Fivebrane instantons and higher derivative couplings in type I theory

Amine B. Hammou, 1
SISSA, Trieste, Italy,

Jose F. Morales, 2
INFN, Sezione di “Tor Vergata”, Roma, Italy,

Abstract

We express the infinite sum of D-fivebrane instanton corrections to $R^2$ couplings in $\mathcal{N} = 4$ type I string vacua, in terms of an elliptic index counting $\frac{1}{2}$-BPS excitations in the effective $Sp(N)$ brane theory. We compute the index explicitly in the infrared, where the effective theory is argued to flow to an orbifold CFT. The form of the instanton sum agrees completely with the predicted formula from a dual one-loop computation in type IIA theory on $K3 \times T^2$. The proposed CFT provides a proper description of the whole spectrum of masses, charges and multiplicities for $\frac{1}{2}$- and $\frac{1}{4}$- BPS states, associated to bound states of D5-branes and KK momenta. These results are applied to show how fivebrane instanton sums, entering higher derivative couplings which are sensitive to $\frac{1}{4}$-BPS contributions, also match the perturbative results in the dual type IIA theory.

PACS: 11.25.-w, 11.25.Hf, 11.25.Sq

Keywords: Instantons, D5-brane, Thresholds, BPS, CFT

1e-mail: amine@sissa.it
2e-mail: morales@roma2.infn.it
1 Introduction

String dualities often map worldsheet instanton effects on one string vacuum to space-time instantons in the dual one [1, 2]. A perturbative computation in one side can then be used to shed light about the general rules of the instanton calculus in string theory. This idea motivates the study of [3], where heterotic world-sheet instanton corrections to four-derivative couplings in toroidal compactifications to eight dimensions were translated into a sum of D-instanton effects in the dual type I side. The analysis was extended to lower dimensional ($D > 4$) heterotic/type I toroidal compactifications [4] and to $\mathcal{N} = 4$ dual pairs of type II string vacua [5, 6]. In each of the cases the N-instanton corrections, associated to bound states of N D-string wrapping the $T^2$-spacetime torus, were expressed in terms of elliptic genera counting the number of $\frac{1}{2}$-BPS excitations in the relevant $O(N)$ gauge theories. The analyses were always restricted to $D > 4$ dimensions where the less understood fivebrane physics is irrelevant. The aim of this paper is to extend to $D = 4$ dimensions these results by explicitly computing the D5-instanton effects in the type I brane theory.

We will consider instanton corrections to higher derivative couplings in toroidally compactified type I theory down to four dimensions. The only sources of non-perturbative corrections (to the kind of couplings we will consider) in these string vacua are associated to Euclidean D5-branes wrapping entirely the $T^6$-torus, whose BPS excitations can be properly described, as we will see, by an $\mathcal{N} = (4,4)$ orbifold CFT. Less supersymmetric instanton configurations would have too many fermionic zero modes to be soak by the vertex insertions (typically $\mathcal{R}^2$), while D-string instantons leads to $\mathcal{N} = (8,0)$ effective sigma models [7], where the vertex insertions can soak at most four of the eight left moving fermionic zero modes.

We will work out the details for $\mathcal{R}^2$ couplings and comment about the generalizations to other four and higher derivative “BPS” saturated terms. These couplings are special in that they receive corrections only from states saturing the BPS bounds and they have been extensively studied in many contexts [8]-[12]. In this paper we will concentrate in the non-perturbative part of the corrections to $D = 4$ effective lagrangians in type I vacua with sixteen supercharges. The instanton sums will be always expressed in terms of an elliptic genus in the effective $Sp(N)$ gauge theory, which encodes the information about masses, charges and multiplicities of $\frac{1}{2}$ BPS excitations in the corresponding D5-brane system. We will compute the “quasi” topological index and show that the form of the instanton corrections to the associated couplings agree with the predictions from duality to type IIA on $K3 \times T^2$. Interestingly, the CFT description of the infrared limit of the D5-brane world-volume theory reproduces the right multiplicities even for $\frac{1}{2}$-BPS states, associated to D5-KK bound states. We apply this result to show that instanton corrections to certain higher derivative couplings, sensitive to these states, agree again
with the fundamental type IIA results\(^1\).

The paper is organized as follows. In section 2 we compute D5-instanton corrections to \(\mathcal{R}^2\)-couplings in type I theory on \(T^6\), and compare it to the perturbative result obtained in the dual type IIA theory. In section 3, we briefly discuss the extension of such results to other higher derivative terms, which are sensitive to \(\frac{1}{4}\) BPS contributions. Section 4 summarizes the results and present some conclusions.

### 2 \(\mathcal{R}^2\) couplings in type IIA/type I \(\mathcal{N} = 4\) string vacua

As we have discussed in the introduction, worldsheet/spacetime instantons in type IIA theory on \(K3 \times T^2\)/type I theory on \(T^6\) are mapped to each other under the triality map

\[
T_{IIA} = S_H = S_I. \tag{2.1}
\]

The subscripts \(IIA, H, I\) refer to the type II, Heterotic and type I theories, while the complex moduli\(^2\)

\[
T_{IIA} = B_{45} + i\sqrt{G} \\
S_H = a + i\frac{V_6}{\lambda^2} \\
S_I = a + i\frac{V_6}{\lambda}. \tag{2.2}
\]

describe the complexified Kahler structure of the \(T^2\) torus in the type IIA side, and the four dimensional complexified string couplings in the heterotic and type I string vacua respectively with \(V_6\) the \(T^6\)-volume. More precisely, \(a = \tilde{B}_{456789}\), where \(\tilde{B}\) is defined by

\[
d\tilde{B} = e^\phi dB,
\]

\(B\) being the second rank, antisymmetric tensor in the corresponding string theory, and \(\lambda = e^\phi\) the ten-dimensional string coupling.

