Comments on 2D dilaton gravity system with a hyperbolic dilaton potential

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Abstract

We proceed to study a (1+1)-dimensional dilaton gravity system with a hyperbolic dilaton potential. Introducing a couple of new variables leads to two copies of Liouville equations with two constraint conditions. In particular, in conformal gauge, the constraints can be expressed with Schwarzian derivatives. We revisit the vacuum solutions in light of the new variables and reveal its dipole-like structure. Then we present a time-dependent solution which describes formation of a black hole with a pulse. Finally, the black hole thermodynamics is considered by taking account of conformal matters from two points of view: 1) the Bekenstein-Hawking entropy and 2) the boundary stress tensor. The former result agrees with the latter one with a certain counter-term.
1 Introduction

The AdS/CFT correspondence [1–3] has been recognized as a realization of the holographic principle [4, 5]. However, the rigorous proof has not been provided yet, although the integrable structure behind the correspondence has led to great advances along this direction (For a comprehensive review see [6]). A recent interest in the study of AdS/CFT is to construct a toy model which realizes a holographic principle at the full quantum level. Recently, Kitaev proposed an intriguing model [7] as a variant of the Sachdev-Ye (SY) model [8]. This is a one-dimensional quantum-mechanical system composed of $N \gg 1$ Majorana fermions with a random, all-to-all quartic interaction. This model is now referred to as the Sachdev-Ye-Kitaev (SYK) model. For some recent progress, see [9–20].

A possible candidate of the gravity dual for the SYK model is a 1+1 dimensional dilaton gravity system with a certain dilaton potential (For a nice review see [21]). This model was originally introduced by Jackiw [22] and Teitelboim [23]. Then it has been further
studied by Almheiri and Polchinski [24] in light of holography [25,27]. This model contains interesting solutions like renormalization group flow solutions, black holes, time-dependent solutions which describe formation of a black hole. Since the black hole is asymptotically AdS$_2$, the boundary stress tensor computed in the standard manner leads to the associated entropy, which agrees with the Bekenstein-Hawking entropy.

In the preceding work [28], we have studied deformations of this dilaton gravity system by employing a Yang-Baxter deformation technique [29,31]. The dilaton potential is deformed from a simple quadratic form to a hyperbolic function-type potential. We have presented the vacuum solutions and studied the associated geometries. As a remarkable feature, the UV region of the geometries is universally deformed to dS$_2$ and a new naked singularity is developed. The vacuum solutions include a deformed black hole solution, which reduces to the original solution [24] in the undeformed limit. We have computed the entropy of the deformed black hole by evaluating the boundary stress tensor with a certain counter-term. The resulting entropy still agrees with the Bekenstein-Hawking entropy.

In this paper, we will further study the dilaton gravity system with the hyperbolic dilaton potential. Introducing a couple of new variables leads to two copies of Liouville equations with two constraint conditions. As a remarkable feature, the constraints can be expressed in terms of Schwarzian derivatives. The new variables are so powerful to study solutions and enable us to reveal the dipole-like structure of the vacuum solutions. As a benefit, we present a time-dependent solution which describes formation of a black hole with a pulse. Finally, the black hole thermodynamics is considered by taking account of conformal matters from two points of view: 1) the Bekenstein-Hawking entropy and 2) the boundary stress tensor. The former result agrees with the latter one with a counter-term modified in a certain way.

This paper is organized as follows. In section 2, we give a short review of the deformed dilaton gravity system. Then we revisit the vacuum solutions by introducing a couple of new variables. In section 3, we consider how to treat matter fields and derive a time-dependent solution which describes formation of a black hole with a pulse. In section 4, adding conformal matters, we derive a deformed black hole solution. Then we reproduce the Bekenstein-Hawking entropy by computing the boundary stress tensor with a certain counter-term. Section 5 is devoted to conclusion and discussion.

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1It is worth noting that an interesting work along a similar line has been done in [32], in which the UV region is AdS$_2$ and the IR region is deformed to dS$_2$. 

2
2 Hyperbolic dilaton potential and new variables

In this section, we revisit a (1+1)-dimensional dilaton gravity system with a hyperbolic dilaton potential. This system has been introduced in [28] as a Yang-Baxter deformation [29–31] of the Jackiw-Teitelboim (JT) model [22, 23].

2.1 A dilaton gravity system with a hyperbolic potential

In the following, we will work in the Lorentzian signature and the (1+1)-dimensional space-time is described by the coordinates $x^\mu = (t, x) \ (\mu = 0, 1)$. This system contains the metric $g_{\mu\nu}$ and the dilaton $\Phi$ as the basic ingredients. We may add other matter fields but will not do that in section 2.

The classical action for $g_{\mu\nu}$ and $\Phi$ is given by [28]

$$S_\Phi = \frac{1}{16\pi G} \int d^2 x \sqrt{-g} \left[ \Phi^2 R + \frac{1}{\eta} \sinh (2\eta \Phi^2) \right] + \frac{1}{8\pi G} \int dt \sqrt{-\gamma_{tt}} \Phi^2 K , \quad (2.1)$$

where $G$ is a two-dimensional Newton constant, and $R$ and $g$ are Ricci scalar and determinant of $g_{\mu\nu}$. The last term is the Gibbons-Hawking term that contains an extrinsic metric $\gamma_{tt}$ and an extrinsic curvature $K$. A remarkable point of this action is the second term. This is a dilaton potential of hyperbolic function, where $\eta$ is a real constant parameter $^2$.

In the $\eta \to 0$ limit, the classical action (2.1) reduces to the JT model (without matter fields)

$$S_\Phi = \frac{1}{16\pi G} \int d^2 x \sqrt{-g} \left[ \Phi^2 R + 2\Phi^2 \right] + \frac{1}{8\pi G} \int dt \sqrt{-\gamma_{tt}} \Phi^2 K . \quad (2.2)$$

Thus the classical action (2.1) can be regarded as a deformation of the JT model.

The vacuum solutions

The deformed model (2.1) gives rise a three parameter family of vacuum solutions $^3$ [28],

$$ds^2 = \frac{1 + \eta^2 P^2}{1 - \eta^2 (X \cdot P)^2} \frac{-dt^2 + dz^2}{z^2} , \quad (2.3)$$

$^2$ In this paper, we slightly changed the normalization of the dilaton potential from [28]. Therefore the constant factors of solutions are also changed.

$^3$ Here the dilaton is turned on, but the solution is still called “vacuum” solution, according to the custom.
\[ \Phi^2 = \frac{1}{2\eta} \log \left| \frac{1 + \eta \langle X \cdot P \rangle}{1 - \eta \langle X \cdot P \rangle} \right| , \]  

where \( X \) and \( P \) are defined as

\[ X^I \equiv \frac{1}{z} \left( t, \frac{1}{2}(1 + t^2 - z^2), \frac{1}{2}(1 - t^2 + z^2) \right) , \]
\[ P_I \equiv (\beta, \alpha - \gamma, \alpha + \gamma) \quad (I, J = 1, 2, 3) , \]

and the products \( X \cdot P \) and \( P^2 \) are given by

\[ X \cdot P \equiv X^I P_I = \frac{\alpha + \beta t + \gamma (-t^2 + z^2)}{z} , \]
\[ P^2 = \eta^{IJ} P_I P_J = - (\beta^2 + 4\alpha\gamma) . \]

Here the metric of the embedding space \( \mathbb{M}^{2,1} \) is taken as \( \eta_{IJ} = \text{diag}(-1, 1, -1) \). This family labeled by \( \alpha, \beta, \) and \( \gamma \) is associated with the most general Yang-Baxter deformation. In other words, the effect of Yang-Baxter deformation appears only through the factor \( \eta \langle X \cdot P \rangle \). It should also be remarked that a black hole solution is contained as a special case \[28\].

