CAPUTO-HADAMARD APPROACH APPLICATIONS: SOLVABILITY FOR AN INTEGRO-DIFFERENTIAL PROBLEM OF LANE AND EMDEN TYPE

MOHAMED BEZZIOU¹, ZOUBIR DAHMANI²∗, IQBAL JEBRIL³, MOHAMED KAID²

¹UDBKM University, Laboratory LPAM of Mathematics, UMAB University of Mostaganem 27000, Algeria
²Laboratory LMPA, Faculty of SEI, UMAB, University Abdelhamid Bni Badis of Mostaganem 27000, Algeria
³Department of Mathematics, Faculty of Science and Information Technology, Al-Zaytoonah University of Jordan, P.O. Box 130 Amman 11733, Jordan

Abstract. The present paper is dealing with a new direction in the Caputo-Hadamard approach. It is concerned with the solvability of an integro differential problem of type Lane and Emden. The studied problem involves Caputo-Hadamard derivative with new different fractional orders. The main results of existence of solutions are based on the contraction principle of Banach, however, for the existence of solutions, the use of Scheafer fixed point theorem is applied to prove the result. Three examples are discussed at the end of this work.

Keywords: Caputo-Hadamard derivative; existence of solution; fixed point.

2010 AMS Subject Classification: 26A33, 34C15.

1. INTRODUCTION

In the present paper, we are concentrating on the investigation of the existence and uniqueness of solutions for the following problem of nonlinear fractional differential equation of Lane-Emden singular type:
\[
\begin{aligned}
C_D^\beta \left( C_D^\alpha + \frac{A}{(\log t)^\mu} \right) x(t) + B f(t,x(t),C_D^\sigma x(t)) + g(t,x(t),I^\rho x(t)) &= h(t), t \in [1,e], \\
(C_D^\alpha + A) x(e) &= 0, x(1) = x(e) = \sum_{i=1}^n \lambda_i I^\delta x(\eta_i), \\
0 < \mu, \sigma < \alpha < 1, 1 < \beta < 2, 1 < \eta_i < e, \rho, \delta, \lambda_i > 0,
\end{aligned}
\]

where \( C_D^\beta, C_D^\alpha \) and \( C_D^\sigma \) are the derivatives in the sense of Caputo-Hadamard, \( I^\rho \) denotes the Hadamard integral of order \( \rho \), with: \( A, B > 0, J = [1,e] \), the functions \( f, g \in C(J \times \mathbb{R}^2, \mathbb{R}) \) and \( h \) is defined over \( J \). To the best of our knowledge, this is the first time where such problem is investigated.

The structure of our paper is as follows: In Section 2, we will recall some preliminary related to fractional calculus and Caputo-Hadamard derivatives. In Section 3, we apply the integral inequality theory combined with the fixed point theory for study the questions of existence and uniqueness of solutions for the considered problem. In Section 4, three illustrative examples are presented and discussed in details.

## 2. Preliminaries on Fractional Calculus

In this section, we recall the basic definitions, properties and lemmas involving Caputo-Hadamard derivatives, for more details, one can consult the references [15, 16, 18]. We begin this section by the following definition:

**Definition 2.1:** The Hadamard fractional integral of order \( \alpha > 0 \), for a function \( f \in L^1(J) \), is defined as

\[
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} f(s) \frac{ds}{s},
\]

where

\[
\Gamma(\alpha) := \int_0^{+\infty} e^{-s}s^{\alpha-1}ds.
\]
Let
\[ \delta = t \frac{d}{dt}, \alpha > 0, n = [\alpha] + 1, \]
with \([\alpha]\) denotes the integer part of a real number \(\alpha\). Define the space
\[ AC^n_\delta ([a, b]) = \{ f : [a, b] \to \mathbb{R}, \delta^{n-1}f \in AC [a, b] \}. \]

**Definition 2.2:** The Caputo-Hadamard fractional derivative of order \(\alpha\) for a function \(f \in AC^n_\delta ([a, b], \mathbb{R})\) is defined by:
\[
C_H D^\alpha_{a+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left( \log \frac{t}{s} \right)^{n-\alpha-1} \left( t \frac{dt}{t} \right)^n f(s) \frac{ds}{s},
\]
provided that the right-hand side integral exists.

Now, we recall the following lemmas:

**Lemma 2.1:** Let \(\alpha, \beta > 0\) and \(f \in L^1 ([a, b], \mathbb{R})\). Then \(I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)\) and \(D^\alpha I^\alpha f(t) = f(t)\).

**Lemma 2.2:** Let \(\beta > \alpha > 0\) and \(f \in L^1 ([a, b], \mathbb{R})\). Then \(D^\alpha I^\beta f(t) = I^{\beta-\alpha} f(t)\).

**Lemma 2.3:** Let \(x \in AC^n_\delta ([a, b], \mathbb{R})\), \(n-1 < \alpha < n\). Then the Caputo–Hadamard fractional differential equation
\[ C_H D^\alpha x(t) = 0, \]
has a solution
\[ x(t) = \sum_{k=0}^{n-1} c_k \left( \log \frac{t}{a} \right)^k, t > a > 0, \]
and the following formula holds
\[ I^\alpha C_H D^\alpha x(t) = x(t) + \sum_{k=0}^{n-1} c_k \left( \log \frac{t}{a} \right)^k, \]
where \(c_k \in \mathbb{R}, k = 0, 1, 2, \ldots, n-1\).

Before proving our main results, we introduce the following important result. It deals with the integral representation of the above considered problem. We have:
Lemma 2.4: Let us take a function $G \in C(J, \mathbb{R})$. Therefore, the unique integral solution of the problem

\[
\begin{align*}
\begin{cases}
\frac{\mathcal{C}_H D^\beta \left( \mathcal{C}_H D^\alpha + \frac{A}{(\log t)^\mu} \right)}{x(t)} x(t) = G(t), t \in ]1, e[,
\end{cases}
\end{align*}
\]  
(2)

\[
\begin{align*}
(\mathcal{C}_H D^\alpha + A) x(e) = 0, x(e) = x(1) = \sum_{i=1}^{n} \lambda_i \tau \delta_i x(\eta_i),
\end{align*}
\]

\[
0 < \mu < \alpha < 1, 1 < \beta < 2, 1 < \eta_i < e, \quad \delta_i, \lambda_i > 0,
\]

is given by

\[
x(t) := \frac{1}{\Gamma(\alpha)} \int_{1}^{t} \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_{1}^{s} \left( \log \frac{s}{\tau} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} - \frac{A}{(\log s)^\mu x(s)} \frac{ds}{s}
\]

