The $f$-divergence of a von Mises-Fisher distribution from some reference distributions

Toru Kitagawa† Jeff Rowley‡

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Abstract

The von Mises-Fisher family is a parametric family of distributions on the surface of the unit ball, summarised by a concentration parameter and a mean direction. As a quasi-Bayesian prior, a von Mises-Fisher distribution is a convenient and parsimonious choice when parameter spaces are isomorphic to the hypersphere (e.g., maximum score estimation in semi-parametric discrete choice, estimation of single-index treatment assignment rules via empirical welfare maximisation, under-identifying linear simultaneous equation models). Despite a long history of application, measures of statistical divergence have not been analytically characterised for von Mises-Fisher distributions. This paper provides analytical expressions for the $f$-divergence of a von Mises-Fisher distribution from another, distinct, von Mises-Fisher distribution in $\mathbb{R}^p$ and from the uniform distribution on the hypersphere. This paper also collects several other results pertaining to the von Mises-Fisher family of distributions, and characterises the limiting behaviour of the measures of divergence that we consider.

Keywords: $f$-divergence, von Mises-Fisher, directional statistics.

1 The von Mises-Fisher family of distributions

The von Mises-Fisher family of distributions is well-known in the field of directional statistics but is foreign to economics and, as such, warrants some introduction. Also known as the Langevin family [Watson, 1984], the von Mises-Fisher family recognises those two titans of statistics, Sir Ronald Fisher and Richard von Mises, for their seminal contributions in considering Gaussianity on the circle [von Mises, 1918] and on the sphere [Fisher, 1953]. Subsequent work has generalised the von Mises-Fisher family to $\mathbb{R}^p$, and has led to the definition of other related parametric distributions such as the Bingham family [Bingham, 1974] and the Fisher-Bingham or Kent family [Kent, 1982].

A von Mises-Fisher distribution assigns probability mass to the surface of the unit ball – the hypersphere. As such, the von Mises-Fisher family is relevant to situations where the researcher...
is interested in either the sampling of directional vectors – i.e., vectors of unit length – or in the clustering of some phenomenon on a circular object, such as occurs if data is periodic. Applications range from the study of sea turtle navigation (Hillen et al., 2017), to the study of perihelia of long-tailed comets (Mardia, 1975) and near-earth objects (Sei et al., 2013), as well as to the study of patient arrival data (Mardia, 1975). Sabelfeld (2018) even links the von Mises-Fisher family to the solving of high-dimensional diffusion-advection-reaction equations. The von Mises-Fisher family is a two-parameter family, summarised by a concentration parameter (or, simply, concentration), which we denote by $\kappa > 0$, and a mean direction, which we denote by $\mu \in \mathbb{R}^p$ and which is of unit length.

The main contribution of this paper is to provide analytical expressions for the $f$-divergence of a von Mises-Fisher distribution from two relevant reference distributions given several common choices of function. We study the broad class of Rényi divergence of simple order as well as several other measures of (statistical) divergence that relate to the Rényi class – the $\chi^2$-square distance, the squared-Hellinger distance and the Kullback-Leibler divergence. Each is, of course, a measure of the difference between two probability distributions. Several well-known inequalities relate these measures to the total variation distance (Bretagnolle and Huber, 1978; Pinsker, 1964), which is often of interest. The reference distributions that we specify are another, distinct, von Mises-Fisher distribution in $\mathbb{R}^p$ and the uniform distribution on the hypersphere. We are unaware of such expressions being available elsewhere. Alongside these expressions, we characterise how the various measures of divergence that we consider change as a von Mises-Fisher distribution becomes increasingly concentrated and degenerate. In particular, we provide asymptotic expansions that complement results given in Kitagawa et al. (2022), which utilise Hankel expansions and which also rely on results in Amos (1974). These asymptotic expansions offer analytically tractable polynomial approximations of each of the measures of divergence that we consider in terms of the concentration parameter, with these approximations accurate when this parameter takes large values.

Obtaining analytical expressions of statistical divergence is useful for implementing minimum distance-type or penalised estimation methods, and also for characterising the statistical performance of these procedures. See, for instance, Kitagawa et al. (2022), which builds upon the analytical expression of the Kullback-Leibler divergence that is derived in this paper to estimate the randomised treatment assignment rule that minimises a penalised empirical welfare criterion.

An intention of this paper is to provide an extensive list of analytical expressions for the moments and other known distributional features of the von Mises-Fisher family. With this objective in mind, we collect several results pertaining to the von Mises-Fisher family that are, to varying extents, available elsewhere. These include statement of the first two moments of a von Mises-Fisher distribution (available in or adaptable from Mardia and Jupp, 2009, §9.3 and §9.6, respectively) and its associated Fisher information matrix (Hornik and Grün, 2013). Dhillon and Sra (2003), Hillen et al. (2017), Hornik and Grün (2013) demonstrate three distinct approaches to obtaining expressions for these moments. These are integration by substitution following transformation to spherical coordinates, application of the divergence theorem and differentiation of the moment-generating function, respectively. We use several of the results in these papers directly, whilst others are included for reference only. In particular, knowledge of the first moment is essential to characterising the divergence of a von Mises-Fisher distribution from our chosen reference distributions.

The divergence theorem relates the area of a surface integral to a volume integral.
Directional objects are common in economics, and the von Mises-Fisher family of distributions is relevant to many environments and several methods. For instance, consider the canonical binary choice model with latent random utility. The rational choice of the individual, which we denote by $D_i \in \{0, 1\}$, is determined according to the linear index equation

$$D_i = 1(X'_i\beta - U_i \geq 0),$$  \hspace{1cm} (1)$$

where $X_i \in \mathbb{R}^p$ and $U_i \in \mathbb{R}^p$ denote the individual’s observable characteristics (including an intercept) and latent heterogeneity, respectively. The conditional zero-median restriction inherent in the semiparametric maximum score approach of [Manski (1975), 1985], which translates here as $\text{median}(U_i|X_i) = 0$, does not identify the scale of the utility coefficients $\beta \in \mathbb{R}^p$. It is common to normalise the parameter space of $\beta$ to the collection of vectors satisfying $\|\beta\|_2 = 1$ – i.e., to the hypersphere defined by the collection of vectors with unit Euclidean length. Similarly, in the context of statistical treatment choice ([Manski, 2004], [Kitagawa and Tetenov, 2018]) considers individualised treatment assignment rules based upon a linear index,

$$D_i = 1(X'_i\beta \geq 0),$$  \hspace{1cm} (2)$$

where $X'_i\beta \in \mathbb{R}$ is an eligibility score that aggregates the individual’s observable characteristics and determines whether she should be assigned to treatment – i.e., if $X_i \in \mathbb{R}^p$ maps to $D_i = 1$ – or to non-treatment – i.e., if $X_i \in \mathbb{R}^p$ maps to $D_i = 0$. Such assignment rules are invariant to multiplication of the eligibility score by a positive constant and can be uniquely indexed by a parameter vector on the hypersphere.

Optimising maximum score or an empirical welfare criterion is difficult, however, and complicates estimation of and inference on $\beta$. This motivates a quasi-Bayesian approach as considered in [Chernozhukov and Hong, 2003]. The von Mises-Fisher family offers a parsimonious and convenient prior specification for $\beta$ within the quasi-Bayesian framework, with prior elicitation facilitated by knowledge of the moments of the distribution. In a related but different context, PAC-Bayesian analysis, which is widely studied in machine learning ([Alquier et al., 2016], [Catoni, 2007], [Germain et al., 2009], [McAllester, 2003], to name but a few relevant papers), considers exponentiated negative empirical risk as a quasi-likelihood, and forms a posterior distribution over prediction rules. The von Mises-Fisher family is then not only useful as a specified prior over directional parameters, but can also be used to approximate a posterior distribution for $\beta$ in the linear classification rule or linear index treatment assignment rule settings ([Kitagawa et al., 2022]).

Another context where the parameter space is isomorphic to the hypersphere is the class of underidentifying linear simultaneous equation models in which the imposed model restrictions identify structural parameters up to sets of orthonormal transformations. See, for instance, [Arias et al., 2018], [Giacomini and Kitagawa, 2021], [Uhlig, 2005] for a class of set-identified structural vector autoregressions in which the identified set of an impulse-response is spanned by the class of orthonormal matrices. The isomorphism of the hypersphere and the orthogonal group suggests that a spherical distribution – and a von Mises-Fisher distribution in particular – can be used as a prior distribution for the non-identified orthonormal matrices. When combined with prior elicitation of the reduced-form parameters, the moments of a von Mises-Fisher distribution can be used to translate a belief about the structural parameters into a prior distribution over the non-identified orthonormal matrices. Like the Gaussian family of distributions on the hyperplane from which it can be derived, the von Mises-Fisher family is highly restrictive \(\text{but, nonetheless,}\)

\[\text{The von Mises-Fisher family is akin to the class of Gaussian distributions that feature a diagonal variance matrix – i.e., statistically independent Gaussian random variables – with constant entries on the diagonal.}\]
forms an interesting baseline case to study.

We are unaware of any paper that characterises the $f$-divergence of a von Mises-Fisher distribution as we do. For instance, Diethe (2015) similarly studies the Kullback-Leibler divergence of von Mises-Fisher distributions. Whereas we provide exact analytical expressions, Diethe (2015) either provides upper bounds on the Kullback-Leibler divergence, or else provides analytical expressions that rely on an approximation that is valid only when the von Mises-Fisher distribution is close to the uniform distribution over the hypersphere, something that we do not rely on. Where updating of the von Mises-Fisher distribution has been considered, this appears to have mainly centred on the likelihood function and its characterisation rather than on measures of divergence per se. We refer to Lin et al. (2017), Mardia and El-Atoum (1976) as pertinent examples.

A von Mises-Fisher distribution constitutes a conjugate prior (Mardia and El-Atoum, 1976). Our choice of reference distributions – another, distinct, von Mises-Fisher distribution in $\mathbb{R}^p$ and the uniform distribution on the hypersphere – reflects both this and the prevalence of the uniform distribution in practice. Another commonly invoked choice that we do not consider is the Jeffreys prior, which is proportional to the square root of the determinant of the Fisher information matrix relative to the parametrisation employed. Hornik and Grün (2013) derive the Fisher information matrix and its determinant and show that the Jeffreys prior is improper in this setting.

