Interplay of the Scaling Limit and the Renormalization Group: Implications for Symmetry Restoration

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Symmetry restoration is usually understood as a renormalization group induced phenomenon. In this context, the issue of whether one-loop RG equations can be trusted in predicting symmetry restoration has recently been the subject of much debate. Here we advocate a more pragmatic point of view and expand the definition of symmetry restoration to encompass all situations where the physical properties have only a weak dependence upon an anisotropy in the bare couplings. Moreover we concentrate on universal properties, and so take a scaling limit where the physics is well described by a field theory. In this context, we find a large variety of models that exhibit, for all practical purposes, symmetry restoration: even if symmetry is not restored in a strict sense, physical properties are surprisingly insensitive to the remaining anisotropy.

Although we have adopted an expanded notion of symmetry restoration, we nonetheless emphasize that the scaling limit also has implications for symmetry restoration as a renormalization group induced phenomenon. In all the models we considered, the scaling limit turns out to only permit bare couplings which are nearly isotropic and small. Then the one-loop beta-function should contain all the physics and higher loop orders can be neglected. We suggest that this feature generalizes to more complex models. We exhibit a large class of theories with current-current perturbations (of which the SO(8) model of interest in two-leg Hubbard ladders/armchair carbon nanotubes is one) where the one-loop beta-functions indicates symmetry restoration and so argue that these results can be trusted within the scaling limit.

I. INTRODUCTION: SYMMETRY RESTORATION

Under the most prevalent definition, symmetry restoration occurs as a Hamiltonian is attracted under a renormalization group (RG) flow to a manifold possessing a higher symmetry than indicated by the original bare or microscopic theory. A recent example in the literature of this phenomenon is found in the work of Lin et al.\textsuperscript{[1]} This work addresses the low energy behaviour of a two-leg Hubbard ladder (or, equivalently, the armchair carbon nanotube). These authors argued that a Hamiltonian for these systems with ‘generic’ (short-range) interactions flow under an 1-loop RG towards the SO(8) Gross-Neveu model, or more specifically, the set of couplings \{\lambda_{i}\} flow onto fixed ratios indicative of SO(8) Gross-Neveu.

Two more examples of symmetry restoration as suggested by the 1-loop RG are provided by the anisotropic Kondo model and the U(1) Thirring model. The anisotropic Kondo model is described by the Hamiltonian

\[
H = \int dx \sum_{\sigma} -i c^\dagger_{\sigma}(x) \partial_x c_{\sigma}(x) + \frac{1}{2} \sum_{\sigma\sigma'} c^\dagger_{\sigma}(0) \left\{ g_{||} \sigma^x_{\sigma\sigma'} s^x + g_{\perp} \left( \sigma^x_{\sigma\sigma'} s^x + \sigma^y_{\sigma\sigma'} s^y \right) \right\} c_{\sigma'}(0),
\]

where \( c_{\sigma}(x) \) is a Fermi field of spin, \( \sigma \), while the U(1) Thirring model is given by the Lagrangian

\[
\mathcal{L} = i \bar{\psi}_\alpha \gamma_{\mu} \partial^\mu \psi_\alpha + \frac{1}{4} g_\parallel (j_x)^2 + \frac{1}{4} g_\perp ((j_x)^2 + (j_y)^2),
\]

where \( j_\mu = \bar{\psi}_\alpha \gamma_\mu \tau_\alpha \beta \psi_\beta \), and \( \psi \) is a doublet of Dirac spinors. In both these models the 1-loop RG equations read,

\[
\begin{align*}
\frac{dg_{||}}{dt} &= -cg_{\perp}^2; \\
\frac{dg_{\perp}}{dt} &= -cg_{||}g_{\perp},
\end{align*}
\]

where \( c \) is a model dependent constant\textsuperscript{[1]}. Thus in the regions where the trajectories flow to strong coupling, both models are attracted to the diagonal \( g_{||} = g_{\perp} \) (the SU(2) invariant line): more precisely, the ratio \( |g_{\perp}/g_{||}| \) converges to unity. The natural conclusion one derives from this feature is that the physics at large distances (large compared with the UV cut-off), or low energies, is well described by an isotropic model (usually an isotropic field theory).

In drawing this conclusion, two immediate difficulties present themselves. There is the possibility that higher loop orders make the diagonal unstable in which case, of

\textsuperscript{1}In the conventions we chose in this paper, \( c > 0 \) for the U(1) Thirring model and \( c < 0 \) for the Kondo model.
course, the symmetry restoration will not in fact occur. This possibility will not be relevant for the examples and discussion in the paper at hand, although, in general, it is a genuine concern. Even upon excluding this scenario, RG flows towards the diagonal do not necessarily indicate that symmetry restoration takes place. Indeed, in all the models of interest here, the interactions are of current-current form and the bare coupling constants are dimensionless. Therefore the physics does not have to depend merely on the ratios of these coupling constants; it might also depend on other combinations that do not flow under the RG. A case in point is precisely the $U(1)$ Thirring model. In this model, there is a quantity $\mu (\mu^2 = g^2_\parallel - g^2_\perp$ at lowest order) that is an RG invariant, and on which physical quantities do depend in a non-trivial way. Therefore, if initially $g_\parallel \neq g_\perp$, symmetry restoration as defined above does not ever occur. Another example is provided by the anisotropic Kondo model. Although the physics of the fixed point of this model is isotropic, the physics of the approach to this same fixed point is not and depends on the same RG invariant, $\mu$. (It governs the amplitudes but not the exponents of the operators controlling the approach to the fixed point.)

This observation in the case of the $U(1)$ Thirring model is one of the main points of Ref. 3. It is further pointed out in this work that the dependence upon $\mu$ can nevertheless be different for different parameter regimes. In the cases of the $U(1)$ Thirring model, it is essentially polynomial in $\mu$ for the region termed AF ($g_\parallel < 0, g_\perp > 0, |g_\parallel| < g_\perp < \pi - |g_\parallel|), but exponential $(\exp(-c s^2 / \mu))$ in the region termed C ($\pi / 2 > g_\perp > 0, g_\perp > |g_\parallel|)$. In the latter case therefore, the dependence on $\mu$ is weak, and so symmetry is, in practice, certainly restored. The authors of Ref. 3 then carry on to conclude that the one loop RG is not reliable, as symmetry restoration does sometimes in fact occur (in region C), while sometimes it does not (region AF).

It is here we come to the crux of this work. As in Ref. 2, we consider an alternative definition of symmetry restoration to include all situations where the low energy behaviour of the theory has only a weak dependence upon the bare anisotropy. To be clear, we now have two definitions of symmetry restoration in play:

1. **Symmetry restoration induced through the renormalization group.**

2. **Symmetry restoration meaning a weak dependence of physical quantities on the bare anisotropy.**

We consider now the consequences of this second, expanded definition. In particular we consider the consequences of combining this definition with insisting upon a field theoretic description of the system. (For the systems discussed in this paper, these turn out to be relativistic field theories.)

Field theoretic descriptions of condensed matter systems are desirable both in that they provide a powerful set of tools and techniques by which the relevant physics can be extracted, and because they represent the physics that is universal in nature, i.e. that carries no dependence upon the microscopic details present in the actual system. In general, a field theoretical description requires the parameters of the model to be in a regime where the correlation length is much larger than UV cut-off (the ‘lattice spacing’). This regime is known as the scaling limit. The anisotropic models considered in this paper are such that they can be studied exactly in a range of parameters including, but by far exceeding, the regime where the theory is in the scaling limit.

We will see for the systems considered in this paper that in order to achieve the scaling limit the bare parameters must be such that the effects of the anisotropy are small. The considerations leading to this conclusion can be phrased in general terms as follows. We begin with some (possibly free) theory governed by an underlying continuous symmetry, some simple Lie-group, $G$. Denote the corresponding Lagrangian, $\mathcal{L}_G$. We will then consider perturbations to $\mathcal{L}_G$ that, in general, break the symmetry, $G$, or in an alternative language, anisotropically deform $G$. The perturbations will typically take the form,

$$\mathcal{L}_{\text{pert}} = \sum_{i=1}^{n} \lambda_i \mathcal{O}_i,$$

where $n$ is the number of generators of the group, $G$, and typically $\mathcal{O}_i$ is an operator associated with the $i$-th generator. If $\lambda_i = \lambda_j$ for every pair $(i, j)$, the symmetry $G$ is preserved, while for differing $\lambda_i$, $G$ is broken. The initial examples mooted in this paper (interacting Hubbard ladders, anisotropic Kondo model, and the $U(1)$ Thirring model) all take this form.

The question of when the theory appears relativistic is then rephrased: for what values of the couplings, $\lambda_i$, does the theory have a relativistic low energy sector? As we will see, this requires for the cases considered here a subset of the bare couplings, $\{\lambda_j\}$, to be taken to 0, while the disjoint subset, $\{\lambda_k\}$, is permitted to be finite. If it did remain finite, we would have the rather odd situation that

$$\frac{\lambda_k}{\lambda_j} = \infty,$$

i.e. the theory would have an “infinite bare anisotropy” (defined here by considering the ratios of the bare coupling constants). This is unphysical: no such ratios would be found in a physical system unless enforced by some symmetry. But the only possible symmetry, $G$, has by
presumption been broken. In order to then remove this pathology we need to take $\lambda_k \to 0$ as well. Now, in the large variety of cases we have considered, it turns out that the physical measures of the anisotropy in the relativistic limit are determined not by the ratios of bare coupling constants, but by quantities which are of $O(\lambda)$ or higher. If all the couplings go to zero, it therefore follows that the relativistic limit will be isotropic, even for arbitrary but finite ratios of these couplings. The lesson we learn is that if a model has a relativistic field theoretic description, the latter must be isotropic. Anisotropic relativistic limits will occur only if one is willing to accept models with infinite bare anisotropy, a rather unphysical requirement. If we return to our definition of symmetry restoration, we thus see in taking the scaling limit, symmetry is restored perfectly, i.e. there is no dependence on any bare anisotropy.

This, of course, is true only when we apply the scaling limit in the strict sense. When we required $\{\lambda_j\}$ to be zero, we did so to ensure the theory looked relativistic at all possible energy scales. However this is needlessly restrictive. We are only interested in the theory at low energy scales and so are only interested in it looking relativistic at these same scales. For example, in the Kondo model, all we would want is the theory to appear relativistic on scales less than some large multiple of the Kondo temperature, $T_k$. Or in the Thirring model, we would only want the model to appear relativistic for scales less than some multiple of the fermion mass. With such a revised criterion, we find that instead of $\{\lambda_j\}$ being 0, they need only be small and finite. Consequently, the set of couplings, $\{\lambda_k\}$, only need to be made small and finite in order for the ratios, $\frac{\lambda_k}{\lambda_j}$, to take on reasonable (i.e. physical) values. This then modifies the previous conclusion. Models that appear anisotropic and relativistic at low energies can exist even with finite bare anisotropy, provided all couplings are small. However, the anisotropy relevant for determining physical properties is then extremely small. This will be made clear in the examples that are found in Section II.

