The Pure Leptonic Decays of $B_c$ Meson and Their Radiative Corrections

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Abstract

The radiative corrections to the pure leptonic decay $B_c \rightarrow \ell \nu_\ell$ up-to one-loop order is presented. How to cancel the infrared divergences appearing in the loop calculations, and the radiative decay $B_c \rightarrow \ell \nu_\ell \gamma$ is shown precisely. It is emphasized that the radiative decay may be separated properly and may compare with measurements directly as long as the theoretical ‘softness’ of the photon corresponds to the experimental resolutions. Furthermore with a kind of non-relativistic constituent quark model, a kind of typical long distance contributions to the radiative decays is estimated, and it is shown that the contributions are negligible in comparison with the accuracy of one-loop corrections and the expected experimental measurements.

13.30.Ce, 13.40.Ks, 13.38.Lg, 12.39.Jh
I. INTRODUCTION

The pure leptonic decays of the heavy meson $B_c$ are very interesting \cite{1, 2}. In principle, the pure-leptonic decay $B_c \rightarrow \ell \nu \ell$ (see, Fig.1) can be used to determine the decay constant $f_{B_c}$ if the fundamental Cabibbo-Kobayashi-Maskawa matrix element $V_{bc}$ of Standard Model (SM) is known. Conversely if we know the value of decay constant $f_{B_c}$ from other method, these process also can be used to extract the matrix element $V_{bc}$. The $B_c$ meson has recently been observed at Fermilab \cite{8}, that opens a ‘new page’ for the experimental study of $B_c$ meson. Based on the estimates in Refs. \cite{9, 10}, numerous $B_c$ mesons will be produced in Tevatron and also in LHC. Considering the schedule for the new runs at Tevatron and LHC constructing, we may expect that experimental studies on the $B_c$ meson with more care and much higher statistics will be accessible in the foreseeable future.

The pure leptonic weak decays of the pseudoscalar meson corresponding to Fig.1 are helicity suppressed by factor of $m_{\ell}^2/m_{B_c}^2$: 

$$\Gamma(B_c \rightarrow \ell \nu \ell) = \frac{G_F^2}{8\pi} |V_{bc}|^2 f_{B_c}^2 m_{B_c}^3 \frac{m_{\ell}^2}{m_{B_c}^2} \left(1 - \frac{m_{\ell}^2}{m_{B_c}^2}\right)^2,$$  

(1)

whereas to study them with their radiative decays $B_c \rightarrow l \nu \gamma$ simultaneously is attractive \cite{3, 4}. Of them only the process $B_c \rightarrow \tau \nu \tau$ is special i.e. it does not suffer so much from the helicity suppression thus its branching ratio may reach to 1.5\% \cite{1} in SM. However the produced $\tau$ will decay promptly and one more neutrino is generated in the cascade decay at least, thus it makes the decay channel difficult to be observed, especially, in such a strong background of hadronic collisions.

Fortunately, having an extra real photon emitted in the leptonic decays, the radiative pure leptonic decays can escape from the suppression, furthermore, as pointed out in \cite{3}, with the extra photon to identify the produced meson $B_c$ experimentally in hadronic collisions from the backgrounds has advantages, namely in Tevatron and LHC to observe the radiative decays certainly has advantages to compare with the pure leptonic ones. Although the
radiative corrections are suppressed by an extra electromagnetic coupling constant $\alpha$, it will not be suppressed by the helicity suppression. Therefore, the radiative decay may be comparable, even larger than the corresponding pure leptonic decays. Considering the possibility to measure the decay constant $f_{B_c}$ and the CKM matrix element $V_{cb}$ in LHC in the foreseeable future, the problem to increase the accuracy of the theoretical calculation, at least up to the first order radiative corrections, emerges.

The radiative pure leptonic decays, theoretically, have inferred divergences and will be canceled with those from loop corrections of the pure leptonic decays, thus we are also interested in considering the radiative decays and the pure leptonic decays with one-loop radiative corrections together. Moreover, all of them can be divided into two components: 1). The so-called short distance contributions, for instance, those of the radiative pure leptonic decays correspond to the four diagrams in Fig.2, i.e. a real photon is attached to any of the charged lines of the pure-leptonic decay Feynman diagrams Fig.1.

2). The so-called long distance contributions which are relevant to a virtual heavy hadronic state as an intermediate state. To increase the accuracy of the theoretical calculation, both components should consider precisely.

