The Spectrum of the Singular Values of Z-Shaped Graph Matrices

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Abstract

Graph matrices are a type of matrix which appears when analyzing the sum of squares hierarchy and other methods using higher moments. However, except for rough norm bounds, little is known about graph matrices. In this paper, we take a step towards better understanding graph matrices by determining the spectrum of the singular values of Z-shaped graph matrices.

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1 Introduction

Graph matrices are a type of matrix which appears naturally when analyzing the sum of squares hierarchy and other methods which analyze higher moments. In the paper “Graph Matrices: Norm Bounds and Applications” [1], Ahn, Medarametla, and Potechin proved rough norm bounds on all graph matrices and described several applications of graph matrices in analyzing random subspaces and analyzing tensor decomposition. However, beyond these rough norm bounds, little is known about graph matrices.

A natural question is whether we can analyze graph matrices more carefully. Can we make the norm bounds on graph matrices tight up to a factor of $(1 \pm o(1))$ rather than polylog$(n)$? More ambitiously, can we determine the spectrum of graph matrices? In other words, can we find analogues of Wigner’s semicircle law [3, 4] and Girko’s Circular Law [2] for graph matrices?

In this paper, we take a step towards this goal by analyzing the spectrum of the singular values of $Z$-shaped and multi-$Z$-shaped graph matrices.

1.1 Definitions

In order to state our results, we need a few definitions.

Definition 1.1 (Fourier Characters). Given a graph $G = (V(G), E(G))$ and a multi-set of possible edges $E \subseteq \binom{V(G)}{2}$, define $\chi_E(G) = \prod_{e \in E} e(G)$ where the edge variable $e(G)$ is $e(G) = 1$ if $e \in E(G)$ and $-1$ otherwise.

Definition 1.2 (Shapes). We define a shape $\alpha$ to be a graph with vertices $V(\alpha)$, edges $E(\alpha)$, and distinguished tuples of vertices $U_\alpha = (u_1, \ldots, u_{|U_\alpha|})$ and $V_\alpha = (v_1, \ldots, v_{|V_\alpha|})$.

Definition 1.3 (Bipartite Shapes). We say that a shape $\alpha$ is bipartite if $U_\alpha \cap V_\alpha = \emptyset$, $V(\alpha) = U_\alpha \cup V_\alpha$, and all edges in $E(\alpha)$ are between $U_\alpha$ and $V_\alpha$.

Definition 1.4 (Graph Matrices). Given a shape $\alpha$, we define the graph matrix $M_\alpha$ (which depends on the input graph $G$) to be the $\frac{n!}{(n-|U_\alpha|)!} \times \frac{n!}{(n-|V_\alpha|)!}$ matrix with rows indexed by tuples of $|U_\alpha|$ distinct vertices, columns indexed by tuples of $|V_\alpha|$ distinct vertices, and entries

$$M_\alpha(A, B) = \sum_{\sigma: V(\alpha) \to V(G), \sigma \text{ is injective, } \sigma(U_\alpha) = A, \sigma(V_\alpha) = B} \chi_\sigma(E(\alpha))(G).$$

1.2 Our Results

Our main result is determining the spectrum of the singular values of the $Z$-shaped graph matrix.

Definition 1.5. Let $\alpha_Z$ be the bipartite shape with vertices $V(\alpha_Z) = \{u_1, u_2, v_1, v_2\}$ and edges $E(\alpha_Z) = \{\{u_1, v_1\}, \{u_2, v_1\}, \{u_2, v_2\}\}$ with distinguished tuples of vertices $U_{\alpha_Z} = (u_1, u_2)$ and $V_{\alpha_Z} = (v_1, v_2)$. See Figure 1.1 for an illustration. We call $\alpha_Z$ the $Z$-shape.
Definition 1.6. Let $a = \frac{3\sqrt{3}}{2}$ and define $g_{\alpha Z} : (0, \infty) \rightarrow \mathbb{R}$ be the function such that

$$g_{\alpha Z}(x) = \frac{i}{\pi} \cdot \left( \sqrt{3} \cdot \sin \left( \frac{1}{3} \cdot \arctan \left( \frac{3}{\sqrt{4x^2/3 - 9}} \right) \right) + \cos \left( \frac{1}{3} \cdot \arctan \left( \frac{3}{\sqrt{4x^2/3 - 9}} \right) \right) \right)$$

if $x \in (0, a]$ and $g_{\alpha Z}(x) = 0$ if $x > a$. See Figure 1.2 for an illustration.

Figure 1.2: The limiting distribution of the singular values of $\frac{1}{n} M_{\alpha Z}$ as $n \rightarrow \infty$

Theorem 1.7. As $n \rightarrow \infty$, the spectrum of the singular values of $\frac{1}{n} M_{\alpha Z}$ approaches $g_{\alpha Z}$.

After proving this result, we apply our techniques to give a partial analysis of the spectrum of the singular values of multi-Z-shaped graph matrices.

1.3 Paper Outline

The remainder of the paper is organized as follows. In section 2, we describe the main techniques we use to prove our results. In section 3, we calculate the trace powers for the Z-shaped graph matrix. In section 4, we use these calculations to determine the limiting distribution of the singular values of $\frac{1}{n} M_{\alpha Z}$ as $n$ goes to infinity. In section 5, we generalize the results in Section 3 to calculate the trace powers for multi-Z shaped graph matrices. In section 6, we use these calculations to determine a differential equation for the limiting distribution of the singular values of one multi-Z shaped graph matrix.
2 Techniques

In this section, we describe our techniques, namely the trace power method and constraint graphs.

2.1 The trace power method

To analyze the spectrum of the singular values of graph matrices \( M \), we use the trace power method, i.e. we compute \( E_{\alpha,\beta} \) for all \( q \in \mathbb{N} \) and use this to deduce what the spectrum of the singular values of \( M \) must be.

**Lemma 2.1.** Let \( \{ M_n : n \in \mathbb{N} \} \) be a family of random matrices where each matrix \( M_n \) is an \( a(n) \times b(n) \) matrix which depends on a random input \( G_n \). If \( r(n) = \min \{ a(n), b(n) \} \), \( \lambda_{\text{max}} > 0 \) is a probabilistic upper bound on almost all singular values of \( M \), then the distribution of the singular values of \( M \) must be.

Proof sketch. Given \( \epsilon, \delta > 0 \) and \( \forall a \in \mathbb{R}, \forall \epsilon > 0, \exists \delta' > 0 : \int_{a-\delta'}^{a+\delta'} g(x)dx < \epsilon \)

then the distribution of the singular values of \( M_n \) approaches \( g(x) \) as \( n \to \infty \). More precisely, for all \( \epsilon, \delta > 0 \) and \( \forall a \geq 0 \),

\[
\lim_{n \to \infty} \frac{1}{r(n)^2} \mathbb{E} \left[ \left( \text{tr} \left( (M_n M_n^T)^q \right) \right) - \mathbb{E} \left[ \text{tr} \left( (M_n M_n^T)^q \right) \right] \right]^2 = 0,
\]

Proof sketch. Given \( \epsilon, \delta > 0 \) and \( a \geq 0 \), choose \( \delta' > 0 \) so that \( \int_{a-\delta'}^{a+\delta'} g(x)dx < \epsilon/8 \) and \( \int_{a+\delta-\delta'}^{a+\delta+\delta'} g(x)dx < \epsilon/8 \). Now take polynomials \( p_1(x) \) and \( p_2(x) \) which only have monomials of even degree, approximate the indicator function \( 1_{x \in [a,a+\delta]} \) on the interval \( [0, \lambda_{\text{max}}] \), and bound it from below and above. More precisely, take \( p_1(x) \) and \( p_2(x) \) so that

1. \( p_1(x) = \sum_{q=0}^{d} c_{1q} x^{2q} \) and \( p_2(x) = \sum_{q=0}^{d} c_{2q} x^{2q} \) for some \( d \in \mathbb{N} \).

2. \( \forall x \in [0, a - \delta], p_1(x) \in [-\epsilon/8, 0] \) and \( p_2(x) \in [0, \epsilon/8] \)

3. \( \forall x \in [a - \delta, a], p_1(x) \in [-\epsilon/8, 0] \) and \( p_2(x) \in [0, 1 + \epsilon/8] \)
4. \( \forall x \in [a, a + \delta'], p_1(x) \in [-\varepsilon/8, 1] \) and \( p_2(x) \in [1, 1 + \varepsilon/8] \)

5. \( \forall x \in [a, a + \delta - \delta'], p_1(x) \in [1 - \varepsilon/8, 1] \) and \( p_2(x) \in [1, 1 + \varepsilon/8] \)

6. \( \forall x \in [a + \delta - \delta', a + \delta], p_1(x) \in [-\varepsilon/8, 1] \) and \( p_2(x) \in [1, 1 + \varepsilon/8] \)

7. \( \forall x \in [a + \delta, a + \delta + \delta'], p_1(x) \in [-\varepsilon/8, 0] \) and \( p_2(x) \in [0, 1 + \varepsilon/8] \)

8. \( \forall x \in [a + \delta + \delta', \max \{a + \delta + \delta', \lambda_{\max}\}], p_1(x) \in [-\varepsilon/8, 0] \) and \( p_2(x) \in [0, \varepsilon/8] \)

9. \( \forall x \geq \max \{a + \delta + \delta', \lambda_{\max}\}, p_1(x) \leq 0 \) and \( p_2(x) \geq 0 \)

Letting \( \{\lambda_i(M_n) : i \in [r(n)]\} \) be the singular values of \( M_n \), we make the following observations about the sums \( \frac{1}{r(n)} \sum_{i \in [r(n)]} p_1(\lambda_i(M_n)) \) and \( \frac{1}{r(n)} \sum_{i \in [r(n)]} p_2(\lambda_i(M_n)) \):

1. Since \( \forall x \geq 0, p_1(x) \leq \mathbb{1}_{x \in [a, a + \delta]} \leq p_2(x) \),

\[
\frac{1}{r(n)} \sum_{i \in [r(n)]} p_1(\lambda_i(M_n)) \leq \left\lfloor \frac{i \in [r(n)] : \lambda_i \in [a, a + \delta]}{r(n)} \right\rfloor \leq \frac{1}{r(n)} \sum_{i \in [r(n)]} p_2(\lambda_i(M_n)).
\]

2. For \( j \in \{1, 2\} \), recalling that \( p_j(x) = \sum_{q=0}^d c_{jq} x^{2q} \),

\[
\frac{1}{r(n)} \sum_{i \in [r(n)]} p_j(\lambda_i(M_n)) = \frac{1}{r(n)} \sum_{q=0}^d c_{jq} \text{tr} \left( (M_n M_n^T)^q \right).
\]

3. For \( j \in \{1, 2\} \), the first condition implies that

\[
\lim_{n \to \infty} \frac{1}{r(n)} \mathbb{E} \left[ \sum_{i \in [r(n)]} p_j(\lambda_i(M_n)) \right] = \sum_{q=0}^d c_{jq} \lim_{n \to \infty} \frac{1}{r(n)} \mathbb{E} \left[ \text{tr} \left( (M_n M_n^T)^q \right) \right] = \sum_{q=0}^d c_{jq} \int_0^\infty x^{2q} g(x) dx = \int_0^\infty p_j(x) g(x) dx.
\]

4. The third condition implies that as \( n \to \infty \), with high probability \( \frac{1}{r(n)} \sum_{i \in [r(n)]} p_1(\lambda_i(M_n)) \) and \( \frac{1}{r(n)} \sum_{i \in [r(n)]} p_2(\lambda_i(M_n)) \) do not deviate too much from their expected values.

Combining these observations, it is sufficient to show that for \( j \in \{1, 2\} \),

\[
\left| \int_{x=0}^\infty p_j(x) g(x) dx - \int_a^{a+\delta} g(x) dx \right| = \int_{x=0}^\infty \left( p_j(x) - \mathbb{1}_{x \in [a, a + \delta]} \right) g(x) dx \leq \frac{\varepsilon}{2}.
\]

To show this, observe that
1. \( \forall x \in [0, \lambda_{\text{max}}] \setminus \left( [a - \delta', a + \delta'] \cup [a + \delta - \delta', a + \delta + \delta'] \right) \), \( |p_j(x) - \mathbb{1}_{x \in [a,a+\delta]}| \leq \epsilon/8. \)

2. \( \forall x \in [a - \delta', a + \delta'] \cup [a + \delta - \delta', a + \delta + \delta'] \), \( |p_j(x) - \mathbb{1}_{x \in [a,a+\delta]}| \leq 1 + \epsilon/8. \)

Thus,

\[
\left| \int_{x=0}^{\infty} \left( p_j(x) - \mathbb{1}_{x \in [a,a+\delta]} \right) g(x)dx \right| \leq \frac{\epsilon}{8} \int_{0}^{\infty} g(x)dx + \int_{a-\delta'}^{a+\delta'} g(x)dx + \int_{a+\delta-\delta'}^{a+\delta+\delta'} g(x)dx < \frac{\epsilon}{2}.
\]

In this draft, we focus on finding the distribution \( g_\alpha(x) \) which satisfies the first condition for a given \( \alpha \). We defer verifying the third condition to the full version of this paper.

### 2.2 Constraint graphs

To use the trace power method to analyze \( M_\alpha \), we use several definitions and results from Section 3 of [1].

**Definition 2.2** (Definition 3.2 of [1]). Given a shape \( \alpha \) and a \( q \in \mathbb{N} \), we define \( H(\alpha, 2q) \) to be the multi-graph which is formed as follows:

1. Take \( q \) copies \( \alpha_1, \ldots, \alpha_q \) of \( \alpha \) and take \( q \) copies \( \alpha_1^T, \ldots, \alpha_q^T \) of \( \alpha^T \), where \( \alpha^T \) is the shape obtained from \( \alpha \) by switching the role of \( U_\alpha \) and \( V_\alpha \).

2. For all \( i \in [q] \), we glue them together by setting \( V_{\alpha_i} = U_{\alpha_i^T} \) and \( V_{\alpha_i^T} = U_{\alpha_{i+1}} \) (where \( \alpha_{q+1} = \alpha_1 \)).

We define \( V(\alpha, 2q) = V(H(\alpha, 2q)) \) and we define \( E(\alpha, 2q) = E(H(\alpha, 2q)) \). See Figure 2.1 for an illustration.

**Remark 2.3.** \( H(\alpha, 2q) \) is defined as a multi-graph because edges will be duplicated if \( U_\alpha \) or \( V_\alpha \) contains one or more edges. That said, in this paper we only consider \( \alpha \) such that \( U_\alpha \) and \( V_\alpha \) do not contain any edges, so here \( H(\alpha, 2q) \) will always be a graph.

**Definition 2.4** (Definition 3.4 of [1]: Piecewise injectivity). We say that a map \( \phi : V(\alpha, 2q) \to [n] \) is piecewise injective if \( \phi \) is injective on each piece \( V(\alpha_i) \) and each piece \( V(\alpha_i^T) \) for all \( i \in [q] \). In other words, \( \phi(u) \neq \phi(v) \) whenever \( u, v \in V(\alpha_i) \) for some \( i \in [q] \) or \( u, v \in V(\alpha_i^T) \) for some \( i \in [q] \).

As observed in [1], with these definitions \( \mathbb{E} \left[ \text{tr} \left( (M_\alpha M_\alpha^T)^q \right) \right] \) can be reexpressed as follows

**Proposition 2.5** (Proposition 3.5 of [1]). For all shapes \( \alpha \) and all \( q \in \mathbb{N} \),

\[
\mathbb{E} \left[ \text{tr} \left( (M_\alpha M_\alpha^T)^q \right) \right] = \sum_{\phi : V(\alpha, 2q) \to [n]: \phi \text{ is piecewise injective}} \mathbb{E} \left[ \chi_{\phi(E(\alpha, 2q))} (G) \right].
\]
To analyze this expression, we use constraint graphs.

**Definition 2.6.** We define a relation $\equiv$ on the set of acyclic graphs where $G \equiv G'$ if

1. $G$ and $G'$ have the same vertex set $V$.
2. For all $u, v \in V$, there is a path from $u$ to $v$ in $G$ if and only if there is a path from $u$ to $v$ in $G'$.

**Proposition 2.7.** $\equiv$ is an equivalence relation.

**Proposition 2.8.** If $G \equiv G'$ and $V$ is their vertex set, then $v \in V$ is isolated in $G$ if and only if $v$ is isolated in $G'$.

**Proposition 2.9.** If $G \equiv G'$, then $|E(G)| = |E(G')|$. 

*Proof.* Let $V$ be the vertex set for $G, G'$. Since $G \equiv G'$, they have the same vertex sets of the connected components, $V = V_1 \sqcup \cdots \sqcup V_k$. Let $T_1, \ldots, T_k$ and $T'_1, \ldots, T'_k$ be the connected components of $G$ and $G'$, respectively, where $T_i, T'_i$ are induced by $V_i$. Since $G$ and $G'$ are acyclic graphs, $T_i, T'_i$ are trees for all $i \in [k]$. Thus $|E(G)| = \sum_{i=1}^{k} |E(T_i)| = \sum_{i=1}^{k} |V_i| - 1 = \sum_{i=1}^{k} |E(T'_i)| = |E(G')|$. 

**Definition 2.10.** Given a set of vertices $V$, a constraint graph $C$ on $V$ (represented by $G$) is the equivalence class of an acyclic graph $G$ on $V$. i.e. $C = [G] = \{ G' \text{ acyclic : } G' \equiv G \}$.

We define $V(C)$, the vertices of $C$, to be $V$. We say two vertices $u, v$ in $C$ are constrained together if for some representative graph $G \in C$, there is a path between $u$ and $v$ in $G$. Denote this as $u \leftrightarrow v$ in $C$.

We define $|E(C)|$, the number of edges of $C$ to be $|E(G)|$ for any $G \in C$. By Proposition 2.9, this is well-defined.
Given a representative graph $G \in C$, we call the edges of $G$ constraint edges.

**Proposition 2.11.** Given a set of vertices, let $C$ be a constraint graph on $V$. If $u \leftrightarrow v$ in $C$, then for all $G \in C$, there is a path between $u$ and $v$ in $G$.

**Definition 2.12.** Given a set of vertices $V$ and a map $\phi : V \to [n]$, we construct an acyclic graph $G(\phi)$ as follows:

1. We take $V(G(\phi)) = V$.
2. For each pair of vertices $u, v \in V$ such that $\phi(u) = \phi(v)$, we add an edge between $u$ and $v$.
3. As long as there is a cycle, we delete one edge of this cycle (this choice is arbitrary). We do this until there are no cycles left.

We define the constraint graph $C(\phi)$ on $V$ associated to $\phi$ to be the equivalence class of $G(\phi)$ under $\equiv$.

**Proposition 2.13.** Let $C(\phi)$ be a constraint graph on $V$ associated to $\phi : V \to [n]$, then two vertices $u, v$ in $C(\phi)$ are constrained together if and only if $\phi(u) = \phi(v)$.

**Definition 2.14** (Definition 3.8 of [1]: Constraint graphs on $H(\alpha, 2q)$). We define $C_{(\alpha, 2q)} = \{C(\phi) : \phi : V(\alpha, 2q) \to [n] \text{ is piecewise injective}\}$ to be the set of all possible constraint graphs on $V(\alpha, 2q)$ which come from a piecewise injective map $\phi : V(\alpha, 2q) \to [n]$.

Given a constraint graph $C \in C_{(\alpha, 2q)}$, we make the following definitions:

1. We define $N(C) = \{|\{\phi : V(\alpha, 2q) \to [n] : \phi \text{ is piecewise injective}, C(\phi) = C\}|\}$.
2. We define $\text{val}(C) = \mathbb{E} \left[ X_{\phi(E(\alpha, 2q))}(G) \right]$ where $\phi : V(\alpha, 2q) \to [n]$ is any piecewise injective map such that $C(\phi) = C$.

We say that a constraint graph $C$ on $H(\alpha, 2q)$ is nonzero-valued if $\text{val}(C) \neq 0$.

As observed in [1], with these definitions $\mathbb{E} \left[ \text{tr} \left( (M_\alpha M_\alpha^T)^q \right) \right]$ can be re-expressed as follows.

**Proposition 2.15** (Proposition 3.9 of [1]). For all shapes $\alpha$ and all $q \in \mathbb{N}$,

\[
\mathbb{E} \left[ \text{tr} \left( (M_\alpha M_\alpha^T)^q \right) \right] = \sum_{C \in C_{(\alpha, 2q)}} N(C) \text{val}(C).
\]

**Definition 2.16.** Let $H$ be a multi-graph and $C$ a constraint graph on $V(H)$. For $e, e'$ two edges in $H$, we say that $e$ and $e'$ are made equal by $C$ if $\phi(e) = \phi(e')$ where $\phi : |V(H)| \to [n]$ is any map such that $C(\phi) = C$. We denote this as $e \leftrightarrow e'$ by $C$.

