CATEGORICAL WALL-CROSSING FORMULA FOR DONALDSON-THOMAS
THEORY ON THE RESOLVED CONIFOLD

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Abstract. We prove wall-crossing formula for categorical Donaldson-Thomas invariants on the resolved conifold, which categorifies Nagao-Nakajima wall-crossing formula for numerical DT invariants on it. The categorified Hall products are used to describe the wall-crossing formula as semiorthogonal decompositions. A successive application of categorical wall-crossing formula yields semiorthogonal decompositions of categorical Pandharipande-Thomas stable pair invariants on the resolved conifold, which categorify the product expansion formula of the generating series of numerical PT invariants on it.

1. Introduction

1.1. Background and summary of the paper. In this paper, we establish wall-crossing formula for categorical Donaldson-Thomas invariants on the resolved conifold, and apply it to give a complete description of categorical Pandharipande-Thomas (PT) stable pair invariants on it.

The PT invariants count stable pairs on CY 3-folds, which were introduced in [PT09] in order to give a better formulation of GW/DT correspondence conjecture [MNOP06]. They are special cases of Donaldson-Thomas (DT) type invariants counting stable objects in the derived category, and are now understood as fundamental enumerative invariants of curves on CY 3-folds as well as Gromov-Witten invariants and Gopakumar-Vafa invariants. Now by efforts from derived algebraic geometry [PTVV13, BBJ19], the moduli spaces which define DT (in particular PT) invariants are known to be locally written as critical loci. In [Todb], we proposed a study of categorical DT theory by gluing locally defined dg-categories of matrix factorizations on these moduli spaces. A definition of categorical DT invariants is introduced in the case of local surfaces in [Todb] via Koszul duality and singular support quotients. We also proposed several conjectures on wall-crossing of categorical DT invariants on local surfaces, motivated by d-critical analogue of Bondal-Orlov and Kawamata’s D/K equivalence conjecture [BO, Kaw02], and also categorifications of wall-crossing formulas of numerical DT invariants [JS12, KS]. In [Todb], we also derived wall-crossing formula of categorical PT invariants on local surfaces in the setting of simple wall-crossing (i.e. there are at most two Jordan-Hölder factors at the wall).

The purpose of this paper is to prove wall-crossing formula for categorical DT invariants on the resolved conifold, which categorifies Nagao-Nakajima wall-crossing formula [NN11] for numerical DT invariants on it. In this case the relevant moduli spaces are global critical loci, so there is no issue on gluing dg-categories of matrix factorizations. However wall-crossing is not necessary a simple wall-crossing, and the analysis of categorical wall-crossing is much harder. Our strategy is to use categorified Hall products for quivers with super-potentials introduced by Pădurariu [Păd1, Păd2]. A key observation is that, up to Knörrer periodicity, a wall-crossing diagram for the resolved conifold locally looks like a Grassmannian flip together with some super-potential (d-critical Grassmannian flip in the sense of d-critical birational geometry [Toda]). We refine the result of [BCF+21] on derived categories of Grassmannian flips via categorified Hall products, and compare them with more global categorified Hall products under the Knörrer periodicity. The above approach via categorified Hall products yields a desired categorical wall-crossing formula. A successive iteration of wall-crossing gives a semiorthogonal decomposition of categorical PT invariants on the resolved conifold whose semiorthogonal summands are the simplest categories of matrix factorizations over a point. We
emphasize that the result of this paper is a first instance where categorical wall-crossing formula is obtained for non-simple wall-crossing in the context of categorical DT theory.

1.2. Categorical PT stable pair theory on the resolved conifold. The resolved conifold $X$ is defined by

$$X := \text{Tot}_{P^1}(\mathcal{O}_{P^1}(-1)^{\oplus 2}),$$

which is also obtained as a crepant small resolution of the conifold singularity $\{xy + zw = 0\} \subset \mathbb{C}^4$.

The resolved conifold is a non-compact CY 3-fold, and an important toy model for enumerative geometry on CY 3-folds such as PT invariants.

For each $(\beta, n) \in \mathbb{Z}^2$, we denote by

$$P_n(X, \beta)$$

the moduli space of PT stable pairs $(F, s)$ on $X$, i.e. $F$ is a pure one dimensional coherent sheaf on $X$ and $s : \mathcal{O}_X \to F$ is surjective in dimension one, satisfying $[F] = \beta[C]$ and $\chi(F) = n$. Here $C \subset X$ is the zero section of the projection $X \to \mathbb{P}^1$, and $[F]$ is the fundamental one cycle of $F$. The PT invariant $P_{n, \beta} \in \mathbb{Z}$ is defined by either taking the integration over the zero dimensional virtual fundamental class on $P_n(X, \beta)$, or weighted Euler characteristic of the Behrend constructible function $[\text{Beh09}]$ on it. It is well-known that the generating series of PT invariants on $X$ is given by the formula:

$$\sum_{n, \beta} P_{n, \beta} q^n t^\beta = \prod_{m \geq 1} (1 - (-q)^m t)^m. \quad (1.1)$$

The above formula is available in $[\text{NN11} \text{ Theorem 3.15}]$, which is also obtained from the DT calculation in $[\text{BB07}]$ together with the DT/PT correspondence $[\text{Bri11} \text{ Tod10} \text{ ST11}]$.

The purpose of this paper is to give a categorification of the formula (1.1). In the case of the resolved conifold, the moduli space $P_n(X, \beta)$ is written as a global critical locus, i.e. there is a pair $(M, w)$ where $M$ is a smooth quasi-projective scheme and $w : M \to \mathbb{A}^1$ is a regular function such that $P_n(X, \beta)$ is isomorphic to the critical locus of $w$. A choice of $(M, w)$ is not unique, and we take it using Van den Bergh’s non-commutative crepant resolution of $X$ $[\text{VdB04}]$ (see Subsection 5.10). We define the categorical PT invariant on $X$ to be the triangulated category of matrix factorizations $\mathcal{D}T(P_n(X, \beta)) := \text{MF}(M, w)$.

The above triangulated category (or more precisely its dg-enhancement) recovers $P_{n, \beta}$ by taking the Euler characteristic of the periodic cyclic homology (see Equation (5.59)). The following is a consequence of the main result in this paper:

**Theorem 1.1.** (Corollary 5.22) There exists a semiorthogonal decomposition of the form

$$\mathcal{D}T(P_n(X, \beta)) = \langle a_{n, \beta} \text{-copies of } \text{MF}(\text{Spec } \mathbb{C}, 0) \rangle. \quad (1.2)$$

Here $a_{n, \beta}$ is defined by

$$a_{n, \beta} := \sum_{l : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 0}} \prod_{m \geq 1} \left(\frac{m}{l(m)}\right)^{l(m)} \sum_{m \geq 1, l(m)l(m) = (n, \beta)}.$$

(1.3)

Here $\text{MF}(\text{Spec } \mathbb{C}, 0)$ is the category of matrix factorizations of the zero super-potential over the point, which is equivalent to the $\mathbb{Z}/2$-periodic derived category of finite dimensional $\mathbb{C}$-vector spaces. As the formula (1.1) is equivalent to $P_{n, \beta} = (-1)^{n+\beta} a_{n, \beta}$, by taking the periodic cyclic homologies of both sides and Euler characteristics, the result of Theorem 1.1 recovers the formula (1.1) (see Remark 5.23).
1.3. **Categorical wall-crossing formula.** In [NN11 Theorem 3.15], Nagao-Nakajima derived the formula (1.1) by proving wall-crossing formula for stable perverse coherent systems on $X$. Under a derived equivalence of $X$ with a non-commutative crepant resolution of the conifold [VdB04], the category of perverse coherent systems on $X$ is equivalent to the category of representations of the following quiver with super-potential $(Q^\dagger, W)$

$$Q^\dagger = \begin{array}{c}
\bullet_\infty \\
\bullet_0 \\
\bullet_1 \\
\bullet_2
\end{array}
\quad \quad W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1. $$

For $v = (v_0, v_1) \in \mathbb{Z}_{\geq 0}^2$, we denote by $\mathcal{M}^\dagger_Q(v)$ the $\mathbb{C}^*$-rigidified moduli stack of $Q^\dagger$-representations with dimension vector $(1, v_0, v_1)$, where $1$ is the dimension vector at $\infty$. It is equipped with a super-potential

$$w = \text{Tr}(W): \mathcal{M}^\dagger_Q(v) \to \mathbb{A}^1$$

whose critical locus is isomorphic to the moduli stack of $(Q^\dagger, W)$-representations with dimension vector $(1, v_0, v_1)$. There is also a stability parameter $\theta = (\theta_0, \theta_1) \in \mathbb{R}^2$ of $(Q^\dagger, W)$-representations, whose wall-chamber structure is given by the following picture (see [NN11 Figure 1]):

![Wall-chamber structures](image)

**Figure 1. Wall-chamber structures**

For $m \in \mathbb{Z}_{\geq 1}$, there is a wall in the second quadrant in the above picture

$$W_m := \mathbb{R}_{>0}(1 - m, m) \subset \mathbb{R}^2.$$  

We take a stability condition on the wall $\theta \in W_m$ and $\theta_\pm = \theta \pm (-\varepsilon, \varepsilon)$ for $\varepsilon > 0$ which lie on its adjacent chambers. Let $\text{DT}^\theta_{\pm}(v_0, v_1) \in \mathbb{Z}$ be the DT invariant counting $\theta_\pm$-stable $(Q^\dagger, W)$-representations with dimension vector $(1, v_0, v_1)$. We have the following wall-crossing formula proved in [NN11 Theorem 3.12]

$$\sum_{(v_0, v_1) \in \mathbb{Z}_{\geq 0}^2} \text{DT}^\theta_{\pm}(v_0, v_1) q_0^{v_0} q_1^{v_1} = \left( \sum_{(v_0, v_1) \in \mathbb{Z}_{\geq 0}^2} \text{DT}^\theta_{\pm}(v_0, v_1) q_0^{v_0} q_1^{v_1} \right) \cdot (1 + q_0^m (-q_1)^{m-1})^m. $$

The formula (1.1) is obtained from the above wall-crossing formula by applying it from $m = 1$ to $m \gg 0$, and noting that the PT invariants correspond to a chamber which is sufficiently close to (and above) the wall $\mathbb{R}_{>0}(-1, 1)$. 
We prove Theorem 1.1 by giving a categorification of the formula (1.4). For \( \theta \in \mathbb{R}^2 \), we denote by 
\[
\mathcal{M}_Q^{\theta, \text{ss}}(v) \subset \mathcal{M}_Q^\dagger(v)
\]
the open substack of \( \theta \)-semistable \( Q^\dagger \)-representations. The following is the main result of this paper, which gives a categorification of the formula (1.4):

**Theorem 1.2.** (Corollary 5.17) For \( \theta \in W_m \), by setting \( s_m = (m, m - 1) \), there exists a semiorthogonal decomposition
\[
\text{MF}(\mathcal{M}_Q^{\theta_+, \text{ss}}(v), w) = \left\langle \text{copies of } \text{MF}(\mathcal{M}_Q^{\theta_{-}, \text{ss}}(v - l s_m), w) : l \geq 0 \right\rangle.
\]

There is also a precisely defined order among semiorthogonal summands in (1.2), see Corollary 5.17 for the precise statement. Again by taking the periodic cyclic homologies and the Euler characteristics, the result of Theorem 1.2 recovers the Nagao-Nakajima formula (1.4) (see Remark 5.18). The result of Theorem 1.1 follows by applying Theorem 1.2 from \( m = 1 \) to \( m \gg 0 \).

1.4. Outline of the proof of Theorem 1.2. The strategy of the proof of Theorem 1.2 is to use the following ingredients:

(i) The window subcategories for GIT quotient stacks developed by Halpern-Leistner [HL15] and Ballard-Favero-Katzarkov [BFK19].

(ii) The categorized Hall products for quivers with super-potentials introduced and studied by Pădurariu [Pădu, Pădb, Pădc].

(iii) The descriptions of derived categories under Grassmannian flips by Ballard-Chidambaram-Favero-McFaddin-Vandermolen [BCF+21], which itself relies on an earlier work by Donovan-Segal [DS14] for Grassmannian flops.

For \( \theta \in W_m \), let \( \mathcal{M}_Q^{\theta_+, \text{ss}}(v) \rightarrow \mathcal{M}_Q^{\theta_{-}, \text{ss}}(v) \) be the good moduli space [Alp13]. We have the wall-crossing diagram
\[
\begin{array}{ccc}
\mathcal{M}_Q^{\theta_+, \text{ss}}(v) & \xrightarrow{\text{wall-crossing}} & \mathcal{M}_Q^{\theta_{-}, \text{ss}}(v) \\
\downarrow & & \downarrow \\
\mathcal{M}_Q^{\theta_{-}, \text{ss}}(v) & & \mathcal{M}_Q^{\theta_{-}, \text{ss}}(v)
\end{array}
\]

which is shown to be a flip of smooth quasi-projective varieties. The D/K principle by Bondal-Orlov [BO] and Kawamata [Kaw02] predicts the existence of a fully-faithful functor of their derived categories or categories of matrix factorizations.

The window subcategories have been used to investigate the D/K conjecture under variations of GIT quotients. In the above setting, there exist subcategories (called **window subcategories**) \( \mathcal{W}_\text{glob}(v) \subset \text{MF}(\mathcal{M}_Q^{\theta_+, \text{ss}}(v), w) \) such that the compositions
\[
\mathcal{W}_\text{glob}(v) \hookrightarrow \text{MF}(\mathcal{M}_Q^{\theta_+, \text{ss}}(v), w) \rightarrow \text{MF}(\mathcal{M}_Q^{\theta_{-}, \text{ss}}(v), w)
\]
are equivalences. If we can show that \( \mathcal{W}_\text{glob}(v) \subset \mathcal{W}_\text{glob}(v) \) for some choice of window subcategories, then we have a desired fully-faithful functor
\[
\text{MF}(\mathcal{M}_Q^{\theta_{-}, \text{ss}}(v), w) \hookrightarrow \text{MF}(\mathcal{M}_Q^{\theta_+, \text{ss}}(v), w).
\]

In fact, the above argument is used in [Todb, Theorem 4.3.5] to show the existence of a fully-faithful functor (1.7).
We are interested in the semiorthogonal complement of the fully-faithful functor (1.7). If the wall-crossing is enough simple, e.g. satisfying the DHT condition in [BCF19] Definition 4.1.4, then the above window subcategory argument also describes the semiorthogonal complement (see [BCF19] Theorem 4.2.1). However our wall-crossing (1.6) does not necessarily satisfy the DHT condition, and we cannot directly apply it. Instead we use categorified Hall products to describe the semiorthogonal complement of (1.7).

The categorified Hall product for quivers with super-potentials is introduced in [Padc] in order to give a K-theoretic version of critical COHA, which was introduced in [KS11] and developed in [Dav17]. For $v = v_1 + v_2$ with $\theta(v_1) = 0$, it is a functor
\[ * : \text{MF}(\mathcal{M}_Q^{θ,ss}(v_1), w) \boxtimes \text{MF}(\mathcal{M}_Q^{θ,ss}(v_2), w) \to \text{MF}(\mathcal{M}_Q^{θ,ss}(v), w) \]
defined by the pull-back/push-forward with respect to the stack of short exact sequences of $Q^l$-representations. We will show that, for $l \geq 0$ and a sequence of integers $0 \leq j_1 \leq \cdots \leq j_l \leq m - l$, the categorified Hall product gives a fully-faithful functor
\[ (1.8) \quad \boxtimes_{i=1}^l \text{MF}(\mathcal{M}_Q^{θ,ss}(s_m), w)_{j_i+(2i-1)(m^2-m)} \boxtimes (\mathcal{W}^{θ,\text{glob}}(v - ls_m) \otimes \chi_{0}^{ji+2l(m^2-m)}) \to \mathcal{W}^{θ,\text{glob}}(v) \]
whose essential images form a semiorthogonal decomposition. Here the subscript $j_i+(2i-1)(m^2-m)$ indicates the fixed $\mathbb{C}^*$-weight part, and $\chi_0$ is some character regarded as a line bundle on $\mathcal{M}_Q^l(v)$ (see Theorem 5.1.16 for details). It follows that the categorified Hall products describe the semiorthogonal complement of (1.7), which lead to a proof of Theorem 1.2.

In order to show that the functor (1.8) is fully-faithful and they form a semiorthogonal decomposition, we prove these statements formally locally on the good moduli space $\mathcal{M}_Q^{θ,ss}(v)$ at any point $p$ corresponding to a $θ$-polystable $(Q^l, W)$-representation $R$. By the étale slice theorem, one can describe the formal fibers of the diagram (1.6) at $p$ in terms of a wall-crossing diagram of the Ext-quiver $Q^l$ associated with $R$, which is much simpler than $Q^l$. After removing a quadratic part of the super-potential, one observes that the wall-crossing diagram for $Q^l$-representations is the product of a Grassmannian flip with some trivial part. Here a Grassmannian flip is a birational map
\[ G_{a,b}^+(d) \to G_{a,b}^-(d) \]
given by two GIT stable loci of the quotient stack
\[ G_{a,b}(d) = \{(\text{Hom}(A, V) \oplus \text{Hom}(V, B))/\text{GL}(V)\} \]
where $d = \dim V$, $a = \dim A$, $b = \dim B$ with $a \geq b$.

Donovan-Segal [DS14] proved a derived equivalence $D^b(G_{a,b}^-(d)) \simeq D^b(G_{a,b}^+(d))$ in the case of $a = b$ (i.e. Grassmannian flop) using window subcategories, and the same argument also applies to construct a fully-faithful functor $D^b(G_{a,b}^-(d)) \to D^b(G_{a,b}^+(d))$. However it is in a rather recent work [BCF+21] where the semiorthogonal complement of the above fully-faithful functor is considered. We will interpret the description of semiorthogonal complement in [BCF+21] in terms of categorified Hall products, and refine it as a semiorthogonal decomposition (see Corollary 4.18)
\[ (1.9) \quad D^b(G_{a,b}^+(d)) = \left\{(a-b)\left(\begin{array}{c} l \\ m \end{array}\right)\right. \text{-copies of } D^b(G_{a,b}^-(d-l)) : 0 \leq l \leq d \right\}. \]

The above semiorthogonal decomposition unifies Kapranov’s exceptional collections of derived categories of Grassmannians, and also semiorthogonal decompositions of standard toric flips, so it may be of independent interest (see Remark 4.19 Remark 4.20).

A semiorthogonal decomposition similar to (1.9) also holds for categories of factorizations of a super-potential of $G_{a,b}(d)$. Under the Knörrer periodicity, we compare global categorified Hall products (1.8) with local categorified Hall products giving the semiorthogonal decomposition (1.9). By combining these arguments, we see that the functor (1.8) is fully-faithful and they form a semiorthogonal decomposition formally locally on $\mathcal{M}_Q^{θ,ss}(v)$, hence they also hold globally.
1.5. Related works. The wall-crossing formula (1.4) was proved by Nagao-Nakajima [NN11] in order to give an understanding of the product expansion formula of non-commutative DT invariants of the conifold studied by Szendrői [Sze08]. The wall-crossing formula (1.4) was later extended to the case of a global flopping contraction in [Tod13, Cal16], and to the motivic DT invariants in [MMNS12]. Recently Tasuki Kinjo studies cohomological DT theory on the resolved conifold and proves a cohomological version of DT/PT correspondence in this case [Kin]. It would be interesting to extend the argument in this paper and categorify his cohomological DT/PT correspondence.

As we already mentioned, the study of wall-crossing of categorical PT invariants was posed in [Todb]. In the case of local surfaces, a categorical wall-crossing formula is conjectured in [Todb, Conjecture 6.2.6] in the case of simple wall-crossing, and proved in some cases in [Todb, Theorem 6.3.19] using Porta-Sala categorified Hall products for surfaces [PS]. The wall-crossing we consider in this paper is not necessary simple, so it is beyond the cases we considered in [Todb, Conjecture 6.2.6]. A similar wall-crossing at \((-1,-1)\)-curve is also considered in [Todc, Section 7], but we only proved the existence of fully-faithful functors and their semiorthogonal complements are not considered.

The categorified (K-theoretic) Hall algebras for quivers with super-potentials was introduced and studied by Tudor Pădurariu [Păd1, Păd2]. He also proved the PBW theorem for K-theoretic Hall algebras [Păd1, Păd2] via much more sophisticated combinatorial arguments (based on earlier works [vVdB17, HLS20]). We expect that his arguments proving the K-theoretic PBW theorem can be applied to prove categorical (or K-theoretic) wall-crossing formula in a broader setting, including DT/PT wall-crossing in this paper.

Recently Qingyuan Jiang [Jia] studies derived categories of Quot schemes of locally free quotients, and proposed conjectural semiorthogonal decompositions of them (see [Jia, Conjecture A.5]). He proved the above conjecture in the case of rank two quotients. His conjectural semiorthogonal decompositions resemble the one in Theorem 1.2. It would be interesting to see whether the technique in this paper can be applied to his conjecture.

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1.7. Notation and Convention. In this paper, all the schemes or stacks are defined over \(\mathbb{C}\). For an Artin stack \(\mathcal{Y}\), we denote by \(\mathcal{D}^b(\mathcal{Y})\) the bounded derived category of coherent sheaves on \(\mathcal{Y}\). For an algebraic group \(G\) and its representation \(V\), we regard it as a vector bundle on \(BG\). For a variety \(Y\) on which \(G\) acts, we denote by \(V \otimes \mathcal{O}_{[Y/G]}\) the vector bundle given by the pull-back of \(V\) by \([Y/G] \to BG\). For a morphism \(\mathcal{M} \to M\) from a stack \(\mathcal{M}\) to a scheme \(M\) and a closed point \(y \in M\), the formal fiber at \(y\) is defined by

\[ \mathcal{M}_y := \mathcal{M} \times_M \text{Spec} \mathcal{O}_{M,y} \to \tilde{M}_y := \text{Spec} \mathcal{O}_{M,y}. \]

For a triangulated category \(\mathcal{D}\), its triangulated subcategory \(\mathcal{D}' \subset \mathcal{D}\) is called dense if any object in \(\mathcal{D}\) is a direct summand of an object in \(\mathcal{D}'\).

2. Preliminary

In this section, we review triangulated categories of factorizations, the window theorem for categories of factorizations over GIT quotient stacks, and the Knörrer periodicity.

2.1. The category of factorizations. Let \(\mathcal{Y}\) be a noetherian algebraic stack over \(\mathbb{C}\) and take \(w \in \Gamma(\mathcal{O}_Y)\). A (coherent) factorization of \(w\) consists of

\[ \mathcal{P}_0 \overset{\alpha_0}{\longrightarrow} \mathcal{P}_1, \quad \alpha_0 \circ \alpha_1 = \cdot w, \quad \alpha_1 \circ \alpha_0 = \cdot w, \]

where
where each \( P_i \) is a coherent sheaf on \( \mathcal{V} \) and \( \alpha_i \) are morphisms of coherent sheaves. The category of coherent factorizations naturally forms a dg-category, whose homotopy category is denoted by \( \text{HMF}(\mathcal{V}, w) \). The subcategory of absolutely acyclic objects

\[ \text{Acy}^{\text{abs}} \subset \text{HMF}(\mathcal{V}, w) \]

is defined to be the minimum thick triangulated subcategory which contains totalizations of short exact sequences of coherent factorizations of \( w \). The triangulated category of factorizations of \( w \) is defined by (cf. [Orl12, EP15, PV11])

\[ \text{MF}(\mathcal{V}, w) := \text{HMF}(\mathcal{V}, w)/\text{Acy}^{\text{abs}}. \]

If \( \mathcal{V} \) is an affine scheme, then \( \text{MF}(\mathcal{V}, w) \) is equivalent to Orlov’s triangulated category of matrix factorizations of \( w \) [Orl09]. For two pairs \((\mathcal{V}_1, w_1), (\mathcal{V}_2, w_2)\) for \( i = 1, 2 \), we use the notation

\[ \text{MF}(\mathcal{V}_1, w_1) \boxtimes \text{MF}(\mathcal{V}_2, w_2) := \text{MF}(\mathcal{V}_1 \times \mathcal{V}_2, w_1 + w_2). \]

It is well-known that \( \text{MF}(\mathcal{V}, w) \) only depends on an open neighborhood of \( \text{Crit}(w) \subset \mathcal{V} \). Namely, let \( \mathcal{V}' \subset \mathcal{V} \) be an open substack such that \( \text{Crit}(w) \subset \mathcal{V}' \). Then the restriction functor gives an equivalence (see [PV11, Corollary 5.3], [HLS20, Lemma 5.5])

\[ \text{MF}(\mathcal{V}, w) \cong \text{MF}(\mathcal{V}', w|_{\mathcal{V}'}). \tag{2.1} \]

Suppose that \( \mathcal{V} = [Y/G] \) where \( G \) is an algebraic group which acts on a scheme \( Y \). Assume that \( \mathbb{C}^* \subset G \) lies in the center of \( G \) which acts on \( Y \) trivially. Then \( \text{MF}(\mathcal{V}, w) \) decomposes into the direct sum

\[ \text{MF}(\mathcal{V}, w) = \bigoplus_{j \in \mathbb{Z}} \text{MF}(\mathcal{V}, w)_j \tag{2.2} \]

where each summand corresponds to the \( \mathbb{C}^* \)-weight \( j \)-part.

