Light-Front spin-dependent Spectral Function and Nucleon Momentum Distributions for a Three-Body System

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Poincaré covariant definitions for the spin-dependent spectral function and for the momentum distributions within the light-front Hamiltonian dynamics are proposed for a three-fermion bound system, starting from the light-front wave function of the system. The adopted approach is based on the Bakamjian-Thomas construction of the Poincaré generators, that allows one to easily import the familiar and wide knowledge on the nuclear interaction into a light-front framework. The proposed formalism can find useful applications in refined nuclear calculations, like the ones needed for evaluating the EMC effect or the semi-inclusive deep inelastic cross sections with polarized nuclear targets, since remarkably the light-front unpolarized momentum distribution by definition fulfills both normalization and momentum sum rules. It is also shown a straightforward generalization of the definition of the light-front spectral function to an A-nucleon system.

I. INTRODUCTION

In the analysis of the next generation of high-energy electron-nucleus scattering experiments, planned at the Jefferson Laboratory (JLab) upgraded at 12 GeV\cite{1}, as well as at the future Electron-Ion Collider\cite{2}, refined description of nuclei will play a relevant role\cite{3}, with a particular interest to the polarized $^3$He target at JLab12. High precision experiments, involving both protons and neutrons, are in fact necessary to clarify the flavour dependence of (i) Parton Distribution Functions (PDFs), measured in inclusive Deep Inelastic Scattering (DIS), and (ii) Transverse-Momentum Dependent Parton Distribution (TMDs, see, e.g., Ref.\cite{4} for a general introduction), accessed through Semi Inclusive DIS (SIDIS). In the next few years, several experiments involving an $^3$He nuclear target will be performed at JLab12, with the aim at extracting information on the parton structure of the neutron. New DIS measurements are planned\cite{2,6} and, in particular, the three-dimensional neutron structure in momentum space, described in terms of quark TMDs, will be probed through SIDIS off polarized $^3$He, where a high-energy pion (kaon) is detected in coincidence with the scattered electron\cite{7,8}.

To be able to extract PDFs and TMDs in the neutron from DIS and SIDIS off $^3$He, accurate theoretical descriptions of the structure of $^3$He and of the scattering process are also needed. Initial studies of DIS and SIDIS off $^3$He were performed in Refs.\cite{9} and \cite{10}, respectively, where the plane wave impulse approximation (PWIA) was adopted to describe the reaction mechanism, namely the interaction in the final state (FSI) was considered only within the two-nucleon spectator pair which recoils. The $^3$He structure was treated non-relativistically using the AV18 NN interaction\cite{11}.

In a recent paper\cite{12}, the spectator SIDIS process off polarized $^3$He, where a deuteron in the final state is detected, was studied taking into account for the first time the FSI between the hadronizing quark and the detected deuteron through a distorted spin dependent spectral function of $^3$He. The study of the standard SIDIS process off transversely polarized $^3$He with a fast detected pion including the FSI is presented in Ref.\cite{13}, where the FSI between the observed pion and the remnant is again taken into account through a distorted spin-dependent spectral function (preliminary results can be found in Ref.\cite{14}). However, the description of the internal nuclear dynamics in \cite{13,14}, is still non-relativistic, or more appropriately non Poincaré covariant, while the high energies involved in the forthcoming SIDIS experiments\cite{7,8} should
require at least a proper treatment of Poincaré covariance. In this paper, the structure of a spin 1/2 three-nucleon system will be investigated within a relativistic, Poincaré invariant framework (see Refs. [13, 16] for early studies). Indeed, our approach can be straightforwardly generalized to other spin 1/2 three-body systems and even to complex nuclei. To develop a Poincaré covariant framework that allows one to embed the large amount of knowledge on the nuclear interaction obtained from the non relativistic description of nuclei, we adopt the Relativistic Hamiltonian Dynamics (RHD) [17] with a fixed number of on-mass-shell constituents in its light-front (LF) version [18–21]. Within the LF form of RHD, the Poincaré group has a subgroup given by the LF boosts, which allows a kinematical separation of the intrinsic motion from the global one. Such a property plays a very important role for the relativistic description of DIS, SIDIS and deeply virtual Compton scattering (DVCS) processes, where the final states can have a fast recoil. Furthermore the LF field theory allows a meaningful Fock expansion of the interacting system state [22] (with the caveat of zero-modes). It has to be noted that in a field-theoretical framework, explicitly covariant, the constituent masses are on-shell and the four-momenta are conserved, but the interaction must be introduced perturbatively. On the contrary, in a RHD framework: (i) the explicit covariance is lost, (ii) the constituent masses are on-mass-shell and only three component of the momenta are conserved, but (iii) the Poincaré covariance fully holds, and (iv) the interaction can be introduced non perturbatively through the Bakamjian-Thomas construction of the Poincaré generators [23]. This last feature is essential for a realistic description of nuclei. In this paper only the valence component of the LF wave function of the system is considered and for the sake of definiteness we consider the case of the three-nucleon systems, i.e., $^3$He and $^3$H.

The key quantity to be considered is the LF spectral function, depending on (i) spin and intrinsic momentum of the nucleon and (ii) the removal energy of the two-nucleon spectator system (for the definition of the non relativistic spin-dependent spectral function see, e.g., Ref. [24]). With respect to previous attempts to describe DIS processes off $^3$He in a LF framework (see, e.g., the one in Ref. [25]), in our approach a special care is devoted to the definition of the intrinsic LF variables of the problem, as well as to the spin degrees of freedom in the definition of the spin-dependent spectral function. In general, for an $A$-body system the spin-dependent spectral function yields the probability distribution to find a constituent with a definite value of spin and momentum, while the $(A-1)$-constituent spectator system has a definite value of its mass. Such a distribution, properly convoluted with the probe-nucleon elementary cross-section, leads to the description of scattering processes off nuclei in impulse approximation. In this case, the motion of the knocked out nucleon is free, while the spectator system is fully interacting. Therefore one has to relativistically describe a final state where the cluster separability should be implemented. As shown in Ref. [18], this can be achieved by adopting the tensor product of a plane wave for the knocked out constituent and a fully interacting intrinsic state for the spectator system, with given mass, all moving in their intrinsic reference frame. In order to build the spin-dependent spectral function, one needs to evaluate overlaps between the final state, previously described, and the ground state of the three-nucleon system. As a consequence a crucial part of the paper is devoted to carefully define interacting and non interacting two- and three-body LF states, also providing the detailed link with the instant form counterparts. Notably, given the BT framework we have assumed, the instant form states in turn can be safely approximated by the corresponding non relativistic quantities, as explained in what follows. It should be pointed out that in order to describe the needed states, three reference frames are considered: i) the laboratory frame of the fully interacting three-body system; ii) the intrinsic LF frame of three free particles; iii) the intrinsic LF frame of a cluster of a free particle and an interacting two-particle subsystem.

Our paper is organized as follows: in Section II the LF kinematics is summarized and in Section III the LF dynamics of two- and three-particle systems is briefly described, and whenever possible, use has been made of appendices to collect and discuss in detail the relevant formal results. In Section IV the definition of the LF spin-dependent spectral function in terms of the above mentioned overlaps, as well as the LF momentum distributions and their sum rules are presented. Conclusion and perspectives are discussed in Section V.

### II. LIGHT-FRONT KINEMATICS

In this Section, for the sake of completeness and to establish the formalism, we briefly review the LF kinematics [15]. A generic LF four vector is $v = (v^-, \mathbf{v})$, with $\mathbf{v} = (v^+, v_\perp)$ and $v^\pm = v^0 \pm v^3$; moreover the scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{a} \cdot \mathbf{b} = (a^- b^+ + a^+ b^-)/2 - a_\perp \cdot b_\perp$. 

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$= (\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.
Let us consider a system of mass $M$ of $n$ on-mass-shell interacting particles of mass $m_i$, momenta $p_i$ $(i = 1, ..., n)$ and total momentum $P$ in the laboratory frame ($P^2 = M^2$). The minus components of the momenta are

$$ p_i^- = \frac{m_i^2 + |\mathbf{p}_i\perp|^2}{\not{p}_i^+} $$

and the following intrinsic variables (invariant under a LF boost) can be introduced

$$ \xi_i = \frac{p_i^+}{P^+}, \quad k_i\perp = p_i\perp - \frac{p_i^+}{P^+} \mathbf{P}_\perp = \mathbf{p}_i\perp - \xi_i \mathbf{P}_\perp. $$

The conserved total LF-momentum of the system (a three-dimensional one!) is given by

$$ P^+ = \sum_{i=1,n} p_i^+ \quad , \quad \mathbf{P}_\perp = \sum_{i=1,n} \mathbf{p}_i\perp $$

and as a consequence one has

$$ \sum_i \xi_i = 1 \quad , \quad \sum_i k_i\perp = 0. \quad (4) $$

One can complete the intrinsic variables adding the plus and minus components of the intrinsic momenta as follows:

$$ k_i^+ = \xi_i M_0 $$

$$ k_i^- = \frac{P^+}{M_0} \left[ p_i^- - 2 \mathbf{p}_\perp \cdot \frac{\mathbf{p}_\perp}{P^+} + \frac{p_i^+}{P^+} \left( \frac{\mathbf{P}_\perp}{P^+} \right)^2 \right] = \frac{m_i^2 + |\mathbf{k}_i\perp|^2}{k_i^-} \quad , $$

where $M_0$ is the invariant (for LF boosts) free mass, given by

$$ M_0^2 = P^+ \sum_i m_i^2 + |\mathbf{p}_i\perp|^2 - |\mathbf{P}_\perp|^2 = \sum_i \left( \frac{m_i^2 + |\mathbf{k}_i\perp|^2}{\xi_i} \right). \quad (6) $$

Then, in a more compact form

$$ k_i^\mu = \left[ B_{LF}^{-1} \left( \tilde{\mathbf{P}}/M_0 \right) \right]^{\mu}_{\nu} p_i^\nu $$

with $B_{LF} \left( \tilde{\mathbf{P}}/M_0 \right)$ a LF boost to the intrinsic rest frame of the system of $n$ free particles of momenta $p_i$.

Such a frame is defined by a total LF-momentum $\tilde{\mathbf{P}}_{\text{int}} = \{ \sum_i k_i^+ = M_0 \mathbf{0}_\perp \}$. Notice that $k_i^2 = p_i^2 = m_i^2$, since in the LFHD the constituents are put on the mass shell, as already mentioned. This feature, with the nice separation of the intrinsic motion from the global one, as shown in Eqs. (4) and (6) (see below), make straightforward the analogy with the non relativistic case.

Instead of the intrinsic variables $\xi_i$, one can introduce an alternative set of variables, namely

$$ k_{iz} = \frac{1}{2} \left[ k_i^+ - k_i^- \right] = \frac{M_0}{2} \left[ \xi_i - \frac{m_i^2 + |\mathbf{k}_i\perp|^2}{M_0^2 \xi_i} \right] $$

that fulfill the following constraint (cf Eqs. (4) and (6))

$$ \sum_{i=1,n} k_{iz} = 0. \quad (9) $$

Then, one can equally well use the LF intrinsic variables, $\{k_i^+, k_i\perp\}$, or the Cartesian intrinsic variables, $\mathbf{k}_i$, that fulfill

$$ \sum_{i=1,n} \mathbf{k}_i = 0. \quad (10) $$
To adopt the variables \( \mathbf{k} \), is useful for making evident the analogy with the non relativistic framework, still remaining in LFHD approach. In the case of free particles the intrinsic LF frame, defined by \( \mathbf{P}_{\text{intr}} \equiv \{ M_0, \mathbf{0} \} \), can be also defined by \( \mathbf{P} \equiv \mathbf{0} \). Let us recall that the bold character indicates a Cartesian vector, while the added tilde symbols indicates a LF three-vector.

Because of the positivity of \( \xi_i \), one can invert Eq. (8) obtaining

\[
\xi_i = \frac{k_{iz} + \sqrt{m_i^2 + |k_i|^2}}{M_0} = \frac{k_{iz} + E_i}{M_0},
\]

where \( E_i = \sqrt{m_i^2 + |k_i|^2} \). Then

\[
\sum_{i=1,n} E_i = M_0.
\]  

Let us stress that the minus component of the total momentum, \( P^- \), is different from the free one \( \text{[18]} \)

\[
P^- = \frac{M^2 + \sum_{i=1,n} p_i^-}{P^+} = \frac{m^2 + \sum_{i=1,n} p_{i\perp}^2}{P^+} = \frac{1}{P^+} \sum_{i=1,n} m_i^2 + \sum_{i=1,n} \xi_i^2 = P_{\text{free}}^-. \]

In terms of the free mass, one can rewrite \( P_{\text{free}}^- \) as follows

\[
P_{\text{free}}^- = \frac{1}{P^+} [M_0^2 + |\mathbf{P}_\perp|^2].
\]

For a particle of mass \( m \), the LF spin, that has the three components \( s_{\perp LF} \) in the particle rest frame, yields the Pauli-Lubanski four-vector in the reference where the particle has LF-momentum \( \mathbf{p} \), by applying a proper LF boost, \( B_{LF}(\mathbf{p}/m) \) (see, e.g., Ref. \( \text{[21]} \) for a detailed discussion of the LF spin). On the other hand, the canonical spin (instant form), \( s^c \), is obtained through a canonical boost, \( B_{c^{-1}}(\mathbf{p}/m) \), applied to the same Pauli-Lubanski four-vector. Therefore, the relation between the two spins is given by

\[
s^c = \left[ B_{c^{-1}}(\mathbf{p}/m) \right]_j \left[ B_{LF}(\mathbf{p}/m) \right]_\nu s^j_{\perp LF} = \left[ \mathcal{R}_{M}(\mathbf{p}) \right]_j \left[ \mathcal{R}_{M}(\mathbf{p}) \right]_\nu s^\nu_{\perp LF},
\]

where \( \mathcal{R}_{M}(\mathbf{p}) \), called Melosh rotation \( \text{[26, 27]} \), is the rotation between the two rest frames reachable through LF and canonical boosts, respectively \( \text{[18]} \). This rotation of spins implies the following relation between the plane wave states of a particle with spin \( s \) (notice that the squared spin does not depend on the chosen RHD form) in the instant form and the LF one

\[
|\mathbf{p}; s\sigma\rangle_c = \sum_\sigma D^s_{\sigma\sigma} [\mathcal{R}_{M}(\mathbf{p})] |\tilde{\mathbf{p}}; s\sigma\rangle_{LF},
\]

where \( D^s_{\sigma\sigma} [\mathcal{R}_{M}(\tilde{\mathbf{p}})] \) is the Wigner function for a spin \( s \). Within \( \text{SL}(2C) \), the covering set of the four-dimensional Poincaré group, the representation of the Melosh rotation for \( s = 1/2 \), relevant in what follows, is a \( 2 \times 2 \) matrix and reads as follow

\[
D^s_{\sigma\sigma} [\mathcal{R}_{M}(\mathbf{p})] = \chi^s_{\sigma} \frac{m + k^+ - i\sigma \cdot (\hat{z} \times \mathbf{k}_\perp)}{\sqrt{(m + k^+)^2 + |\mathbf{k}_\perp|^2}}, \quad \chi_{\sigma} =_{LF} \langle \mathbf{k}; s\sigma| \mathbf{k}; s\sigma\rangle_c,
\]

where \( \chi_{\sigma} \) is a two-dimensional spinor. The main feature of LF rotations, \( \mathcal{R}_{LF} \), is given by the difference between the corresponding Wigner rotations (that occurs when the state \( |\mathbf{k}; s\sigma\rangle_{LF} \) has to be transformed) and the rotations itself, differently from the case of instant-form rotations \( R_{IF} \) (where \( R_{IF} \) coincides with the associated Wigner rotation) \( \text{[18, 21]} \). This prevents the use of the usual Clebsch-Gordan coefficients for constructing the spin-spin and orbital-spin couplings within a LF framework, and therefore one has to exploit the relation \( \text{[16]} \) with the canonical spin.

