On-line Learning of an Unlearnable True Teacher through Mobile Ensemble Teachers

Takeshi Hirama and Koji Hukushima

Department of Basic Science, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8902, Japan

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On-line learning of a hierarchical learning model is studied by a method from statistical mechanics. In our model a student of a simple perceptron learns from not a true teacher directly, but ensemble teachers who learn from the true teacher with a perceptron learning rule. Since the true teacher and the ensemble teachers are expressed as non-monotonic perceptron and simple ones, respectively, the ensemble teachers go around the unlearnable true teacher with the distance between them fixed in an asymptotic steady state. The generalization performance of the student is shown to exceed that of the ensemble teachers in a transient state, as was shown in similar ensemble-teachers models. Further, it is found that moving the ensemble teachers even in the steady state, in contrast to the fixed ensemble teachers, is efficient for the performance of the student.

KEYWORDS: online learning, ensemble teachers, generalization error, statistical mechanics

1. Introduction

Learning is an inference problem of inhered rules from a given set of examples which consist of input data and corresponding output data generated by the rules. In practice, the examples are often supplied inexhaustibly and then the learning must proceed by using each example just once. Such learning is called on-line learning. On the contrary, the learning in which all the examples are presented repeatedly at anytime is called off-line or batch learning.

The on-line learning as well as the off-line one has been extensively studied by using statistical-mechanical methods so far and many extensions of the on-line learning scheme have been made in order to improve a generalization performance. Recently, Miyoshi and Okada and Urakami, Miyoshi and Okada analyzed the generalization performance of a student supervised by a moving teacher that goes around a fixed true teacher in a framework of the on-line learning using the statistical mechanical method. In their model, the student is not directly given the outputs by the true teacher. The moving teacher learns from the true teacher and provides its output to the student. In this sense, the model is a kind of hierarchical learning. In ref. 5, the true teacher is a non-monotonic perceptron, while the moving teacher and the student are simple perceptron using perceptron learning, which could not infer the true teacher completely in principle. The theoretical bound of the generalization error of a simple perceptron learner has been obtained. In that case, the moving teacher goes around
the true teacher with a fixed distance between them. Interestingly, it turned out that when the student’s learning rate is relatively small, the student’s generalization error can temporally become smaller than that of the moving teacher, even if the student only uses the examples from the moving teacher.

Subsequently, Miyoshi and Okada\textsuperscript{7)} and Utsumi, Miyoshi and Okada\textsuperscript{8)} analyzed the generalization performance of an extended model of the on-line learning with multiple teachers, which would be called ensemble-teachers learning model. This model is also regarded as an extension of the ensemble learning\textsuperscript{9,10)} because the ensemble teachers and the student in the ensemble-teachers model can be interpreted as the ensemble students and their integrating mechanism, respectively. In particular, ref. 8 discussed the model in which the true teacher, the ensemble teachers and the student are all simple perceptrons. In this model the true teacher and the ensemble teachers are fixed. The student adopts the Hebbian learning or the perceptron learning as a learning rule and uses examples from the ensemble teachers in turn or randomly. As a result, it was clarified that the Hebbian learning and the perceptron learning show qualitatively different behavior from each other. In the Hebbian learning, the generalization error monotonically decreases during the learning process and its asymptotic value is independent of the learning rate. The asymptotic value is reduced as the number of the ensemble teachers increases since the ensemble teachers have more variety in their representations. On the other hand, in the perceptron learning, the generalization error shows non-monotonic behavior and exhibits a minimum at a certain step in the learning. The minimum value of the generalization error decreases as the learning rate decreases and the total number of the teachers increases.

In ref. 5 and ref. 8, it was shown that the generalization error of a student could be smaller than that of a moving teacher or fixed ensemble teachers. A comparison between the generalization performance with a fixed teacher and that with a mobile teacher, however, has not been made directly. Furthermore, in the on-line learning with the ensemble teachers it is not trivial that either the mobility or the multiplicity of the ensemble teachers is effective for the learning performance of the student. In this paper, we study the on-line learning for the ensemble teachers which can move around a true teacher. We discuss a model in which the fixed true teacher is non-monotonic perceptron and the ensemble moving teachers and the student are a simple perceptron. This is a generalized version of the model studied in ref. 5. Adopting the perceptron learning as a learning rule for the ensemble teachers, they go around the true teacher with constant order parameters in the steady state. Then we analyze the generalization performance of the student which learns from the mobile ensemble teachers using the Hebbian and the perceptron rules. We also study the model with the ensemble teachers fixed in their steady state. It is thus clarified that the movement of the ensemble teachers , in comparison with the fixed ensemble case, significantly improves the
generalization performance of the student as a transient state in the learning process.

The paper is organized as follows: In sec. 2, we introduce the model with the ensemble moving teachers going around the unlearnable true teacher. In sec. 3, based on the statistical-mechanical idea, we theoretically derive the ordinal differential equations of order parameters and an explicit formula of the generalization error of our model in terms of the order parameters. In sec. 4, we show the theoretical and numerical results of the generalization performance of the student with the Hebbian and perceptron rules. The last section is devoted to our conclusion. In the appendixes, the derivations of the differential equations discussed in sec. 3 are presented in detail.

