Brane Realizations of Quantum Hall Solitons and Kac-Moody Lie Algebras

A. Belhaj$^{1,2,5}$, A. ElRhalami$^{1,5}$, N.-E. Fahssi$^{1,3,5}$, M. J. I. Khan$^{1,5}$, E. H. Saidi$^{1,5}$, A. Segui$^{4,5}$

$^1$Lab. Phys Hautes Energies, Modélisation et Simulation, Faculté des Sciences, Rabat, Morocco
$^2$Centre National de l’Énergie, des Sciences et des Techniques Nucléaires, Rabat, Morocco
$^3$Département de Mathématiques, Faculté des Sciences et Techniques, Mohammedia, Morocco
$^4$Departamento de Física Teórica, Universidad de Zaragoza, E-50009-Zaragoza, Spain
$^5$Groupement National de Physique des Hautes Energies, Siège focal, FSR, Rabat, Morocco

August 3, 2010

Abstract

Using quiver gauge theories in (1+2)-dimensions, we give brane realizations of a class of Quantum Hall Solitons (QHS) embedded in Type IIA superstring on the ALE spaces with exotic singularities. These systems are obtained by considering two sets of wrapped D4-branes on 2-spheres. The space-time on which the QHS live is identified with the world-volume of D4-branes wrapped on a collection of intersecting 2-spheres arranged as extended Dynkin diagrams of Kac-Moody Lie algebras. The magnetic source is given by an extra orthogonal D4-brane wrapping a generic 2-cycle in the ALE spaces. It is shown as well that data on the representations of Kac-Moody Lie algebras fix the filling factor of the QHS. In case of finite Dynkin diagrams, we recover results on QHS with integer and fractional filling factors known in the literature. In case of hyperbolic bilayer models, we obtain amongst others filling factors describing holes in the graphene.

Keywords: Quantum Hall Effect, Type IIA Superstring on ALE spaces, Kac-Moody Lie algebras.

$^*$belhaj@unizar.es
$^†$fahssi@uh2m.ac.ma
$^‡$h-saidi@fsr.ac.ma
$^§$segui@unizar.es
1 Introduction

Three dimensional Quantum Hall Systems of condensed matter physics can be engineered by low energy dynamics of D-branes embedded in Type II superstrings [1]–[15]. In particular, a ten dimensional superstring picture of Quantum Hall Solitons (QHS) in (1+2)-dimensions has been given in terms of a stack of $K$ D6-branes and a spherical D2-brane as well as dissolved D0-branes in D2 and a stack of $N$ F-strings stretching between D2 and D6-branes [1]. The world-volume of this D2-brane plays the role of the 3-dimensional space-time, while the external source of the magnetic charges is identified with the stack of the $K$ D6-branes which are placed perpendicular to the spherical D2-brane. When the D6-branes cross the D2-brane, the Hanany-Witten effect produces fundamental strings (F-strings) which are stretched between D2 and D6-branes [16]. The F-strings ending on the D2-brane have an interpretation in terms of the fractional quantum Hall particles (Hall electrons), and they are charged under the U(1) world-volume gauge field associated with the D0-branes which behave as magnetic flux quanta dissolved in the D2 world-volume.

In this paper, we elaborate the brane construction of a class of QHS that are embedded in Type IIA superstring moving on asymptotically locally Euclidean (ALE) spaces with non-trivial singularities [12, 15]. The (1+2)-dimensional QHS are engineered from a stack of $n$ D4-branes wrapping intersecting 2-spheres arranged as Dynkin diagrams of the Kac-Moody Lie algebras. These branes are coupled to an external gauge field described by an extra D4-brane wrapping a particular 2-cycle in ALE spaces playing the role of a magnetic source. Its unwrapped directions are placed perpendicularly to the uncompactified part of the word-volume of D4-branes on which a U(1)$^n$ gauge theory lives. Among our results, we show that the computation of the filling factors can be converted to a task of studying representation theory of the Kac-Moody Lie algebras. In this work, we examine models concerning quivers associated with finite and hyperbolic sectors. Representations of finite algebras classify models with integer, half-integer and fractional filling factors. For the fundamental representation of finite $A_n$ algebra for instance, we can recover celebrated filling factor series. As for the case of bilayer hyperbolic models, we find several subsequences of known series describing holes in the graphene [17]. Throughout this work, we give also multilayer QHS models, based on the hyperbolic $\hat{H}A_n$, with positive, negative or zero filling factor values, depending on its rank and vector charges.

