New results on existence of Atangana-Baleanu fractional differential equations with dependence on the Lipschitz first derivatives

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Abstract
We study on the existence and uniqueness of solutions for a Atangana-Baleanu fractional differential equations with dependence on the Lipschitz first derivative conditions. We develop a Gronwall inequality in the frame of Atangana-Baleanu fractional integral. An example is given to illustrate the main results and investigate the stability in the sense of Ulam.

Keywords
Fractional differential equations; Atangana-Baleanu fractional derivative; Lipschitz first derivatives; Gronwall inequality; Ulam-Hyers stability.

AMS Subject Classification
34A08, 34K37, 58C30.

1. Introduction
In the past decades, there has been a growing interest in the study of the fractional differential equations due to the rigorous growth of the fractional calculus theory itself and its applications in various fields such as chemistry, physics, engineering, control theory, aerodynamics, electrodynamics of complex medium and control of dynamical systems and so on. In consequence, fractional differential equations is obtaining much significance and attention. For details, we refer readers to [12, 18, 19, 23] and references therein.

There are some approaches to the fractional derivatives such as Riemann-Liouville, Caputo, Weyl, Hadamard and Grunwald-Letnikov, etc. The most well known fractional operator are perhaps the Riemann-Liouville fractional integral and derivatives. Fractional operators are act as an magnificent tools for the mathematical modeling of the real world problems. Later, Atangana and Baleanu proposed two new fractional derivatives based on the Caputo and the Riemann-Liouville definitions of fractional-order derivatives. Other kinds of fractional derivative that look like the Riemann-Liouville and Caputo ones can be seen in [6, 21].

In recent years, many researchers paid much attention to ABC-derivative with several conditions in various spaces. The Atangana-Baleanu fractional derivative is familiar to followings nonsingularity as well as nonlocality of the kernel, which acquires the generalized Mittag-Leffler function. Some of the latest studies on ABC-derivatives such as, Jarad et al. investigated a ODE’s in the form of AB-derivative [20]. Ravichandran et al. discussed in details the AB-fractional integro-differential equations. Atangana and Koca find the chaos in a simple nonlinear system with AB-fractional derivatives [10].

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More precisely in [11], Khan et al. investigated Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel. Sene discussed Stokes’ first problem for heated flat plate with AB-derivative [31]. Owolabi studied the modelling and simulation of a dynamical
system with the Atangana-Baleanu fractional derivative [30].

To be concise, in this paper we are concerned with the study of the existence and uniqueness of solutions of the Atangana-Baleanu fractional derivative equation in the sense of Caputo as follows

\[
\left(\frac{ABC}{a}D^{\alpha}u\right)(t) = \int_{0}^{t} \frac{\alpha - 1}{\Gamma(\alpha)} u'(s) ds, \quad 0 < \alpha \leq 2,
\]

\[
u(a) = u_0, \quad t \in [a, b],
\]

where \( t \in C[a, b] \), where \( a \) is the left Caputo AB fractional derivative, \( u(t), (\frac{ABC}{a}D^{\alpha}u) \), \( f \in C[a, b], f(a, u(a), u'(a, u(a))) = 0 \). Consider \( \mathcal{U} u(t) = u'(t, u(t)) \) Then (1.1) becomes

\[
\left(\frac{ABC}{a}D^{\alpha}u\right)(t) = f(t, u(t), \mathcal{U} u(t)), \quad 1 < \alpha \leq 2,
\]

\[
u(a) = u_0
\]

The rest of this paper is organized as follows: In Section 2, we review some useful properties, definitions, propositions and lemmas of fractional calculus. The existence and uniqueness of solutions for AB-fractional derivative results are proved in Section 3. In section 4 is devoted to illustrate an example numerically solved. Ulam-Hyers stability analysis is considered in section 5.

### 2. Preliminaries

In this section, we present some definitions, lemmas and propositions of fractional calculus, which will be used throughout this paper.

