Self-Similar Solutions to Curvature Flow of Convex Hypersurfaces

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Abstract

We classify the self-similar solutions to a class of Weingarten curvature flow of connected compact convex hypersurfaces, isometrically immersed into space forms with non-positive curvature, and obtain a new characterization of a sphere in a Euclidean space $\mathbb{R}^{n+1}$.

1 Introduction

It is a fundamental problem to classifying hypersurfaces in a Euclidean space in classical differential geometry. For a compact and connected hypersurface in $\mathbb{R}^{n+1}$, various conditions have been obtained to guarantee that it is a standard Euclidean sphere, and thus various characterizations of spheres have been given.

Let $X : M \to \mathbb{R}^{n+1}$ be a hypersurface immersed in a Euclidean space $\mathbb{R}^{n+1}$. Denote by $A$ and $H$ the Weingarten transformation and mean curvature of $M$, respectively. Assume $v$ is the unit normal vector field, then the support function of the hypersurface $M$ is defined by

$$\mathcal{Z} = \langle X, v \rangle.$$  

It is known that $M$ is a Euclidean sphere if and only if its support function $\mathcal{Z}$ is constant and its Weingarten transformation $A$ is not degenerate. When $M$ is oriented, Liebmann-Süss’ theorem implies that, it is a Euclidean sphere if and only if it has constant mean curvature and its support function $\mathcal{Z}$ does not change sign. If $M$ is closed and strictly convex, the constant

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mean curvature can guarantee that it is a standard sphere. For an embedded closed hypersurface, an interesting result of Ros [15] shows that $M$ is a standard sphere if its scalar curvature is constant. For hypersurfaces of constant Weingarten curvatures (see below for definitions) immersed into space forms, Ecker-Huisken [9] completely classify such hypersurfaces with non-negative sectional curvatures.

In 1990, Huisken [11] obtained a different characterization of standard spheres in Euclidean spaces by studying self-similar solutions to mean curvature flow of hypersurfaces. The mean curvature flow is a family of evolving hypersurfaces in normal vectors, such that the evolution speed is the mean curvature. More precisely, for a smooth oriented $n$-dimensional manifold $M$ and $X : M \rightarrow \mathbb{R}^{n+1}$ the smooth hypersurface immersed in a Euclidean space $\mathbb{R}^{n+1}$, the mean curvature flow is the following evolving problem (cf. [10])

$$\frac{\partial}{\partial t}X(x,t) = H(x,t)v(x,t), \quad x \in M,$$

(1.1)

satisfying the initial condition $X(x,0) = X_0(x) = X(x)$, $x \in M$, where $H(x,t)$ is the mean curvature and $v(x,t)$ the inward unit normal vector of $M_t = X_t(M)$ at $X(x,t)$.

It is known that, (1.1) is a contracting curvature flow, and when the initial hypersurface is convex, Huisken [10] proved that the solutions exist in a maximal finite time interval and converge to a standard sphere by scaling. Later, Andrews [1] extended this phenomenon to a class of general curvature flow, i.e.

$$\frac{\partial}{\partial t}X(x,t) = F(A(x,t))v(x,t), \quad x \in M,$$

(1.2)

where $F$ is a curvature function (i.e. positive and elliptic) of homogeneous degree one of the evolving hypersurfaces satisfying standard conditions, and $A(x,t)$ the Weingarten form of the corresponding evolving hypersurfaces $M_t$.

If we only assume that the initial hypersurface $X_0$ has non-negative mean curvature, the Type (I) solution to the evolving problem (1.1) is asymptotically self-similar, i.e. the limit hypersurface $X$ of the scaling solutions satisfies the following equation (cf. [11])

$$H_+ < X, v > = 0.$$  

(1.3)

This is a fully nonlinear elliptic equation which relates the support function $\mathcal{Z}$ and the mean curvature $H$ of the hypersurface $X$. Huisken [11] completely classified such self-similar solutions to (1.3). When $M$ is compact, the only possible case is a standard sphere, i.e. Huisken gave a new characterization of Euclidean spheres:

**Proposition 1.1** A compact and connected hypersurface with non-negative mean curvature immersed in a Euclidean space is a standard sphere if and only if (1.3) holds.

Let $f(\lambda)$ be a function defined on a symmetric region in $\mathbb{R}^n$. It is easy to see that $f$ induces a function $F(A) = f(\lambda(A))$ defined in the set of symmetric matrices with eigenvalues $\lambda$. When $f$ is evaluated at the vector $\lambda(x) = \{\lambda_1(x), \cdots, \lambda_n(x)\}$, the components of which are the principal curvatures of $M$, the hypersurface $M$ with curvature $F$ is the so-called Weingarten hypersurface.
If \( X : M \rightarrow \mathbb{R}^{n+1} \) is a sphere immersed in \( \mathbb{R}^{n+1} \), then there exists a constant \( \tau \) such that
\[
F + \tau < X, v > = 0, \tag{1.4}
\]
is trivially satisfied for any symmetric and homogeneous curvature function \( F \) defined in the space of positive transformations, for, in this case the principal curvatures are all equal. We also call hypersurfaces satisfying the condition (1.4) the self-similar solutions of the convex curvature flow (1.2).