It is worth noting that the conformal factor of the metric (2.3) can be expressed as a Schwarzian derivative of the dilaton (2.4) like

\[ \sqrt{\frac{1}{2} \text{Sch}\{ \Phi^2, \eta \langle X \cdot P \rangle \}} = \frac{z^2}{1 + \eta^2 P^2} e^{2\omega} . \]

Here \( \text{Sch}\{X, x\} \) denotes the Schwarzian derivative defined as

\[ \text{Sch}\{X, x\} \equiv \frac{X'''}{X'} - \frac{3}{2} \left( \frac{X''}{X'} \right)^2 . \]

### 2.2 Introducing a couple of new variables

Let us first rewrite the metric into the following form:

\[ ds^2 = e^{2\omega} \tilde{g}_{\mu\nu} dx^\mu dx^\nu . \]

Then the classical action (2.1) can be rewritten as

\[ S_\Phi = \frac{1}{16\pi G} \int d^2x \sqrt{-\tilde{g}} \left[ \Phi^2 \tilde{R} + 2\tilde{\nabla} \Phi \tilde{\nabla} \omega + \frac{e^{2\omega}}{\eta} \sinh \left( 2\eta \Phi^2 \right) \right] . \]

In order to simplify this expression, it is helpful to introduce a couple of new valuables:

\[ \omega_1 \equiv \omega + \eta \Phi^2 , \quad \omega_2 \equiv \omega - \eta \Phi^2 . \]
Then the action (2.10) becomes the sum of two Liouville systems:

\[
S_\Phi = \frac{1}{32\pi G\eta} \int d^2x \sqrt{-\tilde{g}} \left[ (\omega_1 \tilde{R} + (\tilde{\nabla} \omega_1)^2 + e^{2\omega_1}) - (\omega_2 \tilde{R} + (\tilde{\nabla} \omega_2)^2 + e^{2\omega_2}) \right] \quad (2.12)
\]

\[
\equiv S_{\omega_1} + S_{\omega_2}.
\]

By taking variations of the action (2.12) with respect to \(\omega_1\) and \(\omega_2\), it is easy to derive the following equations of motion:

\[
\tilde{R} - 2(\tilde{\nabla} \omega_1)^2 + 2e^{2\omega_1} = 0,
\]

\[
\tilde{R} - 2(\tilde{\nabla} \omega_2)^2 + 2e^{2\omega_2} = 0. \quad (2.13)
\]

Taking a variation with \(\tilde{g}_{\mu \nu}\) gives rise to the constraints

\[
\tilde{T}^{(1)}_{\mu \nu} + \tilde{T}^{(2)}_{\mu \nu} = 0, \quad (2.14)
\]

where \(\tilde{T}^{(1)}_{\mu \nu}\) and \(\tilde{T}^{(2)}_{\mu \nu}\) are the energy-momentum tensors defined as, respectively,

\[
\tilde{T}^{(1)}_{\mu \nu} \equiv -\frac{2}{\sqrt{-\tilde{g}}} \delta S_{\omega_1} / \delta \tilde{g}^{\mu \nu}, \quad \tilde{T}^{(2)}_{\mu \nu} \equiv -\frac{2}{\sqrt{-\tilde{g}}} \delta S_{\omega_2} / \delta \tilde{g}^{\mu \nu}, \quad (2.15)
\]

and the explicit forms are given by

\[
\tilde{T}^{(1)}_{\mu \nu} = \frac{1}{16\pi G\eta} \left[ \tilde{\nabla}_\mu (\tilde{\nabla}_\nu \omega_1 - (\tilde{\nabla}_\mu \omega_1)(\tilde{\nabla}_\nu \omega_1)) - \frac{1}{2} \tilde{g}_{\mu \nu}(2\tilde{\nabla}^2 \omega_1 - (\tilde{\nabla} \omega_1)^2 - e^{2\omega_1}) \right],
\]

\[
\tilde{T}^{(2)}_{\mu \nu} = -\frac{1}{16\pi G\eta} \left[ \tilde{\nabla}_\mu (\tilde{\nabla}_\nu \omega_2 - (\tilde{\nabla}_\mu \omega_2)(\tilde{\nabla}_\nu \omega_2)) - \frac{1}{2} \tilde{g}_{\mu \nu}(2\tilde{\nabla}^2 \omega_2 - (\tilde{\nabla} \omega_2)^2 - e^{2\omega_2}) \right]. \quad (2.16)
\]

Thus, by employing the new variables \(\omega_1\) and \(\omega_2\), the deformed system (2.1) has been simplified drastically.

### 2.3 Conformal gauge and Schwarzian derivatives

In the following, we will work with the usual conformal gauge

\[
ds^2 = -e^{2\omega} dx^+ dx^- \quad (x^\pm \equiv t \pm z). \quad (2.17)
\]

Then the equations of motion obtained from (2.1) are given by

\[
4\partial_+ \partial_- \Phi^2 + \frac{1}{\eta} e^{2\omega} \sinh \left(2\eta \Phi^2\right) = 0, \quad (2.18)
\]

\[
4\partial_+ \partial_- \omega + e^{2\omega} \cosh \left(2\eta \Phi^2\right) = 0, \quad (2.19)
\]
\[-e^{2\omega} \partial_+ (e^{-2\omega} \partial_+ \Phi^2) = 0, \quad (2.20)\]
\[-e^{2\omega} \partial_- (e^{-2\omega} \partial_- \Phi^2) = 0. \quad (2.21)\]

By solving the above equations, the general vacuum solution has been discussed in [28].

However, as we will show below, the deformed model (2.1) has a nice property, with which we can discuss classical solutions in a more systematic way.

**New variables revisited**

In conformal gauge, the classical action for $\omega_1$ and $\omega_2$ is further simplified as

\[S_\Phi = \frac{1}{8\pi G \eta} \int d^2 x \left[ \left( -\partial_+ \omega_1 \partial_- \omega_1 + \frac{1}{4} e^{2\omega_1} \right) - \left( -\partial_+ \omega_2 \partial_- \omega_2 + \frac{1}{4} e^{2\omega_2} \right) \right]. \quad (2.22)\]

The equations of motion take the standard forms of the Liouville equation

\[4 \partial_+ \partial_- \omega_1 + e^{2\omega_1} = 0, \quad 4 \partial_+ \partial_- \omega_2 + e^{2\omega_2} = 0. \quad (2.23)\]

The general solutions of Liouville equation are given by\(^4\)

\[e^{2\omega_1} = \frac{4 \partial_+ X_1^+ \partial_- X_1^-}{(X_1^+ - X_1^-)^2}, \quad e^{2\omega_2} = \frac{4 \partial_+ X_2^+ \partial_- X_2^-}{(X_2^+ - X_2^-)^2}, \quad (2.24)\]

where $X_i^+ = X_i^+(x^+)$ and $X_i^- = X_i^-(x^-)$ are arbitrary holomorphic and anti-holomorphic functions, respectively.

