Abstract

The interface tension between $Z(N)$ vacua in a hot $SU(N)$ gauge theory (without dynamical fermions) is computed at next to leading order in weak coupling. The $Z(N)$ interface tension is related to the instanton of an effective action, which includes both classical and quantum terms; a general technique for treating consistently the saddle points of such effective actions is developed. Loop integrals which arise in the calculation are evaluated by means of zeta function techniques. As a byproduct, up to two loop order we find that the stable vacuum is always equivalent to the trivial one, and so respects charge conjugation symmetry.
I. Introduction

In the absence of dynamical fermions, $SU(N)$ gauge theories possess a global $Z(N)$ symmetry associated with the center of the gauge group $[1]$. Confinement implies that at zero temperature the vacuum is $Z(N)$ symmetric, but at nonzero temperature there is a phase transition to a deconfining phase, where the $Z(N)$ symmetry is spontaneously broken $[2]$. In the deconfined phase, a system of infinite extent falls into one of the $N$ degenerate vacua, but in a finite volume bubbles of different vacua form $[3]$, separated by domain walls. The action of these domain walls is proportional to the interface tension, and so controls the dynamics of large $Z(N)$ bubbles.

We previously discussed how to compute the interface tension $\alpha$ in terms of the temperature $T$ and the coupling constant $g$ $[4]$. In weak coupling, 

$$\alpha \simeq \frac{4(N-1)\pi^2}{3\sqrt{3}N} \frac{T^3}{g} \left(1 - (15.27853\ldots)\frac{g^2N}{16\pi^2}\right).$$  

The term at leading order, $T^3/g$, is the result of $[4]$; the principal result of this paper is the computation of $\alpha$ at next to leading order, $gT^3$. In this expression the coupling $g$ represents a running coupling constant at a temperature $T$; the exact relationship of this $g$ to the bare coupling constant is given in (4.11).

The physical applications of our results is at best indirect, for it only applies to a world devoid of dynamical quarks. It can be compared to numerical simulations of lattice gauge theories $[3]$; indeed, it was these measurements of $\alpha$ which led us to ask if it is computable in weak coupling.

Nevertheless, the methods and techniques which we have developed to compute the interface tension are, we believe, of general interest. The calculation of the interface tension reduces to an instanton problem in an effective theory. This effective theory is one dimensional $[5, 6]$, as it describes the profile of the interface in the direction perpendicular to the domain wall. Remember how a standard instanton calculation proceeds $[6]$: the instanton is the solution of the classical equations of motion, with the action for the domain wall proportional to $1/g^2$. The corrections to the classical action are given by expanding the classical action in a background instanton field. Corrections at next to leading order are given by integrating over terms of quadratic order, which gives corrections to the action of order one.

Contrast this with the interface tension in (1.1). The action of the domain wall is
\( \alpha \) times the transverse volume. By its mass dimensionality, \( \alpha \) is proportional to \( T^3 \), while the constants, and the factors of \( N \), result from the detailed path the \( Z(N) \) interface takes in the space of \( SU(N) \) gauge fields. What is surprising is that \( \alpha \) starts out as \( 1/g \) in weak coupling, for if the \( Z(N) \) interface were a solution to the classical equations of motion, it would start instead as \( 1/g^2 \). This is because the effective action (in one dimension) which controls the \( Z(N) \) interface is the sum of the classical action plus a quantum term, obtained by by integrating out fluctuations at one loop order. In the effective action the classical piece acts like a kinetic term, and the quantum piece like a potential. The \( Z(N) \) instanton is a stationary point only of the full effective action, and this balance between classical and quantum terms transforms the usual factor of \( 1/g^2 \) into a \( 1/g \).

It is then unclear how to compute corrections to \( \alpha \) beyond leading order. Surely one cannot compute blindly: if the effective action is expanded in fluctuations about a background instanton field, and these fluctuations integrated out, how is double counting avoided? That is, how to differentiate between the quantum fluctuations which generate the effective action in the first place, from the quantum fluctuations which are properly included in the expansion about the instanton?

In this paper we solve this problem for the \( Z(N) \) interface in a manner applicable to arbitrary effective actions. The method generalizes what is known as the “constrained” effective potential [7]. Here we use it to reduce the four dimensional gauge theory to an effective scalar theory in one dimension. The method is trivial in design: a delta function constraint for new degrees of freedom is inserted into the functional integral. The original degrees of freedom are then integrated out, producing an effective theory for the new field. There is obviously no problem with double counting, since extra degrees of freedom are introduced in the first place.

For the interface tension, this method shows that the leading corrections are of order \( g^2 \) times that at leading order. These effects are due entirely to corrections in going from the four dimensional theory to the effective, one dimensional theory. They enter both for the kinetic and potential terms in the effective action. For example, the corrections to the kinetic term transform the bare into a renormalized coupling constant.

The order parameter which distinguishes different \( Z(N) \) vacua is the Wilson line at nonzero temperature [2]. It is convenient to parametrize a vacuum expectation value for the Wilson line by giving the gauge field \( A_0 \) a nonzero value; the effective
action for a $Z(N)$ instanton is then related to the free energy in such a background field. The $Z(N)$ instanton is slowly varying in space, so that previous results in a constant $A_0$ field \[8–14\], especially those by Belyaev and Eletsky \[11\], Enqvist and Kajantie \[12\], and Belyaev \[14\], can be used. The technical problem of computing the free energy in a background $A_0$ field is done most easily by using zeta function techniques \[15\].

Much of the interest in considering a hot theory with $A_0 \neq 0$ concerns the possibility of the vacuum spontaneously generating an expectation value for $A_0$, and so for the Wilson line \[9–14\]. If this happens for $N \geq 3$, it implies that the vacuum at nonzero temperature spontaneously breaks charge conjugation symmetry. For theories without dynamical fermions, following Belyaev \[14\] we do not find evidence for the spontaneous breaking of charge conjugation symmetry at two loop order: up to the usual $Z(N)$ rotations, the stable vacuum is the trivial state, with $A_0 = 0$ and the Wilson line equal to one.

The outline of the paper is as follows. In sec. II we review the calculations of \[4\]. Sec. III outlines the general calculation of the interface tension. The calculation of the interface tension at next to leading order in Feynman gauge is given in sec. IV. Sec. V considers the computation in arbitrary covariant gauges. There are two appendices. In appendix A we prove that the path chosen for the $Z(N)$ instanton has minimal action for $N = 3$ and $\infty$. Appendix B summarizes various integrals needed in a constant background $A_0$ field.

II. $Z(N)$ interface at leading order

We begin by rederiving the results for the $Z(N)$ interface tension at leading order \[4\]. This is not mere repetition, for here we use a more natural basis for the generators of $SU(N)$ matrices than before. The subtleties of how to derive the effective action are deferred until sec. III.

We work in euclidean spacetime at a temperature $T$, so the euclidean time $\tau$ varies from 0 to $\beta = 1/T$. In the spatial directions the system is a long tube, of length $L$ in the $z$ direction, of length $L_t$ in the two remaining spatial directions, $\vec{x}_t$, with $L \gg L_t \gg \beta$. The volume in the directions transverse to $z$ is $V_{tr} = \beta L_t^2$.

A $Z(N)$ interface is constructed by assuming that the system is in one $Z(N)$ phase at one end of the tube, $z = 0$, and in another $Z(N)$ phase at the other end, for $z = L$. 3
This forces a $Z(N)$ interface along the $z$ direction, with the action of the interface equal to the interface tension, $\alpha$, times the transverse volume, $V_{tr}$. As a practical matter, while $L \gg L_t$, in the end both are taken to infinity.

The $Z(N)$ symmetry is determined by the the trace of the Wilson line in the fundamental representation,

$$tr \Omega(A) = \frac{1}{N} tr \left( \mathcal{P} \exp \left( ig \int_0^\beta A_0(x) d\tau \right) \right); \quad (2.1)$$

$\mathcal{P}$ refers to path ordering. Let $A_0$ have the value,

$$A_0^cl(x) = \frac{2\pi T}{gcN} q \ t_N; \quad (2.2)$$

where $t_N$ is the diagonal matrix

$$t_N = c \begin{pmatrix} 1_{N-1} & 0 \\ 0 & -(N-1) \end{pmatrix}, \quad c = \frac{1}{\sqrt{2N(N-1)}}. \quad (2.3)$$

In the presence of this $A_0$ field the Wilson line equals

$$tr \Omega(A^cl) = e^{2\pi iq/N} \left( 1 - \frac{1}{N} \left( 1 - e^{-2\pi iq} \right) \right). \quad (2.4)$$

The trivial vacuum is $A_0 = q = 0$, with $tr \Omega = 1$; $Z(N)$ transforms of the trivial vacuum occur for $q = j$, with $tr \Omega = e^{2\pi ij/N}$; $j$ is an integer between 1 and $(N-1)$. Thus the simplest $Z(N)$ interface is constructed by promoting the parameter $q$ in (2.2) to a function of $z$, satisfying $q(0) = 0$ and $q(L) = 1$, so that $tr \Omega = 1$ at $z = 0$, and $tr \Omega = e^{2\pi i/N}$ at $z = L$.

