This paper studies a general class of stochastic population processes in which agents interact with one another over a network. Agents update their behaviors in a random and decentralized manner based only on their current state and the states of their neighbors. It is well known that when the number of agents is large and the network is a complete graph (has all-to-all information access), the macroscopic behavior of the population converges to a differential equation called a mean-field approximation. When the network is not complete, it is unclear in general whether there exists a suitable mean-field approximation for the macroscopic behavior of the population. This paper provides general conditions on the network and policy dynamics for which a suitable mean-field approximation exists. First, we show that as long as the network is well-connected, the macroscopic behavior of the population concentrates around the same mean-field system as the complete-graph case. Next, we show that as long as the network is sufficiently dense, the macroscopic behavior of the population concentrates around a mean-field system that is, in general, different from the mean-field system obtained in the complete-graph case. Finally, we provide conditions under which the mean-field approximation is equivalent to the one obtained in the complete-graph case.

**Keywords**: mean-field approximation, population processes, population games, network-based interaction.
1. INTRODUCTION

We consider the problem of analysis and approximation of stochastic dynamics that emerge in large-scale networks of interacting agents. This general problem has applications to many disciplines including evolutionary biology (Taylor and Jonker (1978), Smith (1982)), optimization and controls (Barreiro-Gomez, Ocampo-Martinez, and Quijano (2017), Quijano, Ocampo-Martinez, Barreiro-Gomez, Obando, Pantoja, and Mojica-Nava (2017), Barreiro-Gomez, Obando, and Quijano (2017), Barreiro-Gomez, Quijano, and Ocampo-Martinez (2016)), economics (Sandholm (2010), Levine and Modica (2016)) and social behavior (Tan, Lü, and Lin (2016), Bauso, Tembine, and Başar (2016), Acemoglu, Bimpikis, and Ozdaglar (2014), Foerster (2019)). Typically, the agent-level dynamics of such systems are too complex to understand directly from a computational or theoretical lens. If we assume that (1) the number of agents is infinite and (2) agents have complete access to all other agents’ actions when changing their behavior, the evolution of the population can be described by a system of deterministic differential equations. These equations are known as a mean-field approximation, and they provide an important first step for all the aforementioned models in understanding the dynamics of the complex system. A natural question is: how accurate is the mean-field approximation when assumptions (1) and (2) are relaxed? Most of the previous literature has analyzed the case when (1) is relaxed: the number of agents is large, but finite instead of infinite. In this case, the process governing the finite-agent population dynamics is typically Markov and converges to a suitable mean-field approximation as the number of agents tends to infinity. When (2) is relaxed, agents use imperfect or incomplete estimates of the state of the population to make decisions. We study in detail the case where agents have partial information, and may only observe the actions of a subset of the population. We call this the distributed-information setting. We can mathematically describe this information structure via a network topology, where agents can only observe the actions of their neighbors.

An important example in which the distributed-information setting appears in practice is the spread of a virus. In many epidemiological models, agents are initially susceptible until they are exposed to the virus. The infected agent then interact with others in the population,
thereby exposing them to the virus, until the agent recovers (see, e.g., Brauer and Castillo-Chavez (2012)). In particular, whether or not an agent becomes infected at a certain point in time depends not on the infections in the full population, but rather in the limited subset of individuals the agent interacts with. Here, the information structure is represented by a so-called contact network, a graph which describes the possible interactions between agents in the population. A large body of literature has investigated how the structure of the contact network impacts the spread of the virus; see Section 1.2 for a more detailed discussion of this literature.

In general, it is unclear whether a suitable mean-field equation exists that approximates the large-scale behavior of the population state in the distributed-information setting. Although recent theoretical work has shown that a mean-field approximation does yield accurate predictions for certain highly-structured networks (see Section 1.2 for a detailed discussion), empirical studies hint that such approximations are accurate for a much larger class of (less-structured) real-world networks (Gleeson, Melnik, Ward, Porter, and Mucha, 2012). Furthermore, in such network-based partial information scenarios, assuming a mean-field behavior exists, it is worthwhile to understand how the mean-field approximation relates to the classical (all-to-all or full information access) case and how it is influenced by the underlying information structure.

In this paper, we establish general conditions on both the information structure and agent dynamics that ensure the stochastic population process is well-approximated by a mean-field equation in the distributed-information setting. Broadly speaking (see precise statements in subsequent sections), our results are threefold. First, we show that if a certain adjacency-like matrix associated with the network topology that governs the information access among agents has a nearly maximal spectral gap\(^1\), the population is well-approximated by a mean-field differential equation. Moreover, the aforementioned mean-field equation is exactly the classical mean-field equation derived when the number of individuals is infinite and all agents have complete access to the entire population when up-

\(^1\)At a high level, this means the network is very well-connected – examples include random regular graphs and Erdős-Rényi graphs.


dating their behavior. Next, we study the case of a general network topology. If the graph is sufficiently dense, we show that the population process concentrates around a different mean-field equation which, unlike the classical mean-field equation, strongly depends on the structure of the network topology. Unlike prior literature which has studied specific families of graphs, this result only depends on a very simple density measure and not on finer properties of the graph. A drawback, however, is that this new mean-field equation has a somewhat complex description. Finally, we identify several conditions under which the new mean-field equation reduces to the classical mean-field equation, hence showing that the classical mean-field analysis well-approximates the stochastic population process even in the distributed-information setting.

1.1. Contributions

We begin by introducing a general class of population processes which extends well-studied models in the literature (see for instance Benaïm and Weibull (2003), Sandholm (2010)). There are \( N \) agents in the population, each of which initially plays a pure strategy from some finite set. At the time indices \( 0, 1/N, 2/N, \ldots \), exactly one agent is chosen uniformly at random to revise their strategy. The selected agent does so in a probabilistic manner via a policy \( \rho \) that depends on the current strategy of the agent and a local estimate of the (macroscopic) population state, which summarizes the agent’s information about the population. Ideally, if agents can perfectly observe the strategies of other agents, the local estimate of an agent at time \( t \) is a vector detailing the proportion of agents playing a particular strategy at time \( t \) - we call this the (true) population state. We refer to the latter model as the perfect information setting. The key novelty in our model is that we allow agents to have different local estimates. This is an important generalization, since agents may not always have perfect access to population statistics especially in real-time, and agents may also compute different population estimates due to inherent biases. We focus our attention to the case where the structure of the local estimates are tied to a graph \( G \) that models inter-agent information access or communication; henceforth, we refer to \( G \) as the communication graph. The vertex set of \( G \) is the set of agents, and the presence of an edge (or
communication link) between two agents indicates that the two agents can observe each other’s strategies at any point in time. The local estimate of an agent is assumed to be a weighted average of the strategies within the agent’s neighborhood.\(^2\) It will convenient to collect these weights into a single matrix \(W\) whose sparsity structure aligns with the structure of \(G\). We call this model the \textit{distributed information setting}. We remark that the perfect information setting may also be viewed as a specific instance of the distributed information setting in which the inter-agent information access graph is a complete graph, i.e., with all-to-all communication links.

The assumption of perfect information, while simplistic, leads to an elegant characterization of the population state evolution via a mean-field approximation, which is a solution to a deterministic ordinary differential equation (ODE) with fixed dimension that depends on the policy \(\rho\). We refer to this mean-field equation, derived from the perfect-information setting, as the \textit{classical} mean-field equation. Unfortunately, when agents have different local estimates the analysis breaks down completely, and it not clear whether the classical mean-field ODE approximates the population state well in this case. Our first main result (Theorem 3.6) shows that the classical mean-field approximation is \textit{robust}: if the structure of the underlying graph ensures that agents’ local estimates are \textit{close} to the true population state\(^3\), the population process is well-approximated by the classical mean-field ODE. Surprisingly, our result shows that the classical mean-field equation is not a fundamental property of the all-to-all information structure, but rather a property of the generic information weighting process that is used to construct local estimates. We emphasize this point by showing that Theorem 3.6 holds for several types of communication graphs – includ-

\(^2\)The idea of locally estimating a global statistic (the macroscopic population state in this case) through neighborhood averaging in a multi-agent or networked setup is a simplistic yet powerful technique that forms the basis of several information processing and opinion formation models, notably the DeGroot model of consensus formation (DeGroot, 1974) as well as other models in optimization, control and game-theoretic computations in networks, see (Tsitsiklis, Bertsekas, and Athans, 1986, Jadbabaie, Lin, and Morse, 2003, Dimakis, Kar, Moura, Rabbat, and Scaglione, 2010, Swenson, Kar, and Xavier, 2015).

\(^3\)Formally, we require the spectral gap associated with \(W\) to be sufficiently small.
ing complete graphs with link failures, random regular graphs, and Erdős-Rényi graphs of appropriate parameters.

We next consider the case of an arbitrary communication graph $G$, and establish general conditions under which the corresponding population dynamics can be approximated by a deterministic process. Our second main result (Theorem 4.2) extends the analysis of Theorem 3.6 to show that as long as the graph is sufficiently dense, the stochastic population dynamics concentrates around a deterministic process that is, in general, different from the classical mean-field ODE and depends strongly on the topology of the communication graph. Much of the prior literature concerning the distributed information case assumes that the underlying graph is part of a highly-structured family, yet Theorem 4.2 holds for any sufficiently dense graph. Furthermore, the new deterministic process that appears in Theorem 4.2 has not been studied before, to the best of our knowledge.

While Theorem 4.2 is quite powerful, the drawback is that the deterministic process has a complicated description. Together, Theorem 4.6 and Corollaries 4.7 and 4.8 provide general conditions under which the deterministic process reduces to the classical mean-field ODE. First, we show that if agents’ local estimates are initially sufficiently close to the true population state (which is possible in many cases if agents choose their initial actions at random), the deterministic process reduces to the mean-field ODE. To handle the case of arbitrary initial conditions, we introduce an additional feature into the distributed information model, termed as neighborhood exploration. Formally, given a non-increasing function $r(t)$ taking values in the interval $[0, 1]$, we make the following change to agent dynamics: when an agent is selected to update, they make the usual policy-based update with probability $r(t)$; with probability $1 - r(t)$, the agent instead imitates the action of a randomly chosen neighbor. When $r(t) = 1$ we recover the classical dynamics, and when $r(t) = 0$, agents always imitate the actions of neighbors. We show that neighborhood exploration serves to “mix” the population so that if agents update via this process sufficiently often, agents’ local estimates asymptotically tend to the true population state, and the deterministic process reduces to the classical mean-field ODE. The idea of embedding this mixing process into agent interaction is inspired by diverse literature in distributed information processing and
control (see for example Sayed (2014) and the references therein) and $\varepsilon$-greedy algorithms for bandits (Bubeck and Nicolò, 2012, Sutton and Barto, 1998, Auer, Cesa-Bianchi, and Fischer, 2002, Vermorel and Mohri, 2005). The unifying idea behind these applications is that while making utility-based decisions, agents also sample from other information at a certain rate that allows them to simultaneously learn or estimate statistics of the macroscopic system through their actions. It is in this same spirit that we use exploration in our dynamics so that agents can estimate global statistics and act on that information.

1.2. Related work

The study of ODE approximations to finite-agent population processes goes back to Kurtz (1970, 1976), who proved that a general class of Markov population processes converges in probability to the mean-field ODE when the number of agents in the population grows large. Sandholm (2003) applied this theory to the perfect information setting of evolutionary games. For the same setting, Benaïm and Weibull (2003, 2009) proved a stronger result, showing the probability of the finite-agent process deviating from the mean-field ODE is an exponentially decaying function of the number of agents. The techniques of the latter work form the basis of our analysis; indeed, Theorems 3.6 and 4.2 can be viewed as generalizations of their work to the distributed information setting. We discuss additional related work on finite agent population processes in the perfect information setting in Section 2.1, where we present details about our model.

The literature surrounding the distributed information setting is quite recent. Since the population-level dynamics are no longer Markov in this setting, prior work has mostly focused on cases where the communication graph, $G$, comes from a highly structured family of graphs. We review the results for different graph families below. The following literature is based on a wide range of population models, including stochastic networks (Kar and Moura (2011)), epidemics (Santos, Moura, and Xavier (2013), Santos, Kar, Moura, and Xavier (2016)), Kuramoto models (Chiba and Medvedev (2019), Medvedev (2019), Chiba, Medvedev, and Mizuhara (2018)) and interacting diffusions (Delattre, Giacomin, and Luçon (2016), Coppini, Dietert, and Giacomin (2020), Olivera and Reis (2019),
Oliveira, Reis, and Stolerman (2020), Lacker, Ramanan, and Wu (2019)). The details and applications of each of these models are different, but the fundamental hurdles of the distributed information setting is present in all of them. We therefore expect that methods developed for one type of model can be extended to others.

1.2.0.1. **Hierarchical graphs.** Some authors assume that $G$ has the following two-level hierarchical structure. A sparse graph connects several “supernodes”, where each supernode itself is a collection of many vertices endowed with its own graph topology – see Example 4.5 for a formal description. In this setting, a mean-field ODE based on the average behavior of agents within each supernode as well as the connectivities between supernodes is usually studied. Kar and Moura (2011) as well as Santos, Moura, and Xavier (2013) further show that the finite-agent population dynamics converges in probability to the alternate deterministic process.

1.2.0.2. **Regular graphs.** Santos, Kar, Moura, and Xavier (2016) study an epidemic process where a new underlying graph is chosen uniformly from the set of $d$-regular graphs whenever two agents interact. As the number of agents increases, the authors show that the population process converges in probability to the solution of an ODE. On the other hand, Farkhooi and Stannat (2017) study the case of static, $d$-regular graphs and show that if the graph satisfies certain balancedness conditions, the finite-agent population process converges in probability to the classical mean-field ODE. The balancedness conditions are expected to be satisfied for graphs with i.i.d. structure, though this is not formalized in their paper. Furthermore, Farkhooi and Stannat extend their analysis to the setting where $d$ increases to infinity as the number of agents increases.

1.2.0.3. **Graphs with independent random structure.** Many authors consider the case where $G$ is a random graph constructed by adding an edge between vertices $i$ and $j$ with some probability $p_{ij}$, independently across all pairs of vertices. When $p_{ij}$ is constant over $i,j$, $G$ is an Erdős-Rényi graph, but generalizations such as graphons (Borgs, Chayes, Lovász, Sós, Szegedy, and Vesztergombi (2006)) are also natural. The works (Chiba and Medvedev (2019), Medvedev (2019), Chiba, Medvedev, and Mizuhara (2018), Delattre,
Giacomin, and Luçon (2016), Coppini, Dietert, and Giacomin (2020), Olivera and Reis (2019)) study properties of the finite-agent population process on random graphs with large average degree. Notably, Olivera and Reis (2019) as well as Coppini, Dietert, and Giacomin (2020) show that if the average degree of an Erdős-Rényi graph tends to infinity, the classical mean-field ODE is the limiting object of the stochastic population process. For comparison, our results show that for any fixed graph that is sufficiently dense, if random initial conditions are used, the finite-agent population process concentrates around the mean-field ODE. If the initial conditions are arbitrary, we directly show the same result in the special case where the average degree of the Erdős-Rényi graph grows faster than the logarithm of the number of vertices. With a more careful analysis, we may be able to show that Erdős-Rényi graphs with diverging average degree is another special case of our results, but we defer this to future work as it is not the focus of our paper.

Recently, Oliveira, Reis, and Stolerman (2020) as well as Lacker, Ramanan, and Wu (2019) have examined the sparse case where the average degree $G$ is bounded. The techniques in this regime are much more complicated and are outside the purview of our results. An important future line of work is to study the case of bounded-degree graphs in our setting as well.

Finally, we discuss a different line of work which shares some similar aspects to ours but is fundamentally different. Instead of constraining peer-to-peer interactions via the graph structure, one may instead constrain the set of actions. Formally, given a finite set of actions $A$, consider a graph $H$ with vertex set $A$. An agent playing action $\alpha \in A$ may only change their action to $\beta \in A$ if the edge $(\alpha, \beta)$ is present in $H$. Thus there is no restriction on agents’ information; rather, the limitation lies in the possible strategy revisions. This formulation has been studied for population games as well as other models (Barreiro-Gomez, Obando, and Quijano (2017), Barreiro-Gomez, Quijano, and Ocampo-Martinez (2016), Quijano, Ocampo-Martinez, Barreiro-Gomez, Obando, Pantoja, and Mojica-Nava (2017), Guéant (2015)).
1.3. *Organization of the paper*

In Section 2 we recap the perfect-information setting studied in prior work and introduce the distributed-information model. In Section 3, we present our first main result, Theorem 3.6, which studies the case where agents’ local estimates are close to the true population state. Section 4 studies the case where $G$ is a general graph. We present a formal justification for our new deterministic process in Section 4.1 and discuss our main concentration result, Theorem 4.2, in Section 4.2. The remainder of Section 4 is devoted to studying further properties of the new deterministic process. We conclude and state some avenues for future work in Section 5. The proofs of our results are deferred to the appendices.

2. *FINITE-AGENT POPULATION PROCESSES*

In Section 2.1, we review the general class of finite-agent population processes studied in detail by Benaïm and Weibull (2003, 2009), as well as the corresponding ODE approximation analysis. As we have mentioned earlier, the key enabler of this analysis is the assumption that each agent can observe the true state of the population when making decisions. Next, in Section 2.2, we extend the model of Section 2.1 to the case where agent information is constrained via a network topology.

2.1. *The perfect information setting*

Consider a population of $N$ agents, where each agent initially plays a pure strategy from a finite set $\mathcal{A}$, with $|\mathcal{A}| = m$, where $m$ is fixed and assumed to be much smaller than $N$. At each time step in the following set of discretized transition times

$$
T^N := \left\{ \frac{1}{N}, \frac{2}{N}, \ldots \right\} = \left\{ \frac{\ell}{N} : \ell \in \mathbb{N} \right\},
$$

exactly one agent is selected uniformly at random, independently chosen at each time, to revise their strategy (usually in a utility-driven manner). The stochastic process of interest is the *population state* $Y^N_{av}(t) = \{Y^N_{av,\alpha}(t)\}_{\alpha \in \mathcal{A}}$, where $Y^N_{av,\alpha}(t)$ is the fraction of agents in the population who play action $\alpha$. The state space of $Y^N_{av}(t)$ is contained within the simplex $\mathbb{T}^N$.
\( \Delta_m \), given formally by
\[
\Delta_m := \left\{ x \in \mathbb{R}^m_\geq 0 : \sum_{i=1}^m x_i = 1 \right\}.
\]

We label the stochastic process \( Y^N_{av}(t) \) with the subscript \( av \) to indicate that is an average or global description of the local states. Based on our description of how agents update their strategies, we can see that \( Y^N_{av}(t) \) is updated as follows. Suppose that \( t \in \mathbb{T}^N \), and that \( Y^N_{av}(t) \) is the population state after the transition at time \( t \). At the next transition time \( t + \frac{1}{N} \), an agent is selected uniformly at random to update their strategy; call them agent \( i \). Then we have
\[
Y^N_{av}\left( t + \frac{1}{N} \right) = \begin{cases} 
Y^N_{av}(t) + \frac{1}{N}(e_\alpha - e_\beta) & \text{if agent } i \text{ switches from } \beta \text{ to } \alpha \\
Y^N_{av}(t) & \text{if agent } i \text{ decides not to switch actions,}
\end{cases}
\]
where \( e_\alpha, \alpha \in A \), denotes the \( \alpha \)-th canonical vector in \( \mathbb{R}^m \). We assume that agents make decisions according to a common, time-invariant probabilistic policy \( \rho \), which depends only on the current action of the agent and the observed information of the agent. In this model, all agents can perfectly observe \( Y^N_{av}(t) \) at any time (we relax this assumption in the current work based on the distributed-information model; see Section 2.2). The probability that an agent switches from action \( \alpha \) to \( \beta \) is given by \( \rho^{\alpha\beta}(Y^N_{av}(t)) \). We emphasize that this is the case for any agent, since all agents have the same observed information. Thus for any two actions \( \alpha \) and \( \beta \), we can write the transition probabilities of the stochastic process \( Y^N_{av}(t) \) as:
\[
\mathbb{P}\left( Y^N_{av}\left( t + \frac{1}{N} \right) = x + \frac{1}{N}(e_\alpha - e_\beta) \mid Y^N_{av}(t) = x \right) = \frac{1}{N} \sum_{i:a_i(t)=\beta} \rho^{\beta\alpha}(x) = x^{\beta} \rho^{\beta\alpha}(x),
\]
where \( a_i(t) \in A \) is the action of agent \( i \) at time \( t \). Note that the right hand side depends only on \( x \), so the population process is Markov in this case. Remarkably, this implies that one can track the macroscopic population behavior \( Y^N_{av}(t) \) without considering the local state changes of individual agents at all. It remains to study the trajectory of \( Y^N_{av}(t) \).
From the functions \( \{\rho^{\beta\alpha}\}_{\alpha,\beta \in A} \) we can also compute the expected net increase \( \Lambda : \Delta_m \rightarrow \mathbb{R}^m \), with components \( \Lambda^\alpha \) for \( \alpha \in A \) given by

\[
\Lambda^\alpha(x) := \mathbb{E} \left[ \frac{Y^{N,\alpha}_{av}(t + 1) - x^\alpha}{1/N} \mid Y^{N}_{av}(t) = x \right] = \sum_{\beta \in A: \beta \neq \alpha} x^\beta \rho^{\beta\alpha}(x) - \sum_{\beta \in A: \beta \neq \alpha} x^\alpha \rho^{\alpha\beta}(x).
\]

Above, for \( x \in \mathbb{R}^m \), we let \( x^\alpha \) be the corresponding component of \( x \). Conditioned on \( Y^{N}_{av}(t) = x \), (3) is the expected rate of change in the proportion of agents playing \( \alpha \). The first term represents the probability of an agent switching to \( \alpha \), and the second term represents the probability of switching from \( \alpha \) to another action. As \( N \) grows larger and the transition times become more and more frequent, it is natural to approximate the stochastic evolution of \( Y^{N}_{av}(t) \) by the deterministic coupled differential equations

\[
\begin{cases}
\dot{x}^\alpha = \Lambda^\alpha(x) & \forall \alpha \in A \\
 x(0) = Y^{N}_{av}(0).
\end{cases}
\]

Under the following assumption, the solution to (4) is unique for any given initial condition.

**Assumption 2.1:** The functions \( \rho^{\alpha\beta} : \Delta_m \rightarrow \mathbb{R} \) are continuously differentiable and \( L_{\rho} \)-Lipschitz with respect to the norm \( \| \cdot \|_{\infty} \) for all \( \alpha, \beta \in A \).

The assumption implies \( \Lambda \) is Lipschitz on the compact invariant set \( \Delta_m \), and uniqueness follows from the Picard-Lindelöf theorem. The first main result of Benaïm and Weibull shows that \( Y^{N}_{av}(t) \) concentrates around the unique solution to (4).

**Theorem 2.2**—Lemma 1 of Benaïm and Weibull (2003): Let \( \hat{Y}^{N}_{av}(t) \) denote the continuous-time version of \( Y^{N}_{av}(t) \) that is obtained by linear interpolation between consecutive time-steps in \( \mathbb{T}^N \). Let \( x(t) \) be the solution to (4). For any \( \epsilon > 0 \), \( T > 0 \), there is a constant \( c = c(T) \) and \( N(\epsilon, T) \) large enough such that for \( N \geq N(\epsilon, T) \),

\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| \hat{Y}^{N}_{av}(t) - x(t) \right\|_{\infty} > \epsilon \right) \leq 2me^{-cN\epsilon^2}.
\]
The above concentration bound also holds for other models of asynchronous updates. For instance, the continuous-time analog of choosing an agent uniformly at random to update at each transition time is to assign each agent an independent Poisson clock with rate 1 (i.e. a Poisson process on $\mathbb{R}_{\geq 0}$ with rate 1); when an agent’s clock rings, the agent may revise their strategy. Benaïm and Weibull also extend Theorem 2.2 to the Poisson clock model (see Appendix I in (Benaïm and Weibull, 2003)).

This type of law of large numbers result for Markov population processes has been developed before by several authors (Kurtz (1970), Boylan (1995), Binmore and Samuelson (1997), Corradi and Sarin (2000), Sandholm (2003)). Recently, some authors have further developed the law of large numbers result from a large-deviations perspective (Sandholm and Staudigl (2018), DeVille and Galiardi (2017), Sawa (2016)). All of these works rely fundamentally on the Markov nature of the population process $Y_{av}^N(t)$. However, as we will see in the following section, the distributed-information version of this formulation, in which agents may not be able to directly estimate $Y_{av}^N(t)$, is no longer Markov, hence usual techniques do not apply.

2.1.0.1. Connections to game theory. A key application of our results is to population games (see for instance Sandholm (2010)), where agents act according to a policy $\rho$ that will hopefully improve their utility at every time they update.

In population games, a common utility function is assigned to all agents and is given by $U : \Delta_m \rightarrow \mathbb{R}^m$. The utility for an agent playing action $\alpha$ is $U_\alpha(x)$, where $x$ is the global population state. Once agents observe $x$ and the utility function $U(x)$, there are several policies that they may use to update their strategy:

- (Best response) $\text{supp } \rho^{\beta\alpha}(x) \subseteq \arg \max_{\alpha \in A} U_\alpha(x)$;
- (Smoothed best response) $\rho^{\beta\alpha}(x) \propto \exp(\theta \cdot U_\alpha(x))$ for some $\theta \geq 0$;
- (Imitation by pairwise comparison) $\rho^{\beta\alpha}(x) \propto x^\alpha \cdot \max \{U_\alpha(x) - U_\beta(x), 0\}$.

