ABHYANKAR’S WORK ON DICRITICAL DIVISORS

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Abstract. We discuss the work of Abhyankar on dicritical divisors with a special focus on the algebraic aspects of this work. We also discuss related work on local quadratic transforms, infinitely near points and Rees valuation rings of an ideal.

1. Introduction

Early in his career, in the context of working on the problem of the resolution of singularities, Abhyankar published in [1, 1956] a paper that has turned out to be one of his most cited papers, “On the Valuations Centered in a Local Domain”. In this paper he proves a theorem (Proposition 3 of [1]) that characterizes prime divisors of a regular local domain. The characterization may be described as follows.

Theorem 1.1. Let $R$ be an $n$-dimensional regular local domain with maximal ideal $M(R) = M$ and assume that $n \geq 2$. Let $V$ be a prime divisor of $R$ with center $M$ in $R$. There exists a unique finite sequence

\[ R = R_0 \subset R_1 \subset \cdots \subset R_h \subset R_{h+1} = V \]

of regular local rings $R_j$, where $\dim R_h \geq 2$ and $R_{j+1}$ is the first local quadratic transform of $R_j$ along $V$ for each $j \in \{0, \ldots, h\}$, and $\ord R_h = V$.

It follows from Theorem 1.1 that the residue field $V/M(V)$ of $V$ is a pure transcendental extension of the field $R_h/M(R_h)$ of transcendence degree one less than...
dim \( R_h \). Therefore the residue field of \( V \) is ruled as an extension field of the residue field of \( R \).

The association of the prime divisor \( V \) with the regular local ring \( R_h \) in Equation \([\text{I}]\) and the uniqueness of the sequence in Equation \([\text{I}]\) establishes a one-to-one correspondence between the prime divisors \( V \) dominating the regular local ring \( R \) and the regular local rings \( S \) of dimension at least 2 that dominate \( R \) and are obtained from \( R \) by a finite sequence of local quadratic transforms as in Equation \([\text{I}]\).

The regular local rings \( R_j \) with \( j \leq h \) displayed in Equation \([\text{I}]\) are the infinitely near points to \( R \) along \( V \). In general, a regular local ring \( S \) of dimension at least 2 is called an infinitely near point to \( R \) if there exists a sequence

\[
R = R_0 \subset R_1 \subset \cdots \subset R_h = S, \quad h \geq 0
\]

of regular local rings \( R_j \) of dimension at least 2, where \( R_{j+1} \) is the first local quadratic transform of \( R_j \) for each \( j \) with \( 0 \leq j \leq h - 1 \). \([\text{II}]\) Definition 1.6).

The Zariski-Abhyankar Factorization Theorem \([\text{I}]\) Theorem 3] implies that if \( \text{dim} \, R = 2 \), then every 2-dimensional regular local ring \( S \) that birationally dominates \( R \) is an infinitely near point to \( R \). We record in Theorem 1.2 implications of \([\text{I}]\) Theorem 3).

**Theorem 1.2.** Let \( R \hookrightarrow S \) be a birational extension of 2-dimensional regular local domains.

1. If \( R \neq S \), then \( M(R)S \) is a proper principal ideal of \( S \). Therefore \( S \) dominates a unique local quadratic transform \( R_1 \) of \( R \).
2. There exists for some positive integer \( \nu \) a sequence

\[
R = R_0 \subset R_1 \subset \cdots \subset R_\nu = S,
\]

where \( R_i \) is a local quadratic transform of \( R_{i-1} \) for each \( i \in \{1, \ldots, \nu\} \). The rings \( R_i \) are precisely the regular local domains that are subrings of \( S \) and contain \( R \).

In the case where \( R \) is a regular local domain with \( \text{dim} \, R \geq 3 \), there are many regular local rings \( S \) that birationally dominate \( R \) with \( \text{dim} \, R = \text{dim} \, S \) such that \( S \)

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4 A field extension \( F \subset L \) is said to be ruled if \( L \) is a simple transcendental extension of a subfield \( K \) such that \( F \subset K \).

5 A quasilocal extension domain \( S \) birationally dominates \( R \) if \( S \) is contained in the quotient field of \( R \) and \( M(S) \cap R = M \).
is not an infinitely near point to \( R \). Perhaps the simplest examples are provided by local monoidal transforms, cf. [41].

**Example 1.3.** Let \((R, \mathfrak{m})\) be a 3-dimensional regular local ring with maximal ideal \( \mathfrak{m} = (x, y, z)R \), and let \( S = R[\frac{y}{x}, \frac{z}{x}, \frac{y}{z}]R[\frac{y}{x}] \). Then \( S \) is a 3-dimensional regular local ring that birationally dominates \( R \) and \( \mathfrak{m}S = (x, z)S \) is a prime ideal of \( S \) of height 2. It follows that \( S \) does not dominate a local quadratic transform of \( R \). Therefore \( S \) is not an infinitely near point of \( R \). Thus \( S \) is a 3-dimensional regular local domain that birationally dominates \( R \), but \( S \) is not an infinitely near point to \( R \).

Consider the blowup \( \text{Proj} \ R[\mathfrak{m} \mathfrak{t}] \) of the maximal ideal \( \mathfrak{m} \) of \( R \). In the notation of Abhyankar, \( \text{Proj} \ R[\mathfrak{m} \mathfrak{t}] \) is the modelic blowup of \( R \) at \( \mathfrak{m} \) and is denoted \( \mathfrak{W}(R, \mathfrak{m}) \) in Definition 3.1. Let \( V \) denote the order valuation ring of \( S \). The center of \( V \) on the modelic blowup \( \mathfrak{W}(R, \mathfrak{m}) \) is the maximal ideal of the 2-dimensional regular local ring

\[
T := R[x, \frac{y}{x}, \frac{z}{x}, \frac{y}{z}, \frac{z}{y}]R[x, \frac{y}{x}, \frac{z}{x}] = R[\frac{x}{z}, \frac{y}{z}, \frac{z}{x}]R[\frac{y}{x}, \frac{z}{y}, \frac{z}{x}].
\]

The regular local ring \( T \) is infinitely near to \( R \); indeed, \( T \) is a point in the first neighborhood of \( R \). However, if \( J \) is an \( \mathfrak{m} \)-primary ideal of \( R \) that has \( T \) as a base point, then \( J \) is not finitely supported [35, Corollary 1.22]. Thus an \( \mathfrak{m} \)-primary ideal of \( R \) such as \( J = (x^2, y, z^2)R \) is not finitely supported.