In order to proceed we need a useful parametrization for the remaining generators of $SU(N)$. The diagonal generators are chosen similarly to (2.3). For example,

$$t_{(N-1)} = \frac{1}{\sqrt{2(N-1)(N-2)}} \begin{pmatrix} 1_{N-2} \\ -(N-2) & 0 \\ \vdots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \quad (2.5)$$

Altogether there are $N - 1$ diagonal generators, with $t_i$ from $i = 2$ to $N$. In $SU(2)$ $t_2$ is proportional to the Pauli matrix $\sigma_3$, while in $SU(3)$ $t_3$ is proportional to the Gell–Mann matrix $\lambda_8$. 

4
For the off-diagonal generators we follow Belyaev and Eletsky [11] and use a ladder basis. For example,

\[
t^+_{N,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cdots & 0 & 1 \\ \cdots & 0 & 0 \\ \vdots & \vdots \end{pmatrix}, \quad (2.6)
\]

\[
t^-_{N,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \vdots & \vdots \\ 0 & 0 & \cdots \\ 1 & 0 & \cdots \end{pmatrix}, \quad (2.7)
\]

All elements not indicated vanish. Each diagonal generator \( t_i \) has \( 2(i - 1) \) ladder generators associated with it: \( t^\pm_{i,j} \), with \( j \) running from \( j = 1 \) to \( i - 1 \). For a ladder generator only one element is nonzero,

\[
(t^+_{i,j})_{mn} = \frac{1}{\sqrt{2}} \delta^{in} \delta^{jm}, \quad (t^-_{i,j})_{mn} = \frac{1}{\sqrt{2}} \delta^{im} \delta^{jn}. \quad (2.8)
\]

We are working in the fundamental representation, so the matrix indices \( m \) and \( n \) vary from 1 to \( N \).

These generators form an orthogonal set, with the diagonal generators normalized as

\[
tr(t_i t_{i'}) = \frac{1}{2} \delta^{ii'}, \quad (2.9)
\]

and the off diagonal generators as

\[
tr(t^+_{i,j} t^-_{i',j'}) = \frac{1}{2} \delta^{ii'} \delta^{jj'}, \quad tr(t^+_{i,j} t^+_{i',j'}) = tr(t^-_{i,j} t^-_{i',j'}) = 0. \quad (2.10)
\]

Observe that the metric in the ladder basis is diagonal in the \( i \) and \( j \) indices, but off-diagonal in the \( \pm \) indices.

The advantage of the ladder basis is the simplicity of the commutation relations. For the generator \( t_N \), since from (2.3) it involves the unit matrix in the first \( N - 1 \) components, \( 1_{N-1} \), the only nontrivial commutator of \( t_N \) is with its associated ladder generators, \( t^\pm_{N,j} \). This commutator is just a constant times the same ladder generator:

\[
[t_N, t^\pm_{N,j}] = \pm cN t^\pm_{N,j}. \quad (2.11)
\]

This relation is familiar from \( SU(2) \), from where up to overall constants, \( t_2 \) is \( \sigma^3 \), and \( t^\pm_{2,1} \) are the matrices \( \sigma^\pm \).
The Wilson line in the adjoint representation can be computed using (2.11). This is defined as

$$\Omega_{a \text{adj}}(A) = \frac{2}{N^2 - 1} \left( t^a \mathcal{P} \exp \left( ig \int_0^\beta [A_0(x),] t^b \right) \right),$$  

(2.12)

where “a” and “b” refer to the \((N^2 - 1)\) adjoint indices, and \([A_0,]\) denotes the adjoint operator, \([A_0,]X = [A_0, X]\). For the constant \(A_0\) field of (2.2), the only nontrivial elements of the adjoint Wilson line are those involving the ladder operators \(t^\pm_{N,j}\), and so its trace is

$$tr \Omega_{\text{adj}}(A^{cl}) = 1 - \frac{2}{N + 1} (1 - \cos(2\pi q)).$$  

(2.13)

The adjoint Wilson line is unaffected by the \(Z(N)\) symmetry: for the \(Z(N)\) degenerate vacua, where \(q\) is an integer, \(tr \Omega_{\text{adj}} = 1\).

The classical action is

$$S^{cl}(A) = \int_0^\beta d\tau \int d^3x \frac{1}{2} tr \left( G_{\mu\nu}^2 \right),$$  

(2.14)

where \(G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\) is the field strength tensor. For the interface problem the field \(q\) of (2.2) is assumed to be a function only of \(z\), the length along the tube. For this field the classical action reduces to

$$S^{cl}(A^{cl}) = V_{tr} \frac{4\pi^2 T^2}{g^2 N} (N - 1) \int dz \left( \frac{dq}{dz} \right)^2.$$  

(2.15)

Using the classical action, a solution to the equations of motion is \(q(z) = z/L\). There is no true interface, since the action vanishes like \(1/L\) as \(L \to \infty\). But this is misleading, for classically there is no sign of the \(Z(N)\) symmetry, as all values of \(q\) degenerate.

This classical degeneracy is lifted by quantum effects [8]. This is shown by calculating the action in the presence of the background field in (2.2). For the time being we assume that \(q\) is independent of \(z\), and concentrate on the \(q\)-dependent terms which lift the degeneracy in \(q\). With \(A_\mu = A^{cl}_\mu + A^{qu}_\mu\), in background field gauge [16] the gauge fixing and ghost terms are

$$S^{gf}(A, \eta) = \int_0^\beta d\tau \int d^3x \left( \frac{1}{\xi} tr \left( D^{cl}_{\mu} A^{qu}_{\mu} \right)^2 + \bar{\eta} \left( -D^{cl}_{\mu} D_{\mu} \right) \eta \right),$$  

(2.16)

where \(D_{\mu} = \partial_{\mu} - ig[A_{\mu},]\) is the covariant derivative in the adjoint representation, \(D^{cl}_{\mu} = \partial_{\mu} - ig[A^{cl}_{\mu},]\), and \(\eta\) is the ghost field.
For a constant field \( q \) it is especially easy to expand the full action, \( S^{cl} + S^{gf} \), in quadratic order in the fluctuations \( A^{qu} \), and then integrate them out:

\[
S_{1}^{qu}(A^{cl}) = \frac{1}{2} \, tr \ln \left( -D_{cl}^{2}\delta_{\mu\nu} + \left( 1 - \frac{1}{\xi} \right) D_{\mu}^{cl} D_{\nu}^{cl} \right) - tr \ln \left( -D_{cl}^{2} \right) . \tag{2.17}
\]

The subscript on \( S_{1}^{qu} \) indicates the quantum action at one loop order. The first term on the right hand side is from the integration over the gauge fields, the second from that over the ghosts. Note that because \( A_{\mu}^{cl} \) is assumed to be independent of spacetime, there are no terms in the inverse gauge propagator proportional to \( G_{\mu\nu}^{cl} \).

This quantum action is independent of the gauge fixing parameter, \( \xi \). To see this, note that the derivative of \( S_{1}^{qu} \) with respect of \( \xi^{-1} \) is

\[
\frac{\partial S_{1}^{qu}(A^{cl})}{\partial \xi^{-1}} = \frac{1}{2} \, tr \left( -D_{\mu}^{cl} D_{\nu}^{cl} \left( \delta_{\mu\nu} - \frac{1}{\xi} \frac{D_{\mu}^{cl} D_{\nu}^{cl}}{(-D_{cl}^{2})^{2}} \right) \right) . \tag{2.18}
\]

That is, the derivative is \( -D_{\mu}^{cl} D_{\nu}^{cl} \) times the gauge propagator in the background field. Normally this propagator is difficult to compute because the covariant derivative doesn’t commute with itself. For a constant background field, however, it does, and so the ordering of the \( D_{\mu}^{cl} \)'s is inconsequential. Then (2.18) reduces to

\[
\frac{\partial S_{1}^{qu}(A^{cl})}{\partial \xi^{-1}} = \frac{\xi}{2} \, tr(1) . \tag{2.19}
\]

Thus the variation of the quantum action with respect to \( \xi \) is a constant independent of the background field, which can be dropped.

Adopting Feynman gauge, \( \xi = 1 \), the commutation relations of the ladder basis reduce the color trace in \( S_{1}^{qu} \) to an abelian problem. As \( D_{0}^{cl} \) is the adjoint covariant derivative, it is independent of the background field unless it acts upon the ladder matrices \( t_{N,j}^{\pm} \):

\[
D_{0}^{cl} t_{N,j}^{\pm} = (\partial_{0} \mp 2\pi T q i) t_{N,j}^{\pm} \equiv D_{\pm}^{0} t_{N,j}^{\pm} . \tag{2.20}
\]

With the euclidean four momentum equal to \((k^{0}, \vec{k})\), at nonzero temperature \( k^{0} = 2\pi n T \) for integral \( n \), and the covariant derivative becomes

\[
i D_{\pm}^{0} \rightarrow k_{\pm}^{0} \equiv 2\pi T (n \pm q) . \tag{2.21}
\]

The sum over \( n \) implicit in the trace includes both positive and negative values, with the sum over \( k_{-}^{0} \) equal to that for \( k_{+}^{0} \). Hence the quantum action reduces to

\[
S_{1}^{qu}(A^{cl}) = 2(N - 1) \, tr \ln \left( (k_{\pm}^{0})^{2} + k^{2} \right) , \tag{2.22}
\]

7
\[ k^2 = \tilde{k}^2. \]

This result is typical of loop effects in a constant background field. For the degrees of freedom along the ladder operators \( t_{k,j}^\pm \), the propagators are as in zero background field, except that \( k^0 \) is shifted by a constant amount, to \( k^0_\pm \). The propagators for the remaining degrees of freedom are unaffected by the background field.

