Bosonization and QCD in Two Dimensions

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ABSTRACT

Quantum Chromodynamics (QCD) is believed to describe the strong interactions. In the asymptotic domain of large momenta, improved perturbation theory describes phenomena by use of point like quarks and gluons. But the spectrum and wave functions are in the non-perturbative domain, for which not much can be done analytically in four dimensions. In order to develop analytical methods physicists turned to simpler models, like $QCD_2$, the theory in one-space and one-time dimensions.

This review is devoted to the application of bosonization techniques to two dimensional QCD. We start with a description of the “abelian bosonization”. The methods of the abelian bosonization are applied to several examples like the Thirring model, the Schwinger model and QCD$_2$. The failure of this scheme to handle flavored fermions is explained. Witten’s non-abelian bosonization rules are summarized including the generalization to the case of fermions with color and flavor degrees of freedom. We discuss in details the bosonic version of the mass bilinear of colored-flavored fermions in various schemes. The color group is gauged and the full bosonized version of massive multiflavor QCD is written down. The strong coupling limit is taken in the “product scheme” and then in the $U(N_F \times N_C)$ scheme. Once the multiflavor $QCD_2$ action in the interesting region of the low energies is written down, we extract the semiclassical low lying baryonic spectrum. First classical soliton solutions of the bosonic action are derived. Quantizing the flavor space around those classical solutions produces the masses as well as the flavor properties of the two dimensional baryons. In addition low lying multibaryonic solutions are presented, as well as wave functions and matrix elements of interest, like $q\bar{q}$ content.
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1. Introduction

In this review we discuss methods of writing theories with fermions in terms of bosonic variables only, with the aim of arriving as close as possible to analytic solutions of Quantum Chromodynamics (QCD). So far, except for some very special situations involving very massive objects (see below), these methods have not been applicable to four dimensional space-time, but have been very successful in the case of two dimensions.

Bosonic variables describe various interactions that involve bosons as well as fermions. The interactions of the standard model are all carried by currents which are bosonic variables. The electro-weak and strong interactions are carried by $SU_W(2) \times U(1)$ and $SU_C(3)$ currents. In fact, even before QCD, the importance of weak and electromagnetic currents was realized in terms of Gell-Mann’s current algebra.\textsuperscript{[1,2]} This led the way to attempts at expressing the strong energy-momentum tensor in terms of these currents\textsuperscript{[3,4]}, and it turned out to be as a quadratic form. These approaches were extended to include electromagnetic and other interactions.\textsuperscript{[5]} However, the coefficient\textsuperscript{[3]} in the energy-momentum tensor was proportional to the inverse of $k$, the Kac-Moody level. In four dimensions the analog is infinite and cut-off dependent. The handling in four dimensions was not precise. The situation was clarified when two-dimensional models with a non-abelian symmetry group were considered. It was then found that the coefficient actually is proportional to the inverse of $k + C_G$, where $C_G$ is the second Casimir operator of the adjoint representation, rather than to the inverse of $k$.\textsuperscript{[6,7]} This came out after a careful handling of the operator products by normal ordering. The exact expression for the energy momentum tensor in terms of currents, first found in ref. [6] for $SU(N)$ (eqs.(5)&(12) there), is being widely used in conformal field theory as well as string theory.

Sometime before that, operator product expansions near the light-cone were investigated \textsuperscript{[8,9,10]}. Products of currents, in this domain, were instrumental in relation with deep-inelastic scattering of the leptons ($e, \mu, \nu$) off hadrons. Afterwards,
the Thirring model was solved in terms of currents\cite{11} signaling the possibility of having fermionic theories in terms of bosonic variables.

An important step forward in solving non-abelian gauge theories was achieved by 't Hooft\cite{12}, who found the meson spectrum in $QCD_2$ in the limit of large number of colors $N_C$, with $e^2 N_C$ fixed. (Many works on that followed, but this is not part of our review). The baryon spectrum, however, remained unsolved until the methods of bosonization were developed.

The start of bosonization was with the now called “abelian bosonization” of Coleman\cite{13} and Mandelstam\cite{14}, generalized to many fermions by Halpern\cite{15}. Attempts of solving $QCD_2$ using these schemes met with technical difficulties\cite{16}. A fundamental step forward was obtained by Witten\cite{17} introducing “non-abelian bosonization”. This opened the way to analyzing $QCD_2$ in the semi-classical limit obtaining the low-lying baryon spectrum,\cite{18,19} multibaryons\cite{20} and matrix elements of interest, like for example various quarks content.\cite{21} The treatment is in the spirit of Skyrme model,\cite{22} which in four dimensions is an “educated guess” of the low energy effective action.\cite{23} A full discussion and references are in the review.

Bosonization techniques have been widely applied and developed in the context of string and superstring theories. The applications varied from bosonization of world-sheet fermions in various superstring theories through the bosonic construction of fermion vertex operators and all the way to the bosonization of anticommuting and commuting ghost systems of dimension $\lambda, 1 - \lambda$. The bosonic description in the heterotic string motivated the discussion of bosonization of chiral fermions. Various formulations were suggested for chiral bosons and a large body of literature deals with this topic. Another development important for string theories was the derivation of bosonization rules for higher genus Riemann surfaces. The action of the “non-abelian bosonization” which is now referred to as the WZW model is a very essential tool in the realization of rational conformal fields theories which relate to the Kac-Moody algebras. Gauged WZW models were invoked to describe conformal coset space models and led to interesting string models like the
two dimensional black hole background. Since all these string and conformal field theory related topics are a large subject matter by themselves and since they are not directly related to the topic of the present review, they are not discussed here and the reference list does not cover them.

As for the four dimensional cases, the treatment was in connection with monopole induced proton decay\cite{24} and fractional charges induced on monopoles by light fermions.\cite{25} In these cases the light fermions move in the field of a very heavy particle, so that only the lowest wave is considered, thus obtaining dynamics in the radial variables only, namely, one space dimension. One can then use the bosonization techniques. To get a theory on a whole \((-\infty, \infty)\) line, a “reflection principle” was used\cite{25} as the boundary conditions at \(r = 0\) were suitable.

We hope that some of methods discussed in this review will find application in four dimensional physics, as some feature of the quark content and spectrum indicate. Let us also mention recent work\cite{26} (that we will not review), to get the “constituent quarks” out of the basic \(QCD_2\) Lagrangian.

This review is organized as follows. Chapter 2 is devoted to abelian bosonization. We start with Mandelstam’s bosonic expressions for free massless Weyl fermions. The associated chiral currents and energy momentum tensor are discussed via the corresponding Kac-Moody and Virasoro algebras. The ideas of bosonization are applied to the examples of the Thirring model and then to massive fermion. We also describe the coupling of the bosonized theory to an electromagnetic field, namely, the Schwinger model. Next we introduce flavor degrees of freedom and mention some related problems. Finally the abelian bosonization of flavored QCD is written down and discussed. The second topic, presented in chapter 3, is the non-abelian bosonization of colored-flavored fermions. Witten’s non-abelian bosonization rules are summarized followed by the generalization for the case of fermions with color and flavor degrees of freedom. Whereas a consistent bosonization of the mass term of unflavored fermions is quite straightforward, the analog for both color and flavor groups, faces some difficulties. It is shown that the
bosonic mass term in what is referred to as the “product scheme” fails to reproduce fermionic correlation functions. We then discuss a solution to this difficulty via the bosonization of $U(N_F \times N_C)$ WZW action at level $k = 1$. Chapter 4 deals with the bosonic version of massive multiflavor QCD$_2$. We first describe a “Hybrid” approach where abelian bosonization is invoked to take care of the color degrees of freedom and a WZW picture to cope with the flavor ones. Coming back to the full non-abelian bosonization approach, two prescriptions of gauging the corresponding WZW action are described. The strong coupling limit is taken in the “product scheme” and then in the $U(N_F \times N_C)$ scheme. Once the multiflavor QCD$_2$ action in the interesting region of the low energies is written down, we devote chapter 5 to the semiclassical extraction of the low lying baryonic spectrum. Classical soliton solutions of the bosonic action are derived. Quantizing the flavor space around those classical solutions produces the masses as well as the flavor properties of the two dimensional baryons. In addition low lying multibaryonic solutions are presented. We also review baryonic wave functions and matrix elements of interest, like various quarks content. Chapter 6 includes a summary of the review together with some concluding remarks and some open problems.

Notations

We use Minkowski metric with $g_{00} = 1$ and $g_{11} = -1$. Light-cone coordinates are written as $x_{\pm} = \frac{1}{\sqrt{2}}(x_0 \pm x_1)$, and $\epsilon_{01} = 1$.

$\frac{1}{2} T^A$ are the matrices representing $SU(N)$ generators, with $Tr(T^A T^B) = 2 \delta^{AB}$.

$e_c$ is the QCD$_2$ gauge coupling. It has a dimension of mass, thus being the analog of the QCD scale in four dimensions.

The constant $c$ appearing in the bosonization formulae is $c = \frac{1}{2} e^\gamma \simeq 0.891$, where $\gamma$ is the Euler constant. $\tilde{c}$ is given by $\tilde{c} = \frac{c}{2\pi}$.

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2. Abelian Bosonization

The heart of the bosonization idea is the equivalence of the massless Dirac field and the massless scalar field in two space-time dimensions. There is a correspondence at the operator level between all operators of one theory and those of the other. This correspondence allows one to relate also the mass term and possible interaction terms and thereby to extend the equivalence beyond the free massless level.

2.1. The Equivalence of the Free Massless Theories

The explicit construction of fermion fields in terms of boson fields is due to Mandelstam.\[^{14}\] Left and right Weyl fermions $\Psi_L$ and $\Psi_R$ are given by

$$
\Psi_L = \sqrt{\frac{c\mu}{2\pi}} : \exp \left[ -i\sqrt{\pi} \left( \int_{-\infty}^{x} d\xi \pi(x) + \phi(x) \right) \right] :
$$

$$
\Psi_R = \sqrt{\frac{c\mu}{2\pi}} : \exp \left[ -i\sqrt{\pi} \left( \int_{-\infty}^{x} d\xi \pi(x) - \phi(x) \right) \right] :
$$

where $c$ is a constant. A computation yields $c = \frac{1}{2}e^\gamma \sim 0.891$, where $\gamma$ is the Euler constant. The normal ordering denoted by $: :$ is performed with respect to the scale $\mu$.

The equal time commutation relations of the $\phi$-field

$$
[\phi(x, t), \pi(y, t)] = i\delta(x - y)
$$

imply the canonical anti-commutation relations for the $\Psi$ field:

$$
\{\Psi_{L,R}^\dagger(x, t), \Psi_{L,R}(y, t)\} = \delta(x - y) .
$$

as can be verified using the explicit construction eq. (2.1). The fermion field $\Psi$ is therefore, an inherently non-local functional of the scalar field. However fermion
bilinears, such as currents or masses, are local functionals. The vector current, for example

\[ J^\mu =: \bar{\Psi} \gamma^\mu \Psi := \frac{-1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi \]  

(2.4)

This identification of \( J^\mu \) leads automatically to a conserved current

\[ \partial_\mu J^\mu = 0 \]

independent of the equations for \( \phi \). This is a “topological” conservation, connected with choosing the “vector conservation” scheme (see also 3.2). In the later applications, we will demand more freedom in the renormalization of interacting theories, in particular the possibility to have a vector current anomaly. The bosonization procedure there will therefore be somewhat modified. The modification will correspond to a change of renormalization scheme.

The overall coefficient of the current is such that the fermion number charge

\[ Q = \int_{-\infty}^{\infty} dx j_0(x) = 1 \]  

(2.5)

for the \( \Psi \)-field. This is the normalization condition that we will follow throughout.

The equivalence of the bosonic and fermionic descriptions is manifested in the fact that the two theories have the same current algebra and the same Virasoro algebra for the energy-momentum. The later is constructed from the currents in a Sugawara form.

Let us start with the current algebra. In addition to the “topologically” conserved vector current the bosonic theory has an axial current \( J_5^\mu = \frac{1}{\sqrt{\pi}} \partial_\mu \phi \) which is the Neother current associated with the invariance of the bosonic action under the global shift \( \delta \phi = \epsilon \). One can then define the left and right chiral currents
\[ J^\mu_\pm = J^\mu \pm \frac{J_3^\mu}{2} \] which correspond to shifts with \( \epsilon(x_+) \) and \( \epsilon(x_-) \). Using the commutation relation eqn.(2.2) the following current algebra is found

\[
\left[ J_\pm(x_\pm), J_\pm(x'_\pm) \right] = \frac{2i}{\pi} \delta'(x_\pm - x'_\pm) \tag{2.6}
\]

This is the same algebra as that of the fermionic chiral currents. The algebra of the currents and the energy-momentum tensor take a more familiar form once expressed in terms of the Laurent modes of operators. Passing to the euclidean plane, using complex coordinates \( z = x + it \), and defining \( J = \sqrt{\pi} J_- \), \( \bar{J} = \sqrt{\pi} J_+ \) one immediately realizes that the later currents are holomorphic and anti-holomorphic functions respectively \( \partial \bar{J} = \partial J = 0 \). Expanding \( J \) in a Laurent series \( J = \sum_n J_n z^{-(n+1)} \) leads to the Kac-Moody form of the current algebra

\[
[J_n, J_m] = + \frac{i}{2} n \delta_{n+m,0} \tag{2.7}
\]

and a similar algebra for \( \bar{J} \). The bosonic energy-momentum tensor which is constructed from the chiral currents \( T_{\pm \pm} = \pi : J_\pm J_\pm : \) obeys the identical Virasoro algebra

\[
\left[ T_\pm(x_\pm), T_\pm(x'_\pm) \right] = 2i \left[ T_\pm(x_\pm) + T_\pm(x'_\pm) \right] \delta'(x_\pm - x'_\pm) - \frac{i}{6\pi} \delta'''(x_\pm - x'_\pm) \tag{2.8}
\]

which is identical to that of the fermionic energy-momentum tensor. Translation to the Laurent modes \( T = \sum_n L_n z^{-(n+2)} \) leads to leads to the well known form of the Virasoro algebra

\[
[L_n, L_m] = (m - n)L_{m+n} + \frac{1}{6} (m-1)m(m+1)\delta_{n+m,0} \tag{2.9}
\]

Coupling the theory to an abelian gauge field, the equivalence of the bosonic and the fermionic formulations can be demonstrated by showing that the corresponding effective actions are identical. This is presented in section 3 for the non-abelian case.
The equivalence of the bosons and the fermion bilinears is not only mathematical. The fermion Fock-space contains those bosons as physical states.\cite{11,13} The reason for this is that in one space dimension a massless field can move either to the left or to the right. A Dirac fermion and its anti-particle having together zero fermionic charge and moving in the same direction will never separate. They are therefore indistinguishable from a free massless boson. This picture changes when masses are introduced, and the above relations will be approached at energies high compared to the mass scale.

2.2. The Thirring Model

The Thirring model is a current-current interaction, given by the Lagrangian density\cite{27}

\[ \mathcal{L} = i \bar{\Psi} \phi \Psi - \frac{1}{2} g J^\mu J_\mu \]  

(2.10)

where \( J_\mu = : \bar{\Psi} \gamma_\mu \Psi : \). The model is exactly solvable \cite{28,29} and meaningful for \( g > -\pi \). Dell’Antonio, Frishman and Zwanziger\cite{11} studied the model further by means of the operator product expansion on the light-cone. They expressed the fermionic bilinears of the model as function of the current, and obtained expressions which seem very natural in the light of the bosonization procedure which was discovered later. The limit of \( g > -\pi \) (see also eq. (2.13)) follows from the scheme \( a = 1 \) in ref.\cite{11}, namely that \( \Psi \) has a vector charge one.

