THE FUNDAMENTAL GROUP OF GALOIS COVER OF THE SURFACE $T \times T$

MEIRAV AMRAM, MINA TEICHER, AND UZI VISHNE

Abstract. This is the final paper in a series of four, concerning the surface $T \times T$ embedded in $\mathbb{CP}^N$, where $T$ is a the one dimensional torus. In this paper we compute the fundamental group of the Galois cover of the surface with respect to a generic projection onto $\mathbb{CP}^2$, and show that it is nilpotent of class 3. This is the first time such a group is presented as the fundamental group of a Galois cover of a surface.

1. Introduction

Studying Galois covers of surfaces proved to be a very useful tool in understanding the structure of moduli spaces of surfaces of general type. In 1987 ([MT1]), such a construction gave the first counterexample to the ‘Bogomolov watershed conjecture’ that surfaces of general type with positive signature have infinite fundamental groups.

The Galois cover $X_{\text{Gal}}$ of an algebraic surface $X$ embedded in a projective space $\mathbb{CP}^N$ is the Zariski closure of the fibred product (with respect to a generic projection to $\mathbb{CP}^2$) of $n = \text{deg}(X)$ copies of $X$ where the generalized diagonal is excluded. The fundamental group of the Galois cover was studied in a series of papers by Moishezon and Teicher, and other authors. References can be found in the recent paper [ADKY]. In the current paper we study $(T \times T)_{\text{Gal}}$, which is a moduli space of surfaces of general type.

The first step in the investigation of $\pi_1(X_{\text{Gal}})$ is a short exact sequence (see [MT1])

$1 \longrightarrow \pi_1(X_{\text{Gal}}) \longrightarrow \tilde{\Pi}_1 \longrightarrow S_n \longrightarrow 1$
where $\tilde{\Pi}_1$ is a certain quotient of a fundamental group of the complement of the branch curve $S$ of the projection $f$. A presentation of $\tilde{\Pi}_1$ can be obtained from the braid monodromy of the curve $S$, and the van Kampen theorem. In principle a presentation of $\pi_1(X_{\text{Gal}})$ can be constructed from that of $\tilde{\Pi}_1$ via the Reidmeister-Schreier method [MKS]. However this is not enough, since the number of generators in $\tilde{\Pi}_1$ is multiplied by $|S_n| = n!$, and moreover it is usually quite difficult to identify the group from its presentation.

In this paper we compute $\pi_1(X_{\text{Gal}})$ for the surface $X = T \times T$, where $T$ is the torus of dimension 1 (i.e. an elliptic curve over $\mathbb{C}$), and the embedding is obtained from a fixed embedding $T \hookrightarrow \mathbb{C}P^2$, and the induced Segre map $X \hookrightarrow \mathbb{C}P^8$. We fix a generic projection $f : \mathbb{C}P^8 \to \mathbb{C}P^2$, and let $S$ be its branch curve.

In [A], [AT1] and [AT2] we gave a finite presentation for the group $\tilde{\Pi}_1$ of Equation (1) for the case of $X = T \times T$. In this paper we use this presentation, the nature of the embedding of $X$ in $\mathbb{C}P^8$, a degeneration of the embedding $T \times T \hookrightarrow \mathbb{C}P^8$ to a union of 18 planes (see [AT1]), to obtain a concise presentation of $\pi_1(X_{\text{Gal}})$. Moreover, we apply group theoretic techniques from [RTV] based on ideas from the Reidmeister-Schreier method [MKS], and geometric methods which were developed in [ATV] for a certain quotient of $\pi_1(X_{\text{Gal}})$, to compute $\pi_1(X_{\text{Gal}})$.

The main result of this paper is that $\pi_1(X_{\text{Gal}})$ is nilpotent of class 3 (Theorem 5.12). In fact we obtain an explicit description of $\pi_1(X_{\text{Gal}})$ as a quotient of a group $K$ by a central element $p$, the projective relation, computed in Equation (21). This group $K$ is given as the kernel of an explicit epimorphism from a group $K^*$ to $\mathbb{Z}^{10}$, where a presentation for $K^*$ is given in Corollary 5.1.

2. Overview of new results related to $T \times T$

As before, let $S$ be the branch curve of $X$ under a generic projection $f : \mathbb{C}P^8 \to \mathbb{C}P^2$. It is very difficult to deal with $\pi_1(\mathbb{C}^2 - S)$ directly, so we start by considering $S_0$ where $S_0$ is the branch of a degenerated object. We construct a degeneration of $X$ into $X_0$, a union of planes, where no three planes meet in a line. The branch curve of the generic projection of $X_0$, denoted by $S_0$, is a union of lines.
The surface $X_0$ is composed of $n = \deg(X_0) = 18$ planes with 27 intersection lines. The curve $S_0$ is the union of the 27 intersection lines. It has 9 intersection points, as depicted in Figure 1. This diagram by itself is a topological torus: vertices with the same number are identified. The vertices are numbered from left to right and bottom up, and the lines are lexicographically numbered from 1 to 27. (Incidentally we comment that this graph is a $(9, 6, 2, 2)$ strongly regular graph, namely the valency of each of the 9 points is 6, and any two neighborhoods intersect at 2 points).

The group $\pi_1(\mathbb{C}^2 - S_0)$ is generated by loops $\Gamma_1, \ldots, \Gamma_{27}$, which correspond to the 27 intersection lines in $S_0$. In the regeneration process (‘inverse’ to the degeneration), lines are doubled so $\pi_1(\mathbb{C}^2 - S)$ is generated by 54 generators, denoted $\Gamma_1, \Gamma_1', \ldots, \Gamma_{27}, \Gamma_{27'}$.

To compute a complete list of relations, one needs first to compute the braid monodromy of $S$. To do this, we first compute the braid monodromy of $S_0$ using the regeneration rules from [MT4] and the braid monodromy algorithm for line arrangements from [MT3]. Then get a braid monodromy (factorization) of $S$ from the one of $S_0$ (as in [AT1]). A finite presentation of $\pi_1(\mathbb{C}^2 - S)$ is obtained from the braid monodromy factorization, via the van Kampen Theorem [AT2]. A presentation of $\pi_1(\mathbb{CP}^2 - S)$ is obtained by adding the ‘projective
relation’

\[ \Gamma_{27} \Gamma_{27} \cdots \Gamma_{1} \Gamma_{1} = 1. \]

The presentation of \( \pi_1(\mathbb{C}^2 - S) \) involves the above mentioned 54 generators, and close to 1700 relations. This was laid out in [AT2]. A complete presentation is given in the appendix of [A], and can be accessed in [App1] (see the Appendix).

There is a natural projection from \( \pi_1(\mathbb{C}P^2 - S) \) to the symmetric group \( S_n \), where \( n = 18 \). The map is defined by sending \( \Gamma_j \) and \( \Gamma_{j'} \) to the transposition \((ab)\) where \( a, b \) are the planes intersecting in the line \( j \). Since \((ab)^2 = 1\), the map splits through \( \pi_1(\mathbb{C}P^2 - S)/\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle \), which we henceforth denote by \( \tilde{\Pi}_1 \). We also let \( \tilde{\Pi}_{1\text{Aff}} = \pi_1(\mathbb{C}^2 - S)/\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle \).

It is known that \( \pi_1(X_{\text{Gal}}) \) is isomorphic to the kernel of the projection \( \tilde{\Pi}_1 \rightarrow S_n \). Our aim in this paper is to compute the group \( \pi_1(X_{\text{Gal}}) \).

Naturally, it is very difficult to get concrete information on a group with a presentation of this size. In [ATV] we computed the quotient \( C \) of \( \tilde{\Pi}_1 \) obtained by identifying \( \Gamma_{j'} = \Gamma_j \) for every \( j \). (The reader might want to check this paper in order to get a feeling of the type of presentation we are dealing with, and the difficulties involved in the computation.)

We remark that for most previous surfaces, the quotient \( C \) defined in the same manner was computed (implicitly) to be the symmetric group \( S_n \) (for example, see [MT1], [MT5] and [MRT]). The first cases in which \( C \) is a larger group are \( \mathbb{C}P^1 \times \mathbb{T} \) ([AGTV] and [A]), of the current surface. See [ATV] for details on this general approach. In fact these are the only known cases in which the fundamental group of a Galois cover of a surface is infinite. In all other cases the fundamental group was shown to be a product of finite cyclic groups. In particular \( \mathbb{T} \times \mathbb{T} \) is the first case of a group which is not abelian by abelian.

3. The Presentation of \( \tilde{\Pi}_{1\text{Aff}} \)

We start with a group \( \tilde{\Pi}_{1\text{Aff}} = \pi_1(\mathbb{C}^2 - S)/\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle \) generated by the 54 generators \( \Gamma_1, \Gamma_{1'}, \ldots, \Gamma_{27}, \Gamma_{27'} \), and subject to about 1700 relations.
These relations are obtained from applying Van Kapmen’s theorem for plane curves to the braid monodromy factorization of $S$. The process is described in details in [AT2] and in [A], and, as mentioned above, the presentation can be accessed in [App1]. However for the sake of this paper it is enough to know that they all have the following forms:

1. order two relations: $\Gamma_j^2 = \Gamma_j'^2 = 1$,
2. conjugates of the generators commute, such as $[\Gamma_3, \Gamma_4] = 1$ or $[\Gamma_8, \Gamma_{13}\Gamma_{13}'\Gamma_{14}\Gamma_{13}'] = 1$,
3. products of conjugates of generators, which have order 3, such as $(\Gamma_{15} \cdot \Gamma_{16}\Gamma_{21}\Gamma_{16})^3 = 1$,
4. $\Gamma_j'$ as a conjugate of $\Gamma_j$, such as $\Gamma_9' = \Gamma_3\Gamma_3'\Gamma_7\Gamma_7'\Gamma_9\Gamma_7'\Gamma_7\Gamma_3'$.