The need to make the couplings all small in order to realize a physical field theoretic description greatly restricts the amount the low energy behaviour of the theory can deviate from its isotropic limit. Thus in taking the scaling limit we have achieved symmetry restoration under its expanded definition (definition 2). In the cases of $U(1)$ Thirring and anisotropic Kondo (considered in detail in the next section together with several other examples) we know that the RG invariant $\mu^2 = g_{\parallel}^2 - g_{\perp}^2$ governs the effect of the anisotropy upon physical quantities (at least at sufficiently small energies). As $g_{\parallel}$ and $g_{\perp}$ are both required to be small by the scaling limit, $\mu$ is small and so the possible anisotropy is correspondingly small. For $U(1)$ Thirring, we thus find that symmetry restoration occurs in both the AF sector (unlike $\mathfrak{g}$) and the C sector of the theory. Although the two definitions 1 and 2 of symmetry restoration look distinct, they do share some commonalities. In both cases there is a concern for the low energy behaviour of the theory. And in certain circumstances, the taking of the scaling limit can be thought of as a crude running of the RG backward. In both the AF sector of $U(1)$ Thirring and anisotropic Kondo, running the RG backwards amounts to taking $g_{\perp} \to 0$. This is precisely what the scaling limit requires in both these cases. This is not as paradoxical as it might at first seem (indeed, a reverse RG seems to imply a focus upon the UV not the IR degrees of freedom in the theory). In running the RG backwards, i.e. increasing rather than decreasing the UV cutoff, one removes any distortions the UV cutoff creates in the low energy sector of the theory. It precisely such distortions that render this sector non-relativistic. We again see a certain complementarity between the two definitions of symmetry restoration when we understand that the scaling limit implies a more favourable scenario for symmetry restoration as understood strictly as an RG induced phenomena. From the above argument, we know that to even write down a field theoretic description, we require the couplings to all be small (all, so as to avoid an unphysically large anisotropy). Thus the bare theory is only weakly anisotropic. As such, we expect, with certain caveats, an 1-loop RG predicting an enhanced symmetry to be trustworthy: the theory, because it is already close to being isotropic, will flow onto the manifold of higher symmetry while the 1-loop description is still valid.

We have already seen one of these caveats in operation in the case of $U(1)$ Thirring. Here the fact that physical quantities depend upon the RG invariant, $\mu$, means the indicated symmetry restoration does not actually occur. We, however, conjecture this is something particular to $U(1)$ Thirring. The $U(1)$ Thirring model possesses a $q$-deformed quantum group symmetry, $sl(2)_q$. The parameter, $q$, describing the symmetry is a function of the parameter, $\mu$, and so does not change under the RG. We thus conjecture that the lack of symmetry restoration is a reflection of the presence of the $sl(2)_q$ symmetry. We will argue (Section IV), however, that if such a $sl(2)_q$ is explicitly broken through breaking additionally the $U(1)$ symmetry, an increase in the symmetry of the problem

\[5\] It is not however a free theory – taking this limit is not the same as setting the couplings to zero.

**However running the RG backwards does not always mimic the results of the scaling limit. In sector C of $U(1)$ Thirring, the backwards RG flows to a UV fixed point different from the point to which the ‘flow’ of the scaling limit takes one (see the next section).**
A second caveat appears when the RG flow is governed by an anisotropic (IR) fixed point. Indeed, here the two definitions of symmetry restoration do differ in the role assigned to the fixed point of the theory.

The sine qua non of symmetry restoration as an RG-induced phenomena lies in the nature of the fixed point. If we have an anisotropic IR fixed point, we have no expectation that symmetry restoration according to definition 1 will occur. However it may well occur in certain special cases using our expanded definition 2 of symmetry restoration. This occurs for instance in the context of the deformed O(3) sigma model (the sausage model) with topological term \( \theta = \pi \), which has a line of fixed points. Here the RG equations promise symmetry restoration. However we know that the particular point on the fixed line to which the theory flows is determined by the amount of anisotropy in the theory. Therefore, symmetry restoration according to definition 1 does not occur. However physical quantities depend only weakly (continuously) on the anisotropy, at least when it is small, so that symmetry is restored according to definition 2. A similar situation occurs in the spin-exchange anisotropic \( s = 1 \), 1-channel Kondo model, as discussed in the concluding section.

If the fixed point is isotropic however, we allow for the possibility of symmetry restoration under the RG. This possibility is realized if the isotropic fixed point is massless as it is then non-trivial and so determines a set of physical quantities. This is the situation found in the anisotropic \( s = 1/2 \), 1-channel Kondo model. However if the fixed point is massive and so trivial, we cannot say with certainty if symmetry restoration under the RG is realized. With massive flows, it is the approach to the fixed point that matters, controlled by the nature of the massive excitations, and it is not possible in general to characterize this approach. We do know through the example of U(1) Thirring that a massive symmetric fixed point does not guarantee symmetry restoration under the RG.

However if we define symmetry restoration merely as a weak dependence upon the bare couplings in the low energy sector, the fixed point need not play an important role. In both the U(1) Thirring and the anisotropic Kondo model we have symmetry restoration so defined regardless of the nature of the fixed point. Indeed we need not even have an isotropic fixed point (massless or massive) in order to have symmetry restoration. One can easily imagine a scenario in which the fixed point is anisotropic but the scaling limit restricts the anisotropy to be weak with a consequent weak dependence of any physical quantity upon the anisotropy.

The outline of the paper is as follows. In Section II, we consider a series of examples illustrating how the scaling limit induces symmetry restoration (in our expanded sense). These include the anisotropic Kondo model, the U(1) Thirring model, a multi-species version of the U(1) Thirring model, the anisotropic principal chiral model, and the deformed O(3) sigma model with \( \theta = 0 \). All of these models possess isotropic fixed points. To explore whether symmetry restoration occurs when the fixed point is anisotropic, we also consider in the concluding section the spin \( s > 1/2 \) anisotropic, 1-channel Kondo model together with the deformed O(3) sigma model with \( \theta = \pi \).

As we indicated, the scaling limit, in providing weak, bare couplings, promises a more favourable environment for RG-induced symmetry restoration. To exploit this conclusion, we consider in Section III a set of theories based upon all possible simple groups (the analysis does not extend to semi-simple groups). In doing this we extend the results of Lin et al. We will show that the 1-loop beta functions for such theories all imply an enhancement in the symmetry. Given the constraints the scaling limit places upon the bare couplings, we argue that this enhancement should be realized in physical models with small enough bare couplings to admit a field theoretic description.

Finally in Section IV, we return to the model that lies at the heart of much of the discussion surrounding the reliability of 1-loop RG equations, the U(1) Thirring model. Specifically we discuss the possibility that symmetry restoration in the RG sense might take place in non-integrable variants of the U(1) Thirring model. We have conjectured the U(1) Thirring model does not experience symmetry restoration under the RG flow because of the presence of a \( s(2)_q \) symmetry. If this symmetry is then explicitly broken, an RG induced restoration should be possible. And indeed we find it is, although we demonstrate in the course of the discussion that the matter is a delicate one.

II. INTERPLAY OF SCALING LIMIT AND SYMMETRY RESTORATION

A. The Kondo model

We start with the anisotropic Kondo Hamiltonian after bosonization: \( \phi \) is the right moving spin field, the charge field having totally decoupled. After standard manipulations (see e.g. [1]), the Hamiltonian reads

\[
H = H_0 + \frac{g}{4\pi a} \left[ s^+ e^{i\sqrt{\pi} \phi(0)} + s^- e^{-i\sqrt{\pi} \phi(0)} \right] + s^n \sqrt{2\pi} \partial_x \phi(0). \tag{2.1}
\]

††We can argue much the same for the low energy sector of anisotropic Kondo. However because the IR fixed point is massless, with the anisotropy being \textit{irrelevant}, this example is not as compelling.
Here, $s^\pm$ are spin 1/2 generators, $g_\perp, g_\parallel$ the bare Kondo couplings. In evaluating the propagators, we have used \( \langle \phi(x) \phi(y) \rangle = -\frac{1}{T} \log \frac{x-y}{\Lambda} \), and thus have adopted condensed matter conventions where the cut-off is left explicitly in the Hamiltonian and the propagator. The bulk Hamiltonian is \( H_0 = v_F \int dx [\partial \phi(x)]^2 \).

The quickest way to analyze (2.1) is to perform a canonical transformation with \( U = \exp \left[ \frac{g_\parallel}{v_F \sqrt{2\pi}} s^+ e^{i\beta \phi(0)} + s^- e^{-i\beta \phi(0)} \right] \), in order to eliminate the \( s_z \) term. The resulting Hamiltonian then reads

\[
H = H_0 + \frac{g_\perp}{4\pi a} \left[ s^+ e^{i\beta \phi(0)} + s^- e^{-i\beta \phi(0)} \right],
\]

where \( \beta = \sqrt{8\pi - \frac{g_0}{v_F \sqrt{2\pi}}} \). In the following, we set \( v_F = 1 \).

Let us now consider \( g_\parallel \) in (2.2) as a parameter, and set \( \frac{\beta^2}{8\pi} = x < 1 \). Here, \( x \) is the scaling dimension of the exponential operators in (2.2). The problem described by (2.2) will exhibit screening, and by dimensional analysis, the Kondo temperature (the temperature that sets the exponential operators in (2.2)) is \( T_K \propto \frac{1}{a} (g_\parallel)^{1/2} \) where again \( a \) is the UV cutoff. To make the spectrum of this theory purely relativistic we need to take the field theory - also called scaling - limit, \( a \to 0 \). At the same time, since we need to keep an observably finite \( T_K \), it is necessary to also take \( g_\perp \to 0 \).

It is useful to stress here that the canonical transformation used in going from (2.1) to (2.2) is valid in the scaling limit only. In this limit, the somewhat different aspects of the \( SU(2) \) Kondo Hamiltonian are readily explained by the foregoing points: in (2.1), the isotropic theory has \( g_\parallel = g_\perp \), while in (2.2), it is described by \( \beta = \sqrt{8\pi} \), that is, \( g_\parallel = 0 \). However, as one requires \( g_\perp \to 0 \), these two points of view are completely equivalent.

In the scaling limit, the Kondo problem becomes describable by an integrable massless field theory. The bulk excitations have dispersion relations \( p = e = Me^\theta \), with \( \theta \) the rapidity, \( M \) an energy like parameter which has no physical significance since the bulk theory is massless. They have a factorized scattering described by a solution of the Yang-Baxter and bootstrap equations. Among these excitations, the most important are the kink and antikink. Physical properties depend only on the ratio \( T/T_K \), where \( T \) is the physical temperature. The kinks/antikinks scatter off the Kondo impurity with an amplitude described by \( R = -i \tanh \left( \frac{\theta - \theta_0}{2} - i\frac{\pi}{4} \right) \), where \( T_K = Me^{\theta K} \) (here we see that changes in the mass scale \( M \) can be readily absorbed in a shift of the rapidities).

They also scatter among one another, their scattering being determined by a six vertex like solution of the Yang-Baxter equation. This solution, in turn, has a quantum affine algebra symmetry \( \tilde{s}(2) \) (we follow here the conventions of [5], for which \( q = -1 \) is the isotropic limit), where in this case \( q = e^{-\frac{i\pi}{2\theta}} \). For small values of \( g_\parallel \), \( q \approx -e^{i\theta_0}/2 \).

To summarize, the Kondo anisotropic field theory is obtained as \( g_\perp \to 0 \), and it is fully characterized by the value of \( x < 1 \), that is the value of \( g_\parallel \). We thus come to the conclusion as stated in the introduction:

\[\text{To observe a finite anisotropy in the field theory (} x \neq 1 \text{), one needs to have an infinite anisotropy in the bare theory} \]

\[\mathcal{A} = g_\parallel / g_\perp \to \infty \text{ as} \ a \to 0.\]

We also note that if we maintain a finite anisotropy in taking the scaling limit, that is, \( \mathcal{A} = g_\parallel / g_\perp = \text{constant} \) (and \( g_\parallel > 0 \) as \( g_\perp \to 0 \) while \( a \to 0 \)), the model will be described by the same isotropic \( SU(2) \) Kondo field (scattering) theory.

The foregoing discussion avoided the issue of regularization, about which we would like to comment now. When \( g_\parallel \) is large enough, i.e., \( \beta < \sqrt{4\pi} \), no further renormalization is needed besides the usual normal ordering for vertex operators (implicit in all our notations). When \( \sqrt{4\pi} < \beta < \sqrt{8\pi} \), an extra subtraction is necessary, since divergences appear in the computation of the impurity free energy (in complete analogy with the bulk sine-Gordon model). Our conclusions then apply to the universal part of the physical properties of interest.

