A PRIORI ESTIMATES FOR SOME ELLIPTIC EQUATIONS INVOLVING THE $p$-LAPLACIAN

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Abstract. We consider the Dirichlet problem for positive solutions of the equation $-\Delta_p(u) = f(u)$ in a convex, bounded, smooth domain $\Omega \subset \mathbb{R}^N$, with $f$ locally Lipschitz continuous.

We provide sufficient conditions guaranteeing $L^\infty$ a priori bounds for positive solutions of some elliptic equations involving the $p$-Laplace operator and extend the class of known nonlinearities for which the solutions are $L^\infty$ a priori bounded. As a consequence we prove the existence of positive solutions in convex bounded domains.

1. Introduction and statement of the results.

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is strictly convex. We are interested in proving $L^\infty(\Omega)$ a priori bounds for $C^1(\bar{\Omega})$ weak solutions of the problem

$$
\begin{cases}
-\Delta_p(u) = f(u) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(1.1)

where $\Delta_p(u) = \text{div}(|Du|^{p-2}Du)$ is the $p$-Laplace operator, $1 < p < \infty$, and

$H_1)$ $f : [0, \infty) \to \mathbb{R}$ is a locally Lipschitz continuous function and $f(0) \geq 0$.

If $p > 2$ we also assume that $f(s) > 0$ if $s > 0$.

The equation $-\Delta_p u = f(u)$ is the $L^p$ counterpart to the classical semilinear elliptic equation $-\Delta u = f(u)$, and appears e.g. in the theory of non-Newtonian fluids (dilatant fluids in the case $p \geq 2$, pseudo-plastic fluids in the case $1 < p < 2$), see [4, 41, 42].

If $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of the problem (1.1), then $u \in C^{1,\tau}(\bar{\Omega})$ with $\tau < 1$ (see [25], [40]), so that we suppose from the

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beginning a $C^1$ regularity for the solution (which is anyway in general only a weak solution).

Moreover in the applications of a priori estimates to existence of solutions to elliptic problems, a standard setting is the space of continuous functions, and if $u \in C^0(\Omega)$ then also $f(u)$ is continuous and the solution $u$ belongs by the cited regularity results to the space $C^{1,r}(\Omega)$.

In the case $p = 2$, i.e. when the equation under investigation is $-\Delta(u) = f(u)$, the problem has been widely studied.

After previous classical partial results (see in particular [9] and the references therein), in 1981 in two celebrated papers Gidas and Spruck [31], [32] proved a priori bounds for nonlinearities $f(u)$ that for $N \geq 3$ behave as a subcritical power at infinity, introducing the blow-up method together with Liouville type theorems for solutions in $\mathbb{R}^N$ and in the half space $\mathbb{R}^N_+$. 

In the same years De Figueiredo, Lions and Nussbaum [27] obtained a similar result using a different method. In convex domains in particular, it is based on the monotonicity results by Gidas, Ni and Nirenberg ([33]) obtained by using the Alexandrov-Serrin moving plane method ([53]), (providing a priori bounds in a neighborhood of the boundary), on the Pohozaev identity ([48]) and on the $L^p$ theory for Laplace equations given by Calderon-Zygmund estimates, see e.g. [34]. They extend then some of the results to arbitrary smooth domains through the Kelvin transform.

In recent years Castro and Pardo ([13]) proved, using the techniques introduced in [27], an extension of the results to the case of nonlinearities $f(u)$ more general than functions similar to subcritical powers, showing the flexibility of the method, see also [14, 43]. Their arguments rely on estimating from below the radius $R$ such that $u(x) \geq \frac{1}{2} \|u\|_{\infty} = \frac{1}{2} u(x_0)$ for any $x \in B(x_0, R)$.

For the case of the $p$-laplacian, Liouville theorems for quasilinear elliptic inequalities in $\mathbb{R}^N$ involving the $p$-laplacian were proved by Mitidieri and Pohozaev ([45, 46]), and later for more general operators by Serrin and Zou ([54]).

Using these and other methods Azizieh and Clement and Ruiz proved in very interesting papers, different versions of a priori estimates for equations of the type (1.1).

With the help of the blow-up procedure, Azizieh and Clement ([5], see also [6] for the case of systems) proved a priori estimates for the equation (1.1) in the case of $\Omega$ being a strictly convex domain, $1 < p < 2$ and $f(u)$ growing not faster than a power $u^{\vartheta}$ at infinity, with

$$1 < \vartheta < p_* - 1 \ , \ p_* = \frac{(N - 1)p}{N - p}$$
The exponent \( p_* = \frac{(N-1)p}{N-p} \) is the optimal exponent for Liouville theorems for elliptic inequalities and observe that

\[
p_* = \frac{(N-1)p}{N-p} < p^* = \frac{Np}{N-p}
\]

where \( p^* \) is the critical exponent for the Sobolev’s embeddings. The restriction \( 1 < p < 2 \) depends on the fact that using a blow-up procedure and Liouville theorems on the whole space, they need to exclude concentration of maximum points of the solutions at the boundary, and they use some result proved in \([21, 22]\) on the symmetry and monotonicity of solutions to \( p \)-Laplace equations in the singular case \( 1 < p < 2 \), results that were later extended to the case \( p > 2 \) in the papers \([23, 24]\).

Ruiz \((52)\) proved, using a different technique based among other tools on Harnack type inequalities, a priori estimates for equation more general than \((1.1)\), therein \( f = f(x, u, Du) \) can depend on \( x \) and on the gradient, for any \( 1 < p < N \) and for general domains. Once again the growth at infinity with respect to \( u \) must be less than powers with exponent \( \vartheta < p_* - 1 \).

In both papers, there is also a general discussion on how the existence of solutions follows from the a priori estimates, using some abstract results by Krasnoselskij already used in \([27]\).

Later, Zou \((57)\) proved Liouville theorems in half spaces that, together with the results in \([54]\), allow him to use the blow-up method and prove, in case \( 1 < p < N \), a priori estimates for equation more general than \((1.1)\), therein \( f = f(x, u, Du) \) can depend on \( x \) and on the gradient, and under various hypotheses on the nonlinearities. In particular, it is supposed that \( f = f(x, u, Du) \) grows with respect to \( u \) as a subcritical power at infinity and zero.

For monotonicity and Liouville type theorems in half spaces see also the papers of Farina, Montoro and Sciunzi \((28, 29)\).

In recent years related a priori estimates for general operators were established by D’Ambrosio and Mitidieri in \([18, 19]\).

The aim of this note is to prove a priori estimates for solutions of \((1.1)\) in the case of \( \Omega \) being a smooth bounded strictly convex domain, for any value of \( p > 1 \). In the case \( 1 < p < N \), \( f(u) \) is supposed to have a subcritical grow at infinity, but allowing more general functions than merely subcritical powers, see Example \(3.1\).

We use the technique introduced in \([27]\), that allows to give the same proof in case \( 1 < p < N \), case \( p = N \) and case \( p > N \).

Of course in the latter cases, much weaker hypotheses are needed in order to obtain the desired estimates (in particular in case \( p > N \) we only need that \( f(u) \) grows faster than \( u^{p-1} \) at infinity, condition that for \( p = 2 \) is the superlinearity at infinity).
The proofs in [27] for the case $p = 2$ rely deeply (among many other tools that we had to modify to handle in our case) on the $C^2$ regularity of the solutions, which are then classical solutions, and on the $W^{2,q}$ estimates based on the Calderon-Zygmund estimates.

These estimates are not available in the singular/degenerate case of $p \neq 2$, and we think that the use of regularity properties and other tools, that we exploit to implement the method, could be interesting also for other problems.

Let us state in more detail the results that we prove.

**Theorem 1.1** (Case $p > N$). Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is strictly convex.

Suppose that $p > N$, the condition $H_1$ holds and

$H_2)$ There exist $\tau > 0$ and $C_1 > 0$ such that

$$\liminf_{s \to +\infty} \frac{f(s)}{s^{p-1+\tau}} > C_1 > 0,$$

Then, the solutions of (1.1) are a priori bounded in $L^\infty$: there exists a constant $C$, depending on $p$, $\Omega$ and $f$, but independent of the solution $u$, such that $\|u\|_{L^\infty(\Omega)} \leq C$ for any solution of (1.1).

**Remark 1.1.** For the ordinary laplacian ($p = 2$), the above theorem corresponds to the case of dimension $N = 1$, and in [27, Remark 1.3] it was observed that if $N = 1$, solutions are uniformly bounded under the only hypothesis of superlinearity at infinity, which corresponds for $p = 2$ to the hypothesis $H_2$) with $\tau = 0$, and the bound from below strictly bigger than $\lambda_1$, the first eigenvalue for the Laplacian operator.

We need the slightly stronger form (with $\tau > 0$ but arbitrarily small) for technical reasons (use of the Picone’s Identity for the $p$-laplacian).

