MULTIDIMENSIONAL REISSNER–NORDSTRÖM PROBLEM
WITH A GENERALIZED MAXWELL FIELD

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We obtain and study static, spherically symmetric solutions for the Einstein — generalized Maxwell field system in 2n dimensions, with possible inclusion of a massless scalar field. The generalization preserves the conformal invariance of the Maxwell field in higher dimensions. Almost all solutions exhibit naked singularities, but there are some classes of black hole solutions. For these cases the Hawking temperature is found and its charge/mass and dimension dependence is discussed. It is shown that, unlike the previously known multidimensional black hole solutions, in our case the black hole temperature may infinitely grow in the extreme case (that of minimum mass for given charges).

1. Introduction

Coupling between gravity and gauge fields is a common feature of many unification theories in higher dimensions [1]. In supergravity theories, such gauge fields are frequently essential in order to complete the multiplet structure and to guarantee the invariance of the Lagrangian with respect to local supersymmetry transformations [2]. They can provide, in some cases, a dynamical mechanism for the compactification of extra dimensions [3]. In this supersymmetric context, they can lead to a geometric interpretation in the superspace.

The simplest example of coupling of a gauge field to gravity is the Einstein-Maxwell system in four dimensions, whose extension to higher dimensions can lead to some interesting features [3]. Its reduction to four dimensions implies a non-trivial coupling of the electromagnetic and scalar fields to gravity. In the conventional, straightforward generalization of the Maxwell field to gravity is the Einstein-Maxwell system in four dimensions, whose extension to higher dimensions can lead to some interesting features [3]. Its reduction to four dimensions implies a non-trivial coupling of the electromagnetic and scalar fields to gravity. In this paper we analyze D-dimensional spherically symmetric solutions for the Einstein-GMF system. We find analytical solutions for many field configurations.

\begin{equation}
F = dU, \quad \text{or} \quad F_{A_1...A_n} = n! \partial_{[A_1} U_{A_2...A_n]} \tag{1}
\end{equation}

where \( U \) is a potential \((n - 1)\)-form and square brackets denote alternation. The field \( F \) is invariant with respect to the gauge transformation

\begin{equation}
U \rightarrow U + dZ \tag{2}
\end{equation}

where \( Z \) is an arbitrary \((n - 2)\)-form; in 4 dimensions this is the conventional gradient transformation of the Maxwell field.

This combination of gravity and the GMF in higher dimensions can potentially form the bosonic sector (or part of the bosonic sector) of some supersymmetric theory. For example, in eight dimensions the metric tensor — GMF system has the same number of degrees of freedom as the Rarita-Schwinger field, and the Lagrangian has a form very close to the one stemming from truncation of the Lagrangian of \( N = 1, D = 11 \) supergravity. Fields represented by different \( m \)-forms like the GMF are also considered in modern unified models like M-theory — see Ref. [5] in this number of the journal.

In the previous papers some cosmological manifestations of the field (1) were investigated [4, 7, 8]. One of the main features of the obtained models is the possible existence of an initial contracting phase; in some cases, the solutions are singularity-free. These results were, of course, obtained with just the 4-dimensional scalar sector of the GMF.

In this paper we analyze \( D \)-dimensional spherically symmetric solutions for the Einstein-GMF system. We find analytical solutions for many field configurations,
but not for the most general case. In general, we consider electric, magnetic and scalar charges associated with the GMF. The internal spaces are assumed to be Ricci-flat.

The great majority of the solutions exhibit naked singularities, but there are also some classes of black hole solutions. We find the corresponding Hawking temperature and discuss its change/mass and dimension dependence. A feature of interest is its possible infinite growth in the extreme black hole limit.

The presence of naked singularities is a common feature of spherically symmetric configurations in multidimensional theories, both in pure gravity and gravity coupled to gauge and dilaton fields \(\text{[4][10][12]}\), quite similarly to the case of minimal coupling between gravity and scalar fields in four dimensions. In the latter case the absence of event horizons is known to be a manifestation of the so-called no-hair theorems, claiming, in particular, that a black hole cannot have a nonzero scalar charge \(\text{[4]}\). Likewise, in pure multidimensional gravity spherically symmetric black solutions are only possible when the internal spaces are “frozen”, i.e., the extra-dimension scale factors, which behave as 4-dimensional scalars, are constant \(\text{[12]}\). No black hole solutions exist with a scalar-type component of a multidimensional Maxwell field \(\text{[10][12]}\). In dilaton gravity, where special black hole solutions exist in the presence of a nontrivial scalar (dilaton) field, the extra dimensions in the “string metric” of these solutions are again “frozen” (see for instance \(\text{[11]}\)). Accordingly, the black hole Hawking temperatures in both Einstein and dilaton gravity do not depend on space-time dimension. We shall see that such a dependence exists in the case under consideration in this paper.

A natural next step (going, however, beyond the scope of this paper) is to study multidimensional configurations with two conformally invariant fields — the GMF and the conformal scalar field. There is a well-known conformal mapping that reduces theories with nonminimally coupled scalar fields to those with a minimally coupled one (denoted \(\phi^\text{min}\)) — see \(\text{[13]}\) for \(D = 4\) and, for instance, \(\text{[14]}\) for \(D \geq 4\). Keeping in mind this mapping, we seek all solutions in the presence of \(\phi^\text{min}\). This addition does not complicate the solution process and, moreover, the influence of \(\phi^\text{min}\) upon the properties of the system is also of certain interest.

