Plane Gravitational Waves in String Theory

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Abstract

We analyze the coset model $(E_2^2 \times E_2^2)/E_2^2$ and construct a class of exact string vacua which describe plane gravitational waves and their duals, generalizing the plane wave background found by Nappi and Witten. In particular, the vector gauging describes a two-parameter family of singular geometries with two isometries, which is dual to plane gravitational waves. In addition, there is a mixed vector-axial gauging which describes a one-parameter family of plane waves with five isometries. These two backgrounds are related by a duality transformation which generalizes the known axial-vector duality for abelian subgroups.

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1 Introduction

The search of exact non-trivial background solutions in string theory plays an important role in understanding the structure of space time at short distances and how string dynamics modifies gravitational interactions. In particular, one is interested to discover how string theory affects space time singularities that are present in many solutions of classical general relativity, such as black holes. On the one hand one would like to find string generalizations of known non-trivial solutions of Einstein’s equations, while on the other hand one would like to determine the corresponding conformal field theories which allows in principle to study the string excitations and their interactions around these backgrounds.

Although this background identification is useful to establish the connection of string solutions with known spaces in general relativity, its validity is restricted only to the region of small curvatures where string interactions can be neglected. An important tool for relating different background solutions which are valid in separate kinematic regions of the same string vacuum is provided by duality transformations (see e.g. [1] for a review). In this way, one may relate spaces which look very different from a geometric and topological point of view, in particular exhibiting different singularity structure, though originating from the same conformal field theory (CFT). Duality was shown to be an exact symmetry of string theory in the case of compact spaces [2] as well as in non-compact abelian coset models [3], a property which is believed to be valid generally.

An interesting class of solutions of Einstein’s equations consists of plane gravitational waves [4, 5]. It was recently shown [6] that the most symmetric gravitational wave (with seven isometries) can be extended to an exact string background, described by a Wess-Zumino-Witten (WZW) model on the non-semi-simple group $E_{2}^{c}$ which is the central extension of the Euclidean group in two dimensions. Furthermore, it was shown that any plane gravitational wave can be extended to an exact string background [7], although the underlying conformal field theories remain to be discovered. Finally, several generalizations [8] of the $E_{2}^{c}$ WZW model and various gaugings [9, 10] on non-semi-simple groups were considered, while in Ref. [10] the representations of affine $E_{2}^{c}$ were constructed in terms of free fields.

In this work, we identify a CFT which describes a large class of plane
gravitational waves and their duals, depending on continuous parameters. It is based on the gauged WZW model \((E_2^c \times E_2^c)/E_2^c\) which contains an arbitrary parameter corresponding to the continuous embedding of the subgroup \(H = E_2^c\) in the product group \(E_2^c \times E_2^c\). Moreover, we find two inequivalent ways of gauging:

a) The vector gauging depends on an additional continuous parameter and it describes a singular geometry with two isometries, which is dual to plane gravitational waves.

b) A mixed vector-axial type gauging, based on the existence of a non-trivial outer automorphism of the subgroup \(H = E_2^c\), which describes a one-parameter family of plane gravitational waves with 5 isometries.

These two geometries are shown to be dual to the same background which corresponds again to a class of plane gravitational waves. The latter, however, depends on two parameters and has 6 Killing symmetries. This duality generalizes the known axial-vector duality for abelian subgroups.

The paper is organized as follows. In Section 2, we present a general discussion of string backgrounds describing plane gravitational waves which can be classified according to their number of Killing symmetries. We also compute their duals with respect to the abelian non-null isometries. In Section 3, we review the WZW model on \(E_2^c\), which describes the most symmetric case of plane waves with 7 isometries. In Section 4, we show how outer automorphisms of the subgroup \(H\) generate inequivalent \(G/H\) coset constructions, and we derive the corresponding general formulae for the \((E_2^c \times E_2^c)/E_2^c\) coset model. The resulting backgrounds emerging from the vector- and vector-axial-gauged models are worked out and studied in Sections 5 and 6, respectively. In Section 7, the connection between these two backgrounds is established by computing their dual geometry. Finally, Section 8 contains conclusions and comments for open directions.
2 Plane Gravitational Waves

We start with a discussion of the most general class of conformally invariant \( \sigma \)-models

\[
I = \int d^2 z [(G_{\mu\nu}(x) + B_{\mu\nu}(x)) \tilde{\partial} x^\mu \partial x^\nu + \alpha' R^{(2)}(x)]
\]

(2.1)

that correspond to strings propagating in a background of plane gravitational waves. Here \( G_{\mu\nu} \) is the background metric, \( B_{\mu\nu} \) the antisymmetric tensor and \( \Phi \) the dilaton.

It was shown in Ref.[7] (though in a different basis) that the metric, antisymmetric tensor and dilaton given by

\[
dS^2 = 2d\zeta d\bar{\zeta} - 2(f(u)\zeta^2 + \bar{f}(u)\bar{\zeta}^2 + F(u)\zeta \bar{\zeta})du^2 - 2du dv
\]

(2.2a)

\[
B_{\zeta \bar{\zeta}} = ib(u), \quad \Phi = -\ln g(u)
\]

(2.2b)

\[
F(u) = \frac{g(u)''}{g(u)} - \left(\frac{g(u)'}{g(u)}\right)^2 + \frac{1}{4}(b(u)')^2
\]

(2.2c)

satisfy the one-loop beta functions of conformal invariance, where \( f(u) \) is an arbitrary complex function, and \( g(u) \) and \( b(u) \) are arbitrary real functions. Primes denote derivatives with respect to \( u \).

The metric (2.2a) describes a plane gravitational wave, and the antisymmetric tensor and dilaton are chosen such that they respect the Killing symmetries of the gravitational wave. In fact, eq.(2.2c) on \( F(u) \) is a consequence of the one-loop beta functions [11]

\[
\beta^G_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}H_{\mu}^{\phantom{\mu}\lambda\sigma}H_{\nu\lambda\sigma} + 2\nabla_\mu \nabla_\nu \Phi = 0
\]

(2.3a)

\[
\beta^B_{\mu\nu} = \nabla_\lambda H^{\lambda}_{\mu\nu} - 2(\nabla_\lambda \Phi) H^{\lambda}_{\mu\nu} = 0
\]

(2.3b)

\[
\beta^\Phi = \frac{c - 4}{3\alpha^2} + 4(\nabla \Phi)^2 - 4\nabla^2 \Phi - R + \frac{1}{12}H^2 = 0
\]

(2.3c)

where \( H_{\mu\nu\lambda} = 3\nabla_{[\mu}B_{\nu\lambda]} \) is the antisymmetric tensor field strength and \( R_{\mu\nu} \) is the Ricci tensor. One finds that for general \( F(u) \) all equations are satisfied, except \( \beta^G_{\mu \mu} \), whose vanishing then results in eq.(2.2c). Moreover, because all scalar invariants vanish the central charge deficit \( c - 4 \) in (2.3c) is zero.
The non-zero components of the Riemann tensor, Ricci tensor, antisymmetric field strength, and squared antisymmetric field strength, for the the background in (2.2) are

\[ R_{\zeta u \bar{\zeta} u} = 2f(u) , \quad R_{\zeta u \zeta u} = 2\bar{f}(u) , \quad R_{\zeta u \bar{\zeta} u} = F(u) , \quad R_{uu} = 2F(u) \]  

(2.4a)

\[ H_{\zeta \bar{\zeta} u} = ib(u)' , \quad H_{uu}^2 = 2(b(u)')^2 \]  

(2.4b)

and the scalar curvature is obviously zero. Moreover, it was argued \[6, 7\] that the 1-loop solution is an exact background to all orders in \(\alpha'\), as one can check that all higher-order contractions of the relevant tensors vanish.

