Visible stripe phases in spin–orbital-angular-momentum coupled Bose–Einstein condensates

N-C Chiu, Y Kawaguchi, S-K Yip and Y-J Lin

1 Institute of Atomic and Molecular Sciences, Academia Sinica, Taipei, 10617, Taiwan
2 Department of Applied Physics, Nagoya University, Nagoya, 464-8603, Japan
3 Institute of Physics, Academia Sinica, Taipei, 11529, Taiwan

E-mail: linyj@gate.sinica.edu.tw and kawaguchi@nuap.nagoya-u.ac.jp

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Abstract

Recently, stripe phases in spin–orbit coupled Bose–Einstein condensates (BECs) have attracted much attention since they are identified as supersolid phases. In this paper, we exploit experimentally reachable parameters and show that annular stripe phases with large stripe spacing and high stripe contrast can be achieved in spin–orbital-angular-momentum coupled (SOAMC) BECs. In addition to using Gross–Pitaevskii numerical simulations, we develop a variational ansatz that captures the essential interaction effects to first order, which are not present in the ansatz employed in previous literature. Our work should open the possibility towards directly observing stripe phases in SOAMC BECs in experiments.

1. Introduction

The realization of synthetic gauge fields and spin–orbit coupling for ultracold atoms has opened new opportunities for creating and investigating topological matters in a clean and easy-to-manipulate environment [1–4]. In the spin–orbit coupled Bose–Einstein condensates (BEC) realized in early works [5–7], the internal spin states are coupled to the center-of-mass linear momentum of the atoms via Raman laser dressing. There, the Raman beams transfer photon momentum to the atoms as the spin state changes. By using a similar method with Laguerre–Gaussian (LG) Raman beams which transfer orbital-angular-momentum (OAM) between atomic spin states, physicists recently demonstrated a coupling between internal spin states and the center-of-mass OAM [8–10]. In the following, we refer to the former as spin-linear-momentum coupling (SLMC) and the latter as spin–orbital-angular-momentum coupling (SOAMC).

The interplay between interactions and SLMC leads to interesting quantum phases [11–15]. For a pseudospin 1/2 system, by tuning the Raman coupling strength from large to small values, the energy-versus-momentum dispersion transforms from a single minimum to double minima (see figure 1(b)). For the latter case, whether the atoms occupy one of the minima or both minima is determined by a competition between inter- and intra-species interactions, where the two species refer to atoms which occupy the two respective minima. When the atoms occupy both minima associated with different quasimomentum, the interference results in density modulations in the position space, which is known as a stripe phase. This is a spatially mixed phase, i.e., miscible phase, of the two quasimomentum states. When only one of the minima is occupied, it is the separated phase, i.e., the plane-wave phase with single quasimomentum, and is immiscible for the two quasimomentum states. When the Raman coupling is below (above) the critical value, the ground state is the stripe phase (separated phase); see figure 1(b). The miscibility is indicated by the critical Raman coupling strength and the magnitude of stripe contrast, where both are determined by the spin-dependent interaction.

For simplicity, we use SLMC for both ‘spin-linear-momentum coupling’ and ‘spin-linear-momentum coupled’, and use SOAMC for both ‘spin–orbital-angular-momentum coupling’ and ‘spin–orbital-angular-momentum coupled’.
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1. Introduction

The stripe phase in SLMC BECs is intriguing since it spontaneously breaks the translational symmetry (being a solid) and the U(1) gauge symmetry (being a superfluid) simultaneously, leading to a so-called supersolid [16]. Analogously, the ground state of SOAMC BECs also has an annular stripe phase and separated phases, which are theoretically studied in references [17–20]. The annular stripe phase of SOAMC BECs corresponds to occupying both energy minima with different quasiangular-momentum. The stripe spatial period is then \( \approx 2\pi R/\Delta \ell \), where \( R \) is a typical length scale smaller than the BEC size \( R_{\text{BEC}} \), and \( \Delta \ell \) is the transferred OAM between spin states in units of \( h \). Since \( R \) is the order of micrometers, the spatial period can be made larger than that in SLMC, which is \( \lambda/2 \) with \( \lambda \) being the optical wavelength of the Raman laser. The submicron stripe period of SLMC BECs is difficult to resolve even with the state-of-the-art quantum gas microscope [21].

Due to both the small spatial period and small contrast resulting from small miscibility, direct observations of stripe phases in position space remain elusive to date. Recent experimental works have demonstrated signatures of stripe phases in spin-linear-momentum-coupled BECs [22–24], where Bragg spectroscopy is employed to detect the stripe density modulation in references [22, 23]. In the experiment with Raman-coupled internal spin states [22], the spatial phase coherence of both the stripe and separated phase is demonstrated interferometrically. In reference [23], atoms localized within each side of a double well serve as two pseudospin states. This circumvents the problem of detuning noises owing to the magnetic field noises for internal spin states, and enhances the miscibility. The observed stripe contrast is \( \sim 8\% \) limited by heatings from the Raman driving fields which create SLMC.

In this paper, we exploit the advantages of SOAMC systems and demonstrate the feasibility to directly observe annular stripe phases in situ with practical experimental parameters. We observe that interactions reduce the stripe density contrast. Here the interaction strength is \( \varepsilon_{\text{int}}/E_{\text{L}} < 1 \), where \( \varepsilon_{\text{int}} \) is the mean field interaction energy and \( E_{\text{L}} = h^2 \Delta \ell^2/2mR^2 \) is the characteristic energy scale of SOAMC systems. The effects of interactions discussed in previous papers [14, 18, 20, 25] are based on the wave function ansatz that is not fully self-consistent in the presence of interaction. Even within first order in interaction strength, we find that the results of references [14, 18, 20, 25] are subject to significant corrections. We use an improved ansatz and obtain results that are correct to first order in interaction. While in SLMC systems, the analogous interaction strength is \( \varepsilon_{\text{int}}/4E_{\text{L}} \) and is typically small, where the photon recoil energy \( E_{\text{r}} \) is larger than \( E_{\text{L}} \). We investigate how the stripe density contrast depends on experimentally accessible parameters: the transferred angular momentum \( \Delta \ell \), the size of the OAM-carrying LG Raman beam, the BEC cloud size, and the mean field energy. By optimizing these parameters, we achieve a stripe period of \( \sim 2\mu \text{m} \) and a \( \lesssim 30\% \) contrast of density modulations. This is detectable using high-resolution imaging with about 1 \( \mu \text{m} \) resolutions [21]. Further, the contrast can be made larger than 30\% by increasing the BEC cloud size. Finally, we point out that by using synthetic clock states [26], the stripe phase of the thermodynamic ground state can be stable against external magnetic field noises despite the narrow detuning window within which the stripe phase exits.

2. Formalism

We consider pseudospin 1/2 atoms tightly confined along \( z \) in a quasi-two-dimensional (quasi-2D) geometry, where \( \hbar \omega_z > \mu \) with \( \omega_z \) being the trap frequency along \( z \) and \( \mu \) the chemical potential. Two
Raman beams couple the two spin states with a transfer of OAM $\Delta \ell$ in unit of $\hbar$, and the frequency difference between the two beams is $\Delta \omega_L$. In the rotating frame at frequency $\Delta \omega_L$ with rotating wave approximation, the single-particle Hamiltonian is

$$\hat{H}_0 = \left[ -\hbar^2 \frac{\partial}{2m \partial r} \left( \frac{\partial}{\partial r} \right) + \frac{L_z^2}{2mr^2} + V(r) \right] \otimes \hat{I} + \frac{\hbar}{2} \hat{\sigma}_z + \frac{\hbar \Omega(r)}{2} \left[ \cos(\Delta \ell \phi) \hat{\sigma}_x - \sin(\Delta \ell \phi) \hat{\sigma}_y \right],$$

where $L_z = -i\hbar \partial_\phi$ is the angular momentum operator, $V(r)$ is the spin-independent trapping potential, $\delta = \Delta \omega_L - \omega_0$ is the Raman detuning, and $\hbar \omega_0 = E_i - E_f$ is the energy splitting between $|\downarrow\rangle$ and $|\uparrow\rangle$. The Raman beams are two Laguerre–Gaussian beams of order $\Delta \ell/2$ and $-\Delta \ell/2$, and the coupling strength is

$$\Omega(r) = e^{\Delta \ell/2} \Omega_M \left( \frac{r}{r_M} \right)^{\Delta \ell} \exp \left[ -\frac{\Delta \ell^2}{2} \frac{r^2}{r_M^2} \right],$$

where the peak coupling $\Omega_M$ is at $r = r_M$, and the waist of each beam is $w = 2r_M/\sqrt{\Delta \ell}$.

In addition to $\hat{H}_0$, we have the mean field energy

$$E_{\text{int}} = \int d^3r \left( \frac{g_{11}}{2} |\psi_1|^4 + \frac{g_{22}}{2} |\psi_2|^4 + g_{12} |\psi_1|^2 |\psi_2|^2 \right),$$

where $|\psi_1|^2, |\psi_2|^2$ are the 2D density of $|\downarrow\rangle, |\uparrow\rangle$. The wave functions are normalized as

$$\int d^3r \int d\phi n(r, \phi) = N$$

where $n = |\psi_1|^2 + |\psi_2|^2$ and $N$ is the atom number. The 2D interaction strengths are $g = g_{11} = g_{12}$ and $g_{22}$. We define $g_1 = (g + g_{22})/2, g_2 = (g - g_{22})/2$, $g_2$ being the spin-dependent interaction strength. We use real experimental parameters by taking the pseudospin states as $|\uparrow\rangle = |F = 1, m_F = 0 \rangle$ and $|\downarrow\rangle = |1, -1 \rangle$ of $^{87}$Rb atoms, for which $g = (g_{00} + g_{-1,-1})/2, g_{11} = g_{00}$ with $g_{00} = 4\pi\hbar^2 a_0/(mn\sqrt{2\pi}R), g_{-1,-1} = g_{00} - 4\pi\hbar^2 a_0/(mn\sqrt{2\pi}R)$, and $R = \hbar/m\omega_0$. The scattering lengths are $a_0 = 100.86 a_0$ and $a_{-1,-1} = 100.40 a_0$, where $a_0$ is the Bohr radius [27]. This gives $g > g_1$ and positive $g_2/g_1 = 0.00114$. As compared to the realistic case with $g_{11} \neq g_{12}$, here our simplification of using $g = g_{11} = g_{22}$ is based on the results of uniform SLMC systems in the absence of trapping potentials in reference [14], which is just a shift in detuning for the ground state. We show that this is a good approximation for the trapped atoms with inhomogeneous $n(r)$ under SOAM.