Huisken’s theorem \cite{11} says that the inverse is true for \( F = H \). In this paper, we will consider closed immersed hypersurfaces in a space form \( \mathbb{N}^{n+1}(c) \) with non-positive curvature \( c \), and show that the inverse of (1.4) is true for a large class of Weingarten hypersurface with Weingarten curvature \( F \) satisfying some given conditions.

For this purpose, we first introduce the following functions for any real number \( c \) (cf. \cite{6})
\[
sh_c(t) = \begin{cases} 
\frac{\sin(\sqrt{c}t)}{\sqrt{c}} & \text{if } c > 0 \\
t & \text{if } c = 0 \\
\frac{\sinh(\sqrt{-c}t)}{\sqrt{-c}} & \text{if } c < 0
\end{cases}, \quad \text{and} \quad ch_c(t) = \begin{cases} 
\cos(\sqrt{c}t) & \text{if } c > 0 \\
1 & \text{if } c = 0 \\
\cosh(\sqrt{-c}t) & \text{if } c < 0
\end{cases}.
\]

Given any fixed point in the ambient space \( \mathbb{N}^{n+1}(c) \), we shall denote by \( \rho \) the distance function to the fixed point in \( \mathbb{N}^{n+1}(c) \), and denote by \( \partial \rho \) the gradient of \( \rho \) in \( \mathbb{N}^{n+1}(c) \). For a hypersurface \( X : M \rightarrow \mathbb{N}^{n+1}(c) \), let
\[
\mathcal{Z} = sh_c(\rho) < \partial \rho, v >, \tag{1.5}
\]
where \( <\cdot, \cdot> \) is the metric of the ambient space, and \( v \) is again the inward unit normal vector of \( M \). It is easy to see that when \( c = 0 \), \( sh_c(\rho)\partial \rho = \rho \partial \rho \) is the position vector, and therefore \( \mathcal{Z} \) in (1.5) coincides with the support function of the hypersurface.

Denote by \( \Gamma_+ \), the positive cone of \( \mathbb{R^n} \), and \( \Gamma(F) \) a component of \( \{ \lambda : f(\lambda) \neq 0 \} \) containing \( \Gamma_+ \).

**Theorem 1.2** Let \( F(A) = f(\lambda(A)) \) be a smooth symmetric function of homogeneous degree \( m \in \mathbb{R}/\{0\} \), and \( \mathbb{N}^{n+1}(c) \) a Riemannian manifold of non-positive constant curvature \( c \). Suppose \( X : M \rightarrow \mathbb{N}^{n+1}(c) \) is a smooth connected compact convex hypersurface with principal curvatures \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_+ \). Assume the following conditions are satisfied:

1. On \( \Gamma(F) \), \( F \) is elliptic, i.e. \( \partial f / \partial \lambda_i > 0, \forall i = 1, 2 \cdots, n \).
2. One of the following holds: (i) \( m \geq 1 \) and \( f \) is convex or concave; (ii) \( m < 0 \) and \( f \) is convex or concave; (iii) \( n = 2 \) and either \( m = 1 \), or \( -7 \leq m < 0 \), or \( m > 1 \) and \( r_{\max} \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{8}{m-1}} \right) \), where \( r = \frac{\lambda_1}{\lambda_i} \geq 1 \) is the pinching ratio of the principal curvatures, or \( m < -7 \) and \( r_{\max} \leq 2/ \left( 1 + \sqrt{1 - \frac{8}{1-m}} \right) \).

Then, if
\[
F + \tau \mathcal{Z} = 0, \tag{1.6}
\]
holds for a nonzero constant \( \tau \) depending only on \( n \), \( X(M) \) is an umbilical sphere.
We remark that spheres are stable solutions to contracting as well as expanding curvature flows of convex hypersurfaces in Euclidean spaces, and therefore for $m = 1$ or $n = 2$, Theorem 1.2 in fact follows from [10, 1, 2, 5].

**Corollary 1.3** A connected compact convex hypersurface immersed in a Euclidean space is a standard Euclidean sphere if and only if (1.6) holds for some curvature function given in Theorem 1.2.

For self-similar solutions of mean curvature flow on arbitrary codimension, Smoczyk [16] classifies such self-shrinkers with parallel principal normal vector field. In the case of isotropic curve flow, Andrews [3] completely classified the homothetically shrinking solutions of (1.2), even the curvature function is not homogeneous of degree one. For the behavior of embedded expanding convex solutions to (1.2), there are also complete descriptions, see [2, 8, 17], and so on.