Following Mandelstam\cite{14}, we generalize the bosonization formula (1) to include a parameter \( \beta \), whose role will become clear later.

\[ \Psi_L = \sqrt{\frac{c_\mu}{2\pi}} : \exp \left[ -i \sqrt{\pi} \left( \frac{2\sqrt{\pi}}{\beta} \int_{-\infty}^{x} d\xi \pi(\xi) + \frac{\beta}{2\sqrt{\pi}} \phi(x) \right) \right] : \]  

\[ \Psi_R = \sqrt{\frac{c_\mu}{2\pi}} : \exp \left[ -i \sqrt{\pi} \left( \frac{2\sqrt{\pi}}{\beta} \int_{-\infty}^{x} d\xi \pi(\xi) - \frac{\beta}{2\sqrt{\pi}} \phi(x) \right) \right] : \]  

(2.11)

The equal-time anti-commutation relations are still obeyed. Eq. (3) for the
vector current is now modified to:

\[ J^\mu = -\frac{\beta}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi, \]  

(2.12)

where the normalization corresponds again to the charge being the fermion number appropriate to the field \( \Psi \). Eq. (2.11) provides the operatoric solution to the Thirring model where \( \Psi \) is a free massless field. The constant \( \beta \) is related to the coupling constant \( g \) through the formula\[^{[13]}\]

\[ \frac{\beta^2}{4\pi} = \frac{1}{1 + g/\pi} \]  

(2.13)

Eq. (2.12) implies that the central term in the Kac-Moddy algebra will be \( \frac{\beta^2}{4\pi^2} \), as compared to the \( \frac{1}{\pi} \) of the free case.

The analogy between the fermionic and bosonic theory therefore holds even when an interaction is introduced. The equivalent bosonic theory is still free, however.

2.3. ABEILAN BOSONIZATION OF THE MASS TERM

We now introduce a mass term for the fermion. The mapping to a bosonic theory still holds. Moreover, its form is precisely (2.15) below. The bosonic field will now be interacting, and the interaction will be precisely the mapping of the fermionic mass term.

The definition of the mass term, as that of the current, requires some care due to the appearance of the products of operators at the same point. In fact, when \( x \) approaches \( y \) one gets\[^{[14]}\] the operator product expansion

\[ \Psi_R^\dagger(x)\Psi_L(y) = \frac{c\mu}{2\pi} |c\mu(x - y)|^\delta : e^{-i\beta\phi} : \]
\[ \Psi_L^\dagger(x)\Psi_R(y) = \frac{c\mu}{2\pi} |c\mu(x - y)|^\delta : e^{i\beta\phi} : \]  

(2.14)

where \( \delta = -\frac{g}{2\pi}(1 + \frac{\beta^2}{4\pi}) \). The proper fermion mass term will therefore be defined
by

\[
\lim_{y \to x} \int_{-\infty}^{\infty} dx \left| c\mu(x - y) \right|^{-\delta} m \bar{\Psi}(x) \Psi(y) = \frac{c\mu}{\pi} m \int_{-\infty}^{\infty} dx : \cos \beta \phi(x) : \tag{2.15}
\]

With \( \mu \) chosen such that \( m = \frac{\mu \pi}{c\beta^2} \), the mass term transforms in the bosonic language to

\[
\Delta L = \frac{\mu^2}{\beta^2} : \cos \beta \phi :
\]

We remind the reader that the normal ordering is performed with respect to \( \mu \).

The bosonic model at which we arrived is the Sine-Gordon interaction. The classical Sine-Gordon model admits a soliton solution. It is time-independent and interpolates between adjacent wells of the scalar potential with finite energy. In the quantum theory this classical solution becomes a particle. (For a review of the Sine-Gordon system see ref.[30]. The classical equation of motion is

\[
\phi'' - \frac{\mu^2}{\beta^2} \sin(\beta \phi) = 0 \tag{2.16}
\]

The static soliton solution is given by

\[
\phi = \frac{4}{\beta} \tan^{-1}\left[ \exp \mu(x - x_0) \right]
\]

where \( x_0 \) is the “center” of the soliton.

Using eq. (2.5) we can compute the fermionic number of this solution. It is given by

\[
Q = \frac{\beta}{2\pi} \left[ \phi(\infty) - \phi(-\infty) \right] = 1 \tag{2.17}
\]

The fermion has therefore a direct physical interpretation as the Sine-Gordon soliton.
It is interesting to note a remarkable property of eq. (2.13) which relates
the coupling constants of the Thirring model and its bosonic equivalent, the Sine-
Gordon model. The weak coupling of one theory is the strong coupling of the other.
This property occurs often in bosonized theories and hints at the usefulness of the
method in dealing with theories at strong coupling, where perturbative methods
tail.

2.4. The Massive Schwinger Model

The bosonization technique has turned out to be a useful tool in the investiga-
tion of other two dimensional models. Our next example is the massive Schwinger
model.\textsuperscript{[31,32]} The Lagrangian of this model is given by

\begin{equation}
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \bar{\Psi} (i \partial - e \mathcal{A} - m) \Psi
\end{equation}

As is well-known, the gauge field in two space-time dimensions is not dynamical.
Following Coleman\textsuperscript{[32]} we choose the axial gauge \( A_1 = 0 \), and the other component
\( A_0 \) can be solved as a function of the electric current. The resulting electric field
will be

\begin{equation}
F_{01} = e \partial_1^{-1} j_0 + \frac{e \theta}{2\pi} \quad J^\mu = : \bar{\Psi} \gamma^\mu \Psi : \tag{2.19}
\end{equation}

\( \theta \) is a new parameter in the theory. It is a vacuum angle, analogous to the vacuum
angle due to instanton tunnelling in 4-dimensional QCD. It is the conjugate to the
winding number, appearing in two dimensions for the abelian case, as follows from
\( \Pi_1[U(1)] = \mathcal{Z} \) ( looking at a circle of large radius in the two dimensional plane).
Physics is invariant under \( \theta \to \theta + 2\pi \). From the expression above it is clear that
\( \frac{e \theta}{2\pi} \) corresponds to a background electric field. The periodicity is due to the ability
to produce electron-positron pairs is the vacuum when \( |\frac{e \theta}{2\pi}| > \frac{1}{2} \epsilon \), and these pairs
create their own electric field which reduces the original one.
The Hamiltonian density in our gauge is

\[ H = \bar{\Psi}(i\gamma_1 \partial_1 + m)\Psi + \frac{1}{2}(F_{01})^2 \]  \hspace{1cm} (2.20)

Bosonizing the current according to eq. (2.4) we find

\[ H = : \left[ \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1 \phi)^2 - \frac{cm^2}{\pi}\cos(2\sqrt{\pi}\phi) + \frac{e^2}{2\pi}\left(\frac{1}{2}\frac{\theta}{\sqrt{\pi}} - \phi\right)^2 \right] : \]  \hspace{1cm} (2.21)

where the normal ordering is with respect to the mass \( m \).

After a shift in the definition of \( \phi \)

\[ \phi \rightarrow \phi + \frac{1}{2}\frac{\theta}{\sqrt{\pi}} \]  \hspace{1cm} (2.22)

and normal ordering with respect to \( \mu^2 = e^2/\pi \) one finds

\[ H = : \left[ \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1 \phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{cm\mu}{\pi}\cos(\theta + 2\sqrt{\pi}\phi) \right] : \]  \hspace{1cm} (2.23)

From this expression the periodicity in \( \theta \) is manifest. In the strong coupling limit, the bosonized form of the Hamiltonian is very useful. The theory contains a meson of a mass that is approximately \( \mu \), and depending on the value of \( \theta \), the number of excitations of that meson is zero, finite or infinite.

For convenience, we will denote

\[ \tilde{c} = \frac{c}{\pi} \]  \hspace{1cm} (2.24)

from here on.
2.5. **Bosonization of Flavored fermions**

So far we have treated the case of one fermion flavor. The several flavor case is in fact more subtle than it appears at first sight, due to the different realization of global symmetries in the fermionic or bosonic languages. In the next chapter a “natural” scheme for bosonizing flavoured fermions, the “non-abelian bosonization”, will be introduced. Here we review the “abelian bosonization” attempt.

Flavor was first introduced in the bosonization procedure by Halpern.\[15\] The straightforward application of the procedure in eq. (2.1) for each one of the flavors will result in commuting fields $\Psi$ for different flavors. They can be made anticommuting by the introduction of a phase factor $K$ defined as a function of the fermion fields on which it acts.

$$K_i = \prod_{j<i} (-1)^{n_j}$$  \hspace{1cm} (2.25)

where $i,j = 1, \ldots, N_F$ are flavor indices. $n_j$ is the number of fermi fields with index $j$ on which $K_i$ acts. The bosonization formula eq. (1) is modified to

$$\Psi_L^i = \sqrt{\frac{c \mu}{2\pi}} K_i : \exp \left[ -i \sqrt{\pi} \left( \int_{-\infty}^{x} d\xi \dot{\phi}^i(\xi) + \phi^i(x) \right) \right] :$$

$$\Psi_R^i = \sqrt{\frac{c \mu}{2\pi}} K_i : \exp \left[ -i \sqrt{\pi} \left( \int_{\infty}^{x} d\xi \dot{\phi}^i(\xi) - \phi^i(x) \right) \right] :$$  \hspace{1cm} (2.26)

This method of bosonization, when applied to the flavored Thirring model, for example, transforms a fermionic system whose symmetry is $U(N_F)$ to a bosonic system whose manifest symmetry is only $[U(1)]^{N_F}$, corresponding to adding constant values to the $\phi$ fields and discrete $P_{N_F}$. The scalar fields transform non-linearly under the isopin, as induced from eqs. (2.26) and the linear transformation properties of the $\Psi$s. The isopin quantum numbers of the soliton solutions are of topological nature.\[15\] In the next chapter we will discuss the non-abelian $U(N_F)$ bosonization, where the full group structure remains manifest.
The flavored Thirring model, both massless and massive, was thoroughly investigated by Dashen and Frishman. In particular, they derived a “Sugawara-like” expression for the energy-momentum tensor in terms of the currents. They separated the energy-momentum tensor into a singlet part $T^{(B)}_{\mu\nu}$, and an $SU(N_F)$ part $T^{(V)}_{\mu\nu}$.

$$T_{\mu\nu} = T^{(B)}_{\mu\nu} + T^{(V)}_{\mu\nu}$$

$$T^{(B)}_{\mu\nu} = \frac{1}{2C_0} \left[ 2 : J_\mu J_\nu : -g_{\mu\nu} : J_\lambda J^\lambda : \right]$$

$$T^{(V)}_{\mu\nu} = \frac{1}{2C_1} \left[ 2 : J^A_\mu J^A_\nu : -g_{\mu\nu} : J^A_\lambda J^{A\lambda} : \right]$$

$C_0, C_1$ are constants.

$\tilde{C}_0 = C_0$ is the coefficient of the Schwinger term in the equal-time continuation relation of the single vector current. In light cone coordinates

$$\left[ J_\pm(u), J_\pm(u') \right] = 2iC_0 \delta'(u - u')$$

It is equal to $\frac{N_F}{\pi}$ in the free field case, and in general serves as a normalization of the current. $\tilde{C}_1 = \frac{N_F}{2\pi} + C_1$, where $C_1$ is the coefficient of the Schwinger term in the equal-time communication relation of the $SU(N_F)$ vector currents.

$$\left[ J^A_\pm(u), J^B_\pm(u') \right] = 2if^{ABC} J^C_\pm(u) \delta(u - u') +$$

$$+ 2iC_1 \delta^A \delta'(u - u')$$

$f^{ABC}$ being the $SU(N_F)$ structure constants. It turns out that $C_1 = \frac{1}{2\pi}$ and thus

$$\tilde{C}_1 = \frac{N_F + 1}{2\pi}$$

The singlet energy momentum tensor equal-time commutes contains Schwinger-type anomaly due to the normal ordering

$$\left[ T^{(B)}_{\pm}(u), T^{(B)}_{\pm}(u') \right] = 2i \left[ T^{(B)}_{\pm}(u) + T^{(B)}_{\pm}(u') \right] \delta'(u - u') - \frac{i}{6\pi} \delta'''(u - u')$$

So does the $T^V$, with a factor of $(N_F - 1)$ multiplying the $\delta'''$ factor. The coefficient
in eqn. (2.27) for $T^{(V)}_{\mu\nu}$, was first obtained correctly in ref. [7]. The original expression of Sugawara [3,4] did not include the part $N_F$ in $\tilde{C}_1$, which comes from careful normal ordering of the products of currents in the expressions for the energy-momentum (what comes up is actually the quadratic Casimir of the adjoint representation, which for $SU(N_F)$ is equal to $N_F$). Eqs. (2.27) are of basic use in conformal field theories as well as string theories.

2.6. Abelian Bosonization of flavored QCD

Let us now apply the prescription for flavored Dirac fermions for the analysis of QCD$_2$. It is convenient to start with the following Hamiltonian [33] [34]

$$H = (e_c)^2 \sum_{a,b=1}^{N_C} (E^a_b)^2 + \sum_{a,b=1}^{N_C} \sum_{i=1}^{N_F} \bar{\Psi}^a_i \gamma_1 (i\partial^a_i \partial^b_1 - A^b_0) \Psi^b_i + m \sum_{a=1}^{N_C} \sum_{i=1}^{N_F} \bar{\Psi}^a_i \Psi^a_i \quad (2.32)$$

in the gauge

$$A_0 = 0; \quad A^a_b = 0 \quad \text{for} \ a = b; \quad E^a_b = 0 \quad \text{for} \ a \neq b \quad (2.33)$$

The Gauss law of the system is given by

$$\partial^1 E^a_b = i[A, E]^a_b + \frac{1}{2} \sum_{i=1}^{N_F} \bar{\Psi}^a_i \Psi^b_i - \frac{\delta^a_b}{2N_C} \sum_{i=1}^{N_C} \sum_{d=1}^{N_C} \bar{\Psi}^a_d \Psi^d \quad (2.34)$$

Bosonizing now the various parts of the Hamiltonian one then gets:

$$H = H^0_{\Psi} + H_E - H^I$$

$$H^0_{\Psi} = \sum_{a,b=1}^{N_C} \left[ \frac{1}{2} \left( \partial^2 \phi_{ai} + (\partial^b \phi_{ai})^2 \right) + \frac{cm\mu}{\pi} : (1 - \cos 2\sqrt{\pi} \phi_{ai}) : \right]$$

$$H_E = \frac{e_c^2}{8\pi N_C} \sum_{a,b} \left[ \sum_i (\phi_{ai} - \phi_{bi}) \right]^2$$

$$H^I = \frac{2c^2\mu^2}{\pi^2} \sum_{a,b \neq \mu} \sum_{i,j} K_{ij,ab}^a N_{\mu} \left[ \cos \sqrt{\pi} \int_{-\infty}^{x} (\phi_{ai} - \phi_{aj} + \phi_{bi} - \phi_{bj})(\xi) d\xi \right]$$

$$\left[ \sin \sqrt{\pi} (\phi_{ai} + \phi_{aj} - \phi_{bi} - \phi_{bj})(\xi) \right] \sum_{k} (\phi_{ka} - \phi_{kb})^{-1}$$

$H^0_{\Psi}$ is the free “fermionic” part, after bosonization, thus in terms of bosonic vari-
ables. $H_E$ is the first term of the Hamiltonian eqn. (2.32) rewritten in terms of the boson variables corresponding to the fermions, by eliminating the electric fields through the Gauss law. Thus although originally coming from the kinetic part of the gauge potentials, it actually involves the interactions. This is a result of the fact that there are no transverse vectors in $1+1$ dimensions. $K_{ij}^{ab}$ is the generalized ordering operator (below).

