xi_U, \xi_V \) which can be easily derived from the fact that in the Schwarzschild coordinate basis \( \xi = \partial_t \). One obtains \( \xi_U = \cos^2 U, \quad \xi_V = \sin^2 V \). Thus, in this coordinate chart, \( E \) can be expressed as

\[ E = -(A + B/2) \left[ \sec^2 U u_U + \csc^2 V u_V \right] + 2D u^\phi. \tag{42} \]

There is also other isometry for rotational symmetry i.e. the ‘angular momentum’ \( L \equiv \zeta \cdot u \) (where \( \zeta \equiv \partial_\phi \)) which is similarly conserved. It can be also expressed as

\[ L = D \left[ \sec^2 U u_U + \csc^2 V u_V \right] + C u^\phi. \tag{43} \]

From the norm of four velocity

\[ u^2 = A \left[ \sec^4 U (u_U)^2 + \csc^4 V (u_V)^2 \right] + B \sec^2 U \csc^2 V u_U u^V \]

\[ + D \left[ \sec^2 U u_U + \csc^2 V u_V \right] u^\phi + C (u^\phi)^2. \tag{44} \]

Now, for circular geodesics; the radial component of the 4-velocity vanishes: \( u^r = 0 \). Therefore, this translates into \( r_U u_U + r_V u_V = 0 \) where \( r_U \equiv \partial r / \partial U \) etc. These derivatives of \( r(U,V) \) can be calculated from equation (46), so that one obtain for circular geodesics

\[ \frac{u^U}{u^V} = \cos^2 U \csc^2 V. \tag{45} \]

\[ E = - \left[ 2(A + B/2) \sec^2 U u_U + 2D u^\phi \right]. \tag{46} \]

\[ L = 2D \sec^2 U u_U + C u^\phi. \tag{47} \]

Solving equations (46), (47) one obtains

\[ u^U \sec^2 U = \frac{2DL + C E}{4D^2 - 2C(A + B/2)}. \tag{48} \]

\[ u^\phi = \frac{L(A + B/2) + D E}{C(A + B/2) - 4D^2}. \tag{49} \]

From (41)

\[ u^2 = -(u^U \sec^2 U) E + C (u^\phi)^2. \tag{50} \]

Now we would like to see what happens for the peculiar geodesics \( r = M \) in this fully well behaved coordinates. Does it coincide with the null generators
of the horizon? Note that in this coordinates the horizon is at \( r(U, V) = M \), \( U = \pi/2 \) and for circular geodesics on the future horizon

\[
\mathcal{E} \to 0,
\]

\[
\frac{dU}{dV} = \cos^2 U \csc^2 V \to 0.
\]

and the norm of the four velocity may be defined as

\[
\mathbf{u} \cdot \mathbf{u} = \frac{L^2}{4M^2}
\]

Once again, as in the Doran frame for the exactly extremal Kerr geometry, the geodesic on the horizon will be a null geodesic generator only if \( L = 0 \). This also implies that the energy \( E \) must also vanish for the geodesic on the horizon.

Alternatively for timelike circular geodesics \( u^2 = -1 \), Using (48, 49) one obtains the energy equation for timelike circular geodesics as

\[
\alpha \mathcal{E}^2 + \beta \mathcal{E} + \gamma = 0.
\]

where

\[
\alpha = G C D^2 - C H^2,
\]

\[
\beta = 2C DGL (A + B/2) - 2DLH^2,
\]

\[
\gamma = C GL^2 (A + B/2)^2 - G H^2,
\]

\[
G = 4D^2 - 2C (A + B/2),
\]

\[
H = C (A + B/2) - 4D^2.
\]

Therefore the effective potential for timelike circular geodesics may be written as

\[
\mathcal{E} = (\mathcal{V}_{\text{eff}})_{\text{Horizon}} = -\beta + \sqrt{\beta^2 - 4\alpha \gamma}.
\]

Similarly the effective potential for null circular geodesics can be written as

\[
\mathcal{E} = (\mathcal{U}_{\text{eff}})_{\text{Horizon}} = -\beta + \sqrt{\beta^2 - 4\alpha \gamma_0}.
\]

where

\[
\gamma_0 = C GL^2 (A + B/2)^2.
\]