## 2 Preliminaries

Let $N^{n+1}(c)$ be an $(n+1)$-dimensional space form of constant curvature $c$, and $M$ a smooth hypersurface immersion in $N$. We will use the same notation as in [10, 1, 14]. In particular, $\nabla$ is the induced connection on $M$, and for a local coordinate system $\{x^1, \cdots, x^n\}$ of $M$, $g = g_{ij}$ and $A = h_{ij}$ denote respectively the metric and second fundamental form of $M$. Let $g^{ij}$ denote the $(i,j)$-entry of the inverse of the matrix $(g_{ij})$. Then $\{h^i_j\}$ where $h^i_j = h_{ik}g^{kj}$ is the Weingarten map. The mean curvature and the squared norm of the second fundamental form of $M$ are given by

$$H = g^{ij}h_{ij} = h^i_i, \quad |A|^2 = g^{ij}g^{kl}h_{ik}h_{jl}.$$ 

In the sequel we will use $\lambda_i$ to denote the $i$-th principal curvature of the hypersurface. Throughout this paper we sum over repeated indices from 1 to $n$ unless otherwise indicated. Raised indices indicate contraction with the metric.

Given a symmetric smooth function $f(\lambda)$ defined in the symmetric region of $\mathbb{R}^n$, the induced function $F(A) = f(\lambda(A))$ defined in the set of symmetric matrices with eigenvalues $\lambda$ is as smooth as $f$ and symmetric of homogeneous degree $m$, if $f$ is so. We denote by $(\hat{F}^{ij})$ the matrix of the first partial derivatives of $F$ with respect to the components of its arguments:

$$\frac{\partial}{\partial s}F(A + sB)\big|_{s=0} = \hat{F}^{ij}(A)B_{ij},$$

where $A$ and $B$ are any symmetric matrices. Similarly for the second partial derivatives of $F$, we write

$$\frac{\partial^2}{\partial s^2}F(A + sB)\big|_{s=0} = \hat{F}^{ij,kl}(A)B_{ij}B_{kl}.$$ 

We also use the notation

$$\hat{f}_i(\lambda) = \frac{\partial f}{\partial \lambda_i}(\lambda), \quad \text{and} \quad \hat{f}_{ij}(\lambda) = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}(\lambda).$$
Recall that the homogeneity of $F = F(h_{ij})$ implies the following

$$
\dot{F}^{ij} h_{ij} = mF \quad \text{and} \quad \ddot{F}^{ij, rs} h_{ij} h_{rs} = (m - 1) \dot{F}^{rs} h_{rs}.
$$

(2.1)

The following proposition is well known (see e.g. [4, 12])

**Proposition 2.1** Let $f$ and $F$ be as above. If $f$ is $C^2$ and symmetric, then at any diagonal matrix $A$ with distinct eigenvalues, the second-order derivative of $F$ in direction $B$ is given by

$$
\ddot{F}(B, B) = \sum_{k, l} \dot{f}_{kl} B_{kk} B_{ll} + 2 \sum_{k < l} \frac{\dot{f}_k - \dot{f}_l}{\lambda_k - \lambda_l} B_{kl}^2.
$$

The following corollary follows immediately,

**Corollary 2.2** If $f$ is convex (concave) at $\lambda(A)$, then $F$ is convex (concave) at $A$. Moreover $f$ is convex (concave) if and only if

$$
\frac{\dot{f}_i - \dot{f}_j}{\lambda_i - \lambda_j} \geq (\leq) 0 \quad \text{for all } i \neq j.
$$

Let $\mathcal{F}(A) = \hat{f}(\lambda(A))$ be another homogeneous function defined in $\mathbb{R}^n$. The first part of the next lemma is in fact in [13], where only the symmetric and homogeneous degree one function is considered.

**Lemma 2.3** Assume $F$ and $\mathcal{F}$ are elliptic and of homogeneous degree $m$, and the eigenvalues $\lambda(A)$ of $A$ are non-negative. If $f$ is convex (concave), and $\hat{f}$ concave (convex), then

$$
m(\mathcal{F} \dot{F}^{ij} h_{ik} h_{j}^k - F \dot{\mathcal{F}}^{ij} h_{ik} h_{j}^k) \geq (\leq) 0,
$$

and

$$
m \sum_j (F \dot{f}_j - \mathcal{F} \dot{f}_j) \geq (\leq) 0.
$$

**Proof.** For the first inequality, using the homogeneity of $F$ and $\mathcal{F}$, we compute as in [13]

$$
m(\mathcal{F} \dot{F}^{ij} h_{ik} h_{j}^k - F \dot{\mathcal{F}}^{ij} h_{ik} h_{j}^k) = \sum_{i,j} (\hat{f}_{ij} \hat{f}_i \lambda_i \lambda_j^2 - \hat{f}_{ij} \hat{f}_j \lambda_i \lambda_j^2)
$$

$$
= \frac{1}{2} \sum_{i \neq j} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2 \left( \dot{f}_j \left( \frac{\dot{f}_i - \dot{f}_j}{\lambda_i - \lambda_j} \right) - \dot{f}_i \left( \frac{\dot{f}_i - \dot{f}_j}{\lambda_i - \lambda_j} \right) \right).
$$