This paper is organized as follows. In section 2, we review briefly some basic facts about Kac-Moody Lie algebras. In section 3, we present our brane construction of QHS in (1+2)-dimensions from Type IIA superstring on the ALE spaces with deformed simply laced singularities. In section 4, we compute the filling factors in terms of weights and in section 5, we give explicit models that are associated with finite and hyperbolic sectors of Kac-Moody
Lie algebras. For the finite case, we first list ADE quiver models with integer, half-integer and fractional filling factors. Then, we examine the hyperbolic bilayered and multilayered systems with positive, negative or zero filling factor values. Concluding discussion and open questions are given in section 6.

2 Generalized Cartan matrices

Kac-Moody Lie algebras and their representations have been extensively studied as they play a crucial role in many areas of classical and quantum physics. They are behind the derivation of several exact results in quantum field theory such as in 4D $\mathcal{N} = 2$ conformal quiver gauge theories and also in the compactification of superstring theory on Calabi-Yau manifolds by using the geometric engineering method [18]. They will be used in this paper to develop new classes of quantum Hall solitons embedded in six dimensional Type IIA superstring. A nice way to introduce the Kac-Moody Lie algebras is in terms of the generalized Cartan matrix [19, 20, 21, 22] and the Dynkin diagrams. For later use, let us give a brief review of some basic facts about this method. The entries of the generalized Cartan matrix satisfy

$$K_{ii} = 2, \quad K_{ij} < 0, \quad \text{if} \quad K_{ij} = 0 \quad \text{then} \quad K_{ji} = 0. \quad (2.1)$$

According to Vinberg theorem on the classification of matrices, we learn that one should distinguish three of generalized Kac-Moody Lie algebras:

1. **Finite type**: $K_{ij}$ is a finite type matrix if there exists a positive definite $u$ ($u_i > 0; i = 1, 2, \ldots$) such that

   $$K_{ij}^{(+)} u_j > 0 \quad (2.2)$$

2. **Affine type**: $K_{ij}$ is an affine type matrix if there exists a unique, up to a multiplicative factor, positive integer definite vector $u$ ($u_i > 0; i = 1, 2, \ldots$) such that

   $$K_{ij}^{(0)} u_j = 0 \quad (2.3)$$

3. **Indefinite type**: $K_{ij}$ is an indefinite type matrix if there exists a positive definite $u$ ($u_i > 0; i = 1, 2, \ldots$) such that

   $$K_{ij}^{(-)} u_j < 0. \quad (2.4)$$

As the third sector is still an open problem in mathematics, we shall restrict ourselves here below to its hyperbolic subset obtained from the affine sector by adding an extra node to the affine Dynkin diagram. The upper indices $\pm$ and 0 carried by the generalized Cartan matrices

2
\(K_{ij}^{(q)}\) introduced above are used to distinguish the three sector which takes the following equivalent form:

(a) Cartan matrices of finite dimensional Lie algebras satisfy \(\det K^{(+)} > 0\),

(b) Cartan matrices of affine Kac-Moody Lie algebras are singular since \(\det K^{(0)} = 0\),

(c) Cartan matrices of the hyperbolic subset obey the property \(\det K^{(-)} < 0\).

Like in the finite dimensional case, we associate to any \(n \times n\) Cartan matrix \(K_{ij}^{(q)}\) a generalized Dynkin diagram which consists of \(n\) nodes; and for each pair of nodes \((i, j)\) we have \(m_{ij}\) lines between them with \(m_{ij}\) precisely given by \(K_{ij}\). This graph plays an important role in the study of quiver gauge theories embedded into Type II superstrings compactified on the K3 surface whose singularities are precisely classified by Kac-Moody Lie algebras [21, 22]. The deformation of these singularities consists on blowing up the singular point by a collection of intersecting 2-spheres with the intersection matrix \(I_{ij} = -K_{ij}\).

3 Brane construction of QHS

Numerous connections between string theory and QHS in (1+2)-dimensions which were found during the last years. The first of them was a construction of Bernevig, Brodie, Susskind and Toumbas describing the QHE on a 2-sphere [1] using a spherical D2-brane and dissolved \(N\) D0-branes moving on it. This construction has been described in the introduction.