Recently, there are a few papers [3–5] on the radiative processes with different methods in literature, but inconsistent results have been obtained no matter all of them just the lowest order calculations. To clarify the inconsistency, i.e. further to re-examine the process is certainly needed. Furthermore, since in Ref. [4] an approach of QCD sum rule is adopted and in Ref. [3] a kind of quark model is adopted, QCD sum rule is supposed some long distance effects may be taken into account, so it is reasonable to doubt that the disagreement between the results of Refs. [3,4] could be due to the long distance contributions. We essentially will adopt the same approach as Ref. [3], thus in the paper we especially make some estimates on the long distance contributions precisely.

Namely in the present paper, baring the problems and the situation pointed out above in mind, we will investigate the $B_c$ leptonic decays up-to one loop radiative corrections carefully.
The paper is organized as follow: in Section II, we present the calculations of all the short distance contributions: i) besides the leptonic decays shown as Fig.1, the radiative decays into three family leptons with a real photon appearing in the final state as shown in Fig.2; ii) the virtual photon corrections to the pure leptonic decays i.e. a photon in loop as shown in Figs.3.1, 3.2, 3.3, 3.4. In Section III, we estimate a typical contributions from the long distance contributions. In Sec.IV, we evaluate the values of the decays. As uncertainties the dependence of the results on the parameters appearing in the considered model is discussed. Comparison with the others' results are made. Finally, preliminary conclusion is obtained.

II. THE SHORT DISTANCE CONTRIBUTIONS

Let us start with the radiative decay. The short distance contributions are corresponding to the four diagrams in Fig.2. According to the constituent quark model which is formulated by Bethe-Salpeter (B.-S.) equation, the amplitude turns out to be the four terms $M_i (i = 1, 2, 3, 4)$:

$$M_1 = Tr \left[ \int \frac{d^4q}{(2\pi)^4} \chi(p, q) i \left( \frac{G_F m_w^2}{\sqrt{2}} \right)^\frac{i}{2} \gamma_\mu (1 - \gamma_5) V_{bc} \right] \times$$

$$\frac{i (-g^{\mu\nu} + \frac{p^\mu p^\nu}{m_w^2})}{p^2 - m_w^2} \epsilon \left[ (p' + p) \lambda g_{\nu\rho} + (k - p')_\rho g_{\rho\lambda} + (-p - k)_\rho g_{\nu\lambda} \right] \epsilon^\lambda \times$$

$$\frac{i (-g^{\rho\sigma} + \frac{(p - k)^\rho (p - k)^\sigma}{m_w^2})}{(p - k)^2 - m_w^2} \epsilon \left[ i \frac{ig}{2\sqrt{2}} \gamma_\sigma (1 - \gamma_5) \nu_\ell, \right]$$

$$M_2 = Tr \left[ \int \frac{d^4q}{(2\pi)^4} \chi(p, q) i \left( \frac{G_F m_w^2}{\sqrt{2}} \right)^\frac{i}{2} \gamma_\mu (1 - \gamma_5) V_{bc} \right] \frac{i (-g^{\mu\nu} + \frac{p_\mu p_\nu}{m_w^2})}{p^2 - m_w^2}$$

$$\times \bar{\ell} (-ie) \frac{i}{k_\ell - m_\ell} \frac{ig}{2\sqrt{2}} \gamma_\nu (1 - \gamma_5) \nu_\ell,$$

$$M_3 = Tr \left[ \int \frac{d^4q}{(2\pi)^4} \chi(p, q) i \left( \frac{G_F m_w^2}{\sqrt{2}} \right)^\frac{i}{2} \gamma_\mu (1 - \gamma_5) V_{bc} \frac{m_b}{m_b + m_c} \frac{i}{p^+ q^- k^- - m_b} \left(-\frac{i}{3} \epsilon \right) \right]$$
\[ M_4 = Tr \left[ \int \frac{d^4q}{(2\pi)^4} \chi(p, q) \left( \frac{2e}{3} \gamma \right) \left( -\frac{m_c}{m_b+m_c} p + \frac{q}{\sqrt{2}} \right) \gamma_{\mu} \left( \frac{G_F m_w^2}{\sqrt{2}} \right)^{1/2} \gamma_{\nu}(1-\gamma_5) V_{bc} \right] \]

\[ \times \frac{i \left( -g^{\mu\sigma} + \frac{(p-k)^{\mu}(p-k)^{\sigma}}{m_w^2} \right)}{(p-k)^2 - m_w^2} \gamma_{\rho} \frac{i g}{2\sqrt{2}} \gamma_{\nu}(1-\gamma_5) \nu \epsilon, \]

