Task-Agnostic Continual Learning Using Online Variational Bayes with Fixed-Point Updates

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Catastrophic forgetting is the notorious vulnerability of neural networks to the changes in the data distribution during learning. This phenomenon has long been considered a major obstacle for using learning agents in realistic continual learning settings. A large body of continual learning research assumes that task boundaries are known during training. However, only a few works consider scenarios in which task boundaries are unknown or not well defined: task-agnostic scenarios. The optimal Bayesian solution for this requires an intractable online Bayes update to the weights posterior.

We aim to approximate the online Bayes update as accurately as possible. To do so, we derive novel fixed-point equations for the online variational Bayes optimization problem for multivariate gaussian parametric distributions. By iterating the posterior through these fixed-point equations, we obtain an algorithm (FOO-VB) for continual learning that can handle nonstationary data distribution using a fixed architecture and without using external memory (i.e., without access to previous data). We demonstrate that our method (FOO-VB) outperforms existing methods in task-agnostic scenarios. FOO-VB Pytorch implementation is available at https://github.com/chenzeno/FOO-VB.
1 Introduction

Continual learning (CL) is the ability of an algorithm to learn from nonstationary data while reusing past knowledge and exploiting it to better adapt to a changing environment. A major challenge in continual learning is to overcome catastrophic forgetting (McCloskey & Cohen, 1989). Catastrophic forgetting is the tendency of neural networks to rapidly lose previously learned knowledge when the input distribution is changed abruptly, for example, when changing a task or a source.

Neural networks are commonly used machine learning models for solving a variety of tasks. They are notorious for being vulnerable to changes in the data during learning. Various methods for preventing catastrophic forgetting in neural networks have been suggested in the literature. Most of these methods assume relaxed conditions in which the tasks arrive sequentially and the data distribution changes only upon task switches (i.e., piecewise stationary). Therefore, they are inapplicable in many realistic task-agnostic applications, in which the task boundaries are unknown or when such boundaries are not well defined (e.g., the data distribution continuously changes in a nonstationary manner). For example, in image classification tasks in real-world scenarios, input images may exhibit several gradual changes through time, such as the zoom, illumination, or the angle of objects in the image. In this letter, we aim to reduce catastrophic forgetting in such difficult and relevant task-agnostic cases.

It is long known that estimating the underlying posterior distribution can help mitigate catastrophic forgetting in neural networks (McCloskey & Cohen, 1989). More recently, it was found (Kirkpatrick et al., 2017) that one can use the posterior to find confidence levels for weights, which can be used to affect weight plasticity. That is, weights with a lower confidence value may be changed more in subsequent tasks (i.e., since they are “less important” for previous tasks). This allows a natural transition between learned tasks while reducing the ill effect of catastrophic forgetting. Nguyen, Li, Bui, and Turner (2017) also used variational Bayes to prevent catastrophic forgetting when the tasks arrive sequentially. In these works, each time a task switches, a new prior (or a regularization term) is added, which restricts the change of weights with high confidence (i.e., since they are “important” for previous tasks). However, in task-agnostic scenarios this approach cannot be used since the task boundary is unknown or does not exist.

To prevent catastrophic forgetting in task-agnostic scenarios, we propose using the online version of variational Bayes, which updates the approximate posterior using only the current data sample without any knowledge on task switches. For the method to be effective, we aim to approximate the Bayes update as accurately as possible. Therefore, we derive novel fixed-point equations for the online variational Bayes optimization problem under gaussian parametric distributions (with either full, matrix variate, or
diagonal covariance). Based on our theoretical results, for each gaussian distribution, we propose the fixed-point operator for online variational Bayes (FOO-VB) algorithm. Our experiments demonstrate significant improvement over existing algorithms in task-agnostic continual learning scenarios.

2 Related Work

2.1 Bayesian Neural Networks. Bayesian inference for neural networks has been a subject of interest over many years. As exact Bayesian inference is intractable (for any realistic network size), much research has focused on approximation techniques. Most modern techniques stem from previous seminal works that used a Laplace approximation (MacKay, 1992), variational methods (Hinton & Camp, 1993), or Monte Carlo methods (Neal, 1994). In recent years, many methods for approximating the posterior distribution have been suggested, falling into one of these categories. Those methods include assumed density filtering (Soudry, Hubara, & Meir, 2014; Hernández-Lobato & Adams, 2015), approximate power expectation propagation (Hernández-Lobato et al., 2016), stochastic Langevin gradient descent (Welling & Teh, 2011; Balan, Rathod, Murphy, & Welling, 2015), incremental moment matching (Lee, Kim, Ha, & Zhang, 2017), or variational Bayes (Graves, 2011; Blundell et al., 2015).

In this work, we focus on a variational Bayes approach. Practical variational Bayes for modern neural networks was introduced by Graves (2011), where a parametric distribution was used to approximate the posterior distribution by minimizing the variational free energy. Calculating the variational free energy is intractable for general neural networks. Thus, Graves (2011) estimated the gradients using a biased Monte Carlo method and used stochastic gradient descent (SGD) to perform minimization. In a later work, Blundell et al. (2015) used a re-parameterization trick to yield an unbiased estimator for the gradients. Variational Bayes methods were also used extensively on various probabilistic models, including recurrent neural networks (Graves, 2011), autoencoders (Kingma & Welling, 2013), and fully connected networks (Blundell et al., 2015). Martens and Grosse (2015), Zhang, Sun, Duvenaud, and Grosse (2017), and Khan et al. (2018) suggested using the connection between natural gradient descent (Amari, 1998) and variational inference to perform natural gradient optimization in deep neural networks.

2.2 Fixed-Point Equations for the Variation Bayes. In this work, we derive a novel fixed-point equation for the online variational Bayes optimization problem under gaussian parametric distributions (with either full, matrix variate, or diagonal covariance) in the case of Bayesian neural networks. Fixed-point equations for the variational Bayes optimization problems have been used in the cases of linear models (Knowles & Minka, 2011; Cseke, Opper, & Sanguinetti, 2013) and gaussian processes (Opper &
Archambeau, 2009). Those derivations cannot be used in the case of Bayesian neural networks. For instance, in the case of Gaussian processes, the negative log likelihood of a single data point is a function of only one random variable (the parameter of the kernel function). So, the nondiagonal elements of the posterior precision matrix are equal to those of the prior precision matrix, resulting in a closed-form solution to the posterior distribution. However, in the case of Bayesian neural networks, the negative log likelihood of a single data point is a function of a random vector (the parameters vector). Another related work is Kurle, Cseke, Klushyn, van der Smagt, and Günnemann (2019), who suggested an algorithm for online inference for nonstationary streaming data. They derived the first-order necessary conditions for the online variational Bayes optimization problem in the case of Bayesian neural networks and multivariate Gaussian distribution. They used the first-order necessary conditions alongside a running memory to sequentially update the posterior parameters—but only in the case of a diagonal Gaussian distribution. However, as we will show here, using nondiagonal covariance can significantly improve performance.

2.3 Continual Learning. Approaches for continual learning can be generally divided into four main categories: (1) architectural approaches, (2) rehearsal approaches, (3) regularization approaches, and (4) Bayesian approaches.

Architectural approaches alter the network architecture to adapt to new tasks (e.g., Rusu et al., 2016; Nagabandi, Finn, & Levine, 2018). Rehearsal approaches use external memory to allow retraining on stored examples from previous tasks (e.g., Shin, Lee, Kim, & Kim, 2017). Regularization approaches use some penalty on deviations from previous task weights. Bayesian approaches use Bayes’ rule with the previous task posterior as the current prior. Each approach has pros and cons. For example, architectural approaches may result in extremely large architectures after multiple task switches. Rehearsal approaches may be problematic due to memory limitations or data availability (e.g., data from previous tasks may not be stored with the model due to privacy or proprietary reasons). For an extensive review of continual learning methods see Parisi, Kemker, Part, Kanan, and Wermter (2018). Our letter, however, focuses on regularization and Bayesian approaches where the architecture is fixed and no external memory is used to retrain on data from previous tasks.

In regularization approaches, a regularization term is added to the loss function. Elastic weight consolidation (EWC), proposed by Kirkpatrick et al. (2017), slows changes in parameters that are important to the previous tasks by penalizing the difference between the previous task’s optimal parameters and the current parameters. The importance of each parameter

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*Rehearsal approaches may include GANs, which can be properly used as memory.*
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is measured using the diagonal of the Fisher information matrix. Synaptic intelligence (SI), proposed by Zenke, Poole, and Ganguli (2017), also uses a penalty term; however, the importance is measured by the path length of updates on the previous task. Chaudhry, Dokania, Ajanthan, and Torr (2018) propose an online generalization of EWC and SI to achieve better performance. Progress and compress, proposed by Schwarz et al. (2018), uses a network with two components (a knowledge base and an active column) with EWC for continual learning in reinforcement learning. Memory aware synapses (MAS) (Aljundi, Babiloni Elhoseiny, Rohrbach, & Tuytel-laars, 2018) also uses a penalty term, but measures weight importance by the sensitivity of the output function. Learning without forgetting (LwF), proposed by Li and Hoiem (2017), uses knowledge distillation to enforce the network outputs of the new task to be similar to the network outputs of previous tasks.

