Suppression of singularities of solutions of the Euler-Poisson system with density-dependent damping

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Abstract

We find a sharp condition on the density-dependent coefficient of damping of a one-dimensional repulsive Euler-Poisson system, which makes it possible to suppress the formation of singularities in the solution of the Cauchy problem with arbitrary smooth data. In the context of plasma physics, this means the possibility of suppressing the breakdown of arbitrary oscillations of cold plasma.

Keywords: Euler-Poisson system, equations of cold plasma, singularity formation, density dependent damping

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1. Introduction

We consider the system of Euler-Poisson equations describing the behavior of cold plasma for velocity $V$, electron density $n > 0$ and electric field potential $\Psi$ in the following form:

$$\frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\nabla \Psi - \nu V,$$
$$\frac{\partial n}{\partial t} + \text{div} (nV) = 0,$$
$$\Delta \Psi = 1 - n. \quad (1)$$

All components of the solution are assumed to be functions of time $t \geq 0$ and point $x \in \mathbb{R}^n$, $\int (n - 1) \, dx = \text{const}$, $\nu \geq 0$ is the damping factor.

This system is a pressureless variant of the general Euler-Poisson system, having numerous physical applications, see [2] for references. One of the crucial questions is the study of the Cauchy problem and the analysis of the possibility...
of the existence of a globally in time smooth solution. The model without pressure is somewhat simpler from a mathematical point of view, since it allows one to obtain criteria for the formation of a singularity from the initial data.

In [2], many versions of the model without pressure, including those with constant damping and viscosity, with both zero and non-zero backgrounds, have been studied. In all these cases, it is possible to find the initial data leading to a blow-up in a finite time. Moreover, this possibility still remains if pressure and heat diffusion are added to the model [3].

In recent years, the pressureless model has attracted great interest, since it is very convenient to describe the wake wave in the cold plasma generated by a laser pulse in order to create a new type of accelerator, [4] and references therein. It is generally known that the plasma oscillation tends to blow-up, forming a gradient catastrophe in the velocity component and a delta singularity in the density component. After the moment of the singularity formation, the cold plasma model loses its relevance; therefore, the conditions on the initial data or other parameters that make it possible to maintain a smooth solution as long as possible or, possibly, guaranteeing a global in time smooth solution is a key question for all the theory.

In plasma physics, the damping factor $\nu$ corresponds to the frequency of the electron-ion collisions, this value is very small from the physical point of view. Depending on the model, the electron-ion collisions either can be neglected or taken into account. In the recent paper [5], for a very particular solution in the 1D case the authors showed numerically that if $\nu = \nu_0 n$, where $\nu_0$ is a positive constant, than the oscillations never blow up.

Our main question is whether this result is valid for all possible initial data and can it be substantiated analytically? In addition, if $\nu$ is a smooth function of $n$, what conditions must be imposed on $\nu(n)$ to ensure the global in time smoothness of the solution to the Cauchy problem for any given data?

In this paper, we focus on the 1D case, since explicit analytical results can
be obtained here, and we rewrite (1) in a form more accepted in plasma physics:

\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = -E - \nu(n)V, \quad \frac{\partial E}{\partial t} + V \frac{\partial E}{\partial x} = V, \quad n = 1 - \frac{\partial E}{\partial x},
\] (2)

see [4] for details. Here \( E = \nabla \Psi \) is the vector of electric field. System (2) will be considered together with the Cauchy data

\[(V, E)|_{t=0} = (V_0(x), E_0(x)) \in A(\mathbb{R}).\] (3)

For \( \nu \neq \text{const} \) system (2), do not belong to the symmetric hyperbolic type, therefore we cannot guarantee that the solution to the Cauchy problem a local solution as smooth as initial data in the Sobolev norm. Therefore we have to prescribe the analyticity to initial data to use the Cauchy-Kovalevskaya theorem. to show that problem (2), (3) a local in time unique analytical solution.

