An Algebraic Construction of Generalized Coherent States for Shape-Invariant Potentials

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Generalized coherent states for shape invariant potentials are constructed using an algebraic approach based on supersymmetric quantum mechanics. We show this generalized formalism is able to: a) supply the essential requirements necessary to establish a connection between classical and quantum formulations of a given system (continuity of labeling, resolution of unity, temporal stability, and action identity); b) reproduce results already known for shape-invariant systems, like harmonic oscillator, double anharmonic, Pöschl-Teller and self-similar potentials and; c) point to a formalism that provides an unified description of the different kind of coherent states for quantum systems.

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I. INTRODUCTION

Coherent states were first introduced by Schrödinger [1], who was interested in finding quantum-mechanical states which provide a close connection between quantum and classical formulations of a given physical system. Based on the Heisenberg-Weyl group and applied specifically to the harmonic oscillator system, the original coherent state introduced by Schrödinger has been extended to a large number of Lie groups with square integrable representations [2,3]. Today these extensions represent many applications in a number of fields of quantum theory, and especially in quantum optics and radiophysics. In particular they are used as bases of coherent state path integrals [4] or dynamical wavepackets for describing the quantum systems in semi-classical approximations [5]. There are different definitions of coherent states. The first one, often called Barut-Girardello coherent states [6], assumes the coherent states are eigenstates with complex eigenvalues of an annihilation group operator. The second definition, often called Perelomov coherent states [7], assumes the existence of an unitary $z$-displacement operator whose action on the ground state of the system gives the coherent state parameterized by $z$, with $z \in \mathbb{C}$. The last definition, based on the Heisenberg uncertainty relation, often called intelligent coherent states [8], assumes that the coherent state gives the minimum-uncertainty value $\Delta x \Delta p = \frac{\hbar}{2}$, and maintains this relation in time because its temporal stability. These three different definitions are equivalent only in the special case of the Heisenberg-Weyl group, the dynamical symmetry group of the harmonic oscillator.

The extension of coherent states for systems other than harmonic oscillator has attracted much attention for the past several years [9–15]. There are a large number of different approaches to this problem and the one to be presented here is based on the supersymmetric quantum mechanics. Supersymmetric quantum mechanics [16] deals with pairs of Hamiltonians which have the same energy spectra, but different eigenstates. A number of such pairs of Hamiltonians share an integrability condition called shape invariance [17]. Although not all exactly-solvable problems are shape-invariant [18], shape invariance, especially in its algebraic formulation [19,20], is a powerful technique to study exactly-solvable systems.

Supersymmetric quantum mechanics is generally studied in the context of one-dimensional systems. The partner Hamiltonians

$$\hat{H}_1 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-(x) = \hbar \Omega \hat{A}^\dagger \hat{A} \quad \text{and} \quad \hat{H}_z = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_+(x) = \hbar \Omega \hat{A} \hat{A}^\dagger$$

are most readily written in terms of one-dimensional operators

$$\hat{A} \equiv \frac{1}{\sqrt{\hbar \Omega}} \left(W(x) + \frac{i}{\sqrt{2m}} \hat{p}\right) \quad \text{and} \quad \hat{A}^\dagger \equiv \frac{1}{\sqrt{\hbar \Omega}} \left(W(x) - \frac{i}{\sqrt{2m}} \hat{p}\right)$$

where $\hbar \Omega$ is a constant energy scale factor, introduced to permit working with dimensionless quantities, and $W(x)$ is the superpotential which is related to the potentials $V_{\pm}(x)$ via

$$V_{\pm}(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx}.$$  \hspace{1cm} (1.3)

In earlier works, by using an algebraic approach, we introduced coherent states for self-similar potentials [13], a class of shape-invariant systems, and presented a possible generalization of these coherent-states and its relation with the Ramanujan’s integrals [14]. In the present paper we extend this generalized formalism to all shape-invariant systems and show that the generalized coherent-states then obtained satisfy the essential requirements necessary to provide the basic principles [21] embodied in Schrödinger’s original idea. This paper is organized as follows. In Section II we present the algebraic formulation to shape invariance and introduce the fundamental principles of our generalized coherent states and its basic properties; in Section III we apply our general formalism to shape-invariant systems classified using the factorization method introduced by Infeld and Hull [22] and work out some possible examples of coherent states for these systems. Finally, brief remarks close the paper in Section IV.

II. GENERALIZED COHERENT STATES FOR SHAPE-INVARIANT SYSTEMS

The Hamiltonian $\hat{H}_1$ of Eq. (1.1) is called shape-invariant if the condition

$$\hat{A}(a_z) \hat{A}^\dagger(a_z) = \hat{A}^\dagger(a_z) \hat{A}(a_z) + R(a_z),$$  \hspace{1cm} (2.1)

is satisfied [17]. In this equation $a_z$ and $a_z$ represent parameters of the Hamiltonian. The parameter $a_z$ is a function of $a_z$ and the remainder $R(a_z)$ is independent of the dynamical variables such as position and momentum. As it is
written the condition of Eq. (2.1) does not require the Hamiltonian to be one-dimensional, and one does not need to choose the ansatz of Eq. (1.2). In the cases studied so far the parameters \(a_i\) and \(a_s\) are either related by a translation [19,23] or a scaling [13,14,24]. Introducing the similarity transformation that replace \(a_i\) with \(a_s\) in a given operator \(\hat{T}(a_i)\hat{O}(a_s)\hat{T}^\dagger(a_i) = \hat{O}(a_s)\) and the operators \(\hat{B}_+ = \hat{A}^\dagger(a_i)\hat{T}(a_i)\) and \(\hat{B}_- = \hat{T}^\dagger(a_s)\hat{A}(a_s)\), the Hamiltonians of Eq. (1.1) take the forms \(\hat{H} = \hbar\Omega \hat{B}_+\hat{B}_-\) and \(\hat{H}_s = \hbar\Omega \hat{T}^\dagger\hat{B}_-\hat{B}_+\hat{T}\). As shown in [19], with Eq. (2.1) one can also easily prove the commutation relation \([\hat{B}_-,\hat{B}_+] = \hat{T}^\dagger(a_s)\hat{R}(a_s)\hat{T}(a_i)\hat{R}(a_i)\hat{T}\), where we used the identity \(\hat{R}(a_n) = \hat{T}(a_s)\hat{R}(a_{n-1})\hat{T}^\dagger(a_s)\), valid for any \(n \in \mathbb{Z}\). This commutation relation suggests that \(\hat{B}_-\) and \(\hat{B}_+\) are the appropriate creation and annihilation operators for the spectra of the shape-invariant potentials provided that their non-commutativity with \(\hat{R}(a_s)\) is taken into account. The additional relations
\[
\hat{R}(a_n)\hat{B}_+ = \hat{B}_+\hat{R}(a_{n-1}) \quad \text{and} \quad \hat{R}(a_n)\hat{B}_- = \hat{B}_-\hat{R}(a_{n+1}),
\]
readily follow from these results. Considering that the ground state of the Hamiltonian \(\hat{H}\) satisfies the condition
\[
\hat{A}|\Psi_o\rangle = 0 = \hat{B}_-|\Psi_o\rangle,
\]
using the relations above it is possible to find the \(n\)-th excited state of \(\hat{H}\)
\[
\hat{H}|\Psi_n\rangle \equiv \hbar\Omega \left(\hat{B}_+\hat{B}_-\right)|\Psi_n\rangle = \hbar\Omega e_n|\Psi_n\rangle \quad \text{and} \quad \hat{B}_-\hat{B}_+|\Psi_n\rangle = \{e_n + \hat{R}(a_s)\}|\Psi_n\rangle.
\]
where these eigenstates can be written in a normalized form as
\[
|\Psi_n\rangle = \frac{1}{\sqrt{\hat{R}(a_1) + \hat{R}(a_2) + \cdots + \hat{R}(a_n)}} \hat{B}_+ \cdots \frac{1}{\sqrt{\hat{R}(a_1) + \hat{R}(a_2) + \cdots + \hat{R}(a_n)}} \hat{B}_+ |\Psi_o\rangle
\]
with the eigenvalues \(E_n = \hbar\Omega e_n\), being
\[
e_n = \sum_{k=1}^{n} \hat{R}(a_k).
\]