**Theorem 1.2** (Case $p = N$). Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is strictly convex.

Let $p = N$ and suppose that $H_1$ and $H_2$ hold, as well as

$H_3)$ There exists $C_2 > 0$ such that

$$\liminf_{s \to +\infty} \frac{F(s)}{f(s)} > C_2 > 0,$$

where $F(t) = \int_0^t f(s) \, ds$ is a primitive of the function $f$.

$H_4)$ There exists $\theta > 0$ such that

$$\limsup_{s \to +\infty} \frac{|f(s)|}{s^\theta} < +\infty$$

(of course this is equivalent to

There exists $\eta > 0$ such that $\lim_{s \to +\infty} \frac{|f(s)|}{s^\eta} = 0$)

Then, the solutions of (1.1) are a priori bounded in $L^\infty$: there exists a constant $C$, depending on $p$, $\Omega$ and $f$, but independent of the solution $u$, such that $\|u\|_{L^\infty(\Omega)} \leq C$ for any solution of (1.1).
Remark 1.2. If $p = 2$ (the ordinary laplacian), the above theorem corresponds to the case of dimension $N = 2$, and in that case solutions are uniformly bounded under the only hypotheses of superlinearity together with polynomial growth at infinity, cf. [27, Theorem 1.1]. All the functions $f$ growing polynomially at infinity are included in hypotheses $H_3$), and $H_4$).

Nevertheless, when $p = N$ the critical embedding is of exponential type, and those hypotheses are not optimal (neither are the hypotheses in [27]), and we think that they can be improved.

**Theorem 1.3** (Case 1 < $p$ < $N$, first result). Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is strictly convex.

Let 1 < $p$ < $N$, and suppose that $H_1$) and $H_2$) hold, as well as

$H_3'$) There exists $C_3 > 0$ such that

$$\liminf_{s \to +\infty} \frac{p^* F(s) - sf(s)}{s f(s)} > C_3 > 0$$

$H_4') \lim_{s \to +\infty} \frac{f(s)}{s^{p^* - 1}} = 0$

where $p^* = \frac{Np}{N-p}$.

Then, the solutions of (1.1) are a priori bounded in $L^\infty$: there exists a constant $C$, depending on $p$, $\Omega$ and the function $f$, but independent of the solution $u$, such that $\|u\|_{L^\infty(\Omega)} \leq C$ for any solution of (1.1).

Remark 1.3. In case of the ordinary laplacian ($p = 2$), the above theorem corresponds to the case of dimension $N \geq 3$, and in [27] it was proved under similar hypotheses.

We include this version here, because the first three theorems share the same proof (and it corresponds to the [27] hypotheses).

Nevertheless, we also prove another result (see Theorem 1.4 that follows), weakening the hypotheses needed for the result (except for a further technical hypothesis, $H_5$), which is satisfied for a general class of nonlinearities) and extends the class of nonlinearities allowed, including functions $f$ more general than subcritical powers, and can be seen as the counterpart for $p \neq 2$ to the results in [13] in case $p = 2$.

**Theorem 1.4** (Case 1 < $p$ < $N$ second result). Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is strictly convex.

Let 1 < $p$ < $N$, and suppose that $H_1$) and $H_2$) hold, as well as

$H_3”)$) There exists a nonincreasing positive function $H : [0, +\infty) \to \mathbb{R}$ such that

$$\liminf_{s \to +\infty} \frac{p^* F(s) - sf(s)}{H(s)f(s)} > 0$$

$H_4”)$) \lim_{s \to +\infty} \frac{f(s)}{s^{p^* - 1} H(s) \sqrt{s}} = 0
There exist $C_4 > 0$, $C_5 > 0$ such that
\[
\lim \inf_{s \to +\infty} \min_{\Omega} \frac{f(s)}{f(s)} \geq C_4 > 0
\]
\[
\lim \sup_{s \to +\infty} \max_{\Omega} \frac{f(s)}{f(s)} \leq C_5
\]
Then, the solutions of (1.1) are a priori bounded in $L^\infty$: there exists a constant $C$, depending on $p$, $\Omega$ and the function $f$, but independent of the solution $u$, such that $\|u\|_{L^\infty(\Omega)} \leq C$ for any solution of (1.1).

The existence of solutions for (1.1) follows from the a priori estimates, with a further hypothesis about the behavior of the nonlinearity at zero. This was proved in [27] (with the hypothesis $H_0$ below) for $p = 2$, using some variants of topological arguments, connected with theorems of Krasnoselskii ([37]) and Rabinowitz ([51]) based on degree theory. It was extended to the case of $p$-Laplace equations in [5, 52, 57].

It also can be adapted to our hypotheses. More precisely, the following result holds.

**Theorem 1.5.** Let us suppose that the hypotheses of one of the previous theorem hold, and assume also that
\[
H_0 \quad \lim \sup_{s \to 0^+} \frac{f(s)}{s} < \lambda_1
\]
where $\lambda_1$ is the first eigenvalue for the $p$-Laplacian (see Section 2).

(since $f(0) \geq 0$ by $H_1$), this hypothesis implies that $f(0) = 0$)

Then, there exists a positive solution of (1.1).

The paper is organized as follows. In Section 2 we recall, and in some cases prove, all the auxiliary results that we need in the sequel.

In Section 3 we give the proofs of the a priori estimates stated in Theorems 1.1-1.4 and we give an example of an almost critical nonlinearity in the case $1 < p < N$ that can be handled with the help of Theorem 1.4 but not with the previous theorems, nor with the blow-up method that relies on Liouville theorems with functions $f$ that behave exactly as a subcritical power at infinity.

Finally in Section 4 we prove Theorem 1.5.

2. Preliminaries

2.1. Strong maximum principles and Hopf’s Lemma. Monotonicity of the solutions and moving planes method.

Let us first recall a particular version of the Strong Maximum Principle and of the Hopf’s Lemma for the $p$-laplacian (see [50] for the case of the $p$-laplacian and [19], [50] for general quasilinear elliptic operators).
Theorem 2.1. (Strong Maximum Principle and Hopf’s Lemma). Let Ω be a domain in \( \mathbb{R}^N \) and suppose that \( u \in C^1(\Omega) \), \( u \geq 0 \) in \( \Omega \), weakly solves
\[
-\Delta_p u + cu^q = g \geq 0 \quad \text{in} \quad \Omega
\]
with \( 1 < p < \infty \), \( q \geq p - 1 \), \( c \geq 0 \) and \( g \in L^\infty_{\text{loc}}(\Omega) \). If \( u \neq 0 \) then \( u > 0 \) in \( \Omega \). Moreover for any point \( x_0 \in \partial \Omega \) where the interior sphere condition is satisfied, and such that \( u \in C^1(\Omega \cup \{x_0\}) \) and \( u(x_0) = 0 \) we have that \( \frac{\partial u}{\partial \nu}(x_0) > 0 \) for any inward directional derivative (this means that if \( y \) approaches \( x_0 \) in a ball \( B \subseteq \Omega \) that has \( x_0 \) on its boundary, then \( \lim_{y \to x_0} \frac{u(y) - u(x_0)}{|y - x_0|} > 0 \)).

Concerning weak and strong comparison principles for the \( p \)-laplacian, see e.g. [17], [20], [23], [24], [35], [49]. Here we will only need the following elementary case of (weak) comparison principle.

Theorem 2.2. (Weak comparison principle) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \), \( 1 < p < \infty \), and suppose that \( u, v \in W^{1,p}(\Omega) \) weakly satisfy
\[
\begin{cases}
-\Delta_p u \leq -\Delta_p v, & \text{in } \Omega \\
u \leq v, & \text{on } \partial \Omega
\end{cases}
\]
i.e. \( (u - v)^+ \in W^{1,p}(\Omega) \), and for any \( \varphi \in C^1_c(\Omega) \), \( \varphi \geq 0 \), (and by density for any nonnegative \( \varphi \in W^{1,p}(\Omega) \))
\[
\int_{\Omega} |Du|^{p-2}Du \cdot D\varphi \leq \int_{\Omega} |Dv|^{p-2}Dv \cdot D\varphi
\]
Then, \( u \leq v \) in \( \Omega \).

The proof is elementary, taking \( \varphi = (u - v)^+ \) as a test function.

Next, we recall some results on the monotonicity of solutions of the \( p \)-Laplace equations. We consider the following problem
\[
\begin{cases}
-\Delta_p(u) = f(u), & \text{in } \Omega \\
u > 0, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( N \geq 2 \), \( 1 < p < \infty \), and we have the following hypotheses on \( f \):
(*) \( f : [0, \infty) \to \mathbb{R} \) is a continuous function which is locally Lipschitz continuous in \((0, \infty)\).

