Duality of Tropical Curves

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Abstract

Duality of curves is an important aspect of the “classical” algebraic geometry. In this paper, using this foundation, the duality of tropical polynomials is constructed to introduce the duality of Non-Archimedean curves. Using the development of an algebraic “mechanism”, based on “distortion” values, geometric and convexity properties are analyzed. Specifically, we discuss some significant aspects referring to quadrics with respect to their dual objects. This topic also includes the induced dual subdivision of the corresponding Newton Polytope and its compatible properties. Finally, a regularity of tropical curves in the duality sense is generally defined and studied for families of tropical quadrics.

Instruction

Over the last years, intensive development in the studies of the tropical algebraic geometry has been made. The special “nature” of the objects belonging to this geometry enables progress in varied directions and by using different concepts. The main significance of these tropical entities is their being geometric “images” of algebraic objects and they concurrently comprise combinatorial attributes. Recently, much efforts have been invested to characterize the tropical analogous to “classical” results and to determine the various connections between these two “worlds”. The purpose of this paper is to introduce the tropical analogous to the “classical” curves’ duality in algebraic geometry, and to outline the connection between these dualities.

As known, duality between algebraic objects is a powerful tool with many applications. This is the motivation of our work – adapting this method to the tropical case as well. Thus, understanding the geometric linkage between algebraic and tropical objects, especially between dual objects, is a preliminary step toward developing a geometric duality over polyhedra. This duality should also agree with the algebraic duality.

Algebraic objects are formally, elements of the geometry over the fields (K, +, ·), where the tropical ones are those over the geometry of the semi-ring (R, max, +) – the semi-ring which contains the Max-Plus Algebra [1, 17]. The fundamental objects in this geometry are Polyhedral Complexes, where their behavior resembles the complex algebraic varieties [11, 18]. Moreover, they may be concerned as the “images” of the Non-Archimedean valuation

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of some “superior” algebraic varieties. Historically, these objects carry the name Non-Archimedean Amebas. Using this approach, a direct relationship exits between the following involved components, the tropical hyper-surface \( V(\mathcal{F}) \) the corresponding Newton Polytope \( \Delta_F \) and the induced subdivision \( S_F \) over \( \Delta_F \) \([14, 16]\). Namely, in this theory, algebraic and combinatorial considerations together with geometric observations are all composed together.

This area of research was formally introduced for complex amebas in 1994 by Gelfand, Kapranov and Zelevinsky \([5]\). Later Kapranov firstly presented the notion of Non-Archimedean Amebas \([10]\) which may be understood as the “spines” of the complex amebas \([11]\). Since then, miscellaneous aspects within this topic have been dealt within several papers. Sturmfels, Speyer and Develin studied matters of tropical algebra \([2, 19, 20]\). Enumerative geometry has been dealt by Mikhalkin \([12, 13]\). Shustin discussed the patchworking of Non-Archimedean amoebas \([18]\) in the algebraic manner, while Itenberg regarded this issues combinatorially \([8]\). These are few of the many directions which recently concerned researchers and are spread along varied fields of study.

The goal of this paper is to introduce the geometric duality of tropical curves and the conceptional connection between this duality and that which is applied to algebraic objects. Since tropical objects are basically piece-wise linear, aiming to develop a duality we join together algebraical and combinatorial methods. Despite the fact that tropical notion of tangency has not been phrased properly yet, by relying on the duality of complex algebraic varieties and their linkage to tropical objects we overcome this obstacle and define the duality for tropical polynomials. That is our point of departure for the discussion in this paper.

Specifically, for a given tropical curve, \( V(\mathcal{F}) \) the properties of the Non-Archimedean valuation are used to “produce” compatible algebraic varieties and so does their dual, and these are “translated backward” to attain the tropical dual objects described by \( V(\mathcal{F}^*) \). Eventually, we show that this whole procedure is reduced to pure computation in terms of tropical operations. This approach disregards convexity considerations and yields only the pre-tropical duality (i.e. the dual “tropical polynomial”, \( \mathcal{F}^* \)). Thus, in order to obtain the proper tropical varieties we should involve geometric considerations which include convexity properties.

In order to observe the convexity properties of a given object we develop an algebraic “mechanism”, expressed in the sense of “distortion” values, which serves to refine the required convexity attributes. Moreover, this “mechanism” appears to be useful for additional geometric analysis. Using these methods, we study the geometric duality of tropical curves via the subdivisions of their corresponding Newton polytopes (i.e. the dual subdivision \( S_{\mathcal{F}^*} \)). This will be done mainly by focusing on “interesting” families of subdivisions which are significant for other researches. The speciality of these subdivisions is that they can be described in the terms of the polytope’s nodes, and this makes the convexity analysis more convenient.

Explicitly for the case of quadrics, we prove that a subdivision which is induced by a tropical polynomial has a one to one correspondence with the tropical curve. Moreover

- The dual subdivision \( S_{\mathcal{F}^*} \) of a subdivision \( S_{\mathcal{F}} \) that is maximal in nodes (minimal in nodes) is also maximal in nodes (minimal in nodes).
These types of subdivisions preserve their preliminary properties of convexity under the dual transformation. Advancing these ideas and enforcing the additional requirement in which not only that the resulting subdivision is regarded but also its specifier, we characterize the regularity of tropical curves in the sense of duality. This notion refers to general curves and it is discussed in detail for the case of quadrics.

Using the above notion, the regularity occurs when we apply the duality twice for a curve, and the difference between the corresponding specifiers of the result and those of the primal curve is a constant. Specifically we show that

- a curve that corresponds to a subdivision which is minimal in nodes is never regular while,
- by enforcing additional restrictions, a subdivision which is maximal in nodes can correspond to a regular curve

In this paper we present some ideas referring to general tropical curves and their duals. This includes aspects which subject to the correspondence that occurs between properties of primal and dual objects. For a deeper discussion we mainly focus on the family of quadrics which reflects the brought ideas in a comprehensible manner and also has significance for other fields of research.

**Organization:** To make this paper reasonably self-contained we provide a short overview of tropical varieties and their linkage to complex algebraic varieties. This overview includes also the construction of the associate Newton polytopes. Using this basis, in section 2 we discuss the duality of tropical polynomials (denoted as pre-tropical duality). Advancing this idea, the duality of tropical curves is introduced in section 3. We close (Sec. 4) by defining the regularity of curves in the duality sense and exam it agains families of quadrics. The appendix contains the detailed definitions and constructions of the classical duality in general case. Specifically, for the case of quadrics the dual map in matrices’ notion is developed (Sec. A.2).

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1. **Tropical Varieties**

For fluent reading, we open with a review of the fundamentals of Tropical Varieties and Non-Archimedean Amoeba, these fundamentals are spread among several different works [9, 10, 11, 12, 13, 16, 18].

1.1 **General Tropical Variety (or Non-Archimedean Amoeba)**

Let \(\mathbb{K}\) be an algebraically closed field with a valuation

\[
\text{Val} : \mathbb{K} \rightarrow \mathbb{R} \cup \{-\infty\}.
\]

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For our concern the field $K$ is assumed to be the field of convergent Puiseux Series, over the complex numbers $\mathbb{C}$, of the form

$$a(t) = \sum_{\tau \in R} c_{\tau}t^{\tau}, \quad (1.1.1)$$

where $R \subset \mathbb{Q}$ is bounded from below and the elements of $R$ have a bounded denominator. More specifically, $R$ is contained in the sum of finitely many arithmetic progressions which are bounded from below and satisfy,

$$\sum_{\tau \in R} |c_{\tau}|t^{\tau} < \infty,$$

for sufficiently small positive $t \neq 0$.

The induced Non-Archimedean Valuation over this field is defined as minus the smallest $\tau$ for which the coefficient $c_{\tau} \neq 0$, formally,

$$Val(a(t)) = -\min\{\tau \in R : c_{\tau} \neq 0\}. \quad (1.1.2)$$

This valuation takes $K^*$ onto $\mathbb{Q}$ and one can verify that for any $a, b \in K^*$ it satisfies the required relations,

$$Val(a \cdot b) = Val(a) + Val(b),$$

$$Val(a + b) \leq \max\{Val(a), Val(b)\}, \quad (1.1.3)$$

of being Non-Archimedean. Thus, the valuation maps a series $a(t)$ into an element of the semi-ring which defined over $\mathbb{R} \cup \{-\infty\}$. Further on, this semi-ring will be described explicitly. Note that, the symbol $-\infty$ is used as the image of the zero element of $K$ under the valuation $Val$.

Based on the above valuation we define of the Tropical Varieties (i.e. the Non-Archimedean Amoeba) as the closure in $\mathbb{R}^n$ of the image of some “superior” algebraic varieties that were placed on $K^n$. This image corresponds to the map defined by $Val$ while the primal elements (i.e. the algebraic variety) are contained in the zero set of a system of Laurent Polynomials with coefficients in $K^*$. Formally, let $I \subset \mathbb{Z}^n$ be a non-empty set, denote by $F_{K^*}(I)$ the family of Laurent polynomials with coefficients in $K^*$ having the form,

$$f(z) = \sum_{\omega \in I} c_{\omega}z^{\omega}, \quad (1.1.4)$$

where $z$ stands for the $n$-tuple $(z_1, \ldots, z_n)$. Let, $f$ be a polynomial in $F_{K^*}(I)$, and let

$$Z_f = \{z \mid f(z) = 0\} \subset (K^*)^n,$$

be its zero locus. The Non-Archimedean amoeba $A_f$ that corresponds to $f$ is defined to be

$$A_f = \overline{Val(Z_f)} \subset \mathbb{R}^n, \quad (1.1.5)$$

where $Val(z_1, \ldots, z_n) = (Val(z_1), \ldots, Val(z_n))$ and $\overline{Val(Z_f)}$ is the closure of $Val(Z_f)$. The set of all amoebas $A_f$, while $f \in F_{K^*}(I)$, is denoted by $A(I)$. In case $I$ is the complete set of the integral points included in a given lattice polygon $\Delta$ the amoebas’ set is signed by $A(\Delta)$. 

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1.2 Tropical Hyper-Surfaces

Regarding tropical hyper-surfaces, the first fundamental result that is brought is due to Kapranov [10]. This result associates between “classical” algebraic objects and “tropical” ones (the proof can also be found in [3] or [18]).

**Theorem 1.1. (Kapranov).** The Non-Archimedean amoeba $A_f$ coincides with the corner locus of the convex piece-wise affine linear function

$$\mathcal{N}_f(x) = \max_{\omega \in I}(\text{Val}(c_\omega) + \omega \cdot x),$$

where $x \in \mathbb{R}^n$.

**Notation & terminology:**

- The term corner locus means domain of non-smoothness.
- Unless otherwise specified, the product of the vectors as it appears here (i.e. $\omega \cdot x$), is holds as the standard scalar product.

Moreover, Kapranov’s theorem implies that not only that every tropical hyper-surface is a corner locus of a tropical polynomial, but also that any corner locus of a tropical polynomial $\mathcal{F}$ is a tropical hyper-surface: simple take $f = \sum_{\omega \in I} c_\omega z^\omega$ such that $\text{Val}(c_\omega) = a_\omega$, this kind of setting is always possible.

The above theorem lays the foundation for understanding the means of what is called “Tropical Geometry”. By this manner, $\mathcal{N}$ is regarded as a “polynomial” description of a geometric structure which is embedded in the semi-ring defined over $\mathbb{R} \cup \{-\infty\}$. Hence, $\mathbb{R}$ can be considered as associated with the two following “tropical operations”,

$$x \oplus y = \max\{x, y\}, \quad x \odot y = x + y.$$  

The triple $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ is referred to as the tropical semi-ring.

**Remark 1.2.** In the above definition of the tropical addition, one may equivalently replace the $\max$ operation by $\min$ operation. However, as will be seen later on, using the operation $\max$ appears to be more convenient for the duality considerations, especially when the corresponding geometric objects are constructed in respect to Newton Polytope.

