A handle is enough for a hard game of Pull

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Abstract

We are going to show that some variants of a puzzle called pull in which the boxes have handles (i.e. we can only pull the boxes in certain directions) are NP-hard.

1 Introduction

In [1], Marcus ritt proved that two variants of pushpush [2], called pull and pullpull, were np-hard. He asked about the complexity of a variant of pull in which the boxes have either horizontal (left and right) or vertical (up and down) handles. We will solve that problem in this document, by showing that the problem is NP-hard. We will also show that other variants of pull are NP-hard as well.

In section two of this document we are going to prove that the storage version of pull is NP-hard. In section three we are going to prove that the variant of pull that the authors ask about in [1] is NP-hard as well.

In section four we are going to prove that the version of pull in which the boxes have handles in opposite directions (i.e left and right) is NP-hard.

In section five we will prove that the version of pull in which the boxes have two handles, one for the vertical direction and another one for the horizontal direction is NP-hard. Finally, in section six, we will prove that the variant of pull where the player can only pull boxes in a single direction is NP-hard. We will show that the previous variants are NP-hard by reducing satisfiability to these problems.

The proofs in this paper are for variants of pull with fixed blocks (Pull-F), but they can be easily adapted so that they work for variants of pull without fixed blocks. This can be done by filling with blocks all the positions of the board that are inaccessible in the corresponding Pull-F variants, by making the separation between the "gaps" and the boxes inside the crossovers bigger and by making the walls thicker for those gadgets for which it is necessary. We will represent movable blocks(boxes) by orange/brown squares and fixed blocks by red squares.
2 The storage version of pull is NP-hard

Here we will present the main components that we will use for reducing SAT to the storage version of pull.

2.1 No return gadget

This is a no return gadget, similar to the ones in [1] and [2], that we can use for the storage version of pull.

We can also use these gadget for those variants in which all the boxes have all the handles in both the vertical and horizontal directions. And also in those variants where boxes have handles in opposite directions (i.e. left and right). For those variants, the gadget is the same, but the player is not forced to place the boxes in the green squares in order to win.

This gadget allows passage from A to B, forbids passage from B to A, and can only be traversed once

Lemma 1. The no return gadget can be traversed from A to B, but after it is traversed, the player cannot go back from B to A.

Proof. To go to B, the player has to move the boxes 1 and 2 out of the way. In figure 2, the only way to do it is by pulling the boxes to the left (it can be any other direction if the gadget is rotated and/or reflected). To be able to pull box 2 out of the narrow corridor, the player has to pull box 1 to the position marked by the leftmost green square, blocking the way to A, the player can then pull the other box to the other green square and exit through B.

2.2 Crossover

This is a crossover for the storage version of pull. If the player enters into the gadget from A1, he can only exit through A2. And if the player comes from B1 he can only exit through B2.
We are now going to describe how to cross the gadget. We will suppose that the player enters the gadget from A1, the other case is symmetrical.

When the player enters, he pulls down the boxes 1 and 2

Figure 3: The player has pulled down boxes 1 and 2

After doing that, the player has opened a gap in which he can place box 3

Figure 4: The player has moved box 3 to the left gap

Now, the player can exit through A2. When he returns to the crossover, he has to place all boxes in their original positions (marked by green squares in the previous figures).

To do that, the first thing he does is pull box 3 to the gap at the right:

Figure 5: The player has moved box 3 to the right gap

Then he can pull to the left this box until he reaches the rightmost green
square, its original position.

Then he pulls upwards boxes 2 and 1 to its original positions, after doing this he can return through A1

**Lemma 2.** *In its original state, the crossover gadget can be traversed from A1 to A2 or it can be traversed from B1 to B2. Those are the only possible traversals of the gadget.*

*Proof.* Let’s suppose that the player enters the gadget from A1, the other direction is symmetrical. When the player enters, the only thing he can do is pull down the two boxes that he finds at the entrance. From there, he cannot pull down the box blocking the way to B2 because he will get trapped between that box and the two boxes below him. He can’t exit through B1 because he can only pull upwards one of the two boxes that are blocking the way to B1.

**2.3 Clause gadget**

This is a clause gadget for the storage version of pull

The gadget has $n$ entrances where $n$ is the number of literals in the clause.

To win the game, the player has to place the O box in Figure 6 to the green square inside every clause gadget.

![Figure 6: A picture of a clause gadget with three literals, the entrances A1, A2 and A3 correspond to entrances that are connected to a literal wire.](image)

**Lemma 3.** *When the player enters the gadget from an entrance that belongs to a literal wire he can’t exit through a literal wire that hasn’t been visited before.*

*Proof.* When the player enters the gadget, he has to pull down two boxes to enter (when he returns he has to pull them up again). When he enters the gadget, he cannot exit through a literal wire that hasn’t been visited before because there are two boxes blocking the way to that wire. The player can only pull one of the two obstacles out.

**2.4 Variable gadget**

This gadget represents one of the variables that belong to the SAT instance being reduced.
When the player enters the gadget, there is a bifurcation that divides the gadget in two paths: a path to the left and another path to the right.

If the player takes the path on the right, that means the player has chosen to assign the variable a negative value. If he chooses the left path, that means the player has chosen to assign the variable a positive value.

When the player enters the path to the left or the path to the right, he crosses a no return gadget.

There will be a single literal wire for each occurrence of that literal in the SAT instance. Once he has traversed the literal wires, he has to keep going down the variable gadget, there he will traverse another no return gadget. This is to ensure that the player cannot access the literal wires of the other path.

Finally, the player traverses a no return gadget at the end of the variable gadget.

2.5 General view of the board

We have seen the gadgets that we are going to use. Now we are going to shown how to arrange them to create a board.

In the board, the variable gadgets are at the left of the board and the clause gadgets are at the right of the board. Each variable gadget is put below the next one, the same for the clause gadgets. For the first variable in the SAT formula, we put a variable gadget in the left and we put a clause gadget for every clause in which the variable appears. First we put a clause gadget for every clause in which there is a negated literal of that variable and then we put the clauses in which there is a positive literal of that variable. Then we connect the corresponding literals to the clauses. We connect a literal wire to every clause gadget in which the variable appears.

