On L-Functions of Cyclotomic Function Fields

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Abstract

We study two criterions of cyclicity for divisor class groups of
function fields, the first one involves Artin L-functions and the second
one involves ”affine” class groups. We show that, in general, these
two criterions are not linked.

Let $P$ be a prime of $\mathbb{F}_q[T]$ of degree $d$ and let $K_P$ be the $P$th cyclotomic
function field. In this paper we study the relation between the $p$-part of
$\text{Cl}^0(K_P)$ and the zeta function of $K_P$,
where $p$ is the characteristic of $\mathbb{F}_q$.

Let $\chi$ be an even character of the Galois group of $K_P/\mathbb{F}_q(T)$, $\chi \neq 1$.
Let $g(X, \chi)$ be the ”congruent to one modulo $p$” part of the L-function of
$K_P/\mathbb{F}_q(T)$ associated to the character $\chi$.

We have two criterions of cyclicity ([2], chapter 8): if $\deg g(X, \chi) \leq 1$ then $\text{Cl}^0(K_P)_p(\chi)$ is a cyclic $\mathbb{Z}_p[\mu_{q^d-1}]$-
module, and if $\text{Cl}(O_{K_P})_p(\chi) = \{0\}$ then $\text{Cl}(K_P)_p(\chi)$ is a cyclic $\mathbb{Z}_p[\mu_{q^d-1}]$-
module. David Goss has obtained that if $\text{Cl}(O_{K_P})_p(\chi)$ is trivial then $g(X, \chi)$
is of degree at most one ([2], Theorem 8.21.2). Unfortunately, there is a gap
in the proof of this result. In fact, we show that in general $\text{Cl}(O_{K_P})_p(\chi) = \{0\}$ does not imply $\deg g(X, \chi) \leq 1$ (Proposition 3.4). We also prove that
if $i$ is a $q$-magic number and if $\omega_P$ is the Teichmüller character at $P$, then
$g(X, \omega_P^i)$ has simple roots when $i \equiv 0 \pmod{q-1}$ (Proposition 5.1).

Note that Goss conjectures that if $i$ is a $q$-magic number then $\deg g(X, \omega_P^i) \leq 1$. This problem is still open and can be viewed as an analogue of Vandiver’s
Conjecture for function fields (see section 5).
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1 Notations

Let $\mathbb{F}_q$ be a finite field having $q$ elements, $q = p^s$ where $p$ is the characteristic of $\mathbb{F}_q$. Let $T$ be an indeterminate over $\mathbb{F}_q$ and set $A = \mathbb{F}_q[T]$, $k = \mathbb{F}_q(T)$. We denote the set of monic elements of $A$ by $A^+$. A prime of $A$ is a monic irreducible polynomial in $A$. We fix $\overline{k}$ an algebraic closure of $k$. We denote the unique place of $k$ which is a pole of $T$ by $\infty$.

Let $L/k$ be a finite geometric extension of $k$, $L \subset \overline{k}$. We set:

- $O_L$: the integral closure of $A$ in $L$,
- $O_L^*$: the group of units of $O_L$,
- $S_\infty(L)$: the set of places of $L$ above $\infty$,
- $\text{Cl}^0(L)$: the group of divisors of degree zero of $L$ modulo the group of principal divisors,
- $\text{Cl}(O_L)$: the ideal class group of $O_L$,
- $R(L)$: the groupe of divisors of degree zero with supports in $S_\infty(L)$ modulo the group of principal divisors with supports in $S_\infty(L)$.

If $d$ is the greatest common divisor of the degrees of the elements in $S_\infty(L)$, we have the following exact sequence:

$$0 \to R(L) \to \text{Cl}^0(L) \to \text{Cl}(O_L) \to \frac{\mathbb{Z}}{d\mathbb{Z}} \to 0.$$ 

Let $P$ be a prime of $A$ of degree $d$. We denote the $P$th cyclotomic function field by $K_P$ (see [2], chapter 7, and [4]). Recall that $K_P/k$ is the maximal abelian extension of $k$ contained in $\overline{k}$ such that:

- $K_P/k$ is unramified outside of $P, \infty$,
- $K_P/k$ is tamely ramified at $P, \infty$,
- for every place $v$ of $K_P$ above $\infty$, the completion of $K_P$ at $v$ is equal to $\mathbb{F}_q((\frac{1}{T}))((q-1)\sqrt{-T})$.

We recall that $\text{Gal}(K_P/k) \cong (\mathbb{A} / \mathbb{P}^* A)^*$, and that the decomposition group of $\infty$ in $K_P/k$ is equal to its inertia group and is isomorphic to $\mathbb{F}_q^*$. 

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Let $E/F_q$ be a global function field and let $F/E$ be a finite geometric abelian extension. Set $G = \text{Gal}(F/E)$ and $\hat{G} = \text{Hom}(G, \mathbb{C}^*)$.

Let $\chi \in \hat{G}$, $\chi \neq 1$, we set:

$$L(X, \chi) = \prod_{v \text{ place of } E} (1 - \chi(v)X^{\deg v})^{-1},$$

Where $\chi(v) = 0$ if $v$ is ramified in $F^{\text{Ker}(\chi)}/E$, and if $v$ is unramified in $F^{\text{Ker}(\chi)}/E$, $\chi(v) = \chi((v, F^{\text{Ker}(\chi)}/E))$, where $(, F^{\text{Ker}(\chi)}/E)$ is the global reciprocity map. If $\chi = 1$, we set $L(X, \chi) = L_E(X)$ where $L_E(X)$ is the numerator of the zeta function of $E$.

