Fixed-point Quantum Walk Search

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Abstract

In this paper, we consider the problem of searching a marked vertex in a graph based on quantum walks. A very large body of research exists on this issue, from both theoretical and experimental studies, yet the existing quantum walk-based search algorithms require knowing exactly how many marked vertices there are before starting the search process. In addition, even if the precise knowledge of the number of marked vertices is known, the success probability of these algorithms could shrink dramatically when the number of search steps is greater than the right one. For avoiding these defects, in this paper we present a new quantum walk-based search algorithm which need not know any prior information about the marked vertices, but still keeps a quadratic speedup over classical ones and ensures that the error is bounded by a tunable parameter $\epsilon$. More specifically, for an $N$-vertex complete bipartite graph with marked vertices but without knowing the number of marked vertices, if the number of search steps $h$ satisfies $h \geq \ln(\frac{2}{\sqrt{\epsilon}})\sqrt{N}$, then the algorithm will output a marked vertex with probability at least $1 - \epsilon$ for any given $\epsilon \in (0, 1]$. This feature leads to quantum search algorithms with stronger robustness and a wider scope of application.

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1 Introduction

Quantum walks, the analogues of classical random walks in quantum realm, were first introduced by Aharonov, Davidovich and Zagury [ADZ93] in 1993. In the last nearly thirty years, much progress about quantum walks has been made from theory to experiments. Quantum walks have become a basic tool for designing quantum algorithms to settle a series of problems such as element distinctness [Amb07], triangle finding [MSS07], quantum backtracking [Mon18], and so on [BS06, MN07, CSV07, JKM13, BCJ+13]. Furthermore, they are a universal model of quantum computation [Chi09, LCE+10]. In the aspect of experiment study, various hardware platforms have been used to demonstrate results of quantum walks. e.g., [ZKG+10, PLM+10, CDQ+18, GWZ+21]. There are two types of quantum walks: discrete-time quantum walks [Mey93, Mey96, Wat01, ABN+01, AAKV01] and continuous-time quantum walks [FG98, CFG02]. In this paper, we are concerned with the discrete-time model.

A central topic in quantum walk-based algorithms is to develop efficient quantum algorithms for searching a marked vertex in a graph. This idea was initially proposed by Shenvi, Kempe, and Whaley [SKW03] in 2003 who constructed a quantum walk search algorithm on the Boolean hypercube for finding a marked item in a dataset. Later, Ambainis, Kempe and Rivosh [AKR05] proposed search algorithms based on quantum walks on $d$-dimensional lattices ($d \geq 2$). A major breakthrough was made in 2004: Ambainis [Amb07] obtained the optimal query complexity of the element distinctness problem by employing quantum walk search on Johnson graphs. In 2004, Szegedy [Sze04] studied the general theory of quantum walk search algorithms from the point of view of Markov chains. In this direction, a series of work [MNRS11, KMOR16, AGJK20, AGJ21] was put forward for searching a marked state in different Markov chains using phase estimation, interpolated quantum walks and quantum fast-forwarding.

Grover’s algorithm [Gro96] can be regarded as a quantum walk search algorithm on the complete graph [AKR05]. As pointed out by Brassard [Bra97], Grover’s quantum searching technique suffers from the soufflé problem. As a result, if the exact number of marked items is not known in advance, then one does not know when to stop the search iteration. Even if the number is known, the success probability of the algorithm could shrink dramatically when the number of search steps is greater than the right one. In order to overcome the soufflé problem, Grover [Gro05] proposed a fixed-point quantum search algorithm that converges monotonically to the target, i.e., avoid overcooking by always amplifying the marked items. Yet, a price paid for this monotonicity is that the quadratic speedup of the original quantum search is lost. In 2014, Yoder, Low and Chuang [YLC14] presented a new fixed-point quantum search algorithm that achieves both goals—the search cannot be overcooked and also achieves optimal time scaling, a quadratic speedup over classical unordered search. This algorithm does not require that the error monotonically improves, but ensures that the error becomes bounded by a tunable parameter $\epsilon$. In this paper, when we mention the fixed-point property, it always means the property possessed by this algorithm.
Grover’s quantum searching technique is like cooking a soufflé. You put the state obtained by quantum parallelism in a “quantum oven” and let the desired answer rise slowly. Success is almost guaranteed if you open the oven at just the right time. But the soufflé is very likely to fall—the amplitude of the correct answer drops to zero—if you open the oven too early. Furthermore, the soufflé could burn if you overcook it: strangely, the amplitude of the desired state starts shrinking after reaching its maximum. After twice the optimal number of shakes, you are no more likely to succeed than before the first shake.

Currently, various quantum walk search algorithms also suffer from the soufflé problem. To the best of our knowledge, there has been no work considering how to avoid the soufflé problem for quantum walk search. Therefore, it is natural to ask: Does the fixed-point property exist in quantum walk search? This problem becomes more challenge than in Grover’s quantum search. There are at least three reasons for the difficulty of designing fixed-point quantum walk search algorithms. Firstly, the search space is more complicated because of the diversity of topological structure of graphs. Secondly, more operations are involved in quantum walk search. Thirdly, it is generally difficult to get an analytical expression for the success probability and to analyze the computational complexity theoretically.

1.1 Our contributions

In this paper, we try to consider how to address the soufflé problem confronted by quantum walk search. We take the first step towards this direction by designing a fixed-point quantum walk search algorithm on complete bipartite graphs. Note that this kind of graphs was extensively studied in quantum walk search algorithms [RHFB09, RW19]. We present a quantum walk search algorithm with the fixed-point property that for an $N$-vertex complete bipartite graph with marked vertices but without knowing the number of marked vertices, if the number of search steps $h$ satisfies $h \geq \ln(\frac{2}{\sqrt{\epsilon}})\sqrt{N}$, then the algorithm will output a marked vertex with probability at least $1-\epsilon$ for any given $\epsilon \in (0, 1]$ (the formal statement can be found in Theorem 1). Thus, the algorithm both avoids overcooking and keeps quadratic speedup over classical ones. Also note that our algorithm need not know the number of target vertices.

The relationship among the theorems and technical lemmas obtained in this paper are depicted in Fig. 1. Theorem 1 giving the main result of this paper comes from Theorems 2 and 3 which corresponds respectively to the two cases: the marked vertices are in one side and in two sides of a complete bipartite graph. Furthermore, Lemma 2 (Lemma 3) is crucial for proving Theorem 2 (Theorem 3). Thus, most space of this paper is devoted to the proof of Lemmas 2 and 3.
The rest of this paper is organized as follows. In Section 2, some necessary notations and models are given. Our algorithm and main results are presented in Section 3. Section 4 provides the proof for Theorem 2, and Section 5 is for the proof of Theorem 3. Section 6 concludes the paper. The missing proofs are given in Appendix.

2 Preliminaries

Graph Notation. Let $G = (V, E)$ be an undirected, unweighted graph with $N = |V|$ vertices and $m = |E|$ edges. An edge between $u$ and $v$ is denoted by $(u, v)$. For $u \in V$, $deg(u) = \{v : (u, v) \in E\}$ denotes the set of neighbors of $u$, and the degree of $u$ is denoted as $d_u = |deg(u)|$. A bipartite graph is represented as $G = (V = \{V_l \cup V_r\}, E)$, where $V_l (V_r)$ denotes the set of vertices in the left (right) side, with $V_l \cap V_r = \emptyset$. We use $N_l$ and $N_r$ to denote the number of left and right vertices, respectively. The number of the marked vertices in the left (right) side is $n_l (n_r)$. A complete bipartite graph is a bipartite graph where every vertex in the left side is connected to every vertex in the right side. For example, a complete bipartite graph in Fig. 2 contains 6 vertices in the left side and 4 vertices in the right side.

Coined Quantum Walk. In this model, the walker’s Hilbert space associated with an $N$-vertex graph $G = (V, E)$ is the following:
\[ \mathbb{H}^{N^2} = \text{span}\{ |uv\rangle, u, v \in V \}, \]

where \( u \) is the position and \( v \) is the coin value representing one neighbor of \( u \). The evolution operator of the coined quantum walk at each step is

\[ U_{\text{walk}} = SC, \]

where the flip-flop shift operator \( S \) is defined as

\[ S |uv\rangle = |vu\rangle, \]

and the coin operator is

\[ C = \sum_u |u\rangle \langle u| \otimes C_u. \]

The Grover diffusion coin operator \( C_u \) often used is

\[ C_u = 2 |s_u\rangle \langle s_u| - I, \]

where

\[ |s_u\rangle = \frac{1}{\sqrt{d_u}} \sum_{v \in \text{deg}(u)} |v\rangle. \]

Given the initial state \( |\Psi_0\rangle \), the walker’s state after \( t \) steps is \( |\Psi_t\rangle = U_{\text{walk}}^t |\Psi_0\rangle \).

**Quantum Walk Search.** In the quantum walk search framework, to find a marked vertex in a graph, the query oracle \( Q \) is given by

\[ Q |uv\rangle = \begin{cases} -|uv\rangle & \text{if } u \text{ is marked,} \\ |uv\rangle & \text{if } u \text{ is not marked.} \end{cases} \]

The evolution operator corresponding to one step of the quantum walk search is

\[ U = SCQ. \]

Given the initial state \( |\Psi_0\rangle \), the walker’s state after \( t \) steps is \( |\Psi_t\rangle = U^t |\Psi_0\rangle \). Finally, the first register is measured and the measurement result is output.

**Quasi-Chebyshev polynomial.** The Chebyshev polynomials of the first kind \( T_n(x) \) are defined by initial values \( T_0(x) = 1, T_1(x) = x, \) and for an integer \( n \geq 2, \)

\[ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x). \]

The trigonometric identity \( T_n(x) = \cos(n \arccos(x)) \) is well known.

A result of one Quasi-Chebyshev polynomial implied in \([YLC14]\) is stated in the following lemma.

**Lemma 1.** Let \( x = \cos(\theta) \) for \( \theta \in [0, 2\pi] \). Let \( h \geq 3 \) be an odd integer. One Quasi-Chebyshev polynomial \( a_k(x) \) is defined by initial values \( a_0(x) = 1, a_1(x) = x, \) and for \( k = 2, \ldots, h, \)

\[ a_k(x) = x(1 + e^{-i(\zeta_k - \zeta_{k-1})})a_{k-1}(x) - e^{-i(\zeta_k - \zeta_{k-1})}a_{k-2}(x). \]

When \( \zeta_{k+1} - \zeta_k = (-1)^k \pi - 2 \arccot \left( \tan(k\pi/h) \sqrt{1 - \gamma^2} \right) \) for \( k = 1, \ldots, h - 1, \) where \( \gamma = \frac{1}{\cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{\epsilon}}))} \) with \( \epsilon \in (0, 1], \) we have \( a_h(x) = \frac{T_{\gamma h}(x/\gamma)}{T_{h}(1/\gamma)} \) with \( T_{h}(1/\gamma) = 1/\sqrt{\epsilon}. \)
3 Fixed-point quantum walk search on complete bipartite graphs

As mentioned before, the already existing quantum walk search algorithms suffer from the soufflé problem. Thus, this section devotes to addressing this problem by considering the case of searching a marked vertex in a complete bipartite graph. For that, first the coin operator $C$ and the query oracle $Q$ have to be adjusted, but the flip-flop shift operator $S$ can remain unchanged. The new evolution operator of one search step is

$$U(\alpha, \beta) = SC(\alpha)Q(\beta),$$

where the coin operator $C$ is changed to

$$C(\alpha) = \sum_{u} |u\rangle \langle u| \otimes [(1 - e^{-i\alpha}) |s_u\rangle \langle s_u| - I],$$

and the query oracle $Q$ is replace by

$$Q(\beta) |uv\rangle = \begin{cases} e^{i\beta} |uv\rangle & \text{if } u \text{ is marked}, \\ |uv\rangle & \text{if } u \text{ is not marked}. \end{cases}$$

The algorithm of search on a complete bipartite graph is as follows. In order to satisfy the fixed-point property and maintain quadratic speedup, the key point is to find a suitable set of parameters $\alpha, \beta$.

**Algorithm 1** Fixed-point quantum walk search

**Inputs:** an $N$-vertex complete bipartite graph with marked vertices, $\epsilon \in (0, 1]$.

**Outputs:** a marked vertex $x_0$ (if $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \max(\sqrt{N_l}, \sqrt{N_r}) + 1$, it outputs a marked vertex with probability at least $1 - \epsilon$).