We adopt the following normalization for the LF states \( |\tilde{\mathbf{p}}; s\sigma\rangle_{LF} \)

\[
_{LF}\langle \sigma' s, \tilde{\mathbf{p}}'| \tilde{\mathbf{p}}; s\sigma\rangle_{LF} = 2p^+(2\pi)^3 \delta^{3}(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}}) \sum_{\mu'\mu} D^s_{\sigma'\sigma} [\mathcal{R}_{M}(\mathbf{p})] D^\dagger_{\mu'\mu} \langle \mathcal{R}_{M}(\mathbf{p}) \rangle c(\mu'|s\mu)_c = \]

\[
= 2p^+(2\pi)^3 \delta^{3}(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}}) \delta_{\sigma'\sigma}
\]
and for the instant form states and spinors
\[
\langle p' | p \rangle = 2E (2\pi)^3 \delta(p'_z - p_z) \delta(p'_\perp - p_\perp), \quad \bar{u} u = 2m, \quad \bar{u}^\dagger u = 2E
\]
with \( E(p) = \sqrt{m^2 + |p|^2} \) and \( \partial p^+ / \partial p_z = 1 + p_z / p_0 = p^+ / p_0 \).

### III. LIGHT-FRONT DYNAMICS FOR TWO- AND THREE-PARTICLE SYSTEMS

In this Section a resumé of the main features of the BT construction, that allows one to consistently include the interaction in the generators of the Poincaré group (see, e.g., \[13\]), is presented. In particular, since for defining the LF spectral function one needs overlaps between the three-nucleon ground state and three-nucleon states composed by the tensor product of a plane wave for one of the particles and a two-body interacting state for the spectator pair, we will focus on two- and three-body cases.

#### A. Dynamics of two interacting particles

In the case of a system of two identical particles, the LFHD leads to an Ansatz for the two-body mass operator able to naturally embed a description based on the Schrödinger equation into a Poincaré-covariant framework (see, e.g., \[28–30\] for an application).

By eliminating the longitudinal LF variable \( \xi \) in favor of the third Cartesian component of the intrinsic momentum
\[
k_z = k_{1z} = M_0(1, 2) \left( \xi - \frac{1}{2} \right),
\]
where \( M_0^2(1, 2) \) is given by
\[
M_0^2(1, 2) = \frac{m^2 + |k_\perp|^2}{\xi(1 - \xi)} = 4 \left| E(k) \right|^2 = 4 \left( m^2 + |k|^2 \right),
\]
one can show the formal equivalence between a non relativistic description and a LF one. Moreover, one has
\[
k_1^+ = \xi M_0(1, 2) = k^+, \quad k_1^+ = (1 - \xi) M_0(1, 2) = M_0(1, 2) - k^+.
\]

The two-body Hamiltonian, with an interaction that depends upon intrinsic variables and fulfills the correct invariance properties under rotations and translations, leads to a square mass operator suitable for a Bakamjian-Thomas (BT) construction of the Poincaré generators \[23\]. This construction gives a simple way to introduce the interaction in the generators, while satisfying the correct commutation rules. As a matter of fact, within the BT framework the two-body mass equation can be written as follows (see, e.g., \[18–20\])
\[
\langle \sigma_1, \sigma_2; \tau_1, \tau_2; k | M_0^2(1, 2) + U(|k|) \rangle | j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle = M^2 \langle \sigma_1, \sigma_2; \tau_1, \tau_2; k | j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle
\]
\[
\langle \sigma_1, \sigma_2; \tau_1, \tau_2; k | \frac{4m^2 + 4|k|^2}{m^2} + U(|k|) \rangle | j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle = M^2 \langle \sigma_1, \sigma_2; \tau_1, \tau_2; k | j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle
\]
\[
\langle \sigma_1, \sigma_2; \tau_1, \tau_2; k | \frac{k^2}{m} + V(|k|) \rangle | j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle = \epsilon_{\text{int}} \langle \sigma_1, \sigma_2; \tau_1, \tau_2; k | j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle,
\]
where \( V = U/(4m) \) and
\[
\epsilon_{\text{int}} = \frac{M^2 - 4m^2}{4m}.
\]

In the last line of Eq. \[23\] one formally recovers the Schrödinger equation for a two-body intrinsic eigenstate (that does not depend upon the chosen RHD) of angular momentum \( (j, j_z) \), intrinsic energy \( \epsilon_{\text{int}} \) (negative for bound states and positive for the scattering ones) and isospin \( (T, T_z) \). The symbol \( \alpha \) represents the quantum numbers needed to completely define the state of the system. For the bound state (the deuteron in our case) one has \( M = 2m - B \), and then
reads (see Eq. (A18))

\[ \epsilon_{\text{int}} = -B + \frac{B^2}{4m} \sim -B \quad , \]

given the small binding energy of the deuteron with respect to its mass. For the scattering states, one has \( M^2 = s \), with \( s \) one of the Mandelstam variables, and asymptotically \( M^2 = 4m^2 + 4|t|^2 \) with \( t \) the asymptotic Cartesian momentum in the intrinsic frame. Then, one can write

\[ \epsilon_{\text{int}} = \frac{M^2 - 4m^2}{4m} = \frac{|t|^2}{m} . \]

Therefore the intrinsic eigenstates of Eq. (23) (i.e. of a Poincaré covariant mass operator) can be safely identified with the usual non relativistic two-body eigenstates (only for bound states one disregards terms \( O(B/4m) \)) and the overlap \( \langle \sigma_1, \sigma_2; \tau_1, \tau_2; k|j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle \), that contains canonical spins, with its non relativistic counterpart.

As discussed in Appendix A, the normalized LF two-body wave function is

\[ l_{\text{F}} \langle \sigma_1, \sigma_2; \tau_1, \tau_2; k, \tilde{P}; j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle_{\text{LF}} = 2 P^+ (2\pi)^3 \delta^3(\tilde{P}' - \tilde{P}) \sqrt{(2\pi)^3 E(k)} \times \]
\[ \sum_{\sigma'_1, \sigma'_2} D^+[R_M( \tilde{k} )]_{\sigma_1, \sigma'_1} D^+[R_M( - \tilde{k} )]_{\sigma_2, \sigma'_2} \langle \sigma'_1, \sigma'_2; \tau_1, \tau_2; k|j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle \quad , \]

where we define (cf Eq. (22))

\[ -\tilde{k} \equiv ((M_0 - k^+), -k_\perp) \quad . \]

It has to be emphasized that in the intrinsic two-body wave function \( \langle \sigma'_1, \sigma'_2; \tau_1, \tau_2; k|j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle \) the canonical spins can be composed with the orbital angular momenta by using the familiar Clebsh-Gordan coefficients. The state \( |k\rangle \) (with Cartesian variables) is normalized as follows

\[ \langle k'|k \rangle = \delta(k' - k) \quad . \]

Notice the difference with Eq. (19). Furthermore, for the two-body interacting case the LF completeness reads (see Eq. (A18))

\[ \int \frac{d\tilde{P}}{2P^+(2\pi)^3} \sum_{j, j_z, \alpha} \sum_{TT_z} \sum_{t} \lambda(t) \ dt |\tilde{P}; j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle_{\text{LF}} l_{\text{F}} \langle T_z T; \alpha, \epsilon_{\text{int}}; j_z, j; \tilde{P} | = \mathbf{I} \quad , \]

where the symbol \( \sum \) means a sum over the bound states of the pair (namely the deuteron in the present case) and an integration over the continuum. Notice the choice of the Cartesian \( t \) momentum to label the intrinsic energy. The quantity \( \lambda(t) \) is the t-density of the two-body states (\( \lambda(t) = 1 \) for the bound states and \( \lambda(t) = t^2 \) for the continuum). Such a completeness follows from the one fulfilled by the eigensolutions of Eq. (23), i.e.

\[ \sum_{j, j_z, \alpha} \sum_{TT_z} \sum_{t} \lambda(t) \ dt \langle k'|j, j_z; \epsilon_{\text{int}}, \alpha; TT_z \rangle \langle T_z T; \alpha, \epsilon_{\text{int}}; j_z, j; k \rangle = \delta^3(k' - k) \quad . \]

**B. Three interacting particle systems**

In order to have a Poincaré covariant description of an interacting system, like the \(^3\)He nucleus, it seems appropriate to adopt the LFHD framework, combined with a Bakamjian-Thomas (BT) construction of the Poincaré generators. With a suitable Ansatz for the interaction (see e.g. [18, 20]), the mass operator is

\[ M(1, 2, 3) = M_0(1, 2, 3) + V(1, 2, 3) = \sum_{i=1,3} \sqrt{m_i^2 + |k_i|^2 + V(k_i \cdot k_j)} \quad , \]

where \( V \) is the interaction potential.
where \( \mathbf{k}_i \) are the intrinsic momenta defined in Sect. II and the interaction \( \mathcal{V} \) is invariant for rotations and translations. The ground state can be written as the product of a plane wave describing the global motion with LF momentum \( \mathbf{P} \) times eigenvectors of the three-body mass operator in Eq. (32). It reads

\[
|\mathbf{P} ; j, j_z, \xi_3, \eta; \frac{1}{2}, T_z\rangle_{LF},
\]

where \( \epsilon_{\text{int}} = M_3 - 3m \) is the energy, \( j \) the total angular momentum, \( 1/2 \) the isospin of the system and \( \Pi \) the parity. From now on, we assume that the three particles have the same mass.

When applications like DIS or SIDIS processes are concerned, the issue of macrocausality has to be considered, i.e. if the subsystems which compose a system are brought far apart, the Poincaré generators of the system have to become the sum of the Poincaré generators corresponding to the subsystems in which the system is asymptotically separated. It is important to notice that the packing operators [18, 31], that make it possible to include the macrocausality, are not considered in the present approximation for the description of the bound state. However, we implement macrocausality in the tensor product of a plane wave for the knocked out constituent times a fully interacting intrinsic state for the spectator pair. This tensor product is needed for the definition of the LF Spectral Function, as shown below.

In a given frame, the LF three-body wave function can be expressed in terms of the intrinsic wave function, with canonical spins. Therefore, as in the two-body case, one can approximate such an intrinsic wave function by the corresponding non relativistic wave function, after checking that the non relativistic Schrödinger operator can be properly identified with a BT mass operator. Then the key point for actual calculations is

\[
M_{NR}(1, 2, 3) = 3m + \sum_{i=1,3} \frac{k_i^2}{2m} + V_{12}^{NR} + V_{23}^{NR} + V_{31}^{NR} + V_{123}^{NR}
\]

fulfills rotational and translational invariance, namely the general properties for making a mass operator acceptable as a BT mass operator. As a matter of fact, those properties are just the ones satisfied by the non relativistic nuclear interactions that give an accurate description of two- and three-nucleon data (see, e.g., [11, 32]). An early investigation of the electromagnetic tri-nucleon systems, within the above illustrated approach and using the refined non relativistic ground states of Ref. [33], can be found in Ref. [34].

### 1. Non-symmetric intrinsic variables

To define the LF spectral function one needs the overlaps between the ground state of the three-body system and the states composed by the tensor product of a free nucleon and a fully interacting two-body system. Therefore proper variables, suited to describe these states, have to be introduced. Instead of the symmetric intrinsic variables \( \mathbf{k}_i \) (\( i = 1, 2, 3 \)) that refer to the three particles moving in the three-body intrinsic frame, it is more suitable to introduce non symmetric variables. Let us consider the intrinsic variable \( \tilde{\mathbf{k}}_j \) for particle \( j \) and the intrinsic variables for the internal motion of the spectator pair. For the sake of concreteness, let us take \( j = 1 \) and focus on the kinematics of the (2,3) pair, that globally moves in the three-body intrinsic frame with total LF momentum \( (K^+_{23}, K^\perp_{23}) \). A set of intrinsic variables for the internal motion of the (2,3) pair can be defined as follows

\[
\eta = \frac{k_1^+}{k_2^+ + k_3^+} = \frac{\xi_2}{\xi_2 + \xi_3} = \frac{\xi_2}{1 - \xi_1} = \frac{p_2^+}{p_2^+ + p_3^+},
\]

\[
k_{23\perp} = k_{2\perp} - \eta (k_{2\perp} + k_{3\perp}) = k_{2\perp} + \eta k_{1\perp}
\]

\[
k_{23}^+ = \eta M_{23}
\]

\[
k_{23z} = M_{23} \left( \eta - \frac{1}{2} \right)
\]

where \( k_1^+ = \sqrt{m^2 + |\mathbf{k}_1|^2} + k_{1z} \) and \( M_{23} \) is the free mass for the (2,3) pair, defined as in Eq. (21),

\[
M_{23}^2 = \frac{m^2 + |k_{23\perp}|^2}{\eta(1 - \eta)} = \left[ 2\sqrt{m^2 + |k_{23\perp}|^2} \right]^2.
\]
Furthermore, the total LF momentum of the free \((2,3)\) pair in the laboratory frame is
\[
\begin{align*}
P_{23}^+ &= p_2^+ + p_3^+ \\
P_{23\perp} &= p_{2\perp} + p_{3\perp}
\end{align*}
\]  
(37)
while in the intrinsic three-body frame the total LF momentum is
\[
\begin{align*}
K_{23}^+ &= k_2^+ + k_3^+ \\
K_{23\perp} &= k_{2\perp} + k_{3\perp} = -k_{1\perp}
\end{align*}
\]  
(38)
In terms of the non-symmetric intrinsic variables, the free mass of the three-particle system can be written as follows
\[
M_0(1,2,3) = \sum_{i=1,3} \sqrt{m^2 + |k_i|^2} = \sqrt{m^2 + |k_1|^2} + \sqrt{M_{23}^2 + |k_1|^2} = \frac{m^2 + |k_1|^2}{k_1^+} + \frac{M_{23}^2 + |k_1|^2}{K_{23}^+} .
\]  
(39)
Then one has
\[
\frac{m^2 + |k_{2\perp}|^2}{k_2^+} + \frac{m^2 + |k_{3\perp}|^2}{k_3^+} = \frac{1}{k_2^+ + k_3^+} \left[ M_{23}^2 + |k_{2\perp} + k_{3\perp}|^2 \right] ,
\]  
(40)
and therefore
\[
M_{23}^2 = \frac{m^2 + |k_{2\perp}|^2}{\eta} + \frac{m^2 + |k_{3\perp}|^2}{(1-\eta)} - |k_1|^2 .
\]  
(41)