Therefore, if $L_F(X)$ is the numerator of the zeta function of $F$, we get:

$$L_F(X) = \prod_{\chi \in \hat{G}} L(X, \chi).$$

Let $\Delta$ be a finite abelian group and let $M$ be a $\Delta$-module. Let $\ell$ be a prime number such that $|\Delta| \not\equiv 0 \pmod{\ell}$. We fix an embedding of $\mathbb{Q}$ in $\mathbb{Q}_{\ell}$. Let $W = \mathbb{Z}_\ell[\mu_{|\Delta|}]$. For $\chi \in \hat{\Delta}$, we set:

$$e_\chi = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \chi(\delta)\delta^{-1} \in W[\Delta],$$

and:

$$M_\ell(\chi) = e_\chi (M \otimes_{\mathbb{Z}} W).$$

Thus, we have:

$$M \otimes_{\mathbb{Z}} W = \bigoplus_{\chi \in \hat{\Delta}} M_\ell(\chi).$$

## 2 Cyclotomic Function Fields and Artin-Schreier Extensions

Let $Q$ be a prime of $A$ of degree $n$, write $Q(T) = T^n + \alpha T^{n-1} + \cdots$, $\alpha \in \mathbb{F}_q$. We set: $i(Q) = tr_{\mathbb{F}_q/\mathbb{F}_p}(\alpha)$. Let $a \in A$, $a \neq 0$, we set:

$$i(a) = \sum_{Q \text{ prime of } A} v_Q(a)i(Q) \in \mathbb{F}_p.$$
where $v_Q$ is the normalized $Q$-adic valuation on $k$.

Let $\theta \in \overline{k}$ such that $\theta^p - \theta = T$. Set $\tilde{A} = \mathbb{F}_q[\theta]$, $\tilde{k} = \mathbb{F}_q(\theta)$ and $G = \text{Gal}(\tilde{k}/k)$. Note that $\tilde{k}/k$ is unramified outside $\infty$ and totally ramified at $\infty$. Let $\tilde{\infty}$ be the unique place of $\tilde{k}$ above $\infty$.

**Lemma 2.1** Let $(., \tilde{k}/k)$ be the usual Artin symbol. For $a \in A \setminus \{0\}$:

$$(a, \tilde{k}/k)(\theta) = \theta - i(a).$$

**Proof** By the classical properties of the Artin symbol, it is enough to prove the Lemma when $a$ is a prime of $A$. Thus, let $P$ be a prime of $A$ of degree $d$. We have:

$$(P, \tilde{k}/k)(\theta) \equiv \theta^q^d \pmod{P}.$$ 

But, for $n \geq 0$, we have:

$$\theta^{p^n} = \theta + T + T^p + \cdots + T^{p^{n-1}}.$$ 

Therefore:

$$\theta^q^d \equiv \theta - i(P) \pmod{P}.$$ 

The Lemma follows. $\diamond$

**Lemma 2.2** Let $P$ be a prime of $A$ of degree $d$ such that $i(P) \neq 0$. Then $P$ is a prime of $\tilde{A}$ of degree $pd$. Let $\tilde{K}_P$ be the $P$th cyclotomic function field for the ring $\tilde{A}$, then $K_P \subset \tilde{K}_P$.

**Proof** We have $-T = -\theta^p(1 - \theta^{1-p})$. Note that:

$$1 - \theta^{1-p} \in (F_q((\frac{1}{\theta})))^{q-1}.$$ 

Therefore:

$$q^{-1}\sqrt{-T} \in F_q((\frac{1}{\theta}))(q^{-1}\sqrt{-\theta}).$$

Thus:

- $\tilde{k}K_P/\tilde{k}$ is unramified outside $P, \tilde{\infty}$,
- $\tilde{k}K_P/\tilde{k}$ is tamely ramified at $P, \tilde{\infty}$,
- for every place $w$ of $\tilde{k}K_P$ above $\tilde{\infty}$, the completion of $\tilde{k}K_P$ at $w$ is contained in $F_q((\frac{1}{\theta}))(q^{-1}\sqrt{-\theta})$. 

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The Lemma follows by class field theory. 

Let $P$ be a prime of $A$, $\deg_T P(T) = d$ and $i(P) \neq 0$. Let $L = \overline{k}K_P \subset \overline{K}_P$. Let $\Delta = \text{Gal}(K_P/k) \simeq \text{Gal}(L/k)$. We have an isomorphism compatible to class field theory: $\Delta \rightarrow \text{Gal}(L/k)$, $\chi \mapsto \overline{\chi} = \chi \circ N_{k/k}$. We fix $\zeta_p \in \overline{\mathbb{Q}}$ a primitive $p$th root of unity.

**Lemma 2.3**

1. Let $\chi \in \Delta$, $\chi \neq 1$. Let $L(X, \overline{\chi})$ be the Artin L-function relative to $L/\overline{k}$ and to the character $\overline{\chi}$. We have:

$$L(X, \overline{\chi}) = \prod_{\phi \in \hat{G}} L(X, \phi \chi),$$

where $L(X, \phi \chi)$ is the Artin L-function relative to $L/k$ and the character $\phi \chi$.