2 The probability density function and moments of the von Mises-Fisher family

Throughout this paper, we exploit the fact that the von Mises-Fisher family is an exponential family and, accordingly, we adopt the terminology that is used in conjunction with that well-known class. Defining $\nu = p/2 - 1$ for convenience and maintaining $\kappa > 0$ and $\|\mu\|_2 = 1$, we write the probability density function of a von Mises-Fisher random vector, which we denote by $x \in \mathbb{R}^p$, for integer $p > 1$ and satisfying $\|x\|_2 = 1$, as

$$f_p(x; \kappa, \mu) \equiv \frac{\exp(\kappa \mu^\prime x)}{C_\nu(\kappa)}, \quad (3)$$

where we reiterate that $\kappa$ and $\mu$ are the concentration and mean direction, respectively, and where

$$C_\nu(\kappa) \equiv \frac{(2\pi)^{\nu+1} I_\nu(\kappa)}{\kappa^{\nu}} = \int_{S^{p-1}} \exp(\kappa \mu^\prime x) \, dx, \quad (4)$$

thereby guaranteeing that the density function integrates to one. We use $I_\nu(z) : \mathbb{R}^2 \to \mathbb{R}$ to denote the modified Bessel function of the first kind (see Appendix A for further details; hereafter, where we say modified Bessel function, we intend this to mean the modified Bessel function of the first kind), and $S^p$ to denote the $p$-sphere – the collection of unit vectors in $\mathbb{R}^{p+1}$. We refer to the exponentiation – i.e., the numerator in Equation (3) – as the kernel of the density function, and to the normalising constant – i.e., the denominator in Equation (3) – as the partition function. We emphasise that the integral in Equation (4) is over the hypersphere.
and it is this fact that makes derivation of statistical features of the von Mises-Fisher random vector difficult. Moreover, we note that our choice of parametrisation is but one way that a von Mises-Fisher distribution can be parametrised. Another parametrisation that is better suited to certain analyses of von Mises-Fisher distributions (in particular, derivation of their moments) is presented in [Hornik and Grün, 2013].

The von Mises-Fisher family is the hyperspherical analogue of the Gaussian family, which is informative as to its shape. This relationship is shown by appropriately normalising the probability density function of statistically independent Gaussian random variables with variance $1/\kappa$ that are distributed on the hypersphere,

$$
\left( \int_{S^{p-1}} \frac{\kappa^{p+1} \exp \left( -\kappa (\mathbf{x} - \mu)^T (\mathbf{x} - \mu)/2 \right)}{(2\pi)^{\nu + 1}} d\mathbf{x} \right)^{-1} \frac{\kappa^{p+1} \exp \left( -\kappa (\mathbf{x} - \mu)^T (\mathbf{x} - \mu)/2 \right)}{(2\pi)^{\nu + 1}} = f_p(\mathbf{x}; \kappa, \mu).$$

We plot the density function and contours of the cumulative distribution function for several values of the concentration parameter and a common orientation in Figures 1 and 2 for the circular case (when $\mathbb{R}^p = \mathbb{R}^2$) and the spherical case (when $\mathbb{R}^p = \mathbb{R}^3$), respectively. As is to be expected, the von Mises-Fisher family is unimodal and symmetric about its mean direction, with the concentration parameter determining the degeneracy (when $\kappa \to \infty$) and uniformity (when $\kappa \to 0$) of the distribution, and it assigns positive density to the entirety of the hypersphere. Importantly, the von Mises-Fisher family is rotationally invariant, which is the hyperspherical analogue of the location invariance property that is exhibited by the Gaussian family.

The von Mises-Fisher family of distributions coincides with the von Mises and Fisher families in the circular and spherical cases, respectively. It is this coincidence that explains the nomenclature.

As explained in [Mardia and Jupp, 2009], the first moment of the von Mises-Fisher family is in the interior of the unit ball, something which is also true for other named directional distributions – i.e., it is not on the hypersphere. Specifically, the first moment of the von Mises-Fisher family has the form

$$\mathbb{E}(\mathbf{x}) = \rho \mu,$$

where $\rho$ is the (population) mean resultant length and satisfies $0 < \rho < 1$. As such, providing a general characterisation of the first and, to a lesser extent, higher-order moments of the von Mises-Fisher family then amounts to characterising the constant of proportionality, the mean resultant length.

We obtain the (centred) moments of the von Mises-Fisher family by differentiating its moment-generating function. Since the von Mises-Fisher family is an exponential family, the moment-generating function is equal to the log-partition function. Whilst the partition function of the von Mises-Fisher family has a closed-form expression, this is not necessarily true for other named directional families. For instance, we are unaware of any closed-form expressions for the partition functions of both the Kent and von Mises families of distributions beyond the bivariate case.

**Proposition 2.1.** Given $\mathbf{x} \in S^{p-1}$ is distributed as a $p$-variate von Mises-Fisher random vector with concentration $\kappa > 0$ and mean direction $\mu \in S^{p-1}$, and recalling that $\nu = p/2 - 1$, its first

\[89x731]\begin{align*}
\text{Proposition 2.1.} & \quad \text{Given } \mathbf{x} \in S^{p-1} \text{ is distributed as a } p\text{-variate von Mises-Fisher random vector with concentration } \kappa > 0 \text{ and mean direction } \mu \in S^{p-1}, \text{ and recalling that } \nu = p/2 - 1, \text{ its first}
\end{align*}

\[89x731\]
and second moments are

\[ \mathbb{E}(x) = r_\nu(\kappa) \mu, \quad (7) \]

and

\[
\text{Variance}(x) = r_\nu(\kappa) \left( \frac{1}{\kappa} I_p + (r_{\nu+1}(\kappa) - r_\nu(\kappa)) \mu \mu' \right), \quad (8)
\]

\[
= \frac{1}{\kappa} r_\nu(\kappa) I_p + \left( 1 - \frac{p}{\kappa} r_\nu(\kappa) - r_\nu^2(\kappa) \right) \mu \mu', \quad (9)
\]

respectively, where, for comparability with Amos (1974),

\[
r_\nu(\kappa) = I_{\nu+1}(\kappa) / I_\nu(\kappa), \quad (10)
\]

which is the ratio of consecutive modified Bessel functions.

Although we defer proof to Appendix B, we emphasise that Proposition 2.1 is adapted from results available elsewhere (Mardia and Jupp, 2009). We also note that Mardia and Jupp (2009) discuss several other important results aside from the first and second moments, including the asymptotic and high-concentration behaviour of von Mises-Fisher random vectors and their tangent normal vectors. We emphasise that the expression for the variance that we present here is distinct from the circular variance, which is simply the distance of the mean resultant length from the surface – i.e., one minus the mean resultant length. Moreover, and as noted in Hornik and Grün (2013), given that the von Mises-Fisher family is an exponential family, its second moment coincides with its Fisher information matrix under the parametrisation considered in that paper.

Although we do not present expressions for the higher-order moments of the von Mises-Fisher family, these can be obtained via iterative differentiation. We present several results in Appendix B that factor in such a process, and which are central to proving Proposition 2.1. These expressions and the modified Bessel functions that underpin them can be implemented in several common statistical programming languages for small to moderately large values of the concentration parameter, with recursive expressions appearing in Amos (1974) facilitating implementation for larger values.

### 3 Measuring the f-divergence of an obtained distribution from a reference distribution

The f-divergence measures the divergence of one probability distribution from another. In what follows, we suppose that

\[ y \sim_{f_p} (y; \kappa_y, \mu_y), \quad (11) \]

\[ z \sim_{f_p} (z; \kappa_z, \mu_z), \]

...
such that \( y \in S^{p-1} \) and \( z \in S^{p-1} \) are two von Mises-Fisher random vectors satisfying all of the usual properties that we maintain. We use the notation above to distinguish the two von Mises-Fisher random vectors from the \( \kappa \)-concentrated \( \mu \)-oriented random vector \( x \) that is discussed in the previous sections.

We adopt the convention of referring to the probability distribution of the random vector \( y \) as the \textit{obtained distribution} and to the probability distribution of the random vector \( z \) as the \textit{reference distribution}, and of measuring the divergence of the \textit{obtained distribution} from the \textit{reference distribution}.

We reiterate that the uniform distribution on the hypersphere corresponds to the limiting case where the concentration of the von Mises-Fisher distribution is zero. As such, we emphasise that our framework is also compatible with the reference distribution being equal to the uniform distribution on the hypersphere.

For all of the measures that we consider, each measure approaches its maximum possible value when the obtained distribution is degenerate. We use Bachmann-Landau notation (otherwise known as \( \text{Big O notation} \) to characterise the limiting behaviour of these measures with respect to the concentration. We defer proof of each proposition in this section to Appendix B.

### 3.1 R\"enyi divergence

Following van Erven and Harremo"es (2014), we define the R\"enyi divergence of simple order \( \alpha \in (0, 1) \cup (1, \infty) \) as

\[
d_\alpha (y, z) = \frac{\nu}{\alpha - 1} \ln \left( \frac{1}{\Gamma \left( \nu \right)} I_\nu \left( \kappa_0 \right) \frac{1}{\Gamma \left( \nu \right)} I_\nu \left( \kappa_1 \right) \frac{1}{\Gamma \left( \nu \right)} I_\nu \left( \kappa_2 \right) \right).
\]

The R\"enyi divergence is the most general measure of \( f \)-divergence that we consider and describes a broad class that relates to the other measures that we study – namely, the \( \chi \)-squared distance, squared-Hellinger distance and Kullback-Leibler divergence.