In the case of one flavor, $i = j = 1$, $H^I$ does not involve the $\pi$ variables. This is the case that first appeared in [33]. The generalization to many flavors, as in ref.[34], did not have the $H^I$ correctly (for example the $\pi'$s did not appear at all). The full expression first appears in ref. [16], where the reader can also find the definition of $K_{ij}^{ab}$ (generalizing the $K_i$ of eqn.(2.26)).

The interaction term involves non-local terms which relate to colour non-singlets. For static and $e_c \to \infty$ approximation one finds[16] that for $N_F = 2$ the interaction is field independent. For $N_F \geq 3$, on the other hand, the limit is singular. This singularity should not be there in the predictions of physical quantities, but it renders further treatment very complicated.[16]

It is thus clear that a different method of bosonization is required for the treatment of flavored $QCD_2$. In the following chapters it will be shown that the so called “non-abelian bosonization” which was introduced by Witten[17] is an adequate tool for this purpose.
3. Non-Abelian Bosonization of colored flavored Fermions

3.1. Introduction

The non-abelian bosonization introduced by Witten is a set of rules assigning bosonic operators to fermionic ones, in a theory of free fermions invariant under a global non-abelian symmetry. In [17] the symmetry considered was $O(N)$ The bosonic operators are not expressed in terms of free bosonic fields like in the abelian bosonization but rather in terms of interacting group elements. In particular, bosonic expressions can be written for the energy-momentum tensor, the various chiral currents, the mass term and the complete action.

3.2. Witten’s Non-abelian bosonization

Let us start with $N$ free Majorana fermions governed by the action

$$ S_\Psi = \frac{i}{2} \int d^2 x (\Psi_- k \partial_+ \Psi_- k + \Psi_+ k \partial_- \Psi_+ k) $$ (3.1)

where $\Psi_-, \Psi_+$ are left and right Weyl-Majorana spinor fields, $\partial_\pm = \frac{1}{\sqrt{2}} (\partial_0 \pm \partial_1)$ and $k = 1, \ldots, N$. The corresponding bosonic action is the Wess-Zumino-Witten (WZW) action:

$$ S[u] = \frac{1}{2} S[u] $$

$$ S[u] = \frac{1}{8\pi} \int d^2 x Tr (\partial_\mu u \partial^\mu u^{-1}) + \frac{1}{12\pi} \int_B d^3 y \varepsilon^{ijk} Tr (u^{-1} \partial_i u)(u^{-1} \partial_j u)(u^{-1} \partial_k u) $$ (3.2)

where $u$ is a matrix in $O(N)$ whose elements are bosonic fields. The second term, the Wess-Zumino (WZ) term, is defined on the ball $B$ whose boundary $S$ is taken to be the euclidian two dimensional space-time. Now, since $\pi_2 [O(N)] = 0$, a
mapping $u$ from $S$ into the $O(N)$ manifold can be extended to a mapping of the solid ball $B$ into $O(N)$. The WZ term however is well-defined only modulo a constant. It was normalized so that if $u$ is a matrix in the fundamental representation of $O(N)$ the WZW term is well defined modulo $WZ \to WZ + 2\pi$. The source of the ambiguity is that $\pi_3[O(N)] \simeq Z$, namely there are topologically inequivalent ways to extend $u$ into a mapping from $B$ into $O(N)$.

The bosonic action is invariant under the chiral $O_L(N) \times O_R(N)$ symmetries just as the fermionic action. The associated transformations are:

$$u \to u' = Au \quad u \to u' = uB \quad A, B \subset O(N).$$  \hspace{1cm} (3.3)

In fact, the invariance holds also for transformation with $A(x_+)$ and $B(x_-)$, leading to the Kac-Moody algebras of left and right currents. The discussion of this algebra and the associated currents will be postponed to the next section.

The action (3.2) is conformally invariant as well. This property was proven by Witten\textsuperscript{17} who originally showed that if one generalizes (3.2) by taking a coupling $\frac{1}{4\lambda^2}$ as a coefficient of the first term and $\frac{k}{24\pi}$ of the WZ term (k integer), the $\beta$ function associated with $\lambda$ is given at the one loop level by

$$\beta = \frac{d\lambda^2}{d\ln \Lambda} = -\frac{(N-2)\lambda^2}{4\pi}[1 - (\frac{\lambda^2 k}{4\pi})^2],$$  \hspace{1cm} (3.4)

namely eq.(3.2) is at a fixed point $\lambda^2 = \frac{4\pi}{k}$ and hence exhibits conformal invariance. By showing that the energy momentum tensor obeys the Virasoro algebra, one can show that this property is in fact exact.

The bosonic picture for the theory of $N$ free massless Dirac fermions is built from a boson matrix $g \subset SU(N)$ and a real boson $\phi$. The bosonized action has
now the form

$$S[g, \phi] = S[g] + \frac{1}{2} \int d^2 x \partial_\mu \phi \partial^\mu \phi$$

The construction of the WZ term discussed above applies just as well to the $SU(N)$ case. Transformation with respect to global $SU_L(N) \times SU_R(N) \times U(1)$, $SU_R(N) \times SU_L(N) \times U(1)$ Kac-Moody, and conformal transformations leave the action invariant. One way to prove the equivalence of the theories now, for $N$ free massless Dirac fermions and the $k = 1$ WZW theory on $U(N)$ group manifold, is by showing that the generating functional of the current Green functions of the two theories are the same. For the fermions we have

$$e^{iW_\Psi(A_\mu)} = \int (d\Psi_+ d\Psi_- d\bar{\Psi}_+ d\bar{\Psi}_-) e^{i \int d^2 x \bar{\Psi}_i \tilde{D} \Psi}$$

where $\tilde{D}_\mu = \partial_\mu + i A_\mu$, $A_\mu = A_\mu^A (\frac{1}{2} T^A) + A_\mu^{(1)} \times 1$ and $(\frac{1}{2} T^A) \subset SU(N)$. $W_\Psi(A_\mu)$ was calculated by Polyakov and Weigmann [36] in a regularization scheme which preserves the global chiral $SU(N_L) \times SU(N_R)$ symmetry and the local $U(1)$ diagonal symmetry, [37] [38] leading to

$$W_\Psi(A_\mu) = -S[\tilde{A}] - S[\tilde{B}] - \frac{1}{4\pi N} \int d^2 x A_\mu^{(1)} A^{(1)}_\mu$$

where $\tilde{A}, \tilde{B} \subset SU(N)$ are related to the gauge fields $A_\mu^A$ by $i A_\mu^A = (\tilde{A}^{-1} \partial_+ \tilde{A})^A$, $i A^A = (\tilde{B}^{-1} \partial_- \tilde{B})^A$.

In the bosonic theory one calculates

$$e^{iW_B(A_\mu^A)} = \int [du] e^{iS[u] + i \int d^2 x (J^B_+ A_+^B + J_+^B A_+^B)}$$

$$e^{iW_B(A_\mu^{(1)})} = \int [d\phi] e^{\frac{i}{2} \int d^2 x [(\partial \phi)^2 + (J_- A_-^{(1)} + J_+ A_+^{(1)})]}$$

where $J_+ A_+^B$ and $J_+ A_+^{(1)}$ are the appropriate parts of $\frac{1}{4\pi} Tr[(g^{-1} \partial_+ g) A_-]$, and similarly for the $(- +)$ case and with $A_\pm^{(1)} = Tr(A_\pm)$. This functional integrals
can be performed exactly,[37] leading to

\[
W_B(A_\mu^4) = -S[\tilde{A}] - S[\tilde{B}] \quad W_B(A^{(1)}) = -\frac{1}{4\pi N} \int d^2x A_\mu^{(1)} A^{\mu(1)}
\] (3.8)

Thus the bosonic current Green functions are identical to those of the fermionic theory, the latter regulated in the way mentioned above. That \( S[u] \) of eqn.(3.2) leads to correlation functions for the currents in the left-right symmetric scheme, can be seen directly[38] without performing the functional integral explicitly.

3.3. Non-Abelian bosonization of Dirac fermions with color and flavor

In his pioneering work on non-abelian bosonization Witten [17] also proposed a prescription for bosonizing Majorana fermions which carry both \( N_F \) “flavors” as well as \( N_C \) “colors”, namely transform under the group \([O(N_F) \times O(N_C)]_L \times [O(N_F) \times O(N_C)]_R\). The action for free fermions is

\[
S_\Psi = \frac{i}{2} \int d^2x (\Psi_{-ai} \partial_+ \Psi_{-ai} + \Psi_{+ai} \partial_- \Psi_{+ai})
\] (3.9)

where now \( a=1,\ldots,N_C, i=1,\ldots,N_F \) are the color and flavor indices respectively. The equivalent bosonic action is

\[
\tilde{S}[g, h] = \frac{1}{2} N_c S[g] + \frac{1}{2} N_F S[h]
\] (3.10)

The bosonic fields \( g \) and \( h \) take their values in \( O(N_F) \) and \( O(N_C) \) respectively and \( S[u] \) is the WZW action given in (3.2) .
The bosonization dictionary for the currents was shown to be:

\[
J_{+ij} =: \Psi_{+ai}\Psi_{+aj} := \frac{iN_C}{2\pi}(g^{-1}\partial_+ g)_{ij} \quad J_{-ij} =: \Psi_{-ai}\Psi_{-aj} := \frac{iN_C}{2\pi}(g\partial_- g^{-1})_{ij}
\]

\[
J_{+ab} =: \Psi_{+ai}\Psi_{+bi} := \frac{iN_F}{2\pi}(h^{-1}\partial_+ h)_{ab} \quad J_{-ab} =: \Psi_{-ai}\Psi_{-bi} := \frac{iN_F}{2\pi}(h\partial_- h^{-1})_{ab}
\]

where \( : : \) stands for normal ordering with respect to fermion creation and annihilation operators. As for the bosonic expressions for the currents, regularization is obtained by subtracting the appropriate singular parts.

This procedure was already used in connection with the Thirring model in ref. [11], and for the case at hand in ref.[39]. It is actually much older, as in Schwinger’s “point splitting” technique\cite{40}, and in general from Wilson’s operator product expansion\cite{41}.

In terms of the complex coordinates \( z = \xi_1 + i\xi_2, \quad \bar{z} = \xi_1 - i\xi_2 \) (where \( \xi_1 \) and \( \xi_2 \) are complex coordinates spanning \( \mathbb{C}^2 \), and the Euclidian plane \( (\xi_1 \rightarrow x, \xi_2 \rightarrow -t) \) and Minkowski space-time \( (\xi_1 \rightarrow x, \xi_2 \rightarrow -it) \) can be obtained as appropriate real sections), one can express the currents as

\[
J(z)_{ij} \equiv \pi J_{-ij} = \frac{iN_C}{2}(g(z)\partial g^{-1})_{ij} \quad \bar{J}(\bar{z})_{ij} \equiv \pi J_{+ij} = \frac{iN_C}{2}(g^{-1}\partial_+ g)_{ij}
\]

and similarly for the colored currents.

In a complete analogy the theory of \( N_F \times N_C \) Dirac fermions can be expressed in terms of the bosonic fields \( g, h, \ e^{-i\sqrt{\frac{4\pi}{N_FN_C}}\phi} \) now in \( SU(N_F), SU(N_C) \) and \( U(1) \) group manifolds respectively. \cite{42,43} The corresponding action is now:

\[
S[g, h] = N_CS[g] + N_FS[h] + \frac{1}{2}\int d^2x \partial_\mu \phi \partial^\mu \phi
\]
This action is derived simply by substituting $ghe^{-i\sqrt{\frac{4\pi}{N_C N_F}}\phi}$ instead of $u$ in (3.2).

As for the equivalence between the bosonic and fermionic theories, we note that in both theories the commutators of the various currents have the same current algebra, and the energy-momentum tensor is the same when expressed in terms of the currents. But the situation changes when mass term are introduced (see next section 3.4). The bosonization rules for the color and flavor currents are obtained from (3.11) and (3.12) by replacing the Weyl-Majorana spinors with Weyl ones, and in addition we have the $U(1)$ current

\[ J^{(1)}(z) \equiv \sqrt{\pi} J^{(1)}_{-} = :\bar{\Psi}_{-ai} \Psi_{-ai} : = \sqrt{\frac{N_F N_C}{\pi}} \partial_{-} \phi \]

\[ J^{(1)}(\bar{z}) \equiv \sqrt{\pi} J^{(1)}_{+} = :\bar{\Psi}_{+ai} \Psi_{+ai} : = \sqrt{\frac{N_F N_C}{\pi}} \partial_{+} \phi \]  

(3.15)

The Kac-Moody algebras are given by:

\[ [J_n^A, J_m^B] = i f^{ABC} J_n^C + \frac{i}{2} k n \delta^{AB} \delta_{n+m,0} \]

(3.16)

where $J^A = Tr(T^A J)$, $T^A$ the matrices of $SU(N_C)$, $k = N_F$ for the colored currents and $J(z)$ is expanded in a Laurent series as $J(z) = \sum z^{-n-1} J_n$. Similar expression will apply for the flavor currents with $T^I$ the matrices of $SU(N_F)$, and the central charge $k = N_C$ instead of $N_F$. The commutation relation for $\bar{J}(\bar{z})$ will have the same form.

Generalizing the case of $SU(N) \times U(1)$\cite{7} to our case, the Sugawara form\cite{3} for the energy momentum tensor of the WZW action is given by:

\[ T(z) = \frac{1}{2N_C} \sum_A : J^A(z) J^A(z) : + \frac{1}{2N_F} \sum_I : J^I(z) J^I(z) : + \frac{1}{2\kappa} : J^{(1)}(z) J^{(1)}(z) : \]  

(3.17)

where the dots denote normal ordering with respect to $n$ ($n > 0$ meaning anihil-
The $\kappa$’s are constants yet to be determined. In terms of the Kac-Moody generators this can be written as

$$L_n = \frac{1}{2\kappa_C} \sum_{m=-\infty}^{\infty} : J^A_m J^A_{n-m} : + \frac{1}{2\kappa_F} \sum_{m=-\infty}^{\infty} : J^I_m J^I_{n-m} :$$

$$+ \frac{1}{2\kappa} \sum_{m=-\infty}^{\infty} : J_m^{(1)} J_m^{(1)} :$$

(3.18)

Now, by applying the last expression on any primary field $\phi_l$ we can get a set of infinitely many “null vectors” of the form

$$\chi^n_l = [L_n - \frac{1}{2\kappa_C} \sum_{m=n}^{0} : J^A_m J^A_{n-m} : - \frac{1}{2\kappa_F} \sum_{m=n}^{0} : J^I_m J^I_{n-m} : - \frac{1}{2\kappa} \sum_{m=n}^{0} : J_m^{(1)} J_m^{(1)} :] \phi_l$$

(3.19)

for any $n \leq 0$ (for $n > 0$ holds immediately). Since each of these vectors must certainly be a primary field, $L_m \chi^n = J^A_m \chi^n = J^I_m \chi^n = J_m \chi^n = O$ for $m > 0$ holds.

