ANNULUS DECOMPOSITION OF HANDLEBODY-KNOTS

YI-SHENG WANG

Abstract. By Thurston’s hyperbolization theorem, handlebody-knots of genus 2, like classical knots, can be classified into four classes based on essential surfaces of non-negative Euler characteristic in their exteriors. The class analogous to that of torus knots comprises irreducible, atoroidal, cylindrical handlebody-knots, which are characterized by the absence of essential disks and tori and the existence of an essential annulus in their exteriors. By Funayoshi-Koda, they are precisely those non-hyperbolic handlebody-knots with a finite symmetry group, and are the main subject of the paper.

The paper purposes a classification scheme for such handlebody-knots building on Johannson’s characteristic submanifold theory and the Koda-Ozawa classification of essential annuli in handlebody-knot exteriors. We introduce the notion of annulus diagrams for such handlebody-knots, and classify their possible shapes. In terms of annulus diagrams, a classification result for irreducible, atoroidal handlebody-knots whose exteriors contain a type 2 annulus and a characterization of the simplest non-trivial handlebody-knot are obtained. Implications thereof for handlebody-knot symmetries are also discussed.

1. Introduction

A genus \( g \) handlebody-knot \( (\mathbb{S}^3, \text{HK}) \) is a genus \( g \) handlebody \( \text{HK} \) embedded in an oriented 3-sphere \( \mathbb{S}^3 \); when \( g = 1 \), it is equivalent to a classical knot. The present work concerns genus 2 handlebody-knots, abbreviated to handlebody-knots hereafter.

It follows from Thurston’s hyperbolization theorem \([26],[24],[17]\) that classical knots are divided into four categories: trivial, torus, satellite and hyperbolic knots, based on essential surfaces with non-negative Euler characteristics in knot exteriors. In a similar manner, by the hyperbolization theorem, together with the equivariant torus theorem by Holzmann \([13]\) and the fixed point theorem by Tollefson \([27]\), handlebody-knots can be classified into four categories:

- \( \text{I reducible;} \)
- \( \text{II irreducible, toroidal;} \)
- \( \text{III irreducible, atoroidal, cylindrical;} \)
- \( \text{IV hyperbolic.} \)

A reducible handlebody-knot is one whose exterior \( E(\text{HK}) := \mathbb{S}^3 - \text{HK} \) contains an essential disk; it is an analogue of the trivial knot. A cylindrical (resp. toroidal) handlebody-knot is characterized by the existence of an essential annulus (resp. torus) in its exterior, and a handlebody-knot is hyperbolic if its exterior admits a hyperbolic metric with totally geodesic boundary \([7],[3]\).

The present paper, a sequel of \([28],[29]\), studies handlebody-knots in Category \( \text{III} \) an analogue to torus knots. Handlebody-knots in this group are of particular interest as they are precisely those non-hyperbolic handlebody-knots with a finite symmetry group by Funayoshi-Koda \([8]\); the (positive) symmetry group \( \text{MCG}_{+}(\mathbb{S}^3, \text{HK}) \) of \( (\mathbb{S}^3, \text{HK}) \), introduced in Koda \([19]\), is the (positive) mapping class group of \( (\mathbb{S}^3, \text{HK}) \). The present work concerns the characterization and classification of irreducible, atoroidal, cylindrical handlebody-knots, and seeks to address the following two questions:

**Question 1.1.** When is a cylindrical handlebody-knot irreducible and atoroidal?

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Question 1.2. Given an irreducible, atoroidal, cylindrical handlebody-knot ($S^3$, HK), how essential annuli are configured in $E(HK)$?

Annuli in a handlebody-knot exterior $E(HK)$ are classified into seven types by Kodama-Ozawa [18], and as observed in Funayoshi-Koda [8] Lemma 3.2], only four of them can exist in an irreducible-atoroidal-handlebody-knot exterior. Conversely, their existence does not entail the irreducibility and atoroidality of ($S^3$, HK) in general.

Essential annuli of these four types are characterized by their boundary in relation to the handlebody $HK$ [18] Proof of Theorem 3.3]. A type 2 annulus $A$ has exactly one component of $\partial A$ that bounds a disk in HK, while a type 3-2 (resp. type 3-3) annulus $A$ has parallel (resp. non-parallel) boundary components in $\partial HK$ which bound no disk in HK and are disjoint from an essential disk in HK. A type 4-1 annulus $A$ also has parallel $\partial A \subset \partial HK$, but no essential disk in HK disjoint from $\partial A$ exists. Non-separating essential annuli, that is, essential annuli of type 2 and type 3-3 by the definition, are the primary focus of the paper.

Type 2 annuli can be classified into two subtypes: $A$ is of type 2-1 if the disk bounded by a component of $\partial A$ is non-separating in HK; otherwise $A$ is of type 2-2. Type 3-3 annuli can also be divided into subfamilies as follows: By the definition, there is a disk $D_A \subset HK$ disjoint from $\partial A$. Let $W_1, W_2$ be the components of $HK - \mathfrak{N}(A)$, and $r_1, r_2$ be the boundary slope of $\partial A$ with respect to ($S^3, W_1$), ($S^3, W_2$). The unordered pair $(r_1, r_2)$ is well-defined since $D_A \subset HK$ is unique, up to isotopy [29] Lemma 2.9], and is called the boundary slope pair of $A$. Furthermore, it is either of the form ($\frac{p}{q}, \frac{p}{q}$), $pq \neq 0$ or ($\frac{p}{q}, pq$), $q \neq 0$, where $p, q$ are coprime integers [29] Lemma 2.12]. When $(r_1, r_2) = (0, 0)$, we say $A$ has a trivial boundary slope.

Given an annulus $A \subset E(HK)$, consider the new pair ($S^3, HK_A$) induced by the union $HK_A$ of HK and a regular neighborhood $\mathfrak{N}(A)$ of $A$ in $E(HK)$. Then ($S^3, HK_A$) is a handlebody-knot if and only if $A$ is of type 2 or of type 3-3 with a boundary slope pair ($\frac{p}{q}, pq$) [29] Corollary 2.13]. ($S^3, HK_A$) is often “simpler” than ($S^3$, HK), yet at the same time, it retains many topological properties of ($S^3$, HK). For instance, in terms of irreducibility and atoroidality of ($S^3$, HK$_A$) and the cores $l_+, l_- \subset \partial E(HK_A)$ of the two annuli in the frontier of $\mathfrak{N}(A)$ in $E(HK)$, [29] develops irreducibility and atoroidality tests for handlebody-knots ($S^3$, HK) whose exteriors admit a type 3-3 annulus with a boundary slope pair $(p, p)$, $p \neq 0$.

Following the same line of thoughts, we consider here type 2 annuli, and obtain Theorem 1.1 in answering to Question 1.1. To state the result, recall that a set of disjoint simple loops $\{l_1, \ldots, l_n\}$ in the boundary of a 3-manifold $M$ is primitive if there exists a set of disjoint disks $\{D_1, \ldots, D_n\}$ in $M$ such that $l_i \cap \partial D_j$ is a point when $i = j$ and empty otherwise.

Theorem 1.1 (Propositions 2.10 and 2.11). Let $A$ be an annulus in $E(HK)$.

(i) Suppose $A$ is of type 2-1. Then ($S^3$, HK) is irreducible and atoroidal if and only if ($S^3$, HK$_A$) is either irreducible and atoroidal or trivial with $(l_+, l_-)$ not primitive in $E(HK_A)$.

(ii) Suppose $A$ is of type 2-2. Then ($S^3$, HK) is irreducible and atoroidal if and only if ($S^3$, HK$_A$) is either irreducible and atoroidal or trivial with $l_+, l_-$ not homotopically trivial in $E(HK_A)$.

Theorem 1.1 is used to construct irreducible, atoroidal handlebody-knots whose exteriors contain a type 2 annulus in Section 2; these examples show the existence of various configurations of annuli in a handlebody-knot exterior. Other irreducibility tests are developed by Ishii-Kishimoto [13] via quandle invariants, Bellettini-Paolini-Wang [2] via homomorphisms from fundamental groups, and Okazaki [23] via Alexander polynomial.

To tackle Question 1.2 we employ Johannson’s characteristic submanifold theory [16] (see also [4]), which guarantees the existence and uniqueness of a characteristic annulus.
decomposition of $E(HK)$, namely, a union $S$ of disjoint essential annuli that satisfies (I) every component of the exterior $E(S) := \overline{E(HK)} \setminus \mathcal{N}(S)$ is either admissibly Seifert/I-fibered or simple, where $\mathcal{N}(S)$ is a regular neighborhood of $S \subset E(HK)$, and (II) the condition (I) fails if any component of $S$ is removed [3]. The characteristic diagram $\Lambda_{ext}$ of $E(HK)$ is then a labeled graph defined as follows. Assign a node to every component of $E(S)$, and to each component of $\mathcal{N}(S)$, assign an edge that joins node(s) corresponding to component(s) of $E(S)$ meeting the component of $\mathcal{N}(S)$. In addition, we label each node $v$ with the genus $g(\partial C_v)$ of the boundary of the corresponding component $C_v \subset E(S)$ when $g(\partial C_v) > 1$, and use a hollow node ◦ if $C_v$ is simple and a solid node • otherwise.

**Theorem 1.2** (Theorem 4.2). Table 1 lists all possible characteristic diagrams of an irreducible-atoroidal-cylindrical-handlebody-knot exterior $E(HK)$.

We remark that Theorem 4.2 (see also Proposition 4.1) implies that the notion of $W$-system introduced in Neumann-Swarup [22] and Johannson’s characteristic submanifold coincide in the case of $E(HK)$, where a $W$-system of $E(HK)$ is a maximal set of canonical annuli in $E(HK)$, and an essential annulus is canonical if any other essential annulus can be isotoped away from it.

**Corollary 1.3.** The components of $S$ constitutes a $W$-system of $E(HK)$, and $\#S \leq 3$.

Theorem 4.2 and Proposition 4.1(v) entail the following.

**Corollary 1.4.** Let $E$ be the set of isotopy classes of essential annuli in $E(HK)$. Then

(i) $|E| \leq 3$ if and only if $\Lambda_{ext}$ is not Figs. 1c, 1g;
(ii) $|E| = 5$ if and only if $\Lambda_{ext}$ is Fig. 1i;
(iii) $|E| = \infty$ if and only if $\Lambda_{ext}$ is Fig. 1g.

The characteristic diagram $\Lambda_{ext}$ depends only on the topology of $E(HK)$ and is unable to differentiate handlebody-knots with homeomorphic exteriors—such examples have been constructed, for instance, in [21], [15], [20], [11]. We can nevertheless enhance the characteristic diagram $\Lambda_{ext}$ by labeling each edge with the type of the corresponding annulus in $S$. The resulting edge-labeled diagram, denoted by $\Lambda_{mk}$, is called the annulus diagram of $(S^3, HK)$. $\Lambda_{mk}$ contains finer information; for example, $(S^3, 5_1)$ and $(S^3, 6_4)$ in the Ishii-Kishimoto-Moriuchi-Suzuki handlebody-kont table [15] have homeomorphic exteriors but different annulus diagrams:

![Figure 1.1](image)

One main result of the paper is the following classification of irreducible, atoroidal handlebody-knots whose exteriors contain a type 2 annulus in terms of annulus diagrams.

**Theorem 1.5** (Theorem 4.19, Proposition 5.9). Suppose $(S^3, HK)$ is irreducible and atoroidal and $E(HK)$ admits a type 2 annulus $A$.

(i) If $A$ is of type 2-1, then $\Lambda_{mk}$ is

![Figure 1.1](image)
(ii) If $A$ is of type $2-2$, then $\Lambda_{\text{HK}}$ is one of the following:

\[
\begin{array}{c}
\begin{array}{c}
\text{(i)}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{(ii)}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{(iii)}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{(iv)}
\end{array}
\end{array}
\end{array}
\]

where $\Box$ can be a solid or hollow node.

(iii) Conversely, all diagrams in (i) and (ii) can be realized by some irreducible, atoroidal handlebody-knots.

In the case the characteristic diagram $\Lambda_{\text{ext}}$ of $E(\text{HK})$ is of $\theta$-shape, namely (1i) and (1j) in Table 1, a sharper statement can be obtained.

**Theorem 1.6** (Theorems 4.15, 4.22). Suppose $(S^3, \text{HK})$ is irreducible and atoroidal. Then

(i) If $A$ is of type $3-3$ with a boundary slope pair $(p, q)$, $pq \neq 0$, then $A$ is the unique annulus in $E(\text{HK})$, up to isotopy.

(ii) If $A$ is of type $3-3$ with a trivial boundary slope pair, then $A$ is the unique type $3-3$ annulus in $E(\text{HK})$, up to isotopy.

(iii) $E(\text{HK})$ contains at most two non-isotopic type $3-3$ annuli.

As an application of Theorem 1.5, we compute the symmetry groups of handlebody-knots whose exteriors contain a type 2 annulus.

**Theorem 1.8** (Theorems 6.9, 6.10, 6.11). Suppose $(S^3, \text{HK})$ is irreducible and atoroidal and $A \subset E(\text{HK})$ a type 2 annulus.

(i) If $A$ is of type $2-1$, then $\text{MCG}(S^3, \text{HK}) < \mathbb{Z}_2$ and $\text{MCG}(S^3, \text{HK}) < \mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) If $A$ is the unique type $2-2$ annulus in $E(\text{HK})$, up to isotopy, then $\text{MCG}(S^3, \text{HK}) \approx \{1\}$ and $\text{MCG}(S^3, \text{HK}) < \mathbb{Z}_2$.

(iii) If $A$ is the unique type $2-2$ annulus, but not the unique annulus in $E(\text{HK})$, up to isotopy, then $\text{MCG}(S^3, \text{HK}) \approx \{1\} \approx \text{MCG}(S^3, \text{HK})$.

(iv) If $E(\text{HK})$ admits a type 2-2 annulus non-isotopic to $A$, then $\text{MCG}(S^3, \text{HK}) < \mathbb{Z}_2$ and $\text{MCG}(S^3, \text{HK}) < \mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 1.8 implies $\text{MCG}(S^3, 4_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\text{MCG}(S^3, 4_1) = \mathbb{Z}_2$ as the reflection against the $xy$-plane and rotation around the $z$-axis by $\pi$ in Fig. 1.9 represents two non-trivial mapping classes. To our knowledge, $(S^3, 4_1)$ is the only known example that attains the upper bound in Theorem 1.8. On the other hand, no handlebody-knot whose exterior contains a unique type 2 annulus has been found to have a non-trivial symmetry group so far. We speculate the following sharper statements both are true.

**Problem 1.3.** Under the same assumption as in Theorem 1.8, $\text{MCG}(S^3, \text{HK}) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if $(S^3, \text{HK})$ is equivalent to $(S^3, 4_1)$. 

Note the distinction between “$A$ is the unique type $XXX$ annulus” and “$A$ is the unique annulus” in $E(\text{HK})$, up to isotopy; in the former, $E(\text{HK})$ may contain annuli of other types. We remark that (ii) is Corollary 4.1, while (ii) follows from Lemma 4.8 and Corollary 4.17 and (iii) from Lemma 4.16, Theorem 1.2, and Theorem 1.6.
Problem 1.4. Under the same assumption as in Theorem 1.8, suppose $A$ is the unique type 2 annulus in $E(HK)$, up to isotopy. Then $\text{MCG}(\mathbb{S}^3, HK) \cong \{1\}$.

The rigid motions shown in Fig. 1.3 suggest the following variant of the Nielsen realization problem.

Problem 1.5. Let $(\mathbb{S}^3, HK)$ be an irreducible, atoroidal handlebody-knots. Then there exists a subgroup $G < \text{Homeo}(\mathbb{S}^3, HK)$ such that $\pi_0 : \text{Homeo}(\mathbb{S}^3, HK) \to \text{MCG}(\mathbb{S}^3, HK)$ restricts to an isomorphism on $G$.

Handlebody-knot symmetries is itself a topic of independent interest. To our knowledge, apart from $(\mathbb{S}^3, 4_{11})$, for only five others in the handlebody-knot table [15], the symmetry group is computed:

\[ \text{MCG}(\mathbb{S}^3, 5_{1}) \cong \text{MCG}(\mathbb{S}^3, 6_{1}) \cong \text{MCG}(\mathbb{S}^3, 6_{11}) \cong \{1\}, \]
\[ \text{MCG}(\mathbb{S}^3, 5_{2}) \cong \text{MCG}(\mathbb{S}^3, 5_{2}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \text{MCG}(\mathbb{S}^3, 6_{1}) \cong \text{MCG}(\mathbb{S}^3, 6_{2}) \cong \mathbb{Z}_2. \]

The first two are computed by Koda [19] using results from Motto [21] and Lee-Lee [20], while the third one follows from [28] and Theorem 1.5; the last two are computed in [29]. They can all be realized as subgroups of homeomorphism groups.