From (2.22), at one loop order the \( q \)-dependence of the free energy reduces to \((N - 1)\) copies of that for \( SU(2) \). To isolate the \( q \)-dependence in \( S_1^{qu} \), consider its derivative with respect to \( q \):

\[
\frac{\partial S_1^{qu}(A^{cl})}{\partial q} = 4(N - 1)(2\pi T) \text{ tr } \left( \frac{k^0_+}{(k^0_+)^2 + k^2} \right). \tag{2.23}
\]

The integral is most easily done by integrating first over the spatial momenta:

\[
(V_{tr} T \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \left( \frac{k^0_+}{(k^0_+)^2 + k^2} \right) = -(V_{tr} T) \pi T^3 \sum_{n=-\infty}^{+\infty} (n + q)|n + q|. \tag{2.24}
\]

This sum, while formally divergent, is interpreted using zeta function regularization [13]. The zeta function \( \zeta(p, q) \) is defined as

\[
\zeta(p, q) = \sum_{n=0}^{+\infty} \frac{1}{(n + q)^p}. \tag{2.25}
\]

Hence

\[
\text{tr} \left( \frac{k^0_+}{(k^0_+)^2 + k^2} \right) = -(V_{tr} T) \pi T^3 (\zeta(-2, q) - \zeta(-2, 1 - q)) . \tag{2.26}
\]

Using

\[
\zeta(-2, q) = -\frac{1}{12} \frac{d}{dq} (q^2 (1 - q)^2) , \tag{2.27}
\]

integration of (2.23) gives

\[
S_1^{qu}(A^{cl}) = V_{tr} \frac{4\pi^2 T^4}{3} (N - 1) \int dz q^2 (1 - q)^2 . \tag{2.28}
\]

From the nature of the sum in (2.24), \( S_1^{qu} \) is periodic in \( q \), and is invariant under shifts of \( q \rightarrow q + l \) for any integer \( l \). Thus in (2.28) \( q \) is defined modulo one. Also, a \( q \)-independent constant in (2.28) was dropped; this constant is just the free energy of an ideal gas of \( N^2 - 1 \) gluons at a temperature \( T \).

As promised, the classical degeneracy in \( q \) is lifted by quantum effects: the minima of the theory are now at integral values of \( q = j \), where \( tr \Omega = exp(2\pi ij/N) \).
In (2.28) the length in the $z$ direction, $L$, is replaced by the integral over $z$. Of course for a constant field this substitution doesn’t matter, but consider the interface problem. Introducing the dimensionless coordinate $z'$,

$$z' = \sqrt{\frac{N}{3}} gT z,$$

the sum of the classical and quantum actions becomes

$$S^{cl}(A^{cl}) + S^{qu}(A^{cl}) = V_{tr} \frac{4\pi^2(N-1)}{\sqrt{3N}} \frac{T^3}{g} \int dz' \left( \left( \frac{dq}{dz'} \right)^2 + q^2(1 - q)^2 \right).$$

We can view minimization of this effective action as a problem in mechanics, with the coordinate $z'$ as the “time”. The classical action contributes the kinetic energy, $(dq/dz')^2$, while the quantum action produces a standard double well potential, $q^2(1 - q)^2$. For any solution to the equations of motion the energy, $E = (dq/dz')^2 - q^2(1 - q)^2$ is conserved, $dE/dz' = 0$. With the $Z(N)$ interface we want a solution which obeys the boundary conditions $q(0) = 0$, $q(L') = 1$, as $L' = (\sqrt{N/3}) gT L \rightarrow \infty$ \[5,6\]. By the boundary conditions the instanton has zero energy, $E = 0$. Consequently, for the $Z(N)$ instanton

$$\int dz' \left( \left( \frac{dq}{dz'} \right)^2 + q^2(1 - q)^2 \right) = 2 \int_0^1 dq \ q(1 - q) = \frac{1}{3}.$$

If the total action of the interface is the transverse volume $V_{tr}$ times the interface tension $\alpha$, then at leading order \[4\]

$$\alpha = \frac{4\pi^2(N-1)}{3\sqrt{3N}} \frac{T^3}{g}.$$ 

Two assumptions must be justified. The first is why $S^{qu}$ can be computed for a constant field $q$, and then applied to the $Z(N)$ instanton, where $q$ is clearly a function of $z$. The reason can be seen from the definition of $z'$ in (2.29). As there is no length scale in the rescaled action, in terms of $z'$ the instanton’s width is of order one. For the original coordinate, $z$, this implies that the $Z(N)$ instanton is “fat”, with a width of order $1/(gT)$. In weak coupling this is much larger than the natural length scale in a gas of massless, nearly ideal gluons, which is $1/T$. Thus while the $Z(N)$ instanton field is large in magnitude, $A^{cl} \sim T/g$, it varies slowly in space, and at leading order this variation can be neglected. Corrections to this approximation do enter beyond leading order.
The second assumption is whether in the space of $SU(N)$ gauge fields the path chosen is of minimal action. The $Z(N)$ instanton interpolates between $tr\Omega = 1$ and $tr\Omega = exp(2\pi i/N)$. By a global gauge rotation $\Omega$ can be chosen as a diagonal matrix, involving the $(N - 1)$ diagonal generators of $SU(N)$, the $t_i$’s. The quantum action for a general (constant) field can be computed directly [8]. We chose the simplest path possible — straight along the $t_N$ direction — but it is not obvious that other paths, which wander off into the direction of the other $(N - 2)$ $t_i$’s, might not have lower action. (Our path is at least a local minimum.) For $SU(2)$ there is only one path possible. In appendix A we show that for $N = 3$ and $N = \infty$, the path along $t_N$ is minimal. On this basis we conjecture that this remains true for all $4 \leq N < \infty$.

III. General analysis of the $Z(N)$ interface

The partition function of an $SU(N)$ gauge theory is

$$Z = \int [dA_\mu(x)] [d\eta(x)] e^{-S^{cl}(A) - S^{gf}(A,\eta)},$$

where $S^{cl}$ and $S^{gf}$ are the classical and gauge fixing actions of (2.14) and (2.16). The gauge field $A_\mu = A^{cl}_\mu + A^\mu_0$; for the time being the choice of $A^{cl}_\mu$ is left open. The coordinate of four dimensional spacetime is $x = (\tau, \vec{x}, z)$

In order to reduce this four dimensional theory to an effective theory in one dimension we introduce the field

$$q(z) = \frac{1}{V_{tr}} \int_0^\beta d\tau \int d^2x_i \frac{gcN}{2\pi T} A^{cl}_0(x).$$

This choice is obviously motivated by the definition of $A_0^{cl}$, unlike the scalar field $q$ of (2.2), $q$ is a matrix valued field in the adjoint representation of $SU(N)$. The only subtlety in the introduction of $q(z)$ is our insistence on defining it not just as the average of $A^0_\mu$ over time, but over the spatially transverse directions as well. The reason for this will become apparent shortly.

The field $q(z)$ is introduced into the functional integral by a delta function constraint:

$$Z = \int [dA_\mu(x)] [d\eta(x)] [d\lambda(z)] [dq(z)] e^{-S^{cl}(A) - S^{gf}(A,\eta) - S^{con}(\lambda,q,A)},$$

$$S^{con}(\lambda,q,A) = \int dz \ 2i tr \left( \lambda(z) \left( q(z) - \frac{1}{V_{tr}} \int_0^\beta d\tau \int d^2x_i \frac{gcN}{2\pi T} A^{cl}_0(x) \right) \right).$$
The constraint field $\lambda(z)$ is introduced to enforce the definition of $q(z)$ in (3.2), and so is also an adjoint matrix. With the overall factor of $i$ in $S^{\text{con}}$, the original contour of integration for the constraint field $\lambda(z)$ is along the real axis.

As an aside, note that $q(z)$ does not transform in a simple fashion under gauge transformations. For an infinitesimal gauge transformation $\omega$, where $A_{\mu} \rightarrow \partial_{\mu} \omega - ig[A_{\mu}, \omega]$,

$$q(z) \rightarrow \frac{1}{V_{\text{tr}}} \int_{0}^{\beta} d\tau \int d^{2}x_{t} \frac{g c N}{2\pi T} (-ig)[A_{0}(x), \omega(x)],$$

assuming that $\omega(x) = \omega(\tau, \vec{x}_{t}, z)$ is periodic in $\tau$. Nevertheless, no further terms besides those in (3.3) are required in the measure of the functional integral: gauge fixing for $A_{\mu}$ takes care of that.

The effective action $S^{\text{eff}}(q)$, is defined as the integral over all fields, excepting $q(z)$:

$$Z = \int [dq(z)] e^{-S^{\text{eff}}(q)}.$$  

(3.6)

How the effective action is computed in practice depends upon the problem at hand. But by introducing $q(z)$ as an extra field into the functional integral, clearly there is no confusion possible about double counting degrees of freedom.

