Cosmological dynamics of brane f(R) gravity

Zahra Haghani *, Hamid Reza Sepangi † and Shahab Shahidi ‡

Department of Physics, Shahid Beheshti University, G. C., Evin, Tehran 19839, Iran

ABSTRACT: The cosmological dynamics of a brane world scenario where the bulk action is taken as a generic function of the Ricci scalar is considered in a framework where the use of the $\mathbb{Z}_2$ symmetry and Israel junction conditions are relaxed. The corresponding cosmological solutions for some specific forms of $f(R)$ are obtained and shown to be in the form of exponential as well as power law for a vacuum brane space-time. It is shown that the existence of matter dominated epoch for a bulk action in the form of a power law for $R$ can only be obtained in the presence of ordinary matter. Using phase space analysis, we show that the universe must start from an unstable matter dominated epoch and eventually falls into a stable accelerated expanding phase.
1. Introduction

In 1999, Randall and Sundrum (RS) [1] proposed a model in which our 4D universe, the brane, is embedded in an AdS\(_5\) bulk space. All gauge fields are confined to the brane while gravity can propagate into the extra dimension and thus into the bulk space. As is now well-known, such a proposal opened a new window through which a fresh look at the universe has become a possibility. One outcome of the RS model was that it offered a rational explanation to the question of hierarchy; the enormous disparity between the fundamental forces. Since its appearance, numerous scenarios based on the RS setup have been proposed, each dealing with some particular aspects of the structure and evolution of the universe. An immediate issue after the introduction of the RS models was the question of what the form of the field equations on the brane is, knowing their form in the bulk space. An answer came along in the elegant work of Shirumizu, Maeda and Sasaki (SMS) [2] where they showed how to, as it were, project the field equations in the bulk onto the brane in a covariant manner. The subsequent avalanche of research works have been somewhat overwhelming. Numerous papers have appeared and tried to remedy the shortcomings of, generally speaking, the standard model of gravity, including the late-time acceleration of the universe, the galaxy rotation curves, the virial mass discrepancy, etc [3]. For a review on the brane world models see [4].

In the SMS method, the bulk geometry is projected onto the brane using the Gauss-Coddazi equations and Israel junction conditions [5]. By means of the Gauss-Coddazi equations, the intrinsic curvature of the bulk is related to the intrinsic and extrinsic curvatures of the brane. The Israel junction conditions then relate the extrinsic curvature of the brane to the energy-momentum content of the model. In this method, the brane equations of motion reflect the global geometry of the bulk space-time through the electric part of the Weyl tensor. However, there are situations where the use of the Israel junction conditions become somewhat restrictive. In cases where more than one extra dimension is involved their use is not well understood. Also, there are certain matters that are not compatible with these junction conditions [6]. In the latter, one cannot relate the extrinsic curvature to the matter content of the brane. Such considerations motivated the idea presented in
and further developed in [8], where the extrinsic curvature from the Gauss-Codazzi equations is calculated geometrically and the assumption of $\mathbb{Z}_2$ symmetry is relaxed. However, the effects of the global structure of the bulk space is retained by the field equations. This model has been largely investigated in the literature [9].

Somewhat parallel to the brane-world development, $f(R)$ theories of gravity appeared on the scene to deal with some of the shortcomings of the cosmological implications of the standard general relativity. In such theories, one modifies the Einstein-Hilbert action by replacing the Ricci scalar $R$ with a generic function $f(R)$. It is now well-known that such theories are equivalent to scalar-tensor theories. In fact, both the metric and Palatini formulations of $f(R)$ gravity can be rewritten in terms of certain versions of the Brans-Dicke theory [10]. These theories with many variations have been successful, for example, in describing galaxy rotation curves which is generally believed to be the result of the existence of dark matter. For a review on $f(R)$ gravities see [11].

Within the context of a brane-world scenario, $f(R)$ theories have been studied recently in [12] where the authors follow the SMS procedure to project the bulk field equations onto the brane hypersurface. The field equations thus obtained show that the matter content of the brane-world is not coupled to the bulk $f(R)$ term. In this paper, we generalize the work [12] to a theory where the use of $\mathbb{Z}_2$ symmetry and the Israel junction condition is relaxed. As it turns out, since the junction conditions are not used, the field equations on the brane contain a new tensor $Q_{\mu\nu}$ which is made out of the extrinsic curvature and should be calculated through the Gauss-Codazzi equations. The effect of the non-trivial bulk action is reflected in a new tensor which is made of the bulk $f(R)$ function. One must only project the scalar $f(R)$ in an obvious manner onto the brane in order to calculate this tensor. The resulting field equations show that the matter content of the brane is non-minimally coupled to the bulk $f(R)$. Therefore, in contrast to the work presented in [12], the non-trivial bulk action can influence the brane energy-momentum tensor directly.

Recent observations provided by the supernovae legacy survey data [13] shows that, taking $\omega_{DE} = \text{const.}$, the equation of the state parameter is found to be $\omega_{DE} = -1.04 \pm 0.06$. This data shows that the de-Sitter accelerating phase is a very good candidate for the late time acceleration of our universe. In this sense, one may expect that a good cosmological model should predict some matter dominated epoch followed by an accelerated expanding phase. In this paper we consider this issue for the brane world model with general bulk action using the phase space analysis. The method of phase space analysis for cosmological models is very well known and is used for many alternative theories [14, 15, 16, 17]. As it turns out, the model can explain both the matter dominated and accelerated expanding phases, resulting in a universe which starts from an unstable matter dominated phase and eventually falls into the accelerated expanding phase.

The paper is organized as follows: In the next section we construct the model from a general bulk action in $d$ dimensions, relaxing the use of Israel junction condition and the $\mathbb{Z}_2$ symmetry. The model is then restricted to a $5D$ bulk and cosmological solutions are obtained in section 3. In section 4 a dynamical system analysis is presented which shows that the model can in principle explain a well founded cosmological evolution. Conclusions and final remarks are drawn in the last section.

