Precise Error Analysis of the LASSO under Correlated Designs

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Abstract

In this paper, we consider the problem of recovering a sparse signal from noisy linear measurements using the so called LASSO formulation. We assume a correlated Gaussian design matrix with additive Gaussian noise. We precisely analyze the high dimensional asymptotic performance of the LASSO under correlated design matrices using the Convex Gaussian Min-max Theorem (CGMT). We define appropriate performance measures such as the mean-square error (MSE), probability of support recovery, element error rate (EER) and cosine similarity. Numerical simulations are presented to validate the derived theoretical results.

Index Terms

LASSO, MSE, element error rate, probability of support recovery, cosine similarity, correlated designs, asymptotic performance

I. INTRODUCTION

The LASSO is one of the most celebrated methods in statistics and signal processing [1]. Given a noisy linear measurements \( y = Ax_0 + z \), it recovers the unknown \( k \)-sparse signal \( x_0 \in \mathbb{R}^n \) by solving the following optimization problem:

\[
\hat{x} := \arg\min_{x} \|y - Ax\|_2^2 + \lambda \|x\|_1,
\]

where \( A \in \mathbb{R}^{m \times n} \) is the design (measurement) matrix, \( z \in \mathbb{R}^m \) is the noise vector that has iid entries \( \mathcal{N}(0, \sigma^2) \), \( \lambda > 0 \) is the regularization parameter, \( \| \cdot \|_2 \) denotes the \( \ell_2 \)-norm of a vector, and \( \| \cdot \|_1 \) represents its \( \ell_1 \)-norm.

The asymptotic performance of the LASSO has been recently extensively studied in many works. One approach that is based on the Approximate Massage Passing (AMP) framework was used in series of papers to study the LASSO under the assumption of iid design matrix [2]–[4]. Another approach used the Convex Gaussian min-max Theorem (CGMT) to derive sharp performance guarantees of the LASSO for iid design matrices [5]–[9]. In addition, in [10], [11] the LASSO was analyzed for imperfect designs. In many practical situations, the design matrix has correlated entries [12] so it is important to take into account correlations in the analysis. Very recently, [13] used
the CGMT framework to analyze the Box-Least Squares decoder under the presence of correlations. To the best of our knowledge, the precise error analysis of the LASSO under correlated designs has not been explicitly derived in this context before.

To close this gap, this paper derives precise asymptotic error analysis of the LASSO with correlated Gaussian design matrix. In particular, we provide asymptotic expressions of the mean squared error (MSE) of the LASSO. In addition, we study other interesting performance measures such as the probability of support recovery, the element error rate (EER) and the cosine similarity.

II. PROBLEM FORMULATION

A. System Model

We consider a noisy linear measurements system \( y = Ax_0 + z \in \mathbb{R}^m \). The unknown signal vector \( x_0 \in \mathbb{R}^n \) is assumed to be \( k \)-sparse, i.e., only \( k \) of its entries are sampled iid from a distribution \( p x_0 \) and the remaining entries are zeros. The noise vector \( z \in \mathbb{R}^m \) is assumed to have iid entries \( \mathcal{N}(0, \sigma^2) \). In this work, we consider a correlated Gaussian design matrix which can be modeled as [14], [15]

\[
A = \Sigma^{1/2} H,
\]

where \( \Sigma \in \mathbb{R}^{m \times m} \) is known Hermitian nonnegative left correlation matrix, satisfying \( \frac{1}{m} \text{tr}(\Sigma) = O(1) \), while \( H \in \mathbb{R}^{m \times n} \) is a Gaussian matrix with iid entries \( \mathcal{N}(0, \frac{1}{m}) \). The analysis is performed when the system dimensions grow simultaneously to infinity \( (m, n, k \to \infty) \) at fixed rates: \( \frac{m}{n} \to \delta \in (0, \infty) \) and \( \frac{k}{n} \to \kappa \in (0, 1) \). The signal-to-noise ratio (SNR) is assumed to be constant and given as \( \text{SNR} = \frac{\kappa}{\sigma^2} \).

B. Performance Metrics

We consider the following performance metrics:

Mean squared error: The recovery mean squared error (MSE) measures the deviation of \( \hat{x} \) from the true signal \( x_0 \). Formally, it is defined as

\[
\text{MSE} := \frac{1}{n} \| \hat{x} - x_0 \|_2^2.
\]

Support Recovery: In the problem of sparse recovery, a natural measure of performance that is used in many applications is support recovery, which is defined as identifying whether an entry of \( x_0 \) is on the support (i.e., non-zero), or it is off the support (i.e., zero). The decision is based on the LASSO solution \( \hat{x} \): we say the \( i^{th} \) entry of \( \hat{x} \) is on the support if \( |\hat{x}_i| \geq \xi \), where \( \xi > 0 \) is a user-defined hard threshold on the entries of \( \hat{x} \). Formally, let

\[
\Phi_{\xi, \text{on}}(\hat{x}) := \frac{1}{k} \sum_{i \in S(x_0)} \mathbb{I}(|\hat{x}_i| \geq \xi),
\]

\[
\Phi_{\xi, \text{off}}(\hat{x}) := \frac{1}{n - k} \sum_{i \notin S(x_0)} \mathbb{I}(|\hat{x}_i| \leq \xi),
\]

where \( \mathbb{I}(\cdot) \) is the indicator function of a set \( \mathcal{A} \), and \( S(x_0) \) is the support of \( x_0 \), i.e., the set of the non-zero entries of \( x_0 \). In Theorem 2, we precisely predict the per-entry rate of successful on-support and off-support recovery.

