Higher weight Gel’fand-Kalinin-Fuks classes of formal Hamiltonian vector fields of symplectic $\mathbb{R}^2$

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Abstract

In “The Gel’fand-Kalinin-Fuks class and characteristic classes of transversely symplectic foliations”, arXiv:0910.3414, (October 2009) by D. Kotschick and S. Morita, the relative Gel’fand-Kalinin-Fuks cohomology groups of the formal Hamiltonian vector fields without constant vector fields on $2n$-plane were characterized by two parameters, one is degree and the other is weight. And they obtained those cohomology groups of the 2-plane while their weight $\leq 10$.

In this paper, for those cohomology groups of the 2-plane, we succeeded in determining the dimension of cochain complexes by $Sp(2,\mathbb{R})$-representation theory for their weight even less than 50, thus, we manipulate the Euler characteristic numbers. We also decide our relative Gel’fand-Kalinin-Fuks cohomology groups until whose weight $< 20$ by getting a concrete matrix representation of the coboundary operator.

1 Introduction

Since 2000, many works have been achieved on Gel’fand-Kalinin-Fuks cohomology. S. Metoki in his doctoral thesis [5] found a new exotic class, which is now called the Metoki class. M. Takamura [7] showed that the relative cohomology of Lie algebra of formal contact vector fields with respect to formal Poisson vector fields is trivial. In the paper of D. Kotschick and S. Morita [4], the relative Gel’fand-Fuks cohomology groups were characterized by two parameters; the degree and the weight, and they obtained those cohomology groups of the 2-plane with weight $\leq 10$. We are ambitious to compute more higher weight cases.

In this paper, we recall fundamental facts on Hamiltonian formalism; particularly we review group actions very carefully. And then we concentrate on the weight. We show that the weight corresponds to Young diagrams, and we construct the generating function for the weight. We make use of representation theory of $Sp(2n)$, especially the fact that the irreducible representations are parameterized by the Young diagrams of height at most $n$. In the final section we exhibit our main result, which we obtain by means of computer calculation.

Points of this revised version (on Feb 2014):

(1) Fixing notations of Lie algebras of formal Hamiltonian vector fields:

|        | ham$_{2n}^0$ | ham$_{2n}^1$ | ham$_{2n}^1$ |
|--------|--------------|--------------|--------------|
| older  | ham$_{2n}^0$ | ham$_{2n}^1$ |              |
|        | $\oplus_{\ell=1}^{\infty} S^{\ell}$ | $\oplus_{\ell=2}^{\infty} S^{\ell}$ | $\oplus_{\ell=3}^{\infty} S^{\ell}$ |

where $S^{\ell}$ means the set of $\ell$-th homogeneous polynomials, to [3] or [4].

(2) An announcement of Betti numbers for the weight 20.

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2 Preliminaries

2.1 Lie algebra Cohomology and Gel’fand-Kalinin-Fuks cohomology

Take a Lie algebra \( g \) over \( \mathbb{R} \). Let \( (\rho, W) \) be a representation of \( g \). Namely, \( \rho \) is a Lie algebra homomorphism of \( g \) into the Lie algebra \( \text{End}(W) \). For each \( k \in \mathbb{Z}_{\geq 0} \),

\[
C^k(g) := \{ \sigma : g \times \cdots \times g \to W | \text{alternative and } \mathbb{R}\text{-multilinear} \}.
\]

For each \( k \)-th cochain \( \sigma \in C^k(g) \), we define

\[
(d\sigma)(X_0, \ldots, X_k) := \sum_{i=0}^{k} (-1)^i \rho(X_i)\sigma(\ldots, \hat{X}_i, \ldots) + \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], \ldots, \hat{X_i}, \ldots, \hat{X_j}, \ldots)
\]

it is known that \( d \) satisfies \( d^2 = 0 \) and defines the cohomology groups of a Lie algebra \( g \) with respect to \( (\rho, W) \). In this paper, hereafter we only deal with the trivial representation of Lie algebra, i.e., \( W = \mathbb{R} \) and \( \rho = 0 \).

Let \( \mathfrak{t} \) be a subalgebra of \( g \). Define

\[
C^m(g, \mathfrak{t}) := \{ \sigma \in C^m(g) | i_X \sigma = 0, i_X d\sigma = 0 \ (\forall X \in \mathfrak{t}) \}
\]

and we get the relative cohomology groups \( H^m(g, \mathfrak{t}) \). Let \( K \) be a Lie group of \( \mathfrak{t} \). Then we also consider

\[
C^m(g, K) := \{ \sigma \in C^m(g) | i_X \sigma = 0 \ (\forall X \in \mathfrak{t}), Ad_f^k \sigma = \sigma \ (\forall k \in K) \}
\]

and we get the relative cohomology groups \( H^m(g, K) \). If \( K \) is connected, those are identical. If \( K \) is a closed subgroup of \( G \), then \( C^* (g, K) = \Lambda^* (G/K)^G \) (the exterior algebra of \( G \)-invariant differential forms on \( G/K \)).

Take a differentiable manifold \( M \) and consider the space \( \mathfrak{X}(M) \) of vector fields of \( M \). Then \( \mathfrak{X}(M) \) forms a Lie algebra by Jacobi-Lie bracket. Thus, we can consider the Lie algebra cohomology of \( \mathfrak{X}(M) \). But, the cochain complex is huge, so we add some restriction, “continuity” by \( C^\infty\)-topology. Let \( g \) be a subalgebra of \( \mathfrak{X}(M) \). For each \( k \in \mathbb{Z}_{\geq 0} \),

\[
C^k(g) := \{ \sigma : g \times \cdots \times g \to \mathbb{R} | \text{continuous, alternative and } \mathbb{R}\text{-multilinear} \}.
\]

For each \( k \)-th cochain \( \sigma \in C^k(g) \), we define

\[
(d\sigma)(X_0, \ldots, X_k) := \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], \ldots, \hat{X_i}, \ldots, \hat{X_j}, \ldots).
\]

It is known that \( d \) satisfies \( d^2 = 0 \) and defines the cohomology group, called Gel’fand-Kalinin-Fuks cohomology of \( \mathfrak{X}(M) \). There are also relative versions.

2.2 Recall of Hamilton formalism

Let \( (M, \omega) \) be a symplectic manifold, namely, \( \omega \) is a non-degenerate closed 2-form on \( M \), and so \( \dim M \) is even. We denote the group of symplectic automorphisms of \( (M, \omega) \) by \( Aut(M, \omega) \). By \( \mathfrak{aut}(M, \omega) \), we denote the space of vector fields on \( M \) satisfying \( \mathcal{L}_X \omega = 0 \) (infinitesimal automorphism of \( \omega \)).

Since \( \mathcal{L}_{[X, Y]} = [\mathcal{L}_X, \mathcal{L}_Y] \) holds, \( \mathfrak{aut}(M, \omega) \) forms a Lie algebra.

For \( \forall f \) on \( M \), we have the Hamiltonian vector field on \( (M, \omega) \) defined by \( \omega(\mathcal{H}_f, \cdot) = df \), and the Poisson bracket given by \( \{f, g\} := \omega(\mathcal{H}_f, \mathcal{H}_g) \). Since \( \mathcal{H}_{\{f, g\}} = -[\mathcal{H}_f, \mathcal{H}_g] \), holds, the Hamiltonian
vector fields of $(M, \omega)$ form a Lie subalgebra of $\text{aut}(M, \omega)$. The correspondence $f \mapsto -\mathcal{H}_f$ is a Lie algebra homomorphism and the kernel is $\mathbb{R}$ when $M$ is connected.

It holds $\varphi \mathcal{H}_f = \mathcal{H}_{f \circ \varphi^{-1}}$ for each $\varphi \in \text{Aut}(M, \omega)$, and $\{f, g\} \circ \varphi = \{f \circ \varphi, g \circ \varphi\}$ for $f, g \in C^\infty(M)$.

Let $K$ be a Lie subalgebra of $\text{Aut}(M, \omega)$ with its Lie algebra $\mathfrak{k}$. For each $\xi \in \mathfrak{k}$, the fundamental vector field on $M$, say $\xi_M$, is defined by $\xi_M := \frac{d}{dt} \exp(t\xi)|_{t=0}$. They satisfy $\mathcal{L}_{\xi_M} \omega = 0$ and $[\xi, \eta]_M = -[\xi_M, \eta_M]$, thus form a subalgebra of $\text{aut}(M, \omega)$. The momentum mapping $J$ (if exists) is a map from $M \rightarrow \mathfrak{k}^*$ satisfying

$$d\dot{J}(\xi) = \omega(\xi_M, \cdot), \quad \text{i.e.,} \quad \xi_M = \mathcal{H}_J(\xi)$$

where $\dot{J}(\xi)$ is defined as $\langle \dot{J}(\xi), m \rangle := \langle \xi, J(m) \rangle$ for $\xi \in \mathfrak{k}, m \in M$. The above definition means $J$ provides with a Hamiltonian potential for each fundamental vector field of $K$ and $J(\xi)$ is defined as $(\dot{J}(\xi), m) := (\xi, J(m))$ for $\xi \in \mathfrak{k}, m \in M$. If $J$ is $K$-equivariant, i.e., $J(a \cdot m) = Ad_a^*(J(m))$ for $\forall a \in K, m \in M$, then $\dot{J}([\xi, \eta]) = \{\dot{J}(\xi), \dot{J}(\eta)\}$ hold for $\xi, \eta \in \mathfrak{k}$, and vice versa if $K$ is connected.

In local, by Darboux’s theorem we always have a local coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$ such that $\omega = \sum_{i=1}^n \left( \frac{\partial}{\partial q^i} \right) \wedge \left( \frac{\partial}{\partial p_i} \right) = -1$, the others are 0 and so the Hamiltonian vector field is given by

$$\mathcal{H}_f = \sum_{i=1}^n \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

and the Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^n \frac{\partial (f, g)}{\partial (q^i, p_i)} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right).$$

If $M = (\mathbb{R}^{2n}, \text{linear symplectic structure})$, the space of the Hamiltonian vector fields of $(M, \omega)$ coincides with $\text{aut}(M, \omega)$, because of the space is connected and 1-connected. Let $K$ be the linear symplectic group of $(\mathbb{R}^{2n}, \omega)$ with its Lie algebra $\mathfrak{k}$, i.e., $K = \text{Sp}(2n, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{sp}(2n, \mathbb{R})$. The equivariant (co-)momentum mapping is given by

$$\dot{J}(\xi)(q, p) = -\frac{1}{2} (q, p) (\text{matrix representation of } \omega) \xi \begin{pmatrix} q \\ p \end{pmatrix}$$

here $q$ is the natural coordinate of $\mathbb{R}^n$, $\dot{J}$ is a Lie algebra homomorphism from the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ into $C^\infty(M)$, with the Poisson bracket. The Hamilton potential of Hamiltonian vector field $[\xi_M, \mathcal{H}_f]$ is given by $-\{\dot{J}(\xi), f\}$, because of $[\xi_M, \mathcal{H}_f] = [\mathcal{H}_J(\xi), \mathcal{H}_f] = -\mathcal{H}_{\{\dot{J}(\xi), f\}}$.