2. Let $\chi \in \Delta$, $\chi \neq 1$, $\chi$ even (i.e. $\chi(\mathbb{F}_q^*) = \{1\}$). Then:

$$\frac{L(X, \overline{\chi})}{L(X, \chi)} \equiv (1 - X)^{p-1} L(X, \chi)^{p-1} \pmod{(1 - \zeta_p)}.$$

**Proof** The assertion (1) is a consequence of the usual properties of Artin L-functions. Now, let $\phi \in \hat{G}$, $\phi \neq 1$. Since $\phi \chi$ is ramified at $\infty$, we get:

$$L(X, \phi \chi) = \sum_{n \geq 0} \left( \sum_{a \in A^+, \deg(a) = n} \phi(a) \chi(a) \right) X^n.$$

Thus:

$$L(X, \phi \chi) \equiv \sum_{n \geq 0} \left( \sum_{a \in A^+, \deg(a) = n} \chi(a) \right) X^n \pmod{(1 - \zeta_p)}.$$

But, since $\chi$ is even, we have $\chi(\infty) = 1$. Therefore:

$$L(X, \phi \chi) \equiv (1 - X) L(X, \chi) \pmod{(1 - \zeta_p)}.$$

The Lemma follows. 

Let $i \in \mathbb{F}_p$ and let $\sigma_i \in G$ such that $\sigma_i(\theta) = \theta - i$. Let $\psi \in \hat{G}$ given by $\psi(\sigma_i) = \zeta_p^i$. 

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Lemma 2.4 Let $\chi \in \hat{\Delta}$, $\chi$ even and non-trivial.

(1) Let $\phi \in \hat{G}$, $\phi \neq 1$. Let $\sigma \in \text{Gal} (\mathbb{Q}(\zeta_p)/\mathbb{Q})$ such that $\phi = \psi^\sigma$. Then:

$$L(X, \phi \chi) = L(X, \psi \chi)^\sigma.$$ 

Furthermore $\deg_X L(X, \phi \chi) = d$.

(2) We have:

$$L(1, \psi \chi) \equiv \left( \sum_{a \in A^+, \deg(a) \leq d} i(a) \chi(a)(\zeta_p - 1) \right) (\mod (1 - \zeta_p)^2).$$

Proof Let $\mathbb{Q}(\chi)$ be the abelian extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ the values of $\chi$. Let $\mathbb{Z}[\chi]$ be the ring of integers of $\mathbb{Q}(\chi)$. Note that $p$ is unramified in $\mathbb{Q}(\chi)$ and:

$$\text{Gal}(\mathbb{Q}(\chi)(\zeta_p)/\mathbb{Q}(\chi)) \simeq \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}).$$

Since $L(X, \phi \chi)$ is a polynomial in $\mathbb{Z} [\chi][\zeta_p][X]$, we have:

$$L(X, \phi \chi) = L(X, \psi \chi)^\sigma.$$ 

Since $\chi$ and $\tilde{\chi}$ are non-trivial even characters, we have:

$$\deg_X L(X, \tilde{\chi}) = pd - 2,$$

and:

$$\deg_X L(X, \chi) = d - 2.$$ 

Therefore $\deg_X L(X, \phi \chi) = d$.

Now, we have:

$$L(X, \psi \chi) = \sum_{n=0}^{d} \left( \sum_{a \in A^+, \deg(a) = n} \zeta_p^{i(a)} \chi(a) \right) X^n.$$ 

But recall that:

$$\zeta_p^{i(a)} \equiv 1 + i(a)(\zeta_p - 1) \pmod{(1 - \zeta_p)^2}.$$ 

Thus, since $\chi$ is even and non-trivial, we get:

$$L(X, \psi \chi) \equiv L(X, \chi)(1 - X) + (\zeta_p - 1) \left( \sum_{n=1}^{d} \left( \sum_{a \in A^+, \deg(a) = n} i(a) \chi(a) \right) X^n \right) \pmod{(1 - \zeta_p)^2}.$$ 

The Lemma follows. ♦

We are now ready to prove the main result of this section:
Proposition 2.5 Let $\chi \in \hat{\Delta}$, $\chi \neq 1$, $\chi$ even. Let $W = \mathbb{Z}_p[\mu_{q^d - 1}]$. We have:

$$\text{Long}_W(\frac{\text{Cl}(O_L)_p(\tilde{\chi})}{\text{Cl}(O_{K_P})_p(\chi)}) \geq 1 \iff \sum_{a \in A^{+}, \deg(a) \leq d} i(a)\bar{\chi}(a) \equiv 0 \pmod{p}.$$ 

Proof Fix $\tau$ a generator of $G \simeq \text{Gal}(L/K_P)$. Let $\varepsilon \in O_L^*$. Since $L/K_P$ is totally ramified at any prime above $\infty$, there exists $\zeta \in \mathbb{F}_q^*$ such that $\tau(\varepsilon) = \zeta \varepsilon$. But $\tau^p(\varepsilon) = \zeta^p \varepsilon = \varepsilon$. Since we are in characteristic $p$, we deduce that $\varepsilon \in O_{K_P}^*$. Therefore:

$$O_L^* = O_{K_P}^*.$$ 

Let $I$ be an ideal of $O_{K_P}$ such that $IO_L = \alpha O_L$ for some $\alpha \in O_L$. Then, there exists $\varepsilon \in O_L^*$ such that $\tau(\alpha) = \varepsilon \alpha$. Since $O_L^* = O_{K_P}^*$ and since $\tau$ is of order $p$, we deduce that $\alpha \in O_{K_P}$. This implies that:

$$\text{Cl}(O_{K_P}) \rightarrow \text{Cl}(O_L).$$

One can also show that:

$$\text{Cl}^0(K_P) \rightarrow \text{Cl}^0(L).$$

Set $\Delta^+ = \frac{\Delta}{\mathbb{F}_q^*}$. Let $\mathcal{I}$ be the augmentation ideal of $\mathbb{F}_p[\Delta^+]$. One sees that we have the following isomorphism of $\Delta$-modules:

$$\frac{R(L)}{R(K_P)} \otimes_{\mathbb{F}_p} \mathbb{Z}_p \simeq \mathcal{I}.$$ 

This implie that we have the following exact sequence of $W$-modules:

$$0 \rightarrow \frac{W}{pW} \rightarrow \text{Cl}^0(L)_p(\tilde{\chi}) \rightarrow \text{Cl}^0(K_P)_p(\chi) \rightarrow \text{Cl}(O_L)_p(\tilde{\chi}) \rightarrow \text{Cl}(O_{K_P})_p(\chi) \rightarrow 0.$$ 