The paper is organized as follows. In Section 2 we introduce the GMF and classify the possible GMF configurations compatible with spherical symmetry. In Sections 3, 4 and 5 we study the Einstein-GMF-\(\phi^\text{min}\) system, for each GMF configuration separately. The Hawking temperature for the black hole solutions is determined and discussed in Section 6. Section 7 contains a brief discussion.

Throughout the paper capital Latin indices range over all \(D\) coordinates, Greek ones take the values 0, 1, 2, 3 and small Latin ones refer to extra dimensions \((i\) is the number of a factor space). The units where \(c = \hbar = 1\) are used.

2. Generalized Maxwell field and spherical symmetry

We consider general relativity in a Riemannian space-time \(V^D\) \((D = 2n, n \geq 2)\) in the presence of two minimally coupled massless fields, with the action

\[
S = \int d^Dx \sqrt{|g|} \left[ R + (-1)^{n-1} F^2 + \phi^A \phi_A \right],
\]

\[
F^2 = F^{A_1...A_n} F_{A_1...A_n},
\]

where \(R = D R\) is the scalar curvature corresponding to the \(D\)-metric \(g_{AB}\), \(g = | \det g_{AB} |\), \(\phi = \phi^\text{min}\), and \(F\) is the GMF.

The corresponding field equations are

\[
G^B_A = R^B_A - \frac{1}{2} \delta^B_A R = -T^B_A,
\]

\[
\nabla_A F^{AA_1...A_n} = 0,
\]

\[
\nabla_A \nabla_A \phi = 0.
\]

where the energy-momentum tensor (EMT) \(T^B_A\) is a sum of contributions from \(\phi^\text{min}\) and \(F\). The EMT of \(\phi^\text{min}\) has its usual form, while that of the \(F\) field is

\[
T^B_F = (-1)^{n-1} (n F_{AC_2...C_n} F^{BC_2...C_n} - \frac{1}{2} \delta^B_A F^2).
\]

The \(F\)-sector of the action is invariant under conformal mappings of \(V^D\) \((g_{AB} \rightarrow f(x^A) g_{AB})\) provided the components \(F_{A_1...A_n}\) are unchanged in such transformations (the potential \(U\) from \(\text{[4]}\) may experience a gauge transformation \(\text{[8]}\) where \(Z\) is some \((n-2)\)-form depending on the conformal factor \(f\).

Consider \(V^D\) with the structure

\[
V^D = M^4 \times V_1 \times ... \times V_s, \quad \dim V_i = N_i,
\]

where \(s\) is the (so far unspecified) number of Ricci-flat internal spaces; \(s = 0\) corresponds to conventional 4-dimensional theory.

The static, spherically symmetric metric in \(V^D\) may be written in the form

\[
ds^2_D = e^{2\gamma(u)} dt^2 - e^{2\alpha(u)} du^2 - e^{2\beta(u)} d\Omega^2
\]

\[
+ \sum_i e^{2\delta_i(u)} ds_i^2,
\]

where \(x^i = u\) is the radial coordinate and \(d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2\).

The following nonzero components of \(F\) are compatible with spherical symmetry:

- \(F_{0i_1...i_n}\), an electric field,
- \(F_{2i_3...i_n}\), a magnetic field,
- \(F_{1e_2...e_n}\), a “quasiscalar” component, behaving as a scalar field in \(M^4\).
where the indices $a_i$, $b_i$, $c_i$ belong to the internal spaces.

**Remark.** Several components of each type can exist simultaneously, with different sets of indices $a$, $b$, $c$; the only restriction is that neither pair of such sets may contain $n - 1$ common indices; in such cases off-diagonal components of the $F$ field EMT would come into play, which is incompatible with the Einstein equations for (12).

We will look for solutions in the following cases:

- $E\phi_{\text{min}}$: electric and magnetic field components of $F$, plus $\phi_{\text{min}}$;
- $S\phi_{\text{min}}$: a quasiscalar component of $F$, plus $\phi_{\text{min}}$.
- $EMS\phi_{\text{min}}$: electric, magnetic and quasiscalar field components of $F$, plus $\phi_{\text{min}}$;

Let us choose the harmonic radial coordinate $u$ specified by the condition (12)

$$\alpha = \gamma + 2\beta + \sigma, \quad \sigma \equiv \sum_{i=1}^{n} N_i \beta_i. \tag{10}$$

Then the nonzero components of the Ricci tensor may be written in the form

$$e^{2\alpha} R^1_1 = -\alpha'' + \alpha'^2 - \sum_{i=1}^{n} N_i \beta_i'^2, \tag{11}$$

$$e^{2\alpha} R^2_2 = e^{2\alpha} R^3_3 = e^{2\alpha} - 2\beta' - \beta'', \tag{11}$$

$$e^{-2\alpha} R^b_i = -\delta^a_{b_i} \beta^a_i, \quad i = -1, 1, \ldots, s, \tag{11}$$

where we have denoted

$$\beta_{-1} = \gamma, \quad N_{-1} = 1, \quad \beta_0 = \beta, \quad N_0 = 2. \tag{12}$$

The Einstein tensor component $G^1_1$ is

$$e^{2\alpha} G^1_1 = -e^{2\alpha} - 2\beta + \frac{1}{2} \alpha'^2 - \frac{1}{2} \sum_{i=1}^{n} N_i \beta_i'^2 \tag{13}$$

and does not contain second-order derivatives. The corresponding component of the Einstein equations is an integral to the other components, similar to the energy integral in cosmology.