Regularization issues are most cleanly controlled in the framework of the exact solution of the Kondo model [5], for which we now would like to recast our arguments. The Hamiltonian is as in (1.1). In order to do so we first need to consider the excitations in this theory. It is well understood that the lowest energy spin and charge excitations in the model are decoupled with the charge excitations completely trivial (as is to be expected given the bosonization of the problem). We thus focus on the spin excitations. At \( T = 0 \), their energy, \( e(\lambda) \), and momentum, \( p(\lambda) \), are given by

\[
e(\lambda) = -\int_0^\infty d\lambda' e(\lambda') \partial_{\lambda'} \theta_0(\lambda' - \lambda) - \frac{N}{L}(\theta_0(\lambda) + \pi),
\]

and

\[\text{§§Here} \ \lambda \text{ is a rapidity and is not to be confused with the coupling constants discussed in the introduction.}\]
\[ p(\lambda) = -\int_Q d\lambda' p(\lambda') \partial_{\lambda'} \theta_2(\lambda' - \lambda) + \frac{N}{L} \theta_1(\lambda), \tag{2.4} \]

where

\[ \theta_n(\lambda) = 2 \tan^{-1} [\tanh(\mu \lambda) \cot(n \mu / 2)]. \tag{2.5} \]

*N* is the number of particles in the system, and *L* is the length of the system. Here *Q* represents the interval of the spectral parameter, \( \lambda \), (or, more intuitively, the spin rapidity) over which (spin) excitations are present in the ground state. The parameter, \( \mu \), together with another parameter, \( f \), serve to characterize the anisotropy in the system. As functions of \( g_\perp \) and \( g_\parallel \) they are given by

\[
(\coth f)^2 = \frac{\sin^2 \left( \frac{g_\perp}{\mu} \right)}{\sin \left( \frac{g_\perp + g_\parallel}{\mu} \right) \sin \left( \frac{g_\perp - g_\parallel}{\mu} \right)}, \tag{2.6}
\]

and

\[
\cos \mu = \frac{\cos(g_\parallel/2)}{\cos(g_\perp/2)}. \tag{2.7}
\]

Isotropy is reached by taking \( \mu, f \to 0 \) with \( f/\mu \) constant. We restrict ourselves to the region where \( \pi - g_\perp > g_\parallel > g_\perp > 0 \) such that \( \mu \) is a real parameter. Finally, we can express the Kondo temperature in terms of the anisotropy:

\[
T_K = \frac{2E_F}{\pi} e^{-\frac{\pi g_\perp}{2}}, \tag{2.8}
\]

where \( E_F \) is the Fermi energy (proportional to the inverse of the UV cutoff) of the theory.

In taking the scaling limit, we need to ensure the low energy excitations look relativistic. We see that the energy-momentum already obey a relativistic dispersion relation. However we still need to ensure the energy scale in the system, \( T_K \), is far below the bandwidth. Thus we need to take \( T_K/E_F \to 0 \) in order to achieve the scaling limit. For finite \( \mu \) this requires \( f \to \infty \). In that limit, the foregoing equations simplify and one finds:

\[
e^{-f} \approx \frac{g_\perp}{4} \cot \frac{g_\parallel}{2}, \quad \mu \approx \frac{g_\parallel}{2}. \tag{2.10}
\]

The same qualitative conclusions therefore hold: that is, to have a finite anisotropy in the scaling limit, one needs to send \( g_\perp \to 0 \), while \( g_\parallel \) remains finite, hence giving rise to an infinite bare anisotropy. One also checks that \( T_K \propto 1/\pi^2 (g_\perp) \propto 1/\pi (g_\perp) \). If instead we have a finite anisotropy, i.e., let \( g_\parallel \) and \( g_\perp \) both go to zero while \( g_\parallel/g_\perp \to \gamma \), \( \gamma \) a finite number, we get that \( \coth^2 f \to \frac{\pi}{2}, \) while \( \mu \to 0 \), corresponding to an isotropic field theory whatever the value of \( \gamma \). In that limit, one has \( T_K \propto \frac{1}{\pi^2} e^{-\frac{\pi g_\perp}{2}} \propto \frac{1}{\pi} e^{-\frac{\pi g_\parallel}{2}} \), a well known result for the isotropic Kondo model. The exact solution thus fully confirms the foregoing field theoretic analysis.

Now, as we indicated in the introduction, enforcing the scaling limit strictly is unnecessary from the condensed matter point of view. All we require is that the low energy excitations in the theory appear relativistic. Returning then to the exact solution of the Kondo model, instead of \( T_K/E_F \to 0 \), we require that

\[
\frac{T_K}{E_F} \leq \frac{1}{A} \tag{2.11}
\]

where \( A \) is some large number. In terms of the parameters \( f \) and \( \mu \), this translates into the condition,

\[
\frac{f}{\mu} \geq \frac{1}{\pi} \log \frac{2A}{\pi}. \tag{2.12}
\]

With this condition it no longer is necessary to have \( g_\perp \to 0 \). Rather, a region of parameter space of finite area satisfies the above constraint. We have plotted this region in Figure 1 for \( A = 50 \) (that is, the theory is relativistic for scales up to \( 50T_K \)).

We observe that the region in Figure 1 is restricted to less than one-half of the possible parameter space. If we insist that \( g_\parallel \geq g_\perp \geq g_\parallel/2 \) (what we may call a maximal ‘reasonable’ value for the ratio of couplings), the region of parameter space is restricted to the gray shaded region between the two lines. We thus see that any physical realization of the Kondo model with a scaling limit will not be far from the isotropic ray \( g_\parallel = g_\perp \).

Yet the fraction of parameter space is still appreciable. In part this is an artifact of the initial data of the analysis of the Kondo model [5]: this analysis begins with a linear spectrum, and as there are no bulk interactions in this model, the spectrum remains linear as evinced in (2.9). It is possible to consider a version of the Kondo model that is equipped with band structure. One can do this straightforwardly by analyzing a lattice system with a Kondo impurity so that the initial bulk electron spectrum obeys a dispersion relation of the form, \( \epsilon(k) = -2t \cos(k) \). Or one can do this more abstractly by turning to an integrable lattice regularization of the Kondo spin dynamics [6]. The latter follows from a general construction, where one introduces a line of spectral parameter defects in an otherwise homogeneous 6-vertex model [6]. The equations that arise from this latter construction have a form similar to those of [6], except for the fact that the bare energy, instead of being given by \( \epsilon \propto \theta_1(\lambda) \), now has the form \( \epsilon \propto \frac{1}{\lambda^2} \theta_1(\lambda) \). The introduction of this curvature shrinks the scaling region considerably, leading to results similar to those of the \( U(1) \) Thirring model to be discussed next where the bulk perturbation of this system also yields a non-linear dispersion relation.
FIG. 1. The portion of parameter space where the Kondo model is well described by a field theory. The lower triangle marks out the total region of considered parameter space (a quarter of the total). The shaded region marks the area where the theory is relativistic while the more lightly shaded subarea is characterized by \( g_\perp > g_\parallel / 2 \).

Given the available parameter space in Figure 1, we can now ask how the physics of the problem varies as we vary the anisotropy. With this in mind we consider the impurity susceptibility, \( \chi(H) \), as a function of magnetic field. From [1], we know it is given by

\[
\chi(H < T_H) = \frac{1}{H\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{(2n+1)\pi a(\mu)} \times \\
\left( \frac{H}{T_H} \right)^{2n+1} \frac{\Gamma(1 + \frac{\pi}{\mu}(n + \frac{1}{2}))}{\Gamma(1 + (\frac{\pi}{\mu} - 1)(n + \frac{1}{2}))} ;
\]

\[
\chi(H > T_H) = \frac{\mu}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \sin(\mu n) \times \\
\Gamma(\frac{1}{2} + \frac{\mu n}{\pi}) \Gamma(n(1 - \frac{\mu}{\pi})) e^{-2\mu a(\mu)} \left( \frac{H}{T_H} \right)^{2n-1} , \tag{2.13}
\]

where

\[
a(\mu) = \frac{1}{2\mu} \log \left( 1 - \frac{\mu}{\pi} \right) - \frac{1}{2\pi} \log \left( \frac{\pi}{\mu} - 1 \right) , \tag{2.14}
\]

and \( T_H \) is related to the Kondo temperature, \( T_K \), via

\[
T_H = 2\sqrt{\pi} \frac{\Gamma(1 + \frac{\pi}{2\mu})}{\Gamma(1/2 + \frac{\pi}{2\mu})} e^{\pi a(\mu)} T_K . \tag{2.15}
\]

For small fields, \( \chi(H) \sim T_K^{-1} \). We thus scale out this factor and plot \( \chi(H)/T_K \) as a function of \( H/T_K \) in Figures 2 and 3 for different pairs of \((g_\parallel, g_\perp)\). The points chosen fall equidistantly on a line connecting \((g_\parallel = g_\perp = \pi/5)\) with \( g_\parallel = 2\pi/5, g_\perp = 0 \). (This line is marked on Figure 1). We see that the appropriately scaled \( \chi \) varies only slightly when the isotropy of the model is deformed, even if the deformation is strong.

FIG. 2. Behaviour of the magnetic susceptibility as a function of the anisotropy for small magnetic field, i.e. \( H < T_H \). From bottom to top, the plots correspond to \((g_\parallel, g_\perp) = (1, 1), (6/5, 4/5), (7/5, 3/5), (8/5, 2/5), (9/5, 1/5), \) and \((2, 0)\).

FIG. 3. Behaviour of the magnetic susceptibility as a function of the anisotropy for large magnetic field, i.e. \( H > T_H \). From bottom to top, the plots correspond to \((g_\parallel, g_\perp) = (1, 1), (6/5, 4/5), (7/5, 3/5), (8/5, 2/5), (9/5, 1/5), \) and \((2, 0)\).

It is unsurprising that the low field susceptibility in Figure 2 varies only a little given the anisotropic Kondo model shares the same fixed point as its isotropic counterpart. One can see the curves collapse upon one another as \( H \to 0 \) indicative of the flow to this same IR fixed point. However the variation is also small for the
high field case in Figure 3 where one ostensibly expects to be far from the fixed point.

To conclude, we see that the scaling limit has drastic consequences for how isotropic a theory is. In order for a system with given bare coupling constants and finite bare anisotropy, to be reasonably described by the Kondo field theory, it must be only weakly anisotropic. The variation of the physical quantities over the allowed anisotropy is correspondingly small. Excluding the possibility of extraordinarily (i.e. unphysically) strong anisotropy in the bare coupling constants, the scaling limit thus enforces a strong restoration of symmetry in its expanded sense.

B. The U(1) Thirring Model

We now move on to consider the U(1) Thirring model where an analogous series of conclusions to those of the Kondo model will be drawn. In fact, this model is essentially a bulk version of the anisotropic Kondo model of the previous section.

The U(1) Thirring model is described by the Lagrangian

\[
\mathcal{L} = i \bar{\psi}_\alpha \gamma_\mu \partial^\mu \psi_\alpha + \frac{1}{4} g_\parallel (j_z)^2 + \frac{1}{2} g_\perp [(j_x)^2 + (j_y)^2],
\]

where \( j^\mu = \bar{\psi}_\alpha \gamma_\mu \tau^\alpha \beta \psi_\beta \) and \( \psi \) is a doublet of Dirac spinors. This model was completely solved by algebraic Bethe ansatz [1].

There, in accordance with earlier perturbative work, it was established that according to the values of the couplings \( g_\perp \) and \( g_\parallel \), the model exhibits different infrared behaviour. The phase diagram, pictured in Figure 4, divides into three sectors of interest. In sectors AF (asymptotic freedom) and C (crossover), a strong coupling regime appears and there is dynamical mass generation. The flows in both these sectors are towards a stable IR fixed point (located at \((g_\parallel, g_\perp) = (-\pi/2, \pi/2)\) in Fig. 4).

The two sectors AF and C are distinguished by their behaviour in the UV: in the AF region the theory is asymptotically free while the UV flow in region C is to a strongly coupled fixed point. In the AF sector (at weak coupling) the model maps onto the sine-Gordon theory. The sine-Gordon theory can be thought of as an anisotropic current-current perturbation of a free boson theory. Thus the U(1) Thirring model in the AF sector bears some resemblance to the anisotropic Kondo model, itself representable as a free boson perturbed by an anisotropic current-spin interaction.