2. Three-body light-front wave function with non-symmetric intrinsic variables

For the fully interacting case, \(V(1,2,3) \neq 0\), the three-body LF wave function, can be expressed through (i) the non-symmetric intrinsic variables \(\{\tilde{k}_1, \tilde{k}_{23}\}\) introduced in the previous subsection, instead of using the three-body standard Jacobi coordinates (defined through \(k_1, k_2, k_3\)), and (ii) canonical spins in the reference frame where \(P^+ = M_0(1,2,3)\). Therefore, by repeating analogous steps as in the two-body case (cf. Eq. (27)) one has
\[
\begin{align*}
&\langle LF|\sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3; \tilde{k}_1, \tilde{k}_{23}, \tilde{P}; j, j_z; c_{int}^3; \Pi; \frac{1}{2} T_z | LF \rangle = 2 P^+ (2\pi)^3 \delta^3(\tilde{P} - \tilde{P}) \\
&\times \sum_{\sigma_1'} \sum_{\sigma_2'} \sum_{\sigma_3'} \frac{D^{\frac{1}{2}}[\mathcal{R}_M(\tilde{k}_1)]_{\sigma_1, \sigma_1'} D^{\frac{1}{2}}[\mathcal{R}_M(\tilde{k}_2)]_{\sigma_2, \sigma_2'} D^{\frac{1}{2}}[\mathcal{R}_M(\tilde{k}_3)]_{\sigma_3, \sigma_3'}}{\sqrt{(2\pi)^6 2E_1 E_2 M_2}} \\
&\times \langle \sigma_1', \sigma_2', \sigma_3'; \tau_1, \tau_2, \tau_3; k_1, k_{23}; j, j_z; c_{int}^3; \Pi; \frac{1}{2} T_z \rangle ,
\end{align*}
\]  
(42)
where \(E_{23} = \sqrt{M_{23}^2 + |k_1|^2}\) and \(M_{23} = |m^2 + |k_{23}\perp|^2 + (k_{23\perp})^2) / k_{23\perp}\). The LF variables \(\tilde{k}_2\) and \(\tilde{k}_3\) can be easily obtained from \(\tilde{k}_1\) and \(\tilde{k}_{23}\). Indeed one has (i) \(\eta = k_{23\perp} / M_{23}\), (ii) \(k_{2\perp} = k_{23\perp} - \eta k_{1\perp}\), (iii) \(k_{3\perp} = -k_{1\perp} - k_{2\perp}\), (iv) \(k_2^+ + k_3^+ = M_0(1,2,3) - k_1^+\). In Eq. (42), the intrinsic wave function with canonical spins \(\langle \sigma_1', \sigma_2', \sigma_3'; \tau_1, \tau_2, \tau_3; k_1, k_{23}; j, j_z; c_{int}^3; \Pi; \frac{1}{2} T_z \rangle\) is the eigensolution of the mass operator \(M(1,2,3)\) of Eq. (42), that in actual calculation can be approximated by the non relativistic Hamiltonian operator (since, we repeat, the symmetry requirements are the same). As shown in Appendix A (see Eq. (219)), the factors in Eq. (42) allow one to recover the normalization for the intrinsic part of the three-body bound state according to
\[
\sum_{\tau_1, \tau_2, \tau_3} \sum_{\sigma_1, \sigma_2, \sigma_3} \int d\tilde{k}_1 \int d\tilde{k}_{23} \left| \langle \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3; k_1, k_{23}; j, j_z; c_{int}^3; \Pi; \frac{1}{2} T_z \rangle \right|^2 = 1 ,
\]  
(43)
like in the non relativistic case.
Because of our interest in constructing the overlap between the three-nucleon ground state and a state where only the pair (2, 3) is interacting, while the third nucleon is free, in what follows we investigate the corresponding mass operator, whose eigenstates are the tensor product we have already mentioned. By using the intrinsic variables \( \{ \xi_1, \kappa_{1\perp} \} \), one can introduce the squared free-mass, \( M_0^2(1, 23) \), for the cluster (1, 2, 3), when the mass eigenvalue of the interacting (2, 3) pair is \( M_S \)

\[
M^2_0(1, 23) = \frac{m^2 + |k_{1\perp}|^2}{\xi_1} + \frac{M_S^2 + |k_{1\perp}|^2}{(1 - \xi_1)} .
\]

The intrinsic frame of the cluster (1, 23) is defined by \( \tilde{\Pi}_{int}(1, 23) \equiv \{ M_0, 0_{\perp} \} \). In this frame, the LF momentum of the nucleon 1 is given by

\[
\kappa_1^\perp = \xi_1 M_0(1, 23) \quad \kappa_{1\perp} = p_{1\perp} - \xi_1 p_\perp = k_{1\perp} ,
\]

while the \( z \) Cartesian component reads (see Eq. (48))

\[
\kappa_{1z} = \frac{1}{2} [\kappa_1^+ - \kappa_1^-] = \frac{M_0(1, 23)}{2} \left[ \xi_1 - \frac{m^2 + |k_{1\perp}|^2}{M_0(1, 23)^2} \right] \xi_1 .
\]

As a consequence one has

\[
M_0(1, 23) = E(\kappa_1) + E_S \quad (47)
\]

with \( E(\kappa_1) = \sqrt{m^2 + |\kappa_1|^2} \) and \( E_S = \sqrt{M_S^2 + |\kappa_1|^2} \).

The total momentum of the (2, 3) pair in the same frame is

\[
K^+_S = (1 - \xi_1) M_0(1, 23) \quad K_{S\perp} = -\kappa_{1\perp} = -k_{1\perp} + k_{2\perp} + k_{3\perp} \quad K_{Sz} = -\kappa_{1z} \quad K_{Son}^- = M_S^2 + |k_{1\perp}|^2 \frac{M_0}{K^+_S} \quad (48)
\]

Summarizing the pair (2, 3), with internal variables \( \{ \eta, k_{23\perp} \} \) and mass eigenvalue \( M_S \) (cf Eqs. \( \ref{EQS} \), \( \ref{EQS} \) ), is moving with LF momentum \( \tilde{\Pi}_S \) in the intrinsic frame of the three-particle cluster (1, 23).

It should be pointed out that the intrinsic frame for the three-body system (1, 2, 3) and the intrinsic frame of the (1, 23) cluster are related by a proper longitudinal LF boost that makes the change \( P^+_\text{int}(1, 2, 3) = M_0(1, 23) \rightarrow P^+_\text{int}(1, 2, 3) = M_0(1, 2, 3) \).

4. Non-symmetric basis for three interacting particle systems

In the 1 + (23) cluster only the interaction \( U_{23} \) between particles 2 and 3 is active; then one can introduce a three-body state given by the tensor product of an eigenstate of the total LF momentum, \( \tilde{\Pi} \), times the intrinsic state of the cluster with a given mass for the interacting pair. In turn, such an intrinsic state, that fulfills the macrocausality \( \ref{EQS} \), is given by the tensor product of a plane wave for particle 1 with LF momentum \( \kappa_1 \), times the fully interacting state of the pair corresponding to the given energy eigenvalue. Therefore, one can write

\[
|\tilde{\Pi}; \kappa_1, \sigma_1; j_{23}, j_{23\perp}, \kappa_{23}, \alpha; T_{23}, \tau_{23}\rangle_{LF} \quad (49)
\]

which is an eigenstate of the mass operator

\[
M'(1, 23) = E(\kappa_1) + \sqrt{M^2_{23}(\kappa_{23}) + U_{23} + |\kappa_1|^2} = E(\kappa_1) + \sqrt{M^2_{23}(|\kappa_{23}|) + |\kappa_1|^2} \quad (50)
\]
with eigenvalue $M_0(1, 23) = E(\kappa_1) + E_S$ ($E_S = \sqrt{M_S^2 + |\kappa_1|^2}$). The operator $M_{23}^2(|k_{23}|) = M_0^2(|k_{23}|) + U_{23}(|k_{23}|)$ is the square of the intrinsic mass operator of the interacting $(2, 3)$ pair, with eigenvalue $M_0^2 = 4(m^2 + m_\epsilon)$ (see Eq. (23)).  

The set of eigenstates $|\tilde{k}\rangle$ is complete with the following completeness relation

$$\int \frac{d\tilde{k}}{2P^+(2\pi)^3} \sum_{\tau_1} \sum_{\lambda_1} \sum_{\sigma_1} \int \frac{d\kappa_1}{2\kappa_1^+(2\pi)^3} \times |\tilde{P}; 1, \sigma_1; 1; \tau_1; 1; \alpha; T_{23}, \tau_{23} \rangle_{LF} L_F(T_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23}; \tau_1; \alpha_1; \kappa_1; \tilde{P}| = I \quad (51)$$

Since it will play a relevant role for a proper definition of the LF spectral function, let us consider the overlap between the eigenstates $|\tilde{k}\rangle$ and the product of plane waves for (i) the total LF momentum $P'$ for a system of three free particles, (ii) the LF momentum of particle 1, $\tilde{k}'_1$, in the intrinsic frame of the three free particles and (iii) the LF momentum, $\tilde{k}'_{23}$, for the intrinsic motion of the free subsystem $(2, 3)$. One has

$$L_F(\sigma'_1, \sigma'_2, \sigma'_3; \tau'_1, \tau'_2, \tau'_3; \tilde{k}'_1, \tilde{k}'_{23}; \tilde{P}; \kappa_1 \sigma_1; 1; j_{23}, j_{23}; \epsilon_{23}; \alpha; T_{23}, \tau_{23})_{LF} =$$

$$= 2 P^+(2\pi)^3 \delta^3(\tilde{P}' - \tilde{P}) \delta_{\tau_1, \tau'_1} \delta_{\sigma_1, \sigma'_1} \times \sum_{\sigma_2} \sum_{\sigma_3} D^+[R_M(\tilde{k}'_{23})]|_{\sigma'_2, \sigma'_3} D^+[R_M(-\tilde{k}'_{23})]|_{\sigma_2, \sigma_3} \left(\sqrt{(2\pi)^3 \frac{E_{23}^2 M_{23}^2}{2M_0^2(1, 2, 3)}}\right) \left(\sqrt{(2\pi)^3 \frac{E_{23}^2 M_{23}^2}{2M_0^2(1, 2, 3)}}\right) \quad (52)$$

where $E_{23} = \sqrt{M_{23}^2 + k_{23}^2}$. $M_{23}^2 = [m^2 + |k'_{23}]^2 + (k'_{23}^2)^2 / k'_{23}^2$, 

$$M_0(1, 2, 3) = \sqrt{m^2 + |k'_{23}]^2 + \sqrt{M_{23}^2 + |k'_{23}]^2} \quad (53)$$

and $-\tilde{k}'_{23} = ((M_{23} - k_{23}^2), -k_{23}^2)$. The right-hand side of Eq. (52) reflects: (i) the normalization properties of $|\tilde{k}'_1\rangle_{LF}$ and $|\tilde{k}_1\rangle_{LF}$; (ii) the expression for the intrinsic wave function of the interacting pair $(2, 3)$; (iii) the proper overall normalization factors.

In Appendix C the correctness of the normalization factors in Eq. (52) is checked. To obtain the last step in Eq. (52), one has to notice that the states $|\kappa_1 \sigma_1\rangle_{LF}$ and $|\tilde{k}_1\rangle_{LF}$ are immediately related to the same LF state $|\tilde{k}_1, \kappa_1 = k_{1L}, \sigma_1\rangle$, since $\xi_1 = k_1^2 / M_0(1, 23) = k_1^2 / M_0(1, 2, 3)$. The two states differ for their normalization, i.e.

$$L_F(|\tilde{k}'_1\rangle |\kappa_1\rangle_{LF}) = (2\pi)^3 2k_1^+ \delta^3(\tilde{k}'_1 - \tilde{k}_1) \quad (54)$$

and

$$L_F(|\tilde{k}'_1\rangle |\kappa_1\rangle_{LF}) = (2\pi)^3 2k_1^+ \delta^3(\tilde{k}'_1 - \tilde{k}_1) \quad . \quad (55)$$

In Eq. (52) $k_1^{(a)}$ is obtained by transforming $k_1^+$ from the frame where $P^+ = M_0(1, 23)$ to the frame where $P^+ = M_0(1, 2, 3)$ through a longitudinal LF boost, while $k_{1L}^{(a)}$ remains unchanged, i.e. one has $k_{1L}^{(a)} = \tilde{k}_{1L}$ (see Eq. (47)). To determine $k_1^{(a)}$ one can first evaluate $M_0(1, 23)$ from Eq. (47)

$$M_0(1, 23) = \frac{(k_1^+)^2 + (m^2 + k_{1L}^2)}{2k_1^+} + \left[\frac{(k_1^+)^2 + (m^2 + k_{1L}^2)}{2k_1^+}\right]^2 + M_0^2 - m^2 \quad (56)$$

Then one can obtain $\xi_1$

$$\xi_1 = \frac{k_1^+}{M_0(1, 23)} \quad , \quad (57)$$
the three-body system free mass \( M_0(1, 2, 3) \)

\[
M_0^2(1, 2, 3) = \frac{m^2 + k_1^2}{\xi_1} + \frac{M_0^2 + k_1^2}{1 - \xi_1}
\]

and

\[
k_1^{(a)} = \xi_1 M_0(1, 2, 3).
\]

5. Overlaps between cluster states and the bound-state of the three-particle system

The overlap between a state of the cluster 1 + (23) and the bound state of the three-particle system is the needed quantity for defining the LF spin-dependent spectral function. As a matter of fact, from Eqs. (58) and (59), one has

\[
LF\langle T_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23z}; \tau_1, \sigma_1; k_1; \bar{\mathbf{P}}'; j, j_z; \epsilon_{int}; \Pi; \frac{1}{2} T_z \rangle =
\]

\[
= 2P^+(2\pi)^3 \delta^3(\mathbf{P}' - \mathbf{P}) \langle T_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23z}; \tau_1, \sigma_1; k_1| j, j_z; \epsilon_{int}; \Pi; \frac{1}{2} T_z \rangle.
\]

As shown in Appendix C.2, after inserting in the intrinsic part of the overlap (60) (i) the completeness operator expressed through plane waves, i.e. (cf Eq. (B10))

\[
\int \frac{d\mathbf{k}_{23}}{k_{23}^2(2\pi)^3} | \mathbf{k}_{23} \rangle \langle \mathbf{k}_{23}| = I,
\]

and (ii) Eqs. (61) and (62), one gets

\[
LF\langle T_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23z}; \tau_1, \sigma_1; k_1| j, j_z; \epsilon_{int}; \Pi; \frac{1}{2} T_z \rangle = \sum_{\tau_2 \tau_3} \int d\mathbf{k}_{23} \sum_{\sigma'_1} D^\dagger_R | R_M(\mathbf{k}_{1}^{(a)}) \rangle_{\sigma_1} \sigma'_1
\]

\[
\times \sqrt{(2\pi)^3 2E(\mathbf{k}_{1}^{(a)})} \sqrt{k_{1}^2(2\pi)^3} \prod_{\sigma'_2, \sigma'_3} \frac{E_{23}}{E_S} \sum_{\sigma'_2, \sigma'_3} D_{\sigma'_2, \sigma'_3}(\mathbf{k}_{23}, \mathbf{k}_2) D_{\sigma'_2, \sigma'_3}(-\mathbf{k}_{23}, \mathbf{k}_3)
\]

\[
\times \langle T_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23z}; \mathbf{k}_{23}, \sigma'_2, \sigma'_3; \tau_2, \tau_3; \mathbf{k}_2, \mathbf{k}_1| j, j_z; \epsilon_{int}; \Pi; \frac{1}{2} T_z \rangle,
\]

where the unitary matrices \( D_{\sigma'_2, \sigma'_3} \) are defined by the equation

\[
D_{\sigma'_2, \sigma'_3}(\pm \mathbf{k}, \mathbf{k}_i) = \sum_{\sigma_i} D^\dagger_R | R_M(\pm \mathbf{k}_{23}) \rangle_{\sigma'_2} D_{\sigma'_2, \sigma'_3}(-\mathbf{k}_{23}, \mathbf{k}_3)
\]

with the + sign corresponding to \( i = 2 \) and the – sign corresponding to \( i = 3 \).

Then the overlap of Eq. (60) can be evaluated by approximating \( \langle T_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23z}; \mathbf{k}_{23}, \sigma'_2, \sigma'_3; \tau_2, \tau_3 \rangle \)

and \( \langle \sigma'_2, \sigma'_3; \tau_2, \tau_3; \mathbf{k}_{23}, \mathbf{k}_1| j, j_z; \epsilon_{int}; \Pi; \frac{1}{2} T_z \rangle \) with the corresponding non relativistic quantities. It should be recalled that the spins involved are canonical spins.