Composing all the above settings together, the obtained result is a Tropical Polynomials in $n$ variables which are defined over $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, and have the form

$$\mathcal{F}(x) = \bigoplus_{\omega \in I} a_\omega \odot x^\omega,$$

where $x_{i_1}^{\omega_1} \ldots \odot x_{i_k}^{\omega_k}$ stands for $x_{i_1} \odot \ldots \odot x_{i_k}$, $\omega_k$ times. Note that a tropical polynomial can actually be interpreted (using the “regular” operation of addition and multiplication) as

$$\mathcal{F}(x) = \max_{\omega \in I}(a_\omega + x \cdot \omega),$$

which is a convex function on $\mathbb{R}^n$. As a result $\mathcal{F}$ can be considered as functional $\mathbb{R}^n \rightarrow \mathbb{R}$ and one can observe its corner locus (see Fig. 1).

**Comment 1.3.** Geometrically, using the above notion, polynomial descriptions of piece-wise linear objects are produced. These descriptions will serve us later in order to manipulate the corresponding objects.
Figure 1: The tropical curve is the projection of the corner locus of $\Delta_{\nu_f}$, which is “placed” above.

1.3 Amoebas, Newton Polytope and Subdivisions

For a given polynomial equation (either, tropical or classical) over a set of indices $I \subset \mathbb{Z}^n$ the Newton Polytope $\Delta$ (polygon in the planar case) is defined to be the convex hull, $\mathcal{C}H(I)$, of all the elements of $I$ [5]. Note that, when concerting a tropical polynomial in respect to the previous definition, all the coefficients take part. Particularly, zero coefficients should also appear. As will be explained, this construction of Newton Polytopes validates the linkage between the algebraic entities and their corresponding geometry objects.

Let $f \in F_k(I)$ be any polynomial in $n$ variable of the form (1.2.1), and let $\Delta_f \subset \mathbb{R}^n$ be its corresponding Newton polytope, then $f$ induces a natural subdivision $S_f$ on $\Delta_f$. This subdivision is obtained as follows. First, over the points of $\Delta_f$ define the Lifting Map

$$\nu_f : \Delta_f \rightarrow \mathbb{R},$$

where, $\nu_f(\omega) = -\text{Val}(c_\omega) = -a_\omega$ (remember that the $\omega$’s are integral points). Then, construct the set

$$J_f = \{(\omega, u) \in I \times \mathbb{R} \mid u > \nu_f(\omega)\},$$

this set is contained in $\mathbb{R}^{n+1}$. Next, compute the convex hull, $\Delta_{\nu_f}$, of $J$ and define the function

$$\tilde{\nu}_f : \Delta_f \rightarrow \mathbb{R},$$

according to the relation

$$\tilde{\nu}_f(\omega) = \min\{u \mid (\omega, u) \in \Delta_{\nu_f}\}.$$  

This is a convex piece-wise linear function whose domain is the preliminary polytope $\Delta_f$ while its vertices are in $I$. Note that, when an $\omega \in I$ is placed on the “lower part” of $\Delta_{\nu_f}$ we have $\tilde{\nu}_f(\omega) = \nu_f(\omega) = -a_\omega$.

Eventually, in order to obtain the required subdivision $S_f$ of $\Delta_f$, take the “lower part” $\partial\mathcal{C}H(\Delta_{\nu_f})$ of this convex hull, i.e. the graph of the function $\tilde{\nu}_f$, and project its non linear part onto $\Delta_f$ to obtain $S_f$ – the Polytope’s Subdivision is induced by $f$ (see Fig. 2).

Using the previous construction of the subdivision we can link between the algebraic description and the geometric notion of the corresponding tropical object. Globally, there are three different kinds of entities with reciprocal relationships which play a role in this
view. The “types” of entities are, polynomial equations, polyhedral objects and tropical curves. The linkage between the two first “types” has been already described, and it remains to understand how do the tropical objects fit in. For this part we will be assisted by subdivisions of Newton polytopes which have been defined previously.

In general, any component of a subdivided polyhedron is a polyhedron by itself. Let $\sigma$ be such a component in respect to the subdivision $S_f$ of $\Delta_f$ (i.e. $\sigma \subseteq \Delta_f$) and let $\sigma_\nu \subseteq \Delta_\nu f$ be the polyhedron above it (i.e. $\text{Proj}(\sigma_\nu) = \sigma$). Assume $\sigma_\nu$ and $\tau_\nu$ to be two polyhedrons of the same subdivision $S_f$ such that $\sigma_\nu \subset \tau_\nu$, and denote by

$$Cone(\tau_\nu, \sigma_\nu) = \{ t(x - y) \mid t \geq 0, x \in \sigma_\nu, y \in \tau_\nu \},$$

the cone of vectors pointing from $\tau_\nu$ into $\sigma_\nu$. Using regular duality in cones’ sense (for more details see [4, 15]), the dual set can be defined as

$$DualCone(\tau_\nu, \sigma_\nu) = \{ \xi \in \mathbb{R}^{n+1} \mid \xi \cdot x \leq 0, \forall x \in Cone(\tau_\nu, \sigma_\nu) \}.$$

This set appeared to be also a proper cone. Clearly, using the same manner, such dual cones can be defined for any pair of polyhedrons compounded in the subdivision $S_f$.

Since $\Delta_\nu f$ is by itself a cone (i.e the maximal cone), we can apply the above considerations to obtain its common dual cone, $DualCone(\sigma_\nu, \Delta_\nu f)$, with respect to any “smaller” $\sigma_\nu$. One can easily verified that this cone is contained in the set $\{ x_{n+1} \leq 0 \}$ and intersects the hyper-plane $\{ x_{n+1} = -1 \}$. The $Cone(\sigma_\nu, \Delta_\nu f)$ is contained in a half space (the cone of a non vertical $n$-face of $\Delta_\nu f$ containing $\sigma_\nu$), which is always the half-space above the (non-vertical) $n$-space cutting $\mathbb{R}^{n+1}$. Thus, $DualCone(\sigma_\nu, \Delta_\nu f)$ contains at least the dual of this half space, which is the semi-line intersecting $\{ x_{n+1} = -1 \}$.

Additionally, from the definition of $\Delta_\nu f$, $Cone(\sigma_\nu, \Delta_\nu f)$ always contains a vertical semi-line (upper part), hence, $DualCone(\sigma_\nu, \Delta_\nu f)$ is comprised in the half space $\{ x_{n+1} \leq 0 \}$. Thus a corresponding polyhedron

$$\sigma^* = \{ x \in \mathbb{R}^n \mid (x, -1) \in DualCone(\sigma_\nu, \Delta_\nu f) \}$$

can be defined. Fig. 3 shows this relation for one dimensional space. In fact, with this construction, a subdivision of $\mathbb{R}^n$ is defined, which in the cones’ sense is dual to the subdivision $S_f$ of $\Delta_f$. Namely, $\sigma \subset \tau$ if and only if $\tau^* \subset \sigma^*$ and for this case $Cone(\sigma, \tau)$ is the dual
of $\text{DualCone}(\tau^*, \sigma^*)$. Here, finally, the required connection between the different objects is cleared up.

Lemma 1.4. The polyhedral subdivision of $\mathbb{R}^n$ given by the $\sigma^*$ where $\sigma$ ranges through the polyhedra of the subdivision $S_f$ of $\Delta_f$ is exactly the subdivision obtained by the corner locus of the Legendre transform of $\nu_f$, which is

$$\tilde{\nu}_f(x) = \max_{\omega \in I} \{ \omega \cdot x - \nu_f(\omega) \}.$$ 

It has already been shown that $\tilde{\nu}_f(x)$ is exactly the evaluation of the tropical polynomial $F$ in $x$. This means that the subdivision of $\mathbb{R}^n$ is determined by the tropical varieties which correspond to $F$ is dual to the subdivision of the compatible Newton polytope of $F$, see Fig. 3. One can also see that $\sigma^*$ is orthogonal to $\sigma$ while the dimension of $\sigma$ equals the codimension of $\sigma^*$ (the detailed proof appears in [16]).

Remark 1.5. Concerning the above geometric observation, characterizing relations over subdivisions of Newton polytope are equivalent to specifying relations over tropical objects. Hence, during our discussion we can base our self on this insight and develop our theory in respect to polyhedral objects (i.e. components of subdivisions of Newton polytope) where the exact translation to tropical objects is that which is described above.

2 Pre–Tropical Duality

As will be discussed later, the tropical duality is practically induced by the duality which occurs between algebraic verities. In general, any polynomial $f$ over $\mathbb{K}$ has a dual polynomial $f^*$, each has its own lifting (i.e. support) function $\nu_f: \Delta_f \to \mathbb{R}$ (res. $\nu_{f^*}: \Delta_{f^*} \to \mathbb{R}$) defined over the integral points of its Newton polytope $\Delta_f$ (res. $\Delta_{f^*}$). Using this notion we establish a duality between the lifting functions and between their subdivisions $S_f$ ($S_{f^*}$) respectively. Moreover, since the values of these lifting functions over integral points are the coefficients of the compatible tropical polynomials $F$ and $F^*$, a duality between tropical polynomials is eventually attained. This duality is called Pre–Tropical Duality while the tropical duality is the duality which occurs between their corresponding convex functions $\tilde{\nu}_f$ and $\tilde{\nu}_{f^*}$. The last duality can be understood as the result of applying convexity considerations to the
pre-tropical duality. At the very end, we will show that the primal algebraic duality can be disregarded, and we can remain with duality in tropical terms only.

2.1 Pre-Tropical Duality – Construction

Let $F$ be a tropical polynomial of the form (1.2.1), we intend to define its dual $F^*$. In order to apply the above considerations, we should specify a corresponding polynomial $f \in F_K(I)$ of the form (1.1.4) such that the coefficients of $F$ are the valuations of the coefficients of $f$ (Theorem 1.1). This, specification can be realized as the “inverse” valuation. Recall that, the values of these valuations are the images of the lifting function $\nu_f$ (1.3.1) in respect to the points of $I$. However, concerning the definition of the Non-Archimedean valuation (1.1.2) it is clear that such $f$ in not unique and there is a complete family of such polynomials. Moreover, with this limited information (i.e. the valuations’ results) we can’t “recover” an explicit equation of a polynomial in $K$. But as will be seen, due to the reapplying of the valuation, the precise determination of such $f$ is not necessary, and we can avoid it. The general view over the construction we intend to introduce can be outlined by the following diagram:

$$
\begin{array}{c}
f \xrightarrow{\text{Dual}} f^*
\end{array}
$$

where the “missing” part is $Val^{-1}$.

Before getting on to our development let us first emphasize an additional important property that relates to Non-Archimedean valuation.

**Remark 2.1.** Let $Q = \sum_{\xi \in J} \pm \epsilon^\xi$ be a polynomial, by applying the valuation to $F$ the following relation is obtained

$$
Val(F) = Val(\sum_{\xi \in J} \pm \epsilon^\xi) \leq \bigoplus_{\xi \in J} Val(\epsilon^\xi). \tag{2.1.1}
$$
In the case that only single component achieves the maximum, the weak inequality is inverted to be equality.

Now let us begin our discussion. Let $F$ be a tropical polynomial of the form (1.2.1), the lifting function $\nu_\bullet : \Delta^\nu \to \mathbb{R}$ is defined as $\nu_\bullet(\omega) = -a_\omega$ for any $\omega \in I$. Note that, the sign "•" is being used since we don’t have an explicit polynomial $f$. According to the previous values we have to produce a Laurent polynomial $f \in F_K(I)$ of the from (1.1.4) where each of its coefficients, $c_\omega$, satisfies $Val(c_\omega) = a_\omega$. Namely, any “tropical coefficient” $a_\omega$ should be mapped as following

$$a_\omega \mapsto c_\omega = \sum_{\tau \in R} c_\tau^{(\omega)} t^\tau,$$

where $Val(c_\omega) = a_\omega$.

Indeed, in order to achieve the above, an additional requirement should be imposed, the coefficients $c_\omega$’s of $f$ should be generic. Specifically, generic means that any two $c_\omega_1$ and $c_\omega_2$ which satisfy

$$Val(c_\omega_1) = Val(c_\omega_2) = \min \{ \tau \in R : c_\tau \neq 0 \}$$

have different coefficients in their leading monomials. Namely, $c_\tau^{(\omega_1)} \neq c_\tau^{(\omega_2)}$ for $\tau = -a_\omega$. Moreover, any combination in $c_\omega$’s dose not zero their leading monomials. This requirement meant to enable an accurate transform to the tropical semi-ring even when combined expressions over these $c_\omega$’s are concerned.