The calculation of $S^{\text{eff}}(q)$ is straightforward at one loop order [7], and so our discussion is brief. The field $q(z)$ is held fixed, while the gauge field and the constraint field $\lambda$ are separated into classical plus quantum terms: $A_{\mu} = A_{\mu}^{cl} + A_{\mu}^{qu}$ and $\lambda = \lambda^{cl} + \lambda^{qu}$. Expanding the action of (3.3) to quadratic order,

$$S^{cl}(A) + S^{gf}(A, \eta) + S^{\text{con}}(\lambda, q, A) \approx S_{0} + S_{1} + S_{2}. $$

(3.7)

The leading term is the sum of the classical action plus the constraint:

$$S_{0} = S^{cl}(A^{cl}) + S^{\text{con}}(\lambda^{cl}, q, A^{cl}).$$

(3.8)

The linear terms determine the equations of motion:

$$S_{1} = S^{\text{con}}(\lambda^{qu}, q, A^{cl}) + 2i \int d^{4}x \text{tr} \left( \lambda^{cl} \delta_{\nu 0} + iD_{\mu}^{cl} G_{\mu\nu}^{cl} \right) A_{\nu}^{qu}. $$

(3.9)

The constraint term, $S^{\text{con}}(\lambda^{qu}, q, A^{cl})$, is linear in $\lambda^{qu}$, and so determines $A^{cl}(x)$. The obvious choice is to take $A^{cl}(x)$ to be a function only of $z$,

$$A^{cl}(z) = \frac{2\pi T}{gc N} q(z).$$

(3.10)
The constraint term in $S_0$ then vanishes, $S^{\text{con}}(\lambda^{cl}, q, A^{cl}) = 0$.

The second term in $S_1$ modifies the equations of motion for the gauge field: the constraint term acts as a source for the gauge field, proportional to $\lambda^{cl}$. As $A^{cl}$ is determined, let

$$\lambda^{cl}(z) = -i D_\mu^{cl} G^{cl}_{\mu0}(z),$$

from which $S_1 = 0$. (As $A^{cl}(x)$ is assumed to depend only upon $z$, the other components of the equations of motion for the gauge field are automatically satisfied.) At the stationary point $\lambda^{cl}$ is purely imaginary, while the fluctuations $\lambda^{qu}$ remain real. This shift in the contour for the constraint field $\lambda$ — by an imaginary amount, keeping it parallel to the real axis — is standard.

The quadratic terms in the action are

$$S_2 = -\frac{g_c N}{2\pi TV_{tr}} \int d^4 x (2i) \text{tr} \left( \lambda^{qu}(z) A^{qu}_0(x) \right)$$

$$+ \int d^4 x \int d^4 y \text{tr} \left( A^{qu}_\mu \Delta^{cl \mu \nu} A^{qu}_\nu + \bar{\eta} \left( -D^{2}_{\mu} \right) \eta \right).$$

(3.12)

The first term in $S_2$, involving $\lambda^{qu}$, is special to the constraint action. Since $\lambda^{qu}$ only enters into $S_2$ linearly, it can be integrated out. This introduces a constraint for the integration over $A^{qu}_0(x)$:

$$\frac{1}{V_{tr}} \int_0^\beta d\tau \int d^2x_t A^{qu}_0(\tau, \vec{x}_t, z) = 0.$$  

(3.13)

This constraint is completely innocuous. Remember that for the two spatial directions $\vec{x}_t$, each length $L_t$ is taken to infinity. Integration is over all modes of $\lambda^{qu}(z)$ which obey (3.13), but the only modes which don’t are those constant both in $\tau$ and $\vec{x}_t$ — in momentum space, modes with $k^0 = \vec{k}_t = 0$. If the length in the $\vec{x}_t$ directions are infinite, the corresponding momenta $\vec{k}_t$ take on all continuous values, and those with $\vec{k}_t = 0$ have zero measure, and can be ignored.

Being able to drop the constraint is an important point. Suppose the constraint field is defined not as an integral over $\tau$ and $\vec{x}_t$, but just as an integral over $\tau$: then $q$ is a field in three, instead of one, dimension. Going through the same procedure as above, at quadratic order integration is over all fields constant in time. But these modes have nonzero measure in the functional integral. At finite temperature, the momentum $k^0 = 2\pi n T$ for integral $n$, and the constant modes, with $n = 0$, are of countable extent.
For situations in which the constraint doesn’t matter — that is, where the effective fields are of zero measure in the space of the original fields — the constraint methods give the same result as for the usual effective potential \([7]\). The constraint field \(\lambda^{cl}\) plays the role of the external source, while the exchange of \(A^{cl}\) for \(q\) mimics precisely the process of Legendre transformation. The effective potential is the energy of the vacuum in the presence of the external source, and so is properly minimized.

The meaning of the effective action when the effective fields are of nonzero measure in the space of the original fields is unclear. We dwell on this point because of such an analysis by Oleszczuk and Polonyi \([10]\), who introduce an effective field in three dimensions, as the integral of \(A^0\) with respect to time. Integrating out modes with \(k^0 \neq 0\), they find a potential different from that of (2.28). The potential in (2.28) is a constant times \(q^2(1-q)^2 = q^2 - 2q^3 + q^4\). The potential for the three dimensional field of \([10]\) is just \(q^2 + q^4\); the term \(-2q^3\) is missing, as that arises from the \(k^0 = 0\) mode of the integral. More generally, with an effective three dimensional theory, the manifest \(Z(N)\) symmetry of the original theory is broken by the separation into modes with zero and nonzero \(k^0\). The loss of the \(Z(N)\) symmetry seems a grievous price to pay.

Returning to (3.12), the remaining terms are standard in a background field expansion. In background field gauge, (2.16), the inverse gauge field propagator is

\[
\Delta^{cl}_{\mu\nu}^{-1} = -D^2_{cl} \delta^{\mu\nu} + D^\mu_{\nu} D^\nu_{\mu} - \frac{1}{\xi} D^\mu_{\mu} D^\nu_{\nu} + i g [G^{cl}_{\mu\nu}] ,
\]

After integrating out \(\lambda^{qu}\), \(A^{qu}\), and \(\eta\),

\[
S^{eff}(q) = S^{cl}(A^{cl}) + S^{qu}_1(A^{cl}) ,
\]

where \(A^{cl}\) is related by \(q(z)\) by (3.10), and

\[
S^{qu}_1(A^{cl}) = \frac{1}{2} tr \log \left( \Delta^{cl}_{\mu\nu}^{-1} \right) - tr \log \left( -D^2_{cl} \right) .
\]

For the \(Z(N)\) interface let

\[
q(z) = q(z) t_N ,
\]

and then repeat the analysis of sec. II. Taking the field of the \(Z(N)\) instanton as slowly varying, to leading order in \(g^2\) the effective action for \(q(z)\), \(S^{eff}(q)\), is given by (2.30).
At next to leading order corrections arise from two sources. Viewing the action of (2.30) as a type of quantum mechanics, these terms can be understood as corrections to the potential and kinetic terms. As the potential was first generated by the free energy at one loop order, so corrections to this potential are produced by the free energy at two loop order. These two loop effects are \(g^2\) times those at one loop order. Like the calculation at leading order in sec. II, for the two loop potential the background field can be taken as constant. Secondly, at next to leading order it is necessary to account for the spatial variation of the \(Z(N)\) instanton. For this only the free energy at one loop order is required, expanding to leading order in \((dq/dz)^2\); thus these terms correct the kinetic term in the effective action. With our definition of \(q\), the classical action is proportional to \(1/g^2\) times \((dq/dz)^2\), (2.15); the terms from the free energy are of order one times \((dq/dz)^2\), and so are smaller by \(g^2\).

Ultimately, corrections are small because the \(Z(N)\) instanton is fat: the ratio of its size to the thermal wavelength is of order \(1/g\). Hence an expansion in the derivatives of the instanton field is automatically an expansion in \(g^2\). This is what makes the problem tractable.

Both of these effects are due to the effects of fluctuations in four dimensions as they generate the effective, one dimensional action \(S_{\text{eff}}(q)\). The functional integral over \(q\) in (3.6), however, is still treated classically. When do fluctuations in \(q(z)\) enter?

Remember that the quantity of physical interest \([5]\) is the partition function, \(Z\), for a system with the appropriate boundary conditions to enforce a domain wall in the spacetime tube:

\[
Z = c e^{-\alpha V_{tr}}. 
\]  

(3.18)

While we have concentrated on the interface tension, \(\alpha\), of course the prefactor “\(c\)” is also of significance.

Integration over fluctuations in four dimensions generate the effective theory in one dimension, \(S_{\text{eff}}\), and so determine \(\alpha\). Fluctuations in \(S_{\text{eff}}\), though, do not contribute to \(\alpha\), only to the prefactor. This is simply because \(S_{\text{eff}}\) itself is proportional to \(V_{tr}\), and so the integral over the effective, one dimensional fields cannot generate a constant times \(V_{tr}\) in the exponent, but merely powers of \(V_{tr}\) in the prefactor \([6]\). The prefactor for a \(Z(N)\) domain wall is given by the integral over \(S_{\text{eff}}\) at one loop order; however, it is necessary to compute \(S_{\text{eff}}\) for a general path in group space, \([a,1]\), instead of the “classical” path of (3.17). This we defer.
Consequently, we confess that the machinery of the effective action $S_{\text{eff}}(q)$ developed in this section is not essential for what follows. We discussed it at such length in order to ensure that there are no problems of principle, and to emphasize the generality of the method.

