Binary LCD Codes and Self-orthogonal Codes via Simplicial Complexes

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Abstract—Due to some practical applications, linear complementary dual (LCD) codes and self-orthogonal codes have attracted wide attention in recent years. In this paper, we use simplicial complexes for construction of an infinite family of binary LCD codes and two infinite families of binary self-orthogonal codes. Moreover, we explicitly determine the weight distributions of these codes. We obtain binary LCD codes which have minimum weights two or three, and we also find some self-orthogonal codes meeting the Griesmer bound. As examples, we also present some (almost) optimal binary self-orthogonal codes and LCD distance optimal codes.

Index Terms—simplicial complex, weight distribution, LCD code, self-orthogonal code.

I. INTRODUCTION

The concept of linear complementary dual (LCD) codes was introduced by Massey [18] in 1992. For implementations against side-channel and fault injection attacks, a new application of binary LCD codes was found by Carlet and Guilley ([11], [12]). Since then, LCD codes have attracted wide attention from the coding research community ([5], [8], [15]-[17], [19], [20]). Carlet et al. [8] proved that for \( q \geq 3 \) any \( q \)-ary linear code is equivalent to an LCD code over \( \mathbb{F}_q \); therefore, it is sufficient to investigate binary LCD codes and ternary LCD codes. Self-orthogonal codes are very important for the study of quantum communications and quantum computations since they can be applied to the classical construction of quantum error-correcting codes ([2], [3]).

In this paper, we mainly use simplicial complexes for constructing binary LCD codes and binary self-orthogonal codes. For the definition of simplicial complexes, we need the following notations. Let \( \mathbb{F}_2 \) be the finite field of order 2 and \( m \) be a positive integer. The support \( \text{supp}(v) \) of a vector \( v \) in \( \mathbb{F}_2^m \) is defined by the set of nonzero coordinate positions of \( v \). Let \( 2^{|m|} \) denote the power set of \( |m| = \{1, \ldots, m\} \). It is easy to check that there is a bijection between \( \mathbb{F}_2^m \) and \( 2^{|m|} \), defined by \( v \mapsto \text{supp}(v) \); hence, due to this bijection, a vector \( v \) in \( \mathbb{F}_2^m \) is identified with its support \( \text{supp}(v) \). For two sets \( A \) and \( B \), the set \( \{x : x \in A \text{ and } x \notin B\} \) is denoted by \( A \setminus B \), and the size of \( A \) is denoted by \( |A| \).

Definition 1.1: A subset \( \Delta \) of \( \mathbb{F}_2^m \) is called a simplicial complex if \( u \in \Delta \) and \( \text{supp}(v) \subseteq \text{supp}(u) \) imply \( v \in \Delta \) for any \( u, v \in \mathbb{F}_2^m \).

An element of a simplicial complex \( \Delta \) is called maximal if it is not properly contained in the others in \( \Delta \). Let \( \mathcal{F} \) be the set of maximal elements of a simplicial complex \( \Delta \). Especially, \( \Delta_F \) denotes the simplicial complex generated by a nonzero vector \( F \) in \( \mathbb{F}_2^m \).

In this paper, we use a typical construction of a linear code given in [14]. Let \( D = \{g_1, g_2, \ldots, g_n\} \subseteq \mathbb{F}_p^m \). Then a linear code \( C_D \) of length \( n = |D| \) over \( \mathbb{F}_p \) can be defined by

\[
C_D = \{c_u = (u \cdot g_1, u \cdot g_2, \ldots, u \cdot g_n) : u \in \mathbb{F}_p^m\},
\]

where \( \cdot \) denotes the Euclidean inner product of two elements in \( \mathbb{F}_p^m \). The set \( D \) is called the defining set of \( C_D \). Let \( G \) be the \( m \times n \) matrix as follows:

\[
G = [g_1^T, g_2^T, \ldots, g_n^T],
\]