**Procedure:**

1. Prepare the initial state $|\Psi_0\rangle = \frac{1}{2N_lN_r} (\sum_u |u\rangle \otimes \sum_{v \in \text{deg}(u)} |v\rangle)$.
2. Apply $U(\alpha_1, \beta_1), ..., U(\alpha_h, \beta_h)$ in turn, where the parameters $\alpha_i, \beta_i$ will determined later by $\epsilon$.
3. Measure the first register in the computational basis. If the result vertex is not marked, then the second register is measured. Output the measurement result.

In the first step, the initial state $|\Psi_0\rangle = \frac{1}{2N_lN_r} (\sum_u |u\rangle \otimes \sum_{v \in \text{deg}(u)} |v\rangle)$ is prepared. Then, $U(\alpha_1, \beta_1), ..., U(\alpha_h, \beta_h)$ with appropriate parameters are applied to $|\Psi_0\rangle$ in turn. The whole operator that performs $h$ steps is

$$\Gamma_h = U(\alpha_h, \beta_h) ... U(\alpha_1, \beta_1) = SC(\alpha_h)Q(\beta_h) ... SC(\alpha_1)Q(\beta_1).$$ (1)
The walker’s state after $h$ steps is

$$|\Psi_h\rangle = \Gamma_h |\Psi_0\rangle.$$  

Finally, the two registers are measured. Note that in the previous work generally only the first register is measured, whereas here we measure the two registers. This will double the success probability for our problems as shown later. The success probability of getting a marked vertex is

$$P_h = \sum_{u \text{ or } v \text{ is marked}} |\langle uv | \Gamma_h |\Psi_0\rangle|^2.$$  

Our main results are as follows.

**Theorem 1.** Given an $N$-vertex complete bipartite graph with marked vertices but without knowing the number of marked vertices, there exists a quantum walk-based algorithm such that if the number of queries $h$ satisfies $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \max(\sqrt{N_l}, \sqrt{N_r}) + 1$, then the algorithm will output a marked vertex with probability at least $1 - \epsilon$ for any given $\epsilon \in (0, 1]$, where $N_l$ ($N_r$) is the number of the left (right) vertices in the complete bipartite graph.

**Proof.** According to the arrangement of marked vertices, two cases are discussed: they are in one side and two sides, as shown in Fig. 3(a) and Fig. 3(b), respectively.

(i) In the first case, without loss of generality, suppose that all the marked vertices lie in the left side $V_l$. Then by Theorem 2, if $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \sqrt{\frac{N_l}{n_l}} + 1$, then Algorithm 1 will output a marked vertex with probability at least $1 - \epsilon$.

(ii) In the second case, by Theorem 3, if $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \max(\sqrt{\frac{N_l}{n_l}}, \sqrt{\frac{N_r}{n_r}}) + 1$, then Algorithm 1 will output a marked vertex with probability at least $1 - \epsilon^2$.

Note that the parameters $\alpha_i, \beta_i$ will be assigned with the same values in the two cases (this can be seen from the proof of Theorems 2 and 3). Thus, we need not know the arrangement of marked vertices. In addition, in the two cases, both $\sqrt{\frac{N_l}{n_l}}$ and $\max(\sqrt{\frac{N_l}{n_l}}, \sqrt{\frac{N_r}{n_r}})$ have the same upper bound $\max(\sqrt{N_l}, \sqrt{N_r})$. Therefore, if $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \max(\sqrt{N_l}, \sqrt{N_r}) + 1$, then Algorithm 1 will output a marked vertex with probability at least $1 - \epsilon$. This completes the proof. 

**Theorem 2.** In Algorithm 1, suppose that all the marked vertices are in the left side. There exists a set of parameters $\alpha_i, \beta_i$, such that if $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \sqrt{\frac{N_l}{n_l}} + 1$, then the algorithm will output a marked vertex with probability at least $1 - \epsilon$, where $N_l$ is the number of left vertices and $n_l$ is the total number of marked vertices.

**Theorem 3.** In Algorithm 1, suppose that the marked vertices are in both the left and right sides. There exists a set of parameters $\alpha_i, \beta_i$, such that if $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \max(\sqrt{\frac{N_l}{n_l}}, \sqrt{\frac{N_r}{n_r}}) + 1$, then the algorithm will output a marked vertex with probability at least $1 - \epsilon^2$, where $N_l$ ($N_r$) is the number of left (right) vertices and $n_l$ ($n_r$) is the number of marked vertices in the left (right) side.

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In the following, the proofs of Theorems 2 and 3 are given in Sections 4 and 5, respectively. The procedures are similar.

4 Marked vertices in one side

The section is devoted to the proof of Theorem 2, of which a key step is Lemma 2 which will be proved in detail.

**Theorem 2 (restated).** In Algorithm 1, suppose that all the marked vertices are in the left side. There exists a set of parameters $\alpha_i, \beta_i$, such that if $h \geq \ln(\frac{2}{\sqrt{\epsilon}})\sqrt{\frac{N_l}{n_l}} + 1$, then the algorithm will output a marked vertex with probability at least $1 - \epsilon$, where $N_l$ is the number of left vertices and $n_l$ is the total number of marked vertices.

**Proof.** According to Lemma 2, when $h$ is odd, in order to ensure $P_h \geq 1 - \epsilon$, it suffices to satisfy

$$|\cos(\frac{1}{h} \arccos(1/\sqrt{\epsilon})))\sqrt{1 - \frac{n_l}{N_l}}| \leq 1,$$

that is,

$$\frac{n_l}{N_l} \geq 1 - \cos^{-2}(\frac{1}{h} \arccos(1/\sqrt{\epsilon}))).$$

(2)

Note that the following functions will be used:

$$\arccos(z) = \frac{1}{i} \ln(z + \sqrt{z^2 - 1}), \tan(iz) = i \tanh(z), \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$
where $\ln(\cdot)$ is the natural logarithm function and $i$ denotes the imaginary number. Now we have

\[
1 - \cos^{-2}\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\epsilon}}\right)\right) = -\tan^{2}\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\epsilon}}\right)\right)
= -\tan^{2}\left(\frac{1}{h} i \ln\left(\frac{1}{\sqrt{\epsilon}} + \sqrt{\left(\frac{1}{\sqrt{\epsilon}}\right)^2 - 1}\right)\right)
= -\tan^{2}\left(\frac{i}{h} \ln\left(\frac{1}{\sqrt{\epsilon}} + \sqrt{\left(\frac{1}{\sqrt{\epsilon}}\right)^2 - 1}\right)\right)
= \tanh^{2}\left(\frac{1}{h} \ln\left(\frac{2}{\sqrt{\epsilon}}\right)\right)
< \left(\frac{\ln(2/\sqrt{\epsilon})}{h}\right)^{2},
\]

where the last inequality follows from $x \geq \tanh(x)$ for $x \geq 0$. Thus, in order to ensure the inequality (2), it suffices to set $\frac{n}{N_l} \geq \left(\frac{\ln(2/\sqrt{\epsilon})}{h}\right)^{2}$, which leads to $h \geq \ln\left(\frac{2}{\sqrt{\epsilon}}\right) \sqrt{\frac{N_l}{n_l}}$.

Similarly, when $h$ is even, in order to ensure $P_h \geq 1 - \epsilon$, it suffices to satisfy

\[
|\cos\left(\frac{1}{h+1} \arccos\left(\frac{1}{\sqrt{\epsilon}}\right)\right)| \sqrt{1 - \frac{n_l}{N_l}} \leq 1,
\]

and

\[
|\cos\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\epsilon}}\right)\right)| \sqrt{1 - \frac{n_l}{N_l}} | \leq 1,
\]

which implies $h + 1 \geq \ln\left(\frac{2}{\sqrt{\epsilon}}\right) \sqrt{\frac{N_l}{n_l}}$ and $h - 1 \geq \ln\left(\frac{2}{\sqrt{\epsilon}}\right) \sqrt{\frac{N_l}{n_l}}$, respectively.

Thus, no matter $h$ is odd or even, $P_h \geq 1 - \epsilon$ holds for $h \geq \ln\left(\frac{2}{\sqrt{\epsilon}}\right) \sqrt{\frac{N_l}{n_l}} + 1$. This completes the proof of Theorem 2.

As shown above, Lemma 2 is a key result. The rest of this section is to prove it.

**Lemma 2.** In Algorithm 1, suppose that all the marked vertices are in the left side. There exists a set of parameters $\alpha_i, \beta_i$, such that the success probability satisfies

\[
P_h = \begin{cases} 
1 - \epsilon T_h^2(\cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{\epsilon}}))) \sqrt{1 - \frac{n_l}{N_l}} & \text{h is odd,} \\
1 - \frac{\epsilon}{2}(T_{h+1}^2(\cos(\frac{1}{h+1} \arccos(\frac{1}{\sqrt{\epsilon}}))) \sqrt{1 - \frac{n_l}{N_l}}) + T_{h-1}^2(\cos(\frac{1}{h-1} \arccos(\frac{1}{\sqrt{\epsilon}}))) \sqrt{1 - \frac{n_l}{N_l}}) & \text{h is even.}
\end{cases}
\]
Proof. As shown in Fig. 3(a), vertices can be classified into three types: the marked vertices denoted by $u$ in the left, the unmarked vertices denoted by $v$ in the left and $s$ in the right. Therefore, our analysis can be simplified in a four-dimensional subspace with the orthogonal basis \{\ket{us}, \ket{su}, \ket{sv}, \ket{vs}\} given below:

\[
\begin{align*}
\ket{us} &= \frac{1}{\sqrt{\sum_u |u|}} \sum_u |u| \otimes \frac{1}{\sqrt{\sum_s |s|}} \sum_s |s|, \\
\ket{sv} &= \frac{1}{\sqrt{\sum_s |s|}} \sum_s |s| \otimes \frac{1}{\sqrt{\sum_v |v|}} \sum_v |v|, \\
\ket{su} &= \frac{1}{\sqrt{\sum_s |s|}} \sum_s |s| \otimes \frac{1}{\sqrt{\sum_u |u|}} \sum_u |u|, \\
\ket{vs} &= \frac{1}{\sqrt{\sum_v |v|}} \sum_v |v| \otimes \frac{1}{\sqrt{\sum_s |s|}} \sum_s |s|.
\end{align*}
\]

Note that $\ket{\Psi_0}$ can be rewritten in the above basis as

\[
\ket{\Psi_0} = \frac{1}{\sqrt{2 N_l N_r}} \left[ \sqrt{n_l N_r} \ket{us} + \sqrt{n_l N_r} \ket{su} + \sqrt{N_r (N_l - n_l)} \ket{sv} + \sqrt{N_r (N_l - n_l)} \ket{vs} \right].
\]

Hence, it can be expressed as a 4-dimensional vector

\[
\ket{\Psi_0} = \frac{1}{\sqrt{2 N_l N_r}} \left( \begin{array}{c}
\sqrt{n_l N_r} \\
\sqrt{n_l N_r} \\
\sqrt{N_r (N_l - n_l)} \\
\sqrt{N_r (N_l - n_l)}
\end{array} \right).
\]

Furthermore, we have

\[
S = \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array} \right), \quad Q(\beta) = \left( \begin{array}{cccc}
e^{i\beta} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right),
\]

and

\[
C(\alpha) = \left( \begin{array}{cccc}
-e^{-i\alpha} & 0 & 0 & 0 \\
0 & \frac{1-e^{-i\alpha}(1-\cos(\omega))}{2} & \frac{(1-e^{-i\alpha})\sin(\omega)}{2} & 0 \\
0 & \frac{(1-e^{-i\alpha})\sin(\omega)}{2} & \frac{1-e^{-i\alpha}(1+\cos(\omega))}{2} & 0 \\
0 & 0 & 0 & -e^{-i\alpha}
\end{array} \right),
\]

where $\cos(\omega) = 1 - \frac{2n_l}{N_l}$, $\sin(\omega) = \frac{2}{N_l} \sqrt{n_l \ast (N_l - n_l)}$.

Let

\[
R(\theta) = -\left( \begin{array}{cccc}
e^{-\frac{i\theta}{2}} & 0 & 0 & 0 \\
0 & e^{\frac{i\theta}{2}} & 0 & 0 \\
0 & 0 & e^{-\frac{i\theta}{2}} & 0 \\
0 & 0 & 0 & e^{\frac{i\theta}{2}}
\end{array} \right),
\]

and

\[
A(\theta) = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \cos\left(\frac{\omega}{2}\right) & -ie^{i\theta} \sin\left(\frac{\omega}{2}\right) & 0 \\
0 & -ie^{-i\theta} \sin\left(\frac{\omega}{2}\right) & \cos\left(\frac{\omega}{2}\right) & 0 \\
0 & 0 & 0 & 1
\end{array} \right).
\]
One can verify the following identities:

\[ C(\alpha) = e^{-i\alpha} A(\frac{\alpha}{2}) R(\alpha) A(-\frac{\alpha}{2}), \]  
\[ Q(\beta)S = -e^{i\beta} SR(\beta), \]  
\[ A(\alpha + \beta) = R(\beta) A(\alpha) R(-\beta), \]  
\[ R(\theta)R(-\theta) = I, \]  
\[ |\Psi_0\rangle = A(\frac{\pi}{2}) S A(\frac{\pi}{2}) |\bar{0}\rangle, \]

where \(|\bar{0}\rangle\) denotes \((0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T\). An important observation is the following equation which will be useful later:

\[ SB_1 S B_2 S = B_2 S B_1, \]

where \(B_1 = \prod_{i=0}^{n} D_i\) and \(B_2 = \prod_{i=0}^{m} D_i\) for \(D_i \in \{A(\theta_i), R(\theta_i)\}\).

