General relativistic tidal heating for the Møller pseudotensor

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Abstract

In his study of tidal stabilization of fully relativistic neutron stars Thorne showed that the fully relativistic expression for tidal heating is the same as in non-relativistic Newtonian theory. Furthermore, Thorne also noted that the tidal heating must be independent of how one localizes gravitational energy and is unambiguously given by that expression. Purdue and Favata calculated the tidal heating for a number of classical gravitational pseudotensors including that of Møller, and obtained the result that all of them produced the same (Newtonian) value. However, in a re-examination of the calculation using the Møller pseudotensor we find that there is no tidal heating. This leads us to the conclusion that Thorne’s assertion needs a minor modification: the relativistic tidal heating is pseudotensor independent only if the pseudotensor is derived from a Freud type superpotential.

1 Introduction

Tidal heating is an empirical physical phenomenon resulting from the net work done by an external tidal field on an isolated body. The ocean tides on Earth provide a familiar example of this kind of phenomenon. However, a more dramatical example is the Jupiter-Io system, where the moon Io’s active volcanoes are the result of tidal heating [1]. In 1998 Thorne demonstrated that the expected tidal heating rate is the same both in relativistic and Newtonian gravity [2]: \( \dot{\mathcal{W}} = -\frac{1}{2} \dot{I}_{ij} E^{ij} \), where \( \dot{\mathcal{W}} \) refers to the work rate, the dot indicates the time derivative, \( I_{ij} \) is the mass quadrupole moment of the isolated body and \( E_{ij} \) is the tidal field of the external universe. Both \( I_{ij} \) and \( E_{ij} \) are time dependent, symmetric and trace free. Moreover, Thorne also noted that this tidal heating is independent of how one localizes the gravitational energy and is unambiguously given by a certain value. This has been verified by calculating \( \dot{\mathcal{W}} \) explicitly using various gravitational pseudotensors to represent the gravitational energy-momentum density.

In 1999, Purdue used the Landau-Lifshitz pseudotensor to calculate the tidal heating and confirmed that the result agreed with the Newtonian perspective [3, 4]. Later in 2001, Favata [5] employed the same method to verify that the Einstein, Bergmann-Thomson and Møller pseudotensors [6, 7, 8] give the same result as Purdue found. Moreover, Booth and Creighton used the quasi-local mass formalism of Brown and York to demonstrate the same subject [9]. All of them give the same value as the Newtonian perspective. Referring to the work of Purdue and Favata, this seems have completed the verification that the tidal heating is indeed independent of the gravitational pseudotensor.

Nevertheless, our re-examination of the calculation for the Møller pseudotensor shows zero gravitational energy and vanishing tidal heating. We suspected that Favata had used an unsuitable extra gauge (see (73) in [5]) and misinterpreted the rate of change of the constant mass \( \dot{M} \neq 0 \). Here we argue that obtaining the energy-momentum pseudotensor through a Freud type superpotential guarantees the expected tidal heating [10] but the converse is not true (see Sec. 3). This means that Thorne’s assertion needs a minor modification. The present paper illustrates that the relativistic tidal heating is indeed pseudotensor independent, but only under the condition that the pseudotensor is one of those that comes from a superpotential which agrees with the Freud superpotential to linear order (i.e., see (25)).
Dirac [11] mentioned that it is not possible to obtain a gravitational field energy expression that satisfies both conditions: (1) when added to other forms of energy the total energy is conserved, and (2) the energy within a definite (three-dimensional) region at a certain time is independent of the coordinate system. For the classical pseudotensors, in general, the first condition can be satisfied but the second does not. The nice property of the Møller energy-momentum complex [12] is that the energy content of a hypersurface does not depend on the chosen spatial coordinates, while the complexes proposed by Einstein, Landau-Lifshitz, Bergmann-Thomson and Goldberg do. Perhaps this may be the reason there were many investigators [13, 14, 15, 16] studying this energy-momentum prescription in the past couple of decades. Thus it is worthwhile to investigate the tidal heating using this Møller pseudotensor.

2 Technical background

We will use \( \eta_{\mu\nu} = (-1,1,1,1) \) as our spacetime signature [17] and let the geometrical units \( G = c = 1 \), where \( G \) is the Newtonian gravitational constant and \( c \) the speed of light. We adopt the convention that Greek letters indicate space time indices and Latin letters refer to spatial indices. In principle, the classical pseudotensors [18] can be obtained from a rearrangement of the Einstein equation: \( G_{\mu\nu} = \kappa T_{\mu\nu} \), where the constant \( \kappa = 8\pi G/c^4 \) and \( T_{\mu\nu} \) is the material energy tensor. This is a basic requirement for pseudotensors (see Ch. 20 in [17]). One can define the gravitational energy-momentum pseudotensor in terms of a suitable superpotential \( U_{\alpha}^{[\mu\nu]} \):

\[
2\kappa \sqrt{-g} t_\alpha^\mu := \partial_\nu U_{\alpha}^{[\mu\nu]} - 2\sqrt{-g} G_{\alpha}^\mu.
\]