where the column vector \( g_i^T \) denotes the transpose of a row vector \( g_i \). Zhou et al. [20] obtained some simple conditions under which the linear codes defined in Eq. (1) are LCD or self-orthogonal, and they also presented four infinite families of binary linear codes. For any positive integers \( m \) and \( t \) with \( 1 \leq t \leq m - 1 \), two defining sets are given as follows:

\[
D_t = \{g \in \mathbb{F}_p^m : \text{wt}(g) = t\},
\]

and \( D_{\leq t} = \{g \in \mathbb{F}_p^m : 1 \leq \text{wt}(g) \leq t\} \),

where \( \text{wt}(v) \) denotes the Hamming weight of \( v \in \mathbb{F}_p^m \). We note that the two sets can also be expressed by using simplicial complexes in the following way:

\[
D_t = \Delta_{D_t} \setminus \Delta_{D_{t-1}}, \quad \text{and} \quad D_{\leq t} = \Delta_{D_t} \setminus \{0\}.
\]

Note that here \( D_t \) denotes a set of maximal elements for any \( t \geq 1 \), and \( \Delta_{D_t} \) and \( \Delta_{D_{t-1}} \) are simplicial complexes. For example, if \( m = 3 \), then \( D_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \) and \( D_2 = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} \); hence, \( \Delta_{D_1} = D_1 \cup \{0\} \) and \( \Delta_{D_2} = \{0\} \cup D_1 \cup D_2 \). It is easy to check that \( \Delta_{D_1} \) and \( \Delta_{D_2} \) are simplicial complexes.

Inspired by [20], we employ the difference of two distinct simplicial complexes for construction of an infinite family of binary LCD codes and two infinite families of binary self-orthogonal codes. This paper is organized as follows. In Section II we introduce some basic concepts on generating functions, LCD codes, and self-orthogonal codes. In Section III we determine the weight distributions of some binary linear codes.

Manuscript received December 30, 2019; accepted March 16, 2020. The paper is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (Grant No. 2019R1A6A1A11051177) and also by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST/NRF-2017R1A2B2004574). The associate editor coordinating the review of this letter and approving it for publication was Marco Baldi. (Corresponding author: Yoonjin Lee.)

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Digital Object Identifier
codes and discuss the minimum distances of their dual codes. Section IV presents a class of binary LCD codes and two classes of binary self-orthogonal codes. Section V concludes this work.

II. PRELIMINARIES

A. Generating functions

The following $m$-variable generating function associated with a subset $X$ of $\mathbb{F}_q^m$ was introduced by Chang et al. [9].

$$H_X(x_1, x_2, \ldots, x_m) = ∏_{u \in X} ∑_{i=1}^{m} x_i^{|u_i|} \in \mathbb{Z}[x_1, x_2, \ldots, x_m],$$

where $u = (u_1, u_2, \ldots, u_m) \in \mathbb{F}_q^m$; here, for $u_i$, we use the identification of $0, 1 \in \mathbb{F}_2$ with $0, 1 \in \mathbb{Z}_2$, respectively (Note that this is just a formal definition and there should be no confusion because we do not make addition operation on the powers of $x_i$).

The following lemma will be used in Section III.

Lemma 2.1: [9 Theorem 1] Suppose that $Δ$ is a simplicial complex of $\mathbb{F}_q^m$ and $F$ is the set of maximal elements of $Δ$. Then

$$H_Δ(x_1, x_2, \ldots, x_m) = ∑_{0 \neq S \subseteq F} (-1)^{|S|+1} ∏_{i \in S} (1 + x_i).$$

Remark 2.2: Recall that there is a bijection between $\mathbb{F}_2^n$ and $2^{|S|}$. Hence, the set $\cap S$ in Lemma 2.1 can be understood as the intersection of the elements of $S$ in $2^{|S|}$. We have the following example. Let $Δ = \{(1,1,0), (0,1,1)\}$ be a simplicial complex in $\mathbb{F}_2^3$. By Lemma 2.1, we have

$$H_Δ(x_1, x_2, x_3) = (1 + x_1)(1 + x_2) + (1 + x_2)(1 + x_3) - (1 + x_2)
= 1 + x_1 + x_2 + x_3 + x_1x_2 + x_2x_3.$$