The GMF components specified before are easily found from the field equations (12):

Electric: \[ F^{01a_1 \ldots a_n} = q_e e^{-2\alpha}, \quad q_e = \text{const}; \]
Magnetic: \[ F^{23b_1 \ldots b_n} = q_m \sin \beta, \quad q_m = \text{const}; \]
Quasiscalar: \[ F^{12c_1 \ldots c_n} = q_s e^{-2\alpha}, \quad q_s = \text{const}. \tag{14} \]

The EMT (12), subject to the above restrictions upon the set of indices, forms a sum of EMTs calculated for each of the components separately.

It should be noted that $q_e$ and $q_m$ are not the physical electric and magnetic charges in conventional units, corresponding to the 4-dimensional Maxwell field. Indeed, the 4-dimensional Einstein-Maxwell Lagrangian (multiplied by $16\pi G$, $G$ being the Newton constant of gravity) is

$$4R - GF^{\mu\nu} F_{\mu\nu}, \tag{15}$$

where $F_{\mu\nu}$ is the Maxwell field. Let us assume that there is a flat space-time asymptotic, where all extradimension scale factors are normalized to unity. Then, comparing (13) with the 4-dimensional reduction of (12) without $\phi_{\text{min}}$, one arrives at the asymptotic identification

$$F^{\mu\nu} = \sqrt{n!/(2G)} F^{\mu\nu a_1 \ldots a_n} \tag{16}$$

where the indices $a_j$ correspond to a nonzero component of the GMF. Accordingly, the charges in conventional units $q_e \text{phys}$ and $q_m \text{phys}$ are connected with $q_e$ and $q_m$ by

$$q_e \text{phys} = \sqrt{n!/(2G)} q_e, \quad q_m \text{phys} = \sqrt{n!/(2G)} q_m. \tag{17}$$

The scalar field $\phi$ satisfying the D'Alembert equation (12) under the above coordinate condition may be written as

$$\phi = \phi_0 + \phi_1 u, \quad \phi_0, \phi_1 = \text{const}, \tag{18}$$

and its EMT takes the form

$$e^{2\alpha} T^R_A \frac{\phi^2}{\phi_A} = \phi_1^2 \text{ diag } (1, -1, [1]_{D-2}). \tag{19}$$

Here and henceforth the notation $[A]_k$ means that a quantity $A$ is repeated $k$ times along the diagonal.

Eq. (13) means, in particular, that if we write the $D$-dimensional Einstein equations in the form

$$R^B_A = -T^B_A + \frac{1}{D-2} \delta^B_A T^C_C, \tag{20}$$

then a term connected with $\phi$ appears only in the ($1$) component, which is reasonably replaced by the integral coming from (13). So the field $\phi_{\text{min}}$ does not affect the equations for $A$, $B \neq 1$; the presence of $\phi_{\text{min}}$ only modifies a relation between integration constants appearing from the equation ($1$) (the first integral).

### 3. Problem EM$\phi_{\text{min}}$: integrable cases

Given the electric and magnetic type fields, with arbitrary $n$, the whole set $I$ of internal indices naturally splits into four subsets, to be labelled by different $i$:

$$a \cap b \mapsto i = 1; \quad a \setminus b \mapsto i = 2; \quad b \setminus a \mapsto i = 3; \quad I \setminus (a \cup b) \mapsto i = 4. \tag{21}$$
where \( a = \{ a_k \} \) and \( b = \{ b_k \} \) are the sets of indices of the nonzero “electric” and “magnetic” components [14] of the tensor \( F \).

It is easy to verify that the corresponding dimensions \( N_i \) are connected by the constraints

\[
N_1 + N_2 = N_3 + N_4 = n - 2; \\
N_1 = N_4; \\
N_2 = N_3. 
\]

(22)

Accordingly, the EMT of the \( F \) field takes the form

\[
T^B_{\lambda A} = Q^2 e^{2x} \text{diag}(1, 1, -1, -1, 1_{[n-2]}, -1_{[n-2]}) \\
+ Q^2_m e^{2y} \text{diag}(1, 1, -1, -1, 1_{N_4}, 1_{N_2}, -1_{N_3}, 1_{N_4}) 
\]

(23)

where

\[
Q^2_e = \frac{1}{2} t! q^2_e, \\
Q^2_m = \frac{1}{2} t! q^2_m; \\
x = \gamma + N_1 \beta_1 + N_2 \beta_2, \\
y = \gamma + N_2 \beta_2 + N_3 \beta_4. 
\]

(24)

(one sees that the \( Q \)'s coincide with the “physical” charges \([117]\) up to \( \sqrt{G} \)). With these expressions the Einstein equations \((A, B \neq 1)\) read:

\[
e^{\alpha - 2\beta} = e^{\alpha - 2\beta}, \\
\gamma'' = Q^2 e^{2x} + Q^2_m e^{2y}; \\
\beta_1'' = Q^2 e^{2x} - Q^2_m e^{2y}; \\
\beta_2'' = \gamma''; \\
\beta_3'' = -\gamma''; \\
\beta_4'' = -\beta_1''. 
\]

(25)

Certain combinations of these equations can be written in terms of \((\alpha - \beta), x \) and \( y \):

\[
(\alpha - \beta)'' = e^{2\alpha - 2\beta}; \\
x'' = (n - 1)Q^2 e^{2x} + (1 - N_1 + N_2)Q^2_m e^{2y}; \\
y'' = (1 - N_1 + N_2)e^{2x} + (n - 1)Q^2_m e^{2y}. 
\]

(26)

Eq. (27) immediately gives

\[
e^{\beta - \alpha} = s(k, u) 
\]

(29)

where the function \( s(k, u) \) is defined as follows:

\[
s(k, u) \equiv \begin{cases} 
(1/k) \sinh ku, & k > 0, \\
u, & k = 0, \\
(1/k) \sin ku, & k < 0. 
\end{cases} 
\]

(30)

Here \( k = \text{const} \) and another integration constant is suppressed by a choice of the origin of \( u \).