The lemma now follows by using Corollary 2.2. For the second inequality, we similarly have

$$
m \sum_j (F \dot{f}_j - \mathcal{F} \dot{f}_j) = \frac{1}{2} \sum_{i,j} (\hat{f}_j (\lambda_i - \lambda_j)(\dot{f}_i - \dot{f}_j) - \dot{f}_i (\lambda_i - \lambda_j)(\dot{f}_i - \dot{f}_j)),$$

the required inequality follows. □
Since $N$ is of constant curvature $c$, we have the Codazzi equation

$$\nabla_k h_{ij} = \nabla_j h_{ik}.$$  

The Codazzi’s equation implies the Ricci identity

$$\nabla_i \nabla_j h_{kl} = \nabla_k \nabla_l h_{ij} + h_{il} A^2_{kj} - h_{kl} A^2_{ij} - c(g_{il} h_{kj} - g_{kj} h_{il}),$$  

(2.2)

where $A^s_{ij} = h^l h_i h_k \cdots h_{kj} (s \text{ factors}).$

### 3 Computations on Curvature Functions

Let $\{\frac{\partial}{\partial x^i}\}$ be the natural frame field on $M^n$. Denote by $\tilde{\nabla}$ the covariant derivative of $N^{n+1}(c)$.

The following is well-known

$$<\tilde{\nabla}_X \partial_\rho, Y> = \tilde{\nabla}^2 \rho(X, Y) = \begin{cases} 0 & \text{if } X = \partial_\rho \\ \frac{ch_c(\rho)}{sh_c(\rho)} < X, Y > & \text{if } < X, \partial_\rho >= 0 \end{cases}.$$  

(3.1)

By (3.1) and Codazzi equation, we have the following lemma [7]

**Lemma 3.1** The second order derivative of $\mathcal{Z}$ is given by

$$\nabla_i \nabla_j \mathcal{Z} = -ch_c(\rho) h_{ij} - sh_c(\rho) < \partial_\rho^T, \nabla h_{ij} > - \mathcal{Z} A^2_{ij},$$

where $\partial_\rho^T$ is the component of $\partial_\rho$ tangent to $M$.

We differentiate the equation (1.6) to get

$$\nabla_j F = F^{kl} \nabla_j h_{kl} = -\tau \nabla_j \mathcal{Z}.$$  

Taking derivative of the above equation again in a normal coordinate system with the help of Lemma 3.1, we have

$$\dot{F}^{kl,rs} \nabla_i h_{rs} \nabla_j h_{kl} + \dot{F}^{kl} \nabla_i \nabla_j h_{kl} = \nabla_i \nabla_j F = -\tau \nabla_i \nabla_j \mathcal{Z}$$

$$= -\tau \left( -ch_c(\rho) h_{ij} - sh_c(\rho) < \partial_\rho^T, \nabla h_{ij} > - \mathcal{Z} A^2_{ij} \right)$$

$$= \tau ch_c(\rho) h_{ij} - FA^2_{ij} + \tau sh_c(\rho) < \partial_\rho, \frac{\partial}{\partial x^l} > \nabla^l h_{ij},$$

which implies

$$\dot{F}^{kl} \nabla_i \nabla_j h_{kl} = \tau ch_c(\rho) h_{ij} - FA^2_{ij} + \tau sh_c(\rho) < \partial_\rho, \frac{\partial}{\partial x^l} > \nabla^l h_{ij} - \dot{F}^{kl,rs} \nabla_i h_{rs} \nabla_j h_{kl}. \quad (3.2)$$
Using the Euler relation (2.1), we have by (3.2)
\[
\dot{F}^{kl}\nabla_k\nabla_l F = \dot{F}^{kl}\nabla_k(\dot{F}^{ij}\nabla_i h_{ij})
\]
\[
= \dot{F}^{kl}\dot{F}^{ij,rs}\nabla_i h_{ij} \nabla_r h_{rs} + \dot{F}^{ij}\dot{F}^{kl}\nabla_k \nabla_l h_{ij}
\]
\[
= \dot{F}^{kl}\dot{F}^{ij,rs}\nabla_i h_{ij} \nabla_r h_{rs} + \dot{F}^{kl}(\tau ch_c(\rho) h_{kl} - FA_{kl}^2
\]
\[
+ \tau sh_c(\rho) < \partial_{\rho}, \frac{\partial}{\partial x^i} > \nabla^i h_{kl} - \dot{F}^{ij,rs}\nabla_k h_{rs} \nabla_l h_{ij}
\]
\[
= m \tau ch_c(\rho) F + \tau sh_c(\rho) < \partial_{\rho}, \frac{\partial}{\partial x^i} > \nabla^i F - F \dot{F}^{kl} A_{kl}^2. \quad (3.3)
\]
Here \( m \) is the degree of \( F \).