**Remark 2.17.** Given a multi-graph $H$ and a constraint graph $C$ on $V(H)$, it is convenient to take a representative graph $G_C$ for $C$ and overlay $H$ and $G_C$ for analysis. See Figure 2.2b for an illustration. We draw $E(G_C)$ with different colors/patterns to distinguish it from $E(H)$. 8
Definition 2.18. Given a multi-graph $H$ and a constraint graph $C$ on $V(H)$, we pick a canonical $\phi : V(H) \to [n]$ such that $C(\phi) = C$. We define $H/C$ to be the multi-graph with vertices $V(H/C) = \{ \phi(v) : v \in H \}$ and edges $E(H/C) = \{ \phi(e) : e \in E(H) \}$ (note that this is a multi-set). The idea is that $H/C$ is obtained by starting with the graph $H$ and contracting along the constraint edges in $C$. See Figure 2.2b for an illustration.

(a) $H(\alpha, 2q)$ and a representative graph $G_C$ of a constraint graph $C \in \mathcal{C}(\alpha, 2q)$.

(b) Left: Overlay of $H(\alpha, 2q)$ and $G_C$ Right: $H(\alpha, 2q)/C$

Figure 2.2

Definition 2.19 (Induced constraint graphs). Given a multi-graph $H$, a constraint graph $C$ on $V(H)$, and a set of vertices $V \subseteq V(H)$, we define the induced constraint graph $C'$ on $V$ to be the constraint graph such that $V(C') = V$ and for all $u, v \in V$, $u \leftrightarrow v$ in $C'$ if and only if $u \leftrightarrow v$ in $C$.

Proposition 2.20 (Proposition 3.10 of [1]). For every constraint graph $C \in \mathcal{C}(\alpha, 2q)$, $\text{val}(C) = 1$ if every edge in $\phi \left( E(\alpha, 2q) \right)$ appears an even number of times and $\text{val}(C) = 0$ otherwise (where $\phi : V(\alpha, 2q) \to [n]$ is any piecewise injective map such that $C(\phi) = C$). Alternatively, we can say that $\text{val}(C) = 1$ if every edge in $H(\alpha, 2q)/C$ appears an even number of times and $\text{val}(C) = 0$ otherwise.

Proposition 2.21. For every constraint graph $C \in \mathcal{C}(\alpha, 2q)$, $N(C) = \frac{n!}{(n - |V(\alpha, 2q)| + |E(C)|)!}$.

Proof. Observe that choosing a piecewise injective map $\phi$ such that $C(\phi) = C$ is equivalent to choosing a distinct element of $[n]$ for each of the $n - |V(\alpha, 2q)| + |E(C)|$ connected components of $C$.

Since the number of constraint graphs in $\mathcal{C}(\alpha, 2q)$ depends on $q$ but not on $n$, as $n \to \infty$ we only care about the nonzero-valued constraint graphs in $\mathcal{C}(\alpha, 2q)$ which have the minimum possible number of edges. We call such constraint graphs dominant.

Definition 2.22 (Dominant Constraint Graphs). we say a constraint graph $C \in \mathcal{C}(\alpha, 2q)$ is a dominant constraint graph if $\text{val}(C) \neq 0$ and $|E(C)| = \min \{ |E(C')| : C' \in \mathcal{C}(\alpha, 2q), \text{val}(C') \neq 0 \}$.

We now state the number of edges in dominant constraint graphs.
Definition 2.23 (Vertex Separators). We say that \( S \subseteq V(\alpha) \) is a vertex separator of \( \alpha \) if every path from a vertex \( u \in U_\alpha \) to a vertex \( v \in V_\alpha \) contains at least one vertex in \( S \).

Definition 2.24. Given a shape \( \alpha \), define \( s_\alpha \) to be the minimum size of a vertex separator of \( \alpha \).

Lemma 2.25 (Follows from Lemma 6.4 of [1]). For any bipartite shape \( \alpha \), for any nonzero-valued \( C \in \mathcal{C}(\alpha,2q) \), \( |E(C)| \geq (q-1)s_\alpha \). Moreover, the bound is tight, i.e. There exists a nonzero-valued \( C \in \mathcal{C}(\alpha,2q) \) such that \( |E(C)| = (q-1)s_\alpha \).

Remark 2.26. In [1], this result was only proved for well-behaved constraint graphs (see Definition 3.22). That said, using the ideas in Appendix B of [1], it can be shown for all constraint graphs \( C \in \mathcal{C}(\alpha,2q) \). For details, see the appendix.

Corollary 2.27. For all bipartite shapes \( \alpha \), for all dominant constraint graphs \( C \in \mathcal{C}(\alpha,2q) \), \( |E(C)| = (q-1)s_\alpha \).

The following Corollary follows from Proposition 2.15, Proposition 2.21 and Corollary 2.27.

Corollary 2.28. For all bipartite shapes \( \alpha \), taking \( r_{\text{approx}}(n) = \frac{n!}{(n-s_\alpha)!} \),

\[
\lim_{n \to \infty} \frac{1}{r_{\text{approx}}(n)} \mathbb{E} \left[ \text{tr} \left( \left( \frac{M_\alpha M_\alpha^T}{n^{|V(\alpha)|-s_\alpha}} \right)^q \right) \right] = \left| \left\{ C \in \mathcal{C}(\alpha,2q) : C \text{ is dominant} \right\} \right|.
\]

Thus, to determine the spectrum of the singular values of \( M_\alpha \) for a bipartite shape \( \alpha \), we need to count the number of constraint graphs \( C \in \mathcal{C}(\alpha,2q) \) such that \( C \) is dominant.

Remark 2.29. We write \( r_{\text{approx}} \) rather than \( r \) here because if \( s_\alpha \leq \min \{|U_\alpha|,|V_\alpha|\} \) then the rank of \( M_\alpha \) will generally be \( \frac{n!}{(n-\min \{|U_\alpha|,|V_\alpha|\})!} \) rather than \( \frac{n!}{(n-s_\alpha)!} \).

Remark 2.30. The same statement is true for general \( \alpha \) except that the number of edges in a dominant constraint graph \( C \in \mathcal{C}(\alpha,2q) \) is \( q|V(\alpha)\setminus(U_\alpha\cup V_\alpha)| + (q-1)(s_\alpha-|U_\alpha\cap V_\alpha|) \) rather than \( (q-1)s_\alpha \).

3 Trace Powers of the Z-shaped Graph Matrix

Recall that \( \alpha_Z \) is the bipartite shape with vertices \( V(\alpha_Z) = \{u_1, u_2, v_1, v_2\} \), distinguished tuples of vertices \( U_{\alpha_Z} = \{u_1, u_2\} \) and \( V_{\alpha_Z} = \{v_1, v_2\} \), and edges \( E(\alpha_Z) = \{\{u_1, v_1\}, \{u_2, v_1\}, \{u_2, v_2\}\} \) (see Definition 1.5 and Figure 1.1). \( M_{\alpha_Z} \) is a graph matrix with dimension \( r(n) = n(n-1) \) (see Definition 1.4). In this section, we determine

\[
\lim_{n \to \infty} \frac{1}{r(n)} \mathbb{E}_{G \sim G(n,1/2)} \left[ \text{tr} \left( \left( \frac{M_{\alpha_Z} M_{\alpha_Z}^T}{n^2} \right)^q \right) \right]
\]

by counting the number of dominant constraint graphs in \( \mathcal{C}(\alpha_Z,2q) \).
Remark 3.1. For \( \alpha_Z \), the size of the minimum separator is \( s_{\alpha_Z} = 2 \). By Corollary 2.27, dominant constraint graphs \( C \in \mathcal{C}_{(\alpha_Z,2q)} \) have \( 2(q-1) \) edges.

Definition 3.2.

\[
D_n = \frac{1}{2n+1} \binom{3n}{n}. \tag{3.1}
\]

Remark 3.3. \( D_n \) is a special case of the generalized Catalan number, which is defined as

\[
A_n(k,r) = \frac{r}{nk+r} \binom{nk+r}{n}. \tag{3.2}
\]

Note that \( A_n(2,1) = \frac{1}{2n+1} \binom{2n+1}{n} = \frac{1}{n+1} \binom{2n}{n} \) is the Catalan number we know.

\[
A_n(3,1) = \frac{1}{3n+1} \binom{3n+1}{n} = \frac{1}{2n+1} \binom{3n}{n} \text{ is the } D_n \text{ defined above.}
\]

Below is the main result of this section.

**Theorem 3.4.** For all \( q \in \mathbb{N} \), the number of dominant constraint graphs \( C \in \mathcal{C}_{(\alpha_Z,2q)} \) is \( D_q \).

As a direct result of Theorem 3.4 and Corollary 2.28, we get the following corollary.

**Corollary 3.5.** Let \( M_n = \frac{1}{n} M_{\alpha_Z}(G) \) where \( G \sim G(n,1/2) \) and let \( r(n) = n(n-1) \) be the dimension of \( M_{\alpha_Z} \). Recall that \( D_q = \frac{1}{2q+1} \binom{3q}{q} \).

Then

\[
\lim_{n \to \infty} \frac{1}{r(n)} \mathbb{E}_{G \sim G(n,1/2)} \left[ \text{tr} \left( \left( M_n(G) M_n(G)^T \right)^q \right) \right] = D_q. \tag{3.3}
\]

**Proof.** By Corollary 2.28

\[
\lim_{n \to \infty} \frac{1}{r_{\text{approx}}(n)} \mathbb{E} \left[ \text{tr} \left( \left( M_{\alpha_Z} M_{\alpha_Z}^T \right)^q \right) \right] = \left| \left\{ C \in \mathcal{C}_{(\alpha_Z,2q)} : C \text{ is dominant} \right\} \right|.
\]

Since \( s_{\alpha_Z} = 2 \) and \( |V(\alpha_Z)| = 4 \), \( r_{\text{approx}}(n) = \frac{n!}{(n-s_{\alpha_Z})!} = n(n-1) = r(n) \) and \( \frac{M_{\alpha_Z} M_{\alpha_Z}^T}{n |V(\alpha_Z)| - s_{\alpha_Z}} = M_n M_n^T \). By Theorem 3.4, \( \left| \left\{ C \in \mathcal{C}_{(\alpha_Z,2q)} : C \text{ is dominant} \right\} \right| = D_q \) and the result follows.

### 3.1 Recurrence Relation for \( D_n \)

One of the key ingredients for proving Theorem 3.4 is the following recurrence relation on \( D_n \).

**Theorem 3.6.**

\[
D_{n+1} = \sum_{i,j,k \geq 0 \atop i+j+k=n} D_i D_j D_k = \sum_{i=0}^{n} D_i \left( \sum_{j=0}^{n-i} D_j D_{n-i-j} \right). \tag{3.3}
\]

To prove this recurrence relation, we consider walks on grids. This proof is a generalization of the third proof in the Wikipedia article on Catalan numbers.
Definition 3.7 (Grid Walk). Let $m, n$ be two positive integers. A grid walk from $(0, 0)$ to $(m, n)$ is a sequence of $(m + n)$ coordinates $(z_0, z_1, z_2, \ldots, z_{m+n})$ where

1. $z_i = (x_i, y_i)$ where $x_i \in [m], y_i \in [n]$ for each $i \in [m+n],$
2. $z_0 = (0, 0)$ and $z_{m+n} = (m, n),$
3. $z_{i+1} - z_i = (1, 0)$ or $(0, 1)$ for any $i \in [m+n].$

Pictorially, a grid is a walk from $(0, 0)$ to $(m, n)$ that steps on integer coordinates and only moves straight up or straight right.

A grid walk from $(0, 0)$ to $(m, n)$ weakly below the diagonal is a grid walk $(z_1, \ldots, z_{m+n})$ where $z_i = (x_i, y_i)$ and for all $i$, $y_i/x_i \leq m/n.$

Proof of Theorem 3.6. Let $W_n$ be the set of all grid walks from $(0, 0)$ to $(n, 2n)$ weakly below the diagonal and let $d_n = |W_n|.$ We will prove that $d_n$ satisfies the recurrence relation in theorem 3.6

\[ d_{n+1} = \sum_{i,j,k \geq 0: i+j+k = n} d_i d_j d_k = \sum_{i=0}^{n} d_i \left( \sum_{j=0}^{n-i} d_j d_{n-i-j} \right). \]

1. $d_n = \sum_{i,j,k \geq 0: i+j+k = n-1} d_i d_j d_k = \sum_{i=0}^{n-1} d_i \left( \sum_{j=0}^{n-i} d_j d_{n-i-j} \right):$

We will establish a bijection between $W_n$ and $W_n' := \bigcup_{i,j,k \geq 0: i+j+k = n-1} W_i \times W_j \times W_k.$

- Let $w = (z_1, \ldots, z_{3n})$ be a grid walk from $(0, 0)$ to $(n, 2n)$ weakly below the diagonal. Consider the first point that $w$ touches the diagonal i.e. let $a \in [n]$ be smallest such that $z_i = (a, 2a)$ for some $i \in [3n].$ Then $w_1 = (z_i, z_{i+1}, \ldots, z_{3n})$ is a grid walk from $(a, 2a)$ to $(n, 2n)$ weakly below the diagonal. After translation $w_1 \in W_{n-a}.$

Let $d'$ be the line parallel to the diagonal which passes $(a, 2a - 1).$ Since $z_i$ is the first point touching the diagonal, $(z_1, \ldots, z_{i-1})$ is weakly below $d'.$ Let $z_j = (b, 2b - 1)$ be the first point touching $d'.$ Then $w_2 = (z_j, \ldots, z_{i-1})$ is a grid walk from $(b, 2b - 1)$ to $(a, 2a - 1)$ weakly below the diagonal. After translation $w_2 \in W_{a-b}.$

Let $d''$ be the line parallel to the diagonal which passes $(b, 2b - 2).$ Since $z_j$ is the first point touching $d',$ $(z_2, \ldots, z_{j-1})$ is weakly below $d''.$ Then $w_3 = (z_2, \ldots, z_{j-1})$ is a grid walk from $(1, 0)$ to $(b, 2b - 2)$ weakly below the diagonal $d''.$ After translation $w_3 \in W_{b-1}.$

Thus from $w \in W_n$ we get a tuple $(w_1, w_2, w_3) \in W_{n-a} \times W_{a-b} \times W_{b-1}$ where $a, b$ are uniquely determined by $w.$ Note $(n-a) + (a-b) + (b-1) = n-1,$ thus $(w_1, w_2, w_3) \in W_n'.

- Conversely, given a $(w_1, w_2, w_3) \in W_n'$, let $(a_i, 2a_i)$ be the last coordinate point of $w_i.$ Let $w = ((0,0), w_1 + (1,0), w_2 + (a_1 + 1, 2a_1 + 1), w_3 + (a_1 + a_2 + 1, 2(a_1 + a_2 + 1)))$
where if \( w = (z_1, \ldots, z_k) \) is a grid walk then \( w + (s, t) \) means translate every coordinate point \( z_i \) in \( w \) by \( (s, t) \). We can easily check that \( w \in W_n \).

- It is not hard to check this is a bijection.

2. \( D_n = d_n = \frac{1}{2n + 1} \binom{3n}{n} \):

For \( i \in \{0, 1, \ldots, 2n\} \), let \( V_r \) be the set of grid walks from \((0, 0)\) to \((n, 2n)\) that has \( r \) vertical steps above the diagonal. i.e. for \( w = (z_1, \ldots, z_{3n}) \in V_r \), there are \( r \) \( z_j = (x_j, y_j) \)'s such that \( y_j/x_j > 2 \). Let \( G_n \) be the set of all grid walk from \((0, 0)\) to \((n, 2n)\). We have that \( |G_n| = \binom{3n}{n} \). Note that \( V_0 = W_n \) and \( \bigcup_{r=0}^{2n} V_r = G_n \). We will prove that \( |V_r| = |V_{r-1}| \) for all \( r \in [2n] \), then \( |V_0| = d_n = \frac{1}{2n + 1} \binom{3n}{n} \) as needed.

**Claim 3.8.** \( |V_{r-1}| = |V_r| \) for all \( r \in [2n] \).

**Proof.** We will find a bijection between \( V_r \) and \( V_{r-1} \) for each \( r \in [2n] \).

- Let \( w = (z_0, \ldots, z_{3n}) \in V_r \) and let \( z_k \) be the last point where the walk is on the diagonal and then takes a step upwards. i.e. \( z_k \) is the last point such that \( z_k = (a, 2a) \) for some \( a \in [n] \) and \( z_{k+1} - z_k = (0, 1) \). Let \( w_1 = (z_0, \ldots, z_k) \) and \( w_2 = (z_{k+1}, \ldots, z_{3n}) \). Let \( w' = (w'_1, w'_2) \) where \( w'_1 = w_2 - (a, 2a + 1) \) and \( w'_2 = w_1 + (n - a, 2n - 2a) \) (see Figure 3.1, \( w' \) exchanges the green and blue part of \( w \)). Then \( w'_2 \) has the same number of steps above the diagonal as \( w_1 \) does. Moreover, since \( z_k \) is the last point such that \( w \) passes the diagonal vertically
through it, $w'_1$ has exactly one less vertical step above the diagonal than $(z_k, w_2)$ does. Thus $w' \in V_{r-1}$.

- Let $w = (z_0, \ldots, z_{3n}) \in V_{r-1}$. Let $z_i$ be the first point such that $w$ touches the diagonal from below. i.e. $z_i$ is the first such that $z_i = (b, 2b)$ for some $b \in [n]$ and $z_i - z_{i-1} = (0, 1)$.

Let $w_1 = (z_0, \ldots, z_{i-1})$ and $w_2 = (z_i, \ldots, z_{3n})$. Let $w' = (w'_1, w'_2)$ where $w'_1 = w_2 - (b, 2b)$ and $w'_2 = (n - b, 2n - 2b + 1) + w_1$. Then $w'_1$ has the same number of steps above the diagonal as $w_2$ does. Moreover, since $z_i$ is the first such that $w$ touches the diagonal from below, $((n - b, 2n - 2b), w'_2)$ has exactly one more step above the diagonal than $w_1$ does. Thus $w' \in V_r$.

- It is not hard to check that this gives a bijection.

To conclude, we proved that $D_n = \frac{1}{2n + 1} \binom{3n}{n} = d_n$ = the number of grid walks from $(0, 0)$ to $(n, 2n)$ that are weakly below the diagonal. □

3.2 Properties of Dominant Constraint Graphs on a Cycle

In order to count the number of dominant constraint graphs in $C(\alpha, 2q)$, we need a few properties of these constraint graphs. As a warm-up, we first consider dominant constraint graphs on a cycle of length $2q$. The first part of this analysis is essentially the same as Lemma 4.4 of [1], but we will need a few additional properties.

Definition 3.9. Let $\alpha_0$ be the bipartite shape with vertices $V(\alpha_0) = \{u, v\}$ and a single edge $\{u, v\}$ with distinguished tuples of vertices $U_{\alpha_0} = (u)$ and $V_{\alpha_0} = (v)$. We call $\alpha_0$ the **line shape**.

![Figure 3.2](image)

Figure 3.2: $\alpha_0$ is the line shape. $H(\alpha_0, 2q)$ is a cycle of length $2q$.

Definition 3.10. Let $\alpha_0$ be the line shape as in definition 3.9. Let $H(\alpha_0, 2q)$ be the multi-graph as in definition 2.2. We label the vertices of $H(\alpha_0, 2q)$ as $\{i_j : j \in [2q]\}$.

We say a representative graph $G$ of a constraint graph $C \in C(\alpha_0, 2q)$ is explicitly non-crossing if no two constraint edges of $G$ cross. Note: constraint edges $\{i_x, i_y\}$ and $\{i_s, i_t\}$ where $x < y$
and \( s < t \) cross if \( x < s < y < t \) or \( s < x < t < y \). We say \( G \) is crossing if it is not explicitly non-crossing.

We say a constraint graph \( C \in \mathcal{C}_{(\alpha_0, 2q)} \) is non-crossing if there is a representative graph \( G \in C \) that is explicitly non-crossing. We say \( C \) is crossing if it is not non-crossing. See Figure 3.3 for an illustration.

![Figure 3.3: C1, C2 ∈ \( \mathcal{C}_{(\alpha_0, 2q)} \). C1 is crossing; C2 is non-crossing since \( G'_{C_2} \) is explicitly non-crossing even though \( G_{C_2} \) is crossing.](image)

**Definition 3.11.** Let \( \alpha_0 \) be the line shape. We say a constraint graph \( C \in \mathcal{C}_{(\alpha_0, 2q)} \) is parity preserving if for all \( i_x, i_y \in V(\alpha_0, 2q) \) such that \( i_x \leftrightarrow i_y, |x - y| \) is even.

**Lemma 3.12.** All dominant constraint graphs in \( \mathcal{C}_{(\alpha_0, 2q)} \) are non-crossing and parity-preserving.

To prove this lemma, we need the following observation about isolated vertices.

**Definition 3.13.** Given a multi-graph \( H \) and a constraint graph \( C \) on \( H \), we say that a vertex \( v \in V(C) = V(H) \) is isolated if for any \( G \in C \), \( v \) is not incident to any constraint edges in \( G \). Note that by Proposition 2.8, this is well-defined.