It shows that the future horizon of the spacetime is given by \( U = \pi/2 \) with \( V \) arbitrary; in other words \( r(\pi/2, V) = M \). One can compute the derivatives in the equations (48, 49); it turns out that \( r_U(\pi/2, V) = 2M^2 \), while the other derivative of \( r \) is regular on the horizon. Now on the future horizon

\[
A \to 0, \quad B \to 0, \quad C \to 4M^2, \quad D \to 0
\]

\[
\alpha \to 0, \quad \beta \to 0, \quad \gamma \to 0, \quad G \to 0, \quad H \to 0
\]
\[ \mathcal{E} \rightarrow 0, \quad \frac{u^U}{u^V} = \cos^2 U \csc^2 V \rightarrow 0. \]  

(60)

which implies \( u^U \rightarrow 0 \) and \( u^V \rightarrow \infty \) on the horizon. It follows that \( L \) is a finite quantity which is vanishing on the horizon. It is also further observed that timelike circular geodesics and null circular geodesics coalesce into a zero energy trajectory (as in the RN case [1])

\[ \mathcal{E} = (V_{\text{eff}})_{\text{Horizon}} = (U_{\text{eff}})_{\text{Horizon}} \rightarrow 0. \]  

(61)

Thus, the geodesic on the horizon must coincide with the principal null geodesic generator. The existence of a timelike circular orbit turning into the null geodesic generator on the event horizon is a peculiar feature of exactly extremal Kerr spacetime.

Another view of this discontinuity is gleaned from the absence of outer trapped surfaces within the horizon in the extremal geometry in contrast to a more generic situation, as we now discuss.

### 6 Absence of Trapped Surfaces in Extremal Kerr Spacetime:

In a most general spacetime \((M, g_{\mu\nu})\) with the metric \(g_{\mu\nu}\) having signature \((- + + +)\), one can define two future directed null vectors \(l^\mu\) and \(n^\mu\) whose expansion scalars are given by

\[ \theta_{(l)} = q^{\mu\nu} \nabla_\mu l_\nu, \quad \theta_{(n)} = q^{\mu\nu} \nabla_\mu n_\nu. \]  

(62)

where \(q_{\mu\nu} = g_{\mu\nu} + l_\mu l_\nu + n_\mu n_\nu\) is the metric induced by \(g_{\mu\nu}\) on the two dimensional spacelike surface formed by spatial foliation of the null hypersurface generated by \(l^\mu\) and \(n^\mu\).

Then (i) a two dimensional spacelike surface \(S\) is said to be a trapped surface if both \(\theta_{(l)} < 0 \) and \(\theta_{(n)} < 0\); (ii) \(S\) is to be marginally trapped surface if one of two null expansions vanish i.e. \(\theta_{(l)} = 0\) or \(\theta_{(n)} = 0\). The null vectors for non-extremal Kerr black hole are given by

\[ l^\mu = \frac{1}{\Delta}(r^2 + a^2, -\Delta, 0, a), \quad n^\mu = \frac{1}{2r^2}(-r^2 + a^2, 0, a\Delta) \]  

(63)

\[ l_\mu = \frac{1}{\Delta}(-\Delta, -(r^2 + a^2), 0, a\Delta), \quad n_\mu = \frac{1}{2r^2}(-\Delta, r^2 + a^2, 0, a\Delta) \]  

(64)

where \(\Delta = (r - r_+)(r - r_-)\) and \(r_{\pm} = M \pm \sqrt{M^2 - a^2}\). The null vectors satisfy the following conditions :

\[ l^\mu n_\mu = -1, \quad l_\mu l_\nu = 0, \quad n_\mu n_\mu = 0 \]  

(65)

Using (62), one obtains

\[ \theta_{(l)} = -\frac{2}{r}, \quad \theta_{(n)} = \frac{(r - r_+)(r - r_-)}{r^3} \]  

(66)
In the region \((r_- < r < r_+)\), \(\theta(l) < 0\) and \(\theta(n) < 0\). This implies that trapped surfaces exist for non-extreme Kerr black hole in this region. In contrast, for the extreme Kerr black hole

\[
\theta(l) = -\frac{2}{r}, \quad \theta(n) = \frac{(r - M)^2}{r^3} \tag{67}
\]

Here inside or outside extremal horizon \(r < M\) or \(r > M\), \(\theta(l) < 0\) and \(\theta(n) > 0\). This implies that there are no trapped surfaces for extremal Kerr black hole beyond the event horizon.