For any other curvature function \( \mathcal{F} \) of homogeneous degree \( m \), we compute similarly
\[
\nabla_k \nabla_l \mathcal{F} = \dot{\mathcal{F}}^{ij,rs} \nabla_k h_{rs} \nabla_l h_{ij} + \dot{\mathcal{F}}^{ij} \nabla_k \nabla_l h_{ij}.
\]

Using the Ricci identity (2.2) and inserting (3.2) into the above equation, we obtain
\[
\dot{F}^{kl}\nabla_k\nabla_l \mathcal{F} = \dot{F}^{kl}\dot{\mathcal{F}}^{ij,rs}\nabla_k h_{rs} \nabla_l h_{ij} + \dot{\mathcal{F}}^{ij} \dot{\mathcal{F}}^{kl}\nabla_k \nabla_l h_{ij}
\]
\[
+ \dot{\mathcal{F}}^{ij} \left[ \dot{\mathcal{F}}^{kl} \nabla_i h_{kl} - \dot{F}^{kl}(h_{il}A_{kl}^2 - h_{kl}A_{il}^2 + h_{ij}A_{kl}^2 - h_{kl}A_{il}^2) \right]
\]
\[
+ \dot{\mathcal{F}}^{ij} \left[ \tau ch_c(\rho) h_{ij} - FA_{ij}^2 + \tau sh_c(\rho) < \partial_{\rho}, \frac{\partial}{\partial x^i} > \nabla^i h_{ij} - \dot{F}^{kl,rs}\nabla_k h_{rs} \nabla_l h_{kl} \right]
\]
\[
= m \tau ch_c(\rho) \mathcal{F} + \tau sh_c(\rho) < \partial_{\rho}, \frac{\partial}{\partial x^i} > \nabla^i \mathcal{F} + (m - 1)F \dot{\mathcal{F}}^{ij} A_{ij}^2 - m \dot{\mathcal{F}}^{ij} A_{ij}^2
\]
\[
+ (\dot{\mathcal{F}}^{ij} \dot{\mathcal{F}}^{kl,rs} - \dot{\mathcal{F}}^{ij} \dot{\mathcal{F}}^{kl,rs}) \nabla_i h_{rs} \nabla_j h_{kl} - cm(\mathcal{F} \dot{F}^{kk} - F \dot{\mathcal{F}}^{kk}). \quad (3.4)
\]

Direct computation gives
\[
\dot{F}^{kl}\nabla_k\nabla_l \left( \frac{F}{\mathcal{F}} \right) = \frac{1}{\mathcal{F}} \dot{F}^{kl}\nabla_k F - \frac{F}{\mathcal{F}^2} \dot{F}^{kl}\nabla_k \nabla_l \mathcal{F}
\]
\[
- \frac{2}{\mathcal{F}^2} \dot{F}^{kl}\dot{F}^{ij,rs}\nabla_i h_{ij} \nabla_r h_{rs} + \frac{2F}{\mathcal{F}^3} \dot{F}^{kl}\nabla_k \mathcal{F} \nabla_l \mathcal{F},
\]

which implies by (3.3) and (3.4)
\[
\dot{F}^{kl}\nabla_k\nabla_l \left( \frac{F}{\mathcal{F}} \right) = \frac{1}{\mathcal{F}} \left( m \tau ch_c(\rho) F + \tau sh_c(\rho) < \partial_{\rho}, \frac{\partial}{\partial x^i} > \nabla^i F - F \dot{F}^{kl} A_{kl}^2 \right)
\]
\[
- \frac{F}{\mathcal{F}^2} \left[ m \tau ch_c(\rho) \mathcal{F} + \tau sh_c(\rho) < \partial_{\rho}, \frac{\partial}{\partial x^i} > \nabla^i \mathcal{F}
\]
\[
+ (m - 1)F \dot{\mathcal{F}}^{ij} A_{ij}^2 - m \dot{\mathcal{F}}^{ij} A_{ij}^2
\]
\[
+ (\dot{\mathcal{F}}^{ij} \dot{\mathcal{F}}^{kl,rs} - \dot{\mathcal{F}}^{ij} \dot{\mathcal{F}}^{kl,rs}) \nabla_i h_{rs} \nabla_j h_{kl} - cm(\mathcal{F} \dot{F}^{kk} - F \dot{\mathcal{F}}^{kk}) \right]
\]
\[
- \frac{2}{\mathcal{F}^2} \dot{F}^{kl}\dot{F}^{ij,rs}\nabla_i h_{ij} \nabla_r h_{rs} + \frac{2F}{\mathcal{F}^3} \dot{F}^{kl}\nabla_k \mathcal{F} \nabla_l \mathcal{F}. \quad (3.5)
\]
Note that the last two terms in (3.5) are equal to $-\frac{2}{F} F^{kl} \nabla_k \nabla_l \left( \frac{F}{F} \right)$, and so we at last arrive at the lemma