Hence in the case where \( R \) is a regular local domain of dimension at least three, the one-to-one correspondence between prime divisors birationally dominating \( R \) and regular local rings infinitely near to \( R \) fails to include many of the regular local rings that birationally dominate \( R \). Indeed, a prime divisor \( V \) birationally dominating a 3-dimensional regular local ring \( R \) may be such that there exist infinitely many 3-dimensional regular local rings that birationally dominate \( R \) and have \( V \) as their order valuation ring, cf. [39, Lemma 4.2 and Corollary 4.5] and [29, Example 2.6]. There can be, however, no proper inclusion relations among these 3-dimensional regular local rings that have the same order valuation because of Theorem 1.4, a result of Sally [40, Corollary 2.6]. Theorem 1.4 extends to higher dimensional regular local rings a result that is true for 2-dimensional regular local rings by the Zariski-Abhyankar factorization theorem.

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6See Section 5 for the definitions of base point and finitely supported.
Theorem 1.4. Let $R \twoheadrightarrow T$ be a birational extension of $d$-dimensional regular local domain, and let $V = \text{ord}_R$ denote the rank one discrete valuation domain defined by the powers of the maximal ideal of $R$. If $V$ dominates $T$, then $R = T$.

During the early 1970’s Abhyankar worked on the Jacobian problem that conjectures: for polynomials $f_1, \ldots, f_n$ in the polynomial ring $k[x_1, \ldots, x_n]$ over a field $k$ with char $k = 0$, if the determinant of the Jacobian matrix $(\partial f_i/\partial x_j)$ is a nonzero constant, then $k[f_1, \ldots, f_n] = k[x_1, \ldots, x_n]$. Abhyankar returned to the Jacobian problem in the early 2000’s. Using techniques involving characteristic sequences, approximate roots and Newton polygons (techniques described carefully in [5]), Abhyankar developed solutions to the two variable Jacobian problem for various cases, the most general case being what he called the characteristic pair $2 + \epsilon$ case. These results are described in detail in the three papers [10], [11], [12].

Then in June and July of 2008 in visits to Spain and France, Abhyankar was introduced by Artal and Bodin to some topological methods for analyzing the Jacobian problem, namely methods focused on the concept of dicritical divisors. Based on these initial conversations, he was inspired to algebraic dicritical divisors. He hoped that this would lead to a fresh and powerful method to attack the Jacobian problem, as well as other related problems, even problems in mixed characteristic.

In a series of papers, some by himself and some in collaboration with others, Abhyankar created a precise and general algebraic theory of dicritical divisors. A key component in this development is the characterization of prime divisors obtained in his paper [1, 1956]. Other fundamental components include work Abhyankar had done on: resolution of singularities, quadratic transformations, characteristic sequences and Newton polygons. The classical work of Zariski on the theory of complete ideals in 2-dimensional regular local domains and related work in ideal theory of Northcott and Rees were also used.

2. What is a dicritical divisor

In Definition 2.1 we define prime divisors of a regular local ring and the dicritical divisors of the nonzero elements in the quotient field of a two-dimensional regular local domain.
**Definition 2.1.** Let $R$ be a regular local ring with maximal ideal $M(R) = M$ and quotient field $L$.

1. A valuation domain $V$ with quotient field $L$ that dominates $R$ is said to be a **prime divisor** of $R$ if the residue field $V/M(V)$ has transcendence degree $\dim R - 1$ over the field $R/M$. Let $D(R)^\Delta$ denote the set of prime divisors of $R$.

2. Assume $\dim R = 2$, and let $z$ be a nonzero element in $L$. A prime divisor $V \in D(R)^\Delta$ is a **dicritical divisor** of $z$ in $R$ if the image of $z$ in $V/M(V)$ is transcendental over the field $R/M$. Let $\mathfrak{D}(R, z)$ denote the set of dicritical divisors of $z$ in $R$.

3. With $z$ and $R$ as in item 2, we say that $z$ **generates a special pencil** if there exists $x \in M \setminus M^2$ such that $x^m z \in R$ for some $m \in \mathbb{N}$.

**Remark 2.2.** Let $R$ be a 2-dimensional regular local domain with maximal ideal $M$ and quotient field $L$, and let $z$ be a nonzero element of $L$. If $z$ or $1/z$ is in $R$, then the set $\mathfrak{D}(R, z)$ of dicritical divisors of $z$ in $R$ is empty. Assume this does not hold and write $z = a/b$, where $a$ and $b$ are in $R$ and have no common factors. Then $J = (a, b)R$ is $M(R)$-primary. In this situation the dicritical divisors of $z$ are precisely the Rees valuations of the ideal $J$ as defined for example in Swanson and Huneke [42, pages 187-210].

The set $\mathfrak{D}(R, z)$ of dicritical divisors of $z$ in $R$ is a finite set. An easy way to describe this set is to consider the extension ring $R[a/b]$. Let $t$ be an indeterminate over $R$ and consider the surjective $R$-homomorphism $R[t] \to R[a/b]$ defined by $t \mapsto a/b$. The kernel of this homomorphism is the prime ideal $(bt - a)R[t]$ and is contained in the extension $MR[t]$ of the maximal ideal $M$ of $R$ to $R[t]$. It follows that $MR[a/b]$ is a prime ideal of height one, and by the Krull-Akizuki Theorem, the integral closure of $R[a/b]_{MR[a/b]}$ is a semilocal PID. The dicritical divisors of $J = (a, b)R$ are precisely the DVRs that birationally dominate $R[a/b]_{MR[a/b]}$.

**Example 2.3.** Let $M(R) = M = (x, y)R$ and let $z = \frac{y^2}{x^3}$ and $J = (x^3, y^2)R$.

Consider the local ring

$$S = R[y^2/x^3]_{MR[y^2/x^3]} = \frac{R(t)}{(x^3t - y^2)R(t)}.$$
Here \( t \) is an indeterminate and \( R(t) \) denotes the polynomial ring \( R[t] \) localized at the prime ideal \( MR[t] \). The ring \( S \) is a one-dimensional local domain. It is not integrally closed. The element \( \frac{y}{x} \) is integral over \( S \). This is related to the fact that the ideal \( I = (x^3, y^2, x^2y)R \) is integral over the ideal \( J = (x^3, y^2)R \), or, in the terminology of Northcott-Rees, the ideal \( J \) is a reduction\(^7\) of \( I \). Indeed, one readily sees that \( JI = I^2 \). Moreover, the ideal \( I \) is what Zariski calls a simple complete ideal.\(^8\)

The integral extension \( S[\frac{y}{x}] = V \) can be seen to be a DVR, and \( V \) is the unique dicritical divisor of \( z = \frac{x^2}{y} \). Let \( v \) denote the valuation with value group \( \mathbb{Z} \) associated to the valuation ring \( V \). Then \( v(y) = 3 \) and \( v(x) = 2 \) and the image \( \varpi \) of \( z \) in \( V/M(V) \) is transcendental over \( R/M \) and generates \( V/M(V) \) over \( R/M \).

In Definition 2.4 we define the dicritical divisors of a nonconstant bivariate polynomial.

