Bianchi groups are conjugacy separable

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Abstract

We prove that non-uniform arithmetic lattices of $SL_2(\mathbb{C})$ and consequently the Bianchi groups are conjugacy separable. The proof is based on recent deep results of Agol, Long, Reid and Minasyan. The conjugacy separability of groups commensurable with Limit groups is also established.

1 Introduction

The Bianchi groups are defined as $PSL_2(O_d)$, where $O_d$ denotes the ring of integers of the field $\mathbb{Q}(\sqrt{-d})$ for each square-free positive integer $d$. These groups are classical objects investigated first time in 1872 by Luigi Bianchi. The Bianchi groups have long been of interest, not only because of their intrinsic interest as abstract groups, but also because they arise naturally in number theory and geometry. They are discrete subgroups of $PSL_2(\mathbb{C}) \cong Isom^+(\mathbb{H}_3)$, and the quotient $\mathbb{H}_3$ modulo $PSL_2(O_d)$ is a finite volume hyperbolic 3-orbifold. We refer to [EGM-98] and [F-89] for further information about Bianchi groups.

The recent big advance in the study of Bianchi groups was the proof that they are subgroup separable (see Theorem 3.4 in [LR-08]). Another important residual property is conjugacy separability on which we concentrate in this paper.

A group $G$ is conjugacy separable if whenever $x$ and $y$ are non-conjugate elements of $G$, there exists some finite quotient of $G$ in which the images of

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$x$ and $y$ are non-conjugate. The notion of the conjugacy separability owes its importance to the fact, first pointed out by Mal’cev [M-58], that the conjugacy problem has a positive solution in finitely presented conjugacy separable groups.

It follows from a recent work of Minasyan [M] combined with the work of Agol, Long and Reid [ALR-01], [LR-08] that Bianchi groups are virtually conjugacy separable, i.e., contain a finite index subgroup that are conjugacy separable\(^*\). This does not prove however the conjugacy separability of Bianchi groups, since (in contrast with subgroup separability) conjugacy separability is not preserved by commensurability (see [G-86, CZ1-09, MM]).

We say that two groups of $PSL_2(\mathbb{C})$ are commensurable if their intersection has finite index in both of them. More generally, two groups are (abstractly) commensurable if they contain isomorphic subgroups of finite index. In the rest of the paper we shall always use commensurability in the sense of abstract commensurability (after all two notions are equivalent since abstractly commensurable groups are commensurable in some group).

In this paper we establish that (abstract) commensurability invariance of conjugacy separability within torsion free groups very much depends on the centralizers of elements (see Theorem 2.4). Using it and different methods for torsion elements we prove our main

\textbf{Theorem 1.1.} Non-uniform arithmetic lattices of $SL_2(\mathbb{C})$ are conjugacy separable.

The conjugacy separability of Bianchi groups was conjectured in [WZ-98] where the conjugacy separability of Euclidean Bianchi groups was proved (more precisely, the cases $d = 1, 2, 7, 11$ were established there and $d = 3$ was completed in [LZ-09]). Recall that non-uniform arithmetic lattices in $PSL_2(\mathbb{C})$ (or equivalently in $SL_2(\mathbb{C})$) are precisely the subgroups commensurable with Bianchi groups (see Theorem 8.2.3 [MR-03]). Therefore Theorem 1.1 imply conjugacy separability of all Bianchi groups.

\textbf{Theorem 1.2.} The Bianchi groups are conjugacy separable.

The same methods also allow us to prove conjugacy separability of virtually Limit groups.

\(^*\)This important observation was made to us by Henry Wilton in private communication.
Theorem 1.3. Virtually Limit groups are conjugacy separable.