### 2.2. Attracting loci

Let \( G \) be a reductive algebraic group with maximal torus \( T \), which acts on a smooth affine scheme \( Y \). We denote by \( M \) the character lattice of \( T \) and \( N \) the cocharacter lattice of \( T \). There is a perfect pairing

\[ \langle -, - \rangle : N \times M \to \mathbb{Z}. \]

For a one parameter subgroup \( \lambda : \mathbb{C}^* \to G \), let \( Y^{\lambda \geq 0}, Y^{\lambda = 0} \) be defined by

\[ Y^{\lambda \geq 0} := \{ y \in Y : \lim_{t \to 0} \lambda(t)(y) \text{ exists} \}, \]

\[ Y^{\lambda = 0} := \{ y \in Y : \lambda(t)(y) = y \text{ for all } t \in \mathbb{C}^* \}. \]

The Levi subgroup and the parabolic subgroup

\[ G^{\lambda = 0} \subset G^{\lambda \geq 0} \subset G \]

are also similarly defined by the conjugate \( G \)-action on \( G \), i.e. \( g \cdot (-) = g(-)g^{-1} \). The \( G \)-action on \( Y \) restricts to the \( G^{\lambda \geq 0} \)-action on \( Y^{\lambda \geq 0} \), and the \( G^{\lambda = 0} \)-action on \( Y^{\lambda = 0} \). We note that \( \lambda \) factors through \( \lambda : \mathbb{C}^* \to G^{\lambda = 0} \), and it acts on \( Y^{\lambda = 0} \) trivially. So we have the decomposition into \( \lambda \)-weight spaces

\[ D^b([Y^{\lambda = 0}/G^{\lambda = 0}]) = \bigoplus_{j \in \mathbb{Z}} D^b([Y^{\lambda = 0}/G^{\lambda = 0}])_{\lambda\text{-wt} = j}. \]

We have the diagram of attracting loci

\[ \begin{array}{ccc} [Y^{\lambda \geq 0}/G^{\lambda \geq 0}] & \xrightarrow{p_{\lambda}} & [Y/G] \\ \sigma_j \downarrow \scriptstyle{\varphi_j} & & \downarrow \scriptstyle{\varphi_{\lambda}} \\ [Y^{\lambda = 0}/G^{\lambda = 0}] \end{array} \tag{2.3} \]
Moreover by setting the slope to be $1$ for each $1$ of (2.4) $Z$, a closed subset of $[HL15, Section 2.1]$. Let $Y$, $G$ be as in the previous subsection. For an element $l \in \text{Pic}([Y/G]_R)$, we have the open subset of $l$-semistable points

$$Y^{l-ss} \subset Y$$

characterized by the set of points $y \in Y$ such that for any one parameter subgroup $\lambda: \mathbb{C}^* \to G$ such that the limit $z = \lim_{t \to 0} \lambda(t)(y)$ exists in $Y$, we have $\text{wt}(l|_z) \geq 0$. Let $|*|$ be the Weyl-invariant norm on $N_R$. The above subset of $l$-semistable points fits into the Kempf-Ness (KN) stratification

$$Y = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_N \sqcup Y^{l-ss}.$$ (2.4)

Here for each $1 \leq i \leq N$ there exists a one parameter subgroup $\lambda_i: \mathbb{C}^* \to T \subset G$, an open and closed subset $Z_i$ of $(Y \setminus \cup_{i' < i} S_{i'})^{\lambda_i = 0}$ (called center of $S_i$) such that

$$S_i = G \cdot Y_i, \quad Y_i := \{ y \in Y^{\lambda_i \geq 0} : \lim_{t \to 0} \lambda_i(t)(y) \in Z_i \}.$$

Moreover by setting the slope to be

$$\mu_i := -\frac{\text{wt}(l|_{Z_i})}{|\lambda_i|} \in \mathbb{R}$$

we have the inequalities $\mu_1 > \mu_2 > \cdots > 0$. We have the following diagram (see [HL15 Definition 2.2])

$$[Y_i/G^{\lambda_i \geq 0}] \xrightarrow{\xi_i} [S_i/G] \xrightarrow{q_i} [(Y \setminus \cup_{i' < i} S_{i'})/G] \xrightarrow{\tau_i} [Z_i/G^{\lambda_i = 0}].$$ (2.5)

Here the left vertical arrow is given by taking the $t \to 0$ limit of the action of $\lambda_i(t)$ for $t \in \mathbb{C}^*$, and $\tau_i, q_i$ are induced by the embedding $Z_i \hookrightarrow Y$, $S_i \hookrightarrow Y$ respectively. Let $\eta_i \in \mathbb{Z}$ be defined by

$$\eta_i := \text{wt}_{\lambda_i}(\det(N_{S_i/Y}^Y|_{Z_i})).$$ (2.6)

In the case that $Y$ is a $G$-representation, it is also written as

$$\eta_i = \langle \lambda_i, (Y^\vee)^{\lambda_i > 0} - (\mathfrak{g}_{Y}^\vee)^{\lambda_i > 0} \rangle.$$
Theorem 2.2. ([HL13 BFK19]) For each $i$, we take $m_i \in \mathbb{R}$. For $N' \leq N$, let
\begin{equation}
\mathcal{W}_{m_i}^d([(Y \setminus \cup_{1 \leq i \leq N'} S_i)/G]) \subset D^b([(Y \setminus \cup_{1 \leq i \leq N'} S_i)/G])
\end{equation}
be the subcategory of objects $\mathcal{P}$ satisfying the condition
\begin{equation}
\tau_i^*(\mathcal{P}) \in \bigoplus_{j \in [m_i, m_i + \pi_i]} D^b([Z_i/G^{\lambda_i=0}])_{\lambda_i, \text{wt}=j}
\end{equation}
for all $N' < i \leq N$. Then the composition functor
\[ \mathcal{W}_{m_i}^d([(Y \setminus \cup_{1 \leq i \leq N'} S_i)/G]) \to D^b([(Y \setminus \cup_{1 \leq i \leq N'} S_i)/G]) \to D^b([Y^{\text{I-ss}}/G])\]
is an equivalence.

Let $w: Y \to \mathbb{A}^1$ be a $G$-invariant function. We will apply Theorem 2.2 for a KN stratification of $\text{Crit}(w)$
\[ \text{Crit}(w) = S'_1 \sqcup S'_2 \sqcup \cdots \sqcup S'_N \sqcup \text{Crit}(w)^{\text{I-ss}} \]
in the following way. After discarding KN strata $S_i \subset Y$ with $\text{Crit}(w) \cap S_i = \emptyset$, the above stratification is obtained by restricting a KN stratification \[\mathbf{2.4}\] for $Y$ to $\text{Crit}(w)$. Let $\lambda_i: \mathbb{C}^* \to G$ be a one parameter subgroup for $S'_i$ with center $Z'_i \subset S'_i$. We define $Y_i \subset Y$ to be the union of connected components of the $\lambda_i$-fixed part of $Y$ which contains $Z'_i$, and $\bar{Y}_i \subset Y$ is the set of points $y \in Y$ with $\lim_{t \to 0} \lambda_i(t)y \in Z'_i$. Similarly to \[\mathbf{2.5}\], we have the diagram
\[ \begin{array}{ccc}
\bar{Y}_i/G^{\lambda_i=0} & \overset{\pi}{\longrightarrow} & \bar{Y}_i^{\lambda_i=0}/G^{\lambda_i=0} \\
\downarrow & & \downarrow \\
[0,1] & \overset{\pi}{\longrightarrow} & [0,1].
\end{array} \]
Here the left horizontal arrows are open and closed immersions. Using the equivalence \[\mathbf{2.1}\], we also have the following version of window theorem for factorization categories (see [Todc Subsection 2.4])

Theorem 2.3. For each $i$, we take $m_i \in \mathbb{R}$. For $N' \leq N$, let
\[ \mathcal{W}_{m_i}^d([(Y \setminus \cup_{1 \leq i \leq N'} S_i)/G], w) \subset \text{MF}([(Y \setminus \cup_{1 \leq i \leq N'} S_i)/G], w) \]
be the subcategory consisting of factorizations $(\mathcal{P}, d\mathcal{P})$ such that
\begin{equation}
(\mathcal{P}, d\mathcal{P})|_{([Z_i \setminus \cup_{1 \leq i \leq N'} S_i^j)/G^{\lambda_i=0}]} \in \bigoplus_{j \in [m_i, m_i + \pi_i]} \text{MF}([Z_i \setminus \cup_{1 \leq i \leq N'} S_i^j)/G^{\lambda_i=0}], w|_{Z_i})_{\lambda_i, \text{wt}=j}
\end{equation}
for all $N' < i \leq N$. Here $\pi_i = \text{wt}_{\lambda_i} = \det(L_{\lambda_i})^{\text{tr}}_{Z_i}$. Then the composition functor
\[ \mathcal{W}_{m_i}^d([(Y \setminus \cup_{1 \leq i \leq N'} S_i)/G], w) \to \text{MF}([(Y \setminus \cup_{1 \leq i \leq N'} S_i)/G], w) \to \text{MF}([Y^{\text{I-ss}}/G], w) \]
is an equivalence.

2.4. Knörrer periodicity. Let $Y$ be a smooth affine scheme and $G$ be an affine algebraic group which acts on $Y$. Let $W$ be a $G$-representation, which determines a vector bundle $W \to Y := [Y/G]$. Given a function $w: Y \to \mathbb{A}^1$, we have another function on the total space of $W \oplus W^v$
\[ w + q: W \oplus W^v \to \mathbb{A}^1, \quad q(x, x') = \langle x, x' \rangle. \]
We have the following diagram
\[ \begin{array}{ccc}
W^N & \overset{i}{\longrightarrow} & W \oplus W^v \\
\downarrow{\text{w}} & & \downarrow{\text{w+q}} \\
Y & \overset{w}{\longrightarrow} & \mathbb{A}^1.
\end{array} \]
Here $i(x) = (0, x)$. The following is a version of the Knörrer periodicity (cf. [Hir17a Theorem 4.2]):
**Theorem 2.4.** The following composition functor is an equivalence

\[ \Phi := i_* \text{pr}^*: \text{MF}(\mathcal{Y}, w)^{\text{pr}^*} \rightarrow \text{MF}(\mathcal{W}^\vee, w)^{\text{pr}^*} \rightarrow \text{MF}(\mathcal{W} \oplus \mathcal{W}^\vee, w + q). \]

The equivalence \((2.10)\) is given by taking the tensor product over \(\mathcal{O}_\mathcal{Y}\) with the following factorization of \(q\) on \(\mathcal{W} \oplus \mathcal{W}^\vee\)

\[ i_* \mathcal{O}_{\mathcal{W}^\vee} \rightarrow 0. \]

The above factorization is isomorphic to the Koszul factorization of \(q\) on \(\mathcal{W} \oplus \mathcal{W}^\vee\), which is of the form (see [BFK14, Proposition 3.20])

\[ \bigwedge^{\text{even}} \mathcal{W}^\vee \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{\mathcal{W} \oplus \mathcal{W}^\vee} \rightarrow \bigwedge^{\text{odd}} \mathcal{W}^\vee \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{\mathcal{W} \oplus \mathcal{W}^\vee}. \]

Let \(\lambda: \mathbb{C}^* \rightarrow G\) be a one parameter subgroup. We have the following diagrams of attracting loci

\[ \begin{array}{ccc}
\mathcal{Y}^{\lambda \geq 0} & & \mathcal{Y}^\lambda, \\
\mathcal{W}^{\lambda \geq 0} & \xrightarrow{p^{\lambda}} & \mathcal{W}^\lambda \oplus \mathcal{W}^\vee. \\
\mathcal{Y}^{\lambda = 0} & \xrightarrow{w^{\lambda = 0}} & \mathcal{A}^1, \\
\mathcal{W}^{\lambda = 0} & \xrightarrow{w^{\lambda = 0}} & \mathcal{A}^1.
\end{array} \]

Note that we have equivalences

\[ \Phi^{\lambda = 0}: \text{MF}(\mathcal{Y}^{\lambda = 0}, w^{\lambda = 0}) \sim \text{MF}((\mathcal{W} \oplus \mathcal{W}^\vee)^{\lambda = 0}, w^{\lambda = 0} + q^{\lambda = 0}), \]

\[ \Phi^{\lambda \geq 0}: \text{MF}(\mathcal{Y}^{\lambda \geq 0}, w^{\lambda \geq 0}) \sim \text{MF}((\mathcal{W} \oplus \mathcal{W}^\vee)^{\lambda \geq 0}, w^{\lambda \geq 0} + q^{\lambda \geq 0}) \]

by applying Theorem 2.3 for \(\mathcal{W}^{\lambda = 0} \rightarrow \mathcal{Y}^{\lambda = 0}, \mathcal{W}^{\lambda \geq 0} \rightarrow \mathcal{Y}^{\lambda \geq 0}\) respectively.

**Proposition 2.5.** The following diagram commutes:

\[ \begin{array}{ccc}
\text{MF}(\mathcal{Y}^{\lambda = 0}, w^{\lambda = 0}) & \xrightarrow{p^{\lambda} \cdot q^{\lambda}_*} & \text{MF}(\mathcal{Y}, w) \\
\Phi^{\lambda = 0} \circ (\det \mathcal{W}^{\lambda \geq 0})^{\vee} [\dim W^{\lambda > 0}] & \xrightarrow{p^{\lambda}_* \cdot q^{\lambda}_*} & \text{MF}(\mathcal{W} \oplus \mathcal{W}^\vee, w + q).
\end{array} \]

**Proof.** We have the following diagram

\[ \begin{array}{ccc}
\mathcal{W}^{\lambda = 0} & \xrightarrow{q^{\lambda}} & \mathcal{W}^{\lambda \geq 0} \\
\mathcal{Y}^{\lambda = 0} & \xrightarrow{q^{\lambda}} & \mathcal{Y}^{\lambda \geq 0} \\
\mathcal{Y}^{\lambda = 0} & \xrightarrow{p^{\lambda}} & \mathcal{Y}.
\end{array} \]
Here each horizontal diagrams are diagrams of attracting loci, and vertical arrows are projections. From the above diagram, we construct the following diagram

\[(W \oplus W^\vee)^{\lambda \geq 0} \xrightarrow{r_1} W^{\lambda \geq 0} \oplus p_\lambda^* W^\vee \xrightarrow{f_2} W \oplus W^\vee\]

Here \((f_1, f_2)\) is induced by the top horizontal diagram in (2.12), \((g_1, g_2)\) is induced by the duals of the morphisms of vector bundles

\[q_\lambda^* W^{\lambda=0} \leftarrow W^{\lambda \geq 0} \rightarrow p_\lambda^* W\]

on \(Y^{\lambda \geq 0}\), and \((r_1, r_2)\) is induced by the diagram of attracting loci \((W^\vee)^{\lambda=0} \leftarrow (W^\vee)^{\lambda \geq 0} \rightarrow W^\vee\) for \(W^\vee\).

By applying Lemma 6.1 for the right square of (2.12) (and also noting that \(p_\lambda, f_2\) are proper), we have the commutative diagram:

\[\begin{array}{ccc}
\text{MF}(Y^{\lambda \geq 0}, w^{\lambda \geq 0}) & \xrightarrow{p_\lambda} & \text{MF}(W, w) \\
\phi_{\lambda \geq 0} & & \phi \\
\text{MF}(W^{\lambda \geq 0} \oplus (W^{\lambda \geq 0})^\vee, w^{\lambda \geq 0} + q^{\lambda \geq 0}) & \xrightarrow{f_2 \circ g_2} & \text{MF}(W \oplus W^\vee, w + q). 
\end{array}\]

Similarly, by applying Lemma 6.2 for the left square of (2.12), we have the commutative diagram:

\[\begin{array}{ccc}
\text{MF}(Y^{\lambda=0}, w^{\lambda=0}) & \xrightarrow{q_\lambda} & \text{MF}(Y^{\lambda \geq 0}, w^{\lambda \geq 0}) \\
\phi_{\lambda=0} & & \phi_{\lambda \geq 0} \\
\text{MF}((W \oplus W^\vee)^{\lambda=0}, w^{\lambda=0} + q^{\lambda=0}) & \xrightarrow{g_1 \circ f_1} & \text{MF}(W^{\lambda \geq 0} \oplus (W^{\lambda \geq 0})^\vee, w^{\lambda \geq 0} + q^{\lambda \geq 0}). 
\end{array}\]

Note that we have

\[g_{1!}(-) = g_1*(- \otimes f_1^* \mathfrak{p} \otimes \text{det}(W^{\lambda > 0})^{\vee} \text{dim} W^{\lambda > 0}).\]

Here \(\mathfrak{p}: (W \oplus W^\vee)^{\lambda=0} \rightarrow Y^{\lambda=0}\) is the projection. By the diagram (2.13) and the base change, we have the isomorphism of functors

\[f_2 \circ g_2 \circ g_1 \circ f_1 \cong p^{\lambda}_\lambda \circ q^{\lambda}_\lambda: \text{MF}(W^{\lambda \geq 0} \oplus (W^{\lambda \geq 0})^\vee, w^{\lambda \geq 0} + q^{\lambda \geq 0}) \rightarrow \text{MF}(W \oplus W^\vee, w + q).\]

Therefore the proposition holds.

3. CATEGORIZED HALL PRODUCTS FOR QUIVERS WITH SUPER-POTENTIALS

In this section, we review categorified Hall products for quivers with super-potentials introduced in [Paďc Paďa].
3.1. Moduli stacks of representations of quivers. A quiver consists of data $Q = (Q_0, Q_1, s, t)$, where $Q_0$, $Q_1$ are finite sets and $s, t$: $Q_1 \to Q_0$ are maps. The set $Q_0$ is the set of vertices, $Q_1$ is the set of edges, and $s, t$ are maps which assign source and target of each edge. A $Q$-representation consists of data

$$V = \{(V_i, u_e) : i \in Q_0, u_e \in \text{Hom}(V_{s(e)}, V_{t(e)})\}$$

where each $V_i$ is a finite dimensional vector space. The dimension vector $v(V)$ of $V$ is $(\dim V_i)_{i \in Q_0}$. For $v = (v_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$, let $R_Q(v)$ be the vector space

$$R_Q(v) = \bigoplus_{e \in Q_1} \text{Hom}(V_{s(e)}, V_{t(e)})$$

where $\dim V_i = v_i$. The algebraic group $G(v) := \prod_{i \in Q_0} GL(V_i)$ acts on $R_Q(v)$ by conjugation. The stack of $Q$-representations of dimension vector $v$ is given by the quotient stack

$$\mathcal{M}_Q(v) := [R_Q(v)/G(v)].$$

We discuss King’s $\theta$-stability condition on $Q^1$-representations [Kin94]. We take

$$\theta = (\theta_i)_{i \in Q_0} \in \mathbb{R}^{Q_0}.$$

For a dimension vector $d \in \mathbb{Z}_{\geq 0}^{Q_0}$, we set $\theta(v) = \sum_{i \in Q_0} \theta_i v_i$. For a $Q$-representation $V$, we set $\theta(V) := \theta(v(V))$.

**Definition 3.1.** A $Q$-representation $V$ is called $\theta$-(semi)stable if $\theta(V) = 0$ and for any non-zero subobject $V' \subsetneq V$ we have $\theta(V') < (\leq) 0$.

There is an open substack

$$\mathcal{M}^{\theta, \text{ss}}_Q(v) \subset \mathcal{M}_Q(v)$$

corresponding to $\theta$-semistable representations. By [Kin94 Proposition 3.1], if each $\theta_i$ is an integer, the above open substack corresponds to the GIT semistable locus with respect to the character

$$\chi_\theta: G(v) \to \mathbb{C}^*, \ (g_i)_{i \in Q_0} \mapsto \prod_{i \in Q_0} \det g_i^{-\theta_i}.$$

By taking the GIT quotient, it admits a good moduli space [Apa13]

$$\pi_M: \mathcal{M}^{\theta, \text{ss}}_Q(v) \to \mathcal{M}^{\theta, \text{ss}}_Q(v)$$

such that each closed point of $\mathcal{M}^{\theta, \text{ss}}_Q(v)$ corresponds to a $\theta$-polystable $Q$-representation.

Let $(a_i, b_i) \in \mathbb{Z}_{\geq 0}^2$ be a pair of non-negative integers for each vertex $i \in Q_0$, and take $c \in \mathbb{Z}_{\geq 0}$. We define the extended quiver $Q^1$ so that its vertex set is $\{\infty\} \cup Q_0$, with edges consist of edges in $Q$ and

$$\sharp(\infty \to i) = a_i, \ \sharp(i \to \infty) = b_i, \ \sharp(\infty \to \infty) = c.$$

The $\mathbb{C}^*$-rigidified moduli stack of $Q^1$-representations of dimension vector $(1, d)$ is given by

$$\mathcal{M}^{1, \theta, \text{ss}}_Q(v) := [R_{Q^1}(1, v)/G(v)]$$

where 1 is the dimension vector at $\infty$. Note that there is a natural morphism $\mathcal{M}_{Q^1}(1, v) \to \mathcal{M}_Q(v)$ which is a trivial $\mathbb{C}^*$-gerbe. For $\theta = (\theta_\infty, \theta_i)_{i \in Q_0}$ with $\theta(1, v) = 0$, the open substack of $\theta$-semistable representations

$$\mathcal{M}^{1, \theta, \text{ss}}_Q(v) \subset \mathcal{M}^1_Q(v)$$

is defined in a similar way. The condition $\theta(1, v) = 0$ determines $\theta_\infty$ by $\theta_\infty = -\sum_{i \in Q_0} \theta_i v_i$, so we just write $\theta = (\theta_i)_{i \in Q_0}$.
3.2. Categorified Hall products. For a dimension vector \( v \in \mathbb{Z}_{\geq 0}^Q \), let us take a decomposition
\[
v = v^{(1)} + \cdots + v^{(l)}, \quad v^{(j)} \in \mathbb{Z}_{\geq 0}.
\]

Let \( V_i = \oplus_{j=1}^l V_i^{(j)} \) be a direct sum decomposition such that \( \{V_i^{(j)}\}_{i \in Q_0} \) has dimension vector \( v^{(j)} \).

We take integers \( \lambda^{(1)} > \cdots > \lambda^{(l)} \), and a one parameter subgroup \( \lambda : \mathbb{C}^* \to G(v) \) which acts on \( V_i^{(j)} \) by weight \( \lambda^{(j)} \). We have the stack of attracting loci
\[
\mathcal{M}_Q(v^\bullet) := [R_Q(v)^{\lambda \geq 0}/G(v)^{\lambda \geq 0}].
\]

The above stack is isomorphic to the stack of filtrations of \( Q \)-representations
\[
(3.2) \quad 0 = \mathcal{V}(0) \subset \mathcal{V}(1) \subset \cdots \subset \mathcal{V}(v) = \mathcal{V}
\]
such that each \( \mathcal{V}^{(j)}/\mathcal{V}^{(j-1)} \) has dimension vector \( v^{(j)} \). Moreover we have
\[
\prod_{j=1}^l \mathcal{M}_Q(v^{(j)}) = [R_Q(v)^{\lambda = 0}/G(v)^{\lambda = 0}]
\]
and we have the diagram of attracting loci (see Subsection 2.3)
\[
(3.3) \quad \mathcal{M}_Q(v^\bullet) \xymatrix{\ar[r]^{p_\lambda} & \mathcal{M}_Q(v) \
\prod_{j=1}^l \mathcal{M}_Q(v^{(j)}) \ar[u]_{q_\lambda}}.
\]

Here \( p_\lambda \) sends a filtration \( 3.2 \) to \( \mathcal{V} \), and \( q_\lambda \) sends a filtration \( 3.2 \) to its associated graded \( Q \)-representation. Since \( p_\lambda \) is proper, we have the functor (called categorified Hall product)
\[
(3.4) \quad p_\lambda \circ q_\lambda^* : \bigotimes_{j=1}^l D^b(\mathcal{M}_Q(v^{(j)})) \to D^b(\mathcal{M}_Q(v)).
\]

For \( \mathcal{E}^{(j)} \in D^b(\mathcal{M}_Q(v^{(j)})) \), we set
\[
\mathcal{E}^{(1)} \star \cdots \star \mathcal{E}^{(l)} := p_\lambda \circ q_\lambda^*(\mathcal{E}^{(1)} \boxtimes \cdots \boxtimes \mathcal{E}^{(l)}).
\]

The above \( \star \)-product is associative, i.e.
\[
(\mathcal{E}^{(1)} \star \mathcal{E}^{(2)}) \star \mathcal{E}^{(3)} \cong \mathcal{E}^{(1)} \star (\mathcal{E}^{(2)} \star \mathcal{E}^{(3)}) \cong \mathcal{E}^{(1)} \star \mathcal{E}^{(2)} \star \mathcal{E}^{(3)}.
\]

We take \( \theta = (\theta_i)_{i \in Q_0} \) such that \( \theta(v^{(j)}) = 0 \) for all \( j \). Then the diagram \( 3.3 \) restricts to the diagram
\[
(3.5) \quad \mathcal{M}^{\theta_{ss}}_Q(v^\bullet) \xymatrix{\ar[r]^{p_\lambda} & \mathcal{M}^{\theta_{ss}}_Q(v) \
\prod_{j=1}^l \mathcal{M}^{\theta_{ss}}_Q(v^{(j)}) \ar[u]_{q_\lambda}}.
\]

which is a diagram of attracting loci for \( \mathcal{M}^{\theta_{ss}}_Q(v) \). Similarly we have the functor
\[
p_\lambda \circ q_\lambda^* : \bigotimes_{j=1}^l D^b(\mathcal{M}^{\theta_{ss}}_Q(v^{(j)})) \to D^b(\mathcal{M}^{\theta_{ss}}_Q(v)),
\]
which coincides with \( 3.3 \), when \( \theta = 0 \).

Similarly applying the above construction for the extended quiver \( Q^\dagger \), for a decomposition
\[
v = v^{(1)} + \cdots + v^{(l)} + v^{(\infty)}, \quad v^{(j)} \in \mathbb{Z}_{\geq 0}
\]
such that \( \theta(v^{(j)}) = 0 \) for \( 1 \leq j \leq l \), we have the functor
\[
\bigotimes_{j=1}^l D^b(\mathcal{M}^{\theta_{ss}}_Q(v^{(j)})) \otimes D^b(\mathcal{M}^{\theta_{ss}}_Q(v^{(\infty)})) \to D^b(\mathcal{M}^{\theta_{ss}}_Q(v))
\]
which gives a left action of \( \bigoplus_{v} D^b(\mathcal{M}^{\theta_{ss}}_Q(v)) \) on \( \bigoplus_{v} D^b(\mathcal{M}^{\theta_{ss}}_Q(v)) \).
3.3. Categorified Hall products for quivers with super-potentials. Let $W$ be a super-potential of a quiver $Q$, i.e. $W \in \mathbb{C}[Q]/[\mathbb{C}[Q], \mathbb{C}[Q]]$ where $\mathbb{C}[Q]$ is the path algebra of $Q$. Then there is a function

(3.8) \[ w := \text{Tr}(W) : \mathcal{M}_Q(v) \to \mathbb{A}^1 \]

whose critical locus is identified with the moduli stack of $(Q,W)$-representations $\mathcal{M}_{(Q,W)}(v)$, i.e. $Q$-representations satisfying the relation $\partial W$.

The diagram (3.8) is extended to the diagram

(3.9) \[ \begin{array}{ccc}
\mathcal{M}^{\beta-ss}(v^\bullet) & \xrightarrow{p_{\lambda}} & \mathcal{M}^{\beta-ss}(v) \\
\pi_{\lambda} \downarrow & & \downarrow w \\
\prod_{j=1}^l \mathcal{M}^{\beta-ss}(v^{(j)}) & \xrightarrow{\sum_{j=1}^l w^{(j)}} & \mathbb{A}^1.
\end{array} \]

Here $w^{(j)}$ is the function (3.8) on $\mathcal{M}_Q(v^{(j)})$. Similarly to (3.4), we have the functor between triangulated categories of factorizations

\[ p_{\lambda} \circ q_{\lambda}^* : \bigoplus_{j=1}^l \text{MF}(\mathcal{M}^{\beta-ss}(v^{(j)}), w^{(j)}) \to \text{MF}(\mathcal{M}^{\beta-ss}(v), w), \]

called the categorified Hall products for representations of quivers with super-potentials.