The results that we are going to recall, can be briefly rephrased saying that all the conclusions of Gidas, Ni, and Nirenberg’s Theorem (see [33], [7]) hold for the \( p \)-Laplacian, provided \( f \) is only locally Lipschitz continuous if \( 1 < p < 2 \), and moreover \( f(s) > 0 \) if \( s > 0 \) for \( p > 2 \) (see [20], [21], [22], [23], [24]).
To state more precisely some known result about the monotonicity and symmetry of solutions of $p$-Laplace equations, we need some notations.

Let $\nu$ be a direction in $\mathbb{R}^N$. For a real number $\lambda$ we define

\begin{equation}
T_\lambda^{\nu} = \{ x \in \mathbb{R} : x \cdot \nu = \lambda \} 
\end{equation}

\begin{equation}
\Omega_\lambda^{\nu} = \{ x \in \Omega : x \cdot \nu < \lambda \} 
\end{equation}

\begin{equation}
x_\lambda^{\nu} = R_\lambda^{\nu}(x) = x + 2(\lambda - x \cdot \nu)\nu, \quad x \in \mathbb{R}^N 
\end{equation}

and

\begin{equation}
a(\nu) = \inf_{x \in \Omega} x \cdot \nu. 
\end{equation}

If $\lambda > a(\nu)$ then $\Omega_\lambda^{\nu}$ is nonempty, thus we set

\begin{equation}
(\Omega_\lambda^{\nu})' = R_\lambda^{\nu}(\Omega_\lambda^{\nu}). 
\end{equation}

Following [33, 53] we observe that for $\lambda - a(\nu)$ small then $(\Omega_\lambda^{\nu})'$ is contained in $\Omega$ and will remain in it, at least until one of the following occurs:

(i) $(\Omega_\lambda^{\nu})'$ becomes internally tangent to $\partial \Omega$.

(ii) $T_\lambda^{\nu}$ is orthogonal to $\partial \Omega$.

Let $\Lambda_1(\nu)$ be the set of those $\lambda > a(\nu)$ such that for each $\mu < \lambda$ none of the conditions (i) and (ii) holds and define

\begin{equation}
\lambda_1(\nu) = \sup \Lambda_1(\nu). 
\end{equation}

Moreover let

\begin{equation}
\Lambda_2(\nu) = \{ \lambda > a(\nu) : (\Omega_\mu^{\nu})' \subseteq \Omega \quad \forall \mu \in (a(\nu), \lambda] \}, 
\end{equation}

and

\begin{equation}
\lambda_2(\nu) = \sup \Lambda_2(\nu). 
\end{equation}

Since $\Omega$ is supposed to be smooth, note that neither $\Lambda_1(\nu)$ nor $\Lambda_2(\nu)$ are empty, and $\Lambda_1(\nu) \subseteq \Lambda_2(\nu)$, so that $\lambda_1(\nu) \leq \lambda_2(\nu)$ (in the terminology of [33], $\Omega_\lambda^{\nu}(\nu)$ and $\Omega_{a(\nu)}^{\nu}(\nu)$ correspond to the 'maximal cap', and the 'optimal cap' respectively). Finally define

\begin{equation}
\Lambda_0(\nu) = \{ \lambda > a(\nu) : u \leq u_\lambda^{\nu} \quad \forall \mu \in (a(\nu), \lambda] \}, 
\end{equation}

and

\begin{equation}
\lambda_0(\nu) = \sup \Lambda_0(\nu). 
\end{equation}

**Theorem 2.3.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, $1 < p < \infty$, $f : [0, \infty) \to \mathbb{R}$ a continuous function which is locally Lipschitz continuous in $(0, \infty)$ and strictly positive in $(0, \infty)$ if $p > 2$. Let $u \in C^1(\overline{\Omega})$ be a weak solution of (2.2).

For any direction $\nu$ and for $\lambda$ in the interval $(a(\nu), \lambda_1(\nu)]$ we have

\begin{equation}
u(x) \leq u(x) \quad \forall x \in \Omega_\lambda^{\nu}. 
\end{equation}
If \( f \) is locally Lipschitz continuous in the closed interval \([0, \infty)\) then (2.13) holds for any \( \lambda \) in the interval \((a(\nu), \lambda_2(\nu))\).

**Corollary 2.1.** If \( f \) is locally Lipschitz continuous in the closed interval \([0, \infty)\) and strictly positive in \((0, \infty)\), and the domain \( \Omega \) is convex with respect to a direction \( \nu \) and symmetric with respect to the hyperplane \( T^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\} \), then \( u \) is symmetric, i.e., \( u(x) = u(x_0^\nu) \), and nondecreasing in the \( \nu \)-direction in \( \Omega_0^\nu \). In particular if \( \Omega \) is a ball then \( u \) is radially symmetric and radially decreasing.

**Remark 2.1.** In this paper we assume \( f \) positive for \( p > 2 \) only because we are going to exploit the monotonicity results stated in the previous theorem, obtained in [21, 22, 23, 24], and in these papers the positivity of \( f \) is assumed when \( p > 2 \).

In any case, in all the results that we prove we always assume that \( f \) satisfies \( H_1) \).

As a consequence of the previous theorem, solutions are monotone increasing from the points on the boundary along directions that belong to a neighborhood of directions close to the inner boundary.

As a further consequence we have the following property, as observed in [27] for \( p = 2 \), which can be deduced by contradiction using the strict convexity of the domain and the monotonicity of the solutions provided by the previous cited papers (see [5] for a related geometric discussion).

**Lemma 2.1.** Let \( \Omega \) be a strictly convex bounded smooth domain, and define \( \Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \delta\} \) for \( \delta > 0 \).

Then the following holds for a weak solution \( u \in C^1(\Omega) \) of the problem (1.1), where \( f \) satisfies the condition \( H_1) \)

\begin{equation}
\begin{cases}
\exists \gamma, \epsilon > 0, \text{ depending only on } \Omega, \text{ such that } \\
\forall x \in \Omega \setminus \Omega_\delta \text{ there is a part of a cone } I_x \text{ with } \\
1) \quad u(\xi) \geq u(x) \quad \forall \xi \in I_x, \\
2) \quad I_x \subset \Omega^\xi_\delta, \\
3) \quad \text{meas (}I_x\text{)} \geq \gamma.
\end{cases}
\end{equation}

Geometrically \( I_x \) is a part of a cone \( K_x \) with vertex in \( x \), where all the \( K_x \) are congruent to a fixed cone \( K \), and if \( x \in \Omega \setminus \Omega_\delta \), then \( I_x = K_x \cap \Omega^\xi_\delta \).

Let us emphasize that \( \epsilon \) and \( \gamma \) depend only on the geometry of the strictly convex, bounded, smooth domain \( \Omega \).

We will use this conditions to get \( L^\infty \) a priori bounds in a neighborhood of the boundary, for the solutions on a strictly convex, bounded, smooth domain \( \Omega \).
2.2. First eigenvalue and eigenfunction.

Let $\Omega$ be a bounded domain and $1 < p < \infty$. A real number $\lambda$ is a (nonlinear) eigenvalue of the $p$-laplacian, with associated eigenfunction $u$ if $u \in W^{1,p}_0(\Omega)$, $u \not\equiv 0$, solves the equation $-\Delta_p u = \lambda |u|^{p-2}u$ in $\Omega$.

Although the general theory of eigenvalues for the $p$-Laplacian is far from complete (see [30] and the survey [47]), the properties of the first eigenvalue are known and are the same as in the case $p = 2$. Namely the following result holds (see [3], [39], [47]).

**Theorem 2.4.** Let us define

$$\lambda_1 = \lambda_1(p, \Omega) = \inf \left\{ \int_{\Omega} |Dv|^p \, dx : v \in W^{1,p}_0(\Omega), \int_{\Omega} |v|^p \, dx = 1 \right\}$$

Then, $\lambda_1$ is the first eigenvalue (i.e. $\lambda_1 \leq \lambda$ for any eigenvalue $\lambda$), it is simple, i.e. there is only an eigenfunction up to multiplication by a constant, and it is isolated.

Moreover a first eigenfunction does not change sign in $\Omega$ and by the strong maximum principle it is in fact either strictly positive or strictly negative in $\Omega$.

So we can select a unique eigenfunction $\phi_1$ such that $\int_{\Omega} |\phi_1|^p \, dx = 1$ and $\phi_1 > 0$ in $\Omega$.

2.3. Picone’s identity and inequality.

The following extension of the Picone’s identity for the $p$-Laplacian has been proved by Allegretto and Huang.

**Theorem 2.5 ([1]).** Let $v_1, v_2 \geq 0$ be differentiable functions in an open set $\Omega$, with $v_2 > 0$ and $p > 1$. Put

$$L(v_1, v_2) = |\nabla v_1|^p + (p - 1) \frac{v_1^p}{v_2} |\nabla v_2|^p - p \frac{v_1^{p-1}}{v_2} \nabla v_1 \cdot |\nabla v_2|^{p-2} \nabla v_2,$$

$$R(v_1, v_2) = |\nabla v_1|^p - \nabla \left( \frac{v_1^p}{v_2^{p-1}} \right) \cdot |\nabla v_2|^{p-2} \nabla v_2.$$

Then, $R(v_1, v_2) = L(v_1, v_2)$ and $L(v_1, v_2) \geq 0$.