The Bayesian approaches provide a solution to continual learning in the form of Bayes’ rule. As data arrive sequentially, the posterior distribution of the parameters for the previous task is used as a prior for the new task. Variational continual learning (VCL), proposed by Nguyen et al. (2017), uses online variational inference combined with the standard variational Bayes approach, Bayes By Backprop (BBB) by Blundell et al. (2015). They reduce catastrophic forgetting by replacing the prior on each task switch. In the BBB approach, a mean-field approximation is applied, assuming weights are independent of each other (i.e., the covariance matrix is diagonal). Ritter, Botev, and Barber (2018) suggest using Bayesian online learning with a Kronecker-factored Laplace approximation to attain a nondiagonal method for reducing catastrophic forgetting. This allows the algorithm to exploit interactions between weights within the same layer.

2.4 Task-Agnostic Continual Learning. Most of these methods assume relaxed conditions, where tasks arrive sequentially and the data distribution changes only on task switches. Little previous work is with task-agnostic scenarios (when task boundaries are unknown or not defined); however, they all use the rehearsal approach (Rao et al., 2019; Aljundi, Lin, Goujaud, & Bengio, 2019; Achille et al., 2018). We present an algorithm for task-agnostic continual learning scenarios using a fixed architecture and without using external memory. Our work is orthogonal to those rehearsal approaches and can potentially be combined with them.

3 General Theoretical Background

Bayesian inference (Gelman et al., 2013; Bishop, 1995) requires a joint probability distribution over the target set $D$ and the model parameters $\theta$ given the input set $X$. This distribution can be written as

$$p(D, \theta | X) = p(D | \theta, X) p(\theta | X),$$

(3.1)
where $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}^T$ is the input set, $D = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}^T$ is the target set, and $\theta$ is the model parameters vector. $p(D|\theta, X)$ is the likelihood function of the target set $D$, and $p(\theta|X)$ is the prior distribution of the model parameters $\theta$. The posterior distribution of the model parameters can be calculated using Bayes’ rule

$$p(\theta|D, X) = \frac{p(D|\theta, X) p(\theta|X)}{p(D|X)},$$

(3.2)

where $p(D|X)$ is calculated using the sum rule. To simplify the notations we omit the conditioning on $X$ for the remainder of this letter.

We focus on the online version of Bayesian inference, in which the data arrive sequentially, and we update the posterior distribution whenever new data arrive. In each step, the previous posterior distribution is used as the new prior distribution. Therefore, according to Bayes’ rule, the posterior distribution at time $n$ is given by

$$p(\theta|D_n) = \frac{p(D_n|\theta) p(\theta|D_1, \ldots, D_{n-1})}{p(D_n)},$$

(3.3)

Unfortunately, calculating the posterior distribution is intractable for most practical probability models, and especially when using deep neural networks. Therefore, we will use variational methods to approximate the true posterior.

3.1 Variational Bayes. In variational Bayes (Graves, 2011), a parametric distribution $q(\theta|\phi)$ is used for approximating the true posterior distribution $p(\theta|D)$ by (indirectly) minimizing the Kullback-Leibler (KL) divergence with the true posterior distribution:

$$D_{KL}(q(\theta|\phi) || p(\theta|D)) = \mathbb{E}_{q(\theta|\phi)} \left[ \log \frac{q(\theta|\phi)}{p(\theta|D)} \right].$$

(3.4)

The optimal variational parameters $(\phi)$ are the solution of the following optimization problem:

$$\arg\min_{\phi} \int q(\theta|\phi) \log \frac{q(\theta|\phi)}{p(\theta|D)} d\theta = \arg\min_{\phi} \int q(\theta|\phi) \log \frac{q(\theta|\phi)}{p(D|\theta) p(\theta)} d\theta = \arg\min_{\phi} \mathbb{E}_{q(\theta|\phi)} \left[ \log q(\theta|\phi) - \log (p(\theta)) + L(\theta) \right],$$

(3.5)

where $L(\theta) = -\log (p(D|\theta))$ is the log-likelihood cost function.\(^2\) The KL divergence between the parametric distribution (approximate posterior) and

\(^2\) Note that we define a cumulative log-likelihood cost function over the data.
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the true posterior distribution, equation 3.4, is also known as the variational free energy.

In online variational Bayes (Broderick, Boyd, Wibisono, Wilson, & Jordan, 2013), one aims to find the posterior in an online setting, where data arrive sequentially. Similar to Bayesian inference, we use the previous approximated posterior as the new prior distribution, and the optimization problem becomes

$$\arg\min_{\phi} \mathbb{E}_{\phi \sim q_{n-1}} \left[ \log \left( q_{n}(\theta|\phi) \right) - \log \left( q_{n-1}(\theta) \right) + 1_{n}(\theta) \right].$$

(3.6)

4 Proposed Theoretical Approach

We present a method to mitigate catastrophic forgetting in task-agnostic continual learning. We aim to approximate the intractable (exact) online Bayes update rule, equation 3.3, using the online variational Bayes optimization problem, equation 3.6. Therefore, we use a new prior for each mini-batch (as in online variational Bayes) instead of using one prior for all the data (as in variational Bayes). When a Gaussian distribution is used as the parametric distribution $q_{n}(\theta|\phi)$, one can find the fixed-point equations for the online variational Bayes optimization problem, equation 3.6, that is, the first-order necessary conditions. The fixed-point equations define the relation between the prior parameters and the posterior parameters. Using the fixed-point equations we derive algorithms for task-agnostic continual learning (see section 5).

We now derive novel fixed-point equations for the online variational Bayes, equation 3.6, for multivariate Gaussian, matrix variate Gaussian, and diagonal Gaussian distributions.

4.1 Fixed-Point Equations for Multivariate Gaussian. Here we focus on our most general case, in which the parametric distribution $q_{n}(\theta|\phi)$ and the prior distribution $q_{n-1}(\theta)$ are multivariate Gaussian:

$$q_{n}(\theta|\phi) = \mathcal{N}(\theta|\mu, \Sigma), \quad q_{n-1}(\theta) = \mathcal{N}(\theta|m, V).$$

(4.1)

To find the fixed-point equations of the optimization problem in equation 3.6 in the case of a Gaussian distribution, we define the following deterministic transformation,

$$\theta = \mu + A\epsilon,$$

(4.2)

where $\phi = (\mu, \Sigma), \quad \Sigma = AA^T, \quad \epsilon \sim \mathcal{N}(0, I)$.

Using the first-order necessary conditions on equation 3.6 for the optimal $\mu$ and $A$ (see appendix G for details), we obtain the following equations:

$$\mu = m - V\mathbb{E}_{\epsilon} [\nabla L_n(\theta)], \quad AA^T + V\mathbb{E}_{\epsilon} [\nabla L_n(\theta) \epsilon^T] A^T - V = 0.$$
In lemma 1, we characterize the full set of solutions of the above quadratic equation (the proof is in appendix H).

**Lemma 1.** Let $T \in \mathbb{R}^{N \times N}$, $M \in \mathbb{R}^{N \times N}$, $M = M^\top$, $M > 0$, $X \in \mathbb{R}^{N \times N}$. The full set of solutions of

$$XX^\top + MTX^\top - M = 0 \quad (4.4)$$

is given by

$$X = DQ - \frac{1}{2}MT,$$

where

$$B = M + \frac{1}{4}MT^\top M, \quad D = B^{1/2},$$

and $Q \in \mathbb{R}^{N \times N}$ is an orthogonal matrix such that $DQ^\top M$ is a symmetric matrix.

In corollary 1 we demonstrate that $\Sigma$ has a set of multiple solutions.

**Corollary 1.** The full set of solutions for the optimal covariance matrix of the posterior distribution is given by

$$\Sigma = V + \frac{1}{2}VE_{\epsilon x}[\nabla L_n(\theta) \epsilon^\top]V + VE_{\epsilon x}[\nabla L_n(\theta) \epsilon^\top]Q^\top D^\top \quad (4.5)$$

where

$$D = \left( V + \frac{1}{4}VE_{\epsilon x}[\nabla L_n(\theta) \epsilon^\top]V + VE_{\epsilon x}[\nabla L_n(\theta) \epsilon^\top]Q^\top D^\top \right)^{1/2} \quad (4.6)$$

and $Q$ is an orthogonal matrix such that $DQ^\epsilon x[\nabla L_n(\theta) \epsilon^\top]V$ is a symmetric matrix.

Next, we characterize a single solution for the quadratic equation using the following Lemma (the proof is in appendix I):

**Lemma 2.** In this lemma we use the notations of lemma 1. Let $Q = SW^\top$ such that $S$, $W$ are the left and right singular matrices of the singular value decomposition (SVD) of $D^{-1}MT$. Then $Q$ is an orthogonal matrix and $DQ^\top M$ is a symmetric matrix.
Lemmas 1 and 2 together reveal the fixed-point equations (we substitute $M = V$ and $T = E_\epsilon [\nabla L_n (\theta) \epsilon^\top]$):

$$\mu = m - V E_\epsilon [\nabla L_n (\theta)], \quad A = DQ - \frac{1}{2} V E_\epsilon [\nabla L_n (\theta) \epsilon'^\top]. \quad (4.7)$$

In case $B$ is not invertible (or, more often, has a large condition number), the fixed-point equations can be derived using lemma 3, in appendix J.