### 3 Cochain complexes, weight and relative cochains

#### 3.1 Cochain complexes

When $M = \mathbb{R}^{2n}$, the Hamiltonian potential for each Hamiltonian vector field is unique up to constant, we consider the Lie subalgebra $\text{ham}_{2n}$ of Hamiltonian vector fields is isomorphic with the formal polynomial space. Then the space is Lie algebra isomorphic with the space of $\mathbb{R}[[q^1, \ldots, q^n, p_1, \ldots, p_n]]/\mathbb{R}$ quotiented by $\mathbb{R}$, where the Lie bracket is given by the Poisson bracket. We see that

$$\mathbb{R}[[q^1, \ldots, q^n, p_1, \ldots, p_n]]/\mathbb{R} = \text{completion of } \bigoplus_{\ell=1}^{\infty} S^\ell$$

where $S^\ell$ is the $\ell$-th symmetric power of $q^1, \ldots, q^n, p_1, \ldots, p_n$. It holds $\{S^k, S^\ell\} \subset S^{k+\ell-2}$, because the Poisson bracket satisfy $\{q^i, p_j\} = -\{p_j, q^i\} = \delta^i_j$ others are 0, and $\dot{J} : \mathfrak{sp}(2n, \mathbb{R}) \rightarrow S^2$ is a Lie algebra isomorphism.
If we denote the dual of $S^\ell$ by $\mathcal{S}_\ell$, then we see the first cochain complex is

$$C^1_{\text{GF}}(\mathfrak{ham}_{2n}) \cong \mathfrak{ham}_{2n}^* = \text{the completion of } \bigoplus_{\ell=1}^{\infty} \mathcal{S}_\ell$$

and the second cochain complex is

$$C^2_{\text{GF}}(\mathfrak{ham}_{2n}) \cong \mathfrak{ham}_{2n}^* \wedge \mathfrak{ham}_{2n}^* = \bigoplus_{\ell=1}^{\infty} \mathcal{S}_\ell \wedge \bigoplus_{k=1}^{\infty} \mathcal{S}_k = \bigoplus_{1 \leq k \leq \ell} \mathcal{S}_k \wedge \mathcal{S}_\ell$$

and so on, now on we omit the comment of “the completion” for simplicity.

### 3.2 Weight of cochains

**Definition 3.1 (cf.[4])** Define the **weight** of each element of $\mathcal{S}_\ell$ by $\ell - 2$.

For each element of $\mathcal{S}_{\ell_1} \wedge \mathcal{S}_{\ell_2} \wedge \cdots \mathcal{S}_{\ell_s}$, define its **weight** by $\sum_{i=1}^{s} (\ell_i - 2)$.

**Remark 3.2** Let $\sigma \in \mathcal{S}_\ell$ be a 1-cochain. Since $(d \sigma)(f_0, f_1) = -\langle \sigma, \{f_0, f_1\} \rangle$, the contribution of $\sigma$ is when the case of $\{f_0, f_1\} \in S^{\ell}$. If $f_0 \in S^{p_0}$ and $f_1 \in S^{p_1}$, then it must hold $p_0 + p_1 = \ell$, namely, $d \mathcal{S}_\ell \subset \sum_{p_0 + p_1 = 2 + \ell} \mathcal{S}_{p_0} \wedge \mathcal{S}_{p_1} \wedge \mathcal{S}_{p_2}$.

$p_0 + p_1 - 2 = \ell$ is equivalent to $(p_0 - 2) + (p_1 - 2) = \ell - 2$, and $p_0 + p_1 + p_2 = k + \ell + 2$ is $(p_0 - 2) + (p_1 - 2) + (p_2 - 2) = (k - 2) + (\ell - 2)$. These show the reason of the definition of **weight** above. And we also see that the coboundary operator $d$ preserve the weight, namely if a cochain $\sigma$ is of weight $w$, then $d(\sigma)$ is also of weight $w$.

Now we can decompose the cochain complex by the weight $w$ as follows:

$$C^*_{\text{GF}}(\mathfrak{ham}_{2n}) := \sum_{\text{w-condition}} \Lambda^{k_1} \mathcal{S}_1 \otimes \Lambda^{k_2} \mathcal{S}_2 \otimes \cdots$$

where w-condition is $\sum_{j=1}^{\infty} (j - 2)k_j = w$. $C^*_{\text{GF}}(\mathfrak{ham}_{2n}) \cong \sum_{w=-2n}^{\infty} C^*_{\text{GF}}(\mathfrak{ham}_{2n})_w$. Since the coboundary operator $d$ preserves the weights and so we have the natural splitting of cohomology groups like

$$H^*_{\text{GF}}(\mathfrak{ham}_{2n}) \cong \sum_{w=-2n}^{\infty} H^*_{\text{GF}}(\mathfrak{ham}_{2n})_w$$

In order to investigate $H^m_{\text{GF}}(\mathfrak{ham}_{2n})_w$, we have to handle

$$C^m_{\text{GF}}(\mathfrak{ham}_{2n})_w := \sum \Lambda^{k_1} \mathcal{S}_1 \otimes \Lambda^{k_2} \mathcal{S}_2 \otimes \cdots$$

$\sum_{j=1}^{\infty} k_j = m$ and $\sum_{j=1}^{\infty} (j - 2)k_j = w$. We have to be careful of no contribution of $(k_2)$-term to $w$.

Let $\mathfrak{ham}^0_{2n}$ be the space of the Hamiltonian vector fields which vanish at the origin of $\mathbb{R}^{2n}$. Then we have

$$\mathfrak{ham}_{2n} \cong \bigoplus_{\ell=1}^{\infty} S^\ell, \quad \mathfrak{ham}^0_{2n} \cong \bigoplus_{\ell=2}^{\infty} S^\ell \quad \text{and} \quad \mathfrak{ham}^1_{2n} \cong \bigoplus_{\ell=3}^{\infty} S^\ell.$$  

We look for the relative $H^*_{\text{GF}}(\mathfrak{ham}_{2n}^0, \mathfrak{sp}(2n, \mathbb{R}))_w$.  


Remark 3.3 ([4]) If $w$ is odd then $C^*_\text{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w = \{0\}$.

\[
\begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} = -\text{Id} \text{ is an element of } \mathfrak{sp}(2n, \mathbb{R}) \text{ and each cochain } \sigma \text{ should be invariant under the action of } -\text{Id}.
\]

Thus

\[
\forall \sigma \in C^m_{\text{GF}}(\mathfrak{ham}_{2n})_w := \sum \Lambda^{k_1} \mathfrak{g}_1 \otimes \Lambda^{k_2} \mathfrak{g}_2 \otimes \cdots , \text{ where } \sum_{j=1}^\infty k_j = m \text{ and } \sum_{j=1}^\infty (j-2)k_j = w
\]

we see that

\[
\sigma = (-\text{Id}) \cdot \sigma = (-1)^{k_1 +1} (-1)^{k_2 +1} \cdots \sigma
\]

\[
= (-1)^{k_1 + k_3 + k_5 + k_7 + \cdots} \sigma = (-1)^{-k_1 + k_3 + k_5 + 5k_7 + \cdots} \sigma
\]

\[
= (-1)^{-k_1 + 0k_2 + k_3 + 2k_4 + 3k_5 + 4k_6 + 5k_7 + \cdots} \sigma = (-1)^w \sigma
\]

thus $\sigma = 0$ if $w$ is odd.

\[\square\]

Remark 3.4 When $n = 1$, the type of Metoki ([5]) and the weight here are related by 2 times of type is equal to weight.

We remember that

\[
C^m_{\text{GF}}(\mathfrak{ham}^0_{2n})_w = \sum \Lambda^{k_1} \mathfrak{g}_1 \otimes \Lambda^{k_2} \mathfrak{g}_2 \otimes \cdots , \text{ where } \sum_{j=1}^\infty k_j = m \text{ and } \sum_{j=1}^\infty (j-2)k_j = w \text{ and } k_1 = 0
\]

\[
C^m_{\text{GF}}(\mathfrak{ham}^1_{2n})_w = \sum \Lambda^{k_1} \mathfrak{g}_1 \otimes \Lambda^{k_2} \mathfrak{g}_2 \otimes \cdots , \text{ where } \sum_{j=1}^\infty k_j = m \text{ and } \sum_{j=1}^\infty (j-2)k_j = w \text{ and } k_1 = k_2 = 0
\]

3.3 Relativity

On our original Lie algebra of Hamiltonian vector fields $\mathcal{H}_f$ of polynomial potential functions $f$, we have the natural group action of $\mathfrak{sp}(2n, \mathbb{R})$

\[
a \cdot \mathcal{H}_f = \mathcal{H}_{f \circ a^{-1}}, \quad a \cdot f := f \circ a^{-1}.
\]

Definition 3.5 On the exterior algebra of the space of polynomial functions of $\mathbb{R}^{2n}$, the group $\mathfrak{sp}(2n, \mathbb{R})$ acts naturally by

\[
a \cdot (f_1 \wedge f_2 \wedge \cdots \wedge f_m) := (a \cdot f_1) \wedge \cdots \wedge (a \cdot f_m)
\]

where $f_j$ are polynomials. We define the infinitesimal action of $\xi \in \mathfrak{sp}(2n, \mathbb{R})$ by

\[
\xi \cdot (f_1 \wedge f_2 \wedge \cdots \wedge f_m) := \frac{d}{dt} (\exp(t\xi) \cdot (f_1 \wedge f_2 \wedge \cdots \wedge f_m))|_{t=0}.
\]

Remark 3.6 $f_1 \wedge f_2 \wedge \cdots \wedge f_m$ looks strange, but the corresponding real object is $\mathcal{H}_{-f_1} \wedge \mathcal{H}_{-f_2} \wedge \cdots \wedge \mathcal{H}_{-f_m}$. Thus the degree of $f_j$ is 1, and so $f \wedge g = -g \wedge f$ for each functions $f, g$.

Proposition 3.7 The above infinitesimal action by $\xi \in \mathfrak{sp}(2n, \mathbb{R})$ is a derivation of degree 0 and

\[
\xi \cdot f = \{\hat{J}(\xi), f\} \quad \text{for each polynomial } f.
\]

Proof: The first assertion of being derivation of degree 0 is trivial. The second assertion is:

\[
\xi \cdot f := \frac{d}{dt} \exp(t\xi) \cdot f|_{t=0} = \frac{d}{dt} f \circ \exp(-t\xi)|_{t=0} = \langle df, -\xi_M \rangle = \omega(\mathcal{H}_f, -\mathcal{H}_{\hat{J}(\xi)}) = \{\hat{J}(\xi), f\}.
\]
This corresponds to \( \xi \cdot \mathcal{H}_f := \frac{d}{dt} \exp(t \xi) \cdot \mathcal{H}_{f(t)} = 0 = \mathcal{L}_{-\xi \mathcal{H}_f} = -[\mathcal{H}_{J(\xi)}, \mathcal{H}_f] = \mathcal{H}_{(J(\xi), f)} \).