**Lemma 3.14.** If \( C \) is a nonzero-valued constraint graph on \( H(\alpha_0, 2q) \) and \( C \) has an isolated vertex \( i_j \), then \( i_{j-1} \leftrightarrow i_{j+1} \). In the cases when \( j = 1 \) or \( j = 2q \), \( i_0 = i_{2q} \) and \( i_{2q+1} = i_1 \) respectively.

**Proof.** Recall that by Proposition 2.20, \( C \) is nonzero-valued if and only if each edge in \( H(\alpha_0, 2q)/C \) appears an even number of times. Since \( i_j \) is isolated, the only way this can happen is if \( i_{j-1} \leftrightarrow i_{j+1} \). □

With this observation, we can now prove Lemma 3.12.

**Proof of Lemma 3.12.** Since \( C \) is dominant, each edge appears an even number of times in \( H(\alpha_0, 2q)/C \) and there are exactly \( (q - 1) \) constraint edges in \( C \). We prove the lemma by induction on \( q \).

- When \( q = 1 \), there are no constraint edge so the lemma trivially holds. For \( q = 2 \), \( |E(C)| = 1 \).
  In order for each edge to appear even number of times in \( H(\alpha_0, 2q)/C \), either \( i_1 \leftrightarrow i_2 \) or
$i_2 \leftrightarrow i_4$, which implies that $C$ is parity preserving. If $i_1 \leftrightarrow i_3$ in $C$, we choose $G_C \in C$ to have a single constraint edge $\{i_1, i_3\}$. If $i_2 \leftrightarrow i_4$ in $C$, we choose $G_C \in C$ to have a single constraint edge $\{i_2, i_4\}$. In either case $G_C$ is explicitly non-crossing, thus $C$ is non-crossing.

Figure 3.4: Illustration of base case of the proof: $H(\alpha_0, 2)$ overlay with $G_C \in C$ where $G_C$ consists of a single constraint edge, either $\{i_1, i_3\}$ or $\{i_2, i_4\}$.

- $q \Rightarrow (q + 1)$: Consider a constraint graph $C$ on $H(\alpha_0, 2q + 2)$ with vertices $\{i_1, \ldots, i_{2q+2}\}$. Since $C$ is dominant by assumption, there are only $q$ constraint edges in $C$, so $C$ must have an isolated vertex. Without loss of generality assume this vertex is $i_{2q+2}$. Then by Lemma 3.14, $i_1 \leftrightarrow i_{2q+1}$ and there exists $G \in C$ such that $\{i_1, i_{2q+1}\}$ is a constraint edge in $G$. Note that $(2q + 1) - 1 = 2q$ is even. Contracting the constraint edge $\{i_{2q+1}, i_1\}$ (identifying $i_1$ with $i_{2q+1}$) results in $H(\alpha_0, 2q)$ with vertices $\{i_1, \ldots, i_{2q}\}$ and two edges $\{i_{2q+1}, i_{2q}\} = \{i_1, i_{2q}\}$ attached to $H(\alpha_0, 2q)$. See Figure 3.5 for an illustration.

Let $G'$ be the induced subgraph of $G$ on $H(\alpha_0, 2q)$. Since $G'$ has one less edge than $G$, and edges in $H(\alpha_0, 2q)$ are only made equal to edges in $H(\alpha_0, 2q)$ by $C$, the constraint graph $C' = [G'] \in C(\alpha_0, 2q)$ represented by $G'$ is dominant. By the inductive hypothesis, $C'$ is non-crossing and parity preserving, which implies that $C$ is parity preserving. Choosing a representative graph of $C'$ that is explicitly non-crossing and adding in the constraint edge $\{i_1, i_{2q-1}\}$, we get an explicitly non-crossing representative graph of $C$, which implies that $C$ is non-crossing, as needed.

Figure 3.5: Illustration of the inductive step of the proof for Lemma 3.12 $i_{2q+2}$ is isolated and $\{i_1, i_{2q+1}\}$ is a constraint edge in a representative graph $G \in C \in C(\alpha_0, 2q)$. 
We now show a few additional properties of dominant constraint graphs in \(C_{(\alpha_0,2q)}\).

**Corollary 3.15.** Let \(C \in C_{(\alpha_0,2q)}\) be a dominant constraint graph. If \(i_x \leftrightarrow i_y\) and \(i_v \leftrightarrow i_w\) for some \(x \leq v \leq y \leq w\), then \(i_x \leftrightarrow i_y \leftrightarrow i_v \leftrightarrow i_w\).

**Proof.** If \(x = v, v = y, or y = w\) then \(i_x \leftrightarrow i_y \leftrightarrow i_v \leftrightarrow i_w\) so we can assume that \(x < v < y < w\). By Lemma 3.12 \(C\) is non-crossing, so there exists \(G \in C\) that is explicitly non-crossing. We think of \(H(\alpha_0,2q)\) as a circle, vertices of \(G\) as points on the circle and edges of \(G\) as chords. Since \(i_x \leftrightarrow i_y\) and \(i_v \leftrightarrow i_w\), there is a path from \(x\) to \(y\) and a path from \(v\) to \(w\). These paths do not leave the circle, so they must intersect. Since there are no crossings, they must intersect at an index which implies that \(i_v \leftrightarrow i_x \leftrightarrow i_y \leftrightarrow i_w\), as needed.

**Corollary 3.16.** Let \(C \in C_{(\alpha_0,2q)}\) be a dominant constraint graph. If \(i_s \leftrightarrow i_t\) for some \(1 \leq s < t \leq 2q\), then there exists an explicitly non-crossing \(G \in C\) such that \(\{i_s,i_t\}\) is an edge in \(G\). Moreover, if we let \(R = \{\{i_2, i_{2x+1}\}: s \leq x < t\}\) and \(L = E(\alpha_0,2q) \setminus R\) then edges in \(R\) can only be made equal to edges in \(R\) by \(C\) and edges in \(L\) can only be made equal to edges in \(L\) by \(C\).

**Proof.** For the first part, let \(G\) be an explicitly non-crossing representative graph of \(C\), we will adjust \(G\) as follows. Let \(V\) be the connected component of \(G\) which contains \(i_s\). Delete all edges between vertices in \(V\) and then add an edge from each vertex in \(V \setminus \{i_s\}\) to \(i_s\). We claim that the adjusted \(G\) is still explicitly non-crossing. Assume not. Then there is an edge \(\{i_x,i_y\}\) which crosses one of these new edges \(\{i_s,i_v\}\). Since \(i_x \leftrightarrow i_y, i_s \leftrightarrow i_v\) and these edges cross, by Corollary 3.15 \(i_x \leftrightarrow i_y \leftrightarrow i_s \leftrightarrow i_v\). But then \(x, y \in V\) so we would have deleted the edge \(\{i_x,i_y\}\). Which is a contradiction.

For the second part, assume not and let \(e_1 = \{i_x,i_y\} \in R\) and \(e_2 = \{i_v,i_w\} \in L\) be edges such that \(e_1 \leftrightarrow e_2\). Since \(e_1, e_2\) are edges, \(|x-y| = |v-w| = 1\). Without loss of generality, assume \(x, v\) are even and \(y, w\) are odd. Since \(C\) is parity preserving, \(i_x \leftrightarrow i_v\) and \(i_y \leftrightarrow i_w\). Since \(e_1 \in R\) and \(e_2 \in L\), \(s \leq x \leq t \leq v\) or \(v \leq s \leq x \leq t\). By Corollary 3.15 \(i_s \leftrightarrow i_x \leftrightarrow i_t \leftrightarrow i_v\). Following similar logic, \(i_s \leftrightarrow i_y \leftrightarrow i_t \leftrightarrow i_w\). Thus \(i_s \leftrightarrow i_t \leftrightarrow i_x \leftrightarrow i_y \leftrightarrow i_v \leftrightarrow i_w\), contradicting that \(C\) is parity preserving.

**Corollary 3.17.** Let \(C \in C_{(\alpha_0,2q)}\) be a dominant constraint graph. If \(i_s \leftrightarrow i_t\) for some \(1 \leq s < t \leq 2q\), contracting \(i_s\) and \(i_t\) splits \(H(\alpha_0,2q)\) into \(H(\alpha_0, t-s)\) and \(H(\alpha_0, 2q-(t-s))\). Letting \(C'\) and \(C''\) be the induced constraint graphs on \(H(\alpha_0, t-s)\) and \(H(\alpha_0, 2q-(t-s))\) respectively, \(C'\) and \(C''\) are dominant. See Figure 3.6b for an illustration.

**Proof.** By Corollary 3.16 no edge in \(H(\alpha_0, t-s)\) can be made equal to an edge in \(H(\alpha_0, 2q-(t-s))\), so \(C'\) and \(C''\) are nonzero-valued constraint graphs in \(C_{(\alpha_0, t-s)}\) and \(C_{(\alpha_0, 2q-(t-s))}\) respectively. This implies that \(|E(C')| \geq (t-s)/2 - 1\) and \(|E(C')| \geq q - (t-s)/2 - 1\). Since \(|E(C)| = q - 1\) as \(C\) is dominant and \(|E(C)| = |E(C')| + |E(C'')| + 1\) (here the additional edge is \(\{i_s,i_t\}\)), we must
Figure 3.6: Illustration of Corollary 3.16 and Corollary 3.17: $i_s \leftrightarrow i_t$ in a dominant constraint graph $C$, contracting $i_s$ and $i_t$ splits $C$ into two dominant constraint graphs.

have that $|E(C')| = (t - s)/2 - 1$ and $|E(C'')| = q - (t - s)/2 - 1$, so $C'$ and $C''$ are dominant, as needed.

**Lemma 3.18.** Consider a dominant constraint graph $C$ on $H(\alpha_0, 2q)$. If $i_j$ is the first vertex $i_1$ is constrained to \(i.e. i_j\) is the smallest index such that $i_1 \leftrightarrow i_j$), then $i_2 \leftrightarrow i_{j-1}$.

Figure 3.7: Illustration of Lemma 3.18: $i_j$ is the first vertex $i_1$ is constrained to.

**Proof.** Contract the edge \(\{i_1, i_j\}\), splitting $H(\alpha_0, 2q)$ into $H(\alpha_0, j - 1)$ and $H(\alpha_0, 2q - j + 1)$. Let $C'$ be the induced constraint graph on $H(\alpha_0, j - 1)$. Since $i_j$ is the first vertex $i_1$ is constrained to, $i_1 = i_j$ is isolated in $H(\alpha_0, j - 1)$. By Lemma 3.14, $i_2 \leftrightarrow i_{j-1}$ in $C'$ and thus $i_2 \leftrightarrow i_{j-1}$ in $C$, as needed.

**3.2.1 List of Properties of Dominant Constraint Graphs on a Cycle**

For convenience, here is a list of the properties we have shown. If $C \in C_{(\alpha_0, 2q)}$ is a dominant constraint graph then

1. $|E(C)| = q - 1$.
2. $C$ is parity-preserving.
3. \( C \) is non-crossing.

4. If \( i_x \leftarrow i_y \) and \( i_v \leftarrow i_w \) for some \( x \leq v \leq y \leq w \), then \( i_x \leftarrow i_y \leftarrow i_v \leftarrow i_w \).

5. If \( i_s \leftarrow i_t \) for some \( 1 \leq s < t \leq 2q \), then there is an explicitly non-crossing representative graph \( G_C \in \mathcal{C} \) so that it includes the edge \( \{i_s, i_t\} \). Moreover, if we let \( \mathcal{R} = \{i_x, i_x+1\} : s \leq x < t \} \) and \( \mathcal{L} = E(\alpha_0, 2q) \setminus \mathcal{R} \), then edges in \( \mathcal{R} \) can only be made equal to edges in \( \mathcal{R} \) by \( C \) and edges in \( \mathcal{L} \) can only be made equal to edges in \( \mathcal{L} \) by \( C \).

6. If \( i_s \leftarrow i_t \) for some \( 1 \leq s < t \leq 2q \), contracting \( i_s \) and \( i_t \) splits \( H(\alpha_0, 2q) \) into \( H(\alpha_0, t-s) \) and \( H(\alpha_0, 2q - (t-s)) \). Letting \( C' \) and \( C'' \) be the induced constraint graphs on \( H(\alpha_0, t-s) \) and \( H(\alpha_0, 2q - (t-s)) \) respectively, \( C' \) and \( C'' \) are dominant.

7. If \( i_j \) is the first vertex \( i_1 \) is constrained to (i.e. if \( j \) is the smallest index such that \( i_1 \leftarrow i_j \)), then \( i_2 \leftarrow i_{j-1} \).

### 3.3 Properties of Dominant Constraint Graphs on \( H(\alpha_Z, 2q) \)

Now that we have analyzed dominant constraint graphs in \( \mathcal{C}(\alpha_0, 2q) \), we can analyze dominant constraint graphs in \( \mathcal{C}(\alpha_Z, 2q) \).

**Definition 3.19.** Let \( \alpha_Z \) be the Z-shape as defined in [1.5] and let \( H(\alpha_Z, 2q) \) be the multi-graph as defined in [2.2]. We label the vertices of \( V(\alpha_Z) \) as \( \{a_{i1}, a_{i2}, b_{i1}, b_{i2}\} \) and the vertices of \( V(\alpha_Z^2) \) as \( \{b_{i1}, b_{i2}, a_{(i+1)1}, a_{(i+1)2}\} \). We call the induced subgraph of \( H(\alpha_Z, 2q) \) on vertices \( \{a_{i1}, b_{i1} : i \in [q]\} \) the *outer wheel* \( W_1 \) and the induced subgraph on vertices \( \{a_{i2}, b_{i2} : i \in [q]\} \) the *inner wheel* \( W_2 \). We denote the vertices of \( W_1 \) as \( V_i \) and edges as \( E_i \).

We label the “middle edges” of \( H(\alpha, 2q) \) in the following way: let \( e_{2i-1} = \{a_{i2}, b_{i1}\} \) and \( e_{2i} = \{b_{i1}, a_{(i+1)2}\} \) for \( i = 1, \ldots, q \). We call the edges \( \{e_i : i \in [2q]\} \) the *spokes* of \( H(\alpha_Z, 2q) \). See Figure 3.8 for an illustration.

![Figure 3.8: H(\alpha_Z, 2q) where q = 4.](image)

**Definition 3.20.** Let \( C \) be a constraint graph in \( \mathcal{C}(\alpha_Z, 2q) \). For \( i = 1, 2 \), we take \( C_i \) to be the induced constraint graph of \( C \) on the vertices \( V_i \).
Remark 3.21. Each wheel $W_i$ can be viewed as $H(\alpha_0, 2q)$. The induced constraint graph $C_i$ can be viewed as a constraint graph on $H(\alpha_0, 2q)$.

The key property that we need about dominant constraint graphs in $\mathcal{C}_{(\alpha, 2q)}$ is that they are well-behaved. This implies that the induced constraint graphs $C_1, C_2$ are dominant constraint graphs in $\mathcal{C}_{(\alpha, 2q)}$.

Definition 3.22. Given a shape $\alpha$, we say that a constraint graph $C \in \mathcal{C}_{(\alpha, 2q)}$ is well-behaved if whenever $u \leftrightarrow v$ in $C$, $u$ and $v$ are copies of the same vertex in $\alpha$ or $\alpha^T$.

Theorem 3.23. All dominant constraint graphs in $\mathcal{C}_{(\alpha, 2q)}$ are well-behaved.

This theorem is surprisingly tricky to prove, so we defer its proof to the appendix.

Remark 3.24. This theorem is not true for all shapes $\alpha$. In particular, this theorem is false for the bipartite shape $\alpha$ with $U_\alpha = (u_1, u_2)$, $V_\alpha = (v_1, v_2)$, $V(\alpha) = U_\alpha \cup V_\alpha$, and $E(\alpha) = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2)\}$.

Definition 3.25. Let $C$ be a constraint graph in $\mathcal{C}_{(\alpha, 2q)}$.

1. We say $C$ is wheel-respecting if whenever $u \leftrightarrow v$, $u, v \in V_i$ for some $i \in \{1, 2\}$ (i.e. no two vertices on different wheels are constrained together). Note that if $C$ is wheel-respecting then if $G_1$ and $G_2$ are representatives of $C_1$ and $C_2$, the graph $G$ with $V(G) = V_1 \cup V_2$ and $E(G) = E(G_1) \cup E(G_2)$ is a representative of $C$.

2. We say $C$ is parity-preserving if for each $i \in \{1, 2\}$, the induced constraint graphs $C_i$ of $C$ on $V_i$ is parity-preserving.

3. We say that $C$ is non-crossing if the induced constraint graphs $C_1$ and $C_2$ are non-crossing.

Proposition 3.26. Let $C$ be a constraint graph in $\mathcal{C}_{(\alpha, 2q)}$. $C$ is well-behaved if and only if $C$ is wheel-respecting and parity-preserving.

Corollary 3.27. If $C$ is a dominant constraint graph in $\mathcal{C}_{(\alpha, 2q)}$ then

1. $C$ is wheel-respecting, parity-preserving, and non-crossing.

2. The induced constraint graphs $C_1, C_2$ are dominant constraint graphs in $\mathcal{C}_{(\alpha, 2q)}$.

Proof. Since dominant constraint graphs in $\mathcal{C}_{(\alpha, 2q)}$ are well-behaved, $C$ is wheel-respecting and parity-preserving. Since $C$ is wheel-respecting, $C$ can only make edges in $W_i$ equal to other edges in $W_i$, so $C_1$ and $C_2$ must be nonzero-valued constraint graphs in $\mathcal{C}_{(\alpha, 2q)}$. Since $|E(C)| = 2q - 2 = |E(C_1)| + |E(C_2)|$, we must have that $|E(C_1)| = |E(C_2)| = q - 1$ and thus $C_1$ and $C_2$ are dominant. This implies that $C_1$ and $C_2$ are non-crossing, so $C$ is non-crossing.

We now show a few additional properties of dominant constraint graphs in $\mathcal{C}_{(\alpha, 2q)}$. We start with the following fact about the spokes of $H(\alpha_2, 2q)$. 
Lemma 3.28. Let $C$ be a dominant constraint graph in $\mathcal{C}_{(\alpha Z, 2q)}$ and consider the spokes $\{e_i : i \in [2q]\}$ of $H(\alpha Z, 2q)$. If $e_i \leftrightarrow e_j$ and $e_s \leftrightarrow e_t$ for some $i < s < j < t$, then $e_i \leftrightarrow e_s \leftrightarrow e_j \leftrightarrow e_t$.

![Figure 3.9: Illustration of Lemma 3.28](image)

Proof. By the definition of the spokes $e_i$’s, one of the endpoints of $e_i$ is $a_{x2}$ where $x = \lfloor i/2 \rfloor + 1$. Since $C$ is well-behaved and $e_i \leftrightarrow e_j$, $a_{x2} \leftrightarrow a_{y2}$ where $x = \lfloor i/2 \rfloor + 1$ and $y = \lfloor j/2 \rfloor + 1$. Similarly since $e_s \leftrightarrow e_t$, $a_{v2} \leftrightarrow a_{w2}$ where $v = \lfloor s/2 \rfloor + 1$ and $w = \lfloor t/2 \rfloor + 1$. Since $i < s < j < t$, we have $x \leq v \leq y \leq w$. By Corollary 3.27, $C_2$ is a dominant constraint graph on $W_2$. By Corollary 3.15, $a_{x2} \leftrightarrow a_{y2} \leftrightarrow a_{v2} \leftrightarrow a_{w2}$. Similarly we can argue that $b_{z'1} \leftrightarrow b_{y'1} \leftrightarrow b_{w'1} \leftrightarrow b_{x'1}$ where $b_{z'1}, b_{y'1}, b_{w'1}, b_{x'1}$ are the endpoints of $e_i, e_s, e_j, e_t$, respectively. Thus $e_i \leftrightarrow e_s \leftrightarrow e_j \leftrightarrow e_t$. □

Combining this fact about the spokes of $H(\alpha Z, 2q)$ with the following lemma, we can show that constraint edges between vertices which are not incident to any spokes split $H(\alpha Z, 2q)$ into two parts, which is the main result needed to prove Theorem 3.4.

Lemma 3.29. For all $m \in \mathbb{N}$, if $M$ is a perfect matching on the indices $[2m]$ such that no two edges of $M$ cross (i.e. there is no pair of edges $\{i, j\}, \{k, l\} \in M$ such that $i < k < j < l$) then either $\{1, 2m\} \in M$ or there is a sequence of indices $i_1 < \ldots < i_k$ such that

1. For all $j \in [k]$, $i_j$ is even.
2. $\{1, i_1\} \in M$ and $\{i_k + 1, 2m\} \in M$.
3. For all $j \in [k - 1]$, $\{i_j + 1, i_{j+1}\} \in M$.

See Figure 3.10a for an illustration.