The definition of Riemann-Liouville fractional integral and derivatives are given as follows:

- For \( \alpha > 0 \), the left Riemann-Liouville fractional integral of order \( \alpha \) is given as [20]

\[
\left(\frac{AL}{a}I^{\alpha}u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} u(s) ds.
\]

- For \( 0 < \alpha < 1 \), the left Riemann-Liouville fractional derivative of order \( \alpha \) is given as [20]

\[
\left(\frac{AL}{a}D^{\alpha}u\right)(t) = \frac{d}{dt} \left( \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} (t-s)^{-\alpha} u(s) ds \right).
\]

- For \( 0 \leq \alpha \leq 1 \), the Caputo fractional derivative of order \( \alpha \) is given as [20]

\[
\left(\frac{AL}{a}D^{\alpha}u\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} (t-s)^{-\alpha} u'(s) ds.
\]

**Definition 2.1.** [7] Let \( u \in H^1(a, b), \ b < a < b \) and \( \alpha \in [0, 1] \). The Caputo Atangana-Baleanu fractional derivative of \( u \) of order \( \alpha \) is defined by

\[
\left(\frac{ABC}{a}D^{\alpha}u\right)(t) = \frac{B(\alpha)}{1-\alpha} \int_{0}^{t} u'(s) E_{\alpha} \left[ -\alpha (t-s)^{\alpha} \right] ds,
\]

where \( E_{\alpha} \) is the Mittag-Leffler function defined by \( E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} \) [25, 32] and \( B(\alpha) > 0 \) is a normalizing function satisfying \( B(0) = B(1) = 1 \). The Riemann Atangana-Baleanu fractional derivative of \( u \) of order \( \alpha \) is defined by

\[
\left(\frac{ABR}{a}D^{\alpha}u\right)(t) = \frac{B(\alpha)}{1-\alpha} \int_{0}^{t} u(s) E_{\alpha} \left[ -\alpha \frac{(t-s)^{\alpha}}{1-\alpha} \right] ds.
\]

The associative fractional integral is defined by

\[
\left(\frac{ABR}{a}I^{\alpha}u\right)(t) = \frac{1}{B(\alpha)} \int_{0}^{t} u(t) + \alpha B(\alpha) \left(\frac{AL}{a}D^{\alpha}u\right)(t)
\]

where \( d^{\alpha} \) is the left Riemann-Liouville fractional integral given in (2.1).

**Lemma 2.2.** [7] Let \( u \in H^1(a, b) \) and \( \alpha \in [0, 1] \). Then the following relation holds.

\[
\left(\frac{ABR}{a}D^{\alpha}u\right)(t) = \left(\frac{ABC}{a}D^{\alpha}u\right)(t) - \frac{B(\alpha)}{1-\alpha} u(a) E_{\alpha} \left[ -\alpha \frac{(t-a)^{\alpha}}{1-\alpha} \right].
\]

**Lemma 2.3.** [20] Suppose that \( \alpha > 0 \), \( c(t)(1-\frac{\alpha}{B(\alpha)} d(t))^{-1} \) is a nonnegative, nondecreasing and locally integrable function on \( [a, b] \), \( \frac{d(t)}{B(\alpha)}(1-\frac{\alpha}{B(\alpha)} d(t))^{-1} \) is non-negative and bounded on \( [a, b] \) and \( u(t) \) is nonnegative and locally integrable \( [a, b] \) with

\[
u(t) \leq c(t) + d(t) \left(\frac{ABR}{a}I^{\alpha}u\right)(t),
\]

then

\[
u(t) \leq \frac{c(t) B(\alpha)}{B(\alpha) - (1-\alpha) d(t)} E_{\alpha} \left( \frac{\alpha}{B(\alpha) - (1-\alpha) d(t)} \right).
\]

**Theorem 2.4.** (Ascoli-Arzela Theorem)[15] Let \( S \) be a compact metric space. Then \( M \subset C(\Omega) \) is relatively compact iff \( M \) is uniformly bounded and uniformly equicontinuous.