Below we will prove Eq. (3) by two cases in Sections 4.1 and 4.2. \(\square\)

### 4.1 Case 1: \(h\) is an odd integer

First we set

\[ \alpha_k = \begin{cases} 
-\beta_{h+2-k} & k = 2, 4, \ldots, h - 1, \\
-\beta_{h-k} & k = 3, 5, \ldots, h.
\end{cases} \]  

Then \(|\Psi_h\rangle\) reduces to

\[ |\Psi_h\rangle \sim S[A(\eta_h) \ldots A(\eta_1)]R(\alpha_1)SR(\beta_h)[A(\zeta_h) \ldots A(\zeta_1)]|\bar{0}\rangle, \]

which will be proven in Appendix A by using Eqs. (4)-(9). Here \(\eta_k = \eta_{h+1-k}\) and \(\zeta_k = \zeta_{h+1-k}\) for \(k = 1, 2, \ldots, h\), and

\[ \eta_{k+1} - \eta_k = \begin{cases} 
\pi - \alpha_{k+1} & k = 2, 4, \ldots, h - 1, \\
-\pi + \alpha_{h-k+1} & k = 3, 5, \ldots, h,
\end{cases} \]  

\[ \zeta_{k+1} - \zeta_k = \begin{cases} 
\pi - \alpha_k & k = 2, 4, \ldots, h - 1, \\
-\pi + \alpha_{h-k} & k = 3, 5, \ldots, h.
\end{cases} \]

Let us have a more detailed analysis at the state evolution in Eq. (11), which can be divided into four stages as follows.
**Stage 1:** apply $A(\zeta_h)\ldots A(\zeta_1)$ to the initial state. Let $(a_0, b_0, c_0, d_0) = (0, 0, 1, 1)$ and
\[
|\mu_k\rangle = \begin{pmatrix} a_k \\ b_k \\ c_k \\ d_k \end{pmatrix} = A(\zeta_k)\ldots A(\zeta_1) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
\]
for $k = 1, 2, \ldots, h$. First note that $A(\zeta_i)$ makes effect only to the 2th and 3th dimension of a 4-dimensional vector. Thus, $a_k = 0$ and $d_k = 1$ for all $k$. Furthermore, we have
\[
|\mu_k\rangle = A(\zeta_k) |\mu_{k-1}\rangle = \begin{pmatrix} b_{k-1} \cos(\frac{\omega}{2}) - ic_{k-1}e^{i\zeta_k} \sin(\frac{\omega}{2}) \\ -ib_{k-1}e^{-i\zeta_k} \sin(\frac{\omega}{2}) + c_{k-1} \cos(\frac{\omega}{2}) \\ 1 \end{pmatrix}, \quad (14)
\]
and
\[
|\mu_{k-2}\rangle = A(\zeta_{k-1})^{-1} |\mu_{k-1}\rangle = \begin{pmatrix} b_{k-1} \cos(\frac{\omega}{2}) + ic_{k-1}e^{i\zeta_{k-1}} \sin(\frac{\omega}{2}) \\ ib_{k-1}e^{-i\zeta_{k-1}} \sin(\frac{\omega}{2}) + c_{k-1} \cos(\frac{\omega}{2}) \\ 1 \end{pmatrix}. \quad (15)
\]
Combined with Eqs. (14) and (15), we have
\[
c_k = -ib_{k-1}e^{-i\zeta_k} \sin(\frac{\omega}{2}) + c_{k-1} \cos(\frac{\omega}{2}),
\]
\[
c_{k-2} = ib_{k-1}e^{-i\zeta_{k-1}} \sin(\frac{\omega}{2}) + c_{k-1} \cos(\frac{\omega}{2}).
\]
The recurrence formula of $c_k(x)$ is defined by $c_0(x) = 1, c_1(x) = x$, and for $k = 2, \ldots, h,
\[
c_k(x) = x(1 + e^{-i(\zeta_k - \zeta_{k-1})})c_{k-1}(x) - e^{-i(\zeta_k - \zeta_{k-1})}c_{k-2}(x),
\]
with $x = \cos(\frac{\omega}{2})$. By Lemma 1, when
\[
\zeta_{k+1} - \zeta_k = (-1)^k \pi - 2\arccot(\tan(\frac{k\pi}{h})\sqrt{1 - \gamma^2}) \quad (16)
\]
for $k = 1, \ldots, h - 1$, where $\gamma^{-1} = \cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{\epsilon}}))$, we have
\[
c_h(x) = \frac{T_h(\frac{x}{\gamma})}{T_h(\frac{1}{\gamma})}
\]
with $T_h(\frac{1}{\gamma}) = \frac{1}{\sqrt{\epsilon}}$. Moreover, $b_h(x)$ is determined by $|b_h(x)|^2 + |c_h(x)|^2 = 1$. Therefore, the state after $A(\zeta_h)\ldots A(\zeta_1)$ applied to the initial state is
\[
\frac{1}{\sqrt{2}} \begin{pmatrix} b_h(x) \\ c_h(x) \end{pmatrix}.
\]
Stage (2): apply $R(\alpha_1)SR(\beta_h)$ to the above state. After that, the state is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\beta_h}b_h(x) \\ 0 \\ 1 \\ c_h(x) \end{pmatrix}.$$

Stage (3): perform $A(\eta_h)\ldots A(\eta_1)$. Let

$$\begin{pmatrix} \bar{a}_k \\ b_k \\ \bar{c}_k \\ \bar{d}_k \end{pmatrix} = A(\eta_k)\ldots A(\eta_1) \begin{pmatrix} e^{i\beta_h}b_h(x) \\ 0 \\ 1 \\ c_h(x) \end{pmatrix},$$

for $k = 1, 2, \ldots, h$. By the property of the matrix $A(\eta_k)$, we have $\bar{a}_k = e^{i\beta_h}b_h(x)$ and $\bar{d}_k = c_h(x)$ for all $k$. The recurrence formula of $\bar{c}_k(x)$ is defined by $\bar{c}_0(x) = 1$, $\bar{c}_1(x) = x$ and for $k = 2, \ldots, h$,

$$\bar{c}_k(x) = x(1 + e^{-i(\eta_k - \eta_{k-1})})\bar{c}_{k-1}(x) - e^{-i(\eta_k - \eta_{k-1})}\bar{c}_{k-2}(x),$$

with $x = \cos(\frac{\omega}{2})$. By Lemma 1, when

$$\eta_{k+1} - \eta_k = (-1)^k \pi - 2\cot^{-1}(\tan(\frac{k\pi}{h})\sqrt{1 - \gamma^2})$$

for $k = 1, \ldots, h - 1$, where $\gamma^{-1} = \cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{\epsilon}}))$, we have

$$\bar{c}_h(x) = \frac{T_h(\frac{x}{\gamma})}{T_h(\frac{1}{\gamma})},$$

with $T_h(\frac{1}{\gamma}) = \frac{1}{\sqrt{\epsilon}}$. Moreover, $\bar{b}_h(x)$ is determined by $|\bar{b}_h(x)|^2 + |\bar{c}_h(x)|^2 = 1$. Hence, the result state is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\beta_h}b_h(x) \\ \bar{b}_h(x) \\ \bar{c}_h(x) \\ c_h(x) \end{pmatrix}.$$

Stage (4): perform the final operation $S$. The final state is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \bar{b}_h(x) \\ e^{i\beta_h}b_h(x) \\ c_h(x) \\ \bar{c}_h(x) \end{pmatrix}.$$

Therefore, the success probability $P_h$ is

$$P_h = 1 - \frac{1}{2}(|c_h(x)|^2 + |\bar{c}_h(x)|^2) = 1 - \epsilon T_h^2(\frac{x}{\gamma}) = 1 - \epsilon T_h^2(\cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{\epsilon}}))\sqrt{1 - \frac{n}{N_l}}).$$
By Eqs. (10), (12), (13), (16) and (17), $\alpha_k, \beta_k$ can be chosen such that
\[
\alpha_k = -\beta_{h+2-k} = \pi + (\eta_{k+1} - \eta_k) = 2\arccot \left(\tan\left(\frac{k\pi}{h}\right)\sqrt{1 - \gamma^2}\right)
\]
for $k = 2, 4, \ldots, h - 1$, and
\[
\alpha_k = -\beta_{h-k} = \pi - (\zeta_{k+1} - \zeta_k) = 2\arccot \left(\tan\left(\frac{(k - 1)\pi}{h}\right)\sqrt{1 - \gamma^2}\right)
\]
for $k = 3, 5, \ldots, h$. In addition, $\alpha_1$ and $\beta_h$ can be any value.

### 4.2 Case 2: $h$ is an even integer

First we set
\[
\beta_k = -\alpha_{h+1-k} \quad k = 1, 2, \ldots, h - 1.
\] (18)

Then $|\Psi_h\rangle$ reduces to

\[
|\Psi_h\rangle \sim R(\beta_h)[A(\phi_{h-1})...A(\phi_1)]R(\alpha_1)S[A(\psi_{h+1})...A(\psi_1)] |\bar{0}\rangle,
\] (19)

which will be proven in Appendix A by using Eqs. (4)-(9). Here $\phi_k = \phi_{h-k}$ for $k = 1, 2, \ldots, h - 1$, $\psi_k = \psi_{h+2-k}$ for $k = 1, 2, \ldots, h + 1$, and

\[
\phi_{k+1} - \phi_k = \begin{cases}
\pi - \alpha_{k+1} & k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k} & k = 1, 3, \ldots, h - 3,
\end{cases}
\] (20)

\[
\psi_{k+1} - \psi_k = \begin{cases}
\pi - \alpha_{k} & k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k+1} & k = 1, 3, \ldots, h.
\end{cases}
\] (21)

Similar to Eq. (11), the final state of Eq. (19) can be gotten by the following four stages:

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
A(\psi_{h+1})...A(\psi_1) \equiv 1 
\frac{1}{\sqrt{2}} \begin{pmatrix}
0 & b_{h+1}(x) \\
0 & c_{h+1}(x)
\end{pmatrix}
R(\alpha_1)S 
\frac{1}{\sqrt{2}} 
\begin{pmatrix}
b_{h+1}(x) \\
ob_{h+1}(x)
\end{pmatrix}
\]

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
A(\phi_{h-1})...A(\phi_1) \\
A(\phi_{h-1})...A(\phi_1)
\end{pmatrix}
\]

\[
\frac{1}{\sqrt{2}} 
\begin{pmatrix}
b_{h+1}(x) & b_{h+1}(x) \\
c_{h+1}(x) & c_{h+1}(x)
\end{pmatrix}
R(\beta_h) 
\frac{1}{\sqrt{2}} 
\begin{pmatrix}
0 & b_{h+1}(x) \\
1 & c_{h+1}(x)
\end{pmatrix}
\]

\[
\frac{1}{\sqrt{2}} 
\begin{pmatrix}
0 & b_{h+1}(x) \\
0 & c_{h+1}(x)
\end{pmatrix}
R(\beta_h) 
\frac{1}{\sqrt{2}} 
\begin{pmatrix}
0 & b_{h+1}(x) \\
0 & c_{h+1}(x)
\end{pmatrix}
.
\]