The paper is organized as follows. Basic properties of a type 2 annulus $A$ in $E(HK)$ is reviewed in Section 2.1, which is followed by a discussion on the reducibility and triviality of $(\mathbb{S}^3, HK)$ in relation to the essentiality of $A \subset E(HK)$; building on this, Theorem 1.1 is proved in Section 2.3. A parallel investigation for type 3-3 annuli is carried out in [29, Section 4].

Section 3 reviews Johannson’s characteristic submanifold theory, and develops a completeness criterion (Theorem 3.10) needed later for our analysis of configurations of essential annuli of various types in a handlebody-knot exterior. Properties of the characteristic diagram of the exterior of an irreducible, atoroidal handlebody-knot is detailed in Section 4.1, where we prove Theorem 1.2 and define the annulus diagram for such a handlebody-knot via the Koda-Ozawa classification of essential annuli. Theorems 1.3, 1.10, 1.6 and 1.7 are proved in Section 4.2 after an extensive search for possible combinations of non-separating annuli.

Section 5 introduces the looping construction, which along with the criteria developed in Section 2, allows us to produce irreducible, atoroidal handlebody-knots whose exterior contain a type 2 annulus with a given annulus diagram, and thus prove Theorem 1.3(ii) Section 6 closes the paper with some applications of the main results to handlebody-knot symmetries—in particular, Theorem 1.8.

2. Irreducibility and Atoroidality

We work in the piecewise linear category. Given a subpolyhedron $X$ of $M$, we denote by $\overline{X}$, $\mathring{X}$, $\partial X$, and $\partial_M X$ the closure, the interior, a regular neighborhood, and the frontier of $X$ in $M$, respectively. The exterior $E(X)$ of $X$ in $M$ is defined to be the complement of $\mathring{X}$.
Table 1. Complete list of characteristic diagrams of \( E(HK) \).

\[
\begin{array}{ccc}
(a) (1, 1, 0, \circ) & (b) (1, 0, 0, \circ) & (c) (1, 0, 0, \bullet) \\
(d) (2, 1, 0, \circ) & (e) (2, 0, 1, \circ) \\
(f) (2, 0, 0, \circ) & (g) (2, 0, 0, \bullet) & (h) (2, 0, 1, \bullet) \\
(i) (3, 0, 3, \bullet) & (j) (3, 0, 1, \circ) & (k) (3, 0, 1, \bullet) \\
(l) (3, 0, 0, \circ) & (m) (3, 0, 0, \bullet) \\
\end{array}
\]

if \( X \subset M \) is of positive codimension, and defined to be the closure of \( M - X \) otherwise. Submanifolds of a manifold \( M \) are assumed to be proper and in general position except in some obvious cases where submanifolds are in \( \partial M \). A surface \( S \) other than a disk in a three-manifold \( M \) is essential if it is incompressible and \( \partial \)-incompressible. A disk \( D \subset M \) is essential if \( D \) does not cut a 3-ball off from \( M \). When \( M \) is a handlebody-knot, an essential disk is also called a meridian disk. We denote by \( (S^3, X) \) an embedding of \( X \) in the oriented 3-sphere \( S^3 \), and given a loop \( l \subset X \), \([l]\) denotes the homology class represented by \( l \) in \( H_1(X) \). All three-manifolds here are assumed to be orientable.

2.1. Type 2 annuli. Here we review the definition of a type 2 annulus \( A \) in the exterior \( E(HK) \) of a handlebody-knot \( (S^3, HK) \), and fix the notation used throughout the paper. Recall that \( HK_A \) denotes the union \( HK \cup \mathcal{R}(A) \), and the frontier of \( \mathcal{R}(A) \) in \( HK \) are two annuli \( A_+, A_- \). Denote by \( l_+, l_- \) the essential loops of \( A_+, A_- \), respectively, and orient them so that \([l_+] = [l_-] \in H_1(\mathcal{R}(A)) \).

**Definition 2.1.** \( A \) is of type 2 if \( \partial A \) is essential in \( \partial HK \) and exactly one component of \( \partial A \) bounds an essential disk in \( HK \). \( A \) is said to be unknotting if \( (S^3, HK_A) \) is trivial.

We denote by \( l_A \) the component of \( \partial A \) that bounds a disk in \( HK \) and by \( D_A \subset HK \) the disk it bounds.

**Definition 2.2.** A type 2 annulus \( A \subset E(HK) \) is of type 2-1 if \( D_A \) non-separating; otherwise it is of type 2-2.

If \( A \) is of type 2-1, then \( \partial HK_A - l_+ \cup l_- \) is a four-times punctured sphere, so they are non-separating, whereas if \( A \) is of type 2-2, \( \partial HK_A - l_+ \cup l_- \) is the disjoint union of a once-punctured torus and twice-punctured disk; in particular, one of \( l_+, l_- \) is separating and the
other is not. By convention, we assume $l_\pm$ is separating. The observation implies the next lemma.

**Lemma 2.1.** If $A$ is of type 2-1, then $\langle [l_+], [l_-] \rangle$ is a basis of $H_1(E(HK_A))$. If $A$ is of type 2-2, then $[l_-]$ is trivial while the quotient $H_1(E(HK_A))/\langle [l_+] \rangle \cong \mathbb{Z}$.

2.2. **Essentiality, triviality and reducibility.** Here we examine the relation between essentiality of a type 2 annulus $A \subset E(HK)$ and the reducibility and triviality of $(S^3, HK)$.

**Lemma 2.2.** Suppose $(S^3, HK)$ is reducible. Then it is trivial if and only if it is atoroidal.

**Proof.** Observe first that there exists a separating essential disk $D \subset E(HK)$. The disk $D$ splits $E(HK)$ into two knot exteriors $E(K_1), E(K_2)$, for some knots $K_1, K_2$ in $S^3$. Then $(S^3, HK)$ is trivial if and only if both $K_1, K_2$ are trivial and therefore if and only if $(S^3, HK)$ is atoroidal. □

**Lemma 2.3.** If $A$ is of type 2-1, then the following are equivalent:

(i) $(S^3, HK)$ is reducible.

(ii) $A$ is inessential.

(iii) $(S^3, HK_A)$ is reducible and there exists a disk $D$ meeting $l_+ \cup l_- \text{ at one point.}$

**Proof.** Note first that by the definition $A$ is incompressible.

$[i] \Rightarrow [ii]$ Let $D \subset E(HK)$ be a compressing disk of $\partial E(HK)$. Minimize $\# D \cap A$ in the isotopy class of $A$. If $D \cap A = \emptyset$, then $\partial D \subset \partial E(HK)$ is separating, and hence $D \subset E(HK)$ is separating. Since $\partial A \subset \partial HK$ is non-parallel and non-separating, components of $\partial A$ lie in different components of $\partial E(HK) - \partial D$, contradicting that $A$ is connected. If $D \cap A \neq \emptyset$, then, since $A$ is incompressible, any outermost disk in $D$ cut off by $D \cap A$ is a $\partial$-compressing disk of $A$ by the minimality.

$[ii] \Rightarrow [iii] \& [ii] \Rightarrow [i]$ Since $A$ is incompressible, it is $\partial$-compressible. Let $D$ be a $\partial$-compressing disk of $A$. Then $D$ induces a disk in $E(HK_A)$ meeting $l_+ \cup l_- \text{ at one point, and}$

$(S^3, HK_A)$ is reducible. On the other hand, the disk component of the frontier of a regular neighborhood $\mathfrak{H}(A \cup D)$ of $A \cup D \subset E(HK)$ implies that $(S^3, HK)$ is reducible.

$[iii] \Rightarrow [ii]$ The disk $D$ induces a $\partial$-compressing disk of $A$. □

**Remark 2.4.** The union $l_+ \cup l_-$ in the condition $[iii]$ of Lemma 2.3 is necessary, there exists an irreducible handlebody-knot $(S^3, HK)$ with $(S^3, HK_A)$ trivial, and one of $l_+, l_-$ is primitive; for instance, $(S^3, S_2)$ in the handlebody-knot table [15].

**Lemma 2.5.** Let $A$ be of type 2-1. Then $(S^3, HK)$ is trivial if and only if $(S^3, HK_A)$ is trivial and $(l_+, l_-)$ is primitive.

**Proof.** “⇒” By Lemma 2.3 there exists a disk $D$ meeting $l_+ \cup l_- \text{ at one point, say } D \cap l_+ \neq \emptyset$. Then the frontier of a regular neighborhood $\mathfrak{H}(A_+ \cup D)$ of $A_+ \cup D \subset E(HK_A) - l_-$ is an essential separating disk $D' \subset E(HK_A)$, which splits $E(HK_A)$ into two parts: a solid torus where $l_+$ lies and $D$ is a meridian disk and the exterior $E(K)$ of a knot $(S^3, K)$ where $l_- \subset \partial E(K)$ is a meridian of $(S^3, K)$. If $(S^3, HK_A)$ is non-trivial, then $(S^3, K)$ is non-trivial, and $\partial E(K)$ induces an incompressible torus $T$ in $E(HK_A)$. $T$ is also incompressible in $E(HK)$, for given any compressing disk of $T$, one can always isotope $A$ away from $D$, given the incompressibility of $A$, contradicting $(S^3, HK)$ is trivial. So $(S^3, K)$ is trivial, and $E(K)$ is a solid torus with $l_-$ primitive in $E(K)$, and hence the assertion.

“⇐”: By [30] (see also [10]), there exists a basis \{x, x_\pm\} of $\pi_1(E(HK_A))$ with $x_\pm$ in the conjugate classes determined by $l_\pm$, respectively. Since $\pi_1(E(HK))$ is the HNN extension of $\pi_1(E(HK_A))$ with respect to $\pi_1(A)$, $\pi_1(E(HK))$ is free, so $(S^3, HK)$ is trivial. □

**Lemma 2.6.** If $A$ is of type 2-2, then the following are equivalent:

(i) $(S^3, HK)$ is reducible.

(ii) $A$ is inessential.

...
Proposition 2.10. Suppose \( A \) is of type 

\[
\text{(iii) } (S^3, HK_A) \text{ is reducible and } l_+ \text{ is homotopically trivial in } E(HK_A).
\]

Proof. \( \Rightarrow \) Let \( D \) be an essential disk in \( E(HK) \). Minimize \( \partial D \cap A \) in the isotopy class of \( A \). Suppose \( D \cap A = \emptyset \). Then \( \partial D \) lies in the one-punctured torus \( T \) in \( \partial HK_A - l_+ \cup l_- \). If \( \partial D \) is separating, then \( \partial D \) is parallel to \( l_- \), and so \( A \) is compressible. If \( \partial D \) is non-separating, then there is a loop \( l \) in \( T \) meeting \( \partial D \) once. The frontier of a regular neighborhood of \( D \cup l \) in \( E(HK_A) - l_+ \) is an essential separating disk disjoint from \( A \), and therefore, as in the previous case, \( A \) compressible. Suppose \( D \cap A \) contains a circle, then any innermost disk in \( D \) cut off by \( D \cap A \) is a compressing disk of \( A \). If \( D \cap A \) contains only arcs, then an outermost disk \( D' \) in \( D \) cut off by \( D \cap A \) is a \( \partial \)-compressing disk of \( A \) or induces an essential disk \( D'' \) disjoint from \( A \) in \( E(HK) \); either way implies \( A \) is inessential.

\[ (\text{iii}) \Rightarrow (\text{ii}) \& (\text{ii}) \Rightarrow (\text{i}) \] Suppose \( A \) is compressible. Then any compressing disk \( D \) induces a disk \( D' \subset E(HK_A) \) with \( \partial D' = l_- \) and a disk \( D'' \subset E(HK) \) with \( \partial D'' = l_+ \), and therefore \( (\text{iii}) \& (\text{i}) \).

Suppose \( A \) is \( \partial \)-compressible and \( D \) is a \( \partial \)-compressing disk of \( A \). Then \( D \) induces a disk \( D' \subset E(HK_A) \) with \( D' \cap l_+ = \emptyset \); the frontier of a regular neighborhood \( \partial (A \cup D') \) in \( E(HK_A) - A \) is a separating disk \( D'' \) with \( \partial D'' \) parallel to \( l_- \); this implies \( A \) is compressible, and so \( (\text{ii}) \) and \( (\text{iii}) \) follow by the previous case.

\[ (\text{iii}) \Rightarrow (\text{i}) \& (\text{ii}) \Rightarrow (\text{i}) \] follow from Dehn’s lemma. \( \square \)

Lemma 2.7. If \( (S^3, HK_A) \) is trivial and \( l_+ \) is homotopically trivial, then \( (S^3, HK) \) is trivial.

Proof. Denote by \( D \subset E(HK_A) \) a disk bounded by \( l_+ \). Then \( D \) splits \( E(HK_A) \) into two solid tori, in one of which \( l_+ \) is primitive. Therefore \( \pi_1(E(HK_A)) \) has a basis \( \{x, y\} \) with \( x \) in the conjugacy class determined by \( l_+ \). The assertion then follows from the fact that \( \pi_1(E(HK)) \) is the HNN extension of \( \pi_1(E(HK_A)) \) with respect to \( \pi_1(A) \). \( \square \)

The converse of Lemma 2.7 is not true in general.

2.3. Irreducibility and atoroidality. Here we present criteria for \( (S^3, HK) \) to be irreducible and atoroidal in terms of \( (S^3, HK_A) \) and \( l_+ \). Recall the results on atoroidality: One is a corollary of [29, Lemma 4.1], and can also be derived from Lemmas 2.3 and 2.6 while the other, a converse of the first, is a direct consequence of [29, Lemma 4.9].

Corollary 2.8. If \( (S^3, HK) \) is irreducible and atoroidal, then \( (S^3, HK_A) \) is atoroidal.

Proof. By Lemmas 2.3 and 2.6 \( A \) is incompressible, and therefore if there exists any incompressible torus \( T \subset E(HK_A) \), then any compressing disk of \( T \) can be isotoped away from \( A \), contradicting the atoroidality of \( (S^3, HK) \). \( \square \)

Corollary 2.9. Suppose \( (S^3, HK_A) \) is atoroidal and \( l_- \subset E(HK_A) \) are not homotopically trivial if \( A \) is of type \( 2-2 \). Then \( (S^3, HK) \) is atoroidal.

Proposition 2.10. Suppose \( A \) is of type \( 2-1 \). Then \( (S^3, HK) \) is irreducible and atoroidal if and only if \( (S^3, HK_A) \) either is irreducible and atoroidal or is trivial with \( \{l_+, l_-\} \) not primitive in \( E(HK_A) \).

Proof. “\( \Rightarrow \)” By Corollary 2.8 \((S^3, HK_A)\) is atoroidal, and hence it is irreducible or trivial by Lemma 2.2. Since \((S^3, HK)\) is non-trivial if \((S^3, HK_A)\) is trivial, \( \{l_+, l_-\} \) is not primitive in \( E(HK_A) \) by Lemma 2.5.

“\( \Leftarrow \)” By Corollary 2.9 \((S^3, HK)\) is atoroidal, and therefore \((S^3, HK)\) is either irreducible or trivial by Lemma 2.2. The latter is not possible, owing to Lemmas 2.3 or 2.5. \( \square \)

Proposition 2.11. Suppose \( A \) is of type \( 2-2 \). Then \( (S^3, HK) \) is irreducible and atoroidal if and only if \((S^3, HK_A)\) either is irreducible and atoroidal or is trivial with \( l_- \subset E(HK_A) \) not homotopically trivial.
Proof. “⇒”: By Corollary 2.8 and Lemma 2.2 (S^3, HK_A) is irreducible or trivial; for the latter, I- cannot be homotopically trivial since (S^3, HK) is irreducible by Lemma 2.6
“⇐”: By Lemma 2.2 and Corollary 2.9 (S^3, HK) is either irreducible or trivial; the latter is ruled out by Lemma 2.6.

3. Characteristic Submanifolds

Here we review Johannson’s characteristic submanifold theory [16] (see also [4]), and define the notion of a characteristic diagram, which we employ to study irreducible, atoroidal handlebody-knot exteriors in Section 4.

3.1. Characteristic Submanifold Theory.

Definition 3.1. Given a compact n-manifold M, a boundary-pattern \( m \) for \( M \) is a finite set of compact, connected \((n-1)\)-submanifolds of \( \partial M \) such that the intersection of any \( i \) of them is empty or an \((n-i)\)-manifold.

Basic and yet important examples include the following. An \( i \)-faced disk is a disk \( D \) whose boundary-pattern \( \partial D \) consists of \( i \) elements with their union \( |\partial D| \) being \( \partial D \). When \( i \leq 3 \) (resp. \( i = 4 \), \( D, \partial D \) is called a small-faced disk (resp. square). The empty boundary-pattern is denoted by \( \phi \), and the completion \( \overline{m} \) of a boundary-pattern \( m \) for \( M \) is the boundary-pattern given by

\[
\overline{m} := \{ G \in m \} \cup \{ \text{components of } M - |\partial m| \},
\]

where \( |\partial m| \) is the union of \( G, G \in m \). Throughout the paper, an annulus (or arc) is assumed to carry the boundary-pattern \( \overline{\phi} \).

