EXPONENTIAL HIGHER DIMENSIONAL
ISOPERIMETRIC INEQUALITIES FOR SOME
ARITHMETIC GROUPS

KEVIN WORTMAN

Abstract. We show that arithmetic subgroups of semisimple groups
of relative $\mathbb{Q}$-type $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, or $E_7$ have an exponential
lower bound to their isoperimetric inequality in the dimension that
is 1 less than the real rank of the semisimple group.

Let $G$ be a connected, semisimple, $\mathbb{Q}$-group that is almost simple
over $\mathbb{Q}$. Let $X$ be the symmetric space of noncompact type associated
with $G(\mathbb{R})$ and let $X_\mathbb{Z}$ be a contractible subspace of $X$ that is a finite
Hausdorff distance from some $G(\mathbb{Z})$-orbit in $X$; Raghunathan proved
that such a space exists [Ra 1]. We denote the $\mathbb{R}$-rank of $G$ by $rk_{\mathbb{R}}G$.

Given a homology $n$-cycle $Y \subseteq X_\mathbb{Z}$ we let $v_X(Y)$ be the infimum of
the volumes of all $(n+1)$-chains $B \subseteq X$ such that $\partial B = Y$. Similarly,
we let $v_\mathbb{Z}(Y)$ be the infimum of the volumes of all $(n+1)$-chains $B \subseteq X_\mathbb{Z}$
such that $\partial B = Y$. We define the ratio

$$R_n(Y) = \frac{v_\mathbb{Z}(Y)}{v_X(Y)}$$

and we let $R_n(G(\mathbb{Z})): \mathbb{R}_{>0} \to \mathbb{R}_{\geq 1}$ be the function

$$R_n(G(\mathbb{Z}))(L) = \sup \{ R_n(Y) \mid \text{vol}(Y) \leq L \}$$

These functions measure a contrast between the geometries of $G(\mathbb{Z})$
and $X$.

Clearly if $G$ is $\mathbb{Q}$-anisotropic (or equivalently, if $G(\mathbb{Z})$ is cocompact
in $G(\mathbb{R})$) then we may take $X_\mathbb{Z} = X$ so that $R_n(G(\mathbb{Z})) = 1$ for all $n$.

The case is different when $G$ is $\mathbb{Q}$-isotropic, or equivalently, if $G(\mathbb{Z})$
is non-cocompact in $G(\mathbb{R})$.

Leuzinger-Pittet conjectured that $R_{rk_{\mathbb{R}}G-1}(G(\mathbb{Z}))$ is bounded below
by an exponential when $G$ is $\mathbb{Q}$-isotropic [L-P]. The conjecture in
the case $rk_{\mathbb{R}}G = 1$ is equivalent to the well-known observation that
the word metric for non-cocompact lattices in rank one real simple
Lie groups is exponentially distorted in its corresponding symmetric
space. Prior to [L-P], the conjecture was evidenced by other authors in
some cases. It was proved by Epstein-Thurston when $G(\mathbb{Z}) = SL_k(\mathbb{Z})$
[Ep et al.], by Pittet when $G(\mathbb{Z}) = \text{SL}_2(\mathcal{O})$ and $\mathcal{O}$ is a ring of integers in a totally real number field [Pa], by Hattori when $G(\mathbb{Z}) = \text{SL}_k(\mathcal{O})$ and $\mathcal{O}$ is a ring of integers in a totally real number field [Ha 1], and by Leuzinger-Pittet when $\text{rk}_R G = 2$ [L-P].

This paper contributes to the verification of the Leuzinger-Pittet conjecture by proving

**Theorem 1.** Let $G$ be as in the introductory paragraph and assume that $G$ is $\mathbb{Q}$-isotropic. Furthermore, suppose the $\mathbb{Q}$-relative root system of $G$ is of type $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, or $E_7$. Then there exist constants $C > 0$ and $L_0 > 0$ such that

$$R_{\text{rk}_G - 1}(G(\mathbb{Z}))(L) \geq e^{CL}$$

for any $L > L_0$.

0.1. **Example.** Let $\mathcal{O}$ be the ring of integers in a number field $K$, and let $G = R_{K/\mathbb{Q}}\text{SL}_k$ where $R_{K/\mathbb{Q}}$ is the restriction of scalars functor. Then $G(\mathbb{Z}) = \text{SL}_k(\mathcal{O})$, $G$ is $\mathbb{Q}$-isotropic, $G$ has a $\mathbb{Q}$-relative root system of type $A_{k-1}$, and $\text{rk}_\mathbb{Q} G = (k-1)S$ where $S$ is the number of inequivalent archimedean valuations on $K$. Therefore, $R_{(k-1)S-1}(\text{SL}_k(\mathcal{O}))$ is bounded below by an exponential.

0.2. **Non-nonpositive curvature of arithmetic groups.** If $G(\mathbb{Z})$ satisfied a reasonable notion of nonpositive curvature (including CAT(0) or combable, for example), we would expect polynomial bounds on isoperimetric inequalities for $G(\mathbb{Z})$. Thus, not only does Theorem 1 provide a measure of the difference between $G(\mathbb{Z})$ and $X$, it also exhibits non-nonpositive curvature tendencies for $G(\mathbb{Z})$ when $G$ is $\mathbb{Q}$-isotropic and $\text{rk}_\mathbb{Q} G > 1$.

0.3. **Type restriction.** Our proof of Theorem 1 excludes the remaining types – $G_2$, $F_4$, $E_8$, and $BC_n$ – because groups of these types do not contain proper parabolic subgroups whose unipotent radicals are abelian. Our techniques require an abelian unipotent radical of a maximal $\mathbb{Q}$-parabolic subgroup of $G$ to construct cycles in $X_\mathbb{Z}$.

0.4. **Related results.** It is an open question whether $R_n(G(\mathbb{Z}))$ is bounded above by a constant when $n < \text{rk}_\mathbb{Q} G - 1$. When $n = 0$ it is; this is a theorem of Lubotzky-Mozes-Raghunathan [L-M-R].