Element Error Rate: We also consider the ad-hoc performance metric that we call (per) Element-Error Rate (EER)
that is the opposite of the probability of successful recovery. After hard thresholding the entries of \( \hat{x} \) by \( \xi \) as before, we define the EER as follows

\[
\text{EER}_\xi := \frac{1}{k} \sum_{i \in \mathcal{S}(x_0)} \mathbb{I}_{\{|\hat{x}_i| < \xi\}} + \frac{1}{n-k} \sum_{i \notin \mathcal{S}(x_0)} \mathbb{I}_{\{|\hat{x}_i| > \xi\}}.
\]

As we can see, this metric can be linked to the support recovery metrics defined before as follows:

\[
\text{EER}_\xi = 2 - \Phi_{\xi,\text{on}}(\hat{x}) - \Phi_{\xi,\text{off}}(\hat{x}).
\] (4)

**Cosine Similarity:** We define another metric that is widely used in machine learning which is the cosine similarity between \( \hat{x} \) and \( x_0 \). It is defined as

\[
\cos(\angle(\hat{x}, x_0)) := \frac{\hat{x}^T x_0}{\|\hat{x}\|_2 \|x_0\|_2} \in [-1, +1].
\]

Obviously, we seek estimates that maximize similarity (correlation).

The rest of the paper is organized as follows. In Section III, we present and discuss the main results of the paper. Numerical results are included in Section IV, while a proof outline is given in Section V. Finally, the paper is concluded in Section VI.

### III. MAIN RESULTS

In this section, we summarize the asymptotic analysis of the LASSO in (1) in terms of its MSE, probability of support recovery, EER and cosine similarity. We use the standard notation \( \operatorname{plim}_{n \to \infty} X_n = X \) to denote that a sequence of random variables \( X_n \) converges in probability towards a constant \( X \). Define the spectral decomposition of \( \Sigma \) as \( \Sigma = U\Gamma U^T \). Finally, let \( Q(\cdot) \) denote the Gaussian \( Q \)-function associated with the standard normal probability density function (pdf) \( \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \).

**Theorem 1** (MSE of the LASSO). Let \( \text{MSE} \) denote the mean squared error of the LASSO in (1) for some fixed but unknown \( k \)-sparse signal \( x_0 \), then in the limit of \( m, n, k \to \infty, m/n \to \delta, \) and \( k/n \to \kappa \), it holds

\[
\operatorname{plim}_{n \to \infty} \text{MSE} = \alpha^*_*,
\] (5)

where \( \alpha^*_* \) is the unique solution to the following:

\[
\min_{\alpha > 0} \max_{\beta > 0} \sup_{\chi > 0} D(\alpha, \beta, \chi) := \frac{1}{n} \sum_{j=1}^{m} \frac{\gamma_j \alpha + \sigma^2}{1 - \gamma_j \mu(\alpha, \beta)} - \left( \frac{\beta^2}{4} \mu(\alpha, \beta) + \frac{\chi}{2} + \frac{\alpha \beta^2}{2 \chi} \right) + \frac{\chi}{\alpha} \mathbb{E}_{X_0 \sim p_{X_0}} \left[ e \left( X_0 + \frac{\alpha \beta}{\chi} Z ; \lambda \alpha \right) \right],
\]

\[
e(\alpha; \beta) = \begin{cases} ba - \frac{1}{2} b^2, & \text{if } a > b \\ \frac{1}{2} a^2, & \text{if } |a| \leq b \\ -ba - \frac{1}{2} b^2, & \text{if } a < -b, \end{cases}
\] (6)

\( \gamma_j \) is the \( j \)-th eigenvalue of the matrix \( \Sigma \), and \( \mu(\alpha, \beta) \) satisfies:

\[
\frac{1}{n} \sum_{j=1}^{m} \frac{\alpha + \sigma^2}{\gamma_j} = \beta^2 = 0.
\]

\(^1\)This performance measure can also be seen as the **correlation** between the estimator \( \hat{x} \) and \( x_0 \).
Proof. A proof outline of this theorem is given in Section V.

**Remark 1.** The optimal solutions $\alpha^*_\star, \beta^*_\star, \chi^*_\star$ can be computed numerically by writing the first order optimality conditions, i.e., by solving $\nabla_{(\alpha, \beta, \chi)} D(\alpha, \beta, \chi) = 0$.

**Remark 2.** Theorem 1 allows us to optimally tune the involved parameter such as the regularizer $\lambda$ or the number of normalized measurements $\delta$, etc.. See Fig.1 for an illustration. Note that the MSE expression in (5) requires the knowledge of the noise variance $\sigma^2$. However, even if the noise variance is unknown, we can use some algorithm to estimate the SNR such as in [16].

The following Theorem precisely characterizes the support recovery metrics introduced in (3).

**Theorem 2** (Probability of support recovery). Under the same settings of Theorem 1 and for any fixed $\xi > 0$, and in the limit of $m, n, k \to \infty, m/n \to \delta$, and $k/n \to \kappa$, it holds that:

$$
\text{plim}_{n \to \infty} \Phi_{\xi, \text{on}}(\mathbf{x}) = \mathbb{P} \left[ \left| \frac{\eta \left( X_0 + \frac{\alpha^*_\star, \beta^*_\star, \chi^*_\star}{\lambda^*_\star} Z; \lambda^*_\star \right)}{\sqrt{\kappa}} \right| \geq \xi \right],
$$

and

$$
\text{plim}_{n \to \infty} \Phi_{\xi, \text{off}}(\mathbf{x}) = \mathbb{P} \left[ \left| \frac{\eta \left( \frac{\alpha^*_\star, \beta^*_\star, \chi^*_\star}{\lambda^*_\star} Z; \lambda^*_\star \right)}{\sqrt{\kappa}} \right| \leq \xi \right],
$$

where $\eta(a; b)$ is the soft-thresholding function defined as

$$
\eta(a; b) = \begin{cases} 
  a - b, & \text{if } a > b \\
  0, & \text{if } |a| \leq b \\
  a + b, & \text{if } a < -b.
\end{cases}
$$

**Proof.** An overview of the proof is given in Section V.

**Remark 3.** It should be clear that these probabilities are taken over the randomness of $X_0$ and $Z$.

The following proposition derives a precise asymptotic characterization of the EER.

