Extension of KNTZ trick to non-rectangular representations

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ABSTRACT

We claim that the recently discovered universal-matrix precursor for the $F$ functions, which define the differential expansion of colored polynomials for twist and double braid knots, can be extended from rectangular to non-rectangular representations. This case is far more interesting, because it involves multiplicities and associated mysterious gauge invariance of arborescent calculus. In this paper we make the very first step – reformulate in this form the previously known formulas for the simplest non-rectangular representations $[r, 1]$ and demonstrate their drastic simplification after this reformulation.

Spectacular success \cite{1, 2} of the lasting program \cite{3, 9} to calculate colored knot polynomials \cite{10} for antiparallel double braids (double twist knots) and Racah matrices \cite{11} in all rectangular representations $R$ from the evolution properties \cite{12, 16} of their differential expansions \cite{13, 14, 17, 18}, opens a way to attack the main problem of arborescent calculus \cite{19, 20}: understanding of non-rectangular representations. The main difference from rectangular case is that multiplicities occur in the product of representations, and this makes the notion of Racah matrices ambiguous. In the language of \cite{20} this is described as the new gauge invariance and one of the problems is to define gauge-invariant arborescent vertices. However, before that there is a problem to calculate the Racah matrices $\overline{S}$ and $S$, which enter the definition of "fingers" and "propagators", connected by these vertices. These problems, are not fully unrelated, because $S$ and $\overline{S}$ in non-rectangular case are not gauge invariant – still one can ask what they are in a particular gauge. As suggested in \cite{4}, the key to evaluation of $\overline{S}$ is differential expansion (DE) for twist knots \cite{14} – which, once known, straightforwardly produces $S$ for rectangular $R$, because of spectacular (and still unexplained!) factorization property of the DE coefficients for double braids. $S$ are then easily extractable as a diagonalizing matrix for $\overline{S}$ – it is enough to solve a system of linear equations. However, for non-rectangular $R$ the situation is worse: differential expansion for double braids includes not $\overline{S}$ itself, but some gauge-invariant combination of its matrix elements, and also the linear system for $S$ is degenerate and again provides only the information about gauge-invariant quantities. The problem therefore is to extract at this stages exactly the combinations, needed for arborescent calculus – and we do not yet know what they are. In other words, for non-rectangular $R$ we face a whole complex of related problems, which is partly surveyed in \cite{20, 7} and, especially, \cite{8}. Whatever will be the resolution, the first step is going to be the differential expansion for twist knots – and it is still not fully known for non-rectangular $R$. It is the goal of the present paper to suggest a mixture of the results of \cite{7} and \cite{1, 2} to advance in this direction.

We avoid repeating the whole story and refer to \cite{2} for the latest summary and references. The crucial facts are the observation of \cite{4} for the antiparallel double braid in Fig.1:

$$H_R^{(m,n)} = \sum_{X \subset R \otimes \overline{R}} Z_X^R \cdot F_X^{(m,n)} = \sum_{X \subset R \otimes \overline{R}} Z_X^R \cdot F_X^{(m)} F_X^{(n)}$$

(1)

and the second observation of \cite{1, 2}, that for rectangular $R$:

$$F_X^{(m)} = \sum_{Y \in X} B_{X}^{m+1}$$

(2)

where $B$ is a universal triangular "embedding" matrix with $Y \subset X \subset R \otimes \overline{R}$. In this paper we consider the possibility for (2) to hold also for non-rectangular $R$. We do not discuss what are the $m$-independent differential combinations $Z_X^R$, which is also a highly non-trivial story in this case, see \cite{7} and a number of preceding papers, cited therein. This $Z$-story actually belongs to the theory of a single figure-eight knot, $4_1$ and is well separated from the problem of $m$-dependence, which we address now – though both are equally relevant for the next step towards Racah matrices.
Representations $X$ and $Y$ in (1) are composite, see Fig.2. For rectangular representations $R = [r^s] = [r_i, \ldots, r_s]$ only very special diagonal composites $\{(\lambda, \mu)\}$ contribute to $R \otimes \bar{R} \subset \mathcal{R}$ and they are in one-to-one correspondence with the Young sub-diagrams of $\lambda \subset R$, and ”embedding” for diagonal composites is understood as embedding of the corresponding $\lambda$:

$$ (\mu, \mu) \prec (\lambda, \lambda) \iff \mu \subset \lambda \tag{3} $$

The entries of the matrix $\mathcal{B}$ in (2) are expressed through the skew Schur functions:

$$ \mathcal{B}_{\lambda\mu} = \Lambda_{\mu} \cdot \frac{\chi_{\mu}^o \cdot \chi_{\lambda^\vee}^o / \mu^\vee}{\chi_{\lambda}^o} \tag{4} $$

where $\vee$ stands for transposition of the Young diagram, and $\Lambda_{\mu}$ are the eigenvalues of $\mathcal{R}$-matrix in the channel

$$ R \otimes \bar{R} = \oplus_{\mu \in R} (\mu, \mu) \tag{5} $$

best expressed through the hook parameters of $\lambda = (a_1, b_1[a_2, b_2], \ldots)$:

$$ \Lambda_{\lambda} = \prod_i \{q^{a_i-b_i} A^{2(a_i+b_i+1)}\} \tag{6} $$

Index $\circ$ means that Schur functions are evaluated at the ”unit” locus in the space of time-variables,

$$ \chi_{\lambda}^o = \chi_{\lambda} \{p_k = \frac{(q^k-g^k)}{q^k-g^k}\} \tag{7} $$

At $q = 1$ this is equivalent to putting $p_k = \delta_{k,1}$, and there is even a a special notation for the result: $\chi_{\lambda} \{\delta_{k,1}\} = a_{\lambda}$. The value of skew Schur at the unit locus at $q = 1$ can be also expressed through shifted Schur functions $\bar{\chi}_{\mu} \{\bar{p}^{\lambda}\}$ evaluated at

$$ p_k = \bar{p}_k^{\lambda} = \sum_{i=1}^{l_{\lambda}} (a_i - i)^k - (-i)^k \tag{9} $$

where $a_i$ denotes the lengths of $l_{\lambda}$ lines of the Young diagram $\lambda$. According to this definition, the shifted $\bar{\chi}_{\mu} \{\bar{p}^{\lambda}\}$ vanishes at the $\lambda$-locus whenever $\mu$ is not a sub-diagram of $\lambda$. Since shifted Macdonald functions can be defined in just the same way as Schurs $\bar{\chi}_{\lambda} \{\bar{p}^{\lambda}\}$ can be immediately used to define a ”refined” matrix $\mathcal{B}$ and thus, through (2), the hyper-polynomials (by definition of $\mathcal{R}$) they are result of a clever substitution of Schur by Macdonald functions in HOMPLY-PT polynomials, see also [18] and [21]. It was demonstrated in [1] that they are indeed positive Laurent polynomials, presumably in all rectangular representations and for all double twist knots.

For non-rectangular $R$ expressions for $Z_X$ and $F_X$ become somewhat complicated, and one can expect that expression (2) of $F_X$ through an auxiliary(?) matrix $\mathcal{B}$ once again leads to drastic simplification. As we will see, this is indeed the case. Note that of the three properties

$$ F_{X}^{m=-1} = 1, \quad F_{X}^{m=0} = 0, \quad F_{X}^{m=1} = \prod_i (-q^{a_i-b_i} A^{2a_i+b_i+1}), \tag{10} $$

for the figure-eight knot, unknot and trefoil respectively, the first one is automatic in (2), the second one requires that sum of the entries is zero along each line of $\mathcal{B}$, and only the third one still remains a non-trivial constraint.