Druțu showed that if the $\mathbb{Q}$-relative root system of $G$ is of type $A_1$ or $BC_1$, then for any $\varepsilon > 0$, $G(\mathbb{Z})$ has a Dehn function that is bounded above by $L^{2+\varepsilon}$ for $L$ sufficiently large [Dr].

Young proved that $G(\mathbb{Z}) = \text{SL}_k(\mathbb{Z})$ has a quadratic Dehn function if $k \geq 5$ [Yo].
Gromov proved that all of the functions $R_n(G(\mathbb{Z}))$ are bounded above by an exponential function, and Leuzinger later provided a more detailed proof of this fact (5.4 [Gr] and Corollary 5.4 [Le]).

1. Choice of parabolic

Let $T \leq G$ be a maximal $\mathbb{Q}$-split torus in $G$. We let $\Phi_\mathbb{Q}$ be the roots of $G$ with respect to $T$. Choose an ordering on $\Phi_\mathbb{Q}$. We denote the corresponding sets of simple and positive roots by $\Delta_\mathbb{Q}$ and $\Phi_\mathbb{Q}^+$ respectively.

If $I \subseteq \Delta_\mathbb{Q}$, we let $[I] \subseteq \Phi_\mathbb{Q}$ be the set of roots that are linear combinations of elements in $I$, and we let $\Phi_\mathbb{Q}(I)^+ = \Phi_\mathbb{Q}^+ - [I]$.

For each $\alpha \in \Phi_\mathbb{Q}$, we let $U_\alpha \leq G$ be the root subgroup associated with $\alpha$. For $J \subseteq \Phi_\mathbb{Q}$, we let $U_J = \prod_{\alpha \in J} U_\alpha$.

We define $T_I = \cap_{\alpha \in I} \text{Ker}(\alpha)^\circ$ where the superscript $\circ$ denotes the connected component of the identity, and we label the centralizer of $T_I$ in $G$ by $Z_G(T_I)$.

1.1. Maximal parabolics with abelian unipotent radicals. For any $\alpha_0 \in \Delta_\mathbb{Q}$, we let $P_{\alpha_0}$ be the maximal proper parabolic subgroup of $G$ given by $U_{\Phi_\mathbb{Q}(\Delta_\mathbb{Q} - \alpha_0)^+}Z_G(T_{\Delta_\mathbb{Q} - \alpha_0})$. The unipotent radical of $P_{\alpha_0}$ is $U_{\Phi_\mathbb{Q}(\Delta_\mathbb{Q} - \alpha_0)^+}$.

**Lemma 2.** There is some $\alpha_0 \in \Delta_\mathbb{Q}$ such that $U_{\Phi_\mathbb{Q}(\Delta_\mathbb{Q} - \alpha_0)^+}$ is abelian.

**Proof.** Suppose $\Delta_\mathbb{Q} = \{\alpha_1, \alpha_2, ..., \alpha_k\}$. The set of positive roots $\Phi_\mathbb{Q}^+$ contains a “highest root” $\sum n_i \alpha_i$ for positive integers $n_i$ such that if $\sum m_i \alpha_i \in \Phi_\mathbb{Q}^+$, then $m_i \leq n_i$ (Bou, VI 1 8).

Given that $\Phi_\mathbb{Q}$ is a root system of type $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, or $E_7$, there is some $\alpha_0 \in \{\alpha_1, \alpha_2, ..., \alpha_k\}$ such that $n_0 = 1$; consult the list of root systems in the appendix of [Bou].

Since any $\sum m_i \alpha_i \in \Phi_\mathbb{Q}(\Delta_\mathbb{Q} - \alpha_0)^+$ has $m_0 > 0$, it follows that any $\sum m_i \alpha_i \in \Phi_\mathbb{Q}(\Delta_\mathbb{Q} - \alpha_0)^+$ has $m_0 = 1$, and thus the sum of two elements in $\Phi_\mathbb{Q}(\Delta_\mathbb{Q} - \alpha_0)^+$ is not a root.

Therefore, given $\tau_1, \tau_2 \in \Phi_\mathbb{Q}(\Delta_\mathbb{Q} - \alpha_0)^+$, we have

$$[U_{\tau_1}, U_{\tau_2}] \subseteq U_{\tau_1 + \tau_2} = 1$$

$\square$

In what remains, we let $P = P_{\alpha_0}$, we let $U_P$ be the real points of $U_{\Phi_\mathbb{Q}(\Delta_\mathbb{Q} - \alpha_0)^+}$. Thus, we can rephrase Lemma 2 as

**Lemma 3.** $U_P$ is abelian.
1.2. A contracting ray. Recall that $T_{\Delta_Q-\alpha_0} \leq Z_G(T_{\Delta_Q-\alpha_0}) \leq P$ is a 1-dimensional $Q$-split torus. Choose $a_+ \in T_{\Delta_Q-\alpha_0}(\mathbb{R})$ such that $\alpha_0(a_+) > 1$ and such that the distance in $T_{\Delta_Q-\alpha_0}(\mathbb{R})$ between 1 and $a_+$ equals 1.

The Lie algebra of $U_P$ is $u$.

**Lemma 4.** There is some $s > 0$ such that for any $v \in u$

$$Ad(a_+^s)v = e^{sv}$$

**Proof.** Recall that

$$u = \prod_{\beta \in \Phi_Q(\Delta_Q-\alpha_0)^+} u_\beta$$

where

$$u_\beta = \{ v \in u \mid Ad(x)v = \beta(x)v \text{ for all } x \in T \}$$

If $\beta \in \Phi_Q(\Delta_Q-\alpha_0)^+$, then $\beta = \alpha_0 + \sum_{\alpha_i \in \Delta_Q-\alpha_0} n_i \alpha_i$. Since $a_+ \in \cap_{\alpha_i \in \Delta_Q-\alpha_0} \text{Ker}(\alpha_i)^\circ$, we have $\beta(a_+) = \alpha_0(a_+)$ and thus for $v \in u$, it follows that $Ad(a_+^s)v = \alpha_0(a_+^s)v$.

