SCALAR CURVATURE AND PROJECTIVE COMPACTNESS

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Abstract. Consider a manifold with boundary, and such that the interior is equipped with a pseudo-Riemannian metric. We prove that, under mild asymptotic non-vanishing conditions on the scalar curvature, if the Levi-Civita connection of the interior does not extend to the boundary (because for example the interior is complete) whereas its projective structure does, then the metric is projectively compact of order 2. This implies a host of results including that the metric satisfies asymptotic Einstein conditions, and induces a canonical conformal structure on the boundary.

1. Introduction

Throughout this article we consider a smooth manifold $\overline{M}$ of dimension $n + 1$ with boundary $\partial M$ and interior $M$, and the basic topic is that of relating geometric structures on $M$ to geometric structures (in general of a different type) on $\partial M$. Apart from their intrinsic interest in differential geometry, questions of this type play an important role in several other areas of mathematics (e.g. scattering theory) and theoretical physics (e.g. general relativity and the AdS/CFT–correspondence), see the introduction of [2] for a more detailed discussion.

In particular we are interested in a problem of the following nature. Suppose we start with a geometric structure on $M$, which on its own does not admit a smooth extension to $\overline{M}$, in such a way that the boundary $\partial M$ is “at infinity” in a suitable sense. Then one may ask whether some weakening of the structure in question does admit such an extension, and whether this extension induces a structure on $\partial M$ linked to the interior geometry. A classical example of this situation, with many applications, involves a notion of conformal extension. In this case, one starts with a pseudo–Riemannian metric $g$ on $M$ that does not admit a smooth extension to $\overline{M}$, for example because it is complete. Then one may first ask whether the conformal structure $[g]$ on $M$, determined by $g$, admits a smooth extension to all of $\overline{M}$. Explicitly, this means that, for each boundary point $x \in \partial M$, there is an open neighborhood $U \subset \overline{M}$ of $x$ and a smooth nowhere vanishing function $f : U \cap M \to \mathbb{R}_{>0}$ such that the pseudo–Riemannian metric $fg$ on $U \cap M$ admits a smooth extension to all of $U$ for which the values on $U \cap \partial M$ are non-degenerate as bilinear forms on tangent spaces.

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In most applications, this idea is refined to the more restrictive, but also more useful, concept of conformal compactness. Rather than assuming an arbitrary smooth rescaling of $g$ extends to the boundary, one requires that $r^2g$ admits a smooth extension to the boundary, where $r : U \to \mathbb{R}_{>0}$ is a local defining function for the boundary, see section 2.1 for the formal definition. This enforces a certain uniformity in the growth rate of the metric $g$ as it approaches the boundary. The property that $r^2g$ admits a smooth extension to the boundary is independent of the specific defining function $r$, and so is the conformal class of the induced pseudo–Riemannian metric on $\partial M$. This conformal class is then the induced structure on the boundary, and the boundary so equipped is then referred to as the conformal infinity of the interior.

Alternative to the underlying conformal structure of a pseudo–Riemannian metric $g$, one may also consider the underlying projective structure of its Levi–Civita connection $\nabla^g$ as an interesting weakening. The resulting applications of projective differential geometry to pseudo–Riemannian geometry have been intensively and successfully studied during the last years. Because of the resulting emphasis on geodesic paths, this approach should be particularly useful for applications in general relativity and scattering. Indeed, as brought to our attention by P. Nurowski, there have been attempts to associate a future time–like projective infinity to space–times, see [5].

In the setting of a manifold with boundary $\overline{M} = M \cup \partial M$ as above, one can start from a torsion free linear connection on $TM$, which does not extend to $M$ and assume that its underlying projective structure extends to $\overline{M}$. Via the Levi–Civita connection, this concept is then automatically defined for pseudo–Riemannian metrics. Our first result in this article is an explicit characterization of extendability of the projective structure in Proposition 2.