In the final sector, W, the weak coupling sector, the \( jj \) perturbation is irrelevant and the theory can be described perturbatively. In this sector, the excitations are all massless. As we are ultimately interested in looking at theories with flows implying symmetry restoration, we will focus on regions AF and C.

1. AF Region

We start with the AF region. As with the Kondo model, the excitations divide into decoupled spin and charge sectors, with the charge sector trivial. The spin excitations, at least in the regime that will be of interest, are given from the following integral equations:

\[
e(\lambda) + \int_{-B}^{B} R(\lambda - \lambda') e(\lambda')d\lambda' = \frac{H}{2(1 - \frac{\lambda}{\pi})} - e_0(\lambda),
\]

\[
p(\lambda) + \int_{-B}^{B} R(\lambda - \lambda') p(\lambda')d\lambda' = -p_0(\lambda),
\]

where \( e_0(\lambda) \) and \( p_0(\lambda) \) are

\[
e_0(\lambda) = \frac{\Lambda_e}{2} \tan^{-1}\left[\frac{\cosh(\frac{\lambda}{\pi})}{\sinh(\frac{\lambda}{\pi})}\right];
\]

\[
p_0(\lambda) = \frac{\Lambda_e}{2} \tan^{-1}\left[\frac{\sin(\frac{\lambda}{\pi})}{\cos(\frac{\lambda}{\pi})}\right].
\]

Here \( \Lambda_e \) is proportional to the bandwidth and \( f \) and \( \mu \) are related to the bare parameters of the model, \( g_\perp \) and \( g_\parallel \), in a near identical fashion to the relations (2.6) and (2.7):

\[
\text{cosh}^2(f) = \frac{\sin^2(g_\parallel)}{\sin(g_\parallel + g_\perp) \sin(g_\parallel - g_\perp)}
\]

\[
\cos(\mu) = \frac{\cos(g_\parallel)}{\cos(g_\perp)}.
\]

\( H \) is the magnetic field in the problem, and \( R(\lambda) \), the kernel of the integral equations, is given by

\[
R(\lambda) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh(\frac{\pi}{\mu} - 2\omega)}{2 \cosh(\omega) \sinh]\left(\frac{\pi}{\mu} - 1\right)\omega].
\]

The limit, \( B \), the Fermi rapidity, in the integral equation is such that

\[
e(\lambda) \geq 0 \quad \text{for} \ |\lambda| \leq B.
\]
When \( H = 0 \), \( e(\lambda) \leq 0 \) and \( B = 0 \). In this case the ground state of the system is filled with states for all \( \lambda \). Elementary excitations are obtained by creating holes with energy-momentum \((e_0(\lambda), \rho_0(\lambda))\) in this filled sea.

We must consider values of the parameters \((|g_{\perp}| < |g_{\parallel}| < \pi - |g_{\perp}|)\) such that \( f \) and \( \mu \) are real. To make the spectrum relativistic, we need to again take the scaling limit. If we hold the spectrum relativistic, we need to again take the scaling limit, in which case the bare anisotropy. Meanwhile, if we keep a finite bare anisotropy even in the scaling limit, we require \( f \rightarrow \infty \).

This, as before, requires \( g_{\perp} \rightarrow 0 \), and thus a ratio of bare coupling constants \( g_{\parallel}/g_{\perp} \rightarrow \infty \), i.e. an infinite bare anisotropy. Meanwhile, if we keep a finite bare anisotropy, both bare couplings have to go to zero in the scaling limit, in which case \( f \) remains finite while \( \mu \rightarrow 0 \), i.e. the field theory is isotropic. These conclusions are exactly the same as the ones obtained in the Kondo case.

In Figure 5 the shaded region marks the values \((g_{\parallel}, g_{\perp})\) permitted by the above relationship for \( A = C = 50 \). Unlike Figure 1 describing the allowed parameter space for the Kondo model, the region in Figure 5 is far smaller. This is a consequence of the exact expressions for energy-momentum not being relativistic from the start. The need to make the correction terms small leads to the large reduction in allowed parameter space. Because of the log dependence in the above constraint, changing \( A \) and \( C \) does not drastically affect the allowed region of parameter space.

FIG. 6. The dependence of the susceptibility in the AF region upon the anisotropy. From bottom to top, the plots correspond to six equally spaced values of \((g_{\parallel}, g_{\perp})\) along the line \((-2, 2)\) to \((-4, 0)\).

Again we ask how the magnetic susceptibility \( \chi \) depends upon the anisotropy. The magnetic susceptibility in the AF sector can be given in terms of the energy:

\[
\chi(H) = -\partial^2_H E(H),
\]

where \( E(H) \) is the ground state energy per unit length as a function of \( H \):

\[
E(H) - E(0) = -\frac{\Lambda_c}{2} \int_{-B}^{B} d\lambda e(\lambda) \rho_0(\lambda),
\]

where \( \rho_0 \) is the bare density of states whose low energy behaviour (valid whenever \((22)\) holds) is governed by \( \rho_0 = (m/2\Lambda_c) \cos(\pi \lambda/2) \). For small \( H \), the integral equation for \( e(\lambda) \) can be solved iteratively with the result

\[
e(\lambda) = m \cos(\frac{\Lambda_c}{2})(1 + \text{correction terms});
\]

\[
\rho(\lambda) = m \sinh(\frac{\Lambda_c}{2})(1 + \text{correction terms}),
\]

where \( m = \Lambda_c e^{-\pi f/2\mu} \). In relaxing the scaling limit, we require instead that \( f/\mu \gg 1 \), and \( H = 0 \), the above dispersion relation becomes

\[
e(\lambda) = m \cosh(\frac{\Lambda_c}{2})(1 + \text{correction terms});
\]

\[
\rho(\lambda) = m \sinh(\frac{\Lambda_c}{2})(1 + \text{correction terms}),
\]

for all \( \lambda \) such that \( e(\lambda) < C m \), that is the spectrum appears relativistic to one part in \( A \) up to scales of \( C \times m \). Imposing the constraints leads to the condition

\[
\frac{f}{\mu} \geq \frac{1}{\pi} \log\left(\frac{4C^2A}{3}\right).
\]

FIG. 5. Region in the (AF) portion of parameter space where the U(1) Thirring model is well described by a field theory. The lower triangle marks the total region of parameter space considered while the shaded region marks the portion where the theory is relativistic. The more lightly shaded subregion is given by the condition that \( g_{\perp} > |g_{\parallel}|/2 \).

However as before, we do not need to take a strict scaling limit. With \( f/\mu \gg 1 \) and \( H = 0 \), the above dispersion relation becomes

\[
e(\lambda) = m \cosh(\frac{\Lambda_c}{2})(1 + \text{correction terms});
\]

\[
\rho(\lambda) = m \sinh(\frac{\Lambda_c}{2})(1 + \text{correction terms}),
\]

where \( m = \Lambda_c e^{-\pi f/2\mu} \). In relaxing the scaling limit, we require instead that\[
\text{correction terms} < \frac{1}{A},
\]

\[
\text{correction terms} < \frac{1}{A},
\]

\[
\text{correction terms} < \frac{1}{A},
\]

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\text{correction terms} < \frac{1}{A},
\]

\[
\text{correction terms} < \frac{1}{A},
\]

\[
\text{correction terms} < \frac{1}{A},
\]
\[
\chi(H) = \frac{1}{2^{5/2} \pi} \frac{1}{(1 - \mu/\pi)^2} \left( \frac{H - H_c}{H_c} \right)^{1/2}
- \frac{2^{9/2}}{\pi} R(0) + \mathcal{O}\left( \frac{H - H_c}{H_c} \right) \right],
\]  
(2.27)

where \( R \) refers to the kernel in (2.17) and
\[ H_c = 2n(1 - \mu/\pi). \]  
(2.28)

In Figure 6 we plot \( \chi(H) \) for different values of \( g_\parallel \) and \( g_\perp \) ranging from \( g_\perp = -0.4 \) along a straight line to \( g_\perp = 0, -g_\parallel = 0.4 \) (shown in Figure 5). We see the plots as a function of \( H/H_c \) fall nearly on top of one another. Thus in the region where a field theoretic description is possible, the allowed anisotropy does not lead to drastic variations in the physics.

2. Region C

We now consider sector C. For the region \( |g_\parallel| < |g_\perp| < \pi/2 \), the same relations as found in (2.19) hold, but here the couplings \( \mu \) and \( f_1 \) are purely imaginary. We thus set \( \mu_1 = -i\mu \) and \( f_1 = if \) with the result
\[ \cot^2(f_1) = \frac{\sin^2(g_\parallel)}{\sin(g_\parallel + g_\perp) \sin(g_\parallel - g_\perp)}, \]
\[ \cosh(\mu_1) = \frac{\cos(g_\parallel)}{\cos(g_\perp)}. \]  
(2.29)

In this sector the spin excitations are described by analogous integral equations as those for the AF sector (2.17), but with \( e_0 \) and \( p_0 \) replaced by
\[ e_0(\lambda) = \frac{\Lambda}{2} \sum_{n=-\infty}^{\infty} \tan^{-1} \left[ \frac{\sinh(\frac{\pi n}{2\mu_1})}{\cosh(\frac{\pi n}{2\mu_1})} \right], \]
\[ p_0(\lambda) = \frac{\Lambda}{2} \sum_{n=-\infty}^{\infty} \tan^{-1} \left[ \frac{\cosh(\frac{\pi n}{2\mu_1})}{\sinh(\frac{\pi n}{2\mu_1})} \right], \]  
(2.30)

and \( R \), the kernel, now given by
\[ R(\lambda) = \sum_{n=-\infty}^{\infty} \tilde{R}(\lambda - \frac{2\pi n}{\mu_1}); \]
\[ \tilde{R}(\lambda) = -\frac{1}{\pi} \int_{0}^{\infty} d\omega \frac{\cos(\omega\lambda)}{1 + e^{2\omega}}. \]  
(2.31)

Again when \( H = 0 \), the energy-momentum of the excitations is simply \( (e_0(\lambda), p_0(\lambda)) \). Note that \( \tilde{R}(\lambda) \) is the isotropic limit of the kernel \( R \) in the AF sector. Thus the sum forming \( R \) here represents the isotropic limit \( n = 0 \) plus what will turn out to be exponentially small corrections \( n \neq 0 \). Because the structure of these expressions for the energy-momentum is considerably more complicated, we need to break down the analysis into two cases: \( -\pi/2 < f_1 < 0 \) and \( 0 < f_1 < \pi/2 \). We will always assume \( \mu \) is such that \( \exp(-\pi^2/2\mu) \ll 1 \).

In the case, \( -\pi/2 < f_1 < 0 \), the lowest energy excitations correspond to those for \( \lambda \sim 0 \). Expanding about this point, \( e(\lambda) \) becomes
\[ e(\lambda) = -\Lambda_c \pi + m \cosh\left( \frac{\pi \lambda}{2} \right) \times (1 + \text{correction terms}), \]  
(2.32)

where \( m = \Lambda_c \exp(-\pi|f_1|/2\mu_1) \). In this case some of the correction terms are of \( \mathcal{O}(\exp(-\pi^2/\mu_1)) \). Thus in order to take the true scaling limit here (i.e. take the correction terms to zero), we need to take \( \mu_1 \to 0 \). In the process, we see from (2.29) that both \( g_\parallel \) and \( g_\perp \) have to go to zero, while their ratio can be an arbitrary number. In this region of coupling constants therefore, the continuum limit, when it exists, is always isotropic. This is not so different that the results encountered in the AF region of parameter space. In region C the ratio of couplings, \( |g_\parallel| \leq |g_\perp| \), is always finite. In region AF the finiteness of this ratio leads to an isotropic scaling limit.

