Cutoff-independent ergodic approximation to the
generic three-body problem

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Abstract: The gravitational three-body problem is generically chaotic and negative energy motions generically decay to a binary + free body. Within the ergodic approximation a statistical theory was introduced and a probability distribution over outcomes was determined. That distribution depends on a single adjustable parameter, the strong interaction radius. Here this cutoff is removed while keeping the probabilities finite. The associated expression and its derivation simplify. As an application, marginalized probability distributions are determined for energy, angular momentum and eccentricity as well as the ejection probability for each one of the masses. Two simpler limits are discussed corresponding to low and high dimensionless total angular momentum.
1 Introduction

The three body problem is concerned with the study of the motion of three point masses moving under the influence of their mutual gravitational attractions.

This problem has a long and venerable history. It was appreciated already by Newton and discussed in his 1687 Principia [1]. Its name began to be common in the 1740s in connection to a rivalry between d’Alembert and Clairaut who studied it. Euler [2] and Lagrange [3] obtained special solutions, in 1767 and 1772 respectively, and their restriction became known as Lagrangian points.

The problem has perturbative limits when masses or orbit sizes are hierarchical. Considerable important work was done in this context, but in this paper we shall focus on the generic problem.

By 1887 the problem has become so celebrated that a prize was offered over it by Oscar II, king of Sweden, advised by the mathematician Mittag-Leffler. Poincare (born 1854) took this challenge and after first submitting a faulty submission finally came to realize that the problem exhibits a sensitivity to initial conditions, in the sense that a small change in initial conditions grows fast in time [4]. This makes a general analytic solution impossible and breaks the mechanical paradigm that knowledge of a system’s forces and initial conditions allows to predict its future for arbitrary late times, because the initial conditions are always known only up to some accuracy, but then the accuracy of the prediction becomes useless after some time. In modern terms this was the first example of a non-integrable, chaotic system, see e. g. [5]. Moreover, this realization won Poincare the prize. At the beginning of the 20th century, this was the status of the three-body problem.

While it is impossible to predict the general motion of the three body system up to the far future, it still makes sense to ask what would be the likely result of any given initial conditions. This suggests to seek a statistical solution. This is not the usual Statistical Physics approach which holds for a large collection of particles, but rather one considers an ensemble of initial conditions for a system with few degrees of freedom. Moreover, it is
natural to define probability to be proportional to phase space volume, namely, an ergodic approximation. In general, the more chaotic a system is, the more limited in time is the validity of solutions to the equations of motion, while at the same the statistical analysis becomes more accurate. In this way, the non-integrability is turned from a liability into an advantage.

The development of computers and computational physics allowed to integrate the three-body equations of motion numerically \[6, 7, 8\] and to do so for large numbers of initial conditions thereby providing much data to the study of such systems. In 1976 Monaghan suggested the statistical approach to the three-body problem \[9\], which was later developed in \[10, 11\]. See also the book \[12\] and the review \[13\].

Recently, Stone and Leigh \[14\] presented the distribution over outcomes in closed form within the ergodic approximation. However, their expression depends on a spurious parameter, the strong interaction radius, which plays the role of a long-distance cutoff. The derivation uses the canonical transformations to elliptic/hyperbolic Delaunay elements.

This paper reports a cutoff-independent outcome distribution. In section 2 we set-up the problem and present the outcome distribution, which is the main result. Its derivation relies only on the phase-space volumes of the Keplerian binary and that of free motion. In section 3 we discuss the dimensionless parameters of the problem and their limits. Then we employ the outcome distribution to determine via marginalization marginal distributions for the energy, angular momentum, eccentricity and escaper identity.

This is an announcement version and it is planned to be replaced by a version with more information on the derivation, including the simple physics that goes into the results for the limits in parameter space.

2 Outcome distribution

Setup

Consider a three-body problem with masses

\[ m_1, m_2, m_3. \] (2.1)

The center of mass moves with constant velocity, and hence we shall work in the center of mass frame, where the system has 6 degrees of freedom. The conserved quantities are

\[ E, \vec{L}. \] (2.2)

the total energy and the total angular momentum. For

\[ E < 0 \] (2.3)

which we henceforth assume, the decay into 3 freely moving bodies is energetically forbidden. However, a decay into a binary + free body is possible and moreover, \textit{its phase space volume contains a divergent factor since the two subsystems could be separated by any distance}. This suggests that any negative \( E \) motion will ultimately decay in this way and
indeed this was found to be the case in numerical integrations. The situation is similar to a body moving inside a finite box that has a small hole in its wall.