Proof. We prove this by induction on $m$. The base case $m = 1$ is trivial. For the inductive step, assume the result is true for $m$ and consider a matching $M$ on $[2m + 2]$ such that no edges of $M$ cross. If $\{1, 2m + 2\} \in M$ then we are done, so we can assume that $\{1, 2m + 2\} \notin M$.
Choose $s < t \in [2m + 2]$ such that $\{s, t\} \in M$ and $t - s$ is minimized. We claim that $t = s + 1$.

To see this, assume that $t > s + 1$. Since $M$ is a perfect matching, $\{s + 1, x\} \in M$ for some $x \in [2m + 2]$. Since no two edges of $M$ cross, we must have that $s + 1 < x < t$. However, this implies that $x - (s + 1) < t - s$, contradicting our choice of $s$ and $t$.

Now consider the matching $M'$ obtained from $M$ by deleting the indices $s, s + 1$ and decreasing all indices greater than $s + 1$ by 2. By the inductive hypothesis, either $\{i'_1, i'_{2m}\} \in M$, or there is a sequence $i'_1 < \ldots < i'_k$ such that for all $j \in [k]$, $i'_j$ is even, $\{1, i'_1\} \in M$ and $\{i'_k, 2m\} \in M$, and for all $j \in [k - 1]$, $\{i'_{j+1} + 1, i'_{j+1}\} \in M$. In the later case we can modify this sequence as follows to obtain the desired sequence:

1. Increase all indices in this sequence which are greater than or equal to $s$ by 2.
2. If $s - 1$ is in this sequence, insert $s + 1$ after it.

If $\{i'_1, i'_{2m}\} \in M$, then there are three cases:

1. When $1 < s < 2m + 1$, $i'_1 = i_1$ and $i'_{2m} = i_{2m+2}$, then we are done.
2. When $s = 1$, $i'_1 = i_3$ and $i'_{2m} = i_{2m+2}$, then we have a sequence with single element $i_1 = 2$ such that $\{1, 2\} \in M$ and $\{3, 2m + 2\} \in M$.
3. When $s = 2m + 1$, $i'_1 = i_1$ and $i'_{2m} = i_{2m}$, then we have a sequence with single element $i_1 = 2m$ such that $\{1, 2m\} \in M$ and $\{2m + 1, 2m + 2\} \in M$.

Lemma 3.30. Let $C$ be a dominant constraint graph in $\mathcal{C}_{(\alpha_Z, 2q)}$. If $a_{i_1} \leftrightarrow a_{j_1}$ for some $1 \leq i < j \leq q$, then $a_{i_2} \leftrightarrow a_{j_2}$.

Moreover, the spokes $\{e_x : x \in [2i - 1, 2j - 2]\}$ can only be made equal to each other. Similarly, if $b_{i_1} \leftrightarrow b_{j_1}$ for some $1 \leq i < j \leq q$, then $b_{i_2} \leftrightarrow b_{j_2}$ and the spokes $\{e_x : x \in [2i, 2j - 1]\}$ can only be made equal to each other.
Proof. We prove the first statement as the proof for the second statement is similar. Without loss of
generality, assume \( i = 1 \). Observe that for all \( x \in [j - 1] \) and all \( y \in [j, q] \), we cannot have
that \( b_{x1} \leftrightarrow b_{y1} \). Otherwise, by Corollary 3.15 we would have that \( a_{11} \leftrightarrow a_{j1} \leftrightarrow b_{x1} \leftrightarrow b_{y1} \),
contradicting the fact that \( C \) is well-behaved.

This implies that the spokes \( \{ e_x : x \in [2j - 2] \} \) can only be made equal to each other. By
Lemma 3.28, there must be a perfect matching \( M \) on the indices \([2j - 2]\) such that

1. If \( \{x, y\} \in M \) then \( e_x \leftrightarrow e_y \) (\( M \) describes how the spokes \( \{e_x : x \in [2j - 2]\} \) are paired up).
2. No two edges of \( M \) cross (there is no pair of edges \( \{x, y\}, \{z, w\} \in M \) such that \( x < z < y < w \)).

By Lemma 3.29, either \( \{1, 2j - 2\} \in M \) or there is a sequence of indices \( i_1 < \ldots < i_k \) such that

1. For all \( l \in [k], i_l \) is even.
2. \( \{1, i_1\} \in M \) and \( \{i_k + 1, 2j - 2\} \in M \).
3. For all \( l \in [k - 1], \{i_l + 1, i_{l+1}\} \in M \).

If \( \{1, 2j - 2\} \in M \) then \( a_{12} \leftrightarrow a_{j2} \). Otherwise, we make the following observations (see Figure
3.10b for an illustration):

1. Since \( \{1, i_1\} \in M \), \( a_{12} \leftrightarrow a_{(i_1/2+1)2} \).
2. For all \( l \in [k - 1], \{i_l + 1, i_{l+1}\} \in M \), \( a_{(i_l/2+1)2} \leftrightarrow a_{(i_{l+1}/2+1)2} \).
3. Since \( \{i_k + 1, 2(j - 1)\} \in M \), \( a_{(i_k/2+1)2} \leftrightarrow a_{j2} \).

Putting these observations together, \( a_{12} \leftrightarrow a_{j2} \), as needed. \( \square \)

Corollary 3.31. If \( C \) is a dominant constraint graph in \( \mathcal{C}_{(\alpha Z, 2q)} \) then the following statements are
true:

1. If \( a_{i1} \leftrightarrow a_{j1} \) for some \( 1 \leq i < j \leq q \) then \( a_{i2} \leftrightarrow a_{j2} \). Moreover, contracting \( a_{i1} \)
and \( a_{j1} \) together and contracting \( a_{i2} \) and \( a_{j2} \) together splits \( H(\alpha Z, 2q) \) into \( H(\alpha Z, 2(j - i)) \)
and \( H(\alpha Z, 2(q - j + i)) \), and the induced constraint graphs \( C' \in \mathcal{C}_{(\alpha Z, 2(j - i))} \) and \( C'' \in \mathcal{C}_{(\alpha Z, 2(q - j + i))} \) are dominant.
2. Similarly, if \( b_{i2} \leftrightarrow b_{j2} \) for some \( 1 \leq i < j \leq q \) then \( b_{i1} \leftrightarrow b_{j1} \). Moreover, contracting \( b_{i1} \)
and \( a_{j1} \) together and contracting \( b_{i2} \) and \( b_{j2} \) together splits \( H(\alpha Z, 2q) \) into \( H(\alpha Z, 2(j - i)) \)
and \( H(\alpha Z, 2(q - j + i)) \), and the induced constraint graphs \( C' \in \mathcal{C}_{(\alpha Z, 2(j - i))} \) and \( C'' \in \mathcal{C}_{(\alpha Z, 2(q - j + i))} \) are dominant.
Figure 3.11: An edge can only be made equal to the edges with the same color.

3.3.1 List of Properties of Dominant Constraint Graphs on $H(\alpha Z, 2q)$

For convenience, here is a list of the properties we have shown. If $C \in C(\alpha Z, 2q)$ is a dominant constraint graph then

1. $|E(C)| = 2q - 2$.

2. $C$ is wheel-respecting, parity-preserving, and non-crossing.

3. The induced constraint graphs $C'$ and $C''$ on the two wheels $W_1$ and $W_2$ are dominant constraint graphs in $C(\alpha Z, 2q)$.

4. If $e_i \leftrightarrow e_j$ and $e_s \leftrightarrow e_t$ for some $i < s < j < t$, then $e_i \leftrightarrow e_s \leftrightarrow e_j \leftrightarrow e_t$.

5. If $a_{i1} \leftrightarrow a_{j1}$ for some $1 \leq j < i \leq q$ then $a_{i2} \leftrightarrow a_{j2}$. Moreover, contracting $a_{i1}$ and $a_{j1}$ together and contracting $a_{i2}$ and $a_{j2}$ together splits $H(\alpha Z, 2q)$ into $H(\alpha Z, 2(j - i))$ and $H(\alpha Z, 2(q - j + i))$ and the induced constraint graphs $C' \in C(\alpha Z, 2(j - i))$ and $C'' \in C(\alpha Z, 2(q - j + i))$ are dominant.

6. Similarly, if $b_{i2} \leftrightarrow b_{j2}$ for some $1 \leq j < i \leq q$ then $b_{i1} \leftrightarrow b_{j1}$. Moreover, contracting $b_{i1}$ and $a_{j1}$ together and contracting $b_{i2}$ and $b_{j2}$ together splits $H(\alpha Z, 2q)$ into $H(\alpha Z, 2(j - i))$ and $H(\alpha Z, 2(q - j + i))$ and the induced constraint graphs $C' \in C(\alpha Z, 2(j - i))$ and $C'' \in C(\alpha Z, 2(q - j + i))$ are dominant.

3.4 Proof of Theorem 3.4

Now we are ready to prove the main result of this section, Theorem 3.4.

Definition 3.32. Define $D(q, m)$ to be the number of dominant constraint graphs $C$ in $C(\alpha Z, 2(q + m))$ such that $a_{12} \leftrightarrow a_{22} \leftrightarrow \ldots \leftrightarrow a_{(m+1)2}$.
Remark 3.33. Notice that for the case when \( m = 0 \), \( D(q, 0) = \left\{ C \in \mathcal{C}_{(\alpha_Z, 2q)} : C \text{ is dominant} \right\} \). For the case when \( q = 0 \), we consider constraint graphs \( C \) on \( H(\alpha_Z, 2m) \) where all the vertices \( a_{i_2} \) on \( W_2 \) are constrained together by \( C \). i.e. \( C \) can be viewed as a dominant constraint graph on \( W_1 \). Thus \( D(0, m) = \left\{ C \in \mathcal{C}_{(\alpha_0, 2m)} : C \text{ is dominant} \right\} \), which is the Catalan numbers.

Proof of Theorem 3.4. To prove Theorem 3.4, we prove the following two statements:

1. For all \( q \in \mathbb{N} \), \( D(q, 0) = \sum_{i=1}^{q} D(q - i, 1) \cdot D(i - 1, 0) \).

2. For all \( q \in \mathbb{N} \cup \{0\} \), \( D(q, 1) = \sum_{i=0}^{q} D(i, 0) \cdot D(q - i, 0) \).

Combining these two statements, for all \( n \in \mathbb{N} \cup \{0\} \),

\[
D(n + 1, 0) = \sum_{i,j,k \geq 0} D(i, 0)D(j, 0)D(k, 0).
\]

This is the same recurrence relation as we have for \( D_n \) and we have that \( D(0, 0) = D_0 = 1 \), so these two statements imply that \( D(n, 0) = D_n = \frac{1}{2n + 1} \binom{3n}{n} \), as needed.

To prove the first statement, given a dominant constraint graph \( C \) in \( \mathcal{C}_{(\alpha_Z, 2q)} \), if \( a_{i_2} \) is not isolated then let \( i \in [q - 1] \) be the first index such that \( a_{i_2} \leftrightarrow a_{(i+1)/2} \). By Lemma 3.18, \( b_{i_2} \leftrightarrow b_{i_2} \). By Corollary 3.31 \( b_{11} \leftrightarrow b_{11} \). Moreover, contracting \( b_{11} \) and \( b_{11} \) together and contracting \( b_{i_2} \) and \( b_{i_2} \) together splits \( H(\alpha_Z, 2q) \) into \( H(\alpha_Z, 2(i - 1)) \) and \( H(\alpha_Z, 2(q - i + 1)) \) and the induced constraint graphs \( C' \in \mathcal{C}_{(\alpha_Z, 2(i-1))} \) and \( C'' \in \mathcal{C}_{(\alpha_Z, 2(q-i+1))} \) are dominant. Now observe that

1. Since \( a_{i_2} \leftrightarrow a_{(i+1)/2} \) in \( C \), \( a_{i_2} \leftrightarrow a_{22} \) in \( C'' \).

2. If we are given dominant constraint graphs \( C' \in \mathcal{C}_{(\alpha_Z, 2(i-1))} \) and \( C'' \in \mathcal{C}_{(\alpha_Z, 2(q-i+1))} \) such that \( a_{i_2} \leftrightarrow a_{22} \) in \( C'' \), we can recover \( C \) and \( i \) by reversing this process. Thus, this map is a bijection.

This implies that the number of dominant constraint graphs \( C \) in \( \mathcal{C}_{(\alpha_Z, 2q)} \) such that \( i \in [q - 1] \) is the first index such that \( a_{i_2} \leftrightarrow a_{(i+1)/2} \) is \( D(q - i, 1) \cdot D(i - 1, 0) \). For an illustration of this argument, see Figure 3.12.

If \( a_{i_2} \) is isolated then we must have that \( b_{i_2} \leftrightarrow b_{q2} \). In this case, \( b_{i_1} \leftrightarrow b_{q1} \) and contracting along these edges gives us \( H(\alpha_Z, 2(q - 1)) \). Thus, the number of dominant constraint graphs \( C \) in \( \mathcal{C}_{(\alpha_Z, 2q)} \) such that \( a_{i_2} \) is isolated is \( D(q - 1, 0) = D(q - 1, 0)D(0, 1) \) as \( D(0, 1) = 1 \). Putting everything together,

\[
D(q, 0) = \sum_{i=1}^{q} D(q - i, 1) \cdot D(i - 1, 0).
\]

To prove the second statement, given a dominant constraint graph \( C \) in \( \mathcal{C}_{(\alpha_Z, 2(q+1))} \) such that \( a_{i_2} \leftrightarrow a_{22} \), consider the first index \( i \) such that \( b_{i_1} \leftrightarrow b_{(i+1)/2} \). If \( b_{i_1} \) is isolated then we take \( i = 0 \). We have the following cases:
Figure 3.12: \( a_{(i+1)2} \) is the first vertex that \( a_{12} \) is constrained to. As a result \( H(\alpha Z, 2q) \) is split into \( H(\alpha Z, 2(i-1)) \) and \( H(\alpha Z, 2(q - i + 1)) \).

1. \( i = 0 \): if \( b_{11} \) is not constrained to any vertex, then it implies that \( a_{11} \leftarrow a_{21} \). Merging \( a_{11} \) and \( a_{21} \), \( a_{12} \) and \( a_{22} \) and deleting spokes \( e_1 \) and \( e_2 \), we get \( H(\alpha Z, 2q) \). The induced constraint graph \( C' \) of \( C \) on \( H(\alpha Z, 2q) \) is dominant, so this gives \( D(q, 0) = D(q, 0) \cdot D(0, 0) \) possible constraint graphs.

2. \( i \in [q] \): By Lemma 3.18 \( a_{21} \leftarrow a_{(i+1)1} \). By Corollary 3.31 \( a_{22} \leftarrow a_{(i+1)2} \). Moreover, contracting \( a_{21} \) and \( a_{(i+1)1} \) together and contracting \( a_{22} \) and \( a_{(i+1)2} \) together splits \( H(\alpha Z, 2(q+1)) \) into \( H(\alpha Z, 2(i-1)) \) and \( H(\alpha Z, 2(q-i+2)) \) and the induced constraint graphs \( C' \in C(\alpha Z, 2(i-1)) \) and \( C'' \in C(\alpha Z, 2(q-i+2)) \) are dominant. Now observe that

(a) Since \( a_{12} \leftarrow a_{22} \) in \( C \) and \( b_{11} \leftarrow b_{(i+1)1} \) in \( C \), \( a_{12} \leftarrow a_{22} \) in \( C'' \) and \( b_{11} \leftarrow b_{21} \) in \( C'' \). Contracting these edges gives us \( H(\alpha Z, 2(q-i+1)) \), so \( C'' \) corresponds to a dominant constraint graph in \( C(\alpha Z, 2(q-i+1)) \).

(b) If we are given dominant constraint graphs \( C' \in C(\alpha Z, 2(i-1)) \) and \( C'' \in C(\alpha Z, 2(q-i+1)) \), we can recover \( C \) and \( i \) by reversing this process. Thus, this map is a bijection.

This gives \( D(i - 1, 0) \cdot D(q - i + 1, 0) \) dominant constraint graphs. For an illustration of this argument, see Figure 3.13.

Putting everything together,

\[
D(q, 1) = D(q, 0) \cdot D(0, 0) + \sum_{i=1}^{q} D(i - 1, 0) \cdot D(q - i + 1, 0) = \sum_{i=0}^{q} D(i, 0) \cdot D(q - i, 0)
\]

as needed.

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Figure 3.13: \( C \) is a dominant constraint graph in \( C(\alpha Z, 2(q+1)) \) such that \( \alpha_{12} \leftarrow a_{22} \). \( b_{(i+1)1} \) is the first vertex that \( b_{11} \) is constrained to.

4 The Spectrum of the Z-shaped Graph Matrix

We now find the limiting distribution of the spectrum of the singular values of \( \frac{1}{n} M_{\alpha Z} \) as \( n \to \infty \)

Definition 4.1. Let \( a = \frac{3\sqrt{3}}{2} \) and define \( g_{\alpha Z} : (0, \infty) \to \mathbb{R} \) be the function such that

\[
g_{\alpha Z}(x) = \frac{i}{\pi} \cdot \left( \sqrt{3} \cdot \sin \left( \frac{1}{3} \cdot \arctan \left( \frac{3}{\sqrt{4x^2/3 - 9}} \right) \right) + \cos \left( \frac{1}{3} \cdot \arctan \left( \frac{3}{\sqrt{4x^2/3 - 9}} \right) \right) \right)
\]

if \( x \in (0, a] \) and \( g_{\alpha Z}(x) = 0 \) if \( x > a \).

Theorem 4.2. As \( n \to \infty \), the spectrum of the singular values of \( \frac{1}{n} M_{\alpha Z} \) approaches \( g_{\alpha Z} \).

Proof. To prove this, we need to show that \( g_{\alpha Z} \) satisfies the conditions of Lemma 2.1. Here we focus on the condition that for all \( k \in \mathbb{N} \),

\[
\int_{x=0}^{a} x^{2k} g_{\alpha Z}(x) dx = \lim_{n \to \infty} \frac{1}{r(n)} \mathbb{E} \left[ \text{tr} \left( (M_{\alpha Z} M_{\alpha Z}^T)^k \right) \right] = D_k
\]

where \( r(n) = n(n-1) \) is the dimension of the graph matrix \( M_{\alpha Z} \). We defer the third condition of Lemma 2.1 to the full version of this paper.

To prove that \( \int_{x=0}^{a} x^{2k} g_{\alpha Z}(x) dx = D_k \), we proceed as follows:

1. We derive a differential equation for \( g_{\alpha Z} \) based on a recurrence relation for \( D_k \) (see Theorem 4.4).

2. We prove that if \( g_{\alpha Z} \) satisfies this differential equation and some conditions at \( x = 0 \) and \( x = a \) then \( \int_{x=0}^{a} x^{2k} g_{\alpha Z}(x) dx = D_k \) (see Theorem 4.11).
3. We verify that \( g_{\alpha x} \) satisfies the required conditions (see Theorem 4.12).

**Remark 4.3.** Technically, only steps 2 and 3 are needed. We include the first step because it gives better intuition for where the differential equation comes from.

**Theorem 4.4.** Let \( D_k = \frac{1}{2k+1} \binom{3k}{k} \) and \( a = \lim_{k \to \infty} D_{k+1}/D_k = 3\sqrt{3}/2 \). Assume \( f(x) \) is a function satisfying that for all \( k \in \mathbb{N} \),

\[
\int_0^a x^{2k} \cdot f(x)\,dx = D_k
\]  

(4.1)

and moreover,

1. \( f(x) \) is twice continuously differentiable on \((0, a)\).
2. \( \lim_{x \to 0^+} xf(x) = 0 \) and \( \lim_{x \to 0^+} x^2 f'(x) = 0 \).
3. \( \lim_{x \to a^-} f(x) = 0 \) and \( \lim_{x \to a^-} f'(x)(4x^2 - 27) = \lim_{x \to a^-} 8af'(x)(x-a) = 0 \).
4. \( \lim_{x \to 0^+} x^3 f''(x) = 0 \) and \( \lim_{x \to a^-} (a-x)^2 f''(x) = 0 \).

Then \( f(x) \) satisfies the following differential equation on \((0, a)\):

\[
(4x^4 - 27x^2)f''(x) + (8x^3 - 27x)f'(x) + 3f(x) = 0.
\]  

(4.2)

**Proof.** To prove this, we use the following recurrence relation for \( D_k = \frac{1}{2k+1} \binom{3k}{k} \).

**Proposition 4.5.**

\[
\frac{D_k}{D_{k-1}} = \frac{3(3k-1)(3k-2)}{2k(2k+1)}.
\]  

(4.3)

**Proof.** Observe that

\[
\frac{D_k}{D_{k-1}} = \frac{2k-1}{2k+1} \cdot \frac{(3k-1)!/(2k)!/(k-1)!}{(3k-3)!/(2k)!/k!} = \frac{2k-1}{2k+1} \cdot \frac{(3k)(3k-1)(3k-2)}{(2k)(2k-1)k} = \frac{3(3k-1)(3k-2)}{2k(2k+1)}.
\]

We also need the following relationship between the moments of \( f \) and the moments of its derivatives.