\[
- \frac{1}{\sum_{i=1}^{n} \lambda_i (\log \eta_i)^{\delta_i} - 1} \sum_{i=1}^{n} \lambda_i \frac{1}{\Gamma(\alpha + \delta_i)} \int_{0}^{\eta_i} \left( \log \frac{\eta_i}{s} \right)^{\alpha + \delta_i - 1}
\]

\[
\times \left( \frac{1}{\Gamma(\beta)} \int_{1}^{s} \left( \log \frac{s}{\tau} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} - \frac{A}{(\log s)^\mu x(s)} \right) \frac{ds}{s}
\]

\[
+ \left[ \frac{(\log t)^{\alpha+1}}{\Gamma(\alpha + 2)} - \frac{(\log t)^{\alpha}}{\Gamma(\alpha + 1)} + \frac{1}{\sum_{i=0}^{n} \lambda_i (\log \eta_i)^{\delta_i} - 1} \left[ \sum_{i=1}^{n} \lambda_i \frac{1}{\Gamma(\alpha + \delta_i + 1)} \right] \times \Gamma(\alpha + 2) \frac{(\log e)^{\alpha-1}}{\alpha \Gamma(\alpha)} \int_{1}^{e} \left( \log \frac{e}{s} \right)^{\alpha-1}
\]

\[
\times \left( \frac{1}{\Gamma(\beta)} \int_{1}^{s} \left( \log \frac{s}{\tau} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} - \frac{A}{(\log s)^\mu x(s)} \right) \frac{ds}{s}
\]

\[
+ \left( \frac{\log t)^{\alpha}}{\Gamma(\alpha + 1)} - \frac{(1 - \frac{1}{\alpha}) (\log t)^{\alpha+1}}{\Gamma(\alpha + 2)} + \left( 1 - \frac{1}{\alpha} \right) \sum_{i=1}^{n} \lambda_i \frac{1}{\Gamma(\alpha + \delta_i + 2)}
\]

\[
\times \frac{1}{\sum_{i=1}^{n} \lambda_i (\log \eta_i)^{\delta_i} - 1} \right) \frac{1}{\Gamma(\beta)} \int_{1}^{e} \left( \log \frac{\tau}{\tau} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau},
\]

such that $\sum_{i=1}^{n} \lambda_i (\log \eta_i)^{\delta_i} / \Gamma(\delta_i + 1) \neq 1$. 

---

MOHAMED BEZZIOU, ZOUBIR DAHMANI, IQBAL JEBRIL, MOHAMED KAIĐ
**Proof:** By applying Lemma 2.3, we can reduce (2) to the following equivalent integral problem:

\[
x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} \right) \, ds \quad - \frac{A}{(\log s)^\mu x(s)}
\]

for some real constants \(c_1, c_2, c_3\).

Since \((\frac{D^\alpha}{\Gamma(\beta)} + A)x(e) = 0\), we get

\[
c_1 + c_2 = \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau},
\]

now, by using the condition \(x(e) = x(1)\) and (5), we obtain

\[
c_1 = \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} \right) \, ds \quad - \frac{A}{(\log s)^\mu x(s)}
\]

and

\[
c_2 = \frac{1 - \frac{1}{\alpha}}{\Gamma(\beta)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} - \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1}
\]

\[
\times \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} - \frac{A}{(\log s)^\mu x(s)} \right) \, ds
\]

Now, the fact that \(x(1) = \sum_{i=0}^n \lambda_i l^\beta x(\eta_i)\) will allow us to get:

\[
c_3 = \frac{1}{\sum_{i=0}^n \lambda_i (\log \eta_i)^{\delta_i}} \left[ \sum_{i=1}^n \lambda_i \frac{1}{\Gamma(\alpha + \delta_i)} \int_0^{\eta_i} \left( \log \frac{\eta_i}{s} \right)^{\alpha + \delta_i - 1} \right.
\]

\[
\times \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} - \frac{A}{(\log s)^\mu x(s)} \right) \, ds
\]

\[
- c_1 \sum_{i=1}^n \lambda_i \frac{(\log \eta_i)^{\alpha + \delta_i}}{\Gamma(\alpha + \delta_i + 1)} - c_2 \sum_{i=1}^n \lambda_i \frac{(\log \eta_i)^{\alpha + \delta_i + 1}}{\Gamma(\alpha + \delta_i + 2)}
\]

...
Substituting (6) and (7) in (8), we get

\[ c_3 = \frac{1}{\sum_{i=1}^{n} \lambda_i \left( \frac{\log \eta_i}{\delta_i} \right)} - 1 \left\{ \sum_{i=1}^{n} \lambda_i \left( \frac{1}{\Gamma(\alpha + \delta_i)} \int_{0}^{\infty} \left( \frac{\log \eta_i}{s} \right)^{\alpha + \delta_i - 1} \right) \times \left( \frac{1}{\Gamma(\beta)} \int_{1}^{s} \left( \frac{\log \tau}{\tau} \right)^{\beta - 1} G(\tau) \frac{d\tau}{\tau} - \frac{A}{(\log s)^\mu x(s)} \right) \right\} \]

(9)

Then, replacing \( c_1, c_2, c_3 \) in (4), we obtain (3).

3. Main Results

For computational convenience, we need to introduce the following notions:

Let \( E = \{ x : x \in C(J, \mathbb{R}) \text{ and } \int_{J}^{D^\sigma x(t)} \in C(J, \mathbb{R}) \} \).

The space \( (E, \| x \|_E) \) is a Banach space endowed with the norm

\[ \| x \|_E = \max \left\{ \| x \|_\infty, \| C^\sigma x \|_\infty \right\}, \text{ such that} \]

\[ \| x \|_\infty = \sup_{t \in J} |x(t)| \text{ and } \| C^\sigma x \|_\infty = \sup_{t \in J} |C^\sigma x(t)|. \]

Now, we define \( S : E \to E \) as follows:

\[ Sx(t) = \frac{1}{\Gamma(\alpha)} \int_{1}^{t} \left( \frac{\log \tau}{\tau} \right)^{\alpha - 1} \left( \frac{1}{\Gamma(\beta)} \int_{1}^{s} \left( \frac{\log \tau}{\tau} \right)^{\beta - 1} G(\tau) \frac{d\tau}{\tau} \right) \]

(10)

\[ - \frac{A}{(\log s)^\mu x(s)} \frac{ds}{s} - \frac{1}{\sum_{i=1}^{n} \lambda_i \left( \frac{\log \eta_i}{\delta_i} \right)} \sum_{i=1}^{n} \lambda_i \frac{1}{\Gamma(\alpha + \delta_i)} \]

\[ \times \int_{0}^{\eta_i} \left( \frac{\log \eta_i}{s} \right)^{\alpha + \delta_i - 1} \left( \frac{1}{\Gamma(\beta)} \int_{1}^{s} \left( \frac{\log \tau}{\tau} \right)^{\beta - 1} G(\tau) \frac{d\tau}{\tau} \right) \]
We begin by taking into account the following hypotheses:

(\(H_1\)) : The functions \(f, g\) are continuous over \(J \times \mathbb{R}^2\) and \(h\) is continuous over \(J\).

