The fourth root of the Kogut-Susskind determinant via infinite component fields.

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An example of interpolation by means of local field theories between the case of normal Kogut-Susskind fermions and the case of keeping just the fourth root of the Kogut-Susskind determinant is given. For the fourth root trick to be a valid approximation certain limits need to be smooth. The question about the validity of the fourth root trick is not resolved, only cast into a local field theoretical framework.

1. Introduction.
Recent simulations of QCD have been claimed to correctly include sea quark effects by eliminating the extra tastes coming with Kogut-Susskind lattice fermions with the help of the so called “fourth root trick”. This trick amounts to replacing the local lattice field theory, which would include all tastes, with one in which the determinant of the gauge dependent Kogut-Susskind ory, which would include all tastes, with one in which the determinant of the gauge dependent Kogut-Susskind fermion matrix, $K_s$, is taken at the power of $\frac{1}{4}$. The objective of this letter is to propose a class of embeddings of the 4D lattice fermions into six dimensions, four of which are the original lattice axes. These embeddings can be deformed by a parameter of mass dimension, $\Lambda$, so that they look local from the four dimensional viewpoint, so long as $\Lambda$ is of the order of the inverse four dimensional lattice spacing $a$. Formally, if one takes $\Lambda \to 0$ at fixed $a$ one recovers the fourth root trick. If one takes $\Lambda \to \infty$ at fixed $a$ one recovers a local theory with unmolested four tastes per species. Whatever scenario one has in mind for the validity of the fourth root trick, it seems plausible that it should boil down to some robustness statement concerning the combined limits $a \to 0$ and $\Lambda \to 0$.

2. General structure.
The basic idea is a generalization of the work of Slavnov and Frolov. As is well known one could have proceeded from this work alone to construct the overlap Dirac operator, and below I try to follow some of the steps that would have achieved this. However, the problem we are looking at here is substantially different, and by no means is it obvious what the final conclusion (if any) about the validity of the fourth root trick would end up being. This letter is limited in scope to merely setting the problem up in the language of infinite component fermi fields.

We replace each original Kogut-Susskind fermion pair $\bar{\chi}, \chi$ by an infinite tower $\chi^\alpha_n, \bar{\chi}^\alpha_n$, labeled by indices $n, \alpha$, with the range of $\alpha$ being given by a “degeneracy” $g_n$, for any given $n$. $n$ runs over all positive integers. We are looking for a set of integers $g_n$, for which, formally at least, the following holds:

$$\det^{\frac{1}{4}}(K_s) = \prod_n \det^{g_n}(K_s)$$  \hspace{1cm} (1)

Neglecting questions of absolute convergence, we have the requirement

$$\sum_n g_n = \frac{1}{4}$$  \hspace{1cm} (2)

Obviously, with the $g_n$ all positive integers we can’t have even conditional convergence with the desired result. However, if we allow the statistics of the fields to vary among the members of the tower, alternating signs might make (2) hold under conditional convergence. This is easily achieved by

$$\frac{1}{(1 + x)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}x^{n-1},$$  \hspace{1cm} (3)

and setting $x = 1$.

We can view the members of the towers as components of a vector in an infinite Hilbert space. Each component is a vector in itself, representing an ordinary Kogut-Susskind fermion. The infinite Hilbert space is defined as the Hilbert space associated with a two dimensional harmonic oscillator, whose Hamiltonian is $H$.

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 = -\frac{1}{2} \left( \frac{\partial}{\partial q} \right)^2 + \frac{1}{2}q^2 = a^\dagger a + b^\dagger b + 1$$

$$[a^\dagger, a] = 1 = [b^\dagger, b], \quad [a, b] = [a, b^\dagger] = 0$$

$$PaP = -a, \quad PbP = -b, \quad [H, P] = 0$$

$$H\, n; \alpha > = n|n; \alpha >, \quad n \geq 1, \quad g_n = n$$

$$a|1\rangle = b|1\rangle = 0, \quad P|n; \alpha \rangle = (-)^{n-1}|n; \alpha \rangle$$  \hspace{1cm} (4)

$P$ is the parity operation. We now declare any component consisting of an ordinary Kogut-Susskind structure to obey fermi statistics if it has positive parity and boson statistics if its parity is negative. A component is fermionic if $n$ is odd and bosonic if $n$ is even. Thus, the bosonic components can be fully paired up, providing a way to make the path integrals over them convergent in spite of $K_s$ having a spectrum that includes positive and negative values. The ground state of $H$ has eigenvalue $1$, is non-degenerate and labels a field of fermionic character. At a fixed $n > 1$, the statistics is the same for all $g(n)$ vector components labeled by $\alpha$. If one deforms $H$, the deformation should preserve parity so that fields corresponding to different statistics do not get mixed. We
denote the inner product in the internal Hilbert space by $(, )$. With the fermion action,
\[
(\bar{\chi}, K_s\chi) = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{2g_n} \bar{\chi}_n^\alpha K_s \chi_n^\alpha
\]
we formally have a local action where the entire contribution of fermion loops is given by the fourth root of the determinant of $K_s$.