**Definition 2.4.** Let \( B \) denote the bivariate polynomial ring \( k[X, Y] \) over a field \( k \), and let \( L \) denote the quotient field of \( B \). Let \( I(B/k) \) denote the set of DVRs \( V \) on \( L \) such that (i) \( k \subset V \), (ii) the residue field of \( V \) is transcendental over \( k \), (iii) \( B \not\subset V \).

Let \( f \in B \setminus k \) and let \( I(B/k, f) \) denote the set of \( V \in I(B/k) \) such that the image of \( f \) in the residue field of \( V \) is transcendental over \( k \).

Let \( B_f \) denote the localization of \( B \) at the multiplicative set of nonzero elements of \( k[f] \). We observe that:

1. \( B_f = k(f)[X, Y] \) can be identified with the affine coordinate ring of the generic curve \( f^g = 0 \) where we take an indeterminate \( u \) over \( k \) and put \( f^g = f - u \). This identifies \( B_f \) as the affine coordinate ring of \( f^g \) over the field \( k(f) \);
2. the quotient field of \( B_f \) is \( L \); and
3. \( \text{tr. deg}_{k(f)} L = 1 \).

Consequently, \( B_f \) is a one-dimensional UFD and hence a PID. Therefore the affine curve associated to \( B_f \) is irreducible and nonsingular. Let \( I(B_f/k(f)) \) be the set of all DVRs \( V \) on \( L \) that contain \( k(f) \) and are such that \( B_f \not\subset V \). The elements in

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\(^7\)An ideal \( J \) is a *reduction* of an ideal \( I \) if \( J \) is contained in \( I \) and \( JI^n = I^{n+1} \) for some nonnegative integer \( n \).

\(^8\)An ideal \( I \) of an integral domain \( R \) is said to be a *simple ideal* if \( I \) is not the unit ideal and \( I \) has no nontrivial factorization, that is, \( I = JK \) implies either \( J \) or \( K \) is the unit ideal of \( R \). The ideal \( I \) is *complete* if it is integrally closed. For integral closure of ideals, see for example [12].
the set \( I(B_f/k(f)) \) are called the **dicritical divisors of \( f \) with respect to the polynomial ring \( B \)**.

**Remark 2.5.** For \( f \in B \setminus k \), the set \( I(B_f/k(f)) \) of dicritical divisors of \( f \) is a finite set of DVRs. It is equal to the set \( I(B/k, f) \) of DVRs defined in Definition 2.4.

The dicritical divisors of \( f \) are the places at infinity for the affine plane curve having coordinate ring \( B_f \), cf. [23, Remark 1, page 57].

**Example 2.6.** Let \( f = X^n \in B \) with \( n \) a positive integer. Then the relative algebraic closure of the field \( k(f) \) in the field \( L \) is the field \( k(X) \). It is straightforward to see that \( f \) has one dicritical divisor \( V \) with respect to the polynomial ring \( B \). Moreover

\[
V := k(X)[Y^{-1}]_{Y^{-1}k(X)[Y^{-1}]}
\]

is that dicritical divisor. If \( n > 1 \), then \( f \) is not a field generator.

**Example 2.7.** Let \( f = X^mY^n \in B \), where \( m \) and \( n \) are positive integers such that \( \gcd(m, n) = 1 \). If \( a \) and \( b \) are integers such that \( mb - na = 1 \) and \( \tau \) is the rational function \( X^aY^b \in L \), then \( f \) and \( \tau \) generate the field \( L \) over \( k \), that is \( k(f, \tau) = L \).

Hence \( f \) is a field generator. It follows that the field \( k(f) \) is relatively algebraically closed in the field \( L \). If \( V \) is a dicritical divisor of the polynomial \( f = X^mY^n \) with respect to the polynomial ring \( B \), then \( X^mY^n \) is a unit of \( V \) with the property that the image of \( X^mY^n \) in the residue field of \( V \) is algebraically independent over \( k \). Since \( V \) does not contain \( B \), either \( X \not\in V \) or \( Y \not\in V \). If \( X \not\in V \), then \( Y \) is in the maximal ideal of \( V \) and vice versa. We conclude that \( X^mV = Y^{-n}V \), and the polynomial \( f \) has two dicritical divisors with respect to the polynomial ring \( B \), one that contains \( X \) and another that contains \( Y \). It follows that \( f \) is not a ring generator, for if there exists an element \( g \) so that \( k[f, g] = B \), the \( f \) has only one dicritical divisor \( V \) with respect to ring \( B \), namely \( V = k(f)[g^{-1}]_{g^{-1}k(f)[g^{-1}]} \).

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\(^9\)Abhyankar elaborates in [13] pp. 147-156] on this equality of the set \( I(B/k, f) \) from surface theory with the set \( I(B_f/k(f)) \) from curve theory. The set \( I(B/k, f) \) represents points at infinity of a projective plane curve, while \( I(B_f/k(f)) \) is the set of all branches at infinity of the generic curve \( f^2 \).

\(^{10}\)An element \( f \in B \) is a **field generator** if there exists \( \tau \in L \) such that \( f \) and \( \tau \) generate \( L \) as an extension field of \( k \), that is \( k(f, \tau) = L \). An element \( f \in B \) is a **ring generator** if there exists \( g \in B \) such that \( k[f, g] = B \).
Example 2.8. Let \( f = X^3 - Y^2 \). Then \( f \) is irreducible in \( B = k[X,Y] \) and the affine coordinate ring \( \frac{B}{fB} \) of \( f \) may be identified with the subring \( k[t^2,t^3] \) of the polynomial ring \( k[t] \) by means of the \( k \)-algebra homomorphism that maps \( x \mapsto t^2 \) and \( y \mapsto t^3 \). Thus \( f \) defines a rational curve, and the function field over \( k \) of the affine coordinate ring \( \frac{B}{fB} \) is a simple transcendental field extension \( k(t) \). Also the field \( k(f) \) is relatively algebraically closed in \( L \) and \( B_f = k(f)[X,Y] \) has one place at infinity. Thus the polynomial \( f \) has one dicritical divisor \( V \) with respect to the polynomial ring \( B \). The fractional ideal \( X^3V = Y^2V \) has \( V \)-value \(-6\), and \( f \) is a unit in \( V \) such that the image of \( f \) is the residue field of \( V \) is transcendental over the field \( k \). The associated algebraic function field \( L/k(f) \) is of genus one. Hence \( L/k(f) \) is not a simple transcendental field extension. The polynomial \( f \) is not a field generator of \( L/k \).