For general \(f(u), g(u)\) and \(b(u)\), the geometry in (2.2) has five Killing vectors, one of which is the null Killing vector

\[ k^\mu = (0, 0, 0, 1) \]  

(2.5)

which is characteristic for gravitational waves. In eq.(2.5), \(\mu\) refers to \(\zeta, \bar{\zeta}, u, v\), in that order. In fact, the metric (2.2a) is of pure radiation type \[4\], which describes a solution of Einstein’s equations with energy momentum tensor

\[ T_{\mu \nu} = T(u)k_\mu k_\nu , \quad k^\mu k_\mu = 0 \]  

(2.6)

Moreover, the null Killing vector is covariantly constant, which defines the subset of plane-fronted gravitational waves, discovered by Brinkmann \[4\], which in turn includes the plane wave solution (2.2a) as a special case. Its additional four space-like Killing vectors are given by

\[ \xi^\mu = (\lambda, \bar{\lambda}, 0, \zeta \bar{\lambda}' + \bar{\zeta} \lambda') \]  

(2.7)

where \(\lambda(u)\) satisfies the differential equation \(\lambda'' + 2\bar{f}(u)\bar{\lambda}' + F(u)\lambda = 0\), which gives rise to four integration constants in (2.7).

Among the five Killing vectors one can find a subset of three commuting Killing vectors, one null and two space-like, which are manifest in the following alternate form of the background

\[ ds^2 = g_{mn}(u)dx^m dx^n - 2du dv' , \quad g_{mn} = 2\bar{\alpha}(m\alpha_n) \]  

(2.8a)

\[ B_{12} = i \int du[(\alpha_1 \bar{\alpha}_2 - \bar{\alpha}_1 \alpha_2)b(u)'] , \quad \Phi(u) = -\ln(g(u)) \]  

(2.8b)
where we have made the coordinate transformation

\[ \zeta = \alpha_m(u)x^m, \quad v = v' + \frac{1}{4}g_{mn}(u)'x^m x^n \]  

(2.9a)

\[ \text{Re}[\bar{\alpha}_m(\alpha''_m + 2f(u)\alpha_m\alpha_n + F(u)\bar{\alpha}_m\alpha_n)] = 0 \]  

(2.9b)

with \( \alpha_m(u), \ m = 1, 2 \) complex functions satisfying (2.9b). This coordinate transformation also generates non zero components \( B_{1u} \) and \( B_{2u} \), which were gauged away by a gauge transformation \( B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \).

A special case of the metric is obtained when \( f = 0 \). In this case, we have at least six Killing vectors, the null Killing vector (2.3) and five space-like Killing vectors which are given by

\[ \xi^\mu = (h, \pm h, 0, (\bar{\zeta} \pm \zeta)h') \quad h'' + Fh = 0 \]  

(2.10a)

\[ \xi^\mu = (\zeta, -\bar{\zeta}, 0, 0) \]  

(2.10b)

where \( h(u) \) contains two integration constants. Note that compared to the general case, we have the extra “rotational” symmetry (2.10b). If in addition \( F \) is independent of \( u \), so that \( m^\mu = (0, 0, 1, 0) \) is also a Killing vector, we arrive at the most symmetric (non-trivial) plane gravitational wave, with seven isometries. The background found by Nappi and Witten [6] for the \( E_2^c \) WZW model corresponds to this case (see Section 3).

For use below, we give here the basis in which the rotational symmetry is manifest. First, we perform the basis transformation in (2.9), with \( \alpha_1 = -i\alpha_2 = h(u)/\sqrt{2} \), where \( h(u) \) is a solution of the differential equation (2.10a). Subsequently, we define polar coordinates \( x^1 = r \cos \theta, \ x^2 = r \sin \theta \), and find

\[ \text{d}s^2 = h(u)^2(dr^2 + r^2d\theta^2) - 2du'dv' \]  

(2.11a)

\[ B_{r\theta} = r \int du[h(u)^2b(u')] \quad \Phi = -\ln(g(u)) \]  

(2.11b)

showing the abelian isometry in the \( \theta \)-direction.

The various gravitational waves backgrounds, classified according to the number of Killing symmetries they possess, are summarized in Table 1. Here \( f, F \) without arguments stand for \( u \)-independent constants.
Table 1. Number of Killing symmetries for plane gravitational waves

|   | \(f(u)\) | \(F(u)\) | \(\text{dim(Killing sym.)}\) |
|---|---|---|---|
| I | 0 | \(F \neq 0\) | 7 |
| II | 0 | \(F(u)\) | 6 |
| III | \(f \neq 0\) | \(F\) | 6 |
| IV | \(f(u)\) | \(F(u)\) | 5 |

We next turn to the study of the background geometries that can be obtained from duality transformations of the plane gravitational wave background. We will restrict to the following two cases:

a) \(O(2,2)\) duality corresponding to the two space-like Killing symmetries that are manifest in (2.8).

b) \(O(1,1)\) duality in the case of \(f = 0\), corresponding to the (rotational) space-like Killing vector manifest in (2.11).

These possibilities were also discussed in Ref. [7], and, in particular, applied to the plane gravitational wave of Ref. [6]. Moreover, a more general possibility of including the null-isometry (combined with other non-null isometries as to avoid singularities in the duality inversion) was considered to show that the Nappi and Witten plane wave is dual to flat space [10, 7] with constant antisymmetric tensor and dilaton, by an \(O(3,3)\) rotation. Finally, we note that in the cases I and III the \(u\)-isometry can be incorporated in the duality transformations, though we will not work this out explicitly.

For the case of \(d\) abelian isometries in a \(D\)-dimensional background geometry, the duality transformations read [1]

\[
Q'_{ij} = Q_{ij} - Q_{ia}(Q^{-1})^{ab}Q_{bj} \quad , \quad Q'_{ab} = (Q^{-1})_{ab} \tag{2.12a}
\]

\[
Q'_{ia} = -Q_{ib}(Q^{-1})^{b} \quad , \quad Q'_{ai} = (Q^{-1})_{a}^{b}Q_{bi} \tag{2.12b}
\]

\[
\Phi' = \Phi - \frac{1}{2} \ln(\det(G_{ab})) \tag{2.12c}
\]

\[
Q_{\mu
\nu} = G_{\mu\nu} + B_{\mu\nu} = 
\begin{pmatrix}
Q_{ij}(x^i) & Q_{ia}(x^i) \\
Q_{ai}(x^i) & Q_{ab}(x^i)
\end{pmatrix} \tag{2.12d}
\]
where the matrix $Q$ and the dilaton $\Phi$ are independent of the $d$ coordinates $x^a, a = 1, \ldots d$, and the remaining $D - d$ coordinates are labelled by $x^i$.

First, it is not difficult to see using eq.(2.12) that the dual of (2.8) with respect to either the two commuting space-like isometries, or with respect to any linear combination of these, is again a gravitational wave.

On the other hand, the dual geometry with respect to the $\theta$-isometry in (2.11) is

$$dS^2 = h(u)^2(1 + l(u)^2)dr^2 - \frac{2l(u)}{r}drd\theta + \frac{1}{r^2h(u)^2}d\theta^2 - 2dudv$$

(2.13a)

$$l(u) = \frac{1}{h(u)^2} \int du[h(u)^2b(u)']$$

(2.13b)

$$B_{\mu\nu} = 0, \quad \Phi = -\ln(g(u)) - \ln(rh(u))$$

(2.13c)

which is clearly not anymore of the type (2.2), and corresponds to a curved background with Ricci tensor and scalar curvature,

$$R_{rr} = -2\frac{1 + l^2}{r^2}, \quad R_{r\theta} = \frac{2g}{r^3h^2}, \quad R_{\theta\theta} = -\frac{2}{r^4h^4}$$

(2.14a)

$$R_{ru} = -\frac{2(1 + l^2)(h'/h) + ll'}{r}, \quad R_{\theta u} = \frac{l' + 2l(h'/h)}{r^2h^2}$$

(2.14b)

$$R_{uu} = -2(1 + l^2)(h'/h)^2 - 2ll'(h'/h) - \frac{1}{2}(l')^2$$

(2.14c)

$$R = -\frac{4}{r^2h^2}$$

(2.14d)

showing singularities at $r = 0$. This solution can be viewed as a new singular solution to Einstein’s equations, with non-trivial matter. We remind the reader that the function $h(u)$ is dependent on $g(u)$ and $b(u)$, through eq.(2.2c) and the differential equation in (2.10a). We have checked that the metric has in general no other Killing symmetries, besides the manifest space-like and null Killing isometry.