Note that the equations (2.23) can be expressed by using the metric and dilaton.

\[4 \partial_+ \partial_- (\omega + \eta \Phi) + e^{2\omega + 2\eta \Phi} = 0, \quad (2.25)\]

By summing and subtracting them each other, the equations of motion (2.18) and (2.19) can be reproduced.

By taking $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ in (2.16), the energy-momentum tensors are also rewritten as

\[\tilde{T}_{++}^{(1)} + \tilde{T}_{++}^{(2)} = \frac{-1}{16\pi G \eta} \left( \partial_+ \partial_- \omega_1 + \frac{1}{4} e^{2\omega_1} - \partial_+ \partial_- \omega_2 - \frac{1}{4} e^{2\omega_2} \right), \quad (2.26)\]

\[\tilde{T}_{\pm\pm}^{(1)} + \tilde{T}_{\pm\pm}^{(2)} = \frac{1}{16\pi G \eta} \left( \partial_\pm \partial_\mp \omega_1 - \partial_\pm \omega_1 \partial_\mp \omega_1 - \partial_\pm \partial_\pm \omega_2 + \partial_\pm \omega_1 \partial_\pm \omega_1 \right). \quad (2.27)\]

\(^4\)A similar structure was found in a two-dimensional non-linear sigma model whose target space is given by a three-dimensional Schrödinger spacetime [33].
The first condition (2.26) vanishes automatically due to the equations of motion. Two conditions in (2.27) give rise to nontrivial constraints for the solutions of the equations of motion (2.23). By using the definitions of $\omega_1$ and $\omega_2$ in (2.11), it is easy to directly see that the constraints in (2.26) and (2.27) are equivalent to the ones in (2.20) and (2.21).

By using the general solutions (2.24), the constraint conditions for the holomorphic (antiholomorphic) functions $X_i^+$ ($X_i^-$) can be rewritten as

$$\text{Sch}\{X_1^+, x^+\} - \text{Sch}\{X_2^+, x^+\} = 0,$$

$$\text{Sch}\{X_1^-, x^-\} - \text{Sch}\{X_2^-, x^-\} = 0. \quad (2.28)$$

These constraints mean that the holomorphic (antiholomorphic) functions should be the same functions, up to linear fractional transformations

$$X_1^+(x^+) = \frac{aX_1^+ + b}{cX_1^+ + d}, \quad (a, b, c, d \in \mathbb{R}),$$

$$X_1^-(x^-) = \frac{a'X_1^- + b'}{c'X_1^- + d'}, \quad (a', b', c', d' \in \mathbb{R}). \quad (2.29)$$

Because $e^{2\omega_1} > 0$ and $e^{2\omega_2} > 0$, determinants of the transformations must be positive:

$$ad - bc > 0, \quad a'd' - b'c' > 0. \quad (2.30)$$

This ambiguity comes from the appearance of Schwarzian derivatives.

### 2.4 Vacuum solutions revisited

In this subsection, let us revisit the vacuum solutions by employing a couple of the new variables (2.11). Before going to the detail, it is helpful to recall that the original metric and dilaton can be reconstructed from $\omega_1$ and $\omega_2$ through the following relations:

$$e^{2\omega} = \sqrt{e^{2\omega_1} e^{2\omega_2}}, \quad \Phi^2 = \frac{1}{2\eta} (\omega_1 - \omega_2) = \frac{1}{4\eta} \log \left( \frac{e^{2\omega_1}}{e^{2\omega_2}} \right). \quad (2.31)$$

Here let us take a parametrization for the linear fractional transformations, which come from (2.28) as follows:

$$X_1^+(x^+) = \frac{(1 - \eta \beta)X^+(x^+) - 2\eta \alpha}{-2\eta \gamma X^+(x^+) + (1 + \eta \beta)}, \quad X_1^-(x^-) = X^-(x^-),$$

Note that we can take this parametrization without loss of generality.
Because of the constraint (2.30), we have to work in a restricted parameter region with
\[ 1 - \eta^2 (\beta^2 + 4\alpha\gamma) > 0. \]  
(2.33)

Then the solutions in (2.24) are expressed as
\[ e^{2\omega_1} = \frac{4 (1 - \eta^2 (\beta^2 + 4\alpha\gamma)) \partial_+ X^+ \partial_- X^-}{(X^+ - X^- - \eta(2\alpha + \beta(X^+ + X^-) - 2\gamma X^+ X^-))^2}, \]
\[ e^{2\omega_2} = \frac{4 (1 - \eta^2 (\beta^2 + 4\alpha\gamma)) \partial_+ X^+ \partial_- X^-}{(X^+ - X^- + \eta(2\alpha + \beta(X^+ + X^-) - 2\gamma X^+ X^-))^2}. \]  
(2.34)

Thus the general solution of \( \omega \) and \( \Phi^2 \) are also determined through the relation (2.31). Given that \( X^\pm(x^\pm) = x^\pm \), the deformed metric and dilaton become
\[ e^{2\omega} = \frac{1 - \eta^2 (\beta^2 + 4\alpha\gamma)}{z^2 - \eta^2 (\alpha + \beta t + \gamma(-t^2 + z^2))^2}; \]
\[ \Phi^2 = \frac{1}{2\eta} \log \left| \frac{z + \eta(\alpha + \beta t + \gamma(-t^2 + z^2))}{z - \eta(\alpha + \beta t + \gamma(-t^2 + z^2))} \right|. \]  
(2.35)

This metric is the same as the result obtained in [28] as a Yang-Baxter deformation of AdS\(_2\), up to a scaling factor.

For concreteness, let consider a simple case of (2.32) with \( \alpha = 1, \beta = \gamma = 0 \). Then conformal factors of the metrics for \( X_1 \) and \( X_2 \) are given by, respectively,
\[ e^{2\omega_1} = \frac{1}{(z - \eta)^2}, \quad e^{2\omega_2} = \frac{1}{(z + \eta)^2}. \]  
(2.36)

For each of the AdS\(_2\) factors, the origin of the \( z \)-direction is shifted by \( \pm \eta \). Another example is the case with \( \alpha = 1/2, \beta = 0, \gamma = \mu/2 \) (where \( \mu \) is a positive), in which we have considered a deformed black hole solution [28].

6 Here the condition (2.33) is consistent with the positivity of \( e^{2\omega_1} \) and \( e^{2\omega_2} \).

7 Note that for arbitrary values of \( \alpha, \beta \) and \( \gamma \), black hole solutions can be realized by employing the following coordinate transformation,
\[ X^\pm(x^\pm) = \frac{\sqrt{\mu}}{2\gamma} \tanh \left( \frac{\sqrt{\mu} x^\pm}{2\gamma} \right) + \frac{\beta}{2\gamma} \quad (\mu = \beta^2 + 4\alpha\gamma). \]  
(2.37)
3 Solutions with matter fields

In this section, we shall include additional matter fields. Then the action is given by a sum of the dilaton part $S_\Phi$ and the matter part $S_{\text{matter}}$ like

$$S = S_\Phi + S_{\text{matter}}.$$  (3.1)

Note here that we have not specified the concrete expression of the matter action $S_{\text{matter}}$ yet. In general, $S_{\text{matter}}$ may depend on the metric, dilaton as well as additional matter fields. Hence the inclusion of matter fields leads to the modified equations:

$$4\partial_+ \partial_- \Phi^2 + \frac{1}{\eta} e^{2\omega} \sinh \left(2\eta\Phi^2\right) = 32\pi G T_{++},$$

$$4\partial_+ \partial_- \omega + e^{2\omega} \cosh \left(2\eta\Phi^2\right) = -16\pi G \frac{\delta S_{\text{matter}}}{\delta \Phi^2},$$

$$-e^{2\omega} \partial_+ (e^{-2\omega} \partial_+ \Phi^2) = 8\pi G T_{++},$$

$$-e^{2\omega} \partial_- (e^{-2\omega} \partial_- \Phi^2) = 8\pi G T_{--}. \quad (3.2)$$

Furthermore, one needs to take account of the equation of motion for the matter fields, which is provided as the conservation law of the energy-momentum tensor $T_{\mu\nu}$ defined as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \quad (3.3)$$

So far, it seems difficult to treat the general expression of $T_{\mu\nu}$. Hence we will impose some conditions for $T_{\mu\nu}$ hereafter.