where \( \chi(p, q) \) is Bethe-Salpeter wave function of the meson \( B_c \); \( p \) is the momentum of \( B_c \); \( \epsilon, k \) are the polarization vector and momentum of the emitted photon. In the quark model, the momenta of \( \bar{b}, c \)-quarks inside the bound state i.e. the \( B_c \) meson, are:

\[ p_b = \frac{m_b}{m_b+m_c} p + q; \quad p_c = \frac{m_c}{m_b+m_c} p - q, \]

where \( q \) is the relative momentum of the two quarks inside the \( B_c \) meson. As the \( B_c \) meson is a nonrelativistic bound state in nature, so the higher order relativistic corrections may be computed precisely, but being an approximation for a \( S \)-wave state, and focusing the light on the radiative decay corrections only at this moment now, we ignore \( q \) dependence i.e. we may still have the non-relativistic spin structure for the wave function of the meson \( B_c \) (a \( ^1S_0 \) state) correspondingly:

\[ \int \frac{d^4q}{(2\pi)^4} \chi(p, q) = \frac{\gamma_5(p+m)}{2\sqrt{m}} \psi(0). \]

Here \( \psi(0) \) is the wave function at origin, and by definitions it connects to the decay constant \( f_{B_c} \):

\[ f_{B_c} = \frac{\psi(0)}{2\sqrt{m}} \]

where \( m \) is the mass of \( B_c \) meson. Moreover we note that for convenience we take unitary gauge for weak bosons to do the calculations throughout the paper. With a straightforward computation, the amplitude can be simplified as:

\[ M_1 = \frac{-4Ai}{(m^2 - m_w^2)(m^2 - 2p \cdot k - m_w^2)} \times \]
\[ \ell \left( -1 + \frac{m^2}{m_w^2} \right) \left[ \frac{p \cdot (\not{k} - \not{p}) - (2p \cdot k - m^2) \not{\varphi}}{\not{k}} \right] (1 - \gamma_5) \nu_\ell, \]  

(6)

\[ M_2 = \frac{4A_i}{(m^2 - m_w^2)(2k_1 \cdot k)} \ell \left( -1 + \frac{m^2}{m_w^2} \right) \varphi(\not{k}_1 + \not{k} + m_e) \not{p}(1 - \gamma_5) \nu_\ell, \]  

(7)

\[ M_3 + M_4 = \frac{-4A_i}{(-p \cdot k)(m^2 - 2p \cdot k - m_w^2)} \ell \left\{ \left[ -(p \cdot \epsilon) \not{p} + (p \cdot \epsilon \cdot k - p \cdot k \varphi) s_2 \right] 
+ i s_1 \varepsilon^{\alpha \beta \gamma \nu} p_\alpha \epsilon_\mu k_\beta \gamma_\nu \right\} \frac{\not{p} \cdot \epsilon (m^2 - p \cdot k)}{m_w^2} (1 - \gamma_5) \nu_\ell, \]  

(8)

where \( k_1 \) is the momentum of the charged lepton, and

\[ A = \psi(0) \left( \frac{G_F m^2}{\sqrt{2}} \right)^{1/2} V_{bc} e g = \psi(0) \left( \frac{G_F m^2}{\sqrt{2}} \right) V_{bc} e; \]

\[ s_1 = -\frac{m_b + m_c}{6m_b} + \frac{m_b + m_c}{3m_c}; \quad s_2 = \frac{m_b + m_c}{6m_b} + \frac{m_b + m_c}{3m_c}. \]

As a matter of fact, there is infrared infinity when performing phase space integral about the square of matrix element at the soft photon limit. It is known that the infrared infinity can be cancelled completely by that of the loop corrections to the corresponding pure leptonic decay \( B_c \rightarrow l\nu \).

In Eqs.(7) and (8), the infrared terms can be read out:

\[ M^i = M_2^i + M_3^i + M_4^i = \frac{4A_i}{m_w^2} \ell \left[ \frac{k_1 \cdot \epsilon \not{p}}{k_1 \cdot k} - \frac{p \cdot \epsilon \not{p}}{p \cdot k} \right] (1 - \gamma_5) \nu_\ell. \]  

(9)

As the diagrams (g), (h), (i), (j) in Figs.3.3, 3.4 always have a further suppression factor \( m^2/m_w^2 \) to compare with the other loop diagrams, we may ignore the contributions from these four diagrams safely. Furthermore we should note that in our calculations throughout the paper, the dimensional regularization to regularize both infrared and ultraviolet divergences is adopted, while the on-mass-shell renormalization for the ultraviolet divergence is used.