Given a manifold \( (M, m) \) with boundary-pattern, and a submanifold \( N \subset M \) of positive codimension, if \( N \cap \partial M \) meets every intersection of elements of \( \overline{m} \) transversely, then \( N \) inherits a natural boundary-pattern given by

\[
m := \{ G \cap \partial N \mid \forall G \in m \}.
\]

Similarly, \( g \) defines a boundary-pattern for a codimension-zero submanifold \( N \subset M \), provided the intersection \( \partial_m N \cap \partial M \) meets every intersection of elements in \( \overline{m} \) transversely. The boundary-pattern \( g \) for \( N \) is called the submanifold boundary-pattern, while in the codimension-zero case, its completion \( \overline{g} \) is called a proper boundary-pattern for \( N \). Throughout the paper, a submanifold \( N \subset M \) is assumed to satisfy the transversality condition, and unless otherwise specified, \( N \) always carries the submanifold boundary-pattern \( g \). We drop \( g \) from the notation when there is no risk of confusion, but specify in the notation the proper boundary-pattern \( \overline{g} \) whenever useful. When \( N \) is considered as the exterior \( E(W) \) of some submanifold \( W \subset M \), the proper boundary-pattern is always assumed and denoted by \( \overline{m} \).

Definition 3.2. An arc \( \gamma \) in a surface \((S, g)\) with boundary-pattern is essential if no component of \((E(\gamma), \overline{g})\) is a small-faced disk.

A surface \( S \) in a 3-manifold \((M, m)\) with boundary-pattern is essential if no component \( X \) of \((E(S), \overline{m})\) contains a small-faced disk that meets the frontier \( \partial_m X \) in an essential arc in \( \partial M \). A codimension-zero submanifold \( N \subset (M, m) \) is essential if its frontier \( \partial_m N \) is essential in \((M, m)\).

In the case \( m = \partial \phi \), the definition is equivalent to the one in terms of incompressibility and \( \partial \)-incompressibility.

A 3-manifold \((M, m)\) with boundary-pattern can be I-fibered (resp. Seifert fibered) if it admits an I-bundle (resp. Seifert bundle) structure \( X \xrightarrow{\pi} B \) with \( B \) equipped with a boundary-pattern \( \overline{b} \) such that

\[
m = \{ \pi^{-1}(G) \mid G \in b \} \cup \{ \text{components of } \partial M - \pi^{-1}(\partial B) \}.
\]

If \((M, m)\) is I-fibered over \((B, b)\), a component of \( \partial M - \pi^{-1}(\partial B) \) is called a lid of \((M, m)\) (with respect to \( \pi \), and any other element in \( m \) is called a side of \((M, m)\) (with respect to \( \pi \).
\((M, \mathfrak{m})\) is called a cylindrical shell if it can be I-fibered over an annulus. An annulus \(A\) in \((M, \mathfrak{m})\) is parallel to an element \(\mathfrak{A} \in \mathfrak{m}\) (resp. another annulus \(A'\) in \((M, \mathfrak{m})\)) if a component of \((E(\mathfrak{A} \cup A), \mathfrak{m}')\) (resp. \((E(\mathfrak{A} \cup A'), \mathfrak{m}')\)) is a cylindrical shell meeting both the regular neighborhoods of \(A\) and of \(A\) (resp. of \(A'\)). The following is a corollary of the vertical-horizontal theorem \cite[Proposition 5.6; Corollary 5.7]{16}.

**Lemma 3.1.** Suppose \((M, \mathfrak{m})\) is I-fibered over \((B, \mathfrak{p})\) with \(\chi(B) < 0\). Let \(A\) be an essential annulus in \((M, \mathfrak{m})\). Then the boundary \(\partial A\) is in the \(\text{id}(s)\) \(L \in \mathfrak{m}\), and there exists an isotopy \(F_t : (A, \partial A) \to (M, L)\) with \(F_0 = \text{id}\) and \(F_1(A)\) the preimage of an essential loop in \(B\).

**Definition 3.3.** An \(\mathcal{F}\)-manifold \(W\) in \((M, \mathfrak{m})\) is a codimension-zero essential submanifold of \(M\) such that each component of \(W\) can be I- or Seifert fibered. An \(\mathcal{F}\)-manifold \(W\) in \(M\) is full if there exists no component \(Y\) of \((E(W), \mathfrak{m}')\) and any essential square, annulus or torus \(S\) in \(Y\), one of the following holds.

If \(S \cap \partial Y \neq \emptyset\), then \(Y\) can be fibered as a product bundle or \(S^1\)-bundle over \(S\). (C1)

If \(S \cap \partial Y = \emptyset\), then \(S\) is parallel to a component of \(\partial Y\) in \(Y\). (C2)

**Definition 3.5.** A characteristic submanifold \(W\) for \((M, \mathfrak{m})\) is a full, complete \(\mathcal{F}\)-manifold in \((M, \mathfrak{m})\).

### 3.2. Characteristic submanifolds of atoroidal manifolds

The subsection develops completeness criteria needed in the subsequent sections. As irreducible atoroidal handlebody-knots are our primary concern, we henceforth assume \(M\) is orientable, irreducible, \(\partial\)-irreducible, atoroidal, and equipped with the boundary-pattern \(\mathfrak{B}\), and drop \(\mathfrak{B}\) from the notation when no confusion may arise.

In this case, the existence and uniqueness of characteristic submanifolds are guaranteed, and they enjoy the engulfing property \cite{16}.

**Theorem 3.2.** \cite[Proposition 9.4; Corollary 10.9]{16}. There exists a characteristic submanifold \(W\) for \(M\), and two characteristic submanifolds \(W_1, W_2\) for \(M\) are ambient isotopic.

**Theorem 3.3.** \cite[Proposition 10.8]{16}. Let \(W\) be a characteristic submanifold for \(M\). Then, for every \(\mathcal{F}\)-manifold \(X \subset M\), there exists an ambient isotopy \(F_t\) such that \(F_1(X) \subset W\).

We now describe an alternative characterization of characteristic submanifolds, a corollary of \cite[Theorem 2.9.3]{4}; we first recall the notion of simpleness.

**Definition 3.6.** A manifold \((X, \mathfrak{x})\) with boundary-pattern \(\mathfrak{x}\) is simple if any component of a characteristic submanifold of \((X, \mathfrak{x})\) is a regular neighborhood of a square, annulus or torus in \(\mathfrak{x}\).

**Theorem 3.4.** Given a full \(\mathcal{F}\)-manifold \(W \subset M\), then \(W\) is a characteristic submanifold for \(M\) if and only if, for every component \(Y \subset (E(W), \mathfrak{m}')\), \(Y\) either is simple or is a cylindrical shell.

Below we examine topological properties of submanifolds of \(M\) that can be I- or Seifert fibered.

**Lemma 3.5.** Let \(X\) be an essential codimension-zero submanifold of \(M\). Then \(g(\partial X) = 1\) if and only if \((X, \mathfrak{x})\) can be Seifert fibered over an \(n\)-faced disk with at most one exceptional fiber, and \(\mathfrak{x}\) non-empty and containing disjoint elements; additionally, it has exactly one exceptional fiber when \(n = 2\).

**Proof.** The direction “\(\Rightarrow\)” is clear. To see the direction “\(\Rightarrow\)”, note first that by the essentiality of \(X\) and the boundary-pattern \(\mathfrak{x}\) on \(M\), the intersection \(X \cap \partial M\) is non-empty and consists of disjoint annuli \(A_1, \ldots, A_m\) in \(\partial X\). This implies \(X\) is a solid torus by the atoroidality of \(M\). Since \(M\) is \(\partial\)-irreducible, \(H_1(A_i) \to H_1(X)\) cannot be trivial, and therefore, \((X, \mathfrak{x})\)
can be Seifert fibered over an n-faced disk \((D_n, \phi)\) with \(n = 2m > 0\). In the case \(n = 2\), by the essentiality of \(\partial_M X\), the Seifert fiberings must contain at least one exceptional fiber. □

**Corollary 3.6.** Let \(X \subset M\) be an essential codimension-zero submanifold with \(g(\partial X) = 1\). Then \((X, \varphi)\) admits an essential annulus meeting \(\partial_M X\).

**Proof.** By Lemma 3.5, the frontier of a regular neighborhood of an element in \(\varphi\) is an essential annulus meeting \(\partial_M X\). □

**Lemma 3.7.** Given an essential codimension-zero submanifold \(X \subset M\), if \((X, \varphi)\) is \(I\)-fibered over \((B, \bar{\varphi})\), then \(\bar{\varphi} = \varphi\), that is, \(\varphi\) consists of only lids.

**Proof.** By the definition \(\varphi = (\varphi_x)\), the lid(s) of \((X, \varphi)\) is(are) element(s) in \(\varphi\). On the other hand, since the boundary pattern on \(M\) is \(\varphi\), the submanifold boundary-pattern \(\varphi\) consists of disjoint elements. Thus \(\varphi\) can contain only the lid(s). □

**Lemma 3.8.** Let \((X, \varphi) \to (B, \bar{\varphi})\) be an \(I\)-bundle and \(g(\partial X) > 1\). Then every essential annulus in \((X, \varphi)\) disjoint from the sides of \((X, \varphi)\) is parallel to a side \(\kappa \in \varphi\) if and only if \(B\) is a pair of pants.

**Proof.** The direction “⇐” follows from Lemma 3.1. We prove the direction “⇒” by contradiction. Observe first that since \(g(\partial X) > 1\), the Euler characteristic \(\chi(B) \leq -1\) by the equality \(2\chi(B) = 2 - 2g(\partial X)\). In particular, \(B\) is a closed surface \(\tilde{B}\) with \(k\) open disks removed such that \(k\) and the genus \(g(\tilde{B})\) satisfy \(3 - 2g(\tilde{B}) \leq k\) when \(B\) is orientable and \(3 - g(\tilde{B}) \leq k\) otherwise. Take \(l\) to be a non-separating loop in \(B\) if \(B\) is orientable with \(g(\tilde{B}) > 0\) or is non-orientable with \(g(\tilde{B}) > 1\), and to be a loop cutting a projective space, and to be a loop cutting a pair of pants from \(B\) if \(\tilde{B}\) is a 2-sphere. Then if \(B\) is not a pair of pants, the preimage of \(l\) is an essential annulus in \(X\) disjoint from the sides and not parallel to any side of \((X, \varphi)\). □

The following is a corollary of [16 Proposition 4.6].

**Lemma 3.9.** Let \(S \subset M\) be a surface consisting of essential annuli, and \(X\) a component of \((E(S), \tilde{\varphi})\). Then \(X\) is atoroidal, and given an annulus \(A \subset X\) disjoint from \(\partial_M X\), \(A\) is essential in \(X\) if and only if \(A\) is essential in \(M\).

**Theorem 3.10 (Completeness Criterion).** Let \(W \subset M\) be a full \(\mathcal{F}\)-manifold. Then \(W\) is complete if and only if, for every component \(Y\) of \((E(W), \tilde{\varphi})\), either \(Y\) is a cylindrical shell or \(g(\partial Y) > 1\). \(Y\) cannot be \(I\)-fibered over a pair of pants, and every essential annulus in \(Y\) disjoint from \(\partial_M Y\) is parallel to a component of \(\partial_M Y\).

**Proof.** “⇒”: Given a component \(Y\) of \((E(W), \tilde{\varphi})\), either \(Y\) admits an essential annulus or a pair of pants that must meet \(\partial_M Y\) or it does not. By \(\varphi\) in Definition 3.4, \(Y\) is a cylindrical shell if \(Y\) is the former. Suppose it is the latter. Then, since \(Y\) contains no essential square, it cannot be \(I\)-fibered over a pair of pants, and by Corollary 3.6, \(g(\partial Y)\) cannot be 1. The rest follows directly from \(\varphi\) of Definition 3.4.

“⇐”: It is clear that the conditions \(\varphi\) and \(\varphi\) in Definition 3.4 are satisfied if \(Y\) is a cylindrical shell. So, we suppose otherwise; by Theorem 3.4, \(Y\) must be a simple. Let \(W_i\) be the characteristic submanifold of \(Y\); note that since \(Y\) is a component of \((E(W), \tilde{\varphi})\), \(Y \subset M\) is equipped with the proper boundary-pattern. If \(W_i \neq \emptyset\), then \(Y\) is simple by the definition. If \(W_i \neq \emptyset\) but \(\partial Y \neq \emptyset\), then \(Y = W_i\). Since \(g(\partial Y) > 1\), by Lemma 3.8, it cannot be Seifert fibered, so \(Y\) admits an \(I\)-bundle structure, contradicting the assumption by Lemma 3.8.

Suppose \(\partial Y \neq \emptyset\), and let \(X_i\) be a component of \(W_i\), and \(A\) be a component of the frontier \(\partial Y\). Then \(A\) is disjoint from \(\partial_M Y\) since \(W_i\) contains a regular neighborhood of \(\partial_M Y\) by Theorem 3.3. Therefore \(A\) cannot be a square, given the boundary-pattern \(\varphi\) on \(M\); it cannot be a torus either by Lemma 3.9. A thereby is an annulus. By the assumption, \(A\) is
parallel to a component $A'$ of $\partial_M Y$ in $Y$. Let $P \subset Y$ be the cylindrical shell between $A$ and $A'$. Then $P \cap X_i$ is non-empty by the fullness of $W$, so $P \subset X_i$, and hence $X_i$ is a regular neighborhood of a component of $\partial_M Y$. Therefore $Y$ is simple. \hfill \Box

3.3. **Characteristic diagram.** Let $M$ be as in the previous subsection.

**Definition 3.7** (**Characteristic Surfaces**). A characteristic surface $S$ of $M$ is a union of components of $\partial_M W$ such that

- no two components of $S$ are parallel, and
- every component of $\partial_M W$ is parallel to some component of $S$,

where $W \subset M$ is a characteristic submanifold.

The existence of a characteristic surface follows from the existence of a characteristic submanifold of $M$. More precisely, let $C$ be the set consisting of the closures of components of $M - \partial_M W$, each carrying the proper boundary-pattern, and $S_0$ be the union of components of $\partial_M W$ not adjacent to a cylindrical shell $C \in C$. For each cylindrical shell $C \in C$, choose a component $A_c$ of $\partial_M C$, and let $S_c$ be the union of $A_c$ for all cylindrical shells $C \in C$. Since cylindrical shells in $C$ are disjoint, the union $S := S_0 \cup S_c$ is a characteristic surface of $M$. The uniqueness of $S$ follows from Theorem 3.2.

**Corollary 3.11.** Given two characteristic surfaces $S_1, S_2$ of $M$, there exists an ambient isotope $F_t$ such that $F_1(S_1) = S_2$.

**Definition 3.8.** An annulus $A \subset M$ is characteristic if it is isotopic to a component of a characteristic surface $S$ of $M$.

**Definition 3.9.** Given a characteristic surface $S$ of $M$, denote by $E(S)$ the complement $M - S$. Then the associated characteristic diagram is a labeled graph defined as follows:

- Assign a solid node $\bullet$ to each component $E(S)$ that can be I-or Seifert fibered.
- Assign a hollow node $\circ$ to each component $E(S)$ that is simple.
- To each component of $\partial(S)$, assign an edge that joins node(s) corresponding to component(s) of $E(S)$ meeting the component of $\partial(S)$.
- If the genus $g(\partial X)$ of the component $X \subset E(S)$ is larger than 1, then label the corresponding node with $g(\partial X)$.

Two characteristic diagrams are isomorphic if there is an isomorphism between their underlying graphs sending solid (resp. hollow) nodes to solid (resp. hollow) nodes with the same label.

For the sake of simplicity, we say a component $X \subset E(S)$ is of genus $g$ if $g(\partial X) = g$. By Corollary 3.11, we have the following.

**Corollary 3.12.** Two characteristic diagrams of $M$ are isomorphic.

4. **Classification**

Throughout the section, $(S^3, HK)$ is an irreducible, cylindrical, atoroidal handlebody-knot, and we denote by $\Lambda_{ext}$ the characteristic diagram of $E(HK)$.

4.1. **Annulus diagram of handlebody-knots.**

**Proposition 4.1.**

(i) There exists exactly one labeled node in $\Lambda_{ext}$, and it is labeled with 2.
(ii) Unlabeled nodes in $\Lambda_{ext}$ are solid, and each corresponds to a Seifert-fibered solid torus that is not a cylindrical shell.
(iii) No loop in $\Lambda_{ext}$ contains a solid node.
(iv) All edges are adjacent to the labeled node.
(v) If the labeled node is solid, it corresponds to an I-bundle over a pair of pants or a Mobius band or Klein bottle with one open disk removed.
(vi) If the labeled node is solid, then $\Lambda_{\text{ext}}$ is not a bigon.

(vii) Every node in $\Lambda_{\text{ext}}$ is at most trivalent.