3.1 A certain class of matter fields

For simplicity, let us consider a certain class of matter fields by supposing the following properties:

$$T_{+-} = 0, \quad \frac{\delta S_{\text{matter}}}{\delta \Phi^2} = 0. \quad (3.4)$$

This case is very special because the equations of motion for $\omega_1$ and $\omega_2$ remain to be a pair of Liouville equations because the right-hand sides of the first and second equations in (3.2) vanish. Hence one can still use the general solutions (2.24). The constraints are also still written in terms of Schwarzian derivatives, but slightly modified like

$$\text{Sch}\{X_1^\pm, x^\pm\} - \text{Sch}\{X_2^\pm, x^\pm\} = -32\pi G \eta T_{\pm\pm}. \quad (3.5)$$
That is, the right-hand side does not vanish.

To solve the set of equations, it is helpful to introduce new functions \( \varphi_\pm = \varphi_\pm(x^\pm) \) defined as

\[
\varphi_\pm \equiv \frac{1}{\sqrt{|\partial_\pm X_2^\pm|}}.
\]  

(3.6)

Note here that \( X_2^\pm \) only have been utilized. Then by using \( \varphi_\pm \), the Schwarzian derivatives can be rewritten as

\[
\text{Sch}\{X_2^\pm, x^\pm\} = -2\frac{\partial^2 \varphi_\pm}{\varphi_\pm}.
\]  

(3.7)

When the coordinates are taken as

\[
X_1^\pm(x^\pm) = x^\pm,
\]  

(3.8)

the constraints become Schrödinger equations as follows:

\[
\left(-\partial^2 + 16\pi G \eta T_\pm\right) \varphi_\pm(x^\pm) = 0.
\]  

(3.9)

Thus, for the simple class of matter fields, the constraints have been drastically simplified.

### 3.2 A solution describing formation of a black hole

As an example in the simple class, let us consider an ingoing matter pulse of energy \( E/(8\pi G) \):

\[
T_{-\neg} = \frac{E}{8\pi G} \delta(x^-), \quad T_{++} = T_{+-} = 0.
\]  

(3.10)

Note here that \( T_{\mu\nu} \) does not depend on the dilaton \( \Phi^2 \) and hence this case belongs to the simple class (3.4). This pulse causes a shock-wave traveling on the null curve \( x^- = 0 \).

Then the constraint for the anti-holomorphic part is written as

\[
\left(-\partial_-^2 - 2E\eta \delta(x^-)\right) \varphi_-(x^-) = 0.
\]  

(3.11)

By solving this equation, we obtain the following solution:

\[
\varphi_-(x^-) = \varphi_-(0) \left(1 - 2E\eta x^- \theta(x^-)\right).
\]  

(3.12)
Assuming the continuity, $X_2^-$ is given by

$$X_2^-(x^-) = \begin{cases} 
(1 - 4\eta^2 Ea)x^- - 2\eta a & \text{(for } x^- < 0) \\
\frac{x^- - 2\eta a}{1 - 2Ea x^-} & \text{(for } x^- > 0)
\end{cases}. \quad (3.13)$$

Here $a$ is an arbitrary integral constant and the scaling factor $\varphi_-(0)$ is fixed as

$$\varphi_-(0)^2 = \frac{1}{1 - 4\eta^2 Ea}. \quad (3.14)$$

The remaining task is to determine $X_2^+(x^+)$. The constraint for $\varphi_+(x^+)$ is given by

$$-\partial_+^2 \varphi_+(x^+) = 0. \quad (3.15)$$

Thus one can determine $\varphi_+(x^+)$ and $\partial_+ X_2^+(x^+)$ as

$$\varphi_+(x^+) = \gamma x^+ + \delta, \quad \partial_+ X_2^+(x^+) = \frac{1}{(\gamma x^+ + \delta)^2},$$

where $\gamma$ and $\delta$ are constants. Hence $X_2^+$ is obtained as

$$X_2^+(x^+) = \frac{\alpha x^+ + \beta}{\gamma x^+ + \delta} \quad (\alpha\delta - \beta\gamma = 1) \quad (3.16)$$

with new constants $\alpha$ and $\beta$. For simplicity, we will set $\alpha = \delta = 1$, $\beta = \gamma = 0$. That is, $X_2^+ = x^+$.

Thus one can obtain a solution of the two Liouville equations as follows:

$$e^{2\omega_1} = \frac{4}{(x^+ - x^-)^2},$$

$$e^{2\omega_2} = \begin{cases} 
\frac{4(1 - 4\eta^2 Ea)}{(x^+ - x^-)^2(1 + 2\eta(1 + 2Ea x^-))^{\frac{1}{2}}} & \text{(for } x^- < 0) \\
\frac{4(1 - 4\eta^2 Ea)}{(x^+ - x^-)^2(1 + 2\eta(a - Ex^+ - x^-))^{\frac{1}{2}}} & \text{(for } x^- > 0)
\end{cases}. \quad (3.17)$$

As a result, the original metric and dilaton are given by

$$e^{2\omega} = \begin{cases} 
\frac{4\sqrt{1 - 4\eta^2 Ea}}{(x^+ - x^-)^2(1 + 2\eta(1 + 2Ea x^-))^{\frac{1}{2}}} & \text{(for } x^- < 0) \\
\frac{4\sqrt{1 - 4\eta^2 Ea}}{(x^+ - x^-)^2(1 + 2\eta(a - Ex^+ - x^-))^{\frac{1}{2}}} & \text{(for } x^- > 0)
\end{cases},$$

$$\Phi^2 = \begin{cases} 
\frac{1}{2\eta} \log \left[ 1 + \frac{2\eta(1 + 2Ea x^-)}{x^+ - x^-} \right] - \frac{1}{2\eta} \log(1 - 4\eta^2 Ea) & \text{(for } x^- < 0) \\
\frac{1}{2\eta} \log \left[ 1 + \frac{2\eta(a - Ex^+ - x^-)}{x^+ - x^-} \right] - \frac{1}{2\eta} \log(1 - 4\eta^2 Ea) & \text{(for } x^- > 0)
\end{cases}. \quad (3.18)$$

The undeformed limit $\eta \to 0$ leads to a solution describing formation of a black hole in the undeformed model [24]. Note here that the energy-dependent constant in $\Phi^2$ vanishes in the undeformed limit. At least so far, we have no idea for the physical interpretation of this constant.

