Nested Sequents for Intuitionistic Grammar Logics via Structural Refinement*

Tim S. Lyon[0000–0003–3214–0828]
Computational Logic Group, Institute of Artificial Intelligence, Technische Universität Dresden, Germany
timothy_stephen.lyon@tu-dresden.de

Abstract. Intuitionistic grammar logics fuse constructive and multi-modal reasoning while permitting the use of converse modalities, serving as a generalization of standard intuitionistic modal logics. In this paper, we provide definitions of these logics as well as establish a suitable proof theory thereof. In particular, we show how to apply the structural refinement methodology to extract cut-free nested sequent calculi for intuitionistic grammar logics from their semantics. This method proceeds by first transforming the semantics of these logics into sound and complete labeled sequent systems, which we prove have favorable proof-theoretic properties such as syntactic cut-elimination. We then transform these labeled systems into nested sequent systems via the introduction of propagation rules and the elimination of structural rules. Our derived proof systems are then put to use, whereby we prove the conservativity of intuitionistic grammar logics over their modal counterparts, establish the general undecidability of these logics, and recognize a decidable subclass, referred to as simple intuitionistic grammar logics.

Keywords: Bi-relational model · Conservativity · Decidability · Grammar logic · Intuitionistic logic · Labeled sequent · Modal logic · Nested sequent · Proof search · Proof theory · Propagation rule · Structural refinement

1 Introduction

Grammar logics form a prominent class of normal, multi-modal logics extending classical propositional logic with a set of modalities indexed by characters from a given alphabet. These logics obtain their name on the basis of their relationship to context-free grammars. In particular, grammar logics incorporate axioms which may be viewed as production rules in a context-free grammar, and which generate sequences of edges indexed with characters (and thus may be viewed as words) in corresponding relational models. Due to the generality of this class of logics, it has been found that this class includes many well-known

* Work supported by the European Research Council (ERC) Consolidator Grant 771779 (DeciGUT).
and useful logics such as description logics [29], epistemic logics [17], information logics [51], temporal logics [8], and standard modal logics (e.g. K, S4, and S5) [14]. Despite the various modes of reasoning offered within the class of grammar logics, such logics are nevertheless classical at their core, being defined atop classical propositional logic. Thus, it is interesting to place such logics on an intuitionistic, rather than a classical, footing.

Intuitionistic logic serves as one of the most eminent formulations of constructive reasoning, that is, reasoning where the claimed existence of an object implies its constructibility [3]. Resting on the philosophical work of L.E.J. Brouwer, propositional intuitionistic logic was axiomatized in the early 20th century by Kolmogorov [31], Orlov [42], and Glivenko [25], with a first-order axiomatization given by Heyting [28]. Rather naturally, as the paradigm of intuitionistic reasoning evolved, it was eventually integrated with the paradigm of modal reasoning, begetting so-called intuitionistic modal logics.

A plethora of intuitionistic modal logics have been proposed in the literature [1,2,15,19,44,46,47], though the class of logics introduced by Plotkin and Stirling [44] has become (most notably through the work of Simpson [47]) one of the most popular formulations. In the same year that Plotkin and Stirling [44] introduced their intuitionistic modal logics, Ewald introduced intuitionistic tense logic [16], which not only includes modalities that make reference to future states in a relational model, but also includes modalities that make reference to past states. As with (multi-)modal and intuitionistic logics, intuitionistic modal logics have proven useful in computer science; e.g. such logics have been used to design verification techniques [18], in reasoning about functional programs [43], and in the definition of programming languages [13]. Continuing in the same vein, the first contribution of this paper is to generalize both the intuitionistic modal logics of Plotkin and Stirling [44] and Ewald’s intuitionistic tense logic, giving rise to the class of intuitionistic grammar logics. We provide a semantics for this class of logics, along with axiomatizations, and confirm soundness and completeness thereof.

Beyond defining the class of intuitionistic grammar logics, we also provide a suitable proof theory for this class of logics. As with any logic, proof calculi are indispensable for facilitating reasoning and prove valuable in establishing non-trivial properties of logics. A prominent formalism that has arisen within the last 30 or so years for formalizing reasoning for modal and intuitionistic logics is the nested sequent formalism. Initiated by Bull [5] and Kashima [30], nested sequent systems perform reasoning over trees of (pairs of) multisets of formulae, proving worthwhile in developing automated reasoning techniques for logics. Such systems have been used to write decision algorithms for logics supporting automated counter-model extraction [22,49], have been employed in constructive proofs of interpolation [21,55], and have even been applied in knowledge integration scenarios [39].

The value of nested sequent systems comes from the fact that such systems normally exhibit useful proof-theoretic properties. For instance, the well-known cut rule (cf. [23,24]), which serves as a generalization of modus ponens, and is
frequently employed in proofs of completeness, tends to be admissible (i.e. redundant) in nested sequent calculi. As the cut rule deletes formulae from the premises to the conclusion when applied, the rule is unsuitable for bottom-up proof-search as the deleted formula must be guessed during proof-search, which can obstruct proofs of termination. Since cut is not required to appear in most nested sequent calculi, such calculi tend to be analytic, i.e. one can observe that any formula occurring in a proof occurs as a subformula of the conclusion, which bounds the space of possible proofs of a given formula, proving beneficial in establishing decidability [4,34]. Still, nested sequent systems possess favorable properties beyond analyticity. For instance, such systems reason within the economical structure of trees, easing proofs of termination of associated proof-search algorithms, such systems permit the admissibility of useful structural rules (e.g. contractions and weakenings), and such systems tend to have invertible rules, which is helpful in extracting counter-models from failed proof-search.

Recently, the structural refinement methodology was developed as a means of generating nested sequent systems for diverse classes of logics [33]. The methodology exploits the formalism of labeled sequents whereby calculi are obtained by transforming the semantics of a logic into inference rules [47,52]. As such, labeled sequent systems perform reasoning within the semantics of a logic, and reason over structures closely resembling a logic’s models, with sequents encoding arbitrary graphs of (pairs of) multisets of formulae (thus generalizing the data structure used in nested sequents). A nice feature of labeled sequent systems is that general results exist for their construction [12,47] and such systems tend to exhibit desirable properties such as cut-elimination, admissibility of certain structural rules, and invertibility of rules [41]. However, labeled sequent systems have a variety of drawbacks as such systems typically involve superfluous structures in sequents, yielding larger proofs than necessary, making proof-search algorithms less efficient, and obfuscating proofs of termination of associated proof-search algorithms (cf. [33]).

To circumvent the drawbacks of labeled sequent systems, the structural refinement methodology (in a broad sense) leverages the general construction techniques in the labeled setting to extract labeled systems from a logic’s semantics, and then systematically transforms these systems into nested systems, which are more economical and better suited to applied scenarios. In a narrow sense, structural refinement consists of transforming a labeled sequent system into a nested system through the introduction of propagation rules (cf. [7,20]) or reachability rules [33,38] and the elimination of structural rules, followed by a notational translation. The propagation rules operate by viewing labeled sequents (which encode binary labeled graphs) as automata, allowing for formulæ to be propagated along a path in the underlying graph of a labeled sequent, so long as the path is encoded by a string derivable in a certain formal grammar. The refinement methodology grew out of works relating labeled systems to ‘more refined’ or nested systems [10,27,34]. The propagation rules we use are largely based upon the work of [26,39], where such rules were used in the setting of display and nested calculi. These rules were then transported to the labeled setting to prove
the decidability of agency (STIT) logics \[34\], to establish translations between calculi within various proof-theoretic formalisms \[11\], and to provide a basis for the structural refinement methodology \[32\]. In this paper, we apply this methodology in the setting of intuitionistic grammar logics, obtaining analytic nested systems for these logics, which are then put to use to establish conservativity and (un)decidability results.

This paper accomplishes the following: In Section 2, we define intuitionistic grammar logics, providing a semantics, axiomatizations, and confirming soundness and completeness results. We also introduce the grammar-theoretic foundations necessary to define propagation rules. In Section 3 we define labeled sequent calculi for intuitionistic grammar logics (which generalize Simpson’s labeled systems for intuitionistic modal logics characterized by Horn properties \[47\]) and provide admissibility, invertibility, and cut-elimination results, which also establishes syntactic cut-elimination for the labeled systems of Simpson mentioned above. In Sections 4 and 5 we demonstrate how to apply the structural refinement method to extract ‘refined’ labeled and analytic nested sequent systems for intuitionistic grammar logics. In Section 6 we leverage our nested and refined labeled systems to show three results: (1) Conservativity: we prove that intuitionistic grammar logics are conservative over intuitionistic modal logics, (2) Undecidability: we prove the undecidability of determining if a formula is a theorem of an arbitrary intuitionistic grammar logic by giving a proof-theoretic reduction of the problem from classical grammar logics, and (3) Decidability: we adapt a method due to Simpson \[47\] to our setting, recognizing that a subclass of intuitionistic grammar logics, referred to \textit{simple}, is decidable. Last, Section 7 concludes and discusses future work.

This paper serves as a journal version extending the conference papers \[36\] and \[37\]. The first conference paper \[36\] introduces intuitionistic grammar logics and provides sound and complete axiomatizations. The second conference paper \[37\] shows how to derive nested sequent systems for intuitionistic modal logics with seriality and Horn-Scott-Lemmon axioms by means of structural refinement, solving an open problem in \[40\].

2 Preliminaries

2.1 Intuitionistic Grammar Logics

The language of each intuitionistic grammar logic is defined relative to an alphabet \(\Sigma\), which is a non-empty countable set of characters, used to index modalities. As in \[14\], we stipulate that each alphabet \(\Sigma\) can be partitioned into a \textit{forward part} \(\Sigma^+ := \{a, b, c, \ldots\}\) and a \textit{backward part} \(\Sigma^- := \{\bar{a}, \bar{b}, \bar{c}, \ldots\}\) where the following is satisfied:

\[\Sigma := \Sigma^+ \cup \Sigma^- \text{ where } \Sigma^+ \cap \Sigma^- = \emptyset \text{ and } a \in \Sigma^+ \iff \bar{a} \in \Sigma^-.
\]

\(\Sigma^+\) contains \textit{forward characters}, which we denote by \(a, b, c, \ldots\) (possibly annotated), and \(\Sigma^-\) contains \textit{backward characters}, which we denote by \(\bar{a}, \bar{b}, \bar{c}, \ldots\).
(possibly annotated). A character is defined to be either a forward or backward character, and we use \( x, y, z, \ldots \) (possibly annotated) to denote them. In what follows, modalities indexed with forward characters will be interpreted as making reference to future states along the accessibility relation within a relational model, and modalities indexed with backward characters will make reference to past states. We define the converse operation to be a function \( \tau \) mapping each forward character \( a \in \Sigma^+ \) to its converse \( \overline{a} \in \Sigma^- \) and vice versa; hence, the converse operation is its own inverse, i.e. for any \( x \in \Sigma \), \( x = \overline{\overline{x}} \).

We let \( \Phi := \{ p, q, r, \ldots \} \) be a denumerable set of propositional atoms and define the language \( L(\Sigma) \) relative to a given alphabet \( \Sigma \) via the following grammar in BNF:

\[
A ::= p \mid \bot \mid A \lor A \mid A \land A \mid A \supset A \mid \langle x \rangle A \mid \lbrack x \rbrack A
\]

where \( p \) ranges over the set \( \Phi \) of propositional atoms and \( x \) ranges over the characters in the alphabet \( \Sigma \). We use \( A, B, C, \ldots \) (possibly annotated) to denote formulae in \( L(\Sigma) \) and define \( \neg A := A \supset \bot \). Formulae are interpreted over bi-relational \( \Sigma \)-models [36], which are inspired by the models for intuitionistic modal and tense logics presented in [2,15,16,44].

**Definition 1 (Bi-relational \( \Sigma \)-Model [36]).** We define a bi-relational \( \Sigma \)-model to be a tuple \( M = (W, \leq, \{ R_x \mid x \in \Sigma \}, V) \) such that:

- \( W \) is a non-empty set of worlds \( \{ w, u, v, \ldots \} \);
- The intuitionistic relation \( \leq \subseteq W \times W \) is a preorder, i.e. it is reflexive and transitive;
- The accessibility relation \( R_x \subseteq W \times W \) satisfies:
  - \( (F1) \) For all \( w, v, v' \in W \), if \( wR_x v \) and \( v \leq v' \), then there exists a \( w' \in W \) such that \( w \leq w' \) and \( w' R_x v' \);
  - \( (F2) \) For all \( w, w', v \in W \), if \( w \leq w' \) and \( wR_x v \), then there exists a \( v' \in W \) such that \( w' R_x v' \) and \( v \leq v' \);
  - \( (F3) \) \( wR_x u \) iff \( uR_{\overline{x}} w \);
- \( V : W \to 2^\Phi \) is a valuation function satisfying the monotonicity condition: for each \( w, u \in W \), if \( w \leq u \), then \( V(w) \subseteq V(u) \).

The (F1) and (F2) conditions are depicted in Figure 2.1 and ensure the monotonicity of complex formulae (see Lemma 1) in our models, which is a property

![Fig. 1. Depictions of the (F1) and (F2) conditions imposed on bi-relational \( \Sigma \)-models. Dotted arrows indicate the relations implied by the presence of the solid arrows.](image-url)
characteristic of intuitionistic logics\footnote{For a discussion of these conditions and related literature, see \cite{ch3}.} If an accessibility relation $R_a$ is indexed with a forward character, then we interpret it as a relation to \textit{future} worlds, and if an accessibility relation $R_\ast$ is indexed with a backward character, then we interpret it as a relation to \textit{past} worlds. Hence, our formulae and related models have a tense character, showing that our logics generalize the intuitionistic tense logics of Ewald \cite{16}.

We interpret formulae from $\mathcal{L}(\Sigma)$ over bi-relational models via the following clauses.

**Definition 2 (Semantic Clauses \cite{36}).** Let $M$ be a bi-relational $\Sigma$-model with $w \in W$. The satisfaction relation $M, w \vDash A$ between $w \in W$ of $M$ and a formula $A \in \mathcal{L}(\Sigma)$ is inductively defined as follows:

- $M, w \vDash \top$;
- $M, w \nvDash \bot$;
- $M, w \vDash A \lor B$ iff $M, w \vDash A$ or $M, w \vDash B$;
- $M, w \vDash A \land B$ iff $M, w \vDash A$ and $M, w \vDash B$;
- $M, w \vDash A \supset B$ iff for all $w' \in W$, if $w \leq w'$ and $M, w' \vDash A$, then $M, w' \vDash B$;
- $M, w \vDash (\langle x \rangle A)$ iff there exists a $v \in W$ such that $w \vDash R_x v$ and $M, v \vDash \Sigma A$;
- $M, w \vDash [x]A$ iff for all $w', v' \in W$, if $w \leq w'$ and $w'R_x v'$, then $M, v' \vDash \Sigma A$.

**Lemma 1 (Persistence).** Let $M$ be a bi-relational $\Sigma$-model with $w, u \in W$ of $M$. If $w \leq u$ and $M, w \vDash \Sigma A$, then $M, u \vDash \Sigma A$.

*Proof.* By induction on the complexity of $A$. \hfill \Box

Given an alphabet $\Sigma$, the set of formulae valid with respect to the class of bi-relational $\Sigma$-models is axiomatizable \cite{36}. We refer to the axiomatization as $\text{HIK}_\Sigma$ (with H denoting the fact that the axiomatization is a \textit{Hilbert calculus}), and call the corresponding logic that it generates $\text{IK}_\Sigma$.

**Definition 3 (Axiomatization).** We define our axiomatization $\text{HIK}_\Sigma$ below, where we have an axiom and inference rule for each $x \in \Sigma$.

\begin{align*}
A0 & \text{ Any axiomatization for intuitionistic propositional logic} \\
A1 & [x](A \supset B) \supset ([x]A \supset [x]B) \\
A2 & [x](A \land B) \equiv ([x]A \land [x]B) \\
A3 & \langle x \rangle (A \lor B) \equiv \langle x \rangle A \lor \langle x \rangle B \\
A4 & [x](A \supset B) \supset \langle x \rangle [A \supset \langle x \rangle B] \\
A5 & [x]A \land \langle x \rangle B \supset \langle x \rangle (A \land B) \\
A6 & \neg(x) \bot \\
A7 & (A \supset [x](\overline{\tau})A) \land (\langle x \rangle \overline{\tau} A \supset A) \\
A8 & ([x]A \supset [x]B) \supset [x](A \supset B) \\
A9 & (\langle x \rangle A \supset B) \supset ([x]A \supset (\langle x \rangle B) \\
R1 & \frac{\frac{A}{[x]A}}{[x]A} (n)
\end{align*}

We define the logic $\text{IK}_\Sigma$ to be the smallest set of formulae from $\mathcal{L}(\Sigma)$ closed under substitutions of the axioms and applications of the inference rules. A formula $A$ is defined to be a theorem of $\text{IK}_\Sigma$ iff $A \in \text{IK}_\Sigma$. 
Fig. 2. Axioms and their related frame conditions. We note that when \( n = 0 \), the related frame condition is taken to be \( wR_a w \).

We note that if we let \( \Sigma := \{ a, \pi \} \), then the resulting logic is a notational variant of Ewald’s intuitionistic tense logic \( \text{IKt} \) [10], which is a conservative extension of the mono-modal intuitionistic modal logic \( \text{IK} \) [14]. In our setting, we let \( \text{IK}_m(\Sigma) \) be the base intuitionistic grammar logic relative to \( \Sigma \), and consider extensions of \( \text{IK}_m(\Sigma) \) with sets \( A \) of the following axioms.

\[
\begin{align*}
D_x : & \quad [x]A \supset (x)A \\
\text{IPA : } & \quad ((x_1) \cdots (x_n)A \supset (x)A) \land ([x]A \supset [x_1] \cdots [x_n]A)
\end{align*}
\]

We refer to axioms of the form shown above left as seriality axioms, and axioms of the form shown above right as intuitionistic path axioms (IPAs). We use \( A \) to denote any arbitrary collection of axioms of the above forms. Moreover, we note that the collection of IPAs subsumes the class of Horn-Scott-Lemmon axioms [27] and includes multi-modal and intuitionistic variants of standard axioms such as \( T_x, B_x, 4_x, \) and \( 5_x \).

\[
\begin{align*}
T_x : & \quad (A \supset (x)A) \land ([x]A \supset A) \\
B_x : & \quad (\pi)A \supset (x)A \land ([x]A \supset [\pi]A) \\
4_x : & \quad ((x)(x)A \supset (x)A) \land ([x]A \supset [x][x]A) \\
5_x : & \quad ([\pi](x)A \supset (x)A) \land ([x]A \supset [\pi][x]A)
\end{align*}
\]

It was proven that any extension of \( \text{HIK}_m(\Sigma) \) with a set \( A \) of axioms is sound and complete relative to a sub-class of the bi-relational \( \Sigma \)-models satisfying frame conditions related to each axiom [30]. Axioms and related frame conditions are displayed in Figure 2 and extensions of \( \text{HIK}_m(\Sigma) \) with seriality and IPA axioms, along with their corresponding models, are defined below.

