Spin-orbit- and coherently-coupled spin-1 Bose-Einstein condensates under rotation

Paranjeet Banger1‡*, R. Kishor Kumar2†, Ashton S. Bradley2‡, and Sandeep Gautam3

1 Department of Physics, Indian Institute of Technology Ropar, Rupnagar 140001, Punjab, India and 2 Department of Physics, Centre for Quantum Science, and Dodd-Walls Centre for Photonic and Quantum Technologies, University of Otago, Dunedin 9054, New Zealand

We theoretically study the stationary-state vortex lattice configurations of rotating spin-orbit- and coherently-coupled spin-1 Bose-Einstein condensates in quasi-two-dimensional harmonic potentials. The combined effect of interactions, spin-orbit and coherent couplings with moderate to high rotation frequencies are analyzed systematically by the variational method and exact numerical solutions of the single-particle Hamiltonian. An analysis of the relevant functions of the single-particle Hamiltonian shows that a boson in these rotating spin-orbit- and coherently-coupled condensates can be subjected to rotating effective potentials equivalent to a toroidal, a symmetric double-well, an asymmetric double-well potentials, etc. These effective potentials are further related with the vector and scalar potentials experienced by the particle. At moderate to high rotation frequencies, we find that the spin-expectation per particle of even an antiferromagnetic spin-1 Bose-Einstein condensate approaches unity indicating a similarity in the response of ferromagnetic and antiferromagnetic spin-orbit-coupled condensates at moderate to fast rotations. This is further confirmed by the similar static vortex-lattice patterns, associated spin-textures, and mass-currents for the two systems at fast rotations.

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I. INTRODUCTION

Depending upon the nature of spin-exchange interactions, values of longitudinal magnetization $M$ and rotational frequency $\Omega_{\text{rot}}$, a wide assortment of topological structures can appear in the ground states of the spin-1 Bose-Einstein condensate (BEC) [1-3]. Among these are thermodynamically stable Mermin-Ho and Anderson-Toulouse coreless vortices in a ferromagnetic spin-1 BEC [1-2]. The phase diagram in $M-\Omega_{\text{rot}}$ plane reveals that the spin-1 Bose-Einstein condensates (BECs) can host different types of both axisymmetric as well as asymmetric vortices [2]. The stability of former types with a phase-winding number of individual components between $-1$ to $1$ has also been studied in $M-\Omega_{\text{rot}}$ plane [4]. The two distinct vortex-lattice phases of a rotating ferromagnetic spin-1 BEC are a lattice of coreless vortices with non-singular spin-texture and a lattice with polar cores [3]. Coreless axisymmetric and non-axisymmetric vortices with spin-texture, respectively, corresponding to a skyrmion and a meron-pair also emerge in pseudospin-1/2 BECs under rotation [4]. As the rotation frequency approaches the trapping frequency, the BEC enters into a regime where the non-interacting part of the Hamiltonian becomes equivalent to the Hamiltonian of a charged particle with mass $m$ and charge $q$ in a magnetic field of magnitude $B = 2m\Omega_{\text{rot}}/q$ [5]. In this regime, where mean-field interaction energy is smaller than $2\hbar\Omega_{\text{rot}}$ and coherence length is of the order of inter-vortex separation, the bosons occupy the single particle states corresponding to the lowest Landau level [6-7] and the Gross-Pitaevskii approximation is valid [8]. With a further increase in rotation frequency, mean-field interaction energy becomes much smaller than $\hbar\Omega_{\text{rot}}$, resulting in highly correlated vortex-liquid states which are no longer described by the mean-field Gross-Pitaevskii equation [7-9]. An important parameter which characterizes the transition between these two regimes is the ratio of the number of bosons to the number of vortices termed as filling fraction [9]. In a single-component scalar BEC, a zero-temperature phase transition corresponding to the transition from vortex-lattice phase to an incompressible vortex-liquid phase occurs when the filling fraction approximately falls below 6 [9]. On the experimental front, the interlaced square vortex lattice in a rotating two-component pseudospin-1/2 BEC has been experimentally observed [11]. The experiment also confirmed the stability of the square vortex lattice by exciting Tkachenko modes [12] and their subsequent relaxation [11]. Spin-1 bosons under fast rotations in the lowest Landau level regime have been theoretically investigated [13], and it has been demonstrated that similar to scalar BECs, a transition from vortex-lattice phase to quantum Hall liquid states at ultra-fast rotations occurs [13].

An important research-direction in the field of quantum degenerate gases in the last two decades or so has been the exploration of atom-light coupling to generate artificial Abelian and non-Abelian gauge fields [14], and these research endeavours culminated with the experimental realization of artificial gauge fields [15] and spin-orbit (SO) coupling between the linear momentum and the spin [17] of electrically neutral bosons. The SO coupling with an equal-strength mixture of Rashba [18] and Dresselhaus [19] couplings, which couples three mag-
netic sublevels of spin-1 $^{87}$Rb, has been realized in an experiment [20]. More recently, using an optical Raman lattice, two-dimensional SO coupling and the resulting topological bands have been experimentally realized with $^{87}$Rb involving two of its hyperfine spin states [21]. In spin-1 BECs, SO coupling leads to a variety of ground-state phases like plane-wave, stripe or standing-wave, vortex-lattice, zero-momentum phases, etc. [22–25]. Besides these unusual phases, self-trapped vortex solitons [26, 27], knotted solitons [28], super-stripes, and super-lattices [29] can also emerge as the ground state solutions of SO-coupled spin-1 BECs in different parameter domains. Recently, coupling between the spin and orbital angular momentum of a neutral bosonic atom has also been experimentally realized [30–31].

The realization of SO coupling paved the way to explore the interplay of an artificial non-Abelian gauge field resulting in SO coupling and rotation which itself is equivalent to an Abelian gauge potential of $A = m\Omega_{\text{rot}} \times r$ [14]. For an SO-coupled BEC, merely rotating the trapping potential will result in a time-dependent Hamiltonian which can not have stationary vortex-lattice solutions [32]. This can be remedied by rotating both the (anisotropic) trap and the laser fields responsible for the creation of SO and coherent couplings in which case the Hamiltonian in the rotating frame is time-independent [32]. The interplay of rotation and SO coupling in such a set-up has been studied for two-component pseudospin-1/2 BECs [33–36]. The ground-state phase diagram of a coherently-coupled pseudospin-1/2 BEC has also been studied [37].

The interplay of isotropic SO coupling and rotation leading to half-skyrmion excitations in rotating and rapidly quenched spin-1 BECs using stochastic projected Gross-Pitaevskii equations (SPGPEs) has been studied in Ref. [39]. A similar study using SPGPEs has shown that a hexagonal lattice of skyrmions and a square lattice of half-quantized vortices can emerge as the equilibrium configurations in a ferromagnetic and an antiferromagnetic SO-coupled BECs, respectively, on rapid quenching [40]. It has also been shown that a rotating spin-1 BEC in the presence of an anisotropic SO coupling can support vortex-chain solutions [11, 12]. In a recent theoretical investigation, it has been shown that spin-1 BECs with a weak isotropic Rashba SO coupling can host vortices with hexagonal or square arrangement [43]. The ground state of a ferromagnetic spin-1 BEC with an isotropic three-dimensional SO coupling has also been theoretically investigated [44]. The other theoretical studies include a study of the ground states of Rashba SO-coupled spin-1 BEC in a concentrically coupled toroidal traps [45], and a recent study of rotating spin-1 BEC with SU(3) coupling and subjected to a magnetic-field gradient [46]. These studies on spin-1 BECs primarily focused on the numerical results of the emergent density patterns and (or) spin textures. More recently, topological vortical phase transitions in an SO-coupled spin-2 BEC under rotation have been theoretically studied [17].

In this paper, we study the combined effect of spin-exchange interactions, SO and coherent couplings, and rotation frequency on spin-1 condensates. We consider a quasi-two-dimensional harmonically trapped spin-1 condensate with a Rashba SO and a coherent coupling in this work. We consider $^{87}$Rb and $^{23}$Na BECs as the prototypical examples of systems with ferromagnetic and antiferromagnetic spin-exchange interactions. We consider a generic Rashba SO-coupling $\propto (\gamma_x S_x \partial/\partial x + \gamma_y S_y \partial/\partial y)$ [15], where $S_x$ and $S_y$ are spin operators for spin-1 system and $\gamma_x$ and $\gamma_y$ are SO-coupling strengths, and study the stationary-state solutions of the aforementioned BECs with a focus primarily on moderate to large rotation frequencies of up to 0.95 times of the trapping frequency and within the domain of validity of the mean-field model. We examine in detail the effective potential experienced by the boson using an analysis based on a non-interacting Hamiltonian and argue that the rotation, SO, and coherent couplings can combine together to result in a variety of effective potentials. We use the variational method to calculate the eigen functions of the non-interacting Hamiltonian. The density profiles corresponding to these eigen functions are used to interpret the effective potentials. We also relate the effective potential calculation to the scalar and vector potentials experienced by a particle in these systems. The effective potentials evaluated from the variational method and from the calculation of gauge potentials are further corroborated by exact numerical solutions of the single particle Hamiltonian. We study the spin-expectation per particle as a function of rotation frequency and find a similarity in the response of SO-coupled $^{87}$Rb and $^{23}$Na BECs at moderate to large rotations; we also examine the spin-texture at small and large rotation frequencies to buttress this point. The inclusion of anisotropic SO coupling, e.g. $\gamma_x \neq 0, \gamma_y = 0$, coherent coupling, and a complete theoretical analysis of an SO-coupled single-particle Hamiltonian leading to the evaluation of effective potentials makes this study distinct from earlier studies on SO-coupled BECs under rotation [33, 43]. In addition, the similarity in the response of experimentally realizable ferromagnetic and antiferromagnetic spin-1 BECs at moderate to high rotations could be of experimental interest. It is also pertinent to point out that the celebrated experimental realization of SO coupling [17] corresponds to one dimensional coupling $\gamma_x S_x p_x$ with non-zero coherent coupling.

The paper is organized as follows. In the Sec [11] we describe the coupled Gross-Pitaevskii equations (CGPEs) for a rotating SO- and coherently-coupled spin-1 BEC in a quasi-two-dimensional trapping potential. In Sec. [13] we report stationary-state solutions of an SO- and a coherently-coupled spin-1 BEC, with either ferromagnetic or antiferromagnetic spin-exchange interactions, when rotated with moderate to high rotation frequencies. In Sec. [15], we calculate the eigen functions and eigen energies of the single-particle Hamiltonian using the variational method and exact numerical solutions. The effective potentials leading to the emergent vortex-
lattice patterns are analyzed via a calculation of gauge potentials in Sec. IV B. This is followed by a study of the magnitude of spin-expectation per particle for SO-coupled $^{87}$Rb and $^{23}$Na BECs as a function of rotation frequency in Sec. IV C. We conclude with a summary of results in Sec. V.