The total energy-momentum density complex can then be defined as

\[
\sqrt{-g} T_\alpha^\mu := \sqrt{-g} (T_\alpha^\mu + t_\alpha^\mu) = (2\kappa)^{-1} \partial_\nu U_{\alpha}^{[\mu\nu]},
\]

where to get the last equality we used (1) and the Einstein equation. In vacuum it reduces to the energy conservation relations: \( \partial_\mu (\sqrt{-g} t_0^\mu) = 0 \). The quantity \( t_0^0 \) and \( t_0^j \) can be interpreted as the gravitational energy density and energy flux. The tidal heating can be computed as

\[
\dot{W} = -\int_V \partial_0 (\sqrt{-g} t_0^0) d^3x = \int_V \partial_j (\sqrt{-g} t_0^j) d^3x.
\]

One needs to note carefully the sign because Favata had used a different sign for the tidal heating formula indicated in [3]. As the energy-momentum can be expressed as \( P_\mu = (-E, \vec{P}) = \int_V \sqrt{-g} t_0^0 d^3x \), Favata apparently used the wrong sign (see (58) in [3]) for calculating the tidal work. However Favata obtained the correct sign for the Einstein pseudotensor simply because he had included another negative sign for the standard Freud superpotential (see (17) in [3]). The sign for the Freud superpotential is important and in fact it can be fixed by evaluating the value of the ADM mass [19, 20]. Using Gauss’s theorem, the last integral in (3) can be converted into a surface integral of the form:

\[
\dot{W} = \oint_{\partial V} \sqrt{-g} t_0^j dS_j,
\]

where \( dS_j = \hat{n}_j r^2 d\Omega \), \( \hat{n}_j \equiv x_j/r \) is the unit radial normal vector and \( r \equiv \sqrt{\delta_{ab} x^a x^b} \) is the distance from the body in its local asymptotic rest frame.

For the tidal heating calculation, we adopt the harmonic gauge

\[
0 = \partial_\beta (\sqrt{-g} g^{\alpha\beta}) = -\sqrt{-g} \Gamma^{\alpha\beta}_\beta.
\]
This harmonic coordinate condition provides the closest approximation to rectilinear coordinates in curved space and is suitable for studying gravitational waves [11]. The metric tensor can be decomposed as [5]

\[ g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} + \epsilon^2 k_{\mu\nu} + \ldots, \]  

(6)

where \( \epsilon \) is a parameter denoting the ordering: we classify \( \eta_{\mu\nu} \) as the zeroth order, \( h_{\mu\nu} \) as the 1st order and \( k_{\mu\nu} \) as the 2nd order. The traces are \( h := \eta_{\alpha\beta} k^{\alpha\beta} \) and \( k := \eta_{\alpha\beta} k^{\alpha\beta} \). For our tidal heating calculation purpose, we only pay attention to the lowest non-vanishing order; to that order we will get a relation of the form [3]

\[ W = k_1 \partial_0 (I_{ij} E^{ij}) + k_2 \dot{I}_{ij} E^{ij}, \]  

(7)

where \( k_1, k_2 \) are constants. The coefficient \( k_1 \) is related to a specific choice for the energy localization where \( \partial_0 (I_{ij} E^{ij}) \) is an ambiguous reversible tidal-quadrupole interaction process. We expect to get \( k_2 = -\frac{1}{2} \) so that \( -\frac{1}{2} \dot{I}_{ij} E^{ij} \) is the unambiguous irreversible tidal heating dissipation process that we are interested in. Therefore we only look for the tidal heating coming from the external tidal field \( E_{ij} \) interacting with the evolving quadrupole moment \( I_{ij} \) of an isolated body. The related expressions (adopted from (38) – (40) in [3]) of the gravitational field tensors are:

\[ h_{00} = \frac{2M}{r} + \frac{3}{r^3} I_{ij} x^i x^j - E_{ij} x^i x^j, \quad h_{0i} = -\frac{2}{r^3} I_{ij} x^i - \frac{10}{21} E_{ik} x^i x^k x_j + \frac{4}{21} E_{ij} x^i r^2, \]  

(8)

and \( h_{ij} = \delta_{ij} h_{00} \). From (8), it is easy to verify that the value of the weighting factor \( \sqrt{-g} = 1 + h_{00} + \ldots \) and \( 2h_{00,0} - \eta^{cd} h_{0c,d} = 4M r^{-1} \).

According to Thorne’s argument the mass \( M \) is constant in time [2] and indeed the harmonic gauge at the lowest order \( \Gamma^{0\beta}_\beta = 2h_{00,0} - \eta^{cd} h_{0c,d} \) demands that \( \dot{M} \) is vanishing. If the isolated body is absorbing the external quadrupolar field, its quadrupole moment \( \dot{I}_{ij} \neq 0 \) in general and this can generate tidal work. As the Einstein field equation is vanishing in vacuum, the plane wave equation \( \partial_\lambda \partial^\lambda h_{\mu\nu} = 0 \) appears under the criterion that the harmonic gauge is chosen. Within our tidal heating approximation limit [3], one should keep \( \partial_0 h_{\mu\nu} \) but can ignore \( \partial_0^2 h_{\mu\nu} \) as being of higher order, so \( \nabla^2 h_{\mu\nu} \simeq 0 \) (see (51) in [5]). We will come to this in Sec. 3 at (40).

