Beyond the Frenkel–Kac–Segal construction of affine Lie algebras

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This contribution reviews recent progress in constructing affine Lie algebras at arbitrary level in terms of vertex operators. The string model describes a completely compactified subcritical chiral bosonic string whose momentum lattice is taken to be the (Lorentzian) affine weight lattice. The main feature of the new realization is the replacement of the ordinary string oscillators by physical DDF operators, whereas the unphysical position operators are substituted by certain linear combinations of the Lorentz generators. As a side result we obtain simple expressions for the affine Weyl translations as Lorentz boosts. Various applications of the construction are discussed.

1. MOTIVATION

Recent developments indicate that finite and affine Lie algebras are not sufficient for the description of symmetries in string theory. It is quite likely that hyperbolic Kac–Moody algebras and their (super)extensions to Borcherds (super)algebras, also called generalized Kac–Moody (super)algebras, will play an important role for the understanding of certain string symmetries.

Unfortunately, from the mathematical point of view not much is known about these infinite-dimensional Lie algebras. One could also turn this into an advantage by interpreting it as a promising sign since history has taught us that new fundamental developments in physics very often involve new mathematics.

At the physical side, there is a class of certain (completely toroidally compactified) string models where hyperbolic and Lorentzian Kac–Moody algebras and some of their Borcherds extensions are explicitly realized. More specifically, they arise as Lie algebras of physical string states (see e.g. the review) with Lie bracket given by \[[\psi, \varphi] := \oint \frac{dz}{2\pi i} V(\psi, z)\varphi\] for physical states \(\psi, \varphi\) and associated vertex operator \(V(\ , z)\). At first sight it may seem rather awkward that the physical states themselves form a Lie algebra. Nonetheless, the bracket has a simple interpretation in terms of tree level scattering of physical string states.

The general feature of hyperbolic Kac–Moody algebras is that they can be decomposed into an infinite direct sum of highest (and of lowest) weight modules for the underlying affine subalgebra. In the above string realization this entails that one has to deal with infinitely many affine highest weight modules within a single physical state space. To handle this problem one would need a physical string vertex operator construction of affine Lie algebras at arbitrary level. This contribution reports on recent progress in this question which was obtained in collaboration with H. Nicolai.

2. COMPACTIFIED BOSONIC STRING

We consider a (chiral half of a) closed bosonic string moving on a \(d\)-dim Minkowskian torus as spacetime. Uniqueness of the quantum mechanical wave function then forces the center of mass momenta of the string to lie on an even Lorentzian lattice \(\Lambda\). From a phenomenological point of view it is certainly not very plausible to com-
pect to the string oscillator modes. Due to the

\[ \alpha_m^\mu \text{ and }\alpha_n^{\nu} = m \kappa^{\mu\nu} \delta_{m+n,0} , \]

and the groundstates \(|\lambda\rangle = e^{i\lambda \cdot q}|0\rangle\) for \(\lambda \in \Lambda\) satisfy \((|q^\mu, p^\nu\rangle = i \kappa^{\mu\nu}, \ p^{\mu} \equiv \alpha_0^{\mu}\)

\[ p^\mu |\lambda\rangle = \lambda^\mu |\lambda\rangle , \quad \alpha_m^\mu |\lambda\rangle = 0 \quad \forall m > 0 . \]

In order to define physical states one has to implement the Virasoro constraints (with central charge \(c = d\))

\[ L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} :\alpha_m, \alpha_{n-m}: , \]

where \(\ldots\ldots\) denotes normal-ordering with respect to the string oscillator modes. Due to the anomaly, however, we proceed à la Gupta–Bleuler in electromagnetism which means that we can impose only half of the constraints. Hence physical states are conformal primary states of weight 1,

\[ \mathcal{P} := \bigoplus_{\lambda \in \mathcal{Q}} \mathcal{P}(\lambda) , \]

where

\[ \mathcal{P}(\lambda) := \{ \psi | L_n \psi = \delta_{n0} \psi \ \forall n \geq 0, \ p^\mu \psi = \lambda^\mu \psi \} . \]

The simplest examples of physical states are tachyons \(|a\rangle\) with \(a \in \Lambda\) and \(a^2 = 2\), photons \(\mathbf{\xi} \cdot \alpha_{-1}|k\rangle\) with \(\mathbf{\xi} \in \mathbb{R}^{d-1,1}, \ k \in \Lambda\) and \(\mathbf{\xi} \cdot k = k^2 = 0\), etc..