Eqs. (27) and (28) decouple and are then easily solved in the following cases:

(i) \( N_1 = N_4 = 0 \), that is, \( a \cap b = \emptyset \). This happens when the electric and magnetic fields are specified by mutually dual components of the GMF, as is the case in conventional 4-dimensional electrodynamics. Then the functions \( x \) and \( y \) coincide and the equations for \( \beta_1 \) and \( \beta_4 \) disappear, since the corresponding subspaces are absent. This defines what we call the EM(dual)-\( \phi_{\text{min}} \) system.

(ii) \( N_1 = 1 + N_2 \). Since \( N_1 + N_2 = n - 2 \), one can write:

\[
n = 2m + 1, \\
m = 1, 2, ..., \\
N_1 = N_4 = m; \\
N_2 = N_3 = m - 1. 
\]

(31)

So this solvable case occurs with the dimensions \( D = 2n = 6, 10, ... \). This defines another integrable case, labelled EM(non-dual)-\( \phi_{\text{min}} \).

For \( D = 6 \) one has only two variants, either (i), with \( F_{015} \neq 0 \) and \( F_{236} \neq 0 \), or (ii) with \( F_{016} \neq 0 \) and \( F_{235} \neq 0 \). For \( n > 3 \) \((D > 6)\) there exist other variants, when our equations are not so easily (if at all) integrable.

3.1. Solutions EM(dual)-\( \phi_{\text{min}} \)

In the previously described case (i) we have

\[
x'' = \gamma'' + (n - 2)\beta_2'' = Q^2 e^{2x}, \\
Q^2 = (n - 1)(Q e^2 + Q_m^2), 
\]

(32)

whence

\[
e^{-x} = Q s(h_1, u + u_1) 
\]

(33)

where \( h \) and \( u_1 \) are constants and the function \( s \) is defined in (29). It is convenient to normalize all the functions \( \gamma \) and \( \beta_i \) by the conditions

\[
\gamma(0) = \beta_i(0) = 0, 
\]

(34)

i.e., at the flat-space asymptotic all the exponential functions in the metric tend to unity (the real sizes of the internal spaces are thus hidden in \( ds_i^2 \)). Then, in particular,

\[
Q s(h_1, u_1) = 1. 
\]

(35)

Further simple integration finally gives

\[
ds_D^2 = e^{2x} dt^2 - \frac{e^{2\beta_0}}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + dQ^2 \right] \\
+ e^{2\beta_2} ds_2^2 + e^{2\beta_3} ds_3^2, 
\]

(36)

with (33) and

\[
(n - 1)\gamma = x - (n - 2)h_2 u, \\
(n - 1)\beta_2 = x + h_2 u, \\
(n - 1)\beta_3 = -x + h_3 u, \\
(n - 1)\beta_0 = -x - (n - 2)h_3 u 
\]

(37)

where \( h_i = \text{const} \); the \( F \) field components are

\[
F_{01a...a_n} = (-1)^{n-1} q e^{2x}, \\
F_{23b...b_n} = q_m \sin \theta. 
\]

(38)
A substitution of (36), (33) to the constraint equation leads to the following relation between the integration constants:

\[ 2k^2 \text{sign } k = \frac{2}{n-1} h_1^2 \text{sign } h_1 + \frac{n-2}{n-1} (h_2^2 + h_3^2) + 2\phi_1^2, \tag{39} \]

so that there are six independent parameters in the solution: \( q_1, q_2, h_1, h_2, h_3 \) and \( \phi_1^2 \).

The solution generalizes the Reissner–Nordström one and reduces to the latter in the case \( n = 2 \) (when the internal spaces disappear), \( \phi_1 = 0 \).

An analysis of the solution for \( n \geq 3 \) reveals naked singularities in most cases, but under some special conditions, in the absence of a \( \phi \) field, one finds a black hole. Indeed, if

\[ \phi_1 = 0, \quad k = h_1 = h_2 = -h_3, \]
\[ u_1 > 0, \quad k > 0 \tag{40} \]

Eq. (33) is satisfied, the functions \( \beta, \beta_1, \beta_2 \) remain finite as \( u \to \infty \), while \( e^\gamma \to 0 \), and the light travel time \( \int e^{\alpha-\gamma} du \), taken from any finite \( u \) to \( u = \infty \), diverges.

The corresponding black-hole metric looks simpler after the transformation \( u \to R \)

\[ e^{-2ku} = 1 - 2k/R, \tag{41} \]

so that \( u \to \infty \) corresponds to \( R \to 2k \). One obtains

\[ ds_D^2 = \frac{1-2k/R}{(1+p/R)^\xi} dt^2 - (1+p/R)^\xi \left( \frac{dR^2}{1-2k/R} - R^2 d\Omega^2 \right) \]
\[ + \frac{ds_2^2}{(1+p/R)^\xi} + (1+p/R)^\xi ds_3^2 \tag{42} \]

where

\[ \xi = 2/(n-1), \quad p = \sqrt{k^2 + Q^2} - k. \tag{43} \]

In the absence of charges (\( Q = 0, \ p = 0 \)) the solution becomes Schwarzschild’s, with \( GM = k \) (\( G \) is the gravitational constant and \( M \) is the mass), with trivial extra dimensions.