II. GROSS-PITAEVSKII EQUATIONS FOR A ROTATING SO-COUPLED SPIN-1 BEC

In a mean-field approximation rotating SO-coupled spin-1 BEC in a quasi-two-dimensional trapping potential can be described by three CGPEs [22, 18] which in the dimensionless form are

\begin{align}
(i - \gamma_{\text{diss}}) \frac{\partial \phi_{\pm \pm}}{\partial t} &= \mathcal{H} \phi_{\pm \pm} + c_2 (\rho_0 \pm \rho_-) \phi_{\pm \pm} + c_2 \phi_{\pm \pm}^* \phi_0^* \\
&- \frac{i}{\sqrt{2}} (\gamma_x \partial_x \phi_0 + i \gamma_y \partial_y \phi_0) + \frac{\Omega_{\text{coh}}}{\sqrt{2}} \phi_0 (1a) \\
(i - \gamma_{\text{diss}}) \frac{\partial \phi_0}{\partial t} &= \mathcal{H} \phi_0 + c_2 \rho_\pm \phi_0 + 2 c_2 \phi_\pm \phi_0 - \frac{i}{\sqrt{2}} (\gamma_x \partial_x \phi_0 + i \gamma_y \partial_y \phi_0) \\
&+ \frac{\Omega_{\text{coh}}}{\sqrt{2}} (\phi_+ + \phi_-), (1b)
\end{align}

where $\Omega_{\text{coh}}$ is the coherent-coupling strength and $\gamma_{\text{diss}}$ is a phenomenological dissipation parameter [49, 50].

\begin{align}
\mathcal{H} &= \sum_{\nu=x,y} \frac{p_\nu^2}{2} + V(x, y) + c_0 \rho - \Omega_{\text{rot}} L_z, \quad \partial_{\nu} = \frac{\partial}{\partial \nu} \\
\rho &= \sum_{j = \pm 1} \rho_j, \quad \rho_j = |\phi_j|^2, \quad \rho_\pm = \rho_+ \pm \rho_-,
\end{align}

$\Omega_{\text{rot}}$ is the angular frequency of rotation around z axis, $L_z = (xp_y - y p_x)$ is the z component of the angular momentum operator, $p_\nu = -i \partial_{\nu}$, and $\nu = x, y$. The interaction parameters $c_0, c_2$ and trapping potential $V(x, y)$ are defined as

\begin{align}
c_0 &= \sqrt{8 \pi \alpha_z} \frac{N(a_0 + 2 a_2)}{3 a_{\text{osc}}} , \quad c_2 = \sqrt{8 \pi \alpha_z} \frac{N(a_2 - a_0)}{3 a_{\text{osc}}}, (2) \\
V(x, y) &= \frac{1}{2} (\alpha_x x^2 + \alpha_y y^2), (3)
\end{align}

where $\alpha_\nu = \omega_\nu / \omega_x$ is the ratio of trapping frequency along $\nu$ direction to $x$ direction, $N$ is the total number of atoms in the BEC, and $a_0$ and $a_2$ are the s-wave scattering lengths in total spin 0 and 2 channels, respectively. The units of spatial coordinates, time, energy, and wavefunction components considered to write Eqs. (1a)-(1b) are $a_{\text{osc}} = \sqrt{\hbar/(m \omega_z)}, \omega_z^{-1}, \hbar \omega_z$, and $a_{\text{osc}}^{-1}$, respectively.

The CGPEs (1a)-(1b) can be written in a matrix form as

\begin{align}
(i - \gamma_{\text{diss}}) \frac{\partial \Phi}{\partial t} &= (H_0^x + H_0^y + H_d + H_{\text{nd}} + H_{\text{SOC}}) \Phi, \\
&= \mathbf{H} \Phi, (4)
\end{align}

where $\Phi = (\phi_{+1}, \phi_0, \phi_{-1})^T$, and $H_0^x$ and $H_0^y$ are $3 \times 3$ diagonal matrices of form

\begin{align}
H_0^x &= \text{diag} \left( \frac{p_x^2}{2} + \Omega_{\text{rot}} y p_\nu , \frac{p_y^2}{2} + \Omega_{\text{rot}} y p_\nu , \frac{p_\nu^2}{2} + \Omega_{\text{rot}} y p_\nu \right), \\
H_0^y &= \text{diag} \left( \frac{p_y^2}{2} - \Omega_{\text{rot}} x p_\nu , \frac{p_x^2}{2} - \Omega_{\text{rot}} x p_\nu , \frac{p_\nu^2}{2} - \Omega_{\text{rot}} x p_\nu \right).
\end{align}

In Eq. (4), $H_d$ consists of trapping potential and diagonal interaction terms, $H_{\text{nd}}$ consists of coherent coupling and off-diagonal interaction terms, and $H_{\text{SOC}}$ consists of SO-coupling terms of the Hamiltonian [51, 52]. We use time-splitting Fourier spectral method [51] to solve Eq. (4) which is briefly discussed in Appendix. In CGPEs (1a)-(1b), $\gamma_{\text{diss}}$ is a phenomenological dissipation parameter which facilitates the vortex-lattice to relax into an equilibrium configuration. The dissipation is present in the experiments on the trapped BECs [51], and the phenomenological dissipation value of 0.03 considered in the present work has earlier been calculated by fitting the numerical results with the experimental results [49]. The dissipation to aid in the faster convergence to the equilibrium configuration has been used by several authors [50, 55].

III. NUMERICAL RESULTS

We consider $10^5$ atoms of spin-1 BECs like $^{87}$Rb and $^{23}$Na in an isotropic quasi-two-dimensional harmonic trap with $\alpha_x = \alpha_y = 1$, and $\alpha_z = 10$. The trapping frequencies $\omega_x = \omega_y = 2\pi \times 10$ Hz resulting in $a_{\text{osc}} = 3.41$ m and $a_{\text{osc}} = 6.63$ m, respectively, for $^{87}$Rb and $^{23}$Na spinor BECs. The ferromagnetic $^{87}$Rb has $a_0 = 101.8 a_B$ and $a_2 = 101.4 a_B$ [55], and anti-ferromagnetic $^{23}$Na has $a_0 = 50 a_B$ and $a_2 = 55.01 a_B$ [57], here $a_B$ is the Bohr radius. The resultant dimensionless interaction strengths for $^{87}$Rb are $c_0 = 2482.21$ and $c_2 = -11.47$, and the same for $^{23}$Na are $c_0 = 674.91$ and $c_2 = 21.12$. We solve CGPEs (1a)-(1b) on a two-dimensional 512x512 spatial grid with a space-step $\Delta x = \Delta y = 0.1$ and a time step $\Delta t = 0.005$ using a time-splitting Fourier spectral method [51]. We start with an initial guess solution corresponding to the ground state solution for an SOC-coupled spin-1 BEC without rotation using imaginary-time propagation, and then CGPEs (1a)-(1b) are solved in real time till the equilibrium solution is achieved. As the norm is not preserved due the dissipation in a real-time propagation, order parameter is normalized to unity after each time step, i.e.,

$$\int_{-\infty}^{\infty} dx \sum_{j = -1}^{1} |\phi_j(x)|^2 = 1,$$

where $x = (x, y)$.

In order to study the vortex-lattice states that can emerge as the minimum energy solutions of an SOC-coupled spin-1 BEC in a rotating frame, we consider the
following SO-coupling strengths

\[ (a) \gamma_x = \gamma_y = 0.5, \quad (b) \gamma_x = \gamma_y = 1, \quad (c) \gamma_x = 0.5, \gamma_y = 0.25, \quad (d) \gamma_x = 1, \gamma_y = 0. \]  

(7) (8)

We first study the SO-coupled $^{87}\text{Rb}$ and $^{23}\text{Na}$ spinor BECs with these SO-coupling strengths without coherent coupling.

**A. Ferromagnetic SO-coupled spin-1 BEC**

The component densities in the ground state of the SO-coupled $^{87}\text{Rb}$ with $(\gamma_x, \gamma_y) = (0.5, 0.5); (1, 1); (0.5, 0.25); (1, 0)$ in the absence of rotation are shown, respectively, in Figs. 1(a1)-(a3), (d1)-(d3), (g1)-(g3), and (j1)-(j3). The component density distributions are asymmetric with $(\gamma_x, \gamma_y) = (0.5, 0.5); (1, 1); (0.5, 0.25)$, whereas it is circularly symmetric for $(\gamma_x, \gamma_y) = (1, 0)$. The ‘+’ and ‘−’ signs in Figs. 1(a1)-(a3) and 1(d1)-(d3) marking the locations of the phase singularities are consistent with the asymmetric density distributions. In one of our previous studies, we showed the emergence of similar asymmetric (-1,0,+1)-type self-trapped solutions in an SO-coupled ferromagnetic spin-1 BEC [26]. Similarly, there are phase singularities along y-axis, which are not shown as these lie just outside the spatial domain considered in the figures, leading to the asymmetric density distributions in Figs. 1(g1)-(g3). In contrast to these, the phase corresponding to Fig. 1(j1)-(j3) is a perfect plane-wave phase of $−\gamma_x x$ consistent with the plane-wave solutions predicted in Ref. [22]. These ground state solutions in the absence of rotation are used as the initial guess solutions to evolve CGPEs with $\Omega_{\text{rot}} \neq 0$ so as to obtain the stationary-state solutions. The component ground-state densities of $^{87}\text{Rb}$ spin-1 BEC with $\gamma_x = \gamma_y = 0.5$ and rotated with $\Omega_{\text{rot}} = 0.5$ are shown in Figs. 1(b1)-(b3). At moderate rotation frequency of 0.5, the small number of vortices which nucleate are unable to crystallize in a triangular vortex-lattice pattern consistent with the observation in Refs. [13, 55]. The same for condensate rotated at a high rotation frequency $\Omega_{\text{rot}} = 0.95$ are shown in Figs. 1(c1)-(c3). In the latter case, phase profile of the condensate (which is not shown) reveals that the center of the condensate has phase singularities of charges $(0, +1, +2)$ in $m_j = +1, 0, \text{ and } -1$ components, respectively. At this large rotation frequency of $\Omega_{\text{rot}} = 0.95$ more vortices are created in condensate and relax in a triangular lattice pattern. The coarse-grained peak value of the total density lies along a circle of radius 4.9 as shown in each component density in Figs. 1(c1)-(c3). With $\gamma_x = \gamma_y = 1$ and $\Omega_{\text{rot}} = 0.5$, the component ground-state densities are shown in Figs. 1(c1)-(c3). The center of the condensate in this case has phase singularities of charges $(0, +1, +2)$, respectively, in $m_j = +1, 0, \text{ and } -1$ components respectively. When rotated with higher frequency of $\Omega_{\text{rot}} = 0.95$, the condensate density acquires a giant hole at the center as shown in Figs. 1(f1)-(f3) with an annulus of triangular vortex-lattice pattern in each component. The coarse-grained peak value of the total density in this case too is along a circle of radius 9.7 which is shown by a blue circle in the component density. The appearance of a giant vortex at the trap center in the component densities surrounded by singly charged vortices arranged in an annulus for sufficiently strong isotropic SO-coupling strengths at fast rotations is a generic feature of these systems [33, 36]. Next we consider the $^{87}\text{Rb}$ spin-1 BEC with anisotropic SO-coupling strengths under different rotation frequencies. With $\gamma_x = 0.5, \gamma_y = 0.25$, at small rotation frequencies of $\Omega_{\text{rot}} \leq 0.16$, the phase-singularities in the component wave functions, not shown here, exclusively align along the x axis. When we increase the rotation frequencies $\Omega_{\text{rot}} > 0.16$, phase-singularities tend to distribute throughout the condensate. At $\Omega_{\text{rot}} = 0.5$ and $\Omega_{\text{rot}} = 0.95$, the component densities are shown in Figs. 1(h1)-(h3) and Figs. 1(i1)-(i3), respectively. In the former case, there is a chain of vortices aligned along x axis in each component which does not persist at a faster rotation in the later case. Lastly with $\gamma_x = 1, \gamma_y = 0$, the exclusive alignment of phase-singularities along x axis persists up to $\Omega_{\text{rot}} = 0.25$. The central-chain of holes in the component densities arising due to these phase-singularities is also evident in Figs. 1(k1)-(k3) for $\Omega_{\text{rot}} = 0.5$. At still higher rotation frequency of $\Omega_{\text{rot}} = 0.95$, majority of vortices arrange themselves on both the sides of the central chain of vortices as shown in Fig. 1(l1-13). The appearance of central chain of vortices is a generic feature of these systems with a sufficiently strong anisotropic SO coupling [11, 42].

