Refined regularity analysis for a Keller-Segel-consumption system involving signal-dependent motilities

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\textbf{ABSTRACT}

We consider the Keller-Segel-type migration-consumption system involving signal-dependent motilities,

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta (u\phi(v)), \\
\frac{\partial v}{\partial t} &= \Delta v - uv,
\end{align*}
\]

in smoothly bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Under the assumption that $\phi \in C^1([0,\infty))$ is positive on $[0,\infty)$, and for nonnegative initial data from $(C^0(\overline{\Omega}))^* \times L^\infty(\Omega)$, previous literature has provided results on global existence of certain very weak solutions with possibly quite poor regularity properties, and on large time stabilization toward semitrivial equilibria with respect to the topology in $(W^{1,2}(\Omega))^* \times L^\infty(\Omega)$.

The present study reveals that solutions in fact enjoy significantly stronger regularity features when $0 < \phi \in C^3([0,\infty))$ and the initial data belong to $(W^{1,\infty}(\Omega))^2$: It is firstly shown, namely, that then in the case $n \leq 2$ an associated no-flux initial-boundary value problem even admits a global classical solution, and that each of these solutions smoothly stabilizes in the sense that as $t \to \infty$ we have

\[
\begin{align*}
\frac{\partial u}{\partial t} &\to \frac{1}{|\Omega|} \int_{\Omega} u_0 \quad \text{and} \quad \frac{\partial v}{\partial t} \to 0 \quad (\star)
\end{align*}
\]
even with respect to the norm in $L^\infty(\Omega)$ in both components.

In the case when $n \geq 3$, secondly, some genuine weak solutions are found to exist globally, inter alia satisfying $\nabla u \in L^{4/3}_{\text{loc}}(\overline{\Omega} \times [0,\infty); \mathbb{R}^n)$. In the particular three-dimensional setting, any such solution is seen to become eventually smooth and to satisfy $(\star)$.

\section{Introduction}

In the macroscopic modeling of bacterial migration influenced by local sensing mechanisms, a prominent role is played by parabolic systems of the form

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta (u\phi(v)), \\
\frac{\partial v}{\partial t} &= \Delta v + f(u,v)
\end{align*}
\] (1)

[1–3]. By featuring a particular link between diffusion and cross-diffusion in the migration operator determining its first equation, (1) can be considered a special case of more general chemotaxis systems...
of Keller-Segel type \([4]\); indeed, suitably making use of this characteristic structure has been underlying substantial parts of the existing literature on subclasses of \((1)\). In the case when \(f(u, v) = u - v\), for instance, within large classes of nonlinearities \(\phi\) an appropriate exploitation of this core property has been forming the respective key in a considerable collection of studies concerned with basic solution theories for the corresponding chemotaxis-production version of \((1)\) \([5-10]\), as well as some close variants \([11-18,35,36]\); beyond this, results on spatially structured large-time behavior, and even on the occurrence of infinite-time blow-up \([5,9,19-21]\), provide conclusive evidence for significant structure-supporting features of such representatives of \((1)\) that were already observed in numerical simulations \([7]\).

In comparison to this, the corresponding chemotaxis-consumption counterpart, as given by

\[
\begin{align*}
  u_t &= \Delta (u\phi(v)), \\
  v_t &= \Delta v - uv,
\end{align*}
\]

\((2)\)

appears to exhibit a significantly weaker tendency toward pattern generation: Here the underlying hypothesis that the considered signal is consumed by cells, rather than produced, does not only go along with an evident nonexistence of inhomogeneous steady states; indeed, as has recently been shown in \([22]\), indeed, when determined by strictly positive migration rate functions \(\phi \in C^1([0, \infty))\), the taxis-absorption interaction in \((2)\) even enforces certain global very weak solutions, known to exist for all initial data from \((C^0(\Omega))^* \times L^\infty(\Omega)\) \([22]\); cf. also \([23]\), to approach the semi-trivial equilibrium \((\frac{1}{|\Omega|} \int_{\Omega} u_0, 0)\) in the topology of \((W^{1,2}(\Omega))^* \times L^\infty(\Omega)\) in the large time limit when \(n \leq 3\) (see \([22]\) and also Lemma 2.1 below); nontrivial large time dynamics in \((2)\) seems possible only in degenerate cases in which \(\phi(0) = 0\) \([24,25]\).

**Main results.** The objective of this manuscript is to undertake a refined regularity analysis for solutions of \((2)\) in said non-degenerate setting involving strictly positive \(\phi\); in fact, according to the mild assumptions on regularity of \(\phi\) and the initial data made in \([22]\), only quite poor regularity information on the global solutions constructed there seems available so far: The corresponding first components, for instance, are merely known to belong to \(L^\infty((0, \infty); L^1(\Omega)) \cap L^2_{loc}(\Omega \times (0, \infty))\), and to be continuous on \([0, \infty)\) as \(C^0(\Omega))^*\)-valued functions with respect to the weak topology therein; in particular, this does not rule out the emergence of transient singularities, and due to the absence of integrability information on \(\nabla u\), this regularity class is far too large to let these objects play the role of genuine weak solutions.

Our results will inter alia reveal that under moderately stronger assumptions than those from \([22]\), particularly still not relying on any requirement on \(\phi\) that goes beyond smoothness and positivity, the problem \((2)\) actually admits global classical solutions when \(n \leq 2\), and eventually smooth weak solutions when \(n = 3\), and that in both these cases the obtained solutions smoothly stabilize. In this regard, \((2)\) will thus be seen to share essential features of blow-up exclusion with the related classical chemotaxis-consumption system with its first equation, \(u_t = \Delta u - \nabla \cdot (u \nabla v)\), reflecting interplay between linear diffusion and standard Keller-Segel type cross-diffusion \([26]\).

Indeed, for the initial-boundary value problem

\[
\begin{align*}
  u_t &= \Delta (u\phi(v)), & x \in \Omega, \ t > 0, \\
  v_t &= \Delta v - uv, & x \in \Omega, \ t > 0, \\
  \nabla (u\phi(v)) \cdot v &= \nabla v \cdot v = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x), & v(x, 0) = v_0(x), & x \in \Omega,
\end{align*}
\]

\((3)\)

the first of our main results addresses the corresponding low-dimensional framework:

**Theorem 1.1:** Let \(n \in \{1, 2\}\) and \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary, and suppose that

\[
\phi \in C^3([0, \infty)) \text{ is such that } \phi(\xi) > 0 \text{ for all } \xi \geq 0,
\]

\((4)\)
and that
\[
\begin{align*}
  u_0 &\in W^{1,\infty}(\Omega) \text{ is nonnegative with } u_0 \not\equiv 0, \quad \text{and} \\
  v_0 &\in W^{1,\infty}(\Omega) \text{ is nonnegative}. 
\end{align*}
\] (5)

Then one can find nonnegative functions
\[
\begin{align*}
  u &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \quad \text{and} \\
  v &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) 
\end{align*}
\] (6)
such that \((u, v)\) solves (3) in the classical sense. Moreover,
\[
\begin{align*}
  u(:, t) &\to \frac{1}{|\Omega|} \int_{\Omega} u_0 \quad \text{in } L^\infty(\Omega) \quad \text{and} \\
  v(:, t) &\to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty. 
\end{align*}
\] (7)

Under the same hypotheses, our result concerning the higher-dimensional version of (3) can be stated as follows.

**Theorem 1.2:** Let \( n \geq 3 \) and \( \Omega \subset \mathbb{R}^n \) be a smoothly bounded domain, and assume (4) and (5). Then one can find
\[
\begin{align*}
  u &\in L^2_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \cap L^\frac{4}{n}_{\text{loc}}([0, \infty); W^{1,\frac{4}{n}}(\Omega)) \quad \text{and} \\
  v &\in L^\infty(\Omega \times (0, \infty)) \cap L^2_{\text{loc}}([0, \infty); W^{2,2}(\Omega)) 
\end{align*}
\] (8)
such that \( u \geq 0 \) and \( v \geq 0 \) a.e. in \( \Omega \times (0, \infty) \), and that \((u, v)\) forms a global weak solution of (3) in the sense that
\[
\begin{align*}
  -\int_0^\infty \int_{\Omega} u \phi_t - \int_{\Omega} u_0 \phi(\cdot, 0) = -\int_0^\infty \int_{\Omega} \nabla (u \phi(v)) \cdot \nabla \phi 
\end{align*}
\] (9)
and
\[
\begin{align*}
  \int_0^\infty \int_{\Omega} v \phi_t + \int_{\Omega} v_0 \phi(\cdot, 0) = \int_0^\infty \int_{\Omega} \nabla v \cdot \nabla \phi + \int_{\Omega} uv \phi 
\end{align*}
\] (10)
for all \( \phi \in C^\infty_0(\overline{\Omega} \times [0, \infty)) \).