The super-potential naturally defines the super-potential of the extended quiver $Q^\dagger$, so we have the regular function $w : \mathcal{M}_Q(v) \to \mathbb{A}^1$ as in (3.8). Similarly to (3.7), for a decomposition (3.6) we have the left action

(3.10) \[ \bigoplus_{j=1}^l \text{MF}(\mathcal{M}^{\beta-ss}(v^{(j)}), w^{(j)}) \cong \text{MF}(\mathcal{M}^{1, \beta-ss}(v^{(\infty)}), w^{(\infty)}) \to \text{MF}(\mathcal{M}^{1, \beta-ss}(v), w). \]

Note that we have the decomposition (2.22) with respect to the diagonal torus $\mathbb{C}^* \subset G(v)$

\[ \text{MF}(\mathcal{M}^{\beta-ss}(v), w) = \bigoplus_{j \in \mathbb{Z}} \text{MF}(\mathcal{M}^{\beta-ss}(v), w)_j. \]

We will often restrict the functor (3.10) to the fixed weight spaces

\[ \bigoplus_{j=1}^l \text{MF}(\mathcal{M}^{\beta-ss}(v^{(j)}), w^{(j)}) \cong \text{MF}(\mathcal{M}^{1, \beta-ss}(v^{(\infty)}), w^{(\infty)}) \to \text{MF}(\mathcal{M}^{1, \beta-ss}(v), w). \]

3.4. Base change to formal fibers. Later we will take a base change of the categorified Hall product to a formal neighborhood of a point in the good moduli space (3.11). Note that the diagram (3.9) extends to the commutative diagram

(3.11) \[ \begin{array}{ccc}
\mathcal{M}^{\beta-ss}(v^\bullet) & \xrightarrow{p_{\lambda}} & \mathcal{M}^{\beta-ss}(v) \\
\pi_{\lambda} \downarrow & & \downarrow w \\
\prod_{j=1}^l \mathcal{M}^{\beta-ss}(v^{(j)}) & \xrightarrow{\sum_{j=1}^l w^{(j)}} & \mathbb{A}^1.
\end{array} \]

Here the bottom arrow is the morphism taking the direct sum of $\theta$-polystable representations which is a finite morphism (see [MR19, Lemma 2.1]), and the left bottom vertical arrow is the good moduli space morphism. For a closed point $p \in M^{\beta-ss}(v)$, we consider the following formal fiber

\[ \widehat{M}^{\beta-ss}(v)_p := \mathcal{M}^{\beta-ss}(v) \times_{M^{\beta-ss}(v)} \widehat{M}^{\beta-ss}(v)_p \to \widehat{M}^{\beta-ss}(v)_p := \text{Spec} \widehat{O}_{\mathcal{M}^{\beta-ss}(v),p}. \]
Let \((p^{(1)}, \ldots, p^{(l)}) \in \prod_{j=1}^{l} M_{Q}^{\theta, ss}(v^{(j)})\) be a point such that \(\oplus (p^{(1)}, \ldots, p^{(l)}) = p\). By taking the fiber product of the diagram (3.11) by \(\widehat{M}_{Q}^{\theta, ss}(v) \to M_{Q}^{\theta, ss}(v)\), we obtain the diagram

\[
\begin{array}{c}
\prod_{p^{(j)} \in \oplus^{-1}(p)} \prod_{j=1}^{l} \widehat{M}_{Q}^{\theta, ss}(v^{(j)})_{p^{(j)}} \downarrow \prod_{p^{(j)} \in \oplus^{-1}(p)} \prod_{j=1}^{l} \widehat{M}_{Q}^{\theta, ss}(v^{(j)})_{p^{(j)}} \downarrow \prod_{p^{(j)} \in \oplus^{-1}(p)} \prod_{j=1}^{l} \widehat{M}_{Q}^{\theta, ss}(v^{(j)})_{p^{(j)}} \\
\end{array}
\]

The above diagram is a diagram of attracting loci for \(\widehat{M}_{Q}^{\theta, ss}(v)\) (see [Toda Lemma 4.11]). By the derived base change, we have the commutative diagram

\[
\begin{array}{c}
\bigotimes_{j=1}^{l} D^b(\mathcal{M}_{Q}^{\theta, ss}(v^{(j)})) \xrightarrow{\oplus_{p^{(j)} \in \oplus^{-1}(p)} \bigotimes_{j=1}^{l} D^b(\widehat{M}_{Q}^{\theta, ss}(v^{(j)})_{p^{(j)}})} D^b(\mathcal{M}_{Q}^{\theta, ss}(v)) \\
\end{array}
\]

Here the vertical arrows are pull-backs to formal fibers.

We denote by \(\widehat{\omega}_p : \widehat{M}_{Q}^{\theta, ss}(v) \to A^1\) the pull-back of the function \(3.8\) to the formal fiber. Similarly to (3.12), we have the commutative diagram

\[
\begin{array}{c}
\bigotimes_{j=1}^{l} \text{MF}(\mathcal{M}_{Q}^{\theta, ss}(v^{(j)}), w) \xrightarrow{\oplus_{p^{(j)} \in \oplus^{-1}(p)} \bigotimes_{j=1}^{l} \text{MF}(\widehat{M}_{Q}^{\theta, ss}(v^{(j)})_{p^{(j)}}, \widehat{\omega}_{p^{(j)}})} \text{MF}(\mathcal{M}_{Q}^{\theta, ss}(v)) \\
\end{array}
\]

4. Derived categories of Grassmannian flips

In this section, we use categorified Hall products to refine the result of [BCF+21 Theorem 5.4.4] on variation of derived categories under Grassmannian flips.

4.1. Grassmannian flips. Let \(V\) be a vector space with dimension \(d\), and \(A, B\) be another vector spaces such that

\[a := \dim A, \quad b := \dim B, \quad a \geq b.\]

We form the following quotient stack

\[(4.1) \quad \mathcal{G}_{a,b}(d) := [(\text{Hom}(A, V) \oplus \text{Hom}(V, B)) / \text{GL}(V)].\]

**Remark 4.1.** The stack \(\mathcal{G}_{a,b}(d)\) is the \(\mathbb{C}^*\)-rigified moduli stack of representations of the quiver \(Q_{a,b}\) of dimension vector \((1, d)\), where the vertex set is \(\{\infty, 1\}\), the number of arrows from \(\infty\) to \(1\) is \(a\), that from \(1\) to \(\infty\) is \(b\), and there are no self loops (see Subsection 3.1), e.g. the quiver \(Q_{3,2}\) is described below:

\[(4.2) \quad Q_{3,2} = \bullet \xrightarrow{\infty} \bullet \xrightarrow{1} \bullet\]

Below we fix a basis of \(V\), and take the maximal torus \(T \subset \text{GL}(V)\) to be consisting of diagonal matrices. For a one parameter subgroup \(\lambda : \mathbb{C}^* \to T\), we use the following notation for the diagram
of attracting loci.

\[(4.3)\]

\[\mathcal{G}_{a,b}(d)^{\lambda \geq 0} \xrightarrow{\psi_\lambda} \mathcal{G}_{a,b}(d) \]

\[\mathcal{G}_{a,b}(d)^{\lambda = 0} \]

We use the following determinant character

\[(4.4)\]

\[\chi_0 : \text{GL}(V) \to \mathbb{C}^*, \ g \mapsto \det(g),\]

and often regard it as a line bundle on \(\mathcal{G}_{a,b}(d)\). There exist two GIT quotients with respect to \(\chi_0^{\pm 1}\) given by open substacks

\[G_{a,b}^{\pm}(d) \subset \mathcal{G}_{a,b}(d).\]

Here \(\chi_0\)-semistable locus \(G_{a,b}^+(d)\) consists of \((\alpha, \beta) \in \text{Hom}(A, V) \oplus \text{Hom}(V, B)\) such that \(\alpha : A \to V\) is surjective, and \(\chi_0^{-1}\)-semistable locus \(G_{a,b}^-(d)\) consists of \((\alpha, \beta)\) such that \(\beta : V \to B\) is injective. We have the following diagram

\[(4.5)\]

\[\mathcal{G}_{a,b}(d) \xrightarrow{G_{a,b}^+(d)} \mathcal{G}_{a,b}(d) \xrightarrow{\mathcal{G}_{a,b}^-(d)} \mathcal{G}_{a,b}^0(d).\]

Here the middle vertical arrow is the good moduli space for \(\mathcal{G}_{a,b}(d)\).

**Remark 4.2.** When \(a \geq d\) and \(b = 0\), then \(G_{a,0}(d) = \emptyset\) and \(G_{a,0}^+(d)\) is the Grassmannian parameterizing surjections \(A \to V\). If \(a \geq b \geq d\), then \(G_{a,b}^\pm(d)\) are birational and \(G_{a,b}^+(d) \to G_{a,b}^0(d)\) is a flip \((a > b)\), flop \((a = b)\).

We have the KN stratifications with respect to \(\chi_0^{\pm 1}\)

\[G_{a,b}(d) = S^+_0 \sqcup S^+_1 \sqcup \cdots \sqcup S^+_{d-1} \sqcup G^+_a(d)\]

where \(S_i^+\) consists of \((\alpha, \beta)\) such that the image of \(\alpha : A \to V\) has dimension \(i\), and \(S_i^-\) consists of \((\alpha, \beta)\) such that the kernel of \(\beta : V \to B\) has dimension \(d - i\). The associated one parameter subgroups \(\lambda_i^{\pm} : \mathbb{C}^* \to T\) are taken as (see [HL13 Example 4.12])

\[(4.6)\]

\[\lambda_i^+(t) = (1, \ldots, 1, t^{-1}, \ldots, t^{-1}), \quad \lambda_i^-(t) = (t, \ldots, t, 1, \ldots, 1).\]

**4.2. Window subcategories for Grassmannian flips.** We fix a Borel subgroup \(B \subset \text{GL}(V)\) to be consisting of upper triangular matrices, and set roots of \(B\) to be negative roots. Let \(M = \mathbb{Z}^d\) be the character lattice for \(\text{GL}(V)\), and \(M^+_R \subset M_R\) the dominant chamber. By the above choice of negative roots, we have

\[M^+_R = \{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_1 \leq x_2 \leq \cdots \leq x_d \}.\]

We set \(M^+ = M^+_R \cap M\). For \(c \in \mathbb{Z}\), we set

\[(4.7)\]

\[\mathbb{B}_c(d) := \{ (x_1, x_2, \ldots, x_d) \in M^+ : 0 \leq x_i \leq c - d \}.\]

Here \(\mathbb{B}_c(d) = \emptyset\) if \(c < d\). For \(\chi \in \mathbb{B}_c(d)\), we assign the Young diagram whose number of boxes at the \(i\)-th row is \(x_{d-i+1}\). The above assignment identifies \(\mathbb{B}_c(d)\) with the set of Young diagrams with height less than or equal to \(d\), width less than or equal to \(c - d\). For example, the following picture illustrates the case of \((3, 7, 7, 10, 15) \in \mathbb{B}_c(d)\) for \(d = 5\) and \(c \geq 20\):
Figure 2. $(3, 7, 7, 10, 15) \in \mathbb{B}_c(d), d = 5, c \geq 20$

We define the triangulated subcategory

\begin{equation}
\mathcal{W}_c(d) \subset D^b(G_{a,b}(d))
\end{equation}

to be the smallest thick triangulated subcategory which contains $V(\chi) \otimes \mathcal{O}_{G_{a,b}(d)}$ for $\chi \in \mathbb{B}_c(d)$. Here $V(\chi)$ is the irreducible $\text{GL}(V)$ representation with highest weight $\chi$, i.e. it is a Schur power of $V$ associated with the Young diagram corresponding to $\chi$. The following proposition is well-known (see [DS14, Proposition 3.6]), which give window subcategories for Grassmannian flips. We reprove it here using Theorem 2.2:

**Proposition 4.3.** The following composition functors are equivalences

\begin{equation}
\mathcal{W}_b(d) \subset D^b(G_{a,b}(d)) \rightarrow D^b(G_{a,b}^{-}(d))
\end{equation}

\begin{equation}
\mathcal{W}_a(d) \subset D^b(G_{a,b}(d)) \rightarrow D^b(G_{a,b}^{+}(d)).
\end{equation}

**Proof.** We only prove the statement for +. Let $\lambda_i^+$ be the one parameter subgroup in (4.6). Then $\eta_i^+$ given in (2.6) is

\begin{equation}
\eta_i^+ = (\lambda_i^+, (\text{Hom}(A, V)^{\vee} \oplus \text{Hom}(V, B)^{\vee})^{\lambda_i^+ > 0} - \text{End}(V)^{\lambda_i^+ > 0})
\end{equation}

\begin{equation}
= (a - i)(d - i).
\end{equation}

Let $\chi' = (x'_1, \ldots , x'_d)$ be a $T$-weight of $V(\chi)$ for $\chi \in \mathbb{B}_a(d)$. Then we have $0 \leq x'_j \leq a - d$ for $1 \leq j \leq d$, so

\begin{equation}
-\eta_i^+ < -(a - d)(d - i) \leq (\chi', \lambda_i^+) = - \sum_{j=i+1}^{d} x'_j \leq 0.
\end{equation}

Therefore by setting $m_i = -\eta_i^+ + \varepsilon$ for $0 < \varepsilon \ll 1$ and $l = \chi_0$ in (2.7), we have

\begin{equation}
\mathcal{W}_a(d) \subset \mathcal{W}_{\chi_0}(G_{a,b}(d)) \subset D^b(G_{a,b}(d)).
\end{equation}

It follows that the second composition functor in (4.9) is fully-faithful.

In order to show that it is essentially surjective, note that the projection to $\text{Hom}(A, V)$ defines a morphism

\begin{equation}
G_{a,b}^+(d) \rightarrow \text{Gr}(a, d)
\end{equation}

where $\text{Gr}(a, d)$ is the Grassmannian which parametrizes $d$-dimensional quotients of $A$. By the above morphism, $G_{a,b}^+(d)$ is the total space of a vector bundle over $\text{Gr}(a, d)$. The objects $V(\chi) \otimes \mathcal{O}_{G_{a,b}(d)}$ for $\chi \in \mathbb{B}_a(d)$ restricted to the zero section of (4.10) forms Kapranov’s exceptional collection [Kap84]. Since $D^b(G_{a,b}^+(d))$ is generated by pull-backs of objects from $D^b(Gr(a, d))$, the essentially surjectivity of (4.9) holds. □
4.3. Resolutions of categorified Hall products. Let \( d = d^{(1)} + \cdots + d^{(l)} + d^{(\infty)} \) be a decomposition of \( d \). Note that from Subsection 5.1, we have the categorified Hall product

\[
\boxtimes_{j=1}^l D^b(B, GL(d^{(j)})) \boxtimes D^b(G_{a,b}(d^{(\infty)})) \to D^b(G_{a,b}(d)).
\]

In particular by setting \( d^{(1)} = 1 \) and \( d^{(\infty)} = d - 1 \), we have the functor

\[
\ast : D^b(B C^*) \boxtimes D^b(G_{a,b}(d - 1)) \to D^b(G_{a,b}(d)).
\]

It is explicitly given as follows. Let \( \lambda : \mathbb{C}^* \to T \) be given by

\[
\lambda(t) = (t, 1, \ldots, 1).
\]

Then we have the decomposition \( V = V^{\lambda > 0} \oplus V^{\lambda = 0} \) where \( V^{\lambda > 0} \) is one dimensional. Then

\[
G_{a,b}(d)^{\lambda = 0} = [B, GL(V^{\lambda > 0})] \times [(\text{Hom}(A, V^{\lambda = 0}) \oplus \text{Hom}(V^{\lambda = 0}, B))/\text{GL}(V^{\lambda = 0})]
\]

\[
= B C^* \times G_{a,b}(d - 1).
\]

The functor \( (4.11) \) is given by \( p_* q^!(-) \) in the diagram \( (4.3) \). The stack \( G_{a,b}(d)^{\lambda \geq 0} \) is the moduli stack of exact sequences of \( Q_{a,b} \)-representations

\[
0 \to \mathbb{V}^{\lambda > 0} \to V \to \mathbb{V}^{\lambda = 0} \to 0
\]

such that \( \mathbb{V}^{\lambda > 0} \) has dimension vector \((0, 1)\). We will often use the following lemmas:

**Lemma 4.4.** For \( \mathcal{E}_1 \in D^b(B C^*) \) and \( \mathcal{E}_2 \in D^b(G_{a,b}(d - 1)) \), we have

\[
(\mathcal{E}_1 * \mathcal{E}_2) \otimes \chi_0^d = (\mathcal{E}_1 \otimes \mathcal{O}_{B C^*}(j)) * (\mathcal{E}_2 \otimes \chi_0^d).
\]

Here we have used the same symbol \( \chi_0 \) for the determinant character of \( GL(V^{\lambda = 0}) \).

**Proof.** The lemma follows since \( p^!_\lambda \chi_0 = \mathcal{O}_{B C^*}(1) \otimes \chi_0 \) and the definition of the functor \( (4.11) \). \( \square \)

**Lemma 4.5.** For \( \mathcal{E} \in \mathcal{W}_c(d) \) and \( j \geq 0 \), we have \( \mathcal{E} \otimes \chi_0^d \in \mathcal{W}_{c+j}(d) \).

**Proof.** The lemma follows since \( V(\chi) \otimes \chi_0^d = V(\chi') \) where \( \chi' = \chi + (j, j, \ldots, j) \). \( \square \)

The following proposition is essentially proved in [DS14, BCF+21], which we interpret in terms of categorified Hall products:

**Proposition 4.6.** [DS14 Theorem A.7, BCF+21 Proposition 5.4.6] For \( \chi \in \mathbb{B}_c(d - 1) \) with \( c \geq b \), let \( \delta \) be the corresponding Young diagram. Then the object \( \mathcal{O}_{B C^*} * (V(\chi) \otimes \mathcal{O}_{G_{a,b}(d - 1)}) \) is a sheaf which fits into an exact sequence

\[
0 \to V(\chi_K) \otimes \mathcal{O}_{G_{a,b}(d)}^{\oplus m_K} \to \cdots \to V(\chi_1) \otimes \mathcal{O}_{G_{a,b}(d)}^{\oplus m_1} \to V(\chi) \otimes \mathcal{O}_{G_{a,b}(d)} \to \mathcal{O}_{B C^*} * (V(\chi) \otimes \mathcal{O}_{G_{a,b}(d - 1)}) \to 0.
\]

Here \( \chi \in \mathbb{B}_c(d - 1) \) is regarded as an element of \( \mathbb{B}_{c+1}(d) \) by \( (x_2, \ldots, x_d) \mapsto (0, x_2, \ldots, x_d) \), and each \( \chi_i \in \mathbb{B}_{c+1}(d) \) in \( (4.14) \) corresponds to a Young diagram \( \delta_i \) obtained from \( \delta \) by the following algorithm (see Figure 3):

- The Young diagram \( \delta_1 \) is obtained from \( \delta \) by adding boxes to the first column until it reaches to height \( d \).
- \( \delta_i \) is obtained from \( \delta_{i-1} \) by adding boxes to the \( i \)-th column until its height is one more than the height of the \((i - 1)\)-th column of \( \delta \).

Moreover \( m_i = \dim \chi_i B \) for \( s_i = |\delta_i| - |\delta| \), and the sequence \( (4.14) \) terminates when we reach a positive integer \( K \) such that \( s_{K+1} > b \).

**Proof.** Let \( \lambda \) be the one parameter subgroup \( (4.12) \). Then we have

\[
(\text{Hom}(A, V) \oplus \text{Hom}(V, B))^{\lambda \geq 0} = \text{Hom}(A, V) \oplus \text{Hom}(V^{\lambda = 0}, B)
\]

\[
\cong \text{Hom}(V^\vee, A^\vee) \oplus \text{Hom}(B^\vee, (V^{\lambda = 0})^\vee).
\]
The parabolic subgroup $GL(V)^{λ ≥ 0}$ is the subgroup of $GL(V)$ which preserves $V^{λ > 0} ⊂ V$. Therefore there is an isomorphism of quotient stacks

$$\left[ \frac{\text{Hom}(A, V) \oplus \text{Hom}(V, B))^{λ ≥ 0}}{GL(V)^{λ ≥ 0}} \right]$$

$$\cong \left[ \frac{\text{Hom}(V^λ, A^λ) \oplus \text{Hom}(B^λ, (V^{λ = 0})^λ) \oplus \text{Hom}^{\text{inj}}((V^{λ = 0})^λ, V^λ)/ GL(V) \times GL(V^{λ = 0})} {GL(V^{λ = 0})} \right].$$

Here $\text{Hom}^{\text{inj}}((V^{λ = 0})^λ, V^λ) \subset \text{Hom}(V^{λ = 0})^λ, V^λ)$ is the subset consisting of injective homomorphisms. The above isomorphism is induced by the embedding into the direct summand $(V^{λ = 0})^λ \hookrightarrow V^λ$ together with the natural inclusion $GL(V)^{λ ≥ 0} \rightarrow GL(V) \times GL(V^{λ = 0})$. Under the above isomorphism, the morphism

$$p_λ : \left[ \frac{\text{Hom}(A, V) \oplus \text{Hom}(V, B))^{λ ≥ 0}}{GL(V)^{λ ≥ 0}} \right] \rightarrow \mathcal{G}_{a, b}(d)$$

from the diagram (5.3) is identified with the one

$$\left[ \frac{\text{Hom}(V^λ, A^λ) \oplus \text{Hom}(B^λ, (V^{λ = 0})^λ) \oplus \text{Hom}^{\text{inj}}((V^{λ = 0})^λ, V^λ)/ GL(V) \times GL(V^{λ = 0})} {GL(V)} \right]$$

induced by the composition of maps. The above morphism is nothing but the one considered in [DS14 Theorem A.7], [BCF21 Proposition 5.4.6]. We then directly apply the computation of $p_λ(−)$ for vector bundles given by Schur powers in [DS14 Theorem A.7], [BCF21 Proposition 5.4.6] to obtain the resolution (4,13).

We also check that each $\chi_λ$ in (4,14) is an element of $\mathbb{B}_{c+1}(d)$, i.e. $δ_i$ has at most height $d$ and width $c - d + 1$. It is obvious that $δ_i$ has at most height $d$. Let $μ_j$ be the number of boxes of $δ$ at the $j$-th column. Then from the algorithm defining $δ_i$, we have

$$s_i = (d - μ_1) + (μ_1 + 1 - μ_2) + \cdots (μ_{i-1} + 1 - μ_i) = d + i - 1 - μ_i.$$

Since $\chi_λ ∈ \mathbb{B}_c(d − 1)$, we have $μ_{c-d+2} = 0$, so $s_{c-d+2} = c + 1 > b$. Therefore we have $K ≤ c - d + 1$. Since $δ$ has width at most $c - d + 1$, it follows that $δ_i$ also has width at most $c - d + 1$.

Using the above proposition, we have the following lemma:

**Lemma 4.7.** For $0 ≤ j ≤ c - 1$, we have

$$\mathcal{O}_{BC^*}(j) * (\mathcal{W}_{c−1−j} (d−1) ⊗ \chi_0^j) ⊂ \mathcal{W}_c(d).$$

**Proof.** We have

$$\mathcal{O}_{BC^*}(j) * (\mathcal{W}_{c−1−j} (d−1) ⊗ \chi_0^j) = (\mathcal{O}_{BC^*} * \mathcal{W}_{c−1−j} (d−1)) ⊗ \chi_0^j$$

$$⊂ \mathcal{W}_{c−j} (d) ⊗ \chi_0^j$$

$$⊂ \mathcal{W}_c(d).$$

Here we have used Lemma 4.4 for the first identity, Proposition 4.6 for the first inclusion and Lemma 4.5 for the last inclusion. □
4.4. Generation of window subcategories. We show that for $c \geq b$ the category $\mathcal{W}_c(d)$ is generated by its subcategory $\mathcal{W}_b(d)$ and subcategories (4.15) for $0 \leq j \leq c - b - 1$. We first prove the case of $c = b + 1$, which is a variant of [BCF + 21, Lemma 5.4.9].

**Lemma 4.8.** The subcategory $\mathcal{W}_{b+1}(d) \subset D^b(G_{a,b}(d))$ is generated by $\mathcal{W}_b(d)$ and $\mathcal{O}_{BC\cdot} \ast \mathcal{W}_b(d-1)$.

**Proof.** For $\chi \in \mathbb{B}_{b+1}(d)$, it is enough to show that $V(\chi) \otimes O_{G_{a,b}(d)}$ is generated by $\mathcal{W}_b(d)$ and $\mathcal{O}_{BC\cdot} \ast \mathcal{W}_b(d-1)$. Let $\delta$ be the Young diagram corresponding to $\chi$, and we denote by $\mu_j$ the number of boxes of $\delta$ at the $j$-th column. We may assume that the width of $\delta$ is exactly $b - d + 1$, i.e. $\mu_j \geq 1$ for $1 \leq j \leq b - d + 1$ and $\mu_{b-d+2} = 0$.

Suppose that the height of $\delta$ is exactly $d$, i.e. $\mu_1 = d$. We define another Young diagram $\delta'$ whose number of boxes at the $j$-th column is $\mu_{j+1} - 1$. Then the height of $\delta'$ is at most $d - 1$, and the width of $\delta'$ is at most $b - d$ (see Figure 4). Let $\chi' \in \mathbb{B}_{b+1}(d-1)$ be the character corresponding to $\delta'$. As $\mathbb{B}_{b+1}(d-1) \subset \mathcal{B}(d-1)$, we apply Proposition 4.6 to obtain a resolution

$$0 \to V(\chi'_{K}) \otimes O_{G_{a,b}(d)}^{(m_k)} \to \cdots \to V(\chi'_{1}) \otimes O_{G_{a,b}(d)}^{(m_1)} \to V(\chi') \otimes O_{G_{a,b}(d)} \to \mathcal{O}_{BC\cdot} \ast (V(\chi') \otimes O_{G_{a,b}(d-1)}) \to 0. \quad (4.16)$$

for $\chi'_i \in \mathbb{B}_{b+1}(d)$. Note that we have

$$|\delta| - |\delta'| = (d - \mu_2 + 1) + (\mu_2 - \mu_3 + 1) + \cdots + (\mu_{b-d} - \mu_{b-d+1} + 1) + \mu_{b-d+1} = b.$$

From the construction of $\delta'$, the Young diagram $\delta$ is reconstructed from $\delta'$ by the algorithm in Proposition 4.5 at the $(b - d + 1)$-th step. Therefore from the above identity, it follows that there are exactly $(b - d + 1)$-terms of the resolution (4.16), i.e. $K = b - d + 1$, and also $m_K = 1$, $\chi'_K = \chi$. Moreover since the width of $\delta$ is as most $b - d$, we also have $\chi'_i \in \mathbb{B}_b(d)$ for $0 \leq i < b - d + 1$. Therefore $V(\chi) \otimes O_{G_{a,b}(d)}$ is generated by objects in $\mathcal{W}_b(d)$ and $\mathcal{O}_{BC\cdot} \ast (V(\chi) \otimes O_{G_{a,b}(d-1)}) \in \mathcal{O}_{BC\cdot} \ast \mathcal{W}_b(d-1)$.

Suppose that the height of $\delta$ is less than $d$. Then we have $\chi \in \mathbb{B}_b(d-1)$. By applying Proposition 4.6 we see that $V(\chi) \otimes O_{G_{a,b}(d)}$ is generated by $\mathcal{O}_{BC\cdot} \ast (V(\chi) \otimes O_{G_{a,b}(d-1)}) \in \mathcal{O}_{BC\cdot} \ast \mathcal{W}_b(d-1)$ and $V(\chi) \otimes O_{G_{a,b}(d)}$ for $\chi \in \mathbb{B}_{b+1}(d)$. By the algorithm in Proposition 4.6 the Young diagram corresponding to $\chi$ has a full column, i.e. the height of $\chi$ is exactly $d$. Therefore by the above argument, each $V(\chi) \otimes O_{G_{a,b}(d)}$ is generated by $\mathcal{W}_b(d)$ and $\mathcal{O}_{BC\cdot} \ast \mathcal{W}_b(d-1)$.

We then show the generation for $\mathcal{W}_c(d)$:

**Lemma 4.9.** For $c \geq b$, the subcategory $\mathcal{W}_c(d) \subset D^b(G_{a,b}(d))$ is generated by $\mathcal{W}_b(d)$ and $\mathcal{O}_{BC\cdot}(j) \ast (\mathcal{W}_{c-1-j}(d-1) \otimes \chi_0')$ for $0 \leq j \leq c - b - 1$.