Problem (2), (3) was completely analyzed for \( \nu = 0 \) in [6] and \( \nu = \text{const} > 0 \) in [7], where sharp conditions on initial data to guarantee a globally in time smooth solution were found (see the analogous result for another context in [2]).

It was found that even for an arbitrarily large constant frequency of collisions there exist data implying a finite time singularity formation.

In the present work, we show that by choosing an appropriate density-dependent damping factor one can obtain a globally smooth solution for any smooth Cauchy data, i.e. completely remove the singularity formation. In particular, for our prototypic function \( \nu(n) = \nu_0 n^\gamma \) the threshold value is \( \gamma = 1 \). For \( \gamma > 1 \) the solution to (2), (3) does not form the gradient catastrophe for any choice of initial data.

The paper is organized as follows. In Section 2 we consider a special solution, linear with respect to the spatial variable (the so-called affine solutions) and prove the exact condition for eliminating blow-up. In Section 3 we prove a similar result for arbitrary initial data. Section 4 discusses issues related to this problem.

3
2. Affine solutions

First, we consider a special form of solutions:

\[ V = a(t)x + A(t), \quad E = b(t)x + B(t). \] (4)

**Theorem 2.1.** Let \( f(n) \in \mathcal{A}(\mathbb{R}_+) \) be a nonnegative function satisfying condition

\[ \int_{\eta_0 > 0}^{+\infty} \frac{f(\eta)}{\eta^2} \, d\eta = \infty. \] (5)

It \( \nu(n) = \epsilon f(n), \epsilon = \text{const} > 0, \) then derivatives of the solution to problem (2), (3), with data (4) are bounded in time for any choice of the data. Otherwise, one can find the data such that the derivatives of solution blow up in a finite time.

**Proof.** We substitute the ansatz (4) in (1) to obtain

\[ \dot{a} = -a^2 - b - \epsilon f(1 - b)a, \quad \dot{b} = (1 - b)a, \] (6)

\[ \dot{A} = -A(a - \epsilon f(1 - b)) - B, \quad \dot{B} = (1 - b)A, \] (7)

Since \( n = 1 - b > 0, \) we consider the domain \( b < 1. \)

The couple of equations (6) splits off from the system, and the second couple (7) is linear with respect to \( A, B \) with the coefficients found in the previous step. Therefore, if we want to study conditions for a blowup of the solution (4), it is sufficient to consider the behavior of the phase curve of the autonomous system (6), given as

\[ \frac{da}{db} = -\frac{a^2 + b}{(1-b)a} - \epsilon \frac{f(1-b)}{1-b}. \] (8)

For an arbitrary \( f \) equation (8) cannot be integrated explicitly, however, it can be considered as a regular perturbation of (8) at \( \epsilon = 0, \) which solution is

\[ a = \pm \sqrt{1 - 2b + C(1-b)^2}. \] (9)

Here the constant \( C = \frac{a_0^2 + 2b_0 - 1}{(1-b_0)^2}, \) with \( a_0 = a(0), b_0 = b(0). \)
The analysis of the phase plane shows that a point on the phase plane moves from the upper half-plane $a > 0$ to the lower half-plane $a < 0$ and there can come back to the upper half-plane or go to minus-infinity. The latter signifies the blowup of derivatives of the solution. If $C < 0$, the curve on the phase plane is bounded (it is ellipse), otherwise $a(t)$ and $b(t)$ move along a parabola ($C = 0$) or hyperbola ($C > 0$) and therefore go to minus-infinity within a finite time (see [6] for details).

Our main question is whether correctors due to parameter $\epsilon$ can change the behavior of trajectory going to infinity and turn it to the upper half-plane $a > 0$.

When analyzing the phase portrait of the perturbed system, we point out the following elementary facts, illustrated in Figure 1.