2. Model

In this paper, we consider a true teacher, $K$ ensemble moving teachers and a student, whose connection weights are expressed as $N$ dimensional vectors, $A$, $B_k$ and $J$, respectively, with $k = 1, 2, \cdots, K$. For simplicity, each component $A_i$ of $A$ with $i = 1, \cdots, N$ is assumed to be drawn from $\mathcal{N}(0, 1)$ independently and fixed, where $\mathcal{N}(m, \sigma^2)$ denotes the Gaussian distribution with $m$ and $\sigma^2$ being a mean and variance, respectively. As an initial condition of the learning process, each of the components $B_{0k}$ and $J_0$ are also assumed to be drawn from $\mathcal{N}(0, 1)$ independently. Input $x$ is also the $N$-dimensional vector and the component $x_i$ follows from $\mathcal{N}(0, 1/N)$ independently. Thus, we have

$$\langle A_i \rangle = \langle B_{0k} \rangle = \langle J_0 \rangle = \langle x_i \rangle = 0,$$

$$\langle (A_i)^2 \rangle = \langle (B_{0k})^2 \rangle = \langle (J_0)^2 \rangle = 1,$$

and

$$\langle (x_i)^2 \rangle = \frac{1}{N},$$

where $\langle \cdots \rangle$ denotes an average over the Gaussian distribution.

In the statistical mechanics of the learning, we are interested in asymptotic behavior of $A$, $B$ and $J$ in a thermodynamics limit $N \to \infty$. Then, one finds that the norms of the vectors are

$$\|A\| = \sqrt{N}, \quad \|B_k\| = \sqrt{N}, \quad \|J\| = \sqrt{N}, \quad \|x\| = 1.$$  

(2.4)

The norms, $\|B_k\|$ and $\|J\|$, of the ensemble moving teachers and the student change during the learning process from their initial values. The normalized length of these vectors is introduced as $l_{B_k} = \|B_k\|/\|B_{0k}\|$ for the ensemble teachers and $l_J = \|J\|/\|J_0\|$ for the student. In the thermodynamic limit, the direction cosines between these vectors are a relevant extensive quantity, denoted for $A$ and $B_k$, $A$ and $J$, $B_k$ and $B_{k'}$, and $B_k$ and $J$ respectively as

$$R_{B_k} = \frac{A \cdot B_k}{\|A\| \|B_k\|}, \quad R_J = \frac{A \cdot J}{\|A\| \|J\|}.$$  

(2.5)
\[ q_{kk'} = \frac{B_k \cdot B'_{k'}}{\|B_k\| \|B'_{k'}\|}, \quad R_{B_kJ} = \frac{B_k \cdot J}{\|B_k\| \|J\|}. \tag{2.6} \]

In the present study, we assume that the true teacher is a non-monotonic perceptron and the ensemble moving teachers and the student are a simple perceptron. The output for a given input \( x \) of the true teacher is defined by a non-monotonic function

\[ o = \text{sgn} \left( (A \cdot x - a) A \cdot x (A \cdot x + a) \right) \tag{2.7} \]

with a fixed threshold \( a \), while those of the ensemble moving teachers and the student are simply given by \( \text{sgn}(B_k \cdot x) \) and \( \text{sgn}(J \cdot x) \), respectively. Here, \( \text{sgn}(\cdot) \) is the sign function defined as

\[ \text{sgn}(s) = \begin{cases} +1, & s \geq 0, \\ -1, & s < 0. \end{cases} \tag{2.8} \]

A measure of dissimilarity between the true teacher and the ensemble teachers or the student is defined by using their outputs as

\[ \epsilon_{B_k} \equiv \Theta \left( -o \cdot \text{sgn} (B_k \cdot x) \right) \tag{2.9} \]

for \( k \)th ensemble teacher and

\[ \epsilon_J \equiv \Theta \left( -o \cdot \text{sgn} (J \cdot x) \right) \tag{2.10} \]

for the student, where \( \Theta(\cdot) \) is the step function defined as

\[ \Theta(s) = \begin{cases} +1, & s \geq 0, \\ 0, & s < 0. \end{cases} \tag{2.11} \]

One of the main purposes of the statistical learning theory is to obtain theoretically the generalization errors \( \epsilon^g_{B_k} \) and \( \epsilon^g_J \), which are defined as the average of the errors, \( \epsilon_{B_k} \) and \( \epsilon_J \) over the whole set of possible inputs \( x \). Since the input \( x \) appears in Eq. (2.9) and Eq. (2.10) as inner products \( A \cdot x \), \( B_k \cdot x \) and \( J \cdot x \), the average over Gaussian vector \( x \) could be reduced to an average over correlated Gaussian variables. When one defines a set of variables, \( v, v_{B_k} \) and \( u \) as

\[ v = A \cdot x, \tag{2.12} \]
\[ v_{B_k} l_{B_k} = B_k \cdot x, \tag{2.13} \]
\[ u l_J = J \cdot x, \tag{2.14} \]

they obey the multiple Gaussian distribution

\[ P(v, \{v_{B_k}\}, u) = \frac{1}{(2\pi)^{(K+2)/2} |\Sigma|^{1/2}} \exp \left( -\frac{(v, \{v_{B_k}\}, u) \Sigma^{-1}(v, \{v_{B_k}\}, u)^T}{2} \right), \tag{2.15} \]
with zero means and the covariance matrix \( \Sigma \)

\[
\Sigma = \begin{pmatrix}
1 & R_{B_1} & R_{B_2} & \cdots & R_{B_K} & R_J \\
R_{B_1} & 1 & q_{1,2} & \cdots & q_{1,K} & R_{B_1J} \\
R_{B_2} & q_{2,1} & 1 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & q_{K-1,K} & R_{BK-1J} \\
R_{BK} & q_{K,1} & q_{K,K-1} & \cdots & 1 & R_{BKJ} \\
R_J & R_{BJ_1} & \cdots & R_{BKJ-1} & R_{BKJ} & 1
\end{pmatrix},
\]

