Comparison of Qubit and Qutrit Like Entangled Squeezed and Coherent States of Light

G. Najarbashi *, S. Mirzaei †

Department of Physics, University of Mohaghegh Ardabili, P.O. Box 179, Ardabil, Iran.

May 31, 2016

Abstract

Squeezed state of light is one of the important subjects in quantum optics which is generated by optical nonlinear interactions. In this paper, we especially focus on qubit like entangled squeezed states (ESS’s) generated by beam splitters, phase-shifter and cross Kerr nonlinearity. Moreover the Wigner function of two-mode qubit and qutrit like ESS are investigated. We will show that the distances of peaks of Wigner functions for two-mode ESS are entanglement sensitive and can be a witness for entanglement. Like the qubit cases, monogamy inequality is fulfilled for qutrit like ESS. These trends are compared with those obtained for qubit and qutrit like entangled coherent states (ECS).

Keywords: squeezed state; coherent state; entanglement; Wigner function.

PACs Index: 03.65.Ud

*E-mail: Najarbashi@uma.ac.ir
†E-mail: SMirzaei@uma.ac.ir
1 Introduction

Squeezed states are a general class of minimum-uncertainty states, for which the noise in one quadrature is reduced compared with a coherent state. The squeezed state of the electromagnetic field can be generated in many optical processes \[1\, 2\, 3\]. The first two experiments used third-order nonlinear optical media. Nonlinear effects such as the Kerr effect can be used to control a light signal by means of another light signal, and thus play an important role in optical data processing. This nonlinear effect exists in all materials, even isotropic ones, like glass or fused silica. On the other hand, the measurement of the properties of a light field without disturbing it is a necessary requirement of many optical quantum communication and computation protocols which can be achieved by means of the optical Kerr medium with large third-order nonlinearity \[4\]. Quantum non-demolition detectors can detect a single photon without destroying it in an absorption process. This detection relies on the phase shift produced by transmitting a single photon through a nonlinear Kerr medium. Such detectors have recently been implemented in the laboratory \[5\]. The optical Kerr effect also allows the propagation of ultrashort soliton-type pulses without spreading or optical bistability \[6\, 7\]. In Ref. \[8\] the authors have produced large cross-phase shifts of 0.3 mrad per photon with a fast response time using rubidium atoms restricted to a hollow-core photonic bandgap fibre, which shows the largest nonlinear phase shift induced in a single pass through a room temperature medium.

Squeezed state finds a wide range of applications in quantum information processing \[9\]. A superposition of odd photon number states for quantum information networks has been generated by photon subtraction from a squeezed vacuum state produced by a continuous wave optical parametric amplifier \[10\]. In Ref. \[11\], orthogonal Bell states with entangled squeezed vacuum states have been constructed and a scheme for teleportation a superposition of squeezed states based on the Bell state measurement have been presented. An analysis
of squeezed single photon states as a resource for teleportation of coherent state qubits has been investigated in Ref. [12]. Ref. [13] has shown that non-Gaussian entangled states are good resources for quantum information processing protocols, such as, quantum teleportation. The problem of generating ESS’s has been discussed in [14, 15, 16, 17, 18]. The new physical interpretation of the generalized two-mode squeezing operator has been studied in [19] which is useful to design optical devices for generating various squeezed states of light. Quantum storage of squeezed state light through an electromagnetically induced transparency (EIT) medium has been experimentally realized in hot atoms and magneto-optical traps in which the atomic quantum memory was verified by measuring the quantum noise of the retrieved states [20, 21, 22, 23, 24].

Coherent states, originally introduced by Schrodinger in 1926 [25]. However, multi-mode quantum states of radiation fields which present the opportunity to be a resource in quantum information theory have been the subject of considerable interest in the experimental and theoretical literature [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. Entanglement concentration for W-type entangled coherent states have been investigated in Ref. [41]. Generation of multipartite ECS’s and entanglement of multipartite states constructed by linearly independent coherent states have been investigated in [42, 43]. In Ref. [44] the production and entanglement properties of the generalized balanced N-mode Glauber coherent states of the form

\[ |\Psi^{(d)}_N\rangle_c = \frac{1}{\sqrt{M_N^{(d)}}} \sum_{i=0}^{d-1} f_i |\alpha_i\rangle \cdots |\alpha_i\rangle, \]

have been stated which is a general form of the balanced two-mode ECS \( |\Psi^{(2)}_c\rangle = \frac{1}{\sqrt{M_2^{(2)}}} (|\alpha\rangle |\alpha\rangle + f|\beta\rangle |\beta\rangle) \) and \( |\Psi^{(3)}_c\rangle = \frac{1}{\sqrt{M_3^{(3)}}} (|\alpha\rangle |\alpha\rangle + f_1 |\beta\rangle |\beta\rangle + f_2 |\gamma\rangle |\gamma\rangle) \). Assuming that the coherent states are linearly independent, these states recast in two qubit and qutrit form respectively. Then the entanglement of these states has been evaluated by concurrence measure. In Ref. [45], the noise effect on two modes qubit and qutrit like ECS’s has been studied.

In 1932, Wigner introduced a distribution function in mechanics that permitted a descrip-
tion of mechanical phenomena in a phase space [46, 47]. Wigner functions have been especially used for describing the quadratures of the electrical field with coherent and squeezed states or single photon states [48, 49, 50]. The Bell inequality based on a generalized quasi probability Wigner function and its violation for single photon entangled states and two-mode squeezed vacuum states have been investigated in Ref. [51]. The negativity of the Wigner function [52] is a reason why the Wigner function can not be regarded as a real probability distribution but it is a quasi-probability distribution function. This character is a good indication of the possibility of the occurrence of nonclassical properties of quantum states. The Wigner function of two-mode qubit and qutrit like ECS’s have been investigated in [53].