If Feynman gauge for photon is taken (we always do so in the paper), the amplitude corresponding to the diagrams (a), (b) of Fig.3.1 can be written as:
\[ M_{(2)}(a) = \frac{2}{3} eA \int \frac{d^4l}{(2\pi)^4} \left[ -4i\epsilon^{\alpha\beta\mu\nu} p_\alpha l_\beta - 4(p_\mu l_\nu - p \cdot l g_{\mu\nu} + p_\nu l_\mu) + \frac{8m_\nu}{m_\nu + m_c} p_\mu p_\nu \right] \]
\[
\times \ell [2(p - k_2)_\mu - \gamma_\mu \ell (-\gamma_\nu)(1 - \gamma_5)\nu_\ell, \tag{10} \]

\[ M_{(2)}(b) = -\frac{1}{3} eA \int \frac{d^4l}{(2\pi)^4} \left[ -4i\epsilon^{\alpha\beta\mu\nu} p_\alpha l_\beta + 4(p_\mu l_\nu - p \cdot l g_{\mu\nu} + p_\nu l_\mu) - \frac{8m_b}{m_b + m_c} p_\mu p_\nu \right] \]
\[
\times \ell [2(p - k_2)_\mu - \gamma_\mu \ell (-\gamma_\nu)(1 - \gamma_5)\nu_\ell, \tag{11} \]

where the \( l, k_2 \) denote the momenta of the loop and the neutrino respectively. These two terms also have infrared infinity when integrating out the loop momentum \( l \).

After doing the on-mass-shell subtraction, the terms corresponding to vertex and self-energy diagrams (c), (d), (e), (f) can be written as:

\[ M_{(2)}(c + d + e + f) = \frac{ieA}{4\pi^2} \ell \ell (1 - \gamma_5)\nu_\ell \times \left[ \ln(4) - \frac{8}{9} + \frac{2}{9} \frac{m_b - m_c}{m_b + m_c} \ln \left( \frac{m_b}{m_c} \right) \right] \]
\[
+ \left( \frac{2}{9} + \frac{8}{9} \frac{m_c}{m_b + m_c} \right) \ln \left( \frac{m_b + m_c}{m_b} \right) + \left( \frac{8}{9} + \frac{8}{9} \frac{m_c}{m_b + m_c} \right) \ln \left( \frac{m_b + m_c}{m_c} \right) \]
\[
+ \frac{2}{\varepsilon} + 2\gamma + \ln \left( \frac{4\pi^2 \mu^2}{m^2} \right) + \ln \left( \frac{4\pi^2 \mu^2}{m_c^2} \right) \right]. \tag{12} \]

Now let us see the cancellation of the infrared divergencies precisely. The infrared parts of the decay widths which are from the interference of the self-energy and vertex correction diagrams with the tree diagrams:

\[ \delta\Gamma_{\text{infrared}}^{s,v} = \left( \frac{\alpha V_{ec}^2 f_{\beta \nu}^2 G_F^2 M m^2}{16\pi^2} \right) \left[ -\frac{34}{9} - \frac{4}{9} \frac{m_b - m_c}{m_b + m_c} \ln \left( \frac{m_b}{m_c} \right) \right] \]
\[
- \left( \frac{2}{9} + \frac{4}{9} \frac{m_c}{m_b + m_c} \right) \ln \left( \frac{m_b + m_c}{m_b} \right) - \left( \frac{4}{9} + \frac{4}{9} \frac{m_c}{m_b + m_c} \right) \ln \left( \frac{m_b + m_c}{m_c} \right) \]
\[
- \frac{2}{\varepsilon} + 2\gamma - \ln \left( \frac{4\pi^2 \mu^2}{m_c^2} \right) \right]; \tag{13} \]

the infrared part of the decay widths from the interference of the "box" correction diagrams with the tree diagrams:
\[ \delta \Gamma_{\text{infrared}}^{\text{box}} = \left( \frac{\alpha V_{bc} f_{B_c}^2 G_F^2 M m_I^2}{16 \pi^2} \right) \left[ - \left( \frac{1}{\varepsilon_I} - \gamma \right) \ln \left( \frac{m_I^2}{M^2} \right) - \ln \left( \frac{m_I^2}{M^2} \right) \ln \left( \frac{4 \pi \mu^2}{M^2} \right) \right]; \quad (14) \]

the infrared part of the decay width from the real photon emission:

\[ \delta \Gamma_{\text{infrared}}^{\text{real}} = \left( \frac{\alpha V_{bc} f_{B_c}^2 G_F^2 M m_I^2}{16 \pi^2} \right) \] \[ \cdot \left[ \frac{2}{\varepsilon_I} - 2 \gamma + \left( \frac{1}{\varepsilon_I} - \gamma \right) \ln \left( \frac{m_I^2}{M^2} \right) + \left( 2 + \ln \left( \frac{m_I^2}{M^2} \right) \right) \ln \left( \frac{4 \pi \mu^2}{4(\Delta E)^2} \right) \right]. \quad (15) \]