Similarly to conformal compactness, as discussed above, a more restrictive concept of projective compactness was introduced in our article [2] and studied further in [3]. This involves an additional parameter $\alpha > 0$, called the order of projective compactness, and one usually assumes that $\alpha \leq 2$. This ensures that $\partial M$ is at infinity according to the parameters of geodesics approaching the boundary, see Proposition 2.4 in [2].

There are two results in [2] which motivate the developments in this article. Assume that $\nabla$ is a linear connection on $TM$, which does not admit a smooth extension to any neighborhood of a boundary point, but whose projective structure does admit a smooth extension to all of $\overline{M}$. Then in Theorem 3.3 of [2] it is shown that if $\nabla$ preserves a volume density and is Ricci flat, then it is projectively compact of order $\alpha = 1$. On the other hand, if $\nabla$ is the Levi–Civita connection of a non–Ricci–flat Einstein metric, then Theorem 3.5 of [2] shows that $\nabla$ is projectively compact of order $\alpha = 2$. In both cases, one actually obtains reductions of projective holonomy, which lead to much more specific information.

The main result of this article is Theorem 5 which provides a vast generalization of the second of these results. Here being Einstein is replaced by a much weaker condition on the asymptotics of the scalar curvature of $g$, but we still can conclude
projective compactness of order $\alpha = 2$. Via the results of [3], this provides a number of further facts about $g$, including a certain asymptotic form, an asymptotic version of the Einstein property, and the fact that $\partial M$ inherits a canonical conformal structure determined by $g$. Several of the arguments used in proving this result should be of considerable independent interest.

2. Results

2.1. Projective structures and projective compactness. Two torsion free linear connections $\nabla$ and $\hat{\nabla}$ on the tangent bundle of a smooth manifold $N$ are called projectively equivalent if they have the same geodesics up to parametrisation or, equivalently, if there is a one–form $\Upsilon \in \Omega^1(N)$ such that

\[ \hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi) \eta + \Upsilon(\eta) \xi, \]

for vector fields $\xi, \eta \in \mathcal{X}(N)$. A projective structure on $N$ is a projective equivalence class of such connections.

Assume that $\overline{M}$ is a smooth manifold with boundary $\partial M$ and interior $M$. Given a torsion free linear connection $\nabla$ on $TM$, we can define what it means for the projective structure determined by $\nabla$ to admit a smooth extension to all of $\overline{M}$. Explicitly, this is property that, for any boundary point $x \in \partial M$, there is an open neighborhood $U$ of $x$ in $\overline{M}$ and a one–form $\Upsilon \in \Omega^1(U \cap M)$ such that for all vector fields $\xi, \eta \in \mathcal{X}(U)$ (so these are smooth up to the boundary), also $\hat{\nabla}_\xi \eta$ as defined in (1) admits a smooth extension to the boundary. It is then clear that the resulting connection $\hat{\nabla}$ on $TU$ is uniquely determined up to projective equivalence, so in this way one indeed obtains an extension of the projective structure to $\overline{M}$.

More restrictively, for a constant $\alpha > 0$ we say that $\nabla$ is projectively compact of order $\alpha$, if in the above considerations the one–form $\Upsilon$ can be taken to be $dr^\alpha$ for a smooth defining function $r : U \to \mathbb{R}_{\geq 0}$ for the boundary. By definition, the latter condition means that $r^{-1}(\{0\}) = U \cap \partial M$ and $dr$ is nowhere vanishing on $U \cap \partial M$.

If $\nabla$ preserves a volume density, then it has been established in [2] that, in addition to the fact that the projective structure defined by $\nabla$ extends to $\overline{M}$, projective compactness of order $\alpha$ of $\nabla$ only requires a specific growth rate (related to $\alpha$) of the parallel volume form towards the boundary. This can be most conveniently formulated in terms of defining densities as follows. Observe first, that the notion of a defining function (for a hypersurface) can be extended to the notion of a defining section of any real line bundle without problems. The point here is that for a section of a line bundle, the derivative with respect to a linear connection is, along the zero–set of the section, independent of the connection. Hence one can simply require that one has a section for which the zero–set coincides with the hypersurface in question and that the derivative of the section with respect to some linear connection is nowhere vanishing along the zero–set.