For $\delta = 0$ in the non-interacting limit, the ground state may be expressed as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \sqrt{n(r)} \begin{pmatrix} C_+ e^{i \phi} \left( \sin \theta(r)e^{i\Delta \ell \phi} + \cos \theta(r)e^{-i\Delta \ell \phi} \right) + C_- e^{-i \phi} \left( \cos \theta(r)e^{i\Delta \ell \phi} - \sin \theta(r)e^{-i\Delta \ell \phi} \right) \\ 0 \end{pmatrix},$$

where $\ell > 0, |C_+|^2 + |C_-|^2 = 1$, and $\bar{n}(r)$ is the density after azimuthal average with $\int dr 2\pi r \bar{n} = N$. Since the Raman coupling has a phase winding number $\Delta \ell$, i.e., an OAM of light, the Raman beams couple $|\uparrow\rangle, |\ell_\uparrow\rangle$ to $|\downarrow\rangle, |\ell_\downarrow\rangle$ where the OAM difference between the spin states is $\ell_\uparrow - \ell_\downarrow = \Delta \ell$. By introducing the quasiangular momentum $\ell_\uparrow, \ell_\downarrow$ and $\ell_\downarrow$ are rewritten as $\ell_\uparrow = \ell + \Delta \ell/2$ and $\ell_\downarrow = \ell - \Delta \ell/2$. Then, equation (4) is referred as ‘two-quasiangular-momentum ansatz’, which have two running wave components along $\phi$ with quasiangular momentum $\pm \ell$. For sufficiently small Raman coupling, the ground state has $\ell = \Delta \ell/2$ [17, 18], and we focus on this regime throughout this paper. With $\ell = \Delta \ell/2$, there is a critical coupling $\Omega_M$ below which the ground state has $|C_+||C_-| > 0$. For $\Omega_M < \Omega_M$, equation (4) shows that the spin component $|\uparrow\rangle (|\downarrow\rangle)$ has an OAM superposition of $\ell_\uparrow = \Delta \ell/2$ and $\ell_\downarrow = 0$ and $\ell_\downarrow = 0$ and $-\Delta \ell$, leading to a density modulation along $\phi$, which is then called stripe phase (see figure 2(a)). Here,

$$|\psi_1|^2/\bar{n} = \left[ |C_+|^2 \sin^2 \theta + |C_-|^2 \cos^2 \theta + |C_+||C_-| \sin 2\theta \cos(\Delta \ell \phi + \phi) \right],$$

$$|\psi_2|^2/\bar{n} = \left[ |C_+|^2 \cos^2 \theta + |C_-|^2 \sin^2 \theta + |C_+||C_-| \sin 2\theta \cos(\Delta \ell \phi + \phi) \right],$$

$$n = \bar{n} \left[ 1 + 2 |C_+|^2 |C_-|^2 \sin 2\theta \cos(\Delta \ell \phi + \phi) \right],$$

where $\phi$ is the relative phase between $C_+$ and $C_-$. With $\Omega_M > \Omega_M$, the ground state is the separated phase with $|C_+|^2 = 0, i.e., |C_-|^2 = 1$ or $|C_+|^2 = 1, |C_-|^2 = 0$ (see figure 2(b)), which are equivalent for $\delta = 0$. For the stripe phase with $|C_+|^2 |C_-|^2 > 0$, the ground state has $|C_+|^2 = |C_-|^2 = 1/2$ and $|C_+| = |C_-|$ where $|C_+|^2 |C_-|^2$ is maximized. Note that at $|C_+| = |C_-|$ the wave function equation (4) is an eigenstate of the time-reversal operator $T = \hat{\sigma}_x K$ with $K$ being the complex-conjugate operator, which is possible because the Hamiltonian commutes with $T$. At radial position $r$, the contrast of the azimuthal density modulation is
The interaction term, and equation (8) is similar to that in reference [28] for spin-linear-momentum coupling.

Then, the contrast of both $\uparrow$ and $\downarrow$ from equation (5) is

$$\eta(r) = \frac{n_{\text{max}}(r) - n_{\text{min}}(r)}{2n_{\text{avg}}(r)},$$

where $n_{\text{max}}, n_{\text{min}}$ and $n_{\text{avg}}$ are the maximum, minimum and average of the density along $\phi$, respectively. Then, the contrast of both $\uparrow$ and $\downarrow$ from equation (5) is

$$\eta(r) = \sin 2\theta(r),$$

and the spatial period of the density stripe is $2\pi r/\Delta \ell$.

Now we consider the general form of the spinor wave function for all interaction strengths, which is

$$(\psi_\uparrow, \psi_\downarrow) = \sum_p \left( \begin{array}{cc} a_{\Delta \ell + p\Delta \ell} & e^{i\Delta \ell \phi} \\ b_{\Delta \ell + p\Delta \ell} & e^{-i\Delta \ell \phi} \end{array} \right) \left( \begin{array}{c} 0 \\ \sin \theta(r) e^{i\Delta \ell \phi} \end{array} \right) + C_+ e^{i\Delta \ell \phi} \left( \begin{array}{c} \cos \theta(r) e^{i\Delta \ell \phi} \\ -\sin \theta(r) e^{-i\Delta \ell \phi} \end{array} \right) + C_- e^{-i\Delta \ell \phi} \left( \begin{array}{c} \cos \theta(r) e^{-i\Delta \ell \phi} \\ -\sin \theta(r) e^{i\Delta \ell \phi} \end{array} \right).$$

This is referred as ‘four-quasiangular-momentum ansatz’, which has quasiangular momentum of $\pm \Delta \ell/2, \pm 3\Delta \ell/2$. Since the ground state energy is independent of the relative phase between $C_+$ and $C_-$, we can take $C_+ < 0, C_- > 0$ as real values without loss of generality. Similar to the case with two-quasiangular-momentum ansatz, we consider states that are eigenstates of the time-reversal operator $T$, which will be confirmed by numerical simulations. The variational parameters then satisfy

$$|C_+| = |C_-|, |A_+| = |A_-|, \alpha_+ + \alpha_- = 0 \mod 2\pi,$$

where $A_+ = |A_+|e^{i\alpha_+}, A_- = |A_-|e^{i\alpha_-}$. The $\cos(\Delta \ell \phi)$ term in the density modulation is $C_- \cos \theta(C_+ \sin \theta + A_+^* e^{i\Delta \ell \phi} + \text{c.c. for } |\uparrow\rangle$, and $-C_+ \cos \theta(C_- \sin \theta + A_-) e^{i\Delta \ell \phi} + \text{c.c. for } |\downarrow\rangle$. Thus, for the ground state with minimized density modulation, the relative phase between $C_+ \sin \theta$ and $A_+$ and between $-C_- \sin \theta$ and $A_+$ is $\pi$. This gives real and positive $A_+, A_-$. We then have

$$-C_+ = C_- > 0, A_+ = A_- = A_\pm > 0.$$
Similarly, if we choose $C_+, C_- > 0$, the condition becomes $C_+ = C_- > 0, A_+ = -A_- > 0$. With $C_+, C_- > 0, A_-, -A_+ > 0$. As real numbers, the densities of $|\uparrow\rangle$ and $|\downarrow\rangle$ are
\[
|\psi_{\uparrow}|^2/\hbar = C_+^2 \sin^2 \theta + C_-^2 \cos^2 \theta + A_+^2 + 2 \cos \theta (C_+ C_- \sin \theta + A_+ C_- \cos (\Delta \ell \phi)) \\
+ 2A_+ C_- \sin \theta \cos (2\Delta \ell \phi),
\]
\[
|\psi_{\downarrow}|^2/\hbar = C_+^2 \sin^2 \theta + C_-^2 \cos^2 \theta + A_-^2 + 2 \cos \theta (C_+ C_- \sin \theta - A_+ C_- \cos (\Delta \ell \phi)) \\
- 2A_+ C_- \sin \theta \cos (2\Delta \ell \phi).
\]
(12)
The normalization is $C_+^2 + C_-^2 + A_+^2 + A_-^2 = 1$, leading to
\[
C_+^2 = C_-^2 = -(1 - A_+^2 - A_-^2)/2,
\]
(13) and for small $A_+, A_-$, $C_+^2 \leq 1/2$ and $C_-^2 \sin^2 \theta + C_+^2 \cos^2 \theta \approx 1/2$. Thus, the density contrast of the $\cos(\Delta \ell \phi)$ term in equation (12) is
\[
\eta_{\uparrow} \approx |2 \sin 2\theta C_+ C_- + 4 \cos \theta A_- C_-|,
\]
\[
\eta_{\downarrow} \approx |2 \sin 2\theta C_+ C_- - 4 \cos \theta A_+ C_-|,
\]
(14) by using equation (6) with $(n_{\text{max}} - n_{\text{min}})/2$ equal to the amplitude of the $\cos(\Delta \ell \phi)$ term and $n_{\text{avg}} \approx 1/2$ for each spin component. With equation (11), the contrast is
\[
\eta_{\uparrow} = \eta_{\downarrow} \approx |\sin 2\theta - 2\sqrt{2} \cos \theta A_+|.
\]
(15) Since the density modulation of the $\cos(2\Delta \ell \phi)$ term is much smaller than $\eta_{\uparrow,\downarrow}$, we use equation (15) as the contrast in our simulations.