**Stage 1:** apply $A(\psi_{h+1})...A(\psi_1)$ to the initial state. Let $(a_0, b_0, c_0, d_0) = (0, 0, 1, 1)$ and

\[
\begin{pmatrix}
a_k \\
b_k \\
c_k \\
d_k
\end{pmatrix} = A(\psi_k)...A(\psi_1) \begin{pmatrix}
0 \\
0 \\
1 \\
1
\end{pmatrix}
\]
for $k = 1, 2, \ldots, h + 1$. By the property of matrix $A(\psi_k)$, we have $a_k = 0$ and $d_k = 1$ for all $k$. The recurrence formula of $c_k(x)$ is defined by $c_0(x) = 1, c_1(x) = x$ and for $k = 2, \ldots, h + 1, $

$$c_k(x) = x(1 + e^{-i(\psi_k-\psi_{k-1})})c_{k-1}(x) - e^{-i(\psi_k-\psi_{k-1})}c_{k-2}(x),$$

with $x = \cos\left(\frac{\omega}{2}\right)$. By Lemma 1, when

$$\psi_{k+1} - \psi_k = (-1)^k \pi - 2\arccot \left(\tan\left(\frac{k\pi}{h+1}\right)\sqrt{1 - \gamma_1^2}\right) \quad (22)$$

with $\gamma_1^{-1} = \cos\left(\frac{1}{h+1} \arccos\left(\frac{1}{\sqrt{\epsilon}}\right)\right)$ for $k = 1, 2, \ldots, h$, we have

$$c_{h+1}(x) = \frac{T_{h+1}\left(\frac{x}{\sqrt{\epsilon}}\right)}{T_{h+1}\left(\frac{1}{\gamma_1}\right)},$$

with $T_{h+1}\left(\frac{1}{\gamma_1}\right) = \frac{1}{\sqrt{\epsilon}}$. Moreover, $b_{h+1}(x)$ is determined by $|b_{h+1}(x)|^2 + |c_{h+1}(x)|^2 = 1$. Therefore, the state after $A(\psi_{h+1})...A(\psi_1)$ applied to the initial state is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ b_{h+1}(x) \\ c_{h+1}(x) \\ 1 \end{pmatrix}.$$

Stage (2): apply $R(\alpha_1)S$ to the above state. After that, the state is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} b_{h+1}(x) \\ 0 \\ 1 \\ c_{h+1}(x) \end{pmatrix}.$$

Stage (3): perform $A(\phi_{h-1})...A(\phi_1)$. Let

$$\begin{pmatrix} \bar{a}_k \\ \bar{b}_k \\ \bar{c}_k \\ \bar{d}_k \end{pmatrix} = A(\phi_k)...A(\phi_1) \begin{pmatrix} b_{h+1}(x) \\ 0 \\ 1 \\ c_{h+1}(x) \end{pmatrix}$$

for $k = 1, 2, \ldots, h - 1$. By the property of matrix $A(\phi_k)$, we have $\bar{a}_k = b_{h+1}(x)$ and $\bar{d}_k = c_{h+1}(x)$ for all $k$. The recurrence formula of $\bar{c}_k(x)$ is defined by $\bar{c}_0(x) = 1, \bar{c}_1(x) = x$ and for $k = 2, \ldots, h - 1, $

$$\bar{c}_k(x) = x(1 + e^{-i(\phi_k-\phi_{k-1})})\bar{c}_{k-1}(x) - e^{-i(\phi_k-\phi_{k-1})}\bar{c}_{k-2}(x)$$

with $x = \cos\left(\frac{\omega}{2}\right)$. By Lemma 1, when

$$\phi_{k+1} - \phi_k = (-1)^k \pi - 2\arccot \left(\tan\left(\frac{k\pi}{h+1}\right)\sqrt{1 - \gamma_2^2}\right) \quad (23)$$
with $\gamma^{-1} = \cos(\frac{1}{h-1} \arccos(\frac{1}{\sqrt{\epsilon}}))$ for $k = 1, 2, \ldots, h-2$, we have

$$c_{h-1}(x) = \frac{T_{h-1}(\frac{x}{\gamma_2})}{T_{h-1}(\frac{1}{\gamma_2})},$$

with $T_{h-1}(\frac{1}{\gamma_2}) = \frac{1}{\sqrt{\epsilon}}$. Moreover, $\bar{b}_{h-1}(x)$ is determined by $|\bar{b}_{h-1}(x)|^2 + |\bar{c}_{h-1}(x)|^2 = 1$. Hence, the result state is

$$\bar{\alpha}(\bar{b}_{h-1}(x)) = \sqrt{\frac{1}{2}} \left( \begin{array}{c} b_{h+1}(x) \\ \bar{b}_{h-1}(x) \\ \bar{c}_{h-1}(x) \\ c_{h+1}(x) \end{array} \right).$$

**Stage 4**: perform the final operation $R(\beta_h)$. The final state is

$$\frac{1}{\sqrt{2}} \left( \begin{array}{c} b_{h+1}(x) \\ e^{i\beta_h} \bar{b}_{h-1}(x) \\ \bar{c}_{h-1}(x) \\ c_{h+1}(x) \end{array} \right).$$

Therefore, the success probability $P_h$ is

$$P_h = 1 - \frac{1}{2}(|\bar{c}_{h-1}(x)|^2 + |c_{h-1}(x)|^2)
= 1 - \frac{\epsilon}{2} \left( T_{h+1}^2 \left( \frac{x}{\gamma_1} \right) + T_{h-1}^2 \left( \frac{x}{\gamma_2} \right) \right)
= 1 - \frac{\epsilon}{2} \left[ T_{h+1} \left( \cos \left( \frac{1}{h+1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \sqrt{1 - \frac{m \epsilon}{N_l}} \right) + T_{h-1} \left( \cos \left( \frac{1}{h-1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \sqrt{1 - \frac{m \epsilon}{N_l}} \right) \right].$$

By Eq. (18) and Eqs. (20)-(23), $\alpha_k, \beta_k$ can be chosen such that

$$\alpha_k = -\beta_{h+1-k} = \pi - (\phi_k - \phi_{k-1}) = 2\arccot \left( \tan \left( \frac{k\pi}{h+1} \right) \sqrt{1 - \gamma_1^2} \right)$$
for $k = 2, 4, \ldots, h-1$, and

$$\alpha_k = -\beta_{h+1-k} = \pi - (\psi_{k+1} - \psi_k) = 2\arccot \left( \tan \left( \frac{(k-1)\pi}{h-1} \right) \sqrt{1 - \gamma_2^2} \right)$$
for $k = 3, 5, \ldots, h$. In addition, $\alpha_1$ and $\beta_h$ can be any value.

## 5 Marked vertices in two sides

The section is devoted to the proof of Theorem 3, of which a key step is Lemma 3 which will be proved in detail.
Theorem 3 (restated). In Algorithm 1, suppose that the marked vertices are in both the left and right sides. There exists a set of parameters $\alpha_i, \beta_i$, such that if $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \max(\sqrt{\frac{N_l}{n_l}}, \sqrt{\frac{N_r}{n_r}}) + 1$, then the algorithm will output a marked vertex with probability at least $1 - \epsilon^2$, where $N_l$ ($N_r$) is the number of left (right) vertices and $n_l$ ($n_r$) is the number of marked vertices in the left (right) side.

Proof. Similar to the proof of Theorem 2, by Lemma 3 one can show that

- When $h$ is odd, in order to ensure $P_h \geq 1 - \epsilon^2$, it suffices to satisfy $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \sqrt{\frac{N_l}{n_l}}$ and $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \sqrt{\frac{N_r}{n_r}}$.

- When $h$ is even, in order to ensure $P_h \geq 1 - \epsilon^2$, it suffices to satisfy $h + 1 \geq \ln(\frac{2}{\sqrt{\epsilon}}) \sqrt{\frac{N_l}{n_l}}$, $h - 1 \geq \ln(\frac{2}{\sqrt{\epsilon}}) \sqrt{\frac{N_r}{n_r}}$ and $h + 1 \geq \ln(\frac{2}{\sqrt{\epsilon}}) \sqrt{\frac{N_r}{n_r}}$.

Therefore, no matter $h$ is even or odd, $P_h \geq 1 - \epsilon^2$ holds for $h \geq \ln(\frac{2}{\sqrt{\epsilon}}) \max(\sqrt{\frac{N_l}{n_l}}, \sqrt{\frac{N_r}{n_r}}) + 1$. This completes the proof of Theorem 3.

As shown above, Lemma 3 is a key result. The rest of this section is to prove it.

Lemma 3. In Algorithm 1, suppose that the marked vertices are in both the left and right sides. There exists a set of parameters $\alpha_i, \beta_i$, such that the success probability satisfies

$$P_h = 1 - \epsilon^2 T_h^2(\cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{\epsilon}})) \sqrt{1 - \frac{n_l}{N_l}}) T_h^2(\cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{\epsilon}})) \sqrt{1 - \frac{n_r}{N_r}})$$

for odd $h$, and

$$P_h = 1 - \epsilon^2 \left[ T_{h+1}^2 \left( \cos \left( \frac{1}{h+1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \sqrt{1 - \frac{n_l}{N_l}} \right) T_{h-1}^2 \left( \cos \left( \frac{1}{h-1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \sqrt{1 - \frac{n_r}{N_r}} \right) \right]$$

for even $h$.

Proof. Vertices in Fig. 3(b) can be divided into four types: the marked vertices denoted by $u$ in the left and $t$ in the right, the unmarked vertices denoted by $v$ in the left and $s$ in the
right. Hence, our analysis can be simplified in an 8-dimensional subspace defined by the following orthogonal basis \( \{|ut\}, |us\}, |tu\}, |tv\}, |vs\}, |su\}, |sv\}\):

\[
\begin{align*}
|ut\rangle &= \frac{1}{\sqrt{N_l}} \sum_u |u\rangle \otimes \frac{1}{\sqrt{N_r}} \sum_t |t\rangle, \\
|us\rangle &= \frac{1}{\sqrt{N_l}} \sum_u |u\rangle \otimes \frac{1}{N_r - n_r} \sum_s |s\rangle, \\
|tu\rangle &= \frac{1}{\sqrt{N_r}} \sum_t |t\rangle \otimes \frac{1}{\sqrt{N_l}} \sum_u |u\rangle, \\
|tv\rangle &= \frac{1}{\sqrt{N_r}} \sum_t |t\rangle \otimes \frac{1}{N_l - n_l} \sum_v |v\rangle, \\
|vt\rangle &= \frac{1}{\sqrt{N_l}} \sum_v |v\rangle \otimes \frac{1}{\sqrt{N_r}} \sum_t |t\rangle, \\
|vs\rangle &= \frac{1}{\sqrt{N_r - n_r}} \sum_v |v\rangle \otimes \frac{1}{\sqrt{N_r - n_r}} \sum_s |s\rangle, \\
|su\rangle &= \frac{1}{\sqrt{N_r - n_r}} \sum_s |s\rangle \otimes \frac{1}{\sqrt{N_l}} \sum_u |u\rangle, \\
|sv\rangle &= \frac{1}{\sqrt{N_l - n_l}} \sum_s |s\rangle \otimes \frac{1}{\sqrt{N_r - n_r}} \sum_v |v\rangle.
\end{align*}
\]

Then \(|\Psi_0\rangle\) can be rewritten in the above basis as

\[
|\Psi_0\rangle = \frac{1}{\sqrt{2N_lN_r}} \left[ \sqrt{n_lN_r} |ut\rangle + \sqrt{n_l(N_r - n_r)} |us\rangle + \sqrt{n_lN_r} |tu\rangle \\
+ \sqrt{n_r(N_l - n_l)} |tv\rangle + \sqrt{n_r(N_l - n_l)} |vt\rangle + \sqrt{(N_l - n_l)(N_r - n_r)} |vs\rangle \\
+ \sqrt{n_l(N_l - n_l)} |su\rangle + \sqrt{(N_l - n_l)(N_r - n_r)} |sv\rangle \right].
\]

Thus, \(|\Psi_0\rangle\) can be expressed as a 8-dimensional vector

\[
|\Psi_0\rangle = \frac{1}{\sqrt{2N_lN_r}} \begin{pmatrix}
\sqrt{n_lN_r} \\
\sqrt{n_l(N_r - n_r)} \\
\sqrt{n_r(N_l - n_l)} \\
\sqrt{(N_l - n_l)(N_r - n_r)} \\
\sqrt{n_lN_r} \\
\sqrt{n_r(N_l - n_l)} \\
\sqrt{(N_l - n_l)(N_r - n_r)} \\
\sqrt{n_r(N_l - n_l)}
\end{pmatrix}.
\]

Furthermore, we have

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
Q(\beta) = \begin{pmatrix}
e^{i\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{i\beta} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{i\beta} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

and

\[
C(\alpha) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \otimes \begin{pmatrix}
M_1 & M_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
M_2 & M_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_4 & M_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_5 & M_6 & 0 & 0 & 0
\end{pmatrix},
\]

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with
\[ M_1 = \frac{(1 - e^{-i\alpha})(1 - \cos(\omega_2))}{2} - 1, \quad M_2 = \frac{(1 - e^{-i\alpha})\sin(\omega_2)}{2}, \]
\[ M_3 = \frac{(1 - e^{-i\alpha})(1 + \cos(\omega_2))}{2} - 1, \quad M_4 = \frac{(1 - e^{-i\alpha})(1 - \cos(\omega_1))}{2} - 1, \]
\[ M_5 = \frac{(1 - e^{-i\alpha})\sin(\omega_1)}{2}, \quad M_6 = \frac{(1 - e^{-i\alpha})(1 + \cos(\omega_1))}{2} - 1, \]
where \(\cos(\omega_1) = 1 - \frac{2n_l}{N_l}, \quad \sin(\omega_1) = \frac{2N_l}{\sqrt{n_l(N_l - n_l)}}, \quad \cos(\omega_2) = 1 - \frac{2n_r}{N_r}, \quad \text{and} \quad \sin(\omega_2) = \frac{2N_r}{\sqrt{n_r(N_r - n_r)}}.