In the remainder of the paper, we will find explicit conformal field theoretic realizations of the plane wave geometries and their duals, discussed above.
3 The WZW Model on $\mathbb{E}_2^c$

In this section we review the WZW action on the non-semi-simple group $\mathbb{E}_2^c$, whose algebra is given by

$$[J, P_i] = \epsilon_{ij} P_j, \quad [P_i, P_j] = \epsilon_{ij} T, \quad [T, J] = [T, P_i] = 0 \quad (3.1a)$$

$$[T_a, T_b] = f_{ab}^c T_c, \quad \{T_a | a = 1, \ldots, 4\} = \{P_1, P_2, J, T\} \quad (3.1b)$$

which is a central extension of the two-dimensional Poincare algebra. Although the Killing metric $\eta_{ab} = f_{ac}^d f_{bd}^e$ is degenerate, there exists a non-degenerate invariant bilinear form,

$$\Omega_{ab} = k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (3.2)$$

which is symmetric and satisfies $\Omega_{ad} f_{dc}^e + \Omega_{bd} f_{ac}^d = 0$. Due to these properties, the bilinear form can be used to construct the WZW action

$$I_{WZW}(g) = -\frac{1}{4\pi} \int_S d^2 z \text{Tr}(g^{-1} \partial g g^{-1} \bar{\partial} g) + \frac{i}{12\pi} \int_B d^3 z \text{Tr}(g^{-1} dg)^3 \quad (3.3)$$

by replacing $\text{Tr}(T_a T_b)$ with $\Omega_{ab}$.

To evaluate the action explicitly one may use the parametrization

$$g = e^{x P_1} e^{y J} e^{P_1 + v T} \quad (3.4)$$

in which case we find,

$$g^{-1} \partial g = J^a T_a, \quad \partial g g^{-1} = \bar{J}^a T_a, \quad g T_a g^{-1} = \omega_a^b T_b \quad (3.5a)$$

$$J^a = (c \partial x + \partial y, -s \partial x + y \partial u, \partial u, \partial v + sy \partial x - \frac{1}{2} y^2 \partial u) \quad (3.5b)$$

$$\bar{J}^a = (c \partial y + \partial x, s \partial y - x \partial u, \partial u, \partial v + sx \partial y - \frac{1}{2} x^2 \partial u) \quad (3.5c)$$

$$\omega_a^b = \begin{pmatrix} c & s & 0 & sx \\ -s & c & 0 & cx + y \\ sy & -x - cy & 1 & -\frac{1}{2} (x^2 + y^2 + 2cxy) \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c \equiv \cos u, s \equiv \sin u \quad (3.5d)$$
Then, the resulting WZW action is given by

$$I(E_2^c) = -\frac{k}{4\pi} \int d^2z (\partial x \bar{x} + \partial y \bar{y} + 2c \partial x \bar{y} + \partial u \bar{v} + \partial v \bar{u} + b \partial u \partial u)$$  \hspace{1cm} \text{(3.6)}$$

By rescaling the space-time coordinates, it is possible to scale out the parameter $k$, so that we will choose $k = 1$ in the remainder of the paper. Moreover, the parameter $b$ can also be removed by coordinate transformations, so we will set $b = 0$ throughout the paper as well.

The corresponding metric in the $\sigma$-model description is a conformally invariant plane gravitational wave. Changing coordinates,

$$x = \frac{i}{\sqrt{2s}} (e^{-iu/2} \zeta - e^{iu/2} \bar{\zeta}) \hspace{1cm} y = \frac{i}{\sqrt{2s}} (-e^{iu/2} \bar{\zeta} + e^{-iu/2} \zeta)$$  \hspace{1cm} \text{(3.7a)}$$

$$v = -v' - \frac{bu}{2} - \frac{1}{4s} (\zeta^2 + \bar{\zeta}^2 - 2c\zeta \bar{\zeta})$$  \hspace{1cm} \text{(3.7b)}$$

the geometry reads

$$dS^2 = 2d\zeta d\bar{\zeta} - \frac{1}{2} \zeta \bar{\zeta} du^2 - 2dudv'$$  \hspace{1cm} \text{(3.8a)}$$

$$B_{\zeta \bar{\zeta}} = iu \hspace{1cm} \Phi = \text{const.}$$  \hspace{1cm} \text{(3.8b)}$$

where we also applied a gauge transformation to the antisymmetric tensor field. This background is of the form (2.2), with $f(u) = 0$, $b(u) = u$, $g(u) = \text{const.}$, and $F(u) = \frac{1}{4}$, in agreement with (2.2c).

Comparing with Table 1, we establish that this is the most symmetric plane gravitational wave, corresponding to case I, whose seven Killing vectors are in the basis $x^\mu = (\zeta, \bar{\zeta}, u, v)$ of (3.8) given by,

$$T^\mu = (0, 0, 0, 1) \hspace{1cm} J^\mu = (-\zeta, \bar{\zeta}, 0, 0) \hspace{1cm} M^\mu = (0, 0, -2, 0)$$  \hspace{1cm} \text{(3.9a)}$$

$$(P^{(i)}_\alpha)^\mu = (h^{(i)}, \alpha h^{(i)}, 0, (\bar{\zeta} + \alpha \zeta) (h^{(i)})') \hspace{1cm} i = 1, 2 \hspace{1cm} \alpha = \pm$$  \hspace{1cm} \text{(3.9b)}$$

$$h^{(1)}(u) = \cos \frac{u}{2} \hspace{1cm} h^{(2)}(u) = \sin \frac{u}{2}$$  \hspace{1cm} \text{(3.9c)}$$

$$[P^{(i)}_\alpha, P^{(j)}_\beta] = \alpha \delta_{\alpha\beta} e^{ij} T \hspace{1cm} [J, P^{(i)}_\alpha] = P^{(i)}_\alpha \hspace{1cm} [M, P^{(i)}_\alpha] = \epsilon_{ij} P^{(i)}_\alpha$$  \hspace{1cm} \text{(3.9d)}$$

where we used the general expressions in (2.10). Here, a summation over any pair of lower and upper indices is understood, while we have used the same notation for the Killing vectors $\xi^\mu$ and the corresponding Lie algebra generators.
The form (2.11) of the metric is also easily obtained by taking e.g. \( h(u) = \cos(u/2) \). As can be seen explicitly from the Lie algebra satisfied by these Killing vectors, one can find a subalgebra of three commuting generators.

We also remark that the corresponding exact CFT of this model was identified as a solution of the Virasoro master equation [12], with central charge \( c_G = 4 \). In particular, the stress tensor on \( G = E_2^c \)

\[
T_G = L_G^{ab} J_a J_b^* \quad , \quad L_G^{ab} = \frac{1}{2} (\Omega^{-1})^{ab} + \frac{1}{2k^2} \delta^a_4 \delta^b_4 \quad (3.10)
\]

is the natural generalization of the affine-Sugawara construction [13, 14] on \( E_2^c \), satisfying the properties:

a) all the currents \( J_a \) are primary with conformal weight \( \Delta = (1, 0) \) with respect to \( T_G \),

b) for any solution \( L^{ab} \) of the master equation, the K-conjugate construction

\[
\tilde{L}^{ab} = L_G^{ab} - L^{ab} \quad , \quad \tilde{c} = c_G - c \quad (3.11)
\]

is also a solution.

A systematic approach to construct affine-Sugawara constructions on non-semisimple groups was given in Ref.[15], and further exploited [16] to show that the central charge of these constructions is always an integer equal to the dimension of the Lie algebra.