Proof. Let $S$ be the characteristic surface of $E(HK)$ that induces $\Lambda_{\text{ext}}$, and $W$ a characteristic submanifold of $E(HK)$ inducing $S$. Suppose the complement $E_{bh}(S) := E(HK) - \mathcal{R}(S)$ contains $n$ components $X_1, \ldots, X_n$. Then the equality of Euler characteristic numbers

$$-2 = 2 - 2g(\partial E(HK)) = \chi(\partial E(HK)) = \sum_{i=1}^n \chi(\partial X_i) = \sum_{i=1}^n (2 - 2g(\partial X_i))$$

implies that

$$\sum_{i=1}^n (g(\partial X_i) - 1) = 1.$$

In particular, there exists exactly one genus 2 component in $E_{bh}(S)$; every other component is of genus 1, and therefore is Seifert-fibered by Lemma 3.5 but is not a cylindrical shells by the definition of $S$. This proves (i) and (ii).

We prove (iii) by contradiction. Suppose there is a loop with a solid node in $\Lambda_{\text{ext}}$, and denote by $A$ the annulus corresponding to the loop, and $X \subset E_{bh}(S)$ the component corresponding to the solid node. Also, let $X'$ be the union $X$ and $\mathcal{R}(A)$, the component of $\mathcal{R}(S)$ containing $A$. If the node is unlabeled, then $\partial X'$ consists of two tori $T_1, T_2$, each meets both $\partial gX'$ and $\partial E(HK)$ since $\partial E(HK)$ contains no torus. This implies the frontier of a regular neighborhood of $T_1 \cup T_2$ in $X'$ are incompressible tori, contradicting the atoroidality.

If the node is labeled, then $X'$ can be I-fibered, contradicting the fullness of $W \subset E(HK)$.

To see (iv), it suffices to show there is no edge connecting two unlabeled solid nodes, given (iii) and (vi). Suppose such an edge exists, and $X_1, X_2 \subset E_{bh}(S)$ are the components corresponding to the solid nodes and $A$ the annulus corresponding to the edge between. Then $X_1 \cup \mathcal{R}(A) \cup X_2$ is I-fibered, contradicting the fullness of $W \subset E(HK)$.

Coming now to (v) we observe first that the component $U$ in $E_{bh}(S)$ corresponding to a labeled solid node cannot be Seifert fibered by Lemma 3.5 and hence is I-fibered. Since the lid(s) of $U$ has Euler characteristic $-2$, the base is either a pair of pants or a Mobius band, torus, or Klein bottle with one open disk removed. Suppose the base is a torus with one open disk removed. Then the characteristic diagram is $\bullet \bullet \circ$ by (iv). Denote by $A$ the annulus corresponding to the edge, and let $V$ be the solid torus corresponding to the unlabeled node. Choose generators of $H_1(A)$ and $H_1(V)$ so that the homomorphism $H_1(A) \cong \mathbb{Z} \overset{m}{\rightarrow} \mathbb{Z} \cong H_1(V)$ has $m \geq 0$. Since $A$ is essential, $m \neq 0, 1$. The short exact sequence

$$0 \rightarrow H_1(A) \overset{(m,0)}{\rightarrow} H_1(V) \oplus H_1(U) \rightarrow H_1(E(HK)) \rightarrow 0$$

then implies $H_1(E(HK)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, a contradiction.

We prove (vi) by contradiction. Suppose $\Lambda_{\text{ext}}$ is a bigon, and let $U$ (resp. $V$) be the components of $E_{bh}(S)$ corresponding to the labeled (resp. unlabeled) node, and $A_1, A_2$ the annuli corresponding to the edges. Then by (v) $U$ is an admissible I-bundle over a Mobius band with one open disk removed. Choose generators of $H_1(A_1), H_1(V), H_1(U)$ so that $H_1(A_i) \cong \mathbb{Z} \overset{n_i}{\rightarrow} \mathbb{Z} \cong H_1(U), i = 1, 2$, and

$$H_1(A_1) \cong \mathbb{Z} \overset{(b_1)}{\rightarrow} \mathbb{Z} \oplus \mathbb{Z} \cong H_1(U) \quad \text{and} \quad H_1(A_2) \cong \mathbb{Z} \overset{(b_2)}{\rightarrow} \mathbb{Z} \oplus \mathbb{Z} \cong H_1(U).$$

Then by the exact sequence

$$0 \rightarrow H_1(A_1 \cup A_2) \rightarrow H_1(V) \oplus H_1(U) \rightarrow H_1(E(HK)) \rightarrow \tilde{H}_1(A_1 \cup A_2) \rightarrow 0,$$

either $(m, 0, 1)$ or $(0, 0, 1)$ in $H_1(V) \oplus H_1(U)$ induces an element of order 2 in $H_1(E(HK))$, a contradiction.

Lastly, for (vii) in view of (iv) it suffices to consider the labeled node. The case where the labeled node is solid follows from (v) so we assume the labeled node is hollow, and $Y \subset$
is the corresponding component. Suppose \( \partial_{E(HK)} Y \) has more than 3 components. Then there exists an annular component \( A \) in \( \partial Y - \partial_{E(HK)} Y \). Let \( A_1, A_2 \) be the components of \( \partial_{E(HK)} Y \) that meet \( \partial A \). Then consider the frontier \( A' \) of a regular neighborhood of \( A_1 \cup A_2 \cup A_3 \) in \( Y \). \( A' \) cannot be inessential, for otherwise there exists an essential square in \( Y \), contradicting the completeness of \( W \subset E(HK) \), but \( A' \) cannot be essential either by the simpleness of \( Y \), given Theorem 3.3.

**Definition 4.1.** We say the characteristic diagram of \( E(HK) \) is of type \((e, l, b, \Box)\) if \( \Lambda_{ext} \) has \( e \) edges, \( l \) loops, and \( b \) bigons, and \( \Box = \bullet \) or \( \circ \) if the labeled node in \( \Lambda_{ext} \) is solid or hollow, respectively.

**Theorem 4.2.** Characteristic diagrams of \( E(HK) \) are classified, up to isomorphism, by their types \((e, l, b, \Box)\) into 13 classes in Table 7.

**Proof.** By Proposition 4.1, we have \( 1 \leq e \leq 3 \), \( l = 0 \) or \( 1 \), and \( b = 0, 1, 3 \), and two characteristic diagrams are isomorphic if and only if they have the same type. \( \Box \)

**Remark 4.3.** Table 1 lists all possible characteristic diagrams \( \Lambda_{ext} \). It is however not clear whether all configurations do occur. To our knowledge, no known example has been found so far with a characteristic diagram of the type

\[
(2, 0, 0, \bullet), (3, 0, 1, \Box), \text{ or } (3, 0, 0, \Box), \quad \Box = \bullet \text{ or } \circ.
\]

The characteristic diagram \( \Lambda_{ext} \) of \( E(HK) \) can be enhanced by adding to each edge the type of the annulus it corresponds to. Recall that essential annuli in \( E(HK) \) are classified into four types [18], [8]: type 2, type 3-2, type 3-3, and type 4-1 as described in Section 1. Type 2 annuli are divided into two subfamilies: types 2-1 and 2-2, while type 3-3 annuli can be further classified based on their boundary slope pairs.

There is also a finer classification for a type 3-2 annulus \( A \) [8]. Recall that by the definition, there exists an essential disk \( D_A \subset HK \) disjoint from \( \partial A \), and it is shown that, the disk \( D_A \subset HK \) is uniquely determined by \( A \), up to isotopy [8 Corollary 2.2], and hence \( A \) determines the solid torus \( V_A := HK - \hat{\eta}(D_A) \subset S^3 \), up to isotopy. There are two possibilities: either \( A \) is essential in the exterior \( E(V_A) \) — that is, the core of \( V_A \) is a non-trivial torus or cable knot in \( S^3 \) with \( A \) the cabling annulus—or \( A \) is \( \partial \)-compressible in \( E(V_A) \) [8 Lemma 3.5]. \( A \) is said to be of type 3-2i if it is the former, and of type 3-2ii otherwise. In the latter case, \( V_A \) is trivially embedded in \( S^3 \), namely \( E(V_A) \) being a solid torus, by the atoroidality of \( (S^3, HK) \), and if \( r \) is the boundary slope of \( A \) with respect to \( (S^3, V_A) \), then \( r \) is never a reciprocal of an integer; otherwise \( A \) is \( \partial \)-compressible in \( E(HK) \).

Following the designation in [18], we label each edge in \( \Lambda_{ext} \) as follows.

**Definition 4.2 (Labeling Scheme).** The label \( h_i \) is assigned to each edge corresponding to a type 2-\( i \) annulus, \( i = 1, 2 \).

The label \( k_i \) or \( k_2(\tau) \) is assigned to each edge corresponding to a type 3-2i or 3-2ii annulus, respectively, where \( \tau \) is the boundary slope of \( A \) with respect to \( (S^3, V_A) \).

The label \( l(r_1, r_2) \) is assigned to each edge corresponding to a type 3-3 annulus with the boundary slope pair \( (r_1, r_2) \). If \( (r_1, r_2) = (0, 0) \), the label is shortened to \( l_i \).

The label \( m \) is assigned to each edge corresponding to a type 4-1 annulus.

The resulting labeled graph, denoted by \( \Lambda_{nk} \), is called the annulus diagram of \((S^3, HK)\).

Note that an edge in \( \Lambda_{nk} \) has the label \( h_i, l(r_1, r_2), \) or \( l_i \) if and only if it is not a cut-edge.

### 4.2. Handlebody-knots that admit a non-separating annulus

Here we investigate different combinations of disjoint non-separating essential annuli in \( E(HK) \). Throughout the subsection, \( A \subset E(HK) \) is an essential annulus, and \( A_+, A_- \) denote the annuli in the frontier of \( \partial A \) in \( E(HK) \) with \( l_+, l_- \) their cores, respectively. We orient \( l_+, l_- \) so that \( [l_+] = [l_-] \in H_1(\partial(A)) \), and denote the components of \( \partial A \) by \( l_1, l_2 \) unless \( A \) is of type
Lemma 4.4. If $A$ is of type 4-1, then there exists no non-separating annulus disjoint from $A$ in $E(HK)$.

We start our investigation with type 3-3 annuli. Given a type 3-3 annulus, we fix an oriented disk $D_A \subset HK$ disjoint from $\partial A$. Recall also the definition of meridional basis from [29].

Definition 4.3. Suppose $A$ is of type 3-3 with a boundary slope pair $(\frac{p}{q}, pq)$. Then a meridional basis of $H_1(E(HK_A))$ is a basis given by the homology classes of the boundary of two oriented, disjoint, non-parallel meridian disks $D_1, D_2 \subset HK_A$ disjoint from $D_A$ with $[\partial D_1] - [\partial D_2] = [\partial D_A] \in H_1(E(HK_A))$.

Lemma 4.5. Suppose $A$ is of type 3-3 with a boundary slope pair $(\frac{p}{q}, pq)$ and $\{b_1, b_2\}$ a meridional basis of $H_1(E(HK_A))$. If $\{l_1, l_2\} = (p_1, p_2)$ in terms of $\{b_1, b_2\}$, then $[l_1] = (p_1 \pm 1, p_2 \pm 1)$ and $p_1 + p_2 = \pm p$.

Proof. Denote by $V_1, V_2$ the solid tori in $HK - \tilde{\partial}(D_A)$, and by $U$ the solid torus $V_1 \cup V_2 \cup \partial(U)$. Then $l_1, l_2$ are two parallel curves in $\partial U$, and they separate the two disks in $\partial(U) = \partial U \cap \partial(U)$, so $[l_1] - [l_2] = \pm [\partial D_A] \in H_1(E(HK_A))$ and therefore the first assertion. Consider the short exact sequence

$$0 \to \langle [\partial D_A] \rangle \to H_1(E(HK_A)) \to H_1(U) \cong \langle b_1 = b_2 \rangle \to 0,$$

and note that the slopes of $l_1, l_2 \subset \partial U$ are $\frac{p}{q}$ with respect to $(S^3, U)$. Hence $p_1 + p_2 = \pm p$. □

Lemma 4.6. Suppose $A$ is of type 3-3 with a boundary slope pair $(r_1, r_2)$.

If $(r_1, r_2) = (\frac{p}{q}, pq)$, $pq \neq 0$, then $\{l_1, l_2\} = [l_1, l_2] = \langle p, q \rangle$ is a basis of $H_1(E(HK_A))$.

If $(r_1, r_2) = (\frac{p}{q}, pq)$, $pq \neq 0$, then $\langle [l_1], [l_2] \rangle$ is a subgroup of $H_1(E(HK_A))$ with index $|p|$.

If $(r_1, r_2) = (0, 0)$, then $\langle [l_1], [l_2] \rangle$ is a rank one subgroup of $H_1(E(HK_A))$.

Proof. Denote by $V_1, V_2$ the solid tori in $HK - \tilde{\partial}(D_A)$, and by $U$ the union $V_1 \cup V_2 \cup \partial(U)$.

Suppose $(r_1, r_2) = (\frac{p}{q}, pq)$, $pq > 1$. Then $U$ is a Seifert fibered space with two exceptional fibers, and therefore the exterior $E(U)$ of $U$ in $S^3$ is a solid torus $W$, whose core is a $(p, q)$-torus knot in $S^3$. Since $l_1, l_2$ are parallel to the core of $W$ in $W$ by [25], $[l_1] = [l_2]$ generates $H_1(W)$. On the other hand, we have $E(\partial D_A) = W - \partial(D_A)$, that is, $E(\partial D_A)$ is obtained by removing a regular neighborhood of an arc in $W$ dual to $D_A$, so $H_1(W, E(\partial D_A)) = 0$. This together with $H_2(W) = 0$ implies the short exact sequence

$$0 \to H_2(W, E(\partial D_A)) \to H_1(E(HK_A)) \to H_1(W) \to 0$$

given by the inclusion $E(\partial D_A) \hookrightarrow W$. Since $[\partial D_A]$ generates $H_2(W, E(\partial D_A))$, $\pm [\partial D_A] = [l_1] - [l_2]$, and $[l_1] = [l_2]$ generates $H_1(W)$, we have $\langle [l_1], [l_2] \rangle$ is a basis of $H_1(E(HK_A))$.

Suppose $(r_1, r_2) = (\frac{p}{q}, pq)$, $pq \neq 0$. Then by Lemma 4.5, $\{l_1, l_2\} = (p_1, p_2)$ and $[l_1] = (p_1 \pm 1, p_2 \pm 1)$ with $p_1 + p_2 = p$ in terms of a meridional basis of $H_1(E(HK_A))$, and hence the determinant

$$\begin{vmatrix} p_1 & p_2 \\ p_1 \pm 1 & p_2 \pm 1 \end{vmatrix} = \pm (p_1 + p_2) = \pm p$$

In other words, when $p \neq 0$, $\langle [l_1], [l_2] \rangle < H_1(E(HK_A))$ is a subgroup of rank two with index $|p|$. When $p = 0$, since $[l_1] - [l_2] = \mp (1, -1)$, at least one of $[l_1], [l_2] \in H_1(E(HK_A))$ is non-trivial, so $\langle [l_1], [l_2] \rangle$ is a subgroup isomorphic to $\mathbb{Z}$. □
Corollary 4.7. Suppose $A$ is a type 3-3 annulus with a non-trivial boundary slope pair, and $A'$ is a non-separating annulus disjoint from $A$. Then $\partial A, \partial A'$ are parallel in $\partial \mathcal{H}K$. In particular, $A'$ is of type 3-3 with the same boundary slope pair.

Proof. Let $P$ be the planar surface $\partial \mathcal{H}(\mathcal{H}K) - A, \cup A'$. Denote by $l_1, l_2$, the components of $\partial A$, and by $l_1', l_2'$, the components of $\partial A'$. Since $l_1, l_2' \subset P$, one of $l_1', l_2'$ is parallel to one of $l_1, l_2$. It may be assumed that $l_1'$ is parallel to $l_1$. By Lemma 4.6, $[l_1] \neq [l_2]$ and none of $[l_1], [l_2]$ is trivial in $H_1(\mathcal{H}(\mathcal{H}K))$. These, together with $[l_1'] \neq [l_2'] \in H_1(\mathcal{H}(\mathcal{H}K)), \gamma \in \mathcal{H}(\mathcal{H}K)$, imply that $l_2'$ is parallel to $l_2$ or $l_1$. The latter is impossible since $l_1, l_2'$ are not parallel in $\partial \mathcal{H}K$ and hence not parallel in $P$. Therefore $\partial A'$ is parallel to $\partial A$, and hence to $\partial A$.

Lemma 4.8. Suppose $A$ is of type 3-3 annulus with a trivial boundary slope, and $A'$ is a type 3-3 annulus disjoint from $A$. Then $A, A'$ are parallel in $\mathcal{H}(\mathcal{H}K)$.