In the general case the gravitating mass of the configuration is obtained from a comparison with the Schwarzschild metric, so that

\[ GM = k + p/(n-1). \tag{44} \]

The charge combination \( Q \) is restricted by

\[ Q^2 < (n-1)^2 G^2 M^2. \tag{45} \]

In the extreme case \( |Q| = GM(n-1), \ k = 0 \), the horizon \( R = 2k \) disappears and, as is easily seen from (32), (33), a naked singularity occurs at \( R = 0 \), the center of symmetry. An exception is the “old” case \( n = 2, \ D = 4 \), when (32) acquires the familiar Reissner–Nordström form after the substitution \( R + p = r \).

### 3.2. Solution EM(non-dual)-\( \phi^{\text{min}} \)

Now consider the case (ii). It is slightly more complex since there are four internal spaces for \( n > 3 \). With (31), Eqs. (27) and (28) yield

\[ e^{-z} = \sqrt{2mQ^2} \, s(h_1, u + u_1), \]
\[ e^{-y} = \sqrt{2mQ^2_m} \, s(h_2, u + u_2), \tag{46} \]

and, after simple integration of the remaining equations, the solution takes the form

\[ ds_D^2 = e^{2\gamma} dt^2 - \frac{e^{2\beta_0}}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + d\Omega^2 \right] \]
\[ + \sum_{i=1}^{4} e^{2\beta_i(u)} ds_i^2, \tag{47} \]

with

\[ \gamma = (x + y)/(2m) - co, \]
\[ \beta_1 = (x - y)/(2m) - c_1 u, \]
\[ \beta_2 = (x + y)/(2m) - c_2 u, \]
\[ \beta_3 = -(x + y)/(2m) - c_3 u, \]
\[ \beta_4 = (y-x)/(2m) - c_4 u, \]
\[ \gamma_0 = -(x + y)/(2m) + mc_1 u + (n-1)c_2 u, \tag{48} \]

where the integration constants \( k, h_i \) and \( c_i \) are connected by the relations

\[ c_0 + mc_1 + (m-1) = 0, \]
\[ k^2 \text{sign } k = \frac{1}{m} (k_1^2 \text{sign } h_1 + k_2^2 \text{sign } h_2) + e^2 \]
\[ + [mc_1 - (m-1)c_2]^2 + 2mc_1^2 \]
\[ + (m-1)(c_2^2 + c_3^2) + 2\phi_1^2. \tag{50} \]

The \( F \) field components retain the same form (53). Thus there are eight independent integration constants: \((q_e, q_m, h_1, h_2, c_1, c_2, c_3, \phi_1)\).

Like that of Subsec. 3.1, this solution possesses naked singularities in most cases, but there is a 3-parameter set of black holes specified by the conditions

\[ \phi_1 = 0, \quad c_1 = 0, \quad h_1 = h_2 = k, \quad -c_2 = c_3 = k/m. \tag{51} \]

Indeed, after the same transformation \( u \to R \) the metric looks as follows:

\[ ds_D^2 = \frac{1 - 2k/R}{(P_e P_m)^{1/m}} dt^2 \]
\[ - (P_e P_m)^{1/m} \left( \frac{dR^2}{1-2k/R} + R^2 d\Omega^2 \right) + (P_m P_e)^{1/m} ds_2^2 \]
\[ + (P_e P_m)^{-1/m} ds_2^2 + (P_e P_m)^{1/m} ds_3^2 + (P_m P_e)^{1/m} ds_4^2 \tag{52} \]

with

\[ P_{e,m} = 1 + p_{e,m}/R; \]
\[ p_{e,m} = \sqrt{2mQ^2_{e,m} + k^2 - k}. \tag{53} \]
Like (42), the metric (52) has the property
\[ \sigma = \sum_i N_i \beta_i \equiv 0, \]  
so that the volume element of the extra dimensions is \(R\)-independent.

The gravitating mass is calculated for (53) similarly to (42):
\[ GM = k + \frac{1}{2m}(p_e + p_m). \]  
The charge parameters \(Q_e\) and \(Q_m\) are constrained by
\[ Q_e^2 + Q_m^2 < 2mG^2M^2. \]  
Again, in the extreme case \(k = 0\), when we have an equality in (56), the metric has a singular center \(R = 0\).

Unlike (42), (52) contains three instead of two parameters since the electric and magnetic charges enter into it separately (although in a symmetric manner). Despite the close similarity between these metrics, (45) has no Reissner–Nordström special case, since the total space-time dimension is \(D = 2(2m + 1)\), \(m \geq 1\). For \(m = 1\), \(n = 3\), the subspaces with \(ds_2^2\) and \(ds_3^2\) are absent; by (51) they have zero dimension.

### 4. Solutions with a scalar-type component

Let us now try to find solutions containing the scalar component of the \(F\) field. We will also assume that there are mutually dual electric and magnetic fields, so that their sets of indices are completely different, \(a \cap c = 0\). We ascribe the following labels \(i\) to the internal factor spaces (where \(a = \{a_j\}\), etc., see (14):
\[ \begin{align*}
    a \cap c & \mapsto i = 1; \\
    a \setminus c & \mapsto i = 2; \\
    b \cap c & \mapsto i = 3; \\
    b \setminus c & \mapsto i = 4. 
\end{align*} \]  
The EMT of the \(F\) field takes the form
\[ e^{2\alpha}T_{A}^{B} = Q_{em}^{2} \text{diag}(1, 1, -1, -1, [1]_{n-2}, [-1]_{n-2}) e^{2x} + Q_{s}^{2} \text{diag}(1, -1, 1, 1, [1]_{N_1}, [1]_{N_2+N_3}, [-1]_{N_2+N_3}) e^{2y}, \]  
where
\[ \begin{align*}
    Q_{em} &= \frac{1}{2}n!(q_e^2 + q_m^2), \\
    x &= \gamma + N_1 \beta_1 + N_2 \beta_2, \\
    y &= N_3 \beta_1 + N_4 \beta_4. 
\end{align*} \]  
Denoting \(N_1 = m \geq 1\), we obtain the following \(N_i\):
\[ \begin{align*}
    N_1 &= m, \\
    N_2 &= n - m - 2, \\
    N_3 &= m - 1, \\
    N_4 &= n - m - 1. 
\end{align*} \]  