**B. Antiferromagnetic SO-coupled spin-1 BECs**

For the SO-coupled $^{23}\text{Na}$ spin-1 BEC with $(\gamma_x, \gamma_y) = (0.5, 0.5); (1, 1); (0.5, 0.25); (1, 0)$, the ground state component densities are shown in Figs. 2(a1)-(a3), Figs. 2(d1)-(d3), Figs. 2(g1)-(g3), and Figs. 2(j1)-(j3), respectively. The ground state hosts a polar-core vortex state [28, 51] with $(\gamma_x, \gamma_y) = (0.5, 0.5)$, and has a stripe density [22] pattern for $(\gamma_x, \gamma_y) = (1, 1); (0.5, 0.25); (1, 0)$. The polar-core vortex state in Figs. 2(a1)-(a3) has phase singularities of winding numbers $(-1, 0, +1)$, respectively in $m_j = +1, 0, -1$ components.

With $\gamma_x = \gamma_y = 0.5$, the component densities when rotated with $\Omega_{\text{rot}} = 0.5$ and $\Omega_{\text{rot}} = 0.95$ are shown in Figs. 2(b1)-(b3) and Figs. 2(c1)-(c3), respectively. In the former case with $\Omega_{\text{rot}} = 0.5$, the vortex patterns in the component densities near the center are more like a square lattices consistent with a similar observation in Ref. [43], which at the faster rotation of $\Omega_{\text{rot}} = 0.95$ in the latter case acquire triangular-lattice patterns as
FIG. 1: (Color online) Contour plot of component densities of the SO-coupled \(^{87}\text{Rb}\) spin-1 BEC with \(\gamma_x = \gamma_y = 2482.21\), \(c_1 = -11.47\). (a1)-(a3),(b1)-(b3) and (c1)-(c3) have been obtained with \(\Omega_{\text{rot}} = 0, 0.5, 0.95\), respectively, and SO-coupling strength of \(\gamma_x = \gamma_y = 0.5\). Similarly, (d1)-(d3), (e1)-(e3) and (f1)-(f3) correspond to SO-coupling strength of \(\gamma_x = \gamma_y = 1.0\) with \(\Omega_{\text{rot}} = 0, 0.5, 0.95\), respectively; (g1)-(g3), (h1)-(h3), (i1)-(i3) correspond to \(\gamma_x = 0.5, \gamma_y = 0.25\), and \(\Omega_{\text{rot}} = 0, 0.5, 0.95\), respectively; and (j1)-(j3),(k1)-(k3), (l1)-(l3) correspond to \(\gamma_x = 1, \gamma_y = 0\) with \(\Omega_{\text{rot}} = 0, 0.5, 0.95\), respectively. The spatial coordinates and densities are in the units of \(a_{\text{Rb osc}}\) and \([a_{\text{Rb osc}}]^{-2}\); where \(a_{\text{Rb osc}} = 3.41\ \mu\text{m}\). The coherence lengths corresponding to the peak total density in (f1)-(f3) and (l1)-(l3) are \(0.32\) and \(0.26\), respectively, which are much less than the intervortex separations of \(\approx 1.8 - 1.9\), and the respective filling fractions are \(\approx 326\) and 469.

More vortices have now entered the system. Moreover at \(\Omega_{\text{rot}} = 0.95\), similar to the \(^{87}\text{Rb}\) BEC, the \(^{23}\text{Na}\) BEC has a \((0, +1, +2)\) vortex at the center and coarse-grained peak value of the total density lies along a circle of radius 4.9 which is shown by a blue circle in the component density. The response of the \(^{23}\text{Na}\) BEC with \((\gamma_x, \gamma_y) = (1, 1); (0.5, 0.25); (1, 0)\) when rotated with moderate to large rotation frequencies is similar to the \(^{87}\text{Rb}\) BEC.

The ground state component densities of the \(^{23}\text{Na}\) BEC with \((\gamma_x, \gamma_y) = (1, 1)\) when rotated with \(\Omega_{\text{rot}} = 0.5\) and
FIG. 2: (Color online) Contour plot of component densities of the SO-coupled $^{23}$Na spin-1 BEC with $c_0 = 674.91$ and $c_2 = 21.12$. (a1)-(a3), (b1)-(b3) and (c1)-(c3) have been obtained with $\Omega_{\text{rot}} = 0, 0.5, 0.95$, respectively, and SO-coupling strength of $\gamma_x = \gamma_y = 0.5$. Similarly, (d1)-(d3), (e1)-(e3) and (f1)-(f3) correspond to SO-coupling strength of $\gamma_x = \gamma_y = 1.0$ with $\Omega_{\text{rot}} = 0, 0.5, 0.95$, respectively; (g1)-(g3), (h1)-(h3), (i1)-(i3) correspond to $\gamma_x = 0.5, \gamma_y = 0.25$ with $\Omega_{\text{rot}} = 0, 0.5, 0.95$, respectively; and (j1)-(j3), (k1)-(k3), (l1)-(l3) correspond to $\gamma_x = 1$, $\gamma_y = 0$ with $\Omega_{\text{rot}} = 0, 0.5, 0.95$, respectively. The spatial coordinates and densities are in the units of $a^{\text{Na}}_{\text{osc}}$ and $[a^{\text{Na}}_{\text{osc}}]^{-2}$ respectively, where $a^{\text{Na}}_{\text{osc}} = 6.63\, \mu\text{m}$. The coherence lengths corresponding to the peak total density in (f1)-(f3) and (l1)-(l3) are 0.48 and 0.35, respectively, which are much less than the respective intervortex separations of $\approx 3.8$ and 2.5, and the respective filling fractions are $\approx 465$ and 1041.

$\Omega_{\text{rot}} = 0.95$ are shown in Figs. 2(e1)-(e3) and Figs. 2(f1)-(f3), respectively. The qualitative features like presence of $(0, +1, +2)$ vortex at the center in the former case and a giant hole at the center in the latter are similar to $^{87}$Rb BEC under rotation. Again, coarse-grained peak value of the total density is along a circle of radius 9.7 which is shown by a blue circle in the component density. The component densities for $(\gamma_x, \gamma_y) = (0.5, 0.25)$ are shown in Figs. 2(h1)-(h3). Figs. 2(j1)-(j3) when rotated, respectively with $\Omega_{\text{rot}} = 0.5, 0.95$ and for $(\gamma_x, \gamma_y) = (1, 0)$ are
shown in Figs. 2(k1)-(k3), Figs. 2(l1)-(l3) when rotated respectively with $\Omega_{\text{rot}} = 0.5, 0.95$. The quantitative differences in respective component densities of $^{87}\text{Rb}$ and $^{23}\text{Na}$ when rotated with $\Omega_{\text{rot}} = 0.95$ is primarily a consequence of $c_0$ for the two BECs being 2482.21 and 674.91, respectively.

**Mixture of Rashba and Dresselhaus couplings:** An anisotropic Rashba SO coupling $\gamma_{x}S_{z}p_{x}$ is equivalent to an equal strength mixture of an isotropic Rashba [15], $(p_{x}S_{x} + p_{y}S_{y})$, and an isotropic Dresselhaus [18], $\gamma(p_{x}S_{x} - p_{y}S_{y})$, couplings. In literature, this coupling is many a times identified with an equivalent $\gamma_{x}S_{z}p_{x}$ coupling [17] as $\gamma_{x}S_{z}p_{x} = \gamma_{x}U^\dagger S_{z}U p_{x}$, where $U$ is a rotation operator which rotates the spin state about $y$ in anticlockwise direction by an angle $\pi/2$. Under this unitary transformation, solutions of Eqs. (1a)-(1b), say $(\phi_{+1}, \phi_{0}, \phi_{-1})^T$, will transform as

$$(\phi_{+1}, \phi_{0}, \phi_{-1}) \rightarrow U^\dagger (\phi_{+1}, \phi_{0}, \phi_{-1}),$$

where $U^\dagger$ is

$$U^\dagger = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}. \quad (10)$$

For $\gamma_{x} = 1$, $\gamma_{y} = 0$, $\Omega_{\text{coh}} = 0$, and $\Omega_{\text{rot}} = 0.5$ and 0.95, the component densities corresponding to $H_{\text{SOC}} \rightarrow \gamma_{x}p_{x}S_{z}$, obtained by using Eq. (9) are shown in Figs. 3(a1)-(a3) and (c1)-(c3) for $^{87}\text{Rb}$ and Figs. 3(b1)-(b3) and (d1)-(d3) for $^{23}\text{Na}$. It is known that in the absence of rotation, $\gamma_{x}S_{z}p_{x}$ SO coupling favors miscibility of $m_{j} = \pm1$ components for anti-ferromagnetic interactions, whereas it leads to phase-separation if the coupling strength is above a critical value for ferromagnetic interactions [20]. In the presence of rotation, however $\gamma_{x}S_{z}p_{x}$ SO coupling can lead to the phase-separation not only for a ferromagnetic $^{87}\text{Rb}$ but also for an antiferromagnetic $^{23}\text{Na}$ as is seen in the component density profiles shown in Figs. 3(a1)-(a3) and (c1)-(c3) for $^{87}\text{Rb}$ and Figs. 3(b1)-(b3) and (d1)-(d3) for $^{23}\text{Na}$. The $m_{j} = 0$ component occupies the cores of vortices in $m_{j} = \pm1$ component at $\Omega_{\text{rot}} = 0.5$ in Figs. 3(a1)-(a3) and (b1)-(b3). With an increase in rotation frequency number of atoms in $m_{j} = \pm1$ components keep on increasing at the cost of atoms in $m_{j} = 0$ component. Hence at larger rotation frequency of $\Omega_{\text{rot}} = 0.95$ in Figs. 3(c1)-(c3) and (d1)-(d3), there are no atoms in $m_{j} = 0$ component. Later in Sec. IV, it will be shown that densities of $m_{j} = \pm1$ components peak at $\pm\gamma_{x}\Omega_{\text{rot}}/(1 - \Omega_{\text{coh}}^{2}) \approx 0$, leading to a spatial segregation of $\pm1$ components. Another consequence of the phase-separation is that the spin-expectation per particle tends to approach one for all the results shown in Fig. 3. The $\gamma_{x}S_{z}p_{x}$ SO coupling provides a simpler description of the results in Figs. 3(k1)-(k3), (l1)-(l3), Figs. 2(k1)-(k3), and (l1)-(l3).