Let
\[
R(\theta) = -\begin{pmatrix}
e^{i\theta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{-i\theta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{i\theta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{-i\theta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{i\theta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{-i\theta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{i\theta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{-i\theta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]
and
\[
A(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \cos(\omega_1/2) & 0 & 0 \\ -ie^{i\theta\sin(\omega_1/2)} & 0 & 0 \\ 0 & 0 & \cos(\omega_1/2) \end{pmatrix},
\]
We have
\[
C(\alpha) = e^{-\frac{i\alpha}{2}} A\left(\frac{\pi}{2}\right) R(\alpha) A\left(-\frac{\pi}{2}\right),
\]
\[
Q(\beta)S = -e^{i\frac{\beta}{2}} SR(\beta),
\]
\[
A(\alpha + \beta) = R(\beta) A(\alpha) R(-\beta),
\]
\[
R(\theta) R(-\theta) = I,
\]
\[
|\Psi_0\rangle = A\left(\frac{\pi}{2}\right) S A\left(\frac{\pi}{2}\right) \overline{|0\rangle},
\]
where \(|\overline{0}\rangle\) denotes \((0, 0, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^T\). Similarly to Eq. (9), the following equation holds.
\[
SB_1 SB_2 S = B_2 B_1,
\]
where \(B_1 = \prod_{i=0}^n D_i\) and \(B_2 = \prod_{i=0}^m D_i\) for \(D_i \in \{A(\theta_i), R(\theta_i)\}\).

Below we will prove Eqs. (24) and (25) in Sections 5.1 and 5.2, respectively.

\[\square\]

### 5.1 Case 1: \(h\) is an odd integer

First we set
\[
\alpha_k = \begin{cases} 
-\beta_{h+2-k} & k = 2, 4, \ldots, h - 1, \\
-\beta_{h-k} & k = 3, 5, \ldots, h.
\end{cases}
\]
Then $|\Psi_h\rangle$ reduces to

$$|\Psi_h\rangle \sim S[A(\eta_h)\ldots A(\eta_1)] R(\alpha_1)SR(\beta_h) [A(\zeta_h) \ldots A(\zeta_1)] |0\rangle,$$  \hspace{1cm} (33)

which will be proven in Appendix B by using Eqs. (26)-(31). Here $\eta_k = \eta_{h+1-k}$, $\zeta_k = \zeta_{h+1-k}$ for $k = 1, 2, \ldots, h$, and

$$\eta_{k+1} - \eta_k = \begin{cases} \pi - \alpha_{k+1} & k = 2, 4, \ldots, h - 1, \\ -\pi + \alpha_{k+1} & k = 3, 5, \ldots, h, \end{cases}$$  \hspace{1cm} (34)

$$\zeta_{k+1} - \zeta_k = \begin{cases} \pi - \alpha_k & k = 2, 4, \ldots, h - 1, \\ -\pi + \alpha_{h-k} & k = 3, 5, \ldots, h, \end{cases}$$  \hspace{1cm} (35)

The final state of Eq. (33) can be gotten by the following four stages:

1. $\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} A(\zeta_h) \ldots A(\zeta_1) \rightarrow \frac{1}{\sqrt{2}} R(\alpha_1)SR(\beta_h) \rightarrow \frac{1}{\sqrt{2}} S R(\beta_h) A(\eta_h) \ldots A(\eta_1) \\
2. \begin{pmatrix} e^{i\beta_h} g_h(x_1) \\ e^{i\beta_h} g_h(x_2) \\ e^{i\beta_h} h(x_1) \\ e^{i\beta_h} h(x_2) \\ e^{i\beta_h} e_h(x_1) \\ e^{i\beta_h} e_h(x_2) \\ g_h(x_1) e_h(x_2) \\ g_h(x_2) e_h(x_1) \\ f_h(x_1) g_h(x_2) \\ f_h(x_2) g_h(x_1) \end{pmatrix} \\
3. \begin{pmatrix} e^{i\beta_h} h(x_1) e_h(x_2) \\ e^{i\beta_h} h(x_2) e_h(x_1) \\ g_h(x_1) f_h(x_2) \\ g_h(x_2) f_h(x_1) \\ f_h(x_1) g_h(x_2) \\ f_h(x_2) g_h(x_1) \end{pmatrix} \\
4. \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Stage (1): apply $A(\zeta_h) \ldots A(\zeta_1)$ to the initial state. Let

$$(a_0, b_0, c_0, d_0, e_0, f_0, g_0, l_0) = (0, 0, 0, 0, 1, 0, 1)$$

and

$$\begin{pmatrix} a_k \\ b_k \\ c_k \\ d_k \\ e_k \\ f_k \\ g_k \\ l_k \end{pmatrix} = A(\zeta_k) \ldots A(\zeta_1) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

for $k = 1, 2, \ldots, h$. The elements of a 8-dimensional vector are divided into four groups: (1th, 2th), (3th, 4th), (5th, 6th), and (7th, 8th). Note that each block of matrix $A(\zeta_i)$ acts on a corresponding group of a 8-dimensional vector. Thus, $a_k = b_k = c_k = d_k = 0$ for all $k$. The recurrence formula of $l_k(x_1)$ is defined by $l_0(x_1) = 1, l_1(x_1) = x_1$ and for $k = 2, \ldots, h$

$$l_k(x_1) = x_1(1 + e^{-i(\zeta_k - \zeta_{k-1})})l_{k-1}(x_1) - e^{-i(\zeta_k - \zeta_{k-1})}l_{k-2}(x_1)$$
with \( x_1 = \cos\left(\frac{\omega_1}{2}\right) \). The recurrence formula of \( f_k(x_2) \) is defined by \( f_0(x_2) = 1, f_1(x_2) = x_2 \) and for \( k = 2, \ldots, h \)

\[
f_k(x_2) = x_2(1 + e^{-i(\delta_k - \zeta_k - 1)})f_{k-1}(x_2) - e^{-i(\delta_k - \zeta_k - 1)}f_{k-2}(x_2)
\]

with \( x_2 = \cos\left(\frac{\omega_2}{2}\right) \). By Lemma 1, when

\[
\zeta_{k+1} - \zeta_k = (-1)^k \pi - 2\arccot \left(\tan\left(\frac{k\pi}{h}\right)\sqrt{1 - \gamma^2}\right),
\]

with \( \gamma^{-1} = \cos\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\gamma}}\right)\right) \) for \( k = 1, 2, \ldots, h - 1 \), we have

\[
l_h(x_1) = \frac{T_h(x_1)}{T_h\left(\frac{\gamma}{h}\right)}, \quad f_h(x_2) = \frac{T_h(x_2)}{T_h\left(\frac{\gamma}{h}\right)},
\]

with \( T_h(\frac{1}{h}) = \frac{1}{\sqrt{\gamma}} \). Moreover, \( e_h(x_2) \) and \( g_h(x_1) \) are determined by \(|e_h(x_2)|^2 + |f_h(x_2)|^2 = 1\) and \(|g_h(x_1)|^2 + |l_h(x_1)|^2 = 1\), respectively. Therefore, the state after \( A(\zeta_h) \ldots A(\zeta_1) \) applied to the initial state is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & e_{h}(x_2) & g_{h}(x_1) & l_h(x_1) \\ 0 & 0 & 0 & e_{h}(x_2) & g_{h}(x_1) & l_h(x_1) \end{pmatrix}
\]

Stage (2): apply \( R(\alpha_1)SR(\beta_h) \) to the above state. After that, the state is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\delta_h g_h(x_1)} & 0 & 0 & 0 & e^{i\delta_h g_h(x_1)} & 0 \\ 0 & e^{i\delta_h e_h(x_2)} & 0 & 0 & e^{i\delta_h e_h(x_2)} & 0 \\ 0 & 0 & g_h(x_1) & 0 & g_h(x_1) & 0 \\ 0 & 0 & l_h(x_1) & 0 & l_h(x_1) & 0 \end{pmatrix}
\]

Stage (3): perform \( A(\eta_h) \ldots A(\eta_1) \). Let \((\bar{a}_0, \bar{b}_0, \bar{c}_0, \bar{d}_0, \bar{e}_0, \bar{f}_0, \bar{g}_0, \bar{l}_0) = (0, 1, 0, 1, 0, 1, 0, 1)\) and

\[
\begin{pmatrix} e^{i\delta_h g_h(x_1)} & 0 & 0 & 0 & e^{i\delta_h g_h(x_1)} & 0 \\ 0 & e^{i\delta_h e_h(x_2)} & 0 & 0 & e^{i\delta_h e_h(x_2)} & 0 \\ 0 & 0 & g_h(x_1) & 0 & g_h(x_1) & 0 \\ 0 & 0 & l_h(x_1) & 0 & l_h(x_1) & 0 \end{pmatrix}
\]

for \( k = 1, \ldots, h \). The recurrence formula of \( \bar{d}_k(x_1) \) is defined by \( \bar{d}_0(x_1) = 1, \bar{d}_1(x_1) = x_1 \) and for \( k = 2, \ldots, h \)

\[
\bar{d}_k(x_1) = x_1(1 + e^{-i(\eta_k - \eta_k - 1)})\bar{d}_{k-1}(x_1) - e^{-i(\eta_k - \eta_k - 1)}\bar{d}_{k-2}(x_1),
\]

with \( x_1 = \cos\left(\frac{\omega_1}{2}\right) \), and the recurrence formula of \( \bar{l}_k(x_1) \) is defined by \( \bar{l}_0(x_1) = 1, \bar{l}_1(x_1) = x_1 \) and for \( k = 2, \ldots, h \)

\[
\bar{l}_k(x_1) = x_1(1 + e^{-i(\eta_k - \eta_k - 1)})\bar{l}_{k-1}(x_1) - e^{-i(\eta_k - \eta_k - 1)}\bar{l}_{k-2}(x_1).
\]
The recurrence formula of $\bar{b}_k(x_2)$ is defined by $\bar{b}_0(x_2) = 1, \bar{b}_1(x_2) = x_2$ and for $k = 2, \ldots, h$

$$
\bar{b}_k(x_2) = x_2(1 + e^{-i(\eta_k - \eta_{k-1})})\bar{b}_{k-1}(x_2) - e^{-i(\eta_k - \eta_{k-1})}\bar{b}_{k-2}(x_2)
$$

with $x_2 = \cos(\frac{\alpha_2}{\gamma})$, and the recurrence formula of $\bar{f}_k(x_2)$ is defined by $\bar{f}_0(x_2) = 1, \bar{f}_1(x_2) = x_2$

and for $k = 2, \ldots, h$

$$
\bar{f}_k(x_2) = x_2(1 + e^{-i(\eta_k - \eta_{k-1})})\bar{f}_{k-1}(x_2) - e^{-i(\eta_k - \eta_{k-1})}\bar{f}_{k-2}(x_2).
$$

By Lemma 1, when

$$
\eta_{k+1} - \eta_k = (-1)^k \pi - 2\arccot(\tan(\frac{k\pi}{h})\sqrt{1 - \gamma^2}), \tag{37}
$$

with $\gamma^{-1} = \cos(\frac{1}{k} \arccos(\frac{1}{\sqrt{\epsilon}}))$ for $k = 1, 2, \ldots, h - 1$, we have

$$
\bar{d}_h(x_1) = \bar{d}_h(x_1) = \frac{T_h(\frac{\pi}{\gamma})}{T_h(\frac{\pi}{\gamma})}, \quad \bar{b}_h(x_2) = \bar{f}_h(x_2) = \frac{T_h(\frac{\pi}{\gamma})}{T_h(\frac{\pi}{\gamma})},
$$

with $T_h(\frac{\pi}{\gamma}) = \frac{1}{\sqrt{\epsilon}}$. Moreover, $\bar{a}_h(x_2), \bar{c}_h(x_1), \bar{e}_h(x_2)$, and $\bar{g}_h(x_1)$ are determined by $|\bar{a}_h(x_2)|^2 + |\bar{b}_h(x_2)|^2 = 1, |\bar{c}_h(x_1)|^2 + |\bar{e}_h(x_2)|^2 = 1$, and $|\bar{g}_h(x_1)|^2 + |\bar{b}_h(x_2)|^2 = 1$, respectively. Hence, the result state is

$$
\frac{1}{\sqrt{2}} \begin{pmatrix}
\bar{a}_h(x_2) \\
\bar{e}_{h}(x_2) \\
\bar{e}_{h}(x_2) \\
\bar{g}_h(x_1) \\
\bar{f}_h(x_2) \\
\bar{f}_h(x_2)
\end{pmatrix}.
$$