4 The Gauged WZW Model \((E_2^c \times E_2^c)/E_2^c\)

Our aim in this paper is to compute and examine the geometry of the gauged WZW model \((E_2^c \times E_2^c)/E_2^c\). To this end we first recall a result obtained in Ref.[17], concerning different ways of gauging a WZW model, which will turn out to be relevant for the particular coset theory that we wish to investigate.

Given a gauged WZW model \( G/H \), for each outer automorphism of the \( H \) algebra there is an inequivalent way of choosing the world-sheet gauge group in an anomaly-free way. More precisely, let \( S \) be an outer automorphism of \( H \), so that

\[
S_a^d S_b^e f_{de}^g (S^{-1})_g^c = f_{ab}^c \quad , \quad S_a^c S_b^d \Omega_{cd} = \Omega_{ab} \quad a = 1, \ldots, \dim H \quad (4.1)
\]
where \( f^{a}_{bc} \) are the structure constants of \( H \) and \( \Omega_{ab} \) the Killing metric on \( H \), or, more generally, the invariant bilinear form on \( H \) when \( H \) is non-semisimple. Then the world-sheet gauge group can be chosen to be

\[
\mathcal{J}^H_a = J^H_a + S^b_a \bar{J}^H_b
\]

(4.2)

where \( J^H_a \) and \( \bar{J}^H_a \) are the left- and right-moving world-sheet currents of \( H \) respectively. It is easy to see using the properties in (4.1) that the currents \( \mathcal{J}^H_a \) in (4.2) form a closed algebra.

The corresponding action may then be written as,

\[
I(g, A) = I_{WZW}(g) + I_{gauge}(g, A)
\]

(4.3a)

\[
I_{gauge}(g, A) = \frac{1}{2\pi} \int_{\Sigma} d^2z \text{Tr} [A_l \bar{\partial} gg^{-1} - \bar{A}_r g^{-1} \partial g + g^{-1} A_l g \bar{A}_r - A'_l \bar{A}_r]
\]

(4.3b)

\[
A_l = A_l^a T^H_a , \quad \bar{A}_r = \bar{A}_r^a T^H_a , \quad A'_l = A'_l^a (T^{'})^H_a , \quad (T^{'})^H_a = S^b_a T^H_b
\]

(4.3c)

where the WZW action \( I_{WZW}(g) \) is defined in eq.(3.3), and the gauge fields \( A_l \) and \( \bar{A}_r \) take values in the subgroup \( H \), as indicated. It is not difficult to check, using the Polyakov-Wiegmann identity and the properties in (4.1) that this action is invariant under the gauge transformations

\[
g \to h_l^{-1} g h_r , \quad h_l = e^{x^a T^H_a} , \quad h_r = e^{x^a (T^{'})^H_a}
\]

(4.4a)

\[
A_l \to h_l^{-1} (A_l - \partial) h_l , \quad \bar{A}_r \to h_r^{-1} (\bar{A}_r - \bar{\partial}) h_r
\]

(4.4b)

where \( h_l \) and \( h_r \) are elements of the subgroup \( H \).

Here, the usual vector gauging corresponds to the trivial automorphism \( S = 1 \). The axial gauging (which is anomaly-free for abelian subgroups) corresponds to \( S = -1 \), which is clearly an outer automorphism for abelian groups. However, the result above implies that even when the subgroup is non-abelian, the existence of non-trivial outer automorphisms gives rise to non-equivalent ways of gauging besides the vector gauging, which are typically of a mixed vector-axial type. Such outer automorphisms occur for example in \( SU(n) \), \( SO(2n) \) with \( n \geq 3 \), and \( E_6 \) groups (when \( H \) is a compact non-abelian subgroup). In analogy with the duality between the vector- and axial gauging \([8]\) that was found for abelian \( H \), one similarly expects a duality between the vector and the mixed
vector-axial gaugings for non-abelian $H$. This is indeed the case, in the particular gauged WZW model that will be discussed below.

To obtain more general non-trivial four-dimensional string backgrounds, we now turn to the gauged WZW model on the product group $G = E^c_2 \times E^c_2$. Using (3.4), the group elements of $G$ can be parametrized as

$$g = g_1 \times g_2, \quad g_i = e^{x_i P^{(i)}_1} e^{u_i J^{(i)}_1} e^{y_i P^{(i)}_1 + v_i T^{(i)}_1}, \quad i = 1, 2$$

(4.5)

where $T^{(i)}_a, i = 1, 2$ are the generators of each of the two copies of $E^c_2$.

The gauge group we take is $H = E^c_2$, so that a four-dimensional target space is obtained. There is a continuous embedding of $E^c_2$ in $G$, which is given by

$$T^H_a = T^{(1)}_a + R^b_a T^{(2)}_b$$

(4.6a)

$$R^b_a = r^b_a g^b_a, \quad r = (\sqrt{\nu}, \epsilon \sqrt{\nu}, \epsilon, \epsilon \nu) \quad \epsilon = \pm 1$$

(4.6b)

where $\nu$ is an arbitrary parameter, and $\epsilon$ labels two distinct sectors of the embedding. The case $\epsilon = \nu = 1$ corresponds to the diagonal subgroup. Note that the same model is found, when one leaves the levels $k_1, k_2$ arbitrary and one gauges the diagonal subgroup, with the identification $\nu = k_2/k_1$.

Moreover, the subgroup $H = E^c_2$ has a non-trivial outer automorphism,

$$S^b_a = \sigma_a g^b_a, \quad \sigma = (1, -1, -1, -1)$$

(4.7)

so that, according to the general result above, we can distinguish two different world-sheet gaugings, the vector gauging (corresponding to $S = 1$) and the mixed vector-axial type, with $S$ given in (4.7).

To compute the action in (4.3) explicitly, we use the subgroup generators (4.6), and the parametrization (4.5) to obtain

$$g^{-1} \partial g = g_1^{-1} \partial g_1 + g_2^{-1} \partial g_2 = J^a_a T^{(1)}_a + J^a_a T^{(2)}_a$$

(4.8a)

$$\bar{\partial} g g^{-1} = \bar{\partial} g_1 g_1^{-1} + \bar{\partial} g_2 g_2^{-1} = \bar{J}^a_a T^{(1)}_a + \bar{J}^a_a T^{(2)}_a$$

(4.8b)

$$g T^{(i)}_a g^{-1} = (\omega^{(i)})_a b T^{(i)}_b, \quad i = 1, 2$$

(4.8c)

where the currents $J^a_a, \bar{J}^a_a$ and the matrices $\omega^{(i)}$ are given in (3.5), with $x \to x_i, y \to y_i, u \to u_i$ and $v \to v_i, i = 1, 2$. Moreover, we use the non-degenerate
bilinear form (3.2) to perform the traces over the representation matrices, so that
\[ \text{Tr}(T^{(i)}_a T^{(j)}_b) \rightarrow \delta_{ij} \Omega_{ab}. \] (4.9)
Recall that we have chosen \( k_1 = k_2 = 1 \) and \( b_1 = b_2 = 0 \) for the arbitrary constants in the bilinear forms.

Then, using eq. (4.3) we obtain
\[ I(g, A) = I_1(E_2^c) + I_2(E_2^c) + \frac{1}{2\pi} \int d^2z [B_a \tilde{J}^a_H - \tilde{B}_a J^a_H + \tilde{B}_a M^{ab} B_b] \] (4.10a)
\[ B_a \equiv A^b_b \Omega_{ba}, \quad \tilde{B}_a \equiv \bar{A}^b_b \Omega_{ba} \] (4.10b)
\[ \tilde{J}^a_H = \tilde{J}^a_{(1)} + \tilde{J}^a_{(2)} \tilde{R}_b \tilde{J}^b_a, \quad J^a_H = J^a_{(1)} + J^a_{(2)} \tilde{R}_b \tilde{J}^b_a \] (4.10c)
\[ M = \Omega^{-1}(\omega^{(1)} - S^{-1} + R \omega^{(2)} \tilde{R} - R \tilde{R} S^{-1}) \] (4.10d)
\[ \tilde{R}_a^b = (\Omega^{-1} R \Omega)_a^b = \hat{r} \delta^b_a, \quad \hat{r} = (\sqrt{\nu}, \epsilon \sqrt{\nu}, \epsilon \nu, \epsilon) \] (4.10e)
where \( I_i(E_2^c), \ i = 1, 2 \) denotes the two copies of the WZW action (3.6) on each of the \( E_2^c \) factors.