Let $s = \log(\alpha_0(a_+))$. \hfill $\square$

2. A horoball in the symmetric space, disjoint from $X_Z$.

**Lemma 5.** There is a maximal $Q$-torus $A \leq G$ such that the maximal $Q$-split torus of $A$ is $T_{\Delta_Q-\alpha_0}$ and such that $A$ contains a maximal $R$-split torus of $G$.

**Proof.** See Proposition 3.3 in [B-W] where $K = Q$, $H = G$, $T_1 = T_{\Delta_Q-\alpha_0}$, $S = \{v\}$, and $K_v = \mathbb{R}$. \hfill $\square$

Let $Q$ be a minimal parabolic that contains $A$ and is contained in $P$. We let $\Phi_\mathbb{R}$ be the roots of $G$ with respect to the maximal $\mathbb{R}$-split subtorus of $A$, $\Delta_\mathbb{R}$ be the collection of simple roots given by $Q$, and $\Phi_\mathbb{R}^+$ be the corresponding positive roots.

2.1. Alternate descriptions of the symmetric space. Let $G = G(\mathbb{R})$ and let $A \leq G$ be the $\mathbb{R}$-points of the maximal $\mathbb{R}$-split subtorus of $A$. Recall that $A(\mathbb{R}) = AB$ for some compact group $B \leq A(\mathbb{R})$.

Choose a maximal compact subgroup $K \leq G$ that contains $B$. Then $G/K$ is a symmetric space that $G$ acts on by isometries. We name this symmetric space $X$.

Let $U_Q$ be the group of real points of the unipotent radical of $Q$. By the Iwasawa decomposition, $U_QA$ acts simply transitively on $X$ and we identify $X$ with $U_QA$. In this description of $X$, $A$ is a flat.
2.2. **Integral translations in a flat.** By Dirichlet’s units theorem, \( A(\mathbb{Z}) \) contains a finite index free abelian subgroup of rank \( \text{rk}_R(G) - 1 = \dim(A) - 1 \). Thus, if \( A_\mathbb{Z} \) is the convex hull in \( X \) of the \( A(\mathbb{Z}) \)-orbit of the point \( 1 \in U_Q A = X \), then \( A_\mathbb{Z} \) is a codimension-1 Euclidean subspace of the flat \( A \), and \( A(\mathbb{Z}) \) acts cocompactly on \( A_\mathbb{Z} \). We may assume \( A_\mathbb{Z} \subseteq X_\mathbb{Z} \).

2.3. **Horoballs.** Notice that \( \{ a_+^t \}_{t>0} \) defines a unit-speed geodesic ray that limits to a point in \( A^\infty \). We let \( b_{a_+^t} : U_Q A \to \mathbb{R} \) be the Busemann function corresponding to the geodesic ray \( \{ a_+^t \}_{t>0} \). That is,

\[
b_{a_+^t}(x) = \lim_{t \to \infty} [d(x, a_+^t) - t]
\]

We let \( A_0 \leq A \) be the codimension-1 subspace of \( A \) consisting of those \( a \in A \) for which \( b_{a_+^t}(a) = 0 \). Thus, \( A_0 \) is orthogonal to \( a_+^\mathbb{R} \).

**Lemma 6.** For \( T \in \mathbb{R} \), \( (b_{a_+^t})^{-1}(-T) = U_Q A_0 a_+^T \).

**Proof.** We first show that for \( u \in U_Q \) and \( x \in X \), \( b_{a_+^t}(x) = b_{ua_+^t}(x) \)

Where \( b_{ua_+^t} \) is the Busemann function for the ray \( \{ ua_+^t \}_{t>0} \).

Notice that \( U_Q = U_p U_a \) where \( U_a \leq Z_G(T_{\Delta_Q - a_0})(\mathbb{R}) \) is a unipotent group whose elements commute with \( a_+ \).

If \( u \in U_p \), then Lemma 4 implies

\[
d(a_+^t, ua_+^t) = d(1, a_+^{-t}ua_+^t) \to 0
\]

Therefore,

\[
b_{a_+^t}(x) = \lim_{t \to \infty} [d(x, a_+^t) - t] = \lim_{t \to \infty} [d(x, ua_+^t) - t] = b_{ua_+^t}(x)
\]

The quotient map of a Lie group by a normal subgroup is distance nonincreasing. Because \( U_p \) is normal in \( U_Q A \), and because \( a_+^\mathbb{R} \) is normal in \( U_a A \), the following composition is distance nonincreasing

\[
U_Q A \to U_a A \to U_a A_0
\]

We denote the geodesic between points \( z, w \in X \) by \( \overline{zw} \). Orthogonality of \( A_0 \) and \( a_+^\mathbb{R} \) and the conclusion of the above paragraph show that for any \( u \in U_a \), \( \overline{1u} \) is orthogonal to \( a_+^\mathbb{R} \) at 1 and to \( ua_+^\mathbb{R} \) at \( u \) and thus that \( a_+^t, ua_+^t \) is orthogonal to \( a_+^\mathbb{R} \) at \( a_+^t \) and to \( ua_+^R \) at \( ua_+^t \). Furthermore, the length of \( a_+^t, ua_+^t \) is independent of \( t \) since \( u \) commutes with \( a_+^t \).