**FIG. 3:** (Color online) The component densities of the SO-coupled spin-1 BECs of $^{87}\text{Rb}$ and $^{23}\text{Na}$ corresponding to $H_{\text{SOC}} = \gamma_{x}p_{x}S_{z}$. (a1)-(a3) are the component densities of $^{87}\text{Rb}$ BEC with $\gamma_{x} = 1$ and $\Omega_{\text{rot}} = 0.5$, the same for $^{23}\text{Na}$ BEC are in (b1)-(b3). (c1)-(c3) are the component densities of $^{87}\text{Rb}$ BEC with $\gamma_{x} = 1$ and $\Omega_{\text{rot}} = 0.95$, and the same for $^{23}\text{Na}$ BEC are in (d1)-(d3). The $m_{j} = 0$ component is fully absent in (c2) and (d2). The solutions have been obtained by operating $U^\dagger$ in Eq. (10) on the solutions corresponding to $H_{\text{SOC}} = \gamma_{x}p_{x}S_{z}$ shown in Figs. 1(k1)-(k3) and Figs. 1(l1)-(l3) for $^{87}\text{Rb}$ BEC, and Figs. 2(k1)-(k3) and Figs. 2(l1)-(l3) for $^{23}\text{Na}$ BEC. The spatial coordinates and densities are in the units of $a_{\text{osc}}$ and $a_{\text{osc}}^{2}$, respectively, where $a_{\text{osc}} = 3.41 \mu m$ and $6.63 \mu m$ for $^{87}\text{Rb}$ and $^{23}\text{Na}$, respectively.
C. Effect of Coherent coupling

Coherent coupling, like SO coupling, strongly affects the equilibrium vortex-lattice configurations of the spin-1 BEC under rotation. As an example, we first consider $^{23}$Na BEC without and with coherent coupling at rotation frequency of $\Omega_{\text{rot}} = 0.95$ in the absence of SO coupling to highlight the effects which can solely be attributed to an interplay of rotation, coherent coupling and interactions. Here without coherent coupling, the BEC supports an array of double-core vortices in each component which arrange themselves in a square-lattice pattern as is shown in Figs. 4(a1)-(a3). Each double-core vortex core consists of two non-overlapping phase singularities of unit charge each which have been marked with white dots in Figs. 4(a1)-(a3). With coherent coupling of $\Omega_{\text{coh}} = 1$, the system at the same rotation frequency of $\Omega_{\text{rot}} = 0.95$ hosts a triangular-lattice pattern in each component as shown in Fig. 4(b1)-(b3), where a typical vortex core in each component consists of a single phase singularity.

Next we consider the combined effect of SO and coherent couplings on the ground-state vortex configurations. Here we consider two parameter sets- first with $\gamma_x = \gamma_y = 1$, $\Omega_{\text{coh}} = 1$, and second with $\gamma_x = 1$, $\gamma_y = 0$, $\Omega_{\text{coh}} = 1$. In the former case, the ground state density has a hole whose center is shifted along $+y$ direction as shown in Figs. 4(c1)-(c3); in the latter case, the component densities distribute in two unequal triangular lattice patterns above and below $x$-axis as shown in Figs. 4(d1)-(d3). With an increase in $\Omega_{\text{coh}}$ with $\gamma_x = 1$, $\gamma_y = 0$, and $\Omega_{\text{rot}} = 0.95$, the size of smaller triangular lattice pattern in the upper-half plane decreases further with a corresponding increase in the size of one in the lower-half plane. The appearance of hole or splitting of the component densities in two unequal parts can be attributed to the effective potential experienced by the system in the presence of both SO and coherent couplings as is argued in the following section. We obtain similar results for $^{87}$Rb spin-1 BEC at $\Omega_{\text{rot}} = 0.95$ which have not been shown here. As discussed in previous subsection, the component densities corresponding to $H_{\text{SOC}} + H_{\text{coh}} \rightarrow U^\dagger(H_{\text{SOC}} + H_{\text{coh}})U$ can be obtained using the transformation (9) and are not shown here.

IV. ANALYSIS BASED ON SINGLE PARTICLE HAMILTONIAN

After demonstrating the numerically obtained results, we now corroborate the present study’s new findings by the variational analysis and exact numerical solutions of the single-particle Hamiltonian. In addition, we also relate these solutions with the effective potential experienced by a boson in these systems.

A. Variational Analysis

We use the variational method to study the single-particle SO-coupled Hamiltonian which for a spin-1 par-
ticle in the rotating frame is

\[ H_0 = \left[ \frac{p_x^2 + p_y^2}{2} + V(x,y) - \Omega_{\text{rot}} L_z \right] \mathbf{1} + H_{\text{SOC}} \]  

where \( \mathbf{1} \) is a 3 x 3 identity matrix and \( S_z \) as defined earlier is the \( x \) component of angular momentum operator for a spin-1 particle. We estimate the minimum energy eigenfunction(s) and eigen energies of the single particle SO-coupled Hamiltonian for two illustrative cases.

(a) \( \gamma_x \neq 0, \gamma_y = 0, \Omega_{\text{coh}} \neq 0 \) \hspace{1cm} (12a)

(b) \( \gamma_x = \gamma_y \neq 0, \Omega_{\text{coh}} = 0 \) \hspace{1cm} (12b)

A more generic version of (b), say case (c), where

\[ \gamma_x = \gamma_y \neq 0, \Omega_{\text{coh}} \neq 0 \] \hspace{1cm} (12c)

will be considered in the next subsection. In case (a), we consider the following normalized variational ansatz

\[ \Phi_{\text{var}}^\pm = \exp\left[ -\frac{(x-x_0)^2}{2} - \frac{(y-y_0)^2}{2} + i(k_x x + k_y y) \right] \times \left( 1,\pm\sqrt{2},1 \right)^T, \hspace{1cm} (13a) \]

\[ \Phi_{\text{var}}^0 = \exp\left[ -\frac{(x-x_0)^2}{2} - \frac{(y-y_0)^2}{2} + i(k_x x + k_y y) \right] \times (-1,0,1)^T, \hspace{1cm} (13b) \]

where \( x_0, y_0, k_x \) and \( k_y \) are the variational parameters. The variational energies are

\[ E_{\text{var}}^\pm (x_0, y_0, k_x, k_y) = \int dx dy \Phi_{\text{var}}^\pm* H_0 \Phi_{\text{var}}^\pm, \]

\[ = \frac{1}{2} \left[ \gamma_x^2 + \gamma_y^2 + k_x^2 + k_y^2 \pm 2(\Omega_{\text{coh}} \pm k_x \gamma_x + \Omega_{\text{rot}} \{ y_0 k_x - x_0 k_y \}) \right], \hspace{1cm} (14a) \]

\[ E_{\text{var}}^0 (x_0, y_0, k_x, k_y) = \int dx dy \Phi_{\text{var}}^0* H_0 \Phi_{\text{var}}^0, \]

\[ = \frac{1}{2} \left[ \gamma_x^2 + \gamma_y^2 + k_x^2 + k_y^2 + 2\Omega_{\text{rot}} \{ y_0 k_x - x_0 k_y \} \right], \hspace{1cm} (14b) \]

where \( \Phi_{\text{var}}^\pm* (\Phi_{\text{var}}^0*) \) is the conjugate transpose of \( \Phi_{\text{var}}^\pm (\Phi_{\text{var}}^0) \). The variational parameters can be fixed by minimizing \( E_{\text{var}}^\pm (E_{\text{var}}^0) \) with respect to \( (x_0, y_0, k_x, k_y) \). The location(s) of minima thus obtained are

\[ x_0 = 0, \quad y_0 = \pm \frac{\gamma_x}{1 - \Omega_{\text{rot}}^2}, \quad k_x = \pm \frac{\gamma_x}{1 + \Omega_{\text{rot}}^2}, \quad k_y = 0, \]

corresponding to the energy minima of

\[ E_{\text{var}}^\pm_{\text{min}} = \frac{2(1 \pm \Omega_{\text{coh}})(1 - \Omega_{\text{rot}}^2) - \gamma_x^2}{2(1 - \Omega_{\text{rot}}^2)}, \hspace{1cm} (15) \]

for \( \Phi_{\text{var}}^\pm \) and \( x_0 = y_0 = k_x = k_y = 0 \), corresponding to the energy minima of \( E_{\text{var}}^0_{\text{min}} = 1 \) for \( \Phi_{\text{var}}^0 \). The \( \Phi_{\text{var}}^\pm \) is the zero-momentum solution of \( H_0 \). If \( \gamma_x \neq 0 \) and \( \gamma_y = \Omega_{\text{coh}} = 0 \), Eq. (16) yields \( E_{\text{var}}^+_{\text{min}} = E_{\text{var}}^-_{\text{min}} \) which implies that \( \Phi_{\text{var}}^+ \) and \( \Phi_{\text{var}}^- \) are the degenerate eigen functions of \( H_0 \), the so called plane-wave solutions. The degeneracy between \( \Phi_{\text{var}}^+ \) and \( \Phi_{\text{var}}^- \) is lifted in the presence of \( \Omega_{\text{coh}} \) with \( \Phi_{\text{var}}^- \) becoming the lower energy solution. We consider following four parameters’ sets to demonstrate the validity of the variational method for case (a):

\[ \gamma_x = 1, \gamma_y = 0, \Omega_{\text{coh}} = 0, \Omega_{\text{rot}} = 0.5; \hspace{1cm} (17a) \]

\[ \gamma_x = 1, \gamma_y = 0, \Omega_{\text{coh}} = 0, \Omega_{\text{rot}} = 0.95; \hspace{1cm} (17b) \]

\[ \gamma_x = 0, \gamma_y = 0, \Omega_{\text{coh}} = 1, \Omega_{\text{rot}} = 0.95; \hspace{1cm} (17c) \]

\[ \gamma_x = 1, \gamma_y = 0, \Omega_{\text{coh}} = 1, \Omega_{\text{rot}} = 0.95. \hspace{1cm} (17d) \]

For these parameters’ sets \( \Phi_{\text{var}}^0 \) has a higher energy than \( \Phi_{\text{var}}^\pm \). For parameters’ sets (17a) and (17b), the variational solutions \( \Phi_{\text{var}}^\pm \) are degenerate. The most generic variational solution for these parameters’ sets is \( c_+ \Phi_{\text{var}}^+ + c_- \Phi_{\text{var}}^- \), where \( c_\pm \) are coefficients of superposition with \( |c_+|^2 + |c_-|^2 = 1 \). The variational minimum-energy eigen functions and eigen energies are in excellent agreement with numerical solutions of the eigen value problem for the single particle Hamiltonian. This is evident from the density profiles of the variational and numerical solutions shown in Figs. (5a),(c) and (b),(d), respectively for parameters’ set (17a). The same for parameters’ set (17b) are shown in Figs. (5f),(h) and (g),(i). The \( E_{\text{var}}^-_{\text{min}} = 0.333 \) for set (17a) and \( E_{\text{var}}^-_{\text{min}} = -4.128 \) for set (17b); these energies match perfectly with the numerical energies unto three decimal places. The perfect agreement of the variational solutions for sets (17a) and (17b) with the numerical solutions suggests that variational solutions are exact in this case. This is further validated by the fact the variational solutions, after variational parameters have been fixed, satisfy the eigen value equation for \( H_0 \) with eigen energies \( E_{\text{var}}^-_{\text{min}} \). In the presence of infinitesimally small repulsive interactions, say spin-independent interactions, the interaction energy is minimized if \( |c_+| = |c_-| = 1/\sqrt{2} \) resulting in a equal-strength mixture of \( \Phi_{\text{var}}^\pm \). The resultant density profiles in Figs. (5c) and (j) for parameters’ sets (17a) and (17b), respectively, are consistent with effective two-well potentials with two minima at \( (x_0 = 0, y_0 = \pm \gamma_x \Omega_{\text{rot}}/(1 - \Omega_{\text{rot}}^2)) \). The effective two-well potential with minima \( (x_0 = 0, y_0 = \pm 9.7 \Omega_{\text{rot}}) \) is in very good agreement with components densities in Figs. (11-13) and Figs. (211-13), respectively. For parameters’ sets (17a) and (17b), as \( \Omega_{\text{coh}} \neq 0 \), it results in the lifting of the degeneracy between \( \Phi_{\text{var}}^+ \) and \( \Phi_{\text{var}}^- \) with

\[ \Delta E = E_{\text{var}}^+_{\text{min}} - E_{\text{var}}^-_{\text{min}} = 2\Omega_{\text{coh}}, \hspace{1cm} (18) \]