For the second and remaining variables, we follow this procedure:

We define two kind of clauses, "old clauses" and "new clauses". Old clauses are those that have a clause gadget already in the board and new clauses are those that dont have a clause gadget in the board yet. That is, old clauses are those that belong to a variable gadget that has already been put
in the board and new clauses are those in which the literals of the current variable appear but not the literals of the previous variables.

For the negated literals of the variable, we connect the wires in the variable gadget to the old clauses if there is any old clause that contains the negated literal. Then we put the variable gadgets corresponding to the new clauses (if there is any) and we connect the literals to them.

We follow the same procedure for the positive literals of the variable.

**Theorem 1.** The storage version of pull is NP-hard

Figure 8: An example board. The variables at the left are $x_1$, $x_2$, and $x_3$. The clauses at the right are $(\neg x_1 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2) \land (x_1 \lor \neg x_2)$.
3 Pull where each box has handles in either vertical or horizontal directions is NP-hard)

Here we are going to solve the open problem in [1], we are going to show that the variant proposed in [1] is NP-hard.

To do that, we will reuse the gadgets from previous section and we will only introduce the gadgets that are unique for this variant. For the other gadgets of the previous section, we will only describe the changes that we have to do to the gadget so that they work for this variant. If a gadget has not changed, it will not be shown here.

3.1 Unidirectional crossover

This crossover works like the crossover in [2], but it doesn’t matter the order in which the player uses the crossover, he can traverse the crossover horizontally and then vertically or vice versa. This crossover allows the player to traverse it horizontally without leakage to the vertical direction or vice versa. If the player enters the gadget from the entry A1, he can only exit through A2. And if the player enters from B1 he can only exit through B2 (see Figure 9).

Figure 9: The crossover in its original state

If the player enters the gadget from A1, he can reach A2 by pulling to the left all boxes that are blocking the way to A2.

If the player enters the gadget from B1, he can reach B2 by pulling downwards all boxes that are blocking the way to B2.

Figure 10: A crossover that has been crossed from both directions

Lemma 4. In its original state, the crossover gadget can be traversed from A1 to A2 or it can be traversed from
B1 to B2. Those are the only possible traversals of the gadget

Proof. Let’s suppose that the player enters the gadget from A1, the other direction is symmetrical. When the player enters, the only thing he can do is pull the two boxes that he finds at the entrance. From there, he cannot go to B2 or B1 because the player has not enough space to pull down the boxes that are blocking the path to B2, and he has not enough space to pull up the boxes that are blocking the way to B1.

This gadget will form an important component of the different crossovers that we will use for the different path variants of pull.

3.2 Bidirectional crossover

This is a crossover that is formed by four limited unidirectional crossovers, rotated and placed as is necessary to make the gadget work. At the entries and exits of the unidirectional crossovers that make the gadget, there are no return gadgets.

This gadget is similar to the bidirectional gadget of [2] and allows the player to traverse it in the horizontal direction and then the vertical direction or vice versa. This gadget only works for the path versions of pull that let the player move the boxes in all directions.

3.3 Variable gadget

For this variant of the game and the following ones, we replace every literal wire inside the variable gadget with two parallel wires. One is used to go to the clause and the other is used to return from the clause. We have to ensure that the player can only use these wires if he comes from the right place.

If the player comes from the path of a positive literal, we don’t have to do any change, since the one way gadget that is just after the literal wires of the negative literal path prevent him from accessing those literal wires.

Now if the player comes from the negative literal path, we have to ensure that the player cannot access any of the literal wires of the positive literal path.

To do that, we simply place an unidirectional crossover for each crossing of the wires inside the variable gadget.
3.4 Clause gadget

The gadget has \( n + 1 \) entrances and \( n \) exits where \( n \) is the number of literals in the clause.

Every entrance that comes from a literal wire is adjacent to an exit. The entrances and the exits are connected by a corridor.

The gadget has a no return gadget before all entrances and exits, including the C1 entrance and the C2 exit.

**Lemma 5.** If the player wants to go from C1 to C2 in figure 11, the player had to access the gadget previously from an entrance that belongs to a literal wire.

**Proof.** When the player enters the gadget, there is a box (with the label O in figure 11) that it’s blocking the way to B2. In order to get this box out of the way, the player has to access the gadget from one of the other entrances (A1 or B1) and pull the box one square to the left, so that the box doesn’t block the corridor anymore. \( \square \)

**Lemma 6.** When the player enters the gadget from an entrance that belongs to a literal wire he has to exit through the adjacent exit.

**Proof.** When the player comes from an entrance wire (A1, for instance) he can go to A2 wire using the corridor, then he can move the box labelled "1" to the center of the "gap" at his left, because there is a no return gadget at the end of the A2 wire, he has to return to the A1 wire and cross a no return gadget in order to get to the clause. Then he has to cross a no return gadget and move the box labelled "3" to the "gap" at the left of the box, then he can enter the gadget and move the box labelled "O" to the left. The no return gadgets that are at the entrance wire (or the boxes at the end of the wires, if the entrances hasn’t been traversed before) doesn’t allow the player to return from these wires. If he tries to
exit from another return wire different from the one that he has unlocked (for instance B2) he will get trapped between a no return gadget and a box labelled "2" in figure 11 (in case the wire had not been traversed before), or he won’t be able to go through the wire, because the no return gadget has already been traversed before. He also won’t be able to exit from the wire he entered because a no return gadget is blocking the way.

Theorem 2. The path version of pull in which each box has handles only in either the vertical or the horizontal directions is NP-hard

4 Pull where boxes have handles in opposite directions is NP-hard

We are going now to prove that all variants where boxes have handles in opposite directions (i.e. left and right or up and down) are NP-hard

As we did in the previous section, we will describe now the changes that we must do to the gadgets we presented in the previous section so that they work correctly for this variant of pull.