To determine relative cochain complex \( C^*_\text{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R})) \), there are two conditions, one is \( i_\xi \sigma = 0 \), and the other is \( i_\xi d \sigma = 0 \) for each \( m \)-cochain \( \sigma \), where \( i_\xi \) is the interior product with respect to \( \xi = \mathfrak{t} = \mathfrak{sp}(2n, \mathbb{R}) = S^2 \). Since \( i_\xi \) is a skew-derivation of degree \(-1\), in order to know the effect of \( i_\xi \), it is enough to know the operation of \( i_\xi \sigma \) for \( 1 \)-cochain \( \sigma \). Going back to Hamiltonian vector fields, we see that

\[
i_\xi \sigma = \langle \sigma, \xi_M \rangle = \langle \sigma, \mathcal{H}_{J(\xi)} \rangle = \langle \sigma, -\mathcal{J}(\xi) \rangle.
\]

In general, it holds

\[
(i_\xi \sigma)(f_1, f_2, \ldots) = \sigma(-\mathcal{J}(\xi), f_1, f_2, \ldots) \quad \text{for} \quad \sigma \in C^m_\text{GF}(\mathfrak{ham}_{2n}).
\]

**Proposition 3.8** Let \( \sigma \neq 0 \) and \( \sigma \in \Lambda^k \mathfrak{g}_1 \otimes \Lambda^k \mathfrak{g}_2 \otimes \Lambda^k \mathfrak{g}_3 \otimes \cdots \) (where \( \sum k_j = m \)). If \( i_\xi \sigma = 0 \) for \( \forall \xi \in \mathfrak{t} = \mathfrak{sp}(2n, \mathbb{R}) \), then \( \sigma \in \Lambda^k \mathfrak{g}_1 \otimes \Lambda^k \mathfrak{g}_3 \otimes \Lambda^k \mathfrak{g}_4 \otimes \cdots \), i.e., \( k_2 = 0 \).

**Proof:** Since \( i_\xi \sigma = -\langle \sigma, \mathcal{J}(\xi) \rangle \) for 1-cochain and \( \mathcal{J}(\xi) \in S^2 \), we see that \( i_\xi \sigma = 0 \) if \( \sigma \in \mathfrak{g}_2 \) with \( \ell \neq 2 \). If \( \sigma \in \mathfrak{g}_2 \) satisfies \( i_\xi \sigma = 0 \) for \( \forall \xi \in \mathfrak{sp}(2n, \mathbb{R}) \), then \( \sigma = 0 \in \mathfrak{g}_2 \) because \( S^2 = \mathfrak{sp}(2n, \mathbb{R}) \) is semi-simple and \( \mathcal{J}(\xi) \) generate \( S^2 \). Now we may rewrite

\[
\sigma = \sum_A \tau_A \wedge \rho_A
\]

where \( \tau_A \in \Lambda^k \mathfrak{g}_2, \rho_A \in \Lambda^k \mathfrak{g}_3 \otimes \Lambda^k \mathfrak{g}_4 \otimes \cdots \) and \( \rho_A \) are linearly independent. Using the fact \( i_\xi \rho_A = 0 \) for \( \forall \xi \in \mathfrak{t} \), we have \( 0 = i_\xi \sigma = \sum_A i_\xi \tau_A \wedge \rho_A \) and those imply that \( i_\xi \tau_A = 0 \) for \( \forall \xi \in \mathfrak{sp}(2n, \mathbb{R}) \) and \( A \).

If \( 0 < k_2 \leq \dim \mathfrak{g}_2 = n(2n + 1) \) then \( \tau_A = 0 \) for \( \forall A \) if necessary we can use \( i_{\xi_2} \cdots i_{\xi_1} \tau_A = 0 \). Thus \( \sigma = 0 \).

Thus, the condition \( i_\xi \sigma = 0 \) for \( \xi \in \mathfrak{sp}(2n, \mathbb{R}) \) imply that \( \mathfrak{g}_2 \) does not appear in \( C^*_\text{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R})) \). The other condition of being relative cochain is \( i_\xi d \sigma = 0 \) for each \( m \)-cochain \( \sigma \). Again going back to Hamiltonian vector fields, for each 1-cochain \( \sigma \), we see that

\[
\langle i_\xi d \sigma, \mathcal{H}_f \rangle = \langle (d \sigma)(\xi_M, \mathcal{H}_f) \rangle = -\langle \sigma, \xi_M, \mathcal{H}_f \rangle = -\langle \sigma, [\mathcal{H}_{J(\xi)}, \mathcal{H}_f] \rangle = \langle \sigma, \mathcal{H}_{(J(\xi), f)} \rangle
\]

and so

\[
\langle i_\xi d \sigma, f \rangle = \langle \sigma, \{ \mathcal{J}(\xi), f \} \rangle = \langle \sigma, \xi \cdot f \rangle
\]

for 1-cochain \( \sigma \), and \( \xi \in \mathfrak{sp}(2n, \mathbb{R}), f \in C^\infty(M) \).

For \( m \)-cochain \( \tau = \sigma_1 \wedge \cdots \wedge \sigma_m \) (\( \sigma_j \) are 1-cochains), since \( d \) is a skew-derivation of degree \(+1\), we have

\[
d\tau = \sum_{j=1}^m (-1)^{j+1} \sigma_1 \wedge \cdots \wedge d \sigma_j \wedge \cdots \wedge \sigma_m,
\]

and since the interior product \( i_\xi \) for each \( \xi \in \mathfrak{sp}(2n, \mathbb{R}) \) is a skew-derivation of degree \(-1\), we have

\[
(i_\xi \circ d) \tau = i_\xi \sum_{j=1}^m (-1)^{j+1} \sigma_1 \wedge \cdots \wedge d \sigma_j \wedge \cdots \wedge \sigma_m
\]

\[
= \sum_{\ell < j} (-1)^{\ell+1} (-1)^{j+1} \sigma_1 \wedge \cdots \wedge i_\xi \sigma_\ell \wedge \cdots \wedge d \sigma_j \wedge \cdots \wedge \sigma_m
\]

\[
+ \sum_j (-1)^{j+1} (-1)^{j+1} \sigma_1 \wedge \cdots \wedge (i_\xi \circ d) \sigma_j \wedge \cdots \wedge \sigma_m
\]

\[
+ \sum_{\ell > j} (-1)^{\ell+1} (-1)^{\ell+2} \sigma_1 \wedge \cdots \wedge d \sigma_j \wedge \cdots \wedge i_\xi \sigma_\ell \wedge \cdots \wedge \sigma_m
\]
If \( i_\xi \sigma_j = 0 \) \((j = 1, \ldots, m)\), then we have
\[
(i_\xi \circ d) \tau = \sum_j \sigma_1 \land \cdots \land ((i_\xi \circ d) \sigma_j) \land \cdots \land \sigma_m,
\]
namely, \( i_\xi \circ d \) is a derivation of degree 0. Thus, under the condition of \( i_\xi \sigma = 0 \) for any cochain \( \sigma \) and \( \xi \in \mathfrak{sp}(2n, \mathbb{R}) \), \( i_\xi \circ d \) becomes an ordinary derivation of degree 0.

**Proposition 3.9** Let \( \sigma \) be a m-cochain with \( i_\xi \sigma = 0 \) for \( \forall \xi \in \mathfrak{sp}(2n, \mathbb{R}) \). Then \( (i_\xi \circ d) \) behaves as derivation of degree 0 and characterized by \((i_\xi \circ d) \sigma, f\rangle = \langle \sigma, \{ \tilde{J}(\xi), f\} \rangle\) for each 1-cochain \( \sigma \) and \( \xi \in \mathfrak{sp}(2n, \mathbb{R}) \).

**Remark 3.10** It may be a better way to recall Cartan’s formula \( L_\xi = d \circ i_\xi + i_\xi \circ d \) in order to prove the above.

From Proposition 3.8 and the discussion above, we see that
\[
C_{GF}^m(\mathfrak{ham}_0^{2n}, \mathfrak{sp}(2n, \mathbb{R})) = \Lambda^m(\mathfrak{e}_3 \oplus \cdots) \mathfrak{Sp}(2n, \mathbb{R}) = \{ \sigma \in C_{GF}^m(\mathfrak{ham}_0^{2n}) | \mathfrak{Sp}(2n, \mathbb{R}) \text{-invariant} \}
\]
and so
\[
C_{GF}^m(\mathfrak{ham}_0^{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w = \{ \sigma \in C_{GF}^m(\mathfrak{ham}_0^{2n})_w | \mathfrak{Sp}(2n, \mathbb{R}) \text{-invariant} \}
\]
The group \( \mathfrak{Sp}(2n, \mathbb{R}) \) acts on cochain complexes as the dual action of that of on the exterior algebra of polynomial functions on \( \mathbb{R}^{2n} \). Thus, the precise definition is
\[
(a \cdot \sigma)(f_1, \ldots, f_m) = \sigma(a^{-1} \cdot f_1, \ldots, a^{-1} \cdot f_m) = \sigma(f_1 \circ a, \ldots, f_m \circ a)
\]
for a general m-cochain \( \sigma \).

**Remark 3.11** We re-confirm here that our coboundary operator \( d \) is compatible with the group-action of \( \mathfrak{Sp}(2n, \mathbb{R}) \), i.e., \( a \cdot d = d \circ a \) holds for each symplectic automorphism \( a \). It is enough only to show for 1-cochain \( \sigma \). We see that
\[
(d (a \cdot \sigma))(f, g) = -\langle a \cdot \sigma, \{ f, g \} \rangle = -\langle \sigma, \{ f, g \} \circ a^{-1} \rangle = -\langle \sigma, \{ f \circ a^{-1}, g \circ a^{-1} \} \rangle
\]
\[
= (d \sigma)(f \circ a^{-1}, g \circ a^{-1}) = (d \sigma)(a \cdot f, a \cdot g).
\]
for \( \forall a \in \mathfrak{Sp}(2n, \mathbb{R}) \).

**Definition 3.12** We define the infinitesimal action of \( \xi \in \mathfrak{sp}(2n, \mathbb{R}) \) for each cochain \( \sigma \) by
\[
\xi \cdot \sigma = \frac{d}{dt} \exp(\xi) \cdot \sigma |_{t=0}.
\]

**Proposition 3.13** The infinitesimal action \( \xi \in \mathfrak{sp}(2n, \mathbb{R}) \) preserves the cochain complex \( C_{GF}^m(\mathfrak{ham}_0^{2n}) \), behaves as an ordinary derivation of degree 0 and
\[
\langle \xi \cdot \sigma, f \rangle = -\langle \sigma, \xi \cdot f \rangle = -\langle \sigma, \{ \tilde{J}(\xi), f\} \rangle = -\langle (i_\xi d) \sigma, f \rangle.
\]
Thus, \( (i_\xi d) \sigma = -\xi \cdot \sigma \) holds for each m-cochain \( \sigma \) with \( i_\xi \sigma = 0 \) \((\xi \in \mathfrak{sp}(2n, \mathbb{R})\)\).