As explained above, there is a natural projection from $\tilde{\Pi}^{Aff}_{18}$ to $S_{18}$. A similar and more abstract situation was studied in [RTV]. Given a simple connected graph $T$ on $n$ vertices, a group $C_Y(T)$ was defined by taking the edges of $T$ as generators, with the following relations:

1. Every generator has order 2 (namely $u^2 = 1$);
2. Two generators commute if their edges are disjoint (so that $[u, v] = 1$);
3. The product of generators $u, v$ has order 3 if their edges intersect (so $(uv)^3 = 1$);
4. If $u, v, w$ are edges meeting in a point then $[u, vvw] = 1$ (denoting by $u, v, w$ the corresponding generators).

We can rephrase the families $(R_2)$ and $(R_3)$ as $uv = vu$ or $uvu = vuv$, and in this form they are called the ‘braid relations’. It is easy to see that such a group naturally maps onto $S_n$ (sending a generator $u$ to the transposition $(ab)$ where $a, b$ are the vertices of $u$). The main result of [RTV] is that $C_Y(T)$ is a semidirect product of $S_n$ and a certain normal co-abelian subgroup of $\pi_1(T)^n$, where of course $\pi_1(T) \cong F_t$ is the free group on $t$ generators (i.e. $t$ is the number of fundamental cycles in $T$). This result was later generalized to the case where $T$ is not a simple graph. In this case $(R_3)$ refers to $u, v$ if they intersect in one vertex, $(R_4)$ refers to three edges $u, v, w$ which intersect in one vertex but cover four vertices together, and there is a fifth family $(R_5)$ of relations, associated to every quadruple of edges of which two edges
connect the same two vertices, and each of the other two edges touches one of these two vertices.

Our group $\tilde{\Pi}^{\text{Aff}}_1$ fits into this general framework, as follows. The action of $\tilde{\Pi}^{\text{Aff}}_1$ on the 18 planes is defined by $\Gamma_j$ and $\Gamma_{j'}$ switching the two planes intersecting in the line $j$. Therefore, let $\hat{T}$ be the dual of the skeleton, namely the graph whose vertices are the planes of $X_0$, with two edges connecting every two intersecting planes (one edge for $\Gamma_j$ and one for $\Gamma_{j'}$), depicted as the solid lines in Figure 2 (there are 18 vertices and 54 edges). In [ATV] we discovered that $\tilde{\Pi}^{\text{Aff}}_1/\langle \Gamma_j = \Gamma_{j'} \rangle$ is a quotient of $C_\gamma(T)$ where $T$ is the graph obtained from $\hat{T}$ by identifying pairs of edges connecting the same two vertices. Therefore it was expected that $\tilde{\Pi}^{\text{Aff}}_1$ is a quotient of $C_\gamma(\hat{T})$.

3.1. **Trivial simplification.** Before we start treating the presentation, we notice that the computations performed in [A] frequently produce unnecessarily long relations. Taking into account the fact that all generators have order 2, we may remove subwords of the form $\gamma\gamma$ from

![Figure 2. the graph $\hat{T}$](image-url)
the relations; furthermore, whenever $\gamma$ and $\delta$ are known to commute, we may replace any subword of the form $\gamma\delta\gamma$ by $\delta$.

Applying these simple observations reduce the number of distinct relations from 1692 (with total length 34644) to 1599 (with total length 18186).

3.2. The braid relations. A possible attack on $\tilde{\Pi}_{1}^{\text{Aff}}$ would be to prove that it satisfies all the five families of relations $(R_1)$–$(R_5)$ defining $C_Y(\hat{T})$, making $\tilde{\Pi}_{1}^{\text{Aff}}$ a quotient of $C_Y(\hat{T})$, which is very well understood. This would enable us to get a satisfying description of $\tilde{\Pi}_{1}^{\text{Aff}}$. However, while we do know that $\Gamma_{2j}^2 = \Gamma_{2j'}^2 = 1$ and some of the braid relations are also given, it is difficult to prove the missing ones (in particular the families $(R_4)$ and $(R_5)$) from the known relations. We went half-way in this direction, as follows.

Our first step was to prove that in $\tilde{\Pi}_{1}^{\text{Aff}}$ the generators $\Gamma_j, \Gamma_{j'}$ satisfy the braid relations $(R_2)$ and $(R_3)$, namely that $\Gamma_{(j)}\Gamma_{(i)}$ has order 2 or 3 depending on whether or not the corresponding edges intersect (for $i \neq j$). Here, $\Gamma_{(j)}$ stands for either $\Gamma_j$ or $\Gamma_{j'}$. There are $4\binom{27}{2} = 1404$ braid relations to prove, out of which we expect $4 \cdot 27 \cdot (27 - 5)/2 = 1188$ pairs to commute (for every $j$ there are four lines $i$ sharing a common plane with the line $j$); the other $4 \cdot 27 \cdot 4/2 = 216$ products are expected to have order 3. In practice, only 775 commutators of the expected 1188 ones, and 152 of the expected 216 products of order 3, are given as relations in the presentation obtained above. This leaves 477 braid relations which we proved one by one, by hand. This part of the computation was the most time consuming. The method behind this lengthy computation is explained in details in [ATV], where the same computation was performed on a smaller scale (after identifying $\Gamma_{j'} = \Gamma_j$).

3.3. Shortening the presentation. Now that the braid relations are known to hold, our next step involves an automatic simplification of the relations using a computer program.

It is well known that the word problem is not solvable in a general presented by generators and relations. Our algorithm is therefore much less ambitious: given a presentation, we are looking for relations which
can be replaced by shorter ones. For every two relations, the algorithm is searching for the intersection of the two words (allowing rotations). Letting $w$ denote the intersection, the two relations can be written as $ww_1 = ww_2 = 1$ for suitable words $w_1, w_2$. If $\text{len}(w) > \text{len}(w_1)$, then the relation $ww_2 = 1$ can be replaced by $w_1^{-1}w_2 = 1$, which is shorter. Furthermore, to somewhat reduced the chances of getting stuck in local minima, the algorithm tosses a coin and replaces $ww_2 = 1$ by $w_1^{-1}w_2 = 1$ with probability one half, if $\text{len}(w) = \text{len}(w_1)$.

To speed things up, we use the fact that many pairs of generators commute. Every few cycles, we scan all the relations for subwords of the form $\Gamma_jw\Gamma_j^{-1}$, in which all the generators composing $w$ are known to commute with $\Gamma_j$. In this case we replace $\Gamma_jw\Gamma_j^{-1}$ by $w$.

The algorithm keeps scanning all pairs of relations as long as possible reductions are found, and then stops. As an indication of the performance of this simplistic algorithm in our situation, we note that the 2331 relations of total length 21272 mentioned before were transformed to a list of 1662 relations, of total length 9312.

It should be noted that the same algorithm could be ran before adding the braid relations; however at that stage almost no shortening occurs since most of the original relations are fairly complicated.

3.4. Removing generators. When the algorithm described above stops, there is no apparent way to shorten the presentation further. The problem is that in order to apply what we know on $C_{\gamma}(\hat{T})$ we still have to prove the relations in families $(R_4)$ and $(R_5)$, and this is quite difficult.

Some of the relations express a generator in terms of others, so it is possible to remove this generator by substitution. For example, we find that

$$\Gamma_{5'} = \Gamma_{11}\Gamma_{11}'\Gamma_4\Gamma_5\Gamma_4'\Gamma_5\Gamma_4\Gamma_{11}'\Gamma_{11}.$$ 

The ‘algorithm’ applied here was to locate a single generator $\Gamma_j'$ (for $j = 1, \ldots, 27$) that can be replaced by a relatively short product, and perform the substitution. Then we run the shortening algorithm of the previous subsection again. To save space in what follows, we write $j$
for $\Gamma_j$ and $j'$ for $\Gamma_{j'}$. The first substitution was

$$27' = 27 \ 18 \ 17' \ 20 \ 21 \ 19' \ 21 \ 20 \ 17' \ 18 \ 27,$$

then

$$26' = 22' \ 23' \ 25' \ 24' \ 27 \ 24' \ 25' \ 23' \ 22',$$

and then the following:

$$
\begin{align*}
25' &= 16' \ 11' \ 10' \ 12' \ 9' \ 12' \ 10' \ 11' \ 16' \\
21' &= 17 \ 19 \ 19' \ 21 \ 17' \ 21 \ 19' \ 19 \ 17 \\
17' &= 17 \ 14 \ 1 \ 7 \ 14 \ 17 \ 14 \ 1 \ 7' \ 14 \ 17 \ 14 \ 17 \ 14 \ 17 \\
15' &= 16 \ 21 \ 26 \ 13 \ 14' \ 13 \ 26 \ 21 \ 16 \\
9' &= 3 \ 7 \ 7' \ 3' \ 9 \ 3' \ 7' \ 7 \\
24' &= 12 \ 7 \ 20 \ 6' \ 8' \ 6' \ 20 \ 7 \ 12 \\
22' &= 22 \ 2' \ 13' \ 13 \ 22 \ 22 \ 13 \ 13' \ 2' \ 22 \\
20' &= 8' \ 6' \ 6 \ 20 \ 8' \ 8' \ 6 \ 20 \ 6' \ 8' \\
14' &= 13 \ 21 \ 15 \ 13' \ 13 \ 16 \ 26 \ 16 \ 13 \ 13' \ 15 \ 21 \ 13 \\
7' &= 6 \ 8' \ 8 \ 7 \ 6' \ 7 \ 8 \ 8' \ 6 \\
5' &= 2 \ 10 \ 2' \ 5 \ 10' \ 5 \ 2' \ 10 \ 2 \\
23' &= 22 \ 13 \ 1 \ 6 \ 4 \ 5 \ 10' \ 18 \ 17 \ 9 \ 7 \ 14 \ 1 \ 7 \ 9 \ 3 \\
&\quad \cdot \ 17 \ 14 \ 17 \ 3 \ 9 \ 7 \ 1 \ 14 \ 7 \ 9 \ 17 \ 18 \ 10' \ 5 \ 4 \ 6 \ 1 \ 13 \ 22 \\
16' &= 11 \ 25 \ 24 \ 20 \ 10' \ 8' \ 8 \ 8' \ 6 \ 7 \ 9 \ 7 \ 6 \ 8' \ 8 \ 8' \ 10' \ 20 \ 24 \ 25 \ 11 \\
13' &= 1 \ 6 \ 4' \ 13 \ 4 \ 6 \ 1 \ 6 \ 4 \ 13 \ 4' \ 6 \ 1 \\
12' &= 7 \ 6 \ 24 \ 20 \ 8' \ 8 \ 8' \ 20 \ 24 \ 6 \ 7 \\
6' &= 6 \ 1 \ 13 \ 22 \ 4' \ 6 \ 4 \ 2 \ 6 \ 4' \ 6 \ 4 \ 2 \ 4 \ 6 \ 22 \ 13 \ 1 \ 6 \\
2' &= 22 \ 4 \ 6 \ 13 \ 1 \ 13 \ 6 \ 4 \ 22 \\
1' &= 1 \ 6 \ 4 \ 13 \ 2 \ 22 \ 2 \ 13 \ 4 \ 6 \ 1 \\
18' &= 23 \ 3 \ 10' \ 2 \ 5 \ 2 \ 10' \ 3 \ 23 \\
11' &= 15 \ 19' \ 8 \ 5 \ 4' \ 5 \ 8 \ 19' \ 15 \\
8' &= 6 \ 7 \ 12 \ 20 \ 24 \ 20 \ 12 \ 7 \ 6 \\
3' &= 17 \ 9 \ 7 \ 14 \ 1 \ 14 \ 7 \ 9 \ 17
\end{align*}
$$

The original plan was to replace all the generators $\Gamma_1, \ldots, \Gamma_{27'}$, so
the group will be generated by $\Gamma_1, \ldots, \Gamma_{27}$ which correspond to the sim-
ple graph $T$. However after the above listed substitutions, $\Gamma_1, \ldots, \Gamma_{27}$
remain, together with $\Gamma_{4'}, \Gamma_{10'}$ and $\Gamma_{19'}$. According to the current rela-
tions, each one of these three generators could be replaced by a product
of few hundred generators, a substitution we did not want to perform.
Fortunately, at this stage there were fairly short substitutions for $\Gamma_4, \Gamma_{10}$ and $\Gamma_{19}$, as follows:

\begin{align*}
19 &= 15 8 4 11 5 11 4 8 15 \\
4 &= 2 22 6 13 7 9 14 17 3 17 14 9 7 13 6 22 2 \\
10 &= 5 9 7 1 14 17 18 2 23 2 18 17 14 1 7 9 5.
\end{align*}

The group $\tilde{\Pi}_1^{\text{Aff}}$ is now shown to be generated by

$$\Delta = \{\Gamma_j\}_{j \neq 4,10,19} \cup \{\Gamma_4', \Gamma_{10}', \Gamma_{19}'\}.$$ 

These generators are in a natural one to one correspondence with the edges of the simple graph $T$.

### 3.5. Quotient of $C_Y(T)$.

In the previous subsection we expressed half of the original generators of $\tilde{\Pi}_1^{\text{Aff}}$ as words on the set $\Delta$, containing only 27 generators. During the process, the number of relations was reduced to 797 (however the total length went up to 92610). The complete presentation on the 27 generators can be found in [App2] (see the Appendix). We already know that the elements of $\Delta$ have order 2 (so $(R_1)$ is satisfied), and it is now easy to verify that also satisfy the braid relations $(R_2)$ and $(R_3)$. In order to prove that $\tilde{\Pi}_1^{\text{Aff}}$ is a quotient of $C_Y(T)$, we only need to show that the fourth family $(R_4)$ of relations holds (relations from $(R_5)$ no longer exist, since the current graph of generators, $T$, is simple).

For example, we need to show that $[\Gamma_4', \Gamma_6 \Gamma_8 \Gamma_6] = 1$ (lines 4,6, and 8 of the skeleton bound plane numbered 3 in Figure 1, dually, which is the language we prefer here, edges 4, 6, 8 of $T$ intersect at the point numbered 3 in Figure 2). Likewise we need $[\Gamma_1, \Gamma_6 \Gamma_7 \Gamma_6] = 1$ and so on, one relation for each of the 18 triangles of Figure 1.

It so happens that these relations are all present in the current list of relations. Thus we proved the claim: $\tilde{\Pi}_1^{\text{Aff}}$ is a quotient of $C_Y(T)$ for the graph $T$ defined before. We note that $\pi_1(T)$ is the free group of rank 10, and so $C_Y(T)$ is a semidirect product of $S_{18}$ and a normal co-abelian subgroup of $\mathbb{F}_{10}^{\text{pl}}$ (details are given below).
4. Simplifying the presentation

4.1. The structure of $C_Y(T)$. The fundamental group of $T$ is freely generated by 10 generators. To see this, choose a spanning subtree $T_0$ (which will contain $18 - 1 = 17$ edges since $T$ connects 18 vertices); then there are $27 - 17 = 10$ basic cycles, since $T$ has 27 edges. For the purpose of this paper we choose the subtree to be the complement of $\Omega = \{1, 2, 3, 4, 7, 10, 13, 15, 17, 23\}$, as in Figure 3 ($T_0$ is shown in double solid lines). Although arbitrary, the same choice was made in [ATV]. Figure 3 also provides direction for the edges outside of $T_0$, to be used later. These arrows are labelled by the edge numbers, taken from Figure 1.

Recall that we set $n = 18$. For every $\alpha \in \{1, \ldots, n\}$, let $F_\alpha$ denote the free group generated by the symbols $1_\alpha, 2_\alpha, \ldots, 17_\alpha, 23_\alpha$, namely $\{\omega_\alpha\}_{\omega \in \Omega}$. Let $F^*$ denote the direct product of the groups $F_1, \ldots, F_n$, so that $F^* \cong (\mathbb{F}_{10})^n$. The symmetric group $S_n$ acts on $F^*$ by its action on the indices.
Let $e_1, e_2, \ldots, e_{17}, e_{23}$ be generators of a free abelian group $\mathbb{Z}^{10}$, and define a map $ab : F^* \to \mathbb{Z}^{10}$ by $ab(\omega_\alpha) = e_\omega$ for every $\omega \in \Omega$ and $\alpha = 1, \ldots, n$. Let $F$ denote the kernel of this map, and note that $\omega_\alpha \omega_\beta^{-1} \in F$ for every $\omega \in \Omega$ and $\alpha, \beta = 1, \ldots, n$; in fact these elements generate $F$. Obviously $F$ is preserved under the action of $S_n$.

For an edge $u \in T$, let $\alpha_u, \beta_u$ denote the end vertices of $u$ (in this order, if $u$ is ordered).

**Definition 4.1.** Define a map

$$(4) \quad \Phi : C_Y(T) \to S_n \rtimes F$$

by sending $u$ to the transposition $(\alpha_u, \beta_u)$ if $u \in T_0$, and to $(\alpha_\omega, \beta_\omega) \omega_\beta^{-1} \omega_\alpha$ if $u = \omega \in \Omega = T - T_0$.

For example, $\Phi(\Gamma_6) = (2 \ 3)$, $\Phi(\Gamma_1) = (2 \ 7) 1_7^{-1} 1_2$ and $\Phi(\Gamma_7) = (2 \ 6) 7_6^{-1} 7_2$. In [RTV] it is shown that $\Phi$ is a well defined isomorphism. Moreover, the natural projection from $C_Y(T)$ to $S_n$ becomes the projection onto the first component of $S_n \rtimes F$.

**4.2. $\tilde{\Pi}_1^{\text{Aff}}$ as a quotient of $S_n \rtimes F$.** We proved in Subsection 3.5 that $\tilde{\Pi}_1^{\text{Aff}}$ is a quotient of $C_Y(T)$, more precisely there is a set $R$ of 797 elements of $C_Y(T)$ such that $\tilde{\Pi}_1^{\text{Aff}}$ is isomorphic to $C_Y(T)$ modulo the normal subgroup generated by $R$.