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8 Note here that for simplicity, we dropped and tuned some integration constants in (3.12) and (3.13).
4 The deformed system with a conformal matter

In this section, we will consider conformal matters, which do not belong to the previous class (3.4), and discuss the effect of them to thermodynamic quantities associated with a black hole solution.

Let us study a conformal matter whose dynamics is governed by the classical action:

\[ S_{\text{matter}} = -\frac{N}{24\pi} \int d^2x \sqrt{-g} \left[ \chi (R - 2\eta \nabla^2 \Phi) + (\nabla \chi)^2 \right] - \frac{N}{12\pi} \int dt \sqrt{-\gamma} \mu K. \]  

(4.1)

Here \( N \) denotes the central charge of \( \chi \). It is worth noting that the conformal matter couples to dilaton as well as the Ricci scalar, in comparison to the undeformed case \([24]\). Then the energy-momentum tensor and a variation of \( S_{\text{matter}} \) with respect to the dilaton are given by

\[ T_{+-} = \frac{N}{12\pi} \partial_+ \partial_- \chi, \]

\[ T_{\pm\pm} = \frac{N}{12\pi} (-\partial_\pm \partial_\pm \chi + \partial_\pm \chi \partial_\pm \chi + 2\partial_\pm \chi \partial_\pm \omega_1), \]

\[ \frac{\delta S_{\text{matter}}}{\delta \Phi^2} = -\frac{N}{6\pi} \eta \partial_+ \partial_- \chi. \]  

(4.2)

Hence the equations of motion are given by

\[ \partial_+ \partial_- (\omega_1 + \chi) = 0, \]

\[ 4\partial_+ \partial_- \omega_1 + e^{2\omega_1} = \frac{16}{3} GN \eta \partial_+ \partial_- \chi, \]

\[ 4\partial_+ \partial_- \omega_2 + e^{2\omega_2} = 0, \]

\[ e^{\omega_1} \partial_\pm \partial_\pm e^{-\omega_1} - e^{\omega_2} \partial_\pm \partial_\pm e^{-\omega_2} = \frac{2}{3} GN (-\partial_\pm \partial_\pm \chi + \partial_\pm \chi \partial_\pm \chi + 2\partial_\pm \chi \partial_\pm \omega_1). \]  

(4.3)

Note here that the third equation is still the Liouville equation, while the second equation acquired the source term due to the matter contribution.

As we will see below, the system of equations (4.3) is still tractable and one can readily find out a black hole solution including the back-reaction from the conformal matter \( \chi \).

4.1 A black hole solution with a conformal matter

Let us derive a black hole solution.

Given that the solution is static, \( \chi \) can be expressed as

\[ \chi = -\omega_1 - \sqrt{\mu} (x^+ - x^-). \]  

(4.4)
By eliminating $\chi$ from the other equations, one can derive a couple of Liouville equations and the constraint conditions:

$$
4 \left( 1 + \frac{4}{3} GN \eta \right) \partial_+ \partial_- \omega_1 + e^{2\omega_1} = 0,
$$

$$
4 \partial_+ \partial_- \omega_2 + e^{2\omega_2} = 0,
$$

$$
\left( 1 + \frac{2}{3} GN \right) e^{\omega_1} \partial_+ \partial_- e^{-\omega_1} - e^{\omega_2} \partial_+ \partial_- e^{-\omega_2} = \frac{2}{3} GN \mu.
$$

(4.5)

Note that a numerical coefficient in the first equation is shifted by a certain constant as a non-trivial contribution of the conformal matter.

Still, we can use the general solutions of Liouville equations given by

$$
e^{2\omega_1} = \frac{4 \left( 1 + \frac{4}{3} GN \eta \right)}{(X_1^+ - X_1^-)^2} \partial_+ X_1^+ \partial_- X_1^-,
$$

$$
e^{2\omega_2} = \frac{4}{(X_2^+ - X_2^-)^2} \partial_+ X_2^+ \partial_- X_2^-.
$$

(4.6)

By using $X_i^\pm$ ($i = 1, 2$) and the Schwarzian derivative, the constraints can be rewritten as

$$
\left( 1 + \frac{2}{3} GN \right) \text{Sch}\{X_1^+, x^+\} - \text{Sch}\{X_2^+, x^+\} = -\frac{4}{3} GN \mu,
$$

$$
\left( 1 + \frac{2}{3} GN \right) \text{Sch}\{X_1^-, x^-\} - \text{Sch}\{X_2^-, x^-\} = -\frac{4}{3} GN \mu.
$$

(4.7)

It is an easy task to see that the hyperbolic-type coordinates

$$X_{1,2}^\pm = L_{1,2}^\pm \left( \tanh(\sqrt{\mu} x^\pm) \right),
$$

(4.8)

satisfy the constraints (4.7), where $L_{1,2}^\pm$ denote linear fractional transformations as in (2.29).

Note that each of $X_{1,2}^\pm$ covers a partial region of the original spacetime. Hence the coordinate transformations (4.8) may lead to a black hole solution [24, 28]. In fact, the Schwarzian derivatives have particular values like

$$\text{Sch}\{L_{1,2}^\pm \left( \tanh(\sqrt{\mu} x^\pm) \right), x^\pm\} = -2\mu,
$$

(4.9)

and hence these coordinates satisfy the constraints.

Here we choose the following linear transformations $L_{1,2}^\pm$:

$$X_1^+(x^+) = \frac{\tanh(\sqrt{\mu} x^+) - \eta \sqrt{\mu}}{-\eta \mu \tanh(\sqrt{\mu} x^+) + \sqrt{\mu}},
$$

$$\frac{2}{3} GN \mu.
$$
\[ X_2^+(x^+) = \frac{\tanh(\sqrt{\mu} x^+) + \eta \sqrt{\mu}}{\eta \mu \tanh(\sqrt{\mu} x^+) + \sqrt{\mu}}, \]
\[ X_1^-(x^-) = X_2^-(x^-) = \frac{1}{\sqrt{\mu}} \tanh(\sqrt{\mu} x^-), \tag{4.10} \]

one can derive a deformed black hole solution with conformal matters:
\[ e^{2\omega} = \frac{4\mu(1 - \eta^2 \mu) \sqrt{1 + \frac{4}{3} G_N \eta}}{\sinh^2(2 \sqrt{\mu} Z) - \eta^2 \mu \cosh^2(2 \sqrt{\mu} Z)}, \tag{4.11} \]
\[ \Phi^2 = \frac{1}{2\eta} \log \left| \frac{1 + \eta \sqrt{\mu} \coth(2 \sqrt{\mu} Z)}{1 - \eta \sqrt{\mu} \coth(2 \sqrt{\mu} Z)} \right| + \frac{1}{4\eta} \log \left( 1 + \frac{4}{3} G_N \eta \right). \tag{4.12} \]

The matter effect just changes the overall factor of the metric and shifts the dilaton by a constant. In the undeformed limit \( \eta \to 0 \), this solution reduces to a black hole solution with conformal matters presented in [24]:
\[ e^{2\omega} = \frac{4\mu}{\sinh^2(2 \sqrt{\mu} Z)}, \quad \Phi^2 = \sqrt{\mu} \coth(2 \sqrt{\mu} Z) + \frac{1}{3} G_N. \tag{4.13} \]

### 4.2 Black hole entropy

In this subsection, we shall compute the entropy of the black hole solution with a conformal matter given in (4.11) and (4.12) from two points of view: 1) the Bekenstein-Hawking entropy and 2) the boundary stress tensor with a certain counter-term.