B. LCD codes and self-orthogonal codes

Let $F_q$ be the finite field of order $q$, where $q$ is a power of a prime. Let $C$ be an $[n, k]$ code over $F_q$. The dual code $C^⊥$ of $C$ is defined by $C^⊥ = \{w \in F_q^n : w : c = 0 \text{ for every } c \in C\}$. If $C \cap C^⊥ = \{0\}$, then $C$ is called a linear complementary dual (LCD) code; if $C \subseteq C^⊥$, then $C$ is called self-orthogonal.

Regarding the codes defined in Eq. (1), Zhou et al. [20] obtained the following lemma.

Lemma 2.3: [20 Corollary 16] Let $C_D$ be the linear code defined in Eq. (1). Let $\text{Rank}(G)$ denote the rank of the matrix $G$ in Eq. (2). Then $C_D$ is self-orthogonal (LCD, respectively) if and only if $\text{GFT} = 0$ ($\text{Rank}(\text{GFT}) = \text{Rank}(G)$, respectively).

Let $C$ be an $[n, k, d]$ linear code over $F_q$. Assume that there are $A_i$ codewords in $C$ with Hamming weight $i$ for $1 \leq i \leq n$. Then $C$ has weight distribution $(1, A_1, \ldots, A_n)$ and weight enumerator $1 + A_1z + \cdots + A_nz^n$. Moreover, if the number of nonzero $A_i$’s in the sequence $(A_1, \ldots, A_n)$ is exactly equal to $t$, then the code is called $t$-weight. An $[n, k, d]$ code $\hat{C}$ is called distance optimal if there is no $[n, k, d+1]$ code (that is, this code has the largest minimum distance for given length $n$ and dimension $k$), and it is called almost optimal if an $[n, k, d+1]$ code is distance optimal (refer to [14 Chapter 2]). On the other hand, the Griesmer bound [(12) on an $[n, k, d]$ linear code over $\mathbb{F}_q$ is given by $∑_{i=0}^{k-1} \left\lfloor \frac{d}{q^i} \right\rfloor \leq n$, where $\left\lfloor \cdot \right\rfloor$ is the ceiling function.

Furthermore, a binary $[n, k, d]$ LCD code $C$ is called LCD distance optimal if there is no $[n, k, d+1]$ LCD code (that is, this LCD code has the largest minimum distance among $[n, k]$ LCD codes for given length $n$ and dimension $k$), and it is called LCD almost optimal if an $[n, k, d+1]$ code is LCD distance optimal.

III. Weights distributions of binary linear codes arising from simplicial complexes

We will determine the weight distributions of the codes defined in Eq. (1), noting that their defining sets are expressed as the differences of two simplicial complexes.

Let $Δ_1$ and $Δ_2$ with $Δ_2 \subseteq Δ_1$ be two distinct simplicial complexes of $\mathbb{F}_2^n$. Let $p = 2$ and $D = Δ_1 \setminus Δ_2$ in Eq. (1). Note that if $u = 0$ in Eq. (1), then $wt(c_u) = 0$. From now on, we assume that $u \neq 0$. Then

$$wt(c_u) = |D| - \frac{1}{2} ∑_{d \in Δ_2} (-1)^{|u_i|} \cdot D$$

where $u = (u_1, u_2, \ldots, u_m) \in \mathbb{F}_2^m$.