In this section we will show how the perturbative, world-sheet instanton contribution to the \(\mathcal{R}^2\) coupling in the type IIA theory, can be directly reproduced in type I theory, using the effective six dimensional, \(\mathcal{N} = 1\) gauge theory of D5 branes, in the limit where it flows to a two dimensional \(\mathcal{N} = (4, 4)\) orbifold Conformal Field Theory, after dimensional reduction on a four-torus.

#### 2.1 One-loop \(\mathcal{R}^2\) couplings in type IIA

Let us start by recalling the results for the moduli dependence \(\Delta_{str}(T, U)\) of \(\mathcal{R}^2\)-couplings in type IIA theory on \(K3 \times T^2\) \[8, 9\]\(^3\). Perturbative corrections to \(\mathcal{R}^2\) terms in (2, 2) string vacua are expected to arise only at one-loop level, and depend only on the \(T(U)\)

\(^1\)The couplings under consideration, are closely related to those studied in [12].

\(^2\)In this paper, all quantities having dimension of a length will be understood in units of \(2\pi \sqrt{\alpha'}\).

\(^3\)We will follow the notations and normalizations of [9].
complex modulus describing the Kahler (complex) structure of the $T^2$ torus in the type IIA (IIB) string compactifications. The type IIA result can then be written as \[8, 9\]

$$\partial_T \Delta_{gT}(T) = \int_F \frac{d^2\tau}{\tau_2} \partial_T B_4,$$  

(2.3)

where $B_4$, is an index counting the number of $\frac{1}{2}$-BPS string states (the helicity supertrace). The index can be defined as \[9\]

$$B_4 = \left\langle (\lambda_L + \lambda_R)^4 \right\rangle = \left(\frac{4}{2}\right) \frac{1}{(2\pi i)^2} \left. \theta^2 \chi(v|\tau) \right|_{v=0} = \frac{6}{16\pi^4} \left. \partial_\tau^2 \partial_{\bar{\tau}}^2 Z(v, \bar{v}) \right|_{v=\bar{v}=0},$$  

(2.4)

in terms of the helicity supertrace generating function

$$Z(v, \bar{v}) = Tr' q^{L_0 - \frac{c}{24}} q^{L_0 - \frac{c}{24}} e^{2\pi i (v\lambda_L - \bar{v}\lambda_R)},$$  

(2.5)

$\lambda_L(R)$ being the left(right) moving helicity operators, and the prime on $Tr$ means the omission of the space-time bosonic zero modes contributions, and $q \equiv e^{2\pi i \tau}$. One can see that $Z(v, \bar{v})$ receives in general contributions from both BPS and non-BPS string states. This is not the case for the chiral supertrace $\chi(v|\tau)$, introduced in (2.4). Indeed, the insertion of two $\lambda_R$’s in (2.5) soaks precisely four right-moving fermionic zero modes (after spin structure sums), while massive bosonic and fermionic right moving excitations, cancel against each other by supersymmetry, giving as a result the holomorphic function $\chi(v|\tau)^4$. This chiral supertrace encodes all the information about BPS multiplicities and charges, and we will refer to it as the “elliptic genus” of the corresponding conformal field theory. Specializing to the case of type IIA string theory on $K3 \times T^2$, we get, after summing over the spin structures,

$$Z(v, \bar{v}) = 8 \left| \xi(v) \frac{\partial^2 (\frac{v}{2})}{\eta^2} \right|^2 \sum_{i=1}^{4} \left| \frac{\partial^2 (\frac{v}{2})}{\partial^2 (0)} \right|^2 \Gamma_{2,2},$$  

(2.6)

where

$$\xi(v) = \frac{\sin \pi v}{\pi} \frac{\theta'_4(0)}{\theta_4(v)},$$  

(2.7)

is the contribution of the spacetime boson coupled to the helicity. Substituting (2.6) in (2.4), one finds \[9\]

$$B_4 = 36 \Gamma_{2,2}.$$  

(2.8)

The $\Gamma_{2,2}$ lattice sum can be written as

$$\Gamma_{2,2} = \frac{T_2}{\tau_2} \sum_{M \in GL(2, \mathbb{Z})} e^{2\pi i T \text{det} M} e^{-\frac{\pi T_2}{\tau_2} \left| (1 U) M \left( \frac{1}{2} \right) \right|^2},$$  

(2.9)

where the sum runs over all possible world-sheet instantons

$$\left( \begin{array}{c} X^4 \\ X^5 \end{array} \right) = M \left( \begin{array}{c} \sigma^1 \\ \sigma^2 \end{array} \right) \equiv \left( \begin{array}{cc} m_1 & n_1 \\ m_2 & n_2 \end{array} \right) \left( \begin{array}{c} \sigma^1 \\ \sigma^2 \end{array} \right),$$  

(2.10)

\footnote{The chiral trace $\chi(v|\tau)$ can be considered as the generating function for the asymmetric supertraces introduced in \[12\].}
with worldsheet and target space coordinates $\sigma^1, \sigma^2$ and $X^4, X^5$ respectively. The modular integration in (2.3) can be done using the standard trick \cite{13}, where fundamental domain integrals are unfolded to the strip (degenerated orbits $\text{Det} M = 0$) and to the whole upper half plane (non-degenerated orbits $\text{Det} M \neq 0$) integrals of certain $SL(2, \mathbb{Z})$ representatives. The final result is \cite{8,9}

$$
\Delta_{gr}(T) = -36 \log \left( T_2 |\eta(T)|^4 \right). \tag{2.11}
$$

Using the duality relations (2.1) we can rewrite this in terms of the type I variables:

$$
\Delta_{gr}(S) = -36 \log \left( S_2 |\eta(S)|^4 \right) = -36 \log S_2 + 12\pi S_2
+ 72 \sum_{N=1} \left( \sum_{M \mid N} \frac{1}{M} \right) \left[ e^{2\pi i NS} + e^{-2\pi i NS} \right]. \tag{2.12}
$$

where $N \mid M$ stands for the partitions of $N$ ($N = LM$ with $N, L, M \in \mathbb{Z}$). The first term in (2.12) corresponds to a logarithmic divergence in the weak coupling limit $S_2 = \frac{V^4}{\lambda} \to \infty$, the second term come from a disk diagram contribution and the rest is an infinite sum of D-instanton corrections. The logarithmic divergent term in (2.12) is attributed to an IR divergence, but a complete understanding is still missing (see \cite{14,15} for details).