Now, by the results of Goss and Sinnott (3):

$$\text{Long}_W \text{Cl}^0(L)_p(\tilde{\chi}) = v_p(L(1, \bar{\chi})), $$

and

$$\text{Long}_W \text{Cl}^0(K_P)_p(\chi) = v_p(L(1, \bar{\chi})).$$

Thus by Lemma 2.3

$$\text{Long}_W(\frac{\text{Cl}(O_L)_p(\tilde{\chi})}{\text{Cl}(O_{K_P})_p(\chi)}) = (p - 1)v_p(L(1, \psi \bar{\chi})) - 1.$$ 

It remains to apply Lemma 2.4 ◀
3 Derivatives of L-functions

Let $P$ be a prime of $A$ of degree $d$. We fix an embedding of $\mathbb{Q}$ in $\mathbb{Q}_p$. Set $\Delta = \text{Gal}(K_P/k)$ and $W = \mathbb{Z}_p[\mu_{q^d-1}]$. We fix an isomorphism $\Phi_P : A/PA \rightarrow W/pW$. Then $\Phi_P$ induces an isomorphism:

$$\omega_P : \Delta \rightarrow \mu_{q^d-1} \subset W^*.$$  

The morphism $\omega_P$ is called ”the” Teichmüller character at $P$. Note that $\hat{\Delta}$ is a cyclic group and $\omega_P$ is a generator of this group.

Let $i \in \mathbb{N}$, set:

- $\beta(0) = 1$,
- $\beta(i) = \sum_{a \in A^+} a^i$ if $i \geq 1$, $i \not\equiv 0 \pmod{q-1}$,
- $\beta(i) = -\sum_{a \in A^+} \deg(a)a^i$ if $i \geq 1$, $i \equiv 0 \pmod{q-1}$.

One can prove that for all $i \in \mathbb{N}$, $\beta(i) \in A$. We also see that:

$$\forall i \in \mathbb{N}, 0 \leq i \leq q^d - 2, \Phi_P(\beta(i)) \equiv L(1, \omega^i_P) \pmod{p}.$$  

Therefore, if $1 \leq i \leq q^d - 2$, by the results of Goss and Sinnott (3), we have:

$$\text{Long}_W \mathcal{C}l^0(K_P)_p(\omega^i_P) \geq 1 \iff \beta(i) \equiv 0 \pmod{p}.$$  

The numbers $\beta(i)$ are called the Bernoulli-Goss polynomials.

Recall that we have a surjective morphism of $\Delta$-modules:

$$W[\Delta^+] \rightarrow R(K_P) \otimes_{\mathbb{Z}} W,$$

where $\Delta^+ = \Delta/F_q^*$. Thus for $\chi \in \hat{\Delta}$, $\chi$ even, $R(K_P)_p(\chi)$ is a cyclic $W$-module. But, for such a character, we have the exact sequence of $W$-modules:

$$0 \rightarrow R(K_P)_p(\chi) \rightarrow Cl^0(K_P)_p(\chi) \rightarrow Cl(O_K)_p(\chi) \rightarrow 0.$$  

This implies that, if $Cl(O_K)_p(\chi) = \{0\}$, $Cl^0(K_P)_p(\chi)$ is a cyclic $W$-module.

David Goss has shown (2, Corollary 8.16.2) that for $\chi$ is even, $\chi \neq 1$, if $L'(1, \overline{\chi}) \not\equiv 0 \pmod{p}$ (here $L'(1, \overline{\chi})$ is the derivative of $L(X, \overline{\chi})$ taken at $X = 1$), then $Cl^0(K_P)_p(\chi)$ is a cyclic $W$-module.

Therefore a natural question arise. Let $\chi \in \hat{\Delta}$, $\chi \neq 1$, $\chi$ even. Assume that $L(1, \overline{\chi}) \equiv 0 \pmod{p}$. Do we have:

$$Cl(O_K)_p(\chi) = \{0\} \Rightarrow L'(1, \overline{\chi}) \not\equiv 0 \pmod{p}?$$
Our aim in this section is to show that in general the answer is no.

Let $d$ be an integer, $d \geq 1$. For $i \in \{1, \cdots, q^d - 2\}$, we set:

$$\gamma(d, i) = \sum_{a \in A^+, \deg(a) \leq d} i(a) a^i.$$

**Lemma 3.1** Let $\tau \in \text{Gal}(\mathbb{F}_q(T)/\mathbb{F}_q(T^p - T))$ such that $\tau(T) = T + 1$. Let $i \in \{1, \cdots, q^d - 2\}$, $i \equiv 0 \pmod{q - 1}$. Recall that $q = p^s$. We have:

$$\tau(\gamma(d, i)) = \gamma(d, i) + s\beta(i).$$

**Proof** Let $Q$ be a prime of $A$ of degree $n$. Write $Q = T^n + \alpha T^{n-1} + \cdots$, where $\alpha \in \mathbb{F}_q$. Then $\tau(Q) = T^n + (\alpha + n)T^{n-1} + \cdots$. Therefore $i(\tau(Q)) = i(Q) + s\deg(Q)$. This implies that:

$$\forall a \in A \setminus \{0\}, i(\tau(a)) = i(a) + s\deg(a).$$

Now:

$$\tau(\gamma(d, i)) = \sum_{a \in A^+, \deg(a) \leq d} i(a) \tau(a)^i.$$

Therefore:

$$\tau(\gamma(d, i)) = \sum_{a \in A^+, \deg(a) \leq d} (i(\tau(a)) - s\deg(a)) \tau(a)^i.$$