The normalization for the intrinsic LF overlap in Eq. (60) follows immediately from the completeness relation

\[
\sum_{T_{23} \tau_{23}} \int \frac{d\mathbf{k}}{2k_1^2(2\pi)^3} \sum_{\sigma_1} \sum_{j_{23}, j_{23z}, \alpha} LF\langle T_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23z}; \tau_1, \sigma_1; k_1| j, j_z; \epsilon_{int}; \Pi; \frac{1}{2} T_z \rangle^2 =
\]

\[
= \left| j, j_z; \epsilon_{int}; \Pi; \frac{1}{2} T_z \right|^2 = 1.
\]
IV. THE LF SPIN-DEPENDENT SPECTRAL FUNCTION

The non relativistic spin-dependent spectral function $\hat{P}_{\mathcal{M}}(\vec{p}, E)$ for a nucleus of mass number $A$ is a $2 \times 2$ matrix, whose elements are

$$P_{\sigma, \sigma'; \mathcal{M}}^\tau(\vec{p}, E) = \sum_{\Gamma_{A-1}} \langle \tilde{\psi}, \sigma; \psi_{\Gamma_{A-1}} \rangle \langle \psi_{\Gamma_{A-1}}; \tilde{\psi}, \sigma' \rangle \delta(E - E_{\Gamma_{A-1}} + E_A),$$

where $|\psi_{\mathcal{M}}\rangle$ is the ground state of the nucleus with energy $E_A$ and polarized along $\vec{S}$. $|\psi_{\Gamma_{A-1}}\rangle$ is an eigenstate of the $(A-1)$ nucleon system with energy $E_{\Gamma_{A-1}}$, interacting with the same interaction of the nucleus, $|\vec{p}, \sigma \rangle$ the plane wave for the nucleon $\tau = \pm 1/2$, with momentum $\vec{p}$ in the nucleon rest frame and spin along the $\mathbf{z}$-axis equal to $\sigma$. Let us recall that the rotations involved act on the three-nucleon bound system as a whole, and therefore they are interaction-free.

In a more compact form, for $\mathcal{J} = 1/2$, the $2 \times 2$ matrix $\hat{P}_{\mathcal{M}}(\vec{p}, E)$ is given by

$$\hat{P}_{\mathcal{M}}(\vec{p}, E) = \frac{1}{2} \left[B_{0,\mathcal{M}}^\tau(\vec{p}, E) + \vec{\sigma} \cdot \vec{f}_{\mathcal{M}}(\vec{p}, E)\right],$$

where the function $B_{0,\mathcal{M}}^\tau(\vec{p}, E)$ is the trace of $\hat{P}_{\mathcal{M}}(\vec{p}, E)$ and yields the usual unpolarized spectral function $P^\tau(\vec{p}, E)$. It should be noticed that the matrix $\hat{P}_{\mathcal{M}}(\vec{p}, E)$ and the pseudovector $\vec{f}_{\mathcal{M}}(\vec{p}, E)$ depend on the direction of the polarization vector $\vec{S}$. Since $\vec{f}_{\mathcal{M}}(\vec{p}, E)$ is a pseudovector, it is a linear combination of the pseudovectors at our disposal, viz. $\vec{S}$ and $\vec{p} \cdot \vec{S}$, and therefore it can be put in the following form, where any angular dependence is explicitly given,

$$\vec{f}_{\mathcal{M}}(\vec{p}, E) = \vec{S} B_{1,\mathcal{M}}(\vec{p}, E) + \vec{p} \cdot \vec{S} B_{2,\mathcal{M}}(\vec{p}, E).$$

Let us focus on the $A = 3$ case. To obtain a Poincaré covariant definition of the spin-dependent spectral function for a three-particle system within the LF dynamics, one replaces the non relativistic overlaps $\langle \tilde{\psi}, \sigma; \psi_{\Gamma_{A-1}} \rangle$, which define the non relativistic spectral function, with their LF counterparts $\text{LF}_{}(\tau_S, T_S; \alpha, \epsilon, J_j J_j; \sigma, \kappa; |\Psi_0; S, T_z\rangle)$, dependent upon the energy $\epsilon$ of the two-body system and upon the intrinsic momentum, $\kappa$, of the third particle in the intrinsic reference frame of the cluster $1 + (23)$ (cf Sec. III D 5). The LF overlaps $\text{LF}_{}(\tau_S, T_S; \alpha, \epsilon, J_j J_j; \sigma, \kappa; |\Psi_0; S, T_z\rangle)$ can be easily obtained from the overlaps of Eq. 62, writing through Eq. 66 the ground state $|\Psi_0; S, T_z\rangle$ of the three-body system, polarized along $\vec{S}$, in terms of the states $|j, j_z; \epsilon_m, \Pi; \frac{1}{2} T_z\rangle$, polarized along the $\mathbf{z}$ axis.

Then, within the LFHD one can define the spin-dependent nuclear spectral function for the three-nucleon system ($^3\text{He}$ or $^3\text{H}$) in the bound state $|\Psi_0; S, T_z\rangle$, as follows

$$P_{\sigma, \sigma'}(\kappa^+, \kappa_-, \kappa^-, S) = \int d\kappa \rho(\kappa) \left(\kappa^- - M_3 + \frac{|\kappa_+|^2}{1 - \xi}\right)$$

$$\times \sum_{J_j, \alpha} \sum_{T_S T_S} \text{LF}_{}(\tau_S, T_S; \alpha, \epsilon; J_j J_j; \sigma, \kappa; |\Psi_0; S, T_z\rangle) \langle S, T_z; \Psi_0 | \kappa, \sigma; J_j J_j; \epsilon, \alpha, T_S, \tau_S\rangle_{\text{LF}}$$

$$= \frac{1}{\partial \kappa^-} \rho(\kappa) \sum_{J_j, \alpha} \sum_{T_S T_S} \text{LF}_{}(\tau_S, T_S; \alpha, \epsilon; J_j J_j; \sigma, \kappa; |\Psi_0; S, T_z\rangle) \langle S, T_z; \Psi_0 | \kappa, \sigma; J_j J_j; \epsilon, \alpha; T_S, \tau_S\rangle_{\text{LF}}$$

$$= \frac{\partial}{\partial \kappa^-} P_{\sigma, \sigma'}(\kappa, \epsilon, S),$$
where

\[ \epsilon = \frac{(M_3 - \kappa^-)(1 - \xi)M_3 - |\kappa_\perp|^2}{4m} - m \]  

(70)

is the intrinsic energy of the fully interacting two-nucleon eigenstate, \( \rho(\epsilon) \) the density of the two-body states (\( \rho(\epsilon) = \frac{tm}{2} \) for the two-body continuum states and \( \rho(\epsilon) = 1 \) for the deuteron bound state), \( M_3 \) the nucleus mass, \( \xi = \frac{\kappa^+}{M_0(1,23)} \) (cf Eqs. (15) and (63)) and

\[ \frac{\partial \epsilon}{\partial \kappa^-} = \frac{(1 - \xi)M_3}{4m} . \]  

(71)

Let us notice that the variable \( \kappa^- \) is the \(-\) component of the momentum of an off mass shell nucleon, as it is clear from the \( \delta \) function in Eq. (19). In Eq. (19) \( \tau = \pm 1/2, J, J_z \) is the spin, \( T_S, \tau_S \) the isospin, \( \alpha \) the set of quantum numbers needed to completely specify the two-body eigenstate, and \( M_3^2 = (m^2 + me) \).

The overlap \( L_F \langle \tau_S, T_S; \alpha, \epsilon; J, J_z; \tau, \sigma, \kappa | \Psi_0, S, T_z \rangle \) is the one defined by Eqs. (91) and (92). In the special case where \( \bar{S} \) is along the z-axis, one obtains

\[ P^\tau_{\kappa, \sigma}(\kappa, \epsilon, S) = \rho(\epsilon) \times \sum_{JJ_\tau, \tau_S T_S} L_F \langle \tau_S, T_S; \alpha, \epsilon; JJ_\tau; \tau, \sigma, \kappa | j, j_z; \epsilon, \alpha, \tau; T_S, \tau_S \rangle_{\kappa, \sigma} \]  

(72)

and the LF spectral function can be evaluated through the explicit expression (52) for the overlap \( L_F \langle \tau_S, T_S; \alpha, \epsilon; J, J_z; \tau, \sigma, \kappa | \Psi_0, S, T_z \rangle \) in terms of canonical two- and three-body wave functions. In turn, these wave functions can be replaced by the non relativistic ones. We emphasize once more that the two- and three-body non relativistic wave functions have all the needed properties with respect to rotations and translations of the corresponding canonical wave functions.

According to the completeness relation (21), the normalization of the spectral function reads (see also Eq. (64) and Appendix [C])

\[ \int \frac{d\kappa}{2E(\kappa)(2\pi)^3} \sum_\tau Tr P^\tau(\kappa, \epsilon, S) = 1 . \]  

(73)

However, in applications one can normalize the spectral function \( P^\tau(\kappa, \epsilon, S) \) for each isospin channel, i.e.,

\[ \int \frac{d\kappa}{2E(\kappa)(2\pi)^3} Tr P^\tau(\kappa, \epsilon, S) = 1 . \]  

(74)

As it occurs for the non relativistic spectral function (see Eqs. (57) and (58)), the LF nucleon spin-dependent spectral function can be expressed by means of three scalar functions, \( B^\tau_{0,S}(\kappa, \epsilon) \), \( B^\tau_{1,S}(\kappa, \epsilon) \) and \( B^\tau_{2,S}(\kappa, \epsilon) \),

\[ P^\tau_{\kappa, \sigma}(\kappa, \epsilon, S) = \frac{1}{2} \left( B^\tau_{0,S}(\kappa, \epsilon) + \sigma \cdot f^\tau_S(\kappa, \epsilon) \right)_{\sigma, \sigma} , \]  

(75)

where

\[ f^\tau_S(\kappa, \epsilon) = \frac{S B^\tau_{1,S}(\kappa, \epsilon) + \kappa \cdot (S) B^\tau_{2,S}(\kappa, \epsilon)}{} . \]  

(76)

The function \( B^\tau_{0,S}(\kappa, \epsilon) \) is the trace of \( P^\tau_{\kappa, \sigma}(\kappa, \epsilon, S) \) and yields the unpolarized spectral function.

A. The LF nucleon momentum distributions and momentum sum rule

Within the LFHD, one can define the LF spin-independent nucleon momentum distribution, averaged on the spin directions, through the spectral function \( P^\tau_{\kappa, \sigma}(\kappa, \epsilon, S) \) as follows

\[ n^\tau(\xi, \kappa_\perp) = \int \frac{d\kappa^+}{2(2\pi)^3} \int \frac{d\xi}{2(2\pi)^3} \frac{\partial \kappa^+}{\partial \xi} Tr P^\tau(\kappa, \epsilon, S) \times \rho(\epsilon) \sum_\sigma \sum_{JJ_\tau, \tau_S T_S} L_F \langle \tau_S, T_S; \alpha, \epsilon; J, J_z; \tau, \sigma, \kappa | \Psi_0, S, T_z \rangle \langle S, T_z; \Psi_0 | \kappa, \sigma; J, J_z; \epsilon, \alpha; T_S, \tau_S \rangle_{\kappa, \sigma} \]  

(77)
where Eq. (B17) has been used. From the completeness relation (51), one gets immediately the normalization of the nucleon momentum distribution

$$\int d\xi \int d\mathbf{k}_\perp n^\tau(\xi, \mathbf{k}_\perp) = 1 \quad .$$  \hspace{1cm} (78)

An explicit expression for the spin-averaged momentum distribution can be obtained inserting in Eq. (77) the LF spectral function as written in Eq. (72) and in turn the expression for the overlaps given in Eq. (62). Then, using again the two-body completeness of Eq. (51) and the unitarity of the $D$ and $D^{1/2}$ matrices, one obtains

$$n^\tau(\xi, \mathbf{k}_\perp) = \frac{1}{1 - \xi} \sum_{\sigma, \tau_2, \tau_3, \sigma_2, \sigma_3} \int d\mathbf{k}_{23} \frac{E(k_1) E_{23}}{k_1^+} \left| \langle \sigma_3', \sigma_2', \sigma; \tau_3', \tau_2, \tau; k_{23}, k_1 | j, j_z; \epsilon_{int}^3, \Pi; \frac{1}{2} T_z \rangle \right|^2 , \quad (79)$$

where $k_{1\perp} = k_\perp$ and $k_1^+ = \xi M_0(1, 2, 3)$ (see Eq. (59)). Combining Eqs. (B11) and (B14), the normalization of the LF nucleon momentum distribution (23) can be rewritten as follows

$$\int d\xi \int d\mathbf{k}_\perp n^\tau(\xi, \mathbf{k}_\perp) = \int d\mathbf{k}_\perp \sum_{\tau_2, \tau_3, \sigma_2, \sigma_3} \int d\mathbf{k}_{23} \int \frac{\partial k_z}{\partial k_\perp} d\mathbf{k}_z \frac{E_{23}}{(1 - \xi)} \left| \langle \sigma_3, \sigma_2, \sigma; \tau_3, \tau_2, \tau; k_{23}, k_1 | j, j_z; \epsilon_{int}^3, \Pi; \frac{1}{2} T_z \rangle \right|^2 =$$

$$= \int d\mathbf{k}_\perp \sum_{\tau_2, \tau_3, \sigma_2, \sigma_3} \int d\mathbf{k}_{23} \int d\mathbf{k}_z \left| \langle \sigma_3, \sigma_2, \sigma; \tau_3, \tau_2, \tau; k_{23}, k_1 | j, j_z; \epsilon_{int}^3, \Pi; \frac{1}{2} T_z \rangle \right|^2 =$$

$$= \int d\mathbf{k}_\perp \int d\mathbf{k}_z f^\tau(k_z, \mathbf{k}_\perp) = 1 \quad , \quad (80)$$

where $f^\tau(k_z, \mathbf{k}_\perp)$ is the instant form momentum distribution in terms of the intrinsic nucleon momentum $k = k_1$, defined by Eqs. (2), and (5) of Sec. III.

$$f^\tau(k_z, \mathbf{k}_\perp) = \sum_{\tau_2, \tau_3, \sigma_2, \sigma_3} \int d\mathbf{k}_{23} \left| \langle \sigma_3, \sigma_2, \sigma; \tau_3, \tau_2, \tau; k_{23}, k_1 | j, j_z; \epsilon_{int}^3, \Pi; \frac{1}{2} T_z \rangle \right|^2 . \quad (81)$$

Let us show that the momentum sum rule

$$\int \xi \ d\xi \int d\mathbf{k}_\perp n^\tau(\xi, \mathbf{k}_\perp) = \frac{1}{3} \quad \hspace{1cm} (82)$$

is satisfied by the LF momentum distribution $n^\tau(\xi, \mathbf{k}_\perp)$. Indeed, because of the symmetry of the three-body bound state, one has

$$\int \xi \ d\xi \int d\mathbf{k}_\perp n^\tau(\xi, \mathbf{k}_\perp) =$$

$$= \sum_{\tau_2, \tau_3, \sigma_2, \sigma_3} \int d\mathbf{k}_1 \int d\mathbf{k}_{23} \frac{k_1^+}{M_0(1, 2, 3)} \left| \langle \sigma_3, \sigma_2, \sigma; \tau_3, \tau_2, \tau; k_{23}, k_1 | j, j_z; \epsilon_{int}^3, \Pi; \frac{1}{2} T_z \rangle \right|^2 =$$

$$= \sum_{\tau_2, \tau_3, \sigma_2, \sigma_3} \int d\mathbf{k}_2 \int d\mathbf{k}_{31} \frac{k_2^+}{M_0(1, 2, 3)} \left| \langle \sigma_3, \sigma_2, \sigma; \tau_3, \tau_2, \tau; k_{31}, k_2 | j, j_z; \epsilon_{int}^3, \Pi; \frac{1}{2} T_z \rangle \right|^2 =$$

$$= \sum_{\tau_2, \tau_3, \sigma_2, \sigma_3} \int d\mathbf{k}_3 \int d\mathbf{k}_{12} \frac{k_3^+}{M_0(1, 2, 3)} \left| \langle \sigma_3, \sigma_2, \sigma; \tau_3, \tau_2, \tau; k_{12}, k_3 | j, j_z; \epsilon_{int}^3, \Pi; \frac{1}{2} T_z \rangle \right|^2 =$$

$$= \frac{1}{3} \sum_{\tau_2, \tau_3, \sigma_2, \sigma_3} \int d\mathbf{k}_1 \int d\mathbf{k}_{23} \left( k_1^+ + k_2^+ + k_3^+ \right) \frac{k_1^+}{M_0(1, 2, 3)} \left| \langle \sigma_3, \sigma_2, \sigma; \tau_3, \tau_2, \tau; k_{23}, k_1 | j, j_z; \epsilon_{int}^3, \Pi; \frac{1}{2} T_z \rangle \right|^2 =$$

$$= \frac{1}{3} \quad . \quad (83)$$
since (see Eqs. \[313\], and \[42\])

\[
\frac{\partial (k_1, k_2, k_3)}{\partial (k_2, k_3)} = \frac{M_{23} E_1 E_{23}}{M_{31} E_2 E_{31}}, \quad \frac{\partial (k_1, k_2)}{\partial (k_3, k_1)} = \frac{M_{23} E_1 E_{23}}{M_{12} E_3 E_{12}},
\]

(84)

\[
\sqrt{E_1 E_{23} M_{23}} |k_1, k_2, k_3)\rangle = \sqrt{E_2 E_{31} M_{31}} |k_2, k_3)\rangle = \sqrt{E_3 E_{12} M_{12}} |k_3, k_1)\rangle,
\]

(85)

and \(k_1^+ + k_2^+ + k_3^+ = M_0(1, 2, 3)\). The momentum sum rule, Eq. \[82\] has also been successfully checked calculating numerically Eq. \[133\] in an actual case using the three-body wave-function of Ref. \[33\] with the nuclear interaction of Ref. \[11\]. In the case of the proton (with accuracy produced by the normalization of the non relativistic wave function) we obtain 0.9989 for the normalization and 0.3324 for the sum rule, while for the neutron we have 0.9981 and 0.3336, respectively (see also Ref. \[40\]).