**Definition 4 (Syntactic Notions for Extensions).** We define the axiomatization \( \text{HIK}_m(\Sigma, A) \) to be \( \text{HIK}_m(\Sigma) \cup A \), and define the logic \( \text{IK}_m(\Sigma, A) \) to be the smallest set of formulae from \( L(\Sigma) \) closed under substitutions of the axioms and applications of the inference rules. A formula \( A \) is defined to be an \( \text{IK}_m(\Sigma, A) \)-theorem, written \( \vdash_A^\Sigma A \), iff \( A \in \text{IK}_m(\Sigma, A) \), and a formula \( A \) is said to be derivable from a set of formulae \( \mathcal{A} \subseteq L(\Sigma) \), written \( \mathcal{A} \vdash_A^\Sigma A \), iff for some \( B_1, \ldots, B_n \in \mathcal{A} \), \( \vdash_A^\Sigma B_1 \land \cdots \land B_n \supset A \).

**Definition 5 (Semantic Notions for Extensions).** We define a bi-relational \( (\Sigma, A) \)-model to be a bi-relational \( \Sigma \)-model satisfying each frame condition related to an axiom \( A \in A \). A formula \( A \) is defined to be globally true on a bi-relational \( (\Sigma, A) \)-model \( M \), written \( M \models^\Sigma A \), iff \( M, u \models^\Sigma A \) for all worlds \( u \in W \) of \( M \). A formula \( A \) is defined to be \( (\Sigma, A) \)-valid, written \( \models^\Sigma_A A \), iff \( A \) is globally true on every bi-relational \( (\Sigma, A) \)-model. Last, we say that a set \( \mathcal{A} \) of formulae semantically implies a formula \( A \), written \( \mathcal{A} \models^\Sigma_A A \), iff for all bi-relational \( (\Sigma, A) \)-models \( M \) and each \( w \in W \) of \( M \), if \( M, w \models^\Sigma B \) for each \( B \in \mathcal{A} \), then \( M, w \models^\Sigma A \).
Remark 1. Note that the axiomatization \( \text{HIK}_m(\Sigma) = \text{HIK}_m(\Sigma, \emptyset) \) and that a bi-relational \((\Sigma, \emptyset)\)-model is a bi-relational \(\Sigma\)-model.

One can prove strong soundness for each intuitionistic grammar logic (i.e. for any \( \mathcal{A} \subseteq \mathcal{L}(\Sigma) \) with \( A \in \mathcal{L}(\Sigma) \), if \( \mathcal{A} \vdash A \), then \( \mathcal{A} \models A \)) by showing that each axiom in \( \text{HIK}_m(\Sigma, \mathcal{A}) \) is \((\Sigma, \mathcal{A})\)-valid and that each inference rule preserves \(\Sigma, \mathcal{A}\)-validity. The converse of this result, strong completeness, is shown by means of a typical canonical model construction. Proofs of both facts can be found in Lyon [36].

Theorem 1 (Soundness and Completeness [36]). \( \mathcal{A} \vdash A \) iff \( \mathcal{A} \models A \).

2.2 Formal Grammars and Languages

A central component to the structural refinement methodology—i.e. the extraction of nested calculi from labeled—is the use of inference rules (viz. reachability rules) whose applicability depends on strings generated by formal grammars [33]. We therefore introduce grammar-theoretic notions that are essential to the functionality of such rules, and formally define such rules in Section 4.

Definition 6 (\(\Sigma^*\)). We let \(\cdot\) be the concatenation operation with \(\varepsilon\) the empty string. We define the set \(\Sigma^*\) of strings over \(\Sigma\) to be the smallest set such that:

\[
\begin{align*}
\Sigma \cup \{\varepsilon\} & \subseteq \Sigma^* \\
\text{If } s \in \Sigma^* \text{ and } x \in \Sigma, \text{ then } s \cdot x & \in \Sigma^*
\end{align*}
\]

For a set \(\Sigma^*\) of strings, we use \(s, t, r, \ldots\) (potentially annotated) to represent strings in \(\Sigma^*\). As usual, the empty string \(\varepsilon\) is taken to be the identity element for the concatenation operation, i.e. \(s \cdot \varepsilon = \varepsilon \cdot s = s\) for \(s \in \Sigma^*\). Furthermore, we will not usually mention the concatenation operation in practice and will let \(st := s \cdot t\), that is, we denote concatenation by simply gluing two strings together. Beyond concatenation, another useful operation to define on strings is the converse operation.

Definition 7 (String Converse [14]). We extend the converse operation to strings as follows:

\[
\begin{align*}
\varepsilon & := \varepsilon; \\
\text{If } s = x_1 \cdots x_n, \text{ then } s & := x_n \cdots x_1.
\end{align*}
\]

We define strings of modalities accordingly: if \(s = a_0a_1 \cdots a_n\), then \([s] = [a_0][a_1] \cdots [a_n]\) and \(\langle s \rangle = \langle a_0 \rangle \langle a_1 \rangle \cdots \langle a_n \rangle\), with \([s]\phi = \langle s \rangle \phi = \phi \) when \(s = \varepsilon\). Hence, every IPA may be written in the form \((\langle s \rangle \mathcal{A} \supset \langle x \rangle \mathcal{A}) \land ([x] \mathcal{A} \supset [s] \mathcal{A})\), where \(\langle s \rangle = \langle x_1 \rangle \cdots \langle x_n \rangle\) and \([s] = [x_1] \cdots [x_n]\). We make use of this notation to compactly define \(\mathcal{A}\)-grammars, which are types of Semi-Thue systems [45], encoding information contained in a set \(\mathcal{A}\) of axioms, and employed later on in the definition of our reachability rules.

Definition 8 (\(\mathcal{A}\)-grammar [37]). An \(\mathcal{A}\)-grammar is a set \(g(\mathcal{A})\) such that:
An $A$-grammar $g(A)$ is a type of string re-writing system. For example, if \( x \rightarrow s \in g(A) \), we may derive the string $tsr$ from $txr$ in one-step by applying the mentioned production rule. Through repeated applications of production rules to a given string $s \in \Sigma^*$, one derives new strings, the collection of which, determines a language. Let us make such notions precise by means of the following definition:

**Definition 9 (Derivation, Language [37]).** Let $g(A)$ be an $A$-grammar. The one-step derivation relation $\rightarrow_{g(A)}^*$ holds between two strings $s$ and $t$ in $\Sigma^*$, written $s \rightarrow_{g(A)} t$, iff there exist $s', t' \in \Sigma^*$ and $x \rightarrow r \in g(A)$ such that $s = s'xt'$ and $t = s'rt'$. The derivation relation $\rightarrow_{g(A)}^*$ is defined to be the reflexive and transitive closure of $\rightarrow_{g(A)}$. For two strings $s, t \in \Sigma^*$, we refer to $s \rightarrow_{g(A)}^* t$ as a derivation of $t$ from $s$, and define its length to be equal to the minimal number of one-step derivations needed to derive $t$ from $s$ in $g(A)$. Last, for a string $s \in \Sigma^*$, the language of $s$ relative to $g(A)$ is defined to be the set $\mathcal{L}_{g(A)}(s) := \{ t \mid s \rightarrow_{g(A)}^* t \}$.

### 3 Labeled Sequent Systems

We generalize Simpson’s labeled sequent systems for intuitionistic modal logics [47] to the multi-modal case with converse modalities. Our systems make use of *labels* (which we occasionally annotate) from a denumerable set $\text{Lab} := \{ w, u, v, \ldots \}$, as well as two distinct types of formulae: (i) *labeled formulae*, which are of the form $w : A$ with $w \in \text{Lab}$ and $A \in \mathcal{L}(\Sigma)$, and (ii) *relational atoms*, which are of the form $wR_1u$ for $w, u \in \text{Lab}$ and $x \in \Sigma$. We define a labeled sequent to be a formula of the form $\mathcal{R}, \Gamma \vdash w : A$, where $\mathcal{R}$ is a (potentially empty) multiset of relational atoms, and $\Gamma$ is a (potentially empty) multiset of labeled formulae. We will occasionally use $A$ and annotated versions thereof to denote labeled sequents, and we let $\text{Lab}(\mathcal{R})$, $\text{Lab}(\Gamma)$, and $\text{Lab}(A)$ be the set of labels occurring in a multiset of relational atoms $\mathcal{R}$, a multiset of labeled formulae $\Gamma$, and a labeled sequent $A$, respectively. For a string $s = x_1 \cdots x_n$, we let $wR_1u := wR_{x_1}w_1R_{x_2}w_2, \ldots, w_{n-1}R_{x_n}u$, and note that $wR_eu := (w = u)$. For a multiset $\Gamma$ of labeled formulae, we let $\Gamma \mid w$ be the multiset $\{ A \mid w : A \in \Gamma \}$, which is equal to the empty multiset $\emptyset$ when $w \not\in \text{Lab}(\Gamma)$.

For any alphabet $\Sigma$ and set $A$ of axioms, we obtain a calculus $\mathcal{L}_A(\sim)$, which is displayed in Figure 3. Such systems can be seen as intuitionistic variants of the labeled systems $\mathcal{G}3KM(S)$ for classical grammar logics [33], obtained by fixing a single labeled formula on the right-hand-side of the sequent arrow.

We classify the (id) and $(\perp_i)$ rules as *initial rules*, the $(\sim_e)$, $(\ast_i)$, and $(\ast_e)$ rules as *structural rules*, and the remaining rules in Figure 3 as *logical rules*. Initial rules serve as the axioms of our labeled systems and initiate proofs, structural rules operate on relational atoms, and logical rules construct complex logical formulae. The $(\sim_e)$, $(\langle x \rangle i)$, and $(\langle x \rangle_r)$ rules possess a side condition, which

\[(x \rightarrow s), (\tau \rightarrow \tau) \in g(A) \text{ if } ((s)A \supset (x)A) \land ((x)A \supset [s]A) \in A.\]
defined accordingly: \(\vdash\)

\[
\begin{align*}
\frac{\mathcal{R}, \Gamma; w : p \vdash w : p}{\mathcal{R}, \Gamma; w : \bot, \Gamma \vdash u : A} & \quad (id) \\
\frac{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C}{\mathcal{R}, \Gamma; w : A \land B \vdash u : C} & \quad (\land i) \\
\frac{\mathcal{R}, \Gamma; w : A \vdash u : A}{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C} & \quad (\land e) \\
\frac{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C}{\mathcal{R}, \Gamma; w : A \land B \vdash u : C} & \quad (\lor r) \\
\frac{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C}{\mathcal{R}, \Gamma; w : A \land B \vdash u : C} & \quad (\lor l) \\
\frac{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C}{\mathcal{R}, \Gamma; w : A \land B \vdash u : C} & \quad (\lor r) \\
\frac{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C}{\mathcal{R}, \Gamma; w : A \land B \vdash u : C} & \quad (\lor l) \\
\frac{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C}{\mathcal{R}, \Gamma; w : A \land B \vdash u : C} & \quad (\lor r) \\
\frac{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C}{\mathcal{R}, \Gamma; w : A \land B \vdash u : C} & \quad (\lor l) \\
\frac{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C}{\mathcal{R}, \Gamma; w : A \land B \vdash u : C} & \quad (\lor r) \\
\frac{\mathcal{R}, \Gamma; w : A \lor B \vdash u : C}{\mathcal{R}, \Gamma; w : A \land B \vdash u : C} & \quad (\lor l)
\end{align*}
\]

Fig. 3. The labeled calculi \(L_\Sigma(A)\). We have \((d_u)\) as a rule in the calculus, if \(D_u \in A\), and an \((i_x)\) rule in the calculus, for each IPA of the form \(((s)A \supset (x)A) \land ([x]A \supset [s]A) \in A\). Furthermore, we have a \((⟨x⟩)\), \((⟨x⟩)\), \(([x])\), \(([x])\), and \((c_u)\) rule for each \(x \in \Sigma\). The side condition \(\dagger\) states that the rule is applicable only if \(u\) is fresh.

stipulates that the rule is applicable only if the label \(u\) is fresh, i.e. if the rule applied, then the label \(u\) will not occur in the conclusion. A proof or derivation in \(L_\Sigma(A)\) is constructed in the usual fashion by successively applying logical or structural rules to initial rules, and the height of a proof is defined to be the longest sequence of sequents from the conclusion of the proof to an initial rule.

We point out that the \((d_u)\) and \((i_x)\) structural rules form a proper subclass of Simpson’s geometric structural rules (see [47] p. 126) used to generate labeled sequent systems for \(\mathbf{IK}\) extended with any number of geometric axioms. When \(s = \varepsilon\) in \((i_x)\), i.e. when \((A \supset (x)A) \land ([x]A \supset [s]A) \in A\), the structural rule \((i_x)\) is defined accordingly:

\[
\frac{\mathcal{R}, \Gamma; w : A \vdash u : A}{\mathcal{R}, \Gamma; w : A \vdash u : A} \quad (i_x)
\]

The semantics for our labeled sequents is given below, and we will utilize the semantics to establish the soundness of each calculus \(L_\Sigma(A)\) subsequently.

**Definition 10 (Sequent Semantics).** Let \(M := (W, \leq, \{R_x \mid x \in \Sigma\}, V)\) be a bi-relational \((\Sigma, A)\)-model with \(I : \text{Lab} \rightarrow W\) an interpretation function mapping labels to worlds. We define the satisfaction of relational atoms and labeled formulae as follows:

\[
\begin{align*}
\mathcal{R}, \Gamma; w : A \vdash \mathcal{R}, \Gamma; u : A & \quad (d_u) \\
\mathcal{R}, \Gamma; w : A \lor B \vdash \mathcal{R}, \Gamma; u : C & \quad (v_i)
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}, \Gamma; w : A \lor B \vdash \mathcal{R}, \Gamma; u : C & \quad (v_i)
\end{align*}
\]
\begin{itemize}
  \item $M, I \models^\Sigma A$ iff for all $wRu \in \mathcal{R}$, $I(w)RI(u)$;
  \item $M, I \models^\Sigma \Gamma$ iff for all $w : A \in \Gamma$, $M, I(w) \vdash A$.
\end{itemize}

A labeled sequent $\Lambda := \mathcal{R}, \Gamma \vdash w : B$ is satisfied in $M$ with $I$, written $M, I \models^\Sigma \Lambda$, iff if $M, I \models^\Sigma \mathcal{R}$ and $M, I \models^\Sigma \Gamma$, then $M, I \models^\Sigma w : B$. A labeled sequent $\Lambda$ is falsified in $M$ with $I$ iff $M, I \not\models^\Sigma \Lambda$, that is, $\Lambda$ is not satisfied by $M$ with $I$.

Last, a labeled sequent $\Lambda$ is $(\Sigma, \mathcal{A})$-valid, written $\models^\Sigma \Lambda$, iff it is satisfiable in every bi-relational $(\Sigma, \mathcal{A})$-model $M$ with every interpretation function $I$. We say that a labeled sequent $\Lambda$ is $(\Sigma, \mathcal{A})$-invalid iff $\not\models^\Sigma \Lambda$, i.e. $\Lambda$ is not $\mathcal{A}$-valid.

\begin{theorem}[Soundness] If $\mathcal{R}, \Gamma \vdash w : A$ is derivable in $L_\Sigma(\mathcal{A})$, then $\models^\Sigma \mathcal{R}, \Gamma \vdash w : A$.
\end{theorem}

\begin{proof}
We prove the result by induction on the height of the given derivation. As the base cases are trivial, we omit them, and with the exception of $(\circ_r)$, all cases of the inductive step are straightforward and are similar to those given for classical grammar logics \[33]\ Theorem 5. Hence, we prove the $(\circ_r)$ case of the inductive step.

We argue by contraposition, that is, we show that if the conclusion is $(\Sigma, \mathcal{A})$-invalid, then the premise is $(\Sigma, \mathcal{A})$-invalid. Let $M$ be a $(\Sigma, \mathcal{A})$-model and $I$ an interpretation function such that $M, I \not\models^\Sigma \mathcal{R}, \Gamma \vdash w : A \supset B$. Then, $M, I \not\models^\Sigma w : A \supset B$, which implies that there exists a world $u \in W$ of $M$ such that $I(w) \leq u$, $M, u \models^\Sigma A$, and $M, u \not\models^\Sigma B$. We now define a new interpretation $I'$ such that $M, I' \not\models^\Sigma \mathcal{R}, \Gamma', w : A \vdash w : B$.

First, we set $I'(w) = u$. Second, we let $wR_{x_1}v_1, \ldots, wR_{x_n}v_n$ be all relational atoms in $\mathcal{R}$ of the form $wR_{x}v$ (containing all successors of $w$) and $u_1R_{x_1}w, \ldots, u_nR_{x_n}w$ be all relational atoms in $\mathcal{R}$ of the form $uR_{x}w$ (containing all predecessors of $w$). Let us now pick an arbitrary $wR_{x_1}v_i$ from the first collection and an arbitrary $u_jR_{x_2}w$ from the second collection. Since $I(w)R_{x_1}I(v_i)$ and $I(u_j)R_{x_2}I(w)$ hold in $M$ and $I(w) \leq u$, we know there exists a $v'_i$ and $u'_j$ such that $I(v_i) \leq v'_i$ and $uR_{x}v'_i$ by condition (F2), and $I(u_j) \leq u'_j$ and $u'R_{x}u'_j$ by condition (F1) (see Definition [1] for these conditions). Thus, if we set $I'(v_i) = v'_i$ and $I'(u_j) = u'_j$, then it follows that $I'(w)R_{x_1}I'(v_i)$ and $I'(u_j)R_{x_2}I'(w)$. Moreover, since $M, I \models^\Sigma \Gamma \vdash v_i$ and $M, I \models^\Sigma \Gamma \vdash u_j$, it follows that $M, I' \models^\Sigma \Gamma \vdash v_i$ and $M, I' \models^\Sigma \Gamma \vdash u_j$ by Lemma [1] as $I(v_i) \leq I'(v_i)$ and $I(u_j) \leq I'(u_j)$. We continue in this fashion, defining $I'$ for all $R_{x}$-successors and predecessors of $w$, for all $x \in \Sigma$, and then successively repeat this process for each of their successors and predecessors, until all descendants and ancestors are of $w$ in $\mathcal{R}$ are defined for $I'$. For all remaining labels $z \in \text{Lab}(\mathcal{R}, \Gamma \vdash w : A \supset B)$, we let $I'(z) = I(z)$.