It has been shown in \cite{11} that $R^2$ couplings receive a resulting non-vanishing one-loop contributions from the Klein, annulus and Moebius in type I theory. The absence of the one-loop term in the perturbative formula (2.12) suggests that the $R^2$ result in type IIA should correspond to a combination of $R^2$ together with other four derivative couplings in the type I side. For instance, the authors of \cite{11} have shown that a suitable combination of $R^2$ and $F^2_1F^2_2$ can account for the discrepancy. One can see that four-derivative couplings involving the dilaton field will lead again to a one-loop expression similar to its closely related partner $R^2$, making them potential sources to take into account the perturbative discrepancy. Moreover, the duality relation $G^{IIA}_{\mu \nu} = \sqrt{4} e^{-\phi} G^{I}_{\mu \nu}$ suggests that in the translation of the type IIA $R^2$ term in the type I variables four-derivative couplings constructed out of the dilaton and volume modulus should indeed be relevant. Our instanton computation will not give a definite answer to this question, but it will support these latter sources. The comparison with the perturbative formula will account then only for the form of the instanton sums, leaving an overall coefficient to be accounted for by the right combination of four derivative terms in type I dual to the type IIA $R^2$ coupling.

A similar formula (with $S$ replaced by $S_H$ in (2.12)) describe the instanton corrections in the heterotic side. The correct instanton action weights $e^{2\pi i NS_H \mu}$ were reproduced in \cite{14} from the classical action of heterotic NS-fivebranes wrapped on the $T^6$ torus, however it is hard to see how the determinant factor $\sum_{N \mid M} \frac{1}{M}$ can be computed with our current understanding of the NS-fivebrane physics. Fortunately, type I spacetime
instantons are associated to the more tractable D-branes, for which a CFT description is at our disposal [16].

2.2 D5-brane instanton corrections to $R^2$-couplings

We are interested in computing the two-graviton correlation function

$$\langle V_g V_g \rangle_{D5\text{-inst}}$$

in the background of $N$ Euclidean D5 branes wrapping the $T^6$ spacetime torus. We take for the spacetime torus the limit in which the volume of a $T^4$ torus in $T^6$ becomes very small ($R_i \sim \sqrt{\alpha'}, i = 6, 7, 8, 9$) keeping $R_4, R_5$ fixed when the D5-brane theory decouples from gravity $\alpha' \to 0$. We will reduce the six-dimensional world-volume theory onto this two-dimensional plane with light cone coordinates $X^\pm = X^4 \pm X^5$.

The classical part of the computation in the D5-instanton background closely follows the one for the heterotic NS-fivebrane [14] with obvious modifications. The low energy effective action describing $N$ spinless Euclidean D5-branes wrapping once the $T^6$ spacetime torus can be written as [17]

$$S_{\text{ND5}} = NT_5 \int d^6 \xi \left[ e^{-\phi} \sqrt{\det \hat{G}} + i \tilde{B}_{456789} \right] + \cdots = -2\pi i NS_I + \frac{2\pi \mathcal{V}_4 U^2}{\lambda} \int d^2 h_{\mu\nu} \sum_{t=1}^N D_z X_t^\mu D_{\bar{z}} X_t^\nu + \cdots$$

(2.14)

where $\mathcal{V}_4$ is the volume of the small $T^4$ torus, $U$ the complex structure of the large $T^2$-torus, $T_5 = \frac{(2\pi)^4}{\alpha'}$ the fivebrane tension $^5$, $S_I$ is the complexified string coupling constant given by (2.2) and $h_{\mu\nu}$ the quantum fluctuation ($G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) around the flat metric, $\mu, \nu = 0, 1, 2, 3$. We choose the static gauge $X^a = \xi^a$ ($a = 4, 5, \ldots, 9$) for the fivebrane coordinates, where the six-dimensional pullback metric $\hat{G}$ is identified with the spacetime torus metric. The last term in (2.14) represents the coupling of the graviton to the instanton background with covariant derivatives

$$D_z X_t^\mu = \partial_z X_t^\mu - \frac{1}{4} p_\nu S_t \gamma^{\nu\mu} S_t$$

$$D_{\bar{z}} X_t^\mu = \partial_{\bar{z}} X_t^\mu - \frac{1}{4} p_\nu \bar{S}_t \gamma^{\nu\mu} \bar{S}_t$$

(2.15)

in the $(4, 5)$ plane with complex coordinates $z = \xi^4 + U \xi^5$, $\partial_z = \frac{1}{U^2} (\partial_4 + U \partial_5)^6$. Quantum corrections around this background are described by an $Sp(N)$ gauge theory

\footnote{Notice that the $T_1, T_3$ tensions of type I D-branes are $\frac{1}{\sqrt{2}}$ the corresponding tension in type IIB. The electric-magnetic quantization condition implies then that D5 branes come always in pairs to account for the extra factor of $\sqrt{2}$ [18]. $N$ here counts the number of type I D5-branes which halves the number in the parent type IIB theory.}

\footnote{Strictly speaking, our analysis will be performed in the Minkowski world-volume with time like coordinate $\xi^5$. Only at the end we will come back to the Euclidean plane by an analytic continuation.}
defined by the quantization of the massless modes of unoriented open strings ending on the D5 branes.