If moreover \( n = 3 \), then there exists \( T > 0 \) such that after re-definition of \((u, v)\) on a null set we can achieve that
\[
\begin{align*}
  u &\in C^{2,1}(\overline{\Omega} \times (T, \infty)) \quad \text{and} \\
  v &\in C^{2,1}(\overline{\Omega} \times (T, \infty)), 
\end{align*}
\] (11)
and that (7) holds.

**Remark.** As has already been said, in their final shape the findings of this paper parallel some results known for the classical chemotaxis-consumption system given by
\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (u \nabla v), \\
  v_t &= \Delta v - uv. 
\end{align*}
\] (12)
[26], both revealing the explosion-inhibiting effects in the sense that when \( n \leq 2 \), (3) and (12) are classically solved by global solutions enjoying comparable convergence property (7), while in three-dimensional frameworks only global weak solutions are found to exist, and each of these solutions becomes eventually smooth with stabilization properties consistent with the corresponding low-dimensional setting. However, thanks to the peculiar diffusion-taxis interplay included in the first equation of (3), (3) gets an edge over (12) by featuring a spatio-temporal a priori estimate for \( u \), as shown in Lemma 2.2, which is uniformly true for all spatial dimensions but is apparently absent for
$n \geq 3$ for (12) and which ultimately leads to the global existence of weak solutions to (3) in higher-dimensional cases, the latter thereby slightly setting the existing solution theories of (3) apart from those of (12).

**Main ideas.** The starting point of our analysis consists in making appropriate use of the non-degenerate migration operator contained in the first equation in (3) to derive a spatio-temporal a priori estimate for $u$, which is accomplished by a duality-based argument (Lemma 2.2), thereby facilitating the establishment of certain regularity information on $v$ (Lemma 2.3).

In light of these estimates, in the lower-dimensional setting of Theorem 1.1, boundedness of $u$ in $L^p(\Omega)$ for any $p \geq 1$ can be established through a straightforward testing procedure (Lemma 3.1), providing the first step in a bootstrap-type argument that involves standard regularity theory for parabolic equations and yields smoothness of solutions and the stabilization result recorded in Theorem 1.1 (Lemmata 3.2–3.5).

For spatially higher-dimensional versions of (3), another application of the estimates obtained in Section 2 leads to time-independent bounds for expressions of the form $\int_\Omega u \ln u$, an estimate for $\nabla u$ (Lemma 4.1), which imply that the limit function $(u, v)$ constructed in [22] actually is a global weak solution as claimed in Theorem 1.2. Furthermore, in the three-dimensional case an eventual smallness property of $\|v\|_{L^\infty(\Omega)}$ (Lemma 4.2) facilitates an analysis of functionals of the form $\int_\Omega \frac{u^p(x)}{(\delta - v(x))^\kappa}$, with arbitrary $p > 1$ and suitably chosen $\delta = \delta(p) > 0$ and $\kappa = \kappa(p) > 0$, to establish $L^p$ bounds for $u$ (Lemma 4.3). Subsequent higher order regularity analysis (Lemmata 4.4–4.6) will lead to the conclusion of Theorem 1.2.

2. Preliminaries

In order to appropriately regularize (3), let us consider the approximate variants of (3) given by

\[
\begin{align*}
\frac{u_\varepsilon}{x} &= \Delta (u_\varepsilon \phi (v_\varepsilon)), \\
\frac{v_\varepsilon}{x} &= \Delta v_\varepsilon - \frac{u_\varepsilon v_\varepsilon}{1 + \varepsilon u_\varepsilon}, \\
\frac{\partial u_\varepsilon}{\partial v} &= \frac{\partial v_\varepsilon}{\partial v} = 0, \\
u_\varepsilon (x, 0) &= u_0(x), \quad v_\varepsilon (x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

for $\varepsilon \in (0, 1)$. Then in the context of actually less restrictive assumptions on $\phi$ and the initial data, the following collection of findings extracted from [22, Theorems 1.1 and 1.2 and Lemma 2.2] will be needed for our subsequent analysis. In formulating this, given a Banach space $X$, we let $C^{0}_{\text{w*-}}([0, \infty); X)$ denote the space of functions that are continuous on $[0, \infty)$ with respect to the weak-$\star$ topology in $X$.

**Lemma 2.1:** Let $\varepsilon \in (0, 1)$. Then there exist nonnegative functions

\[
\begin{align*}
u_\varepsilon &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \quad \text{and} \\
\nu_\varepsilon &\in \bigcap_{q > n} C^0([0, \infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))
\end{align*}
\]

such that $(u_\varepsilon, v_\varepsilon)$ solves (13) in the classical sense, and that

\[
\int_{\Omega} u_\varepsilon (\cdot, t) = \int_{\Omega} u_0 \quad \text{for all } t > 0
\]

as well as

\[
\|v_\varepsilon (\cdot, t)\|_{L^\infty(\Omega)} \leq \|v_\varepsilon (\cdot, t_0)\|_{L^\infty(\Omega)} \quad \text{for all } t_0 \geq 0 \quad \text{and} \quad t > t_0.
\]
Moreover, there exist \((e_j)_{j \in \mathbb{N}} \subset (0, 1)\) as well as nonnegative functions
\[
\begin{aligned}
u &\in C_w^0([0, \infty); (C^0(\overline{\Omega}))^*) \cap L^\infty((0, \infty); L^1(\Omega)) \cap L^2_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \\
\end{aligned}
\]
(17)
such that \(e_j \searrow 0\) as \(j \to \infty\), that
\[
u_e \to \nu \quad \text{and} \quad \nu_e \to \nu \quad \text{a.e. in } \Omega \times (0, \infty)
\]
(18)
as \(\varepsilon = e_j \searrow 0\), and that \((u, \nu)\) forms a global very weak solution of \((3)\) in the sense that \(u(\cdot, 0) = u_0\) in \((C^0(\overline{\Omega}))^*\) and \(v(\cdot, 0) = v_0\) in \(L^\infty(\Omega)\), and that for each \(\varphi \in C^0_0(\overline{\Omega} \times (0, \infty))\) fulfilling \(\frac{\partial \varphi}{\partial \nu} = 0\) on \(\partial \Omega \times (0, \infty)\) we have
\[
\begin{array}{l}
-\int_0^\infty \int_\Omega u \varphi_t = \int_0^\infty \int_\Omega u \varphi(\nu) \Delta \varphi \\
\end{array}
\]
(19)
and
\[
\begin{array}{l}
-\int_0^\infty \int_\Omega v \varphi_t = \int_0^\infty \int_\Omega v \Delta \varphi - \int_0^\infty \int_\Omega uv \varphi.
\end{array}
\]
(20)
Finally, if \(n \leq 3\) then there exists a null set \(N \subset (0, \infty)\) such that
\[
u(\cdot, t) \to \frac{1}{|\Omega|} \int_\Omega \nu_0 \quad \text{in } (W^{1,2}(\Omega))^* \quad \text{and} \quad v(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } (0, \infty) \setminus N \ni t \to \infty.
\]
(21)
A yet fairly elementary but crucial regularity feature of the first solution component can be obtained by means of a duality-based reasoning in the spirit of a precedent contained, e.g. in [10].

**Lemma 2.2:** There exists \(C > 0\) such that
\[
\int_{(t-1)^+}^t \int_\Omega u_e^2 \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1).
\]
(22)

**Proof:** Following [10], we let \(A\) denote the realization of \(-\Delta + 1\) under homogeneous Neumann boundary conditions in \(L^2(\Omega)\), and multiply the identity \(\partial_t A^{-1} u_e + u_e \phi(v_e) = A^{-1}(u_e \phi(v_e))\) by \(u_e\) to see that by self-adjointness of \(A^{-1}\), and by Young’s inequality,
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left| A^{-\frac{1}{2}} u_e \right|^2 + \int_\Omega u_e^2 \phi(v_e)
\]
\[
= \int_\Omega u_e A^{-1} (u_e \phi(v_e))
\]
\[
= \int_\Omega u_e \phi(v_e) A^{-1} u_e
\]
\[
\leq \frac{1}{2} \int_\Omega u_e^2 \phi(v_e) + \frac{1}{2} \int_\Omega \phi(v_e) \left| A^{-1} u_e \right|^2 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1).
\]
\[
\]
Since (4) together with (16) yields \(c_1 > 0\) and \(c_2 > 0\) such that \(c_1 \leq \phi(v_e) \leq c_2\) in \(\Omega \times (0, \infty)\) for all \(\varepsilon \in (0, 1)\), this implies that \(y_e(t) := \int_\Omega |A^{-\frac{1}{2}} u_e(\cdot, t)|^2, t \geq 0, \varepsilon \in (0, 1),\) satisfies
\[
y_e'(t) + y_e(t) + c_1 \int_\Omega u_e^2 \leq c_2 \int_\Omega |A^{-1} u_e|^2 + \int_\Omega \left| A^{-\frac{1}{2}} u_e \right|^2 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1).
\]
(23)
Here we use that elliptic regularity theory and the compactness of the embedding \(W^{2,2}(\Omega) \hookrightarrow L^2(\Omega)\) ensure compactness of the operator \(A^{-1}\), and hence also of the fractional powers \(A^{-\vartheta}\) for any \(\vartheta > 0\).
Accordingly, \( L^2(\Omega) \) is compactly embedded into both domains \( D(A^{-1}) \) and \( D(A^{-\frac{1}{2}}) \), so that by means of associated Ehrling inequalities and (15) we can readily find \( c_3 > 0 \) and \( c_4 > 0 \) such that

\[
c_2 \int_{\Omega} |A^{-1} u_e|^2 + \int_{\Omega} |A^{-\frac{1}{2}} u_e|^2 \leq \frac{c_1}{2} \int_{\Omega} u_e^2 + c_3 \left( \int_{\Omega} u_e \right)^2 \leq \frac{c_1}{2} \int_{\Omega} u_e^2 + c_4 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1),
\]

so that from (23) we obtain that

\[
y'_e(t) + y_e(t) + \frac{c_1}{2} \int_{\Omega} u_e^2 \leq c_4 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1).
\]