This leads to expressions for the various $\kappa$, for the central charge $c$ of the Virasoro Algebra and for the dimensions of the primary fields $\Delta_l = \Delta_{l+} + \Delta_{l-}$, in terms of $N_C, N_F$ and the group properties of the primary fields:

$$\kappa_C = \kappa_F = \frac{1}{2}(N_C + N_F), \quad \kappa = N_F N_C$$

(3.20)

$$c = \frac{N_C(N_F^2 - 1)}{(N_C + N_F)} + \frac{N_F(N_C^2 - 1)}{(N_C + N_F)} + 1 = N_F N_C.$$  

(3.21)

$$\Delta_l = \frac{(c^2_{l\pm})^F}{(N_F + N_C)} + \frac{(c^2_{l\pm})^C}{(N_F + N_C)} + \frac{(c^2_{l\pm})^{(1)}}{N_C N_F}.$$  

(3.22)

where $(c^2_{l\pm})^C$ is the eigenvalue of the $SU_{R,L}(N_C)$ second Casimir operator in the representation of the primary field $\phi_l$, namely $(1/2 T^A)(1/2 T^A) = (c^2_l)^C I$, and similarly.
for the flavor group. In the cases of $SU(N_C)$ and $SU(N_F)$ the discussion applies to $\Delta_{l+}$ or $\Delta_{l-}$ separately, with $C_{l+}^2$ and $C_{l-}^2$ respectively. For the $U(1)$ factor we discuss only the total dimension, as we do not wish to enter here into the issue of chiral bosons ([44,45] and references therein). Note that the expressions for $\kappa_F$ and $\kappa_C$ of eqn. (3.20) are an immediate generalization of ref.[7] in which the group $SU(N)$ was discussed with central term equal to one. There the factor was $N + 1$, the $N$ being the second Casimir of the adjoint representation, and the 1 being the central term.

The equivalence of the bosonic and fermionic Hilbert spaces was demonstrated by showing that the two theories have the same current algebra (Kac-Moody algebra), and that the energy-momentum tensor can be constructed from the currents in a Sugawara form. Goddard et al [46] showed that a necessary and sufficient condition for such a construction of the fermionic $T(z)$, in a theory with a symmetry group $G$, is the existence of a larger group $G \subset G'$ such that $G'/G$ is a symmetric space with the fermions transforming under $G$ just as the the tangent space to $G'/G$ does. Based on this theorem they found all the fermionic theories for which an equivalent WZW bosonic action can be constructed. The cases stated above fit in this category. Note in passing that this does not hold for cases where the symmetry group includes more non abelian group factors, like for example $SU(N_A) \times SU(N_F) \times SU(N_C) \times U(1)$. [47]

The prescription eqn. (3.14) described above, for the bosonic action that is equivalent to that of colored and flavored Dirac fermions, is by no means unique. In fact it will be shown that this prescription will turn out to be inconvenient once mass terms are introduced. Another scheme, based on the WZW theory of $U(N_FN_C)$ will be recommended.
3.4. The bosonization of a mass bilinear of Dirac fermions

A further bosonization rule has to be invoked for the mass bilinear. For a theory with a U(N) symmetry group the rule is the following:

\[ \Psi_+^\dagger \Psi_- = \bar{c} \mu N_\mu g_j^l e^{-i \sqrt{\frac{2}{N}} \phi} \] (3.23)

where \( N_\mu \) denotes normal ordering at mass scale \( \mu \). The fermion mass term \( m_q \bar{\Psi}^i \Psi_i \) is therefore

\[ m_c^2 N_\mu \int d^2 x Tr(g + g^\dagger) \] (3.24)

where \( m_c^2 = m_q \bar{c} \mu \), \( m_q \) is the quark mass, and \( c \) is the same constant as in eqn. (2.1). It is straightforward to show that the above bosonic operator transforms correctly under the \( U(N)_L \times U(N)_R \) chiral transformations. On top of that it has the correct total dimension

\[ \Delta = \Delta_g + \Delta_\phi = \left( \frac{N-1}{N} + \frac{1}{N} \right) = 1 \] (3.25)

where \( \Delta_g = \frac{N-1}{N} \) and \( \Delta_\phi = \frac{1}{N} \) are the dimensions associated with the SU(N) and U(1) group factors respectively. Moreover it was explicitly shown that the four point function

\[ G(z_1, z_2, z_3, z_4) = < g(z_1, \bar{z}_1) g^{-1}(z_2, \bar{z}_2) g^{-1}(z_3, \bar{z}_3) g(z_4, \bar{z}_4) > \] (3.26)

is given by\[^{[39]}\]:

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\begin{equation}
G(z_i, \bar{z}_i) = \left[ (z_1 - z_4)(z_2 - z_3)(\bar{z}_1 - \bar{z}_4)(\bar{z}_2 - \bar{z}_3) \right]^{-\Delta_g} G(x, \bar{x}) \tag{3.27}
\end{equation}

\[
G(x, \bar{x}) \text{ is the following function of the harmonic quotients}
\]

\[
x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \quad \text{and} \quad \bar{x} = \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_4 - \bar{z}_3)}{\bar{(z_1 - z_4)(z_2 - z_3)}},
\]

\begin{equation}
G(x, \bar{x}) = [x \bar{x}(1 - x)(1 - \bar{x})]^\frac{1}{N} \times \left[ I_1 \frac{1}{x} + I_2 \frac{1}{1 - x} \right] \left[ I_1 \frac{1}{\bar{x}} + I_2 \frac{1}{1 - \bar{x}} \right], \tag{3.28}
\end{equation}

where \( I_1, I_2, \bar{I}_1, \bar{I}_2 \) are group invariant factors. This result for the correlation function, combined with the \( U(1) \) part gives an expression identical to that for the fermionic bilinears. Moreover the result can be generalized to an \( n \)-point function.

The method of computing four point functions in conformally invariant theories was suggested already in ref. [7].

### 3.5. Bosonization of mass bilinears in the product scheme

A natural question here is how to generalize the rule (3.23) to theories given by (3.10) and its analog for the case of \( SU(N_F) \times SU(N_C) \times U(1) \) given in (3.14).

We call the latter the \textbf{product scheme}. It was argued that the bosonization rule for the latter case is \cite{42}

\[
\Psi_+^{ia} \Psi_{-bj} = \tilde{c}_\mu N_f \nu g_j^i h_b^a e^{-i \sqrt{4\pi \frac{N_f N_C}{N_f N_C}} \phi} \tag{3.29}
\]

Consequently, the bosonic form of the fermion mass term \( m_q \bar{\Psi}^{ia} \Psi_{ia} \) is therefore

\begin{equation}
m'^2 N_f \int d^2 x (Trg Trh + Trh^\dagger Trg^\dagger) e^{-i \sqrt{\frac{4\pi}{N_f N_C}} \phi} \tag{3.30}
\end{equation}

with \( m'^2 = m_q \tilde{c}_\mu \). Once again the bosonic operator (3.29) has the correct chiral
transformations and the proper dimension:

\[ \Delta = \Delta_g + \Delta_h + \Delta_\phi = \frac{N_F^2 - 1}{N_F(N_F + N_C)} + \frac{N_C^2 - 1}{N_C(N_C + N_F)} + \frac{1}{N_C N_F} = 1 \quad (3.31) \]

Unfortunately, the explicit calculation of the four point function\[39\] reveals a discrepancy between the fermionic and bosonic results. This can actually be understood directly. Since \( g \) and \( h \) are fields defined on entirely independent group manifolds, then (ignoring for a moment the U(1) factor) the analog of (3.26) can be written as:

\[
<g(z_1, \bar{z}_1)g^{-1}(z_2, \bar{z}_2)g^{-1}(z_3, \bar{z}_3)g(z_4, \bar{z}_4)> <h(z_1, \bar{z}_1)h^{-1}(z_2, \bar{z}_2)h^{-1}(z_3, \bar{z}_3)h(z_4, \bar{z}_4)> \]

(3.32)

This expression differs from the corresponding fermionic Green’s function in at least two aspects: (i) It includes independent “contractions” for the \( g \) and \( h \) factors, whereas in the fermionic correlation function the flavor and color contractions are correlated. (ii) The result (3.28) is true only for a bosonic field associated with Kac-Moody central charge \( k = 1 \). For \( g \) and \( h \), however, the central charges are \( N_C \) and \( N_F \) respectively. For such cases the expression for the Green’s function is much more complicated (expressed in terms of hyper-geometric functions) and does not resemble the case of free fermions.

3.6. Bosonization using the \( U(N_F \times N_C) \) WZW action

It is clear from the previous discussion that the bosonization prescription for our case needs an alteration. Apriori there can be two ways out: modifying the rule for the bosonization of the mass bilinear or using a different bosonic theory altogether. As for the first approach, eq. (3.29) preserves the proper chiral transformation laws under the product group \( SU(N_F) \times SU(N_C) \times U(1) \) as well as the
correct dimension, and therefore the number of possible modifications is very limited. For example one might think of multiplying the expression in eq. (3.29) by an operator which is a chiral singlet under the above group, with zero dimension. We do not know of such a modification. It may possibly be non-local. Therefore we are going to try a different bosonic theory than eq. (3.14). The symmetry of the free fermionic theory can actually be taken as $U_L(N_F \times N_C) \times U_R(N_F \times N_C)$ rather than $[SU(N_F) \times SU(N_C) \times U(1)]_L \times [SU(N_F) \times SU(N_C) \times U(1)]_R$. The natural bosonic action is hence a WZW theory of $u \subset U(N_F N_C)$ and with $k = 1$. The action will be given by eqn. (3.2). The currents are now

$$J(z)_{\alpha \beta} = \frac{i}{2}(u \partial_z u^{-1})_{\alpha \beta} \quad \tilde{J}(\tilde{z})_{\alpha \beta} = \frac{i}{2}(u^{-1} \partial_{\tilde{z}} u)_{\alpha \beta}$$ (3.33)

with $\alpha, \beta$ running from 1 to $N_F \times N_C$. The formulas (3.11-3.12) can be obtained from (3.33) by appropriate traces, over color for (3.11) and over flavor for (3.12). The mass bilinear is now

$$\Psi^\dagger_{+\alpha} \Psi_{-\beta} = \tilde{c} \mu N_\mu u_{\alpha \beta}$$ (3.34)

which is as (3.23) but with the $U(1)$ term absorbed in $u$.

Clearly the requirement for Sugawara construction of $T$, for proper chiral transformations of all the operators and for a correct dimension for the mass bilinear are fulfilled. Since now the flavor and color degrees of freedom are attached to the same bosonic field, the previous “contraction problem” in the n point functions is automatically resolved. Moreover as stated above the four-point function and in fact any Green’s function will now reproduce the results of the fermionic calculation.

The currents constructed from $u$ obey the Kac-Moody algebra with $k = 1$. The color currents, for instance, are $J^A = Tr(T^A J)$, where $T^A$ are expressed as
\((N_C N_F) \times (N_C N_F)\) matrices defined by \(\lambda^A \otimes 1\), with \(\lambda^A\) the Gell-Mann matrices in color space and 1 stands for a unit \(N_F \times N_F\) matrix. The central charge is \(k = N_F\). The same arguments will apply for the flavor currents, now with \(k = N_C\). The central charge for the \(U(1)\) current is \(N_C N_F\).

To see the difference between the present theory and the previous one let us express \(u\) in terms of \((N_F N_C) \times (N_F N_C)\) matrices \(\tilde{g}, \tilde{h}\), and \(\tilde{l}\) in \(SU(N_F), SU(N_C)\) and the coset-space \(SU(N_F \times N_C)/\{SU(N_F) \times SU(N_C) \times U(1)\}\) respectively, through \(u = \tilde{g} \tilde{h} e^{-i \frac{4\pi}{N_F N_C} \phi}\). Using the formula for expressing an action of the form \(S[A g B^{-1}]\) \([36]\) we get:

\[
S[u] = S[\tilde{g} \tilde{h}] + \frac{1}{2} \int d^2 x \partial_{\mu} \phi \partial^{\mu} \phi \\
S[\tilde{g} \tilde{h}] = S[\tilde{g}] + S[\tilde{l}] + S[\tilde{h}] + \frac{1}{2\pi} \int d^2 x Tr(g^\dagger \partial_+ \tilde{g} \partial_\perp \tilde{l} + \tilde{h}^\dagger \partial_+ \tilde{h} \partial_\perp \tilde{l})
\]

We can now choose \(\tilde{l} = l\) so that \(l \partial_\perp \partial_\dagger l\) will be spanned by the generators that are only in \(SU(N_F \times N_C)/\{SU(N_F) \times SU(N_C) \times U(1)\}\). This can be achieved by taking \(\tilde{u} = \tilde{g} \tilde{h} l\) (namely \(u\) but without the \(U(1)\) part), and then taking for \(\tilde{h} = h \otimes 1\) a solution of the equation \(\partial_\perp h \partial_\dagger l = \frac{1}{N_F} Tr_F[(\partial_\perp \tilde{u}) \tilde{l}]\), and similarly for \(g\) with \(\frac{1}{N_C} Tr_C\). These are also the conditions that the flavor currents be expressed in terms of \(\tilde{g}\) and the color currents in terms of \(\tilde{h}\). For this choice, the mixed term in the action \((3.35)\), the one involving products of \(\tilde{l}\)'s with \(\tilde{g}'s\) or \(\tilde{h}'s\) is zero, and so the new action is

\[
S[u] = N_C S[g] + N_F S[h] + \frac{1}{2} \int d^2 x \partial_{\mu} \phi \partial^{\mu} \phi + S[l]
\]

\((3.36)\)

Note that \(l\) is still an \(SU(N_C N_F)\) matrix while \(g\) and \(h\) are expressed now as \(SU(N_F)\) and \(SU(N_C)\) matrices respectively, but the matrix \(l\) involves only products of color and flavor matrices (not any of them separately).
4. Bosonized Massive Multiflavor QCD\textsubscript{2}

In the last chapter we derived the bosonization rules for massive fermions which transform under flavor and color symmetry groups. Here we develop the bosonic version of multiflavor massive QCD\textsubscript{2}. Applying the strong coupling limit we deduce the low energy effective action\cite{18,19}. In the product scheme\cite{18} this effective action comes out to depend on color singlet variables only, the $g$’s. It presents a derivation for a two dimensional “Skyrme model”,\cite{22} which in four dimensions was not derived directly,\cite{23} but argued on general grounds.

4.1. The “Hybrid” Approach

Before proceeding to gauge the color symmetry group of the colored flavored WZW model, we describe briefly another approach introduced by Gepner\cite{48} in which the flavor sector appears in the form of a WZW model but for the color degrees of freedom the gauged abelian bosonization is invoked. Recall the multiflavor QCD\textsubscript{2} in the abelian formulation (see section 2.6). In the gauge eqn. (2.32) one can use the Gauss law eqn. (2.33) to express the gauge fields in terms of the appropriate fermionic bilinears, which translate into bosonic group elements as follows:

$$\partial_1 e_a = \sqrt{\pi} \Sigma_i \Psi^\dagger_i a \Psi_i a = \frac{i}{\sqrt{\pi}} \partial_1 \text{Tr} \log g$$

(4.1)

where $g_a \in U(N_F)$ is one out of $N_C$ such matrices, and $e_a$ connects to the original electric fields by $2\sqrt{\pi} E_a^a = (e_a - \frac{1}{N_C} \Sigma_k e_k)$. One can now use again eqn. (2.33) again to also express $A_b^a$ for $a \neq b$ in terms of fermion densities. Inserting these
into the $QCD_2$ Hamiltonian given in eqn. (2.31) Gepner gets\textsuperscript{[48]}

$$\mathcal{H} = \mathcal{H}^0 + \mathcal{H}^I$$

$$\mathcal{H}^I = -\sum_{a,b} \frac{(e_c)^2}{32\pi^2 N_C} [\text{Tr}\log(g_a g_b^{-1})]^2 - \sum_{a,b} \pi\mu^2 \frac{\text{Tr}(g_a g_b^{-1})}{\text{Tr}\log(g_a g_b^{-1})} + \sum_a mc\sqrt{N_F}\text{Tr}(g_a)$$

\(\mathcal{H}^0\) includes the fermion kinetic term. For \(N_F = 2\) the potential is free from singularities, for \(N_F \geq 3\) it is not. In ref. \[48\] the case of \(N_F = 2\) was analyzed thoroughly deducing the corresponding low lying baryonic spectrum. In the present review the full WZW is emphasized, so we refer the reader to \[48\] for details in the framework of the “Hybrid” approach.