**Theorem 2.5.** (Krasnoselskii Fixed Point Theorem)[15] Let \( S \) be a closed, bounded and convex subset of a real Banach space \( X \) and let \( T_1 \) and \( T_2 \) be operators on \( S \) satisfying the following conditions

\[
T_1(s) + T_2(s) \subset S
\]

- \( T_1 \) is a strict contraction on \( S \), i.e., there exist a \( k \in [a, b] \) such that

\[
||T_1(u) - T_1(v)|| \leq k ||u - v|| \quad \forall u, v \in S
\]

- \( T_2 \) is continuous on \( S \) and \( T_2(s) \) is a relatively compact subset of \( X \).

Then, there exist a \( u \in S \) such that \( T_1 u + T_2 u = u \)

**Proposition 2.6.** [14] For \( 0 \leq \alpha \leq 1 \),

\[
\left(\frac{ABR}{a}I^{\alpha}u\right)(t) = u(t) - u(a) E_{\alpha}(\lambda t^{\alpha}) - \frac{\alpha}{1-\alpha} u(a) E_{\alpha,\alpha+1}(\lambda t^{\alpha})
\]

\[
u(t) - u(a).
\]
Proposition 2.7. ([22, 28]) \( f'(u) \in D \) satisfy the Lipschitz condition, i.e., there exist a constant \( k > 0 \) such that
\[
\|f'(u) - f'(v)\| \leq k \|u - v\|, \quad u, v \in D.
\] (2.10)

Definition 2.8. A continuous function \( u : [a, b] \to \mathbb{R} \) is called a mild solution of the following Atangana-Baleanu fractional derivative equation in the sense of Caputo
\[
\begin{aligned}
(ABC^\alpha D^\alpha u)(t) &= g(t), \quad 1 < \alpha \leq 2, \\
u(a) &= u_0
\end{aligned}
\]
for each \( t \in C[a, b] \), \( u(t) \) satisfies the following integral equation
\[
u(t) = u_0 +_a AB I^\alpha g(t)
\]

3. Existence and Uniqueness

In this section, we prove the existence and uniqueness of (1.3) and (1.4).

We need the following assumptions to prove the existence and uniqueness results for the problem (1.3) and (1.4) by using the Banach contraction principle.

\textbf{A}_1 Let \( u \in C[a, b] \) and \( f \in (C[a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) is continuous function and there exist a positive constants \( M_1, M_2 \) and \( M \) such that
\[
\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq M_1 (\|u_1 - u_2\| + \|v_1 - v_2\|)
\]
for all \( u_1, v_1, u_2, v_2 \in Y \), \( M_2 = \max_{\lambda \in [a, b]} \|f(\lambda, 0, 0)\| \) and \( M = \max\{M_1, M_2\} \). Let \( Y = C[\mathbb{R}, X] \) be the set of continuous functions on \( \mathbb{R} \) with values in the Banach spaces \( X \).

\textbf{A}_2 Let \( u' \in C[a, b] \) satisfy the Lipschitz condition, i.e., there exist a positive constants \( N_1, N_2 \) and \( N \) such that
\[
\|D(t, u) - D(t, v)\| \leq N_1 (\|u - v\|),
\]
for all \( u, v \) in \( Y \). \( N_2 = \max_{t \in D} \|D(t, 0)\| \) and \( N = \max\{N_1, N_2\} \).

\textbf{A}_3 For each \( \lambda > 0 \), let \( B_1 = \{u \in Y : \|u\| \leq \lambda\} \subset Y \) where \( \lambda = ((1 - \rho)^{-1} \|u_0\|) \) and take \( \rho = (M(\|u\| + \|u_0\|)).

\textbf{A}_4 For each \( \lambda_0 > 0 \), let \( B_2 = \{u \in C([a, b], \mathbb{R}) : \|u\| \leq \lambda\} \) then \( B_2 \) is clearly bounded, closed and convex subset in \( C([a, b], \mathbb{R}) \).

Lemma 3.1. If \( \text{A}_1 \) and \( \text{A}_2 \) are satisfied, then the estimate
\[
\|D(u(t))\| \leq t(N_1 \|u\| + N_2), \quad \|D(u(t) - Dv(t))\| \leq N \|u - v\|,
\]
are satisfied for any \( t \in \mathbb{R} \), and \( u, v \in Y \).

