A Simple Lack-of-Fit Test for Regression Models

Aubin Jean-Baptiste\textsuperscript{a}, Leoni-Aubin Samuela\textsuperscript{b}

\textsuperscript{a}Univ. de Technologie de Compiègne, Rue Personne de Roberval - BP 20529, 60205 Compiègne, France.
\textsuperscript{b}INSA Lyon, ICJ, 20, Rue Albert Einstein, 69621 Villeurbanne Cedex, France.

Reçu le ***** ; accepté après révision le +++++
Présenté par £££££

Abstract

A simple test is proposed for examining the correctness of a given completely specified response function against unspecified general alternatives in the context of univariate regression. The usual diagnostic tools based on residuals plots are useful but heuristic. We introduce a formal statistical test supplementing the graphical analysis. Technically, the test statistic is the maximum length of the sequences of ordered (with respect to the covariate) observations that are consecutively overestimated or underestimated by the candidate regression function. Note that the testing procedure can cope with heteroscedastic errors and no replicates. Recursive formulae allowing to calculate the exact distribution of the test statistic under the null hypothesis and under a class of alternative hypotheses are given. To cite this article: A. Nom1, A. Nom2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).
1. Introduction

Regression is one of the most widely used statistical tools to examine how one variable is related to another. Statisticians usually begin their work by proposing a model for their observations. Then, they have to check on whether this model is correct. The graphical analysis of the residuals is an important step of this process since the detection of a systematic pattern would indicate a misspecified model. Unfortunately, this procedure is heuristic and could lead to errors of interpretation since it is often difficult to determine whether the observed pattern reflects model misspecification or random fluctuations. So it is of interest to complement such an analysis by a formal test. A large literature in this area can be found in Hart (1997). A review of statistical tests and procedures to determine lack of fit associated with the deterministic portion of a proposed linear regression model is presented in Neill and Johnson (1984). We propose a new approach based on maximum length of sequences of consecutive overestimated (or underestimated) observations by the model. This test is very simple and can be computed visually if the sample size is small enough. This test is a modification of a nonrandomness test (see Bradley 1968, chap. 11). In other words, we use this it to detect whether residuals are randomly distributed or not.

In Section 2, the Length of the Longest Run Test is presented. Section 3 is devoted to the law of the test statistic under the null hypothesis. In Section 4, the power of the test for a class of fixed alternatives is given.

2. The Length of the Longest Run Test Statistic

Consider a collection of \( n \) random variables \( Y_i \) generated as

\[
Y_i = m_0(x_i) + \varepsilon_i, \quad i = 1, \ldots, n,
\]

where the \( x_i \) are fixed design points and \( m_0 \) is the true regression function. Moreover, the \( \varepsilon_i \) are independent and centered random variables such that:

\[
\forall i = 1, \ldots, n, \quad Pr(\varepsilon_i > 0) = Pr(\varepsilon_i < 0) = \frac{1}{2}.
\]  

(1)

Note that no hypothesis is made on the regularity of the function \( m_0 \) or on the fact that errors must be identically distributed or homoscedastic, and that normality of \( \varepsilon_i \) implies Condition (1). Moreover, contrary to other classical tests (like the F-test), no replicates are needed to compute our test statistic. We address the problem of testing the null hypothesis

\[
H_0 : m_0 = m \quad \text{vs.} \quad H_1 : m_0 \neq m,
\]

where \( m \) is a completely specified function.

The \( i \)-th residual, \( \hat{\varepsilon}_i \), may be seen as substitute for the realisation of the random variable \( \varepsilon_i \), thus comprising clues for adequacy or inadequacy of the model assumptions related to the distribution of \( \varepsilon_i \). Some classical lack-of-fit test statistics are based on squared residuals, hence their signs are neglected, and we can expect to loose some information. We propose a test statistic that takes these signs into account. This test statistic, \( L_n \), is the maximum length of the sequences of ordered (with respect to the covariate) observations that are consecutively overestimated (or underestimated) by the candidate \( m \). Formally, we define \( Z_i := 1(\hat{\varepsilon}_i > 0), 1 \leq i \leq n \), \( S_0 := 0 \), \( S_l := Z_1 + \ldots + Z_l \), and put for \( 0 \leq K \leq n \),

\[
I^+(n, K) := \max_{0 \leq l \leq n-K} (S_{l+K} - S_l).
\]
Let $L^+_n$ be the largest integer $K$ for which $I^+(n,K) = K$. $L^+_n$ is the length of the longest run of 1’s in $Z_1, \ldots, Z_n$, i.e. the length of the longest run of positive residuals. By analogy, we define $L^-_n$ as the length of the longest run of 0’s in $Z_1, \ldots, Z_n$, that is $L^-_n$ is the largest integer $K$ for which $I^-(n,K) = K$, where