In this paper we will consider two-mode qubit like ESS $|\Psi^{(2)}\rangle_s = \frac{1}{\sqrt{M^{(2)}_s}}(|\xi\xi\rangle + f|\eta\eta\rangle)$, in which two squeezed states $|\xi\rangle$ and $|\eta\rangle$ are in general nonorthogonal and span a two dimensional qubit like Hilbert space $\{|0\rangle, |1\rangle\}$. Therefore, two-mode squeezed state $|\Psi^{(2)}\rangle_s$ can be recast in two qubit form. Moreover as an example we introduce a method for producing qubit like ESS using kerr medium and beam splitters [16]. The same argument can be formulated for other two-mode squeezed states such as $|\Psi^{(3)}\rangle_s = \frac{1}{\sqrt{M^{(3)}_s}}(|\xi\xi\rangle + f_1|\eta\eta\rangle + f_2|\tau\tau\rangle)$, in which the set $\{|\xi\rangle, |\eta\rangle, |\tau\rangle\}$ are linearly independent and span the three dimensional qutrit like Hilbert space $\{|0\rangle, |1\rangle, |2\rangle\}$, implying that $|\Psi^{(3)}\rangle_s$ can be recast in two qutrit form. The entanglement of the state $|\Psi^{(2)}\rangle_s$ can be calculated by concurrence measure introduced by Wootters [54, 55] and in the same manner, its generalized version [56] can be used to measure the entanglement of $|\Psi^{(3)}\rangle_s$. Moreover we investigate the Wigner quasi-probability distribution function for qubit and qutrit like ESS. For both qubit like ESS and ECS the distance of peaks in Wigner function can be an evidence for amount of entanglement. While for qutrit like ESS this argument seems to be rather pale. On the other hand, the monogamy inequality for multi-qutrit like ESS and ECS are discussed and it is shown that there exist qutrit like states violating monogamy inequality.

The outline of this paper is as follows: In section 2 we investigate the entanglement of
two-mode qubit like ESS and compare it with ECS. Moreover the Wigner quasi-probability distribution function for qubit like ESS is studied in this section. The behavior of Wigner function and monogamy inequality for multi-qutrit like ESS are discussed in section 3. Our conclusions are summarized in section 4.

2 Two-mode Qubit like ESS’s

A rather more exotic set of states of the electromagnetic field are the squeezed states. The squeezed state of light is two-photon coherent state that photons will be created or destroyed in pairs. They may be generated through the action of a squeeze operator defined as

\[ \hat{S}(\xi) = \exp[\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^\dagger 2)], \]  

(2.2)

where \( \xi = r_1 e^{i\theta_1} \), with \( 0 \leq r_1 < \infty \), \( 0 \leq \theta_1 \leq 2\pi \) and \( r_1 \) is known as the squeeze parameter. We note that the squeeze operator is unitary and \( \hat{S}^\dagger(\xi) = \hat{S}^{-1}(\xi) = \hat{S}(-\xi) \). The operator \( \hat{S}(\xi) \) is a kind of two-photon generalization of the displacement operator used to define the usual coherent states of a single-mode field. Acting squeeze operator on vacuum states would create some sort of two photon coherent states:

\[ |\xi\rangle = \hat{S}(\xi)|0\rangle, \]  

(2.3)

namely squeezed states. One can write the squeezed states in terms of Fock states \( |n\rangle \) as

\[ |\xi\rangle = \frac{1}{\sqrt{\cosh r_1}} \sum_{n=0}^{\infty} (-e^{i\theta_1} \tanh r_1)^n \frac{\sqrt{(2n)!}}{2^n n!} |2n\rangle. \]  

(2.4)

The overlap of two squeezed state reads

\[ \langle \xi|\eta \rangle = \frac{1}{\sqrt{\cosh r_1 \cosh r_2 (1 - e^{-i\Delta \theta} \tanh r_1 \tanh r_2)}}, \]  

(2.5)

in which \( \xi = r_1 e^{i\theta_1}, \eta = r_2 e^{i\theta_2} \) and \( \Delta \theta = \theta_2 - \theta_1 \).
2.1 Entanglement of Two-Mode Qubit like ESS

Let us consider two-mode qubit like ESS as

$$|\Psi^{(2)}\rangle_s = \frac{1}{\sqrt{M_s^{(2)}}}(|\xi\rangle + f|\eta\rangle),$$

(2.6)

where $f$, $\xi$ and $\eta$ are generally complex numbers and $M_s^{(2)}$ is a normalization factor, i.e.

$$M_s^{(2)} = 1 + |f|^2 + 2\text{Re}(fp^2),$$

(2.7)

in which $p = \langle \xi | \eta \rangle$. We used the superscript $(2)$ for qubit-like states to distinguish it from that of qutrit like states in the next section. Moreover, subscript $s$ is referred to squeezed states to distinguish it from coherent states. Two nonorthogonal squeezed states $|\xi\rangle$ and $|\eta\rangle$ are assumed to be linearly independent and span a two-dimensional subspace of the Hilbert space $\{ |0\rangle, |1\rangle \}$. The state $|\Psi^{(2)}\rangle_s$ is in general an entangled state. To show this avowal we use a measure of entanglement called concurrence which is introduced by Wooters [54, 55]. For any two qubit pure state in the form $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$, the concurrence is defined as $C = 2|a_{00}a_{11} - a_{01}a_{10}|$. By defining the orthonormal basis as

$$|0\rangle = |\xi\rangle, \quad |1\rangle = \frac{1}{\sqrt{1-|p|^2}}(|\eta\rangle - p|\xi\rangle),$$