An important goal is to understand the dynamics of the macrostate $Y_{av}^N(t)$ when agents act according to a given policy $\rho$. Using Theorem 2.2, Benaïm and Weibull characterize the behavior of the finite-agent process $Y_{av}^N(t)$ in the perfect-information setting using prop-
erties of the deterministic mean-field solution. The same can be done for the distributed-information setting using our results, Theorems 3.6 and 4.2.

2.2. The distributed-information setting

A key assumption of the model in Section 2.1 is that all agents have perfect access to the true state of the population. In many cases of practical interest, however, agents may only be able to observe a subset of agents to inform their decisions. In this section, we generalize the model of Section 2.1 to account for this information structure.

Suppose that we have $N$ agents. We represent the collection of all agents by $[N] := \{1, \ldots, N\}$. Agent information is represented by a fixed, undirected graph $G$ on $N$ vertices labelled $1, \ldots, N$, where the edge $(i, j)$ exists if and only if agents $i$ and $j$ can observe each other’s actions at any point in time. We make the following assumption on $G$.

Assumption 2.3: The graph $G$ is connected and has self-loops for each vertex.

As will be shown below, the first condition ensures that information diffuses across the network. The assumption that $G$ is connected can be assumed without loss of generality; if the network was not connected, we could consider the connected components individually. The second assumption follows naturally from the interpretation of $G$: each agent can observe and use their own information, hence there is a self-loop for each vertex. Note that when agents can perfectly observe the true population state, which is the case in the model of Section 2.1, $G$ corresponds to a complete graph (with self-loops for each vertex), i.e., with all-to-all communication links.

Since agents only have partial information in our setting, they use a local population state based on aggregating neighborhood information as a surrogate for the true, global population state. For each $i \in [N]$, let $w_{i1}, \ldots, w_{iN} \in [0, 1]$ be constants satisfying $\sum_{j=1}^{N} w_{ij} = 1$, where $w_{ij}$ informally represents the (direct) influence the actions of agent $j$ have on agent $i$ in terms of estimating the population state. To write these ideas formally, we first denote, for $j \in [N]$, $Y^N_j(t) = \{Y_j^{N,\alpha}(t)\}_{\alpha \in A}$ to be the vectorized action of agent $j$: if $a_i(t) = \alpha$ then $Y_j^{N,\alpha}(t) = 1$ and $Y_j^{N,\beta}(t) = 0$ for $\beta \neq \alpha$. We may also concisely write $Y_j^{N}(t) = e_{\alpha}$,
where $e_{\alpha}$ is a vector indexed by $A$ where only the $\alpha$-component is 1 while the rest are zero. The local estimate of agent $i$ at time $t$ is then defined as

$$Y_i^N(t) := \sum_{j=1}^N w_{ij} Y_j^N(t).$$

(6)

For convenience, we may compile all the weights $w_{ij}$ into a single matrix $W \in \mathbb{R}^{N \times N}$, which we call the combination matrix. We assume the following about $W$.

**Assumption 2.4:** The combination matrix $W = [w_{ij}]_{1 \leq i, j \leq N}$ is non-negative, doubly-stochastic, and conforms to the structure of $G$ in the following sense: $w_{ij} > 0$ if and only if $(i, j)$ is an edge in $G$.

The condition that $W$ is doubly stochastic is common in distributed algorithms (Boyd, Ghosh, Prabhakar, and Shah (2006)). Furthermore, if $G$ is connected and has self-loops for every vertex, it is always possible to find a doubly-stochastic combination matrix (see Corollary 3.3 in Gharesifard and Cortes (2012)). In this paper, we use the *equal weights design* (see (7)), but there are other natural constructions that may be considered, such as Metropolis weights (Nedić, Olshevsky, and Rabbat (2018)) and Sinkhorn scaling (Sinkhorn (1964)). For doubly-stochastic $W$, we also have the following useful relation:

$$\frac{1}{N} \sum_{i=1}^N Y_i^N(t) = \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N w_{ij} Y_j^N(t) = \frac{1}{N} \sum_{j=1}^N \left( \sum_{i=1}^N w_{ij} \right) Y_j^N(t) = Y_{av}^N(t).$$

The last condition in Assumption 2.4 – that $w_{ij} > 0$ if and only if $(i, j)$ is an edge in $G$ – is quite natural. If the edge $(i, j)$ does not exist, agents $i$ and $j$ cannot directly observe

---

4In this model, the local estimates are linear functions of the neighborhood states. In principle, one could use a nonlinear aggregation function, but we focus on the linear case as it is simple and natural to study. This form of local estimate formation through neighborhood averaging is relevant to and arises in problems of distributed information processing, see for example, DeGroot’s consensus formation model (DeGroot (1974)) as well as other models in optimization, control and game-theoretic computations in networks, see (Tsitsiklis, Bertsekas, and Athans (1986), Jadbabaie, Lin, and Morse (2003), Dimakis, Kar, Moura, Rabbat, and Scaglione (2010), Swenson, Kar, and Xavier (2015)).
each other, so their respective actions are not considered when agents $i$ and $j$ form local estimates.

We now describe the agent dynamics in our setting. As in the model in Section 2.1, at each time in $\mathbb{T}^N$, a single agent is chosen uniformly at random to update their action. Suppose that at time $t$, agent $i$ is chosen to update their strategy and the agent currently plays action $\alpha$. When agent $i$ revises their strategy, we obtain the state of the system at time $t + 1/N$. There are two types of updates that can happen:

1. **Policy-based update.** Agent $i$ switches from $\alpha$ to $\beta$ with probability $\rho^\alpha \beta(\bar{Y}_i^N(t))$. In other words, the agent uses their local estimate as a surrogate for the true population state.

2. **Neighborhood exploration.** Agent $i$ chooses to imitate a neighboring agent $j$ with probability $w_{ij}$.

The first type of update is a natural generalization of the model in Section 2.1. The second type of update enables information to diffuse across the network. Formally, the choice of update at time $t$ is controlled by a random parameter $u \in \{0, 1\}$. If an agent $i$ is chosen to update, we sample $u \sim \text{Bernoulli}(r(t))$, where the function $r : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$ is a given decreasing function whose purpose will be described in the following paragraph. If $u = 1$, then agent $i$ performs a policy-based update; else if $u = 0$ the agent updates according to neighborhood exploration. It is important to note that, in this case, the process $Y_{\text{av}}^N(t)$ is not Markov. That is, while the value of $Y_{\text{av}}^N(t + 1/N)$ can be simulated given only $Y_{\text{av}}^N(t)$ in the perfect-information setting, one needs to also know all of the local estimates $\{\bar{Y}_i^N(t)\}_{i \in [N]}$ in order to simulate $Y_{\text{av}}^N(t + 1/N)$ in the distributed-information setting.

We remark that if the communication graph is complete and $r(t) = 1$ for all $t \geq 0$, we recover the classical model described in Section 2.1. While the motivation for the policy-based update in the distributed-information setting is clear, the usefulness of the neighborhood exploration step is more subtle. As we shall see in Section 4, the mean-field approx-

---

5When we explain the dynamics formally in Section 4.1, we will use the more informative notation $u_i^N(t)$, which refers to the type of update that agent $i$ does in order to construct the state of the stochastic process at time $t + 1/N$. 
imation for general communication graphs $G$ is, in general, not the same as the classical mean-field equation derived in the perfect-information setting. However, we shall show that if the neighborhood exploration step is performed often enough, the mean-field approximation for general $G$ converges, asymptotically in time, to the classical mean-field equation. To see why this occurs on a more technical level, note that if an agent performs the neighborhood exploration step, the expected value of their new action is a weighted average of their neighbors’ states. If this weighted averaging is repeated by all agents sufficiently often, $\overline{Y}_i^N(t)$ can be shown to converge to $Y_{av}^N(t)$ for every $i \in [N]$. This implies, once $t$ is large enough, that the population dynamics in the distributed-information case will resemble that of the perfect-information case. In Theorem 3.6, we assume that $r(t) = 1$, since the classical mean-field ODE is shown to approximate $Y_{av}(t)$ well in this case. In our second concentration result, Theorem 4.2, we place no restrictions on $r(t)$. In Corollary 4.8 we show how one may choose $r(t)$ so that the mean-field equation for the distributed-information setting converges to the classical mean-field equation.

In the following sections, we study the evolution of the population process $Y_{av}^N(t)$ in the distributed-information setting over bounded time intervals and as $N \rightarrow \infty$. Since the dynamics depend heavily on the underlying graph $G$, our first main result (Theorem 3.6 in Section 3) will consider sequences $\{(G_n, W_n)\}_n$ of graphs and combination matrices (where $G_n$ is a graph on $n$ vertices and $W_n \in \mathbb{R}^{n \times n}$) to obtain mean-field-like characterizations of $Y_{av}^N(t)$. In Section 4, we also consider fixed graphs that have a sufficiently large number of vertices.

### 3. MEAN-FIELD ANALYSIS FOR RAPIDLY-MIXING GRAPHS

In this section we derive mean-field approximation results for the distributed information setting (see Section 2.2) for graphs that have a “rapidly mixing” property: the spectral gap of the combination matrix $W$ tends to 1 as the number of vertices tends to infinity. This is the case in many families of graphs, including classes of random regular graphs and Erdős-Rényi graphs. Under this assumption, we show that the same probability bound of Theorem 2.2 holds. Remarkably, the only properties of the graph used are the number of vertices and the spectral gap, which indicates that these are exactly the correct quantities to
consider when evaluating the correctness of a mean-field approximation. Prior literature, on the other hand, has mostly established mean-field characterizations when the communication graph $G$ is highly-structured (e.g., is regular or generated according to some probability distribution).

We remark that while our ultimate goal is to establish concentration results for fixed graphs with no special structure, the rapidly mixing setting is of independent interest and serves as an introduction to more involved techniques later used in this paper. In addition, we assume in this section that there is no neighborhood exploration (equivalently, $r(t) = 1$). The reason for this is that we will show the classical mean-field ODE approximates $Y^N_{av}(t)$ well if the "rapidly mixing" property (to be rigorously defined) holds. As explained in Section 2.2, our interest in neighborhood exploration stems from the observation that the classical mean-field ODE is not always a good approximation for the evolution of $Y^N_{av}(t)$ -- a point that will be explored in more detail in Section 4.

Before beginning our analysis, we summarize the evolution of the stochastic process (without neighborhood exploration):

- Initially, at time $t = 0$, each agent chooses an action from the set $\mathcal{A}$. For each $i \in [N]$, $Y^N_{i,\alpha}(0) = 1$ if agent $i$ initially plays action $\alpha$, else $Y^N_{i,\alpha} = 0$.
- Given the collection $\{Y^N_i(t)\}_{i \in [N]}$, the construct the state of the system at time $t + 1/N$ as follows:
  - Choose an agent $i$ uniformly at random, who will be allowed to revise their action.
  - If agent $i$ plays action $\alpha$ at time $t$, they switch to playing action $\beta$ at time $t + 1/N$ with probability $\rho^{\alpha\beta}(Y^N_i(t))$. The behavior of all other agents remain unchanged from their state at time $t$.

We next introduce a few important definitions pertaining to the graph $G$ and combination matrix $W$. For two vectors $u, v \in \mathbb{R}^k$, denote the inner product by $\langle u, v \rangle := \sum_{\ell=1}^k u_\ell v_\ell$ as well as the $\ell_2$-norm $\|u\|_2 := \sqrt{\langle u, u \rangle}$. Let $1$ denote a vector of all ones whose dimension is
the number of columns of $W$, and define the quantity

$$\lambda(W) := \sup_{x: \|x\|_2=1, \langle 1, x \rangle = 0} \| Wx \|_2.$$  

The quantity $1 - \lambda(W)$ is often referred to as the spectral gap of $W$. Since $W$ is non-negative and doubly-stochastic, $0 \leq \lambda(W) \leq 1$. Noting that $W$ is primitive due to the connectivity of $G$ (see Assumptions 2.3 and 2.4), the Perron-Frobenius theorem implies that $\lambda(W) < 1$ (see Berman and Plemmons (1994)).

On a high level, $\lambda(W)$ measures the average similarity between agents’ local population states; we illustrate this idea with two extreme examples. If the combination matrix is $W = I$, then $\lambda(W) = 1$ and corresponds to the case where the agents are completely decoupled in the information sense. On the other hand, if the combination matrix is $W = \frac{1}{N} 1 1^{	op}$ (which is the case when $G$ is the complete graph with self-loops, or equivalently, in the model of Section 2.1), then $\lambda(W) = 0$ and all agents perfectly estimate the true population state.

Another useful perspective is that $\lambda(W)$ represents the “Markovianity” of the population process. When $\lambda(W) = 0$, the underlying graph is complete and the population process $Y_{av}^N(t)$ is Markov as discussed in Section 2.1. Taking this idea one step further, if $\lambda(W) \to 0$ as $N \to \infty$, we can think of $Y_{av}^N(t)$ as being asymptotically Markov. The goal of our first result is to show that concentration results which hold for Markov population processes also hold for these asymptotically Markov population processes. While this may seem like a simple generalization of Theorem 2.2, it broadens our understanding of the mean-field approximation tremendously. We will show, for instance, that random regular graphs and Erdős-Rényi graphs satisfy such a condition for an appropriate but general choice of parameters. To formalize these ideas, we will introduce the following definition.

**Definition 3.1—Rapidly mixing sequence:** A sequence $\{(G_n, W_n)\}_n$ is rapidly mixing if the following conditions are satisfied:

1. $G_n$ is a graph on $n$ vertices satisfying Assumption 2.3, and $W_n$ is a combination matrix for $G_n$ satisfying Assumption 2.4.
2. $\lim_{n \to \infty} \lambda(W_n) = 0$. 

There are several natural ways to construct rapidly mixing sequences from well-understood families of graphs. One such example is an equal weights design. Let $d_i$ be the number of neighbors of vertex $i$ in $G$, not including vertex $i$ itself, and let $d_{\text{max}} := \max_{i \in [N]} d_i$. Given a parameter $a < d_{\text{max}}^{-1}$, the entries of $W$ under the equal weights design are given by

$$W_{ij} := \begin{cases} 
  a & (i, j) \text{ is an edge in } G \text{ and } i \neq j; \\
  1 - a d_i & i = j \\
  0 & \text{else.}
\end{cases}$$

(7)

It can be readily seen that $W$ is a symmetric, doubly-stochastic matrix satisfying Assumption (2.4). This in particular implies that $\lambda(W)$ is the second-largest eigenvalue of $W$. We remark that there several other ways to construct doubly-stochastic combination matrices, such as Metropolis weights (Nedić, Olshevsky, and Rabbat (2018)) and Sinkhorn scaling (Sinkhorn (1964)).

**Example 3.2—Complete graphs:** In the classical setup of Benaim and Weibull’s model, we can interpret the underlying graph topology as being complete with self-loops for each vertex. Let $G_n$ be such a graph on $n$ vertices, and let $W_n$ be the corresponding combination matrix given by an equal weights design with $a = 1/n$. Then $W_n = \frac{1}{n} \mathbf{1} \mathbf{1}^\top$, so $\lambda(W_n) = 0$ for all $n$. Thus the sequence $\{(G_n, W_n)\}_n$ formed by complete graphs with self-loops is rapidly mixing.

**Example 3.3—Link failures in the complete graph:** Suppose that for each vertex $i$, there is a set $S_i \subset [n]$ such that for each $j \in S_i$, the edge $(i, j)$ is removed from the complete graph to form the graph $G_n$. This models, for instance, link failures in networks. The entries of the resulting combination matrix $W_n$ under the equal weights design with $a = 1/n$ are then given by

$$(W_n)_{ij} := \begin{cases} 
  \frac{1}{n} & j \notin S_i, j \neq i \\
  \frac{|S_i|}{n} & i = j \\
  0 & j \in S_i.
\end{cases}$$
We can also write \( W = \frac{1}{n} \mathbf{1} \mathbf{1}^\top + H \), where

\[
H_{ij} := \begin{cases} 
0 & j \notin S_i, j \neq i \\
\frac{|S_i| - 1}{n} & i = j \\
\frac{1}{n} & j \in S_i.
\end{cases}
\]

By the triangle inequality, \( \lambda(W) \leq \lambda(H) \). By the Gershgorin circle theorem, \( \lambda(H) \) can be bounded by \( \max_{i \in [n]} \frac{2|S_i|}{n} \). Hence, as long as \( \max_{i \in [n]} |S_i| \ll n \) (in other words, if the number of link failures is sublinear in the number of vertices), the sequence \( \{(G_n, W_n)\}_n \) is rapidly mixing.

**Example 3.4—Random regular graphs:** In a \( d \)-regular graph, each vertex has degree \( d \) and there are no self-loops or multiple edges. Fix \( n \), and construct \( G_n \) so that it is a \( dn \)-regular graph with self-loops added for each vertex. If \( G_n \) is chosen uniformly at random from the set of \( dn \)-regular graphs, the combination matrix \( W_n \) constructed according to the equal weights design and \( a = \frac{1}{dn+1} \) satisfies \( \lambda(W_n) \leq \frac{C_{\text{reg}}}{\sqrt{n}} \) for some universal constant \( C_{\text{reg}} \) (for details, see Proposition E.1 in Appendix E). Then for \( n_0 \) sufficiently large, \( \{(G_n, W_n)\}_{n \geq n_0} \) is a rapidly mixing family if \( dn \to \infty \) as \( n \to \infty \).

**Example 3.5—Erdős-Rényi graphs:** Given \( p \in (0, 1) \), an Erdős-Rényi graph \( G \sim G(n, p) \) on \( n \) vertices is sampled by creating an edge between a pair of vertices with probability \( p \), independently across all pairs of vertices in the graph. We will also add self-loops for every vertex so that our network assumptions are satisfied. Let \( W_n \) be the associated combination matrix with uniform weights. If \( p = p_n \) satisfies \( \frac{np_n}{\log n} \to \infty \), then with probability tending to 1 as \( n \to \infty \), \( \lambda(W_n) \leq C_{\text{ER}} \sqrt{\frac{\log n}{(n-1)p_n}} \) where \( C_{\text{ER}} \) is a universal constant (see Proposition E.4 in Appendix E for details). Then for sufficiently large \( n_0 \), \( \{(G_n, W_n)\}_{n \geq n_0} \) is a rapidly mixing family for \( G_n \) chosen in this manner.

The term “rapidly mixing” is inspired by the following observation. Fix an action \( \alpha \in \mathcal{A} \) and define the \( n \)-dimensional vector \( Y^{n, \alpha}(t) := \{Y_i^{n, \alpha}(t)\}_{i=1}^n \), where we recall that \( Y_i^{n, \alpha}(t) = 1 \) if agent \( i \) plays \( \alpha \) at time \( t \), else \( Y_i^{n, \alpha}(t) = 0 \). Similarly define \( Y^{n, \alpha}(t) \), and
note that $\bar{Y}_{n,\alpha}^{\alpha}(t) = W_n Y_{n,\alpha}^{\alpha}(t)$ in light of (6). We can decompose $Y_{n,\alpha}^{\alpha}(t)$ as

$$Y_{n,\alpha}^{\alpha}(t) = \frac{1}{n} \langle 1, Y_{n,\alpha}^{\alpha}(t) \rangle + z = Y_{av}^{\alpha}(t) 1 + z,$$

where $\langle 1, z \rangle = 0$. Since $W_n$ is doubly stochastic,

$$\bar{Y}_{n,\alpha}^{\alpha}(t) = W_n Y_{n,\alpha}^{\alpha}(t) = Y_{av}^{\alpha}(t) W_n 1 + W_n z = Y_{av}^{\alpha}(t) 1 + W_n z.$$

If $\lambda(W_n) \approx 0$, then

$$\left\| \bar{Y}_{n,\alpha}^{\alpha}(t) - Y_{av}^{\alpha}(t) 1 \right\|_2 = \| W_n z \|_2 \leq \lambda(W_n) \| z \|_2 \approx 0,$$

which in turn implies that $\bar{Y}_{n,\alpha}^{\alpha}(t) \approx Y_{av}^{\alpha}(t) 1$. In words, agents’ local estimates are, on average, close to the global population state. We can therefore treat the evolution of the population state in the distributed information case as a perturbed version of the classical dynamics, where the magnitude of the perturbations can be bounded by $\lambda(W_n)$, a quantity that tends to zero in the limit of large networks. This leads us to the following result.

**Theorem 3.6:** Let $\hat{Y}_{av}^{N}(t)$ denote the continuous-time version of $Y_{av}^{N}(t)$ that is obtained by linear interpolation between consecutive time-steps in $T^N$. Let $\{(G_n, W_n)\}_n$ be a rapidly mixing sequence, and let $x$ be the solution to the mean-field ODE (4). Fix any $\epsilon > 0$ and $T \geq 0$. For $N$ sufficiently large but depending on $T, \epsilon$ and the rapidly mixing sequence itself, if $Y_{av}^{N}(t)$ is the population process evolving on the graph $G_N$ with combination matrix $W_N$,

$$\mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| \hat{Y}_{av}^{N}(t) - x(t) \right\|_\infty > \epsilon \right) \leq 2me^{-c(T)N\epsilon^2}. \quad (8)$$

The proof can be found in Appendix A. We make a few important remarks about the theorem. The constant $c(T)$ which appears in the exponent of the probability bound in (8) depends only on $T$ and not in any way on the graph $G_N$. Furthermore, the constant $c(T)$ is the same constant that appears in the exponent of (5) in Theorem 2.2. To summarize, Theorems 2.2 and 3.6 yield identical concentration bounds.
Remark 3.7: Theorem 3.6 may be viewed as a robust version of Theorem 2.2 in the following sense. If the underlying topology \((G, W)\) is complete but is perturbed to \((G', W')\) (due to the failure or deletion of inter-agent links, for instance), the mean-field characterization of the population process still holds as long as \(N\) is sufficiently large and \(\lambda(W')\) is close to \(\lambda(W) = 0\).

Simulations. We illustrate our theoretical results with numerical simulations, using an example taken from population games. In this setting, there is a common payoff function \(U\) which maps the population state to a vector of rewards, each entry corresponding to a different strategy. For simplicity, we assume that there are two actions, and that the payoff has the form

\[ U(x_1, x_2) := (-2x_1 + x_2, x_1 - 2x_2) = (U_1(x_1, x_2), U_2(x_1, x_2)). \]

Above, \(x_1\) represents the fraction of the population currently playing strategy 1, and \(x_2\) represents the fraction of the population currently playing strategy 2. Based on the payoff function, there are numerous reasonable ways agents may update their actions. We assume the revision policy is a smoothed best response, given by

\[ \rho^{12}(x_1, x_2) := \frac{\exp(10 \cdot U_2(x_1, x_2))}{\exp(10 \cdot U_1(x_1, x_2)) + \exp(10 \cdot U_2(x_1, x_2))}, \]

\[ \rho^{21}(x_1, x_2) := \frac{\exp(10 \cdot U_1(x_1, x_2))}{\exp(10 \cdot U_1(x_1, x_2)) + \exp(10 \cdot U_2(x_1, x_2))}. \]

We simulate the evolution of \(Y_{av}^N(t)\) when the underlying topology \(G\) is an Erdős-Rényi graph. We set \(p = 0.1\) and \(N = 100, 250, 1000\); our results are displayed in Figure 1. As predicted by Theorem 3.6, the population process enjoys better concentration around the classical mean-field equation as the number of vertices increases.

4. MEAN-FIELD ANALYSIS FOR GENERAL GRAPHS

When \(G\) is an arbitrary graph, the previous techniques for showing concentration around the mean-field ODE no longer apply since agents’ local estimates are not necessarily perturbations of the true population state. Interestingly, we find that the natural mean-field
Figure 1.—Realizations of $Y_{av}^N(t)$ for $N = 100, 250, 1000$ when $G \sim G(N, p = 0.1)$. As the number of vertices increases, $Y_{av}^N(t)$ enjoys a tighter concentration around the mean-field ODE.

approximation in this more general setting could be different from the classical mean-field ODE obtained from the perfect-information case; we derive this new mean-field equation in Section 4.1. We next show in Section 4.2 that if $G$ is sufficiently dense (technically, we assume that the normalized Frobenius norm of $W$ is sufficiently small), the population process $Y_{av}^N(t)$ concentrates around the new mean-field approximation we derive. While the techniques use the same basic arguments as Theorems 2.2 and 3.6, the proof is much more involved due to the non-Markov nature of the process $Y_{av}^N(t)$. Finally, in Section 4.3, we study properties of the mean-field equation, and identify conditions under which it reduces to the classical mean-field ODE.

4.1. Deriving the deterministic process

Assume that all random objects are defined on a common measurable space equipped with a filtration $\{\mathcal{F}_t\}_t$, where $\mathcal{F}_t$ consists of all measurable events formed by the random
variables used in the evolution of the stochastic process until and including time \( t \). Recall that to derive the deterministic approximation for the classical dynamics, we computed the expected change from time \( t \) to \( t + \frac{1}{N} \) conditioned on \( \mathcal{F}_t \). For the distributed setting, agent interactions are non-homogenous, so we first compute the expected change for an individual, and then average to obtain the expected change for the population process.

At each time \( t \in \mathbb{T}^N \), exactly one agent is chosen, uniformly at random, to update their action. Formally, we assign to each agent \( i \in [N] \) a sequence of random variables \( \{\text{Clock}_i^N(t)\}_{t \in \mathbb{T}^N} \), where \( \text{Clock}_i^N(t) = 1 \) if and only if agent \( i \) is chosen to update their action at time \( t \). Furthermore, \( \text{Clock}_i^N(t) \) is \( \mathcal{F}_t \)-measurable, and is independent of \( \mathcal{F}_{t-1/N} \). Since the updating agent is chosen independently at random, we have \( \text{Clock}_i^N(t) \sim \text{Bernoulli}(1/N) \) (independently of all other random objects) and \( \sum_{i=1}^N \text{Clock}_i^N(t) = 1 \) for every \( t \).