Discussion 2.9. We describe how the dicritical divisors of a bivariate polynomial are related to the dicritical divisors of an element in the quotient field of a two-dimensional regular local ring. The polynomial ring \( B = k[X,Y] \) is an affine component of the modelic projective plane \( \mathbb{P}^2_k = \mathcal{M}(k;X,Y,1) \) of the field \( L \), cf. \[14\] Section 5] or \[23\] pages 116-119]. Let \( \ell_\infty \) denote the line at infinity with respect to \( B \). Let \( f = f(X,Y) = \sum_{i+j \leq N} a_{ij}X^iY^j \in k[X,Y] \) be a nonconstant polynomial, and let \( V \in I(B_f/k(f)) \). Since \( B \not\subset V \), the center of \( V \) on \( \mathbb{P}^2_k \) is a point on \( \ell_\infty \). Let \( R \) denote the two-dimensional regular local ring associated to this point. Then \( V \in \mathcal{D}(R,f) \), that is, \( V \) is a dicritical divisor of \( f \) in \( R \). Moreover, \( W \in \mathcal{D}(R,f) \implies W \in I(B_f/k(f)) \). Thus the dicritical divisors in \( I(B_f/k(f)) \) may be partitioned as follows: Let \( R_1, \ldots, R_m \) be the two-dimensional regular local rings associated with points on \( \ell_\infty \) that are the center of some \( V \in I(B_f/k(f)) \). Then \( I(B_f/k(f)) \) is the disjoint union of the sets \( \mathcal{D}(R_i,f) \).

Let \( v \) denote a valuation associated to the valuation ring \( V \in I(B_f/k(f)) \). Since \( B \not\subset V \), either \( v(X) < 0 \) or \( v(Y) < 0 \). Without loss of generality, we may assume \( v(X) < 0 \) and \( v(Y) \leq v(Y) \). Let \( x := 1/X \) and \( y := Y/X \). Then \( R = k[x,y]_P \), where \( P = (x,\zeta(y))k[x,y] \) and \( \zeta(y) \in k[y] \) is an irreducible monic polynomial. Let \( \phi(X,Y) = \sum_{i+j=N} a_{ij}X^iY^j \) denote the degree form of \( f(X,Y) \). Then \( \phi(1,y) \in \zeta(y)k[y] \) and \( x^N f \in R \). Therefore \( f \) generates a special pencil.
**Definition 2.10.** Let $V \in \mathcal{D}(R, z)$ be a dicritical divisor, and let $k'$ denote the relative algebraic closure of the field $k := R/M$ in the residue field of $V$. Then $k'/k$ is a finite algebraic field extension, and $V/M(V)$ is a simple transcendental extension $k'/(\tau)$ of $k'$. Moreover, the image $\overline{z}$ of $z$ in $k'/(\tau)$ is a nontrivial rational function in $\tau$. The degree of the field extension $[k'/(\tau) : k(\overline{z})]$ is called the **degree of $V$ as a dicritical divisor in $\mathcal{D}(R, z)$**.

In Example 2.3, the degree of $V$ is one. Example 2.11 describes a situation where the degree of a dicritical divisor $V$ is greater than one. Example 2.11 also illustrates how the local algebraic theory of dicritical divisors connects with that of polynomials in $\mathbb{C}[x, y]$.

**Example 2.11.** Consider the polynomial $p(x, y) = x^4y^4 - x \in \mathbb{C}[x, y]$ and for each $t \in \mathbb{C}$, let $C_t$ denote the curve $x^4y^4 - x - t = 0$. Notice that each point $(a, b) \in \mathbb{C}^2$ is on precisely one of the curves $C_t$ of the pencil $\{C_t\}_{t \in \mathbb{C}}$. Hence the polynomial $p$ defines a map $f_p : \mathbb{C}^2 \to \mathbb{C}$. Let $\ell_\infty$ denote the line at infinity in $\mathbb{P}^2$ for the natural embedding of $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$ as in Discussion 2.3. The curves $C_t$ have two points on the line $\ell_\infty$, the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$. Let $\tilde{p}(x, y, z) = x^4y^4 - xz^7 - tz^8 \in \mathbb{C}[x, y, z]$ denote the homogenization of the polynomial $p - t$. We consider each of the two points on $\ell_\infty$:

1. For the point $[1 : 0 : 0]$, set $x = 1$ to get the rational function $t = \frac{y^4 - z^7}{z^8}$. Let $R_1$ be the 2-dimensional regular local ring with maximal ideal $(y, z)R_1$ associated to this point. We are interested in the dicritical divisors of the ideal $J_1 = (y^4 - z^7, z^8)R_1$. The integral closure of the ideal $J_1$ is a simple complete ideal associated to a prime divisor $V$ with associated valuation $v$, where $v(y) = 7$, $v(z) = 4$ and $v(y^4 - z^7) = 32 = v(z^8)$. Thus the ideal $J_1$ has precisely one dicritical divisor and this dicritical divisor has degree one. The integral closure of $J_1$ is the simple complete ideal

$$\overline{J}_1 = (y^4 - z^7, z^8, y^3z^3, y^2z^5, y^5)R_1.$$  

2. For the point $[0 : 1 : 0]$, set $y = 1$ to get the rational function $t = \frac{x^4 - xz^7}{z^8}$. Let $R_2$ be the 2-dimensional regular local ring with maximal ideal $(x, z)R_2$ associated to this point. We are interested in the dicritical divisors of the ideal

$$J_2 = (x^4 - xz^7, z^8)R_2.$$
Consider the extension $R_2 \hookrightarrow R_2[\frac{z}{x}] := S$, and define $x_1 := \frac{z}{x}$. We have

$$J_2S = (x^4 - xz^7, z^8)S = (z^4x_1^4 - z^8x_1, z^8)S = z^4(x_1^4 - z^4x_1, z^4)S.$$ 

Thus the ideal $(x_1^4 - z^4x_1, z^4)S = (x_1^4, z^4)S$ is the transform$^{11}$ of $J_2$ in $S$. Since the transform of $J_2$ in $R_2[\frac{z}{x}]$ is the ring $R_2[\frac{z}{x}]$, the ideal $J_2$ has a unique base point in the blowup of the maximal ideal of $R_2$ with this unique base point being the 2-dimensional regular local ring $R_3 := S_{(x_1,x_1)}$. The ideal $(x_1^4 - z^4x_1, z^4)R_3 = (x_1^4, z^4)R_3$ is the transform of $J_2$ in $R_3$. The integral closure of $(x_1^4, z^4)R_3$ is $(x_1, z^4)R_3$, the 4-th power of the maximal ideal of $R_3$. The integral closure of $J_2$ is the ideal

$$\overline{J_2} = (x, z^2)^4R_2,$$

and $(x, z^2)R_2$ is a simple complete ideal associated to a prime divisor $W$, with associated valuation $w$, where $w(x) = 2$ and $w(z) = 1$. Thus the ideal $J_2$ has precisely one dicritical divisor and this dicritical divisor has degree four.