4.1 Half unidirectional crossover

This is a crossover that works like the unidirectional crossover of previous section, but it only works in a single direction. That is, it is a crossover that can, for instance, prevent a player from accessing the horizontal direction if the player enters the gadget from the vertical direction. But it can’t prevent the player from entering the vertical direction from the horizontal direction.

When the gadget prevents the player from traversing it vertically from the horizontal wires, we will say that the gadget is a "vertical crossover".

When the gadget prevents the player from traversing it horizontally from the vertical wires, we will say that the gadget is an "horizontal crossover".

Figure 12: Example picture of an horizontal crossover. If we rotate the gadget 90 degrees, it becomes a vertical crossover.

We will use this crossover for all those variants in which we cannot create unidirectional crossovers like the ones of the previous section. This crossover will form a component of bigger crossovers with more functionality.
4.2 Variable gadget

This gadget is the same variable gadget the one used in the previous section, but we cannot use the same crossovers we used in the previous section. We have to replace the crossovers of the previous section with half crossovers. See figure 13

Figure 13: This picture represents the intersection point between an horizontal and a vertical wire inside a variable gadget. The two horizontal wires are a closeup of the two literal wires that the player has to use to go to a clause (up wire) and return from that clause (down wire)

4.3 Two way gadget

This is a gadget that allows the player to go to a position and return back. Once the player returns, the player cant go back to the position he visited previously.

This gadget is formed by two no return gadgets as it is shown in figure 14

4.4 Crossover gadget

This is a crossover, similar to the one shown in previous section, but it only works for variants of pull in which the player can only pull boxes in opposite directions

Figure 14: Two way gadget, the direction in which the arrow is pointing represents the direction we must follow to pass through the wire

We will only use this gadget inside crossovers

Figure 15: Schematic drawing of the crossover

Lemma 7. The crossover allows the player to traverse the horizontal wires without letting the player traverse the vertical wires

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Proof. See figure 16 to see a picture of the crossover. We will assume in these proofs that the boxes inside the crossover can only be moved in the horizontal direction. The proof is symmetrical for the variant of pull in which boxes have only vertical handles. If the player has to traverse the crossover horizontally, he will enter the crossover from the A1 entry. When the player enters the crossover, he crosses a no return gadget. After crossing this gadget, he cannot go back. There he has to open an horizontal crossover in front of him. After opening this crossover, notice that he cannot go up towards B2 because there is a box just after a no return gadget that is blocking the way to B2, either the box or the no return gadget will prevent him from moving upwards. He can’t move down the vertical wire towards B1 because there is a no return gadget blocking his way down. So the player has to keep moving forward. After crossing the horizontal crossover, he has to cross another no return gadget. And after crossing it, he will have to open another horizontal crossover in front of him. After opening this horizontal crossover, he will be able to go upwards through a corridor above him. From this corridor, he will be able to move the box at the end of this corridor and unlock the way to B2 (if it is not unlocked already). This box prevents the player from entering into the vertical wire from that part of the corridor. The player also cannot enter inside the vertical wire from the bottom part of the corridor because there will be a box inside the corridor blocking his way (only if the player has not traversed the vertical wires before, see next lemma), this box prevents the player from entering into the bottom part of the corridor (assuming it has not been moved). If the box has been moved out of the way, that will mean that the two way gadget inside the corridor has already been traversed (see next lemma for a proof) and, once again, the player cannot enter the vertical wire from inside the bottom part of the corridor. So he cannot enter inside the vertical wire from inside this corridor and he will have to keep moving forward through the horizontal wire to A2. There, he has to traverse another no return gadget. After traversing this gadget, he can only keep moving forward. There, he has to open an horizontal crossover. After opening the crossover, in the center of this gadget there are four possible directions the player can take. Upwards, downwards, back to the left or forward right. From here, the player cannot move upwards towards B3 because there is a no return gadget that is blocking his way upwards, this gadget can only be traversed if the player accesses it from B3. If the player has not traversed the crossover vertically before traversing it horizontally, there will be an obstacle blocking his way downwards, so the player cannot go down this wire. If the player has
already traversed the crossover vertically, he will have traversed the no return gadget and once again the player will not be able to move down and reach the second horizontal wire from here. So the only option left for the player is to keep moving forward. After crossing the last no return gadget in this wire, the only option left for the player is to keep moving forward and exit the gadget from A2.

The player will enter into the gadget again from A3, after crossing a no return gadget, he will have to open an horizontal crossover. Once this crossover is opened, the center of the crossover intersects with the vertical wire. From here, he can’t move vertically towards B3 because a no return gadget is blocking his way up. If, instead, the player goes down through the vertical wire, either an obstacle box (if the player has not crossed the crossover vertically) or a no return gadget that has been traversed (if the player has crossed the crossover vertically) will block his way down to B4. So he will have to keep moving forward. After traversing another no return gadget, the player will reach another horizontal crossover. This crossover intersects with a vertical wire, but he will not be able to traverse the vertical wire down towards B1, because there is a no return gadget blocking his way down. He also cannot move upwards through the vertical wire, because, after traversing a no return gadget, he will get trapped between the no return gadget and a box (if the crossover hasn’t been traversed vertically). If the crossover has been traversed vertically, the no return gadget will have already been crossed by the player and therefore the player can’t go upwards either. The only option left for him is to keep moving forward through the horizontal wire. After crossing another no return gadget he reaches the last horizontal crossover that he will have to open inside this horizontal wire. This crossover intersects with a vertical corridor. Once he opens the crossover, the player can move up and down through this corridor, but he won’t be able to go into the vertical wire. If he traverses the corridor upwards he will be prevented from entering the vertical wire by a box that is blocking his way up towards the vertical wire. He can move this box out of the vertical wire (In case it hasn’t be done already), but he won’t be able to enter the vertical wire because the no return gadget will prevent him from returning to the previous horizontal crossover. If, instead, he goes down, there will be a box preventing him from entering the vertical wire. This box can only be moved out of the way if the player has entered into the crossover from the B1 entry. Notice that if the box has already been moved out of the way of the player the two way gadget will have been used and therefore the player will not be able to cross this corridor to reach the vertical wire from here. So the player will have to keep
moving forward through the horizontal wire. After crossing the last no return gadget, he reaches A4.