3. Simulations methods

We perform both the Gross–Pitaevskii (GP) simulations and the variational calculations to find the ground state in the SOAMC system. The GP simulation gives the ground state with the full Hamiltonian including both $\hat{H}_0$ and the interaction energy. Additionally, we perform the variational calculations with a simplified picture: we neglect the radial kinetic energy associated with $\partial_r$ in $\hat{H}_0$, thus the rest of all the energy terms are functions of radial position $r$. We find the results of GP and variational methods have good agreements.

3.1. Gross–Pitaevskii ground state

We use the GP simulations to find the ground state by numerically solving the Gross–Pitaevskii equation (GPE). We perform imaginary time propagations, where the initial state of the stripe phase for the imaginary time propagation is
\[
\begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \end{pmatrix} = \sqrt{\frac{\eta_{\text{TF}}(r)}{2}} \begin{pmatrix} 1 + e^{i\Delta \ell \phi} \\ 1 + e^{-i\Delta \ell \phi} \end{pmatrix},
\]
(16) and for the separated phase it is
\[
\begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \end{pmatrix} = \sqrt{\frac{\eta_{\text{TF}}(r)}{2}} \begin{pmatrix} e^{i\Delta \ell \phi} \\ 1 \end{pmatrix}.
\]
(17)
The initial state of the stripe phase has a superposition of OAM differing by $\Delta \ell$ in either spin up and down, such that all the OAM components differing by $\Delta \ell$ [see equation (8)] can be reached in the final ground state. The initial state of the separated phase corresponds to $C_+ = 1, C_- = 0$. After the numerical computation, we compare the energy differences between the two phases and determine the ground state from the lower energy state.

3.2. Variational method

We adopt a variational method to minimize the energy $E_{\text{var}}$ for $\delta = 0$ and obtain the variational ground state. $E_{\text{var}}$ includes the single particle Hamiltonian in equation (1) but excluding the radial kinetic energy from $\partial_r$, and the mean field interaction equation (3). This gives $E_{\text{var}} = \int dr r 2\pi n(r) \varepsilon(r)$ where $\varepsilon(r)$ is the energy per atom after the azimuthal average.

We discuss calculations based on the two-quasiangular-momentum ansatz, equation (4), and four-quasiangular-momentum ansatz, equation (9), respectively. The variational ground state from the simple equation (4) agrees with our GP simulation in the non-interacting limit. This variational form,
equation (4), is used in earlier papers [14, 18, 20, 25]; [14] for SLMC and [18, 20, 25] for SOAMC BECs. In our simulations, we find the variational ground state from equation (4) is inconsistent with the GP result in the non-negligible interaction regime, where the additional OAM components must be taken into account as the ansatz equation (9).

3.2.1. Two-quasiangular-momentum ansatz
With the ansatz of equation (4), the variational energy per atom $\varepsilon^{\text{var}}$ is given by

$$\varepsilon^{\text{var}} = \varepsilon_0^{\text{var}} + \varepsilon_{\text{int}},$$

$$\varepsilon_0^{\text{var}} = -\frac{\hbar \Omega(r)}{2} \sin 2\theta + \frac{E_L}{2} (1 - \cos 2\theta),$$

$$\varepsilon_{\text{int}}^{\text{var}} = \frac{\bar{n}(r) g_1}{2} + \frac{\bar{n}(r) g_2}{2} \cos 2\theta + \beta \left[ \bar{n}(r) g_1 \sin^2 2\theta - 2\bar{n}(r) g_2 \cos^2 2\theta \right].$$

$\varepsilon_0^{\text{var}}$ is the single-particle energy arising from the Raman coupling and the centrifugal potential $L_z^2/2mr^2$, where the latter is characterized by $E_L = \hbar^2 (\Delta \ell)^2/2mr^2$ at position $r$. Here we exclude the trap energy $V(r)$ in $\hat{H}_0$ since $V(r)$ does not depend on any variational parameters and is simply an offset. $\varepsilon_{\text{int}}^{\text{var}}$ is the mean field interaction energy with $\beta = |C_+|^2/C_-^2$ satisfying $0 \leq \beta \leq 1/4$. Given the local energy $\varepsilon^{\text{var}}(r)$ at a radial position $r$ with the averaged density $\bar{n}(r)$, we take $\theta(r)$ and $\beta$ as variational parameters. $\beta = 0$ for the separated phase with $|C_+| = 1, |C_-| = 0$ or $|C_+| = 0, |C_-| = 1$, and $\beta = 1/4$ for the stripe phase with $|C_+| = |C_-| = 1/\sqrt{2}$. Within $0 \leq \beta \leq 1/4$, the energy difference between the stripe and the separated phase is lowest at either $\beta = 0$ or $\beta = 1/4$ (see appendix A). At a given $\beta$, minimizing $\varepsilon^{\text{var}}(r)$ with respect to $\theta$ determines $\theta$ as a solution of the following equation:

$$\sin 2\theta - \cos 2\theta \left[ \frac{\hbar \Omega(r)}{E_L(r)} - \frac{\bar{n}(r) g_1}{E_L(r)} \left( 4\beta + 8\beta \frac{g_2}{g_1} \right) \sin 2\theta \right] = 0. \tag{19}$$

In the non-interacting case, the solution of equation (19) is $\sin 2\theta_0 = \hbar \Omega/E_L$, or equivalently

$$\sin 2\theta_0 = \frac{\hbar \Omega/E_L}{\sqrt{1 + (\hbar \Omega/E_L)^2}},$$

which can be approximated for small $\hbar \Omega/E_L$ as

$$\sin 2\theta_0 \approx \frac{\hbar \Omega}{E_L}.$$

With interactions, the solution $\theta^{\text{var}}$ of the stripe phase with $\beta = 1/4$ is smaller than $\theta_0$, given by

$$\sin 2\theta^{\text{var}} - \cos 2\theta^{\text{var}} \left( \frac{\hbar \Omega(r)}{E_L(r)} - \frac{\bar{n}(r) g_1}{E_L(r)} \sin 2\theta^{\text{var}} \right) = 0,$$  

from which the contrast is given by

$$\eta^{\text{var}}(r) = \sin 2\theta^{\text{var}}(r) \tag{21}$$

as derived in equation (7). We use $\bar{n}(r)$ obtained from the GP simulation, which is the same for the stripe phase and separated phase and is well approximated by the Thomas–Fermi (TF) profile except for small $\gamma_M$. By expanding to first order in $\bar{n} g_1/E_L$ and $\hbar \Omega/E_L$,

$$\theta^{\text{var}} \approx \frac{\hbar \Omega}{2E_L} \left( 1 - \frac{\bar{n} g_1}{E_L} \right),$$

$$\eta^{\text{var}} \approx \frac{\hbar \Omega}{E_L} \left( 1 - \frac{\bar{n} g_1}{E_L} \right).$$

For the separated phase with $\beta = 0$, $\theta_{\text{sep}}$ is well approximated with $\theta_0$ owing to $\bar{n} g_2 \ll E_L$.

3.2.2. Four-quasiangular-momentum ansatz
With the ansatz of equation (9), the single-particle part of the variational energy is given by

$$\varepsilon_0^{\text{var}} = \left[ -\frac{\hbar \Omega(r)}{2} \sin 2\theta + \frac{E_L}{2} (1 - \cos 2\theta) \right] \left( C_+^2 + C_-^2 \right) + E_L (A_+^2 + A_-^2), \tag{23}$$
where $C_+, C_-, A_+, A_-$ are real. The interaction energy $\varepsilon^{\text{int}}_{\text{var}}$ is also a function of $\theta, C_+, C_-, A_+, A_-$; by using equation (11) for the stripe phase, we plug in $C_\pm = \mp \sqrt{(1 - 2A_\pm^2)/2}, A_+ = A_- = A_\pm$ and obtain

$$\varepsilon^{\text{var}} \equiv -\frac{1}{8}\bar{n}_g \left[-5 - 12A_+^4 + 28A_+^2 + 8A_+ (2 - 4A_+^2)^{3/2}\cos^2 \theta \sin \theta \right.$$

$$\left. + (1 - 4A_+^2 + 4A_+^4)(1 - 2\sin^2 2\theta)\right].$$

(24)

Then we minimize $\varepsilon^{\text{var}} = \varepsilon^{\text{var}}_0 + \varepsilon^{\text{var}}_{\text{int}}$ with respect to $(\theta, A_\pm)$, respectively, giving the numerical solutions for the stripe phase, $\theta^{\text{var}}$ and $A^{\text{var}}_\pm > 0$, where the sign of $A^{\text{var}}_\pm$ agrees with equation (11). The contrast of the $\cos(\Delta \ell(\phi))$ term is

$$\eta^{\text{var}}(r) = \left|\sin 2\theta^{\text{var}}(r) - 2\sqrt{2} \cos \theta^{\text{var}}(r) A^{\text{var}}_\pm(r)\right|$$

(25)

for both $|\uparrow\rangle, |\downarrow\rangle$ following equation (15).

By expanding $A^{\text{var}}_\pm$ to first order in $\bar{n}_g/E_\ell$ and $\hbar \Omega/E_\ell$, we obtain $A^{\text{var}}_\pm \approx \bar{n}_g \theta / \sqrt{2}\epsilon_\ell$, which also agrees with the result using perturbation (see appendix A). After plugging it into equation (25) and expand $\theta^{\text{var}}$, we have

$$\theta^{\text{var}} \approx \frac{\hbar \Omega}{2E_\ell} \left(1 - \frac{\bar{n}_g}{E_\ell}\right),$$

$$\eta^{\text{var}} \approx \frac{\hbar \Omega}{E_\ell} \left(1 - 2\frac{\bar{n}_g}{E_\ell}\right).$$

(26)

Comparing to $\eta^{\text{var}}_0$ in equation (22), we find the coefficient of $\bar{n}_g/E_\ell$ in $\eta^{\text{var}}$ is $-2$, twice as that in $\eta^{\text{var}}_0$. That is, including the additional OAM $\pm 3\Delta \ell/2$ in equation (9) is necessary for correct results to first order in $\bar{n}_g/E_\ell$. For the separated phase with $C_+ = 1, C_- = 0$ and $A_+ = A_- = 0$, it is identical to that using the two-quasiangular-momentum ansatz.