The detailed expansion for the harmonic gauge in terms of \( h, \dot{h} \) and \( k \) terms is

\[ \Gamma^{\alpha\beta}_\beta = \left( \epsilon - \frac{\epsilon^2}{2} \right) \partial_\beta \tilde{h}^{\alpha\beta} + \epsilon^2 \partial_\beta \left( \tilde{h}^{\alpha\beta} - h^{\alpha\beta} \tilde{h}^{\beta\lambda} + \frac{1}{4} \eta^{\alpha\beta\lambda\sigma} \tilde{h}_{\lambda\sigma} \right), \]  

(10)

where \( \tilde{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} h \) and likewise \( \tilde{k}^{\alpha\beta} = k^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} k \). This equation gives the usual first order harmonic gauge \( \partial_\alpha \tilde{h}^{\alpha\beta} = 0 \) and the second order

\[ \partial_\beta (\tilde{k}^{\alpha\beta} - h^{\alpha\beta} \tilde{h}^{\beta\lambda} + \frac{1}{4} \eta^{\alpha\beta\lambda\sigma} \tilde{h}_{\lambda\sigma}) = 0. \]  

(11)

We split the time and spatial components for (10) as follows:

\[ \Gamma^{0\beta}_\beta = \left( 2h_{00,0} - \eta^{cd} h_{0c,d} \right) - \frac{1}{2} \eta^{\beta\lambda} (2k_{0\beta,\lambda} - k_{\beta\lambda,0}) + h_{00} h_{00,0} - \eta^{cd} h_{0c} (h_{00,d} + h_{0d,0}), \]  

(12)

\[ \Gamma^{ij}_\beta = \eta^{ja} \left[ \frac{1}{2} \eta^{\beta\lambda} (2k_{a\beta,\lambda} - k_{\beta\lambda,a}) - h_{0a,0} + h_{00} h_{0a,0} + h_{0c} (h_{0c,a} - h_{0a,c}) \right] \]  

\[ + h^{ja} (h_{00,0} - \eta^{cd} h_{0c,d}). \]  

(13)
Including only the terms that will contribute to the tidal heating to the order of accuracy of our interest (17), which are shown in (12) and (13), the 2nd order harmonic gauge components of interest are

\[ 2\Gamma^{0\beta}_\beta \simeq -\eta^{\beta\lambda}(2k_{\beta\lambda,\lambda} - k_{\beta\lambda,0}) + 2h^{0\beta}h_{00,\beta}, \quad 2\Gamma^{j\beta}_\beta \simeq \eta^{ja}[\eta^{\beta\lambda}(2k_{a\beta,\lambda} - k_{\beta\lambda,a}) + \partial_\alpha h^2_{00}], \]  

(14)

where \( h^2_{00} \) means the square of \( h_{00} \). Taking an integration for \( \Gamma^{j\beta}_\beta \) in (14) gives

\[ \int_V \eta^{ja}\eta^{\beta\lambda}(2k_{a\beta,\lambda} - k_{\beta\lambda,a})d^3x = 0, \]  

(15)

where \( \int_V \eta^{ja}h^2_{00}dS_a \) is vanishing since the integrand is an even function. Recall the Ricci tensor

\[ R_\alpha^\mu = \Gamma^\beta_\alpha \Gamma^\mu_\beta - \Gamma^\beta_\mu \Gamma^\alpha_\beta + \frac{1}{2}g^{\mu\rho}g^{\beta\nu}(g_{\alpha\beta,\rho\nu} + g_{\rho\beta,\alpha\nu} - g_{\alpha\rho,\beta\nu} - g_{\beta\nu,\alpha\rho}). \]  

(16)

For the first order \( h_{\beta\lambda} \) in vacuum, there is an identity using (16) (i.e., see (70) in [5]):

\[ 0 = \eta^{\beta\nu}(h_{\alpha\beta,\rho\nu} + h_{\rho\beta,\alpha\nu} - h_{\alpha\rho,\beta\nu} - h_{\beta\nu,\alpha\rho}). \]  

(17)

Meanwhile, for the 2nd order, referring to (16) again when \( (\alpha, \mu) = (0, j) \)

\[ 0 = \eta^a[3h_{00,0}h_{00,a} - \eta^{cd}h_{00,c}h_{00,d} - h^{0b}h_{00,ab} + \eta^{\beta\lambda}(k_{a\beta,0\lambda} + k_{0\beta,a\lambda} - k_{\beta\lambda,0a} - k_{0a,\beta\lambda})]. \]  

(18)

Favata proposed an extra gauge (see (73) in [5])

\[ k^\beta_\nu,\beta = k^\beta_\nu, \]  

(19)

but we argue that this is invalid. Here we give three explanations does not agree: (i) equation (15) shows that \( \int_V \eta^{ja}h^2_{00}dS_a \) vanishes, but our calculation gives an inconsistent value

\[ \int_V (\eta^{ja}\partial_a\Gamma^{0\beta}_\beta + \partial_\alpha\Gamma^{j\beta}_\beta)dS_j = \frac{4\pi}{21} \left[ \frac{48}{5} \partial_0(I_{ij}E^{ij}) + I_{ij}E^{ij} \right]. \]  

