Refractive index perturbations – Unruh effect, Hawking radiation or dynamical Casimir effect?

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1 Introduction

Motivated by recent experimental progress to manipulate the refractive index of dielectric materials by strong laser beams, we study some aspects of the quantum radiation created by such refractive index perturbations. The Kerr effect in non-linear dielectrics describes a change of the refractive index $n$ due to electromagnetic radiation with the intensity $I$

$$n = n_0 + n_2 I,$$

(1)

where $n_0$ is the unperturbed index of refraction and $n_2$ is the Kerr coefficient. For example, for fused silica, it is $n_2 = 3 \times 10^{-16} \text{W}^{-1}\text{cm}^2$. Since the intensity $I$ will be far below $1/n_2$, the change $\delta n$ in the refractive index $n$ will be small. However, with a space-time dependent intensity $I(t, r)$, we can generate a varying refractive index perturbation

$$n(t, r) = n_0 + \delta n(t, r), \quad \delta n \ll 1.$$

(2)

Such a perturbation is able to create photon pairs out of the vacuum – more precisely, the ground state of the quantized electromagnetic field in the dielectric medium. In order to investigate this effect, we will make the following simplifications:

- We assume zero temperature and thus start in the ground state.
- We omit the photon polarizations and consider scalar fields.
- We neglect the dispersion $n(\omega)$ of the medium.
- Since $\delta n \ll 1$ is small, we use perturbation theory.

The first two assumptions are not critical and not difficult to fix. However, taking into account the dispersion $n(\omega)$ of the medium is rather non-trivial and calculations beyond perturbation theory are also far more involved in general.
2 Perturbation Theory

Given the aforementioned assumptions, we may start from the Lagrangian ($\hbar = c_0 = 1$)

$$\mathcal{L} = \frac{1}{2} (\varepsilon E^2 - B^2) = \frac{1}{2} ([n_0 + \delta n]^2 E^2 - B^2) ,$$  \hspace{1cm} (3)

where $\varepsilon$ is the dielectric permittivity of the medium and $E$ and $B$ are the microscopic electric and magnetic fields, respectively. Now standard quantization procedure yields the Hamiltonian

$$\hat{H} = \frac{1}{2} \int d^3 r \left( \frac{\hat{D}^2}{[n_0 + \delta n]^2} + \hat{B}^2 \right) = \hat{H}_0 + \hat{H}_1 ,$$  \hspace{1cm} (4)

where $D = \varepsilon E$ is the electric displacement field including the polarization $P$ of the medium $D = E + P$. Using $\delta n \ll 1$, we expand $\hat{H}$ into powers of $\delta n$ such that the lowest order $\hat{H}_0$ is the undisturbed Hamiltonian and the rest $\hat{H}_1$ contains the effect of the refractive index perturbation. Then we may use usual time-dependent perturbation theory in the interaction picture to calculate the final quantum state $|\Psi_{\text{out}}\rangle$ starting from the initial vacuum state $|0\rangle$

$$|\Psi_{\text{out}}\rangle = |0\rangle - i \int dt \hat{H}_1(t) |0\rangle + \mathcal{O}(\delta n^2) .$$  \hspace{1cm} (5)

Inserting the expansion into creation and annihilation operators for the quantized electromagnetic field, we find the production of photon pairs (to lowest order in $\delta n$)

$$|\Psi_{\text{out}}\rangle = |0\rangle + \sum_{k, k'} A_{k, k'} |k, k'\rangle + \mathcal{O}(\delta n^2) ,$$  \hspace{1cm} (6)

with the two-photon amplitude/probability (see, e.g., [1] and [2])

$$|A_{k, k'}|^2 = \frac{\omega_k \omega_{k'}}{n_0^4} \left| \tilde{n}(\omega_k + \omega_{k'}, k + k') \right|^2 .$$  \hspace{1cm} (7)

Here $k$ and $k'$ are the wave-numbers of the created photon pair and $\omega_k, \omega_{k'}$ their frequencies.

3 Single One-Parameter Pulse

As a first example, let us consider a simple pulse whose spatial and temporal extent are given by the same parameter $\Omega$, i.e.,

$$\delta n(t, r) = \delta \bar{n} f(\Omega t, \Omega r/c) ,$$  \hspace{1cm} (8)

with a well behaved function $f = \mathcal{O}(1)$ (e.g., Gaussian). Then it turns out that the total emission probability is independent of $\Omega$

$$P = \sum_{k, k'} |A_{k, k'}|^2 = \mathcal{O}(\delta \bar{n}^2) \ll 1 ,$$  \hspace{1cm} (9)
where the precise pre-factor in front of $\delta \bar{n}^2$ depends on the precise pulse shape. The typical photon energy, however, is given by

$$E = \frac{1}{P} \sum_{k,k'} |A_{k,k'}|^2 \omega_k = \mathcal{O}(\Omega).$$

(10)

Nevertheless, the total probability is bound to be small. For an experimental verification, therefore, a pulse with more than one parameter might be more useful.