Fix \( i \in [N] \), and suppose that agent \( i \) is chosen to update their action at time \( t + 1/N \) (in terms of our notation, \( \text{Clock}_i^N(t + 1/N) = 1 \)). The type of update that agent \( i \) performs (policy-based or neighborhood exploration) is dictated by a sequence of random variables \( \{u_i^N(t)\}_{t \in \mathbb{T}^N} \), where \( u_i^N(t) \) is \( \mathcal{F}_{t+1/N} \)-measurable and is independent of \( \mathcal{F}_t \); that is, conditioned on the state of the system at time \( t \), \( u_i^N(t) \) is a random variable whose value leads to the state of the system at time \( t + 1/N \). If \( u_i^N(t) = 1 \), then agent \( i \) performs a policy-based update; otherwise they perform a neighborhood exploration step. We also have \( u_i^N(t) \sim \text{Bernoulli}(r(t)) \), where \( r(t) \) is a function that satisfies the following assumption:

\textbf{Assumption 4.1:} The function \( r(t) \) satisfies \( r : \mathbb{R}_{\geq 0} \rightarrow (0, 1] \), \( r \) is \( L_r \)-Lipschitz and \( \int_0^\infty r(t)dt = \infty \).

Above, \( \mathbb{R}_{\geq 0} \) denotes the set of non-negative real numbers. On a high level, \( r(t) \) can be thought of as the rate at which agents make policy-based decisions; in classical settings, \( r(t) = 1 \). The assumption that the integral of \( r(t) \) diverges essentially ensures that neigh-

---

6Recall that \([N] := \{1, \ldots, N\}\).

7We name the variable “Clock” to emphasize its similarity with with the Poisson clock model, which is often used in continuous-time processes involving interacting systems of agents (see, e.g., Sandholm (2010)).
borhood exploration does not significantly change the population dynamics - we formalize this idea in Section 4.1.1. We will next formalize the transitions of the stochastic process. From the description in Section 2.2, we can summarize the stochastic evolution as follows:

- Initially, at time $t = 0$, each agent chooses an action from the set $A$. For each $i \in [N]$, $Y_{iN,\alpha}(0) = 1$ if agent $i$ initially plays $\alpha$ and $Y_{iN,\alpha}(0) = 0$ otherwise.
- Given the collection $\{Y_{iN}(t)\}_{i \in [N]}$, construct the state of the system at time $t + 1/N$ as follows:
  - Choose an agent $i$ uniformly at random, independently of all other randomness, to revise their action (formally, $Clock_i(t + 1/N) = 1$).
  - Sample $u_i(t) \sim$ Bernoulli($r(t)$).
  - If $u_i(t) = 1$ (policy-based update) and agent $i$ plays action $\alpha$, they switch to playing action $\beta$ at time $t + 1/N$ with probability $\rho_{\alpha\beta}(Y_{iN}(t))$. The behavior of all other agents remains unchanged.
  - If $u_i(t) = 0$ (neighborhood exploration) then $Y_{iN}(t + 1/N) = Y_{jN}(t)$ with probability $w_{ij}$ – that is, agent $i$ imitates the action of a neighbor, agent $j$. The behavior of all other agents remains unchanged.

We now formalize the description provided above. Assume that $u_i(t) = 1$ so that agent $i$ performs a policy-based update. Define the $\mathcal{F}_{t + 1/N}$-measurable indicator variable $R_i^{\alpha\beta}(t)$, where

$$R_i^{\alpha\beta}(t) := \begin{cases} 1 & \text{agent } i \text{ switches from } \alpha \text{ to } \beta \text{ at time } t + 1/N \\ 0 & \text{else.} \end{cases}$$

Conditioned on $\mathcal{F}_t$, $R_i^{\alpha\beta}(t) \sim$ Bernoulli($\rho_{\alpha\beta}(Y_{iN}(t))$). We can then write, for every $\alpha \in A$,
Else, if $Clock^N_i (t + 1/N) = 1$ and $u^N_i (t) = 0$, agent $i$ instead imitates a randomly selected neighbor. Define the $\mathcal{F}_{t+1/N}$-measurable indicator variable $\mathcal{W}_{ij}(t)$, where

$$\mathcal{W}_{ij}(t) := \begin{cases} 1 & \text{agent } i \text{ imitates agent } j \text{ at time } t + 1/N \\ 0 & \text{else.} \end{cases}$$

For every $t$ and $i$, there is a unique $j$ such that $\mathcal{W}_{ij}(t) = 1$. Furthermore, conditioned on $\mathcal{F}_t$, $\mathcal{W}_{ij}(t) \sim \text{Bernoulli}(w_{ij})$. We can then write, for every $\alpha \in A$,

$$Y^N,\alpha_i \left( t + \frac{1}{N} \right) - Y^N,\alpha_i (t) = \sum_{j=1}^{N} \mathcal{W}_{ij}(t) Y^N,\alpha_j (t) - Y^N,\alpha_i (t). \quad (10)$$

Combining (9) and (10), we can write, for every $\alpha \in A$,

$$Y^N,\alpha_i \left( t + \frac{1}{N} \right) - Y^N,\alpha_i (t) = Clock^N_i \left( t + \frac{1}{N} \right) \left[ \left( 1 - u^N_i (t) \right) \sum_{j=1}^{N} \mathcal{W}_{ij}(t) Y^N,\alpha_j (t) - Y^N,\alpha_i (t) \right]$$

$$+ u^N_i (t) \left[ \sum_{\beta \in A : \beta \neq \alpha} Y^N,\beta_i (t) R^\beta_{\alpha i} (Y^N_i (t)) - \sum_{\beta \in A : \beta \neq \alpha} Y^N,\alpha_i (t) R^\beta_{\alpha i} (Y^N_i (t)) \right]. \quad (11)$$

Take an expectation with respect to $\mathcal{F}_t$ to obtain

$$\mathbb{E} \left[ Y^N,\alpha_i \left( t + \frac{1}{N} \right) - Y^N,\alpha_i (t) \mid \mathcal{F}_t \right] = \frac{1 - r(t)}{N} \left( Y^N,\alpha_i (t) - Y^N,\alpha_i (t) \right)$$

$$+ \frac{r(t)}{N} \left( \sum_{\beta \in A : \beta \neq \alpha} Y^N,\beta_i (t) \rho^\beta (Y^N_i (t)) - \sum_{\beta \in A : \beta \neq \alpha} Y^N,\alpha_i (t) \rho^\beta (Y^N_i (t)) \right) \quad (12)$$

This suggests a deterministic approximation of

$$y^N,\alpha_i \left( t + \frac{1}{N} \right) - y^N,\alpha_i (t) = \frac{1 - r(t)}{N} \left( y^N,\alpha_i (t) - y^N,\alpha_i (t) \right)$$

$$+ \frac{r(t)}{N} \left( \sum_{\beta \in A : \beta \neq \alpha} y^N,\beta_i (t) \rho^\beta (y^N_i (t)) - \sum_{\beta \in A : \beta \neq \alpha} y^N,\alpha_i (t) \rho^\beta (y^N_i (t)) \right)$$
where $y_i^N(0) := Y_i^N(0)$. We remark that (13) has a useful consensus-preserving property: if $y_i^N(0) = y_j^N(0)$ for all $i, j \in [N]$, then we have $y_i^N(t) = y_j^N(t)$ for all $i, j \in [N]$ and all $t \in \mathbb{T}^N$; this can be easily seen by induction.

Averaging over $i \in [N]$ in (13) and noting that $\frac{1}{N} \sum_{i=1}^{N} y_i^N(t) = y_{av}^N(t)$ due to $W$ being doubly stochastic suggests the following deterministic approximation for $Y_{av}^N(t)$:

$$y_{av}^N(t + \frac{1}{N}) - y_{av}^N(t) = \frac{r(t)}{N^2} \sum_{i=1}^{N} \left( \sum_{\beta \in \mathcal{A}, \beta \neq \alpha} y_i^{N,\beta}(t) \rho^{\beta\alpha}(y_i^N(t)) - \sum_{\beta \in \mathcal{A}, \beta \neq \alpha} y_i^{N,\alpha}(t) \rho^{\alpha\beta}(y_i^N(t)) \right).$$

(14)

4.1.1. Consensus and ODE approximation

In an ideal case of (14), suppose that for all $i$, $\bar{y}_i^N(t) = y_{av}^N(t)$ (this is the case, for instance, in the model of Section 2.1). Then (14) becomes

$$y_{av}^{N,\alpha} \left( t + \frac{1}{N} \right) - y_{av}^{N,\alpha}(t) = \frac{r(t)}{N^2} \sum_{i=1}^{N} \left( \sum_{\beta \in \mathcal{A}, \beta \neq \alpha} y_i^{N,\beta}(t) \rho^{\beta\alpha}(y_{av}^N(t)) - y_i^{N,\alpha}(t) \rho^{\alpha\beta}(y_{av}^N(t)) \right)$$

$$= \frac{r(t)}{N} \sum_{\beta \in \mathcal{A}, \beta \neq \alpha} \left( y_{av}^{N,\beta}(t) \rho^{\beta\alpha}(y_{av}^N(t)) - y_{av}^{N,\alpha}(t) \rho^{\alpha\beta}(y_{av}^N(t)) \right)$$

$$= \frac{r(t)}{N} \Lambda^\alpha(y_{av}^N(t)).$$

(15)

where $\Lambda$ is the vector field for the (classical) mean-field dynamics of Section 2.1. We see that $y_{av}^{N,\alpha}(t)$ is exactly the Euler Approximation to the system of ODEs

$$\begin{cases}
\dot{x}^\alpha(t) = r(t) \Lambda^\alpha(x(t)) & \forall \alpha \in \mathcal{A} \\
x(0) = Y_{av}^N(0)
\end{cases}$$

(16)
It turns out that the solution of (16) is simply a time-reparametrization of the solution of (4). To see this, let \( R(t) := \int_0^t r(s)ds \) and note that, in light of Assumption 4.1, \( R : [0, \infty) \rightarrow [0, \infty) \), \( R(t) \) is strictly increasing and bijective. Let \( z \) be the solution to (4) and define \( x(t) := z(R(t)) \). Then for any \( \alpha \in \mathcal{A} \),

\[
\dot{x}^{\alpha}(t) = \dot{R}(t) \dot{z}^{\alpha}(R(t)) = r(t)\Lambda^\alpha(z(R(t))) = r(t)\Lambda^\alpha(x(t))
\]

so \( x(t) \) solves (16). Furthermore, since \( r(t) \) is bounded and \( \Lambda \) is Lipschitz, it is the unique solution to (16). Since we also have \( R(t) \rightarrow \infty \) as \( t \rightarrow \infty \), then the time-asymptotic behavior of the solutions to (4) and (16) are the same.

Based on these observations, we will show that \( Y_{av}^N(t) \) is well-concentrated around a solution to (16) in two main steps. First, we show that \( Y_{av}^N(t) \) is well-concentrated around the deterministic process \( y_{av}^N(t) \). We then study the process \( y_{av}^N(t) \) and provide general conditions under which the \( y_{av}^N(t) \)'s are close to \( y_{av}^N(t) \) (possibly asymptotically in time), which enables us to formalize the heuristic arguments outlined above. Combining the two results shows that \( Y_{av}^N(t) \) is concentrated around a solution of (16).

### 4.2. Concentration around a deterministic process

We now establish a version of Theorems 2.2 and 3.6 that holds for any sufficiently dense graph. As discussed in the results below, the population process \( Y_{av}^N(t) \) will be shown to concentrate around the deterministic process \( y_{av}^N(t) \) rather than the classical mean-field ODE. We obtain large-deviations type probability estimates, where the focus is to obtain an exponentially-decaying bound with respect to \( N \). The proof of the theorem can be found in Appendix B.

**Theorem 4.2:** Let \((G, W)\) be a graph and combination matrix that satisfy Assumptions 2.3 and 2.4. Fix \( \epsilon > 0 \) and a time horizon \( T \). For \( L = L(\rho, m) \), further suppose that \( N \geq \frac{4e^{LT}}{\epsilon}, \frac{1}{N} \|W\|_F^2 \leq \frac{\epsilon^2}{8Te^{LT}} \) and that

\[
\rho_{max} := \max_{\alpha, \beta \in \mathcal{A}} \max_{x \in \Delta_m} \rho^{\alpha \beta}(x) < 1.
\]

(17)
Let $\hat{Y}^N_{av}(t)$ and $\hat{y}^N_{av}(t)$ be the continuous-time versions of $Y^N_{av}(t)$ and $y^N_{av}(t)$, respectively, that are obtained by linear interpolation between consecutive time-steps in $\mathbb{T}^N$. Then we have

$$\mathbb{P}\left(\max_{0 \leq t \leq T} \left| \hat{Y}^N_{av}(t) - \hat{y}^N_{av}(t) \right|_\infty > \epsilon \right) \leq c_1 e^{-c_2 N \epsilon^2},$$

where $c_1 = c_1(m, T, \epsilon, \rho)$ and $c_2 = c_2(m, T, \rho)$.

We make a few remarks about the assumptions we make in the theorem. Equation (17) essentially states that when any agent updates their action, they do not do so deterministically; there are always at least two actions $\alpha, \beta$ where the probability of switching to either $\alpha$ or to $\beta$ is positive. This is the case, for instance, if agents are stubborn, and keep their current action with positive probability. On a more technical note, (17) leads to useful contractive inequalities in the proof of the theorem (for details, see the proof of Lemma B.1 in Section B.1) which are needed if the $Y^N_i(t)$’s are not close to a consensus. It can also be seen that the constant $c_1$ blows up as $\rho_{\text{max}}$ tends to 1. The constant $c_1$ blows up as $\epsilon \to 0$ as well, increasing at a rate of $\exp((\log 1/\epsilon)^2)$. The quantity $\frac{1}{N} \|W\|_F^2$, on a high level, measures the edge-density of the weighted network; we will illustrate this qualitatively with several examples.

**Example 4.3**—Regular graphs: Let $G$ be a regular $d$-regular graph that has self-loops for every vertex. Define the combination matrix $W$ via

$$W_{ij} := \begin{cases} \frac{1}{d} & (i, j) \in E \\ 0 & \text{else}. \end{cases}$$

Then $W$ is doubly-stochastic, and $\frac{1}{N} \|W\|_F^2 = \frac{1}{d}$. Theorem 4.2 applies to $(G, W)$ as long as

$$d \geq \frac{8T e^{2LT}}{\epsilon^2}.$$

**Example 4.4**—Graphs with small degree variations: Let $d_i$ be the degree of vertex $i$ and fix $\delta > 0$. Let $\bar{d} := \frac{1}{N} \sum_{j=1}^N d_j$ be the average degree and assume that for every
$i \in [N]$, $(1 - \delta)\bar{d} \leq d_i \leq (1 + \delta)\bar{d}$. This condition holds with high probability, for instance, in homogenous random graphs such as Erdős-Rényi random graphs and Random Geometric graphs with average degree growing faster than $\log n$.$^8$ We can construct a symmetric combination matrix $W$ according to the equal weights design with $a = \frac{1}{d_{\text{max}} + 1}$:

$$W_{ij} := \begin{cases} 1 - \frac{d_i}{d_{\text{max}} + 1} & i = j \\ \frac{1}{d_{\text{max}} + 1} & (i, j) \in E, i \neq j \\ 0 & \text{else.} \end{cases}$$

In particular, we have the bounds

$$W_{ij} \leq \begin{cases} \frac{2\delta}{1 - \delta} + \frac{1}{(1 - \delta)d} & i = j \\ \frac{1}{(1 - \delta)d} & (i, j) \in E, i \neq j \\ 0 & \text{else.} \end{cases}$$

Since there are $\bar{d}N$ edges (not counting self-loops), we have the bound

$$\frac{1}{N} \|W\|_F^2 \leq \left(\frac{2\delta}{1 - \delta} + \frac{1}{(1 - \delta)d}\right)^2 + \bar{d} \left(\frac{1}{(1 - \delta)d}\right)^2 \leq \frac{8\delta^2}{(1 - \delta)^2} + \frac{2}{(1 - \delta)^2\bar{d}^2} + \frac{1}{(1 - \delta)^2\bar{d}}.$$ 

Thus Theorem 4.2 applies to $(G, W)$ as long as $\delta$ is a sufficiently small function of $\epsilon, T$ and $\bar{d}$ is a sufficiently large function of $\epsilon, T$.

**Example 4.5**—Supernode graphs: A multipartite or supernode graph on $N$ vertices is specified by a smaller graph $\tilde{G}$ on $M \leq N$ vertices, along with the combination matrix $\tilde{W} \in \mathbb{R}^{M \times M}$ associated with $\tilde{G}$. For convenience assume that $N$ is a multiple of $M$. To construct the graph, partition

$$[N] = S_1 \cup \ldots \cup S_M,$$

$^8$This is readily obtained by Bernstein’s inequality - see Proposition E.4 in Appendix E for a proof for Erdős-Rényi graphs.
where \(|S_i| = \frac{N}{M}\) for all \(i \in [M]\). Each vertex in \(S_i\) shares an edge with each vertex in \(S_j\) if and only if \((i, j)\) is an edge in \(\tilde{G}\). We construct the combination matrix for \(G\) as

\[
W_{uv} := \frac{\tilde{W}_{ij}}{|S_i|} = \frac{M}{N} \tilde{W}_{ij} \text{ if } u \in S_i, v \in S_j.
\]

In particular, \(W\) is doubly-stochastic as long as \(\tilde{W}\) is doubly-stochastic. Then

\[
\frac{1}{N} \|W\|_F^2 = \frac{1}{N} \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{u \in S_i, v \in S_j} \frac{M^2}{N^2} (\tilde{W}_{ij})^2 = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (\tilde{W}_{ij})^2 = \frac{1}{N} \|\tilde{W}\|_F^2.
\]

It follows that Theorem 4.2 applies to \((G, W)\) as long as

\[
N \geq \frac{8Te^{2LT}}{\epsilon^2} \|\tilde{W}\|_F^2.
\]

4.3. **Consensus analysis of the deterministic process**

While Theorem 4.2 significantly generalizes Theorems 2.2 and 3.6, the drawback is that \(y^N_{av}(t)\) is quite complex. If we can show that the \(\tilde{y}_i^N(t)\)’s stay close to a consensus, possibly asymptotically, the heuristic arguments at the end of Section 4.1 imply that the evolution of \(y^N_{av}(t)\) closely follows a solution to the ODE defined in (16). The following theorem provides general conditions under which we can expect the \(\tilde{y}_i^N(t)\)’s to be close to a consensus; the proof can be found in Appendix C.

**Theorem 4.6—Convergence to consensus:** Suppose that the pair \((G, W)\) satisfies Assumptions 2.3 and 2.4. Let \(\lambda := \lambda(W)\) and \(R(T) := \int_0^T r(s)ds\). There is a constant \(c_3 = c_3(m, \rho)\) such that for every \(T \in \mathbb{T}^N\),

\[
\sum_{\alpha \in A} \left( \sum_{i=1}^{N} |\tilde{y}_i^{N,\alpha}(T) - y^N_{av}(T)|^2 \right)^{1/2} \leq e^{-(1-\lambda)T+c_3R(T)} \sum_{\alpha \in A} \left( \sum_{i=1}^{N} |\tilde{y}_i^{N,\alpha}(0) - y^N_{av}(0)|^2 \right)^{1/2}
\]

\[
\leq e^{-(1-\lambda)T+c_3R(T)} m \sqrt{N}.
\]

We make a few important remarks about the theorem. Since \(W\) is doubly stochastic, we have \(\lambda(W) < 1\). Hence we can always ensure that the exponent \(-(1 - \lambda)T + c_3R(T)\) is
negative by making the right choice of $r(t)$. In particular, we need $(1 - \lambda)t - c_3 R(t) \to \infty$ as $t \to \infty$ if we want the $\bar{y}_i^N(t)$’s to tend to a consensus. In other words, the policy-based updates should not overpower neighborhood exploration if we want to guarantee convergence to consensus under general conditions.

We also briefly remark on the two upper bounds in the theorem statement. The first upper bound depends on the initial configuration, and the second follows from bounding the first upper bound in a crude manner. The advantage of the second bound is that we can characterize consensus of the $\bar{y}_i^N(t)$’s independently of the initial condition. However, if we know that the process starts close to a consensus (which is possible, for instance, if the initial conditions are chosen uniformly at random for each agent) then the first bound will be much more useful.

### 4.4. Approximation by the classical mean-field ODE

In this section we apply the consensus analysis of Theorem 4.6 to argue that $y^N_{av}(t)$ approximates a solution to the ODE (16) in certain cases. Once this is established, we can control the deviation of $\bar{y}_{av}^N(t)$ from a solution of (16) by applying Theorem 4.2. The proofs can be found in Appendix D.

Our first result shows that if agents’ local estimates are initially close to a consensus, the population state concentrates around the solution of (16) with the same initial conditions.

**Corollary 4.7:** Fix $\epsilon > 0$ and a time horizon $T \geq 0$. In addition to the conditions of Theorem 4.2, assume that

$$\sum_{\alpha \in \mathcal{A}} \left[ \frac{1}{N} \sum_{i=1}^{N} \left| Y_i^{N,\alpha}(0) - Y_{av}^N(0) \right|^2 \right] \leq \frac{\epsilon}{8mL \rho Te^{(L_\Lambda + c_3)T}}.$$  

Let $x(t)$ solve the ODE

$$\begin{align*}
\dot{x}^\alpha(t) &= r(t) \Lambda^\alpha(x(t)) & \alpha \in \mathcal{A}; \\
 x(0) &= Y_{av}^N(0).
\end{align*}$$

Then if $L = L(\rho, m)$ is the same constant from Theorem 4.2 and
\[
N \geq \max \left\{ \frac{4(L_r + L_\Lambda)T e^{L_rT}}{\epsilon}, \frac{8e^{LT}}{\epsilon} \right\}, \quad \text{and} \quad \frac{1}{N} \|W\|_F^2 \leq \frac{\epsilon^2}{32T e^{2LT}},
\]
then
\[
\mathbb{P} \left( \max_{0 \leq t \leq T} \left\| \hat{Y}_{av}^N(t) - x(t) \right\|_\infty > \epsilon \right) \leq c_1 e^{-\frac{1}{2} c_2 N \epsilon^2}, \quad (19)
\]
where the constants $c_1, c_2$ are the same as in Theorem 4.2.

Corollary 4.7 leverages the consensus-preserving property of the deterministic dynamics: if agents’ local estimates are initially at a consensus, then the deterministic process stays at a consensus. The condition (18) ensures that the initial conditions are close to a consensus, hence the deterministic process stays close to a consensus. From the arguments of Section 4.1.1, this in turn implies that the deterministic dynamics behave like the mean-field ODE. We also note that when $N$ is sufficiently large, the condition (18) is easily satisfied, for example, if each agent chooses their initial action independently from some probability distribution over $\mathcal{A}$.

However there are many scenarios of practical interest where (18) is not applicable. Using Theorem 4.6, we show that with the right choice of $r(t)$ we can achieve consensus asymptotically. Formally, we show there is a time index $T_c \geq 0$ (called the consensus time) such that $y_{av}^N(T_c + t)$ approximates a solution to the ODE (16) for $t \geq 0$. Hence we can expect to approximate $Y_{av}^N(T_c + t)$ by the ODE for $t \geq 0$; see the following result for details.

**Corollary 4.8:** Fix $\epsilon > 0$ and a time horizon $T \geq 0$. Assume that there exists $t_0$ such that for $t \geq t_0$,
\[
(1 - \lambda)t - c_3 \mathcal{R}(t) \geq \left( \frac{1 - \lambda}{2} \right) t. \quad (20)
\]
For a given positive number $s$, let $x_s(t)$ solve the ODE
\[
\begin{cases}
\dot{x}_s(t) = r(t) \Lambda^\alpha(x_s(t)) & \alpha \in \mathcal{A} \\
x_s(s) = Y_{av}^N(s).
\end{cases}
\]
There exists $T_c = T_c(m, T, \epsilon, \rho, \lambda, t_0)$ such that, if

$$N \geq \max \left\{ \frac{4(L_r + L_A)T e^{L \Lambda T}}{\epsilon}, \frac{8e^{L(T_c + T)}}{\epsilon} \right\}$$

and

$$\frac{1}{N} \|W\|_F^2 \leq \frac{\epsilon^2}{8(T_c + T)e^{2L(T_c + T)}},$$

then we have the bound

$$\mathbb{P}\left( \max_{0 \leq t \leq T} \left\| \hat{Y}_{av}^N(T_c + t) - x_{T_c}(T_c + t) \right\|_{\infty} > \epsilon \right) \leq c_1(T_c)e^{-\frac{1}{4}c_2(T_c)N\epsilon^2}.$$  

Above, we have $c_1(T_c) := c_1(m, T_c + T, \epsilon, \rho)$ and $c_2(T_c) := c_2(m, T_c + T, \rho)$, where the constants $c_1$ and $c_2$ are the constants from Theorem 4.2.

The condition (20) can be satisfied if $r(t) < \frac{1-\lambda}{c_3}$ or if $r(t) \to 0$ as $t \to \infty$. A subtle point in Corollary 4.8 is that (21) requires $N$ to be greater than a function of $T_c$, but $T_c$ in turn depends on $\lambda(W)$. We have an analogous situation relating $\frac{1}{N} \|W\|_F^2$ and $T_c$ due to the condition (22). Hence it may be problematic if we consider families of graphs where, as the number of vertices increases, $\lambda(W)$ tends to 1. Through the following examples, we illustrate that in many cases of interest, $\lambda(W)$ is bounded away from 1 even as the number of vertices tends to infinity, showing that the conditions (21) and (22) are satisfiable.