Proof. Suppose $\partial A$ and $\partial A'$ are parallel in $\partial \mathcal{H}K$. Let $B_1, B_2 \subset \partial \mathcal{H}K$ be the annuli cut off by $\partial A, \partial A'$. Then $A \cup A' \cup B_1 \cup B_2$ bounds a solid torus $V$ in $\mathcal{H}(\mathcal{H}K)$ by the atoroidality of $(\mathbb{S}^3, \mathcal{H}K)$. Since $A$ has a trivial boundary slope pair, the linking number $\langle k(l_1, l_2) \rangle$ is 0 and hence the core of $A$ is a preferred longitude with respect to $(\mathbb{S}^3, \mathcal{H}K)$. Let $A'$ be the planar surface $\mathcal{H}(\mathcal{H}K) - A$. Then $\partial A, \partial A'$ are parallel through $\mathcal{H}(\mathcal{H}K)$.

Suppose $\partial A$ and $\partial A'$ are not parallel. Let $l_1', l_2'$ be the components of $\partial A'$. Then $\partial \mathcal{H}K = \partial A$ is a four-times punctured sphere, may be assumed that $l_1, l_1'$ are parallel in $\partial \mathcal{H}K$, and $l_2, l_2'$ are not. Let $B_1 \subset \partial \mathcal{H}K$ be the annulus cut off by $l_1, l_1'$. Then $B_1 \cup A \cup A'$ induces an annulus $A'' \subset \mathcal{H}(\mathcal{H}K)$ disjoint from $A \cup A'$ with $\partial A''$ parallel to $l_2, l_2'$. Let $B_2, B_3 \subset \partial \mathcal{H}K$ be the annuli cut off by $\partial A''$ and $l_2, l_2'$. Then the torus $B_1 \cup B_2 \cup B_3 \cup A \cup A' \cup A''$ bounds a solid torus $W$ in $\mathcal{H}(\mathcal{H}K)$ since $(\mathbb{S}^3, \mathcal{H}K)$ is atoroidal.

Let $P_1, P_2 \subset \partial \mathcal{H}K$ be the pairs of pants cut off by $B_1 \cup B_2 \cup B_3$. Then $P_1, P_2$ can be regarded as a planar surface in $E(W)$. By [18] Lemma 3.5, $P_1, P_2$ are inessential in $E(W)$.

Case 1: $P_1$ is compressible. Let $D$ be a compressing disk of $P_1$ that minimizes 

$$\#\{D \cap P_2 \mid D \text{ a compressing disk of } P_1\}.$$

Subcase 1.1: $D \cap P_2 = \emptyset$. $D$ is either in $\mathcal{H}K$ or in $E(\mathcal{H}K)$. Since $\partial D$ is essential in $P_1$, $\partial D$ is essential in $\partial \mathcal{H}K$, so $D$ is a compressing disk of $\partial \mathcal{H}K$ in $\mathbb{S}^3$. On the other hand, $\partial A \cup \partial A' \cup \partial A''$ contains three mutually non-parallel simple loops in $\partial \mathcal{H}K$ that bound no disks in $\mathcal{H}K$, so every meridian disk in $\mathcal{H}K$ meets $\partial A \cup \partial A' \cup \partial A''$, and hence $D \subset E(\mathcal{H}K)$, but this contradicts $(\mathbb{S}^3, \mathcal{H}K)$ is irreducible.

Subcase 1.2: $D \cap P_2 \neq \emptyset$. Note first that $D \cap P_2$ only contains circles. Let $D' \subset D$ be the disk cut off by a circle in $D \cap P_2$ innermost in $D$. By the minimality $\partial D'$ is essential in $P_2$; hence $D'$ is a compressing disk of $\partial \mathcal{H}K$ in $\mathbb{S}^3$, a contradiction as in Subcase 1.1.

The same argument applies to the case where $P_2$ is compressible.

Case 2: $P_1, P_2$ are incompressible. First observe that, since none of the components of $\partial A \cup \partial A' \cup \partial A''$ is separating in $\partial \mathcal{H}K$, $P_1$ (resp. $P_2$) meets $B_i$ for each $i$. Let $D$ be a $\partial$-compressing disk of $P_1$ that minimizes

$$\#\{D \cap P_2 \mid D \text{ a } \partial \text{-compressing disk of } P_1\}.$$

Then by the minimality and incompressibility of $P_2$, $D \cap P_2$ is either empty or some arcs.

Subcase 2.1: $D \cap P_2 = \emptyset$. Denote by $\gamma$ the arc $D \cap E(W)$, and note that $\gamma \subset B_i := B_1 \cup B_2 \cup B_3$ if $D \subset \mathcal{H}K$; otherwise $\gamma \subset A_i := A \cup A' \cup A''$. Also, $\gamma$ is inessential in each case: in the former case, it follows from the fact that none of $B_i, i = 1, 2, 3,$ has two boundary components lying in $P_1$, whereas for the latter, it results from the $\partial$-incompressibility of $A, A', A''$.

Let $D'$ be the disk cut off from $B_i$ (resp. $A_i$). Then $D \cup D'$ induces a disk $D''$ disjoint from $B_i$ (resp. $A_i$). Since $D$ is a $\partial$-compressing disk of $P_1$ in $E(W)$, $\partial D''$ is essential in $P_1$, contradicting the incompressibility of $P_1$.

Subcase 2.2: $D \cap P_2 \neq \emptyset$. Let $D' \subset D$ be a disk cut off by an arc in $D \cap P_2$ outermost in $D$. Denote by $\gamma$ the arc $D \cap \partial W$; as with Subcase 2.1, $\gamma$ is either in $A_i$ or in $B_i$, and is
Lemma 4.9. If \([l_+], [l_-]\) is a basis of \(H_1(E(HK_a))\), then \(A\) is the unique annulus in \(E(HK)\).

Proof. By Theorem \ref{thm:characteristic}, it suffices to show that \(\mathcal{R}(A) \subset E(HK)\) is a characteristic submanifold of \((E(HK), \partial)\), and we employ Theorem \ref{thm:characteristic}. Since \(\mathcal{R}(A)\) is a full \(\mathcal{F}\)-manifold of \((E(HK), \partial)\), it amounts to show that every essential annulus \(A'\) in \((E(HK), \partial)\) disjoint from \(A_+, A_-\) is parallel to \(A_+, A_-\), where \((E(HK), \partial)\) is endowed with the proper boundary pattern. Denote by \(\ell'\) a core of \(A'\).

Case 1: \(A'\) is non-separating in \(E(HK)\). Note that the same argument for Corollary \ref{corollary:non-separating} implies that \(\partial A'\) is parallel to \(\partial A_+ \) or \(\partial A_-\) in \(E(HK)\); it may be assumed that it is the former. Denote by \(B_1, B_2\) the annuli cut off by \(\partial A_-, \partial A'\) from \(E(HK_\lambda)\). Then \(A \cup A' \cup B_1 \cup B_2\) bounds a solid torus \(W\) in \((E(HK_\lambda), \partial)\) by Corollary \ref{corollary:essential}. Let \(X\) be the closure of the complement \(E(HK_\lambda) - W\) and \(l_w\) a core of \(W\), and orient \(\ell' l_w\) so that \([\ell'] = [l_+]\) and \([\ell] = [l_-]\) with \(k > 0\) in \(H_1(W)\). Consider the short exact sequence

\[0 \rightarrow H_1(A') \xrightarrow{(\iota_1, \iota_2)} H_1(W) \oplus H_1(X) \xrightarrow{\iota_1 + \iota_2} H_1(E(HK_\lambda)) \rightarrow 0,\]

where \(\iota_i, i = 1, 2, 3, 4\), are induced by the inclusions. Note that \(\iota_4\) sends \([\ell']\) to \([l_+]\) and \([l_-]\) to itself, and \(\iota_1\) sends \([\ell']\) to \([l_+]\). Since \([l_+]\) is a basis of \(H_1(E(HK_\lambda))\), the image of \([l_-]\) under \(\iota_4\) is \(m[l_+] + n[l_-]\), for some \(m, n \in \mathbb{Z}\). Then the identity \(\iota_3 \circ \iota_1 = \iota_4 \circ \iota_2\) gives us \(km[l_+] + kn[l_-] = [l_-]\), and therefore \(n = 0, k = m = 1\). This implies \(H_1(A') \xrightarrow{\iota_1} H_1(W)\) is an isomorphism, and hence \(A'\) is parallel to \(A_+\) through \(W\) in \((E(HK_\lambda), \partial)\).

Case 2: \(A'\) is separating in \(E(HK)\), \(\partial A'\) being parallel in \(\partial HK\) and being not separating components of \(\partial A\) imply that \(\partial A'\) is parallel in \(E(HK_\lambda)\). Let \(B \subset E(HK_\lambda)\) be the annulus cut off by \(\partial A'\). Then \(B \cup A'\) bounds a solid torus \(W\) in \((E(HK_\lambda), \partial)\) by Corollary \ref{corollary:essential}. Set \(X := \overline{E(HK_\lambda) - W}\), and consider the short exact sequence

\[0 \rightarrow H_1(A') \xrightarrow{(\iota_1, \iota_2)} H_1(W) \oplus H_1(X) \xrightarrow{\iota_1 + \iota_2} H_1(E(HK_\lambda)) \rightarrow 0,\]

where \(\iota_i, i = 1, 2, 3, 4\), are induced by the inclusions. Let \(l_w\) be a core of \(W\), and orient \(\ell' l_w\) so that \([\ell'] = [l_+]\) with \(k > 0\). Note that \(k\) is necessarily larger than 1 by the essentiality of \(A'\). Since \([l_+]\), \([l_-]\) \(\in H_1(X)\) and \(H_2(E(HK_\lambda), X) = 0, \iota_4 : H_1(X) \rightarrow H_1(E(HK_\lambda))\) is an isomorphism. Let the image of \([l_-]\) under \(\iota_3\) is \(m[l_+] + n[l_-]\), and the image of \([\ell']\) under \(\iota_2\) be \(m'[l_+] + n'[l_-]\), for some \(m, n, m', n' \in \mathbb{Z}\). Then \(x = ([l_+], [l_+] + n[l_-]) \in H_1(W) \oplus H_1(X)\) is in the kernel of \(\iota_3 - \iota_4\), and therefore, there exists \(c \in \mathbb{Z}\) such that the image of \(c[\ell']\) under \((\iota_1, \iota_2)\) is \(x\); in other words, we have the equality

\[(kc[l_+], m'[l_+] + n'c[l_-]) = ([l_+], m[l_+] + n[l_-]) \in H_1(W) \oplus H_1(X),\]

but this implies \(k = c = 1, m = m', n = n'\), contradicting \(k > 1\).

Lemma 4.10. \([l_+], [l_-]\) is a basis of \(H_1(E(HK_\lambda))\) if and only if \(A\) is of type 2-1 or of type 3-3 with the boundary slope pair \((\frac{2}{p}, \frac{2}{q})\), \(pq \neq 0\).

Proof. “\(\Leftarrow\)” follows from Lemma \ref{lemma:characteristic} and \ref{lemma:non-separating} “\(\Rightarrow\)” also results from the same lemmas as \([l_+], [l_-]\) can form a basis of \(H_1(E(HK_\lambda))\) only if \(A\) is of type 2 or of type 3-3.

Lemmas \ref{lemma:characteristic} and \ref{lemma:non-separating} give us the following uniqueness result.

Corollary 4.11. If \(A\) is of type 2-1 or of type 3-3 with the boundary slope pair \((\frac{2}{p}, \frac{2}{q})\), \(pq \neq 0\), then \(A\) is the unique annulus in \(E(HK)\).

Lemma 4.12. Let \(A, A'\) be two disjoint type 2-2 annuli in \(E(HK)\). If \(\partial A, \partial A'\) are parallel in \(\partial HK\), then \(A, A'\) are parallel in \(E(HK)\).
Proof. Let $B_1, B_2 \subset \partial HK$ be the annuli cut off by $\partial A, \partial A'$. Then $B_1 \cup B_2 \cup A \cup A'$ bounds a solid torus $W$ in $E(HK)$ by the atoroidality of $(S^3, HK)$. Observe that $l_A$ is a longitude of $(S^3, W)$ since it bounds a disk in HK. This implies $H_1(A) \rightarrow H_1(W)$ is an isomorphism, and hence $A, A'$ are parallel through $W$ in $E(HK)$.

**Lemma 4.13.** Let $A$ be a type 2-2 annulus. Then there exists another type 2-2 annulus $A'$ disjoint from and non-parallel to $A$ if and only if there exists a type 3-3 annulus $A''$ with a trivial slope pair disjoint from $A$.

**Proof.** “$\Rightarrow$”: Let $l_A \subset \partial A'$ be the component that bounds a disk in HK and $l' \subset \partial A'$ another component. Then $l_A, l_{A'}$ are parallel and bound an annulus $B$ in $\partial HK$, and $l, l'$ are non-parallel in $\partial HK$ by Lemma 4.12. The union $A \cup A' \cup B$ induces a type 3-3 annulus, which has a trivial boundary slope pair since $(k(l, l')) = (k(l_A, l_{A'})) = 0$.

“$\Leftarrow$”: Let $l''_A, l''_{A'}$ be components of $\partial A''$. Then one of them, say $l''_A$, is parallel to $l$ in $\partial HK$. Let $B \subset \partial HK$ be the annulus cut off by $l, l''_A$. Then the union $A \cup B \cup A''$ induces an annulus whose boundary components parallel to $l_A, l''_{A'}$ and hence an annulus of type 2-2.

**Corollary 4.14.** Let $A, A', A''$ be three disjoint type 2-2 annuli in $E(HK)$. Then at least two of them are parallel in $E(HK)$.

**Proof.** Let $l' \subset \partial A, l'' \subset \partial A'$ be the components that do not bound a disk in HK, and $l_A \subset \partial A', l_{A'} \subset \partial A''$ the other components. Then $l_A, l_{A'}, l_{A''}$ are parallel in $\partial HK$ by the definition of a type 2-2 annulus.

Suppose $A, A'$ are not parallel in $E(HK)$. Then $l, l'$ are longitudes of the solid tori $V, V''$ in $HK - U$, where $U \subset HK$ is the 3-ball cut off by the disks bounded by $l_A, l_{A'}$. In particular, $l''$ is parallel to either $l$ or $l'$, so by Lemma 4.12 $A''$ is parallel to $A$ or $A'$.

Recall that $\Lambda_{ext}$ denotes the characteristic diagram of $E(HK)$ and $\Lambda_{lk}$ the annulus diagram of $(S^3, HK)$.

**Theorem 4.15 (Characteristic diagram of a type $(3, 0, 3, \square)$).** If $\Lambda_{ext}$ is of type $(3, 0, 3, \square)$, then $\Lambda_{lk}$ is \begin{align*} \begin{array}{c} \circ \end{array}, \end{align*} where $\square = \bullet$ or $\circ$, and the unlabeled node corresponds to a Seifert fibered solid torus without exceptional fiber.

**Proof.** Let $A, A', A''$ be the non-separating annuli corresponding to the edges of $\Gamma$. None of them is of type 2-1 by Corollary 4.11 or of type 3-3 with a non-trivial boundary slope pair by Corollaries 4.11 and 4.7 since no two of them separate $E(HK)$. Therefore, $A, A', A''$ are of type 2-2 or of type 3-3 with a trivial boundary slope. By Lemma 4.8 at most one of them is of type 3-3, whereas by Corollary 4.14 at most two of them is of type 2-2. This implies $\Lambda_{lk}$ is \begin{align*} \begin{array}{c} \circ \end{array} \end{align*}.

Let $W$ be the component corresponding to the unlabeled node, and $A$ the type 3-3 annulus. If a core of $A$ is a $(p, q)$-curve with respect to $(S^3, W)$, then the linking number of the components of $\partial A$ in $S^3$ is $\pm pq$. Since $A$ has a trivial boundary slope, $pq = 0$, and by the essentiality of $A$, $q \neq 0$ and therefore $(p, q) = (0, \pm 1)$. Thus $W$ has no exceptional fiber.

**Lemma 4.16.** $E(HK)$ contains a non-characteristic, non-separating annulus $A$ if and only if $\Lambda_{ext}$ is of type $(1, 0, 0, \bullet)$. In addition, $A$ is of type 3-3 with a boundary slope pair $(\frac{p}{q}, pq)$, $pq \neq 0$, and is the unique non-separating annulus in $E(HK)$.

**Proof.** “$\Rightarrow$”: Let $X$ be the component corresponding to the labeled node. By Proposition 4.7 $X$ is I-fibered over a Klein bottle $B$ with an open disk removed. Any non-separating simple loop $l$ in $B$ induces an essential annulus $A$ in $X$ and hence in $E(HK)$ by Lemma 3.9.
Since \( l \) cannot be isotoped away from essential separating loops that are not parallel to \( \partial B \) in \( B \) by [9, Theorem 3.3], \( A \) is not characteristic.

\[ \Rightarrow \] By Theorem 5.5 and Lemma 5.9, we may assume \( A \) is an essential annulus in a component \( X \) of a characteristic submanifold of \( E(HK) \) with \( A \) non-parallel to any component of \( \partial E(HK) \). By Proposition 4.11 \( X \) is either an I-bundle with \( x(\partial X) < 0 \) or a Seifert fibered solid torus. The latter is impossible because \( \#\partial E(HK)X \leq 3 \) by Theorem 4.2 and, when \( \#\partial E(HK)X = 3 \), it has no exceptional fiber by Theorem 4.15.