### 3. DDF CONSTRUCTION

It would be nice to have an explicit description of the space of physical states in the same way as the Fock space is built as an infinite sum of Heisenberg modules. Such a description is indeed possible and is provided by the so-called DDF construction \[ \mathcal{F} \] adjusted to the discrete model \[ \mathcal{F} \]. For a given physical momentum \(\lambda \in \Lambda\), \(\lambda^2 \leq 2\), one first has to find a tachyon \(|a\rangle\) and a lightlike vector \(k\) such that \(a \cdot k = 1\) and

\[ \lambda = a - nk \quad \text{with } n := 1 - \frac{1}{2} \lambda^2 . \]

Such a pair \((a, k)\) always exists as long as we do not require the vectors to lie on the lattice. We refer to it as a DDF decomposition of \(\lambda\). Next we choose \(d - 2\) orthonormal polarization vectors \(\mathbf{\xi}_i(a, k) \in \mathbb{R}^{d-1,1}\) satisfying \(\mathbf{\xi}_i a = \mathbf{\xi}_i k = 0\), and define the transversal DDF operators

\[ A_m = A_m^i(a, k) := \int_0^\infty \frac{dz}{2\pi i} \mathbf{\xi}_i P(z) e^{imk \cdot X(z)} , \]

in terms of the Fubini–Veneziano fields

\[ X^\mu(z) := q^\mu - iy^\mu \ln z + i \sum_{m \neq 0} \frac{1}{m} \alpha_m z^{-m} , \]

\[ P^\mu(z) := i \frac{d}{dz} X^\mu(z) = \sum_{m \in \mathbb{Z}} \alpha_m z^{-m-1} . \]

Using operator product techniques it is straightforward to show that the \(A_m^i\)'s obey a \((d-2)\)-fold "transversal" oscillator algebra,

\[ [A_m^i, A_n^j] = m \delta_{ij} \delta_{m+n,0} . \]

There are also longitudinal DDF operators \(A_m^{-} = A_m^{-i}(a, k)\) whose complicated expressions are not needed here. They form a "longitudinal" Virasoro algebra with central charge \(c = 26 - d\) and commute with the transversal DDF operators.

The nice thing about the DDF operators is that they constitute a spectrum-generating algebra for the string; for it can be shown that they commute with the Virasoro constraints \(L_n\) (and hence map physical states into physical states) and \(\mathcal{P}(\lambda)\) is spanned by

\[ A_{n_1}^1 \cdots A_{n_N}^N A_{-m_1}^{-} \cdots A_{-m_M}^{-} |a\rangle , \]

where \(n_1 + \ldots + m_M = 1 - \frac{1}{2} \lambda^2\). Note that the DDF operators also account for an explicit description of the null physical states contained in
\( \mathcal{P}^{(\lambda)} \) due to the relation \( A_{-1}(\mathbf{a}) \propto L_{-1}(\mathbf{a} - \mathbf{k}) \), which can be verified by a straightforward computation.

4. AFFINE LIE ALGEBRA

Let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra of type ADE and rank \( d - 2 \) (\( d \geq 3 \)). We denote the root system of \( \mathfrak{g} \) by \( \Delta \). Consider the associated non-twisted affine Lie algebra \( \mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{n}_- \) (see e.g. [10, 11]). The \( d \)-dimensional affine Cartan subalgebra \( \mathfrak{h} \) then contains a central element and a scaling element denoted by \( K \) and \( d \), respectively. The corresponding elements in \( \mathfrak{h}^* \) are called null root \( \delta \) and basic fundamental weight \( \Lambda_0 \), respectively. We have the scalar products \( \delta \cdot \Lambda_0 = 1 \) and \( \delta^2 = \Lambda_0^2 = \delta \cdot \mathbf{r}_i = \Lambda_0 \cdot \mathbf{r}_i = 0 \) for the real simple roots \( \mathbf{r}_i \) (\( 1 \leq i \leq d - 2 \)) of \( \mathfrak{g} \).

In the above string model we shall now choose for the momentum lattice \( \Lambda \) the affine weight lattice \( Q^* \), which is even Lorentzian as required.

Recall that an irreducible level-\( \ell \)-highest weight module \( L(\Lambda) \) for an affine Lie algebra \( \mathfrak{g} \) is determined by the following data: a vacuum vector \( v_\Lambda \), a dominant integral weight \( \Lambda \in Q^* \), and a weight system \( \Omega(\Lambda) \) with appropriate weight multiplicities. Without loss of generality we may assume that \( \Lambda^2 = 2 \), because \( \Lambda + z \delta \) for any \( z \in \mathbb{R} \) gives rise to an isomorphic \( \mathfrak{g} \)-module.

