Handle decomposition for a class of compact orientable PL 4-manifolds

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Abstract

In this article we study a particular class of compact connected orientable PL 4-manifolds with empty or connected boundary which have infinite cyclic fundamental group. We show that the manifold in the class admits a handle decomposition in which number of 2-handles depends upon its second Betti number and other $h$-handles ($h \leq 4$) are at most 2. In particular, our main result is that if $M$ is a closed connected orientable PL 4-manifold with fundamental group as $\mathbb{Z}$, then $M$ admits either of the following handle decompositions:

1. one 0-handle, two 1-handles, $1 + \beta_2(M)$ 2-handles, one 3-handle and one 4-handle,
2. one 0-handle, one 1-handle, $\beta_2(M)$ 2-handles, one 3-handle and one 4-handle,

where $\beta_2(M)$ denotes the second Betti number of manifold $M$ with $\mathbb{Z}$ coefficients. Further, we extend this result to any compact connected orientable 4-manifold $M$ with boundary and give three possible representations of $M$ in terms of handles.

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1 Introduction

A crystallization $(\Gamma, \gamma)$ of a connected compact PL $d$-manifold is a certain type of edge colored graph which represents the manifold (details provided in Subsection 2.1). The journey of crystallization theory has begun due to Pezzana who gives the existence of a crystallization for every closed connected PL $d$-manifold (see [22]). Later the existence of a crystallization has been proved for every connected compact PL $d$-manifold with boundary (see [13, 15]).

Extending the notion of genus in 2 dimension, the term regular genus for a closed connected PL $d$-manifold has been introduced in [19], which is related to the existence of regular embeddings of graphs representing the manifold into surfaces (cf. Subsection 2.2 for details). Later, in [17], the concept of regular genus has been extended for compact PL $d$-manifolds with boundary, for $d \geq 2$. The same terminology is available for singular manifolds.

For compact PL 4-manifolds with empty or non-spherical boundary, there is a one-one correspondence between singular manifolds and compact 4-manifolds with empty or non-spherical boundary. In [9], the class of semi-simple gems has been introduced for compact
4-manifold with empty or connected boundary. In this paper, we particularly work on compact connected PL 4-manifolds admitting semi-simple crystallizations.

A problem for closed 4-manifolds was posed by Kirby and which can be formulated as: “Does every simply connected closed 4-manifold have a handlebody decomposition without 1-handles?” Many researchers worked on it for the decades, in manifold with boundary as well, like Trace’s work in [24] and [25]. It is also known that every contractible 4-manifold with boundary other than \( D^4 \) must have 1- or 3-handles.

In this paper, we extend the earlier known work to the compact 4-manifolds with empty or connected boundary with the fundamental group \( \mathbb{Z} \). We particularly work on compact connected PL 4-manifolds admitting semi-simple crystallizations. Also, the class of PL 4-manifolds admitting semi-simple crystallizations is not completely known by now. Recently in [11], the authors gave a class of compact 4-manifolds with empty or connected boundary which admit a special handle decomposition lacking in 1-handles and 3-handles. In this article, we show that the closed 4-manifolds of this class admit a handle decomposition which must have 1- and 3-handles. In particularly, we give exact number of each index handles in Theorem [14]. Then, we give all possible ways in which a manifold with connected boundary can be represented in terms of handles.

2 Preliminaries

Crystallization theory provides a combinatorial tool for representing piecewise-linear (PL) manifolds of arbitrary dimension via colored graphs and is used to study geometrical and topological properties of manifolds.

2.1 Crystallization

For a multigraph \( \Gamma = (V(\Gamma), E(\Gamma)) \) without loops, a surjective map \( \gamma : E(\Gamma) \to \Delta_d := \{0, 1, \ldots, d\} \) is called a proper edge-coloring if \( \gamma(e) \neq \gamma(f) \) for any two adjacent edges \( e \) and \( f \). The elements of the set \( \Delta_d \) are called the colors of \( \Gamma \). A graph \( (\Gamma, \gamma) \) is called \((d + 1)\)-regular if degree of each vertex is \( d + 1 \) and is said to be \((d + 1)\)-regular with respect to a color \( c \) if the graph is \( d \)-regular after removing all the edges of color \( c \) from \( \Gamma \). We refer to [7] for standard terminology on graphs. All spaces and maps will be considered in PL-category.

A regular \((d + 1)\)-colored graph is a pair \((\Gamma, \gamma)\), where \( \Gamma \) is \((d + 1)\)-regular and \( \gamma \) is a proper edge-coloring. A \((d + 1)\)-colored graph with boundary is a pair \((\Gamma, \gamma)\), where \( \Gamma \) is not a \((d + 1)\)-regular graph but a \((d + 1)\)-regular with respect to a color \( c \) and \( \gamma \) is a proper edge-coloring. If there is no confusion with coloration, one can use \( \Gamma \) for \((d + 1)\)-colored graphs instead of \((\Gamma, \gamma)\). For each \( B \subseteq \Delta_d \) with \( h \) elements, the graph \( \Gamma_B = (V(\Gamma), \gamma^{-1}(B)) \) is an \( h \)-colored graph with edge-coloring \( \gamma_{|\gamma^{-1}(B)} \). For a color set \( \{j_1, j_2, \ldots, j_k\} \subset \Delta_d \), \( g(\Gamma_{\{j_1, j_2, \ldots, j_k\}}) \) or \( g_{j_1j_2\ldots j_k} \) denotes the number of connected components of the graph \( \Gamma_{\{j_1, j_2, \ldots, j_k\}} \). Let \( g_{j_1j_2\ldots j_k} \) denote the number of regular components of \( \Gamma_{\{j_1, j_2, \ldots, j_k\}} \). A graph \((\Gamma, \gamma)\) is called contracted if subgraph \( \Gamma_c := \Gamma_{|\Delta_d \setminus c} \) is connected for all \( c \).

