Analysis on the synchronized network of Hindmarsh–Rose neuronal models

Youceng Feng 1, Wei Li 2
1 College of Electronics and Information Engineering, South-Central University for Nationalities, Wuhan, 430074, China
2 Complexity Science Center, and Institute of Particle Physics, Hua-Zhong Normal University, Wuhan, 430079, China
E-mail: fengyc@mail.scuec.edu.cn, liw@mail.ccnu.edu.cn

Abstract. We studied a network of pulse-coupled Hindmarsh-Rose neurons and discovered that all that matters for the onset of complete synchrony is the number of signals, \( k \), received by each neuron. This is independent of all other details of the network structure.

1. Introduction
The problem of synchronization of coupled neurons of the system is the key to study the brain information processing. The collective behaviour of various coupled neural networks is the focus of the research [1]. Model studies of neuronal synchronization can be separated into those where threshold models of integrate-and-fire type are used [2] and those where conductance-based spiking and bursting models are employed [3]. In contrast to coupled spiking neurons, whose synchronous dynamics is relatively simple, interacting bursting neurons may exhibit different forms of synchrony [4]. Typically, burst synchronization arises at a low coupling strength whereas complete synchrony, which involves both burst and spike synchronization regimes, requires a stronger coupling. The coupling is pulsatile and often modeled as a static sigmoidal nonlinear input-output function with a threshold and saturation [5].

One important question about interacting neurons with such linear and pulsatile couplings is that of complete synchronization. What are the conditions for the stability of the complete synchronous state, especially with respect to coupling strengths and coupling configurations of the network? This problem was intensively studied for linearly coupled networks of bursting neurons [6, 7], and more generally, of limit-cycle and chaotic oscillators [8, 9]. In particular, it has been shown that synchrony in such networks strongly depends on the structure and size of the network.

The purpose of this letter is to investigate the synchronous stability of a network of pulse-coupled Hindmarsh-Rose (HR) neurons [10]. We discovered that all that matters for the onset of complete synchrony is the number of signals, \( k \), received by each neuron.

2. The Hindmarsh-Rose neuronal model
The HR model of neuronal activity is aimed to study the spiking-bursting behaviour of the potential membrane experimentally observed in a single neuron. In fact, the original Hodgkin-Huxley equations cannot describe bursting but only reproduce spiking, while the HR model describes the bursting by taking into account slow currents. Concerning HR model, the relevant variable is the membrane
potential, \( x(t) \) (dimensionless units). There are two more variables, \( y(t) \) and \( z(t) \), which take into account the transport of ions across the membrane through the ion channels. The transport of sodium and potassium ions occurs through fast ion channels, and its rate is described by \( y(t) \), labelled as the spiking variable. The transport of other ions occurs by slow channels, and is taken into account through \( z(t) \), the bursting variable. The corresponding mathematical model is a system of three first-order ordinary differential equations with dimensionless dynamical variables \( x(t) \), \( y(t) \) and \( z(t) \). They read:

\[
\begin{align*}
\dot{x} &= ax^2 - x^3 - y - z, \\
\dot{y} &= (a + \alpha) x^2 - y, \\
\dot{z} &= \mu (bx + c - z).
\end{align*}
\] (1)

It is well known that the HR neuronal model presents a chaotic behaviour for many neuronal parameters. For the variables in Equation (1), the physical parameters generating a chaotic behaviour are: \( a=2.8 \), \( \alpha=1.6 \), \( c=5 \), \( b=9 \), \( \mu=0.001 \). Figure 1 shows the corresponding time series. The chaotic transition between bursting periodic and spiking periodic solutions, the strange attractor and the time evolution of the action potential, can be found in Ref. [11].

**Figure 1.** The square-wave bursting of the HR model and the corresponding time series.

**Figure 2.** The phase graph of the HR model and the synaptic thresholds \( \Theta \).

3. The network of identical Hindmarsh-Rose neuronal models

Consider now a network of \( n \) synaptically coupled HR models. The equations of motion are the following:

\[
\begin{align*}
\dot{x}_i &= ax_i^2 - x_i^3 - y_i - z_i - g_x(x_i - V_s) \sum_{j=1}^{n} c_{ij} \Gamma(x_j), \\
\dot{y}_i &= (a + \alpha)x_i^2 - y_i, \\
\dot{z}_i &= \mu(bx_i + c - z_i).
\end{align*}
\] (2)

Here, \( i, j = 1, \ldots, n \), the neurons are identical and the synapses are fast and instantaneous. The parameter \( g_x \) is the synaptic coupling strength. The reversal potential \( V_s > x_i(t) \) for \( \forall x_i \) and \( \forall t \), i.e., the synapse is excitatory. The synaptic coupling function is modeled by the sigmoid function \( \Gamma(x_j) = 1/[1 + \exp(-\lambda(x_j - \Theta_s))] \) (a limiting version of \( \Gamma(x_j) \) is the Heaviside function). This oft-used coupling form was called fast threshold modulation by Somers and Kopell [12]. The threshold \( \Theta_s \) is chosen such that every spike in the single neuron burst can reach a threshold (see Figure 2).