Now on any smooth manifold $N$, there is a family of natural line bundles, obtained from the (trivial) bundle of volume densities by forming real powers. In the presence of a projective structure, there is an established convention of projective weight for these line bundles, which are then denoted by $\mathcal{E}(w)$ with
Lemma 1. Let $M$ be a smooth manifold of dimension $n + 1$ with boundary $\partial M$ and interior $M$. Let $\nabla$ be a linear connection on $TM$ whose projective structure admits a smooth extension to $\overline{M}$ and which preserves a volume density on $M$.

Then $\nabla$ is projectively compact of order $\alpha$ if and only if there is a defining density $\sigma \in \Gamma(E(\alpha))$ for $\partial M$ such that $\sigma$ is parallel for $\nabla$ on $M$.

2.2. Extension of the projective structure. Before we move to the main subject of the article, we prove a result showing that the condition that the projective class of a linear connection $\nabla$ on $TM$ admits a smooth extension to $\overline{M}$ can be easily verified explicitly. To the best of our knowledge this result is not in the literature, although an essentially equivalent observation is implicit in the approach of [5].

Given a linear connection $\nabla$ on $TM$ and a chart $U$ with local coordinates $x^0, \ldots, x^n$ for $\overline{M}$, we obtain the connection coefficients (or Christoffel symbols) $\Gamma^k_{ij} \in C^\infty(U \cap M, \mathbb{R})$ for $\nabla$. Denoting by $\partial_i := \frac{\partial}{\partial x^i}$ the coordinate vector fields determined by the chart, the connection coefficients are characterized by $\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$, so they are symmetric in $i$ and $j$. Fixing the local coordinates, the connections coefficients may be viewed as giving the contorsion tensor that distinguishes $\nabla$ from the flat connection determined by the coordinate frame. We can thus form the trace of the connection coefficients $\gamma_i := \sum_k \Gamma^k_{ik}$, as well as form the tracefree part $\Psi^k_{ij} := \Gamma^k_{ij} - \frac{1}{n+2}(\gamma_i \delta^k_j + \gamma_j \delta^k_i)$. Both $\gamma_i$ and $\Psi^k_{ij}$ are smooth, real valued functions on $U \cap M$ for all $i, j, k = 0, \ldots, n$.

Proposition 2. Let $\overline{M}$ be a smooth manifold of dimension $n + 1$, with boundary $\partial M$ and interior $M$, and let $\nabla$ be a linear connection on $TM$.

Then the projective class determined by $\nabla$ admits a smooth extension to $\overline{M}$ if and only if for any point $x \in \partial M$, there is a local chart $U$ for $\overline{M}$, with $x \in U$, such that the components of the tracefree part of the connection coefficients of $\nabla$, with respect to the local coordinates determined by $U$, admit a smooth extension from $U \cap M$ to all of $U$.

Proof. The fact that a linear connection $\hat{\nabla}$ on $TM$ admits a smooth extension to all of $\overline{M}$ is clearly equivalent to the fact that its connection coefficients in any local chart admit a smooth extension to the whole domain of the chart. This in turn is equivalent to the same fact in at least one local chart around each boundary point. Now suppose that $\nabla$ and $\hat{\nabla}$ are projectively equivalent as in [1] and $\Upsilon$ is the corresponding one–form. Then the connection coefficients in a chart $U$ are clearly related by $\hat{\Gamma}^k_{ij} = \Gamma^k_{ij} + \Upsilon_i \delta^k_j + \Upsilon_j \delta^k_i$,