**Lemma 3.2** Let $F$ and $\mathcal{F}$ be two nonzero curvature functions on $M$, which are homogeneous of degree $m$. If $F$ satisfies (1.6), then the following holds

\[ \dot{F}^{kl} \nabla_k \nabla_l \left( \frac{F}{\mathcal{F}} \right) = \tau \mathrm{sh}(\rho) < \frac{\partial}{\partial x} > \nabla^l \left( \frac{F}{\mathcal{F}} \right) - \frac{2}{\mathcal{F}} \dot{F}^{kl} \nabla_k \nabla_l \left( \frac{F}{\mathcal{F}} \right) \]

\[ + \frac{F}{\mathcal{F}^2} (\mathcal{F}^{ij} \dot{F}^{kl,rs} - \dot{F}^{ij} \dot{\mathcal{F}}^{kl,rs}) \nabla_i h_{rs} \nabla_j h_{kl} \]

\[ + (m-1) \frac{F}{\mathcal{F}^2} (\mathcal{F}^{ij} A^2_{ij} - F \dot{\mathcal{F}}^{ij} A^2_{ij}) \quad (3.6) \]

\[ - \frac{cmF}{\mathcal{F}^2} (F \dot{\mathcal{F}}^{kk} - \mathcal{F} F^{kk}). \quad (3.7) \]

\[ \frac{cmF}{\mathcal{F}^2} (F \dot{\mathcal{F}}^{kk} - \mathcal{F} F^{kk}). \quad (3.8) \]

## 4 Proof of the Main Theorem

Firstly we consider the case $m \geq 1$. It is clear, in this case, $F$ is positive restricting to $M$. We only prove the theorem for $f$ concave. It is similar for $f$ convex. Taking an elliptic convex curvature function $\mathcal{F}(A) = f(\lambda(A))$ of homogeneous degree $m$, the homogeneity implies that $\mathcal{F}$ is also positive as $F$. By Corollary 2.2, Lemma 2.3 and the homogeneity of $F$ and $\mathcal{F}$, we see that (3.6), (3.7) and (3.8) are non-positive since $c \leq 0$. Then applying the strong maximum principle to $\frac{F}{\mathcal{F}}$ in Lemma 3.2 yields $\frac{F}{\mathcal{F}} = c_1$, a positive constant, on $M$. Therefore by assumption, either $F$ and $\mathcal{F}$ are constant restricting to $M$ or

\[ 0 \geq \dot{F}^{ij,kl} \eta_{ij} \eta_{kl} = c_1 \mathcal{F}^{ij,kl} \eta_{ij} \eta_{kl} \geq 0, \]

for any real symmetric matrix $\eta$. Especially,

\[ \dot{F}^{ij} \dot{F}^{kl,rs} \nabla_i h_{rs} \nabla_j h_{kl} = 0, \]

and

\[ \dot{F}^{kl,rs} h_{rs} h_{kl} = 0. \quad (4.1) \]

If $\nabla_i h_{kl} = 0$ for any $i, j, k = 1, 2, \cdots, n$, then the mean curvature $H$ is constant and we have done. Otherwise by (2.1), (4.1) implies $m(m-1)F = 0$, and so $m = 1$. Since $H$ is concave as well as convex, taking $\mathcal{F} = H$, we have $F = c_1 H$. The theorem now follows from Proposition 4.1 below.