Therefore, \( y_e(t) \leq c_5 := \max[\int_{\Omega} |A^{-\frac{1}{2}} u_0|^2, c_4] \) for all \( t \geq 0 \) and \( \varepsilon \in (0, 1) \), and thus

\[
\frac{c_1}{2} \int_{t-(t-1)+}^{t} u_e^2 \leq (t-1+) + c_4 \leq c_5 + c_4
\]

for all \( t > 0 \) and \( \varepsilon \in (0, 1) \).

As a first application, when combined with a standard testing procedure the above yields some basic regularity information on \( v_e \), as the previous estimate available without any restriction on the spatial dimension.

**Lemma 2.3:** There exists \( C > 0 \) such that

\[
\int_{\Omega} |\nabla v_e(\cdot, t)|^2 \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1),
\]

and

\[
\int_{(t-1)+}^{t} \int_{\Omega} |\Delta v_e|^2 \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1)
\]

as well as

\[
\int_{(t-1)+}^{t} \int_{\Omega} |\nabla v_e|^4 \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1)
\]

and

\[
\int_{(t-1)+}^{t} \int_{\Omega} v_{e,t}^2 \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1).
\]

**Proof:** According to the second equation in (13), Young’s inequality and (16),

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v_e|^2 + \int_{\Omega} |\Delta v_e|^2 = \int_{\Omega} \frac{u_e v_e}{1 + \varepsilon u_e} \Delta v_e
\]

\[
\leq \frac{1}{2} \int_{\Omega} |\Delta v_e|^2 + \frac{1}{2} \int_{\Omega} \left( \frac{u_e v_e}{1 + \varepsilon u_e} \right)^2
\]

\[
\leq \frac{1}{2} \int_{\Omega} |\Delta v_e|^2 + \frac{c_1^2}{2} \int_{\Omega} u_e^2 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1),
\]
where \(c_1 := \|v_0\|_{L^\infty(\Omega)}\). Since the Gagliardo-Nirenberg inequality combined with elliptic regularity theory (cf. [27, Lemma 3.3]) provides \(c_2 > 0\) such that again by (16) we have

\[
\int \Omega |\nabla v_\varepsilon|^4 \leq c_2 \|\Delta v_\varepsilon\|^2_{L^2(\Omega)} \|v_\varepsilon\|^2_{L^\infty(\Omega)} \leq c_1^2 c_2 \int \Omega |\Delta v_\varepsilon|^2 \quad \text{for all } t > 0 \quad \text{and } \varepsilon \in (0, 1),
\]

and since thus also

\[
\int \Omega |\nabla v_\varepsilon|^2 \leq \frac{1}{4} \int \Omega |\Delta v_\varepsilon|^2 + c_1^2 c_2 |\Omega| \quad \text{for all } t > 0 \quad \text{and } \varepsilon \in (0, 1)
\]

thanks to Young’s inequality, this implies that

\[
y_\varepsilon(t) := \int \Omega |\nabla v_\varepsilon(\cdot, t)|^2, \quad t \geq 0, \; \varepsilon \in (0, 1),
\]

as well as

\[
g_\varepsilon(t) := \frac{1}{2} \int \Omega |\Delta v_\varepsilon(\cdot, t)|^2 + \frac{1}{4c_1^2 c_2} \int \Omega |\nabla v_\varepsilon(\cdot, t)|^4 \quad \text{and}
\]

\[
h_\varepsilon(t) := c_1^2 \int \Omega u_\varepsilon^2(\cdot, t) + c_1^2 c_2 |\Omega|, \quad t \geq 0, \; \varepsilon \in (0, 1),
\]

satisfy

\[
y_\varepsilon'(t) + y_\varepsilon(t) \leq \left\{ - \int \Omega |\Delta v_\varepsilon|^2 + c_1^2 \int \Omega u_\varepsilon^2 \right\} + \left\{ \frac{1}{4} \int \Omega |\Delta v_\varepsilon|^2 + c_1^2 c_2 |\Omega| \right\}
\]

\[
\leq -\frac{1}{2} \int \Omega |\Delta v_\varepsilon|^2 - \frac{1}{4} \int \Omega |\Delta v_\varepsilon|^2 + h_\varepsilon(t)
\]

\[
\leq -g_\varepsilon(t) + h_\varepsilon(t) \quad \text{for all } t > 0 \quad \text{and } \varepsilon \in (0, 1). \quad (28)
\]

As from Lemma 2.2 we obtain \(c_3 > 0\) such that

\[
\int_{(t-1)+}^t h_\varepsilon(s) \, ds \leq c_3 \quad \text{for all } t > 0 \quad \text{and } \varepsilon \in (0, 1), \quad (29)
\]

using that

\[
\int_0^t e^{-\lambda(t-s)} h_\varepsilon(s) \, ds \leq \frac{1}{1 - e^{-\lambda}} \cdot \sup_{s \geq 0} \int_{(s-1)+}^s h_\varepsilon(\sigma) \, d\sigma \quad \text{for all } t > 0, \; \varepsilon \in (0, 1) \quad \text{and } \lambda > 0
\]

(cf. [28, Lemma 3.4]), from (28) we infer that \(y_\varepsilon(t) \leq c_4 := \int \Omega |\nabla v_0|^2 + \frac{c_1}{1 - e^{-\lambda}}\) for all \(t \geq 0\) and \(\varepsilon \in (0, 1)\), and that hence, again by (29),

\[
\int_{(t-1)+}^t g_\varepsilon(s) \, ds \leq y_\varepsilon ((t - 1) +) + \int_{(t-1)+}^t h_\varepsilon(s) \, ds \leq c_4 + c_3 \quad \text{for all } t > 0 \quad \text{and } \varepsilon \in (0, 1).
\]

In view of the definitions of \((y_\varepsilon)_{\varepsilon \in (0, 1)}\) and \((g_\varepsilon)_{\varepsilon \in (0, 1)}\), these inequalities establish both (24) and (25)–(26), whereupon (27) immediately results upon estimating \(v_{\varepsilon t}^2 \leq 2|\Delta v_\varepsilon|^2 + 2u_{\varepsilon t}^2 v_\varepsilon^2\) for \(\varepsilon \in (0, 1)\), and combining (25) with Lemma 2.2 and (16). \(\blacksquare\)
3. The case \( n \leq 2 \). Proof of Theorem 1.1

In this section, the global classical solvability for problem (3) as well as large time convergence of solutions is examined in the case when \( n \leq 2 \). The following lemma already contains the main step toward this, establishing \( L^p \) bounds for \( u_\varepsilon \) on the basis of the spatio-temporal \( L^4 \) estimate for \( \nabla v_\varepsilon \) provided by (26):

**Lemma 3.1:** Let \( n \leq 2 \). Then for all \( p > 1 \) there exists \( C(p) > 0 \) such that

\[
\int_\Omega u_\varepsilon^p(\cdot, t) \leq C(p) \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1).
\]  

**Proof:** We detail the proof only for \( n = 2 \), and remark that the one-dimensional case can be addressed by very minor modification of the argument.