As a consequence we have

$$\nabla \left( \frac{v_1^p}{v_2^{p-1}} \right) \cdot |\nabla v_2|^{p-2} \nabla v_2 \leq |\nabla v_1|^p$$

2.4. Pohozaev’s Identity for the $p$-Laplacian.

The following extension of the Pohozaev’s identity for the $p$-Laplacian has been proved by Guedda and Veron.
Theorem 2.6 (\[35\]). Let \( u \in W^{1,p}_{0}(\Omega) \cap L^{\infty}(\Omega) \) be a weak solution of the problem
\[
(2.17) \quad \begin{cases}
-\Delta_p(u) = f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( N \geq 2 \), \( p > 1 \) and \( f : [0, \infty) \to \mathbb{R} \) is a continuous function.

Let \( F(t) = \int_0^t f(s) \, ds \) be a primitive of the function \( f \). Then,
\[
(2.18) \quad N \int_{\Omega} F(u) \, dx - \frac{N-p}{p} \int_{\Omega} f(u) u \, dx = \frac{p-1}{p} \int_{\partial \Omega} |\frac{\partial u}{\partial \nu}|^p (x \cdot \nu) \, ds
\]
where \( \nu \) is the unit exterior normal on \( \partial \Omega \).

2.5. Gradient Regularity.

We are going to use the following result about the summability of the gradient for solutions to equations involving the \( p \)-Laplace operator.

Theorem 2.7 (Gradient Regularity). Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \), and let \( u \in W^{1,p}_{0}(\Omega) \), \( 1 < p < \infty \), be a solution of the problem
\[
(2.19) \quad \begin{cases}
-\Delta_p(u) = g & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with \( g \in L^q(\Omega) \).

We suppose that
\[
(2.20) \quad \begin{cases}
1 < q < \infty & \text{if } p \geq N, \\
(p^*)' \leq q < \infty & \text{if } 1 < p < N.
\end{cases}
\]
Here \( p^* = \frac{Np}{N-p} \) is the critical exponent for Sobolev embedding, and \((p^*)' = \frac{p^*}{p^* - 1} = \frac{Np}{Np-N+p} \) is its conjugate exponent.

i) If \( q < N \), then \( \|\nabla u\|_{L^{\frac{N(p-1)}{N-p}}(\Omega)} \leq C \|g\|_{L^{q'}(\Omega)}^{\frac{1}{p-1}} \)

ii) If \( q \geq N \), then \( \|\nabla u\|_{L^p(\Omega)} \leq C \|g\|_{L^q(\Omega)}^{\frac{1}{p-1}} \) for any \( \sigma < \infty \).

Here \( C \) is a constant that depends on \( p, N, q \).

Remark 2.2. The exponent \((p^*)'\) is called the duality exponent, and the condition \( q \geq (p^*)' \) if \( 1 < p < N \) guarantees by Sobolev’s embeddings that \( g \in L^q(\Omega) \) belongs to the dual space \( W^{-1,p'}(\Omega) \). If this is not the case, then we enter into the field of problems with measure data, and other notions of solutions have been proposed (see \([8], [44]\)).
The previous theorem follows from different results proved in several papers (see [8], [10], [15], [25], [26], [36], [44], the survey [16], and the references therein), where also much more general situations are considered.

In the form that we need it is a consequence of the following

**Theorem 2.8.** Let \( \Omega \) be a bounded (smooth) domain in \( \mathbb{R}^N \), \( 1 < p < \infty \) and let \( u \in W^{1,p}_0(\Omega) \) be a solution of the Dirichlet problem connected to the equation \( -\Delta_p u = \text{div } F \) in \( \Omega \), where \( F \in L^{p'}(\Omega; \mathbb{R}^N) \), with \( p' = \frac{p}{p-1} \).

If \( F \in L^{p'}(\Omega; \mathbb{R}^N) \cap L^s(\Omega; \mathbb{R}^N) \), with \( s \geq p' \), then \( \nabla u \in L^{(p-1)s}(\Omega; \mathbb{R}^N) \), and
\[
\| \nabla u \|_{L^{p-1}(\Omega; \mathbb{R}^N)} = \| \nabla u \|^{p-1}_{L^s} \leq C(s, p, N) \| F \|_{L^s}.
\]

In [36] the theorem is proved in the case \( \Omega = \mathbb{R}^N \) (Theorem 2), and generalized in [26] for systems in bounded domains \( \Omega \) to obtain interior estimates of the gradient.

The global estimates provided by Theorem 2.8 are then proved in the paper [10] (see Theorem 1.8), where also much more irregular domains are considered.

Let us show how Theorem 2.8 follows from Theorem 2.7.

**Proof of Theorem 2.7.** Let \( u \in W^{1,p}_0(\Omega) \) be a solution of (2.19), with \( g \in L^q(\Omega) \), \( 1 < q < N \) (and \( q \geq (p^*)' \) if \( 1 < p < N \)).

Then, we claim that \( g = \text{div } F \) in \( \Omega \), with \( F \in L^s(\Omega; \mathbb{R}^N) \) with \( s = q^* = \frac{Nq}{N-q} \).

In fact, if \( z \) solves the Dirichlet problem for \( -\Delta z = g \in L^q(\Omega) \), then \( z \in W^{2,q}(\Omega) \) by the Calderon-Zygmund estimates (see [34]), so that \( \nabla z \in L^s(\Omega; \mathbb{R}^N) \), \( s = q^* \), and \( g = \text{div } F \) in \( \Omega \), with \( F := -\nabla z \in L^s(\Omega; \mathbb{R}^N) \), and \( \| F \|_{L^s} = \| F \|_{L^{q^*}} \leq C(\| g \|_{L^q}) \).

Moreover \( q^* \geq p' \).

Indeed, if \( p \geq N \) then \( p' \leq N' = \frac{N}{N-1} = 1^* < q^* \) for any \( q > 1 \).

On the other side, if \( 1 < p < N \), then it is straightforward to check that condition \( q^* \geq p' \) is equivalent to our hypothesis \( q \geq (p^*)' \).

Therefore by Theorem 2.8 we get that \( \nabla u \in L^{(p-1)s}(\Omega; \mathbb{R}^N) \) with \( s = q^* \).

If \( q \geq N \) then by the same method \( g = \text{div } (F) = -\text{div } (\nabla z) \) in \( \Omega \), with \( F \in L^s(\Omega; \mathbb{R}^N) \) for any \( s > 1 \), in particular for any \( s \geq p' \).

By exploiting Theorem 2.8 in the same way, we get the result. \( \square \)

We will need also a local \( W^{1,\infty} \) result at the boundary, namely the following

...
Theorem 2.9. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, and let $u \in C^1(\overline{\Omega})$ be a solution of the problem

$$
\begin{aligned}
-\Delta_p(u) &= g & \text{in } \Omega \\
u &= 0 & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
$$

with $g \in L^{p^*'}(\Omega)$.

For $\delta > 0$, let $\Omega_\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \}$, and suppose that $u, g \in L^\infty(\Omega \setminus \Omega_\delta)$ with $\| g \|_{L^\infty(\Omega \setminus \Omega_\delta)} \leq M$, $\| u \|_{L^\infty(\Omega \setminus \Omega_\delta)} \leq M$.

Then, there exists a constant $C > 0$ only depending on $M$ and $\delta$ such that $\| \nabla u \|_{L^\infty(\partial \Omega)} \leq C$

Although similar estimates can be found in the literature, it was difficult for us to find the exact reference.

Therefore, we provide an ad-hoc proof based on comparison with the solution to a simple $p$-Laplace Dirichlet problem.

Proof. Let us consider the solution $v \in W^{1,p}(\Omega)$ of the problem

$$
\begin{aligned}
-\Delta_p(v) &= 1 & \text{in } \Omega \\
v &= 0 & \text{on } \partial \Omega
\end{aligned}
$$

By the regularity results in [25, 40] $v \in C^1(\overline{\Omega})$, and by the weak and the strong maximum principle $v > 0$ in $\Omega$.