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2 General Results

The profinite topology on a group $G$ is the topology where the collection of all finite index normal subgroups of $G$ serves as a fundamental system of neighborhoods of the identity element $1 \in G$, turning $G$ into a topological group. The completion $\hat{G}$ of $G$ with respect to this topology is called the profinite completion of $G$ and can be expressed as an inverse limit

$$\hat{G} = \lim_{\leftarrow} G/N,$$

where $N$ ranges over all finite index normal subgroups of $G$. Thus $\hat{G}$ becomes a profinite group, i.e. a compact totally disconnected topological group. Moreover, there exists a natural homomorphism $\iota : G \rightarrow \hat{G}$ that sends $g \mapsto (gN)$, this homomorphism is a monomorphism when $G$ is residually finite. If $S$ is a subset of a topological group $\hat{G}$, we denote by $\overline{S}$ its closure in $\hat{G}$. The profinite topology on $G$ is induced by the topology of $\hat{G}$. Note that for a subgroup $H$ of $G$, the profinite topology of $H$ can be stronger than the topology induced by the profinite topology of $G$; these topologies coincide iff $\overline{H} = H$, for example this is the case if $G$ is finitely generated and $H$ is of finite index.

We say that $g \in G$ is conjugacy distinguished if its conjugacy class $g^G$ is closed in the profinite topology of $G$. For residually finite $G$ this exactly means $g^{\hat{G}} \cap G = g^G$. Note that $G$ is conjugacy separable iff every element of $G$ is conjugacy distinguished.

Proposition 2.1. Let $G$ be a finitely generated group containing a conjugacy separable normal subgroup $H$ of finite index. Let $a \in G$ be an element such that there exists a natural number $m$ with $a^m \in H$ and the following conditions hold:

- $C_G(a^m)$ is conjugacy separable;
- $\overline{C_H(a^m)} = \overline{C_H(a^m)} = \overline{C_H(a^m)}$. 

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Then $a$ is conjugacy distinguished.

**Proof.** Since $H$ is conjugacy separable it is residually finite and therefore so is $G$. Suppose $b = γ^{-1}aγ$, for some $γ ∈ \hat{G}$, it means that $b ∈ a\hat{G} ∩ G$ and then we need to prove that $b = a^γ$, for some $g ∈ G$. Observe that $\hat{G} = G\hat{H}$, so that we can write $γ = δγ_0$, where $γ_0 ∈ \hat{H}$ and $δ ∈ G$. Therefore $b^m = (a^γ)^m = (a^m)^γ = (a^m)^δγ_0$. Now substituting $a$ by $aδ$, we can suppose that $γ ∈ \hat{H}$. Thus $a^m$ and $b^m$ are conjugate in $\hat{H}$, and since $H$ is conjugacy separable there exists $h ∈ H$ such that $a^m = (b^h)^m$. Hence $a^m = (b^m)^h$, so $a^m$ and $b^m$ are conjugate in $H$. Thus we can suppose that $a^m = b^m$ and so $γ ∈ C_{\hat{H}}(a^m)$.

Let $C = C_G(a^m)$ be the centralizer of $a^m$ in $G$, so $b ∈ C$. By the second hypothesis $C_H(a^m) = C_{\hat{H}}(a^m) = C_{\hat{H}}(a^m)$, so $γ ∈ C_G(a^m)$. Now observe that $a, b ∈ C$ and $γ ∈ C_G(a^m)$, so by the first hypothesis there exists $g ∈ G$ such that $a^g = b$.

**Definition 2.2.** We say that a group $G$ is hereditarily conjugacy separable if every finite index subgroup of $G$ is conjugacy separable.

**Remark 2.3.** By Proposition 3.2 in [M] $H$ is hereditarily conjugacy separable iff $C_H(h) = C_{\pi}(h)$ for every $h ∈ H$, so hereditary conjugacy separability of $H$ would imply the second equality of the second condition of Proposition 2.1.

Proposition 2.1 implies the following

**Theorem 2.4.** Let $G$ be a finitely generated torsion free group containing a conjugacy separable normal subgroup $H$ of finite index. Suppose that for every $1 ≠ h ∈ H$,

- $C_G(h)$ is conjugacy separable;
- $C_{\hat{H}}(h) = C_{\hat{G}}(h) = C_{\hat{H}}(h)$.

Then $G$ is conjugacy separable.

**Theorem 2.5.** Let $G$ be a finitely generated subgroup separable group having a finite index hereditarily conjugacy separable subgroup $H$ and a free normal subgroup $F$ of $H$ such that $H/F$ is polycyclic. Then every element of prime order $p$ in $G$ is conjugacy distinguished in $G$. 