An affine Cartan–Weyl basis in terms of integrated vertex operators can be introduced as follows. Let \( \xi^i \) be a set of \( d - 2 \) orthonormal polarization vectors associated with the DDF decomposition \( (\Lambda, \mathbf{k}_\ell) \) where \( \mathbf{k}_\ell := \frac{1}{\ell} \delta \). On

\[ \mathcal{P}(\Lambda) := \bigoplus_{\lambda \in \Omega(\Lambda)} \mathcal{P}^{(\lambda)}, \]

which is a subspace of the space of physical string states, \( \mathcal{P} \), we define

\[ K := \delta \cdot \mathbf{p}, \quad d := \Lambda_0 \cdot \mathbf{p}, \quad (1a) \]

\[ H^i_m := \oint_{0}^{\frac{1}{2\pi \imath}} \xi^i \cdot \mathcal{P}(z) \mathbf{e}^{m \delta \cdot \mathbf{X}(z)}, \quad (1b) \]

\[ E^*_m := \oint_{0}^{\frac{1}{2\pi \imath}} \mathbf{e}^{(\mathbf{r} + m \delta) \cdot \mathbf{X}(z)} \cdot e_{\mathbf{r}}, \quad (1c) \]

for all \( \mathbf{r} \in \bar{\Delta} \), where \( c_{\mathbf{r}} \) denotes some appropriate cocycle factor satisfying \( c_{\mathbf{r}} | \mathbf{s} \rangle = \epsilon(\mathbf{r}, \mathbf{s}) | \mathbf{s} \rangle \) for some 2-cocycle \( \epsilon \). One can show ([12, 13]) that these operators obey the commutation relations

\[ [H^i_m, H^j_n] = \ell m \delta^{ij} \delta_{m+n, 0}, \]

\[ [H^i_m, E^r_n] = (\xi^i \cdot \mathbf{r}) E^r_{m+n}, \]

\[ [E^r_m, E^s_n] = \begin{cases} 0 & \text{if } r \cdot s \geq 0, \\ \epsilon(\mathbf{r}, \mathbf{s}) E^r_{m+n} & \text{if } r \cdot s = -1, \\ H^r_{m+n} + \ell m \delta_{m+n, 0} & \text{if } r \cdot s = -2, \end{cases} \]

\[ [d, H^i_m] = m H^i_m, \quad [d, E^r_m] = m E^r_m, \]

\[ [K, x] = 0 \quad \forall x \in \mathfrak{g}. \]

This gives a level-\( \ell \) vertex operator realization of \( \mathfrak{g} \) on \( \mathcal{P}(\Lambda) \). In fact, we can identify the vacuum vector \( v_\Lambda \) in \( L(\Lambda) \) with the tachyonic groundstate \( | \Lambda \rangle \) in \( \mathcal{P}(\Lambda) \) to conclude that

\[ L(\Lambda) \hookrightarrow \mathcal{P}(\Lambda), \quad L(\Lambda)_{\Lambda} \hookrightarrow \mathcal{P}(\lambda). \]

Let us make some remarks.

We observe that the \( H^i_m \)'s, which make up the homogeneous Heisenberg subalgebra of \( \mathfrak{g} \), are nothing but the transversal DDF operators \( A^i_m \) and thus play the role of spectrum-generating elements. Only for \( \ell = 1 \) the Heisenberg subalgebra and the transversal oscillator algebra are identical. One may wonder why there is only one set of polarization vectors \( \xi^i(\Lambda, \mathbf{k}_\ell) \) and DDF operators \( A^i_m(\Lambda, \mathbf{k}_\ell) \) although one has such data for each \( \lambda \in \Omega(\Lambda) \). One easily shows, however, that the polarization vectors can always be chosen such that they differ only by vectors proportional to \( \delta \) when going from one \( \lambda \) to another. But since \( \delta \cdot \mathcal{P}(z) \mathbf{e}^{m \delta \cdot \mathbf{X}(z)} \) for \( m \neq 0 \) is a total derivative, we conclude that the operators \( H^i_m \) are indeed universally defined on \( \mathcal{P}(\Lambda) \).

Below, we will see that only transversal physical states can occur in the affine highest weight module \( L(\Lambda) \). Hence we effectively deal with the embedding \( L(\Lambda)_{\Lambda} \hookrightarrow \mathcal{P}(\lambda) \) and have the following universal estimate for affine weight multi-

\( ^3 \)An especially nice feature of this string realization is the fact that both the vacuum vector and the null vector conditions in \( L(\Lambda) \) immediately follow from momentum conservation and the physical state condition \( L_0 \psi = \psi \).
plexities at arbitrary level:
\[ \text{mult}_{L(\Lambda)}(\lambda) \leq \dim P(\Lambda)_{\text{transv}} = p_{d-2}(1 - \frac{1}{2} \lambda^2), \]
where \( p_{d-2}(n) \) counts the partition of \( n \) into “parts” of \( d - 2 \) “colours”.