**Definition 4.6.** For all \( j \in \{0, 1, 2\} \) and \( k \in \mathbb{Z} \) such that \( k \geq j \), we define \( A(j, k) \) to be \( A(j, k) := \int_0^a f^{(j)}(x) \cdot x^k \,dx \) where \( f^{(j)}(x) \) denotes the \( j^{th} \) derivation of \( f \). Notice that \( A(0, 2k) = D_k \).

**Lemma 4.7.** For all \( j \in \{1, 2\} \) and \( k \in \mathbb{Z} \) such that \( k \geq j \),

\[
A(j, k) = \left[ f^{(j-1)}(x) \cdot x^k \right]_0^a - kA(j-1, k-1).
\]
Proof. Using integration by parts, we have that

\[
A(j, k) = \int_0^a f^{(j)}(x) \cdot x^k \, dx = \left[ f^{(j-1)}(x) \cdot x^k \right]_0^a - \int_0^a k f^{(j-1)}(x) \cdot x^{k-1} \, dx \\
= \left[ f^{(j-1)}(x) \cdot x^k \right]_0^a - k A(j - 1, k - 1).
\]

\[\square\]

**Corollary 4.8.** If \( \lim_{x \to 0^+} x f(x) = 0 \) and \( \lim_{x \to a^-} f(x) = 0 \) then

1. For all \( k \in \mathbb{N} \), \( A(1, k) = k A(0, k - 1) \)

2. For all \( k \in \mathbb{N} \) such that \( k \geq 2 \),

\[
A(2, k + 2) = \left[ f'(x) \cdot x^{2k+2} \right]_0^a - k A(1, 2k + 1) = \left[ f'(x) \cdot x^k \right]_0^a + k(2k + 1) A(0, 2k) - 2 A(1, 2k + 1) + 3(2k - 1)(k - 2) A(0, 2k - 2) - 2 A(1, 2k + 1).
\]

Using Corollary 4.8, Proposition 4.5, and the fact that \( A(0, 2k) = D_k \), we have that for all \( k \in \mathbb{N} \),

\[
A(2, 2k + 2) = \left[ f'(x) \cdot x^{2k+2} \right]_0^a - 2k A(1, 2k + 1) - 2 A(1, 2k + 1) + 27(2k + 1) A(0, 2k - 2) - 8 A(1, 2k + 1)
\]

Multiplying both sides by 4 and repeatedly applying Corollary 4.8, we get

\[
4A(2, 2k + 2) = 4 \left[ f'(x) \cdot x^{2k+2} \right]_0^a + 27(2k)(2k - 1) A(0, 2k - 2) + (-54k + 24) A(0, 2k - 2) - 8 A(1, 2k + 1)
\]

\[
= 4 \left[ f'(x) \cdot x^{2k+2} \right]_0^a + 27 \left( - \left[ f'(x) \cdot x^{2k} \right]_0^a + A(2, 2k) \right) - 27(2k - 1) A(0, 2k - 2) - 3 A(0, 2k - 2) - 8 A(1, 2k + 1)
\]

\[
= 27 A(2, 2k) + 27 A(1, 2k - 1) - 3 A(0, 2k - 2) - 8 A(1, 2k + 1).
\]

where the last inequality holds because \( \lim_{x \to a^-} (4x^2 - 27) f'(x) = 0 \) and \( \lim_{x \to 0^+} x^2 f'(x) = 0 \) by assumption.

Writing the \( A(j, k) \)'s above as integrals, we get that for all \( k \in \mathbb{N} \)

\[
\int_0^a \left( 4 f''(x) \cdot x^4 - 27 f''(x) \cdot x^2 - 27 f'(x) \cdot x + 8 f'(x) \cdot x^3 + 3 f(x) \right) \cdot x^{2k-2} \, dx = 0.
\]

One way for this equation to be true is if \( (4x^4 - 27x^2) f''(x) + (8x^3 - 27x) f'(x) + 3 f(x) = 0 \) on \((0, a)\). As shown by the following lemma and corollary, this is the only way for this equation to be true for all \( k \in \mathbb{Z} \), which completes the proof of Theorem 4.4.

**Lemma 4.9.** Let \( a \) be some positive constant. If \( f \) is continuous on \([0, a]\) and \( \int_0^a f(x) x^{2k} \, dx = 0 \) for all nonnegative integers \( k \), then \( f = 0 \) on \((0, a)\).
Proof. Let \( M > 0 \) be an upper bound of \( f \) on \([0, a]\). For an arbitrary \( \epsilon > 0 \), let \( p(x) \) be a polynomial such that \(|p(x) - f(x)| < \frac{\epsilon}{M \cdot a}\) for all \( x \in (0, a^2)\). Taking \( p_\epsilon(x) = p(x^2) \), \( p_\epsilon \) is a linear combination of monomials of even power and \(|p_\epsilon(x) - f(x)| < \frac{\epsilon}{M \cdot a}\) for all \( x \in (0, a)\). Thus
\[
\int_0^a (f(x) - p_\epsilon(x)) \cdot f(x) \, dx < \epsilon.
\]
On the other hand, since all even moments of \( f \) are zero,
\[
\int_0^a (f(x) - p_\epsilon(x)) \cdot f(x) \, dx = \int_0^a f(x)^2 - p_\epsilon(x)f(x) \, dx = \int_0^a f(x)^2 \, dx.
\]
Thus \( \int_0^a f(x)^2 \, dx < \epsilon \) for all \( \epsilon > 0 \) and we conclude that \( f(x) = 0 \) on \((0, a)\). \( \square \)

**Corollary 4.10.** Let \( a \) be some positive constant. If \( f \) is continuous on \((0, a)\), \( \lim_{x \to a^+} x^2 f(x) = 0 \), \( \lim_{x \to a^-} (a - x)^2 f(x) = 0 \), and \( \int_0^a f(x)x^{2k} \, dx = 0 \) for all nonnegative integers \( k \), then \( f = 0 \) on \((0, a)\).

**Proof.** This follows by applying Lemma 4.9 to the function \( f(x)x^{2(a^2 - x^2)} \). \( \square \)

We now confirm that if \( f \) satisfies the differential equation \((4x^4 - 27x^2)f''(x) + (8x^3 - 27x)f'(x) + 3f(x) = 0\), the conditions of Theorem 4.4, and the condition that \( \int_0^a f(x) \, dx = 1 \), then \( \int_0^a x^{2k} \cdot f(x) \, dx = D_k \).

**Theorem 4.11.** Let \( a = \lim_{k \to \infty} D_k / D_{k-1} = 3\sqrt{3}/2 \). Let \( f \) be a function satisfying the following ODE on \((0, a)\)
\[
(4x^4 - 27x^2)f''(x) + (8x^3 - 27x)f'(x) + 3f(x) = 0 \tag{4.4}
\]
and the first three conditions in Theorem 4.4, i.e.

1. \( f(x) \) is twice continuously differentiable on \((0, a)\).
2. \( \lim_{x \to a^+} x f(x) = 0 \) and \( \lim_{x \to 0^+} x^2 f'(x) = 0 \).
3. \( \lim_{x \to a^-} f(x) = 0 \) and \( \lim_{x \to a^-} f'(x)(4x^2 - 27) = \lim_{x \to a^-} 8af'(x)(x - a) = 0 \).

Moreover, assume that \( \int_0^a f(x) \, dx = 1 \). Then for all \( k \in \mathbb{N} \cup \{0\} \),
\[
\int_0^a x^{2k} \cdot f(x) \, dx = D_k \tag{4.5}
\]

**Proof.** Notice that \( \int_0^a f(x) \, dx = 1 = D_0 \) by assumption. We aim to prove that for all \( k \in \mathbb{N} \cup \{0\} \),
\[
(2k + 3)(2k + 2) \int_0^a x^{2k+2} \cdot f(x) \, dx = 3(3k + 2)(3k + 1) \int_0^a x^{2k} \cdot f(x) \, dx.
\]
We first verify that this

Theorem 4.12.

If so, then since \((2k + 3)(2k + 2)D_{k+1} = 3(3k + 2)(3k + 1)D_k\), we can prove Theorem 4.11 by induction on \(k\).

We multiply \(4.4\) by \(x^{2k}\) and integrate from 0 to \(a\):

\[
0 = \int_0^a (4x^4 - 27x^2) f''(x) \cdot x^{2k} + (8x^3 - 27x) f'(x) \cdot x^{2k} + 3x^{2k} f(x) \, dx
\]

\[
= \left( \int_0^a (4x^4 - 27x^2) f''(x) \cdot x^{2k} + (8x^3 - 27x) f'(x) \cdot x^{2k} + 3x^{2k} f(x) \, dx \right)
\]

\[
+ \int_0^a (8x^{2k+3} - 27x^{2k+1}) f'(x) \, dx + \int_0^a \frac{3x^{2k} f(x)}{\pi} \, dx
\]

\[
= - \int_0^a \left( 8(k + 1)x^{2k+3} - 27(2k + 1)x^{2k+1} \right) f'(x) \, dx + \int_0^a 3x^{2k} f(x) \, dx
\]

\[
= - \left[ f(x) \left( 8(k + 1)x^{2k+3} - 27(2k + 1)x^{2k+1} \right) \right]_0^a
\]

\[
+ \int_0^a \left( 8(k + 1)(2k + 3)x^{2k+2} - 3(36k^2 + 36k + 8)x^{2k} \right) f(x) \, dx + \int_0^a 3x^{2k} f(x) \, dx
\]

\[
= \int_0^a \left( 8(k + 1)(2k + 3)x^{2k+2} - 3(36k^2 + 36k + 8)x^{2k} \right) f(x) \, dx
\]

\[
= \int_0^a \left[ f'(x)(4x^4 - 27x^2)x^{2k} \right]_0^a \text{ and } \left[ f(x) \left( 8(k + 1)x^{2k+3} - 27(2k + 1)x^{2k+1} \right) \right]_0^a \text{ are zero by the assumed conditions on } f.
\]

Rearranging the last step we get

\[
(2k + 2)(2k + 3) \int_0^a f(x)x^{2k+2} \, dx = 3(3k + 1)(3k + 2) \int_0^a f(x)x^{2k} \, dx
\]

as needed.

Using WolframAlpha to solve the above ODE and analyzing the constant coefficient by the imposed initial conditions, we can get an explicit solution for \(f(x)\). We verify the solution below.

**Theorem 4.12.** Let \(a = 3\sqrt{3}/2\) and \(f(x)\) be such that

\[
f(x) = \frac{i}{\pi} \cdot \left( \sqrt{3} \cdot \sin \left( \frac{1}{3} \cdot \arctan \left( \frac{3}{\sqrt{4x^2/3 - 9}} \right) \right) + \cos \left( \frac{1}{3} \cdot \arctan \left( \frac{3}{\sqrt{4x^2/3 - 9}} \right) \right) \right) (4.6)
\]

for \(0 < x < a\). Then \(f(x)\) is an solution to the ODE \(4.2\) on \((0, a)\). Moreover, \(f\) satisfies the conditions listed in Theorem 4.11.

**Proof.** We first verify that this \(f(x)\) satisfies the ODE \(4.2\)

\[
(4x^4 - 27x^2) f''(x) + (8x^3 - 27x) f'(x) + 3f(x) = 0.
\]

on \((0, a)\).

For simplicity, we will denote \(g(x) = \frac{1}{3} \cdot \arctan \left( \frac{3}{\sqrt{4x^2/3 - 9}} \right)\). Then

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1. \( f(x) = \frac{i}{\pi} \left( \sqrt{3} \sin(g(x)) + \cos(g(x)) \right) \).

2. \( f'(x) = \frac{i}{\pi} \left( \sqrt{3} \cos(g(x)) - \sin(g(x)) \right) \cdot g'(x) = \frac{i}{\pi} \left( \sqrt{3} \cos(g(x)) - \sin(g(x)) \right) \cdot \frac{-1}{x \sqrt{4x^2/3 - 9}} \).

3. \( f''(x) = \frac{i}{\pi} \left( -\sqrt{3} \sin(g(x)) - \cos(g(x)) \right) \cdot (g'(x))^2 + \frac{i}{\pi} \left( \sqrt{3} \cos(g(x)) - \sin(g(x)) \right) \cdot g''(x) \\
   = \frac{-i}{\pi} \left( \sqrt{3} \sin(g(x)) + \cos(g(x)) \right) \cdot \frac{1}{x^2(4x^2/3 - 9)} + \frac{i}{\pi} \left( \sqrt{3} \cos(g(x)) - \sin(g(x)) \right) \cdot \frac{8x^2/3 - 9}{x^2(4x^2/3 - 9)^{3/2}}. \)

Plugging the above into the LHS of (4.2), one can verify that \( (4x^4 - 27)f''(x) + (8x^3 - 27x)f'(x) + 3f(x) = 0 \).

Now we check the conditions listed in Theorem 4.11. For this purpose, it is more convenient to re-write \( f(x) \) as a function all of real terms.

We will use the following facts:

1. \( \arctan(ix) = \frac{i}{2} \ln \left( \frac{1 + x}{1 - x} \right) \).

2. \( \sin(ix) = i \cdot \sinh(x) = \frac{i}{2} (e^x - e^{-x}), \cos(ix) = \cosh(x) = \frac{e^x + e^{-x}}{2} \).

Recall that the domain for \( f(x) \) is \( 0 < x \leq 3\sqrt{3}/2 \). Let \( y = \frac{3}{\sqrt{-4x^2/3 + 9}} \). Note that \( y \) is a real variable and \( y \geq 1 \). Also note that \( \frac{3}{\sqrt{4x^2/3 - 9}} = -iy \) and \( g(x) = \frac{1}{3} \cdot \arctan(-iy) = \frac{i}{6} \ln \left( \frac{1 - y}{1 + y} \right) \).

Let \( z = \frac{y - 1}{y + 1} = \frac{27 - 2x^2 - 9\sqrt{9 - 4x^2/3}}{2x^2} \). Note that \( z \) is a real variable and \( z \geq 0 \). Now observe that

1. \( g(x) = \frac{i}{6} \ln (-z) \)

2. \( \sin(g(x)) = \frac{i}{2} \left( (-z)^{1/6} - (-z)^{-1/6} \right) = \frac{i}{2} \left( \left( \frac{\sqrt{3} + i}{2} \right) z^{1/6} - \left( \frac{\sqrt{3} - i}{2} \right) z^{-1/6} \right) \)

3. \( \cos(g(x)) = \frac{1}{2} \left( (-z)^{1/6} + (-z)^{-1/6} \right) = \frac{1}{2} \left( \left( \frac{\sqrt{3} + i}{2} \right) z^{1/6} + \left( \frac{\sqrt{3} - i}{2} \right) z^{-1/6} \right) \).

Plugging in the above equations to \( f(x) \) and simplifying, we get that

\( f(x) = \frac{i}{\pi} \left( \sqrt{3} \sin(g(x)) + \cos(g(x)) \right) = \frac{-1}{\pi} \cdot (z^{1/6} - z^{-1/6}), \)

\( f'(x) = \frac{i}{\pi} \left( \sqrt{3} \cos(g(x)) - \sin(g(x)) \right) \cdot \frac{-1}{x \sqrt{4x^2/3 - 9}} = \frac{-1}{\pi} \cdot (z^{1/6} + z^{-1/6}) \cdot \frac{1}{x \sqrt{9 - 4x^2/3}}. \)

Recall that \( y = \frac{3}{\sqrt{-4x^2/3 + 9}} \) and \( z = \frac{y - 1}{y + 1} \). Observe that
1. As $x \to 0^+$, $y \approx 1 + \frac{2x^2}{27}$. Thus, $\lim_{x \to 0^+} \frac{z}{x^2} = \frac{1}{27}$.

2. As $x \to a^-$, $y \to \infty$. Thus, $\lim_{x \to a^-} z = 1$.

Thus,

1. $f$ is twice differentiable on $(0, a)$.

2. $\lim_{x \to 0^+} xf(x) = \lim_{x \to 0} x \cdot \left(\frac{z^{-1/6}}{\pi}\right) = 0$.

3. $\lim_{x \to 0^+} x^2 f'(x) = \lim_{x \to 0} x \cdot \left(\frac{-z^{-1/6}}{3\pi}\right) = 0$.

4. $\lim_{x \to a^-} f(x) = \frac{-1}{\pi} (1 - 1) = 0$.

5. $\lim_{x \to a^-} (4x^2 - 27)f'(x) = \lim_{x \to a^-} \frac{1}{\pi}(z^{1/6} + z^{-1/6}) \cdot \left(\frac{\sqrt{3(27 - 4x^2)}}{x}\right) = 0$.

Now we will prove the last piece of this Theorem: $\int_0^a f(x) \, dx = 1$.

We have that $a = 3\sqrt{3}/2$, $z = \frac{y - 1}{y + 1} = \frac{27 - 2x^2 - 9 \left(9 - 4x^2/3\right)^{1/2}}{2x^2} = \frac{27 - 27 \left(1 - x^2/a^2\right)^{1/2}}{2x^2}$.

Let $x = a \sin \theta$. Then $z = \frac{27 - 27 \cos \theta}{2a^2 \sin^2 \theta} - 1 = \frac{2(1 - \cos \theta)}{\sin^2 \theta} - 1 = \frac{1 - \cos \theta}{1 + \cos \theta}$. Expressing $\cos \theta$ in terms of $z$, we get $\cos \theta = \frac{1 - z}{1 + z}$, thus $\sin \theta = \frac{2\sqrt{z}}{1 + z}$. Moreover,

$$dz = \left(\frac{1 - \cos \theta}{1 + \cos \theta}\right) \frac{d\theta}{(1 + \cos \theta)^2} = \frac{2 \sin \theta}{\sin^2 \theta(1 + \cos \theta)} = \frac{2z \sin \theta \, d\theta}{\sin^2 \theta(1 + \cos \theta)} = \frac{2z}{\sin \theta} \, d\theta \implies d\theta = \frac{\sqrt{z}}{z(1 + z)} \, dz.$$ 

Thus

$$\int_0^a f(x) \, dx = \frac{-1}{\pi} \int_{\pi/2}^{\pi/2} \left(\frac{1 - \cos \theta}{1 + \cos \theta}\right)^{1/6} - \left(\frac{1 - \cos \theta}{1 + \cos \theta}\right)^{-1/6} \, a \cos \theta \, d\theta$$

$$= \frac{-a}{\pi} \int_0^{\pi/2} \left(\frac{z^{1/6} - z^{-1/6}}{z^{1/6} - z^{-1/6}}\right) \cdot \left(\frac{1 - z}{1 + z}\right) \cdot \frac{\sqrt{z}}{z(1 + z)} \, dz$$

$$= \frac{-a}{\pi} \int_0^{\pi/2} \frac{(1 - z)(z^{2/3} - z^{1/3})}{z(1 + z)^2} \, dz.$$
Let \( w = z^3 \), then

\[
\int_0^a f(x) \, dx = \frac{-a}{\pi} \int_0^1 \frac{(1 - w^3)(w - 1)}{(1 + w^3)^2} \, dw
\]

\[
= \frac{-a}{\pi} \int_0^1 \frac{-4/3}{(1 + w^3)^2} + \frac{2w}{(w^2 - w + 1)^2} + \frac{-5/3}{w^2 - w + 1} \, dw
\]

\[
= \frac{-a}{\pi} \left( \frac{4}{3} \left[ \frac{1}{1 + w} \right]_0^1 + \int_0^1 \frac{2w - 1}{(w^2 - w + 1)^2} \, dw + \int_0^1 \frac{1}{(w^2 - w + 1)^2} \, dw + \int_0^1 \frac{-5/3}{w^2 - w + 1} \, dw \right)
\]

\[
= \frac{-a}{\pi} \left( \frac{-2}{3} + \left[ \frac{-1}{(w^2 - w + 1)} \right]_0^1 + \int_0^1 \frac{1}{(w^2 - w + 1)^2} \, dw + \int_0^1 \frac{-5/3}{w^2 - w + 1} \, dw \right)
\]

\[
= \frac{-a}{\pi} \left( \frac{-2}{3} + \int_0^1 \frac{1}{((w - 1/2)^2 + 3/4) \, dw} \, dw \right).
\]

Lemma 4.13. For any \( b \neq 0 \),

\[
\int \frac{1}{(x^2 + b^2)^2} \, dx = \frac{1}{2b^2} \left( \int \frac{1}{x^2 + b^2} \, dx - \frac{x}{x^2 + b^2} \right).
\] (4.7)

Proof.

\[
\int \frac{1}{(x^2 + b^2)^2} \, dx = \frac{1}{b^2} \int \frac{x^2 + b^2}{(x^2 + b^2)^2} + \frac{-x^2}{x^2 + b^2} \, dx
\]

\[
= \frac{1}{b^2} \left( \int \frac{1}{x^2 + b^2} \, dx + \int \frac{-x}{2} \frac{1}{x^2 + b^2} \, dx \right)
\]

\[
= \frac{1}{b^2} \left( \int \frac{1}{x^2 + b^2} \, dx - \frac{x}{2(x^2 + b^2)} + \int \frac{-1}{2(x^2 + b^2)} \, dx \right)
\]

\[
= \frac{1}{2b^2} \left( \int \frac{1}{x^2 + b^2} \, dx - \frac{x}{x^2 + b^2} \right).
\]

Apply the lemma to the \( \int_0^1 \frac{1}{((w - 1/2)^2 + 3/4) \, dw} \) term, we get that

\[
\int_0^a f(x) \, dx = \frac{-a}{\pi} \left( \frac{-2}{3} + \frac{2}{3} \left( \int_0^1 \frac{1}{(w - 1/2)^2 + 3/4} \, dw - \left[ \frac{w - 1/2}{w^2 - w + 1} \right]_0^1 \right) + \int_0^1 \frac{-5/3}{(w - 1/2)^2 + 3/4} \, dw \right)
\]

\[
= \frac{-a}{\pi} \left( \frac{-2}{3} + \frac{2}{3} + \int_0^1 \frac{-1}{(w - 1/2)^2 + 3/4} \, dw \right)
\]

\[
= \frac{a}{\pi} \left( \frac{1}{\sqrt{3/2}} \arctan \left( \frac{w - 1/2}{\sqrt{3/2}} \right) \right)_0^1 = \frac{3\sqrt{3}/2}{\pi} \cdot \frac{2\pi}{3\sqrt{3}} = 1.
\]

\[
\]
Figure 4.1 shows some graphs of $g_{\alpha Z}(x)$ and samplings of singular values of $M_{\alpha Z}$ for $n = 20, 30, 70$. We can see that the sampled distribution of the singular values of $M_{\alpha Z}$ gets closer to $g_{\alpha Z}(x)$ as $n$ gets bigger.