Since the map from $C_Y(T)$ to $S_n$ splits through $\tilde{\Pi}_1^{\text{Aff}}$, $\Phi$ (Definition 4.1) maps the defining relations $R$ of $\tilde{\Pi}_1^{\text{Aff}}$ into $F$. Let $N$ denote the normal subgroup of $S_n \rtimes F$ generated by these images. Obviously $N$ is a normal subgroup of $F$, which is moreover invariant under the action of $S_n$. See Figure 4. It follows that $\Phi$ restricts to an isomorphism

$$(5) \quad \Phi : \tilde{\Pi}_1^{\text{Aff}} \to S_n \rtimes F/N.$$ 

We can now rewrite the defining relations in terms of the generators $\omega_\alpha$ of $F$ ($\omega \in \Omega$, $\alpha = 1, \ldots, n$). Following the example given after Definition 4.1, the relation $(\Gamma_6 \Gamma_7 \Gamma_6 \Gamma_1)^2 = e$ translates to

$$(2 \ 3)(2 \ 6)7_6^{-1} 7_2(2 \ 7) 1_7^{-1} 1_2)^2 = e,$$

which is equivalent to $7_3^{-1} 17_6^{-1} 1_2^{-1} 1_7 7_6^{-1} 17_3^{-1} 1_2 = e$, trivially holding by the definition of $F$. 

When this process is applied to the 797 defining relations, almost all vanish (namely they express the trivial word in \(F\)). We are left with 102 relations, from which the relation below is one of the shortest:

\[
1^{-1}13_1^{-1}2_1^{-1}10_4^{-1}17_10_4^{-1}10_13_10_7_10_14_113_14_12_14_17_14_14_14_14 = e.
\]

These 102 elements generate \(N\) (as a normal and invariant subgroup). Since \(N\) is invariant under the action of \(S_n\), the indices in the relations can be made arbitrary. Taking the indices to be the ‘generic’ \(i, j, k, l\) and removing duplicates, the 102 relations collapse into a list of 66 relations. This list can be found in [App3] (see the Appendix).

Next, if a relation has the form \(\omega_iu_i\omega_i^{-1}w_j = e\) for \(u_i, \omega_i \in F_i, w_j \in F_j\) and \(i \neq j\), then \(\omega_i\) commutes with \(w_j\) and so it is equivalent to \(u_iw_j = e\). For example,

\[
7_1^{-1}13_1^{-1}2_1^{-1}4_1^{-1}7_14_2^{-1}13_14_17_14^{-1} \cdot 7_1^{-1}13_1^{-1}2_1^{-1}4_1^{-1}7_14_2^{-1}13_14_1 = e.
\]

becomes

\[
(6) \quad 1_1^{-1}13_1^{-1}2_1^{-1}4_1^{-1}7_14_2^{-1}13_14_17_14^{-1} \cdot 7_1^{-1}13_1^{-1}2_1^{-1}4_1^{-1}7_14_2^{-1}13_14_1 = e.
\]

‘Cleaning’ the 66 relations from such conjugations (which can be done by hand), we obtain a list of 49 relations, all of the forms \(u_i = e\) (a single relation), \(u_i \cdot w_j = e\) (29 relations), \(u_i \cdot w_j \cdot v_k = e\) (13 relations) or \(u_i \cdot w_j \cdot v_k \cdot x_l = e\) (6 relations) for distinct \(i, j, k, l\) and various elements \(u_i \in F_i, w_j \in F_j, v_k \in F_k, x_l \in F_l\).

4.3. Breaking \(N\) into sections. The subgroup \(N\) defined above is generated, as a normal subgroup of \(F\) invariant under the action of \(S_n\), by 49 elements. Let \(K = F/N\), so that

\[
\tilde{\Pi}_1^\text{Aff} \cong S_n \rtimes K.
\]
For convenience, we would like to work in $F^*$ rather than $F$ (see Subsection 4.1 for the definitions). It is easy to see that a normal subgroup of $F$ which is generated by elements with at least one trivial entry in $F^* = F_1 \times \cdots \times F_n$, is also normal in $F^*$. In particular $N$ is normal in $F^*$. We now define

\begin{equation}
K^* = F^*/N.
\end{equation}

Since $N \leq F$, the map $ab : K^* \to \mathbb{Z}^{10}$ is well defined, and has $K$ as its kernel. As in $F$, $K$ is generated by the elements $\omega_\alpha \omega_\beta^{-1}$ (i.e. their images in $K^*$).

The images of a word in $F^* = F_1 \times \cdots \times F_{18}$ under the projections to the $F_i$ are called ‘sections’; e.g. the sections of $u_i w_j v_k$ are $u_i, w_j$ and $v_k$ (where $u_i \in F_i$, $w_j \in F_j$ and $v_k \in F_k$). Let $\hat{N}$ be the group generated by sections of elements in $N$. Alternatively, $\hat{N}$ is generated (as a normal and invariant subgroup of $F^*$) by the $122 = 1 \cdot 1 + 29 \cdot 2 + 13 \cdot 3 + 6 \cdot 4$ sections of the 49 generators of $N$. Obviously $N \subseteq \hat{N}$.

**Proposition 4.2.** $\hat{N}/N$ is central in $F^*/N$. Moreover, $\hat{N}/N$ is fixed (element-wise) under the action of $S_n$.

**Proof.** The first claim holds for any normal subgroup of $F^*$. Indeed, let $u = u_1 u_2 \cdots u_n$ be an element of $N$ (with $u_i \in F_i$), and consider the section $u_i$. By definition of $F^*$, $u_i$ commutes with every generator $\omega_j$ for $j \neq i$. But at the same time, $u_i$ can be expressed (modulo $N$) as a product of the other sections of $u$, and so it also commutes with the generators $\omega_i$.

For the second claim we need to know that $N$ is invariant and generated by words with less than $n$ non-trivial sections (this is indeed the case in $N$, where the generators have at most 4 sections). Without loss of generality let $u_1 u_2 \cdots u_k \in N$ where $k < n$, then $\tau(u_1) u_2 \cdots u_k \in N$ for $\tau$ the transposition $(1 \, n)$, so modulo $N$ we have that $\tau(u_1) \equiv u_1$. Acting now with an arbitrary permutation $\sigma \in S_n$ which fixes the index 1, we get that $\sigma(u_1) \equiv u_1$ and so $u_1$ is fixed under $S_n$ (as an element of $\hat{N}/N$). \qed
By definition, every section of an element of $\hat{N}$ is also in $\hat{N}$, so that $\hat{N} = (F_1 \cap \hat{N}) \cdots (F_n \cap \hat{N})$. It follows that

$$F^*/\hat{N} = (F_1 \times \cdots \times F_n)/\hat{N} \cong H_1 \times \cdots \times H_n,$$

where $H_i = F_i/(F_i \cap \hat{N})$. Moreover, the groups $H_i$ are all isomorphic, being permuted by $S_n$.

As an illustration, recall the generator of $N$ from Equation (6). The first section is

$$1^{-1} \cdot 13^{-1} \cdot 2 \cdot 4^{-1} \cdot 7 \cdot 4 \cdot 2^{-1} \cdot 13 \cdot 1 \cdot 7^{-1} \in \hat{N},$$

which can written shortly as

$$1^{-1} \cdot 13^{-1} \cdot 2 \cdot 4^{-1} \cdot 7 \cdot 4 \cdot 2^{-1} \cdot 13 \cdot 1 \cdot 7^{-1},$$

to represent an element of $(N \cdot F_i \cap \hat{N})/N$ for arbitrary $i = 1, \ldots, n$.

4.4. The group $H$. The decomposition of Equation (8) presents $K^* = F^*/N$ modulo its central subgroup $\hat{N}/N$ as a product of $n = 18$ isomorphic groups, which from now on we denote by $H$. This group is generated by $\Omega = \{1, 2, \ldots, 17, 23\}$ with 122 relations obtained from the sections of the previous 49 generators of $N$. It turns out that the
122 relations include some repetition, so we only have 97 relations. Here are the shortest seven of them:

\[
10 \cdot 7 \cdot 10^{-1} \cdot 4^{-1} \cdot 15 = e
\]
\[
17^{-1} \cdot 1 \cdot 3^{-1} \cdot 17^{-1} \cdot 13^{-1} \cdot 15^{-1} \cdot 1 \cdot 7^{-1} \cdot 3^{-1} = e
\]
\[
17 \cdot 3 \cdot 1^{-1} \cdot 17 \cdot 3 \cdot 7 \cdot 1^{-1} \cdot 15 \cdot 13 = e
\]
\[
4^{-1} \cdot 1^{-1} \cdot 17 \cdot 15 \cdot 13 \cdot 1 \cdot 7^{-1} \cdot 1^{-1} \cdot 13^{-1} \cdot 2 = e
\]
\[
4^{-1} \cdot 7^{-1} \cdot 4 \cdot 2^{-1} \cdot 13 \cdot 1 \cdot 7 \cdot 1^{-1} \cdot 13^{-1} \cdot 2 = e
\]
\[
4^{-1} \cdot 7 \cdot 4 \cdot 2^{-1} \cdot 13 \cdot 1 \cdot 7^{-1} \cdot 1^{-1} \cdot 13^{-1} \cdot 2 = e
\]
\[
4 \cdot 2^{-1} \cdot 13 \cdot 1 \cdot 7 \cdot 1^{-1} \cdot 13^{-1} \cdot 15^{-1} \cdot 17^{-1} \cdot 1 = e
\]