#### 1) the Bekenstein-Hawking entropy

Let us first compute the Bekenstein-Hawking entropy. From the metric (4.11), one can compute the Hawking temperature as
\[ T_H = \frac{\sqrt{\mu}}{\pi}. \tag{4.14} \]

From the classical action, the effective Newton constant \( G_{\text{eff}} \) is determined as
\[ \frac{1}{G_{\text{eff}}} = \frac{\Phi^2}{G} - \frac{2}{3} N \chi. \tag{4.15} \]

Note that the presence of the conformal matter fields is reflected as a shift of \( G_{\text{eff}} \). Given that the horizon area \( A \) is 1, the Bekenstein-Hawking entropy \( S_{\text{BH}} \) is computed as
\[ S_{\text{BH}} = \frac{A}{4G_{\text{eff}} \bigg|_{Z \to \infty}}. \]
\[
= \frac{1 + \frac{2}{3} G N \eta}{4 G \eta} \arctanh(\pi T_H \eta) + \frac{N}{6} \log(T_H)
+ \frac{1 + \frac{2}{3} G N \eta}{16 G \eta} \log \left(1 + \frac{4}{3} G N \eta\right) + \frac{N}{6} \log(4\pi).
\] (4.16)

The terms in the last line are constants independent of the Hawking temperature.

2) the boundary stress tensor

The next is to evaluate the entropy by computing the boundary stress tensor with a certain
counter-term.

In conformal gauge, the total action including the Gibbons-Hawking term can be rewrit-
ten as
\[
S_\Phi = \frac{1}{8\pi G} \int d^2 x \left[ -4 \partial_+ \Phi^2 \partial_- \omega + \frac{1}{2\eta} \sinh(2\eta \Phi^2) e^{2\omega} \right],
\]
\[
S_{\text{matter}} = \frac{N}{6\pi} \int d^2 x \left[ \partial_+ \chi \partial_- \chi + 2 \partial_+ \chi \partial_- \omega + 2 \eta \partial_+ \chi \partial_- \Phi^2 \right].
\] (4.17)

By using the explicit expression of the black hole solution in (4.11) and (4.12), the on-shell
bulk action can be evaluated on the boundary,
\[
S_\Phi + S_{\text{matter}} = \int dt \left. \frac{-(1 + \frac{2}{3} G N \eta)\mu - \sqrt{\pi G N}}{2\pi G (1 + \eta^2 \mu)} \left(1 - \frac{\eta^2 \mu}{\cosh(4\sqrt{\mu} Z)}\right) \right|_{Z \to Z_0}.
\] (4.18)

As argued in [23], the singularity of (4.11) is identified as the boundary \(Z_0\):
\[
Z_0 \equiv \frac{1}{2\sqrt{\mu}} \arctanh(\eta \sqrt{\mu}).
\] (4.19)

As the bulk action approaches the boundary \((Z \to Z_0)\), the bulk action (4.18) diverges
and hence one needs to introduce a cut-off. When the regulator \(\epsilon\) is introduced such that
\(Z - Z_0 = \epsilon\), the on-shell action is expanded as
\[
S_\Phi + S_{\text{matter}} = \int dt \left[ \frac{1 + \frac{4}{3} G N \eta}{16\pi G \eta} - \frac{1 + \eta^2 \mu}{16\pi G \eta^2} + O(\epsilon^1) \right].
\] (4.20)

To cancel the divergence, it is appropriate to add the following counter-term
\[
S_{\text{ct}} = \frac{-1}{8\pi G} \int dt \frac{\sqrt{-g} \mu}{L} \left[ \sqrt{F(\Phi^2) - \frac{1}{\eta^2} \log(1 - \eta^2 \mu)} \right].
\]

\[\text{For an earlier argument for the relation between the singularity and the holographic screen, see [24].}\]

\[\text{Note that in the undeformed limit } \eta \to 0, \text{ this counter-term reduces to the one in [24]. When } \mu = N = 0, \text{ this term becomes the dilaton potential } \frac{1}{\eta} \sinh(2\eta \Phi^2).\]
\[ + \frac{4}{3} GN \sqrt{G(\Phi^2) - \frac{1}{2\eta^2} \log(1 - \eta^2 \mu)} \]. \quad (4.21) \]

Here \( L \) is the overall factor of the metric defined as
\[ L^2 \equiv (1 - \eta^2 \mu) \sqrt{1 + \frac{4}{3} GN \eta}, \quad (4.22) \]

and scalar functions \( F \) and \( G \) are defined as
\[
F(\Phi^2) \equiv - \frac{1 - \eta^2 \mu + (1 - \eta^2 \mu) \cosh \left[ \frac{2\eta}{2\eta^2} \left( \Phi^2 - \frac{\log(1 + \frac{4}{3} GN \eta)}{4\eta} \right) \right]}{2\eta^2},
\]
\[
G(\Phi^2) \equiv \frac{8\eta \sqrt{\mu} + (1 - \eta^2 \mu) \exp \left[ \frac{2\eta}{4\eta^2} \left( \Phi^2 - \frac{\log(1 + \frac{4}{3} GN \eta)}{4\eta} \right) \right]}{4\eta^2}. \quad (4.23) \]

The extrinsic metric \( \gamma_{tt} \) on the boundary is evaluated as
\[ \gamma_{tt} = -e^{2\omega} \big|_{Z \to Z_0}. \]

In the undeformed limit \( \eta \to 0 \), this counter-term reduces to
\[ S_{ct}^{(\eta = 0)} = \int dt \sqrt{-\gamma_{tt}} \left( -\frac{\Phi^2}{8\pi G} - \frac{N}{24\pi} \right). \quad (4.24) \]

This is nothing but the counter-term utilized in the undeformed model \[24\].

It is straightforward to check that the sum \( S = S_\Phi + S_{\text{matter}} + S_{ct} \) becomes finite on the boundary by using the expanded form of the counter-term (4.21):
\[ S_{ct} = \int dt \left[ -\frac{1 + \frac{4}{3} GN \eta}{16\pi G \eta \epsilon} + \frac{1 + \eta^2 \mu - \frac{16}{3} GN \eta^2 + 2(1 + \frac{2}{3} GN \eta) \log(1 - \eta^2 \mu)}{16\pi G \eta^2} + O(\epsilon) \right]. \]

In a region near the boundary, the warped factor of the metric (4.11) is expanded as
\[ e^{2\omega} = \frac{L}{\eta \epsilon} + O(\epsilon^0). \quad (4.25) \]

Hence, by normalizing the boundary metric as
\[ \hat{\gamma}_{tt} = \frac{\eta \epsilon}{L} \gamma_{tt}, \]
the boundary stress tensor is defined as
\[ \langle \hat{T}_{tt} \rangle \equiv \frac{-2 \sqrt{-\hat{\gamma}_{tt}}}{\delta \hat{\gamma}_{tt}} \frac{\delta S}{\delta \hat{\gamma}_{tt}} = \lim_{\epsilon \to 0} \sqrt{\frac{\eta \epsilon}{L}} \frac{-2 \sqrt{-\gamma_{tt}}}{\sqrt{-\gamma_{tt}}} \frac{\delta S}{\delta \gamma_{tt}}. \quad (4.26) \]