**Proposition 1** (Element Error Rate). Under the same settings of Theorem 1 and for any fixed $\xi > 0$, it holds that:

$$
\text{plim}_{n \to \infty} \text{EER}_\xi = \mathbb{P} \left[ \left| \eta \left( X_0 + \frac{\alpha^*_\star, \beta^*_\star, \chi^*_\star}{\lambda^*_\star} Z; \lambda^*_\star \right) \right| < \xi \right] + \mathbb{P} \left[ \left| \eta \left( \frac{\alpha^*_\star, \beta^*_\star, \chi^*_\star}{\lambda^*_\star} Z; \lambda^*_\star \right) \right| > \xi \right].
$$

**Proof.** The proof follows from Theorem 2 and the definition of the EER as given by (4).

In all of the above metrics, we care about the magnitude of the LASSO estimate. However, in many applications, the orientation of the solution matters as well. This is the objective of the next proposition that characterizes the cosine similarity of the LASSO.

**Proposition 2** (Cosine Similarity). Under the same settings of Theorem 1 it holds that:

$$
\text{plim}_{n \to \infty} \cos(\angle(\mathbf{x}_0, \mathbf{x})) = \frac{\mathbb{E}_{X_0, Z} \left[ \eta \left( X_0 + \frac{\alpha^*_\star, \beta^*_\star, \chi^*_\star}{\lambda^*_\star} Z; \lambda^*_\star \right) X_0 \right]}{\sqrt{\kappa \mathbb{E}_{X_0, Z} \left[ \eta^2 \left( \frac{\alpha^*_\star, \beta^*_\star, \chi^*_\star}{\lambda^*_\star} Z; \lambda^*_\star \right) \right]}}.
$$
Fig. 1. MSE performance of the LASSO decoder for the exponential correlation model in (9), $\delta = 0.7$, $n = 400$, $\rho = 0.7$, $\sigma^2 = 0.01$, and $\kappa = 0.1$. For each $\lambda$ value, the data are averaged over 500 independent realizations of the channel matrix, the signal vector and the noise vector.

**Proof.** The proof is based on the CGMT to derive asymptotic predictions of the numerator and the denominator of the cosine similarity expression separately and then use the Continuous Mapping Theorem to arrive at this proposition. Details are omitted for space limitations.

IV. Numerical Results

To validate the provided theoretical results of the MSE as given by Theorem 1 and probability of support recovery as stated in Theorem 2, we consider the following example for the correlation matrix $\Sigma$ [12]:

$$\Sigma(\rho) = \left[ |\rho|^{i-j} \right]_{i,j=1,2,\cdots,m}, \rho \in [0,1).$$  (9)

For illustration, we focus only on the case where $x_0$ has entries that are sampled iid from a sparse-Bernoulli distribution $p_{x_0} = (1-\kappa)\delta_0 + \kappa\delta_1$, where $\delta$ is the Dirac delta function. Fig. 1 shows the MSE performance of the LASSO for different values of the regularizer $\lambda$. Monte Carlo Simulations are used to validate the theoretical prediction of Theorem 1. Comparing the simulation results to the asymptotic MSE prediction of Theorem 1 shows the close match between the two. We used $\delta = 0.7$, $n = 400$, $\rho = 0.7$, $\sigma^2 = 0.01$, and $\kappa = 0.1$, and the data are averaged over 500 realizations of the channel matrix and the noise vector.

In Fig 2 and Fig 3, we provide the comparison between simulation and theory for the probability of successful on-support and off-support recovery respectively. We used the same values as in Fig 1. Again these figures show the preciseness of our results.

Fig 4 validates the prediction of Proposition 1 for the EER. This figure shows the close agreement between simulation and Proposition 1. From this figure we can see that there is an optimal value of the regularizer $\lambda$ for which the EER is minimized.
Fig. 2. The probability of on-support recovery performance of the LASSO with $x_0$ being a sparse-Bernoulli vector. For simulations $\kappa = 0.1, \delta = 0.7, n = 400, \xi = 0.001, \text{SNR} = 10 \text{dB}$.

Finally, Fig 5 shows the cosine similarity metric. As before, this figure show the precise nature of our results. As discussed earlier, we seek estimates that maximizes this measure and form this figure we can see a clear maximum value of the measure for some value of $\lambda$ around 0.14.

V. Approach and Proof Overview

In this section, we provide a proof outline of Theorems 1 and 2. The proof idea is mainly based on the framework of the CGMT which is summarized next.

A. Convex Gaussian Min-max Theorem (CGMT)

The key ingredient of the analysis is the CGMT. Here, we recall the statement of the theorem, and we refer the reader to [5, 17] for the complete technical details. Consider the following two min-max problems, which we refer to, respectively, as the Primary Optimization (PO) and Auxiliary Optimization (AO):

$$\Phi(C) := \min_{w \in S_w} \max_{u \in S_u} u^T C w + \psi(w, u), \quad (10a)$$

$$\phi(g, h) := \min_{w \in S_w} \max_{u \in S_u} \|w\|g^Tu - \|u\|h^Tw + \psi(w, u), \quad (10b)$$

where $C \in \mathbb{R}^{m \times n}, g \in \mathbb{R}^m, h \in \mathbb{R}^n, S_w \subset \mathbb{R}^n, S_u \subset \mathbb{R}^m$ and $\psi : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$. Denote by $w_{\Phi} := w_{\Phi}(C)$ and $w_{\phi} := w_{\phi}(g, h)$ any optimal minimizers of (10a) and (10b), respectively. Further let $S_w, S_u$ be convex and compact sets, $\psi(\cdot, \cdot)$ is convex-concave continuous on $S_w \times S_u$ and, $C, g$ and $h$ all have iid standard normal entries.
Fig. 3. The probability of off-support recovery performance of the LASSO with $x_0$ being a sparse-Bernoulli vector. For simulations $\kappa = 0.1, \delta = 0.7, n = 400, \xi = 0.001$, SNR = 10 dB.