(\(H_2\)) : There exist nonnegative constants \(M_i\) and \(N_i, i = 1, 2\), such that for all \(t \in J, x_i, y_i \in \mathbb{R}\) :

\[
|f(t, y_1, y_2) - f(t, x_1, x_2)| \leq \sum_{i=1}^{2} M_i |y_i - x_i|,
\]

\[
|g(t, y_1, y_2) - g(t, x_1, x_2)| \leq \sum_{i=1}^{2} N_i |y_i - x_i|,
\]

we put

\[
\Delta_f = \max_{i=1,2} \{M_i\} \quad \text{and} \quad \Delta_g = \max_{i=1,2} \{N_i\}.
\]

(\(H_3\)) : There exist nonnegative constants \(L_1, L_2, L_3\), such that for all \(t \in J, x_i \in \mathbb{R}, i = 1, 2\) :

\[
|f(t, x_1, x_2)| \leq L_1, \quad |g(t, x_1, x_2)| \leq L_2, \quad |h(t)| \leq L_3.
\]
Setting the following quantities:

\[
\Omega_1 = \frac{1}{\Gamma(\alpha + \beta + 1)} + \frac{\sum_{i=1}^{n} \lambda_i (\log \eta_i)^{\alpha + \beta + \delta_i}}{\sum_{i=1}^{n} \lambda_i \Gamma(\delta_i + 1) - 1}
\]

\[
\Omega_2 = \frac{\Gamma(1 - \mu)}{\Gamma(\alpha - \mu + 1)} + \frac{\sum_{i=1}^{n} \lambda_i \frac{\Gamma(1 - \mu) (\log \eta_i)^{\alpha + \delta_i - \mu}}{\Gamma(\delta_i - \mu + 1) - 1} + \left[ \frac{\alpha + 2}{\Gamma(\alpha + 2)} \right]
\]

\[
\frac{1}{\sum_{i=1}^{n} \lambda_i \Gamma(\delta_i + 1) - 1} \left( \sum_{i=1}^{n} \lambda_i (\log \eta_i)^{\alpha + \delta_i} \right) \Gamma(\alpha + \delta_i + 1)
\]

\[
+ \sum_{i=1}^{n} \lambda_i (\log \eta_i)^{\alpha + \delta_i + 1} \left[ \frac{\Gamma(\alpha + 2) \Gamma(1 - \mu)}{\alpha \Gamma(\alpha - \mu + 1)} \right],
\]

\[
\bar{\Omega}_1 = \frac{1}{\Gamma(\alpha + \beta - \sigma + 1)} + \left( \frac{1}{\Gamma(\alpha - \sigma + 1)} \right)
\]

\[
+ \left( \frac{1}{\Gamma(\alpha - \sigma + 1)} \right) \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha + \beta + 1)}
\]

\[
+ \left( \frac{1}{\Gamma(\alpha - \sigma + 1)} \right) \frac{|1 - \frac{1}{\alpha}|}{\Gamma(\alpha - \sigma + 2)} \frac{1}{\Gamma(\beta + 1)},
\]
\( \Omega_2 = \frac{\Gamma(1 - \mu)}{\Gamma(\alpha - \mu - \sigma + 1)} + \left( \frac{1}{\Gamma(\alpha - \sigma + 2)} + \frac{1}{\Gamma(\alpha - \sigma + 1)} \right) \times \frac{\Gamma(\alpha + 2) \Gamma(1 - \mu)}{\alpha \Gamma(\alpha - \mu + 1)}. \)

Now, we are in a good position to present our first uniqueness of solution for (1):

**Theorem 3.1:** If \((H_1)\) and \((H_2)\) are satisfied and we have also:

\[
\max \left\{ (B\Delta_f + \Delta_g) \Omega_1 + A\Omega_2 ; (B\Delta_f + \Delta_g) \Omega_1 + A\Omega_2 \right\} < 1.
\]

Then, (1) has a unique solution on \(J\).

**Proof:** We shall proceed to prove that \(S\) is contractive. For \(x, y \in E\), we can write

\[
\|Sy - Sx\|_\infty \leq \sup_{t \in J} \left\{ \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta - 1} \right. \right. \\
\times \left. \left. \left. B \left| f \left( \tau, y(\tau) \right) \cdot C_1 D^\sigma y(\tau) \right| - f \left( \tau, x(\tau) \cdot C_1 D^\sigma x(\tau) \right) \right| \\
+ \left. |g(\tau, y(\tau), I^\sigma y(\tau)) - g(\tau, x(\tau), I^\sigma x(\tau))| \right| \right. \right. \frac{d\tau}{\tau} \\
+ \left. \frac{A}{(\log s)^\mu} |y(s) - x(s)| \right\} \frac{ds}{s} \\
+ \frac{1}{\sum_{i=1}^n \lambda_i (\log \eta_i)^{\delta_i}} \frac{1}{\Gamma(\alpha + \delta_i)} \\
\times \left. \int_1^\eta_i \left( \log \frac{\eta_i}{s} \right)^{\alpha + \delta_i - 1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta - 1} \right. \right. \\
\times \left. \left. \left. B \left| f \left( \tau, y(\tau) \right) \cdot C_1 D^\sigma y(\tau) \right| - f \left( \tau, x(\tau) \cdot C_1 D^\sigma x(\tau) \right) \right| \\
+ \left. |g(\tau, y(\tau), I^\sigma y(\tau)) - g(\tau, x(\tau), I^\sigma x(\tau))| \right| \right. \right. \frac{d\tau}{\tau} \\
+ \left. \frac{A}{(\log s)^\mu} |y(s) - x(s)| \right\} \frac{ds}{s} + \left[ \left( \log t \right)^{1+1} \Gamma(\alpha+2) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} \right. \\
+ \frac{1}{\sum_{i=1}^n \lambda_i (\log \eta_i)^{\delta_i}} \frac{1}{\Gamma(\alpha + \delta_i + 1)} \\
\sum_{i=1}^n \lambda_i (\log \eta_i)^{\alpha + \delta_i} \right] \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha - 1}.
\]
\[
\times \left( \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{s}{\tau} \right)^{\beta - 1} \right)
\times \left( B \left| f \left( \tau, y(\tau) \cdot \int_0^\tau d^\sigma y(\tau) \right) - f \left( \tau, x(\tau) \cdot \int_0^\tau d^\sigma x(\tau) \right) \right| 
+ \left| g \left( \tau, y(\tau) \cdot \int_0^\tau D^\sigma y(\tau) \right) - g \left( \tau, x(\tau) \cdot \int_0^\tau D^\sigma x(\tau) \right) \right| \frac{d\tau}{\tau} \right) 
+ \frac{A}{(\log s)^\mu} \left| y(s) - x(s) \right| \frac{ds}{s} + \left( \frac{(\log r)^\alpha}{\Gamma(\alpha + 1)} \right)
+ \frac{1 - \frac{1}{\alpha}}{\Gamma(\alpha + 2)} \left| 1 - \frac{1}{\alpha} \sum_{i=1}^n \lambda_i (\log \eta_i)^{\alpha + \delta + 1} \right| \Gamma(\alpha + \delta_i + 2) 
\times \left[ \frac{1}{\sum_{i=1}^n \lambda_i \left( \frac{\log \eta_i}{\Gamma(\alpha + \delta_i + 1)} - 1 \right)} \right]
\times \left( B \left| f \left( \tau, y(\tau) \cdot \int_0^\tau d^\sigma y(\tau) \right) - f \left( \tau, x(\tau) \cdot \int_0^\tau d^\sigma x(\tau) \right) \right| 
+ \left| g \left( \tau, y(\tau) \cdot \int_0^\tau D^\sigma y(\tau) \right) - g \left( \tau, x(\tau) \cdot \int_0^\tau D^\sigma x(\tau) \right) \right| \frac{d\tau}{\tau} \right)
\right)
\]