7 Discussion:

The study reveals the disparity between precisely extremely and nearly extremely geometry manifested in the fact that the near-extremal and near-horizon limits do not commute, and also the singular nature of the extremal limit of Carter’s maximal analytic extension of a generic Kerr geometry. We also showed that to study the geodesics close to the horizon, one must first go to the precisely extremal geometry, before considering geodesics (near or on) the horizon by using Carter’s frame which is well-behaved on the horizon. While Doran and ingoing-Kerr coordinates can also be used to reach the same conclusion, this is arrived at in the latter frames only through careful limiting procedures.

Another feature of our work is that using Carter’s frame the direct ISCO in extremal Kerr spacetime, which lies on the event horizon, coincides with the principal null geodesic generator; such an ISCO is non-existent in the near-extremal geometry.

We have compared the results here with that in the work of Jacobson [6] where it is also inferred that the ISCO on the horizon in the extremal case coincides with the null geodesic generator. For the near-extremal geometry, our results for the energy and angular momentum per unit mass agree with those in this work, and also with the earlier results of [9] (which is, surprisingly, not referred to in [6]). However, the demonstration that the norm of the 4-velocity of the geodesic on the horizon must vanish in the extremal limit poses some challenging manipulations involving limits. In [6], a way around this problem has been sought by considering the proper spatial separation between the geodesic in question, and the bifurcation sphere which does indeed exist in the near-extremal geometry. However, as emphasized in section IVB, it is not enough to argue that the extremal limit of a few quantities like the energy and momentum exist and are non-zero. One must analyze separately the exactly extremal geometry and compare the results with those in the extremal limit to exhibit the absence of subtleties in that limit. This has not been done in [6], since the exactly extremal situation has not been considered there. Indeed, for this geometry, there is no bifurcation sphere, and therefore arguments involving the proper distance between the geodesic and the bifurcation sphere cannot be made in this case. Rather, as we have unambiguously demonstrated
in sections IVA and V, for the exactly extremal geometry, the geodesic on the horizon has vanishing norm provided the energy and angular momentum per unit mass vanish. The demonstrations in this case are far more straightforward, requiring no subtle manipulations of limits.

In sum, we hope to have persuaded the reader that the extremal limit of a generic non-extremal Kerr spacetime has subtle disparities from the precisely extremal situation. The evidence presented in favour of our contention is threefold: first of all, it is the absence of a unique maximal analytic extension of the non-extremal Kerr spacetime covering both the event and Cauchy horizons, thus leading to a divergent metric in the extremal limit of such an extension. In other words, the maximal analytic extension of the extremal Kerr geometry had to be worked out separately, rather than by a limiting procedure on the generic spacetime. Secondly, the existence of an ISCO which turns into the null geodesic generator of the horizon in the extremal case, with vanishing energy and angular momentum per unit mass. In the near-extremal situation, this demonstration is complicated because to show that the norm of the geodesic vanishes on the horizon requires careful handling of limits. Finally, our demonstration of the absence of trapped surfaces in the extremal spacetime, which tallies well with the other properties of the extremal spacetime noted here.

What we have apparently established has implications for extant approaches to computing the Wald entropy function for extremal black holes. It is not clear that current assays in this direction actually compute extremal black hole entropy; rather, the results are most likely for some sort of entanglement entropy of ambient matter in the field of such black holes.

8 Appendix: Maximal Analytical Extension of Kerr Spacetime (Off Axis of Symmetry)

The maximal analytic extension of the non-extremal Kerr spacetime along the off axis of symmetry was first reported by Carter [7]. As we showed for symmetry axis the analytic extension is not continuous at the extremal limit $r_+ \to r_-$ in section 3, here we implemented for the off axis symmetry and only derive the extremal case. For non-extremal case see Carter’s [7] paper.