(2.16)

Evaluating the correlated Gaussian integrations, the generalization errors \( \epsilon^g_{B_k} \) and \( \epsilon^g_J \) are obtained as

\[
\epsilon^g_{B_k} = 2 \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) D_v H\left( \frac{-R_{B_k} v}{\sqrt{1 - R^2_{B_k}}} \right),
\]

(2.17)

and

\[
\epsilon^g_J = 2 \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) D_v H\left( \frac{-R_{J} v}{\sqrt{1 - R^2_{J}}} \right),
\]

(2.18)

where \( Ds \) is the Gaussian measure defined as

\[
Ds \equiv \frac{ds}{\sqrt{2\pi}} \exp\left( -\frac{s^2}{2} \right),
\]

(2.19)

and \( H(\cdot) \) is the error function defined as

\[
H(s) \equiv \int_{s}^{\infty} Dx.
\]

(2.20)

It should be noted that the dynamical effect of the generalization errors appears only through \( R_{B_k} \) and \( R_J \). This implies that the generalization errors have a fundamental minimum as a function of \( R_{B_k} \) and \( R_J \), irrespective of the matter if the values of \( R_{B_k} \) and \( R_J \) which give the minimum value of the generalization error appear in a particular chosen learning rule of the student and the ensemble teachers. An efficient learning rule might realize the fundamental minimum for a given learning model.

Let us defined the update rule in the on-line learning. The ensemble moving teachers \( B_k \) are updated from the current state \( B_k^{m'} \) using an input \( x \) and output of the true teacher \( A \) for the input \( x^{m'} \), independently as

\[
B_k^{m'+1} = B_k^{m'} + f_k^{m'}(x^{m'}, B_k^{m'}, o^{m'})x^{m'},
\]

(2.21)

where \( f^{m'} \) is an update function of the ensemble moving teachers and \( m' \) denotes the time step of the ensemble moving teachers. In particular, we choose the perceptron learning for the update function \( f_k \), which is given by

\[
f_k^{m'} = \eta_B \Theta \left( -v^{m'}_{B_k} o^{m'} \right) o^{m'}.
\]

(2.22)

Here, \( \eta_B \) is the learning rate of the ensemble moving teachers. In our analysis, the learning rate
\( \eta_B \) is independent of the teachers and is fixed during the learning process. After a sufficient long learning process using the perceptron rule, the ensemble moving teachers reach steady state with \( R_{Bk}, l_{Bk} \) and \( q_{kk'} \) fixed. In the present study, we focus our attention to dynamical effect of the ensemble teachers for the learning performance of the student. In order to separate off a transient effect of the ensemble teachers, the student learns from the ensemble teachers in the steady state. The student \( J \) is updated using an input \( x \) and an output of one of the \( K \) ensemble moving teachers \( B_k \) chosen randomly. The explicit recursion formula for \( J^m \) with \( m \) being the time step of the student is given by

\[
J^{m+1} = J^m + g^m_k(x^m, J^m, \text{sgn}(v_{B_k} l_{B_k}))x^m,
\]

where \( g^m_k \) is an update function of the student and \( k \) is a uniform random integer chosen from 1 to \( K \). Note that the ensemble moving teachers are also updated using the same input.

We particularly discuss two different learning rules for the student, which are the Hebbian learning

\[
g^m_k = \eta \text{sgn} \left( v_{B_k} l_{B_k} \right),
\]

and the perceptron learning

\[
g^m_k = \eta \Theta \left( -v_{B_k} u^m \right) \text{sgn} \left( v_{B_k} l_{B_k} \right).
\]

The learning rate of the student \( \eta \) is also constant during the learning process.

3. Order-parameter theory

As shown in the previous section, the generalization errors of the ensemble teachers and the student are expressed in terms of the parameter \( R_{Bk} \) and \( R_J \) and evolve only through a few parameters associated with the learning of \( B_k \) and \( J \) in the thermodynamic limit. It has been shown that a class of the on-line learning can be characterized by a few extensive parameters, called order parameter. In this section, following ref. 3, a set of ordinal differential equations of the order parameters are obtained in our model by taking the thermodynamic limit.