4.2. Gauging the WZW action

Coming back to the full WZW description of colored and flavored fermions, the next step toward the bosonized version of QCD$_2$ is obviously the introduction of the color interaction. This is achieved by gauging the the vector subgroup SU$_V(N_C)$ of SU$_L(N_C) \times$ SU$_R(N_C)$. There are various methods to gauge the model. Here we present two of them: a trial and error method,\textsuperscript{[49]} and gauging via covariantizing the current. Those methods are applicable also in the U($N_F N_C$) bosonization scheme.

(4.2a) Trial and error Noether method

The WZW action on SU($N_C$) group manifold is, as stated above, invariant under the global vector transformation \(h \rightarrow UhU^{-1}\) where \(U \subset SU(N_C)\). Now we want to vary \(h\) with respect to the associated local infinitesimal transformation \(U = 1 + i\epsilon(x) = 1 + iT^A\epsilon^A(x)\)

$$\delta_\epsilon h = i[\epsilon, h] \quad \delta_\epsilon h^{-1} = i[\epsilon, h^{-1}]$$

(4.3)
The variation of the action $S^{(0)}[h] \equiv S[h]$ under such a transformation is obviously given by:

$$\delta_\epsilon S^{(0)}[h] = - \int d^2x Tr(\partial_\mu \epsilon J^\mu) \quad (4.4)$$

where the Noether vector current is given by:

$$J_\mu = \frac{i}{4\pi} \{ [h^\dagger \partial_\mu h + h \partial_\mu h^\dagger] - \varepsilon_{\mu\nu} [h^\dagger \partial^\nu h - h \partial^\nu h^\dagger] \} \quad (4.5)$$

We introduce now the first correction term $S^{(1)}$ given by

$$S^{(1)} = \int d^2x Tr(A_\mu J^\mu) \quad \delta_\epsilon S^{(1)}[h] = - \int d^2x Tr[\partial_\mu \epsilon (J^\mu + J'^\mu)] \quad (4.6)$$

The second part of eqn. (4.6) is the variation of $S^{(1)}$ which is derived using the infinitesimal variation of the gauge field $\delta A_\mu = -D_\mu \epsilon = -(\partial_\mu \epsilon + i[A_\mu, \epsilon])$. $J'^\mu$ is found to be:

$$J'^\mu = -\frac{1}{4\pi} \{ [h^\dagger A_\mu h + h A_\mu h^\dagger - 2A_\mu] - \varepsilon_{\mu\nu} [h^\dagger A^\nu h - h A^\nu h^\dagger] \} \quad (4.7)$$

The second iteration will be by adding $S^{(2)}$ where now $J'^\mu$ is replacing $J^\mu$.

$$S^{(2)} = \int d^2x Tr(A_\mu J'^\mu) \quad \delta_\epsilon S^{(2)}[h] = -2 \int d^2x Tr(\partial_\mu \epsilon J'^\mu). \quad (4.8)$$

It is therefore obvious that

$$\delta_\epsilon [S^{(0)} + S^{(1)} - \frac{1}{2} S^{(2)}] = 0 \quad (4.9)$$

Hence $S[h, A_\mu] \equiv [S^{(0)} + S^{(1)} - \frac{1}{2} S^{(2)}]$ is given by

$$S[h, A_\mu] = \frac{1}{8\pi} \int d^2x Tr(D_\mu h D^\mu h^\dagger) + \frac{1}{12\pi} \int d^3y \varepsilon^{ijk} Tr(h^\dagger \partial_i h)(h^\dagger \partial_j h)(h^\dagger \partial_k h)$$

$$- \frac{1}{4\pi} \int d^2x \varepsilon_{\mu\nu} Tr[iA_\mu (h^\dagger \partial^\nu h - h \partial^\nu h^\dagger + ih^\dagger A^\nu h)] \quad (4.10)$$
which can also be written in light cone coordinates

\[ S[h, A_+, A_-] = S[h] + \frac{i}{2\pi} \int d^2 x Tr(A_+ h \partial_- h^\dagger + A_- h^\dagger \partial_+ h) \]

\[ - \frac{1}{2\pi} \int d^2 x Tr(A_+ h A_- h^\dagger - A_- A_+) \]  

(4.11)

(4.2b) Gauging via covariantization of the Neother current

In four space-time dimensions the current, in terms of bosonic matrices, involves up to third power\textsuperscript{[23]} in the gauge potentials. In D space-time dimensions the bare current will contain (D-1) derivatives and is gauged by replacing the ordinary derivatives with covariant derivatives and by adding terms which contain products of \( F_{\mu\nu} \) with powers of \( h \) and \( h^\dagger \) and covariant derivatives \( D_\mu h \) and \( D_\mu h^\dagger \). In two dimensions, however, there is no room for such terms in the gauge covariant current, as these terms involve \( \epsilon_{\mu_1 \ldots \mu_D} \) in D dimensions with one free index and the others contracted with \( F_{\mu\nu}'s \) and \( D_\mu's \), and in two dimensions they cannot be constructed. Therefore the covariantized current is given by:

\[ J_\mu(h, A_\mu) = \frac{i}{4\pi} \{ [h^\dagger D_\mu h + h D_\mu h^\dagger] - \epsilon_{\mu\nu}[h^\dagger D^\nu h - h D^\nu h^\dagger] \} \]  

(4.12)

Knowing the current we deduce the action via \( J_\mu = \frac{\partial S}{\partial A_\mu} \) getting straightforwardly eq. (4.11). Finally, we combine the gauged WZW action of the color group manifold, the WZW of the flavor group manifold and the action term for the gauge fields, to get the bosonic form of the action of massless QCD\textsubscript{2}. The well known fermionic form of the action is (a mass term will be added later),

\[ S_F[\Psi, A_\mu] = \int d^2 x \left\{ -\frac{1}{2e_c^2} Tr(F_{\mu\nu} F^{\mu\nu}) - \bar{\Psi}^a \left[ (i \not\partial + \not{\Lambda}) \psi \right]_a \right\} \]  

(4.13)

where \( e_c \) is the coupling constant to the color potentials (note it has mass dimen-

36
sions in 1+1 space-time), and

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \]

The bosonized action is

\[
S[g, h, A_+, A_-] = N_c S[g] + N_F S[h] \\
+ \frac{N_F}{2\pi} \int d^2x T r [i(A_+ h \partial_- h^\dagger + A_- h^\dagger \partial_+ h) - (A_+ h A_- h^\dagger - A_- A_+)] \\
- \frac{1}{2e_c^2} \int d^2x T r F_{\mu\nu} F^{\mu\nu}.
\]

(4.14)

4.3. The strong coupling limit

So far we wrote the bosonic version of massless QCD$_2$. It turns out that the mass term plays an important role in the determination of classical soliton solutions in 1+1 space-time dimensions. It is therefore required to switch on this term before deducing the low energy effective action. As was explained in the last chapter this we know how to do rigorously only in the scheme of $U(N_F N_C)$. Nevertheless, we want first to integrate over the gauge fields and high energy modes in the product scheme eqn. (4.13). In the next section we repeat the derivation in the other scheme. It will turn out that the product scheme can be used for the low mass states in the strong coupling limit.

We first have to choose a gauge. We choose a light-cone gauge $A_- = 0$. This has two advantages, namely no Fadeev-Popov ghosts and also no interaction terms quadratic in $A$. The latter fact makes it immediate to integrate out the gauge potentials. Define $H(x)$ by $\partial_- H = ih \partial_- h^\dagger$ with $H(-\infty, x_-) = 0$. Then also $H(\infty, x_-) = 0$ for colorless states. Now we add the mass term as in eqn. (3.30),
absorbing the $U(1)$ factor in $g$, and then integrate out $A_+^{[18]}$ to obtain

$$S[g, h] = N_c S[g] + N_F S[h] - \frac{1}{2} \left( \frac{N_F e_c}{2\pi} \right)^2 \int d^2 x Tr(H^2) + m^2 N_\mu \int d^2 x (TrgTrh + Trh^\dagger Trg^\dagger) \quad (4.15)$$

where $Trh$ is over color, $Trg$ over flavor, etc. In the strong coupling limit $e_c/m_q \to \infty$ the low energy effective action reads:

$$S[g] = N_c S[g] + m^2 N_m \int d^2 x (Trg + Trg^\dagger) \quad (4.16)$$

The determination of the parameter $m$ obtained after appropriate normal ordering, will be explained in the next section. Note that the analog of our strong coupling to the case of 3+1 space time, would be that of light current quarks compared to the QCD scale $\Lambda_{QCD}$.

4.4. Multiflavor QCD$_2$ using the $U(N_F \times N_C)$ Scheme

Let us now repeat the gauging of the $SU_V(N_C)$ subgroup in the framework of the $U(N_F \times N_C)$ bosonization procedure.

Using the gauging prescription discussed in section 4.2 we get first the action where the whole $SU(N_C N_F)$ is gauged, namely

$$S[u, A_+, A_-] = S[u] + \frac{i}{2\pi} \int d^2 x Tr(A_+ u \partial_- u^\dagger + A_- u^\dagger \partial_+ u) - \frac{1}{2\pi} \int d^2 x Tr(A_+ u A_- u^\dagger - A_- A_+) + m'^2 N_\tilde{m} \int d^2 x Tr(u + u^\dagger) \quad (4.17)$$

where we have also added a mass term with $m'^2 = m_q \tilde{m}$. Now since we are interested in gauging only the $SU(N_C)$ subgroup of $U(N_F N_C)$, we take $A_\mu$ to be related to the generator $T^D \subset SU(N_C)$ via $A_\mu = e_c A^D_\mu T^D$. We then add to this
action the kinetic term for the gauge fields $-\frac{1}{2e_c} \int d^2 x Tr(F_{\mu\nu}F^{\mu\nu})$. $e'_c$ has the appropriate form so that after tracing in flavor space the color gauge coupling is obtained, namely $e'_c = \sqrt{N_F} e_c$. The resulting action is invariant under:

$$u \to V(x)uV^{-1}(x) \quad A_\mu \to V(x)(A_\mu - i\partial_\mu)V^{-1}(x) \quad V(x) \subset SU_V(N_C)$$

(4.18)

$$u \to WuW^{-1} \quad W \subset U(N_F)$$

(4.19)

The symmetry group is now $SU_V(N_C) \times U(N_F)$, just as for the gauged fermionic theory. We choose the gauge $A_- = 0$, so now the action takes the form:

$$S[u, A_+] = S[u] + \frac{1}{e'^2} \int d^2 x Tr(\partial_- A_+)^2 + \frac{i}{2\pi} \int d^2 x Tr(A_+ u \partial_- u^\dagger) + m'^2 N_{\tilde{m}} \int d^2 x Tr(u + u^\dagger)$$

(4.20)

Upon the decomposition $u = g \tilde{h} e^{-i\sqrt{\frac{4\pi}{N_F N_C}} \phi}$ we see that the current that couples to $A_+$ is $\tilde{h} \partial_- \tilde{h}^\dagger$. In terms of $u$ it is the color projection $(u \partial_- u^\dagger)_C = \frac{1}{N_F} Tr_F[u \partial_- u^\dagger] - \frac{1}{N_C} Tr_C u \partial_- u^\dagger].$ Thus the coupling of the current to the gauge field $\frac{i}{2\pi} \int d^2 x Tr(A_+ \tilde{h} \partial_- \tilde{h}^\dagger)$. To proceed we define $\tilde{H}(x)$ by $\partial_\mu \tilde{H} = i\tilde{h} \partial_- \tilde{h}^\dagger$. We take the boundary conditions to be $\tilde{H}(-\infty, x_-) = 0$ and then we integrate out $A_+$ obtaining

$$\tilde{S}[u] = S[u] - \left(\frac{e_c}{4\pi}\right)^2 N_F \int d^2 x Tr(\tilde{H}^2) + m'^2 N_{\tilde{m}} \int d^2 x Tr(u + u^\dagger)$$

(4.21)

In the strong coupling limit $\frac{e_c}{m_q} \to \infty$ the fields in $\tilde{h}$ which contribute to $\tilde{H}$ will become infinitely heavy. The sector $\tilde{g}l \subset \frac{SU(N_F N_C)}{SU(N_C)}$, however, will not acquire mass
from the gauge interaction term. Since we are interested only in the light particles we can in the strong coupling limit ignore the heavy fields, if we first normal order the heavy fields at the mass scale \( \tilde{\mu} = \frac{e_c \sqrt{N_F}}{\sqrt{2\pi}} \). Using the relation, for a given operator \( O \),

\[
\left( \frac{\tilde{\mu}}{m} \right)^\Delta N_\tilde{\mu} O = N_m O
\]  

(4.22)

to perform the change in the scale of normal ordering, and then substituting \( h_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} \), we get for the low energy effective action:

\[
S_{eff}[u] = S[\tilde{g}] + S[l] + \frac{1}{2} \int d^2x \partial_\mu \phi \partial^\mu \phi + \int d^2x \text{Tr} \left( e^{-i \sqrt{\frac{4\pi}{N_C N_F}} \tilde{\phi} g l + e^{+i \sqrt{\frac{4\pi}{N_C N_F}} \phi l^\dagger g^\dagger} \right).
\]  

(4.23)

We can now replace the two mass scales \( m_q \) and \( \tilde{\mu} \) by a single scale by normal ordering at a certain \( m \) so the final form of the effective action becomes

\[
S_{eff}[u] = S[\tilde{g}] + S[l] + \frac{1}{2} \int d^2x \partial_\mu \phi \partial^\mu \phi + m^2 N_m \int d^2x \text{Tr} \left( e^{-i \sqrt{\frac{4\pi}{N_C N_F}} \tilde{\phi} g l + e^{+i \sqrt{\frac{4\pi}{N_C N_F}} \phi l^\dagger g^\dagger} \right)
\]  

(4.24)

with \( m \) given by:

\[
m = [N_C cm_q (\frac{e_c \sqrt{N_F}}{\sqrt{2\pi}})^\Delta_C]^{\frac{1}{1+\Delta_C}}
\]  

(4.25)

here \( \Delta_C \), the dimension of \( \tilde{h} \), is \( \frac{N_C^2 - 1}{N_C(N_C + N_F)} \). For the \( l = 1 \) sector, defining \( g' = \tilde{g} e^{-i \sqrt{\frac{4\pi}{N_C N_F}} \phi} \subset U(N_F) \) one gets the effective action

\[
S_{eff}[g'] = N_CS[g'] + m^2 N_m \int d^2x \text{Tr}_F (g' + g'^\dagger)
\]  

(4.26)