An advantage of the relation in Proposition 3.13 is that \( \xi : C_{GF}^m(\mathfrak{ham}_0^{2n}) \rightarrow C_{GF}^m(\mathfrak{ham}_0^{2n}) \) is a derivation of degree 0 with respect to the wedge product but also a derivation of degree 0 inside of 1-cochain. Namely, each 1-cochain is a linear combination of symmetric powers and since \( f \mapsto \{ \tilde{J}(\xi), f\} \) is a derivation for each polynomial function \( f \), we have
\[
\xi \cdot (e_1^{k_1} \cdots e_\ell^{k_\ell} \cdots e_{2n}^{k_{2n}}) = \sum_\ell e_1^{k_1} \cdots \xi \cdot (e_\ell^{k_\ell}) \cdots e_{2n}^{k_{2n}}
\]
\[
= \sum_\ell e_1^{k_1} \cdots (k_\ell \xi e_\ell^{k_\ell-1} \xi(e_\ell)) \cdots e_{2n}^{k_{2n}}.
\]
4 Decomposition of $m$-th cochain complex of weight $w$

Our concern in this section is, given a pair of positive integers $(m, w)$, find all possibilities of cochain complex by denoting the sequences $(k_3, k_4, \ldots)$ of non-negative integers of multiplicity which satisfies $\sum_{j \geq 3} k_j = m$ and $\sum_{j \geq 3} (j - 2)k_j = w$. We have to be careful about $\Lambda^k \mathfrak{S}_\ell = \{0\}$ may be happen when $k_{\text{ent}} > \dim \mathfrak{S}_\ell$, where $\dim \mathfrak{S}_\ell = (\ell + 2n - 1)!/((2n - 1)!)$.

By shifting the indices by $-2$, we rearrange our situation as below. Given a pair of non-negative integers $(m, w)$, we would like to find all sequences $(\hat{k}_1, \hat{k}_2, \ldots)$ of non-negative integers satisfying

$$\sum_{j \geq 1} \hat{k}_j = m \quad \text{and} \quad \sum_{j \geq 1} j\hat{k}_j = w \quad (3)$$

$m \leq w$ is a necessary condition and denote the set of sequences $(\hat{k}_1, \hat{k}_2, \ldots)$ satisfying (3) by $F(m, w)$, by subtracting the first equation of (3) from the second one of (3), we have

$$\sum_{j \geq 2} (j - 1)\hat{k}_j = w - m .$$

Thus, we have the all solutions of (3), $F(m, w)$ by the next recursive formula:

$$F(m, w) = \bigcup_{\hat{k}_1 = \max(0, 2m - w)}^m \{(\hat{k}_1, x) \mid x \in F(m - \hat{k}_1, w - m)\}$$

For a given $(m, w)$, we have the Maple script `gkf_act-1.mpl`, which shows us the all solutions of (3).

There is a primitive question “do the solutions exist for all $m \leq w$? or how many?”. We will give an answer to this question, here. We join our two equations into one equation as below:

$$w = \hat{k}_1 + 2\hat{k}_2 + \cdots + s\hat{k}_s$$

$$= (1 + \cdots + 1) + (2 + \cdots + 2) + \cdots + (s + \cdots + s)$$

$$= (s + \cdots + s) + \cdots + (2 + \cdots + 2) + (1 + \cdots + 1)$$

$$= \ell_1 + \ell_2 + \cdots + \ell_m$$

where $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m \geq 1$. This is a partition of $w$ with length $m$ or a Young diagram of height $m$ with $w$ cells. Conversely, for a partition of $w$

$$w = \ell_1 + \ell_2 + \cdots + \ell_m$$

$$\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m \geq 1$$

$$\hat{k}_i := \#\{j \mid \ell_j = i\}$$ gives a solution of (3). That means there is a one-to-one correspondence between the solution of (3) and all the partitions of $w$ with length $m$ or a Young diagram of height $m$ with $w$ cells.

Remark 4.1 Be careful of difference of definition of $\hat{k}_i := \#\{j \mid \ell_j = i\}$ and the conjugate Young diagram $c_i := \#\{j \mid \ell_j \geq i\}$.

Proposition 4.2 By $r(m, w)$ we mean the number of solutions of (3).

For integers $m > 0$, $w \geq 0$, we define $\bar{r}(m, w)$ the number of solutions of

$$w = \ell_1 + \ell_2 + \cdots + \ell_m$$

$$\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m \geq 0$$
By an elementary observation below

\[ w = \ell_1 + \ell_2 + \cdots + \ell_m \quad \text{with} \quad \ell_i \geq 0 \quad \text{and} \quad w + m = (\ell_1 + 1) + (\ell_2 + 1) + \cdots + (\ell_m + 1) \quad \text{with} \quad (\ell_i + 1) > 0 \]

we see \( \tilde{r}(m, w) = r(m, w + m) \). The generating function of \( \tilde{r}(m, w) \) is known as

\[
\sum_{c=0}^{\infty} \tilde{r}(m, c)x^c = \prod_{k=1}^{m} \left( \frac{1}{1 - x^k} \right).
\]

**Example 4.3** Let us try to know all possibilities when weight=2 case.

When \( m = 1 \), then \( \ell_1 = 2 \), so we have \( k_2 = 1 \) and \( k_j = 0 \) \((j \neq 2)\). Thus, \( k_4 = 1 \).

When \( m = 2 \), then \( 2 = \ell_1 + \ell_2 \ (\ell_1 \geq \ell_2 \geq 1) \), \( \ell_1 = \ell_2 = 1 \), so we have \( \hat{k}_1 = 2 \) and \( \hat{k}_j = 0 \) \((j \neq 1)\). Thus, \( k_3 = 2 \). These show

\[
\begin{align*}
C_{GF}^1(\text{ham}_2^1) &= \mathcal{G}_4 \\
C_{GF}^2(\text{ham}_2^2, \text{sp}(2n, \mathbb{R})) &= \mathcal{G}_4^{Sp(2n, \mathbb{R})} = \{0\} \\
C_{GF}^1(\text{ham}_2^1) &= \Lambda^2 \mathcal{G}_3 \\
C_{GF}^2(\text{ham}_2^1, \text{sp}(2n, \mathbb{R})) &= (\Lambda^2 \mathcal{G}_3)^{Sp(2n, \mathbb{R})}
\end{align*}
\]

In the above, \( \mathcal{G}_4 \) is an irreducible representation of \( Sp(2n, \mathbb{R}) \) and so \( \mathcal{G}_4^{Sp(2n, \mathbb{R})} = \{0\} \). Also, concerning \( \Lambda^2 \mathcal{G}_3 \), if \( n=1 \) we know \( \Lambda^2 \mathcal{G}_3 = \mathcal{G}_6 \oplus \mathcal{G}_4 \) as we will see in Example 5.1. If \( n=2 \) then by the help of Littlewood-Richardson rule, we get a little complex expression \( \Lambda^2 \mathcal{G}_3 = \mathcal{G}_9 \oplus \mathcal{G}_7 \oplus V_{(1,1)} \oplus V_{(2,2)} \oplus V_{(3,3)} \oplus V_{(5,1)} \), where \( V_{(p,q)} \) is the irreducible representation of the natural action of \( Sp(4, \mathbb{R}) \) on \( \mathbb{R}^4 \), corresponding the Young diagram \((p,q)\), and \( V_{(p,0)} = \mathcal{G}_p \). Thus, if \( n=1 \) or \( 2 \), then we see \( \text{dim} \mathcal{C}_{GF}^2(\text{ham}_2^1, \text{sp}(2n, \mathbb{R})) = 1 \). Actually, we get a lot of help from representation theory in this project.

**Example 4.4** weight=4 case:

When \( m = 2 \), i.e., \( 4 = \ell_1 + \ell_2 \ (\ell_1 \geq \ell_2 \geq 1) \), then \( (\ell_1, \ell_2) = (3,1) \) or \( (2,2) \), so we have \( (\hat{k}_1 = 1, \hat{k}_3 = 1) \), or \( \hat{k}_2 = 2 \). Thus, \( k_3 = 1 \) or \( k_4 = 2 \). When \( m = 3 \), i.e., \( 4 = \ell_1 + \ell_2 + \ell_3 \ (\ell_1 \geq \ell_2 \geq \ell_3 \geq 1) \), \( \ell_1 = 2, \ell_2 = 1, \ell_3 = 1 \). Thus \( (\hat{k}_2 = 1, \hat{k}_3 = 2) \), so \( k_3 = 2, k_4 = 1 \).

These show

\[
\begin{align*}
C_{GF}^1(\text{ham}_2^1) &= \mathcal{G}_6 \\
C_{GF}^2(\text{ham}_2^1) &= \mathcal{G}_4 \oplus \mathcal{G}_9 \oplus \Lambda^2 \mathcal{G}_4 \\
C_{GF}^4(\text{ham}_2^1) &= \Lambda^2 \mathcal{G}_3 \oplus \mathcal{G}_4 \\
C_{GF}^5(\text{ham}_2^1) &= \Lambda^4 \mathcal{G}_3
\end{align*}
\]

**Example 4.5** By the same way, we have weight=6 case:

\[
\begin{align*}
C_{GF}^1(\text{ham}_2^1) &= \mathcal{G}_8 \\
C_{GF}^2(\text{ham}_2^1) &= (\mathcal{G}_3 \oplus \mathcal{G}_7) \oplus (\mathcal{G}_4 \oplus \mathcal{G}_6) \oplus \Lambda^2 \mathcal{G}_5 \\
C_{GF}^3(\text{ham}_2^1) &= (\Lambda^2 \mathcal{G}_3 \oplus \mathcal{G}_6) \oplus (\mathcal{G}_3 \oplus \mathcal{G}_4 \oplus \mathcal{G}_5) \oplus \Lambda^3 \mathcal{G}_4 \\
C_{GF}^4(\text{ham}_2^1) &= (\Lambda^3 \mathcal{G}_3 \oplus \mathcal{G}_5) \oplus (\Lambda^2 \mathcal{G}_3 \oplus \Lambda^2 \mathcal{G}_4) \\
C_{GF}^5(\text{ham}_2^1) &= \Lambda^4 \mathcal{G}_3 \oplus \mathcal{G}_4 \\
C_{GF}^6(\text{ham}_2^1) &= \Lambda^6 \mathcal{G}_3
\end{align*}
\]

If \( n = 1 \), then \( \text{dim} \mathcal{G}_3 = 4 \) and so we have \( \text{dim} \mathcal{C}_{GF}^6(\text{ham}_2^1) = \{0\} \).