4 Single Two-Parameter Pulse

Thus, as a second example, let us assume that the spatial and temporal scales of the pulse are given by two different parameters, $\Omega_1$ and $\Omega_2$

$$\delta n(t, r) = \delta \bar{n} f(\Omega_1 t, \Omega_2 r/c).$$

(11)

If $\Omega_1$ and $\Omega_2$ are of the same order, we basically reproduce the previous case. Thus, let us consider the two limiting cases $\Omega_1 \ll \Omega_2$ and $\Omega_1 \gg \Omega_2$.

- For $\Omega_1 \ll \Omega_2$, the spatial extend is very small and we have a “point-like pulse”.
- For $\Omega_1 \gg \Omega_2$, the pulse is almost spatially homogeneous and thus the situation is very similar to “cosmological particle creation”.

In the first case (“point-like pulse”), one can derive a closed expression for the average energy emitted by the refractive index perturbation [1]

$$\langle E \rangle \propto \int dt \left( \frac{d^4}{dt^4} \left[ \int d^3 r \delta n(t, r) \right] \right)^2 = \mathcal{O} \left( \frac{\Omega_1^7}{\Omega_2^5} \delta \bar{n}^2 \right).$$

(12)

However, since the typical photon energy is set by the temporal frequency $\Omega_1$ and not by the spatial scale $\Omega_2$, the total probability is extremely small

$$E = \mathcal{O}(\Omega_1) \sim P = \mathcal{O}\left( \frac{\Omega_1^6}{\Omega_2^6} \delta \bar{n}^2 \right) \ll 1.$$  

(13)

Thus this effect is also probably very hard to detect.

5 Cosmological Particle Creation

In the opposite limit $\Omega_1 \gg \Omega_2$, we may approximate the pulse with

$$\delta n(t, r) = \delta \bar{n} f(\Omega_1 t, \Omega_2 r/c)$$

(14)

by a effectively spatially homogeneous profile $\delta n(t, r) \approx \delta n(t)$ and thus the Lagrangian reads

$$\mathcal{L} \approx \frac{1}{2} \left( \epsilon(t) E^2 - B^2 \right) = \frac{1}{2} \left( n^2(t) E^2 - B^2 \right).$$

(15)
This is completely analogous to an expanding/contracting universe where the scale factor $a^2(t)$
corresponds to the refractive index $n(t)$. Hence we can immediately apply the results known for
this situation, see, e.g., [3] and references therein.

- Photons are created in pairs with nearly opposite momenta $k' \approx -k$.
- Thus, they are in an entangled (squeezed) state.
- The typical photon energy scales with $E = \mathcal{O}(\Omega_1)$.
- The total probability is enlarged by a volume enhancement factor $\Omega_1^3/\Omega_2^3$
  \[ P = \mathcal{O}\left(\frac{\Omega_1^3}{\Omega_2^3} \delta n^2\right). \] (16)

In view of this enhancement, this effect might well be observable.

6 Mixed Case

For completeness, let us briefly discuss the scenario with three different parameters
\[ \delta n(t, r) = \delta \bar{n} f(\Omega_1 t, \Omega_2 x/c, \Omega_3 y/c, \Omega_3 z/c), \] (17)
where one spatial scale, say $\Omega_2$, is much smaller than the temporal frequency $\Omega_1$ while the other,
$\Omega_3$, is much larger – corresponding to a long “needle-like pulse” pointing in $x$-direction
\[ \Omega_2 \ll \Omega_1 \ll \Omega_3. \] (18)
Again, the typical photon energy is set by the temporal frequency $\Omega_1$ (which is the generic case).
The total probability is enhanced by a length factor $\Omega_1/\Omega_2$ but suppressed by the small cross
section area $\propto 1/\Omega_3^2$ of the “needle”
\[ P = \mathcal{O}\left(\frac{\Omega_1}{\Omega_2 \Omega_3^2} \delta \bar{n}^2\right). \] (19)

7 Moving Pulse

So far, we have discussed the situation of a refractive index perturbation $\delta n(t, r)$ appearing in
a localized spatial region and then disappearing again. Now, let us consider the scenario of a
moving pulse – where we first study the case of constant velocity $v$
\[ \delta n(t, r) = \delta \bar{n} f(\Omega[r - vt] c). \] (20)
The resulting quantum radiation crucially depends on whether the velocity $v$ is smaller or larger
than the speed of light $c = 1/n$ in the medium, see also [4].
For a “sub-luminal” pulse $v < c$, there is no effect (to lowest order in $\delta n$).