where $\Upsilon = \sum \Upsilon_i dx^i$. In particular, the tracefree parts of $\hat{\Gamma}^k_{ij}$ and $\Gamma^k_{ij}$ agree. So if the projective class of $\nabla$ admits a smooth extension to $\overline{M}$, the tracefree parts of its connection coefficients, with respect to any local chart for $\overline{M}$ that includes a boundary point, admit a smooth extension to the boundary.
Conversely, assume that $x \in \partial M$ is a boundary point, and $U$ is a local chart for $M$ that contains $x$, and on which the tracefree parts of the connection coefficients for $\nabla$ admit a smooth extension to the boundary. Let us denote by $\Gamma_{jk}^i$ these connection coefficients, by $\gamma_i$ the their trace, and by $\Psi_{ij}^k$ their tracefree part, as defined above. As mentioned above, we may interpret the connection coefficients $\Gamma_{jk}^i$ as the coordinate components of the contorsion tensor needed to modify the flat connection determined by the chart to the connection $\nabla$. Hence we can interpret the components $\Psi_{ij}^k$ of the tracefree part in exactly the same way, i.e. define a connection $\tilde{\nabla}$ by $\tilde{\nabla}_a \partial_j := \sum_k \Psi_{ij}^k \partial_k$ and by assumption, this admits a smooth extension to all of $U$. But the fact that $\Psi_{ij}^k := \Gamma_{ij}^k - \frac{1}{n+2} (\gamma_i \delta_j^k + \gamma_j \delta_i^k)$ on $U \cap M$ extends to any neighborhood of a boundary point. This is for example implied by completeness of $\nabla$.

Hence we can interpret the restriction of $\tilde{\nabla}$ to $U \cap M$ is projectively equivalent to $\nabla$. 

\[ \nabla \partial_i \partial_j := \sum_k \Psi_{ij}^k \partial_k \]

by completeness of $\nabla$. Our standing assumption will be that $\nabla$ does not extend to any neighborhood of a boundary point. This is for example implied by completeness of $g$, since this is defined as geodesic completeness of $\nabla$. Starting from here, we will not use local coordinates any more and indices showing up will be abstract indices as introduced by R. Penrose. In particular, we will denote the metric $g$ by $g_{ij}$ and its inverse by $g^{ij}$. The Riemann curvature tensor of $g$ will be denoted by $R_{ij}^{\ k\ l}$, the Ricci-curvature of $g$ is $Ric_{ij} = R_{kij}^{\ k}$ and its scalar curvature is $S := R_{ij}^{\ ij}$. We will also use the projective Schouten tensor $P_{ij}$ which in this simple setting satisfies $P_{ij} = \frac{1}{n} Ric_{ij}$ and its trace $P = g^{ij} P_{ij} = \frac{1}{n} S$.

In general, an affine connection is not projectively equivalent to the Levi–Civita connection of any pseudo–Riemannian metric. The fact that a projective class contains a Levi–Civita connection is equivalent to the existence of a non–degenerate solution of a certain projectively equivalent differential equation [8, 9], which is sometimes referred to as the metricity equation. The details of the equation are not important to us, but a discussion in the notation used here may be found in [4, 6]. Let us briefly discuss some implications of the existence of the solution to the metricity equation determined by $g$, as well as its interpretation in terms of tractor bundles.

A projective structure on an $(n+1)$-manifold canonically determines an invariant linear connection $\nabla^T$. This tractor connection is not defined on the tangent bundle but rather on a rank $(n+2)$-vector bundle known as the standard tractor bundle $T$, see [1]. We write $T^\ast$ for its dual, and $S^2 T$ and $S^2 T^\ast$ for the symmetric squares of these bundles. Each of these bundles comes with a composition series in terms of weighted tensor bundles, which we write as

\[ T = \mathcal{E}(-1) \oplus \mathcal{E}^\ast(-1) \]

\[ S^2 T = \mathcal{E}(-2) \oplus \mathcal{E}^i(-2) \oplus \mathcal{E}^{ij}(-2) \]

\[ T^\ast = \mathcal{E}_i(1) \oplus \mathcal{E}(1) \]

\[ S^2 T^\ast = \mathcal{E}_{ij}(2) \oplus \mathcal{E}_i(2) \oplus \mathcal{E}(2) \]

Here we use the usual conventions of abstract index notation for tensor bundles, as well as the convention that adding “(w)” to the name of a bundle indicates a tensor product with $\mathcal{E}(w)$. The composition series for $S^2 T$, for example, means that there are smooth subbundles $\mathcal{F}_1 \subset \mathcal{F}_2 \subset S^2 T$ such that $\mathcal{F}_1 \cong \mathcal{E}(\cdot - 2)$, $\mathcal{F}_2/\mathcal{F}_1 \cong \mathcal{E}^i(\cdot - 2)$.
and \( S^2\mathcal{T}/\mathcal{F}_2 \cong \mathcal{E}^{(ij)}(-2) \). Choosing a connection in the projective class, one obtains an isomorphism between each of the tractor bundles and the direct sum of its composition factors, and there are explicit formulae in [1] and [2] for the relation between the splittings corresponding to different connections in the projective class.