Secondly, we consider the case $m < 0$. Again, the homogeneity and ellipticity of $F$ imply that $F < 0$ since $m < 0$. As in the first case, we only consider the case for $f$ concave, and it is similar for $f$ convex. As before we take an elliptic and convex curvature function $\mathcal{F}$ which is homogeneous of degree $m$. Then terms in (3.6)-(3.8) are non-negative. Applying again the strong maximum principle to $\frac{F}{\mathcal{F}}$ in Lemma 3.2 yields $\frac{F}{\mathcal{F}} = c_2$, a positive constant.
As in [5] again, we can work at a maximum point of the principal curvatures, which is homogeneous of degree zero, we compute as in (3.4) to obtain

\[
\dot{F}^{kl} \nabla_k \nabla_l \mathcal{F} = \dot{F}^{kl} \dot{\mathcal{F}}_{ij,rs} \nabla_k h_{rs} \nabla_l h_{ij} + \dot{\mathcal{F}}^{ij} \dot{F}^{kl} \nabla_k h_{ij} = \nabla^l \mathcal{F} + (m-1) F \dot{\mathcal{F}}^{ij} A_{ij}^2 + (\dot{F}^{ij} \dot{\mathcal{F}}^{kl,rs} - \dot{\mathcal{F}}^{ij} \dot{F}^{kl,rs}) \nabla_i h_{rs} \nabla_j h_{kl} + cmF \dot{\mathcal{F}}^{kk}. \tag{4.2}
\]

We now compute the second-order derivatives in terms of Proposition 2.1. Since \( n = 2 \), it’s not difficult to check as in [5] that, the terms in (4.2) containing the second-order derivatives of \( F \) and \( \mathcal{F} \) are given by in a frame diagonalizing the second fundamental form

\[
Q = (\dot{F}^{ij} \dot{\mathcal{F}}^{kl,rs} - \dot{\mathcal{F}}^{ij} \dot{F}^{kl,rs}) \nabla_i h_{rs} \nabla_j h_{kl}
\]

\[
= (\dot{f}_1 \dot{f}_{11} - \dot{\mathcal{F}}_{11}) (\nabla_1 h_{11})^2 + (\dot{f}_1 \dot{f}_{22} - \dot{\mathcal{F}}_{22}) (\nabla_1 h_{22})^2 + (\dot{f}_2 \dot{f}_{11} - \dot{\mathcal{F}}_{11}) (\nabla_2 h_{11})^2 + (\dot{f}_2 \dot{f}_{22} - \dot{\mathcal{F}}_{22}) (\nabla_2 h_{22})^2 + 2(\dot{f}_1 \dot{f}_{12} - \dot{\mathcal{F}}_{12}) (\nabla_1 h_{11} \nabla_1 h_{22} + 2(\dot{f}_2 \dot{f}_{12} - \dot{\mathcal{F}}_{12}) (\nabla_2 h_{11} \nabla_2 h_{22}) + 2 \frac{\dot{f}_1 \dot{f}_2 - \dot{\mathcal{F}}_{12}}{\lambda_2 - \lambda_1} (\nabla_1 h_{12})^2 + 2 \frac{\dot{f}_1 \dot{f}_2 - \dot{\mathcal{F}}_{12}}{\lambda_2 - \lambda_1} (\nabla_2 h_{12})^2.
\]

As in [5] again, we can work at a maximum point of \( \mathcal{F} \). Then using the gradient conditions and the homogeneity of \( F \), we have

\[
Q = -mF \dot{\mathcal{F}}_1 \left( \frac{m-1}{\lambda_2} + \frac{2}{\lambda_1 (\lambda_2 - \lambda_1)} \right) (\nabla_1 h_{22})^2
\]

\[
- mF \dot{\mathcal{F}}_2 \left( \frac{m-1}{\lambda_1} - \frac{2}{\lambda_2 (\lambda_2 - \lambda_1)} \right) (\nabla_2 h_{11})^2. \tag{4.3}
\]

By Euler identity we also compute

\[
F \dot{\mathcal{F}}^{ij} A_{ij}^2 = F \dot{\mathcal{F}}_2 (\lambda_2 - \lambda_1).
\]

Similarly for the last term in (4.2)

\[
cmF \dot{\mathcal{F}}^{kk} = -cmF \dot{\mathcal{F}}_2 \frac{\lambda_2 - \lambda_1}{\lambda_1}.
\]
Putting these formulae into (4.2), we have at a maximum point of $\mathcal{F}$

$$F^{kl}\nabla_k\nabla_l\mathcal{F} = \tau sh_\nu(p) < \partial_{\rho}, \frac{\partial}{\partial x} > \nabla^\nu\mathcal{F}$$

$$+(m - 1)F\dot{\lambda_1}\dot{\lambda_1} - cmF\dot{\lambda_2}\dot{\lambda_2} = \frac{\lambda_2 - \lambda_1}{\lambda_1}$$

$$-mF\dot{\lambda_1} \left( \frac{m - 1}{\lambda_1^2} + \frac{2}{\lambda_1(\lambda_2 - \lambda_1)} \right)(\nabla_1 h_{22})^2$$

$$-mF\dot{\lambda_2} \left( \frac{m - 1}{\lambda_2^2} - \frac{2}{\lambda_2(\lambda_2 - \lambda_1)} \right)(\nabla_2 h_{11})^2.$$  

(4.4)

(4.5)

(4.6)