For $u \in \mathbb{F}_2^m$ and $X \subseteq \mathbb{F}_2^m$, a Boolean function $χ(u|X)$ in $m$-variable is defined by $χ(u|X) = 1$ if and only if $u \cap X = \emptyset$. If a simplicial complex is generated by a maximal element $A$ (denoted by $Δ_A$), then by Lemma 2.1 we have

$$H_Δ((−1)^{u_1}, \ldots, (−1)^{u_m}) = ∏_{i \in A} (1 + (−1)^{u_i}) \geq 2(|A| − u_i).$$

Theorem 3.1: Let $m \geq 3$ be a positive integer. Suppose that $A$ and $B$ are two elements of $\mathbb{F}_2^m$ with $B \subseteq A$. Let $D = Δ_A \setminus Δ_B$. Then the code $C_D$ defined in Eq. (1) meets the Griesmer bound.

(1) If $|B| = 0$, then $C_D$ is a $[2^{|A|}−1, |A|, 2^{|A|}−1]$ one-weight code with weight enumerator $1 + (2^{|A|−1}) \cdot z^{2^{|A|−1}}$.

(2) If $|B| \geq 1$, then $C_D$ is a $[2^{|A|}−2|B|, |A|, 2^{|A|}−2|B|−1]$ two-weight code with weight enumerator

$$1 + (2^{|A|−2|B|}) \cdot z^{2^{|A|−2|B|−1}} + (2^{|A|−|B|}) \cdot z^{2^{|A|−1}}.$$
if and only if $\chi(u|A) = \chi(u|B) = 1$: that is, $u \cap A = \emptyset$. Since $u \in \mathbb{F}_q^m$, every codeword is repeated $2m-|A|$ times. Hence, we see that the code $C_D$ has dimension $|A|$. Furthermore, if $|B| \geq 1$, then we have

$$\sum_{i=0}^{\frac{|A|-1}{2}} \left( \frac{|A|-1-2|B|-1}{2^i} \right) = (2^{|A|}-1) - (2^{|B|}-1) = 2^{|A|} - 2^{|B|}.$$ 

Hence, we conclude that $C_D$ meets the Griesmer bound. Similarly, the result holds for the case where $|B| = 0$.

**Theorem 3.2:** Let $D$ be defined as in Theorem 3.1. Then $C_D$ is a $[2^{|A|} - 2|B|, 2^{|A|} - 2|B| - |A|, \delta]$ linear code, where

$$\delta = \begin{cases} 3 & \text{if } |A| > |B| + 1, \\ 4 & \text{if } |A| = |B| + 1 \geq 3. \end{cases}$$

**Proof** Assume that $D = \{g_1, \ldots, g_n\} \subseteq \mathbb{F}_2^n$ with $n = |D|$. The generator matrix $G'$ of $C_D$ can be induced by the matrix $G$ in Eq. (2) by deleting all the zero row vectors of $G$. Clearly, $G'$ is the parity-check matrix of $C_D$. The minimum distance of $C_D$ is greater than 2. We divide the proof into two parts.

(1) If $|A| > |B| + 1$, then there are two distinct positive integers $i$ and $j$ in $A \setminus B$. Let $e_k = (e_{1j}, e_{2j}, \ldots, e_{mj}) \in \mathbb{F}_2^m$, where $e_{ij} = 1$ and $e_{ij} = 0$ if $i \neq k$. Then it is easy to check that $e_{ij}^T$, $e_{ij}^T$, and $e_{ij}^T + e_{ij}^T$ are three different columns of $G'$; therefore, the minimum distance of $C_D$ is 3.

(2) If $|A| = |B| + 1$, then we assume that $A \setminus B = \{i\}$ without loss of generality. We note that any three columns of $G'$ are linearly independent. Since $|B| \geq 2$, there are two integers $i$ and $j$ in $B$. Then $e_{ij}^T$, $e_{ij}^T + e_{ij}^T$, and $e_{ij}^T + e_{ij}^T + e_{ij}^T$ are four linearly dependent columns of $G'$.

Corollary 3.3: Let $|B| = 0$ and $|A| > 1$ in Theorem 3.2. Then $C_D$ is a $[2^{|A|} - 1, 2^{|A|} - 1 - |A|, 3]$ Hamming code.