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Proof. Pick $a, b \in G$ such that $a = b^\gamma$, where $\gamma \in \widehat{G}$ and $a, b$ have order $p$. We need to prove that $a$ and $b$ are conjugate in $G$.

Without loss of generality we can suppose that $H$ and $F$ are normal in $G$. Indeed we can replace $H$ and $F$ by their cores (the core of a subgroup is the intersection of all its conjugates) $H_G$ and $F_G = \cap_{g \in G} F^g$; then since $H/F_G$ is a subgroup of $\prod_r H/F^r$ and so is nilpotent, where $r$ ranges over representatives of $G/H$, we can take the core $(H/F_G)_{G/F_G}$ of $H/F_G$ in $G/F_G$ and replace $H$ by the preimage of $(H/F_G)_{G/F_G}$ in $G$.

Since $\widehat{G} = G\widehat{H}$ we can write $\gamma = g\delta$, where $\delta \in \widehat{H}$ and $g \in G$ and so $a = b^\gamma = b^{g\delta}$. Therefore, changing $b$ to $b^g$ we may assume that $\gamma \in \widehat{H}$ and so $b \in \langle \widehat{H}, a \rangle \cap G = \langle H, a \rangle$. Thus we can suppose that $G = \langle H, a \rangle$.

Put $G_0 = G/F$. We shall use subindex 0 for the images in $G_0$. Since $G_0$ is polycyclic group it is conjugacy separable (cf. [Fo-89], [R-69] and [S-83]), so we may assume that $a_0 = b_0$ and so $\gamma_0 \in C_{H_0}(a_0)$.

Let $U$ be the preimage of the centralizer $C_{G_0}(a_0) = C_{H_0}(a_0) \times \langle a_0 \rangle$ and let $V$ be the preimage of $\langle a_0 \rangle$ in $G$. By Theorem of Dyer and Scott [DS-75] $V = \ast_{i \in I} (F_i \times T_i) \ast L$, where $T_i = \langle t_i \rangle$ and $t_i$ has order $p$ and $F_i, L$ are free groups. Since torsion elements are conjugate to elements of free factors, we may assume that $a = t_k$ and $b = t_j$ for some $k, j \in I$. Moreover, since $H_0$ is polycyclic by Proposition 3.3 in [RSZ-98] $C_{H_0}(a_0) = C_{\widehat{H}_0}(a_0)$. Thus using equalities $\widehat{G}_0 = \widehat{G}/F$ and $\widehat{U}/F = \widehat{C}_{G_0}(a_0)$ we deduce that $\gamma \in \widehat{U}$. We shall prove that $a$ and $b$ are conjugate in $U$.

Consider the abelianization $V/V'$ of $V$ and observe that the torsion elements are conjugated in $V$ if and only if they coincide in $V/V'$. Therefore, $a$ and $b$ are conjugate in $U$ if, and only if, their images are conjugated in $V/V'$. Thus it is enough to prove that the images of $a$ and $b$ are conjugate in $U/V'$.

Suppose not. We use bar to denote the images of elements in $U/V'$. Observe that $N := U/V = C_{H_0}(a_0)$ acts on $V/V'$ and $\bar{a}$ and $\bar{b}$ are conjugate in $U/V'$ iff $\bar{a}$ and $\bar{b}$ are in the same $N$-orbit. Note also that since $U$ permutes torsion elements of $V$, $N$-submodule $M$ of torsion elements in $V/V'$ is permutation, i.e. is isomorphic to $\oplus_i \mathbb{F}_p[N/N_i]$, where $N_i$ runs via subgroups of $N$ (namely $N_i = C_N(t_i)$ in our case) and the images of non-conjugate torsion elements $t_i$ are canonical generators of different summands (here $\mathbb{F}_p[N/N_i]$ means $\mathbb{F}_p$-vector space with the basis $[N/N_i]$). Since $a$ and $b$ are not conjugate, $\bar{a}$ and $\bar{b}$ are in different summands $M_a = \mathbb{F}_p[N/N_i]$ and $M_b = \mathbb{F}_p[N/N_j]$. Put $M_{ab} = M_a \oplus M_b$.