**Proposition 3.1.** Given \( y \in S^{p-1} \) and \( z \in S^{p-1} \) are distributed as \( p \)-variate von Mises-Fisher random vectors with concentrations \( \kappa_y > 0 \) and \( \kappa_z > 0 \), respectively, and mean directions \( \mu_y \in S^{p-1} \) and \( \mu_z \in S^{p-1} \), respectively, and recalling that \( \nu = p/2 - 1 \), the R\"enyi divergence of simple order \( \alpha \in (0, 1) \cup (1, \infty) \) is

\[
d_\alpha (y, z) = \frac{\nu}{\alpha - 1} \ln \left( \frac{\kappa_0^{\nu_2 - \alpha}}{\kappa_2^{\alpha - \nu_2}} \right) + \frac{\alpha}{\alpha - 1} \ln \left( \frac{I_\nu (\kappa_0)}{I_\nu (\kappa_y)} \right) - \ln \left( \frac{1}{\Gamma \left( \nu \right)} I_\nu (\kappa_1) \right),
\]

where

\[
\kappa_\alpha = \left\| \alpha \kappa_y \mu_y + (1 - \alpha) \kappa_z \mu_z \right\|_2.
\]

In the special case where \( \kappa_0 = 0 \),

\[
d_\alpha (y, z) = \frac{\nu}{\alpha - 1} \ln \left( \frac{\kappa_0^{\nu_2 - \alpha}}{2} \right) + \frac{\alpha}{\alpha - 1} \ln \left( \frac{1/\Gamma \left( \nu + 1 \right)}{I_\nu \left( \kappa_0 \right)} \right) - \ln \left( \frac{1/\Gamma \left( \nu + 1 \right)}{I_\nu \left( \kappa_1 \right)} \right).
\]

7A necessary condition that we require is that the probability distribution of the random vector \( y \) is absolutely continuous with respect to the probability distribution of the random vector \( z \) (Lattimore and Szepesvári, 2020, §2.7 and §14.5), thereby guaranteeing that each of the measures that we consider is well-defined. The von Mises-Fisher family clearly satisfies this requirement, assigning positive density to all points on the hypersphere.
with
\[
\kappa_z = \frac{\alpha}{1 - \alpha} \kappa_y \quad \text{and} \quad \mu_z + \text{Sign} \left( 1 - \alpha \right) \mu_y = 0. \tag{16}
\]

In the special case where \( z \) is, instead, uniformly distributed on the hypersphere,
\[
d_\alpha (y, z) = \frac{\nu}{\alpha - 1} \ln \left( \frac{2^{1 - \alpha}}{\alpha \kappa_y^{1 - \alpha}} \right) + \frac{\alpha}{\alpha - 1} \ln \left( \frac{I_\nu (\alpha \kappa_y)}{I_\nu (\kappa_y)} \right) - \ln \left( \frac{I_\nu (\alpha \kappa_y)}{1 / \Gamma (\nu + 1)} \right). \tag{17}
\]
The Rényi divergence is an \( O (\ln (\kappa_y)) \) function.

### 3.2 \( \chi \)-square distance

We define the \( \chi \)-square distance as
\[
d_\chi (y, z) = \int_{S^{p-1}} \frac{\left( f_p (y; \kappa_y, \mu_y) - f_p (z; \kappa_z, \mu_z) \right)^2}{f_p (z; \kappa_z, \mu_z)} \, dx,
\]
\[
d_\chi (y, z) = \int_{S^{p-1}} f_p (y; \kappa_y, \mu_y)^2 \, dx - 1, \tag{19}
\]
with Equation (19) being the more convenient definition to work with. We observe that the \( \chi \)-square distance relates to the Rényi class of measures via the equality relation
\[
d_\alpha (y, z) |_{\alpha=2} = \ln \left( 1 + d_\chi (y, z) \right), \tag{20}
\]
as per van Erven and Harremoës (2014), which serves to illustrate the breadth of the Rényi class.

**Proposition 3.2.** Given \( y \in S^{p-1} \) and \( z \in S^{p-1} \) are distributed as \( p \)-variate von Mises-Fisher random vectors with concentrations \( \kappa_y > 0 \) and \( \kappa_z > 0 \), respectively, and mean directions \( \mu_y \in S^{p-1} \) and \( \mu_z \in S^{p-1} \), respectively, and recalling that \( \nu = p/2 - 1 \), the \( \chi \)-square distance is
\[
d_\chi (y, z) = \frac{\kappa_y^{2 \nu} I_\nu (\kappa_y) I_\nu (\kappa_z)}{\kappa_y^{\nu} \kappa_z^{\nu} I_\nu (\kappa_y)^2} - 1, \tag{21}
\]
where
\[
\kappa_\chi = \left\| 2 \kappa_y \mu_y - \kappa_z \mu_z \right\|_2. \tag{22}
\]

In the special case where \( \kappa_\chi = 0 \),
\[
d_\chi (y, z) = \frac{\kappa_y^{2 \nu} I_\nu (2 \kappa_y)}{1 / \Gamma (\nu + 1) I_\nu (\kappa_y)^2} - 1, \tag{23}
\]
with
\[
\kappa_z = 2 \kappa_y \quad \text{and} \quad \mu_z - \mu_y = 0. \tag{24}
\]
In the special case where \( z \) is, instead, uniformly distributed on the hypersphere, the \( \chi \)-square distance coincides with Equation (23). The \( \chi \)-square distance is an \( O(\kappa_y) \) function.

### 3.3 Squared-Hellinger distance

We define the squared-Hellinger distance as

\[
d_h(y, z)^2 = \int_{S^{p-1}} \left( \sqrt{f_p(x; \kappa_y, \mu_y)} - \sqrt{f_p(x; \kappa_z, \mu_z)} \right)^2 \, dx,
\]

\[
= 2 \left( 1 - \int_{S^{p-1}} \sqrt{f_p(x; \kappa_y, \mu_y)} \, df_p(x; \kappa_z, \mu_z) \right),
\]

with Equation (26) being the more convenient definition to work with. We observe that the squared-Hellinger distance relates to the Rényi class of measures via the equality relation

\[
d_{1/2} (y, z) = 2 \ln \left( \frac{1 - d_h(y, z)^2}{2} \right),
\]

as per van Erven and Harremoës (2014), which is also interpretable as twice the negative logarithm of the Bhattacharyya coefficient. The Bhattacharyya coefficient is an approximate measure of the amount of overlap between two probability distributions, such that when the obtained and reference distributions are close (i.e., they have a similar concentration and mean direction) then the squared-Hellinger distance is small in absolute value, as is to be expected.

**Proposition 3.3.** Given \( y \in S^{p-1} \) and \( z \in S^{p-1} \) are distributed as \( p \)-variate von Mises-Fisher random vectors with concentrations \( \kappa_y > 0 \) and \( \kappa_z > 0 \), respectively, and mean directions \( \mu_y \in S^{p-1} \) and \( \mu_z \in S^{p-1} \), respectively, and recalling that \( \nu = p/2 - 1 \), the squared-Hellinger distance is

\[
d_h(y, z)^2 = 2 \left( 1 - \sqrt{\frac{\kappa_y \kappa_z I_{\nu}^2 (\kappa_h)}{\kappa_h I_{\nu} (\kappa_y) I_{\nu} (\kappa_z)}} \right),
\]

where we define

\[
\kappa_h = \frac{1}{2} \left\| \kappa_y \mu_y + \kappa_z \mu_z \right\|_2^2.
\]

In the special case where \( \kappa_h = 0 \),

\[
d_h(y, z)^2 = 2 \left( 1 - \frac{\kappa_y^\nu}{2^\nu I_{\nu} (\kappa_y) \Gamma (\nu + 1)} \right),
\]

with

\[
\kappa_z = \kappa_y \text{ and } \mu_z + \mu_y = 0.
\]
In the special case where \( z \) is, instead, uniformly distributed on the hypersphere,

\[
d_h(y, z)^2 = 2 \left( 1 - \sqrt{\frac{2^{\nu} I_\nu^2 (\kappa_y/2) \Gamma (\nu + 1)}{\kappa_y I_\nu (\kappa_y)}} \right). 
\]

(32)

The squared-Hellinger distance is an \( O (1 - 1/\kappa_y) \) function.

### 3.4 Kullback-Leibler divergence

We define the Kullback-Leibler divergence as

\[
d_\ell (y, z) \doteq \int_{S^p} \ln \left( \frac{f_p (x; \kappa_y, \mu_y)}{f_p (x; \kappa_z, \mu_z)} \right) f_p (x; \kappa_y, \mu_y) \, dx. 
\]

(33)

We observe that the Kullback-Leibler divergence is a limiting case of the Rényi divergence. That is,

\[
\lim_{\alpha \to 1} d_\alpha (y, z) = d_\ell (y, z), 
\]

(34)

as per van Erven and Harremoës (2014).

**Proposition 3.4.** Given \( y \in S^{p-1} \) and \( z \in S^{p-1} \) are distributed as \( p \)-variate von Mises-Fisher random vectors with concentrations \( \kappa_y > 0 \) and \( \kappa_z > 0 \), respectively, and mean directions \( \mu_y \in S^{p-1} \) and \( \mu_z \in S^{p-1} \), respectively, and recalling that \( \nu = p/2 - 1 \), the Kullback-Leibler divergence is

\[
d_\ell (y, z) = \nu \ln \left( \frac{\kappa_y}{\kappa_z} \right) - \ln \left( I_\nu (\kappa_y) \right) + r_\nu (\kappa_y) (\kappa_y \mu_y - \kappa_z \mu_z)' \mu_y. 
\]

(35)

In the special case where \( z \) is, instead, uniformly distributed on the hypersphere,

\[
d_\ell (y, z) = \nu \ln \left( \frac{\kappa_y}{2} \right) - \ln \left( I_\nu (\kappa_y) \right) - \ln (\Gamma (\nu + 1)) + r_\nu (\kappa_y) \kappa_y. 
\]

(36)

The Kullback-Leibler divergence is an \( O (\ln (\kappa_y)) \) function.