Let \( \mathcal{G}_d \) denote the set of graphs \((\Gamma, \gamma)\) which are \((d + 1)\)-regular with respect to the fixed color \( d \). Also, if \((\Gamma, \gamma)\) is \((d + 1)\)-regular then \((\Gamma, \gamma) \in \mathcal{G}_d \). For each \((\Gamma, \gamma) \in \mathcal{G}_d \), a corresponding \(d\)-dimensional simplicial cell-complex \( K(\Gamma) \) is determined as follows:

- for each vertex \( u \in V(\Gamma) \), take a \( d \)-simplex \( \sigma(u) \) and label its vertices by \( \Delta_d \);
• corresponding to each edge of color \( j \) between \( u, v \in V(\Gamma) \), identify the \((d-1)\)-faces of \( \sigma(u) \) and \( \sigma(v) \) opposite to \( j \)-labeled vertices such that the vertices with same label coincide.

The geometric carrier \( |\mathcal{K}(\Gamma)| \) is a \( d \)-pseudomanifold and \((\Gamma, \gamma)\) is said to be a gem (graph encoded manifold) of any \( d \)-pseudomanifold homeomorphic to \( |\mathcal{K}(\Gamma)| \) or simply is said to represent the \( d \)-pseudomanifold. We refer to [6] for CW-complexes and related notions. It is known via the construction that for \( \mathcal{B} \subset \Delta_d \) of cardinality \( h + 1 \), \( \mathcal{K}(\Gamma) \) has as many \( h \)-simplices with vertices labeled by \( \mathcal{B} \) as many connected components of \( \Gamma_{\Delta_d \setminus \mathcal{B}} \) are (cf. [15]).

For a \( k \)-simplex \( \lambda \) of \( \mathcal{K}(\Gamma) \), \( 0 \leq k \leq d \), the star of \( \lambda \) in \( \mathcal{K}(\Gamma) \) is the pseudocomplex obtained by taking the \( d \)-simplices of \( \mathcal{K}(\Gamma) \) which contain \( \lambda \) and identifying only their \((d-1)\)-faces containing \( \lambda \) as per gluings in \( \mathcal{K}(\Gamma) \). The link of \( \lambda \) in \( \mathcal{K}(\Gamma) \) is the subcomplex of its star obtained by the simplices that do not contain \( \lambda \).

**Definition 1.** A closed connected PL \( d \)-manifold is a compact \( d \)-dimensional polyhedron which has a simplicial triangulation such that the link of each vertex is \( S^{d-1} \).

A connected compact PL \( d \)-manifold with boundary is a compact \( d \)-dimensional polyhedron which has a simplicial triangulation where the link of each vertex is either a \( S^{d-1} \) or a \( \mathbb{B}^{d-1} \).

A singular PL \( d \)-manifold is a compact \( d \)-dimensional polyhedron which has a simplicial triangulation where the links of vertices are closed connected \((d-1)\)-manifolds while, for each \( h \geq 1 \), the link of any \( h \)-simplex is a PL \((d-h-1)\) sphere. A vertex whose link is not a sphere is called a singular vertex. Clearly, A closed (PL) \( d \)-manifold is a singular (PL) \( d \)-manifold with no singular vertices.

It is known that the \(|\mathcal{K}(\Gamma_c)|\) is homeomorphic to the link of vertex \( c \) of \( \mathcal{K}(\Gamma) \) in the first barycentric subdivision of \( \mathcal{K}(\Gamma) \). And from the correspondence between \((d+1)\)-regular colored graphs and \( d \)-pseudomanifolds, we have that:

1. \(|\mathcal{K}(\Gamma)|\) is a closed connected PL \( d \)-manifold if and only if for each \( c \in \Delta_d \), \( \Gamma_c \) represents \( S^{d-1} \).

2. \(|\mathcal{K}(\Gamma)|\) is a connected compact PL \( d \)-manifold with boundary if and only if for each \( c \in \Delta_d \), \( \Gamma_c \) represents either \( S^{d-1} \) or \( \mathbb{B}^{d-1} \).

3. \(|\mathcal{K}(\Gamma)|\) is a singular (PL) \( d \)-manifold if and only if for each \( c \in \Delta_d \), \( \Gamma_c \) represents closed connected PL \((d-1)\)-manifold.

If \( \Gamma_c \) does not represent \((d-1)\)-sphere then the color \( c \) is called singular color.

**Definition 2.** A \((d+1)\)-colored graph \((\Gamma, \gamma)\) which is a gem of a singular manifold or compact (PL) \( d \)-manifold \( M \) with empty or connected boundary is called a crystallization of \( M \) if it is contracted. In this case, there are exactly \( d+1 \) number of vertices in the corresponding colored triangulation.

The initial point of the crystallization theory is the Pezzana’s existence theorem (cf. [22]) which gives existence of a crystallization for a closed connected PL \( n \)-manifold. Later, it has been extended to the boundary case (cf [13, 18]). Further, the existence of crystallizations has been extended for singular (PL) \( d \)-manifolds (cf. [12]).
Remark 3 (9). There is a bijection between the class of connected singular (PL) \(d\)-manifolds and the class of connected closed (PL) \(d\)-manifolds union with the class of connected compact (PL) \(d\)-manifolds with non-spherical boundary components. For, if \(M\) is a singular \(d\)-manifold then removing small open neighbourhood of each of its singular vertices (if possible), a compact \(d\)-manifold \(\hat{M}\) (with non-spherical boundary components) is obtained. It is obvious that \(M = \hat{M}\) if and only if \(M\) is a closed \(d\)-manifold.

Conversely, if \(M\) is a compact \(d\)-manifold with non-spherical boundary components then a singular \(d\)-manifold \(\hat{\tilde{M}}\) is obtained by coning off each component of \(\partial M\). If \(M\) is a closed \(d\)-manifold then \(\hat{\tilde{M}} = M\).