Using the form of the currents in (3.5b),(3.5c), the matrix \( \omega \) in (3.5d), and the action (3.6), it is not difficult to check that the choice \( \epsilon \rightarrow -\epsilon \) in the subgroup generators (4.6) corresponds to the coordinate transformation \( u_2 \rightarrow -u_2, \ v_2 \rightarrow -v_2 \) so that, without loss of generality, we can take \( \epsilon = 1 \) in the following.

The next step is to choose a gauge fixing, integrate out the gauge fields and determine the background geometry by identifying the action with the \( \sigma \)-model form in (2.1), and reading off the metric \( G_{\mu \nu} \), the antisymmetric tensor \( B_{\mu \nu} \) and the dilaton \( \Phi \). This is done for the vector gauging in Section 5 and for the vector-axial gauging in Section 6. The geometries that we will find are all in the general class of plane gravitational waves and their duals, discussed in Section 2, and they are accompanied by a non-constant dilaton.

Of course, we know that this model is conformally invariant to all orders, since there is an underlying CFT based on the \( G/H \) coset construction [13, 19]. For completeness, we give here the form of the corresponding stress tensor
\[ T_{G/H} = T_G - T_H, \quad c_{G/H} = c_G - c_H = 4 \] (4.11)
where $T_G$ and $T_H$ are the affine-Sugawara constructions on $G = E_2^c \times E_2^c$ and $H = E_2^c$, respectively:

\[
T_G = \frac{1}{2} \sum_{i=1}^{2} [((\Omega^{-1})^{ab} + \delta_{\delta_4}^{c_4})_a J^{(i)}_a J^{(i)}_b] \ast, \quad c_G = 8 \quad (4.12a)
\]

\[
T_H = \frac{1}{2} [((\Omega_H^{-1})^{ab} + \frac{1}{(1 + \nu)^2} \delta_1^{c_1})_a J^{(i)}_a J^{(i)}_b] \ast, \quad c_H = 4 \quad (4.12b)
\]

\[
J^H_a = J_a^{(1)} + R_a^b J_a^{(2)} \quad (4.12c)
\]

with $J^{(i)}_a$, $a = 1, \ldots, 4$ the currents of the two copies of the affine Lie algebra $E_2^c$. Here, we have used the $E_2^c$ stress tensor in (3.10) (with $k_1 = k_2 = 1$ and $b_1 = b_2 = 0$) and we have introduced the induced bilinear form on the algebra generated by $J^H_a$:

\[
\Omega_H^{ab} = (1 + \nu) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (4.13)
\]

We have verified explicitly that the coset stress tensor in (4.11) satisfies the Virasoro master equation, as it should according to the K-conjugation property given in (3.11).

Note that the central charge of the construction is exactly equal to 4, so that, in particular, the central charge deficit appearing in the one-loop dilaton beta function (2.3c) vanishes.

We finally remark that the gauged WZW model should have a remaining chiral $U(1)$ current, since

\[
J = J^{(1)}_4 - J^{(2)}_4 \quad (4.14)
\]

is a dimension $\Delta = (1, 0)$ operator of the coset construction in (4.11). In fact, this chiral symmetry, is the origin of the existence of a null Killing vector in the geometries we will find below.

## 5 The Vector-Gauged Model

In this section, we discuss the evaluation of the action (4.10) for the vector gauging, corresponding to $S = 1$. First, we find a gauge-fixing by studying the
transformations in (4.4a) which show that we may choose
\[ x_1 = \frac{x_2}{\sqrt{\nu}} \equiv r \quad , \quad y_1 = y_2 = 0 \quad . \] (5.1)

Note that it is only possible to eliminate three from the eight degrees of freedom, so that naively a five dimensional target space in the gauged model is expected. However, when integrating out the gauge fields in (4.10) we also find an additional constraint, which eliminates a fourth degree of freedom.

In fact, the matrix \( M \) is singular in this case with
\[ M_a^3 = M_3^a = 0 \quad , \quad a = 1, 2, 3, 4 \] (5.2)
so that when integrating out the fields \( B_3 \) and \( \bar{B}_3 \) one finds the constraints
\[ \tilde{J}_H^3 = J_H^3 = 0 \quad . \] (5.3)

Then, substituting the explicit form of the currents in (4.10c), (3.5b), (3.5c) and using \( \hat{R} \) from (4.10c), we have
\[ \tilde{\partial}(u_1 + \nu u_2) = \partial(u_1 + \nu u_2) = 0 \quad . \] (5.4)
The general solution is
\[ u_1 = u(1 - \kappa) + \rho \quad , \quad u_2 = -u(1 + \kappa) + \rho \] (5.5)
where we have defined
\[ \kappa \equiv \frac{1 - \nu}{1 + \nu} \] (5.6)
while \( \rho \) is an arbitrary integration constant. Moreover, for the remaining two variables it is useful to perform the change of basis:
\[ v_1 = \frac{1}{2} \left( \theta - \frac{2}{(1 - \kappa)(1 + \kappa)^2} \right) \quad , \quad v_2 = \frac{1}{2} \left( \frac{1 - \kappa}{1 + \kappa} \theta + \frac{2}{(1 - \kappa)(1 + \kappa)^2} \right) \] (5.7)

Then we can integrate the remaining gauge fields, and obtain
\[ I_{\text{vector}} = I_1(E_2^c) + I_2(E_2^c) + \frac{1}{2\pi} \int d^2 z \sum_{a, b = 1, 2, 4} \tilde{J}_H^a (\hat{M}^{-1})_{ab} J_H^b \] (5.8)
where the non-singular reduced $3 \times 3$ matrix $\hat{M}$ is

$$
\hat{M} = \begin{pmatrix}
c_+ - 1 + \nu(c_- - 1) & s_+ - \nu s_- & (s_+ - \nu s_-)r \\
-s_+ + \nu s_- & c_+ - 1 + \nu(c_- - 1) & (c_+ + \nu c_-)r \\
0 & -(1 + \nu)r & -(1 + \nu)\frac{r^2}{2}
\end{pmatrix}
$$

(5.9a)

$$
\det \hat{M} = -4r^2s^2\frac{(1 - \kappa)}{(1 + \kappa)^2}, \quad c_\pm \equiv \cos(u(1 \mp \kappa) \pm \rho).
$$

(5.9b)

Moreover, as a result of the integration, we also get a non-trivial dilaton field $\Phi = -\frac{1}{2} \ln(\det \hat{M})$.

The final result for the background geometry (up to a common multiplicative factor) is

$$
dS^2 = \frac{1}{s^2} \left[ F_+(u, \rho)dr^2 + \frac{4}{r}(\kappa s\tilde{c}_r - c\tilde{s}_\rho)d\theta + \frac{F_-(u, \rho)}{r^2}d\theta^2 \right] - 2dudv
$$

(5.10a)

$$
B_{\mu\nu} = 0, \quad \Phi = -\ln(rs)
$$

(5.10b)

where we defined

$$
c = \cos u, \quad s = \sin u, \quad \tilde{c}_\rho = \cos(\kappa u - \rho), \quad \tilde{s}_\rho = \sin(\kappa u - \rho)
$$

(5.11a)

$$
F_\pm(u, \rho) = 1 + \kappa^2 + (1 - \kappa^2)c^2 \pm 2(c\tilde{c}_\rho + \kappa c\tilde{s}_\rho).
$$

(5.11b)

The geometry (5.11) depends on the two arbitrary parameters $\kappa$ and $\rho$.