Notice that the angle between \( \overline{a_+^t, x} \) and \( \overline{a_+^t, 1} \) limits to 0 as \( t \to \infty \). Similarly, the angle between \( \overline{ua_+^t, x} \) and \( \overline{ua_+^t, u} \) limits to 0. Hence, the
triangle in $X$ with vertices $a'_+, u a'_+$, and $x$ approaches a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, and $0$. That is
\[d(x, a'_+) - d(x, u a'_+) \to 0\]
Consequently, for $u \in U_a$ we have
\[b_{a'_+}(x) = \lim_{t \to \infty} [d(x, a'_+) - t] = \lim_{t \to \infty} [d(x, u a'_+) - t] = b_{u a'_+}(x)\]
Therefore, for $u \in U_q$, $b_{a'_+}(u^{-1} x) = b_{u a'_+}(x) = b_{a'_+}(x)$, and it follows that $U_q(b_{a'_+})^{-1}(-T) = (b_{a'_+})^{-1}(-T)$. The lemma is a combination of this last fact together with $A_0 a^T_+ \subseteq (b_{a'_+})^{-1}(-T)$.
\[\square\]

**Lemma 8.** For some $T > 0$, $X_Z \subseteq U_Q A_0 a^T_+^{(-\infty, T]}$.

**Proof.** Theorem A of [Ha 2] states that $X_Z \subseteq (b_{a'_+})^{-1}[-T, \infty)$ for some $T > 0$, and $(b_{a'_+})^{-1}[-T, \infty) = U_Q A_0 a^{(-\infty, T]}_+$ by Lemma 6.

\[\square\]

2.4. **Projecting onto a horosphere.** Let $\pi : U_Q A_0 a^{\infty}_+ \to U_Q A_0$ be the obvious projection of $X$ onto the horosphere $(a_{a'_+}^+)^{-1}(0)$.

**Lemma 8.** There is some $M > 0$ such that for any $x_1, x_2 \in X_Z$, we have $d(x_1, x_2) + M \geq d(\pi(x_1), \pi(x_2))$.

**Proof.** Recall that $U_Q = U_P U_a$ where elements of $U_a \leq P$, and elements of $A_0$, commute with $a_+$. Similar to Lemma 4 we have that for any $t > 0$ and any $v$ in the Lie algebra of $U_Q A$ that
\[||Ad(a^{+t}_+)v|| \leq ||v||\]
Let $T$ be as in Lemma 7 and define $\pi_T : U_Q A_0 a^T_+^{(-\infty, T]} \to U_Q A_0 a^T_+$ by $\pi_T = R_{a^T_+} \circ \pi$ where $R_{a^T_+}$ is right multiplication by $a^T_+$.

We claim that $\pi_T$ is distance nonincreasing. To see this, first let $v$ be a tangent vector to $X$ at the point $a'_+$ for some $t \leq T$. With $|| : ||_x$ as the norm at $x$, and $f_*$ as the differential of $f$, we have
\[||(|\pi_T)_*v||_{\pi_T(a'_+)} = ||(R_{a^{T-t}_+})_*v||_{a^{T-t}_+}\]
\[= ||(L_{a^{T-t}_+})_*v||_{a^{T-t}_+}\]
\[= ||Ad(a^{T-t}_+)v||_{a^{T-t}_+}\]
\[\leq ||v||_{a^{T-t}_+}\]
Left-translations by $U_Q A_0$ show that for any $x \in U_Q A_0 a_+^{(-\infty,T]}$, and any $v \in T_x X$,

$$||(\pi_T)_* v||_{\pi_T(x)} \leq ||v||_x$$

For any path $c : [0, 1] \to U_Q A_0 a_+^{(-\infty,T]}$, apply $\pi_T$ to those segments contained in $U_Q A_0 a_+^{(-\infty,T]}$ to define a path between $\pi_T(c(0))$ and $\pi_T(c(1))$. This new path will have its length bounded above by the length of $c$ as is easily verified from the inequality on norms of vectors from above. This confirms our claim that $\pi_T$ is distance nonincreasing.

To confirm the lemma, notice that similarly, the map $R_{a_+^T} : U_Q A_0 a_+^{T} \to U_Q A_0$ translates all point in $X$ a distance of

$$d(x, R_{a_+^T}(x)) = d(1, a_+^T)$$

Therefore,

$$d(R_{a_+^T}(x_1), R_{a_+^T}(x_2)) \leq d(x_1, x_2) + 2d(1, a_+^T)$$

The lemma follows as $\pi = R_{a_+^T} \circ \pi_T$. 

\[\square\]

3. Choice of a Cell in $X_Z$

We want to construct a cycle $Y \subseteq X_Z$. In this section we begin by constructing a cell $F \subseteq A_0$ that will be used in the construction of $Y$.

**Lemma 9.** $A_0 \subseteq X_Z$.

*Proof.* Both $A_0$ and the convex hull of $A_Z$ are codimension 1 subspaces of $A$. Since $A_Z \subseteq X_Z \subseteq U_Q A_0 a_+^{(-\infty,T]}$ we have that $A_Z \subseteq A_0 a_+^{(-\infty,T]}$. Therefore $A_Z$ and $A_0$ are parallel hyperplanes. Since the both contain 1, they are equal.

\[\square\]

Let $X^\infty$ be the spherical Tits building for $X = U_Q A$, and let $A^\infty \subseteq X^\infty$ be the apartment given by $A$. Let $\Pi^\infty \subseteq X^\infty$ be the simplex given by $P$ and let $\Pi^\infty \subseteq X^\infty$ be the simplex opposite of $\Pi^\infty$ in $A^\infty$, or equivalently, $\Pi^\infty$ is the simplex given by the parabolic group $P^- = U_{\Phi(Q^{-\infty})} Z_G(T_{\Delta_Q^{-\infty}})$.

Denote the star of $\Pi^\infty$ in $A^\infty$ by $\Sigma \subseteq A^\infty$. Note that $\Sigma$ is homeomorphic to a $\text{rk}_R(G) - 1$ ball. We denote the codimension 1 faces of $\Sigma$ as $\Sigma_1, ..., \Sigma_n$. 
3.1. \( A_0^\infty \) and \( \Sigma \) are disjoint. Let \( \Psi \subseteq \Phi_\tilde{\mathfrak{g}} \) be such that \( U_\Psi = R_u(\mathbb{P}^-) \). Given \( b \in A_0 \) we define the following sets of roots:

\[
C(b) = \{ \beta \in \Psi \mid \beta(b) > 1 \}
\]

\[
Z(b) = \{ \beta \in \Psi \mid \beta(b) = 1 \}
\]

\[
E(b) = \{ \beta \in \Psi \mid \beta(b) < 1 \}
\]

Thus, if \( U_{C(b)} \) are the real points of \( U_{C(b)} \) etc., then \( R_u(\mathbb{P}^-)(\mathbb{R}) = U_{C(b)}U_{Z(b)}U_{E(b)} \).