Therefore, \( X \) is an I-bundle over a Mobius band or Klein bottle with an open disk removed; in particular, \( \Lambda_{ext} \) is of type \((2,0,0,\bullet)\) or of type \((1,0,0,\bullet)\). It may be assumed that \( A \) is vertical with respect to the I-bundle structure by [16, Corollary 5.7]. In the former case, up to isotopy, there are only two essential annuli disjoint from \( \partial E(HK) \) and not parallel to any component of \( \partial E(HK)X \). Since both are separating, they cannot be \( A \). Thus \( X \) is an I-bundle over a Klein bottle with an opened disk removed \( B \), and \( \Lambda_{ext} \) is of type \((1,0,0,\bullet)\).

By [9, Theorem 3.3], every two non-separating simple loops in a Klein bottle with an opened disk removed are isotopic, so \( A \) is the unique non-separating annulus in \( E(HK) \). Now, to determine the type of \( A \), first note that the annulus \( A' := \partial E(HK)X \subset E(HK) \) is an annulus non-isotopic to \( A \), so \( A \) is not of type 2-1 or of type 3-3 with a boundary slope pair \( (\frac{\ell_1}{\ell_2}, \frac{\ell_3}{\ell_4}) \), \( \ell_2 \neq 0 \), by Corollary 4.11. Denote by \( X' \) the solid torus \( E(HK) - X \) and observe that, by the essentiality of \( A' = X \cap X' \subset E(HK) \), the homomorphism

\[ H_1(A') \cong \mathbb{Z} \to \mathbb{Z} \cong H_1(X') \]

induced by the inclusion neither is trivial nor is an isomorphism, namely \( k \neq 0, \pm 1 \). On the other hand, the decomposition \( E(HK_A) = (X - \mathfrak{R}(A)) \cup X' \) gives us the isomorphism:

\[ H_1(E(HK_A)) \cong \langle v_+, v_- \rangle/v_+ + v_- = \pm ku, \quad (4.1) \]

where \( u \) is a generator of \( H_1(X') \), \( \langle v_\pm \rangle = \langle l_\pm \rangle \), and \( l_\pm \) are the cores of the frontier \( \partial E(HK)\mathfrak{R}(A) \). If \( A \) is of type 2-2, then by Lemma 4.3 \( v_- \) is trivial in \( H_1(E(HK_A)) \), so \( H_1(E(HK_A)) \cong \mathbb{Z} \), a contradiction. If \( A \) is of type 3-3 with a trivial boundary slope, then at least one of \( v_+ \), \( v_- \) is not a generator by Lemma 4.3, contradicting (4.1), as both \( \langle v_+ \rangle \) and \( \langle v_-, u \rangle \) form a basis of \( H_1(E(HK_A)) \). Therefore \( A \) is of type 3-3 with a boundary slope pair \( (\frac{\ell_1}{\ell_2}, \frac{\ell_3}{\ell_4}), \ell_2 \neq 0 \).

The following is an immediate consequence of Lemma 4.16.

**Corollary 4.17.** If \( A \) is of type 2 or of type 3-3 with a trivial boundary slope or a boundary slope pair \( (\frac{\ell_1}{\ell_2}, \frac{\ell_3}{\ell_4}), \ell_2 \neq 0 \), then \( A \) is characteristic.

**Corollary 4.18.** Up to isotopy, non-separating annuli in \( E(HK) \) are mutually disjoint.

**Theorem 4.19 (Classification Theorem).**

(i) If \( E(HK) \) admits a type 2-1 annulus, then \( \Lambda_{inst} \) is \( h_1 \).

(ii) If \( E(HK) \) admits a type 2-2 annulus, then \( \Lambda_{inst} \) is one of the following:

\[ h_1, h_2, h_1 k_1, h_2 k_1, h_2 k_2, h_2 k_1 h_1, \quad \text{or} \quad h_1 h_2. \]

**Proof.** (i) follows from Corollary 4.11 To see (ii) let \( S \) be a characteristic surface of \( E(HK) \). By Proposition 4.11 \( S \) consists of at most three annuli, and by the assumption and Corollary 4.17 it may be assumed that \( S \) contains a type 2-2 annulus.

**Case 1:** \( \#S = 1 \). This implies \( \Lambda_{inst} \) is \( h_1 \).

**Case 2:** \( \#S = 2 \). Let \( A' \in S \) be the other annulus. Then by Corollaries 4.11 and 4.7 it is not of type 2-1, or of type 3-3 with a non-trivial boundary slope. By Lemma 4.13 and...
Corollary 4.17, it is not of type 2-2 or of type 3-3 with a trivial boundary slope pair since \#S = 2. Therefore \( A' \) is separating, and by Lemma 4.13 it is not of type 4-1, so \( \Lambda_{mk} \) is \( \kappa \) or \( \Lambda \).

**Case 3:** \#S = 3. Let \( A', A'' \) be the other two annuli. Then at least one of them, say \( A' \), is non-separating by Theorem 4.2. On the other hand, \( A' \) cannot be of type 2-1 or of type 3-3 with a non-trivial boundary slope by Corollaries 4.11 and 4.7, so \( A' \) is of type 2-2 or of type 3-3 with a trivial boundary slope; this implies that \( A'' \) is of type 3-3 with a trivial boundary slope or of type 2-2, respectively, by Lemma 4.13 and Corollary 4.17. Therefore \( \Lambda_{mk} \) is \( \kappa \) or \( \kappa \).

\[ \square \]

We now give a characterization of \((S^3, 4_1)\) in terms of characteristic diagrams.

**Lemma 4.20.** Suppose the annulus diagrams of the handlebody-knots \((S^3, HK), (S^3, \overline{HK})\) both are \( \kappa \). Then \((S^3, HK)\) and \((S^3, \overline{HK})\) are equivalent.

**Proof.** Let \( A \) (resp. \( \overline{A} \)) and \( A_0, A_1 \) (resp. \( \overline{A}_0, \overline{A}_1 \)) be the type 3-3 annulus and the two type 2-2 annuli in \( E(HK) \) (resp. \( E(\overline{HK}) \)), respectively, and denote by \( I_{A_0}, I_{A_1} \) (resp. \( \overline{I}_{A_0}, \overline{I}_{A_1} \)) the boundary components of \( A_0, A_1 \) (resp. \( \overline{A}_0, \overline{A}_1 \)) that bound disks \( D_{A_0}, D_{A_1} \) (resp. \( \overline{D}_{A_0}, \overline{D}_{A_1} \)) in \( HK \) (resp. \( \overline{HK} \)), respectively. Also, let \( U \subseteq E(HK), \overline{U} \subseteq E(\overline{HK}) \) be the 1-bundles and \( W, \overline{W} \) their exteriors in \( E(HK), E(\overline{HK}) \), respectively. Note that \( W \) (resp. \( \overline{W} \)) are Seifert fibered solid torus with frontier in \( E(HK) A \cup A_0 \cup A_1 \) (resp. \( \overline{A} \cup A_0 \cup A_1 \)), and \( I_{A_0}, I_{A_1} \) (resp. \( \overline{I}_{A_0}, \overline{I}_{A_1} \)) lie in different lids of \( U \) (resp. \( \overline{U} \)); see Fig. 4.1.

To show \((S^3, HK), (S^3, \overline{HK})\) are equivalent, we first construct a homeomorphism

\[ f_0 : (U, A, A_0, A_1, I_{A_0}, I_{A_1}) \rightarrow (\overline{U}, \overline{A}, \overline{A}_0, \overline{A}_1, I_{\overline{A}_0}, I_{\overline{A}_1}). \]

To do this, we identify \( U, \overline{U} \) with \( P \times I, \overline{P} \times I \), respectively, where \( P, \overline{P} \) are pairs of pants. Let \( C, C_0, C_1 \) (resp. \( \overline{C}, \overline{C}_0, \overline{C}_1 \)) be the components of \( \partial P \) (resp. \( \partial \overline{P} \)), and identify \((C_0 \times I, C_0 \times 0) \) and \((C_1 \times I, \overline{C}_1 \times 1) \) with \((A_0, I_{A_0}) \) and \((A_1, I_{A_1}) \) (resp. \((\overline{C}_0 \times I, \overline{C}_0 \times 0) \) and \((\overline{C}_1 \times I, \overline{C}_1 \times 1) \) with \((\overline{A}_0, \overline{I}_{A_0}) \) and \((\overline{A}_1, \overline{I}_{A_1}) \)), respectively.

It is not difficult to see there exist homeomorphisms \( g_i : P \times i \rightarrow \overline{P} \times i \) that map \((C \times i, C_0 \times i, C_1 \times i) \) to \((\overline{C} \times i, \overline{C}_0 \times i, \overline{C}_1 \times i) \), \( i = 0, 1 \). On the other hand, since the mapping class group of a three-times punctured sphere is given by the permutation group on the punctures, \( g_0, g_1 \) can be extended to \( f_0 \).

**Figure 4.1.** Decompose \( E(HK) \) and \( \overline{HK} \).
Now, let $V, V_0, V_1 \subset \text{HK}$ (resp. $\tilde{V}, \tilde{V}_0, \tilde{V}_1 \subset \tilde{\text{HK}}$) be the 3-ball and two solid tori cut off by $D_{A_0}, D_{A_1}$ (resp. $D_{\tilde{A}_0}, D_{\tilde{A}_1}$) such that $D_{A_i}, P \times i \subset \partial V_i$ (resp. $D_{\tilde{A}_i}, \tilde{P} \times i \subset \partial \tilde{V}_i$), $i = 0, 1$. Then the exterior $E(V \cup W)$ (resp. $E(\tilde{V} \cup \tilde{W})$) of $V \cup W$ (resp. $\tilde{V} \cup \tilde{W}$) in $S^3$ is $U \cup V_0 \cup V_1$ (resp. $\tilde{U} \cup \tilde{V}_0 \cup \tilde{V}_1$); see Fig. 4.1, and $f_0$ can be extended to a homeomorphism

$$f_1 : (E(V \cup W), U, V_0, V_1) \rightarrow (E(\tilde{V} \cup \tilde{W}), \tilde{U}, \tilde{V}_0, \tilde{V}_1)$$

as follows. Extend first the restriction $f_0|_{\partial V_0}$ to a homeomorphism

$$f_0 : \partial(V_0 \cup V_1) \rightarrow \partial(\tilde{V}_0 \cup \tilde{V}_1)$$

that sends a meridian of $V_i$ to a meridian of $\tilde{V}_i$, $i = 0, 1$; this can be done because $\partial V_i - P \times i$ consists of an annulus and the disk $D_{A_i}$. Then extend $f_0$ to a homeomorphism from $V_0 \cup V_1$ to $\tilde{V}_0 \cup \tilde{V}_1$, which, together with $f_0$, induces $f_1$.

Next, observe that $\partial E(V \cup W)$ (resp. $\partial E(\tilde{V} \cup \tilde{W})$) consists of an annulus $A_i := \partial W \cap (U \cup V_0 \cup V_1) \subset \partial W$ (resp. $\tilde{A}_i := \partial \tilde{W} \cap (\tilde{U} \cup \tilde{V}_0 \cup \tilde{V}_1) \subset \partial \tilde{W}$) and the disks $D_{A_0}, D_{A_1} \subset \partial V$ (resp. $D_{\tilde{A}_0}, D_{\tilde{A}_1} \subset \partial \tilde{V}$). Thus we can extend the restriction $f_1|_{A_0}$ to a homeomorphism

$$f_1 : (W, A^0) \rightarrow (\tilde{W}, \tilde{A}^0).$$

Gluing $\tilde{f}_1$ and $f_1$ together yields a homeomorphism

$$f_2 : (E(V, U, V_1, V_2, W) \rightarrow (E(\tilde{V}, \tilde{U}, \tilde{V}_1, \tilde{V}_2, \tilde{W}).$$

Since $V \subset \text{HK}, \tilde{V} \subset \tilde{\text{HK}}$ are 3-balls, by the Alexander trick, $f_2|_{\partial V}$ can be extended to a homeomorphism

$$\tilde{f}_2 : (V, \partial V) \rightarrow (\tilde{V}, \partial \tilde{V}).$$

Gluing $\tilde{f}_2$ and $f_2$ together yields a homeomorphism

$$(S^3, U, W, V_1, V_2, V) \rightarrow (S^3, \tilde{U}, \tilde{W}, \tilde{V}_1, \tilde{V}_2, \tilde{V}),$$

and hence an equivalence between $(S^3, \text{HK})$ and $(S^3, \tilde{\text{HK}})$. \hfill \Box

![Figure 4.2. Annulus diagram of $(S^3, A_1)$](image-url)
Lemma 4.21. The annulus diagram of \((S^3, 4_1)\) is \(
abla_{3, 3}, a_3, a_0, a_0\).

Proof. Recall that \((S^3, 4_1)\) is equivalent to the handlebody-knot in Fig. 4.2a and its exterior admits three annuli \(A, A', A''\) as depicted in Fig. 4.2b, where \(A\) is of type 3-3, and \(A', A''\) are of type 2-2. By Corollary 4.17, they are characteristic and hence the characteristic diagram of \(E(4_1)\) is either of type \((3, 0, 3, \bullet)\) or of type \((3, 0, 3, \circ)\). Let \(W \subset E(HK)\) be the Seifert fibered solid torus cut off by \(A \cup A' \cup A''\) (Fig. 4.2b). Then as shown in Fig. 4.2c and 4.2d the exterior of \(W \subset E(HK)\) together with \(A \cup A' \cup A''\) is an I-bundle over a pair of pants, and hence the assertion.

\[\square\]

Theorem 4.22. \(\Lambda_{ext}\) is of type \((3, 0, 3, \bullet)\) if and only if \((S^3, HK)\) is equivalent to \((S^3, 4_1)\).

Proof. This follows from Theorem 4.15 and Lemmas 4.20 and 4.21.

\[\square\]

5. Examples

In this section, we present a construction of handlebody-knots whose exteriors admit a type 2 annulus, and show that the annulus diagrams in Theorem 4.19 can all be realized by some handlebody-knots.

5.1. Looping trivalent spatial graphs. Let \((S^3, \Gamma)\) be a spatial graph with \(\Gamma\) either a \(\theta\)-graph or a handcuff graph. Then we can produce a new spatial graph \((S^3, \Gamma')\) by replacing a small neighborhood of a trivalent node\(^1\) in \(\Gamma\) with a loop as shown in Fig. 5.1.

Label the trivalent node with \(v\) and its three adjacent edges with \(e_1, e_2, e_3\) as in Fig. 5.1. Then the new spatial graph \((S^3, \Gamma')\) is said to be obtained by looping \(e_1 e_2\) at \(v\). We call \((S^3, \Gamma')\) a looping of \((S^3, \Gamma)\); there are six possible loopings for each spatial \(\theta\)-graph.

Similarly, if \(\Gamma\) is a handcuff graph, then a looping \((S^3, \Gamma')\) of \((S^3, \Gamma)\) is obtained by replacing a neighborhood of a trivalent node with a loop as shown in Fig. 5.2 provided the resulting spatial graph remains connected. For each spatial handcuff graph, there are four possible loopings.

\[\square\]

\(^1\)A neighborhood \(\mathcal{N}(v) \subset \Gamma\) of the trivalent node \(v\) such that \((\mathcal{N}(v), \mathcal{N}(v) \cap \Gamma)\) is homeomorphic to a unit 3-ball with three non-negative axes.
A double looping \((\mathbb{S}^3, \Gamma^\circ)\) of \((\mathbb{S}^3, \Gamma)\) is the spatial graph obtained by looping at both trivalent nodes of \(\Gamma\). Taking a regular neighborhood of a looping \(\Gamma^\circ\) (resp. double looping \(\Gamma^\circ\)) in \(\mathbb{S}^3\), we obtained a handlebody-knot, denoted by \((\mathbb{S}^3, \text{HK}_\Gamma^\circ)\) (resp. \((\mathbb{S}^3, \text{HK}_\Gamma^\circ)\)), whose exterior contains a canonical type 2 annulus \(A^\circ_i\) induced by the new loop in \((\mathbb{S}^3, \Gamma^\circ)\).

A spatial graph \((\mathbb{S}^3, \Gamma)\) is said to be irreducible and atoroidal if the induced handlebody-knot \((\mathbb{S}^3, \theta(\Gamma))\) is irreducible and atoroidal.