**Proof.** The case of $c = b + 1$ is proved in Lemma 4.8 We prove the lemma for $c > b + 1$ by the induction of $c$. For $\chi \in \mathcal{W}_c(d)$, suppose that the corresponding Young diagram $\delta$ has a full column. Let $\delta''$ be the Young diagram obtained by removing the first column, and $\chi''$ the corresponding character. Then $\chi'' \in \mathcal{W}_{c-1}(d)$, so by the induction hypothesis $V(\chi'') \otimes O_{G_{a,b}(d)}$ is generated by $\mathcal{W}_b(d)$ and $\mathcal{O}_{BC\cdot}(j) \ast (\mathcal{W}_{c-2-j}(d-1) \otimes \chi_0')$ for $0 \leq j \leq c - b - 2$. By taking the tensor product with $\chi_0$ and setting $j' = j + 1$, we see that $V(\chi) \otimes O_{G_{a,b}(d)}$ is generated by $\mathcal{W}_b(d) \otimes \chi_0$ and $\mathcal{O}_{BC\cdot}(j') \ast (\mathcal{W}_{c-1-j'}(d-1) \otimes \chi_0')$ for $1 \leq j' \leq c - b - 1$. Since $\mathcal{W}_b(d) \otimes \chi_0 \subset \mathcal{W}_{b+1}(d)$, and the
latter is generated by $\mathcal{W}_b(d)$ and $O_{BC^*} \ast \mathcal{W}_b(d-1) \subset O_{BC^*} \ast \mathcal{W}_{c-1}(d-1)$ by Lemma 4.8, we have the desired generation for $V(\chi) \otimes O_{G_{a,b}}(d)$ when $\delta$ has a full column.

If $\delta$ does not have a full column, then $\chi \in \mathcal{B}_{c-1}(d-1)$. By applying Proposition 4.6, we see that $V(\chi) \otimes O_{G_{a,b}}(d)$ is generated by $O_{BC^*} \ast (V(\chi) \otimes O_{G_{a,b}}(d-1))$ and $V(\chi_i) \otimes O_{G_{a,b}}(d)$ for $\chi_i \in \mathcal{B}_c(d)$. Since each Young diagram corresponding to $\chi_i$ has a full column, the desired generation of $V(\chi) \otimes O_{G_{a,b}}(d)$ is reduced to the case of the existence of full column which is proved above.

The above generation result is stated in terms of iterated Hall products as follows:

**Proposition 4.10.** For $c \geq b$, the subcategory $\mathcal{W}_c(d) \subset D^b(G_{a,b}(d))$ is generated by the subcategories

\begin{equation}
C_j := O_{BC^*} \ast \cdots \ast O_{BC^*} \ast (\mathcal{W}_b(d-l) \otimes \chi_l^{j_i-j_1}) \subset D^b(G_{a,b}(d))
\end{equation}

for $0 \leq l \leq d$ and $0 \leq j_1 \leq \cdots \leq j_l \leq c-b-l$. Here when $l = 0$, the above subcategory is set to be $\mathcal{W}_b(d)$.

**Proof.** We first show that (4.17) are subcategories of $\mathcal{W}_c(d)$ by the induction on $c$. By Lemma 4.4, the subcategory (4.17) is written as

$$O_{BC^*} \ast \left( (O_{BC^*} \ast (j_2-j_1) \ast \cdots \ast O_{BC^*} \ast (j_l-j_1) \ast (\mathcal{W}_b((d-1)-(l-1)) \otimes \chi_0^{l-j_1}) \otimes \chi_0^j) \right).$$

Since $j_l - j_1 \leq (c-1 - j_1) - b - (l-1)$, by the induction hypothesis we have

$$O_{BC^*} \ast (j_2-j_1) \ast \cdots \ast O_{BC^*} \ast (j_l-j_1) \ast (\mathcal{W}_b((d-1)-(l-1)) \otimes \chi_0^{l-j_1}) \in \mathcal{W}_{c-1-j_1}(d-1).$$

Therefore (4.17) is a subcategory of $\mathcal{W}_c(d)$ by Lemma 4.4.

We next show that $\mathcal{W}_c(d)$ is generated by subcategories (4.17) by the induction on $c$. By Lemma 4.9, the subcategory $\mathcal{W}_c(d)$ is generated by $\mathcal{W}_b(d)$ and $O_{BC^*} \ast (\mathcal{W}_{c-1-j_1}(d-1) \otimes \chi_0^j)$ for $0 \leq j \leq c-b-1$. By the induction hypothesis and Lemma 4.4, $O_{BC^*} \ast (\mathcal{W}_{c-1-j_1}(d-1) \otimes \chi_0^j)$ is generated by

\begin{align*}
O_{BC^*} \ast (j) \ast \left( (O_{BC^*} \ast (j_1) \ast \cdots \ast O_{BC^*} \ast (j_l) \ast (\mathcal{W}_b((d-1)-(l'-1)) \otimes \chi_0^{l-j_1}) \right) \otimes \chi_0^j) \\
= O_{BC^*} \ast (j) \ast O_{BC^*} \ast (j_1) \ast \cdots \ast O_{BC^*} \ast (j_l) \ast (\mathcal{W}_b((d-1)-(l'-1)) \otimes \chi_0^{l+j-l'})
\end{align*}

for $0 \leq l' \leq d-1$ and $0 \leq j_1 \leq \cdots \leq j_l' \leq (c-1-j_1) - b - l'$. Since $j + j_{l'} \leq c - b - (l' - 1)$, the subcategory is of the form (4.17) for $l' = l + 1$. Therefore we obtain the desired generation. \hfill \Box

**Remark 4.11.** Let $F_j(\cdot) := (O_{BC^*} \ast (\cdot) \otimes \chi_0^j = O_{BC^*} \ast (\cdot) \otimes \chi_0^j)$. Then the repeated use of Lemma 4.4 implies that

$$C_j \ast = F_{j_1} \circ F_{j_2-j_1} \circ \cdots \circ F_{j_l-j_{l-1}} (\mathcal{W}_b(d-l)).$$

Similarly for an intermediate step, we have

$$O_{BC^*} \ast (j_1) \ast \cdots \ast O_{BC^*} \ast (j_l) \ast (\mathcal{W}_b(d-l) \otimes \chi_0^j) = F_{j_1} \circ F_{j_2-j_1} \circ \cdots \circ F_{j_l-j_{l-1}} (\mathcal{W}_b(d-l)),$$

By the repeated use of Lemma 4.4, the above category is a subcategory of $\mathcal{W}_b(d-l-i+1+j_i)$ for $i = l + 1$.

**4.5. Semiorthogonal decompositions under Grassmannian flips.** We show that the subcategories in Proposition 4.10 form a semiorthogonal decomposition. We prepare some lemmas:

**Lemma 4.12.** For any $\chi \in \mathcal{B}_b(d)$ and $\chi' \in \mathcal{B}_c(d-1)$ for some $c \geq 0$, we have the vanishing for $j \geq 0$

\begin{equation}
\text{Hom}_{G_{a,b}}(O_{BC^*} \ast (\cdot) \otimes O_{G_{a,b}}(d-1)), V(\chi) \otimes O_{G_{a,b}}(d)) = 0.
\end{equation}

**Proof.** Let $\lambda : C^* \to T$ be the one parameter subgroup given by (4.12). Using the notation of the diagram (4.3), the LHS of (4.18) is

$$\text{Hom}(p_{x-b}(O_{BC^*} \ast (\cdot) \otimes O_{G_{a,b}}(d-1)), V(\chi) \otimes O_{G_{a,b}}(d))$$

\begin{equation}
= \text{Hom}(q_b^*(O_{BC^*} \ast (\cdot) \otimes O_{G_{a,b}}(d-1)), p_b^*(V(\chi) \otimes O_{G_{a,b}}(d))).
\end{equation}
We have the formula for $p_j^\lambda$ (cf. [DS14] Section A.1, [BCF+21] (5.8))

$$p_j^\lambda(-) = (-) \otimes (\det V^{\lambda _{>0}})^{d-b -1} \otimes (\det V^{\lambda _{=0}})^{-1}[d-b -1].$$

Since $\chi \in \mathbb{B}_d(d)$ and it is a highest weight of $V(\chi)$, any $T$-weight $\chi'' = (x_1'', \ldots, x_n'')$ of $V(\chi)$ satisfies $x''_j \leq b - d$. Therefore any $T$-weight of $V(\chi) \otimes (\det V^{\lambda _{>0}})^{d-b -1} \otimes (\det V^{\lambda _{=0}})^{-1}$ pair negatively with $\lambda$. On the other hand, a pairing of $\lambda$ with any $T$-weight of the GL($V$)$^{\lambda =0}$ representation $(\det V^{\lambda _{>0}})^l \boxtimes V(\chi')$ is $j \geq 0$. Therefore we have the vanishing of (4.19) by Lemma 2.1.

□

**Lemma 4.13.** For $\chi, \chi' \in \mathbb{B}_c(d-1)$ for some $c \geq 0$, we have the vanishing for $j > j'$

$$(4.20) \quad \text{Hom}_{\mathcal{G}_{a,b}(d)}(\mathcal{O}_{BC^*}(j) \otimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}), \mathcal{O}_{BC^*}(j') \otimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})) = 0.$$

*Proof.* By Lemma 4.12 we may assume that $j > j'$. We use the notation in the proof of Lemma 4.12 Using Lemma 4.4 and the adjunction, the LHS of (4.20) is

$$\text{Hom}(p_{\lambda *}(q_{\lambda}^*(\mathcal{O}_{BC^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})))) \otimes (\lambda_0^0, p_{\lambda *}(q_{\lambda}^*(\mathcal{O}_{BC^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}))))$$

$$\cong \text{Hom}(p_{\lambda *}(q_{\lambda}^*(\mathcal{O}_{BC^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})))) \otimes (\lambda_0^0, q_{\lambda}^*(\mathcal{O}_{BC^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}))).$$

By Proposition 4.6 the object

$$p_{\lambda *}(q_{\lambda}^*(\mathcal{O}_{BC^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}))) \in D^b(\mathcal{G}_{a,b}(d))$$

is resolved by vector bundles of the form $V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}$ where $\chi''$ is either $\chi'' = \chi$, or $\chi'' \in \mathbb{B}_c(d-1)$ whose corresponding Young diagram has a full column. In the latter case, any $T$-weight of $V(\chi')$ pair positively with $\lambda$. Therefore in both cases, any $T$-weight of $V(\chi')$ for $j > 0$ pair positively with $\lambda$. On the other hand the $\lambda$-weight of $\mathcal{O}_{BC^*} \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})$ is zero so the desired vanishing (4.20) follows from Lemma 2.1. □

**Lemma 4.14.** In the situation of Lemma 4.13, we have the isomorphism for $j \in \mathbb{Z}$

$$(4.21) \quad \text{Hom}_{\mathcal{G}_{a,b}(d-1)}(V(\chi) \otimes \chi_0^0, V(\chi') \otimes \chi_0^0, \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})$$

$$\cong \text{Hom}_{\mathcal{G}_{a,b}(d)}(\mathcal{O}_{BC^*}(j) \otimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}), \mathcal{O}_{BC^*}(j) \otimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})).$$

*Proof.* By Lemma 4.12 we may assume that $j > 0$. Let $\lambda''$ be a weight which appeared in the proof of Lemma 4.13. Note that we observed that any $T$-weight of $V(\chi')$ pair positively with $\lambda$ except $\chi'' = \chi$. Therefore by Lemma 2.1(i), the RHS of (4.21) is isomorphic to

$$(4.22) \quad \text{Hom}(p_{\lambda *}(V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}), q_{\lambda}^*(\mathcal{O}_{BC^*} \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}))).$$

Since $\mathcal{G}_{a,b}(d)^{\lambda \geq 0}$ parametrizes exact sequences (4.13), the object $p_{\lambda *}(V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)})$ admits a filtration whose associated graded is of the form $q_{\lambda}^*(\mathcal{O}_{BC^*}(j) \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}))$ for $j \geq 0$ and $\chi'' \in \mathbb{B}_c(d-1)$, and $j = 0$ if and only if $\chi'' = \chi$. Therefore by Lemma 2.1(i), (ii), the above (4.22) is isomorphic to

$$(4.23) \quad \text{Hom}(q_{\lambda}^*(\mathcal{O}_{BC^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})), q_{\lambda}^*(\mathcal{O}_{BC^*} \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})))$$

$$\cong \text{Hom}_{\mathcal{G}_{a,b}(d-1)}(V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}, V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}).$$

□

In order to state the order of semiorthogonal decompositions, we take a lexicographical order on $\mathbb{Z}^d$, i.e. for $m_* = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ and $m'_* = (m'_1, \ldots, m'_d) \in \mathbb{Z}^d$, we write $m_* > m'_*$ if $m_i > m'_i$ for $1 \leq i \leq k$ for some $k \geq 0$ and $m_{k+1} > m'_{k+1}$.

**Definition 4.15.** For $j_* = (j_1, j_2, \ldots, j_l)$ and $j'_* = (j'_1, j'_2, \ldots, j'_l)$ with $l, l' \leq d$, we define $j_* > j'_*$ if we have $j_*> j'_*$, where $j_*$ is defined by

$$(4.24) \quad j_* = (j_1, j_2, \ldots, j_l, -1, \ldots, -1) \in \mathbb{Z}^d.$$
Corollary 4.18. There exists a semi-orthogonal decomposition

\[ D^b(G^+_{a,b}(d)) = \left\langle D^b(G^-_{a,b}(d-l))_{j_1,\ldots,j_l} : 0 \leq l \leq d, 0 \leq j_i \leq a-b-l \right\rangle. \]

Here \( D^b(G^-_{a,b}(d-l))_{j_1,\ldots,j_l} \) is a copy of \( D^b(G^+_{a,b}(d-l)). \)
we obtain the diagram

Each factor $D^b(\text{Spec } \mathbb{C})_{j_1, \ldots, j_d}$ is generated by a vector bundle which forms Kapranov’s exceptional collection $\text{Kap84}$ of the Grassmannian $G_{a,0}(d)$.}

**Remark 4.20.** When $d = 1$, the birational map $G_{a,b}(1) \to G_{a,b}(1)$ is a standard toric flip. In this case, the semiorthogonal decomposition in Corollary 4.18 is

$$D^b(G_{a,b}(1)) = \langle D^b(G_{a,b}(1)), D^b(pt)_{(0)}, \ldots, D^b(pt)_{(a-b-1)} \rangle.$$ The above semiorthogonal decomposition is a (mutation of) well-known semiorthogonal decomposition for a standard flip (see [Kaw, Example 8.8 (2)]).

**Remark 4.21.** For a fixed $(a,b,l)$, the set of sequences of integers $(j_1, \ldots, j_l)$ satisfying $0 \leq j_1 \leq \cdots \leq j_l \leq a - b - l$ consists of $\binom{a-b}{l} + 1$ elements. Therefore Corollary 4.18 implies (1.9). The same applies to Corollary 5.17, Corollary 5.22 below so that they imply (1.2), (1.3) respectively.

### 4.6. Applications to categories of factorizations

We will use the following variant of Corollary 4.17: Let $Z$ be a smooth scheme with a closed point $z \in Z$. Let us take the formal completion of $G_{a,b}^0(d) \times Z$ where $G_{a,b}^0(d)$ is the good moduli space for $G_{a,b}(d)$,

$$\widehat{G}_{a,b}^0(d)_Z := \text{Spec } \widehat{G}_{a,b}^0(d)_{X, (0, z)}.$$ We also take a regular function $w$ on it

$$w : \widehat{G}_{a,b}^0(d)_Z \to \mathbb{A}^1, w(0, z) = 0.$$ By taking the product of the diagram (4.3) with $Z$ and pulling it back via $\widehat{G}_{a,b}^0(d)_Z \to G_{a,b}^0(d) \times Z$, we obtain the diagram

$$\begin{array}{ccc}
\widehat{G}_{a,b}^+(d)_Z & \xrightarrow{\partial} & \widehat{G}_{a,b}^0(d)_Z \\
\downarrow w & & \downarrow w \\
G_{a,b}^0(d)_Z & \xrightarrow{\partial} & G_{a,b}^0(d)_Z 
\end{array}$$

Similarly to (4.11), we have the categorified Hall product for formal fibers (see Subsection 3.4)

$$* : \text{MF}(B\mathcal{C}^*, 0) \boxtimes \text{MF}(\widehat{G}_{a,b}(d-1)_Z, w) \to \text{MF}(\widehat{G}_{a,b}(d)_Z, w).$$

The subcategory

$$\widehat{W}_c(d) \subset \text{MF}(\widehat{G}_{a,b}(d)_Z, w)$$

is also defined similarly to (4.18) to be the smallest thick triangulated subcategory which contains factorizations with entries $V(\chi) \otimes \mathcal{O}$ for $\chi \in \mathbb{B}_c(d)$. Note that we have the decomposition (2.2)

$$\text{MF}(B\mathcal{C}^*, 0) = \bigoplus_{j \in \mathbb{Z}} \text{MF}(\text{Spec } \mathbb{C}, 0)_j$$

such that $\text{MF}(\text{Spec } \mathbb{C}, 0)_j$ is equivalent to $\text{MF}(\text{Spec } \mathbb{C}, 0)$. We then define

$$\widehat{C}_{j_l} := \text{MF}(\text{Spec } \mathbb{C}, 0)_{j_1} * \cdots * \text{MF}(\text{Spec } \mathbb{C}, 0)_{j_l} * (\widehat{W}_b(d - l) \otimes \chi^{\otimes j_{l+1}}) \subset \text{MF}(\widehat{G}_{a,b}(d)_Z, w))$$

for $0 \leq l \leq d$ and $0 \leq j_1 \leq \cdots \leq j_l \leq c - b - l$. We have the following variant of Theorem 4.17.
Corollary 4.22. For $c \geq b$, there exists a semi-orthogonal decomposition
\[
\mathcal{W}_c(d) = \left\{ \mathcal{C}_j : 0 \leq j \leq d, j_0 = (0 \leq j_1 \leq \cdots \leq j_i \leq c - b - l) \right\}
\]
where \( \text{Hom}(\mathcal{C}_j, \mathcal{C}_{j'}) = 0 \) for \( j_0 > j'_0 \), and for each \( j_0 \) we have an equivalence \( \mathcal{W}_0(d - l) \cong \mathcal{C}_{j_0} \).

Proof. The argument of Theorem 4.17 implies an analogous semiorthogonal decomposition for \( D^b(G_{a,b}(d)_Z) \). Then it is well-known that the above semiorthogonal decomposition induces the one for categories of factorizations (cf. [HLP20, Lemma 1.17, 1.18], [Orl06, Proposition 1.10], [Pâdc, Proposition 2.7], [Pâda, Proposition 2.1]). \qed

5. CATEGORICAL WALL-CROSSING FORMULA FOR THE RESOLVED CONIFOLD

In this section, we use the result in the previous section to prove Theorem 1.2.

5.1. Geometry and algebras for the resolved conifold. Let \( X \) be the resolved conifold
\[ X := \text{Tot}_p(\mathcal{O}_p(-1)^{\oplus 2}). \]
Here we recall some well-known geometry and algebras for the resolved conifold (see [VdB04, NNT1] for details). There is a birational contraction
\[ f: X \to Y := \{ xy + zw = 0 \} \subset \mathbb{C}^4 \]
which contracts the zero section \( C = \mathbb{P}^1 \subset X \) to the conifold singularity \( 0 \in Y \). Let \( \mathcal{E} := \mathcal{O}_X \oplus \mathcal{O}_X(1) \), and \( A := \text{End}(\mathcal{E}) \). Then there is an equivalence by Van den Bergh [VdB04]
\[
\Phi := \text{RHom}(\mathcal{E}, -): D^b(X) \cong D^b(\text{mod} \ A).
\]
Here \( \text{mod} \ A \) is the abelian category finitely generated right \( A \)-modules. The non-commutative algebra \( A \) is isomorphic to the path algebra associated with a quiver with super-potential \( (Q, W) \), given below

\[
Q = \begin{array}{c}
\bullet_0 \\
\circ a_1 \\
\circ a_2 \\
\circ b_1 \\
\circ b_2
\end{array}
\quad W = a_1b_1a_2b_2 - a_1a_2b_1.
\]

The equivalence (5.1) restricts to the equivalences of abelian subcategories
\[ \Phi: \text{Per}(X/Y) \xrightarrow{\sim} \text{mod} \ A, \ \Phi: \text{Per}_c(X/Y) \xrightarrow{\sim} \text{mod}_{id}(A). \]
Here \( \text{Per}(X/Y) \) is the abelian category of Bridgeland’s perverse coherent sheaves [Br02], explicitly given by
\[ \text{Per}(X/Y) = \left\{ E \in D^b(X) : \begin{array}{l}
\mathcal{H}^i(E) = 0 \ for \ i \neq -1, 0, R^1f_*\mathcal{H}^0(E) = f_*\mathcal{H}^{-1}(E) = 0 \\
\text{Hom}(\mathcal{H}^0(E), \mathcal{O}_C(-1)) = 0
\end{array} \right\}. \]
The subcategory \( \text{Per}_c(X/Y) \subset \text{Per}(X/Y) \) consists of compactly supported objects, and \( \text{mod}_{id}(A) \subset \text{mod}(A) \) consists of finite dimensional \( A \)-modules. The simple \( (Q, W) \)-representations corresponding to the vertex \( \{0, 1\} \) are given by
\[ \{ \mathcal{O}_C, \mathcal{O}_C(-1)[1] \} \subset \text{Per}_c(X/Y). \]
An object \( F \in \text{Per}_c(X/Y) \) is supported on \( C \) or zero dimensional subscheme in \( X \). For \( F \in \text{Per}_c(X/Y) \), we set
\[ \text{cl}(F) := (\beta, n) \in \mathbb{Z}^{\oplus 2}, \ [F] = \beta[C], \chi(F) = n \]
where \([F]\) is the fundamental one cycle of \( F \). Under the equivalence \( \Phi \), an object \( F \in \text{Per}_c(X/Y) \) with \( \text{cl}(F) = (\beta, n) \) corresponds to a \( (Q,W) \)-representation with dimension vector \( (n, n - \beta) \).
Following \cite[Section 1]{NN11}, a \textit{perverse coherent system} is defined to be a pair
\begin{equation}
(F, s), \quad F \in \text{Per}_c(X/Y), \quad s : O_X \rightarrow F.
\end{equation}
Let \((Q^\dagger, W)\) be a quiver with super-potential, given below
\[
Q^\dagger = \begin{tikzpicture}
\node (a1) at (0,0) [label=left:a_1] {1};
\node (a2) at (1,0) [label=left:a_2] {2};
\node (b1) at (0,-1) [label=below:b_1] {0};
\node (b2) at (1,-1) [label=below:b_2] {1};
\draw (a1) to [in=180, out=0, looseness=1] (b1);
\draw (a2) to [in=180, out=0, looseness=1] (b2);
\end{tikzpicture}
\]
\begin{equation}
W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1.
\end{equation}

Note that \(Q^\dagger\) is an extended quiver obtained from \(Q\) as in Subsection \cite[5.1]{NN11}. By the equivalence \cite[(5.1)]{NN11},
giving a perverse coherent system with \(\text{cl}(F) = (\beta, n)\) is equivalent to giving a representation of
\((Q^\dagger, W)\) with dimension vector \((v_\infty, v_0, v_1) = (1, n, n - \beta)\).

\subsection{Categorical DT invariants for the resolved conifold.}
For a dimension vector \(v = (v_0, v_1)\) of \(Q\), let \(V_0, V_1\) be vector spaces with dimension \(v_0, v_1\) respectively. The \(\mathbb{C}^*\)-rigidified moduli stack
of \(Q^\dagger\)-representations of dimension vector \((1, v)\) in Subsection \cite[3.1]{NN11} is explicitly written as
\[
\mathcal{M}^\dagger_Q(v) = [R_{Q^\dagger}(v)/G(v)] = \left[ V_0 \oplus \text{Hom}(V_0, V_1)^{\oplus 2} \oplus \text{Hom}(V_1, V_0)^{\oplus 2} / \text{GL}(V_0) \times \text{GL}(V_1) \right].
\]
Let \(w\) be the function
\begin{equation}
w = \text{Tr}(W) : \mathcal{M}^\dagger_Q(v) \rightarrow \mathbb{A}^1, \quad w(v, A_1, A_2, B_1, B_2) = \text{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1).
\end{equation}
Then its critical locus
\begin{equation}
\mathcal{M}^\dagger_{(Q, W)}(v) := \text{Crit}(w) \subset w^{-1}(0) \subset \mathcal{M}^\dagger_Q(v)
\end{equation}
is the \(\mathbb{C}^*\)-rigidified moduli stack of \((Q^\dagger, W)\)-representations of dimension vector \((1, v)\). Here the first inclusion follows from the fact that \(w\) is a homogeneous function on \(R_{Q^\dagger}(v)\) of degree four. By the equivalence \cite[(5.1)]{NN11}, \(\mathcal{M}^\dagger_{(Q, W)}(v)\) is isomorphic to the moduli stack of perverse coherent systems \cite[(5.2)]{NN11}
satisfying \(\text{cl}(F) = (v_0 - v_1, v_0)\).

For \(\theta = (\theta_0, \theta_1) \in \mathbb{R}^2\), we denote by
\[
\mathcal{M}^\dagger_{\theta, \text{ss}}(v) = [R^\theta_{Q^\dagger}(v)/G(v)] \subset \mathcal{M}^\dagger_Q(v)
\]
the open substack of \(\theta\)-semistable \(Q^\dagger\)-representations. We also have the open substack
\[
\mathcal{M}^\dagger_{\theta, \text{ss}}_{(Q, W)}(v) := \mathcal{M}^\dagger_{\theta, \text{ss}}(v) \cap \mathcal{M}^\dagger_{(Q, W)}(v) \subset \mathcal{M}^\dagger_{(Q, W)}(v)
\]
corresponding to \(\theta\)-semistable \((Q^\dagger, W)\)-representations. If \(\theta_i \in \mathbb{Z}\), as mentioned in Subsection \cite[3.1]{NN11},
these open substacks are GIT semistable loci with respect to the character
\begin{equation}
\chi_\theta : G(v) = \text{GL}(V_0) \times \text{GL}(V_1) \rightarrow \mathbb{C}^*, \quad (g_0, g_1) \mapsto \det(g_0)^{-\theta_0} \det(g_1)^{-\theta_1}.
\end{equation}
We have the good moduli spaces by taking GIT quotients
\begin{equation}
\pi^\dagger_Q : \mathcal{M}^\dagger_{\theta, \text{ss}}(v) \rightarrow \mathcal{M}^\dagger_{\theta, \text{ss}}(v), \quad \pi^\dagger_{(Q, W)} : \mathcal{M}^\dagger_{\theta, \text{ss}}_{(Q, W)}(v) \rightarrow \mathcal{M}^\dagger_{(Q, W)}(v).
\end{equation}
We will consider the triangulated category
\[
\text{MF}(\mathcal{M}^\dagger_{\theta, \text{ss}}(v), w)
\]
and call it the \textit{categorical DT invariant} for the conifold quiver \((Q^\dagger, W)\). The above triangulated category (or more precisely its dg-enhancement) recovers the numerical DT invariant considered in \cite[NN11]{NN11}:
Lemma 5.1. For a generic $\theta \in \mathbb{R}^2$, there is an equality
\[ e_{\mathbb{C}[[u]]}(\text{HP}_*(\text{MF}(\mathcal{M}^{1,\theta\text{-ss}}_Q(v), w))) = (-1)^{v_1} \text{DT}^\theta(v). \]

Here $\text{HP}_*(-)$ is the periodic cyclic homology which is a $\mathbb{Z}/2$-graded $\mathbb{C}[[u]]$-vector space (see [Kel99]), $e_{\mathbb{C}[[u]]}(-)$ is the Euler characteristic of $\mathbb{Z}/2$-graded $\mathbb{C}[[u]]$-vector space, and $\text{DT}^\theta(v) \in \mathbb{Z}$ is the numerical DT invariant counting $(Q^1, w)$-representations with dimension vector $(1, v)$.