At this point we apply the shortening algorithm described in Subsection 3.3 and obtain 67 relations on the ten generators, of total length 382. See [App4] (see the Appendix) for the complete list. Some of the relations are given below:

\[
10 \cdot 17^{-1} \cdot 7 = e
\]
\[
10 \cdot 7 \cdot 17^{-1} = e
\]
\[
13 \cdot 23^{-1} \cdot 15 = e
\]
\[
4 \cdot 7^{-1} \cdot 15^{-1} = e
\]
\[
4 \cdot 15^{-1} \cdot 7^{-1} = e
\]
\[
7 \cdot 23 \cdot 2 \cdot 17 \cdot 1^{-1} = e
\]
\[
7^{-1} \cdot 23^{-1} \cdot 2^{-1} \cdot 1 \cdot 17^{-1} = e
\]
\[
1 \cdot 7 \cdot 10^{-1} \cdot 4 \cdot 2^{-1} \cdot 4^{-1} = e.
\]

Substituting some of the generators in terms of others and simplifying the relations further, we obtain the following equivalent description of \( H \). Set \( c = 13 \cdot 4 \), so the generator 13 can be replaced by \( c \cdot 4^{-1} \).
Then we obtain the following equalities in $H$:

\begin{align*}
15 &= 7^{-1} \cdot 4, \\
17 &= 10 \cdot 7, \\
23 &= c \cdot 7^{-1}, \\
2 &= c^2 \cdot 7^{-1} \cdot 10^{-1} \cdot 1, \\
3 &= c \cdot 7^{-1} \cdot 10^{-1} \cdot 1.
\end{align*}

(9)

In particular $H = \langle 1, 4, 7, 10, c \rangle$. The shortening algorithm yields the following presentation on these five generators:

\begin{align*}
[1, c] &= [7, c] = [4, c] = [10, c] = e, \\
[10, 1] &= [10, 7] = [4, 7] = e, \\
[1, 7]^{-1} &= [4, 10] = c^3, \\
[4, 1] &= c^{-2}r^2.
\end{align*}

(10)

**Corollary 4.3.** $H$ is nilpotent of class 3.

*Proof.* The commutator subgroup of $H$ is generated by $c^3$ and $c^{-2}r^2$, so $H' = \langle c, 7^2 \rangle$ (isomorphic to $\mathbb{Z}^2$). It follows that $[H, H'] = [H, \langle 7^2 \rangle^H] = [(1)^H, \langle 7^2 \rangle^H] = \langle e^6 \rangle$ which is central. \hfill $\square$

We note that the subgroup $\langle c, 7, 10 \rangle$ is isomorphic to $\mathbb{Z}^3$, and that $H/\langle c, 7, 10 \rangle \cong \mathbb{Z}^2$. Thus $H$ is an extension

\[ 1 \longrightarrow \mathbb{Z}^3 \longrightarrow H \longrightarrow \mathbb{Z}^2 \longrightarrow 1, \]

and so $H$ has derived length 2.

4.5. **Simplifying $N$.** In Subsection 4.3 we proved that $F^*/\hat{N}$ decomposes in a natural way into a product $H_1 \times \cdots \times H_{18}$, with each $H_i$ isomorphic to the group $H$ studied in Subsection 4.4. This group is generated by the 10 generators $\Omega$, with 15 defining relations (out of which, five relations express redundant generators, and the other ten are related to commutators of the remaining generators).

We lift this description back to $K^* = F^*/N$, recalling that $N$ is generated by 49 relations (as a normal subgroup of $F^*$ which is invariant under $S_{18}$). In order to simplify these relations, we apply the defining relations of $H$ — this time not as relations, but by equating each one of them to a newly defined element of $\hat{N}/N \subseteq Z(K^*)$. In other words
(after replacing 13 by \(c_i 4^{-1}\) throughout), we define for \(i = 1, \ldots, 18\),
\[
\theta_i^{(15)} = 15_i 4^{-1} 17_i, \theta_i^{(17)} = 17_i 7_i 10_i 1^{-1},
\]
and likewise \(\theta_i^{(23)}\), \(\theta_i^{(2)}\) and \(\theta_i^{(3)}\) (see Equation (9)). In a similar manner we set \(\theta_i^{(1, c)} = [1_i, c_i]\), \(\theta_i^{(7, e)} = [7_i, c_i]\)
eq 15, \theta_i^{(4, 1)} = [4_i, 1_i] 7_i 2^{-2} c_i^2 (see Equation (10)). The 18-10 equations
of the form \([1_i, c_i] = \theta_i^{(1, c)}\) are called the commutator relations of \(N^\ast\).
Since modulo \(\hat{N}\) all these products become trivial, all the 18 \cdot 15 new
elements \(\theta_i^{(c)}\) just defined, belong to \(\hat{N}/\hat{N}\). It follows from Proposition
4.2 that the \(\theta_i^{(c)}\) are in fact independent of the index \(i\), being fixed under
\(S_n\). Nevertheless, since we anyway have to simplify the 49 relations,
we keep the indices to help tracking the computation.

The first step in simplifying the generators of \(N\) is to express
the ‘redundant generators’ \(15_i, 17_i, 23_i, 2_i, 3_i\) in terms of \(1_i, 4_i, 7_i, 10_i, c_i\)
and the variables \(\theta_i^{(c)}\). For example, the relation given in Equation (6)
becomes
\[
1_i^{-1} 4_i c_i^{-1} \theta_i^{(2)} c_i 2^{7_i} 10_i^{-1} 1_i 4_i^{-1} 10_i 7_i c_i^{-2} \theta_i^{(2)}^{-1} c_i 4_i^{-1} 1_i 7_i^{-1}
\]
\[
\cdot 7_i 1_i^{-1} 4_i c_j^{-1} \theta_j^{(2)} c_j 2^{7_i} 10_i^{-1} 1_j 4_j^{-1} 1_j 7_j c_j^{-2} \theta_j^{(2)}^{-1} c_j 4_j^{-1} 1_j = e,
\]
and then, since \(\theta_i^{(2)}\) is central,
\[
1_i^{-1} 4_i c_i 7_i^{-1} 10_i^{-1} 1_i 4_i^{-1} 10_i 7_i c_i^{-1} 4_i^{-1} 1_i 7_i^{-1}
\]
\[
\cdot 7_i 1_i^{-1} 4_i c_j 7_j^{-1} 10_i^{-1} 1_j 4_j^{-1} 1_j 7_j c_j^{-1} 4_j^{-1} 1_j = e
\]
(it so happens that almost all the relations are balanced in terms the
\(\theta_i^{(c)}\), and so they rarely show up in the simplified relations).

The fact that \(\theta_i^{(c)}\) are all central enables a rather quick simplification
of the generators of \(N\), by translating the commutator relations into
action by conjugation. First,
\[
\begin{align*}
4_i & \cdot c_i 4_i^{-1} = \theta_i^{(4, c)} c_i, \\
4_i & \cdot 7_i 4_i^{-1} = \theta_i^{(4, 7)} 7_i, \\
4_i & \cdot 10_i 4_i^{-1} = \theta_i^{(4, 10)} c_i^2 10_i, \\
4_i & \cdot 1_i 4_i^{-1} = \theta_i^{(4, 1)} c_i^{-2} 7_i 2_i 1_i,
\end{align*}
\]
with trivial action of \(4_i\) on every \(\theta_i^{(c)}\). Using this we can remove \(4_i\) from
the relations; for example (11) becomes
\[
1_i^{-1} c_i 7_i^{-1} 10_i^{-1} c_i^{-5} 7_i^2 1_i 7_i 1_i^{-1} 7_i^{-2} c_i^5 10_i 7_i c_i^{-1} 1_i 7_i^{-1}
\]
\[
\cdot 7_i 1_i^{-1} c_j 7_j^{-1} 10_j^{-1} c_j^{-5} 7_j^2 1_j 7_j^{-1} 1_j^{-1} 7_j^{-2} c_j^5 10_j 7_j c_j^{-1} 1_j = e.
\]
Again the first section is balanced on \( \theta_i^{(4,c)}, \theta_i^{(4,7)}, \theta_i^{(4,1)} \) and \( \theta_i^{(4,10)} \) so they do not show up in the simplified relation, and likewise for the \( \theta_j^{(c)} \) in the second section. This is common among the 49 generators of \( N \), but there are some exception. The effect of these exceptions is described later.