Within the BT framework one can obtain LF momentum distributions dependent upon the spin directions, \(n_{\alpha'} (\xi, k_1; \bar{S})\), for any direction of the polarization vector \(\bar{S}\) of the three-body system, using Eq. \[66\] and the expression for the LF spin-dependent spectral function given by Eq. \[72\]

\[
n_{\alpha'} (\xi, k_1; \bar{S}) = \frac{1}{1 - \xi} \sum_{\tau_2 \tau_3} \int dk_{23} \sum_{\alpha''} D^+_{M}(\bar{k}_1^{(a)}) |\alpha'\rangle \langle \alpha''| E(k_1^{(a)})
\]

\[
\times \sum_{\sigma_2 \sigma_3} D^3_{m', M}(\alpha, \beta, \gamma) |\sigma_2' \rangle \langle \sigma_3' | \langle \tau_2 | \sigma_3 \rangle \langle \tau_2 | \bar{k}_{23}, k_1^{(a)} | j, j_z = m; \epsilon_{int}^z; \Pi; \frac{1}{2} T_z)
\]

\[
\times \sum_{\sigma_1} D^3_{m', M}(\alpha, \beta, \gamma) |\sigma_1'\rangle \langle \sigma_1 | \langle m' j, j_z = m'; \epsilon_{int}^z; \Pi; \frac{1}{2} T_z^*) \rangle.
\]

(86)

We remind that \(\alpha, \beta, \gamma\) are the Euler angles describing the rotation from the \(z\)-axis to the polarization vector \(\bar{S}\). In Eq. \[86\] the explicit expression \[62\] for the overlaps is used, as well as the two-body completeness and once again the unitarity of the \(D\) and \(D^{1/2}\) matrices.

V. CONCLUSIONS AND PERSPECTIVES

In this paper, within the BT approach for the Poincaré generators, a LF spin-dependent spectral function and LF spin-dependent momentum distributions have been defined starting from the LF wave function for a three-body system, having in mind the \(^3\text{He}\) and the \(^3\text{H}\) nuclei. The spectral function is defined through the overlaps between the ground state wave function of the three-body system and the tensor product of a plane wave for one of the nucleons in the intrinsic reference frame of the cluster (1,23) and the state which describes the intrinsic motion of the fully interacting two-nucleon spectator subsystem. In the present approach the packing operators, needed to implement the macrocausality, are not considered in the description of the ground state of the three-body system, but the macrocausality is fully considered in the mentioned tensor product.

A generalization to \(A\)-nucleon nuclei is straightforward: one has only to generalize the definition of the intrinsic momentum \(\kappa\) as the momentum of one of the nucleons in the intrinsic reference frame of the cluster composed by this free nucleon and by the fully interacting system of the remaining \(A - 1\) nucleons. Then the LF spin-dependent spectral function for the \(A\)-nucleon nucleus is

\[
\mathcal{P}_{\kappa} (\kappa^+, \kappa_1, \kappa^-, S, A) = \sum_{A-1} d\epsilon_{A-1} \rho(\epsilon_{A-1}) A_{A-1} \delta \left( \kappa^- - M_A + \frac{M_{A-1}^2 + |\kappa_1|^2}{(1 - \xi) M_A} \right) \times
\]

\[
\sum_{A-1} \sum_{A-1} \left\langle J J_z; \tau \sigma' ; k | A, \Psi_0; S, T_z \rangle \langle S, T_z; \Psi_0, A| k, \sigma \tau; J J_z; \epsilon_{A-1}, \alpha, T_{A-1}, \tau_{A-1} \right\rangle \times
\]

\[
\left( A - 1 \right) \text{LF}
\]

(87)

where \(|A, \Psi_0; S, T_z\rangle\) is the ground-state of the \(A\)-nucleon nucleus, while \(M_{A-1}\) and \(\epsilon_{A-1}\) are the mass and the intrinsic energy, \(\rho(\epsilon_{A-1}) A_{A-1}\) is the density, \(J, J_z\) the spin, \(T_{A-1}, \tau_{A-1}\) the isospin of the \((A - 1)\)-nucleon system and \(\alpha\) the set of quantum numbers needed to fully specify this system.
Notably within the LF Hamiltonian dynamics, both normalization and momentum sum rule can be exactly satisfied at the same time. With respect to previous attempts to describe DIS processes off $^3$He in a LF framework (see, e.g., the one in Ref. [25]), in our approach for the spin-dependent spectral function a special care is devoted to the definition of the intrinsic LF variables of the problem, as well as to the spin degrees of freedom through the Melosh rotations. Our approach allows one to embed in a Poincaré covariant framework the large amount of knowledge on the nuclear interaction obtained from the non relativistic description of nuclei, since we adopt the LF version of the relativistic Hamiltonian dynamics with a fixed number of on-mass-shell constituents. The LF form of RHD has a sub-group composed by the LF boosts, which allows a separation of the intrinsic motion from the global one, very important for the description of DIS, SIDIS and deeply virtual Compton scattering processes, since it is possible to unambiguously identify the effects due to the inner dynamics. Therefore our LF spectral functions can be useful in many problems that require both a proper relativistic treatment and at the same time a good description of the internal structure of the system.

As a first example of forthcoming applications, we can mention the study of the effect of relativity in the evaluation of SIDIS cross section off $^3$He, taking into account both the relativity and the interaction in the final state between the observed pion and the remnant. In Refs. [13, 14], by adopting a non relativistic spectral function evaluated from the $^3$He wave function of Ref. [33], a distorted spin-dependent spectral function was obtained using a generalized eikonal approximation to deal with the final state interaction, and it was shown that within this framework it is actually possible to get reliable information on the quark TMDs in the neutron from SIDIS experiments off $^3$He. By considering the new LF spin-dependent spectral function, we plan to evaluate SIDIS cross sections off $^3$He through a LF distorted spin-dependent spectral function obtained applying again the generalized eikonal approximation for the description of the final state interaction. Preliminary results can be found in Ref. [16].

A second example for an application of the LF approach proposed in this paper, is the study of the role played by relativity in the EMC effect on $^3$He, for which JLab data have been taken at 6 GeV [39] in the standard inclusive DIS sector. Encouraging results including an exact treatment of the deuteron channel and an approximated treatment for the continuum of the LF spectral function can be found in Ref. [40].

In view of the large efforts in the determination of the TMDs to study the three-dimensional structure of the nucleon, the same concepts and definitions that are used in this paper to build up the LF spin-dependent spectral function for a three-nucleon system could be tentatively applied to a system of three valence quarks to define a nucleon spectral function in valence approximation and then to describe the nucleon TMDs in terms of a valence wave function for the nucleon.

It will be also interesting to study in detail the relation between the LF spin-dependent spectral function and the correlator, $\Phi(k, P, S)$, of a nucleon of momentum $k$ in a nucleus of momentum $P$ and spin polarization $S$, defined in terms of the nucleon fields, in analogy to the quark correlator in a nucleon, defined in terms of the quark fields [4]. In Refs. [15, 16] preliminary results were presented and it was shown that, in the valence approximation, a simple relation between the correlator and the LF spin-dependent spectral function naturally emerges and that only three of the six time-reversal even TMDs at the leading twist [4] are independent. The relations among these TMDs could be experimentally checked to test our LF description of the spin-dependent spectral function.
Appendix A: Two-body light-front wave function

In this Appendix, some details are given on the two-body light-front wave function that are useful for the general discussion presented in Sect. III.

1. Completeness of two-body free states

Let \( \tilde{P} \) be the total LF momentum for a two-particle system

\[
\tilde{P} = \tilde{p}_1 + \tilde{p}_2 .
\]

The Jacobian from \{\( \tilde{p}_1, \tilde{p}_2 \)\} to \{\( \tilde{P}, \xi, k_\perp \)\} is

\[
\frac{\partial (\tilde{p}_1, \tilde{p}_2)}{\partial (\tilde{P}, \xi, k_\perp)} = P^+ , \tag{A1}
\]

and the Jacobian from \{\( \tilde{p}_1, \tilde{p}_2 \)\} to \{\( \tilde{P}, k^+, k_\perp \)\} is given by

\[
\frac{\partial (\tilde{p}_1, \tilde{p}_2)}{\partial (\tilde{P}, k^+, k_\perp)} = 2(1 - \xi) M_0(1, 2) P^+ = \frac{2\xi(1 - \xi)}{k^+} P^+ , \tag{A2}
\]

with \( M_0(1, 2) \) defined by Eq. (21), since

\[
\begin{align*}
\frac{\partial k^+}{\partial \xi} &= M_0(1, 2) - \xi \frac{1}{2 M_0(1, 2)} m^2 + |k_\perp|^2 \frac{1}{\xi^2(1 - \xi)^2} (1 - 2\xi) = \frac{M_0(1, 2)}{2(1 - \xi)} = \frac{k^+}{2(1 - \xi)} . \tag{A3}
\end{align*}
\]

Furthermore the Jacobian from \{\( \tilde{p}_1, \tilde{p}_2 \)\} to \{\( \tilde{P}, k_z, k_\perp \)\} is given by

\[
\frac{\partial (\tilde{p}_1, \tilde{p}_2)}{\partial (\tilde{P}, k_z, k_\perp)} = \frac{2\xi(1 - \xi)}{E(k)} P^+ , \tag{A4}
\]

since (cf Eq. (3))

\[
\begin{align*}
\frac{\partial k_z}{\partial \xi} &= M_0(1, 2) - \left( \xi - \frac{1}{2} \right) \frac{1}{2 M_0(1, 2)} m^2 + |k_\perp|^2 \frac{1}{\xi^2(1 - \xi)^2} (1 - 2\xi) = \frac{E(k)}{2(1 - \xi)} . \tag{A5}
\end{align*}
\]

From Eqs. (A4) and (A6) one has

\[
\begin{align*}
\frac{\partial k_z}{\partial k^+} &= \frac{\partial k_z}{\partial \xi} \frac{\partial \xi}{\partial k^+} = \frac{E(k)}{k^+} . \tag{A7}
\end{align*}
\]

Keeping separate the global motion from the intrinsic one, the completeness reads

\[
I = \int \frac{d\tilde{P}_1}{2P_1^+ (2\pi)^3} \frac{d\tilde{P}_2}{2P_2^+ (2\pi)^3} |\tilde{p}_1\rangle |\tilde{p}_2\rangle \langle \tilde{p}_1| \langle \tilde{p}_2| =
\]

\[
= 2 \int \frac{d\tilde{P}}{2P^+ (2\pi)^3} |\tilde{P}\rangle \langle \tilde{P}| \int \frac{d\xi}{(2\pi)^3} \frac{d\xi}{4\xi(1 - \xi)} \int d|\tilde{k}\rangle |\tilde{k}\rangle \langle \tilde{P}| =
\]

\[
= 2 \int \frac{d\tilde{P}}{2P^+ (2\pi)^3} |\tilde{P}\rangle \langle \tilde{P}| \int \frac{d|\tilde{k}\rangle}{(2\pi)^3} \frac{d|\tilde{k}\rangle}{2k^+(2\pi)^3} |\tilde{k}\rangle \langle \tilde{P}| =
\]

\[
= 2 \int \frac{d\tilde{P}}{2P^+ (2\pi)^3} |\tilde{P}\rangle \langle \tilde{P}| \int \frac{d|\tilde{k}\rangle}{(2\pi)^3} \frac{d|\tilde{k}\rangle}{2E(k)(2\pi)^3} |\tilde{k}\rangle \langle \tilde{k}| (A8)
\]

Notice in the last step the hybrid notation in the intrinsic part. It will be used in what follows.
The normalization of the free state $|\tilde{P}\rangle|\tilde{k}\rangle = |\tilde{p}_1\rangle|\tilde{p}_2\rangle$ is

$$\langle \tilde{p}_2|\tilde{P}|\tilde{p}_1\rangle = 2p_1^+ (2\pi)^3 \delta^3(\tilde{p}_1 - \tilde{p}_2) 2p_2^+ (2\pi)^3 \delta^3(\tilde{p}_2 - \tilde{p}_2) = \left[ \frac{\partial (\tilde{P}, k^+, k_{\perp})}{\partial (\tilde{P}_1, \tilde{P}_2)} \right] 2p_1^+ (2\pi)^3 2p_2^+ (2\pi)^3 \delta^3(\tilde{P}' - \tilde{P}) \delta^3(\tilde{k}' - \tilde{k}) = 2P^+ (2\pi)^3 \delta^3(\tilde{P}' - \tilde{P}) k^+ (2\pi)^3 \delta^3(\tilde{k}' - \tilde{k}) = \langle \tilde{P}|\tilde{P}\rangle \langle \tilde{k}'|\tilde{k}\rangle . \quad (A9)$$

It should be pointed out that $\langle \tilde{k}'|\tilde{k}\rangle = k^+ (2\pi)^3 \delta^3(\tilde{k}' - \tilde{k})$, i.e. without a factor of two, since it refers to a two-body intrinsic state.

The overlap between the free two-body intrinsic states $|\tilde{k}; \sigma_2, \sigma_1\rangle_{LF}$ and the corresponding ones with canonical spin and Cartesian coordinates is relevant for the following discussion. Reminding that $\delta(k'^+ - k^+) = \delta(k'_z - k_z)/\partial k^+/\partial k_z$ and using Eq. (10) one has

$$\varepsilon\langle \sigma_1', \sigma_2'; k|\tilde{k}; \sigma_2, \sigma_1\rangle_{LF} = \sqrt{(2\pi)^3} k^+ \frac{\partial k_z}{\partial k^+} \delta(k' - k) D^\frac{\pi}{\lambda} |\mathcal{R}_M(\tilde{k})|\sigma_1 \sigma_1' D^\frac{\pi}{\lambda} |\mathcal{R}_M(-\tilde{k})|\sigma_2 \sigma_2' , \quad (A10)$$

where the normalization and the completeness of the plane waves with Cartesian variables, $|k\rangle$ are

$$\langle k'|k\rangle = \delta(k' - k) \int dk |k\rangle \langle k| = 1 , \quad (A11)$$

and

$$\tilde{k} \equiv ((M_0 - k^+), -k_{\perp}) . \quad (A12)$$

2. Light-front wave function for a system of two interacting particles

By using the subgroup properties of the LF boosts, the LF wave function for an interacting two-body system, in a given frame, can be expressed through the intrinsic variables as follows (see Eq. (A10))

$$L.F\langle \sigma_1, \sigma_2; \tau_1, \tau_2; \tilde{k}; \tilde{j}, \tilde{j}; \epsilon_{int}, \alpha; TT_z \rangle_{LF} = 2P^+ (2\pi)^3 \delta^3(\tilde{P}' - \tilde{P}) \sqrt{(2\pi)^3} k^+ \frac{\partial k_z}{\partial k^+}$$

$$\times \sum_{\sigma_1', \sigma_2'} D^\frac{\pi}{\lambda} |\mathcal{R}_M(\tilde{k})|\sigma_1 \sigma_1' D^\frac{\pi}{\lambda} |\mathcal{R}_M(-\tilde{k})|\sigma_2 \sigma_2' \langle \sigma_1', \sigma_2'; \tau_1, \tau_2; k|j, j; \epsilon_{int}, \alpha; TT_z \rangle , \quad (A13)$$

where a canonical completeness has been inserted for obtaining the final step.