We comment on earlier theoretical papers on SOAMC systems [18, 20, 25]. We examine the dimensionless interaction strength $\bar{n}_g/E_\ell$ in these papers. In reference [20], $\bar{n}_g/E_\ell$ is $\lesssim 0.01$, and single-particle eigenstates are taken as the basis of the variational method, i.e., $\theta = \theta_0$. References [18, 25] use variational methods with the wave function ansatz equation (4), where reference [18] has ring traps with $\bar{n}_g/E_\ell > 100$, and the interaction $\bar{n}_g$ is not specified in reference [25].

4. Results and discussions

We consider practical experimental parameters to maximize the density contrast of the stripe phase. We first discuss BECs in harmonic traps in the Thomas–Fermi regime along the radial direction with the Thomas–Fermi radius $R_T$. We study how the GP stripe phase contrast depends on $(\Delta \ell, \mu, R_T, \mu)$; $\mu$ is the chemical potential and the peak mean field energy in the harmonic trap.

For comparisons, we also consider atoms in a ring trap with radius $r_0$. Here $R_T = r_0$ is the only length scale, unlike the harmonically trapped systems where there are two relevant length scales, $(R_T, R_T)$.

4.1. Harmonic traps

We first obtain the GP ground state phase diagram as shown in figure 1(c). We then focus on the GP stripe phase at $\delta = 0$, setting $\Omega_M = \Omega_c$. We run simulations for $\Delta \ell$ between 2 and 30, all with $r_0 = 17 \mu$m, $R_T = 46 \mu$m, and $\mu = h \times 21$ Hz. $\Delta \ell = 30$ corresponds to the LG beam with phase winding number of $\pm 15$, which can be achieved experimentally (in reference [29], LG beams with phase winding number of 45 are realized). From the GP wave function $\psi(r, \phi)$, we evaluate the density contrast $\eta^{\text{GP}}(r)$ from the normalized Fourier components $\tilde{\psi}(r, \phi)$ following equation (15). For $|\downarrow\rangle$, $\psi_{\downarrow,4}$ are given by [see equation (9)]

$$\psi_{\downarrow,4} = -C_+ \cos \theta^{\text{GP}}, \quad \psi_{\downarrow,20,4} = -C_- \sin \theta^{\text{GP}}, \quad \tilde{\psi}_{\downarrow,4} = A^{\text{GP}}_{\downarrow,4},$$

(27)

from which the contrast $\eta^{\text{GP}}_{\downarrow,4} = \eta^{\text{GP}}_{\downarrow,20,4}$ (the first term) and $4\bar{\psi}_{\downarrow,4} \psi_{\downarrow,20,4}$ (the second term). We then compare $\eta^{\text{GP}}(r)$ to the variational solutions of the contrast, which are sin $2\theta_0(r)$ for the non-interacting case, $\eta^{\text{var}}(r)$ for using the two-quasiangular-momentum ansatz [equation (21) from the ansatz equation (4)] and $\eta^{\text{var}}(r)$ for using the four-quasiangular-momentum ansatz [equation (25) from the ansatz equation (9)]. In figure 3(a), we plot sin $2\theta_0(r)$, $\eta^{\text{var}}(r)$, $\eta^{\text{var}}(r)$ and $\eta^{\text{GP}}(r)$ for the example value $\Delta \ell = 20$; their maxima are at $r \gtrsim R_T$. In figure 3(b), We plot the peak values of sin $2\theta_0(r)$, $\eta^{\text{var}}(r)$, $\eta^{\text{var}}(r)$ and $\eta^{\text{GP}}(r)$ versus $\Delta \ell$, which are denoted as sin $2\theta_0$, $\eta^{\text{var}}$, $\eta^{\text{var}}$ and $\eta^{\text{GP}}$, respectively. We observe that the single-particle contrast sin $2\theta_0$ is significantly larger than $\eta^{\text{GP}}$ for small $\Delta \ell$, while sin $2\theta_0$ and $\eta^{\text{GP}}$ are close for $\Delta \ell \gtrsim 20$. As the dimensionless interaction $\bar{n}_g/E_\ell$ increases with decreasing $\Delta \ell$, the contrast $\eta^{\text{GP}}$ decreases. When the interaction is taken into
consideration using the ansatz equation (4), the resulting $\eta_{\text{var}}$ overestimates $\eta_{\text{GP}}$, indicating that equation (4) is insufficient. We can understand this from the annular Fourier transform of the GP wave function $\psi_1$ for $\Delta \ell = 20$. Figure 3(c) shows the power spectrum of the normalized Fourier components $\psi_{\ell=\pm1}$, where there are $\ell_1 = 0, \pm20$ components, and the $\ell_1 = -20$ component signifies that the more general equation (9) should be used in the variation method with the $A_\pm$ term accounting for $\ell_1 = 20$, while $A_+, A_-$ are absent in the simple equation (4). The spectrum for $|\uparrow>$ is also displayed in figure 3(c). The GP results have $C_\pm \approx \mp 1/\sqrt{2}$ and $A_\pm = A_{\pm \text{GP}}$, which confirms the time-reversal symmetry condition, equation (11). From the GP results, the signs of $C_\pm \approx \mp 1/\sqrt{2}$ are applied in equation (11) and in the variational method using equation (9). We also show the peak values, $\eta_{\text{var}}$ of $\eta_{\text{GP}}(r)$, in figure 3(b), where $\eta_{\text{var}} > \eta_{\text{GP}} > \eta_{\text{GP}}$ and $\eta_{\text{GP}}$ fits well with $\eta_{\text{GP}}$.

From the above studies, we find the maximum of the stripe contrast is at $r = r_{\text{peak}} \approx r_M$, where the spatial period is $\approx 2\pi r_M/\Delta \ell$. ($r_{\text{peak}}$ is only slightly larger than $r_M$ for all the contrasts, e.g., $r_{\text{peak}} = 17.6 \mu m$ for $\eta_{\text{GP}}$ in figure 3(a)). The peak value of the contrast increases with $\Delta \ell$, and thus a larger contrast corresponds to a smaller stripe period $\Delta \ell$. To observe the stripe phase in experiments, we note that with state-of-the-art imaging techniques in ultracold atoms, e.g., those using quantum gas microscopes, one can resolve as small as $0.5 \mu m$ with $\lambda = 0.78 \mu m$ for $^{87}\text{Rb}$ [21]. This sets the lower bound on $2\pi r_M/\Delta \ell$ in our simulations. Since the peak contrast is the signal we optimize, $r_{\text{peak}} \approx r_M$, and thus $E_L(r_{\text{peak}}) \approx E_L(r_M)$ are the relevant length and energy scale for SOACM, respectively.

Next, we fix $\Delta \ell = 20$ and $\mu = h \times 93$ Hz, and vary $(r_M, R_{TF})$. We study $r_M < R_{TF}$ where there is sufficient atomic density at $r = r_M$ for various combinations of $(r_M, R_{TF})$. Here we set the smallest $r_M = 5 \mu m$ for $\Delta \ell = 20$, where the spatial period of the stripe is $\approx 1.6 \mu m$ and is larger than the diffraction limit of the imaging, $0.5 \mu m$. With $\Omega_M = \Omega_c$, the GP results have the peak contrast $\eta_{\text{GP}}$ increasing with decreasing $r_M$ and increasing $R_{TF}$, as shown in figure 4(b). We then compare the GP to the variational calculations with four-quasiangular-momentum ansatz. In figure 4(a), we plot the density contrast contributed from $\theta, A_\pm$ and the sum, respectively. These are $-\sin 2\theta_{\text{GP}}$, $2\sqrt{2}\cos \theta_{\text{GP}}A_{\uparrow \text{GP}}$, and $\eta_{\text{GP}}$ versus $r$ for GP, and $-\sin 2\theta_{\text{var}}$, $2\sqrt{2}\cos \theta_{\text{var}}A_{\uparrow \text{var}}$, and $\eta_{\text{var}}$ versus $r$ for the variational calculations. We find the contrast of GP obtained from equation (6) versus $r$ agrees well with $\eta_{\text{GP}}(r)$, showing that $\eta_{\text{GP}}$ from the $\cos(\Delta \ell \phi)$ term dominates the contrast in equation (12), where the second harmonics is negligible. We display the peak values $\eta_{\text{var}}$ versus $r_M$ for all $R_{TF}$ in figure 4(b) and compare them to the peak values $\eta_{\text{GP}}$, where $\eta_{\text{var}}$ overestimates $\eta_{\text{GP}}$ by $3\%-8\%$. For all $(r_M, R_{TF})$, the Fourier spectrum of GP has $\ell_\uparrow = 0, \pm20$ components and $\ell_\downarrow = 40, \ell_\downarrow = -40$ are negligible, being consistent with equation (9). We then compare the peak values $(\theta_{\text{GP}}, A_{\uparrow \text{GP}})$ to $(\theta_{\text{var}}, A_{\uparrow \text{var}})$ versus $r_M$ for $R_{TF} = 50 \mu m$ in figure 4(c). The peaks of $\theta$ and $A_\pm$ are at $r \approx r_M$, as well as that of the contrast $\eta, \theta_{\text{var}}$ and $A_{\uparrow \text{var}}$ slightly overestimate $\theta_{\text{GP}}$ and $A_{\uparrow \text{GP}}$, respectively: both $\theta_{\text{GP}}/\theta_{\text{var}}$ and $A_{\uparrow \text{GP}}/A_{\uparrow \text{var}}$ are between 0.90–0.93. This is attributed to the radial kinetic energy that is neglected in the variational calculations. The GP ground state has smaller $(\theta_{\text{GP}}, A_{\uparrow \text{GP}})$ than $(\theta_{\text{var}}, A_{\uparrow \text{var}})$ where the smaller radial spin gradient corresponds to a smaller radial kinetic energy, and thus the lowest overall energy.