**Lemma 10.** There is a sequence \( \gamma_n \in R_u(\mathbb{P}^-)(\mathbb{Z}) - 1 \) such that \( d(\gamma_n, U_{C(b)}) \to 0 \).

**Proof.** There is a \( \mathbb{Q} \)-isomorphism of the variety \( R_u(\mathbb{P}^-) \) with affine space that maps \( U_{C(b)} \) onto an affine subspace. Therefore, the problem reduces to showing that the distance between \( \mathbb{Z}^n - 1 \) and a line in \( \mathbb{R}^n \) that passes through the origin is bounded above by any positive number, and this is well known.

\[ \square \]

**Lemma 11.** \( A_0^\infty \cap \Sigma = \emptyset \)

**Proof.** Suppose \( A_0^\infty \cap \Sigma \neq \emptyset \). Then there is some \( b \in A_0 \) such that \( b^\infty \in \Sigma \) where \( b^\infty = \lim_{t \to \infty} b^t \).

If \( \mathcal{C} \subseteq \Sigma \) is a chamber, then \( \Pi_{\mathcal{C}} \subseteq \mathcal{C} \). Hence, the minimal \( \mathbb{R} \)-parabolic subgroup corresponding to \( \mathcal{C} \) contains \( R_u(\mathbb{P}^-) \) and thus elements of \( R_u(\mathbb{P}^-)(\mathbb{R}) \) fix \( \mathcal{C} \) pointwise. That is, elements of \( R_u(\mathbb{P}^-)(\mathbb{R}) \) fix \( \Sigma \) pointwise, so they fix \( b^\infty \).

Let \( u \in R_u(\mathbb{P}^-)(\mathbb{R}) \). Then \( ub^\infty = b^\infty \), so \( d(ub^t, b^t) \) is bounded, so \( \{b^{-t}ub^t\}_{t>0} \) is bounded. It follows that \( \beta(b^{-1}) \leq 1 \) for all \( \beta \in \Psi \), or equivalently that \( \beta(b) \geq 1 \). Hence, \( E(b) = \emptyset \) and \( R_u(\mathbb{P}^-)(\mathbb{R}) = U_{C(b)}U_{Z(b)} \).

Now we use Lemma [10]. For any \( n \in \mathbb{N} \), there exists \( \gamma_n \in R_u(\mathbb{P}^-)(\mathbb{Z}) - 1 \) with \( d(\gamma_n, U_{C(b)}) < 1/n \). Let \( \gamma_n = c_n z_n \) where \( c_n \in U_{C(b)} \), and \( z_n \in U_{Z(b)} \). Notice that \( z_n \to 1 \), \( bz_n = z_n b \), and that \( b^{-t}c_n b^t \to 1 \) as \( t \to \infty \).

Choose \( t_n > 0 \) such that \( d(b^{-t_n}c_n b^{t_n}, 1) < 1/n \). Then

\[
b^{-t_n}\gamma_n b^{t_n} = (b^{-t_n}c_n b^{t_n})z_n \to 1
\]

By Theorem 1.12 of [Ra 2], \( \{b^{-t}\}_{t>0} \) is not contained in any compact subset of \( G(\mathbb{Z}) \backslash G(\mathbb{R}) \), which contradicts that \( b^{-t} \in A_0 \subseteq X_\mathbb{Z} \) (Lemma [9]).

\[ \square \]
3.2. $L > 0$ and choice of cell in $A_0$. At this point, we fix $L > 0$ to be sufficiently large. We will use this fixed $L$ for our proof of the Theorem \[1\]

Let $W_i \subseteq A$ be the kernel of a root $\beta_i \in \Phi^+_R$ such that the visual image of $W_i$ in $A^\infty$ is a great sphere that contains $\Sigma_i$.

We let $F$ be the component of $A_0 - \cup_i a^L_iW_i$ that contains 1.

**Lemma 12.** $F$ is compact Euclidean polygon with volume $O(L^{rkG-1})$.

**Proof.** The visual cone of $\Sigma$ in $A$ based at $a^L_iW_i$ is abelian.

The lemma follows if $\Sigma$ and $a^\infty_+$ are contained in distinct components of $A^\infty - A^\infty_0$, and if $a_-^\infty = \lim_{t \to \infty} a_-^{-t} \in \Sigma$. That is indeed the case: $\alpha(a_+) > 1$ for all $\alpha \in \Phi_\infty(Q - a_0)\cup \Phi_\infty(Q - a_0)\cup \Phi_\infty(Q - a_0)$, and $Z_G(T_{\Delta a - a_0})$ fixes $a_+^\infty$. Hence, $a^\infty_+ \in \Pi^\infty$. The antipodal map on $A^\infty$ stabilizes $A^\infty_0$, transposes $a^\infty_+$ and $a^-\infty_+$, and maps $\Pi^\infty$ onto $\Pi^\infty_\Sigma$.

We denote the face of $F$ given by $a^L_+W_i \cap F$ as $F_i$, so that the topological boundary of $F$ equals $\cup_{i=1}^n F_i$.

4. Other cells in $X_Z$ and their homological boundaries

We denote the real points of the root group $U_{(\beta_i)}$ as $U_i$, and $\langle U_i \rangle_i$ is the group generated by the $U_i$ for $i \in \{1, 2, ..., n\}$.