Since $u = |u|$ and $|u|, |g| \leq M$ in $\Omega \setminus \Omega_\delta$, there exists $N = N_{M,\delta} > 0$ such that

$$
\begin{aligned}
-\Delta_p(Nv) &= N^{p-1} \geq g = -\Delta_p(u) & \text{in } \Omega \setminus \Omega_\delta \\
v &= u & \text{on } \partial(\Omega \setminus \Omega_\delta)
\end{aligned}
$$

(it suffices to take

$$
N \geq M \frac{1}{\| g \|_{L^\infty(\Omega \setminus \Omega_\delta)}} \quad \text{and} \quad N \geq \frac{M}{\inf_{\partial \Omega_\delta} v} \geq \frac{\| u \|_{L^\infty(\partial \Omega_\delta)}}{\inf_{\partial \Omega_\delta} v}.
$$

Putting $v_N = N_{M,\delta} v$ and $C = \sup_{\partial \Omega} |Dv|$, by the weak comparison principle we obtain that

$$
u_N = v_N \in \Omega \setminus \Omega_\delta.$$

If $x \in \partial \Omega$ and $\nu = \nu_i$ is the inner normal and $\alpha(t) = u(x + t\nu)$, $\beta(t) = v_N(x + t\nu)$, it follows that $\alpha(t) \leq \beta(t)$ if $t \in [0, \delta)$, so that $\alpha'(0) = \beta'(0)$, i.e. $\frac{\partial u}{\partial \nu} (x) \leq \frac{\partial v}{\partial \nu} (x)$.

Moreover since $u = v = 0$ on $\partial \Omega$, the size of the gradient coincides with the normal derivative, in the interior direction since $u$ and $v$ are positive inside $\Omega$, so that for any $x \in \partial \Omega$ we have that

$$
|Du(x)| = \frac{\partial u}{\partial \nu} (x) \leq \frac{\partial v}{\partial \nu} (x) = |Dv_N(x)| = N|Dv(x)| \leq NC \quad \text{and} \quad \| Du \|_{L^\infty(\partial \Omega)} \leq N_{M,\delta} C =: C_{M,\delta}.
\]
3. Proofs of Theorems 1.1–1.4

Proof of Theorems 1.1–1.3.

Let us start with a consequence of hypotheses $H_1$) and $H_2$), which are assumed in all the theorems that we are proving.

By hypothesis $H_2$) there exist $u_0 > 0$ and $C_1 > 0$ such that

$$u^\tau \leq C_1 \frac{f(u)}{u^{p-1}}, \quad \text{if } u \geq u_0.$$  \hfill (3.1)

In particular (if $1 < p < 2$, it is part of the hypothesis $H_1$) if $p > 2$) we have that

$$f(u) > 0, \quad \text{if } u \geq u_0.$$  \hfill (3.2)

On the other hand, by hypothesis $H_1$) there exists $\Lambda \geq 0$ such that

$$\frac{f(u)}{u^{p-1}} \geq -\Lambda, \quad \text{for every } u > 0.$$  \hfill (3.3)

In fact if $p > 2$ then $f$ is supposed to be positive in $(0, +\infty)$ and (3.3) is immediate.

If instead $1 < p < 2$, as observed we have that $f(u)$ is positive in $[u_0, +\infty)$.

Step 1 - Uniform $L^1$ estimates of $u^\tau \varphi_p$ for some $\tau > 0$

Since Picone’s inequality (2.16) with $v_2 = u$, $v_1 = \varphi_1$, the unique first eigenfunction which is positive in $\Omega$ and normalized in the $L^p$ norm, we obtain that

$$\int_\Omega \frac{f(u)}{u^{p-1}} \varphi_1^p \leq \lambda_1 \int_\Omega \varphi_1^p = \lambda_1.$$  \hfill (3.4)

Therefore, taking into account (3.1) and (3.3), we have that

$$\int_\Omega u^\tau \varphi_1^p \, dx = \int_{[0 \leq u < u_0]} u^\tau \varphi_1^p \, dx + \int_{[u \geq u_0]} u^\tau \varphi_1^p \, dx$$

$$\leq u_0^\tau \int_\Omega \varphi_1^p \, dx + C_1 \int_{[u \geq u_0]} \frac{f(u)}{u^{p-1}} \varphi_1^p \, dx$$

$$= u_0^\tau + C_1 \int_{[0 \leq u < u_0]} \varphi_1^p \, dx - C_1 \int_{[0 \leq u < u_0]} \frac{f(u)}{u^{p-1}} \varphi_1^p \, dx$$

$$\leq u_0^\tau + C_1 \lambda_1 + C_1 \Lambda = C,$$  \hfill (3.5)

for a constant $C$ independent of $u$.

Step 2 - Uniform $L^\infty$ estimates near the boundary and uniform $W^{1,\infty}$ estimates at the boundary
Let \( \phi_1 \) be the first eigenfunction, positive in \( \Omega \) and normalized in the \( L^p \) norm. By (2.14) for any \( x \in \Omega \setminus \Omega_\delta \) we have that 
\[
\gamma (\inf_{\Omega_\delta} \phi_1^p) u(x)^p \leq \int_{\Omega} u^p \phi_1^p(\xi) \leq \int_{\Omega} u^p \phi_1^p \, dx \leq C
\]
where \( C \) is the uniform constant obtained in the previous step.

This gives the uniform \( L^\infty \) bounds near the boundary: there exists a \( \delta > 0 \) (depending on the geometry of \( \Omega \) through (2.14)) and a constant \( C > 0 \) independent of the solution \( u \), such that
\[
\|u\|_{L^\infty(\Omega \setminus \Omega_\delta)} \leq C
\]
for any solution \( u \) of (1.1).

Using Theorem 2.9 we get
\[
\left\| \frac{\partial u}{\partial \nu} \right\|_{L^\infty(\partial \Omega)} \leq C
\]
for a uniform constant \( C \) independent of the solution.

**Step 3 - Uniform \( W^{1,p}_0(\Omega) \) estimates**

If \( p > N \), since \( N \geq 2 \), by hypothesis \( H_1 \) we have that \( f \geq 0 \), so that also \( F \geq 0 \). By the Pohozaev’s identity (2.18) we have that
\[
\frac{N}{p} \int_{\Omega} f(u) \, u \, dx \leq \frac{p-1}{p} \int_{\Omega} |\frac{\partial u}{\partial \nu}|^p (x \cdot \nu) \, ds \quad \text{and} \quad |\frac{\partial u}{\partial \nu}|^p \leq C
\]
by the previous step, so that
\[
\int_{\Omega} |\nabla u|^p \, dx = \int f(u) \, u \, dx \leq C.
\]

If \( p = N \), by hypothesis \( H_3 \) there exist \( C_2 > 0 \) and \( s_0 > 0 \) such that \( uf(u) \leq C_2 F(u) \) if \( u \geq s_0 \), so that by the Pohozaev’s identity (2.18) (which in this case reduces to
\[
N \int_{\Omega} F(u) \, dx = \frac{p-1}{p} \int_{\Omega} |\frac{\partial u}{\partial \nu}|^p (x \cdot \nu) \, ds
\]
we have again that
\[
\int_{\Omega} |\nabla u|^N \, dx = \int f(u) \, u \, dx \leq C.
\]

If \( 1 < p < N \), again by Pohozaev’s identity (which in this case can be written as
\[
p^* \int_{\Omega} F(u) \, dx - \int_{\Omega} f(u) \, u \, dx = \frac{p-1}{N-p} \int_{\Omega} |\frac{\partial u}{\partial \nu}|^p (x \cdot \nu) \, ds
\]
and since hypothesis \( H'_3 \), there exists \( C_3 > 0 \) and \( s_0 > 0 \) such that \( uf(u) \leq C_3 (p^* F(u) - f(u)) \) if \( u \geq s_0 \), so that
\[
\int_{\Omega} |\nabla u|^p \, dx = \int f(u) \, u \, dx \leq C.
\]

This step ends the proof in the case \( p > N \), since \( W^{1,p}_0(\Omega) \) is continuously injected in \( L^\infty(\Omega) \) if \( p > N \).

From now on we suppose that \( 1 < p \leq N \).

**Step 4 - Uniform \( L^q(\Omega) \) estimates for any \( q < \infty \).**

If \( p = N \), the uniform \( W^{1,p}(\Omega) \) estimate implies, by Sobolev’s embedding, that \( u \) is uniformly bounded in \( L^q(\Omega) \) for any \( q < \infty \).
If $1 < p < N$, we adapt to the $p$-laplacian the technique used in [27] (that goes back to Brezis-Kato, see [12]) to get the $L^q$ estimates for any finite $q \geq 1$.

Testing the equation $-\Delta_p u = f(u)$ with $u^t$, $t \geq 1$, we get that
t\int_\Omega |\nabla u|^{p(t^{-1})} dx = \int f(u) u^t dx.