As mentioned in the introduction, a possible way to define a coherent state is to find a quantum state annihilated by the lowering operator. Annihilation-operator coherent states for shape-invariant potentials were introduced in [10,13]. Here we follow the notation of [13]. Our first step is to introduce the necessary tools to be used in this construction. After we obtain the coherent-state we must verify if this state satisfies the set of four essential requirements, introduced and discussed in [21], necessary for a close connection between classical and quantum formulations of a given system: a) label continuity; b) overcompleteness or resolution of unity; c) temporal stability and; d) action identity. Indeed the first two requirements are standard and rely on the algebraic structure behind the system in question, while the last two are more general and relate to the classical-quantum connection question.

### A. Construction

To remove the energy scale we rewrite the shape-invariant Hamiltonian as
\[
\hat{H} = \hbar\Omega \hat{\mathcal{H}}, \quad \text{with} \quad \hat{\mathcal{H}} = \hat{B}_+\hat{B}_-\cdot
\]
The operator \(\hat{B}_-\) does not have a left inverse in the Hilbert space of the eigenstates of the Hamiltonian \(\hat{H}\). However, a right inverse for \(\hat{B}_-\) (\(\hat{B}_-\hat{B}_-^{-1} = 1\)), can be defined. Similarly the inverse of \(\hat{\mathcal{H}}\) does not exist, but \(\hat{\mathcal{H}}^{-1}\hat{B}_+ = \hat{B}_-^{-1}\) does. Therefore, if we define the Hermitian conjugate operators \(\hat{Q} = \hat{B}_-\hat{\mathcal{H}}^{-1/2}\), and \(\hat{Q}^\dagger = \hat{\mathcal{H}}^{-1/2}\hat{B}_+\) we can easily show that \(\hat{B}_-^{-1} = \hat{\mathcal{H}}^{-1/2}\hat{B}_+\) and the normalized form of the \(n\)-th excited state of \(\hat{\mathcal{H}}\), given by (2.5), can be rewritten as \(|\Psi_n\rangle = (\hat{Q}^\dagger)^n|\Psi_o\rangle\). Then, taking into account Eqs. (2.4), (2.6) and these two last relations, we can prove that
\[
\hat{B}_-^{-n}|\Psi_o\rangle = C_n|\Psi_n\rangle, \quad \text{where} \quad C_n = \left\{\prod_{k=0}^{n-1} (e_n - e_k)\right\}^{-1/2} = \left\{\prod_{k=1}^{n} \left[\sum_{s=k}^{n} \hat{R}(a_s)\right]\right\}^{-1/2},
\]

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since \( e_0 = 0 \). After these preliminary considerations we are ready to define our generalized expression for the coherent state of shape-invariant systems as

\[
|z; a_j\rangle = \sum_{n=0}^{\infty} \left\{ z Z_j \hat{B}^{-1}_j \right\}^n |\Psi_0\rangle, \quad z, Z_j \in \mathbb{C},
\]  

(2.9)

where we used the shorthand notation \( Z_j = Z(a_j) = Z(a_1, a_2, a_3, \ldots) \) for an arbitrary functional of the potential parameters, introduced to establish a more general approach. As one will see in the applications below, for harmonic oscillator system, the presence of the functional \( Z_j \) introduces only a constant scale factor in the complex expansion variable \( z \) that can be absorbed by a redefinition of this constant and, thus, we get back to the standard results for this system. Formally the definition (2.9) can be expressed as

\[
|z; a_j\rangle = \left[ \frac{1}{1 - z Z_j \hat{B}^{-1}_j} \right] |\Psi_0\rangle.
\]  

(2.10)

Using the relation (2.2) we can prove this coherent state is eigenstate of the operator \( \hat{B}_- \) since

\[
\hat{B}_- |z; a_j\rangle = z \hat{Z}_{j-1} |z; a_j\rangle.
\]  

(2.11)

This state also satisfies the additional condition

\[
\left\{ \hat{B}_- - z \hat{Z}_{j-1} \right\} \frac{\partial}{\partial z} |z; a_j\rangle = Z_{j-1} |z; a_j\rangle,
\]  

(2.12)

where \( Z_{j-1} = \hat{T}^\dagger(a_j) Z_j \hat{T}(a_j) \). An important observation is that the coherent state definition (2.9) satisfies the continuity of labeling requirements since the transformation of the variables \( (z, a_j) \rightarrow (z', a_j') \) leads to the transformation of the states \( |z; a_j\rangle \rightarrow |z'; a_j'\rangle \). This is the first standard property required for coherent states. The other three we take up in the next subsections. Together with the resolution of unity, the continuity of labeling represents the minimal condition to be satisfied for a set of coherent states to be represented by a Lie algebraic group.