### 4.2 Fixed-Point Equations for Matrix Variate Gaussian.

We now focus on the case in which the parametric distribution $q_n (\theta | \phi)$ and the prior distribution $q_{n-1} (\theta)$ are multivariate gaussian whose covariance matrix is a Kronecker product of two PD matrices (Gupta & Nagar, 2018). This type of distribution is also known as Kronecker-factored gaussian. Therefore:

$$q_n (\theta | \phi) = \mathcal{N} (\theta | \mu, \Sigma_1 \otimes \Sigma_2), \quad q_{n-1} (\theta) = \mathcal{N} (\theta | m, V_1 \otimes V_2), \quad (4.8)$$

where $V_1, \Sigma_1 \in \mathbb{R}^{d_1 \times d_1}$ (variance among-column) and $V_2, \Sigma_2 \in \mathbb{R}^{d_2 \times d_2}$ (variance among-row). To find the fixed-point equations of the optimization problem in equation 3.6 in the case of a matrix variate gaussian distribution, we define a deterministic transformation,

$$\theta = \mu + (A \otimes B) \epsilon, \quad (4.9)$$

where the distribution parameters are $\phi = (\mu, \Sigma_1, \Sigma_2)$ and $\Sigma_1 = AA^\top, \Sigma_2 = BB^\top, \epsilon \sim \mathcal{N} (0, I)$.

We use the first-order necessary conditions for the optimal $\mu, A, B$ (see appendix K for additional details)

$$\mu = m - (V_1 \otimes V_2) E_\epsilon [\nabla L_n (\theta)], \quad (4.10)$$

$$\text{Tr} (V_2^{-1} \Sigma_2) AA^\top + V_1 E_\epsilon [\Psi^T B \Phi] A^\top - p V_1 = 0,$$

$$\text{Tr} (V_1^{-1} \Sigma_1) BB^\top + V_2 E_\epsilon [\Psi A \Phi^\top] B^\top - n V_2 = 0,$$

where $\Psi, \Phi \in \mathbb{R}^{p \times n}$ such that vec$(\Psi) = \nabla L_n (\theta)$ and vec$(\Phi) = \epsilon$. In appendix M we use lemma 2 to derive the fixed-point equations for equation 4.10.

### 4.3 Fixed-Point Equations for Diagonal Gaussian.

Now consider a parametric distribution $q (\theta | \phi)$ and a prior distribution that are both gaussian with a diagonal covariance matrix (i.e., mean-field approximation). Therefore,

$$q_n (\theta | \phi) = \prod_i \mathcal{N} (\theta_i | \mu_i, \sigma_i^2), \quad q_{n-1} (\theta) = \prod_i \mathcal{N} (\theta_i | m_i, v_i^2). \quad (4.11)$$
To derive the fixed-point equations of the optimization problem in equation 3.6, we define a deterministic transformation,

$$\theta_i = \mu_i + \epsilon_i \sigma_i,$$

where $\phi = (\mu, \sigma)$, $\epsilon_i \sim N(0, 1)$.

We use the first-order necessary conditions for the optimal $\mu_i$ and $\sigma_i$:

$$\mu_i = m_i - v_i^2 \mathbb{E}_\epsilon \left[ \frac{\partial \mathcal{L}_n(\theta)}{\partial \theta_i} \right], \quad \sigma_i^2 + \sigma_i v_i^2 \mathbb{E}_\epsilon \left[ \frac{\partial \mathcal{L}_n(\theta)}{\partial \theta_i} \epsilon_i \right] - v_i^2 = 0. \quad (4.13)$$

Since this is a special case of equation 4.3 (where the covariance matrix is diagonal), one can use equation 4.7 to derive the following fixed-point equations:

$$\mu_i = m_i - v_i^2 \mathbb{E}_\epsilon \left[ \frac{\partial \mathcal{L}_n(\theta)}{\partial \theta_i} \right], \quad \sigma_i = v_i \sqrt{1 + \left( \frac{1}{2} v_i^2 \mathbb{E}_\epsilon \left[ \frac{\partial \mathcal{L}_n(\theta)}{\partial \theta_i} \epsilon_i \right] \right)^2 - \frac{1}{2} v_i^2 \mathbb{E}_\epsilon \left[ \frac{\partial \mathcal{L}_n(\theta)}{\partial \theta_i} \epsilon_i \right]^2}. \quad (4.14)$$

5 Proposed Algorithms

5.1 From Theory to Practice. The fixed-point equations we derived in section to the optimization problem, equation 3.6, are only implicit solutions. Note, for instance, that equations 4.7 include the derivative $\nabla \mathcal{L}_n(\theta)$, which is a function of $\phi$ (the unknown posterior parameters). One possible approach to find a solution for the fixed-point equations is to iterate them. In certain simple linear models in offline variational Bayes, a similar approach can be proven to converge (Sheth & Khardon, 2016). Here, since we are at an online setting, we take a single explicit iteration of the fixed-point equation for each mini-batch, that is, we evaluate the derivative $\nabla \mathcal{L}_n(\theta)$ using the prior parameters. This is done during the multiple passes through the data, as in assumed density filter (ADF) (Soudry, Hubara, & Meir, 2014; Hernández-Lobato & Adams, 2015).

The fixed-point equations iteration consists of an expectation term with regard to $\epsilon$. We use Monte Carlo samples to estimate those expectations.

5.2 Algorithms.

5.2.1 FOO-VB. Using the relaxations above, we present the fixed-point operator for online variational Bayes (FOO-VB) algorithmic framework (algorithm 1), including three specific variants: multivariate gaussian, matrix variate gaussian, and diagonal gaussian (algorithms 2, 3, and 4; the last two variants are in appendix I).
Algorithm 1: Fixed-Point Operator for Online Variational Bayes (FOO-VB).

**Initialize** Prior parameters $\phi_0$, Number of iterations $N_{max}$.

Number of Monte Carlo samples $K$

for $n = 1, \ldots, N$ sample a mini-batch

for $k = 1, \ldots, K$

Sample $\epsilon^{(k)} \sim \mathcal{N}(0, I)$ \hfill (step 1)

$\theta^{(k)} = \text{TRANSFORM} (\epsilon^{(k)}, \phi_{n-1})$ \hfill (step 2)

$g^{(k)} = \nabla L_n(\theta^{(k)})$ \hfill (step 3)

end

$\phi_n \leftarrow \text{UPDATE} (\phi_{n-1}, \epsilon^{(1 \ldots K)}, g^{(1 \ldots K)})$ \hfill (step 4)

Algorithm 1 describes the general framework in which we update the posterior distribution as we iterate over the data. For each mini-batch, we use $K$ Monte Carlo samples of the neural network weights using the current prior distribution $\phi_{n-1}$ (steps 1 and 2), and then calculate the gradient with regard to these randomized weights (step 3). Finally, we update the posterior parameters using the estimations and the update rules derived in section 4 (step 4).

The deterministic transformation (step 2) and the posterior update rule (step 4) differ for each distribution. Algorithms 2, 3, and 4 describe those steps for multivariate gaussian, matrix variate gaussian, and diagonal gaussian, respectively (algorithms 3 and 4 are in appendix I).

5.2.2 Feasibility and Complexity. As deep neural networks (DNN) often contain millions of parameters, it is infeasible to store the full covariance matrix between all parameters. A very common relaxation is to assume this distribution is factored between the layers (i.e., independent layers), so that the covariance matrix is a block diagonal matrix between layers. Even with this relaxation, the multivariate gaussian version of FOO-VB is impractical due to memory limitations. For example, storing matrix $A$ for a fully connected layer with 400 inputs and 400 outputs will require approximately 95 GB (using 32-bit floating point).
We avoid large covariance matrices by factoring them into a Kronecker product of two much smaller matrices (matrix variate gaussian). Alternatively, we employ a diagonal covariance matrix (diagonal gaussian). The matrix variate gaussian approximation enables us to store and apply mathematical operations on matrices of a practical size. For example, storing matrices $A$ and $B$ of the matrix variate version for a fully connected layer with 400 inputs and 400 outputs will require approximately 1.2 MB. In terms of runtime, this version requires 4 SVD decompositions for each layer in addition to the $K = 2500$ Monte-Carlo (MC) samples for each iteration. To reduce the runtime, one can parallelize the SVD (as in Berry & Sameh, 1989) and the MC sampling.

The diagonal approximation is our lightest version, with a memory footprint of only twice that of the regular network (to store the mean and variance of every weight). The $K = 10$ MC samples are the only overhead over a standard SGD optimizer, and so the runtime is linear with regard to the number of MC samples. See appendix K for computational complexity analysis.

5.3 Theoretical Properties of Diagonal FOO-VB. We present theoretical properties of FOO-VB. For simplicity, we focus on the diagonal gaussian approximation of the transformation $\theta = \mu + A\epsilon$. The updating equations for FOO-VB are given by:

**Algorithm 2: Methods for the Multivariate Gaussian Version.**

**Transform**($\epsilon, \phi = (\mu, A)$) :

$\theta = \mu + A\epsilon$

$\Sigma = AA^T$

**Update**($\phi_{n-1} = (\mu, A), \epsilon^{[1:K]}, e^{[1:K]}$) :

$E_1 = \frac{1}{K} \sum_{k=1}^{K} e^{(k)}$

$E_2 = \frac{1}{K} \sum_{k=1}^{K} e^{(k)} (e^{(k)})^T$

$\mu \leftarrow \mu - AA^T E_1$

$A \leftarrow X s.t. XX^T + AA^T E_2 X^T = AA^T = 0$

(This matrix equation is solved using lemma 2)
version. This version consists of a gradient descent algorithm for \( \mu \) and a recursive update rule for \( \sigma \). The learning rate of \( \mu_i \) is proportional to the uncertainty in the parameter \( \theta_i \) according to the prior distribution. During the learning process, as more data are seen, the learning rate decreases for parameters with a high degree of certainty and increases for parameters with a high degree of uncertainty. Next, we establish this intuitive idea more precisely.