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After all, $\langle \hat{T}_{tt} \rangle$ is evaluated as

$$\langle \hat{T}_{tt} \rangle = \frac{-(1 + \frac{2}{3}GN\eta) \log(1 - \eta^2\mu)}{8\pi G \eta^2} + \frac{N\sqrt{\mu}}{6\pi}. \quad (4.27)$$

To compute the associated entropy, $\langle \hat{T}_{tt} \rangle$ should be identified with energy $E$ like

$$E = \frac{-(1 + \frac{2}{3}GN\eta) \log(1 - \pi^2T_H^2\eta^2)}{8\pi G \eta^2} + \frac{N}{6}T_H, \quad (4.28)$$

where we have used the expression of the Hawking temperature (4.14). Then by solving the thermodynamic relation,

$$dE = \frac{dS}{T_H}, \quad (4.29)$$

the associated entropy is obtained as

$$S = \frac{(1 + \frac{2}{3}GN\eta)}{4G\eta} \text{arctanh}(\pi T_H\eta) + \frac{N}{6} \log(T_H) + S_{T_H=0}. \quad (4.30)$$

Here $S_{T_H=0}$ has appeared as an integration constant that measures the entropy at zero temperature. Thus the resulting entropy precisely agrees with the Bekenstein-Hawking entropy (4.16), up to the temperature-independent constant.

### 5 Conclusion and discussion

In this paper, we have considered some matter contributions to a (1+1)-dimensional dilaton gravity system with a hyperbolic dilaton potential. By introducing a couple of new variables, this system has been rewritten into a pair of Liouville equations with two constraints. In particular, the constraints in conformal gauge can be expressed in terms of Schwarzian derivatives. We have revisited the vacuum solutions and revealed its dipole-like structure. The new variables are so powerful in studying solutions. As a benefit, we have constructed a time-dependent solution which describes formation of a black hole with a pulse. Finally, the black hole entropy has been considered by taking account of conformal matters. The Bekenstein-Hawking entropy agrees with the entropy computed from the boundary stress tensor with a certain counter-term.

There are some future directions. The first is to clarify a connection between the system considered here and the doubled formalism such as Double Field Theory (DFT) \cite{35,37} and
Double Sigma Model (DSM) [38–41]. As well recognized, Yang-Baxter deformations of type IIB string theory defined on AdS$_5 \times$S$^5$ [42,43] are closely related to DFT and DSM [44–46] via the generalized supergravity [47,48]. A similar connection may be expected in the present lower-dimensional case as well, because the present system was originally constructed by employing the Yang-Baxter deformation technique. The second is to reveal the underlying symmetry. By following a nice work by Ikeda and Izawa [49], the hyperbolic dilaton potential leads to the expected $q$-deformed $sl(2)$ algebra realized in the associated non-linear gauge theory. Elaborating this symmetry algebra helps us to identify the holographic dual. The third is to a generalization to include arbitrary matter fields and discuss the associated one-dimensional boundary theory by following [25,26]. It seems likely that the anticipated system is a deformed Schwarzian theory. Finally, it is interesting to consider a similar deformation of the asymptotically flat case [50] by following [51]. It is also nice to study how the holographic relation should be modified in the case with a reflecting dynamical boundary by generalizing [52]. The integrability techniques discussed there would still be useful even after performing Yang-Baxter deformations.

We hope that the dipole-like structure uncovered here would shed light on a new aspect of the 2D dilaton gravity system and further the holographic principle as well.

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References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Int. J. Theor. Phys. 38 (1999) 1113 [Adv. Theor. Math. Phys. 2 (1998) 231] [hep-th/9711200].

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[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428 (1998) 105 [arXiv:hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253 [arXiv:hep-th/9802150].

[4] G. 't Hooft, “Dimensional reduction in quantum gravity,” Salamfest 1993:0284-296 [gr-qc/9310026].

[5] L. Susskind, “The World as a hologram,” J. Math. Phys. 36 (1995) 6377 [hep-th/9409089].

[6] N. Beisert et al., “Review of AdS/CFT Integrability: An Overview,” Lett. Math. Phys. 99 (2012) 3 [arXiv:1012.3982 [hep-th]].

[7] A. Kitaev, “A simple model of quantum holography.”
http://online.kitp.ucsb.edu/online/entangled15/kitaev/
http://online.kitp.ucsb.edu/online/entangled15/kitaev2/
Talks at KITP, April 7, 2015 and May 27, 2015.

[8] S. Sachdev and J.-w. Ye, “Gapless spin fluid ground state in a random, quantum Heisenberg magnet,” Phys. Rev. Lett. 70 (1993) 3339 [arXiv:cond-mat/9212030 [cond-mat]].

[9] J. Polchinski and V. Rosenhaus, “The Spectrum in the Sachdev-Ye-Kitaev Model,” JHEP 1604 (2016) 001 [arXiv:1601.06768 [hep-th]].

[10] J. Maldacena and D. Stanford, “Remarks on the Sachdev-Ye-Kitaev model,” Phys. Rev. D 94 (2016) no.10, 106002 [arXiv:1604.07818 [hep-th]].

[11] K. Jensen, “Chaos in AdS2 Holography,” Phys. Rev. Lett. 117 (2016) no.11, 111601 [arXiv:1605.06098 [hep-th]].

[12] D. J. Gross and V. Rosenhaus, “A Generalization of Sachdev-Ye-Kitaev,” JHEP 1702 (2017) 093 [arXiv:1610.01569 [hep-th]].

[13] W. Fu, D. Gaiotto, J. Maldacena and S. Sachdev, “Supersymmetric SYK models,” Phys. Rev. D 95 (2017) no.2, 026009 [arXiv:1610.08917 [hep-th]].

[14] E. Witten, “An SYK-Like Model Without Disorder,” [arXiv:1610.09758 [hep-th]].
[15] J. S. Cotler et al., “Black Holes and Random Matrices,” arXiv:1611.04650 [hep-th].

[16] I. R. Klebanov and G. Tarnopolsky, “Uncolored Random Tensors, Melon Diagrams, and the SYK Models,” Phys. Rev. D 95 (2017) no.4, 046004 arXiv:1611.08915 [hep-th].

[17] T. Nishinaka and S. Terashima, “A Note on Sachdev-Ye-Kitaev Like Model without Random Coupling,” arXiv:1611.10290 [hep-th].

[18] A. M. García-García and J. J. M. Verbaarschot, “Analytical Spectral Density of the Sachdev-Ye-Kitaev Model at finite $N$,” arXiv:1701.06593 [hep-th].

[19] D. Stanford and E. Witten, “Fermionic Localization of the Schwarzian Theory,” arXiv:1703.04612 [hep-th].

[20] H. Itoyama, A. Mironov and A. Morozov, “Rainbow tensor model with enhanced symmetry and extreme melonic dominance,” arXiv:1703.04983 [hep-th].

[21] D. Grumiller, W. Kummer and D. V. Vassilevich, “Dilaton gravity in two-dimensions,” Phys. Rept. 369 (2002) 327 [hep-th/0204253].

[22] R. Jackiw, “Lower Dimensional Gravity,” Nucl. Phys. B 252 (1985) 343-356.

[23] C. Teitelboim, “Gravitation and Hamiltonian Structure in Two Space-Time Dimensions,” Phys. Lett. B 126 (1983) 41-45.

[24] A. Almheiri and J. Polchinski, “Models of AdS$_2$ backreaction and holography,” JHEP 1511 (2015) 014 arXiv:1402.6334 [hep-th].