### B. Normalization

At this stage we can use the action of the \( \hat{B}^{-1} \) operator on the Hilbert space of the eigenstates \( \{|\Psi_n\rangle, n = 0, 1, 2, \ldots\} \), and (2.2) to get the generalized Glauber’s form [25] of the coherent state \( |z; a_j\rangle \) based in its expansion in the eigenstates of the Hamiltonian \( \hat{H} \):

\[
|z; a_r\rangle = \mathcal{N} (|z|^2; a_r) \sum_{n=0}^{\infty} \frac{z^n}{h_n(a_r)} |\Psi_n\rangle,
\]  

(2.13)

where we used the shorthand notation \( (a_r) \equiv \{ R(a_1), R(a_2), \ldots, R(a_n); a_j, a_{j+1}, \ldots, a_{j+n-1} \} \) for the expansion coefficients, which are given by \( h_n(a_r) = 1 \) and

\[
h_n(a_r) = \frac{1}{\sqrt{\prod_{k=1}^{n} \sum_{s=k}^{n} R(a_s) / \prod_{k=0}^{n-1} Z_{j+k}}}, \quad \text{for} \quad n \geq 1
\]  

(2.14)

with \( Z_{j+k} = \{ \hat{T}^\dagger(a_j) \}^k Z_j \{ \hat{T}^\dagger(a_i) \}^k \), as well for the real normalization factor

\[
\mathcal{N} (x; a_r) = 1 / \sqrt{\sum_{n=0}^{\infty} |x^n h_n(a_r)|^2}.
\]  

(2.15)

At this point we observe that the transformation properties between the potential parameters \( a_n \), imposed by shape invariance, constrains the freedom to define \( Z_j \). Besides that, when we consider the relation (2.14), this potential parameter dependence in \( Z_j \) shows strong influence in the final expression of the expansion coefficient \( h_n(a_r) \). Another thing to observe about \( Z_j \) is its importance in the determination of the radius of convergence in the series defining \( \mathcal{N} (|z|^2; a_r) \) since this radius is given by \( \mathcal{R} = \limsup_{n \rightarrow +\infty} \sqrt{|h_n(a_r)|^2} \).
It should be noted that this normalized coherent state has an \( \hat{B}_- \) operator eigenvalue different from the unnormalized one since the potential parameters in the normalization factor are changed by the action of that operator. Indeed we can prove that in this case Eq. \( \text{(2.11)} \) must assume the form

\[
\hat{B}_- | z; a_r \rangle = z Z_{\gamma-1} \left[ \frac{N(a_{r-1}; |z|^2)}{N(a_r; |z|^2)} \right] | z; a_r \rangle ,
\]

where \( N(a_{r-1}; |z|^2) = \hat{T}^\dagger(a_r) N(a_r; |z|^2) \hat{T}(a_r). \) Although they are normalized, the coherent states \( | z; a_r \rangle \) are not orthogonal to each other since

\[
\langle z'; a_r | z; a_r \rangle = \frac{N(a_r; |z'|^2) N(a_r; |z|^2)}{N^2(a_r; z z^*)} .
\]

So we conclude that they form an over-complete linearly dependent set.

C. Overcompleteness

Now we can investigate the overcompleteness or resolution of unity property of the generalized coherent states introduced by equation \( \text{(2.9)} \). To this end we assume the existence of a positive-definite weight function \( w(|z|^2; a_r) \) so that an integral over the complex plane exists and gives the result

\[
\int_\mathbb{C} d^2 z |z; a_r \rangle \langle z; a_r| w(|z|^2; a_r) = \hat{1}_N ,
\]

where \( \hat{1}_N \) is the identity operator in the Hilbert space of the \( \hat{H} \)-eigenstates. Inserting Eq. \( \text{(2.13)} \) into Eq. \( \text{(2.18)} \) the resolution of unity can be expressed by

\[
\int_\mathbb{C} d^2 z N^2(|z|^2; a_r) \sum_{m,n=0}^\infty \frac{z^m z^n}{h_m(a_r) h_n(a_r)} | \Psi_m \rangle \langle \Psi_n | w(|z|^2; a_r) = \hat{1}_N .
\]

At this point we can use the orthonormality of the eigenstates \( | \Psi_n \rangle \) to show that the diagonal matrix elements of Eq. \( \text{(2.19)} \) can be written as

\[
\int_\mathbb{C} d^2 z N^2(|z|^2; a_r) (z^* z)^n w(|z|^2; a_r) = | h_n(a_r) |^2 .
\]

Therefore, assuming the polar coordinate representation \( z = r e^{i\phi} \) of complex numbers we must have \( d^2 z = r dr d\phi \) and using the result

\[
\frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i(n-m)\phi} = \delta_{n,m}
\]

we conclude that to get a resolution of unity we must require

\[
\int_0^\infty d\rho \rho^n W(\rho; a_r) = | h_n(a_r) |^2 , \quad \text{where} \quad W(\rho; a_r) = \pi N^2(\rho; a_r) \ w(\rho; a_r) ,
\]

and \( \rho \) stands for \( r^2 \). In other words, equation \( \text{(2.22)} \) provides the set of moments \( \{ \rho_n \} \) of the distribution function \( W(\rho; a_r) \), since we assume all moments exist and have finite values. Therefore, as pointed in Ref. [26], the problem of finding a suitable measure \( w(\rho; a_r) \) reduces to a moment distribution problem. After this point there are several possible ways to get the measure \( w(\rho; a_r) \). We can choose a possible form of \( w(\rho; a_r) \) by using the result of a known integral. Another possibility is to use a transformation procedure, like Mellin [27] or Fourier, to determine the form of the measure \( w(\rho; a_r) \). For example, in the Fourier transformation case we can multiply Eq. \( \text{(2.22)} \) by the sum factor \( \sum_{n=0}^\infty (i\xi)^n/n! \) and use the series expansion of the exponential function to obtain

\[
\int_0^\infty d\rho W(\rho; a_r) e^{i\rho \xi} = \Phi(\xi; a_r) = \sum_{n=0}^\infty | h_n(a_r) |^2 (i\xi)^n/n! .
\]
Thus, taking the inverse Fourier transformation of Eq. (2.23) we can show that

\[ \mathcal{W}(\rho; a_r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, \Phi(\xi; a_r) e^{-i\rho\xi}. \]  

(2.24)

In the applications of the next section we will use several different procedures to get the resolution of unity. To conclude this part, we note that explicit computation of the weight function \( w(|z|^2; a_r) \) requires the knowledge of the spectrum of the quantum mechanical system under consideration and the form of the functional \( Z_j \).