**Theorem 3.4:** Let $m \geq 3$ be a positive integer. Suppose that $A$ and $B$ are two distinct elements of $\mathbb{F}_2^m$ such that $0 < |B| < |A|$ and $A \cap B = \emptyset$. Let $D = (\Delta_A \cup \Delta_B) \setminus \{0\}$. Then $C_D$ in Eq. (1) is a $[2^{|A|} + 2^{|B|} - 2, |A| + |B|, 2^{|B|}-1]$ three-weight code with weight enumerator

$$1 + (2^{|B|}-1)z^{2^{|B|}-1} + (2^{|A|}-1)z^{2^{|A|}-1}.$$ 

**Proof** The length of $C_D$ is $2^{|A|} + 2^{|B|} - 2$. By Eqs. (3) and (4), we have

$$wt(c_u) = 2^{|A|}-1(1 - \chi(u|A)) + 2^{|B|}-1(1 - \chi(u|B)).$$

The frequency of each codeword in $C_D$ can be determined by the vector $u$, and so the result follows immediately.

In a similar way to Theorem 3.2, we have:

**Theorem 3.5:** Let $D$ be defined as in Theorem 3.4. Then $C_D$ is a $[2^{|A|} + 2^{|B|} - 2, 2^{|A|} + 2^{|B|} - 2 - |A| - |B|, 3]$ code.

**Theorem 3.6:** Let $m$ be a positive even integer and $k = \frac{m}{2}$. Let $\{A_1, \ldots, A_k\}$ be a partition of $\{1, 2, \ldots, m\}$, where $|A_i| = 2$ for $1 \leq i \leq k$. Let $D = (\Delta_{A_1} \cup \cdots \cup \Delta_{A_k}) \setminus \{0\}$ in Eq. (1). Then $C_D$ is a $[3m/2, m, 2]$ code and its weight enumerator is given by

$$w(t) = \prod_{i=0}^{k-1} \left( \frac{3}{1} \right) z^{m-2t}.$$ 

**Proof** The length of $C_D$ is $\frac{3}{2}m$. By Eqs. (3) and (4),

$$wt(c_u) = m - 2(\chi(u|A_1) + \cdots + \chi(u|A_k)).$$

The frequency of each codeword in $C_D$ can be determined by the vector $u$, and hence the result follows right away.

We obtain the following theorem in a similar way to Theorem 3.2.

**Theorem 3.8:** Let $D$ be defined as in Theorem 3.6. Then $C_D$ is a $[3m/2, m/2, 3]$ code.

IV. BINARY LCD CODES AND SELF-ORTHOGONAL CODES

We present some binary LCD codes and binary self-orthogonal codes in this section.

**Lemma 4.1:** Let $\Delta_A$ be a simplicial complex generated by a nonzero element $A \in \mathbb{F}_2^n$ and $\Delta_A \setminus \{0\} = \{g_1, g_2, \ldots, g_n\} \subseteq \mathbb{F}_2^n$, where $n = 2^{|A|}-1$. Let $G = [g_1^T g_2^T \cdots g_n^T]$ be the $m \times n$ matrix in Eq. (2).

Then $\text{Rank}(G) = |A|$ and

$$\text{Rank}(G G^T) = \begin{cases} 0 & \text{if } |A| \geq 3, \\ |A| & \text{if } |A| < 3. \end{cases}$$

**Proof** Note that $\text{Rank}(G) = |A|$. Let $M = (m_{ij})_{m \times m} = G G^T$. By [20], Lemma 18, assume that $c_i$ is the $i$-th row vector of $G$. Then $m_{ij} = c_i G_{ij}$. Let $U_{ij} = \{g = (g_1, g_2, \ldots, g_n) \in D : g_i = g_j = 1\}$. Then $m_{ij} = |U_{ij}|$ (mod 2). Then the result follows from Lemma 2.2 and

$$U_{ij} = \begin{cases} 2^{|A|-1} & \text{if } i = j \in A, \\ 2^{|A|-2} & \text{if } i \neq j, i, j \in A, \\ 0 & \text{otherwise}. \end{cases}$$

**Theorem 4.2:** Let $D$ be defined as in Theorem 3.1. Then the code $C_D$ defined in Eq. (1) is self-orthogonal if and only if one of the followings holds:

(1) $|B| = 0$ and $|A| \geq 3$.