We note that the final terms of Equations (35) and (36) are proportional to the first moment of a von Mises-Fisher distribution (specifically, the first moment of the obtained distribution). In particular, we note that the final term of Equation (35) can otherwise be written as

\[
r_\nu (\kappa_y) (\kappa_y \mu_y - \kappa_z \mu_z)' \mu_y = r_\nu (\kappa_y) (\kappa_y - \kappa_z \mu_z)' \mu_y, 
\]

(37)

which we contrast with the corresponding term in Equation (36). The difference between Equations (36) and (37) arises because the mean direction does not enter the probability density function of the uniform distribution on the hypersphere. The intuition here is that, in the special case where the reference distribution is the uniform distribution on the hypersphere, the mean direction can always be taken to be equal to the mean direction of the obtained distribution such that the information gain is interpretable as learning about the concentration only. In the more general case where the reference distribution is another, distinct, von Mises-Fisher
distribution, the information gain is interpretable as learning about the concentration and the mean direction depending upon their respective values.

3.5 Total variation distance
We define the total variation distance, for all measurable $A \subset \mathbb{S}^{p-1}$, as

$$d_t (y, z) = \sup_{A} \left| \int_{A} f_p (x; \kappa_y, \mu_y) - f_p (x; \kappa_z, \mu_z) \, dx \right|,$$  \hspace{1cm} (38)

which is well-defined if the underlying probability space is endowed with the Borel $\sigma$-algebra – something that we, implicitly, maintain throughout all parts of our analysis.

Corollary 3.1. Given $y \in \mathbb{S}^{p-1}$ and $z \in \mathbb{S}^{p-1}$ are distributed as $p$-variate von Mises-Fisher random vectors with concentrations $\kappa_y > 0$ and $\kappa_z > 0$, respectively, and mean directions $\mu_y \in \mathbb{S}^{p-1}$ and $\mu_z \in \mathbb{S}^{p-1}$, respectively, and recalling that $\nu = p/2 - 1$, the total variation distance satisfies the inequality relations

$$d_t (y, z)^2 \leq d_h (y, z)^2 \leq d_\ell (y, z)^2 \leq d_\chi (y, z)^2,$$  \hspace{1cm} (39)

and, for all orders $\alpha \in (0, 1]$,

$$d_t (y, z)^2 \leq \frac{\alpha}{2} d_\alpha (y, z),$$  \hspace{1cm} (40)

where each measure of divergence is as defined in Section 3.

The inequalities in Corollary 3.1 are well-known (see, for instance, Lattimore and Szepesvári, 2020, §14.3 and references therein, and van Erven and Harremoës, 2014). The total variation distance is difficult to characterise in this setting due to the need to integrate over a region of the hypersphere, which is non-trivial. The inequality relations stated in Corollary 3.1 provide a means to bound the total variation distance from above using measures that are more easily characterised. In particular, Equation (41) is a generalisation of Pinsker’s inequality, yielding the classic statement of that inequality at the limit (see Equation (34)).

4 Limiting behaviour of the modified Bessel function and its ratios
We now discuss the limiting behaviour of the modified Bessel function and its ratios as a von Mises-Fisher distribution becomes increasingly concentrated around its mean direction. Such discussion is useful, in particular, for understanding how the various measures of divergence that we consider respond to changes – specifically, an increase – in the value of the concentration parameter.

We begin by referring to Amos (1974) and introducing several results that are stated therein, adapting the notation of that paper to suit our purposes. Firstly, Amos (1974) shows that, for all $\kappa_y \geq \kappa_z > 0$,

$$L_\nu (\kappa_y, \kappa_z) \leq \ln (I_\nu (\kappa_y)) \leq U_\nu (\kappa_y, \kappa_z),$$  \hspace{1cm} (42)
where

\[
L_\nu (\kappa_y, \kappa_z) \doteq \ln (I_\nu (\kappa_z)) + \nu \ln \left( \frac{\kappa_y}{\kappa_z} \right) + \xi_\nu \ln \left( \frac{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}}{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}} \right) + \frac{\kappa_y^2 - \kappa_z^2}{\sqrt{\kappa_y^2 + \xi_\nu^2} + \sqrt{\kappa_z^2 + \xi_\nu^2}},
\]

(43)

and

\[
U_\nu (\kappa_y, \kappa_z) = \ln (I_\nu (\kappa_z)) + \nu \ln \left( \frac{\kappa_y}{\kappa_z} \right) + \xi_\nu \ln \left( \frac{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}}{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}} \right) + \frac{\kappa_y^2 - \kappa_z^2}{\sqrt{\kappa_y^2 + \xi_\nu^2} + \sqrt{\kappa_z^2 + \xi_\nu^2}},
\]

(44)

given \( \xi_\nu \doteq \nu + 1/2 \) and \( \tau_\nu \doteq \nu + 3/2 \). We note that Equation (42) is reversed when the concentrations of the obtained and reference distributions instead satisfies \( \kappa_z > \kappa_y \). In the special case where \( \kappa_z \to 0 \), which we reiterate corresponds to the case where the reference distribution is the uniform distribution on the hypersphere, Amos (1974) shows that Equations (43) and (44) reduce to

\[
L_\nu (\kappa_y, 0) = \frac{1}{2} \ln \left( \frac{2}{\kappa_y} \right) - \ln (\Gamma (\nu + 1)) + \xi_\nu \ln \left( \frac{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}}{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}} \right) + \frac{\kappa_y^2 - 1}{\sqrt{\kappa_y^2 + \xi_\nu^2} + \sqrt{\kappa_y^2 + \xi_\nu^2}},
\]

(45)

and

\[
U_\nu (\kappa_y, 0) = \frac{1}{2} \ln \left( \frac{2}{\kappa_y} \right) - \ln (\Gamma (\nu + 1)) + \xi_\nu \ln \left( \frac{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}}{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}} \right) + \frac{\kappa_y^2 - 1}{\sqrt{\kappa_y^2 + \xi_\nu^2} + \sqrt{\kappa_y^2 + \xi_\nu^2}},
\]

(46)

respectively. Secondly, Amos (1974) shows that\n
\[
\frac{\kappa_y}{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}} \leq \tau_\nu (\kappa_y) \leq \frac{\kappa_y}{\xi_\nu + \sqrt{\kappa_y^2 + \xi_\nu^2}},
\]

(47)

where we again rely on the preceding definitions. This inequality defines a subset of the unit interval.

**Corollary 4.1.** Given that \( \kappa > 0 \), then for \( L_\nu (\kappa, 0) \) and \( U_\nu (\kappa, 0) \) defined as in Equations (45) and (46) and recalling that \( \nu = p/2 - 1 \),

\[
L_\nu (\kappa, 0) \geq \kappa - \frac{1}{2} \ln (\kappa) - \nu \ln (2) - \ln (\Gamma (\nu + 1)) - \tau_\nu,
\]

(48)

and

\[
U_\nu (\kappa, 0) \leq \kappa - \frac{1}{2} \ln (\kappa) + \frac{1}{2} \ln (2) - \ln (\Gamma (\nu + 1)) + \xi_\nu \ln (\xi_\nu)
\]

(49)

such that \( \ln (I_\nu (\kappa)) \) is an \( O (\kappa - \ln (\kappa)/2) \) function.

Although we defer proof to Appendix B, we emphasise that Corollary 4.1 follows from Equations (45) and (46) and the simple exploitation of the properties of increasing concave functions.
Moreover, that we choose to state that the logarithm of the modified Bessel function is an \( O(\kappa - \ln(\kappa) / 2) \) function rather than a linear function is for practical reasons. Specifically, we rely on Corollary 4.1 to prove many of the statements in Section 3, for which we find that the linear component typically cancels. We observe that Corollary 4.1 aligns with results elsewhere (NIST 2021 §10.30).

One further quantity that is often of interest is the circular variance of the von Mises-Fisher family of distributions which is defined as one minus the ratio of modified Bessel functions – i.e., one minus the mean resultant length. Kitagawa et al. (2022) demonstrates that the circular variance is an \( O(1/\kappa) \) function. We observe that the same result can also be attained via Hankel series expansion (see Equation (A.6) for a definition and Appendix C for a demonstration). We note that Hankel series expansion is appropriate when \( \kappa \to \infty \), which is the limiting behaviour that we are interested in. We emphasise that the circular variance is strictly contained in the unit interval, which is asymptotically guaranteed by the fact that \( p \geq 2 \) – i.e., the minimal hypersphere (the circle) is defined on the real plane.

5 Conclusions

The main contribution of this paper is to provide analytical expressions for the \( f \)-divergence of a von Mises-Fisher distribution from two relevant reference distributions given several common choices of function. These expressions can be input into several common statistical programming languages to form part of an estimation routine – for instance, to optimise maximum score or an empirical welfare criterion. In characterising the limiting behaviour of our chosen measures of divergence, we provide results that are useful for the theoretical development of econometric methods, especially those that involve penalisation.

We suggest that there are several directions for further research. Firstly, the von Mises-Fisher family is highly restrictive. It would be useful to provide similar characterisations for related parametric distributions such as the Bingham family and the Fisher-Bingham or Kent family that are less restrictive and allow for asymmetry and correlation. Secondly, directional objects are common in economics but use of the von Mises-Fisher family and related distributions is extremely rare. We point to several examples where the von Mises-Fisher family would augment or be an attractive choice for existing economic theory, with one example being as a choice of prior in structural vector autoregression.