$$I^-(n,K) := \max_{0 \leq I \leq n-K} (K - S_{I+k} + S_I).$$

Clearly, $L^-_n$ is the length of the longest run of negative residuals. Finally, we define $L_n := \max(L^+_n, L^-_n)$. For a fixed nominal level $\alpha > 0$, we obtain the following unilateral rejection regions $W_{n,\alpha} = \{L_n > c_{n,\alpha}\}$, where $c_{n,\alpha}$ is the largest integer such that $Pr(L_n > c_{n,\alpha}) \geq \alpha$. The corresponding bilateral rejection regions are $W_{n,\alpha} = \{L_n \notin [c_{n,1-\alpha/2}, c_{n,\alpha/2}]\}$.

3. Distribution of $L_n$ under the null hypothesis

If $m$ is equal to $m_0$, then, the residuals $\hat{\varepsilon}_i$ are the true errors $\varepsilon_i$. Since Condition (1) holds, we can apply the following recursive formula (Riordan (1958), p.153, Problem 13):

$$(n-1)! \Pr(L_n = k) = 2(n-2)! \Pr(L_{n-1} = k) - (n-k-2)! \Pr(L_{n-k-1} = k) + (n-2)! \Pr(L_{n-1} = k-1) - 2(n-3)! \Pr(L_{n-2} = k-1) + (n-k-1)! \Pr(L_{n-k} = k-1).$$

By using $Pr(L_2 = 2) = 1/2$ and $\forall n > 0$, $Pr(L_n = 1) = 1/2^{n-1}$, the entire exact distribution of $L_n$ and critical values for every nominal level can be deduced from the above formula.

For most of practical cases of interest, $m$ is estimated. For example, if $m$ is estimated by OLS, an unfortunate property of residuals is that they are autocorrelated even when the true errors are white noise. This divergence from the assumptions disappears in large samples, but may be a problem when performing diagnostic tests in small samples. One way of handling this problem is to transform the OLS residuals so that they do satisfy the LS assumptions when these are correct. One of the most common of these transformations are the so called recursive residuals (see Kianifard and Swallow (1996) among others).

Another possibility is to estimate $m$ on a subset of the data and to test it on the rest of the data.

In a coin tossing experiment, $L_n$, $L^+_n$, and $L^-_n$ can be seen as the length of the longest run of heads or tails, heads and tails, respectively. The length of the longest head run in a coin tossing experiment was investigated in the early days of probability theory. Later, Deheuvels (1985) gives upper and lower bounds for $L^+_n$ for a biased coin.

Schilling (1990) discusses the distributions of $L_n$ for unbiased coins, and remarks that for $n$ tosses of a fair coin the length of the longest run of heads or tails, statistically speaking, tends to be about one longer than the length of the longest run of heads only. For a biased coin, when $n$ is very large, if head is more likely than tail, the distribution function of $L^+_n$ is well approximated by an extreme value distribution (see Gordon et al. (1986)).

4. Distribution of $L_n$ under fixed alternative hypotheses

The distribution of the Length of the Longest Run Test statistic can be calculated under some fixed alternative hypotheses. First of all, we suppose that Condition (1) is fulfilled, and that errors are identically distributed.
Moreover, if we test

\[ H_0 : \forall x, \ m_0(x) = m(x) \quad \text{vs.} \quad H_{1,c} : \forall x, \ m_0(x) = m(x) + c, \ c \neq 0 \]

then, under \( H_{1,c} \), the probability for an observation to be underestimated (respectively, overestimated), \( p(c) \neq \frac{1}{2} \), is constant for all the observations. By considering the total number of positive residuals, \( k \), in the sequence, the cumulative distribution of \( L_n \) can be expressed as:

\[
P(L_n \leq x) = \sum_{k=0}^{n} S_n^{(k)}(x) p(c)^k (1 - p(c))^{n-k},
\]

where \( S_n^{(k)}(x) \) is the number of sequences of length \( n \) that contain \( k \) positive residuals in which the length of the longest run of positive or negative residuals does not exceed \( x \). Analogously, Schilling (1990) studied the cumulative distribution of \( L_n^* \).