The ten dimensional string picture of QHE in (1+2)-dimensions has been extended to the compactification of Type IIA superstring theory on the ALE spaces [12, 15]. In particular, one can geometrically engineer at least two Type IIA brane realizations of QHS [15]. The first realization uses D2 and wrapped D6-branes wrapping the K3 surface, while the second one deals with only D4-branes which we are interested in here. This approach is based on the study of quiver gauge theories living on the world-volume of the wrapped D4-branes on intersecting 2-spheres \(S^2_i\) arranged as extended Dynkin Diagrams. On the world-volume of theses wrapped branes lives a U(1)^n gauge symmetry in 3-dimensional space-time on which our QHS will appear.

To couple the system to an external gauge field, one needs an extra D4-brane wrapping a generic 2-cycle described by a linear combination of \(S^2_i\) as follows

\[
[C_2] = \sum_i q_i [S^2_i],
\]  

(3.1)
where $S^2_i$ denote a basis of $H_2(ALE, \mathbb{Z})$. A priori there are two different ways in which the D4-brane is wrapped on each $S^2$. This can be supported by the fourth homotopy group of $S^2$

$$\Pi_4(S^2) = \mathbb{Z}_2.$$  

(3.2)

Wrapping a D4-brane over such a geometry gives two possible D2-brane configurations. Each one corresponds to an equivalent class of the $\mathbb{Z}_2$ group. The wrapped D4-brane over a $S^2$ is sensitive to this $\mathbb{Z}_2$ symmetry and, thus, it carries two charges $\pm 1$. A general value of the charge $\pm q_i$ can be obtained by wrapping a D4-brane $q_i$ times over a $S^2_i$ in two possible orientations producing membranes which play the role of the magnetic source in six dimensions. These membranes should be placed perpendicularly to the uncompactified part of the world-volume of D4-branes on which the $U(1)^n$ gauge theory lives. In this realization of QHS, the $q_i$ charges play the same role as the D6-brane ones in ten dimensions and can be interpreted as the vector charges of D-particles living on the space-time. As in ten dimensional case [1], the missing spatial dimension is filled by F-strings stretching between the orthogonal D4-branes (the ones on which the $U(1)^n$ gauge theory lives and the one playing the role of the magnetic source). The full brane system can be described by the $U(1)^n$ Chern-Simons gauge theory with the following action

$$S \sim \frac{1}{4\pi} \int \sum_{ij} K_{ij} A^i \wedge dA^j + 2 \sum_i q_i \tilde{A} \wedge dA^i. \quad (3.3)$$

In this action, $\tilde{A}$ is an external $U(1)$ gauge field and $A^i$ are dynamical gauge ones. $K_{ij}$ is a $n \times n$ matrix which can be related to the intersection matrix of 2-spheres of ALE spaces [12, 15]. Following Wen-Zee model [23, 24], $K_{ij}$ and $q_i$ are interpreted as order parameters classifying the various QHS states and are related to the filling factor via the relation

$$\nu = q_i K^{-1} q_i, \quad (3.4)$$

which is invariant under the above $\mathbb{Z}_2$ symmetry. This expression involves the inverse of the extended Cartan matrix being symmetric as required by simply laced singularities of the ALE spaces. We will show that the computation of the filling factors can be converted to a task of studying representation theory of Kac-Moody Lie algebras. This means we will solve (3.4) in terms of Dynkin geometry data on which QHS reside. In particular, we will demonstrate that (3.4) can be solved using weight equations. Then we illustrate our results for special choices for the vector charge corresponding to known representations.
4 Algebraic solution of (3.4)

In this section, we give an algebraic solution of (3.4) in terms of weight space data of the corresponding algebra. This is not a surprising result since the expression of the filling factors involves the Cartan matrix of Kac-Moody Lie algebras. Thus, it is natural to expect that representation theory will play an important role for solving the equation (3.4). For simplicity, let us restrict our self to finite dimensional algebras defined by (2.2). They are classified in two types: simply laced $ADE$ algebras having a symmetric Cartan matrix and non simply laced $BCFG$ ones having a non symmetric Cartan matrix. The last feature is the main reason behind the complexity of the analysis of the blow ups of the $BCFG$ singularities of the ALE surface. For this reason, let us restrict our self to the simply laced case with

$$K_{ij} = (\alpha_i, \alpha_j)$$

(4.1)

where $\alpha_i$ are simple roots forming an orthonormal basis of the root space. In this space, a weight vector $\Lambda$ can be expressed as follows

$$\Lambda = \sum \bar{\lambda}_i \alpha_i$$

(4.2)

where $\bar{\lambda}_i$ are the coordinates of $\Lambda$ in the dual basis. It turns out that, one can define, for any $\Lambda$, the so-called Dynkin components $a_i$

$$a_i = \sum_j \bar{\lambda}_j K_{ij}$$

(4.3)

which are integer numbers [25]. In Dynkin formalism, the scalar product is given by the expression

$$(\Lambda, \Lambda') = (\Lambda', \Lambda) = \sum_{i,j} a_i' K_{ij}^{-1} a_j.$$  

(4.4)