The tractor connection on \( \mathcal{T} \) induces induces linear connections on the other tractor bundles, for which we use analogous notation. Again, explicit formulae for the tractor connections are available in the references cited above.

Now given the metric \( g \) on \( M \), we denote by \( \text{vol}(g) \) its volume density and we write \( \tau := \text{vol}(g)^{-\frac{n}{n+2}} \), which is a section of \( \mathcal{E}(2) \) defined over \( M \) and nowhere vanishing there. In particular, \( \tau^{-1}g^{ij} \in \Gamma(\mathcal{E}^{(ij)}(-2)) \) is well defined over \( M \), and this is the solution to the metricity equation determined by \( g \). The crucial fact for our purposes is that there is a corresponding section \( L(\tau^{-1}g^{ij}) \) of \( S^2\mathcal{T} \), which projects onto \( \tau^{-1}g^{ij} \) under the canonical projection, see [6] or e.g. [4, Proposition 3.1]. The fact that \( \tau^{-1}g^{ij} \) satisfies the metricity equation can be equivalently characterized as \( L(\tau^{-1}g^{ij}) \) being parallel for a natural modification \( \nabla^p \) of the tractor connection \( \nabla^{S^2\mathcal{T}} \) which is discussed in more detail below.

**Proposition 3.** Let \( g = g_{ij} \) be a pseudo–Riemannian metric on \( M \) such that the projective structure determined by the Levi–Civita connection \( \nabla \) of \( g \) admits a smooth extension to all of \( \overline{M} \).

Then the sections \( \tau^{-1}g^{ij} \in \Gamma(\mathcal{E}^{(ij)}(-2)) \) and \( L(\tau^{-1}g^{ij}) \in \Gamma(S^2\mathcal{T}) \) and the scalar curvature \( S \) of \( g \) admit smooth extensions to all of \( \overline{M} \).

**Proof.** Since we have a well-defined projective structure on \( \overline{M} \), all tractor bundles and tractor connections, as well as the natural modification \( \nabla \) of the tractor connection on \( S^2\mathcal{T} M \), are also well-defined on all of \( \overline{M} \). By assumption, \( L(\tau^{-1}g^{ij}) \) is a section of \( S^2\mathcal{T} \) defined on the dense open subset \( M \subset \overline{M} \) and parallel for the connection \( \nabla \) there. Hence it can be extended by parallel transport to a smooth parallel section over all of \( \overline{M} \). The projection of this extension to the quotient bundle \( \mathcal{E}^{(ij)}(-2) \) then provides the claimed extension of \( \tau^{-1}g^{ij} \).

Finally, we can view \( L(\tau^{-1}g^{ij}) \) as a bundle metric on \( \mathcal{T} \). Similarly as discussed in the proof of Theorem 12 of [3], the fact that the square of the top exterior power of the tractor bundle is canonically trivial implies that there is a well defined determinant for this bilinear form, which is a smooth function on \( \overline{M} \). It is also computed in that proof that, over \( M \), \( \det(L(\tau^{-1}g^{ij})) = \frac{1}{n!} g^{ij} \mathcal{P}_{ij} \). Hence up to a constant factor, this coincides with \( S \), thus providing the claimed smooth extension. \( \square \)