**Example 4.9**—Supernode graphs: Let $G$ be a supernode graph specified by a smaller graph $\tilde{G}$ on $M$ vertices, where $M$ is fixed with respect to $N$. By Proposition E.2 in Appendix E, $\lambda(W) = \lambda(\tilde{W})$, which is constant with respect to $N$. Hence (21) and (22) are satisfied for $N$ sufficiently large.

**Example 4.10**—Nearest-neighbor graphs on the unit circle: A nearest-neighbor graph $G_N$ is specified by the number of vertices $N$ as well as an odd positive integer $k \leq N$. Vertices are placed at equidistant positions on the unit circle and each vertex shares an edge with the $k$ closest vertices (including the vertex itself so that self-loops are added). The combination matrix $W_n$ is constructed with uniform weights. If $k/N \to \gamma \in (0, 1)$ as
\( N \to \infty \) then \( \lim_{N \to \infty} \lambda(W_N) \in (0, 1) \); see Proposition E.3 in Appendix E for details. In particular, since \( \lambda(W_N) \) is bounded away from 1 for \( N \) large, (21) and (22) hold for sufficiently large \( N \).

4.5. Simulations

We consider the same agent dynamics presented at the end of Section 3. The underlying graph \( G_n \) is set to be a nearest-neighbor graph, with uniform weights for \( W_n \). We will illustrate our results for various initial conditions and values of \( \gamma \).

The deterministic approximation and concentration. In Section 4.1, we derived a different deterministic approximation \( y_{av}^N(t) \) whose structure is tied to the graph topology. To illustrate this, we simulate \( y_{av}^N(t) \) for \( N = 1000 \) and \( \gamma = 0.1, 0.5, 0.9 \). We set the initial condition as follows. The agents labelled 1, 2, \ldots, 0.4 \* \( N \) start with action 2, and the rest start with action 1. The initial conditions for \( y_{av}^N(t) \) are therefore \((0.6, 0.4)\). We plot these curves and also display for comparison the mean-field ODE in Figure 2a. In Figure 2b, we set \( \gamma = 0.8 \) and plot \( Y_{av}^N(t), y_{av}^N(t) \) as well as the mean-field ODE, where the initial conditions of \( Y_{av}^N(t) \) and \( y_{av}^N(t) \) are the same. We see that \( Y_{av}^N(t) \) indeed concentrates around \( y_{av}^N(t) \) rather than the mean-field approximation, confirming the predictions of Theorem 4.2.

Random initial conditions. We consider the case of random initial conditions: initially, each agent independently starts with action 1 with probability 0.6, and action 2 otherwise. Using Hoeffding or Bernstein inequalities, it is straightforward to see that (18) is satisfied with high probability as the number of agents tends to infinity. Corollary 4.7 then implies that the population process \( Y_{av}^N(t) \) instead concentrates around the mean-field ODE. This is illustrated in Figure 3.

Dynamics with neighborhood imitation. Corollary 4.8 shows that when there are arbitrary initial conditions, we can set \( r(t) \) to be a function decaying to 0 as \( t \to \infty \); this implies that the population process \( Y_{av}^N(t) \) concentrates around the modified mean-field ODE for \( t \) sufficiently large. We use the same structure of initial conditions as the first example and set \( r(t) = \frac{1}{t+1} \). As predicted by Corollary 4.8, we find that the population processes con-
Figure 2.—Figure 2a contains plots of $y^{N,2}_{av}(t)$ for $N = 1000$ and various $\gamma$. The initial condition is $y^{N}_{av}(0) = (0.6, 0.4)$. As $\gamma$ tends closer to 1, the curves become closer to the mean-field approximation. Figure 2b shows that the population process $Y^{N}_{av}(t)$ concentrates around the corresponding deterministic approximation; the same initial conditions are used.
Figure 3.—Realizations of the population process $Y^N_{av}(t)$ for $N = 500, 1000, 1500$ with random initial conditions. As $N$ grows larger, we see that $Y^N_{av}(t)$ enjoys tighter concentration around the mean-field ODE.

Figure 4.—Plots of the population process $Y^N_{av}(t)$ for various $\gamma$ when $r(t) = \frac{1}{1+t}$. The solutions to the mean-field ODE around which the stochastic processes are concentrated are shown in dotted lines. The concentration around the dotted curves does hold after a sufficient length of time that is longer as $\gamma$ decreases.

centrate around solutions of the mean-field ODE after a sufficient length of time; see Figure 4.
5. CONCLUSION AND FUTURE DIRECTIONS

In this paper, we study a general class of population processes where agents have partial information, rather than perfect information about the state of all agents. Specifically, we assume that agent information is given by an underlying graph $G$ on the set of agents, where the presence of an edge means that the two corresponding agents can observe each other’s actions. In a broad sense, we prove a law of large numbers result for population processes of this type. When $G$ is constructed so that agent information is nearly perfect, we find that the classical mean-field ODE obtained when agents have perfect information is the limiting process as the number of agents tends to infinity. Our results here are relatively generic, and captures several examples of interest including complete graphs with link failures, random regular graphs and Erdős-Rényi graphs. We next study the case of a general graph $G$. Interestingly, we show that if $G$ is sufficiently dense, the population process concentrates around a mean-field approximation which depends strongly on the graph topology. We study this new mean-field approximation in more detail, and establish general conditions for which it reduces to the classical mean-field ODE.

There are a number of important open questions worth pursuing. Our work establishes concentration of the population process over bounded time intervals; a natural next step is to characterize the time-asymptotic properties. Benaïm and Weibull (2003) study such questions in detail in the perfect information case, using the bounded interval concentration inequality, Theorem 2.2 as a starting point. Similar time-asymptotic properties of the general partial information based population processes considered in this work would be of interest, especially in application contexts such as population games in terms of characterizing equilibria and relevant limiting behavior. Another important avenue of future work is a better characterization of the deterministic process $y_{\text{av}}^N(t)$. We have shown that under certain conditions, it reduces to the classical mean-field ODE. In general, however, the evolution of $y_{\text{av}}^N(t)$ is quite complex but a simpler representation may exist for certain families of graphs. Yet another goal is to consider similar law of large numbers results for sparse graphs. Some recent work has explored this direction for sparse random graphs (see
Lacker, Ramanan, and Wu (2019), Oliveira, Reis, and Stolerman (2020)), and it would be interesting to see if our methods can be adapted to that setting as well.

**APPENDIX A: PROOF OF THEOREM 3.6**

We begin by defining some notation. Let $F_{av}^N(t)$ denote the expected rate of change in the population state, conditioned on current information, given explicitly by

$$F_{av}^{N,\alpha}(t) := \mathbb{E}\left[ N \left( Y_{av}^N \left( t + \frac{1}{N} \right) - Y_{av}^N(t) \right) | \mathcal{F}_t \right].$$

Although $F_{av}^N(t)$ is only well-defined for $t \in \mathbb{T}^N$, we will abuse notation and also write $F_{av}^N(t)$ to mean the right-continuous step process where $F_{av}^N(s) = F_{av}^N(t)$ for $t \leq s < t + \frac{1}{N}$ when $t \in \mathbb{T}^N$. We do the same abuse of notation for the process $Y_{av}^N(t)$. We also define $\hat{Y}_{av}^N(t)$ to be the continuous process where $\hat{Y}_{av}^N(t) = Y_{av}^N(t)$ for $t \in \mathbb{T}^N$ and $\hat{Y}_{av}^N(s)$ is linearly interpolated between $Y_{av}^N(t)$ and $Y_{av}^N \left( t + \frac{1}{N} \right)$ when $t \leq s \leq t + \frac{1}{N}$ and $t \in \mathbb{T}^N$. We use the same conventions for any other process defined over $\mathbb{T}^N$.

The following lemma gives a useful relation between $\hat{F}_{av}^N(t)$ and the classical mean-field ODE (4).

**Lemma A.1:** Let $\lambda_N := \lambda(W_N)$ and let $L_\Lambda$ be the Lipschitz constant of $\Lambda$ (defined in (3)). For all $t \geq 0$,

$$\left\| \hat{F}_{av}^N(t) - \Lambda \left( \hat{Y}_{av}^N(t) \right) \right\|_{\infty} \leq 2L_\rho m^2 \lambda_N + \frac{L_\Lambda}{N}.$$

On a high level, this lemma will allow us to show that in expectation, $Y_{av}^N(t)$ is well-approximated by the mean-field ODE as $\lambda_N \to 0$. It will also be important to study the behavior of the empirical jumps, given for $t \in \mathbb{T}^N$ by

$$U_{av}^N(t) := N \left( Y_{av}^N \left( t + \frac{1}{N} \right) - Y_{av}^N(t) \right) - F_{av}^N(t).$$

In particular, $U_{av}^N(t)$ is $\mathcal{F}_{t+\frac{1}{N}}$-measurable and $\mathbb{E} \left[ U_{av}^N(t) | \mathcal{F}_t \right] = 0$. Furthermore, since $\left\| Y_{av}^N \left( t + \frac{1}{N} \right) - Y_{av}^N(t) \right\|_{\infty} \leq 1/N$, $U_{av}^N(t)$ is almost surely bounded. As an abuse of notation, we will also write $U_{av}^N(t)$ to mean the right-continuous step process where $U_{av}^N(s) = U_{av}^N(t)$ for $t \leq s < t + \frac{1}{N}$. 
As a fundamental step to proving Theorem 2.2, Benaïm and Weibull proved the following concentration inequality for $U_{av}^N(t)$. The proof is relatively generic, and uses only the fact that $\{U_{av}^N(t)\}_{t \in T^N}$ is a martingale difference sequence with bounded increments. We omit the proof here - for details, see Lemma 3 and the proof of Lemma 1 in Benaïm and Weibull (2003).

**Lemma A.2:** For any $\epsilon > 0$,  
\[
\mathbb{P} \left( \max_{0 \leq t \leq T} \left\| \int_0^t U_{av}^N(s) ds \right\|_\infty \geq \epsilon \right) \leq 2m \cdot \exp \left( -\frac{c(\Lambda)N\epsilon^2}{T} \right),
\]
where $c(\Lambda)$ is a constant depending only on $\Lambda$ through $L_\Lambda$ and $\max_{x \in \Delta_m} \|\Lambda(x)\|_2$.

We are now in a position to prove the theorem. We remark that the structure of the proof closely follows the proof of Lemma 1 in Benaïm and Weibull (2003).

**Proof of Theorem 3.6.** For any $t \geq 0$, we can write
\[
\dot{Y}_{av}^N(t) - \dot{Y}_{av}^N(0) = \int_0^t \left( F_{av}^N(s) + U_{av}^N(s) \right) ds
\]

\[
= \int_0^t \Lambda \left( \dot{Y}_{av}^N(s) \right) ds + \int_0^t \left( F_{av}^N(s) - \Lambda \left( \dot{Y}_{av}^N(s) \right) \right) ds + \int_0^t U_{av}^N(s) ds
\]

Similarly, we can write
\[
x(t) - x(0) = \int_0^t \Lambda(x(s)) ds.
\]

Noting that $x(0) = Y_{av}^N(0)$, we can subtract the two equations, take the norm of both sides and apply Lemma A.1 to show that for $0 \leq t \leq T$,
\[
\left\| Y_{av}^N(t) - x(t) \right\|_\infty \leq L_\Lambda \int_0^t \left\| \dot{Y}_{av}^N(s) - x(s) \right\|_\infty ds + \Psi(T),
\]

where
\[
\Psi(T) := 2L_\rho m^2 T \lambda_N + \frac{L_\Lambda T}{N} + \max_{0 \leq t \leq T} \left\| \int_0^t U_{av}^N(s) ds \right\|_\infty.
\]

Grönwall’s inequality then implies
\[
\max_{0 \leq t \leq T} \left\| Y_{av}^N(t) - x(t) \right\|_\infty \leq \Psi(T)e^{L_\Lambda T}.
\]
Since \((G_N, W_N)\) is an element of a rapidly mixing sequence, \(\lambda_N \to 0\) as \(N \to \infty\). Hence, for \(N\) sufficiently large (but depending on \(T, \epsilon\) and the rapidly-mixing sequence), Lemma A.2 implies

\[
\mathbb{P} \left( \max_{0 \leq t \leq T} \left\| \bar{Y}_{av}^N(t) - x(t) \right\|_\infty > \epsilon \right) \leq \mathbb{P} \left( \max_{0 \leq t \leq T} \left\| \int_0^t U_{av}^N(s) \, ds \right\|_\infty > \frac{\epsilon}{2eL\Lambda T} \right)
\]

\[
\leq 2m \cdot \exp \left( -\frac{c(\Lambda)N \epsilon^2}{4T e^{2L\Lambda T}} \right).
\]

\[\square\]

A.1. Proof of Lemma A.1

Let \(t \in T^N\), and denote

\[
e^N(t) := F_{av}^N(t) - A \left( Y_{av}^N(t) \right).
\]

From (12), we can write \(e^{N,\alpha}(t)\) more explicitly as

\[
e^{N,\alpha}(t) = \sum_{\beta \in A; \beta \neq \alpha} \left( \frac{1}{N} \sum_{i=1}^N Y_i^{N,\beta}(t) \left( \rho^{\beta\alpha} \left( Y_i^N(t) \right) - \rho^{\beta\alpha} \left( Y_{av}^N(t) \right) \right) \right)
\]

\[\quad - \sum_{\beta \in A; \beta \neq \alpha} \left( \frac{1}{N} \sum_{i=1}^N Y_i^{N,\alpha}(t) \left( \rho^{\alpha\beta} \left( Y_i^N(t) \right) - \rho^{\alpha\beta} \left( Y_{av}^N(t) \right) \right) \right). \tag{25}\]

Our goal is to bound \(|e^{N,\alpha}(t)|\); to do so, we will bound the absolute value of each line above separately. We will make use of the following inequality: if \(a_1, \ldots, a_m\) are nonnegative real numbers, then Jensen’s inequality implies

\[
\sqrt{a_1 + \ldots + \sqrt{a_m} \leq \sqrt{m} \sqrt{a_1 + \ldots + a_m}}. \tag{26}\]

We can now bound

\[
\frac{1}{N} \sum_{i=1}^N \left| \rho^{\beta\alpha} \left( Y_i^N(t) \right) - \rho^{\beta\alpha} \left( Y_{av}^N(t) \right) \right| \leq \frac{L}{N} \sum_{i=1}^N \left\| Y_i^N(t) - Y_{av}^N(t) \right\|_\infty \]

\[\quad \leq \frac{L}{N} \sum_{i=1}^N \sum_{\beta \in A} \left| Y_i^{N,\beta}(t) - Y_{av}^{N,\beta}(t) \right| \tag{27}\]
\[ \leq \frac{L \rho \sqrt{m}}{\sqrt{N}} \sum_{i=1}^{N} \sqrt{\sum_{\beta \in A} \left( Y_{i}^{N,\beta}(t) - Y_{av}^{N,\beta}(t) \right)^2} \]
\[ \leq L \rho \sqrt{m} \sqrt{\frac{1}{N} \sum_{\beta \in A} \sum_{i=1}^{N} \left( Y_{i}^{N,\beta}(t) - Y_{av}^{N,\beta}(t) \right)^2} \]
\[ \leq L \rho \sqrt{m} \sqrt{\sum_{\beta \in A} \left\| W_N \left( Y_{N,\beta}(t) - Y_{av}^{N,\beta}(t) \right) \right\|^2}. \]

(28)

The first inequality comes from the Lipschitz property of the \( \rho^{\alpha\beta} \)'s. The second inequality follows from bounding the maximum of the entries of \( Y_{i}^{N}(t) - Y_{av}^{N}(t) \) by the absolute sum. The third inequality is an application of (26). The fourth inequality is an application of Jensen’s inequality since the square root function is concave. Next, noting that the vector \( Y_{N,\beta}(t) - Y_{av}^{N,\beta}(t) \mathbf{1} \) is orthogonal to \( \mathbf{1} \) and has a 2-norm of at most \( \sqrt{N} \), we can bound
\[ \left\| W_N (Y_{N,\beta}(t) - Y_{av}^{N,\beta}(t) \mathbf{1}) \right\|_2 \leq \frac{\lambda^2}{N}. \] (29)

Substituting (29) into (28) gives the following bound:
\[ \frac{1}{N} \sum_{i=1}^{N} \left| \rho^{\alpha\beta} \left( Y_{i}^{N}(t) \right) - \rho^{\alpha\beta} \left( Y_{av}^{N}(t) \right) \right| \leq L \rho m \lambda_N. \]

Next, since \( |Y_{i}^{N,\beta}(t)| \leq 1 \), we can bound the absolute value of the first line in (25) by
\[ \sum_{\beta \in A; \beta \neq \alpha} \frac{1}{N} \sum_{i=1}^{N} \left| \rho^{\beta\alpha} \left( Y_{i}^{N}(t) \right) - \rho^{\beta\alpha} \left( Y_{av}^{N}(t) \right) \right| \leq L \rho m^2 \lambda_N. \]

Using identical arguments, the absolute value of the third line in (25) is at most \( L \rho m^2 \lambda_N \) as well. Putting everything together, we see that
\[ |e_{N,\alpha}^{N}(t)| \leq 2L \rho m^2 \lambda_N. \]

As this holds for every \( \alpha \in A \), we have that
\[ \left\| F_{av}(t) - r(t) \Lambda \left( Y_{av}^{N}(t) \right) \right\|_{\infty} \leq 2L \rho m^2 \lambda_N. \]
We will now extend this to the case where $t$ is not an element of $\mathbb{T}^N$. Specifically, suppose that $s \leq t < s + \frac{1}{N}$ where $s \in \mathbb{T}^N$. Then by the triangle inequality,

$$
\left\| F_{av}^N(t) - \Lambda \left( \dot{Y}_{av}^N(t) \right) \right\|_{\infty} = \left\| F_{av}^N(s) - \Lambda \left( \dot{Y}_{av}^N(t) \right) \right\|_{\infty} \\
\leq \left\| F_{av}^N(s) - \Lambda \left( Y_{av}^N(s) \right) \right\|_{\infty} + \left\| \Lambda \left( Y_{av}^N(s) \right) - \Lambda \left( \dot{Y}_{av}^N(t) \right) \right\|_{\infty}.
$$
(30)

We have already shown that the first term above is at most $2L\rho m^2\lambda_N$. To bound the second term above, first note that $\left\| Y_{av}^N(s + \frac{1}{N}) - Y_{av}^N(s) \right\|_{\infty} \leq 1/N$ for any $s \in \mathbb{T}^N$. Hence, we have the bound

$$
\left\| \Lambda \left( Y_{av}^N(s) \right) - \Lambda \left( \dot{Y}_{av}^N(t) \right) \right\|_{\infty} \leq \frac{L\Lambda}{N}.
$$

Putting everything together, the right hand side of (30) is at most $2L\rho m^2\lambda_N + \frac{L\Lambda}{N}$.

**APPENDIX B: PROOF OF THEOREM 4.2**

We introduce some notation. Let $v \in \mathbb{R}^N$ be a vector with nonnegative entries and define

$$
Y_v^N(t) := \sum_{i=1}^{N} v_i Y_i^N(t).
$$

Note in particular if $v = \frac{1}{N} \mathbf{1}$, then $Y_v^N(t) = Y_{av}^N(t)$. Analogously define $y_v^N(t)$. We can then define the following processes:

$$
F_v^N(t) := \mathbb{E} \left[ N \left( Y_v^N \left( t + \frac{1}{N} \right) - Y_v^N(t) \right) \mid \mathcal{F}_t \right],
$$
$$
U_v^N(t) := N \left( Y_v^N \left( t + \frac{1}{N} \right) - Y_v^N(t) \right) - F_v^N(t),
$$
$$
f_v^N(t) := N \left( y_v^N \left( t + \frac{1}{N} \right) - y_v^N(t) \right).
$$

The stochastic process $F_v^N(t)$ is the expected change in $Y_v^N(t)$, $U_v^N(t)$ is the empirical jump centered at 0, and $f_v$ is the change in the deterministic process $y_v^N(t)$. Technically speaking, all the above processes are well-defined only for $t \in \mathbb{T}^N$, but we will abuse notation and treat the processes as right-continuous step functions in the same manner as in Section A.
We also write \( \tilde{Y}_v^N(t) \) to denote the continuous, linearly interpolated version of \( Y_v^N(t) \) that is defined in the same manner as in Section A.

We now introduce the main class of stochastic processes we will study in the proof of the theorem. Let \( P \) be a doubly sub-stochastic matrix.\(^9\) For each \( i \in [N] \), let \( p_i \) be the \( i \)th row vector of \( P \). Define the random variable

\[
A(P, t) := \frac{1}{N} \sum_{i=1}^{N} \left\| \tilde{Y}_p_i^N(t) - \hat{y}_{p_i}^N(t) \right\|_{\infty}.
\]

We are particularly interested in the behavior of \( A(W, t) \). Since \( W \) is doubly-stochastic, the convexity of the infinity norm implies

\[
\left\| \tilde{Y}_{av}^N(t) - \hat{y}_{av}^N(t) \right\|_{\infty} \leq \frac{1}{N} \sum_{i=1}^{N} \left\| \tilde{Y}_i^N(t) - \hat{y}_i^N(t) \right\|_{\infty} = A(W, t).
\]

Hence any upper tail bounds we establish for \( A(W, t) \) will also hold for the process \( \|Y_{av}^N(t) - y_{av}^N(t)\|_{\infty} \). The main challenge in applying Benaïm and Weibull’s techniques to \( A(W, t) \) is the absence of a recursive inequality of the type in (24). To remedy this, we will consider \( A(P, t) \) for \( P \) in some finite set \( \Gamma \) containing \( W \) such that \( \max_{P \in \Gamma} A(P, t) \) exhibits a recursive inequality of a similar form as (24). Carrying out the same techniques as the proof of Theorem 3.6 will lead to the desired concentration inequality. Our first step, therefore, is to find a set \( \Gamma \) that will enable this analysis. This is the purpose of the following lemma. We defer the proof to Section B.1.

**Lemma B.1:** Fix \( \delta > 0 \) and a time horizon \( T \), and suppose that \( N \geq \frac{4}{\delta} \). There exists a subset \( \Gamma \) of doubly sub-stochastic matrices containing the combination matrix \( W \) such that for any \( P \in \Gamma \), \( 0 \leq P_{ij} \leq W_{ij} \) for all \( i, j \in [N] \) and

\[
\frac{1}{N} \sum_{i=1}^{N} \left\| f_{p_i}^N(t) - f_{p_i}^N(t) \right\|_{\infty} \leq (2L_\rho + m^2 + 2) \max_{Q \in \Gamma} A(Q, t) + \delta.
\]

Furthermore, \( |\Gamma| \) depends only on \( \delta, T, m \) and the functions \( \{\rho^\alpha\}_{\alpha, \beta \in A} \).

\(^9\)A matrix is doubly sub-stochastic all entries are non-negative and the sum of the entries in each column and each row is at most 1.
We also have the following tail bound, which serves the same purpose as Lemma A.2 in the proof of Theorem 3.6. The proof is involved so we defer it to Section B.2.

**Lemma B.2:** Fix $\epsilon > 0$ and a time horizon $T \geq 0$. Let $P \in \mathbb{R}^{N \times N}$ be a doubly sub-stochastic matrix satisfying $\frac{1}{N} \|P\|_F^2 \leq \frac{\epsilon^2}{2T}$ and suppose $N \geq \frac{2}{\epsilon^2}$. Then

$$
\mathbb{P} \left( \max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^{N} \left\| \int_0^t U_{p_i}^N(s) ds \right\|_{\infty} > \epsilon \right) \leq m \cdot \exp \left( - \frac{N \epsilon^2}{1000 m^2 (T+1)} \right).
$$

We now turn to the proof of the theorem.