Fig. 4. The EER performance of the LASSO. For simulations $\kappa = 0.1, \delta = 0.7, n = 400, \xi = 0.001$, SNR = 10 dB, and the data are averaged over 500 independent realizations of problem.
Fig. 5. The cosine similarity performance of the LASSO with $x_0$ being a sparse-Bernoulli vector. For simulations $\kappa = 0.1, \delta = 0.7, n = 400$, SNR = 10 dB.

Let $S$ be any arbitrary open subset of $S_w$, and $S^c = S_w \setminus S$. Denote $\phi_{S^c}(g, h)$ the optimal cost of the optimization in (10b), when the minimization over $w$ is constrained over $w \in S^c$. Suppose that there exist constants $\tilde{\phi}$ and $\eta > 0$ such that in the limit as $n \to +\infty$, it holds with probability approaching one: (i) $\phi(g, h) \leq \tilde{\phi} + \eta$, and, (ii) $\phi_{S^c}(g, h) \geq \tilde{\phi} + 2\eta$. Then, $\lim_{n \to \infty} P[w \phi \in S] = 1$, and $\lim_{n \to \infty} P[w \Phi \in S] = 1$.

B. Identifying the PO and the AO

For notational convenience, we consider the error vector $w := x - x_0$, then the problem in (11) (after proper normalization by $n$) can be reformulated as

$$
\hat{w} := \arg\min_w \frac{1}{n} \| A w - z \|_2^2 + \frac{\lambda}{n} \| w + x_0 \|_1. \tag{11}
$$

Using the invariance of the Gaussian distribution under orthogonal transformations, we have

$$
\hat{w} = \arg\min_w \frac{1}{n} \| \Gamma^{\frac{1}{2}} G w - z \|_2^2 + \frac{\lambda}{n} \| w + x_0 \|_1, \tag{12}
$$

where $G$ has iid Gaussian entries $N(0, \frac{1}{n})$. The loss function can be expressed in its dual form through the Fenchel conjugate as

$$
\| \Gamma^{\frac{1}{2}} G w - z \|_2^2 = \max_u \sqrt{n} u^T (\Gamma^{\frac{1}{2}} G w - z) - \frac{n \| u \|_2^2}{4}.
$$

Then, the PO can be written as

$$
\Phi(n) = \frac{1}{n} \min_w \max_u \sqrt{n} u^T \Gamma^{\frac{1}{2}} G w
- \sqrt{n} u^T z - \frac{n \| u \|_2^2}{4} + \lambda \| w + x_0 \|_1. \tag{13}
$$
Redefining \( u \) as \( \Gamma^{\frac{1}{2}} u \) yields
\[
\Phi^{(n)} = \frac{1}{n} \min_{\bar{w}} \max_{u} \sqrt{n} u^T G w - \sqrt{n} u^T \Gamma^{-\frac{1}{2}} z - \frac{n}{4} u^T \Gamma^{-1} u + \lambda \| w + x_0 \|_1.
\] (14)

The above optimization is in a PO form, and its corresponding AO is
\[
\phi^{(n)} = \frac{1}{n} \min_{\bar{w}} \max_{u} \| w \|_2 g^T u - \| u \|_2 h^T w - \sqrt{n} u^T \Gamma^{-\frac{1}{2}} z - \frac{n}{4} u^T \Gamma^{-1} u + \lambda (w + x_0)^T v.
\] (15)

Recalling that
\[
\| a \|_1 = \max_{\| v \|_2 \leq 1} a^T v,
\]
for any vector \( a \), we have
\[
\phi^{(n)} = \frac{1}{n} \min_{\bar{w}} \max_{u} \max_{\| v \|_2 \leq 1} \| w \|_2 g^T u - \| u \|_2 h^T w - \sqrt{n} u^T \Gamma^{-\frac{1}{2}} z - \frac{n}{4} u^T \Gamma^{-1} u + \lambda (w + x_0)^T v.
\] (16)

Fixing the normalized norm of \( w \) to \( \sqrt{\alpha} = \| w \|_2 / \sqrt{\alpha} \), the AO can be expressed as
\[
\phi^{(n)} = \min_{\alpha \geq 0} \max_{\| u \|_2 \leq 1} \sqrt{\alpha} g^T u - \frac{1}{\sqrt{n}} u^T \Gamma^{-\frac{1}{2}} z - \frac{1}{4} u^T \Gamma^{-1} u + \lambda \| v \|_2\| u \|_2 h + \min_{\| v \|_2 = 1} \sqrt{\alpha} (\lambda v - \| u \|_2 h)^T \bar{w}.
\] (17)

The last minimization is easy to perform as
\[
\min_{\| v \|_2 = 1} \sqrt{\alpha} (\lambda v - \| u \|_2 h)^T \bar{w} = - \sqrt{\alpha} \| \lambda v - \| u \|_2 h \|_2^2,
\]
with the optimal solution
\[
\bar{w} = - \frac{\lambda v - \| u \|_2 h}{\| \lambda v - \| u \|_2 h \|_2^2}.
\] (18)

Then, we have the following
\[
\phi^{(n)} = \min_{\alpha \geq 0} \max_{\| u \|_2 \leq 1} \sqrt{\alpha} g^T u - \frac{1}{\sqrt{n}} u^T \Gamma^{-\frac{1}{2}} z - \frac{1}{4} u^T \Gamma^{-1} u + \lambda \| v \|_2\| u \|_2 h - \sqrt{\alpha} \| \lambda v - \| u \|_2 h \|_2^2.
\] (19)

Let \( \tilde{g} = \sqrt{\alpha} g - \Gamma^{-\frac{1}{2}} z \), and fixing the norm of \( u \) to \( \beta = \| u \|_2 \), then
\[
\phi^{(n)} = \min_{\alpha \geq 0} \max_{\| u \|_2 = 1} \sqrt{\alpha} \tilde{g}^T \tilde{u} - \frac{\beta^2}{4} \tilde{u}^T \Gamma^{-1} \tilde{u} + \lambda \| v \|_2\| u \|_2 h - \sqrt{\alpha} \| \lambda v - \beta h \|_2^2.
\] (20)