which implies that

\[
\|Sy - Sx\|_\infty \leq (B\Delta f + \Delta g) \left\{ \frac{1}{\Gamma(\alpha + \beta + 1)} + \frac{\sum_{i=1}^n \lambda_i (\log \eta_i)^{\alpha + \beta + \delta_i}}{\left[ \sum_{i=1}^n \lambda_i \left( \frac{\log \eta_i}{\Gamma(\alpha + \delta_i + 1)} - 1 \right) \Gamma(\alpha + \delta_i + 1) \right]} \right\}
\times \frac{1}{\sum_{i=1}^n \lambda_i \left( \frac{\log \eta_i}{\Gamma(\alpha + \delta_i + 1)} - 1 \right)} \left\{ \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha + \beta + 1)} \right\}
\times \frac{1}{\Gamma(\beta + 1)} \left\{ 1 - \frac{1}{\alpha} \sum_{i=1}^n \lambda_i (\log \eta_i)^{\alpha + \delta + 1} \Gamma(\alpha + \delta_i + 2) \right\}
\times \left\{ 1 - \frac{1}{\alpha} \sum_{i=1}^n \lambda_i \left( \frac{\log \eta_i}{\Gamma(\alpha + \delta_i + 1)} - 1 \right) \Gamma(\alpha + \delta_i + 1) \right\}
\times \left\{ \frac{1}{\sum_{i=1}^n \lambda_i \left( \frac{\log \eta_i}{\Gamma(\alpha + \delta_i + 1)} - 1 \right) \Gamma(\alpha + \delta_i + 1)} \right\}
\times \|y - x\|_E
\]

(17)
Also, one can see that

\[
D^\sigma Sx(t) = \frac{1}{\Gamma(\alpha - \sigma)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - \sigma - 1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta - 1} \right) \left( h(\tau) - Bf \left( \tau, x(\tau), C_H D^\sigma x(\tau) \right) - g(\tau, x(\tau), p^0 x(\tau)) \right) \frac{d\tau}{\tau} \frac{ds}{s} + \left( \frac{\log t}{\Gamma(\alpha - \sigma + 1)} - \frac{(\log t)^{\alpha - \sigma}}{\Gamma(\alpha - \sigma + 2)} \right) \\
\times \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha - 1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta - 1} \right) \left( h(\tau) - Bf \left( \tau, x(\tau), C_H D^\sigma x(\tau) \right) - g(\tau, x(\tau), p^0 x(\tau)) \right) \frac{d\tau}{\tau} \\
- \frac{A}{(\log s)^{\alpha - \sigma}} \frac{ds}{s} + \left( \frac{(\log t)^{\alpha - \sigma}}{\Gamma(\alpha - \sigma + 1)} - \frac{(1 - \frac{1}{\alpha}) (\log t)^{\alpha - \sigma + 1}}{\Gamma(\alpha - \sigma + 2)} \right) \\
\times \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{e}{\tau} \right)^{\beta - 1} \left( h(\tau) - Bf \left( \tau, x(\tau), C_H D^\sigma x(\tau) \right) - g(\tau, x(\tau), p^0 x(\tau)) \right) \frac{d\tau}{\tau}. 
\]
We have also

\[
\|D^\sigma S y - D^\sigma S x\|_\infty \leq \sup_{t \in J} \left\{ \frac{1}{\Gamma(\alpha - \sigma)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - \sigma - 1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta - 1} \right) \times \left| B \right| f \left( \tau, y(\tau) \right) - f \left( \tau, x(\tau) \right) \right| d\tau \right. \\
+ \left. \frac{A}{(\log s)^{\mu}} \left| y(s) - x(s) \right| \frac{d s}{s} + \left( \frac{(\log t)^{\alpha - \sigma + 1}}{\Gamma(\alpha - \sigma + 2)} + \frac{(\log t)^{\alpha - \sigma}}{\Gamma(\alpha - \sigma + 1)} \right) \right. \\
\times \left. \left| B \right| f \left( \tau, y(\tau) \right) - f \left( \tau, x(\tau) \right) \right| d\tau \right. \\
+ \left. \left[ B \right| f \left( \tau, y(\tau) \right) - f \left( \tau, x(\tau) \right) \right| d\tau \right. \\
\left. + \left| g \left( \tau, y(\tau), P^t y(\tau) \right) - g \left( \tau, x(\tau), P^t x(\tau) \right) \right| d\tau \right. \\
+ \left. \left[ A + \right| f \left( \tau, y(\tau), P^t y(\tau) \right) - f \left( \tau, x(\tau), P^t x(\tau) \right) \right| d\tau \right. \\
\left. + \left| g \left( \tau, y(\tau), P^t y(\tau) \right) - g \left( \tau, x(\tau), P^t x(\tau) \right) \right| d\tau \right. \\
\left. \left( B \Delta f + \Delta g \right) \left[ \frac{1}{\Gamma(\alpha + \beta - \sigma + 1)} + \left( \frac{1}{\Gamma(\alpha - \sigma + 1)} \right) \right] \right. \\
+ \left. \frac{1}{\Gamma(\alpha - \sigma + 2)} \frac{1}{\alpha \Gamma(\alpha + \beta + 1)} \right. \\
\left. + \left( \frac{1}{\Gamma(\alpha - \sigma + 1)} + \frac{1}{\Gamma(\alpha - \sigma + 2)} \right) \frac{1}{\Gamma(\beta + 1)} \right. \left. \right| y - x \right|_E \\
\left. + A \frac{\Gamma(1 - \mu)}{\Gamma(\alpha - \mu - \sigma + 1)} + \left( \frac{1}{\Gamma(\alpha - \sigma + 2)} + \frac{1}{\Gamma(\alpha - \sigma + 1)} \right) \right. \\
\left. \times \frac{\Gamma(\alpha + 2) \Gamma(1 - \mu)}{\alpha \Gamma(\alpha - \mu - 1)} \right. \left. \| y - x \|_E \right. \\
\leq \left( (B \Delta f + \Delta g) \bar{\Omega} + A \bar{\Omega}_2 \right) \| y - x \|_E.
\]