The learning process of the ensemble moving teachers are described by the three order parameter \( R_{Bk}, l_k \) and \( q_{kk'} \), which are assumed to be self-averaging. It is sufficient to consider the evolution of \( R_{Bk} \) and \( l_k \) in order to describe the dynamics of the ensemble teachers, but that of the overlap \( q_{kk'} \) between two different teachers is necessary for the student dynamics as seen later. From the update rules of the ensemble teachers in eq. (2.21), one finds a closed formula of the ordinal differential equations of the order parameters as,

\[
\frac{dl_B}{dt'} = \frac{\eta_B}{\sqrt{2\pi}} \left[ R_B \left\{ 2 \exp \left( -\frac{a^2}{2} \right) - 1 \right\} - 1 \right] + \frac{1}{2 \eta_B} \left( 2 \eta^2_B \left( \int_{-\infty}^{-a} + \int_0^a \right) DvH \left( -\frac{R_B v}{\sqrt{1 - R_B^2}} \right) \right),
\]

(3.1)
\[
\frac{dR_B}{dt'} = \frac{R_B}{l_B} \frac{dl_B}{dt'} + \frac{1}{l_B} \left( \frac{\eta_B}{\sqrt{2\pi}} \left\{ 2\exp\left(-\frac{a^2}{2}\right) - R_B - 1 \right\} \right), \tag{3.2}
\]

\[
\frac{dq}{dt'} = -\frac{q}{l_B} \frac{dl_B}{dt'} - \frac{q}{l_B} \frac{dl_B}{dt'} + \frac{2}{l_B} \left( \frac{\eta_B}{\sqrt{2\pi}} \left[ R_B \left\{ 2\exp\left(-\frac{a^2}{2}\right) - 1 \right\} - q \right] \right)
+ \frac{1}{l_B^2} \left( 2\eta_B^2 \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) Dv \int_{-\infty}^{\infty} \frac{R_B v}{\sqrt{1-R_B^2}} DxH(z) \right), \tag{3.3}
\]

where

\[
z \equiv -\frac{(q - R_B^2)x + R_B \sqrt{1-R_B^2} v}{\sqrt{(1-q)(1+q-2R_B^2)}}, \tag{3.4}
\]

and \( t' \) denotes continuous time. We omit the subscript \( k \) from the order parameters, because the differential equations including their initial conditions have a permutation symmetry for the subscript \( k \). Derivation of the differential equation is given in the appendix A.

From these equation one easily obtain the steady solutions of \( R_B, l_B \) and of \( q \) as follows:

\[
R_B = 2\exp\left(-\frac{a^2}{2}\right) - 1, \tag{3.5}
\]

\[
l_B = \sqrt{2\pi \eta_B} \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) DvH \left( -\frac{R_B v}{\sqrt{1-R_B^2}} \right), \tag{3.6}
\]

\[
q = R_B^2 + \frac{(1-R_B^2) \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) Dv \int_{-\infty}^{\infty} \frac{R_B v}{\sqrt{1-R_B^2}} DxH(z)}{\left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) DvH \left( -\frac{R_B v}{\sqrt{1-R_B^2}} \right)}. \tag{3.7}
\]

Note that \( R_B, q \) and \( l_B/\eta_B \) depend only on the threshold \( a \) of the true teacher. In our study, the ensemble teachers are assumed to take the steady state before the student begins to learn in order to make the dynamical effect of the ensemble teachers clear. Therefore these solutions of \( R_B, l_B \) and \( q \) are used as an initial condition of the learning dynamics of the student discussed below.

The learning dynamics of the student is also described by a set of ordinal differential equations of a few order parameters, which is derived from the update functions for the Hebbian rule (2.24) and the perceptron one (2.25). We refer to the appendix B for the derivation of the dynamical equations. A straightforward calculation for the Hebbian rule leads to

\[
\frac{dl}{dt} = \eta \left( \sqrt{\frac{2}{\pi}} R_B + \frac{\eta}{2l} \right), \tag{3.8}
\]

\[
\frac{dR_I}{dt} = -\frac{R_I}{l} \frac{dl}{dt} + \eta \sqrt{\frac{2}{\pi}} R_B, \tag{3.9}
\]
\[
\frac{dR_{BJ}}{dt} = -R_{BJ}\left(\frac{1}{l} \frac{dl}{dt} + \frac{1}{l_B} \frac{dl_B}{dt}\right) + \frac{\eta_B}{l_B \sqrt{2\pi}} \left\{ R_J \left(2e^{-\frac{a^2}{2}} - 1\right) - R_{BJ}\right\} \\
+ \frac{\eta}{lK} \sqrt{\frac{2}{\pi}} q \frac{2\eta B}{K l_B} \left( (K - 1) \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) Dv \int_{-\frac{R_{BJ}^2}{\sqrt{1-R_B^2}}}^{\infty} Dx \left\{ 2H(z) - 1 \right\} \right) \\
+ \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) Dv H\left( -\frac{R_{Bv}}{\sqrt{1-R_B^2}} \right) \right) . 
\]

(3.10)

Corresponding differential equations for the perceptron rule are given as

\[
\frac{dl}{dt} = \eta \left( R_{BJ} - 1 \right) \frac{1}{\sqrt{2\pi}} + \frac{\eta \tan^{-1} \left( \sqrt{1-R_{BJ}^2} \right)}{\pi R_{BJ}} , \tag{3.11}
\]

\[
\frac{dR_J}{dt} = -\frac{R_J}{l} \frac{dl}{dt} + \eta \frac{l}{l_B \sqrt{2\pi}} (R_B - R_J) , \tag{3.12}
\]

\[
\frac{dR_{BJ}}{dt} = -R_{BJ} \left(\frac{1}{l} \frac{dl}{dt} + \frac{1}{l_B} \frac{dl_B}{dt}\right) + \frac{\eta_B}{l_B \sqrt{2\pi}} \left\{ R_J \left(2e^{-\frac{a^2}{2}} - 1\right) - R_{BJ}\right\} \\
+ \frac{\eta q}{lK \sqrt{2\pi}} \left( \frac{q}{K} - R_{BJ} \right) \\
+ \frac{2\eta_B}{K l_B} \left( (K - 1) \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) Dv \int_{-\frac{R_{BJ}^2}{\sqrt{1-R_B^2}}}^{\infty} Dx \left\{ -\int_{z}^{\infty} DyH(-z_1) + \int_{-\infty}^{z_1} DyH(z_1) \right\} \right) \\
+ \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) Dv \int_{-\frac{R_{BJ}^2}{\sqrt{1-R_B^2}}}^{\infty} Dx \left\{ 2H(z_2) - 1 \right\} \right) . \tag{3.13}
\]

Solving these differential equations for the student and the ensemble teachers, we can obtain the generalization errors \( \epsilon^g_J \) and \( R_J \) as a function of time step.

