On the energy density of linearly polarized, plane gravitational wave

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ABSTRACT

In this article, the energy density of plane gravitational wave is studied by using Einstein and Møller’s prescription of energy-momentum pseudotensors. The linearly polarized plan gravitational wave solution of Einstein field equation, which has been defined by Bondi et al., is represented by four kinds of different coordinates. The energy distribution of gravitational wave solution in Einstein and Møller’s prescription are obtained. Particularly the energy component is zero in null coordinates.

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Since Einstein [1] has predicted the existence of gravitational wave in 1916, a century after the first direct observation of gravitational waves had been made by LIGO and Virgo scientific collaboration [2]. But actually, in 1905 Poincaré [3] first proposed the existence of gravitational wave and suggested gravitational waves would be generated by accelerating masses in the same way electromagnetic waves are generated by accelerating charges.

To consider the weak-field approximation

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \]  

(1)

where \( \eta_{\mu\nu} \) is Minkowski metric and \( |h_{\mu\nu}| \ll 1 \), Einstein [1] obtained the gravitational wave equation in vacuum from Einstein field equations

\[ \Box^2 h_{\mu\nu} = 0 \]  

(2)

with the harmonic gauge condition \( \partial^\mu h_{\mu\nu} = 0 \). It resemble to the electromagnetic wave equation in vacuum

\[ \Box^2 A_\mu = 0 \]  

(3)

with the Lorentz gauge condition \( \partial^\mu A_\mu = 0 \). Because of a great many similarities between gravitation and electromagnetism, it should be therefore expected that Einstein field equations, like Maxwell field equation, have radiative solutions.

In theory of electromagnetism, the electromagnetic waves can transport energy, so it is believed that the gravitational waves also can transport energy in general relativity. However, the notion of energy is one of the oldest and most controversial problems in general relativity. Nester et al. [4] show that quasilocal energy-momentum can be obtained from the Hamiltonian.

\[ H(N, \Sigma) = \oint_{\partial\Sigma} \mathcal{B}(N) = -\frac{1}{2\kappa} \oint_{\partial\Sigma} N^\mu U^{\nu\lambda}_\mu dS_{\nu\lambda}. \]  

(4)

Hence, energy-momentum pseudotensors, which are associated with a legitimate Hamiltonian boundary term, are acceptable. So, in this article, Einstein and Møller’s prescriptions of energy-momentum pseudotensors are used to study the energy density of plan gravitational waves, which are the simplest solutions of wave equation.
As a compromise between realism and complexity, the linearly polarized, plane gravitational wave solution, which has been defined by Bondi et al. \[5, 6\], have been chosen in Cartesian coordinates \((t, x, y, z)\)

\[
ds_{(I)}^2 = dt^2 - L^2 \left( e^{2\beta} dx^2 + e^{-2\beta} dy^2 \right) - dz^2. \tag{5}\]

In order to study wave propagation, the light-cone coordinates are always used to express the mathematical formula of traveling wave. In the light-cone coordinates \((u, v, x, y)\), the line element will become \[6\]

\[
ds_{(II)}^2 = du dv - L^2 \left( e^{2\beta} dx^2 + e^{-2\beta} dy^2 \right). \tag{6}\]

Here

\[
u = (t + z)/\sqrt{2}, \tag{7}\]
\[
v = (t - z)/\sqrt{2}, \tag{8}\]

and \(L\) and \(\beta\) are functions of \(u\). In addition, the so-called plane-fronted gravitational wave with parallel rays (pp-wave) was introduced by Ehlers and Kundt \[7\] in Brinkmann coordinates \((U, V, X, Y)\) as

\[
ds_{(III)}^2 = 2H(U, X, Y)dU^2 + 2dUdV - dX^2 - dY^2. \tag{9}\]