In this paper we consider the simplest case of $R = [r,1]$, for which the answers are already known from [7]. In this case in addition to the $2r+1$ diagrams $X = (\lambda,\lambda)$ with $\lambda \subset R = [r,1]$, i.e. $\lambda = 0, [i], [i,1], i = 1, \ldots, r$ there are $r-1$ additional composite pairs $\tilde{X}_i = ([i-1,1], [i]) \oplus ([i],[i-1,1])$ with the same dimensions and eigenvalues

$$ \tilde{\lambda}_i = (q^{i-2} A^{2i})^{2i}, \quad i = 2, \ldots, r \tag{11} $$

each contributing once to the differential expansion. These $\tilde{X}_i$ contribute $r-1$ additional lines to the matrix $\mathcal{B}$, which thus becomes of the size $2r+1 + r - 1 = 3r$. Remarkably, $\mathcal{B}$ remains triangular, though a notion of embedding for generic composites $X$ gets somewhat more subtle than [5]. The first $2r+1$ lines remain as they were in (4). The new entries in the new $r-1$ lines $\tilde{X}_i$ with $i = 2, \ldots, r$ are:
\[ \mathcal{B}_{\tilde{X}_i,0} = \frac{(-i)^{i+1} A_i}{q^{i+1-\lambda_i}}, \ A^2 \]
\[ \mathcal{B}_{\tilde{X}_i,[j]} = \frac{(-i)^{i+1} A_i}{q^{i+1-\lambda_i}}, \frac{[i-2]!}{[j-2]!}, \frac{[i-1]q^{i+j-1} A^2 - [j-1]q^{-3} A^2 - [j-1]}{q^{j-1}} \quad j=1,\ldots,i \]
\[ \mathcal{B}_{\tilde{X}_i,[j],1} = \frac{(-i)^{i+1} A_i}{q^{i-j-2+i+j}}, \frac{[i-2]!}{[j-1]!}, \frac{(A^2 - q^6)(A^2 - q^6)}{q^{j+2-1}(A^2 q^{j-1}-1)} \quad j=1,\ldots,i-1 \]
\[ \mathcal{B}_{\tilde{X}_i,\tilde{X}_j} = \frac{(-i)^{i+1} A_i}{q^{i+1-\lambda_i}}, \frac{[i-2]!}{[j-2]!}, \frac{A^2 q^{3i-4} - 1}{A^2 q^{j-1}} \quad j=2,\ldots,i \]

In particular,
\[ \begin{align*}
\mathcal{B}_{\tilde{X}_i,1} &= (-i)^{i+1} A_i, \frac{[i+1] A^2}{q^{i-1}} \\
\mathcal{B}_{\tilde{X}_i,[1]} &= (-i)^{i+1} A_i, \frac{A^2 - q^6}{q^{i+1+i}} \\
\mathcal{B}_{\tilde{X}_i,1} &= (-i)^{i+1} A_i, \frac{A^2 q^{3i-4} - 1}{q^{j-1}} \\
\mathcal{B}_{\tilde{X}_i,1} &= 0 \quad (12) 
\end{align*} \]

As we see, these entries essentially depend on \( A \), and are therefore sensitive to characters (or something else) beyond the unit locus \( (7) \).

In the simplest case of \( R = [2,1] \) the matrix is
\[
\mathcal{B}^{[2,1]} = \begin{pmatrix}
\emptyset & [1] & [1,1] & [2] & [2,1] & \tilde{X}_2 \\
\emptyset & 1 & 0 & 0 & 0 & 0 \\
[1] & -A^2 & A^2 & 0 & 0 & 0 \\
[1,1] & \frac{A^4}{q^7} & -[2] A^2 & \frac{A^4}{q^7} & 0 & 0 \\
[2] & q^2 A^4 & -[2] q^2 A^4 & 0 & q^4 A^4 & 0 \\
[2,1] & -A^6 & [3] A^6 & -\frac{[3] A^6}{[2]} & -\frac{[3] q^6}{[2]} & A^6 \\
\tilde{X}_2 & -A^6 & [3] A^6 & \frac{(A^2 - q^6) A^4}{q^7(q^2-1)} & -\frac{(A^2 q^6 - 1) A^4}{q^{7-1}} & 0 \\
\end{pmatrix} \quad (13)
\]

The new one – revealed by consideration of the non-rectangular \( R \) – is the last line.