After studying the dependence of the peak contrast $\eta_{\text{GP}}$ on $(\Delta \ell, r_M, R_{TF})$, we vary $\mu$ and thus the interaction strength $\tilde{n}_g/E_L$ at $r = r_{\text{peak}} \approx r_M$, where $\tilde{n}_g/E_L = \mu [1 - (r_{\text{peak}}/R_{TF})^2] / E_L(r_{\text{peak}})$. We fix $\Delta \ell = 20, R_{TF} = 50 \mu m$ for $r_M = 5$ and $15 \mu m$, respectively. We study $\mu = h \times 21, 46, 93$ Hz for both $r_M$, and additionally $\mu = h \times 600$ Hz for $r_M = 5 \mu m$. Figure 5 shows $\eta_{\text{GP}}$ weakly depends on $\tilde{n}_g/E_L$. 

---

Figure 3. Stripe contrast from the GP and variational calculations with varying $\Delta \ell$. $\Omega_M = \Omega_c$, $r_M = 17 \mu m, R_{TF} = 46 \mu m$ and $\mu/h = 21$ Hz. (a) Contrast versus radial position $r$ for $\Delta \ell = 20$. From variational solutions: $\sin 2\theta(r)$ for non-interacting atoms (grey curve), $\eta_{\text{var}}(r)$ for using two-quasiangular-momentum ansatz (green), and $\eta_{\text{GP}}(r)$ for using four-quasiangular-momentum ansatz (red). Blue curve indicates $\eta_{\text{GP}}(r)$ for the GP. (b) Peak values of the contrast vs $\Delta \ell$. Grey, green, red, and blue symbols denote $\sin 2\theta, \eta_{\text{var}}, \eta_{\text{GP}},$ respectively. (c) Annular Fourier power spectrum after integration along $r$ for $|\uparrow>$ (dark grey) and $|\uparrow>$ (light grey).
Figure 4. Stripe contrast and $\theta_A$, of the variational results using four-quasiangular-momentum ansatz and of GP with varying $(r_M, R_{TF})$. $\Omega_\text{TF} = \Omega_c$, $\Delta \ell = 20$ and $\mu/\hbar = 93$ Hz. (a) Comparison of the contrast $\eta$ vs $r$ for $r_M = 15$ $\mu$m and $R_{TF} = 50$ $\mu$m. Grey, blue and green curves indicate $-\sin 2\theta_A$, $2\sqrt{2} \cos \theta_A$, and $\eta$, respectively. Solid (dashed) curves denote the variational (GP) simulations. (b) Peak contrast vs $r_M$. Red (blue) symbols indicate $\eta_{\text{var}}$ ($\eta_{\text{GP}}$). Circles, squares and triangles for $R_{TF} = 50, 25, 12.5 \mu$m, respectively. (c) Peak values of $\theta_A$, vs $r_M$ for $R_{TF} = 50 \mu$m. Red (blue) symbols indicate the variational (GP) results; circles (triangles) for $\theta_A$. 

Figure 5. Peak stripe contrast $\eta_{\text{GP}}$ versus interaction strength $\bar{n}(r_{\text{peak}})g_{1}/E_{L}(r_{\text{peak}})$ with varying $\mu$, $\Delta \ell = 20$ and $R_{TF} = 50 \mu$m; $r_{\text{peak}} \gtrsim r_M$. Circles and square symbols indicate $r_M = 5, 15$ $\mu$m, respectively.

Besides the density contrast, we compare the critical coupling $\Omega_c$ of the GP results to those given by the variational methods. Using the two-quasiangular-momentum ansatz, equation (4), the critical coupling $\Omega_{c\text{var}}$ is given by

$$\int dr 2\pi \bar{n}(r) \Delta \epsilon_{\text{var}} = 0,$$

$$\Delta \epsilon_{\text{var}} = \epsilon_{\text{var}}(\theta_{\text{var}}, \beta = 1/4) - \epsilon_{\text{sep}},$$

(28)

$\Delta \epsilon_{\text{var}}$ is the energy difference between the stripe phase and the separated phase, $\theta_{\text{var}}$ is the variational solution of $\theta$ for the stripe phase, and $\epsilon_{\text{sep}}$ is the energy of the separated phase with $\theta = \theta_{\text{sep}}$ and $\beta = 0$. When the integral is $<0$ at $\Omega_M < \Omega_c$, the ground state is the stripe (separated) phase. Similarly, by using the four-quasiangular-momentum ansatz, equation (9), the critical coupling $\Omega_{c\text{var}}$ is given by

$$\int dr 2\pi \bar{n}(r) \Delta \epsilon_{\text{var}} = 0,$$

$$\Delta \epsilon_{\text{var}} = \epsilon_{\text{var}}(\theta_{\text{var}}, C_\pm = \mp \sqrt{1 - 2 (A_{\pm \text{var}})^2} / 2) - \epsilon_{\text{sep}}.$$

(29)

In figure 6, we plot $\Omega_c$ of the GP and from the solutions of equations (28) and (29) versus $(r_M, R_{TF})$, where $\Omega_c$ from GP have good agreements with $\Omega_{c\text{var}}$.

We can understand that $\Omega_c$ increases with increasing $R_{TF}$ and decreasing $r_M$ from a geometric argument. Such dependence on $(r_M, R_{TF})$ is crucial since a larger $\hbar \Omega_c/E_L$ leads to larger stripe contrast $\eta_{\text{var}}$, see equation (26). We find numerically that the energy difference between the stripe phase and separated phase can be approximated as

$$\Delta \epsilon_{\text{var}} = f(r) - \bar{n}(r) g_1^2,$$

$$f(r) = \bar{n}(r) g_1 H \left( \bar{n} g_1 / E_L \right) \frac{\hbar^2 \Omega^2(r)}{E_L^2}$$

(30)
for small $\theta$, where $H$ is a dimensionless function of $\bar{g}_1/E_L$ and $H \to 1/4$ as $\bar{g}_1/E_L \to 0$. The peak value of $f(r)$ is

$$f_M = \bar{g}_1 H \left( \frac{\bar{g}_1}{E_L} \right) \frac{\hbar^2 \Omega_M^2}{E_L^2},$$

where $\bar{g}_1$, $E_L$, and $\Omega_M$ are evaluated at the peak position of $f(r)$, $r_{\text{peak}} > r_M$. To make a geometric analysis, we simplify $f(r)$ as a flat impulse function centered at $r = r_M$ with full width $\Delta r = (\Delta \ell/2)^{-1/2} r_M$: $f(r) = f_M$ for $r_M - \Delta r/2 < r < r_M + \Delta r/2$. Assuming a cylindrical box trap with uniform $n$ within $r = R_{\text{box}}$, the integral in equation (29) then gives

$$\bar{n} 2 \pi r_M \Delta r f_M = \frac{\bar{n} \pi R_{\text{box}}^2}{2},$$

and thus

$$f_M = \frac{\bar{g}_1}{2} \left( \frac{\pi R_{\text{box}}^2}{2 \pi r_M \Delta r} \right),$$

where the number in the parentheses is a ‘geometric’ area ratio of the box to that of the distribution $f(r)$; the ratio scales as $R_{\text{box}}^2 r_M^{-2} \Delta \ell^{1/2}$. By equating equations (31) and (32), assuming a fixed interaction strength $\bar{g}_1/E_L$ and using $E_L \propto \Delta \ell^2 r_M^2$, we obtain $\Omega_M = \Omega_c \propto R_{\text{box}} r_M$ which increases with increasing $R_{\text{box}}$ and decreasing $r_M$; see figure 6.

We then derive the stripe contrast $\eta^{\text{var}}$ versus $(\Delta \ell, r_M, R_{\text{box}})$. First, $\hbar \Omega_c/E_L$ is given by equating equations (31) and (32),

$$h \Omega_c/E_L \propto \left( \frac{\bar{g}_1}{g_1} \right)^{1/2} \left[ H \left( \frac{\bar{g}_1}{E_L} \right) \right]^{-1/2} \Delta \ell^{1/4} \frac{R_{\text{box}}}{r_M}.$$

The contrast $\eta^{\text{var}}$ can be expressed as

$$\eta^{\text{var}} = \frac{h \Omega_c/E_L}{\hbar \Omega_c/E_L},$$

where $h \Omega_c/E_L$ is approximately the non-interacting contrast $\sin 2\theta_0$ in section 3.2.1, and $\eta^{\text{var}}/(h \Omega_c/E_L) < 1$ is the contrast-reduction ratio due to interactions; see equation (26) (with expansion to first order in interaction). $\eta^{\text{var}}/(h \Omega_c/E_L)$ decreases with increasing interaction $\bar{g}_1/E_L$, which is proportional to $\Delta \ell^{-2} r_M^2$ with $\bar{g}_1$ approximately fixed. Therefore, $\eta^{\text{var}}/(h \Omega_c/E_L)$ increases with increasing $\Delta \ell$ and decreasing $r_M$. In equation (33), $H$ decreases with $\bar{g}_1/E_L$, and thus $H^{-1/2}$ decreases with increasing $\Delta \ell$ and decreasing $r_M$. The overall $h \Omega_c/E_L$ decreases with increasing $\Delta \ell$ and increases with decreasing $r_M$. When combining equations (33) and (34), $\eta^{\text{var}}$ increases with increasing $\Delta \ell$ given fixed $(r_M, R_{\text{box}}, \bar{g}_1)$, and increases with decreasing $r_M$ and increasing $R_{\text{box}}$. Such dependence on $\Delta \ell$ and $r_M$ is shown in figures 3(b) and 4(b), respectively, where the system has harmonic traps with $R_{\text{TF}}$ replacing $R_{\text{box}}$.