\[ (19) \]
Thus, we obtain the inequality:

\[(20) \quad \|Sy - Sx\|_\infty \leq \max \left\{ \left( B\Delta f + \Delta g \right) \Omega_1 + A\Omega_2 ; \left( B\Delta f + \Delta g \right) \overline{\Omega}_1 + A\overline{\Omega}_2 \right\} \|y - x\|_E. \]

We conclude that \( S \) is contractive. As a consequence of Banach contraction principle, we deduce that \( S \) has a unique fixed point which is the exact solution of \((1)\).

Now, we shall use the following lemma to prove the second main result.

**Lemma 3.1:** Let \( E \) be a Banach space and \( S : E \to E \) be a completely continuous operator. If one has bounded the set

\[ F = \{ x \in E : x = \mu Sx, \ 0 < \mu < 1 \}, \]

then \( S \) has a fixed point in \( E \).

**Theorem 3.2:** Assume that the hypotheses \((H_1)\) and \((H_3)\) are satisfied. Then, \((1)\) has at least one solution defined over \( J \).

**Proof:** We need the bounded ball: \( B_r = \{ x \in E : \|x\|_E \leq r \} \) and we proceed as follows:

**Step 1:** The application \( S \) is continuous on \( E \). The proof is trivial and then it can be omitted.

**Step 2:** Uniform boundedness: For all \( x \in C_r \) and by \((H_3)\) we have

\[ \|Sx\|_\infty \leq [BL_1 + L_2 + L_3] \left\{ \frac{1}{\Gamma(\alpha + \beta + 1)} + \frac{\sum_{i=1}^{n} \lambda_i (\log \eta_i) (\alpha + \beta + \delta_i + 1)}{\sum_{i=1}^{n} \lambda_i (\log \eta_i) (\delta_i + 1) - 1} \right\} \]

\[ + \left[ \frac{\alpha + 2}{\Gamma(\alpha + 2)} + \frac{1}{\sum_{i=1}^{n} \lambda_i (\log \eta_i) (\delta_i + 1)} \right] \left( \frac{\sum_{i=1}^{n} \lambda_i (\log \eta_i) (\alpha + \delta_i + 1)}{\Gamma(\alpha + \delta_i + 2)} \right) \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha + \beta + 1)} \]

\[ + \left( \frac{\alpha + 1 + \frac{1}{\alpha}}{\Gamma(\alpha + 2)} \right) \frac{1 - \frac{1}{\alpha}}{\sum_{i=1}^{n} \lambda_i (\log \eta_i) (\alpha + \delta_i + 1)} \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha + \beta + 1)} \]
\[\times \left( \sum_{i=1}^{n} \frac{1}{\lambda_i} \left( \log \eta_i \right)^{\alpha + \delta_i} \Gamma(\alpha + \delta_i + 1) \right) \left( \sum_{i=1}^{n} \lambda_i \frac{1}{\Gamma(\alpha + \delta_i + 1)} - 1 \right) \]

\[+ Ar \left\{ \frac{1}{\Gamma(\alpha + 2)} + \sum_{i=1}^{n} \frac{\lambda_i}{\Gamma(\delta_i + 1)} - 1 \right\} \]

\[\times \left( \sum_{i=1}^{n} \lambda_i \frac{1}{\Gamma(\alpha + \delta_i + 1)} \right) \]

\[\times \frac{\Gamma(\alpha + 2) \Gamma(1 - \mu) \Gamma(\alpha - \mu + 1) \Gamma(\alpha - \sigma + 1)}{\alpha \Gamma(\alpha - \mu + 1) \Gamma(\alpha - \sigma + 1) \Gamma(\beta + 1)} \]

\[\leq (BL_1 + L_2 + L_3) \Omega_1 + Ar \Omega_2 < +\infty, \]

and

\[\|D^\sigma Sx\|_\infty \leq (BL_1 + L_2 + L_3) \left[ \frac{1}{\Gamma(\alpha + \beta - \sigma + 1)} + \left( \frac{1}{\Gamma(\alpha - \sigma + 1)} \right) \right. \]

\[+ \left. \frac{1}{\Gamma(\alpha - \sigma + 2)} \right] \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha + \beta + 1)} \]

\[+ \left( \frac{1}{\Gamma(\alpha - \sigma + 1)} + \frac{1 - \frac{1}{\alpha}}{\Gamma(\alpha - \sigma + 2)} \right) \frac{1}{\Gamma(\beta + 1)} \]

\[\leq (BL_1 + L_2 + L_3) \Omega_1 + Ar \Omega_2 < +\infty. \]

Hence, for any \(x \in C_r\), we obtain \(\|Sx\|_E < +\infty\), which means in particular that the operator \(S\) is uniformly bounded on \(C_r\).