The densities obtained by the variational method for parameters’ sets (17a) and (17b) shown, respectively, in Figs. (5a) and (c) are in excellent agreement with...
the numerical results shown, respectively, in Figs. 6(b) and (d). The variational energies $E_{\text{var}}$ are 0 and -5.128, respectively, for sets (17a) and (17b), and perfectly match with the numerical results. The numerical value of $\Delta E = 2$, energy difference between the two lowest eigen states of $H_0$ for parameters’ sets (17a)-(17b), matches with variational result in Eq. (18). In the presence of interaction, mean-field interaction energy can exceed the $\Delta E$ which can lead to condensate again asymmetrically occupying the two minima corresponding to $\Phi_{\text{var}}^\pm$. We also consider parameters’ sets (17a) and (17b) with $H_{\text{SO}} = \gamma_x S_x p_x$. The two low lying analytic eigen functions in this case can simply be obtained by operating $U^\dagger$ on previously calculated $\Phi_{\text{var}}^\pm$ in Eqs. (13)(a) and (15). The resultant degenerate minimum energy eigen functions corresponding to $\Phi_{\text{var}}$ and $\Phi_{\text{var}}^+$ are, respectively,

$$\frac{1}{\sqrt{\pi}} \exp \left[ -\frac{x^2 + \left( y + \frac{\gamma_x \Omega_{\text{rot}}}{2(1 - \Omega_{\text{rot}})^2} \right)^2}{2} + i \frac{\gamma_x}{2(1 - \Omega_{\text{rot}})^2} x \right] (0, 0, 1)^T,$$

(19)

$$\frac{1}{\sqrt{\pi}} \exp \left[ -\frac{x^2 + \left( y - \frac{\gamma_x \Omega_{\text{rot}}}{2(1 - \Omega_{\text{rot}})^2} \right)^2}{2} - i \frac{\gamma_x}{2(1 - \Omega_{\text{rot}})^2} x \right] (1, 0, 0)^T,$$

(20)

with the same energy as given in Eq. (16) with $\Omega_{\text{coh}} = 0$. The most generic solution is a linear superposition of (19) and (20). In presence of infinitesimally small interactions, an equal strength mixture of (19) and (20) minimizes the energy; the resultant solutions, which have the total densities identical to the ones shown in Figs. 5(c) and (j) for parameters’ sets (17a) and (17b), are in perfect agreement with the numerical results (not shown here). The total density in $y > 0$ region is now fully contributed by $m_j = +1$ component, whereas in $y < 0$ region it is because of $m_j = -1$ component. The results clearly illustrate the phase-separation between $\pm 1$ components in this model with component densities of $\pm 1$ components centered around $(0, \pm \gamma_x \Omega_{\text{rot}}/(1 - \Omega_{\text{rot}}^2))$. This is consistent with the phase-separation and peaks of densities of $\pm 1$ components in Figs. 3(a1)-(a3), (b1)-(b3), (c1)-(c3), and (d1)-(d3). Especially, it can be noted that $\pm 1$ components in Figs. 3(a1)-(a3), (b1)-(b3) peak around $\pm 0.67$, whereas in Figs. 3(c1)-(c3), (d1)-(d3) the respective peaks are at $\pm 0.7$ in excellent agreement with the variational results. The exact analytic solutions of single-particle Hamiltonian with a one-dimensional SO coupling are discussed in the Appendix.

Next we consider case (b), i.e., $\gamma_z = \gamma_y = \gamma$ and $\Omega_{\text{coh}} = 0$. Here we consider the following variational ansatz

$$\Phi_{\text{var}} = \frac{\exp \left( -\frac{x^2}{2 \sigma^2} \right)}{\sqrt{\pi} \sigma^{2n+4} \Gamma(n+2)} \times (i A_1 e^{i n \phi},$$

$$- A_2 e^{i (n+1) \phi}, i A_3 e^{i (n+2) \phi})^T,$$

(21)

where $A_1, A_2, A_3$ are the variational amplitudes, $\sigma$ is the variational width of the ansatz, and $n$ is a variational integer. In the absence of rotation, the ground state of the single particle Hamiltonian is a circularly symmetric $(-1, 0, +1)$ type multi-ring solution with $\pm 1$ components hosting $\pm 1$ phase-singularities. This allows us to fix the integer $n \geq -1$. The normalization condition imposes the constraint

$$\frac{A_1^2 \Gamma(n+1)}{\sigma^{2(1-|n|+n)} \Gamma(n+2)} + [A_2^2(n+2)\sigma^2 + A_3^2] = 1,$$

(22)
FIG. 6: (Color online) (a) and (c) show the total densities obtained by variational method corresponding to $\rho_{\text{var}}$ for $\gamma_x = \gamma_y = 0, \Omega_{\text{coh}} = 1, \Omega_{\text{rot}} = 0.95$ and $\gamma_y = 0, \gamma_x = \Omega_{\text{coh}} = 1, \Omega_{\text{rot}} = 0.95$; (b) and (d) are the numerical densities ($\rho_{\text{num}}$) corresponding to the analytic solutions in (a) and (c). (e) shows the densities corresponding to variational and numerical solutions for $\gamma_x = \gamma_y = 1, \Omega_{\text{coh}} = 0, \Omega_{\text{rot}} = 0.5$ and (f) shows the same for $\gamma_x = \gamma_y = 1, \Omega_{\text{coh}} = 0, \Omega_{\text{rot}} = 0.95$. The charges of phase singularities in the component wavefunctions corresponding to the total densities in (e) and (f) are $(0, +1, +2)$ and $(+98, +99, +100)$, respectively.

This energy can be minimized with respect to all variational parameters subject to the constraint in Eq. (22) to fix the variational parameters. To illustrate the validity of the variational method in this case, we consider three parameters’ sets

$$\gamma_x = \gamma_y = 0.5, \quad \Omega_{\text{coh}} = 0, \quad \Omega_{\text{rot}} = 0.95,$$  
(24)

$$\gamma_x = \gamma_y = 1, \quad \Omega_{\text{coh}} = 0, \quad \Omega_{\text{rot}} = 0.5,$$  
(25)

$$\gamma_x = \gamma_y = 1, \quad \Omega_{\text{coh}} = 0, \quad \Omega_{\text{rot}} = 0.95.$$  
(26)

The minimization of Eq. (23) results in $(A_1, A_2, A_3, n, \sigma) = (-2.512, 0.707, 0.097, 25, 0.975)$ for parameters’ set (24). $(A_1, A_2, A_3, n, \sigma) = (0.517, -0.675, -0.336, 0, 0.828)$ for parameters’ set (25), and $(A_1, A_2, A_3, n, \sigma) = (-4.861, 0.707, 0.051, 98, 0.975)$ for parameters’ set (26). As examples, the comparison of variational and numerical single-particle density profiles for (25) and (26) are shown in Fig. 6(e) and (f), respectively. The total density profiles as shown in Fig. 6(f1) is consistent with the effective toroidal potential experienced by the particle; a similar total density profile for (26) with a peak single-particle density at $r = 4.95$ is not shown here. The charges of phase singularities in the component wavefunctions obtained with the variational analysis, i.e., $(+25, +26, +27)$ for set (24), $(+98, +99, +100)$ for (26), are in perfect agreement with the single-particle numerical results. For parameters’ sets (24) and (26), the peak of total variational density lies along circles of radii 4.95 and 9.74, respectively, in agreement with the numerical results. For parameters’ set (24), the effective potential results in central holes appearing in the component densities as shown in Figs. 1(c1)-(c3) and 2(c1)-(c3) with total density having a peak along a circle of radius 4.9. The circle contains approximately 20 phase singularities as shown via the phase profiles of $m_j = 0$ components in Figs. 1(c1)-(c3) and 2(c1)-(c3), which respectively correspond to the densities of $m_j = 0$ components in Figs. 1(c1)-(c3) and 2(c1)-(c3). Similarly, for set (26), the effective potential results in giant holes appearing in the component densities as shown in Figs. 1(c1)-(c3) and 2(c1)-(c3) with the total density having a peak along a circle of radius 9.7. The circle in this case contains approximately 100 phase singularities as is shown via the phase profiles of $m_j = 0$ components in Figs. 1(c1)-(c3) and 2(c1)-(c3), which respectively correspond to the densities of $m_j = 0$ components in Figs. 1(c1)-(c3) and 2(c1)-(c3). These results are thus in agreement with the variational results. For case (c), i.e., $\gamma_x = \gamma_y \neq 0$ and $\Omega_{\text{coh}} \neq 0$, variational analysis is not pursued due to the lack a simple variational ansatz.

B. Effective potential

The effective potential experienced by the boson can also be calculated from a calculation vector and scalar potentials [32]. To this end, we consider the free single-particle SO-coupled Hamiltonian for a spin-1 BEC in the
Considering a unitary operator $U$ singularities inside the circle along which the coarse-grained ing to Figs. 1(f2) and 2(f2) with approximately 100 phase profiles.

**FIG. 7:** (Color online) (a) and (b) are the phase profiles corresponding to the component densities in Figs. 1(c2) and 2(c2), respectively. The circle corresponds to the peak in coarse-grained total density and encloses approximately 26 phase singularities. Similarly (c) and (d) are the same corresponding to Figs. 1(d2) and 2(d2) with approximately 100 phase singularities inside the circle along which the coarse-grained peak total density lies.

In the laboratory frame

$$H_f = \frac{p_x^2 + p_y^2}{2} + H_{SOC} + \Omega_{coh} S_x. \quad (27)$$

Considering a unitary operator $U = \exp[-i(k_x x + k_y y)]$, the excitation spectrum of $H_f$ is same as that of $U H_f U^{-1}$, and the eigen vectors are $U^{-1}$ times the eigen vectors of $U H_f U^{-1}$. We again consider the single particle SO-coupled Hamiltonian for the two cases in Eqs. (12a)-(12b). In case (a) defined by Eq. (12a), the three eigen functions of the single particle Hamiltonian are

$$\Phi_{\pm} = \frac{e^{i(k_x x + k_y y)}}{2} \begin{pmatrix} 1, \pm \sqrt{2}, 1 \end{pmatrix}^T, \quad (28a)$$

$$\Phi_0 = \frac{e^{i(k_x x + k_y y)}}{\sqrt{2}} \begin{pmatrix} -1, 0, 1 \end{pmatrix}^T, \quad (28b)$$

with eigen energies

$$E_{\pm}(k_x, k_y) = \frac{1}{2} \left[ k_x^2 + k_y^2 \pm 2(\gamma_x k_x + \Omega_{coh}) \right], \quad (29a)$$

$$E_0(k_x, k_y) = \frac{1}{2} (k_x^2 + k_y^2), \quad (29b)$$

respectively. In absence of confining potential and rotation, the minimum energy eigen function(s) can be obtained by minimizing eigen energies in Eqs. (29a)-(29b) with respect to $(k_x, k_y)$ and substituting the respective locations of minima thus obtained in Eqs. (25a)-Eqs. (25b). The minima of $E_{\pm}$ are at $(k_x = \mp \gamma_x, k_y = 0)$ corresponding to minimum energies of $-\gamma_x^2/2 \pm \Omega_{coh}$ and that of $E_0$ is at $(k_x = 0, k_y = 0)$ corresponding to a minimum energy of 0. In the absence of $\Omega_{coh}$, plane-wave solutions $\Phi_{\pm}$ are degenerate. The minimization of spin-dependent interaction energy leads to a plane-wave solution for a ferromagnetic spin-1 BEC which is consistent with the density patterns in Figs. 1(j1)-(j3), whereas for an antiferromagnetic system superposition of $\Phi_{\pm}$ results in a stripe phase in agreement with the density profiles in Figs. 2(j1)-(j3) [22, 59]. This description of the eigen functions and eigen energies of the free single-particle SO-coupled Hamiltonian (27) is exact.