Now, the optimization over \( \tilde{u} \) becomes separable, hence we need to solve the following non-convex problem
\[
\max_{\| u \|_2 = 1} \frac{1}{\sqrt{n}} \beta \tilde{g}^T \tilde{u} - \frac{\beta^2}{4} \tilde{u}^T \Gamma^{-1} \tilde{u},
\] (21)
which is a standard optimization that has been extensively studied \cite{18,19}. Its solution is \( \tilde{u}_* = \frac{2}{\beta \sqrt{n}} \left( \Gamma^{-1} - \mu(\alpha, \beta) I \right)^{-1} \bar{g} \), with \( \mu(\alpha, \beta) \) satisfying
\[
\frac{1}{n} \tilde{g}^T \left( \Gamma^{-1} - \mu(\alpha, \beta) I \right)^{-2} \tilde{g} - \frac{\beta^2}{4} = 0. \tag{22}
\]
Substituting \( \tilde{u}_* \) into (21) gives:
\[
-\frac{\beta^2}{4} \mu(\alpha, \beta) + \frac{1}{n} \tilde{g}^T \left( \Gamma^{-1} - \mu(\alpha, \beta) I \right)^{-1} \tilde{g}.
\]
Therefore, the AO becomes
\[
\phi^{(n)} = \min_{\alpha \geq 0} \max_{\beta \geq 0} -\frac{\beta^2}{4} \mu(\alpha, \beta) + \frac{1}{n} \tilde{g}^T \left( \Gamma^{-1} - \mu(\alpha, \beta) I \right)^{-1} \tilde{g}
+ \frac{\lambda}{n} x_0^T v - \frac{\alpha}{2\chi} \left\| \sqrt{n} (\lambda v - \beta h) \right\|_2^2,
\tag{23}
\]
where \( \mu(\alpha, \beta) \) satisfies (22). We proceed by expressing the \( \ell_2 \)-norm in (23) using the following variational form
\[
\|s\|_2 = \inf_{\chi > 0} \frac{\chi}{2} + \frac{\|s\|_2^2}{2\chi},
\]
for any vector \( s \).
\[
\phi^{(n)} = \min_{\alpha \geq 0} \max_{\beta \geq 0} -\frac{\beta^2}{4} \mu(\alpha, \beta) + \frac{1}{n} \tilde{g}^T \left( \Gamma^{-1} - \mu(\alpha, \beta) I \right)^{-1} \tilde{g}
+ \frac{\lambda}{n} x_0^T v - \frac{\alpha}{2\chi} \left\| \sqrt{n} (\lambda v - \beta h) \right\|_2^2.
\tag{24}
\]
Next, we perform the optimization over \( v \), since it is separable now. So, we need to solve:
\[
\max_{\|v\|_\infty \leq 1} \frac{\lambda}{n} x_0^T v - \frac{\alpha}{2\chi} \left\| \sqrt{n} (\lambda v - \beta h) \right\|_2^2.
\tag{25}
\]
This can be rewritten as
\[
\frac{\chi}{\alpha n} \sum_{i=1}^n \max_{1 \leq i \leq 1} \left( \frac{\lambda}{\chi} x_{0,i} + \frac{\alpha \beta}{\chi^2} h_i \right) v_i - \frac{\alpha \beta}{2\chi^2} v_i^2 - \frac{\alpha^2 \beta^2}{2\chi^2} h_i^2.
\tag{26}
\]
Let \( a_i = \frac{\lambda}{\chi} x_{0,i} + \frac{\alpha \beta}{\chi^2} h_i \). Then, the optimal solution is given by:
\[
v_i^* = \begin{cases} 
-1, & \text{if } \left( \frac{\lambda}{\alpha n} \right)^2 a_i < -1 \\
\left( \frac{\lambda}{\alpha n} \right)^2 a_i, & \text{if } -1 \leq \left( \frac{\lambda}{\alpha n} \right)^2 a_i \leq 1 \\
1, & \text{if } \left( \frac{\lambda}{\alpha n} \right)^2 a_i > 1.
\end{cases}
\tag{27}
\]
Substituting \( v_i^* \) into (26) and after some algebraic manipulations we get
\[
\frac{\chi}{\alpha n} \left[ \sum_{i=1}^n e \left( x_{0,i} + \frac{\alpha \beta}{\chi} h_i; \frac{\lambda}{\chi} \right) - \frac{\alpha \beta^2}{2\chi^2} h_i^2 \right],
\]
where \( e(\cdot, \cdot) \) is as defined in (6). Now, the AO becomes
\[
\tilde{\phi}^{(n)} = \min_{\alpha \geq 0} \max_{\beta \geq 0} \sup_{\chi > 0} -\frac{\beta^2}{4} \mu(\alpha, \beta) + \frac{1}{n} \tilde{g}^T \left( \Gamma^{-1} - \mu(\alpha, \beta) I \right)^{-1} \tilde{g}
- \frac{\chi}{2} + \frac{1}{n} \sum_{i=1}^n \frac{\chi}{\alpha} e \left( x_{0,i} + \frac{\alpha \beta}{\chi} h_i; \frac{\lambda}{\chi} \right) - \frac{\alpha \beta^2}{2\chi^2} h_i^2.
\tag{28}
\]
C. Probabilistic Asymptotic Analysis of the AO

Note that \( \hat{g} \sim \mathcal{N}(0, C_{\hat{g}}) \), where \( C_{\hat{g}} \) is given by
\[
C_{\hat{g}} = \alpha I_m + \sigma^2 \Gamma^{-1}.
\]

Thus, applying the trace lemma \[20\], we have
\[
\frac{1}{n} \hat{g}^T \left( \Gamma^{-1} - \mu(\alpha, \beta) I \right)^{-1} \hat{g} - \frac{1}{n} \text{tr} \left( C_{\hat{g}} \left( \Gamma^{-1} - \mu(\alpha, \beta) I \right)^{-1} \right) \xrightarrow{P} 0.
\]

Also, using the WLLN, \( \frac{1}{n} \sum_i h_i^2 \xrightarrow{P} 1 \), and for all \( \alpha \geq 0, \beta > 0 \) and \( \chi > 0 \), we have \( \frac{1}{n} \sum_i e \left( x_{0,i} + \frac{\alpha^2}{\chi} h_i; \frac{\lambda \alpha}{\chi} \right) \xrightarrow{P} \mathbb{E}_{X_0 \sim p_{X_0}} \mathbb{E}_{X_0 \sim \mathcal{N}(0,1)} \left[ e \left( X_0 + \frac{\alpha^2}{\chi} Z; \frac{\lambda \alpha}{\chi} \right) \right] \).