**Lemma 13.** For each $i \in \{1, 2, ..., n\}$, $U_i \leq U_P$, and thus $\langle U_i \rangle_i \leq U_P$ is abelian.

**Proof.** Since $\beta_i \in \Phi^+_R$, we have $U_i \leq U_Q = U_P U_a$. Either $U_i \leq U_P$ or $U_i \leq U_a \leq Z_G(T_{\Delta a - a_0})$.

Because $Z_G(T_{\Delta a - a_0})$ is contained in both $P$ and $P^-$, the latter case implies that $U_i$ fixes the antipodal cells $\Pi^\infty$ and $\Pi^-\infty$. The fixed point set of $U_i$ is a hemisphere in $A^\infty$ with boundary equal to $W^\infty_i$. Thus, $\Pi^\infty$ and $\Pi^-\infty$ are contained in $W^\infty_i$, which contradicts that $\Sigma_i = \Sigma \cap W^\infty_i$ does not contain $\Pi^-\infty$.

Having ruled out the latter case, $U_i \leq U_P$ and the lemma follows from Lemma \[3\].

4.1. A space for making cycles in $X_Z$.

**Lemma 14.** $\langle U_i \rangle_i F \subseteq X_Z$.

**Proof.** Because $R_u(P)$ is unipotent, $R_u(P)(Z)$ is a cocompact lattice in $U_P$. We choose a compact fundamental domain $D \subseteq U_P$ for the $R_u(P)(Z)$-action.
There is also a compact set $C \subseteq A_0 = A_2$ such that $A(\mathbb{Z})C = A_2 = A_0$. As $DC$ is compact, we may assume that $G(\mathbb{Z})DC \subseteq X_Z$.

Recall that $A$ is contained in $P$, so $A$ normalizes $R_u(P)$. Hence,

\[
\langle U_i \rangle A_0 \subseteq U_PA(\mathbb{Z})C
\]
\[
\subseteq A(\mathbb{Z})U_PC
\]
\[
\subseteq A(\mathbb{Z})R_u(P)(\mathbb{Z})DC
\]
\[
\subseteq G(\mathbb{Z})DC
\]
\[
\subseteq X_Z
\]

\[\square\]

4.2. Description of cells used to build our cycle. Given $i \in \{1, ..., n\}$, let $f_i$ be a point in $F_i$ that minimizes the distance to $1 \in A$, and let $u_i \in U_i$ be such that $d(u_if_i, f_i) = 1$. Since $F_i \subseteq a_L + W_i$, any $f_i \in F_i$ can be expressed as $f = w f_i$ for some $w \in \text{Ker}(\beta_i)$. It follows that $Ad(w)$ acts trivially on the Lie algebra of $U_i$, that $u_i$ commutes with $w$, and that

\[
d(u_i f_i, f_i) = d(u_i w f_i, w f_i) = d(w u_i f_i, w f_i) = d(u_i f_i, f_i) = 1
\]

Setting $u_i = \{u_i^t\}_{t=0}^1$, the space $\overline{u_i}F_i$ is a metric direct product of volume $O(L^{\dim(F_i)})$.

For $I \subseteq \{1, ..., n\}$, let $F_I = \cap_{i \in I} F_i$ with $F_\emptyset = F$. And let $u_I = \prod_{i \in I} u_i$ and $\overline{u_I} = \prod_{i \in I} \overline{u_i}$ with $\overline{u_\emptyset} = u_\emptyset = 1$.

Similar to the case when $|I| = 1$, $\overline{u_I}F_I$ is a metric direct product of volume $O(L^{\dim(F_I)})$.

4.3. Homological boundaries of the cells. We endow each interval $\overline{u_i} = [0, u_i]$ with the standard orientation on the closed interval, and we orient each $\overline{u_I}$ with the product orientation, where the product is taken over ascending order in $\mathbb{N}$. Given $m \in I$, we let $s_I(m)$ be the ordinal of $m$ assigned by the order on $I$ induced by $\mathbb{N}$. Notice that the standard formula for the homological boundary of a cube then becomes

\[
\partial(\overline{u_I}) = \sum_{m \in I} (-1)^{s_I(m)} (\overline{u_{I-m}} - u_m \overline{u_{I-m}})
\]

We assign an orientation to $F$, and then assign the orientation to each $F_i$ such that

\[
\partial(F) = \sum_{i=1}^n F_i
\]

In what follows, if we are given a set $I \subseteq \{1, ..., n\}$ with an ordering (which may differ from the standard order on $\mathbb{N}$), and if $m \in \{1, ..., n\}$ with $m \notin I$, then the set $I \cup m$ is ordered such that the original order on
I is preserved and \( m \) is the “greatest” element of \( I \cup m \). For example, \( \{1, 7, 5\} \cup 3 = \{1, 7, 5, 3\} \).

If \( m \in I \), for some ordered set \( I \subseteq \{1, \ldots, n\} \), then we endow \( I - m \) with the order restricted from \( I \).

For an ordered \( I \) and \( m \in I \), let \( r_I(m) = 1 \) if an even number of transpositions are required to transform the order on \( I \) to the order on \( (I - m) \cup m \). Let \( r_I(m) = -1 \) otherwise.

Given an ordering on a set \( I \subseteq \{1, \ldots, n\} \), an orientation on \( F_I \), and some \( m \in \{1, \ldots, n\} \) with \( m \notin I \), we define the orientation of \( F_{I \cup m} \) to be such that \( F_{I \cup m} \), and not \( -F_{I \cup m} \), is the oriented cell that appears as a summand in \( \partial(F_I) \). Therefore

\[
\partial(F_I) = \sum_{m \notin I} F_{I \cup m}
\]

In what follows, whenever we write the exact symbols \( F_I \) or \( F_I' \) – but not necessarily the symbol \( F_{I \cup m} \) – the order on \( I \) or \( I' \) will be the order from \( \mathbb{N} \). Thus, the orientation on \( F_I \) and \( F_I' \) can be unambiguously determined from the above paragraph.