In the rotating frame, the single-particle Hamiltonian becomes

$$H_f' = H_f + \Omega_{rot}(y p_x - x p_y) \mathbb{1}. \quad (30)$$

The $H_f'$ under the unitary transformation $R = (\Phi_1, \Phi_2, \Phi_3)$, which diagonalizes $H_f$ in (27), transforms to $R^\dagger H_f' R$ with eigen values

$$E_{\pm}(k_x, k_y) = \frac{1}{2} \left[ k_x^2 + k_y^2 \pm 2(\gamma_x k_x + \Omega_{coh}) \right] - \Omega_{rot}(x k_y - y k_x), \quad (31a)$$

$$E_0(k_x, k_y) = \frac{1}{2} (k_x^2 + k_y^2) - \Omega_{rot}(x k_y - y k_x), \quad (31b)$$

where we have treated the $k$ as a classical variable. A full quantum mechanical model can be recovered by raising $(k_x, k_y)$ to their operator analogues, i.e. $(-i\partial/\partial x, -i\partial/\partial y)$, which can then be solved by the variational method or numerically as has been done in the previous subsection. Nevertheless, this is not necessary as far as calculation of effective potential is concerned as we shall demonstrate in the following discussion. The spectrum in Eqs. (31a)-(31b) around a minimum can be simply described by a parabola of form $(k_x - k_{\text{min}})^2 + (k_y - k_{\text{min}})^2 + E_{\text{min}}$, which describes the particle moving in an effective gauge field $(A, \Phi) = (\{k_{\text{min}}, k_{\text{min}}, 0\}, E_{\text{min}})$, where $A$ and $\Phi$ are vector and scalar potentials, respectively [32]. In the presence of rotation term $k_{\text{min}}$, $k_{\text{min}}$ and $E_{\text{min}}$ become spatially dependent [32]. Rewriting Eqs. (31a)-(31b) as

$$E_{\pm}(k_x, k_y) = \frac{1}{2} \left[ (k_x \pm \gamma_x y_{\text{rot}})^2 + (k_y - x_{\text{rot}})^2 \right] + E_{\text{min}}(x, y), \quad (32a)$$

$$E_0(k_x, k_y) = \frac{1}{2} \left[ (k_x + y_{\text{rot}})^2 + (k_y - x_{\text{rot}})^2 \right] + E_{\text{min}}(x, y), \quad (32b)$$

where $A_{\pm} = \mp (\gamma_x y_{\text{rot}} - x_{\text{rot}})$, $A_0 = -y_{\text{rot}} x_{\text{rot}}$, and $\Phi_{\pm} = E_{\text{max}}$, $A_0 = E_{\text{min}}$. The effective potentials, which are the sums of trapping
and scalar potentials [32], can now be written as
\[
V_{\pm}^\pm(x, y) = \frac{1}{2} \left[ (1 - \Omega_{\text{rot}}^2)(x^2 + y^2) - \gamma_x^2 \pm 2\gamma_x \Omega_{\text{rot}} \right],
\]
\[
V_{\text{eff}}^0(x, y) = \frac{1}{2} \left[ (1 - \Omega_{\text{rot}}^2)(x^2 + y^2) \right].
\]
For the parameters’ sets (17a)-(17d), the effective potential energy curves are shown in Figs. 8(a)-(d). From Eqs. (33a)-(33b), \( V_{\text{eff}}(x, y) \) and \( V_{\text{eff}}^0(x, y) \) overlap at \( y = \Omega_{\text{coh}}/\gamma_x \Omega_{\text{rot}} \) for \( \gamma_x \neq 0 \) and \( \Omega_{\text{rot}} \neq 0 \). In the region, \( y < \Omega_{\text{coh}}/\gamma_x \Omega_{\text{rot}} \), \( V_{\text{eff}} \) is lower than the other two, and with a minima at \( -\gamma_x \Omega_{\text{rot}}/(\Omega_{\text{rot}}^2 - 1) \), whereas for \( y > \Omega_{\text{coh}}/\gamma_x \Omega_{\text{rot}} \), \( V_{\text{eff}}^0 \) is the low lying potential curve with a minima at \( \gamma_x \Omega_{\text{rot}}/(\Omega_{\text{rot}}^2 - 1) \). The minima of \( V_{\text{eff}}^\pm \) are in perfect agreement with \( y_0 \) in Eq. (15) in Sec. IV A from the variational analysis. The potential experienced by the boson is effectively equivalent to a symmetric double-well potential for (17a) and (17b), with minima at \( (x = 0, y = 0.67) \) and \( (x = 0, y = 9.7) \), respectively, a harmonic potential with a minima at \( x = y = 0 \) for (17c), and an asymmetric double well potential for (17d) with a global minima at \( (x = 0, y = -9.7) \). These results are in agreement with the variational densities of minimum energy solutions shown in Figs. 8(e),(j) for parameter’s sets (17a) and (17b), respectively, and Figs. 8(a) and (c) for parameters’ sets (17c) and (17d), respectively. The symmetric effective double well potential leads to the condensate occupying the two minima at \( (x = 0, y = 0.67) \) for (17a) and \( (x = 0, y = 9.7) \) for (17d), the later is evident Figs. 4(11)-(13) and Figs. 2(11)-(13). The central chain of vortices may also get formed along the line of overlap of \( V_{\text{eff}} \) and \( V_{\text{eff}}^0 \), which is \( y = 0 \) for (17a) and (17b), consistent with the component densities in Figs. 1(k1)-(k3), (11)-(13) and 2(k1)-(k3),(11)-(13). Similarly, a harmonic potential with a minima at \( (x = 0, y = 0) \) for (17d) leads to a symmetric condensate density distribution shown in Figs. 4(b1)-(b3), and the asymmetric effective double well potential with a global minima at \( (x = 0, y = -9.7) \) and a local minima at \( (x = 0, y = 9.7) \) for (17d) leads to condensate density distribution as shown in Figs. 4(d1)-(d3). The spin-independent interaction energy in this case is more than the potential barrier between the two minima, and thus results in the occupation of both the potential minima. At moderate to high rotation frequencies, another consequence of the effective potential experienced by the particle as shown in Figs. 8(a), (b), and (d) is the state of the particle becomes \( \Phi_{\pm k} - k_{\min} \), in Eq. (38) for \( y < \Omega_{\text{coh}}/\gamma_x \Omega_{\text{rot}} \) where \( k_{\min} = (\gamma_x - y \Omega_{\text{rot}}, x \Omega_{\text{rot}}) \) corresponds to minima of (31b) and \( \Phi_{\mp k} - k_{\min} \) in Eq. (38) for \( y > \Omega_{\text{coh}}/\gamma_x \Omega_{\text{rot}} \) where \( k_{\min} = (\gamma_x - y \Omega_{\text{rot}}, x \Omega_{\text{rot}}) \). This results in spin-expectation per particle \( |f| \) becoming equal to one for even antiferromagnetic BEC. As previously stated, in the absence of rotation, antiferromagnetic interactions favor the equal strength mixing of the \( \Phi_{\pm k} - k_{\min} \) and \( \Phi_{\mp k} - k_{\min} \), which are degenerate, resulting in \( |f| = 0 \) and stripe density pattern [22, 61]. At small rotation frequencies, the antiferromagnetic interactions still will lead to unequal strength mixing of \( \Phi_{\pm} \), which has lower energy for \( y < \Omega_{\text{coh}}/\gamma_x \Omega_{\text{rot}} \) and \( \Phi_{\pm} \) which has a lower energy for \( y > \Omega_{\text{coh}}/\gamma_x \Omega_{\text{rot}} \) leading to \( |f| \neq 0 \). Similarly in case (b) defined by Eq. (12b), the three eigen functions and eigen energies of the single particle SO-coupled Hamiltonian \( H_f \) are
\[
\Phi_{\pm} = \frac{e^{i(k_x x + k_y y)}}{2} \left( e^{-i\theta}, \pm \sqrt{2}, e^{i\theta} \right)^T,
\]
\[
\Phi_0 = \frac{e^{i(k_x x + k_y y)}}{\sqrt{2}} \left( -e^{-i\theta}, 0, e^{i\theta} \right)^T,
\]
and
\[
E_\pm(k_x, k_y) = \frac{1}{2} \left(k_x^2 + 2\gamma_x k_y \right),
\]
\[
E_0(k_x, k_y) = \frac{1}{2} k_x^2,
\]
where \( k = \sqrt{k_x^2 + k_y^2} \) and \( \theta = \tan^{-1}(k_y/k_x) \). The lowest
lying dispersion corresponds to \( E_\gamma (k_x, k_y) \) which has a minima at \( k = \gamma_x \). The eigen energy is minimum along a circle of radius \( \gamma_x \). The eigen functions with different orientations of the vector \( k = (k_x, k_y) \) are all degenerate. It was shown in Ref. [23] that only energetically favorable superpositions of these degenerate plane-wave states correspond either to single or multiple pairs of counter-propagating plane waves. A single pair of counter-propagating plane waves leads to the stripe phase as shown in Figs. 2(d1)-(d3). Another solution to the single-particle Hamiltonian can be obtained by considering the superposition of the eigen functions \( \Phi_\gamma \), with \( k \) allowed to point along directions with letting \( \theta \) vary from zero to \( 2\pi \). The solution thus obtained is

\[
\Phi_{\gamma}^{\text{gen}} = \frac{1}{4\pi} \int_0^{2\pi} d\theta \left( \frac{e^{-i\theta}}{\sqrt{2}} \right) e^{-i\gamma_x r \cos(\theta - \Theta)} (36a)
\]

\[
= \frac{1}{2} \left[ \frac{ie^{-i\theta} J_0(\gamma_x r)}{ie^{i\theta} J_1(\gamma_x r)} \right] (36b)
\]

where \( \Theta = \tan^{-1}(y/x) \), \( J_n(\gamma_x r) \) with \( n = 0, 1 \) is the Bessel function of first kind of order \( n \), and \( \Phi_{\gamma}^{\text{gen}} \) has the phase singularities of \((-1,0,+1)\)-type. For \( c_2 > 0 \) the superposition of counter-propagating plane waves corresponding to the stripe phase or \( \Phi_{\gamma}^{\text{gen}} \) may result in the minimization interaction energy and hence can emerge as the ground state solution depending on the strength of SO coupling, whereas for \( c_2 < 0 \), stripe phase or \( \Phi_{\gamma}^{\text{gen}} \) are energetically not favored. This explains the \((-1,0,+1)\)-type vortex solution shown in Figs. 2(a1)-(a3) for \(^{23}\text{Na} \) at small SO coupling strength, which at larger SO coupling strength gives way to the stripe phase in Figs. 2(d1)-(d3) as the ground state. These are in agreement with Ref. [23]. In the rotating frame, the corrected dispersion is

\[
E_c^\gamma (k_x, k_y) = \frac{1}{2} \left( k^2 - 2\gamma_x k \right) - \Omega_{\text{rot}}(xk_y - yk_x). (37)\]

Minimizing \( E_c^\gamma \) with respect to \((k_x, k_y)\) and adding the resultant scalar potential to the trapping potential, the effective trapping potential experienced by a particle is

\[
V_{\text{eff}}(r) = \frac{1}{2} \left[ r^2 - (\gamma_x + r\Omega_{\text{rot}})^2 \right], (38)\]

where \( r = \sqrt{x^2 + y^2} \). The effective potential in Eq. (38) has a minima along a circle of radius \( \gamma_x \Omega_{\text{rot}}/(1 - \Omega_{\text{rot}}) \) like a toroidal trap. For \( \gamma_x = \gamma_y = 1 \) and \( \Omega_{\text{rot}} = 0.5 \) and \( \gamma_x = \gamma_y = 1 \) and \( \Omega_{\text{rot}} = 0.95 \), the potential minima lie along a circle of radii 0.67 and 9.7, respectively as shown in Fig. 3(e). The latter is in very good agreement with the density distribution shown in Figs. 1(f1-f3) and Figs. 2(f1-f3). The absence of hole in the total density for \( \gamma_x = \gamma_y = 1 \) and \( \Omega_{\text{rot}} = 0.5 \) in Figs. 1(e1-e3) and Figs. 2(e1-e3) is due to the fact that the potential barrier at the center is too small in comparison to mean-field energy. The effective potential minima are consistent with the circularly symmetric variational densities for parameters sets’ [25] and [26] as shown in Fig. 6(e) and (f), respectively.