**Step 3:** The application \(S\) maps bounded sets into equicontinuous sets of \(E\):
Let $t_1, t_2 \in J$ with $t_1 < t_2$ and let $C_r$ be the above bounded set of $E$, for all $x \in C_r$, we have

$$
|Sx(t_2) - Sx(t_1)| \leq \left| \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left( \log \frac{t_2}{s} \right)^{\alpha-1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{-1} \right.ight.
$$

$$
\times \left[ h(\tau) - Bf\left(\tau, x(\tau), C H D^{\sigma} x(\tau)\right) - g\left(\tau, x(\tau), C H P x(\tau)\right) \right] \frac{d\tau}{\tau}
$$

$$
- \frac{A}{(\log s)^\beta} x(s) \int_1^{t_1} \left( \log \frac{t_1}{s} \right)^{\alpha-1} d\tau
$$

$$
\times \frac{1}{\Gamma(\alpha + 2)} \int_1^{e} \left( \log \frac{e}{s} \right)^{\alpha-1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{-1} \right)
$$

$$
\times \left[ h(\tau) - Bf\left(\tau, x(\tau), C H D^{\sigma} x(\tau)\right) - g\left(\tau, x(\tau), C H P x(\tau)\right) \right] \frac{d\tau}{\tau}
$$

$$
- \frac{A}{(\log s)^\beta} x(s) \int_1^{t_1} \left( \log \frac{t_1}{s} \right)^{\alpha-1} d\tau
$$

$$
+ \left[ \frac{(\log t_2)^{\alpha+1} - (\log t_1)^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{(\log t_1)^{\alpha} - (\log t_2)^{\alpha}}{\Gamma(\alpha + 1)} \right]
$$

$$
\times \frac{1}{\Gamma(\alpha + 2)} \int_1^{e} \left( \log \frac{e}{s} \right)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{-1} \right.
$$

$$
\times \left[ h(\tau) - Bf\left(\tau, x(\tau), C H D^{\sigma} x(\tau)\right) - g\left(\tau, x(\tau), C H P x(\tau)\right) \right] \frac{d\tau}{\tau}
$$

$$
\leq \left[ BL_1 + L_2 + L_3 \right] \left\{ \left[ \left( \log t_2 \right)^{\alpha+\beta} - \left( \log t_1 \right)^{\alpha+\beta} \right] \right.
$$

$$
+ \left( \frac{(\log t_2)^{\alpha+1} - (\log t_1)^{\alpha+1}}{\Gamma(\alpha + 2)} \right)
$$

$$
+ \left( \frac{(\log t_1)^{\alpha} - (\log t_2)^{\alpha}}{\Gamma(\alpha + 1)} \right) \frac{1}{\alpha} \frac{1}{\Gamma(\alpha + \beta + 1)}
$$

$$
+ \left( \frac{(\log t_2)^{\alpha} - (\log t_1)^{\alpha}}{\Gamma(\alpha + 1)} \right)
$$

$$
+ \left( \frac{1 - \frac{1}{\alpha}}{(\log t_1)^{\alpha+1} - (\log t_2)^{\alpha+1}} \right) \frac{1}{\Gamma(\beta + 1)} \right\} + \left[ \left( \log t_2 \right)^{\alpha-\mu} - \left( \log t_1 \right)^{\alpha-\mu} \right]
$$

$$
+ \left( \frac{(\log t_2)^{\alpha+1} - (\log t_1)^{\alpha+1}}{\Gamma(\alpha + 2)} \right)
$$
Similarly as before, we have

$$\left| D^\sigma S_x(t_2) - D^\sigma S_x(t_1) \right| \leq \left| \frac{1}{\Gamma(\alpha - \sigma)} \int_1^{t_2} \left( \log \frac{t_2}{s} \right)^{\alpha - \sigma - 1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right) \beta^{-1} \right) dh(\tau) - B f \left( \tau, x(\tau), C_{ij} D^\sigma x(\tau) \right) \right| \frac{d\tau}{\tau}$$

- \frac{A}{(\log s)^\mu x(s)} \frac{ds}{s}

$$\left[ \frac{1}{\Gamma(\alpha - \sigma)} \int_1^{t_1} \left( \log \frac{t_1}{s} \right)^{\alpha - \sigma - 1} \left( \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right) \beta^{-1} \right) dh(\tau) - B f \left( \tau, x(\tau), C_{ij} D^\sigma x(\tau) \right) \right| \frac{d\tau}{\tau}$$

- \frac{A}{(\log s)^\mu x(s)} \frac{ds}{s}

$$\left( \frac{\log t_2^{\alpha - \sigma + 1} - \log t_1^{\alpha - \sigma + 1}}{\Gamma(\alpha - \sigma + 2)} + \frac{(\log t_1^{\alpha - \sigma} - \log t_2^{\alpha - \sigma})}{\Gamma(\alpha - \sigma + 1)} \right) \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha + \beta + 1)}$$

$$\left( \log t_2^{\alpha + \mu - \sigma} - \log t_1^{\alpha + \mu - \sigma} \right) \frac{\Gamma(1 - \mu)}{\Gamma(\alpha - \mu + 1)}$$

$$+ \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha + \beta + 1)}$$

$$+ \frac{\Gamma(1 - \mu)}{\Gamma(\alpha - \mu + 1)}$$

$$+ \frac{\Gamma(1 - \mu)}{\Gamma(\alpha - \mu + 1)}$$

$$+ \frac{\Gamma(1 - \mu)}{\Gamma(\alpha - \mu + 1)}.$$
The right hand sides of (22) and (23) tend to zero independently of \((u_1, u_2)\) as \(t_1 \to t_2\).

As a consequence of Steps 1, 2 and 3, thanks to Ascoli-Arzela theorem, we conclude that \(S\) is completely continuous.

**Step 4:** The set

\[
F = \{ x \in X : x = \omega Sx, \ 0 < \omega < 1 \}
\]

is bounded.

Let \(x \in F\), then, we have \(x = \omega Sx\) for some \(0 < \omega < 1\). Hence, we can write

\[
\| \omega Sx \|_\infty \leq \omega [BL_1 + L_2 + L_3] \left\{ \frac{1}{\Gamma(\alpha + \beta + 1)} + \sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha + \beta + \delta_i}}{\Gamma(\alpha + \beta + \delta_i + 1)} \right\}
+ \left[ \frac{\alpha + 2}{\Gamma(\alpha + 2)} + \frac{1}{\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i + 1)} - 1} \right] \left( \frac{n}{\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i + 1)} - 1} \right) \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha + \beta + 1)}
+ \left( \frac{\alpha + 1 + 1 - \frac{1}{\Gamma(\alpha + 2)}}{\Gamma(\alpha + 2)} + \left| 1 - \frac{1}{\alpha} \sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha + \delta_i + 1}}{\Gamma(\alpha + \delta_i + 2)} \right| \right) \frac{\Gamma(\alpha + 1 + 1 - \frac{1}{\Gamma(\alpha + 2)})}{\Gamma(\alpha + 2)}
\times \left( \frac{n}{\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i + 1)} - 1} \right) \frac{1}{\Gamma(\beta + 1)}
\times \left( \frac{\Gamma(1 - \mu)}{\Gamma(\alpha - \mu + 1)} + \frac{\sum_{i=1}^{n} \lambda_i \frac{\Gamma(1 - \mu)(\log \eta_i)^{\alpha + \delta_i - \mu}}{\Gamma(\alpha + \delta_i - \mu + 1)}}{\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i + 1)} - 1} \right)
+ \omega A r \left\{ \frac{\Gamma(1 - \mu)}{\Gamma(\alpha - \mu + 1)} + \frac{\sum_{i=1}^{n} \lambda_i \frac{\Gamma(1 - \mu)(\log \eta_i)^{\alpha + \delta_i - \mu}}{\Gamma(\alpha + \delta_i - \mu + 1)}}{\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i + 1)} - 1} \right\}
\times \left( \frac{\alpha + 2}{\Gamma(\alpha + 2)} + \frac{1}{\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i + 1)} - 1} \right) \left[ \sum_{i=1}^{n} \lambda_i \frac{\Gamma(1 - \mu)(\log \eta_i)^{\alpha + \delta_i}}{\Gamma(\alpha + \delta_i + 1)} + \sum_{i=1}^{n} \lambda_i \frac{\Gamma(1 - \mu)(\log \eta_i)^{\alpha + \delta_i + 1}}{\Gamma(\alpha + \delta_i + 2)} \right]
\times \frac{\Gamma(\alpha + 2) \Gamma(1 - \mu)}{\alpha \Gamma(\alpha - \mu + 1)}
\]