2. The model

We consider a pseudo-Riemannian manifold $V_m$ in which the background manifold $\bar{V}_4$ is isometrically embedded by the map $Y: \bar{V}_4 \to V_m$ such that

$$G_{AB}Y_{A\mu}Y_{B\nu} = \tilde{g}_{\mu\nu}, \quad G_{AB}Y_{A\mu}N^B_a = 0, \quad G_{AB}N^A_aN^B_b = g_{ab} = \pm 1.$$  \hfill (2.1)

where $G_{AB}(\tilde{g}_{\mu\nu})$ is the bulk (brane) metric, $\{Y^A\}(\{x^\mu\})$ are bulk (brane) coordinates and $N^A_a$ are $(m - 4)$ unit vectors, orthogonal to the brane. In order to investigate the effects of the general bulk
geometry, we assume a non-trivial bulk action of the form
\[ I = \frac{1}{2 \kappa_m^2} \int d^m x \sqrt{-G} \left[ f(\mathcal{R}) - 2\Lambda \right] + I_M, \]  
(2.2)
where
\[ I_M = \int d^m x \sqrt{-\mathcal{G}} \mathcal{L}_M, \]  
(2.3)
and \( \mathcal{L}_M \) is the matter Lagrangian and \( \mathcal{R} \) is the bulk Ricci scalar. After variation of \( I \) with respect to the bulk metric, \( G_{AB} \), we obtain
\[ f'(\mathcal{R}) R_{AB} - \frac{1}{2} G_{AB} f(\mathcal{R}) + \nabla_A \nabla_B f'(\mathcal{R}) + \Lambda G_{AB} = \kappa_m^2 T_{AB}. \]  
(2.4)
where prime represents derivative with respect to the argument. Rearranging the above equation, we obtain the effective Einstein field equations in the bulk
\[ G_{AB} \equiv R_{AB} - \frac{1}{2} \mathcal{R} G_{AB} = S_{AB}, \]  
(2.5)
where
\[ S_{AB} = \frac{1}{f'(\mathcal{R})} \left[ \kappa_m^2 T_{AB} - \Lambda G_{AB} - \left( \frac{1}{2} \mathcal{R} f'(\mathcal{R}) - \frac{1}{2} f'(\mathcal{R}) + \Box f'(\mathcal{R}) \right) G_{AB} + \nabla_A \nabla_B f'(\mathcal{R}) \right]. \]  
(2.6)
As was mentioned in the Introduction, we use the method introduced in [7] to project the bulk geometry onto the brane. In this sense, perturbations of \( \bar{V}_4 \) in a small neighborhood of the brane along a generic transverse direction \( \xi = \xi^a N_a \) \( (a = 1, 2, ..., m - 4) \) orthogonal to the brane are considered as follows
\[ Z^A(x^{\mu}, \xi^a) = Y^A + (\mathcal{L}_\xi Y)^A, \]  
(2.7)
where \( \mathcal{L} \) represents the Lie derivative and \( \xi^a \) denotes a small parameter along \( N^A_a \) that parameterizes the extra noncompact dimensions. The presence of tangent components of the vector \( \xi \) along the submanifold \( \bar{V}_4 \) can cause some difficulties because it can induce some undesirable coordinate gauges, but, as is shown in the theory of geometric perturbations, it is possible to choose this vector to be orthogonal to the background [18].

Let us now consider the perturbation of the embedding map along the orthogonal extra dimension \( N_a \), giving local coordinates of the perturbed brane as
\[ Z^A_{\mu}(x^{\mu}, \xi^a) = Y^A_{\mu} + \xi^a N^A_{a,\mu}(x^{\nu}). \]  
(2.8)
Using the above assumptions, the embedding equations of the perturbed geometry are given by
\[ \mathcal{G}_{\mu\nu} = \mathcal{G}_{AB} Z^A_{\mu} Z^B_{\nu}, \quad \mathcal{G}_{\mu a} = \mathcal{G}_{AB} Z^A_{\mu a} N^B_b, \quad \mathcal{G}_{AB} N^A_a N^B_b = g_{ab}. \]  
(2.9)
Now, use of the embedding equations and the relation
\[ g^{AB} = Y^A_{\mu} Y^B_{\nu} g^{\mu\nu} + N^A_a N^B_b g^{ab}, \]  
(2.10)
allow us to write the metric of the bulk space in the following form
\[ \mathcal{G}_{AB} = \left( g_{\mu\nu} + A_{\mu c} A^c_{\nu a} \right) N_{a b}, \]  
(2.11)
where
\[
g_{\mu\nu} = \bar{g}_{\mu\nu} - 2 \xi^a \bar{K}_{\mu\nu} + \xi^a \xi^b \bar{K}_{\mu\alpha} \bar{K}_{\nu\beta},
\]
\[(2.12)\]
is the metric of the perturbed brane and \(\bar{K}_{\mu\nu}\) is the extrinsic curvature of the original brane defined as
\[
\bar{K}_{\mu\nu} = -\mathcal{G}_A \chi^{A}_{,\mu} \chi^{B}_{,\nu}.
\]
\[(2.13)\]
We also use the notation \(A_{\mu c} = \xi^d A_{\mu c d}\)
\[(2.14)\]
represent the twisting vector fields. The presence of gauge fields \(A_{\mu a}\) tilts the embedded family of sub-manifolds with respect to the normal vector \(N^A\). According to our construction, the original brane is orthogonal to the normal vectors \(N^A\). However, from equation (2.9), it can be seen that this is not true for the deformed geometry. Let us introduce
\[
X^A_{,\mu} = Z^A_{,\mu} - g_{ab} N^A_{,\mu} A_{\mu b}.
\]
\[(2.15)\]
One can easily verify that for the set \(\{X^A_{,\mu}, N^A_{,\mu}\}\) we have the following projection relations
\[
\mathcal{G}_{AB} X^A_{,\mu} X^B_{,\nu} = g_{\mu\nu}, \quad \mathcal{G}_{AB} X^A_{,\mu} N^B_{,\mu} = 0, \quad \mathcal{G}_{AB} N^A_{,\mu} N^B_{,\nu} = g_{ab} = \pm 1.
\]
\[(2.16)\]
These define a new family of embedded manifolds whose members are always orthogonal to \(N^A\). This new embedding of the local coordinates can be suitably used for obtaining induced Einstein field equations on the brane. The extrinsic curvature of a perturbed brane in the coordinates \(\{X^A_{,\mu}, N^A_{,\mu}\}\) becomes
\[
K_{\mu\nu} = -\mathcal{G}_{AB} X^A_{,\mu} X^B_{,\nu} = \bar{K}_{\mu\nu} - \xi^a \bar{K}_{\mu\gamma} \bar{K}_{\nu\delta} \bar{g}^\gamma^\delta
\]
\[
= -\frac{1}{2} \partial g_{\mu\nu}.
\]
\[(2.17)\]
which is the generalized York relation. In the basis \(\{X^A_{,\mu}, N^A_{,\mu}\}\), the Gauss, Codazzi and Ricci equations are given by
\[
R^a_{\alpha\beta\gamma\delta} = 2 g^{ab} K_{[\alpha a} K_{\beta]b} + \mathcal{R}_{ABCD} X^A_{,\alpha} X^B_{,\beta} X^C_{,\gamma} X^D_{,\delta},
\]
\[(2.18)\]
\[
2K_{[\alpha|\gamma c] = 2 g^{ab} A_{[\gamma a c} K_{\delta]b} + \mathcal{R}_{ABCD} X^A_{,\alpha} N^B_{,\beta} X^C_{,\gamma} X^D_{,\delta},
\]
\[(2.19)\]
and
\[
2A_{[\mu a b,\nu]} = -2 g^{mn} A_{[\mu m a} A_{\nu n b} - g^{\sigma\rho} K_{[\mu\sigma a} K_{\nu\rho b} - \mathcal{R}_{ABCD} X^C_{,\mu} X^D_{,\nu} N^A_{,a} N^B_{,b},
\]
\[(2.20)\]
where \(\mathcal{R}_{ABCD}\) and \(R_{\alpha\beta\gamma\delta}\) are the Riemann tensors for the bulk and the brane respectively. After contracting equation (2.18) and using the relation
\[
\mathcal{G}^{AB} = X^A_{,\mu} X^B_{,\nu} g^{\mu\nu} + N^A_{,a} N^B_{,b} g^{ab},
\]
\[(2.21)\]
one obtains
\[
R_{\mu\nu} = (K_{\mu c} K^{\nu c} - K_{\mu a} K^{\nu a}) + \mathcal{R}_{AB} X^A_{,\mu} X^B_{,\nu} - g^{ab} \mathcal{R}_{ABCD} X^A_{,\mu} X^B_{,\nu} X^C_{,\rho} X^D_{,\delta},
\]
\[(2.22)\]
So, the Einstein tensor on the brane is in the following form