We express the data which we got by the next table in short form, dividing into direct sum components. Our abbreviation rule is that

1. only pick up the \( i \)'s with \( k_i > 0 \),
2. if \( k_i > 1 \) then express the multiplicity by the power like \( i^{k_i} \), i.e., \((i^k, j^k, \ldots)\),
3. if \( k_i = 1 \) then only write \( i \)
Using the rule, Example 4.5 above with weight=6 can be written in the next table:

| degree | ref. # | type |
|--------|--------|------|
| 1      | 1      | (8)  |
| 2      | 1      | (3 7) |
| 2      | 2      | (4 6) |
| 3      | 3      | (5^3) |

| degree | ref. # | type |
|--------|--------|------|
| 3      | 1      | (3^2 6) |
| 2      | 2      | (3 4 5) |
| 3      | 3      | (4^3)  |
| 4      | 1      | (3^2 5) |
| 5      | 1      | (3^2 4) |
| 6      | 1      | (3^3)  |

5 Aid from Representation theory of $Sp(2n, \mathbb{R})$

In order to compute the relative cohomology groups, we have to know some basis of the cochain complex and the concrete matrix representation of the coboundary operator, and that rank. Fortunately, in the case of $C^m_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w$, there is a very sophisticated way to know the dimension without knowing basis. Namely, we consider the natural representation of $Sp(2n, \mathbb{R})$ acting on $\mathbb{R}^{2n}$.

Then the all irreducible representations are known by Young diagram of length at most $n$. Also, the $p$-th symmetric tensor product of $\mathbb{R}^{2n}$ is an irreducible representation and is identified with $\mathfrak{S}_p$ or its dual in this paper. The corresponding Young diagram is $(p) = \begin{array}{cccc} \cdot & \cdot & \cdots & \cdot \end{array}$. Since we have (2), if we know the index of the trivial representation in $C^m_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w$ by some help of representation theory of $Sp(2n, \mathbb{R})$, that is just equal to $\dim C^m_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w$. Thus, we can calculate the Euler characteristic number without knowing cohomology data at least theoretically.

When $n = 1$, the representation theory is rather clear and that is very helpful as we see later.

5.1 Assistance of $Sp(2, \mathbb{R})$

Instead of the variables $(q^1, p_1)$ in $\mathbb{R}^2$, we use the classical notation $(x, y)$, and let $z^a_k \in S^k$ and the dual basis in $\mathfrak{S}_k$ of $z^a_k$ by the notation $z^a_k$. Now, the Poisson bracket $\{f, g\}$ is the Jacobian $\frac{\partial(f, g)}{\partial(x, y)}$ and get the relations $\{xy, x^2\} = -2x^2$, $\{xy, y^2\} = 2y^2$, $\{x^2, y^2\} = 4xy$ and so we have the correspondence below with the famous matrices through the momentum mapping $J$ of the natural symplectic action $Sp(2, \mathbb{R})$:

\[
xy \leftrightarrow H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{y^2}{2} \leftrightarrow X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \frac{x^2}{2} \leftrightarrow Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

and we have already seen that the component $S^2$ is a subalgebra of $\mathfrak{ham}^0_2$ and isomorphic to $\mathfrak{sp}(\mathbb{R}^2, \omega) \cong \mathfrak{sl}(2, \mathbb{R})$.

It is well-known that $S^\ell$ or the (symplectic) dual space $\mathfrak{S}_\ell$ are the irreducible representations of $Sp(2, \mathbb{R})$ and Clebsch-Gordan rule, for instance, an irreducible decomposition of tensor product is the following:

\[
\mathfrak{S}_k \otimes \mathfrak{S}_\ell = \mathfrak{S}_{k+\ell} \oplus \mathfrak{S}_{k+\ell-2} \oplus \cdots \oplus \mathfrak{S}_{|k-\ell|}
\]

Since $C^m_{GF}(\mathfrak{ham}^0_2)_w = \sum_{k_j=m, \sum(j-2)k_j=w} (\Lambda^{k_2} \mathfrak{S}_2 \otimes \Lambda^{k_4} \mathfrak{S}_4 \otimes \cdots)$, if we know the irreducible decomposition of $\Lambda^{k} \mathfrak{S}_\ell$, then after tensor product-ing those, we have the complete irreducible decomposition, and can pick up the trivial representation.

Example 5.1 As shown in Example 4.3, $\Lambda^2 \mathfrak{S}_3$ is a component of $C^2_{GF}(\mathfrak{ham}^1_{2n})_2$, we decompose $\Lambda^2 \mathfrak{S}_3$ into irreducible components when $n = 1$. $z^0_3 \wedge z^{\ell_3}_3 \quad (0 \leq \ell_1 < \ell_2 < 3)$ are a basis of $\Lambda^2 \mathfrak{S}_3$. We will find the weight vector space from

\[
T = \sum_{0 \leq \ell_1 < \ell_2 < 3} c_{\ell_1, \ell_2} z^{\ell_1}_3 \wedge z^{\ell_2}_3
\]
which must be zero by the dual action of $X$. Since our action is $z_3^0 \mapsto 3z_3^1$, $z_3^1 \mapsto 2z_3^2$, $z_3^2 \mapsto 1z_3^3$, and $z_3^3 \mapsto 0$, we have

$$0 = c_{01}(3z_3^1 \wedge z_3^1 + z_3^2 \wedge 2z_3^2) + c_{02}(3z_3^1 \wedge z_3^1 + z_3^3 \wedge 3z_3^3) + c_{03}(3z_3^1 \wedge z_3^1 + 3z_3^2 \wedge 3z_3^2) + \cdots$$

By solving a homogeneous linear equations, we get $T = c_{03}(z_3^0 \wedge z_3^3 - 3z_3^1 \wedge z_3^2) + c_{23}z_3^2 \wedge z_3^3$, and $z_3^0 \wedge z_3^3 - 3z_3^1 \wedge z_3^2$ is the highest weight vector of $S_0$ and $z_3^2 \wedge z_3^3$ is of $S_4$, and we have $\Lambda^2S_3 = S_0 \oplus S_4$. Thus, when $n = 1$ and the weight=2 case, we add our decomposition and we see

$$C_{GF}^1(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_2 = S_{Sp}^{2p(2, \mathbb{R})} = \{0\}$$

$$C_{GF}^0(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_2 = (\Lambda^0S_3)^{Sp(2, \mathbb{R})} = (S_0 \oplus S_4)^{Sp(2, \mathbb{R})} = S_0 = \mathbb{R}$$

since $S_{q}^{Sp(2n, \mathbb{R})} = \{0\}$ for $q > 2$.

**Example 5.2** $\Lambda^2S_4$ is a component of $C_{GF}^2(\text{ham}_2^0, S_4)$ in Example 4.4. When $n = 1$, we can decompose it into the irreducible components as the same way above, and get $\Lambda^2S_4 = S_2 \oplus S_6$.

When $n = 1$, since we have the next decomposition $\Lambda^qS_5 \cong S_9 \oplus S_5 \oplus S_3$, we have no rule of negative 4-step descending nor 2-step descending for $\Lambda^qS_5$.

We would like to see the effect of tensor product.

**Example 5.3** When weight=6, we know that $C_{GF}^2(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_6$ has 3 components $(3, 7), (4, 6), (5, 2)$, and $C_{GF}^4(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_6$ has 3 components $(3^2, 3), (4^2, 1), (5^2, 1)$, and $C_{GF}^6(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_6$ has 1 component $(3^4, 1)$. Assume $n = 1$. Concerning with degree 2 cases, since $S_3 \otimes S_7 = S_4 \oplus \cdots \oplus S_{10}$ and $S_4 \otimes S_6 = S_2 \oplus \cdots \oplus S_{10}$, we see that

$$[S_3 \otimes S_7, S_9] = 0 \quad \text{and} \quad [S_4 \otimes S_6, S_9] = 0$$

On the other hand, we get $\Lambda^2S_5 = S_9 \oplus S_4 \oplus S_8$, we have

$$[\Lambda^2S_5, S_9] = 1$$

degree 3: (3.1)-case: $\Lambda^2(S_3) \otimes S_6 = (S_0 \oplus S_4) \otimes S_6 = S_6 + (S_2 + \cdots + S_{10})$, and so

$$[\Lambda^2(S_3) \otimes S_6, S_9] = 0$$

(3.2)-case: $S_3 \otimes S_4 \otimes S_5 = (S_1 + S_3 + S_5 + S_7) \otimes S_5 = (S_4 + S_6) + (S_2 + S_4 + S_6 + S_8) + (S_0 + S_2 + S_4 + S_6 + S_8 + S_{10})$

$$+ (S_2 + S_4 + S_6 + S_{10} + S_{12})$$

we see that

$$[S_3 \otimes S_4 \otimes S_5, S_9] = 1$$

Since $\Lambda^2S_4 = S_2 + S_6$,

$$[\Lambda^2S_4, S_9] = 0$$

degree 4: (4.1)-case: $\Lambda^4S_3 \otimes S_5 = S_3 \otimes S_5 = S_2 + \cdots + S_8$, here we used the property that $\Lambda^4W = \Lambda^{\dim W-k}W$ in general, and

$$[(\Lambda^3S_3) \otimes S_5, S_9] = 0$$
(4.2)-case: \((\Lambda^2 S_3) \otimes (\Lambda^2 S_4) = (S_0 + S_4) \otimes (S_2 + S_6) = (S_2 + S_6) + (S_2 + S_4 + S_6) + (S_2 + \cdots + S_{10})\),
\[ [\Lambda^2 S_3] \otimes (\Lambda^2 S_4), S_0] = 0 \]

(5)-case: \((\Lambda^4 S_3) \otimes S_4 = S_0 \otimes S_4 = S_4\) because of \(\dim S_3 = 4\),
\[ [\Lambda^4 S_3] \otimes S_4, S_0] = 0 \]

We add those facts in the table in Example 4.5.

| degree | ref.# | type | dim |
|--------|-------|------|-----|
| 1      | 1     | (8)  | 0   |
| 2      | 1     | (3 7)| 0   |
| 2      | 1     | (4 6)| 0   |
| 3      | 1     | (5^2)| 1   |
| 3      | 1     | (3^2 6)| 0   |
| 4      | 1     | (3^4 5)| 1   |
| 5      | 1     | (3^4 4)| 0   |
| 6      | 1     | (3^6)| 0   |

We summarize the ideas, so far. For a given weight \(w\) and degree \(m\), we have the complete list of subcomplex of type \((k_3, k_3, \ldots, k_n)\) such that \(\sum_j (j - 2)k_j = w\) and \(\sum_j k_j = w\) and

\[ C^m(\text{ham}_2^0)w = \oplus \Lambda^{k_3} S_3 \otimes \Lambda^{k_4} S_4 \otimes \cdots \Lambda^{k_n} S_n \]

It is possible to decompose \(\Lambda^p S_q\) into irreducible subspaces, say, \(\Lambda^p S_q = \sum \alpha^p_{qr} S_r\), where \(\alpha^p_{qr} \in \mathbb{Z}_{\geq 0}\).