It follows that the vector norm reads as

$$(\Lambda, \Lambda) = \sum_{i,j} a_i K_{ij}^{-1} a_j.$$  

(4.5)

Now we take the vector charge $q_i$, which is the electromagnetic charge carried by the extra D4-brane, to be the Dynkin labels $a_i$. This observation can be supported by a special choice of a divisor in $H_2(ALE, \mathbb{Z})$ vector space. In this case, we should consider a D4-brane in class
\[ C_2 \in H_2(ALE, \mathbb{Z}) \] carrying magnetic charges under \( U(1)^n \) gauge symmetry described by

\[ [C_2] = \sum_i a_i [S_i^2]. \]  

(4.6)

From representation theory, it follows that the expression of the filling fractions (3.4) becomes

\[ \nu = (\Lambda, \Lambda). \]  

(4.7)

Generally, one may have three situations classified by the signature of the norm as given below

\( (i) \ \nu > 0, \ (ii) \ \nu = 0, \ (iii) \ \nu < 0. \)  

(4.8)

It should be interesting to note that the above results can have different situations on weight lattices. The most interesting are self-dual ones used in the construction of the heterotic string theory. They can exist only in spaces with dimension \( n = 8d \) where \( d \) is a positive integer. For \( d = 1 \), there is just one lattice called \( E_8 \), while in \( d = 2 \) there are two special lattices which are \( E_8 \times E_8 \) and \( D_{16} \). In the case of \( d = 3 \) there is a special lattice called Leach lattice which has a minimal norm equal 4. Beside self-dual cases, there exit also integer-valued lattices with minimal norm equal 1 and 2 which are called respectively odd and even lattices.

In the sequel, we will adopt the solution (4.7) in our illustrating examples.

5 Explicit quiver models

In this section, we apply to stringy QHS some ideas familiar from the study of the extended Cartan matrix associated with quiver gauge theories to compute the filling factors. In particular, we give some illustrating models for simply laced finite and hyperbolic Kac-Moody Lie algebras. For these algebras, the quadratic form (3.4) can be written as follows

\[ \nu = \sum_i K_{ii}^{-1} a_i^2 + 2 \sum_{i<j} K_{ij}^{-1} a_i a_j, \]  

(5.1)

5.1 Finite quiver models

Here, we consider QHS associated with finite dimensional simply laced algebras of type \( A_n, D_n \) and \( E_s \) \( (s = 6, 7, 8) \) for which all roots have the same length squared. In particular, we describe the \( A_n \) case in detail. The results for the other algebras may be obtained without difficulty. To that purpose, we first collect some useful information. We stress that generally the determinant of the Cartan matrix of all these series is always positive [20]. They are given
by
\[ A_n : n + 1 ; \quad D_n : 4 ; \quad E_6 : 3 ; \quad E_7 : 2 ; \quad E_8 : 1. \]

In the residue of this section, we will give some concrete examples to illustrate our results for integer and fractional values for the \( \nu \)-factor.

### 5.1.1 \( A_n \) quivers

Consider a \( U(1)^n \) quiver gauge theory with finite \( A_n \) Cartan matrix. In the context of string theory, this model appears as the world-volume of \( n \) D4-branes wrapping separately \( n \) 2-cycles of the deformed \( \mathbb{C}^2/\mathbb{Z}_{n+1} \) singularity. The Kähler deformation of this geometry is obtained by blowing up the singular point using \( n \) intersecting 2-spheres, \( (S^2_i) \), arranged as follows

\[
A_n : \quad \circ - \bigcirc - \cdots \quad \circ - \circ \circ \quad (5.2)
\]

The elements of the corresponding Cartan matrix are given by
\[ K_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j}, \]
from which the inverse matrix can be evaluated to be
\[ K^{-1}_{ij} = \frac{i(n - j + 1)}{n + 1}, \quad (i \leq j). \]

Thus, the filling factor (3.4) reduces to the following quadratic form
 \[
\nu(A_n) = \frac{1}{n + 1} \left( \sum_{i=1}^{n} i(n - i + 1) a_i^2 + 2\sum_{i<j} i(n - j + 1)a_i a_j \right). \tag{5.3}
\]