Therefore, again using Lemma 10 of \[5\], \( \hat{\phi}(n) - \phi(n) \xrightarrow{P} 0 \), where
\[
\frac{\alpha}{\beta} = \min_{\alpha \geq 0, \beta > 0} \sup_{\chi > 0} \frac{-\beta^2}{4} \mu(\alpha, \beta) - \frac{\chi}{2} - \frac{\alpha \beta^2}{2 \chi} + \frac{1}{n} \sum_{j=1}^m \frac{\gamma_j \alpha + \sigma^2}{1 - \gamma_j \mu(\alpha, \beta)} + \frac{\chi}{\alpha} \mathbb{E}_{X_0 \sim p_{X_0}} \mathbb{E}_{X_0 \sim \mathcal{N}(0,1)} \left[ e \left( X_0 + \frac{\alpha^2}{\chi} Z; \frac{\lambda \alpha}{\chi} \right) \right],
\]
where \( \mu(\alpha, \beta) \) satisfies (from \[22\] and using the trace Lemma)
\[
\frac{1}{n} \sum_{j=1}^m \frac{\alpha + \sigma^2}{\gamma_j - \mu(\alpha, \beta)} - \frac{\beta^2}{4} = 0.
\]

D. Applying the CGMT

Now we will evaluate the performance of the LASSO using the different metrics introduced earlier. We begin with the MSE analysis. Let \( \hat{w} \) be the optimal solution to the AO defined as the solution to (17). Let \( \alpha_* \) be the optimal solution to (29). For any \( \epsilon > 0 \), define the set:
\[
S_\epsilon = \left\{ r : \left| \frac{1}{n} \| r \|_2^2 - \alpha_* \right| < \epsilon \right\}.
\]

Define \( \hat{\alpha}_n \) as the minimizer of (17). Then, by definition, \( \hat{\alpha}_n = \frac{\| \hat{w} \|_2^2}{n} \). In the previous section, we showed that \( \hat{\phi}(n) - \phi(n) \xrightarrow{P} 0 \). Hence, we can show that \( \hat{\alpha}_n - \alpha_* \xrightarrow{P} 0 \), which implies
\[
\left| \frac{1}{n} \| \hat{w} \|_2^2 - \alpha_* \right| \xrightarrow{P} 0.
\]

Therefore, \( \hat{w} \in S_\epsilon \) with probability approaching 1. Then, applying the CGMT yields that \( \hat{w} \in S_\epsilon \) with probability approaching 1 as well. This completes the proof of Theorem \[1\]

We proceed now to the proof of the probabilities of support recovery. First, for the on-support recovery probability, change the set to the following:
\[
S_\epsilon = \left\{ r : \frac{1}{k} \sum_{i \in \mathcal{S}(\mathbf{x}_o)} \mathbb{1}_{\{|r_i| \geq \xi\}} - \mathbb{P} \left[ \eta \left( X_0 + \frac{\alpha_* \beta_*}{\chi_*} Z; \frac{\lambda \alpha_*}{\chi_*} \right) \geq \xi \right] < \epsilon \right\},
\]
for any \( \xi > 0 \). Note that it can be shown, based on \[18\], that for all \( i = 1, 2, \ldots, n \):
\[
\hat{w}_i = \frac{\hat{\alpha}(\lambda \nu_* - \beta h_i)}{\sqrt{\frac{2}{n} \| \lambda \nu_* - \beta h_i \|_2}},
\]

(31)
where $\tilde{\alpha}, \tilde{\beta}$ are the solutions of (28). Note that $\sqrt{\frac{\mu}{n}}\|\lambda v_i^* - \tilde{\beta} h_i\|_2 = \tilde{\chi}$ which is the solution of (28) as well. Then

$$\tilde{w}_i = -\frac{\tilde{\alpha}(\lambda v_i^* - \tilde{\beta} h_i)}{\tilde{\chi}}.$$ 

Recall that $\tilde{w} = \tilde{x} - x_0$, where $\tilde{w}$ is the AO solution. Hence

$$\tilde{x}_i = \tilde{w}_i + x_{0,i}.$$ 

Substituting the values of $\tilde{w}_i$ and $v_i^*$ and after some algebraic manipulations, it can be shown that $\tilde{x}_i = \eta(x_{0,i} + \frac{\tilde{\alpha}\beta}{\chi} h_i; \frac{\lambda\alpha}{\chi})$. Since $\tilde{\phi}^{(n)} - \phi^{(n)} \xrightarrow{P} 0$, it can be shown that $\tilde{\alpha} - \alpha_* \xrightarrow{P} 0, \tilde{\beta} - \beta_* \xrightarrow{P} 0$, and $\tilde{\chi} - \chi_* \xrightarrow{P} 0$. Then, after some simple calculations, it holds

$$\left| \frac{1}{k} \sum_{i \in S(x_0)} \mathbb{1}_{\{|\tilde{x}_i| \geq \xi\}} - P\left[ \mathbb{1}_{\{|x_0 + \frac{\alpha_*\beta_*}{\sqrt{\lambda_*\chi_*}} Z; \frac{\lambda_*}{\chi_*}| \geq \xi\}} \right] \right| \xrightarrow{P} 0.$$ 

This proves that $\tilde{x} \in S_e$ with probability approaching 1. Note that the indicator function $\mathbb{1}_{\{|\tilde{x}_i| \geq \xi\}}$ is not Lipschitz, so we cannot directly apply the CGMT. However, as discussed in [17, Lemma A.4] and [21], this function can be appropriately approximated with Lipschitz functions. Therefore, we can conclude by applying the CGMT that $\tilde{x} \in S_e$ with probability approaching 1, which proves the first result of Theorem 2. The off-support recovery probability can be derived in a similar manner and details are thus omitted.