In case (c), the lowest eigen energy of free single-particle the SO-coupled Hamiltonian \( H_j \) is

\[
E = \frac{1}{2} \left( k_x^2 + k_y^2 - 2\sqrt{\gamma_x^2(k_x^2 + k_y^2) + 2\gamma_x k_x \Omega_{\text{coh}} + \Omega_{\text{coh}}^2} \right), (39)\]

corresponding to eigen function

\[
\Phi = \frac{1}{2} \left( \frac{\gamma_x k e^{-i\theta} + \Omega_{\text{coh}}}{\sqrt{\gamma_x^2 k^2 + 2\gamma_x \Omega_{\text{coh}} k \cos \theta + \Omega_{\text{coh}}^2}} - \sqrt{2}, \right) \frac{\gamma_x k e^{i\theta} + \Omega_{\text{coh}}}{\sqrt{\gamma_x^2 k^2 + 2\gamma_x \Omega_{\text{coh}} k \cos \theta + \Omega_{\text{coh}}^2}} \right). (40)\]

Taking into account the correction from the rotation terms, the corrected energy is

\[
E_c = \frac{1}{2} \left( k_x^2 + k_y^2 - 2\sqrt{\gamma_x^2(k_x^2 + k_y^2) + 2\gamma_x k_x \Omega_{\text{coh}} + \Omega_{\text{coh}}^2} \right) - \Omega_{\text{rot}}(xk_y - yk_x). (41)\]

The effective potential, \( E_{\text{min}}^c + V(x,y) \), where \( E_{\text{min}}^c \) (scalar potential) is the numerically obtained minima of \( E \) with respect to \((k_x, k_y)\). For \( \gamma_x = \gamma_y = 1 \), \( \Omega_{\text{coh}} = 1 \), and \( \Omega_{\text{rot}} = 0.95 \), the effective potential thus obtained is shown in Fig. 3(f). The effective potential is consistent with the condensate density profile shown in Figs. 2(c1)-(c3). The effective potential analysis for \( \gamma_x \neq \gamma_y \neq 0 \) and \( \Omega_{\text{rot}} \neq 0 \) corresponding to the density profiles in Figs. 2(h1)-(h3), 2(i1)-(i3) and Figs. 2(h1)-(h3), 2(i1)-(i3) is not pursued due to the lack of analytical or semi-analytic description.

C. Spin-expectation per particle and spin-texture

As noted previously, the ground state solutions of the rotating SO-coupled \(^{87}\text{Rb} \) and \(^{23}\text{Na} \) BECs at moderate to high rotation frequencies are qualitatively similar, and the quantitative differences are mainly the result of different magnitudes of \( c_0 \). To ascertain this further, here we consider SO-coupled \(^{87}\text{Rb} \) and \(^{23}\text{Na} \) spin-1 BECs with \( \gamma_x = \gamma_y = 0.5 \) or 1, \( \Omega_{\text{coh}} = 0 \) and (approximately) same \( c_0 \) but with different atom numbers. For \(^{87}\text{Rb} \), as in subsection IIIA, we consider \( c_0 = 2482.21 \) and \( c_2 = -11.47 \) corresponding to \( 10^5 \) atoms, whereas for \(^{23}\text{Na} \) we consider \( 3.68 \times 10^5 \) atoms resulting in \( c_0 = 2482.35 \) and \( c_2 = 77.68 \). We define spin-density vector \( \mathbf{F} = (F_x, F_y, F_z) \) where

\[
F_\nu(x, y) = \sum_{m,m'} \phi_m^*(x, y)(S_\nu)_{mm'} \phi_{m'}(x, y), (42)\]

and \( f = \int |\mathbf{F}(x, y)| dx / \int \rho(x, y) dx \), which serves as a measure of spin-expectation per particle for an inhomogeneous system. We examine the angular momentum per particle, \( f \), and spin-texture [15] \( I(x, y) = F(x, y)/\rho(x, y) \)
as a function of rotation frequency. In the absence of rotation with \( \gamma_x = \gamma_y = \gamma, \Omega_{\text{coh}} = 0 \), and \( c_2 < 0 \) (ferromagnetic interactions), the state of the particle is \( \sim \Phi_{-|\mathbf{k} = \mathbf{k}_{\text{min}} \rangle} \) where \( \mathbf{k}_{\text{min}} = (\gamma \cos \theta, \gamma \sin \theta) \) resulting in \( |f| = 1 \). The same for an antiferromagnetic BEC in the absence of rotations is \( \sim \int_0^{2\pi} \Phi_{-|\mathbf{k} = \mathbf{k}_{\text{min}} \rangle} d\theta/2\pi \) resulting in \( f = 0 \). This leads to a plane-wave solution for a ferromagnetic BEC and a circularly symmetric \((-1, 0, +1)\)-type multi-ring solution for an antiferromagnetic BEC. The \( f \) as a function of rotation frequency \( \Omega_{\text{rot}} \) for \( ^{87}\text{Rb} \) and \( ^{23}\text{Na} \) systems is shown in Fig. 9(a), which illustrates that with increase in \( \Omega_{\text{rot}} \), \( f \to 1 \) for \( ^{23}\text{Na} \) whereas it remains close to 1 for \( ^{87}\text{Rb} \). We also analyse spin-expectation per particle using the single-particle variational solution \( \Phi_{\text{var}} \) in Eq. (21) to evaluate \( f \). The variational analysis predicts \( f \approx 1 \) for \( \gamma_x = \gamma_y = 0.5 \) (1) and \( \Omega_{\text{rot}} \geq 0.6 \) (0.4), which is consistent with the numerical results for \( ^{87}\text{Rb} \) and \( ^{23}\text{Na} \) BECs at moderate to high rotations as is shown in Fig. 9(a). The differences in numerical and variational \( f \) values for \( \Omega_{\text{rot}} \leq 0.6 \) (0.4) are mainly because of spin-dependent interactions. The similarity in the response of the two systems at faster rotation is also evident from the percentage difference in angular momentum acquired by the two systems at different rotation frequencies as shown in Fig. 9(b). Next we consider the spin-texture of \( ^{87}\text{Rb} \), and \( ^{23}\text{Na} \) BEC with \( \gamma_x = \gamma_y = 0.5, \Omega_{\text{coh}} = 0 \) when rotated with \( \Omega_{\text{rot}} = 0.1 \) and 0.95. The spin-texture of the two systems when rotated with \( \Omega_{\text{rot}} = 0.1 \) is shown in Fig. 10. At this fre-

![Fig. 9](image)

**FIG. 9:** (Color online) (a) The numerical and variational (var.) \( f = \int \rho(x,y)|F(x,y)|d\mathbf{r} \) as a function of rotation frequency for SO-coupled \( ^{87}\text{Rb} \) and \( ^{23}\text{Na} \) BECs. (b) The percentage difference in the angular momentum acquired by the two SO-coupled BECs as a function of rotation frequency. We also analyse spin-expectation per particle using the single-particle variational solution \( \Phi_{\text{var}} \) in Eq. (21) to evaluate \( f \). The variational analysis predicts \( f \approx 1 \) for \( \gamma_x = \gamma_y = 0.5 \) (1) and \( \Omega_{\text{rot}} \geq 0.6 \) (0.4), which is consistent with the numerical results for \( ^{87}\text{Rb} \) and \( ^{23}\text{Na} \) BECs at moderate to high rotations as is shown in Fig. 9(a). The differences in numerical and variational \( f \) values for \( \Omega_{\text{rot}} \leq 0.6 \) (0.4) are mainly because of spin-dependent interactions. The similarity in the response of the two systems at faster rotation is also evident from the percentage difference in angular momentum acquired by the two systems at different rotation frequencies as shown in Fig. 9(b). Next we consider the spin-texture of \( ^{87}\text{Rb} \), and \( ^{23}\text{Na} \) BEC with \( \gamma_x = \gamma_y = 0.5, \Omega_{\text{coh}} = 0 \) when rotated with \( \Omega_{\text{rot}} = 0.1 \) and 0.95. The spin-texture of the two systems when rotated with \( \Omega_{\text{rot}} = 0.1 \) is shown in Fig. 10. At this fre-

![Fig. 10](image)

**FIG. 10:** (Color online) (a) shows the spin-texture of an SO-coupled \( ^{87}\text{Rb} \) BEC with \( \gamma_x = \gamma_y = 0.5, \Omega_{\text{coh}} = 0 \) when rotated with \( \Omega_{\text{rot}} = 0.1 \) and (b) shows the same for \( ^{23}\text{Na} \) system. (a) have the three skyrmions, whereas (b) shows has two near the center of the trap in addition to two cross-disgyrations in spin-texture along y-axis coinciding with +1 phase singularity in \( m_y \) component. The interaction strengths \( (c_\theta, c_2) \) for (a) and (b) are (2482.21,-11.47) and (2482.35,77.68), respectively.

87Rb hosts three skyrmions as compared to two for \( ^{23}\text{Na} \) (near the center of the trap). The generation of skyrmion and half-skyrmion excitations in rotating SO-coupled BECs is discussed in Refs. [35, 39]. The component densities corresponding to spin-texture in Fig. 10 are shown in Figs. 11(a1)-(a3) for \( ^{87}\text{Rb} \) and Figs. 11(b1)-(b3) for \( ^{23}\text{Na} \). The spin-textures at \( \Omega_{\text{rot}} = 0.95 \) are shown in Fig. 12(a) for \( ^{87}\text{Rb} \) and Fig. 12(b) for \( ^{23}\text{Na} \), here both the systems have a skyrmion at the center sur-
rounded by a lattice of half-skyrmions. The similarity of the two systems at faster rotation is therefore also reflected in the spin-textures. The spin-texture of $^{87}$Rb corresponds to the component densities shown in Figs. 1(c1)-(c3), whereas the component densities of $^{23}$Na which are indistinguishable $^{87}$Rb are not shown here. The similarity in the response of the two systems at fast rotations has also been confirmed based upon their mass and spin-currents.