If the boundary of connected compact PL 4-manifold \(M\) is connected then, by a graph representing \(M\) we mean the graph representing its corresponding singular manifold \(\hat{M}\) obtained from \(M\) by coning off the boundary \(\partial M\) with a cone. Thus, for connected boundary case, we need the colored graphs representing singular manifolds with at most one singular color throughout the paper and without loss of generality we will assume 4 as its singular color.

2.2 Regular Genus of closed PL \(d\)-manifolds and singular \(d\)-manifolds

In [19], the author extended the notion of genus to arbitrary dimension as regular genus. Roughly, if \((\Gamma, \gamma) \in G_d\) is a bipartite (resp. non bipartite) \((d + 1)\)-regular colored graph which represents a closed connected PL \(d\)-manifold \(M\) then for each cyclic permutation \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_d)\) of \(\Delta_d\), there exists a regular imbedding of \(\Gamma\) into an orientable (resp. non-orientable) surface \(F_\varepsilon\). Moreover, the Euler characteristic \(\chi_\varepsilon(\Gamma)\) of \(F_\varepsilon\) satisfies

\[\chi_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}_{d+1}} g_{\varepsilon_i\varepsilon_{i+1}} + (1 - d)p.\]

and the genus (resp. half of genus) \(\rho_\varepsilon\) of \(F_\varepsilon\) satisfies

\[\rho_\varepsilon(\Gamma) = 1 - \frac{\chi_\varepsilon(\Gamma)}{2}\]

where \(2p\) is the total number of vertices of \(\Gamma\).

The regular genus \(\rho(\Gamma)\) of \((\Gamma, \gamma)\) is defined as

\[\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) \mid \varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d) \text{ is a cyclic permutation of } \Delta_d\}.\]

The regular genus of \(M\) is defined as

\[G(M) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \in G_d \text{ represents } M\}.\]

Similar steps are followed for singular \(d\)-manifolds. So, on the same lines the definition of regular genus of graphs representing singular and closed \(d\)-manifold is formulated as follows:

Definition 4. The regular genus \(\rho(\Gamma)\) of \((\Gamma, \gamma)\) is the minimum genus (resp. half of genus) of an orientable (resp. non-orientable) surface into which \((\Gamma, \gamma)\) embeds regularly.

Definition 5. The regular genus \(G(M)\) of a connected singular and closed \(d\)-manifold \(M\) is defined as the least regular genus of its crystallizations.

We need the concept of regular genus of graphs \(\rho(\Gamma)\) only throughout the paper. Also, we will use the result in section [3] by Montesinos and Laudenbach-Poenaru ([20] and [21]) ensuring that the 3-handles (if any) and the 4-handle are added in a unique way to obtain the closed 4-manifold. Further in [25], Trace proved that 3-handles can be attached uniquely in the simply connected manifolds with connected boundary.
3 Semi simple crystallizations of closed 4-manifolds

In [4], semi-simple crystallizations of closed 4-manifolds have been introduced and they are proved to be minimal with respect to regular genus among the graphs representing the same manifold. The notion of semi-simple crystallizations is generalisation of the simple crystallizations of closed simply-connected 4-manifolds (see [5]).

Definition 6. Let $M$ be a closed 4-manifold. A 5-colored graph $\Gamma$ representing $M$ is called semi-simple if $g_{ijk} = m + 1 \forall i,j,k \in \Delta_4$, where $m$ is the rank of fundamental group of $M$. In other words, the 1-skeleton of the associated colored triangulation contains exactly $m+1$ number of 1-simplices for each pair of 0-simplices.

From [8], we have the following result on the number of components of crystallization representing closed 4-manifolds and a relation between Euler characteristic and regular genus of crystallizations.

Proposition 7 ([8]). Let $M$ be a closed 4-manifold and $(\Gamma, \gamma)$ be a crystallization of $M$. Then

\begin{align}
g_{j-1,j+1} &= g_{j-1,j,j+1} + \rho - \rho_j \forall j \in \Delta_4, \quad (1) \\
g_{j-1,j+1} &= 1 + \rho - \rho_{j-1} - \rho_{j+1} \forall j \in \Delta_4, \quad (2)
\end{align}

and

\[ \chi(M) = 2 - 2\rho + \sum_{i \in \Delta_4} \rho_i, \quad (3) \]

where $\rho$ and $\rho_i$ denote the regular genus of $\Gamma$ and $\Gamma_i$ respectively, and $\chi(M)$ is the Euler characteristic of $M$.

Lemma 8. Let $M$ be a closed connected orientable 4-manifold. Let $(\Gamma, \gamma)$ be a 5-colored semi-simple crystallization for $M$. Let $\beta_i(M)$ denotes the $i^{th}$ Betti number of manifold $M$ with $\mathbb{Z}$ coefficients. Then $g_{j-1,j+1} = 4m + \beta_2 - 2\beta_1 + 1, \forall j \in \Delta_4$.

Proof. Let $(\Gamma, \gamma)$ be a semi-simple crystallization representing $M$. From Equation (2), for $j = k, k + 2 \pmod{5}$, we get $\rho_{k-2} = \rho_{k+3}$. This is true for each $k \in \Delta_4$ which implies $\rho_i = \rho_0 \forall i \in \Delta_4$. Then, by adding all the equations in (2) for each $j \in \Delta_4$, we have

\[ 5m = 5\rho - 10\rho_0 \Rightarrow \rho = m + 2\rho_0. \]

From Equation (3), $\chi(M) = 2 - 2\rho + 5\rho_0$. This implies $\chi(M) = 2 - 2m + \rho_0$. Further, from Equation (1) for $j = 0$, we have $g_{14} = 2m + \rho_0 + 1$ and $g_{14} = 4m + \chi(M) - 1$. It follows from Poincaré duality that $\chi(M) = 2 + \beta_2 - 2\beta_1$. This follows the result.

From now onwards, we particularly take the manifolds admitting semi-simple crystallizations and with fundamental group $\mathbb{Z}$. This implies, $\beta_1 = 1$, $m = 1$ and $g_{ijk} = 2$. It follows from Lemma 8 that $g_{14} = \beta_2 + 3$.