\[ G_{\mu\nu} = G_{AB} \chi^A_{\mu} \chi^B_{\nu} + Q_{\mu\nu} + g^{ab} R_{AB} N_a^A N_b^B g_{\mu\nu} - g^{ab} R_{ABCD} N_a^A N_b^B N_c^C N_d^D - \frac{1}{2} C g_{\mu\nu}, \quad (2.23) \]

where

\[ Q_{\mu\nu} = -g^{ab} (K_{a\mu} K_{b\nu} - K_a K_{\mu\nu}) + \frac{1}{2} (K_{a\beta a} K^a_{\mu\nu} - K_a K^a_{\mu\nu}) g_{\mu\nu}. \quad (2.24) \]

and

\[ C = g^{ab} g^{cd} R_{ABCD} N_a^A N_b^B N_c^C N_d^D. \quad (2.25) \]

Using the decomposition of the Riemann tensor into the Ricci tensor, Ricci scalar and Weyl tensor

\[ R_{ABCD} = C_{ABCD} + \frac{2}{(m-2)} (g_{B[D} R_{C]A} - g_{A[D} R_{C]B}) + \frac{2}{(m-1)(m-2)} R (g_{A[D} g_{C]B}), \quad (2.26) \]

one obtains the generalized Einstein equations on the brane as

\[ G_{\mu\nu} = G_{AB} \chi^A_{\mu} \chi^B_{\nu} - \varepsilon_{\mu\nu} + Q_{\mu\nu} - \frac{1}{2} C g_{\mu\nu} + \frac{(m-3)}{(m-2)} g^{ab} R_{AB} N_a^A N_b^B g_{\mu\nu} \]
\[ - \frac{(m-4)}{(m-2)} R_{AB} \chi^A_{\mu} \chi^B_{\nu} + \frac{(m-4)}{(m-1)(m-2)} R g_{\mu\nu}, \quad (2.27) \]

where

\[ \varepsilon_{\mu\nu} = g^{ab} C_{ABCD} N_a^A N_b^B \chi^C_{\mu} \chi^D_{\nu}, \quad (2.28) \]

is the electric part of the Weyl tensor \( C_{ABCD} \).

In order to obtain the field equations corresponding to action (2.2), we follow Dvali and Shifman [19] which have proposed a mechanism to localize the standard model gauge fields to the brane. Using this idea, we may decompose the components of the energy-momentum tensor of the matter as

\[ \kappa^2 \tau_{\mu\nu} = \frac{2\kappa^2}{m-2} T_{\mu\nu}, \quad T_{ab} = 0, \quad T_{\mu a} = 0. \quad (2.29) \]

However, the geometric part of the generalized energy-momentum tensor includes the bulk-bulk and the bulk-brane components, and one must take this into account when writing the Einstein field equations. We thus define the components of the total energy-momentum tensor in the basis \( \{ \chi^A_{\mu}, N_a^A \} \) as

\[ S_{\mu\nu} = S_{AB} \chi^A_{\mu} \chi^B_{\nu}, \quad S_{\mu a} = S_{AB} \chi^A_{\mu} N_a^B, \quad S_{ab} = S_{AB} N_a^A N_b^B. \quad (2.30) \]

After contracting equation (2.5), one obtains

\[ R = -\frac{2}{m-2} S, \quad (2.31) \]

where \( S = S_A^A \) is the trace of the generalized bulk energy-momentum tensor. The \( R_{AB} \) is then given by

\[ R_{AB} = -\frac{1}{m-2} g_{AB} S + S_{AB}. \quad (2.32) \]

Substituting \( R_{AB} \) and \( R \) in equation (2.27) and using equation (2.30), we find
\[ G_{\mu\nu} = Q_{\mu\nu} - \mathcal{E}_{\mu\nu} - \frac{1}{2} \mathcal{C} g_{\mu\nu} + \frac{2}{m-2} S_{\mu\nu} + \frac{m-3}{m-2} g^{ab} S_{ab} g_{\mu\nu} - \frac{(m-4)(m-3)}{(m-2)(m-1)} g_{\mu\nu} S. \] (2.33)

Finally, using equation (2.29), we obtain the effective Einstein equations on the brane as

\[ G_{\mu\nu} - Q_{\mu\nu} + \mathcal{E}_{\mu\nu} - \frac{\kappa^2}{f'(\mathcal{R})} \mathcal{T}_{\mu\nu} + \Pi_{\mu\nu} + \frac{1}{2} \mathcal{C} g_{\mu\nu} = 0, \] (2.34)

where \( Q_{\mu\nu} \) is defined by equation (2.24) and

\[ \Pi_{\mu\nu} = W g_{\mu\nu} - \frac{2}{m-2} \frac{\nabla A \nabla B f'(\mathcal{R})}{f'(\mathcal{R})} \chi^A_{\mu} \chi^B_{\nu} \]
\[ = W g_{\mu\nu} - \frac{2}{m-2} \frac{\nabla \mu \nabla \nu f'(\mathcal{R})}{f'(\mathcal{R})}, \] (2.35)

is a new tensor that reflects the effects of the non-trivial bulk action on the brane. In the second line of the above equation we have assumed \( \chi^A_{\mu} = \delta^A_\mu \), so that \( \nabla_\mu \) is the \( \mu = 0, 1, 2, 3 \) components of the \( m \) dimensional covariant derivative. We also define

\[ W = \left( \frac{m^2 - 5m + 10}{2m^2 - 6m + 4} \right) \mathcal{R} + \left( \frac{m^2 - 7m + 14}{2(m-2)} \right) \left( \frac{2\Lambda - f}{f'} \right) \]
\[ + \left( \frac{m^2 - 8m + 17}{m-2} \right) \left( \frac{m\Box f'}{f'} + \frac{m - 3 \nabla A \nabla B f'}{m-2} \right) \chi^A_{\mu} \chi^B_{\nu} g_{\mu\nu}. \] (2.36)

These tensors must be evaluated on the brane. We also note that in the limit \( f(\mathcal{R}) = \mathcal{R} \) the tensor \( \Pi_{\mu\nu} \) can be converted to the trace of brane energy-momentum tensor, and the field equation (2.34) reduces to [7].