Comparing with the general background (2.13) dual to the plane wave, we see that (5.10) is precisely of this form (up to a constant rescaling $r \to r/(1 - \kappa^2)$), with the identifications

$$
h(u) = \frac{s}{F_-(u, \rho)^{1/2}}, \quad l(u) = \frac{2(c\tilde{s}_\rho - \kappa s\tilde{c}_\rho)}{(1 - \kappa^2)s^2}, \quad g(u) = F_-(u, \rho)^{1/2}
$$

(5.12)

where we used the relation

$$
F_-(u, \rho)F_+(u, \rho) - 4(c\tilde{s}_\rho - \kappa s\tilde{c}_\rho)^2 = (1 - \kappa^2)^2s^4.
$$

(5.13)

It follows from the discussion of Section 2, that to check the one-loop beta functions, we may use the results in (5.12) to compute $F(u) = h(u)''/h(u)$ and $b(u)' = [h(u)^2l(u)']/h(u)^2$ (see (2.10a) and (2.13b)). Using also $g(u)$, one can verify that the condition (2.2d) is indeed satisfied as it should.
The Ricci tensor for the metric can be obtained using (2.14) and (5.12) and, in particular, the curvature scalar of the metric is

\[ R = -\frac{4F_-(u, \rho)}{(1 - \kappa^2)^2 r^2 s^2} \]  

which has singularities at \( r = 0 \) and \( u = n\pi, n \in \mathbb{Z} \).

6 The Mixed Vector-Axial Gauged Model

In this section, we evaluate the action (4.10) for the mixed vector-axial gauging, which corresponds to the non-trivial outer automorphism \( S \) in (4.4). First, by studying the gauge transformations in (4.4a), we make the gauge-fixing choice:

\[
\begin{align*}
    x_1 &= \frac{x_2}{\sqrt{\nu}} \equiv x, \\
    y_1 &= -\frac{y_2}{\sqrt{\nu}} \equiv y, \\
    u_1 &= -u_2 \equiv u, \\
    v_1 &= -\frac{v_2}{\nu} \equiv v.
\end{align*}
\]

Note that, in contrast to the vector gauging of the previous section, the gauge fixing now eliminates four degrees of freedom, as expected generically. Indeed, in this case the matrix \( M \) in eq.(4.10d) is non-degenerate, so that no additional constraints arise from integrating out the gauge fields.

The resulting action is

\[
I_{\text{vector-axial}} = I_1(E_2^a) + I_2(E_2^b) + \frac{1}{2\pi} \int d^2 z \tilde{J}_H^a (M^{-1})_{ab} \tilde{J}_H^b \quad (6.2)
\]

where the matrix \( M \), in terms of the parameter \( \kappa \) defined in (5.6), takes the form:

\[
M = \frac{2}{1 + \kappa} \begin{pmatrix}
    c - 1 & \kappa s & 0 & \kappa s x \\
    -\kappa s & c + 1 & 0 & cx + \kappa y \\
    0 & 0 & 0 & 2 \\
    sy & -x + \kappa cy & 2 & -\frac{1}{2}(x^2 + y^2) - cxy
\end{pmatrix} \quad (6.3a)
\]

\[
\det M = \frac{64(1 - \kappa)s^2}{(1 + \kappa)^3}. \quad (6.3b)
\]

After some algebra, one obtains the following background geometry (up to a multiplicative factor):

\[
dS^2 = \frac{2}{1 - c}[1 + \kappa^2 + (1 - \kappa^2)c]dx^2 + \frac{8\kappa}{1 - c}dx dy + \frac{2}{1 - c}[1 + \kappa^2 - (1 - \kappa^2)c]dy^2.
\]
\[ \frac{\kappa}{2(1+c)}[(1 + \kappa^2 + (\kappa^2 - 1)c)\kappa x^2 + 4xy + \kappa(5 - 3\kappa^2 + c(1 - \kappa^2))y^2]du^2 \\
+ 4(1 - \kappa^2)dudv + \frac{2\kappa}{s}[-2\kappa x + (\kappa^2 - 3 + (\kappa^2 - 1)c)y]dxdu \\
- \frac{2}{s}[(1 + \kappa^2 + (\kappa^2 - 1)c)\kappa x + 2(1 - c(1 - \kappa^2))y]dydu \]  
(6.4a)

\[ \Phi = -\ln s \]  
(6.4b)

\[ B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \]  
(6.4c)

so that the antisymmetric tensor field is pure gauge and can be discarded.

After a tedious but straightforward calculation, whose details can be found in Appendix A, one finds that the metric (6.4a) can be transformed into the two alternate forms

\[ dS^2 = 2d\zeta d\bar{\zeta} - 2(f(u))\zeta^2 + \bar{f}(u)\bar{\zeta}^2 + F(u)\zeta\bar{\zeta})du^2 - 2dudv \]  
(6.5a)

\[ f(u) = \frac{1}{2s^2}(c - i\kappa s)e^{i\kappa u} \]  
(6.5b)

or

\[ dS^2 = \frac{1}{s^2}[F_(u,0)dx^2 + 4(c\tilde{s}_0 - \kappa s\tilde{c}_0)dx dy + F_+(u,0)dy^2] - 2dudv \]  
(6.6)

where in the last form we have used the definitions \( \tilde{c}_\rho, \tilde{s}_\rho \) and \( F_\pm(u, \rho) \) in (5.11) at \( \rho = 0 \). Note that this class of backgrounds depends on one continuous parameter \( \kappa \).

Comparison of (6.5) with Table 1, shows that this is a generic plane gravitational wave with five Killing vectors, corresponding to case IV. It is easy to check that the relation (2.2c) is obeyed using \( F(u) = (\sin u)^{-2}, g(u) = \sin u, \) and \( b(u) = 0, \) so that the background is conformally invariant to all orders. The explicit form of the non-vanishing component of the Ricci tensor and the curvature scalar are

\[ R_{uu} = -\frac{2}{s^2} \]  
(6.7)

where we used eq.(2.4) and \( F(u) \) in (6.5b).

As an example, we have computed the five Killing vectors when \( \nu = 1 (\kappa = 0) \). In the basis \( x^\mu = (\zeta, \bar{\zeta}, u, v) \) of (6.5) they are given by

\[ T^\mu = (0, 0, 0, 2) \]  
(6.8a)
\[(P^{(i)})^{\mu} = (f^{(i)}_\alpha, \alpha f^{(i)}_\alpha, 0, (\bar{\zeta} + \alpha \zeta)(f^{(i)}_\alpha)' \quad i = 1, 2 \quad \alpha = \pm \quad (6.8b)\]
\[f^{(1)}_\alpha(u) = \frac{\alpha s}{1 - \alpha c} \quad , \quad f^{(2)}_\alpha(u) = 2 - \alpha \frac{us}{1 - \alpha c} \quad (6.8c)\]

where we used the general formula in (2.7). This set of Killing vectors satisfies the commutation relations \[\{P^{(i)}_\alpha, P^{(j)}_\beta\} = \alpha \delta^{ij} \epsilon_{\alpha \beta T} \] which defines a subalgebra of the 7-dimensional algebra in (3.9).

\section{The Dual Backgrounds}

As announced in Section 4, one expects the different geometries obtained by the vector and mixed vector-axial gauging of the \((E^c_2 \times E^c_2)/E^c_2\) model, to be related by duality. In this section, we establish this connection.