It is known that every closed 4-manifold $M$ admits a handle decomposition, i.e.,

\[ M = H^{(0)} \cup (H^{(1)}_1 \cup \cdots \cup H^{(1)}_{d_1}) \cup (H^{(2)}_1 \cup \cdots \cup H^{(2)}_{d_2}) \cup (H^{(3)}_1 \cup \cdots \cup H^{(3)}_{d_3}) \cup H^{(4)}. \]
where $H^{(0)} = \mathbb{D}^4$ and each $k$-handle $H_i^{(k)} = \mathbb{D}^k \times \mathbb{D}^{4-k}$ (for $1 \leq k \leq 4$, $1 \leq i \leq d_k$), is attached with a map $f_i^{(k)} : \partial \mathbb{D}^k \times \mathbb{D}^{4-k} \to \partial (H^{(0)} \cup \cdots \cup (H_i^{(k-1)} \cup \cdots \cup H_{d_k-1}^{(k-1)}))$.

Let $(\Gamma, \gamma)$ be a crystallization of a closed PL 4-manifold $M$ and $\mathcal{K}(\Gamma)$ be the corresponding triangulation with the vertex set $\Delta_4$. If $B \subset \Delta_4$, then $\mathcal{K}(B)$ denotes the subcomplex of $\mathcal{K}(\Gamma)$ generated by the vertices $i \in B$. If $\text{Sd} \mathcal{K}(\Gamma)$ is the first barycentric subdivision of $\mathcal{K}(\Gamma)$, then $F(i,j)$ (resp. $F(i,j,k)$) is the largest subcomplex of $\text{Sd} \mathcal{K}(\Gamma)$, disjoint from $\text{Sd} \mathcal{K}(i,j) \cup \text{Sd} \mathcal{K}(\Delta_4 \setminus \{i,j\})$ (resp. $\text{Sd} \mathcal{K}(i,j,k) \cup \text{Sd} \mathcal{K}(\Delta_4 \setminus \{i,j,k\})$). Then the polyhedron $|F(i,j)|$ (resp. $|F(i,j,k)|$) is a closed 3-manifold which partitions $M$ into two 4-manifolds $N(i,j)$ (resp. $N(\Delta_4 \setminus \{i,j\})$) with $|F(i,j)|$ (resp. $|F(i,j,k)|$) as common boundary. Further, $N(i,j)$ (resp. $N(i,j,k)$) is regular neighbourhood of the subcomplex $|\mathcal{K}(i,j)|$ (resp. $|\mathcal{K}(i,j,k)|$) in $|\mathcal{K}(\Gamma)|$. See [14] and [16] for more details. Thus, $M$ has a decomposition of type $M = N(i,j) \cup_\phi N(\Delta_4 \setminus \{i,j\})$, where $\phi$ is a boundary identification.

**Remark 9.** Let $M$ be a closed connected orientable 4-manifold with fundamental group $\mathbb{Z}$ and which admits semi simple crystallization. Without loss of generality, we write $M = N(1,4) \sqcup N(0,2,3)$. We denote $N(1,4)$ and $N(0,2,3)$ by $V$ and $V'$ respectively. Since number of $\{14\}$-colored edges is 2, $V$ is either $\mathbb{S}^1 \times \mathbb{B}^3$ or $\mathbb{S}^1 \tilde{\times} \mathbb{B}^3$, where $\mathbb{S}^1 \times \mathbb{B}^3$ and $\mathbb{S}^1 \tilde{\times} \mathbb{B}^3$ denote direct and twisted product of spaces $\mathbb{S}^1$, $\mathbb{B}^3$ respectively. From Mayer Vietoris exact sequence of the triples $(M,V,V')$, we have

$$0 \to H_4(M) \to H_3(\partial V) \to 0.$$ 

This implies $M$ is orientable if and only if $\partial V$ is orientable. Thus, $V = \mathbb{S}^1 \times \mathbb{B}^3$ and $\partial V = \mathbb{S}^1 \times \mathbb{S}^2$.

**Lemma 10.** Let $M$, $V$ and $V'$ be the spaces as in remark 9. Then

$$\beta_2(V') - \beta_2(V) - \beta_1(V') + 1 = 0. \quad (4)$$

**Proof.** Since $V'$ collapses onto the 2-dimensional complex $\mathcal{K}(0,2,3)$, the Mayer Vietoris sequence of the triple $(M,V,V')$ gives the following long exact sequence.

$$0 \to H_3(M) \to H_2(\partial V) \to H_2(V) \oplus H_2(V') \to H_2(M) \to H_1(\partial V) \to 0$$

$$0 \leftarrow H_1(M) \leftarrow H_1(V) \oplus H_1(V')$$

By assumption $\pi_1(M) \cong \mathbb{Z}$ which implies $H_1(M) \cong \mathbb{Z}$. By Poincaré duality and Universal Coefficient theorem, $H_3(M) \cong H^1(M) \cong PH_1(M) \cong \mathbb{Z}$. Remark 9 gives $V = \mathbb{S}^1 \times \mathbb{B}^3$ and $\partial V = \mathbb{S}^1 \times \mathbb{S}^2$. Now, $H_2(\partial V) \cong H_1(\partial V) \cong \mathbb{Z}$, $H_2(V) \cong 0$ and $H_1(V) \cong \mathbb{Z}$. Thus above exact sequence reduces to

$$0 \to \mathbb{Z} \to \mathbb{Z} \to H_2(V') \to H_2(M) \to \mathbb{Z} \to \mathbb{Z} \oplus H_1(V') \to \mathbb{Z} \to 0.$$ 

Since the alternate sum of the rank of finitely generated abelian groups in an exact sequence is zero, the result follows.

**Lemma 11.** Let $M$, $V$ and $V'$ be as in remark 9, where $V$ and $V'$ are regular neighbourhoods $N(1,4)$ and $N(0,2,3)$ respectively. Let $\pi_1(M) = \mathbb{Z}$. Then the the fundamental group of $V'$ is neither trivial nor $\mathbb{Z}_k$ for any $k$. 