Thus:

$$\tau(\gamma(d, i)) = \sum_{a \in A^+, \deg(a) \leq d} i(\tau(a)) \tau(a)^i - s \sum_{a \in A^+, \deg(a) \leq d} \deg(\tau(a)) \tau(a)^i.$$

Observe that $\sum_{a \in A^+, \deg(a) \leq d} i(\tau(a)) \tau(a)^i = \gamma(d, i)$ and $-\sum_{a \in A^+, \deg(a) \leq d} \deg(\tau(a)) \tau(a)^i = \beta(i)$. \(\diamondsuit\)

**Proposition 3.2** Let $P$ be a prime of $A$ of degree $d$ such that $i(P) \neq 0$. Set $Q(T) = P(T^p - T)$. Then $Q$ is a prime of $A$ of degree $pd$. Let $i$ be an integer such that $1 \leq i \leq q^d - 2$, $i \equiv 0 \pmod{q - 1}$ and $\text{Cl}(O_{K_P})_p(\omega_P^{-i}) = \{0\}$. Then:

$$\text{Long}_{W}(\text{Cl}(O_{K_Q})_p(\omega_Q^{-i(q^d - 1)/(q^d - 1)})) \geq 1 \iff \gamma(d, i) \equiv 0 \pmod{P}.\$$
Lemma 3.3 Assume \( p \neq 2 \). Let \( d \geq 1 \) be an integer. There exists a prime \( P \) in \( A \), \( \deg(P) = d \), such that \( i(P(T))i(P(T + 1)) \neq 0 \).

Proof Let \( Q \) be a prime of \( A \) of degree \( d \) such that \( i(Q) \neq 0 \). Such a prime exists by the normal basis Theorem. Fix \( \theta \in \mathbb{F}_q \) an algebraic closure of \( \mathbb{F}_q \). We assume that \( i(Q(T + 1)) = 0 \). Write \( Q = T^d + \alpha T^{d-1} + \cdots \). Then \( Tr_{\mathbb{F}_q/\mathbb{F}_p}(\alpha) = -sd \). Therefore \( sd \neq 0 \pmod{p} \). Let \( \theta \in \mathbb{F}_q \) such that \( Q(\theta) = 0 \). We observe that:

\[
\forall \zeta \in \mathbb{F}_p, \quad Tr_{\mathbb{F}_q/\mathbb{F}_p}(\zeta \theta) = -\zeta sd.
\]

Since \( p \geq 3 \), we can find \( \zeta \in \mathbb{F}_p^* \) such that \( -\zeta sd \neq -sd \). Set \( P(T) = \text{Irr}(\zeta \theta, \mathbb{F}_q; T) \). Then \( P \) is a prime of degree \( d \) such that \( i(P)i(\tau(P)) \neq 0 \).

Proposition 3.4 Assume that \( p \neq 2 \) and \( s \not\equiv 0 \pmod{p} \). Let \( d \) be an integer, \( d \geq 2 \), and let \( P \) be a prime of degree \( d \) such that \( i(P(T))i(P(T + 1)) \neq 0 \). Set \( Q(T) = P(T^p - T) \). Then:

- \( L(1, \omega_Q^{-(q-1)(q^{pd-1})/(q^{d-1})}) \equiv 0 \pmod{p} \),
- \( L'(1, \omega_Q^{-(q-1)(q^{pd-1})/(q^{d-1})}) \equiv 0 \pmod{p} \),
- \( Cl(O_{K_Q})_p(\omega_Q^{-(q-1)(q^{pd-1})/(q^{d-1})}) = \{0\} \).

Proof Set \( R = P(T + 1) \) and \( Z = R(T^p - T) \). We observe that we have an isomorphism:

\[
Cl(O_{K_Q})_p(\omega_Q^{-(q-1)(q^{pd-1})/(q^{d-1})}) \cong Cl(O_{K_Z})_p(\omega_Z^{-(q-1)(q^{pd-1})/(q^{d-1})}).
\]

Not also that \( \beta(q - 1) = 1 \). Thus:

\[
Cl(O_{K_Q})_p(\omega_Q^{-(q-1)}) = Cl(O_{K_R})_p(\omega_R^{-(q-1)}) = \{0\}.
\]

We have:

\[
L(1, \omega_Q^{-(q-1)(q^{pd-1})/(q^{d-1})}) \equiv L(1, \omega_Z^{-(q-1)(q^{pd-1})/(q^{d-1})}) \equiv 0 \pmod{p}.
\]
And, by Lemma 2.3, since $p \geq 3$:

$$L'(1, \omega_Q^{-(q-1)(d^d-1)/(d^d-1)}) \equiv L'(1, \omega_Z^{-(q-1)(d^d-1)/(d^d-1)}) \equiv 0 \pmod{p}.$$ 

Suppose that we have $Cl(O_{K_Q})_p(\omega_Q^{-(q-1)(d^d-1)/(d^d-1)}) \neq \{0\}$. Then by Proposition 3.2:

$$\gamma(d, q - 1) \equiv 0 \pmod{P},$$

and also:

$$\gamma(d, q - 1) \equiv 0 \pmod{R}.$$ 

Thus:

$$\tau(\gamma(d, q - 1)) \equiv 0 \pmod{\tau(P)}.$$ 

Now, by Lemma 3.1 and the fact that $\tau(P) = R$, we get:

$$\gamma(d, q - 1) + s\beta(q - 1) \equiv 0 \pmod{R}.$$ 

Therefore we get $s \equiv 0 \pmod{p}$ which is a contradiction. The Proposition follows. ♦

4 Cyclicity of Class Groups and L-Functions

Let $E/F_q$ be a global function field and let $F/E$ be a finite geometric abelian extension. Set $\Delta = \text{Gal}(F/E)$. Let $\ell$ be a prime number. Let’s recall some well-known facts about $L$-functions.