(But, it is not clear if those \(\alpha^p_{qr}\) have some “rule”.) Since we have the tensor product formula, called Clebsch-Gordan rule,

\[ \mathcal{S}_p \otimes \mathcal{S}_q = \mathcal{S}_{|p-q|} \oplus \mathcal{S}_{|p-q|+2} \oplus \cdots \mathcal{S}_{p+q} \]

we can decompose \(\Lambda^{k_3} S_3 \otimes \Lambda^{k_4} S_4 \otimes \cdots \Lambda^{k_n} S_n\) into irreducible components, thus we can divide \(C^m_{GF}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))w\) into irreducible components, and therefore we can pick up the multiplicity of the trivial representation, and we know the dimension of \(C^m_{GF}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))w\) as that multiplicity.

**Theorem 5.4** There is a sequence of computer programs which follow the mathematical story above.

If we input the weight \(w\), then we can get \(\dim C^m_{GF}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))w\) and the precise contributions of subcomplex, and the Euler characteristic number. If \(w\) is big, we may face some trouble of shortage of memory and so on. Nevertheless, so for we have a table of dimensions of relative cochain complex until \(w \leq 20\). The table below means the horizontal direction is the degree of cochain complex and the descending vertical direction means increasing weight.

| \(w\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | \(\chi\) |
|------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|------|
| 2    | 0 | 1 |   |   |   |   |   |   |   |    |    |    |    |    | 2    |
| 4    | 0 | 0 | 1 | 1 |   |   |   |   |   |    |    |    |    |    | 1    |
| 6    | 0 | 1 | 1 | 0 | 0 |   |   |   |   |    |    |    |    |    | 1    |
| 8    | 0 | 0 | 4 | 5 | 1 | 0 |   |   |   |    |    |    |    |    | 1    |
| 10   | 0 | 1 | 3 | 9 | 12| 4 | 0 |   |   |    |    |    |    |    | 0    |
| 12   | 0 | 0 | 8 | 23| 22| 13| 5 | 0 |   |    |    |    |    |    | 2    |
| 14   | 0 | 1 | 6 | 31| 71| 58| 15| 2 | 1 |    |    |    |    |    | 0    |
| 16   | 0 | 1 | 2 | 6 | 12| 126|147|95 |24 |0   |    |    |    |    | 0    |
| 18   | 0 | 1 | 10| 80| 262|380|268|100|21 |1   |    |    |    |    | 2    |
| 20   | 0 | 0 | 17| 124|423|791|801|414|96 |9   | 1  |    |    |    | 1    |
| 22   | 0 | 1 | 14| 163|738|1586|1874|1276|479|82 |3   |    |    |    | 1    |
| 24   | 0 | 0 | 23| 229|1091|2897|4281|3554|1628|408 |49 |1   |    |    | 1    |
| 26   | 0 | 1 | 19| 285|1722|5102|8613|8735|5222|1703|266 |19 | 1  | 3    |
| 28   | 0 | 0 | 29| 385|2428|8465|16905|19930|14133|5981|1408|144|2   | 1    |
| 30   | 0 | 1 | 25| 468|3541|13661|30687|42291|35986|18457|5431|855|63 | 0    |

Here, \(\chi\) means the alternating sum of the dimension of relative cochain complex, including 0-dimensional cochain complex \(\mathbb{R}\). Thus, \(\chi \neq 1\) means there is non-trivial Betti number.
5.2 How to use computer in order to get the dimension of relative cochain complex (brief summary)

1. For a given weight $w$, edit and fix the weight $w$ in $./gkf$.act-1.mpl and run this maple script in order to get possible direct summands of $C^m_{GF}(\mathfrak{ham}_2^1)$, for each degree $dp$.

   \% maple $./gkf$.act-1.mpl > OUT-w

2. Using the output file OUT-w, we run the next perl script

   \% perl $./gkf$.act-2.prl OUT-w

   After this job, we have a couple of output range-w and files cases_w.dp.txt.

   Also, we run

   \% perl $./mk-tex-gkf.prl$ OUT-w

   we get a TeX-file gkf. We follow the requirements for relative cochain complex (brief summary). In this section, we only deal with $\mathfrak{sp}$-theory. We have already observed that $i_\xi d = \xi$, we get

   \begin{align*}
   (i_H \circ d) z^r_R &= -H \cdot z^r_R = -(2r - R) z^r_R \\
   (i_X \circ d) z^r_R &= -X \cdot z^r_R = -(R - r) z^{r+1}_R \\
   (i_Y \circ d) z^r_R &= -Y \cdot z^r_R - rz^{r-1}_R
   \end{align*}

   for each generators of 1-cochain complex. $d$ is a skew-derivation of degree +1 and

   \[ dz^r_R = \frac{r!(R-r)!}{2} \sum_{a+b=1+r, A+B=2+r} \begin{vmatrix} a & b \\ A & B \end{vmatrix} \frac{z^a_A}{a!(A-a)!} \wedge \frac{z^b_B}{b!(B-b)!} \]

   where $0 \leq r \leq R$, $0 \leq a \leq A$, $0 \leq b \leq B$, $A \geq 2$, $B \geq 2$.

6 Getting concrete bases and matrix representation of $d$

Even though we know all the dimensions of subcomplex of relative cochain complex $C^m_{GF}(\mathfrak{ham}_2^1)$, it is not enough to investigate the coboundary operator $d$ itself. We have to know concrete bases of $C^m_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2n, \mathbb{R}))/w$ and $C^{m+1}_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2n, \mathbb{R}))/w$, and the matrix representation of $d$ with respect to those bases. In this section, we only deal with $n = 1$ and $M = \mathbb{R}^2$.

We follow the requirements for relative cochain complex. The first one is $i_\xi d = \xi$, and we already have obtained that $i_\xi d = -\xi$, we get

\begin{align*}
(i_H \circ d) z^r_R &= -H \cdot z^r_R = -(2r - R) z^r_R \\
(i_X \circ d) z^r_R &= -X \cdot z^r_R = -(R - r) z^{r+1}_R \\
(i_Y \circ d) z^r_R &= -Y \cdot z^r_R - rz^{r-1}_R
\end{align*}

for each generators of 1-cochain complex. $d$ is a skew-derivation of degree +1 and

\[ dz^r_R = \frac{r!(R-r)!}{2} \sum_{a+b=1+r, A+B=2+r} \begin{vmatrix} a & b \\ A & B \end{vmatrix} \frac{z^a_A}{a!(A-a)!} \wedge \frac{z^b_B}{b!(B-b)!} \]

where $0 \leq r \leq R$, $0 \leq a \leq A$, $0 \leq b \leq B$, $A \geq 2$, $B \geq 2$.

6.1 An easy example

Here, we show a small practice of getting concrete basis of relative cochain complex of the case of $n = 1$ and the weight 6. As shown in Example 5.3 by some help of $Sp(2, \mathbb{R})$-theory, we have already known that $\dim C^2_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_6 = \dim C^3_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_6 = 1$ and the others are 0-dimensional, and furthermore the corresponding basis lives in $\Lambda^2 \mathfrak{g}_5$ when $C^2_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_6$, and
this discussion. By taking the example of weight = 8, we emphasize that how getting concrete basis is tough job. 

For 3-cochain, let \( \tau = \sum_{i=0}^{3} \sum_{j=0}^{4} \sum_{k=0}^{5} c_{ijk} z_3^i \wedge z_4^j \wedge z_5^k \), and again solve the equations \( i_\xi \circ d \sigma = 0 \) for \( \xi \in \{X,Y,H\} \). Then we have

\[
\tau = c_{2,3,1} (z_3^2 \wedge z_4^3 \wedge z_5^1 + 8z_3^4 \wedge z_4^4 \wedge z_5^1 - 5/2z_3^3 \wedge z_4^3 \wedge z_5^2 + z_3^4 \wedge z_4^3 \wedge z_5^2 - 6z_3^0 \wedge z_4^2 \wedge z_5^4 \\
+ 15z_3^1 \wedge z_4^4 \wedge z_5^1 + z_3^3 \wedge z_4^3 \wedge z_5^2 - 20z_3^1 \wedge z_4^0 \wedge z_5^0 - 6z_3^0 \wedge z_4^1 \wedge z_5^2 + 9/2z_3^0 \wedge z_4^3 \wedge z_5^3 \\
+ 8z_3^0 \wedge z_4^4 \wedge z_5^0 - 5/2z_3^2 \wedge z_4^4 \wedge z_5^2 + z_3^3 \wedge z_4^4 \wedge z_5^2 - 20z_3^2 \wedge z_4^2 \wedge z_5^2 - 6z_3^3 \wedge z_4^1 \wedge z_5^3 \\
+ 9/2z_3^2 \wedge z_4^1 \wedge z_5^2 - 6z_3^3 \wedge z_4^2 \wedge z_5^1 + 15z_3^3 \wedge z_4^1 \wedge z_5^0 )
\]

and putting \( c_{2,3,1} = 4 \), we have a basic vector \( \tau_1 \) in \( C_{GF}(\text{ham}_0, \text{sp}(2, \mathbb{R})))_6 \). We see that

\[
d \sigma_1 = \tau_1
\]

thus, we get \( H_{GF}^3(\text{ham}_0, \text{sp}(2, \mathbb{R})))_6 = \{0\} \), and so we have \( H_{GF}^3(\text{ham}_0, \text{sp}(2, \mathbb{R})))_6 = \{0\} \). Therefore, the trivial 0-th Betti number number 1, is 1 as we have seen before.

### 6.2 More complicated example

By taking the example of weight = 8, we emphasize that how getting concrete basis is tough job. Later, we stress that our job sequences, which complete those jobs automatically, are how useful in this discussion.

| deg | ref.# | type | dim |
|-----|-------|------|-----|
| 1   | 1     | (10) |     |
| 2   | 1     | (3 9 )|     |
| 2   | 2     | (4 8 )|     |
| 2   | 3     | (5 7 )|     |
| 2   | 4     | (6^2)|     |
| 3   | 1     | (3^2 8) |     |
| 3   | 2     | (3 4 7) | 1   |
| 3   | 3     | (3 5 6) | 1   |
| 3   | 4     | (4^2 6) | 1   |
| 3   | 5     | (4 5^2) |     |

The cochain complex of deg = 3 is 4-dim, that of deg = 4 is 5-dim, and that of deg = 5 is 1-dimensional.

Here, we will see the whole process of getting a basis of each cochain complex.

1. 3-cochains

(a) type (3, 4, 7)-case: The candidate is \( \sigma_1 = \sum_{i=0}^{3} \sum_{j=0}^{4} \sum_{k=0}^{7} c_{ijk} z_3^i \wedge z_4^j \wedge z_7^k \) with 160 unknown variables \( c_{ijk} \). From the three conditions \( i_\xi \circ d \sigma = 0 \) for \( \xi = X, Y, H \), we have a homogeneous linear equations (the number of equations is 455). Solving these equations, we have one undetermined parameter \( c_{007} \), and we put it to be 1, and get a 1-dimensional basis:

(b) type (3, 5, 6)-case: The method is completely same and get a basis \( \sigma_2 \) by putting \( c_{016} = 1 \).
(c) type \((4^2, 6)\)-case: Here we have to be careful for dealing with exterior product elements as below. The candidate is \(\sigma = \sum_{0 \leq i < j \leq 4} \sum_{k=0}^{6} c_{ijk} z_i^1 \wedge z_j^1 \wedge z_k^6 \) with 70 unknown variables \(c_{ijk}\). Putting \(c_{016} = 1\), we have a basis \(\sigma_3\).