One can confirm that $M, I' \models^\Sigma \mathcal{R}, \Gamma$, and since $I'(w) = u$, we also know that $M, I \models^\Sigma w : A$, but $M, I \not\models^\Sigma w : B$. Hence, the premise of $(\circ_r)$ is $(\Sigma, \mathcal{A})$-invalid.

With soundness confirmed, we now aim to prove that each calculus $L_\Sigma(\mathcal{A})$ is complete, that is, every $(\Sigma, \mathcal{A})$-valid formula is provable in $L_\Sigma(\mathcal{A})$. To accomplish this, we first prove a sequence of hp-admissibility and hp-invertibility
results, which will be helpful in proving cut-elimination (see Theorem 3), and ultimately, in establishing completeness. A rule is \textit{(hp-)admissible} iff each premise \( \Lambda_1, \ldots, \Lambda_n \) of the rule has a proof (of heights \( h_1, \ldots, h_n \), respectively), then the conclusion has a proof (of height \( h \leq \max\{h_1, \ldots, h_n\} \)). The \textit{(hp-)admissible} rules we consider are shown in Figure 3. A rule is \textit{hp-invertible} iff if the conclusion has a proof of height \( h \), then each premise has a proof of height \( h \) or less. Let us now establish a variety of proof-theoretic properties, which hold for each calculus \( L_\Sigma(A) \).

**Lemma 2.** If \( A \in L(\Sigma) \), then \( \mathcal{R}, \Gamma, w : A \vdash w : A \) is derivable in \( L_\Sigma(A) \).

**Proof.** By induction on the complexity of \( A \).

**Lemma 3.** The \((\bot_l)\) rule is \textit{hp-admissible} in \( L_\Sigma(A) \).

**Proof.** We prove the result by induction on the height of the given derivation.

\textit{Base case.} The \((\bot_l)\) case is trivial, so we show the \((id)\) case. Observe that the conclusion of \((\bot_r)\) in the proof shown below left is an instance of \((\bot_l)\), and hence, we can derive the same conclusion with a single application of \((\bot_l)\) as shown below right.

\[
\frac{\mathcal{R}, \Gamma, w : \bot \vdash w : \bot \quad (id)}{\mathcal{R}, \Gamma, w : \bot \vdash u : A \quad (\bot_l)}
\]

\textit{Inductive step.} We note that \((\bot_r)\) cannot be applied after an application of \((\lor_r),(\land_r),(\Rightarrow_r),(\langle x \rangle_r)\), or \((\lceil x \rceil_r)\) as the labeled formula on the right of the sequent arrow cannot be of the form \( w : \bot \). Hence, these cases need not be considered. For all remaining cases, the \((\bot_r)\) rule freely permutes above any other rule instance, thus establishing the inductive step. For instance, if \((\lor_l)\) is followed by an application of \((\bot_r)\), then \((\bot_l)\) may be permuted above the right premise of \((\lor_l)\) as shown below:

\[
\frac{\mathcal{R}, \Gamma, w : A \lor B \vdash w : A \quad \mathcal{R}, \Gamma, w : B \vdash u : \bot \quad (\lor_l)}{\mathcal{R}, \Gamma, w : A \lor B \vdash u : \bot \quad (\bot_l)} \quad \frac{\mathcal{R}, \Gamma, w : A \lor B \vdash v : C \quad (\bot_l)}{\mathcal{R}, \Gamma, w : A \lor B \vdash v : C \quad (\bot_r)}
\]
Lemma 4. The \( (s) \) and \( (w_l) \) rules are hp-admissible in \( L_\Sigma(A) \).

Proof. The hp-admissibility of each rule is shown by induction on the height of the given derivation. The base cases are simple as applying \((s)\) or \((w_l)\) to an instance of \((id)\) or \((\bot_l)\) yields another instance of the rule. In the inductive step, we make a case distinction based on the last rule applied above \((s)\) or \((w_l)\). With the exception of \((d_x)\), \((\langle x \rangle_l)\), and \((\lceil x \rceil_l)\) cases, all cases are handled by permuting \((s)\) or \((w_l)\) above the rule. In the \((d_x)\), \((\langle x \rangle_l)\), and \((\lceil x \rceil_l)\) cases, we must ensure that if \((s)\) or \((w_l)\) is applied to the premise of the rule that the freshness condition still holds. This can be ensured in the \((s)\) case by invoking IH twice and in the \((w_l)\) case by invoking the hp-admissibility of \((s)\) before invoking IH.

For instance, in the proof shown below left, the fresh label \( u \) is substituted for \( v \) after \((d_x)\) is applied, and therefore, we must invoke IH once to replace \( u \) by a fresh label \( z \), and then invoke IH a second time to substitute \( u \) for \( v \), as shown in the proof below right. Applying \((d_x)\) after both substitutions gives the desired conclusion.

\[
\begin{align*}
\frac{\mathcal{R}, \Gamma, w : A \supset B \vdash w : A}{\mathcal{R}, \Gamma, w : B \vdash v : C} & \quad (\bot_l) \\
\frac{\mathcal{R}, \Gamma, w : B \vdash v : C}{\mathcal{R}, \Gamma, w : A \supset B \vdash v : C} & \quad (\supset_l)
\end{align*}
\]

Lemma 5. The following properties hold for \( L_\Sigma(A) \):

1. The \((\lor_l)\), \((\land_l)\), \((\langle x \rangle_l)\), and \((\lceil x \rceil_l)\) rules are hp-invertible;
2. The \((\supset_l)\) rule is hp-invertible in the right premise;
3. All structural rules are hp-invertible.

Proof. The proofs of claims 1 and 2 are standard and are shown by induction on the height of the given derivation. Claim 3 follows immediately from the hp-admissibility of \((w_l)\).

Lemma 6. The \((ctr_{l1})\) and \((ctr_{l2})\) rules are hp-admissible in \( L_\Sigma(A) \).

Proof. The hp-admissibility of \((ctr_{l1})\) and \((ctr_{l2})\) is shown by induction on the height of the given derivation. The base cases for both rules are trivial since applying either rule to any instance of \((id)\) or \((\bot_l)\) gives another instance of the rule. The inductive step for \((ctr_{l1})\) is also trivial as the rule freely permutes above any other rule of \( L_\Sigma(A) \).

The inductive step for \((ctr_{l2})\) requires slightly more work: we assume that a rule \((r)\) was applied, followed by an application of \((ctr_{l2})\), yielding a labeled sequent \( \mathcal{R}, \Gamma, w : A, w : A \vdash u : B \). If neither contraction formula \( w : A \) is principal in \((r)\), then we may resolve the case by invoking IH, and then applying
(r). If, however, a contraction formula \( w : A \) is principal in (r), then we need to use Lemma 5. We show how to resolve the case where (r) is \( \langle x \rangle_l \) and \( w : A = \langle x \rangle_l \). The remaining cases are similar.

\[
\begin{align*}
R, wR_x u, \Gamma, w : \langle x \rangle_l, C, u : C \vdash u : B & \quad \vdash \langle x \rangle_l \\
R, \Gamma, w : \langle x \rangle_l, C, w : \langle x \rangle_l C \vdash u : B & \quad (ctr_1) \\
R, \Gamma, w : \langle x \rangle_l C \vdash u : B & \quad \left\| \begin{array}{l}
R, wR_x u, \Gamma, w : \langle x \rangle_l, C, u : C \vdash u : B \\
R, wR_x u, \Gamma, v : C, u : C \vdash u : B \\
R, wR_x u, \Gamma, u : C \vdash u : B \\
R, \Gamma, w : \langle x \rangle_l C \vdash u : B \\
\end{array} \right\|
\end{align*}
\]

In the output proof, shown above right, the \( (s) \) rule substitutes the label \( u \) for the fresh label \( v \) and we apply the hp-admissibility of \( (ctr_1) \). Since all operations are height-preserving, we can apply IH after all such operations.

Theorem 3 (Cut-elimination). The \( (cut) \) rule is admissible in \( L_\Sigma (A) \).

Proof. We prove the result by induction on the lexicographic ordering of pairs \( (|A|, h_1 + h_2) \), where \( |A| \) is the complexity of the cut formula \( A \) and \( h_1 + h_2 \) is the sum of the heights of the proofs deriving the left and right premises of \( (cut) \), respectively.

Let us suppose first that one of the premises of \( (cut) \) is an initial rule. If the left premise of \( (cut) \) is an instance of \( (\bot_l) \), then the conclusion will be an instance of \( (\bot_l) \), and thus, the conclusion is derivable by \( (\bot_l) \) only, without the use of \( (cut) \). If the left premise of \( (cut) \) is an instance of \( (id) \), then our \( (cut) \) is of the following shape:

\[
\begin{align*}
R, \Gamma, w : p \vdash w : p & \quad (id) \\
R, R', \Gamma, w : p, \Gamma' \vdash u : A & \quad (cut)
\end{align*}
\]

Observe that the conclusion may be derived without \( (cut) \) by applying the hp-admissibility of \( (w_l) \) as shown below:

\[
R, R', \Gamma, w : p, \Gamma' \vdash u : A \quad (w_l)
\]

If the right premise of \( (cut) \) is an instance of \( (\bot_l) \) with \( w : \bot \) principal, then there are two cases to consider. Either, \( w : \bot \) is not the cut formula, in which case, the conclusion is an instance of \( (\bot_l) \), or \( w : \bot \) is the cut formula, in which case our \( (cut) \) is of the following shape:

\[
\begin{align*}
R, \Gamma \vdash w : \bot & \quad (\bot_l) \\
R, R', \Gamma, w : \bot \vdash u : A & \quad (cut)
\end{align*}
\]

Then, we may derive the conclusion without the use of \( (cut) \) by applying the hp-admissibility of \( (\bot_r) \) and \( (w_l) \).
\[
\frac{\mathcal{R}, \Gamma \vdash w : \bot}{\mathcal{R}, \Gamma \vdash u : A} \quad (\bot_r)
\]
\[
\frac{\mathcal{R}}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \vdash u : A} \quad (w_1)
\]

If the right premise of \(\textsc{cut}\) is an instance of \(\text{id}\) with \(w : p\) principal, then we have two cases to consider. Either, \(w : p\) is not the cut formula, in which case, the conclusion \(A\) is an instance of \(\text{id}\), or \(w : p\) is the cut formula, meaning that our \(\textsc{cut}\) is of the following shape:

\[
\frac{\mathcal{R}, \Gamma \vdash w : p}{\mathcal{R}', \Gamma' \vdash w : p} \quad (\text{id})
\]
\[
\frac{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \vdash w : p}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \vdash w : p} \quad (\text{cut})
\]

Observe that the conclusion can be derived from the left premise of \(\textsc{cut}\) by applying the \(\text{hp-}\)admissibility of \(w_1\). Hence, \(\text{cut}\) is eliminable whenever one premise of \(\text{cut}\) is an initial rule.

Let us now suppose that no premise of \(\text{cut}\) is an initial rule. We have two cases to consider: either, the cut formula is not principal in at least one premise of \(\text{cut}\) or the cut formula is principal in both premises. We consider each case in turn.

We assume that neither cut formula is principal and consider the case when the left premise of \(\text{cut}\) is derived via a unary rule and the right premise of \(\text{cut}\) is derived via a binary rule. All remaining cases are shown similarly, and thus, we omit them.

\[
\frac{\mathcal{R}_1, \Gamma_1 \vdash w : A}{\mathcal{R}'_1, \Gamma'_1 \vdash w : A} \quad (r_1)
\]
\[
\frac{\mathcal{R}_2, \Gamma_2, w : A \vdash \Delta_2}{\mathcal{R}'_2, \Gamma'_2, w : A \vdash \Delta'_2} \quad (r_2)
\]
\[
\frac{\mathcal{R}_1, \mathcal{R}'_2, \Gamma_1, \Gamma'_2 \vdash \Delta_2 \vdash \Delta'_2}{\mathcal{R}'_1, \Gamma'_1 \vdash \Delta'_2} \quad (\text{cut})
\]

The case is resolved as shown below by lifting the \(\text{cut}\) to apply to the premise of \(r_1\). We note that if \(r_1\) must satisfy a freshness condition (e.g. as with the \((\langle x \rangle l)\) rule), then it may be necessary to apply the \(\text{hp-}\)admissibility of \((s)\) to the premise of \((r_1)\) prior to lifting the \(\text{cut}\) upward to ensure the condition will be met after applying \(\text{cut}\). However, this causes no issues as \(h_1 + h_2\) will still decrease.

\[
\frac{\mathcal{R}_1, \Gamma_1 \vdash w : A}{\mathcal{R}_2, \Gamma_2, w : A \vdash \Delta_2}{\mathcal{R}'_2, \Gamma'_2, w : A \vdash \Delta'_2} \quad (r_2)
\]
\[
\frac{\mathcal{R}_1, \mathcal{R}'_2, \Gamma_1, \Gamma'_2 \vdash \Delta_2 \vdash \Delta'_2}{\mathcal{R}'_1, \Gamma'_1 \vdash \Delta'_2} \quad (r_1)
\]

We now consider the case where the cut formula is principal in both premises of \(\text{cut}\). We consider the cases where the cut formula is of the form \(A \supset B\) or \(\langle x \rangle A\) and omit the remaining cases as they are similar. In the first case, our \(\text{cut}\) is of the following shape:

\[
\mathcal{D} = \frac{\mathcal{R}', \Gamma', w : A \supset B \vdash w : A}{\mathcal{R}', \Gamma', w : A \supset B \vdash u : C} \quad (\supset_l)
\]
\[
\frac{\mathcal{R}, \Gamma, w : A \vdash B}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \vdash w : A \supset B} \quad (\supset_r)
\]
\[
\frac{\mathcal{R}, \Gamma \vdash w : B}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \vdash u : C} \quad \mathcal{D} \quad (\text{cut})
\]
We first apply \((\text{cut})\) to the conclusion of \((\supset r)\) and to the left premise of \((\supset l)\), thus reducing the height of \((\text{cut})\), and then apply \((\text{cut})\) two additional times to the formulae \(A\) and \(B\) of less complexity.

\[
\begin{align*}
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{\Gamma', w : A \supset B \vdash w : A}{\Gamma', \Gamma'' \vdash w : A} \quad \text{(cut)} \\
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)} \\
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)} \\
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)} \\
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)}
\end{align*}
\]

For the second case, when our cut formula is of the form \(\langle x \rangle A\), our \((\text{cut})\) is of the following shape:

\[
\begin{align*}
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)} \\
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)} \\
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)}
\end{align*}
\]

The case is resolved as shown below, where the hp-admissibility of \((s)\) is applied to replace the fresh label \(v\) by \(u\).

\[
\begin{align*}
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)} \\
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)} \\
D' &= \frac{\Gamma, w : A \vdash w : B}{R, \Gamma, w \vdash w : A} \quad \frac{R', \Gamma', w : A \vdash u : C}{R, \Gamma', w \vdash u : C} \quad \text{(cut)}
\end{align*}
\]

Finally, we are in a position to confirm the completeness of each calculus \(L_\Sigma(A)\). Completeness relies in part on Theorem 1 that is, we assume that a formulae \(A\) is \((\Sigma, A)\)-valid, implying that \(A\) is derivable in \(HIK_\Sigma(A)\). We then establish completeness by showing that all axioms of \(HIK_\Sigma(A)\) are derivable in \(L_\Sigma(A)\) and that all inference rules are admissible, which implies that \(A\) is derivable in \(L_\Sigma(A)\). It is a straightforward exercise to show that all axioms of \(HIK_\Sigma(A)\) are derivable in \(L_\Sigma(A)\). The admissibility of \((n)\) is shown by invoking the hp-admissibility of \((w)\) and \((s)\), and since standard axiomatizations for propositional intuitionistic logic (axiom \(A_0\) in Definition 3) include \textit{modus ponens} as an inference rule [50], we use cut-elimination (Theorem 3) to simulate the rule.

\textbf{Theorem 4 (Completeness).} If \(\vdash_{\Sigma} A\), then \(\vdash w : A\) is derivable in \(L_\Sigma(A)\).

\section{Structural Refinement}

We now show how to \textit{structurally refine} the labeled systems introduced in the previous section, that is, we implement a methodology introduced and applied in [11,32,33,34,37] (referred to as \textit{structural refinement}, or \textit{refinement} more
simply) for simplifying labeled systems and/or permitting the extraction of nested systems. The methodology consists of eliminating structural rules (viz. the \(i^s\) and \(e_x\) rules in our setting) through the addition of propagation rules (cf. \[7,20,47\]) or reachability rules \[33,38\] to labeled calculi, begetting systems that are translatable into nested systems.

The propagation rules we introduce are based on those of \[33,37,49\], and operate by viewing a labeled sequent as an automaton, allowing for the propagation of a formula (when applied bottom-up) from a label \(w\) to a label \(u\) given that a certain path of relational atoms exists between \(w\) and \(u\) (corresponding to a string generated by an \(A\)-grammar).

The definition of our propagation rules is built atop the notions introduced in the following two definitions:

**Definition 11 (Propagation Graph).** The propagation graph \(PG(R)\) of a multiset of relational atoms \(R\) is defined recursively on the structure of \(R\):

- \(PG(\emptyset) := (\emptyset, \emptyset)\);
- \(PG(wRw) := (\{w, u\}, \{(w, x, u), (u, x, w)\});\)
- \(PG(R_1, R_2) := (V_1 \cup V_2, E_1 \cup E_2)\) where \(PG(R_i) = (V_i, E_i)\).

We will often write \(w \in PG(R)\) to mean \(w \in V\), and \((w, x, u) \in PG(R)\) to mean \((w, x, u) \in E\).