VI. CONCLUSION

In this paper, we derived precise asymptotic error performance analysis of LASSO under the assumption that the design matrix has correlated entries. In particular, we derived precise expressions of the MSE, probability of support recovery, EER, and cosine similarity. Numerical simulations show the close agreement to the theory even for low dimensions of the problem. Possible future extensions include the double-sided correlation model, imperfect channel models and analyzing the box variant of the LASSO.

APPENDIX

The expectation in (29) can be evaluated in closed form for any distribution. For example, take the case of a sparse-Bernoulli vector $x_0$, i.e., the entries of $x_0$ are sampled iid from a distribution $p_{x_0} = (1 - \kappa)\delta_0 + \kappa\delta_1$, then

$$\mathbb{E}_{X_0 \sim p_{X_0}} \left[ e \left( X_0 + \frac{\alpha\beta}{\chi} Z; \frac{\lambda\alpha}{\chi} \right) \right] =$$

$$\frac{\alpha(1 - \kappa)}{\chi} \left( \frac{\beta^2}{2} + \beta \lambda \varphi \left( \frac{\lambda}{\beta} \right) \right) - \left( \lambda^2 + \beta^2 \right) Q \left( \frac{\lambda}{\beta} \right)$$

$$+ \kappa \left( \lambda - \frac{\alpha\lambda^2}{2\chi} \right) Q \left( \frac{\lambda}{\beta} \right) - \kappa \left( \lambda + \frac{\alpha\lambda^2}{2\chi} \right) Q \left( \frac{\lambda}{\beta} \right)$$

$$+ \frac{\alpha\beta\lambda\kappa}{\chi} \left( \varphi \left( \frac{\lambda}{\beta} + \frac{\chi}{\alpha\beta} \right) + \varphi \left( \frac{\lambda}{\beta} - \frac{\chi}{\alpha\beta} \right) \right)$$

$$- \frac{\kappa\beta}{\chi} \varphi \left( \frac{\alpha\lambda + \chi}{\alpha\beta} \right) \left( \alpha\lambda - \chi + (\alpha\lambda + \chi) \exp \left( \frac{2\lambda\chi}{\alpha\beta^2} \right) \right)$$

$$+ \kappa \left( \frac{\alpha\beta^2}{\chi} + \frac{\lambda}{\alpha} \right) \left( \text{erf} \left( \frac{\alpha\lambda + \chi}{\sqrt{2\alpha\beta}} \right) + \text{erf} \left( \frac{\alpha\lambda - \chi}{\sqrt{2\alpha\beta}} \right) \right),$$

where $\text{erf}(x)$ is the error function defined as $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. 


Also, for the sparse-Bernoulli distribution, we have
\[
\operatorname{plim}_{n \to \infty} \Phi_{\xi,\omega} (\tilde{x}) = Q \left( \frac{\lambda}{\beta_x} + \frac{\chi_x (\xi + 1)}{\alpha_x \beta_x^2} \right) + Q \left( \frac{\lambda}{\beta_x} + \frac{\chi_x (\xi - 1)}{\alpha_x \beta_x^2} \right),
\]
where
\[
\operatorname{plim}_{n \to \infty} \Phi_{\xi,\omega} (\tilde{x}) = 1 - 2Q \left( \frac{\chi_x \xi}{\alpha_x \beta_x^2} + \frac{\lambda}{\beta_x} \right).
\]
and
\[
\operatorname{plim}_{n \to \infty} \operatorname{EER}_\xi = Q \left( \frac{\chi_x (1 - \xi)}{\alpha_x \beta_x^2} - \frac{\lambda}{\beta_x} \right) - Q \left( \frac{\chi_x (1 + \xi)}{\alpha_x \beta_x^2} + \frac{\lambda}{\beta_x} \right) + 2Q \left( \frac{\chi_x \xi}{\alpha_x \beta_x^2} + \frac{\lambda}{\beta_x} \right).
\]
Finally, for the cosine similarity, we have for the sparse-Bernoulli distribution:
\[
\operatorname{plim}_{n \to \infty} \cos (\angle (\tilde{x}, x_0)) = \frac{I_0}{\sqrt{\kappa (I_1 + I_2)}},
\]
where
\[
I_0 = \frac{\kappa}{\chi_x} \left[ \alpha_x \beta_x \left( \frac{\chi_x \xi}{\alpha_x \beta_x^2} - \frac{\lambda}{\beta_x} \right) - \varphi \left( \frac{\chi_x \xi}{\alpha_x \beta_x^2} + \frac{\lambda}{\beta_x} \right) \right] + (\chi_x - \lambda \alpha_x) Q \left( \frac{\lambda}{\beta_x} - \frac{\chi_x}{\alpha_x \beta_x^2} \right) + (\chi_x + \lambda \alpha_x) Q \left( \frac{\lambda}{\beta_x} + \frac{\chi_x}{\alpha_x \beta_x^2} \right),
\]
\[
I_1 = \frac{\alpha_x \beta_x}{\chi_x} \left[ (\alpha_x \beta_x^2 + (\lambda \alpha_x - \chi_x) \chi_x) Q \left( \frac{\lambda}{\beta_x} - \frac{\chi_x}{\alpha_x \beta_x^2} \right) + (\alpha_x \beta_x^2 + (\lambda \alpha_x + \chi_x) \chi_x) Q \left( \frac{\lambda}{\beta_x} + \frac{\chi_x}{\alpha_x \beta_x^2} \right) - \alpha_x \beta_x \left( (\lambda \alpha_x - \chi_x) \exp \left( \frac{2 \lambda \alpha_x}{\alpha_x \beta_x^2} \right) + \lambda \alpha_x + \chi_x \right) \varphi \left( \frac{\lambda}{\beta_x} + \frac{\chi_x}{\alpha_x \beta_x^2} \right) \right],
\]
and
\[
I_2 = \frac{2(1 - \kappa) \alpha_x^2}{\chi_x} \left[ (\lambda^2 + \beta_x^2) Q \left( \frac{\lambda}{\beta_x} \right) - \lambda \beta_x \varphi \left( \frac{\lambda}{\beta_x} \right) \right].
\]
These expressions were used in Section V for the provided numerical results.