The complete analytic extension of the extremal Kerr spacetime thus cannot be obtained as a limiting case of the non-extremal geometry as previously, one needs to treat the extremal case separately [7]. Defining the double null coordinates and angular coordinates are

$$du + dv = 2\frac{r^2 + M^2}{\Delta} dr, \quad d\phi + d\tilde{\phi} = 2\frac{M}{\Delta} dr.$$  \hspace{1cm} (68)

Define ignorable angle coordinates $\phi^*$ given by

$$2d\phi^* = d\phi - d\tilde{\phi} - \frac{(du - dv)}{2M},$$ \hspace{1cm} (69)
which are constant on the null generator of the horizon at \( r = M \). The metric is thus given by

\[
 ds^2 = \frac{\Delta}{8\rho^2} \left[ \frac{\rho^2}{r^2 + M^2} + \frac{\rho^2}{2M^2} \right] \frac{(r^2 - M^2)\sin^2 \theta}{(r^2 + M^2)} (du^2 + dv^2) + \rho^2 d\theta^2 \\
+ \frac{\Delta}{2\rho^2} \left[ \frac{\rho^4}{(r^2 + M^2)^2} + \frac{\rho^4}{4M^4} \right] dudv - \frac{\Delta M \sin^2 \theta}{\rho^2} \left[ M \sin^2 \theta d\phi^* - \frac{\rho^2}{2M^2} (du - dv) \right] d\phi^* \\
+ \frac{\sin^2 \theta}{\rho^2} \frac{(M^2 - r^2)}{4M} (du - dv) - (r^2 + M^2) d\phi^* d\phi^* ,
\]

\( \Delta = (r - M)^2, \rho^2 = (r^2 + M^2 \cos^2 \theta) \). \( r \) is now defined implicitly as a function of \( u \) and \( v \) by

\[
 F(r) = u + v = 2r^*,
\]

where \( r^* \) is called ‘tortoise’ coordinate, determined by

\[
 dr^* = \frac{(r^2 + M^2)dr}{\Delta} = \frac{(r^2 + M^2)dr}{(r - M)^2}.
\]

Integrating (72) yields

\[
 r^* = \int \frac{(r^2 + M^2)dr}{(r - M)^2} = r + 2M \left[ \ln|r - M| - \frac{M}{2(r - M)} \right] .
\]

Near the horizon \( r = M \) this has a leading pole-type singularity

\[
 r^* \approx \frac{M^2}{(r - M)}
\]

instead of a logarithmic one. To locate the event horizon at a finite region in the coordinate chart, we follow ref. [7] and introduce null coordinates \( U, V \) such that

\[
 u = \tan U, \quad v = \cot V.
\]

This implies that

\[
 \tan U + \cot V = 2r^*(U, V).
\]

Therefore the complete extremal Kerr metric in \( (U, V, \theta, \phi^*) \) is given by

\[
 ds^2 = \frac{\Delta}{8\rho^2} \left[ \frac{\rho^2}{r^2 + M^2} + \frac{\rho^2}{2M^2} \right] \frac{(r^2 - M^2)\sin^2 \theta}{(r^2 + M^2)} (\sec^4 U dU^2 + \csc^4 V dV^2) + \rho^2 d\theta^2 \\
- \frac{\Delta}{2\rho^2} \left[ \frac{\rho^4}{(r^2 + M^2)^2} + \frac{\rho^4}{4M^4} \right] \sec^2 U \csc^2 V dU dV
\]
\[
- \frac{\Delta M \sin^2 \theta}{\rho^2} \left[ M \sin^2 \theta d\phi^* - \frac{\rho^2}{2M^2} (\sec^2 UdU + \csc^2 V dV) \right] d\phi^*
+ \frac{\sin^2 \theta}{\rho^2} \left[ \frac{(M^2 - \rho^2)}{4M} (\sec^2 UdU + \csc^2 V dV) - (\rho^2 + M^2) d\phi^* \right]^2.
\] (77)

It can be easily check that in the limit $\theta = 0$, we obtain the metric (18) for symmetry axis.

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