1. If the initial point of a phase curve is situated in the upper half-plane $a > 0$, within a finite time point $(b, a)$ turns in the lower half-plane $a < 0$, therefore a possible blowup can happen only for $a < 0$ (see Figure 1, left);

2. Since $\frac{da}{db} = -\frac{a^2 + b - \epsilon f(1 - b)}{(1 - b)a} - \frac{a^2 + b}{(1 - b)a}$, and $b$ decreases with $t$ as $a < 0$, then the Chaplygin theorem implies that the phase curve of the perturbed equation, $a_\epsilon(b)$ lies higher that the phase curve of non-perturbed equation, $a_0(b) = -\sqrt{1 - 2b + C(1 - b)^2}$. Therefore for $C < 0$ the curve $a_\epsilon(b)$ always comes back to the upper half-plane $a > 0$. So, for a possible blow-up we have to consider only the initial data corresponding to $C \geq 0$ (see Figure 1, right).

3. Analogously, the Chaplygin theorem implies that if the data are such that the phase curve $a_{\epsilon_1}(b)$ does not go to infinity, then $a_{\epsilon_2}(b)$, $\epsilon_2 > \epsilon_1$ does not go to infinity as well. Therefore, we can consider for the proof arbitrarily small $\epsilon$.

Since the perturbation by means of parameter $\epsilon$ is regular, in a neighborhood $U_\epsilon(0)$ we can expand the solution in a series

$$a_\epsilon(b) = a_0(b) + \sum_{k=1}^{\infty} \epsilon^k a_k(b) = a_0(b) + \epsilon a_1(b) + o(\epsilon),$$

converging at any fixed $b$.

Thus, if the first corrector $a_1(b)$ is such that $a_0(b) + \epsilon a_1(b) > 0$ for some $b_*$ and arbitrary small positive $\epsilon$, then we can guarantee that $a_\epsilon(b_*) > 0$, in other
words, the trajectory came back to the upper half-plane. The linear equations for the correctors are the following:

\[
\frac{d\alpha_1}{db} = -\alpha_1 Q - \frac{f(1-b)}{1-b}, \quad (10)
\]

\[
\frac{d\alpha_k}{db} = -\alpha_k Q + \frac{b\phi_k(a_0, \alpha_1, \ldots, \alpha_{k-1})}{(1-b)a_0^k}, \quad k = 2, \ldots, \quad (11)
\]

where \( Q = \frac{a_0^2 - b}{(1-b)a_0^2} = \frac{1 - 3b + C(1-b)^2}{(1-b)(1-b)^2}, \) and \( \phi_k \) is a homogeneous polynomial of order \( k \) from its arguments, \( a_0 \) is found in (9). It can be readily found from (10) that

\[
\alpha_1(b) = \frac{(1-b)^2}{a_0(b)} b_0 \int_b^{b_0} a_0(\beta) f(1-\beta) (1-\beta)^3 d\beta > 0, \quad b < b_0 = \text{const.}
\]

As for \( C > 0 \)

\[
a_0(b) + \epsilon \alpha_1(b) \sim -\sqrt{C}(1-b) + \epsilon \frac{(1-b)^2}{a_0(b)} \int_b^{b_0} f(1-\beta) (1-\beta)^2 d\beta, \quad b \to -\infty,
\]

then condition (5) guarantees boundedness of the phase trajectory \( a_\epsilon(b) \). If the integral (5) converges, then for sufficiently small \( \epsilon \) the prevailing term is \( a_0 \), and the trajectory goes to infinity. Moreover, for sufficiently small \( \epsilon \), satisfying condition

\[
-\sqrt{C} + \epsilon \int_{1-b_0>0}^{+\infty} \frac{f(\eta)}{\eta^2} d\eta < 0, \quad (12)
\]

the solution blowup for the same initial data as in the non-perturbed case.