2.4. **The non–degenerate case.** Let \( g \) be a pseudo–Riemannian metric on \( M \) such that the projective structure defined by the Levi–Civita connection \( \nabla \) of \( g \) admits a smooth extension to \( \overline{M} \). Then from Proposition 3 we know that the scalar curvature \( S \) of \( g \) admits a smooth extension to \( \overline{M} \). In what follows, we will primarily be interested in the case that the resulting boundary value is nowhere vanishing. This has to happen if, for example, \( S \) is bounded away from zero on \( M \). The proof of Proposition 3 also gives a conceptual explanation for the
relevance of this condition. We have seen that $S$ arises as the determinant of the section $L(\tau^{-1}g^{ij}) \in \Gamma(S^2T^*M)$ associated to the solution $\tau^{-1}g^{ij}$ of the metricity equation determined by $g$. Hence our condition exactly means that $L(\tau^{-1}g^{ij})$ is non-degenerate as a bilinear form on the standard cotractor bundle along $\partial M$ and hence locally around $\partial M$.

If this condition is satisfied, we can thus consider the inverse $\Phi$ of $L(\tau^{-1}g^{ij})$, which is a smooth section of $S^2T^*$. In the proof of Proposition 3 we have also seen that there is a natural connection on the vector bundle $S^2\mathcal{T}$, for which $L(\tau^{-1}g^{ij})$ is parallel. Consequently, $\Phi$ is parallel for the dual connection on $S^2T^*$.

To compute this dual connection explicitly, we use the notation for elements of $S^2T$ and $S^2T^*$ as triples according to the splittings into a direct sum induced by the choice of a connection in the projective class as described in Section 2.3. This notation is chosen in such a way that, for the composition series (2), the projecting slot is on top, while the injecting slot is in the bottom. The necessary information to compute the dual pairing between $S^2T^*$ and $S^2T$ in this notation is contained in the proof of Proposition 17 of [3]. In the splitting determined by a connection in the projective class, we have the following correspondence between triples representing elements of $S^2T^*$ and linear maps $\mathcal{T} \to \mathcal{T}^*$:

\[
\begin{pmatrix}
\tau \\
\varphi_i \\
\psi_{jk}
\end{pmatrix} \leftrightarrow \begin{pmatrix}
\nu^k \\
\alpha \\
\end{pmatrix} \mapsto \begin{pmatrix}
\tau \alpha + \varphi_k \nu^k \\
\alpha \varphi_i + \psi_{ik} \nu^k
\end{pmatrix}.
\]

On the other hand, the correspondence between triples representing elements of $S^2T$ and linear maps $\mathcal{T}^* \to \mathcal{T}$ looks as follows.

\[
\begin{pmatrix}
\sigma^{ij} \\
\xi^k \\
\eta
\end{pmatrix} \leftrightarrow \begin{pmatrix}
\beta \\
\mu_i \\
\end{pmatrix} \mapsto \begin{pmatrix}
\sigma^{ik} \mu_k + \beta \xi^i \\
\beta \eta + \xi^k \mu_k
\end{pmatrix}.
\]

The pairing between elements of $S^2T^*$ and $S^2T$ can then be computed by taking the trace of the composition of the corresponding maps. Hence this is given by

\[
\left\langle \begin{pmatrix}
\tau \\
\varphi_i \\
\psi_{jk}
\end{pmatrix}, \begin{pmatrix}
\sigma^{ij} \\
\xi^k \\
\eta
\end{pmatrix} \right\rangle = \sigma^{ij} \psi_{ij} + 2\varphi_i \xi^i + \tau \eta.
\]

As mentioned in Section 2.3, there is a natural linear connection $\nabla^p$ on the vector bundle $S^2\mathcal{T}$ whose parallel sections are in bijective correspondence with solutions of the metricity equation. The solution of the metricity equation corresponding to a parallel section is simply the top slot of the corresponding tractor (which is independent of the choice of splitting). This connection has been constructed in [3], a general version for arbitrary first BGG–operators is available in [7]. The formula for this connection in the conventions for the splitting we use (which is slightly different from the one in [6]), is given in Theorem 4.1 of [3]. In the splitting
determined by a connection $\tilde{\nabla}$ from the projective class, one has

$$\nabla^p_i \left( \begin{array}{c} \sigma^{jk} \\ \xi \end{array} \right) = \left( \begin{array}{c} \tilde{\nabla}_i \sigma^{jk} + \delta_i^j \xi^k + \delta_i^k \xi^j \\ \tilde{\nabla}_i \xi - 2P_{ik} \xi^k - \frac{2}{n+1} Y_{ik} \xi^k \end{array} \right).$$

(6)

Here the tensors $P_{ij}, W_{ijk}^{\ell}$, and $Y_{ijk}$ denote the projective Schouten–tensor, the projective Weyl curvature, and the projective Cotton–tensor, respectively, see [1, 4].