**Definition 12 (Propagation Path).** We define a propagation path from \(w_1\) to \(w_n\) in \(PG(R) := (V, E)\) to be a sequence of the following form:

\[\pi(w_1, w_n) := w_1, x_1, w_2, x_2, \ldots, x_{n-1}, w_n\]

such that \((w_1, x_1, w_2), (w_2, x_2, w_3), \ldots, (w_{n-1}, x_{n-1}, w_n) \in E\). Given a propagation path of the above form, we define its converse as shown below top and its string as shown below bottom:

\[\pi(w_n, w_1) := w_n, x_{n-1}, w_{n-1}, x_{n-2}, \ldots, x_1, w_1\]

\[s_{\pi}(w_1, w_n) := x_1 x_2 \cdots x_{n-1}\]

Last, we let \(\varepsilon(w, w) := w\) represent an empty path with the string of the empty path defined as \(s_\varepsilon(w, w) := \varepsilon\).

We are now in a position to define the operation of our propagation rules \((p_{(x)})\) and \((p_{(w)})\), which are displayed in Figure 5. Each propagation rule \((p_{(x)})\) and \((p_{(w)})\) is applicable only if there exists a propagation path \(\pi(w, u)\) from \(w\) to \(u\) in the propagation graph \(PG(R)\) such that the string \(s_{\pi}(w, u)\) is in the language \(L_{g(A)}(x)\). We express this statement compactly by making use of its equivalent first-order representation:

\[\exists \pi(w, u) \in PG(R) (s_{\pi}(w, u) \in L_{g(A)}(x))\]

We provide further intuition regarding such rules by means of an example:
Lemma 7. Let \( \mathcal{R}, \Gamma \vdash u : A \) (\( p_{(z)} \)) only if \( \exists \pi(w, u) \in PG(\mathcal{R})(s_{\pi}(w, u) \in L_{g(A)}(x)) \)

\[
\frac{\mathcal{R}, \Gamma \vdash w : \langle x \rangle A}{\mathcal{R}, \Gamma, w : [x]A \vdash v : B} \quad (p_{[v]}) \text{ only if } \exists \pi(w, u) \in PG(\mathcal{R})(s_{\pi}(w, u) \in L_{g(A)}(x))
\]

Fig. 5. Propagation rules.

Example 1. Let \( \mathcal{R} := vR_xu, uR_yw \). We give a graphical depiction of \( PG(\mathcal{R}) \):

Let \( A := \{(y \langle x \rangle A \supset \langle z \rangle A) \land \{[y]A \supset \langle \pi \rangle A\}\} \), so that the corresponding \( \mathcal{A} \)-grammar is \( g(A) = \{z \rightarrow y\pi, \pi \rightarrow \pi x\} \). Then, the path \( \pi(w, u) := w, y, u, \pi, v \) exists between \( w \) and \( v \). The first production rule of \( g(A) \) implies that \( s_{\pi}(w, v) = y\pi \in L_{g(A)}(z) \). Therefore, we are permitted to (top-down) apply the propagation rule \( (p_{[z]}) \) to \( A \) to delete the labeled formula \( v : p \), letting us derive \( vR_xu, uR_yw, w : [z]p, v : p \vdash v : q \).

Remark 2. The \( (\langle x \rangle_r) \) and \( (\langle x \rangle_l) \) rules are instances of \( (p_{(z)}) \) and \( (p_{[z]}) \), respectively.

Definition 13 (Refined Labeled Calculus). We define the refined labeled calculus \( L^*_L(A) := (L_L(A) \setminus R) \cup \{(p_{(z)}), (p_{[z]})\} \), where

\[
R := \{(i^*_z) \mid \langle s \rangle A \supset \langle x \rangle A \land \{[x]A \supset [s]A \in \mathcal{A}\} \cup \{(\langle x \rangle_r), (\langle x \rangle_l), (c_x) \mid x \in \Sigma\}.
\]

We show that each calculus \( L^*_L(A) \) is complete by means of a proof transformation procedure. That is, we show that through the elimination of structural rules we can transform a proof in \( L_L(A) \) into a proof in \( L^*_L(A) \). We first prove three crucial lemmata, and then argue the elimination result.

Lemma 7. Let \( \mathcal{R}_1 := \mathcal{R}, wR_xu, uR_yu \) and \( \mathcal{R}_2 := \mathcal{R}, wR_xu \). Suppose we are given a derivation in \( L_L(A) \cup L^*_L(A) \) ending with:

\[
\begin{align*}
\frac{\mathcal{R}, wR_xu, wR_yu, \Gamma \vdash v : A}{\mathcal{R}, wR_xu, wR_yu, \Gamma \vdash z : \langle x \rangle A} \quad (p_{(z)}) \\
\frac{\mathcal{R}, wR_xu, \Gamma \vdash z : \langle x \rangle A}{\mathcal{R}, wR_xu, \Gamma \vdash z : \langle x \rangle A} \quad (i^*_z)
\end{align*}
\]

where the side condition \( \exists \pi(z, v) \in PG(\mathcal{R}_1)(s_{\pi}(z, v) \in L_{g(A)}(x)) \) holds due to \( (p_{(z)}) \). Then, \( \exists \pi'(z, v) \in PG(\mathcal{R}_2)(s_{\pi'}(z, v) \in L_{g(A)}(x)) \), that is to say, the \( (i^*_z) \) rule is permutable with the \( (p_{(z)}) \) rule.

Proof. Let \( wR_xu = wR_xu_1, \ldots, u_nR_xu \). We have two cases: either (i) the relational atom \( wR_yu \) is not active in the \( (p_{(z)}) \) inference, or (ii) it is. As case (i) is easily resolved, we show (ii).
We suppose that the relational atom \( wR_yu \) is active in \( (p_{(z)}) \), i.e. \( wR_yu \) occurs along the propagation path \( \pi(z, v) \). To prove the claim, we need to show that \( \exists \pi'(z, v) \in PG(\mathcal{R}_2)((s_{\pi'}(z, v) \in L_{g(\mathcal{A})}(x)) \). We construct such a propagation path by performing the following operations on \( \pi(z, v) \): replace each occurrence of \( w, y, u \) in \( PG(\mathcal{R}_1) \) with the propagation path \( \pi''(w, u) \) shown below left and replace each occurrence of \( u, \overline{y}, w \) in \( PG(\mathcal{R}_1) \) with the propagation path \( \pi''(u, w) \) shown below right:

\[
\pi''(w, u) = w, x_1, u_1, \ldots, u_n, x_n, u \quad \pi''(u, w) = u, \overline{x}_n, u_n, \ldots, u_1, \overline{x}_1, w
\]

We let \( \pi'(z, v) \) denote the path obtained by performing the above operations on \( \pi(z, v) \). Note that \( \pi''(w, u) \) corresponds to the edges \( (w, x_1, u_1), \ldots, (u_n, x_n, u) \in PG(\mathcal{R}_1) \) and \( \pi''(u, w) \) corresponds to the edges \( (u, \overline{x}_n, u_n), \ldots, (u_1, \overline{x}_1, w) \in PG(\mathcal{R}_1) \), obtained from the relational atoms \( wR_yu \in \mathcal{R}_1 \) (by Definition [14]). Since the sole difference between \( PG(\mathcal{R}_1) \) and \( PG(\mathcal{R}_2) \) is that the former is guaranteed to contain the edges \( (w, y, u) \) and \( (u, \overline{y}, w) \) obtained from \( wR_yu \), while the latter is not, and since \( \pi'(z, v) \) omits the use of such edges (i.e. \( w, y, u \) and \( u, \overline{y}, w \) do not occur in \( \pi'(z, v) \)), we have that \( \pi'(z, v) \) is a propagation path in \( PG(\mathcal{R}_2) \).

To complete the proof, we need to additionally show that \( s_{\pi'}(z, v) \in L_{g(\mathcal{A})}(x) \). By assumption, \( s_{\pi}(z, v) \in L_{g(\mathcal{A})}(x) \), which implies that \( x \rightarrow s_{g(\mathcal{A})} s_{\pi}(z, v) \) by Definition [8]. Since \( (i_y^\pi) \) is a rule in \( L^\pi(A) \), it follows that \( (y \rightarrow s), (\overline{y} \rightarrow \overline{s}) \) in \( g(A) \) by Definition [8]. If we apply \( y \rightarrow s \) to each occurrence of \( y \) in \( s_{\pi}(z, v) \) corresponding to the edge \( (w, y, u) \) (and relational atom \( wR_yu \)), and apply \( \overline{y} \rightarrow \overline{s} \) to each occurrence of \( \overline{y} \) in \( s_{\pi}(z, v) \) corresponding to the edge \( (u, \overline{y}, w) \) (and relational atom \( wR_yu \)), we obtain the string \( s_{\pi'}(z, v) \) and show that \( x \rightarrow s_{g(\mathcal{A})} s_{\pi'}(z, v) \), i.e. \( s_{\pi'}(z, v) \in L_{g(\mathcal{A})}(x) \). 

\[\square\]

**Lemma 8.** Let \( \mathcal{R}_1 := \mathcal{R}, wR_yu, wR_yu \) and \( \mathcal{R}_2 := \mathcal{R}, wR_yu \). Suppose we are given a derivation in \( L^\pi(A) \cup L^\pi(A) \) ending with:

\[
\frac{\mathcal{R}, wR_yu, wR_yu, \Gamma, z : [x]A, v : A \vdash t : B}{\mathcal{R}, wR_yu, wR_yu, \Gamma, z : [x]A, v : A \vdash t : B} (p_{(x)})
\]

where the side condition \( \exists \pi(z, v) \in PG(\mathcal{R}_1)((s_{\pi}(z, v) \in L_{g(\mathcal{A})}(x)) \) holds due to \( (p_{(x)}) \). Then, \( \exists \pi'(z, v) \in PG(\mathcal{R}_2)((s_{\pi'}(z, v) \in L_{g(\mathcal{A})}(x)) \), that is to say, the \( (i_y^\pi) \) rule is permutable with the \( (p_{(x)}) \) rule.

**Proof.** Similar to the proof of Lemma [7] above. \(\square\)

**Lemma 9.** For each \( x \in \Sigma \), the \( (c_x) \) rule is \( hp \)-admissible in \( L^\pi(A) \).

**Proof.** Let \( x \in \Sigma \). We prove the result by induction on the height of the given derivation. The bases cases are trivial as any application of \( (c_x) \) to \( (id) \) or \( (\bot_1) \) yields another instance of the rule. For the inductive step, \( (c_x) \) freely permutes above every rule of \( L^\pi(A) \) with the exception of \( (p_{(x)}) \) and \( (p_{[x]}) \). We show the \( (p_{(x)}) \) case as the \( (p_{[x]}) \) case is similar.
Let us suppose that our proof ends with an application of \((p_{(x)})\) followed by an application of \((c_y)\) as shown below left. We let \(\mathcal{R}_1 = \mathcal{R}, wR_yu, uR_fw\) and \(\mathcal{R}_2 = \mathcal{R}, wR_yu\). We know that \(\exists \pi(z, v) \in PG(\mathcal{R}_1)(s_{(z, v)} \in L_{g(A)}(x))\) by the side condition on \((p_{(z)}))\). Moreover, we may assume that \(uR_fw\) is active in the application of \((p_{(z)}))\) because the two rules freely permute in the alternative case. If we apply \((c_y)\) first, as shown in the derivation below right, observe that since \(wR_yu\) still occurs in the conclusion, \(PG(\mathcal{R}_1) = PG(\mathcal{R}_2)\). Hence, the side condition of \((p_{(z)}))\) still holds, showing that the rule may be applied after \((c_y)\).

\[
\frac{\mathcal{R}, wR_yu, uR_fw, \Gamma \vdash v : A}{\mathcal{R}, wR_yu, uR_fw, \Gamma \vdash z : \langle x \rangle A} \quad (p_{(x)}) \quad \frac{\mathcal{R}, wR_yu, \Gamma \vdash v : A}{\mathcal{R}, wR_yu, \Gamma \vdash z : \langle x \rangle A} \quad (c_y) \quad \frac{\mathcal{R}, wR_yu, \Gamma \vdash v : A}{\mathcal{R}, wR_yu, \Gamma \vdash z : \langle x \rangle A} \quad (p_{(z)})
\]

To improve the comprehensibility of the above lemmata, we provide an example of permuting instances of structural rules above an instance of a propagation rule.

**Example 2.** Let \(A := \{([y]x A \vdash \langle x \rangle A) \land ([x]A \vdash [y][z]A)\}\) so that the \(A\)-grammar \(g(A) = \{x \rightarrow \overline{y}z, x \rightarrow \overline{z}y\}\). In the top derivation below, we let \(\mathcal{R}_1 = wR_zu, wR_yv, vR_fw, uR_xv\) and assume that \((p_{(z)})\) is applied due to the existence of the propagation path \(\pi(u, v) = u, v\) in \(PG(\mathcal{R}_1)\), where \(s_{(u)}(u, v) = x \in L_{g(A)}(x)\) by Definition 9. The propagation graph \(PG(\mathcal{R}_1)\) corresponding to the top sequent of the derivation shown below left is shown below right:

\[
\begin{align*}
\frac{wR_zu, wR_yv, vR_fw, uR_xv, v : p \vdash v : p}{wR_fw, wR_yu, uR_fw, uR_xv, v : p \vdash u : \langle x \rangle p} & \quad (id) \\
\frac{wR_yv, wR_yu, uR_fw, uR_xv, v : p \vdash u : \langle x \rangle p}{wR_fw, wR_yu, v : p \vdash u : \langle x \rangle p} & \quad (c_y)
\end{align*}
\]

Let \(\mathcal{R}_2 = wR_zu, wR_yv, vR_fw\). If we apply \(x \rightarrow \overline{y}z\) to \(s_{(u)}(u, v) = x\), then we obtain the string \(\overline{y}z\). Hence, \(x \rightarrow \overline{y}z\) \(\overline{y}z\), i.e. \(\overline{y}z \in L_{g(A)}(x)\), meaning that a propagation path \(\pi'(u, v) = u, \overline{y}z, w, z, v\) exists in \(PG(\mathcal{R}_2)\) such that \(s_{(u)}(u, v) = \overline{y}z \in L_{g(A)}(x)\). We may therefore apply \((i \overline{y}z)\) and then \((p_{(z)})\) as shown below left; the propagation graph \(PG(\mathcal{R}_2)\) is shown below right:

\[
\begin{align*}
\frac{wR_zu, wR_yv, vR_fw, uR_xv, v : p \vdash v : p}{wR_fw, wR_yu, uR_fw, uR_xv, v : p \vdash v : p} & \quad (i \overline{y}z) \\
\frac{wR_fw, wR_yu, v : p \vdash u : \langle x \rangle p}{wR_fw, wR_yu, v : p \vdash u : \langle x \rangle p} & \quad (p_{(z)}) \\
\frac{wR_fw, wR_yu, v : p \vdash u : \langle x \rangle p}{wR_fw, wR_yu, v : p \vdash u : \langle x \rangle p} & \quad (c_y)
\end{align*}
\]

Furthermore, if we let \(\mathcal{R}_3 = wR_zu, wR_yv\), then observe that \(PG(\mathcal{R}_3) = PG(\mathcal{R}_4)\), and thus, the propagation path \(\pi'(u, v)\) exists in \(PG(\mathcal{R}_4)\) as well, showing that \((c_y)\) can be permuted above \((p_{(z)})\), as shown below:
\[
\frac{wR_xu, wR_yv, vR_yw, uR_xv, v : p \vdash v : p}{(id)}
\]
\[
\frac{wR_yv, wR_yu, uR_yw, v : p \vdash v : p}{(i^2)}
\]
\[
\frac{wR_yv, wR_yu, v : p \vdash v : p}{(c_y)}
\]
\[
\frac{wR_y v, wR_y u, u : p \vdash u : (x)p}{(p(x))}
\]

Since the conclusion of \((c_y)\) is an instance of \((id)\), we can delete the applications of \((i^2)\) and \((c_y)\), thus giving the proof shown below, which is free of structural rules and exists in \(L^*_L(L)\). Thus, we have shown how a proof in \(L_L(A)\) can be transformed into a proof in \(L^*_L(L)\).

\[\frac{wR_y v, wR_y u, v : p \vdash v : p}{(id)}\]
\[\frac{wR_y v, wR_y u, u : p \vdash u : (x)p}{(p(x))}\]

**Theorem 5.** Every derivation in \(L_L(A)\) can be transformed into a derivation in \(L^*_L(L)\).

**Proof.** We consider a derivation in \(L_L(A)\), which is a derivation in \(L_L(A) \cup L^*_L(L)\) (meaning Lemma [11] and [12] are applicable). By Remark [12] each instance of \((\langle x \rangle_r)\) and \((\langle x \rangle_l)\) can be replaced by a \((p(x))\) and \((p(x)|x)\) instance, respectively, meaning we may assume our derivation is free of \((\langle x \rangle_r)\) and \((\langle x \rangle_l)\) instances. We show that the derivation can be transformed into a derivation in \(L^*_L(L)\) by induction on its height, that is, we consider a topmost occurrence of a structural rule \((i^*_y)\) or \((c_x)\) and show that it can be eliminated. Through successively eliminating topmost instances of structural rules, we obtain a proof in \(L^*_L(L)\).

**Base case.** The base case is trivial as any application of \((i^*_y)\) or \((c_x)\) to \((id)\) or \((\langle l \rangle)\) yields another instance of the rule.

**Inductive step.** It is straightforward to verify that \((i^*_y)\) freely permutes above all rules in \(L_L(A) \cup L^*_L(L)\) with the exception of \((i^*_y), (c_x), (p(x)), and (p(x))\) (this follows from the fact that all other rules do not have active relational atoms in their conclusion). Since we are considering a topmost application of \((i^*_y)\), we need not consider the permutation of \((i^*_y)\) above \((i^*_y)\) or \((c_x)\). The last two cases of permuting \((i^*_y)\) above \((p(x))\) and \((p(x))\) follow from Lemma [11] and [12] respectively. Last, by Lemma [12] we know that \((c_x)\) can be eliminated as we are considering a topmost occurrence of a structural rule, and thus, the derivation above \((c_x)\) is a derivation in \(L^*_L(L)\).

**Lemma 10.** If \(R, uR_xv, \Gamma \vdash w : A\) is derivable in \(L_L(A)\) and \(\exists \pi(u, v) \in PG(R)(s_\pi(u, v) \in L_{g(A)}(x))\), then \(R, \Gamma \vdash w : A\) is derivable in \(L_L(A)\).