**Lemma 5.1.** If \((\mathbb{S}^3, \Gamma)\) is irreducible and atoroidal, then the handlebody-knot \((\mathbb{S}^3, \text{HK}_\Gamma^\circ)\) induced by a looping of \((\mathbb{S}^3, \Gamma)\) is irreducible and atoroidal, and \(A^\circ_i \subset E(\text{HK}_\Gamma^\circ)\) is of type 2-1 and is the unique annulus if \(\Gamma\) is a \(\theta\)-graph, and is of type 2-2 if \(\Gamma\) is a handcuff graph.

**Proof.** The disk bounded by a component of \(\partial A^\circ_i\) in \(\text{HK}_\Gamma^\circ\) is dual to the two edges being looped, so \(A^\circ_i\) is of type 2-1 if \(\Gamma\) is a \(\theta\)-graph and of type 2-2 otherwise. The essentiality of \(A^\circ_i\) follows from Lemmas 2.3 and 2.6, and the irreducibility and atoroidality of \((\mathbb{S}^3, \text{HK}_\Gamma^\circ)\) from Propositions 2.10 and 2.11. \(\Box\)

**Corollary 5.2.** If \((\mathbb{S}^3, \Gamma)\) is irreducible and atoroidal, then any handlebody-knot \((\mathbb{S}^3, \text{HK}_\Gamma^\circ)\) obtained by a double looping of \((\mathbb{S}^3, \Gamma)\) is irreducible and atoroidal, and its exterior contains two non-isotopic type 2-2 annuli.

**Proof.** The two canonical annuli are of type 2-2 since any looping \((\mathbb{S}^3, \Gamma^\circ)\) is a spatial handcuff graph. The rest follows from Lemma 5.1. \(\Box\)

As an application of Lemma 5.1 and Corollary 5.2, we consider the spine \((\mathbb{S}^3, \Gamma)\) of \((\mathbb{S}^3, \mathcal{S}_2^3)\) as shown in Fig. 5.3a. Then Fig. 5.3b is a looping of \((\mathbb{S}^3, \Gamma)\) and its associated handlebody-knot has the annulus diagram \(\scriptstyle \kappa \bigcirc \bigcirc_{\text{2}}\); on the other hand, the double looping of \((\mathbb{S}^3, \Gamma)\) in Fig. 5.3c induces a handlebody-knot whose annulus diagram is \(\bigcirc_{\text{1}} \bigcirc_{\text{2}} \bigcirc_{\text{2}}\).

![Figure 5.3. Handlebody-knots with unknotting type 2 annuli.](image)

(a) Spine of \((\mathbb{S}^3, \mathcal{S}_2^3)\). (b) Looping of \((\mathbb{S}^3, \mathcal{S}_2^3)\). (c) Double looping of \((\mathbb{S}^3, \mathcal{S}_2^3)\).

### 5.2. Unknotting annuli of type two.

In this subsection, we construct handlebody-knots whose exterior contain an **unknotting** annulus of type 2, in contrast to Lemma 5.1 and Corollary 5.2.

Let \((\mathbb{S}^3, \Gamma)\) be a spatial \(\theta\)-graph that is a union of a non-trivial knot \((\mathbb{S}^3, K)\) and a tunnel \(\tau\) of \((\mathbb{S}^3, K)\). Let \(k_1, k_2\) be the arcs of \(K\) cut off by \(\tau\). Then a **tunnel** looping of \((\mathbb{S}^3, K \cup \tau)\) is a looping obtained by looping \(k_i\tau\) at a trivalent node of \(\Gamma = K \cup \tau\), \(i = 1\) or 2.

**Lemma 5.3.** The handlebody-knot \((\mathbb{S}^3, \text{HK}_\Gamma^\circ)\) induced by a tunnel looping of \((\mathbb{S}^3, \Gamma)\) is irreducible and atoroidal, and \(E(\text{HK}_\Gamma^\circ)\) admits an unknotting type 2-1 annulus.

**Proof.** It follows from the construction itself and Proposition 2.10 since \((\mathbb{S}^3, K)\) is non-trivial. \(\Box\)
To produce unknotting type 2-2 annuli, we let \((S^3, \Gamma)\) be the union of a non-split link \((S^3, L)\) and a tunnel \(\tau\) of \((S^3, L)\).

**Lemma 5.4.** The handlebody-knot \((S^3, HK^n)\) induced by a looping of \((S^3, \Gamma)\) is irreducible and atoroidal, and \(E(HK^n)\) admits an unknotting type 2-2 annulus.

**Proof.** It follows from Proposition 2.11 since \((S^3, L)\) is non-split. \(\square\)

Now, to show that all annulus diagrams in Theorem 4.19 can be realized by some handlebody-knots, we consider first the union of an \((n, 2)\)-torus link \((S^3, L_n = l_1 \cup l_2), n \in \mathbb{Z}\), with a tunnel \(\tau\) as depicted in Fig. 5.4a. Denote by \((S^3, HK_n)\) the handlebody-knot induced by the looping of \((S^3, L_n \cup \tau)\) in Fig. 5.4b. Note that \((S^3, HK_2)\) is equivalent to \((S^3, 4_1)\), while \(\{HK_n\}_{n \geq 2}\) is Koda’s handlebody-knot family in [19, Example 3]; in particular, Lemmas 5.3 and 5.4 give an alternative way to see they are all irreducible.

![Figure 5.4](image)

(a) \((n, 2)\)-torus link \((S^3, L_n)\) with a tunnel \(\tau\).  
(b) (Tunnel) looping of \((S^3, L_n \cup \tau)\).

**Corollary 5.5.** Suppose \(n > 2\) and is even. Then the annulus diagram of the handlebody-knot \((S^3, HK_n)\) obtained by the looping of \((S^3, L_n \cup \tau)\) in Fig. 5.4b is \(2\hbar_k(h)\).

![Figure 5.5](image)

(a) Union of \((S^3, L_n)\) with \(n\) odd and a tunnel \(\tau\).  
(b) Looping of \((S^3, L_n \cup \tau)\).

Next we consider the union of the 2-component link \((S^3, L_n)\) with \(n\) odd and the tunnel \(\tau\) in Fig. 5.5a. Then the looping of \((S^3, L_n \cup \tau)\) in Fig. 5.5b induces a handlebody-knot \((S^3, HK_n)\) whose exterior contains a type 3-2i annulus given by the cabling annulus of the \((n, 2)\)-torus knot component of \((S^3, L_n)\), so we have the following.
Corollary 5.6. The annulus diagram of the handlebody-knot \((S^3, HK_n)\) obtained by the looping of \((S^3, L_n \cup \tau)\) in Fig. 5.5b is \(\bigotimes_n^{k_2}\).

Lastly, to produce handlebody-knots with the annulus diagram \(2h_2k_1\), we recall that given a handlebody-knot \((S^3, HK)\) with a type 2-2 annulus \(A \subset E(HK)\), the loops \(l_1, l_2\) determines a spine \(\Gamma_A\) of \(HK_A\). Denote by \(l_1, l_2\) the constituent loops in \(\Gamma_A\) with \(l_2\) disjoint from a disk in \(HK_A\) bounded by \(l_+\). Then we have the following criterion for the non-uniqueness of \(A \subset E(HK)\).

Lemma 5.7. (1) Suppose \(E(HK)\) contains a type 3-2 annulus \(A'\). Then \(\ell_k(l_1, l_2) \neq \pm 1\).

(2) Suppose \(E(HK)\) contains a type 2-2 annulus \(A'\) not isotopic to \(A\), and \((S^3, l_1)\) is a trivial knot. Then \((S^3, l_1 \cup l_2)\) is either a trivial link or a Hopf link.

Proof. (1) Let \(W \subset E(HK)\) be the solid torus cut off by \(A'\), and \(l_w\) an oriented core of \(W\). Note that the core \(l_w' \subset \partial W\) of \(A'\) is a \((p, q)\)-curve with respect to \((S^3, l_w)\) with \(|q| > 1\) since \(A' \subset E(HK)\) is essential. Orient \(l_1, l_2\) so that \([l_2] = [l_1'] \in H_1(HK_A)\). Then if the linking number \(\ell_k(l_1, l_2) = m\), the linking number \(\ell_k(l_1, l_2) = qm \neq \pm 1\).

(2) Observe first that \((S^3, l_2)\) is trivial by the existence of \(A'\). Therefore, \((S^3, l_1 \cup l_2)\) is trivial if it is split. Suppose it is non-split. Then the disk \(D \subset E(l_2)\) induced by \(A'\) meets \(l_1\) at exactly one point. Denote by \(W\) the exterior \(E(l_2) - \mathfrak{N}(D)\), and observe that, since \((S^3, l_1)\) is trivial, the ball-arc pair \((W, l_1 \cap W)\) is trivial, so \((S^3, l_1 \cup l_2)\) is a Hopf link. □

Consider now the handcuff graph given by the union of the 2-component link \((S^3, L_n)\) with \(n\) odd and the tunnel \(\tau\) in Fig. 5.6a.

Corollary 5.8. The handlebody-knot induced by the looping of \((S^3, L_n \cup \tau)\) in Fig. 5.6b is irreducible and atoroidal with the annulus diagram \(\bigotimes_n^{2h_2k_1}\).

Proof. It follows from Lemmas 5.4 and 5.7 since the linking number of \((S^3, L_n)\) is \(\pm 1\), and it is not a Hopf link, for every odd \(n\). □

The handlebody-knots induced by the loopings in Figs. 5.3b, 5.3c, 5.4b, 5.5b and 5.6b imply the following.

Proposition 5.9. Annulus diagrams in Theorem 4.19 can all be realized by some handlebody-knots.
6. Handlebody-knot symmetries and geometric realization

In this section, we compute the symmetry groups of handlebody-knots whose exteriors contain a type 2 annulus, based on the classification in Theorem 4.19. We start by fixing the notation and reviewing some properties regarding mapping class groups.

6.1. Mapping class group. Given subpolyhedra $X_1, \ldots, X_n$ of a manifold $M$, the space of self-homeomorphisms of $M$ preserving $X_i$, $i = 1, \ldots, n$, setwise (resp. pointwise) is denoted by

\[
\text{Homeo}(M, X_1, \ldots, X_n) \quad \text{(resp. Homeo(\rel X_1, \ldots, X_n))}
\]

and the mapping class group of $(M, X_1, \ldots, X_n)$ is defined as

\[
\MCG(M, X_1, \ldots, X_n) := \pi_0(\text{Homeo}(M, X_1, \ldots, X_n))
\]

(respectively \[
\MCG(\rel X_1, \ldots, X_n) := \pi_0(\text{Homeo}(\rel X_1, \ldots, X_n))
\].

The “+” subscript is added when considering only orientation-preserving homeomorphisms:

\[
\text{Homeo}^+\,(M, X_1, \ldots, X_n) \quad \text{(resp. Homeo}^+\,(\rel X_1, \ldots, X_n))
\]

\[
\MCG^+\,(M, X_1, \ldots, X_n) \quad \text{(resp. MCG}^+\,(\rel X_1, \ldots, X_n))
\].

Given $f \in \text{Homeo}(M, X_1, \cdots, X_n)$, $[f]$ denotes the mapping class it represents. In the case $M = \mathbb{S}^3$, the mapping class group is also called the symmetry group. Below we recall some properties of mapping class groups of surfaces and three-manifolds.

Lemma 6.1 (Cutting Homomorphism, \cite{[6] Proposition 3.20}). Let $\Sigma$ be a closed surface and $\alpha_1, \ldots, \alpha_n$ mutually disjoint and non-homotopic simple loops in $\Sigma$. Then there is a well-defined homomorphism

\[
\text{cut} : \MCG_\Sigma([\alpha_1], \ldots, [\alpha_n]) \to \MCG_\Sigma(\Sigma - \gamma(\alpha_1 \cup \cdots \cup \alpha_n))
\]

whose kernel is generated by the Dehn twists about $\alpha_1, \ldots, \alpha_n$, where the group

\[
\MCG_\Sigma([\alpha_1], \ldots, [\alpha_n])
\]

is the subgroup of $\MCG_\Sigma(\Sigma)$ given by homeomorphisms that preserve the isotopy classes of $\alpha_1, \ldots, \alpha_n$, respectively.

Then next two lemmas are proved in \cite{[5]} and \cite{[8]} (see also \cite{[28] Remark 2.1])

Lemma 6.2 (\cite{[5]} Lemma 2.3). If $(\mathbb{S}^3, \text{HK})$ is atoroidal, then

\[
\MCG_\Sigma(E(\text{HK}), \rel \partial E(\text{HK})) \approx \{1\}.
\]

Lemma 6.3 (\cite{[8]}). $(\mathbb{S}^3, \text{HK})$ is irreducible and atoroidal if and only if $\MCG(\mathbb{S}^3, \text{HK})$ is finite.

Lemma 6.4. Let $(W, w)$ be a solid torus with boundary pattern, where $w = [G_1, G_2, \ldots, G_n]$ with $G_i$, $i = 1, \ldots, n$, all annuli and $\partial w = \partial W$. Suppose $f \in \text{Homeo}_\Sigma(W, G_1, \ldots, G_n)$ does not swap the components of $\partial G_1$—which holds automatically when $n > 2$. Then $f$ is isotopic to id in $\text{Homeo}_\Sigma(W, G_1, \ldots, G_n)$.

Proof. Without loss of generality, it may be assumed that $G_i \cap G_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Denote by $U_k$ the union $G_1 \cup \cdots \cup G_k$ and set $U_0 = \emptyset$. Observe that, if $f|_{U_{k-1}} = \text{id}$, $1 \leq k \leq n$, then $f$ can be isotoped in

\[
\text{Homeo}_\Sigma(W, G_1, \ldots, G_n, \rel U_{k-1}),
\]

so that $f|_{U_k} = \text{id}$. To see this, we first isotope $f|_{U_k}$ to id in $\text{Homeo}(U_k, \rel U_{k-1})$ as follows: In the case $k = 1$, it results from the assumption that $f$ does not swap components of $\partial G_1$, whereas if $1 < k < n$, it follows from the fact that $\MCG(U_k, \rel U_{k-1}) = \{1\}$. If $k = n$, it is a consequence of $f$ sending meridian disks of $W$ to themselves. Via a regular neighborhood of $U_k$ in $W$, the isotopy of $f|_{U_k}$ can be extended to an isotopy in $\text{Homeo}_\Sigma(W, G_1, \ldots, G_n)$ that isotopes $f$ so that
\( f_{|\delta} = \text{id}. \) Hence by induction, we may assume \( f \in 'Homeo(W,\partial\delta W) \), and the assertion follows since \( \text{MCG}(W,\partial W) \simeq \{1\}. \)

**Lemma 6.5.** Let \( W \) be a solid torus and \( A \subset \partial W \) an annulus with \( H_1(A) \to H_1(W) \) non-trivial and not an isomorphism. Then \( \text{MCG}(W,A) \simeq \text{MCG}_s(W,A) \simeq \mathbb{Z}_2. \)

**Proof.** Identify \( W \) with the subspace of \( \mathbb{C}^2 \)

\[
(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = re^{i\theta}, \quad \sqrt{2-r^2}e^{i\phi}, \quad -\pi \leq \theta, \phi \leq \pi, \quad 0 \leq r \leq 1,
\]

and \( A \subset W \) with

\[
[z_1 = e^{i(qx+s)}, z_2 = e^{i(qx-s)}, -\pi \leq t \leq \pi, -\epsilon \leq s \leq \epsilon],
\]

where \( p, q \) are coprime integers with \( q > 0 \) and \( 0 < \epsilon < \frac{\pi}{q} \). Let \( c_A, c_W \) be the cores of \( A, W \) given by \( s = 0, r = 0 \), respectively, and orient them so that \([c_A] = q[c_W] \in H_1(W)\) By the assumption, \( q \neq 0 \) or \( \pm 1 \), and therefore \( f(c_A, c_W) \) is non-trivial and every homeomorphism \( f \) of \((W,A)\) either preserves the orientations of both \( c_A, c_W \) or reverses them. This implies \( f \) is orientation-preserving.

On the other hand, the conjugation

\[
c : (z_1, z_2) \in W \mapsto (\bar{z}_1, \bar{z}_2) \in W
\]

preserves \( A \) but swaps its boundary components, so it is is non-trivial in \( \text{MCG}_s(W,A) \). By Lemma 6.4, it generates the entire group since any homeomorphism \( f \in 'Homeo(W,A) \) either swap components of \( \partial A \) or preserve them.

**Lemma 6.6.** Let \( W \) be a solid torus and \( A_1, A_2 \subset \partial W \) two disjoint annuli with \( H_1(A_i) \to H_1(W), i = 1, 2, \) isomorphisms. Then \( \text{MCG}_s(W, A_1 \cup A_2) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( \text{MCG}(W, A_1 \cup A_2) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \)

**Proof.** Identify \( W \) with \( Q \times S^1 \subset \mathbb{R}^2 \times \mathbb{C} \), where \( S^1 \) is the unit circle \([z = e^{i\theta}]\) and \( Q \) is the square given by

\[
\{(x, y) \mid -1 \leq x, y \leq 1\}.
\]

Identify \( A_1, A_2 \) with the annuli given by \( y = \pm 1 \), and their cores \( c_1, c_2 \) with the loops given by \( x = 0 \), and denote by \( B_1, B_2 \) the annuli in the closure of \( W - A \).