The field equations similar to (27) are
\[ \begin{align*}
    e^{2\alpha-2\beta} - \beta'' &= Q_{em}^2 e^{2x} - Q_{s}^2 e^{2y}, \\
    \gamma'' &= Q_{em}^2 e^{2x} + Q_{s}^2 e^{2y}, \\
    \beta''_1 &= Q_{em}^2 e^{2x} - Q_{s}^2 e^{2y}, \\
    \beta''_2 &= \gamma'', \\
    \beta''_3 &= -\beta''_1, \\
    \beta''_4 &= -\gamma''. 
\end{align*} \]  
As in Subsection 3.1, we obtain (26), while the equations for \(x\) and \(y\) are
\[ \begin{align*}
    x'' &= (n - 1)Q_{em}^2 e^{2x} + (n - 2m - 1)Q_{s}^2 e^{2y}, \\
    y'' &= -(n - 2m - 1)Q_{em}^2 e^{2x} - (n - 1)Q_{s}^2 e^{2y}. 
\end{align*} \]  
The situation is somewhat similar to that in equations (27, 28), with the difference that we cannot put \(x = y\): the set of equations (61) then becomes inconsistent. So the only case when Eqs. (62) decouple is
\[ n = 2m + 1, \quad m = 1, 2, ... \]  
and we again deal only with \(D = 2n = 6, 10, 14, \ldots\), so that, by (50),
\[ N_1 = N_4 = m, \quad N_2 = N_3 = m - 1. \]  
It is easy to verify that the case of coinciding indices indicated in the Remark in Sec. 2 \((a \cap c = a)\) occurs here only for \(m = 1, n = 3\). In this case our solution cannot contain an electric component \(F_{01a}\) and incorporates only magnetic and scalar charges.

After integration we obtain
\[ \begin{align*}
    e^{-x} &= \frac{1}{h_1} \sqrt{n - 1}Q_{em}^2 s(h_1 + u + u_1), \\
    e^{-y} &= \frac{1}{h_2} \sqrt{n - 1}Q_{s}^2 \cosh[h_2(u + u_2)], 
\end{align*} \]  
where, as before, \(h_1, h_2, u_1, u_2\) are constants constrained by the conditions \(x(0) = y(0) = 0\).

Further integration is quite simple and results in a metric having the form (17), but with the following exponents instead of (48):
\[ \begin{align*}
    \gamma &= x - y/(2m) - c_0 u, \\
    \beta_1 &= x + y/(2m) - c_1 u, \\
    \beta_2 &= -x + y/(2m) - c_2 u, \\
    \beta_3 &= -x + y/(2m) - c_3 u, \\
    \beta_4 &= y - x/(2m) + c_4 u, \\
    \beta_5 &= -x + y/(2m) - mc_1 u + (m - 1)c_3 u, 
\end{align*} \]  
with constraints upon the constants very similar to (13) and (56):
\[ \begin{align*}
    c_0 + mc_1 + (m - 1)c_2 &= 0, \\
    2k^2 \text{sign} k &= \frac{1}{m}(h_1^2 \text{sign} h_1 + h_2^2) + c_0^2 + 2mc_1^2 \]
\[ + [mc_1 - (m - 1)c_3]^2 + (m - 1)(c_1^2 + c_3^2) + 2c_1^2. \]  

Unlike the solutions EM(dual)-$\phi_{min}$ and EM(nondual)-$\phi_{min}$, this one, labelled EM(dual)-$S\phi_{min}$, does not contain a black-hole case. Indeed, as is easily verified, all solutions with the coordinate $u$ specified in a finite range $0 < u < u_{\text{max}} < \infty$ (this happens when $h_1 < 0$ and/or $u_1 < 0$) have naked singularities. As for $u \rightarrow \infty$, the place for a horizon in the previous solutions, the requirement that $\beta_1, ..., \beta_4$ tend to finite limits cannot be fulfilled since now $\beta_1(u) + \beta_4(u) = y/m$, while $y \rightarrow -\infty$ as $u \rightarrow \infty$.

In the case $m = 1$, $n = 3$, the functions $\beta_2$ and $\beta_3$ disappear along with the corresponding factor spaces, and the solution is considerably simplified.