REFERENCES

[1] Robert Tibshirani, “Regression shrinkage and selection via the lasso,” Journal of the Royal Statistical Society: Series B (Methodological), vol. 58, no. 1, pp. 267–288, 1996.
[2] Mohsen Bayati and Andrea Montanari, “The lasso risk for gaussian matrices,” IEEE Transactions on Information Theory, vol. 58, no. 4, pp. 1997–2017, 2011.
[3] Mohsen Bayati and Andrea Montanari, “The dynamics of message passing on dense graphs, with applications to compressed sensing,” IEEE Transactions on Information Theory, vol. 57, no. 2, pp. 764–785, 2011.
[4] David L. Donoho, Arian Maleki, and Andrea Montanari, “Message-passing algorithms for compressed sensing,” Proceedings of the National Academy of Sciences, vol. 106, no. 45, pp. 18914–18919, 2009.
[5] Christos Thrampoulidis, Ehsan Abbasi, and Babak Hassibi, “Precise error analysis of regularized m-estimators in high dimensions,” IEEE Transactions on Information Theory, vol. 64, no. 8, pp. 5592–5628, 2018.
[6] Ismail Ben Atiallah, Christos Thrampoulidis, Abla Kammoun, Tareq Y Al-Naffouri, Mohamed-Slim Alouini, and Babak Hassibi, “The box-lasso with application to gsk modulation in massive mimo systems,” in 2017 IEEE International Symposium on Information Theory (ISIT), IEEE, 2017, pp. 1082–1086.
[7] Mihailo Stojnic, “A framework to characterize performance of lasso algorithms,” arXiv preprint arXiv:1303.7291, 2013.
[8] Mihailo Stojnic, “Recovery thresholds for 1 optimization in binary compressed sensing,” in 2010 IEEE International Symposium on Information Theory, IEEE, 2010, pp. 1593–1597.
[9] Christos Thrampoulidis, Ehsan Abbasi, and Babak Hassibi, “Lasso with non-linear measurements is equivalent to one with linear measurements,” in Advances in Neural Information Processing Systems, 2015, pp. 3420–3428.
[10] Ayed M. Alrashdi, Ismail Ben Atiallah, Tareq Y Al-Naffouri, and Mohamed-Slim Alouini, “Precise performance analysis of the lasso under matrix uncertainties,” in 2017 IEEE Global Conference on Signal and Information Processing (GlobalSIP), IEEE, 2017, pp. 1290–1294.
[11] Ayed M. Alrashdi, Ismail Ben Atiallah, and Tareq Y Al-Naffouri, “Precise performance analysis of the box-elastic net under matrix uncertainties,” IEEE Signal Processing Letters, vol. 26, no. 5, pp. 655–659, 2019.
[12] Hyundong Shin, Moe Z Win, Jae Hong Lee, and Marco Chiani, “On the capacity of doubly correlated mimo channels,” *IEEE Transactions on Wireless Communications*, vol. 5, no. 8, pp. 2253–2265, 2006.

[13] Ayed M Alrashdi, Houssem Sifaou, Abla Kammoun, Mohamed-Slim Alouini, and Tareq Y Al-Naffouri, “Box-relaxation for bpsk recovery in massive mimo: A precise analysis under correlated channels,” in *ICC 2020-2020 IEEE International Conference on Communications (ICC)*. IEEE, 2020, pp. 1–6.

[14] Ansuman Adhikary, Junyoung Nam, Jae-Young Ahn, and Giuseppe Caire, “Joint spatial division and multiplexing the large-scale array regime,” *IEEE transactions on information theory*, vol. 59, no. 10, pp. 6441–6463, 2013.

[15] Axel Mueller, Abla Kammoun, Emil Björnson, and Mérouane Debbah, “Linear precoding based on polynomial expansion: Reducing complexity in massive mimo,” *EURASIP journal on wireless communications and networking*, vol. 2016, no. 1, pp. 63, 2016.

[16] Mohamed A Suliman, Ayed M Alrashdi, Tarig Ballal, and Tareq Y Al-Naffouri, “Snr estimation in linear systems with gaussian matrices,” *IEEE Signal Processing Letters*, vol. 24, no. 12, pp. 1867–1871, 2017.

[17] Christos Thrampoulidis, Weiyu Xu, and Babak Hassibi, “Symbol error rate performance of box-relaxation decoders in massive mimo,” *IEEE Transactions on Signal Processing*, vol. 66, no. 13, pp. 3377–3392, 2018.

[18] Walter Gander, Gene H Golub, and Urs Von Matt, “A constrained eigenvalue problem,” *Linear Algebra and its applications*, vol. 114, pp. 815–839, 1989.

[19] Pham Dinh Tao and Le Thi Hoai An, “A dc optimization algorithm for solving the trust-region subproblem,” *SIAM Journal on Optimization*, vol. 8, no. 2, pp. 476–505, 1998.

[20] T. Couillet and M. Debbah, *Random Matrix Methods for Wireless Communications*, U.K., Cambridge: Cambridge Univ. Press, 2011.

[21] Ayed M Alrashdi, Abla Kammoun, Ali H Muqaibel, and Tareq Y Al-Naffouri, “Optimum m-pam transmission for massive mimo systems with channel uncertainty,” *arXiv preprint arXiv:2008.06993*, 2020.