![FIG. 7. The portion of region C where the U(1) Thirring model is well described by a field theory. The lower triangle marks out the total considered parameter region while the shaded region is where the theory is relativistic. The more lightly shaded area meets the additional condition that \( g_\parallel > g_\perp/2 \).](attachment:image.png)

In relaxing the scaling limit to insisting the correction terms are \( < 1/A \) for scales up to \( C \times m \), we instead find that the following two conditions need to be satisfied:
\[ \frac{f_1}{\mu_1} \geq \frac{1}{\pi} \log \frac{4C \mu^2}{A}; \]
\[ \frac{\pi - f_1}{\mu_1} \geq \frac{1}{\pi} \log A. \]  
(2.33)
The region of parameter space satisfying these conditions for $A = C = 50$ is a portion of the shaded region in Figure 7.

In the case $0 < f_1 < \pi/2$, the lowest energy excitations occur nearest $\lambda \sim \pm \pi/\mu_1$. Near these points, $e(\lambda)$ takes the form

$$e(\pm \frac{\pi}{\mu} - \lambda) = m \cosh\left(\frac{\pi \lambda}{2}\right)(1 + \text{correction terms}),$$

(2.34)

with $m = \Lambda_\epsilon \exp(\frac{\pi}{2\mu_1}(\pi - f_1))$. Again the correction terms are of $O(\exp(-\pi^2/\mu_1))$ and hence $\mu_1 \to 0$ in the scaling limit thus restoring isotropy. If we relax the scaling limit we find an analogous set of constraints as above:

$$\frac{\pi - f_1}{\mu_1} \geq \frac{1}{\pi} \log 4C^2A; \quad \frac{f_1}{\mu_1} \geq \frac{1}{\pi} \log A.$$  

(2.35)

The region of parameter space satisfying these constraints completes the shaded region in Figure 7.

We see that the total permitted region is asymmetric under $g_\parallel \to -g_\parallel$. This underlying asymmetry is reflected in the RG: the RG flows from an UV fixed point at negative $g_\parallel$ to an IR fixed point at positive $g_\parallel$. The scaling region is weighted towards a parameter regime away from the UV fixed point as it is in this regime that the low energy sector of the theory is least distorted by the UV cutoff. Unlike the AF sector, a good part of the region meeting these constraints is such that $g_\perp/g_\parallel < 2$ (the region shaded gray in Figure 7), that is, the couplings here take on reasonable values.

We again ask how the magnetic susceptibility varies over the permitted anisotropy. The susceptibility takes on a similar form to that in (2.27).

$$\chi(H) = \frac{1}{25/2\pi} \left\{ \left(\frac{H - H_c}{H_c}\right)^{-1/2} - \frac{29/2}{\pi} R(0) + O\left(\frac{H - H_c}{H_c}\right) \right\},$$

(2.36)

but with $H_c = 2m$ (its isotropic value) and $R$ the kernel in (2.31). We note that the first term in the susceptibility has no dependence upon $\mu$. To leading order, region C is thus identical to the isotropic model. In Figure 8 we plot how $\chi(H)$ varies when the couplings are changed for six equally spaced values of $(g_\perp = A,g_\parallel = A)$ to $(g_\perp = A,g_\parallel = 0)$ (plotted in Figure 7). We see there is no discernible variation in $\chi$ from its isotropic value. This is a consequence of the kernel in (2.31). Terms for $n \neq 0$ correspond to exponentially small corrections to the isotropic limit.

C. MultiFlavour Fermion Model

In this section we consider a multiflavour fermion variant of the U(1) Thirring model and show that the same conclusions hold. In particular, we demonstrate that the symmetry restoration is not affected by the number of fermion species in the theory.

The action, involving $2s + 1$ fermion species, that we consider is,

$$S = \int d^2x \left\{ \sum_{a=1}^{2s+1} i\bar{\psi}_a \gamma_\mu \partial^\mu \psi_a + \bar{\psi}_a \gamma_\mu \psi_a^{a1} V_{a1b1} V^{a2b2} \bar{\psi}_b \gamma^\mu \psi_b \right\}.$$  

(2.37)

The interaction potential, $V$, is chosen such that the model is integrable. For $s > 1/2$, $V$ has a complicated functional form. But in the scaling limit, the interaction takes on the simple form,

$$V = J_\parallel S_\parallel^1 S_\parallel^2 + J_\perp S_\perp^1 S_\perp^2,$$

(2.38)

where $S^a$ are the generators of the spin-$s$ representation of $SU(2)$.

The fermionic model in (2.37) has been solved with Bethe ansatz [8]. The structure of the solution mimics that of U(1) Thirring in the AF region. Thus there are spin and charge excitations with the charge excitations trivial. The lowest energy excitations are given by

$$e(\lambda) + \int_{-B}^{B} d\lambda' R(\lambda - \lambda') e(\lambda') = \frac{H}{2(1 - 2s\mu/\pi)} - e_0(\lambda);$$

$$p(\lambda) + \int_{-B}^{B} d\lambda' R(\lambda - \lambda') p(\lambda') = -p_0(\lambda),$$

(2.39)

where

$$R(\lambda) = \frac{1}{2\pi} \int e^{-i\omega\lambda} \left(-1 + \frac{1}{\sinh(\pi\omega/\mu) \sinh(\omega)} \right) \sinh(2s\omega)$$

$$\left(\frac{\sinh(\pi\omega/\mu) \sinh(\omega)}{2 \cosh(\omega) \sinh(\omega/\mu - 2s) \sinh(2s\omega)}\right).$$
\[ e_0 = \frac{\Lambda_c}{2} \tan^{-1} \left( \frac{\cosh \left( \frac{\pi f}{2} \right)}{\sinh \left( \frac{\pi f}{2} \right)} \right) \]

\[ p_0 = \frac{\Lambda_c}{2} \tan^{-1} \left( \frac{\cosh \left( \frac{\pi f}{2} \right)}{\sinh \left( \frac{\pi f}{2} \right)} \right). \]

(2.40)

These equations are derived assuming \( \pi / \mu > 2s \). They, as they should, reduce to those of the U(1) Thirring model in the AF region when \( s = 1/2 \). Here again we should demand that the spectrum appear relativistic to one part in \( A \) up to scales of \( C \times m \), where \( A \) is the fermion mass scale given by \( m \sim \Lambda_c e^{-f/\mu} \). The region of parameter space satisfying these constraints for \( s = 4 \) and \( A = C = 50 \) is plotted in Figure 9. We see that the allowed region of bare parameters with a ‘physical’ anisotropy is small compared to the entire allowed parameter space.

As with the U(1) Thirring model, we ask how the magnetic susceptibility depends upon the anisotropy. Given the functional similarity of (2.40) with (2.17), the susceptibility is given (almost) identically by (2.27):

\[ \chi(H) = \frac{1}{25/\pi} \left( \frac{1}{1 - 2\mu/\pi} \right)^2 \left( \frac{H - H_c}{H_c} \right)^{-1/2} \]

\[ - \frac{9}{\pi} R(0) + O \left( \frac{H - H_c}{H_c} \right), \quad (2.43) \]

where \( R \) is now the kernel in (2.40) and

\[ H_c = 2m(1 - 2s\mu). \quad (2.44) \]

In Figure 10 we plot \( \chi(H) \) for values of the parameters lying along the small diagonal line shown in Figure 9. We see, as with the U(1) Thirring AF region, the various plots lie essentially on top of one another. Again, the allowed ‘physical’ anisotropy does not lead to large variations in the physics.

\[ \text{FIG. 9. The portion of the region of the multi-flavour fermion model that is well described by a field theory for } s = 4. \]

\[ \text{The two diagonal lines delimit the region } J_{\parallel} > J_{\perp}/2. \]

\[ \text{The more lightly shaded area marks the intersection of the region satisfying (2.12) with the region given by } 2J_{\perp} > J_{\parallel} > J_{\perp}/2. \]

\[ \text{FIG. 10. The dependence of the susceptibility upon the anisotropy for } s = 4. \]

\[ \text{From bottom to top, the plots correspond to six equally spaced values of } (J_{\parallel}, J_{\perp}) \text{ along the line } (-2, 2) \text{ to } (-4, 0). \]

D. Anisotropic principal chiral model \( U(1) \times SU(2) \)

A model closely related to the previous one is the anisotropic \( SU(2) \) principal chiral model (APCM), with the action

\[ S_{\text{APCM}} = \int d^2 x \left[ \frac{1}{J_{\perp}} (\omega_\mu^a)^2 + (\omega_\mu^a)^2 \right] + \frac{1}{J_{\parallel}} (\omega_\mu^a)^2; \]

\[ \omega_\mu^a = \text{Tr}(\sigma^a g^{-1} \partial_\mu g), \quad (2.45) \]
where $g$ are matrices in the fundamental representation of SU(2).

The APCM is intimately related to models of N fermions as $N \to \infty$ [3]. In particular, the multifermion model of the previous section was shown to be equivalent to the APCM when $s \to \infty$ provided $\mu \to 0$ (as the Bethe ansatz solution is periodic in $\mu s$) while $\pi/\mu - 2s$ is held constant. In this limit the bare parameters of the APCM model, $J_\perp$ and $J_\parallel$, are given in terms of the parameters $\mu$ and $f$ by

$$J_\parallel = -2\mu$$

$$J_\perp = \frac{16\mu}{\pi^2} e^{-f}.$$  

Given the form of these equations, all the conclusions regarding the scaling limit of the multiflavour fermion model hold. In particular, we imagine holding $s$ large and $\mu$ small but both finite. Then for the spectrum of excitations to be purely relativistic, we must take the limit $f \to \infty$ thus implying $J_\perp \to 0$. To then avoid an infinite bare anisotropy, we have $J_\parallel \to 0$ and we end up with an isotropic theory. If we relax the scaling limit to the same degree done for the multiflavour model, we again find a finite region of parameter space admitting a relativistic sector. This region, determined by (2.42) with $A = C = 50$, is plotted in Figure 11. Unlike the multiflavour model for finite $s$, this region does not see a ratio, $J_\perp/J_\parallel$, of the bare parameters larger than 1/2.

E. The O(3) Sigma Model and the sausage model at $\theta = 0$

An amusing illustration of the interplay between anisotropy in the bare theory and in the scaling limit is provided by the so called sausage model [10], an anisotropic deformation of the $O(3)$ sigma model. Setting $\phi_i \phi_i = 1$, the action of the usual $O(3)$ sigma model reads

$$S = \frac{1}{2g} \int (\partial_\mu \phi_i)^2,$$

where $g$ is a running coupling constant given at 1-loop by

$$g = -\frac{2\pi}{t},$$

and $t$ is the ‘RG time’. In terms of the cutoff, $t = \ln e$, and so $t \to -\infty$ in the limit when the cutoff is sent to zero. The S matrix describing this model has of course an $O(3)$ symmetry. There exists, as usual, an anisotropic deformation of this S matrix, which is characterized in [10] by a parameter $\nu$ which in turn determines a quantum group parameter, $q \approx -e^{4\nu}$ for small $\nu$. As argued in [10], the one loop action for this model then reads

$$S = \frac{1}{2g(t)} \int d^2x \frac{(\partial_\mu \phi_i)^2}{1 - \frac{\nu^2}{2g^2} \phi_i^2},$$

where $g(t) = \frac{\nu}{2} \coth \frac{\nu(t_0 - t)}{2}$. This action also corresponds to a sigma model, but in this case the target space has the shape of a “sausage”. In order for this action to reproduce the physics contained in the corresponding S-matrix, the authors of [10] found it necessary to take $t \to -\infty$ or equivalently, the cutoff, $a$, to 0. In this limit, the sausage becomes a very long cylinder of length $L = \frac{2\pi t}{2\nu}(t_0 - t)$, and circumference, $l = 2\pi \sqrt{2/\nu}$. For any $\nu \neq 0$, it thus follows that the asymptotic UV shape of the sausage as $a \to 0$ is infinitely elongated, i.e. $L/l \to \infty$. Once again, a finite anisotropy, $\nu \neq 0$, in the scaling limit requires an infinite bare anisotropy.