4. Results and Discussion

In this section we present dynamical behavior of the order parameter \( R_J \) and the generalization error \( \epsilon^g_J \) obtained by solving numerically the set of the differential equations obtained in the previous section. In order to study “dynamical” effect of the ensemble teachers, we compare results of two different cases; one with the teachers fixed to a steady state and the other with the teachers kept to learn in the steady state sharing the same inputs with the student. In this study, we choose the threshold value \( a = 0.5 \) of the non-monotonic perceptron for the true teacher, yielding \( l_B/\eta_B \approx 0.93, R_B \approx 0.76 \) and \( q \approx 0.91 \) in the steady state for the ensemble teachers. We also perform direct simulations of the given update rules for the finite-size perceptrons. In the simulations we use the dimension of vectors \( N = 10^4 \) and perform \( 10^5 \) trajectories of the learning process for taking the average over the random inputs. As shown in figures below, although a limited case with \( \eta = 0.1 \) is only shown for avoiding crowded plots, the results of \( R_J \) and \( \epsilon^g_J \) obtained by the simulations for all the parameter
studied agree with the theoretical ones by the order-parameter differential equations, This confirms that the assumption of the self-averaging is appropriate in our model.

Figure 1 shows time dependence of $R_J$ for the Hebbian learning when the ensemble teachers stop to learn and take a steady-state vector. The transient process of $R_J$ depends on the learning rate $\eta$ of the student and the number $K$ of the ensemble teachers. The value of $R_J$ gets larger with increasing the number $K$ and the learning rate $\eta$, meaning that the student comes close to the true teacher. As the time $t$ goes on, it approaches monotonically a steady value, which increases as $K$ increases. Interestingly, the steady value of $R_J$ exceeds the value of $R_B$ when the number $K$ of the ensemble teachers is greater than 1. This is similar to that shown in ref. 8. Figure 2, on the other hand, shows the corresponding time dependence of $R_J$ when the ensemble teachers continue to learn in their steady state. While at the very beginning of the learning process the value of $R_J$ shows monotonic time development similar to the case that the ensemble teachers are fixed, it is larger than that with the fixed teachers after a certain time and eventually approaches unity, which is independent of the learning rate, even if the number $K$ is one. It should be noted that the value of $R_B$ is common in two cases of Figs. 1 and 2. This implies that the number $K$ of the ensemble teachers is not efficient for the learning of the student, but their continuous learning even with a fixed similarity to the true teacher is significantly important.

Figure 3 shows dynamical behavior of the generalization error of the student for the Hebbian learning, which monotonically decreases and eventually converges to the steady value when the ensemble teachers are fixed. The steady value of $\epsilon^g_J$ only depends on the number $K$ and not the learning rate $\eta$. As $K$ increases, the value decreases and furthermore it can be smaller than that of the generalization error $\epsilon^g_B$ of the ensemble teachers when $K$ is larger than one, reflecting the behavior of $R_J$. This means that the performance of the student becomes better than the ensemble teachers when $K \geq 2$. The obtained value of $\epsilon^g_J$, however, does not reach the fundamental minimum value of the generalization error in this case even when $K$ increases to infinity. In Fig. 4 the dynamical behavior of $\epsilon^g_J$ is shown in the case where the ensemble teachers are moving. In contrast to the case of the fixed ensemble teachers, $\epsilon^g_J$ shows non-monotonic behavior in the learning process and the steady value of independent of both $K$ and $\eta$ while it is quite larger than $\epsilon^g_B$. The minimum value of $\epsilon^g_J$ reaches the fundamental minimum value at a certain time step, depending on the learning rate $\eta$. In a sense, the mobile ensemble teachers is a better on-line learning model, while the best performance occurs only at a transient state unfortunately.

Let us turn to the perceptron learning of the student. We show the time development of $R_J$ for the fixed and moving ensemble teachers in Figs. 5 and 6, respectively. The steady values of $R_J$ coincide with $R_B$ both for the two cases and it is independent of $K$ and $\eta$. Further non-monotonic behavior is found for small $\eta$ and large $K$ and then the value of $R_J$
takes a maximum value at a certain time step, which exceeds $R_B$ certainly as a transient state. Moving the ensemble teachers enhances significantly the maximum value, meaning that the student is closer to the true teacher. In particular, for small value of $\eta$ the maximum value of $R_J$ for the unique moving teacher is larger than that for the $K = \infty$ fixed ensemble teachers.