For the sake of illustration we shall now calculate this \( \nu \)-factor for some irreps of \( A_n \) algebras. The simplest ones are the so-called basic representations whose Dynkin components are zeros except one entry which is equal 1. In the quantum Hall literature, this example is related to single layer states. If we take \( a_i = \delta_{ip} \) for some \( p = 1, \ldots, n \), then equation (5.3) reduces to
\[ \nu(A_n) = \nu_{p,n} = \frac{p(n + 1 - p)}{n + 1} \tag{5.4} \]

exhibiting the obvious symmetry \( \nu_{p,n} = \nu_{n+1-p,n} \). In the fundamental representation (where \( p = 1 \) or \( n \)), we find \( \nu_{1,n} = 1 - \frac{1}{n+1} \) which coincides with known filling fractions in the literature [24]. In table 1, we list the values of \( \nu \) for the fundamental, the adjoint and the \( \rho \)-representation (that is irreps whose highest weight is the Weyl vector \( \rho \) which is half sum of positive roots).

At this stage, two points are worthy of notes

- If we fix the normalization of the inner product (4.4) to be such that the highest weight
of the adjoint representation has norm 1, then, in the fundamental representation of $A_{2n}$ we have $(\Lambda, \Lambda) = \frac{n}{2n+1}$, which coincides with a subsequence of the celebrated Jain’s series. This series can besides be recovered as an exact value of the $\nu$-factor (3.4) if we have in the action (3.3), $2K_{ij}$ instead of $K_{ij}$ [26] (or more generally $K_{ij} + K_{ji}$ in place of $K_{ij}$ if we wish to cover non symmetric Cartan matrices).

- The $\nu$-factor is independent of the rank $n$ in the adjoint representation. This fact is general for vector charges of the form $a_i = \delta_{ip} + \delta_{i,n+1-p}$. We find in this case $\nu = 2p$.

### 5.1.2 DE quivers

The results for DE quiver gauge models can straightforwardly be obtained in similar way. We just display, in the end of this subsection, in Table 2 the values of the $\nu$-factor for simple irreps, which are representations from which all other irreps can be constructed by tensor products.

| Irreps       | Vector charge | $\nu$          | Comments                      |
|--------------|---------------|----------------|-------------------------------|
| fundamental  | (10...00) or (00...01) | $\frac{n}{n+1}$ | fractional                    |
| adjoint      | (10...01)     | 2              | for all $n$                   |
| $\rho$-representation | (11...11) | $\frac{1}{2}(\frac{n+2}{3})$ | integer if $n \equiv 1 \mod 4$ |
|              |               |                | half-integer otherwise       |

Table 1: The $\nu$-factor associated with some irreducible representations of $A_n$.

| Dynkin designation | $\nu$ |
|--------------------|-------|
| $D_n$ (10...00) | 1     |
| (00...01) or (0...010) | $n/4$ |
| $E_6$ (100000) or (000010) | $4/3$ |
| $E_7$ (0000010) | $3/2$ |
| $E_8$ (0000010) | 2     |

Table 2: The $\nu$-factor associated with simple irreducible representations of $D$ and $E$ algebras.

### 5.2 Hyperbolic quiver models

In this section, we consider indefinite quiver gauge theories describing multilayers states in QHS. In this case, we will see that the filling factor (3.4) can take positive, negative or zero...
values, depending on the vector charges and the rank of the algebras. To start, recall first that there is no full classification for indefinite Cartan matrices and very little is known about the corresponding representation theory. For this reason, we will not refer to representation theory and we will restrict our selves to quiver gauge theories with a hyperbolic Cartan matrix (matrices with a single negative eigenvalue and all the other positive). Recall that hyperbolic Kac-Moody Lie algebras are by definition Lorentzian Kac-Moody Lie algebras with the property that cutting any node from their Dynkin diagrams leaves one with a Dynkin diagram of affine or finite type.

The general study is beyond the scope of the present work, though we will consider two explicit examples. The first example corresponds to the case of two layers QHS, while the second one will be associated with multilayer systems.