Notice that the intrinsic two-body wave function $\langle \sigma_1', \sigma_2'; \tau_1, \tau_2; k|j, j; \epsilon_{int}, \alpha; TT_z \rangle$ contains canonical spins, and therefore it can be composed by using the Clebsch-Gordan coefficients. Moreover, $j$ is the total angular momentum of the pair, $T$ the isospin, $\alpha$ the set of the parity and quantum numbers that label the coupled waves, and $\epsilon_{int}$ is the eigenvalue of the mass operator (see Eqs. (24-25)).

The normalization of the intrinsic part of a LF bound state follows from the normalization fulfilled by $\langle \sigma_1, \sigma_2; \tau_1, \tau_2; k|j, j; \epsilon_{int}, \alpha; TT_z \rangle$. Indeed, if we adopt the following normalization, suitable for bound states,

$$\sum_{\tau_1, \tau_2} \sum_{\sigma_1, \sigma_2} \int dk \langle \sigma_1, \sigma_2; \tau_1, \tau_2; k|j, j; \epsilon_{int}, \alpha; TT_z \rangle^2 = 1 , \quad (A14)$$

from Eq. (A13) one has for the intrinsic part of the two-body LF wave function

$$\sum_{\tau_1, \tau_2} \sum_{\sigma_1, \sigma_2} \int \frac{dk}{k^+ (2\pi)^3} |L.F\langle \sigma_1, \sigma_2; \tau_1, \tau_2; \tilde{k}|j, j; \epsilon_{int}, \alpha; TT_z \rangle|^2 =$$

$$= \sum_{\tau_1, \tau_2} \sum_{\sigma_1, \sigma_2} \int \frac{dk}{E(k)(2\pi)^3} |L.F\langle \sigma_1, \sigma_2; \tau_1, \tau_2; \tilde{k}|j, j; \epsilon_{int}, \alpha; TT_z \rangle|^2 =$$
\[
\sum_{\tau_1, \tau_2} \sum_{\sigma_1', \sigma_2'} \int \frac{d\mathbf{k}}{E(k)} \left( \sum_{\sigma_1, \sigma_2} D^\frac{1}{2}_L[R_M(\mathbf{k})]_{\sigma_1, \sigma_1'} D^\frac{1}{2}_L[R_M(-\mathbf{k})]_{\sigma_2, \sigma_2'} \langle \sigma_1', \sigma_2'; \tau_1, \tau_2; j, j; \epsilon_{\text{int}}, \alpha; TTz \rangle \right)^2 = \\
= \sum_{\tau_1, \tau_2} \sum_{\sigma_1, \sigma_2} \int d\mathbf{k} \ |\langle \sigma_1, \sigma_2; \tau_1, \tau_2; j, j; \epsilon_{\text{int}}, \alpha; TTz \rangle|^2 = 1 .
\]

(A15)

In the last step of (A15) the unitarity of the \( D^{1/2} \) matrices has been used. The normalization for the LF scattering states follows from: (i) the orthogonality condition adopted for the canonical scattering wave function \( \langle \sigma_1, \sigma_2; \tau_1, \tau_2; j, j; \epsilon_{\text{int}}, \alpha; TTz \rangle \), given by (see also Eq. (A19) below for the completeness of the canonical states)

\[
\sum_{\sigma_1', \sigma_2'} \sum_{\tau_1', \tau_2'} \int \frac{d\mathbf{k}}{E(k)} \langle T_{\tau_1'}^{\sigma_1'} T_{\tau_2'}^{\sigma_2'} | \tilde{\mathbf{P}}_\sigma' | \tilde{\mathbf{k}} \rangle \langle \tilde{\mathbf{k}} | \tilde{\mathbf{P}}_\sigma'' | T_{\tau_1''} T_{\tau_2''} \rangle = \\
= \delta_{T_1', T_2', \tau_1', \tau_2'} \delta_{\alpha', \alpha} \delta_{\gamma', \gamma} \frac{\delta(t' - t)}{t^2} ,
\]

(A16)

where \( t = \sqrt{m \epsilon_{\text{int}}} \), and (ii) the orthogonality adopted for the LF scattering states, that reads (see also the completeness of the free states for a two-body system \( | \tilde{\mathbf{P}} | \tilde{\mathbf{k}} \) in Eq. (A5),

\[
LF(T_{\tau_1} T_{\tau_2} | \alpha' \epsilon_{\text{int}} j' \tilde{\mathbf{j}}'; | \tilde{\mathbf{P}} | \tilde{\mathbf{k}}; j, j; \epsilon_{\text{int}}, \alpha; TTz)_{LF} = \\
= \sum_{\sigma_1', \sigma_2'} \sum_{\tau_1', \tau_2'} \int \frac{d\mathbf{k}}{2E(k)(2\pi)^3} \int \frac{d\mathbf{k}}{E(k)} \langle T_{\tau_1'}^{\sigma_1'} T_{\tau_2'}^{\sigma_2'} | \tilde{\mathbf{P}}_\sigma' | \tilde{\mathbf{k}} \rangle \langle \tilde{\mathbf{k}} | \tilde{\mathbf{P}}_\sigma'' | T_{\tau_1''} T_{\tau_2''} \rangle_{LF} = \\
= 2 P^+ (2\pi)^3 \delta^3(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}}) \delta_{T_1', T_2', \tau_1', \tau_2'} \delta_{\alpha', \alpha} \delta_{\gamma', \gamma} \frac{\delta(t' - t)}{t^2} ,
\]

(A17)

Then for the two-body interacting case the LF completeness reads

\[
\int \frac{d\mathbf{k}}{2E(k)(2\pi)^3} \sum_{j, j', \alpha} \sum_{TTz} \sum_{\lambda(t)} \lambda(t) \ dt \ LF(\sigma_1, \sigma_2; \tau_1, \tau_2; \tilde{\mathbf{k}}; | \tilde{\mathbf{P}} | \tilde{\mathbf{k}}; j, j; \epsilon_{\text{int}}, \alpha; TTz)_{LF} \\
\times LF(T_{\tau_1} T_{\tau_2} | \alpha' \epsilon_{\text{int}} j' \tilde{\mathbf{j}}'; | \tilde{\mathbf{P}}_\sigma' | \tilde{\mathbf{k}}; j, j; \epsilon_{\text{int}}, \alpha; TTz)_{LF} = \\
= 2 P^+ (2\pi)^3 \delta^3(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}}) \sum_{j, j, \alpha} \sum_{TTz} \sum_{\lambda(t)} \lambda(t) \ dt \\
\times \sqrt{(2\pi)^3 E(k)} \sum_{\sigma_1, \sigma_2} D^\frac{1}{2}_L[R_M(\mathbf{k})]_{\sigma_1, \sigma_1'} D^\frac{1}{2}_L[R_M(-\mathbf{k})]_{\sigma_2, \sigma_2'} \langle \tilde{\mathbf{P}}_\sigma' | \tilde{\mathbf{k}} \rangle \langle \tilde{\mathbf{k}} | \tilde{\mathbf{P}}_\sigma'' \rangle = \\
= 2 P^+ (2\pi)^3 \delta^3(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}}) \delta_{\lambda(t)} \delta_{T_1', T_2', \tau_1', \tau_2'} \delta_{\alpha', \alpha} \delta_{\gamma', \gamma} \frac{\delta(t' - t)}{t^2} ,
\]

(A18)

where the symbol \( \sum \) means a sum over the bound states of the pair (namely the deuteron in the present case) and the integration over the continuum. The quantity \( \lambda(t) \) is the \( t \)-density of the two-body states (\( \lambda(t) = 1 \) for the bound states and \( \lambda(t) = t^2 \) for the continuum). To obtain Eq. (A15), one has to use: (i) the expression (A19) for the LF wave function, (ii) the unitarity of the \( D^{1/2} \) matrices, (iii) the completeness for the eigensolutions of Eq. (23), i.e.,

\[
\sum_{j, j, \alpha} \sum_{TTz} \sum_{\lambda(t)} \lambda(t) \ dt \langle \mathbf{k}' | j, j; \epsilon_{\text{int}}, \alpha; TTz \rangle \langle T_{\tau_1} T_{\tau_2} | \alpha, \epsilon_{\text{int}}; j, j; \mathbf{k} \rangle = \delta^3(\mathbf{k}' - \mathbf{k}) ,
\]

(A19)

and (iv) Eq. (A7).
Appendix B: Three-body states

In this Appendix, the three-body free and interacting states are analyzed in analogy to the two-body case.

1. Completeness of three-body free states with symmetric intrinsic variables

Let \( \tilde{\mathbf{P}} \) be the total LF momentum for a three-particle system

\[
\tilde{\mathbf{P}} = \tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 + \tilde{\mathbf{p}}_3
\]

of free mass \( M_0(1, 2, 3) \)

\[
M_0^2(1, 2, 3) = \frac{m^2 + |k_{1\perp}|^2}{\xi_1} + \frac{m^2 + |k_{2\perp}|^2}{\xi_2} + \frac{m^2 + |k_{3\perp}|^2}{\xi_3} = (E_1 + E_2 + E_3)^2
\]

where \( E_i = \sqrt{m^2 + |k_i|^2} \) and \( \sum_i k_i = 0 \).

The completeness for the different set of variables, \{\tilde{\mathbf{p}}_i\} \rightarrow \{\xi_i, k_{i\perp}\} \rightarrow k_i, is given by

\[
I = \int \frac{d\tilde{\mathbf{P}}_1}{2p_1^+(2\pi)^3} \frac{d\tilde{\mathbf{P}}_2}{2p_2^+(2\pi)^3} \frac{d\tilde{\mathbf{P}}_3}{2p_3^+(2\pi)^3} |\tilde{\mathbf{P}}_3⟩⟨\tilde{\mathbf{P}}_2|⟨\tilde{\mathbf{P}}_1|\tilde{\mathbf{P}}_1⟩\tilde{\mathbf{P}}_1⟩ = \int \frac{d\tilde{\mathbf{P}}}{2P^+(2\pi)^3} |\tilde{\mathbf{P}}⟩⟨\tilde{\mathbf{P}}|\int \frac{dk_{1\perp} \xi_1}{2\xi_1(2\pi)^3} |k_{1\perp}\xi_1⟩⟨k_{1\perp}\xi_1| \int \frac{dk_{2\perp} \xi_2}{2\xi_2(2\pi)^3} |k_{2\perp}\xi_2⟩⟨k_{2\perp}\xi_2| = \int \frac{d\tilde{\mathbf{P}}}{2P^+(2\pi)^3} |\tilde{\mathbf{P}}⟩⟨\tilde{\mathbf{P}}|\int \frac{dk_1}{2E_1(2\pi)^3} \int \frac{dk_2}{2E_2(2\pi)^3} \frac{M_0(1, 2, 3)}{E_3}|k_1⟩|k_2⟩|k_2⟩|k_1|,
\]

where \( |\tilde{\mathbf{P}}_3⟩⟨\tilde{\mathbf{P}}_2|⟨\tilde{\mathbf{P}}_1|\tilde{\mathbf{P}}_1⟩\tilde{\mathbf{P}}_1⟩ = |\tilde{\mathbf{P}}⟩⟨\xi_1, k_{1\perp}|⟨\xi_2, k_{2\perp}| \) and the Jacobians

\[
\left[ \frac{\partial}{\partial (\tilde{\mathbf{P}}, \xi_1, k_{1\perp}, k_{2\perp})} \right] = (P^+)^2
\]

\[
\left[ \frac{\partial}{\partial (\tilde{\mathbf{P}}, k_1, k_2)} \right] = \frac{p_1^+ p_2^+ p_3^+ M_0(1, 2, 3)}{P^+ E_1 E_2 E_3}
\]

have been used.

2. Completeness of three-body free states with non-symmetric intrinsic variables

Instead of the symmetric intrinsic variables in the 3-body frame, one can introduce non-symmetric intrinsic variables, corresponding to the intrinsic frame of the (2,3) pair, i.e. \{\tilde{\mathbf{p}}_{23}, \tilde{\mathbf{p}}_3\} \rightarrow \{\tilde{\mathbf{P}}_{23}, \eta, k_{23\perp}\} (see Eqs. \[B5, B7\]).

The completeness

\[
\int \frac{d\tilde{\mathbf{P}}_1}{2p_1^+(2\pi)^3} \frac{d\tilde{\mathbf{P}}_2}{2p_2^+(2\pi)^3} \frac{d\tilde{\mathbf{P}}_3}{2p_3^+(2\pi)^3} |\tilde{\mathbf{P}}_1⟩⟨\tilde{\mathbf{P}}_2|⟨\tilde{\mathbf{P}}_3|⟨\tilde{\mathbf{P}}_1| = I
\]

can be arranged in different ways, depending upon the the choice of variables one needs. In particular,

1. for the variables \( \tilde{\mathbf{p}}_1, \tilde{\mathbf{P}}_{23} \) and \( \tilde{\mathbf{k}}_{23} \) one can exploit Eq. \[A6\], obtaining

\[
I = \int \frac{d\tilde{\mathbf{P}}_1}{2p_1^+(2\pi)^3} |\tilde{\mathbf{P}}_1⟩⟨\tilde{\mathbf{P}}_2|⟨\tilde{\mathbf{P}}_3|⟨\tilde{\mathbf{P}}_1| = \int \frac{d\tilde{\mathbf{P}}_{23}}{2P_{23}^+(2\pi)^3} |\tilde{\mathbf{P}}_{23}⟩⟨\tilde{\mathbf{P}}_{23}| \int \frac{d\tilde{\mathbf{k}}_{23}}{k_{23\perp}^+(2\pi)^3} |\tilde{\mathbf{k}}_{23}⟩⟨\tilde{\mathbf{k}}_{23}|\]
2. for the variables \( \tilde{P}, \{\xi, k_{1\perp}\} \) and \( \{\eta, k_{23\perp}\} \) one has from Eq. (B3)

\[
I = \int \frac{d\tilde{P}}{2P^+(2\pi)^3} |\tilde{P}\rangle \langle \tilde{P}| \int \frac{d\xi_1}{2\xi_1(1-\xi_1)(2\pi)^3} |\tilde{\xi}_1 k_{1\perp}\rangle \langle k_{1\perp}\xi_1|
\times \int \frac{d\eta \, d\tilde{k}_{23\perp}}{2\eta(1-\eta)(2\pi)^3} |\tilde{\eta} k_{23\perp}\rangle \langle k_{23\perp}\eta| ,
\] (B8)

after recalling Eq. (35) that yields

\[
\frac{d\xi_2}{\xi_2 \xi_3} = \frac{d\eta}{\eta(1-\eta)(1-\xi_1)} \quad \text{and} \quad d\tilde{k}_{2\perp} = d\tilde{k}_{23\perp}
\] (B9)

3. for the variables \( \tilde{P}, \tilde{k}_1 \) and \( k_{23\perp} \) one has

\[
I = \int \frac{d\tilde{P}}{2P^+(2\pi)^3} |\tilde{P}\rangle \langle \tilde{P}| \int \frac{d\tilde{k}_{23}}{k^+_3(2\pi)^3} |\tilde{k}_{23}\rangle \langle \tilde{k}_{23}| \int \frac{M_0(1, 2, 3) \, d\tilde{k}_1}{2k^+_1 E_23(2\pi)^3} |\tilde{k}_1\rangle \langle \tilde{k}_1| ,
\] (B10)

where the following relations have been used (remind that \( k^+_1 = \xi_1 M_{0}(1, 2, 3) \) and \( k^+_3 = \eta M_{23} \))

\[
\frac{\partial k^+_1}{\partial \xi_1} = M_0(1, 2, 3) + \xi_1 \frac{\partial M_0(1, 2, 3)}{\partial \xi_1} = M_0(1, 2, 3) = \frac{\xi_1}{2M_0(1, 2, 3)} \frac{\partial M_0^2(1, 2, 3)}{\partial \xi_1} = \nonumber
\]

\[
= \frac{1}{2M_0(1, 2, 3)} \left[ M_0^2(1, 2, 3) + \frac{M_0^2 + |\tilde{k}_{1\perp}|^2}{(1-\xi_1)^2} \right] = \frac{1}{2(1-\xi_1)} \left[ M_0(1, 2, 3)(1-\xi_1) + \frac{M_0^2 + |\tilde{k}_{1\perp}|^2}{K_{23}^2} \right] =
\]

\[
= \frac{1}{2(1-\xi_1)} \left[ K_{23}^+ + K_{23\perp} \right] = \frac{E_{23}}{(1-\xi_1)}
\] (B11)

\[
\frac{\partial k^+_3}{\partial \eta} = M_{23} - \eta \frac{1}{2M_23} \frac{m^2 + |\tilde{k}_{1\perp}|^2}{\eta^2(1-\eta)^2} (1-2\eta) = \nonumber
\]

\[
= \frac{M_23}{2(1-\eta)} \left[ 2(1-\eta) - 1 + 2\eta \right] \frac{M_23}{2(1-\eta)} = \frac{k^+_3}{2\eta(1-\eta)}
\] (B12)

with \( K_{23}^+ \) the total momentum of the free \( (2,3) \) pair in the intrinsic frame of the three particles, i.e. \( K_{23}^+ = M_0(1, 2, 3)(1-\xi_1) \), \( K_{23\perp} = k_{2\perp} + k_{3\perp} = -k_{1\perp} \), \( K_{23\perp} = (M_0^2 + |\tilde{k}_{1\perp}|^2)/K_{23}^+ \), and \( E_{23} = \sqrt{M_23^2 + |k_1|^2} \).