5. Solutions $S\phi_{min}$

Equations (61) are easily solved for any $n$ if the electric and magnetic components of the $F$ field are absent. In this case, the factor spaces (1 and 4) and (2 and 3) unify and the resulting solution, obtained just as the previous ones, has the form,

$$ds_D^2 = e^{2\gamma}dt^2 - e^{2\beta_0} \frac{du^2}{s^2(k,u)} + d\Omega^2$$

$$+ e^{2\beta_1}d\bar{s}_1^2 + e^{2\beta_2}d\bar{s}_2^2$$

(68)

where

$$e^{-(n-1)\beta_1} = \frac{1}{h} \sqrt{(n-1)Q_y^2} \cosh[h(u+u_1)],$$

$$\beta_1(0) = 0, \quad k > 0,$$

$$\gamma = -\beta_1 - cu, \quad \beta_2 = -\beta_1 - cu,$$

$$\beta_0 = -\beta_1 + [c_0 + (n-3)c_2]u.$$  

(69)

The constants are related by

$$2k^2 \text{sign} k = -\frac{2}{n-1}h^2 + c_0^2 + [c_0 + (n-3)c_2]^2$$

$$+ (n-3)c_2^2 + 2\phi_1^2.$$  

(70)

There is no black-hole case in this solution as well since in all cases $\beta_1 \rightarrow -\infty$ as $u \rightarrow \infty$.

6. The Hawking temperature for black-hole solutions

Event horizons are known to induce quantum vacuum instability, leading to creation of particle-antiparticle pairs [21, 22], also interpreted as black hole evaporation. The latter is observable from infinity as black-body radiation, whose temperature, called the Hawking temperature, is thus one of the key parameters of a black hole. In non-black-hole cases the notion of a temperature is apparently unapplicable.

According to [22], the Hawking temperature of a spherically symmetric black hole can be found in the form

$$k_{\text{rmB}}T = \hbar \alpha e/2\pi,$$

$$\alpha = \frac{\sqrt{g_{00}^\gamma}}{\sqrt{-g_{11}}} - \gamma_{\text{horizon}},$$

(71)

where $k_B$ is the Boltzmann constant and the notations $\alpha$, $\gamma$ and “prime” correspond to Eq. (8).

It should be noted that the expression (71) is not only invariant under reparametrization of the radial coordinate (as is necessary for any quantity having a direct physical meaning), but also conformal gauge independent, or, in other words, invariant under conformal mappings of the 4-dimensional metric provided the conformal factor is regular on the horizon. Indeed, due to the above reparametrization invariance, we may safely assume that the horizon takes place at a finite value of the radial coordinate. Then, from $e^{\gamma} \rightarrow 0$ it follows $\gamma' \rightarrow \infty$ on the horizon, and in the expression (71) any finite contribution to $\gamma'$ may be ignored; but, on the other hand, a regular conformal factor results in just a finite contribution to $\gamma'$ and does not affect the expression $e^{\alpha-\gamma}$ at all.

Thus a black hole has the same temperature for observers using different sets of instruments (such that they see the space-time in different conformal gauges). The same is true for black hole electric and magnetic charges (if any) but not for the mass, which is determined by the 4-dimensional metric at the asymptotic and is thus sensitive to conformal factors. In particular, the mass dependence of the temperature is conformal gauge dependent.

We have obtained here two black hole solutions [22], [23]. Let us find their temperatures, comparing them with some other known solutions describing multidimensional black holes. We begin with referring to known results.

Conventional Einstein-Maxwell fields in $D$ dimensions

The metric of an electrically charged black hole may be written in the form [17, 23]

$$ds_D^2 = \frac{1 - 2k/R}{(1 + p/R)^2} dt^2$$

$$- (1 + p/R)^2/N \left[ \frac{dR^2}{1 - 2k/R} + R^2 d\Omega^2 - \sum_{i=1}^s ds_i^2 \right]$$

(72)

where the integration constant $k \geq 0$ has the same meaning as in the present paper,

$$N = D - 3, \quad p = \sqrt{k^2 + \vec{q}^2} - k, \quad \vec{q}^2 = 2Nq^2/(N+1)$$

and $q$ is the electric charge. According to (71),

$$\alpha = \frac{1}{4k} \left( 1 + \frac{p}{2k} \right)^{(N+1)/N}.$$  

(73)
On the other hand, the black-hole mass is (up to the factor \( G \)) \( M = p + k = \sqrt{k^2 + q^2} \), so that \( \alpha \) can be expressed in terms of the mass and the charge:

\[
\alpha = \frac{1}{2}(2\sqrt{M^2 - q^2})^{1/N}(M + \sqrt{M^2 - q^2})^{-(N+1)/N}. \tag{74}
\]

The minimal mass for a given charge \( q \) corresponds to \( k = 0 \) and zero temperature (the extreme case). For \( N > 1 \) \( (D > 4) \) the metric possesses a naked singularity instead of a horizon. In other respects the above formulas are direct generalizations of those well-known for the Reissner–Nordström case.

**Dyon black holes in multidimensional dilaton gravity**

In the case of dilaton gravity, known as a field limit of string theory, the most general spherically symmetric black hole solution contains both electric \((q_e)\) and magnetic \((q_m)\) charges; the string metric (i.e., the metric in the string conformal gauge, fundamental for the underlying theory) reads \( [11] \):

\[
ds^2_D = \frac{(r - r_e)(r - r_m)}{(r + r_e - r_m)^2}dt^2 - \frac{r^2dr^2}{(r - r_e)(r - r_m)} - r^2d\Omega^2 + \sum_{i=1}^{s} ds_i^2 \tag{75}
\]

with the notations

\[r_e = 2k + r_m \geq r_m, \quad r_{e,m} = \sqrt{k^2 + 2q_e q_m} - k.\]

The event horizon takes place at \( r = r_+ \), except for the purely magnetic extreme case \( k = r_e = 0 \), when the metric is regular and \( g_{00} = \text{const.} \).

The “temperature factor” \( \alpha \) is

\[
\alpha = \frac{k}{r_+(2k + r_e)} = \frac{r_+ - r_m}{r_+(r_+ + r_e - r_m)}. \tag{76}
\]

It is dimension-independent and vanishes for \( k = 0 \).