**Remark 2.12.** Let $R$ be a 2-dimensional regular local ring and let $J = (a, b)R$ be an ideal primary for the maximal ideal of $R$. Let $V$ be a dicritical divisor of $J$. Let $k$ denote the residue field of $R$ and assume that $k(\tau)$ is the residue field of $V$. For $z = a/b$, let $\overline{z}$ denote the image of $z$ in $V/M(V) = k(\tau)$. It is natural to ask for conditions in order that $\overline{z}$ be a polynomial in $\tau$ as opposed to just being a rational function in $\tau$. Abhyankar and Luengo in [24] prove this for the prime divisors at infinity of a nontrivial polynomial in a polynomial ring in 2 variables over a field. More generally, they prove in [24] for $R$ a 2-dimensional regular local domain that if $z$ generates a special pencil at $R$, then $z$ generates a polynomial pencil in $R$.

### 3. Modelic spectrum and modelic blowup

To describe a more general algebraic definition of dicritical divisors, we review the following concepts. Let $R$ be an integral domain. The **modelic spectrum** of $R$ is

$$\mathfrak{M}(R) = \{ R_P \mid P \text{ is a prime ideal of } R \}.$$ 

Here we are identifying the prime ideals of $R$ with the local rings obtained by localizing the integral domain $R$ at these prime ideals.

$^{11}$If $A \hookrightarrow B$ is a birational extension of unique factorization domains and $I$ is an ideal of $A$ not contained in any proper principal ideal, the **transform** of $I$ in $B$ is the ideal $a^{-1}IB$, where $aB$ is the smallest principal ideal in $B$ that contains $IB$ [35, Definition 1.4].
Definition 3.1. Assume that $R$ is a subring of a field $K$ and let $x_0, \ldots, x_n$ be nonzero elements of $K$. The modelic proj of $R$ with respect to $x_0, \ldots, x_n$ is

$$\mathfrak{P}(R; x_0, \ldots, x_n) = \bigcup_{i=0}^{n} \mathfrak{P}(R\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right])$$

Let $I$ be a nonzero ideal in the integral domain $R$ and assume that $x_0, \ldots, x_n$ are nonzero elements of $I$ that generate $I$. The modelic blowup of $R$ at $I$ is

$$\mathfrak{P}(R, I) = \mathfrak{P}(R; x_0, \ldots, x_n) = \bigcup_{i=0}^{n} \mathfrak{P}(R\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]) = \bigcup_{0 \neq a \in I} \mathfrak{P}(R[I/a]).$$

The modelic blowup of $R$ at $I$ is independent of the nonzero generators chosen for $I$. Notice that $IR[I/a] = aR[I/a]$ is a principal ideal.

Let $t$ be an indeterminate over $R$ and let $R[It]$ denote the graded subring of the polynomial ring $R[t]$ generated by $It := \{at \mid a \in I\}$. In other notation, the modelic blowup of $R$ at $I$ is

$$\text{Proj } R[It] := \bigcup_{0 \neq a \in I} \text{Spec } R[I/a].$$

Discussion 3.2. Consider the family of all quasilocal domains on the quotient field $K$ of $R$ that contain $R$, and define a partial order on this set with respect to domination, i.e., for $S_1$ and $S_2$ define $S_1 \leq S_2$ if $S_1 \subset S_2$ and $M(S_2) \cap S_1 = M(S_1)$.

The quasilocal rings in the modelic blowup $\mathfrak{P}(R, I)$ of $R$ at $I$ are the minimal elements $S$ in this partial order such that $IS$ is principal. Every valuation domain $V$ containing $R$ has a unique center on the modelic blowup of $R$ at $I$, i.e., there exists a unique quaslocal ring $S \in \mathfrak{P}(R, I)$ such that $V$ dominates $S$.

Let $S$ be a quasilocal domain with quotient field $K$ and let $\overline{S}$ denote the integral closure of $S$ in $K$. Let $S^{\Omega}$ denote the set of all members of the modelic spectrum $\mathfrak{P}(\overline{S})$ of $\overline{S}$ that dominate $S$. The Lying Above Theorem for integral extensions [9, T40, page 244] implies that the set $S^{\Omega}$ is nonempty, and the Proper Containment Lemma [9, T42, page 245] implies that every quasilocal ring $T$ in the set $S^{\Omega}$ satisfies $\dim T \leq \dim S$. If $\dim S$ is finite, then there exists a quasilocal ring $T$ in $S^{\Omega}$ with $\dim T = \dim S$ by the Going Up Theorem [9, T41, page 245]. If $S$ is Noetherian, then $\dim S$ is finite by the Generalized Principal Ideal Theorem [9, T27, page 232]. Moreover, by results proved by Mori and Nagata [37, Theorem 33.10], the integral closure $\overline{S}$ of $S$ is a Krull domain and the set $S^{\Omega}$ is finite; however, the quasilocal rings $T$ in $S^{\Omega}$ may fail to be Noetherian. This is because the integral closure of a
Noetherian local domain $S$ with $\dim S \geq 3$ may fail to be Noetherian [37, Example 5, p. 207].

**Definition 3.3.** Let $I$ be a nonzero ideal in the integral domain $R$ and assume that $x_0, \ldots, x_n$ are nonzero elements of $I$ that generate $I$. The **normalized modelic blowup** of $R$ at $I$ is

$$W(R, I)^N = W(R; x_0, \ldots, x_n)^N = \bigcup_{i=0}^n W(R[x_0/x_i, \ldots, x_n/x_i])^N = \bigcup_{0 \neq a \in I} W(R[I/a])^N.$$

Concerning the modelic blowup and normalized modelic blowup of ideals, it is natural to ask:

**Question 3.4.** Let $I$ and $J$ be nonzero finitely generated ideals of an integral domain $R$.

(1) Under what conditions does one have $W(R, I) = W(R, J)$?

(2) Under what conditions does one have $W(R, I)^N = W(R, J)^N$?

Thus we are asking for conditions in order that the ideals $I$ and $J$ have the same blowup or the same normalized blowup.

**Remark 3.5.** It is straightforward to see that an ideal $I$ and a power $I^n$ of $I$ have the same modelic blowup and the same normalized modelic blowup. Moreover, if there exist nonzero elements $a$ and $b$ in $R$ such that $aI = bJ$, then the ideals $I$ and $J$ have the same modelic blowup and the same normalized modelic blowup.

**Example 3.6.** Let $x$ and $y$ be indeterminates over a field $k$ and let $R = k[x^2, xy, y^2]$. The ring $R$ is the coordinate ring of an affine surface that has an ordinary double point singularity at the origin. Let $M := (x^2, xy, y^2)R$ and let $P := (x^2, xy)R$. Notice that $M$ is a maximal ideal and $P$ is a height-one prime ideal. Moreover, we have $x^2M = P^2$. Hence by Remark 3.5, we have $W(R, M) = W(R, P)$. The ideals $M$ and $P$ are both normal ideals [12]. Therefore the modelic blowups of $M$ and $P$ are also their normalized modelic blowups and we have $W(R, M)^N = W(R, P)^N$.