Lemma 8. The crossover allows the player to traverse the vertical wires without letting the player cross the horizontal wires

Proof. See figure 16 to see a picture of the crossover. We will assume in these proofs that the boxes inside the crossover can only be moved in the horizontal direction. The proof is symmetrical for the variant of pull in which the boxes have only vertical handles. The player enters the gadget from the A1 entry. After entering the crossover, the player will be able to enter a corridor to the left and another one to the right of the vertical wire. In order to traverse the crossover, he will have to go into both corridors. Inside the corridor that is at the right of the vertical wire there is a box that is at the end of the corridor. This box has to be moved to the left, this is to unlock the return wire so that the player can return through the other vertical wire. Notice that this box prevents the player from entering inside the other vertical wire. After returning from this corridor, the player can keep going upwards, he will only be able to do this if the corridor to his left has been visited before, because if the corridor has not been visited before, there will be a box keeping the player from moving upwards. This box can only be moved out of the way only if the player enters into this corridor. Assuming that the player has not moved the box out of the vertical wire, he will have to enter the corridor at his left. Inside the corridor there is a two way gadget. He has to traverse one of the two no return gadgets that form the two way gadget. After passing this gadget the player can keep moving forward through the corridor. Notice that, from here, the player can use the other no return gadget to come back to the wire, but then he will have closed the access to the corridor without having unlocked his way upwards through the vertical wire. In his way through this corridor, he will have to move a box to a "gap" that is near the box in order to keep moving forward through the corridor. This box is placed at a corner of the corridor to prevent the player from entering the vertical wire from the horizontal wire, see previous lemma. If he goes upwards through the corridor, his path will intersect with an horizontal wire. If the player has already traversed the horizontal wire, the no return gadgets will prevent him from entering this wire. If the player has not traversed the horizontal wire before, the horizontal crossover will be closed and, again, he won’t be able to enter inside the horizontal wire. So the player cannot enter into the horizontal wire from this corridor. The only thing that the player can do in this corridor, is to move to the left the box at the end of the corridor. This will allow him to cross the ver-
Figure 16: Complete drawing of the crossover. This drawing depicts the intersection between two horizontal wires and two vertical wires. In this crossover, the player can only pull the boxes left and right. The image is rotated 90 degrees, A1 is at the bottom left
tial wire towards B2. This box also prevents him from entering the vertical wire from this part of the corridor. The only thing he can do now is return to the part of the vertical wire he came from. To do that, he traverses the other no return crossover inside the two way gadget. After this, the player cannot enter this corridor from inside the vertical wire anymore and the player cannot enter into the vertical wire from inside the corridor. Now the only option for the player is to keep moving upwards through the vertical wire. After crossing a no return gadget, he will be in the middle of an intersection with the horizontal wire. Either, the no return gadgets or the horizontal crossover, will prevent him from entering this wire, so he will have to keep moving upwards. After traversing another no return gadget, the vertical wire the player is traversing will have an entry to the left (to the corridor that the player traversed before), the player cannot enter the corridor from here because of the box inside the corridor. If he keeps going upwards, he will find another entry to a corridor to the right of the vertical wire. He will have to enter this corridor if he wants to return through the other vertical wire. From inside this corridor, he can move to the left a box that is inside the other vertical wire. He will have to do this in order to be able to return from the other vertical wire without getting trapped. Notice that this box prevents the player from entering the other vertical wire from this corridor. After moving this box, the only thing he can do is return to the vertical wire and keep moving upwards towards B2. After crossing another no return gadget in his way upwards, he will find another corridor to the right of the vertical wire. He has to enter this corridor in order to move a box at the end of the corridor (if the box hasn’t been moved already), he has to do this to be able to keep moving upwards through the vertical wire without getting trapped. Assuming that the player has not moved this box already, he will have to enter inside this corridor. Inside the corridor, he will have to cross a no return gadget to advance into the corridor, this no return gadget forms part of a two way gadget. From there he can go back through the other no return gadget and come back to the vertical wire. But if he does this he won’t be able to unlock the box at the end of the corridor. So he will have to keep moving forward through the corridor. Just before reaching a corner inside the corridor, he will find a box that is just in that corner. He will have to move to the left this box and bring the box to a gap that is just before the box (this box prevents the player from entering to the vertical wire from inside the horizontal wire above the corridor). Finally, he can reach the end of the corridor and move the box that is at the end of the corridor. Notice that this box prevents the player from entering the vertical wire from this corridor. This
vertical corridor intersects with an horizontal wire, but the no return gadgets or the horizontal crossover make sure that the player cannot enter the horizontal wire from this vertical corridor. After unlocking the path towards B2, the only thing he can do is go back to the vertical wire. To do that, he traverses the other no return gadget inside the two way gadget. Once he returns to the vertical wire, this corridor becomes inaccessible to the player because the no return gadgets prevent him to move inside this corridor. So the only option left to the player is to keep moving upwards towards B2. After traversing another no return gadget, his path will intersect with an horizontal wire. He can’t enter to the horizontal wire, because of the horizontal crossover (in case the wire hasn’t been traversed before) or because the no return gadgets have been traversed already (in case the wire has already been crossed). So he can only move downwards. After crossing another no return gadget, the vertical wire he is traversing will have an entrance to a corridor at the right of the wire. A box inside the corridor prevents the player from reaching the other vertical wire from this corridor. So he will have to keep moving downwards. After crossing another no return gadget, the vertical wire he is traversing is crossed by another horizontal wire. He can’t enter to the horizontal wire, because of the horizontal crossover (in case the wire hasn’t been traversed before) or because the no return gadgets have been traversed already (in case the wire has already been crossed). So he can only move downwards. Finally, after crossing the last no return gadget inside the crossover, the vertical wire will have an entry to a corridor at the right of the wire. A box inside the corridor prevents the player from reaching the other vertical wire from this corridor. So he will have to keep moving downwards until he reaches B4.