Using the first equation in (13) along with Young’s inequality, we then see that

\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u_\varepsilon^p = -(p-1) \int_\Omega u_\varepsilon^{p-2} \phi(v_\varepsilon) |\nabla u_\varepsilon|^2 - (p-1) \int_\Omega u_\varepsilon^{p-1} \phi'(v_\varepsilon) \nabla u_\varepsilon \cdot \nabla v_\varepsilon
\]

\[
\leq -\frac{p-1}{2} \int_\Omega u_\varepsilon^{p-2} \phi(v_\varepsilon) |\nabla u_\varepsilon|^2 + \frac{p-1}{2} \int_\Omega u_\varepsilon^p \frac{\phi^2(v_\varepsilon)}{\phi(v_\varepsilon)} |\nabla v_\varepsilon|^2 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1),
\]

whence, again combining (4) with (16), we obtain \( c_1 = c_1(p) > 0 \) and \( c_2 = c_2(p) > 0 \) such that

\[
\frac{d}{dt} \int_\Omega u_\varepsilon + c_1 \int_\Omega |\nabla u_\varepsilon|^2 \leq c_2 \int_\Omega |\nabla v_\varepsilon|^2 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1).
\]

Here by the Cauchy-Schwarz inequality, the Gagliardo-Nirenberg inequality, (15) and Young’s inequality, we find that with some positive constants \( c_i = c_i(p), i \in \{3, 4, 5, 6\} \), we have

\[
c_2 \int_\Omega u_\varepsilon^p |\nabla v_\varepsilon|^2 \leq c_2 \|\nabla v_\varepsilon\|^2_{L^4(\Omega)} \|u_\varepsilon^p\|^2_{L^2(\Omega)}
\]

\[
\leq c_3 \|\nabla v_\varepsilon\|^2_{L^4(\Omega)} \|\nabla u_\varepsilon^p\|^2_{L^2(\Omega)} \|u_\varepsilon^p\|^2_{L^2(\Omega)} + c_3 \|\nabla v_\varepsilon\|^2_{L^4(\Omega)} \|u_\varepsilon^p\|^2_{L^2(\Omega)}
\]

\[
\leq c_3 \|\nabla v_\varepsilon\|^2_{L^4(\Omega)} \|\nabla u_\varepsilon^p\|^2_{L^2(\Omega)} \|u_\varepsilon^p\|^2_{L^2(\Omega)} + c_4 \|\nabla v_\varepsilon\|^2_{L^4(\Omega)}
\]

\[
\leq \frac{c_1}{2} \int_\Omega |\nabla u_\varepsilon^p|^2 + c_5 \|\nabla v_\varepsilon\|^4_{L^4(\Omega)} \int_\Omega u_\varepsilon^p + c_4 \|\nabla v_\varepsilon\|^2_{L^4(\Omega)}
\]

\[
\leq \frac{c_1}{2} \int_\Omega |\nabla u_\varepsilon^p|^2 + c_6 \cdot \left\{ \int_\Omega |\nabla v_\varepsilon|^4 + 1 \right\} \cdot \left\{ \int_\Omega u_\varepsilon^p + 1 \right\}
\]

for all \( t > 0 \) and \( \varepsilon \in (0, 1) \). Since we can similarly apply Young’s inequality, the Gagliardo-Nirenberg inequality and (15) to fix \( c_7 = c_7(p) > 0 \) and \( c_8 = c_8(p) > 0 \) such that

\[
\left\{ \int_\Omega u_\varepsilon^p + 1 \right\} \leq 2 \frac{1}{p-1} \|u_\varepsilon^p\|_{L^2(\Omega)}^{2p} + 2 \frac{1}{p-1}
\]

\[
\leq c_7 \|\nabla u_\varepsilon^p\|^2_{L^2(\Omega)} \|u_\varepsilon^p\|_{L^2(\Omega)}^{2p} + c_7 \|u_\varepsilon^p\|_{L^2(\Omega)}^{2p} + 2 \frac{1}{p-1}
\]
from (31) and (32) we thus infer that for
\[ y_\epsilon(t) := \int_{\Omega_1} u_\epsilon^p(x,t) + 1, \quad t \geq 0, \quad \epsilon \in (0,1), \quad \text{and} \]
\[ h_\epsilon(t) := c_6 \int_{\Omega_1} |\nabla v_\epsilon(x,t)|^4 + c_6 + \frac{c_1}{2}, \quad t > 0, \quad \epsilon \in (0,1), \]
we have
\[ y_\epsilon'(t) + \frac{c_1}{2c_8} y_\epsilon^{p-1}(t) \leq h_\epsilon(t) y_\epsilon(t) \quad \text{for all } t > 0 \quad \text{and} \quad \epsilon \in (0,1). \]

As \( \sup_{\epsilon \in (0,1)} \sup_{t>0} \int_{(t-1)^+}^t h_\epsilon(s) \, ds \) is finite thanks to Lemma 2.3, from Lemma A.2 we directly obtain (30) with some suitably large \( C(p) > 0 \), because \( \frac{p}{p-1} > 1 \).

Combining the latter with well-known smoothing estimates for the heat semigroup, we can derive uniform boundedness of \( \nabla v_\epsilon \).

**Lemma 3.2:** Let \( n \leq 2 \). Then there exists \( C > 0 \) such that
\[ \| \nabla v_\epsilon(\cdot,t) \|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \epsilon \in (0,1). \] (33)

**Proof:** This follows from Lemma 3.1 when applied to any fixed \( p > 2 \) and combined with (16) and standard smoothing estimates for the Neumann heat semigroup on \( \Omega \) [29].

This in turn enables us to improve our knowledge on the first solution components:

**Lemma 3.3:** When \( n \leq 2 \), one can find \( C > 0 \) with the property that
\[ \| u_\epsilon(\cdot,t) \|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \epsilon \in (0,1). \] (34)

**Proof:** On the basis of Lemmata 3.1, 3.2, (4) and (16), this can be obtained by means of a Moser-type iterative argument, as recorded in [30, Lemma A.1].

As a direct consequence of standard parabolic regularity theory, we next obtain Hölder bounds for \( u_\epsilon \) and \( v_\epsilon \).

**Lemma 3.4:** Let \( n \leq 2 \). Then there exist \( \theta \in (0,1) \) and \( C > 0 \) such that
\[ \| u_\epsilon \|_{C^{\theta, \frac{\theta}{2}}(\Omega \times [t,t+1])} \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \epsilon \in (0,1), \] (35)
and that
\[ \| v_\epsilon \|_{C^{\theta, \frac{\theta}{2}}(\Omega \times [t,t+1])} \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \epsilon \in (0,1). \] (36)

**Proof:** For \( \epsilon \in (0,1) \), rewriting the first equation in (13) as \( u_{\epsilon t} - \nabla \cdot (\phi(v_\epsilon) \nabla u_\epsilon + f_\epsilon(x,t)) = 0, \quad x \in \Omega, \quad t > 0 \), with \( f_\epsilon(x,t) := \phi'(v_\epsilon(x,t)) u_\epsilon(x,t) \nabla v_\epsilon(x,t) \), from (4), (16) as well as Lemmata 3.2 and 3.3, we can find \( c_1 > 0, c_2 > 0 \) and \( c_3 > 0 \) such that \( c_1 \leq \phi'(v_\epsilon) \leq c_2 \) and \( |f_\epsilon| \leq c_3 \), which, by (5) and an application of [31, Theorem 1.3, Remark 1.3 and 1.4], implies (35). By means of a similar argument, we may also prove (36).
The derivation of higher order Hölder bounds, locally in time, thereupon becomes straightforward as well.

**Lemma 3.5**: If \( n \leq 2 \), then for all \( \tau > 0 \) and any \( T > \tau \) there exist \( \theta = \theta(\tau, T) \in (0, 1) \) and \( C(\tau, T) > 0 \) such that

\[
\| u_\varepsilon \|_{C^{2+\theta} \times [\tau, T]} \leq C(\tau, T) \quad \text{for all } \varepsilon \in (0, 1),
\]

and

\[
\| v_\varepsilon \|_{C^{2+\theta} \times [\tau, T]} \leq C(\tau, T) \quad \text{for all } \varepsilon \in (0, 1).
\]

**Proof**: For any \( \varepsilon \in (0, 1) \) and \( \tau > 0 \), letting \( v_\varepsilon := \zeta_\tau v_\varepsilon \), we infer from the second equation in (13) that \( v_\varepsilon \) solves

\[
\tilde{v}_{\varepsilon t} = \Delta \tilde{v}_\varepsilon - \frac{u_\varepsilon \tilde{v}_\varepsilon}{1 + \varepsilon u_\varepsilon} + \zeta_\tau' \tilde{v}_\varepsilon,
\]

in \( \Omega \times (0, \infty) \), where \( \zeta_\tau \) is a smooth cut-off function such that \( \zeta_\tau(t) = 0 \) if \( 0 \leq t \leq \frac{\tau}{2} \), \( \zeta_\tau(t) = 1 \) if \( t \geq \tau \), and, moreover, \( ||\zeta_\tau||_{W^{1,\infty}(0,\infty)} \leq 1 + \frac{\tau}{2} \) with \( c_1 > 0 \). Based on Lemma 3.4, the estimate in (38) follows from parabolic Schauder theory [32, Theorem 5.3] applied to (39) and the definition of \( \zeta_\tau \). Having (38) at hand, we can thereupon employ the same token again to readily derive (37), again relying on (4) and (16).

Our main result on the existence of global classical solutions to the one- and two-dimensional versions of (3), as well as their large-time behavior, can now be established by using the boundedness properties collected above with the approximation and stabilization results known from Lemma 2.1.