The phase space volume of the decay configurations is dominated by large separations between the binary and the ejected free body, which is known as the escaper. In fact, due to the divergent factor, *the three-body phase space volume for decay is equal to that of a decoupled system composed of the free relative motion and the Keplerian binary motion*. In particular, the gravitational interaction of the binary with the escaper can be neglected. We denote the escaper mass by \( m_s \) and the binary masses by \( m_a, m_b \).

The conserved quantities for the decoupled decay products are the union of those for each motion separately. For the free motion in the center of mass frame, with masses \( m_s, m_a + m_b \) the conserved quantities are

\[
\epsilon_F, \vec{l}_F \tag{2.4}
\]

the energy and angular momentum (or equivalently, the relative velocity and the impact parameter) and the reduced mass is

\[
\mu_F = \frac{m_s(m_a + m_b)}{M} \tag{2.5}
\]

where \( M = m_1 + m_2 + m_3 \) is the total mass. For the Keplerian motion the conserved quantities are

\[
\epsilon, \vec{l}, \vec{A} \tag{2.6}
\]

the energy, angular momentum and Laplace-Runge-Lenz (LRL) vector, which are subject to the relations

\[
A^2 = \mu_B^2 \alpha^2 + 2 \mu_B \epsilon l^2 \tag{2.7}
\]

\[
0 = \vec{l} \cdot \vec{A} \tag{2.8}
\]

where

\[
\mu_B := \frac{m_a m_b}{m_a + m_b} \tag{2.9}
\]

is the reduced binary mass and

\[
\alpha := G m_a m_b \tag{2.10}
\]

where \( G \) is Newton’s gravitational constant, is the coefficient of gravitational attraction. \( A^2 \geq 0 \) and (2.7) imply the inequality

\[
-2 \epsilon l^2 \leq \mu_B \alpha^2 \tag{2.11}
\]

which will play an important role.

The phase space distribution for the free motion is

\[
d\sigma_F = \rho_F(\epsilon_F, \vec{l}_F) d\epsilon_F \, d^3l_F = 2\pi R_\infty \sqrt{\mu_F} \frac{d\epsilon_F}{\sqrt{2\epsilon_F}} \frac{d^3l_F}{l_F} \tag{2.12}
\]
where $d\sigma$ is the phase space element, $\rho$ is the distribution density, and $R_\infty$ is the size strong interaction region, which is taken to infinity and cancels out in all probabilities. The phase space distributions for the Keplerian binary motion is

$$d\sigma_B = \rho_B(\epsilon, \vec{l}) \, d\epsilon \, d^3l = 4\pi^2 \sqrt{\mu_B} \alpha \frac{d\epsilon}{(-2\epsilon)^{3/2}} \frac{d^3l}{l}$$

(2.13)

We note that in the free distribution $\epsilon_F, \vec{l}_F$ are independent, while in the Keplerian distribution they couple only through (2.11), which defines the allowed region in $\epsilon, l$ space. Also, the binary phase space distribution is uniform over (independent of) the direction of $\vec{A}$, namely the direction of perihelion, and it is not mentioned in (2.13).

Given the conserved charges $E, \vec{L}$ the three-body phase space element in the center of mass frame is defined by

$$d\sigma = \prod_{a=1}^{3} (d^3r_a \, d^3p_a) \delta(3) \left( \frac{1}{M} \sum_{a=1}^{3} m_a \vec{r}_a \right) \delta(3) \left( \sum_{a=1}^{3} \vec{p}_a - \vec{r}_a \right) \delta(H(\vec{r}_a, \vec{p}_a)_{a=1}^3 - E) \delta(3) \left( \vec{j}(\vec{r}_a, \vec{p}_a)_{a=1}^3 - \vec{L} \right)$$

(2.14)