**Proof.** By our assumption that \(\exists \pi(u, v) \in PG(R)(s_\pi(u, v) \in L_{g(A)}(x))\), we know that \(x \rightarrow^{*}_{g(A)} s_\pi(u, v)\). We show the result by induction on the length of the derivation \(x \rightarrow^{*}_{g(A)} s_\pi(u, v)\).

**Base case.** If the derivation \(x \rightarrow^{*}_{g(A)} s_\pi(u, v)\) has length zero, then it follows that \(x \rightarrow^{*}_{g(A)} x\), implying that \(R, uR_xv, uR_xv, \Gamma \vdash w : A\) is derivable in \(L_L(A)\).

A single application of the hp-admissible \((ctr)\) rule (see Lemma [8]) gives the

\[\text{The length of a derivation in an } A\text{-grammar is defined in Definition [9].}\]
desired result. If the derivation \( x \rightarrow^{g(A)} s_\pi(u, v) \) has a length of one, then \( x \rightarrow s_\pi(u, v) \) is a production rule in \( g(A) \). Hence, \( R, uR_x v, \Gamma \vdash w : A \) of \( \pi ' \R, uR_x v, \Gamma \vdash w : A \) where the relational atoms \( uR_x v \) give rise to the propagation path \( \pi(u, v) \). We note that \( \pi ' \R, uR_x v, \Gamma \vdash w : A \) can be derived from the above labeled sequent by applying the hp-admissible \( (w_l) \) rule (see Lemma 4) and the \( (e_c) \) rules a sufficient number of times, from which \( \pi ' \R, uR_x v, \Gamma \vdash w : A \) is derivable by a single application of \( (\pi _c^* \pi ) \). Again, by applying \( (w_l) \) and the \( (e_c) \) rules a sufficient number of times, we derive \( R, \Gamma \vdash w : A \).

**Inductive step.** Suppose the derivation \( x \rightarrow^{g(A)} s_\pi(u, v) \) has a length of \( n + 1 \). This implies the existence of a string \( t \) such that \( x \rightarrow^{g(A)} t \) and \( t \rightarrow \gamma \). We show the non-trivial \( (p_{(u,v)}) \) case of the inductive step as the remaining cases are simple or similar.

\[
\begin{align*}
\frac{R, \Gamma \vdash u : A}{R, \Gamma \vdash w : \langle x \rangle A} & \quad (p_{(u,v)}) & \frac{R, \Gamma \vdash u : A}{R, w \Gamma \vdash u : A} & \quad (w_l) \\
\frac{R, \Gamma \vdash u : A}{R, w \Gamma \vdash w : \langle x \rangle A} & \quad (\langle x \rangle_r) & \frac{R, w \Gamma \vdash w : \langle x \rangle A} & \quad \text{Lemma 10}
\end{align*}
\]

By the side condition on \( (p_{(u,v)}) \), we know that Lemma 10 is applicable in the proof shown above right.

**Theorem 6.** Every derivation in \( L^*_E(A) \) can be transformed into a derivation in \( L^*_E(A) \).

**Proof.** The theorem is shown by induction on the height of the given derivation. We show the non-trivial \( (p_{(u,v)}) \) case of the inductive step as the remaining cases are simple or similar.

**Theorem 7 (\( L^*_E(A) \) Soundness and Completeness).**

1. If \( R, \Gamma \vdash w : A \) is derivable in \( L^*_E(A) \), then \( R, \Gamma \vdash w : A \) is \( A \)-valid;
2. If \( \vdash A \), then \( \Gamma \vdash w : A \) is derivable in \( L^*_E(A) \).

**Proof.** Follows from Theorems 2 and 4 and 6.

**Corollary 1.** The following properties hold for \( L^*_E(A) \):

1. The \( (\bot_r) \), \( (s) \), \( (w) \), \( (ctr_1) \), \( (ctr_2) \), and \( (cut) \) rules are admissible;
2. The \( (\lor_l) \), \( (\land_l) \), \( (\bot_r) \), and \( (p_{(u,v)}) \) rules are invertible;
3. The \( (\top_l) \) rule is invertible in the right premise.
Example 3. We note that all of the above terminology is due to [48].

5 Nested Sequent Systems

In our setting, nested sequents are taken to be trees of multisets of formulae containing a unique formula that occupies a special status. We utilize a syntax which incorporates components from the nested sequents given for classical grammar logics [19] and intuitionistic modal logics [48]. Following [48], we mark a special, unique formula with a white circle \( \circ \) indicating that the formula is of output polarity, and mark the other formulae with a black circle \( \bullet \) indicating that the formulae are of input polarity. A nested sequent \( \Sigma \) is defined via the following BNF grammars:

\[
\Sigma ::= \Delta, \Pi \quad \Delta ::= A^*_1, \ldots, A^*_n, (x_1)\{\Delta_1\}, \ldots, (x_k)\{\Delta_k\} \quad \Pi ::= A^* \mid [\Sigma]
\]

where \( x_1, \ldots, x_k \in \Sigma \) and \( A_1, \ldots, A_n, A \in \mathcal{L}(\Sigma) \).

We assume that the comma operator associates and commutes, implying that such sequents are truly trees of multisets of formulae, and we let the empty sequent \( \emptyset \) be the empty multiset \( \emptyset \). We refer to a sequent in the shape of \( \Delta \) (which contains only input formulae) as an LHS-sequent, a sequent in the shape of \( \Pi \) as an RHS-sequent, and a sequent \( \Sigma \) as a full sequent. We use both \( \Sigma \) and \( \Delta \) to denote LHS- and full sequents with the context differentiating the usage.

As for classical modal logics (e.g. [4,26]), we define a context \( \Sigma \{ \} \cdots \{ \} \) to be a nested sequent with some number of holes \( \{ \} \) in the place of formulae. This gives rise to two types of contexts: input contexts, which require holes to be filled with LHS-sequents to obtain a full sequent, and output contexts, which require a single hole to be filled with an RHS-sequent and the remaining holes to be filled with LHS-sequents to obtain a full sequent. We also define the output pruning of an input context \( \Sigma \{ \} \cdots \{ \} \) or full sequent \( \Sigma \), denoted \( \Sigma^l \{ \} \cdots \{ \} \) and \( \Sigma^l \) respectively, to be the same context or sequent with the unique output formula deleted. We note that all of the above terminology is due to [48].

Example 3. Let \( \Sigma_1 \{ \} ::= p^*, (a)[(a)q^*, \{ \} \} \) be an output context and \( \Sigma_2 \{ \} ::= p^*, (\overline{p}][(a)q^*, \{ \} \} \) be an input context. Also, we let \( \Delta_1 := \bot^*, (b)[q \supset r^*] \) be a full sequent and \( \Delta_2 := \bot^*, (a)[q \supset r^*] \) be an LHS-sequent. Observe that neither \( \Sigma_1 \{ \Delta_2 \} \) (shown below left) nor \( \Sigma_2 \{ \Delta_1 \} \) (shown below right) are full sequents since the former has no output formula and the latter has two output formulae.

\[
p^*, (a)[(a)q^*, \bot^*, (a)[q \supset r^*]] \quad p^*, (\overline{p}][(a)q^*, \bot^*, (b)[q \supset r^*]]
\]

Conversely, both \( \Sigma_1 \{ \Delta_1 \} \) (shown below left) and \( \Sigma_2 \{ \Delta_2 \} \) (shown below right) are full sequents.

\[
p^*, (a)[(a)q^*, \bot^*, (b)[q \supset r^*]] \quad p^*, (\overline{p}][(a)q^*, \bot^*, (a)[q \supset r^*]]
\]

Proof. Follows from Theorems [5] and [6]. For the admissible rules, suppose the premise(s) is (are) derivable in \( \mathbf{L}^+_\Sigma (\mathcal{A}) \). Then, by Theorem [6] these labeled sequents are derivable in \( \mathbf{L}^+_\Sigma (\mathcal{A}) \). Since all such rules are admissible in \( \mathbf{L}^+_\Sigma (\mathcal{A}) \), the rule may be applied and the desired conclusion is derivable in \( \mathbf{L}^+_\Sigma (\mathcal{A}) \) by Theorem [5]. The invertibility results are argued in a similar fashion. \( \square \)
Our nested sequent systems are presented in Figure 6 and are generalizations of those given for extensions of the intuitionistic modal logic \( \text{IK} \) with seriality and Horn-Scott-Lemmon axioms of the form \((\Box^n A \supset \Box^k A) \land (\Box^k A \supset \Box^n \Box A)\) with \(n, k \in \mathbb{N}\) [36], which includes the logics of the intuitionistic modal cube [48]. These systems can also be seen as intuitionistic variants of the nested sequent systems given for classical grammar logics [49]. Our nested systems incorporate the propagation rules \((p(x))\) and \((p[x])\), which rely on auxiliary notions (e.g., propagation graphs and paths) that we now define.

**Definition 14 (Propagation Graph/Path).** Let \(w\) be the label assigned to the root of the nested sequent \(\Sigma\). We define the propagation graph \(PG(\Sigma) := PG_u(\Sigma)\) of a nested sequent \(\Sigma\) recursively on the structure of the nested sequent.

- \(PG_u(\emptyset) := (\emptyset, \emptyset, \emptyset)\);
- \(PG_u(A) := \{(u), \emptyset, \{(u, A)\}\} \text{ with } A \in \{A^*, A^\circ\}\);
- \(PG_u(\Delta_1, \Delta_2) := (V_1 \cup V_2, E_1 \cup E_2, L_1 \cup L_2) \text{ where } PG_u(\Delta_i) = (V_i, E_i, L_i)\);
- \(PG_u((x)(\Sigma)) := (\bigcup \{u\}, E \cup \{(u, x, v), (v, x, u)\}, L) \text{ where } v \text{ is a fresh label and } PG_v(\Sigma) = (V, E, L)\).

We will often write \(u \in PG(\Sigma)\) to mean \(u \in V\), and \((u, x, v) \in PG(\Sigma)\) to mean \((u, x, v) \in E\). We define propagation paths, converses of propagation paths, and strings of propagation paths as in Definition 15.

For input or output formulae \(A\) and \(B\), we use the notation \(\Sigma\{A\}_w\) and \(\Sigma\{A\}_w\{B\}_u\) to mean that \((w, A) \in L\) and \((w, A), (u, B) \in L\) in \(PG(\Sigma)\), respectively. For example, if \(\Sigma := p \supset q^o, (b)p^*, \Pi, [a]p^*\), \(PG(\Sigma) := (V, E, L)\), and \((v, p \supset q^o), (u, p^*), (w, [a]p^*) \in L\), then both \(\Sigma\{p \supset q^o\}_v\{[a]p^*\}_w\) and \(\Sigma\{p^*\}_u\{v \supset q^o\}_v\) are valid representations of \(\Sigma\) in our notation.

We now prove that proofs can be translated between our refined labeled and nested systems. In order to prove this fact, we make use of the following definitions, which are based on the work of [27,32].

**Definition 15 (Labeled Tree Sequent/Derivation).** We define a labeled tree sequent to be a labeled sequent \(\Lambda := \mathcal{R}, \Gamma \vdash \Delta\) such that \(\mathcal{R}\) forms a tree and all labels in \(\Gamma, \Delta\) occur in \(\mathcal{R}\) (unless \(\mathcal{R}\) is empty, in which case every labeled formula in \(\Gamma, \Delta\) must share the same label).

We define a labeled tree derivation to be a proof containing only labeled tree sequents. We say that a labeled tree derivation has the fixed root property iff every labeled sequent in the derivation has the same root.

We now define our translation functions which transform a full nested sequent into a labeled tree sequent, and vice-versa. Our translations additionally depend on sequent compositions and labeled restrictions. If \(A_1 := \mathcal{R}_1, \Gamma_1 \vdash \Gamma_1'\) and \(A_2 := \mathcal{R}_2, \Gamma_2 \vdash \Gamma_2'\), then we define its sequent composition \(A_1 \otimes A_2 := \mathcal{R}_1, \mathcal{R}_2, \Gamma_1, \Gamma_2 \vdash \Gamma_1', \Gamma_2'\). Given that \(\Gamma\) is a multiset of labeled formulae, we define the labeled restriction \(\Gamma \upharpoonright w := \{A \mid w : A \in \Gamma\}\), and if \(w\) is not a label in \(\Gamma\), then \(\Gamma \upharpoonright w := \emptyset\). Moreover, for a multiset \(A_1, \ldots, A_n\) of formulae, we define \((A_1, \ldots, A_n)^* := A_1^*, \ldots, A_n^*\) and \((\emptyset)^* := \emptyset\), where \(* \in \{\bullet, \circ\}\).
Fig. 6. The nested sequent calculi $\mathcal{N}^*_i(A)$. We have a copy of $((x)\ast)$, $([x]\ast)$, $(p(x))$, and $(p_{[x]})$ for each $x \in \Sigma$, and the $(d_x)$ rule occurs in a calculus $\mathcal{N}^*_i(A)$ iff $D_x \in A$.

Definition 16 (Translation $\mathcal{L}$). We define $\mathcal{L}_w(\Sigma) := \mathcal{R}, \Gamma \vdash v : A$ as follows:

- $\mathcal{L}_w(\emptyset) := \emptyset \vdash \emptyset$
- $\mathcal{L}_w(A) := v : A \vdash \emptyset$
- $\mathcal{L}_w(A^\ast) := 0 \vdash v : A$

We note that since $\Sigma$ is a full sequent, the obtained labeled sequent will contain a single labeled formula in its consequent.

Example 4. We let $\Sigma := p \supset q, (b)[p^\ast, (\overline{\pi})[[a]p^\ast]$ and show the output labeled sequent under the translation $\mathcal{L}$.

$\mathcal{L}_w(\Sigma) = wR_b v, v R_{\overline{\pi}} u, v : p u : [a]p \vdash w : p \supset q$

Definition 17 (Translation $\mathcal{R}$). Let $\Lambda := \mathcal{R}, \Gamma \vdash w : A$ be a labeled tree sequent with root $u$. We define $\Lambda' \subseteq \Lambda$ iff there exists a labeled tree sequent $\Lambda''$ such that $\Lambda = \Lambda' \ast \Lambda''$. Let us define $\Lambda_u := \mathcal{R}^u, \Gamma^u \vdash \Delta'$ to be the unique labeled tree sequent rooted at $u$ such that $\Lambda_u \subseteq \Lambda, \Gamma^u \upharpoonright u = \Gamma \upharpoonright u$, and $\Delta' \upharpoonright u = \Delta \upharpoonright u$. We define $\mathcal{R}(\Lambda) := \mathcal{R}_u(\Lambda)$ recursively on the tree structure of $\Lambda$:

- $\mathcal{R}_u(\emptyset) := \emptyset$
- $\mathcal{R}_u(A) := v : A \vdash \emptyset$
- $\mathcal{R}_u(A^\ast) := 0 \vdash v : A$

In the second case above, we assume that $v R_{z_1} \ldots v R_{z_n}$ are all of the relational atoms occurring in the input sequent which have the form $vRx$.

Example 5. We let $\Lambda := wR_v v R_{\overline{\pi}} u, v : p u : [a]p \vdash w : p \supset q$ and show the output nested sequent under the translation $\mathcal{R}$.

$\mathcal{R}(\Lambda) = p \supset q, (b)[p^\ast, (\overline{\pi})[[a]p^\ast]$
Lemma 11. Every proof in \( L^*_\Sigma(A) \) of a labeled tree sequent is a labeled tree proof with the fixed root property.

Proof. Let us consider a proof in \( L^*_\Sigma(A) \) of a labeled tree sequent in a bottom-up fashion. With the exception of \( (\langle x \rangle_l) \) and \( ([x]_r) \), every rule of \( L^*_\Sigma(A) \) bottom-up preserves the relational atoms present in the conclusion of the rule (only changing the formulae associated with labels), showing that if the conclusion is a labeled tree sequent, then the premises will be labeled tree sequents. In the case of \( (\langle x \rangle_l) \) and \( ([x]_r) \), a new relational atom is added to the premise, thus creating a new branching edge due to the freshness condition of the rule, but demonstrating that the premise is a labeled tree sequent nonetheless. Hence, the property of being a labeled tree sequent will be inherited by all labeled sequents occurring in the proof, and since no rule of \( L^*_\Sigma(A) \) changes the ‘root’ of the conclusion as only forward edges are bottom-up added to labeled tree sequents, the proof will have the fixed-root property. \( \square \)

Theorem 8. Every proof of a labeled tree sequent in \( L^*_A \) is transformable into a proof in \( N^*_\Sigma(A) \), and vice-versa.

Proof. Follows from Lemma 11 and the fact that the rules of \( L^*_A \) and \( N^*_\Sigma(A) \) are translations of one another under the \( \mathcal{R} \) and \( \Sigma \) functions. \( \square \)

Theorem 9 (\( N^*_\Sigma(A) \) Soundness and Completeness). A formula \( A \) is derivable in \( N^*_\Sigma(A) \) iff \( A \) is \( A \)-valid.

Proof. Follows from Theorem 8 and 7. \( \square \)

We end this section by establishing a collection of proof-theoretic properties satisfied by each nested calculus \( N^*_\Sigma(A) \). In particular, we argue that certain rules of \( N^*_\Sigma(A) \) are hp-invertible and that the rules displayed in Figure 7 are hp-admissible.

Theorem 10. The following properties hold for \( N^*_\Sigma(A) \):  

1. The \( (\bot_r) \), \( (n) \), and \( (w) \) rules are hp-admissible;  
2. The \( (\lor_l) \), \( (\land_l) \), \( (\langle x \rangle_l) \), and \( (p_{[x]_l}) \) rules are hp-invertible;  
3. The \( (\bot_l) \) rule is hp-invertible in the right premise;  
4. The \( (m) \) and \( (c) \) rules are hp-admissible.

Proof. Every claim is shown by induction on the height of the given derivation. As the proofs of claims 1 - 3 are standard, we omit them, and only show the

| Rule | Premise | Conclusion |
|------|---------|------------|
| \( \Sigma(\bot) \) | \( \Sigma(\bot_r) \) | \( \Sigma(\bot) \) |
| \( \Sigma(n) \) | \( \Sigma(\Delta) \) | \( \Sigma([x]_l, [x]_r) \) |
| \( \Sigma(w) \) | \( \Sigma([x]_l, [x]_r) \) | \( \Sigma([x]_l, [x]_r) \) |
| \( \Sigma(A^*, A^*) \) | \( \Sigma([x]_l, [x]_r) \) | \( \Sigma([x]_l, [x]_r) \) |

Fig. 7. Hp-admissible rules.
proof of claim 4. We first argue the hp-admissibility of \((m)\), and then use this to demonstrate the hp-admissibility of \((c)\).