The above restriction also insures that the relations which are satisfied by the valuation are all interpreted as strong equalities. Obviously, such generic specification of $f$ is possible and it is not unique. Thus, once we select such generic polynomial $f$ it is to be regarded as the representative of the complete family of such polynomials having generic coefficients $c_\omega$’s and satisfy $Val(c_\omega) = a_\omega$. Therefore, using such representative $f$, the following correspondences can be outlined

$$F \rightarrow \nu_\bullet \rightarrow f.$$

Namely, $f$ can be related to as the image of $F$ under the “inverse” valuation $Val^{-1}$ (i.e. $Val^{-1}(F) = f$).

In the next step, one can apply the “classical” dual transform$^2$ to $f$, i.e. Dual : $f \rightarrow f^\ast$. Once, the dual $f^\ast$ had been characterized, the valuation can be applied to $f^\ast$ which will yield $F^\ast$. Let us develop this part explicitly. Let $f^\ast$ be the dual polynomial of the form,

$$f^\ast(z) = \sum_{\omega \in I^\ast} c_\omega^\ast z^\omega,$$

by the properties of “classical” duality, any $c_\omega^\ast$ is a polynomial in $c_\omega$’s (i.e. $c_\omega^\ast \in Pol \omega \in I(c_\omega)$ where $c_\omega \in \mathbb{K}$). Assume $m = |I \cap \mathbb{Z}|$ then, by applying the Non-Archimedean valuation to $c_\omega^\ast$ we have,

$$Val(c_\omega^\ast) = Val(\sum_{\xi \in J} \pm c_\xi) \leq \bigoplus_{\xi \in J} Val(c_\xi) = \bigoplus_{\xi \in J} \bigcap_{\omega \in I} \xi_\omega Val(c_\omega),$$

$^2$For those who are not familiar with this topic, a detailed description, including the explicit construction, appears in the appendix.
where $J \subset \mathbb{Z}^m$ and $c = (c_{\omega_1}, \ldots, c_{\omega_m})$. Since the preliminary selection of $f$ was to be generic, referring to Remark 2.1 the weak inequality is inverted to equality. Let $Val(c_{\omega}) = a_{\omega}$, then the above equation can be written as

$$Val(c^*_\varpi) = \bigoplus_{\xi \in J, \omega \in I} \xi_{\omega} a_{\omega} = a^*_\varpi, \quad (2.1.3)$$

where $a^*_\varpi$ is a proper tropical equation in $a_{\omega}$'s for any $\varpi \in J$. Thus, the valuation of any coefficient $c^*_\varpi$ of $f^*$ can be described in tropical terms of the valuations of the coefficients $c_{\omega}$ of $f$. Thus, the involved values are only the $a_{\omega}$'s up to the multiplications by integers.

**Corollary 2.2.** Any $a^*_\varpi \in TropPoly_{\omega \in I}(a_{\omega})$ and thus we construct a direct map

$$\mu : \{a_{\omega} \mid \omega \in I\} \longrightarrow \{a^*_\varpi \mid \varpi \in J\}.$$ 

Clearly, this map is independent in a specific selection of a representative $f$ unless it has generic coefficients. Eventually, according to $\mu$ we can define the tropical duality

$$Dual : F \longrightarrow F^*.$$ 

where

$$F^*(x) = \bigoplus_{\varpi \in I} a^*_\varpi \odot x^\varpi. \quad (2.1.4)$$

**Observation 2.3.** In fact, we have shown that in order to characterize the $F^*$ we can disregard the returning “backward” to computation over the elements of $\mathbb{K}$ and remain only in the tropical semi-ring. In particular, the explicit construction of $\mu$ is done similarly to that which is applied to the members of $\mathbb{K}$ (for the complete algorithm, see Appendix) only by replacing the standard operations by tropical ones.

Based on the above observation, assume that $c^*_\varpi \in Poly_{\omega \in I}(c_{\omega})$ was computed, we will outline the rules of operations’ translation,

$$Val : c^*_\varpi \longrightarrow a^*_\varpi$$

where $a^*_\varpi \in TropPoly_{\omega \in I}(a_{\omega})$. In general, the basic operations are switched by tropical ones,

$$c_{\omega_1} + c_{\omega_2} \longrightarrow a_{\omega_1} \oplus a_{\omega_2},$$
$$c_{\omega_1} \cdot c_{\omega_2} \longrightarrow a_{\omega_1} \odot a_{\omega_2},$$

where $Val(c_{\omega_1}) = a_{\omega_1}$. Note that any constant multiplier (including negative signs) of $c_{\omega_i}$ is neglected, and the following maps are valid

$$n \cdot c_{\omega} \longrightarrow a_{\omega},$$
$$-c_{\omega} \longrightarrow a_{\omega},$$

where $n \in \mathbb{Z}$. 

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Conclusion 2.4. Two polynomials in the tropical view are equivalent up to the tropical addition of similar monomials, namely any sequence of similar monomials,

\[(a_{\omega_0} \odot x^{\omega_0}) \oplus \ldots \oplus (a_{\omega_0} \odot x^{\omega_0}) \simeq a_{\omega_0} \odot x^{\omega_0},\] (2.1.5)
can be regarded as a single appearance.

Note that in this case, since polynomials equations are involved, this consideration is allowed. This operation is interpreted differently while constant values are concerned, and in that case the tropical addition of equal values is regarded as minus infinity.

To summarize, using the previous rules, we can perform all the calculations over the tropical semi-ring. This occurs after applying the valuation to dual polynomial of that have been constructed using the “inverse valuation”,

\[Val(\ Poly_{\omega \in I}(Val^{-1}(a_\omega))) = Val(\ Daul(Val^{-1}(a_\omega))) .\]

In the next section these ideas are demonstrated for conics.

Example 2.5. Let \( F \) be the tropical polynomial of the form

\[F = a_1 \odot x_1 \odot x_2^2 \oplus a_2.\]

Apply \( Val^{-1} \) to obtain \( f = c_1(t)z_1z_2^2 + c_2(t) \) in \( \mathbb{K} \) such that \( Val(c_1(t)) = a_1, \) thus \( F = c_1(t)z_1z_2^2 + c_2(t)z_0^3. \) Substitute

\[z_0 \mapsto -(\frac{a_1 u + a_2 v}{a_0}), \quad z_1 \mapsto u, \quad z_2 \mapsto v,\]

so that

\[\tilde{F}(u, v)_{a_0, a_1, a_2} = F(-\frac{a_1 u + a_2 v}{a_0}, a_0 u, a_0 v) = c_2a_1^3u^3 + 3c_2a_1^2a_2u^2v + (c_1a_0^3 + 3c_2a_1a_2^2)v^2 + c_2a_2^3v^3.\]

and its two derivatives are

\[\frac{\partial \tilde{F}}{\partial u} = 3c_2a_1^3u^2 + 6c_2a_1^2a_2uv + (c_1a_0^3 + 3c_2a_1a_2^2)v^2,\]
\[\frac{\partial \tilde{F}}{\partial v} = 3c_2a_1^2a_2u^2 + 2(c_1a_0^3 + 3c_2a_1a_2^2)vu + 3c_2a_2^3v^2.\]

Their resultant is

\[Res(a_0, a_1, a_2) = 3a_0^6a_1^3(27c_1^2c_2^2a_1a_2^2 + 4c_1^3c_2a_0^3)\]

neglect the multiplier and compute

\[f^* = Res(-a_0, -a_1, 1) = 27c_1^2c_2^2a_1 + 4c_1^3c_2a_0^3.\]

Finally, the dual tropical polynomial \( F^* \) is received by applying \( Val \) to \( c_i, \)

\[F^* = 2a_1 \odot 2a_2 \odot x_1 \oplus 3a_1 \odot a_2 \odot x_2^3\]

The analogous computation for the general case “classical” can be found in the appendix.
2.2 Duality of Pre-Tropical Conics

Next we develop the general form of tropical duality in the case of conics. Let $F$ be a tropical polynomial of degree 2 defined as

$$F(\bar{x}) = \bar{x}^T \hat{A} \bar{x} = 0,$$  \hspace{1cm} (2.2.1)

where $\bar{x} = [x_1, x_2, 1]$ and $\hat{A}$ is the symmetric matrix

$$\hat{A} = \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix}. \hspace{1cm} (2.2.2)$$

Remark: This setting is proper due to conclusion 2.4, despite the fact that the tropical monomials $(a_4 \odot x_1 \odot x_2)$ and $(a_5 \odot x_1) \oplus (a_6 \odot x_2)$ appear twice.

For this matrix we can fit the matrix

$$C = \begin{bmatrix} c_1 & c_4 & c_5 \\ c_4 & c_2 & c_6 \\ c_5 & c_6 & c_3 \end{bmatrix},$$

(i.e. $Val(c_i) = a_i$) that corresponds to a polynomial $f$ over $K$ (see (A.2.2) in the appendix). Using the same methods as used for the classical duality, (see appendix (A.2.3)) the dual of $C$ is given as

$$C^* = \begin{bmatrix} (c_6^2 - c_3c_2) & (c_6c_5 - c_4c_3) & (c_5c_2 - c_4c_6) \\ (c_6c_5 - c_4c_3) & (c_3^2 - c_1c_3) & (c_4c_5 - c_1c_6) \\ (c_5c_2 - c_4c_6) & (c_4c_5 - c_1c_6) & (c_2^2 - c_1c_2) \end{bmatrix}.$$  \hspace{1cm} (2.2.3)

Taking $Val(C^*)$ with respect to all restrictions mentioned formerly we get

$$\hat{A}^* = \begin{bmatrix} (2a_6 \oplus a_3 \odot a_2) & (a_6 \odot a_5 \oplus a_4 \odot a_3) & (a_5 \odot a_2 \oplus a_4 \odot a_6) \\ (a_6 \odot a_5 \oplus a_4 \odot a_3) & (2a_5 \oplus a_1 \odot a_3) & (a_4 \odot a_5 \oplus a_1 \odot a_6) \\ (a_5 \odot a_2 \oplus a_4 \odot a_6) & (a_4 \odot a_5 \oplus a_1 \odot a_6) & (2a_4 \oplus a_1 \odot a_2) \end{bmatrix}. \hspace{1cm} (2.2.3)$$

This completes the required dual transform and makes the below diagram commutative:

$$\begin{array}{ccc} \, & \, & \, \\ C \xrightarrow{\text{Dual}} & C^* \xrightarrow{\text{Val}^{-1}} & \, \\ \, & \, & \, \\ Val \downarrow & \text{Val} \uparrow & \, \\ \, & \, & \, \\ \hat{A} \longrightarrow & \hat{A}^* & \end{array}$$

the missing path is constructed by:

$$\hat{A} \longrightarrow C \longrightarrow C^* \longrightarrow \hat{A}^*.$$  \hspace{1cm} (2.2.4)

and $\hat{A}^*$ defines the dual tropical conics $F^*$.

In the case of applying the dual transform twice we have an extra multiplier which is $Det(C)$, this multiplier is now interpreted in the tropical sense,

$$Val(Det(C)) = \text{TropDet}(\hat{A}) = \max (a_1 + a_3 + a_2, c_1 + 2a_6, a_4 + a_5 + a_6, a_2 + 2a_5, 2a_4 + a_3). \hspace{1cm} (2.2.4)$$
Note that, the above tropical determinant can be calculated equivalently by using the primitive tropical operations. This is done in a similar manner as is done in the classical determinant, by switching the “regular” operations by tropical ones. However, it might happen that the tropical determinant is invalid, i.e. two of its components simultaneously achieve the maximum.

2.3 Pre-Tropical Duality of Quadrics

This part is mostly technical and based on quadrics’ duality in terms of matrices as being developed in appendix A.2. Hence, we will not get into details, and just review the final result. Let \( F \) be a tropical quadric characterized by the tropical matrix \( A \) then its dual is defined by \( A^* \) – the tropical adjoint matrix of \( A \).