Using the decoupling of decay products together with (2.12,2.13) we find that the phase space distribution over outcomes is

$$d\sigma = d\sigma(\epsilon, \vec{l}, \epsilon_F, \vec{l}_F; E, \vec{L}; m_s, m_a, m_b) =$$

$$= C_0 \frac{d\epsilon \, d\epsilon_F \, \delta(\epsilon + \epsilon_F - E) \, d^2l \, d^2l_F \, l \cdot l_F}{\sqrt{(-\epsilon)^3 \epsilon_F}} \frac{\delta(2) \left( \vec{i} + \vec{i}_F - \vec{l}_F \right)}{l \cdot l_F}$$

(2.15)

where we proceed to define the notation: $s = 1, 2, 3$ is the identity of the escaper ; $C_0$ is a constant, which is independent of the variables $\epsilon, \vec{i}, \epsilon_F, \vec{i}_F$ and of the identity of the escaper, and hence does not affect probabilities. It is given by

$$C_0 := \frac{1}{4} (2\pi)^4 R_\infty G \sqrt{\frac{(m_1 m_2 m_3)^3}{m_1 + m_2 + m_3}}$$

(2.16)

and finally $d^2l := dl || dl_\perp$, where $l ||$ is the component of $\vec{l}$ in the direction of $\vec{L}$ and $l_\perp$ is the magnitude of the perpendicular component, and similarly for $d^2l_F$. Within the $\epsilon, l$ plane, the distribution is confined to the region (2.11) together with $\epsilon \leq E$ (which follows from $\epsilon_F \geq 0$).

Normalized outcome probabilities $dP$ are given by

$$dP(\epsilon, \vec{l}, \epsilon_F, \vec{i}_F; E, \vec{L}) = \frac{1}{Z(E, \vec{L})} \, d\sigma(\epsilon, \vec{l}, \epsilon_F, \vec{i}_F; E, \vec{L})$$

(2.17)

where

$$Z(E, \vec{L}) := \sum_{s=1,2,3} \int d\sigma_s$$

(2.18)

where the sum is over the identity of the escaper, and the subscript $s$ was added to $d\sigma$ to stress its $s$ dependence.
The outcome phase space distribution (2.15) is the main result of this paper.

**Confirmation.** The distribution (2.15) coincides with the large cutoff limit, $R \to \infty$ of the distribution of [14] (eq. 2), upon the replacement $R \to R_\infty$ and up to an overall prefactor independent of the variables $\epsilon, \vec{l}, \epsilon_F, \vec{l}_F$ and the identity of the escaper. In fact, the distribution of [14] seems to have incorrect dimensions, but such a prefactor does not affect probabilities.

This agreement confirms our result, which was obtained through a simpler derivation, without using Delaunay elements.

The usefulness of the ergodic approximation was studied in depth in [14] by comparing with numerical simulations. In general, reasonable (and sometimes very close) agreement was found for a certain choice of $R$, the strong interaction region, and as long as the considered processes underwent 2 or more scrambles, which are periods of time in which no pairwise binaries exist. The dependence on the choice of $R$ was studied there as well.

### 3 Marginalization

We now proceed to analyze the distribution (2.15).

**Dimensionless parameters.** The distribution depends on the following parameters: the conserved charges $E, \vec{L}$, the masses $m_1, m_2, m_3$ and $G$. We denote

$$k_s := \mu_B \alpha^2.$$  \hfill (3.1)

This is the RHS of the binary inequality (2.11), and the subscript $s = 1, 2, 3$ stresses that it depends on the identity of the escaper. Now we define the following 3 dimensionless parameters

$$x_s^2 := \frac{-2EL^2}{k_s}.$$  \hfill (3.2)

These dimensionless quantities are independent and complete (any other dimensionless quantity must be a function of these).

We shall use the following re-definitions of the variable $x$

- **Eccentricity** $e$ defined by

  $$e^2 = 1 - x^2.$$  \hfill (3.3)

  $e$ has the interpretation of binary eccentricity for $\epsilon = E, \vec{l} = \vec{L}$, namely if all the energy and angular momentum of the system goes to the binary.

- **An angle** $\theta$ and rapidity $w$ defined by

  $$e = \sin \theta = \tanh w.$$  \hfill (3.4)

  This implies $x = \cos \theta$.