The computation of the scattering amplitude (2.13) is very similar to the perturbative computation in the one-loop worldsheet instanton background of the last section, with the worldsheet parameter $\tau$ replaced by the complex structure $U$ of the target space $(4,5)$ torus. Graviton insertions can be expressed as derivatives of the instanton action (2.14) with respect to the metric fluctuations $h_{\mu\nu}$, bringing down the $(4,4)$ needed fermionic zero modes $S_{0cm}$, the right power of momenta to reproduce the $R^2$ kinematics and an overall $V^2 U^2 / N^2 \lambda^2$ factor. Unlike in references [5, 6], we adopt the canonical normalization $S_t = \frac{1}{\sqrt{N}} S_{cm} + ...$ for the fermionic center of mass $(\sum_{t=1}^{N} S_t \partial S_t = S_{cm} \partial S_{cm} + ...)$, which is responsible for the additional factor of $1 / \sqrt{N}$. The final result can then be written as

$$\Delta_{\text{inst}}^{\text{gr}} = \frac{V^2 U^2}{N^2 \lambda^2} \langle e^{-S_{\text{class}} - S_{\text{SYM}}} \rangle' = \frac{V^2}{\lambda^2} \sum_{N} B_4^{ND5}(e^{2\pi i N S} + e^{-2\pi i \bar{S}})$$

in terms of the $\frac{1}{2}$-BPS index $B_4$ (2.4) of the $Sp(N)$ gauge theory. The prime in (2.16) means that the trace does not include the fermionic zero mode part already taken into account by the vertex insertions, while the spacetime bosonic zero mode contribution cancel the $U^2_2$ factor in the numerator. The overall $1 / \sqrt{N}$ factor in (2.16) has been reabsorbed in $B_4^{ND5}$ in order to make more transparent the comparison with the IIA perturbative result. Finally the $S_{\text{class}} = -2\pi i N S(2\pi i \bar{S})$ represents the classical (anti)instanton action.

The rest of this section will be devoted to the computation of the $B_4^{ND5}$ index of the $Sp(N)$ gauge theory. For future reference, we will be slightly more general than what we really need, by determining the whole BPS elliptic genus $\chi(v|\tau)$. We will follow the strategy of [7, 5]. The elliptic genus, being invariant under any deformation of the gauge theory, in particular under variation of the string coupling constant, can be evaluated in the regime which is more convenient for our purposes. We will compute it explicitly in the infrared limit, where the theory will be argued to flow to an orbifold conformal fixed point.

The low energy effective action associated to a system of $N$ parallel D5-branes in type I theory can be obtained from the more familiar $U(2N)$ gauge theory describing $2N$ D5-branes in the parent type IIB theory. Type IIB D5 brane fields $\Phi$ are projected onto $2N \times 2N$ matrices satisfying the $\Omega$-even simplectic condition [19]

$$\Phi = \pm \Omega \Phi^T \Omega^{-1} \quad \Omega = \sigma_2 \times I_N$$

where $+(-)$ stands for the DD(NN) directions and $\sigma_i$ are the Pauli matrices. In addition, anomaly cancellation requires the inclusion of 32 D9-branes and the corresponding open string sectors. The resulting two-dimensional field content after dimensional reduction on $T^4$ is defined by all possible open strings ending on the D5-branes, and is
given by

| Sector | Bosons | Fermions | $Sp(N) \times SO(32)$ |
|--------|--------|----------|----------------------|
| 5-5 $NN$ | $A_\alpha$ | $\epsilon^{AA'}_-, \epsilon^{AA'}_+$ | $N(2N + 1), 1$ |
| 5-5 $NN_I$ | $a^{A'A'}$ | $\epsilon^{A'A'}_-, \epsilon^{A'A'}_+$ | $N(2N + 1), 1$ |
| 5-5 $DE$ | $X^{AY}$ | $\eta^{AY}_-, \eta^{AY}_+$ | $N(2N - 1), 1$ |
| 5-9 $ND$ | $h^A$ | $\rho^A_-, \rho^A_+$ | $2N, 32$ |

Here the subscript $E$ refers to the 4 directions transverse to the D5-brane whereas $I$ refers to the 4 directions corresponding to the $T^4$ along which the dimensional reduction is performed. The corresponding isometry groups $SO(4)_E$ and $SO(4)_I$ are decomposed as $SO(4)_E = SU(2)_A \times SU(2)'_Y$ and $SO(4)_I = SU(2)_{A'} \times SU(2)_{\tilde{A}}$. $A_\alpha$ is a two-dimensional $Sp$ gauge field and $+(-)$ refer to left-(right-) moving fermions. Fields in the same line are related by the $\mathcal{N} = (4,4)$ supersymmetry. We will consider a generic type I background involving Wilson lines on $T^4$, which break $SO(32)$ down to $U(1)^{16}$.