In the following Proposition, we give a recursive formula to compute the \( S_n^{(k)}(x) \):

**Proposition 4.1** Let \( n \) and \( x \) such that \( 0 < x \leq n \). Then,

(i) If \( n - k \leq x \) and \( k \leq x \), \( S_n^{(k)}(x) = C_n^k \).
(ii) If \( n - k \leq x \) and \( k > x \), \( S_n^{(k)}(x) = \sum_{j=0}^{x} S_{n-j}^{(k)}(x) \).
(iii) If \( n - k > x \) and \( k \leq x \), \( S_n^{(k)}(x) = \sum_{j=0}^{x} S_{n-j}^{(k+1)}(x) \).
(iv) If \( n - k > x \) and \( k > x \), let

\[
R_n^{(k)}(x) = \sum_{j \geq 0} \left\{ \sum_{i=1}^{x} \left\{ S_{n-1-i-2j}^{(k-1-j)}(x) + S_{n-1-i-2j}^{(k-1-j)}(x) \right\} \right\}
\]

(2)

with the following conventions: \( \forall x \in \mathbb{N}^* \), \( R_{0}^{(0)}(x) = 1 \) and \( \forall n \in \mathbb{N}^* \), \( k \in \mathbb{N}^* \), \( R_{-n}^{(-k)}(x) = R_{-n}^{(k)}(x) = R_{-n}^{(0)}(x) = 0 \). Finally,

- If \( \exists (i, j) \in \{1, \ldots, x\} \times \mathbb{N}^* \) such that \( (k, n) = (2j(x+1) + i, j(x+1)) \) or \( (k, n) = (2j(x+1) + i, j(x+1) + i) \), then \( S_n^{(k)}(x) = R_n^{(k)}(x) + 1 \).
- If \( \exists (i, j) \in \{1, \ldots, x\} \times \mathbb{N}^* \) such that \( (k, n) = ((2j+1)(x+1) + i, j(x+1) + i) \) or \( (k, n) = ((2j+1)(x+1) + i, j(x+1) + i) \), then \( S_n^{(k)}(x) = R_n^{(k)}(x) - 1 \).
- Else, \( S_n^{(k)}(x) = R_n^{(k)}(x) \).

From this result, one can deduce the exact law of the test-statistic under \( H_{1,c} \), and the power of the test follows. In the next Proposition, we show that, for \( n \) large enough, the distribution function of \( L_n \) is well approximated by the distribution function of \( L_n^+ \) (or \( L_n^- \), depending on the value of \( p(c) \)):

**Proposition 4.2** If \( \forall i = 1, \ldots, n \), \( P(r_i > 0) = p(c) \), \( p(c) > \frac{1}{2} \), (resp. \( p(c) < \frac{1}{2} \)), then

\[
\forall k, \ P(L_n \leq k) = P(L_n^+ \leq k) + o(1) \quad \text{when} \ n \to \infty
\]

(resp. \( P(L_n \leq k) = P(L_n^- \leq k) + o(1) \)).

5. Proofs.

**Proof of Proposition 4.1:**
The recursive formula to compute \( S_n^{(k)}(x) \), the number of sequences of length \( n \) that contain \( k \) positive residuals
residuals in which the length of the longest run of positive or negative residuals does not exceed $x$, is found through a direct combinatorial analysis.

We distinguish the following cases:

(i) For $n - k \leq x$ and $k \leq x$, $S_n^{(k)}(x)$ is equal to the binomial coefficient $\binom{n}{k}$.

(ii) When $n - k \leq x$ and $k > x$, all the not-favorable sequences (that is, the sequences of length $n$ that contain $k$ positive residuals in which the length of the longest run of residuals having the same sign exceeds $x$) will contain at least a run of consecutively positive residuals (and no run of consecutively negative residuals) of length larger than $x$. In this particular case, we want to study the length of the longest head run in $n$ tosses of a biased coin including $k$ heads, problem solved by [9].

(iii) In a similar way, when $n - k > x$ and $k \leq x$, the problem is the same, swapping heads and tails.

