Effective embedding of residually hyperbolic groups into direct products of extensions of centralizers

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Abstract

For any torsion-free hyperbolic group Γ and any group G that is fully residually Γ, we construct algorithmically a finite collection of homomorphisms from G to groups obtained from Γ by extensions of centralizers, at least one of which is injective. When G is residually Γ, this gives an effective embedding of G into a direct product of such groups. We also give an algorithmic construction of a diagram encoding the set of homomorphisms from a given finitely presented group to Γ.

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1 Introduction

A group $G$ is discriminated by another group $\Gamma$ (or is fully residually $\Gamma$) if for every finite set of non-trivial elements $\{g_1, \ldots, g_n\}$ of $G$ there exists a homomorphism $\phi : G \to \Gamma$ such that $\phi(g_i) \neq 1$ for $i = 1, \ldots, n$. If this condition is only required to hold for $n = 1$ we say that $G$ is separated by $\Gamma$ (or is residually $\Gamma$).

The class of fully residually free groups (when $\Gamma$ is a non-abelian free group) has been extensively studied in the last 15 years, particularly in connection with Tarski’s problems on the elementary theory of a free group ([KM06], [Sel06]), and now have a well-developed theory. Further, many algorithmic problems related to these groups have been solved in recent years.

Generalizing to the case when $\Gamma$ is a hyperbolic group, much of the theory has been developed and is similar, but many algorithmic questions remain open. This paper is motivated by the following problem.

**Problem.** Is the elementary theory $\text{Th}(G)$ of a (torsion-free) hyperbolic group $G$ decidable?

Notice, that it was proved in [KM05a] that the universal theory of a finitely generated fully residually free group is decidable and in [Dah09] that the universal theory of a hyperbolic group is decidable. We will give another proof of this result (for torsion-free hyperbolic groups) in Corollary 3.20.

Fix throughout this paper a torsion-free hyperbolic group $\Gamma = \langle A \mid R \rangle$. One of the characterizations of finitely generated groups $G$ discriminated by $\Gamma$ is that they embed into the Lyndon completion $\Gamma\mathbb{Z}[l]$ of $\Gamma$, or equivalently, into a group obtained from $\Gamma$ by a series of extensions of centralizers [KM09]. If $G$ is separated by $\Gamma$, there is an embedding into a finite direct product of such groups. The case when $\Gamma$ is a free group was proved by Kharlampovich and Myasnikov, who also provided an algorithm to construct the embedding (in both the discriminated and separated cases) [KM98b].

For the general case, however, the embedding described in [KM09] is not effective. We provide an algorithm to construct the embedding of any residually $\Gamma$ group $G$ into a direct product of groups obtained from $\Gamma$ by extensions of centralizers. When $G$ is fully residually $\Gamma$, we effectively construct a finite collection of homomorphisms from $G$ into groups obtained from $\Gamma$ by extensions of centralizers, at least one of which is an embedding (Theorem 3.17).

The first step of our approach is to use canonical representatives for certain elements of $\Gamma$, developed by Rips and Sela in their study of equations over hyperbolic groups [RS05], to reduce part of the problem to the free group case. As a corollary of this reduction, we are able to effectively construct a finite diagram that describes the complete set $\text{Hom}(G, \Gamma)$ of homomorphisms from an arbitrary finitely presented group $G$ to $\Gamma$ (Theorem 2.3).
1.1 Algebraic geometry over groups

We will use throughout the language of algebraic geometry over groups [BMR99].

We recall here some important notions and establish notation. Let \( \Gamma \) be a finite group generated by \( A \) (‘constants’) and \( X \) a finite set (‘variables’) and set \( \Gamma[X] = \Gamma \ast F(X) \). Let \( \text{Hom}_\Gamma(\Gamma[X], \Gamma) \) denote the set of homomorphisms from \( \Gamma[X] \) to \( \Gamma \) that are identical on \( \Gamma \) (‘\( \Gamma \)-homomorphisms’).

To each element \( s \) of \( \Gamma[X] \) we associate a formal expression ‘\( s = 1 \)’ called an equation over \( \Gamma \). A solution to this equation is a homomorphism \( \phi \in \text{Hom}_\Gamma(\Gamma[X], \Gamma) \) such that \( s\phi = 1 \). A subset \( S \) of \( \Gamma[X] \) corresponds to the system of equations ‘\( S = 1 \)’ which we also denote ‘\( S(X, A) = 1 \)’. Define \( \Gamma_S = \Gamma[X]/\text{ncl}(S) \) and note that every solution to the system \( S \) factors through \( \Gamma_S \). If \( \Gamma \) has the presentation \( \langle A \mid R \rangle \), then \( \Gamma_S \simeq \langle X, A \mid S, R \rangle \).