### D. Temporal Stability

Let us now investigate the dynamical evolution of the generalized coherent state (2.13). To do that we must remember that the time evolution of this generalized coherent state can be obtained by

\[ |z; a_r\rangle_t = \hat{U}(t, 0)|z; a_r\rangle, \]  

(2.25)

where the time evolution operator fulfills the differential equation

\[ i\hbar \frac{\partial \hat{U}(t, 0)}{\partial t} = \hat{H} \hat{U}(t, 0), \]  

(2.26)

with the initial condition \( \hat{U}(0, 0) = \hat{1}_n \). Thus,

\[ |z; a_r\rangle_t = \exp \left( -i\hat{H}t/\hbar \right) |z; a_r\rangle. \]  

(2.27)

At this point if we consider the expansion (2.13), the results of Eqs. (2.4) and the commutation between any function of the potential parameters \( a \) and the Hamiltonian \( \hat{H} \) in the equation (2.27) we obtain

\[ |z; a_r\rangle_t = \mathcal{N} \left( |z|^2; a_r \right) \sum_{n=0}^{\infty} \frac{z^n}{h_n(a_r)} e^{-i\Omega n t} |\Psi_n\rangle. \]  

(2.28)

To establish the temporal stability of this coherent state we utilize the freedom in the choice of the functional \( Z(a_j) \) to redefine it as \( \tilde{Z}(a_j) = Z(a_j) e^{-i\alpha R(a_j)} \) where \( \alpha \) is a real constant. This redefinition implies \( \tilde{h}_n(a_r) = h_n(a_r) e^{i\alpha e_n} \), where \( e_n \) is given by (2.6) and \( h_n(a_r) \) still given by equation (2.14). Therefore we can write the coherent state \( |z; a_r\rangle \) as

\[ |z; a_r\rangle \implies |z, \alpha; a_r\rangle = \mathcal{N} \left( |z|^2; a_r \right) \sum_{n=0}^{\infty} \frac{z^n}{h_n(a_r)} e^{-i\alpha e_n} |\Psi_n\rangle, \]  

(2.29)

and its time-evolved form as

\[ |z, \alpha; a_r\rangle_t = \mathcal{N} \left( |z|^2; a_r \right) \sum_{n=0}^{\infty} \frac{z^n}{h_n(a_r)} e^{-i(\alpha + \Omega t)e_n} |\Psi_n\rangle \equiv |z, \alpha + \Omega t; a_r\rangle, \]  

(2.30)

illustrating the fact that the time evolution of any such generalized coherent state remains within the family of generalized coherent states. In other words, the generalized coherent states \( |z, \alpha; a_r\rangle \) show temporal stability under \( \hat{H} \). To conclude this part, note that the polar coordinates representation of the redefined complex functional \( \tilde{Z}(a_j) \) imply that in the coherent state time evolution its real modulus remains constant while its complex phase increases linearly. These properties are similar to the classical behaviour of canonical action-angle variables.

### E. Action Identity

The last property to be satisfied for the coherent state \( |z; a_r\rangle \) is the action identity. To verify this identity we take the conjugate of Eq. (2.11) and use the definition of the operator \( \hat{B}_+ \) to get

\[ \langle z; a_r | \hat{B}_+ = \langle z; a_r | z^* Z_{j-1} \rangle. \]  

(2.31)
Now with this result, Eq. (2.11) and the expression of the Hamiltonian $\hat{H}$ we can calculate the expectation value

$$
\langle \hat{H} \rangle = \frac{\langle z; a_r | \hat{H} | z; a_r \rangle}{\langle z; a_r | z; a_r \rangle} = \hbar \Omega \frac{\langle z; a_r | \hat{B}_x \hat{B}_z | z; a_r \rangle}{\langle z; a_r | z; a_r \rangle} = \hbar \Omega | z; Z_{j-1} \rangle^2.
$$

(2.32)

Using this result we can define a canonical action variable $J = \hbar \beta_j^* \beta_j$, with $\beta_j = z Z_{j-1}$, such that $\langle \hat{H} \rangle = \nu J$, so that $\nu = \partial \langle \hat{H} \rangle / \partial J = \Omega \Rightarrow \nu = \Omega \hbar + \alpha$, as required for a couple of canonical conjugate action-angle variables. Note that the normalized form (2.13) of the coherent state $|z; a_r \rangle$ requests the definition $\beta_j = z Z_{j-1} N (|z|^2; a_{r-1}) / N (|z|^2; a_r)$.

With these properties we showed that the generalized coherent state $|z; a_r \rangle$ satisfies the set of basic requirements we enumerated.

**III. SOME EXAMPLES OF GENERALIZED COHERENT STATE SYSTEMS**

Using the definition presented in the previous section we now illustrate the concept of generalized coherent states for shape invariant systems using some known shape invariant potential systems. As in reference [10], for these applications we follow the classification based on the factorization method introduced by Infeld and Hull [22] in which six possible types of shape-invariant systems are grouped when its potential parameters are related by a translation. We also study the case of self-similar potential system as an example of shape invariant potential with potential parameters related by scaling.

**A. Types (C) and (D) shape-invariant systems**

We begin with these systems because they are the simplest cases among the shape-invariant potential systems. The partner potentials $V_\pm (x)$ for these systems are obtained with the superpotentials

$$
W_c (x, a_1) = \sqrt{\hbar \Omega} \left( \frac{a_1 + \delta}{x} + \frac{\beta}{2} x \right), \quad \text{and} \quad W_d (x, a_1) = \sqrt{\hbar \Omega} (\beta x + \delta),
$$

(3.1)

where $\beta$ and $\delta$ are real constants, while the remainders in the shape-invariant condition (2.1) are given by [16]

$$
R_c (a_n) = \beta \left( a_n - a_{n+1} + \sqrt{\hbar / 2m \Omega} \right), \quad \text{and} \quad R_d (a_n) = \sqrt{\hbar / 2m \Omega} (a_n + a_{n+1}).
$$

(3.2)

Taking into account that the parameters for these potentials are related by

$$
\begin{cases}
a_{n+1} = a_n - \sqrt{\hbar / (2m \Omega)}, & \text{for (C)}, \\
a_1 = a_2 = \ldots = a_n = \beta, & \text{(C)}, \\
(\forall n \in \mathbb{Z}),
\end{cases}
$$

(3.3)

we conclude that for both shape-invariant systems the remainders (3.2) can be written as $R(a_n) = \gamma$, with $\gamma = \sqrt{2 \hbar / (m \Omega)} \beta$, and thus

$$
\prod_{k=1}^{n} \left[ \sum_{s=k}^{n} R(a_s) \right] = \prod_{k=1}^{n} [\gamma (n-k+1)] = \gamma^n n!.
$$

(3.4)