Consider \( r_i \in 'Homeo_s(W, A_1 \cup A_2), i = 1, 2, \) defined by the assignments:

\[
Q \times S^1 \to Q \times S^1 \setminus 1
\]

\[
(x, y, z) \mapsto (-x, -y, z),
\]

\[
(x, y, z) \mapsto (-x, y, \bar{z})
\]

respectively. Note that \( r_1, r_2 \) both are of order 2 and commute with each other. In addition, \( r_1 \) swaps \( A_1, A_2 \) and also \( B_1, B_2 \), whereas \( r_2 \) swaps \( A_1, A_2 \) but preserves \( B_1, B_2 \), so their composition \( r_1 \circ r_2 \) swaps \( B_1, B_2 \) but preserves \( A_1, A_2 \). Since every \( f \in 'Homeo(W, A_1 \cup A_2) \) either swaps \( A_1, A_2 \) (resp. \( B_1, B_2 \)) or preserves them. By Lemma 6.4 \([(r_1, [r_2])\) generates \( \text{MCG}_s(W, A_1 \cup A_2) \).

To see \( \text{MCG}(W, A_1 \cup A_2) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \) consider \( m \in 'Homeo(W, A_1 \cup A_2) \) defined by the assignment

\[
Q \times S^1 \to Q \times S^1 \setminus 1
\]

\[
(x, y, z) \mapsto (-x, y, z),
\]

which is orientation-reversing, commutes with \( r_i, i = 1, 2 \), and together with \( r_i, i = 1, 2 \), generates \( \text{MCG}(W, A_1 \cup A_2) \).

**Lemma 6.7.** Let \( W \) be a solid torus and \( A_1, A_2, A_3 \subset \partial W \) three disjoint annuli with \( H_1(A_i) \to H_1(W), i = 1, 2, 3, \) isomorphisms. Then \( \text{MCG}_s(W, A_1 \cup A_2 \cup A_3) \simeq \mathbb{Z}_2 \) and \( \text{MCG}(W, A_1 \cup A_2 \cup A_3) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2. \)
Proof. Identify $W$ with $\mathcal{H} \times S^1 \subset \mathbb{C} \times \mathbb{C}$, where $S^1 \subset \mathbb{C}$ is the unit circle, and $\mathcal{H} \subset \mathbb{C}$ the regular hexagon with center at origin and vertices $v_k = \frac{e^{2\pi i k}}{2}$, $k = 1, \ldots, 6$. Identify $A_i$ with the product of the edge $e_k$ connecting $v_{2k-1}$, $v_{2k}$ and $S^1$, $k = 1, 2, 3$. Denote by $r \in \text{Homeo}_*(W, A_1, A_2, A_3)$ the homeomorphism given by

$$\mathcal{H} \times S^1 \rightarrow \mathcal{H} \times S^1$$

$$(z_1, z_2) \mapsto (-\bar{z}_1, \bar{z}_2);$$

$r$ swaps $A_2, A_3$ and hence represents a non-trivial mapping class in $\text{MC}_+(W, A_1, A_2, A_3)$. Since every $f \in \text{Homeo}_*(W, A_1, A_2, A_3)$ either swaps $A_2, A_3$ or preserves them, by Lemma 6.4 either $[f] = [r]$ or $[f]$ is trivial, so $\text{MC}_+(W, A_1, A_2, A_3) \cong \mathbb{Z}_2$. On the other hand, there is an orientation-reversing homeomorphism $m \in \text{Homeo}(W, A_1, A_2, A_3)$ defined by

$$\mathcal{H} \times S^1 \rightarrow \mathcal{H} \times S^1$$

$$(z_1, z_2) \mapsto (z_1, \bar{z}_2),$$

which is of order 2 and commutes with $r$, and they together generate $\text{MC}(W, A_1, A_2, A_3)$.

The next lemma follows from [11, Section 2] (see also [12, Theorem 1]).

Lemma 6.8. Given a handlebody-knot $(\mathbb{S}^3, \text{HK})$ and an essential surface $S$ in $E(\text{HK})$, the natural homomorphisms

$$\text{MC}(\mathbb{S}^3, \text{HK}, S) \rightarrow \text{MC}(\mathbb{S}^3, \text{HK})$$

$$\text{MC}(\mathbb{S}^3, \text{HK}, \partial(S)) \rightarrow \text{MC}(\mathbb{S}^3, \text{HK})$$

are injective.

6.2. Symmetry groups of handlebody-knots. Throughout the subsection, $(\mathbb{S}^3, \text{HK})$ is an irreducible, atoroidal handlebody-knot, and $A$ is a type 2 annulus in $E(\text{HK})$. Recall from Section 2, $l_1, l_2$ denote the boundary components of $A$ with $l_2$ bounding a disk in $\text{HK}$. Also, we identify the intersection $\partial(A) \cap \partial HK$ with $\partial(l \cup l_3) = \partial(l) \cup \partial(l_3)$.

Theorem 6.9. If $A$ is of type 2-1, then $\text{MC}_+(\mathbb{S}^3, \text{HK}) < \mathbb{Z}_2$ and $\text{MC}(\mathbb{S}^3, \text{HK}) < \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Note first that the injection $\text{MC}_+(\mathbb{S}^3, \text{HK}, \partial(A)) \rightarrow \text{MC}_+(\mathbb{S}^3, \text{HK})$ in Lemma 6.8 is an isomorphism since $A$ is unique by Theorem 4.19 composing its inverse with the homomorphism $\text{MC}_+(\mathbb{S}^3, \text{HK}, \partial(A)) \xrightarrow{\Phi} \text{MC}_+(\partial(A), A_+ \cup A_-)$ given by restriction to $\partial(A)$ yields the homomorphism

$$\text{MC}_+(\mathbb{S}^3, \text{HK}) = \text{MC}_+(\mathbb{S}^3, \text{HK}, \partial(A)) \rightarrow \text{MC}_+(\partial(A), A_+ \cup A_-).$$

By Lemma 6.6, it then suffices to show the injectivity of $\Phi$ as it entails the injectivity of

$$\text{MC}_+(\mathbb{S}^3, \text{HK}, \partial(A)) \rightarrow \text{MC}(\partial(A), A_+ \cup A_-).$$

To see $\Phi$ is injective, let $[f] \in \text{MC}_+(\mathbb{S}^3, \text{HK}, \partial(A))$ with $\Phi([f]) = 1$. This implies $f|_{\partial HK | l \cup l_3}$ does not permute punctures of the four-times punctured sphere $\partial HK - \partial(l \cup l_3)$, and thus $[f|_{\partial HK | l \cup l_3}] = 1 \in \text{MC}_+(\partial HK - \partial(l \cup l_3))$ since $[f|_{\partial HK | l \cup l_3}]$ is of finite order by Lemma 6.3. Similarly, $[f|_{\partial HK}]$ is of finite order in $\text{MC}_+(\partial HK, [l], [l_3])$, and hence by Lemma 6.1 it is the identity. Because $f|_{\partial HK}$ is isotopic to id, $f$ can be isotoped in $\text{Homeo}(\mathbb{S}^3, \text{HK})$ so that $f|_{\partial HK} = \text{id}$. Applying Lemma 6.2 one can further isotope $f$ to id in $\text{Homeo}(\mathbb{S}^3, \text{rel} \partial HK)$.

Theorem 6.10. If $A \subset E(\text{HK})$ is the unique type 2-2 annulus, then $\text{MC}_+(\mathbb{S}^3, \text{HK}) = \{1\}$ and $\text{MC}(\mathbb{S}^3, \text{HK}) < \mathbb{Z}_2$. If in addition $E(\text{HK})$ admits an annulus $A'$ of another type, then $\text{MC}(\mathbb{S}^3, \text{HK}) \cong \text{MC}_+(\mathbb{S}^3, \text{HK}) = \{1\}$. 
Proof. As in the previous case, the uniqueness of \( A \) gives us the homomorphism
\[
\text{MCG}_4(\mathbb{S}^3, \partial \text{HK}) \rightarrow \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}) \rightarrow \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}).
\]
The first assertion follows from the injectivity of \( \Phi \) because a homeomorphism \( f \in \text{Homeo}_4(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}) \) can neither swap \( A_i, A_j \) or swap \( \partial \text{HK} \), \( \partial \text{HK} \) by the definition of a type 2-2 annulus. On the other hand, the second assertion can be derived from the first as follows: by Theorem 4.19 \( A' \) is the unique type 3-2 annulus in \( E(\text{HK}) \). Let \( W \subset E(\text{HK}) \) be the solid torus cut off by \( A' \). Then by the essentiality of \( A, H_l(A) \rightarrow H_l(W) \) is non-trivial and not an isomorphism. On the other hand, by Lemma 6.8 there is a homomorphism
\[
\text{MCG}(\mathbb{S}^3, \partial \text{HK}) \rightarrow \text{MCG}(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}) \rightarrow \text{MCG}(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK} \rightarrow \text{MCG}(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}).
\]
Now, if \( \text{MCG}(\mathbb{S}^3, \partial \text{HK}) \) is non-trivial, then by the first assertion, \( \text{MCG}(\mathbb{S}^3, \partial \text{HK}) \) contains a mapping class represented by an orientation-reversing homeomorphism, contradicting Lemma 6.5.

It therefore suffices to show \( \Phi \) is injective. Let \( [\partial \text{HK}] \in \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}) \) is of finite order, so \( [\partial \text{HK}] \in \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}) \) is also of finite order. The group \( \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}), \partial \text{HK} \) is isomorphic to \( \text{id} \) in \( \text{Homeo}_4(\mathbb{S}^3, \partial \text{HK}) \). We may then isotope \( f \) in \( \text{Homeo}_4(\mathbb{S}^3, \partial \text{HK}) \) so that \( \partial \text{HK} = \text{id} \). By Lemma 6.2, \( f \) may be further isotoped to \( \text{id} \) in \( \text{Homeo}_4(\mathbb{S}^3, \partial \text{HK}) \).

Theorem 6.11. If \( A \subset E(\text{HK}) \) is of type 2-2 but not the unique type 2-2 annulus, then \( \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}) < \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Proof. By Theorem 4.19, \( E(\text{HK}) \) admits a unique type 3-3 annulus \( A_0 \), and exactly two non-isotopic annuli \( A, A' \), which cut off a solid torus \( W \subset E(\text{HK}) \) and form a characteristic surface of \( E(\text{HK}) \); this together with Lemma 6.8 gives us the homomorphism
\[
\text{MCG}_4(\mathbb{S}^3, \partial \text{HK}) \rightarrow \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}) \rightarrow \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}).
\]

Let \( f \in \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}) \) and \( \Phi(f) = 1 \in \text{MCG}_4(\mathbb{S}^3, \partial \text{HK}, \partial \text{HK}) \). Note that \( \partial \text{HK} \cap W \) consists of three annuli \( B_0, B_1, B_2 \); denote by \( c_0, c_1, c_2 \) their cores, respectively.

Since \( \Phi(f) = 1 \), \( f_{\partial \text{HK}}(c_0, c_1, c_2) \) does not permute punctures of \( \partial \text{HK} \setminus (B_0 \cup B_1 \cup B_2) \), which is two copies of the three-times punctured sphere, and therefore \( f_{\partial \text{HK}}(c_0, c_1, c_2) \in \text{MCG}_4(\partial \text{HK} \setminus (B_0 \cup B_1 \cup B_2)) \). On the other hand by Lemma 6.3, \( f_{\partial \text{HK}} \) is of finite order in \( \text{MCG}_4(\partial \text{HK}, \partial \text{HK}) \), and hence is trivial therein by Lemma 6.1 in particular, \( f_{\partial \text{HK}} \) is isotopic to \( \text{id} \) in \( \text{Homeo}_4(\partial \text{HK}) \). We can thereby isotope \( f \) in \( \text{Homeo}_4(\mathbb{S}^3, \partial \text{HK}) \) so that \( f_{\partial \text{HK}} = \text{id} \); applying Lemma 6.2, we may further isotope \( f \) in \( \text{id} \) in \( \text{Homeo}_4(\mathbb{S}^3, \partial \text{HK}) \).

References

1. G. Bellettini, M. Paolini, Y.-S. Wang: A complete invariant for connected surfaces in the 3-sphere, J. Knot Theory Ramifications 29 (2020), 1950091.
2. G. Bellettini, M. Paolini, Y.-S. Wang: Numerical irreducibility criteria for handlebody links, Topology Appl. 284 (2020), 107361.
[3] F. Bonahon: Geometric structure on 3-manifolds, Handbook of Geometric Topology, edited by R.J. Daverman and R.B. Sher, Elsevier (2001), 93–164.

[4] R. D. Canary, D. McCullough: Homotopy Equivalences of 3-manifold and Deformation Theory of Kleinian Groups, Mem. Amer. Math. Soc. 172 (2004).

[5] S. Cho, Y. Koda: Topological symmetry groups and mapping class groups for spatial graphs, Michigan Math. J. 62 (2013), 131–142.

[6] B. Farb, D. Margalit: A Primer on Mapping Class Groups, Princeton University Press, (2011).

[7] R. Frigerio, C. Petronio: Construction and recognition of hyperbolic 3-manifolds with geodesic boundary Trans. Amer. Math. Soc. 356 (2004), 3243–3282.

[8] K. Funayoshi, Y. Koda: Extending automorphisms of the genus-2 surface over the 3-sphere, Q. J. Math. 71 (2020), 175–196.

[9] D. Gomez: The fundamental group of the punctured Klein bottle and the simple loop conjecture, Graduate J. Math. 2 (2017), 59–65.

[10] C. Gordon: On primitive sets of loops in the boundary of a handlebody, Topology Appl. 27 (1987), 285–299.

[11] A. Hatcher: Homeomorphisms of sufficiently large P^2-irreducible 3-manifolds, Topology 15 (1976) 343–347.

[12] A. Hatcher: Spaces of incompressible surfaces, arXiv:math/9906074 [math.GT].

[13] W. H. Holzmann: An equivariant torus theorem for involutions, Trans. Amer. Math. Soc. 326 (1991), 887–906.

[14] A. Ishii, K. Kishimoto: The quandle coloring invariant of a reducible handlebody-knot, Tsukuba J. Math. 35 (2011), 131–141.

[15] A. Ishii, K. Kishimoto, H. Moriuchi, M. Suzuki: A table of genus two handlebody-knots up to six crossings, J. Knot Theory Ramifications 21(4), (2012) 1250035.

[16] K. Johannson: Homotopy Equivalences of 3-Manifolds with Boundaries, Lecture Notes in Math. 761, Springer, Berlin, Heidelberg, 1979.

[17] M. Kapovich: Hyperbolic Manifolds and Discrete Groups, Modern Birkhäuser Classics, Birkhäuser Boston, 2010.

[18] Y. Koda, M. Ozawa, with an appendix by C. Gordon: Essential surfaces of non-negative Euler characteristic in genus two handlebody exteriors, Trans. Amer. Math. Soc. 367 (2015), no. 4, 2875–2904.

[19] Y. Koda: Automorphisms of the 3-sphere that preserve spatial graphs and handlebody-knots, Math. Proc. Cambridge Philos. Soc. 159 (2015), 1–22.

[20] J. H. Lee, S. Lee: Inequivalent handlebody-knots with homeomorphic complements, Algebr. Geom. Topol. 12 (2012), 1059–1079.

[21] M. Motto: Inequivalent genus two handlebodies in $S^3$ with homeomorphic complements, Topology Appl. 36, (1990), 283–290.

[22] W. D. Neumann, G. A. Swarup: Canonical decompositions of 3-manifolds, Geom. Topol. 1 (1997), 21–40.

[23] S. Okazaki: An invariant derived from the Alexander polynomial for handlebody-knots, Osaka J. Math. 57 (2020), 737–750.

[24] J.-P. Otal: Le théorème d’hyperbolisation pour les variétés fibrées de dimension 3, Astérisque 235, Société Mathématique de France, Paris, 1996.

[25] H. Seifert: Topologie dreidimensionaler gefaserter Räume, Acta Math. 60 (1933), 147–288.

[26] W. P. Thurston: Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 6 (1982), 357–381.

[27] J. L. Tollefson: Involutions of sufficiently large 3-manifolds, Topology 20 (1981), 323–352.

[28] Y.-S. Wang: Unknotting annuli and handlebody-knot symmetry, Topology Appl. 305 (2021), 107884.

[29] Y.-S. Wang: Rigidity and symmetry of cylindrical handlebody-knots, to appear in Osaka J. Math.

[30] H. Zieschang: On simple systems of paths on complete pretzels, Amer. Math. Soc. Transl., 92 (1970), 127–137.

Institute of Mathematics, Academia Sinica, Taipei City 106, Taiwan
Email address: yisheng@gate.sinica.edu.tw