**Proof of Theorem 4.2.** Fix a doubly sub-stochastic matrix $P$ and let $p_i$ denote the $i$th row vector of $P$. Using identical arguments as the proof of Theorem 3.6, we can write for each $i \in [N],$

$$
\hat{Y}^N_{p_i}(t) - \hat{Y}^N_{p_i}(0) = \int_0^t F^N_{p_i}(s) + U^N_{p_i}(s) ds
$$

$$
\hat{y}^N_{p_i}(t) - \hat{y}^N_{p_i}(0) = \int_0^t f^N_{p_i}(s) ds.
$$

Noting that $Y^N_{p_i}(0) = y^N_{p_i}(0)$, we can subtract the two equations and take the infinity norm of both sides to obtain

$$
\left\| \hat{Y}^N_{p_i}(t) - \hat{y}^N_{p_i}(t) \right\|_{\infty} \leq \int_0^t \left\| F^N_{p_i}(s) - f^N_{p_i}(s) \right\|_{\infty} ds + \left\| \int_0^t U^N_{p_i}(s) ds \right\|_{\infty}.
$$

(31)

Average over $i$ to obtain

$$
A(P, t) \leq \int_0^t \frac{1}{N} \sum_{i=1}^{N} \left\| F^N_{p_i}(s) - f^N_{p_i}(s) \right\|_{\infty} ds + \frac{1}{N} \sum_{i=1}^{N} \left\| \int_0^t U^N_{p_i}(s) ds \right\|_{\infty}.
$$

(32)

Set $L := 2L_\rho + m^2 + 2$ and fix $\delta > 0$ to be determined later. Next, apply Lemma B.1 to the first term on the right hand side of (32) and maximize over $0 \leq t \leq T$ in the second term on the right hand side of (32) to obtain, for $P \in \Gamma,$

$$
A(P, t) \leq L \int_0^t \max_{Q \in \Gamma} A(Q, s) ds + \delta T + \max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^{N} \left\| \int_0^t U^N_{p_i}(s) ds \right\|_{\infty}.
$$

(33)
Maximize over elements of \( \Gamma \) on both sides to obtain

\[
\max_{Q \in \Gamma} A(Q, t) \leq L \int_0^t \max_{Q \in \Gamma} A(Q, s) ds + \delta T + \max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^N \left\| \int_0^t U_{N}_q(s) ds \right\|_{\infty}.
\]

We remark that \( A(Q, t) \) is continuous in \( t \) (in particular, it is piecewise linear) so \( \max_{Q \in \Gamma} A(Q, t) \) is also continuous in \( t \) since \( \Gamma \) is a finite set. We can therefore apply Grönwall’s inequality to get

\[
\max_{0 \leq t \leq T} A(Q, t) \leq \left( \delta T + \max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^N \left\| \int_0^t U_{N}_q(s) ds \right\|_{\infty} \right) e^{LT}.
\]

Setting \( \delta = \frac{\epsilon}{2Te^{LT}} \) implies

\[
\mathbb{P} \left( \max_{0 \leq t \leq T} A(Q, t) > \epsilon \right) \leq \mathbb{P} \left( \max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^N \left\| \int_0^t U_{N}_q(s) ds \right\|_{\infty} > \frac{\epsilon}{2e^{LT}} \right)
\]

\[
\leq \sum_{Q \in \Gamma} \mathbb{P} \left( \max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^N \left\| \int_0^t U_{N}_q(s) ds \right\|_{\infty} > \frac{\epsilon}{2e^{LT}} \right)
\]

\[
\leq m|\Gamma| \exp \left( - \frac{N \epsilon^2}{4000m^2 e^{2LT}(T + 1)} \right). \tag{34}
\]

Above, the second inequality is due to a union bound over elements of \(|\Gamma|\). The third inequality follows from an application of Lemma B.2 which is valid for \( N \) sufficiently large and since, for \( Q \in \Gamma \), we have

\[
\frac{1}{N} \|Q\|_F^2 \leq \frac{1}{N} \|W\|_F^2 \leq \frac{\epsilon^2}{4e^{2LT}(3T + 1)}.
\]

Finally, noting that \( W \in \Gamma \), the doubly-stochastic property of \( W \) implies

\[
\left\| Y_{av}^N(t) - y_{av}^N(t) \right\|_{\infty} \leq A(W, t) \leq \max_{Q \in \Gamma} A(Q, t).
\]

The desired result then follows from (34). \( \square \)
B.1. Proof of Lemma B.1

In this subsection, we will prove two intermediate results, from which Lemma B.1 will directly follow. In our first result, we provide a bound for \( \frac{1}{N} \sum_{i=1}^{N} \| F_{p_i}^N (t) - f_{p_i}^N (t) \|_{\infty} \) for a general doubly sub-stochastic matrix \( P \).

**Lemma B.3:** Let \( P \) be a doubly sub-stochastic matrix. Then for every \( t \in T^N \), there is a finite set of sub-stochastic matrices \( \Gamma(P,t) \) with \( |\Gamma(P,t)| \leq m^2 \) such that

\[
\frac{1}{N} \sum_{i=1}^{N} \left\| F_{p_i}^N (t) - f_{p_i}^N (t) \right\|_{\infty} \leq (2L^2 + m^2 + 2) \max_{Q \in \Gamma(P,t)} A(Q, t).
\]

**Proof.** Recall from (12) and (13) that for each \( \alpha \in A \),

\[
F_{j}^{N,\alpha}(t) = (1 - r(t)) \left( Y_{j}^{N,\alpha}(t) - Y_{j}^{N,\alpha}(t) \right) + r(t) \sum_{\beta: \beta \neq \alpha} \left( Y_{j}^{N,\beta}(t) \rho^{\beta \alpha} \left( \overline{Y}_{j}^{N}(t) \right) - Y_{j}^{N,\alpha}(t) \rho^{\alpha \beta} \left( \overline{Y}_{j}^{N}(t) \right) \right);
\]

\[
f_{j}^{N,\alpha}(t) = (1 - r(t)) \left( \overline{Y}_{j}^{N,\alpha}(t) - \overline{Y}_{j}^{N,\alpha}(t) \right) + r(t) \sum_{\beta: \beta \neq \alpha} \left( \overline{Y}_{j}^{N,\beta}(t) \rho^{\beta \alpha} \left( \overline{Y}_{j}^{N}(t) \right) - \overline{Y}_{j}^{N,\alpha}(t) \rho^{\alpha \beta} \left( \overline{Y}_{j}^{N}(t) \right) \right).
\]

By adding and subtracting terms, we can write

\[
F_{j}^{N,\alpha}(t) - f_{j}^{N,\alpha}(t) = (1 - r(t)) \left( Y_{j}^{N,\alpha}(t) - \overline{Y}_{j}^{N,\alpha}(t) - (Y_{j}^{N,\alpha}(t) - Y_{j}^{N,\alpha}(t)) \right) + r(t) \sum_{\beta: \beta \neq \alpha} \left[ (Y_{j}^{N,\beta}(t) - Y_{j}^{N,\alpha}(t)) \rho^{\beta \alpha} (\overline{Y}_{j}^{N}(t)) - (Y_{j}^{N,\alpha}(t) - Y_{j}^{N,\alpha}(t)) \rho^{\alpha \beta} (\overline{Y}_{j}^{N}(t)) \right] + Y_{j}^{N,\beta}(t) \left( \rho^{\beta \alpha} (\overline{Y}_{j}^{N}(t)) - \rho^{\beta \alpha} (\overline{Y}_{j}^{N}(t)) \right) - Y_{j}^{N,\alpha}(t) \left( \rho^{\alpha \beta} (\overline{Y}_{j}^{N}(t)) - \rho^{\alpha \beta} (\overline{Y}_{j}^{N}(t)) \right).
\]

Recall that our goal is to study

\[
\frac{1}{N} \sum_{i=1}^{N} \left| F_{p_i}^{N,\alpha}(t) - f_{p_i}^{N,\alpha}(t) \right|.
\]
Thus we will take a weighted sum over $j$ with respect to the vector $p_i = \{p_{ij}\}_{j=1}^N$ in (36), take an absolute value, then average over $i$. There are two main terms in the first line that we need to bound. First, we can write

$$
\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N p_{ij} \left| Y_j^{N,\alpha}(t) - \bar{y}_j^{N,\alpha}(t) \right| = \frac{1}{N} \sum_{j=1}^N \left( \sum_{i=1}^N p_{ij} \right) \left| Y_j^{N,\alpha}(t) - \bar{y}_j^{N,\alpha}(t) \right| \\
\leq \frac{1}{N} \sum_{j=1}^N \left| Y_j^{N,\alpha}(t) - \bar{y}_j^{N,\alpha}(t) \right| = A(W, t).
$$

Above, we have used the inequality $\sum_{i=1}^N p_{ij} \leq 1$, which is due to $P$ being doubly sub-stochastic. Hence after taking the weighted sums and absolute value, the first line of (36) can be bounded by $A(W, t) + A(P, t)$. We now turn to the second line of (36). Taking a weighted sum over $j$ and then averaging over $i$, we obtain terms of the form

$$
\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N p_{ij} \rho^{\beta\alpha}(\bar{y}_j^{N}(t)) \left( Y_j^{N,\beta}(t) - y_j^{N,\beta}(t) \right) = A(P \rho^{\beta\alpha}(t), t),
$$

where $\rho^{\beta\alpha}(t)$ is the diagonal matrix with the $j$th diagonal entry equal to $[\rho^{\beta\alpha}(t)]_j := \rho^{\beta\alpha}(\bar{y}_j^{N}(t))$. Hence the second line of (36) can be bounded by

$$
\sum_{\beta \in \mathcal{A} : \beta \neq \alpha} \left( A(P \rho^{\beta\alpha}(t), t) + A(P \rho^{\alpha\beta}(t), t) \right) \leq \sum_{\alpha, \beta \in \mathcal{A} : \alpha \neq \beta} A(P \rho^{\beta\alpha}(t), t).
$$

For the third line in (36), we use the Lipschitz property of the $\rho^{\beta\alpha}$’s. Using the fact that $|Y_j^{N,\alpha}(t)| \leq 1$, we obtain bounds of the form

$$
\frac{L_\rho}{N} \sum_{i=1}^N \sum_{j=1}^N p_{ij} \left\| Y_j^{N}(t) - \bar{y}_j^{N}(t) \right\|_{\infty} = \frac{L_\rho}{N} \sum_{j=1}^N \left\| Y_j^{N}(t) - \bar{y}_j^{N}(t) \right\|_{\infty} = L_\rho \cdot A(W, t).
$$

Above, we have used the fact that $P$ is doubly sub-stochastic. Putting everything together, consider the following set of matrices:

$$
\Gamma(P, t) := \{W\} \cup \{P \rho^{\beta\alpha}(t) : \alpha, \beta \in \mathcal{A}, \alpha \neq \beta\}.
$$
As the bounds we have derived do not depend on $\alpha$, we have therefore shown that for $t \in \mathbb{T}^N$,

$$\frac{1}{N} \sum_{i=1}^{N} \left\| F^{N}_{p_i}(t) - f^{N}_{p_i}(t) \right\|_\infty \leq (2L_\rho + m^2 + 2) \max_{Q \in \Gamma(P,t)} A(Q, t).$$

The disadvantage of Lemma B.3 is that the right hand side of (35) depends heavily on $P$ and $t$ through the set of doubly sub-stochastic matrices $\Gamma(P,t)$; this prevents us from using Benaïm and Weibull’s techniques in the proof of Theorem 4.2. The next result will remedy this problem by effectively allowing us to replace $\Gamma(P,t)$ with a fixed set $\Gamma$.

**Lemma B.4:** For every $\delta > 0$ and time horizon $T$, there is a finite set of sub-stochastic matrices $\Gamma$ such that for every $P \in \Gamma$ and $t \in \mathbb{T}^N \cap [0,T)$,

$$\max_{Q \in \Gamma(P,t)} A(Q, t) \leq \max_{Q' \in \Gamma} A(Q', t) + \delta.$$

Furthermore, $|\Gamma|$ depends only on $T, \delta, m$ and the functions $\rho^{\beta \alpha}$.

**Proof.** Let $\ell_\delta$ be the largest positive integer such that $\frac{\delta}{4L_\rho} \leq \frac{\ell_\delta}{N} \leq \frac{\delta}{2L_\rho}$; such an $\ell_\delta$ will always exist for $N$ sufficiently large (as long as $\delta$ does not depend on $N$). Define the set of transition times

$$\mathbb{T}^N_\delta := \left\{ \frac{\ell_\delta}{N} n : n \in \mathbb{N} \cup \{0\} \right\}. $$

Further, recall the definition of the diagonal matrix $\rho^{\beta \alpha}(t)$ from the proof of Lemma B.3. Also recall that $W$ is the doubly-stochastic combination matrix used by agents to compute their local population estimates. Consider the following family of subsets of doubly sub-stochastic matrices on $\mathbb{R}^{N \times N}$ constructed as follows:

$$\Gamma_{\delta,0} := \{W\}$$

$$\Gamma_{\delta,k} := \left\{ W \left( \prod_{i=1}^{k} \rho^{\beta \alpha}(t_i) \right) : \alpha_i, \beta_i \in A, \alpha_i \neq \beta_i, t_i \in \mathbb{T}^N_\delta \cap [0,T) \right\} \text{ for } k \geq 1.$$
A useful consequence of the construction is that \( P \in \Gamma_{\delta,k} \) implies that, for any pair of distinct actions \( \alpha, \beta \in A \) and any \( s \in \mathbb{T}^N_\delta \), \( P^\beta \alpha (s) \in \Gamma_{\delta,k+1} \). Moreover, it is simple to confirm that all matrices in \( \Gamma_{\delta,k} \) are doubly sub-stochastic since \( \rho^\beta \alpha (x) \in [0,1) \) for all \( x \in \Delta_m \). Furthermore, if we denote \( \rho_{\text{max}} < 1 \) to be an upper bound on the maximum value of \( \rho^\beta \alpha \) over all \( \alpha, \beta \in A \), it follows that all row and column sums of matrices in \( \Gamma_{\delta,k} \) are bounded by \( \rho_{\text{max}}^k \), which tends to 0 as \( k \) grows large. We also have the bound

\[
|\Gamma_{\delta,k}| \leq \left( \frac{4L\rho m^2 T}{\delta} \right)^k.
\]

The bound above follows from the fact that there are at most \( m^2 \) pairs of \( (\alpha, \beta) \) such that \( \alpha, \beta \) are distinct, and there are at most \( \frac{4L\rho T}{\delta} \) elements of \( \mathbb{T}^N_\delta \cap [0,T) \).

Next, let \( M \) be a fixed positive integer. For any \( P \in \bigcup_{k=0}^M \Gamma_{\delta,k} \), and for any \( t \in \mathbb{T}^N \cap [0,T) \), we claim that

\[
\max_{Q \in \Gamma(P,t)} A(Q,t) \leq \max_{Q' \in \bigcup_{k=0}^M \Gamma_{\delta,k}} A(Q',t) + \max \left\{ \delta, \rho_{\text{max}}^{M+1} \right\}.
\]

**Case 1:** \( P \in \Gamma_{\delta,k} \) for some \( k < M \).

Let \( Q \in \Gamma(P,t) \). If \( Q = W \), then we trivially have

\[
A(W,t) \leq \max_{Q' \in \bigcup_{k=0}^M \Gamma_{\delta,k}} A(Q',t)
\]

since \( W \in \Gamma_{\delta,0} \subset \bigcup_{k=0}^M \Gamma_{\delta,k} \). Else, assume that \( Q \neq W \), so by the definition of \( \Gamma(P,t) \) there exists distinct \( \alpha, \beta \in A \) such that \( Q = P \rho^\beta \alpha (t) \). Let \( t' \) be the closest element of \( \mathbb{T}^N_\delta \) to \( t \) with ties broken arbitrarily, so that \( |t - t'| \leq \frac{\delta}{2T_\rho} \). Define \( Q' := P \rho^\beta \alpha (t') \). Since \( P \in \Gamma_{\delta,k} \) for some \( k < M \), it follows that \( Q' \in \bigcup_{k=0}^M \Gamma_{\delta,k} \). Let \( \{q_i\}_{i \in [N]} \) denote the row-vectors of \( Q \) and let \( \{q'_i\}_{i \in [N]} \) denote the row-vectors of \( Q' \). By the triangle inequality, we can write, for any \( i \in [N] \),

\[
\left\| Y^N_{q_i}(t) - y^N_{q_i}(t) \right\|_{\infty} \leq \left\| Y^N_{q_i}(t) - y^N_{q'_i}(t) \right\|_{\infty} + \left\| Y^N_{q'_i}(t) - y^N_{q'_i}(t) \right\|_{\infty} + \left\| y^N_{q'_i}(t) - y^N_{q_i}(t) \right\|_{\infty}.
\]

(39)
For any $j \in [N]$, we also have
\[
|q_{ij} - q'_{ij}| = p_{ij} \left| \rho^{\beta \alpha} \left( \eta_j^N(t) \right) - \rho^{\beta \alpha} \left( \eta_j^N(t') \right) \right| \\
\leq p_{ij} L_{\rho} \left\| \eta_j^N(t) - \eta_j^N(t') \right\|_{\infty} \\
\leq p_{ij} \delta / 2.
\]

Above, the first inequality follows from the Lipschitz property of the $\rho^{\beta \alpha}$’s and the second inequality follows from the fact that the deterministic process $y_i^N(t)$ is 1-Lipschitz, which can be directly seen from the definition in (13). Next, since the entries of $Y_i^N(t)$ and $y_i^N(t)$ are all at most 1 and $\sum_{j=1}^N p_{ij} \leq 1$,
\[
\left\| Y_{\text{q}_i}^N(t) - Y_{\text{q}'_i}^N(t) \right\|_{\infty} = \max_{\alpha \in A} \left| Y_{\text{q}_i}^{N,\alpha}(t) - Y_{\text{q}'_i}^{N,\alpha}(t) \right| = \max_{\alpha \in A} \left| \sum_{j=1}^N Y_j^{N,\alpha}(t)(q_{ij} - q'_{ij}) \right| \\
\leq \sum_{j=1}^N |q_{ij} - q'_{ij}| \leq \sum_{j=1}^N p_{ij} \delta / 2 \leq \delta / 2.
\]

Identical reasoning shows that $\left\| y_{\text{q}_i}^N(t) - y_{\text{q}'_i}^N(t) \right\|_{\infty} \leq \delta / 2$. Substituting these bounds into (39) shows that
\[
\left\| Y_{\text{q}_i}^N(t) - y_{\text{q}_i}^N(t) \right\|_{\infty} \leq \left\| Y_{\text{q}'_i}^N(t) - y_{\text{q}'_i}^N(t) \right\|_{\infty} + \delta.
\]

Averaging over $i \in [N]$ yields $A(Q, t) \leq A(Q', t) + \delta$. This proves (38) for the case $P \in \Gamma_{\delta,k}$ for $k < M$.

**Case 2:** $P \in \Gamma_{k,M}$.

As we have noted, the row and column sums of $P$ are at most $\rho_{\text{max}}^M$. Hence if $Q \in \Gamma(P, t)$ and $Q \neq W$ (the case $Q = W$ has already been shown in the previous case) then all row and column sums of $Q$ are at most $\rho_{\text{max}}^{M+1}$, so
\[
A(Q, t) = \frac{1}{N} \sum_{i=1}^N \max_{\alpha \in A} \left| Y_{\text{q}_i}^{N,\alpha}(t) - y_{\text{q}_i}^{N,\alpha}(t) \right|
\]
\[
\leq \frac{1}{N} \sum_{i=1}^{N} \left( \max_{\alpha \in A, j \in [N]} |Y_{j}^{N,\alpha}(t) - y_{j}^{N,\alpha}(t)| \right) \sum_{j=1}^{N} q_{ij} \\
\leq \rho_{\max}^{M+1}.
\]

To obtain the final inequality above, we have used the fact that \(|a - b| \leq 1\) for \(a, b \in [0, 1]\). Equation (38) follows.

Both cases have shown that (38) holds, so

\[
\Gamma := \bigcup_{k=0}^{M} \Gamma_{\delta,k}
\]

is the desired subset of doubly sub-stochastic matrices in \(\mathbb{R}^{N \times N}\), with \(M := \left\lfloor \frac{\log 1/\delta}{\log 1/\rho_{\max}} \right\rfloor\) to ensure that \(\rho_{\max}^{M+1} \leq \delta\). Equation (37) then implies

\[
|\Gamma| \leq (M + 1) \left( \frac{4L \rho m^2 T}{\delta} \right)^M,
\]

which is a function of only \(\delta, T, m\), and the policy update functions \(\rho^{\beta\alpha}\). □

We now put the two lemmas together to prove the main result of this subsection.

**Proof of Lemma B.1.** Replacing \(\delta\) with \(\frac{\delta}{2(2L\rho + m^2 + 2)}\) in Lemma B.4 and then applying Lemma B.3 shows that there is a finite set of doubly sub-stochastic matrices \(\Gamma\) such that, for any \(t \in \mathbb{T}^{N} \cap [0, T)\) and \(P \in \Gamma\),

\[
\frac{1}{N} \sum_{i=1}^{N} \left\| F_{p_i}(t) - f_{p_i}(t) \right\|_{\infty} \leq (2L\rho + m^2 + 2) \max_{Q \in \Gamma} A(Q, t) + \frac{\delta}{2}.
\]

It remains to see what changes when \(t\) is not an element of \(\mathbb{T}^{N}\). Suppose that \(s \leq t < s + 1/N\), where \(s \in \mathbb{T}^{N}\). Then \(F_{p_i}(t) = F_{p_i}(s)\) and \(f_{p_i}(t) = f_{p_i}(s)\), so the left hand side above is unchanged. We now turn to the right hand side. By the triangle inequality, for any doubly sub-stochastic \(Q\) with row vectors \(\{q_i\}_{i \in [N]}\) we can write

\[
\left\| Y_{q_i}(t) - \hat{y}_{q_i}(t) \right\|_{\infty} \leq \left\| Y_{q_i}(s) - y_{q_i}(s) \right\|_{\infty} - \left\| Y_{q_i} s + \frac{1}{N} - y_{q_i}(s) \right\|_{\infty} - \left\| Y_{q_i}(s) - y_{q_i}(s) \right\|_{\infty}
\]
If agent $j$ is given the opportunity to revise their action at time $s + \frac{1}{N}$, then the last two terms on the right hand side above are each bounded by $q_{ij}$. Averaging over $i$, we see that the sub-stochastic property of $Q$ implies

$$|A(Q, t) - A(Q, s)| \leq \frac{2}{N} \sum_{i=1}^{N} q_{ij} \leq \frac{2}{N}.$$ 

Hence $A(Q, s) \leq A(Q, t) + \frac{2}{N}$, and the desired claim follows if $N$ is sufficiently large so that $\frac{2}{N} \leq \frac{\delta}{2}$.

**B.2. Proof of Lemma B.2**

Our focus will be on proving the following lemma.

**Lemma B.5:** Under the same assumptions as Lemma B.2, it holds for every $\alpha \in A$ that

$$\mathbb{P}\left( \max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^{N} \left| \int_{0}^{t} U_{\alpha}^{N, i}(s) ds \right| > \epsilon \right) \leq \exp \left( - \frac{N \epsilon^2}{1000(T+1)} \right).$$

Lemma B.2 follows as a corollary using simple union bounds.

**Proof of Lemma B.2.** Using the inequality $\|\cdot\|_{\infty} \leq \|\cdot\|_1$, we have

$$\frac{1}{N} \sum_{i=1}^{N} \left\| \int_{0}^{t} U_{\alpha}^{N, i}(s) ds \right\|_{\infty} \leq \sum_{\alpha \in A} \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \int_{0}^{t} U_{\alpha}^{N, i}(s) ds \right\| \right).$$

Hence a union bound gives

$$\mathbb{P}\left( \max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^{N} \left\| \int_{0}^{t} U_{\alpha}^{N, i}(s) ds \right\|_{\infty} > \epsilon \right) \leq \sum_{\alpha \in A} \mathbb{P}\left( \max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^{N} \left\| \int_{0}^{t} U_{\alpha}^{N, i}(s) ds \right\|_{\infty} > \epsilon \right) \leq m \cdot \exp \left( - \frac{N \epsilon^2}{1000m^2(T+1)} \right).$$

For the remainder of the section, we focus on the proof of Lemma B.5. We proceed by proving the following lemma, which generalizes Lemma 3 in Benaïm and Weibull (2003).
**Lemma B.6:** Let \( \{A(t)\}_{t \in \mathbb{T}^N} \) be a random process where \( A(t) \) is \( \mathcal{F}_{t+1/N} \)-measurable, \( |A(t)| \leq B(t) \) a.s. for some \( \mathcal{F}_t \)-measurable process \( B(t) \), and \( \mathbb{E}[A(t) | \mathcal{F}_t] \leq 0 \). Then for all \( \theta \geq 0 \),

\[
\mathbb{E} \left[ \exp(\theta A(t)) | \mathcal{F}_t \right] \leq \exp \left( \frac{1}{2} \theta^2 B(t)^2 \right).
\]

**Proof.** Consider the function \( g(p) := \log \mathbb{E}[e^{p\theta A(t)} | \mathcal{F}_t] \). This is convex, since the log moment generating function of any random variable (whenever it exists) is convex. Since \( A(t) \) is bounded with respect to \( \mathcal{F}_t \), \( g \) is indeed finite for any \( p \in \mathbb{R} \), and therefore convex. Furthermore, we have \( g(0) = 0 \). Taking a derivative with respect to \( p \), we can compute

\[
g'(p) = \frac{\mathbb{E}[\theta A(t)e^{p\theta A(t)} | \mathcal{F}_t]}{\mathbb{E}[e^{p\theta A(t)} | \mathcal{F}_t]}.
\]

Hence \( g'(0) = \mathbb{E}[\theta A(t)] \leq 0 \) for \( \theta \geq 0 \). We can also compute explicitly the second derivative of \( g \) with respect to \( p \):

\[
g''(p) = \frac{\mathbb{E}[e^{p\theta A(t)} | \mathcal{F}_t]\mathbb{E}[\theta^2 A(t)^2 e^{p\theta A(t)} | \mathcal{F}_t] - \mathbb{E}[\theta A(t)e^{p\theta A(t)} | \mathcal{F}_t]^2}{\mathbb{E}[e^{p\theta A(t)} | \mathcal{F}_t]^2} \leq \theta^2 B(t)^2.
\]

By the fundamental theorem of calculus, for \( p \geq 0 \) we have

\[
g'(p) = g'(0) + \int_0^p g''(s)ds \leq p\theta^2 B(t)^2.
\]

Applying the fundamental theorem of calculus again, we see that

\[
g(1) = g(0) + \int_0^1 g'(p)dp \leq \frac{1}{2} \theta^2 B(t)^2.
\]

\( \square \)

We next prove the processes of interest are bounded.