Proof. Since $\theta$ is generic and the dimension vector $(1, v)$ of $Q^1$ is primitive, the stack $\mathcal{M}^{1,\theta\text{-ss}}_Q(v)$ consists of only $\theta$-stable objects and it is a smooth quasi-projective scheme. By [Bri12, Corollary 1.4.3], there is an isomorphism of $\mathbb{Z}/2$-graded vector spaces over $\mathbb{C}[[u]]$.

\[ \text{HP}_*(\text{MF}(M, w)) \cong H^*(M, \phi_w(Q_M)) \otimes_{\mathbb{C}} \mathbb{C}[[u]] \]
\[ \cong H^{*+\dim M}(M, \phi_w(\text{IC}_M)) \otimes_{\mathbb{C}} \mathbb{C}[[u]]. \]

Here $\phi_w(-)$ is the vanishing cycle functor and $u$ has degree two, and $\text{IC}_M = Q_M[\dim M]$. We take the Euler characteristics of both sides as $\mathbb{Z}/2$-graded vector spaces over $\mathbb{C}[[u]]$. Since we have
\[ e(H^*(M, \phi_w(\text{IC}_M)))) = \int_M \chi_B \, de =: \text{DT}^\theta(v), \]
where $\chi_B$ is the Behrend function [Beh09] on $M$, it is enough to show that $(-1)^{v_1} = (-1)^{\dim M}$. Let $E$ be a $Q^1$-representation with dimension vector $(1, v_0, v_1)$. Then we have
\[ \dim M = 1 + \dim \text{Ext}^1_{Q^1}(E, E) - \dim \text{Hom}_{Q^1}(E, E) \]
\[ = v_0 - v_0^2 - v_1^2 + 4v_0v_1. \]
Here we have used Lemma 5.2 below for the second identity. Therefore $(-1)^{v_1} = (-1)^{\dim M}$ holds. \hfill \Box

The following lemma follows immediately from the Euler pairing computations of quiver representations (see [Bri12, Corollary 1.4.3]):

Lemma 5.2. For $Q^1$-representations $E, E'$ with dimension vector $(v_{\infty}, v_0, v_1)$, $(v'_{\infty}, v'_0, v'_1)$, we have
\[ \dim \text{Hom}_{Q^1}(E, E') - \dim \text{Ext}^1_{Q^1}(E, E') = v_{\infty}v'_{\infty} - v_{\infty}v'_0 + v_0v'_1 - 2v_0v'_1 - 2v_1v'_0 + v_1v'_1. \]

We have the following unstable locus
\[ \mathcal{M}^{1,\theta\text{-us}}_{(Q,W)}(v) := \mathcal{M}^{1}_{(Q,W)}(v) \setminus \mathcal{M}^{1,\theta\text{-ss}}_{(Q,W)}(v). \]
Then we have the open immersion
\[ \mathcal{M}^{1,\theta\text{-ss}}_Q(v) \subset \mathcal{M}^{1}_{Q}(v) \setminus \mathcal{M}^{1,\theta\text{-ss}}_{(Q,W)}(v). \]

The following lemma shows that the categorical DT invariant can be also defined on a bigger ambient space:

Lemma 5.3. The following restriction functor is an equivalence
\[ \text{MF}(\mathcal{M}^{1}_Q(v) \setminus \mathcal{M}^{1,\theta\text{-ss}}_{(Q,W)}(v), w) \cong_{\sim} \text{MF}(\mathcal{M}^{1,\theta\text{-ss}}_Q(v), w). \]

Proof. The lemma follows since the category of factorizations only depends on an open neighborhood of the critical locus (see the equivalence (2.1)). \hfill \Box
5.3. **Wall-chamber structure.** There is a wall-chamber structure for the $\theta$-stability as in Figure 5 (see [NN11, Figure 1]):

![Wall-chamber structure diagram]

**Figure 5.** Wall-chamber structures

In Figure 5, if $\theta$ lies in the first quadrant then $\mathcal{M}^{(Q,W)}_{(Q,W)}(v) = \emptyset$ unless $v = 0$, so it is called an *empty chamber*. In this case, the categorical DT invariants are given in the following lemma:

**Lemma 5.4.** Let $\theta_{en} \in \mathbb{R}^2$ lies in an empty chamber. Then

$$
\text{MF}(\mathcal{M}^{(Q,W)}_{(Q,W)}(v), w) = \begin{cases} 
\text{MF}(\text{Spec } \mathbb{C}, 0), & v = 0 \\
0, & v \neq 0.
\end{cases}
$$

**Proof.** If $v \neq 0$, then $\text{MF}(\mathcal{M}^{(Q,W)}_{(Q,W)}(v), w) = 0$ by the equivalence (2.1), since the critical locus of $w$ is empty. If $v = 0$, then $\mathcal{M}^{(Q,W)}_{(Q,W)}(v) = \text{Spec } \mathbb{C}$ and $w = 0$. \hfill \Box

We focus on the wall in the second quadrant, classified by $m \in \mathbb{Z}_{\geq 1}$

$$
W_m := \mathbb{R}_{>0} \cdot (1 - m, m) \subset \mathbb{R}^2.
$$

If $\theta$ lies between $W_m$ and $W_{m+1}$, then the moduli stack $\mathcal{M}^{(Q,W)}_{(Q,W)}(v)$ is constant, consisting of $\theta$-stable objects. So $\mathcal{M}^{(Q,W)}_{(Q,W)}(v)$ is a quasi-projective scheme, and the good moduli space morphism $\pi_{(Q,W)}^1$ in (5.6) is an isomorphism. If $\theta$ is also sufficiently close to the wall $W_m$, then $\mathcal{M}^{(Q,W)}_{(Q,W)}(v)$ also consists of $\theta$-stable objects and the morphism $\pi_{(Q,W)}^1$ in (5.6) is an isomorphism. The categorical DT invariant is also constant when $\theta$ deforms inside a chamber:

**Lemma 5.5.** The triangulated category $\text{MF}(\mathcal{M}^{(Q,W)}_{(Q,W)}(v), w)$ is constant (up to equivalence) when $\theta$ deforms inside a chamber in Figure 5.

**Proof.** Suppose that $\theta$ lies in a chamber in Figure 5. Although $\theta$ does not lie in a wall for $(Q^+, W)$-representations, it may lie on a wall for $Q^+$-representations. However the destabilizing locus in $\mathcal{M}^{(Q,W)}_{(Q,W)}(v)$ is disjoint from $\text{Crit}(w) = \mathcal{M}^{(Q,W)}_{(Q,W)}(v)$, so by (2.1) the triangulated categories $\text{MF}(\mathcal{M}^{(Q,W)}_{(Q,W)}(v), w)$ are equivalent under wall-crossing inside a chamber of Figure 5. \hfill \Box
For \( \theta \in W_m \), there is a unique (up to isomorphism) \( \theta \)-stable \((Q,W)\)-representation \( S_m \) (i.e. \((Q^1,W)\)-representation whose dimension vector at \( \infty \) is zero), given by (see [NNII Theorem 3.5])

\[
S_m := \begin{pmatrix}
0 & & & 0 \\
& 0 & & \\
& & & \\
& & & \\
B_1^0 & & & B_2^0
\end{pmatrix}, \quad B_1^0(f_i) = e_i, \quad B_2^0(f_i) = e_{i+1}.
\]

(5.7)

Here \( \{e_1, \ldots, e_m\}, \{f_1, \ldots, f_{m-1}\} \) are basis of \( \mathbb{C}^m, \mathbb{C}^{m-1} \) respectively. Note that \( S_m \) has dimension vector \( s_m = (m, m-1) \) so that \( \theta(S_m) = 0 \) when \( \theta \in W_m \). Under the equivalence \( \Phi \) in (5.11), we have the following relation (see [NNII Remark 3.6])

\[
\Phi(\mathcal{O}_C(m-1)) = S_m.
\]

(5.8)

Since \( s_m = (m, m-1) \) is primitive, the moduli stack \( \mathcal{M}^\theta_{Q}(s_m) \) consists of \( \theta \)-stable \( Q \)-representations, and the good moduli space morphism

\[
\mathcal{M}^\theta_{Q}(s_m) \to \mathcal{M}^\theta_{Q}(s_m)
\]

is a \( \mathbb{C}^* \)-gerbe. There is a function defined similarly to (5.3)

\[
w = \text{Tr}(W) : \mathcal{M}^\theta_{Q}(s_m) \to \mathbb{A}^1
\]

whose critical locus \( \mathcal{M}^\theta_{Q,W}(s_m) \) is the moduli stack of \( \theta \)-stable \((Q,W)\)-representation. Note that \( \mathcal{M}^\theta_{Q,W}(s_m) \) consists of a one point corresponding to the unique \( \theta \)-stable \((Q,W)\)-representation \( S_m \).

**Lemma 5.6.** For any \( j \in \mathbb{Z} \), there is an equivalence

\[
\text{MF}(\mathcal{M}^\theta_{Q}(s_m), w)_j \simeq \text{MF}(\text{Spec } \mathbb{C}, 0).
\]

**Proof.** Let \( V_0 = \mathbb{C}^m, V_1 = \mathbb{C}^{m-1} \) and \( B_1^0, B_2^0 : V_1 \to V_0 \) be maps as in (5.7). Note that we have

\[
\mathcal{M}^\theta_{Q}(s_m) = \left[ (\text{Hom}(V_0, V_1)^{\oplus 2} \oplus \text{Hom}(V_1, V_0)^{\oplus 2}) / \text{GL}(V_0) \times \text{GL}(V_1) \right].
\]

It admits a projection

\[
\mathcal{M}^\theta_{Q}(s_m) \to \left[ \text{Hom}(V_1, V_0)^{\oplus 2} / \text{GL}(V_0) \times \text{GL}(V_1) \right].
\]

(5.10)

The target of the above morphism is identified with the moduli stack of representations of the Kronecker quiver \( Q_K \) (i.e. two vertices \( \{0, 1\} \) with two arrows from \( 1 \) to \( 0 \)), so it contains an open substack corresponding to stable coherent sheaves on \( \mathbb{P}^1 \) under the Beilinson equivalence

\[
\text{RHom}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), -) : D^b(\mathbb{P}^1) \xrightarrow{\sim} D^b(\text{Rep}(Q_K)).
\]

Under the above equivalence, \( \mathcal{O}_{\mathbb{P}^1}(m-1) \) corresponds to \( (B_1^0, B_2^0) \). Since \( \mathcal{O}_{\mathbb{P}^1}(m-1) \) is rigid in \( \mathbb{P}^1 \), there is a \( \text{GL}(V_0) \times \text{GL}(V_1) \)-invariant open neighborhood

\[
(B_1^0, B_2^0) \in \mathcal{U} \subset \text{Hom}(V_1, V_0)^{\oplus 2}
\]

such that \( \mathcal{U} / \text{GL}(V_0) \times \text{GL}(V_1) \) is isomorphic to \( BC^* \). By pulling it back by the projection (5.10), we see that there is an open immersion

\[
\text{Hom}(V_0, V_1)^{\oplus 2} \times BC^* \subset \mathcal{M}^\theta_{Q}(s_m), \quad (A_1, A_2) \mapsto (A_1, A_2, B_1^0, B_2^0),
\]

whose image contains \( \{S_m\} = \text{Crit}(w) \) as \( 0 \in \text{Hom}(V_0, V_1)^{\oplus 2} \). By the equivalence (2.1), the restriction functor gives an equivalence

\[
\text{MF}(\mathcal{M}^\theta_{Q}(s_m), w) \xrightarrow{\sim} \text{MF}(\text{Hom}(V_0, V_1)^{\oplus 2} \times BC^*, w).
\]

(5.12)

The function \( w \) restricted (5.11) is a quadratic function by the definition of \( w \), which must be non-degenerate as its critical locus is one point. Since \( \text{Hom}(V_0, V_1)^{\oplus 2} \) is even dimensional, the RHS of (5.12) is equivalent to \( \text{MF}(BC^*, w) \) by the Knörrer periodicity in Theorem 2.10.
5.4. **Descriptions of formal fibers.** By the above classification of \(\theta\)-stable \((Q, W)\)-representations, a \(\theta\)-polystable \((Q^1, W)\)-representation of dimension vector \((1, v_0, v_1)\) at the wall \(\theta \in W_m\) is of the form

\[
R = R_\infty \oplus (V \otimes S_m)
\]

where \(V\) is a finite dimensional vector space and \(R_\infty\) is a \(\theta\)-stable \((Q^1, W)\)-representation. By setting \(d := \dim V\), the dimension vector of \(R_\infty\) is \((1, v_0 - dm, v_1 - d(m - 1))\). By regarding \(R\) as a \(\theta\)-polystable \(Q^1\)-representation, it determines a point \(p \in M^1_{Q^1, \theta}\)-stable \((Q, W)\)-representation, it determines a point \(p \in M^1_{Q^1, \theta}(v)\).

**Remark 5.7.** The vector space \(V\) in (5.13) will play the same role of the vector space \(V\) in Section 4. Below we fix a basis of \(V\) and use the same convention of the dominant chamber in Subsection 4.3.

Below we fix a \(\theta\)-polystable object \((5.13)\), and \(p \in M^1_{Q^1, \theta}(v)\) is the corresponding point as above. We will give a description of the formal fiber of the good moduli space morphism \(\pi_Q^1: M_{Q^1, \theta}(v) \rightarrow M^1_{Q^1, \theta}(v)\) at \(p\). We set

\[
G_p := \text{Aut}(R) = \text{GL}(V).
\]

It acts on \(\text{Ext}^1_{Q^1}(R, R)\) by the conjugation, and we have the good moduli space morphism

\[
[\text{Ext}^1_{Q^1}(R, R)/G_p] \rightarrow \text{Ext}^1_{Q^1}(R, R)/\!/G_p.
\]

Let \(q \in R_{Q^1}(v)\) be a point corresponding to the polystable object \((5.13)\). Note that \(\text{Ext}^1_{Q^1}(R, R)\) is the tangent space of the stack \(M^1_{Q^1, \theta}(v)\) at \(q\). By Luna’s étale slice theorem, there exists a \(G_p\)-invariant locally closed subset \(q \in W_p \subset R_{Q^1}(v)\) and a commutative diagram

\[
\begin{array}{ccc}
([\text{Ext}^1_{Q^1}(R, R)/G_p], 0) & \longrightarrow & ([W_p/G_p], q) \\
\downarrow & & \downarrow \\
(M^1_{Q^1, \theta}(v), q) & \longrightarrow & (M^1_{Q^1, \theta}(v), p)
\end{array}
\]

such that each horizontal arrows are étale.

We have the following decomposition of \(\text{Ext}^1_{Q^1}(R, R)\) as \(G_p\)-representations,

\[
\text{Ext}^1_{Q^1}(R, R) = \text{Ext}^1_{Q^1}(R_\infty, R_\infty) \oplus (V \otimes \text{Ext}^1_{Q^1}(R_\infty, S_m)) \\
\oplus (V^\vee \otimes \text{Ext}^1_{Q^1}(S_m, R_\infty)) \oplus (\text{End}(V) \otimes \text{Ext}^1_{Q^1}(S_m, S_m)) \\
= (\text{Ext}^1_{Q^1}(R_\infty, R_\infty) \oplus \text{Ext}^1_{Q^1}(S_m, S_m)) \oplus (V \otimes \text{Ext}^1_{Q^1}(R_\infty, S_m)) \\
\oplus (V^\vee \otimes \text{Ext}^1_{Q^1}(S_m, R_\infty)) \oplus (\text{End}_0(V) \otimes \text{Ext}^1_{Q^1}(S_m, S_m)).
\]

Here \(\text{End}_0(V)\) is the kernel of the trace map \(\text{End}(V) \rightarrow \mathbb{C}\) which is an irreducible \(G_p\)-representation, and the last identity gives a direct sum decomposition of \(\text{Ext}^1_{Q^1}(R, R)\) into its irreducible \(G_p\)-representations whose irreducible factors are \(\mathbb{C}\) (trivial representation), \(V\), \(V^\vee\) and \(\text{End}_0(V)\). The number of summands is calculated as follows:

**Lemma 5.8.** We have the following identities:

\[
a_{v,m,d} := \text{ext}^1_{Q^1}(R_\infty, S_m) = C_{v,m} + m + d(-2m^2 + 2m + 1),
\]

\[
b_{v,m,d} := \text{ext}^1_{Q^1}(S_m, R_\infty) = C_{v,m} + d(-2m^2 + 2m + 1),
\]

\[
C_{v,m} := (m - 2)v_0 + (m + 1)v_1,
\]

\[
c_m := \text{ext}^1_{Q^1}(S_m, S_m) = 2m^2 - 2m.
\]

**Proof.** The lemma easily follows from Lemma 4.2 noting that

\[
\text{Hom}(R_\infty, S_m) = \text{Hom}(S_m, R_\infty) = 0, \text{Hom}(T_m, S_m) = \mathbb{C}.
\]
For example, since the dimension vectors of $R_\infty$, $S_m$ are $(1, v_0 - md, v_1 - (m - 1)d)$, $(0, m, m - 1)$ respectively, we have

$$-a_{v,m,d} = \text{hom}(R_\infty, S_m) - \text{ext}^1(R_\infty, S_m)$$

$$= -m + (v_0 - md)m - 2(v_0 - md)(m - 1) - 2(v_1 - (m - 1)d)m + (m - 1)(v_1 - (m - 1)d)$$

$$= -(m - 2)v_0 - (m + 1)v_1 - m - d(-2m^2 + 2m + 1).$$

The left vertical arrow in (5.13) is also identified with a moduli stack of some quiver representations and its good moduli space. We define $Q_p$ to be the Ext-quiver for $\{S_m\}$ and $Q_p^1$ to be the Ext-quiver for $\{R_\infty, S_m\}$. Namely $Q_p$ is the quiver with one vertex $\{1\}$ and the number of loops at $1$ is $c_m$. The quiver $Q_p^1$ consists of two vertices $\{\infty, 1\}$, the number of arrows from $\infty$ to $1$ is $a_{v,m,d}$, from $1$ to $\infty$ is $b_{v,m,d}$, and the number of loops at $\infty$ (resp. $1$) is $\text{ext}^1_{Q_p^1}(R_\infty, R_\infty)$ (resp. $c_m$). From (5.10), we have the identification

$$\text{Ext}^1_{Q_p^1}(R, R) G_p,$$

By combining the diagrams (5.15), (5.18) and taking the formal fibers, we have a commutative diagram

$$(5.19)$$

Here each vertical arrow is a good moduli space morphism, the vertical arrow second from the right (resp. left) is the formal fiber of the morphism (5.14) at the origin, and the square second from the right is obtained by the formal completions of good moduli spaces in the diagram (5.15).

We then compare the semistable loci under the isomorphism $\eta_p$ in the diagram (5.19). We take $\theta = (\theta_0, \theta_1) \in W_m$ and $\theta_{\pm}$ of the form

$$\theta_{\pm} = (\theta_0 \mp \varepsilon, \varepsilon, \theta_1 \pm \varepsilon), \, \varepsilon > 0.$$

We take $\theta_0, \theta_1$ and $\varepsilon$ to be integers and $\theta_{\pm}$ lie on chambers adjacent to $W_m$ which are sufficiently close to $W_m$, e.g. take $\varepsilon = 1$ and $(\theta_0, \theta_1) = N \cdot (1 - m, m)$ for a sufficiently large integer $N$. We have the open substacks

$$\mathcal{M}_{Q_p}^{1, \theta_{\pm} - \text{ss}}(v) \subset \mathcal{M}_{Q_p}^{1, \theta - \text{ss}}(v), \, \hat{\mathcal{M}}_{Q_p}^{1, \theta_{\pm} - \text{ss}}(v)_p \subset \hat{\mathcal{M}}_{Q_p}^{1, \theta - \text{ss}}(v)_p$$

corresponding to $\theta_{\pm}$-semistable representations.

On the other hand, as in (4.1) we set $\chi_0: \text{GL}(V) \to \mathbb{C}^*$ to be the determinant character $g \mapsto \text{det}(g)$. We have the open substacks

$$\mathcal{M}_{Q_p}^{1, \chi_0^{\pm 1} - \text{ss}}(d) \subset \mathcal{M}_{Q_p}^{1, \chi_0 - \text{ss}}(d), \, \hat{\mathcal{M}}_{Q_p}^{1, \chi_0^{\pm 1} - \text{ss}}(d) \subset \hat{\mathcal{M}}_{Q_p}^{1, \chi_0 - \text{ss}}(d)$$

corresponding to $\chi_0^{\pm 1}$-semistable $Q_p^1$-representations. We have the following lemma:
Lemma 5.9. The isomorphism \( \eta_p \) in (5.19) restricts to the isomorphisms

\[
\eta_p : \hat{\mathcal{M}}_{Q_p}^{\chi_{\theta_{\pm}}^{-s}}(d) \cong \hat{\mathcal{M}}_{Q_p}^{\chi_{\theta_{\pm}}^{-s}}(v)_p.
\]

Proof. Let us consider the composition

\[
G_p = \text{GL}(V) \hookrightarrow \text{GL}(V_0) \times \text{GL}(V_1) \xrightarrow{\chi_{\theta_{\pm}}} C^*.
\]

We see that the above function is a sum of a function from \( \hat{\mathcal{M}}_{Q_p}^{\chi_{\theta_{\pm}}^{-s}} \) pulled back by the isomorphism at the origin.

\[
(5.25)
\]

\[
\text{Reduced Ext-quiver.}
\]

We define the reduced Ext-quiver \( Q_p^{\text{red}, \dagger} \) to be the quiver obtained from \( Q_p^{\dagger} \) by removing all the loops at the vertex \( \{1\} \), and adding \( c_m \)-loops at the vertex \( \{\infty\} \), where \( c_m \) is given in (5.17). It contains the full sub quiver (5.23)

\[
Q_p^{\text{red}, \dagger} \subset Q_p^{\text{red}, \dagger}
\]

consisting of the vertex \( \{1\} \) and no loops. See the following picture:

\[
Q_p^{\dagger} = \begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array} \quad Q_p^{\text{red}, \dagger} = \begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array}
\]

Let \( \mathcal{M}_{Q_p}^{\text{red}, \dagger}(d) \) be the \( C^* \)-rigidified moduli stack of \( Q_p^{\text{red}, \dagger} \)-representations with dimension vector \((1, d)\). It is described as

\[
(5.24) \quad \mathcal{M}_{Q_p}^{\text{red}, \dagger}(d) = \left( \textnormal{C}^{\text{ext}^1_{Q_p}((R_\infty, R_\infty)+c_m \oplus V^{a_{a,m,d}} \oplus C_{a_{a,m,d}}(V^d) \oplus b_{b,m,d}) / \text{GL}(V) \right),
\]

by composing it with \( \chi_{\theta_{\pm}} \), we see that the composition (5.22) is given by \( g \mapsto \det(g)^{\pm \varepsilon} \). Therefore the lemma holds.

\[
\square
\]

By restricting the function (5.3) to the formal fiber of the good moduli space morphism (5.6) and pulling it back by the isomorphism \( \eta_p \) in (5.19), we have the function

\[
w_p : \hat{\mathcal{M}}_{Q_p}^{\dagger}(d) = [\text{Ext}^1_{Q_p}(R, R)/G_p] \to \mathbb{A}^1.
\]

We see that the above function is a sum of a function from \( \hat{\mathcal{M}}_{Q_p}^{\dagger}(d) \) and some non-degenerate quadratic form. Let us take a (non-canonical) isomorphism of \( \mathbb{C} \)-vector spaces

\[
(5.25) \quad \text{Ext}^1_{Q}(S_m, S_m) \cong H \oplus H^\vee
\]
where the dimension of $H$ is $m^2 - m$. There is also an isomorphism $\text{End}_0(V) \cong \text{End}_0(V)^\vee$ of $G_p$-representations, so we have an isomorphism of $G_p$-representations

$$\text{End}_0(V) \otimes \text{Ext}^1_Q(S_m, S_m) \cong W \oplus W^\vee$$

where $W = \text{End}_0(V) \otimes H$. In particular we have the non-degenerate symmetric quadratic form

$$q = (-,-) : \text{End}_0(V) \otimes \text{Ext}^1_Q(S_m, S_m) \to \mathbb{A}^1$$

defined to be the natural pairing on $W$ and $W^\vee$. Note that (5.25) is a summand of $\text{Ext}^1_Q(R, R)$ by the decomposition (5.16). We will use the following proposition, whose proof will be given in Subsection 6.4.

**Proposition 5.10.** By replacing the isomorphisms in (5.19) and (5.25) if necessary, the function $w_p$ is written as

$$w_p = w_{\text{red}} + q.$$  

Here $w_{\text{red}}$ is non-zero and does not contain variables from $\text{End}_0(V) \otimes \text{Ext}^1_Q(S_m, S_m)$ under the decomposition (5.16).

The GL($V$)-representation $W$ determines the vector bundle $\mathcal{W}$ on $\mathcal{M}_{Q_p}^\dagger (d)$. By Proposition 5.10 we have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_{Q_p}^\dagger (d) & \overset{\eta_p}{\rightarrow} & \mathcal{M}_{Q}^\dagger, \\
\downarrow w_{\text{red}} & & \downarrow w_p \\
\mathcal{M}_{Q_p}^\dagger (d) & \overset{\eta_p}{\rightarrow} & \mathcal{M}_{Q}^\dagger, \\
\downarrow w_{\text{red}} & & \downarrow w_p \\
\mathbb{A}^1 & & \mathbb{A}^1.
\end{array}
$$

Here $\text{pr}$ is the projection, $\iota$ is given by $\iota(x) = (0, x)$, $\iota$ is the natural morphism by the formal completion (see Lemma 5.4) and $\eta_p$ is the isomorphism in (5.19). We have the following proposition:

**Proposition 5.11.** There is an equivalence

$$\Phi_p := \iota^* \iota_* \text{pr}^*: \text{MF}(\mathcal{M}_{Q_p}^\dagger (d), w_{\text{red}}) \xrightarrow{\sim} \text{MF}(\mathcal{M}_{Q_p}^\dagger (d), w_p).$$

Proof. The composition functor

$$\iota_* \text{pr}^*: \text{MF}(\mathcal{M}_{Q_p}^\dagger (d), w_{\text{red}}) \xrightarrow{\text{pr}^*} \text{MF}(\mathcal{W}, w_{\text{red}}) \xrightarrow{\iota_*} \text{MF}(\mathcal{W} \oplus \mathcal{W}, w_{\text{red}} + q)$$

is an equivalence by Theorem 2.4. By Lemma 6.3 the functor

$$\iota^* : \text{MF}(\mathcal{W} \oplus \mathcal{W}, w_{\text{red}} + q) \to \text{MF}(\mathcal{M}_{Q_p}^\dagger (d), w_p)$$

is fully-faithful with dense image. By Lemma 6.3 and the equivalence (5.30), the LHS of (5.31) is idempotent complete, so the functor (5.31) is an equivalence. Therefore we obtain the proposition.