Just like $x$ depends on $s$, so do $e$, $\theta$ and $w$.

The problem has two limits
• $L = 0$ and hence $x_s = 0, s = 1, 2, 3$

• $x_s \geq 1$. Strictly speaking this is a region in parameter space, but from the perspective of the interval $0 \leq x_s \leq 1$ is can also be considered to be a limit.

In these two limits the marginalization simplifies. It is instructive to consider them before considering the general case, but in the interest of brevity we shall determine the general marginalization and obtain the simpler result in these limits by specializing.

**Energy distribution.** After integrating over angular momenta we find that for $x_s \leq 1$ the energy distribution is

$$d\sigma = \frac{C_0}{2L} \frac{k_s}{m_s} \frac{d\epsilon}{(-\epsilon)^{3/2}\sqrt{\epsilon_F}} \left\{ \begin{array}{ll} 2\sqrt{\epsilon_s - \tilde{\epsilon}_s} & x_s^2 \leq \tilde{\epsilon}_s \leq 1 \\ 1 & 1 \leq \tilde{\epsilon}_s \leq \infty \end{array} \right. \tag{3.5}$$

where we define

$$\epsilon_s := -\frac{k_s}{2L^2} \tag{3.6}$$

and

$$\epsilon_F := E - \epsilon \quad \tilde{\epsilon}_s := \frac{\epsilon}{\epsilon_s} \tag{3.7}$$

For $L = 0$ we have $\epsilon_s \to -\infty$, only the first term in the first range is relevant and the energy distribution simplifies to

$$d\sigma = C_0 \frac{\sqrt{2k_s}}{m_s} \frac{d\epsilon}{(-\epsilon)^{3/2}\sqrt{\epsilon_F}} \tag{3.8}$$

while for $x_s \geq 1$ the distribution naturally simplifies to

$$d\sigma = \frac{C_0}{2L} \frac{k_s}{m_s} \frac{d\epsilon}{(-\epsilon)^{3/2}\sqrt{\epsilon_F}}. \tag{3.9}$$

**Average binary energy.** Using this energy distribution we determined the average binary energy with known escaper identity and $x_s \leq 1$ to be

$$\langle \epsilon \rangle = 3 \frac{1 - \sin \theta + 2\theta \cos \theta - w \cos^2 \theta}{2 - (3 - \sin^2 \theta) \sin \theta + 3\theta \cos \theta} E \tag{3.10}$$

where $\theta \equiv \theta_s, w \equiv w_s$ were defined in (3.4).

In the $L = 0$ limit where $0 = \theta = w$ we have

$$\langle \epsilon \rangle = \frac{3}{2} E, \tag{3.11}$$

while for $x_s \geq 1$ (including in the $x_s \to 1$ limit where $\theta \to \pi/2$) we have

$$\langle \epsilon \rangle \to 2 E. \tag{3.12}$$
In the intermediate range $0 \leq \theta/2$, $\langle \epsilon \rangle/E$ is a monotonous rising function, as can be confirmed by plotting it [15].

**Angular momentum distribution.** After integrating over energy and $\vec{l}$ directions we find that for $x_0 \leq 1$ the distribution for $l \equiv |\vec{l}|$ is given by

$$d\sigma = \frac{4}{(-E)L} \frac{C_0}{m_s} \frac{1}{\sqrt{1 - \frac{l^2}{l_s^2}}} \left\{ \begin{array}{ll}
1 & l \leq L \\
L & L \leq l \leq l_s
\end{array} \right. \quad (3.13)$$

where

$$l_s^2 := -\frac{k_s}{-2E} \quad (3.14)$$

For $L = 0$ the distribution simplifies to

$$d\sigma = \frac{4}{(-E)} \frac{C_0}{m_s} \frac{1}{\sqrt{1 - \frac{l^2}{l_s^2}}} dl \quad (3.15)$$

in the whole range $0 \leq l \leq l_s$, while for $x_s \geq 1$ we have $l_s \leq L$ and the distribution becomes

$$d\sigma = \frac{4}{(-E)L} \frac{C_0}{m_s} \frac{1}{l} \sqrt{1 - \frac{l^2}{l_s^2}} dl \quad (3.16)$$

in the same range.