IV. $Z(N)$ interface at next to leading order: Feynman gauge

In this section we compute the leading corrections to the interface tension in background field Feynman gauge, $\xi = 1$; in the next section, for arbitrary $\xi$. While the methods are the same for all $\xi$, technically the calculations are simpler in Feynman gauge, and so in this section we discuss our methods in some detail. In sec. V the results are merely summarized, in order to emphasize the physical interpretation of the gauge dependence which arises in the effective action.

As discussed following (3.17), at next to leading order there are two pieces needed for the interface tension. The first is the effective potential in a constant background $A_0^{cl}$ field to two loop order. This was calculated for $SU(3)$ by Belyeav and Eletsky [11] and by Enqvist and Kajantie [12]. We have independently computed the potential for the field of (2.2) at arbitrary $N$, but given previous calculations, are content here to just establish the $N$–dependence at two loop order.

At one loop order the $N$ dependence of $S_{\text{qu}}^{\text{cl}}(A)\mid_{\xi=1}$ is obvious. The background field enters only through adjoint covariant derivatives; from (2.11), $t_N$ only has nontrivial commutators with the generators $t^\pm_{N,j}$. Thus the only $q$–dependence is from the free energy of the $2(N-1)$ fields for these ladder operators, and at one loop order $S_{\text{qu}}^{\text{cl}}(A)\mid_{\xi=1} \sim (N-1)$, (2.22).

The diagrams which enter at two loop order involve either two three–gluon vertices or one four–gluon vertex. Both types of diagrams involve a product of structure constants. The only diagrams that depend nontrivially upon the background field are those in which two lines are along the ladder operators $t^\pm_{N,j}$. Denoting the $q$–dependent terms in the free energy at two loop order as $S_{\text{qu}}^{\text{cl}}(A)\mid_{\xi=1}$, after writing each structure constant as a trace

$$S_{\text{qu}}^{\text{cl}}(A)\mid_{\xi=1} \sim \sum_{a} \left( tr \left( t^a \left[ t^+_N t^{-}_N, t^+_N t^{-}_N \right] \right) \right)^2 = (N-1) \sum_{a} \left( tr \left( t^a \left[ t^+_N t^{-}_N, t^+_N t^{-}_N \right] \right) \right)^2. \tag{4.1}$$

The sum is over all generators $t^a$ with nonzero trace. The last expression follows by
noting that from the form of \( t_N \) in (2.3), each value of \( j \) contributes equally to (4.1). Thus the complete sum over \( j \) is \((N-1)\) times that for any single term, such as \( j = (N - 1) \). The commutator for \( t_{N,(N-1)}^+ \) is:

\[
[t_{N,(N-1)}^+, t_{N,(N-1)}^-] = \frac{1}{2} \begin{pmatrix}
\vdots & \vdots \\
\ldots & -1 & 0 \\
\ldots & 0 & 1
\end{pmatrix}.
\] (4.2)

All elements not indicated vanish. From (4.2) the only terms which contribute to (4.1) are if \( t^a \) is one of two diagonal generators, \( t_N \) or \( t_{N-1} \), (2.3) and (2.5). It is then easy to show that the sum over \( a \) in (4.1) is proportional to \( N \), so that in all \( S^{\text{qu}}_2(A^c) \sim N(N-1) \).

Knowing the \( N \)-dependence, the general result can be read off from that for \( N = 3 \). For the constant field of (2.2), the sum of the the free energies at one loop order, \( S^{\text{qu}}_1(A^c) \) (2.28), and at two loop order in Feynman gauge, \( S^{\text{qu}}_{2,\xi=1}(A^c) \), is

\[
S^{\text{qu}}_1(A^c) + S^{\text{qu}}_{2,\xi=1}(A^c)
= V_{tr} \frac{4\pi^2 T^4}{3} (N-1) \int dz \left( q^2 (1 - q)^2 + \left( \frac{g^2 N}{16\pi^2} \right) \left( 3 q^2 (1 - q)^2 - 2 q (1 - q) \right) \right).
\] (4.3)

The second piece required for the interface tension arises from the free energy at one loop order for a background field which varies in \( z \). This corrects the kinetic term in the effective action, as a function of the background field, \( q \), times \((dq/dz)^2\): once one factor of \((dq/dz)^2\) is extracted, the remaining factors of \( q \) can be taken as constant. At one loop order the quantum action \( S^{\text{qu}}_1(A^c) \) is given by (3.16), with the inverse gluon propagator of (3.14), setting \( \xi = 1 \) in Feynman gauge. To calculate this some, although not all, of the tricks of sec. II can be used. For example, from (2.17)-(2.19), for a constant field the one loop quantum action is independent of \( \xi \). This is no longer true for a spatially varying field.

The ladder basis of sec. II can be used to simplify the color algebra. Defining \( G_{0z}^+ = - G_{z0}^+ = dq/dz \), else zero, and \( D^2_+ = (D_0^+)^2 + \partial_i^2 \), with \( D_0^+ \) as in (2.20), for arbitrary fields \( q(z) \) the one loop quantum action reduces to

\[
S_{1,\xi=1}^{\text{qu}}(A^c) = (N-1) \left( tr \ log \left( - D^2_+ \delta_{\mu\nu} + 4\pi T i G_{\mu
u}^+ \right) - 2 \ tr \ log \left( - D^2_+ \right) \right).
\] (4.4)

One kinetic term arises by expanding to quadratic order in \( G_{\mu\nu}^+ \):

\[
S_{1a,\xi=1}^{\text{qu}}(A^c) = - 16\pi^2 T^2 (N-1) \left( \frac{dq}{dz} \right)^2 \ tr \left( \frac{1}{((k_0^a)^2 + k^2)^2} \right),
\] (4.5)
with \( k_0 \) as in (2.21).

Having extracted this, the remaining kinetic term arises from the expansion of

\[
S_{1_b,\xi=1}^{\text{qu}}(A^\xi) = 2(N - 1) \left(1 - \frac{\epsilon}{2}\right) tr \log(-D_+^2). \tag{4.6}
\]

The factor of \( 1 - \epsilon/2 \) appears because the theory is regularized in \( 4 - \epsilon \) dimensions, and in covariant gauges the number of gluons equals the dimensionality.

Calculating the momentum dependence for such a one loop action is a standard problem: see, for example, the treatment of Iliopoulos, Itzykson, and Martin [17]. The computations of [17], however, are in coordinate space, which for most problems is rather awkward. Instead, it is much simpler to perform the calculations in momentum space. We have checked that for the problem at hand, as well as for the scalar example treated in [17], the results agree.

To work in momentum space, let \( q \to q + \delta q \) in (4.6), and expand to quadratic order in \( \delta q \). The idea is to isolate the momentum dependence in the fluctuation, \( \delta q \):

\[
tr \log(-D_+^2) \sim 8\pi^2 T^2 \left(\frac{1}{2} tr \left(\frac{1}{D_+^2} (\delta q)^2\right) + tr \left(\frac{D_+^0}{-D_+^2} \delta q \frac{D_+^0}{D_+^2} \delta q\right)\right). \tag{4.7}
\]

The momentum dependence only arises through the second term on the right hand side, since the first term is a type of tadpole, independent of the momentum flowing through \( \delta q \). The field \( q \) varies only in \( z \), so its momentum is purely spatial. If \((0, \vec{p})\) is the external momenta, then, and \((k^0, \vec{k})\) the loop momenta, (4.7) becomes

\[
8\pi^2 T^2 \delta q(\vec{p})\delta q(-\vec{p}) \left(\frac{(k_0^0)^2}{((k_+^0)^2 + k^2)((k_+^0)^2 + (\vec{p} - \vec{k})^2)}\right). \tag{4.8}
\]

In this form it is trivial to expand to order \( p^2 \). Trading \( L(\delta q(p^2) \delta q) \) for \( \int dz (dq/dz)^2 \), the second kinetic term is

\[
S_{1_b,\xi=1}^{\text{qu}}(A^\xi) = 16\pi^2 T^2 (N - 1) \left(\frac{dq}{dz}\right)^2 \left(1 - \frac{\epsilon}{2}\right) tr \left(\frac{-k^2}{((k_+^0)^2 + k^2)^3} + \frac{4}{3 - \epsilon} \frac{k^2}{((k_+^0)^2 + k^2)^4}\right). \tag{4.9}
\]

At nonzero temperature dimensional continuation is carried out by changing the number of spatial dimensions to \( 3 - \epsilon \), which produces the factor of \( 4/(3 - \epsilon) \) above.