Let us focus our attention on the five dimensional bulk. In the case of a co-dimension one brane we have \( C = 0 \) due to the symmetries of the Riemann tensor. We also consider the case where \( \chi^A_{\mu} = \delta^A_\mu \). In this case, equation (2.21) reduces to

\[ G_{AB} = g_{AB} + N_A N_B, \] (2.37)

and the field equations (2.34) become

\[ G_{\mu\nu} - Q_{\mu\nu} + \mathcal{E}_{\mu\nu} - \frac{\kappa^2}{f'(\mathcal{R})} \mathcal{T}_{\mu\nu} + \Pi_{\mu\nu} = 0, \] (2.38)

where \( \kappa^2 = \frac{2}{3} \kappa_5^2 \), and

\[ \Pi_{\mu\nu} = W g_{\mu\nu} - L_{\mu\nu}, \] (2.39)

\[ W = \frac{5}{12} \mathcal{R} + L + \frac{2}{3f'} \left[ 5 \Box f' - f - 2\Lambda \right], \] (2.40)

\[ L_{\mu\nu} = \frac{2}{3} \frac{\nabla \mu \nabla \nu f'}{f'}, \quad L = L'_{\mu}, \] (2.41)

with

\[ Q_{\mu\nu} = (K K_{\mu\nu} - K_{\mu\alpha} K'_{\nu}) + \frac{1}{2} (K_{\mu\beta} K'^{\alpha\beta} - K^2) g_{\mu\nu}, \] (2.42)
where $K = g^{\mu \nu} K_{\mu \nu}$. In the case of a constant curvature bulk, the tensor $Q_{\mu \nu}$ is an independently conserved quantity as can be seen easily from the Codazzi equation (2.13). However, we are interested in a general bulk geometry, so we consider $Q_{\mu \nu}$ as a general tensor which reflects the bulk effects on the brane equations of motion. By defining the new tensor

$$M_{\mu \nu} = \frac{\kappa_4^2}{f(R)} \tau_{\mu \nu} + Q_{\mu \nu} - E_{\mu \nu} - \Pi_{\mu \nu}, \quad (2.43)$$

and using the Bianchi identity, we find that $M_{\mu \nu}$ is conserved. It is therefore possible to consider $M_{\mu \nu}$ as a new effective energy-momentum tensor made of the standard conserved matter energy-momentum tensor and terms reflecting the effects of the extra dimension and also the non-trivial bulk action. The field equation on the brane can now be written as

$$G_{\mu \nu} = M_{\mu \nu}. \quad (2.44)$$

3. Cosmological Solutions

To study the time evolution of the universe, we take the bulk metric as

$$ds_B^2 = -dt^2 + a(t, w)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) + dw^2. \quad (3.1)$$

Taking $\mathcal{N}^A = \delta^A_5$ and using the equation (2.37) the brane metric reduces to the usual FRW form

$$ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \quad (3.2)$$

With this choice, the electric part of the Weyl tensor becomes

$$\mathcal{E}^0_0 = \frac{1}{2a^2} \left[ -\dot{a} \ddot{a} + \dot{a}^2 + k + \dot{a} \ddot{a} - \ddot{a}^2 \right] = -3\mathcal{E}^i_i, \quad i = 1, 2, 3, \quad (3.3)$$

where dot represents derivative with respect to $t$ and a hat over an arbitrary function $f(r, w)$ is defined as $\hat{f} = \frac{\partial f}{\partial w} |_{w=0}$. We can also calculate the extrinsic curvature and $Q_{\mu \nu}$ by use of equations (2.17) and (2.42) respectively, with the result

$$K^\mu_\nu = -\left( \frac{\dot{a}}{a} \right) \text{diag} (0, 1, 1, 1), \quad (3.4)$$

and

$$Q^\mu_\nu = -\left( \frac{\dot{a}}{a} \right)^2 \text{diag} (3, 1, 1, 1). \quad (3.5)$$

By assuming that the matter on the brane has the perfect fluid form

$$\tau^\mu_\nu = \text{diag} (-\rho, p, p, p), \quad (3.6)$$

the conserved tensor $M_{\mu \nu}$ can be written as

$$M^\mu_\nu = \text{diag} (-\rho_A, p_A, p_A, p_A), \quad (3.7)$$

which, using equations (2.39), (2.43), (3.3) and (3.6), result in an effective energy-density and pressure on the brane

$$\rho_A = \frac{\kappa_4^2}{f} \rho + 3 \left( \frac{\dot{a}}{a} \right)^2 + \mathcal{E}^0_0 + W - L^0_0, \quad (3.8)$$

$$p_A = \frac{\kappa_4^2}{f} p - \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{3} \mathcal{E}^0_0 - W + L^1_1. \quad (3.8)$$
Now, with the aid of equation (2.41), $L_{\mu\nu}$ becomes

$$L^0_0 = -\frac{2}{3} \dot{f}_i,$$

$$L^i_i = -\frac{2}{3} \frac{1}{af} \left( \dot{f} \dot{a} - \dot{f} a \right), \quad i = 1, 2, 3. \quad (3.9)$$

Using equation (2.44), the Friedmann and the Raychaudhuri equations become

$$H^2 = \frac{1}{3} \left( \kappa^2 \rho + 3 \left( \frac{\dot{a}}{a} \right)^2 + E_0^0 + W + \frac{2}{3} \dot{f} \right) + \frac{3k}{a^2},$$

$$\ddot{a} = -\frac{1}{6} \left[ \kappa^2 f (\rho + 3p) + 2 (E_0^0 - W) - \frac{1}{3} \frac{1}{af} \left( 3 \dot{f} \dot{a} - 3 \dot{f} a - \ddot{f} a \right) \right]. \quad (3.10)$$

### 3.1 The case $f(R) = f_0 e^{\alpha R}$

Let us consider first the case $f(R) = f_0 e^{\alpha R}$ where $f_0$ and $\alpha$ are some constants. Assuming $\tau_{\mu\nu} = 0$ and $\Lambda = 0$, and using the trace of the field equation (2.44)

$$L = \frac{1}{3} \left( R - \mathcal{R} + Q + \frac{1}{\alpha} \right), \quad (3.11)$$

where $R$ is the brane Ricci scalar, $L_{\mu\nu}$ and $W$ take the following forms

$$L_{\mu\nu} = \frac{2}{3} \left( \alpha^2 \nabla_{\mu} \mathcal{R} \nabla_{\nu} \mathcal{R} + \alpha \nabla_{\mu} \nabla_{\nu} \mathcal{R} \right),$$

$$W = \frac{1}{12} \left[ \frac{1}{\alpha} - \mathcal{R} + 4(R + Q) \right]. \quad (3.12)$$