Thus, the low energy effective action in the \( l = 1 \) sector coincide with the result of “naive” approach of the product scheme given by eqn. (4.15).
5. The Baryonic Spectrum of Multiflavor QCD

5.1. Classical Soliton solutions

We now look for static solutions of the classical action. For a static field configuration \( g(x) \), the WZ term does not contribute. One way to see this is by noting that the variation of the WZ term can be written as

\[
\delta WZ \propto \int d^2x \varepsilon^{ij} Tr(\delta g)g^\dagger(\partial_i g)(\partial_j g^\dagger)
\]

(5.1)

and for \( g \) that has only spatial dependence \( \delta WZ=0 \). Without loss of generality we may take, for the lowest energy, a diagonal \( g(x) \) \[50]\,

\[
g(x) = \left( e^{-i\sqrt{\frac{4\pi}{N_C}}\varphi_1},..., e^{-i\sqrt{\frac{4\pi}{N_C}}\varphi_NF} \right)
\]

(5.2)

For this ansatz and with a redefinition of the constant term, the action density reduces to

\[
\tilde{S}_d[g] = -\int dx \sum_{i=1}^{N_F} \left[ \frac{1}{2} \left( \frac{d\varphi_i}{dx} \right)^2 - 2m^2 \left( \cos \sqrt{\frac{4\pi}{N_C}}\varphi_i - 1 \right) \right]
\]

(5.3)

This is a sum of standard Sine Gordon actions \[51]\ For each \( \varphi_i \) the well known solutions of the associated equations of motion are:

\[
\varphi_i(x) = \sqrt{\frac{4N_C}{\pi}} \arctan \left( e^{\frac{8\pi}{N_C}mx} \right)
\]

(5.4)

with the corresponding classical energy,

\[
E_i = 4m \sqrt{\frac{2N_C}{\pi}} \quad i = 1, ..., N_F
\]

(5.5)

Clearly the minimum energy configuration for this class of ansatz is when only one

† From here on we omit the prime from \( g' \) so we denote \( g \in U(N_F) \)
\( \varphi_i \) is nonzero, for example:

\[
g_\circ(x) = \text{Diag}(1, 1, \ldots, e^{-i\sqrt{\frac{4\pi}{N_C}}\varphi(x)})
\]

(5.6)

Conserved charges, corresponding to the vector current eqn.(3.11), can be computed using the definition:

\[
Q^A[g(x)] = \frac{1}{2} \int dx \text{Tr}(J_0 T^A),
\]

(5.7)

where \( T^A/2 \) are the \( SU(N_F) \) generators and the \( U(1) \) baryon number is generated by the unit matrix (instead of \( \frac{1}{2} T^A \) in the equation). This follows from \( J_\mu = J_\mu^A T^A \), and in the fermionic basis \( J_\mu^A = \bar{\psi} \gamma_\mu \frac{1}{2} T^A \psi \).

In particular, for eqn.(5.4), we get charges different from zero only for \( Q_B \) and \( Q_Y \) corresponding to baryon number and “hypercharge” respectively:

\[
Q_B^\circ = N_C \quad Q_Y^\circ = -\frac{1}{2} \sqrt{\frac{2(N_F - 1)}{N_F}} N_C,
\]

(5.8)

these charges are determined solely by the boundary values of \( \varphi(x) \), which are:

\[
\sqrt{\frac{4\pi}{N_C}} \varphi(\infty) = 2\pi \quad \sqrt{\frac{4\pi}{N_C}} \varphi(-\infty) = 0.
\]

(5.9)

Under a general \( U_V(N_F) \) global transformation \( g_\circ(x) \rightarrow \tilde{g}_\circ(x) = A g_\circ(x) A^{-1} \) the energy of the soliton is obviously unchanged, but charges other than \( Q_B \) and \( Q_Y \) will be turned on. Let us introduce a parametrization of \( A \) that will be useful later,

\[
A = \begin{pmatrix}
    A_{i\ell} & z_1 \\
    \vdots & \vdots \\
    \vdots & \vdots \\
    Y_1 & \ldots & Y_{(N_F-1)} & z_{(N_F-1)} \\
    \vdots & \ldots & \vdots & \vdots & \ddots & \ddots & \ddots \\
    Y_{(N_F-1)} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & z_{N_F}
\end{pmatrix}
\]

(5.10)
Now
\[ \tilde{g}_0 = 1 + (e^{-i\sqrt{\frac{4\pi}{N_C}} - 1})z \] (5.11)

where \((z)_{\alpha\beta} = z_{\alpha}z_{\beta}^*\) and from unitarity \(\sum_{\alpha=i}^{N_F} z_{\alpha}z_{\alpha}^*=1\). The charges with \(\tilde{g}_0(x)\) are:

\[ (\tilde{Q}^o)^A = \frac{1}{2} N_C Tr(T^A z) \] (5.12)

Only the baryon number is unchanged. The discussion of the possible \(U(N_F)\) representations is clearly irrelevant here, since we are dealing so far with a classical system. We will return to the question of possible representations after quantizing the system.

5.2. Semi-classical quantization and the Baryons

The next step in the semiclassical analysis is to consider configurations of the form

\[ g(x, t) = A(t)g_o(x)A^{-1}(t) \quad A(t) \in U(N_F), \] (5.13)

and to derive the effective action for \(A(t)\). Quantization of this action corresponds to doing the functional integral over \(g(x,t)\) of the above form. The effective action for \(A(t)\) is derived by substituting \(g(x, t) = A(t)g_o(x)A^{-1}(t)\) in the original action. Here we use the following property of the WZ action\(^{[36,52]}\)

\[ S[AgB^{-1}] = S[AB^{-1}] + S[g, \tilde{A}_\mu] \] (5.14)

where \(S[g]\) and \(S[g, \tilde{A}]\) are given by equations (3.2) and (4.10) respectively, with
the gauge field $\tilde{A}_\mu$ given as:

$$i\tilde{A}_+ = A^{-1} \partial_+ A \quad i\tilde{A}_- = B^{-1} \partial_- B \quad A, B \in U(N_F) \quad .$$ \hspace{1cm} (5.15)

Using the above formula for $A=B$, noting that $S(AA^{-1}=1)=0$, and taking $A = A(t)$

$$\partial_+ A = \partial_- A = \frac{\dot{A}}{\sqrt{2}},$$ \hspace{1cm} (5.16)

we get

$$\tilde{S} \left[ A(t)g_\circ(x)A^{-1}(t) \right] - \tilde{S}[g_\circ] =$$

$$\frac{N_C}{8\pi} \int d^2x Tr \left\{ [A^{-1} \dot{A}, g_\circ][A^{-1} \dot{A}, g_\circ^\dagger] \right\}$$

$$+ \frac{N_C}{2\pi} \int d^2x Tr \left\{ (A^{-1} \dot{A})(g_\circ^\dagger \partial_1 g_\circ) \right\}$$ \hspace{1cm} (5.17)

This action is invariant under global $U(N_F)$ transformations $A \to U A$ where $U \in G = U(N_F)$. This corresponds to the invariance of the original action under $g \to UgU^{-1}$. On top of that it is also invariant under the local changes $A(t) \to A(t)V(t)$ where $V(t) \in H = SU(N_F-1) \times U_B(1) \times U_Y(1)$ with the last two $U(1)$ factors corresponding to baryon number and hypercharge, respectively. This subgroup $H$ of $G$ is nothing but the invariance group of $g_\circ(x)$. In terms of $g_\circ(x)$ and $A(t)$ the charges associated with the global $U(N_F)$ symmetry, eq. (5.7), have the following form:

$$Q^B = i\frac{N_C}{8\pi} \int dx Tr \left\{ T^B A \left( \left[ g_\circ^\dagger \partial_1 g_\circ - g_\circ \partial_1 g_\circ^\dagger \right] + \left[ g_\circ, [A^{-1} \dot{A}, g_\circ^\dagger] \right] \right) A^{-1} \right\}$$ \hspace{1cm} (5.18)

The effective action eq. (5.17) is an action for the coordinates describing the coset-space

$$G/H = SU(N_F) \times U_B(1)/SU(N_F-1) \times U_Y(1) \times U_B(1)$$

$$= SU(N_F)/SU(N_F-1) \times U_Y(1) = CP^N \quad .$$

To see this explicitly we define the Lie algebra valued variables $q^A$ through $A^{-1} \dot{A} =$
\[ i \sum T^A \dot{q}^A. \]

In terms of these variables (5.17) takes the form (the part that depends on \( q^A \)):

\[
S_q = \int dt \left[ \frac{1}{2M} \sum_{A=1}^{2(N_F-1)} (\dot{q}^A)^2 - N_C \sqrt{\frac{2(N_F-1)}{N_F}} \dot{q}^Y \right]
\]

\[
\frac{1}{2M} = \frac{N_C}{2\pi} \int_{-\infty}^{\infty} (1 - \cos \sqrt{\frac{4\pi}{N_C}} \varphi) dx = \frac{\sqrt{2} m}{(N_C \pi)^{3/2}}
\]

(5.19)

The sum is over those \( q^A \) which correspond to \( G/H \) generators and \( q^Y \) is associated with the hypercharge generator. Although the \( q^A \) seem to be a “natural” choice of variables for the action eq. (5.17), which depends only on the combination \( A^{-1} \dot{A} \), they are not a convenient choice of variables. The reason for that is the explicit dependence of the charges (5.18) on \( A^{-1}(t) \) and \( A(t) \) as well as on \( A^{-1} \dot{A}(t) \).

Instead we found that a convenient parametrization is that of (5.10) . One can rewrite the action (5.17), as well as the charges (5.18), in terms of the \( z_1, \ldots, z_{N_F} \) variables, which however are subject to the constraint \( \sum_{\alpha=1}^{N_F} z_\alpha z^*_\alpha = 1 \). Thus

\[
\tilde{S} [A(t)] = \tilde{S} [g_0] = S[z_\alpha(t), \varphi(x)]
\]

(5.20)

where

\[
S[z_\alpha(t), \varphi(x)] = \frac{N_C}{2\pi} \int d^2 x \{ (1 - \cos \sqrt{\frac{4\pi}{N_C}} \varphi) [\dot{z}^*_\alpha \dot{z}_\alpha
\]

\[- (z^*_\gamma \dot{z}_\gamma)(\dot{z}^*_\beta z_\beta) \} \frac{\sqrt{4\pi}}{N_C} \varphi \}
\]

(5.21)

We can do the integral over \( x \) and rewrite (5.21) as follows:

\[
S[z_\alpha(t)] = \frac{1}{2M} \int dt [\dot{z}^*_\alpha \dot{z}_\alpha - (z^*_\gamma \dot{z}_\gamma)(\dot{z}^*_\beta z_\beta)] - \frac{N_C}{2} \int dt (\dot{z}^*_\alpha \dot{z}_\alpha - \dot{z}^*_\alpha z_\alpha)
\]

(5.22)

where 1/M is defined in equation (5.19). The first term in (5.22) is the usual \( \text{CP}^{N_F-1} \) quantum mechanical action, while the second term is a modification due
to the WZ term, as obtained from eq.(4.16). Similarly we express the $U(N_F)$ charges in terms of the $z$ variables, using equation (5.18):

$$Q^C = \frac{1}{2} f^C_{\beta \alpha} Q_{\alpha \beta}$$

$$Q_{\alpha \beta} = N_C z_\alpha z_\beta^* + \frac{i}{2M} [z_\alpha z_\beta^*(\dot{z}_\gamma \dot{z}_\gamma - \dot{z}_\gamma^* z_\gamma) + z_\alpha^* \dot{z}_\beta - z_\beta^* \dot{z}_\alpha]$$

(5.23)

Of course the symmetries of $S[z]$ are the global $U(N_F)$ group under which

$$z_\alpha \rightarrow z'_\alpha = U_{\alpha \beta} z_\beta \quad U \in U(N_F)$$

(5.24)

and a local $U(1)$ subgroup of $H$ under which:

$$z_\alpha \rightarrow z'_\alpha = e^{i\delta(t)} z_\alpha$$

(5.25)

As a consequence of the gauge invariance one can rewrite the action in a covariant form

$$S[z_\alpha] = \frac{1}{2M} \int dt \text{Tr} (Dz)^\dagger Dz + iN_C \int dt \text{Tr} \dot{z}^\dagger z,$$

(5.26)

where

$$(Dz)_\alpha = \dot{z}_\alpha + z_\alpha (\dot{z}_\beta z_\beta),$$

(5.27)

Constructing Noether charges of the $U(N_F)$ global invariance of (5.24) out of the action (5.26) leads to expressions identical with (5.23). Note that in eqn. (5.27) we can view $\dot{z}_\beta^* z_\beta = ia(t)$ as a composite $U(1)$ gauge potential.

Now let us count the degrees of freedom. The local $U(1)$ symmetry allows us to take one of the $z$'s to be real, and the constraint $\sum_\alpha z_\alpha z_\alpha^* = 1$ removes one more degree of freedom, so altogether we are left over with $2N_F - 2 = 2(N_F - 46$
1) physical degrees of freedom. This is exactly the dimension of the coset-space $SU(N_F) / SU(N_F-1) \times U(1)$. The corresponding phase space should have real dimension of $4(N_F - 1)$. Naively, however, we have a phase space of $4N_F$ dimensions and, therefore, we expect 4 constraints. There are several methods of quantizing systems with constraints. Here we choose to eliminate the redundancy in the $z$ variables and then invoke the canonical quantization procedure. (For a different procedure see [53]). But before following these lines let us briefly describe another method, through the use of Dirac’s brackets. We outline the classical case. The quantum case is obtained by replacing $\{ , \}$ with $i[ , ]$. The first step in this prescription is to add to the Lagrangian a term of the form $\lambda(\sum_{\alpha} z_{\alpha} z_{\alpha}^* - 1)$, in which case the conjugate momentum $\Pi_\lambda$ of the Lagrange multiplier vanishes. By requiring that this condition be preserved in time one get the constraint $\Phi_1 = (\sum_{\alpha} z_{\alpha} z_{\alpha}^* - 1) = 0$. Further imposing $\dot{\Phi}_1 = \{ \Phi_1, H \}_P = 0$, where $\{ \}_P$ denotes a Poisson bracket, one finds another second class constraint $\Phi_2 = \Pi \cdot z + z^\dagger \cdot \Pi^\dagger$. In addition there is a first class constraint $\Phi_3 = \Pi \cdot z - z^\dagger \cdot \Pi^\dagger$, which corresponds to the local $U(1)$ invariance of the model. Fixing this symmetry one gets an additional constraint $\Phi_4$. For instance one can choose the unitary gauge $\Phi_4 = z_{N_F} - z_{N_F}^\dagger$. The next step is to compute the constraint matrix $\{ \Phi_i, \Phi_j \}_P = c_{ij}$. In the constrained theory, the brackets between $F$ and $G$ are replaced by the Dirac brackets of those operators, given by:

$$\{ F, G \}_D = \{ F, G \}_P - \{ F, \chi_i \}_P (c_{ij}^{-1}) \{ \chi_j, G \}_P \tag{5.28}$$

where $c_{ij}^{-1}$ is the inverse of the constraint matrix. Imposing the constraints as operator relations it is easy to see that $z_{N_F}, \Pi_{N_F}$ and their complex conjugates can be eliminated. The brackets for the rest of the fields coincide [54] with the results we derive below, when eliminating the constraints explicitly.
We now describe in some details the quantization of the system using unconstrained variables. We want to choose a set of new variables so that the constraint
\[ \sum_{\alpha=1}^{N_F} z_\alpha z_{\alpha}^* = 1 \]
is automatically fulfilled. There is a standard choice of such variables, namely\[55\] (for \( i = 1, \ldots, N_F - 1 \))
\[
z_i = \frac{k_i}{\sqrt{1 + X}} \quad z_i^* = \frac{k_i^*}{\sqrt{1 + X}} \quad z_{N_F} = \frac{e^{i\chi}}{\sqrt{1 + X}}
\]
where
\[
X = \sum_{i=1}^{N_F-1} k_i^* k_i.
\]