It’s easy to check that if \( I \) is ordered by the standard order on \( \mathbb{N} \) and \( m \in I \), then \((-1)^{s_I(m)} r_I(m) = (-1)^{|I|} \) and thus

\[
-(-1)^{s_I(m)} = (-1)^{|I|} -1 \cdot r_I(m)
\]

Suppose \( w_0 \) is an outward normal vector for \( F_{I \cup m} \) with respect to \( F_I \), and \( w_1, \ldots, w_k \) is a collection of vectors tangent to \( F_{I \cup m} \) such that \( \{w_0, w_1, \ldots, w_k\} \) defines the orientation for \( F_I \). Then \( \{w_1, \ldots, w_k\} \) defines the orientation for \( F_{I \cup m} \). If \( \{v_1, \ldots, v_{|I|}\} \) is an ordered basis for the tangent space of \( \overline{\mathcal{F}}_I \) that induces the standard orientation on \( \overline{\mathcal{F}}_I \), then \(|I|\) transpositions are required to arrange the ordered basis

\[
\{w_0, v_1, \ldots, v_{|I|}, w_1, \ldots, w_k\}
\]

into the ordered basis

\[
\{v_1, \ldots, v_{|I|}, w_0, w_1, \ldots, w_k\}
\]

That is, the orientation on \( \overline{\mathcal{F}}_IF_{I \cup m} \) defined above is a \((-1)^{|I|}\)-multiple of the orientation on \( \overline{\mathcal{F}}_IF_{I \cup m} \) assigned by \( \partial(\overline{\mathcal{F}}_IF_I) \).

It follows from this fact and our above formulas for \( \partial(\overline{\mathcal{F}}_I) \) and \( \partial(F_I) \) that

\[
\partial(\overline{\mathcal{F}}_IF_I) = \sum_{m \in I} (-1)^{s_I(m)} (\overline{u_{I-m}} - u_m \overline{u_{I-m}}) F_I + (-1)^{|I|} \sum_{m \notin I} \overline{u_I} F_{I \cup m}
\]
5. A cycle in $X_Z$

Let

$$Y = \sum_{\substack{K, I \subseteq \{1, \ldots, n\} \\ K \cap I = \emptyset}} (-1)^{|K|} u_K u_I F_I$$

**Lemma 15.** $Y$ is a cycle that is contained in $X_Z$ and has volume $O(L^{rkG-1})$.

**Proof.** Each cell of $Y$ is contained in $X_Z$ by Lemma 14 and has volume $O(L^k)$ for $k \leq rkG - 1$, so we have to check that $\partial Y = 0$.

From our formula for $\partial(\overline{u_I F_I})$ we have that

$$\partial Y = \sum_{\substack{K, I \subseteq \{1, \ldots, n\} \\ K \cap I = \emptyset}} (-1)^{|K|} u_K \left[ \sum_{m \in I} (-1)^{s_I(m)} (\overline{u_I - m} - u_m \overline{u_I - m}) F_I \right]$$

$$\quad + (-1)^{|I|} \sum_{m \notin I} u_I F_{I \cup m}$$

$$= \sum_{\substack{K, I \subseteq \{1, \ldots, n\} \\ K \cap I = \emptyset}} \sum_{m \in I} (-1)^{s_I(m)} (-1)^{|K|} u_K (\overline{u_I - m} - u_m \overline{u_I - m}) F_I$$

$$\quad + \sum_{\substack{K, I \subseteq \{1, \ldots, n\} \\ K \cap I = \emptyset}} \sum_{m \notin I} (-1)^{|I|} \sum_{m \notin I} (-1)^{|K|} u_K u_I F_{I \cup m}$$

For $K, I \subseteq \{1, \ldots, n\}$ with $K \cap I = \emptyset$ we have

$$\sum_{m \notin I} (-1)^{|K|} u_K u_I F_{I \cup m}$$

$$= \sum_{m \notin I \cup K} (-1)^{|K|} u_K u_I F_{I \cup m}$$

$$\quad + \sum_{m \in K} (-1)^{|K|} u_K u_I F_{I \cup m}$$

$$= \sum_{m \notin I \cup K} (-1)^{|K|} u_K \overline{u_{I \cup m} - m} F_{I \cup m}$$

$$\quad + \sum_{m \in K} (-1)^{|K|} u_K - m u_m \overline{u_{I \cup m} - m} F_{I \cup m}$$

There is a natural bijection between triples $(I, K, m)$ where $K \cap I = \emptyset$ and $m \notin I \cup K$, and triples $(I', K', m)$ where $K' \cap I' = \emptyset$ and $m \in I'$. To realize the bijection, let $K' = K = K - m$ and $I' = I \cup m$. 
There is also a bijection between triples \((I, K, m)\) where \(K \cap I = \emptyset\) and \(m \in K\), and triples \((I', K', m)\) where \(K' \cap I' = \emptyset\) and \(m \in I'\). This bijection is also realized by setting \(K' = K - m\) and \(I' = I \cup m\).

Therefore, if we let \(K' = K - m\) and \(I' = I \cup m\) then the above equation gives

\[
\sum_{K, I \subseteq \{1, \ldots, n\}} (-1)^{|I|} \sum_{m \not\in I} (-1)^{|K|} u_K u_I u_{I \cup m} F_{I \cup m}
\]

\[
= \sum_{K', I' \subseteq \{1, \ldots, n\}, K' \cap I' = \emptyset} (-1)^{|I'|-1} \left[ \sum_{m \in I'} (-1)^{|K'|} r_{I'}(m) u_{K'} u_{I' - m} F_{I'} \right]
\]

\[
+ \sum_{m \in I'} (-1)^{|K' \cup m|} r_{I'}(m) u_{K'} u_{I' - m} F_{I'}
\]

\[
= \sum_{K, I \subseteq \{1, \ldots, n\}, K \cap I = \emptyset} (-1)^{|I|} \left[ \sum_{m \in I} (-1)^{|K|} r_{I}(m) u_{K} u_{I - m} F_{I} \right]
\]