A Useful functions

The modified Bessel function of the first kind is defined in NIST (2021 §10.25 and §10.32) and, its precursor, Abramowitz and Stegun (1964 §9.6), and we refer the reader to those references for further (detailed) information about this function. The modified Bessel function is defined, for all \( z > 0 \), as

\[
I_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k! \Gamma(\nu + k + 1)} = \frac{(z/2)^\nu}{\sqrt{\pi \Gamma(\nu + 1/2)}} \int_0^\pi \exp(\pm z \cos(\theta)) \sin^{2\nu}(\theta) \, d\theta, \quad (A.1)
\]
or as

\[
I_\nu(z) = \frac{1}{\pi} \left( \int_0^{\pi} \exp(z \cos(\theta)) \cos(\nu \theta) \, d\theta - \sin(\nu \pi) \int_0^{\infty} \exp(-\nu t - z \cosh(t)) \, dt \right), \quad (A.2)
\]

8See Mardia and Jupp 2009 for further, general, details about the circular variance.
where $\nu$ is said to be the order and $z$ is said to be the argument. Various other definitions of the modified Bessel function exist, including as the $g : \mathbb{C} \to \mathbb{R}$ that solves

$$z^2 \frac{d^2 g}{dz^2} + z \frac{dg}{dz} - (z^2 + \nu^2) g = 0,$$

(A.3)

which is a modification of Bessel’s equation. We emphasise that the solutions to the above equation are imaginary Bessel’s equation and the modified Bessel function relate to Laplace’s equation and harmonic functions (that describe the propagation of a wave along a taut string). The modified Bessel function satisfies the recurrence relations (NIST, 2021, §10.29).

$$I_\nu(z) = \frac{z}{2\nu} (I_{\nu-1}(z) - I_{\nu+1}(z)),$$

$$I'_\nu(z) = I_{\nu-1}(z) - \frac{\nu}{z} I_{\nu}(z) = I_{\nu+1}(z) + \frac{\nu}{z} I_{\nu}(z),$$

(A.4)

and the limiting behaviour (NIST, 2021, §10.30)

$$\lim_{z \to 0} \frac{I_\nu(z)}{z^\nu} = \frac{1}{2^n T(n + 1)}.$$

(A.5)

Moreover, the modified Bessel function admits, for all $n \in \mathbb{N}$ as $z \to \infty$, the Poincaré asymptotic expansion (NIST, 2021, §2.1)

$$I_\nu(z) = \exp(z) \sqrt{2\pi z} \sum_{j=0}^{n-1} (-1)^j \frac{a_j(\nu)}{z^j} + O\left(\frac{1}{z^n}\right),$$

(A.6)

where $(a_j(\nu))_{j=0}^{\infty}$ is a sequence of constants satisfying, for all $j \in \mathbb{N}$,

$$a_j(\nu) = \prod_{i=1}^{j} \left(4\nu^2 - (2j - i)^2\right),$$

(A.7)

with $a_0(\nu) = 1$. This sequence of constants is derived from Hankel’s expansion (NIST, 2021, §10.40). The modified Bessel function takes a particularly useful form in certain instances.

### B Proofs

Proof of Proposition 2.2 We reiterate that the von Mises-Fisher family is an exponential family and, as such, its moments can be obtained via differentiation of the log-partition function, with the first moment equal to the first derivative, the second moment equal to the second derivative, and so forth. To facilitate our analysis of the log-partition function and its derivatives, we rewrite the probability density function of the von Mises-Fisher family in terms of a single

---

9Bessel’s equation is generally written with the final term on the left-hand side entering additively. It enters negatively here because we admit purely imaginary solutions. This is what distinguishes a Bessel function (that admits real solutions) from a modified Bessel function (that admits imaginary solutions).
parameter vector. We let
\[
\kappa \doteq \|\eta\|_2, \\
\mu \doteq \|\eta\|_2^{-1} \eta,
\]
where \(\eta \in \mathbb{R}^p\). The log-partition function is equal to
\[
\ln \int_{S^{p-1}} \exp(\kappa \mu' x) \, dx = \ln (C_\nu(\kappa)) = -\nu \ln (\kappa) + (\nu + 1) \ln (2\pi) + \ln (I_\nu(\kappa)).
\]

The moments of the von Mises-Fisher family of distributions are obtained by recursively differentiating this function with respect to the new parameter vector.

We now present some derivatives that are useful for the construction of the first, second and higher-order moments. Firstly,
\[
\frac{d}{d\eta} \kappa = \mu, \\
\frac{d}{d\eta} \mu = \frac{1}{\kappa} (\mathbb{I}_p - \mu \mu').
\]

Secondly, for all \(n \in \mathbb{N}_+\),
\[
\frac{d}{d\eta} \frac{I_{\nu+n}(\kappa)}{I_\nu(\kappa)} = \frac{I_\nu(\kappa) I'_{\nu+n}(\kappa) - I_{\nu+n}(\kappa) I'_\nu(\kappa)}{I_\nu(\kappa)^2} \mu', \\
= \left( \frac{I_{\nu+n+1}(\kappa)}{I_\nu(\kappa)} + \frac{n I_{\nu+n}(\kappa)}{\kappa I_\nu(\kappa)} - \frac{I_{\nu+n}(\kappa) I'_{\nu+1}(\kappa)}{I_\nu(\kappa)} \right) \mu',
\]
which exploits Equation (A.4) and the telescoping property of the ratios. We now differentiate Equation (B.2) (i.e., we take the first derivative of the log-partition function), which yields
\[
\frac{d}{d\eta} \ln (C_\nu(\kappa)) = r_\nu(\kappa) \mu,
\]
where we use Equation (B.3) and Equation (A.4). This proves the first result of the corollary. We then differentiate Equation (B.5) (i.e., we take the second derivative of the log-partition function), which yields
\[
\frac{d}{d\eta} r_\nu(\kappa) \mu = r_\nu(\kappa) \left( \frac{1}{\kappa} I_p + (r_{\nu+1}(\kappa) - r_\nu(\kappa)) \mu \mu' \right),
\]
where we use Equations (A.4), (B.3) and (B.4). Substituting
\[ r_\nu (\kappa) r_{\nu+1} (\kappa) = 1 - \frac{p}{\kappa} r_\nu (\kappa), \]  
(B.7)

which is valid by Equation (A.4), we obtain an alternative expression for the variance. This proves the second result of the corollary. The higher-order moments of the von Mises-Fisher family can be obtained via recursive differentiation of these expressions, with application of the product rule of differentiation and the derivatives that are presented in Equations (B.3) and (B.4) then sufficient to construct these moments.

**Proof of Proposition 3.1.** We focus on the integrand in the definition of the Rényi divergence, and note that
\[
f_p (x; \kappa_\alpha, \mu_y) \alpha f_p (x; \kappa_\alpha, \mu_z)^{1-\alpha} = \exp \left( \left( \alpha \kappa_\alpha \mu_y + (1 - \alpha) \kappa_\alpha \mu_z \right)' x \right) C_\nu (\kappa_\alpha \alpha C_\nu (\kappa_\alpha)^{1-\alpha}. \]  
(B.8)

We then define
\[
\kappa_\alpha = \left\| \alpha \kappa_\alpha \mu_y + (1 - \alpha) \kappa_\alpha \mu_z \right\|_2, \\
\mu_\alpha = \left\| \alpha \kappa_\alpha \mu_y + (1 - \alpha) \kappa_\alpha \mu_z \right\|_2^{1} \left( \alpha \kappa_\alpha \mu_y + (1 - \alpha) \kappa_\alpha \mu_z \right), \]  
(B.9)

such that the resulting expression respects the conventions that we have adopted for the concentration and mean direction – i.e., that the concentration is non-negative and that the mean direction is a unit vector. We substitute Equation (B.9) into Equation (B.8) and integrate over the hypersphere, finding that
\[
\frac{1}{\alpha - 1} \ln \left( \int_{S^{p-1}} \frac{\exp (\kappa_\alpha \mu'_x x)}{C_\nu (\kappa_\alpha) C_\nu (\kappa_\alpha)^{1-\alpha} \, dx} \right) = \frac{1}{\alpha - 1} \ln \left( \frac{C_\nu (\kappa_\alpha)}{C_\nu (\kappa_\alpha)^{\alpha} C_\nu (\kappa_\alpha)^{1-\alpha}} \right). \]  
(B.10)

Hence, Equations (B.8) and (B.10) imply that
\[
d_\alpha (y, z) = \frac{\alpha}{\alpha - 1} \ln \left( \frac{C_\nu (\kappa_\alpha)}{C_\nu (\kappa_\alpha) C_\nu (\kappa_\alpha)^{1-\alpha}} \right), \]  
(B.11)
\[
= \frac{\alpha}{\alpha - 1} \ln \left( \frac{\kappa_\alpha I_\nu (\kappa_\alpha)}{\kappa_\alpha I_\nu (\kappa_\alpha)} \right) - \ln \left( \frac{\kappa_\alpha I_\nu (\kappa_\alpha)}{\kappa_\alpha I_\nu (\kappa_\alpha)} \right), \]  
(B.12)

which yields Equation (13) upon rearrangement.

In the special case where \( \kappa_\alpha = 0 \), the numerator on the right-hand side of Equation (B.8) is one, with the corresponding integral over the hypersphere equal to the surface area of the unit ball. The formula for the surface area of the unit ball is well-known. Hence, Equation (B.10) reduces to
\[
\frac{1}{\alpha - 1} \ln \left( \int_{S^{p-1}} \frac{1}{C_\nu (\kappa_\alpha) C_\nu (\kappa_\alpha)^{1-\alpha} \, dx} \right) = \frac{1}{\alpha - 1} \ln \left( \frac{2\pi^{p+1} / \Gamma (\nu + 1)}{C_\nu (\kappa_\alpha)^{\alpha} C_\nu (\kappa_\alpha)^{1-\alpha}} \right), \]  
(B.13)
which in turn means that
\[
d_\alpha(y, z) = \frac{\nu}{\alpha - 1} \ln \left( \frac{\kappa_y^{\alpha - 1}}{2} \right) + \frac{\alpha}{\alpha - 1} \ln \left( \frac{1}{\Gamma (\nu + 1)} \right) - \ln \left( \frac{1}{\Gamma (\nu + 1)} \right). \tag{B.14}
\]

We reiterate that this special case occurs only when Equation (16) holds, which facilitates the restatement of the Renyi divergence in terms of one of the two concentration parameters. We choose to state the Renyi divergence in terms of both concentration parameters because the expression that we obtain is relatively simple.