Figures 7 and 8 show the corresponding dynamical behavior of the generalization errors $\epsilon_J^\theta$ of the perceptron-learning student with the fixed and mobile ensemble teachers, respectively. As expected from the behavior of $R_J$ in Figs. 5 and 6, the steady value of $\epsilon_J^\theta$ for all the case is the same as that of the ensemble teachers. However, an essential difference is found in transient behavior of $\epsilon_J^\theta$. Although the minimum value does not necessarily achieve the fundamental minimum value of $\epsilon_J^\theta$ in the case of the fixed ensemble teachers, it does for small value of $\eta$ in the moving ensemble teachers with a finite time interval as shown in Fig. 8. This means again that moving the ensemble teachers plays an important role for the learning performance of the student.

![Fig. 1. Time dependence of the direction cosine $R_J$ between the student $J$ with the Hebbian learning and the true teacher $A$ with $a = 0.5$ in the case that the $K$ ensemble teachers are fixed to be a steady state vector. Curves represent numerical solution of the order-parameter differential equations with $K = 1, 50$ and $\infty$ and $\eta = 0.1, 1$ and 10. The straight line is the direction cosine $R_B$ between the fixed ensemble teachers and the true teacher. Symbols represent corresponding results obtained by the direct simulation with system size $N = 10^4$ and $\eta = 0.1$.](image)

5. Conclusion

We have analyzed the generalization performance of a student supervised by ensemble moving teachers in the framework of on-line learning. In this paper we adopted a non-monotonic perceptron as a true teacher and a simple perceptron as the ensemble moving teachers and the student. We have treated the Hebbian learning and the perceptron learning
Fig. 2. Time dependence of the direction cosine $R_J$ between the student $J$ with the Hebbian learning and the true teacher $A$ with $\alpha = 0.5$ in the case that the ensemble teachers continue to learn in their steady state. The symbols and the lines are the same as those in Fig. 1.

Fig. 3. Time dependence of the generalization error of the student $\epsilon_J$ between the student $J$ with the Hebbian learning and the true teacher $A$ with $\alpha = 0.5$ in the case of the fixed ensemble teachers. The symbols and lines are the same as those in Fig. 1.

as a learning rule for the student and have calculated the generalization error of the student with some order parameters analytically or numerically. In this study, we particularly focus on the effect of mobile ensemble teachers on the learning performance of the student. Therefore, it is assumed that the ensemble teachers learn only from the true teacher by using the perceptron learning and reach a steady state before the student begins to learn. This is helpful for separating a transient learning effect of the ensemble teachers from an intrinsic effect.

In the Hebbian learning, it has been proven that the number $K$ of the ensemble teachers is not efficient, but their continuous learning in their steady state is significantly important for the student to come close to the true teacher. In the case that the ensemble teachers continue to learn, the value of $R_J$ eventually approaches unity, which is independent of the learning
Fig. 4. Time dependence of the generalization error of the student $\epsilon_J$ between the student $J$ with the Hebbian learning and the true teacher $A$ with $a = 0.5$ in the case of the mobile ensemble teachers. The symbols of the lines and plots are the same as in Fig. 1.

Fig. 5. Time dependence of the direction cosine $R_J$ between the student $J$ with the perceptron learning and the true teacher $A$ with $a = 0.5$ in the case of the fixed ensemble teachers. The symbols of the lines and plots are the same as in Fig. 1.

rate, even if the number $K$ is one. Although the student with $R_J = 1$ does not always mean a best learning performance in the Hebbian learning, the minimum value of $\epsilon_J^g$ reaches the fundamental minimum value as a transient state, regardless of the number $K$. This is sharp contrast to the case of the fixed ensemble teachers, in which the fundamental minimum value of $\epsilon_J^g$ never occurs. The time step at which $\epsilon_J^g$ has a minimum value decreases with increasing the learning rate $\eta$, but its precise step has not been predicted theoretically at the present moment.

In the perceptron learning, in contrast to the Hebbian learning, no significant difference has been found in the steady states. The steady values of $R_J$ and $\epsilon_J^g$ coincide with those of $R_B$ and $\epsilon_B^g$ in both of the fixed and mobile ensemble teachers. However, the effect of the
Fig. 6. Time dependence of the direction cosine $R_{J}$ between the student $J$ with the perceptron learning and the true teacher $A$ with $a = 0.5$ in the case of the mobile ensemble teachers. The symbols of the lines and plots are the same as in Fig. 1.

Fig. 7. Time dependence of the generalization error of the student $\epsilon_{J}$ between the student $J$ with the perceptron learning and the true teacher $A$ with $a = 0.5$ in the case that the fixed ensemble teachers. The symbols of the lines and plots are the same as in Fig. 1.

movement of the ensemble teachers appears in the transient state in the learning process, where, in particular for the small value of the learning rate $\eta$, the maximum value of $R_{J}$ exceeds the value of $R_B$ and then the minimum value of $\epsilon^g_{J}$ reaches the fundamental minimum value even if the number $K$ is one. In the case of the fixed ensemble teachers, while the former is found only for the large $K$ and small $\eta$, the latter is hardly seen for any parameter observed. It would be interesting to see that the result of the mobile ensemble teachers weakly depends on the number of the ensemble teachers. Further, the minimum value of $\epsilon^g_{J}$ for the $K = 1$ mobile ensemble teacher is smaller than that for $K = \infty$ fixed ensemble teachers. Our study suggests that the movement of the ensemble teachers, rather than the number $K$, is important for the student learning in our model.
Fig. 8. Time dependence of the generalization error of the student $\epsilon_f$ between the student $J$ with the perceptron learning and the true teacher $A$ with $a = 0.5$ in the case of the mobile ensemble teachers. The symbols of the lines and plots are the same as in Fig. 1.