Hereafter, \( \Theta_s = -0.25 \) and \( V_s = 2 \). \( C = (c_{ij}) \) is the \( n \times n \) connectivity matrix: \( c_{ij} = 1 \) if neuron \( i \) is connected to neuron \( j \), \( c_{ij} = 0 \) otherwise, and \( c_{ii} = 0 \). Matrix \( C \) can be asymmetric if both mutual and unidirectional couplings are allowed.
4. Transversal stability of the synchronization manifold

We require equal row sums \( k = \sum_{j=1}^{n} c_{ij}, i = 1, \ldots, n \). This requirement is a necessary condition for the existence of the synchronous solution, namely, the invariance of hyperplane \( D = \{ \xi_1(t) = \xi_2(t) = \cdots = \xi_n(t) \}, \xi_i = (x_i, y_i, z_i) \) and \( i = 1, \ldots, n \).

In fact, the equal row-sum property implies a network where each cell has the same number \( k \) of inputs from other neurons. Synchronous behaviour on the manifold \( D \) is generated by the system:

\[
\begin{align*}
\dot{x} &= ax^2 - x^3 - y - z - kg_1(x-V_x)\Gamma(x), \\
\dot{y} &= (a+\alpha)x^2 - y, \\
\dot{z} &= \mu(bx+c-z).
\end{align*}
\]

By introducing the difference between the neural oscillator coordinates \( \xi_i = x_i - x, \eta_i = y_i - y, \zeta_i = z_i - z, i, j = 1, \ldots, n \), and with the limit of infinitesimal differences, we can change the stability equations for the transverse perturbations to the synchronization manifold \( D \):

\[
\begin{align*}
\dot{\xi_i} &= (2ax - 3x^2)\xi_i - \eta_i - \zeta_i - kg_1\Gamma(x)\xi_i \\
&\quad + g_1(V_x-x)\Gamma'(x)\left( k\xi_i + \sum_{j=1}^{n} \{ c_{ij}\xi_j - c_{ji}\eta_j \} \right), \\
\dot{\eta_i} &= 2(a+\alpha)x\xi_i - \eta_i, \\
\dot{\zeta_i} &= \mu(b\xi_i-c\zeta_i). 
\end{align*}
\]

The derivatives are calculated at the point \( \xi = 0, \eta = 0, \zeta = 0 \), and \( \{ x(t), y(t), z(t) \} \) corresponds to the synchronous bursting solution defined via system (3).

The first coupling term \( S_1 = kg_1\Gamma(x)\xi_i \) accounts for the number of inputs \( k \). At the same time, the contribution of the second coupling term \( S_2 = g_1(V_x-x)\Gamma'(x)(\cdot) \) depends on the coupling configuration. Note that the term \( \sum_{j=1}^{n} \{ c_{ij}\xi_j - c_{ji}\eta_j \} \) is the same as for linear coupling [8].

In terms of the original variables \( x_i \), the corresponding coupling matrix \( G = C - kI \) is the Laplacian of the connected graph, except for a sign change.

It is well known that \( G \) has one zero eigenvalue \( \gamma_1 \) and all other eigenvalues have nonpositive real parts. If the coupling is mutual, \( G \) is symmetric and all eigenvalues are real. For simplicity, suppose that the eigenvalue \( \gamma_2 \) with the largest real part is simple. Then, by applying the linear transform that diagonalizes \( G \) to Eq. (4), we obtain the stability equation for the most unstable transverse mode:

\[
\begin{align*}
\dot{\xi} &= (2ax - 3x^2)\xi - \eta - \zeta - \Omega(x)\xi, \\
\dot{\eta} &= 2(a+\alpha)x\xi - \eta, \\
\dot{\zeta} &= \mu(b\xi - c\zeta), 
\end{align*}
\]

where \( \Omega(x) = kg_1\Gamma(x) - g_1(V_x-x)\Gamma'(x)(k+\gamma_2) \). System (5) is an analogy of the Master Stability function [8] for synaptically coupled networks (2). For a given \( x(t) \), \( \Gamma(x) \) is determinate. For different network structures, the value of \( \gamma_2 \) is variable but determined. The synchronization threshold of \( g_1 \) is inversely proportional to the number of incoming signals \( k \) [13].
5. Conclusion
In this paper, we observe the dynamics of HR model and construct coupling network with identical HR neurons. Through the analysis of transversal stability of the synchronization manifold, we obtain that the synchronization condition depends on the number of inputs $k$ but does not rely on the connectivity matrix. The onset of complete synchrony is independent of all other details of the network structure.

References
[1] Ivanchenko M V et al 2004 Phys. Rev. Lett. 93 134101
[2] Gerstner W and Van Hemmen J L 1993 Phys. Rev. Lett. 71 312
[3] Izhikevich E M 2001 SIAM Rev. 43 315
[4] Dhamala M, Jirsa V K and Ding M 2004 Phys. Rev. Lett. 92 028101
[5] Somers D and Kopel N 1993 Biol. Cybernet. 68 393
[6] Sherman A 1994 Bull. Math. Biol. 56 811
[7] Dhamala M, Jirsa V K and Ding M 2004 Phys. Rev. Lett. 92 074104
[8] Pecora L M and Carroll T L 1998 Phys. Rev. Lett. 80 2109
[9] Belykh V N, Belykh I V and Hasler M 2004 Physica D 195
[10] Hindmarsh J L and Rose M 1984 Proc. R. Soc. B 221 87
[11] Detchtgnia Djeundam S R et al. 2013 Chaos 23 033125
[12] Somers D and Kopell N 1993 Biol. Cybernet. 68 393
[13] Belykh V N, Belykh I V and Halsler M 2004 Physica D 195 159