V. SUMMARY AND CONCLUSIONS

We have studied the SO- and coherently-coupled spin-1 BECs under rotation with an emphasis on moderate to high rotations using a mean-field model. Our analysis of the eigen functions and eigen energies of the non-interacting part of the Hamiltonian, using the variational analysis and exact numerical solutions, shows that a boson in such BECs can be subjected to a variety of rotating effective potentials which include among others potentials resembling symmetric, asymmetric double-well, and toroidal potentials. The shape of these effective potentials is consistent with the component density profiles for SO- and coherently-coupled $^{87}$Rb and $^{23}$Na BECs with experimentally realizable interaction parameters. The analytic expressions for effective potentials have also been calculated via a calculation of effective gauge fields. We have shown the spin-expectation per particle for a ferromagnetic BEC stays close to one as a function of rotation frequency, whereas for an antiferromagnetic BEC it starts increasing with an increase in rotation frequency and tends to approach one at high rotations. For an SO coupling between the spin and the linear momentum along one direction, i.e., $\gamma_z S_z p_x$, spatial segregation of the eigen functions ($\Phi_{var}$) of the single particle Hamiltonian in the presence of moderate to high rotations, which translates to a spatial segregation between $m_j = \pm 1$ components in an equivalent $\gamma_z S_z p_x$ coupling, can result in spin-expectation per particle approaching one for an antiferromagnetic BEC. For an isotropic SO coupling without coherent coupling, our variational analysis also predicts spin-expectation per particle approaching one at moderate to high rotations. This results in a similar response of the two systems at moderate to large rotations as exemplified in similar spin-texture, mass-current, etc. in addition to component-density profiles of the two systems.

![FIG. 11](image1.png)

![FIG. 12](image2.png)
Acknowledgements

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Appendix

Details of the numerical method

The split equations corresponding to $H_0^x$ and $H_0^y$ are

\[
(i - \gamma_{\text{diss}}) \frac{\partial \Phi(x,y,t)}{\partial t} = H_0^x \Phi(x,y,t), \quad (43a)
\]

\[
(i - \gamma_{\text{diss}}) \frac{\partial \Phi(x,y,t)}{\partial t} = H_0^y \Phi(x,y,t). \quad (43b)
\]

The solution of Eq. (43a) in the Fourier space is

\[
\hat{\Phi}(k_x, y, t + \delta t) = \Phi(k_x, y, t) \exp \left( \frac{k_x^2}{2} \right),
\]

where $\hat{\Phi}(k_x, y, t)$ is the Fourier transform of $\Phi(x, y, t)$ corresponding to $x$-coordinate. The solution to the split equation for $\Phi(x, y, t + \delta t)$ is inverse Fourier transformed to obtain the transient wavefunction in coordinate space which is considered as the initial solution of Eq. (43b) at time $t$. Similarly, the solution of Eq. (43b) in Fourier space is

\[
\hat{\Phi}(x, k_y, t + \delta t) = \hat{\Phi}(x, k_y, t) \exp \left( \frac{2 \Omega_{\text{rot}} x k_y}{2(i - \gamma_{\text{diss}})} \delta t, \right)
\]

where $\hat{\Phi}(x, k_y, t)$ is the Fourier transform of $\Phi(x, y, t)$ corresponding to $y$-coordinate. Again, $\hat{\Phi}(x, k_y, t)$ can be inverse Fourier transformed to obtain, $\Phi(x, y, t)$, the transient wavefunction to be used as the input to solve next of the split equations. The procedure to solve the split equations corresponding to $H_0^x$, $H_0^y$, and $H_{\text{SOC}}$ are discussed in detail in Refs. [51][53]. Nonetheless, for the sake of completeness, we provide the solutions to these split equations. The solution to the split equation for $H_0^x$ is

\[
\Phi(x, y, t + \delta t) = \Phi(x, y, t) \exp \left( -i H_0^x \delta t \right), \quad (44)
\]

Similarly, the solution to the split equation for $H_0^y$ is

\[
\Phi(x, y, t + \delta t) = \left[ I + \frac{\cos \zeta - 1}{\zeta^2} \left( \frac{\delta t}{1 + i \gamma_{\text{diss}}} \right)^2 H_0^y \right] \Phi(x, y, t),
\]

where

\[
\zeta = \frac{\delta t}{1 + i \gamma_{\text{diss}}} \sqrt{c_2 \phi_0 \phi_{-1}^* + \frac{\Omega_{\text{coh}}}{\sqrt{2}}^2 + c_2 \phi_0 \phi_1^* + \frac{\Omega_{\text{coh}}}{\sqrt{2}}^2}.
\]

The split equation for $H_{\text{SOC}}$ is again solved in Fourier space with a solution

\[
\hat{\Phi}(k_x, k_y, t + \delta t) = \left( I + \frac{\cos \beta - 1}{\beta^2} \hat{M}^2 - i \frac{1}{\beta} \hat{M} \right) \hat{\Phi}(k_x, k_y, t), \quad (45)
\]

where $\beta = \sqrt{2} |A| \delta t / (1 + i \gamma_{\text{diss}})$, $A = (\gamma_2 k_x - i \gamma_1 k_y) / \sqrt{2}$, and $\hat{M}$ is defined as

\[
\hat{M} = \frac{\delta t}{1 + i \gamma_{\text{diss}}} \begin{pmatrix} 0 & A & 0 \\ A^* & 0 & A \\ 0 & A^* & 0 \end{pmatrix}.
\]

Finally, $\hat{\Phi}(k_x, k_y, t + \delta t)$ is inverse Fourier transformed to obtain the wavefunction in coordinate space at $t + \delta t$ time.

Exact solutions of the single-particle Hamiltonian with one-dimensional SO coupling

We consider the single-particle Hamiltonian $U^\dagger H_0 U$, where $H_0$ is defined in Eq. (43). Then the eigenvalue equation of the single-particle Hamiltonian is

\[
U^\dagger H_0 U \Phi = \text{diag} (h_1, h_0, h_{-1}) (\phi_+ \phi_0 \phi_-)^T, \quad (48)
\]

where $\text{diag} (\ldots)$ stands for a $3 \times 3$ diagonal matrix (operator). In Eq. (48), which now consists of three decoupled equations, the operators $h_j$ with $j = -1, 0, +1$ are

\[
h_j = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} - \Omega_{\text{rot}} (xp_y - yp_x) + j \gamma_x p_x + j \Omega_{\text{coh}}
\]

\[
= \frac{(p_x + j \gamma_x)^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} - \Omega_{\text{rot}} (xp_y - yp_x)
\]

\[
- \frac{j \gamma_x^2}{2} + j \Omega_{\text{coh}}
\]

\[
= \frac{\bar{p}_x^2 + \bar{p}_y^2}{2} + \frac{x^2 + y^2}{2} - \Omega_{\text{rot}} (xp_y - yp_x)
\]

\[
- j \gamma_x \Omega_{\text{rot}} y - \frac{j \gamma_x^2}{2} + j \Omega_{\text{coh}}, \quad (49)
\]

where $\bar{p}_x = (p_x + j \gamma_x)$. Rewriting (49) as

\[
= \frac{(p_x + \Omega_{\text{rot}} x)^2}{2} + \frac{(p_y - \Omega_{\text{rot}} x)^2}{2} - j \gamma_x \Omega_{\text{rot}} y - \frac{j \gamma_x^2}{2}
\]

\[
+ (1 - \Omega_{\text{rot}}^2) \frac{x^2 + y^2}{2} + j \Omega_{\text{coh}}
\]

\[
= \frac{(p_x + j \gamma_x - \Omega_{\text{rot}} x)^2 + \Omega_{\text{rot}} y^2}{2} + (p_y - \Omega_{\text{rot}} x)^2
\]

\[
+ (1 - \Omega_{\text{rot}}^2) \frac{(x^2 + y^2)}{2} - \frac{j \gamma_x^2}{2} + \frac{j \Omega_{\text{coh}}^2}{2}, \quad (50)
\]
where $\bar{y} = y - j\gamma_x \Omega_{\text{rot}}/(1 - \Omega_{\text{rot}}^2)$ and $p_y = p_{\bar{y}} = -i \partial / \partial \bar{y}$ is the canonical conjugate of $\bar{y}$. The decoupled eigen value equation for $h_j$ are

$$h_j \phi_j(x, \bar{y}) = E_j \phi_j(x, \bar{y}), \quad (51)$$

which can be simplified by substituting

$$\phi_j(x, \bar{y}) = \bar{\phi}_j(x, \bar{y}) \exp \left( -i \frac{j\gamma_x}{1 - \Omega_{\text{rot}}^2} x \right) \quad (52)$$

to obtain

$$\left[ \frac{(p_x + \Omega_{\text{rot}} y)^2}{2} + \frac{(p_y - \Omega_{\text{rot}} x)^2}{2} + (1 - \Omega_{\text{rot}}^2) \frac{x^2 + y^2}{2} - \frac{j^2 \gamma_x^2}{2(1 - \Omega_{\text{rot}}^2)} + j\Omega_{\text{coh}} \right] \bar{\phi}_j(x, \bar{y}) = E_j \bar{\phi}_j(x, \bar{y}) \quad (53)$$

which can be rewritten as

$$\left[ \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} - \Omega(x p_y - y p_x) - \frac{j^2 \gamma_x^2}{2(1 - \Omega_{\text{rot}}^2)} + j\Omega_{\text{coh}} \right] \bar{\phi}_j(x, \bar{y}) = E_j \bar{\phi}_j(x, \bar{y}) \quad (54)$$

The Hamiltonian on the left hand side of Eq. (54), barring the constant terms, is that of isotropic harmonic oscillator under rotation. The eigen functions of this Hamiltonian are the eigen functions of the isotropic Harmonic oscillator without rotation which commutes with $L_z$. The ground state $\bar{\phi}_j(x, \bar{y})$ thus obtained is

$$\bar{\phi}_j(x, \bar{y}) = \frac{1}{\sqrt{\pi}} \exp \left( - \frac{x^2 + y^2}{2} \right), \quad (55)$$

which can be substituted in Eq. (52) along with $\bar{y} = y - j\gamma_x \Omega_{\text{rot}}/(1 - \Omega_{\text{rot}}^2)$ to obtain

$$\phi_j(x, y) = \frac{1}{\sqrt{\pi}} \exp \left[ - \frac{x^2 + \left( y - \frac{j\gamma_x \Omega_{\text{rot}}}{1 - \Omega_{\text{rot}}^2} \right)^2}{2} - \frac{j\gamma_x}{1 - \Omega_{\text{rot}}^2} x \right]. \quad (56)$$

And the respective eigen energies in Eq. (51) are

$$E_j = \frac{2(1 + j\Omega_{\text{coh}})(1 - \Omega_{\text{rot}}^2) - j^2 \gamma_x^2}{2(1 - \Omega_{\text{rot}}^2)}. \quad (57)$$

In the absence of $\Omega_{\text{coh}}, E_{+1} = E_{-1}$, and the respective degenerate ground-state eigen-functions of the Hamiltonian in Eq. (48) are

$$\frac{1}{\sqrt{\pi}} \exp \left[ - \frac{x^2 + \left( y - \frac{j\gamma_x \Omega_{\text{rot}}}{1 - \Omega_{\text{rot}}^2} \right)^2}{2} - \frac{j\gamma_x}{1 - \Omega_{\text{rot}}^2} x \right](1, 0, 0)^T (58)$$

$$\frac{1}{\sqrt{\pi}} \exp \left[ - \frac{x^2 + \left( y + \frac{j\gamma_x \Omega_{\text{rot}}}{1 - \Omega_{\text{rot}}^2} \right)^2}{2} + \frac{j\gamma_x}{1 - \Omega_{\text{rot}}^2} x \right](0, 0, 1)^T (59)$$

which were earlier calculated with the variational method in Sec. IV A.

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