For \( R = [3,1] \) the line \( \tilde{X}_2 \) remains the same – this is the universality property of \( \mathcal{B} \) – and there is one more new, as compared to \( [4] \), line for \( \tilde{X}_3 \):
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\emptyset & [1] & [1,1] & [2] & [2,1] & \tilde{X}_2 \\
\emptyset & 1 & 0 & 0 & 0 & 0 \\
[1] & -A^2 & A^2 & 0 & 0 & 0 \\
[1,1] & \frac{A^4}{q^7} & -[2] A^2 & \frac{A^4}{q^7} & 0 & 0 \\
[2] & q^2 A^4 & -[2] q^2 A^4 & 0 & q^4 A^4 & 0 \\
[2,1] & -A^6 & [3] A^6 & -\frac{[3] A^6}{[2]} & -\frac{[3] q^6}{[2]} & A^6 \\
\tilde{X}_2 & -A^6 & [3] A^6 & \frac{(A^2 - q^6) A^4}{q^7(q^2-1)} & -\frac{(A^2 q^6 - 1) A^4}{q^{7-1}} & 0 \\
\tilde{X}_3 & q^4 A^8 & -[4] q^9 A^8 & \frac{A^2 - q^6}{q^{2-1}} & \frac{A^2(q^{10} + q^{8} - 1)}{q^{9-1}} & \frac{A^2 q^{10} - 1}{q^{7-1}} & 0 \\
\end{array}
\]

One can compare with the original formulas for \( F_X \) in \( [7] \) to appreciate the simplification.

For generic \( R \) we should consider all composite \( X = \oplus_{[\lambda]} [\lambda'](\lambda,\lambda') \) with all pairs of the same-size sub-diagrams of \( R \): \( \lambda, \lambda' \subset R \), \( |\lambda| = |\lambda'| \). For example, for \( R = [3,2] \) there will be three non-diagonal \((\lambda' \neq \lambda)\) pairs: the two already familiar \([2], [1,1] \oplus [1,1], [2] \), \([3], [2,1] \oplus [2,1], [3] \) and a new one: \([3,1], [2,2] \oplus [2,2], [3,1] \). For the psychologically important \( R = [4,2] \) we encounter the first triple \([4], [3,1], [2,2] \), with \( X = \oplus_{[4], [3,1]} [2,2] \oplus ([3,1], [4]) \oplus ([3,1], [2,2] \oplus ([2,2], [4]) \oplus ([2,2], [3,1]) \). Formulas \( [12] \) should be straightforwardly extendable to this general case – but it remains to be done, and it remains to be seen if triangular shape of \( B \) persists. Hopefully straightforwardly are also their Macdonald deformations – and it is interesting to see if this leads to hyper-polynomials, but not fully positive – as currently suspected for non-rectangular representations.
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Figure 1: Antiparallel double braid and the two representations of associated HOMFLY-PT polynomial: aboresent one through exclusive Racah matrix $\tilde{S}$ from [19] and the factorized differential expansion from [4].

$$\mathcal{H}_R^{(m,n)} = \sum_{Y' \subset R \otimes R} \sqrt{\frac{D_Y D_{Y'}}{D_{R(N)}}} \tilde{S}^{R}_{Y'} \Lambda^m_Y \Lambda^n_{Y'} = \sum_{X \subset R \otimes \bar{R}} Z^X_{R} \cdot F_X^{(m,n)} = \sum_{X \subset R \otimes \bar{R}} Z^X_{R} \cdot F_X^{(m)} F_X^{(n)}$$

$$F_X^{(m)} = \sum_{Y \subset X} f_{XY} \cdot \Lambda^m_Y$$

Figure 2: Composite representation of $Sl_N$, described by the $N$-dependent Young diagram

$$(\lambda, \mu) = \left[ \lambda_1 + \mu_1, \ldots, \lambda_{l_\lambda} + \mu_1, \mu_1, \ldots, \mu_{l_\mu}, \mu_1 - \mu_1, \ldots, \mu_1 - \mu_2 \right]$$