![Image of graphs showing singular values](image1)

(a) The spectrum of singular values
(b) Sampling of singular values of $M_{\alpha Z}$ where $n = 20$

![Image of graphs showing singular values](image2)

(c) Sampling of singular values of $M_{\alpha Z}$ where $n = 30$
(d) Sampling of singular values of $M_{\alpha Z}$ where $n = 70$

Figure 4.1: The Spectrum of singular values of the Z-shape graph matrix and some samplings of the Z-shape graph matrices with random input graphs on $n$ vertices, for $n = 20, 30, 70$.

4.1 Behavior near $x = 0$ and $x = a$ and numerically solving the differential equation

We now consider the behavior of the differential equation $(4x^4 - 27)f''(x) + \left(8x^3 - 27x\right)f'(x) + 3f(x) = 0$ near $x = 0$ and near $x = a$. While this kind of analysis is not necessary for this differential equation as we were able to find an explicit solution, this kind of analysis is very useful for differential equations where we cannot find an explicit solution.

When $x$ is very close to 0, the differential equation is approximately

$$-27x^2f''(x) - 27xf'(x) + 3f(x) \sim 0.$$
Plugging in $f(x) = x^r$, we obtain that

$$( -27r(r-1) - 27r + 3 ) x^r = ( -27r^2 + 3 ) x^r = 0$$

which is satisfied when $r = \pm 1/3$. Thus, near $x = 0$ the solution to the differential equation is approximately $c_1 x^{1/3} + c_2 x^{-1/3}$.

When $x$ is very close to $a$, we observe that

1. $4x^4 - 27x^2 = 4x^2(x-a)(x+a) \approx 8a^3(x-a)$,
2. $8x^3 - 27x = 4x^2(x-a)(x+a) \approx 4a^3$,

If we further assume that $\lim_{x \to a} f(x) = 0$ then when $x$ is very close to $a$, the differential equation is approximately

$$-8a^3(a-x)f''(x) + 4a^3f'(x) = 0.$$ 

Plugging in $f'(x) = (a-x)^r$, we obtain that

$$8ra^3(a-x)^r + 4a^3(a-x)^r = (8r + 4)a^3(a-x)^r = 0$$

which is satisfied when $r = -\frac{1}{2}$. Thus, the solution to the differential equation near $x = a$ which is 0 at $x = a$ is approximately $c\sqrt{a-x}$.

This analysis helps us solve this differential equation numerically. To solve this differential equation numerically, we need an initial point $x_0$ and the initial conditions $f(x_0)$ and $f'(x_0)$. However, we can’t use $x_0 = 0$ because $\lim_{x \to 0^+} f(x) = \infty$ and we can’t use $x_0 = a$ because $\lim_{x \to a^-} f'(x) = -\infty$. Instead, we proceed as follows:

1. Choose an $\epsilon > 0$ and approximate the solution by $\sqrt{a-x}$ on the interval $[a-\epsilon, a]$.
2. Numerically solve the differential equation on the interval $(0, a-\epsilon)$.
3. Scale the resulting function $f$ so that $\int_{x=0}^{a} f(x) dx = 1$.

Figure 4.2 shows several plots of the explicit solution we get in Theorem 4.12 with the numerical solution we get for various $\epsilon$’s. One can see that as $\epsilon$ gets smaller, the approximated spectrum gets closer to the actual spectrum. When $\epsilon = 0.0001$, the two curves are almost identical.

5 Trace Powers of Multi-Z-shaped Graph Matrices

Now we consider a generalization of the Z-shape graph matrix discussed in Section 3.

**Definition 5.1.** Let $\alpha_{Z(m)}$ be the bipartite shape with vertices $V(\alpha_{Z(m)}) = \{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ and edges $E(\alpha_{Z(m)}) = \{\{u_i, v_i\} : i \in [m]\} \cup \{\{u_{i+1}, v_i\} : i \in [m-1]\}$ with distinguished tuples of vertices $U_{\alpha_{Z(m)}} = (u_1, \ldots, u_m)$ and $V_{\alpha_{Z(m)}} = (v_1, \ldots, v_m)$. See Figures 5.1 for an illustration.

We refer to $\alpha_{Z(m)}$ as the $m-$layer Z-shape. Note that $\alpha_{Z(2)}$ is the Z-shape $\alpha_Z$ as in definition 1.5.
Remark 5.2. For the $m$–layer Z-shape $\alpha_{Z(m)}$, the size of the minimum separator is $m$. By Lemma 2.25, for any nonzero-valued constraint graph $C \in C(\alpha_{Z(m)}, 2q)$, $|E(C)| \geq m \cdot (q - 1)$. By Corollary 2.27, dominant constraint graphs $C \in C(\alpha_{Z(m)}, 2q)$ have $m \cdot (q - 1)$ edges.

Definition 5.3. For $m, n$ positive integers,

$$D(m, n) = \frac{1}{m \cdot n + 1} \binom{(m + 1) \cdot n}{n}.$$  \hspace{1cm} (5.1)

Remark 5.4. The number $D_n = \frac{1}{2n + 1} \binom{3n}{n}$ in Section 3 is $D(2, n)$ here. Also $D(m - 1, n) = A_n(m, 1)$ where the generalized Catalan number $A_n(k, r) = \frac{r}{n k + r} \binom{n k + r}{n}$ is defined in Remark 3.3.

Below is the main result of this section.

Theorem 5.5. Let $\alpha_{Z(m)}$ be the $m$–layer Z-shape as in definition 5.1. Then the number of dominant constraint graphs $C \in C(\alpha_{Z(m)}, 2q)$ is $D(m, q)$.

Remark 5.6. When $m = 2$, $D(m, n) = D(2, n) = D_n$, $\alpha_{Z(m)} = \alpha_Z$, and this theorem is exactly Theorem 3.4.
A direct corollary we get from the above theorem is the following.

**Corollary 5.7.** Let \( \alpha_{Z(m)} \) be the \( m \)-layer Z-shape as in definition 5.1. Let \( M_{n,m} = \frac{1}{n^{m/2}} M_{\alpha_{Z(m)}}(G) \) be the graph matrix where \( G \sim G(n, 1/2) \) and \( r(n, m) = \frac{n!}{(n-m)!} \) be the dimension of \( M_{n,m} \). Recall that \( D(m,q) = \frac{1}{mq + 1} \binom{m+1}{q} \). Then

\[
\lim_{n \to \infty} \frac{1}{r(n, m)} \mathbb{E} \left[ \text{tr} \left( \left( M_{n,m}M_{n,m}^T \right)^q \right) \right] = D(m, q). \tag{5.2}
\]

**Proof.** Recall that Corollary 2.28 says that for any bipartite shape \( \alpha \),

\[
\lim_{n \to \infty} \frac{1}{r_{\text{approx}}(n)} \mathbb{E} \left[ \text{tr} \left( \left( \frac{M_{\alpha}M_{\alpha}^T}{n^{|V(\alpha)|-s_\alpha}} \right)^q \right) \right] = \left| C_{(\alpha, 2q)} : C \text{ is dominant} \right|.
\]

Since \( s_{\alpha_{Z(m)}} = m \) and \( |V(\alpha_{Z(m)})| = 2m \), \( r_{\text{approx}}(n) = \frac{n!}{(n-s_{\alpha_{Z(m)}})!} = \frac{n!}{(n-m)!} = r(n, m) \) and

\[
\frac{M_{\alpha_{Z(m)}}M_{\alpha_{Z(m)}}^T}{n^{|V(\alpha_{Z(m)})|-s_{\alpha_{Z(m)}}}} = \frac{M_{\alpha_{Z(m)}}M_{\alpha_{Z(m)}}^T}{n^m} = M_{n,m}M_{n,m}^T.
\]

By Theorem 5.5, \( \left| C_{(\alpha_{Z(m)}, 2q)} : C \text{ is dominant} \right| = D(m, q) \) and the result follows.

\[
\square
\]

### 5.1 Recurrence Relation for \( D(m, n) \)

To prove the main result for this section, We need the following crucial recurrence relation for \( D(m, n) \).

**Theorem 5.8.**

\[
D(m, n + 1) = \sum_{\substack{i_0, \ldots, i_m \geq 0: \\ i_0 + \cdots + i_m = n}} D(m, i_0) \cdots D(m, i_m). \tag{5.3}
\]
Proof. The proof is similar to the proof of Theorem 3.6. Let $W_{m,n} :=$ the set of all grid walks from $(0,0)$ to $(n, mn)$ that are weakly below the diagonal and $d_{m,n} = |W_{m,n}|$. We will prove that $d_{m,n} = D(m,n)$ and that $d_{m,n}$ satisfies the recurrence relation in the theorem.

1. $d_{m,n} = D(m,n)$:

For $r \in \{0,1,\ldots, mn\}$, let $W_{m,n}(r) :=$ the set of grid walks from $(0,0)$ to $(n, mn)$ that are $r$ steps above the diagonal. Then $\bigcup_{r=0}^{mn} W_{m,n}(r)$ is the set of all grid walks from $(0,0)$ to $(n, mn)$, which has cardinality $\binom{(m+1) \cdot n}{n}$. Also $|W_{m,n}(0)| = |W_{m,n}| = d_{m,n}$. By the same proof as in the Theorem 3.6, there is a bijection between $W_{m,n}(r)$ and $W_{m,n}(r-1)$ for each $r \in [mn]$. Thus $d_{m,n} = \frac{1}{mn+1} \binom{(m+1) \cdot n}{n}$.

2. $d_{m,n} = \sum_{i_0 \ldots i_m \geq 0: \ i_0 + \ldots + i_m = n-1} d_{m,i_1} \ldots d_{m,i_m}$:

Similar to the proof of Theorem 3.6, we now will find a bijection between $W_{m,n}$ and $\bigcup_{i_0 \ldots i_m \geq 0: \ i_0 + \ldots + i_m = n-1} W_{m,i_0} \times \cdots \times W_{m,i_m}$.

Let $d_k$ be the line that is shifted $k$ vertical grids down from the diagonal. i.e. $d_k$ is the line $y = m \cdot x - k$. Let $w = (z_1, \ldots, z_{n \cdot (m+1)}) \in W_{m,n}$.

- Let $z_{i_0} = (a_0, m \cdot a_0)$ be the first point where $w$ touches the diagonal. Then $w_0 := (z_{i_0}, \ldots, z_{n \cdot (m+1)})$ can be viewed as an element in $W_{m,n-a_0}$. Moreover, $w' := (z_2, \ldots, z_{i_0-1})$ is strictly below the diagonal $d_0$, thus weakly below $d_1$.

- Let $z_{i_1} = (a_1, m \cdot a_1 - 1)$ be the first point where $w'$ touches $d_1$. Then $w_1 := (z_1, \ldots, z_{i_0-1})$ can be viewed as an element in $W_{m,a_0-1}$. Moreover, $w' := (z_2, \ldots, z_{i_1-1})$ is strictly below $d_1$, thus weakly below $d_2$.

Figure 5.2: Illustration of the proof part 2 for Theorem 5.8
- Continue this way, we get a sequence of points \( z_{i_0}, \ldots, z_{i_{m-1}} \) and walks \( w_0, \ldots, w_{m-1} \) where each \( z_{ij} = (a_j, m \cdot a_j - j) \) is a point on \( d_j \) and each \( w_i \) can be viewed as an element in \( W_{m,a_i-1-a_i} \).

- Since \( z_{i_{m-1}} \) is the first point touching \( d_{m-1} \), \( w_m = (z_2, \ldots, z_{i_{m-1}} - 1) \) is strictly below \( d_{m-1} \) and thus weakly below \( d_m \). Since \( d_m \) crosses \( (1,0) = z_2 \), \( w_m \) can be viewed as an element in \( W_{m,a_m-1-1} \).

Since \( (n - a_0) + (a_0 - a_1) + \cdots + (a_{m-1} - 1) = n - 1 \), we conclude that

\[
(w_0, \ldots, w_m) \in \bigcup_{i_0, \ldots, i_m \geq 0: i_0 + \cdots + i_m = n-1} W_{m,i_0} \times \cdots \times W_{m,i_m}.
\]

The other direction of the bijection can be constructed in a backward manner. It is not hard to prove this construction gives a bijection.

Combining 1 and 2, we conclude that \( D(m,n) \) satisfies the recurrence relation.

\[ \square \]

### 5.2 Properties of Dominant Constraint Graphs on \( H(\alpha_{Z(m)}, 2q) \)

#### Definition 5.9.

Let \( \alpha_{Z(m)} \) be the multi-layer Z-shape as in Definition 5.1 and let \( H(\alpha_{Z(m)}, 2q) \) be the multi-graph as in definition 2.2. We label the vertices of \( V(\alpha_{Z}) \), as \( \{a_{ij}, b_{ij} : j \in [m]\} \) and the vertices of \( V_{\alpha_{Z}}^{T_j} \), as \( \{b_{ij}, a_{(i+1)j} \} \). Let \( V_i = \{a_{ij}, b_{ij} : i \in [q]\} \). For \( j \in [m] \), we call the induced subgraph of \( H(\alpha_{Z(m)}, 2q) \) on vertices \( V_i \) the \( j^{th} \) wheel \( W_j \).

We label the “middle edges” of \( H(\alpha, 2q) \) in the following way: let \( e_{2i-1,j} = \{a_{i(j+1)}, b_{ij}\} \) and \( e_{2i,j} = \{b_{ij}, a_{(i+1)(j+1)}\} \) for \( i = 1, \ldots, q \). For a fixed \( j \in [m] \), we call the edges \( e_{i,j} \)’s the spokes between wheels \( W_j \) and \( W_{j+1} \) of \( H(\alpha_{Z(m)}, 2q) \). See Figure 5.3 for an illustration.

#### Definition 5.10.

Let \( \alpha_{Z(m)} \) be the multi-layer Z-shape. Let \( C \) be a constraint graph on \( H(\alpha_{Z}, 2q) \). For \( i = 1, 2 \), We denote \( C_i \) the induced subgraph of \( C \) on vertices \( V_i \). We call \( C_i \) the induced constraint graph of \( C \) on \( V_i \).

Recall that a constraint graph \( C \in \mathcal{C}(\alpha, 2q) \) is well-behaved if whenever \( u \leftrightarrow v \) in \( C \), \( u \) and \( v \) are copies of the same vertex in \( \alpha \) or \( \alpha^T \).

#### Theorem 5.11.

All dominant constraint graphs in \( \mathcal{C}(\alpha_{Z(m)}, 2q) \) are well-behaved.

**Proof.** See appendix. \[ \square \]

We extend our definitions of wheel-respecting, parity-preserving, and non-crossing to \( \mathcal{C}(\alpha_{Z(m)}, 2q) \).

#### Definition 5.12.

Let \( C \) be a constraint graph in \( \mathcal{C}(\alpha_{Z(m)}, 2q) \).
Figure 5.3: $H\left(\alpha_{Z(m)}, 2q\right)$, here $m = 3$

1. We say $C$ is wheel-respecting if for all $u \leftrightarrow v$, $u, v \in V_i$ for some $i \in [m]$. (i.e. no two vertices on different wheels are constrained together by $C$).

2. We say $C$ is parity-preserving if for each $i \in [m]$, the induced constraint graphs $C_i$ of $C$ on $V_i$ is parity-preserving.

3. We say $C$ is non-crossing if the induced constraint graphs $C_i$’s are non-crossing.

Remark 5.13. If $C \in C_{\left(\alpha_{Z(m)}, 2q\right)}$ is wheel-respecting and $G_i$ are representatives of $C_i$, then the graph $G$ with $V(G) = V(\alpha_{Z(m)}, 2q)$ and $E(G) = E(G_1) \cup \cdots \cup E(G_m)$ is a representative of $C$.

Proposition 5.14. Let $C$ be a constraint graph in $C_{\left(\alpha_{Z(m)}, 2q\right)}$. $C$ is well-behaved if and only if $C$ is wheel-respecting and parity-preserving.

Corollary 5.15. If $C$ is a dominant constraint graph in $C_{\left(\alpha_{Z(m)}, 2q\right)}$, then

1. $C$ is wheel-respecting, parity-preserving and non-crossing.

2. The induced constraint graphs $C_i$ on wheels $W_i$ are dominant constraint graphs in $C_{(\alpha_0, 2q)}$.

The same proofs for Lemma 3.30 yields the following statement.

Lemma 5.16. Let $\alpha_{Z(m)}$ be as in definition 5.1 and let $C$ be a dominant constraint graph in $C_{\left(\alpha_{Z(m)}, 2q\right)}$. If $a_{sj} \leftrightarrow a_{tj}$ for some $1 \leq s < t \leq q$ and $j \in [m - 1]$, then $a_{s(j+1)} \leftrightarrow a_{t(j+1)}$. Moreover, the spokes $\{e_{x,j} : x \in \{2s - 1, 2t - 2\}\}$ can only be made equal to each other.

Similarly, if $b_{sj} \leftrightarrow b_{tj}$ for some $1 \leq s < t \leq q$ and $j \in \{2, 3, \ldots, m\}$, then $b_{s(j-1)} \leftrightarrow b_{t(j-1)}$. Moreover, the spokes $\{e_{x,j-1} : x \in \{2s, 2t - 1\}\}$ can only be made equal to each other.

Corollary 5.17. If $C$ is a dominant constraint graph in $C_{\left(\alpha_{Z(m)}, 2q\right)}$, then the following statements are true:
Figure 5.4: Illustration of Corollary 5.17: $C \in C(\alpha_{Z(m)}, 2q)$ is dominant. $a_{ik} \leftrightarrow a_{jk}$ and $b_{i(k-1)} \leftrightarrow b_{j(k-1)}$. Here $q = 7$, $m = 4$, $k = 3$ and $j - i = 3$.

1. If $a_{i1} \leftrightarrow a_{j1}$ for some $1 \leq i < j \leq q$, then $a_{ik} \leftrightarrow a_{jk}$ for all $k \in [m]$.
2. Similarly, if $b_{im} \leftrightarrow a_{jm}$ for some $1 \leq i < j \leq q$, then $b_{ik} \leftrightarrow b_{jk}$ for all $k \in [m]$.
3. More generally, if $a_{ik} \leftrightarrow a_{jk}$ and $b_{i(k-1)} \leftrightarrow b_{j(k-1)}$ for some $k \in [m+1]$, then $a_{is} \leftrightarrow a_{js}$ for all $k \leq s \leq m$ and $b_{it} \leftrightarrow b_{jt}$ for all $1 \leq t \leq k - 1$. Note that $k = 1$ corresponds to case 1 above and $k = m + 1$ corresponds to case 2 above. See Figure 5.4 for an illustration.

In all three cases, contracting the constrained vertices splits $H(\alpha_{Z(m)}, 2q)$ into $H(\alpha_{Z(m)}, 2(j - i))$ and $H(\alpha_{Z(m)}, 2q - j + i)$, and the induced constraint graphs $C' \in C(\alpha_{Z(m)}, 2(j - i))$ and $C'' \in C(\alpha_{Z(m)}, 2(q - j + i))$ are dominant.