With these assumptions, equation (3.13) reduces to

$$W = \frac{3}{2} \left[ \frac{\dot{a}}{a} + H^2 - \left( \frac{\dot{a}}{a} \right)^2 \right] - \frac{1}{3} \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{12a^2}. \quad (3.14)$$

where we have assumed $k = 0$. The solutions of equations (3.10) are of an inflationary form, given by

$$a(t) = c_2 e^{c_1 t},$$

$$\dot{a}(t) = \beta a(t), \quad \ddot{a}(t) = -\left( \beta^2 + \frac{1}{12a} \right) a(t), \quad (3.15)$$

where $c_1$ and $c_2$ are constants which we choose to be positive and, $\beta = \beta(\alpha)$ is an arbitrary constant depending only on $\alpha$. As one can see from the above solutions, derivatives of the 5D scale factor with respect to the extra dimension $w$ is proportional to the 4D scale factor.

We should note that in our derivation of the field equations for the brane, we did not use the Codazzi equation. However, the Codazzi equation must be satisfied for all smooth manifolds. One can easily check that the above cosmological solutions satisfie the Codazzi equation.

### 3.2 The case $f(R) = f_0 (R - R_0)^\alpha$

Now, we consider a power law form $f(R) = f_0 (R - R_0)^\alpha$, where $f_0$, $R_0$ and $\alpha$ are some constants. Using the trace of the field equations (2.44), one obtains

$$L = \frac{1}{3} \left( R + Q - (1 - \frac{1}{\alpha})R - \frac{R_0}{\alpha} \right). \quad (3.16)$$
Next, using (2.40) and (2.41), one finds
\[ W = \frac{1}{12} \left( \frac{1}{\alpha - 1} \mathcal{R} + 4(R + Q) - \frac{\mathcal{R}_0}{\alpha} \right). \]  
\[ (3.17) \]
\[ L_{\mu \nu} = \frac{2}{3} (\alpha - 1) \left( (\alpha - 2) \nabla_\mu \mathcal{R} \nabla_\nu \mathcal{R} + (\mathcal{R} - \mathcal{R}_0) \nabla_\mu \nabla_\nu \mathcal{R} \right), \]  
\[ (3.18) \]
For this specific form of \( f(\mathcal{R}) \), we again consider the solutions of the equations (3.10) in the absence of ordinary matter and cosmological constant. For general \( \alpha \) one has a solution
\[ a(t) = c_1 t^{1/2}, \quad \dot{a}(t) = c_2 a(t), \quad \ddot{a}(t) = -\left( c_2^2 + \frac{1}{6} \mathcal{R}_0 \right) a(t), \]  
\[ (3.19) \]
where \( c_1 \) and \( c_2 \) are the constants of integration. This form of the scale factor is similar to the radiation dominated epoch.

We have also a power-law solution
\[ a(t) = c_1 t^n, \quad \dot{a}(t) = \sqrt{\frac{\mathcal{R}_0}{6} a(t)}, \quad \ddot{a}(t) = -\frac{1}{3} \mathcal{R}_0 a(t), \]  
\[ (3.20) \]
where
\[ n = -\frac{2\alpha^3 - 9\alpha^2 + 4\alpha + 6}{\alpha^2 + 7\alpha - 12}, \]  
\[ (3.21) \]
and \( c_1 \) and \( c_2 \) are the constants of integration. This solution shows that the above form for the bulk action cannot admit a matter dominated behavior. However, this can be traced to our assumption of the space-time being empty. In the next section we will see that adding ordinary matter to the system will produce matter dominated solutions. We have also an inflationary solution as follows
\[ a(t) = c_1 e^{nt}, \quad \dot{a}(t) = c_2 a(t), \quad \ddot{a}(t) = \left( 2n^2 - c_2^2 - \frac{1}{6} \mathcal{R}_0 \right) a(t), \]  
\[ (3.22) \]
where \( n, c_1 \) and \( c_2 \) are some constants. The solutions above also satisfy the Codazzi equation and are therefore cosmological solutions of the model.

All the solutions above have the property that derivatives of the scale factor with respect to the extra dimension are proportional to the scale factor itself. In the next section we will use this property to build an autonomous system of equation for the theory.

4. Cosmological Dynamics
In this section, we investigate the cosmological dynamics of the model introduced in the previous section. Substitution of expressions for the electric part of the Weyl tensor and quantity \( W \) in equations (3.10) results in the Friedmann and Raychaudhuri equations for a general \( f(\mathcal{R}) \)
\[ -\dot{H} - \left( \frac{\ddot{a}}{a} \right)^2 + \frac{\dot{a}}{a} + \frac{1}{4F} f + 2 \frac{\dot{F}}{F} \dot{a} - \frac{\dot{F}}{F} \frac{\kappa^2}{F} (\rho_{\text{rad}} + \rho_m) = 0, \]  
\[ (4.1) \]
and
\[ \frac{4}{3} \dot{H} + \frac{2}{3} \frac{\ddot{a}}{a} + \frac{4}{3} \left( \frac{\dot{a}}{a} \right)^2 + \frac{2}{3} \frac{\dot{F}}{F} \ddot{a} - \frac{2}{3} \frac{\dot{F}}{F} \frac{\kappa^2}{F} \left( \frac{4}{3} \rho_{\text{rad}} + \rho_m \right) = 0, \]  
\[ (4.2) \]
where we have defined $F(\mathcal{R}) = f'(\mathcal{R})$. With definition of the bulk Ricci scalar

$$\mathcal{R} = 6 \left[ 2H^2 + \dot{H} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{\ddot{a}}{a} \right],$$

(4.3)

one can write the Friedmann equation as

$$1 = \frac{\mathcal{R}}{12H^2} + \left( \frac{\dot{a}}{Ha} \right)^2 - \frac{1}{8FHF} \frac{\dot{F}}{HF} + \frac{\dot{F}\dot{a}}{H^2Fa} + \frac{\kappa^2}{2FH^2} \left( \rho_{\text{rad}} + \rho_m \right).$$

(4.4)

In order to study the dynamical evolution of the system, we define the following dimensionless quantities \[14\]

$$x_1 = \frac{1}{H^2} \left[ \frac{1}{12} \mathcal{R} + \left( \frac{\dot{a}}{a} \right)^2 \right], \quad x_2 = -\frac{1}{8H^2F}, \quad x_3 = -\frac{\dot{F}}{HF}, \quad x_4 = \frac{\dot{F}\dot{a}}{H^2Fa}, \quad x_5 = \frac{\kappa^2\rho_{\text{rad}}}{2H^2F},$$