Since $|\nabla u|^{\frac{p-1+t}{p}} = (\frac{p-1+t}{p})^t |\nabla u|^{t^{-1}}$, if we put $\alpha_t = (\frac{p}{p-1+t})^t$ we can write the previous equation as

\begin{equation}
\alpha_t t \int_\Omega |\nabla u|^{\frac{p-1+t}{p}} = \int f(u) u^t.
\end{equation}

If $\varepsilon > 0$ by hypothesis $H'_4$ there exists $s_\varepsilon$ such that $|f(s)|s^q \leq \varepsilon s^{q-1+t}$ if $s \geq s_\varepsilon$, so that denoting by $C_t$ a uniform constant depending also on $t$, we get that
\begin{equation}
\int_\Omega |\nabla u|^{\frac{p-1+t}{p}} \leq C_t(C_t + \varepsilon \int_\Omega u^{p-1+t} dx) = C_t + \varepsilon C_t \int u^{t^{-1}p-1} u^{p-1} dx.
\end{equation}

By Sobolev’s inequality, and Hölder’s inequality with exponents $\frac{p}{p-1+t}$, $-\frac{p}{p-1+t}$, we get that
\begin{equation}
\left( \int_\Omega u^{\frac{p-1+t}{p}} dx \right)^\frac{p}{p-1+t} \leq C \int_\Omega \nabla u^{\frac{p-1+t}{p}} \leq C_t + \varepsilon C_t \int u^{t^{-1}p-1} u^{p-1} dx \leq C_t + \varepsilon C_t \left( \int_\Omega u^{\frac{p-1+t}{p}} dx \right)^\frac{p}{p-1+t} \leq C_t + \varepsilon C_t \left( \int_\Omega u^{\frac{p-1+t}{p}} dx \right)^\frac{p}{p-1+t},
\end{equation}

so that by Step 3 we have that \( \int |\nabla u|^p \) is uniformly bounded, hence also \( \int |\nabla u|^p \) is uniformly bounded.

Taking $\varepsilon$ small we get that
\begin{equation}
\int u^{\frac{p-1+t}{p}} dx \text{ is uniformly bounded for any fixed } 1 \leq t < \infty,
\end{equation}

so that
\begin{equation}
\int u^q \text{ is uniformly bounded for any fixed } q \geq p^* \text{ (and since } \Omega \text{ is bounded in fact for any } q \in [1, \infty)).}
\end{equation}

**Step 5 - $L^\infty(\Omega)$ uniform estimates.**

If $p = N$, since $u$ is uniformly bounded in $L^q(\Omega)$ for any fixed $q < \infty$, by hypothesis $H'_4$ also $f(u)$ is uniformly bounded in $L^q(\Omega)$ for any fixed $q < \infty$.

Taking any $q$ with $1 < q < N$, we get that $f(u)$ is uniformly bounded in $L^q(\Omega)$ so that by the regularity results cited before (see Theorem 2.7), $|\nabla u|$ is uniformly bounded in $L^{(N-1)q^*}(\Omega)$, with $(N-1)q^* > (N-1)1^* = N$.

This implies by Sobolev’s embedding that $u$ is uniformly bounded in $L^\infty(\Omega)$.

If instead $1 < p < N$, since $u$ is uniformly bounded in $L^q(\Omega)$ for any fixed $q < \infty$, by hypothesis $H'_4$ also $f(u)$ is uniformly bounded in $L^q(\Omega)$ for any fixed $q < \infty$.

Taking any $q$ with $1 < q < N$ (observe that $(p^*)' < N$ exactly when $1 < p < N$), by the regularity results cited before (see Theorem 2.7), $|\nabla u|$ is uniformly bounded in $L^{(p-1)q^*}(\Omega)$, and $(p-1)q^* > N$ if $q > \frac{N}{p}$. 

This implies again by Sobolev’s embedding that $u$ is uniformly bounded in $L^\infty(\Omega)$.

\[ \square \]

**Proof of Theorem 1.4.**

Proceeding exactly as in the previous theorems we get uniform $L^\infty$ estimates near the boundary, see \[3.6\]. Next, instead of getting uniform estimates of $\int_\Omega |\nabla u|^p \, dx = \int_\Omega uf(u) \, dx$ we get, from the condition $H''_3$ and the Pohozaev’s identity that $\int_\Omega uf(u)H(u) \, dx \leq C$, where $C$ is a constant that does not depend on the solution. From this, since by hypothesis $H_2$ we have that there exists $s_0 > 0$ such that $f(s) > 0$ for $s \geq s_0$, it follows easily that

\[ (3.9) \quad \int_\Omega u|f(u)|H(u) \, dx \leq C. \]

We observe now that by hypothesis $H''_4$, since $p^* - 1 = \frac{N(p-1)+p}{N-p}$, we have that $\lim_{s \to +\infty} \frac{|f(s)|^{1+p^*}}{s^{N(p-1)+p}} = 0$, so that, multiplying numerator and denominator by $|f(s)|H(s)^{\frac{N(p-1)}{N(p-1)+p}}$ we get that there exists a constant $C > 0$ such that

\[ (3.10) \quad |f(s)|^{1+p^*}H(s)^{\frac{N(p-1)}{N(p-1)+p}} \leq s|f(s)|H(s) + C. \]

Using \[3.9\] we get that

\[ (3.11) \quad \int_\Omega |f(u)|^{1+p^*}H(u)^{\frac{N(p-1)}{N(p-1)+p}} \leq C. \]

Consequently, for any $q > N/p$, since $N/p > 1 + \frac{1}{p^*-1}$ precisely when $1 < p < N$, and $H$ is non-increasing,

\[ (3.12) \quad \int_\Omega |f(u(x))|^q \, dx \leq \]

\[ \leq \int_\Omega |f(u(x))|^{1+\frac{1}{p^*-1}}H(u)^{\frac{N(p-1)}{N(p-1)+p}} |f(u(x))|^{q-1-\frac{1}{p^*-1}} \, dx \]

\[ \leq C \frac{\|f(u(\cdot))\|_{L^\infty}^{q-1-\frac{1}{p^*-1}}}{H(\|u\|_\infty)^{\frac{N(p-1)}{N(p-1)+p}}}. \]
Let us restrict $q \in (N/p, N)$. From Theorem 2.7 i), we have that
\[
\|\nabla u\|_{L^{Nq(p-1)}(\Omega)} \leq C \|f(u)\|^{{p-1 \over p}}_{L^{q}(\Omega)}
\leq C \left(\|f(u(\cdot))\|_{\infty}^{1-{1 \over q} + {1 \over N(p-1)}} \left[H\left(\|u\|_{\infty}\right)\right]^{N(p-1) \over (N(p-1)+q)}\right)^{1 \over p-1},
\]
and note that, since $q > N/p$,
\[
r := {Nq(p-1) \over N-q} > N.
\]

From Morrey’s Theorem, (see [11, Theorem 9.12 and Corollary 9.14]), there exists a constant $C$ only dependent on $\Omega$, $r$ and $N$ such that
\[
|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^{1-N/r}\|\nabla u\|_{L^{r}(\Omega)}, \forall x_1, x_2 \in \Omega.
\]
Therefore, for all $x \in B(x_1, R) \subset \Omega$
\[
|u(x) - u(x_1)| \leq C R^{1-{N \over r}}\|\nabla u\|_{L^{r}(\Omega)}.
\]

From now on, we shall argue by contradiction, being the main idea that if a sequence of solutions $u_k$ is unbounded in $L^\infty(\Omega)$, then also the integrals $\int_\Omega u_k|f(u_k)|H(u_k)\,dx$ tends to infinity. We achieve it by estimating the radius $R_k$ such that $u_k(x) \geq {1 \over 2}\|u_k\|_{\infty}$ for any $x \in B(x_k, R_k)$.