Here \( \Delta E \) is a small energy, which corresponds to the experimental resolution of a soft photon so that the phase space of the emitting photon in fact is divided into a soft and a hard part by \( \Delta E \). Where \( \mu \) is the dimensional parameter appearing in the dimensional regularization. It is easy to check that when adding up all the parts: the real photon emission \( \delta \Gamma_{\text{infrared}}^{\text{real}} \) and the virtual photon corrections \( \delta \Gamma_{s,v}^{\text{infrared}}, \delta \Gamma_{\text{box}}^{\text{infrared}} \), the infrared divergences \( \frac{2}{\varepsilon_I} - \gamma \) are canceled totally and the \( \mu \) dependence is also cancelled. Hence we may be sure that we finally obtain the pure leptonic widths for the \( B_c \) meson decays to the three families of leptons which are accurate up-to the ‘next’-order corrections and ‘infrared finite’ but depend on the experimental resolution \( \Delta E \).

III. ESTIMATE OF THE LONG DISTANCE CONTRIBUTIONS

We have done the calculations of the radiative ‘short distance’ corrections corresponding to the energy scale around \( m_{B_c} \), whereas, there are corrections which correspond to much softer nature than what we have considered. Typically, some of them in the radiative decay \( B_c \to l \nu \gamma \) may be described by Fig.4. People, generally, call the soft corrections as the ‘long distance contributions’ correspondingly.

To have a rough estimate on the contributions of the softer part and to be typical, let us only consider those corresponding to Fig.4.

The amplitude for the long distance radiative corrections corresponding to Fig.4 may be written as:

\[ M = \frac{-ie(2\pi)^4 \delta^4(P_{B_c} - k - k_1 - k_\nu)G_F\epsilon_\mu(k)}{\sqrt{2}} \bar{\ell}(k_1) \gamma^\lambda(1 - \gamma_5)\nu_\epsilon(k_\nu) \]
where \( j_{\mu\nu} \), \( q_{\rho\sigma} \) are electromagnetic current and weak current respectively, and \( \epsilon(k) \) is polarization vector of the real photon. The intermediate states \( |p_n\rangle \) are all the possible physical states, and in the summation \( \sum_{\vec{p}_n=k} \) for the first term and \( \sum_{\vec{p}_n=P_{B_c}-k} \) for the second term are kept. Note that for the time-component, generally, \( k_0 \neq p_{n0} \) for the first term and \( p_{n0} \neq P_{B_{c0}} - k_0 \) for the second term. To compute the whole amplitude, let us compute the current matrix elements appearing in Eq.(16) with the intermediate meson states being on mass-shell. However to have a rough estimate instead of a precise one, here only the intermediate meson states \( J/\psi \) and \( B_c^* \), being of typical long distance contributions, are computed by means of the three Feynman diagrams shown in Fig.4.

According to Eq.(16), we have:

\[
\langle 0|j_{\mu\nu}(0)|p_{J/\psi}\rangle = -\frac{2}{3} \frac{1}{\sqrt{2\pi p_{J/\psi}}} \phi_{J/\psi}(0) M_{J/\psi} \epsilon_\mu(J/\psi),
\]

\[
\langle 0|q_{\rho\sigma}(0)|p_{B_c^*}\rangle = -\frac{1}{\sqrt{2\pi p_{B_c^*}}} \phi_{B_c^*}(0) M_{B_c^*} \epsilon_\mu(B_c^*).
\]