By suitable coordinate transformations

\[
X = xLe^\beta, \tag{10}\]
\[
Y = yLe^{-\beta}, \tag{11}\]
\[
U = \sqrt{2}u, \tag{12}\]
\[
V = \sqrt{2}v - x(Le^\beta)' - y(Le^{-\beta})', \tag{13}\]

the Eq.(6) can be rewritten in the form which is similar to Eq.(9)

\[
ds_{(III)}^2 = 2(Y^2 - X^2)\frac{F(U)}{2}dU^2 + 2dUdV - dX^2 - dY^2, \tag{14}\]

where

\[
H(U, X, Y) = \frac{Y^2 - X^2}{2}F(U). \tag{15}\]

And lastly, Bonner \[8\] has described an exact solution of an infinitely straight beam of light in the following form

\[
ds_{(IV)}^2 = dT^2 - dX^2 - dY^2 - dZ^2 + m(dT - dZ)^2, \tag{16}\]
where \( m(T, X, Y, Z) = H(U, X, Y) \). So the null coordinate \((U, V)\) is transferred to new coordinates \((T, Z)\), named Brinkmann-Cartesian coordinates, by

\[
T = \frac{(V + U)}{\sqrt{2}},
\]

\[
Z = \frac{(V - U)}{\sqrt{2}}.
\]

Here, the line element of the gravitational wave solution is represented by four kinds of different coordinates.

In physics, energy-momentum, associated with a continuity equation in the differential form \( \partial_\mu T^{\mu\nu} = 0 \), is regarded as the most fundamental conserved quantity. However, attempts at identifying an energy-momentum density for gravity, led only to the energy-momentum complex which is pseudotensor

\[
\Theta_\nu^{\mu} = \sqrt{-g} \left( T_\nu^{\mu} + t_\nu^{\mu} \right)
\]

as the total energy-momentum of matter \( T_\nu^{\mu} \) and gravitational field \( t_\nu^{\mu} \). After the expression proposed by Einstein \([9, 10]\) for the energy-momentum complexes with the gravitational field

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} t_\nu^{\nu} \right) \equiv -\frac{1}{2} \frac{\partial g^{\mu\alpha}}{\partial x^\mu} T^{\mu\alpha},
\]

various physicists, such as Landau and Lifshitz \([11]\), Papapetrou \([12]\), Bergmann \([13]\), Weinberg \([14]\) and Møller \([15]\), had given different definitions for the energy-momentum complexes. Particularly, the energy-momentum complex will satisfy the continuity equation

\[
\frac{\partial \Theta_\nu^{\mu}}{\partial x^\mu} = 0
\]

Mathematically, antisymmetric \( U_\nu^{\mu\rho} \) in their two indices \( \mu \) and \( \rho \) would be introduced by

\[
\Theta_\nu^{\mu} = \frac{\partial U_\nu^{\mu\rho}}{\partial x^\rho}
\]

So, the definition of the Einstein energy-momentum complex \([9, 10]\) is exhibited as

\[
e \Theta_\nu^{\mu} = \frac{1}{16\pi} \frac{\partial H_\nu^{\mu\sigma}}{\partial x^\sigma},
\]

where the Freud’s superpotential is

\[
H_\nu^{\mu\sigma} = \frac{g_{\nu\rho}}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left[ (-g) (g^{\mu\rho} g^{\sigma\alpha} - g^{\sigma\rho} g^{\mu\alpha}) \right],
\]
and the definition of the Møller energy-momentum complex \[15\]

\[
M_\Theta^{\mu} = \frac{1}{8\pi} \frac{\partial X^\mu}{\partial x^\sigma}.
\] (25)

where the Møller’s superpotential is

\[
\chi^\mu_{\nu} = \sqrt{-g} \left( \frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\nu\beta}}{\partial x^\alpha} \right) g^{\mu\beta} g^{\sigma\alpha}.
\] (26)

Here these gravitational wave solutions, which are represented as Eq. (5), (6), (9) and (16) in four differential coordinates, are considered to investigate their energy distribution. According to the definition of energy-momentum complexes of Einstein and Møller, all components of the Freud’s and Møller’s superpotential \(U_0^i\) is shown in Table 1.