(20)

Obviously, Favata’s \( k \) condition is incompatible with the harmonic gauge. (iii) we get the same non-consistent result if we apply the harmonic gauge and the scalar Riemann curvature \( g^{\mu\nu}R_{\mu\nu} \) together in empty space:

\[ \eta^{\alpha\beta}\eta^{\mu\nu}\partial_\nu(k_{\alpha\beta,\mu} - k_{\alpha\mu,\beta}) = \Gamma^{\alpha\beta\nu}\Gamma_{\alpha\beta\nu}, \]  

(21)

which is not vanishing in general. Explicitly, the LHS in (21) should vanishes according to Favata’s gauge indicated in (19), but the RHS cannot have the same value in principle. One can double check using the tidal heating approximation limit, in detail

\[ \eta^{\alpha\beta}\eta^{\mu\nu}\partial_\nu k_{\alpha\beta,\mu} = 2(2\Gamma^{\mu\alpha}_\alpha \Gamma^{\alpha\mu}_{\mu} + \Gamma^{\alpha\beta\nu}\Gamma_{\alpha\beta\nu}), \quad \eta^{\alpha\beta}\eta^{\mu\nu}\partial_\nu k_{\alpha\mu,\beta} = 4\Gamma^{\mu\nu}_\alpha \Gamma^{\alpha\mu}_{\nu} + \Gamma^{\alpha\beta\nu}\Gamma_{\alpha\beta\nu}, \]  

(22)

where \( \Gamma^{\mu\nu}_\alpha \Gamma^{\alpha\mu}_{\nu} = -\frac{1}{2}\eta^{cd}h_{00,c}h_{00,d} \) and \( \Gamma^{\alpha\beta\nu}\Gamma_{\alpha\beta\nu} = \frac{5}{2}\eta^{cd}h_{00,c}h_{00,d} \).

### 3 Tidal heating from the Freud superpotential

There are an infinite number of superpotentials, the Freud [6] superpotential \( F_U^{[\mu\nu]} := -\sqrt{-g}g^{\beta\gamma}\Gamma^{\gamma}_{\alpha\beta\lambda}k^{\lambda\mu}_{\tau\alpha} \) is a straightforward expression that can be used for illustrating...
the tidal work. Here we use it reproduce the result of the tidal heating for the Einstein pseudotensor. We have mentioned that the energy-momentum complex can be computed as

$$\sqrt{-g} T^\alpha_\mu = \partial_\nu (\mathcal{F}_U^\alpha_{\ [\mu \nu]}).$$

At any point, to lowest order in Riemann normal coordinates inside matter this gives the desired energy-momentum stress tensor

$$T^\alpha_\mu = \kappa^{-1} G^\alpha_\mu. \tag{18}$$

In vacuum, the Einstein pseudotensor becomes

$$2\kappa E_t^\alpha_\mu = \frac{1}{2} \sum \delta^\mu_\nu (\Gamma^\beta_\lambda_\nu \Gamma^\nu_\beta_\lambda - \Gamma^\nu_\beta_\lambda_\nu \Gamma^\nu_\alpha_\beta_\gamma + \Gamma^\nu_\gamma_\alpha_\beta_\lambda + \Gamma^\nu_\alpha_\beta_\gamma - \Gamma^\nu_\alpha_\beta_\lambda - \Gamma^\nu_\alpha_\beta_\gamma) - 2\Gamma^\mu_\nu_\alpha \Gamma^\nu_\beta_\mu. \tag{23}$$

Apply the harmonic gauge, the gravitational energy density and energy flux \[5\] are

$$2\kappa E_t^0_0 = -\frac{1}{2} \eta^{cd} h^{00,c} h^{00,d}, \quad 2\kappa E_t^0_j = \eta^{ja} h^{00,j} h^{00,a}. \tag{24}$$

Note that the sign of the energy

$$E = -\int V \sqrt{-g} E_t^0_0 \, d^3 x$$

is positive. Using (4), we recover the known tidal work

$$\dot{W}_E = \frac{3}{10} \partial_0 I_{ij} E_{ij} - \frac{1}{2} \dot{I}_{ij} E_{ij}. \tag{27}$$

To the order of concern here, it is sufficient to consider superpotentials that are linear in the connection. There are only three possible terms with suitable symmetry, one by itself is the Møller superpotential. The general three parameter expression can be written as

$$U_\alpha_{\ [\mu \nu]} := \sqrt{-g} (a_1 \delta^\alpha_\tau \Gamma^\tau_\rho_\lambda + a_2 \Gamma^\tau_\rho_\alpha + a_3 \delta^\rho_\alpha \Gamma^\lambda_\tau_\lambda) \delta^\mu_\nu, \tag{25}$$

where $a_1, a_2, a_3$ are real.