It is easy to verify that the update rule for \( \sigma \) is a strictly monotonically decreasing function of

\[
E_{\epsilon} \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i \right] > 0 \implies \sigma_i(n) < \sigma_i(n-1),
\]

\[
E_{\epsilon} \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i \right] < 0 \implies \sigma_i(n) > \sigma_i(n-1),
\]

\[
E_{\epsilon} \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i \right] = 0 \implies \sigma_i(n) = \sigma_i(n-1).
\]

Next, using a Taylor expansion for the loss, we show that for small values of \( \sigma \), the quantity

\[
E_{\epsilon} \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i \right] = \frac{\partial^2 L_n(\mu)}{\partial^2 \theta_i} \sigma_i + O(\|\sigma\|^2),
\]

is a finite difference approximation to the component-wise product of the diagonal of the Hessian of the loss and the vector \( \sigma \). Therefore, we expect the uncertainty (learning rate) to decrease in areas with positive curvature (e.g., near local minima) or increase in areas with high negative curvature (e.g., near maxima, or saddles). This seems like a “sensible” behavior of the algorithm, since we wish to converge to local minima but escape saddles. This is in contrast to many common optimization methods, which are either insensitive to the sign of the curvature or use it the wrong way (Dauphin et al., 2014).

In the case of a strongly convex loss, we prove a more rigorous statement in appendix H.

**Theorem 1.** Consider FOO-VB with a diagonal gaussian distribution for \( \theta \). If \( L_n(\theta) \) is a strongly convex function with parameter \( m_n > 0 \) and a continuously differentiable function over \( \mathbb{R}^n \), then

\[
E_{\epsilon} \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i \right] \geq m_n \sigma_i > 0.
\]

**Corollary 2.** If \( L_n(\theta) \) is strongly convex (concave) for all \( n \in \mathbb{N} \), then the sequence \( \{\sigma_i(n)\}_{n=1}^{\infty} \) is strictly monotonically decreasing (increasing).
Furthermore, one can generalize these results and show that if a restriction of $L_n(\theta)$ to an axis $\theta_i$ is strongly convex (concave) for all $n \in \mathbb{N}$, then $\{\sigma_i(n)\}_{n=1}^{\infty}$ is monotonic decreasing (increasing).

Therefore, in the case of a strongly convex loss function, $\sigma_i = 0$ in any stable point of equation 4.14, which means that we collapse to point estimation similar to SGD. However, for neural networks, $\sigma_i$ does not generally converge to zero. In this case, the stable point $\sigma_i = 0$ is generally not unique, since $E[\partial L(\theta) / \partial \theta_i | \epsilon_i] \epsilon_i$ implicitly depends on $\sigma_i$.

In appendix L we show the histogram of STD (standard deviation) values on MNIST when training for 5000 epochs and demonstrate FOO-VB do not collapse to point estimation.

5.3.1 FOO-VB in Continual Learning. In the case of overparameterized models and continual learning, only a part of the weights is essential for each task. We hypothesize that if a weight $\theta_i$ is important to the current task, this implies that near the minimum, the function $L_i = L(\theta)|_{\theta_i = \theta_i}$ is locally convex. Corollary 2 suggests that in this case, $\sigma_i$ would be small. In contrast, the loss will have a flat curvature in the direction of weights that are not important to the task. Therefore, these unimportant weights may have a large uncertainty $\sigma_i$. Since FOO-VB in the diagonal gaussian version introduces the linkage between the learning rate and the uncertainty (STD), the training trajectories in the next task would be restricted along the less important weights, leading to good performance on the new task, while retaining the performance on the current task. The use of FOO-VB to continual learning exploits the inherent features of the algorithm, and it does not need any explicit information on tasks; it is completely unaware of the notion of tasks. We show empirical evidence for our hypothesis in section 6.2.

6 Applications and Experiments

6.1 Inapplicability of Other CL Algorithms. In task-agnostic scenarios, previous optimization-based methods for continual learning are generally inapplicable, as they rely on taking some actions (e.g., changing parameters in the loss function) on task switch, which is undefined in those scenarios. Nevertheless, one possible adaptation is to take the core action at every iteration instead of at every task switch. Doing so is impractical for many algorithms due to the computational complexity, but for a fair comparison, we have succeeded on running both online EWC (Chaudhry et al., 2018) and MAS (Aljundi, Babiloni Elhoseiny, Rohrbach, & Tuytelaars, 2018) with such an adaptation. As for the rehearsal approach, it is orthogonal to our approach and can be combined. However, we focus on the challenging real-world scenario in which we do not have access to any data from previous tasks. Thus, in our experiments, we do not compare to rehearsal algorithms.
Figure 1: Discrete task-agnostic Permuted MNIST. The average test accuracy on all seen tasks as a function of the number of tasks. The hyperparameters of all algorithms were tuned to maximize the average accuracy over all 10 tasks (therefore some of the algorithms have a relatively low accuracy for the first task). Offline (oracle) is a joint (i.e., not continual) training on all tasks.

We experiment on a task-agnostic variations of the Permuted MNIST benchmark for continual learning. The Permuted MNIST benchmark is a set of tasks constructed by a random permutation of MNIST pixels. Each task has a different permutation of pixels from the previous one. For all the experiments shown below, we conducted an extensive hyperparameter search to find the best results for each algorithm. See additional details in appendix J.

6.2 Discrete Task-Agnostic Permuted MNIST. We evaluate the algorithms on a task-agnostic scenario where the task boundaries are unknown. To do so, we use the Permuted MNIST benchmark for continual learning, but without informing the algorithms on task switches. The network architecture is two hidden layers of width 100 (see additional details in appendix P). In Figure 1, we show the average test accuracy over all seen tasks as a function of the number of tasks. Designed for task-agnostic scenarios, our algorithms surpass all other task-agnostic algorithms. The matrix variate gaussian version of FOO-VB experiences only approximately 2% degradation in the average accuracy after 10 tasks. The diagonal gaussian version of FOO-VB attains a good balance between high accuracy (approximately 88% after 10 tasks) and low computational complexity.
The average test accuracy over all tasks at the end of training implies a good balance between remembering previous tasks while adapting to new ones. On the other hand, in Figure 2 we show the test accuracy of the first task as a function of the number of seen tasks, which shows how well the algorithm remembers. The matrix variate gaussian version of FOO-VB is being able to remember the first task almost perfectly (i.e., like the oracle), and the overall performance is limited only by the test accuracy of the current task. The diagonal gaussian version of FOO-VB exhibits the next best performance compared to other algorithms.

We use the discrete task-agnostic Permuted MNIST experiment to examine our hypothesis of how the diagonal version of FOO-VB works in continual learning (see section 5.3). Figure 3 shows the histogram of STD values at the end of the training process of each task. The results show that after the first task, a large portion of the weights have STD values close to the initial value 0.047, while a small fraction of them have a much lower value. As training progresses, more weights are assigned with STD values much lower than the initial value. These results support our hypothesis in section 5.3 that only a small part of the weights is essential for each task, and as training progresses, more weights have low STD values.

6.3 Continuous Task-Agnostic Permuted MNIST. We consider the case where the transition between tasks occurs gradually over time, so
Figure 3: The histogram of STD values at the end of the training process of each task. As training progresses, more weights are assigned with STD values lower than the initial value. The initial STD value is 0.047. Best seen in color.

Figure 4: Distribution of samples from each task as a function of iteration. The tasks are not changed abruptly but gradually that is, boundaries are undefined. Here, the number of samples from each task in each batch is a random variable drawn from this distribution, which changes with time (iterations).

the algorithm gets a mixture of samples from two different tasks during the transition (see Figure 4) so the task boundaries are undefined. In all

\(^3\)The most challenging scenario is when mixing two different tasks. As we add more tasks, we are getting closer to offline (non-CL) training.
The average test accuracy on all seen tasks as a function of the number of tasks. Tasks are changing gradually over time as showed in Figure 4. Offline (oracle) is a joint (not continual) training on all tasks.

The network architecture is two hidden layers of width 200 (see additional details in appendix J). The output heads are shared among all tasks, task duration is $9380 \times T$ iterations, where $T = 10$ is the number of tasks (corresponds to 20 epochs per task), and the algorithms are unaware of the number of tasks or when the tasks are being switched.

The average test accuracy over all tasks for different numbers of tasks is presented in Figure 5. As can be seen, the matrix variate gaussian version of FOO-VB experiences less than 1% degradation in the average accuracy after 10 tasks. Similar to the discrete task-agnostic experiment (see section 6.2), the diagonal gaussian version of FOO-VB maintains a good balance between high accuracy (approximately 94% after 10 tasks) and low computational complexity.

6.4 Task-Aware Continual Learning on Vision Data Sets. We provide additional evaluation of FOO-VB using the experiment of vision data sets conducted in Ritter et al. (2018). This experiment is done in the task-aware scenario, and uses more complex data sets (such as CIFAR-10 and SVHN) and architecture (LeNet). Thus, we use for this experiment the diagonal
version of FOO-VB. The algorithms that FOO-VB is compared with are using the information on task-switch. Nevertheless, FOO-VB results are on par. See appendix M for the full details.

7 Conclusion

In this work, we aim to reduce catastrophic forgetting in task-agnostic scenarios (where task boundaries are unknown or not defined), using fixed architecture and without the use of external memory (i.e., without access to previous data, which can be restricted, for example, due to privacy issues). This can allow deep neural networks to better adapt to new tasks without explicitly instructed to do so, enabling them to learn in real-world continual learning settings.