3 Tropical Duality

In general, the tropical duality is obtained by involving convexity consideration into the pre-tropical duality. As a result we can discuss the “behavior” of dual subdivisions in respect to their sources, and thus deduce the duality of Non-Archimedean Amoebas. After we clarify the connection between the classical and tropical objects for this matter, we can eventually leave the beyond classical construction and persist only with the tropical one.

3.1 Duality of Tropical Objects

As mention earlier, the pre-tropical duality is used as the base for including convexity issues into tropical duality, and it also gives the linkage to the “classical” duality. However, the convexity issues in this case refer to the properties of the polyhedra which are defined by the lifting functions and their corresponding subdivisions. These issues will be studied next.

Assume \( F \) is a tropical polynomial and \( f \in F_K(I) \) is the representative of the polynomials’ family whose members have the valuation correspond to \( F \). Let us remind ourselves that a lifting function, \( \nu_f \), is defined over the Newton polytope \( \Delta_f \) and the values of \( \nu_f \) over the integral points \( \omega \)'s of \( \Delta_f \) relate to the coefficients of \( f \), i.e. \( \nu_f(\omega) = Val(c_\omega) \). But, according to our setting, we have \( Val(c_\omega) = a_\omega \) which are the coefficients of the compatible tropical polynomial \( F \) and thus \( \Delta_f = \Delta_F \). According to this insight, we can identify the lifting function, \( \nu_f \) with this of \( F \) and write \( \nu_f = \nu_F \).

Using this notion, the convex function \( \tilde{\nu}_f(x) = \max_{\omega \in I} \{ \omega \cdot x - \nu_f(\omega) \} \) can be rephrased in terms of the coefficients of \( F \) as

\[
\tilde{\nu}_F(x) = \max_{\omega \in I} \{ \omega \cdot x + a_\omega \}
\]  

(see Lemma [24]). Thus, the convex polyhedrons \( \Delta_{\nu_f} \) and \( \Delta_{\nu_F} \) which are specified by \( \tilde{\nu}_f(x) \) and \( \tilde{\nu}_F(x) \) are the same one. Finally, since \( \Delta_{\nu_F} = \Delta_{\nu_f} \), both induce the similar subdivision \( S_f = S_F \). According the above “translations”, we actually leave behind any involvement of \( f \) and remain only with the tropical polynomial \( F \).

All the previous “translations” have been made for some \( f \in F_K(I) \), clearly the same considerations can be applied to its dual \( f^* \). The dual relation between \( f \) and \( f^* \) induces
the duality on the entire participants, and particularly the duality between subdivisions,

\[ Daul : S_F \rightarrow S_{F^*}, \quad (3.1.2) \]

which is derived from the dual relation between \( \tilde{\nu}_F \) and \( \tilde{\nu}_{F^*} \). In general, this duality is taking place over polyhedra, and we have already explained the linkage between these types of objects to tropical curves. So, in the future, we can concentrate on polyhedra, assuming the map to the tropical curves is known. However, a global analysis of convexity relations between polyhedrons is too complicated and we should restrict ourselves to certain cases. Some of these cases are bought in the remaining sections.

### 3.2 Duality of Tropical Conics

Based on conics' definition in terms of matrices (i.e. \( F(\bar{x}) = \bar{x} A \bar{x}^T \), Sec. 2.2), in order to observe relationships between any subdivisions \( S_F \) and \( S_{F^*} \), we should first define the map \( A \rightarrow J_F \) (\( J_F \) is defined according to (1.3.2)). Specifically, \( J_F \) can be realized as \( \Delta_F \) with attached values over its integral points, thus for convenience we can regard this map as having the correspondence \( A \rightarrow \Delta_F \). The required correspondence can simply be sketched as following:

\[
A = \begin{bmatrix}
a_1 & a_4 & a_5 \\
a_4 & a_2 & a_6 \\
a_5 & a_6 & a_3
\end{bmatrix} \quad \rightarrow \quad \Delta_{\nu_F} : \begin{array}{cc}
a_1 \\
a_5 \\
a_3
\end{array} \begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
a_4 & a_4 \\
a_6 & a_2
\end{array} \quad (3.2.1)
\]

Composing this representation with the result we had developed for the dual conic in matrices form (2.2.3) the following is obtained:

\[
\begin{array}{c}
a_1 \\
a_5 \\
a_3
\end{array} \begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
a_4 & a_4 \\
a_6 & a_2
\end{array} \quad \rightarrow \quad 2a_6 \oplus a_3a_2 \quad \rightarrow \quad a_5a_2 \oplus a_4a_6 \quad \rightarrow \quad a_5a_2 \oplus a_4a_6 \quad (3.2.2)
\]

where the multiplications above are the topical ones. Eventually, the values that are placed over the integral points determine the subdivisions \( S_F \) and \( S_{F^*} \). These subdivisions are determined according to convexity considerations and this issue will be analyzed in the next section.

**Terminology 3.1.** The diagram of Newton polytopes with the above form (3.2.2) can be referred to as geometric graphs. Thus, for convenience in the rest of our discussion, an integral point will be denoted Node and nodes which are not intermediate points are called Vertices. In addition, the term Edge refers to a 1-dimensional face (i.e. geometrically a segment) which connects two nodes.
3.3 Analysis of Conics Duality

Aiming to analyze convexity properties, which eventually determine a subdivision, we will need the following property which regards to arbitrary points whose projections are co-linear. Let \( a_i \) and \( a_j \) be two points which are placed on the “lower part” of the convex hull \( \partial \mathcal{CH}(\Delta_{\nu_F}) \). A middle point \( a_k \in \Delta_{\nu_F} \) appears also on \( \partial \mathcal{CH}(\Delta_{\nu_F}) \) if and only if
\[
a_k \leq ta_i + (1-t)a_j, \quad t \in [0,1].
\]

Note that, this property is valid to any arbitrary triple and not only for internal points.

Now, assume \( \mathcal{F} \) is a conic and apply this observation to points \( a_\omega \)’s that correspond to integral points of \( \Delta_{\mathcal{F}} \) that are not vertices. Since the distances between neighboring integral points of \( \Delta_{\mathcal{F}} \) are all one, \( t \) can be fixed to be 1/2, and all the values that correspond to the integral points can be written in terms of the vertices’ values as:
\[
\begin{align*}
a_4 &= \frac{1}{2}a_1 \odot \frac{1}{2}a_2 \odot \epsilon_4 \\
a_5 &= \frac{1}{2}a_1 \odot \frac{1}{2}a_3 \odot \epsilon_5 \\
a_6 &= \frac{1}{2}a_2 \odot \frac{1}{2}a_3 \odot \epsilon_6
\end{align*}
\]

Observation 3.2. Using the above notations, any \( a_i \) lays on \( \partial \mathcal{CH}(\Delta_{\nu_F}) \) only when \( \epsilon_i \leq 0 \).

Substituting these values in \( \Delta_{\nu_{\mathcal{F}^*}} \), according to (3.2.2) yields
\[
\begin{align*}
a_2 \odot a_3 \odot (2\epsilon_6 \oplus 0) \\
\frac{1}{2}(a_1 \odot a_2 \odot a_3) \odot (\epsilon_5 \oplus (\epsilon_4 \odot \epsilon_6)) - \frac{1}{2}(a_1 \odot a_2 \odot a_3) \odot (\epsilon_4 \odot (\epsilon_5 \oplus \epsilon_6)) \\
a_1 \odot a_2 \odot (2\epsilon_4 \oplus 0) - \frac{1}{2}(2a_1 \odot a_2 \odot a_3) \odot (\epsilon_6 \oplus (\epsilon_4 \odot \epsilon_5)) - a_1 \odot a_3 \odot (2\epsilon_5 \oplus 0).
\end{align*}
\]

As one can observe, the tropical “factor” of any node which is not a vertex is exactly the average of the “factors” of its two neighboring vertices along a same line. Thus, for the convexity matter we can disregard these tropical multipliers and retain only with
\[
\begin{align*}
\epsilon_1^* &= 2\epsilon_6 \oplus 0 \\
\epsilon_5^* &= \epsilon_4^* = \epsilon_5 \oplus (\epsilon_4 \odot \epsilon_6) - \epsilon_4 \odot (\epsilon_5 \oplus \epsilon_6) \\
\epsilon_4^* &= \epsilon_6^* = \epsilon_2^* = 2\epsilon_4 \oplus 0 - \epsilon_6 \oplus (\epsilon_4 \odot \epsilon_5) - 2\epsilon_5 \oplus 0.
\end{align*}
\]

For our purpose this result is called the “Distortions” of \( \nu_{\mathcal{F}^*} \). In addition, by using (3.3.1) this consideration can be generalized to any \( n \times n \) symmetric matrix.

Naturally, the questions that should be asked are, in which scenarios a point \( a_\omega^* \) is contained in \( \partial \mathcal{CH}(\Delta_{\nu_{\mathcal{F}^*}}) \) and, what is the dependency of a dual coefficient on those who appear in its description. In other words, what is the induced dual subdivision \( S_{\mathcal{F}^*} \) with respect to the primal subdivision \( S_{\mathcal{F}} \). Clearly this dependency determines the duality between the induced subdivisions, \( S_{\mathcal{F}^*} \) and its dual \( S_{\mathcal{F}} \). Using this insight, our discussion can be restricted to subdivisions. For this matter we use the notation \( S_\Delta \) to denote a subdivision of a polytope \( \Delta \).
The immediate observation of (3.3.2) yields that, only nodes of $J_F$ which are not vertices involve in determining whether a node of $J_F^*$ is comprised or not in $\partial \mathcal{CH}(\Delta_{\nu F^*})$. These conditions can be summarized using the corresponding “distortions” values (3.3.2) as:

- $a_4^* \in \partial \mathcal{CH}(\Delta_{\nu F^*}) \iff \epsilon_4 < (\epsilon_5 \odot \epsilon_6) \oplus \epsilon_5 \oplus \epsilon_6 \oplus 0$, 
- $a_5^* \in \partial \mathcal{CH}(\Delta_{\nu F^*}) \iff \epsilon_5 < (\epsilon_4 \odot \epsilon_6) \oplus \epsilon_4 \oplus \epsilon_6 \oplus 0$, 
- $a_6^* \in \partial \mathcal{CH}(\Delta_{\nu F^*}) \iff \epsilon_6 < (\epsilon_4 \odot \epsilon_5) \oplus \epsilon_4 \oplus \epsilon_5 \oplus 0$.

**Remark:** The interesting situations are those when an image of a node is vertex of $\partial \mathcal{CH}(\Delta_{\nu F^*})$, thus in the above conditions the inequalities are the strong ones.

**Definition 3.3.** (Complete subdivision) A subdivision $S_\Delta$ of a polytope $\Delta$ is called complete if it cannot be refined into other subdivision $S'_\Delta$.

This definition is given in the sense of the resulting components, namely, none of the subdivision’s components can be subdivided further. On the other hand the next classification is defined in nodes sense, and it can be regarded as an “almost” complete subdivision.

**Definition 3.4.** (maximal in nodes subdivision) A subdivision $S_\Delta$ of a polytope $\Delta$ is called maximal in nodes if all the polytope’s nodes appear in $S_\Delta$. A subdivision $S_\Delta$ is called minimal in nodes if only the vertices of $\Delta$ appear in $S_\Delta$.

Any complete subdivision is always maximal in nodes while the contrary is not necessarily true. As an example, chose the following subdivision,

```
    •
   | \  
  • ... •
  | \  | \ 
• - • - •
```

which is not a “perfect” triangulation (i.e. not all of its components are primitive triangles) and it can be refined, this despite all of the vertices involved in the subdivision.

**Proposition 3.5.** The dual subdivision $S_{F^*}$ of

1. a subdivision $S_F$ which is maximal in nodes is maximal in nodes.
2. a subdivision $S_F$ which is minimal in nodes is minimal in nodes.

**Proof.** The two assertions derived form construction (3.3.2) using the corresponding $\epsilon$’s values as defined in (3.3.1).