Stage (4): perform the final operation $S$. The final state is

$$
\frac{1}{\sqrt{2}} \begin{pmatrix}
\bar{a}_h(x_2) \\
\bar{g}_h(x_1) \\
\bar{f}_h(x_2) \\
\bar{c}_{h}(x_1) \\
\bar{f}_h(x_2) \\
\bar{f}_h(x_2)
\end{pmatrix}.
$$

Hence, the success probability $P_h$ is

$$
P_h = 1 - \frac{1}{2}|f_h(x_2)\bar{l}_h(x_1)|^2 - \frac{1}{2}|g_h(x_1)\bar{f}_h(x_2)|^2 = 1 - \epsilon^2 T_h^2(\frac{x_1}{\gamma})T_h^2(\frac{x_2}{\gamma})
$$

$$
= 1 - \epsilon^2 T_h^2(\cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{\epsilon}}))) T_h^2(\cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{\epsilon}})) 1 - \frac{n_t}{N_t}).
$$

By Eq. (32) and Eqs. (34)-(37), $\alpha_k, \beta_k$ can be chosen such that

$$
\alpha_k = -\beta_{h+2-k} = \pi + (\eta_{k+1} - \eta_k) = 2\arccot(\tan(\frac{k\pi}{h})\sqrt{1 - \gamma^2}).
$$
for \( k = 2, 4, \ldots, h - 1 \), and

\[
\alpha_k = -\beta_{h-k} = \pi - (\zeta_{k+1} - \zeta_k) = 2\arccot \left( \tan \frac{(k-1)\pi}{h} \right) \sqrt{1 - \gamma^2}
\]

for \( k = 3, 5, \ldots, h \). In addition, \( \alpha_1 \) and \( \beta_h \) can be any value.

### 5.2 Case 2: \( h \) is an even integer

First we set

\[
\beta_k = -\alpha_{h+1-k} \quad k = 1, 2, \ldots, h - 1.
\]  

(38)

Then \( |\Psi_h\rangle \) reduces to

\[
|\Psi_h\rangle \sim R(\beta_h) \left[ A(\phi_{h-1}) \ldots A(\phi_1) \right] R(\alpha_1) S \left[ A(\psi_{h+1}) \ldots A(\psi_1) \right] |0\rangle,
\]

which will be proven in Appendix B by using Eqs. (26)-(31). Here \( \phi_k = \phi_{h-k} \) for \( k = 1, 2, \ldots, h - 1 \), \( \psi_k = \psi_{h+2-k} \) for \( k = 1, 2, \ldots, h + 1 \), and

\[
\phi_{k+1} - \phi_k = \begin{cases} 
\pi - \alpha_{k+1} & k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k} & k = 1, 3, \ldots, h - 3,
\end{cases}
\]

(40)

\[
\psi_{k+1} - \psi_k = \begin{cases} 
\pi - \alpha_k & k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k+1} & k = 1, 3, \ldots, h.
\end{cases}
\]

(41)

The final state of Eq. (39) is given by the four stages as follows:

Stage (1): apply \( A(\psi_{h+1}) \ldots A(\psi_1) \) to the initial state. Let

\[
(a_0, b_0, c_0, d_0, e_0, f_0, g_0, l_0) = (0, 0, 0, 0, 1, 0, 1)
\]
and
\[
\begin{pmatrix}
    a_k \\
    b_k \\
    c_k \\
    d_k \\
    e_k \\
    f_k \\
    g_k \\
    l_k
\end{pmatrix} = A(\psi_k) \cdots A(\psi_1) \begin{pmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    1 \\
    0 \\
    1
\end{pmatrix}
\]
for \( k = 1, 2, \ldots, h \). By the property of the matrix \( A(\zeta) \), we have \( a_k = b_k = c_k = d_k = 0 \) for all \( k \). And \( e_k, f_k, g_k, l_k \) can be gotten in the following. The recurrence formula of \( l_k(x_1) \) is defined by \( l_0(x_1) = 1, l_1(x_1) = x_1 \) and for \( k = 2, \ldots, h + 1 \)
\[
l_k(x_1) = x_1(1 + e^{-i(\psi_k - \psi_{k-1})})l_{k-1}(x_1) - e^{-i(\psi_k - \psi_{k-1})}l_{k-2}(x_1)
\]
with \( x_1 = \cos(\frac{\psi_1}{2}) \), and the recurrence formula of \( f_k(x_2) \) is defined by \( f_0(x_2) = 1, f_1(x_2) = x_2 \) and for \( k = 2, \ldots, h + 1 \)
\[
f_k(x_2) = x_2(1 + e^{-i(\psi_k - \psi_{k-1})})f_{k-1}(x_2) - e^{-i(\psi_k - \psi_{k-1})}f_{k-2}(x_2)
\]
with \( x_2 = \cos(\frac{\psi_2}{2}) \). By Lemma 1, when

\[
\psi_{k+1} - \psi_k = (-1)^k \pi - 2 \arccot \left( \tan \left( \frac{k \pi}{h + 1} \right) \sqrt{1 - \gamma_1^2} \right)
\]
with \( \gamma_1^{-1} = \cos(\frac{1}{h+1} \arccos(\frac{1}{\sqrt{h}})) \) for \( k = 1, 2, \ldots, h \), we have
\[
l_{h+1}(x_1) = \frac{T_{h+1}(\frac{\pi}{\gamma_1})}{T_{h+1}(\frac{1}{\gamma_1})}, \quad f_{h+1}(x_2) = \frac{T_{h+1}(\frac{\pi}{\gamma_1})}{T_{h+1}(\frac{1}{\gamma_1})},
\]
with \( T_{h+1}(\frac{1}{\gamma_1}) = \frac{1}{\sqrt{h}} \). Moreover, \( e_{h+1}(x_2) \) and \( g_{h+1}(x_1) \) are determined by \( |e_{h+1}(x_2)|^2 + |f_{h+1}(x_2)|^2 = 1 \) and \( |g_{h+1}(x_1)|^2 + |l_{h+1}(x_1)|^2 = 1 \), respectively. Therefore, the state after \( A(\psi_h) \cdots A(\psi_1) \) applied to the initial state is
\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
    0 \\
    0 \\
    e_{h+1}(x_2) \\
    f_{h+1}(x_2) \\
    g_{h+1}(x_1) \\
    l_{h+1}(x_1)
\end{pmatrix}.
\]

Stage (2): apply \( R(\alpha_1) S \) to the above state. After that, the state is
\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
    0 \\
    e_{h+1}(x_2) \\
    l_{h+1}(x_1) \\
    f_{h+1}(x_2)
\end{pmatrix}.
\]

Stage (3): perform \( A(\phi_{h-1}) \cdots A(\phi_1) \). Let \( (a_0, \bar{b}_0, \bar{c}_0, \bar{d}_0, \bar{e}_0, \bar{f}_0, \bar{g}_0, \bar{l}_0) = (0, 1, 0, 1, 0, 1, 0, 1) \) and
\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
    g_{h+1}(x_1) \\
    g_{h+1}(x_1) \\
    e_{h+1}(x_2) \\
    e_{h+1}(x_2) \\
    l_{h+1}(x_1) \\
    l_{h+1}(x_1) \\
    f_{h+1}(x_2) \\
    f_{h+1}(x_2)
\end{pmatrix} = A(\phi_k) \cdots A(\phi_1) \frac{1}{\sqrt{2}} \begin{pmatrix}
    0 \\
    g_{h+1}(x_1) \\
    e_{h+1}(x_2) \\
    l_{h+1}(x_1)
\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}
    0 \\
    0 \\
    0 \\
    f_{h+1}(x_2)
\end{pmatrix}.
\]
for \( k = 1, \ldots, h - 1 \). The recurrence formula of \( d_k(x_1) \) is defined by \( d_0(x_1) = 1, d_1(x_1) = x_1 \) and for \( k = 2, \ldots, h - 1 \)

\[
\bar{d}_k(x_1) = x_1(1 + e^{-i(\phi_k-\phi_{k-1})}) \bar{d}_{k-1}(x_1) - e^{-i(\phi_k-\phi_{k-1})} \bar{d}_{k-2}(x_1)
\]

with \( x_1 = \cos(\frac{\pi}{2}) \), and the recurrence formula of \( \bar{l}_k(x_1) \) is defined by \( \bar{l}_0(x_1) = 1, \bar{l}_1(x_1) = x_1 \) and for \( k = 2, \ldots, h - 1 \)

\[
\bar{l}_k(x_1) = x_1(1 + e^{-i(\phi_k-\phi_{k-1})}) \bar{l}_{k-1}(x_1) - e^{-i(\phi_k-\phi_{k-1})} \bar{l}_{k-2}(x_1).
\]

The recurrence formula of \( \bar{b}_k(x_2) \) is defined by \( \bar{b}_0(x_2) = 1, \bar{b}_1(x_2) = x_2 \) and for \( k = 2, \ldots, h - 1 \)

\[
\bar{b}_k(x_2) = x_2(1 + e^{-i(\phi_k-\phi_{k-1})}) \bar{b}_{k-1}(x_2) - e^{-i(\phi_k-\phi_{k-1})} \bar{b}_{k-2}(x_2)
\]

with \( x_2 = \cos(\frac{\pi}{2}) \), and the recurrence formula of \( \bar{f}_k(x_2) \) is defined by \( \bar{f}_0(x_2) = 1, \bar{f}_1(x_2) = x_2 \) and for \( k = 2, \ldots, h - 1 \)

\[
\bar{f}_k(x_2) = x_2(1 + e^{-i(\phi_k-\phi_{k-1})}) \bar{f}_{k-1}(x_2) - e^{-i(\phi_k-\phi_{k-1})} \bar{f}_{k-2}(x_2).
\]

By Lemma 1, when

\[
\phi_{k+1} - \phi_k = (-1)^k \pi - 2 \arccot \left( \tan \left( \frac{k \pi}{h-1} \right) \sqrt{1 - \gamma_2^2} \right)
\]

with \( \gamma_2^{-1} = \cos \left( \frac{1}{h-1} \arccos \left( \frac{1}{\sqrt{\bar{e}}} \right) \right) \) for \( k = 1, 2, \ldots, h - 2 \), we have

\[
\bar{d}_{h-1}(x_1) = \bar{l}_{h-1}(x_1) = \frac{T_{h-1}(\frac{\pi}{h-1})}{T_{h-1}(\frac{\pi}{h-2})}, \quad \bar{b}_{h-1}(x_2) = \bar{f}_{h-1}(x_2) = \frac{T_{h-1}(\frac{\pi}{h-2})}{T_{h-1}(\frac{\pi}{h-2})},
\]

with \( T_{h-1}(\frac{\pi}{h-2}) = \frac{1}{\sqrt{\bar{e}}} \). Moreover, \( \bar{a}_h(x_2), \bar{e}_h(x_1), \bar{e}_h(x_2), \) and \( \bar{g}_h(x_1) \) are determined by \( |\bar{a}_h(x_2)|^2 + |\bar{b}_h(x_2)|^2 = 1, |\bar{e}_h(x_1)|^2 + |\bar{d}_h(x_1)|^2 = 1, |\bar{e}_h(x_2)|^2 + |\bar{f}_h(x_2)|^2 = 1, \) and \( |\bar{g}_h(x_1)|^2 + |\bar{l}_h(x_1)|^2 = 1 \), respectively. Hence, the result state is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} g_{h+1}(x_1) \bar{a}_{h-1}(x_2) \\ g_{h+1}(x_1) \bar{b}_{h-1}(x_2) \\ e_{h+1}(x_2) \bar{e}_{h-1}(x_1) \\ e_{h+1}(x_2) \bar{d}_{h-1}(x_1) \\ l_{h+1}(x_1) \bar{e}_{h-1}(x_2) \\ l_{h+1}(x_1) \bar{f}_{h-1}(x_2) \\ f_{h+1}(x_2) \bar{g}_{h-1}(x_1) \\ f_{h+1}(x_2) \bar{l}_{h-1}(x_1) \end{pmatrix}
\]