**Lemma B.7:** Let \( v \in \mathbb{R}^N \) be a vector with \( \|v\|_1 \leq 1 \). Then we have the following almost sure bound:

\[
\left| \int_0^t U_v^{N,\alpha}(s)ds \right| \leq 1 + t.
\]
Proof. Recall that for $s \in \mathbb{T}^N$,

$$U^{N,\alpha}_v(s) = N \left(Y^{N,\alpha}_v \left(s + \frac{1}{N}\right) - Y^{N,\alpha}_v(s) - F^{N,\alpha}_v(s)\right).$$

Hence

$$\int_0^t U^{N,\alpha}_v(s) ds = \frac{1}{N} \sum_{k=0}^{tN-1} U^{N,\alpha}_v \left(\frac{k}{N}\right)$$

$$= \sum_{k=0}^{tN-1} \left(Y^{N,\alpha}_v \left(\frac{k+1}{N}\right) - Y^{N,\alpha}_v \left(\frac{k}{N}\right) - \frac{1}{N} F^{N,\alpha}_v \left(\frac{k}{N}\right)\right)$$

$$= Y^{N,\alpha}_v(t) - Y^{N,\alpha}_v(0) - \frac{1}{N} \sum_{k=0}^{tN-1} F^{N,\alpha}_v \left(\frac{k}{N}\right).$$

Take the absolute value of both sides and apply the triangle inequality to obtain

$$\left|\int_0^t U^{N,\alpha}_v(s) ds\right| \leq |Y^{N,\alpha}_v(t) - Y^{N,\alpha}_v(0)| + \frac{1}{N} \sum_{k=0}^{tN-1} \left|F^{N,\alpha}_v \left(\frac{k}{N}\right)\right| \leq 1 + t.$$

Above, we have used the fact that $Y^{N,\alpha}_i(s), F^{N,\alpha}_i(s) \in [0, 1]$ for all $s \in \mathbb{T}^N$ and all $i \in [N]$. □

Before proving Lemma B.5, we give a high-level overview of the proof as it is more involved. It will be more convenient to study the process

$$V^{N,\alpha}(t) := \frac{1}{N} \sum_{i=1}^{N t} \left(\int_0^t U^{N,\alpha}_{p_i}(s) ds\right)^2 + \epsilon^2,$$

which we define only for $t \in \mathbb{T}^N$. In particular, since the step functions $U_{p_i}(s)$ are right-continuous, $V^{N,\alpha}(t)$ is $\mathcal{F}_t$-measurable. We can relate $V^{N,\alpha}(t)$ to the process of interest by Jensen’s inequality:

$$\sqrt{V^{N,\alpha}(t)} \geq \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left(\int_0^t U^{N,\alpha}_{p_i}(s) ds\right)^2} \geq \frac{1}{N} \sum_{i=1}^{N} \left|\int_0^t U^{N,\alpha}_{p_i}(s) ds\right|. \quad (40)$$
Our first goal is to establish a subgaussian inequality for the process

\[ \frac{V_{N,\alpha}(t + 1/N) - V_{N,\alpha}(t)}{2 \sqrt{V_{N,\alpha}(t)}}. \]  

(41)

Since we divide by \( \sqrt{V_{N,\alpha}(t)} \), the ratio could blow up if \( V_{N,\alpha}(t) \) is too small. This is the reason for adding \( \epsilon^2 \) in the definition of \( V_{N,\alpha}(t) \) - it ensures that \( V_{N,\alpha}(t) \geq \epsilon^2 \), leading to an upper bound on (41). Once we derive the subgaussian inequality, we use standard techniques to obtain an upper tail bound for the sum process

\[ \sum_{t \in T \cap [0,T]} \frac{V_{N,\alpha}(t + 1/N) - V_{N,\alpha}(t)}{2 \sqrt{V_{N,\alpha}(t)}}. \]  

(42)

Then, using the concavity of the square root function, we have the inequality

\[
\sum_{t \in T \cap [0,T]} \frac{V_{N,\alpha}(t + 1/N) - V_{N,\alpha}(t)}{2 \sqrt{V_{N,\alpha}(t)}} \geq \sum_{t \in T \cap [0,T]} \left( \sqrt{V_{N,\alpha}(t + 1/N)} - \sqrt{V_{N,\alpha}(t)} \right) \\
\approx \sqrt{V_{N,\alpha}(T)} - \sqrt{V_{N,\alpha}(0)} \\
= \sqrt{V_{N,\alpha}(T)} - \epsilon.
\]

We note that the \( \approx \) above is equality if \( T \in T^N \), else, the \( \approx \) symbol hides a small error term which is of smaller order than \( \epsilon \). The inequalities in the display above show that the upper tail bounds we develop for (42) will hold for the process

\[
\frac{1}{N} \sum_{i=1}^{N} \left| \int_{0}^{t} U_{p_i}^{N,\alpha}(s)ds \right|
\]

as well, which leads to the desired claim.

**Proof of Lemma B.5.** We begin by studying the expectation of the increments of \( V_{N,\alpha}(t) \). We can explicitly write the increments as

\[
V_{N,\alpha}(t + 1/N) - V_{N,\alpha}(t) = \frac{1}{N} \sum_{i=1}^{N} \left( \left( \int_{0}^{t + 1/N} U_{p_i}^{N,\alpha}(s)ds \right)^2 - \left( \int_{0}^{t} U_{p_i}^{N,\alpha}(s)ds \right)^2 \right).
\]
\[
\frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{N} U_{p_i}^{N,\alpha}(t) \right)^2 \left( 2 \int_0^t U_{p_i}^{N,\alpha}(s) ds + \frac{1}{N} U_{p_i}^{N,\alpha}(t) \right) \cdot
\]

(43)

Take a conditional expectation with respect to \( F_t \) and use the property \( E[U_{N,\alpha}^{p_i}(t) | F_t] = 0 \) to obtain

\[
E \left[ V^{N,\alpha}(t + \frac{1}{N}) - V^{N,\alpha}(t) | F_t \right] = \frac{1}{N} \sum_{i=1}^{N} E \left[ \left( \frac{1}{N} U_{p_i}^{N,\alpha}(t) \right)^2 | F_t \right].
\]

To study the right hand side above, we will condition on the agent that receives an opportunity to revise their action at time \( t + 1/N \). Suppose that this agent is \( j \); in other words, \( \text{Clock}_N^{j}(t + 1/N) = 1 \). Then we can write

\[
\left( \frac{1}{N} U_{p_i}^{N,\alpha}(t) \right)^2 = \left( p_{ij}(Y_j^{N,\alpha}(t + 1/N) - Y_j^{N,\alpha}(t)) - \frac{1}{N} \sum_{k=1}^{N} p_{ik} F_{k}^{N,\alpha}(t) \right)^2
\leq 2 \left(p_{ij}(Y_j^{N,\alpha}(t + 1/N) - Y_j^{N,\alpha}(t))\right)^2 + 2 \left( \frac{1}{N} \sum_{k=1}^{N} p_{ik} F_{k}^{N,\alpha}(t) \right)^2
\leq 2p_{ij}^2 + \frac{2}{N^2}.
\]

Above, the first inequality is a consequence of Jensen’s inequality. The second inequality follows from the fact that \( |Y_j^{N,\alpha}(t + 1/N) - Y_j^{N,\alpha}(t)| \leq 1, |F_{k}^{N,\alpha}(t)| \leq 1 \) and that \( P \) is row sub-stochastic. Next, to remove the conditioning on \( \text{Clock}_N^{j}(t + 1/N) = 1 \), we can write

\[
\frac{1}{N} \sum_{i=1}^{N} E \left[ \left( \frac{1}{N} U_{v_i}^{N,\alpha}(t) \right)^2 | F_t \right] = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E \left[ \left( \frac{1}{N} U_{v_i}^{N,\alpha}(t) \right)^2 | F_t, \text{Clock}_N^{j}(t + 1/N) = 1 \right]
\leq \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( 2p_{ij}^2 + \frac{2}{N^2} \right) = \frac{2}{N^2} \left\| P_{F} \right\|^2 + 1.
\]

Above, the first equality follows since the probability that agent \( j \) is chosen to update is \( 1/N \), and \( \| P \|_F \) denotes the Frobenius norm of \( P \). Putting everything together, we have
shown that
\[
\mathbb{E} \left[ V^{N,\alpha}(t + 1/N) - V^{N,\alpha}(t) - \frac{2}{N^2} \left( \|P\|_F^2 + 1 \right) \mid \mathcal{F}_t \right] \leq 0. \tag{45}
\]

We will next obtain a subgaussian inequality for the above process via Lemma B.6. From (43) we can bound
\[
\begin{aligned}
& \left| V^{N,\alpha}(t + 1/N) - V^{N,\alpha}(t) \right| \\
& \leq \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{N} U_{p_i}^{N,\alpha}(t) \right)^2 + \frac{2}{N} \sum_{i=1}^{N} \left| \frac{1}{N} U_{p_i}^{N,\alpha}(t) \right| \int_0^t U_{p_i}^{N,\alpha}(s) ds \\
& \quad \quad \quad + \frac{2}{N^2} \sum_{i=1}^{N} \left| \int_0^t U_{p_i}^{N,\alpha}(s) ds \right|. \tag{46}
\end{aligned}
\]

If \( \text{Clock}_j^N(t + 1/N) = 1 \), we can use (44) to bound the first term of (46) by
\[
\frac{2}{N} \sum_{i=1}^{N} p_{ij}^2 + \frac{2}{N^2}. \tag{47}
\]

Using the bound
\[
\left| \frac{1}{N} U_{p_i}^{N,\alpha}(t) \right| = \left| p_{ij} \left( Y_j^{N,\alpha}(t + 1/N) - Y_j^{N,\alpha}(t) \right) - \frac{1}{N} \sum_{k=1}^{N} p_{ik} F_k^{N,\alpha}(t) \right| \leq p_{ij} + \frac{1}{N},
\]
we obtain the following bound for the second term in (46):
\[
\frac{2}{N} \sum_{i=1}^{N} p_{ij} \left| \int_0^t U_{p_i}^{N,\alpha}(s) ds \right| + \frac{2}{N^2} \sum_{i=1}^{N} \left| \int_0^t U_{p_i}^{N,\alpha}(s) ds \right|. \tag{48}
\]

Define \( B_j(t) \) to be \( N \) times the summation of (47) and (48), written explicitly as
\[
B_j(t) := 2 \sum_{i=1}^{N} p_{ij}^2 + \frac{2}{N} + 2 \sum_{i=1}^{N} p_{ij} \left| \int_0^t U_{p_i}^{N,\alpha}(s) ds \right| + \frac{2}{N} \sum_{i=1}^{N} \left| \int_0^t U_{p_i}^{N,\alpha}(s) ds \right|. \tag{49}
\]

Moreover, \( B_j(t) \) is almost surely bounded. The last two terms on the right hand side of (49) can be bounded by \( 4(t + 1) \) using Lemma B.6 and the double-stochasticity of \( P \). We also have the following bound for \( N \geq 2 \):
\[
2 \sum_{i=1}^{N} p_{ij}^2 + \frac{2}{N} \leq 2 \sum_{i=1}^{N} p_{ij} + \frac{2}{N} \leq 2 + \frac{2}{N} \leq 3(t + 1).
\]
Putting everything together, \( B_j(t) \leq 7(t + 1) \) almost surely for \( N \geq 2 \). Additionally, \( B_j(t) \) is \( \mathcal{F}_t \)-measurable and we have the following almost sure bound in the case where agent \( j \) is chosen to update:

\[
\left| V_{N,\alpha}^N(t + \frac{1}{N}) - V_{N,\alpha}^N(t) - \frac{2}{N^2} \left( \|P\|_{F}^2 + 1 \right) \right| \leq \frac{1}{N} B_j(t) + \frac{2}{N^2} \left( \|P\|_{F}^2 + 1 \right).
\]

In light of (45), Lemma B.6 implies

\[
\mathbb{E} \left[ e^{\theta \left( V_{N,\alpha}^N(t+1/N) - V_{N,\alpha}^N(t) - \frac{2}{N^2} \left( \|P\|_{F}^2 + 1 \right) \right)} \mid \mathcal{F}_t, \text{Clock}_j^N(t + 1/N) = 1 \right] \leq \exp \left( \frac{\theta^2}{2} \left( \frac{1}{N} B_j(t) + \frac{2}{N^2} \left( \|P\|_{F}^2 + 1 \right) \right)^2 \right)
\]

\[
\leq \exp \left( \frac{\theta^2}{N^2} \left( B_j(t)^2 + 4 \left( \frac{\|P\|_{F}^2 + 1}{N} \right)^2 \right) \right),
\]

where we have used \((a + b)^2 \leq 2a^2 + 2b^2\) to obtain the final inequality. Removing the conditioning on the agent \( j \) who has the opportunity to revise their strategy, we obtain

\[
\mathbb{E} \left[ e^{\theta \left( V_{N,\alpha}^N(t+1/N) - V_{N,\alpha}^N(t) - \frac{2}{N^2} \left( \|P\|_{F}^2 + 1 \right) \right)} \mid \mathcal{F}_t \right] \leq \exp \left( \frac{4\theta^2}{N^2} \left( \frac{\|P\|_{F}^2 + 1}{N} \right)^2 \right) \leq \sum_{j=1}^{N} e^{\frac{\theta^2}{N^2} B_j(t)^2}. \tag{50}
\]

To simplify the summation on the right hand side, we will expand the first few polynomial terms of the exponential function. To this end, we first note that we have the almost sure bound \( B_j(t) \leq 7(T + 1) \) for \( N \) sufficiently large and for all \( t \in \mathbb{T}_N \cap [0, T) \). Next, define the function

\[
\phi(x) := \frac{e^x - 1}{x}.
\]

Note that \( \phi(x) \) is increasing for positive \( x \) and \( \phi(1) \leq 2 \). If \( 0 \leq \frac{\theta}{N} \leq \frac{1}{7(T+1)} \) then \( \frac{\theta}{N} B_j(t) \leq 1 \) for all \( j \in \mathbb{N} \), so

\[
\exp \left( \frac{\theta^2}{N^2} B_j(t)^2 \right) = 1 + \phi \left( \frac{\theta^2}{N^2} B_j(t)^2 \right) \frac{\theta^2}{N^2} B_j(t)^2 \leq 1 + \frac{2\theta^2}{N^2} B_j(t)^2.
\]
Averaging over $j$, we obtain
\[
\frac{1}{N} \sum_{j=1}^{N} \exp \left( \frac{\theta^2}{N^2} B_j(t)^2 \right) \leq 1 + \frac{2\theta^2}{N^2} \left( \frac{1}{N} \sum_{j=1}^{N} B_j(t)^2 \right). \tag{51}
\]

From (49) and the inequality $(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)$, we can bound
\[
B_j(t)^2 \leq 16 \left( \sum_{i=1}^{N} p_{ij}^2 \right) + \frac{16}{N^2} \sum_{i=1}^{N} \left( \int_0^t U_{N,\alpha}^N p_i(s) ds \right)^2 \tag{52}
\]

In obtaining the bound for the third and fourth terms, we have used Jensen’s inequality and the fact that $P$ is doubly sub-stochastic. Next, we average over $j$ for each term on the right hand side of (52). The first term becomes
\[
\frac{16}{N} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} p_{ij}^2 \right) \leq \frac{16}{N} \sum_{j=1}^{N} \sum_{i=1}^{N} p_{ij}^3 \leq \frac{16}{N} \|P\|_F^2.
\]

Above, the first inequality is due to Jensen’s inequality. The second term on the right hand side of (52) remains the same as it does not depend on $j$. For the third term, we use the fact that $P$ is doubly sub-stochastic to obtain
\[
\frac{16}{N} \sum_{j=1}^{N} \sum_{i=1}^{N} p_{ij} \left( \int_0^t U_{N,\alpha}^N p_i(s) ds \right)^2 \leq \frac{16}{N} \sum_{i=1}^{N} \left( \int_0^t U_{N,\alpha}^N p_i(s) ds \right)^2.
\]

The final term in (52) remains the same as it does not depend on $j$. Putting everything together, we have shown that
\[
\frac{1}{N} \sum_{j=1}^{N} B_j(t)^2 \leq \frac{16}{N} \|P\|_F^2 + \frac{16}{N^2} \sum_{i=1}^{N} \left( \int_0^t U_{N,\alpha}^N p_i(s) ds \right)^2 \leq 32 V_{N,\alpha}(t), \tag{53}
\]
where the last inequality above holds if \( N \geq 1/\epsilon \) and if \( \frac{1}{N} \| P \|_F^2 \leq \epsilon^2 \). Combining (53) and (51) shows that, for \( 0 \leq \frac{\theta}{N} \leq \frac{1}{\pi(T+1)} \),

\[
\frac{1}{N} \sum_{j=1}^{N} \exp \left( \frac{\theta^2}{N^2} B_j(t)^2 \right) \leq 1 + \frac{64\theta^2}{N^2} V^{N,\alpha}(t) \leq \exp \left( \frac{64\theta^2}{N^2} V^{N,\alpha}(t) \right). \tag{54}
\]

Substituting (54) into (50) then yields

\[
\mathbb{E} \left[ e^{\theta (V^{N,\alpha}(t+1/N) - V^{N,\alpha}(t) - \frac{2}{N^2} (\| P \|_F^2 + 1))} \mid \mathcal{F}_t \right] \leq \exp \left( \frac{4\theta^2}{N^2} \left( \frac{\| P \|_F^2 + 1}{N} \right)^2 + \frac{64\theta^2}{N^2} V^{N,\alpha}(t) \right).
\]

To simplify the right hand side further, note that if \( N \geq 1/\epsilon \) and \( \frac{1}{N} \| P \|_F^2 \leq \epsilon^2 \), then

\[
\left( \frac{\| P \|_F^2 + 1}{N} \right)^2 \leq 4\epsilon^2 \leq 4V^{N,\alpha}(t).
\]

This leads to the following subgaussian bound for \( 0 \leq \frac{\theta}{N} \leq \frac{1}{\pi(T+1)} \):

\[
\mathbb{E} \left[ e^{\theta (V^{N,\alpha}(t+1/N) - V^{N,\alpha}(t) - \frac{2}{N^2} (\| P \|_F^2 + 1))} \mid \mathcal{F}_t \right] \leq \exp \left( \frac{80\theta^2}{N^2} V^{N,\alpha}(t) \right). \tag{55}
\]

Next, we will replace \( \theta \) with \( \frac{\theta}{2\sqrt{V^{N,\alpha}(t)}} \). Recall that \( \sqrt{V^{N,\alpha}(t)} \geq \epsilon \).

Next, for every \( \theta \) satisfying \( 0 \leq \frac{\theta}{N} \leq \frac{2\epsilon}{\pi(T+1)} \),

\[
\mathbb{E} \left[ \exp \left( \theta \left( \frac{V^{N,\alpha}(t+1/N) - V^{N,\alpha}(t)}{2\sqrt{V^{N,\alpha}(t)}} - \frac{\| P \|_F^2 + 1}{\epsilon N^2} \right) \right) \mid \mathcal{F}_t \right] \leq \mathbb{E} \left[ \exp \left( \theta \left( \frac{V^{N,\alpha}(t+1/N) - V^{N,\alpha}(t)}{2\sqrt{V^{N,\alpha}(t)}} - \frac{\| P \|_F^2 + 1}{N^2 \sqrt{V^{N,\alpha}(t)}} \right) \right) \mid \mathcal{F}_t \right] \leq e^{\frac{20\theta^2}{N^2}}. \tag{56}
\]
Above, the first inequality follows from $\sqrt{V_{N,\alpha}(t)} \geq \epsilon$ and the second inequality is due to replacing $\theta$ with $\frac{\theta}{2\sqrt{V_{N,\alpha}(t)}}$ in (55). This substitution is indeed valid since

$$0 \leq \frac{\theta}{2N\sqrt{V_{N,\alpha}(t)}} \leq \frac{1}{7(T+1)} \cdot \frac{\epsilon}{\sqrt{V_{N,\alpha}(t)}} \leq \frac{1}{7(T+1)}.$$ 

With (56) in hand, we will now derive a maximal Azuma-type inequality following similar steps as the proof of Lemma 1 in Benaïm and Weibull (2003). For $t \in \mathbb{T}^N \cap [0, T)$, define the process

$$Z_\theta(t) := \exp \left( \theta \sum_{k=0}^{Nt-1} \left( \frac{V_{N,\alpha}(k+1/N) - V_{N,\alpha}(k/N)}{2\sqrt{V_{N,\alpha}(k/N)}} - \frac{\|P\|_F^2 + 1}{\epsilon N^2} \right) - 20\theta^2 t \right).$$

Equation (56) implies that $Z_\theta(t)$ is a supermartingale for $\theta$ satisfying $0 \leq \frac{\theta}{N} \leq \frac{2\epsilon}{7(T+1)}$. For such $\theta$, Doob’s supermartingale inequality implies

$$\mathbb{P} \left( \max_{t \in \mathbb{T}^N \cap [0, T)} \sum_{k=0}^{Nt-1} \left( \frac{V_{N,\alpha}(k+1/N) - V_{N,\alpha}(k/N)}{2\sqrt{V_{N,\alpha}(k/N)}} - \frac{\|P\|_F^2 + 1}{\epsilon N^2} \right) \geq \theta \epsilon \right)$$

$$\leq \mathbb{P} \left( \max_{t \in \mathbb{T}^N \cap [0, T)} Z_\theta(t) \geq \exp \left( \theta \epsilon - \frac{20\theta^2 T}{N} \right) \right) \leq \exp \left( \frac{20\theta^2 T}{N} - \theta \epsilon \right).$$

If $T \leq 1$, then we set $\theta = \frac{N\epsilon}{40}$; this is indeed a valid choice, since

$$0 \leq \frac{\theta}{N} = \frac{\epsilon}{40} \leq \frac{\epsilon}{7} \leq \frac{2\epsilon}{7(T+1)}.$$ 

Moreover, this choice of $\epsilon$ leads to a probability bound of $\exp \left( -\frac{N\epsilon^2}{80} \right)$. If $T \geq 1$, we can use $\theta = \frac{N\epsilon}{40T}$ (following identical steps, this is a valid choice of $\theta$) to obtain a probability bound of $\exp \left( -\frac{N\epsilon^2}{80T} \right)$. Combining both cases, we have shown that
\[ \mathbb{P} \left( \max_{t \in \mathbb{T} \cap [0,T]} \sum_{k=0}^{Nt-1} \left( \frac{V^{N,\alpha} \left( \frac{k+1}{N} \right) - V^{N,\alpha} \left( \frac{k}{N} \right)}{2 \sqrt{V^{N,\alpha}(k/N)}} - \frac{\|P\|_F^2 + 1}{\epsilon N^2} \right) \geq \epsilon \right) \leq e^{- \frac{N \epsilon^2}{80(T+1)}}. \]  

(57)

To simplify further, note that if \( \frac{1}{N} \|P\|_F^2 \leq \frac{\epsilon^2}{27} \) and \( N \geq 2/\epsilon \),

\[ \max_{t \in \mathbb{T} \cap [0,T]} \sum_{k=0}^{Nt-1} \frac{\|P\|_F^2 + 1}{\epsilon N^2} \leq \frac{T(\|P\|_F^2 + 1)}{\epsilon N} \leq \epsilon. \]  

(58)

Additionally, by the concavity of the square root function, we have

\[ \sum_{k=0}^{Nt-1} \frac{V^{N,\alpha} \left( \frac{k+1}{N} \right) - V^{N,\alpha} \left( \frac{k}{N} \right)}{2 \sqrt{V^{N,\alpha}(k/N)}} \geq \sum_{k=0}^{Nt-1} \left( \sqrt{V^{N,\alpha} \left( \frac{k+1}{N} \right)} - \sqrt{V^{N,\alpha} \left( \frac{k}{N} \right)} \right) = \sqrt{V^{N,\alpha}(t)} - \epsilon. \]  

(59)

Combining (57), (58) and (59), we have shown that

\[ \mathbb{P} \left( \max_{t \in \mathbb{T} \cap [0,T]} \sqrt{V^{N,\alpha}(t)} \geq 3 \epsilon \right) \leq \exp \left( - \frac{N \epsilon^2}{80(T+1)} \right), \]

which in turn implies that

\[ \mathbb{P} \left( \max_{t \in \mathbb{T} \cap [0,T]} \sqrt{V^{N,\alpha}(t)} \geq \epsilon \right) \leq \exp \left( - \frac{N \epsilon^2}{1000T} \right). \]

The desired result then follows from the concavity of the square root function (see (40) for details).

\[ \square \]

APPENDIX C: PROOF OF THEOREM 4.6

We begin by defining some notation. First, for a given vector \( v \in \mathbb{R}^N \), we let \( \tilde{v} \) be the projection of \( v \) onto the \( \text{span}(1)^\perp \); in other words, \( \tilde{v} = v - \frac{1}{N} v^T 1 \). From (13), we can compactly write

\[ y^{N,\alpha} \left( t + \frac{1}{N} \right) - y^{N,\alpha}(t) = \frac{1 - r(t)}{N} (W - I) y^{N,\alpha}(t) + \frac{r(t)}{N} g^{N,\alpha}(t), \]  

(60)
where \( y_{N,\alpha}^N(t) := [y_i^{N,\alpha}(t)]_{i=1}^N \) and \( g_{N,\alpha}^N(t) := [g_i^{N,\alpha}(t)]_{i=1}^N \), where we define

\[
g_i^{N,\alpha}(t) := \sum_{\beta \in \mathcal{A}; \beta \neq \alpha} y_i^{N,\beta}(t) \rho^{\beta \alpha}(\overline{y}_i^N(t)) - \sum_{\beta \in \mathcal{A}; \beta \neq \alpha} y_i^{N,\alpha}(t) \rho^{\alpha \beta}(\overline{y}_i^N(t)). \tag{61}
\]

The following lemma provides a useful bound for the norm of \( \tilde{g}_{N,\alpha}^N(t) \).