Set $T_\ell = \text{Hom}(\mathbb{Q}/\mathbb{Z}, J)$ where $J$ is the inductive limit of the $Cl^0((\mathbb{F}_q^n)F)$, $n \geq 1$. We fix an embedding of $\mathbb{Q}$ in $\mathbb{Q}_\ell$. Let $\gamma$ be the Frobenius of $\mathbb{F}_q$. Then $\gamma$ and $\Delta$ act on $T_\ell$.

If $\ell \neq p$, we have (see [6], chapter 15):

$$\det(1 - \gamma X |_{T_\ell}) = L_F(X),$$

where $L_F(X)$ is the numerator of the zeta function of $F$.

If $\ell = p$, write $L_F(X) = \prod_i (1 - \alpha_i X)$ and set $L_F^{nr}(X) = \prod_{\nu_p(\alpha_i) = 0}(1 - \alpha_i X$). Then (see [1] and also [3]):

$$\det(1 - \gamma X |_{T_p}) = L_F^{nr}(X).$$
Now assume that $\ell$ does not divide the cardinal of $\Delta$, then the above results are also valid character by character. More precisely, if $\ell \neq p$, we have:

$$\forall \chi \in \hat{\Delta}, \ Det(1 - \gamma X \mid_{T_\ell(\chi)}) = L(X, \chi).$$

If $\ell = p$, for $\chi \in \hat{\Delta}$, write $L(X, \chi) = \prod_i (1 - \alpha_i(\chi)X)$ and set $L^{nr}(X, \chi) = \prod_{\nu_p(\alpha_i(\chi))=0} (1 - \alpha_i(\chi)X)$. Then:

$$\forall \chi \in \hat{\Delta}, \ Det(1 - \gamma X \mid_{T_p(\chi)}) = L^{nr}(X, \chi).$$

Now, let $\chi \in \hat{\Delta}$, write:

$$L(X, \chi) = \prod_i (1 - \alpha_i(\chi)X),$$

and set:

$$g(X, \chi) = \prod_{\nu_\ell(\alpha_i(\chi))>0} (1 - \alpha_i(\chi)X).$$

Set:

$$g(X) = \prod_{\chi \in \hat{\Delta}} g(X, \chi).$$

We also set:

$$\forall \chi \in \hat{\Delta}, \ H(X, \chi) = (1 + X)^{\deg_X g(X, \chi)} g((1 + X)^{-1}, \chi),$$

and:

$$H(X) = \prod_{\chi \in \hat{\Delta}} H(X, \chi).$$

For $n \geq 0$, set $F_n = \mathbb{F}_{q^n}F$, and let $A_n$ be the $\ell$-Sylow subgroup of $C\ell^0(F_n)$. Let $F_\infty = \cup_{n \geq 0} F_n$ and let $A_\infty$ be the inductive limit of the $A_n$, $n \geq 0$. We set:

$$Y = \text{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A_\infty).$$

Set $\Gamma = \text{Gal}(F_\infty/F)$, then $\gamma$ is a topological generator of $\Gamma \simeq \mathbb{Z}_\ell$.

**Lemma 4.1**

(1) For all $n \geq 0$, we have an isomorphism of $\Delta$-modules:

$$\frac{Y}{(\gamma^{\ell^n} - 1)Y} \simeq A_n.$$
Assume \( |\Delta| \neq 0 \pmod{\ell} \). Then, \( \forall \chi \in \hat{\Delta}, \forall n \geq 0 \), we have:

\[
\frac{Y(\chi)}{(\gamma^{\ell n} - 1)Y(\chi)} \simeq A_n(\chi).
\]

**Proof** We prove assertion (1), and note that (2) is a consequence of (1). Recall that \( A_\infty \) is a divisible group (see [8], Proposition 11.16). We start with the following exact sequence:

\[
0 \to A_n \to A_\infty \to A_\infty \to 0,
\]

where the middle map is the multiplication by \( \gamma^{\ell n} - 1 \). We apply \( \text{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, .) \) to this sequence, we get:

\[
0 \to Y \to Y \to \text{Ext}^1(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A_n) \to 0.
\]

we also have the following exact sequence:

\[
0 \to \mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \frac{\mathbb{Q}_\ell}{\mathbb{Z}_\ell} \to 0.
\]

We apply \( \text{Hom}(., A_n) \) to this last sequence, using the fact that:

\[
\text{Ext}^1(\mathbb{Q}_\ell, A_n) = \{0\},
\]

we get:

\[
\text{Hom}(\mathbb{Z}_\ell, A_n) \simeq \text{Ext}^1(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A_n).
\]

The Lemma follows. \( \lozenge \)

**Proposition 4.2**

(1) Let \( \Lambda = \mathbb{Z}_\ell[[X]] \) be the Iwasawa algebra of \( \Gamma \) over \( \mathbb{Z}_\ell \) where \( X \) acts like \( \gamma - 1 \). Then \( Y \) is a finitely generated \( \Lambda \)-module and a torsion \( \Lambda \)-module. The characteristic polynomial of the \( \Lambda \)-module \( Y \) is equal to \( H(X) \).