Stage 4: perform the final operation \( R(\bar{\beta}_h) \). The final state is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\bar{\beta}_h} g_{h+1}(x_1) \bar{a}_{h-1}(x_2) \\ g_{h+1}(x_1) \bar{b}_{h-1}(x_2) \\ e^{i\bar{\beta}_h} e_{h+1}(x_2) \bar{e}_{h-1}(x_1) \\ e_{h+1}(x_2) \bar{d}_{h-1}(x_1) \\ e^{i\bar{\beta}_h} l_{h+1}(x_1) \bar{e}_{h-1}(x_2) \\ l_{h+1}(x_1) \bar{f}_{h-1}(x_2) \\ e^{i\bar{\beta}_h} f_{h+1}(x_2) \bar{g}_{h-1}(x_1) \\ f_{h+1}(x_2) \bar{l}_{h-1}(x_1) \end{pmatrix}
\]
Therefore, the success probability $P_h$ is

\[
P_h = 1 - \frac{1}{2} |c_{h+1}(x_1)d_{h-1}(x_2)|^2 - \frac{1}{2} |c_{h+1}(x_2)d_{h-1}(x_1)|^2
\]

\[
= 1 - \frac{\epsilon^2}{2} \left( T_{h+1}^2 \left( \frac{x_1}{\gamma_h} \right) T_{h-1}^2 \left( \frac{x_2}{\gamma_2} \right) + T_{h+1}^2 \left( \frac{x_2}{\gamma_1} \right) T_{h-1}^2 \left( \frac{x_1}{\gamma_2} \right) \right)
\]

\[
= 1 - \frac{\epsilon^2}{2} \left[ T_{h+1}^2 \left( \cos \left( \frac{1}{h+1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \right) \sqrt{1 - \frac{n_r}{N_r}} \right] *
\]

\[
T_{h+1}^2 \left( \cos \left( \frac{1}{h+1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \right) \sqrt{1 - \frac{n_r}{N_r}} +
\]

\[
T_{h+1}^2 \left( \cos \left( \frac{1}{h+1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \right) \sqrt{1 - \frac{n_r}{N_r}} *
\]

\[
T_{h-1}^2 \left( \cos \left( \frac{1}{h-1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \right) \sqrt{1 - \frac{n_l}{N_l}}
\]

By Eq. (38) and Eqs. (40)-(43), $\alpha_k, \beta_k$ can be chosen such that

\[
\alpha_k = -\beta_{h+1-k} = \pi - (\phi_k - \phi_{k-1}) = 2\arccot \left( \tan \left( \frac{k\pi}{h+1} \right) \sqrt{1 - \gamma_r^2} \right)
\]

for $k = 2, 4, \ldots, h - 1$, and

\[
\alpha_k = -\beta_{h+1-k} = \pi - (\psi_{k+1} - \psi_k) = 2\arccot \left( \tan \left( \frac{(k-1)\pi}{h-1} \right) \sqrt{1 - \gamma_l^2} \right)
\]

for $k = 3, 5, \ldots, h$. In addition, $\alpha_1$ and $\beta_h$ can be any value.

6 Conclusion

In this paper we investigated how to address the soufflé problem of quantum walk search. We presented for the first time a fixed-point quantum walk-based algorithm for searching a marked vertex in a complete bipartite graph. Our algorithm need not know any prior information about the marked vertices (e.g., the number of marked vertices), but keeps a quadratic speedup over classical search algorithms and ensures that the error is bounded by a tunable parameter $\epsilon$.

In this paper we have just taken the first step towards fixed-point quantum walk search. More questions are worthy of further consideration. For example, several interesting questions are listed below:

- Can the fixed-point property be introduced into quantum walk search on other graphs?
- For the framework of searching a marked state in Markov chains, can we propose a fixed-point version?
- Can continuous-time quantum walk search algorithms admit the fixed-point property?
• Another interesting direction would be to explore some important problems in practical scenarios by fixed-point quantum walk search algorithms.

We will try to answer these questions in the further work.

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A Proof of Equations (11) and (19)

In this appendix, we give the detailed proof of Equations (11) and (19).

Recall that

$$|\Psi_h\rangle = U(\alpha_h, \beta_h) \ldots U(\alpha_1, \beta_1) |\Psi_0\rangle = SC(\alpha_h)Q(\beta_1) \ldots SC(\alpha_1)Q(\beta_1) |\Psi_0\rangle .$$

Using $SS = I$, we have

$$|\Psi_h\rangle = SC(\alpha_h)Q(\beta_h) \ldots SC(\alpha_1)Q(\beta_1)SS |\Psi_0\rangle .$$

Using Eq. (4) and Eq. (5), we have

$$|\Psi_h\rangle \sim SA(\frac{\pi}{2})R(\alpha_h)A(-\frac{\pi}{2})SR(\beta_1)A(-\frac{\pi}{2})SR(\beta_h-1)\ldots$$

$$A(-\frac{\pi}{2})A(\alpha_2)A(-\frac{\pi}{2})SR(\beta_2)A(-\frac{\pi}{2})R(\alpha_1)A(-\frac{\pi}{2})SR(\beta_1)S |\Psi_0\rangle .$$

The number of $S$ is $h + 2$. Two cases need to be considered.

**Case 1: $h$ is odd.** By Eq. (9), we initially chose the middle $S$ and the left and right adjacent $S$ to eliminate two $S$. Then, the remaining $S$ are eliminated in turn using Eq. (9).

Hence,

$$|\Psi_h\rangle \sim SA(\frac{\pi}{2})R(\alpha_h)A(-\frac{\pi}{2})...SR(\beta_{\frac{h+1}{2}})A(\frac{\pi}{2})R(\alpha_{\frac{h+1}{2}})A(-\frac{\pi}{2})SR(\beta_{\frac{h+1}{2}})$$

$$A(\frac{\pi}{2})R(\alpha_{\frac{h+1}{2}})A(-\frac{\pi}{2})SR(\beta_{\frac{h+1}{2}})A(\frac{\pi}{2})R(\alpha_{\frac{h-1}{2}})A(-\frac{\pi}{2})SR(\beta_{\frac{h-1}{2}})$$

$$A(-\frac{\pi}{2})SR(\beta_2)A(\frac{\pi}{2})R(\alpha_2)A(-\frac{\pi}{2})SR(\beta_1)S |\Psi_0\rangle$$

Using Eq. (8) and $A(-\frac{\pi}{2})A(\frac{\pi}{2}) = I$, we have

$$|\Psi_h\rangle \sim SA(\frac{\pi}{2})R(\alpha_h)A(-\frac{\pi}{2})R(\beta_h-1)A(\frac{\pi}{2})R(\alpha_1)A(-\frac{\pi}{2})$$

$$SR(\beta_h)A(\frac{\pi}{2})R(\alpha_h-1)A(-\frac{\pi}{2})...R(\alpha_2)A(-\frac{\pi}{2})R(\beta_1)S |\Psi_0\rangle .$$

Using Eq. (8) and $A(-\frac{\pi}{2})A(\frac{\pi}{2}) = I$, we have

$$|\Psi_h\rangle \sim SA(\frac{\pi}{2})R(\alpha_h)A(-\frac{\pi}{2})R(\beta_h-1)A(\frac{\pi}{2})R(\alpha_1)A(-\frac{\pi}{2})$$

$$SR(\beta_h)A(\frac{\pi}{2})R(\alpha_h-1)A(-\frac{\pi}{2})...R(\alpha_2)A(-\frac{\pi}{2})R(\beta_1)S |\Psi_0\rangle .$$

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Here, two cases need to be discussed.

**Case 1.1:** When \( h \equiv 1 \mod 4 \), \( \beta_i = \alpha_{h+2-i} \) for \( i = 2, 4, \ldots, h-1 \), and \( \beta_i = \alpha_{h-i} \) for \( i = 1, 3, \ldots, h-2 \), using Eqs. (6) and (7), we have

\[
|\Psi_h\rangle \sim S A (\frac{\pi}{2}) A (-\frac{\pi}{2} + \alpha_h) A \left( \frac{\pi}{2} - \alpha_3 + \alpha_h \right) A \left( -\frac{\pi}{2} - \alpha_3 + \alpha_h + \alpha_{h-2} \right) \]
\[
\ldots A \left( \frac{\pi}{2} - \alpha_3 - \alpha_5 \cdots - \alpha_{k_1} + \alpha_{k_1+2} \cdots + \alpha_h \right) \]
\[
\ldots A \left( -\frac{\pi}{2} - \alpha_3 + \alpha_h + \alpha_{h-2} \right) A \left( \frac{\pi}{2} - \alpha_3 + \alpha_h \right) A \left( -\frac{\pi}{2} + \alpha_h \right) R(\alpha_1) \]
\[
SR(\beta_h) A \left( \frac{\pi}{2} \right) A \left( -\frac{\pi}{2} + \alpha_{h-1} \right) A \left( \frac{\pi}{2} - \alpha_2 + \alpha_{h-1} \right) A \left( -\frac{\pi}{2} - \alpha_2 + \alpha_{h-1} + \alpha_{h-3} \right) \]
\[
\ldots A \left( -\frac{\pi}{2} - \alpha_2 - \alpha_4 \cdots - \alpha_{k_2} + \alpha_{k_2+2} \cdots + \alpha_{h-1} \right) \]
\[
\ldots A \left( -\frac{\pi}{2} - \alpha_2 + \alpha_{h-1} + \alpha_{h-3} \right) A \left( \frac{\pi}{2} - \alpha_2 + \alpha_{h-1} \right) A \left( -\frac{\pi}{2} + \alpha_{h-1} \right) A \left( \frac{\pi}{2} \right) |\bar{0}\rangle , \]  

(44)

where \( k_1 = (h+1)/2 \), \( k_2 = (h-1)/2 \).

**Case 1.2:** When \( h \equiv 3 \mod 4 \), \( \beta_i = \alpha_{h+2-i} \) for \( i = 2, 4, \ldots, h-1 \), and \( \beta_i = \alpha_{h-i} \) for \( i = 1, 3, \ldots, h-2 \), using Eqs. (6) and (7), we have

\[
|\Psi_h\rangle \sim S A (\frac{\pi}{2}) A (-\frac{\pi}{2} + \alpha_h) A \left( \frac{\pi}{2} - \alpha_3 + \alpha_h \right) A \left( -\frac{\pi}{2} - \alpha_3 + \alpha_h + \alpha_{h-2} \right) \]
\[
\ldots A \left( \frac{\pi}{2} - \alpha_3 - \alpha_5 \cdots - \alpha_{k_1} + \alpha_{k_1+2} \cdots + \alpha_h \right) \]
\[
\ldots A \left( -\frac{\pi}{2} - \alpha_3 + \alpha_h + \alpha_{h-2} \right) A \left( \frac{\pi}{2} - \alpha_3 + \alpha_h \right) A \left( -\frac{\pi}{2} + \alpha_h \right) R(\alpha_1) \]
\[
SR(\beta_h) A \left( \frac{\pi}{2} \right) A \left( -\frac{\pi}{2} + \alpha_{h-1} \right) A \left( \frac{\pi}{2} - \alpha_2 + \alpha_{h-1} \right) A \left( -\frac{\pi}{2} - \alpha_2 + \alpha_{h-1} + \alpha_{h-3} \right) \]
\[
\ldots A \left( -\frac{\pi}{2} - \alpha_2 - \alpha_4 \cdots - \alpha_{k_2} + \alpha_{k_2+2} \cdots + \alpha_{h-1} \right) \]
\[
\ldots A \left( -\frac{\pi}{2} - \alpha_2 + \alpha_{h-1} + \alpha_{h-3} \right) A \left( \frac{\pi}{2} - \alpha_2 + \alpha_{h-1} \right) A \left( -\frac{\pi}{2} + \alpha_{h-1} \right) A \left( \frac{\pi}{2} \right) |\bar{0}\rangle , \]  

(45)

where \( k_1 = (h-1)/2 \), \( k_2 = (h-3)/2 \).