Proof. We have $M = V \cup V' = (S^1 \times B^3) \cup V'$ with $V \cap V' = \partial V = \partial V'$ from Remark 9. We can extend the spaces $V$ and $V'$ by open simplices. Without loss of generality, we can assume that $V$ and $V'$ in the given hypothesis are open. Let $i_1 : \pi_1(V \cap V') \to \pi_1(V)$ and $i_2 : \pi_1(V \cap V') \to \pi_1(V')$ be the maps induced from inclusion maps $j_1 : V \cap V' \to V$ and $j_2 : V \cap V' \to V'$ respectively. Since $V = S^1 \times B^3$ and $V \cap V' = S^1 \times S^2$, if we let $\pi_1(V \cap V') = \langle \alpha \rangle$ then $\pi_1(V) = \langle \alpha \rangle$ and $i_1(\alpha) = \alpha$.

If we assume to the contrary that $\pi_1(V') = \langle \epsilon \rangle(\text{or } \langle \beta|\beta^k \rangle)$ then Seifert-van Kampen Theorem implies $\pi_1(M) = \langle \epsilon \rangle(\text{or } \langle \beta|\beta^k \rangle)$ which is a contradiction as fundamental group of $M$ is $\mathbb{Z}$. Hence, the lemma follows.

Lemma 12. Let $V$ and $V'$ be as in remark 9. Then, $0 \leq \beta_1(V') \leq 2$.

Proof. If $\beta_1(V') = k$ then $g_{14} = \beta_2(V') + 4 - k$ using Equation (4) and Lemma 8 for orientable case. Since each edge is a face of at least one triangle, the result follows.

Proposition 13 (23). Let $M$ be a manifold and $X \subset \text{int } M$ be a polyhedron. If $X$ collapses onto $Y$ then a regular neighbourhood of $X$ is PL-homeomorphic to a regular neighbourhood of $Y$.

Theorem 14. Let $M$ be a closed orientable 4-manifold with fundamental group $\mathbb{Z}$. Let $(\Gamma, \gamma)$ be a semi-simple crystallization representing $M$. Then, $M$ admits either of the following two handle decompositions:

1. one 0-handle, two 1-handles, $1 + \beta_2(M)$ 2-handles, one 3-handle and one 4-handle,

2. one 0-handle, one 1-handle, $\beta_2(M)$ 2-handles, one 3-handle and one 4-handle.

Proof. Let $(\Gamma, \gamma)$ be a semi-simple crystallization representing $M$. We write $M = N(1,4) \cup N(0,2,3) = V \cup V'$, where $V = S^1 \times B^3$ by Remark 9. Now, we have to analyse $V'$. For $i \geq 1$, let $A_i$ be the set of all the triangles which have same boundary in such a way that the triangles in $A_i$ and $A_j$ do not have all the edges same for $i \neq j$. Since the number of edges with the same labeled end vertices is 2, we have 8 triangles such that none of them shares the same boundary. This implies $1 \leq i \leq 8$. Let $k_i$ be the cardinality of $A_i$ for each $i$. Let $A_{j_1}, A_{j_2}, \ldots, A_{j_q}$ be the subcollection of $\{A_i : 1 \leq i \leq 8\}$ in the cell complex $K(0,2,3)$. Since $k + 1$ number of triangles with same boundary contribute $k$ number of 2-dimensional holes,

$$\left( \sum_{r=1}^{q} k_{j_r} \right) - q \leq \beta_2(V').$$

Case A. Let us first consider

$$\left( \sum_{r=1}^{q} k_{j_r} \right) - q = \beta_2(V').$$

It follows from Lemma 12 that $\beta_1(V') = 0, 1$ or 2.

Case 1. Suppose $\beta_1(V') = 2$. By the proof of Lemma 12 we have $g_{14} = \beta_2(V') + 2$. Using Equation (5), we get $q = 2$. Then any triangle in $A_{j_1}$ does not have any common edge with any triangle in $A_{j_2}$ in $K(0,2,3)$ because each edge must be a face of at least one triangle. By collapsing two triangles one from each $A_{j_1}$ and $A_{j_2}$, we observe that $|K(0,2,3)|$ collapses onto a CW complex $K'$; with single 0-cell, two 1-cells and 2-cells consisting of sets $B_{j_1}$ and $B_{j_2}$ with cardinality $k_{j_1} - 1$ and $k_{j_2} - 1$ respectively.
Now, small regular neighbourhood of single 0-cell in geometric carrier of $K'$ is a 0-handle $H^{(0)} = D^4$, two 1-cells contribute two 1-handles and 2-cells give $\beta_2(V')$ number of 2-handles by Equation \(5\). Thus, by using Proposition \(13\)

$$V' = H^{(0)} \cup \left(H_1^{(1)} \cup H_2^{(1)}\right) \cup \left(H_1^{(2)} \cup \cdots \cup H_\beta_2(V')^{(2)}\right).$$

Now, the boundary identification between $V$ and $V'$ is attachment of one 3- and one 4-handle, that is done uniquely from [20] and [21]. Further, $\beta_2(V')$ equals $1 + \beta_2(M)$ from Equation \(4\). Thus,

$$M = H^{(0)} \cup \left(H_1^{(1)} \cup H_2^{(1)}\right) \cup \left(H_1^{(2)} \cup \cdots \cup H_\beta_2(M)+1^{(2)}\right) \cup H^{(3)} \cup H^{(4)}.$$

**Case 2.** Suppose $\beta_1(V') = 1$. By the proof of Lemma \(12\) we have $g_{14} = \beta_2(V') + 3$. Using Equation \(5\), we get $q = 3$. As we approached in Case (1), we collapse three triangles one from each $A_{i_r}, r \in \{1, 2, 3\}$ and observe that $|K(0, 2, 3)|$ collapses onto a CW complex $K'$; with one 0-cell, one 1-cell and 2-cells consisting of sets $B_{i_r}$ with cardinality $\kappa_{i_r} - 1, r \in \{1, 2, 3\}$.