One of the drawbacks of the present model is that the minimum of $\epsilon_f^g$ is given as the transient state in the learning process and that no algorithm is found to stop the learning at the transient state. We point out that the perceptron learning shows a finite time interval of the transient state which gives the minimum of $\epsilon_f^g$ as shown in Fig. 8. This might be convenient in comparison to the Hebbian learning, but the explicit construction of the stopping algorithm, including a practical way, still remains to be solved in further work.

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Appendix A: Derivation of the learning dynamics for the ensemble teachers

In this appendix, we derive a set of the ordinal differential equations (3.1), (3.2) and (3.3) of the order parameters for the ensemble moving teachers in our model. From the update rules of the ensemble teachers of eq. (2.21), a standard calculus leads to the following ordinal differential equations in terms of the average over the correlated Gaussian variables,

$$\frac{dl_{B_k}}{dt'} = \langle f_k v_{B_k} \rangle + \frac{\langle f_k^2 \rangle}{2l_{B_k}}$$  \hspace{1cm} (A-1)

$$\frac{dR_{B_k}}{dt'} = -\frac{R_{B_k}}{l_{B_k}} \frac{dl_{B_k}}{dt'} + \frac{\langle f_k v \rangle}{l_{B_k}}$$ \hspace{1cm} (A-2)

$$\frac{dq_{kk'}}{dt'} = -\frac{q_{kk'}}{l_{B_k}} \frac{dl_{B_k}}{dt'} - \frac{q_{kk'}}{l_{B_k'}} \frac{dl_{B_k'}}{dt'}$$

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where the continuous time $t'$ is defined by the thermodynamic limit of $m'/N$ with $m'$ being the time step of the ensemble teachers in eq. (2.21). The bracket $\langle \cdots \rangle$ denotes the average with respect to the multiple Gaussian distribution given in eq. (2.15). Since each component of $A$ and $B^0_k$ are generated independently from the Gaussian distribution, $A$ and $B^0_k$ with any $k$ are orthogonal to each other in the thermodynamic limit. Then, the initial conditions of the differential equations for $R_{Bk}$ and $q_{kk'}$ are given by

$$R^0_{Bk} = 0, \quad q^0_{kk'} = 0,$$

(A·4)

One easily finds that from eqs. (A·1)-(A·3) and (A·4) that the order parameters $R_{Bk}$, $l_{Bk}$ and $q_{kk'}$ are invariant under a permutation of the index $k$ of the ensemble teachers. Because of the symmetry, we omit the subscripts $k$ from the order parameters. We can calculate sample averages in eqs. (A·1)-(A·3) and obtain

$$\langle f_k v_{Bk} \rangle = \frac{\eta_B}{\sqrt{2\pi}} \left[ R_B \left\{ 2 \exp \left( -\frac{a^2}{2} \right) - 1 \right\} - 1 \right],$$

(A·5)

$$\langle f_k^2 \rangle = 2\eta_B^2 \left( \int_{-\infty}^{-a} + \int_0^a \right) DvH \left( \frac{R_B v}{\sqrt{1 - R_B^2}} \right),$$

(A·6)

$$\langle f_k v \rangle = \frac{\eta_B}{\sqrt{2\pi}} \left\{ 2 \exp \left( -\frac{a^2}{2} \right) - R_B - 1 \right\},$$

(A·7)

$$\langle f_k^2 v_{Bk} \rangle = \langle f_k v_{Bk} \rangle = \frac{\eta_B}{\sqrt{2\pi}} \left[ R_B \left\{ 2 \exp \left( -\frac{a^2}{2} \right) - 1 \right\} - q \right],$$

(A·8)

$$\langle f_k f_{k'} \rangle = 2\eta_B^2 \left( \int_{-\infty}^{-a} + \int_0^a \right) Dv \int_0^\infty \frac{g_{Bk}}{\sqrt{1 - g_{Bk}^2}} DzH(z),$$

(A·9)

where

$$z \equiv -\frac{(q - R_B^2) x + R_B \sqrt{1 - R_B^2} v}{\sqrt{(1 - q)(1 + q - 2R_B^2)}}.$$  

(A·10)

Substituting them into eqs. (A·1), (A·2) and (A·3), the differential equations (3.1), (3.2) and (3.3) are derived.

**Appendix B: Derivation of the learning dynamics for the student**

As in the appendix A, a set of the differential equations for the student dynamics is derived in this appendix. From the update rule (2.23) of the student, the standard calculus again leads to the following equations:

$$\frac{dl}{dt} = \frac{1}{K} \sum_{k=1}^K \left( g_k u + \frac{g_k^2}{2l} \right),$$

(B·1)
\[
\frac{dR_J}{dt} = - \frac{R_J}{L} \frac{dl}{dt} + \frac{1}{K} \sum_{k=1}^{K} \frac{\langle g_k v \rangle}{l}, \quad (B-2)
\]
\[
\frac{dR_{B_kJ}}{dt} = - \frac{R_{B_kJ}}{L} \frac{dl}{dt} + \frac{R_{B_kJ}}{l_{B_k}} \frac{dl_{B_k}}{dt}
\]
\[
+ \frac{1}{K} \sum_{k'=1}^{K} \left( \frac{\langle f_k u \rangle}{l_{B_k}} + \frac{\langle g_k' v_{B_k} \rangle}{l_{B_k}} + \frac{\langle f_k g_k' \rangle}{l_{B_k}} \right), \quad (B-3)
\]

where \( t \) denotes a continuous time defined by \( t = m/N \). As an initial condition of eqs. (B-2) and (B-3), we take
\[
R_J^0 = 0, \quad R_{B_kJ}^0 = 0, \quad (B-4)
\]

since \( A, B_k^0 \) and \( J^0 \) are orthogonal to each other in the thermodynamic limit. It is shown from eqs. (B-4) and (B-3) that the order parameter \( R_{B_kJ} \) does not depend on the index \( k \). Then, one can omit the subscript \( k \) from the order parameter without loss of the generality.