1. According to setting (3.3.1) a subdivision $S_F$ is maximal in nodes if and only if $\epsilon_i < 0$ for all $i = 4, 5, 6$. In respect to these values, $\epsilon_i \oplus 0 = 0$, and any combination which satisfies the relation $\epsilon_i^* = \epsilon_i \oplus (\epsilon_j \odot \epsilon_k) < 0$ where $i, j, k \in \{4, 5, 6\}$. Thus, the corresponding dual image as described in (3.3.2) can be written as

\[
\begin{array}{ccc}
0 & 0 \\
\epsilon_5 \oplus (\epsilon_4 \odot \epsilon_6) & - \epsilon_4 \oplus (\epsilon_5 \odot \epsilon_6) \\
0 & - \epsilon_6 \oplus (\epsilon_4 \odot \epsilon_5) & - 0.
\end{array}
\]

\[
\begin{array}{ccc}
\epsilon_5 < 0 & - \epsilon_4 < 0 \\
- \epsilon_5 < 0 & - \epsilon_4 < 0 \\
0 & - \epsilon_6 < 0 & - 0.
\end{array}
\]
As one can easily observe, any node which is not vertex has a negative value and hence, it appears in the lower part $\mathcal{CH}(\Delta_{\nu_F})$. Since, this is valid for any node, the compatible subdivision, $S_{F^*}$ is maximal in nodes.

2. As defined in (3.3.1) a subdivision $S_F$ is minimal in nodes if and only if $\epsilon_i > 0$ for $i = 4, 5, 6$. In respect to these values $\epsilon_i \oplus 0 = \epsilon_i$ and any combination satisfies the relation $\epsilon_i \oplus (\epsilon_j \odot \epsilon_k) \geq \epsilon_j \odot \epsilon_k$ where $i, j, k \in \{4, 5, 6\}$. The relative values of the corresponding dual polytope can be written via (3.3.2) as

$$
2\epsilon_6 \\
\epsilon_5 \oplus (\epsilon_4 \odot \epsilon_6) \\
2\epsilon_4 \quad - \quad \epsilon_6 \oplus (\epsilon_4 \odot \epsilon_5) \\
\quad - \\
2\epsilon_5.
$$

The obtained result is that, only the vertices appear in the subdivision which means that it is minimal in nodes.

Example 3.6. A subdivision which is maximal in nodes is mapped via the dual tropical transform (3.2.2) into a subdivision which is also maximal in nodes. In addition the diagram on the left is the result of applying the duality twice. This and the calculations of the tropical determinants (2.2.4) will assist us in our later development.

Example 3.7. As done in the previous example the three diagrams below refer to $\nu_F$, $\nu_{F^*}$ and $\nu_{F^{**}}$ for the case that $\nu_F$ induces a subdivision which is minimal in nodes.
For this case the tropical determinant is,
\[ TropDet(A) = \max\{1 + 5 + 2, 1 + 2 \times 4, 4 + 4 + 3, 2 + 2 \times 4, 2 \times 3 + 5\} = -\infty. \]

However, as can be seen this does not harm the result, and the dual of subdivision which is minimal in nodes is also minimal in nodes.

### 3.4 Duality of Tropical Quadrics

In this section we will generalize the previous results of subdivisions corresponding to conics for the case of subdivisions that refer to quadrics. As before, we base ourselves on the linkage between the two classes of associated objects: the classical and the tropical where both are given in matrices form. For this purpose, we will develop assisting methods which lead the further analysis.

As a reminder, let
\[ f(\tilde{z}) = \tilde{z}C\tilde{z}^T = 0 \]
be a quadric in \( \mathbb{K}^n \), where \( C \) is the \((n+1) \times (n+1)\) symmetric matrix and \( \tilde{z} = (z_1, \ldots, z_n, 1) \).
Its dual is defined by,
\[ f^*(\tilde{z}) = \tilde{z}^* [C^{-1}] (\tilde{z}^*)^T = 0. \]

For subdivision issues, which are influenced only by convexity properties, one can rely on the relation
\[ C^{-1} = \text{Adj}(C) \left( \frac{1}{\text{Det}(C)} \right), \]
and write
\[ f^*(\tilde{z}) = \tilde{z}^* [C^*] (\tilde{z}^*)^T = 0 \]
where \( C^* \) denotes the \( \text{Adj}(C) \) the adjoint matrix.

Recall that, an entry \( c_{i,j}^* \) of \( C^* \) is equal to the determinant \( \text{Det}(C_{i,j}) \) of the \( i,j \)-minor \( C_{i,j} \). In general we have the equation,
\[ \text{Det}(C) = \sum_{k=1}^{n+1} (-1)^{h+k} c_{h,k} \text{Det}(C_{h,k}). \] (3.4.1)

For convenience \((-1)^{h+k}\) is sometimes signed as \( \pm \) and \( \text{Det}(C_{h,k}) = |C_{h,k}| \).

Let \( V(F) \) be a tropical quadric defined by the matrix \( A \) and let \( f \) be a quadric over \( \mathbb{K}^n \) defined by the matrix \( C \), where the entries \( a_{i,j} \) of \( A \) are the valuation of \( c_{i,j} \) – the entries of \( C \). Thus, according to this construction, \( A \) can be written as \( A = \text{Val}(C) \) and \( \text{Val}(\text{Det}(C)) = TropDet(A) = |A| \). Clearly, the same holds for any two minors \( A_{h,k} \) and \( C_{h,k} \). These results coincide with the results which are obtained by the “pure” calculation of tropical determinants. The basics of tropical determinants and their properties can be found [2, 9, 17]. Let \( A^* \) be the matrix with entries \( a_{i,j}^* = \text{Val}(c_{i,j}^*) \) thus, its entries are the tropical determinants \( |A_{i,j}| \). Hence, eventually, for our concern we can remain with only the two tropical matrices \( A \) and \( A^* \), and disregard the beyond constructions.
In order to observe convexity properties, as done for conics (see (3.2.1)), we should first map the matrix $A$ into the corresponding Newton Polytope $\Delta_F$, this is done using the following rules:

- $a_{i,i} \mapsto$ Zero Vertex of $\Delta_F \quad i = n + 1$,
- $a_{i,i} \mapsto$ Vertex of $\Delta_F \quad i \neq n + 1$,
- $a_{i,j} \mapsto$ Node of $\Delta_F \quad i \neq j$.

The scheme of this map in respect to one of the polytope’s faces is as follow:

$$
\mathcal{A}:
\begin{bmatrix}
  a_{1,1} & \cdots & a_{n,1} \\
  \vdots & \ddots & \vdots \\
  a_{n,1} & \cdots & a_{n,n}
\end{bmatrix}
\longrightarrow
\Delta_F|_{i,j,k}:
\begin{bmatrix}
  a_{j,j} \\
  a_{j,k} - a_{i,j} \\
  a_{k,k} - a_{i,k} - a_{i,i}
\end{bmatrix}
$$

(3.4.2)

Recall that, in this specific case a node is placed on the middle of some boundary edge. Thus, the convexity of the lifting function over these nodes can be determined via the interior relationships over all the triples,

$$
a^*_{i,i} \leftrightarrow a^*_{i,j} \leftrightarrow a^*_{j,j}.
$$

(3.4.3)

For future development the following notation will be needed, let $[A_{i,j}]_{h,k}$ be the $h, k$-minor of the minor $A_{i,j}$ such that $h \neq i$ and $k \neq j$ with respect to the primal indices of $A$, then $[A_{i,j}]_{h,k}$ is signed shortly as $[A_{ih,jk}]$. In the reminder of our deployment, unless otherwise specified, in order to avoid confusion all indices will be taken relatively to the primal matrix. In addition, by the symmetry of $A$, the following are also valid:

$$
A_{ih,jk} = A^T_{jk,ih}, \quad A_{ih,jk} = A_{hi,kj}.
$$

(3.4.1)

Using the tropical analogue to the determinant construction by minors over $\mathbb{K}$, Eq. (3.4.1), and the above notation any entry $a^*_{i,j}$ can be written as,

$$
a^*_{i,j} = \bigoplus_{h \neq j} a_{h,k} \odot [A_{ih,jk}], \quad h \neq i
$$

(3.4.4)

where $k = 1, \ldots, n + 1$.

Since, the distances between neighboring nodes of the Newton Polytope are fixed to one (as was in (3.3.1)) we generally define the “distortion” value $\epsilon_{i,j}$ of $a_{i,j}$ using the relation

$$
a_{i,j} = \frac{1}{2} a_{i,i} \odot \frac{1}{2} a_{j,j} \odot \epsilon_{i,j},
$$

(3.4.5)

where $\epsilon_{i,i}$ are always zero.

**Remark 3.8.** The above construction is invariant in respect to convexity, i.e. a triple is convex if and only if the triple of its “distortions” is convex.

**Definition 3.9.** Let $A$ be a symmetric tropical matrix, the “distortion” matrix $E$ of $A$ is defined to be $(\epsilon_{i,j})$ where the entries $\epsilon_{i,j}$ are the “distortion” values of each entry which is obtained via (3.4.5).
Using the “distortion” values of \( a_{i,j} \), Eq. (3.4.5), we can represent the entries \( a_{i,j}^{*} \) of \( A^{*} \), this will make the convexity analysis easier. Let \( g_{i,j}(A) \) be the “secondary” expression defined as,

\[
g_{i,j}(A) = \begin{cases} 0, & i = j, \\ \frac{1}{2}a_{i,i} \circ \frac{1}{2}a_{j,j}, & i \neq j. \end{cases} \tag{3.4.6}
\]

and let the “main” expression be

\[
G_{i,j}(A) = g_{i,j} \circ \bigcirc_{l \neq i,j} a_{l,l}.
\tag{3.4.7}
\]

To emphasize, with the above setting when \( h = k \), we have the equalities,

\[
G_{ih,jk}(A) = G_{i,j}(A_{h,k}) = g_{i,j} \circ g_{h,k} \circ \bigcirc_{l \neq i,j,h,k} a_{l,l},
\]

and since for this case \( g_{h,k} = 0 \) then the relation, \( G_{ih,jk}(A) = G_{i,j}(A) \) is obtained.

**Proposition 3.10.** Let \( A \) be the symmetric matrix relates to a tropical quadric \( V(F) \) in \( n \)-dimensional space, and let \( a_{i,j} = \frac{1}{2}a_{i,j} \circ \frac{1}{2}a_{j,j} \circ \epsilon_{i,j} \), then

\[
|A_{i,j}| = G_{i,j}(A) \circ \text{TropDet}(\mathcal{E}_{i,j})
\tag{3.4.8}
\]

where \( \mathcal{E} \) is the “distortion” matrix of \( A \).

**Proof.** This assertion is proved by induction over the space dimension \( n \), i.e. the matrix’s size minus one.

**Base step for \( n = 2 \):** Let \( A \) be the \( 3 \times 3 \) tropical matrix \((a_{i,j})\), and let

\[
A^{*} = \begin{bmatrix}
a_{2,2} \circ a_{3,3} \circ 2a_{2,3} & a_{1,2} \circ a_{3,3} \circ a_{1,3} \circ a_{2,3} & a_{1,2} \circ a_{2,3} \circ a_{1,3} \circ a_{2,2} \\
a_{1,2} \circ a_{3,3} \circ a_{1,3} \circ a_{2,3} & a_{1,1} \circ a_{3,3} \circ 2a_{1,3} & a_{1,1} \circ a_{2,3} \circ a_{1,3} \circ a_{1,1} \\
a_{1,2} \circ a_{2,3} \circ a_{1,3} \circ a_{2,2} & a_{1,1} \circ a_{2,3} \circ a_{1,2} \circ a_{1,3} & a_{1,1} \circ a_{2,2} \circ 2a_{1,2}
\end{bmatrix}.
\]

be its adjoint matrix. Take for instance,

\[
a^{*}_{3,2} = \frac{G_{2,3}(A)}{a_{1,1} \circ \frac{1}{2}a_{2,2} \circ \frac{1}{2}a_{3,3}} \circ \frac{|\mathcal{E}_{2,3}|}{(\epsilon_{1,1} \circ \epsilon_{2,3} \circ \epsilon_{1,2} \circ \epsilon_{2,3})}
\]

then one can easily verify that it satisfies the required. The same can be checked similarly for any pair of indices \( i, j = 1, 2, 3 \).