(2.8)

the state $|\Psi^{(2)}\rangle_s$ is reduced to the state $|\psi\rangle$. Therefore the concurrence is obtained as

$$C^{(2)} = \frac{2|f|(1 - \frac{\cosh r_1 \cosh r_2 \sqrt{1 + \tanh^2 r_1 \tanh^2 r_2 - 2 \cos \Delta \theta \tanh r_1 \tanh r_2}}{\cosh r_1 \cosh r_2 (1 - \cos \Delta \theta \tanh r_1 \tanh r_2)^2 + \sin^2 \Delta \theta \tanh^2 r_1 \tanh^2 r_2})}{1 + f^2 + \frac{2f(1 - \cos \Delta \theta \tanh r_1 \tanh r_2)^2}{\cosh r_1 \cosh r_2 (1 - \cos \Delta \theta \tanh r_1 \tanh r_2)^2 + \sin^2 \Delta \theta \tanh^2 r_1 \tanh^2 r_2}},$$

(2.9)

where for simplicity we assume that $f$ to be a real number. The behavior of concurrence as a function of $f$ is shown in figure [4]. For the sake of comparison, the concurrence of two-mode qubit like ECS,

$$|\Psi^{(2)}\rangle_c = \frac{1}{\sqrt{M_c^{(2)}}}(|\alpha\alpha\rangle + f|\beta\beta\rangle),$$

(2.10)

is also plotted in figure [4] (for further details see [44]). A comparison between full and dashed
Figure 1: (Color online) Concurrence of $|\Psi^{(2)}\rangle_s$ (dashed line) and $|\Psi^{(2)}\rangle_c$ (full line) as a function of $f$ for $r_1 = 1.5$: (a) $r_2 = 2$ and $\Delta \theta = 0$ and (b) $r_2 = 0.5$ and $\Delta \theta = 1.68\pi$.

Line in figure 1 shows that for $f > 0$ the concurrence of squeezed state is less than the concurrence of coherent state. On the other hand for $f < 0$, there are $r_1$, $r_2$ and $\Delta \theta$ for which the cross over occurs. Moreover figure shows that only for $f = 0$ the state $|\Psi^{(2)}\rangle_s$ is separable which is confirmed by Eq. (2.9). If we assume that all parameters are real and $f = 1$, then the concurrence (2.9) can be rewritten as

$$C^{(2)}(\Delta_s) = \frac{1 - \text{sech}[\Delta_s]}{1 + \text{sech}[\Delta_s]},$$

(2.11)

where $\Delta_s = \xi - \eta$. This equation shows that the concurrence is a monotone function of $\Delta_s$ (see figure 2). If $\Delta_s \to \infty$, the concurrence tends to its maximum value ($C^{(2)}_{\text{max}} = 1$), while for small separation (i.e. $\Delta_s \to 0$) the concurrence tends to zero which means the state $|\Psi^{(2)}\rangle_s$ is separable.

Let us introduce a scheme to generate such ESS’s. To this end, consider the interaction Hamiltonian for a cross Kerr nonlinearity as $\hat{H} = \hbar k \hat{n}_a \hat{n}_b$, where $\hat{n}_a$ and $\hat{n}_b$ are photon number operators of mode $a$ and mode $b$, respectively and $k$ is proportional to the third-order nonlinear susceptibility $\chi^{(3)}$. The time evolution operator is

$$\hat{U}(\tau) = e^{-i\tau \hat{n}_a \hat{n}_b},$$

(2.12)
where $\tau = kt = k(l/v)$, in which $l$ is the length of Kerr medium (KM) and $v$ is the velocity of light in the Kerr medium. If we assume that mode $a$ is initially in a squeezed vacuum state $|\xi\rangle_a$, and choose $\tau = \frac{\pi}{2}$, then we obtain the superpositions of squeezed vacuum states $|\varphi\rangle \sim |\xi\rangle_a \pm |-\xi\rangle_a$.

The method proposed here, involves cross-Kerr interactions. But, in general, the strengths of these interactions are far too weak to bring about the required phase shifts. The techniques of EIT, a quantum interference phenomenon that arises when coherent optical fields couple to the states of a material quantum system, have been used to overcome this undesirable problem. Strictly speaking, the squeezing of the delayed output state in an EIT medium has not been retained the same as that of input state [57], and the squeezing property of the retrieved state after a storage time has decreased [22, 23]. To qualify the efficiency of the squeezing propagation or storage in an EIT system, experimental studies have discovered an approach to repress the noise of output delay light via decreasing the detection frequency or operating the system at off one or two-photon resonance [58, 59, 60, 61, 62].

Now, assume that the state $|1\rangle_b|0\rangle_c$ is inserted to BS1, then the output state becomes $\frac{1}{\sqrt{2}}(|10\rangle_{bc} + i|01\rangle_{bc})$ (see figure [3]). On the other hand, the Kerr medium together with phase
Qubit and Qutrit like ESS

Figure 3: (Color online) Experimental set up for generation ESS.

shifter $\hat{P} = e^{iN\theta}$, transform the state $\frac{1}{\sqrt{2}}|\xi\rangle_a (|10\rangle_{bc} + i|01\rangle_{bc})$ to

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\xi e^{-2i\tau}\rangle_a |10\rangle_{bc} + ie^{i\theta}|\xi\rangle_a |01\rangle_{bc}),$$  \hspace{1cm} (2.13)

where we used the fact that

$$\hat{U}(\tau)|n\rangle_a |\xi\rangle_b = |n\rangle_a |\xi e^{2i\tau}\rangle_b.$$  \hspace{1cm} (2.14)

Now let mode $b$ interacts with mode $a'$ (in a squeezed vacuum state $|\eta\rangle_{a'}$), via Kerr medium $KM2$, then the state $|\psi_1\rangle$ after the Kerr medium $KM2$ and $BS2$ reads

$$|\psi_2\rangle = \frac{1}{2}(|\xi e^{-2i\tau}\rangle_a |\eta e^{-2i\tau'}\rangle_{a'} - e^{i\theta}|\xi\rangle_a |\eta\rangle_{a'}|10\rangle_{bc}$$
$$+ \frac{i}{2}(|\xi e^{-2i\tau}\rangle_a |\eta e^{-2i\tau'}\rangle_{a'} + e^{i\theta}|\xi\rangle_a |\eta\rangle_{a'}|01\rangle_{bc}).$$  \hspace{1cm} (2.15)