We also have $W(R, M) = W(R[y^2/x^2]) \cup W(R[y^2/x^2])$, and $R[y^2/x^2] = k[x^2, y^2]$ is isomorphic to a polynomial ring in 2 variables over the field $k$. Similarly, we see that $R[y^2/x^2]$ is equal to $k[y^2, \frac{x}{y}]$. Therefore the modelic blowup of $R$ at $M$ is nonsingular.

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12 An ideal $I$ of an integral domain is said to be a **normal ideal** if all the powers of $I$ are integrally closed.
4. The Rees valuations and the dicritical divisors of an ideal

Let \( J \) be a nonzero ideal of a quasilocal domain \( S \), and let \( \mathfrak{M}(S, J)_{1}^{\Delta} \) denote the set of all 1-dimensional members \( T \) of the blowup \( \mathfrak{M}(S, J) \) of \( S \) at \( J \) such that \( T \) dominates \( S \).

**Definition 4.1.** For a nonzero ideal \( J \) of a quasilocal domain \( S \), the set
\[
\mathfrak{D}(S, J) := (\mathfrak{M}(S, J)_{1}^{\Delta})^{\mathfrak{M}}
\]
is the **dicritical set** of \( J \) in \( S \); members of this set are called **dicritical divisors** of \( J \) in \( S \).

The set \( \mathfrak{D}(S, J) \) may be empty.

For a nonzero ideal \( I \) in a Noetherian integral domain \( R \) one can take the union of the sets \( \mathfrak{D}(R_{P}, IR_{P}) \), where \( P \) varies over the set of all prime ideals \( P \) of \( R \) that contain \( I \). This set may be described as follows:

**Definition 4.2.** Let \( I \) be a nonzero ideal in a Noetherian integral domain \( R \), and let \( \mathfrak{M}(R, I, I)_{1}^{\Delta} \) denote all the one-dimensional members \( S \) of the blowup of \( I \) such that \( IS \neq S \). Then \( (\mathfrak{M}(R, I, I)_{1}^{\Delta})^{\mathfrak{M}} \) is the set of **dicritical divisors** of \( I \).

This gives a finite set of DVRs.

The blowup \( \mathfrak{M}(R, I) \) is a model that has finitely many one-dimensional local rings \( S \) that contain \( I \). For each such \( S \), the set \( S^{\mathfrak{M}} \) is a finite set of DVRs. The union over all \( S \) gives the dicritical divisors of \( I \).

How does this compare with the set Rees \( I \) of Rees valuations rings of \( I \)? The Rees valuation rings of \( I \) are the DVRs that contain \( I \) and arise as one-dimensional members of the normalized blowup \( (\mathfrak{M}(R, I))^{\mathfrak{M}} \).

Thus \( \text{Rees } I = ((\mathfrak{M}(R, I))^{\mathfrak{M}})_{1}^{\Delta} \). The difference is one first takes integral closure and then localizes. For a nonzero ideal \( I \) of a Noetherian integral domain \( R \), the dicritical divisors of \( I \) are Rees valuation rings of \( I \), that is \( (\mathfrak{M}(R, I, I)_{1}^{\Delta})^{\mathfrak{M}} \) is a subset of Rees \( I \). If \( R \) is also universally catenary, then the Rees valuations of \( I \) are precisely the same as the dicritical divisors of \( I \). Thus if \( I \) is an \( \mathfrak{m} \)-primary ideal of a universally catenary Noetherian local domain \((R, \mathfrak{m})\) then the set Rees \( I \) of Rees valuation rings of \( I \) is precisely the set \( \mathfrak{D}(R, I) \) of dicritical divisors of \( I \).
The following three results about Rees valuation rings of an ideal are given in [30].

**Theorem 4.3.** Let \((R, m)\) be a universally catenary analytically unramified Noetherian local domain with \(\dim R = d\), and let \(V\) be a prime divisor of \(R\) centered on \(m\). Let \(I \subseteq m\) be an ideal of \(R\). The following are equivalent

1. \(V \in \text{Rees } I\).
2. There exist elements \(b_1, \ldots, b_d\) in \(I\) such that \(b_1V = \cdots = b_dV = IV\) and the images of \(\frac{b_2}{b_1}, \ldots, \frac{b_d}{b_1}\) in the residue field \(k_v\) of \(V\) are algebraically independent over \(R/m\).
3. If \(I = (a_1, \ldots, a_n)R\), then there exist elements \(b_1, \ldots, b_d\) in \(\{a_i\}_{i=1}^n\) such that \(b_1V = \cdots = b_dV = IV\) and the images of \(\frac{a_2}{a_1}, \ldots, \frac{a_d}{a_1}\) in \(k_v\) are algebraically independent over \(R/m\).

Thus if \(I = (a_1, a_2, \ldots, a_d)R\), then \(V \in \text{Rees } I \iff a_1V = a_2V = \cdots = a_dV\) and the images of \(\frac{a_2}{a_1}, \ldots, \frac{a_d}{a_1}\) in \(k_v\) are algebraically independent over \(R/m\).

**Proposition 4.4.** Let \((R, m)\) be a \(d\)-dimensional regular local ring with \(d \geq 2\), let \(x \in m \setminus m^2\), and let \(S = R[\frac{m}{x}]\). Let \(K\) be an ideal of \(R\) that is contracted from \(S\), and let \(xf \in K\), where \(f \in R\). Then

1. \(gf \in K\) for each \(g \in m\).
2. If \(\text{ord}_R(xf) = \text{ord}_R(K)\), then the order valuation \(\text{ord}_R\) is a Rees valuation of \(K\).

**Proposition 4.5.** Let \((R, m)\) be a \(d\)-dimensional regular local ring and let \((S, n)\) be a \(d\)-dimensional regular local ring that birationally dominates \(R\). Let \(I\) be an \(m\)-primary ideal of \(R\) such that its transform \(J = I^S\) in \(S\) is not equal to \(S\), and let \(V\) be a DVR that birationally dominates \(S\). Then

\[ V \in \text{Rees}_S J \iff V \in \text{Rees}_R I. \]

**Remark 4.6.** With notation as in Theorem 4.3 let \(\frac{b_2}{b_1}, \ldots, \frac{b_d}{b_1}\) denote the images of \(\frac{b_2}{b_1}, \ldots, \frac{b_d}{b_1}\) in the residue field \(k_v\) of \(V\). An interesting integer associated with \(V \in \text{Rees } I\) and \(b_1, \ldots, b_d\) is the field degree

\[ [k_v : (R/m)\left(\frac{b_2}{b_1}, \ldots, \frac{b_d}{b_1}\right)] \]

Notice the analogy with the degree of \(V\) as a dicritical divisor in Definition 2.10.
5. Dicritical divisors and the structure of quadratic sequences

A central question concerning dicritical divisors that interested Abhyankar can be stated as follows:

**Question 5.1.** Let $R$ be a 2-dimensional regular local domain with quotient field $L$ and let $U \subset D(R)\Delta$ be a finite set of prime divisors of $R$.