**Induction step \( (n - 1) \rightarrow n \):** Assuming the correctness for \( n - 1 \) we will prove the assertion for \( n \). Using the minors’ decomposition (3.4.4) we can write

\[
a^{*}_{i,j} = |A_{i,j}| = \bigoplus_{k \neq j} a_{h,k} \circ |A_{ih,jk}|, \quad h \neq i.
\]

Combine this decomposition with the induction assumption and the description of \( a_{h,k} \) in “distortions” terms we get,

\[
|A_{i,j}| = \bigoplus_{k \neq j} \left( \frac{1}{2}a_{h,h} \circ \frac{1}{2}a_{k,k} \circ \epsilon_{h,k} \right) \circ G_{ih,jk}(A) \circ |\mathcal{E}_{ih,jk}|, \quad h \neq i.
\]
By the definition of $G_{i,j}(A)$, Eq. (3.4.7), for any $h \neq i, k \neq j$ we have
\[
\left( \frac{1}{2} a_{h,h} \odot \frac{1}{2} a_{k,k} \right) \odot G_{ih,jk}(A) = G_{i,j}(A)
\]
and hence,
\[
|\mathcal{A}_{i,j}| = G_{i,j}(A) \odot \bigoplus_{k \neq j} \varepsilon_{h,k} \odot |\mathcal{E}_{ih,jk}|, \quad h \neq i.
\]
Clearly the left component is $|\mathcal{E}_{i,j}|$. \hfill \Box

The next two assertions, which refer to some properties of the “distortion” matrix $\mathcal{E}$, pave the way to the forthcoming generalization of proposition 3.5 to quadrics. For this development the following notation is required.

**Notation:** Let $S_n$ be the collection of all the permutations over the set of indices $N = \{1, \ldots, n\}$, then $S_n|_{i \rightarrow j}$ denotes the sub collection of all permutations $\pi \in S_n$ with the fixed correspondence $\pi(i) = j$, namely
\[
S_n|_{i \rightarrow j} = \{ \pi \mid \pi \in S_n \text{ s.t. } \pi(i) = j \}. \quad (3.4.9)
\]

**Claim 3.11.** Let $\mathcal{E} = (\varepsilon_{i,j})$ be a $n \times n$ symmetric matrix where $\varepsilon_{i,i} = 0$ for any $i = 1, \ldots, n$ and $\varepsilon_{i,j} < 0$ for any $i \neq j$ then $|\mathcal{E}_{i,i}| = 0$ and $|\mathcal{E}_{i,j}| < 0$.

**Proof.** Any minor, $\mathcal{E}_{i,i}$, of $\mathcal{E}$ is also a symmetric matrix with zero diagonal. The determinant of any matrix can be fundamentally stated as $|\mathcal{E}| = \bigoplus_{\pi \in S_n} \bigodot \varepsilon_{i,\pi(i)}$.

1. Since all the matrix’s entries are negative except those that lay on the main diagonal which are equal to zero, the single permutation that achieves the maximum is the identity (i.e. $\pi(l) = l$), hence $|\mathcal{E}_{i,i}| = \bigodot_{l \neq i} \varepsilon_{i,\pi(l)} = \bigodot_{l \neq i} \varepsilon_{l,l} = 0$.

2. According to the same considerations as above, without loss of generality assume $i < j$, then both the “new” $j - 1$’th row and the “new” $i$’th column (of $\mathcal{E}_{i,j}$) contain only negative elements. As a result for any permutation $\pi \in S_n$ we have $\bigodot_{l} \varepsilon_{i,\pi(l)} < 0$ and hence use the tropical amount, $|\mathcal{E}_{i,j}| < 0$.

**Claim 3.12.** Let $\mathcal{E} = (\varepsilon_{i,j})$ be a $n \times n$ symmetric matrix where $\varepsilon_{i,i} = 0$ for any $i = 1, \ldots, n$ and $\varepsilon_{i,j} > 0$ for any $i \neq j$ then $|\mathcal{E}_{i,i}| \odot |\mathcal{E}_{j,j}| = 2|\mathcal{E}_{i,j}|$.

**Proof.** Contrarily assume:
\[
2|\mathcal{E}_{i,j}| = |\mathcal{E}_{i,j}| \odot |\mathcal{E}_{j,i}| < |\mathcal{E}_{i,i}| \odot |\mathcal{E}_{j,j}|. \quad (*)
\]
Let $N$ be the set of $n$ indices $N = \{1, \ldots, n\}$, then any minor can be written as
\[
|\mathcal{E}_{i,j}| = \bigoplus_{\pi \in S_n|_{i \rightarrow j}} \bigodot_{k \neq i} \varepsilon_{k,\pi(k)} \quad (\pi(k) \neq j),
\]
where $S_{n|i\rightarrow j}$ is defined according to (3.4.9). Recall that, by the assumption $|E_{i,j}| > 0$ for any $i, j$.

Let $\pi_{i,i} \in S_{n|i\rightarrow i}$, $\pi_{j,j} \in S_{n|j\rightarrow j}$, $\pi_{j,i} \in S_{n|i\rightarrow j}$ and $\pi_{i,j} \in S_{n|i\rightarrow j}$ be the permutations which achieve the maximum for $|E_{i,i}|, |E_{j,j}|, |E_{j,i}|$ and $|E_{i,j}|$ correspondingly. Using the above notation, (*) can be rewritten equivalently as

\[
\begin{array}{cccc}
\text{fixed row} & \text{fixed column} & I & II \\
\hline
\epsilon_{k,\pi_{i,j}(k)} & \epsilon_{\pi_{j,i}(k),k} & \epsilon_{\pi_{i,i}(k),\pi_{i,j}(k)} & \epsilon_{\pi_{j,j}(k),\pi_{j,i}(k)} \\
\hline
k \neq i & k \neq j & \pi_{i,j}(k) \neq i & \pi_{j,j}(k) \neq j
\end{array}
\]

Since the permutations in both sizes of the equation are defined over the same set of indices and are different only in their restrictions over the indices $i,j$ then the distinction among their maximum evaluations is derived only from the elements that correspond to these indices.

Let $k'$ be the index for which $\pi_{i,i}(k') = j$ and let $k''$ be the index for which $\pi_{j,j}(k'') = i$. Set $\tilde{\pi}_{i,i}$ and $\tilde{\pi}_{j,j}$ to be the results of the switching $\pi_{i,i}(k') := i$, $\pi_{i,i}(i) := j$ and $\pi_{j,j}(k'') := j$, $\pi_{j,j}(j) := i$ in the permutations $\pi_{i,i}$ and $\pi_{j,j}$ respectively. Thus, $\tilde{\pi}_{i,i} \in S_{n|i\rightarrow j}$ and $\tilde{\pi}_{j,j} \in S_{n|j\rightarrow i}$. Since the components are all positive and symmetric (i.e. $|E_{i,j}| = |E_{j,i}|$), this yields that either the component that corresponds to the permutation $\tilde{\pi}_{i,i} \in S_{n|i\rightarrow j}$ is bigger than the one that corresponds to the preliminary chosen $\pi_{i,i} \in S_{n|i\rightarrow i}$, $\pi_{j,i} \in S_{n|j\rightarrow i}$ or the second $\tilde{\pi}_{j,j} \in S_{n|j\rightarrow j}$ has this property. Namely, we specify a permutation if $S_{n|i\rightarrow j}$ which contradicts the preliminary selection of the permutation which maximized the determinant $|E_{i,j}|$.

Similarly, we can contrarily assume the inequivalence in the opposite direction $|E_{i,j}| \circ |E_{j,i}| > |E_{i,j}| \circ |E_{j,i}|$, to obtain a contradiction as well. This completes the proof of the required.

Using the above preparation we can now generalize proposition 3.5 to quadrics.

**Theorem 3.13.** Let $F$ be a quadratic tropical polynomial and let $S_F$ be the corresponding induced subdivision of its Newton polytope $\Delta_F$. The dual subdivision $S_{F^*}$ (induced by $F^*$) of

1. a subdivision $S_F$ that is maximal in nodes, is maximal in nodes,
2. a subdivision $S_F$ that is minimal in nodes, is minimal in nodes.

**Proof.** Let $A$ be the tropical symmetric matrix that characterizes $F$ and let $A^*$ be the corresponding matrix that describes $F^*$ – the dual of $F$. The subdivision $S_F$ is induced directly from the translation of $A$ to $\Delta_F$ (3.4.12) and similarly from the dual (i.e. $A^*$ to $\Delta_{F^*}$). To prove these two assertions we will assisted by Proposition 3.10 which is based on the entries’ representation (3.4.15) in “distortions” terms.

First, in both cases, due to the property of $G(A)$ (derived from Def. 3.4.7) which determines

\[
G_{i,i}(A) + G_{j,j}(A) = 2G_{i,j}(A)
\]
when observing the convexity of any triple,

\[ |A_{i,i}| \leftrightarrow |A_{i,j}| \leftrightarrow |A_{j,j}|, \]

the tropical multipliers, \( G_{i,j}(A) \), can be neglected. Thus, convexity examination can be based only on the “distortion” values \( \epsilon_{i,j} \).

1. By the setting (3.3.5), a subdivision \( S_F \) is maximal in nodes if and only if \( \epsilon_{i,j} < 0 \) for all \( i \neq j \). As a result of Claim 3.11 the following translation is received:

\[
\begin{align*}
a^*_i,j &= |A_{i,j}| \mapsto |E_{i,j}| \begin{cases} 
= 0 & i = j, \\
< 0 & i \neq j.
\end{cases}
\end{align*}
\]

As one can easily verify, any node which is not vertex (i.e. is corresponding to indices \( i \neq j \)) has a negative value and hence, it appears in the lower part \( \partial CH(\Delta_{\nu^*_F}) \). This means the it is expressed in the compatible subdivision, \( S_{F*} \), of the dual polytope \( \Delta_{F*} \). This property is valid for any \( i \neq j \), or equivalently for any node which is not vertex, hence, the subdivision is maximal in nodes.

2. A subdivision \( S_F \) is minimal in nodes if and only if \( \epsilon_{i,j} \geq 0 \) for \( i, j = 1, \ldots, n+1 \) while \( \epsilon_{i,i} = 0 \). The same convexity consideration as has been used in the previous part is also standing in this case. Using Claim 3.12 we have the equality \( 2|E_{i,j}| = |E_{i,i}| \circ |E_{j,j}| \). Thus, despite \( a^*_i,j \) being placed on the “lower part” of the convex hull, \( \partial CH(\Delta_{\nu_{F*}}) \), it is not one of the polytopes vertices. This means that any middle node does not appear in the subdivision \( S_{F*} \) which is induced by \( \partial CH(\Delta_{\nu_{F*}}) \). The net result is that only the vertices of Newton Polytope appear in the subdivision, hence the subdivision is minimal in nodes.

\[ \square \]

4 Regularity in Duality Sense

Up to this point, all our development was done for families of objects with respect to their dual, where the referred issues were either the resulting geometric objects or equivalently the subdivisions determining these objects. Here, we intend to advance these idea and to specify additional properties of the subdivision’s generators, namely the lifting functions. Recall that lifting functions are defined by the coefficients of the tropical polynomials. For this purpose we will specify lifting functions \( \nu_F \) which have a “cooperative” behavior with respect to applying duality twice. The corresponding tropical curve, \( V(F) \), of such functions will be denoted as Regular Tropical Curves. The formal meaning is described next.

**Notation:** Let \( F \) be a tropical polynomial defined by the matrix \( A \) and \( F^* \) its dual (determined by \( A^* \)), then \( F^{**} \) denotes the dual of \( F^* \) (i.e. the result of applying twice the duality to \( F \)) and \( A^{**} \) is its corresponding matrix.

**Definition 4.1.** Let \( V(F) \) be a tropical curve where \( F \) is described by the symmetric matrix \( A \). The curve will be called Regular in duality sense if

\[ a^{**}_{i,j} = L_A \odot a_{i,j}, \quad \forall i, j = 1, \ldots, n, \]
where $a_{i,j}^{**}$ is an entry of $(A^*)^*$ and $L_A$ is a fixed value which is called the lifting constant.