The sausage model in fact bears a close resemblance to the anisotropic Kondo and $U(1)$ Thirring model. To see this, one writes the one loop action using stereographic coordinates

$$S = \int d^2x \frac{(\partial_\mu X)^2 + (\partial_\mu Y)^2}{a(t) + b(t) \cosh 2Y},$$

and shows that the standard RG equations for the metric [11] give rise to

$$\frac{da}{dt} = \frac{1}{2\pi} b^2$$

$$\frac{db}{dt} = \frac{1}{2\pi} ab.$$
with the invariant $\nu^2 = a^2 - b^2$. We have not worked out in detail the physical properties of this model, but the similarity of the equations with those of the anisotropic Kondo and $U(1)$ Thirring model (with $\mu$ identified with $\nu$) suggest that the qualitative results in sections A and B above will hold here as well.

III. LESSONS FROM THE SCALING LIMIT: RESTORATION OF GENERAL SYMMETRIES THROUGH THE RG

We now apply the lessons drawn in the previous section to a wider class of theories. Specifically, we consider current-current perturbations of Wess-Zumino-Witten (WZW) models together with a set of Kondo problems where the underlying symmetry is $SU(2)_k$. We show that the parameter spaces of these models possess subspaces of higher symmetry to which the 1-loop RG flows. Given the scaling limit implies that one can only realize a field theory if the bare couplings are weak (unless one is ready to accept unphysically large bare anisotropies), we argue that these 1-loop flows are to be trusted, subject to certain caveats. To begin, we consider current-current perturbations of WZW models.

**A. Current-Current Perturbations of WZW models**

Here we consider a set of theories represented by current-current perturbations of WZW models based upon simple groups, $G$:

$$S = S_{WZW}^G + \sum_{a=1}^n g_a \int d^2 x J_a^L \cdot J_a^R,$$

(3.1)

where $n$ is the dimension of the group. The 1-loop beta function of these theories has the form:

$$\frac{dg_a}{dl} = -\sum_{bc} (f_{abc})^2 g_b g_c,$$

(3.2)

where $f_{abc}$ are the structure constants for $G$. Such beta functions are predicated upon the second term of the current-current operator product expansion (OPE):

$$J_a(x) J_b(y) = \frac{k}{(x-y)^2} \delta_{ab} + \frac{1}{(x-y)} f_{abc} J_c(y),$$

(3.3)

where $k$ is the level of the WZW model. The above beta functions are level independent, a consequence of the second term not depending upon $k$.

***The structure constants are generated by some matrix representation, $t^a$ via $[t^a, t^b] = i f^{abc} t^c$. We suppose the $t^a$'s are such that $Tr(t^a t^b) = 2 \delta^{ab}$.***

It is straightforward to show that there are regions of parameter space where (3.2) implies a restoration of symmetry, i.e. the RG flows onto a ray described by $g_a = g, \forall a$. We clearly see that the ray itself is an RG invariant. Setting $g = g_a$ on the r.h.s. of (3.2), we have

$$\frac{dg_a}{dl} = -g^2 \sum_{bc} (f_{abc})^2 = -c_v g^2,$$

(3.4)

where $c_v$ is the quadratic casimir of the adjoint representation for the group $G$. The question then becomes whether the ray actually attracts the flow.

To answer this we perform a linear stability analysis. Writing

$$g_a = g + \delta g_a,$$

(3.5)

we obtain RG equations for $\delta g_a$:

$$\frac{d\delta g_a}{dl} = -g \sum_{bc} (f_{abc})^2 (\delta g_b + \delta g_c) = -2g \sum_{bc} (f_{abc})^2 \delta g_b.$$

(3.6)

Thus all we need to do is find the eigenvalues/eigenvectors of the above equation. We see that the eigenvector $\delta g_a = 1$ (i.e. directed along the isotropic ray) has an eigenvalue, $\lambda = -2c_v$. If $g < 0$, we claim that it is the largest positive eigenvalue. To see this let

$$v = (\alpha_1, \ldots, \alpha_n),$$

(3.7)

be an eigenvector of (3.6) with eigenvalue, $\lambda_v$, where the couplings $\alpha_i$ are chosen without loss of generality such that $1 \geq |\alpha_1| \geq |\alpha_2| \geq \ldots \geq |\alpha_n|$. Then the eigenvalue $\lambda_v$ is constrained by

$$|\lambda_v \alpha_1| = |2g \sum_{bc} (f_{1bc})^2 \alpha_b| \\
\leq |2g| \sum_{bc} (f_{1bc})^2 |\alpha_b| \\
\leq |2g| \sum_{bc} (f_{1bc})^2 |\alpha_1| \\
= |2g||\alpha_1|c_v.$$

(3.8)

Thus $\lambda_v \leq 2|g|c_v$, and the isotropic ray is the most relevant direction.

Although we have established that the most relevant RG direction for $g < 0$ is the isotropic ray, we have not dealt with the existence of other less (but still) relevant directions in the RG, i.e. beyond $-2g c_v$, there are still a set of positive eigenvalues to the linear analysis of (3.6). Unlike the eigenvalue corresponding to the isotropic ray, these eigenvalues are non-universal in the sense that they depend upon the particular form of the structure constants. Nevertheless it is instructive to compute the value
of the next largest eigenvalue for the various groups. The results are in the table below:

| $G$            | $\gamma = \lambda/[2g_{c,v}]$ |
|----------------|----------------------------------|
| $SO(N \geq 5)$ | $(N-4)/[2(N-4)]$                |
| $SU(N \geq 3)$ | $1/2$                            |
| $Sp(2N)$       | $(2 + N)/(2 + 2N)$               |

Here the second column gives the ratio of the next largest eigenvalue to that of the largest, $2|g_{c,v}|$, for all the non-exceptional groups. Although these ratios are not universal, it is worthwhile to note that none are close to one and so no other ray approaches the isotropic direction in importance.

As discussed by [12] in the context of a current-current perturbation of an SO(8) level 1 WZW model, i.e. an anisotropic SO(8) Gross-Neveu model, the presence of these less relevant directions does not destroy the symmetry restoration. Rather it introduces corrections to the theory that behave as the power $1 - \gamma$ of the bare coupling. As $\gamma$ is not close to unity, small bare couplings induce small corrections. We, following [12], can make this statement precise.

Let $g_i$ be the coupling along the isotropic ray and let $g_b$ be the coupling along the next most relevant direction breaking the symmetry. We determine the $\beta$-functions of these couplings from the $\beta$-functions of the linear stability analysis:

$$\frac{dg_i}{dl} = -g_i^2,$$

$$\frac{dg_b}{dl} = -\gamma g_b g_i; \quad \gamma < 1, \quad (3.9)$$

where $\gamma$ is given in the above table. Integrating these equations we find

$$g_i(l) = \frac{g_i(0)}{1 + g_i(0)l}$$

where $g(0)$ marks out the bare value of the couplings.

In order to ascertain the effect of $g_b$ on the symmetry restoration, we consider its effect on the gaps/masses of the model. To determine the gaps we use the relation relating the bare/physical gaps to the renormalized gaps:

$$\Delta(g_i(0), g_b(0)) = e^{-1}\Delta(g_i(l), g_b(l)). \quad (3.10)$$

Typically the above equation is evaluated at $l = l_c$, where $l_c$, the cutoff scale, is determined by $g_i(l_c) = 1$, the point where the 1-loop RG breaks down. From (3.10), $l_c$ is given by

$$l_c = \frac{1}{g_b(0)} - 1. \quad (3.12)$$

To determine the effect of $g_b$ on the gaps we consider the ratio of two gaps:

$$\frac{\Delta_1(g_i(0), g_b(0))}{\Delta_2(g_i(0), g_b(0))} = \frac{\Delta_1(1, g_b(l_c))}{\Delta_2(1, g_b(l_c))}. \quad (3.13)$$

As $\Delta_1(1,0) = \Delta_2(1,0)$ by the underlying symmetry, this ratio reduces to

$$\frac{\Delta_1(g_i(0), g_b(0))}{\Delta_2(g_i(0), g_b(0))} = 1 + c g_b(l_c), \quad (3.14)$$

where $c$ is some number. Using (3.13) and (3.12) we then have

$$\frac{\Delta_1(g_i(0), g_b(0))}{\Delta_2(g_i(0), g_b(0))} = 1 + c g_b(0)g_i(0)^{-\gamma}. \quad (3.15)$$

Then as $g_i, g_b \to 0$, the symmetry restoration becomes exact. With small but finite couplings the ratio of two gaps is $1 + (U/t)^{-\gamma}$, where $U$ is the typical bare coupling strength and $t$ is the bandwidth.

Thus we have established that the $\beta$-functions of (3.1) indicate a symmetry restoration in a portion of weak coupling parameter space. However this symmetry restoration might in general be much wider. In the work of [1], where the previously mentioned example of a restoration to an SO(8) symmetry in the context of two-leg Hubbard ladders was studied in great detail, it was found that the entire parameter space at weak coupling saw a restoration of symmetry. They found additional rays characterized by $|g_a| = g$ but with sign variations among the $g_a$. These additional rays marked out different phases of the Hubbard ladders. Included among the phases were a D-Mott phase, a phase characterized by an interaction induced charge gap with D-wave symmetry, an S-Mott phase, a CDW phase, and a spin-Peierls phase. Presumably such additional rays are present generically, but lacking physical motivation, we do not search for them here.

Having established the $\beta$-functions in (3.2) imply a restoration of symmetry, we ask if this RG-induced
restoration is actually realized. Although it is impossible to answer this definitively, we can examine this question in the light of the caveats concerning symmetry restoration through an RG. The first of these concerned the presence of an RG invariant that controls the physics. As the RG flow leaves the invariant unchanged, we cannot expect the physics along the flow to become more isotropic. However, we conjecture that the presence of this physics-controlling RG invariant is related to the presence of an additional symmetry in the theory. In the case of the U(1) Thirring model, we found that the RG invariant, \( \mu_i \), which governed the physics, parameterized the deformed quantum group symmetry, \( \hat{sl}(2)_q \). In the U(1) Thirring case, this quantum group symmetry arose as a deformation of an \( sl(2) \) Yangian symmetry present in the isotropic theory. However the U(1) Thirring model is somewhat unique in this regard. The isotropic current-current perturbations of level 1 WZW models all possess similar Yangians. But there is no sensible way in which they can be deformed. There do exist generalizations of the U(1) Thirring model which are characterized by deformed quantum group symmetries \(^{[13]}\). However these models correspond to imaginary coupling Toda theories and on the face of it are not even hermitian. Thus for symmetries larger than SU(2), we do not expect to be faced with this particular problem. Of course, it is conceivable that other hidden symmetries lurk in these models.

We must also be aware of the possibility that the IR fixed point to which the RG flows is not so much a point but a ray, that is, there exists a marginal operator at the fixed point such as in the deformed O(3) sigma model with topological angle \( \theta = \pi \). This is obviously not a concern when the fixed point is massive and so is unlikely to be relevant in the cases of bulk current-current perturbations of WZW models.

**B. SU(2)\(_k\) Overscreened Kondo Problems**

In this subsection we consider a set of overscreened \( SU(2), k \)-channel Kondo models. Such models arise when \( k \) channels of spin-1/2 fermions are coupled to an impurity spin, \( s^a \), of magnitude, \( s < k/2 \). These theories can be represented \(^{[14]}\) as boundary perturbations of a chiral \( SU(2)_k \) WZW model (with Hamiltonian \( H_{WZW}^L \)):

\[
H = H_{WZW}^L + \sum_{a=x,y,z} g_a J_a^L(0) \cdot s^a.
\]  

(3.16)

As usual we have represented these Kondo problems in their unfolded realization: only one chiral (\( L^- \)) component of the local spin density of the bulk theory, \( J_a^L(0) \), couples to the impurity spin, \( s^a \).

In the following we restrict our discussion of these anisotropic models to an \( s = 1/2 \) impurity spin. Indeed, in this case these models share the same 1-loop beta functions as the current-current perturbations of bulk WZW models, considered in the previous subsection. Therefore, the overscreened Kondo models also share the same putative RG-induced symmetry restoration with the current-current perturbations. The question then becomes whether this symmetry restoration suggested by the 1-loop RG equations is genuine.