(4.5)

and

$$\Omega_m = \frac{\kappa^2\rho_m}{2H^2F} = 1 - \sum_{i=1}^{5} x_i, \quad \Omega_{\text{rad}} = x_5, \quad \Omega_{\text{dark}} = \sum_{i=1}^{4} x_i.$$  

(4.6)

with physical constraints represented by $\Omega_m, \Omega_{\text{rad}} \geq 0$. As was mentioned before, the Codazzi equation must be satisfied for any solution to the field equations. In our case, it is reduced to

$$3\frac{\dot{a}}{a} + \frac{\ddot{F}}{F} = 0.$$  

(4.7)

We now assume that the derivatives of the scale factor with respect to the extra dimension are proportional to the scale factor itself, that is

$$\frac{\dot{a}}{a} = b, \quad \frac{\ddot{a}}{a} = c,$$

(4.8)

where $b$ and $c$ are constants which may depend on the detailed functionality of $f(\mathcal{R})$. This, for example, occurs in the cosmological solution obtained in the previous section. We also define the dimensionless quantity

$$x_6 = -\frac{4\mathcal{R} + 6b(b+c)}{\mathcal{R} + 12b^2}.$$  

(4.9)

The resulting dynamical equations become

$$\frac{dx_1}{dA} = 4x_1 - \frac{x_1x_3}{3m} + x_2x_6,$$  

(4.10)

$$\frac{dx_2}{dA} = 4x_2 + x_1x_2x_6 + \frac{x_1x_3}{2m} + x_2x_3,$$  

(4.11)

$$\frac{dx_3}{dA} = -1 + (2\beta + 1)x_1 + \frac{1}{2}\beta x_1x_6 - 3x_2 - 2x_4 + x_5 + x_3^2 + \frac{1}{2}x_1x_3x_6,$$  

(4.12)

$$\frac{dx_4}{dA} = -(2\beta + 6)x_1 - \frac{1}{2}(3 + \beta)x_1x_6 + 4x_4 + x_1x_4x_6 + x_3x_4,$$  

(4.13)
\[
\frac{dx_5}{dA} = x_1 x_5 x_6 + x_3 x_5, \quad (4.14)
\]
and
\[
\frac{dx_6}{dA} = \frac{1}{3m} x_3 (x_6 + 4), \quad (4.15)
\]
where \( A = \ln a \) and we have defined
\[
\beta = \frac{3c}{b - c}. \quad (4.16)
\]

We also define the quantity
\[
m = \left( \frac{R}{3} + 4b^2 \right) \frac{F'}{F}, \quad (4.17)
\]
which depends on the bulk Ricci scalar. From equation (4.9) we see that the Ricci scalar can be represented as a function of the quantity \( x_6 \), so one may consider the quantity \( m \) as a function of \( x_6 \).

Let us define the effective equation of state \( p_{\text{tot}} = \omega_{\text{eff}} \rho_{\text{tot}} \), where
\[
\omega_{\text{eff}} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} = \frac{1}{3} (1 + x_1 x_6). \quad (4.18)
\]

With these definitions, \( \omega_{\text{eff}} = 0 \) gives the behavior \( a \propto t^{\frac{2}{3}} \), which is the standard behavior of the matter dominated epoch. Also \( \omega_{\text{eff}} = -1 \) corresponds to the de Sitter epoch.

In order to find the dynamical behavior of the above cosmological model, one must obtain the critical points of the dynamical system (4.10)-(4.15) followed by expanding the system near the critical point. The resulting equations can be written as
\[
X' = \Xi X, \quad (4.19)
\]
where \( X \) is a column vector made out of \( x_i \)'s, and the prime represents derivative with respect to parameter \( A = \ln a \) while \( \Xi \) is a matrix obtained from linearizing the system (4.10)-(4.15). The eigenvalues of this matrix at each critical point determine the stability of that point. If all the real parts of the eigenvalues are negative, then the point is stable. The appearance of positive eigenvalues makes the point a saddle point. If, however, all real parts of the eigenvalues are positive, we have an unstable point. In the evolution of the universe, one needs a saddle or an unstable point which would correspond to the matter epoch. Because of the instability of the this phase, it can fall into a stable point which would then correspond to an accelerating phase.

The critical points of the dynamical system (4.10)-(4.15) are discussed in what follows:

4.1 Radiation epoch

\[
P_1 : (x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 1, x), \quad (4.20)
\]
where \( x \) is arbitrary. This is the standard radiation dominated epoch, with \( a \propto t^{\frac{2}{3}} \), using equation (4.18). The eigenvalues of this point are
\[
0, \quad 1, \quad -1, \quad 4, \quad 4, \quad 4, \quad (4.21)
\]
making this point a saddle point as discussed above.
4.2 Kinetic epoch

We have three critical points which correspond to the kinetic epoch of the universe as follows

\[ P_2 : (x_1, x_2, x_3, x_4, x_5, x_6) = (0, 5, -4, 0, 0, -4), \]
\[ \omega_{\text{eff}} = \frac{1}{3}, \quad \Omega_m = 0, \quad \Omega_{\text{rad}} = 0, \quad \Omega_{\text{dark}} = 1. \] (4.22)

\[ P_3 : (x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 1, 0, 0, -4), \]
\[ \omega_{\text{eff}} = \frac{1}{3}, \quad \Omega_m = 0, \quad \Omega_{\text{rad}} = 0, \quad \Omega_{\text{dark}} = 1, \] (4.23)

and

\[ P_4 : (x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, -1, 0, 0, -4), \]
\[ \omega_{\text{eff}} = \frac{1}{3}, \quad \Omega_m = 2, \quad \Omega_{\text{rad}} = 0, \quad \Omega_{\text{dark}} = -1. \] (4.24)

The eigenvalues to these points are

\[ P_2 : 0, -5, -4, -3, -\frac{4(1 + 3m)}{3m}, -\frac{4}{3m}, \]
\[ P_3 : 1, 2, \frac{12m - 1}{3m}, 5, 5, \frac{1}{3m}, \]
\[ P_4 : -2, -1, \frac{12m + 1}{3m}, 3, 3, -\frac{1}{3m}. \] (4.25)

As can be seen, point \( P_4 \) is a saddle point. The point \( P_2 \) is either stable or saddle, and the point \( P_3 \) is either saddle or unstable depending on the sign of \( m \).