(2) $|A| > |B| \geq 3$.

**Proof** Let $\Delta_B \setminus \{0\} = \{g_1, g_2, \ldots, g_l\} \subseteq \mathbb{F}_2^n$ and $\Delta_A \setminus \Delta_B = \{g_{l+1}, g_{l+2}, \ldots, g_n\} \subseteq \mathbb{F}_2^n$, and $\Delta_A \setminus \{0\} = \{g_{l+1}, g_{l+2}, \ldots, g_n\} \subseteq \mathbb{F}_2^n$. Let $G_1 = [g_{1}^T g_{2}^T \cdots g_{l}^T]$, $G_2 = [g_{l+1}^T g_{l+2}^T \cdots g_{n}^T]$ and $G = [G_1 G_2]$. By Lemma 2.2, the code $C_D$ is self-orthogonal if and only if $G G^T = 0$. Note that $G G^T = G_1 G_1^T + G_2 G_2^T$. Now, consider the following four cases depending on the value of $|B|$.

(1) If $|B| = 0$, then $C_D$ is self-orthogonal if and only if $|A| \geq 3$ from Lemma 4.1.

(2) If $|B| = 1$, then $m_{ii} = 2^{|A|-1} - 1 \equiv 1$ (mod 2) for $i \in B$. Hence, $C_D$ cannot be self-orthogonal in this case.

(3) If $|B| = 2$, then $m_{ij} = 2^{|A|-2} - 1 \equiv 1$ (mod 2) for $i, j \in B$. Thus, $C_D$ cannot be self-orthogonal in this case.

(4) If $|B| \geq 3$, then we have that $G_1 G_1^T = 0$ and $G G^T = 0$ by Lemma 4.1. Therefore, $C_D$ is self-orthogonal.

**Example 4.3:** Let $|B| = 0$ and $|A| = 3 \leq m$. Then $C_D$ in Theorem 3.1 is a $[7, 3, 4]$ self-orthogonal code, and $C_D^1$ is a $[7, 3, 4]$ code. According to [11], we find that both $C_D$ and $C_D^1$ are distance optimal.

**Example 4.4:** Let $|B| = 3$ and $|A| = 5 \leq m$. Then $C_D$ in Theorem 3.1 is a $[24, 5, 12]$ self-orthogonal code, and $C_D^1$ is a $[24, 19, 3]$ code. We confirm that both $C_D$ and $C_D^1$ are distance optimal according to [11].
Example 4.5: Let $|B| = 4$ and $|A| = 5 \leq m$. Then $C_D$ in Theorem 3.1 is a $[16, 5, 8]$ self-orthogonal code and $C_D^o$ is a $[16, 11, 4]$ code. According to [11], we conclude that both $C_D$ and $C_D^o$ are distance optimal.

Theorem 4.6: Let $D$ be defined as in Theorem 3.4. Then $C_D$ is self-orthogonal if and only if $|A| > |B| \geq 3$.

Proof: Let $\Delta_B \{0\} = \{g_1, \ldots, g_1\} \subseteq \mathbb{F}^{m}_{2}$ and $\Delta_A \{0\} = \{h_1, \ldots, h_n\} \subseteq \mathbb{F}^{m}_{2}$. Let $G_1 = [g_1 \cdots g_1]$, $G_2 = [h_1 \cdots h_n]$, and $G = [G_1G_2]$. From the assumption that $A \cap B = \emptyset$ and Lemma 2.2, it follows that $C_D$ is self-orthogonal if and only if $GG^T = 0$. The result thus follows from the fact that $GG^T = G_1G_1^T + G_2G_2^T$ and Lemma 4.1.