Our method, FOO-VB, outperforms other continual learning methods in task-agnostic scenarios. It relies on solid theoretical foundations, being derived from novel fixed-point equations of the online variational Bayes optimization problem. We derive two practical versions of the algorithm to enable a trade-off between computational complexity versus performance.

There are many possible extensions and use cases for FOO-VB that we did not explore in this work. One possible extension is to incorporate FOO-VB in the framework of meta-learning (Finn, Abbeel, & Levine, 2017) to address more challenging scenarios. Another direction is to use FOO-VB to improve GAN stability (Thanh-Tung & Tran, 2018). During the training process, the discriminators exhibit catastrophic forgetting, since the generator output distribution changes gradually. FOO-VB fit this scenario naturally since it has no well-defined task boundaries. Indeed, some of thus extensions were considered in He et al. (2019) using the diagonal version of FOO-VB (published in our preliminary preprint Zeno, Golan, Hoffer, & Soudry, 2019). Finally, FOO-VB could be potentially useful in reinforcement learning, which often includes nonstationary environments.

Appendix A: Derivation of Equation 4.3

In this section, we provide additional details on the derivation of equation 4.3. The objective function is

\[ f(\mu, \Sigma) = \frac{1}{2} \left[ \log \frac{\det(V)}{\det(\Sigma)} - N + \text{Tr}(V^{-1} \Sigma) + (m - \mu)^\top V^{-1} (m - \mu) \right] 
+ \mathbb{E}_\theta [L_\mu(\theta)] . \]  

To solve the optimization problem in equation 2.6 in the case of gaussian approximation, we use the deterministic transformation, equation 4.2. To
calculate the first derivative of the objective function, we use the following identities:

\[
E_\theta [L_n(\theta)] = E_\epsilon [L_n(\theta)], \quad (A.2)
\]

\[
\frac{\partial E_\epsilon [L_n(\theta)]}{\partial A_{i,j}} = E_\epsilon \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_j \right], \quad (A.3)
\]

\[
\frac{\partial \text{Tr}(V^{-1}\Sigma)}{\partial A_{i,j}} = 2 \sum_n V^{-1}_{i,n} A_{n,j}, \quad (A.4)
\]

\[
\frac{\partial \log |\det(A)|}{\partial A_{i,j}} = A_{i,j}^{-\top}, \quad (A.5)
\]

We use the first-order necessary conditions for the optimal \( \mu \):

\[
-V^{-1}(m - \mu) + E_\epsilon [\nabla L_n(\theta)] = 0. \quad (A.6)
\]

And so we obtained equation 4.3.

Next, we use the first-order necessary conditions for the optimal \( A \):

\[
-(A^{-\top})_{i,j} + V^{-1} \sum_n V^{-1}_{i,n} A_{n,j} + E_\epsilon \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_j \right] = 0. \quad (A.7)
\]

And in matrix form we obtain:

\[
-A^{-\top} + V^{-1} A + E_\epsilon [\nabla L_n(\theta) \epsilon^\top] = 0. \quad (A.8)
\]

And so we obtained equation 4.3.

**Appendix B: Proof of Lemma 1**

The following proof is based on Poloni (2018).

**Proof.** Let

\[
X = DQ - \frac{1}{2}MT, \quad (B.1)
\]

such that

\[
B = M + \frac{1}{4}MTT^\top M \quad (B.2)
\]

\[
D = B^{1/2}. \quad (B.3)
\]
If we compare equation 4.4 with its transpose, we obtain

\[ \text{MTX}^\top = \text{XT}^\top \text{M}, \tag{B.4} \]

so we can rewrite equation 4.4 as follows,

\[ \text{XX}^\top + \frac{1}{2} \text{MTX}^\top + \frac{1}{2} \text{XT}^\top \text{M} - \text{M} = 0. \tag{B.5} \]

Next, we can factor equation B.5 as follows,

\[ \left( \text{X} + \frac{1}{2} \text{MT} \right) \left( \text{X} + \frac{1}{2} \text{T}^\top \text{M} \right)^\top = \text{M} + \frac{1}{4} \text{MTT}^\top \text{M}. \tag{B.6} \]

Since the matrix \( \text{B} = \text{M} + \frac{1}{4} \text{MTT}^\top \text{M} \) is positive definite (PD),

\[ \text{D}^{-1} \left( \text{X} + \frac{1}{2} \text{MT} \right) \left( \text{X} + \frac{1}{2} \text{T}^\top \text{M} \right) \text{D}^{-1} = \text{I}_N. \tag{B.7} \]

the equality holds if and only if \( \text{Q} = \text{D}^{-1} \left( \text{X} + \frac{1}{2} \text{MT} \right) \) is an orthogonal matrix. In addition, the matrix

\[ \text{XT}^\top \text{M} = \left( \text{DQ} + \frac{1}{2} \text{MT} \right) \text{T}^\top \text{M} = \text{DQT}^\top \text{M} - \frac{1}{2} \text{MTT}^\top \text{M} \tag{B.8} \]

is symmetric if and only if \( \text{DQT}^\top \text{M} \) is symmetric. \( \square \)

**Appendix C: Proof of Lemma 2**

The following proof is based on Poloni (2018).

**Proof.** Let

\[ \text{Q} = \text{SW}^\top \tag{C.1} \]

such that \( \text{S}, \text{W} \) are the left and right singular matrices of the singular value decomposition (SVD) of \( \text{D}^{-1} \text{MT} \). Then

\[ \text{Q}^\top \text{Q} = \text{WS}^\top \text{SW}^\top = \text{I} \tag{C.2} \]
and

\[ DQT^\top M = DSW^\top T^\top M = DSW^\top T^\top MD^{-\top}D^\top \]  
\[ \overset{(a)}{=} DSW^\top W\Lambda_1 S^\top D^\top = DS\Lambda_1 S^\top D^\top, \]  
\[ (C.3) \]

where (a) is because \( S, W \) are the left and right singular matrices of the SVD of \( D^{-1}MT \) meaning, \( D^{-1}MT = SAW^\top \) and \( A \) is a diagonal matrix. Therefore, \( Q \) is an orthogonal matrix and \( DQT^\top M \) is a symmetric matrix. \( \square \)

Appendix D: Proof of Lemma 3

In this section, we derive a solution for equation 4.4 in the case where \( B \) is not invertible (or, more often, has a large condition number).

**Lemma 3.** In this lemma, we use the notations of lemma 1. Let \( Q = UZ^\top \) such that \( U, Z \) are the left singular matrices of the generalized singular value decomposition (GSVD) of \((D^\top, T^\top M)\), respectively. Then \( Q \) is an orthogonal matrix and \( DQT^\top M \) is a symmetric matrix.

The following proof is based on Poloni (2018).

**Proof.** Let

\[ Q = SW^\top \]  
\[ (D.1) \]

such that \( U, Z \) are the left singular matrices of the GSVD of \((D^\top, T^\top M)\), respectively. Then

\[ Q^\top Q = ZU^\top UZ^\top = I \]  
\[ (D.2) \]

and

\[ DQT^\top M = DUZ^\top T^\top M \overset{(a)}{=} WA_1 U^\top UZ^\top ZA_2 W^\top \]  
\[ = WA_1 A_2 W^\top, \]  
\[ (D.3) \]

where (a) is because is because \( U, Z \) are the left singular matrices of the GSVD of \((D^\top, T^\top M)\) meaning, \( D^\top = UA_1 W^\top \), \( T^\top M = ZA_2 W^\top \) and \( A_1, A_2 \) are diagonal matrices. Therefore, \( Q \) is an orthogonal matrix, and \( DQT^\top M \) is a symmetric matrix. \( \square \)
Appendix E: Derivation of Equation 4.10

In this section, we provide additional details on the derivation of equation 4.10. The objective function is

\[
  f(\mu, \Sigma) = \frac{1}{2} \left[ \log \det (V_1)^p \det (V_2)^n \right] - np + \text{Tr} \left( (V_1 \otimes V_2)^{-1} (\Sigma_1 \otimes \Sigma_2) \right) + (m - \mu)^\top (V_1 \otimes V_2)^{-1} (m - \mu) + \mathbb{E}_\theta [L_{\theta}(\theta)]. \tag{E.1}
\]

To solve the optimization problem in equation 2.6 in the case of Kronecker-factored approximation, we use the deterministic transformation, equation 4.9. To calculate the first derivative of the objective function we use the following identities (see appendix F for additional details):

\[
  \frac{\partial \mathbb{E}_\epsilon [L_{\epsilon}(\theta)]}{\partial A_{i,j}} = \mathbb{E}_\epsilon \left[ \sum_{\ell=1}^p \sum_{k=1}^p \frac{\partial L_{\epsilon}(\theta)}{\partial \theta_{\ell+\ell-1}p} B_{\ell,k} \epsilon_{k+(j-1)p} \right], \tag{E.3}
\]

\[
  \frac{\partial \mathbb{E}_\epsilon [L_{\epsilon}(\theta)]}{\partial B_{i,j}} = \mathbb{E}_\epsilon \left[ \sum_{\ell=1}^n \sum_{k=1}^n \frac{\partial L_{\epsilon}(\theta)}{\partial \theta_{\ell+\ell-1}n} A_{\ell,k} \epsilon_{j+(k-1)n} \right], \tag{E.4}
\]

\[
  \text{Tr} \left( (V_1 \otimes V_2)^{-1} (\Sigma_1 \otimes \Sigma_2) \right) = \text{Tr} (V_1^{-1} \Sigma_1) \text{Tr} (V_2^{-1} \Sigma_2). \tag{E.5}
\]

We use the first-order necessary conditions for the optimal \( \mu \):

\[
  - (V_1 \otimes V_2)^{-1} (m - \mu) + \mathbb{E}_\epsilon [\nabla L_{\epsilon}(\theta)] = 0. \tag{E.6}
\]

And so we obtained equation 4.10. We use the first-order necessary conditions for the optimal \( A \):

\[
  -p (A^{-\top})_{i,j} + \text{Tr} (V_2^{-1} \Sigma_2) \sum_k (V_1)_{i,k}^{-1} A_{k,j} + \mathbb{E}_\epsilon \left[ \sum_{\ell=1}^p \sum_{k=1}^p \frac{\partial L_{\epsilon}(\theta)}{\partial \theta_{\ell+\ell-1}p} B_{\ell,k} \epsilon_{k+(j-1)p} \right] = 0. \tag{E.7}
\]

And in matrix form we obtain

\[
  -p A^{-\top} + \text{Tr} (V_2^{-1} \Sigma_2) V_1^{-1} A + \mathbb{E}_\epsilon [\Psi^{-1} \Phi] = 0, \tag{E.8}
\]
where $\Psi, \Phi \in \mathbb{R}^{p \times n}$ such that $\text{vec}(\Psi) = \nabla L_m(\theta)$ and $\text{vec}(\Phi) = \epsilon$. And so we obtained equation 4.10.