One limit that should be considered is the small region limit. Around any arbitrary point, one can introduce Riemann normal coordinates \[19, 21\] such that

$$g^\alpha_\beta \big|_0 = \eta^\alpha_\beta, \quad g^\alpha_\beta_\mu \big|_0 = 0, \quad -3 \Gamma^\alpha_\beta_\mu_\nu \big|_0 = R^\alpha_\beta_\mu_\nu + R^\alpha_\mu_\beta_\nu. \tag{26}$$

According to the equivalence principle, to lowest order the pseudotensor associated with the above superpotential should reduce to the interior stress

$$2\kappa T^\alpha_\mu = \frac{1}{3} [(2a_1 + 3a_2 + a_3) R^\alpha_\mu - (2a_1 + a_3) \delta^\alpha_\mu R]. \tag{27}$$

In order for this to agree with the Einstein equation, we have the following constraints \[19\]:

$$2a_1 + a_3 = 3, \quad a_2 = 1. \tag{28}$$

Furthermore we considered the mass at null infinity \[22\] and found

$$\frac{1}{2} (a_1 + a_2) m(u) + \frac{1}{4\kappa} (a_2 - a_3) \frac{d}{du} \int_{\partial V} c^2 \sin \theta \, d\theta \, d\phi \tag{29}$$

where $c$ is the Bondi news function \[23\]. This gives the Bondi mass provided that

$$a_1 + a_2 = 2, \quad a_2 - a_3 = 0. \tag{30}$$

Combining the results inside material and the null infinity from (28) and (30), we have the unique solution for $a_1, a_2, a_3$ are all unity. This choice is the same as that required at spatial infinity, in order to obtain the ADM mass \[17\].

Here we explain what we mean by a Freud type superpotential; we decompose the Freud superpotential as follows

$$\mathcal{F}_U_\alpha_{\ [\mu \nu]} := -\sqrt{-g} (\eta^\beta_\gamma - \epsilon h^\beta_\gamma + ...) \Gamma^\tau_\beta_\lambda \delta^\lambda_\tau_\gamma. \tag{31}$$
Note that the linear in \(\eta\Gamma\) terms give the expected interior mass and tidal heating, while the \(h\Gamma\) terms only alter the value \(\partial_0(I_{ij}E^{ij})\) [10]. Any superpotential that agrees with the Freud superpotential to lowest order in \(h_{\mu \nu} := g_{\mu \nu} - \eta_{\mu \nu}\), is referred to as a Freud type superpotential. We know that the Landau-Lifshitz (LL) superpotential can be identified as a Freud type superpotential since we can raise the indices: \(LLU^\alpha [\mu \nu] = FU_\beta [\mu \nu] \sqrt{-g} g^{\alpha \beta}\), which gives the desired interior mass and tidal work, whereas the extra weighting factor \(\sqrt{-g}\), once again, only affects \(\partial_0(I_{ij}E^{ij})\). There also exists another possibility such as the Papapetrou superpotential [10]:

\[
\rho U^\alpha [\mu \nu] = FU_\beta [\mu \nu] g^{\alpha \beta} - \sqrt{-g}(g^{\rho \sigma} h_{\rho \sigma} \Gamma^\sigma_\lambda \chi + g^{\rho \sigma} h^{\lambda \sigma} \Gamma^\gamma_\lambda \chi_\gamma) \delta_\mu^\mu \delta_\nu^\nu.
\]

(32)

Referring to (25), to lowest order there are just three possible superpotential terms and each term has its characteristic features.

### 3.1 The 1st term of the Freud superpotential

When \((a_1, a_2, a_3) = (1, 0, 0)\) referring to (25), the first term of the Freud type superpotential is \(U^\alpha [\mu \nu] := \sqrt{-g} \Gamma^\alpha_\lambda \delta_\mu^\nu\). The corresponding contribution to the energy-momentum complex is

\[
(2\kappa)T_\alpha^\mu = (\partial_\nu + \Gamma^\sigma_\nu \pi_\nu) \Gamma^\rho^\lambda \chi \delta_\rho^\mu.
\]

(33)

Inside matter at the origin, the energy-momentum \(\kappa T_\alpha^\mu = \frac{1}{3} (R_\alpha^\mu - \delta_\alpha^\mu R)\) in Riemann normal coordinates. Upon applying the harmonic gauge in vacuum, the pseudotensor \(t_\alpha^\mu = 0\), i.e., both the energy density \(E^0\) and energy flux \(E^j\) are zero.

### 3.2 The 2nd term of the Freud superpotential: Møller’s superpotential

When \((a_1, a_2, a_3) = (0, 1, 0)\) for (25), the Møller superpotential [8] is recovered, more precisely it is one-half of the magnitude, i.e., \(MU_\alpha [\mu \nu] := \sqrt{-g} \Gamma^\alpha_\lambda \delta_\mu^\nu\). The associated energy-momentum complex is

\[
(2\kappa)_M T_\alpha^\mu = 2R_\alpha^\mu - \partial_\alpha \Gamma^\mu^\beta + g^\beta^\mu \partial_\alpha \Gamma^\nu_\beta^\nu - 2\Gamma^\beta^\nu_\alpha \Gamma^\mu_\beta^\nu.
\]

(34)