On the other hand, the constant values of the potential parameters for (D) shape-invariant potential imply that for these systems we must have $Z_j = c$, a constant. Using this and Eq. (2.14) we obtain

$$
\prod_{k=0}^{n-1} Z_{j+k} = c^n \quad \Rightarrow \quad h_n (a_r) = \frac{\sqrt{\gamma^n n!}}{c^n}.
$$

(3.5)

Taking into account it in Eqs. (2.15) and (2.13) we find

$$
N (|z|^2; a_r) = \exp \left( -\frac{|z|^2 |z|^2}{2 \gamma} \right), \quad \text{and} \quad |z; a_r \rangle = e^{-\langle |z|/\sqrt{2 \gamma} \rangle^2} \sum_{n=0}^{\infty} \frac{(c/\sqrt{\gamma})^n}{\sqrt{n!}} |\Psi_n\rangle.
$$

(3.6)
With these results we can show that the inner product (2.17) of two coherent states can be readily found as

$$
\langle z'; a_r|z; a_r \rangle = \exp \left[ - \frac{c^2}{2\gamma} \left( \left| z' \right|^2 + \left| z \right|^2 - 2zz' \right) \right].
$$

(3.7)

The overcompleteness property can be verified by using Eq. (2.23). The function

$$
\Phi(\xi; a_r) = \sum_{n=0}^{\infty} \left( \frac{i\gamma\xi}{c^2} \right)^n = \frac{1}{1 - i\gamma\xi/c^2},
$$

(3.8)

that has a pole at $\xi = -ic^2/\gamma$ and its integration in Eq. (2.24) by using the lower-half complex plane enclosing this pole yields

$$
\mathcal{W}(\rho; a_r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \frac{e^{-i\gamma\xi}}{1 - i\gamma\xi/c^2} = e^{-\gamma\rho/\gamma}.
$$

(3.9)

Now, taking into account the result for $\mathcal{N}(\rho; a_r)$ and the relation between the function $\mathcal{W}(\rho; a_r)$ and the weight function, it is possible to show that $w(\rho; a_r) = 1/\pi$. The example of a (D) shape-invariant system is the harmonic oscillator system does not permit any special modification in the standard result with the definition of the generalized coherent state (2.13).

For (C) shape-invariant systems, if we make the choice $Z_j = c$, a constant, and following the steps above it is possible to obtain identical results as (D) shape-invariant systems for the coherent state. However, any other choice would imply different results. Just as an example, let us define the following auxiliary function

$$
g(a_j; c, d) = ca_j + d,
$$

(3.11)

where $c$ and $d$ are constants. With the help of Eq. (3.3) we can show that

$$
\prod_{k=0}^{n-1} g(a_{j+k}; c, d) = \frac{(-c\eta)^n \Gamma[n + j - \rho - d/(c\eta) - 1]}{\Gamma[j - \rho - d/(c\eta) - 1]} = \frac{(c\eta)^n \Gamma[\rho + d/(c\eta) - j + 2]}{\Gamma[\rho + d/(c\eta) - n - j + 2]},
$$

(3.12)

where $\eta = \sqrt{\hbar/(2m\Omega)}$ and $\rho = a_j/\eta$. Taking into account this result and defining the functional $Z_j$ as

$$
Z_j = \sqrt{g(a_j; -\gamma/\eta, 1)} e^{-i\alpha R(a_j)}
$$

(3.13)

we get

$$
\prod_{k=0}^{n-1} Z_{j+k} = \sqrt{\frac{\gamma^n \Gamma(n - \rho)}{\Gamma(-\rho)}} e^{-i\alpha e^n},
$$

(3.14)

where we used that $e_n = n\gamma$. Substituting Eqs. (3.4) and (3.14) in (2.14) we obtain

$$
h_n(a_r) = \sqrt{\frac{\Gamma(-\rho) \Gamma(n + 1)}{\Gamma(n - \rho)}} e^{i\alpha e^n},
$$

(3.15)

and we can show that the normalization factor (2.15) in this case is given by

$$
\mathcal{N}(\left|z\right|^2; a_r) = \left[ \frac{1}{\Gamma(-\rho)} \sum_{n=0}^{\infty} \frac{\Gamma(n - \rho)}{\Gamma(n + 1)} \left|z\right|^{2n} \right]^{-1/2} = (1 - \left|z\right|^2)^{-\rho/2},
$$

(3.16)
with the restriction $|z| < 1$. Thus, the coherent state (2.13) obtained with these results is

$$|z; a_r\rangle = (1 - |z|^2)^{-\rho} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n - \rho)}{\Gamma(-\rho) \Gamma(n + 1)}} e^{-i \alpha \gamma n} z^n |\Psi_n\rangle,$$

(3.17)

where we take $a_i < 0$ implying $\rho < 0$. As it is always possible to get $a_i + \delta > 0$ with an adequate choice of $\delta$, there are no problems with this assumption. In this case the inner product (2.17) of two coherent states will be

$$\langle z'; a_r | z; a_r \rangle = \left[ \frac{\sqrt{(1 - |z|^2)} (1 - |z'|^2)}{(1 - z'^*z)} \right]^{-\rho}.$$

(3.18)

The completeness can be obtained by using the measure $w(|z|^2; a_r) = -(\rho + 1) (1 - |z|^2)^{-2}/\pi$, that is invariant on the disk $|z| < 1$. Example of a shape invariant type (C) system [28] is the double anharmonic potential $V_-(x, a_i)$, obtained with Eq. (1.3) and using the superpotential $W(x, a_i)$.

The coherent state we obtained for the (C)-type shape-invariant system, Eq. (3.17), is the Perelomov coherent state [7] for the group SU(1,1). This is not surprising since the SU(1,1) algebra is both the shape-invariance and spectrum-generating algebra of this shape-invariant system. The appropriate realization of this algebra is

$$\hat{K}_o = \frac{1}{4} \left( \hat{p}^2 + x^2 + \frac{\alpha}{x^2} \right) \quad \text{and} \quad \hat{K}_\pm = \frac{1}{4} \left( \hat{p}^2 - x^2 + \frac{\alpha}{x^2} \right) \pm \frac{i}{4} (\hat{p} x + x \hat{p}).$$

(3.19)

For the (C)-type shape-invariant systems the shape-invariance [19] connects eigenstates of the same system.