Now, small regular neighbourhood of single 0-cell in geometric carrier of $K'$ is a 0-handle $H^{(0)} = D^4$, one 1-cell contribute one 1-handle and 2-cells give $\beta_2(V')$ number of 2-handles by Equation \(5\). Thus,

$$V' = H^{(0)} \cup H_1^{(1)} \cup \left(H_1^{(2)} \cup \cdots \cup H_\beta_2(V')^{(2)}\right).$$

Now, the boundary identification between $V$ and $V'$ is attachment of one 3- and one 4-handle and using $\beta_2(V')$ equals $\beta_2(M)$ from Equation \(4\). Thus, by using Proposition \(13\)

$$M = H^{(0)} \cup H_1^{(1)} \cup \left(H_1^{(2)} \cup \cdots \cup H_\beta_2(M)\right) \cup H^{(3)} \cup H^{(4)}.$$

**Case 3.** Suppose $\beta_1(V') = 0$. Then we have $g_{14} = \beta_2(V') + 4$. Using Equation \(5\), we get $q = 4$. Now, we collapse four triangles one from each $A_{i_r}, r \in \{1, 2, 3, 4\}$ and it can be observed that $|K(0, 2, 3)|$ collapses to a CW complex $K'$ with fundamental group as one of these three: $\mathbb{Z}, \mathbb{Z}_2$ or $\langle e \rangle$. If the fundamental group of $K'$ is $\mathbb{Z}_2$ or $\langle e \rangle$, then the fundamental group of $V'$ is $\mathbb{Z}_2$ or $\langle e \rangle$, which is a contradiction by Lemma \(11\). If the fundamental group of $K'$ is $\mathbb{Z}$ then $\beta_1(K') = \beta_1(V') = 1$, which is again not possible in this case as $\beta_1(V')$ equals zero. Thus, this case is not possible.

**Case B.** Let \(\sum_{r=1}^{8} k_{i_r} - q < \beta_2(V')\). This case considers the subcollection of $\{A_i : 1 \leq i \leq 8\}$ such that the triangles from different $A_i$’s contribute to $\beta_2(V')$. Let $q$ be the cardinality of this subcollection and let $T$ denotes the space formed by these $q$ number of triangles. First, we find the subcollection which gives non-zero second Betti number. So, without loss of generality we assume that $|A_i| = 1$. If $5 \leq q \leq 8$ then it is not difficult to check that $\beta_2(T)$ is non-zero and the fundamental group is trivial, which contradicts the Lemma \(11\). If $q < 4$ then $\beta_2(T)$ is zero and was considered in Case A.

If $q = 4$ then the non-zero $\beta_2$ is only given by the space in Figure \(1\) consisting four triangles, say $A_{i_1}, A_{i_2}, A_{i_3}$ and $A_{i_4}$, which is PL-homeomorphic to a CW-complex consists of one 1-cell, for each $l$, $0 \leq l \leq 2$. Now, to write the Handle decomposition we work in the general case, that is, we do not restrict ourselves to the condition with $|A_i| = 1$.

Now, by collapsing four triangles one from each $A_{i_t}, t \in \{1, 2, 3, 4\}$, we get that $|K(0, 2, 3)|$ collapses to a CW complex $K'$; with one 0-cell, one 1-cell and 2-cells consisting of one more than the sets $B_{i_t}$ with cardinality $i_t - 1, t \in \{1, 2, 3, 4\}$. 

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Now, small regular neighbourhood of single 0-cell in geometric carrier of $K'$ is a 0-handle $H^{(0)} = D^4$, one 1-cell contribute one 1-handle and 2-cells give $\beta_2(V')$ number of 2-handles. Thus,

$$V' = H^{(0)} \cup H_1^{(1)} \cup \left( H_1^{(2)} \cup \cdots \cup H_{\beta_2(V')}^{(2)} \right).$$

Now, the boundary identification between $V$ and $V'$ is attachment of one 3- and one 4-handle and using $\beta_2(V')$ equals $\beta_2(M)$ from Equation (4). Thus,

$$M = H^{(0)} \cup H_1^{(1)} \cup \left( H_1^{(2)} \cup \cdots \cup H_{\beta_2(M)}^{(2)} \right) \cup H^{(3)} \cup H^{(4)}.$$

This completes the proof.

$$\square$$

4 Semi simple crystallizations of compact 4-manifolds with boundary

In [9], the concept of semi-simple crystallization are extended to the compact manifolds with connected boundary.

**Definition 15.** Let $M$ be a 4-manifold with empty or connected boundary. A 5-colored graph $\Gamma$ representing $M$ is called semi-simple if $g_{ijk} = m' + 1 \forall i, j, k \in \Delta_3$ and $g_{ij4} = m + 1 \forall i, j \in \Delta_3$, where $rk(\pi_1(M)) = m$ and $rk(\pi_1(\hat{M})) = m'$.

The definition of semi simple crystallizations is for the compact 4-manifolds with connected boundary. In [3], there is a similar concept for the manifolds with any number of boundary components.

Proposition 10 of [11] gives the following result on isomorphism between certain cohomology and homology groups of 4-manifold with connected boundary and its associated singular manifold respectively.

**Proposition 16.** Let $M$ be any compact connected orientable 4-manifold with connected boundary and $\hat{M}$ be the singular manifold by coning off its boundary. Then

$$H_k(\hat{M}) \cong H^{4-k}(M) \text{ for } k \in \{2, 3\}.$$

**Proposition 17** ([10]). Let $M$ be a compact connected 4-manifold with connected boundary and $(\Gamma, \gamma)$ be a crystallization of $M$. Then

$$g_{j-1,j+1} = g_{j-1,j,j+1} + \rho - \rho_j \forall j \in \Delta_4, \quad (6)$$

$$g_{j+1,j+1} = 1 + \rho - \rho_{j+1} - \rho_{j+1} \forall j \in \Delta_4, \quad (7)$$
Using the value of $\rho \in \{j\}$.