This can be understood in the following way: electrodynamics in a (homogeneous and isotropic) dielectric medium is formally invariant under “Lorentz” transformations with the speed of light $c_0$ in vacuum being replaced by the speed of light $c = 1/n$ in the medium. Thus, after such a “Lorentz” boost, the refractive index perturbation $\delta n(t, r)$ becomes effectively stationary $\delta n(t, r) \to \delta n(r)$ and thus – as we have seen before – does not produce radiation.

For a “super-luminal” pulse $v > c$, we get “quantum Cherenkov” radiation.

In this case, a suitable “Lorentz” boost yields an instantaneous perturbation $\delta n(t) \times f(r)$.

In the case of a moving pulse (without beginning and end), we cannot compute a total emission probability but only the emission probability per unit time (as in Fermi’s golden rule, for example). In the “super-luminal” case, the situation after the “Lorentz” boost is analogous to the previous section with $\Omega_1 = \Omega_3 \to \Omega$ and $\Omega_2 \to 0$ and thus we get

$$\frac{P}{T} = \mathcal{O}(\Omega \delta n^2).$$

(21)

Due to phase matching conditions, the photon pairs are predominantly emitted in forward direction within an emission angle $\vartheta = \mathcal{O}(\sqrt{v - c})$. This angle closes for $v \downarrow c$ and thus the emission probability (per unit time) vanishes in this limit – consistent with the above picture. As usual, the typical photon energy scales with $E = \mathcal{O}(\Omega).

8 Hawking Radiation

Now let us go right to the borderline between the “sub-luminal” and the “super-luminal” case discussed above and study a pulse which is slower than the medium speed of light outside the pulse $c = 1/n$ but faster than the medium speed of light inside $1/(n + \delta n)$

$$\delta n(t, r) = \delta n f(\Omega[r - vt]/c), \quad c = \frac{1}{n} > v > \frac{1}{n + \delta n}.$$  

(22)

In this case, we get the analogue of black hole horizon (light cannot escape) at the front end of the pulse and the analogue of a white hole horizon (light cannot enter) at its back and, see Fig. 1. The idea of creating analogues of black (or white) holes in the laboratory and so to experimentally test Hawking radiation [5, 6] goes back to Bill Unruh [7], see also [8, 9, 10]. Originally, the analogy was developed for sound waves [7], but later electromagnetic waves were considered as well. In [11, 12] proposals based on “slow light” (i.e., electromagnetically induced transparency) where put forward. However, these suggestions had various problems, see, e.g., [13, 14, 15]. A set-up based on moving dielectrics was proposed in [16], but there the difficulty was to have the medium flowing faster than the speed of light in the medium – which is typically
Figure 1: Sketch of the pulse creating analogues of black and white hole horizons.

very fast. Later is was realized [17] that it is not really necessary to actually move the medium – instead a moving pulse which changes the local propagation speed can be enough, see also [18]. The Hawking temperature (in the co-moving frame) is determined by the gradient of the refractive index, see, e.g., [7, 16]

\[ T_{\text{Hawking}} \propto \nabla \delta n \sim \Omega \bar{n} . \]  

Since Hawking radiation is thermal, the total emission probability (per unit time) scales as

\[ \frac{P}{T} = O (A T_{\text{Hawking}}^3) \to O (\Omega \delta \bar{n}^3) . \]  

As we may infer from the \( \delta \bar{n}^3 \) scaling, this effect is beyond lowest order perturbation theory. In one spatial dimension (such as in an optical fibre, see [18]), we would get \( P/T \propto \delta \bar{n} \). This shows that this effect is non-perturbative – since the perturbative expansion always starts with \( \delta \bar{n}^2 \). This can be explained by the fact that Hawking radiation occurs due to the tearing apart of modes at the black-hole horizon, which takes a time duration long compared to \( 1/(\Omega \delta \bar{n}) \). An experiment with a pulse satisfying Eq. (22) in some frequency range has been done [19] and it has been claimed that this was the first observation of the analogue of Hawking radiation. However, comparison with the above estimates casts some doubts at these claims, see [20, 21].