**B. Types (A) and (B) shape-invariant systems**

The partner potentials $V_{\pm}(x)$ for these systems are obtained with the superpotentials

$$W_A(x, a_i) = \sqrt{\hbar \Omega} \{ \beta(a_i + \gamma) \cot [\beta(x + \lambda)] + \delta \csc [\beta(x + \lambda)] \}$$

(3.20)

$$W_B(x, a_i) = \sqrt{\hbar \Omega} \{ \beta(a_i + \gamma) + \delta \exp (-\beta x) \},$$

(3.21)

being $\beta$, $\gamma$, $\delta$ and $\lambda$ real constants. For these systems the remainders in the shape-invariant condition (2.1) are given by $R(a_i) = \pm \beta^2 \eta \{ 2(a_i + \gamma) \pm \eta \}$, with the potential parameters related by $a_{n+1} = a_n \pm \eta$, where $\eta = \sqrt{\hbar/(2m\Omega)}$ and the signs $(\pm)$ and $(\mp)$ stand for (A) and (B) types, respectively. Using these results we can prove that for (A) type systems one has

$$\prod_{k=1}^{n} \left[ \sum_{s=k}^{n} R(a_s) \right] = \frac{\kappa^{2n} \Gamma(n + 1) \Gamma(2\rho + 2n)}{\Gamma(2\rho + n)},$$

(3.22)

with $\kappa = \eta \beta$ and $\rho = (a_i + \gamma)/\eta$. To investigate the consequences of our general approach for this type of systems let us consider some possibilities. First, if we make the choice $Z_j = c$, a constant, and use the result of Eqs. (3.5) and (3.22) we find

$$h_n(a_r) = \sqrt{\frac{\Gamma(n + 1) \Gamma(2\rho + 2n)}{\Gamma(2\rho + n)}}$$

(3.23)

and

$$\mathcal{N}(|z|^2; a_r) = \left[ \sum_{n=0}^{\infty} \frac{\Gamma(2\rho + n)}{\Gamma(n + 1) \Gamma(2\rho + 2n)} |z|^{2n} \right]^{-1/2}$$

(3.24)

for the expansion coefficient and the normalization factor, respectively, after assuming $c = \kappa$. At this point, if we choose $\rho = 1/2$ we get the simple expression found in [10] for the coherent state (2.13) because in this case $h_n(a_r) = \sqrt{(2n)!}$, and since by Eq. (2.15)
\[ \mathcal{N}(|z|^2; a_r)^{-1} = \sqrt{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!}} = \frac{1}{\sqrt{\text{sech}(|z|)}}, \quad \text{and} \quad |z; a_r⟩ = \sqrt{\text{sech}(|z|)} \sum_{n=0}^{\infty} \frac{|a|^n}{(2n)!} |\Psi_n⟩. \] (3.25)

As shown in [10], in this case the identity resolution is obtained with the measure \( w(|z|^2; a_r) = e^{-|z|/2(|z|)} \).

Another interesting possibility is to use the auxiliary function (3.11). In this case, because of the translation relation between the \( a_n \) potential parameters, we can prove that for type (A) systems one gets

\[ \prod_{k=0}^{n-1} g(a_{j+k}; c, d) = \frac{(c/\eta)^n \Gamma \left[ \frac{\nu}{2} + j + n + d/(c\eta) - 1 \right]}{\Gamma \left[ \frac{\nu}{2} + j + d/(c\eta) - 1 \right]}, \] (3.26)

where \( \nu = 2a_\nu/\eta \). If we define the functional \( Z_j = \sqrt{g(a_1; 2\kappa/\eta, \kappa)} g(a_1; 2\kappa/\eta, 2\kappa) e^{-\alpha R(a_1)} \), we obtain

\[ \prod_{k=0}^{n-1} Z_{j+k} = \sqrt{(2\kappa)^{2n} \Gamma \left( \frac{\nu}{2} + n + 1 \right) \Gamma \left( \frac{\nu}{2} + n + \frac{1}{2} \right)} e^{-\alpha e_n} = \sqrt{\kappa^{2n} \Gamma \left( \nu + 2n + 1 \right)} e^{-\alpha e_n}, \] (3.27)

where \( e_n = \kappa^2 n(n + 2\rho) \). Assuming \( \gamma = \eta/2 \) and using Eqs. (3.22) and (3.27) in (2.14) we obtain

\[ h_n(a_r) = \sqrt{\frac{\Gamma(\nu+1)\Gamma(n+1)}{\Gamma(\nu+n+1)}} e^{i\alpha e_n}, \] (3.28)

and we can show that the normalization factor (2.15) in this case is given by

\[ \mathcal{N}(|z|^2; a_r) = \left[ \frac{1}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n+1)}{\Gamma(n+1)} |z|^{2n} \right]^{-1/2} = (1 - |z|^2)^{\nu+1/2}, \] (3.29)

with the restriction \( |z| < 1 \). The coherent state (2.13) obtained with these results is

\[ |z; a_r⟩ = (1 - |z|^2)^{\nu+1/2} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(\nu+n+1)}{\Gamma(\nu+1)\Gamma(n+1)}} e^{-\alpha e_n} z^n |\Psi_n⟩. \] (3.30)

Comparing with [29] one notes that Eq. (3.30) is a form for the coherent state of the Pöschl-Teller potential of first type [30]. This potential \( V(x, a_r) \) obtained with Eq. (1.3) and using the superpotential \( W_\lambda(x, a_r) \) is the example of a shape-invariant system type (A). As shown in [29], in this case, the resolution of unity is obtained with the measure \( w(|z|^2; a_r) = \nu (1 - |z|^2)^{-2} / \pi \).

Finally, note that if one takes

\[ Z_j = \sqrt{g(a_1; 2\eta/\kappa, 1) g(a_1; 2\eta/\kappa, 2 \kappa) \Gamma \left( \frac{\nu}{2} + 1 \right)} e^{-\alpha R(a_1)}, \] (3.31)

and follows the same way used before one get a second possible form for the coherent state of the Pöschl-Teller potential [29,31]

\[ |z; a_r⟩ = \frac{|z|^{\nu/2}}{\sqrt{I_\nu(2|z|)}} \sum_{n=0}^{\infty} \frac{e^{-\alpha e_n} z^n}{\sqrt{\Gamma(n+1)\Gamma(\nu+n+1)}} |\Psi_n⟩, \] (3.32)

where \( J_\nu(x) \) is the modified Bessel function of the first kind. As shown in Ref. [29] and [31], in this case the resolution of unity is given by the measure

\[ w(|z|^2; a_r) = \frac{2}{\pi} K_\nu(2|z|^2), \] with \( K_\nu(x) = \frac{\pi [I_\nu(x) - I_\nu(x)]}{2 \sin(\pi \nu)}, \nu \notin \mathbb{Z}. \] (3.33)

Note that the coherent state in Eq. (3.32) is the Barut-Girardello coherent state for the SU(1,1) algebra [6]. This is not surprising since SU(1,1) is the shape-invariance algebra for the Pöschl-Teller potential as shown in Ref. [19]. Note that in this case the shape-invariant potential relates a series of potentials with different depths, not the quantum states of the given potential, i.e. the shape-invariance algebra is not the spectrum-generating algebra in contrast to
the (C) and (D) type shape-invariant systems. Hence the coherent state corresponds to a non-compact algebra with
infinite number of states representing all possible potentials with different depths.