**Proof of Theorem 1.1**: As a consequence of Lemmata 3.4, 3.5 and the Arzelà-Ascoli theorem, possibly after modification of the functions \( u \) and \( v \) from Lemma 2.1 on a null set we can achieve that in (18), with \( (\varepsilon_j) \in \mathbb{N} \) as provided there we have

\[
u_\varepsilon \rightarrow v \quad \text{in } C_{\text{loc}}^0(\Omega \times [0, \infty)) \quad \text{and} \quad v_\varepsilon \rightarrow v \quad \text{in } C_{\text{loc}}^{2,1}(\Omega \times (0, \infty)) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0,
\]

and that hence \((u, v)\) has the regularity features in (6) and solves (3) classically. Since (40) together with (35) ensures that \((u(\cdot, t))_{t>0}\) is bounded in \( C^\theta(\Omega) \) with some \( \theta \in (0, 1) \), and since due to the boundedness of \( \Omega \), we know that the first of the two continuous embeddings \( C^\theta(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow (W^{1,2}(\Omega))^* \) is compact, once more drawing on an Ehrling-type inequality we infer that writing \( \overline{u}_0 := \lim_{T \rightarrow \tau} \int_\Omega u_0 \) we see that for each \( \eta > 0 \) there exists \( c_1(\eta) > 0 \) fulfilling

\[
\| u(\cdot, t) - \overline{u}_0 \|_{L^\infty(\Omega)} \leq \eta \| u(\cdot, t) - \overline{u}_0 \|_{C^\theta(\Omega)} + c_1(\eta) \| u(\cdot, t) - \overline{u}_0 \|_{(W^{1,2}(\Omega))^*} \quad \text{for all } t \in (0, \infty).
\]

Therefore, by (21) and passing to the limit as \((0, \infty) \setminus N \ni t \rightarrow \infty\), with \( N \) as provided in Lemma 2.1, we conclude that \( u(\cdot, t) \rightarrow \overline{u}_0 \) in \( L^\infty(\Omega) \) as \((0, \infty) \setminus N \ni t \rightarrow \infty\), which in conjunction with the density of \((0, \infty) \setminus N \ni t \rightarrow \infty\) and the continuity of \( u \) in \( \overline{\Omega} \times (0, \infty) \), the latter guaranteed by (40), readily yields

\[
u(\cdot, t) \rightarrow \overline{u}_0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \rightarrow \infty.
\]

Combined with a similar argument for the second solution component, this shows that also (7) is valid.
4. The case $n \geq 3$. Proof of Theorem 1.2

4.1. A space-time $L^4$ estimate for $\nabla u_\varepsilon$

We now turn our attention to the case $n \geq 3$, in which due to lacking favorable embeddings, any expedient analogue of Lemma 3.1 seems absent. To show that nevertheless the pair $(u, v)$ obtained in Lemma 2.1 solves (3) actually in a standard weak sense, we use the $L^4$ estimate for $\nabla v_\varepsilon$ from Lemma 2.3 in the context of an $L \log L$ testing procedure, indeed leading to bounds for $\nabla u_\varepsilon$ in some reflexive $L^p$ space.

**Lemma 4.1:** Let $n \geq 1$. Then there exists $C \geq 0$ such that

$$\int_{\Omega} u_\varepsilon(\cdot, t) \ln u_\varepsilon(\cdot, t) \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1),$$

and that

$$\int_{(t-\varepsilon)}^{t} \int_{\Omega} |\nabla u_\varepsilon|^4 \leq C \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1).$$

**Proof:** Again by (13) and Young’s inequality,

$$\frac{d}{dt} \int_{\Omega} u_\varepsilon \ln u_\varepsilon = - \int_{\Omega} \phi(v_\varepsilon) \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} - \int_{\Omega} \phi'(v_\varepsilon) \nabla u_\varepsilon \cdot \nabla v_\varepsilon$$

$$\leq - \frac{c_1}{2} \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{c_2}{2} \int_{\Omega} |\nabla u_\varepsilon| \cdot |\nabla v_\varepsilon|$$

$$\leq - \frac{c_1}{2} \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{c_2^2}{4c_1} \int_{\Omega} u_\varepsilon |\nabla v_\varepsilon|^2$$

$$\leq - \frac{c_1}{2} \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{c_2^2}{4c_1} \int_{\Omega} u_\varepsilon^2 + \frac{c_2^2}{4c_1} \int_{\Omega} |\nabla v_\varepsilon|^4 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1),$$

where $c_1 := \inf_{\varepsilon \in (0, 1)} \inf_{\Omega \times (0, \infty)} \phi(v_\varepsilon) > 0$ and $c_2 := \sup_{\varepsilon \in (0, 1)} \sup_{\Omega \times (0, \infty)} |\phi'(v_\varepsilon)| < \infty$ by (4) and (16). As furthermore

$$- \frac{\ln u_\varepsilon}{e} \leq \int_{\Omega} u_\varepsilon \ln u_\varepsilon \leq \int_{\Omega} u_\varepsilon^2 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1)$$

due to the fact that $- \frac{1}{e} \leq \xi \ln \xi \leq \xi^2$ for all $\xi > 0$, we thus infer that

$$y_\varepsilon(t) := \int_{\Omega} u_\varepsilon(\cdot, t) \ln u_\varepsilon(\cdot, t) + \frac{\ln u_\varepsilon}{e}, \quad t \geq 0, \quad \varepsilon \in (0, 1),$$

as well as

$$g_\varepsilon(t) := \frac{c_1}{2} \int_{\Omega} \frac{|\nabla u_\varepsilon(\cdot, t)|^2}{u_\varepsilon(\cdot, t)} \quad \text{and} \quad h_\varepsilon(t) := \left( \frac{c_2^2}{4c_1} + 1 \right) \int_{\Omega} u_\varepsilon^2(\cdot, t) + \frac{c_2^2}{4c_1} \int_{\Omega} |\nabla v_\varepsilon(\cdot, t)|^4 + \frac{\ln u_\varepsilon}{e}.$$
are all nonnegative and satisfy
\[ y'_e(t) + y_e(t) + g_e(t) \leq h_e(t) \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1). \]

Since \( c_3 := \sup_{\varepsilon \in (0, 1)} \sup_{t > 0} \int_{(t-1)}^t h_e(s) \, ds \) is finite according to Lemmata 2.2 and 2.3, again in view of Lemma A.1 this implies that
\[ y_e(t) \leq \frac{c_3}{1 - \varepsilon^{-1}} + \int_{\Omega} u_0^2 + \frac{\mid \Omega \mid}{\varepsilon} \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0, 1), \]

hence
\[ \int_{(t-1)_+}^t \int_{\Omega} \frac{|\nabla u_e|^2}{u_e} = \frac{2}{c_1} \int_{(t-1)_+}^t g_e(s) \, ds \leq c_4 := \frac{2}{c_1} \cdot \left( \frac{c_3}{1 - \varepsilon^{-1}} + c_3 + \int_{\Omega} u_0^2 + \frac{\mid \Omega \mid}{\varepsilon} \right) \]

for all \( t > 0 \) and \( \varepsilon \in (0,1) \). While (43) directly yields (41), it is sufficient to derive (42) by employing Young’s inequality in estimating
\[ \int_{(t-1)_+}^t \int_{\Omega} |\nabla u_e|^2 = \int_{(t-1)_+}^t \int_{\Omega} \left( \frac{|\nabla u_e|^2}{u_e} \right)^{\frac{2}{3}} \cdot u_e^{\frac{2}{3}} \leq \int_{(t-1)_+}^t \int_{\Omega} \frac{|\nabla u_e|^2}{u_e} + \int_{(t-1)_+}^t \int_{\Omega} u_e^2 \quad \text{for all } t > 0 \quad \text{and} \quad \varepsilon \in (0,1), \]

and combining (44) with Lemma 2.2.

\[ \Box \]

4.2. Eventual bounds in the case \( n = 3 \)

Of crucial importance for our derivation of the eventual regularity features claimed in Theorem 1.2 will be the information on asymptotic smallness of \( v \) contained in (21). Together with an \( L^\infty \) approximation property implied by Lemma 2.3 in the three-dimensional case, this entails a doubly uniform ultimate smallness property of \( v_e \) in the following sense.