As mentioned, the starting example concerns a bilayer system corresponding to \( U(1) \times U(1) \) quiver gauge theory with the following Cartan matrix

\[
K_{ij}(U(1) \times U(1)) = \begin{pmatrix}
2 & -k \\
-k & 2
\end{pmatrix}
\]

In order to get a hyperbolic Dynkin geometry, one needs to require

\[
4 - k^2 < 0. \tag{5.5}
\]

From string theory point of view, this model can be obtained from two D4-branes wrapping on two intersecting 2-spheres \( S^2_i \) with the following intersection constraint

\[
S^2_i \cdot S^2_j = -K_{ij}(U(1) \times U(1)). \tag{5.6}
\]

Evaluating (3.4) for the charges \( a_i = (1, 1) \) yields

\[
\nu = \frac{2}{2 - k}. \tag{5.7}
\]

This is a general expression containing some known series used in the study of trial functions of QHE in the graphene. These kind of functions have been proposed first by Laughlin [27] and they have been generalized by Halperin in order to understand multi-components with \( SU(K) \) internal symmetries [28]. Based on Halperin ideas on the study of trial wave functions \( (m_1, m_2, m_3) \) of QHE, the results we obtain here may correspond to holes in the graphene structure. In particular, putting \( k = 2m + 1 \), we recover a \( U(1) \times U(1) \) quiver gauge theory
of bilayers QHS with the following filling fraction
\[ v = \frac{2}{1-2m}. \] (5.8)

The opposite value (\( v = \frac{2}{2m-1} \)) has been obtained in the study of \((m,m,m-1)\) trial wave functions given in [29]. One may also get \( v = -\frac{1}{m} \) by taking \( k = 2m + 2 \) and it corresponds to \((m,m,m-1)\) Laughlin’s wave functions describing holes. We note that one can reproduce integer values like \( v = -1 \) and \( v = -2 \) by taking \( k = 3 \) and \( k = 4 \) respectively. These values has been dealt with to study non equilibrium breakdown of QHE in graphene [30].

The next example of hyperbolic Kac-Moody Lie algebras that we give is related the over-extensions of affine ones. A simple situation is to add one extra node to the Dynkin graph of the \( \hat{A}_n \) affine Kac-Moody Lie algebra. In geometric Dynkin language, this corresponds to a particular hyperbolic Dynkin diagram. The derivation of such a hyperbolic geometry is based on the same philosophy one uses in the building of the affine Dynkin diagrams from the finite ones by adding a node. In other words, by cutting this node in such a hyperbolic Dynkin diagram, the resulting sub-diagram coincides with the Dynkin graph of the \( \hat{A}_n \) affine Kac-Moody Lie algebras. We refer to this Dynkin graph as \( H\hat{A}_n \)

\[ H\hat{A}_n : \quad \cdots \cdots \]

The Cartan matrix of this algebra reads as
\[ K_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i+1,j} - \delta_{i,n+2}\delta_{j,2} - \delta_{2,i}\delta_{j,n+2} \quad i,j = 1, \ldots n+2, \] (5.10)
from which we derive, rather laboriously, the inverse matrix elements (for \( i \leq j \))
\[ K^{-1}_{ij} = \begin{cases} 0, & \text{if } j = 1 \\ -1, & \text{if } j > 1 \end{cases} \]
\[ K^{-1}_{ij} = -\frac{1}{n+1} (ij + (4 - i)n - 3i - 2j + 8). \] (5.11)

This expression shows that, interestingly, \( \det K(H\hat{A}_n) = -(n+1) \). Adding two nodes to the Dynkin diagram of the \( A_n \) finite Kac-Moody Lie algebra, as shown in the above figure, makes that the determinant of the Cartan matrix is multiplied by -1. Mathematically, it would be
instructive to examine this property for more general over-extensions. For the present case, (3.4) can be cast in the following form

\[
v(H \hat{A}_n) = -\frac{1}{n+1} \sum_{i=2}^{n+2} (i^2 + (4 - i)n - 5i + 8) a_i^2 - \frac{2}{n+1} \sum_{j=3}^{n+2} \sum_{i=2}^{j-1} (ij + (4 - i)n - 3i - 2j + 8) a_i a_j - 2a_1 \sum_{j=2}^{n+2} a_j. \quad (5.12)
\]

Numerical investigations show that, for any vector charge, the sign of \( \hat{\nu}_p \) is not constant and vanishes for special values of \( n \) and \( a_i \), covering the three signatures of (4.8). This reveals the Lorentzian nature of the weight space for hyperbolic algebras. In Table (3), we will collect some remarkable values of \( \nu \).