Now if one of the detectors $D_c$ or $D_b$ fires then, up to a normalization factor, we have the ESS’s

$$\langle |\xi e^{-2i\tau}\rangle_a |\eta e^{-2i\tau'}\rangle_{a'} - e^{i\theta}|\xi\rangle_a |\eta\rangle_{a'}\rangle,$$  \hspace{1cm} (2.16)

or

$$i(|\xi e^{-2i\tau}\rangle_a |\eta e^{-2i\tau'}\rangle_{a'} + e^{i\theta}|\xi\rangle_a |\eta\rangle_{a'}\rangle).$$  \hspace{1cm} (2.17)

Choosing different parameters $\tau, \tau'$ and $\theta$, one may obtain the various entangled states \[16\].

For example, taking $\tau = \tau' = \frac{\pi}{2}$ and $\theta = 0$ we have $|\psi^{(2)}\rangle_s \sim | - \xi\rangle_a | - \eta\rangle_{a'} - |\xi\rangle_a |\eta\rangle_{a'}$ which
is an entangled state with concurrence $C^{(2)} \neq 0$. To see this, we consider two nonorthogonal squeezed states $\{|\xi\rangle, |\bar{\xi}\rangle\}$ and $\{|\eta\rangle, |\bar{\eta}\rangle\}$, which are linearly independent and span a two-dimensional space $\{|0\rangle, |1\rangle\}$ defined as

\[
|0\rangle = |\xi\rangle, \quad |1\rangle = \frac{|-\xi\rangle - p_1 |\xi\rangle}{N_1} \quad \text{for system 1},
\]

\[
|0\rangle = |\eta\rangle, \quad |1\rangle = \frac{|-\eta\rangle - p_2 |\eta\rangle}{N_2} \quad \text{for system 2},
\]

where

\[
p_1 = \langle \xi | - \xi \rangle, \quad N_1 = \sqrt{1 - p_1^2},
\]

\[
p_2 = \langle \eta | - \eta \rangle, \quad N_2 = \sqrt{1 - p_2^2}.
\]

For simplicity, we again assumed that $\xi$ and $\eta$ are real parameters. Altogether, we obtain the entangled state

\[
|\psi^{(2)}\rangle_s = \frac{1}{\sqrt{2(1 - p_1p_2)}} \{(p_1p_2 - 1)|00\rangle + N_2 p_1 |01\rangle + N_1 p_2 |10\rangle + N_1 N_2 |11\rangle\},
\]

whose concurrence reads

\[
C^{(2)} = \sqrt{(1 - p_1^2)(1 - p_2^2)}.
\]

The behavior of concurrence as a function of $r_1$ and $r_2$ is shown in figure 4. Figure 4 shows that for $r_1 = r_2 \neq 0$ (i.e. $p_1 = p_2$) the state $|\psi^{(2)}\rangle_s$ is maximally entangled state (i.e. $C_{\text{max}}^{(2)} = 1$).
2.2 Wigner Function for Qubit like ESS

One of the important quasi-probability distribution over phase space is the Wigner function defined as 

\[ W(\gamma) = \frac{1}{\pi^2} \int d^2 \lambda C_W(\lambda) e^{\lambda^* \gamma - \lambda \gamma^*}, \]  

(2.22)

where \( C_W(\lambda) = Tr(\rho D(\lambda)) \) in which \( D(\lambda) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \) is displacement operator and \( \rho \) is the reduced density matrix obtained by partially tracing out second mode. Here we investigate the behavior of one-mode Wigner function for qubit like state \([2.6] \). Wigner characteristic function reads

\[ C_W^{(2)}(\lambda) = \frac{1}{M_s^2} \left\{ \langle \xi | D(\lambda) | \xi \rangle + f^2 \langle \eta | D(\lambda) | \eta \rangle + f p(\langle \xi | D(\lambda) | \eta \rangle + \langle \eta | D(\lambda) | \xi \rangle) \right\}. \]  

(2.23)

To calculate the Wigner characteristic function, let us use the fact that commuting displacement operator \( D(\lambda) \) with squeezing operator \( S(\xi) \) yields

\[ S(\xi) D(\lambda) = D(\lambda') S(\xi), \]  

(2.24)

where \( \lambda' = \mu \lambda - \nu \lambda^* \), \( \mu = \cosh r_1 \) and \( \nu = e^{i\theta_1} \sinh r_1 \). By substituting \( C_W^{(2)}(\lambda) \) in Eq.(2.22) the Wigner function can be obtained as

\[ W^{(2)}(\gamma) = \frac{1}{M^2} \left\{ \frac{2}{\pi} \left( e^{-\gamma^2/2} + f^2 e^{-\gamma^2} \right) \right\} + \frac{fp}{\sqrt{2R(\mu^2 - \nu^2)(A - B - D)}} \exp\left[ \frac{\gamma^2}{A - B - D} \left( -1 + \frac{(B-D)^2}{2R(A-B-D)} \right) \right] \]  

(2.25)

where

\[ A = \frac{1}{2} (\mu'^2 + \nu'^2 - 2\frac{\mu'\nu'(\nu' - \mu')}{\mu^2 - \nu^2}), \quad A' = \frac{1}{2} (\mu^2 + \nu^2 - 2\frac{\mu\nu(\nu' - \mu')}{\mu'\nu' - \nu^2}), \]

\[ B = \frac{1}{2} (\mu' - \mu^2 - \nu^2), \quad B' = \frac{1}{2} (\mu' - \mu^2 - \nu^2), \]

\[ D = \frac{1}{2} (\mu' - \mu^2 - \nu^2), \quad D' = \frac{1}{2} (\mu' - \mu^2 - \nu^2), \]

\[ R = \frac{1}{2} (A + B + D + \frac{(B-D)^2}{A - B - D}), \quad R' = \frac{1}{2} (A' + B' + D' + \frac{(B-D)^2}{A' - B' - D'}), \]  