Theorem 3.2. Let \( u(t) \in C[a, b] \) such that \( (ABD_\alpha^\alpha D_\alpha^\alpha u)(t) \in C[a, b] \). Suppose that \( f \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) satisfies \( \text{A}_1 - \text{A}_3 \). Then, if \( f(a, u(a), Du(a)) = 0 \) and
\[
\left(1 - \alpha \right) \left(\frac{1}{B(\alpha)} + \frac{\alpha^2}{B(\alpha)\Gamma(\alpha)}\right) \leq 1
\]
the problem (1.3) and (1.4) has an unique solution.

Proof. First, we have to prove that \( u(t) \) satisfies the problem (1.3) and (1.4) if and only if \( u(t) \) satisfies the integral equation
\[
u(t) = u_0 +_a AB I^\alpha f(t, u(t), Du(t))
\]
Now, constructing use of Proposition 2.6, we get
\[
u(t) - u(a) =_a AB I^\alpha f(t, u(t), Du(t))
\]
Since \( u(a) = u_0 \) from (1.4) and \( f(a, u(a), Du(a)) = 0 \), (3.1) is satisfied. If \( u(t) \) satisfies (3.1), then by using that \( f(a, u(a), Du(a)) = 0 \) it is obvious that \( u(a) = u_0 \).

To apply the Riemann-Liouville Atangana-Baleanu fractional derivative to both sides of (3.1), we get
\[
u(t) = u_0 +_a AB I^\alpha (ABD_\alpha^\alpha D_\alpha^\alpha u)(t)
\]
Thus, we have
\[
u(t) = u_0 E_\alpha \left(-\frac{\alpha}{1 - \alpha}(t - a)^\alpha\right) + f(t, u(t), Du(t))
\]
Then, the result is acquired by benefiting from theorem 3.2 in [7]. Now, we define the operator
\[
u(t) = u_0 +_a AB I^\alpha f(t, u(t), Du(t)).
\]
Then, by $A_3$, $\|u\| \leq \lambda$ we get
\[
\|Tu(t)\| \leq \|u_0\| + \|A^\alpha f(t, u(t), Du(t))\|
\leq \|u_0\| + \left\| \frac{1 - \alpha}{B(\alpha)} f(t, u(t), Du(t)) \right\|
+ \frac{\alpha}{B(\alpha)} D^\alpha f(t, u(t), Du(t))
\leq \|u_0\| + \frac{1 - \alpha}{B(\alpha)} (\|u\| + \|Du\|)(\alpha^\alpha(t - a)\alpha^\alpha)
+ \frac{\alpha}{B(\alpha)} (\|u\| + \|Du\|)(\alpha^\alpha(t - a)\alpha^\alpha)
\leq \|u_0\| + \|u\| + \|Du\|
\leq \lambda (1 - \rho) + \rho \lambda
\leq \lambda
\]
i.e., $\|Tu(t)\| \leq \lambda$. Now to prove uniqueness
\[
\|T(u) - T(v)\|
= \|A^\alpha f(t, u(t), Du(t)) - A^\alpha f(t, v(t), Dv(t))\|
\leq \left\| \frac{1 - \alpha}{B(\alpha)} (\|u\| + \|Du\|)(\alpha^\alpha(t - a)\alpha^\alpha)
+ \frac{\alpha}{B(\alpha)} (\|u\| + \|Du\|)(\alpha^\alpha(t - a)\alpha^\alpha)
\leq \|u\| + \|Du\|
\leq \lambda (1 - \rho) + \rho \lambda
\leq \lambda
\]
Next, we investigate the problem (1.3) and (1.4) has a fixed point by using another fixed point technique, namely Krasnoselskii’s fixed point theorem.

**Theorem 3.3.** If $A_1 - A_4$ are satisfied and
\[
q(t_2 - t_1) = \|M(\|u(t_2) - u(t_1)\| + \|Du(t_2) - Du(t_1)\|),
\]
then the problem (1.3) and (1.4) has a solution.