In the special case where the reference distribution is the uniform distribution on the hypersphere, the integrand in the definition of the Renyi divergence has the specific form
\[
\lim_{\kappa_z \to 0} f_p(x; \kappa_y, \mu_y) \alpha f_p(x; \kappa_z, \mu_z)^{1-\alpha} = \exp \left( \alpha \kappa_y \mu_y \Gamma (\nu + 1)^{1-\alpha} \Gamma (\nu + 1)^{1-\alpha} C_\nu (\kappa_y)^{\alpha} \right) \tag{B.15}
\]
which again utilises the formula for the surface area of the unit ball. As such, Equation (B.10) reduces to
\[
\frac{1}{\alpha - 1} \ln \left( \int_{S^{p-1}} \exp \left( \alpha \kappa_y \mu_y \Gamma (\nu + 1)^{1-\alpha} \Gamma (\nu + 1)^{1-\alpha} C_\nu (\kappa_y)^{\alpha} \right) \, dx \right) = \frac{1}{\alpha - 1} \ln \left( \frac{C_\nu (\alpha \kappa_y)^{\alpha}}{(2\pi^{\nu+1})^{1-\alpha} \Gamma (\nu + 1)^{1-\alpha} C_\nu (\kappa_y)^{\alpha}} \right). \tag{B.16}
\]
We then note that
\[
\frac{1}{\alpha - 1} \ln \left( \frac{C_\nu (\alpha \kappa_y)^{\alpha}}{C_\nu (\kappa_y)^{\alpha}} \right) = \frac{\alpha}{\alpha - 1} \ln \left( \frac{I_{\nu} (\alpha \kappa_y)}{I_{\nu} (\kappa_y)} \right) - \frac{\alpha}{\alpha - 1} \nu \ln (\alpha), \tag{B.17}
\]
and
\[
\frac{1}{\alpha - 1} \ln \left( \frac{C_\nu (\alpha \kappa_y)^{1-\alpha} \Gamma (\nu + 1)^{1-\alpha}}{(2\pi^{\nu+1})^{1-\alpha} \Gamma (\nu + 1)^{1-\alpha} C_\nu (\kappa_y)^{\alpha}} \right) = - \ln \left( \frac{I_{\nu} (\alpha \kappa_y)}{1/\Gamma (\nu + 1)} \right) - \nu \left( \ln \left( \frac{2}{\kappa_y} \right) - \ln (\alpha) \right). \tag{B.18}
\]
Equations (B.17) and (B.18) sum to form Equation (B.16), which we have already established is the formula for the Renyi divergence in this case. We then obtain Equation (17) upon rearrangement.

We now turn our attention to the limiting behaviour of the Renyi divergence with respect to an increase in \( \kappa_y \) given that \( \kappa_y \geq \kappa_z \), holding \( \kappa_z > 0 \) fixed. We rely extensively on Corollary (11) and define
\[
g_\alpha \equiv \kappa_\alpha - \alpha \kappa_y - (1 - \alpha) \kappa_z, \tag{B.19}
\]
for convenience. Given that
\[
\kappa_\alpha \in [\alpha \kappa_y - |1 - \alpha| \kappa_z, \alpha \kappa_y + |1 - \alpha| \kappa_z], \tag{B.20}
\]
so
\[
g_\alpha \in [\min (0, 2 (\alpha - 1) \kappa_z), \max (0, 2 (\alpha - 1) \kappa_z)], \tag{B.21}
\]

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thereby establishing that the Rényi-weighted difference in concentrations is bounded. We recall that

\[ L_\nu (\kappa, 0) \geq \kappa - \frac{1}{2} \ln (\kappa) - g (\nu), \]

\[ U_\nu (\kappa, 0) \leq \kappa - \frac{1}{2} \ln (\kappa) - \overline{g (\nu)}, \]

with

\[ g (\nu) = \ln (\Gamma (\nu + 1)) + \nu \ln (2) + c_\nu, \]

\[ \overline{g (\nu)} = \ln (\Gamma (\nu + 1)) - \frac{1}{2} \ln (2) - c_\nu \ln (c_\nu). \]

We further recall that the Rényi divergence is defined as

\[ d_\alpha (y, z) = \frac{\nu}{\alpha - 1} \ln \left( \frac{\kappa_0 \kappa_z^{1-\alpha}}{\kappa_y} \right) + \frac{\alpha}{\alpha - 1} \ln \left( \frac{\nu (\kappa_\alpha)}{\nu (\kappa_y)} \right) - \ln \left( \frac{\nu (\kappa_\alpha)}{\nu (\kappa_z)} \right), \]

\[ d_\alpha (y, z) = \frac{\nu}{\alpha - 1} \ln \left( \frac{\kappa_0 \kappa_z^{1-\alpha}}{\kappa_y} \right) + \frac{\alpha}{\alpha - 1} \ln \left( \frac{\nu (\kappa_\alpha)}{\nu (\kappa_y)} \right) - \frac{\alpha \ln (\nu (\kappa_y)) - (1 - \alpha) \ln (\nu (\kappa_z))}{\alpha - 1}, \]

and we focus on the final three terms of Equation (B.25). To bound the final three terms of Equation (B.25), we replace each logarithm with one of the two bounds in Equation (B.22). Which bound we use depends upon whether we are interested in bounding the Rényi divergence from below or from above, as well as whether the logarithm is added or subtracted, which itself depends upon the magnitude of \( \alpha \) (specifically, whether \( \alpha \) is less than or greater than one). Applying this strategy to the final three terms of Equation (B.25), we obtain

\[ d_\alpha (y, z) \geq \frac{c_\nu}{\alpha - 1} \ln \left( \frac{\kappa_0 \kappa_z^{1-\alpha}}{\kappa_y} \right) + \frac{1}{\alpha - 1} \left( g_\nu + \max (\alpha, 1) \frac{\nu}{\nu (\kappa_y)} \right), \]

and

\[ d_\alpha (y, z) \leq \frac{c_\nu}{\alpha - 1} \ln \left( \frac{\kappa_0 \kappa_z^{1-\alpha}}{\kappa_y} \right) + \frac{1}{\alpha - 1} \left( g_\nu + \max (\alpha, 1) \frac{\nu}{\nu (\kappa_y)} \right). \]

In view of Equation (B.21), Equations (B.26) and (B.27) are identical up to additive constants. As such, the limiting behaviour of the Rényi divergence is wholly determined by the first term on the right-hand side of Equations (B.26) and (B.27), which is common to both. Equation (B.20) implies that the aforementioned first term is a non-trivial function of \( \kappa_y \). It is then trivial to show that the Rényi divergence is an \( O (\ln (\kappa_y)) \) function. 

\[
\Rightarrow \text{Proof of Proposition 3.2.} \quad \text{We focus on the integrand in the definition of the } \chi\text{-square distance,}
\]

\[ = \]
and note that
\[ \frac{f_p(x; \kappa_y, \mu_y)^2}{f_p(x; \kappa_z, \mu_z)} = \exp \left( \frac{(2\kappa_y \mu_y - \kappa_z \mu_z)'x}{C_\nu(\kappa_y)^2 / C_\nu(\kappa_z)} \right) . \] (B.28)

We then define
\[ \kappa_\chi = \|2\kappa_y \mu_y - \kappa_z \mu_z\|_2, \]
\[ \mu_\chi = \|2\kappa_y \mu_y - \kappa_z \mu_z\|_2^{-1} (2\kappa_y \mu_y - \kappa_z \mu_z) . \] (B.29)

We substitute these definitions into Equation (B.28) and integrate, finding that
\[ \int_{S^{p-1}} \frac{\exp (\kappa_\chi \mu_\chi' x)}{C_\nu(\kappa_y)^2 / C_\nu(\kappa_z)} \, dx - 1 = \frac{C_\nu(\kappa_y)}{C_\nu(\kappa_z)} \frac{C_\nu(\kappa_z)}{C_\nu(\kappa_y)} - 1. \] (B.30)

Hence, Equations (B.28) and (B.30) imply that
\[ d_\chi(y, z) = \frac{\kappa_\chi^p - 2 - 2\nu I_\nu(\kappa_y)}{\kappa_\chi \mu_\chi' \mu_\chi} - 1, \] (B.31)

which is Equation (21). The final expression of the \( \chi \)-square distance can also be obtained via Equation (20), with this relationship easily verified.

In the special case where \( \kappa_\chi = 0 \), Equation (B.30) reduces to
\[ \int_{S^{p-1}} \frac{\exp (\kappa_\chi \mu_\chi' x)}{C_\nu(\kappa_y)^2 / C_\nu(\kappa_z)} \, dx - 1 = 2 \pi^{\nu+1} \Gamma(\nu+1) \frac{C_\nu(\kappa_y)^2}{2\nu I_\nu(\kappa_y)}, \] (B.32)

which in turn means that
\[ d_\chi(y, z) = \frac{\kappa_\chi^p I_\nu(2\kappa_y)}{2\nu I_\nu(\kappa_y)^2 / \Gamma(\nu + 1)} - 1. \] (B.33)

Here, we use the fact that \( \kappa_\chi = 0 \) implies that \( 2\kappa_y = \kappa_z \). In particular, we rely on the fact that
\[ C_\nu(2\kappa_y) = \frac{2\pi^{\nu+1} I_\nu(2\kappa_y)}{\kappa_\chi^p}, \] (B.34)

which facilitates much of the cancellation. We then obtain Equation (23) upon rearrangement of Equation (B.33).

In the special case where the reference distribution is the uniform distribution on the hypersphere, the integrand in the definition of the \( \chi \)-square distance has the specific form
\[ \lim_{\kappa_z \to 0} \int_{S^{p-1}} \frac{f_p(x; \kappa_y, \mu_y)^2}{f_p(x; \kappa_z, \mu_z)} = \frac{2\pi^{\nu+1} \exp (2\kappa_y \mu_y' x)}{C_\nu(\kappa_y)^2} / \Gamma(\nu + 1). \] (B.35)
Integrating, we obtain
\[
\int_{S^p-1} 2\pi^{p+1} \exp \left( \frac{2\kappa_y \mu'_y x}{C'(\kappa_y)^{1/2}} \right) \frac{1}{C' \kappa_y} \, dx = 1 - \frac{2\pi^{p+1} / \Gamma(p + 1)}{C' \kappa_y (2\kappa_y)} = 1, \tag{B.36}
\]
but this coincides with Equation (B.33) and so the expressions of the \(\chi\)-square distance in the two special cases must coincide.

We now turn our attention to the limiting behaviour of the \(\chi\)-square distance with respect to an increase in \(\kappa_y\) given that \(\kappa_y \geq \kappa_z\), holding \(\kappa_z > 0\) fixed. Although the proof of Proposition 3.1 can be used as a prototype for the construction of a proof here, we instead choose to exploit Equation (20). Since Proposition 3.1 establishes that the Rényi divergence is an \(O(\ln(\kappa_y))\) function, the \(\chi\)-square distance is necessarily an \(O(\kappa_y)\) function.