In the strong coupling limit $g \to \infty$ of the Coulomb branch of the above gauge theory, the off-diagonal fields get infinite masses and can be integrated out (see [7] for a detailed analysis) leaving a free (up-to a Weyl group orbifolding) conformal field theory in terms of the Cartan components

$$a^{A'A'}, \epsilon^{A'A'}_-, \epsilon^{A'A'}_+ \quad \sigma_3 \times \lambda_N$$

$$X^{AY}, \eta^{AY}_-, \eta^{AY}_+ \quad I_2 \times \lambda_N$$

with $\lambda_N$ an $N \times N$ matrix with diagonal entries. The Weyl group of $Sp(N)$ is given by the semi-direct product $S_N \ltimes Z_2^N$, with $S_N$ permuting the $Z_2$’s factors, and $Z_2$’s acting as $\sigma_2$, and therefore reflecting the fields proportional to $\sigma_3$ while leaving invariant the ones proportional to the identity $I_2$. The breaking of the $Sp(N)$ gauge group down to $S_N \ltimes Z_2^N$ can be understood in two steps. First by giving generic expectation values to the diagonal entries in $X$ (D5-brane positions) we break the group to $Sp(1)^N$ with the Weyl group $S_N$ permuting the branes. One can then further break each $Sp(1)$ to its Weyl subgroup $Z_2$ by turning on the $SU(2)$ Wilson lines $a^{A'A'}$. Alternatively, one can start from the type I theory and perform four T-dualites on the small $T^4$ directions. The $\Omega$- projection goes under the T-duality map to $\Omega I_4$, introducing 16 5-orientifold planes whose charges are locally canceled with the inclusion of D5-branes symmetrically distributed over the 16 fixed points. Carrying out the same steps as before on this effective $Sp(N)$ gauge theory, we are left with the N D-string sigma model moving on $(R^4 \times T^4/Z_2)^N/S_N$.

The resulting CFT can then be written in terms of a second quantized string theory

\footnote{We assume to be away from loci in the moduli space where (5-9) fields can become massless due to cancellations between $SO(32)$ and $Sp(N)$ Wilson lines [20].}
describing \( N \) copies of a type IIA string moving on the target space

\[
\mathcal{M} = \left[ \mathbb{R}^4 \times T^1/\mathbb{Z}_2 \right]^N / S_N
\]  

(2.18)

The orbifold partition function is given by a sum over twisted sectors labeled by the conjugacy classes of the \( S_N \times \mathbb{Z}_2^N \) orbifold group. In particular the sum over conjugacy classes in the permutation group \( S_N \) runs over the decompositions

\[
[g] = (1)^{N_1}(2)^{N_2}......(s)^{N_s}
\]  

(2.19)

with \( \sum_s sN_s = N \). However, as it has been shown in [7], only the sectors belonging to the conjugacy classes of the kind \([g] = (L)^M\) (with \( N = LM \)) will lead to a non trivial contribution to \( \chi(v|\tau) \). Sectors with strings of different lengths \([g] = (l_1)^{m_1}(l_2)^{m_2}...\) with \( l_1 \neq l_2 \) will contain always additional right moving fermionic zero modes leading to vanishing contributions to \( \chi(v|\tau) \). The elliptic index \( \chi(v|\tau) \) can then be computed using the formula [5] for the \( N \)-symmetric product:

\[
\chi_N(v|U) = \frac{8}{N} \sum_{L,M} \sum_{s=0}^{L-1} M^{-2} \xi(Mv|\tilde{U}) \frac{\theta_1^2(Mv|\tilde{U})}{\eta^6(\tilde{U})} \sum_{i=2,3,4} \frac{\theta_1^2(Mv|\tilde{U})}{\theta_i^2(0|\tilde{U})} (2.20)
\]

with all modular functions evaluated in the induced space-time complex modulus \( \tilde{U} = \frac{MU + s}{L} \). The relative \( M^{-2} \) factor is determined from modular transformations from the untwisted sector [5, 6], once the overall \( N^2 \) factor brought down by the vertex insertions is taken into account. Let us recall how these relative factors arise in the present canonical normalization. One start from \( Tr_{\text{untwisted}}(L)^M \) trace in the untwisted sector where one can unambiguously compute the partition function using the operator formalism. After a modular transformation \( \tau \rightarrow -\frac{1}{\tau} \), to the \( g = (L)^M \)-twisted sector we are left with the partition function for \( M \) copies of strings of length \( L \) \((q \rightarrow q^{1\over L})\) weighted by an \( L^{2M} \) factor. The projection by a \( Z_M \) permutation elements remove the additional fermionic zero modes, apart from the center of mass zero modes, leading finally to an \((L^2)^2\) factor from the uncompact bosons and an \( M^2\)-factor from the right moving fermionic \( Z_M\)-trace. After including the overall \( N^2 \) factor we are left with the \( M^{-2} \) result claimed above.

Substituting (2.20) in (2.4), one can see that oscillator contributions from massive fermionic and bosonic modes cancel against each other leaving the \( 1\over 2 \)-BPS index

\[
B^{ND5}_4 = 36 \sum_{N|M} \frac{1}{N} L = 36 \sum_{N|M} \frac{1}{M} (2.21)
\]

Apart from a factor of two, the type I result (2.16) in terms of this \( B_4 \) index reproduce exactly the instanton sum in the formula (2.12). The extra factor \( \sqrt{\lambda} \), coming from the

\footnote{Alternatively one can compute the helicity generating function \( Z(v, \bar{v}) \) using the more general formula [21] and then derive the BPS index from (2.4).}
coupling of the metric to the instanton background (2.16) is cancelled by a similar one coming from the spacetime measure and the metrics used in the $\mathcal{R}^2$ contractions, once the duality map $G_{\mu \nu}^{II} = \frac{\nu_{\mu} G_{\nu}^{I}}{\chi}$ is taken into account.

One can easily extend the previous results to other four derivative couplings in the $D = 4$ type I effective lagrangians. We can consider for example $F^4$ terms with $F$ the $U(1)$ gauge fields coming from the reduction of the metric on the $T^2$ torus ($F_{\mu \nu} = \partial_{[\mu} G_{\nu]}$, with $i = 4, 5$). The analysis follows closely our previous one for the $\mathcal{R}^2$ computation, with an effective coupling of the gauge field to the instanton background given in this case by

$$S_{N_{D5}} = -2\pi i NS + \frac{2\pi \mathcal{V}_4}{\lambda} \sum_{i=1}^{N} \left[ (h_{\mu 5} + U h_{\mu 4}) D_z X_4^\mu + (h_{\mu 5} - \bar{U} h_{\mu 4}) D_{\bar{z}} X_4^\mu \right] + ...$$

(2.22)

The four vertex insertions provide then the $(4,4)$ fermionic zero modes needed to get a non trivial answer and the four powers of momenta to reproduce the $F^4$ kinematics.