**Eccentricity distribution.** We find that

$$d\sigma = \frac{2C_0}{(-3E)m_s} e_s de_s \left\{ \begin{array}{ll}
f\left(\frac{x_s}{\sqrt{1-e_s^2}}\right) & 0 \leq e_s^2 \leq 1 - x_s^2 \\
\frac{2k_s}{-EL} & 1 - x_s^2 \leq e_s^2 \leq 1
\end{array} \right. \quad (3.17)$$

where

$$f(y) := 4 - (4-y^2)\sqrt{1-y^2} + \frac{3\pi}{2} y - 3 y \arcsin y \quad (3.18)$$

For $L = 0$ only the top line in (3.17) is relevant, $f$ is dominated by the term linear in $y$ and the distribution simplifies to

$$d\sigma = \frac{\pi C_0}{(-E)m_s} \frac{l_s}{\sqrt{1-e_s^2}} e_s de_s \quad (3.19)$$

while for $x_s \geq 1$ only the bottom line is relevant and the distribution becomes

$$d\sigma = \frac{4C_0}{3E^2 L} \frac{k_s}{m_s} e de. \quad (3.20)$$

Both distributions are defined over the whole range $0 \leq e \leq 1$.

**Escaper probability.** After integrating over both energy and angular momentum we find the phase space volume (unnormalized probability) for the escaper identity

$$\sigma_s = \frac{2C_0}{3E^2 L} \frac{k_s}{m_s} \left\{ \begin{array}{ll}
1 - e_s^3 + \frac{3}{2} x_s (\theta_s - e_s x_s) & x_s \leq 1 \\
1 & 1 \leq x_s
\end{array} \right. \quad (3.21)$$
The lightest mass has the highest ejection probability (this appears to hold also for three-body problems outside the realm of physics). This can be argued as follows. \( m_s \sigma_s \) is monotonous in \( k_s \) since the distribution (2.15) is positive and independent of \( k_s \) while its region (2.11,3.1) increases with \( k_s \). Since both \( \mu_B \) and \( \alpha \) are largest for the smallest mass, so is \( k_s \) and hence \( m_s \sigma_s \) is maximal for it, and hence so is \( \sigma_s \).

We note that for \( L = 0 \) the expression is dominated by the term linear in \( x_s \) and hence it reduces to

\[
\sigma_s(x_s = 0) = \frac{2 \pi C_0}{\sqrt{(-2E)^3}} \frac{\sqrt{k_s}}{m_s}.
\]

while for \( x_s \geq 1 \) it becomes

\[
\sigma_s(x_s \geq 1) = \frac{2 C_0}{3 E^2 L} \frac{k_s}{m_s}.
\]

4 Discussion

The main result of the paper is the outcome distribution (2.15). It expresses the phase space volume distributed over the possible binary \(+\) free decay outcomes and within the ergodic approximation it translates into outcome probabilities. This distribution is is cutoff independent unlike that of \([14]\).

The expression for the distribution was confirmed as discussed at the end of section 2 and as far as we know it does not appear in the literature. It presents the following advantages

- No spurious parameter.
- Simpler derivation, not using Delaunay elements.
- Produces convenient analytical marginal distributions, including for the identity of the escaper, as demonstrated in section 3.

Open questions. This paper does not address the validity and usefulness of the ergodic approximation, especially that islands of regular motion are known to exist. It would be interesting to study

- Can the error be estimated, at least in some limit?
- Can corrections be defined and determined?

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A Useful integrals

In this section we collect some useful integrals, obtained e.g. by [15]

\[
\int \frac{dx}{(1 + x)^{3/2}\sqrt{x}} = 2\sqrt{x} + x
\]

\[
\int \frac{dx}{(1 + x)^{2}\sqrt{x}} = \frac{\sqrt{x}}{1 + x} - \arctan \sqrt{x}
\]

\[
\int \frac{dx}{(1 + x)^{5/2}\sqrt{x}} = \frac{2(x+3)}{3\sqrt{(1+x)^3}}
\]

\[
\int_{-1}^{1} \frac{d\cos \theta}{(l^2 + L^2 - 2lL \cos \theta)^{1/2}} = \frac{2}{\max\{l,L\}} \tag{A.1}
\]

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