Having at hand the dual pairing from (5) a direct computation leads to the following result:

**Proposition 4.** The connection $\nabla^{\mu*}$ on $S^2 T^*$, which is dual to the connection $\nabla^p$ on $S^2 T$ is, in the splitting determined by a connection $\tilde{\nabla}$ from the projective class, given by

$$\nabla^{\mu*}_i \left( \begin{array}{c} \tau \\ \varphi_i \\ \psi_{jk} \end{array} \right) = \left( \begin{array}{c} \tilde{\nabla}_i \tau - 2\varphi_i \\ \tilde{\nabla}_i \varphi_j - \psi_{ij} + \tau P_{ij} \\ \tilde{\nabla}_i \psi_{jk} + 2P_{i(j} \varphi_{k)} - \frac{2}{n+1} \varphi_{i} \psi_{\ell}^{\ell} k + \frac{2}{n+1} \tau Y_{ij} k \end{array} \right).$$

(7)

2.5. The main result. Now we are ready to prove our main result.

**Theorem 5.** Let $\overline{M}$ be a smooth manifold of dimension $n+1$ with boundary $\partial M$ and interior $M$. Let $g = g_{ij}$ be a pseudo–Riemannian metric on $M$ with Levi–Civita connection $\nabla$, which has the following properties.

- The projective structure on $M$ defined by $\nabla$ admits a smooth extension to $\overline{M}$.
- The connection $\nabla$ itself does not admit a smooth extension to any neighborhood of a boundary point.
- The scalar curvature $S$ of $g$ is bounded away from zero, or, more generally, the boundary value of the smooth extension of $S$ to $\overline{M}$ guaranteed by Proposition 3 is nowhere vanishing.

Then $g$ is projectively compact of order $\alpha = 2$.

**Proof.** As in Section 2.3, we put $\tau := \text{vol}(g)^{-\frac{2}{n+2}}$, which is a section of $E(2)$ defined over $M$ and nowhere vanishing there. Denoting by $g^{ij}$ the inverse of $g_{ij}$, we know from Section 2.3 and Proposition 3 that the section $\tau^{-1}g^{ij}$ of $E^{ij}(-2)$, which is initially defined over $M$, is a solution of the metricity equation and admits a smooth extension to $\overline{M}$. Then the corresponding section $L(\tau^{-1}g^{ij}) \in \Gamma(S^2 T^*)$ is parallel for the connection $\nabla^\mu$.

In Section 2.4 we have noted that our assumption on $S$ implies that, as a bilinear form on $T^*$, $L(\tau^{-1}g^{ij})$ is non–degenerate along $\partial M$, and hence locally around $\partial M$. In the further considerations, we can restrict to a neighborhood of $\partial M$ where this is true, i.e. we will assume that $L(\tau^{-1}g^{ij})$ is non–degenerate on all of $\overline{M}$. Then we can form the inverse bundle metric $\Phi \in \Gamma(S^2 T^*)$, which by construction is parallel for the dual connection $\nabla^{\mu*}$.