[25] J. Maldacena, D. Stanford and Z. Yang, “Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space,” PTEP 2016 (2016) no.12, 12C104 arXiv:1606.01857 [hep-th].

[26] J. Engelsöy, T. G. Mertens and H. Verlinde, “An investigation of AdS$_2$ backreaction and holography,” JHEP 1607 (2016) 139 arXiv:1606.03438 [hep-th].

[27] M. Cvetič and I. Papadimitriou, “AdS$_2$ holographic dictionary,” JHEP 1612 (2016) 008 arXiv:1608.07018 [hep-th].

[28] H. Kyono, S. Okumura and K. Yoshida, “Deformations of the Almheiri-Polchinski model,” JHEP 1703 (2017) 173 arXiv:1701.06340 [hep-th].
[29] C. Klimcik, “Yang-Baxter sigma models and dS/AdS T duality,” JHEP 0212 (2002) 051 [hep-th/0210095]; “On integrability of the Yang-Baxter sigma-model,” J. Math. Phys. 50 (2009) 043508 [arXiv:0802.3518 [hep-th]].

[30] F. Delduc, M. Magro and B. Vicedo, “On classical q-deformations of integrable sigma-models,” JHEP 1311 (2013) 192 [arXiv:1308.3581 [hep-th]].

[31] T. Matsumoto and K. Yoshida, “Yang-Baxter sigma models based on the CYBE,” Nucl. Phys. B 893 (2015) 287 [arXiv:1501.03665 [hep-th]].

[32] D. Anninos and D. M. Hofman, “Infrared Realization of dS$_2$ in AdS$_2$,” arXiv:1703.04622 [hep-th].

[33] I. Kawaguchi, T. Matsumoto and K. Yoshida, “Schrödinger sigma models and Jordanian twists,” JHEP 1308 (2013) 013 [arXiv:1305.6556 [hep-th]].

[34] T. Kameyama and K. Yoshida, “A new coordinate system for q-deformed AdS$_5 \times$S$^5$ and classical string solutions,” J. Phys. A 48 (2015) 7, 075401 [arXiv:1408.2189 [hep-th]]; “Minimal surfaces in q-deformed AdS$_5 \times$S$^5$ string with Poincare coordinates,” J. Phys. A 48 (2015) 24, 245401 [arXiv:1410.5544 [hep-th]]; “Generalized quark-antiquark potentials from a q-deformed AdS$_5 \times$S$^5$ background,” PTEP 2016 (2016) no. 6, 063B01 [arXiv:1602.06786 [hep-th]].

[35] W. Siegel, “Two vierbein formalism for string inspired axionic gravity,” Phys. Rev. D 47 (1993) 5453 [hep-th/9302036]; “Superspace duality in low-energy superstrings,” Phys. Rev. D 48 (1993) 2826 [hep-th/9305073]; “Manifest duality in low-energy superstrings,” hep-th/9308133.

[36] C. Hull and B. Zwiebach, “Double Field Theory,” JHEP 0909 (2009) 099 [arXiv:0904.4664 [hep-th]]; “The Gauge algebra of double field theory and Courant brackets,” JHEP 0909 (2009) 090 [arXiv:0908.1792 [hep-th]].

[37] O. Hohm, C. Hull and B. Zwiebach, “Background independent action for double field theory,” JHEP 1007 (2010) 016 [arXiv:1003.5027 [hep-th]]; “Generalized metric formulation of double field theory,” JHEP 1008 (2010) 008 [arXiv:1006.4823 [hep-th]].
[38] A. A. Tseytlin, “Duality Symmetric Formulation of String World Sheet Dynamics,” Phys. Lett. B 242 (1990) 163; “Duality symmetric closed string theory and interacting chiral scalars,” Nucl. Phys. B 350 (1991) 395.

[39] C. M. Hull, “A Geometry for non-geometric string backgrounds,” JHEP 0510 (2005) 065 [hep-th/0406102]; “Doubled Geometry and T-Folds,” JHEP 0707 (2007) 080 [hep-th/0605149].

[40] N. B. Copland, “A Double Sigma Model for Double Field Theory,” JHEP 1204 (2012) 044 [arXiv:1111.1828 [hep-th]].

[41] K. Lee and J. H. Park, “Covariant action for a string in ”doubled yet gauged” space-time,” Nucl. Phys. B 880 (2014) 134 [arXiv:1307.8377 [hep-th]].

[42] F. Delduc, M. Magro and B. Vicedo, “An integrable deformation of the AdS$_5 \times S^5$ superstring action,” Phys. Rev. Lett. 112 (2014) 051601 [arXiv:1309.5850 [hep-th]]; “Derivation of the action and symmetries of the $q$-deformed AdS$_5 \times S^5$ superstring,” JHEP 1410 (2014) 132 [arXiv:1406.6286 [hep-th]].

[43] I. Kawaguchi, T. Matsumoto and K. Yoshida, “Jordanian deformations of the AdS$_5 \times S^5$ superstring,” JHEP 1404 (2014) 153 [arXiv:1401.4855 [hep-th]].

[44] Y. Sakatani, S. Uehara and K. Yoshida, “Generalized gravity from modified DFT,” arXiv:1611.05856 [hep-th].

[45] A. Baguet, M. Magro and H. Samtleben, “Generalized IIB supergravity from exceptional field theory,” JHEP 1703 (2017) 100 [arXiv:1612.07210 [hep-th]].

[46] J. Sakamoto, Y. Sakatani and K. Yoshida, “Weyl invariance for generalized supergravity backgrounds from the doubled formalism,” accepted in PTEP [arXiv:1703.09213 [hep-th]].

[47] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban and A. A. Tseytlin, “Scale invariance of the $\eta$-deformed AdS$_5 \times S^5$ superstring, T-duality and modified type II equations,” Nucl. Phys. B 903 (2016) 262 [arXiv:1511.05793 [hep-th]].

[48] L. Wulff and A. A. Tseytlin, “Kappa-symmetry of superstring sigma model and generalized 10d supergravity equations,” JHEP 1606 (2016) 174 [arXiv:1605.04884 [hep-th]].
[49] N. Ikeda and K. I. Izawa, “General form of dilaton gravity and nonlinear gauge theory,” Prog. Theor. Phys. 90 (1993) 237 [hep-th/9304012].

[50] C. G. Callan, Jr., S. B. Giddings, J. A. Harvey and A. Strominger, “Evanescent black holes,” Phys. Rev. D 45 (1992) no.4, R1005 [hep-th/9111056].

[51] T. Matsumoto, D. Orlando, S. Reffert, J. Sakamoto and K. Yoshida, “Yang-Baxter deformations of Minkowski spacetime,” JHEP 1510 (2015) 185 [arXiv:1505.04553 [hep-th]]. A. Borowiec, H. Kyono, J. Lukierski, J. Sakamoto and K. Yoshida, “Yang-Baxter sigma models and Lax pairs arising from $\kappa$-Poincaré $r$-matrices,” JHEP 1604 (2016) 079 [arXiv:1510.03083 [hep-th]]. H. Kyono, J. Sakamoto and K. Yoshida, “Lax pairs for deformed Minkowski spacetimes,” JHEP 1601 (2016) 143 [arXiv:1512.00208 [hep-th]].

[52] M. Fitkevich, D. Levkov and Y. Zenkevich, “Exact solutions and critical chaos in dilaton gravity with a boundary,” [arXiv:1702.02576 [hep-th]].