**Solutions EM(dual), Eq. (42) and EM(non-dual), Eq. (52)**

The field \( \phi^{\text{min}} \) is omitted from the notation since the scalar field is absent in the black hole solutions.

In the dual case the metric and the mass are determined by (42) and (44). The black-hole “temperature factor” \( \alpha \) is

\[
\alpha = \frac{1}{4k} \left(1 + \frac{p}{2k}\right)^{-2/(n-1)}. \tag{77}
\]

Recall that here \( p = \sqrt{k^2 + Q^2} - k \), \( k > 0 \) and \( Q^2 = (n-1)/Q_e^2 + Q_m^2 \), \( Q_e \) and \( Q_m \) being the physical electric and magnetic charges of the black hole. The well-known result for a Reissner–Nordström black hole \( [2] \) is recovered when \( n = 2 \).

In the non-dual case the corresponding relations are (72) and (55). For \( \alpha \) we obtain:

\[
\alpha = \frac{1}{4k} \left(1 + \frac{p_e}{2k}\right)^{-1/m} \left(1 + \frac{p_m}{2k}\right)^{-1/m}. \tag{78}
\]

The quantities \( p_{e,m} \) are expressed in terms of \( k > 0 \) and the charges in Eq. (43). Recall that here the space-time dimension is \( D = 2(2m + 1) \), \( m \) being a positive integer, so that this family of solutions does not include the Reissner–Nordström one.

All the above expressions for \( \alpha \), except (76), depend explicitly on the space-time dimension, so that the temperature tends to a finite limit as \( D \to \infty \). In the dilaton case (43) the extra dimensions are “passive” not only in that their scale factors are constant (in the string gauge), but also in that the \( D \)-dependence falls out of the whole metric. In this respect the dilaton case is the most similar to the uncharged (Schwarzschild) solution — the zero charge limit of all the above solutions.

The expressions for \( \alpha \) exhibit the most interesting distinctions in the extreme limit \( k \to 0 \), describing the minimum possible mass for given charges. Note that this extreme case corresponds to a black hole only in the Reissner–Nordström case (\( D = 4 \)) and for the dilaton solution (73) with \( q_e \neq 0 \).

Thus, for the known solutions (72) and (74), the temperature \( T \) vanishes in this extreme case. Unlike that, the expression (74) for Solution EM(dual) vanishes as \( k \to 0 \) only for \( n = 2 \) (the Reissner–Nordström black hole); for \( n = 3 \) the limiting \( T \) is finite; for greater \( n \), \( T \to \infty \) as \( k \to 0 \). The same picture is observed with the non-dual case: the expression (78) tends to a finite limit as \( k \to 0 \) only if \( m = 1 \) and there is just one nonzero charge, either electric, or magnetic one (this case actually coincides with a special case of EM(dual), \( n = 3 \)); in all other cases of EM(non-dual) we have \( T \to \infty \) as \( k \to 0 \). So in our case the black hole evaporation dynamics should be drastically different from that of Reissner–Nordström or dilaton black holes.

**7. Discussion**

We have found some exact spherically symmetric static solutions for a theory containing gravity and a generalized Maxwell field in higher dimensions in integrable cases which we were able to select.

All the solutions found, except for some special cases, exhibit naked singularities. However, from the standpoint of the well-known no-hair theorems (claiming, in particular, that in general relativity a spherically symmetric black hole cannot have an external minimally coupled scalar field), it is more surprising that there exist black hole subfamilies in our families of solutions, since the extra-dimension scale factors are
4-dimensional scalars. The latter, however, are not minimally coupled to matter, and this is apparently a reason for the appearance of nontrivial scalar fields in multidimensional black hole solutions. Other examples of such solutions are those known in multidimensional Einstein and dilaton gravity, where nontrivial scalar (or 4-dimensional scalar) fields exist only in the presence of a nonzero electric or/and magnetic charge.

The present black-hole solutions are among the simplest ones in the following sense. There are physically different 4-dimensional formulations of the same multidimensional theory, corresponding to different conformal gauges (Einstein gauge, atomic gauge, etc.) — conformal factors depending on the volume of the internal space, which, in general, varies from point to point. In the present notation, such conformal factors depend on \( \sigma \) (see (1)). One can see that in our black-hole solutions \( \sigma = \text{const} \), so for these solutions all conformal gauges coincide.

For the black-hole solutions found, the Hawking temperature depends both on electromagnetic charges and masses and on the space-time dimension \( D \). The \( D \) dependence is somewhat similar to that in the solution for the conventional Einstein-Maxwell system and disappears with switching-off the gauge fields.

A crucial, and potentially observable, difference between our black-hole solutions and the “old” ones, Eqs. (2) and (3), is that for most of our solutions the black-hole temperature grows infinitely in the extreme limit (that of minimal mass for given charges) — for more details see the previous section.

An issue of importance is the classical stability of static configurations. From the previous studies of the stability of static vacuum and electrovacuum solutions in multidimensional Einstein and dilaton gravity it can be concluded that only black-hole solutions are stable, while those with naked singularities are catastrophically unstable. It would be of interest to extend this study to the present solutions with the GMF.

It also makes sense to analyze the weak-field limit of the new solutions to learn their viability range and to try to predict their observational manifestations.

As is the case with other spherically symmetric solutions, their actual significance (in particular, the role of naked singularity solutions and their relation to the cosmic censorship conjecture) can be understood only after a full dynamical study of gravitational collapse.

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