Next, we act by conjugation with \( 10_i \), applying the relations
\[
\begin{align*}
10_i c_i 10_i^{-1} &= \theta_i^{(10,c)} c_i \\
(13) \quad 10_i 1_i 10_i^{-1} &= \theta_i^{(10,1)} 1_i \\
10_i 7_i 10_i^{-1} &= \theta_i^{(10,7)} 7_i,
\end{align*}
\]
so continuing with the example, we obtain
\[
\theta_i^{(10,7)} 1_i^{-1} c_i 7_i^{-1} c_i^{-5} 7_i^2 1_i 7_i 1_i^{-1} 7_i^{-2} c_i 5_i 7_i c_i^{-1} 1_i 7_i^{-1} \\
\cdot \theta_j^{(10,7)} 7_j 1_j^{-1} c_j 7_j^{-1} c_j^{-5} 7_j^2 1_j 7_j 1_j^{-1} 7_j^{-2} c_j 5_j 7_j c_j^{-1} 1_j = e.
\]
(14)

Next we act by conjugation with \( 1_i \), using the relations
\[
\begin{align*}
1_i c_i 1_i^{-1} &= \theta_i^{(1,1)} c_i \\
(15) \quad 1_i 7_i 1_i^{-1} &= \theta_i^{(1,7)} c_i^3 7_i.
\end{align*}
\]
To make the computation easier, we may at times conjugate the relations (i.e. rotate each section), so the acting variables envelope as shortest word as possible. Thus the relation (14) is equivalent to
\[
\begin{align*}
\theta_i^{(10,7)} 1_i 7_i^{-1} 1_i^{-1} c_i 7_i^{-1} c_i^{-5} 7_i^2 1_i 7_i 1_i^{-1} 7_i^{-2} c_i 5_i 7_i c_i^{-1} \\
\cdot \theta_j^{(10,7)} 1_j 7_j 1_j^{-1} c_j 7_j^{-1} c_j^{-5} 7_j^2 1_j 7_j 1_j^{-1} 7_j^{-2} c_j 5_j 7_j c_j^{-1} 1_j = e,
\end{align*}
\]
which becomes
\[
\begin{align*}
\theta_i^{(10,7)} 1_i^{-1} c_i 7_i^{-1} c_i^{-5} 7_i^2 1_i 7_i 1_i^{-1} 7_i^{-2} c_i 5_i 7_i c_i^{-1} \\
\cdot \theta_j^{(10,7)} c_j^{-3} 7_j 1_j^{-1} c_j 7_j^{-1} c_j^{-5} 7_j^2 1_j 7_j 1_j^{-1} 7_j^{-2} c_j 5_j 7_j c_j^{-1} e = e,
\end{align*}
\]
Finally, the last commutator relations translates to
\[
7_i c_i 7_i^{-1} = \theta_i^{(7,c)} c_i,
\]
so we obtain (among the 49 relations)
\[
\theta_i^{(10,7)} 1_i^{-1} \theta_i^{(7,c)} \cdot \theta_j^{(10,7)} \theta_j^{(7,c)}^{-1} = e.
\]
(17)

It should be noted that some relations between the \( \theta_i^{(c)} \) can be proved directly from their definition.
Proposition 4.4. For every $i$, the elements $\theta_i^{(1,c)}$, $\theta_i^{(4,c)}$, $\theta_i^{(7,c)}$, $\theta_i^{(10,c)}$ and $\theta_i^{(10,7)}$ satisfy

\[
\begin{align*}
\theta_i^{(10,c)} &= \theta_i^{(10,7)}^{-2} \theta_i^{(4,c)}^3, \\
\theta_i^{(10,c)}^3 &= e, \\
\theta_i^{(7,c)} &= e, \\
\theta_i^{(4,c)}^3 &= e.
\end{align*}
\]

Proof. Conjugation of (13), (15) and (16) by 4 yields the four equations

\[
\begin{align*}
\theta_i^{(10,c)} - 2 \theta_i^{(10,7)} \theta_i^{(1,c)} - 3 \theta_i^{(7,c)} - 3 &= e, \\
\theta_i^{(7,c)} &= e, \\
\theta_i^{(7,4)} &= e, \\
\theta_i^{(4,c)}^4 &= \theta_i^{(4,c)}^3,
\end{align*}
\]

and conjugation of $1_i \theta_i (1_i)^{-1} = \theta_i^{(1,c)} c_i^3 1_i$ (in (15)) by 10$_i$ yields

\[
\theta_i^{(10,c)}^3 = e.
\]

These five identities are equivalent to the ones listed in the proposition. \hfill \Box

5. The fundamental group of the Galois cover

5.1. A presentation for $K^*$. In the previous subsection we described the simplification process of the 49 relations defining $K^* = F^*/\hat{N}$. Since we apply relations that were already known modulo $\hat{N}$, it is not surprising that the simplification removes all the original generators $1_i$, $4_i$, $7_i$, $10_i$ and $c_i$, and leaves 49 relations on the $\theta_i^{(1)} \in \hat{N}/N$ defined there.

Equation (17) is one example of these. This relation tells us that $\theta_i^{(10,7)} \theta_i^{(7,c)}^{-1}$ is independent of $i$. Working out the relations in this manner, it turns out that all the 15 elements $\theta_i^{(j)}$ are independent of the lower index, namely $\theta_i^{(15)} = \theta_j^{(15)}$, $\theta_i^{(17)} = \theta_j^{(17)}$, and so on. Therefore we may remove the lower index and refer to the elements as $\theta^{(15)}$, $\theta^{(17)}$...
etc. As mentioned above, this was already known when we started the computation, from Proposition 4.2.

Eventually, the 49 relations in $N$ translate to the invariance of all the $\theta_i^{(\cdot)}$, plus three more relations, which refer to the four generators $\theta^{(1,c)}, \theta^{(7,c)}, \theta^{(10,c)}, \theta^{(10,7)}$.

\begin{align*}
\theta^{(10,c)^3} &= e, \\
\theta^{(1,c)^6} &= e, \\
\theta^{(10,c)} &= \theta^{(10,7)^{-2}}\theta^{(7,c)^{-8}}\theta^{(1,c)^{-3}}.
\end{align*}

Combining this with Proposition 4.4, the relations on the $\theta^{(\cdot)}$ can be summarized by

\begin{align*}
\theta^{(10,c)} &= \theta^{(10,7)^4}, \\
\theta^{(10,7)^6} &= \theta^{(1,c)^3}, \\
\theta^{(7,c)} &= e, \\
\theta^{(4,c)^3} &= e, \\
\theta^{(1,c)^6} &= e.
\end{align*}

(18)

It is worth noting that the five elements involved in the above relations are all pure commutators (i.e. $\theta^{(1,c)} = [1_i, c_i]$ etc., unlike $\theta^{(4,1)}$ or $\theta^{(1,7)} = [1_i, 7_i][c_i^3$ which are not commutators). Let $\Omega' = \{1, 2, 3, 4, 7, 10, c, 15, 17, 23\}$ (i.e. 13 is replaced by $c$ in $\Omega$). Also let $\Omega_0 = \{c, 7, 1, 10, 4\}$, ordered in this order (smaller first), and set $\Theta = \{\theta^{(\omega)} : \omega \in \Omega' - \Omega_0\} \cup \{\theta^{(\omega_1, \omega_2)} : \omega_1, \omega_2 \in \Omega_0, \omega_1 < \omega_2\}$, the set of 15 central elements discussed so far. We obtain the following presentation:

Corollary 5.1. The group $K^* = F^*/N$ is generated by the $18 \cdot 10$ generators $\omega_i$ for $\omega \in \Omega'$ and $i = 1, \ldots, 18$ and the 15 central generators
\[ 15_i = \theta^{(15)} 7_{i}^{-1} 4_i, \]
\[ 17_i = \theta^{(17)} 10_i 7_i, \]
\[ 23_i = \theta^{(23)} c_i 7_i^{-1}, \]
\[ 2_i = \theta^{(2)} c_i^2 7_i^{-1} 10_i^{-1} 1_i, \]
\[ 3_i = \theta^{(3)} c_i 7_i^{-1} 10_i^{-1} 1_i, \]
\[ [7_i, c_i] = \theta^{(7,c)}, \]
\[ [1_i, c_i] = \theta^{(1,c)}, \]
\[ [1_i, 7_i] = \theta^{(1,7)} c_i^{-3}, \]
\[ [10_i, c_i] = \theta^{(10,c)}, \]
\[ [10_i, 7_i] = \theta^{(10,7)}, \]
\[ [10_i, 1_i] = \theta^{(10,1)}, \]
\[ [4_i, c_i] = \theta^{(4,c)}, \]
\[ [4_i, 1_i] = \theta^{(4,1)} c_i^{-2} 7_i^2, \]
\[ [4_i, 7_i] = \theta^{(4,7)}, \]
\[ [4_i, 10_i] = \theta^{(4,10)} c_i^3, \]

and
\[ \theta^{(10,c)} = \theta^{(10,7)}^4, \]
\[ \theta^{(10,7)}^6 = \theta^{(1,c)}^3, \]
\[ \theta^{(7,c)} = e, \]
\[ \theta^{(4,c)}^3 = e, \]
\[ \theta^{(1,c)}^6 = e. \]

Proposition 5.2. The subgroup \( \langle \Theta \rangle \) of \( K^* \) is isomorphic to \( \mathbb{Z}^{10} \times (\mathbb{Z}/3)^2 \times \mathbb{Z}/12 \).

Proof. It is easy to see that the abstract group generated by \( \Theta \), subject only to the relations given in Equation (19), is indeed isomorphic to \( \mathbb{Z}^{10} \times (\mathbb{Z}/3)^2 \times \mathbb{Z}/12 \).
In order to show that the natural map from the abstract group \( \langle \Theta \rangle \) into \( K^* \) is indeed an embedding, we need to construct \( K^* \) as an extension of \( \langle \Theta \rangle \). This can easily be done as a sequence of HNN extensions. First adjoin the elements \( c_i \) (all inducing the trivial automorphism on \( \langle \Theta \rangle \)). Then adjoin \( 7_1, \ldots, 7_{18} \) with \( 7_i \) commuting with every generator \( \omega_j (j \neq i) \) and with the elements of \( \Theta \), and acting on \( c_i \) by \( c_i \mapsto \theta^{(7,c)} c_i \). The automorphisms commute, and so we can form the (repeated) HNN extension by \( 7_1, \ldots, 7_{18} \).

In a similar manner we adjoin the \( 1_i \), then the \( 10_i \) and finally the \( 4_i \). In each step one needs to verify that the map we want to be induced is indeed an automorphism, but this was done (implicitly) in the proof Proposition 4.4.