9 Non-Inertial Pulse

After having discussed the case of constant velocity, let us turn to non-inertial pulse motion

\[ \delta n(t, r) = \delta \bar{n} f (\Omega [r - r_P(t)]/c) , \]  

where \( r_P(t) \) is the trajectory of the pulse. For simplicity, let us consider the case of approximately uniform acceleration \( \ddot{r}_P \approx \text{const} \) for some time. In this case, the quantum radiation created by the refractive index perturbation can be nicely understood as a signature of the Unruh effect.
The Unruh effect [22] describes the striking prediction that a uniformly accelerated detector experiences the inertial Minkowski vacuum state as thermal bath with Unruh temperature

\[ T_{\text{Unruh}} = \frac{\hbar}{2\pi k_B c_0} a, \]

where \( a \) is the acceleration. Unfortunately, due to the factors \( \hbar \) and \( c_0 \), this temperature is quite low for everyday accelerations—e.g., the earth’s gravitational acceleration \( a = g = 9.81 \text{ m/s}^2 \) corresponds to \( T_{\text{Unruh}} = \mathcal{O}(10^{-20} \text{ K}) \). This is one of the reasons why this effect has not been directly observed yet, see also [23, 24, 25] and references therein.

![Sketch of the scattering of two photons (red arrows) in the accelerated frame (left) and translation of this event to the inertial frame (right).](image)

Figure 2: Sketch of the scattering of two photons (red arrows) in the accelerated frame (left) and translation of this event to the inertial frame (right).

However, it should be possible to observe a signature of the Unruh effect with accelerated refractive index perturbations, for example. As discussed before, the ground state of a dielectric medium behaves analogous to the Minkowski vacuum with the vacuum speed of light \( c_0 \) being replaced by the medium light velocity \( c \). Thus, in its non-inertial frame, the accelerated refractive index perturbation experiences the ground state of the medium as a thermal bath. Then there is a finite probability that one photon out of this thermal bath is scattered by the refractive index perturbation into another mode. This scattering event in the accelerated frame, when translated back into the inertial (laboratory) frame, corresponds to the emission of real photon pairs, as sketched in Fig. 2. In agreement with this intuitive picture, the emission probability per unit time scale as (again using perturbation theory)

\[ \frac{P}{T} = \mathcal{O} \left( \sigma_{\text{scattering}} T_{\text{Unruh}}^3 \right) \rightarrow \mathcal{O} \left( \frac{\delta n^2}{\Omega^2} \frac{2^3}{P^3} \right). \]  

Note that this expression is valid for \( \Omega \gg |F_P| \) only since otherwise the internal width \( \sim 1/\Omega \) of the pulse smears out its trajectory \( r_P(t) \) and thus the picture based on the Unruh effect does not apply any more (even though there still would be quantum radiation).

10 Dynamical Casimir Effect

Finally, let us discuss the relation between the quantum radiation given off by these refractive index perturbations and the dynamical Casimir effect. In the original setting, the static Casimir effect [26] describes the attraction (or repulsion—depending on the boundary conditions) of two conducting plates at rest in vacuum, which is caused by the distortion of the quantum
vacuum fluctuations of the electromagnetic field. The dynamical Casimir effect then refers to
the creation of photon pairs out of QED vacuum due to the non-inertial motion of one or both
mirrors, see, e.g., [27].
Replacing the two mirrors by two bodies of dielectric material, one can also get a static Casimir
attraction. Thus, the quantum radiation created by the non-inertial motion of a dielectric body
in vacuum could also be called dynamical Casimir effect. As we have discussed before, this
scenario is formally equivalent to a refractive index perturbation moving in a dielectric medium.
Thus, the signatures of the Unruh effect caused by the non-inertial motion of such a perturbation
as discussed in the previous Section can also be viewed as a manifestation\footnote{Note that there is recent experimental evidence [25] for the observation of the dynamical Casimir effect – though not in vacuum, but in a wave-guide (which is analogous to the dielectric medium discussed here).} of the dynamical
Casimir effect.

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References

[1] R. Schützhold, G. Plunien and G. Soff, Quantum Radiation in External Background Fields,
Phys. Rev. A 58, 1783 (1998).

[2] Z. Białynicka-Birula and I. Białynicki-Birula, Space-time description of squeezing, J. Opt.
Soc. Am. B 4, 1621 (1987).

[3] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space, (Cambridge University
Press, Cambridge, England 1982).