The \( k_i, k_i^* \) and \( \chi \) are \( 2N_F - 1 \) real variables with no constraints on them. The phase space will now have dimension \( 2(2N_F - 1) \) and we still have extra two constraints. After some straightforward algebra we can write\[54\]
\[
S[k, k^*, \chi] = \int dt L(k, k^*, \chi)
\]
\[
L(k, k^*, \chi) = \frac{1}{2M} k_i^* h_{ij} k_j - i \frac{N_C}{2} \frac{k_i^* \dot{k}_i - \dot{k}_i^* k_i}{1 + X} + \frac{1}{2M (1 + X)^2} \dot{\chi}^2 + \chi \left\{ \frac{i}{2M} \frac{k_i^* \dot{k}_i - \dot{k}_i^* k_i}{(1 + X)^2} + \frac{N_C}{1 + X} \right\}
\]
where
\[
h_{ij} = \frac{\delta_{ij}}{1 + X} - \frac{k_i k_j^*}{(1 + X)^2}.
\]

The local U(1) transformations of the \( z \) variables transcribe into the transformations
\[
\delta \chi = \epsilon(t); \quad \delta k_i = i\epsilon(t) k_i; \quad \delta k_i^* = -i\epsilon(t) k_i^*
\]
and \( \delta L = -N_C \dot{\epsilon} \) just as in terms of the \( z \) variables. This local U(1) symmetry can
be made manifest by defining the covariant derivatives

\[ Dk_i = \dot{k}_i - i\dot{\chi}k_i \quad \text{and} \quad Dk^*_i = \dot{k}^*_i + i\dot{\chi}_i, \quad (5.33) \]

The lagrangian can then be recast in a manifestly gauge invariant form:

\[ L(k, k^*, x) = \frac{1}{2M} Dk^*_i h_{ij} Dk_j - \frac{iN_C}{2} \frac{k^*_i Dk_i - (Dk^*_i)k_i}{1 + X} + N_C \dot{\chi}, \quad (5.34) \]

Although one can now fix the gauge, for instance \( \dot{\chi} = 0 \), we will continue to work with (5.34). The conjugate momenta are given by

\[ \pi_i = \frac{\partial L}{\partial \dot{k}_i} = \frac{1}{2M} Dk^*_j h_{ji} - \frac{iN_C}{2} \frac{k^*_i}{1 + X} \]
\[ \pi^*_i = \frac{\partial L}{\partial \dot{k}^*_i} = \frac{1}{2M} h_{ij} Dk_j + \frac{iN_C}{2} \frac{k_i}{1 + X} \quad (5.35) \]
\[ \pi_\chi = \frac{\partial L}{\partial \dot{\chi}} = \frac{i}{2M} (k^*_i h_{ij} Dk_j - Dk^*_i h_{ij} k_j) + N_C \frac{1}{1 + X} \]

Since \( h_{ij} \) is invertible we can solve for \( Dk^*_i, Dk_i \) in term of the phase space variables

\[ Dk^*_i = 2M[\pi_j + \frac{iN_C}{2} \frac{k^*_j}{1 + X}] h^{-1}_{ji} \]
\[ Dk_i = 2M h^{-1}_{ij} [\pi^*_j - \frac{iN_C}{2} \frac{k_j}{1 + X}] \quad (5.36) \]

where

\[ h^{-1}_{ij} = (1 + X)(\delta_{ij} + k_ik^*_j) \quad (5.37) \]

Also

\[ \pi_\chi = i(k^*_i \pi^*_i - \pi_i k_i) + N_C \quad (5.38) \]

giving the constraint equation

\[ \psi = \pi_\chi - i(k^*_i \pi^*_i - \pi_i k_i) - N_C = 0 \quad (5.39) \]
The canonical Hamiltonian is given by

\[
H_c = \pi_i k_i + \pi_i^* k_i^* + \pi_\chi \dot{\chi} - L
= 2M[\pi_i + i \frac{N_C k_i^*}{2(1 + X)}]h^{-1}[\pi_j^* - i \frac{N_C k_j}{2(1 + X)}]
+ \dot{\chi}[\pi_\chi - i(\pi_i^* k_i^* - \pi_i k_i) - N_C]
\]

and this can be further simplified to:

\[
H_c = 2M(1 + X)[\pi_i \pi_i^* + (\pi_i k_i)(\pi_i^* k_i^*)]
- i \frac{N_C}{2}(\pi_i k_i - \pi_i^* k_i^*) + \frac{1}{4} \frac{N_C^2 X}{(1 + X)} + \dot{\chi} \psi
\]  

Here \(H_c\) is obtained explicitly in terms of the canonical variables \(k_i, k_i^*, \pi_i, \pi_i^*\). The \(\dot{\chi} \psi\) term indicates that \(\dot{\chi}\) also behaves as a Lagrange multiplier since, following the Dirac procedure, we should define

\[
H_T = H_c + \lambda(t) \psi
\]

where \(\lambda\) is a priori an arbitrary function of \(t\). We could absorb the \(\dot{\chi}\) in \(\lambda\).

Quantization of this Hamiltonian is now essentially straightforward. Let us first consider the symmetry generators \(Q_{\alpha\beta}\), which in terms of the new canonical variables take the form

\[
Q_{ij} = i(k_i \pi_j - \pi_i^* k_j^*)
Q_{i,N_F} = e^{-i\chi}[\frac{N_C k_i}{2} - i(\pi_i^* + k_i \pi_j k_j)]
Q_{N_F,i} = e^{i\chi}[\frac{N_C k_i^*}{2} + i(\pi_i + k_i^* \pi_j k_i^*)] = Q_{i,N_F}^*
Q_{N_F,N_F} = N_C - i(\pi_i k_i - \pi_i^* k_i^*)
\]

We will now show that the \(H_T\) can be expressed in terms of the second Casimir operator of the \(SU(N_F)\) group. 

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The second $U(N_F)$ Casimir operator is related to charge matrix elements $Q_{\alpha\beta}$ in the following way:

$$Q_A Q^A = \frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha}$$  \hfill (5.44)

A straightforward substitution gives:

$$\frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha} = (1 + X)[\pi_i^* \pi_i + \pi_i k_i \pi_j^* k_j^* - i \frac{N_C}{2} (\pi_i k_i - \pi_i^* k_i^*)] + \frac{1}{2} N_C^2 (1 + \frac{X}{2})$$  \hfill (5.45)

Therefore, the Hamiltonian is:

$$H_T = 2M [Q^A Q^A - \frac{N_C^2}{2}] + \lambda(t)\psi$$  \hfill (5.46)

Denoting the $SU(N_F)$ second Casimir operator by $C_2$, and using $Q_A Q^A = C_2 + \frac{1}{2N_F} (Q_B)^2$ we get:

$$H_T = 2M [C_2 - \frac{N_C^2 (N_F - 1)}{2N_F}]$$  \hfill (5.47)

The fact that $H_T$ is, up to a constant, the second Casimir operator, is another way to show that the charges $Q_{\alpha\beta}$ are conserved. These conserved charges will generate symmetry transformations via:

$$\delta k_i = i[Tr(\epsilon Q), k_i] \quad \delta k_i^* = i[Tr(\epsilon Q), k_i^*]$$

$$\delta \chi = i[Tr(\epsilon Q), \chi]$$  \hfill (5.48)

and similar equations for the momenta $\pi_i, \pi_i^*, \pi_\chi$. Here $\epsilon_{ij} = \frac{1}{2} \epsilon^{A T^A}_{ij}$ is the matrix of parameters. The transformation laws are derived using the constraint equation $\psi = 0$ after performing the commutator calculations. Notice that $Q_{ij}$ and $Q_{N_F, N_F}$ are linear in coordinates and momenta and therefore, the $SU(N_F - 1) \times U_Y(1)$ transformations they generate are linear. The $Q_{N_F, i}$ and $Q_{i, N_F}$ charges, on the other
hand, have cubic terms as well (quadratic in coordinates), so that the coset-space transformations of $SU(N_F)$ are non-linear. This is a well known property of $\mathbb{C}P^n$ models. Substitution of $Q_{\alpha\beta}$ in equation (5.48) gives:

$$\delta k_l = i[\epsilon_{ji} k_i \delta_{jl} + e^{i\chi} \epsilon_{iN_F} \delta_{il} - e^{-i\chi} \epsilon_{N_F i} k_i k_l - \epsilon_{N_F N_F} k_l]$$  (5.49)

where we used $i[k, \pi] = 1$. Inversely, starting with these transformation laws it is easy to verify the invariance of the action. The standard Noether procedure then gives the charges $Q_{\alpha\beta}$ in terms of the coordinates and velocities, which (not surprisingly) coincide with those given in equation (5.43). One could also deduce these transformation laws by making the change of variables $z_\alpha, z^*_\alpha \rightarrow k_i, k^*_i, \chi$ in (5.23) directly.

One can verify that

$$[Q^A, Q^B] = i f^{ABC} Q^C$$  (5.50)

where $f^{ABC}$ are the structure constants of the $U(N_F)$ group.

Do we have further restrictions on the physical states? We shall see now that in fact we do have. Remember that our lagrangian (5.34) includes an auxiliary gauge field $A_\circ \equiv \dot{\chi}$ and thus has to obey the associated Gauss law:

$$\frac{\partial L}{\partial A_\circ} = \frac{\partial L}{\partial \dot{\chi}} = \pi_\chi = N_C - i(\pi_i k_i - \pi^*_i k^*_i) = 0$$  (5.51)

Since $\pi_\chi$ is a linear combination of $Q_B$ and $Q_Y$, and the first is constrained to be $Q_B = N_C$, the $Q_Y$ is restricted as well. More specifically, $Q_Y = \bar{Q}_Y$, with

$$\bar{Q}_Y = \frac{1}{2} \sqrt{\frac{2}{(N_F - 1)N_F N_C}} \cdot$$  (5.52)
5.3. THE BARYONIC SPECTRUM

The masses of the baryons (5.5) and (5.47), and the two constraints on the multiplets of the physical states, namely $Q_B = N_C$ and that the multiplets contain $Q_Y = \tilde{Q}_Y = \frac{1}{2} \sqrt{\frac{2}{(N_F-1)N_F} N_C}$, are the main results of the last section. All states of the multiplet with $Q_Y \neq \tilde{Q}_Y$ will be generated from the state $Q_Y = \tilde{Q}_Y$ by $SU(N_F)$ transformations as in eq. (5.13). Using the above constraints we can investigate now what possible representations will appear in the low energy baryon sector. Considering states with quarks only (no antiquarks), the requirement of $Q_B = N_C$ implies that only representations described by Young tableau with $N_C$ boxes appear. The extra constraint $Q_Y = \tilde{Q}_Y$ implies that all $N_C$ quarks are from $SU(N_F - 1)$, not involving the $N_F$ th. These are automatically obeyed in the totally symmetric representation of $N_C$ boxes. In fact, this is the only representation possible for flavor space, since the states have to be constructed out of the components of one complex vector $z$ as $\prod_{i=1}^{N_F} z_i^{n_i}$ with $\sum_i n_i = N_C$. See also more detailed discussion in the next section. For another way of deriving this result see section (5.5). Thus for $N_C = 3, N_F = 3$ we get only 10 of $SU(3)$. This is understandable, since there is no physical spin in two dimensions.

What about the masses of the baryons? The total mass of a baryons is given by the sum of (5.5) and (5.47) namely

$$E = 4m\sqrt{\frac{2N_C}{\pi}} + m\sqrt{2\left(\frac{\pi}{N_C}\right)^3 \left[C_2 - N_C^2 \frac{(N_F - 1)}{2N_F}\right]} \quad . \quad (5.53)$$

For large $N_C$, the classical term behaves like $N_C$, while the quantum correction like 1. This will be worked out in section (5.5). That the total mass goes like $N_C$ for large $N_C$, and that the quantum fluctuations are $\frac{1}{N_C}$ of the classical result, is in accord with general considerations (see ref. [58]).
5.4. Flavor quark content of the baryons

A measure of the quark content of a given flavor $q_i$ in a baryon state $|B\rangle$ is given by

$$\langle \bar{q}_i q_i \rangle_B = \int dx \langle g_{ii} \rangle_B - \int dx \langle g_{ii} \rangle_0$$

$$= \int dx z^*_i z_i \left\langle e^{-i \sqrt{\frac{4\pi}{NC}} \phi_c} - 1 \right\rangle_B$$

$$= \text{const.} \langle z^*_i z_i \rangle_B$$  \hspace{1cm} (5.54)

In order to make contact with the real world, we take here $N_C = 3$ and $N_F = 3$, getting the baryons in the $10$ representation of flavor. Similarly, for $SU_F(2)$ there is only the isospin $\frac{3}{2}$ representation. This is what we would expect from naïve quark model considerations. The total wave function must be antisymmetric. Baryon is a color singlet, so the wavefunction is antisymmetric in color and it must be symmetric in all other degrees of freedom. There is no spin, so the baryon must be in a totally symmetric representation of the flavor group, a $10$ for three flavors. Therefore, strictly speaking there is no state analogous to the proton. On the other hand, there is a state which is the analogue of the $\Delta^+$, namely the charge 1 state in the $10$ representation, $z^2_1 z_2$. The $10$ is the lowest baryon multiplet in QCD$_2$. In the following we shall be dealing with the relative weight of a given flavor in some baryon state. Thus, $\langle \bar{q}q \rangle_B$ will henceforth stand for the ratio

$$\frac{\langle \bar{q}q \rangle_B}{\langle \bar{u}u + \bar{d}d + \bar{s}s \rangle_B}$$

For $\Delta^+ \sim z^2_1 z_2$ we obtain\[^{[21]}\]
\[
\langle \bar{s}s \rangle_{\Delta^+} = \frac{\int d^2 z_1 d^2 z_2 |z_3|^2 (z_1^2 z_2^*) (z_1^2 z_2^*)}{\int d^2 z_1 d^2 z_2 (z_1^2 z_2^*) (z_1^2 z_2^*)} = \frac{1}{6} \tag{5.55}
\]

as well as
\[
\langle \bar{u}u \rangle_{\Delta^+} = \frac{1}{2} \quad \langle \bar{d}d \rangle_{\Delta^+} = \frac{1}{3} \tag{5.56}
\]

In evaluating the integral in the numerator in eq. (5.55) we have used \(|z_3|^2 = 1 - |z_1|^2 - |z_2|^2\), which follows from the unitarity of the matrix \(A\) in eq. (5.13).

Similarly, for \(\Delta^{++} \sim z_3^3\) we have
\[
\langle \bar{u}u \rangle_{\Delta^{++}} = \frac{2}{3} \quad \langle \bar{d}d \rangle_{\Delta^{++}} = \frac{1}{6} \quad \langle \bar{s}s \rangle_{\Delta^{++}} = \frac{1}{6} \tag{5.57}
\]

In the constituent quark picture \(\Delta^{++}\) contains just three \(u\) quarks. Both the \(d\)-quark and the \(s\)-quark content of the \(\Delta^{++}\) come only from virtual quark pairs. Therefore in the \(SU(3)\)-symmetric case \(\langle \bar{s}s \rangle_{\Delta^+} = \langle \bar{d}d \rangle_{\Delta^+}\), and \(\langle \bar{s}s \rangle_{\Delta^+} = \langle \bar{s}s \rangle_{\Delta^{++}}\), as expected.