The integrals required are given in appendix B, (b.4), (b.6), and (b.7). Without worrying about their detailed form, one feature is evident. Each integral is logarithmically divergent in four dimensions, so in \( 4 - \epsilon \) dimensions, there are poles in
These are the standard terms which produce the renormalization of the coupling constant at one loop order. Thus it is instructive to combine the results for (4.5) and (4.9) with the classical action of (2.15) to find

\[ S_{cl}(A_{cl}) + S_{qu}^{a, \xi=1} + S_{qu}^{b, \xi=1} = V_{tr} \frac{4\pi^2 T^2}{g^2(T)N} (N - 1) \int dz \left( \frac{dq}{dz} \right)^2 \left( 1 + \frac{11}{3} \frac{g^2 N}{16\pi^2} \left( \psi(q) + \psi(1 - q) + \frac{1}{11} \right) \right); \]

\[ \psi(q) = \frac{d}{dq} \left( \log \Gamma(q) \right) \]
is the digamma function. The prefactor includes the running coupling constant at a temperature \( T \), \( g^2(T) \), which is related to the bare coupling constant \( g^2 \) as

\[ \frac{1}{g^2(T)} = \frac{1}{g^2} \left( 1 - \frac{11}{3} \frac{g^2 N}{16\pi^2} \left( \frac{2}{\epsilon} + \log \left( \frac{\mu^2}{\pi T^2} \right) + \psi(1/2) \right) \right), \]

with \( \mu \) the renormalization mass scale. The relationship between the bare and renormalized coupling constants in (4.11) is arbitrary up to a constant; our choice is similar but not identical to the modified minimal subtraction scheme, and is convenient at nonzero temperature. At high temperature, (4.11) exhibits the standard logarithmic fall off of the running coupling constant, \( g^2(T) \), with the coefficient of \( 11N/3 \) appropriate for the \( \beta \)-function of an \( SU(N) \) gauge theory at one loop order. Notice that the \( q \)-dependence in (4.10), through the digamma functions of \( q \) and \( 1 - q \), enters with precisely the same coefficient as for the \( \beta \)-function, \( 11N/3 \).

The effective action which governs the \( Z(N) \) instanton at next to leading order is the sum of (4.3) and (4.11). The action is determined by the properties of the solution at leading order. As discussed following (2.30), the \( Z(N) \) instanton has zero energy, and so

\[ \int dz' \left( \frac{dq}{dz'} \right)^2 (\psi(q) + \psi(1 - q)) = 2 \int_0^1 dq \left( q - \frac{1}{2} \right) \log \left( \frac{\Gamma(q)}{\Gamma(1 - q)} \right) \sim -0.995018 \ldots; \]

the value of the integral was determined by numerical integration. Hence at next to leading order, the interface tension \( \alpha \) is

\[ \alpha = \frac{4(N - 1)\pi^2}{3\sqrt{3N}} \frac{T^3}{g(T)} \left( 1 - (15.2785...) \frac{g^2(T)N}{16\pi^2} \right), \]

which is the result quoted in (1.1). We have taken the liberty of writing the corrections as proportional not just to the bare coupling constant, \( g^2 \), but to the running coupling constant, \( g^2(T) \).
V. $Z(N)$ interface at next to leading order: general gauges

The calculation of the interface tension in an arbitrary background field gauge is similar to that for Feynman gauge. Nevertheless, the calculation illuminates some features which are missed by working at fixed $\xi$.

Including the terms at both one and two loop order, the free energy in the constant background field of (2.2) is

\begin{equation}
S_{cl}(A^{cl}) + S_{qu}^{a,\xi}(A^{cl}) + S_{qu}^{b,\xi}(A^{cl}) = V_{tr} \frac{4\pi^2 T^4}{3} (N - 1) \int dz \left( q^2 (1 - q)^2 + \left( \frac{g^2 N}{16\pi^2} \right) \left( (7 - 4\xi) q^2 (1 - q)^2 - (3 - \xi) q(1 - q) \right) \right). \tag{5.1}
\end{equation}

The two loop potential for general $\xi$ was computed first by Enqvist and Kajantie [12]; we agree with their result when $\xi = 1$, but not for $\xi \neq 1$.

The potential changes in a rather dramatic fashion in going from one to two loop order. At one loop order the potential is just a standard double well, $q^2 (1 - q)^2$. At two loop order the potential becomes $\xi$-dependent. This includes a correction to the coefficient of the double well potential, as well as a new term, proportional to $q(1 - q)$. This new term is peculiar, for it controls the behavior of the potential for small $q$. If $\xi < 3$, the stable minima are not at $q = 0$ and $q = 1$, but at $q_0 \sim (3 - \xi) q^2$ and $1 - q_0$. On the other hand, if $\xi > 3$, the stable minima remain $q = 0$ and $q = 1$.

Such a nonzero value of the stable minima would have profound consequences for a gauge theory at high temperature [9-14]. For $N \geq 3$, the trace of the Wilson line in the fundamental representation is a complex number. Under charge conjugation ($C$) or time reversal ($T$) transformations, the trace of the Wilson line goes into its complex conjugate (up to global $Z(N)$ transformations). Thus if the stable vacuum indeed has $q \neq 0$ (modulo 1), then the vacuum spontaneously breaks $C$ and $T$ symmetries, conserving $C T$. While conceivable, it is unexpected to find $C$ symmetry breaking arising spontaneously in a pure gauge theory. Of course a physical phenomenon cannot depend upon the choice of the gauge fixing parameter, while $q_0$ changes with $\xi$.

Leaving these questions aside for the moment, in a general background gauge the kinetic terms in the effective action are, to one loop order,

\begin{equation}
S^{cl}(A^{cl}) + S_{1a,\xi}(A^{cl}) + S_{1b,\xi}(A^{cl})
\end{equation}
\[ V_{tr} \frac{4\pi^2 T^2}{g^2(T)N} (N-1) \int dz \left( \frac{dq}{dz} \right)^2 \left( 1 + \frac{11}{3} \frac{g^2 N}{16\pi^2} \left( \psi(q) + \psi(1-q) + \frac{7-6\xi}{11} \right) \right). \]

The running coupling constant, \( g^2(T) \), remains as in (4.11).

In (5.2) the only \( \xi \) dependence is an overall constant, proportional to \( 7-6\xi \), and is independent of the background field \( q \). As discussed following (5.7) in appendix B, for each of the integrals which contribute to the kinetic term, the coefficient of the pole in \( 1/\epsilon \) is always the same as for the digamma functions of \( q \) and \( 1-q \), which is how the \( q \)-dependence arises. As is customary in background field calculations \([16]\), the poles in \( 1/\epsilon \) generate the \( \beta \)-function at one loop order, and is independent of \( \xi \). Thus if the coefficient of the digamma functions is the same as for \( 1/\epsilon \), at one loop order it also must be independent of \( \xi \). The only remaining \( \xi \)-dependence possible is as a constant, which does appear.

The effective action for arbitrary \( \xi \) is the sum of (5.1) and (5.2). It is easy to show that while each term depends individually upon \( \xi \), for any solution with zero energy, \( \mathcal{E} = 0 \), the \( \xi \)-dependence cancels in the sum. As the \( Z(N) \) instanton has zero energy, the value of the interface tension for \( \xi \neq 1 \) is equal to that for \( \xi = 1 \), (4.13).

The cancellation of \( \xi \)-dependence is a necessary check on the consistency of our method, but by itself is rather unsatisfactory. To understand this better we follow Belyaev \([14]\). In \( SU(2) \) Belyaev showed that the apparent \( \xi \)-dependence in the potential for a constant \( A_0 \) field can be understood as a renormalization of the Wilson line. We now generalize his results to arbitrary \( SU(N) \), and show that they explain the \( \xi \)-dependence of both the potential and kinetic terms in the effective action.

The point is that while the vacuum expectation value of the trace of the Wilson line is a gauge invariant quantity, the fields which we have been using to parametrize the Wilson line — \( A_0^{cl} \), and so \( q \) — are not. For instance, from (3.3), our effective field transforms in a nonlocal manner under infinitesimal gauge transformations. At tree level this doesn’t matter, but it does at one loop order and beyond, as the Wilson line undergoes both infinite and finite renormalizations. To compute these, define

\[ A_0(x) = A_0^{cl} + A_0^{qu}(x), \]

with \( A_0^{cl} \) related to \( q \) as in (2.2). In general the Wilson line is a function of the spatial position, and so we consider the trace of the Wilson line, averaged over space.
Expanding in powers of $A^{qu}$,

$$\int \frac{d^2x_t}{L_t} \int \frac{dz}{L} \langle \text{tr} \Omega(A) \rangle = \text{tr} \Omega(A^{cl}) + \Omega_1 + \Omega_2 + \ldots.$$  \hspace{1cm} (5.4)

The first term on the right hand side, $\text{tr} \Omega(A^{cl})$, is the value in the classical background field, (2.4). The term linear in $A^{qu}$, $\Omega_1$, can be written as

$$\Omega_1 = ig \int \frac{dz}{L} \int_0^\beta d\tau \int \frac{d^2x_t}{L_t^2} \text{tr} \left( A^{qu}_0(\tau, \vec{e}_t, z) \Omega(A^{cl}) \right).$$ \hspace{1cm} (5.5)

Due to the constraint imposed upon the quantum fluctuations in (3.13), this term vanishes.

The term quadratic in quantum fluctuations is nontrivial. Consider first its value in zero background field, $A^{cl} = 0$:

$$\Omega_2 = -\frac{g^2(N^2 - 1)}{2} \frac{1}{\beta \left( \frac{2\pi^2}{2} k^2 \right)}.$$ \hspace{1cm} (5.6)

This is a standard renormalization of the Wilson line, a type of “wave function” renormalization, proportional to $\beta$, the length in euclidean time. In dimensional regularization this vanishes identically.