Proof sketch. We focus on the third case as this is the trickiest case. To split $H(\alpha_{Z(m)}, 2q)$ into $H(\alpha_{Z(m)}, 2(j - i))$ and $H(\alpha_{Z(m)}, 2q - j + i)$, imagine doing the following:

1. Cut towards the center of the wheels through the vertices $b_{i1}, b_{i2}, \ldots, b_{i(k-1)}$, then cut along the spoke $\{a_{ik}, b_{i(k-1)}\}$, and then cut towards the center of the wheels through the vertices $a_{ik}, \ldots, a_{im}$.

2. Similarly, cut towards the center of the wheels through the vertices $b_{j1}, b_{j2}, \ldots, b_{j(k-1)}$, then cut along the spoke $\{a_{jk}, b_{j(k-1)}\}$, and then cut towards the center of the wheels through the vertices $a_{jk}, \ldots, a_{jm}$.

These cuts split $H(\alpha_{Z(m)}, 2q)$ into two halves. Taking each half and gluing it to itself along the cuts gives us $H(\alpha_{Z(m)}, 2(j - i))$ and $H(\alpha_{Z(m)}, 2(q - j + i))$.

To see that these the induced constraint graphs $C'$ and $C''$ are dominant, observe that except for the vertices along the cuts, each vertex appears in one half or the other but not both. Except
for the spokes \( \{a_{ik}, b_{i(k-1)}\} \) and \( \{a_{jk}, b_{j(k-1)}\} \), each edge is incident to a vertex which is not part of the cut, so this implies that all of the edges in \( H(\alpha_{Z(m)}, 2q) \) except for the spokes \( \{a_{ik}, b_{i(k-1)}\} \) and \( \{a_{jk}, b_{j(k-1)}\} \) (which are made equal to each other) appear in one half or the other but not both.

By Corollary 3.17, edges on wheels on one side of the split can only be made equal to the edges on the same side. By Lemma 3.30, \( b_{ir} \leftarrow b_{jr} \) implies that spokes between wheels \( W_r \) and \( W_{r-1} \) on one side of the split can only be made equal to the spokes on the same side. By the same lemma, \( a_{ir} \leftarrow a_{jr} \) implies that spokes between wheels \( W_r \) and \( W_{r+1} \) on one side of the split can only be made equal to the spokes on the same side. By the assumptions on the constrained vertices, we conclude that other than the spokes between wheels \( W_k \) and \( W_{k-1} \), all other spokes are made equal to the spokes on its own side after splitting.

Thus so far, in both \( H(\alpha_{Z(m)}, 2(j - i)) / C' \) and \( H(\alpha_{Z(m)}, 2(q - j + i)) / C'' \), each edge except for the spokes between \( W_k \) and \( W_{k-1} \) appears an even number of times.

For the spokes between \( W_k \) and \( W_{k-1} \), assume there is a spoke \( e_s \) on one side of the split such that \( e_s \not\leftrightarrow \{a_{ik}, b_{i(k-1)}\} \) and \( e_s \leftrightarrow e_t \) for some \( e_t \) on the other side of the split. Then by Lemma 3.28, \( e_s \leftrightarrow e_t \leftrightarrow e_{2i-1,k-1} \leftrightarrow e_{2j-1,k-1} \), a contradiction. Thus spokes on one side that are not made equal to \( \{a_{ik}, b_{i(k-1)}\} \) or \( \{a_{jk}, b_{j(k-1)}\} \) are only made equal to other spokes on the same side. Since the total number of spokes on each side is even, \( \{a_{ik}, b_{i(k-1)}\} \) needs to appear even number of times in \( H(\alpha_{Z(m)}, 2(j - i)) / C' \) and \( \{a_{jk}, b_{j(k-1)}\} \) needs to appear even number of times in \( H(\alpha_{Z(m)}, 2(q - j + i)) / C'' \). Thus in both \( H(\alpha_{Z(m)}, 2(j - i)) / C' \) and \( H(\alpha_{Z(m)}, 2(q - j + i)) / C'' \), each spoke between \( W_k \) and \( W_{k-1} \) appears an even number of times.

This implies that \( C' \) and \( C'' \) are nonzero-valued constraint graphs on \( H(\alpha_{Z(m)}, 2(j - i)) \) and \( H(\alpha_{Z(m)}, 2(q - j + i)) \) and are thus dominant, as needed.

5.3 Proof of Theorem 5.5

Now we are ready to prove the main result of this section, Theorem 5.5. The proof will be similar to the one for the Z-shape case. For the Z-shape, we split \( H(\alpha_Z, 2q) \) two times to get the recurrence relation, where first step is based on the constrained vertices on the inner wheel, and the second step is based on the constrained vertices on the outer wheel. For the multi-Z-shape with \( m \) layers, we will split \( H(\alpha_{Z(m)}, 2q) \) \( m \) times for the recurrence relation, where the \( r \)th step will be based on the constrained vertices on the \( (m - r + 1) \)th wheel, starting from the inner-most wheel.

**Definition 5.18.** For \( r \in [m] \), let \( \mathcal{D}_{m,q,r} \) denote the set of all dominant constraint graphs in \( \mathcal{C}(\alpha_{Z(m)}, 2q) \) such that \( a_{1m} \leftrightarrow a_{2m}, a_{1(m-1)} \leftrightarrow a_{2(m-1)}, \ldots, a_{1(m-r+1)} \leftrightarrow a_{2(m-r+1)} \) in \( C \). When \( r = 0 \), define \( \mathcal{D}_{m,q,0} = \text{the set of all dominant constraint graphs in } \mathcal{C}(\alpha_{Z(m)}, 2q) \). Define \( D(m, q, r) = |\mathcal{D}_{m,q,r}| \) for all \( m, q, r \).

We first consider the first step of the splitting.
Lemma 5.19. Let $D_{m,q,r}$ be as defined in 5.18. There is a bijection between $D_{m,q,0}$ and $D_{m,q-i,1}$. Thus

$$D(m, q, 0) = \sum_{i=0}^{q-1} D(m, i, 0) \cdot D(m, q - i, 1).$$

(5.4)

Proof. Let $C \in D_{m,q,0}$. i.e., $C$ is a dominant constraint graph on $H(\alpha Z(m), 2q)$. Let $i > 0$ be the smallest index such that $a_{1m}$ is constrained to $a_{i+1,m}$. If $a_{1m}$ is isolated then we take $i = q$.

Since $C$ is dominant, by Corollary 5.15, the induced constraint graph $C_m$ on $W_m$ is dominant. By Lemma 3.18 and Lemma 3.14 (in the case when $a_{1m}$ is isolated), $b_{1m} \leftrightarrow b_{i+1,m}$. By Corollary 5.17, $H(\alpha Z(m), 2q)$ splits into two parts, $H(\alpha Z(m), 2(i-1))$ and $H(\alpha Z(m), 2(q-i+1))$ where $a_{1m} \leftrightarrow a_{(i+1)m}$ (see Figure 5.5 for an illustration). Moreover, the induced constraint graphs $C' \in C(\alpha Z(m), 2(i-1))$ and $C'' \in C(\alpha Z(m), 2(q-i+1))$ are dominant.

Since $C'$ is dominant on $H(\alpha Z(m), 2(i-1))$, we have $C' \in D_{m,i-1,0}$. Since $C''$ is dominant on $H(\alpha Z(m), 2(q-i+1))$ and $a_{1m} \leftrightarrow a_{(i+1)m}$ where $a_{1m}$ and $a_{(i+1)m}$ are adjacent in $H(\alpha Z(m), 2(q-i+1))$ (see Figure 5.5 for an illustration), $C'' \in D_{m,q-i+1,1}$.

So far we constructed a mapping $C \in D_{m,q,0} \mapsto (C', C'') \in D_{m,i-1,0} \times D_{m,q-i+1,1}$ given that $i \in [q]$ is the smallest index such that $a_{1m} \leftrightarrow a_{(i+1)m}$ in $C$. Thus by considering the first vertex that $a_{1m}$ is constrained to in $C$, we can construct a map from $D_{m,q,0}$ to $\bigcup_{i=0}^{q-1} D_{m,i,0} \times D_{m,q-i,1}$.

The other direction of the map goes in reverse order of the contracting process. It can be verified that this is a bijection. Thus we proved that $D(m, q, 0) = \sum_{i=0}^{q-1} D(m, i, 0) \cdot D(m, q - i, 1)$.

Now we will consider the later steps of the splitting.
Lemma 5.20. For $r \geq 1$, there exists an bijection between $D_{m,q,r}$ and $\bigcup_{i=0}^{q-1} D_{m,i,0} \times D_{m,q-i,r+1}$.

Thus

$$D(m, q, r) = \sum_{i=0}^{q-1} D(m, i, 0) \cdot D(m, q-i, r+1).$$

(5.5)

---

Figure 5.6: Illustration of proof for Lemma 5.20: On the left is a $C \in D_{m,q,r}$, with constrained vertices indicated by dash lines. Here $q = 7$, $m = 3$ and $r = 1$. $b_{i(m-r)}$ is the first vertex that $b_{1(m-r)}$ is constrained to in $C$. On the right is $H \left( \alpha_{Z(m)}, 2(q-i+1) \right)$ and $H \left( \alpha_{Z(m)}, 2(i-1) \right)$ after splitting.

Proof. Let $C \in D_{m,q,r}$ for some $r \geq 1$. i.e. $C$ is dominant and $a_{1k} \leftrightarrow a_{2k}$ for all $m-r+1 \leq k \leq m$.

Let $i > 1$ be the first index such that $b_{1(m-r)} \leftrightarrow b_{i(m-r)}$. If $b_{1(m-r)}$ is isolated then we take $i = q + 1$. By Corollary 5.15, since $C$ is dominant on $H \left( \alpha_{Z(m)}, 2q \right)$, the induced constraint graph $C_{m-r}$ is dominant, thus by Lemma 3.18 and Lemma 3.14 $a_{2(m-r)} \leftrightarrow a_{i(m-r)}$. By Lemma 5.16 $a_{2k} \leftrightarrow a_{ik}$ for all $m-r \leq k \leq m$. Since $a_{1k} \leftrightarrow a_{2k}$ for all $m-r+1 \leq k \leq m$ by assumption, $a_{1k} \leftrightarrow a_{ik}$ for all $m-r+1 \leq k \leq m$. By Lemma 5.16 since $b_{1(m-r)} \leftrightarrow b_{i(m-r)}$, $b_{1k} \leftrightarrow b_{ik}$ for all $1 \leq k \leq m-r$.

So far we have that

1. $a_{1k} \leftrightarrow a_{2k} \leftrightarrow a_{ik}$ for all $m-r+1 \leq k \leq m$.
2. $b_{1k} \leftrightarrow b_{ik}$ for all $1 \leq k \leq m-r$.
3. $a_{2(m-r)} \leftrightarrow a_{i(m-r)}$.

Since $b_{1(m-r)} \leftrightarrow b_{i(m-r)}$ and $a_{1(m-r+1)} \leftrightarrow a_{i(m-r+1)}$, by Corollary 5.17 $H \left( \alpha_{Z(m)}, 2q \right)$ splits into $H \left( \alpha_{Z(m)}, 2(i-1) \right)$ and $H \left( \alpha_{Z(m)}, 2(q-i+1) \right)$. Let $C' \in C \left( \alpha_{Z(m)}, 2(i-1) \right)$ and $C'' \in C \left( \alpha_{Z(m)}, 2(q-i+1) \right)$ be the induced constraint graphs. By Corollary 3.17 the $C'$ and $C''$ are dominant. Notice that there is no extra required constrained vertices in $C''$, so $C'' \in D_{m,q-i+1,0}$.
a_{2k} \leftrightarrow a_{ik} \text{ for all } m - r + 1 \leq k \leq m \text{ and } a_{2(m-r)} \leftrightarrow a_{i(m-r)}, C' \in D_{m,i-1,r+1} \text{ (see Figure 5.6 for an illustration).}

So far we have constructed a mapping \( C \in D_{m,q,r} \mapsto (C',C'') \in D_{m,q-i+1,0} \times D_{m,i-1,r+1} \) given that \( 1 < i < q + 1 \) is the first index such that \( b_{1(m-r)} \leftrightarrow b_{i(m-r)} \) in \( C \). Thus by considering the first vertex that \( b_{1(m-r)} \) is constrained to in \( C \), we can construct a mapping from \( D_{m,q,r} \) to \( \bigcup_{i=0}^{q-1} D_{m,i,0} \times D_{m,q-i,r+1} \).

The other direction of the map simply reverses the the splitting process. It can be verified that this is a bijection. Thus we proved that \( D(m,q,r) = \sum_{i=0}^{q-1} D(m,i,0) \cdot D(m,q-i,r+1). \) \( \square \)

Finally we arrive at the last step, where we identify the last item from the splitting process \( D_{m,q,m} \) with \( D_{m,q-1,0} \).

**Lemma 5.21.** Let \( D_{m,q,r} \) be defined as in [5.18]. For any \( q \geq 1 \) and \( m \geq 1 \), there is a bijection between \( D_{m,q,m} \) and \( D_{m,q-1,0} \). Thus \( D(m,q,m) = D(m,q-1,0). \)

**Proof.** Let \( C \in D_{m,q,m} \). Then \( a_{1j} \leftrightarrow a_{2j} \) for all \( j \in [m] \). Contracting the constrained vertices we get the induced constraint graph \( C' \) on \( H(\alpha_{Z(m)}, 2(q-1)) \), and \( C' \) is dominant. i.e. \( C' \in D_{m,q-1,0} \).

Given \( C' \in D_{m,q-1,0} \), we first relabel the indices by increasing all of them by 1. We then expand \( a_{2j} \) to \( a_{1j} \) and \( a_{2j} \) and make them constrained, for each \( j \in [m] \). Adding these constraints \( a_{1j} \leftrightarrow a_{2j} \) for all \( j \in [m] \) to \( C' \), we get a new constraint graph \( C \in D_{m,q,m} \).

We have constructed maps between \( D_{m,q,m} \) and \( D_{m,q-1,0} \) in both directions. It is easy to see that this is a bijection. Thus we proved that \( D(m,q,m) = D(m,q-1,0). \) \( \square \)

With the above lemmas, we are ready to prove Theorem 5.5.

**Proof of Theorem 5.5.** Recall that \( D(m,q) = \frac{1}{mq+1} \binom{m+1}{q} \) and we want to prove that \( D(m,q,0) \) is the number of dominant constraint graphs in \( \mathcal{C}_{(\alpha_Z(m),2q)} = D(m,q) \). For \( q = 1 \), we check that \( D(m,1,0) = 1 = D(m,1) \). We will then show that

\[
D(m,q,0) = \sum_{i_1,\ldots,i_{m+1} \geq 0 \atop i_1 + \ldots + i_{m+1} = q-1} D(m,i_1,0) \ldots D(m,i_{m+1},0). \tag{5.6}
\]

which is the same recurrence relation for \( D(m,q) \) shown in Theorem 5.8. Thus by induction on \( q \), we can prove \( D(m,q,0) = D(m,q) \) for all \( q \geq 1 \).

Therefore it suffices to prove Equation (5.6).

Recall that from Lemma 5.19, Lemma 5.20 and Lemma 5.21 we have

1. \( D(m,q,0) = \sum_{i=0}^{q-1} D(m,i,0) \cdot D(m,q-i,1). \)
2. \( D(m, q, r) = \sum_{i=0}^{q-1} D(m, i, 0) \cdot D(m, q - i, r + 1) \).

3. \( D(m, q, m) = D(m, q - 1, 0) \).

Thus

\[
D(m, q, 0) = \sum_{i_1=0}^{q-1} D(m, i_1, 0) \cdot D(m, q - i_1, 1)
\]

\[
= \sum_{i_1=0}^{q-1} D(m, i_1, 0) \cdot \left( \sum_{i_2=0}^{q-i_1-1} D(m, i_2, 0) \cdot D(q - i_1 - i_2, 2) \right)
\]

\[
\vdots
\]

\[
= \sum_{i_1, \ldots, i_m \geq 0, i_{m+1} \geq 1: \ i_1 + \cdots + i_m + i_{m+1} = q} D(m, i_1, 0) \ldots D(m, i_m, 0) \cdot D(m, i_{m+1}, m)
\]

\[
= \sum_{i_1, \ldots, i_m \geq 0, i_{m+1} \geq 1: \ i_1 + \cdots + i_m + i_{m+1} = q} D(m, i_1, 0) \ldots D(m, i_m, 0) \cdot D(m, i_{m+1}, 0)
\]

This proves Equation (5.6), as needed.

\[\square\]

6 The Spectrum of a Multi-Z-shaped Graph Matrix

In this section we aim to find the spectrum of the singular values for m-layer Z-shape graph matrices. Let \( M_{n,m} = \frac{1}{nm^{m/2}} \alpha_Z(m)(G) \) where \( \alpha_Z(m) \) is the m-layer Z-shape as defined in 5.1 and \( G \sim G(n, 1/2) \). Let \( r(n, m) = \frac{n!}{(n-m)!} \) be the dimension of \( M_{n,m} \). By Corollary 5.7, \( \frac{1}{r(n, m)} \cdot E \left[ \text{tr} \left( M_{n,m} M_{n,m}^T \right)^k \right] = D(m, k) \). Thus by Lemma 2.1, if we can find a function \( g_m \) such that

\[
\int_0^\infty g_m(x) x^{2k} dx = D(m, k)
\]

then \( g_m \) describes the limiting spectrum of singular values for the m-layer Z-shape graph matrix as \( n \) goes to \( \infty \).

Recall that \( D(m, n) = \frac{1}{nm+1} \binom{(m+1)n}{n} \) as defined in 5.3. In this section we will generalize the arguments for the \( m = 2 \) case in Section 4 to the case \( m = 3 \). The general steps will be:

1. Assume \( \int_0^\infty f(x) x^{2k} dx = D(3, k) \) and derive a differential equation for \( f(x) \).

2. Prove that under this differential equation, the moments of \( f(x) \) are indeed \( D(3, k) \).

3. Apply Lemma 2.1 to conclude that \( f(x) \) is the desired spectrum.
**Theorem 6.1.** Let \( a = \lim_{n \to \infty} D(3, n + 1)/D(3, n) = \frac{16}{3\sqrt{3}} \). If \( f(x) \) is a function such that
\[
\int_0^a f(x)x^{2k}dx = D(3, k)
\]
for all nonnegative integers \( k \) and

1. \( f \) is three times continuously differentiable on \((0, a)\) and \(\lim_{x \to a^-} f(x) = 0\),
2. \(\lim_{x \to a^-} (x - a)f'(x) = 0\),
3. \(\lim_{x \to a^-} 2(x - a)f''(x) + f'(x) = 0\),
4. \(\lim_{x \to 0^+} xf(x) = 0\), \(\lim_{x \to 0^+} x^2f'(x) = 0\), and \(\lim_{x \to 0^+} x^3f''(x) = 0\).

then \( f \) satisfies the following ODE on \((0, a)\):
\[
(27x^4 - 256x^2)f'''(x) + (162x^3 - 768x)f''(x) + (177x^2 - 192x)f'(x) + 15xf(x) = 0.
\]

(6.1)

**Lemma 6.2.** Let \( a \) be some positive constant. If \( f \) is continuous on \([0, a]\) and \(\int_0^a f(x)x^{2n+1}dx = 0\) for all nonnegative integers \( n \), then \( f = 0 \) on \((0, a)\).

**Proof.** The proof is similar to the proof for Lemma [4.9] for the case of all zero even moments. Here we approximate \( f(\sqrt{x} \cdot \sqrt{x}) \) by a polynomial \( p(x) \), so that the odd polynomial \( p(x^2)/x \) (\( p(x) \) has 0 constant term) approximates \( f(x) \).

**Proof of Theorem 6.1.** Denote the LHS of the ODE by \( G(x) \). Let \( A(k, n) = \int_0^a f(k)(x) \cdot x^n dx \) and \( B(k, n) = \left[ f^k(x) \cdot x^n \right]_0^a \). Repeatedly doing integration by parts we get that

\[
A(m, n) = B(m - 1, n) - n \cdot A(m - 1, n - 1)
\]

\[
= B(m - 1, n) - nB(m - 2, n - 1) + n(n - 1)A(m - 2, n - 2)
\]

\[
= B(m - 1, n) - nB(m - 2, n - 1) + n(n - 1)B(m - 3, n - 2) - n(n - 1)(n - 2)A(m - 3, n - 3).
\]

Also, we have that
\[
\frac{D(3, n)}{D(3, n - 1)} = \frac{A(0, 2n)}{A(0, 2n - 2)} = \frac{4(4n - 3)(4n - 2)(4n - 1)}{(3n + 1)(3n)(3n - 1)}.
\]

(6.2)

The steps for deducing the ODE for \( f(x) \) are very similar to the steps used in the proof of Theorem 4.4 to deduce the ODE for the Z-shaped graph matrix. We first apply \( m = 3 \) and \( n = 2n + 3 \) to the first equation and rewrite the term \( n(n - 1)(n - 2)A(m - 3, n - 3) \) using the second equation (6.2). We then gradually eliminate the non-constant coefficients in front of \( A(m, n) \)'s using the first equation.