**Lemma 4.2:** Let \( n = 3 \), and let \( (\varepsilon_j)_{j \in \mathbb{N}} \) and \((u, v)\) be as in Lemma 2.1. Then there exists a subsequence \((\varepsilon_{j_k})_{k \in \mathbb{N}}\) of \((\varepsilon_j)_{j \in \mathbb{N}}\) with the property that whenever \( \eta > 0 \), one can find \( T(\eta) > 0 \) and \( \varepsilon_*(\eta) \in (0,1) \) such that
\[ \|v_e(\cdot, t)\|_{L^{\infty}(\Omega)} \leq \eta \quad \text{for all } t > T(\eta) \quad \text{and} \quad \varepsilon \in (\varepsilon_{j_k})_{k \in \mathbb{N}} \subset (0, \varepsilon_*(\eta)). \]

**Proof:** From Lemma 2.3 it follows that \((v_e)_{\varepsilon \in (0,1)}\) is bounded in \( L^4((0, T); W^{1,4}(\Omega)) \) and that \((v_{eT})_{\varepsilon \in (0,1)}\) is bounded in \( L^2(\Omega \times (0, T)) \) for all \( T > 0 \), so that since \( W^{1,4}(\Omega) \) is compactly embedded into \( L^{\infty}(\Omega) \) due to our assumption that \( n \leq 3 \), an Aubin-Lions lemma [33] along with (18) asserts that \( v_e \to v \) in \( L^4_{\text{loc}}([0, \infty); L^{\infty}(\Omega)) \) as \( \varepsilon = \varepsilon_j \searrow 0 \); for an appropriate null set \( N_1 \subset (0, \infty) \) and some subsequence \((\varepsilon_{j_k})_{k \in \mathbb{N}}\) of \((\varepsilon_j)_{j \in \mathbb{N}}\), it thus follows that
\[ v_e(\cdot, t) \to v(\cdot, t) \quad \text{in } L^{\infty}(\Omega) \quad \text{for all } t \in (0,\infty) \setminus N_1 \quad \text{as} \quad \varepsilon = \varepsilon_{j_k} \searrow 0. \]

Now if we fix \( \eta > 0 \) and let \( N \subset (0, \infty) \) be as in Lemma 2.1, in line with (21) we can pick \( T(\eta) \in (0, \infty) \setminus (N \cup N_1) \) such that \( \|v(\cdot, T(\eta))\|_{L^{\infty}(\Omega)} \leq \frac{\eta}{2} \), and since then \( T(\eta) \in (0, \infty) \setminus N_1 \), we may rely on (46) to choose \( \varepsilon_*(\eta) \in (0,1) \) in such a way that \( \|v_e(\cdot, T(\eta)) - v(\cdot, T(\eta))\|_{L^{\infty}(\Omega)} \leq \frac{\eta}{2} \) for all
\( \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \) such that \( \varepsilon < \varepsilon_*(\eta) \). Consequently,

\[
\| v_\varepsilon(\cdot, T(\eta)) \|_{L^\infty(\Omega)} \leq \| v_\varepsilon(\cdot, T(\eta)) \|_{L^\infty(\Omega)} + \| v(\cdot, T(\eta)) \|_{L^\infty(\Omega)}
\]

\[
\leq \frac{\eta}{2} + \frac{\eta}{2} = \eta \quad \text{for all} \quad \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*(\eta)),
\]

so that (45) results from the monotonicity property of \( 0 \rightarrow \| v_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \) for \( \varepsilon \in (0, 1) \), as expressed in (16).

By means of an argument dating back to [26], the key step toward our large time analysis in the three-dimensional case can now be accomplished through the study of the evolution of the functionals \( \int_\Omega u_\varepsilon^p(\delta - v_\varepsilon)^{-\kappa} \) with arbitrary \( p > 1 \) and suitably chosen \( \delta = \delta(p) > 0 \) and \( \kappa = \kappa(p) > 0 \), crucially based on the premise that \( v_\varepsilon \) is sufficiently small.

**Lemma 4.3:** Let \( n = 3 \), and let \( (\varepsilon_{jk})_{k \in \mathbb{N}} \) be as provided by Lemma 4.2. Then for each \( p > 1 \) there exist \( T(p) > 0, \varepsilon_*(p) \in (0, 1) \) and \( C(p) > 0 \) such that

\[
\int_\Omega u_\varepsilon^p(\cdot, t) \leq C(p) \quad \text{for all} \quad t > T(p) \quad \text{and} \quad \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \subset (0, \varepsilon_*(p)).
\]

**Proof:** Once more relying on (4) and (16), we fix \( c_1 > 0, c_2 > 0 \) and \( c_3 > 0 \) such that

\[
c_1 \leq \phi(v_\varepsilon) \leq c_2 \quad \text{and} \quad |\phi'(v_\varepsilon)| \leq c_3 \quad \text{in} \quad \Omega \times (0, \infty) \quad \text{for all} \quad \varepsilon \in (0, 1),
\]

and given \( p > 1 \) we then pick \( \kappa = \kappa(p) > 0 \) such that

\[
p(c_2 + 2)\kappa^2 \leq \frac{\kappa}{2}
\]

and choose \( \delta = \delta(p) > 0 \) small enough fulfilling both

\[
p c_3 \delta \leq \frac{1}{2} \quad \text{and} \quad (p - 1) c_3 \delta + \kappa \leq 2 \kappa.
\]

Thanks to Lemma 4.2, it is then possible to find \( T_1 = T_1(p) > 0 \) and \( \varepsilon_* = \varepsilon_*(p) \in (0, 1) \) such that

\[
v_\varepsilon(x, t) \leq \frac{\delta}{2} \quad \text{for all} \quad x \in \Omega, t > T_1 \quad \text{and} \quad \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*),
\]

and to make appropriate use of this, we recall (13) to compute

\[
\frac{d}{dt} \int_\Omega u_\varepsilon^p(\delta - v_\varepsilon)^{-\kappa}
\]

\[
= p \int_\Omega u_\varepsilon^{p-1}(\delta - v_\varepsilon)^{-\kappa} \Delta (u_\varepsilon \phi(v_\varepsilon)) + \kappa \int_\Omega u_\varepsilon^p(\delta - v_\varepsilon)^{-\kappa - 1} \cdot \{ \Delta v_\varepsilon - \frac{u_\varepsilon v_\varepsilon}{1 + \varepsilon u_\varepsilon} \}
\]

\[
= -p \int_\Omega \{(p - 1)u_\varepsilon^{p-2}(\delta - v_\varepsilon)^{-\kappa} \nabla u_\varepsilon + \kappa u_\varepsilon^{p-1}(\delta - v_\varepsilon)^{-\kappa - 1} \nabla v_\varepsilon \} \cdot \{ \phi(v_\varepsilon) \nabla u_\varepsilon + u_\varepsilon \phi'(v_\varepsilon) \nabla v_\varepsilon \}
\]

\[
- \kappa \int_\Omega \{ p u_\varepsilon^{p-1}(\delta - v_\varepsilon)^{-\kappa - 1} \nabla u_\varepsilon + (\kappa + 1) u_\varepsilon^p(\delta - v_\varepsilon)^{-\kappa - 2} \nabla v_\varepsilon \} \cdot \nabla v_\varepsilon
\]

\[
- \kappa \int_\Omega \frac{u_\varepsilon^{p+1} v_\varepsilon(\delta - v_\varepsilon)^{-\kappa - 1}}{1 + \varepsilon u_\varepsilon}
\]
\[
\begin{align*}
&= -p(p - 1) \int_{\Omega} u_e^{p-2} (\delta - v_e)^{-\kappa} \phi(v_e) |\nabla v_e|^2 \\
&\quad - \int_{\Omega} u_e^p \cdot \left\{ \kappa (\kappa + 1)(\delta - v_e)^{-\kappa - 2} + p \kappa (\delta - v_e)^{-\kappa - 1} \phi'(v_e) \right\} |\nabla v_e|^2 \\
&\quad - \int_{\Omega} u_e^{p-1} \cdot \left\{ p \kappa (\delta - v_e)^{-\kappa - 1} \phi(v_e) + p(p - 1)(\delta - v_e)^{-\kappa} \phi'(v_e) + p \kappa (\delta - v_e)^{-\kappa - 1} \right\} \\
&\quad \quad \quad \nabla u_e \cdot \nabla v_e \\
&\quad - \kappa \int_{\Omega} \frac{u_e^{p+1} v_e (\delta - v_e)^{-\kappa - 1}}{1 + \varepsilon u_e} \quad \text{for all } t > T_1 \quad \text{and } \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*). \quad (52)
\end{align*}
\]

Here by (48),
\[
p(p - 1) \int_{\Omega} u_e^{p-2} (\delta - v_e)^{-\kappa} \phi(v_e) |\nabla v_e|^2 \\
\geq p(p - 1)c_1 \int_{\Omega} u_e^{p-2} (\delta - v_e)^{-\kappa} |\nabla v_e|^2 \quad \text{for all } t > T_1 \quad \text{and } \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*), \quad (53)
\]

while due to (48) and (50),
\[
\kappa (\kappa + 1)(\delta - v_e)^{-\kappa - 2} + p \kappa (\delta - v_e)^{-\kappa - 1} \phi'(v_e) \\
\geq \kappa (\kappa + 1)(\delta - v_e)^{-\kappa - 2} - p \kappa c_3 (\delta - v_e)^{-\kappa - 1} \\
= \kappa (\delta - v_e)^{-\kappa - 2} \cdot \left\{ \kappa + 1 - pc_3 \cdot (\delta - v_e) \right\} \\
\geq \kappa (\delta - v_e)^{-\kappa - 2} \cdot \left\{ \kappa + 1 - pc_3 \delta \right\} \\
\geq \frac{\kappa}{2} (\delta - v_e)^{-\kappa - 2} \quad \text{in } \Omega \times (T_1, \infty) \quad \text{for all } \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*)
\]

and hence
\[
\int_{\Omega} u_e^p \cdot \left\{ \kappa (\kappa + 1)(\delta - v_e)^{-\kappa - 2} + p \kappa (\delta - v_e)^{-\kappa - 1} \phi'(v_e) \right\} |\nabla v_e|^2 \\
\geq \frac{\kappa}{2} \int_{\Omega} u_e^p (\delta - v_e)^{-\kappa - 2} |\nabla v_e|^2 \quad \text{for all } t > T_1 \quad \text{and } \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*). \quad (54)
\]