**Proof.** For any constant $\lambda_0 > 0$ and $u \in B_{\lambda_0}$, define two operator $T_1$ and $T_2$ on $B_{\lambda_0}$ as follows
\[
(T_1 u)(t) = u_0
(T_2 u)(t) = \frac{AB}{\lambda_0} f(t, u(t), Du(t)).
\]
Obviously, $u$ is a solution of (1.3) and (1.4) iff the operator $T_1 + T_2 = u$ has a solution $u \in B_{\lambda_0}$

Our proof will be divided into three steps.

**Step 1.** $\|T_1 u + T_2 u\| \leq \lambda_0$ whenever $u \in B_{\lambda_0}$.
For every $u \in B_{\lambda_0}$, we have
\[
\|T_1 u(t) + T_2 u(t)\| = \|Tu(t)\| + \|T_2 u(t)\|
\leq \|u_0\| + \|\frac{AB}{\lambda_0} f(t, u(t), Du(t))\|
\leq \|u_0\| + \frac{1 - \alpha}{B(\alpha)} (\|u\| + \|Du\|)(\alpha^\alpha(t - a)\alpha^\alpha)
+ \frac{\alpha}{B(\alpha)} (\|u\| + \|Du\|)(\alpha^\alpha(t - a)\alpha^\alpha)
\leq \|u\| + \|Du\|
\leq \lambda (1 - \rho) + \rho \lambda
\leq \lambda
\]
Hence, $\|T_1 u + T_2 u\| \leq \lambda_0$ for every $u \in B_{\lambda_0}$.

**Step 2.** $T_1$ is a contraction on $B_{\lambda_0}$ for any $u, v \in B_{\lambda_0}$, according to $A_4$ and (3.6), we have
\[
\|T_1 u(t) - T_1 v(t)\| \leq \|u_0 - v_0\| = R\|u_0 - v_0\|
\]
which implies that $\|T_1 u - T_1 v\| \leq R\|u_0 - v_0\|$, since $R = 1$, $T_1$ is a contraction.

**Step 3.** $T_2$ is completely continuous operator.
First we have to prove that $T_2$ is continuous on $B_{\lambda_0}$. For any $u_n, u \in B_{\lambda_0}$, $n = 1, 2, 3, \ldots$ with $\lim_{n \to \infty} \|u_n - u\| = 0$, we get $\lim_{n \to \infty} u_n(t) = u(t)$, for $t \in [a, b]$. Thus by $A_4$, we have
\[
\lim_{n \to \infty} f(t, u_n(t), Du_n(t)) = f(t, u(t), Du(t))
\]
for $t \in [a, b]$.
We can conclude that
\[
\sup_{x \in [a, b]} \|f(t, u_n(t), Du_n(t)) - f(t, u(t), Du(t))\| \to 0 \text{ as } n \to \infty
\]
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On other hand, for $t \in [a, b]$

$$\| (T_2u_n)(t) - (T_2u)(t) \|$$

$$\leq \| a^{AB} f(t, u_n(t), DU_n(t)) - a^{AB} f(t, u(t), DU(t)) \|$$

$$\leq 1 - \frac{\alpha}{B(\alpha)} ||a^{AB} f(t, u_n(t), DU_n(t))||$$

$$- a^{AB} f(t, u(t), DU(t))||$$

$$+ \frac{\alpha}{B(\alpha)} ||a^{AB} f(t, u_n(t), DU_n(t))||$$

$$- a^{AB} f(t, u(t), DU(t))||$$

$$\sup_{t \in [a, b]} ||a^{AB} f(t, u_n(t), DU_n(t))||$$

$$- f(t, u(t), DU(t))||$$

$$\leq \left( 1 - \frac{\alpha}{B(\alpha)} \right) \left( \frac{|t - a|^\alpha}{B(\alpha) \Gamma(\alpha)} \right)$$

$$\leq \left( 1 - \frac{\alpha}{B(\alpha)} \right) \left( \frac{|t - a|^\alpha}{B(\alpha) \Gamma(\alpha)} \right)$$

$$\sup_{t \in [a, b]} ||a^{AB} f(t, u_n(t), DU_n(t))||$$

$$- f(t, u(t), DU(t))||$$

Hence $\| (T_2u_n)(t) - (T_2u)(t) \| \to 0$ as $n \to \infty$. Therefore $T_2$ is continuous on $B_{\lambda_0}$.