For \( C = 0 \) the analysis is analogous, but the result is different. Namely, \( a_0(b) \sim -(1-b)^\gamma \) as \( b \to -\infty \) and the respective condition for boundedness of trajectory is

\[
\lim_{\eta \to \infty} \eta \int_{\eta_0>0} \frac{f(\tilde{\eta})}{\tilde{\eta}^{5/2}} d\tilde{\eta} = \infty. \quad (13)
\]

Condition (13) is predictively more mild than (5). For example, for \( f(\eta) = \eta^\gamma \), (13) gives \( \gamma > \frac{1}{2} \), whereas (5) gives \( \gamma \geq 1 \). Nevertheless, we have to take into account the worst situation, i.e. \( C > 0 \).
Figure 2 shows the effect of $\epsilon$ on the solution. It can be seen that even for sufficiently large $\epsilon$ the solution first closely mimics the unperturbed case and only after some time sharply changes its behavior (Figure 2, left). In fact, for the threshold value $\gamma = 1$, the difference between the perturbed and unperturbed cases is at first so small that it cannot be detected numerically. Figure 2, right, shows rapidly decaying oscillations for a sufficiently long time.

Thus, since the analytical solution to (2), (3) is unique, if the data belong to the class $\mathbb{A}$, so does the solution. The theorem is proved. □

![Figure 1: $f(n) = n^2$. Left: the direction field to system (6), $\epsilon = 0$. Right: the phase curves starting from the same point for (8) at $\epsilon = 0$, singularity formation (dash) and $\epsilon = 0.8$, smooth solution (solid).]

3. Arbitrary initial data

**Theorem 3.1. (Main theorem)** Let $f(n) \in \mathbb{A}(\mathbb{R}_+)$ be a nonnegative function satisfying conditions

$$\lim_{\eta \to \infty} \frac{\eta f'(\eta)}{f(\eta)} = \text{const} > 1$$

(14)
Figure 2: Solution of (6) for \( f(n) = n^2 \). Left: the behavior of \( b(t) \) at \( \epsilon = 0 \), singularity formation (dash) and \( \epsilon = 1 \), smooth solution (solid). Right: the behavior of \( b(t) \) for \( \epsilon = 1 \) near equilibrium \( b = 0 \).

and (5). If \( \nu(n) = \epsilon f(n) \), \( \epsilon = \text{const} > 0 \), then problem (2), (3) admits a global in time classical (\( C^1 \)-smooth) solution for any choice of the data. Otherwise, one can find the data such that the derivatives of solution blow up in a finite time.

Proof. We denote \( q = V_x \), \( s = E_x \), \( \xi = V_{xx} \), \( \sigma = E_{xx} \) and differentiate (2) with respect to \( x \). Since \( n = 1 - s > 0 \), then it makes sense to consider only the half-plane \( s < 1 \). Along every characteristic line \( x(t) \), starting from point \( x_0 \in \mathbb{R} \) we get system

\[
\begin{align*}
\dot{q} &= -q^2 - s - \epsilon(f(1-s)q + V f'(1-s)\sigma), \\
\dot{s} &= (1-s)q,
\end{align*}
\]

complemented by initial conditions \( q(0) = V_x(x_0), s(0) = E_x(x_0) \). Due to the term \( V f'(1-s)\sigma \) this system is not closed. The dynamics of \( V \) can be found from (2):

\[
\begin{align*}
\dot{V} &= -E - \epsilon f(1-s)V, \\
\dot{E} &= V,
\end{align*}
\]
it implies
\[ \frac{d}{dt}(V^2 + E^2) = -2\epsilon f(1 - s)V^2 \leq 0, \]
therefore \( V \) and \( E \) remain bounded.

However, the equation for \( \sigma \) contains \( \sigma_x \) and the whole system cannot be closed. It is the principal difficulty comparing with the case \( \epsilon = 0 \), treated in [6].

For the solutions [1] this problem does not arise, since \( \sigma = 0 \) for them.

Further, (15), (16) imply
\[ \frac{dq}{ds} = -q^2 + s - s - \frac{\sigma f(1 - s)}{1 - s} V, \tag{19} \]
which coincides with [8], except for the last term. We are going to show that this term is subjected to the previous one as \( s \to -\infty \) and therefore similar to the arguments of Theorem 2.1 along every characteristic \( x = x(t) \) the derivatives of the solution are bounded. Variables \( (s, q) \) correspond to \( (a, b) \) in the proof of Theorem 2.1

Namely, to find the condition for the boundedness of \( q, s \) we consider expansion with respect to the small parameter \( \epsilon \).