Moreover, again by (48) and (50),
\[
|p \kappa (\delta - v_e)^{-\kappa - 1} \phi(v_e) + p(p - 1)(\delta - v_e)^{-\kappa} \phi'(v_e) + p \kappa (\delta - v_e)^{-\kappa - 1}| \\
\leq p \kappa c_2 (\delta - v_e)^{-\kappa - 1} + p(p - 1)c_3 (\delta - v_e)^{-\kappa} + p \kappa (\delta - v_e)^{-\kappa - 1} \\
= p(\delta - v_e)^{-\kappa - 1} \cdot \left\{ \kappa c_2 + (p - 1)c_3 \cdot (\delta - v_e) + \kappa \right\} \\
\leq p(\delta - v_e)^{-\kappa - 1} \cdot \left\{ \kappa c_2 + (p - 1)c_3 \delta + \kappa \right\} \\
\leq p(c_2 + 2) \kappa (\delta - v_e)^{-\kappa - 1} \quad \text{in } \Omega \times (T_1, \infty) \quad \text{for all } \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*),
\]

so that thanks to Young’s inequality, for all \( t > T_1 \) and \( \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*) \) we have
\[
\begin{align*}
&- \int_{\Omega} u_e^{p-1} \cdot \left\{ p \kappa (\delta - v_e)^{-\kappa - 1} \phi(v_e) + p(p - 1)(\delta - v_e)^{-\kappa} \phi'(v_e) + p \kappa (\delta - v_e)^{-\kappa - 1} \right\} \nabla u_e \cdot \nabla v_e \\
&\leq p(c_2 + 2) \kappa \int_{\Omega} u_e^{p-1} (\delta - v_e)^{-\kappa - 1} |\nabla u_e| \cdot |\nabla v_e|
\end{align*}
\]
that, and here a superlinear absorptive summand can be created by observing that due to the Gagliardo- Nirenberg inequality and (15), using (51) and abbreviating \( a \equiv a(p) := \frac{3(p-1)}{2p-1} \in (0,1) \) we can find \( c_4 = c_4(p) > 0 \) and \( c_5 = c_5(p) > 0 \) such that

\[
\left\{ \int \Omega \left\| \nabla u^p - \nabla v^p \right\|^2 \right\}^{\frac{1}{p}} \\
\leq \left( \delta \right)^{-\frac{\kappa}{\alpha}} \| u^p \|_{L^2(\Omega)}^\frac{2}{\alpha} \\
\leq c_4 \| \nabla u^p \|_{L^2(\Omega)}^2 + c_5 \| u^p \|_{L^p(\Omega)}^\frac{2(1-a)}{\alpha} + c_4 \| u^p \|_{L^p(\Omega)}^\frac{2}{\alpha} \\
\leq c_5 \delta^a \int \Omega u^p \left( \delta - v^p \right)^{-\kappa} |\nabla u^p|^2 + c_5 \quad \text{for all } t > T_1 \quad \text{and } \eps \in (\eps_{kj})_{k \in N} \cap (0, \eps_*) \tag{56}
\]

Therefore, (56) implies that if we let \( c_6 \equiv c_6(p) := \frac{p(p-1)c_1}{2c_5\delta} \) and \( c_7 \equiv c_7(p) := \frac{p(p-1)c_1}{2b} \), then

\[
y''(t) + c_6 y^\frac{1}{2}(t) \leq c_7 \quad \text{for all } t > T_1 \quad \text{and } \eps \in (\eps_{kj})_{k \in N} \cap (0, \eps_*) \tag{57}
\]

so that since \( \frac{1}{\alpha} > 1 \), Lemma A.3 applies so as to assert that

\[
y^\frac{1}{a}(t) \leq \left( \frac{c_7}{c_6} \right)^a + \left( \frac{a}{c_6(1-a)} \right)^{\frac{a}{1-a}} \cdot (t - T_1)^{-\frac{a}{1-a}} \leq \left( \frac{c_7}{c_6} \right)^a + \left( \frac{a}{c_6(1-a)} \right)^{\frac{a}{1-a}} \quad \text{for all } t > T_1 + 1 \quad \text{and } \eps \in (\eps_{kj})_{k \in N} \cap (0, \eps_*) \tag{58}
\]

and thus (47) holds with \( T(p) := T_1 + 1 \) upon an evident choice of \( C(p) \).
Proof: Again based on known regularization features of the Neumann heat semigroup on $\Omega$, this can be derived from Lemma 4.3 and (16) in a straightforward manner.

After an adequate $\varepsilon$-independent waiting time, both $L^\infty$ and Hölder estimates for $u_\varepsilon$ can be obtained from Lemmata 4.3 and 4.4 upon a suitable cut-off procedure with respect to the time variable.

**Lemma 4.5**: Let $n = 3$, and let $(\varepsilon_{jk})_{k \in \mathbb{N}}$ be taken from Lemma 4.2. Then one can find $T > 0, \varepsilon_* \in (0, 1), \theta \in (0, 1)$ and $C > 0$ such that

$$
\|u_\varepsilon\|_{C^0([0, T])} \leq C \quad \text{for all } t > T \quad \text{and} \quad \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \subset (0, \varepsilon_*).
$$

**Proof**: For arbitrary $T > 0$ and $\zeta = \zeta^{(T)} \in C^\infty([0, \infty))$ fulfilling $\zeta \equiv 0$ on $[0, T + \frac{1}{2}]$ and $\zeta \equiv 1$ on $[T + 1, \infty)$ as well as $0 \leq \zeta \leq 1$ and $|\zeta'| \leq 4$, the functions given by

$$
w_{k}(x, t) := w_{k}^{(T)}(x, t) := \zeta(t) u_{\varepsilon}(x, t), \quad (x, t) \in \Omega \times [0, \infty), \quad \varepsilon \in (0, 1),
$$
satisfy $w_{k} \equiv 0$ on $\Omega \times [0, T + \frac{1}{2}]$ and

$$
w_{k,t} = \nabla \cdot (D_{\varepsilon}(x, t) \nabla w_{k}) + \nabla \cdot (b_{\varepsilon}(x, t) w_{k}) + h_{\varepsilon}(x, t)
$$

for all $\varepsilon \in (0, 1)$, with $D_{\varepsilon}(x, t) := \phi(v_{\varepsilon}(x, t))$, $b_{\varepsilon}(x, t) := \phi'(v_{\varepsilon}(x, t)) \nabla v_{\varepsilon}(x, t)$ and $h_{\varepsilon}(x, t) \equiv h_{\varepsilon}^{(T)}(x, t) := \zeta'(t) u_{\varepsilon}(x, t)$, $(x, t) \in \Omega \times (0, \infty)$, $\varepsilon \in (0, 1)$. Here we note that $\inf_{\varepsilon \in (0, 1)} \inf_{\Omega \times (0, \infty)} D_{\varepsilon} > 0$ by (4) and (16), and that Lemmata 4.3, 4.4, (4) and (16) assert that for each $p > 1$ we can find $T_1 = T_1(p) > 0$ and $\varepsilon_* = \varepsilon_*(p) \in (0, 1)$ such that $(w_{k})_{\varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}}} \cap (0, \varepsilon_*)$ and $(h_{k})_{\varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}}} \cap (0, \varepsilon_*)$ are bounded in $L^\infty((T_1, \infty); L^p(\Omega))$. Applying this to some suitably large $p$ in the course of a Moser-type iteration in (59) [30, Lemma A.1], we hence infer the existence of $T_2 > 0, \varepsilon_* \in (0, 1)$ and $c_1 > 0$ such that

$$
\|w_{k}^{(T_2)}(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1 \quad \text{for all } t > T_2 \quad \text{and} \quad \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*),
$$

and thus, by definition of $(w_{k}^{(T_2)})_{\varepsilon \in (0, 1)}$,

$$
\|u_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1 \quad \text{for all } t > T_2 + 1 \quad \text{and} \quad \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \cap (0, \varepsilon_*).
$$

Based on this, and again on Lemma 4.4 and (4), we may draw on temporally localized parabolic Hölder estimates [31] to see that, indeed, (58) holds with $\varepsilon_*$ as above, $T := T_2 + 1$ and some suitably large $C > 0$.