(2.26)
in which \( \mu' = \cosh r_2 \) and \( \nu' = e^{i\beta_2} \sinh r_2 \). To explore the behavior of Wigner function in more detail, it is convenient to take all parameters to be real except for \( \lambda \), i.e.

\[
\mu = \cosh \xi, \quad \nu = \sinh \xi, \quad \mu' = \cosh \eta, \quad \nu' = \sinh \eta.
\]

(2.27)

Now we consider two special cases \( f = \pm 1 \). For the case \( f = 1 \) we plot Wigner distribution as a function of \( \gamma \) for fixed \( \eta = 0.5 \) but different \( \xi \) (see figure 5 (a)). Specifically, we wish to compare the results with those obtained for ECS [53]. By increasing the difference of \( \xi \) and \( \eta \) the width of squeezed Wigner function \( W_s^{(2)}(\gamma) \) decreases, while the peaks of \( W_c^{(2)}(\gamma) \) for ECS (2.10) recede by increasing the difference of \( \alpha \) and \( \beta \). Furthermore, in contrast to ESS, for the ECS the height of peaks changes manifestly. The behavior of \( W_s^{(2)}(\gamma) \) and \( W_c^{(2)}(\gamma) \) are depicted in figures 6 for the case \( f = -1 \). Evidently, the peaks of \( W_s^{(2)}(\gamma) \) approach to each other by increasing the difference of \( \Delta_s = \xi - \eta \). This result is different for \( W_c^{(2)}(\gamma) \), that is by increasing the \( \Delta_c = \alpha - \beta \), the peaks of \( W_c^{(2)}(\gamma) \) recede from each other.

Another example is the qubit like ESS \( |\psi^{(2)}\rangle_s \sim | - \xi\rangle_a - \eta\rangle_{a'} - | \xi\rangle_a | \eta\rangle_{a'} \rangle \), generated by the scheme proposed in figure 3, whose Wigner function is given by

\[
W_s^{(2)}(\xi, \eta, \gamma) = \frac{e^{-\frac{1}{2} \gamma^2 e^{2\xi}} (\cosh(2\eta)(e^{2\xi} \sinh(2\xi) + 1) - 2\sqrt{\cosh(2\eta)} \sqrt{\text{sech}(2\xi)} e^{\frac{1}{2} \gamma^2 (4\xi - \gamma)} \text{sech}(2\xi))}{\pi \left( \cosh(2\eta) - \text{sech}(2\xi) \sqrt{\cosh(2\eta) \cosh(2\xi)} \right)}. \]

(2.28)
Figure 6: (Color online) Wigner distribution as a function of $\gamma$ with $f = -1$ (a) $W_s^{(2)}(\gamma)$ for given $\eta, \xi = 0.2$, (b) $W_c^{(2)}(\gamma)$ for given $\beta, \alpha = 0.2$.

Figure 7 shows the behavior of the above Wigner function, as a function of $\gamma$, for given $\xi$ and $\eta$.

Figure 7: (Color online) $W_s^{(2)}(\gamma)$ as a function of $\gamma$ for given: $\xi = \eta = 0.5$ (red), $\xi = 0.5, \eta = 1$ (blue), $\xi = 1, \eta = 2$ (green), $\xi = 1, \eta = 3$ (black).
3 Qutrit like ESS’s

In this section, we consider two and three-mode qutrit like ESS’s. First consider two-mode case:

\[ |\Psi^{(3)}_s\rangle = \frac{1}{\sqrt{M^{(3)}_s}} \left( |\xi\rangle + f_1 |\eta\rangle + f_2 |\tau\rangle \right), \]  

(3.29)

where \( M^{(3)}_s = 1 + f_1^2 + f_2^2 + 2f_1p_2^2 + 2f_2p_3^2 + 2f_1f_2p_2^2 \) is the normalization factor with \( p_1 = \langle \xi |\eta \rangle \), \( p_2 = \langle \tau |\eta \rangle \) and \( p_3 = \langle \tau |\xi \rangle \). For simplicity we assumed that all parameters are real. In general, three squeezed states \( |\xi\rangle, |\eta\rangle \) and \( |\tau\rangle \), can define three-dimensional space spanned by the set \( \{ |0\rangle, |1\rangle, |2\rangle \} \)

\[
|0\rangle = |\xi\rangle,
|1\rangle = \frac{1}{\sqrt{1-p_1^2}} (|\eta\rangle - p_1|\xi\rangle),
|2\rangle = \sqrt{\frac{1-p_1^2}{1-p_1^2-p_2^2-p_3^2+2p_1p_2p_3}} \left( |\tau\rangle + (p_1p_3-p_2)|\eta\rangle + (p_1p_2-p_3)|\xi\rangle \right).
\]

(3.30)