(iv) For a fixed $x$ and $k$, when $n - k > x$ and $k > x$, the key is to partition the set of favorable sequences according to their beginning. Each sequence of length $n$ that contains $k$ positive residuals in which the length of the longest run of residuals having the same sign does not exceed $x$ can begin in at most $2x$ different ways and every beginning is followed by a sub-sequence with no more than $x$ consecutive residuals having the same sign. In Table 1, we introduce the notation for the number of favorable sequences conditionally to the possible beginnings.

| 1 2 3 4 5 6 ... $x$ ($x+1$) | Number of favorable sequences | Upper bound for the number of favorable sequences |
|-------------------------------|-------------------------------|-----------------------------------------------|
| + −                           | $N_{1+}$ | $S_{n-2}^{(k-1)}(x)$ |
| + + −                         | $N_{2+}$ | $S_{n-3}^{(k-2)}(x)$ |
| + + + −                       | $N_{3+}$ | $S_{n-4}^{(k-3)}(x)$ |
| + + + + −                     | $N_{4+}$ | $S_{n-5}^{(k-4)}(x)$ |
| + + + + + −                   | $N_{5+}$ | $S_{n-6}^{(k-5)}(x)$ |
| ...                           | ...   | ...                       |
| + + + + + + ... + −          | $N_{x+}$ | $S_{n-(x+1)}^{(k-2)}(x)$ |
| − +                           | $N_{1−}$ | $S_{n-1}^{(k-1)}(x)$ |
| − − +                         | $N_{2−}$ | $S_{n-3}^{(k-1)}(x)$ |
| − − − +                       | $N_{3−}$ | $S_{n-4}^{(k-1)}(x)$ |
| − − − − +                     | $N_{4−}$ | $S_{n-5}^{(k-1)}(x)$ |
| − − − − − +                   | $N_{5−}$ | $S_{n-6}^{(k-1)}(x)$ |
| ...                           | ...   | ...                       |
| − − − − − − ... − +          | $N_{x−}$ | $S_{n-(x+1)}^{(k-1)}(x)$ |

Table 1
The possible beginnings for a favorable sequence and the associated number of favorable sequences (and upper bounds)

Clearly, $S_n^{(k)} = N_{1+} + N_{2+} + \ldots + N_{x+} + N_{1−} + \ldots + N_{x−}$. Let determine the number of “favorable” sequences beginning by a positive residual and then a negative one $N_{1+}$.
Step 1:

$N_+$ is at most equal to $S_{n-2}^{(k-1)}(x)$ (i.e., the number of favorable ways to complete a sequence beginning by $+-$, see Table 1).

Step 2:

Among these $(n-2)$—sequences (the first two signs of the residuals are fixed), those beginning by $x$ “-” must be taken away because, in this case, the obtained sequences admit $x+1$ consecutive “-” (see Table 2).

There are $S_{n-2-(x+1)}^{(k-2)}(x)$ of them. At this point, $N_+$ is at least equal to $S_{n-2}^{(k-1)}(x) - S_{n-2-(x+1)}^{(k-2)}(x)$.

Further steps:

Analogously, $(n-2-(x+1))$—sequences beginning by $x$ “+” must be subtracted from the $S_{n-2-(x+1)}^{(k-2)}(x)$ sequences taken away previously (see Table 3).

Then $N_+ \leq S_{n-2}^{(k-1)}(x) - \left( S_{n-2-(x+1)}^{(k-2)}(x) - S_{n-2-2(x+1)}^{(k-1)-(x+1)}(x) \right) = S_{n-2}^{(k-1)}(x) - S_{n-2-(x+1)}^{(k-2)}(x) + S_{n-2-2(x+1)}^{(k-1)-(x+1)}(x)$.

Recursively,

$$N_+ = \sum_{j \geq 0} \left( S_{n-2-(2j+1)(x+1)}^{(k-1-j(x+1))}(x) - S_{n-2-2(2j+1)(x+1)}^{(k-2-j(x+1))}(x) \right)$$

Note that for $j$ large enough, indexes become negative. We use the following conventions for all $x$, $S_0(x) = 1$ and $\forall n \in \mathbb{R}^+$, $k \in \mathbb{N}^*$, $S_{-k}(x) = S_k(x) = S_{n-k}(x) = 0$. We use the same method to calculate $N_-$ for every possible beginning, we conclude the proof of Formula (2) by summing them. There are some “special points” that need a correction when applying the Formula (2). These points are such that the quantity $S_0^k(x)$ appears in the formula when $k < \frac{n}{2}$ (or the quantity $S_x^n(x)$ when $k > \frac{n}{2}$).

For example, if $\exists (i,j) \in \{1, \ldots, x\} \times \mathbb{N}^*$ such that $(k,n) = (2j(x+1)+i,j(x+1))$, then the point $S^{(0)}_j(x)$ appears in the term $-\sum_{i=1}^x S^{(k-(j+1)(x+1))}_{n-1-(2j+1)(x+1)}(x)$ of the recursive formula of $S_n^{(k)}(x)$. In this case, $S^{(0)}_j(x)$ represents the number of sequences of length $x$ with $x$ negative residuals and zero positive residuals that must be subtracted when the last residual before the $x$ last ones is negative. So, this sequence $(S^{(0)}_x(x) = 1)$ mustn’t be subtracted since it hadn’t been counted before (because it would have make appear a sequence of $(x+1)$ consecutive negative residuals).
Note that in other cases, $S_x^{(0)}(x)$ has to be taken into account (for example is the $(n - x)$-sequence preceding the $x$ last residuals ends with a positive residual).