Higher-order regularity features can thereupon directly be inferred from standard parabolic theory.

**Lemma 4.6**: Let $n = 3$, and let $(\varepsilon_{jk})_{k \in \mathbb{N}}$ be as in Lemma 4.2. Then there exist $T > 0, \varepsilon_* \in (0, 1), \theta \in (0, 1)$ and $C > 0$ such that

$$
\|u_\varepsilon\|_{C^{2+\theta,1+\theta}([0, T])} \leq C \quad \text{for all } t > T \quad \text{and} \quad \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \subset (0, \varepsilon_*)
$$

as well as

$$
\|v_\varepsilon\|_{C^{2+\theta,1+\theta}([0, T])} \leq C \quad \text{for all } t > T \quad \text{and} \quad \varepsilon \in (\varepsilon_{jk})_{k \in \mathbb{N}} \subset (0, \varepsilon_*).
$$
Proof: By resorting to standard parabolic Schauder theory again [32], we readily obtain first (61) and then (60) using Lemmata 4.5, 4.4 and (4).  

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2: We first focus on the existence of global weak solutions for \( n \geq 3 \) as defined in (9) and (10). From (16), Lemmata 2.2, 2.3 and 4.1, we see that for all \( T > 0 \),

\[
(u_\varepsilon)_\varepsilon \subset (0,1) \text{ is bounded in } L^2(\Omega \times (0, T)),
\]

and

\[
(v_\varepsilon)_\varepsilon \subset (0,1) \text{ is bounded in } L^2((0, T); W^{2,2}(\Omega)), \quad \text{in } L^4((0, T); W^{1,4}(\Omega)) \quad \text{and in } L^\infty(\Omega \times (0, \infty)),
\]

while

\[
(v_{\varepsilon t})_\varepsilon \subset (0,1) \text{ is bounded in } L^2(\Omega \times (0, T)),
\]

whence relying on (18), we conclude by means of an Aubin–Lions lemma that

\[
\begin{align*}
 u_\varepsilon & \xrightarrow{} u \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)), \\
 u_\varepsilon & \xrightarrow{} u \quad \text{in } L^4_{\text{loc}}([0, \infty); W^{1,4}(\Omega)), \\
 v_\varepsilon & \xrightarrow{} v \quad \text{in } L^\infty(\Omega \times (0, \infty)), \\
 v_\varepsilon & \xrightarrow{} v \quad \text{in } L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)) \quad \text{and} \\
 v_\varepsilon & \xrightarrow{} v \quad \text{in } L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)) \quad \text{and a.e. in } \Omega \times (0, \infty),
\end{align*}
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \), with \( (\varepsilon_j)_{j \in \mathbb{N}} \) provided by Lemma 2.1. Furthermore, it follows from (4), (16), (62) and (63) that

\[
(u_\varepsilon \phi(v_\varepsilon))_\varepsilon \subset (0,1) \text{ is bounded in } L^4((0, T); W^{1,4}(\Omega)) \quad \text{for all } T > 0,
\]

which combined with (4) and (18) entails that

\[
\nabla(u_\varepsilon \phi(v_\varepsilon)) \rightharpoonup \nabla(u \phi(v)) \quad \text{in } L^4_{\text{loc}}(\Omega \times [0, \infty)) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0.
\]

Besides implying (8), together with a result on stability of weak \( L^1 \) convergence with respect to certain small nonlinear perturbations of the identity [22, Lemmata 3.7 and 5.1] which ensures that

\[
\frac{u_\varepsilon v_\varepsilon}{1 + \varepsilon u_\varepsilon} \rightharpoonup uv \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0
\]

by (64), (68) and (16), these approximation properties (64)–(68) and (70) enable us to derive (9) and (10) for each \( \varphi \in C^\infty_0(\Omega \times [0, \infty)) \) from the corresponding weak formulations associated with (13) in a straightforward fashion.

In order to thereupon prove our main result on eventual smoothness and asymptotic behavior in the case \( n = 3 \), we only need to observe that for adequately large \( T > 0 \), the additional smoothness features in (11) are consequences of Lemma 4.6 and the Arzelà-Ascoli theorem, whereas (7) results from (18) by interpolation, because if \( T \) is sufficiently large, then \( (u(\cdot, t))_{t > T} \) and \( (v(\cdot, t))_{t > T} \) are both bounded in \( C^1(\overline{\Omega}) \) due to Lemma 4.6.

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Appendix

In this appendix, we briefly collect three statements on quantitative bounds for linearly and superlinearly damped first-order ordinary differential inequalities.

A proof of the following can be found in [28, Lemma 3.4].

**Lemma A.1:** Let \( t_0 \in \mathbb{R}, T > t_0, a > 0 \) and \( b > 0 \), and suppose that \( y \in C^0([t_0, T)) \cap C^1((t_0, T)) \) and \( h \in L^1_{\text{loc}}(\mathbb{R}) \) are such that \( y \geq 0 \) on \((t_0, T)\) and \( h \geq 0 \) a.e. on \( \mathbb{R} \), that

\[
\int_{(t-1)_+}^{t} h(s) \, ds \leq b \quad \text{for all } t \in (t_0, T),
\]

and that

\[
y'(t) + ay(t) \leq h(t) \quad \text{for all } t \in (t_0, T).
\]

Then

\[
y(t) \leq y(t_0) + \frac{b}{1 - e^{-a}} \quad \text{for all } t \in [t_0, T).
\]

The following elementary inequality has been recorded in [34, Lemma 2.2].

**Lemma A.2:** Let \( \lambda > 1 \), and suppose that with some \( T > 0, a > 0 \) and \( b > 0 \), the functions \( y \in C^0([0, T)) \cap C^1((0, T)) \) and \( h \in L^1_{\text{loc}}(\mathbb{R}) \) satisfy \( y(t) > 0 \) for all \( t \in [0, T) \) and \( h(t) \geq 0 \) for a.e. \( t \in \mathbb{R} \), and are such that

\[
\int_{(t-1)_+}^{t} h(s) \, ds \leq b \quad \text{for all } t \in (t_0, T)
\]
as well as

\[
y'(t) + ay^\lambda(t) \leq h(t)y(t) \quad \text{for all } t \in (0, T).
\]

Then

\[
y(t) \leq \max \left\{ y(0) e^b, (a(\lambda - 1))^{-\frac{1}{\lambda}} e^b \right\} \quad \text{for all } t \in [0, T).
\]

Let us finally provide a brief proof for a variant of Lemma A.2, here focusing on an estimate independent of the behavior near the initial instant.

**Lemma A.3:** Let \( \lambda > 1, a > 0, b > 0, t_0 \in \mathbb{R} \), and \( T > t_0 \), and suppose that \( y \in C^1((t_0, T)) \cap L^\infty((t_0, T)) \) be nonnegative and such that

\[
y'(t) + ay^\lambda(t) \leq b \quad \text{for all } t \in (t_0, T).
\]

Then

\[
y(t) \leq \left( \frac{b}{a} \right)^{\frac{1}{\lambda}} + \left( \frac{1}{a(\lambda - 1)} \right)^{\frac{1}{\lambda-1}} (t - t_0)^{-\frac{1}{\lambda-1}} \quad \text{for all } t \in (t_0, T).
\]
Proof: Since

\[ \overline{y}(t) := c_1 + c_2 \cdot (t - t_0)^{-\frac{1}{\lambda-1}}, \quad t > t_0, \]

with \( c_1 := \left( \frac{b}{a} \right)^{\frac{1}{\lambda}} \) and \( c_2 := \left( \frac{1}{a(\lambda-1)} \right)^{\frac{1}{\lambda-1}} \), satisfies

\[ \overline{y}'(t) + a\overline{y}^\lambda(t) - b = -\frac{c_2}{\lambda - 1} (t - t_0)^{-\frac{2}{\lambda-1}} + a \cdot \left\{ c_1 + c_2 \cdot (t - t_0)^{-\frac{1}{\lambda-1}} \right\}^{\frac{\lambda}{\lambda-1}} - b \]

\[ \geq -\frac{c_2}{\lambda - 1} (t - t_0)^{-\frac{2}{\lambda-1}} + ac_1^\lambda + ac_2^\lambda (t - t_0)^{-\frac{1}{\lambda-1}} - b \quad \text{for all } t > t_0 \]

due to the fact that \((\xi + \eta)^\lambda \geq \xi^\lambda + \eta^\lambda\) for all \( \xi \geq 0 \) and \( \eta \geq 0 \), and since here \( ac_1^\lambda - b = 0 \) and \( ac_2^\lambda - \frac{c_2}{\lambda-1} = 0 \), the inequality in (A3) can readily be derived by means of a comparison argument applied to (A2), because \( \overline{y}(t) - y(t) \to +\infty \) as \( t \searrow t_0 \). \( \square \)