Define the radical \( R_\Gamma(S) \) of \( S \) over \( \Gamma \) by
\[
R_\Gamma(S) = \{ t \in \Gamma[X] \mid \forall \phi \in \text{Hom}_\Gamma(\Gamma[X], \Gamma) \forall s \in S \ (s^\phi = 1 \implies t^\phi = 1) \}
\]
and define the coordinate group of \( S \) over \( \Gamma \) by
\[
\Gamma_{R_\Gamma(S)} = \Gamma[X]/R_\Gamma(S).
\]
Every solution to \( S \) factors through \( \Gamma_{R_\Gamma(S)} \).

When \( S \) is a subset of \( F(X) \) we say that the system is coefficient-free and we may consider \( F(X) \) in place of \( \Gamma \ast F(X) \) and the ordinary set of homomorphisms \( \text{Hom}(F(X), \Gamma) \) in place of \( \text{Hom}_\Gamma(\Gamma[X], \Gamma) \). In particular, for any group \( G \) presented by \( \langle Z \mid S \rangle \) we may consider \( S \) as a system of equations in variables \( Z \). In the general case, when \( S \subset \Gamma[X] \), we may consider \( S \) as a system of equations over any group \( G \) that has \( \Gamma \) as a fixed subgroup (i.e. any \( G \) in the ‘category of \( \Gamma \)-groups’).

1.2 Notation

Fix \( \Gamma = \langle A \mid R \rangle \) a finitely presented torsion-free hyperbolic group, \( F \) the free group on \( A \), and \( \pi : F \to \Gamma \) the canonical epimorphism.

The map \( \pi \) induces an epimorphism \( F[X] \to \Gamma[X] \), also denoted \( \pi \), by fixing each \( x \in X \). For a system of equations \( S \subset F[X] \), we study the corresponding system \( S^\pi \subset \Gamma[X] \) which we may denote again by \( S \), depending on context. The radical of \( S^\pi \) over \( \Gamma \) may be denoted \( R_\Gamma(S^\pi) \), \( R_\Gamma(S) \), or \( R(S^\pi) \). Likewise, the coordinate group \( \Gamma_{R_\Gamma(S^\pi)} \) may be denoted simply \( \Gamma_{R(S)} \).

Notice that the relators \( R \) of \( \Gamma \) are in the radical \( R_\Gamma(S) \) for every system of equations \( S \), hence
\[
F_{R_\Gamma(S)} = \Gamma_{R(S)}.
\]
In denoting a coordinate group \( \Gamma_{R(S)} = \Gamma[X]/R(S) \) we always assume that \( X \) is precisely the set of variables appearing in \( S \).
1.3 Toral relatively hyperbolic groups

A group $G$ that is hyperbolic relative to a collection $\{H_1, \ldots, H_k\}$ of subgroups is called toral if $H_1, \ldots, H_k$ are all abelian and $G$ is torsion-free. Many algorithmic problems in (toral) relatively hyperbolic groups are decidable, and in particular we take note of the following for later use.

Lemma 1.1. In every toral relatively hyperbolic group $G$, the following hold.

(1) The conjugacy problem in $G$, and hence the word problem, is decidable.

(2) If $G$ is non-abelian then we may effectively construct two non-commuting elements of $G$.

(3) If $g \in G$ is a hyperbolic element (i.e. not conjugate to any element of any $H_i$), then the centralizer $C(g)$ of $g$ is an infinite cyclic group. Further, a generator for $C(g)$ can be effectively constructed.

Proof. The word problem was solved in [Far98] and the conjugacy problem in [Bum04]. For the second statement, we need only enumerate pairs $(g, h) \in G \times G$ until we find a pair with $[g, h] \neq 1$.

For the third statement, let $G = \langle A \rangle$ and let $g \in G$ be a hyperbolic element. Theorem 4.3 of [Osi06a] shows that the subgroup $E(g) = \{ h \in G : \exists n \in \mathbb{N} : h^{-1}g^n h = g^n \}$ has a cyclic subgroup of finite index. Since $G$ is torsion-free, $E(g)$ must be infinite cyclic (see for example the proof of Proposition 12 of [MR96]). Clearly $C(g) \leq E(g)$, hence $C(g)$ is infinite cyclic.

To construct a generator for $C(g)$, consider the following results of D. Osin (see the proof of Theorem 5.17 and Lemma 5.16 in [Osi06b]):

(i) there exists a constant $N$, which depends on $G$ and the word length $|g|$ and can be computed, such that if $g = f^n$ for some $f \in G$ and with $n$ positive then $n \leq N$;

(ii) there is a computable function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ such that if $f$ is an element of $G$ with $f^n = g$ for some positive $n$, then $f$ is conjugate to some element $f_0$ satisfying $|f_0| \leq \beta(|g|)$.