5.6. Window subcategories. In this subsection, we define several window subcategories for moduli stacks of representations of quivers and their formal fibers discussed in the previous subsection. The notation is summarized in Table 1.

| moduli stack | formal fiber | windows |
|--------------|--------------|---------|
| conifold quiver $Q^1$ | $\mathcal{M}^\text{glob}_{Q}(v)$ | $\mathcal{W}_{\text{glob}}(v), \mathcal{W}_{\text{loc}}(v)_p$ |
| Ext-quiver $Q^1_p$ | $\mathcal{M}_{Q_p}^\dagger (d)$ | $\mathcal{M}_{Q_p}^\dagger (d)$ |
| reduced Ext-quiver $Q^{\text{red},1}_p$ | $\mathcal{M}_{Q_p}^\dagger (d)$ | $\mathcal{W}_{\text{red}}(d)_p$ |
Global window subcategory $\mathcal{W}_\text{glob}^{\theta_+}(v)$. We take $\theta \in W_m$ and $\theta_\pm$ as in (5.20) which are sufficiently close to the wall $W_m$. Then the KN stratification of $M_{Q}^{1}(v)$ for $\chi_{\theta_\pm}$ is finer than those for $\chi_{\theta}$. So we have KN stratifications for $M_{Q}^{1,\theta_{\pm\text{-ss}}}(v)$ with respect to $\chi_{\theta_\pm}$

$$M_{Q}^{1,\theta_{\pm\text{-ss}}}(v) = S^\pm_1 \sqcup \cdots \sqcup S^\pm_{N_\pm} \sqcup M_{Q}^{1,\theta_{\pm\text{-ss}}}(v)$$

(5.32)

with associated one parameter subgroups $\lambda^\pm_i: C^* \to \text{GL}(V_0) \times \text{GL}(V_1)$ and the associated number $\eta^\pm_i \in \mathbb{Z}$ as in (2.6). By Theorem (2.3) (and also noting Lemma 5.3), for each choice of real numbers $m^\pm_\bullet = \{(m^\pm_i)\}_{1 \leq i \leq N_\pm}$ we have the subcategories

$$\mathcal{W}_{m^\pm_\bullet}^{\theta_{\pm\text{-ss}}}(v) \subset \text{MF}(M_{Q}^{1,\theta_{\pm\text{-ss}}}(v), w)$$

(5.33)

such that the compositions

$$\mathcal{W}_{m^\pm_\bullet}^{\theta_{\pm\text{-ss}}}(v) \hookrightarrow \text{MF}(M_{Q}^{1,\theta_{\pm\text{-ss}}}(v), w) \to \text{MF}(M_{Q}^{1,\theta_{\pm\text{-ss}}}(v), w)$$

(5.34)

are equivalences. The subcategory (5.33) consists of objects whose $\lambda^\pm_i$-weights at each center of $S^\pm_i$ are contained in $[m^+_i, m^+_i + \eta^+_i]$. We define the following character

$$\chi_0: \text{GL}(V_0) \times \text{GL}(V_1) \to \mathbb{C}^*, \quad (g_0, g_1) \mapsto \det(g_0) \cdot \det(g_1)^{-1}$$

i.e. $\chi_0 = \chi_{(-1,1)}$ in the notation (5.3). As we discussed in (5.22), the composition

$$G_p = \text{GL}(V) \hookrightarrow \text{GL}(V_0) \times \text{GL}(V_1) \twoheadrightarrow \mathbb{C}^*$$

coincides with the determinant character $\chi_0: \text{GL}(V) \to \mathbb{C}^*$. We use the following special choices for $m^\pm_\bullet$

$$m^+_i = -\frac{1}{2}\eta^+_i + \left(\frac{1}{2}C_{v,m} + \frac{m}{2}\right)\langle \lambda^+_i, \chi_0 \rangle, \quad m^-_i = -\frac{1}{2}\eta^-_i + \frac{1}{2}C_{v,m}\langle \lambda^-_i, \chi_0 \rangle.$$  

(5.35)

Here $C_{v,m}$ is given in (5.17). We then define

$$\mathcal{W}_{m^\pm_\bullet}^{\theta_{\pm\text{-ss}}}(v) \subset \text{MF}(M_{Q}^{1,\theta_{\pm\text{-ss}}}(v), w)$$

(5.36)

to be the window subcategories (5.33) for the choices of $m^\pm_\bullet$ as (5.35).

Local window subcategories $\mathcal{W}_{\text{loc}}^{\theta_{\pm\text{-ss}}}(v)_p$. Let us take a $\theta$-polystable object $R$ as in (5.13), and the corresponding closed point $p \in M_{Q}^{1,\theta_{\pm\text{-ss}}}(v)$. Then we have the diagram of formal fibers (5.19). By restricting the KN stratification (5.32) to the formal fiber, we obtain the KN stratification of $\widehat{M}_{Q}^{1,\theta_{\pm\text{-ss}}}(v)_p$

$$\widehat{M}_{Q}^{1,\theta_{\pm\text{-ss}}}(v)_p = S^\pm_{1,p} \sqcup \cdots \sqcup S^\pm_{N_\pm,p} \sqcup \widehat{M}_{Q}^{1,\theta_{\pm\text{-ss}}}(v)_p$$

(5.37)

We define local window subcategories

$$\mathcal{W}_{\text{loc}}^{\theta_{\pm\text{-ss}}}(v)_p \subset \text{MF}(\widehat{M}_{Q}^{1,\theta_{\pm\text{-ss}}}(v)_p, w_p)$$

similarly to (5.36) as in Theorem (2.3) with respect to the KN stratifications (5.37) and the choices of $m^\pm_\bullet$ in (5.35). The following lemma follows immediately from the definition of window subcategories:

**Lemma 5.12.** An object $E \in \text{MF}(M_{Q}^{1,\theta_{\pm\text{-ss}}}(v), w)$ is an object in $\mathcal{W}_{\text{glob}}^{\theta_{\pm\text{-ss}}}(v)$ if and only if for any closed point $p \in M_{Q}^{1,\theta_{\pm\text{-ss}}}(v)$ represented by an object of the form (5.13) we have $E|_{\widehat{M}_{Q}^{1,\theta_{\pm\text{-ss}}}(v)_p} \in \mathcal{W}_{\text{loc}}^{\theta_{\pm\text{-ss}}}(v)_p$.

**Proof.** The defining conditions of window subcategories $\mathcal{W}_{\text{glob}}^{\theta_{\pm\text{-ss}}}(v)$ is local on the good moduli space, so $E$ is an object in $\mathcal{W}_{\text{glob}}^{\theta_{\pm\text{-ss}}}(v)$ if and only if $E|_{\widehat{M}_{Q}^{1,\theta_{\pm\text{-ss}}}(v)_p} \in \mathcal{W}_{\text{loc}}^{\theta_{\pm\text{-ss}}}(v)_p$ for any $p \in M_{Q}^{1,\theta_{\pm\text{-ss}}}(v)$. If $p$ is not represented by an object of the form (5.13), then the formal fiber $\widehat{M}_{Q}^{1,\theta_{\pm\text{-ss}}}(v)_p$ does not intersect with the critical locus of $w$, so $\text{MF}(\widehat{M}_{Q}^{1,\theta_{\pm\text{-ss}}}(v)_p, w_p) = 0$. \qed
Window subcategories $\mathbb{W}^\pm(d)_p$ for the Ext-quiver. By pulling the KN stratification back to $\mathcal{M}^1_{Q_p}(d)$ by the isomorphism $\eta_p$ in (5.19), we have the KN stratification of $\mathcal{M}^1_{Q_p}(d)$ with respect to $\chi_0^{\pm 1}$

\[(5.38) \quad \mathcal{M}^1_{Q_p}(d) = \mathcal{S}^{\pm}_{1, p} \sqcup \cdots \sqcup \mathcal{S}^{\pm}_{N, p} \sqcup \mathcal{M}^{\pm 1}_{Q_p}(d).\]

We define window subcategories

\[\mathbb{W}^\pm(d)_p \subset \text{MF}(\mathcal{M}^1_{Q_p}(d), w_p)\]

as in Theorem 2.3 with respect to the KN stratifications (5.38) and the choices of $m^\pm$ in (5.35). By the isomorphism $\eta_p$ in (5.19), we have the equivalence

\[(5.39) \quad \eta^*_p : \mathbb{W}^\pm_{\text{loc}}(v)_p \xrightarrow{\sim} \mathbb{W}^\pm(d)_p.\]

Window subcategories $\mathbb{W}_c(d)_p$ for the reduced Ext-quiver. For $c \in \mathbb{Z}_{\geq 0}$, we also define

\[\mathbb{W}_c(d)_p \subset \text{MF}(\mathcal{M}^1_{Q_p}\text{red}(d), w_p^\text{red})\]

to be the thick closure of matrix factorizations whose entries are of the form $V(\chi) \otimes \mathcal{O}$ for $\chi \in \mathcal{B}_c(d)$, where $\mathcal{B}_c(d)$ is defined in (4.7). By the description (5.24) of $\mathcal{M}^1_{Q_p\text{red}}(d)$ in terms of the stack $G_{a,m,d,b,m,d}(d)$, the argument of Proposition 4.3 (also see the argument of Corollary 4.22) implies that the following composition functors are equivalences

\[(5.40) \quad \mathbb{W}_{a,m,d}(d)_p \hookrightarrow \text{MF}(\mathcal{M}^1_{Q_p\text{red}}(d), w_p^\text{red}) \rightarrow \text{MF}(\mathcal{M}^1_{Q_p\text{red}}(d), w_p^\text{red}),\]

\[\mathbb{W}_{b,m,d}(d)_p \hookrightarrow \text{MF}(\mathcal{M}^1_{Q_p\text{red}}(d), w_p^\text{red}) \rightarrow \text{MF}(\mathcal{M}^1_{Q_p\text{red}}(d), w_p^\text{red}).\]

5.7. Comparison of window subcategories. We compare the window subcategories in the previous subsection under the Knörrer periodicity:

**Proposition 5.13.** The equivalence (5.29) restricts to the equivalences

\[\Phi_p : \mathbb{W}_{a,m,d}(d)_p \otimes \chi_0^{d(m^2-m)} \xrightarrow{\sim} \mathbb{W}^+(d)_p,\]

\[\Phi_p : \mathbb{W}_{b,m,d}(d)_p \otimes \chi_0^{d(m^2-m)} \xrightarrow{\sim} \mathbb{W}^-(d)_p.\]

**Proof.** We only give a proof for the $+$ part. Let $\mathcal{W} \rightarrow \mathcal{M}^1_{Q_p\text{red}}(d)$ be the vector bundle as in the diagram (5.28). The KN stratifications (5.32) are pull-back of the KN stratifications

\[(5.41) \quad \mathcal{W} \oplus \mathcal{W}^\vee = \mathcal{S}^\pm_1 \sqcup \cdots \sqcup \mathcal{S}^\pm_{N,\pm} \sqcup (\mathcal{W} \oplus \mathcal{W}^\vee)^{\chi_0^{\pm 1}\text{ss}}\]

of $\mathcal{W} \oplus \mathcal{W}^\vee$ with respect to $\chi_0^{\pm 1}$ by the morphism $\iota$ in (5.28). We denote by

\[\mathbb{W}^\pm(d)_p \subset \text{MF}(\mathcal{W} \oplus \mathcal{W}^\vee, w_p^\text{red} + q)\]

the window subcategories with respect to the above stratifications (5.41) and $m^\pm \in \mathbb{R}$ given by (5.35). By the definition of the above window subcategories, the equivalence (5.31) restricts to the equivalence

\[\iota^* : \mathbb{W}^\pm(d)_p \xrightarrow{\sim} \mathbb{W}^\pm(d)_p.\]

Therefore it is enough to show that the equivalence (5.30) restricts to the equivalence

\[i_* \text{pr}^* : \mathbb{W}_{a,m,d}(d)_p \otimes \chi_0^{d(m^2-m)} \xrightarrow{\sim} \mathbb{W}^+(d)_p.\]
We have the following commutative diagram

\[
\begin{array}{cccc}
\mathbb{W}_{a,v,m,d}(d)_{p} \otimes \chi_{0}^{d(m^{2}-m)} & \rightarrow & \text{MF}(\hat{M}_{Q_{p}}^{\dagger}(d), w_{p}^{\text{red}}) & \rightarrow \text{MF}(\hat{M}_{Q_{p}}^{\dagger,\chi_{0}^{\dagger,\ast}}(d), w_{p}^{\text{red}}) \\
\left\updownarrow i_{\ast}\text{pr} \right. & \sim & \left\updownarrow \sim \right. & \\
\mathbb{W}^{+}(d)_{p} & \rightarrow & \text{MF}(\mathcal{W} \oplus \mathcal{W}^{\vee}, w_{p}^{\text{red}} + q) & \rightarrow \text{MF}((\mathcal{W} \oplus \mathcal{W}^{\vee})^{\chi_{0}^{\dagger,\ast}}, w_{p}^{\text{red}} + q).
\end{array}
\]

The composition of top arrows is an equivalence by the equivalence in (5.40), and that of bottom arrows is also an equivalence by Theorem 2.3. We see that the middle vertical arrow descends to an equivalence of right vertical dotted arrow. Note that we have the isomorphism

\[
\text{Crit}(w_{p}^{\text{red}}) \cap \hat{M}_{Q_{p}}^{\dagger,\chi_{0}^{\dagger,\ast}} \cong \text{Crit}(w_{p}^{\text{red}} + q) \cap (\mathcal{W} \oplus \mathcal{W}^{\vee})^{\chi_{0}^{\dagger,\ast}}
\]

induced by the zero section \(\hat{M}_{Q_{p}}^{\dagger}(d) \hookrightarrow \mathcal{W} \oplus \mathcal{W}^{\vee}\). In particular we have the inclusion

\[
(5.43) \quad \text{Crit}(w_{p}^{\text{red}} + q) \cap (\mathcal{W} \oplus \mathcal{W}^{\vee})^{\chi_{0}^{\dagger,\ast}} \subset (\mathcal{W} \oplus \mathcal{W}^{\vee}) \times \hat{M}_{Q_{p}}^{\dagger,\chi_{0}^{\dagger,\ast}}(d).
\]

The desired equivalence is given by the composition

\[
\text{MF}(\hat{M}_{Q_{p}}^{\dagger,\chi_{0}^{\dagger,\ast}}(d), w_{p}^{\text{red}}) \to \text{MF}((\mathcal{W} \oplus \mathcal{W}^{\vee}) \times \hat{M}_{Q_{p}}^{\dagger,\chi_{0}^{\dagger,\ast}}(d), w_{p}^{\text{red}} + q)
\]

\[
\cong \text{MF}((\mathcal{W} \oplus \mathcal{W}^{\vee})^{\chi_{0}^{\dagger,\ast}}, w_{p}^{\text{red}} + q).
\]

Here the first equivalence is Knörrer periodicity in Theorem 2.3, and the second equivalence follows from (5.43) and the equivalence (2.1).

Therefore it is enough to show that the middle vertical arrow in (5.42) restricts to the left dotted arrow, i.e., for \(P \in \mathbb{W}_{a,v,m,d}(d)_{p} \otimes \chi_{0}^{d(m^{2}-m)}\), we show that the object \(i_{\ast}\text{pr}^{\ast}(P)\) lies in \(\mathbb{W}^{+}(d)_{p}\). Note that the critical locus of \(w_{p}^{\text{red}} + q\) lies in the zero section \(\hat{M}_{Q_{p}}^{\dagger}(d) \subset \mathcal{W} \oplus \mathcal{W}^{\vee}\). From Theorem 2.3 it is enough to that \(i_{\ast}\text{pr}^{\ast}(P)\) satisfies the condition (2.9) for one parameter subgroups which appear in the KN stratification of \(\hat{M}_{Q_{p}}^{\dagger}(d)\). From the description (5.24) of \(M_{Q_{p}}^{\dagger}(d)\), its KN stratifications with respect to \(\chi_{0}^{\pm 1}\) are KN stratifications of \(G_{a,v,m,d}(d)\) discussed in Subsection 1.1 up to a product with a trivial factor. Therefore they are of the form

\[
\hat{M}_{Q_{p}}^{\dagger}(d) = S_{0}^{\pm} \sqcup \cdots \sqcup S_{d-1}^{\pm} \sqcup \hat{M}_{Q_{p}}^{\dagger,\chi_{0}^{\dagger,\ast}}(d)
\]

such that each associated one parameter subgroup \(\lambda_{i}^{\pm}: \mathbb{C}^{\ast} \to G_{p} = \text{GL}(V)\) is given by (4.6), i.e., \(\lambda_{i}^{\pm}\) is

\[
(5.44) \quad \lambda_{i}^{\pm}(t) = (1, \ldots, 1, t^{-1}, \ldots, t^{-1}).
\]

Therefore in order to show that the object \(i_{\ast}\text{pr}^{\ast}(P)\) lies in \(\mathbb{W}^{+}(d)_{p}\), it is enough to check the weight conditions (2.10) for the above \(\lambda_{i}^{\pm}\).

Since the object \(i_{\ast}\text{pr}^{\ast}(P)\) is given by taking the tensor product with the Koszul factorization (2.11), it is isomorphic to a direct summand of a matrix factorization whose entries are of the form

\[
V(\chi) \otimes \bigwedge_{k}^{k} W \otimes \chi_{0}^{d(m^{2}-m)} \otimes O, \quad \chi \in \mathbb{B}_{a,v,m,d}(d), \quad 0 \leq k \leq \dim W.
\]

For each one parameter subgroup \(\lambda: \mathbb{C}^{\ast} \to G_{p}\), we set

\[
\gamma_{\lambda} := (\lambda, W^{\lambda>0}) = -(\lambda, W^{\lambda<0}),
\]
where the second identity holds as \( W = \text{End}_0(V) \otimes \mathbb{C}^{m^2-m} \) is a self-dual \( G_p \)-representation. Then we have the following inclusions of the set of \( \lambda^+_i \)-weights of \( V(\chi) \otimes \bigwedge^k W \otimes \chi_0^{d(m^2-m)} \):

\[
\text{wt}_{\lambda^+_i}(V(\chi) \otimes \bigwedge^k W \otimes \chi_0^{d(m^2-m)}) \\
\subset \bigcup_{\chi' \in \text{wt}(V(\chi))} \left[ -\sum_{j=i+1}^d x'_j - (d-i) \cdot d(m^2-m) - \gamma_{\lambda^+_i}, \quad -\sum_{j=i+1}^d x'_j - (d-i) \cdot d(m^2-m) + \gamma_{\lambda^+_i} \right]
\subset \left[ (d-i)(-a_{v,m,d} + d - dm^2 + dm) - \gamma_{\lambda^+_i}, (d-i)(-dm^2 + dm) + \gamma_{\lambda^+_i} \right].
\]

Here \( \text{wt}(V(\chi)) \) is the set of \( T \)-weights of \( V(\chi) \) for the maximal torus \( T \subset G_p \), and we have written \( \chi' = (x'_1, \ldots, x'_d) \) satisfying \( 0 \leq x'_j \leq a_{v,m,d} - d \).

We show that the above set of weights is contained in \( [m^+_i, m^+_i + \eta^+_i] \). From the decomposition \(4.10\), the \( \eta^+_i \in \mathbb{Z} \) which appears in \(5.35\) for the one parameter subgroup \(5.44\) is calculated as in the proof of Proposition \(4.13\):

\[
\eta^+_i = \langle \lambda^+_i, (\text{Ext}^1_Q(R, R)^{\vee})^{\lambda^+_i > 0} - (\mathfrak{g}^v)^{\lambda^+_i > 0} \rangle \\
= \langle \lambda^+_i, ((V^{\vee})^{\oplus b_{v,m,d}} \otimes T^{\oplus b_{v,m,d}} \oplus W \otimes W^\vee)^{\lambda^+_i > 0} - \text{End}(V)^{\lambda^+_i > 0} \rangle \\
= (a_{v,m,d} - i)(d-i) + 2\gamma_{\lambda^+_i}.
\]

Here \( \mathfrak{g}_p = \text{End}(V) \) is the Lie algebra of \( G_p = \text{GL}(V) \). Therefore we have

\[
[m^+_i, m^+_i + \eta^+_i] \\
= \left[ -\frac{1}{2} m^+_i + \frac{1}{2} C_{v,m} + \frac{m}{2}, \frac{1}{2} \lambda^+_i, \chi_0, \frac{1}{2} \eta^+_i + \left( \frac{1}{2} C_{v,m} + \frac{m}{2} \right) \langle \lambda^+_i, \chi_0 \rangle \right] \\
= \left[ (d-i) \left( -a_{v,m,d} + \frac{i}{2} + \frac{d}{2} - dm^2 + dm \right) - \gamma_{\lambda^+_i}, (d-i) \left( -dm^2 + dm + \frac{d}{2} - \frac{i}{2} \right) + \gamma_{\lambda^+_i} \right].
\]

Since \( 0 \leq i \leq d-1 \), we conclude the inclusion

\[
\text{wt}_{\lambda^+_i}(V(\chi) \otimes \bigwedge^k W \otimes \chi_0^{d(m^2-m)}) \subset [m^+_i, m^+_i + \eta^+_i].
\]

Therefore the weight condition \(4.14\) for \( \text{pr}^* \mathcal{P} \) with respect to \( \lambda^+_i \) is satisfied.

Let \( s_m = (m, m-1) \) be the dimension vector of the stable \( Q \)-representation \( S_m \), defined in \(5.7\). Let \( \eta^+_m \in \tilde{M}^\theta_{Q_0}(s_m) \) be the corresponding closed point. We consider the formal fiber of the good moduli space morphism \(5.11\) at \( s_m \)

\[
\tilde{M}^\theta_{Q_\eta}(s_m) \rightarrow \tilde{M}^\theta_{Q_0}(s_m).
\]

Similarly to \(5.15\), the étale slice theorem implies an isomorphism

\[
\tilde{M}^\theta_{Q_\eta}(1) = \left[ \text{Ext}^1_Q(S_m, S_m)/\text{Aut}(S_m) \right] \xrightarrow{\sim} \tilde{M}^\theta_{Q_0}(s_m).
\]

Here \( \text{Aut}(S_m) = \mathbb{C}^* \) acts on \( \text{Ext}^1_Q(S_m, S_m) \) trivially. We will also use the following lemma, which compares window subcategories for quivers without framings:

**Lemma 5.14.** For any \( j \in \mathbb{Z} \), we have equivalences

\[
\text{MF}(\tilde{M}^\theta_{Q_\eta}(1), u_{\eta}^{\text{red}})_j \rightsquigarrow \text{MF}(\tilde{M}^\theta_{Q_0}(1), u_p)_j \rightsquigarrow \text{MF}(\tilde{M}^\theta_{Q_0}(s_m), w)_j,
\]

and all of them are equivalent to \( \text{MF}(\text{Spec } \mathbb{C}, 0) \). Here the first equivalence is given by the Knörrer periodicity in Theorem 2.4.
Proof. By the definition of $Q_p^{\text{red}}$ in (5.23), we have $(\widehat{\mathcal{M}}_{Q_p}^\text{red}(1), w_p^{\text{red}}) = (B\mathbb{C}^*, 0)$. On the other hand, the isomorphisms (5.24), (5.45) and an argument of Proposition 5.10 imply an isomorphism

$$
(\widehat{\mathcal{M}}_{Q_p}(1), w_p) \cong \left(\left(\mathbb{H} \oplus \tilde{H}^\vee\right)/\mathbb{C}^*, q\right)
$$

where $\mathbb{C}^*$ acts on $H = \mathbb{C}^m$ trivially and $q$ is the natural paring on $H$ and its dual. By the Knörrer periodicity in Theorem 2.3, we have an equivalence

$$
\text{MF}(\widehat{\mathcal{M}}_{Q_p}^\text{red}(1), w_p^{\text{red}})_j = \text{MF}(\text{Spec } \mathbb{C}, 0) \xrightarrow{\sim} \text{MF}(H \oplus \tilde{H}^\vee, q).
$$

The natural functor by the formal completion

$$
\text{MF}(H \oplus \tilde{H}^\vee, q) \rightarrow \text{MF}(H \oplus \tilde{H}^\vee, q)
$$

is an equivalence (see [Bro] Remark 2.18). Therefore we obtain the desired equivalences (5.46). □

5.8. Comparison of Hall products. As in the previous subsections, we take a stability condition on the wall $\theta \in W_m$ for $m \geq 1$. As in Subsection 3.3 we have the categorified Hall product

$$
\text{MF}(\mathcal{M}_{Q_p}^{\theta, \text{ss}}(s_m), w)_{j_1} \otimes \cdots \otimes \text{MF}(\mathcal{M}_{Q_p}^{\theta, \text{ss}}(s_m), w)_{j_l} \otimes \text{MF}(\mathcal{M}_{Q_p}^{\theta, \text{ss}}(v - l s_m), w) \rightarrow \text{MF}(\mathcal{M}_{Q_p}^{\theta, \text{ss}}(v), w).
$$

Here $s_m = (m, m - 1)$ is the dimension vector of $S_m$. We take a $\theta$-polystable representation $Q^l$-representation $R$ of the form (5.13), i.e. $R = R_\infty \oplus (V \otimes S_m)$ with $\dim V' = d$, and the corresponding closed point $p \in M_{Q_p}^{\theta, \text{ss}}(v)$. By taking the base change of the above categorified Hall product to the formal completion at $p$ (see Subsection 5.4), we obtain the functor

$$
\text{MF}(\widehat{\mathcal{M}}_{Q_p}^{\theta, \text{ss}}(s_m), w)_{j_1} \otimes \cdots \otimes \text{MF}(\widehat{\mathcal{M}}_{Q_p}^{\theta, \text{ss}}(s_m), w)_{j_l} \otimes \text{MF}(\widehat{\mathcal{M}}_{Q_p}^{\theta, \text{ss}}(v - l s_m)_p, w) \rightarrow \text{MF}(\widehat{\mathcal{M}}_{Q_p}^{\theta, \text{ss}}(v)_p, w).
$$

Here $p_l \in M_{Q_p}^{\theta, \text{ss}}(v - l s_m)$ corresponds to the $\theta$-polystable representation $R_\infty \oplus (V' \otimes S_m)$ with $\dim V' = d - l$. We note that by the isomorphism $\eta_p$ in (5.19) and the isomorphism (5.45), the above functor is identified with the functor

$$
\text{MF}(\widehat{\mathcal{M}}_{Q_p}(1), w_p)_{j_1} \otimes \cdots \otimes \text{MF}(\widehat{\mathcal{M}}_{Q_p}(1), w_p)_{j_l} \otimes \text{MF}(\widehat{\mathcal{M}}_{Q_p}^l(d - l), w_p) \rightarrow \text{MF}(\widehat{\mathcal{M}}_{Q_p}^l(d), w_p)
$$

obtained by the categorified Hall products for $Q^l_p$-representations and the completions at the origins. A similar construction also gives the categorified Hall product for $Q^\text{red}_p$-representations

$$
\text{MF}(\widehat{\mathcal{M}}_{Q_p}^\text{red}(1), 0)_{j_1} \otimes \cdots \otimes \text{MF}(\widehat{\mathcal{M}}_{Q_p}^\text{red}(1), 0)_{j_l} \otimes \text{MF}(\widehat{\mathcal{M}}_{Q_p}^\text{red}(d - l), w_p^{\text{red}}) \rightarrow \text{MF}(\widehat{\mathcal{M}}_{Q_p}^\text{red}(d), w_p^{\text{red}}).
$$

We compare the above categorified Hall products under the Knörrer periodicity:

**Proposition 5.15.** The following diagram commutes

$$
\begin{array}{ccc}
\otimes_{i=1}^l \text{MF}(\widehat{\mathcal{M}}_{Q_p}^\text{red}(1), 0)_{j_i} \otimes \text{MF}(\widehat{\mathcal{M}}_{Q_p}^l(d - l), w_p^{\text{red}}) & \rightarrow & \text{MF}(\widehat{\mathcal{M}}_{Q_p}^l(d), w_p^{\text{red}}) \\
\downarrow & & \downarrow \\
\otimes_{i=1}^l \text{MF}(\widehat{\mathcal{M}}_{Q_p}(1), w_p)_{j_i + (2i - d - 1)(m^2 - m)} \otimes \text{MF}(\widehat{\mathcal{M}}_{Q_p}^l(d - l), w_p) & \rightarrow & \text{MF}(\widehat{\mathcal{M}}_{Q_p}^l(d), w_p).
\end{array}
$$
Here the horizontal arrows are given by categorized Hall products \((5.60), (5.51)\), the right vertical arrow is given in Proposition \(5.11\) and the left vertical arrow is a composition of the functors in Proposition \(5.11\) Lemma \(5.14\) with the equivalence \(\otimes\). Here the horizontal arrows are given by categorized Hall products \((5.50), (5.51)\), the right vertical arrow is induced by the GL(\(V\))\(-\)representation \(W = \text{End}_0(V) \otimes H\) for \(H = \mathbb{C}^{m^2 - m}\) by its definition. By Proposition \(2.5\) the categorized Hall products in \((5.52)\) commute with Knörrer periodicity equivalences up to the following equivalence \((5.54)\) restricts to the equivalence \((5.53)\).