Now, Equation (9) gives the result.

Equation (8),

Lemma 18. Let $M$ be a compact connected 4-manifold with connected boundary. Let $(\Gamma, \gamma)$ be a 5-colored semi-simple crystallization for $M$. Then

\[ g_{j-1,j+1} = 3m + m' + \chi(\hat{M}) - 1, \text{ if } 4 \in \{j - 1, j + 1\}, \]
\[ g_{j-1,j+1} = 2m + 2m' + \chi(\hat{M}) - 1, \text{ if } 4 \notin \{j - 1, j + 1\}. \]

Proof. Let $(\Gamma, \gamma)$ be a semi-simple crystallization representing $M$. From Equation (7), $\rho_i = \rho_0 \forall i \in \Delta_3$ and $\rho_4 = (m - m') + \rho_0$. Then adding all the equations in (7) for each $j \in \Delta_4$, we have

\[ 5m = 5\rho - 10\rho_0 \Rightarrow \rho = m + 2\rho_0. \]

From Equation (8), $\chi(\hat{M}) = 2 - 2\rho + 4\rho_0 + \rho_4$. This implies

\[ \chi(\hat{M}) = 2 - 2m + \rho_4 = 2 - m - m' + \rho_0. \]

Using the value of $\rho$ in Equation (8) we get that if $j \in \{0, 3\}$ then $g_{j-1,j+1} = 2m + 1 + \rho_j$, if $j \in \{1, 2\}$ then $g_{j-1,j+1} = m + m' + 1 + \rho_j$ and if $j = 4$ then $g_{j-1,j+1} = 2m + 1 + 2\rho_{j+1} - \rho_j$.

Now, Equation (9) gives the result.

Lemma 19. Let $M$ be a compact connected orientable PL 4-manifold with connected boundary and $\hat{M}$ be its corresponding singular manifold. Let $\pi_1(M) = \mathbb{Z}$. Let $(\Gamma, \gamma)$ be a 5-colored semi-simple crystallization for $M$. Then, we have the following relations.

\[ \pi_1(\hat{M}) = 3 + \beta_2(M), \quad \forall j \in \Delta_4 \]
\[ \langle \epsilon \rangle = 3 + \beta_2(M), \quad 4 \in \{j - 1, j + 1\} \]
\[ \langle \epsilon \rangle = 2 + \beta_2(M), \quad 4 \notin \{j - 1, j + 1\}. \]

Proof. If $M$ is orientable then $\hat{M}$ is also orientable. This implies that

\[ \chi(\hat{M}) = 2 - \beta_1(\hat{M}) + \beta_2(M) - \beta_1(M) \]

using Proposition 16. Since $\pi_1(\hat{M})$ is either $\langle \epsilon \rangle$ or $\mathbb{Z}$, $\beta_1(\hat{M}) = m'$. Also, $\pi_1(M) = \mathbb{Z}$ implies $\beta_1(M) = 1$. Now, Lemma 18 gives the required result.

For any crystallization $(\Gamma, \gamma)$ of a PL 4-manifold $M$ and any partition $\{\{l, m\}\{i, j, k\}\}$ of $\Delta_4$, $M$ has a decomposition $\hat{M} = N(\Upsilon, m) \cup \phi N(i, j, k)$, similarly as in closed 4-manifold case, where $N(l, m)$ (resp. $N(i, j, k)$) denotes a regular neighbourhood of the subcomplex $|K(l, m)|$ (resp. $|K(i, j, k)|$) in $|K(\Gamma)|$ generated by the vertices labelled $\{l, m\}$ (resp. $\{i, j, k\}$), where $\phi$ is a boundary identification.

Remark 20. Let $M$ be a compact connected orientable 4-manifold with connected boundary and $\hat{M}$ be its corresponding singular manifold. Let $M$ admits a semi-simple crystallization with $\pi_1(M) = \pi_1(\hat{M}) = \mathbb{Z}$. Without loss of generality, we write $\hat{M} = N(1, 4) \cup N(0, 2, 3)$. We denote $N(1, 4)$ and $N(0, 2, 3)$ by $V$ and $V'$ respectively. Let $m'$ be the rank of fundamental group of $\hat{M}$. 

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From Mayer Vietoris exact sequence of the triples \((\tilde{M}, V, V')\), we have

\[
0 \to H_4(\tilde{M}) \to H_3(\partial V) \to 0.
\]

This implies \(\tilde{M}\) is orientable if and only if \(\partial V\) is orientable. If \(m' = 1\) then the number of \(\{14\}\)-colored edges is 2. This implies \(V\) is boundary connected sum between \(S^1 \times \mathbb{B}^3\) and the cone over \(\partial M\) and \(\partial V = (S^1 \times S^2) \# \partial M\).

**Lemma 21.** Let \(M, V\) and \(V'\) be the spaces as in Remark 20. Then

\[
\beta_2(V') - \beta_2(M) - \beta_1(V') + 1 = 0. \tag{10}
\]

**Proof.** Since \(V'\) collapses onto the 2-dimensional complex \(K(0, 2, 3)\), the Mayer Vietoris sequence of the triple \((\tilde{M}, V, V')\) gives the following long exact sequence.

\[
0 \to H_3(\tilde{M}) \to H_2(\partial V) \to H_2(V) \oplus H_2(V') \to H_2(\tilde{M}) \to H_1(\partial V) \to 0 \quad \text{↓}
\]