We use the first-order necessary conditions for the optimal $B$:

$$
-n (B^{-T})_{i,j} + \text{Tr} \left( V_1^{-1} \Sigma_1 \right) \sum_k (V_2^{-1})_{i,k} B_{k,j}
$$

$$+
E_{\epsilon} \left[ \sum_{\ell=1}^n \sum_{k=1}^n \frac{\partial L_m(\theta)}{\partial \theta_{\ell (\ell-1)p}} A_{\ell,k} \epsilon_{j+(\ell-1)p} \right] = 0.
$$

(E.9)

And in matrix form we obtain

$$
-nB^{-T} + \text{Tr} \left( V_1^{-1} \Sigma_1 \right) V_2^{-1} B + E_{\epsilon} [\Psi^T B \Phi] = 0.
$$

(E.10)

And so we obtained equation 4.10.

**Appendix F: Technical Results for Appendix E**

In this section, we prove the technical results used in Appendix E.

Let $\Delta_{i,j}^{(a)} \in \mathbb{R}^{n \times n}$ such that

$$
(\Delta_{i,j}^{(a)})_{m,n} = \delta_{i,n} \delta_{m,j}.
$$

We then have

$$
\frac{\partial E_{\epsilon} [L_m(\theta)]}{\partial A_{i,j}} = E_{\epsilon} \left[ \frac{\partial L_m(\theta)}{\partial A_{i,j}} \right]
$$

$$
\overset{(a)}{=} E_{\epsilon} \left[ \sum_{\ell=1}^{np} \frac{\partial L_m(\theta)}{\partial \theta_{\ell}} \cdot \frac{\partial \theta_{\ell}}{\partial A_{i,j}} \right]
$$

$$
\overset{(b)}{=} E_{\epsilon} \left[ \sum_{\ell=1}^{np} \frac{\partial L_m(\theta)}{\partial \theta_{\ell}} \cdot \frac{\partial (\mu_{\ell} + \sum_k (A \otimes B)_{\ell,k} \epsilon_k)}{\partial A_{i,j}} \right]
$$

$$
\overset{(c)}{=} E_{\epsilon} \left[ \sum_{\ell=1}^{np} \frac{\partial L_m(\theta)}{\partial \theta_{\ell}} \sum_{k=1}^{np} \left( \Delta_{i,j}^{(a)} \otimes B \right)_{\ell,k} \epsilon_k \right]
$$

$$
\overset{(d)}{=} E_{\epsilon} \left[ \sum_{\ell=(j-1)p+1}^{jp} \frac{\partial L_m(\theta)}{\partial \theta_{\ell}} \sum_{k=(j-1)p+1}^{kp} \left( \Delta_{i,j}^{(a)} \otimes B \right)_{\ell,k} \epsilon_k \right]
$$

$$
= E_{\epsilon} \left[ \sum_{\ell=1}^{p} \sum_{k=1}^{p} \frac{\partial L_m(\theta)}{\partial \theta_{\ell+(j-1)p}} B_{\ell,k} \epsilon_{i+(j-1)p} \right].
$$

(F.1)
where

- (a) is because $\theta$ is a vector of length $n \cdot p$ and by the chain rule for derivatives;
- (b) holds since $\theta_\ell = \mu_\ell + \sum_{k=1}^{np} (A \otimes B)_{\ell,k} \epsilon_k$;
- (c) is by definition of $\Delta_{i,j}^{(p)}$, and since we differentiate by $A_{i,j}$;
- (d) is since $(A \otimes \Delta_{i,j}^{(p)})_{\ell,k} \neq 0$ if $(i-1)p + 1 \leq \ell \leq ip$ or if $(j-1)p + 1 \leq k \leq jp$.

Let $\Delta_{i,j}^{(p)} \in \mathbb{R}^{p \times p}$ such that

$$(\Delta_{i,j}^{(p)})_{n,m} = \delta_{n,i} \delta_{m,j}.$$ 

We then have

$$\frac{\partial \mathbb{E}_\epsilon [L_n(\theta)]}{\partial B_{i,j}} = \mathbb{E}_\epsilon \left[ \frac{\partial L_n(\theta)}{\partial B_{i,j}} \right],$$

$$(a) = \mathbb{E}_\epsilon \left[ \sum_{\ell=1}^{np} \frac{\partial L_n(\theta)}{\partial \theta_\ell} \cdot \frac{\partial \theta_\ell}{\partial B_{i,j}} \right],$$

$$(b) = \mathbb{E}_\epsilon \left[ \sum_{\ell=1}^{np} \frac{\partial L_n(\theta)}{\partial \theta_\ell} \cdot \frac{\partial (\mu_\ell + \sum_{k=1}^{np} (A \otimes B)_{\ell,k} \epsilon_k)}{\partial B_{i,j}} \right],$$

$$(c) = \mathbb{E}_\epsilon \left[ \sum_{\ell=1}^{np} \frac{\partial L_n(\theta)}{\partial \theta_\ell} \sum_{k=1}^{np} (A \otimes \Delta_{i,j}^{(p)})_{\ell,k} \epsilon_k \right],$$

$$(d) = \mathbb{E}_\epsilon \left[ \sum_{\ell=1}^{n} \sum_{k=1}^{n} \frac{\partial L_n(\theta)}{\partial \theta_{\ell+(\ell-1)n}} A_{\ell,k} \epsilon_{j+(k-1)n} \right].$$ 

(2)

where

- (a) is because $\theta$ is a vector of length $n \cdot p$ and by the chain rule for derivatives;
- (b) holds since $\theta_\ell = \mu_\ell + \sum_{k=1}^{np} (A \otimes B)_{\ell,k} \epsilon_k$;
- (c) is by definition of $\Delta_{i,j}^{(p)}$, and since we differentiate by $B_{i,j}$;
- (d) is since $(A \otimes \Delta_{i,j}^{(p)})_{\ell,k} \neq 0$ if $k \mod n = j$ and if $\ell \mod n = i$.
Appendix G: Derivation of the Fixed-Point Equations for the Matrix Variate Gaussian

In this section, we provide additional details on the derivation of the fixed-point equations for the matrix variate Gaussian:

\[ \mu = m - (V_1 \otimes V_2) \mathbb{E}_\epsilon [\nabla L_n(\theta)] , \quad \text{(G.1)} \]

\[ AA^\top + \left( \frac{p}{\text{Tr} (V_2^{-1} \Sigma_2)} \right) V_1 \frac{1}{p} \mathbb{E}_\epsilon [\Psi \top B \Phi] A^\top - \left( \frac{p}{\text{Tr} (V_2^{-1} \Sigma_2)} \right) V_1 = 0, \quad \text{(G.2)} \]

\[ BB^\top + \left( \frac{n}{\text{Tr} (V_1^{-1} \Sigma_1)} \right) V_2 \frac{1}{n} \mathbb{E}_\epsilon [\Psi A \Phi^\top] B^\top - \left( \frac{n}{\text{Tr} (V_1^{-1} \Sigma_1)} \right) V_2 = 0. \quad \text{(G.3)} \]

Lemma 2 reveals the fixed-point equations:

\[ \mu = m - (V_1 \otimes V_2) \mathbb{E}_\epsilon [\nabla L_n(\theta)] , \quad \text{(G.4)} \]

\[ A = D_1 Q_1 - \frac{1}{2} \left( \frac{p}{\text{Tr} (V_2^{-1} \Sigma_2)} \right) V_1 \frac{1}{p} \mathbb{E}_\epsilon [\Psi \top B \Phi] , \quad \text{(G.5)} \]

\[ B = D_2 Q_2 - \frac{1}{2} \left( \frac{n}{\text{Tr} (V_1^{-1} \Sigma_1)} \right) V_2 \frac{1}{n} \mathbb{E}_\epsilon [\Psi A \Phi^\top] . \quad \text{(G.6)} \]

Appendix H: Proof of Theorem 1

**Proof.** We define \( \theta_j = \mu_j + \epsilon_j \sigma_j \) where \( \epsilon_j \sim \mathcal{N}(0, 1) \). According to the smoothing theorem, the following holds:

\[ \mathbb{E}_\epsilon \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i \right] = \mathbb{E}_{\epsilon_{j \neq i}} \left[ \mathbb{E}_{\epsilon_i} \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i \bigg| \epsilon_{j \neq i} \right] \right] . \quad \text{(H.1)} \]