First of all, we introduce a new independent variable as \( s = s(t) \). This is possible if \( \dot{s} \neq 0 \), i.e \( q \neq 0 \). The blow-up implies that \( s \) tends to \( -\infty \) as \( t \to t_* < \infty \), for \( q < 0 \).

Let us set \( q(s) = q_0(s) + \epsilon q_1(s) + o(\epsilon) \), \( \sigma(s) = \sigma_0(s) + \epsilon \sigma_1(s) + o(\epsilon) \), \( \xi(s) = \xi_0(s) + \epsilon \xi_1(s) + o(\epsilon) \). Then as in [9] we find
\[ q_0(s) = \pm \sqrt{1 - 2s + \epsilon f(1 - s)}, \quad C = \frac{q_0^2(0) + 2s(0) - 1}{(1 - s(0))^2} \tag{20} \]
and
\[ \frac{dq_1}{ds} = -q_1 Q(s) - \frac{f(1 - s)}{1 - s} - \frac{\sigma_0 f'(1 - s)}{(1 - s)q_0}, \tag{21} \]
where \( Q(s) = \frac{q_0^2 - s}{(1 - s)q_0} = \frac{1 - 3s + C(1 - s)^2}{(1 - 3s + C(1 - s)^2)(1 - s)}. \)
To find $\sigma_0$, we get the system of linear equations

\[
\frac{d\sigma_0}{ds} = \frac{(1-s)\xi_0 - 2\sigma_0 q_0}{(1-s)q_0}, \tag{22}
\]
\[
\frac{d\xi_0}{ds} = -\frac{3q_0\xi_0 + \sigma_0}{(1-s)q_0}. \tag{23}
\]

Further, taking into account (20), from the system of linear equations (22), (23) we have

\[
\sigma_0(s) = (s-1)^2(C_1 s + C_2 q_0(s)), \tag{24}
\]

with constants $C_1, C_2$, depending on $s(0), q_0(0), \sigma_0(0), \xi_0(0)$. Due to (24) and condition (14) the ratio $\frac{\sigma_0 f'(1-s)}{(1-s)q_0}$ has the same behavior as $f(1-s)$ as $s \to -\infty$.

Let us study the behavior of the term $V(s, q_0(s))$ as $s \to -\infty$.

From (15) – (18) we have

\[
\frac{dV}{ds} = -E(s) - \epsilon f(1-s)V(s),
\]
\[
\frac{dE}{ds} = \frac{V(s)}{(1-s)q(s)}.
\]

Since $V(s, q(s)) = V(s, q_0(s)) + O(\epsilon)$, $E(s, q(s)) = E(s, q_0(s)) + O(\epsilon)$, $\epsilon \to 0$, fixed $s$, then for the zero terms $V_0 = V(s, q_0(s))$, $E_0 = E(s, q_0(s))$ we obtain

\[
\frac{dV_0}{ds} = -E_0(s) - \epsilon f(1-s)V_0(s), \tag{25}
\]
\[
\frac{dE_0}{ds} = \frac{V_0(s)}{(1-s)q_0(s)}. \tag{26}
\]

Further, for convenience, we change the independent variable once again as

\[
s_1(s) = -\arctan\frac{s}{\sqrt{1-2s+C(1-s)^2}} + \arctan\frac{1}{\sqrt{C}} \sim \frac{1}{\sqrt{C}(1-s)}, \quad s \to -\infty.
\]

Then $s_1 \to 0+$ and $f(1-s) \sim f((\sqrt{C}s_1)^{-1})$ as $s \to -\infty$. With the new independent variable (27), (28) take the form

\[
\frac{dV_0}{ds_1} = E_0(s_1) + \epsilon f(1-s)V_0(s_1), \tag{27}
\]
\[
\frac{dE_0}{ds_1} = -V_0(s_1). \tag{28}
\]
Let us study the structure of the solution near the point $s_1 = 0$, the hypothetic point of singularity formation.