\[
- \sum_{m \in I} (-1)^{|K|} r_{I}(m) u_{K} u_{I - m} F_{I}
\]

\[
= \sum_{K, I \subseteq \{1, \ldots, n\}, K \cap I = \emptyset} (-1)^{|I|} \sum_{m \in I} (-1)^{|K|} r_{I}(m) u_{K} \left( u_{I - m} - u_{m} u_{I - m} \right) F_{I}
\]

\[
= \sum_{K, I \subseteq \{1, \ldots, n\}, K \cap I = \emptyset} \sum_{m \in I} (-1)^{|I| - 1} r_{I}(m) (-1)^{|K|} u_{K} \left( u_{I - m} - u_{m} u_{I - m} \right) F_{I}
\]

\[
= - \sum_{K, I \subseteq \{1, \ldots, n\}, K \cap I = \emptyset} \sum_{m \in I} (-1)^{s_{I}(m)} (-1)^{|K|} u_{K} \left( u_{I - m} - u_{m} u_{I - m} \right) F_{I}
\]

Substituting the preceding equation into our equation for \(\partial Y\) proves

\[
\partial Y = 0
\]

\(\square\)
6. Fillings of $Y$

There exists polynomially efficient fillings for $Y$ in the symmetric space $X$.

**Lemma 16.** There exists a chain $Z$ with volume $O(L^{rkG})$ and $\partial Z = Y$.

*Proof.* As $Y \subseteq \overline{\pi_1 F}$, it follows from Lemma 4 that there is some $T = O(L)$ such that $a_+^T Y$ is contained in an $\varepsilon$-neighborhood of $a_+^T F$, which is isometric to $F$. Thus, there is a filling, $Z_0$, of $a_+^T Y$ of volume $O(L^{rkG-1})$.

Let $Z = Z_0 \cup_{t \in \{1, T\}} a_+^T Y$.

\[ \square \]

6.1. Fillings of $Y$ in $X_Z$. In contrast to Lemma 16, the fillings of $Y$ that are contained in $X_Z$ have volumes bounded below by an exponential in $L$. A fact that we will prove after a couple of helpful lemmas.

For $f \in F$, define $d_i(f)$ to be the distance in the flat $A$ between $f$ and $a_+^L W_i$.

**Lemma 17.** There are $s_i > 1$ and $s_0 > 0$ such that the cube $\overline{\pi_1 f}$ with the path metric is isometric to $\prod_{i \in I} [0, e^{s_i d_i(f) + s_0}]$.

*Proof.* It suffices to prove that $\overline{\pi_1 f}$ is isometric to $[0, e^{s_i d_i(f) + s_0}]$.

Choose $b_i \in A$ such that $d(b_i, 1) = d(f, a_+^L W_i) = d_i(f)$ and such that there exists some $w_i \in W_i$ with $f = b_i a_+^L w_i$. Notice that $W_i$ separates $b_i$ from $a_+^L$ in $A$. Since $U_i \leq U_P$, Lemma 4 shows that $\beta_i(a_+^L) > 1$. It follows that $\beta_i(b_i) < 1$.

With $d_\Omega$ as the path metric of a subspace $\Omega \subseteq X$,

\[ d_{U_i,f}(u_i f, f) = d_{U_i,f}(u_i b_i a_+^L w_i, b_i a_+^L w_i) \]

As $W_i$ is the kernel of $\beta_i$, $w_i$ commutes with $u_i$ implying

\[ d_{U_i,f}(u_i f, f) = d_{w_i^{-1} U_i,f}(u_i b_i a_+^L, b_i a_+^L) \]
\[ = d_{U_i}(a_+^{-L} b_i^{-1} u_i b_i a_+^L, 1) \]

On the Lie algebra of $U_i$, $Ad(a_+^{-L} b_i^{-1})$ scales by $\beta_i(a_+^{-L}) \beta_i(b_i)^{-1}$.

\[ \square \]

In the above lemma we may let $f = 1$ and let $I$ be the singleton $i$. It can easily be seen that $d_i(1) = O(L)$ which leaves us

**Lemma 18.** There is some $C > 0$ such that $d_{U_i}(u_i, 1) \geq e^{CL+s_0}$ for any $i$.

We conclude our proof of Theorem 1 with the following
Lemma 19. Suppose there is a chain $B \subseteq X_Z$ such that $\partial B = Y$. Then the volume of $B$ is bounded below by $e^{C_0 L}$ for some $C_0 > 0$.

Proof. Suppose $B$ has volume $\lambda$. By Lemma 8, $\pi(B) \subseteq U_Q A_0$ has volume $O(\lambda)$.

Recall that $Y \subseteq U_Q A_0$, so $\partial \pi(B) = Y$.

After perturbing $\pi(B)$, we may assume that $\pi(B)$ is transverse to $U_Q$, and that the 1-manifold $\pi(B) \cap U_Q$ has length proportional to the volume of $\pi(B)$. Since

$$\partial(\pi(B) \cap U_Q) = \partial \pi(B) \cap U_Q = Y \cap U_Q = \{u_I\}_{I \subseteq \{1, \ldots, n\}}$$

there is an $I \subseteq \{1, \ldots, n\}$ and a path $\rho : [0, 1] \to \pi(B) \cap U_Q$ such that $\rho(0) = 1$ and $\rho(1) = u_I$ with $\text{length}(\rho) = O(\lambda)$.

Choose $i \in I$. $U_Q$ is nilpotent, so the distortion of the projection $q : U_Q \to U_i$ is at most polynomial. Therefore, $q \circ \rho$ is a path in $U_i$ between 1 and $u_i$ with $\text{length}(q \circ \rho) = O(\lambda^k)$ for some $k \in \mathbb{N}$.

The preceding lemma showed $e^{CL + s_0} \leq \text{length}(q \circ \rho)$. Therefore, $\lambda \geq \kappa e^{\frac{CL + s_0}{N}}$ for some $\kappa > 0$.

□

Combining Lemmas 16 and 19 yields Theorem 1.

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