The conditional expectation is

\[ \mathbb{E}_{\epsilon_i} \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i \bigg| \epsilon_{j \neq i} \right] = \int_{-\infty}^{\infty} \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i f_{\epsilon_i}(\epsilon_i) d\epsilon, \quad \text{(H.2)} \]

where \( f_{\epsilon_i} \) is the probability density function of a standard normal distribution. Since \( f_{\epsilon_i} \) is an even function,

\[ \mathbb{E}_{\epsilon_i} \left[ \frac{\partial L_n(\theta)}{\partial \theta_i} \epsilon_i \bigg| \epsilon_{j \neq i} \right] = \int_{0}^{\infty} \frac{\partial L_n(\mu_i + \epsilon_i \sigma_i, \theta_{-i})}{\partial \theta_i} \epsilon_i f_{\epsilon_i}(\epsilon_i) d\epsilon_i \]

\[ - \int_{0}^{\infty} \frac{\partial L_n(\mu_i - \epsilon_i \sigma_i, \theta_{-i})}{\partial \theta_i} \epsilon_i f_{\epsilon_i}(\epsilon_i) d\epsilon_i. \quad \text{(H.3)} \]
Now, since $L_n(\theta)$ is strongly convex function with parameter $m_n > 0$ and continuously differentiable function over $\mathbb{R}^d$, the following holds $\forall \theta_1, \theta_2 \in \mathbb{R}^d$:

$$
(\nabla L_n(\theta_1) - \nabla L_n(\theta_2))^T (\theta_1 - \theta_2) \geq m_n \|\theta_1 - \theta_2\|^2.
$$

(H.4)

For $\theta_1, \theta_2$ such that

$$
(\theta_1)_j = \begin{cases} 
(\theta_2)_j, & j \neq i \\
\mu_i + \epsilon_i \sigma_i, & j = i,
\end{cases}
$$

(H.5)

and

$$
(\theta_2)_j = \begin{cases} 
(\theta_1)_j, & j \neq i \\
\mu_i - \epsilon_i \sigma_i, & j = i,
\end{cases}
$$

(H.6)

the following holds:

$$
\left( \frac{\partial L(\theta_1)}{\partial \theta_i} - \frac{\partial L(\theta_2)}{\partial \theta_i} \right) \epsilon_i \geq 2m_n \sigma^2_i \epsilon^2_i.
$$

(H.7)

Therefore, substituting this inequality into equation H.3, we obtain

$$
E_\epsilon \left[ \frac{\partial L(\theta)}{\partial \theta_i} \epsilon_i \right] \geq m_n \sigma^2_i > 0.
$$

(H.8)

□

Appendix I: FOO-VB Algorithm: Some Special Cases

In this section, we present the FOO-VB algorithms for the matrix variate gaussian and diagonal gaussian variants.

1.1 Matrix Variate Gaussian. In the case of matrix variate gaussian, one can write the deterministic transformation

$$
\theta = \mu + (A \otimes B) \epsilon
$$

(I.1)

in matrix form,

$$
W = M + \Phi A^T,
$$

(I.2)

where $W, \Phi \in \mathbb{R}^{d_2 \times d_1}$ and $\text{vec}(W) = \theta, \text{vec}(\Phi) = \epsilon$. The deterministic transformation and the posterior update rule for the matrix variate gaussian in the matrix form can be found in algorithm 3.
Algorithm 3: Methods for the Matrix Variate Gaussian Version.

**Transform**$(\epsilon, \phi = (M, A, B))$:

$W = M + B\Phi A^T$

$\Sigma_1 = AA^T$

$\Sigma_2 = BB^T$

**Update**$(\phi_{k-1} = (M, A, B), e^{(1..K)}, g^{(1..K)})$:

$\text{vec}(\Psi^{(k)}) = g^{(k)}$

$\text{vec}(\Phi^{(k)}) = e^{(k)}$

$\hat{E}_1 = \frac{1}{K} \sum_{k=1}^{K} \Psi^{(k)}$

$\hat{E}_2 = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{b} \Psi^{(k)^T} B\Phi^{(k)}$

$\hat{E}_3 = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{n} \Psi^{(k)^T} A\Phi^{(k)}$

$M \leftarrow M - BB^T \hat{E}_1 AA^T$

$A \leftarrow X \text{ such that } XX^T + AA^T \hat{E}_2 X^T - AA^T = 0$

$B \leftarrow X \text{ such that } XX^T + BB^T \hat{E}_3 X^T - BB^T = 0$

(The matrix equations are solved using lemma 2)

1.2 Diagonal Gaussian. The deterministic transformation and the posterior update rule for the diagonal gaussian can be found in algorithm 4.

Appendix J: Implementation Details

For the matrix variate version of FOO-VB, we initialize the weights by sampling from a gaussian distribution with zero mean and a variance of
We do so by sampling the mean of the weights from a Gaussian distribution, such that

$$M_{i,j} \sim N(0, \frac{2\alpha}{n_{\text{input}} + 2})$$

where $\alpha \in (0, 1)$. In addition, we sample the diagonal elements of the matrices $A, B$ from a Gaussian distribution, such that

$$\epsilon_{i,i} \sim N\left(0, \frac{2(1-\alpha)}{n_{\text{input}} + 2}\right)$$

$$\epsilon_{i,j} \sim N\left(0, \frac{2(1-\alpha)}{n_{\text{input}} + 2}\right)$$

where the non-diagonal elements are initialized to zero. We use 2500 Monte Carlo samples to estimate the expected gradient during training and average the accuracy of 2500 sampled networks during testing unless stated otherwise.

For the diagonal version of FOO-VB, we initialize the mean of the weights $\mu$ by sampling from a Gaussian distribution with a zero mean and a variance of $2/(n_{\text{input}} + n_{\text{output}})$, unless stated otherwise. We use 10 Monte Carlo samples to estimate the expected gradient during training, and set the weights to the learned mean testing.
Table 1: Hyperparameter Search Results on Discrete Permuted MNIST: EWCOnline

| Algorithm | LR   | Regularization Coefficient | Accuracy |
|-----------|------|----------------------------|----------|
| EWCOnline | 0.01 | 250.0                      | 38.47    |
| EWCOnline | 0.01 | 150.0                      | 48.33    |
| EWCOnline | 0.01 | 10.0                       | 81.33    |
| EWCOnline | 0.01 | 0.1                        | 42.46    |
| EWCOnline | 0.01 | 0.02                       | 39.40    |
| EWCOnline | 0.0001 | 250.0                      | 46.70    |
| EWCOnline | 0.0001 | 150.0                      | 55.58    |
| EWCOnline | 0.0001 | 10.0                       | 84.03    |
| EWCOnline | 0.0001 | 0.1                        | 58.68    |
| EWCOnline | 0.0001 | 0.02                       | 55.17    |
| EWCOnline | 0.0001 | 250.0                      | 46.20    |
| EWCOnline | 0.0001 | 150.0                      | 54.19    |
| EWCOnline | 0.0001 | 10.0                       | 85.06    |
| EWCOnline | 0.0001 | 0.1                        | 52.13    |
| EWCOnline | 0.0001 | 0.02                       | 40.48    |
| EWCOnline | 0.1   | 250.0                      | 19.32    |
| EWCOnline | 0.1   | 150.0                      | 29.05    |
| EWCOnline | 0.1   | 10.0                       | 10.04    |
| EWCOnline | 0.1   | 0.1                        | 10.49    |
| EWCOnline | 0.1   | 0.02                       | 9.68     |

J.1 Discrete Permuted MNIST. We use a fully connected neural network with two hidden layers of 100 width, ReLUs as activation functions, and softmax output layer with 10 units. We used mini-batch of size 128 and 20 epochs. We conducted a hyperparameter search for all algorithms and present the best test accuracy for 10 tasks, with LR ∈ {0.1, 0.01, 0.001, 0.0001} and regularization coefficient ∈ {250, 150, 10, 0.1, 0.02} (see Tables 1, 2, and 3). For FOO-VB, we used α = 0.5 in the matrix variate version and σ_{init} = 0.047 in the diagonal version. For Online EWC, we used LR of 0.001 and regularization coefficient of 10.0, and for MAS, we used LR of 0.001 and regularization coefficient of 0.1.