The answer to this question is affirmative: this follows from the analysis of the isotropic overscreened strong coupling fixed point, done in \(^{[14]}\), where the latter was found to be stable with respect to small spin-exchange anisotropic perturbations \(^{[15]}\). In other words, isotropy in the spin-exchange couplings is restored, and the conclusions obtained from the 1-loop RG analysis can indeed be trusted.

When the number of channels is large, \( k >> s = 1/2 \), this can be easily understood on the basis of the 2-loop RG equation for the isotropic model (i.e. \( g = g_s \)), which has the form \(^{[13],[14]}\)

\[
\frac{dg}{dl} = -\frac{c_v}{2} g^2 - \frac{k c_v}{4} g^3 + ..., 
\]

(3.17)

where here \( c_v \), the casimir of the adjoint representation, is 2. We see that the number, \( k \), of channels appears in the two-loop term. This two loop \( \beta \)-function thus implies a fixed point at \( g_* \sim 1/k \). For large \( k \) this fixed point appears at small coupling, in the range of validity of the perturbative RG. Moreover it is known \(^{[14],[16]}\) that higher loop terms do not destroy this fixed point provided \( k \) is taken to be large compared to 3/4, the value of the quadratic casimir of the representation of spin-1/2.

To exploit this behaviour in the isotropic \( \beta \)-function, we suppose that the initial bare anisotropic couplings are much smaller than the isotropic fixed point at \( O(1/k) \). In this regime the 1-loop term in the \( \beta \)-function controls the flow and symmetry is thus restored. The theory will then proceed to the isotropic fixed point, undisturbed by higher loop terms, as in the purely isotropic case.

We would like to stress that these issues of symmetry restoration become more involved for the RG flows in the overscreened multi-channel Kondo models with impurity spin \( s > 1/2 \). In all those cases (except when \( k/2 = (s - 1/2) \) or \( k \leq 4 \)), the isotropic overscreened strong coupling fixed was found unstable to small anisotropies in the spin-exchange couplings \(^{[13]}\). At first sight the anisotropic 1-loop RG equations appear to be the same as in the case of impurity, spin \( s = 1/2 \). Thus as these predict symmetry restoration, one may then be tempted to conclude that the 1-loop RG in this case is unreliable.

However, a careful look at the 1-loop RG with exchange anisotropic couplings reveals that new terms of zero dimension in the Hamiltonian of the form, \( h_a S^a S^a \),

\[
h_a S^a S^a, 
\]

(3.18)

\(^{[13]}\)Recall that we are discussing the case of an \( s = 1/2 \) impurity spin.
will be generated. (Note that for spin, \( s = 1/2 \), such terms are proportional to the identity and so serve to only renormalize the impurity free energy.) Since these terms are highly relevant, they have a profound effect on the RG flows, which are therefore very different from the case of impurity spin, \( s = 1/2 \). Indeed, in the large-\( k \) limit it is those dimension zero (bare) operators which renormalize into the relevant symmetry breaking perturbation of scaling dimension, \( \Delta = 6/(2 + k) \to 0 \), found in [3].

IV. SYMMETRY RESTORATION IN A BROKEN U(1) THIRRING MODEL

In the previous sections, we have argued that while the U(1) Thirring model sees symmetry restoration as more broadly understood in the course of taking the scaling limit, it does not experience an RG-induced restoration of symmetry. This occurs in the U(1) Thirring model because physical quantities depend upon the RG invariant, \( \mu \). The quantity \( \mu \) is a reflection of the quantum group symmetry \( s(1,1) \), present in U(1) Thirring, and it is natural to determine the consequences of breaking this symmetry. This does not necessarily mean that in breaking the symmetry, we exclude the possibility of a \( \mu \)-like parameter. \( \mu \) is an RG invariant. In breaking the symmetry, one does not eliminate all RG invariants. One merely alters their form. Indeed as RG trajectories are lines, they can generally be parametrized by quantities that will be constant along the RG flow, say the intercept with one of the hyper-planes in the RG space. What is not clear is to what extent the physical properties are going to depend on that parameter. In breaking the symmetry, we will in general lose any known means to connect the unchanging physical properties with the RG invariant. It is akin to finding accidental degeneracies in QM. Such accidental degeneracies are almost always related to hidden symmetries. With no symmetries, there are no such degeneracies. It is thus tempting to propose that with no symmetries around to be associated with the RG invariants, these invariants cannot, in general, govern the physics, and therefore that an RG-induced symmetry restoration should generally occur.

A simple example where this idea can be investigated is a broken U(1) Thirring model. We thus consider what we call the \( g_{xy} \)-Thirring model:

\[
\mathcal{L} = i\bar{\psi}_\alpha \gamma_\mu \partial^\mu \psi_\alpha + \frac{1}{4} g_{\parallel} (j_z)^2 + \frac{1}{4} g_{\perp} ((j_x)^2 + (j_y)^2) + \frac{1}{4} g_{xy} j_x j_y. \tag{4.1}
\]

The one loop RG equations for this model are

\[
\begin{align*}
\frac{dg_{\parallel}}{dl} &= -g_{\parallel}^2 + \frac{g_{xy}^2}{2}; \\
\frac{dg_{\perp}}{dl} &= -g_{\perp} g_{\parallel}; \\
\frac{dg_{xy}}{dl} &= \frac{g_{xy} g_{\parallel}}{2}.
\end{align*}
\tag{4.2}
\]

Recalling that \( \mu^2 = g_{\parallel}^2 - g_{\perp}^2 \), we see that the \( g_{xy} \)-perturbation causes \( \mu^2 \) to flow:

\[
\frac{d\mu^2}{dl} = g_{\parallel} g_{xy}^2. \tag{4.3}
\]

In the AF region, \( g_{\parallel} \) is negative and so \( \mu^2 \) decreases under the RG. However, as \( g_{xy} \) is (marginally) irrelevant in the AF region, we do not necessarily expect a full restoration of the SU(2) symmetry: as \( g_{xy} \) goes to zero, \( \mu^2 \) will stop flowing.

In region C, \( \mu^2 \) is negative while \( g_{\parallel} \) is either positive or negative. Thus depending on the bare value of \( g_{\parallel} \), \( \mu^2 \) either increases or decreases in magnitude. However again \( \mu^2 \) will not change without bound as \( g_{\parallel} \) eventually flows to negative values. Moreover a change of \( \mu^2 \) in region C corrects the physics only minimally due to the exponentially small dependence of physical quantities upon \( \mu \) in this region.

Thus it would seem that in breaking \( s(1,1) \), there is an RG-induced symmetry restoration in the model where there was none before. However we offer a cautionary tale on the interpretation of the above analysis. We now choose to modify the U(1) Thirring model by instead introducing separate couplings for \( j_x^2 \) and \( j_y^2 \): \n
\[
\mathcal{L} = i\bar{\psi}_\alpha \gamma_\mu \partial^\mu \psi_\alpha + \frac{1}{4} g_{\parallel} (j_z)^2 + \frac{1}{4} g_x (j_x)^2 + \frac{1}{4} g_y (j_y)^2. \tag{4.4}
\]

In this XYZ-Thirring model, the one loop RG equations become

\[
\begin{align*}
\frac{dg_{\parallel}}{dl} &= -g_x g_y; \\
\frac{dg_x}{dl} &= -g_y g_{\parallel}; \\
\frac{dg_y}{dl} &= -g_x g_{\parallel}.
\end{align*}
\tag{4.5}
\]

Unlike [1], this model is known to be solvable by Bethe ansatz. The results are similar to those of the U(1) Thirring model, trigonometric functions being replaced by elliptic functions. Following [17] and [18], one finds for instance the dispersion relations

\[
e_0/p_0(\lambda) = \frac{\Lambda_c}{4} \left\{ \tan^{-1} \left[ \frac{\text{cn}(\tilde{f} - \frac{\pi \lambda}{\mu}, \tilde{k})}{\text{sn}(\tilde{f} - \frac{\pi \lambda}{\mu}, \tilde{k})} \right] \pm \tan^{-1} \left[ \frac{\text{cn}(\tilde{f} + \frac{\pi \lambda}{\mu}, \tilde{k})}{\text{sn}(\tilde{f} + \frac{\pi \lambda}{\mu}, \tilde{k})} \right] \right\}. \tag{4.6}
\]

Here \( \text{cn}, \text{sn} \) and \( \text{dn} \) are the usual elliptic functions. The parameter \( \tilde{f} \) is defined by \( \tilde{f} = \frac{\pi}{\mu} K'(\tilde{k}) \), where in turn the dual modulus \( \tilde{k} \) is defined by \( K'(\tilde{k})K'(\tilde{k}) = \mu K(\tilde{k}) \) with \( k \) being the original modulus. The key parameters \( \mu, k \), and \( f \) are obtained in terms of the bare coupling constants as
\[ k \text{sn}^2(\mu, k) = \tan \left( \frac{y_x - y_y}{2} \right) \tan \left( \frac{y_x + y_y}{2} \right) ; \]
\[ \text{cn}(\mu, k) d\mu(\mu, k) = \frac{\cos g}{\cos \left( \frac{y_x - y_y}{2} \right) \cos \left( \frac{y_x + y_y}{2} \right)} ; \]
\[ \frac{\text{sn}(i f, k)}{\text{sn}(\mu - i f, k)} = -\frac{\cos \left( \frac{y_x + y_y}{2} \right)}{\cos \left( \frac{y_x - y_y}{2} \right)} e^{-ig_1} . \quad (4.7) \]

Both \( \mu \) and \( k \) are RG invariants, as readily can be checked. All physical quantities are determined in terms of these parameters and so in the XYZ-Thirring model there is no RG induced symmetry restoration.

The absence of symmetry restoration does not necessarily contradict our notions on accidental degeneracies. Here we have not actually broken the \( \hat{sl}(2)_q \) symmetry. Rather we have only deformed it into its elliptical cousin \( \hat{sl}(2) \). Unfortunately, it is easily possible to convince oneself otherwise. Defining \( \delta g_{\perp} = g_x - g_y \ll g_x + g_y \) and \( g^{av}_{\perp} = \frac{1}{2}(g_x + g_y) \), the former as a measure of how broken the U(1) symmetry is, we find the following RG equations:

\[
\frac{d\delta g_{\perp}}{dl} = g_{\parallel} \delta g_{\perp} ;
\frac{dg_{\parallel}}{dl} = \frac{1}{4}(\delta g_{\perp})^2 - (g^{av}_{\perp})^2 ;
\frac{dg^{av}_{\perp}}{dl} = -g_{\parallel} g^{av}_{\perp} . \quad (4.8)
\]

In region AF we have \( g_{\parallel} < 0 \). If we did not know the model was integrable, it would be natural to define the analog to \( \mu \) in the broken U(1) case to be

\[ \mu^2 = g_{\parallel}^2 - (g^{av}_{\perp})^2 . \quad (4.9) \]

We then see

\[ \frac{d\mu^2}{dl} = \frac{g_{\parallel}}{2}(\delta g_{\perp})^2 . \quad (4.10) \]

Hence under the RG, \( \mu^2 \) decreases. Thus it would seem that in breaking the U(1), an additional increase in symmetry is achieved. However this does not mean the full SU(2) symmetry is restored, as with \( g_{\parallel} < 0 \), we see the quantity driving the flow of \( \mu \), \( \delta g_{\perp} \), is itself in fact irrelevant. Thus we have a conclusion similar in spirit to the \( gx_y \)-Thirring model. In region C, we again see similarities between the XYZ and \( gx_y \)-Thirring models. In this region, \( \mu^2 \) is negative while \( g_{\parallel} \) is either positive or negative. So again \( \mu^2 \) will either increase or decrease, but not without bound. And the apparent physical consequence of this is minimal given the exponentially small dependence of physical quantities upon \( \mu \).