(24)
\[
\leq \omega [(BL_1 + L_2 + L_3) \Omega_1 + Ar\Omega_2] < +\infty,
\]

and

\[
\|D^{\alpha} \omega Sx\| \leq \omega (BL_1 + L_2 + L_3) \left[ \frac{1}{\Gamma(\alpha + \beta - \sigma + 1)} + \frac{1}{\Gamma(\alpha - \sigma + 1)} \right]
\]
\[
+ \frac{1}{\Gamma(\alpha - \sigma + 2)} \left[ \frac{\Gamma(\alpha + 2)}{\alpha \Gamma(\alpha + \beta + 1)} + \frac{1}{\Gamma(\alpha - \sigma + 1)} \right]
\]
\[
+ \left( \frac{1}{\Gamma(\alpha - \sigma + 2)} + \frac{1}{\Gamma(\alpha - \sigma + 1)} \right) \frac{\Gamma(\alpha + 2) \Gamma(1 - \mu)}{\alpha \Gamma(\alpha - \mu + 1)}
\]
\[
\leq \omega [(BL_1 + L_2 + L_3) \Omega_1 + Ar\Omega_2] < +\infty.
\]

From (24) and (25), we see that \(\|x\|_E < +\infty\). Consequently, \(F\) is bounded.

As a consequence of the above Schaefer theorem, we conclude that \(S\) has a fixed point which is a solution of (1).

4. Examples

We discuss the examples. We begin by considering the problem:

Example 4.1:

\[
\left\{ \begin{array}{l}
C_D^{1.7} \left( C_H D^{0.54} + \frac{10^{-2}}{\ln(t + 0.5)} \right) x(t) + \frac{|t - C_H D^{0.5} x(t)|}{30^4 e^{2t + 1}} = e^{4t + 1}, t \in ]1, e[,

\left( C_H D^{0.54} + 3 \right) x(e) = 0, x(1) = x(e) = \sum_{i=1}^{2} \lambda_i I^{3\delta} x(\eta_i),
\end{array} \right.
\]

where \(\beta = 1.7, \alpha = 0.54, \mu = 0.5, \sigma = 0.4, \rho = 0.3, \delta_1 = 0.2, \delta_2 = 0.25, A = 10^{-2}, B = \frac{1}{30\pi}, \lambda_1 = 1.5, \lambda_2 = 2, \eta_1 = 2, \eta_2 = 2.33,

and \(\frac{1}{30\pi^2} \times \frac{1}{e^3} + \frac{1}{99e} \) \(3.7966 + 10^{-2} \times 10.461 : 0.11878\)

\[
f(t, x(t), C_H D^{0.5} x(t)) = \frac{|t - C_H D^{0.4} x(t)|}{e^{2t + 1} \left( 1 + |t - C_H D^{0.4} x(t)| \right)},
\]

\[
g(t, x(t), I^{0.4} x(t)) = \frac{\sin(t + x(t) + 10^3 x(t))}{39e^t},
\]
we have: \( \Delta_f = \frac{1}{e^\tau}, \Delta_g = \frac{1}{9\log e} \). Since, it is found that \( \Omega_1 = 5.6332, \Omega_2 = 20.653, \Omega_1 = 3.7966, \Omega_2 = 10.461 \) and \( \left( B \Delta_f + \Delta_g \right) \Omega_1 + A \Omega_2 = 0.22756, \left( B \Delta_f + \Delta_g \right) \Omega_1 + A \Omega_2 = 0.11878 \). Note that

\[
\max \left\{ \left( B \Delta_f + \Delta_g \right) \Omega_1 + A \Omega_2 ; \left( B \Delta_f + \Delta_g \right) \Omega_1 + A \Omega_2 \right\} = 0.22756 < 1.
\]

Thus, Theorem 3.1 implies that the problem (26) has a unique solution on \([1, e]\).

Next example illustrates Theorem 3.2

**Example 4.2:** Consider the Caputo-Hadamard fractional problem:

\[
\begin{aligned}
\left\{ \begin{array}{l}
C_H D^{1.8} \left[ C_H D^{0.7} + \frac{7 - 2}{(\log t)^{0.4}} \right] x(t) + \frac{1}{4e^t} \sin \left( \frac{C_H D^{0.5} x(t) + 2t + 1}{3t + 1} \right) \\
\cos \left( x(t) - 5^{0.4} x(t) + t \right) = \frac{1}{t(t + 2)^4}, t \in [1, e[,
\end{array} \right.
\end{aligned}
\]

(27)

where \( \beta = 1.8, \alpha = 0.7, \mu = 0.6, \sigma = 0.5, \rho = 0.4, \delta_1 = 0.2, \delta_2 = 0.3, A = 7^{-2}, B = \frac{1}{4e^t}, \lambda_1 = 1, \lambda_2 = 2, \eta_1 = 2.1, \eta_2 = 2.4, r = 0.99 \) and

\[
\begin{aligned}
f(t, x(t), C_H D^{0.5} x(t)) &= \sin \left( \frac{C_H D^{0.5} x(t) + 2t + 1}{3t + 1} \right), \\
g(t, x(t), 5^{0.4} x(t)) &= \cos \left( x(t) - 5^{0.4} x(t) + t \right), h(t) = \frac{1}{t(t + 2)^4},
\end{aligned}
\]

We have: \( L_1 = \frac{1}{4}, L_2 = \frac{1}{10\log 2}, L_3 = \frac{1}{8} \). So, we obtain: \( \Omega_1 = 3.7053, \Omega_2 = 23.155, \Omega_1 = 2.7075, \Omega_2 = 11.763 \) and \( \| Sx \|_\infty \leq 1.0597, \| D^{0.5} Sx \|_\infty \leq 0.67012, \| \omega Sx \|_\infty \leq 1.0597 \omega, \| D^\rho \omega Sx \|_\infty \leq 0.67012 \omega : 0 < \omega < 1 \). Hence, all conditions of theorem 3.2 are holds true, witch implies that

the problem (27) has at least one solution on \([1, e]\).