For the case where \( a_i = \delta_{ip} \) (for some \( p = 1, \ldots, n \)), the filling factors read as

\[
v(H \hat{A}_n) = K_{pp}^{-1} = -\frac{1}{n+1} \left( p^2 - (n + 5)p + 4n + 8 \right) := \hat{\nu}_{p,n}, \quad (5.13)
\]

for \( p = 2, \ldots, n+2 \). It is easy to show that the following symmetry relation is fulfilled

\[
\hat{\nu}_{p,n} = \hat{\nu}_{5+n-p,n}
\]

which may reflect a symmetry in the corresponding lattice weight space. It can be also shown that \( \hat{\nu}_{p,n} \) is zero only for the values \((p, n) = (5, 8), (8, 8), (6, 7)\). Here, we list the following result

| Vector charge | \( \nu \) | Comments |
|---------------|-----------|----------|
| \((10\ldots00)\) | 0 | for all \( n \) |
| \((01\ldots00)\) or \((00\ldots010)\) | -2 | for all \( n \) |
| \((000010\ldots0)\) or \((00\ldots0100)\) | \((n - 8)/(n + 1)\) | integer for \( n = 2, 8 \) |
| \((000010\ldots0)\) or \((00\ldots1000)\) | \(2(n - 7)/(n + 1)\) | integer for \( n = 1, 3, 7, 15 \) |
| \((0010\ldots01)\) | -6 | for all \( n \) |
| \((11\ldots11)\) | \(\frac{1}{n^2}(n - 24)(n + 1)(n + 2)\) | half-integer if \( n \equiv 1 \mod 4 \) integer otherwise |

Table 3: The \( \nu \)-factor associated with \( H \hat{A}_n \).

From the significant observation of the above table we conclude that the value -6 in Table 3 is a particular case of the following more general fact. If \( a_i = \delta_{ip} + \delta_{iq} \) such that \( p + q = n + 5 \),
(p \geq 3), then \( \nu(H\tilde{A}_n) = 2(p - 6) \) clearly vanishes for the vector charge \((000010 \ldots 01000)\). It should be instructive to understand these properties of hyperbolic QHS.

## 6 Discussions and Open questions

In this paper we have discussed brane realizations of QHS associated with extended Cartan matrices of Kac-Moody Lie algebras. The models considered here are described by abelian quiver gauge theories obtained from the compactification of Type IIA superstring theory on the ALE spaces with deformed simply laced singularities. In particular, we have shown that the computation of the filling factors can be converted to a task of studying representation theory of Kac-Moody Lie algebras. For some concrete models involving intersecting D4-branes on two-spheres arranged as hyperbolic Dynkin diagrams with two nodes, we have obtained values for the filling factors which coincide with several subsequences of known series used in graphene. Concerning the finite quiver gauge models, we have found several QHS with integer, half-integer and fractional filling factors (with odd-denominators). In particular, one may recover the celebrated Jain’s series from the fundamental representation of \( A_{2n} \) algebra.

Our work opens up for further studies. One interesting problem is to complete the analysis by considering non abelian quiver gauge theories. It would therefore be of great interest to try to extract information on the exact connection between the graphene and hyperbolic sector of the Kac-Moody Lie algebras. We believe that this observation deserves to be studied further. We hope to report elsewhere on these opens questions.

**Acknowledgments.** This work is supported by the program Protars III D12/25. AS is supported by CICYT (grant FPA-2006-02315 and grant FPA-2009-09638) and DGIID-DGA (grant 2007-E24/2). We thank also the support by grant A/9335/07 and A/024174/09. AB would like to thank M. Asorey for discussions and scientific help and also Departamento de Fisica Teórica, Universidad de Zaragoza for kind hospitality.

**References**

[1] B. A. Bernevig, J. Brodie, L. Susskind and N. Toumbas, *How Bob Laughlin Tamed the Giant Graviton from Taub-NUT space*, JHEP **0102**(2001)003, hep-th/0010105.

[2] I. Bena, A. Nudelman, *On the stability of the Quantum Hall soliton*, JHEP **0012**(2000)017, hep-th/0011155.

[3] S. Hellerman and L. Susskind, *Realizing the quantum Hall system in string theory*, hep-th/0107200.
[4] O. Bergman, Y. Okawa and J. H. Brodie, *The stringy quantum Hall fluid*, JHEP 0111 (2001) 019, hep-th/0107178.

[5] M. Fabinger, *Higher-Dimensional Quantum Hall Effect in String Theory*, JHEP 0205 (2002) 037, hep-th/0201016.