In his return from the variable gadget, the player will have to return inside the crossover from the other vertical wire. After crossing the first no return gadget, the vertical wire he is traversing is crossed by an horizontal wire. He can’t enter to the horizontal wire, because of the horizontal crossover (in case the wire hasn’t been traversed before) or because the no return gadgets have been traversed already (in case the wire has already been crossed). So he can only move downwards. After crossing another no return gadget, the vertical wire will have an entry to a corridor at the right of the wire. A box inside the corridor prevents the player from reaching the other vertical wire from this corridor. So he will have to keep moving downwards. Finally, after crossing the last no return gadget inside the crossover, the vertical wire will have an entry to a corridor at the right of the wire. A box inside the corridor prevents the player from reaching the other vertical wire from this corridor. So he will have to keep moving downwards until he reaches B4.
With these components, we can prove the hardness of pull for those versions in which the boxes can only be pulled from opposite directions.

**Theorem 3.** The path version of pull in which all boxes can only be pulled from opposite directions is NP-hard.

5 Pull variants where boxes have a handle in a vertical direction and another handle in an horizontal direction are NP-hard.

As we did in the previous sections, we will only describe the changes we must do to the gadgets so that they work for this variant. If a gadget doesn’t change with respect to the previous section, we will not describe it in this section.

### 5.1 No return gadget

![Figure 17: No return gadget for variants in which the player can pull a box in a direction but not the opposite direction.](image)

This is a no return gadget like the one in the previous sections, but adapted for this variant of pull and the next one. It allows passage from A to B a single time and once we traverse the gadget we cannot return from B to A.

### 5.2 Variable gadget

We are going to describe the changes we need to make to the variable gadget so that it works for this variant and the next one.

If the player comes from the path of a positive literal, we don’t have to do any changes to the gadget, since the one way gadget that is just after the literal wires of the negative literal prevent him from accessing those literal wires.
Figure 18: This picture represents the intersection point between an horizontal and a vertical wire inside a variable gadget. The two horizontal wires are a closeup of the two literal wires that the player has to use to go to a clause (up wire) and return from that clause (down wire). In this example figure, the player can only pull the boxes left and down.

Now, if the player comes from the negative literal path, we have to ensure that the player cannot access any of the literal wires that belong to the other path.

In the example shown in figure 18, this is done by placing an horizontal crossover in the wire that goes to the clause.

For the return wire, a single box prevents the player from accessing the wire. This box can be unlocked from a corridor that communicates the two wires. This corridor can only be traversed from the interior of the other wire.

5.3 Clause gadget

This clause gadget is similar the one that appears in section 3, but the no return gadgets have been changed so that they work for this variant of pull and the next one.

Figure 19: Clause gadget, it is similar to the one shown in section 3 but changing the no return gadgets.

5.4 Crossover

This is a crossover that works for the path variants of pull where the boxes have only a handle in an horizontal direction and/or have only a handle in the vertical direction.

Here is an schematic drawing of the crossover:
Figure 20: Schematic drawing of the crossover. This drawing shows, in an schematic view, the crossing of two vertical wires by two horizontal wires.

See figure 21 to see a complete drawing of the crossover.

**Lemma 9.** The crossover allows the player to traverse the horizontal wires without letting the player traverse the vertical wires.

**Proof.** For this proof we will assume that the player can only move the boxes to the left or down. The proof for the other directions follows by rotating or reflecting the gadget. See figure 21 to see an image of the crossover. When the player enters from the A1 entry, he crosses a no return gadget, after crossing this gadget, the player can access a corridor that is below him or he can keep going forward. The player has to go into the corridor that is below him and move to the left a box that is inside that corridor. This box is preventing the player from using the second horizontal wire of the crossover to return from the crossover, he will need to move this box so that he can return through that wire without getting trapped inside the crossover. Notice that this box prevents the player from entering to the other horizontal wire from this corridor. So the player is forced to go upwards again to the horizontal wire, and then, he has to keep going forward through that wire. There, he crosses a second no return gadget. After crossing this gadget, he can’t go back, so he is forced to keep going forward towards the A2 exit. He will find an unidirectional crossover in front of him. He can unlock it and keep going forward forward. He will not be able to cross it vertically if that part of the crossover has not been opened. If the vertical part of the crossover is opened, the no return gadgets that are above and below the crossover will have been used, so that he can’t go up towards B2 or down towards B1. After crossing the gadget, he will be forced to pass through a no return gadget in front of him. From here he will have to keep going forward and he will have to cross an other horizontal crossover. After opening this crossover, he will be able to go forward, or downward if the player has not crossed the vertical wires yet (he cannot go upwards towards B3 because a no return gadget is blocking the way). If he decides to go downwards, he will cross a no return gadget that is in his way down. But then he will get trapped between this crossover and a box below him. This
box can only be moved out of the way if the player entered the gadget from B1 on the first vertical wire. So the only option left to the player is to move forward towards A2, cross another no return gadget, and exit the crossover from A2.

After going to the clause gadget, he will have to take the return wire. He will return to the gadget from A3, from there he will have to cross a no return gadget. After this, he will be able to keep going forward or he will be able to cross a no return gadget that is below him (assuming that the player hasn’t crossed the gadget in the vertical direction yet), but if he does this he will get trapped between this gadget and a box below him. This box can only be moved out of the way if the player entered the crossover from B1 before. The player cannot go upwards because of a no return gadget above him. So he is forced to keep moving forward towards A4. After crossing another no return gadget, he will find a vertical crossover. Because of the properties of this crossover, he will not be able to go up or down through this wire if the crossover is closed. If the crossover is open, the no return gadgets will prevent him to go up or down through the vertical wire. So he will be forced to keep moving forward towards A4. After crossing a no return gadget, he will have to go through a corridor in front of him. The corridor will be unlocked if the player decided to move the box inside the corridor to the left when he traversed the first horizontal wire. This box prevents the player from accessing the first horizontal wire. Finally the player can traverse the last no return gadget before exiting the crossover from the A4 exit.