Inside matter this reduces to \(2\kappa T_\alpha^\mu = R_\alpha^\mu\) at the origin in Riemann normal coordinates. Since this result is not compatible with Einstein’s equation, one may have doubts that whether it is meaningful keep calculating the tidal work? Although the Møller pseudotensor has already failed the inside matter requirement, this pseudotensor has the important feature that its gravitational energy is coordinate system independent. In vacuum, using the harmonic gauge condition, the energy density from (34) is

\[
(2\kappa)_M t_0^0 = 3(h_{00,0})^2 + h_{00,0}h_{00,00} + \eta^{cd}[h_{0c,0}h_{0d,0} - h_{00,c}h_{0d,0} - \partial_0(h_{0c,0}h_{0d,0})] - \frac{1}{2}\eta^{\beta\lambda} g_{\beta\lambda,00}.
\]

(35)

Referring to the detailed explanation from Purdue (p.6 in [3]), for a proper gravitational energy, we expect something like \(t_0^0 \sim \eta^{cd}h_{00,c}h_{00,d}\) which means the gravitational energy density should not involve any time derivatives: “This restriction has given us only products of \(\bar{h}^{\mu \nu}\), which will produce terms containing the products \(M^2, M\xi, M\tilde{\xi}, \tilde{\xi}\tilde{\xi}, \tilde{\xi}\xi\) for \((-g)t_0^0\) and \(M\tilde{\xi}, M\xi, \tilde{\xi}\tilde{\xi}, \tilde{\xi}\xi\) for \((-g)t_0^0\).” Thus we can immediately conclude that both gravitational energy and tidal heating vanish to the order considered. Explicitly, according to the accuracy limit involving the types of terms of interest mentioned in connection with (7), the tidal heating is

\[
\dot{W}_M = - \int K_0 \partial_0(\sqrt{-g} M t_0^0) d^3x = 0.
\]

(36)
The question then arises why Favata obtained the desired tidal heating value for the Møller pseudotensor but we get null? To understand this discrepancy, we turn to the analysis of how Favata obtained his expression. Up to a sign Favata used the Møller expression, recall the energy density for this pseudotensor according to his gauge condition (see (74) in [5]):

\[-8\pi M \tau_0^j = -(1 + h_{00})h_{00,00} + 2(h_{00,0})^2 + \eta^{\alpha \beta} [2h_{0\alpha,0\beta} - \partial_0(h_{0\alpha,0\beta})]. \quad (37)\]

This result shows that \( f_V \frac{\partial_0 (M \tau_0^j)}{\partial x} \) can be identified as a null tidal work since the integrand vanishes based on the tidal heating approximation limit. The corresponding energy flux referring to Favata (see (75) in [5]) is:

\[-8\pi M \tau_0^j = \eta^{j\alpha} [(1 - 3h_{00,0a})h_{00,0a} - 2h_{00,0}h_{00,0a}]. \quad (38)\]

The accompanied tidal heating based on Favata (see (77) in [5]) is

\[ M_M = - \int_{\partial V} M \tau_0^j \delta r^2 d\Omega = -M - \int_{\partial V} E^j, \quad (39) \]

where \( M \) can be classified as the tidal work according to Favata’s point of view. He took the part on the RHS and combined it with the LHS part and then divided by 2 to get the desired result. But these \( 2M \) quantities are different, and the one the RHS is, as already explained, a constant (i.e., see (9)). The LHS is written as \( W \) in other words. Checking \( \partial_j (\eta^{j\alpha} h_{00,0a}) \) by using a volume integral

\[ \frac{1}{\kappa} \frac{d}{dt} \int_V \hat{\nabla}^2 h_{00} d^3 x = \frac{d}{dt} \int V M \delta (\hat{r} - \hat{r}_0) d^3 x. \quad (40) \]

Favata argued that he obtained a nice value \( \dot{M} \), but we claim this is invalid. Here we have three objections for Favata’s non-vanishing tidal work. (i) he had included \( \dot{M} \) and this is prohibited by the harmonic gauge condition. (ii) We explained that \( \hat{\nabla}^2 h_{00} \) should be vanishing based on the plane wave equation and the tidal heating approximation limit. (iii) one can apply the Poisson’s equation to RHS in (40) which is fixed at a one particular point, but the tidal heating requires the separation between two different neighbourhood points.

Favata had used an extra gauge \( k^{\alpha}_{\nu, \alpha} = k_{\nu} \) (see (73) in [5]) to simplify the computation of the Møller pseudotensor, but it is not a suitable gauge condition. The detail is as follows: Let

\[ g^{\alpha \beta} = \eta^{\alpha \beta} + \epsilon h^{\alpha \beta} + \epsilon^2 K^{\alpha \beta}, \quad x^{\mu} = x^{\mu} + \epsilon \xi^{\mu} + \epsilon^2 \chi^{\mu}, \quad (41) \]

where \( \xi^{\mu} \) and \( \chi^{\mu} \) are vectors, \( h^{\alpha \beta} \) and \( K^{\alpha \beta} \) (i.e., some linear combination of \( k^{\alpha \beta} \) and quadratic of \( h^{\alpha \beta} \)) are known functions. Consider

\[ g^{\mu \nu'} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} \frac{\partial x^{\nu'}}{\partial x^{\beta}} g^{\alpha \beta} \]