5. RESULT

In view of the fact that the operators of the Cartan–Weyl basis \( \{ \mathbf{1} \} \) are integrated vertex operators associated with physical string states and thus are physical operators by construction, it is sensible to ask whether they may be directly expressed in terms of the DDF operators, i.e., in a manifestly physical form. Note that this is already the case for the \( H_i \)'s (see above remark). So only the step operators \( \{ \mathbf{3} \} \) remain to be dealt with. It is certainly true that given a step operator acting on some physical state, the resulting physical state can be written in the DDF basis. So the precise question is whether there is some unifying formula (independent of the state acted on) for the step operators in terms of physical operators. Note that this is a highly nontrivial problem because of the exponential dependence of both the step operators and the DDF operators on the string oscillators. The main result of \( \{ \mathbf{4} \} \) gives an affirmative answer to the above question.

**Theorem.** On \( P(\Lambda) \), one can rewrite the step operators as follows:

\[ E^r_m \big|_{P(\Lambda)} = \oint_{\mathcal{C}_r} \frac{dz}{2\pi i} \frac{z^{\ell_m \times \mathcal{X}_r}}{x^{\mathcal{X}_r \times \mathcal{C}_r}}, \quad (2) \]

where

\[ \lambda_i(z) := Q_i - iA_{0i} \ln z + i \sum_{m \neq 0} \frac{1}{m} A^i_m z^{-m}, \]

\[ Q_i := (\xi^i)_{\mu}(k_\ell)_\nu M^{\mu\nu}, \]

with Lorentz generators

\[ M^{\mu\nu} := q^\mu p^\nu - q^\nu p^\mu - i \sum_{n \neq 0} \frac{1}{n} q^{[\mu}_n a_{\nu]}^r, \]

and \( \times \ldots \times \) denotes normal-ordering with respect to the mode indices of the transversal DDF operators.

Let us discuss some aspects of this formula. The above vertex operator construction may be characterized as “doubly transcendental” due to the appearance of DDF operators in the exponential. Furthermore, it is “purely transversal” since only transversal DDF operators are involved.

At first sight it is surprising that the Lorentz generators pop up. On the other hand, it is clear that the unphysical position operators \( q^\mu \) in the step operators \( E^r_m \), which generate a momentum shift by \( r^\mu \), must be replaced by some physical operators other than the DDF operators, which shift the momentum only along the direction of the affine null root. Therefore one is inevitably led to consider the Lorentz generators which are physical and which rotate the momentum vectors. If we introduce momentum operators \( \mathcal{P}^\mu \equiv \mathcal{A}^i_0 \) then the Lorentz generators \( \mathcal{Q}^i \), which replace the \( q^\mu \)'s, are canonically conjugate to them and may be regarded as “physical position operators”, viz. \( \mathcal{Q}^i, \mathcal{P}^\mu = i\delta^{ij} \).

As a side result, the new formula for the step operators provides a natural interpretation of affine Weyl translations as Lorentz boosts. One finds that

\[ e^{ir^\nu \mathcal{P}^\nu} \alpha_m e^{-ir^\nu \mathcal{P}^\nu} = t_{\ell}^{[\nu}(\mathcal{v}) \cdot \alpha_m \]

for all \( \mathcal{v} \in h^*, r \in \tilde{Q} \) (finite root lattice), where

\[ t_{\ell}^{[\nu}(\mathcal{v}) := \mathcal{v} + (\mathcal{v} \cdot k_\ell) r - \frac{1}{2} (\mathcal{v} \cdot k_\ell)^2 r^2 + r \cdot \mathcal{v} \cdot k_\ell. \]

Indeed, the last expression is precisely the formula for an affine Weyl translation (see e.g., \( \{ \mathbf{6} \} \)) which arises in the decomposition of the affine Weyl group into a semidirect product of the finite Weyl group and the affine translation group isomorphic to the finite root lattice. Clearly, the affine null root is invariant under the translations \( t_{\ell}^{[\nu}. \) In this way the affine Weyl group becomes a discrete subgroup of ISO\((d - 2)\), the subgroup of the full Lorentz group SO\((d - 1, 1)\) leaving fixed a given lightlike vector. We can thus think of the affine Weyl group as a “dimensional null reduction” of the full Lorentzian Weyl group.