\(\rightarrow\) Proof of Proposition 3.3 We focus on the integrand in the definition of the squared-Hellinger distance, and note that
\[
\sqrt{f_p(x; \kappa_y, \mu_y)} f_p(x; \kappa_z, \mu_z) = \frac{\exp \left( \kappa_y \mu_y + \kappa_z \mu_z \right)}{\sqrt{C' \kappa_y C' \kappa_z}}. \tag{B.37}
\]
We then define
\[
\kappa_h = \frac{1}{2} \left\| \kappa_y \mu_y + \kappa_z \mu_z \right\|_2,
\]
\[
\mu_h = \left\| \kappa_y \mu_y + \kappa_z \mu_z \right\|_2^{-1} (\kappa_y \mu_y + \kappa_z \mu_z).
\]
We substitute these definitions into Equation (B.37) and integrate, finding that
\[
2 \left( 1 - \int_{S^{p-1}} \exp \left( \frac{\kappa_y \mu_y + \kappa_z \mu_z}{2} \right) \, dx \right) = 2 \left( 1 - \sqrt{\frac{C' \kappa_h^2}{C' \kappa_y C' \kappa_z}} \right). \tag{B.39}
\]
Hence, Equations (B.37) and (B.39) imply that
\[
d_h(y, z) = 2 \left( 1 - \sqrt{\frac{\kappa_y \kappa_z^2 I_p(\kappa_y) \Gamma(\kappa_z)}{\kappa_y^2 I_p(\kappa_y) \Gamma(\kappa_z)}} \right). \tag{B.40}
\]
which is Equation (22). The final expression of the squared-Hellinger distance can also be obtained via Equation (27), with this relationship easily verified.

In the special case where \(\kappa_h = 0\), Equation (B.39) reduces to
\[
2 \left( 1 - \int_{S^{p-1}} \frac{1}{\sqrt{C' \kappa_y C' \kappa_z}} \, dx \right) = 2 \left( 1 - \frac{2\pi^{p+1} / \Gamma(p + 1)}{C' \kappa_y} \right), \tag{B.41}
\]
and...
which in turn means that

\[ d_h(y, z) = 2 \left( 1 - \frac{\kappa_y^{2\nu}/\text{I}_\nu(\kappa_y)}{\text{I}_\nu(\kappa_y) \Gamma(\nu + 1)} \right). \]  

(B.42)

Here, we use the fact that \( \kappa_h = 0 \) implies that \( \kappa_y = \kappa_z \). We then obtain Equation (30) upon rearrangement of Equation (B.42).

In the special case where the reference distribution is the uniform distribution on the hypersphere, the integrand in the definition of the squared-Hellinger distance has the specific form

\[ \lim_{\kappa_z \to 0} \sqrt{\int_{S^{n-1}} f_p(x; \kappa_y, \mu_y) f_p(x; \kappa_z, \mu_z)} = \sqrt{\frac{\Gamma(\nu + 1)}{2\pi^{\frac{n\nu+1}{2}} \text{I}_\nu(\kappa_y)} \exp\left(\kappa_y \mu_y' x / 2\right)}. \]  

(B.43)

As such, Equation (B.39) reduces to

\[ 2 \left( 1 - \int_{S^{n-1}} \frac{\Gamma(\nu + 1)}{2\pi^{\frac{n\nu+1}{2}} \text{I}_\nu(\kappa_y)} \exp\left(\kappa_y \mu_y' x / 2\right) \text{d}x \right) = 2 \left( 1 - \frac{\text{I}_\nu(\kappa_y/2)^2 \Gamma(\nu + 1)}{2\pi^{\frac{n\nu+1}{2}} \text{I}_\nu(\kappa_y)} \right). \]  

(B.44)

We then note that

\[ \frac{\text{I}_\nu(\kappa_y/2)^2}{\text{I}_\nu(\kappa_y)} = 2^{2\nu} \frac{\text{I}_\nu(\kappa_y/2)^2}{\text{I}_\nu(\kappa_y)} = 2^{2\nu} \frac{2\pi^{\frac{n\nu+1}{2}} \text{I}_\nu(\kappa_y/2)^2}{\kappa_y^\nu \text{I}_\nu(\kappa_y)}, \]  

(B.45)

which we substitute into Equation (B.44) to obtain Equation (32).

We now turn our attention to the limiting behaviour of the squared-Hellinger distance with respect to an increase in \( \kappa_y \) given that \( \kappa_y \geq \kappa_z \), holding \( \kappa_z > 0 \) fixed. Although the proof of Proposition 3.1 can be used as a prototype for the construction of a proof here, we instead choose to exploit Equation (27). Since Proposition 3.1 establishes that the Rényi divergence is an \( \text{O}(\ln(\kappa_y)) \) function, the squared-Hellinger distance is necessarily an \( \text{O}(1/\kappa_y) \) function.

\( \implies \) Proof of Proposition 3.4. We focus on the integrand in the definition of the Kullback-Leibler divergence and note that

\[ \ln \left( \frac{f_p(x; \kappa_h, \mu_y)}{f_p(x; \kappa_z, \mu_z)} \right) f_p(x; \kappa_y, \mu_y) = \left( (\kappa_y \mu_y - \kappa_z \mu_z)' x + \ln \left( \frac{C_\nu(\kappa_z)}{C_\nu(\kappa_y)} \right) \right) f_p(x; \kappa_y, \mu_y), \]  

(B.46)

We integrate each term on the right-hand side of Equation (B.46), finding that

\[ \int_{S^{n-1}} (\kappa_y \mu_y - \kappa_z \mu_z)' x f_p(x; \kappa_y, \mu_y) \text{d}x = (\kappa_y \mu_y - \kappa_z \mu_z)' \mathbb{E}(y), \]  

(B.47)

and

\[ \int_{S^{n-1}} \ln \left( \frac{C_\nu(\kappa_z)}{C_\nu(\kappa_y)} \right) f_p(x; \kappa_y, \mu_y) \text{d}x = \ln \left( \frac{\kappa_y^\nu \text{I}_\nu(\kappa_z)}{\kappa_z^\nu \text{I}_\nu(\kappa_y)} \right), \]  

(B.48)

respectively, which relies on the fact that the density function integrates to one. Adding Equations (B.47) and (B.48) and substituting the results of Proposition 2.1 – specifically, the result
pertaining to the first moment – we obtain Equation (35).

In the special case where the reference distribution is the uniform distribution on the hypersphere, the integrand in the definition of the Kullback-Leibler divergence has the specific form

$$\lim_{\kappa_z \to 0} \ln \left( \frac{f_p(x; \kappa_y, \mu_y)}{f_p(x; \kappa_z, \mu_z)} \right) f_p(x; \kappa_y, \mu_y) = (\kappa_y \mu_y' x + \ln \left( \frac{2\pi^{\nu+1}}{C_\nu (\kappa_y) \Gamma (\nu + 1)} \right) f_p(x; \kappa_y, \mu_y).$$

We integrate each term on the right-hand side of Equation (B.49), finding that

$$\int_{S_{\nu-1}} \kappa_y \mu_y' f_p(x; \kappa_y, \mu_y) \, dx = \kappa_y \mu_y' E(y),$$

and

$$\int_{S_{\nu-1}} \ln \left( \frac{2\pi^{\nu+1}}{C_\nu (\kappa_y) \Gamma (\nu + 1)} \right) f_p(x; \kappa_y, \mu_y) \, dx = \ln \left( \frac{(\kappa_y/2)^\nu}{I_\nu (\kappa_y) \Gamma (\nu + 1)} \right),$$

respectively. Adding Equations (B.50) and (B.51) and substituting the results of Proposition 2.1 – specifically, the result pertaining to the first moment – we obtain Equation (36). In particular, we rely on the fact that the inner product of a single mean direction is one.

We now turn our attention to the limiting behaviour of the Kullback-Leibler divergence with respect to an increase in $\kappa_y$ given that $\kappa_y \geq \kappa_z$, holding $\kappa_z > 0$ fixed. Given Equation (34) and Proposition 3.1, it is perhaps unsurprising that we find that the Kullback-Leibler divergence is also an $O (\ln (\kappa_y))$ function. To show this, however, requires a slight modification of our approach as compared to the proof of Proposition 3.1. Whilst the main idea remains the same (replace the logarithms of modified Bessel functions with their lower or upper bounds and characterise the limiting behaviour), the Kullback-Leibler divergence is also a function of ratios of modified Bessel functions. We must take this into account when deriving bounds. We rely extensively on Corollary 4.1 and recall that the Kullback-Leibler divergence is defined as

$$d_\ell (y, z) = \nu \ln \left( \frac{\frac{\kappa_y}{\kappa_z}}{\frac{\kappa_y}{\kappa_z}} \right) - \ln \left( \frac{I_\nu (\kappa_y)}{I_\nu (\kappa_z)} \right) + r_\nu (\kappa_y) (\kappa_y - \kappa_z \mu'_z \mu_y).$$

We bound the Kullback-Leibler divergence from below, as

$$d_\ell (y, z) \geq \nu \ln \left( \frac{\kappa_y}{\kappa_z} \right) - U (\kappa_y, \kappa_z) + \frac{\kappa_y^2}{4 \nu \sqrt{\kappa_y^2 + \frac{\kappa_z^2}{\kappa_y}}} \left( \frac{\kappa_y^2}{\kappa_z} \right) + \frac{\kappa_z^2}{4 \nu \sqrt{\kappa_z^2 + \frac{\kappa_y^2}{\kappa_z}}}$$

$$= - \ln \left( I_\nu (\kappa_z) \right) + \zeta_{\nu} \ln \left( \frac{\epsilon_{\nu} + \sqrt{\kappa_y^2 + \epsilon_{\nu}^2}}{\epsilon_{\nu} + \sqrt{\kappa_z^2 + \epsilon_{\nu}^2}} \right) - g_{\ell} (\kappa; \nu),$$

where

$$g_{\ell} (\kappa; \nu) = \sqrt{\kappa_y^2 + \epsilon_{\nu}^2} - \sqrt{\kappa_z^2 + \epsilon_{\nu}^2} - \frac{\kappa_y^2}{\epsilon_{\nu} + \sqrt{\kappa_y^2 + \epsilon_{\nu}^2}}.$$
\[ \sqrt{\kappa_y^2 + \tau_y^2} - \sqrt{\kappa_z^2 + \tau_z^2} - \sqrt{\kappa_y^2 + \tau_y^2} + \tau_y. \]  
(B.56)