Plugging in \( m = 3, n = 2n + 3 \) into the first equation, we get
\[
27A(3, 2n + 3) = 27B(2, 2n + 3) - 27(2n + 3)B(1, 2n + 2) + 27(2n + 3)(2n + 2)B(0, 2n + 1) - 27(2n + 3)(2n + 2)(2n + 1)A(0, 2n).
\]
Rewriting the last term on the RHS above and applying the second equation (6.2), we get

\[
27(2n + 3)(2n + 2)(2n + 1)A(0, 2n) = 8(3n + 1)(3n)(3n - 1)A(0, 2n) - \\
81 \cdot 2(2n + 2)(2n + 1)A(0, 2n) + 59 \cdot 3(2n - 1)A(0, 2n) - 15A(0, 2n) \\
= 32(4n - 3)(4n - 2)(4n - 1)A(0, 2n - 2) - \\
81 \cdot 2(2n + 2)(2n + 1)A(0, 2n) + 59 \cdot 3(2n - 1)A(0, 2n) - 15A(0, 2n)
\]

We can rewrite the first term on the RHS as

\[
32(4n - 3)(4n - 2)(4n - 1)A(0, 2n - 2) = 256(2n + 1)(2n - 1)A(0, 2n - 2) - \\
256 \cdot 3(2n)(2n - 1)A(0, 2n - 2) + 128 \cdot 3(2n - 1)A(0, 2n - 2).
\]

Now apply the first equation to all the \((n + 1)A(m, n), (n + 2)(n + 1)A(m, n)\) and \((n + 3)(n + 2)(n + 1)A(m, n)\) above, group together the \(A(m, n)\) terms and \(B(m, n)\) terms separately, and rewrite the \(B(m, n)\) term using the definition of \(B(m, n)\). We get that for all \(n \geq 1\),

\[
27A(3, 2n + 3) - 256A(3, 2n + 1) + 2 \cdot 81A(2, 2n + 2) - 3 \cdot 256A(2, 2n) \\
+ 3 \cdot 59A(1, 2n + 1) - 3 \cdot 128A(1, 2n - 1) + 15A(0, 2n) \\
= \left[\left((27x^2 - 256) f''(x) + 27xf'(x)\right)x^{2n+1}\right]^a_0 \\
+ \left[\left((2n - 2)(-27x^2 + 256)\right)f'(x)x^{2n}\right]^a_0 + \left[p(n, x) \cdot f(x)x^{2n-1}\right]^a_0
\]

where \(p(n, x)\) is some polynomial in terms of \(n\) and \(x\).

Observe that the last term on the RHS is 0 since \(\lim_{x \to 0^+} xf(x) = 0\) and \(\lim_{x \to a^-} f(x) = 0\) by assumption. The second last term is 0 since \(27x^2 - 256 = 27(x + a)(x - a)\), \(\lim_{x \to a^-} (x - a) f'(x) = 0\) and \(\lim_{x \to 0^+} x^2 f'(x) = 0\). The first term top part is \(\lim_{x \to a^-} 27 ((x + a)(x - a)f''(x) + xf'(x)) x^{2n+1} = \\
\lim_{x \to a^-} 27 (2a(x - a)f''(x) + af'(x)) a^{2n+1} = 0\) since \(\lim_{x \to a^-} 2(x - a)f''(x) + f'(x) = 0\) by assumption. The bottom part is 0 since \(\lim_{x \to 0^+} x^3 f''(x) = 0\) by assumption. Thus the RHS is 0.

Expanding out each \(A(m, n)\) term by using the definition of \(A(m, n)\), we get that \(\int_0^a G(x)x^{2n+1} = 0\) for all \(n \geq 1\). By Lemma 6.2, \(G(x) = 0\) on \((0, a)\), which proves that \(f\) satisfies the ODE.

\[\square\]

**Theorem 6.3.** Let \(\alpha = \lim_{n \to \infty} D(3, n)/D(3, n - 1) = \frac{16}{3\sqrt{3}}\) and let \(f\) be a function satisfying the following ODE

\[
(27x^4 - 256x^2)f'''(x) + (162x^3 - 768x)f''(x) + (177x^2 - 192)f'(x) + 15xf(x) = 0 \quad (6.3)
\]
and the conditions listed in Theorem 6.1. Moreover, assume \( \int_0^a f(x) \, dx = 1 \). Then for any non-negative integer \( k \),

\[
A(0, 2k) = \int_0^a x^{2k} \cdot f(x) \, dx = D(3, k).
\]  
(6.4)

The proof is very similar to the proof for Theorem 4.11. We integrate the ODE from 0 to \( a \), do integration by parts and use the conditions for \( f \) to eliminate the redundant terms and finally arrive at the ratio between \( A(0, 2(k - 1)) \) and \( A(0, 2k) \) which matches the ratio between \( D(3, k - 1) \) and \( D(3, k) \). By induction on \( k \) we conclude (6.4).

**Corollary 6.4.** Let \( M_n \) be the graph matrix \( M_\alpha(3) \) with random input graph \( G \sim G(n, 1/2) \) where \( \alpha(3) \) is the multi-Z-shape defined in 5.1. Let \( r(n) = n(n - 1)(n - 2) \) be the dimension of \( M_n \). Let \( g(x) \) be \( f(x) \) as in Theorem 4.11 on \((0, a)\) and 0 for \( x \geq a \). Then as \( n \to \infty \), the distribution of the singular values of \( M_n \) approaches \( g(x) \). More precisely, for all \( \epsilon, \delta > 0 \) and all \( a \geq 0 \),

\[
\lim_{n \to \infty} \Pr \left[ \left| \text{# of singular values of } M_n \in [a, a + \delta] - \left( \int_{x=a}^{a+\delta} g(x) \, dx \right) r(n) \right| \leq \epsilon r(n) \right] = 1.
\]

**Proof.** This is true by Corollary 5.7, Lemma 2.1, and Theorem 4.11. □

For this ODE (6.1), WolframAlpha fails to give us an explicit solution. Instead, we solve the ODE numerically by approximating the tail segment of \( f(x) \) by \( c \cdot (a - x)^r \) for some constants \( c \) and \( r \).

**Step 1:** We analyze the behaviour of the ODE when \( x \) is very close to \( a \). Notice that \( a = \frac{16}{3\sqrt{3}} \) \( \iff \)

\[
27a^2 - 256 = 0.
\]

1. \( 27x^4 - 256x^2 = 27x^2(x - a)(x + a) \sim 52a^3(x - a) \).
2. \( 162x^3 - 768x = 81x^3 + 3x(27x^2 - 256) \sim 81a^3 \).
3. \( 177x^2 - 192 = \frac{3}{4}(209x^2 + 27x^2 - 256) \sim \frac{3 \cdot 209a^2}{4} \).

Thus when \( x \) is very close to \( a \), the ODE is

\[
52a^3(x - a)f''''(x) + 81a^3f'''(x) + \left( \frac{3 \cdot 209a^2}{4} \right) f'(x) = 0.
\]

One can check that \( f'(x) = C' \left( (a - x)^{-1/2} + \frac{209}{64\sqrt{3}}(a - x)^{1/2} \right) \) is a solution to the above ODE. Thus

\[
f(x) \sim g(x) = C((a - x)^{1/2} + \frac{209}{64 \cdot 3\sqrt{3}}(a - x)^{3/2})
\]  
(6.5)

for some constant \( C \) when \( x \) is very close to \( a \).
**Step 2:** We approximate the solution of \( f(x) \) by approximating the tail segment of \( f(x) \) (where \(|x-a| < \epsilon|\)) by \( g(x) \) and use this approximation to obtain initial conditions for the ODE for \( f \). In particular, we choose a small \( \epsilon > 0 \) and set the initial conditions for the ODE as follows:

\[
f(a - \epsilon) = g(a - \epsilon), \quad f'(a - \epsilon) = g'(a - \epsilon), \quad f''(a - \epsilon) = g''(a - \epsilon).
\]

We calculate the constant \( C \) in \( g \) by noticing that the integration of \( f \) over \((-a, a)\) should be 1.

Setting \( \epsilon = 0.01 \) and solving the ODE in python, we get the following plot for the solution to the ODE (concatenated with a tail segment where we use \( g \) as an approximation):

![Plot of the ODE solution with the approximated tail segment.](image)

(a) Plot of the ODE solution with the approximated tail segment.

![Zoom in at the tail segment.](image)

(b) Zoom in at the tail segment.

Figure 6.1: The plot of the spectrum where \( x > 0 \).

To test this solution experimentally, we can sample from the distribution of singular values of \( M_n \) by sampling a random graph \( G \), computing the resulting matrix \( M_n(G) \), and computing its singular values. See Figure 6.2 for a plot of the approximated spectrum together with the empirical distribution of the singular values of \( M_n \) with \( n = 10 \) and \( n = 12 \), respectively (where we sampled 100 random graphs \( G \)).

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A Dominant Constraint Graphs on $H(\alpha Z_m, 2q)$ are Well-Behaved

In this section, we prove that dominant constraint graphs on $H(\alpha Z_m, 2q)$ are well-behaved.

A.1 The Set of Graphs $R(H(\alpha Z_m, 2q))$

In order to analyze constraint graphs on $H(\alpha Z_m, 2q)$, we need to analyze a more general class of $H$. In particular, we need to analyze all $H$ which can be obtained by taking isolated vertices which are not incident to any spokes in $H(\alpha Z_m, 2q)$ and merging their neighbors together.

**Definition A.1.** Define $R(H(\alpha Z_m, 2q))$ to be the set of graphs $H$ which can be obtained by starting from $H(\alpha Z_m, 2q)$ and repeatedly applying the following operation:

1. Choose a vertex $v \in V(H)$ which is in a wheel with at least 4 vertices and is not incident to any spokes. Merge the two neighbors of $v$, delete $v$ from the graph, and delete any pairs of edges in $H$ which coincide.

**Lemma A.2.** For any $H \in R(H(\alpha Z_m, 2q))$, we can decompose $H$ as $H = \alpha_1 \circ \cdots \circ \alpha_{2q'}$ where the following statements are true:

1. For all odd $i$, $\alpha_i$ consists of trivial top layers, a multi-$Z$ shape in the middle layers, and trivial bottom layers.
2. For all even $i$, $\alpha_i$ consists of trivial top layers, a multi-$Z^T$ shape in the middle layers, and trivial bottom layers.
3. For any two neighboring wheels, the spokes connect with each other and alternate between going up and to the right and down and to the right.
4. For any layer, the intervals where this layer is trivial have even length.

Proof. To prove this, we show that this structure is preserved when we make a new contraction. Without loss of generality, we can assume that the isolated vertex $v$ is the bottom right vertex in some multi-$Z$ shape $\alpha_i$ where $i$ is odd, as cases when the isolated vertex is the top left vertex in a multi-$Z$ shape, the bottom left vertex in a multi-$Z^T$ shape, or the top right vertex in a multi-$Z^T$ shape can be handled in a similar way. Let $j$ be the next index where this wheel is non-trivial. Note that $j$ must be even. Moreover, all layers below $v$ must be trivial in $\alpha_i$ and $\alpha_j$ as otherwise $v$ would not be isolated.

Let $u$ be the vertex preceding $v$ and let $w$ be the vertex after $v$. We merge $u$ and $w$ together and delete $v$. This deletes the edges $(u,v), (v,w)$ and may delete spokes incident to $u$ and $w$. We have the following cases.

1. There is a spoke $(u,t)$ in $\alpha_i$ and a spoke $(t,w)$ in $\alpha_j$. In this case, merging $u$ and $w$ together also deletes the spokes $(u,t), (t,w)$. We account for this by making $v$’s layer trivial in $\alpha_i$ and $\alpha_j$, replacing it with the single vertex $u = w$.

2. There is a spoke $(u,t)$ in $\alpha_i$ but no spoke incident to $w$ in $\alpha_j$. In this case, the edge $(v,w)$ is the only non-trivial part of $\alpha_j$. Let $k$ be the next index such that this layer is non-trivial in $\alpha_k$. Observe that $\alpha_k$ is a multi-$Z$ shape where all layers above this layer are trivial. We account for merging $u$ and $w$ together and deleting $v$ as follows:
   
   (a) Glue $\alpha_i$ and $\alpha_k$ together at the vertex $u = w$.
   
   (b) Create a copy of $\alpha_i \circ \cdots \circ \alpha_{k-1}$ which only contains the part below the current layer and is trivial at this layer and above. Put these copies to the left of the glued shape.
   
   (c) Create a copy of $\alpha_{i+1} \circ \cdots \circ \alpha_k$ which only contains the part above the current layer and is trivial at this layer and above. Put these copies to the right of the glued shape.

3. There is no spoke incident to $u$ in $\alpha_i$ but there is a spoke $(t,w)$ in $\alpha_j$. This can be handled in a similar way to the previous case.

4. There are no spokes incident to $u$ or $w$. In this case, the edges $(u,v)$ and $(v,w)$ are the only non-trivial parts of $\alpha_i$ and $\alpha_j$. We again account for this by making $v$’s layer trivial in $\alpha_i$ and $\alpha_j$, replacing it with the single vertex $u = w$.

\[ \square \]

A.2 Proof that dominant constraint graphs are well-behaved

With this structural result on $R(H(\alpha Z_m, 2q))$, we can now prove that dominant constraint graphs for $H(\alpha Z_m, 2q)$ are well behaved. To do this, we use ideas from Appendix B of [1]. First, we modify our constraint graph as follows:
Definition A.3. Let \( H = \alpha_1 \circ \ldots \circ \alpha_{2q} \) where we set \( V_{\alpha_{2q}} = U_{\alpha_1} \). Given a constraint graph \( C \) on \( H \), we define the constraint graph \( C_{\text{aug}} \) as follows:

1. Draw the constraint edges so that all paths in \( C \) go from left to right.
2. For each vertex \( u \in U_{\alpha_1} \), letting \( v \) be the rightmost vertex such that there is a path of constraint edges from \( u \) to \( v \), we add an auxiliary constraint edge from \( v \) to \( u \). We treat this edge as going from \( v \) on the left to \( u \) on the right (i.e. we think of \( u \) as both on the left side of \( H \) and on the right side of \( H \) as \( u \in U_{\alpha_1} = V_{\alpha_{2q}} \)). If \( u \) is isolated, this means that we add an auxiliary loop from \( u \) to itself.

Definition A.4. Given \( H \) and \( C_{\text{aug}} \) as described above, for each \( \alpha_i \) we define \( S_{\alpha_i} \) to be union of \( U_{\alpha_i} \setminus V_{\alpha_i} \) and the set of vertices \( v \) such that there exists a path \( P \) in \( \alpha_i \) from \( U_{\alpha_i} \) to \( V_{\alpha_i} \) such that \( v \) is the first vertex on \( P \) where either

1. There is a constraint edge from \( v \) to the right (i.e. to a vertex in some \( \alpha_j \) where \( j > i \)).
2. \( v \in V_{\alpha_i} \).

Proposition A.5. \( S_{\alpha_i} \) is a vertex separator of \( \alpha_i \).

Remark A.6. Alternatively, we could have started from \( V_{\alpha_i} \) and taken the first vertex on each path in \( \alpha_i \) from \( V_{\alpha_i} \) to \( U_{\alpha_i} \), which has a constraint edge to the left or is in \( U_{\alpha_i} \).

Lemma A.7. Let \( P \) be a path in \( \alpha_i \) from \( U_{\alpha_i} \) to \( V_{\alpha_i} \) and let \( v \) be the first vertex on \( P \) where either

1. There is a constraint edge from \( v \) to the right (i.e. to a vertex in some \( \alpha_j \) where \( j > i \)).
2. \( v \in V_{\alpha_i} \).

For any vertex \( u \in V(P) \setminus U_{\alpha_i} \), which is equal to \( v \) or comes before \( v \), \( u \) has an edge to the left.

Proof. Let \( l \) be the vertex which comes before \( u \) in \( P \). Observe that \( l \) does not have any constraint edges to the right. Thus, in order for the edge \( (l, u) \) to be duplicated, \( u \) must have a constraint edge to the left.

Corollary A.8. \( C \) is a dominant constraint graph for \( H \) if and only if the following statements are true for each \( \alpha_i \) and \( S_{\alpha_i} \):

1. \( S_{\alpha_i} \) is a minimum vertex separator of \( \alpha_i \).
2. Each vertex in \( V(\alpha_i) \setminus (U_{\alpha_i} \cup V_{\alpha_i} \cup S_{\alpha_i}) \) is incident with exactly one constraint edge.
3. Each vertex in \( U_{\alpha_i} \setminus S_{\alpha_i} \) is not incident with any constraint edges to the right.
4. Each vertex in \( V_{\alpha_i} \setminus S_{\alpha_i} \) is not incident with any constraint edges to the left.
Theorem A.9. For any $H \in R(H(\alpha Z_m, 2q))$, all dominant constraint graphs $C$ on $H$ are well-behaved (i.e. wheel-respecting and parity preserving).

Proof sketch. This theorem can be proved by induction using the following lemma.

Lemma A.10. For any $H \in R(H(\alpha Z_m, 2q))$ and any dominant constraint graph $C$ on $H$, there is a vertex $v$ which is isolated and is not incident to any spokes in $H$.

Proof. We prove this lemma with a series of observations.

Proposition A.11. In any dominant constraint graph $C$, if $u$ precedes $v$ on some wheel then either $u$ does not have a constraint edge to the right or $v$ does not have a constraint edge to the left.

Proof. Assume that $C$ is dominant, $u$ has an edge to the right, and $v$ has an edge to the left. Consider the separator $S$ for the segment containing $u$ and $v$. Since $u$ has a constraint edge to the right, $u \in S$. Now either $v \in S$ or $v \notin S$. If $v \in S$ then $S$ is not a minimum vertex separator so $C$ is not dominant. If $v \notin S$ then the constraint edge to the left from $v$ is not accounted for by $S$ so $C$ is not dominant. Thus, in either case $C$ is not dominant, which is a contradiction.

Corollary A.12. In any dominant constraint graph, if $l$ and $r$ are two vertices on the same wheel such that $l < r$, $l$ has a constraint edge to the right, and $r$ has a constraint edge to the left then there is a vertex $m$ such that $l < m < r$ and $m$ is isolated.

Proof. Assume that there exist two vertices $l$ and $r$ on the same wheel such that $l < r$, $l$ has a constraint edge to the right, $r$ has a constraint edge to the left, and there is no vertex $m$ such that $l < m < r$ and $m$ is isolated. Choose $l$ and $r$ such that $d(l, r)$ is minimized. Let $v$ be the vertex after $l$ on this wheel.

By Proposition A.11, since $l$ precedes $v$ and $l$ has a constraint edge to the right, $v$ does not have a constraint edge to the left. This implies that either $v$ is isolated or $v$ only has a constraint edge to the right. However, we cannot have that $v$ is isolated as otherwise we could take $m = v$ and we would have that $l < m < r$ and $m$ is isolated. Thus, $v$ must only have a constraint edge to the right. But then if we take $l' = v$, $l'$ and $r$ are on the same wheel, $l'$ has a constraint edge to the right, $r$ has a constraint edge to the left, there is no vertex $m$ such that $l < m < r$ and $m$ is isolated, and $d(l', r) < d(l, r)$. This contradicts the fact that we chose $l$ and $r$ to minimize $d(l, r)$.

With these observations, we can now prove Lemma A.10. Consider the highest wheel such that there exist vertices $l < r$ on this wheel satisfying the following properties:

1. $l$ has a constraint edge to the right and $r$ has a constraint edge to the left.
2. For any vertex $m$ between $l$ and $r$, $m$ is not incident to any spokes in $H$ going from $m$ to the wheel below $m$. 
Observe that the bottom wheel has these properties, so this wheel always exists. By Corollary A.12 there is a vertex $m$ such that $l < m < r$ and $m$ is isolated. There are now two cases to consider. Either $m$ is not incident to any spokes in $H$ going from $m$ to the wheel above $m$, or $m$ is incident to two such spokes. Note that $m$ is not incident to any spokes in $H$ going from $m$ to the wheel below $m$, so if $m$ is not incident to any spokes in $H$ going from $m$ to the wheel above $m$, then we have found an isolated vertex which is not incident to any spokes in $H$. If $m$ is incident to two spokes in $H$ from $m$ to the wheel above $m$, let $l' < r'$ be the other endpoints of these spokes. Since $m$ is isolated, because of the structure of $H$, $l'$ must have an edge to the right, $r'$ must have an edge to the left, and there are no vertices $m'$ such that $l' < m' < r'$ and $m'$ is incident to spokes in $H$ going from $m'$ to the wheel below $m'$. However, this is a contradiction, as this implies that we did not start with the highest wheel such that there exist vertices $l < r$ on this wheel, where $l$ has an edge to the right, $r$ has an edge to this left, and for any vertex $m$ between $l$ and $r$, $m$ is not incident to any spokes in $H$ going from $m$ to the wheel below $m$. □