Formula \( \{ \mathbf{8} \} \) for the step operators resembles very much the famous Frenkel–Kac–Segal construction of level-1 affine Lie algebras \( \{ \mathbf{14}, \mathbf{13} \} \). There, the momentum lattice \( \Lambda \) is taken to be the Euclidean finite root lattice \( \tilde{Q} \) and the
6. Sugawara Operators

Given an affine Lie algebra $\mathfrak{g}$ with Cartan–Weyl basis $H_n^i$, $E_n^r$ ($1 \leq i \leq d - 2$, $r \in \Delta$), it is well-known that there is a Virasoro algebra $\text{Vir}_\mathfrak{g}$ associated with it such that any representation of $\mathfrak{g}$ can be extended to a representation of the semidirect product $\mathfrak{g} \rtimes \text{Vir}_\mathfrak{g}$. Indeed, the so-called Sugawara operators

$$L_m := \frac{1}{2(\ell + h^\vee)} \sum_{n \in \mathbb{Z}} \sum_{i=1}^{d-2} \sum_{r \in \Delta} \delta H_n^i H_{m-n}^r + \sum_{r \in \Delta} \delta E_n^r E_{m-n}^r$$

form a Virasoro algebra with central charge $c_\ell := \frac{\ell \dim \mathfrak{g}}{2(\ell + h^\vee)}$, where $\ell$ and $h^\vee$ denote the level and the dual Coxeter number, respectively, and the symbol $\delta$ refers to normal-ordering with respect to the mode indices of the affine generators. If we insert the expression (2) for the step operators, we arrive at the following new formula for the Sugawara operators at arbitrary level in terms of transversal DDF operators [16]:

$$L_m = \frac{1}{2\ell} \sum_{n \in \mathbb{Z}} \sum_{i=1}^{d-2} \sum_{r \in \Delta} \delta H_n^i H_{m-n}^r \times + \frac{h^\vee}{2\ell(\ell + h^\vee)} \sum_{n \not= 0} \sum_{i=1}^{d-2} \delta A_n^i A_{m-n}^i \times + \frac{(\ell^2 - 1)(d - 2)h^\vee}{24\ell(\ell + h^\vee)} \delta_{m,0}$$

$$- \frac{1}{2\ell(\ell + h^\vee)} \sum_{n \not= 0} \sum_{r \in \Delta} \sum_{p=1}^{\ell - 1} \frac{1}{|\zeta^p - 1|^2} \times \times \oint \frac{dz}{2\pi i} (z^{m-1} \times e^{\phi r} [x(z^\vee) - \mathcal{X}(z)] \times x),$$

(4)

where $\zeta := e^{2\pi i/\ell}$.

Only special cases of this formula had been known before. For level $\ell = 1$ one easily finds that

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{i=1}^{d-2} \delta H_n^i H_{m-n}^i \times,$n

which is referred to in the literature as the equivalence of the Virasoro and the Sugawara construction. At arbitrary level, the action of the zero mode operator on an affine highest weight vector $|\Lambda\rangle$ is given by the simple formula

$$L_0|\Lambda\rangle = \frac{(\hat{\Lambda} + 2\hat{\rho}) \cdot \hat{\Lambda}}{2(\ell + h^\vee)} |\Lambda\rangle,$$

where $\hat{\rho}$ denotes the finite Weyl vector and $\hat{\Lambda}$ is the projection of $\Lambda$ onto $\mathfrak{h}$. Previously, one had to rely on properties of the affine Casimir operator in order to derive this result. With the new formula (4) at hand, the above two expressions can be immediately read off (for the second expression one also has to invoke some simple identity for sums over roots of unity).

A remarkable feature of the last term in formula (4) is its nonlocality. Responsible for this is the factor $z^{m-1}$ in (2). Indeed, in evaluating the
expression $\sum_{n \in \mathbb{Z}} \circ E_n \circ E_{-n}$ this leads to a factor of $\sum_{n \geq 0} z^{-\ell n} e^{\ell m+n} = \frac{z^\ell e^m}{z^\ell - w}$ in the corresponding operator product expansion. Consequently, we pick up additional poles at $z = e^{2 \pi i p/\ell} w$ ($1 \leq p \leq \ell$) when we perform the contour integral in the $z$-plane. An interpretation of this is that we are not working on the Riemann sphere but rather on a $\ell$-sheeted covering of it.\footnote{I am grateful to A. M. Semikhatov for pointing this out to me.}

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