Example 4.7: Let $|B| = 2$, $|A| = 3$, and $5 \leq m$. Then $C_D$ in Theorem 3.4 is a $[10, 5, 3]$ self-orthogonal code and $C_D^o$ is a $[10, 5, 3]$ code. According to [11], we find that $C_D$ and $C_D^o$ are both almost optimal.

Theorem 4.8: Let $D$ be defined as in Theorem 3.6. Then $C_D$ is an LCD code.

Proof: By Lemma 4.1, for any $1 \leq i \leq k$ we have $m_{i_1,i_2} = m_{i_2,i_1} = 1$, where $\{i_1,i_2\} = A_i$. Note that $\{A_1, \ldots, A_k\}$ is a partition of $\{1, 2, \ldots, m\}$ and $\text{Rank}(G) = m$. Equivalently, we can write $GG^T = \text{diag}[I_2, I_2, \ldots, I_2]$, where $I_2$ is the identity matrix of order 2. Then $\text{Rank}(G) = \text{Rank}(GG^T) = m$. Then the result follows from Lemma 2.2.

In [10, 13], the authors obtained some bounds on LCD codes, and they also gave a complete classification of binary LCD codes with small lengths.

Example 4.9: Let $m = 4$. Then $C_D$ in Theorem 3.6 is a $[6, 4, 2]$ binary LCD code and $C_D^o$ is a $[6, 2, 3]$ binary LCD code. According to [11], $C_D$ is distance optimal and $C_D^o$ is almost optimal. According to the tables in [10, 13], we see that $C_D$ and $C_D^o$ are both LCD distance optimal codes as well.

Example 4.10: Let $m = 6$. Then $C_D$ in Theorem 3.6 is a $[9, 6, 2]$ binary LCD code and $C_D^o$ is a $[9, 3, 3]$ binary LCD code. According to [11], $C_D$ is distance optimal and $C_D^o$ is almost optimal. Moreover, we conclude that the code $C_D$ is LCD distance optimal based on the tables in [10, 13].

Example 4.11: Let $m = 8$. Then $C_D$ in Theorem 3.6 is a $[12, 8, 2]$ binary LCD code and $C_D^o$ is a $[12, 4, 3]$ binary LCD code. We find that the code $C_D$ is almost optimal according to [11]. Furthermore, we can see that $C_D$ is an LCD distance optimal code according to the tables in [10, 13].

V. CONCLUDING REMARKS

In this paper we obtain an infinite family of binary LCD codes and two infinite families of binary self-orthogonal codes by using simplicial complexes. Weight distributions are explicitly determined for these codes. We also find some (almost) optimal binary self-orthogonal and LCD codes.

It is worth noting that some of our self-orthogonal codes in Theorem 3.1 meet the Griesmer bound. Table I presents some of optimal binary LCD codes obtained by using Theorems 3.6 and 3.7; their optimality is based on the tables in [10, 13]. Their classification in [10, 13] treats binary LCD codes of only small lengths, so that optimality of our LCD codes in Theorems 3.6 and 3.7 is confirmed for only small lengths due to limited current database. However, we believe that our binary LCD codes may include new LCD distance optimal codes of larger lengths provided that the database is supported for larger lengths.

| Parameters | Optimality          |
|------------|---------------------|
| [3, 2, 2]  | LCD distance optimal |
| [6, 4, 2]  | LCD distance optimal |
| [6, 2, 3]  | LCD distance optimal |
| [9, 3, 3]  | LCD almost optimal   |
| [9, 6, 2]  | LCD distance optimal |
| [12, 8, 2]| LCD distance optimal |
| [15, 10, 2]| LCD almost optimal   |

Acknowledgement. We express our gratitude to the reviewers for their very helpful comments, which improved the exposition of this paper.

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