Now, we have to show that $T_2u, u \in B_{\lambda_0}$ is relatively compact which is sufficient to prove that the function $T_2u, u \in B_{\lambda_0}$ uniformly bounded and equicontinuous, and $\forall t \in [a, b]$

$\| T_2u \| \leq \lambda_0$, for any $u \in B_{\lambda_0}$, therefore $(T_2u)(t), u \in B_{\lambda_0}$ is bounded uniformly.

Now, we prove that $(T_2u)(t), u \in B_{\lambda_0}$ is a equicontinuous. For any $u \in B_{\lambda_0}$ and $a \leq t_1 \leq t_2 \leq t$, we get

$$\| (T_2u)(t_2) - (T_2u)(t_1) \|$$

$$\leq \| a^{AB} f(t_2, u(t_2), DU(t_2)) - a^{AB} f(t_1, u(t_1), DU(t_1)) \|$$

$$\leq 1 - \frac{\alpha}{B(\alpha)} ||a^{AB} f(t_2, u(t_2), DU(t_2))||$$

$$- a^{AB} f(t_1, u(t_1), DU(t_1))||$$

$$+ \frac{\alpha}{B(\alpha)} ||a^{AB} f(t_2, u(t_2), DU(t_2))||$$

$$- a^{AB} f(t_1, u(t_1), DU(t_1))||$$

$$\sup_{t \in [a, b]} ||a^{AB} f(t, u(t_2), DU(t_2))||$$

$$- f(t_1, u(t_1), DU(t_1))||$$

$$\leq \left( 1 - \frac{\alpha}{B(\alpha)} \right) \left( \frac{|t_2 - t_1|^\alpha}{B(\alpha) \Gamma(\alpha)} \right)$$

$$\leq \left( 1 - \frac{\alpha}{B(\alpha)} \right) \left( \frac{|t_2 - t_1|^\alpha}{B(\alpha) \Gamma(\alpha)} \right)$$

$$\sup_{t \in [a, b]} ||a^{AB} f(t, u(t_2), DU(t_2))||$$

$$- f(t_1, u(t_1), DU(t_1))||$$

$\| (T_2u)(t_2) - (T_2u)(t_1) \| \to 0$ as $t_2 \to t_1$. Therefore, the operator $T_2$ is an equicontinuous on $B_{\lambda_0}$. Hence, which implies $T_2$ is relatively compact on $B_{\lambda_0}$.

Therefore $T_2$ is relatively compact subset of $X$ by theorem 2.4 and, by theorem 2.5 we can conclude that $T_2$ has at least one fixed point. Therefore the operator $T$ has a fixed point $u$ which is the solution of (1.3) and (1.4).

### 4. Ulam-Hyers stability

In this section, we study the Ulam-Hyers stability of (1.3) and (1.4). Now, we present the definition of Ulam-Hyers stability.

**Definition 4.1.** Equation (1.3) is Ulam-Hyers stable, if for all $v(t)$ satisfying the inequality

$$|\int_{a}^{B} D^\alpha v(t) - f(t, v(t), DU(t))| < \epsilon,$$

there exist a solution $u(t)$ of (1.3) and (1.4) satisfying

$$|v(t) - u(t)| < h_{f, \epsilon}, \quad h_{f, \epsilon} \in \mathbb{R}.$$ (4.2)

**Theorem 4.2.** Suppose that the hypothesis for existence of solutions to (1.3) and (1.4) are satisfied with

$$\mathfrak{M}(1 + \mathfrak{M}) \leq \frac{B(\alpha)}{1 - \alpha}.$$ (4.1)

Then (1.3) and (1.4) is Ulam-Hyers stable.