Let $x_1, \ldots, x_n$ be elements of $U$ that whose images modulo $V$ generate
(recall that $N$ is finitely generated since it is a subgroup of a finitely generated nilpotent group $H_0$). Consider the group $R = \langle a, b, x_1, \ldots, x_n \rangle$. Then $RV/V = N$ and we have the following embedding $M_{ab} \rightarrow RV'/V'$. The elements $\bar{a}$ and $\bar{b}$ are not conjugate in $RV'/V'$ since they are in different direct summands $M_a$ and $M_b$ of $M_{ab}$. Moreover, the images of $\bar{a}$ and $\bar{b}$ in the quotient group $U_N$ of $RV'/V'$ modulo the normal subgroup $[N, M_{ab}]$ are central distinct and so are non-conjugate. Note that $U_N$ is finitely generated abelian and so is residually finite. Since $M_{ab}/[N, M_{ab}] \cong C_p \times C_p$ is finite, there exists a normal subgroup of finite index $B$ in $U_N$ intersecting trivially $M_{ab}/[N, M_{ab}]$. Thus the images of $a$ and $b$ in a finite quotient group $U_N/B$ of $R$ are distinct central elements.

On the other hand, since $G$ is subgroup separable $\hat{R} = \hat{R}$. It follows that there is a natural epimorphism $\hat{R} \rightarrow U_N/B$. Since the intersection $\overline{V'} \cap \overline{R}$ of the closures of $V'$ and $R$ in $\hat{G}$ is in the kernel of the natural epimorphism $\overline{R} \rightarrow U_N/B$ we deduce that this epimorphism factors via $\overline{RV'}/\overline{V'}$. Thus to arrive at contradiction it suffices to show that the images of $a$ and $b$ in $\overline{RV'}/\overline{V'}$ are conjugate.

Observe that $\overline{N} := \overline{U}/\overline{V}$ acts on $\overline{V}/\overline{V'}$. Since a finitely generated abelian-by-polycyclic group is residually finite (see Theorem 7.2.1 in [LR-04]), $H/V'$ is residually finite and so $\overline{V'} \cap H = V'$. This means that $N$-module $V/V'$ embeds naturally in $\overline{N}$-module $\overline{V}/\overline{V'}$. Thus the composition of the natural homomorphisms $M \rightarrow V/V' \rightarrow \overline{V}/\overline{V'}$ is injection and so $M$ embeds naturally in $\overline{V}/\overline{V'}$. Denote by $\overline{M_{ab}}$ the closure of $M_{ab}$ in $\overline{V}/\overline{V'}$. Since $\bar{a}$ and $\bar{b}$ are conjugate in $\overline{U}/\overline{V}$, $\bar{a}$ and $\bar{b}$ are in the same $\overline{N}$-orbit. Since $\overline{RV}/\overline{V} = \overline{N}$, $\bar{a}$ and $\bar{b}$ are conjugate in $\overline{RV'}/\overline{V'}$ as required.

**Corollary 2.6.** Let $G = H \rtimes C_p$ be a semidirect product of a finitely generated hereditarily conjugacy separable subgroup separable free-by-polycyclic group $H$ and a group $C_p$ of prime order $p$. Suppose that for every $1 \neq h \in H$ the centralizer $C_G(h)$ is hereditarily conjugacy separable and $\overline{C_H(h)} = \overline{C_H(h)}$ in $H$. Then $G$ is hereditarily conjugacy separable.

**Proof.** Follows from Theorem 2.5 and Proposition 2.1 combined with Remark 2.3.

**Theorem 2.7.** Let $G$ be a finitely generated subgroup separable group. Suppose there exists a free-by-polycyclic hereditarily conjugacy separable subgroup $H$ of $G$ of finite index. If $C_G(h)$ is hereditarily conjugacy separable for every
Proof. We need to prove that any element \( a \in G \) is conjugacy distinguished. Suppose \( a \) has finite order. Let \( t \) be an integer such that \( c = a^t \) has prime order \( p \). Replacing \( H \) by its core we may assume that \( H \) is normal in \( G \). By Corollary 2.6 \( G_1 := \langle c, H \rangle \) is hereditarily conjugacy separable and so by Remark 2.3 \( C_{G_1}(c) \) is dense in \( C_{G_1}(c) \). Since \( G \) is subgroup separable and \( C_{G_1}(c) \) is finitely generated, \( C_{G_1}(c) = C_{G_1}(c) \). Then by Proposition 2.1 (applied to the pair \( (G, G_1) \)) \( a \) is conjugacy distinguished.