### 4.2. Ring traps

We show the variational results for the ring trap versus the dimensionless interaction strength $g'_1 = \bar{g}_1 E_L (r_0)$. These are the solutions of the critical coupling $\Omega_c$ and the contrast at $\Omega_c$. We solve the dimensionless critical coupling $\Omega'_c = \hbar \Omega_c/E_L$ by using $\Delta^{\text{var}}(r_0) = 0$ and $\Delta^{\text{var}}(r_0) = 0$, respectively, and plot $\Omega'_c$ vs $g'_1$ in figure 7(a). $\Omega''^{\text{var}}$ exceeds $\Omega''^{\text{var}}$ for $g'_1 > 0$, and $h \Omega''^{\text{var}} \approx h \Omega'_c \approx \sqrt{2g_1/g_L}$ $E_L \approx 0.05 E_L$ as $g'_1 \to 0$. We can understand $\Omega''^{\text{var}} > \Omega''^{\text{var}}$ as the following: at a given $\Omega$, the stripe phase energy is $\varepsilon^{\text{var}} < \varepsilon^{\text{var}}$ since the smaller contrast $\eta^{\text{var}}$ [see equations (22) and (26)] corresponds to smaller interaction energy.
In summary, we optimize the density contrast of the ground state stripe phase of $^{87}$Rb SOAMC BECs by and determines the critical coupling $\Omega_c$ by equations (22) and (26). The stripe contrast of SOAMC is at most order in $\mu$. The spin-dependent interaction strength in the clock state basis is close to the $\hbar/\varepsilon$, approximately the same variational solution, $\theta_{\text{var}} \approx \theta_{\text{sep}} \approx \theta_0$, and thus $\Delta \varepsilon_{\text{var}} = (1/4) \mu \left[ \sin^2 2\theta_0 - 2g_l \cos^2 2\theta_0 \right]$ = 0 from equation (18) leads to $\Omega_c^1 = \sqrt{2g_l/g_1}$. The stripe contrasts $\eta_{\text{var}}$ and $\eta_{\text{sep}}$ at respective $\Omega_c$ are displayed in figure 7(b); they are $\approx 5\%$ as $g_l^1 \to 0$, both decreasing with increasing $g_l^1$ and $\eta_{\text{sep}} < \eta_{\text{var}}$. To compare with SLMC systems, $h\Omega_{c,\text{SLMC}}^0/E_L$ for $g_l^1 < 1$ agrees well with that of SLMC (see figure 7(a)), which is $h\Omega_{c,\text{SLMC}}^0/4E_L = \sqrt{2g_l/g_1}$ as $g_l^1 \to 0$ [14], i.e., $h\Omega_{c,\text{SLMC}}^0 = 0.2E_L$; $4E_L$ is equivalent to $E_I$ for SOAMC. The spin-dependent interaction strength $g_l^1/g_1$ determines the critical coupling $\Omega_{c}^1/E_L$ and stripe contrast $\eta$ as $g_l^1 \to 0$; larger $g_l^1/g_1$ gives larger $h\Omega_{c}^1/E_L$ and $\eta$, i.e., larger miscibility. The contrast of SLMC also agrees well with $\eta_{\text{var}}$, see figure 7(b). We find that $\Omega_{c}^1$ and $\eta$ of SOAMC with $\varepsilon_{\text{var}}$ and of SLMC, where both are based on equation (4), are incorrect to first order in $g_l^1$. By including higher order OAM in equation (9), the result is correct to first order, as indicated by equations (22) and (26). The stripe contrast of SOAMC is at most $\approx 5\%$ in the $g_l^1 \to 0$ limit, and it is independent of either the ring radius $r_0$ or $\Delta \ell$. On the other hand, a stripe contrast $\leq 30\%$ for harmonic traps is achieved with a relatively large $R_{TF} = 50 \mu m$ and a relatively small $r_M = 5 \mu m$, and this can be understood from the geometric analysis as shown in equations (31)–(34).

We then perform GP simulations for a ring trap with $r_0 = r_M = 10 \mu m$ and $\Delta \ell = 20$. The atoms are in an annular box potential within $8 \mu m < r < 12 \mu m$, and $n_{R_1}/E_L = 0.63$ at $r = r_0$. The GP result has good agreement with the variational calculation, where $\theta_{\text{GP}}/\theta_{\text{var}} = 0.95$ and $A_{\text{GP}}^1/A_{\text{var}}^1 = 0.91$.

5. Conclusions

In summary, we optimize the density contrast of the ground state stripe phase of $^{87}$Rb SOAMC BECs by tuning experimental parameters. A contrast of nearly $30\%$ is achieved for atoms in harmonic traps, and a larger contrast of about $50\%$ is expected by using a twice larger BEC cloud size based on variational calculations. Such high contrasts are achieved owing to the geometry with two length scales in harmonic traps, the Raman Laguerre–Gaussian beam size and the BEC cloud size. While for ring traps, these two scales are the same, leading to maximal contrast about $5\%$, which is dictated by the spin-dependent interaction strength and is the same as that of the SLMC systems. For both atoms in harmonic traps and ring traps, we perform GP simulations and variational calculations based on the two-quasiangular-momentum ansatz and four-quasiangular-momentum ansatz. We find the results from the simple two-quasiangular-momentum ansatz, which is used in previous papers [14, 18, 20, 25], are consistent with the GP results only in the non-interacting limit. With small interactions, high order OAM components must be included as the four-quasiangular-momentum ansatz; this then leads to correct results to first order in interaction and good agreements with the GP simulations.

We point out that one can improve the stability by using the synthetic clock states instead of bare spin states. The clock states are immune to detuning variations arising from the bias field variations. A $0.1–1$ Hz stability of the clock transition frequency is achieved as shown in reference [26]. Thus, the ground state stripe phase within a narrow detuning window of about 1 Hz may be observed for $^{87}$Rb atoms with mean field energy about 1 kHz. The spin-dependent interaction strength in the clock state basis is close to the

![Figure 7](image-url)
$g_1/g_i$ for bare spin states (see appendix A), leading to similar magnitude of stripe contrast to our simulations using bares spin states. We envision our work to pave the way towards a direct observation of high-contrast stripe phases in spin-orbital-angular-momentum coupled Bose–Einstein condensates, achieving a long-standing goal in quantum gases.

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**Appendix A**

**A.1. Spinor wave function ansatz**

We show that the spinor wave function ansatz equation (9) is valid for small $\theta$ (given small Raman coupling $\hbar\Omega/E_L$) and small interaction $\hbar g_i/E_L$. The GPE for $|\uparrow\rangle, |\downarrow\rangle$ with $\delta = 0$ is

$$
-\frac{\hbar^2}{2m}\nabla^2\psi_\uparrow + \frac{\hbar\Omega(r)}{2}e^{i\Delta\ell/\hbar}\psi_\downarrow - \mu_\uparrow\psi_\uparrow + g|\psi_\uparrow|^2\psi_\uparrow + g_{11}|\psi_\downarrow|^2\psi_\uparrow = 0, \tag{35a}
$$

$$
-\frac{\hbar^2}{2m}\nabla^2\psi_\downarrow + \frac{\hbar\Omega(r)}{2}e^{-i\Delta\ell/\hbar}\psi_\uparrow - \mu_\downarrow\psi_\downarrow + g|\psi_\downarrow|^2\psi_\downarrow + g_{11}|\psi_\uparrow|^2\psi_\downarrow = 0. \tag{35b}
$$

Here we set $V(r) = 0$ to simplify the discussion. For the spatially-mixed stripe phase ground state, $\mu_\uparrow = \mu_\downarrow$. Next we show the nonlinear interaction leads to multiple OAM components in the spinor wave function ansatz. With $\ell = \Delta\ell/2 = 10$, we plug

$$
\psi_\uparrow = \sqrt{n}(r) \left( C_\uparrow \sin \theta e^{i20} + C_\downarrow \cos \theta \right), \quad \psi_\downarrow = \sqrt{n}(r) \left( -C_\uparrow \cos \theta - C_\downarrow \sin \theta e^{-i20} \right)
$$

into equation (35a), neglect radial gradients and keep the expansion terms up to $\theta^2$. The nonlinear interaction terms for $\psi_\uparrow$ are

$$
g|\psi_\uparrow|^2\psi_\uparrow + g_{11}|\psi_\downarrow|^2\psi_\downarrow = \sqrt{n} \left[ C_- C_\uparrow (g + g_{11}) \theta^2 e^{i40} + C_\uparrow \left( 2C_\uparrow^2 g + g_{11} \right) \theta e^{i20} + C_- \left( C_\uparrow^2 g + C_\uparrow g_{11} + O(\theta^2) \right) C_\downarrow + C_- C_\uparrow (g + g_{11}) \theta e^{-i20} \right]. \tag{36}
$$

Besides $\ell_t = 0, 20$, additional OAM terms with $\ell_t = -20, 40$ appear due to the nonlinear interaction, which are of order of $\theta, \theta^3$, respectively, and are not included in equation (4). By keeping up to order $\theta^2$, the spinor wave function has additional variational parameters $A_+, A_-, B_+, B_-$, given by

$$
\begin{pmatrix}
\psi_\uparrow \\
\psi_\downarrow
\end{pmatrix} = \sqrt{n}(r) \left[ B_+ e^{i\Delta\phi} \left( A_+ e^{-i\Delta\phi} \right) + C_+ e^{i\Delta\phi} \left( \sin \theta(r) e^{i\Delta\phi} \right) - \cos \theta(r) e^{-i\Delta\phi} \right) + C_- e^{-i\Delta\phi} \left( \cos \theta(r) e^{i\Delta\phi} \right) - \sin \theta(r) e^{-i\Delta\phi} \right) + e^{-i\Delta\phi} \left( A_- e^{i\Delta\phi} \right) B_- e^{-i\Delta\phi} \right]. \tag{37}
$$

$\ell_t = -20$ is of order $\theta$ and corresponds to $\ell = -3\Delta\ell/2 = -30$ with $A_-$; $\ell_t = 40$ is of order $\theta^2$ and corresponds to $\ell = 3\Delta\ell/2 = 30$ with $B_+$. For small $\theta$, by taking up to order $\theta$ we have the spinor wave function ansatz equation (9) with $A_+, A_- \neq 0$ and $B_+ = B_- = 0$.