The matrix element \( \langle p_{J/\psi}|q_{\rho\sigma}(0)|P_{B_c}\rangle \) can be decomposed into two parts:

\[
\langle p_{J/\psi}|q_{\rho\sigma}(0)|P_{B_c}\rangle = \langle p_{J/\psi}|V_\lambda|P_{B_c}\rangle - \langle p_{J/\psi}|A_\lambda|P_{B_c}\rangle,
\]

namely those of the vector current \( V_\lambda = \frac{1}{2} \gamma_\lambda b \) and the axial current \( A_\lambda = \frac{1}{2} \gamma_\lambda \gamma_5 b \). They are related to the form factors [11][12] as follows:

\[
\langle p_{J/\psi}|V_\mu|P_{B_c}\rangle = ig \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \epsilon^{*\nu}(P_{B_c} + p_{J/\psi})^\rho(P_{B_c} - p_{J/\psi})^\sigma,
\]

\[
\langle p_{J/\psi}|A_\mu|P_{B_c}\rangle = \frac{1}{2} [f \epsilon_\mu + a_+(\epsilon^* \cdot P_{B_c})(P_{B_c} + p_{J/\psi})_\mu + a_- (\epsilon^* \cdot P_{B_c})(P_{B_c} - p_{J/\psi})_\mu],
\]

where \( g, f, a_+, a_- \) are the possible form factors. For convenience, here we actually calculate the matrix element \( \langle p_{B_c}|j_{\mu\nu}(0)|P_{B_c^*}\rangle \) instead of \( \langle p_{B_c^*}|j_{\mu\nu}(0)|P_{B_c}\rangle \):
\[ \langle p_{B_c} | j^{\mu em}(0) | p_{B_c}^* \rangle = i (c_b g_b + c_c g_c) \varepsilon_{\mu \nu \rho \sigma} \epsilon^{\nu \gamma}(B_c^*) (P_{B_c} + p_{B_c})^\rho (P_{B_c} - p_{B_c})^\sigma \] (22)

Now we just present the final results of the form factors \( g, f, a_+, a_-, g_b, g_c \), here, as the detail calculations on them can be found in Ref. [1].

Let us introduce further definitions, which is convenient for presenting the results. If \( p_1, p_2 \) are the momenta of the constituent particles 1 and 2 respectively, the total and the relative momenta \( p \) and \( q \) are defined as:

\[
\begin{align*}
p_1 &= \alpha_1 p + q, \quad \alpha_1 = \frac{m_1}{m_1 + m_2}; \\
p_2 &= \alpha_2 p + q, \quad \alpha_2 = \frac{m_2}{m_1 + m_2}.
\end{align*}
\]

The momenta \( p, p' \) are replaced to denote those for the initial and final mesons \( P_{B_c}, p_{B_c}^* \) (or \( p_{J/\psi} \)), and \( M, M' \) are denoted the masses of the initial and final mesons.

Furthermore \( q_p, q_{p'T} \) are denoted the two Lorentz covariant variables as follows:

\[
q_p = \frac{p \cdot q}{M_p}, \quad q_{p'T} = \sqrt{q_p^2 - q'^2}
\]

and:

\[
\begin{align*}
\omega_{ip} &= \sqrt{m_i^2 + q_p^2} \\
\omega'_{ip'} &= \sqrt{m_i'^2 + q_{p'T}^2}.
\end{align*}
\]

Now we can give the formulas of form factor using the above covariant variables:

\[
\begin{align*}
g &= \xi \frac{\omega'_1 + \omega'_2}{MM' m_1} \\
f &= \xi \left[ \frac{m_1 p_1}{MM' m_1} + 1 \right] \\
a_+ &= \xi \left\{ \frac{2m_2}{M^2 m_1} + \delta + \left[ \frac{\omega'_1 + \omega'_2}{MM' m_1} + \frac{(p \cdot p')}{M'^2} \right] \right\}
\end{align*}
\] (23)

where

\[
\delta = -\frac{C[1 + (p \cdot p')/M m'_1]}{p^2 - (p \cdot p')^2/M'^2}
\]
\[
C = \frac{1}{L_+ \left[ \frac{(p' \cdot p_1)(p' \cdot p_1')}{M^2} - \frac{1}{L_+^2} \right]^{1/2} - 1}
\]

where

\[
L_\pm = \left[ \frac{1}{2}(p_1 \cdot p_1' \pm m_1 m_1') \right]^{-1/2}.
\]

The common factor:

\[
\xi = \frac{2 \omega_2 m_1^2 m_1'}{\left[ (p_1 \cdot p_1') + m_1 m_1' \omega_1 \omega_1' \omega_2 \right]}^{1/2} \times \int \frac{d^3 \overrightarrow{q}}{(2\pi)^3} \phi_{p'}^* (q_{p'}') \cdot \phi_p (|q|)
\]

if it is written in c.m.s. of the initial meson ($\vec{p} = 0$).