4. for the variables \( \tilde{P}, k_1 \) and \( k_{23\perp} \) one has

\[
I = \int \frac{d\tilde{P}}{2P^+(2\pi)^3} |\tilde{P}\rangle \langle \tilde{P}| \int \frac{d\tilde{k}_{23}}{M_{23}(2\pi)^3} |\tilde{k}_{23}\rangle \langle \tilde{k}_{23}| \int \frac{M_0(1, 2, 3) \, d\tilde{k}_1}{2E_1 E_{23}(2\pi)^3} |\tilde{k}_1\rangle \langle \tilde{k}_1| .
\] (B13)

For obtaining the above results, the following properties have been used

\[
\frac{\partial k_{1\perp}}{\partial k_1^+} = \frac{1}{2} \left[ 1 + \frac{m^2 + |\tilde{k}_{1\perp}|^2}{k_1^+} \right] = \frac{E(k_1)}{k_1^+} = \frac{E(k_1)}{M_0(1, 2, 3) \xi_1}
\] (B14)

\[
\frac{\partial k_{23\perp}}{\partial \eta} = M_{23} - \left( \eta - \frac{1}{2} \right) \frac{1}{2M_23} \frac{m^2 + |\tilde{k}_{1\perp}|^2}{\eta^2(1-\eta)^2} (1-2\eta) = \nonumber
\]

\[
= \frac{M_23}{4\eta(1-\eta)} \left[ 4\eta(1-\eta) + (2\eta - 1)^2 \right] = \frac{M_23}{4\eta(1-\eta)}
\] (B15)

5. for the variables \( \tilde{P}, \tilde{k}_1 \) and \( k_{23\perp} \) one has

\[
I = \int \frac{d\tilde{P}}{2P^+(2\pi)^3} |\tilde{P}\rangle \langle \tilde{P}| \int \frac{2 \, d\tilde{k}_{23}}{M_{23}(2\pi)^3} |\tilde{k}_{23}\rangle \langle \tilde{k}_{23}| \int \frac{M_0(1, 2, 3) \, d\tilde{k}_1}{2k_1^+ E_{23}(2\pi)^3} |\tilde{k}_1\rangle \langle \tilde{k}_1| .
\] (B16)
3. Useful derivatives involving non-symmetric intrinsic variables

Let us evaluate the derivatives $\frac{\partial \kappa_1^+}{\partial \xi_1}$ and $\frac{\partial \kappa_{12}}{\partial \kappa_1^+}$:

$$\frac{\partial \kappa_1^+}{\partial \xi_1} = M_0(1, 23) + \xi_1 \frac{\partial M_0(1, 23)}{\partial \xi_1} = M_0(1, 23) + \frac{\xi_1}{2M_0(1, 23)} \frac{\partial M_0(1, 23)^2}{\partial \xi_1} =$$

$$= \frac{1}{2M_0(1, 23)} \left[ M_0(1, 23)^2 + \frac{M_2^2 + |k_{1\perp}|^2}{(1 - \xi_1)^2} \right] = \frac{1}{2(1 - \xi_1)} \left[ M_0(1, 23)(1 - \xi_1) + \frac{M_2^2 + |k_{1\perp}|^2}{P_S^2} \right] =$$

$$= \frac{1}{2(1 - \xi_1)} \left[ P_S^+ + P_{S\perp} \right] = \frac{E_S}{(1 - \xi_1)} \quad \text{(B17)}$$

$$\frac{\partial \kappa_{12}}{\partial \kappa_1^+} = \frac{1}{2} \left[ 1 + \frac{m^2 + |k_{1\perp}|^2}{\kappa_1^+} \right] = \frac{E(\kappa_1)}{\kappa_1^+} = \frac{E(\kappa_1)}{M_0(1, 23)\xi_1} \quad \text{(B18)}$$

4. Normalization of the light-front wave function

Let us check that the factors in the expression of the intrinsic part of the LF wave function given by the second and the third lines of Eq. (52) allow one to obtain the normalization of the bound state $|j, j_z; \epsilon_{\text{int}}, \Pi; \frac{1}{2}, T_z\rangle$. Indeed using Eqs. (B10) and (B13)) one has

$$\langle T_z \frac{1}{2}; \Pi, \epsilon_{\text{int}}^3; j, j_z; \epsilon_{\text{int}}^3; \Pi; \frac{1}{2}, T_z\rangle =$$

$$= \sum_{\tau_1, \tau_2, \tau_3} \sum_{\sigma_1, \sigma_2, \sigma_3} \int \frac{d\bar{k}_1}{2k_1^+(2\pi)^3} \int \frac{d\bar{k}_{23}}{k_{23}^+(2\pi)^3} |_{LF} \langle \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3; \bar{k}_1, \bar{k}_{23}|j, j_z; \epsilon_{\text{int}}^3, \Pi; \frac{1}{2}, T_z\rangle^2 =$$

$$= \sum_{\tau_1, \tau_2, \tau_3} \sum_{\sigma_1, \sigma_2, \sigma_3} \int \frac{d\bar{k}_1}{E_1(2\pi)^3} \int \frac{d\bar{k}_{23}}{E_{23}(2\pi)^3} |_{LF} \langle \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3; \bar{k}_1, \bar{k}_{23}|j, j_z; \epsilon_{\text{int}}^3, \Pi; \frac{1}{2}, T_z\rangle^2 =$$

$$= \sum_{\tau_1, \tau_2, \tau_3} \sum_{\sigma_1, \sigma_2, \sigma_3} \int \frac{d\bar{k}_1}{E_1(2\pi)^3} \int \frac{d\bar{k}_{23}}{E_{23}(2\pi)^3} \frac{M_0(1, 2, 3)}{M_{23}E_{23}(2\pi)^3} 2E_1M_{23}2(2\pi)^3$$

$$\times \left| \sum_{\sigma_1', \sigma_2', \sigma_3'} D_{\sigma_1' \sigma_1}^+ |R_M(k_1)|_{\sigma_1, \sigma_1'} D_{\sigma_2' \sigma_2}^+ |R_M(k_2)|_{\sigma_2, \sigma_2'} D_{\sigma_3' \sigma_3}^+ |R_M(k_3)|_{\sigma_3, \sigma_3'} \right|^2 =$$

$$= \sum_{\tau_1, \tau_2, \tau_3} \sum_{\sigma_1, \sigma_2, \sigma_3} \int d\bar{k}_{23} \left| \langle \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3; k_{23}|j, j_z; \epsilon_{\text{int}}^3, \Pi; \frac{1}{2}, T_z\rangle \right|^2 = 1 \quad \text{(B19)}$$

given the unitarity of the Melosh rotations and the normalization of the canonical wave function [123].

Appendix C: Properties of the basis states of the cluster \{1, (23)\}

In this Appendix, the general formalism, suitable for describing the cluster \{1, (23)\}, is presented. It should be reminded that the final goal is to construct states where the interaction is acting only between the particles 2 and 3, namely the three-body states we are interested in are the tensor product of free one-body states and interacting two-body states.

1. Completeness relation for the non-symmetric basis states and orthogonality properties of three-body free states

The correctness of the normalization factors in Eq. (52) can be checked as follows.
Indeed, let us consider the product of two three-body free states:

\[
A = \langle \sigma_1', \sigma_2', \sigma_3' ; \tau_1', \tau_2', \tau_3' | \tilde{P}'_1 \tilde{P}'_2 \tilde{P}'_3 | \tilde{K}'_{23} \tilde{K}'_1 \tilde{P}'_1'' \tilde{P}'_2'' \tilde{P}'_3'' ; \tau_1'', \tau_2'', \tau_3'' | \sigma_1'', \sigma_2'', \sigma_3'' \rangle_{LF} .
\]

(C1)

Then, let us insert in Eq. (C1) the completeness relation (31) for the non-symmetric basis states (10):

\[
A = \int \frac{d\tilde{P}}{2P'} \int \frac{d\tilde{k}}{2\tilde{k}^1} \int \frac{d\tilde{k}}{2\tilde{k}^1} \sum_{T_2T_3} \sum_{\sigma_1\sigma_2} \sum_{\tau_1\tau_2\tau_3} \sum_{\tilde{P}_1' \tilde{P}_2' \tilde{P}_3'} \sum_{\tilde{K}'_{23} \tilde{K}'_1} \sum_{\tilde{P}'_1'' \tilde{P}'_2'' \tilde{P}'_3''} \sum_{\tau_1'' \tau_2'' \tau_3''} \sum_{\sigma_1'' \sigma_2'' \sigma_3''} \lambda(t) dt
\]

\[
\times \sum_{j23;j23;\alpha} LF(\sigma_1', \sigma_2', \sigma_3'; \tau_1', \tau_2', \tau_3'; \tilde{P}'_1 \tilde{P}'_2 \tilde{P}'_3 \tilde{K}'_{23} \tilde{K}'_1 \tilde{P}'_1'' \tilde{P}'_2'' \tilde{P}'_3'' ; \tau_1'', \tau_2'', \tau_3'' ; \sigma_1'', \sigma_2'', \sigma_3'' ; LF) .
\]

(C2)

With the help of the overlap in Eq. (B2), the above equation reads

\[
A = 2 P' (2\pi)^3 \delta^3(\tilde{P}' - \tilde{P}) \delta_{\tau' \tau''} \delta_{\sigma_1' \sigma_1''} \int d\tilde{k}_{1 \perp} \int \frac{d\xi_1}{(1 - \xi_1)}
\]

\[
\times \sum_{j23;j23;\alpha} LF(\sigma_1', \sigma_2', \sigma_3'; \tau_1', \tau_2', \tau_3'; \tilde{P}'_1 \tilde{P}'_2 \tilde{P}'_3 \tilde{K}'_{23} \tilde{K}'_1 \tilde{P}'_1'' \tilde{P}'_2'' \tilde{P}'_3'' ; \tau_1'', \tau_2'', \tau_3'' ; \sigma_1'', \sigma_2'', \sigma_3'' ; LF) .
\]

(C3)

where the variable integration change \(dk_{1 \perp} = d\xi_1 E_S/(1 - \xi_1)\) was performed (see Eq. (B17)). In Eq. (33) \(k''_{1 \perp} = k_1\) and \(k''_{\perp} = \xi_1 M_0(1, 2, 3)\) with

\[
M_0^2(1, 2, 3) = \frac{m^2 + k_{1 \perp}^2}{\xi_1} + \frac{M_{23}^2 + k_{1 \perp}^2}{1 - \xi_1} .
\]

(C4)

Then, taking into account the completeness for the two-body intrinsic states \(\langle \sigma_2, \sigma_3 ; \tau_2', \tau_3' ; \tilde{K}'_{23} ; j_{23}, j_{23}; \epsilon_{23}, \alpha ; T_{23}, \tau_{23} \rangle\) for the (2,3) pair (see Eqs. (31) and (19)), one obtains

\[
A = 2 P' (2\pi)^3 \delta^3(\tilde{P}' - \tilde{P}) \delta_{\tau' \tau''} \delta_{\sigma_1' \sigma_1''} \int d\tilde{k}_{1 \perp} \int \frac{d\xi_1}{(1 - \xi_1)}
\]

\[
\times \sum_{j23;j23;\alpha} LF(\sigma_1', \sigma_2', \sigma_3'; \tau_1', \tau_2', \tau_3'; \tilde{P}'_1 \tilde{P}'_2 \tilde{P}'_3 \tilde{K}'_{23} \tilde{K}'_1 \tilde{P}'_1'' \tilde{P}'_2'' \tilde{P}'_3'' ; \tau_1'', \tau_2'', \tau_3'' ; \sigma_1'', \sigma_2'', \sigma_3'' ; LF) .
\]
\[
\times (2\pi)^3 k_{1''}^+ \delta^3(\mathbf{k}_{1''} - \mathbf{k}_{1''(a)}) \sqrt{\frac{E''_{k_{1''}}}{k_{1''}^+}} \sqrt{\frac{E'_{k_{1'}} M'_{(1, 2, 3)}}{M''_0}} \sum_{\sigma_2} \sum_{\sigma_3} D^+ [\mathcal{R}_M(\mathbf{k}_{23''})]_{\sigma_2' \sigma_2} D^+ [\mathcal{R}_M(-\mathbf{k}_{23})]_{\sigma_3' \sigma_3} \\
\times \delta_{\tau_1',\tau_2''} \delta_{\tau_1',\tau_3''} \delta_{\sigma_2,\bar{\sigma}_2} \delta_{\sigma_3,\bar{\sigma}_3} \delta^3(\mathbf{k}_{23''} - \mathbf{k}_{23''}) .
\]

Therefore, using the unitarity of the \(D^+\) matrices and changing the integration variable from \(d\xi_1 1/(1 - \xi_1)\) to \(1/E''_{k_{1''}} d\xi_{1''}\) (see Eq. (B11)), one obtains

\[
A = \delta_{\sigma'_1,\sigma''_1} \delta_{\sigma'_2,\sigma''_2} \delta_{\sigma'_3,\sigma''_3} \delta_{\tau_1',\tau_2''} \delta_{\tau_1',\tau_3''} \delta^3(\mathbf{P}'' - \mathbf{P}'') k_{1''}^+ \delta^3(\mathbf{k}_{1''} - \mathbf{k}_1') \frac{E_{k_{1'}} M_{(1, 2, 3)}}{M''_0} \delta^3(\mathbf{k}_{23''} - \mathbf{k}_{23}) = \\
= \delta_{\sigma'_1,\sigma''_1} \delta_{\sigma'_2,\sigma''_2} \delta_{\sigma'_3,\sigma''_3} \delta_{\tau_1',\tau_2''} \delta_{\tau_1',\tau_3''} \delta^3(\mathbf{P}'' - \mathbf{P}') k_{1''}^+ \delta^3(\mathbf{k}_{1''} - \mathbf{k}_1') \frac{E_{k_{1'}} M_{(1, 2, 3)}}{M''_0} \delta^3(\mathbf{k}_{23''} - \mathbf{k}_{23}) = \\
= \delta_{\sigma'_1,\sigma''_1} \delta_{\sigma'_2,\sigma''_2} \delta_{\sigma'_3,\sigma''_3} \delta_{\tau_1',\tau_2''} \delta_{\tau_1',\tau_3''} \delta^3(\mathbf{P}'' - \mathbf{P}') E(\mathbf{k}_1') \delta^3(\mathbf{k}_{1''} - \mathbf{k}_1') \frac{E_{k_{1'}} M_{(1, 2, 3)}}{M''_0} \delta^3(\mathbf{k}_{23''} - \mathbf{k}_{23})
\]

The above expressions are the proper orthogonality relations for the free case, to be related to the completeness relations of Eqs. (B16), (B10), and (B13), respectively.