1. Does there exist an element $z \in L$ such that $D(R, z) = U$?
2. If the answer to item 1 is affirmative, is it possible in some algorithmic way to describe all the $z \in L$ such that $D(R, z) = U$?

In a sequence of papers written with Heinzer [20], [21], [22], Question 5.1 and various related questions are studied. In particular, an affirmative answer to item 1 of Question 5.1 is given in [20]. If the residue field of $R$ is infinite, it is also shown in [20] that the element $z$ can be chosen so that, in the terminology of Definition 2.10, for each $V \in U$, the degree of $V$ as a dicritical divisor in $D(R, z)$ is 1.

A main tool in [20] is the work of Zariski in Appendix 5 of [43] concerning the structure of complete ideals of a 2-dimensional regular local domain. Let $Q(R)$ denote the set of 2-dimensional regular local domains that birationally dominate $R$. For each $S \in Q(R)$, the order valuation domain $ord S$ is a prime divisor on $R$, and Theorem 1.2 implies that the map $Q(R) \rightarrow D(R)\Delta$ that maps $S$ to $ord S$ is a bijection of the sets $Q(R)$ and $D(R)\Delta$.

Let $I$ be a simple complete $M$-primary ideal. If $I \neq M$, Zariski proves the existence of a positive integer $\nu$ and a unique regular local ring $S \in Q(R)$ such that the finite sequence as given in Theorem 1.2

$$R = R_0 \subset R_1 \subset \cdots \subset R_\nu = S,$$

where $R_i$ is a local quadratic transform of $R_{i-1}$ for each $i \in \{1, \ldots, \nu\}$, consists precisely of the regular local rings $T \in Q(R)$ for which the transform of $I$ in $T$ is a proper ideal of $T$. Moreover, the transform of $I$ in $S$ is the maximal ideal of $S$. The regular local domains $R_i$, with $i \in \{1, \ldots, \nu\}$, are the base points of $I$.\(^{13}\)

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\(^{13}\)If $I$ is an $M$-primary ideal of a regular local ring $R$, the base points of $I$ are the regular local rings $S$ infinitely near to $R$ such that the transform of $I$ in $S$ is not the unit ideal. The ideal $I$ is said to be finitely supported if it has only finitely many base points. If $\dim R = 2$, then every $M$-primary ideal is finitely supported. This is no longer true if $\dim R \geq 3$. 
Let $C(R)$ denote the set of $M$-primary simple complete ideals of $R$. The Zariski Quadratic Theorem [20, page 173] asserts that for each $V \in D(R)^\Delta$ there exists at least one and at most a finite number of $V$-ideals in $R$ that are members of $C(R)$. Labeling these $V$-ideals of $C(R)$ as 

$$
 M = J_0 \supset J_1 \supset \cdots \supset J_\nu
$$

one obtains a bijection $\zeta_R : D(R)^\Delta \to C(R)$ by defining $\zeta_R(V) = J_\nu$.

Zariski proves that a product of complete ideals in $R$ is again complete and every complete $M$-primary ideal can be expressed uniquely as a finite product of powers of the ideals in $C(R)$. For a finite subset $U$ of $D(R)^\Delta$, define $\zeta_R(U)$ to be the product $\prod_{V \in U} \zeta_R(V)$. Using results from Northcott-Rees [38], it is shown in [20] that some power of the ideal $\zeta_R(U)$ has a 2-generated reduction. This yields an affirmative answer to item 1 of Question 5.1.

To consider item 2 of Question 5.1, Abhyankar turned to a careful and detailed analysis of quadratic transformations (QDT’s) and inverse QDT’s. Inverse transforms as considered by Zariski [43, pp. 390-391] involve complete ideals. Abhyankar’s interest was more in 2-generated reductions. For a complete ideal $I$ that is primary to the maximal ideal of a 2-dimensional regular local domain, the ideals $J$ that are 2-generated reductions of $I$ are usually very far from being unique. Abhyankar was more interested in specific properties of QDT’s and their inverses. In this connection he coined the term antiquadratic transformation sequence. Example 5.4 below, illustrates the type of results Abhyankar proved to answer item 2 of Question 5.1 in special cases.

To construct examples such as Example 5.4 and to develop the algebraic theory of curvettes as in [19], Abhyankar developed another characterization of dicritical divisors. This characterization depends on the concept of Zariski number and Zariski index that he defined as follows:

**Definition 5.2.** Let $R$ be a 2-dimensional regular local ring with maximal ideal $M$ and let $\operatorname{grad} R = R/M \oplus M/M^2 \oplus \cdots$ denote the associated graded ring of $R$.

14 An ideal $J$ of $R$ is a $V$-ideal if $J = JV \cap R$.

15 The intersection of all the reductions of $I$ is called the core of $I$. Interesting results about the core of a complete ideal $I$ that is primary for the maximal ideal of a 2-dimensional regular local domain with infinite residue field are obtain by Huneke and Swanson in [33]. They prove that the core of $I$ is integrally closed and is the the product of $I$ with its second Fitting ideal.
The ring \( \text{grad } R \) is a polynomial ring in 2 variables over the residue field \( k := R/M \). If \( M = (x, y)R \), then the initial forms \( \text{info } x \) and \( \text{info } y \) of \( x \) and \( y \) in \( M/M^2 \) are algebraically independent over \( k \) and generate \( \text{grad } R \). Let \( J \) be a nonzero ideal in \( R \) and let \( d := \text{ord}_R J \). Thus \( J \subset M^d \) and \( J \not\subset M^{d+1} \). Let \( s \) denote the degree of the GCD of all the initial forms \( \text{info } f \) of elements \( f \in J \) such that \( \text{ord}_R f = d \).

The Zariski number \( m(R, J) \) is defined by \( m(R, J) := d - s \). Notice that the Zariski number of \( J \) is a nonnegative integer less than or equal to \( d \).

If \( J \) is \( M \)-primary, Zariski proved that the power of \( M \) that occurs in the factorization of the integral closure of \( J \) as a product of simple complete ideals is the integer \( m(R, J) \).

Let \( T \in Q(R) \) be a quadratic transform of \( R \). The derived Zariski number \( m(R, J, T) \) is the nonnegative integer \( m(R, J, T) := m(T, J_T) \), where \( J_T \) denotes the transform of the ideal \( J \) in \( T \). For a prime divisor \( V \in D(R)^\Delta \), the Zariski index \( n(R, J, V) \) is the nonnegative integer \( n(R, J, V) = m(R, J, o^{-1}_R(V)) \), where \( o^{-1}_R(V) \) is the unique 2-dimensional regular local domain \( S \in Q(R) \) such that \( \text{ord}_S V = V \).