Similar considerations yield to the three other corrections.

The recursive formula (2) becomes more clearful if we look at the Table (5) which illustrates it. In Table (5), for fixed $n$, $k$ and $x$ we represent the coefficients to assign to each $S_{\bar{n}}^{(k)}(x)$ (where $\bar{k} < k$ and $\bar{n} < n$) in order to compute $S_n^{(k)}$. The sign “+” means that such coefficient equals 1, “−” that such coefficient equals −1, and an empty cell means that the coefficient equals 0.

**Proof of Proposition 4.2:**
This proposition is a direct application of the fact that $Pr(L_n^- < L_n^+)$ tends to 1 when $n$ tends to infinity as shown in Muselli (2000), since, for all $x \geq 1$,

$$ Pr(L_n \leq x) = Pr(L_n < x | L_n^- < L_n^+) Pr(L_n^- < L_n^+) + Pr(L_n < x | L_n^- \geq L_n^+) Pr(L_n^- \geq L_n^+) $$

and $L_n = \max(L_n^-, L_n^+)$. 

**References**

[1] **Bradley, J. V.** *Distribution-free statistical tests.* Prentice-Hall Inc, 1968.

[2] **Deheuvels, P.** On the Erdos-Renyi theorem for random fields and sequences and its relationships with the theory of runs and spacings. *Z. Wahrsch. Verw. Gebiete* 70 (1985) 91–115.

[3] **Gordon, L., Schilling, M.F. and Waterman, M.S.** An Extreme value Theory for Long Head Runs *Probab. Th. Rel. Fields* 72 (1986) 279–287.

[4] **Hart, J.** *Nonparametric Smoothing and Lack-of-Fit Tests.* New York: Springer-Verlag, 1997.

[5] **Kianifard, F.** and **Swallow, W. H.** A review of the development and application of recursive residuals in linear models. *J.A.S.A.* 91 (433) (1996) 391–400.

[6] **Muselli, M.** Useful inequalities for the longest run distribution. *Statistics & Probability Letters* 46 (2000) 239–249.

[7] **Neill, J. W.** and **Johnson, D. E.** Testing for lack of fit in regression – A review. *Comm. Statist. A—Theory Methods* 13 (4) (1984) 485–511.

[8] **Riordan, J.** *An introduction to combinatorial analysis.* John Wiley and Sons, Inc, 1958.

[9] **Schilling, M. F.** The longest run of heads. *College Math. J.* 21 (1990) 196–207.
| k     | 0 | 1 | (k - 2 - x) | (k - 1 - x) | (k - x) | ... | (k - 2) | (k - 1) | (k) |
|-------|---|---|------------|------------|--------|-----|--------|--------|-----|
| n     | 0 |   |            |            |        |     |        |        |     |
|       | 1 |   |            |            |        |     |        |        |     |
| ...   |   |   |            |            |        |     |        |        |     |
|       |   |   |            |            |        |     |        |        |     |
| n - 2 - 3(x + 1) |   |   |            |            |        |     |        |        |     |
| n - 2 - 3x - 1  |   |   |            |            |        |     |        |        |     |
| ...   |   |   |            |            |        |     |        |        |     |
| n - 2 - 2x - 2  |   |   |            |            |        |     |        |        |     |
| n - 2 - 2x - 1  |   |   |            |            |        |     |        |        |     |
| n - 2 - 2x      |   |   |            |            |        |     |        |        |     |
| n - 2 - (x + 1) |   |   |            |            |        |     |        |        |     |
| n - 2 - x      |   |   |            |            |        |     |        |        |     |
| n - 1 - x      |   |   |            |            |        |     |        |        |     |
| n - x          |   |   |            |            |        |     |        |        |     |
| ...            |   |   |            |            |        |     |        |        |     |
| n - 3          |   |   |            |            |        |     |        |        |     |
| n - 2          |   |   |            |            |        |     |        |        |     |
| n - 1          |   |   |            |            |        |     |        |        |     |
| n             |   |   |            |            |        |     |        |        |     |

Table 4
Illustration of Recursive Formula (2), for a fixed $x$. 

8