Here $\phi_{p'}^* (q_{p'}')$ and $\phi_p (|q|)$ \[14\] correspond to the radius wave functions of the mesons in the initial and final states respectively, but both are presented in c.m.s. of the initial meson.

The equations about $g_a, g_b$ are similar as $g$, we will not repeat them here.

**IV. NUMERICAL RESULTS AND DISCUSSION**

First of all, let us show the ‘whole’ leptonic decay widths i.e. the sum of the corresponding radiative decay widths and the corresponding pure leptonic decay widths with radiative corrections, and put them in Table (1). As firstly we would like to see the facts of ‘short distance’ contributions, so here only the ‘short distance’ contributions are taken into account. Why we put the radiative decay and the pure leptonic decay with radiative corrections together here is to make the width not to depend on the experimental resolution for a soft photon at all. To compare with the earlier computations \[3] \[5\] in the numerical evaluation, the values for the parameters $\alpha = 1/132$ and $|V_{bc}| = 0.04$ \[13\] are taken, and two possible values for the rests are selected as bellow:

1. $m_{Bc} = 6.258 \text{ GeV}, m_b = 4.758 \text{ GeV}, m_c = 1.500 \text{ GeV}, f_{Bc} = 0.360 \text{ GeV}$ \[3\];
2. $m_{Bc} = 6.258 \text{ GeV}, m_b = 4.700 \text{ GeV}, m_c = 1.400 \text{ GeV}, f_{Bc} = 0.350 \text{ GeV}$ \[4,5\].

| Table (1) The ‘Whole’ Leptonic Decay Widths (in unit GeV) |
| (short distance contributions only) |
For comparison, the width of each pure leptonic decay at tree level with the same parameters as those in Table (1) is put in Table (2).

Table (2) The Pure Leptonic Decay Widths (in unit GeV) of Tree Level

|       | (1)  | (2)  |
|-------|------|------|
| $\Gamma_e(10^{-17})$ | 6.444 | 6.902 |
| $\Gamma_\mu(10^{-16})$ | 1.383 | 1.389 |
| $\Gamma_\tau(10^{-14})$ | 1.871 | 1.782 |

If the lifetime of $B_c$ meson is (a). $\tau(B_c) = 0.46 \times 10^{-12} s$ as indicated by the first observation \[8\]; (b). $\tau(B_c) = 0.52 \times 10^{-12} s$ as adopted in Ref. \[3\], the corresponding branching ratios are showed in Tables (3), (4).

Table (3) Branching Ratios of the ‘Whole’ Leptonic Decays

|       | (short distance contributions) |
|-------|-------------------------------|
|       | (1-a) | (2-a) | (1-b) | (2-b) |
| $B_e(10^{-9})$ | 5.09  | 5.45  | 4.5   | 4.82  |
| $B_\mu(10^{-5})$ | 10.93 | 10.98 | 9.69  | 9.76  |
| $B_\tau(10^{-2})$ | 1.477 | 1.407 | 1.306 | 1.246 |

Table (4) Tree Level Branching Ratios of The Pure Leptonic Decays

|       | (1-a) | (2-a) | (1-b) | (2-b) |
|-------|-------|-------|-------|-------|
| $B_e(10^{-9})$ | 1.44  | 1.36  | 1.28  | 1.21  |
| $B_\mu(10^{-4})$ | 0.62  | 0.586 | 0.55  | 0.52  |
| $B_\tau(10^{-2})$ | 1.47  | 1.40  | 1.30  | 1.24  |
To see the contributions of the radiative decays precisely we present the radiative decay widths with a cut of the photon energy i.e. the widths of the radiative decays $B_c \to l\nu\gamma$ with the photon energy $E_\gamma \geq k_{min}$ as the follows: $k_{min} = 0.1 \text{ GeV}$, $k_{min} = 0.2 \text{ GeV}$, $k_{min} = 0.5 \text{ GeV}$ and $k_{min} = 1.0 \text{ GeV}$ respectively in Table (5).