Let $\{u_k\}_k$ be a sequence of $C^1(\bar{\Omega})$ positive solutions to (1.1) and assume that
\[
\lim_{k \to \infty} \|u_k\| = +\infty, \quad \text{where } \|u_k\| := \|u_k\|_{\infty}.
\]
From the previous estimate near the boundary, let $C, \delta > 0$ be as in (3.6), see Theorem 2.9. Let $x_k \in \bar{\Omega}_\delta$ be such that
\[
u_k(x_k) = \max_{\Omega_{\delta}} u_k = \max_{\Omega} u_k.
\]
By taking a subsequence if needed, we may assume that there exists $x_0 \in \Omega_\delta$ such that
\[
\lim_{k \to \infty} x_k = x_0 \in \Omega_\delta, \quad \text{and } d_0 := dist(x_0, \partial\Omega) \geq \delta > 0.
\]
Let us choose $R_k$ such that $B_k = B(x_k, R_k) \subset \Omega$, and
\[
u_k(x) \geq {1 \over 2}\|u_k\| \quad \text{for any } x \in B(x_k, R_k),
\]
and there exists $y_k \in \partial B(x_k, R_k)$ such that
\[
u_k(y_k) = {1 \over 2}\|u_k\|.
\]
Let us denote by
\[ m_k := \min_{\|u_k\|^2/\|u_k\|} f, \quad M_k := \max_{0,\|u_k\|} f. \]
From definition, we obtain
\[ m_k \leq f(u_k(x)) \quad \text{if} \quad x \in B_k, \quad f(u_k(x)) \leq M_k \quad \forall x \in \Omega. \]
Then, reasoning as in (3.12), we obtain
\[ \int_\Omega |f(u_k)|^q \, dx \leq C \frac{M_k^{q-1-\frac{1}{p^*-1}}}{H(\|u_k\|)^{N/(p-1)+p}}. \]  
From gradient regularity for \( p \)-laplacian equations, see (3.13), we  
deduce
\[ \|\nabla u_k\|_{L^r(\Omega)} \leq C \left( \frac{M_k^{-\frac{1}{r} - \frac{1}{p^* - 1} q}}{H(\|u_k\|)^{N/(p-1)+p/ q}} \right)^{\frac{1}{p^*-1}}. \]
Therefore, from Morrey’s Theorem, see (3.16), for any \( x \in B(x_k, R_k) \)
\[ |u_k(x) - u_k(x_k)| \leq C (R_k)^{1-\frac{N}{q}} \left( \frac{M_k^{-\frac{1}{r} - \frac{1}{p^* - 1} q}}{H(\|u_k\|)^{N/(p-1)+p/ q}} \right)^{\frac{1}{p^*-1}}. \]
Particularizing \( x = y_k \) in the above inequality and from (3.19) we  
obtain
\[ C (R_k)^{1-\frac{N}{q}} \left( \frac{M_k^{-\frac{1}{r} - \frac{1}{p^* - 1} q}}{H(\|u_k\|)^{N/(p-1)+p/ q}} \right)^{\frac{1}{p^*-1}} \geq \frac{1}{2} \|u_k\|, \]
which implies
\[ (R_k)^{1-\frac{N}{q}} \geq \frac{1}{2C} \left( \frac{\|u_k\|}{M_k^{-\frac{1}{r} - \frac{1}{p^* - 1} q}} \right)^{\frac{N}{N/(p-1)+p/ q}}, \]
or equivalently
\[ R_k \geq \left( \frac{1}{2C} \frac{\|u_k\|}{M_k^{-\frac{1}{r} - \frac{1}{p^* - 1} q}} \right)^{\frac{1}{1-\frac{N}{q}}}. \]
Consequently, taking into account (3.20), and that \( H \) is non-increasing
\[ \int_{B(x_k, R_k)} u_k |f(u_k)| H(u_k) \, dx \geq \frac{1}{2} \|u_k\| H(\|u_k\|) m_k \omega(R_k)^N, \]
where $\omega = \omega_N$ is the volume of the unit ball in $\mathbb{R}^N$.

Due to $B(x_k, R_k) \subset \Omega$, substituting inequality (3.26), and rearranging terms, we obtain

$$
\int_{\Omega} u_k \mid f(u_k) \mid H(u_k) \, dx \\
\geq \frac{1}{2} \| u_k \| H(\| u_k \|) m_k \omega \left( \frac{1}{2C} \left\| u_k \right\| \left[ H(\| u_k \|) \right]_{N(p-1)+p|q|} \frac{1}{\| u_k \|_{N(p-1)+p|q|}} \right)^{\frac{N}{1-k}}
$$

$$
= C m_k \left( \frac{\| u_k \|^{1+\frac{N}{q}+\frac{1}{p}}} {M_k \left[ \frac{1}{1-q} - \frac{1}{(p-1)q} + \frac{1}{p} \right]} \right)^{\frac{1}{N}}
$$

$$
= C \frac{m_k}{M_k} \left( \frac{\| u_k \|^{1+\frac{N}{q}+\frac{1}{p}}} {M_k \left[ \frac{1}{1-q} - \frac{1}{(p-1)q} + \frac{1}{p} \right]} \right)^{\frac{1}{N}}
$$

At this moment, let us observe that from hypothesis $H_5$)

$$
(3.27) \quad \frac{m_k}{M_k} \geq C, \quad \text{for all } k \quad \text{big enough.}
$$

Hence, taking again into account hypothesis $H_3$), we can assert that

$$
\int_{\Omega} u_k \mid f(u_k) \mid H(u_k) \, dx \\
\geq C \left( \frac{\| u_k \|^{1+\frac{N}{q}+\frac{1}{p}}} {M_k \left[ \frac{1}{1-q} - \frac{1}{(p-1)q} + \frac{1}{p} \right]} \right)^{\frac{1}{N}}
$$

Let us denote by

$$
a = 1 + \frac{1}{N} - \frac{1}{r}, \quad b = \frac{1}{N} - \frac{1}{r} + \frac{N}{N(p-1)+p|q|},
$$

$$
c = \left[ 1 - \frac{1}{q} - \frac{1}{(p-1)q} \right] \frac{1}{p} \frac{1}{(p-1)} - \frac{1}{N} + \frac{1}{r}.
$$

By substituting $r = \frac{Nq(p-1)}{N-q}$, see [14], we obtain

$$
a = \frac{q[N(p-1)+p] - N}{qN(p-1)}, \quad b = \frac{pq}{qN(p-1)}, \quad c = \frac{q(N-p) - \frac{N}{p^{p-1}}}{qN(p-1)}.
and it is easy to see that \( a, b, c > 0 \), and \( \frac{a}{c} = p^* - 1, \quad \frac{b}{c} = \frac{p}{N-p} \).

Finally, from hypothesis \( H_4'' \) we deduce

\[
\int_\Omega u_k |f(u_k)| H(u_k) \, dx \\
\geq C \left( \frac{\|u_k\|^{p^*-1} \left[ H \left( \|u_k\| \right) \right]^{\frac{p}{N-p}}}{f(\|u_k\|)} \right) \rightarrow \infty \quad \text{as } k \rightarrow \infty
\]

which contradicts (3.9), ending the proof.

We end this section by giving an example in the case \( 1 < p < N \) of an almost critical nonlinearity \( f \) that cannot be handled with the help of Theorem 1.3, neither with the blow-up methods that rely on the exact behavior of \( f \) at infinity as a subcritical power, neither with the methods of [27] (for \( p = 2 \)), but that fulfills the hypotheses of Theorem 1.4.

**Example 3.1.** Assume that \( \Omega \subset \mathbb{R}^N \) is a smooth bounded convex domain, \( 1 < p < N \), and \( u > 0 \) is a \( C^1(\Omega) \) solution to

\[
\begin{aligned}
-\Delta_p u &= \frac{u^{p^*-1}}{[\ln(e+u)]^{\alpha}}, & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega,
\end{aligned}
\]

with \( \alpha > p/(N-p) \).

Then, there exists a uniform constant \( C \), depending only on \( \Omega \) and \( f \) but not on the solution, such that \( \|u\|_{L^\infty(\Omega)} \leq C \).

**Proof.** We will prove that \( f(s) = s^{p^*-1}/\ln(e+s)^\alpha \) with \( \alpha > p/(N-p) \) satisfies our hypotheses for \( H(s) = 1/\ln(e+s) \). Hypotheses \( H_1-H_2 \) and \( H_3 \) hold trivially. Let us prove \( H_3'' \) and \( H_4'' \).

**\( H_3'' \)** From definition, and integrating by parts

\[
F(t) = \int_0^t \frac{s^{p^*-1}}{[\ln(e+s)]^\alpha} \, ds \\
= \frac{1}{p^*} \frac{t^{p^*}}{[\ln(e+t)]^\alpha} + \frac{\alpha}{p^*} \int_0^t \left( \frac{1}{\ln(e+s)} \right)^{\alpha+1} \frac{s^{p^*}}{e+s} \, ds
\]

(3.29)
Therefore, using also l’Hôpital rule, and simplifying we can write
\[
\lim_{t \to +\infty} \frac{p^* F(t) - tf(t)}{tf(t) H(t)} = \lim_{t \to +\infty} \frac{\alpha}{t^{p^*}} \frac{\int_0^t \left( \frac{1}{\ln(e + s)} \right)^{\alpha + 1} \frac{sp^*}{e + s} ds}{\ln(e + t)^{\alpha + 1}} = \frac{\alpha}{p^*} > 0,
\]
and so \( H''_4 \) holds.

\[ H''_4 \) From definition of \( f \) and \( H \), for any \( \alpha > \frac{p}{N - p} \),
\[
\lim_{t \to +\infty} \frac{f(t)}{t^{p - 1} \left[ H(t) \right]^{\frac{p}{N - p}}} = \lim_{t \to +\infty} \frac{\frac{1}{\ln(e + t)}^{\alpha + 1} \frac{p^*}{e + t}}{(1 + \alpha) \left( \frac{1}{\ln(e + t)} \right)^{\alpha + 2} \frac{p^*}{e + t}} = 0.
\]

\[ \square \]

4. Proof of Theorem 1.5

Here we prove Theorem 1.5 adapting to our case (and in some point simplifying) the proofs given for \( p = 2 \) in [27] and for \( 1 < p < N \) in [5, 52, 57], and referring to this papers for other applications (see e.g. [5] and [27] for the proof of the existence of a continuum of solutions to some related parameter problem).

Throughout the section we suppose that \( f \) satisfies the hypothesis \( H_0 \), as well as the hypotheses of one of the previous Theorems 1.1–1.4.