Suppose now \( a \) has infinite order. Choose any conjugacy separable subgroup \( H \) of finite index in \( G \). Then \( a^m \in H \setminus \{1\} \) for some \( m \in \mathbb{N} \), so similarly to the previous paragraph the result follows from Proposition 2.1 combined with Remark 2.2.

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### 3 Groups commensurable with Bianchi and Limit groups

The proof of the next proposition was communicated to us by Henry Wilton.

**Proposition 3.1.** A non-uniform arithmetic lattice in \( SL_2(\mathbb{C}) \) possesses a (finitely generated free)-by-cyclic hereditarily conjugacy separable subgroup of finite index.

**Proof.** Recall that non-uniform arithmetic lattices in \( SL_2(\mathbb{C}) \) are precisely the subgroups commensurable with Bianchi groups (see Theorem 8.2.3 [MR-03]). Therefore we just need to show the existence of a (finitely generated free)-by-cyclic hereditarily conjugacy separable subgroup of finite index in a Bianchi group \( G \). By [ALR-01] Bianchi groups are (virtually) geometrically finite subgroups of right angled Coxeter groups obtained as groups of isometries of the hyperbolic space \( \mathbb{H}^n \) (for some \( n \)) generated by the reflections about co-dimension one faces of some right angled polyhedron. Therefore by Theorem 1.6 in [LR-08] Bianchi groups have a subgroup \( H \) of finite index that is virtual retract of a right angled Coxeter group. By Corollary 2.3 in [M] a right angled Coxeter group is virtually hereditarily conjugacy separable. Since a virtual retract of a hereditarily conjugacy separable group is hereditarily conjugacy separable (see Theorem 3.4 in [CZ1-09]) it follows that \( H \) is hereditarily conjugacy separable. On the other hand by [A-08] there exist
a finite index surface-by-infinite cyclic subgroup \( U = S \times \mathbb{Z} \) in the Bianchi group \( G \). It is well known that Bianchi groups have virtual cohomological dimension 2 (see Theorem 11.4.4 [BS-73]), so from [B-78, Corollary 1] \( S \) is finitely generated free. Thus \( U \cap H \) is the desired subgroup. \( \square \)

**Theorem 3.2.** Non-uniform arithmetic lattices of \( SL_2(\mathbb{C}) \) are conjugacy separable.

*Proof.* Let \( G \) be a non-uniform arithmetic lattice of \( SL_2(\mathbb{C}) \). By Proposition 3.1 there exists a finite index hereditarily conjugacy separable subgroup \( H \) of \( G \) such that \( H = F \times \mathbb{Z} \) with \( F \) free of finite rank. Then for every \( h \in H \) the centralizer \( C_G(h) \) is finitely generated virtually abelian by Lemma 2.2 in [CZ2-09] and therefore is hereditarily conjugacy separable. The equality \( C_H(h) = \hat{C}_H(h) \) holds since \( H \) is subgroup separable (see Theorem 3.4 in [LR-08]). Thus the result follows from Theorem 2.7. \( \square \)

Since non-uniform arithmetic lattices in \( PSL_2(\mathbb{C}) \) (or equivalently in \( SL_2(\mathbb{C}) \)) are precisely the subgroups commensurable with Bianchi groups (see Theorem 8.2.3 [MR-03]) Theorem 3.2 implies the conjugacy separability of the Bianchi groups.

**Theorem 3.3.** The Bianchi groups are conjugacy separable.

We come to limit groups. Recall that a fully residually free group is a group that satisfies the following condition: for each finite subset \( K \subset G \) of elements, there exist a homomorphism \( \varphi : G \to F \) to some free group \( F \) such that the restriction \( \varphi|_K \) is injective. A finitely generated fully residually free group is called a limit group.