We note that the base cases are trivial, and with the exception of the \((p_{(x)})\) and \((p_{[x]})\) rules, all cases of the inductive step are trivial as \((c)\) freely permutes above each rule instance. Regarding the \((p_{(x)})\) and \((p_{[x]})\) cases, as discussed in Goré et al. [26, Figure 12], the \((m)\) rule preserves propagation paths, and therefore, if we permute \((m)\) above \((p_{(x)})\) or \((p_{[x]})\), then the rule may still be applied afterward.

Let us now argue the hp-admissibility of \((c)\) by induction on the height of the given derivation. The base cases are simple as applying \((c)\) to \((id)\) or \((\bot)\) yields another instance of the rule, thus showing that the conclusion is derivable without the use of \((c)\). For the inductive step, if neither contraction formula \(A^*\) is principal in the last rule applied above \((c)\), then we may freely permute \((c)\) above the rule instance. On the other hand, if one of the contraction formulae \(A^*\) is principal in the rule \((r)\) applied above \((c)\), then we use claim 2, claim 3, or the hp-admissibility of \((m)\), along with the inductive hypothesis to resolve the case. For instance, if \((r)\) is the rule \((\langle x\rangle l)\), then our proof is as shown below left. The desired conclusion may be derived by invoking the hp-admissibility of \((\langle x\rangle l)\), applying the hp-admissibility of \((m)\), and then applying IH, followed by an application of \((\langle x\rangle l)\).

\[
\begin{align*}
\Sigma\{\langle x\rangle A^*, \langle x\rangle A^*\} & \Rightarrow\Sigma\{\langle x\rangle A^*, (\langle x\rangle l)\} \text{ (c)} \\
\Sigma\{\langle x\rangle A^*, (\langle x\rangle l)\} & \Rightarrow\Sigma\{\langle x\rangle A^*\} & \text{IH} \\
\Sigma\{\langle x\rangle A^*\} & \Rightarrow\Sigma\{\langle x\rangle A^*\} & \text{IH}
\end{align*}
\]

\[\square\]

6 Properties of Intuitionistic Grammar Logics

We now put our refined labeled and nested systems to use, proving that intuitionistic grammar logics satisfy a certain collection of properties. We first employ our nested systems in establishing the conservativity of intuitionistic grammar logics over their (mono-)modal restrictions (defined below). In the second subsection, we show that it is undecidable to check if a formula is valid in an arbitrary grammar logic by means of a proof-theoretic reduction from the validity problem for classical grammar logics (which is known to be undecidable [9]). In the third and final section, we recognize that validity can be decided for simple intuitionistic grammar logics, which are defined by restricting the IPA’s that may occur as axioms. In the latter two subsections, we make use of our refined labeled systems as the syntax of such systems is better suited for our purposes.

6.1 Conservativity

Each intuitionistic modal logic exists as a mono-modal fragment of an intuitionistic grammar logic. In [37], (cut-free) nested sequent systems were provided
for an extensive class of intuitionistic modal logics, which can be seen as restricted variants of the nested systems presented in the previous section. We leverage this fact to establish the conservativity of intuitionistic grammar logics over their modal counterparts. Toward this end, we first define the class of intuitionistic modal logics from [37], and subsequently discuss the fundamental concepts required to state our conservativity result, ending the section with a proof thereof.

**Definition 18 (Intuitionistic Modal Logics).** We define the language $\mathcal{L}$ to be the set of all formulae generated via the following grammar in BNF:

$$A ::= p | \bot | A \lor A | A \land A | A \rightarrow A | \lozenge A | \square A$$

where $p$ ranges over the set $\Phi$ of propositional atoms. We define the base intuitionistic modal logic $\mathcal{I}\mathcal{K}$ to be the smallest set of formulae closed under substitutions of the following axioms and applications of the following inference rules.

- **A0** Any set of axioms for propositional intuitionistic logic
- **A1** $\square(A \land B) \supset (\square A \land \square B)$
- **A2** $\square(A \lor B) \supset (\square A \lor \square B)$
- **A3** $\neg \lozenge \bot$
- **A4** $\lozenge(A \lor B) \supset (\lozenge A \lor \lozenge B)$
- **A5** $\lozenge(A \rightarrow \square B) \supset \square(A \rightarrow B)$
- **R0** $A \quad A \supset B \quad (\text{mp})$
- **R1** $A \quad \square A \quad (\text{nec})$

We also consider extensions of $\mathcal{I}\mathcal{K}$ with sets $\mathcal{B}$ of the following axioms, where $n,k \in \mathbb{N}$. We refer to $\mathcal{D}$ as the seriality axiom and to each $\mathcal{H}\mathcal{S}\mathcal{L}$ as a Horn-Scott-Lemmon axiom.

$$\mathcal{D} : \square A \supset \lozenge A \quad \mathcal{H}\mathcal{S}\mathcal{L} : (\lozenge^n \square A \supset \square^k A) \land (\lozenge^k A \supset \square^n \lozenge A)$$

We define the intuitionistic modal logic $\mathcal{I}\mathcal{K}(\mathcal{B})$ to be the smallest set of formulae closed under substitutions of the axioms A0–A5 and $\mathcal{B}$, and closed under the inference rules R0 and R1.

For a given alphabet $\Sigma$, we can encode the language $\mathcal{L}(\Sigma)$ in the language $\mathcal{L}(\Sigma)$ by identifying the modalities $\langle x \rangle = \lozenge$ and $[x] = \square$ for a fixed character $x \in \Sigma$. We can then view the language $\mathcal{L}$ as the subset of $\mathcal{L}(\Sigma)$ containing only $x$-formulae, i.e. formulae from $\mathcal{L}(\Sigma)$ that only use the modalities $\langle x \rangle$ and $[x]$. Similarly, we can view each logic $\mathcal{I}\mathcal{K}(\mathcal{B})$ as a subset of $\mathcal{L}(\Sigma)$ by identifying each formula $A \in \mathcal{I}\mathcal{K}(\mathcal{B})$ with the formula $B \in \mathcal{L}(\Sigma)$ obtained by replacing every $\lozenge$ and $\square$ in $A$ by $\langle x \rangle$ and $[x]$, respectively. For the remainder of the section we view $\mathcal{L}$ and $\mathcal{I}\mathcal{K}(\mathcal{B})$ in the manner just described. For each intuitionistic modal logic, we can then define a corresponding nested calculus as follows:

**Definition 19 (NIK($\mathcal{B}$) [37]).** We define the nested calculus NIK($\mathcal{B}$) for $\mathcal{I}\mathcal{K}(\mathcal{B})$ to be the set of rules (id), ($\bot^*$), ($\lor^*$), ($\land^*$), ($\neg^*$), ($\impliedby^*$), ($\supset^*$), ($\lozenge$, $\square$), ($\langle x \rangle$), ($\lbrack x \rbrack$), ($p(x)$), and ($p_{\mathcal{B}}(x)$), which also contains ($d_x$) iff $D \in \mathcal{B}$. We define the grammar $g(\mathcal{B})$ used in the propagation rules ($p_{\mathcal{B}}(x)$) and ($p_{\mathcal{B}}(x)$) as follows: $\langle x \rangle \rightarrow (\bigtriangledown x)^n$, $\langle \mathcal{F} \rangle \rightarrow (\bigtriangledown x)^k$, $\langle \mathcal{F} \rangle \rightarrow (\bigtriangledown x)^n$ $\in g(\mathcal{B})$ iff $(\lozenge^n \square A \supset \square^k A) \land (\lozenge^k A \supset \square^n \lozenge A) \in \mathcal{B}$. 
For each intuitionistic modal logic $\text{IK}(\mathcal{B})$, the calculus $\text{NIK}(\mathcal{B})$ is isomorphic to and functions precisely as the nested calculus introduced for the same logic in $\text{NIK}(\mathcal{B})$ $3$ Hence, the following soundness and completeness result follows from $\text{NIK}(\mathcal{B})$ Theorem 6].

**Theorem 11 (Soundness and Completeness [37])**. Let $\Sigma$ be an alphabet with $x \in \Sigma$ and $A$ be an $x$-formula. Then, $A$ is derivable in $\text{NIK}(\mathcal{B})$ iff $A \in \text{IK}(\mathcal{B})$.

Let us now define the notion of conservativity in our setting. Afterward, we put our calculi to use and establish the conservativity relation between specific intuitionistic modal and grammar logics.

**Definition 20 (Conservative Extension)**. Let $\Sigma$ be an alphabet with $x \in \Sigma$. We define an intuitionistic grammar logic $\text{IK}_{m}(\Sigma, \mathcal{A})$ to be an $x$-extension of an intuitionistic modal logic $\text{IK}(\mathcal{B})$ iff (1) $D \in \mathcal{B}$ iff $D_x \in \mathcal{A}$, and (2) $\langle \Diamond^n \Box A \rangle \in \mathcal{B}$ iff $\langle \Diamond^n \Box (x)A \rangle$ and $\langle [x]A \rangle$. Last, we say that an intuitionistic grammar logic $\text{IK}_{m}(\Sigma, \mathcal{A})$ is an $x$-conservative extension of an intuitionistic modal logic $\text{IK}(\mathcal{B})$ iff for any $x$-formula $A$, if $A$ is a theorem of $\text{IK}_{m}(\Sigma, \mathcal{A})$, then $A$ is a theorem of $\text{IK}(\mathcal{B})$.

If one observes the rules employed in our nested calculi, they will find that every rule exhibits the sub-formula property, that is, the formulae occurring within the premise of a rule are sub-formule of those occurring in the conclusion. By this observation, along with the observation that no rule changes the character $x$ indexing a nesting $(x)[\Sigma]$, it follows that our nested calculi possess the separation property $\text{NIK}(\mathcal{B})$, summarized in the statement of the theorem below. We note that nested calculi also exhibit this property in the setting of classical grammar and tense logics $\text{NIK}(\mathcal{B})$.

**Theorem 12 (Separation)**. Let $\Sigma$ be an alphabet with $x \in \Sigma$, $A$ be an $x$-formula, and $\text{IK}_{m}(\Sigma, \mathcal{A})$ be an $x$-extension of $\text{IK}(\mathcal{B})$. Then, $A$ is derivable in $\text{NIK}_{\Sigma}(\mathcal{A})$ iff $A$ is derivable in $\text{NIK}(\mathcal{B})$.

**Proof**. First, we note that if $A$ is derivable in $\text{NIK}(\mathcal{B})$, then $A$ is derivable in $\text{NIK}_{\Sigma}(\mathcal{A})$ as the latter calculus is an extension of the former. Thus, the backward implication is trivial. We therefore argue that if $A$ is derivable in $\text{NIK}_{\Sigma}(\mathcal{A})$, then $A$ is derivable in $\text{NIK}(\mathcal{B})$.

Let $D$ be a proof of $A$ in $\text{NIK}_{\Sigma}(\mathcal{A})$. By the subformula property of $\text{NIK}_{\Sigma}(\mathcal{A})$, it follows that if any rule $(y)^{(\Sigma)}$, $(y)^{(\Sigma)}$, $(p_{(x)})$, or $(p_{[y]})$ is applied in $D$ with $y \in \Sigma \setminus \{x\}$, then either $y$ or $[y]$ must occur in $A$. This contradicts the fact that $A$ is an $x$-formula. Hence, $D$ only consists of rules that exist in $\text{NIK}(\mathcal{B})$. Furthermore, since $\text{IK}_{m}(\Sigma, \mathcal{A})$ is an $x$-conservative extension of $\text{IK}(\mathcal{B})$, we know that $(d_x) \in \text{NIK}_{\Sigma}(\mathcal{A})$ iff $(d_x) \in \text{NIK}(\mathcal{B})$. In addition, the side conditions on $(p_{(x)})$ and $(p_{[y]})$ will

---

3 In $\text{NIK}(\mathcal{B})$, each intuitionistic modal logic is denoted $\text{IK}(\mathcal{A})$ and its nested calculus is denoted $\text{NIK}(\mathcal{A})$. We have opted to use $\mathcal{B}$ as opposed to $\mathcal{A}$ in this section however to distinguish sets $\mathcal{A}$ of intuitionistic path axioms and sets $\mathcal{B}$ of Horn-Scott-Lemmon axioms.
be identical in \( N^*_\Sigma(A) \) and \( \text{NIK}(B) \) as each Horn-Scott-Lemmon axiom \((\Diamond^n \Box A \supset \Box^k A) \land (\Diamond^k A \supset \Box^n \Diamond A)\) and intuitionistic path axiom \((\overline{\alpha})^n(x) A \supset (x) A) \land ([x] A \supset [\overline{x}]^n [x]^k A)\) give rise to the same set of production rules. We may conclude that \( D \) is a proof of \( A \) in \( \text{NIK}(B) \).

**Corollary 2.** Let \( \Sigma \) be an alphabet with \( x \in \Sigma \) and \( \text{IK}^m(\Sigma, A) \) be an \( x \)-extension of \( \text{IK}(B) \). Then, \( \text{IK}^m(\Sigma, A) \) is an \( x \)-conservative extension of \( \text{IK}(B) \).

**Proof.** Let \( A \) be a theorem of \( \text{IK}^m(\Sigma, A) \). Then, \( A \) is derivable in \( N^*_\Sigma(A) \) by Theorem 9, from which it follows that \( A \) is derivable in \( \text{NIK}(B) \) by Theorem 12 above. Hence, \( A \) is a theorem of \( \text{IK}(B) \) by Theorem 11.

### 6.2 General Undecidability

We now establish the undecidability of the general validity problem over the class of intuitionistic grammar logics. In other words, we show that given an arbitrary intuitionistic grammar logic \( \text{IK}^m(\Sigma, A) \) and an arbitrary formula \( A \), it is undecidable to determine if \( \vdash_{\Sigma} A \). We prove this result by giving a reduction from classical grammar logics, for which it is known that determining the (in)validity of a formula relative to an arbitrarily given classical grammar logic is undecidable [9, Theorem 3.1]. Hence, we introduce classical grammar logics, and as our reduction is proof-theoretic in nature, we will also introduce their refined labeled systems [33, p. 98], which are based on the nested systems of Tiu et al. [49].

Classical grammar logics utilize a language similar to their intuitionistic counterparts, but where formulae are in negation normal form. We define this language, denoted \( L^C(\Sigma) \), via the following grammar in BNF:

\[
A ::= p \mid \neg p \mid A \lor A \mid A \land A \mid (x)A \mid [x]A
\]

where \( p \in \Phi \) and \( x \in \Sigma \). Although negation is restricted to propositional atoms in \( L^C(\Sigma) \), we can recursively define the negation \( \neg A \) of an arbitrary formula \( A \in L^C(\Sigma) \) as follows:

- \( \neg p ::= \neg p \)
- \( \neg (B \lor C) ::= \neg B \land \neg C \)
- \( \neg (B \land C) ::= \neg B \lor \neg C \)
- \( \neg (x)B ::= [x]\neg B \)
- \( \neg [x]B ::= (x)\neg B \)

We define \( \bot ::= p \land \neg p \) for a fixed \( p \in \Phi \) and \( A \supset B ::= \neg A \lor B \). Therefore, we may assume that \( L(\Sigma) \) and \( L^C(\Sigma) \) use the same signature since every logical connective in one language occurs or can be defined in the other.

**Definition 21 (Classical Grammar Logic).** Let \( \Sigma \) be an alphabet. We define the base classical grammar logic \( K^m(\Sigma) \) to be the smallest set of formulae from \( L^C(\Sigma) \) closed under substitutions of the following axioms and applications of the following inference rules. We note that we have an axiom and inference rule for each \( x \in \Sigma \).
Nested Sequents for Intuitionistic Grammar Logics

\begin{align*}
A0 \quad \text{Any set of axioms for propositional classical logic} & \quad R0 \quad \frac{A}{B} \quad (mp) \\
A1 \quad [x](A \supset B) \supset ([x]A \supset [x]B) & \quad R1 \quad \frac{A}{[x]A} \quad (nec) \\
A2 \quad A \supset [x](\overline{\tau})A & \\
\end{align*}

We also consider extensions of $K_m(\Sigma)$ with sets $\mathcal{A}$ of seriality axioms $D_x = [x]A \supset [x]A$ and path axioms $\langle x_1 \cdots x_n \rangle A \supset [x]A$. We define the classical grammar logic $K_m(\Sigma, \mathcal{A})$ to be the smallest set of formulae closed under substitutions of the axioms $A0$–$A2$ and $\mathcal{A}$, and closed under the inference rules $R0$ and $R1$. Last, for an intuitionistic grammar logic $IK_m(\Sigma, \mathcal{A})$ we define its corresponding classical grammar logic to be $K_m(\Sigma', \mathcal{A'})$ where (1) $\Sigma = \Sigma'$, (2) $D_x \in \mathcal{A}$ iff $D_x \in \mathcal{A}'$, and (3) $\langle x_1 \cdots x_n \rangle A \supset [x]A \Leftrightarrow \langle x_1 \cdots x_n \rangle A \supset [x]A \in \mathcal{A}$ iff $\langle x_1 \cdots x_n \rangle A \supset [x]A \in \mathcal{A}'$. For the remainder of the section, when we refer to $IK_m(\Sigma, \mathcal{A})$ and $K_m(\Sigma, \mathcal{A})$, we assume that the latter is the corresponding classical grammar logic of the former.

For those interested in the semantics of classical grammar logics, consult del Cerro and Penttonen [9]. The refined labeled systems for classical grammar logics are displayed in Figure 8 and employ labeled sequents of the form $R \vdash \Gamma$, where $R$ is a multiset of relational atoms and $\Gamma$ is a multiset of labeled formulae of the form $w : A$ with $w \in \text{Lab}$ and $A \in \mathcal{L}^c(\Sigma)$. We define $\neg \Gamma = w_1 : \neg A_1, \ldots, w_n : \neg A_n$ for $\Gamma = w_1 : A_1, \ldots, w_n : A_n$. Furthermore, we remark that the weakening right rule ($w_r$) and classical cut rule ($cut_c$) (shown in Figure 5) are admissible in each calculus $L_{k}^e(A)$ \[ Corollary 1. \] We also define the grammar $g(A)$ that parameterizes the $((x))$ rule as:

$$((x) \rightarrow \langle x_1 \rangle \cdots \langle x_n \rangle), (\langle \overline{\tau} \rangle \cdots \langle \overline{\tau} \rangle) \in g(A) \text{ iff } \langle x_1 \cdots x_n \rangle A \supset [x]A \in \mathcal{A}.$$  
Observe that if $K_m(\Sigma, \mathcal{A})$ corresponds to $IK_m(\Sigma, \mathcal{A})$, then both logics generate the same grammar $g(A)$ (see Definition 3).