The connection between a priori estimates and existence theorems relies on some topological argument using degree theory.

The abstract result, which goes back to Krasnoselskii [37] and Amann [2], and has been adapted by De Figueiredo et al. and other authors, can be given in several formulation. We will state it as in [52] as follows.

**Theorem 4.1.** Let \( C \) a cone in the Banach space \( X \) and \( K : C \to C \) continuous and compact, with \( K(0) = 0 \). Suppose that there exist \( r > 0 \), \( R > r \) and a compact homotopy \( H : [0, 1] \times C \to C \) such that

\[ H(0, u) = K(u) \text{ for any } u \in C \text{ and }
\]

\begin{enumerate}
  \item \( u \neq sK(u) \) for any \( s \in [0, 1] \), \( u \in C \) with \( \|u\| = r \)
  \item \( H(t, u) \neq u \) for any \( t \in [0, 1] \), \( u \in C \) with \( \|u\| = R \)
  \item \( H(1, u) \neq u \) for any \( u \in C \) with \( \|u\| \leq R \)
\end{enumerate}

If \( D = \{ u \in C : r < \|u\| < R \} \) then \( K \) has a fixed point in \( D \).
In fact, for the proof only the basic properties of topological degree are needed (it is easy to see that if $i_C$ is the topological index we have that $i_C(K, B_R) = 0$ by c) and b), while $i_C(K, B_r) = 1$ by a), so that $i_C(K, D) = -1$ and $K$ has a fixed point in $D$).

Before proving Theorem 1.5, let us first make some remarks.

Let $\lambda \in [0, +\infty)$ and consider the problem

\[
\begin{cases}
-\Delta_p(u) = f(u) + \lambda & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Let us observe that it is easy to see from the proofs in Section 3 that not only $f_\lambda$ fulfills the same hypotheses of $f$, but also for any fixed $\lambda_0 > 0$ there exists a constant $C_{\lambda_0} > 0$ such that

\[
\|u_\lambda\|_{L^\infty(\Omega)} \leq C_{\lambda_0}
\]

for any solution $u_\lambda$ of (4.1) and for any $\lambda \in [0, \lambda_0]$.

Moreover the proofs in Section 3 show that under the hypotheses of one among Theorems 1.1—1.4, there exists an $\varepsilon > 0$ (depending on the geometry of $\Omega$ through (2.14)) and a constant $C > 0$ (the a priori bound in a neighborhood of the boundary obtained in the proof of those theorems, see (3.6) ) such that

\[
\|u_\lambda\|_{L^\infty(\Omega, \Omega_\varepsilon)} \leq C
\]

for any solution $u_\lambda$ of (4.1) and for any $\lambda \in [0, +\infty)$.

In other words the constants $C$ obtained in Step 1 and in (3.6) in the proof of Theorems 1.1—1.4 is independent of $\lambda \in (0, +\infty)$.

Analyzing in particular Step 1 of the proof, where the Picone’s identity is exploited, this assertion can be seen.

The same identity shows then the following property.

**Lemma 4.1.** There exists $\lambda_0 > 0$ such that the problem

\[
\begin{cases}
-\Delta_p(u) = f(u) + \lambda & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

has no solutions if $\lambda \geq \lambda_0$.

**Proof.** As in Step 1 of the proof in Section 3, we have by Picone’s identity that

\[
\int_\Omega \frac{f(u) + \lambda}{u^{p-1}} \phi_1^p \leq \lambda_1 \int_\Omega \phi_1^p = \lambda_1,
\]

and by (3.3) we have that $\frac{f(u)}{u^{p-1}}$ is bounded from below by a constant $-\Lambda$. So we get that

\[
\int_\Omega \frac{\lambda}{w^{p-1}} \phi_1^p \leq (\Lambda + \lambda_1) \int_\Omega \phi_1^p = (\Lambda + \lambda_1).
\]

As a consequence, using (4.3) we have that

\[
\frac{\lambda}{C_{\lambda_0}} \int_{\Omega_\varepsilon} \phi_1^p \leq \int_{\Omega_\varepsilon} \phi_1^p \leq \int_\Omega \phi_1^p \leq (\Lambda + \lambda_1),
\]
where $C$ is the constant in (3.3), so that we get the bound $\lambda_0$ for the existence of a solution. 

\[ \Box \]

**Proof of Theorem 1.5**

To prove Theorem 1.5 we will apply Theorem 4.1.

We consider the Banach space $X = C^0(\Omega)$, and the cone $C$ of non-negative functions.

Let us observe that in previous papers the Banach space used is the space $C^1(\Omega)$, but it is the same (and simpler) dealing with the space of continuous functions.

If $u \in X = C^0(\Omega)$, $\lambda_0 > 0$ is the number provided in Lemma 4.1, and $\Lambda$ is the number in (3.3), let us define $v := H(t, u)$ as the solution in $W^{1,p}_0(\Omega)$ of the problem

\[
\begin{cases}
-\Delta_p(v) + \Lambda v^{p-1} = f(u) + t\lambda_0 + \Lambda u^{p-1} & \text{in } \Omega \\
\quad v = 0 & \text{on } \partial\Omega,
\end{cases}
\]

and $K(u) = H(0, u)$, i.e. $v = K(u)$ solves the problem

\[
\begin{cases}
-\Delta_p(v) + \Lambda v^{p-1} = f(u) + \Lambda u^{p-1} & \text{in } \Omega \\
\quad v = 0 & \text{on } \partial\Omega,
\end{cases}
\]

By the estimates in \([25, 55, 40]\), $K : C^0(\Omega) \rightarrow C^{1,\beta}(\Omega)$ is a continuous operator, and therefore as an operator $K : X \rightarrow X$, $K$ is compact.

Likewise, $H : [0, 1] \times C \rightarrow C$ is a compact homotopy.

By \((3.3)\) we have that

$f(s) + \Lambda s^{p-1} \geq 0$ for every $s \in [0, +\infty)$,

and by the weak and strong maximum principles we deduce that if $u$ is nonnegative in $\Omega$, then $K(u)$ is nonnegative as well, and in fact positive if it does not vanish in $\Omega$. Likewise $H(t, u)$ is positive if $u$ is positive.

This implies that $K : C \rightarrow C$ and $H : [0, 1] \times C \rightarrow C$, where $C$ is the cone of nonnegative functions in $X$.

We have to verify that, for a suitable choice of $\lambda_0$, hypotheses a), b), and c) in Theorem 4.1 are satisfied.

Let us observe that if $0 \neq u = H(t, u)$, then $u$ solves the problem

\[
\begin{cases}
-\Delta_p(u) = f(u) + t\lambda_0 & \text{in } \Omega \\
\quad u > 0 & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Likewise if $0 \neq u = sK(u)$, $0 \leq s \leq 1$, then $u$ solves the problem

\[
\begin{cases}
-\Delta_p(u) = s^{p-1}f(u) - \Lambda(1 - s^{p-1})u^{p-1} & \text{in } \Omega \\
\quad u > 0 & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial\Omega.
\end{cases}
\]
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The property c) (for any choice of $R > 0$) follows from Lemma 4.1, since $u = H(1, u)$ is the equation (4.4) for $\lambda = \lambda_0$.

Property b) follows by observing that the a priori estimates that we proved are satisfied if we take $f(u) + t\lambda_0$ instead of $f(u)$, and uniformly with respect to $t \in [0, 1]$.

So, if $\|u\|_{\infty} \leq C$ and we take $R = C + 1$, there are not solutions of the equation $u = H(t, u)$, i.e. (4.7), with $\|u\| = R$.

Finally property a) follows from Poincaré’s inequality or equivalently from definition of the first eigenvalue for the $p$-Laplacian.

In fact, by hypothesis $H_0$, there exists $0 < \lambda < \lambda_1$ and $r_0 > 0$ such that $\frac{f(s)}{s^{p-1}} \leq \lambda$ for any $s \in (0, r_0]$.

Let $0 < r \leq r_0$ and suppose that $u \neq 0$ solves $u = sK(u)$, i.e. (4.8) with $0 \leq s \leq 1$ and $\|u\|_{\infty} = r$. Then, taking $u$ as a test function we obtain

$$\int_{\Omega} |Du|^p \, dx = s^{p-1} \int_{\Omega} uf(u) \, dx - \Lambda(1 - s^{p-1}) \int_{\Omega} u^p \, dx \leq s^{p-1} \int_{\Omega} uf(u) \, dx$$

$$= s^{p-1} \int_{\Omega} \frac{f(u)}{u^{p-1}} u^p \, dx \leq s^{p-1} \lambda \int_{\Omega} u^p \, dx \leq \lambda \int_{\Omega} u^p \, dx$$

which is a contradiction since $\lambda < \lambda_1$.

This implies that there are not nontrivial solutions of (4.8) with $0 \leq s \leq 1$ and $\|u\|_{\infty} = r \leq r_0$.

\[ \Box \]

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