However these infinite terms are regularized, they are independent of the background field. In addition, there are finite terms which depend upon $A^{cl}$. Using the commutation relations of (2.11), for an arbitrary constant “$y$”

$$e^{yt_N} t^+_N e^{-yt_N} = e^{+ye N} t^+_N.$$ \hspace{1cm} (5.7)

Using this relation, and remembering the path ordering required for the Wilson line, the terms dependent on the background field are

$$\Omega_2 = -\frac{g^2}{N} \sum_{j=1}^{N-1} \int_0^\beta d\tau \int_0^{\tau'} d\tau' \left( A^{qu}_{0,j+}(\tau) A^{qu}_{0,j-}(\tau') e^{-2\pi i q(\tau'-\tau)} \text{tr} \left( t^+_N t^-_{N,j} \Omega(A^{cl}) \right) \right) + A^{qu}_{0,j+}(\tau) A^{qu}_{0,j-}(\tau') e^{2\pi i q(\tau'-\tau)} \text{tr} \left( t^-_{N,j} t^+_{N,j} \Omega(A^{cl}) \right).$$ \hspace{1cm} (5.8)

$A^{qu}_{0,j\pm}$ is the component of $A^{qu}_0$ in the direction of $t^\pm_{N,j}$. The $\tau$ integrals are evaluated using the background field propagator, as in (2.18). After doing the color trace, in momentum space

$$\Omega_2 = \frac{ig^2\beta(N - 1)}{2N} e^{2\pi i q/N} \left( 1 - e^{-2\pi i q} \right) \text{tr} \left( \frac{1}{k_+^0} \Delta_{00}(k_+^0, k) \right).$$ \hspace{1cm} (5.9)
\( \Delta_{00}(k^0, k) \) is the usual covariant gauge propagator,
\[
\Delta_{00}(k^0, k) = \frac{1}{(k^0)^2 + k^2} - (1 - \xi) \frac{(k^0)^2}{((k^0)^2 + k^2)^2},
\]
with the only dependence on the background field in \((5.9)\) through the shifted momentum \(k^0_+\), \((2.21)\).

How then to interpret these corrections at one loop order to the classical value of the Wilson line? By the constraint imposed upon \(A^{qu}\), the linear term in \((5.5)\) vanishes, and so they cannot absorbed in \(A^{qu}\). We then introduce “renormalized” fields for \(A_0\) and \(q\) as
\[
A_0^{\text{ren}} = \frac{2\pi T}{g c N} q^{\text{ren}} t_N, \quad q^{\text{ren}} = q + \delta q,
\]
and require that the spatial average of the vacuum expectation value of the Wilson line be given by \(A^{\text{ren}}\).
\[
\int \frac{d^2 x_t}{L^2_t} \int \frac{dz}{L} \langle \text{tr} \, \Omega(A) \rangle = \text{tr} \, \Omega(A^{\text{ren}}).
\]
Expanding the right hand side to linear order in \(\delta q\), \(\delta q\) is proportional to \(\Omega_2\). The integrals required for \((5.9)\) are \((5.9)\) and \((5.10)\) of appendix B, and give
\[
q^{\text{ren}} = q + \frac{g^2 N}{16\pi^2} (3 - \xi) \left( q - \frac{1}{2} \right).
\]
This relation is valid for \(0 < q < 1\). For \(SU(2)\), it agrees with the result of Belyaev \[14\]. Note that the renormalization from \(q\) to \(q^{\text{ren}}\) is entirely a matter of a finite shift.

The gauge dependent actions, \((5.1)\) and \((5.2)\), can be trivially rewritten in terms of the renormalized fields. The potential term becomes
\[
S^{qu}_1(A^{\text{ren}}) + S^{qu}_2(A^{\text{ren}}) = V_tr \frac{4\pi^2 T^4}{3} (N - 1) \int dz \left( 1 - 5 \frac{g^2 N}{16\pi^2} \right) q^2(1 - q)^2,
\]
while for the kinetic terms,
\[
S^{cl}(A^{\text{ren}}) + S^{qu}_{1a}(A^{\text{ren}}) + S^{qu}_{1b}(A^{\text{ren}})
= V_tr \frac{4\pi^2 T^2}{g^2(T) N} (N - 1) \int dz \left( \frac{dq}{dz} \right)^2 \left( 1 + \frac{11}{2} \frac{g^2 N}{16\pi^2} (\psi(q) + \psi(1 - q) + 1) \right).
\]
With the effective action of (5.14) and (5.15), the corrections to the interface are of course unchanged, equal to (4.11).

Once written in terms of $q^{\text{ren}}$, all of the gauge dependence found previously in the kinetic and potential terms cancels. Further, the effect of two loop terms in the renormalized potential is just to change the coefficient of the one loop terms: the new terms found previously at two loop order, proportional to $q(1-q)$, have all been absorbed by $q^{\text{ren}}$.

Hence the apparent instability of the perturbative vacuum at two loop order merely results from a classical parametrization of the renormalized Wilson line. After correcting for loop effects, the stable vacuum is the trivial one (plus $Z(N)$ transforms thereof) and is $C^*$ symmetric.

It seems unlikely that the cancellations found at two loop order are mere coincidence. We conclude with a conjecture: that the stable vacuum of hot gauge theories — both with and without fermions — is symmetric under charge conjugation to arbitrary loop order.

The research of A.G. and R.D.P. was supported in part by the U.S. Department of Energy under contract DE–AC02–76–CH0016.

Appendix A: Proof of minimal action for $N = 3, \infty$

In this appendix we prove that for $N = 3$ and $\infty$, the path chosen for the $Z(N)$ instanton is of minimal action.

By a global gauge rotation a constant background field $A^{cl}$ can be chosen to be a diagonal matrix. Thus the most general constant field for the $A^{cl}$ of (3.10) is

$$q = \sum_{i=2}^{N} q_i t_i .$$

The $t_i$ are the $N - 1$ diagonal generators of $SU(N)$, as in (2.4) and (2.5). For clarity, in (a.1) the field $q$ is relabeled as $q_N$.

Promoting each $q_i$ to be a function of $z$, for this ansatz the classical action becomes a sum over $N - 2$ independent kinetic terms

$$S^{cl}(A^{cl}) = V_{tr} \frac{4\pi^2 T^2}{g^2 N} (N - 1) \int dz \sum_{i=2}^{N} \left( \frac{dq_i}{dz} \right)^2 .$$

23
For the $Z(N)$ interface the boundary conditions required are

$$q_i(0) = q_i(L) = 0, \quad i = 2 \ldots (N - 1); \quad q_N(0) = 0, \quad q_N(L) = 1. \quad (a.3)$$

We wish to show that the path with $q_2(z) = \ldots q_{N-1}(z) = 0$, and $q_N(z) = q(z)$ as before, is the path of minimal action. To demonstrate this we need the potential generated by fluctuations at one loop order. We do so in two special cases where the analysis is elementary.

$N=3$: There are two independent fields, $q_2$ and $q_3$. The potential term is

$$S^{\mu}_{eu}(A^{cl}) = V_{tr} \frac{4\pi^2 T^4}{3} \int dz \, V_{tot}(q_2, q_3),$$

$$V_{tot}(q_2, q_3) = V(q_2) + V(q_2/2 + q_3) + V(-q_2/2 + q_3), \quad (a.4)$$

where

$$V(q) = [q]^2(1 - [q])^2, \quad [q] = |q|_{Mod 1}. \quad (a.5)$$

Because the $V(q)$ is a function of $[q]$, the absolute value modulo one, it is not quite the simple polynomial form it first appears to be. (This restriction could be ignored before, since the $Z(N)$ instanton only involves $q : 0 \rightarrow 1$.)

The effective action is the sum of (a.3) and (a.4). It is not difficult to see why for the boundary conditions of (a.3), the path with $q_2(z) = 0$ is of minimal action. Considering the problem as classical mechanics in two dimensions, from energy conservation the action for any solution to the equations of motion is proportional to

$$\int ds \, \sqrt{V_{tot}(q_2, q_3)}, \quad (a.6)$$

where $ds$ is the arc length in the space of $q_2$ and $q_3$. It can be shown that for any fixed value of $q_3$, the potential is minimized for $q_2 = 0$: $V_{tot}(q_2, q_3) \geq V_{tot}(0, q_3)$. Given the boundary conditions, the path chosen is clearly of minimal length. As both the arc length and the potential are bounded by the path with $q_2 = 0$, by (a.6), so is the action.

$N = \infty$: The large $N$ limit is taken by holding $\tilde{g}^2 \equiv g^2 N$ fixed as $N \rightarrow \infty$; we work in weak coupling, for small $\tilde{g}$. Then the interface tension is of order $N$, $\alpha \sim N/\tilde{g}$, while the $Z(N)$ instanton remains fat, with a width of order $1/(\tilde{g}T)$.  

24
Besides $q_N$, there are of order $N$ $q_i$'s which can contribute. Assume first that there are a finite fraction of the $q_i$'s for which $q_i(z) \neq 0$. From (a.3), since each field has a kinetic term proportional to $N$, if order $N$ fields contribute, the sum of the kinetic terms for all fields is of order $N^2$. Similarly, the potential term is also of order $N^2$. Because both the kinetic and potential terms are positive definite, any solution has positive action, equal to a pure number times $N^2$. This is $N$ times the action of the path chosen and so is not minimal.