3. Regularization.

We need to control the infinite number of fields; so far the locality is a mere illusion, as we have an infinite number of massless fields from the four dimensional viewpoint. This can be done by giving all the higher members of the towers a large mass, of the order of the ultraviolet cutoff affecting the four dimensional part of the fermion momenta, $\frac{1}{a}$:
\[
K_s \rightarrow K_s + \Lambda f(H)
\]
where $f(x) > c > 0$ for any $x = 2, 3, 4, \ldots$. One can expand the sea quark contribution in Feynman diagrams which now contain also a trace over states in $H$. The convergence of that trace would depend on the number of attached gauge field legs and the asymptotic behavior of $f(x)$ as $x \to \infty$. It is clear that demanding that $f(x)$ behave asymptotically as $x^\kappa$ will make all diagrams converge if $\kappa$ is a large enough integer; for example $\kappa = 4$ is already an overkill, and $\kappa = 2$ seems sufficient.

The extra two $\vec{q}$ dimensions are seen only by the fermions; other fields are oblivious of them. One could try to make up an operator $H$ which creates a point-like defect at the origin of $\vec{q}$ space, so that low energy-momentum fermionic modes are restricted to it. In that case it would suffice to pick $f(x) \to c > 0$ as $x \to \infty$. However, this is not guaranteed to eliminate all ambiguities and some additional interpretation might be needed.

For a finite $\Lambda$, the limit $a \to 0$ will produce, most likely, a theory with undesirable four-fold degeneracy for each fermion species if $f(1) = 0$. Formally, if we take $\Lambda \to 0$ at fixed $a$, we get a model that employs the fourth root trick. These observations reduce the problem to an investigation of the combined limits $a \to 0$ and $\Lambda \to 0$.

4. Discussion.

It is sometimes argued that the fourth root trick is valid because (1), it agrees with experiment, and (2), it agrees with low energy predictions about systems with approximate Goldstone bosons. Point (1) is well taken, and the improvement of the agreement between lattice data and experimental data, in particular in cases where numerical data obtained from quenched simulations showed distinct differences from experiment, is notable. However, eventually, we would like numerical QCD to be so reliable that when one detects a numerically significant discrepancy between its prediction and experimental data, one can interpret it as evidence of new physics. In other words, we shouldn’t rely on experimental data when assessing our calculations. Point (2) is not valid, as far as the author can see: One could have a regime where low energy effective Lagrangians describe the theory well, even if the ultraviolet completion of the theory is non-local. At a more practical level, the number of free parameters in the effective Lagrangian – even ignoring (unjustifiably) some lattice Lorentz-violating terms – is too large to make the agreement credible to such a degree that support for far reaching features can be abstracted from it.

The fourth root trick has been recently criticized in [4], where an operator corresponding to $K_s^+$ was considered. While this is one possible way to get to a theory where the fermion contribution to the partition function is given by the fourth root of the determinant of $K_s$, it does not create an option for the system to become local, and therefore the lack of locality one finds not necessarily implies that something is wrong with the fourth root trick, as used. To substantially increase the amount of intuitive doubt about the fourth root trick one would need to establish non-locality in a scheme that induces the system to select its true low energy modes in a smooth manner.

Thus, it seems that the more theoretical and direct approach outlined above could be of use to those who feel strongly that the fourth root trick does not mislead us. Alternatively, although $a$ priori it would seem to be substantially more difficult to establish a negative, the present approach could yield a proof that the fourth root trick never allows locality to be restored in the continuum limit. Obviously, the preferred outcome would be a positive one, and one would hope that in the future the proponents of the fourth root trick would produce a proof of its validity using the above approach, or possibly some variation of it. The mere absence of a real proof that the fourth root trick is inadequate will never provide satisfactory support for relying on numerical QCD carried out employing the fourth root trick.

5. Acknowledgments.

This research was partially supported by the DOE under grant number DE-FG02-01ER41165 at Rutgers University.

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