**Lemma C.1:**

\[
\sum_{\alpha \in \mathcal{A}} \left\| \tilde{g}_{N,\alpha}^N(t) \right\|_2^2 \leq (8m^3 + 32m^2 L\rho^2) \sum_{\alpha \in \mathcal{A}} \left\| \tilde{y}_{N,\alpha}^N(t) \right\|_2^2. \tag{62}
\]

**Proof.** For each \( i, j \in [N] \), we can write

\[
g_i^{N,\alpha}(t) - g_j^{N,\alpha}(t) = \sum_{\beta \in \mathcal{A}; \beta \neq \alpha} \left( y_i^{N,\beta}(t) \rho^{\beta \alpha}(\overline{y}_i^N(t)) - y_j^{N,\beta}(t) \rho^{\beta \alpha}(\overline{y}_j^N(t)) \right)
- \sum_{\beta \in \mathcal{A}; \beta \neq \alpha} \left( y_i^{N,\alpha}(t) \rho^{\alpha \beta}(\overline{y}_i^N(t)) - y_j^{N,\alpha}(t) \rho^{\alpha \beta}(\overline{y}_j^N(t)) \right). \tag{63}
\]

To bound the individual terms in the summation, we add and subtract terms as follows:

\[
y_i^{N,\beta}(t) \rho^{\beta \alpha}(\overline{y}_i^N(t)) - y_j^{N,\beta}(t) \rho^{\beta \alpha}(\overline{y}_j^N(t))
= (y_i^{N,\beta}(t) - y_j^{N,\beta}(t)) \rho^{\beta \alpha}(\overline{y}_i^N(t)) + y_j^{N,\beta}(t) (\rho^{\beta \alpha}(\overline{y}_i^N(t)) - \rho^{\beta \alpha}(\overline{y}_j^N(t)))
\leq \|y_i^{N,\beta}(t) - y_j^{N,\beta}(t)\| + L\rho \left\| \overline{y}_i^N(t) - \overline{y}_j^N(t) \right\|_\infty. \tag{64}
\]

To obtain the final inequality, we use the fact that \( \rho^{\beta \alpha}(\overline{y}_i^N(t)) \leq 1 \) and that \( \rho^{\beta \alpha} \) is \( L\rho \)-Lipschitz (see Assumption 2.1). Applying the triangle inequality to (63) and applying (64) to the terms in the summation yields

\[
\left| g_i^{N,\alpha}(t) - g_j^{N,\alpha}(t) \right| \leq \sum_{\beta \in \mathcal{A}; \beta \neq \alpha} \left( \left| y_i^{N,\beta}(t) - y_j^{N,\beta}(t) \right| + L\rho \left\| \overline{y}_i^N(t) - \overline{y}_j^N(t) \right\|_\infty \right)
+ \sum_{\beta \in \mathcal{A}; \beta \neq \alpha} \left( \left| y_i^{N,\alpha}(t) - y_j^{N,\alpha}(t) \right| + L\rho \left\| \overline{y}_i^N(t) - \overline{y}_j^N(t) \right\|_\infty \right)
\leq m \left\| y_i^N(t) - y_j^N(t) \right\|_1 + 2mL\rho \left\| \overline{y}_i^N(t) - \overline{y}_j^N(t) \right\|_\infty. \tag{65}
\]
We now turn to bounding the $\ell_2$-norm of $\tilde{g}^{N,\alpha}(t)$. We have

$$
|\tilde{g}_i^{N,\alpha}(t)|^2 = \left| g_i^{N,\alpha}(t) - \frac{1}{N} \sum_{j=1}^{N} g_j^{N,\alpha}(t) \right|^2 \\
\leq \frac{1}{N} \sum_{j=1}^{N} \left| g_i^{N,\alpha}(t) - g_j^{N,\alpha}(t) \right|^2 \\
\leq \frac{1}{N} \sum_{j=1}^{N} \left( m \| y_i^N(t) - y_j^N(t) \|_1 + 2mL \| y_i^N(t) - y_j^N(t) \|_\infty \right)^2 \\
\leq \frac{2}{N} \sum_{j=1}^{N} \left( m^2 \| y_i^N(t) - y_j^N(t) \|_1^2 + 4m^2L^2 \| y_i^N(t) - y_j^N(t) \|_\infty^2 \right).
$$

The first inequality is due to Jensen’s inequality, the second follows from the bound in (64), and the third is due to the inequality $(a+b)^2 \leq 2a^2 + 2b^2$. Next, using the inequalities $\|v\|_1^2 \leq m \|v\|_2^2$ and $\|v\|_\infty \leq \|v\|_1$ for $v \in \mathbb{R}^m$, we can write

$$
|\tilde{g}_i^{N,\alpha}(t)|^2 \leq \frac{2}{N} \sum_{j=1}^{N} \left( m^2 \| y_i^N(t) - y_j^N(t) \|_2^2 + 4m^2L^2 \| y_i^N(t) - y_j^N(t) \|_\infty^2 \right).
$$

(66)

Using the inequality $\|v - w\|_2^2 \leq 2 \|v\|_2^2 + 2 \|w\|_2^2$ for any vectors $v, w$ of the same dimension, (66) implies

$$
|\tilde{g}_i^{N,\alpha}(t)|^2 \leq 4m^3 \left( \| y_i^N(t) - y_{av}^N(t) \|_2^2 + \frac{1}{N} \sum_{j=1}^{N} \| y_j^N(t) - y_{av}^N(t) \|_2^2 \right) \\
+ 16m^2L^2 \left( \| \overline{y}_i^N(t) - y_{av}^N(t) \|_2^2 + \frac{1}{N} \sum_{j=1}^{N} \| \overline{y}_j^N(t) - y_{av}^N(t) \|_2^2 \right).
$$

(67)

To simplify further, note that

$$
\sum_{i=1}^{N} \| y_i^N(t) - y_{av}^N(t) \|_2^2 = \sum_{\alpha \in \mathcal{A}} \sum_{i=1}^{N} |y_i^{N,\alpha}(t) - y_{av}^{N,\alpha}(t)|^2 = \sum_{\alpha \in \mathcal{A}} |\tilde{g}^{N,\alpha}(t)|^2
$$

(68)
and
\[
\sum_{j=1}^{N} \left\| y_j^N(t) - y_{av}^N(t) \right\|_2^2 = \sum_{\alpha \in \mathcal{A}} \left\| y^{N,\alpha}(t) - y_{av}^{N,\alpha}(t) \right\|_2^2
\]
\[
= \sum_{\alpha \in \mathcal{A}} \left\| W \tilde{y}^{N,\alpha}(t) \right\|_2^2 \leq \sum_{\alpha \in \mathcal{A}} \left\| \tilde{y}^{N,\alpha}(t) \right\|_2^2. \tag{69}
\]

To conclude, we sum over \( i \in [N] \) in (67) and apply the relations derived in (68) and (69) to obtain (62).

We now turn to the proof of the theorem.

**Proof of Theorem 4.6.** Since the matrix
\[
I - \frac{1 - r(t)}{N}(I - W)
\]
is doubly stochastic, we consider only the components orthogonal to \( 1 \) in (60) to write
\[
\tilde{y}^{N,\alpha}(t + \frac{1}{N}) = \left( I - \frac{1 - r(t)}{N}(I - W) \right) \tilde{y}^{N,\alpha}(t) + \frac{r(t)}{N} \tilde{g}^{N,\alpha}(t).
\]

Taking the norm of both sides and applying the triangle inequality gives
\[
\left\| \tilde{y}^{N,\alpha}(t + \frac{1}{N}) \right\|_2 \leq \left\| \left( I - \frac{1 - r(t)}{N}(I - W) \right) \tilde{y}^{N,\alpha}(t) \right\|_2 + \frac{r(t)}{N} \left\| \tilde{g}^{N,\alpha}(t) \right\|_2. \tag{70}
\]

To bound the first term on the right hand side, we can write, for any \( x \in \mathbb{R}^N \) satisfying \( \| x \|_2 = 1 \) and \( \langle x, 1 \rangle = 0 \),
\[
\left\| \left( I - \frac{1 - r(t)}{N}(I - W) \right) x \right\|_2 = \left\| \left( 1 - \frac{1 - r(t)}{N} \right) I + \frac{1 - r(t)}{N} W \right\|_2 \| x \|_2
\]
\[
\leq 1 - \frac{1 - r(t)}{N} + \frac{1 - r(t)}{N} \| W x \|_2
\]
\[
\leq 1 - \frac{1 - r(t)}{N} + \frac{1 - r(t)}{N} \lambda = 1 - \frac{(1 - \lambda)(1 - r(t))}{N}.
\]

For the second term in (70), we can bound via Lemma C.1
\[
\left\| \tilde{g}^{N,\alpha}(t) \right\|_2 \leq \sqrt{\sum_{\alpha \in \mathcal{A}} \left\| \tilde{y}^{N,\alpha}(t) \right\|_2^2} \leq c_3 \sqrt{\sum_{\alpha \in \mathcal{A}} \left\| \tilde{g}^{N,\alpha}(t) \right\|_2^2} \leq c_3 \sum_{\alpha \in \mathcal{A}} \left\| \tilde{y}^{N,\alpha}(t) \right\|_2^2.
\]
where $c_3 = c_3(m, \rho)$. Next, define $z^N(t) := \sum_{\alpha \in A} \|\tilde{y}^{N,\alpha}(t)\|_2$. Using the bounds we have just derived in (70) and summing both sides over $\alpha \in A$, we derive the following recursive inequality.

$$z^N(t + \frac{1}{N}) \leq \left(1 - \frac{(1 - \lambda)(1 - r(t)) - mc_3r(t)}{N} \right) z^N(t) = \left(1 - \frac{(1 - \lambda) - (mc_3 + 1 - \lambda)r(t)}{N} \right) z^N(t).$$

Using the inequalities $1 + x \leq e^x$ and

$$\frac{1}{N} \sum_{i=0}^{k} r\left(\frac{i}{N}\right) \leq \frac{r(0)}{N} + \int_0^{k/N} r(s)ds \leq 2 \int_0^{k/N} r(s)ds,$$

it holds for any nonnegative integer $k$ that

$$z^N\left(\frac{k}{N}\right) \leq \exp\left(\frac{(1 - \lambda)k}{N} + 2(mc_3 + 1 - \lambda)\mathcal{R}\left(\frac{k}{N}\right)\right) z^N(0),$$

where above we have used the shorthand $\mathcal{R}(t) := \int_0^t r(s)ds$. Writing $k = NT$ for some $T \in \mathbb{T}^N$ and bounding $mc_3 + 1 - \lambda$ by $mc_3 + 1$, we can rewrite the above equation as

$$z^N(T) \leq \exp\left(-(1 - \lambda)T + 2(mc_3 + 1)\mathcal{R}(T)\right) z^N(0).$$

The first bound in the theorem statement follows from noticing that, for any $t$,

$$\sum_{\alpha \in A} \sqrt{\sum_{i=1}^{N} \left|\tilde{y}^{N,\alpha}_i(t) - \tilde{y}^{N,\alpha}_i(t)\right|^2} \leq \sum_{\alpha \in A} \left\|Wy^{N,\alpha}(t) - y^{N,\alpha}_{av}(t)\right\|_2^2 = \sum_{\alpha \in A} \left\|Wy^{N,\alpha}(t) - y^{N,\alpha}_{av}(t)\right\|_2^2 \leq \lambda \sum_{\alpha \in A} \left\|y^{N,\alpha}(t) - y^{N,\alpha}_{av}(t)\right\|_2 \leq \lambda z^N(t).$$

The second bound in the theorem statement follows from bounding $|y^{N,\alpha}_i(0) - y^{N,\alpha}_{av}(0)|$ in the first bound by 1. $\square$
Proof of Corollary 4.7. For any $t \in \mathbb{T}^N$, from (14) we can write
\[
y_N^{av}\left(t + \frac{1}{N}\right) - y_N^{av}(t) = \frac{r(t)}{N} \Lambda(y_N^{av}(t)) + \frac{r(t)}{N} e^N(t) \tag{71}
\]
where $e^N(t)$ denotes the remaining terms, given explicitly by
\[
e^N,\alpha(t) := \frac{1}{N} \sum_{i=1}^{N} \sum_{\beta \in A} y_i^{N,\beta}(t)(\rho^{\beta\alpha}(\overline{y}_i^{N}(t)) - \rho^{\beta\alpha}(y_N^{av}(t)))
- \frac{1}{N} \sum_{i=1}^{N} \sum_{\beta \in A} y_i^{N,\alpha}(t)(\rho^{\alpha\beta}(\overline{y}_i^{N}(t)) - \rho^{\alpha\beta}(y_N^{av}(t))). \tag{72}
\]
Our goal will be to bound the infinity norm of $e^N(t)$. To this end, we can bound
\[
\left| y_i^{N,\beta}(t)(\rho^{\beta\alpha}(\overline{y}_i^{N}(t)) - \rho^{\beta\alpha}(y_N^{av}(t))) \right| = \left| y_i^{N,\beta}(t) \right| \left| \rho^{\beta\alpha}(\overline{y}_i^{N}(t)) - \rho^{\beta\alpha}(y_N^{av}(t)) \right| \leq L_\rho \left\| \overline{y}_i^{N}(t) - y_N^{av}(t) \right\|_\infty, \tag{73}
\]
where the equality in the first line is due to $|ab| = |a||b|$ and the inequality on the second line is due to $|y_i^{N,\beta}(t)| \leq 1$ and the fact that $\rho^{\beta\alpha}$ is $L_\rho$-Lipschitz. Using identical reasoning, we also have
\[
\left| y_i^{N,\alpha}(t)(\rho^{\alpha\beta}(\overline{y}_i^{N}(t)) - \rho^{\alpha\beta}(y_N^{av}(t))) \right| \leq L_\rho \left\| \overline{y}_i^{N}(t) - y_N^{av}(t) \right\|_\infty. \tag{74}
\]
Applying the triangle inequality to (72) and using the bounds given in (73) and (74) shows that
\[
\left\| e^N(t) \right\|_\infty = \max_{\alpha \in A} \left| e^N,\alpha(t) \right| \leq \frac{2mL_\rho}{N} \sum_{i=1}^{N} \left\| \overline{y}_i^{N}(t) - y_N^{av}(t) \right\|_\infty. \tag{75}
\]
Squaring both sides of (75) gives
\[
\left\| e^N(t) \right\|_\infty^2 \leq 4m^2L_\rho^2 \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \overline{y}_i^{N}(t) - y_N^{av}(t) \right\|_\infty \right)^2 \leq \frac{4m^2L_\rho^2}{N} \sum_{i=1}^{N} \left\| \overline{y}_i^{N}(t) - y_N^{av}(t) \right\|_\infty^2.
\]
where the inequality on the second line is due to Jensen’s inequality and the inequality on the final line follows from bounding the maximum of a collection of non-negative real numbers by their sum.

Next, suppose that

\[
\sum_{\alpha \in A} \left( \frac{1}{N} \sum_{i=1}^{N} |y_{i}^{N,\alpha}(t) - y_{av}^{N,\alpha}(t)| \right)^{2} \leq \delta,
\]

where \( \delta \) will be determined later. By Theorem 4.6 and the subadditivity of the square root function, we have the following bound for \( 0 \leq t \leq T \):

\[
\left( \frac{1}{N} \sum_{\alpha \in A} \sum_{i=1}^{N} |y_{i}^{N,\alpha}(t) - y_{av}^{N,\alpha}(t)| \right)^{2} \leq \left( \frac{1}{N} \sum_{i=1}^{N} |y_{i}^{N,\alpha}(t) - y_{av}^{N,\alpha}(t)| \right)^{2} \leq e^{c_{3}t} \delta,
\]

where, to obtain the final inequality, we have used

\[
e^{-(1-\lambda)t+c_{3}\mathcal{R}(t)} \leq e^{c_{3}\mathcal{R}(t)} \leq e^{c_{3}t} \leq e^{c_{3}T}.
\]

Combining the display above with (75), we obtain

\[
\|e^{N}(t)\|_{\infty} \leq 2mL_{\rho}e^{c_{3}T} \delta. \tag{77}
\]

Next, let \( \hat{y}_{av}^{N}(t) \) be the linearly-interpolated continuous time version of the discrete process \( y_{av}^{N}(t) \). If \( t \in \mathbb{T}^{N} \) and \( t < s < t + 1/N \), (71) implies

\[
\frac{d}{ds} \hat{y}_{av}^{N}(s) = r(t)\Lambda(y_{av}^{N}(t)) + r(t)e^{N}(t) = r\left( \frac{\lfloorNs\rfloor}{N} \right) \Lambda(y_{av}^{N}(s)) + r\left( \frac{\lfloorNs\rfloor}{N} \right) e^{N}(s),
\]
where we have used the fact $y_N^N(s) = y_N^N(t)$ if $t \leq s < t + 1/N$, the same holding for $e_N^N(s)$, and $t = \lfloor Ns \rfloor/N$. For any $t \geq 0$ we can therefore write

$$
\dot{y}_N^N(t) - y_N^N(0) = \int_0^t \left( \frac{[Ns]}{N} \right) \Lambda(y_N^N(s)) + \frac{r(s)}{N} e_N^N(s) ds
$$

$$
= \int_0^t r(s) \Lambda(y_N^N(s)) ds + \int_0^t \left( \frac{[Ns]}{N} \right) e_N^N(s) ds
$$

$$
+ \int_0^t \left( r \left( \frac{[Ns]}{N} \right) \Lambda(y_N^N(s)) - r(s) \Lambda(\hat{y}_N^N(s)) \right) ds. \tag{78}
$$

We can also write a similar equation for the classical mean-field ODE:

$$
x(t) - x(0) = \int_0^t r(s) \Lambda(x(s)) ds. \tag{79}
$$

Noting that $y_N^N(0) = x(0)$ we subtract both sides of (79) from (78), take the infinity norm on both sides and apply the triangle inequality to obtain

$$
\left\| \dot{y}_N^N(t) - x(t) \right\|_{\infty} \leq \int_0^t r(s) \left\| \Lambda(\dot{y}_N^N(s)) - \Lambda(x(s)) \right\|_{\infty} ds + \int_0^t \left( \frac{[Ns]}{N} \right) \left\| e_N^N(t) \right\|_{\infty} ds
$$

$$
+ \int_0^t \left( \frac{[Ns]}{N} \right) \Lambda(y_N^N(s)) - r(s) \Lambda(\hat{y}_N^N(s)) \right\|_{\infty} ds. \tag{80}
$$

Since $\Lambda$ is $L_\Lambda$-Lipschitz, we can bound the first term on the right hand side of (80) by

$$
\int_0^t r(s) L_\Lambda \left\| \dot{y}_N^N(s) - x(s) \right\|_{\infty} ds \leq L_\Lambda \int_0^t \left\| \dot{y}_N^N(s) - x(s) \right\|_{\infty} ds,
$$

where we have used the inequality $r(s) \leq 1$. To bound the second term, we use (75) to obtain a bound of

$$
\int_0^t \left( \frac{[Ns]}{N} \right) 2mL_\rho e^{c_3 T} \delta \leq 2mL_\rho e^{c_3 T} \delta T,
$$

where again we have used $r \leq 1$. To bound the final term of (80), we first note that $\Lambda$ being $L_\Lambda$-Lipschitz and $r \leq 1$ implies that

$$
\left\| r \left( \frac{[Ns]}{N} \right) \left( \Lambda(y_N^N(s)) - \Lambda(\hat{y}_N^N(s)) \right) \right\|_{\infty} \leq L_\Lambda \left\| y_N^N(s) - \hat{y}_N^N(s) \right\|_{\infty} \leq \frac{L_\Lambda}{N}. \tag{81}
$$
Similarly, the fact that \( r \) is \( L_r \)-Lipschitz (see Assumption 4.1) and since \( \| \Lambda \|_{\infty} \leq 1 \) (this is readily seen from the definition of \( \Lambda \) in (3)), we have

\[
\left\| \Lambda(\hat{y}_{av}^N(s)) \left( r \left( \left\lfloor \frac{Ns}{N} \right\rfloor \right) - r(s) \right) \right\|_{\infty} \leq L_r \left\| \frac{Ns}{N} - s \right\| \leq \frac{L_r}{N}.
\]  

(82)

Together, (81) and (82) shows that the third term in (80) is at most \( (L_r + L_\Lambda)T/N \). Putting everything together, we have the following bound for any \( t \leq T \):

\[
\left\| \hat{y}_{av}^N(t) - x(t) \right\|_{\infty} \leq L_\Lambda \int_0^t \left\| \hat{y}_{av}^N(s) - x(s) \right\|_{\infty} ds + 2mL_\rho e^{c_3T}\delta T + \frac{(L_r + L_\Lambda)T}{N}.
\]

Grönwall’s inequality then implies

\[
\max_{0 \leq t \leq T} \left\| \hat{y}_{av}^N(t) - x(t) \right\|_{\infty} \leq \left( 2mL_\rho e^{c_3T}\delta + \frac{L_r + L_\Lambda}{N} \right) Te^{L_\Lambda T}.
\]

If we fix \( \epsilon > 0 \) and let

\[
\delta \leq \frac{\epsilon}{8mL_\rho Te^{(L_\Lambda+c_3)T}} \quad \text{and} \quad N \geq \frac{4(L_r + L_\Lambda)Te^{L_\Lambda T}}{\epsilon},
\]

then \( \max_{0 \leq t \leq T} \left\| \hat{y}_{av}^N(t) - x(t) \right\|_{\infty} \leq \epsilon/2 \). Furthermore, by the triangle inequality we have that

\[
\max_{0 \leq t \leq T} \left\| \hat{Y}_{av}^N(t) - x(t) \right\|_{\infty} \leq \max_{0 \leq t \leq T} \left\| \hat{Y}_{av}^N(t) - \hat{y}_{av}^N(t) \right\|_{\infty} + \max_{0 \leq t \leq T} \left\| \hat{y}_{av}^N(t) - x(t) \right\|_{\infty}.
\]

Hence

\[
\mathbb{P} \left( \max_{0 \leq t \leq T} \left\| \hat{Y}_{av}^N(t) - x(t) \right\|_{\infty} \geq \epsilon \right) \leq \mathbb{P} \left( \max_{0 \leq t \leq T} \left\| \hat{Y}_{av}^N(t) - \hat{y}_{av}^N(t) \right\|_{\infty} \geq \frac{\epsilon}{2} \right).
\]  

(83)

The desired result follows from applying Theorem 4.2 to the right hand side above. \( \square \)

**Proof of Corollary 4.8.** The details are largely the same as the proof of Corollary 4.7, with a few key differences which mostly concern bounds for \( \| e^N(t) \|_{\infty} \).

For any \( t \geq t_0 \), Theorem 4.6 and the sub-additivity of the square root function imply

\[
\sqrt{\sum_{\alpha \in A} \sum_{i=1}^N \left( y_{i,\alpha}^N(t) - y_{av}^N(t) \right)^2} \leq \sum_{\alpha \in A} \sqrt{\sum_{i=1}^N \left( y_{i,\alpha}^N(t) - y_{av}^N(t) \right)^2} \leq m\sqrt{N}e^{-\frac{1}{2}\Delta t}.
\]
Substituting the above bound into (76) gives the following bound for \( t \geq t_0 \):

\[
\| e^N(t) \|_\infty \leq 2m^2 L \rho e^{-\frac{1-\lambda}{2} t}.
\] (84)

Let \( T_c \geq t_0 \) (we will define \( T_c \) explicitly later); then we have, for any \( t \geq 0 \),

\[
\int_0^t \| e^N(T_c + s) \|_\infty \, ds \leq 2m^2 L \rho \int_{T_c}^{T_c + t} e^{-\frac{(1-\lambda)}{2} s} \, ds \leq \frac{4m^2 L \rho}{1-\lambda} e^{-\frac{(1-\lambda)}{2} T_c}.
\]

Let \( x_{T_c} \) be the solution to the ODE (16) satisfying \( x_{T_c}(T_c) = Y_{av}(T_c) \). Following the same steps as the proof of Corollary 4.7, we see that

\[
\| \hat{y}_{av}(T_c + t) - x_{T_c}(T_c + t) \|_\infty \leq \int_0^t r(T_c + s) \left\| \Lambda(\hat{y}_{av}(T_c + s)) - \Lambda(x_{T_c}(T_c + s)) \right\| ds + \int_0^t \left( r \left( \frac{N(T_c + s)}{N} \right) \right) \left\| e^N(T_c + s) \right\|_\infty ds
\]

Bounding each term on the right hand side in an identical manner as the proof of Corollary 4.7, we obtain, for \( t \in \mathbb{T}^N \) satisfying \( t \leq T \),

\[
\| \hat{y}_{av}(T_c + t) - x_{T_c}(T_c + t) \|_\infty \leq L_\Lambda \int_0^t \| \hat{y}_{av}(T_c + s) - x_{T_c}(T_c + s) \|_\infty ds + \frac{4m^2 L \rho}{1-\lambda} T e^{-\frac{(1-\lambda)}{2} T_c} + \frac{(L_r + L_\Lambda) T}{N}.
\]

Grönwall’s inequality implies

\[
\max_{0 \leq t \leq T} \| \hat{y}_{av}(T_c + t) - x_{T_c}(T_c + t) \|_\infty \leq \left( \frac{4m^2 L \rho}{1-\lambda} T e^{-\frac{(1-\lambda)}{2} T_c} + \frac{(L_r + L_\Lambda) T}{N} \right) e^{L_\Lambda T}.
\]

Set \( N \geq \frac{4(L_r + L_\Lambda) T e^{L_\Lambda T}}{\epsilon} \) and set

\[
T_c := \max \left\{ t_0, \frac{2}{1-\lambda} \left( L_\Lambda T + \log \left( \frac{16 T m^2 L \rho}{\epsilon (1-\lambda)} \right) \right) \right\}.
\]
This choice of $T_c$ implies that

$$\max_{0 \leq t \leq T} \left\| \hat{y}_{av}^N(T_c + t) - x_{T_c}(T_c + t) \right\|_{\infty} \leq \frac{\epsilon}{2}. $$

As in the proof of Corollary 4.7, the triangle inequality implies

$$\mathbb{P} \left( \max_{0 \leq t \leq T} \left\| \hat{y}_{av}^N(T_c + t) - x_{T_c}(T_c + t) \right\|_{\infty} > \epsilon \right) \leq \mathbb{P} \left( \max_{0 \leq s \leq T_c + T} \left\| \hat{y}_{av}^N(s) - \hat{y}_{av}^N(s) \right\|_{\infty} > \frac{\epsilon}{2} \right).$$

We conclude by applying Theorem 4.2 to bound the right hand side above. □

APPENDIX E: PROPERTIES OF VARIOUS NETWORK MODELS

In this section, we prove some properties of various common graph families to show the applicability of our results. We emphasize that this list is far from exhaustive.