(2) Assume that \( \ell \) does not divide the cardinal of \( \Delta \). Let \( \Lambda = W[[X]] \) be the Iwasawa algebra of \( \Gamma \) over \( W = \mathbb{Z}_\ell[\mu_{|\Delta|}] \) where \( X \) acts like \( \gamma - 1 \). Then, for \( \chi \in \hat{\Delta}, Y(\chi) \) is a finitely generated \( \Lambda \)-module and a torsion \( \Lambda \)-module. The characteristic polynomial of the \( \Lambda \)-module \( Y \) is equal to \( H(X, \overline{\chi}) \).
Proof We prove (1), the proof of (2) is essentially similar. For all \( n \geq 0 \), we set \( \omega_n(X) = (1 + X)^{\ell^n} - 1 \). By Lemma 4.1, we have:

\[
\forall n \geq 0, \quad \frac{Y}{\omega_n Y} \simeq A_n.
\]

Therefore \( Y \) is a finitely generated \( \Lambda \)-module and a torsion \( \Lambda \)-module. Let \( r \in \mathbb{N} \) such that we have an isomorphism of groups:

\[
Y \simeq \mathbb{Z}_r^\ell.
\]

Then, there exists a constant \( \nu \in \mathbb{Z} \), such that, for all \( n \) sufficiently large:

\[
| \frac{Y}{\omega_n Y} | = \ell^{rn+\nu}.
\]

But, for all \( n \geq 0 \), we have:

\[
| A_n | = \ell^{\nu(L_{\mathbb{F}_r}(1))}.
\]

Therefore, there exists a constant \( \nu' \in \mathbb{Z} \) such that, for all \( n \) sufficiently large:

\[
| A_n | = \ell^{\deg \chi H(X)n + \nu'}.
\]

Thus: \( r = \deg X H(X) \). But let \( V(X) \) be the characteristic polynomial of the \( \Lambda \)-module \( Y \). We know that \( r = \deg X V(X) \), and we also know that \( V(X) \) divides \( (1 + X)^{\deg L_{\mathbb{F}_r}(1)} L_{\mathbb{F}}((1 + X)^{-1}) \). But \( V(X) \) is a distinguished polynomial, thus \( V(X) \) divides \( H(X) \). The Proposition follows. ♦

Proposition 4.3

(1) If \( A_0 \) is a cyclic \( \mathbb{Z}_\ell \)-module then \( g(X) \) has simple roots.

(2) Assume that \( | \Delta | \not\equiv 0 \pmod{\ell} \). Let \( \chi \in \hat{\Delta} \). If \( A_0(\chi) \) is a cyclic \( W \)-module then \( g(X, \chi) \) has simple roots.

Proof We prove (1). By Nakayama’s Lemma, \( Y \) is pseudo-isomorphic to \( \Lambda / H(X) \Lambda \). But, by a result of Tate (3), we know that the action of \( \gamma \) on \( Y \) is semi-simple. This implies that \( H(X) \) has simple roots. ♦

Let’s give an application of this last Proposition.

Proposition 4.4 We assume that \( q \geq 5 \). Let \( E / \mathbb{F}_q(T) \) be a real quadratic field, i.e. \( [E : \mathbb{F}_q(T)] = 2 \) and \( \infty \) splits completely in \( E \). If \( O_E \) is a principal ideal domain then \( L_E(X) \) has simple roots.

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Proof Let \( g \) be the genus of \( E \) and write:

\[
L_E(X) = \prod_{i=1}^{2g} (1 - \alpha_i X).
\]

Let \( K = \mathbb{Q}(\alpha_1, \cdots, \alpha_{2g}) \), then \( K \) is a CM-field. Let \( \alpha \in \{\alpha_1, \cdots, \alpha_{2g}\} \). Then:

\[
(1 - \alpha)(1 - \bar{\alpha}) \geq q + 1 - 2\sqrt{q} > 1.
\]

Therefore:

\[
N_{K/Q}(1 - \alpha) > 1.
\]

Thus \( 1 - \alpha \) is not a unit of \( K \). Let \( \infty_1 \) and \( \infty_2 \) be the places of \( E \) above \( \infty \). Then \( R(E) \) is a quotient of \( \mathbb{Z}(\infty_1 - \infty_2) \) and we have an exact sequence:

\[
0 \to R(E) \to Cl^0(E) \to Cl(O_E) \to 0.
\]

Therefore, if \( O_E \) is a principal ideal domain then \( Cl^0(E) \) is a cyclic group. It remains to apply Proposition 4.3.

It is conjectured that there exists infinitely many real quadratic function fields \( E/F_q(T) \) such that \( O_E \) is a principal ideal domain. In view of this conjecture, it will be interesting to prove that there exists infinitely many real quadratic function fields \( E/F_q(T) \) such that \( L_E(X) \) has simple roots.

5 A Conjecture of Goss

Set \( D_0 = 1 \) and for \( i \geq 1 \), \( D_i = (T^{q^i} - T)D_{i-1}^q \). The Carlitz exponential is defined by:

\[
Exp(X) = \sum_{i \geq 0} \frac{X^{q^i}}{D_i} \in k[[X]].
\]

Let \( n \in \mathbb{N} \), write \( n = a_0 + a_1 q + \cdots + a_r q^r \), where \( a_0, \cdots, a_r \in \{0, \cdots, q-1\} \). We set:

\[
\Gamma_n = \prod_{i=0}^{r} D_i^{a_i}.
\]

The \( i \)th Bernoulli-Carlitz number, \( B(i) \in k \), is defined by:

\[
\frac{X}{Exp(X)} = \sum_{i \geq 0} \frac{B(i)}{\Gamma_i} X^i.
\]
Let $P$ be a prime of $A$ of degree $d$ and let $i \in \{1, \cdots, q^d - 2\}$, $i \equiv 0 \pmod{q - 1}$. We have the following result (5):

$$Cl(O_{K_P})_p(\omega_P^i) \neq \{0\} \Rightarrow B(i) \equiv 0 \pmod{P}.$$  

We fix an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_p$. Let $i \in \{1, \cdots, q^d - 2\}$. Write:

$$L(X, \omega_P^i) = \prod_j (1 - \alpha_j(i)X),$$

and set:

$$g(X, \omega_P^i) = \prod_{v_p(\alpha_j(i)-1)>0} (1 - \alpha_j(i)X).$$

Let $i \in \mathbb{N}$. We say that $i$ is a $q$-magic number if there exist $c \in \{0, \cdots, q - 2\}$ and an integer $n \in \mathbb{N}$ such that $i = cq^n + q^n - 1$.