**Lemma 10.** The crossover allows the player to traverse the vertical wires without letting the player traverse the horizontal wires

*Proof.* For this proof we will assume that the player can only move the boxes to the left or down. The proof for the other directions follow by rotating or reflecting the gadget. See figure 21 to see a picture of the crossover. The player will access to the crossover from B1, there, he will find a no return gadget above him and a corridor to the right. The player can enter this corridor and move a box at the end of the corridor to the left, he has to do this in order to return through the other vertical wire. Notice that this box prevents the player from entering into the other vertical wire from inside this corridor. After returning to the vertical wire from the corridor, he has to keep moving upwards towards B2. There, he crosses a no return gadget above him. After crossing this gadget, he will have to open a vertical crossover in top of him. The crossover intersects with an horizontal wire. After opening this crossover. In the crossover he will be able to move left towards A4 (this is only possible if the player has not
Figure 21: Complete drawing of an example crossover. This picture represents the crossing of two horizontal wires with two vertical wires. In this example, the player can only pull the boxes left and down. Entrance "A1" is in the top left corner of the drawing.
crossed the horizontal wire before) or keep going upwards towards B2, he can’t move right towards A3 because there is a no return gadget blocking his way. If he decides to move to the left inside the horizontal wire, after crossing a no return gadget, he will be trapped between this gadget and a box that is blocking his way. This box can only be moved out of the way only if the player entered into the crossover from A1. So he will be forced to keep going upwards towards B2. After crossing another no return gadget, the player will be able to move into a corridor to his right. In this corridor, he can pull left a box that is at the end of this corridor, he has to move this box in order to be able to return through the second vertical wire. After returning to the vertical wire again, he crosses another no return gadget. After that he will reach a limited unidirectional crossover on top of him, after unlocking the crossover, he will be able to go upwards to B2. If the crossover is closed in the horizontal direction, he will not be able to go left to A1 or right to A2. If it is unlocked, the no return gadgets will prevent him from going to the left towards A1 or going to the right towards A2. So he has to keep moving upwards towards B2. He crosses the last no return gadget before reaching B2.

After going back from the clause gadget, he will have to use the return wire to come back. First, after entering through B3 he will cross a no return gadget. After traversing this gadget, his path will intersect with an horizontal crossover. If the crossover hasn’t been traversed before, then he will not be able to move inside the horizontal wire. If the crossover has been traversed horizontally, the no return gadgets at the left and at the right of the crossover will prevent him from entering into the horizontal wire. So he will have to keep going downwards through the vertical wire towards B4. Then, he traverses another no return gadget. After traversing it, if the player remembered to move the box inside corridor to the left when he was in the other wire, he will be able to keep moving downwards towards B4. There, his path will intersect with the second horizontal wire. He can’t go right to A3 because a no return gadget is blocking the way. If the player has already used this horizontal wire, a no return gadget that has already been traversed will prevent him from traversing the horizontal wire towards A1. If the player didn’t visit this horizontal wire before, he will be able to go to the left towards A1, but he will eventually get trapped. Let’s assume that the player didn’t visited the second horizontal wire yet and that the player decides to go left towards A1: First, the player traverses a no return gadget, after that, he finds himself in the middle of a vertical crossover. The crossover will be opened, because the player has traversed it before, but
the no return gadgets have already been traversed, preventing the player from going upwards towards B2 or downwards towards B1. So the player will have to keep going left towards A1. Then, after crossing another no return gadget, the player will be trapped between this no return gadget and a box that is blocking his way. This box can only be cleared out of the way if the player traversed the crossover first in the horizontal direction, so he will be trapped. So the only option that the player has, is to keep moving downwards towards B4. In his way down, he crosses a last no return gadget inside the crossover. There the player will see be able to access a corridor that it’s at the right of the vertical wire. This corridor connects the two vertical wires, but the player cannot enter inside the other vertical wire from this corridor because there is a box inside the corridor that prevents him from doing so. The only option left to te player is to keep moving downwards and exit the crossover from the B4 exit.

Theorem 4. The path version of pull in which all boxes have only a handle for the horizontal direction and another one for the vertical direction are NP-hard

6 Pull with one handle is NP-hard

In this section, we are going to prove the main claim of our paper. That is we are going to prove that the version of pull where all boxes can only be pulled from one direction (in our examples, the left direction) are NP-hard.

As we did in the previous sections, we are only going to describe the changes that must be done to the gadgets so that they work in this section

6.1 Crossover gadget

This is a crossover that is valid for all variants of pull where the player can only pull the boxes from a single direction. In our examples, the left direction. The crossover can be transformed into a crossover where we can pull the box in another direction by rotating and reflecting the gadget.

Lemma 11. The crossover allows the player to traverse the horizontal wires without letting the player traverse the vertical wires