Our reduction relies on a variant of the well-known double-negation translation, attributed to Gödel, Gentzen, and Kolmogorov [6]. We define a modal version of the translation below, and utilize it in a sequence of subsequent lemmata that are ultimately used to confirm our undecidability result.

**Definition 22 (Double-Negation Translation).** We recursively define the double-negation translation over the set of formulae in $\mathcal{L}(\Sigma) \cup \mathcal{L}^c(\Sigma)$ as follows:

- $p^N = \neg \neg p$
- $\neg p^N = \neg \neg p$
- $\bot^N = \neg \bot$
- $(A \lor B)^N = \neg (\neg A^N \land \neg B^N)$
- $(A \land B)^N = A^N \land B^N$
- $(A \vdash B)^N = A^N \vdash B^N$
- $(\langle x \rangle A)^N = \neg [x] \neg A^N$
- $([x] A)^N = [x] A^N$

**Lemma 12.** For any $A \in \mathcal{L}(\Sigma)$, $w : \neg \neg A^N \vdash w : A^N$ is derivable in $L_{k}^e(A)$. 
\[
\frac{\Gamma, w : p, w : \neg p}{\Gamma, w : p, w : \neg p} \quad (id) \quad \frac{\Gamma, w : A, w : B}{\Gamma, w : A \lor B} \quad (\lor) \quad \frac{\Gamma, w : A}{\Gamma, w : [x]A} \quad ([x])
\]

\[
\frac{\Gamma, w : A}{\Gamma, w : A \land B} \quad (\land) \quad \frac{\Gamma, w : B}{\Gamma, w : [x]A} \quad (\lor)
\]

\[
\frac{\Gamma, w : \langle x \rangle A, u : A}{\Gamma, w : \langle x \rangle A} \quad ((x)) \quad \text{only if } \exists \pi(w, u) \in PG(R)(s_p(w, u) \in L_g(A)(x))
\]

\[
\frac{\Gamma}{\Gamma, w : A} \quad (w_r) \quad \frac{\Gamma, w : A}{\Gamma, w : \neg A} \quad (\text{cutC})
\]

**Lemma 13.** If \( \Gamma \vdash \Gamma \) is derivable in \( \mathbf{L}_\Sigma^C(A) \), then \( \Gamma N \vdash w : \bot \) is derivable in \( \mathbf{L}_\Sigma^C(A) \).

**Proof.** We prove the result by induction on the height of the derivation of \( \Gamma \vdash \Gamma \) in \( \mathbf{L}_\Sigma^C(A) \). Below, we use \( \neg A \) to denote \( A \) prefixed with \( n \in \mathbb{N} \) negation symbols.

**Base case.** We may transform an instance of \((id)\) in \( \mathbf{L}_\Sigma^C(A) \) into a proof of the desired labeled sequent in \( \mathbf{L}_\Sigma^C(A) \). Recall that in the intuitionistic setting \( \neg A = A \supset \bot \), thus explaining the \((\supset)\) inference applied in the output proof, whose left premise is derivable by Lemma 2 and Theorem 3.
\[
\begin{align*}
\overline{R} \vdash \Gamma, u : p, u : \neg p & \quad (id) \\
\overline{R}, \neg \Gamma^N, u : \neg^3 p, u : \neg^4 p \vdash u & \quad (\bot_i) \\
\overline{R}, \neg \Gamma^N, u : \neg^3 p, u : \bot \vdash w & \quad (\lor_i)
\end{align*}
\]

\textbf{Inductive step.} We show the \((\llbracket x \rrbracket)\) case as the remaining cases are shown similarly. Below, the top sequent in \(\mathcal{D}\) is derivable by IH and the right premise of \((cut)\) is derivable by Lemma [12].

\[
\mathcal{D} = \frac{R, uR, v, \neg \Gamma^N, v : \neg A^N \vdash v : \bot}{w} \quad (w)
\]

\[
\mathcal{D} = \frac{R, uR, v, \neg \Gamma^N, u : \neg [x]A^N, v : \neg A^N \vdash v : \bot}{(\lor_i)}
\]

\[
\mathcal{D} = \frac{R, uR, v, \neg \Gamma^N, u : \neg [x]A^N + v : [x]A^N \vdash [x]r}{(cut)}
\]

\[
\frac{R, \neg \Gamma^N, u : \bot \vdash w : \bot}{(\lor_i)}
\]

\textbf{Lemma 14.} For any \(A \in \mathcal{L}_C^\Sigma\), \(\vdash w : \neg A^N, w : A\) is derivable in \(\mathcal{L}_C^\Sigma(A)\).

\textbf{Proof.} The result can be shown by a straightforward induction on the complexity of \(A\). \(\square\)

\textbf{Lemma 15.} If \(R, \Gamma \vdash w : A\) is derivable in \(\mathcal{L}_C^\Sigma(A)\), then \(R, \neg \Gamma, w : A\) is derivable in \(\mathcal{L}_C^\Sigma(A)\).

\textbf{Proof.} We prove the lemma by induction on the height of the given derivation.

\textbf{Base case.} We show the \((\bot_i)\) case as the \((id)\) case is trivial. Observe that in the output proof we use the definition of negation and \(\bot\) as \(p \land \neg p\) in \(\mathcal{L}_C^\Sigma\) to show how the proof is translated from \(\mathcal{L}_C^\Sigma(A)\) to \(\mathcal{L}_C^\Sigma(A)\).

\[
\frac{R, \Gamma, w : \bot \vdash u : A}{(\bot_i)} \quad \frac{R \vdash \neg \Gamma^N, w : \neg p, w : p, w : A}{(id)} \quad \frac{R \vdash \neg \Gamma^N, w : \neg p \lor p, w : A}{(\lor)}
\]

\[
\frac{R \vdash \neg \Gamma^N, w : \bot, w : A}{(\bot_i)}
\]

\textbf{Inductive step.} We show the \((p_{(\varsigma)})\) and \((\llbracket x \rrbracket)_i\) cases as the remaining are similar. For the \((p_{(\varsigma)})\) case, we invoke the admissibility of \((w)\) in \(\mathcal{L}_C^\Sigma(A)\) and note that the side condition of \((p_{(\varsigma)})\) holds in the output proof as \(R\) is unaffected in the translation.

\[
\frac{R, \Gamma \vdash u : A}{\frac{R, \neg \Gamma, u : A}{(p_{(\varsigma)})}} \quad \frac{R \vdash \neg \Gamma, u : A}{(w_r)} \quad \frac{R \vdash \neg \Gamma, w : \llbracket x \rrbracket A, u : A}{(\llbracket x \rrbracket)}
\]

\textbf{In the \((\llbracket x \rrbracket)_i\) case, we apply the definition of negation in \(\mathcal{L}_C^\Sigma\) to obtain the desired result.}
\[ R, wR_xu, u : A, \Gamma \vdash v : B \quad \frac{\implies R, \Gamma, w : (x)A \vdash v : B \quad (x)1} \quad \frac{\implies R, wR_xu \vdash \neg \Gamma, u : \neg A, v : B \quad ([x])} \quad \frac{\implies R \vdash \neg \Gamma, u : [x]A, v : B} \]

\[ \square \]

**Theorem 13.** \( A \in K_m(\Sigma, A) \) iff \( A^N \in I K_m(\Sigma, A) \).

**Proof.** For the forward direction, we assume that \( A \in K_m(\Sigma, A) \), which implies that \( \Gamma \vdash w : A \) is derivable in \( L^C_\Sigma(\mathcal{A}) \) by completeness. By Lemma [13] it follows that \( w : \neg A^N \vdash u : \perp \) is derivable in \( L^C_\Sigma(\mathcal{A}) \), and so, \( w : \neg A^N \vdash w : \perp \) is derivable in \( L^C_\Sigma(\mathcal{A}) \) as \( u : \perp \vdash w : \perp \) is an instance of \( (\perp_\perp) \) and \( (cut) \) is admissible. Therefore, \( \Gamma \vdash w : \neg A^N \) is derivable in \( L^C_\Sigma(\mathcal{A}) \), meaning \( \Gamma \vdash w : A^N \) is derivable in \( L^C_\Sigma(\mathcal{A}) \) by Lemma [12] and the admissibility of \( (cut) \).

For the backward direction, we assume that \( A^N \in I K_m(\Sigma, A) \), which implies that \( \Gamma \vdash w : A^N \) is derivable in \( L^C_\Sigma(\mathcal{A}) \) by completeness. By Lemma [15] \( \vdash w : A^N \) is derivable in \( L^C_\Sigma(\mathcal{A}) \), showing that \( \Gamma \vdash w : A \) is derivable in \( L^C_\Sigma(\mathcal{A}) \) since \( \vdash w : \neg A^N, w : A \) is derivable in \( L^C_\Sigma(\mathcal{A}) \) by Lemma [14] and \( (cutC) \) is admissible.

**Corollary 3.** It is undecidable whether a formula of an arbitrarily given intuitionistic grammar logic is a theorem.

**Proof.** Follows from Theorem [13] and the fact that it is undecidable if a formula of an arbitrary classical grammar logic is a theorem [9, Theorem 3.1]. \( \square \)

### 6.3 Decidability

Despite the undecidability of the general validity problem, certain subclasses of intuitionistic grammar logics remain decidable. In this section, we identify such a subclass by relating it to the class of intuitionistic modal logics proven decidable by Simpson [17, Section 7.3]. Simpson established the decidability of \( I K \) extended with combinations of the seriality, reflexivity, and symmetry axioms by demonstrating that every validity of such a logic has a finite number of proofs within a certain form. Thus, decidability of a formula is obtained by searching through this finite set, and if a proof is found, then the formula is known to be valid, and if a proof is not found, then the formula is known to be invalid.

With basic modifications and extensions, Simpson’s decidability method may be straightforwardly adapted to our setting of intuitionistic grammar logics. For an arbitrary alphabet \( \Sigma \), we obtain the decidability of all intuitionistic grammar logics \( I K_m(\Sigma, A) \) such that

\[ A \subseteq \{(y)^n A \supset (x)A) \land ([x]A \supset (y)^n A) \mid x, y \in \Sigma, 0 \leq n \leq 1 \} \cup \{D_x \mid x \in \Sigma \}. \]

We refer to all such intuitionistic grammar logics as *simple*. Moreover, we explicitly present the propagation rules in Figure [9] that appear in the refined labeled systems for simple intuitionistic grammar logics. For each axiom of the form \( (A \supset (x)A) \land ([x]A \supset A) \), \( L^C_\Sigma(\mathcal{A}) \) includes the rules \( (r_{\perp x}) \) and \( (r_{\neg x}) \), and for each axiom of the form \( ((y)^n A \supset (x)A) \land ([x]A \supset (y)^n A) \), \( L^C_\Sigma(\mathcal{A}) \) includes the...
Fig. 9. Propagation rules for simple intuitionistic grammar logics.

\[
\begin{align*}
\frac{\mathcal{R}, \Gamma \vdash w : A}{\mathcal{R}, \Gamma \vdash w : \langle x \rangle A} & \quad (r_{\langle x \rangle}) \\
\frac{\mathcal{R}, \Gamma, w : [x]A, w : A \vdash v : B}{\mathcal{R}, \Gamma, w : [x]A \vdash v : B} & \quad (r_{[x]}) \\
\frac{\mathcal{R}, wRy, \Gamma \vdash u : A}{\mathcal{R}, wRy, \Gamma, w : \langle x \rangle A} & \quad (p^1_{\langle x \rangle}) \\
\frac{\mathcal{R}, uRyw, \Gamma \vdash u : A}{\mathcal{R}, uRyw, \Gamma, w : \langle x \rangle A} & \quad (p^2_{\langle x \rangle}) \\
\frac{\mathcal{R}, wRy, \Gamma, w : [x]A, u : A \vdash v : B}{\mathcal{R}, wRy, \Gamma, w : [x]A \vdash v : B} & \quad (p^1_{[x]}) \\
\frac{\mathcal{R}, uRyw, \Gamma, w : [x]A, u : A \vdash v : B}{\mathcal{R}, uRyw, \Gamma, w : [x]A \vdash v : B} & \quad (p^2_{[x]})
\end{align*}
\]

We note that the proof of decidability for this class of logics is almost identical to Simpson's proof with the exception that basic modifications are required to handle the use of multiple modalities and converse modalities. By reading through Simpson's proof while taking our simple intuitionistic grammar logics into account, it is straightforward to verify the following theorem:

**Theorem 14.** The validity problem for simple intuitionistic grammar logics is decidable.

Due to the ease with which this proof is adapted to our setting, we omit it from the main text and provide a sketch of the proof in the appendix for the interested reader.

## 7 Concluding Remarks

In this paper we provided an in-depth study of intuitionistic grammar logics and their associated proof theory. We supplied this class of logics with labeled calculi, obtained from each logic’s semantics, and showed that certain $hp$-admissibility and $hp$-invertibility results obtained for these systems with a proof of syntactic cut-elimination. Subsequently, we showed how to apply the structural refinement methodology to derive deductively equivalent ‘refined’ labeled calculi, from which nested calculi were extracted. Moreover, these derivative systems were shown to exhibit favorable admissibility and invertibility properties just as their parent labeled systems. We then concluded by employing our refined labeled and nested systems in the establishment of conservativity and (un)decidability results.

A few interesting open problems still remain. For instance, certain inference rules in our provided systems are not (fully) invertible (e.g. the $(\supset \iota)$ rule), giving rise to the question of if variants of these systems can be produced which admit the complete invertibility of every inference rule. In relation to intuitionistic grammar logics more specifically, it could be worthwhile to investigate if
such logics possess the Craig interpolation property, by adapting proof-theoretic methods of interpolation to our setting [21,35]. Finally, as decidability was only recognized to hold for a relatively small class of intuitionistic grammar logics, it would be of interest to determine decidability for larger classes of such logics.

References

1. Bierman, G.M., de Paiva, V.C.V.: On an intuitionistic modal logic. Studia Logica: An International Journal for Symbolic Logic 65(3), 383–416 (2000), http://www.jstor.org/stable/20016199
2. Bozić, M., Dosen, K.: Models for normal intuitionistic modal logics. Studia Logica 43(3), 217–245 (1984)
3. Brouwer, L.E.J., Heyting, A.: L.E.J. Brouwer: Collected Works, Volume 1: Philosophy and Foundations of Mathematics. North-Holland Publishing Company; New York: American Elsevier Publishing Company (1975)
4. Brinnler, K.: Deep sequent systems for modal logic. Arch. Math. Log. 48(6), 551–577 (2009). https://doi.org/10.1007/s00153-009-0137-3
5. Bull, R.A.: Cut elimination for propositional dynamic logic without * , Z. Math. Logik Grundlag. Math. 38(2), 85–100 (1992)
6. Buss, S.R.: An introduction to proof theory. Handbook of proof theory 137, 1–78 (1998)
7. Castilho, M.A., del Cerro, L.F., Gasquet, O., Herzig, A.: Modal tableaux with propagation rules and structural rules. Fundamenta Informaticae 32(3, 4), 281–297 (1997)
8. del Cerro, L.F.n., Herzig, A.: Modal deduction with applications in epistemic and temporal logics. In: Gabbay, D.M., Hogger, C.J., Robinson, J.A. (eds.) Handbook of Logic in Artificial Intelligence and Logic Programming (Vol. 4): Epistemic and Temporal Reasoning, p. 499–594. Oxford University Press, Inc., USA (1995)
9. del Cerro, L.F., Penttonen, M.: Grammar logics. Logique et Analyse 31(121/122), 123–134 (1988)
10. Ciabattoni, A., Lyon, T., Ramanayake, R.: From display to labelled proofs for tense logics. In: Artëmov, S.N., Nerode, A. (eds.) Logical Foundations of Computer Science - International Symposium, LFCS 2018, Deerfield Beach, FL, USA, January 8-11, 2018, Proceedings. Lecture Notes in Computer Science, vol. 10703, pp. 120–139. Springer (2018). https://doi.org/10.1007/978-3-319-72056-2_8
11. Ciabattoni, A., Lyon, T., Ramanayake, R., Tiu, A.: Display to labelled proofs and back again for tense logics. ACM Transactions on Computational Logic 22(3), 1–31 (2021). https://doi.org/10.1145/3460492
12. Ciabattoni, A., Maletti, P., Sendzi, L.: Hypersequent and labelled calculi for intermediate logics. In: Galmiche, D., Larchey-Wendling, D. (eds.) Automated Reasoning with Analytic Tableaux and Related Methods. Lecture Notes in Computer Science, vol. 8123, pp. 81–96. Springer Berlin Heidelberg, Berlin, Heidelberg (2013)
13. Davies, R., Pfenning, F.: A modal analysis of staged computation. J. ACM 48(3), 555–604 (May 2001). https://doi.org/10.1145/382780.382785
14. Demri, S., de Nivelle, H.: Deciding regular grammar logics with converse through first-order logic. Journal of Logic, Language and Information 14(3), 289–329 (2005). https://doi.org/10.1007/s10849-005-5788-9
15. Došen, K.: Models for stronger normal intuitionistic modal logics. Studia Logica 44(1), 39–70 (1985)
16. Ewald, W.B.: Intuitionistic tense and modal logic. The Journal of Symbolic Logic 51(1), 166–179 (1986), [http://www.jstor.org/stable/2273953](http://www.jstor.org/stable/2273953)
17. Fagin, R., Moses, Y., Halpern, J.Y., Vardi, M.Y.: Reasoning about knowledge. MIT press (1995)
18. Fairtlough, M., Mendler, M.: An intuitionistic modal logic with applications to the formal verification of hardware. In: Pacholski, L., Tiuryn, J. (eds.) Computer Science Logic. pp. 354–368. Springer Berlin Heidelberg, Berlin, Heidelberg (1995)
19. Fitch, F.B.: Intuitionistic modal logic with quantifiers. Portugaliae mathematica 7(2), 113–118 (1948), [http://eudml.org/doc/114664](http://eudml.org/doc/114664)
20. Fitting, M.: Tableau methods of proof for modal logics. Notre Dame Journal of Formal Logic 13(2), 237–247 (1972)
21. Fitting, M., Kuznets, R.: Modal interpolation via nested sequents. Annals of pure and applied logic 166(3), 274–305 (2015). [https://doi.org/10.1016/j.apal.2014.11.002](https://doi.org/10.1016/j.apal.2014.11.002)
22. Galmiche, D., Salhi, Y.: Tree-sequent calculi and decision procedures for intuitionistic modal logics. Journal of Logic and Computation 28(5), 967–989 (06 2015). [https://doi.org/10.1093/logcom/exv039](https://doi.org/10.1093/logcom/exv039)
23. Gentzen, G.: Untersuchungen über das logische schließen. i. Mathematische zeitschrift 39(1), 176–210 (1935)
24. Gentzen, G.: Untersuchungen über das logische schließen. ii. Mathematische Zeitschrift 39(1), 405–431 (1935)
25. Glivenko, V.: Sur quelques points de la logique de m. brouwer. Bulletins de la classe des sciences 15(5), 183–188 (1929)
26. Goré, R., Postniece, L., Tiu, A.: On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics. Log. Methods Comput. Sci. 7(2) (2011). [https://doi.org/10.2168/LMCS-7(2:8)2011](https://doi.org/10.2168/LMCS-7(2:8)2011)
27. Goré, R., Ramanayake, R.: Labelled tree sequents, tree hypersequents and nested (deep) sequents. In: Bolander, T., Braüner, T., Ghilardi, S., Moss, L.S. (eds.) Advances in Modal Logic 9, papers from the ninth conference on “Advances in Modal Logic,” held in Copenhagen, Denmark, 22–25 August 2012. pp. 279–299. College Publications (2012), [http://www.aiml.net/volumes/volume9/Gore-Ramanayake.pdf](http://www.aiml.net/volumes/volume9/Gore-Ramanayake.pdf)
28. Heyting, A.: Die formalen regeln der intuitionistischen logik. Sitzungsbericht Preußische Akademie der Wissenschaften Berlin, physikalisch-mathematische Klasse II pp. 42–56 (1930)
29. Horrocks, I., Sattler, U.: Decidability of shiq with complex role inclusion axioms. Artificial Intelligence 160(1-2), 79–104 (2004)
30. Kashima, R.: Cut-free sequent calculi for some tense logics. Studia Logica 53(1), 119–135 (1994)
31. Kolmogorov, A.: On the principle of tertium non datur. mathematics of the ussr, sbornik 32, 646–667 (1925). In: Van Heijenoort, J. (ed.) From Frege to Gödel: a source book in mathematical logic, 1879-1931, vol. 9. Harvard University Press (1967)
32. Lyon, T.: On the correspondence between nested calculi and semantic systems for intuitionistic logics. Journal of Logic and Computation 31(1), 213–265 (12 2020). [https://doi.org/10.1093/logcom/exaa078](https://doi.org/10.1093/logcom/exaa078)
33. Lyon, T.: Refining Labelled Systems for Modal and Constructive Logics with Applications. Ph.D. thesis, Technische Universität Wien (2021)
34. Lyon, T., van Berkel, K.: Automating agential reasoning: Proof-calculi and syntactic decidability for stit logics. In: Baldoni, M., Dastani, M., Liao, B., Sakurai, Y., Zalila Wenkstern, R. (eds.) PRIMA 2019: Principles and Practice of Multi-Agent Systems - 22nd International Conference, Turin, Italy, October 28-31, 2019, Proceedings. Lecture Notes in Computer Science, vol. 11873, pp. 202–218. Springer International Publishing, Cham (2019)