**Remark 5.3.** The quotient \( K^*/\langle \Theta \rangle \) is isomorphic to \( H^{18} \) (by inspection, see the definition of \( H \)).

**Proposition 5.4.** The center of \( K^* \) is generated by \( \Theta \) and \( c_6^1, \ldots, c_6^{18} \). In particular \( \mathbb{Z}(K^*) \cong \mathbb{Z}^{18} \times \langle \Theta \rangle \), and \( K^*/\mathbb{Z}(K^*) \cong (H/\langle \theta^5 \rangle)^{18} \).

**Proof.** Since \( K^* \) can be viewed as an iterated HNN extension of \( \langle \Theta \rangle \) by the \( c_i, 7_i, 1_i, 10_i, 4_i \), every element can be written as a product \( \theta_0 \prod_{i=1}^{18} c_i^{n_{c,i}} 7_i^{n_{7,i}} 1_i^{n_{1,i}} 10_i^{n_{10,i}} 4_i^{n_{4,i}} \) for \( \theta_0 \in \langle \Theta \rangle \). Assume such an element is central. Conjugating by \( 7_i \) proves that \( n_{1,i} = n_{4,i} = 0 \), since \( \theta^{(1,7)} \) and \( \theta^{(4,7)} \) have infinite order. Likewise conjugation by \( 4_i \) shows \( n_{7,i} = n_{10,i} = 0 \). Eventually conjugation by \( 1_i \) shows \( n_{c,i} \) is divisible by 6, and on the other hand it is easy to see that \( c_i^6 \) are central. \( \square \)

Since \( F^*/N)/(\tilde{N}/N) = F^*/\tilde{N} \) is known to be isomorphic to \( H^{18} \) (Subsection 4.4), we have a short exact sequence

\[
1 \rightarrow \mathbb{Z}^{10} \times (\mathbb{Z}/3)^2 \times \mathbb{Z}/12 \rightarrow K^* \rightarrow H^{18} \rightarrow 1.
\]

In particular from Corollary 4.3 we conclude that \( K^* \) is nilpotent of class at most 4. However, we have

**Proposition 5.5.** \( K^* \) is nilpotent of class 3. In more details:

a. The commutator subgroup of \( K^* \) is

\[
[K^*, K^*] = \langle \theta^{(1,c)}, \theta^{(1,7)} c_i^{-3}, \theta^{(10,7)}, \theta^{(10,1)}, \theta^{(4,c)} c_i^{-2} 7_i^2, \theta^{(4,7)}, \theta^{(4,10)} c_i^3 \rangle,
\]
isomorphic to $\mathbb{Z}^5 \times (\mathbb{Z}/3)^2 \times \mathbb{Z}/12$.

b. The next term in the upper central series is

$$[K^*, [K^*, K^*]] = \langle \theta^{(1,c)}^3, \theta^{(1,c)} \theta^{(1,7)}^2 c_i^{-6}, \theta^{(4,c)} \theta^{(4,7)}^2 \rangle,$$

isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}/2$.

c. $[K^*, [K^*, [K^*, K^*]]] = 1$ and in particular $[[K^*, K^*], [K^*, K^*]] = 1$.

d. $[K^*, K^*]/[K^*, [K^*, K^*]] \cong \mathbb{Z}^3 \times (\mathbb{Z}/6)^3$.

Proof. Direct computation from the defining relations of $K^*$. \qed

Remark 5.6. The abelianization of $K^*$ is isomorphic to $\mathbb{Z}^{61} \times (\mathbb{Z}/6)^{17}$.

Proof. The abelianization is generated by $\Theta$ and the $5 \cdot 18$ elements $c_i, 1_i, 7_i, 10_i, 4_i$. Setting the commutators to be trivial, we are left only with $\theta^{(2)}, \theta^{(3)}, \theta^{(15)}, \theta^{(17)}, \theta^{(23)}$ and $\theta^{(1,7)}, \theta^{(4,1)}, \theta^{(4,10)}$ from $\Theta$, with the relations $\theta^{(4,10)} c_i \equiv \theta^{(1,7)} c_i^3$ and $c_i^2 \equiv \theta^{(4,1)} T_i^2$. This is equivalent to $c_i \equiv \theta^{(1,7)} \theta^{(4,1)} \theta^{(4,1)} T_i^2$ and

$$T_i^6 \equiv \theta^{(1,7)} T_i^2 \theta^{(4,1)}^{-3}.$$

This can further be replaced by $(7_i T_i^{-1})^6 \equiv 1$ for $i > 1$, and $\theta^{(4,1)} \equiv (T_i^{-3} \theta^{(1,7)} \theta^{(4,1)})^{-1}$, so by a base change we obtain the independent generators $\theta^{(2)}, \theta^{(3)}, \theta^{(15)}, \theta^{(17)}, \theta^{(23)}, 1_i, 4_i, 10_i (i = 1, \ldots, 18), 7_i, 7^{-3} \theta^{(1,7)} \theta^{(4,1)}$, and the 17 elements $7_i T_i^{-1}$ ($i > 1$). The result follows. \qed

5.2. Back to $K$. Recall from Subsection 4.3 that $\tilde{\Pi}^{\text{Aff}}_1 \cong S_n \rtimes K$ where $K$ is the kernel of the map $K^* \to \mathbb{Z}^{10}$ defined as the restriction of the map from $F^*$ (see Subsection 4.1).

Corollary 5.7. $K$ is a normal subgroup of the group $K^*$, whose presentation is given in Corollary 5.1. In fact $K$ is the kernel of the map $ab : K^* \to \mathbb{Z}^{10} = \langle e_2, e_3, \ldots, e_{23} \rangle$ induced from $F^* \to \mathbb{Z}^{10}$.
Explicitly, the map $ab$ is defined by sending the generators $\omega_i$ to $e_\omega$ for every $\omega \in \Omega'$, with

\[
\begin{align*}
\theta^{(15)} &\mapsto e_{15} + e_7 - e_4 \\
\theta^{(17)} &\mapsto e_{17} - e_7 - e_{10} \\
\theta^{(23)} &\mapsto e_{23} - e_c + e_7 \\
\theta^{(2)} &\mapsto e_2 - 2e_c + e_7 + e_{10} - e_1 \\
\theta^{(3)} &\mapsto e_2 - e_c + e_7 + e_{10} - e_1 \\
\theta^{(1,7)} &\mapsto 3e_c \\
\theta^{(4,1)} &\mapsto 2e_c - 2e_7 \\
\theta^{(4,10)} &\mapsto -3e_c,
\end{align*}
\]

and the other generators in $\Theta$ map to 0.

**Remark 5.8.** The commutator subgroup of $K$ is $[K, K] = [K^*, K^*]$. It follows that $[[K^*, [K^*, K^*]], K^*] = [K, [K^*, K^*]] = [K, [K, K]]$. Therefore $K$ is nilpotent of class 3 (and not less).

**Proof.** To prove that $[K^*, K^*] \subseteq [K, K]$, let $\omega_i, \omega'_i$ be any two generators of $K^*$. Let $j, k \neq i$ be two distinct indices, then $\omega_i\omega_j^{-1}, \omega'_i\omega'_k^{-1} \in K$ and $[\omega_i\omega_j^{-1}, \omega'_i\omega'_k^{-1}] = [\omega_i, \omega'_i]$.

For the corollary, note that $[B, [A, A]] \subseteq [A, [B, B]]$ for every two groups $A \subseteq B$ by the three subgroups lemma. \hfill \Box

Since $\tilde{\Pi}_1^{\text{Aff}} \cong S_n \rtimes K$, we proved our main result:

**Theorem 5.9.** $\tilde{\Pi}_1^{\text{Aff}}$ is virtually nilpotent of class 3.

**Proposition 5.10.** The centralizer of $K$ in $K^*$ is generated by $Z(K^*)$ and the element $c_1c_2\cdots c_{18}$.

**Proof.** We know that $K$ is generated by the elements $\omega_\alpha\omega_\beta^{-1}$ for $\omega \in \Omega$ and $\alpha, \beta = 1, \ldots, 18$. Mimicking the argument of Proposition 5.4, we conclude that elements of the centralizer are all of the form

\[
z \cdot \left( \prod c_i \right)^{n_c} \left( \prod 7_i \right)^{n_7} \left( \prod 1_i \right)^{n_1} \left( \prod 10_i \right)^{n_{10}} \left( \prod 4_i \right)^{n_4}
\]

for $z \in Z(K^*)$. Conjugating by $7_\alpha 7_\beta^{-1}$, we conclude that $n_1 = 0$. Likewise conjugation by $10_\alpha 10_\beta^{-1}, 1_\alpha 1_\beta^{-1}$ and $4_\alpha 4_\beta^{-1}$ shows that $n_4 = 0, n_{10} = 0$ and $n_7 = 0$. On the other hand, $\prod c_i$ commutes with $K$. \hfill \Box
Corollary 5.11. The centralizer of $K$ in $K^*$ is $C_{K^*}(K) = Z(K) Z(K^*)$.

Proof. The inclusion $\supseteq$ is trivial, and by the last proposition it remains to prove that $\hat{c} = c_1 \ldots c_{18} \in Z(K) Z(K^*)$. But $ab(\hat{c}^{(1,10)^6}) = 0$, so $\hat{c}^{(1,10)^6} \in K \cap Z(K^*) \subseteq Z(K)$ and $\hat{c} = (\hat{c}^{(1,10)^6})^{(4,10)^{-6}} \in Z(K) Z(K^*)$.