Thus, we can get Eq. (11) by rewriting Eq. (44) and Eq. (45) as follows.

\[
|\Psi_h\rangle \sim S \left[ A(\eta_k) \cdots A(\eta_1) \right] R(\alpha_1) SR(\beta_h) \left[ A(\zeta_k) \cdots A(\zeta_1) \right] |\bar{0}\rangle , \]  

(11)

where \( \eta_k = \eta_{h+1-k} \) and \( \zeta_k = \zeta_{h+1-k} \) for \( k = 1, 2, \ldots, h \), and

\[
\eta_{k+1} - \eta_k = \begin{cases} 
\pi - \alpha_{k+1} & k = 2, 4, \ldots, h-1, \\
-\pi + \alpha_{h-k+1} & k = 3, 5, \ldots, h,
\end{cases}
\]

\[
\zeta_{k+1} - \zeta_k = \begin{cases} 
\pi - \alpha_k & k = 2, 4, \ldots, h-1, \\
-\pi + \alpha_{h-k} & k = 3, 5, \ldots, h.
\end{cases}
\]

Here, when \( h \equiv 1 \mod 4 \),

\[
\eta(h+1)/2 = \frac{\pi}{2} + (\alpha(h+5)/2 + \alpha(h+9)/2 + \ldots + \alpha_{h-2} + \alpha_h) \\
- (\alpha_3 + \alpha_5 + \ldots + \alpha_{h-3}/2 + \alpha_{(h+1)/2}),
\]

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\[ \zeta_{(h+1)/2} = \frac{\pi}{2} + (\alpha_{(h+3)/2} + \alpha_{(h+7)/2} + \ldots + \alpha_{h-3} + \alpha_{h-1}) \\
- (\alpha_2 + \alpha_4 + \ldots + \alpha_{(h-5)/2} + \alpha_{(h-1)/2}), \]

and when \( h \% 4 = 3, \)

\[ \eta_{(h+1)/2} = -\frac{\pi}{2} + (\alpha_{(h+3)/2} + \alpha_{(h+7)/2} + \ldots + \alpha_{h-2} + \alpha_h) \\
- (\alpha_3 + \alpha_5 + \ldots + \alpha_{(h-5)/2} + \alpha_{(h-1)/2}), \]

\[ \zeta_{(h+1)/2} = -\frac{\pi}{2} + (\alpha_{(h+1)/2} + \alpha_{(h+5)/2} + \ldots + \alpha_{h-3} + \alpha_{h-1}) \\
- (\alpha_2 + \alpha_4 + \ldots + \alpha_{(h-7)/2} + \alpha_{(h-3)/2}). \]

**Case 2: h is even.** Using Eq. (9), there is

\[
|\Psi_h\rangle \sim SA(\frac{\pi}{2}) R(\alpha_h) A(-\frac{\pi}{2}) \ldots SR(\beta_{\frac{h+2}{2}}) A(\frac{\pi}{2}) R(\alpha_{\frac{h+1}{2}}) A(-\frac{\pi}{2}) \\
SR(\beta_{\frac{h}{2}+1}) A(\frac{\pi}{2}) R(\alpha_{\frac{h}{2}}) A(-\frac{\pi}{2}) SR(\beta_{\frac{h}{2}}) A(\frac{\pi}{2}) R(\alpha_{\frac{h-1}{2}}) A(-\frac{\pi}{2}) \\
SR(\beta_{\frac{h}{2}-1}) A(\frac{\pi}{2}) R(\alpha_{\frac{h-2}{2}}) A(-\frac{\pi}{2}) \ldots SR(\beta_2) A(\frac{\pi}{2}) R(\alpha_1) A(-\frac{\pi}{2}) SR(\beta_1) S |\Psi_0\rangle \\
= SA(\frac{\pi}{2}) R(\alpha_h) A(-\frac{\pi}{2}) \ldots S R(\beta_{\frac{h+2}{2}}) A(\frac{\pi}{2}) R(\alpha_{\frac{h+1}{2}}) A(-\frac{\pi}{2}) \\
SR(\beta_{\frac{h}{2}+1}) A(\frac{\pi}{2}) R(\alpha_{\frac{h}{2}}) A(-\frac{\pi}{2}) SR(\beta_{\frac{h}{2}}) A(\frac{\pi}{2}) R(\alpha_{\frac{h-1}{2}}) A(-\frac{\pi}{2}) S \\
SR(\beta_{\frac{h}{2}-1}) A(\frac{\pi}{2}) R(\alpha_{\frac{h-2}{2}}) A(-\frac{\pi}{2}) \ldots SR(\beta_{\frac{h}{2}-2}) A(\frac{\pi}{2}) R(\alpha_{\frac{h-3}{2}}) A(-\frac{\pi}{2}) S |\Psi_0\rangle \\
= SA(\frac{\pi}{2}) R(\alpha_h) A(-\frac{\pi}{2}) R(\beta_{h-1}) A(\frac{\pi}{2}) \ldots A(\frac{\pi}{2}) R(\alpha_2) A(-\frac{\pi}{2}) R(\beta_1) \\
SR(\beta_{h}) A(\frac{\pi}{2}) R(\alpha_{h-1}) A(-\frac{\pi}{2}) \ldots R(\alpha_1) A(-\frac{\pi}{2}) |\Psi_0\rangle. 
\]

Using Eq. (8) and \( A(-\frac{\pi}{2}) A(\frac{\pi}{2}) = I, \) we have

\[
|\Psi_h\rangle \sim SA(\frac{\pi}{2}) R(\alpha_h) A(-\frac{\pi}{2}) R(\beta_{h-1}) A(\frac{\pi}{2}) \ldots A(\frac{\pi}{2}) R(\alpha_2) A(-\frac{\pi}{2}) R(\beta_1) \\
SR(\beta_{h}) A(\frac{\pi}{2}) R(\alpha_{h-1}) A(-\frac{\pi}{2}) \ldots R(\alpha_1) A(-\frac{\pi}{2}) A(\frac{\pi}{2}) SA(\frac{\pi}{2}) |\bar{0}\rangle \\
\sim R(\beta_{h}) A(\frac{\pi}{2}) R(\alpha_{h-1}) A(-\frac{\pi}{2}) \ldots A(\frac{\pi}{2}) R(\beta_2) A(\frac{\pi}{2}) R(\alpha_1) S \\
A(\frac{\pi}{2}) R(\alpha_h) A(-\frac{\pi}{2}) R(\beta_{h-1}) A(\frac{\pi}{2}) \ldots A(\frac{\pi}{2}) R(\alpha_2) A(-\frac{\pi}{2}) R(\beta_1) A(\frac{\pi}{2}) |\bar{0}\rangle. 
\]

Here, two cases need to be discussed.
Case 2.1: When $h/4 = 0$, and $\beta_i = \alpha_{h+1-i}$ for $i = 1, 2, \ldots, h - 1$, using Eqs. (6) and (7), we have

$$|\Psi_h\rangle \sim R(\beta_h)A(\frac{\pi}{2})A(-\frac{\pi}{2} + \alpha_{h-1})A(\frac{\pi}{2} - \alpha_3 + \alpha_{h-1})A(-\frac{\pi}{2} - \alpha_3 + \alpha_{h-1} + \alpha_{h-3})$$
$$\ldots A(-\frac{\pi}{2} - \alpha_3 - \alpha_5 \cdots - \alpha_{k_1} + \alpha_{k_1+2} \cdots + \alpha_{h-1}) \ldots$$
$$A(-\frac{\pi}{2} - \alpha_3 + \alpha_{h-1} + \alpha_{h-3})A(\frac{\pi}{2} - \alpha_3 + \alpha_{h-1})A(-\frac{\pi}{2} + \alpha_{h-1})A(\frac{\pi}{2})R(\alpha_1)S$$
$$A(\frac{\pi}{2})A(-\frac{\pi}{2} + \alpha_{h})A(\frac{\pi}{2} - \alpha_2 + \alpha_{h})A(-\frac{\pi}{2} - \alpha_2 + \alpha_{h} + \alpha_{h-2})$$
$$\ldots A(-\frac{\pi}{2} - \alpha_2 - \alpha_4 \cdots - \alpha_{k_2} + \alpha_{k_2+2} \cdots + \alpha_{h}) \ldots$$
$$A(-\frac{\pi}{2} - \alpha_2 + \alpha_{h} + \alpha_{h-2})A(\frac{\pi}{2} - \alpha_2 + \alpha_{h})A(-\frac{\pi}{2} + \alpha_{h})A(\frac{\pi}{2}) |\bar{0}\rangle,$$  \hspace{1cm} (46)

where $k_1 = h/2 - 1$, $k_2 = h/2$.

Case 2.2: When $h/4 = 2$, and $\beta_i = \alpha_{h+1-i}$ for $i = 1, 2, \ldots, h - 1$, using Eqs. (6) and (7), we have

$$|\Psi_h\rangle \sim R(\beta_h)A(\frac{\pi}{2})A(-\frac{\pi}{2} + \alpha_{h-1})A(\frac{\pi}{2} - \alpha_3 + \alpha_{h-1})A(-\frac{\pi}{2} - \alpha_3 + \alpha_{h-1} + \alpha_{h-3})$$
$$\ldots A(\frac{\pi}{2} - \alpha_3 - \alpha_5 \cdots - \alpha_{k_1} + \alpha_{k_1+2} \cdots + \alpha_{h-1}) \ldots$$
$$A(-\frac{\pi}{2} - \alpha_3 + \alpha_{h-1} + \alpha_{h-3})A(\frac{\pi}{2} - \alpha_3 + \alpha_{h-1})A(-\frac{\pi}{2} + \alpha_{h-1})A(\frac{\pi}{2})R(\alpha_1)S$$
$$A(\frac{\pi}{2})A(-\frac{\pi}{2} + \alpha_{h})A(\frac{\pi}{2} - \alpha_2 + \alpha_{h})A(-\frac{\pi}{2} - \alpha_2 + \alpha_{h} + \alpha_{h-2})$$
$$\ldots A(-\frac{\pi}{2} - \alpha_2 - \alpha_4 \cdots - \alpha_{k_2} + \alpha_{k_2+2} \cdots + \alpha_{h}) \ldots$$
$$A(-\frac{\pi}{2} - \alpha_2 + \alpha_{h} + \alpha_{h-2})A(\frac{\pi}{2} - \alpha_2 + \alpha_{h})A(-\frac{\pi}{2} + \alpha_{h})A(\frac{\pi}{2}) |\bar{0}\rangle,$$  \hspace{1cm} (47)

where $k_1 = h/2$, $k_2 = h/2 - 1$.

Using Eqs. (46) and (47) we can get Eq. (19),

$$|\Psi_h\rangle \sim R(\beta_h)[A(\phi_{h-1})\ldots A(\phi_1)]R(\alpha_1)S[A(\psi_{h+1})\ldots A(\psi_1)] |\bar{0}\rangle,$$  \hspace{1cm} (19)

where $\phi_k = \phi_{h-k}$ for $k = 1, 2, \ldots, h - 1$, $\psi_k = \psi_{h+2-k}$ for $k = 1, 2, \ldots, h + 1$, and

$$\phi_{k+1} - \phi_k = \begin{cases} 
\pi - \alpha_{k+1} & \text{for } k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k} & \text{for } k = 1, 3, \ldots, h - 3,
\end{cases}$$

$$\psi_{k+1} - \psi_k = \begin{cases} 
\pi - \alpha_k & \text{for } k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k+1} & \text{for } k = 1, 3, \ldots, h.
\end{cases}$$

Here, when $h/4 = 0$, there are

$$\phi_{h/2} = -\frac{\pi}{2} + (\alpha_{h/2+1} + \alpha_{h/2+3} + \ldots + \alpha_{h-3} + \alpha_{h-1}) - (\alpha_3 + \alpha_5 + \ldots + \alpha_{h/2-3} + \alpha_{h/2-1}),$$

$$\psi_{h/2+1} = \frac{\pi}{2} + (\alpha_{h/2+2} + \alpha_{h/2+4} + \ldots + \alpha_{h-2} + \alpha_h) - (\alpha_2 + \alpha_4 + \ldots + \alpha_{h/2-2} + \alpha_{h/2}),$$

$$\psi_{h/2+2} = \frac{\pi}{2} + (\alpha_{h/2+3} + \alpha_{h/2+5} + \ldots + \alpha_{h-1} + \alpha_h) - (\alpha_3 + \alpha_5 + \ldots + \alpha_{h/2-1}),$$

$$\phi_{h/2+1} = -\frac{\pi}{2} + (\alpha_{h/2+2} + \alpha_{h/2+4} + \ldots + \alpha_h) - (\alpha_2 + \alpha_4 + \ldots + \alpha_{h/2-1}).$$
and when $h \% 4 = 2$, there are

$$
\phi_{h/2} = \frac{\pi}{2} + (\alpha_{h/2+2} + \alpha_{h/2+4} + \ldots + \alpha_{h-3} + \alpha_{h-1}) - (\alpha_3 + \alpha_5 + \ldots + \alpha_{h/2-2} + \alpha_{h/2}),
$$

$$
\psi_{h/2+1} = -\frac{\pi}{2} + (\alpha_{h/2+1} + \alpha_{h/2+3} + \ldots + \alpha_{h-2} + \alpha_{h}) - (\alpha_2 + \alpha_4 + \ldots + \alpha_{h/2-3} + \alpha_{h/2-1}).
$$

# B Proof of Equations (33) and (39)

The proofs are similar to the ones in Appendix A.