\[ = \eta^{\alpha \beta} + \epsilon h^{\alpha \beta} + \epsilon^2 K^{\alpha \beta} + \mathcal{O}(\epsilon^3), \quad (42) \]

where

\[ h^{\alpha \beta}_\alpha = h^{\alpha \beta} + \partial_\alpha \xi^\beta + \partial^\beta \xi_\alpha \]

\[ K^{\alpha \beta}_\alpha = K^{\alpha \beta} + \partial_\alpha \chi^\beta + \partial^\beta \chi_\alpha + h_{\alpha \lambda} \partial^\lambda \xi^\beta + h^{\beta \lambda} \partial_\lambda \xi_\alpha + (\partial_\lambda \xi_\alpha) (\partial^\lambda \xi^\beta). \quad (43) \]

We fix the vector \( \xi^{\mu} \) by the condition \( \partial_\beta \hat{h}^{\mu \nu} = 0 \), well known as the (linear order) harmonic condition, and Favata used it. Explicitly

\[ \partial_\beta (h^{\alpha \beta}_\alpha - C_1 \delta^{\beta}_\alpha h') = \partial_\beta (h^{\alpha \beta}_\alpha - C_1 \delta^{\beta}_\alpha h') + \partial_\alpha \partial_\beta (1 - 2C_1) \xi^\beta + \partial_\beta \partial^\beta \xi_\alpha, \quad (45) \]
where the parameter $C_1 = \frac{1}{2}$. Then (as is well known) $\partial_\mu \bar{h}^{\mu \nu} = 0$ leads to a wave equation for $\xi_\alpha$, which has solutions for all given $h_{\mu \nu}$. Now, how about $k_{\mu \nu}$? Favata claimed $k_{\nu \beta} = k_{\nu \beta}$. Let’s consider a similar technique for a one parameter set of conditions to fix $\chi^\mu$. To wit

$$
\partial_\beta (K^{\alpha \beta} - C_2 \delta^{\alpha \beta} \bar{K}) = \partial_\beta (K^{\alpha \beta} - C_2 \delta^{\alpha \beta} K) + (1 - 2C_2) \partial_\alpha \partial_\beta \chi^\alpha + \partial_\beta \partial_\beta \chi_\alpha + h_{\alpha \lambda} \partial_\beta \xi^\beta + h_{\beta \lambda} \partial_\beta \xi_\alpha + (\partial_\lambda \xi_\alpha)(\partial_\lambda \xi^\beta) - C_2 \partial_\alpha \left[ 2h^{\lambda \sigma} \partial_\lambda \xi_\sigma + (\partial_\lambda \xi_\sigma)(\partial_\lambda \xi^\sigma) \right].
$$

This is a 2nd order to be solved for $\chi^\mu$, namely

$$
(1 - 2C_2) \partial_\alpha \partial_\beta \chi^\beta + \partial_\beta \partial_\beta \chi_\alpha = \text{known terms},
$$

where the RHS is made up of some known terms that are independent of $\chi^\mu$. Consider the divergence of this equation

$$
2(1 - C_2) \partial^\alpha \partial_\alpha \partial_\beta \chi^\beta = \partial_\alpha (\text{known terms}).
$$

Obviously that RHS is non-vanishing in general so $C_2 = 1$ is not a viable option. However, choosing $C_2 = \frac{1}{2}$ is especially nice, since it gives a wave equation for $\chi_\alpha$. Thus Favata’s gauge condition cannot be satisfied in general. In fact it seems that any value other than 1 for the parameter $C_2$ could be used. Comparing with (11), we find the 2nd order harmonic condition as follows

$$
\partial_\beta \left( K^{\alpha \beta} - \frac{1}{2} \delta^{\alpha \beta} \bar{K} \right) = -\frac{1}{2} h \partial_\beta \bar{h}^{\alpha \beta} + \partial_\beta \left( \bar{k}^{\alpha \beta} - h_{\alpha \lambda} \bar{h}^{\beta \lambda} + \frac{1}{4} \delta^{\alpha \beta} h^{\lambda \sigma} \bar{h}_{\lambda \sigma} \right) = 0.
$$

For the completeness, referring to [36], using the property of the conservation of energy-momentum $\partial_\mu (\sqrt{-g} t^0_\mu) = 0$. The vanishing tidal work can be calculated from $\int_{\partial V} \sqrt{-g} M_{t_0}^3 dS_j$, where

$$
(2k)_{t_0}^3 = \partial_\beta \Gamma^\beta_\beta + 2\eta^{i a} h_{00,0 a} h^{00,i}.
$$

Then we deduced $\int_{\partial V} \sqrt{-g} \partial_\beta \Gamma^\beta_\beta dS_j = -4k \left[ \frac{3}{10} \partial_\beta (I_{ij} E^{i j}) - \frac{1}{2} I_{ij} E^{i j} \right]$. In particular, one can solve for the value as follows

$$
\frac{1}{2k} \int_{\partial V} \eta^{i a} \eta^{\beta \lambda} k_{0 a \beta \lambda} dS_j = -\frac{1}{5} \partial_\beta (I_{ij} E^{i j}) + \frac{1}{2} I_{ij} E^{i j}.
$$