(d) type \((4, 5^2)\)-case: Here again we have to be careful for dealing with exterior product elements as before. The candidate is \(\sigma = \sum_{i=0}^{4} \sum_{0 \leq j < k \leq 5} c_{ijk} z_i^1 \wedge z_j^5 \wedge z_k^5 \) with 75 unknown variables \(c_{ijk}\). Putting \(c_{025} = 1\), we have a basis

\[
\sigma_1 = z_0^0 \wedge z_1^1 \wedge z_2^7 - 4z_3^3 \wedge z_4^1 \wedge z_5^7 + 6z_3^3 \wedge z_4^1 \wedge z_5^2 - 4z_3^3 \wedge z_4^1 \wedge z_5^4 + 2z_3^3 \wedge z_4^1 \wedge z_5^6 - 8z_3^3 \wedge z_4^1 \wedge z_5^7 - 8z_3^3 \wedge z_4^1 \wedge z_5^2 - 8z_3^3 \wedge z_4^1 \wedge z_5^6 + 4z_3^3 \wedge z_4^1 \wedge z_5^7
\]

\[
\sigma_2 = z_0^0 \wedge z_1^1 \wedge z_2^6 - 4z_3^3 \wedge z_4^1 \wedge z_5^6 + 6z_3^3 \wedge z_4^1 \wedge z_5^4 - 4z_3^3 \wedge z_4^1 \wedge z_5^2 + 4z_3^3 \wedge z_4^1 \wedge z_5^2 - 8z_3^3 \wedge z_4^1 \wedge z_5^3 - 8z_3^3 \wedge z_4^1 \wedge z_5^5 + 4z_3^3 \wedge z_4^1 \wedge z_5^7
\]

\[
\sigma_3 = z_0^0 \wedge z_1^1 \wedge z_2^6 - 3z_3^3 \wedge z_4^1 \wedge z_5^6 + 3z_3^3 \wedge z_4^1 \wedge z_5^4 - 8z_3^3 \wedge z_4^1 \wedge z_5^2 + 4z_3^3 \wedge z_4^1 \wedge z_5^6 - 8z_3^3 \wedge z_4^1 \wedge z_5^7
\]

\[
\sigma_4 = z_0^0 \wedge z_1^1 \wedge z_2^6 - 3z_3^3 \wedge z_4^1 \wedge z_5^6 + 2z_3^3 \wedge z_4^1 \wedge z_5^4 - 8z_3^3 \wedge z_4^1 \wedge z_5^2 + 4z_3^3 \wedge z_4^1 \wedge z_5^6 - 8z_3^3 \wedge z_4^1 \wedge z_5^7
\]

Those \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\) are a basis of \(C^3_{GF}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_8\).

2. 4-cochains

(a) type \((3^2, 4, 6)\)-case: \(c_{0206} \tau_1\), where \(\tau_1\) is below:

(b) type \((3^2, 5^2)\)-case: The general solution is \(c_{0314} \tau_2 + c_{0314} \tau_3\), where \(\tau_2, \tau_3\) are below:

(c) type \((3, 4^2, 5)\)-case: The general solution is \(c_{0125} \tau_4 + c_{0035} \tau_5\), where \(\tau_4, \tau_5\) are below:
\[ \tau_1 = -2z^0_3 \land z^3_3 \land z^4_3 \land z^5_3 + 6z^0_0 \land z^1_3 \land z^2_3 \land z^3_3 - 6z^0_3 \land z^3_3 \land z^2_3 \land z^5_3 + 2z^0_3 \land z^1_3 \land z^4_3 \land z^6_3 \\
+ z^0_3 \land z^3_3 \land z^6_3 - 6z^0_3 \land z^3_3 \land z^2_3 \land z^4_6 + 8z^0_3 \land z^3_3 \land z^3_3 \land z^2_3 - 3z^0_3 \land z^3_3 \land z^1_3 \land z^5_3 \\
- z^0_3 \land z^3_3 \land z^4_3 \land z^5_3 + 2z^0_3 \land z^3_3 \land z^4_3 \land z^6_3 - 2z^0_3 \land z^1_3 \land z^1_3 \land z^5_3 + z^0_3 \land z^3_3 \land z^1_3 \land z^4_3 \\
- 3z^1_3 \land z^2_3 \land z^4_3 \land z^5_3 + 6z^1_3 \land z^2_3 \land z^3_3 \land z^4_3 - 6z^1_3 \land z^3_3 \land z^3_3 \land z^2_3 + 3z^1_3 \land z^3_3 \land z^1_3 \land z^4_3 \\
+ 3z^1_3 \land z^3_3 \land z^0_3 \land z^4_3 - 8z^1_3 \land z^3_3 \land z^3_3 \land z^2_3 + 6z^1_3 \land z^3_3 \land z^3_3 \land z^2_3 - 3z^1_3 \land z^3_3 \land z^1_3 \land z^4_3 \\
- 2z^3_3 \land z^3_3 \land z^0_3 \land z^3_3 + 6z^3_3 \land z^3_3 \land z^3_3 \land z^2_3 - 6z^3_3 \land z^3_3 \land z^3_3 \land z^2_3 + 2z^3_3 \land z^3_3 \land z^3_3 \land z^4_3 \\
- 2z^3_3 \land z^3_3 \land z^0_3 \land z^3_3 - 9z^3_3 \land z^3_3 \land z^3_3 \land z^2_3 - 3z^0_3 \land z^3_3 \land z^3_3 \land z^5_3 + 6z^0_3 \land z^3_3 \land z^3_3 \land z^6_3 \\
+ 3z^0_3 \land z^3_3 \land z^0_3 \land z^4_3 - 8z^0_3 \land z^3_3 \land z^3_3 \land z^2_3 - 3z^0_3 \land z^3_3 \land z^3_3 \land z^5_3 - 6z^0_3 \land z^3_3 \land z^3_3 \land z^6_3. \]

\[ \tau_2 = 3z^0_3 \land z^1_3 \land z^2_3 \land z^0_3 - 3z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 - 3z^0_3 \land z^1_3 \land z^2_3 \land z^4_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 \\
+ 2z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 + 2z^0_3 \land z^1_3 \land z^2_3 \land z^7_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^8_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^9_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 \]

\[ \tau_3 = z^0_3 \land z^1_3 \land z^2_3 \land z^3_3 - 3z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 - 3z^0_3 \land z^1_3 \land z^2_3 \land z^4_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 \\
+ 2z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 + 2z^0_3 \land z^1_3 \land z^2_3 \land z^7_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^8_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^9_3 - 2z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 \]

\[ \tau_4 = - \frac{2}{3}z^0_3 \land z^2_3 \land z^3_3 \land z^4_3 + \frac{2}{3}z^0_3 \land z^2_3 \land z^3_3 \land z^5_3 + z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 + \frac{2}{3}z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 \\
- \frac{5}{3}z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 - z^0_3 \land z^1_3 \land z^2_3 \land z^4_3 + 2z^0_3 \land z^2_3 \land z^3_3 \land z^5_3 - \frac{6}{5}z^0_3 \land z^2_3 \land z^3_3 \land z^6_3 \\
+ z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 - z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 - z^0_3 \land z^1_3 \land z^2_3 \land z^7_3 - z^0_3 \land z^1_3 \land z^2_3 \land z^8_3 - z^0_3 \land z^1_3 \land z^2_3 \land z^9_3 - z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 \]

\[ \tau_5 = z^0_3 \land z^0_3 \land z^1_3 \land z^2_3 \land z^5_3 + z^0_3 \land z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 + z^0_3 \land z^0_3 \land z^1_3 \land z^2_3 \land z^7_3 + z^0_3 \land z^0_3 \land z^1_3 \land z^2_3 \land z^8_3 + z^0_3 \land z^0_3 \land z^1_3 \land z^2_3 \land z^9_3 + z^0_3 \land z^0_3 \land z^1_3 \land z^2_3 \land z^6_3 \]

3. 5-cochain
type (3^1 4 5)-case: 120 unknown variables and solving the linear 339-equations, and putting \(c_{01314} = 1\), we have a basis:
\[ \rho = 3z_3^0 \land z_3^1 \land z_3^2 \land z_3^4 \land z_5^5 - 9z_3^0 \land z_3^1 \land z_3^2 \land z_4^2 \land z_4^4 + 9z_3^0 \land z_3^1 \land z_2^2 \land z_3^2 \land z_3^3 \land z_3^5 \]

\[ - 3z_3^0 \land z_2^1 \land z_3^2 \land z_4^4 + z_3^0 \land z_3^1 \land z_3^2 \land z_3^4 \land z_3^5 + z_3^0 \land z_3^1 \land z_3^2 \land z_3^4 \land z_4^5 \]

\[ + 3z_3^0 \land z_3^1 \land z_3^2 \land z_4^5 - 5z_3^0 \land z_3^1 \land z_3^2 \land z_3^4 \land z_3^5 + 2z_3^1 \land z_3^0 \land z_3^3 \land z_4^4 \land z_3^5 \]

\[ + 2z_3^0 \land z_3^1 \land z_3^2 \land z_3^4 \land z_3^5 - 5z_3^0 \land z_3^1 \land z_3^2 \land z_3^4 \land z_3^5 + z_3^0 \land z_3^1 \land z_3^2 \land z_3^4 \land z_3^5 \]

\[ + z_3^0 \land z_2^3 \land z_3^3 \land z_3^4 \land z_3^5 - 9z_3^1 \land z_3^2 \land z_3^4 \land z_3^5 \]

\[ + 9z_3^0 \land z_3^1 \land z_3^2 \land z_3^4 \land z_3^5 \land z_3^5 \]

With respect to those concrete bases of relative cochain complex, the coboundary operator \(d\) between degree 4 and 5, is of the form \(d \tau_j = b_j \rho\) \((j = 1..5)\) and comparing both sides, we get a matrix expression \(B := (b_1 \ b_2 \ b_3 \ b_4 \ b_5) = (-4 \ -3 \ -3 \ -9 \ 6)\).

About the coboundary operator \(d\) between degree 3 and 4, \(d \sigma_j = \sum_{i=1}^{5} a_{ij}^i \tau_i\) \((j = 1, \ldots, 4)\). For each \(i\), comparing both sides again, we get a matrix expression

\[
A = \begin{pmatrix}
-6 & -12 & 9 & 0 \\
0 & -27 & 0 & 1 \\
0 & 59 & 0 & 3 \\
5 & 4 & 3 & 22 \\
7 & 14 & 21 & 35
\end{pmatrix}
\]

As expected, we see that \(BA = O\), which corresponds to \(d \circ d = 0\). Since \(\text{rank}A = 4\), \(\text{rank}B = 1\), thus we have \(H^3_{(3,6)}(\text{ham}_2, \text{sp}(2, \mathbb{R})) = \{0\}\), \(H^5_{(3,6)}(\text{ham}_2, \text{sp}(2, \mathbb{R})) = \{0\}\), \(H^7_{(3,6)}(\text{ham}_2, \text{sp}(2, \mathbb{R})) = \{0\}\).