By Proposition 16 and Universal Coefficient Theorem, we have \(H_3(\tilde{M}) \cong H^1(M) \cong FH_1(M) \cong Z\) and \(H_2(M) \cong H^2(M)\). Now, \(H_2(\partial V) = H_2(\mathbb{S}^1 \times \mathbb{B}^3) \# \partial M = H_2(\mathbb{S}^1 \times \mathbb{S}^2) \oplus \partial M \cong Z \oplus H^1(\partial M) \cong Z \oplus FH_1(\partial M)\). Also, we have \(H_1(\partial V) = Z \oplus H_1(\partial M)\). Now for \(i = 1, 2, H_i(V) = H_i(\mathbb{S}^1 \times \mathbb{B}^3) \oplus H_i(C)\), where \(C\) is cone over \(\partial M\). This implies \(H_2(V) = 0\) and \(H_1(V) = Z\). Thus above exact sequence reduces to the following.

\[
0 \to Z \to Z \oplus FH_1(\partial M) \to H_2(V') \to H_2(M) \to Z \oplus H_1(\partial M) \to Z \oplus H_1(V') \quad \text{↓}
\]

Thus the alternate sum of rank of finitely generated abelian groups in an exact sequence is zero, the result follows.

**Lemma 22.** Let \(M, \tilde{M}, V\) and \(V'\) be the spaces as in Remark 20. Then the the fundamental group of \(V'\) is neither trivial nor \(Z_k\) for any \(k\).

**Proof.** We can extend the spaces \(V\) and \(V'\) to open simplices. Without loss of generality, we assume that \(V\) and \(V'\) in hypothesis are open. Let \(i_1 : \pi_1(V \cap V') \to \pi_1(V)\) and \(i_2 : \pi_1(V \cap V') \to \pi_1(V')\) be the maps induced from inclusion maps \(j_1 : V \cap V' \to V\) and \(j_2 : V \cap V' \to V'\) respectively. Let \(\pi_1(V \cap V') = \pi_1(\partial V) = \langle \alpha_1, \alpha_2, \ldots, \alpha_{n_1} | d_1, d_2, \ldots, d_{n_2} \rangle\) where generator \(\alpha_1\) comes from \(S^1 \times \mathbb{S}^2\) and remaining generators come from \(\partial M\). Also, \(\pi_1(V) = \langle \alpha_1 \rangle\), \(i_1(\alpha_1) = \alpha_1\) and \(i_1(\alpha_k) = e\).

On the contrary, if we assume that the fundamental group of space \(V'\) is trivial or \(Z_k\) for some \(k\) then Seifert-van Kampen Theorem implies \(\pi_1(M)\) is trivial or a finite cyclic group, which is a contradiction as fundamental group of \(M\) is \(Z\). Therefore, rank of fundamental group of \(V'\) is non zero.

**Remark 23.** Let \(M, \tilde{M}, V\) and \(V'\) be the spaces as in Remark 20. On the similar lines of the proof of Lemma 17, we get \(0 \leq \beta_1(V') \leq 2\) using Equation (10) and Lemma 19.
Theorem 24. Let $M$ be a compact connected orientable PL 4-manifold with $\pi_1(M) = \pi_1(\hat{M}) = \mathbb{Z}$. Let $(\Gamma, \gamma)$ be a semi-simple crystallization of $M$. Then, $M$ can be represented as either of the following:

(1) $H^{(0)} \cup \left( H_{1}^{(1)} \cup H_{2}^{(1)} \right) \cup \left( H_{1}^{(2)} \cup \cdots \cup H_{1+\beta_2(M)}^{(2)} \right) \cup_\phi \left( \mathbb{S}^1 \times \mathbb{B}^3 \right)$,

(2) $H^{(0)} \cup H_{1}^{(1)} \cup \left( H_{1}^{(2)} \cup \cdots \cup H_{\beta_2(M)}^{(2)} \right) \cup_\phi \left( \mathbb{S}^1 \times \mathbb{B}^3 \right)$,

where $\phi$ is boundary identification between $(\mathbb{S}^1 \times \mathbb{B}^3)$ and handle decomposition till 2-handles in the above union.

Proof. We first consider $\hat{M}$. We know that $\hat{M} = V \cup_\phi V'$ where $V$ is boundary connected sum between $(\mathbb{S}^1 \times \mathbb{B}^3)$ and a cone over $\partial M$ from Remark 20. If $m' = 1$ then $V'$ being regular neighbourhood of $|K(0, 2, 3)|$ is same as in proof of theorem 14 by using remark 23. In other words, $\beta_1(V')$ cannot be zero and $\hat{M}$ can be represented as

$$H^{(0)} \cup \left( H_{1}^{(1)} \cup H_{2}^{(1)} \right) \cup \left( H_{1}^{(2)} \cup \cdots \cup H_{1+\beta_2(M)}^{(2)} \right) \cup_\phi \left( (\mathbb{S}^1 \times \mathbb{B}^3) \#_{bd} C \right) \text{ or } (11)$$

$$H^{(0)} \cup H_{1}^{(1)} \cup \left( H_{1}^{(2)} \cup \cdots \cup H_{\beta_2(M)}^{(2)} \right) \cup_\phi \left( (\mathbb{S}^1 \times \mathbb{B}^3) \#_{bd} C \right), \text{ (12)}$$

where $C$ is a cone over $\partial M$, depending upon $\beta_1(V')$ is 2 or 1 respectively. Now, the result follows directly from the Equations (11) and (12). \hfill \Box

Now, if the statement that “3-handles can be attached uniquely for PL 4-manifolds with boundary case” is proved in future then we get the following remark.

Remark 25. If it is proved that 3-handles can be attached uniquely, as possible in closed manifolds case, then $M$ can be represented as either of the following handle decompositions:

(1) $H^{(0)} \cup \left( H_{1}^{(1)} \cup H_{2}^{(1)} \right) \cup \left( H_{1}^{(2)} \cup \cdots \cup H_{1+\beta_2(M)}^{(2)} \right) \cup H^{(3)}$,

(2) $H^{(0)} \cup H_{1}^{(1)} \cup \left( H_{1}^{(2)} \cup \cdots \cup H_{\beta_2(M)}^{(2)} \right) \cup H^{(3)}$.

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