Proof. See figure 22 to see a picture of the crossover. If the player traverses the crossover from the horizontal direction, he has to enter inside the crossover through the A1 entry. After traversing the first no return gadget he can enter inside a corridor that is below him, inside this corridor he can pull a box to the left, he needs to do this in
order to be able to exit the crossover from the other horizontal wire. After that, he can only keep moving right towards A2, there he traverses another no return gadget. After that, he will be in front of an horizontal crossover. This crossover intersects a vertical wire. After opening the crossover, the player won’t be able to traverse the vertical wire upwards towards B2 or down towards B1. If the player has traversed the vertical wire before, no return gadgets will prevent him from moving upwards. If the player has not traversed the vertical wire, he will traverse a no return gadget and he will be trapped between this gadget and a box inside the vertical wire. This box cannot be moved out of the way from inside the vertical wire, so he will be trapped. So he can’t move upwards towards B2 and he can’t also move down towards B1, because a no return gadget below him prevents him from doing so. So the only option left to the player is to keep moving forwards through the horizontal wire. After crossing another no return gadget, he will have to open another horizontal crossover. This horizontal crossover intersects with a vertical corridor. He can’t enter inside the vertical wire from inside this corridor. If he moves upwards through this corridor, he will be able to move out of the vertical wire that goes towards B2 one box that is at the end of the corridor. This will open the way towards B2 from the vertical wire (assuming it wasn’t opened before). This box prevents the player from entering the vertical wire from this part of the corridor. He can’t also enter inside the vertical wire from the bottom part of the corridor. A box just at the end of the corner (assuming that the player has not traversed the crossover in the vertical direction) or a two way gadget that has already been used (assuming that the player has traversed the crossover in the vertical direction), will prevent him from reaching the vertical wire from inside this corridor. Also, he cannot go back to the previous horizontal crossover, the one that intersected with the vertical wire 2, because there is a no return gadget that prevents him from doing so. So he will have to keep forward inside the horizontal wire towards A2. After traversing another no return gadget, he will have to open a last crossover in his way to A2. This crossover intersects with another vertical wire. He cannot go up towards B3 through the vertical wire, because a no return gadget is blocking his way up. If the player has not traversed the crossover vertically before, he can traverse a no return gadget below him, but if he does this, he will get trapped between the no return gadget and a box below him. This box can only be moved out of the way if the player crosses the crossover vertically. So the only thing the player can do is to move forward towards A2. He reaches A2 after traversing one last no return gad-
He will enter into the crossover again from A3. After traversing a no return gadget, his path will intersect with a vertical wire. He cannot go up through the vertical wire towards B3 because there is a no return gadget in his way upwards. If the player has not traversed the crossover vertically before, he can traverse a no return gadget below him in order to reach B4, but if he does this, he will get trapped between the gadget and a box below him. This box can only be moved out of the wire if the player crosses the crossover vertically. So the player will be forced to move forward towards A4. After crossing another no return gadget, the horizontal wire will intersect with the other vertical wire. If the player has traversed the vertical wire before, the no return gadgets will all have been used and the player won’t be able to move up and down through this wire. If the player has not traversed this wire before, he won’t still be able to go down towards B1 or upwards towards B2. The player cannot move downwards towards B1, because there is a no return gadget that will prevent him from moving downwards. If the player moves upwards towards B2 instead, after crossing a no return gadget, he will get trapped between this gadget and a box above him. This box can only be pulled out of the way if the player traverses the crossover vertically. So he will have to keep moving forward through this horizontal wire. After crossing another no return gadget, his path will intersect with a corridor. He can’t use this corridor to enter into the vertical wire. If he tries to move upwards through the corridor, a box inside the corridor will prevent him from moving inside the vertical wire. He can move this box outside the vertical wire and clear the path upwards towards B2, but the box will prevent him from entering the vertical wire from this corridor. If he decides to move down through the corridor, he also won’t be able to enter the vertical wire from below. The two way gadget inside the corridor will prevent him from accessing the vertical wire in case the corridor has already been visited. If the corridor has not been visited before, the player will be stopped by a box that he cannot pull out of the way from this part of the corridor. So the player cannot reach the vertical wire from this corridor, the only option left to the player is to keep moving forward towards A4 through the horizontal wire. After crossing another no return gadget, he will move upwards through a corridor, this corridor is connected to the horizontal wire above it, but the player can’t reach the other horizontal wire from this corridor because there is a box inside the corridor that prevents him from moving upwards to the other horizontal wire. So the only option left to the player is to keep moving through this corridor. After reaching the horizontal wire again,
Lemma 12. The crossover allows the player to traverse the horizontal wires without letting the player traverse the vertical wires