By substituting the two update functions \( g \) of the Hebbian and the perceptron learning respectively, one calculates the Gaussian averages in eqs. (B-1)-(B-3) in the case of the Hebbian learning as
\[
\langle g_k u \rangle = \eta \sqrt{\frac{2}{\pi}} R_{BJ}, \quad (B-5)
\]
\[
\langle g_k^2 \rangle = \eta^2, \quad (B-6)
\]
\[
\langle g_k v \rangle = \eta \sqrt{\frac{2}{\pi}} R_B, \quad (B-7)
\]
\[
\langle f_k u \rangle = \frac{\eta_B}{\sqrt{2\pi}} \left[ R_J \left\{ 2 \exp \left( - \frac{a^2}{2} \right) - 1 \right\} - R_{BJ} \right], \quad (B-8)
\]
\[
\langle g_k' v_{B_k} \rangle = \eta \sqrt{\frac{2}{\pi}} q \delta_{k,k'}, \quad (B-9)
\]
\[
\langle f_k g_k' \rangle = -2\eta \eta_B \left[ \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) Dv \int_{-R_{BJ}}^{-\frac{R_{BV}}{\sqrt{1-R_B^2}}} Dx \{ 2H(z) - 1 \} \right], \quad (B-10)
\]
\[
\langle f_k g_k \rangle = -2\eta \eta_B \left( \int_{-\infty}^{-a} + \int_{0}^{a} \right) Dv H \left( -\frac{R_{BV}}{\sqrt{1-R_B^2}} \right), \quad (B-11)
\]

and in the case of the perceptron learning as
\[
\langle g_k u \rangle = \frac{\eta}{\sqrt{2\pi}} (R_{BJ} - 1), \quad (B-12)
\]
\[
\langle g_k^2 \rangle = \frac{\eta^2}{\pi} \tan^{-1} \left( \frac{\sqrt{1-R_{BJ}^2}}{R_{BJ}} \right), \quad (B-13)
\]
\[
\langle g_k v \rangle = \frac{\eta}{\sqrt{2\pi}} (R_B - R_J), \quad (B-14)
\]
\[
\langle f_k u \rangle = \frac{\eta_B}{\sqrt{2\pi}} \left[ R_J \left\{ 2 \exp \left( -\frac{a^2}{2} \right) - 1 \right\} - R_{BJ} \right], \quad (B-15)
\]

\[
\langle g_k v_{B_k} \rangle = \frac{\eta}{\sqrt{2\pi}} (q\delta_{k,k'} - R_{BJ}), \quad (B-16)
\]

\[
\langle f_k g_{k'} \rangle = 2\eta_B \left( \int_{-a}^{-a} + \int_0^a \right) Dv \int_{-\frac{R_B v}{\sqrt{1-R_B^2}}}^{\infty} Dx \left\{ - \int_{-\infty}^{z} Dy H (-z_1) + \int_{z}^{\infty} Dy H (z_1) \right\}, \quad (B-17)
\]

\[
\langle f_k g_k \rangle = 2\eta_B \left( \int_{-a}^{-a} + \int_0^a \right) Dv \int_{-\frac{R_B v}{\sqrt{1-R_B^2}}}^{\infty} Dx \left\{ 2H(z_2) - 1 \right\}. \quad (B-18)
\]

Here, \(z_1\) and \(z_2\) are defined as
\[
z_1 \equiv -\frac{(R_{BJ} - R_{BR}R_J) \left( \sqrt{1-qy} + \sqrt{1+q-2R_B^2x} \right) + R_J \sqrt{(1-R_B^2)(1+q-2R_B^2)v}}{\sqrt{(1-R_B^2)} \left\{ (1+q)(1-R_J^2) - 2(R_B^2 - 2R_BR_JR_{BJ} + R_{BJ}^2) \right\}} \quad (B-19)
\]

and
\[
z_2 \equiv -\frac{(R_{BJ} - R_{BR}R_J)x + R_J \sqrt{1-R_B^2}v}{\sqrt{1-R_J^2 - R_{BJ}^2 - 2R_BR_JR_{BJ}}}, \quad (B-20)
\]

and \(\delta_{k,k'}\) is the Kronecker delta defined by
\[
\delta_{k,k'} = \begin{cases} +1, & k = k', \\ 0, & k \neq k'. \end{cases} \quad (B-21)
\]

Inserting (B-5)-(B-11) and (B-12)-(B-18) into (B-1)-(B-3) gives the dynamical equations (3.8)-(3.10) for the Hebbian rule and those (3.11)-(3.13) for the perceptron one, respectively.
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