In addition each $F_4$ ($F_5$) insertion carry an additional $\frac{\mathcal{V}_4}{\lambda} (\frac{\mathcal{V}_4 U_2}{\lambda})$ factor from (2.22) (for simplicity we take a rectangular spacetime torus, i.e. $U = iU_2$). The result reproduces exactly the contributions from non-degenerate orbits to the one-loop formula for similar $F^4$ terms in type IIA. In particular, we can see that the $N$-instanton contribution to $F_4^2 F_5^2$ is $N^2$ times the previously found for $\mathcal{R}^2$ (2.16). Therefore the mixing of $\mathcal{R}^2$ and $F_4^2 F_5^2$ cannot account for the absence of the one-loop term in (2.12), without destroying the agreement at the non-perturbative level. On the other hand, one can easily see that four derivative couplings involving the dilaton, mentioned in section (2.1), have the same $N$ dependence of the $\mathcal{R}^2$ term, making them potential candidates to account for the perturbative discrepancy and the factor of 2 in the non-perturbative contribution.

Finally, we would like to stress that the result (2.20) is stronger than what we really need. Indeed, as we have discussed, $\mathcal{R}^2$ couplings in type I theory receive contributions only from the $\frac{1}{4}$-BPS states (2.21). However, one can see that even the right $\frac{1}{4}$-BPS degeneracies, are reproduced by the CFT elliptic genus (2.20). Indeed according to the type IIA/type I duality map, a fundamental type IIA string with winding $N$ and momentum $k$ is mapped in the type I theory to a bound state of $N$ D5 branes and $k$ Kaluza-Klein momenta. The masses of the two objects agree according to (2.2). Multiplicities and charges for the bound states can be read from the longest string sector in the CFT elliptic genus (the only orbifold sector representing a true one-particle state as shown in [7]). Specifying to this sector, $L = N, M = 1$ in (2.20), we are left with

$$\chi_N(v|\tau) = \frac{8}{N} \sum_{s=0}^{N-1} \frac{\xi(v) \theta^2_{\frac{1}{2}}(\frac{v}{2})}{\eta^6} \sum_{i=2,3,4} \frac{\theta^2_{\frac{i}{2}}(\frac{v}{2})}{\theta^2_{\frac{i}{2}}(0)} \left( q^\frac{1}{4} e^{-2\pi i s} \right)$$

(2.23)

The sum over $s$ projects onto states satisfying

$$k \equiv N_L - c_L N \in \mathbb{Z}$$

(2.24)
which reproduce the level matching condition in the fundamental side (2.6) after putting
right moving modes on their ground state \((N_R - c_R = 0)\) and identifying \(k\) with the
KK momentum along the direction where the string wraps. The bound state degeneracies are defined by the coefficients in the expansion of (2.23) in powers of \(q^{N_L - c_L}/q^{N_R - c_R}\).

In particular the ground state multiplicities \((q^0)\ order\) counts the number of ultrashort supermultiplets, associated to bound states of N D5-branes, while degeneracies in the excited left moving part of the CFT (coefficient of \(q^k\ with \(k > 0\)) are associated to bound states of N D5 branes with \(k\) units of Kaluza-Klein longitudinal momenta, which sit in intermediate supermultiplets. The expansion clearly coincide with the similar one for the fundamental result (2.6). We conclude then that the whole spectrum of masses, charges and multiplicities of \(1/2\)- and \(1/4\)-BPS excitations in the D5-brane system agree with the S-duality predictions. In the next section we will give an application of these results.

3 \(1/4\)-BPS saturated couplings in type I theory

In this section we consider higher derivative couplings which are sensitive to \(1/4\)-BPS contributions. Instanton corrections to type I thresholds will translate again into an infinite sum of one-loop contributions coming from \(T^2\)-wrapping modes of type IIA fundamental strings running in the loop, the novelty being in the fact that now \(1/4\)-BPS fundamental strings will be the relevant ones. The type IIA perturbative computation follows with slight modifications from the ones appering in [12], reference that can be consulted for details and a more complete discussion. We will consider in particular the \((4+g)\)-derivative couplings \(\partial F_{\nu L} \partial \bar{F}_{\nu L} F_{L}^0\), with \(F_{\mu\nu L} = \partial_{[\mu}(G_{\nu z] + B_{\nu z]}\) the left moving combination of \(U(1)\ gauge fields arising from the reduction of the metric and antisymmetric tensor on \(T^2\).

The relevant string amplitude is given by

\[
A_g = \left\langle V_L^{g+1} V_L \right\rangle
\]

with vertex operators

\[
V_L = (G_{\mu z} + B_{\mu z}) \int d^2z \left( \partial X^\mu - \frac{1}{4} p_\rho S \gamma^{\mu\rho} S \right) \left( \bar{\partial} Z - \frac{1}{4} p_\sigma \tilde{S} \gamma^{\sigma z} \tilde{S} \right) e^{i p X}
\]

\[
V_\bar{L} = (G_{\mu \bar{z}} + B_{\mu \bar{z}}) \int d^2\bar{z} \left( \partial X^\mu - \frac{1}{4} \bar{p}_\rho S \gamma^{\mu\rho} S \right) \left( \bar{\partial} \bar{Z} - \frac{1}{4} \bar{p}_\sigma \tilde{S} \bar{\gamma}^{\sigma \bar{z}} \tilde{S} \right) e^{i p X}.
\]