Further, from (27) and a linear homogeneous equation (28) we have

$$\frac{d^2V_0}{ds_1^2} - \epsilon f(1 - s) \frac{dV_0}{ds_1} + (1 - \epsilon f'(1 - s))V_0 = 0, \quad s = s(s_1), \quad (29)$$

where $s_1 = 0$ is an irregular singular point. To obtain the asymptotics of $V_0(s_1)$ as $s_1 \to 0^+$, we use the standard theory described, for example, in [1], Sec.3.4.

To find the leading terms of the asymptotic expansion for sufficiently small $s_1 > 0$ we first take into account that (14) implies that there exists $\gamma = \text{const}$ such that $f(\eta) \sim f_0 \eta^\gamma, \eta \to \infty, f_0 = \text{const} > 0$, condition (5) implies that $\gamma \geq 1$.

Thus, two linearly independent solutions to (29) behave as

$$Y_1 \sim C s_1^\gamma, \quad Y_2 \sim C \exp\left(\frac{-\epsilon}{(\gamma - 1)s_1^{\gamma - 1}}\right), \quad \gamma > 1, \quad (30)$$

$$Y_1 \sim C s_1, \quad Y_2 \sim C s_1^\gamma, \quad \gamma = 1, \quad (31)$$

$s_1 \to 0^+$. Thus, if $\gamma > 1$, we see from (30) that $(1 - s)V(s, q_0(s)) \sim \frac{V_0(s_1)}{s_1} = o(s_1), s_1 \to 0^+$ or $s \to -\infty$ and the behavior of $q_1(s)$, given by (21), is defined only by term $f(1 - s)$.

If $\gamma = 1$, then basically the last term in (21) is greater than $f(1 - s) \frac{1}{1 - s}$ as $s \to -\infty$, see (31), and tends to plus or minus infinity depending on the sign of $V$. The initial data can be chosen so that this term tends to plus infinity and changes the behavior of the phase trajectory (21) in such a way that it remains in the lower half-plane, and $q$ tends to minus infinity, and $s$ tends to minus infinity.

As for the higher-order terms, $q_i, i = 2, \ldots$, they obey a system of linear equations, similar to (11), therefore for a fixes $s$, the property to come back in the upper half-plane $q > 0$ is defined only by $q_1(s)$ for sufficiently small $\epsilon$.

However at any $\epsilon > 0$ for $s, q \to -\infty$ the last term in (19) is subjected to the previous one, therefore, equation (19) is equivalent at the point of singularity formation to (8), and the phase trajectory for any initial data turns out in the upper half-plane $q > 0$, where $q, s$ cannot go to infinity. Theorem 3.1 is proved. □
Remark 1. We notice that for the non-perturbed case $\epsilon = 0$ the derivatives of solution can go to infinity at $V_\infty = \lim_{s \to -\infty} V(s)$, this value is defined by initial data and it can be any constant. For the case $\epsilon > 0$, the blow-up necessarily happens for $V_\infty = 0$.

Remark 2. Note that for the case of affine solutions, the threshold friction $\nu(n) = n$ guarantees the global smoothness of the solution, but for arbitrary data it is insufficient.

4. Discussion

1. System (2) has the form $U_t + A(U, U_x)U_x = F(U), U = (V, E)$. The matrix $A$ is a Jordan block, it has multiple eigenvalue $V$, but only one eigenvector. The system does not belong to symmetric hyperbolic one, and nonlinear resonance can occur in the solution [8]. The simplest system of this form is the so-called pressureless gas dynamics. It is commonly known that there the component of density develops the delta-singularity. In our case it happens at the points where $E_x$ tends to $-\infty$.

2. There exist different approaches to the well-posedness of weak solutions to the pressureless Euler-Poisson equations [8], [9], [10].

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