**Proof.** If $v(t)$ satisfies (4.1), there exists a function $\xi(t)$ satisfying $|\xi(t)| < \epsilon$ such that

$$\int_{a}^{B} D^\alpha v(t) - f(t, v(t), DU(t)) = \int_{a}^{B} D^\alpha \xi(t).$$ (4.3)

which is satisfies to

$$v(t) - v(a) - a^{AB} f(t, v(t), DU(t)) = a^{AB} \int_{a}^{B} D^\alpha \xi(t).$$ (4.4)

Therefore, we have

$$|v(t) - v(a) - a^{AB} f(t, v(t), DU(t))|$$

$$= \left| a^{AB} \int_{a}^{B} D^\alpha \xi(t) \right|$$

$$= \left| 1 - \frac{\alpha}{B(\alpha)} \xi(t) + \frac{\alpha}{B(\alpha)} \int_{a}^{B} D^\alpha \xi(t) \right|$$

$$\leq \left( 1 - \frac{\alpha}{B(\alpha)} \right) |\xi(t)| + \frac{\alpha}{B(\alpha)} |\xi(t)|$$

$$\leq \left( 1 - \frac{\alpha}{B(\alpha)} \right) |\xi(t)| + \frac{\alpha}{B(\alpha)} |\xi(t)|$$

$$\leq \epsilon \left( 1 - \alpha + 1 \frac{b - a}{\Gamma(\alpha)} \right)$$

Now let $u(t)$ be the solution of (1.3) satisfies $u(a) = v(a)$.

Then, we have

$$u(t) = v(a) + a^{AB} f(t, u(t), DU(t)).$$ (4.5)
Therefore \(|v(t) - u(t)| \leq h_f \varepsilon\), where

\[
h_f = \left(\frac{1 - \alpha}{B(\alpha)} + \frac{1}{B(\alpha)} \frac{(b-a)^{\alpha}}{\Gamma(\alpha)}\right) \frac{B(\alpha)}{\Gamma(\alpha) - (1 - \alpha)\omega(1 + \frac{1}{\alpha})}.
\]

Hence (1.1) is Ulam-Hyers stable.

5. Example

Consider the following problem

\[
\left(\frac{ABC}{0} \right)^3 D^2 u(t) = \frac{t}{3 \sqrt{(\pi)}} \sin(u(t) + u'(t)), \quad t \in [1, 2],
\]

\[
B(\alpha) = 1 \tag{5.1}
\]

\[
u(0) = 1 \tag{5.2}
\]

Notice that

\[
f(0, u(0), \xi(0)) = 0
\]

and

\[
u'(t) \in C[1, 2]
\]

satisfy the Lipschitz conditions.

Let

\[
\frac{t}{3 \sqrt{(\pi)}} \sin(u + v), \quad t \in [1, 2].
\]

It is easy to see that

\[
\frac{t}{3 \sqrt{(\pi)}} (|u_1 - u_2| + |v_1 - v_2|),
\]

for all \(t \in [1, 2], u_1, u_2, v_1, v_2 \in \mathbb{R}\)

\[
\leq \frac{1}{3 \sqrt{(\pi)}} |u - v|
\]

Thus \(\rho = \frac{1}{3 \sqrt{(\pi)}}\). Now

\[
\rho \left(\frac{1 - \alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha)\Gamma(\alpha)}\right)
\]

\[
= \frac{1}{3 \sqrt{(\pi)}} \left(1 - \alpha + \frac{1}{\Gamma(\alpha)}\right) < 1
\]

By theorem 3.2, (5.1) and (5.2) has a unique solution. It can be written as

\[
u(t) = \lim_{n \to 0} u_n(t),
\]

where

\[
u_n(t) = 1 + \frac{1}{3 \sqrt{(\pi)}} A_{\alpha} F'(tu_{n-1}(t)), \quad n = 0, 1, 2, \ldots
\]

or

\[
u_n(t) = 1 + (1 - \alpha)tu_{n-1}(t) + \alpha \frac{t}{\Gamma(\alpha)} (tu_{n-1}(t))
\]

\[
= 1 + (1 - \alpha)tu_{n-1}(t)
\]

\[
+ \frac{\alpha}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} s u_{n-1}(s) ds, \quad n = 1, 2, 3, \ldots
\]

Solving (5.1) and (5.2), we apply the method proposed by Mekkaoui and Atangana in [29], utilizing from the two-step Lagrange polynomial interpolation.
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