At the order of momenta, we are interested in, one can see that the left moving part in the vertices (3.2) enters only through their zero mode part. Indeed, after soaking the \((4, 4)\) fermionic zero modes the remaining extra \(g\) powers of momenta are necesarly carried out by the right moving pieces of the vertices (3.2) and therefore the left moving part reduces effectively to the \(p_L, \bar{p}_L\) bosonic zero modes of \(\partial Z\) and \(\partial \bar{Z}\) respectively. The
right moving part can then be replaced by the first order in the momentum effective vertex

\[ V_{\text{eff}} = \mathcal{F}_{\mu \nu \ell \ell} \int d^2 z \left( X^\mu \partial X^\nu - \frac{1}{4} S \gamma^{\mu \nu} S \right) \]  

(3.3)

The \( A_g \) string amplitudes reduce then to a correlation function of \( g \) effective vertices (3.3), which exponentiate to [22, 23]

\[ A_g = \partial \mathcal{F}_L \partial \bar{\mathcal{F}}_L \left( \frac{i \sqrt{2 U_2}}{T_2} \right)^g \sum_{m_1, n_1, n_2} d(n) \frac{(T_2 U_2)^g}{\pi^g (m_1 T_2 + \frac{m_1}{m_1} U_1)^{2g}} \frac{\partial^g}{\partial \alpha^g} \frac{\partial^g}{\partial \nu^g} I(\alpha, \nu) \big|_{\nu=0, \alpha=1} \]

with \( f_g(\tau) \) holomorphic indices generated by the oscillator contribution \( \chi_{\text{osc}}(v|\tau) \) to the type IIA elliptic genus \( \chi(v|\tau) = \chi_{\text{osc}}(v|\tau) \Gamma_{2,2} \) through

\[ f_g(\tau) = \frac{\partial^g}{\partial \nu^g} \chi_{\text{osc}}(v|\tau) \big|_{v=0} \]  

(3.5)

We have denoted again by prime, the omission of the fermionic zero mode trace in (3.4).

We can now follow the results of [11, 12], where the modular integral in (3.4) has been performed for an arbitrary holomorphic function \( f_g(\tau) \). The result can be written as

\[ A_g = \partial \mathcal{F}_L \partial \bar{\mathcal{F}}_L \left( \frac{i \sqrt{2 U_2}}{T_2} \right)^g \sum_{m_1, n_1, n_2} d(n) \frac{e^{2 \pi i n_2 m_1 T}}{n_2 n_1} \frac{n_2 f_g(n_2 U + n_1)}{n_1} + h.c + \ldots \]  

(3.6)

The \( T_2^H \) expansion of the above formula translate into a series of perturbative corrections (subleading orders in \( S^L_2 \)) around the instanton background. We will consider in the following only the leading order in this expansion. The analysis of subleading quantum corrections to this result can be done following the techniques in [6]. At this order \( \nu \) derivatives hit always the terms proportional to \( \nu T_2 \) and set the remaining \( \nu \)-dependent terms to zero. The \( \alpha \) derivatives hit all the time the exponential leading to an overall \( (\pi n_2 m_1 T_2)^g \) factor. Altogether we are left with

\[ A_g = \partial \mathcal{F}_L \partial \bar{\mathcal{F}}_L \left( \frac{e^{2 \pi i n_2 m_1 T}}{n_2 n_1} \right) \frac{\partial^g}{\partial \nu^g} \left( \frac{i \sqrt{2 U_2}}{T_2} \frac{n_2 f_g(n_2 U + n_1)}{n_1} \right) \big|_{v=0} + h.c + \ldots \]  

(3.7)
where h.c. stands for hermitian conjugation and dots for the higher orders in $T^2$ expansion. It is now straight to compare this result with the corresponding string amplitudes in the instanton background. Indeed, the partition function for the ND5-instanton in the presence of a background (3.3) is described by the elliptic genus (2.20), whose $v$-derivatives coincide with the fundamental result (3.7) after trivial identifications.

The presence of perturbative corrections around the instanton background is a new feature of these higher derivative couplings, to be contrasted with the $R^2$ case. It would be interesting to compare (along the lines of [6]) the fundamental and D-instanton results for these quantum corrections, where the Born-Infeld nature of the instanton couplings are strongly tested. Notice in particular that insertions of $F_L$ gauge fields appears already at a quantum level. A complete analysis of the quantum subleading terms would determine the moduli dependence of the additional couplings in the effective lagrangian.

Notice also that for the case $g = 4$, the result (3.5) is proportional to $\Gamma_{2,2}$. Indeed $f_4 \Gamma_{2,2}$ is nothing but the helicity supertrace $B_6 \equiv \text{Str} \lambda^6$ for type IIA string theory on $K3 \times T^2$, where $\lambda$ is the 4-dimensional physical helicity. $B_6$ in this case is therefore only sensitive to short multiplets [24], the intermediate ones give vanishing contribution to this helicity supertrace, however this accident will clearly not persist for the rest of the asymmetric supertraces generated by $\chi(v|\tau)$.

4 Conclusions

In this paper we have considered instanton corrections to four and higher derivative couplings in $D = 4$ type I string vacua with sixteen supercharges. We restricted our attention to couplings for which Euclidean D5-brane wrapping the $T^6$ torus represent the only source of instanton corrections. D-string instantons have been extensively studied in [3]-[6] and are fairly better understood. The couplings we consider are also special in the sense that they are sensitive only to states sitting in short and intermediate multiplets of the $N = 4$ supersymmetry.