**Example 4.3:** Consider the following third problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
C_H D^{1.8} \left( C_H D^{1.8} + \frac{1}{(\log t)^{0.4}} \right) x(t) + \frac{1}{4 + e^{1 + C_H D^{1.8}}} \left( C_H D^{1.8} \right) \\
\sin x(t) + \frac{\sqrt{2} x(t)}{1 + |x|} = \frac{e^{-t}}{3 \sqrt{t^2 + 9}}, t \in [1, e[,
\end{array} \right.
\end{aligned}
\]

(28)

For this third example, we note that \( \beta = \frac{3}{2}, \alpha = \frac{1}{2}, \mu = \frac{1}{4}, \rho = \frac{3}{4}, \delta_1 = \delta_2 = 0.1, A = 1, B = \frac{1}{4 + e}, \)
and
\[ f(t, x(t), \frac{C_H}{ \sqrt{1+C_H^2} } D_{-}^{\frac{1}{2}} x(t)) = \frac{C_H D_{-}^{\frac{1}{2}} x(t)}{1+C_H D_{-}^{\frac{1}{2}}}, \]
\[ g(t, x(t), t^{0.4} x(t)) = \frac{1}{\sqrt{25+t^2}} \left( \sin x(t) + \frac{|x|}{1+|x|} + t^{3} x(t) \right), \]
\[ h(t) = \frac{e^{-t}}{3 \sqrt{t^2 + 9}}. \]

It’s clear \( f, g \) and \( h \) are continuous and bounded functions. Thus the conditions of Theorem 3.2 are valid, then the above example 4.3 has at least one solution on \([1, e] \).

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

**REFERENCES**

[1] R.P. Agarwal, D. O’Regan, V. Lakshmikantham, Quadratic forms and nonlinear non-resonant singular second order boundary value problems of limit circle type, Z. Anal. Anwendungen. 20 (2001), 727-737.
[2] R.P. Agarwal, D. O’Regan, Existence Theory for Singular Initial and Boundary Value Problems: A Fixed Point Approach, Appl. Anal. 81(2) (2002), 391-434.
[3] R.P. Agarwal, D. O’Regan, S. Stanček, Positive Solutions for Mixed Problems of Singular Fractional Differential Equations, Math. Nachr. 285(1) (2012), 27-41.
[4] Z. Bai, W. Sun, Existence and Multiplicity of Positive Solutions for Singular Fractional Boundary Value Problems, Computers Math. Appl. 63(9) (2012), 1369-1381.
[5] Y. Bai, H. Kong, Existence of solutions for nonlinear Caputo-Hadamard fractional differential equations via the method of upper and lower solutions, J. Nonlinear Sci. Appl. 10 (2017), 5744-5752.
[6] I.M. Batıha, R.B. Albadarneh, S. Momani, I.H. Jebril, Dynamics analysis of fractional-order Hopfield neural networks, Int. J. Biomath. 2020 (2020), 2050083.
[7] Z. Bekkouche, Z. Dahmani, G. Zhang, Solutions and Stabilities for a 2D-NonHomogeneous Lane-Emden Fractional System, Int. J. Open Probl. Computer Sci. Math. 11 (2018), 1-14.
[8] M. Bezzou, Z. Dahmani, A. Ndiaye, Langevin differential equation of fractional order in non compactness Banach space, J. Interdiscip. Math. 23(4) (2020), 857-876.
[9] Z. Dahmani, A. Taieb, A Coupled System of Fractional Differential Equations Involing Two Fractional Orders, ROMAI J. 11, (2015), 141-177.
[10] L. Ghaffour and Z. Dahmani. Fractional differential equations with arbitrary singularities, J. Inform. Optim. Sci. 39 (2018), 1547-1565.
[11] Y. Gouari, Z. Dahmani, A. Ndiaye, A generalized sequential problem of Lane-Emden type via fractional calculus, Moroccan J. Pure Appl. Anal. 6 (2020), 14-29.

[12] Y. Gouari, Z. Dahmani, M.Z. Sarikaya, A non local multi-point singular fractional integro-differential problem of lane-emden type, Math. Methods Appl. Sci. 43 (2020), 6938-6949.

[13] Y. Gouari, Z. Dahmani, I.H Jebril, Application of fractional calculus on a new differential problem of duffing type, Adv. Math. Sci. J. 9(12) (2020), 10989-11002.

[14] J. Hadamard, Essai sur l’étude des fonctions données par leur développement de Taylor, J. Mat. Pure Appl. Ser. 8 (1892), 101-186.

[15] M. Houas, M. Beuzziou, Existence and Stability Results for Fractional Differential Equations with two Caputo Fractional Derivatives, Facta Univ. Ser. Math. Inform. 34(2) (2019), 341-357.

[16] R.W. Ibrahim, Existence of Nonlinear Lane-Emden Equation of Fractional Order, Miskolc Math. Notes. 13 (2012), 39-52.

[17] F. Jarad, T. Abdeljawad, D. Beleanu, Caputo-type modification of the Hadamard fractional derivative, Adv. Differ. Equations. 2012 (2012), 142.

[18] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam. (2006).

[19] S.M. Mechee, N. Senu, Numerical Study of Fractional Differential Equations of Lane-Emden Type by Method of Collocation, Appl. Math. 3 (2012), 851-856.

[20] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York. (1993).

[21] S.A. Okunuga, J.O. Ehigie, A.B. Sofoluwe, Treatment of Lane-Emden Type Equations via Second Derivative Backward Differentiation Formula Using Boundary Value Technique, Proceedings of the World Congress on Engineering. Vol. IWCE, (2012). July 4-6, 2012, London, U.K.

[22] I. Podlubny, Fractional Differential Equations, Acad. Press, London. 1999.

[23] J. Serrin, H. Zou, Existence of Positive Solutions of Lane-Emden Systems, Atti Del Sem. Mat. Fis. Univ. Modena. 46 (1998), 369-380.

[24] W. Shammakh, A study of Caputo-Hadamard-Type fractional differential equations with nonlocal boundary conditions, Journal of Function Spaces. (2016).

[25] D.R. Smart, Fixed Point Theorems, Cambridge University Press, Cambridge, UK. 1980.

[26] X. Zhang, On impulsive partial differential equations with Caputo-Hadamard fractional derivatives, Adv. Differ. Equ. 2016 (2016), 281.