4.3 de Sitter epoch

This epoch includes four critical points corresponding to the accelerating phase of the universe. The one which can be used to describe the late time acceleration of the universe is the stable point. However, unstable accelerated fixed points can be used to describe the inflation epoch to ensure the end of the inflationary accelerating phase. The critical points corresponding to this epoch are

\[ P_5 : (x_1, x_2, x_3, x_4, x_5, x_6) = \left( \frac{3m}{1 + 3m}, \frac{-9m}{2(1 + 3m)^2}, \frac{12m}{1 + 3m}, 0, -\frac{144m^2 + 21m - 2}{2(1 + 3m)^2}, -4 \right), \]
\[ \omega_{\text{eff}} = 1 - \frac{9m}{3 + 9m}, \quad \Omega_m = \frac{6m(1 + 6m)}{(1 + 3m)^2}, \quad \Omega_{\text{rad}} = -\frac{144m^2 + 21m - 2}{2(1 + 3m)^2}, \quad \Omega_{\text{dark}} = \frac{3m(30m + 7)}{2(1 + 3m)^2}. \] (4.28)

For \( m < -\frac{1}{3} \) this point corresponds to the phantom epoch, and it is also compatible with the condition \( \Omega_m \geq 0 \), but it gives \( \Omega_{\text{rad}} < 0 \). The corresponding eigenvalues are then given by

\[ 4, \quad 4, \quad \frac{4}{1 + 3m}, \quad \frac{\alpha_1 + 4m\gamma}{2\gamma(1 + 3m)}, \quad -\frac{\alpha_1 - 8m\gamma \pm i\sqrt{3}\alpha_2}{4(1 + 3m)\gamma}, \] (4.29)

where we have defined

\[ \alpha_{1,2} = \left( -2240m^3 - 1572m^2 - 156m + 16 + 2\sqrt{M} \right)^\frac{1}{3} \pm \left( 208m^2 + 14m - 6 \right), \] (4.30)

and

\[ \gamma = \left( -2240m^3 - 1572m^2 - 156m + 16 + 2\sqrt{M} \right)^\frac{1}{3}, \] (4.31)
with
\[ M = -995328m^6 + 1306368m^5 + 956628m^4 + 130218m^3 - 11226m^2 - 1626m + 118. \] (4.32)

The critical point \( P_5 \) is thus a saddle point and together with the result \( \Omega_{\text{rad}} < 0 \) cannot serve as a final accelerating phase for the universe. The next point is

\[ P_6 : (x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 0, 0, 0, -4), \]
\[ \omega_{\text{eff}} = -1, \quad \Omega_m = 0, \quad \Omega_{\text{rad}} = 0, \quad \Omega_{\text{dark}} = 1, \] (4.33)

which corresponds to the standard de Sitter epoch. The eigenvalues corresponding to this point are

\[ 0, \quad 0, \quad -4, \quad \delta_1 - 2, \quad -\frac{1}{2}\delta_1 - 2 \pm i\frac{\sqrt{3}}{2}\delta_2, \] (4.34)

where

\[ \delta_{1,2} = \frac{\gamma}{6m} \pm \frac{4m - 11}{3\gamma}, \quad \gamma = m^{\frac{3}{2}} \left( 2\sqrt{\frac{M}{m} - 252} \right)^{\frac{1}{2}}, \] (4.35)

and

\[ M = 2662 - 1548m + 38016m^2 - 27648m^3. \] (4.36)

So for \(-4 < \delta_1 < 2\) we have a stable de Sitter epoch which, when written in terms of \( m \), leads to \( m < -\frac{5}{24} \) or \( 0 < m < \frac{1}{12} \) or \( m > 1.38 \). The point \( P_6 \) corresponds to an accelerating epoch much in the same way as the cosmological constant. As was mentioned in the Introduction, the observational data predict that the accelerating behavior of the late time universe is very close to that of the cosmological constant. Our model is therefore compatible with such predictions. Finally we have two other critical points which can be written as

\[ P_{7,8} : (x_1, x_2, x_3, x_4, x_5, x_6) = \left( \frac{-2}{3}x(1 + 3m), x, 4m[2x(1 + 3m) + 3], 0, 0, -4 \right), \] (4.37)

\[ \omega_{\text{eff}} = \frac{1}{3} \left[ \frac{8}{3}x(1 + 3m) + 1 \right], \]
\[ \Omega_m = 1 - \Omega_{\text{dark}}, \quad \Omega_{\text{rad}} = 0, \quad \Omega_{\text{dark}} = \frac{1}{3}x + 6m \left[ x(1 + 4m) + 2 \right], \] (4.38)

where

\[ x = \frac{1}{64} \frac{11 - 42m - 720m^2 - 1728m^3 \pm \sqrt{20736m^4 - 15552m^3 - 9468m^2 + 540m + 121}}{m(1 + 12m + 45m^2 + 54m^3)}. \] (4.39)

The point \( P_7 \) is interesting since it represents a phantom epoch for \( m < -\frac{1}{3} \) or \( -\frac{1}{6} < m < 0 \). In figure 7 we have shown the behavior of \( \omega_{\text{eff}} \) as a function of \( m \). However, we must impose the physical condition \( \Omega_m \geq 0 \) to this point which leads to \(-0.21 < m < 0 \). Figure 8 shows four different eigenvalues of the critical point \( P_7 \) as a function of \( m \). As we can see from the figure, only in the case \(-\frac{1}{6} < m < 0 \) all eigenvalues are negative. So the critical point \( P_7 \) is stable if \(-\frac{1}{6} < m < 0 \). This is interesting since such a range is also physical and can be used for the stable accelerated expansion phase of the universe. The critical point \( P_8 \) has an accelerating de Sitter phase for \( |m| \gg 1 \).
Figure 1: Plot of $\omega_{\text{eff}}$ as a function of $m$ for $P_7$. The physical range for the stable accelerating epoch is $-\frac{1}{6} < m < 0$.

Figure 2: Plot of the eigenvalues of the critical point $P_7$. As can be seen, for the range $0 < m < -\frac{1}{6}$, all the eigenvalues are negative.

Figure 3: Plot of $\omega_{\text{eff}}$ as a function of $m$ for the critical point $P_8$. The asymptotic value for $\omega_{\text{eff}}$ in the limit $m \to \infty$ represents the de Sitter phase with $\omega_{\text{eff}} = -1$. The matter dominated phase is represented by $m = -0.36$ as shown in the figure.

$$\omega_{\text{eff}} = -1, \quad \Omega_m = -\frac{1}{4m}. \quad (4.40)$$

The eigenvalues for the corresponding point in this limit are

$$\frac{3}{4m}, \quad -2 + \frac{4}{3m}, \quad -4 + \frac{1}{6m}, \quad \frac{3}{4m}, \quad \frac{3}{4m}, \quad 0. \quad (4.41)$$
The point $P_8$ is thus stable provided $m < 0$. In figure 3 we have plotted $\omega_{\text{eff}}$ as a function of $m$ for the point $P_8$. As can be seen, $\omega_{\text{eff}}$ reaches its minimum value at minus infinity which corresponds to the de Sitter accelerating phase. Figure 3 shows another interesting feature of the critical point $P_8$. The value $m = -0.36$ corresponds to $\omega_{\text{eff}} = 0$ which has the behavior of a matter dominated epoch, $a(t) \propto t^2$, with

$$\Omega_m = 0.14, \quad \Omega_{\text{rad}} = 0, \quad \Omega_{\text{dark}} = 0.86.$$  \hspace{1cm} (4.42)

At the fix point, we have from (4.17)

$$f(\mathcal{R}) = f_0 (\mathcal{R} + 2b^2)^{-\frac{3\Delta}{8}}.$$  \hspace{1cm} (4.43)

The eigenvalues corresponding to this point are

$$-0.3, \quad 3, \quad -4.6, \quad -3.5, \quad -0.3, \quad -0.3.$$  \hspace{1cm} (4.44)

This point is therefore a saddle point and represents the matter dominated era of the universe.