E.1. Random regular graphs

**Proposition E.1:** Let $G_n$ be a random $d_n$-regular graph on $n$ vertices with self-loops for each vertex, and let $W_n$ be the corresponding combination matrix constructed according to the equal weights design with $a = \frac{1}{d_{\text{max}} + 1}$. Then there is a universal constant $C_{\text{reg}}$ such that

$$\lim_{n \to \infty} \mathbb{P} \left( \lambda(W_n) \leq \frac{C_{\text{reg}}}{\sqrt{d_n}} \right) = 1.$$

**Proof.** We can construct $G_n$ by sampling a uniformly random $(d_n - 1)$-regular graph $G'$ on $n$ vertices and then adding self-loops for each vertex. Let $A'$ be the adjacency matrix corresponding to $G'$, so that $A'_{ij} = 1$ if $(i, j)$ is an edge in $G'$, else $A'_{ij} = 0$. Then we can write

$$W_n = \frac{1}{d_n} \left( A' + I \right).$$

By the triangle inequality, we can write

$$\lambda(W_n) \leq \frac{1}{d_n} + \frac{1}{d_n} \lambda(A').$$
Since $A'$ is symmetric, $\lambda(A')$ is exactly the second-largest eigenvalue in magnitude of $A'$. The rich prior literature on the spectrum of random regular graphs (see for instance Tikhomirov and Youssef (2019), Cook, Goldstein, and Johnson (2018), Friedman (2008)) implies the existence of an absolute constant $C_{\text{reg}}$ such that $\lambda(A') \leq C_{\text{reg}} \sqrt{d_n}$, and the desired result follows.

\[ \square \]

**E.2. Multipartite graphs**

**Proposition E.2:** Let $(G, W)$ be a multipartite graph along with its combination matrix given by the smaller graph and combination matrix $(\tilde{G}, \tilde{W})$ on $M$ vertices, where $\tilde{G}$ is connected with self-loops, and $\tilde{W}$ is a doubly-stochastic matrix conforming to the sparsity structure of $\tilde{G}$. Furthermore assume that $N$ is a multiple of $M$. Then

$$\lambda(W) \leq \sup_{x : \|x\|_2 = 1, \langle 1, x \rangle = 0} \|\tilde{W}x\|_2 =: \lambda.$$ 

**Proof.** Denote $n := N/M$ and let $1_n \in \mathbb{R}^n$ be the $n$-dimensional (column) vector of all ones, and let $1_{n \times n} \in \mathbb{R}^{n \times n}$ be the $n \times n$ matrix of ones. We can then express the weighted adjacency matrix $W$ in block form as

$$W = \begin{pmatrix}
\frac{\tilde{w}_{11}}{n} 1_{n \times n} & \cdots & \frac{\tilde{w}_{1M}}{n} 1_{n \times n} \\
\vdots & \ddots & \vdots \\
\frac{\tilde{w}_{M1}}{n} 1_{n \times n} & \cdots & \frac{\tilde{w}_{MM}}{n} 1_{n \times n}
\end{pmatrix}$$

Let $v \in \mathbb{R}^N$ be an eigenvector of $W$ with a 2-norm of 1 and eigenvalue $\lambda(W)$ (since $\lambda(W) < 1$, this implies that $\langle 1_N, v \rangle = 0$). Partition $v$ as $v = (v^1, v^2, \ldots, v^M)$, where $v^k \in \mathbb{R}^n$ for each $k \in [M]$. Writing $Wv$ in block form shows that

$$Wv = W \begin{pmatrix}
v^1 \\
\vdots \\
v^M
\end{pmatrix} = \begin{pmatrix}
\sum_{k=1}^M \frac{\tilde{w}_{1k}}{n} 1_{n \times n} v^k \\
\vdots \\
\sum_{k=1}^M \frac{\tilde{w}_{Mk}}{n} 1_{n \times n} v^k
\end{pmatrix} = \begin{pmatrix}
\left( \sum_{k=1}^M \frac{\tilde{w}_{1k}}{n} \cdot \langle 1_n, v^k \rangle \right) 1_n \\
\vdots \\
\left( \sum_{k=1}^M \frac{\tilde{w}_{Mk}}{n} \cdot \langle 1_n, v^k \rangle \right) 1_n
\end{pmatrix}. $$

Taking the 2-norm and squaring,

\[ \|Wv\|_2^2 = \sum_{\ell=1}^{M} n \left( \sum_{k=1}^{M} \bar{w}_{\ell k} \cdot \frac{\langle 1_n, v_k \rangle}{n} \right)^2. \]

Consider the vector \( \tilde{v} \in \mathbb{R}^M \) where \( \tilde{v}_k := \frac{\langle 1_n, v_k \rangle}{n} \). Then

\[ \langle 1_M, \tilde{v} \rangle = \frac{1}{n} \sum_{k=1}^{M} \langle 1_n, v_k \rangle = \frac{1}{n} \langle 1_N, v \rangle = 0 \]

and

\[ \|\tilde{v}\|_2^2 = \frac{1}{n^2} \sum_{k=1}^{M} |\langle 1_n, v_k \rangle|^2 \leq \frac{1}{n^2} \sum_{k=1}^{M} n \cdot \|v_k\|_2^2 = \frac{1}{n} \sum_{k=1}^{M} \|v_k\|_2^2 = \frac{1}{n}. \]

Above, the first inequality is due to the Cauchy-Schwartz inequality. The last equality above follows since the \( v_k \)'s are orthogonal and \( \|v\|_2^2 = 1 \). Putting everything together,

\[ \|Wv\|_2^2 = \sum_{\ell=1}^{M} \left( \sum_{k=1}^{M} \bar{w}_{\ell k} \tilde{v}_k \right)^2 = n \|\tilde{W}\tilde{v}\|_2^2 \leq \lambda^2. \]

\[ \square \]

**E.3. Nearest-neighbor graphs**

**Proposition E.3:** Let \( G_n \) be the nearest-neighbor graph on \( n \) vertices with degree \( \gamma n \), where \( \gamma \in (0, 1) \). Further, let the combination matrix \( W_n \) be constructed according to the equal weights design with \( a = \frac{1}{\gamma n} \). Then

\[ \lim_{n \to \infty} \lambda(W_n) < 1. \]

**Proof of Proposition E.3.** As a shorthand, let \( D := \gamma n \), and assume that \( D \) is an integer. The matrix \( W_n \) is circulant, with each edge weight being \( \frac{1}{D} \). Suppose for simplicity that \( D \) is odd. It is well known (see for instance Gray (2006)) that all eigenvalues have the form

\[ \mu_k = \frac{1}{D} \omega_k^{D-1} \left( 1 + \omega_k + \omega_k^2 + \ldots + \omega_k^{D-1} \right), \quad (85) \]
where \( i \) is the imaginary unit, \( \omega_k := e^{\frac{2\pi n}{N} k} \) is an \( n \)th root of unity, and \( \mu_k \) is the eigenvalue of \( A \) corresponding to \( \omega_k \) via the above equation. Using the formula for a geometric series, we obtain the following:

\[
\mu_k = \frac{1}{D} \omega_k - \frac{D-1}{2} \frac{e^{\frac{2\pi n}{N} k D} - 1}{e^{\frac{2\pi n}{N} k} - 1}
\]

\[
= \frac{1}{D} \omega_k - \frac{D-1}{2} \frac{e^{\frac{2\pi n}{N} k D} - 1}{e^{\frac{2\pi n}{N} k} - 1}
\]

\[
= \frac{1}{D} \omega_k - \frac{D-1}{2} \frac{e^{\frac{2\pi n}{N} k D} - e^{-\frac{2\pi n}{N} k}}{e^{\frac{2\pi n}{N} k} - e^{-\frac{2\pi n}{N} k}}
\]

\[
= \frac{1}{D} \sin \left( \frac{\pi N}{N} k D \right) \sin \left( \frac{\pi N}{N} k \right)
\]

Since \( D = \gamma n \), this becomes

\[
\mu_k = \frac{1}{\gamma n} \sin (\pi k \gamma) = \frac{\sin (\pi k \gamma)}{\pi k} \cdot \frac{\pi k}{n} \sin (\pi k \gamma)
\]

\[
\Rightarrow \lim_{n \to \infty} \mu_k = \frac{\sin (\pi k \gamma)}{\pi k \gamma}
\]

By the Perron-Frobenius theorem, the only eigenvector (up to scaling) that can correspond to \( \mu_0 = 1 \) is the constant vector. The next largest eigenvalue is thus \( \mu_1 \), which, in the \( n \to \infty \) limit, is \( \frac{\sin (\pi \gamma)}{\pi \gamma} \).

\[
\square
\]

E.4. Erdős-Rényi graphs

**Proposition E.4:** Let \( G_n \sim G(n, p_n) \) with added self-loops for each vertex, and suppose that the associated combination matrix \( W_n \) is constructed according to the equal weights design with \( a = \frac{1}{d_{max} + 1} \). If \( \frac{np_n}{\log n} \to \infty \) as \( n \to \infty \), there is a universal constant
$C_{ER}$ such that
\[
\lim_{n \to \infty} \mathbb{P} \left( \lambda(W_n) \leq C_{ER} \sqrt{\frac{\log n}{(n-1)p_n}} \right) = 1.
\]

**Proof.** We first show that $G_n$ is *almost* regular using a simple application of Bernstein’s inequality. Recall that $d_i$ is the degree of vertex $i$, not counting self-loops. Since the graph has self-loops for each vertex, we have for each $i \in [n]$ that
\[
d_i = X_1 + \ldots + X_{n-1},
\]
where the $X_k$’s are independent, and $X_k \sim \text{Bernoulli}(p_n)$ for each $k = 1, \ldots, n-1$. Bernstein’s inequality then implies that for every $\delta > 0$,
\[
\mathbb{P} \left( |d_i - (n-1)p_n| > (n-1)p_n \frac{\delta}{2} \right) \leq 2 \exp \left( -\frac{3}{32} (n-1)p_n \delta^2 \right).
\]
Take a union bound over $i \in [n]$ to obtain
\[
\mathbb{P} \left( \forall i \in [n], |d_i - (n-1)p_n| \leq (n-1)p_n \frac{\delta}{2} \right) \geq 1 - 2n \cdot \exp \left( -\frac{3}{32} (n-1)p_n \delta^2 \right). \tag{86}
\]
Define the event
\[
\mathcal{E}_\delta := \{ \forall i \in [n], (1-\delta)np_n \leq d_i \leq (1+\delta)np_n \}.
\]
Next, denote $A$ to be the unweighted adjacency matrix of $G$ without self-loops; that is,
\[
A_{ij} := \begin{cases} 1 & (i,j) \in E, i \neq j \\ 0 & \text{else}. \end{cases}
\]
We can then write
\[
W = \frac{1}{d_{\max} + 1} A + I - \frac{1}{d_{\max} + 1} D,
\]
where $D$ is a diagonal matrix with $D_{ii} = d_i$. On the event $\mathcal{E}_\delta$,
\[
1 - \frac{d_i}{d_{\max} + 1} = \frac{d_{\max} - d_i + 1}{d_{\max} + 1} \leq \frac{\delta(n-1)p_n + 1}{(1-\delta)(n-1)p_n} = \frac{\delta}{1-\delta} + \frac{1}{(1-\delta)(n-1)p_n}.
\]
Moreover, by the triangle inequality, we have on the event $E_{\delta}$ that

$$
\lambda(W) \leq \frac{1}{d_{\max} + 1} \lambda(A) + \lambda \left( I - \frac{1}{d_{\max} + 1} D \right)
$$

$$
\leq \frac{1}{(1 - \delta)(n - 1)p_n} \lambda(A) + \frac{\delta}{1 - \delta} + \frac{1}{(1 - \delta)(n - 1)p_n}.
$$

If $\delta \leq 1/2$, we have the slightly simpler bound

$$
\lambda(W) \leq \frac{2}{(n - 1)p_n} \lambda(A) + 2\delta + \frac{2}{(n - 1)p_n}.
$$

Next, a result of Feige and Ofek (2005) implies the existence of a positive constant $C$ such that if $\frac{np_n}{\log n}$ is sufficiently large,

$$
\lim_{n \to \infty} \mathbb{P}(\lambda(A) \geq C \sqrt{np_n}) = 0.
$$

(87)

Set $\delta := 4\sqrt{\frac{\log n}{(n-1)p_n}}$ so that $\mathbb{P}(E_{\delta}) \to 0$ as $n \to \infty$ and let $C_{ER} := \max\{C, 9\}$. Putting everything together, we have

$$
\mathbb{P} \left( \lambda(W) \geq C_{ER} \sqrt{\frac{\log n}{(n - 1)p_n}} \right) \leq \mathbb{P}(E_{\delta}) + \mathbb{P} \left( \left\{ \lambda(W) \geq C_{ER} \sqrt{\frac{\log n}{(n - 1)p_n}} \right\} \cap E_{\delta} \right)
$$

$$
\leq \mathbb{P}(E_{\delta}) + \mathbb{P} \left( \frac{2}{(n - 1)p_n} \lambda(A) \geq (C_{ER} - 8) \sqrt{\frac{\log n}{(n - 1)p_n}} - \frac{2}{(n - 1)p_n} \right)
$$

$$
\leq \mathbb{P}(E_{\delta}) + \mathbb{P} \left( \lambda(A) \geq (C_{ER} - 8) \sqrt{\frac{\log n}{(n - 1)p_n}} - 1 \right).
$$

The right hand side above tends to 0 as $n \to \infty$ since $\mathbb{P}(E_{\delta}) \to 0$ due to our choice of $\delta$ and the second term on the right hand side tends to 0 due to (87).

\[\square\]

APPENDIX: REFERENCES

ACEMOGLU, D., K. BIMPIKIS, AND A. OZDAGLAR (2014): “Dynamics of information exchange in endogenous social networks,” *Theoretical Economics*, 9(1), 41–97.

AUER, P., N. CESAR-BIANCHI, AND P. FISCHER (2002): “Finite-time Analysis of the Multiarmed Bandit Problem,” *Machine Learning*, 47(2), 235–256.

BARREIRO-GOMEZ, J., G. OBANDO, AND N. QUIJANO (2017): “Distributed Population Dynamics: Optimization and Control Applications,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 47(2), 304–314.
BARREIRO-GÓMEZ, J., C. OCAMPO-MARTINEZ, AND N. QUIJANO (2017): “Dynamical tuning for MPC using population games: A water supply network application,” ISA Transactions, 69, 175 – 186.

BARREIRO-GÓMEZ, J., N. QUIJANO, AND C. OCAMPO-MARTINEZ (2016): “Constrained distributed optimization: A population dynamics approach,” Automatica, 69, 101 – 116.

BAUSO, D., H. TEMBINE, AND T. BAŞAR (2016): “Opinion Dynamics in Social Networks through Mean-Field Games,” SIAM Journal on Control and Optimization, 54(6), 3225–3257.

BENAIM, M., AND J. WEIBULL (2009): “Mean-field approximation of stochastic population processes in games,” working paper or preprint.

BENAIM, M., AND J. W. WEIBULL (2003): “Deterministic Approximation of Stochastic Evolution in Games,” Econometrica, 71(3), 873–903.

BERMAN, A., AND R. J. PLEMMONS (1994): Nonnegative Matrices in the Mathematical Sciences. Society for Industrial and Applied Mathematics.

BINMORE, K., AND L. SAMUELSON (1997): “Muddling Through: Noisy Equilibrium Selection,” Journal of Economic Theory, 74(2), 235 – 265.

BORG, C., J. CHAYES, L. LOVASZ, V. T. SÓS, B. SZEGEDY, AND K. VESZTERGOMBI (2006): “Graph Limits and Parameter Testing,” in Proceedings of the Thirty-Eighth Annual ACM Symposium on Theory of Computing, pp. 261–270, New York, NY, USA. Association for Computing Machinery.

BOYD, S., A. GHOSH, B. PRABHAKAR, AND D. SHAH (2006): “Randomized gossip algorithms,” IEEE Transactions on Information Theory, 52(6), 2508–2530.

BOYLAN, R. T. (1995): “Continuous Approximation of Dynamical Systems with Randomly Matched Individuals,” Journal of Economic Theory, 66(2), 615 – 625.

BRAUER, F., AND C. CASTILLO-CHAVEZ (2012): Mathematical Models in Population Biology and Epidemiology. Springer.

BUBECK, S., AND C. B. NICOLÒ (2012): Regret Analysis of Stochastic and Nonstochastic Multi-armed Bandit Problems. Now Foundations and Trends.

CHIBA, H., AND G. S. MEDVEDEV (2019): “The mean field analysis of the Kuramoto model on graphs I. The mean field equation and transition point formulas,” Discrete & Continuous Dynamical Systems - A, 39(1078-0947_2019_1_131), 131.

CHIBA, H., G. S. MEDVEDEV, AND M. S. MIZUHARA (2018): “Bifurcations in the Kuramoto model on graphs,” Chaos: An Interdisciplinary Journal of Nonlinear Science, 28(7), 073109.

COOK, N., L. GOLDSTEIN, AND T. JOHNSON (2018): “Size biased couplings and the spectral gap for random regular graphs,” Ann. Probab., 46(1), 72–125.

COPPINI, F., H. DIETERT, AND G. GIACOMIN (2020): “A law of large numbers and large deviations for interacting diffusions on Erdős-Rényi graphs,” Stochastics and Dynamics, 20(02), 2050010.

CORRADI, V., AND R. SARIN (2000): “Continuous Approximations of Stochastic Evolutionary Game Dynamics,” Journal of Economic Theory, 94(2), 163 – 191.
MEAN-FIELD APPROXIMATION FOR POPULATION PROCESSES IN NETWORKS

DEGROOT, M. H. (1974): “Reaching a consensus,” Journal of the American Statistical Association, 69(345), 118–121.

DELATTRE, S., G. GIACOMIN, AND E. A. LUÇON (2016): “A Note on Dynamical Models on Random Graphs and Fokker-Planck Equations,” J. Stat. Phys., 165, 785–798.

DEVILLE, L., AND M. GALIARDI (2017): “Finite-size effects and switching times for Moran dynamics with mutation,” Journal of Mathematical Biology, 74, 1197–1222.

DIMAKIS, A. G., S. KAR, J. M. MOURA, M. G. RABBAT, AND A. SCAGLIONE (2010): “Gossip algorithms for distributed signal processing,” Proceedings of the IEEE, 98(11), 1847–1864.

FARKHOOI, F., AND W. STANNA(T 2017): “Complete Mean-Field Theory for Dynamics of Binary Recurrent Networks,” Phys. Rev. Lett., 119, 208301.

FEIGE, U., AND E. OFEK (2005): “Spectral techniques applied to sparse random graphs,” Random Structures & Algorithms, 27(2), 251–275.

FOERSTER, M. (2019): “Dynamics of strategic information transmission in social networks,” Theoretical Economics, 14(1), 253–295.

FRIEDMAN, J. (2008): A Proof of Alon’s Second Eigenvalue Conjecture and Related Problems. American Mathematical Society, Providence, R.I.

GHARESHIFARD, B., AND J. CORTES (2012): “Distributed Strategies for Generating Weight-Balanced and Doubly Stochastic Digraphs,” European Journal of Control, 18(6), 539 – 557.

GLEESON, J., S. MELNIK, J. A. WARD, M. A. PORTER, AND P. J. MUCHA (2012): “Accuracy of Mean-Field Theory for Dynamics on Real-World Networks,” Physical review. E, Statistical, nonlinear, and soft matter physics, 85, 026106.

GRAY, R. (2006): Toeplitz and Circulant Matrices: A Review, Foundations and Trends in Technology. Now Publishers.

GUÉANT, O. (2015): “Existence and Uniqueness Result for Mean Field Games with Congestion Effect on Graphs,” Applied Mathematics & Optimization, 72(2), 291–303.

JADBABAIE, A., J. LIN, AND A. S. MORSE (2003): “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” IEEE Transactions on automatic control, 48(6), 988–1001.

KAR, S., AND J. M. F. MOURA (2011): “Global emergent behaviors in clouds of agents,” in 2011 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pp. 5796–5799.

KURTZ, T. G. (1970): “Solutions of Ordinary Differential Equations as Limits of Pure Jump Markov Processes,” Journal of Applied Probability, 7(1), 49–58.

——— (1976): Limit theorems and diffusion approximations for density dependent Markov chainspp. 67–78. Springer Berlin Heidelberg, Berlin, Heidelberg.

LACKER, D., K. RAMANAN, AND R. WU (2019): “Large sparse networks of interacting diffusions,” arXiv e-prints, p. arXiv:1904.02585.
LEVINE, D. K., AND S. MODICA (2016): “Dynamics in stochastic evolutionary models,” *Theoretical Economics*, 11(1), 89–131.

MEDVEDEV, G. S. (2019): “The continuum limit of the Kuramoto model on sparse random graphs,” *Communications in Mathematical Sciences*, 17(4), 883 – 898.

NEDIĆ, A., A. OLSHEVSKY, AND M. G. RABBAT (2018): “Network Topology and Communication-Computation Tradeoffs in Decentralized Optimization,” *Proceedings of the IEEE*, 106(5), 953–976.

OLIVEIRA, R. I., G. H. REIS, AND L. M. STOLERMAN (2020): “Interacting diffusions on sparse graphs: hydrodynamics from local weak limits,” *Electron. J. Probab.*, 25, 35 pp.

OLIVERA, R. I., AND G. H. REIS (2019): “Interacting Diffusions on Random Graphs with Diverging Average Degrees: Hydrodynamics and Large Deviations.,” *J. Stat. Phys.*, 176, 1057?1087.

QUIJANO, N., C. OCAMPO-MARTINEZ, J. BARREIRO-GÓMEZ, G. OBANDO, A. PANTOJA, AND E. MOJICA-NAVA (2017): “The Role of Population Games and Evolutionary Dynamics in Distributed Control Systems: The Advantages of Evolutionary Game Theory,” *IEEE Control Systems*, 37(1), 70–97.

SANDHOLM, W. (2010): *Population Games and Evolutionary Dynamics*, Economic Learning and Social Evolution. MIT Press.

SANDHOLM, W. H. (2003): “Evolution and equilibrium under inexact information,” *Games and Economic Behavior*, 44(2), 343 – 378.

SANDHOLM, W. H., AND M. STAUDIGL (2018): “Sample Path Large Deviations for Stochastic Evolutionary Game Dynamics,” *Math. Oper. Res.*, 43(4), 1348?1377.

SANTOS, A., J. M. F. MOURA, AND J. M. F. XAVIER (2013): “Emergent Behavior in Multipartite Large Networks: Multi-virus Epidemics,” *CoRR*, abs/1306.6198.

SANTOS, A. A., S. KAR, J. M. F. MOURA, AND J. XAVIER (2016): “Thermodynamic limit of interacting particle systems over dynamical networks,” in *2016 50th Asilomar Conference on Signals, Systems and Computers*, pp. 997–1000.

SAWA, R. (2016): “Stochastic Stability in the Large Population and Small Mutation Limits for General Normal Form Games,” SSRN.

SAYED, A. H. (2014): “Adaptation, Learning, and Optimization over Networks,” *Foundations and Trends in Machine Learning*, 7(4-5), 311–801.

SINKHORN, R. (1964): “A Relationship Between Arbitrary Positive Matrices and Doubly Stochastic Matrices,” *Ann. Math. Statist.*, 35(2), 876–879.

SMITH, J. M. (1982): *Evolution and the Theory of Games*. Cambridge University Press.

SUTTON, R. S., AND A. G. BARTO (1998): “Reinforcement Learning: An Introduction,” *Trans. Neur. Netw.*, 9(5), 1054–1054.

SWENSON, B., S. KAR, AND J. XAVIER (2015): “Empirical centroid fictitious play: An approach for distributed learning in multi-agent games,” *IEEE Transactions on Signal Processing*, 63(15), 3888–3901.
TAN, S., J. LÜ, AND Z. LIN (2016): “Emerging Behavioral Consensus of Evolutionary Dynamics on Complex Networks,” SIAM Journal on Control and Optimization, 54(6), 3258–3272.

TAYLOR, P. D., AND L. B. JONKER (1978): “Evolutionary stable strategies and game dynamics,” Mathematical Biosciences, 40(1), 145 – 156.

TIKHOMIROV, K., AND P. YOUSSEF (2019): “The spectral gap of dense random regular graphs,” Ann. Probab., 47(1), 362–419.

TSITSIKLIS, J., D. BERTSEKAS, AND M. ATHANS (1986): “Distributed asynchronous deterministic and stochastic gradient optimization algorithms,” IEEE transactions on automatic control, 31(9), 803–812.

VERMOREL, J., AND M. MOHRI (2005): “Multi-armed Bandit Algorithms and Empirical Evaluation,” in Machine Learning: ECML 2005, ed. by J. Gama, R. Camacho, P. B. Brazdil, A. M. Jorge, and L. Torgo, pp. 437–448, Berlin, Heidelberg. Springer Berlin Heidelberg.