| coordinates | Einstein \(H_0^i\) | Møller \(\chi_0^i\) |
|-------------|---------------------|---------------------|
| \((t, x, y, z)\) | \(H_0^{01} = 0\) | \(\chi_0^{01} = 0\) |
|             | \(H_0^{02} = 0\) | \(\chi_0^{02} = 0\) |
|             | \(H_0^{03} = -4L \frac{\partial L}{\partial z}\) | \(\chi_0^{03} = 0\) |
| \((u, v, x, y)\) | \(H_0^{01} = L \frac{\partial L}{\partial u}\) | \(\chi_0^{01} = 0\) |
|             | \(H_0^{02} = 0\) | \(\chi_0^{02} = 0\) |
|             | \(H_0^{03} = 0\) | \(\chi_0^{03} = 0\) |
| \((U, V, X, Y)\) | \(H_0^{01} = 0\) | \(\chi_0^{01} = 0\) |
|             | \(H_0^{02} = 0\) | \(\chi_0^{02} = 0\) |
|             | \(H_0^{03} = 0\) | \(\chi_0^{03} = 0\) |
| \((T, X, Y, Z)\) | \(H_0^{01} = \frac{\partial m}{\partial X}\) | \(\chi_0^{01} = \frac{\partial m}{\partial X}\) |
|             | \(H_0^{02} = \frac{\partial m}{\partial Y}\) | \(\chi_0^{02} = \frac{\partial m}{\partial Y}\) |
|             | \(H_0^{03} = m \frac{\partial m}{\partial Z}\) | \(\chi_0^{03} = \frac{\partial m}{\partial T} + \frac{\partial m}{\partial Z}\) |

Finally, the energy distribution of gravitational wave solutions in Einstein and Møller’s prescriptions are obtained, and the energy component \(\Theta_0^0\) of Einstein and Møller energy-momentum complexes are shown in Table 2. The energy component \(E\Theta_0^0\) and \(M\Theta_0^0\) are both equal to zero in null coordinates \((u, v, x, y)\) and \((U, V, X, Y)\), but not in non-null coordinates \((t, x, y, z)\)
and \((T, X, Y, Z)\). However, in my earlier article [16], the energy components of Einstein energy-momentum complex for the static spherically symmetric space-time with the generalized PG Cartesian coordinates, Kerr-Schild Cartesian coordinates, and Schwarzschild Cartesian coordinates are the same. General relativity was introduced along with the principle that not only is no coordinate system preferred, but that any arbitrary coordinate system would do. But, the energy component \(\Theta^0_0 = 0\) in null coordinates shows this choice of coordinates is particular. It means that the energy of gravitational wave can’t be detect in null coordinates because the observer is moving with the gravitational wave. Therefore, the choice of null coordinates should be unique on the exploration of gravitational wave.

Table 2: The energy component \(\Theta^0_0\) of energy-momentum complexes

| coordinates     | \(\Theta^0_0\)   | \(\Theta^0_{00}\) |
|-----------------|------------------|------------------|
| \((t, x, y, z)\) | \(-\frac{1}{16\pi}\left[\frac{\partial^2 L}{\partial z^2} + \left(\frac{\partial L}{\partial z}\right)^2\right]\) | 0                |
| \((u, v, x, y)\) | 0                | 0                |
| \((U, V, X, Y)\) | 0                | 0                |
| \((T, X, Y, Z)\) | \(-\frac{1}{16\pi}\left[\nabla^2_{XY} m + m \frac{\partial^2 m}{\partial z^2} + \left(\frac{\partial m}{\partial z}\right)^2\right] - \frac{1}{8\pi}\left[\nabla^2_{XY} m + \frac{\partial^2 m}{\partial z^2} + \frac{\partial^2 m}{\partial z \partial T}\right]\) | \(-\frac{1}{16\pi}\left[\nabla^2_{XY} m + \frac{\partial^2 m}{\partial z^2} + \left(\frac{\partial m}{\partial z}\right)^2\right]\) |

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