35. Lyon, T., Tiu, A., Goré, R., Clouston, R.: Syntactic interpolation for tense logics and bi-intuitionistic logic via nested sequents. In: Fernández, M., Muscholl, A. (eds.) 28th EACSL Annual Conference on Computer Science Logic, CSL 2020, January 13-16, 2020, Barcelona, Spain. LIPIcs, vol. 152, pp. 28:1–28:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2020). https://doi.org/10.4230/LIPIcs.CSL.2020.28

36. Lyon, T.S.: A framework for intuitionistic grammar logics. In: Baroni, P., Benzmüller, C., Wáng, Y.N. (eds.) Logic and Argumentation. pp. 495–503. Springer International Publishing, Cham (2021)

37. Lyon, T.S.: Nested sequents for intuitionistic modal logics via structural refinement. In: Das, A., Negri, S. (eds.) Automated Reasoning with Analytic Tableaux and Related Methods. pp. 409–427. Springer International Publishing, Cham (2021)

38. Lyon, T.S.: Nested sequents for first-order modal logics via reachability rules (2022), https://arxiv.org/abs/2210.00789

39. Lyon, T.S., Gómez Álvarez, L.: Automating Reasoning with Standpoint Logic via Nested Sequents. In: Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning. pp. 257–266 (8 2022). https://doi.org/10.24963/kr.2022/26 https://doi.org/10.24963/kr.2022/26

40. Marin, S., Straßburger, L.: Label-free modular systems for classical and intuitionistic modal logics. In: Advances in Modal Logic 10, invited and contributed papers from the tenth conference on "Advances in Modal Logic," held in Groningen, The Netherlands, August 5-8, 2014. pp. 387–406 (2014), http://www.aiml.net/volumes/volume10/Marin-Strassburger.pdf

41. Negri, S.: Proof analysis in modal logic. Journal of Philosophical Logic 34(5-6), 507 (2005)

42. Orlov, I.E.: The calculus of compatibility of propositions. Mathematics of the USSR, Sbornik 35, 263–286 (1928)

43. Pitts, A.M.: Evaluation logic. In: IV Higher Order Workshop, Banff 1990. pp. 162–189. Springer (1991)

44. Plotkin, G., Stirling, C.: A framework for intuitionistic modal logics: Extended abstract. In: Proceedings of the 1986 Conference on Theoretical Aspects of Reasoning about Knowledge. p. 399–406. TARK ’86, Morgan Kaufmann Publishers Inc., San Francisco, CA, USA (1986)

45. Post, E.L.: Recursive unsolvability of a problem of Thue. The Journal of Symbolic Logic 12(1), 1–11 (1947)

46. Servi, G.F.: Axiomatizations for some intuitionistic modal logics. Rend. Sem. Mat. Univ. Politecn. Torino 42(3), 179–194 (1984)

47. Simpson, A.K.: The proof theory and semantics of intuitionistic modal logic. Ph.D. thesis, University of Edinburgh, College of Science and Engineering, School of Informatics (1994)

48. Straßburger, L.: Cut elimination in nested sequents for intuitionistic modal logics. In: Pfennig, F. (ed.) Foundations of Software Science and Computation Structures. Lecture Notes in Computer Science, vol. 7794, pp. 209–224. Springer Berlin Heidelberg, Berlin, Heidelberg (2013)
A Decidability Proof for Simple Intuitionistic Grammar Logics

Recall that for a simple intuitionistic grammar logic $\mathcal{L}_m(\Sigma, \mathcal{A})$ the set $\mathcal{A}$ of axioms is defined as follows:

$$\mathcal{A} \subseteq \{[(y)^n A \triangleright (x) A) \land ([x] A \triangleright [y]^n A) \mid x, y \in \Sigma, 0 \leq n \leq 1 \} \cup \{D_x \mid x \in \Sigma\}.$$

For the remainder of the appendix, we assume that $\mathcal{A}$ is defined as above. Furthermore, recall that for a simple intuitionistic grammar logic, the propagation rules from Figure 9 are used in the refined labeled calculus $L^*_m(\mathcal{A})$.

We will repeat definitions and lemmata due to Simpson to give the reader intuition regarding the proof of decidability, but will omit various details as they can be found in Simpson’s PhD thesis [17 Section 7.3].

We recursively define the modal depth $m_d(A)$ of a formula $A$ as follows:

$$m_d(p) = m_d(\perp) = 0, m_d(B \circ C) = \max\{m_d(B), m_d(C)\} \text{ for } \circ \in \{\land, \lor, \triangleright\}, \text{ and } m_d(\lor B) = m_d(B) + 1 \text{ for } \lor \in \{\land, \lor, \triangleright\}.$$

We also recursively define the set $S(A)$ of subformulas of a formula $A$ as follows:

$$S(p) = \{p\}, S(\perp) = \perp, S(B \circ C) = \{B \circ C\} \cup S(B) \cup S(C) \text{ for } \circ \in \{\land, \lor, \triangleright\}, \text{ and } S(\lor B) = \{\lor B\} \cup S(B) \text{ for } \lor \in \{\land, \lor, \triangleright\}.$$

We define $S(\Gamma)$, the set of subformulas of a formula $\Gamma$ as follows:

$$m_d(\Gamma) = \max\{m_d(A) \mid w : A \in \Gamma\} \text{ and } S(\Gamma) = \bigcup_{w : A \in \Gamma} S(A).$$

For a multiset $R$ of relational atoms, we define an $R$-extension as follows: (1) $\mathcal{R}$ is an $R$-extension, and (2) if $\mathcal{R}'$ is an $R$-extension and $u \not\in \text{Lab}(\mathcal{R})$, then $\mathcal{R}' \cup \{wRu\}$ is an $R$-extension. If $\mathcal{R}'$ is an $R$-extension, then a label $u \in \text{Lab}(\mathcal{R}')$ has depth $n \geq 0$ iff there exists a sequence $w_0 R_{x_1} w_1, \ldots, w_{n-1} R_{x_n} w_n \in \mathcal{R}'$ such that $w_0 \in \text{Lab}(\mathcal{R})$, $w_n \in \text{Lab}(\mathcal{R}' \setminus \text{Lab}(\mathcal{R}))$, and $w_n = u$. The depth of a label is well-defined since every label has only one such accessing sequence by the definition of an $R$-extension. We define an $R$-extension $\mathcal{R}'$ to be bounded relative to a labeled sequent $\mathcal{R}, \Gamma \vdash w : A$ iff the depth of every label in $\mathcal{R}'$ is less than or equal to $m_d(\Gamma, w : A)$, and we define the bounded-restriction of $\mathcal{R}'$
relative to a labeled sequent $R, \Gamma \vdash w : A$ to be the set of all relational atoms from $R'$ whose labels have a depth less than or equal to $m_d(\Gamma, w : A)$. A labeled sequent $R', \Gamma' \vdash w' : A'$ is defined to be \textit{bounded} relative to a labeled sequent $R, \Gamma \vdash w : A$ if one of the following holds:

1. For each $1 \leq i \leq n$, $R_i$ is an $R$-extension and if $u : B \in \Gamma_i \cup \{w_i : A_i\}$, then $u$ has a depth $n \leq d$, $B \in S(\Gamma, w : A)$, and $m_d(B) \leq d - n$.

2. There exists a bounded pseudo-derivation of $R, \Gamma \vdash w : A$ from the labeled sequents in $\{R_i, \Gamma_i \vdash w_i : A_i \mid 1 \leq i \leq n\}$ where each $R_i$ is the bounded-restriction of $R_i$.

\textbf{Lemma 16.} Let $D$ be a pseudo-derivation of $R, \Gamma \vdash w : A$ from the labeled sequents in $\{R_i, \Gamma_i \vdash w_i : A_i \mid 1 \leq i \leq n\}$ and let $d = m_d(\Gamma, w : A)$. Then, the following holds:

1. For each $1 \leq i \leq n$, $R_i$ is an $R$-extension and if $u : B \in \Gamma_i \cup \{w_i : A_i\}$, then $u$ has a depth $n \leq d$, $B \in S(\Gamma, w : A)$, and $m_d(B) \leq d - n$.

2. There exists a bounded pseudo-derivation of $R, \Gamma \vdash w : A$ from the labeled sequents in $\{R_i, \Gamma_i \vdash w_i : A_i \mid 1 \leq i \leq n\}$ where each $R_i$ is the bounded-restriction of $R_i$.

\textit{Proof.} The result is shown by induction on the size of the given pseudo-derivation $D$. We only show the $(p^2_{(x)})$ case of the inductive step as all cases are argued in an almost identical fashion as in [47] Lemma 7.3.5.

Suppose that $(p^2_{(x)})$ was applied at the top of $D$ as shown below.

\[
\frac{R', vR\mu, \Gamma' \vdash \langle x \rangle B}{R', vR\mu, \Gamma' \vdash u : \langle x \rangle B} \quad (p^2_{(x)})
\]

\[
\vdots
\]

\[
R, \Gamma \vdash w : A
\]

We know that $u$ occurs at a depth $n \leq d - 1$ because $m_d(\langle x \rangle B) \leq d - 1$ by the inductive hypothesis. This implies claim 1 since $v$ occurs at a depth $n - 1 < n \leq d - 1$, $B \in S(\Gamma, w : A)$, and $m_d(B) \leq d - (n - 1) \leq d - n$. For claim 2, we suppose that we have a bounded pseudo-derivation of $R'', \Gamma' \vdash u : \langle x \rangle B$ as shown below, where $R''$ is the bounded-restriction of $R'$.

Observe that a single application of $(p^2_{(x)})$ gives the desired bounded pseudo-derivation.

\[
R'', vR\mu, \Gamma' \vdash u : \langle x \rangle B
\]

\[
\vdots
\]

\[
R, \Gamma \vdash w : A
\]

$\square$
We now define a pre-order on all labeled sequents \( \mathcal{R}', \Gamma' \vdash w' : A' \) which are bounded relative to a given labeled sequent \( \mathcal{R}, \Gamma \vdash w : A \). This will be used to define an equivalence relation over the set of such sequents, partitioning the set into a finite number of equivalence classes, and ultimately permitting us to restrict the number of proofs of \( \mathcal{R}, \Gamma \vdash w : A \) considered during proof-search to a finite number. For the remainder of the appendix, we fix the labeled sequent \( \Lambda = \mathcal{R}, \Gamma \vdash w : A \) and only consider labeled sequents bounded relative to this one.

Let \( A_0 = \mathcal{R}_0, \Gamma_0 \vdash w_0 : A_0 \) and \( A_1 = \mathcal{R}_1, \Gamma_1 \vdash w_1 : A_1 \) be bounded relative to \( \mathcal{R}, \Gamma \vdash w : A \). We define a morphism from \( A_0 \) to \( A_1 \) to be a function \( f \) such that (1) for all \( u \in \text{Lab}(A) \), \( f(u) = u \), (2) if \( u : B \in \Gamma_0 \), then \( f(u) \in \Gamma_1 \), (3) \( f(w_0) = w_1 \), and (4) if \( uR_\sim v \in R_0 \), then \( f(u)R_\sim f(v) \in R_1 \). We then define the pre-order \( \preceq \) as follows: \( A_0 \preceq A_1 \) iff \( A_0 = A_1 \) and there exists a morphism \( f \) from \( A_0 \) to \( A_1 \). The equivalence relation \( \equiv \) is then defined as: \( A_0 \equiv A_1 \) iff \( A_0 \preceq A_1 \) and \( A_1 \preceq A_0 \).

**Lemma 17.** The equivalence relation \( \equiv \) partitions the set of labeled sequents bounded relative to \( \Lambda \) into a finite number of classes.

**Proof.** The proof is similar to the proof of [47, Proposition 7.3.6], but with the exception that one must generalize the arguments of Simpson to account for multiple modalities. \( \Box \)

We now define an irredundant pseudo-derivation \( \mathcal{D} \) to be a bounded pseudo-derivation relative to a labeled sequent \( \mathcal{R}, \Gamma \vdash w : A \) such that no two labeled sequents \( A_0 = \mathcal{R}_0, \Gamma_0 \vdash w_0 : A_0 \) and \( A_1 = \mathcal{R}_1, \Gamma_1 \vdash w_1 : A_1 \) occur in \( \mathcal{D} \), with the former above the latter, such that \( A_0 \preceq A_1 \).

**Lemma 18.** Let \( \Lambda = \mathcal{R}, \Gamma \vdash w : A \) with \( A_i = \mathcal{R}_i, \Gamma_i \vdash w_i : A_i \) bounded relative to \( \Lambda \) for \( i \in \{1, 2\} \). Then,

1. If \( A_0 \preceq A_1 \) and \( A_0 \) has a bounded derivation of size \( n \) relative to \( \Lambda \), then \( A_1 \) has a bounded derivation of size \( n \) relative to \( \Lambda \).
2. If \( \Lambda \) is derivable in \( L_{\Sigma}^\mathcal{A}(\mathcal{L}) \), then \( \Lambda \) has an irredundant derivation.

**Proof.** The proof of claim 1 is similar to [47, Lemma 7.3.7] and the proof of claim 2 is similar to [47, Proposition 7.3.8]. \( \Box \)

**Theorem 15.** Let \( L_{\Sigma}^\mathcal{A}(\mathcal{L}) \) be a refined labeled calculus for a simple intuitionistic grammar logic \( \mathcal{W}_{\Sigma}(\mathcal{L}) \). Then, it is decidable to check if \( \models_{\mathcal{A}} \mathcal{R}, \Gamma \vdash w : A \) for any arbitrary labeled sequent \( \mathcal{R}, \Gamma \vdash w : A \).

**Proof.** We decide the validity of a labeled sequent \( \mathcal{R}, \Gamma \vdash w : A \) by searching for an irredundant proof of it in \( L_{\Sigma}^\mathcal{A}(\mathcal{L}) \). The proof-search algorithm is similar to the one given in [47, Section 7.3.3]. First, we take the labeled sequent \( \mathcal{R}, \Gamma \vdash w : A \) as input and check if it is an instance of \((id)\) or \((\bot_l)\). If so, then we know that \( \models_{\mathcal{A}} \mathcal{R}, \Gamma \vdash w : A \), and if not, then we continue proof-search, searching for all irredundant derivations of size 1, and then of size 2, and so forth by
applying all relevant rules of $L^*_\Sigma(A)$ bottom-up. Only a finite number of bottom-up rule applications are possible at each stage modulo the choice of fresh labels in the $(x_l)$, $([x]_r)$, and $(d_x)$ rules. However, this peculiarity can be overcome by fixing how fresh labels are chosen during proof-search. Also, note that it is computable to check irredundancy since the $\preceq$ relation is decidable on labeled sequents and it is decidable to check if a labeled sequent is bounded relative to the input $\mathcal{R}, \Gamma \vdash w : A$. Last, since the equivalence relation $\cong$ partitions the set of all labeled sequent bounded relative to $\mathcal{R}, \Gamma \vdash w : A$ into a finite number of equivalence classes, we know that eventually proof-search will terminate. □

**Theorem 14** The validity problem for simple intuitionistic grammar logics is decidable.

*Proof*. Follows from Theorem 14 above. □