2. Product of the non-symmetric basis states and the bound state of the three-particle system

Let us express the overlaps between the states of the non-symmetric basis \([19]\) and the bound state of the three-particle system in terms of the canonical wave functions for the two-body and the three-body systems. To this end, the plane-wave completeness operator \([61]\) is inserted in the intrinsic part of the overlap \([60]\), viz.

\[
\mathcal{L}_F(\mathbf{T}_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23''}; \tau_1 \sigma_1 \mathbf{k}_1 | j, j_1, \epsilon_{int}; \Pi; \frac{1}{2} T_z) = \sum_{\mathbf{k}_{23}} \sum_{\mathbf{k}_{1'}} \int \frac{d\mathbf{k}_{23}}{E_{k_{23}}^2 (2\pi)^3} \sum_{\sigma_1} \int \frac{M_{(1, 2, 3)}^2 d\mathbf{k}_1'}{2k_{1''}^+ E_{k_{1'}}^2 (2\pi)^3} \\
\times \mathcal{L}_F(\mathbf{T}_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23''}; \tau_1 \sigma_1 \mathbf{k}_1 | \tau_2 \sigma_2 \tau_3; \sigma_2 \sigma_3; \mathbf{k}_{1'} | j, j_1, \epsilon_{int}; \Pi; \frac{1}{2} T_z) .
\]

We can notice that the LF spin states do not change for LF boosts. Therefore, the spin states \([\sigma_2, \sigma_3], \mathcal{L}_F\) in the intrinsic reference frame of the pair \((23)\) or in the intrinsic reference frame of the three-particle system, with momenta related by the LF boost \(B_{LF}^\dagger(\mathbf{k}_{23}/M_{23})\), are equal. Then we can take \(\mathcal{L}_F(\mathbf{T}_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23''}; \tau_1 \sigma_1 \mathbf{k}_1 | \mathcal{L}_F(\mathbf{k}_{23}/M_{23})\) as the intrinsic part of the overlap \([52]\) and \(\mathcal{L}_F(\mathbf{T}_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23''}; \mathbf{k}_{23}; \tau_1 \sigma_1 \mathbf{k}_1 | j, j_1, \epsilon_{int}; \Pi; \frac{1}{2} T_z)\) as the intrinsic three-body wave function of Eq. \([12]\) and we obtain

\[
\mathcal{L}_F(\mathbf{T}_{23}, \tau_{23}; \alpha, \epsilon_{23}; j_{23}, j_{23''}; \tau_1 \sigma_1 \mathbf{k}_1 | j, j_1, \epsilon_{int}; \Pi; \frac{1}{2} T_z) = \\
= \sum_{\tau_2 \tau_3} \sum_{\sigma_2 \sigma_3} \int \frac{d\mathbf{k}_1'}{2k_{1''}^+ (2\pi)^3} \sum_{\sigma_1} \int \frac{2M_{(1, 2, 3)}^2 d\mathbf{k}_{23}}{E_{k_{1'}}^2 E_{k_{23}}^2 (2\pi)^3} D_{\sigma_1, \sigma_2}^\dagger \delta^3(\mathbf{P}'' - \mathbf{P}') \delta^3(\mathbf{k}_{1''} - \mathbf{k}_1') \frac{E_{k_{1'}} M_{(1, 2, 3)}}{M''_0} \delta^3(\mathbf{k}_{23''} - \mathbf{k}_{23}) \\
\times \delta^3(\mathbf{k}_{1} - \mathbf{k}_{1}^{(a)}) \sqrt{\frac{E_{k_{1'}}^+}{k_{1''}^+}} \sqrt{\frac{E_{k_{23}}^+}{E_{k_{1}}^+}} \sqrt{\frac{(2\pi)^3 E_{k_{1'}} M_{(1, 2, 3)}^2}{2M_{(1, 2, 3)}}} \sum_{\sigma_2 \sigma_3} D_{\sigma_2, \sigma_3}^\dagger [\mathcal{R}_M(\mathbf{k}_{23})]_{\sigma_2' \sigma_2} D_{\sigma_3, \sigma_3}^\dagger [\mathcal{R}_M(-\mathbf{k}_{23})]_{\sigma_3' \sigma_3} \\
\times \sum_{\sigma'_1} D_{\sigma'_1, \sigma''_1}^\dagger [\mathcal{R}_M(\mathbf{k}_{1})]_{\sigma'_1 \sigma''_1} D_{\sigma'_1, \sigma''_1}^\dagger [\mathcal{R}_M(-\mathbf{k}_{1})]_{\sigma'_1 \sigma''_1} \\
\times \sqrt{\frac{(2\pi)^6 2E(\mathbf{k}_{1}) M_{(1, 2, 3)}^2}{2M_{(1, 2, 3)}}} (\sigma''_1, \sigma'_2, \sigma'_3; \tau_1, \tau_2, \tau_3; \mathbf{k}_{23}; \mathbf{k}_1 | j, j_1; \epsilon_{int}; \Pi; \frac{1}{2} T_z) .
\]
In the previous equation the integration variable \( k_{23}^+ \) has been changed in \( k_{23z}^+ \), using the equality \( \partial k_{23z}^+ / \partial k_{23z} = 2 k_{23z}^+ / M_{23} \) (see Eqs. (B12) and (B14)). Then one obtains

\[
L_F \langle T_{23}; \tau_{23} \rangle; \alpha, \epsilon_{23}; j_{23}, j_{23z}; \tau_1 \sigma_1 \tilde{k}_1 | j, j_z; \epsilon^3_{\text{int}}, \Pi; \frac{1}{2} T_z \rangle = \]

\[
= \sum_{\tau_2 \tau_3} \int d \tilde{k}_{23} \sum_{\sigma'_i} D^\frac{1}{2} [R_M (\tilde{k}_{23}^{(a)})]_{\sigma_1 \sigma'_i} \sqrt{(2\pi)^3} 2E(\tilde{k}_{1}^{(a)}) \frac{\kappa^+_{1} E_{23}^{\alpha}}{\kappa^{(a)}_{1} E_{23}} \]

\times \sum_{\sigma'_2, \sigma'_3} D^\frac{1}{2} [R_M (\tilde{k}_{22}^{(a)})]_{\sigma'_2 \sigma'_3} D^\frac{1}{2} [R_M (\tilde{k}_{23})]_{\sigma_2 \sigma_3} D^\frac{1}{2} [R_M (\tilde{k}_{2})]_{\sigma_2 \sigma'_3} \]

\times \sum_{\sigma_1', \sigma_2', \sigma_3'} D_{\sigma_1', \sigma_2', \sigma_3'}(\kappa_{23}, \kappa_{2}) D_{\sigma_1', \sigma_2', \sigma_3'}(-\tilde{k}_{23}, \tilde{k}_{3}) \]

\times \langle T_{23}; \tau_{23} \rangle; \alpha, \epsilon_{23}; j_{23}, j_{23z} | k_{23}^+, \sigma'_2, \sigma'_3; \tau_2, \tau_3 \rangle \langle \sigma'_3, \sigma'_2, \sigma'_1; \tau_3, \tau_2, \tau_1 \rangle \kappa_{23}^+(a) | j, j_z; \epsilon^3_{\text{int}}, \Pi; \frac{1}{2} T_z \rangle \right) , \quad (C9)

where

\[
D_{\sigma''_1, \sigma'_1}(\pm \tilde{k}_{23}, \tilde{k}_{4}) = \sum_{\sigma_i} D^\frac{1}{2} [R_M (\pm \tilde{k}_{23})]_{\sigma''_1 \sigma_i} D^\frac{1}{2} [R_M (\tilde{k}_{4})]_{\sigma_i \sigma'_1} \]

with the + corresponding to \( i = 2 \) and the - corresponding to \( i = 3 \).

Let us notice that the matrices \( D_{\sigma''_1, \sigma'_1}(\pm \tilde{k}_{23}, \tilde{k}_{4}) \) are unitary, i.e.

\[
\sum_{\sigma_i} D^\frac{1}{2} [R_M (\pm \tilde{k}_{23})]_{\sigma''_1 \sigma_i} D_{\sigma_i, \sigma'_1}(\pm \tilde{k}_{23}, \tilde{k}_{4}) = \delta_{\sigma''_1, \sigma'_1} \]

because of the unitarity of the \( D^{1/2} \) matrices.

3. Normalization of the overlaps between a state of the cluster \{ 1, (23) \} and the bound state of the three-particle system

The normalization of the intrinsic LF overlaps \( L_F \langle T_{23}; \tau_{23} \rangle; \alpha, \epsilon_{23}; j_{23}, j_{23z} \rangle; \tau_1 \sigma_1 \tilde{k}_1 | j, j_z; \epsilon^3_{\text{int}}, \Pi; \frac{1}{2} T_z \rangle \) can be easily recovered using Eq. (C9), viz.

\[
N = \sum_{T_{23} \tau_{23}} \int \sum_{j_{23}, j_{23z}} \sum_{\sigma_1, \tau_1} \int \frac{d \tilde{k}_{1}}{2 \kappa_{1}^{+}(2\pi)^2} \left| L_F \langle T_{23}; \tau_{23} \rangle; \alpha, \epsilon_{23}; j_{23}, j_{23z} \rangle; \tau_1 \sigma_1 \tilde{k}_1 | j, j_z; \epsilon^3_{\text{int}}, \Pi; \frac{1}{2} T_z \rangle \right|^2 = \]

\[
= \sum_{T_{23} \tau_{23}} \int \sum_{j_{23}, j_{23z}} \sum_{\sigma_1, \tau_1} \int \frac{d \tilde{k}_{1}}{2 \kappa_{1}^{+}(2\pi)^2} \left| \sum_{\tau_2 \tau_3} \int d \tilde{k}_{23} \sum_{\sigma'_i} D^\frac{1}{2} [R_M (\tilde{k}_{23}^{(a)})]_{\sigma_1 \sigma'_i} \sqrt{(2\pi)^3} 2E(\tilde{k}_{1}^{(a)}) \right| \]

\times \sqrt{\frac{\kappa_{1}^{+}(a) E_{23}}{\kappa_{1}^{(a)} E_{23}}} \sum_{\sigma'_2, \sigma'_3} D_{\sigma''_2, \sigma'_2}(\tilde{k}_{23}, \tilde{k}_{2}) D_{\sigma''_3, \sigma'_3}(-\tilde{k}_{23}, \tilde{k}_{3}) \]

\times \langle T_{23}; \tau_{23} \rangle; \alpha, \epsilon_{23}; j_{23}, j_{23z} | k_{23}^+, \sigma'_2, \sigma'_3; \tau_2, \tau_3 \rangle \langle \sigma''_3, \sigma'_2, \sigma'_1; \tau_3, \tau_2, \tau_1 \rangle \kappa_{23}^+(a) | j, j_z; \epsilon^3_{\text{int}}, \Pi; \frac{1}{2} T_z \rangle \right|^2 = \]

\[
= \sum_{T_{23} \tau_{23}} \int \sum_{j_{23}, j_{23z}} \sum_{\sigma_1, \tau_1} \int d \tilde{k}_{23} \int d \tilde{k}_{1} \sum_{\sigma'_i} D^\frac{1}{2} [R_M (\tilde{k}_{23}^{(a)})]_{\sigma_1 \sigma'_i} \sqrt{\frac{E(\tilde{k}_{1}^{(a)})}{E_{23}}} \]
\[ \times \left\{ \sum_{\sigma_2', \sigma_3'} D_{\sigma_2', \sigma_3'} (\tilde{k}_{23}, \tilde{k}_{3}) D_{\sigma_3', \sigma_3'} (-\tilde{k}_{23}, \tilde{k}_{3}) \right\} \]

\[ \times \langle T_{23}, r_{23}; \alpha, \epsilon_{23}: j_2 j_3 j_2 j_3 | k_{23}, \sigma_{23}''; \tilde{r}_2, \tilde{r}_3 \rangle \langle \sigma_3', \sigma_2', \sigma_1'; \tilde{r}_3, \tilde{r}_2, \tilde{r}_1; k_{23}, k_1^{(a)} | j, j_z; \epsilon_{int}, \Pi; \frac{1}{2} T_z \rangle \]

\[ \times \sum_{\sigma_2, \sigma_3} \int d\kappa_2^{(\mu)} D_{\sigma_2, \sigma_2} (\tilde{k}_{23}, \tilde{k}_{2}) D_{\sigma_3, \sigma_3} (-\tilde{k}_{23}, \tilde{k}_{3}) \] \[ \langle \sigma_3', \sigma_2', \sigma_1'; \tilde{r}_3, \tilde{r}_2, \tilde{r}_1; k_{23}, k_1^{(a)} | j, j_z; \epsilon_{int}, \Pi; \frac{1}{2} T_z \rangle^* \] . (C12)

In the last step of Eq. (C12) the change of integration variable \( d\kappa_1^{\perp} = d\kappa_1^{\perp} E_S/E_{23} \) (see Eqs. (B11) and (B14)) was performed.

Then, using the completeness for the two-body system (2,3) (see Eq. (A19)) one obtains

\[ N = \sum_{\sigma_1, \sigma_2, \sigma_3} \sum_{\tau_2, \tau_3} \int d\kappa_2 \int d\kappa_1^{(a)} \frac{E(k_1^{(a)})}{k_1^{(a)}} \left| \sum_{\sigma_1'} D_{\sigma_2, \sigma_3} (\tilde{k}_{23}, \tilde{k}_{2}) D_{\sigma_3, \sigma_3} (-\tilde{k}_{23}, \tilde{k}_{3}) \right|^2 \sum_{\sigma_1'} \langle \sigma_1', \sigma_2', \sigma_1'; \tilde{r}_3, \tilde{r}_2, \tilde{r}_1; k_{23}, k_1^{(a)} | j, j_z; \epsilon_{int}, \Pi; \frac{1}{2} T_z \rangle^* \] . (C13)

Finally, exploiting the unitarity of \( D^{1/2} \) and \( D \) matrices (see Eq. (C11)), one has

\[ N = \sum_{\tau_1, \tau_2, \tau_3} \int d\kappa_2 \int d\kappa_1^{(a)} \frac{E(k_1^{(a)})}{k_1^{(a)}} \sum_{\sigma_1'} \sum_{\sigma_1'} \langle \sigma_3', \sigma_2', \sigma_1'; \tilde{r}_3, \tilde{r}_2, \tilde{r}_1; k_{23}, k_1^{(a)} | j, j_z; \epsilon_{int}, \Pi; \frac{1}{2} T_z \rangle^* \] \[ \times \langle \sigma_3', \sigma_2', \sigma_1'; \tilde{r}_3, \tilde{r}_2, \tilde{r}_1; k_{23}, k_1^{(a)} | j, j_z; \epsilon_{int}, \Pi; \frac{1}{2} T_z \rangle^* = \] \[ \sum_{\sigma_1'} \sum_{\tau_1} \int d\kappa_2 \int d\kappa_1^{(a)} \sum_{\tau_2, \tau_3} \sum_{\sigma_2', \sigma_3'} \langle \sigma_3', \sigma_2', \sigma_1'; \tilde{r}_3, \tilde{r}_2, \tilde{r}_1; k_{23}, k_1^{(a)} | j, j_z; \epsilon_{int}, \Pi; \frac{1}{2} T_z \rangle^* \] \[ = 1 \] . (C14)

where Eqs. (B13) and (43) were used.
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