The following is the semiorthogonal decomposition of global window subcategories. The following is the main result in this section:
Theorem 5.16. For $l \geq 0$ and $0 \leq j_1 \leq \cdots \leq j_l \leq m - l$, the categorified Hall product \((5.49)\) restricts to the fully-faithful functor

\[
(5.55) \quad \mathcal{Y}_{j_*} : \bigoplus_{i=1}^l \text{MF}(\mathcal{M}^{\text{ss}}_{Q}(s_m), w)_{j_i+(2i-1)(m^2-m)} \boxtimes \left( \mathcal{W}^\theta_{\text{glob}}(v - ls_m) \otimes \chi_0^{j_i+2l(m^2-m)} \right) \to \mathcal{W}^\theta_{\text{glob}}(v)
\]

such that, by setting $Q \to \mathcal{Y}_{j_*}$, we obtain the semiorthogonal decomposition

\[
(5.56) \quad \mathcal{W}^\theta_{\text{glob}}(v) = (C_{j_*} : l \geq 0, 0 \leq j_1 \leq \cdots \leq j_l \leq m - l),
\]

where $\text{Hom}(C_{j_*}, C_{j'_*}) = 0$ for $j_* \neq j'_*$ (see Definition 4.15).

Proof. We take a $\theta$-polystable representation $R$ of the form \((5.13)\), i.e. $R = R_\infty \oplus (V \otimes S_m)$ with $\dim V = d$, the corresponding closed point $p$ in $M^1_{Q, \text{ss}}(v)$, and consider the quivers $Q^\dagger$, $Q^\dagger_{\text{red}}$ as in the previous subsections. Note that if we remove the loops at the vertex $\{\infty\}$ from $Q^\dagger$, then we obtain the quiver $Q_{a,b}$ for $a = a_{v,m,d}$ and $b = b_{v,m,d}$ considered in Remark 4.1. By applying Corollary 4.22 for the above $Q_{a,b}$, and then taking the tensor product with $\chi_0^{(m^2-m)}$, we obtain the semiorthogonal decomposition

\[
\mathcal{W}^\theta_{a_{v,m,d}}(d)_p \otimes \chi_0^{(m^2-m)} = \left( \bigoplus_{i=1}^l \text{MF}(\mathcal{M}_{Q^\dagger}(1), 0)_{j_i+d(m^2-m)} \boxtimes \left( \mathcal{W}^{-}(d-l)_p \otimes \chi_0^{d-l(m^2-m)} \otimes \chi_0^{j_i+l(m^2-m)} \right) \right).
\]

Here $l \geq 0$, $p_i \in M^1_{Q, \text{ss}}(v - ls_m)$ corresponds to $R_\infty \oplus (V' \otimes S_m)$ with $\dim V' = d - l$, and

\[
(5.57) \quad 0 \leq j_1 \leq \cdots \leq j_l \leq a_{v,m,d} - b_{v,m,d} - l = m - l.
\]

By applying Proposition 5.13, Lemma 5.14 and Proposition 5.15 we obtain the semiorthogonal decomposition

\[
\mathcal{W}^\theta_{p}(d)_p = \left( \bigoplus_{i=1}^l \text{MF}(\mathcal{M}_{Q^\dagger}(1), w_p)_{j_i+(2i-1)(m^2-m)} \boxtimes \left( \mathcal{W}^{-}(d-l)_p \otimes \chi_0^{j_i+2l(m^2-m)} \right) \right).
\]

By the identification of categorified Hall products \((5.49)\) with \((5.50)\) together with the equivalence \((5.39)\), we obtain the semiorthogonal decomposition

\[
(5.58) \quad \mathcal{W}^\theta_{p}(v)_p = \left( \bigoplus_{i=1}^l \text{MF}(\mathcal{M}^{\theta, \text{ss}}_{Q}(s_m), w)_{j_i+(2i-1)(m^2-m)} \boxtimes \left( \mathcal{W}^\theta_{loc}(v - ls_m)_p \otimes \chi_0^{j_i+2l(m^2-m)} \right) \right).
\]

A key observation is that in the above semiorthogonal decomposition there is no term involving $d = \dim V$ (which depends on a choice of $\theta$-polystable object \((5.13)\) so that we can globalize it. Indeed we have globally defined functors \((5.55)\) and, noting Lemma 5.12, in order to show that they are fully-faithful and forms a semiorthogonal decomposition it is enough to check these properties formally locally at each closed point of $M^1_{Q} \text{ss}(v)$ corresponding to a $\theta$-polystable $(Q^\dagger, W)$-representation (see the arguments in Todc Proposition 6.9, Theorem 6.11 for example).

Here we give some more details for how to derive the global semiorthogonal decomposition \((5.56)\) from the formal local one \((5.58)\). We first note that the categorified Hall product \((5.48)\) restricts to the functor \((5.55)\). This follows from the fact that the categorified Hall products commute with base change to the formal completion of good moduli spaces (see the diagram 4.24), the fact (which follows from \((5.58)\)) that formally locally over $M^1_{Q} \text{ss}(v)$ the categorified Hall product restricts to the functor

\[
\bigoplus_{i=1}^l \text{MF}(\mathcal{M}^{\theta, \text{ss}}_{Q}(s_m), w)_{j_i+(2i-1)(m^2-m)} \boxtimes \left( \mathcal{W}^\theta_{loc}(v - ls_m)_p \otimes \chi_0^{j_i+2l(m^2-m)} \right) \to \mathcal{W}^\theta_{loc}(v)_p
\]

and noting Lemma 5.12.
By Lemma 6.6 below, the functor $\Upsilon_{j\ast}$ admits a right adjoint $\Upsilon_{j\ast}^R$. Now in order to show that $\Upsilon_{j\ast}$ is fully-faithful, it is enough to show that the adjunction morphism
\[
(-) \to \Upsilon_{j\ast}^R \circ \Upsilon_{j\ast}(-)
\]
is an isomorphism. Equivalently, it is enough to show the cone of the above morphism is zero. By Lemma 6.5 this is a property formally locally over $M_Q^{1,\text{ss}}(v)$. So from the semiorthogonal decomposition (5.56) we conclude that $\Upsilon_{j\ast}$ is fully-faithful. A similar argument also shows that $C_{j\ast}$ for $j\ast$ given in (5.57) are semiorthogonal.

In order to show that $C_{j\ast}$ for $j\ast$ given in (5.57) generate $V_{\glob}(v)$, let us take $E \in V_{\glob}(v)$ and $j\ast$ so that $j\ast$ is maximal in the order of Definition 5.17. We have the distinguished triangle
\[
\Upsilon_{j\ast} \Upsilon_{j\ast}^R(E) \to E \to E', \quad E' \in C_{j\ast}^\perp.
\]
By applying the above construction for $E'$ and the second maximal $j\ast$ and repeating, we obtain the distinguished triangle
\[
E_1 \to E \to E_2, \quad E_1 \in \langle C_{j\ast} \rangle, \quad E_2 \in \langle C_{j\ast} \rangle^\perp.
\]
Here $\langle C_{j\ast} \rangle$ is the right hand side of (5.56). From the semiorthogonal decomposition (5.58), we have $E_2|_{M_Q^{1,\text{ss}}(v)_p} = 0$ for any closed point $p \in M_Q^{1,\text{ss}}(v)$, therefore $E_2 = 0$ by Lemma 6.5. Therefore $E \in \langle C_{j\ast} \rangle$, and we have the desired semiorthogonal decomposition (5.56). \hfill $\square$

The following corollary, which is an immediate consequence from Theorem 5.16 categorifies wall-crossing formula of the associated DT invariants in \cite{NNT1}.

**Corollary 5.17.** There exists a semiorthogonal decomposition of the form
\[
\text{MF}(M_Q^{1,\theta}(v), w) = \left\langle \text{MF}(M_Q^{1,\theta^\pm}(v - l s_m), w) : l \geq 0, 0 \leq j_1 \leq \cdots \leq j_l \leq m - l \right\rangle.
\]

Here $\text{MF}(M_Q^{1,\theta^\pm}(v - l s_m), w)_{j\ast}$ is a copy of $\text{MF}(M_Q^{1,\theta^\pm}(v - l s_m), w)$.

**Proof.** By the equivalences (5.34), the LHS of (5.56) is equivalent to $\text{MF}(M_Q^{1,\theta^\pm}(v), w)$. On the other hand, the subcategory $C_{j\ast}$ in (5.56) is equivalent to $\text{MF}(M_Q^{1,\theta^\pm}(v - l s_m), w)$ by the equivalences (5.34) together with Lemma 5.6. \hfill $\square$

**Remark 5.18.** The semiorthogonal decomposition in Corollary 5.17 recovers the numerical wall-crossing formula (1.1). Indeed the periodic cyclic homologies are additive with respect to semiorthogonal decompositions (see \cite{Tab05} Theorem 6.3, Section 6.1), so we have
\[
\text{HP}_\ast(\text{MF}(M_Q^{1,\theta\text{-ss}}(v), w)) = \bigoplus_{l \geq 0} \text{HP}_\ast(\text{MF}(M_Q^{1,\theta\text{-ss}}(v), w))^{\oplus (m)}.
\]
By taking the Euler characteristics and using Lemma 6.1 we obtain the formula (1.1).

By applying Corollary 5.17 from the empty chamber in Figure 6 to the wall-crossing at $W_m$, and noting Lemma 5.5 we obtain the following:

**Corollary 5.19.** For $\theta \in W_m$, there exists a semiorthogonal decomposition
\[
\text{MF}(M_Q^{1,\theta}(v), w) = \left\langle C_{j_{\ast}} \right\rangle.
\]

Here each $C_{j_{\ast}}$ is equivalent to $\text{MF}(\text{Spec } \mathbb{C}, 0)$ and $j_{\ast}$ is a collection of non-positive integers of the form
\[
\{0 \leq j_1^{(i)} \leq \cdots \leq j_l^{(i)} \leq i - l_i \} \leq i \leq m
\]
for some integers $l_i \geq 0$ satisfying
\[
(v_0, v_1) = \sum_{i=1}^m l_i \cdot (i, i-1).
\]
We have $\text{Hom}(C_{j^*(\cdot)}, C_{j^*(\cdot)}) = 0$ if $j^*(i) = j^*(i)$ for $k < i \leq m$ for some $k$ and $j^*(k) > j^*(k)$.

Proof. Let $\theta_m \in \mathbb{R}^2$ lie in the empty chamber in Figure 3. By Lemma 5.5, a successive application of Corollary 5.14 gives the semiorthogonal decomposition

$$\text{MF}(\mathcal{M}^1_{\theta} (v), w) = \Big\{ \text{MF}(\mathcal{M}^1_{\theta} (v - l_m s_m - l_{m-1} s_{m-1} - \cdots - l_1 s_1), w))_{j^*(m-1), \ldots, j^*(1)} \Big\}$$

Here $l_i \geq 0$ are integers and $0 \leq j^*(1) \leq \cdots \leq j^*(i) \leq i - l_i$ for $1 \leq i \leq m$. By applying Lemma 5.4, we obtain the corollary.

Remark 5.20. The arguments of Theorem 5.16 and Corollary 5.17 work for other walls except walls at $\{\theta_0 + \theta_1 = 0\}$. For example, let us consider the wall in Figure 5.

Then for $\theta \in W_m$, there is a unique $\theta$-stable $(Q, W)$-representation $S_m'$ of dimension vector $s_m' = (m, m + 1)$, which corresponds to $O_C(-m - 1)[1]$ under the equivalence $\Phi$ in (5.1) (see [NN11, Remark 3.6]). The arguments of Theorem 5.16 and Corollary 5.17 work verbatim by replacing $S_m$, $s_m$ with $S_m'$, $s_m'$, so that we have the semiorthogonal decomposition

$$\text{MF}(\mathcal{M}^1_{\theta} (v), w) = \Big\{ \text{MF}(\mathcal{M}^1_{\theta} (v - l's_m'), w)_{j^*} : l \geq 0, 0 \leq j^*_1 \leq \cdots \leq j^*_l \leq m - l \Big\}.$$ 

On the other hand, the above arguments do not work at walls in $\{\theta_0 + \theta_1 = 0\}$. For example at the DT/PT wall $\theta \in W_{m}\mathbb{R}_{>0}(-m - 1, m)$, $m \in \mathbb{Z}_{\geq 0}$.

5.10. Semiorthogonal decompositions of categorical stable pair theory. By definition a PT stable pair [PT09] on $X$ is a pair $(F, s)$ where $F$ is a pure one dimensional coherent sheaf on $X$ and $s : O_X \to F$ is surjective in dimension one. For $(\beta, n) \in \mathbb{Z}^2$, we denote by

$$P_n(X, \beta)$$

the moduli space of PT stable pair moduli space $(F, s)$ on $X$ satisfying $[F] = [\beta][C]$ and $\chi(F) = n$, where $[F]$ is the fundamental one cycle of $F$. Since any such a sheaf $F$ is supported on $C$, the moduli space $P_n(X, \beta)$ is a projective scheme.

It is proved in [NN11] Proposition 2.11 that the equivalence (5.1) induces the isomorphism

$$\Phi : P_n(X, \beta) \cong \mathcal{M}^1_{\theta_{PT}} (\beta, n - \beta)$$

where $\theta_{PT} := (-1 + \varepsilon, 1 + \varepsilon)$ for $0 < \varepsilon \ll 1$. The RHS is the critical locus of the function $w : \mathcal{M}^1_{\theta_{PT}} (\beta, n - \beta) \to \mathbb{A}^1$ defined by (5.3). Based on the above isomorphism, the categorical PT invariant is defined as follows:

Definition 5.21. We define the categorical PT invariant for the resolved conifold $X$ to be

$$\mathcal{D}T(P_n(X, \beta)) := \text{MF}(\mathcal{M}^1_{\theta_{PT}} (\beta, n - \beta), w).$$

Similarly to Lemma 5.1, the categorical PT invariant recovers the numerical PT invariant by

$$(5.59) \quad P_{n, \beta} = (-1)^{n+\beta} \epsilon_{C[u]} (\mathcal{D}T(P_n(X, \beta))).$$

By applying Corollary 5.19 for $m \gg 0$, we obtain the following:

Corollary 5.22. For any $(\beta, n) \in \mathbb{Z}^2$, there exists a semiorthogonal decomposition

$$\mathcal{D}T(P_n(X, \beta)) = \langle C_{j^*(\cdot)} \rangle.$$

Here each $C_{j^*(\cdot)}$ is equivalent to $\text{MF}(\text{Spec} \mathbb{C}, 0)$ and $j^*(\cdot)$ is a collection of non-positive integers of the form

$$j^*(\cdot) = \{0 \leq j^*(i) \leq \cdots \leq j^*(i) \leq i - l_i \}_{i \geq 1}$$
for some integers $l_i \geq 0$ satisfying

$$(\beta, n) = \sum_{i \geq 1} l_i \cdot (1, i).$$

We have $\text{Hom}(C_{j^*(\cdot)}, C_{j'^*(\cdot)}) = 0$ if $j^{(i)} = j'^{(i)}$ for $i > k$ for some $k$ and $j^{(k)} > j'^{(k)}$.

**Remark 5.23.** Similarly to Remark 5.18, the semiorthogonal decomposition in Corollary 5.22 implies

$$\text{HP}_*(\mathcal{D}(P_n(X, \beta))) = \text{HP}_*(\text{MF}(\text{Spec } \mathbb{C}, 0)) \oplus a_{n, \beta}$$

where $a_{n, \beta}$ is given by (1.3). By taking the Euler characteristics of both sides, we obtain $P_{n, \beta} = (−1)^{n+\beta} a_{n, \beta}$, which recovers the formula (1.1).

### 6. Some technical lemmas

In this section, we give proofs of some postponed technical lemmas.

#### 6.1. Functoriality of Knörrer periodicity

Let $\mathcal{Y}_1, \mathcal{Y}_2$ be stacks of the form $\mathcal{Y}_i = [Y_i/G_i]$ where $Y_i$ is a smooth affine scheme and $G_i$ is a reductive algebraic group which acts on $Y_i$. Let $\mathcal{W}_i \to \mathcal{Y}_i$ be vector bundles. Then by Theorem 2.4, we have equivalences

$$(6.1) \quad \Phi_i : \text{MF}(\mathcal{Y}_i, w_i) \xrightarrow{\sim} \text{MF}(\mathcal{W}_i \oplus \mathcal{W}_i^\vee, w_i + q_i)$$

where $q_i$ is a natural quadratic form on $\mathcal{W}_i \oplus \mathcal{W}_i^\vee$, i.e. $q_i(x, x') = \langle x, x' \rangle$. On the other hand, the categories of quasi-coherent factorizations $\text{MF}_{qcoh}(\mathcal{Y}_i, w_i)$ are compactly generated by $\text{MF}(\mathcal{Y}_i, w_i)$ (see [BFK14, Proposition 3.15]), so it is equivalent to the ind-completion of $\text{MF}(\mathcal{Y}_i, w_i)$. Therefore by taking ind-completions of both sides in (6.1), the above equivalences extend to equivalences

$$\Phi_i : \text{MF}_{qcoh}(\mathcal{Y}_i, w_i) \xrightarrow{\sim} \text{MF}_{qcoh}(\mathcal{W}_i \oplus \mathcal{W}_i^\vee, w_i + q_i).$$

Suppose that we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{W}_1 & \xrightarrow{g} & \mathcal{W}_2 \\
\downarrow & & \downarrow \\
\mathcal{Y}_1 & \xrightarrow{f} & \mathcal{Y}_2
\end{array}$$

where $f$ is a morphism of stacks, and the top arrow is induced by a morphism of vector bundles $g : \mathcal{W}_1 \to f^* \mathcal{W}_2$. We have the induced diagram

$$(6.2) \quad \mathcal{W}_1 \oplus \mathcal{W}_1^\vee \xrightarrow{h_1} \mathcal{W}_1 \oplus f^* \mathcal{W}_2^\vee \xrightarrow{h_2} \mathcal{W}_2 \oplus \mathcal{W}_2^\vee$$

where $h_1 = (\text{id}_{\mathcal{W}_1}, g^\vee)$ and $h_2 = (g, f)$. The following lemma is a variant of [Toëb] Lemma 2.4.4.

**Lemma 6.1.** The following diagram commutes:

$$(6.3) \quad \begin{array}{ccc}
\text{MF}_{qcoh}(\mathcal{Y}_1, w_1) & \xrightarrow{f^*} & \text{MF}_{qcoh}(\mathcal{Y}_2, w_2) \\
\Phi_1 \sim & & \Phi_2 \\
\text{MF}_{qcoh}(\mathcal{W}_1 \oplus \mathcal{W}_1^\vee, w_1 + q_1) & \xrightarrow{h_2 \cdot h_1^*} & \text{MF}_{qcoh}(\mathcal{W}_2 \oplus \mathcal{W}_2^\vee, w_2 + q_2).
\end{array}$$
\textbf{Proof.} We have the commutative diagram
\begin{equation}
\begin{array}{c}
\xymatrix{ f^*W_2^\vee \ar[r]^{h_4} & W_1 \oplus f^*W_2^\vee \ar[r]^{h_2} & W_2 \oplus W_2^\vee \\
\xymatrix{ & h_3 \ar[u]^{h_5} \ar[d]_{h_1} & \\
\mathcal{Y}_1 \ar[r]_{pr_1} & \mathcal{W}_1^\vee \ar[r]_{i_1} & \mathcal{W}_1 \oplus \mathcal{W}_1^\vee.}
\end{array}
\end{equation}

Here \(pr_1\) is the projection and \(i_1(x) = (0, x)\). By the above diagram together with derived base change, we have
\[
h_2 \ast h_1^! \Phi_1(-) \cong h_2 \ast h_1^! i_1 \ast pr_1^!(-) \cong h_2 \ast h_4 \ast pr_1^!(-) \cong h_6 \ast h_5^!(-).
\]

On the other hand, we have the commutative diagram
\begin{equation}
\begin{array}{c}
\xymatrix{ f^*W_2^\vee \ar[r]^{h_7} & W_2^\vee \oplus W_2^\vee \\
\xymatrix{ & h_6 \ar[u]^{h_5} \ar[d]_{pr_2} & \\
\mathcal{Y}_1 \ar[r]^{f} & \mathcal{Y}_2.}
\end{array}
\end{equation}

Here \(pr_2\) is the projection and \(i_2(x) = (0, x)\). Similarly we have
\[
\Phi_2 f_*(-) \cong i_2 \ast pr_2^! f_* \cong i_2 \ast h_7 \ast h_5^* \cong h_6 \ast h_5^*(-).
\]

Therefore the diagram (6.3) commutes. \(\square\)

We also have the following lemma, which is a variant of \[\text{[TodH, Lemma 2.4.7]}\].

\textbf{Lemma 6.2.} Suppose that \(g: \mathcal{W}_1 \rightarrow f^* \mathcal{W}_2\) is a surjective morphism of vector bundles on \(\mathcal{Y}_1\). Then we have the commutative diagram
\begin{equation}
\begin{array}{c}
\xymatrix{ MF(\mathcal{Y}_2, w_2) \ar[r]^{f^*} & MF(\mathcal{Y}_1, w_1) \\
\Phi_2 \ar[u]^\sim & \Phi_1 \ar[u]^\sim \\
MF(\mathcal{W}_2 \oplus \mathcal{W}_2^\vee, w_2 + q_2) \ar[r]^{h_1 \ast h_5^*} & MF(\mathcal{W}_1 \oplus \mathcal{W}_1^\vee, w_1 + q_1).}
\end{array}
\end{equation}

\textbf{Proof.} The assumption that \(g: \mathcal{W}_1 \rightarrow f^* \mathcal{W}_2\) is surjective implies that the morphism \(h_1\) in (6.2) is a closed immersion, hence \(h_1!\) gives a left adjoint of \(h_1^*\). The lemma now follows by taking left adjoints of horizontal arrows in (6.3) and restrict to coherent factorizations. \(\square\)

\textbf{6.2. The categories of factorizations on formal fibers.}\ Let \(G\) be a reductive algebraic group and \(Y\) be a finite dimensional \(G\)-representation. We denote by \(\hat{Y}\) the formal fiber of the quotient morphism \(Y \rightarrow Y//G\) at the origin (see Subsection 1.7 for the definition of formal fiber). Then
\[
[\hat{Y}/G] \rightarrow \hat{Y}//G = \text{Spec} \hat{O}_{\hat{Y}/G,0}
\]
is a good moduli space for \([\hat{Y}/G]\), and is isomorphic to the formal fiber of the morphism \([Y/G] \rightarrow Y//G\) at 0. We take an element \(w \in \Gamma(\mathcal{O}_{[\hat{Y}/G]} = \hat{O}_{\hat{Y}/G,0}\) with \(w(0) = 0\). We have the following lemma:

\textbf{Lemma 6.3.} For \(w \neq 0\), the triangulated category \(MF([\hat{Y}/G], w)\) is idempotent complete.
Proof. Let \( \tilde{Z} \subset \tilde{Y} \) be the closed subscheme defined by the zero locus of \( w \). We have the following version of Orlov equivalence \([\text{Orl}09]\) relating the categories of factorizations and those of singularities (see \([\text{PV11}]\) Theorem 3.14)

\[
\text{MF}(\tilde{Y}/G, w) \sim \text{D}^b(\tilde{Z}/G)/\text{Perf}(\tilde{Z}/G).
\]

Let \( m_0 \subset O_\tilde{Z} \) be the maximal ideal which defines \( 0 \in \tilde{Z} \), and denote by \( \tilde{O}_\tilde{Z} \) the formal completion of \( O_\tilde{Z} \) at \( m_0 \). Let \( \tilde{Z}^{(n)} := \text{Spec} \tilde{O}_\tilde{Z}/m_0^n \) and \( \mathbb{Z} := \text{Spec} \tilde{O}_\tilde{Z} \). By the coherent completeness for the stacks \( \tilde{Z}/G \) and \( \mathbb{Z}/G \) (see \([\text{AHR}]\) Theorem 1.6)), we have the equivalences

\[
\text{Coh}(\tilde{Z}/G) \sim \lim_{\rightarrow n} \text{Coh}(\tilde{Z}^{(n)}/G) \sim \text{Coh}(\mathbb{Z}/G).
\]

In particular we have an equivalence

\[
\text{D}^b(\tilde{Z}/G) \sim \text{D}^b(\mathbb{Z}/G)
\]

which restricts to the equivalence for subcategories of perfect objects. Therefore we obtain the equivalence

\[
\text{MF}(\tilde{Y}/G, w) \sim \text{D}^b(\mathbb{Z}/G)/\text{Perf}(\mathbb{Z}/G).
\]

Since \( \tilde{O}_\tilde{Z} \) is a complete local ring, the singularity category \( \text{D}^b(\mathbb{Z})/\text{Perf}(\mathbb{Z}) \) is well-known to be idempotent complete (for example, see \([\text{Dyc11}]\) Lemma 5.6, \([\text{MD}]\) Lemma 5.5)). The argument can be easily extended to the \( G \)-equivariant setting. Indeed following the proof of \([\text{MD}]\) Lemma 5.5, it is enough to show that for a \( G \)-equivariant maximal Cohen-Macaulay \( \tilde{O}_\tilde{Z} \)-module \( M \) and an idempotent \( e \in \text{End}^G(\tilde{O}_\tilde{Z}) \), it is lifted to a \( G \)-invariant idempotent in \( \text{End}(M) \). Here \( \text{End}^G(\tilde{O}_\tilde{Z}) \) is the set of morphisms in the \( G \)-equivariant stable category of maximal Cohen-Macaulay modules over \( \tilde{O}_\tilde{Z} \). For an idempotent \( e \in \text{End}^G(\tilde{O}_\tilde{Z}) \), we lift it to \( a \in \text{End}(M) \), which we can assume to be \( G \)-invariant as \( G \) is reductive. Then as in the proof of \([\text{CR81}]\) Theorem 6.7], the limit \( \tilde{e} := \lim f_j(a) \) converges, idempotent in \( \text{End}(M) \) which lifts \( e \). Here \( f_j(x) \) is given by

\[
f_j(x) = \sum_{i=0}^n \binom{2n}{i} x^{2n-i} (1-x)^i.
\]

By the construction \( \tilde{e} \) is \( G \)-invariant, so we obtain the desired lifting property of the idempotents. \( \square \)

Let \( W \) be another finite dimensional \( G \)-representation and \( q: W \to \mathbb{A}^1 \) be a \( G \)-invariant non-degenerate quadratic form. We take \( w \in \tilde{O}_{\tilde{Y}/G,0} \) with \( w(0) = 0 \). We have the following lemma:

**Lemma 6.4.** There is a natural morphism of stacks

\[
(6.4) \quad \iota: \left[ (\tilde{Y} \oplus W)/G \right] \to \left[ (\tilde{Y} \times W)/G \right]
\]

such that the induced functor

\[
(6.5) \quad \iota^*: \text{MF} \left( \left[ (\tilde{Y} \times W)/G \right], w + q \right) \to \text{MF} \left( \left[ (\tilde{Y} \oplus W)/G \right], w + q \right)
\]

is fully-faithful with dense image.