Next we derive $A_-$ for small interactions $\hbar g_i/E_L$ using first order perturbation. We plug

$$
\begin{align*}
\psi_\uparrow &= \sqrt{n} \left( C_\uparrow \sin \theta e^{i20} + C_- \cos \theta + A_- e^{-i20} \right), \\
\psi_\downarrow &= \sqrt{n} \left( A_+ e^{i20} - C_\downarrow \cos \theta - C_- \sin \theta e^{-i20} \right)
\end{align*} \tag{38}
$$

into equation (35a) for $\psi_\uparrow$, and focus on the coefficient of the $e^{-i20}$ term, which is

$$
(E_L - \mu_\uparrow) \sqrt{n} A_- + \sqrt{n} C_- C_\uparrow (g + g_{11}) \theta = 0. \tag{39}
$$

Besides reading out from equation (36), the coefficient of the $e^{-i20}$ term in the nonlinear interaction $g|\psi_\uparrow|^2\psi_\uparrow + g_{11}|\psi_\downarrow|^2\psi_\downarrow$ can be readily found from the Fourier components of $|\psi_\uparrow|^2, |\psi_\downarrow|^2$ in equation (5),
both of which have OAM = 0, ±20. Then, using $\mu_+ \approx \mu_0 (\Omega = 0, g = g_{\pm}) = 0$ for small $\hbar \Omega/E_L$ and $\bar{n}g_{\pm}/E_L$, along with $C_\pm \approx \mp 1/\sqrt{2}$, it gives

$$A_+ \approx \frac{\bar{n}g_{\pm}}{E_L} \theta,$$  \hbox{(40)}

From the GP stripe phase wave function, in figure 8(a) we plot the ratio of the peak values $A_{\pm}^{GP}/\theta^{GP}$ vs $\bar{n}(r_{peak})g_{\pm}/E_L(r_{peak})$, along with the ratio $\bar{n}g_{\pm}/\sqrt{2}E_L$ given by equation (40), which agrees with $A_{\pm}^{GP}/\theta^{GP}$ at small $\bar{n}(r_{peak})g_{\pm}/E_L(r_{peak})$. The dimensionless interaction $\bar{n}g_{\pm}/E_L$ evaluated at $r_{peak} \approx \bar{r}_M$ vs $\bar{r}_M$ for $R_{TF} = 12.5, 25, 50 \mu m$ is shown in figure 8(b), all with $\mu = \hbar/93$ Hz. For a fixed $\mu$, $\bar{n}g_{\pm} = \mu [1 - (r/R_{TF})^2]$ weakly depends on $r$ for $r/R_{TF} \lesssim 0.5$. $\bar{n}(r_{peak})g_{\pm}/E_L(r_{peak})$ increases with increasing $\bar{r}_M$, which is dominated by $E_L \propto \bar{r}_M^2$. Similarly, we derive $B_+$ by plugging $\psi_\uparrow = \sqrt{n} \left( B_+ e^{i20\phi} + C_+ \sin \theta e^{i20\phi} + C_- \cos \theta e^{-i20\phi} \right)$, $\psi_\downarrow = \sqrt{n} \left( A_+ e^{i20\phi} - C_+ \cos \theta e^{i20\phi} - C_- \sin \theta e^{-i20\phi} + B_- e^{-i40\phi} \right)$ into equation (35a). The coefficient of the $e^{i40\phi}$ term is

\begin{equation}
\left(4E_L - \mu_+ \right) \sqrt{n}B_+ + \frac{\hbar \Omega}{2} \sqrt{n}A_+ + \sqrt{n} C_- \left( g + g_{\uparrow \downarrow} \right) \theta^2 = 0.
\end{equation}

With $\theta \approx \hbar \Omega/2E_L (1 - \bar{n}g_{\pm}/E_L)$ from $\theta^{var}$, and $A_+ = A_- \pm 0$ for the ground state, it leads to

$$B_+ \approx -\left( \frac{\hbar \Omega}{8E_L} - \frac{\bar{n}g_{\pm}}{2\sqrt{2}E_L} \theta^2 \right) \approx -\frac{\bar{n}g_{\pm}}{2\sqrt{2}E_L} \theta^2.$$

In our GP data, we have small $\theta^{GP} \approx 0.05 < \theta^{GP} < 0.16$, and small interactions, $\bar{n}(r_{peak})g_{\pm}/E_L(r_{peak}) \approx 0$ for the data with the two smallest $\Delta \ell = 2, 4$ in figure 3. The peak values of $B_+^{GP}$ at $r_{peak} \gtrsim \bar{r}_M$ are small, $0.02 < B_+^{GP}/A_{+}^{GP} < 0.04$ for $\bar{n}(r_{peak})g_{\pm}/E_L(r_{peak}) < 2$. Thus it is valid to neglect $B_+$ by using the wave function ansatz equation (9).

### A.2. Methods for GP ground state simulations

We run the GP simulations with both the open-source GPELab toolbox [30] and Crank–Nicolson method. The grid size is between 0.093–0.55 $\mu m$ depending on the spatial resolution we need.

To do analysis of the GP wave function in the cylindrical coordinate, we first make interpolations of the raw data in the Cartesian coordinate. The annular Fourier transform is performed as

$$\psi_m(r, \phi) = \sum_q \psi_{q,m}(r) e^{iq\phi},$$

$$\psi_{q,m} = (2\pi)^{-1} \int d\phi \psi_m(r, \phi) e^{-iq\phi}.$$
where \( q = \ell_x, \ell_y \) is the OAM and \( m = \uparrow, \downarrow \) is the spin label. For the stripe phase with \( \bar{n}_r(r) = \bar{n}_r(r)/2 \),

\[
\sum_q |\tilde{\psi}_{q,m}(r)|^2 = \bar{n}(r)/2. \tag{45}
\]

We take the normalized Fourier components as \( \tilde{\psi}_{q,m}(r) = \psi_{q,m}(r)/\bar{n}(r)^{-1/2} \), leading to \( \tilde{\psi}_{0,\downarrow} = -C_+ \cos \theta^{GP} \), \( \tilde{\psi}_{-2,\downarrow} = -C_- \sin \theta^{GP} \), \( \tilde{\psi}_{2,\downarrow} = A_\pm^{GP} \). The power spectrum in figure 3(c) is after the integration along \( r \),

\[
\bar{n}_{q,m} \propto \int drr|\tilde{\psi}_{q,m}(r)|^2, \sum_q \bar{n}_{q,m} = 1. \tag{46}
\]

### A.3. Variational calculations

We consider the SOAMC ground state as either the stripe phase with \( |C_+ C_-| > 0 \) or the separated phase with \( |C_+ C_-| = 0 \), i.e., \( C_+ = 1, C_- = 0 \) or \( C_+ = 0, C_- = 1 \). The former corresponds to a density stripe and the latter to no density stripe. In the variational calculation using two-quasiangular-momentum ansatz where \( A_\pm \) in absent in the wave function, \( 0 \leq |C_+|^2 |C_-|^2 \leq 0/4 \) and \( |C_+|^2 = |C_-|^2 \) corresponds to \( |C_+|^2 = |C_-|^2 = |C_+ C_-| = 1/2 \). We compare the energy of \( 1/4 \) and \( 0 \) and take the lower one as the ground state. This is valid because the lowest energy is at either \( \beta = 1/4 \) or \( \beta = 0 \), i.e., no energy minimum within \( 0 \leq |C_+|^2 \leq 1/4 \). We find this condition holds by numerically checking the second order derivative of \( E^{var} \), which is negative for \( 0 \leq |C_+|^2 \leq 1/4 \) and \( \bar{n}_{\bar{g}_l}/E_h < 2 \). As for the calculation using four-quasiangular-momentum ansatz for the stripe phase, we compare the energy \( E^{var} \) of the stripe phase, \( C_\pm = \pm \sqrt{(1 - 2A_\pm^2)/2} \), and of the separated phase with \( C_+ = 1, C_- = 0 \) and \( A_\pm = 0 \). It is valid to take \( C_\pm = \pm \sqrt{(1 - 2A_\pm^2)/2} \) for the stripe phase since the GP results confirm that this condition holds, which has the time reversal symmetry, equation (11).

### A.4. Dependence of stripe contrast on system parameters

We study the peak stripe contrast versus \( (\Delta \ell, r_M, R_TF) \) and find the contrast increases with increasing \( \Delta \ell \), decreasing \( r_M \) and increasing \( R_TF \). We show the contrast \( \eta^{var} \) of the variational calculations has such dependence at the critical Raman coupling \( \Omega_c \). We employ a simplified model with a cylindrical box trap of radius \( R_{box} \) and obtain equations (33) and (34), showing the dependence of \( h\Omega_c/E_h \) and \( \eta^{var}(\Omega = \Omega_c) \) on \( (\Delta \ell, r_M, R_{box}, \bar{n}_{\bar{g}_l}/E_h) \); here the dimensionless interaction \( \bar{n}_{\bar{g}_l}/E_h \) is evaluated at \( r_{peak} \approx r_M \), where \( \bar{n}_{\bar{g}_l}/E_h \) is proportional to \( \Delta \ell^{-2} r_M^{-2} \) with \( \bar{n}_{\bar{g}_l} \) approximately fixed (also see appendix A.1). \( H \) in equation (33) decreases with \( \bar{n}_{\bar{g}_l}/E_h \) as displayed in figure 9(a), and the resulting \( h\Omega_c/E_h \) decreases with increasing \( \Delta \ell \) and increases with decreasing \( r_M \). The contrast-reduction ratio \( \eta^{var}/(h\Omega_c/E_h) \) is unity for non-interacting atoms, and decreases with increasing interaction \( \bar{n}_{\bar{g}_l}/E_h \); see figure 9(b) with simulations of harmonic traps in figures 3 and 4. Therefore, \( \eta^{var}/(h\Omega_c/E_h) \) increases with increasing \( \Delta \ell \) and decreasing \( r_M \). By combining equations (33) and (34), \( \eta^{var} \) increases with increasing \( \Delta \ell \), decreasing \( r_M \) and increasing \( R_{box} \). Note that \( h\Omega_c/E_h \) decreases while \( \eta^{var}/(h\Omega_c/E_h) \) increases with \( \Delta \ell \), and \( h\Omega_c/E_h \) has a weaker dependence, leading to \( \eta^{var} \) increasing with increasing \( \Delta \ell \). In figure 9(b), the two largest \( \bar{n}_{\bar{g}_l}/E_h \) are about 4 and 20, corresponding to \( \Delta \ell = 4, 2 \), respectively. Although we expect the variational result to be valid at small \( \bar{n}_{\bar{g}_l}/E_h \), we find the contrast \( \eta^{GP} \) and \( \eta^{var} \) agree well for \( \Delta \ell = 2, 4 \); see figure 3(b).