Meanwhile, applying the harmonic gauge, comparing (14), (18) and (51), we obtained

$$
\frac{1}{2k} \int_{\partial V} \eta^{i a} \eta^{\beta \lambda} k_{0 a \beta \lambda} dS_j = \frac{3}{70} \partial_\beta (I_{ij} E^{i j}) + \frac{29}{42} I_{ij} E^{i j},
$$

$$
\frac{1}{2k} \int_{\partial V} \eta^{i a} \eta^{\beta \lambda} k_{0 a \beta \lambda} dS_j = \frac{1}{2} \partial_\beta (I_{ij} E^{i j}) + \frac{29}{42} I_{ij} E^{i j},
$$

$$
\frac{1}{2k} \int_{\partial V} \eta^{i a} \eta^{\beta \lambda} k_{0 a \beta \lambda} dS_j = -\frac{1}{5} \partial_\beta (I_{ij} E^{i j}) + \frac{1}{2} I_{ij} E^{i j}.
$$

### 3.3 The 3rd term of the Freud superpotential

When $(a_1, a_2, a_3) = (0, 0, 1)$ for (25), we named this superpotential as $S$ and $S U_{a \mu \nu} := \sqrt{-g} \partial_\mu \eta_{a \nu} \Gamma^{\lambda \mu \nu}$, which is the essential part which gives the desired tidal heating expression. The associated energy-momentum complex is

$$
(2k)_S \mathcal{T}^\alpha_\mu = \delta^\mu_\alpha (-R + \partial_\lambda \Gamma^{\lambda \beta}_\beta + \Gamma^{\beta \nu}_\lambda \Gamma^{\lambda}_\beta \nu - (\partial_\alpha + \Gamma^{\pi}_\pi \alpha) \Gamma^{\lambda \mu}_\lambda).
$$


Inside matter, \( \kappa T_\alpha^\mu = \frac{1}{6} (R_\alpha^\mu - \delta_\alpha^\mu R) \) in Riemann normal coordinates. In vacuum, using the harmonic gauge condition, this \( S \) pseudotensor can be written as
\[
(2\kappa) St_\alpha^\mu = \delta_\alpha^\mu \Gamma^{\beta\nu\lambda}_\beta \Gamma^\lambda_{\beta\nu} - (\partial_\alpha \Gamma^\nu_\pi \pi_\alpha) \Gamma^\lambda_{\mu\lambda}.
\]
(56)
The related gravitational energy density and energy flux are
\[
(2\kappa) St_0^0 = -\frac{1}{2} \eta^{cd} h_{00,c} h_{00,d}, \quad (2\kappa) St_0^j = -\partial_0 \Gamma^j_\beta \beta - \eta^{ja} h_{00,0} h_{00,a}.
\]
(57)
A minor difficulty for the tidal work integration for (57) is the term \( \partial_0 \Gamma^j_\beta \beta \), but this problem can easily be solved using \([51]\). We find that not only this energy density is the same as that of the Einstein pseudotensor, but also it gives the tidal heating. Only this superpotential is the essential part which contributes the desired tidal heating. More accurately, besides failing to meet the inside matter result \( 2G^\mu_\nu \), we discovered that whenever the superpotential includes this with unit magnitude, one can guarantee that the suitable tidal heating value will be achieved. In other words, the tidal heating is pseudotensor dependent, i.e., not pseudotensor independent as Thorne expected and Favata claimed to have verified \([5]\). Thorne wrote: “Similarly, if, in our general relativistic analysis, we were to change our energy localization by switching from the Landau-Lifshitz pseudotensor to some other pseudotensor, or by performing a gauge change on the gravitational field, we thereby would alter \( E_{int} \) but leave \( W \) unchanged” (p.9 in \([2]\)). Perhaps Thorne had assumed that all pseudotensors already had the standard form to linear order (see Ch. 20 in \([17]\)).

4 Conclusion

Thorne argued that tidal heating is independent of how one localizes the gravitational energy and the value is unambiguous. Purdue and Favata used a number of well known pseudotensors to calculate the tidal heating and verify that Thorne’s assertion is correct. However, after a re-examination of the Møller pseudotensor, we found it gives a vanishing value, which suggests that the tidal heating is after all pseudotensor dependent. In coming to this conclusion, we have identified a minor revision of Favata’s calculation. More precisely, the pseudotensor needs to come from a superpotential that agrees with the Freud superpotential to linear order in \( h_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu} \).

All of the famous pseudotensors have this property, with the exception of the Møller pseudotensor.

Here we emphasize that if a suitable gravitational energy-momentum pseudotensor fulfills the Freud type superpotential condition, this requirement ensures the expected tidal heating. Furthermore, the pseudotensor will not be physically satisfactory if it only succeeds in achieving the desired tidal heating, but fails to meet the inside matter requirement (e.g., pseudotensor \( S \)). Therefore Thorne’s assertion needs a minor modification: the relativistic tidal heating is pseudotensor independent under the condition that the pseudotensor is derived from a superpotential which is linearly of the Freud type.

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