Starting with the geometry (5.10), the dual background obtained using the isometry in the \(\theta\)-direction, can be read off immediately from (2.11) and the identifications in (5.12), since we already showed that (5.10) is dual to a plane gravitational wave. In fact, an \(O(1,1)\) duality in the \(\theta\)-variable results in

\[dS^2 = \frac{s^2}{F_-(u, \rho)}[dr^2 + r^2d\theta^2] - 2dudv \quad (7.1a)\]

\[B_{\gamma \theta} = \frac{2r(c\bar{s}_\rho - \kappa s\bar{c}_\rho)}{(1 - \kappa^2)F_-(u, \rho)} \quad , \quad \Phi = -\frac{1}{2} \ln(F_-(u, \rho)) \quad (7.1b)\]

For completeness, we also give the alternate form

\[dS^2 = 2d\zeta d\bar{\zeta} - 2F(u)\zeta \bar{\zeta}du^2 - dudv \quad (7.2a)\]

\[F(u) = \frac{h(u)''}{h(u)} \quad , \quad h(u) = \frac{s}{F_-(u, \rho)^{1/2}} \quad (7.2b)\]

\[B_{\zeta \bar{\zeta}} = \int du \left(\frac{2i(c\bar{s}_\rho - \kappa s\bar{c}_\rho)}{(1 - \kappa^2)F_-(u, \rho)}\right) \frac{F_-(u, \rho)}{s^2} \quad , \quad \Phi = -\frac{1}{2} \ln(F_-(u, \rho)) \quad (7.2c)\]

showing that this is a gravitational wave of the type II in Table 1, with six Killing vectors. One can check conformal invariance by verifying the relation (2.2c), and relevant geometric quantities can be easily read off from (2.7).

As an example, we have computed again the six Killing vectors when \(\nu = 1\) \((\kappa = 0)\). In the basis \(x^\mu = (\zeta, \bar{\zeta}, u, v)\) of (7.2a), these are given by

\[T^\mu = (0, 0, 0, 2) \quad , \quad J^\mu = (-\zeta, \bar{\zeta}, 0, 0) \quad (7.3a)\]
\[(P^{(i)}_{\alpha})^{\mu} = (h^{(i)}, \alpha h^{(i)}, 0, (\bar{\zeta} + \alpha \zeta)(h^{(i)})') \quad , \quad i = 1, 2 \quad , \quad \alpha = \pm 7.3b\]

\[h^{(1)}(u) = \frac{s}{1 - c} \quad , \quad h^{(2)}(u) = 2 - \frac{us}{1 - c} 7.3c\]

where we used (2.10a). The commutation relations of the corresponding generators \(P^{(i)}_{\alpha}, T, J\) define a six-dimensional subalgebra of the Lie algebra in (3.9):

\[[J, P^{(i)}_{\alpha}] = P^{(i)}_{-\alpha} , [P^{(i)}_{\alpha}, P^{(j)}_{\beta}] = \alpha \delta_{\alpha\beta} \epsilon^{ij} T.\]

Turning to the dual of the vector-axial gauged model, we note that we now have three commuting isometries at our disposal, generating an \(O(3,3)\) duality. To establish the connection with the vector-gauged model, it suffices, however, to restrict to the duality transformations corresponding to the space-like Killing symmetries.

Looking at (6.6), we see that we can either compute the dual with respect to the \(O(2,2)\) transformations in the \((x, y)\)-coordinates or we can take the dual with respect to an isometry formed by an arbitrary linear combination of \(x\) and \(y\). In the first case, one finds that the metric is self-dual. In the second case, we first apply the rotation

\[x' = \cos(\rho/2)x + \sin(\rho/2)y \quad , \quad y' = -\sin(\rho/2)x + \cos(\rho/2)y 7.4\]

which leaves the metric (6.6) of the same form, with the substitutions

\[\tilde{c}_0 \rightarrow \tilde{c}_\rho \quad , \quad \tilde{s}_0 \rightarrow \tilde{s}_\rho \quad , \quad F_-(u, 0) \rightarrow F_-(u, \rho) \quad . \quad 7.5\]

Of course, the parameter \(\rho\) introduced in this way is unphysical at this point. However, the dual metric with respect to the \(x'\)-isometry coincides with the background (7.1) obtained as the dual of the vector-gauged model, which has a non-trivial \(\rho\)-dependence.

We recall here that the \(\rho\)-dependence of the dual background (7.1) originates from the constraint (5.3) when viewed as the dual of the vector-gauged model. On the other hand, when viewed as the dual of the vector-axial gauged model, its origin lies in the coordinate transformation (7.4).

Hence, we have established that both the vector and the vector-axial gauged models are mapped onto the same geometry with duality transformations. Note that these transformations modify the number of isometries of the background at each stage of their application.
8 Conclusions

We have discussed the most general class of conformally invariant $\sigma$-models corresponding to strings propagating in a background of plane gravitational waves in four dimensions. These backgrounds can be classified according to their number of Killing symmetries, ranging from the generic number five to the most symmetric case with seven Killing vectors. In all cases the Cartan subgroup consists of three commuting isometries, two space-like and one null.

For a subset of the isometries of these plane gravitational wave backgrounds the dual geometries were computed, showing in particular, that either other plane wave solutions are obtained or geometries with non-zero curvature scalar. The latter correspond to new singular solutions of Einstein’s equations with non-trivial matter, possessing one (“rotational”) space-like and one null Killing vector.

The WZW model on the non-semi-simple group $E_2^c$ provides an explicit conformal field theoretic realization of the most symmetric plane wave. To find realizations of more general conformally invariant plane gravitational waves and their duals, we have considered the gauged WZW model on $(E_2^c \times E_2^c)/E_2^c$. This coset contains an arbitrary parameter corresponding to the continuous embedding of the subgroup $H = E_2^c$ in the product group $E_2^c \times E_2^c$. Moreover, two inequivalent world-sheet gaugings for this subgroup were found, the usual vector-gauging, and a mixed vector-axial gauging which arises due to a non-trivial outer automorphism of $H$.

The vector gauging leads to a singular geometry (with non-zero scalar curvature) which is dual to a plane gravitational wave background, while the mixed vector-axial gauging gives rise to a plane gravitational wave with five isometries. These distinct backgrounds were shown to be related by duality transformations, as one expects by analogy with the axial-vector duality for abelian subgroups.
The results are summarized in the following diagram

\[
\begin{array}{c}
\begin{array}{c}
E_2^c \times E_2^c \\
\Downarrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\Rightarrow \text{singular geometry} \\
(2 \text{ isometries}) \\
\Rightarrow \text{plane gravitational wave} \\
(5 \text{ isometries}) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\Rightarrow \text{mixed vector-axial gauging} \\
\Rightarrow \text{plane gravitational wave} \\
(6 \text{ isometries}) \\
\end{array}
\end{array}
\]

where the dual geometry was obtained via \(O(1,1)\) duality transformations. For the vector gauged model, we employed the rotational isometry, while for the mixed vector-axial gauged model, a one-parameter linear combination of two translational isometries was used. All three geometries displayed in the above diagram exhibit a different number of Killing vectors, as indicated.

It is an open problem to investigate whether this duality symmetry between the vector and vector-axial gauged model is an exact symmetry of the corresponding conformal field theories. It is also interesting to derive the string excitations around these backgrounds and study their interactions by computing physical quantities, such as the partition function and correlation functions.

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Appendix A:  
Coordinate transformations in the vector-axial gauged model

In this Appendix, we give the coordinate transformations involved in trans- 
forming the metric (6.4a) obtained directly from the vector-axial gauged WZW 
model to the simpler forms in (6.5) and (6.6). These computations were per- 
formed using the algebraic manipulation program Maple, and we will give the 
main logical steps, while avoiding lengthy intermediate results.