Using this notation, the Zariski Factorization Theorem concerning complete ideals of the 2-dimensional regular local domain \( R \) may be stated as follows:

\[
J^{-R} = \text{GCD}(J)_R \prod_{V \in D(R)^\Delta} \zeta_R(V)^{n(R, J, V)},
\]

where \( J^{-R} \) denotes the integral closure of \( J \) and \( \text{GCD}(J)_R \) denotes the smallest nonzero principal ideal in \( R \) containing \( J \). The set \( \mathcal{D}(R, J) \) of dicritical divisors of \( J \) is characterized as

\[
\mathcal{D}(R, J) = \{ V \in D(R)^\Delta \mid n(R, J, V) > 0 \}.
\]

A prime divisor \( V \) is a dicritical divisor of the ideal \( J \) if and only if the Zariski index \( n(R, J, V) \) is positive.

In Example 5.3, we illustrate the Zariski number, derived Zariski number and Zariski index in a simple example.

**Example 5.3.** Let \( R \) be a 2-dimensional regular local domain with maximal ideal \( M(R) = M = (x, y)R \), and let \( J := (x^3, x^2y, y^7)R \). We have \( \text{ord}_R J = 3 \), and the GCD of \( \text{info } x^3 \) and \( \text{info } x^2y \) is a polynomial in \( \text{grad } R \) of degree 2. Hence the Zariski number \( m(R, J) = 3 - 2 = 1 \). Thus \( M \) divides the integral closure of \( J \), but \( M^2 \) does not. The ideal \( J \) has one base point in the first neighborhood of
the blowup of $M$, namely $R_1 := R[\frac{z}{y}](y, \frac{z}{y})R[\frac{z}{y}]$. Let $x_1 := \frac{z}{y}$. We have $JR_1 = (y^3x_1^3, y^3x_1^2, y^7)R_1$. The transform of $J$ in $R_1$ is the ideal $J_1 := (x_1^3, x_1^2, y^4)R_1$. We have $\text{ord}_{R_1} J_1 = 2$ and the derived Zariski number $m(R, J, R_1) = 0$. Thus the maximal ideal $M_1$ of $R_1$ does not divide the integral closure of $J_1$ in $R_1$. Let $R_2 := R_1[\frac{z}{y}](y, \frac{z}{y})R_1[\frac{z}{y}]$. It is straightforward to check that $R_2$ is the only base point of $J$ in the second neighborhood of $R$. Let $x_2 := \frac{z}{y}$. We have $J_1R_2 = (y^3x_2^3, y^3x_2^2, y^4)R_2 = y^2(3y^2x_2, x_2^2, y^3)R_2$. The transform of $J$ in $R_2$ is the ideal $J_2 := (x_2^3, y^2)R_2$. We have $\text{ord}_{R_2} J_2 = 2$ and the derived Zariski number $m(R, J, R_2) = 2$. It follows that the integral closure of $J_2$ is $M_2^2$, where $M_2$ is the maximal ideal of $R_2$. The ideal $J$ has 3 base points $R = R_0, R_1, R_2$. Let $V_i := \text{ord}_{R_i}, i \in \{0, 1, 2\}$. The Zariski index $n(R, J, V_0) = 1$, while $n(R, J, V_1) = 0$ and $n(R, J, V_2) = 2$. The ideal $J$ has two dicritical divisors, namely $V_0$ and $V_2$.

**Example 5.4.** Let $R$ be a 2-dimensional regular local domain with maximal ideal $M(R) = M = (x, y)R$. Consider the infinite QDT-sequence $(S_j)_{0 \leq j < \infty}$ such that $S_0 = R$ and $M(S_j) = (x_j, y_j)S_j$, where $x = x_j$ and $y_j = \frac{y}{x}$ for each $j \geq 1$. Let $V_j := \text{ord}_{S_j}$. The simple complete ideal $\zeta_R(V_j) = (x, y^j)R$ for each $j$. For each $m \in \mathbb{N}$, let $I_m = \prod_{0 \leq j < m} \zeta_R(V_j)$. The complete ideal $I_m$ has order $m$ and is minimally generated by $m + 1$ monomials in $x$ and $y$. Let $B(n) = \frac{n(n+1)}{2}$ and consider the polynomials

$$F_m(X, Y) := \sum_{0 \leq p \leq \lfloor \frac{m-1}{2} \rfloor} X^{B(2p+1)} Y^{m-1-2p}, \quad G_m(X, Y) = \sum_{0 \leq p \leq \lfloor \frac{m}{2} \rfloor} X^{B(2p)} Y^{m-2p}$$

over the ring of integers. Abhyankar showed the ideal $J_m := (F_m(x, y), G_m(x, y))R$ is a 2-generated reduction of $I_m$ for each $m \in \mathbb{N}$. Thus, for example, if $m = 5$, then the ideal

$$J_5 = (xy^4 + x^6y^2 + x^{15}, \ y^5 + x^3y^3 + x^{10}y)R$$

is a 2-generated reduction of the complete ideal $I_5$.

Notice that for each prime divisor $V \in D(R)^\Delta$ and each $m \in \mathbb{N}$, the Zariski index $n(R, J_m, V)$ of the ideal $J_m$ is either 0 or 1. Moreover, $n(R, J_m, V) = 1$ if and only if $\sigma_R^{-1}(V) \in \{S_0, \ldots, S_m\}$.

An attractive feature of Example 5.4 is that it applies without any restrictions on the residue field of $R$. The residue field of $R$ could be the finite field with two elements.
Remark 5.5. Let \((R, M)\) be a Noetherian local ring with \(\dim R = d > 0\) and let \(I\) be an \(M\)-primary ideal. If the residue field of \(R\) is infinite, then Northcott and Rees \cite{38} prove that there exist \(d\)-generated ideals \(J\) that are reductions of \(I\). Moreover, each \(d\)-generated reduction \(J\) of \(I\) is a minimal reduction in the sense that there is no ideal properly contained in \(J\) that is a reduction of \(I\). If \(R\) has a finite residue field, it may happen that \(I\) fails to have a \(d\)-generated reduction. Let \(F\) be an arbitrary finite field. In \cite{31} Example 2.3], an example is given of a 2-dimensional Cohen-Macaulay local ring \((R, M)\) such that \(R\) has residue field \(F\) and the maximal ideal \(M\) of \(R\) fails to have a 2-generated reduction.

Specific solutions to item 2 of Question 5.1 such at those obtained by Abhyankar in Example 5.4 and those obtained by Abhyankar and Artal in \cite{19} Theorem 3.6 and Theorem 4.5] indicate that Question 5.6 may possibly have an affirmative answer.

Question 5.6. Let \((R, M)\) be a 2-dimensional regular local domain having a finite residue field, and let \(I\) be a complete \(M\)-primary ideal. Does there always exist a 2-generated ideal \(J = (a, b)R\) such that \(J\) is a reduction of \(I\)?

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