As a last example we obtain a new coherent state for this kind of systems with the introduction of the functional

\[ Z_j = \sqrt{\frac{\sigma, \mu (a_1; \beta, \beta \gamma) g (a_1; \beta, \beta \gamma + \kappa / 2) g (a_1; \beta, \beta \gamma + \kappa / 2)}{g (a_1; \beta, \beta \gamma + \kappa / 2) g (a_1; \beta, \beta \gamma + \kappa / 2)}} e^{-i \alpha R(a_1)}, \]  

(3.34)

which leads to

\[ \prod_{k=0}^{n-1} Z_{j+k} = \sqrt{\frac{\kappa^2 n \Gamma(2n + 2 \rho) \Gamma(n + 2 - \sigma)}{\Gamma(2 - \sigma) \Gamma(n + 2)}} e^{-i \alpha e_n}. \]  

(3.35)

Therefore using Eqs. (3.22) and (3.35) in (2.14) we obtain

\[ h_n (a_r) = \sqrt{\frac{\Gamma(2 - \sigma) \Gamma(n + 1) \Gamma(n + 2)}{\Gamma(n + 2 - \sigma)}} e^{i \alpha e_n}, \]  

(3.36)

and we can show that the normalization factor (2.15) in this case is given by

\[ N (|z|^2; a_r) = \left[ \frac{1}{\Gamma(2 - \sigma)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 - \sigma) |z|^{2n}}{n!} \right]^{-1/2} = \frac{1}{\sqrt{\Phi (2 - \sigma; 2; |z|^2)}}, \]  

(3.37)

where \( \Phi(a; b; x) = F_1(a; b; x) \) is the degenerate hypergeometric function [32]. The coherent state of Eq. (2.13)

obtained with these results is

\[ |z; a_r \rangle = \frac{1}{\sqrt{\Phi (2 - \sigma; 2; |z|^2)}} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n + 2 - \sigma)}{\Gamma(n + 2)}} \frac{e^{-i \alpha e_n} z^n | \Psi_n \rangle}. \]  

(3.38)

With the help of the integral [33]

\[ \int_0^\infty t^{\lambda-1} e^{-t/2} W_{\sigma, \mu} (t) dt = \frac{\Gamma (\lambda - \mu - \frac{1}{2}) \Gamma (\lambda + \mu - \frac{1}{2})}{\Gamma (\lambda - \sigma + 1)}, \]  

(3.39)

it is possible to show that the resolution of the unity can be obtained with the measure

\[ w (|z|^2; a_r) = \frac{\Gamma(2 - \sigma)}{\pi} \frac{e^{-|z|^2/2}}{\Phi (2 - \sigma; 2; |z|^2)} W_{\sigma, 1/2 (|z|^2)}, \]  

(3.40)

where \( W_{\sigma, \mu} (x) \) is the Whittaker function [32,33].

One example of the shape-invariant systems of type (B) is the Morse potential [34]. This potential has a finite

number of normalizable bound states which cannot form a complete set of states in the Hilbert space, the condition

necessary to construct the coherent state using our generalized approach. We nevertheless observe that since the

superpotential \( W_{\sigma} (x, a_r) \) has a special form \((x\)-independent and linear in \( a_r \)-term) it is possible construct coherent

states for Morse potential systems using other sets of eigenstates that form a complete orthonormal basis in Hilbert

space and examples of these procedures can be found in the references [35,36].

In the other hand, the shape-invariant systems classified as types (E) and (F) have superpotentials given by

\[ W_{\kappa} (x, a_r) = \sqrt{d \Omega} \left( \beta a_r \cot \beta (x + \lambda) + \frac{\delta}{a_r} \right) \]  

and \[ W_{\kappa} (x, a_r) = \sqrt{d \Omega} \left( \frac{a_r}{x} + \frac{\delta}{a_r} \right), \]  

(3.41)

where \( \beta, \delta \) and \( \lambda \) are real constants. The systems classified as type (E) only have bound states while the systems type

(F) have continuous as well as bound states. Like the systems type (B), the (E) systems have a finite number of energy

eigenstates. In this case, an alternative way is construct the coherent state using the finite set of energy eigenstates

with an adequate redefinition of the measure \( w (|z|^2; a_r) \) to get a finite number of moments \( \{ \rho_n \} \), as was done in [37]

for Morse potential. The systems of type (F), i.e. the Coulomb potential, because of its three-dimensional character,

present energy-degenerated eigenstates. In this case the expansion (2.10) defined for our generalized coherent state

must be appropriately adjusted for this situation. Our generalized definition (2.9) of coherent states for shape-invariant

systems can be extended to include these alternative approaches for systems type (B), (E) and (F). More details and

further developments on this subject will be published elsewhere.
C. Self-similar potential systems

All previous examples have partner potentials $V_k(x)$ with parameters related by a translation. One class of shape-invariant potentials are given by an infinite chain of reflectionless potentials $V^{(k)}_k(x)$, $(k = 0, 1, 2, \ldots)$, for which associated superpotentials $W_k(x)$ satisfy the self-similar ansatz $W_k(x) = q^k W(q^k x)$, with $0 < q < 1$. These set of partners potentials $V^{(k)}_k(x)$, also called self-similar potentials [38,39], have an infinite number of bound states and its parameters related by a scaling: $a_n = q^{n-1} a_n$. Shape invariance of self-similar potentials was studied in detail in [40,41]. In the simplest case studied by them the remainder of Eq. (2.1) is given by $R(a_n) = c a_n$, where $c$ is a constant. Hence

$$\prod_{k=1}^{n} \left[ \sum_{s=k}^{n} R(a_s) \right] = \left[ \frac{R(a_n)}{1-q} \right]^{n} q^{n(n-1)/2} (q;q)_n^m$$