Substitution these new basis into Eq.(3.29) the state \( |\Psi^{(3)}_s\rangle \) is reduced to a qutrit like ESS. We use the general concurrence measure for bipartite state \( |\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} a_{ij} |e_i \otimes e_j\rangle \) \[36\]. The norm of concurrence vector is obtained as \( C = 2(\sum_{i<j}^{d_1} \sum_{k<l}^{d_2} |a_{ik}^* a_{jl} - a_{ik} a_{jl}^*|^2)^{1/2} \), where \( d_1 \) and \( d_2 \) are dimensions of first and second part respectively (in the present case \( d_1 = d_2 = 3 \)). The concurrence \( |\Psi^{(3)}_s\rangle \) is represented in figure 8. To compare this result with ECS, we also plotted the concurrence of \( |\Psi^{(3)}_c\rangle \) in the figure 8. By \( |\Psi^{(3)}_c\rangle \) we mean that in Eq.(3.29) the squeezed states \( |\xi\rangle, |\eta\rangle \) and \( |\tau\rangle \) must be replaced by coherent states \( |\alpha\rangle, |\beta\rangle \) and \( |\gamma\rangle \). Figure 9(a) displays the Wigner function of \( |\Psi^{(3)}_s\rangle \) as a function of \( \gamma \) with \( f_1 = f_2 = 1 \) for fixed \( \xi = 0.5 \) but various \( \eta \) and \( \tau \). One may immediately deduce that the the width of Wigner function decreases by increasing the distance of \( \xi \) with \( \eta \) and \( \tau \). This result is confirmed by the concurrence measure in a opposite manner which is accompanied by a shift in \( C^{(3)}(\xi) \) (see figure 9(b)). Figure 10 illustrates the \( W^{(3)}_s(\gamma, \xi) \) for given \( \eta = 0.2, \tau = 0.4 \) and \( f_1 = f_2 = -1 \).
3.1 Monogamy Inequality for Multi-Qutrit like ESS

Another problem that arises in multipartite states is monogamy inequality. Here in this section, we examine the concurrence based monogamy inequality for qutrit like ESS. To this end consider multi qutrit like ESS:

$$|\Phi^{(3)}_{\text{s}}\rangle = \frac{1}{\sqrt{M^{(3)}_N}} (|\xi\rangle\cdots|\xi\rangle + f_1|\eta\rangle\cdots|\eta\rangle + f_2|\tau\rangle\cdots|\tau\rangle),$$

where

$$M^{(3)}_N = 1 + f_1^2 + f_2^2 + 2f_1p_1^N + 2f_2p_3^N + 2f_1f_2p_2^N,$$

is normalization factor. Once again, we assume that the all parameters are real. We next assume that the state $|\Phi^{(3)}_{\text{s}}\rangle$ as tripartite $A$, $B$ and $D$ including $m_1$, $m_2$ and $m_3 = N - m_1 - m_2$ modes respectively. For the moment, suppose that $\tau \to \infty$, i.e. $p_2, p_3 = 0$ and $p_1 \neq 0$.

One can easily obtain the reduced density matrices $\rho_{AB} = Tr_D (|\Phi^{(3)}_N\rangle_{ABD}\langle\Phi^{(3)}_N|)$ and $\rho_{AD} = Tr_B (|\Phi^{(3)}_N\rangle_{ABD}\langle\Phi^{(3)}_N|)$ by partially tracing out subsystems $D$ and $B$ respectively. For general mixed states the concurrence is defined as $|C|^2 = \sum_{\alpha\beta} |C_{\alpha\beta}|^2$ where $C^{\alpha\beta} = \lambda^{\alpha\beta}_1 - \sum_{i=2}^{n} \lambda^{\alpha\beta}_i$. 

Figure 8: (Color online) Concurrence of $|\Psi^{(3)}_{\text{s}}\rangle$ (dashed line) and $|\Psi^{(3)}_{\text{c}}\rangle$ (full line) as a function of $f_1$ for $\xi = 6$, $\eta = 2.5$, $\tau = 5$ and $f_2 = -1$. 
Figure 9: (Color online) (a) Wigner function $W_s^{(3)}(\gamma)$ as a function of $\gamma$ with $\xi = 0.5$, (b) concurrence $C^{(3)}(\xi)$ as a function of $\xi$ with $f_1 = f_2 = 1$ for given $\eta = \tau = 0.5$ (red), $\eta = 0.5, \tau = 1$ (blue), $\eta = 1, \tau = 2$ (green).

with $\lambda_1 = \max\{\lambda_i, i = 1, \ldots, d^2\}$ and $\lambda_i^{\alpha\beta}$ are the nonnegative eigenvalues of $\tau \tau^*$ defined as

$$
\tau^{\alpha\beta} \tau^{\alpha'\beta'} = \sqrt{\rho}(E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta}) \rho^*(E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta}) \sqrt{\rho},
$$

in which $\alpha$’s are positive roots of the lie group SU(N) (here for qutrit like SU(3)) and $E_{\pm \alpha}$’s are corresponding raising/lowering operators (like $J_\pm$ of the angular momentum operator).

Let us consider $N = 20, m_1 = 1, m_2 = 2$ and $f_2 = 0.4$ and explore the monogamy inequality

$$
C^2_{A(BD)} \geq C^2_{AB} + C^2_{AD},
$$

where $C_{AB}$ and $C_{AD}$ are the concurrences of the reduced density matrices $\rho_{AB}$ and $\rho_{AD}$ respectively and $C_{A(BD)}$ is the concurrence of pure state $|\Phi^{(3)}\rangle_s$ with respect to the partitions $A$ and $BD$. The concurrence vectors $C_{AB}$ and $C_{AD}$ read

$$
C_{AB} = \frac{f_1 p^{17} \sqrt{(1-p^2)(1-p^4)}}{0.58 + 0.5 f_1^2 + f_1 p^{20}},
$$
$$
C_{AD} = \frac{f_1 p^2 \sqrt{(1-p^2)(1-p^{34})}}{0.58 + 0.5 f_1^2 + f_1 p^{20}},
$$

(3.35)
Figure 10: (Color online) $W_s^{(3)}(\gamma, \xi)$ as a function of $\gamma$ and $\xi$ with $f_1 = f_2 = -1$ for given $\eta = 0.2$ and $\tau = 0.4$.

where $p = \frac{1}{\sqrt{\cosh r_1 \cosh r_2 (1 - \tanh r_1 \tanh r_2)}}$. On the other hand, one finds that

$$C_{A(BD)} = \frac{\sqrt{0.64 + 1.28 f_1 p^{20} + f_1^2 (4.64 - 4p^2 - 4p^{38} + 4p^{10})}}{1.16 + f_1^2 + 2f_1 p^{20}}. \quad (3.36)$$

The behavior of $\tau_{ABD} = C_{A(BD)}^2 - C_{AB}^2 - C_{AD}^2$ as a function of $f_1$, for given $\xi$ and $\eta$, is shown in figure 11. In order to compare this results with ECS we also display the results of $\tau_{ABD}^{(c)}$ calculated in Ref. [44]. Figure 11 shows that in general $\tau_{ABD}^{(c)} > \tau_{ABD}^{(s)}$.