J.2 Continuous Permuted MNIST. We use a fully connected network with two hidden layers of width 200. Following Hsu, Liu, and Kira (2018), the original MNIST images are padded with zeros to match size 32 × 32. The batch size is 128, and we sample with replacement due to the properties of the continuous scenario (no definition for epoch as task boundaries are undefined). We conducted a hyperparameter search for all algorithms and present the best test accuracy for 10 tasks (see Figure 6). For FOO-VB, we used α = 0.6 in the matrix variate version and σ_{init} = 0.06 in the diagonal version. For Online EWC and MAS, we used the following combinations...
Table 2: Hyperparameter Search Results on Discrete Permuted MNIST: MAS

| Algorithm | LR    | Regularization Coefficient | Accuracy |
|-----------|-------|----------------------------|----------|
| MAS       | 0.01  | 250.0                      | 29.73    |
| MAS       | 0.01  | 150.0                      | 29.85    |
| MAS       | 0.01  | 10.0                       | 44.20    |
| MAS       | 0.01  | 0.1                        | 63.50    |
| MAS       | 0.01  | 0.02                       | 63.46    |
| MAS       | 0.0001| 250.0                      | 49.57    |
| MAS       | 0.0001| 150.0                      | 54.13    |
| MAS       | 0.0001| 10.0                       | 78.03    |
| MAS       | 0.0001| 0.1                        | 83.49    |
| MAS       | 0.0001| 0.02                       | 75.32    |
| MAS       | 0.001 | 250.0                      | 46.41    |
| MAS       | 0.001 | 150.0                      | 50.64    |
| MAS       | 0.001 | 10.0                       | 72.59    |
| MAS       | 0.001 | 0.1                        | 84.56    |
| MAS       | 0.001 | 0.02                       | 82.63    |
| MAS       | 0.1   | 250.0                      | 8.948    |
| MAS       | 0.1   | 150.0                      | 9.772    |
| MAS       | 0.1   | 10.0                       | 13.64    |
| MAS       | 0.1   | 0.1                        | 21.45    |
| MAS       | 0.1   | 0.02                       | 19.49    |

Table 3: Hyperparameter Search Results on Discrete Permuted MNIST: Adam, SGD, and Adagrad

| Algorithm | LR    | Regularization Coefficient | Accuracy |
|-----------|-------|----------------------------|----------|
| Adam      | 0.0001| 0.0                        | 52.64    |
| Adam      | 0.01  | 0.0                        | 10.69    |
| Adam      | 0.1   | 0.0                        | 10.31    |
| Adam      | 0.001 | 0.0                        | 27.20    |
| SGD       | 0.01  | 0.0                        | 66.18    |
| SGD       | 0.0001| 0.0                        | 71.17    |
| SGD       | 0.001 | 0.0                        | 76.94    |
| SGD       | 0.1   | 0.0                        | 36.97    |
| Adagrad   | 0.1   | 0.0                        | 51.48    |
| Adagrad   | 0.001 | 0.0                        | 82.42    |
| Adagrad   | 0.0001| 0.0                        | 74.14    |
| Adagrad   | 0.01  | 0.0                        | 75.98    |

of hyperparameters: LR of 0.01 and 0.0001; regularization coefficient of 10, 0.1, 0.001, 0.0001; and optimizer SGD and Adam. The best results for online EWC were achieved using LR of 0.01, regularization coefficient of 0.1, and SGD optimizer. For MAS, best results were achieved using LR of 0.01, regularization coefficient of 0.001 and SGD optimizer.
Figure 6: Hyperparameters search for continuous permuted MNIST experiment.

J.3 Vision Data Sets Mix. We followed the experiment as described in Ritter et al. (2018). We use a batch size of 64, and we normalize the data sets to have zero mean and unit variance. The network architecture is LeNet like with two convolution layers with 20 and 50 channels and kernel size of 5, each convolution layer is followed by a ReLU activations function and max pool, and the two layers are followed with a fully connected layer of size 500 before the last layer. The SGD baseline was trained with a constant learning rate of 0.001 and ADAM used $\epsilon = 10^{-8}$, LR of 0.001 and $(\beta_1, \beta_2) = (0.9, 0.999)$. FOO-VD trained with initial STD of 0.02 and batch size 64.

Appendix K: Complexity

The diagonal version of FOO-VB requires $\times 2$ more parameters compared to SGD, as it stores both the mean and the STD per weight. In terms of time complexity, the major difference between SGD and FOO-VB arises from the estimation of the expected gradients using Monte Carlo samples during training. Since those Monte Carlo samples are completely independent, the algorithm is embarrassingly parallel.

Specifically, given a mini-batch, for each Monte Carlo sample, FOO-VB generates a random network using $\mu$ and $\sigma$, then making a forward-backward pass with the randomized weights.

Two main implementation methods are available (using 10 Monte Carlo samples as an example):

1. Producing the (10) Monte Carlo samples sequentially, thus saving only a single randomized network in memory at a time (decreasing memory usage, increasing runtime)
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Table 4: Average Runtime of a Single Training Epoch with Different Numbers of Monte Carlo Samples.

| Experiment           | MC Iterations | Accuracy | Iteration Runtime [seconds] | Versus SGD |
|----------------------|---------------|----------|----------------------------|------------|
| Continuous SGD       | 77.79%        | 0.0024   | ×1                         |            |
| Task Agnostic (Figure 5) | 2 (FOO-VB) | 92%      | 0.0075 × 3.12              |            |
|                      | 10 (FOO-VB)   | 93.40%   | 0.0287 × 11.95             |            |

Note: The MC iterations have linear effect on runtime for classification and almost no effect in the continuous task-agnostic experiment, probably due to implementation specifics. Using fewer MC iterations does not affect accuracy significantly. Accuracy reported in the table is from the continuous experiment (see Figure 5).

2. Producing the (10) Monte Carlo samples in parallel, thus saving (10) randomized networks in memory (increasing memory usage, decreasing runtime)

We analyzed how the number of Monte Carlo iterations affects the runtime on continuous permuted MNIST using the first method of implementation (sequential MC samples). The results, reported in Table 4, show that runtime is indeed a linear function of the number of MC iterations.

For the matrix variate version of FOO-VB, the main bottleneck is the SVD operation, with the following breakdown:

- A single MC iteration takes 0.002 seconds.
- A single iteration (including all MC iterations and the matrix updates) takes 0.68 seconds (with 10 MC iterations).
- Each SVD takes 0.22 seconds. In this case, we have two SVDs, which takes 0.44 seconds about \( \frac{2}{3} \) of the iteration runtime.

In the experiments, we used a single GPU (GeForce GTX 1080 Ti).

Appendix L: 5000 Epochs Training

L.1 MNIST. We turn to a MNIST classification experiment to demonstrate the convergence of the log-likelihood cost function and the histogram of STD values. We train a fully connected neural network with two hidden layers and a layer width of 400 for 5000 epochs.

Figure 7 shows the log-likelihood cost function of the training set and the test set. As can be seen, the log-likelihood cost function on the training set decreases during the training process and converges to a low value. Thus, FOO-VB does not experience underfitting and overpruning as Trippe and Turner (2018) showed for BBB (Blundell et al., 2015).

Figure 8 shows the histogram of STD values during the training process. As can be seen, the histogram of STD values converges. This demonstrates that \( \sigma_i \) does not collapse to zero even after 5000 epochs.
Figure 7: Average log-likelihood cost function of the train set and the test set: layer width 400.

Figure 8: Histogram of STD values the initial STD value is 0.05.

Figure 9 shows the learning curve of the train set and the test set. As can be seen, the test accuracy does not drop even if we continue to train for 5000 epochs.

Appendix M: Task-Aware Continual Learning on Vision Data Sets

We followed Ritter et al. (2018) and challenged our algorithm with the vision data sets experiment. In this experiment, we train sequentially on
Figure 9: Test accuracy and train accuracy: layer width 400.

Table 5: Accuracy for Each Task after Training Sequentially on All Tasks.

| Method      | Test Accuracy [%] on the End of Last Task (CIFAR-10) |
|-------------|--------------------------------------------------------|
|             | Average | MNIST | notMNIST | F-MNIST | SVHN | CIFAR-10 |
| Diagonal methods |          |        |          |         |      |          |
| FOO-VB      | 81.37   | 86.42  | 89.23    | 83.05   | 82.21 | 65.96    |
| SGD         | 69.64   | 84.79  | 82.12    | 65.91   | 52.31 | 63.08    |
| ADAM        | 29.67   | 17.39  | 26.26    | 25.02   | 15.10 | 64.62    |
| SI          | 77.21   | 87.27  | 79.12    | 84.61   | 77.44 | 57.61    |
| PTL         | 82.96   | 97.83  | 94.73    | 89.13   | 79.80 | 53.29    |
| AL          | 82.55   | 96.56  | 92.33    | 89.27   | 78.00 | 56.57    |
| OL          | 82.71   | 96.48  | 93.41    | 88.09   | 81.79 | 53.80    |
| Nondiagonal methods |          |        |          |         |      |          |
| PTL         | 85.32   | 97.85  | 94.92    | 89.31   | 85.75 | 58.78    |
| AL          | 85.35   | 97.90  | 94.88    | 90.08   | 85.24 | 58.63    |
| OL          | 85.40   | 97.17  | 94.78    | 90.36   | 85.59 | 59.11    |

Note: PTL stands for Per-Task Laplace (one penalty per task), AL is Approximate Laplace (Laplace approximation of the full posterior at the mode of the approximate objective), and OL is Online Laplace approximation. Results for SI, PTL, AL, and OL are as reported in Ritter et al. (2018). We highlight the best accuracy in bold.

MNIST, notMNIST, FashionMNIST, SVHN and CIFAR-10 (LeCun, Bottou, Bengio, & Haffner, 1998; Xiao, Rasul, & Vollgraf, 2017; Netzer et al., 2011; Krizhevsky & Hinton, 2009). Training is done in a sequential way with

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4Originally published at http://yaroslavvb.blogspot.co.uk/2011/09/notmnist-data-set.html and downloaded from https://github.com/davidflanagan/notMNIST-to-MNIST.
20 epochs per task. In epochs 1 to 20 we train on MNIST (first task), and on epochs 81 to 100 we train on CIFAR-10 (last task). All five data sets consist of about 50,000 training images from 10 different classes, but they differ from each other in various ways for example, black and white versus RGB, letters and digits versus vehicles and animals. We use the exact same setup as in Ritter et al. (2018) for the comparison: LeNet-like (LeCun et al., 1998) architecture with separated last layer for each task as in CIFAR-10/CIFAR-100 experiment. Results are reported in Table 5.

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