By the above, the corresponding convex hulls of the lifting functions $\nu_\mathcal{F}$ and $\nu_{\mathcal{F}^{**}}$ are same up to lift by constant. Thus, it is clear that the regularity in this sense means that for the two tropical polynomials $\mathcal{F}$ and $\mathcal{F}^{**}$ the induced subdivisions $S_\mathcal{F}$ and $S_{\mathcal{F}^{**}}$ are identical. In this case we may also say that the polynomial $\mathcal{F}$ and the lifting function $\nu_\mathcal{F}$ are regular.

The two following claims rephrase the regularity in term of “distortion” values, these are a preparation for the main statement.

Claim 4.2. Let $\mathcal{F}$ be a quadratic tropical polynomial and $\mathcal{F}^{**}$ the result of applying twice the duality to $\mathcal{F}$, and let $A, A^{**}$ be their characteristic matrices having the “distortion” matrices $E$ and $E^{**}$ correspondingly. In case $\epsilon_{i,j} < 0$ for all $i \neq j$, $V(\mathcal{F})$ is regular if and only if $E = E^{**}$.

Proof. This assertion is directly derived for Claim 3.11. For this case $\epsilon_{i,i} = \epsilon_{i,i} = \epsilon_{i,i}^{**} = 0$. Thus, the lifting constant is $L_A = 0$ and hence $a_{i,i} = a_{i,i}^{**}$. In addition, since $L_A = 0$ we have $a_{i,i}^{**} = \epsilon_{i,i}^{**}$ for any $i, j = 1, \ldots, n$ then, $a_{i,j}^{**} = \frac{1}{2}a_{i,i}^{**} \odot \frac{1}{2}a_{j,j}^{**} \odot \epsilon_{i,j}^{**} = \frac{1}{2}a_{i,i} \odot \frac{1}{2}a_{j,j} \odot \epsilon_{i,j} = a_{i,j}$.

Claim 4.3. Let $E = (\epsilon_{i,j})$ be a $n \times n$ symmetric matrix where $\epsilon_{i,i} = 0$ for any $i = 1, \ldots, n$ and $\epsilon_{i,j} < 0$ for any $i \neq j$ then

$$|E_{i,j}| = \epsilon_{j,i} \oplus \left( \bigoplus_{k \neq i,j} \epsilon_{j,k} \odot \bigoplus_{h \neq i,j} \epsilon_{h,i} \right) = \left\{ \begin{array}{ll}
|E_{i,i}| = 0, & i = j, \\
|E_{i,j}| < 0, & i \neq j.
\end{array} \right.$$ 

Remark: This assertion is also valid without the symmetry restriction.

Proof. A careful combinatoric observation of the matrix decomposition yields the result. Without loss of generality assume $j > i$, then the minor $E_{i,j}$ is of the form:

$$E_{i,j} = \begin{array}{c|ccc|c}
& i & & \\
\hline
0 & | & | & i \\
\hline
j - 1 & - & - & 0,
\end{array}$$

where, besides the $i$’th column and $(j - 1)$’th row, any row or column contains a zero entry. The tropical determinant “selects” the maximal product that is achieved by some permutation which simultaneously choses one element from each row and each column. Naturally, in this case since all the non-diagonal entries are $< 0$, the aim is to chose the maximal number of zero entries.

However, such choosing should be completed with entries of the $i$’th column and $(j - 1)$’th row. One option is to select their intersection (i.e. the entry $\epsilon_{j,i}$), while the other option is a separated selection, one for the column and one for the row, and then taking their tropical product. Eventually, we take the “best” result that can be achieved from both options. Note that in these two scenarios the zero entries don’t play a role. Formulating the above we get,

$$|E_{i,j}| = \bigoplus_{k \neq j} \epsilon_{j,k} \odot |E_{ij,jk}| = \epsilon_{j,i} \oplus \left( \bigoplus_{k \neq i,j} \epsilon_{j,k} \odot \bigoplus_{h \neq i,j} \epsilon_{h,i} \right),$$
and the required is attained. Moreover, since this is a combination of values \( \epsilon_{i,j} < 0 \) then for \( i \neq j \) \( |\mathcal{E}_{i,j}| < 0 \). In case \( i = j \), each column and each row of \( |\mathcal{E}_{i,i}| \) contains a zero entry, which is the maximal chosen, thus \( |\mathcal{E}_{i,i}| = 0 \).

The next two theorems give the explicit conditions for determining whether or not a curve of the previous families is regular.

**Theorem 4.4.** Let \( \mathcal{F} \) be a quadratic tropical polynomial, \( \mathcal{A} \) its characteristic matrix and let \( \mathcal{E} \) be the “distortion” matrix of \( \mathcal{A} \). Assume \( \epsilon_{i,j} < 0 \) for all \( i, j = 1, \ldots, n \) and \( \epsilon_{i,i} = 0 \), a sufficient and necessary condition for \( V(\mathcal{F}) \) to be regular is

\[
\epsilon_{j,i} > \bigoplus_{k \neq i,j} \epsilon_{j,k} \odot \bigoplus_{h \neq i,j} \epsilon_{h,i}
\]

for any \( i, j = 1, \ldots, n \).

**Proof.** Using Claim 4.3, due to the assumptions, (*) can be stated equally as \( \epsilon^*_{i,j} = \epsilon_{i,j} \).

The sufficiency part is a direct result of Claim 4.3, for this case we have \( \epsilon^*_{i,j} = |\mathcal{E}_{i,j}| = \epsilon_{i,j} \) and thus \( \epsilon^{**}_{i,j} = |(\mathcal{E}^*)_{i,j}| = \epsilon^*_{i,j} = \epsilon_{i,j} \) which yields \( \mathcal{E} = \mathcal{E}^{**} \). Then apply Claim 1.2 to obtain the required.

For the necessary part, contrarily assume there exist indices \( i, j \) such that (*) is not satisfied, namely \( \epsilon^*_{i,j} > \epsilon_{i,j} \), and the equivalence \( \mathcal{E} = \mathcal{E}^{**} \) is still valid. Apply Claim 4.3 for \( \mathcal{E}^* \) to obtain:

\[
\epsilon^{**}_{i,j} = |\mathcal{E}^{**}_{i,j}| = \epsilon^*_{i,j} + \left( \bigoplus_{k \neq i,j} \epsilon^*_{j,k} \odot \bigoplus_{h \neq i,j} \epsilon^*_{h,i} \right) > \epsilon_{i,j}.
\]

That is a contradiction to Claim 1.2 which determines the equivalence \( \epsilon_{i,j} = \epsilon^{**}_{i,j} \).

**Theorem 4.5.** Let \( \mathcal{F} \) be a quadratic tropical polynomial, \( \mathcal{A} \) its \( n \times n \) characteristic matrix and let \( \mathcal{E} \) be the “distortion” matrix of \( \mathcal{A} \). In case \( \epsilon_{i,j} > 0 \) for all \( i, j = 1, \ldots, n \), \( V(\mathcal{F}) \) is never regular.

**Proof.** Using claim 3.12 we obtain that \( \frac{1}{2} |\mathcal{E}_{i,j}| \odot \frac{1}{2} |\mathcal{E}_{j,i}| = |\mathcal{E}_{i,j}| \), for any \( i, j \). As a result, the “distortion” matrix of \( \mathcal{E}^* \), is a complete zero matrix \( \mathcal{Z} \). Since all of its entries are equal, we get \( |\mathcal{Z}_{i,j}| = -\infty \). According to the definition of the “distortion” values the equality \( |\mathcal{E}_{i,j}| = L_{\mathcal{E}} \odot |\mathcal{Z}_{i,j}| = L_{\mathcal{A}} \odot -\infty = -\infty \) is satisfied. Using the same consideration again we obtain \( |\mathcal{A}_{i,j}| = -\infty \), and hence \( a^*_{i,j} = -\infty \) for all \( i, j \). Thus any combination of these entries has the same evaluation and in particular \( |(\mathcal{A}^*)_{i,j}| = a^{**}_{i,j} = -\infty \Rightarrow a_{i,j} \neq L_{\mathcal{A}} \odot a^*_{i,j} \).

Hence, \( V(\mathcal{F}) \) is non-regular.

Appendix

A Duality of Algebraic Varieties

In this appendix we will overview the classical algebraic duality in polynomial equations \[6, 7\], this review will include the explicit construction (i.e. an algorithm) of the polynomial equation which describes the dual curves. We open by the formal definition of duality that occurred in the projective space, which is used later to define the duality over affine spaces.
A.1 Projective Dual Curve

Let \( P^2_K = Proj \mathbb{K}[T_0, T_1, T_2] \) be the projective plane, a curve \( L \) on \( P^2_K \) is said to be a line, if \( \text{Deg}(L) = 1 \) i.e. \( L = V(a_0T_0 + a_1T_1 + a_2T_2) \) for some \((a_0, a_1, a_2) \in (\mathbb{K}^*)^3\). Clearly \( V(a_0T_0 + a_1T_1 + a_2T_2) = V(b_0T_0 + b_1T_1 + b_2T_2) \) \iff \( a_i = \lambda b_i, \quad i = 0, 1, 2, \lambda \in \mathbb{K}^* \).

Therefore a bijection \( \phi : \{\text{lines on } P^2_K\} \rightarrow P^2_K(\mathbb{K}) \) is obtained and defined by \( \phi(V(a_0T_0 + a_1T_1 + a_2T_2)) = (a_0 : a_1 : a_2) \).

Namely, \((a_0 : a_1 : a_2)\) are taken as homogeneous coordinates of the line \( L : a_0T_0 + a_1T_1 + a_2T_2 = 0 \).

Let \( \gamma = V(F) \) be a curve with \( \text{Deg}(\gamma) = \text{Deg}(F) = r \), where \( F \) is homogeneous polynomial of degree \( r \geq 2 \). For a nonsingular point \( p \) of \( \gamma \), let \( l_p \) denote the tangent to \( \gamma \) at \( p \), equivalently the tangent space \( T_p \) at the point \( p \). In case \( p = (p_0, p_1, p_2) \) and \( (\partial F/\partial T_i)(p) = \partial_i F(p_0, p_1, p_2) \) is written as \( (\partial_i F)(p) \) for \( i = 0, 1, 2 \), and

\[
l_p = V \left( \sum_{i=0}^{2} (\partial_i F)(p)T_i \right),
\]

i.e. \( \phi(l_p) = ((\partial_0 F)(p) : (\partial_1 F)(p) : (\partial_2 F)(p)) \). From the rational map \( \theta_\gamma \) defined by \( \theta_\gamma(p) = ((\partial_0 F)(p) : (\partial_1 F)(p) : (\partial_2 F)(p)) \) the relation \( \theta_\gamma(p) = \phi(l_p) \) between the maps’ images is trivially obtained.

Definition A.1. \( \gamma^* = \theta_\gamma(\gamma) \) is said to be the dual curve of \( \gamma \). In case \( \gamma^* \) is expressed as \( V(F^*) \) by an irreducible homogeneous polynomial \( F^* \), then \( F^* \) is said to be the dual homogeneous polynomial of \( F \).

A.1.1 Construction of Dual Curve in \( \mathbb{P}^2 \)

In the previous section we outlined the formal definition of the dual curves with the following relation

\[
\gamma \xrightarrow{\theta_\gamma} \gamma^* \quad V(F) \xrightarrow{\theta_\gamma} V(F^*).
\]

The next goal is, to attain the description of a dual curve \( \gamma^* \) according to the primal curve \( \gamma \). Namely, to write down the explicit equation of \( F^* \) where its coefficients are described in terms of \( c_\omega \)'s (the coefficients of \( F \)). For our concern we may assume that \( F \) is a Laurent polynomial. A number of preparatory steps are needed in order to pave the way for the easier construction. In order to simplify these geometric matters one can refer to the projective
plane as Euclidean 3-dimensional space (in this case the coordinates are signed by \((t_0, t_1, t_2)\) with all the required restrictions.

**Step 1:** In order to define the tangent space \(T_p(\gamma)\) (line in this case), the Gradient \(\nabla F\) is found and denoted by:

\[
\nabla F(p) = ((\partial_0 F)(p), (\partial_1 F)(p), (\partial_2 F)(p)) = (\alpha_0, \alpha_1, \alpha_2).
\]

The derivatives provide the direction of the normal to the tangent space at a fixed point \(p_0 = (p_{00}, p_{01}, p_{02})\) that satisfies \(F\). Taking \(p_0\) as varied along, the curve of the complete family of the tangent spaces \(\Lambda_\gamma\) (i.e. tangent bundle) is obtained.