### A.5. Trap parameters of the simulations

We indicate the trap parameters: for data in figure 1 with \( \mu = h \times 926 \) Hz and \( R_TF = 12.5 \) \( \mu m \), \( N = 2.5 \times 10^4, \omega_r/2\pi = 37.13 \) Hz, and \( \omega_z/2\pi = 1000 \) Hz. For data in figure 3 with \( \mu = h \times 21 \) Hz and \( R_TF = 46 \mu m, N = 10^4, \omega_r/2\pi = 1.5 \) Hz, and \( \omega_z/2\pi = 600 \) Hz. For data in figure 4 with \( \mu = h \times 93 \) Hz and \( R_TF = 12.5, 25, 50 \mu m, N = 0.25 \times 10^4, 10^4, 4 \times 10^4, \omega_r/2\pi = 11.744, 5.872, 2.936 \) Hz respectively; \( \omega_z/2\pi = 1000 \) Hz.

### A.6. Comparison to SLMC systems

We list results of SLMC BECs from reference [14]. Two counter-propagating Raman beams along x transfer linear momentum \( \Delta k_x = 2k_r \) between spin \( \uparrow \) and \( \downarrow \), producing SLMC. The linear momentum transfer \( 2k_r \) is analogous to the OAM transfer \( \Delta \ell \) in SOAMC, and thus \( 4E_r \) is equivalent to \( E_h \) in SOAMC. A spinor wave function ansatz analogous to our two-quasiangular-momentum ansatz, equation (4), is employed. For a uniform system with no trapping potentials, the critical coupling is

\[
h\Omega_c^{SLMC} \approx \sqrt{\frac{2g_2}{3}\frac{3\bar{g}}{4E}}\left(1 + \frac{\bar{g}_{\ell}}{4E_r}\right) \tag{47}
\]
We verify the stripe phases with realistic interactions after expanding to first order in $\eta_{\text{INT}}$ and thus a superposition of bare spin states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. This can be potentially observed given the measured stability of $n_{\text{INT}}$ based on reference [26]. These clock states are $\psi_m = \sum_m r_m |m\rangle$, where $r_m = |G_m|/\sqrt{\bar{N}}$. The critical coupling is $g_{\text{INT}} = 25\, \text{MHz}$, which is about 70 times the critical coupling $g_{\text{INT}} = 40\, \text{MHz}$ at zero detuning where $\bar{N} = n_{\text{INT}}$. The lowest, middle, and highest-energy dressed states correspond to $|z\rangle$, $|x\rangle$, and $|\bar{y}\rangle$, respectively. By choosing proper rf parameters, the $xz$ transition frequency can be made fourth-order sensitive to rf detuning, and thus to the bias field. We consider a two-level system of Raman-coupled $|x\rangle$ and $|z\rangle$.

The mean field energy can be expressed in the basis of $|x\rangle$ and $|z\rangle$, 

$$E_{\text{int}} = \int d^3r \left( \frac{G_{xx}}{2} |\psi_x|^4 + \frac{G_{xz}}{2} |\psi_z|^4 + G_{xz} |\psi_x|^2 |\psi_z|^2 \right)$$

with effective interactions $G_{xx}, G_{xz}, G_{x\bar{z}}$, and $G = (G_{xx} + G_{xz})/2, G_{1} = (G + G_{xz})/2, G_{2} = (G - G_{xz})/2$. We consider rf Rabi coupling $\Omega_{rf} = 2.77 \omega_q$ at zero detuning where $\hbar \omega_q$ is the quadratic Zeeman energy. This gives $G_{xx} = c_0/\sqrt{2\pi R_s}, G_{xz} = (c_0 + 0.97c_2)/\sqrt{2\pi R_s}, G_{x\bar{z}} = (c_0 + 0.825c_2)/\sqrt{2\pi R_s}$, where $c_0 = 4\pi \hbar^2 (a_0 + 2a_1)/3m$ and $c_2 = 4\pi \hbar^2 (d_2 - a_0)/3m < 0$; $a_1$ is the s-wave scattering length in the total spin $f$ channel. (Note that $G_{12} = c_0/(\sqrt{2\pi R_s})$ and $g_{-1,1} = (c_0 + 2c_2)/(\sqrt{2\pi R_s})$. The resulting $G_{1} / G_{12} \approx 1 - 0.17c_2/c_0 > 0$, which is about 70\% of the $g_{13} / g_{1}\approx -0.35c_2/c_0$, even bigger than $g_{13} / g_{1}$. Therefore, the stripe contrast using the synthetic clock states is similar to our simulations using bare spin $|0\rangle, |1\rangle$. If we choose the two levels as $|x\rangle, |y\rangle$ instead, $G_{1} / G_{12} \approx -0.35c_2/c_0$, even bigger than $g_{13} / g_{1}$.

Consider the detuning window within which the stripe phase exist. At $\Omega_M \approx 0$, the window is $\gtrsim 1\, \text{Hz}$ for our data in figure 1(c), where $\mu = n_{3D} = 4.25\, \mu\text{m}$, and $n_{3D}$ is the peak 3D density. This can be potentially observed given the measured stability of $\approx 0.1–1\, \text{Hz}$.

### A.7. Scheme of using synthetic clock states

We propose to use synthetic clock states in the SOAMC system of $^{87}\text{Rb}$ atoms. Here the discussions are based on reference [26]. These clock states are $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, each of which is a radio-frequency-dressed state, and thus a superposition of bare spin states $|m_F = 0, \pm 1\rangle$. The lowest, middle, and highest-energy dressed state corresponds to $|z\rangle$, $|x\rangle$, and $|\bar{y}\rangle$, respectively. By choosing proper rf parameters, the $xz$ transition frequency can be made fourth-order sensitive to rf detuning, and thus to the bias field. We consider a two-level system of Raman-coupled $|x\rangle$ and $|z\rangle$.

The mean field energy can be expressed in the basis of $|x\rangle$ and $|z\rangle$, 

$$E_{\text{int}} = \int d^3r \left( \frac{G_{xx}}{2} |\psi_x|^4 + \frac{G_{xz}}{2} |\psi_z|^4 + G_{xz} |\psi_x|^2 |\psi_z|^2 \right)$$

with effective interactions $G_{xx}, G_{xz}, G_{x\bar{z}}$, and $G = (G_{xx} + G_{xz})/2, G_{1} = (G + G_{xz})/2, G_{2} = (G - G_{xz})/2$. We consider rf Rabi coupling $\Omega_{rf} = 2.77 \omega_q$ at zero detuning where $\hbar \omega_q$ is the quadratic Zeeman energy. This gives $G_{xx} = c_0/\sqrt{2\pi R_s}, G_{xz} = (c_0 + 0.97c_2)/\sqrt{2\pi R_s}, G_{x\bar{z}} = (c_0 + 0.825c_2)/\sqrt{2\pi R_s}$, where $c_0 = 4\pi \hbar^2 (a_0 + 2a_1)/3m$ and $c_2 = 4\pi \hbar^2 (d_2 - a_0)/3m < 0$; $a_1$ is the s-wave scattering length in the total spin $f$ channel. (Note that $G_{12} = c_0/(\sqrt{2\pi R_s})$ and $g_{-1,1} = (c_0 + 2c_2)/(\sqrt{2\pi R_s})$. The resulting $G_{1} / G_{12} \approx 1 - 0.17c_2/c_0 > 0$, which is about 70\% of the $g_{13} / g_{1}\approx -0.35c_2/c_0$, even bigger than $g_{13} / g_{1}$. Therefore, the stripe contrast using the synthetic clock states is similar to our simulations using bare spin $|0\rangle, |1\rangle$. If we choose the two levels as $|x\rangle, |y\rangle$ instead, $G_{1} / G_{12} \approx -0.35c_2/c_0$, even bigger than $g_{13} / g_{1}$.

Consider the detuning window within which the stripe phase exist. At $\Omega_M \approx 0$, the window is $\gtrsim 1\, \text{Hz}$ for our data in figure 1(c), where $\mu = n_{3D} = 4.25\, \mu\text{m}$, and $n_{3D}$ is the peak 3D density. This can be potentially observed given the measured stability of $\approx 0.1–1\, \text{Hz}$.

### A.8. Validity of the symmetric inter-spin interaction

We verify the stripe phases with realistic $g_{1\uparrow} \neq g_{1\downarrow}$ are approximately the same as that with symmetric inter-spin interaction, $g_{1\uparrow} = g_{1\downarrow} = g$, while with a detuning shift. We obtain the phase diagram using realistic $g_{1\uparrow}, g_{1\downarrow}$, and identify the ground state with the maximum stripe contrast is at $\delta/2\pi = 1.4\, \text{Hz}$, instead of $\delta = 0$ for $g_{1\uparrow} = g_{1\downarrow} = g$. The parameters are $\Delta\ell = 40, \gamma_M = 4.25\, \mu\text{m}, R_{TF} = 10\, \mu\text{m}, N = 1.55 \times 10^4, \omega_1/2\pi = 45.746\, \text{Hz}$, and $\omega_2/2\pi = 1000\, \text{Hz}$. The critical coupling is $\Omega_c/2\pi = 731.7\, \text{Hz}$.

![Figure 9](image)
where the peak contrast is \( \eta_{GP} = 0.217 \). This is very close to that with \( g_{\uparrow \uparrow} = g_{\downarrow \downarrow} = g \), where \( \Omega_c/2\pi = 730.0 \text{ Hz} \) and \( \eta_{GP} = 0.210 \).

**ORCID iDs**

N-C Chiu [https://orcid.org/0000-0002-2119-1841](https://orcid.org/0000-0002-2119-1841)

Y Kawaguchi [https://orcid.org/0000-0003-1668-6484](https://orcid.org/0000-0003-1668-6484)

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