**Proof.** Let \( \pi_Y, \pi_{Y \oplus W} \) be the quotient morphisms

\[
\pi_Y: Y \to Y/G, \quad \pi_{Y \oplus W}: Y \oplus W \to (Y \oplus W)/G.
\]

Then we have \( \pi_{Y \oplus W}^{-1}(0,0) \subset \pi_Y^{-1}(0) \times W \), therefore we have the induced natural morphism \((6.4)\) by the definition of formal fibers.

Note that we have \( \text{Crit}(w + q) = \text{Crit}(w) \times \{0\} \), so the morphism \((6.4)\) induces the isomorphism of critical loci of \( w + q \) on \( \tilde{Y} \times W \) and \( \tilde{Y} \oplus W \), and also their formal neighborhoods. Therefore the functor \((6.5)\) is fully-faithful with dense image by \([\text{Orl11}]\) Theorem 2.10] (in loc. cit. it is stated without \( G \)-action, but the same argument applies to the \( G \)-equivariant setting verbatim). \( \square \)
Suppose that \( Y \) is quasi-projective variety with an action of a reductive algebraic group \( G \) such that the good moduli space \( \pi: [Y/G] \to Y/G \) exists. For each closed point \( y \in Y/G \), we denote by \([\hat{Y}_y/G]\) the formal fiber of \( \pi \) at \( y \). For a regular function \( w: [Y/G] \to \mathbb{A}^1 \), we denote by \( \hat{w}_y \) its restriction to \([\hat{Y}_y/G]\), and \( \pi_y: [\hat{Y}_y/G] \to \hat{Y}_y/G \) its good moduli space. We have the following lemma:

**Lemma 6.5.** For \( \mathcal{E} \in \text{MF}([Y/G], w) \), suppose that \( \mathcal{E}|_{[\hat{Y}_y/G]} \in \text{MF}([\hat{Y}_y/G], \hat{w}_y) \) is isomorphic to zero for any closed point \( y \in Y/G \). Then we have \( \mathcal{E} \cong 0 \).

**Proof.** The inner homomorphism \( \mathcal{H}om^\bullet(\mathcal{E}, \mathcal{E}) \) is an object in \( \text{MF}([Y/G], 0) \), which is equivalent to the \( \mathbb{Z}/2 \)-periodic derived category of coherent sheaves on \([Y/G]\). By the derived base change, we have

\[
\pi_\ast \mathcal{H}om^\bullet(\mathcal{E}, \mathcal{E}) \otimes_{\mathcal{O}_{\hat{Y}/G}} \hat{O}_{\hat{Y}/G, y} \cong \hat{\pi}_y \mathcal{H}om^\bullet(\mathcal{E}|_{[\hat{Y}_y/G]}, \mathcal{E}|_{[\hat{Y}_y/G]}) \cong 0
\]

in the \( \mathbb{Z}/2 \)-periodic derived category of quasi-coherent sheaves on \( \hat{Y}/G \). The object \( \pi_\ast \mathcal{H}om^\bullet(\mathcal{E}, \mathcal{E}) \) is an object in the \( \mathbb{Z}/2 \)-periodic derived category of quasi-coherent sheaves on \( Y/G \) whose formal completions at any \( y \in Y/G \) is zero, so it is isomorphic to zero. Then we have \( \mathcal{H}om^\bullet(\mathcal{E}, \mathcal{E}) = \mathcal{R}\Gamma(\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{E})) = 0 \), so \( \mathcal{E} \cong 0 \).

### 6.3. Right adjoint functor.

**Lemma 6.6.** The functor \( Y_j \ast \) in \([5.55]\) admits a right adjoint \( Y_j^R \).

**Proof.** We consider the following diagram

\[
\begin{array}{ccc}
M^\theta_{Q}(v^\bullet) & \xrightarrow{p} & M^\theta_{Q}(v) \\
M^\theta_{Q}(s_m) \times t \times M^\theta_{Q}(v - l s_m) & \xrightarrow{\Delta} & M^\theta_{Q}(s_m) \times t \times M^\theta_{Q}(v - l s_m) & \xrightarrow{\oplus} & M^\theta_{Q}(v) & \xrightarrow{w} & \mathbb{A}^1.
\end{array}
\]

Similarly to \([5.39]\), let

\[
\mathcal{W}^\theta_{\text{glob}}(v) \subset D^b(\mathcal{M}^\theta_{Q}(v))
\]

be the window subcategory \([2.7]\) for the choice \( m^\ast \) in \([5.35]\). We consider the composition functor

\[
D^b(M^\theta_{Q}(s_m)) \boxtimes D^b(M^\theta_{Q}(v - l s_m)) \xrightarrow{\Delta} D^b(M^\theta_{Q}(s_m)_{j_1 + (2i - 1)(m^2 - m)}) \boxtimes \mathcal{W}^\theta_{\text{glob}}(v - l s_m) \xrightarrow{\Delta} D^b(\mathcal{M}^\theta_{Q}(v)) \to D^b(M^\theta_{Q}(v))
\]

\([6.7]\)

Here the first equivalence is due to window theorem in Theorem\([2.2]\) together with the fact that \([6.5]\) is a \( \mathbb{C}^\ast \)-gerbe, the second arrow is the categorified Hall product (i.e. \( p \circ q^\ast \) in the diagram \([6.6]\)), and the last arrow is the restriction to the semistable locus. The first arrow is of Fourier-Mukai type by Lemma \([6.7]\) below, and the second and the third arrows are also of Fourier-Mukai type by their constructions. Therefore the above composition functor is of Fourier-Mukai type. So we have the kernel object

\[
\mathcal{P} \in D^b(M^\theta_{Q}(s_m) \times t \times M^\theta_{Q}(v - l s_m)) \times M^\theta_{Q}(v).
\]
Moreover the kernel objects of the second and the third arrows in (6.7) are push-forward from the fiber products over $M^1_{Q^{\theta, ss} (v)}$ by their constructions. By Lemma 6.7 below, the kernel object of the first arrow in (6.7) is a push-forward from the fiber product over $\mathbb{A}^1$ and supported on the fiber product over $M^1_{Q^{\theta, ss} (v)}$. Therefore the object $\mathcal{P}$ is a push-forward of an object

\[(6.8) \quad \mathcal{P}_w \in D^b((M^0_{Q^{\theta, ss} (s_m)} \times l \times M^1_{Q^{\theta, ss} (v - l_s m)}) \times \mathbb{A}^1, M^1_{Q^{\theta, ss} (v)}(\mathbb{A}^1)) \]

supported on the fiber product over $M^1_{Q^{\theta, ss} (v)}$. Since $M^1_{Q^{\theta, ss} (v)}$ and $M^0_{Q^{\theta, ss} (s_m)} \times l \times M^1_{Q^{\theta, ss} (v - l_s m)}$ are proper over $M^1_{Q^{\theta, ss} (v)}$, the functor (6.8) admits a right adjoint given by the Fourier-Mukai kernel $\mathcal{P}^R$ defined by

\[
\mathcal{P}^R := \mathcal{P} \otimes \omega_{M^0_{Q^{\theta, ss} (s_m)} \times l \times M^1_{Q^{\theta, ss} (v - l_s m)}}[\dim M^0_{Q^{\theta, ss} (s_m)} \times l \times M^1_{Q^{\theta, ss} (v - l_s m)}].
\]

By Theorem 2.8 the functor $\mathcal{Y}_i$ in (5.55) is regarded as a functor

\[(6.9) \quad \mathcal{Y}_i : MF(M^0_{Q^{\theta, ss} (s_m)}, w) \otimes \bigotimes MF(M^1_{Q^{\theta, ss} (v - l_s m)}, w) \to MF(M^1_{Q^{\theta, ss} (v)}(w), w).
\]

The above functor is a Fourier-Mukai functor with kernel given by $\Xi(\mathcal{P}_w)$, where $\Xi$ is the natural functor (see [Hir17b, Theorem 5.5])

\[
\Xi : D^b((M^0_{Q^{\theta, ss} (s_m)} \times l \times M^1_{Q^{\theta, ss} (v - l_s m)}) \times \mathbb{A}^1, M^1_{Q^{\theta, ss} (v)}) \to MF(M^0_{Q^{\theta, ss} (s_m)} \times l \times M^1_{Q^{\theta, ss} (v - l_s m)}) \times M^1_{Q^{\theta, ss} (v)}(w), \mathbb{P}(w)
\]

By the Grothendieck Riemann-Roch theorem, the object $\mathcal{P}^R$ is the push-forward of an object $\mathcal{P}_w^R$ in the RHS of (6.8). Then the right adjoint of (6.9) is obtained by the Fourier-Mukai kernel $\Xi(\mathcal{P}_w^R)$. □

**Lemma 6.7.** In the setting of Theorem 2.2 let $\mathcal{Y} = [Y/G], \mathcal{Y}^{ss} = [Y^{1ss}/G]$, and assume that $\mathcal{Y}^{ss}$ is a projective scheme over $Y/G$. Then the splitting of $D^b(\mathcal{Y}) \to D^b(\mathcal{Y}^{ss})$ in Theorem 2.2 (applied for $N' = 0$) is of Fourier-Mukai type whose kernel object $\mathcal{P} \in D^b(\mathcal{Y} \times \mathcal{Y}^{ss})$ is supported on $\mathcal{Y} \times_{Y/G} \mathcal{Y}^{ss}$. Moreover for any non-constant $w : Y/G \to \mathbb{A}^1$, we have $\mathcal{P} = i_*\mathcal{P}_w$ for some $\mathcal{P}_w \in D^b(\mathcal{Y} \times \mathbb{A}^1, \mathcal{Y}^{ss})$. Here $\mathcal{Y} \times_{\mathbb{A}^1} \mathcal{Y}^{ss}$ is given by the diagram

\[
\begin{array}{c}
\mathcal{Y} \times_{\mathbb{A}^1} \mathcal{Y}^{ss} \xrightarrow{i} \mathcal{Y} \times \mathcal{Y}^{ss} \\
\downarrow \quad \downarrow \mathbb{P}(w) \\
\mathbb{A}^1, \\
0 \end{array}
\]

**Proof.** The KN stratification of $\mathcal{Y}$ pulls back to the one on $\mathcal{Y} \times \mathcal{Y}^{ss}$ via the first projection, thus by a choice of $m_*$ in Theorem 2.2 we have the splitting $\Psi$ of $D^b(\mathcal{Y} \times \mathcal{Y}^{ss}) \to D^b(\mathcal{Y}^{ss} \times \mathcal{Y}^{ss})$. From its construction, $\Psi$ is linear over $\text{Perf}(Y/G \times Y/G)$. Therefore for any non-constant $w$, by [HIT15, Proposition 5.5] there is a splitting $\Phi_w$ of $D^b(\mathcal{Y} \times \mathbb{A}^1, \mathcal{Y}^{ss}) \to D^b(\mathcal{Y}^{ss} \times \mathbb{A}^1, \mathcal{Y}^{ss})$ such that the following diagram commutes:

\[
\begin{array}{c}
D^b(\mathcal{Y}^{ss} \times \mathbb{A}^1, \mathcal{Y}^{ss}) \xrightarrow{\Phi_w} D^b(\mathcal{Y} \times \mathbb{A}^1, \mathcal{Y}^{ss}) \\
\downarrow \quad \downarrow \quad \downarrow \Phi \\
D^b(\mathcal{Y}^{ss} \times \mathcal{Y}^{ss}) \to D^b(\mathcal{Y} \times \mathcal{Y}^{ss}).
\end{array}
\]

Since $\mathcal{Y}^{ss}$ is a quasi-projective scheme, we have $\mathcal{O}_{\Delta} \in D^b(\mathcal{Y}^{ss} \times \mathcal{Y}^{ss})$. We set $\mathcal{P} = \Phi(\mathcal{O}_{\Delta})$ and $\mathcal{P}_w = \Phi_w(\mathcal{O}_{\Delta})$. Then $\mathcal{P} = i_*\mathcal{P}_w$. Since this holds for any $w$, the object $\mathcal{P}$ is supported on $\mathcal{Y} \times_{Y/G} \mathcal{Y}^{ss}$. Then the object $\mathcal{P}$ induces the Fourier-Mukai functor $D^b(\mathcal{Y}^{ss}) \to D^b(\mathcal{Y})$ which gives the splitting in Theorem 2.2 by the argument in [HIT15, Section 2.3]. □
6.4. Proof of Proposition 5.10

Proof. The assertion is trivial if $\dim V \leq 1$. Below we assume that $\dim V \geq 2$. Note that $\ord_0(w_p) \geq 2$ where $\ord_0(w_p)$ is the vanishing order of $w_p$ at $0$. This is because $w_p(0) = 0$ by the first inclusion in (6.10) together with the fact that $0 \in \Crit(w_p) \neq 0$.

Let us consider the Hessian of $w_p$

$$\Hess(w_p): \Ext^1_Q(R, R) \otimes \mathcal{O}_{\mathcal{X}_q}(d) \to \Ext^1_Q(R, R) \otimes \mathcal{O}_{\mathcal{X}_q}(d).$$

The kernel of the above morphism at the origin is $\Ext^1_Q(V, V)(R, R)$. By the relation (5.8), we have

$$\Ext^1_Q(S_m, S_m) = \Ext^1_Q(\mathcal{O}_C(m - 1), \mathcal{O}_C(m - 1)) = 0.$$  

It follows that

$$(6.10) \quad \ker(\Hess(w_p)|_0) \cap (\End(V) \otimes \Ext^1_Q(S_m, S_m)) = 0.$$

By Lemma 6.8 below, by replacing the isomorphism $\eta_0$ in (5.19) if necessary, there exist linear subspaces

$$W_1 \subset \Ext^1_Q(R_\infty, R_\infty), W_2 \subset \Ext^1_Q(R_\infty, S_m), W_3 \subset \Ext^1_Q(S_m, R_\infty)$$

such that $w_p$ is written as $w_p = w_1 + w_2$, where $w_1$ does not contain variables from $\End(V) \otimes \Ext^1_Q(S_m, S_m)$ with $\deg(w_1) \geq 3$, and $w_2$ is a non-degenerate $G$-invariant quadratic form on

$$W_1 \oplus (W_2 \otimes V) \oplus (W_3 \otimes V^\vee) \oplus (\End(V) \otimes \Ext^1_Q(S_m, S_m))$$

$$= (W_1 \oplus \Ext_Q(S_m, S_m)) \oplus (W_2 \otimes V) \oplus (W_3 \otimes V^\vee) \oplus (\End_0(V) \otimes \Ext^1_Q(S_m, S_m)).$$

As we assumed that $\dim V \geq 2$, the $GL(V)$-representation $\End_0(V)$ is a non-trivial irreducible $GL(V)$-representation, and it is not isomorphic to $V$ or $V^\vee$. Therefore $w_2$ is written as $w_2 = w_3 + q$ where $w_3$ does not contain variables from $\End_0(V) \otimes \Ext^1_Q(S_m, S_m)$ and $q$ is a non-degenerate $GL(V)$-invariant quadratic form on $\End_0(V) \otimes \Ext^1_Q(S_m, S_m)$. Moreover $w_3$ is non-zero, since otherwise it contradicts with (6.10) and $\End_0(V) \subset \End(V)$. By replacing the isomorphism (5.24) if necessary, we can also assume that $q$ coincides with (5.27). Therefore we obtain a desired form (5.27).

We have used the following lemma, whose proof is a variant of [Joy15 Proposition 2.24]:

**Lemma 6.8.** Let $G$ be a reductive algebraic group and $V$ be a finite dimensional $G$-representation. Let $w: \hat{V} \to \mathbb{A}^1$ be a $G$-invariant formal function such that $\ord_0(w) \geq 2$. Let $V_1$ be the kernel of the Hessian at the origin

$$V_1 = \ker(\Hess(w)|_0: V \to V^\vee).$$

Then there exists a direct sum decomposition $V = V_1 \oplus V_2$ of $G$-representations and a $G$-equivariant isomorphism $\phi: \hat{V} \xrightarrow{\cong} \hat{V}$ such that $\phi^*w = w_1 + w_2$, where $w_1 \in \mathcal{O}_{\hat{V}_1}$ is $G$-invariant with $\ord_0(w_1) \geq 3$, and $w_2 \in \text{Sym}^2(V_2^\vee)$ is a $G$-invariant non-degenerate quadratic form on $V_2$.

**Proof.** As $w$ is $G$-invariant, the Hessian of $w$ at the origin $\Hess(w)|_0: V \to V^\vee$ is $G$-equivariant. As $G$ is reductive, there is a splitting $V = V_1 \oplus V_2$ as $G$-representations and the Hessian at the origin is written as

$$\Hess(w)|_0 = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}: V_1 \oplus V_2 \to V_1^\vee \oplus V_2^\vee$$

where $q$ is a $G$-equivariant isomorphism $q: V_2 \xrightarrow{\cong} V_2^\vee$ with $q^\vee = q$. We identify $q$ as an element $q \in \text{Sym}^2(V_2^\vee)^G$, which is a $G$-invariant non-degenerate quadratic form $q$ on $V_2$. For $(y_1, y_2) \in V_1 \oplus V_2$, we can write $w(y_1, y_2)$ as

$$w(y_1, y_2) = w_2^\geq 3(y_1, y_2) + q(y_2)$$
where \( w^{\geq 3}(y_1, y_2) \) consists of terms with degrees bigger than or equal to three. We set \( d_i = \text{dim } V_i \) and fix basis of \( V_1, V_2 \) so that we write elements of them as \( y_1 = \{ y_1^{(i)} \}_{1 \leq i \leq d_1}, y_2 = \{ y_2^{(i)} \}_{1 \leq i \leq d_2} \) respectively. Here we take an orthonormal basis for \( V_2 \) so \( q \) is written as
\[
q(y_2) = \frac{1}{2} \sum_{i=1}^{d_1} (y_2^{(i)})^2.
\]

Then the closed subscheme
\[
\left\{ \frac{\partial w}{\partial y_2^{(i)}} = 0 : 1 \leq i \leq d_2 \right\} = \left\{ y_2^{(i)} + \frac{\partial w^{\geq 3}}{\partial y_2^{(i)}} = 0 : 1 \leq i \leq d_2 \right\} \subset \hat{V}
\]
is smooth of codimension \( d_2 \). By the variable change
\[
(6.12)
y_2^{(i)} \mapsto \frac{\partial w}{\partial y_2^{(i)}} = y_2^{(i)} + \frac{\partial w^{\geq 3}}{\partial y_2^{(i)}}
\]
we may assume that \( \text{Crit}(w) \) is contained in \( \{ y_2 = 0 \} \subset \hat{V} \). The variable change \((6.12)\) can be described without coordinates as follows. Let \( dw \) be the morphism given by the derivation of \( w \)
\[
(6.13)
dw : V \otimes \hat{O}_{\hat{V}} \rightarrow \hat{O}_{\hat{V}}.
\]
We have the following morphisms
\[
\phi : V^{\vee} = V_1^{\vee} \oplus V_2^{\vee} \xrightarrow{(id, g^{-1})} V_1^{\vee} \oplus V_2^{(id, dw|_{V_2})} \hat{O}_{\hat{V}}.
\]

The above composition induces the isomorphism \( \hat{O}_{\hat{V}} \xrightarrow{\cong} \hat{O}_{\hat{V}} \), which is identified with the variable change \((6.12)\). The above construction is \( G \)-equivariant, so the variable change \((6.12)\) is \( G \)-equivariant.

The condition that \( \text{Crit}(w) \subset \{ y_2 = 0 \} \) implies that each \( y_2^{(i)} \) is written as
\[
y_2^{(i)} = \sum_{j=1}^{d_1} a_{ij} \frac{\partial w}{\partial y_1^{(j)}} + \sum_{j=1}^{d_2} b_{ij} \frac{\partial w}{\partial y_2^{(j)}}
\]
for some \( a_{ij}, b_{ij} \in \hat{O}_{\hat{V}} \). By writing \( b_{ij} = b_{ij}(0) + b_{ij}^{(1)} \) and comparing the degree one terms for \( y_2 \), we see that \( b_{ij}(0) = \delta_{ij} \). Therefore we obtain the relation
\[
\frac{\partial w^{\geq 3}}{\partial y_2^{(i)}} = \sum_{j=1}^{d_1} a_{ij} \frac{\partial w^{\geq 3}}{\partial y_1^{(j)}} + \sum_{j=1}^{d_2} b_{ij}^{(1)} \left( y_2^{(j)} + \frac{\partial w^{\geq 3}}{\partial y_2^{(j)}} \right).
\]
The Nakayama lemma implies the inclusion of ideals
\[
(6.14) \left( \frac{\partial w^{\geq 3}}{\partial y_2^{(i)}} : 1 \leq i \leq d_2 \right) \subset \left( \frac{\partial w^{\geq 3}}{\partial y_1^{(j)}}, y_2^{(i)} : 1 \leq j \leq d_1, 1 \leq i \leq d_2 \right)
\]
in \( \hat{O}_{\hat{V}} \), the formal completion at the maximal ideal of \( \hat{O}_{\hat{V}} \). Since these are \( G \)-invariant ideals, by the coherent completeness of \( \hat{V}/G \) the inclusion \((6.14)\) also holds in \( \hat{O}_{\hat{V}} \) (see the proof of Lemma \( 6.3 \)). In particular there is a relation of the form
\[
(6.15) \left. \frac{\partial w}{\partial y_2^{(i)}} \right|_{y_2=0} = \sum_{i,j} c_{ij} \left. \frac{\partial w}{\partial y_1^{(j)}} \right|_{y_2=0}
\]
for some \( c_{ij} \in \hat{O}_{\hat{V}} \). We apply the variable change
\[
(6.16) \tilde{y}_1^{(i)} = y_1^{(i)} + \sum_j c_{ij} y_2^{(i)}, \tilde{y}_2^{(i)} = y_2^{(i)}.
\]
Then we have
\[
\frac{\partial w}{\partial y^{(i)}_2} \bigg|_{\tilde{y}_2 = 0} = \left( \sum_j \frac{\partial y^{(j)}_1}{\partial y^{(i)}_2} \frac{\partial w}{\partial y^{(j)}_1} + \sum_j \frac{\partial y^{(j)}_2}{\partial y^{(i)}_2} \frac{\partial w}{\partial y^{(j)}_2} \right) \bigg|_{\tilde{y}_2 = 0} = -\sum_j c_{ij} \frac{\partial w}{\partial y^{(j)}_1} \bigg|_{\tilde{y}_2 = 0} + \frac{\partial w}{\partial y^{(i)}_2} \bigg|_{\tilde{y}_2 = 0} = 0.
\]

It follows that we can assume that \((\partial w/\partial y^{(i)}_2)|_{y_2 = 0} = 0\).

We see that the variable change (6.16) can be taken to be \(G\)-equivariant. For the morphism (6.13), we can write \(dw \otimes \mathcal{O}_{\hat{V}_1}\) as
\[
dw \otimes \mathcal{O}_{\hat{V}_1} = \alpha^{(1)} + \alpha^{(2)}; (V_1 \otimes \mathcal{O}_{\hat{V}_1}) \oplus (V_2 \otimes \mathcal{O}_{\hat{V}_1}) \rightarrow \mathcal{O}_{\hat{V}_1}.
\]

Then the ideals of \(\mathcal{O}_{\hat{V}_1}\)
\[
I_1 = \left( \frac{\partial w}{\partial y^{(i)}_1} \bigg|_{y_2 = 0} \right), \quad I_2 = \left( \frac{\partial w}{\partial y^{(i)}_2} \bigg|_{y_2 = 0} \right)
\]
are generated by the images of \(\alpha^{(1)}, \alpha^{(2)}\) respectively, so in particular they are \(G\)-invariant. By the relation (6.15) we have \(I_2 \subset I_1\). We have the following \(G\)-equivariant diagram
\[
\begin{array}{ccc}
V_2 \otimes \mathcal{O}_{\hat{V}_1} & \xrightarrow{\alpha^{(2)}} & I_2 \\
\phi \downarrow & & \downarrow \\
V_1 \otimes \mathcal{O}_{\hat{V}_1} & \xrightarrow{\alpha^{(1)}} & I_1
\end{array}
\]
where each horizontal arrows are surjections. As \(G\) is reductive, from the above diagram there is a \(G\)-equivariant dotted arrow \(\phi\) which makes the above diagram commutative. A choice of \(\phi\) corresponds to a choice of \(c_{ij}\) in (6.13). Then we have the \(G\)-equivariant morphism
\[
V' = V_1' \oplus V_2' \xrightarrow{\text{id} + \phi'(\text{id})} \mathcal{O}_{\hat{V}}.
\]
The above morphism induces the \(G\)-equivariant isomorphism \(\tilde{O}_V \xrightarrow{\sim} \hat{O}_V\), which corresponds to the variable change (6.16). In particular we can choose \(c_{ij}\) so that (6.10) is \(G\)-equivariant.

Finally we set
\[
g(y_1, y_2) := w(y_1, y_2) - w(y_1, 0).
\]
Then from the above arguments we have \(g(y_1, 0) = 0\) and \((\partial g/\partial y^{(i)}_2)|_{y_2 = 0} = 0\). It follows that \(g(y_1, y_2)\) is written as
\[
g(y_1, y_2) = \sum_{i,j} y^{(i)}_2 y^{(j)}_2 Q_{ij}(y_1, y_2)
\]
for some \(Q_{ij} \in \mathcal{O}_{\hat{V}}\). As the quadratic term of \(g(y_1, y_2)\) coincides with \(q\) by (6.11), we have \(Q_{ij}(0) = 1/2 \cdot \delta_{ij}\). It follows that the critical locus of \(g(y_1, y_2)\) is \(\{y_2 = 0\} \subset \hat{V}\), so the \(G\)-equivariant Morse lemma (see [AGZV85, Section 17.3]) applied for \(g\) implies that by a \(G\)-equivariant variable change of the form \(\tilde{y}^{(i)}_1 = y^{(i)}_1, \tilde{y}^{(i)}_2 = \sum_{i,j} a^{(i)}(y_1, y_2) y^{(j)}_2\) we can make \(g(\tilde{y}_1, \tilde{y}_2) = q(\tilde{y}_2)\). As \(\text{ord}_0(w(y_1, 0)) \geq 3\) from (6.11), the lemma is proved. \(\square\)
A theory of generalized Donaldson-Thomas invariants

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