Restricting to $M$, we can work in the splittings of tractor bundles determined by the Levi–Civita connection $\nabla$. In this splitting, it is easy to describe $L(\tau^{-1}g^{ij})$
explicitly, since both $\tau^{-1}$ and $g^{ij}$ are parallel for $\nabla$, see Theorem 3.3 of [4]. One gets

$$L(\tau^{-1}g^{ij}) = \begin{pmatrix} \tau^{-1}g^{ij} \\ 0 \\ \frac{1}{n+1}\tau^{-1}g^{ij}P_{ij} \end{pmatrix}$$

in this splitting, so the bottom slot is a non–zero multiple of $\tau^{-1}S$. Now from (3) and (4), one immediately concludes that in the splitting determined by $\nabla$, we get

$$\Phi = \begin{pmatrix} (n+1)\tau(g^{ij}P_{ij})^{-1} \\ 0 \\ \tau g_{ij} \end{pmatrix},$$

so the top slot of this is a non–zero multiple of $\tau S^{-1}$. Since this is the projecting slot, it is actually independent of the choice of splitting, and passing to the splitting associated to a connection $\tilde{\nabla}$ in the projective class which is smooth up to the boundary, we conclude that $\tau S^{-1}$ admits a smooth extension to all of $\overline{M}$, so by Proposition 3, $\tau$ admits a smooth extension to $\overline{M}$.

Next, we claim that this smooth extension vanishes along the boundary $\partial M$. Indeed, if $x \in \partial M$ were a point such that $\tau(x) \neq 0$, then choose an open neighborhood $U$ of $x$ in $\overline{M}$, on which $\tau$ is nowhere vanishing. Then it is well known that there is a unique connection $\nabla$ in the restriction of the projective class to $U$, such that $\tau|_U$ is parallel for the induced connection on $\mathcal{E}(2)$. But then over $U \cap M$, both $\nabla$ and $\tilde{\nabla}$ preserve $\tau$ and hence have to agree. This contradicts the assumption that $\nabla$ does not extend smoothly to any neighborhood of a boundary point.

Now we finally claim that $\tau \in \Gamma(\mathcal{E}(2))$ is a defining density for $\partial M$, which in view of Lemma 1 completes the proof. So taking a connection $\tilde{\nabla}$ in the projective class which is smooth up to the boundary as above, we have to prove that $\tilde{\nabla}_\tau$ is nowhere vanishing along $\partial M$. From above we know that $\Phi$ is parallel for the connection $\tilde{\nabla}^\rho$, and its top slot with respect to any splitting is given by a non–zero constant multiple of $\tau S^{-1}$. In particular, this top slot vanishes along $\partial M$. By Proposition 3, we conclude that the middle slot of $\Phi$ in this splitting has to be a non–zero multiple of $\tilde{\nabla}_\tau(\tau S^{-1}) = S^{-1}\tilde{\nabla}_\tau \tau + \tau \tilde{\nabla}_\tau S^{-1}$. Of course, the second summand vanishes along the boundary, so there the middle slot equals, up to a nowhere–vanishing function, $\tilde{\nabla}_\tau \tau$. But we know that $\Phi$ is the inverse of $L(\tau^{-1}g^{ij})$, so in particular it is non–degenerate as a bilinear form (on $T$) over all of $\overline{M}$. By non–degeneracy, vanishing of the top slot along $\partial M$ implies that the middle slot has to be nowhere vanishing along $\partial M$.

Knowing that a metric $g$ satisfying the assumptions of Theorem 5 is projectively compact of order $\alpha = 2$, we can now apply all the results of [3] to $g$; see especially Theorems 4 and 5 in that source. In particular, the smooth extension of the scalar curvature $S$ of $g$ is automatically constant along the boundary. Moreover, in terms of a local defining function $\rho$ for the boundary $\partial M$, $g$ admits an asymptotic form given by

$$g = \frac{C\rho^2}{\rho^2} + \frac{h}{\rho},$$
where $C$ is a constant related to the boundary value of $S$, and $h$ is a symmetric $(0,2)$-tensor field which admits a smooth extension to the boundary with boundary values being non-degenerate on $T\partial M$. One also obtains a well-defined pseudo-Riemannian conformal structure on $\partial M$ induced by $g$, which is given by the conformal class of the restriction of the boundary value of $h$ to $T\partial M$. Finally, $g$ satisfies an asymptotic version of the Einstein equation, and the boundary conformal structure can be described in terms of the asymptotics of the Ricci curvature of $g$.

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