The discussion in this section suggests that the generalized model proposed here has a dominant accelerated expansion behavior. This is desirable since the brane-world models are well-known for the production of late time accelerated behavior of the universe. However as we saw in this section, the effect of the non-trivial bulk action results in changes in the behavior of the scale factor and the dominant contribution is that of dark energy.

5. Conclusions and final remarks

In this work we have considered a brane-world scenario in the context of $f(\mathcal{R})$ gravity. We showed that the change in the dynamics of the bulk space-time leads to changes in the dynamics of the brane space-time. The change to the brane equation of motion is however important, for we have the non-minimal coupling of the brane energy-momentum tensor to the bulk $f(\mathcal{R})$, which is similar to 4-dimensional $f(R)$ gravity models. This would not have been the case had we used Israel junction conditions to project the bulk geometry to the brane. 

![Figure 4: Stream plot of the point $P_8$ and $P_6$ which correspond to the unstable matter dominated phase and stable accelerating phase respectively.](image)
The cosmological solution found with the exponential form of $f(R)$ exhibits an inflationary behavior. The power law assumption for the form of $f(R)$ was also considered. The main point of this ansatz is that if we consider the vacuum space-time one cannot obtain matter dominated solutions for the theory. However, this difficulty can be resolved if one adds an ordinary matter to the theory. On the other hand, the radiation dominated solution always exists. It is worth noting that all the solutions obtained in this paper have the property that the derivative of the scale factor with respect to the extra dimension are always proportional to the scale factor itself. Of course solutions with general extra dimensional derivatives may also exist.

In order to describe the cosmological evolution of the universe, one must consider the general form of $f(R)$. This was done by studying the phase space of the theory. The interesting result was that there is an unstable critical point $P_8$, which can be considered as a matter dominated phase of the universe. This point corresponds to a power law form for $f(R)$ and can fall into a stable accelerating critical point $P_6$ which leads to the value $m = -0.36$. Figure (4) shows a stream plot of these two points. We also note that with the assumption (4.8), we obtained a new dynamical variable which contains derivative of $f(R)$ with respect to the extra dimension. This dynamical variable is a new feature of the model and is responsible for the change of the dynamics of the system relative to what one gets in the corresponding 4D scenario [14]. The complete analysis of the system would be more complicated if we relaxed the assumption (L8) which reflects the full behavior of the non-trivial dynamics of the bulk space-time.

References

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370; ibid, 83 (1999) 4690.
[2] T. Shiromizu, K. Maeda, M. Sasaki, Phys. Rev. D 62 (2000) 024012.
[3] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B 565 (2000) 269; P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477 (2000) 285; M. K. Mak and T. Harko, Phys. Rev. D 70 (2004) 024010; T. Harko and K. S. Cheng, Phys. Rev. D 76 (2007) 044013.
[4] R. Maartens and K. Koyama, Living Rev. Relativity 13 (2010) 5.
[5] W. Israel, Nouvo Cimento B 44 (1966) 1.
[6] Z. Haghani, H. R. Sepangi and S. Shahidi, Phys. Rev. D 83 (2011) 064014 [arXiv:gr-qc/1103.0075].
[7] M. D. Maia, E. M. Monte and J. M. F. Maia, Phys. Lett. B 585 (2004) 11; M. D. Maia, E. M. Monte, J. M. F. Maia and J. S. Alcaniz, Class. Quant. Grav. 22 (2005) 1623.
[8] S. Jalalzadeh and H. R. Sepangi, Class. Quant. Grav. 22 (2005) 2035.
[9] M. Heydari-Fard, H. R. Sepangi, JCAP 0808 (2008) 018; ibid JCAP 02 (2009) 029; ibid JCAP 0808 (2008) 018; M. Heydari-Fard, H. Razmi, H. R. Sepangi, Phys. Rev. D 76 (2007) 066002.
[10] P. Teyssandier and P. Tourrenc, J. Math. Phys. 24 (1983) 2793; D. Wands, Class. Quant. Grav. 11 (1994) 269; J. D. Barrow and S. Cotsakis, Phys. Lett. B 214 (1988) 515; S. Cecotti, Phys. Lett. B 190 (1987) 86.
[11] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82 (2010) 451; T. P. Sotiriou, J. of Phys. Conf. Series 189 (2009) 012039; A. De Felice, S. Tsujikawa, Living Rev. Rel. 13 (2010) 3.
[12] A. Borzou, H. R. Sepangi, S. Shahidi, R. Yousefi, Euro. Phys. Lett. 88 (2009) 29001.
[13] P. Astier et al., Astron. Astrophys. 447 (2006) 31.
[14] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, Phys. Rev. D 75 (2007) 083504.
[15] E. J. Copeland, A. R. Liddle, D. Wands, Phys. Rev. D 57 (1998) 4686.
[16] X. Chen, Y. Gong and E. N. Saridakis, JCAP 04 (2009) 001.
[17] X. Chen, Y. Gong and E. N. Saridakis, JCAP 0904 (2009) 001; M. Alimohammadi, A. Ghalee, Phys.Rev.D80 (2009) 043006; G. Leon, E. N. Saridakis, JCAP 0911 (2009) 006; R. Yang, X. Gao, Class. Quant. Grav. 28 (2011) 065012; N. Mazumder, R. Biswas, S. Chakraborty, arXiv: 1106.4620v1 [gr-qc], arXiv: 1106.4627v2 [gr-qc]; N. Goheer, J. A. Leach, P. K.S. Dunsby, Class. Quant. Grav. 24 (2007) 5689; S. Carloni, P. K. S. Dunsby, J. Phys. A 40 (2007) 6919; A. Coley, S. Hervik, Class. Quant. Grav. 22 (2005) 579.

[18] J. Nash, Ann. Math. 63 (1956) 20.

[19] G. Dvali, M. Shifman, Phys.Lett. B 396 (1997) 64; Erratum: ibid, 407 (1997) 452.

[20] T. P. Sotiriou, Class. Quant. Grav. 23 (2006) 5117.