Recall that homogeneous polynomial of degree \(n\) satisfying the relation

\[
t_0(\partial_0 F) + t_1(\partial_1 F) + t_2(\partial_2 F) = nF. \tag{A.1.1}
\]

Hence, when \(F = 0\) the equations of the tangent lines can be written as

\[
\alpha_0 t_0 + \alpha_1 t_1 + \alpha_2 t_2 = 0. \tag{A.1.2}
\]

**Step 2:** A parameterization of the tangent bundle \(\Lambda_\gamma\) along \(\gamma\) can be obtained due to the substitution

\[
t_0 \mapsto -\left(\frac{\alpha_1 u + \alpha_2 v}{\alpha_0}\right), \quad t_1 \mapsto u, \quad t_2 \mapsto v, \tag{A.1.3}
\]
in \(F\), and by the homogeneity of \(F\) we have

\[
\tilde{F} = F(-\left(\alpha_1 u + \alpha_2 v\right), \alpha_0 u, \alpha_0 v) = 0. \tag{A.1.4}
\]

Note that this polynomial equation is homogeneous as well and of the same degree in its five variables (i.e. \(\tilde{F} \in \mathbb{K}[\alpha_0, \alpha_1, \alpha_2, u, v]_n\)). However, it can also be regarded as polynomial only in \(u\) and \(v\) (i.e. \(\tilde{F} \in \mathbb{K}[\alpha_0, \alpha_1, \alpha_2][u, v]_n\)) where its coefficients are determined by the \(\alpha_i\)'s

\[
\tilde{F}_{(\alpha_0, \alpha_1, \alpha_2)}(u, v) = \sum_{i=0}^{n} a'_i(\alpha_0, \alpha_1, \alpha_2)u^iv^{n-i} = 0. \tag{A.1.5}
\]

**Step 3:** At last, the required dual curve \(\gamma^*\) is the projection of \(\Lambda_\gamma\) into the space which spanned by \(\alpha_i\)'s. Algebraically, this is equivalent to the elimination of \(u\) and \(v\) from \(\tilde{F}\). Here the homogeneity of \(\tilde{F}\) and the property of being a Laurent polynomial becomes useful.

**Lemma A.2.** Let \(F\) be a non-degenerate homogeneous polynomial in \(t_i\) then

\[
F(p) = 0 \quad \iff \quad \frac{\partial F}{\partial t_i}(p) = 0, \forall i.
\]

The lemma is directly derived from the property \(\text{(A.1.1)}\) and the assumption that \(F\) is not a degenerate polynomial.

Suppose that \(A = \{m_1, \ldots, m_s\} \subset \mathbb{Z}^n\) and \(A\) generates \(\mathbb{Z}^n\). Let \(L(A)\) be the set of Laurent polynomials with exponent vectors in \(A\), i.e.,

\[
L(A) = \{a_{1}t^{m_1} + \cdots + a_{s}t^{m_s} : a_{i} \in \mathbb{K}\}
\]
where $t^m = t_1^a \cdots t_n^a$ for $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. Given $n + 1$ Laurent polynomials $f_0, \ldots, f_n \in L(A)$, their $A$-resultant

$$\text{Res}_A(f_0, \ldots, f_n)$$

is a polynomial in the coefficients of the $f_i$.

**Proposition A.3.** The vanishing of $\text{Res}_A(f_0, \ldots, f_n)$ is necessary and sufficient condition for the equations $f_0 = \cdots = f_n = 0$ to have a solution (see [5], Prop. 2.1).

However, one must be careful where the solution lies. The $f_i$ are defined initially on the torus $(\mathbb{C}_m)^n$, but the definition of the generalized corresponding projective toric variety $Y_A$ shows that the equation $f_i = 0$ makes sense on $Y_A$. Then one can prove that

$$\text{Res}_A(f_0, \ldots, f_n) = 0 \iff f_0 = \cdots = f_n = 0$$

have a solution in $Y_A$. This stays true for any $A \subset \mathbb{Z}^n$ finite or not, especially while $A$ is bounded from below which is enough for our purpose.

**Proposition A.4.** Let $\tilde{F}$ be a homogeneous Laurent polynomial then

$$\tilde{F} = 0 \iff \text{Res}(\frac{\partial \tilde{F}}{\partial u}, \frac{\partial \tilde{F}}{\partial v}) = 0.$$ 

As a result the dual curve $\gamma^*$ is described by $V(\text{Res}(\alpha_0, \alpha_1, \alpha_2))$.

### A.1.2 Affine Dual Curve

The dual curve had been defined for the case of projective spaces, next we intend to show the compatible duality over affine spaces. In general, two steps are required, the first to embed the affine curve defined by $f$ in $\mathbb{P}^2$, and the second to “extract” the dual from $\mathbb{P}^2$ after it has been determined. This can be outlined via the following diagram

$$
\begin{array}{ccc}
V(F) & \xrightarrow{\text{Dual}} & V(F^*) \\
\uparrow & & \downarrow \text{extract} \\
V(f) & \longrightarrow & V(f^*).
\end{array}
$$

Note that in this case the dual of a line is defined as follows:

$$l : t_2 = a_1 t_1 + a_0 \quad \longrightarrow \quad (a_1, -a_0), \quad (A.1.6)$$

and the tangent in a given point $p^0 = (p_0^0, p_1^0)$ is

$$l : t_2 = \left( \frac{\partial f}{\partial t_1} \right)(p_0^0)t_1 + p_1^0 - \left( \frac{\partial f}{\partial t_1} \right)(p_0^0)p_0^0. \quad (A.1.7)$$

**Step 1:** Naively embed $((t_1, t_2) \mapsto (1, t_1, t_2))$ the curve $\gamma$ defined by $f$ in the projective plane to obtain the curve described by the homogeneous polynomial $F$ (which is irreducible if $f$ dose) where $F|_{t_0=1} = f$. Then find $F^*$.

**Step 2:** The “extraction” of the dual $f^*$ is based on the tangent space as constructed in [A.1.2] and the restriction of $t_0$ to 1. In this case one has $\alpha_0 = -(a_1 t_1 + a_2 t_2)$. Rewriting
the dual transform (A.1.6) in terms of $F$ and using the Implicit Function Theorem, the image of a tangent to the curve defined by $f$ is

$$a_1 = -\frac{\alpha_1}{\alpha_2}, \quad a_0 = \frac{\alpha_2 t_2 + \alpha_1 t_1}{\alpha_2}.$$ (A.1.8)

Composing this with the restriction of $t_0 = 1$ we obtain

$$a_1 = -\frac{\alpha_1}{\alpha_2}, \quad a_0 = -\frac{\alpha_0}{\alpha_2}.$$ (A.1.9)

Let $c = \alpha_2$ where $0 \neq c \in \mathbb{K}$, so that $\alpha_1 = -ca_1$ and $\alpha_0 = -ca_0$. The substitution of these values provides the transform of the original curve $f(t_1, t_2) = 0$:

$$\text{Res}(\alpha_0, \alpha_1, \alpha_2) = \text{Res}(-ca_0, -ca_1, c) =
\begin{align*}
\text{c}^{m} \text{Res}(-a_0, -a_1, 1) = 0 \quad \Rightarrow \\
\text{Res}(-a_0, -a_1, 1) = 0,
\end{align*}
$$

where $c \neq 0$ and $m \leq n(n - 1)$. The net result is, the dual curve for this case, is described by $V(\text{Res}(-a_0, -a_1, 1))$.

### A.2 Duality of Quadrics in Matrices’ Notion

In this section we apply the complete dual transformation to the general form of conics, then using this base we will construct the duality of quadrics in matrices’ notion.

Let $f$ be a polynomial of degree 2 defined as

$$f(\bar{z}) = \bar{z}C\bar{z}^T = 0$$ (A.2.1)

where $\bar{z} = [z_1, z_2, 1]$ and

$$C = \begin{bmatrix} c_1 & c_4 & c_5 \\
            c_4 & c_2 & c_6 \\
            c_5 & c_6 & c_3 \end{bmatrix},$$ (A.2.2)

the obtained result of applying the duality is $f^* = \bar{a}C^*\bar{a}^T$ where $\bar{a} = [a_1, a_2, 1]$ and

$$C^* = \begin{bmatrix} (c_6^2 - c_3c_2) & (c_6c_5 - c_4c_3) & (c_5c_2 - c_4c_6) \\
(c_6c_5 - c_4c_3) & (c_5^2 - c_1c_3) & (c_4c_5 - c_1c_6) \\
(c_5c_2 - c_4c_6) & (c_4c_5 - c_1c_6) & (c_2^2 - c_1c_2) \end{bmatrix}.$$ (A.2.3)

Since for the case of conics there is a preserving of degrees $21$, the duality induced the mapping of matrices

$$C_{3 \times 3} \mapsto C^*_{3 \times 3}$$

and vise versa. Reapplying the dual transform, one can easily check that

$$(C^*)^* = -\text{Det}(C)C.$$ (A.2.4)

Considering the conics which are varieties of a given describing equation of the form $f = \bar{z}C\bar{z}^T = 0$, a corresponding duality occurs in the space of the compatible matrices $C_{3 \times 3}$. This relation can generalized to quadrics embedded in space of any dimension.
Reobserving \( C^* \) as appears in equation \((A.2.3)\) one can easily verify that \( C^* = -\text{Adj}(C) \) is the standard adjoint matrix of \( C \). Additionally, since only \( V(f^*) \) are considered, any multiplier \( d \) of \( C^* \) is acceptable without harming the result, particularly for the setting \( d := -1/\text{Det}(C) \). Composing this specific setting with the previous observation yields,

\[
C^* = -\text{Adj}(C) \longleftrightarrow -\text{Adj}(C)(-\frac{1}{\text{Det}(C)}) = C^{-1}.
\]  
(A.2.5)

Clearly the above coincides with both the direct calculation as appears in \((A.2.4)\) and the applying of the duality twice

\[
(C^*)^* \longleftrightarrow (C^{-1})^{-1} = C.
\]

Next, we will construct directly the dual of quadrics in terms of matrices, this can be done due to their special properties. Let

\[
f(\bar{z}) = \bar{z}C\bar{z}^T = 0
\]  
(A.2.6)

be a polynomial of degree 2 in \( \mathbb{C}^n \) where \( C \) is \((n+1) \times (n+1)\) symmetric matrix and \( \bar{z} = (z_1, \ldots, z_n, 1) \). Its Gradient \( \nabla f = 2C\bar{z} \) defines the cotangent bundle \( \Lambda^*_V(f) \) of the corresponding manifold \( V(f) \). Hence, the obtained map for each point \( \bar{z} \in V(f) \) is

\[
2C\bar{z}^T|_{\bar{z}_0} \mapsto (\bar{z}_0^*)^T.
\]

Assuming \( C \) is an invertible matrix and since the above is valid for any point we have

\[
\bar{z}^T = \frac{1}{2} C^{-1}(\bar{z}^*)^T.
\]

Substitute this into equation \((A.2.6)\) to get

\[
\left[ \frac{1}{2} C^{-1}(\bar{z}^*)^T \right]^T C \left[ \frac{1}{2} C^{-1}(\bar{z}^*)^T \right] = 0.
\]

Developing the above via,

\[
\left[ \frac{1}{2} C^{-1}(\bar{z}^*)^T \right]^T C \left[ \frac{1}{2} C^{-1}(\bar{z}^*)^T \right] = \frac{1}{4} [(\bar{z}^*)^T]^T [C^{-1}]^T C [C^{-1}] [(\bar{z}^*)^T] = 0
\]
yields

\[
\frac{1}{4} \bar{z}^* [C^{-1}]^T (\bar{z}^*)^T = 0.
\]

Since only the varieties of this equation are concerned we remain with

\[
\bar{z}^* [C^{-1}]^T (\bar{z}^*)^T = 0.
\]  
(A.2.7)

Using the matrices identity \((C^{-1})^T = (C^T)^{-1}\) and the fact that \( C \) is symmetric, this result coincides with the basic demand that the dual of a dual is the source.
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