First, we make the following basis transformation in (6.4a),
\[ x = \frac{1}{1 - \kappa^2} (x_1 - \kappa y_1) \quad \text{,} \quad y = \frac{1}{1 - \kappa^2} (y_1 - \kappa x_1) \]  
(A.1a)
\[ v = v_1 + \frac{1}{s} \left[ \kappa^2 c x_1^2 + \kappa (1 - c)x_1 y_1 + (c - 1) y_1^2 \right] \]  
(A.1b)
and the metric takes the simpler form,
\[ ds^2 = 2 \frac{1 + c}{1 - c} dx_1^2 + 2dy_1^2 - 4\kappa \frac{s}{1 - c} y_1 dx_1 du 
\quad + \frac{1}{2s^2} [\kappa^2 (1 + c)^2 x_1^2 + (1 - c)(4 + \kappa^2 (1 + c)) y_1^2] du^2 - 2dudv_1 \]  
(A.2)

Next, we let
\[ x_1 = x_2 + a(u)y_2 \quad , \quad y_1 = y_2 + b(u)x_2 \quad , \quad 1 - a(u)b(u) \neq 0 \]  
(A.3a)
\[ v_1 = v_2 + d(u)x_2 y_2 \]  
(A.3b)
and we determine \( a(u) \), \( b(u) \), and \( d(u) \) such that the new metric does not contain 
terms of the form \( dx_2 dy_2 \), \( y_2 dx_2 du \) and \( x_2 dy_2 du \). These three conditions result 
in the equations
\[ (1 - c)a(u)' + a(u) + \frac{\kappa}{2} [(1 + c)a(u)^2 + (1 - c)] = 0 \]  
(A.4a)
\[ b(u) = -\frac{1 + c}{1 - c} a(u) \]  
(A.4b)
\[ d(u) = \frac{2}{1 - c} [(1 + c)a(u)' - \kappa s] \]  
(A.4c)
The differential equation (A.4a) is of the Riccati type, and using the particular solution
\[ a(u) = i \frac{1 - c}{s} \left( \frac{\tau e^{iku}}{\tau e^{iku} + 1} \right) \]  
(A.5)
where \( \tau \) is an arbitrary constant. Choosing \( \tau = 1 \) one finds:
\[ a(u) = \frac{1 - c}{s} \tan(\kappa u/2), \quad b(u) = -\frac{1 + c}{s} \tan(\kappa u/2) \]  
(A.6a)
\[ d(u) = \frac{1}{1 - c} \left( \frac{\kappa s}{\cos^2(\kappa u/2)} + 2 \tan(\kappa u/2) - 2\kappa s \right) \]  
(A.6b)

Thus, using (A.3) we obtain a metric of the form:
\[ ds^2 = g_1(u)dx_2^2 + g_2(u)dy_2^2 + +2[f_1(u)x_2dx_2 + f_2(u)y_2dy_2]du \]  
(A.7a)
\[ +[p_1(u)x_2^2 + p_2(u)y_2^2 + p_3(u)x_2y_2]du^2 - 2dudv_2 \]
\[ g_1(u) = 2 \frac{1 + c}{(1 - c) \cos^2(\kappa u/2)}, \quad g_2(u) = 2 \frac{1}{\cos^2(\kappa u/2)} \]  
(A.7b)
\[ f_1(u) = \frac{s}{\cos^2(\kappa u/2)(1 - c)^2}[\kappa s(1 + 2 \cos^2(\kappa u/2)) \tan(\kappa u/2) - 2 \sin^2(\kappa u/2)] \]  
(A.7c)
\[ f_2(u) = \frac{1}{\cos^2(\kappa u/2)s} [\kappa s(1 - 2 \cos^2(\kappa u/2)) \tan(\kappa u/2) + 2 \sin^2(\kappa u/2)] \]  
(A.7d)

where we omit listing the functions \( p_i(u), \ i = 1, 2, 3 \) since they are not needed for the final coordinate transformation. Then, we set
\[ x_2 = \frac{i}{\sqrt{2g_1(u)}}(\zeta - \bar{\zeta}), \quad y_2 = \frac{1}{\sqrt{2g_2(u)}}(\zeta + \bar{\zeta}) \]  
(A.8a)
\[ v_2 = v - \frac{1}{8} \left[ \frac{(2f_1(u) - g_1(u))'}{g_1(u)}(\zeta - \bar{\zeta})^2 - \frac{(2f_2(u) - g_2(u))'}{g_2(u)}(\zeta + \bar{\zeta})^2 \right] \]  
(A.8b)
which leads to the metric given in (6.5).

To show the equivalence with the alternate form in (6.6), we start with that metric and discuss how to get back to the form (6.5). First, we make the following transformation in (6.4),
\[ x = \frac{i}{1 - \kappa^2}(\alpha_2(u)\zeta' - \bar{\alpha}_2(u)\bar{\zeta}') \quad , \quad y = \frac{i}{1 - \kappa^2}(-\bar{\alpha}_1(u)\zeta' + \alpha_1(u)\zeta') \]  
(A.9a)
we find exactly the form (6.5).

We determine the phase \( \alpha \) which was found by determining to vanish, so that \( \phi \).
The metric (6.6) then takes the form (2.9)). The metric (A.12), and subsequently let ting with solution:

\[
\alpha_1(u) = \frac{1}{s[2F_+(u, 0)]^{1/2}} ((1 - \kappa^2) s^2 + 2i(c \bar{s}_0 - \kappa s \bar{c}_0)) , \quad \alpha_2(u) = \frac{i[2F_+(u, 0)]^{1/2}}{2s}
\]

\[
v = v_1 + \frac{1}{4} (q(u) \bar{\zeta} + \bar{q}(u) \zeta)
\]

\[
q(u) = -\frac{2}{sF_+(u, 0)} (2c + \bar{c}_0 + \bar{c}^2 \bar{c}_0 + 2 \kappa c s \bar{s}_0 - \kappa^2 s^2 \bar{c} + i[(1 + \kappa^2) s^2 \bar{s}_0 + 2 \kappa s(1 + c \bar{c}_0)]
\]

which was found by determining \( \alpha_1, \alpha_2 \) such that \( g_{mn} = 2 \alpha_{(m} \alpha_n) \) in the subspace spanned by \( x, y \) (this corresponds to taking the inverse of the transformation (2.9)). The metric (6.7) then takes the form

\[
dS^2 = 2d\zeta d\bar{\zeta} + i \text{Im}[q(u)](\bar{\zeta}' \zeta' - \zeta \bar{\zeta}') du
\]

\[
-2[\hat{f}(u) \zeta' + \bar{\hat{f}}(u) \zeta'] du^2 - 2du dv_1
\]

\[
\hat{f}(u) = |f(u)|e^{i\phi_f(u)} , \quad |f(u)|^2 = (c^2 + \kappa^2 s^2)/(4s^4)
\]

where the angle \( \theta_f(u) \) and function \( \hat{F}(u) \) are complicated functions of \( u \), which we do not list here.

Finally, we perform the rotation

\[
\zeta' = e^{i\phi(u)/2} \zeta , \quad \bar{\zeta}' = e^{-i\phi(u)/2} \bar{\zeta}
\]

so that the metric becomes

\[
dS^2 = 2d\zeta d\bar{\zeta} + i(\text{Im}[q(u)] - \phi(u))(\bar{\zeta} d\zeta - \zeta \bar{d}\zeta) du
\]

\[
-2[|f(u)|e^{i\phi_f(u) + i\phi(u)} \zeta^2 + |f(u)|e^{-i\phi_f(u) - i\phi(u)} \bar{\zeta}^2 + (\hat{F}(u) - \frac{1}{4} (\phi(u))^2) \zeta \bar{\zeta}] du^2 - 2du dv_1
\]

We determine the phase \( \phi(u) \) by requiring the off-diagonal terms \( (\bar{\zeta} d\zeta - \zeta \bar{d}\zeta) du \) to vanish, so that \( \phi(u)' = \text{Im}[q(u)] \). This differential equation can be simplified by first shifting out the phase \( \theta_f(u) \), which leads to

\[
\phi = \bar{\phi} - \theta_f , \quad \bar{\phi}(u)' = \frac{\kappa(1 - \kappa^2) s^2}{c^2 + \kappa^2 s^2}
\]

with solution:

\[
\bar{\phi}(u) = \arctan \left( \frac{-c \bar{s}_0 + \kappa s \bar{c}_0}{\bar{c} \bar{c}_0 + \kappa s \bar{s}_0} \right) .
\]

Inserting this expression in the metric (A.12), and subsequently letting \( u \rightarrow -u \), we find exactly the form (6.7).
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