We have worked it out the details of the instanton sums for $R^2$ thresholds in toroidal compactifications of type I string theory. The instanton sums translate, under the duality map, into a sum over wrapping modes of fundamental type IIA strings on the $T^2$ part of $K3 \times T^2$. We argued that the relevant 6-dimensional $Sp(N)$ gauge theory flows in the infrared to an orbifold conformal fixed point, after dimensional reduction to 1+1 dimensions. The elliptic genus for the orbifold CFT was computed in this limit and shown to reproduce correctly the whole spectrum of $\frac{1}{2}$ BPS masses, charges and multiplicities, as required by type I/type IIA duality. As a consequence, the whole infinite sum of instanton corrections to four derivative couplings agree with the expected result from the IIA fundamental string side.
The proposed CFT reproduce also the right multiplicities for $\frac{1}{4}$-BPS states associated to bound states of D5 branes and KK momenta in the type I theory, providing several more examples of higher derivative couplings where the worldsheet/spacetime instanton correspondence works properly.

Besides supporting type IIA/type I duality in $D = 4$ dimensions, we believe the results reported here motivate a deeper study of D5-brane conformal field theory description in type I theory, rather less understood than its type IIB parents. An exciting direction would be the study of conformal field theory description of other $\frac{1}{4}$-BPS magnetic charges in the type I side. In particular, unlike in the type IIB case where an available orbifold CFT description of BPS excitations in the D1-D5 system exists, the type I analog is still missing. The role of these states in the ADS/CFT correspondence, as well as of the $\frac{1}{4}$-BPS bound states studied here, deserves a deeper attention.

Acknowledgements

We are particularly grateful to E. Gava and K.S. Narain for several discussions and comments. We thank E. Kiritsis for suggestions and M. Bianchi for helpful discussions.

References

[1] E. Witten, String Theory Dynamics In Various Dimensions, Nucl.Phys. B443 (1995) 85-126, hep-th/9503124.

[2] S. Ferrara, J. A. Harvey, A. Strominger, C. Vafa, Second-Quantized Mirror Symmetry, Phys. Lett. B361 (1995) 59, hep-th/9505162 ; S. Kachru and C. Vafa, Exact results for $N = 2$ compactifications of heterotic strings, Nucl. Phys. B450 (1995) 69, hep-th/9505105.

[3] C. Bachas, C. Fabre, E. Kiritsis, N. Obers, and P. Van Hove, Heterotic - Type I Duality and D-brane Instantons, Nucl. Phys. B509 (1998) 33. 

[4] E. Kiritsis and N. Obers, Heterotic - Type I Duality in $D < 10$ Dimensions, Threshold corrections and D-Instantons, J.High Energy Phys. 10 (1997) 004; C. Bachas, Heterotic versus Type I, Nucl. Phys. Proc. Suppl. 68 (1998) 348. N.A. Obers and B. Pioline, Eisenstein Series and String Thresholds, hep-th/9903113.

[5] M. Bianchi, E. Gava, J. F. Morales, K. S. Narain, D-strings in unconventional type I vacuum configurations, Nucl.Phys. B547 (1999) 96-126, hep-th/9811013.

[6] E.Gava, A.Hammou, F.Morales and K.S.Narain, On the perturbative corrections around D-string instantons, hep-th/9902202, JHEP9903 (1999) 023.

[7] E. Gava, J. F. Morales, K. S. Narain, and G. Thompson, Bound States of Type I D-Strings, Nucl. Phys. B528 (1998) 95.

[8] Jeffrey A. Harvey, Gregory Moore, Fivebrane Instantons and $R^2$ couplings in $N = 4$ String Theory,Phys.Rev. D57 (1998) 2323-2328, hep-th/9610237.
[9] A. Gregori, E. Kiritsis, C. Kounnas, N. A. Obers, P. M. Petropoulos, B. Pioline, R² Corrections and Non-perturbative Dualities of N=4 String ground states, Nucl. Phys. B510 (1998) 423-476, hep-th/9708062.

[10] W. Lerche, S. Stieberger, N.P. Warner Quartic Gauge Couplings from K3 Geometry, hep-th/9811228, K. Foerger, On Heterotic/Type I Duality in d=8, hep-th/981215.

[11] K. Foerger, S. Stieberger, Higher Derivative Couplings an Heterotic-Type I Duality in Eight Dimensions, hep-th/9901020.

[12] W. Lerche and S. Stierberger, 1/4 BPS states and non-perturbative couplings in N = 4 string theories, hep-th/9907133.

[13] L.J. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. B355 (1991) 649.

[14] E. Kiritsis, Duality and Instantons in String Theory, Lectures given at the 1999 ICTP Spring School, hep-th/9906018.

[15] E. Kiritsis and B. Pioline, Nucl. Phys. B508 (1997) 509, hep-th/9707018.

[16] J. Polchinski, Dirichlet-branes and Ramond-Ramond Charges, Phys. Rev. Lett. 75 (1995) 4724.

[17] J. Polchinski, TASI Lectures on D-Branes, hep-th/9611050.

[18] E.G. Gimon and J. Polchinski, Consistency conditions for orientifolds and D-manifolds, Phys. Rev. D54 (1996) 1667-1676, hep-th/9601038.

[19] Soo-Jong Rey, Heterotic M(atrix) Strings and Their Interactions, Nucl. Phys. B502 (1997) 170.

[20] E. Witten, Small Instantons in String Theory, Nucl. Phys. B460 (1996) 541-559, hep-th/9511030.

[21] R. Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde, Elliptic Genera of Symmetric Products and Second Quantized Strings, Comm. Math. Phys. 185 (1997) 197.

[22] I. Antoniadis, E. Gava, K. S. Narain, T. R. Taylor, N=2 Type II- Heterotic duality and Higher derivative F-terms, Nucl. Phys. B455 (1995) 109-130, hep-th/9507115.

[23] Jose F. Morales, Marco Serone, Higher Derivative F-terms in N=2 Strings, Nucl. Phys. B481 (1996) 389-402, hep-th/9607193.

[24] E. Kiritsis, Introduction to Superstring Theory, CERN-TH/97-218, hep-th/9709062.