Proof. See figure 22 to see a picture of the crossover. When the player enters inside the crossover, the vertical wire he is traversing will have an entrance to a corridor at the right. Inside the corridor, he will have to pull to the left a box that is inside a corridor that connects the two vertical wires. He must pull this box to be able to go to B4 from the other vertical wire without getting trapped inside the crossover. If he doesn’t pull this box, he will be trapped when he tries to return from the other vertical wire. Notice that this box makes sure that the player can’t use this corridor to travel from one vertical wire to the other. This wire has another entrance to a corridor to the left of the wire. At the end of this corridor, there is a box inside the vertical wire. This box has to be moved out of the wire if the player wants to move upwards through the vertical wire and this box can only be pulled out from inside this corridor. Assuming that the box has not been moved before, the player will have to enter inside this corridor. Inside the corridor, he will traverse the first no return gadget, this no return gadget forms part of a two way gadget. Immediately after, the player can traverse the other no return gadget. But this will left the player outside the corridor without having unlocked his way upwards and because of the properties of the two way gadget, he won’t be able to enter into the corridor anymore. If the player keeps going forward into the corridor, he will find a box that is blocking his way forward inside the corridor. He will have to move this box to a gap behind that box and then he will be able to move forward. This obstacle is inside the corridor to prevent the player from entering this part of the corridor from the horizontal wire. After clearing the obstacle, he can keep moving upwards through the corridor. There, the corridor will intersect with an horizontal wire. He cannot move left through this wire. Assuming that the player has not crossed the horizontal wire before, after traversing one no return gadget, he will get trapped between this no return gadget and inside a corridor where an obstacle will prevent him from moving forward. If the horizontal wire has been traversed, he won’t be able to move to the left inside the horizontal wire because the no return gadgets will have already been traversed. Also, he can’t move right inside the horizontal wire because one no return gadget prevents him from moving right. The only thing the player can do inside the corridor is move the obstacle at the end of the corridor out of the vertical wire and then re-
Figure 22: Complete drawing of the crossover. In this crossover, the player can only pull the boxes to the left (the image is rotated 90 degrees, A1 is at the bottom left)
turn back to the vertical wire. When he goes back, he traverses the other no return gadget that forms part of the two way gadget that is placed at the beginning of the corridor. This closes the access to the corridor from the vertical wire and vice versa. After doing this, the player keeps traversing the vertical wire upwards towards B2. After traversing the first no return gadget in his way upwards, his path will intersect with an horizontal wire once again. From there, the player can traverse a no return gadget that is at his left (if the player has not traversed the crossover horizontally before), if he decides to traverse this gadget, he finds a corridor where a box is blocking his way and he is trapped between this box and the no return gadget he has just traversed. This is because this box can only be moved out of the way only if the player enters the crossover from A1. He can't move right towards A3 because there is a no return gadget blocking his path. If the player decides to keep moving upwards, he traverses a second no return gadget. From there, he has access to a second corridor. Inside the corridor there is another box. He must pull this box out of the wire in order to be able to return to the crossover from B3 without getting trapped inside the crossover. After doing this, he can keep moving upwards towards B2. After traversing another no return gadget, the player will be able to enter a corridor that is at the right of the vertical wire. At the end of this corridor, there is a box inside the vertical wire. He has to pull this box out of the wire in order to be able to keep moving upwards towards B2 without getting trapped. So the player will have to enter the corridor if the box has not been moved out of the wire yet. Inside the corridor, at the beginning, there is a two way gadget. He has to traverse one of the no return gadgets in order to enter the corridor. After traversing one of the no return gadgets, the player can use the other one to return, but then he will get out of the corridor without having moved the box at the end of the corridor out of the vertical wire. And he won't be able to enter inside the corridor anymore. So he will have to keep moving forward inside the corridor. Then there is a box in a corner inside the corridor. This box is preventing the player from going forward inside the corridor. He can move this box to a gap that is behind the box and then move forward through the corridor. The purpose of this box is to prevent the player from using the corridor to enter inside the vertical wire from inside the horizontal wire that crosses the corridor. After clearing the obstacle, the player can keep moving forward inside the corridor. There, his path will intersect with an horizontal wire. If the horizontal wire has not been traversed before, the horizontal crossover will prevent him from entering the wire. If the wire has been traversed before,
the no return gadgets at the left and at the right of the opened crossover will prevent him from entering inside the horizontal wire. So the only thing the player can do inside this corridor is move the box at the end of the corridor outside the vertical wire and return to the vertical wire. In his return, the player will have to traverse the other no return gadget that forms part of the two way gadget, this prevents the player from entering again inside the corridor from the vertical wire and vice versa. So the only option left for the player is to keep moving upwards through the vertical wire. After traversing another no return gadget, the vertical wire will intersect once more with an horizontal wire. If the horizontal wire has not been traversed before, the horizontal crossover will prevent him from entering the wire. If the wire has been traversed before, the no return gadgets at the left and at the right of the opened crossover will prevent him from entering inside the horizontal wire. So the only option left for the player is to keep moving upwards towards B2.

The player will return inside the crossover once more from B3 he will cross the first no return gadget in his way down to B4. There, his path will intersect with an horizontal wire. If the horizontal wire has not been traversed before, the horizontal crossover will prevent him from entering the wire. If the wire has been traversed before, the no return gadgets at the left and at the right of the opened crossover will prevent him from entering inside the horizontal wire. So the only option left for him is to keep moving downwards towards B4. There, the wire he is traversing will have an entry to a corridor that is at the left of the wire. He can’t use the corridor to enter inside the other vertical wire because the box inside the corridor prevents him from entering inside the other vertical wire. So he will have to keep moving downwards towards B4. After crossing another no return gadget the vertical wire the player is traversing will intersect with another horizontal wire again. If the player has traversed the horizontal wire before, the no return gadgets inside the wire will be used and therefore he won’t be able to enter inside the horizontal wire. If the horizontal wire has not been used before, a no return gadget at inside the wire will prevent him from entering to A3. He will still be able to move left towards B3, but if he does, he will get trapped. If he decides to move left inside the horizontal gadget, he will have to traverse a no return gadget. There, the horizontal wire will intersect with the other vertical wire. The no return gadgets will all have been used and the player won’t be able to move up and down through this wire. So he will have to keep moving forward through this horizontal wire. After crossing another no return gadget, his path...
will intersect with a corridor that he used before. He can’t use this corridor to enter into the vertical wire. If he tries to move upwards through the corridor, a box inside the corridor will prevent him from moving into the vertical wire. If he decides to move down through the corridor, the two way gadget inside the corridor will prevent him from accessing the vertical wire from here. So the player cannot reach the vertical wire from this corridor, the only option left to the player is to keep moving forward through the horizontal wire. After crossing another no return gadget, the player will get trapped between the no return gadget and a box that prevents him from moving forward. He has trapped himself. So the player can’t traverse the horizontal wire, the only thing the player can do is to keep moving downwards to B4. After crossing a last no return gadget inside the crossover, the wire will have a last entry to a corridor at the right of the wire. He can’t use the corridor to enter into the other vertical wire because there is a box inside the corridor that prevents him from doing so. So he will have to exit the crossover from the B4 exit.

**Theorem 5.** *The path version of pull in which all boxes can only be pulled from a single direction is NP-hard*

### 7 Open questions and future directions

Can the results proven here for the path versions of pull be proven also for the storage versions of pull? That is, will it be possible to show that the storage versions of pull where the player can only pull boxes in opposite directions are NP-hard? Can it be proven that the storage variants of pull where all boxes can only be pulled in only one direction are NP-hard?

We think that the versions where the player can only pull boxes in opposite directions are NP-hard as well. If there exists a variant of planar SAT where the negative clauses are outside the cycle of variables and the positive clauses are inside the cycle of variables, and this variant of SAT is NP-hard, then we can reduce that variant of SAT to the version of storage pull where the player can only pull boxes in opposite directions.

We think that the storage variant of pull where all boxes can only be pulled in a single direction is tractable. Which variants of Push are NP-hard when the player can only push boxes in certain directions?

Is the variant of pull where the player cannot revisit a square (called pull-X) still hard if the player can only pull boxes in certain directions? If the problem is tractable, this will solve the open question of [3] about finding an interesting but tractable
block-moving puzzle.

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