5.3. The projective relation. We denote the affine part of the Galois cover of $X_{\text{Gal}}$ by $X_{\text{Gal}}^{\text{Aff}}$. Recall from the introduction that the fundamental group of $X_{\text{Gal}}$ is the kernel of the projection $\tilde{\Pi}_1 = \pi_1(\mathbb{C}^2 - S)/\langle \Gamma_j, \Gamma^2_j \rangle \to S_n$, namely the bottom line of the diagram in Figure 7 is exact. In a similar manner, the upper line is also exact, and so we know from Equation (5) that $\pi_1(X_{\text{Gal}}^{\text{Aff}}) = K = F/N$ (where $F$ and $N$ are defined in Subsection 4.2).
Let $p$ denote the `projective product' $p = \Gamma_{27} \Gamma_{27} \ldots \Gamma_1 \Gamma_1$ (see Equation (2)). The general theory guarantees that

$$1 \longrightarrow \langle p \rangle \longrightarrow \tilde{\Pi}^\text{Aff} \longrightarrow \tilde{\Pi}_1 \longrightarrow 1$$

is a short exact sequence (and in fact a central extension, see [T, Prop. 3.2]).

Since $p \in K^*$ is clearly mapped to the trivial permutation in $S_n$, it belongs to the kernel $K$, and so the following is again a central extension:

$$1 \longrightarrow \langle p \rangle \longrightarrow K \longrightarrow \pi_1(X_{\text{Gal}}) \longrightarrow 1,$$

so $\pi_1(X_{\text{Gal}})$ (which is the group we are really after) is isomorphic to $K/\langle p \rangle$.

**Theorem 5.12.** The fundamental group $\pi_1(X_{\text{Gal}})$ is nilpotent, of nilpotency class 3.

**Proof.** In Remark 5.8 we saw that $K$ is nilpotent of class 3, and that $[K,[K,K]] \cong \mathbb{Z}^2 \times \mathbb{Z}/2$ (by Proposition 5.5b). Therefore

$$[K/\langle p \rangle, [K/\langle p \rangle, K/\langle p \rangle]] = [K,[K,K]]\langle p \rangle/\langle p \rangle$$

cannot be trivial. \hfill \Box

In order to compute $p$ explicitly, we first apply the substitutions given in Subsection 3.4 obtaining a word of length 3822 on the 27 generators $\Delta$ of Equation (3). This word can be viewed as an element of $C_Y(T)$ (modulo the relations defining $\tilde{\Pi}_1$). Next, since $p$ maps to the trivial permutation, it is an element of $F^*/N$, namely a word in the generators $\omega_\alpha$ ($\omega \in \Omega$ and $\alpha = 1, \ldots, 18$); in fact $p \in K = F/N$ since $p$ comes from $C_Y(T)$. However, $F/N$ is a central extension of $F/\hat{N} \cong H_{18}$ by $\hat{N}/N$ (see Proposition 4.2), so we can decompose $p$ as a product of elements from these groups. Note that $p$ being central in $\tilde{\Pi}_1$ guarantees that $p$ is invariant under the action of $S_n$, and so it has to have the same component (modulo $\hat{N}$) in each factor of the $H_1 \times \cdots \times H_{18}$. 


The substitutions made so far yield an element $p$ of length 325, as follows:

$$p = 10_17_11_{-1}17_123_1214_{-1}17_{-1}3_117_{-1}13_{-1}21 \cdot 17_23_21_2{^{-1}} \cdot 17_33_31_3{^{-1}} \cdot 10_4{^{-1}}2_4{^{-1}}17_4{^{-1}}147_4{^{-1}}17_4{^{-1}}144_42_4{^{-1}}13_44_41_4{^{-1}}17_43_4 \cdot 4_5{^{-1}}15_52_5{^{-1}}13_517_53_57_515_5{^{-1}}17_5{^{-1}}2_5 \cdot 4_6{^{-1}}1_6{^{-1}}17_63_67_64_62_6{^{-1}}13_61_67_6{^{-1}}16_613_6{^{-1}}2_6 \cdot 4_7{^{-1}}1_7{^{-1}}13_715_7{^{-1}}17_72_7{^{-1}}13_717_7{^{-1}}3_7{^{-1}}17_7{^{-1}}13_7{^{-1}}2_7 \cdot 4_182_1813_181_187_181_{-1}7_181_{-1}13_182_184_182_187_184_182_1813_181_187_1817_18{^{-1}}17_181_18 \cdot 4_182_1813_181_187_181_{-1}7_181_{-1}13_182_184_182_187_184_182_1813_181_187_1817_18{^{-1}}17_181_18.$$  

We then compute each component using the relations of $K^*$ given in Subsection 5.1 and find that

$$p = \theta^{(4,10)}3\theta^{(4,c)}2\theta^{(1,7)}{^{-3}}\theta^{(1,c)}c_1c_2\ldots c_{18}. \tag{21}$$  

As expected, $p$ is invariant under the action of $S_n$, centralizes $K$ by Proposition 5.10 and $ab(p) = 0$ by Corollary 5.7, showing that $p \in K$.

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APPENDIX

The text refers to four files, each containing the presentation of the fundamental group or one of its sections. The files

[App1]: presentation.txt
[App2]: presentation27.txt
[App3]: relations_of_N.txt
[App4]: relations_of_H.txt

can be downloaded from http://www.math.biu.ac.il/~vishne/downloads/TxT/.

REFERENCES

[ADKY] Auroux, D., Donaldson, S. K., Katzarkov, L. and Yotov, M., Fundamental groups of complements of plane curves and symplectic invariants, Topology 43(6) (2004), 1285–1318.

[A] Amram, M., The Galois Covers of Algebraic Surfaces, doctoral dissertation, Bar Ilan, Israel, (2001).

[AGTV] Amram, M., Goldberg, D., Teicher, M., Vishne U., The fundamental group of a Galois cover of $\mathbb{CP}^1 \times \mathbb{T}$, Algebraic and Geometric Topology 2, 403–432, (2002).

[AT1] Amram, M., Teicher, M., On the degeneration, regeneration and braid monodromy of $\mathbb{T} \times \mathbb{T}$, Acta Applicandae mathematicae, 75(1), 195-270, (2003).

[AT2] Amram, M., Teicher, M., The fundamental group of the complement of the branch curve of the surface $\mathbb{T} \times \mathbb{T}$ in $\mathbb{C}^2$, Osaka, Japan, 40(4), 1-37, (2003).

[AT3] Amram, M., Teicher, M., Non trivial fundamental groups for Hirzebruch surface $F_k(a, b)$ branch curve, submitted.

[ATV] Amram, M., Teicher, M., Vishne U., The Coxeter quotient of the fundamental group of a Galois cover of $\mathbb{T} \times \mathbb{T}$, Communications in Algebra, to appear.

[MKS] Magnus, W., Karrass, A. and Solitar, D., Combinatorial group theory (2nd edition), Dover Publications, New-York (1975).

[MRT] Moishezon, B., Robb, A., Teicher, M., On Galois covers of Hirzebruch surfaces, Math. Ann. 305, (1996), 493-539.

[MT1] Moishezon, B., Teicher, M., Simply connected algebraic surfaces of positive index, Invent. Math. 89, 601-643, (1987).

[MT2] Moishezon, B., Teicher, M., Braid group technique in complex geometry I, Line arrangements in $\mathbb{CP}^2$, Contemporary Math. 78, 425-555, (1988).

[MT3] Moishezon, B., Teicher, M., Braid group techniques in complex geometry. II. From arrangements of lines and conics to cuspidal curves, Algebraic Geometry, Lect. Notes in Math. 1479 (1990).
[MT4] Moishezon, B., Teicher, M., *Braid group techniques in complex geometry. IV. Braid monodromy of the branch curve $S_3$ of $V_3 \rightarrow \mathbb{CP}^2$ and application to $\pi_1(\mathbb{CP}^2 - S_3, *)$, Classification of algebraic varieties (L’Aquila, 1992), 333–358, Contemp. Math., 162, Amer. Math. Soc., Providence, RI, 1994.

[MT5] Moishezon, B., Teicher, M., *Finite fundamental groups, free over $\mathbb{Z}/c\mathbb{Z}$, Galois covers of $\mathbb{CP}^2$, Math. Ann. 293, 749-766, (1992).

[MT6] Moishezon, B., Teicher, M., *Braid group technique in complex geometry V: the fundamental group of a complements of a branch curve of a Veronese generic projection, Communications in Analysis and Geometry 4(1), 1–120, (1996).

[RTV] Rowen, L.H., Teicher, M. and Vishne, U., *Coxeter Covers of the Symmetric Groups*, J. Group Theory, 8, 139–169, (2005).

[T] Teicher, M., *The fundamental group of a $\mathbb{CP}^2$ complement of a branch curve as an extension of a solvable group by a symmetric group*, Math. Ann. 314, 19-38, (1999).

Meirav Amram, Einstein Institute of Mathematics, Hebrew University, Jerusalem

E-mail address: ameirav@math.huji.ac.il

Mina Teicher, department of mathematics, Bar-Ilan university, Ramat-Gan 52900, Israel

E-mail address: teicher@math.biu.ac.il

Uzi Vishne, department of mathematics, Bar-Ilan university, Ramat-Gan 52900, Israel

E-mail address: vishne@math.biu.ac.il