Thus given our definition of \( \mu \), we see a putative partial restoration of symmetry under the RG. But as we know from the exact solution this does not actually occur. It is merely an artifact of our definition of \( \mu \). Thus in saying a (partial) RG-induced symmetry restoration occurs in the \( gx_y \)-Thirring model, we are implicitly supposing that this XYZ-Thirring scenario is not applicable. In particular, we are supposing that the \( gx_y \)-perturbation has genuinely broken the \( sl(2)_q \) symmetry and not merely deformed it, thus forbidding an exact solution.

Although there is no RG-induced symmetry restoration of the XYZ-Thirring model, the above 1-loop RG analysis (4.8-4.10) is reflected in the exact solution. At weak coupling appropriate to the 1-loop analysis, one must take the modulus, \( k \), of the elliptic functions in the relations (4.7) to zero, leaving

\[ \cos \mu \approx \frac{\cos 2g_{\parallel}}{\cos(g_x - g_y) \cos(g_x + g_y)} . \quad (4.11) \]

For a fixed \( g^{av}_{\parallel} \), increasing the measure of the U(1) breaking, \( \delta g_{\perp} \) leads to smaller values of \( \mu \), confirming the one loop RG results.

The 1-loop RG is also reflected in a more elemental analysis of the XYZ problem. For simplicity consider an XYZ Kondo model instead, with couplings \( g_x, g_y \), and \( g_{\parallel} \). After bosonization, and forgetting inessential constant factors, the Hamiltonian reads

\[ H = H_0 + (g_x + g_y) \left[ s^+ e^{i\sqrt{3} \phi(0)} + s^- e^{-i\sqrt{3} \phi(0)} \right] 
+ (g_x - g_y) \left[ s^+ e^{-i\sqrt{3} \phi(0)} + s^- e^{i\sqrt{3} \phi(0)} \right] + g_{\parallel} \partial_x \phi . \quad (4.12) \]

Suppose we now perform the canonical transformation, \( U \), again (see text before (7)). This time, the vertex operators involved in the second part of the Hamiltonian (that break the U(1) symmetry) see their dimension increased: they thus become irrelevant, and disappear in the scaling limit. Thus the 1-loop RG seems to correctly predict that the symmetry breaking term, \( g_x \neq g_y \), vanishes in the low energy limit. Of course, bosonization combined with a canonical transformation gives similar results for the XYZ Thirring model.

**V. CONCLUSION**

This article has two overarching themes, both interconnected through the scaling limit. Firstly, we expanded the notion of symmetry restoration to include all situations in which the physical properties of a model have a weak dependence upon an anisotropy. When this definition was combined with constraints coming from taking a scaling limit, we found (as discussed in Section II) that a wide variety of models in fact see symmetry restoration. Secondly, we observed that the scaling limit in general restricts the range of bare parameters to be small. We then exploited this fact in Section III to argue that a 1-loop RG should accurately describe the physics.

Perhaps an underemphasized result of this work is that the trustworthiness of the 1-loop RG has been previously underestimated. Even in cases where it seemed to promise symmetry restoration which in fact did not occur
(the U(1) Thirring model), the fault lay not in the 1-loop RG but in its interpretation. Indeed the 1-loop RG of the U(1) Thirring model correctly predicts the parameter \( \mu \) to be an RG-invariant. The problem, in contrast, was in assuming that the physics will be governed by a ratio of couplings, \( g_\parallel /g_\perp \), and not \( \mu \). When the 1-loop RG does indeed fail, for example in the O(3) sigma model with \( \theta = \pi \), the reason is readily apparent: the topological term is manifestly non-perturbative.

On a finishing note, the reader should certainly not be left with the impression that symmetry restoration always occurs in all possible models. There are many situations where it does not. In many cases this is again faithfully represented by the one loop RG.

Indeed, in all the examples discussed in the bulk of the paper, the IR strong coupling fixed point was “truly isotropic”, in the sense that it was \( \mu \)-independent. This is trivial for the massive cases where there are no leftover massless degrees of freedom in the IR; this is also true for the spin 1/2, 1-channel Kondo problem where the impurity spin is entirely screened by the conduction electrons. In the latter case the isotropy of the fixed point manifests itself technically in the independence of the impurity scattering (reflection) matrix upon \( \mu \). The operators determining the approach to the fixed point have \( \mu \)-independent dimensions, and it is only their respective amplitudes that depend upon the anisotropy.

As an obvious candidate lacking symmetry restoration according to either definition, consider the channel-anisotropic multichannel Kondo problem with impurity spin, \( s = 1/2 \). This model possesses an anisotropic (massless) IR fixed point. Here we find the exact opposite to what we have seen so far: that is, instead of a symmetry restoration, there is an enhancement in asymmetry. The corresponding Hamiltonian is given by:

\[
H = H_0 + 2 \sum_{a,b=1,2} \sum_{m=1}^f J_m \langle \Psi_{a,m} \sigma_{ab} \Psi_{b,m} \rangle (0) \cdot \hat{s}. \tag{5.1}
\]

Here \( a,b \) is the spin index, and \( m \) the channel index.

\( H_0 = -i \sum_{a,m} \int \Psi_{a,m}^\dagger \partial_x \Psi_{a,m} (x) \) describes non-interacting partial-wave electrons.

Consider the simplest case with \( f = 2 \) channels (generalizations to more channels, \( f > 2 \), are straightforward with the same conclusions). The physics of the channel-anisotropic case is simple (see also [13]): the more strongly coupled channel undergoes an ordinary (1-channel) Kondo effect, screening the spin-1/2 impurity, whereas the more weakly coupled channel decouples from the impurity. Thus, the IR behavior of the channel-anisotropic case is completely different from the (over-screened) channel-isotropic situation. Clearly, channel-anisotropy is not restored, according to either definition (1 or 2 of the introduction).

This feature is easily seen in the one loop RG equations, which read in this case,

\[
\frac{dJ_2}{dl} = -C J_1^2.
\]

Thus the ratio \( A \equiv \frac{J_2}{J_1} \) obeys

\[
\frac{dA}{dl} = C J_1 (1 - A), \tag{5.2}
\]

and so grows under the RG. Observe also that at the channel-isotropic low energy fixed point (the non Fermi liquid Kondo fixed point) channel anisotropy is strongly relevant [14]. There are also known cases where the anisotropy is exactly marginal, i.e. a finite bare anisotropy leads to a finite anisotropy in the scaling limit [21].

The channel anisotropic model is also integrable [20], allowing us to relate to our discussion of the scaling limit. In the simplest case of two channels, \( f = 2 \), the Bethe ansatz solution leads to expressions for the two characteristic energy scales in the problem: \( T_1 \equiv D e^{\frac{\pi}{2}} \), \( T_\infty \equiv D \cos \left( \frac{\pi}{2} \frac{\pi}{2} \right) e^{\frac{\pi}{2}} \) \((J_1 \leq J_2)\), where \( D \) is the bandwidth that has to be taken to infinity in the scaling limit. The ratio \( \Delta \equiv \frac{T_1}{T_\infty} \) is the physical measure of anisotropy, the channel isotropic case corresponding to \( \Delta = 0 \).

The scaling limit is obtained by letting \( D \to \infty \), \( J_1 \to 0 \). One checks then that keeping a ratio \( \frac{J_2}{J_1} \) finite, i.e. a finite bare channel anisotropy, leads to \( \Delta \to \infty \), i.e. an infinite channel anisotropy in the scaling limit. The only way to have a finite anisotropy in the scaling limit is to start with an infinitesimally small bare anisotropy, explicitly setting \( J_1 \approx \frac{\Delta}{\pi^2} \), \( J_2 = J_1 / (1 - \epsilon) \), as \( \epsilon \to 0 \). The situation is thus the exact opposite of what we observed for the spin anisotropy: a finite channel anisotropy in the continuum limit requires an infinitesimally small bare channel anisotropy!

Another example of an anisotropic (massless) IR fixed point is the spin \( s = 1 \), 1-channel Kondo problem. The general \( s > 1/2 \), \( k = 1 \) Kondo problem is described by the Hamiltonian [23]. Consider first the regime where \( g_\parallel > g_\perp > 0 \). Performing the same canonical transformation as in Section II.A leads to a Hamiltonian as in (22). This model is integrable if the impurity spin transforms in a spin \( s \) representation of \( sl(2)_a, q \approx -e^{-i\pi s_1} \) (a quantum deformed version of \( SU(2) \)) [23]. For \( s = 1/2 \) or \( s = 1 \) an ordinary spin can be used, and no quantum deformation is necessary. The resulting IR fixed point can be studied exactly. For \( s = 1 \) it is found to consist of a left-over conserved spin \( s' = 1/2 \). The spin’s z-component induces on the electron degrees of freedom a phase shift depending weakly upon the original anisotropy. This is the usual anisotropic ferromagnetic Kondo effect, representing a ‘line of anisotropic fixed points’ continuously deformable into the isotropic one. Physical quantities at the fixed point, such as the cross section for electron scattering of spin up or down, do depend upon the anisotropy, and so do the operators governing the approach to the fixed point. However all these dependencies upon the anisotropy are weak (continuous), and therefore we find
symmetry restoration in the expanded sense of definition 2.

When the size of the $sl(2)_q$ impurity is $s > 1$, the left over impurity spin obeys $sl(2)_q' = 0$ commutation relations, with $q' \approx q$ for small anisotropies, and the same conclusions hold as for the $s = 1$ result.

The above nature of the flow can also be understood on the basis of the weak coupling RG. Consider spin $s = 1$. As discussed earlier, a highly relevant zero dimensional operator, $hs^2 s^2$, as in (3.18) will be generated. When the renormalized coupling $\mu$ reaches a magnitude of order unity, spin flip processes stop, and so does the RG flow. For an anisotropy, $\mu^2 = g^2 - g^2 > 0$, as above, the induced coupling $\mu$ is negative, and hence the impurity spin can be only in two states, $s^2 = \pm 1$, at those scales. At the same time, the residual renormalized coupling, $g_{||}$, amounts to a phase shift on the electrons, as in the integrable formalism. As $\mu^2 \to 0$, the phase shift becomes $\pi/2$, appropriate for the underscreened isotropic IR fixed point.

In summary, the region, $\mu^2 = g^2 - g^2 > 0$, of the $s = 1$ Kondo model flows into the ferromagnetic region of the $s = 1/2$ Kondo problem. This flow can be represented through Figure 4. Although Figure 4 is sketched for the conventions of the U(1) Thirring model, by taking $g_{||} \to -g_{||}$ it can be understood as that of the $s = 1$ Kondo model. Then the $s = 1$, $\mu^2 > 0$, Kondo model flows into region $W$ of Figure 4. Region $\lambda F$ is the usual anti-ferromagnetic Kondo regime. Moreover, the boundary between regions $C$ and $W$ describes the isotropic underscreened flow of the $s = 1$ Kondo model.

There are however potential subtleties when $\mu^2 < 0$. A weak coupling analysis including the zero dimensional operators as above indicates that the region, $\mu^2 = g^2 - g^2 < 0$, of the $s = 1$ Kondo model flows into region $C$ of Figure 4. This is so since the induced coupling constant $\mu$ is positive, freezing the impurity spin into the single state, $s^2 = 0$. Hence, the ultimate IR fixed point in this region would consist of a completely eliminated impurity spin and no phase shift for any non-vanishing amount of anisotropy, in contrast to the underscreened isotropic fixed point with phase shift, $\pi/2$. Note that this lack of symmetry restoration is already visible in the weak coupling RG. This situation requires further study.

Another example that would be interesting to study in more detail is the sausage model at $\theta = \pi$. In that case, it is expected by analogy with the usual $O(3)$ model that the theory flows to a non trivial fixed point at finite distance that is not accessible perturbatively and is described by a compactified free boson with radius $R \approx 1 + \frac{\mu^2}{2\pi}$. No simple bare equivalent of this model is known, and it is not clear that symmetry restoration in its expanded sense does occur: however, formal regularizations based on the Bethe ansatz equations certainly show a behaviour in all points identical with the previous cases.

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