These mean that except \(H^3_{(3,6)}(\text{ham}_2, \text{sp}(2, \mathbb{R})) = \mathbb{R}\), the other cohomologies are zero and the Euler characteristic number is 1.

### 6.3 How tuning computer to subscribe our job

We here show our typical strategy for getting a concrete basis of relative cochain complex. Take an example with weight 8, which we have seen above. We follow the above discussion in the case of type \((3, 5, 6)\) and type \((4^2, 6)\) in order to emphasize some difference. We need to distinguish scalar and vectors in general. For that purpose, we use \texttt{diffforms} package. In the type \((3, 5, 7)\) of degree 3 with weight 8, a cochain in general is

\[
\sigma = \sum_{i=0}^{3} \sum_{i=0}^{5} \sum_{i=0}^{7} c_{i_1,i_2,i_3} z_{i_1} \hat{\otimes} z_{i_2} \otimes z_{i_3}
\]

(in this case, \(\land\) and \(\otimes\) have the same meaning), and we declare \(c_{i_1,i_2,i_3}\) are scalar. For that purpose we prepare \texttt{in\_\$\_3-3\_a.txt}.

```plaintext
uke31 := NULL: uke32 := NULL:
for i1 from 0 to d3 do
for i2 from 0 to d5 do
for i3 from 0 to d6 do
 uke31 := uke31, cat(c3, "_", i1, "_", i2, "_", i3) = const ;
 uke32 := uke32, cat(c3, "_", i1, "_", i2, "_", i3) ;
od od od:
deform(uke31):
```

To construct \(\sigma\) above, we prepare \texttt{in\_\$\_3-3\_b.txt}.

```plaintext
# deform( A3 = 3); A3 := 0:
for i1 from 0 to d3 do
for i2 from 0 to d5 do
for i3 from 0 to d6 do
A3 := A3 + cat(c3, "_", i1, "_", i2, "_", i3) *
& ( z[i1,d3-i1], z[i2,d5-i2], z[i3,d6-i3]) ;
od od od:
```

17
In the type \((4^2,6)\) of degree 3 with weight 8, a general cochain is
\[
\sigma = \sum_{i_1=0}^{4} \sum_{i_2=0}^{4} \sum_{i_3=0}^{6} c_{i_1,i_2,i_3} z_{14}^{i_1} \wedge z_{14}^{i_2} \wedge z_{16}^{i_3},
\]
where \(c_{i_1,i_2,i_3}\) are skewsymmetric in 3 indices.

To avoid complicated requirement, we restrict range of indices as
\[
\sigma = \sum_{0 \leq i_1 < i_2 \leq 4} \sum_{i_3=0}^{6} c_{i_1,i_2,i_3} z_{14}^{i_1} \wedge z_{14}^{i_2} \otimes z_{16}^{i_3},
\]
(we need some scale factor, in the case above, 2-times, but we ignore them here after.)

This time, a declaration that coefficients are constant, is \textit{difforms}, we prepare \texttt{in.B.3-4.a.txt}.

\[
\text{uke41} := \text{NULL}; \text{uke42} := \text{NULL};
\]
\[
\text{for } i_1 \text{ from 0 to } d_4 \text{ do}
\]
\[
\text{for } i_2 \text{ from } 1+i_1 \text{ to } d_4 \text{ do}
\]
\[
\text{for } i_3 \text{ from 0 to } d_6 \text{ do}
\]
\[
\text{uke41} := \text{uke41}, \text{cat}(c_{4}, "\_", i_1, "\_", i_2, "\_", i_3) = \text{const} ;
\]
\[
\text{uke42} := \text{uke42}, \text{cat}(c_{4}, "\_", i_1, "\_", i_2, "\_", i_3) ;
\]
\[
\text{od} \text{ od} \text{ od}:
\]
\[
\text{defform(uke41)}:
\]

To construct \(\sigma\) above, we prepare \texttt{in.B.3-4.b.txt}.

\[
\text{# defform( A4 = 3): A4 :=0}:
\]
\[
\text{for } i_1 \text{ from 0 to } d_4 \text{ do}
\]
\[
\text{for } i_2 \text{ from } 1+i_1 \text{ to } d_4 \text{ do}
\]
\[
\text{for } i_3 \text{ from 0 to } d_6 \text{ do}
\]
\[
A_4 := A_4 + \text{cat}(c_{4}, "\_", i_1, "\_", i_2, "\_", i_3) * \\
&^ ( z[i_1,d_4-i_1], z[i_2,d_4-i_2], z[i_3,d_6-i_3] ) ;
\]
\[
\text{od} \text{ od} \text{ od}:
\]

To pick up “coefficients” of \(i_\xi d(\sigma) (\forall \xi \in \text{sp}(2,\mathbb{R}))\) with respect “some generators”, we prepare \texttt{in.B.3-4.c.txt}.

\[
\text{for } i_1 \text{ from 0 to } d_4 \text{ do}
\]
\[
\text{for } i_2 \text{ from } 1+i_1 \text{ to } d_4 \text{ do}
\]
\[
\text{for } i_3 \text{ from 0 to } d_6 \text{ do}
\]
\[
\text{ukez} := \text{ukez}, \text{seq(}
\text{mInnd([z[i_1,d_4-i_1], z[i_2,d_4-i_2], z[i_3,d_6-i_3]],}
\text{W W[s]),s=0..d_0});
\]
\[
\text{od} \text{ od} \text{ od}:
\]

For the main purpose to solve \(i_\xi d(\sigma) = 0 (\forall \xi \in \text{sp}(2,\mathbb{R}))\) and to determine \(\sigma\) and \(d(\sigma)\), we have a core Maple script \textit{action\_new.mpl}, we omit it because it will occupy one and a half pages.

\textbf{Remark 6.1} When \(w \geq 16\), we encounter some trouble of kernel panic in processing the above steps. Main problem seems shortage of CPU memory. To recover this trouble, we change our process slightly. A big guarantee is \textit{linearity} of \(d\). Namely, suppose for a given “huge” cochain \(A\) we cannot compute \(d(A)\). Then, we divide \(A\) into “small” pieces like \(A = a_1 + a_2 + \cdots\). Instead of handling \(d(A)\) itself, we manipulate \(d(a_i)\), and by \(d(a_1) + d(a_2) + \cdots\), we get the whole \(d(A)\).

\textbf{6.4 Sequence of computer process to get concrete basis of cochain complex (brief summary)}

1. The same (1) of §5.2.
2. The same (2) of §5.2.
3. We prepare a plain text file `in_dp-new.txt`, this is a prototype of cochain of degree $dp$, independent of $w$. In which we prepare coefficients of degree $dp$ cochain, and ready to handle to deform() of difforms Maple package, and (Of course, the range of $dp$ depends on $w$.) We manipulate this `in_dp-new.txt` and `cases_w_dp.txt` in step 2 by the perl script `gkf_act-3.prl`. % perl ../gkf_act-3.prl

Out put file are `out_w_dp-ref#`.  

4. Important! For `out_w_dp-ref#`, we have to revise it in order to guarantee the skew-symmetry. For instance, the right side is desired form:

```
for i1 from 0 to d4 do  
for i2 from 0 to d4 do
```

For that purpose, we prepare a perl script `gkf_act-4.prl` and run for all files:
% ../gkf_act-4.prl out_w_dp-ref#

Output are `out_w_dp-ref#.txt`. In §6.2, we encountered with `in_8_3-4_[abc].txt`. 

5. For the revised `out_w_dp-ref#.txt`, we apply `gkf_act-5.prl`.
% ../gkf_act-5.prl out_w_dp-ref#.txt

Output are `in_w_dp-ref_[abc].txt`. In §6.2, we encountered with `in_8_3-4_[abc].txt`. 

6. We prepare a maple script `action_new.mpl`, in which for fixed weight $w$ and degree $dp$, get a basis of cocycles and $d$-image of them for each direct sum component labeled by ref#.

We omitted to explain about this key job in §6.2.

## 7 Main result

While weight is less that 12, the whole cohomology groups were studied in [4].

**Theorem 7.1** Using those simple but powerful tricks explained above, we could finish all computations for weight from 12 to 18 of $H^\cdot_\cdot_\cdot_{GF}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_{wt}$. We will show the results a table below. The abbreviations in the table mean degree $k$ is $\text{Sp}(2, \mathbb{R})$-invariant cochain complex $C^k$, dim is $\text{dim}C^k$, and $\text{rank}(d)$ is the rank of $d : C^k \rightarrow C^{1+k}$.

| weight | degree | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|--------|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 12     | dim    | 0   | 0   | 8   | 23  | 22  | 13  | 5   | 0   | 0   | 0   |
|        | rank($d$) | 0   | 0   | 8   | 14  | 8   | 5   | 0   | 0   | 0   | 0   |
|        | Betti#  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 14     | dim    | 0   | 1   | 6   | 31  | 71  | 58  | 15  | 2   | 1   | 0   |
|        | rank($d$) | 0   | 1   | 5   | 26  | 44  | 14  | 1   | 1   | 0   | 0   |
|        | Betti#  | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   |
| 16     | dim    | 0   | 0   | 12  | 61  | 126 | 147 | 95  | 24  | 0   | 0   |
|        | rank($d$) | 0   | 0   | 12  | 49  | 77  | 70  | 24  | 0   | 0   | 0   |
|        | Betti#  | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   |
| 18     | dim    | 0   | 1   | 10  | 80  | 262 | 380 | 268 | 100 | 21  | 1   |
|        | rank($d$) | 0   | 1   | 9   | 71  | 191 | 188 | 80  | 20  | 1   | 0   |
|        | Betti#  | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   |
Remark 7.2 The Euler characteristic number is the alternating sum of $\dim$ of cochain complexes or Betti numbers, including 0-dimensional. The tables above show that $\chi(\text{weight } = 12) = 2$, $\chi(\text{weight } = 14) = 0$, $\chi(\text{weight } = 16) = 0$, and $\chi(\text{weight } = 18) = 2$.

When the weight = 20, we have the complete list of degree of all relative cochain complexes as below:

| degree | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|--------|---|---|---|---|---|---|---|---|---|----|----|
| dim    | 0 | 0 | 17 | 124 | 423 | 791 | 801 | 414 | 96 | 9 | 1 |

and the Euler characteristic number, i.e., the alternating sum including 0-dimensional, is 1. It will be interesting to check if there are non-trivial cohomology groups with weight 20.

Remark 7.3 The above remark was noted on October, 2012. Recently (February 2014) we finished computing the Betti numbers when weight 20, using Gröbner Basis theory for linear polynomials. Our result is that the Betti numbers for weight 20 are all trivial, except 0-dimensional.

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