[6] O. Bergman, *Quantum Hall Physics in String Theory*, hep-th/0401106.

[7] J. L. Davis, P. Kraus and A. Shah, *Gravity Dual of a Quantum Hall Plateau Transition*, JHEP 0811 (2008) 020, arXiv:0809.1876 [hep-th].

[8] S. A. Hartnoll and P. Kovtun, *Hall conductivity from dyonic black holes*, Phys. Rev. D76 (2007) 066001, arXiv:0704.1160 [hep-th].

[9] A. El Rhalami, E.H. Saidi, *NC effective gauge model for multilayer FQH states*, JHEP 0210(2002)039, hep-th/0208144.

[10] A. El Rhalami, E.M. Sahraoui, E. H. Saidi, *NC Branes and Hierarchies in Quantum Hall Fluids*, JHEP 0205 (2002) 004, hep-th/0108096.

[11] R. Abounasr, M. Ait Ben Haddou, A. El Rhalami, E.H. Saidi, *Algebraic geometry realization of quantum Hall soliton*, J. Math. Phys. 46(2005)022302, hep-th/0406036.

[12] M. Fujita, W. Li, S. Ryu, T. Takayanagi, *Fractional Quantum Hall Effect via Holography: Chern-Simons, Edge States, and Hierarchy*, arXiv:0901.0924[hep-th].

[13] D. Bak, S. J. Rey, *Composite Fermion Metals from Dyon Black Holes and S-Duality*, arXiv:0912.0939[hep-th].

[14] S. J. Rey, *String Theory on Thin Semiconductors: Holographic Realization of Fermi Points and Surfaces*, Prog. of Theo. Phy. [Supp.] 177(2009)128-142, arXiv:0911.5295[hep-th].

[15] A. Belhaj, A. Segui, *Engineering of Quantum Hall Effect from Type IIA String Theory on The K3 Surface*, Phys. Lett. B691 (2010)261-267, arXiv:1002.2067[hep-th].

[16] A. Hanany, E. Witten, *Type IIB Superstrings, BPS Monopoles, And Three-Dimensional Gauge Dynamics*, Nucl. Phys. B492 (1997) 152, hep-th/9611230.

[17] L. B. Drissi, E. H. Saidi, M. Bousmina, *Electronic properties and hidden symmetries of graphene*, Nucl. Phys. B829, (2010)523-533.

[18] S. Katz, P. Mayr, C. Vafa, *Mirror symmetry and exact solution of 4d N = 2 gauge theories I*, Adv. Theor. Math. Phys. 1 (1998) 53, hep-th/9706110.
[19] V. G. Kac, *Infinite dimensional Lie algebras*, third edition, Cambridge University Press (1990).

[20] J-S. Huang, *Lectures on representation theory*, World scientific 1999.

[21] M. Ait Ben Haddou, A. Belhaj, E.H. Saidi, *Classification of N=2 supersymmetric CFT(4)s: Indefinite series*, J. Phys. A*38*(2005)1793-1806, hep-th/0308005.

[22] M. Ait Ben Haddou, A. Belhaj, E.H. Saidi, *Geometric Engineering of N=2 CFT4’s based on Indefinite Singularities: Hyperbolic Case*, Nucl. Phys. B*674* (2003) 593-614, hep-th/0307244.

[23] X.G. Wen, A. Zee, *Classification of Abelian quantum Hall states and matrix formulation of topological fluids*, Phys. Rev. B*46* (1992)2290-2301.

[24] X-G. Wen, *Quantum Field Theory of Many-body Systems*, Oxford University Press, 2004.

[25] R. Slansky, *Group theory for unified theory model building*, Physics report 79 No. 1 (1981)1-128

[26] A. Belhaj, N-E. Fahssi, E.H. Saidi, A. Segui, *Embedding Fractional Quantum Hall Solitons in M-theory Compactifications*, arXiv:1007.4485[hep-th].

[27] R. B. Laughlin, *Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations*, Phys. Rev. Lett. *50*(1983)1395.

[28] B. I. Halperin, Helv. Phys. Acta *56* (1983)75.

[29] Z. Papica, M. O. Goerbiga, N. Regnault, *Theoretical expectations for a fractional quantum Hall effect in graphene*, arXiv:0902.3233.

[30] V. Singh, M. M. Deshmukh, *Non-equilibrium breakdown of quantum Hall state in graphene*, Phys. Rev. B *80*(2009) 081404, arXiv:0908.0074[cond-mat].