Hence we can assume that at large $N$, only a finite number of the $q_i$'s contribute. For simplicity, assume that they are just $q_N$ and $q_{N-1}$. The generators for $t_N$, (2.3), and $t_{N-1}$, (2.5), simplify greatly at large $N$, reducing to just one diagonal element. (Essentially, the terms which enforce tracelessness can be ignored, as a correction in $1/N$.) In this case it is easy to work out the full potential term, which is proportional to

$$V_{tot}(q_{N-1}, q_N) = (N-2)V(q_N) + (N-2)V(q_{N-1}) + V(q_{N-1} - q_N)$$

$$\sim N (V(q_N) + V(q_{N-1})). \quad (a.7)$$

The last term is the leading term at large $N$; like the kinetic energy, it is of order $N$. But notice that at large $N$, in the potential the coupling between $q_N$ and $q_{N-1}$ has dropped out. Thus even though the potential for $q_{N-1}$ is nontrivial, because it is positive definite, the path of minimal action which satisfies $q_{N-1}(0) = q_{N-1}(L) = 0$ is just $q_{N-1}(z) = 0$.

This argument generalizes immediately to any finite number of $q_i$'s: the term of order $N$ in the potential is a sum over decoupled potentials, and the path of minimal action excites only one field, $q_N$.

Appendix B: Integrals in a constant $A_0$ field

In this appendix we catalog the integrals required for the computation of quantum actions in the constant background field of (2.2).

The trace is defined as

$$tr = V_T \sum_{n=-\infty}^{+\infty} \int \frac{d^{3-\epsilon}k}{(2\pi)^{3-\epsilon}}.$$

(b.1)

Remember that the system is of length $L_t$ in the $\vec{x}_t$ directions, and $L$ in the $z$ direction. At a temperature $T$, $\beta = 1/T$, and the momentum $k^0 = 2\pi n T$ for integral $n$. The
volume of spacetime $V = \beta L_t^2 L$. Dimensional continuation in $3 - \epsilon$ dimensions is used.

As discussed following (2.22), after using the ladder basis to reduce the color algebra, the propagators in the background field of (2.2) are given by their values in zero field, except for the replacement of $k^0 \rightarrow k^0_+ = 2\pi T(n + q)$. Following the example of (2.22)–(2.28), the integrals are most easily computed by zeta function techniques [15]. First, the spatial integrals over $d^3-k$ are done using the standard formulas of dimensional regularization. This leaves a sum over $n$, which is evaluated in terms of zeta functions. For instance, the simplest integral which arises in the expansion of the kinetic term at one loop order is

$$tr \frac{1}{((k_0^0)^2 + k^2)^2} = \frac{V}{16\pi^2} \left( 1 + \frac{\epsilon}{2} \right) \left( \psi(1/2) - \log(\pi T^2) \right) \sum_{n=-\infty}^{+\infty} \frac{1}{|n + q|^{1+\epsilon}}. \quad (b.2)$$

Using the definition of the zeta function, (2.23), and then expanding to order $\epsilon$,

$$\sum_{n=-\infty}^{+\infty} \frac{1}{|n + q|^{1+\epsilon}} = \zeta(1+\epsilon, q) + \zeta(1+\epsilon, 1-q) \sim 2 \frac{\epsilon}{\epsilon'} - (\psi(q) + \psi(1-q)), \quad (b.3)$$

with $\psi(q) = d(\log \Gamma(q))/dq$ the digamma function. Hence in all

$$tr \frac{1}{((k_0^0)^2 + k^2)^2} = \frac{V}{16\pi^2} \left( \frac{2}{\epsilon'} - (\psi(q) + \psi(1-q)) \right), \quad (b.4)$$

where

$$\frac{2}{\epsilon'} = 2 \frac{\epsilon}{\epsilon} + \psi(1/2) - \log(\pi T^2). \quad (b.5)$$

Similarly,

$$tr \frac{(k_0^0)^2}{((k_0^0)^2 + k^2)^3} = \frac{V}{64\pi^2} \left( \frac{2}{\epsilon'} - (\psi(q) + \psi(1-q)) + 2 \right), \quad (b.6)$$

$$tr \frac{(k_0^0)^4}{((k_0^0)^2 + k^2)^4} = \frac{V}{128\pi^2} \left( \frac{2}{\epsilon'} - (\psi(q) + \psi(1-q)) + \frac{8}{3} \right). \quad (b.7)$$

These integrals are all those required to compute the one loop corrections to the kinetic term in sec.’s IV and V. The factors of the renormalization group scale $\mu$ which enter into the running coupling constant, (4.11), arise by taking $g^2 \rightarrow g^2 \mu^\epsilon$ in the classical action, and expanding in $\epsilon$.

Notice that for the three integrals of (b.4), (b.6), and (b.7), the dependence on the background field $q$, through the digamma functions, always has the same coefficient
as the pole in $1/\epsilon$. This is because in all of these integrals, $q$ enters through the zeta functions $\zeta(1 + \epsilon, q) + \zeta(1 + \epsilon, 1 - q)$. Expanding about small $\epsilon$, (b.3), generates a common factor of $1/\epsilon$ and digamma functions in each case, so that up to their overall normalization, these integrals only differ by a constant.

Other integrals required for the potential at two loop order, and the renormalization of the Wilson line at one loop order, include (2.22), (2.26), and the following. These are all finite integrals, so it is safe to set $\epsilon = 0$.

\[
tr \frac{1}{(k^0_+)^2 + k^2} = \frac{VT^2}{2} \left( q^2 - q + \frac{1}{6} \right) , \quad (b.8)
\]
\[
tr \frac{k^0_+}{((k^0_+)^2 + k^2)} = \frac{VT}{2\pi} \left( q - \frac{1}{2} \right) , \quad (b.9)
\]
\[
tr \frac{k^0_+}{((k^0_+)^2 + k^2)^2} = -\frac{VT}{4\pi} \left( q - \frac{1}{2} \right) . \quad (b.10)
\]

The last two integrals, (b.9) and (b.10), are singular for $q = 0$ or 1, and as written are defined over $0 < q < 1$. Properly defined, they are discontinuous and vanish for $q = 0$ or 1.
References

[1] G. ’t Hooft, Nucl. Phys. B138 (1978) 1.

[2] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B210 (1982) 423.

[3] K. Kajantie, L. Kärkkäinen, Phys. Lett. B214 (1988) 595; K. Kajantie, L. Kärkkäinen, K. Rummukainen, Nucl. Phys. B333 (1990) 100; K. Kajantie, L. Kärkkäinen, K. Rummukainen, Nucl. Phys. B357 (1991) 693; S. Huang, J. Potvin, C. Rebbi and S. Sanielevici, Phys. Rev. D 42 (1990) 2864, (E) Phys. Rev. D 43 (1991) 2056; J. Potvin and C. Rebbi, Boston University preprint 91-0069 (Jan. 1991); R. Brower, S. Huang, J. Potvin, C. Rebbi, and J. Ross, Boston University preprint 91-22 (Nov. 1991); R. Brower, S. Huang, J. Potvin, and C. Rebbi, Boston University preprint 92-3 (Jan. 1992).

[4] T. Bhattacharya, A. Gocksch, C. P. Korthals Altes and R. D. Pisarski, Phys. Rev. Lett. 66 (1991) 998.

[5] E. Brézin and J. Zinn-Justin, Nucl. Phys. B257 [FS14] (1985) 867.

[6] G. Münster, Nucl. Phys. B324 (1989) 630; Nucl. Phys. B340 (1990) 559.

[7] R. Fukuda and E. Kyriakopoulos, Nucl. Phys. B85 (19354) 354; L. O’Raifeartaigh, A. Wipf, and H. Yoneyama, Nucl. Phys. B271 (1986) 653; C. Wetterich, Nucl. Phys. B352 (1991) 529.

[8] D. Gross, R.D. Pisarski, L.G. Yaffe, Rev. Mod. Phys. 53 (1981) 43; N. Weiss, Phys. Rev. D 24 (1981) 75; Phys. Rev. D 25 (1982) 2667;

[9] R. Anishetty, Jour. of Phys. G10 (1984) 439; K. J. Dahlem, Z. Phys. C 29 (1985) 553; S. Nadkarni, Phys. Rev. Lett. 60 (1988) 491;

[10] J. Polonyi, Nucl. Phys. A461 (1987) 279; J. Polonyi and S. Vazquez; Phys. Lett. B240 (1990) 183; M. Oleszczuk and J. Polonyi, MIT preprint MIT-CTP-1984 (June, 1991).

[11] V. M. Belyaev and V. L. Eletsky, Z. Phys. C 45 (1990) 355.

[12] K. Enqvist and K. Kajantie, Z. Phys. C 47 (1990) 291.

[13] V. M. Belyaev, Phys. Lett. B241 (1990) 91.
[14] V. M. Belyaev, Phys. Lett. B254 (1991) 153.

[15] A. Actor, Nucl. Phys. B265 [FS15] (1986) 689; Fortschr. Phys. 35 (1987) 793; H. A. Weldon, Nucl. Phys. B270 (1986) 79; R. V. Konoplich, Theor. Math. Phys. 78 (1989) 315; F. T. Brandt, J. Frenkel, and J. C. Taylor, Phys. Rev. D 44 (1991) 1801.

[16] G. ’t Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189; L. F. Abott, Nucl. Phys. B185 (1981) 189.

[17] J. Iliopoulos, C. Itzykson, and A. Martin, Rev. Mod. Phys. 47 (1975) 165.