4 Conclusion

In summary, we introduced two-mode qubit and qutrit like ESS’s. For qubit like state $|\Psi^{(2)}\rangle_s$, a comparison between ESS and ECS showed that for $f > 0$ the concurrence of squeezed state is smaller than the concurrence of coherent state without crossing. On the other hand for $f < 0$, there are crossing for some parameters $r_1$, $r_2$ and $\Delta \theta$. It was shown the concurrence is a monotone function of $\Delta_s$ for all real parameters and $f = 1$ with the properties that when $\Delta_s \to \infty$, the concurrence tends to its maximum value ($C_{max}^{(2)} = 1$), while for small separation (i.e. $\Delta_s \to 0$) the concurrence tends to zero and the state becomes separable. Moreover we investigated the ESS generated by beam splitters, phase-shifter and cross Kerr nonlinearity.
Figure 11: (Color online) $\tau_{ABD}^{(s)}$ for qutrit like ESS (dashed line) and $\tau_{ABD}^{(c)}$ for qutrit like ECS (full line) as a function of $f_1$ for given $N = 20, m_1 = 1, m_2 = 2$ and $f_2 = 0.4$ for $\xi = 3, \eta = 2$.

and showed that for $r_1 = r_2 \neq 0$ (i.e. $p_1 = p_2$) the state $|\psi^{(2)}\rangle_s$ is maximally entangled state (i.e. $C_{max}^{(2)} = 1$). Comparing Wigner functions $W_s^{(2)}(\gamma)$ and $W_c^{(2)}(\gamma)$ revealed that for $f = 1$ and $f = -1$, unlike the $W_c^{(2)}(\gamma)$, the peaks of $W_s^{(2)}(\gamma)$ approached to each other by increasing the difference of $\Delta_s = \xi - \eta$. This behavior is different for $W_c^{(2)}(\gamma)$, i.e. the peaks of $W_c^{(2)}(\gamma)$ recede from each other by increasing the $\Delta_c = \alpha - \beta$. Similar results hold for qutrit like ESS, $|\Psi^{(3)}\rangle_s$. Comparing the the concurrences of ESS to that of the ECS, showed that for $f_2 = -1$ there are $\xi, \eta$ and $\tau$ which cross over occurs. Furthermore, for $f_1 = f_2 = 1$, it was shown that by increasing the distances of $\xi, \eta$ and $\tau$, the width of $W_s^{(3)}(\gamma)$ decreases whereas the concurrence $C^{(3)}(\xi)$ has upper-shift. Finally, we explored the monogamy inequality for both qutrit like ESS and ECS. Although in both cases the monogamy inequality was not violated, but in general the inequality $\tau_{ABD}^{(c)} > \tau_{ABD}^{(s)}$ is satisfied.
Acknowledgments

The authors also acknowledge the support from the University of Mohaghegh Ardabili.

References

[1] R.E. Slusher, L.W. Hollberg, B. Yurke, J.C. Mertz, J.F. Valleys, Phys. Rev. Lett. 55, 2409 (1985).
[2] R.M. Shelby, M.D. Levenson, S.H. Perlmutter, R.G. DeVoe, D.F. Walls, Phys. Rev. Lett. 57, 691 (1986).
[3] L.A. Wu, H.J. Kimble, J.L. Hall, H. Wu, Phys. Rev. Lett. 57, 2520 (1986).
[4] N. Imoto, H.A. Haus, Y. Yamamoto, Phys. Rev. A 32, 2287-2292 (1985).
[5] Y. Yamamoto, N. Imoto, S. Machida, Phys. Rev. A 33, 3243 (1986).
[6] Y. Wu, L. Deng, Opt. Lett. 29, 2064 (2004).
[7] H.R. Hamedi, G. Juzeliunas, Phys. Rev. A 91, 053823 (2015).
[8] V. Venkataraman, K. Saha, A. L. Gaeta, Nature photonics, 7, 138141 (2013).
[9] S.J. van Enk, Phys. Rev. A 60, 5095 (1999).
[10] J.S. Neergaard-Nielsen, B.M. Nielsen, C. Hettich, K. Moelmer, E.S. Polzik, Phys. Rev. Lett. 97, 083604 (2006).
[11] H. Lu, L. Chen, J. Lin, Chinese Optics Letters, Vol. 2, No. 10 (2004).
[12] A.M. Branczyk, T.C. Ralph, Phys. Rev. A 78, 052304 (2008).
[13] K.P. Seshadreesan, J.P. Dowling, G.S. Agarwal, Physica Scripta, Vol. 90, No. 7, 074029 (2015).
[14] M.S. Kim, W. Son, V. Buzek, P.L. Knight, Phys. Rev. A 65, 032323 (2002).

[15] L.M. Kuang, A.H. Zeng, Z.H. Kuang, Phys. Lett. A, Vol. 319 24-31 (2003).

[16] Z.M. Zhang, [arxiv:quant-ph/0604128v1] (2006).

[17] A. Tipsmark, J.S. Neergaard-Nielsen, U.L. Andersen, Optics Express, Vol. 21 6670-6680 (2013).

[18] X. Wang, Phys. Rev. A, 66, 064304 (2002).

[19] F. Gao, Y.X. Wang, H.Y. Fan, X.B. Tang, arxiv: quant-ph/1511.04833v1 (2015).