We recall now a construction of a limit group. Let \( F \) be a free group of finite rank and put \( \mathcal{Y}_1 = F \). For \( i > 1 \), define the class \( \mathcal{Y}_i \) to consist of all groups that are free products \( G_i = G_{i-1} \ast_C A \) of a group \( G_{i-1} \in \mathcal{Y}_{i-1} \) and a free abelian group \( A \) of finite rank amalgamating self-centralizing cyclic subgroup of \( G_{i-1} \) with a subgroup of \( A \) generated by a generator of \( A \) (this construction is known as an extension of the centralizer). If \( L \) is a limit group then by Theorem 4 in [KM-98], there exists \( n \) such that \( L \) embeds in some \( G_n \in \mathcal{Y}_n \). We shall need now the following abstract analogue of Theorem 1 in [KZ-07].

**Theorem 3.4.** Let \( H \) be a group of finite cohomological dimension \( cd(H) = d \) such that for every \( i \geq 1 \) the cohomology group \( H^i(H, \mathbb{F}_p) \) is finite and \( \sigma \) be
an automorphism of $H$ of order $p$. Then for the group $P$ of fixed points of $\sigma$ we have
\[ \sum_{j \geq 0} \dim H^j(P, \mathbb{F}_p) \leq \sum_{j \geq 0} \dim H^j(H, \mathbb{F}_p). \]

Proof. The proof is a repetition of the proof of Theorem 1 in [KZ-07] with the use of Remark after Theorem 7.4 in [Br-82] instead of Theorem 2 in [KZ-07].

Theorem 3.5. A group commensurable with a limit group is conjugacy separable.

Proof. Let $G$ be a group commensurable with a limit group $L$. We have to show that every element $g$ of $G$ is conjugacy distinguished. Replacing $L$ by a common subgroup of finite index in $G$ and $L$ we may assume that $L$ is a subgroup of finite index of $G$. Then replacing $L$ by its core in $G$ we may assume that $L$ is normal in $G$. We shall show that a pair $(G, L)$ satisfy hypothesis of Theorem 2.7.

Subgroup separability of limit groups [W-2008] implies that $G$ is subgroup separable. Since a subgroup of finite index of a limit group is a limit group the main result of [CZ-07] shows that $L$ is hereditarily conjugacy separable. By Theorem 2 in [K-09] there is a term of lower central series $\gamma_i(L)$ which is free, so $L$ is finitely generated free-by-nilpotent and in particular free-by-polycyclic. By commutative transitivity property $C_L(h)$ is abelian for every $h \in L$ and it is well-known that an abelian subgroup of a limit group is finitely generated (follows from finiteness of cohomological dimension for example). Therefore $C_G(h)$ is finitely generated virtually abelian for every $h \in L$ and hence is hereditarily conjugacy separable. We are left with checking that $C_G(g)$ is finitely generated for every $g \in G$. Clearly it is equivalent to $C_L(g)$ being finitely generated.

If $g$ has infinite order then some of its power is in $L$ and so (as was explained above) $C_G(g)$ is finitely generated virtually abelian. Suppose now $g$ is of finite order and $h := g^m$ is of prime order $p$ for some natural $m$.

Since limit groups are of finite cohomological dimension and is of type $FP_\infty$ that in turn implies finiteness of cohomology with coefficients in a finite field we can apply Theorem 3.3 to deduce that $\dim H^1(C_L(h), \mathbb{F}_p)$ is finite. Since $H^1(C_L(h), \mathbb{F}_p) = \text{Hom}(C_L(h), \mathbb{F}_p) \cong C_L(h)/(C_L(h)[C_L(h)]C_L(h)^p) \cong H_1(C_L(h), \mathbb{F}_p)$, it follows from Theorem 2 together with the last line of the first paragraph of Section 4 in [BH-07] that $C_L(h)$ is finitely generated. Since
$g$ normalizes $C_L(h)$ we can apply induction on the order of $g$ to deduce that $C_{C_L(h)}(\langle g \rangle/\langle h \rangle) = C_L(g)$ is finitely generated, as required.

Thus the pair $(G, L)$ satisfies the premises of Theorem 2.7 according to which $G$ is conjugacy separable. The proof is complete. □

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