To move from Equation (B.55) to Equation (B.56), we replace \(\xi\) with \(\tau\) where it is appropriate to do so and apply the formula for the difference of two squares (the other square being zero) to the final term. It then follows that

\[ d_\ell (y, z) \geq -\ln (I_\nu (\kappa_z)) + \xi \ln \left( \frac{\xi + \sqrt{\kappa_y^2 + \tau_y^2}}{\xi + \sqrt{\kappa_z^2 + \tau_z^2}} \right) - \tau + \sqrt{\kappa_y^2 + \tau_y^2}. \]  
(B.57)

We bound the Kullback-Leibler divergence from above, as

\[ d_\ell (y, z) \leq \nu \ln \left( \frac{\kappa_y}{\kappa_z} \right) - L (\kappa_y, \kappa_z) + \frac{\kappa_y^2}{\xi + (\kappa_y^2 + \tau_y^2)^{1/2}}, \]  
(B.58)

\[ = -\ln (I_\nu (\kappa_z)) + \xi \ln \left( \frac{\xi + (\kappa_y^2 + \tau_y^2)^{1/2}}{\xi + (\kappa_z^2 + \tau_z^2)^{1/2}} \right) - g_\ell (\kappa; \nu), \]  
(B.59)

where

\[ g_\ell (\kappa; \nu) \doteq \sqrt{\kappa_y^2 + \tau_y^2} - \sqrt{\kappa_z^2 + \tau_z^2} - \frac{\kappa_y^2}{\xi + \sqrt{\kappa_y^2 + \tau_y^2}}. \]  
(B.60)

\[ \geq \sqrt{\kappa_y^2 + \tau_y^2} - \sqrt{\kappa_z^2 + \tau_z^2} - \sqrt{\kappa_y^2 + \tau_y^2} + \xi. \]  
(B.61)

To move from Equation (B.60) to Equation (B.61), we replace \(\tau\) with \(\xi\) where it is appropriate to do so and apply the formula for the difference of two squares (the other square being zero) to the final term. It then follows that

\[ d_\ell (y, z) \leq -\ln (I_\nu (\kappa_z)) + \xi \ln \left( \frac{\xi + \sqrt{\kappa_y^2 + \tau_y^2}}{\xi + \sqrt{\kappa_z^2 + \tau_z^2}} \right) - \xi + \sqrt{\kappa_y^2 + \tau_y^2}. \]  
(B.62)

Together, Equations (B.57) and (B.62) imply that

\[ |d_\ell (\kappa_y, \kappa_z)| \leq \xi \ln \left( \frac{\xi + \sqrt{\kappa_y^2 + \tau_y^2}}{\xi + \sqrt{\kappa_z^2 + \tau_z^2}} \right) + \left| \sqrt{\kappa_y^2 + \tau_y^2} - \xi - \ln (I_\nu (\kappa_z)) \right|, \]  
(B.63)

and so it is trivial to show that the Kullback-Leibler divergence is an \(O(\ln (\kappa_y))\) function.

\[ \Rightarrow \text{Proof of Corollary 4.7} \] We rewrite Equations (45) and (46) using the formula for the differ-
ence of two squares as

\[
L_{\nu}(\kappa, 0) = \frac{1}{2} \ln \left( \frac{2}{\kappa} \right) - \ln (\Gamma (\nu + 1)) + \xi_{\nu} \ln \left( \frac{\kappa (\xi_{\nu} + \tau_{\nu}) / 2}{\xi_{\nu} + \sqrt{\kappa^2 + \tau_{\nu}^2}} \right) + \sqrt{\kappa^2 + \tau_{\nu}^2} - \tau_{\nu},
\]  

(B.64)

\[
\geq \frac{1}{2} \ln \left( \frac{2}{\kappa} \right) - \ln (\Gamma (\nu + 1)) + \xi_{\nu} \ln \left( \frac{\kappa (\xi_{\nu} + \tau_{\nu}) / 2}{\xi_{\nu} + \kappa + \tau_{\nu}} \right) + \kappa - \tau_{\nu},
\]  

(B.65)

\[
= \frac{1}{2} \ln \left( \frac{2}{\kappa} \right) - \ln (\Gamma (\nu + 1)) + \xi_{\nu} \ln (\kappa) + \xi_{\nu} \ln \left( \frac{\xi_{\nu} + \tau_{\nu}}{2 (\xi_{\nu} + \tau_{\nu}) + \kappa} \right) + \kappa - \tau_{\nu},
\]  

(B.66)

\[
\geq \frac{1}{2} \ln \left( \frac{2}{\kappa} \right) - \ln (\Gamma (\nu + 1)) + (\xi_{\nu} - \xi_{\nu}) \ln (\kappa) + \xi_{\nu} \ln \left( \frac{\xi_{\nu} + \tau_{\nu}}{2 (\xi_{\nu} + \tau_{\nu})} \right) + \kappa - \tau_{\nu},
\]  

(B.67)

\[
= \kappa - \frac{1}{2} \ln (\nu) - \nu \ln (2) - \ln (\Gamma (\nu + 1)) - \tau_{\nu},
\]  

(B.68)

and

\[
U_{\nu}(\kappa y, 0) = \frac{1}{2} \ln \left( \frac{2}{\kappa y} \right) - \ln (\Gamma (\nu + 1)) + \xi_{\nu} \ln \left( \frac{\kappa (\xi_{\nu} + \tau_{\nu}) / 2}{\xi_{\nu} + \sqrt{\kappa^2 + \xi_{\nu}^2}} \right) + \sqrt{\kappa^2 + \xi_{\nu}^2} - \xi_{\nu},
\]  

(B.69)

\[
\leq \frac{1}{2} \ln \left( \frac{2}{\kappa y} \right) - \ln (\Gamma (\nu + 1)) + \xi_{\nu} \ln \left( \frac{\kappa \xi_{\nu} / \kappa}{\xi_{\nu} + \sqrt{\kappa^2 + \xi_{\nu}^2}} \right) + \kappa + \xi_{\nu} - \xi_{\nu},
\]  

(B.70)

\[
= \kappa - \frac{1}{2} \ln (\kappa) + \frac{1}{2} \ln (2) - \ln (\Gamma (\nu + 1)) + \xi_{\nu} \ln (\xi_{\nu}),
\]  

(B.71)

respectively. Here, we exploit the concavity of the logarithmic and square root functions, and set some terms equal to zero where it is useful to do so. This establishes the veracity of the first part of Corollary 4.1. We now need to show that the logarithm of the modified Bessel function, which we know lies between these two bounds for all \( \kappa > 0 \), is an \( \Omega (\kappa - \ln (\kappa)/2) \) function. To do so, we show that the maximum of the absolute value of Equations (B.68) and (B.71) is bounded from above by a function that has the required order. Specifically, for all \( \kappa > 0 \),

\[
|\ln (L_{\nu}(\kappa))| \leq \kappa - \frac{1}{2} \ln (\kappa) + \max \left( \frac{1}{2}, \nu \right) \ln (2) + \ln (\Gamma (\nu + 1)) + \max (\tau_{\nu}, \xi_{\nu} \ln (\xi_{\nu})),
\]  

(B.72)

\[
\leq \left( \kappa - \frac{1}{2} \ln (\kappa) \right) \left( 1 + \frac{\max (1/2, \nu) \ln (2) + \ln (\Gamma (\nu + 1)) + \max (\tau_{\nu}, \xi_{\nu} \ln (\xi_{\nu}))}{(1 + \ln (2)) / 2} \right),
\]  

(B.73)

where the denominator in the second term of Equation (B.73) is the minimum value that the first term of Equation (B.73) can attain, which establishes the result. 

\[ \blacksquare \]
C Hankel expansion of the circular variance

One quantity that is often of interest is the circular variance of the von Mises-Fisher family of distributions, which is defined as one minus the ratio of modified Bessel functions – i.e., one minus the mean resultant length. Kitagawa et al. (2022) demonstrates that the circular variance is an $O(1/\kappa)$ function. Here, we show that the same result can be obtained via Hankel series expansion (see Equation (A.6) for a definition). We note that Hankel series expansion is appropriate when $\kappa \to \infty$, which is the limiting behaviour that we are interested in. We begin by writing the circular variance in the alternative form

$$1 - r_\nu(\kappa) = 1 - \frac{I_{\nu+1}(\kappa)}{I_\nu(\kappa)} = \frac{I_\nu(\kappa) - I_{\nu+1}(\kappa)}{I_\nu(\kappa)},$$

(C.1)

to which we apply the expansion. Recalling Equations (A.6) and (A.7),

$$1 - r_\nu(\kappa) = \left(1 - \frac{a_1(\nu)}{\kappa} + O\left(\frac{1}{\kappa^2}\right)\right)^{-1} \left(\frac{a_1(\nu + 1)}{\kappa} - \frac{a_1(\nu)}{\kappa} + O\left(\frac{1}{\kappa^2}\right)\right),$$

(C.2)

$$= \frac{a_1(\nu + 1) - a_1(\nu) + O(\kappa)}{\kappa - a_1(\nu) + O(\kappa)},$$

(C.3)

which is, trivially, an $O(1/\kappa)$ function.

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The density function of a von Mises-Fisher distribution on the circle with mean direction \((-1, 0)\) – i.e., a polar orientation with reference angle equal to \(\pi\) radians – for several values of the concentration parameter. A spherical coordinate system is used.

Orthogonal projection of the sphere, oriented to the mean direction, for several values of the concentration parameter. Each contour describes a region in which the von Mises-Fisher distribution assigns 10% mass, with contours distinguished by their shading. As the value of the concentration parameter is increased, more mass is assigned to the vicinity of the pole.