From eqn. (5.57) one can also read the results for \(\Omega^- \sim z_3^3\), by replacing \(u \leftrightarrow s\).

In the general case of \(N_F\) flavors and \(N_C\) colors, one obtains\(^{[21]}\)
\[
\langle (\bar{q}q)_{\text{sea}} \rangle_B = \frac{1}{N_C + N_F}, \tag{5.58}
\]

where \((\bar{q}q)_{\text{sea}}\) refers to the non-valence quarks in the baryon \(B\). Moreover, one can also compute flavor content of valence quarks. Consider a baryon \(B\) containing \(k\) quarks of flavor \(v\). The \(v\)-flavor content of such a baryon is
\[
\langle \bar{v}v \rangle_B = \frac{k + 1}{N_C + N_F} \tag{5.59}
\]

This implies an “equipartition” for valence and sea, each with a content of \(1/(N_C + \ldots\)
It also follows that the total sea content of \( N_F \) flavors is

\[
\sum_{q=1}^{N_F} \langle \bar{q}q \rangle_{\text{sea}} B = \frac{N_F}{N_C + N_F}
\]

which goes to zero for fixed \( N \) and \( N_C \to \infty \), as expected.

It is interesting to compare these results with the Skyrme model in 3+1 dimensions.\(^{[59-61]}\) For the proton\(^{[59]}\)

\[
\langle \bar{u}u \rangle_p^{3+1} = \frac{2}{5} \quad \langle \bar{d}d \rangle_p^{3+1} = \frac{11}{30} \quad \langle \bar{s}s \rangle_p^{3+1} = \frac{7}{30}
\]

and for the \( \Delta \)\(^{[61]}\)

\[
\langle \bar{s}s \rangle_{\Delta}^{3+1} = \frac{7}{24} \quad \langle \bar{s}s \rangle_{\Omega^-}^{3+1} = \frac{5}{12}
\]

The qualitative picture is similar, although the \( \bar{s}s \) content in the non-strange baryons is lower in 1+1 dimensions. One may speculate that in 1+1 dimensions the effects of loops are smaller than in 3+1 dimensions, since the theory is super-renormalizable and there are only longitudinal gluons. In the \( SU_F(3) \)-symmetric limit the strange quark content of baryons with zero net strangeness is significant, albeit smaller than that of either of the other two flavors. The situation obviously is reversed for \( \Omega^- \).

In the real world the current mass of the strange quark is much larger than the current masses of \( u \) and \( d \) quarks. It is natural to expect that this will have the effect of decreasing the strange quark content from its value in the \( SU_F(3) \) symmetry limit. We do not know the exact extent of this effect, but it is likely that the strange content decreases by a factor which is less than two. This estimate is based on both explicit model calculations\(^{[60-61]}\) and what we know from PCAC,
namely that the analogous quark bilinear expectation values in the vacuum are not dramatically different from their $SU(3)$ symmetric values:

$$0.5 \lesssim \frac{\langle \bar{s}s \rangle_0}{\langle \bar{u}u \rangle_0} \lesssim 1 \quad (5.63)$$

5.5. Multibaryons

Let us now explore the possibility of having multi-baryons states. The procedure follows similar lines to that of the baryonic spectrum, namely, we look for classical solution of the equation of motions with baryon number $kN_C$ and then we semiclassically quantize it. The ansatz for the classical solution of the low lying $k$-baryon state is\textsuperscript{[20]} taken now to be

$$g_0(k) = \begin{pmatrix}
(N_F - k) \\
1 \\
\vdots \\
k \\
\exp[-i(\frac{4\pi}{N_C})\frac{1}{2} \varphi_c] \\
\vdots \\
\end{pmatrix}.$$

(5.64)

For the semi-classical quantization we generalize the parametrization given in (5.10) to the following

$$A = \begin{pmatrix}
A_{ij} \\
z_{i\alpha}
\end{pmatrix},$$

(5.65)

where $i$ represents the rows $(1, \ldots, N_F)$ and $\alpha$ the columns $(N_F - k + 1, \ldots, N_F)$. The effective action in its covariant form (5.26) becomes\textsuperscript{[20]}

$$S[z_\alpha] = \frac{1}{2M} \int dt \text{Tr} (Dz)^\dagger Dz + iN_C \int dt \text{Tr} \dot{z}^\dagger z,$$

(5.66)
where now, instead of eqn. (5.27)

$$ (Dz)_{i\alpha} = \dot{z}_{i\alpha} + z_{i\beta}(\dot{z}^{*}_{j\beta}z_{j\alpha}), $$

Using the same steps as those which led to (5.47) one finds now the Hamiltonian \cite{20}

$$ H = 2M\left[C_2(N_F) - \frac{N^2_{C}}{2N_{F}}k(N_F - k)\right] + kE_c, $$

with $E_c$ the classical contribution for one baryon, the first term in (5.53).

It was shown in ref. [20] that the allowed $k$-baryon states contain $(kN_C)$ boxes in the Young tableaux representation of the flavour group $SU(N_F)$. Let us recall that this result followed from the constraint implied by the local invariance $z_{i\alpha} \rightarrow e^{i\delta(t)}z_{i\alpha}$. \hspace{1cm} (5.69)

Performing a variation corresponding to this invariance we find that the action $S$ changes by

$$ \Delta S = (kN_C)\int \dot{\delta} dt. $$

This means that the $N_z$ number is equal to $(kN_C)$. Thus for any wave function, written as a polynomial in $z$ and $z^*$, the number of $z$’s minus the number of $z^*$’s must equal $(kN_C)$. Note that for $k = 1$ the transformation (5.69) represents also the $N^th$ flavor number. Thus (5.70) entails that the representation contains a state with $N_C$ boxes of the $N_F$ flavor, and therefore must be the totally symmetric representation.

Now, the effective action (5.66) is invariant under a larger group of local transformations. In fact, we have extra $(k^2 - 1)$ generators, which correspond to $SU(k)$...
under which (5.66) is locally invariant. This can be exhibited by defining \[^{[56]}\text{“local gauge potentials”}\]

$$
\tilde{A}_{\beta\alpha}(t) = -(z^\dagger \dot{z})_{\beta\alpha}.
$$

(5.71)

Then

$$
Dz = \dot{z} + z \tilde{A}.
$$

(5.72)

Under the local gauge transformation corresponding to $\Lambda(t)$, $\tilde{A}$ transforms as

$$
\tilde{A}(t) \rightarrow e^{i\Lambda} \tilde{A} e^{-i\Lambda} + (\partial_t e^{i\Lambda}) e^{-i\Lambda}.
$$

(5.73)

Then we have

$$
(Dz)_{i\alpha} \rightarrow (Dz)_{i\beta}(e^{-i\Lambda})_{\beta\alpha}
$$

(5.74)

and so $\Delta S = 0$. If we perform the $U(1)$ transformation (5.69) we obtain a contribution (5.70) from the Wess-Zumino term, which implies $N_z = (kN_C)$. But due to the larger local symmetry we have more restrictions; they imply that the allowed states have to be singlets under the above mentioned $SU(k)$ symmetry. This is analogous to the confinement property of QCD, which tells that, due to the non-abelian gauge invariance, the physical states have to be colour singlets. Here we have analogous singlet structure of the $SU(k)$ in the flavour space. Taking a wave function that has $z$’s only (analogous to quarks only for QCD), it must be of the form

$$
\psi_k(z) = \prod_{i=1}^{N_C} (\epsilon_{\alpha_1...\alpha_k} z_{i_1 \alpha_1} ... z_{i_k \alpha_k}),
$$

(5.75)

for a given set of $1 \leq i_1, ..i_k \leq N_F$.  

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The most general state will be then of the form

$$\tilde{\psi}(z, z^*) = \psi_k(z)[\prod_{\{i,j\}} (z_i^* z_j)^n_{ij}], \quad (5.76)$$

and the products are over given sets of indices.

Let us now compute the mass of the state represented by (5.75). Using the explicit formula from ref. [57], we obtain

$$E[\psi_k] = 2M \left[ \frac{k(N_F - k)}{2N_F} N_C^2 + \frac{k(N_F - k)}{2} N_C ight. \left. - \frac{N_C^2}{2N_F} k(N_F - k) \right] + kE_c$$

$$= Mk(N_F - k)N_C + kE_c. \quad (5.77)$$

To obtain binding energies, consider our $k$-baryon as built from constituents $k_r$, such that $k = \sum_r k_r$. Then

$$B[k|k_r] = -(MN_C)[k^2 - \sum_i k_i^2]$$

$$= -(2MN_C) \sum_{r>s} k_r k_s \quad (5.78)$$

When all $k_r = 1$, the sum gives us $\frac{1}{2}k(k - 1)$, i.e. the number of one-baryon pairs in the $k$-baryon state. Note that the binding energy is always negative, thus the $k$-baryon is stable. The maximal binding corresponds to the case when all $k_r = 1$.

Note also that in the $N_C \to \infty$ limit, the binding tends to a finite value, since then

$$\lim_{N_C \to \infty} (2MN_C) = (C_{meC})^\frac{1}{2} \left( \frac{2N_F}{\pi} \right)^\frac{1}{2} N_C^\frac{1}{2}. \quad (5.79)$$

Let us take as an example an analogue of a deuteron i.e. a di-baryon $k = 2$. Then taking $N_C = 3$, $N_F = 2$ we find that its representation is a flavour singlet.
(this is the limiting case of $k = N_F$). The ratio of the binding to twice the baryon mass is given by

$$\epsilon_2 = \frac{1}{1 + \frac{24}{\pi^2}} = 0.29. \quad (5.80)$$

For $k = 2$, $N_C = 3$ and $N_F = 3$ we find that the di-baryon is represented by 10 and the ratio is given by

$$\epsilon_3 = \frac{1}{2 + \frac{24}{\pi^2}} = 0.23. \quad (5.81)$$

For general $N_F$ we obtain

$$\epsilon_F = \frac{1}{(N_F - 1) + \frac{24}{\pi^2}} = \frac{1}{N_F + 1.43}. \quad (5.82)$$

Finally, let us make the following comment. The ratio of the quantum fluctuations term to the classical term, in the expression for the mass eq.(5.77), is given by

$$\frac{Quantum \ Corrections}{Classical \ Term} = \left(\frac{\pi^2}{8}\right) \frac{N_F - k}{N_C}. \quad (5.83)$$

Thus, we do not expect our approximations to hold in the region $N_F \geq (N_C + 1)$. We expect it to start for $N_C \geq N_F$, and to be good in the region $N_C \gg N_F$. 

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6. Summary and Conclusions

One of the outstanding problems of high energy physics is the derivation of the hadronic spectrum from QCD, the underlying theory. A large variety of methods have been used to address this question, including lattice gauge simulations, low energy effective Lagrangians like the Skyrme model and chiral Lagrangians, QCD sum rules etc. In spite of this major effort the gap between the phenomenology and the basic theory has been only partially bridged. Another direction that has been taken in order to gain insight into the problem is a lower dimensional analogous system $QCD_2$.

The study of two dimensional problems to improve the understanding of four dimensional physical systems was found to be fruitful. For example the study of spin systems in two dimensions shed light on four dimensional gauge field theories. Obviously, physics in two dimensions is simpler than that of the real world since the underlying manifold is simpler and since the number of degrees of freedom of each field is smaller. There are some additional simplifying features in two dimensional physics. In one space dimension there is no rotation symmetry and no angular momentum. The light cone is disconnected and is composed of left moving and right moving branches. Therefore, massless particles are either on one branch or the other. These two properties are the basic building blocks of the idea of transmutation between systems of different statistics. Also, the ultra-violet behaviour is more convergent in two dimensions, making for instance $QCD_2$ a superconvergent theory.

Bosonization is the formulation of fermionic systems only in terms of bosonic variables and fermionization is just the opposite process. The study of bosonized physical systems offers several advantages:
(1) It is usually easier to deal with commuting fields rather than anti-commuting ones.

(2) In certain examples like the Thirring model the fermionic strong coupling regime turns into the weak coupling one in its bosonic version, the Sine-Gordon model.

(3) The non-abelian bosonization, especially in the product scheme, offers a separation between colored and flavored degrees of freedom, which is very convenient for the analyzing low lying spectrum.

(4) Baryons composed of $N_C$ quarks are a many-body problem in the fermion language, while simple solitons in the boson language.

(5) One loop fermionic computations involving the currents turn into tree level consideration in the bosonized version. The best known example of the latter are the chiral (or axial) anomalies.

An important question that has to be addressed when applying bosonization methods is to what extent are the two formulations equivalent. Let us first demonstrate this equivalence in simple “physical” terms for the example of the zero charge sector of a massless Dirac fermion Fock space. A wave function of a state composed of a fermion and its anti-particle having together zero fermionic charge and moving in the same direction, never spreads and the two particles will never separate. They are therefore indistinguishable from a free massless boson. A more rigorous argument of the equivalence of the fermionic and bosonic formulations of colored-flavored Dirac fermions was given in terms of their algebraic structure. It was shown that the currents and energy-momentum tensor of the two pictures correspond to the same representation of the Kac-Moody and Virasoro algebras. In both cases the energy momentum tensor is quadratic in the currents. Using current algebra and conformal symmetry Ward identities guarantee that correlation
functions of currents in the two formulations coincide. The equivalence properties for the massive theory are less obvious. As reviewed in this article, the bosonization of the mass term in the “product scheme” failed to reproduce the fermionic correlators of mass bilinears. The $U(N_C \times N_F)$ scheme, however, is free from such a problem. The equivalence of the gauged theories was argued by integrating the matter degrees of freedom in the bosonic and fermionic versions. The derived effective actions in the bosonic description where equal to those of the fermion models in various different regularization schemes.

$QCD_2$ was addressed first in the fermionic formulation. In his seminal work 't Hooft deduced the mesonic spectrum in the large $N_c$ limit. However, it seems that it is easier to analyze the baryonic physical states in the bosonic language. The extraction of the baryonic spectrum and wave functions from the low energy effective action is similar to the steps taken in the Skyrme model. It is worth mentioning again that whereas in the four dimensional case it is only an approximated model derived by an “educational guess”, in two dimension the action at the strong coupling regime is exact.

In spite of the progress that has been made in the understanding of $QCD_2$ there are still several interesting open questions. The incorporation of more complicated mass matrices and higher order corrections to the $\frac{m_q}{e_c} \rightarrow 0$ limit, are examples of such questions that are intimately related to the analysis presented in the review. Possible relations of the low-energy effective action to some massive two dimensional integrable models requires further exploration since it may lead to the full solution (not semi-classical) of the strong coupled $QCD_2$. One may attempt to write down the full bosonized standard model in two dimension namely to incorporate the electroweak interaction as well. This extension of $QCD_2$ faces
the difficulty of chiral bosonization. Nevertheless, one may gain in this way some further insight on the real world standard model. Obviously, the most interesting task is to extract useful methods and that are applicable to the four dimensional theory. As mentioned in the introduction, bosonization techniques were applied to four dimensional systems, like monopole induced proton decay and fractional charges induced on monopoles by light fermions. So far not much has been achieved in the application to $QCD_4$. The identification of hadronic systems that may be approximated by a bosonized two dimensional systems is still an open question which deserves further investigation.
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