[4] F. Belgiorno, S.L. Cacciatori, G. Ortenzi, V.G. Sala, and D. Faccio, Quantum Radiation
from Superluminal Refractive-Index Perturbations, Phys. Rev. Lett. 104, 140403 (2010).

[5] S. W. Hawking, Black Hole Explosions, Nature 248, 30 (1974).

[6] S. W. Hawking, Particle Creation by Black Holes, Commun. Math. Phys. 43, 199 (1975).

[7] W. G. Unruh, Experimental Black Hole Evaporation?, Phys. Rev. Lett. 46, 1351 (1981).

[8] M. Novello, M. Visser, and G. Volovik (editors), Artificial Black Holes (World Scientific,
Singapore, 2002).
[9] G. E. Volovik, *Universe in a Helium Droplet* (Oxford University Press, Oxford, 2003).

[10] R. Schützhold and W. G. Unruh (editors), *Quantum Analogues: From Phase Transitions to Black Holes & Cosmology*, Springer Lecture Notes in Physics 718 (2007).

[11] U. Leonhardt and P. Piwnicki, *Relativistic Effects of Light in Moving Media with Extremely Low Group Velocity*, Phys. Rev. Lett. 84, 822 (2000).

[12] U. Leonhardt, *A laboratory analogue of the event horizon using slow light in an atomic medium*, Nature 415, 406 (2002).

[13] M. Visser, *Comment on “Relativistic Effects of Light in Moving Media with Extremely Low Group Velocity”*, Phys. Rev. Lett. 85, 5252 (2000).

[14] U. Leonhardt and P. Piwnicki, *Reply to comment on “Relativistic Effects of Light in Moving Media with Extremely Low Group Velocity”*, Phys. Rev. Lett. 85, 5253 (2000).

[15] W. G. Unruh and R. Schützhold, *On Slow Light as a Black Hole Analogue*, Phys. Rev. D 68, 024008 (2003).

[16] R. Schützhold, G. Plunien, and G. Soff, *Dielectric black hole analogs*, Phys. Rev. Lett. 88, 061101 (2002).

[17] R. Schützhold and W. G. Unruh, *Hawking radiation in an electro-magnetic wave-guide?*, Phys. Rev. Lett. 95, 031301 (2005).

[18] T.G. Philbin, C. Kuklewicz, S. Robertson, S. Hill, F. König, U. Leonhardt, *Fiber-Optical Analog of the Event Horizon*, Science 319, 1367 (2008).

[19] F. Belgiorno, S.L. Cacciatori, M. Clerici, V. Gorini, G. Ortenzi, L. Rizzi, E. Rubino, V.G. Sala, and D. Faccio, *Hawking Radiation from Ultrashort Laser Pulse Filaments*, Phys. Rev. Lett. 105, 203901 (2010).

[20] Ralf Schützhold and William Unruh, *Comment on: Hawking Radiation from Ultrashort Laser Pulse Filaments*, Phys. Rev. Lett. 107, 149401 (2011).

[21] F. Belgiorno, S.L. Cacciatori, M. Clerici, V. Gorini, G. Ortenzi, L. Rizzi, E. Rubino, V.G. Sala, and D. Faccio, *Reply to Comment on: Hawking radiation from ultrashort laser pulse filaments*, Phys. Rev. Lett. 107, 149402 (2011).

[22] W. G. Unruh, *Notes on Black Hole Evaporation*, Phys. Rev. D 14, 870 (1976).

[23] R. Schützhold and C. Maia, *Quantum radiation by electrons in lasers and the Unruh effect*, Eur. Phys. J. D 55, 375 (2009).
[24] R. Schützhold, G. Schaller, and D. Habs, *Table-top creation of entangled multi-keV photon pairs and the Unruh effect*, Phys. Rev. Lett. **100**, 091301 (2008).

[25] R. Schützhold, G. Schaller, and D. Habs, *Signatures of the Unruh effect from electrons accelerated by ultra-strong laser fields*, Phys. Rev. Lett. **97**, 121302 (2006).

[26] H.B.G. Casimir, *On the attraction between two perfectly conducting plates*, Proc. Roy. Netherl. Acad. Arts Sci. **51**, 793 (1948).

[27] V.V. Dodonov, *Current status of the Dynamical Casimir Effect*, Physica Scripta **82**, 038105 (2010).

[28] C.M. Wilson, G. Johansson, A. Pourkabirian, J.R. Johansson, T. Duty, F. Nori, P. Delsing, *Observation of the Dynamical Casimir Effect in a Superconducting Circuit*, arXiv:1105.4714.