Now take $\mathcal{F} = \frac{2|A|^2 - H^2}{H^2} = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}$, we have $\dot{\lambda_1} = \frac{4\lambda_2(\lambda_1 - \lambda_2)}{(\lambda_1 + \lambda_2)^3}$, and $\dot{\lambda_2} = -\frac{4\lambda_1(\lambda_1 - \lambda_2)}{(\lambda_1 + \lambda_2)^3}$. We assume $\lambda_1 \leq \lambda_2$ at the maximum point of $\mathcal{F}$. Then when $m \geq 1$ and $c \leq 0$, (4.4) is non-negative. If $m = 1$, (4.5) and (4.6) are all non-negative, we have immediately by strong maximum principle $\mathcal{F}$ is a constant, and therefore $X(M)$ is an umbilical sphere.

If $m > 1$, in order to apply the maximum principle, we require $\frac{m - 1}{\lambda_1^2} + \frac{2}{\lambda_1(\lambda_2 - \lambda_1)}$ is non-negative, and $\frac{m - 1}{\lambda_2^2} - \frac{2}{\lambda_2(\lambda_2 - \lambda_1)}$ is non-positive. Thus the pinching ratio $r = \frac{\lambda_2}{\lambda_1}$ must satisfy the conditions

$$2r^2 + (m - 1)r - (m - 1) > 0,$$  

(4.7)

and

$$(m - 1)r^2 - (m - 1)r - 2 \leq 0.$$  

(4.8)

The first is always true since $r \geq 1$, and the second is true if and only if

$$r \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{8}{m - 1}} \right).$$

Then when $r$ satisfies the above inequality, by maximum principle, $\mathcal{F}$ is a constant and therefore $X(M)$ is an umbilical sphere.

If $m < 0$, since (4.5) and (4.6) are non-negative, we also require (4.7) and (4.8) hold since $F < 0$. It is easy to check that when $-7 \leq m < 0$, (4.7) and (4.8) are always satisfied. For $m < -7$, (4.8) is always true, and (4.7) is true if and only if

$$r \leq \frac{1}{4} \left( (1 - m) - \sqrt{(1 - m)^2 - 8(1 - m)} \right) = \frac{2}{1 + \sqrt{1 - \frac{8}{1 - m}}}.$$  

Therefore when $-7 \leq m < 0$ or when $m < -7$ and $r \leq 2/\left( 1 + \sqrt{1 - \frac{8}{1 - m}} \right)$, the maximum principle implies that $X(M)$ is an umbilical sphere.  

$\square$

When $c = 0$, the following Proposition 4.1 is essentially a result of Huisken [11], for it differs from his result only by a constant $\tau$. For completeness we give the proof.
Proposition 4.1 If \( X : M^n \to N^{n+1} (c \leq 0) \) is compact, connected, with non negative mean curvature and satisfies \( H + \tau \mathcal{Z} = 0 \) for some positive constant \( \tau \) depending only on \( n \), then \( X(M) \) is an umbilical sphere.

Proof. By (3.2) with \( F = H \), we have

\[
\triangle H = \tau ch_c(\rho)H - H|A|^2 + \tau sh_c(\rho) < \partial^i, \frac{\partial}{\partial x^i} > \nabla^i H,
\]  

which implies that \( H > 0 \) by strong maximum principle, and from Ricci identity (2.2)

\[
\triangle |A|^2 = 2|\nabla A|^2 + 2\tau ch_c(\rho)|A|^2 - 2|A|^4 + \tau sh_c(\rho) < \partial^i, \frac{\partial}{\partial x^i} > \nabla^i |A|^2 - 2c(n|A|^2 - H^2).
\]  

Using (4.9) and (4.10), by similar calculation as in section 3, we have

\[
\triangle \left( \frac{|A|^2}{H^2} \right) = \tau sh_c(\rho) < \partial^i, \frac{\partial}{\partial x^i} > \nabla^i \left( \frac{|A|^2}{H^2} \right) + \frac{2}{H^4} |h_{ij} \nabla_i H - \nabla_i h_{ij} H|^2
\]

\[- \frac{1}{H^2} < \nabla H^2, \nabla \left( \frac{|A|^2}{H^2} \right) > - 2c \frac{n}{H^2}(n|A|^2 - H^2). \]  

(4.11)

Since \( M \) is compact, the strong maximum principle implies that

\[
\frac{|A|^2}{H^2} = \text{constant} \quad \text{and} \quad |H \nabla_i h_{ij} - \nabla_i H h_{ij}|^2 \equiv 0. \]  

(4.12)

Then if \( c = 0 \), Huiskens’ theorem implies that \( X(M) \) is a sphere. If \( c < 0 \), we have by (4.11) and (4.12), \( n|A|^2 - H^2 = 0 \). It follows immediately that \( X(M) \) is an umbilical sphere.

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