Table (5): The Radiative Decay Widths (in unit $10^{-17}\text{GeV}$)
(with cuts of the photon momentum and the angle between photon and lepton)

| $k_{min}(\text{GeV})$ | (1) | (2) |
|------------------------|-----|-----|
|                        | $5^0$ | $15^0$ | $30^0$ | $5^0$ | $15^0$ | $30^0$ |
| 0.1 $\Gamma_e$        | 6.384 | 6.370 | 6.297 | 6.832 | 6.819 | 6.752 |
| 0.2 $\Gamma_e$        | 6.317 | 6.303 | 6.242 | 6.762 | 6.750 | 6.693 |
| 0.5 $\Gamma_e$        | 5.931 | 5.918 | 5.883 | 6.351 | 6.340 | 6.307 |
| 1.0 $\Gamma_e$        | 4.807 | 4.800 | 4.790 | 5.151 | 5.143 | 5.136 |
| 0.1 $\Gamma_{\mu}$   | 6.613 | 6.518 | 6.385 | 7.049 | 6.958 | 6.834 |
| 0.2 $\Gamma_{\mu}$   | 6.484 | 6.412 | 6.306 | 6.920 | 6.850 | 6.753 |
| 0.5 $\Gamma_{\mu}$   | 6.018 | 5.977 | 5.917 | 6.433 | 6.394 | 6.340 |
| 1.0 $\Gamma_{\mu}$   | 4.843 | 4.824 | 4.802 | 5.184 | 5.165 | 5.146 |
| 0.1 $\Gamma_\tau$    | 13.75 | 13.66 | 12.88 | 13.60 | 13.52 | 12.78 |
| 0.2 $\Gamma_\tau$    | 10.87 | 10.82 | 10.34 | 10.86 | 10.81 | 10.36 |
| 0.5 $\Gamma_\tau$    | 7.139 | 7.121 | 6.970 | 7.282 | 7.265 | 7.122 |
| 1.0 $\Gamma_\tau$    | 4.169 | 4.165 | 4.146 | 4.340 | 4.335 | 4.318 |

For the convenience to compare with experiments, we present the photon spectrum of the radiative decays in Fig.6 and give the decay widths of the so-called physical pure leptonic decays: $\Gamma_{\text{phys}} = \Gamma_{\text{whole}} - \Gamma_{\text{cut}}$ where $\Gamma_{\text{cut}}$ are showed in Table (5); $\Gamma_{\text{whole}}$ are showed in Table (1). We put the values of $\Gamma_{\text{phys}}$ in Table (6).

Table (6): Decay widths of $\Gamma_{\text{phys}}$ ($10^{-17}\text{GeV}$)
Let us select the parameters as in Ref. [3]: $m_{B_c} = 6.258 \text{ GeV}$, $m_b = 4.758 \text{ GeV}$, $m_c = 1.500 \text{ GeV}$, $f_{B_c} = 0.360 \text{ GeV}$, so as to compare with the short distance contributions and to show the contributions from the typical long distance component concerned in the paper.

For each of the family, the contribution to the width:

$$\delta \Gamma_e = 1.828 \times 10^{-18} \text{GeV},$$

$$\delta \Gamma_\mu = 1.827 \times 10^{-18} \text{GeV},$$

$$\delta \Gamma_\tau = 1.190 \times 10^{-18} \text{GeV}.$$
c quark in $B_c$ meson are heavy. Their heavy mass causes the long distance contributions small. Therefore, it seems that we can conclude that the disagreement between Ref. [3] and Refs. [4,5] is not due to the long distance effects.

In addition, we should note that the widths are quite sensitive to the decay constant $f_{B_c}$, and the values of the quark masses $m_b$ and $m_c$.

Note that when we almost completed this paper, one similar paper [15] appeared and its results supported those in Ref. [3] that is in agreement with this paper.

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FIG. 1. Tree diagram for $B_c \rightarrow \ell \nu$. 

FIG. 3. 1. Box-loop diagrams for $B_c \rightarrow \ell \nu$. 

FIG. 2. Diagrams for $B_c \rightarrow \ell \nu \gamma$. 

FIGURES
FIG. 3. 2. Self-energy and vertex diagrams for $B_c \to \ell \nu$.

FIG. 3. 3. Vertex diagrams for $B_c \to \ell \nu$. 
FIG. 3.4. Self-energy diagrams for $B_c \rightarrow \ell \nu$. 

FIG. 3.4. Self-energy diagrams for $B_c \rightarrow \ell \nu$. 

(a) 

(b) 

(c)
FIG. 4. Diagrams: (a) $J/\psi$ as an intermediate state; (b) and (c) $B_c^*$ as an intermediate state.

![Diagram of $J/\psi$ and $B_c^*$](image)

FIG. 5. A Feynman diagram corresponding to the relevant weak or electromagnetic current sandwiched by the $B_c$ meson and a suitable single-particle state.

![Feynman diagram](image)

FIG. 6. Photon energy spectra of radiative decays $B_c \rightarrow \ell \nu \gamma (\ell = e, \mu, \tau)$.