(3.42)

where the $q$-shifted factorial $(q;q)_n^m$ is defined as $(p;q)_0 = 1$ and $(p;q)_n = \prod_{j=0}^{n-1} (1 - pq^j)$, with $n \in \mathbb{Z}$. Coherent states for self-similar potentials were introduced in [10,13,14]. Before to apply our generalized approach for this system let us first assume the choice $Z_j = 1$, and use it and the result of Eqs. (3.42) in the the expansion coefficient (2.14) to show the coherent state (2.13) in this case is given by

$$|z; a_r \rangle = \frac{1}{\sqrt{E_q^{(1/2)}(|\xi_o|^2)}} \sum_{n=0}^{\infty} \frac{q^{-n^2/4}}{(q;q)_n^m} \xi_o^n |\Psi_n \rangle .$$

(3.43)

where $\xi_o = z \sqrt{(1-q)/[\sqrt{q} R(a_1)]}$ and the $q$-exponential is defined by [42-44]

$$E_q^{(1/2)}(x) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^m} x^n .$$

(3.44)

The result (3.43) is the normalized form of the initial expression we obtained in our previous paper [13] for the coherent states of the self-similar potentials. To apply our generalized approach for this kind of potential system we assume

$$Z_j = R(a_1) e^{-i \alpha R(a_1)} \quad \text{yielding} \quad \prod_{k=0}^{n-1} Z_{j+k} = [R(a_1)]^n q^{n(n-1)/2} e^{-i \alpha e_n} ,$$

(3.45)

where $e_n = R(a_1) (1 - q^n)/(1 - q)$. Substituting Eqs. (3.42) and (3.45) in (2.14) we find

$$h_n (a_r) = \frac{(q;q)_n}{[R(a_1) (1-q)]^n q^{n(n-1)/2}} e^{i \alpha e_n}, \quad \text{and} \quad \mathcal{N}(|z|^2; a_r) = \frac{1}{\sqrt{E_q^{(1/2)}(|\xi|^2)}} ,$$

(3.46)

where $\xi = z \sqrt{R(a_1) (1-q)/\sqrt{q}}$. The coherent state (2.13) obtained with these results is

$$|z; a_r \rangle = \frac{1}{\sqrt{E_q^{(1/2)}(|\xi|^2)}} \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n^m} e^{-i \alpha e_n} \xi^n |\Psi_n \rangle .$$

(3.47)

In this case we can show that the inner product (2.17) of two coherent states can be readily found as

$$\langle z'; a_r | z; a_r \rangle = \frac{E_q^{(1/2)}(|\xi|^2)}{\sqrt{E_q^{(1/2)}(|\xi|^2) E_q^{(1/2)}(|\xi|^2)}} .$$

(3.48)

This result still valid for the first expression obtained for coherent state of self-similar potentials (3.43) since we change $\xi \rightarrow \xi_o$, and $E_q^{(1/2)}(x) \rightarrow E_q^{(1/2)}(x)$. Equation (3.47) is the temporally stable version of the coherent state found in our previous paper [14] (see also Ref. [45]). As shown in that paper, this choice for $Z_j$ makes possible to establish
an overcompleteness relation for the coherent state $|z; a_r\rangle$ using Ramanujan’s integral extension of the beta function [46] and the measure, in this case, is given by

$$w\left(|z|^2; a_r\right) = \frac{1}{2\pi} \frac{1}{(-|\xi_1|^2; q)_\infty \log (1/q)}.$$  \hspace{1cm} (3.49)

Finally we introduce a possible new coherent state for this class of shape invariant potentials by using the functional definition

$$Z_j = R(a_i) \sqrt{1 - c/a_z} e^{-i\alpha R(a_i)} \quad \Longrightarrow \quad \prod_{k=0}^{n-1} Z_{j+k} = [R(a_i)]^n q^{n(n-1)/2} \frac{(c; q^{-1})_{n+1}}{1 - c} e^{-i\alpha \xi_n},$$  \hspace{1cm} (3.50)

where $c$ is an arbitrary constant. Substituting Eqs. (3.42) and (3.50) in (2.14) we find

$$h_n (a_r) = \sqrt{\frac{(1-c) (q; q)_n}{[R(a_i) (1-q)]^n q^{n(n-1)/2} (c; q^{-1})_{n+1}}} e^{i\alpha \xi_n}, \quad \text{and} \quad \mathcal{N} \left(|z|^2; a_r\right) = \frac{(-c|\xi_2|^2 q^{-1}; q)_\infty}{(-|\xi_2|^2; q)_\infty},$$  \hspace{1cm} (3.51)

where $\xi_2 = z \sqrt{R(a_i)(1-q)}$. The coherent state (2.13) obtained with these results is

$$|z; a_r\rangle = \sqrt{\frac{(-c|\xi_2|^2 q^{-1}; q)_\infty}{(1-c) (-|\xi_2|^2; q)_\infty}} \sum_{n=0}^{\infty} q^{n(n-1)/4} \frac{(c; q^{-1})_{n+1}}{(q; q)_n} e^{-i\alpha \xi_n} \xi_2^n |\Psi_n\rangle.$$  \hspace{1cm} (3.52)

In this case it is possible to establish an overcompleteness relation for the coherent state $|z; a_r\rangle$ with the introduction of the measure

$$w\left(|\xi_2|^2; a_r\right) = \frac{R(a_i)(1-q)(1-c)}{\pi q \log (1/q)} \left[ (-|\xi_2|^2; q)_\infty \right]/\left( (-|\xi_2|^2 q^{-1}; q)_\infty \right),$$  \hspace{1cm} (3.53)

and using the Ramanujan integral given by [46]

$$\int_0^\infty t^{k-1} \frac{(-a t; q)_\infty}{(-t; q)_\infty} dt = \frac{\log (1/q) (q; q)_{k-1}}{q^{k(k-1)/2} (a; q^{-1})_k},$$  \hspace{1cm} (3.54)

(An elementary proof of (3.54) was given by Askey [47].) Note that it is straightforward to show that the results for this last example reduce to the previous one when we take the limit $c \to 0$ and consider the properties of the $(a; q)_\infty$ functions and the relations between the $\xi_1$ and $\xi_2$ complex variables. Indeed the Ramanujan integral and its integral extension of the beta function [46] used in the previous example is a particular case of the more general Ramanujan integral (3.54).

IV. FINAL REMARKS

In this article, using an algebraic approach, we constructed generalized coherent states for shape-invariant systems. This generalization based on the introduction of a factor functional $Z_j$ of the potential parameters in the coherent state: a) satisfies the set of essential requirements we enumerated in the Introduction to establish classical quantum correspondence; b) reproduces results already known for shape-invariant potential systems; c) gives new possible expressions for coherent states. Another aspect to emphasize is that our generalized construction of coherent states gives some insight on the question of relating different sets of coherent states found in the literature for such systems.

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