[20] S.E. Harris, J.E. Field, A. Imamoglu, Phys. Rev. Lett. 64(10), 1107 (1990).

[21] C.H. Van der Wal, M.D. Eisaman, A. Andre, R.L. Walsworth, D.F. Phillips, A.S. Zibrov, and M.D. Lukin, Science 301, 196200 (2003).

[22] J. Appel, E. Figueroa, D. Korystov, M. Lobino, A.I. Lvovsky, Phys. Rev. Lett. 100, 093602 (2008).

[23] K. Honda, et al. Phys. Rev. Lett., 100, 093602 (2008).

[24] Y. Li, Z. Li, D. Wang, J. Gao, J. Zhang, J. Opt. Soc. Am. B, Vol. 29, No. 11 (2012).

[25] E. Schrödinger Naturwissenschaften 14, 664 (1926).

[26] P.T. Cochrane, G.J. Milburn, and W.J. Munro, Phys. Rev. A 59, 2631 (1999).

[27] M.C. de Oliveira and W. J. Munro, Phys. Rev. A 61, 042309 (2000).

[28] H. Jeong and M.S. Kim, Phys. Rev. A 65, 042305 (2002).

[29] T.C. Ralph, A. Gilchrist and G.J. Milburn, Phys. Rev. A 68, 042319 (2003).

[30] W.J. Munro, G.J. Milburn, and B.C. Sanders, Phys. Rev. A 62, 052108 (2001).
[31] X. Wang, B.C. Sanders, S.H. Pan, J. Phys. A 33, 7451 (2000).

[32] X. Wang, Phys. Rev. A 64, 022302 (2001).

[33] X. Wang, B.C. Sanders, Phys. Rev. A 65, 012303 (2002).

[34] W. Vogel, J. Sperling, Phys. Rev. A 89, 052302 (2014).

[35] S. Salimi, A. Mohammadzade, Int J. Theor. Phys 50: 1444-1450 (2011).

[36] S.J. van Enk, O. Hirota, Phys. Rev. A 67, 022313 (2001).

[37] X. Wang, J. Phys. A, Math. Gen. 35(1), 165173 (2002).

[38] H. Fu, X. Wang, A.I. Solomon, Phys. Lett. A 291, 7376 (2001).

[39] G. Najarbashi, Y. Maleki, Int J. Theor. Phys. 50, 2601-2608 (2011).

[40] Y.B. Sheng, C.C. Qu, O.Y. Yang, Z.F. Feng, L. Zhou, Int J. Theor. Phys. 53, 2033-2040 (2014).

[41] Y.B. Sheng, J. Liu, S.Y. Zhao, L. Wang, L. Zhou, Chin. Phys. B, Vol. 23, No. 8, 080305 (2014).

[42] B.C. Sanders, Phys. Rev. A 45, 9 (1992).

[43] S. J. van Enk, Phys. Rev. Lett., Vol. 91, No. 1 (2003).

[44] G. Najarbashi, S. Mirzaei, Int J. Theor. Phys, Vol. 54, No. 10 (2015).

[45] G. Najarbashi, S. Mirzaei, Int J. Theor. Phys, Vol. 55, No. 5 (2016).

[46] E. Wigner, Phys. Rev, Vol. 40, 749 (1932).

[47] M. Hillery, R.F. O’Connell, M.O. Scully, E.P. Wigner, Physics Reports, 106, 121-167 (1984).
[48] G. Breitenbach, S. Schiller, J. Mlynek, Nature London, 387, 471 (1997).

[49] A.I. Lvovsky, H. Hansen, T. Aichele, O. Benson, J. Mlynek, S. Schiller, Phys. Rev. Lett. 87, 050402 (2001).

[50] G.S. Agarwal, New J. Phys. 13, 073008 (2011).

[51] S.W. Lee, H. Jeong, D. Jaksch, Phys. Rev. A 80, 022104 (2009).

[52] A. Banerji, R.P. Singh, A. Bandyopadhyay, Optics Communications 330, 8590 (2014).

[53] G. Najarbashi, S. Mirzaei, arxiv: quant-ph/1601.00143v1 (2016).

[54] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[55] W.K. Wootters, Quantum Information and Computation, Vol. 1, No. 1 (2001).

[56] S.J. Akhtarshenas, J. Phys. A: Math. Gen. 38, 6777-6784 (2005).

[57] D. Akamatsu, Y. Yokoi, M. Arikawa, S. Nagatsuka, T. Tanimura, A. Furusawa, M. Kozuma, Phys. Rev. Lett 99, 153602 (2007).

[58] J. Zhang, J. Cai, Y. Bai, J. Gao, and S.Y. Zhu, Phys. Rev. A 76, 033814 (2007).

[59] G. Hetet, B.C. Buchler, O. Glockl, M.T.L. Hsu, A.M. Akulshin, H.A. Bachor, P.K. Lam, Opt. Express 16, 73697381 (2008).

[60] Y. Xiao, et al. Phys. Rev. A 80, 041805(R) (2009).

[61] M. Fleischhauer, M.D. Lukin, Phys. Rev. Lett. 84, 5094 (2000).

[62] M. Fleischhauer, A. Imamoglu, J.P. Marangos, Rev. Mod. Phys. 77, 633 (2005).

[63] C. Gerry, P. Knight, Introductory Quantum Optics, Cambridge University Press (2005).

[64] Y.Q. Li, G.Q. Zhu, Front. Phys. China, 3(3): 250257 (2008).
[65] V. Coffman, J. Kundu and W.K. Wootters, Phys. Rev. A 61, 052306 (2000).

[66] J.S. Kim, A. Das, B.C. Sanders, Phys. Rev. A 79, 012329 (2009).

[67] B.C. Sanders, J.S. Kim, Applied Mathematics and Information Sciences, 4(3) (2010).