**Proposition 5.1** Let $P$ be a prime of $A$ of degree $d$. Let $i$ be a $q$-magic number, $1 \leq i \leq q^d - 2$, $i \equiv 0 \pmod{q - 1}$. Then $g(X, \omega_P^i)$ has simple roots.

**Proof** We have $i = q^n - 1$ for some integer $n$, $1 \leq n \leq d - 1$. By a result of Carlitz (2, Lemma 8.22.4):

$$B(q^d - 1 - i) = \frac{(-1)^{d-n}}{L_d^{q^n}},$$

where $L_0 = 1$ and for $j \geq 1$, $L_j = (T^{q^j} - T)L_{j-1}$. Therefore:

$$Cl(O_{K_P})_p(\omega^{-i}) = \{0\}.$$

It remains to apply Proposition 4.3.

In (2), David Goss makes the following conjecture: let $P$ be a prime of degree $d$ and let $i$ be a $q$-magic number, $1 \leq i \leq q^d - 2$. Then $\deg_X g(X, \omega_P^i) \leq 1$.

It is natural to ask if there exist primes $P$ and $q$-magic numbers $i$, $1 \leq i \leq q^{\deg P} - 2$, such that $\deg_X g(X, \omega_P^i) \geq 1$. This is the case.
Proposition 5.2 Let \( c \in \{0, \ldots, q-2\} \). There exist infinitely many primes \( P \) such that:

\[
\prod_{n=1}^{\deg P-1} \beta(cq^n + q^n - 1) \equiv 0 \pmod{P}.
\]

Proof We prove this Proposition for \( c \neq 0 \). The proof for \( c = 0 \) is very similar. If we apply the results in [7], we get:

\[
\forall n \geq 0, \deg T \beta(cq^n + q^n - 1) = n(c+1)q^n - \frac{q^{n+1} - q}{q-1}.
\]

Let \( S \) be the set of primes \( P \) in \( A \) such that:

\[
\prod_{i=1}^{\deg P-1} \beta(cq^n + q^n - 1) \equiv 0 \pmod{P}.
\]

Let’s assume that \( S \) is a finite set. We set:

\[
D = \prod_{P \in S} \deg P,
\]

and \( D = 1 \) if \( S = \emptyset \). Note that:

\[
\forall P \in S, q^D \equiv 1 \pmod{q^{\deg P} - 1}.
\]

Therefore, since \( \beta(c) = 1 \), we have:

\[
\forall P \in S, \beta(cq^D + q^D - 1) \equiv 1 \pmod{P}.
\]

But \( \deg_T \beta(cq^D + q^D - 1) \geq 1 \), thus we can select a prime \( Q \) of \( A \) such that \( \beta(cq^D + q^D - 1) \equiv 0 \pmod{Q} \). Note that \( Q \notin S \). Set \( d = \deg Q \). Since \( d \) does not divide \( D \), there exists an integer \( r, 1 \leq r \leq d-1 \), such that \( D \equiv r \pmod{d} \). Therefore:

\[
\beta(cq^D + q^D - 1) \equiv \beta(cq^r + q^r - 1) \equiv 0 \pmod{Q}.
\]

But this implies that \( Q \in S \), which is a contradiction. \( \diamond \)

Let \( P \) be a prime of \( A \) of degree \( d \). Let \( J \) be the jacobian of \( K_P \), i.e., \( J \) is the inductive limit of the \( Cl^{0}(\mathbb{F}_{q^n}K_P) \), \( n \geq 1 \). Set \( \mathbb{F}_{q^\infty} = \bigcup_{n \geq 0} \mathbb{F}_{q^n} \subset \overline{\mathbb{F}_q} \),
where $\overline{\mathbb{F}_q}$ is the algebraic closure of $\mathbb{F}_q$ in $k$. We consider the $\Delta = \text{Gal}(K_P/k)$-module:

$$\mathcal{A}_P = \frac{J[p]^\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^\infty})}{\text{Cl}^0(K_P)[p]}.$$ 

As a consequence of the results in section 4, we get:

**Proposition 5.3** Let $W = \mathbb{Z}_p[\mu_{q^d-1}]$ and let $\chi \in \hat{\Delta}$. We have:

$$\dim_{\mathbb{F}_p} \mathcal{A}_P(\chi) = \deg \chi g(X, \overline{\chi}) - \dim_{\mathbb{F}_p} \text{Cl}^0(K_P)_p(\chi).$$

Note that in general, by Proposition 3.4 we do not have $\mathcal{A}_P = \{0\}$. But Goss conjecture implies the following:

let $P$ be a prime of $A$ of degree $d$ and let $i$ be a $q$-magic number, $1 \leq i \leq q^d - 2$, then $\mathcal{A}_P(\omega_P^{-i}) = \{0\}.$

It would be interesting to prove (or find a counter-example) to this weak form of Goss conjecture.

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