We proceed as follows. Let $\mathcal{F}$ be the set of all $f \in G$ such that $|f| \leq \beta(|g|)$ and $h^{-1}f^n h = g$ for some $h \in G$ and $1 \leq n \leq N$. It is finite, non-empty (since $g$ is an element), and can be computed (since conjugacy is decidable). Let $f$ be an element of $\mathcal{F}$ such that the exponent $n$ is maximum amongst elements of $\mathcal{F}$ and find an element $h \in G$ such that $h^{-1}f^n h = g$ (we may find $h$ by enumeration).

We claim that if $\overline{g}$ is a generator of $C(g)$ then either $h^{-1}f h = \overline{g}$ or $h^{-1}f h = \overline{g}^{-1}$. Indeed, $h^{-1}f h \in C(g)$ since it commutes with $g = (h^{-1}f h)^n$, hence $h^{-1}f h = \overline{g}^k$ for some $k$ and so

$$g = (h^{-1}f h)^n = \overline{g}^{kn}.$$
Suppose $k > 0$. Since $g^n = g$, it implies that $g$ is conjugate to some element $g_0$ with $|g_0| \leq \beta(|g|)$. Then $g_0^n$ is conjugate to $g$, so by $k$, $kn \leq N$ hence $g_0 \in F$. By maximality of the exponent in the choice of $f$, $k$ must be 1 and $h^{-1}fh = g$. If $k < 0$, a similar argument shows that $h^{-1}fh = g^{-1}$.

2 Effective description of homomorphisms to $\Gamma$

In this section, we describe an algorithm that takes as input a system of equations $S$ over $\Gamma$ and produces a tree diagram $T$ that encodes the set $\text{Hom}_\Gamma(\Gamma_{R(S)}, \Gamma)$. When $S$ is a system without coefficients, we interpret $S$ as relators for a finitely presented group $G = \langle Z \mid S \rangle$ and the diagram $T$ encodes instead the set $\text{Hom}(G, \Gamma)$.

Though the diagram $T$ will give a finite description of $\text{Hom}_\Gamma(\Gamma_{R(S)}, \Gamma)$, it is not a ‘Makanin-Razborov diagram’ in the sense of [Gro05]. We discuss this further at the end of this section.

There are two ingredients in this construction: first, the reduction of the system $S$ over $\Gamma$ to finitely many systems of equations over free groups, and second, the construction of Hom-diagrams (Makanin-Razborov diagrams) for systems of equations over free groups.

Notation 2.1. Let $\pi$ denote the canonical epimorphism $F(A) \to \Gamma_{R(S)}$. For a homomorphism $\phi : F(Z, A) \to K$ we define $\overline{\phi} : \Gamma_{R(S)} \to K$ by

$$\overline{\phi}(w) = \phi(w),$$

where any preimage $w$ of $\overline{w}$ may be used. We will always ensure that $\overline{\phi}$ is a well-defined homomorphism.

2.1 Reduction to systems of equations over free groups

In [RS95], the problem of deciding whether or not a system of equations $S$ over a torsion-free hyperbolic group $\Gamma$ has a solution was solved by constructing canonical representatives for certain elements of $\Gamma$. This construction reduced the problem to deciding the existence of solutions in finitely many systems of equations over free groups, which had been previously solved. The reduction may also be used to find all solutions to $S$ over $\Gamma$, as described below.

Lemma 2.2. Let $\Gamma = \langle A \mid R \rangle$ be a torsion-free $\delta$-hyperbolic group and $\pi : F(A) \to \Gamma$ the canonical epimorphism. There is an algorithm that, given a system $S(Z, A) = 1$ of equations over $\Gamma$, produces finitely many systems of equations

$$S_1(X_1, A) = 1, \ldots, S_n(X_n, A) = 1 \quad (1)$$

over $F$ and homomorphisms $\rho_i : F(Z, A) \to F_{R(S_i)}$ for $i = 1, \ldots, n$ such that

(i) for every $F$-homomorphism $\phi : F_{R(S_i)} \to F$, the map $\overline{\rho_i\phi\pi} : \Gamma_{R(S_i)} \to \Gamma$ is a $\Gamma$-homomorphism, and
(ii) for every $\Gamma$-homomorphism $\psi : \Gamma_{R(S)} \to \Gamma$ there is an integer $i$ and an $F$-homomorphism $\phi : F_{R(S_i)} \to F(A)$ such that $\rho_i \phi \pi = \psi$.

Further, if $S(Z) = 1$ is a system without coefficients, the above holds with $G = \langle Z \mid S \rangle$ in place of $\Gamma_{R(S)}$ and ‘homomorphism’ in place of $\Gamma$-homomorphism’.

Proof. The result is an easy corollary of Theorem 4.5 of [RS95], but we will provide a few details.

We may assume that the system $S(Z, A)$, in variables $z_1, \ldots, z_l$, consists of $m$ constant equations and $q - m$ triangular equations, i.e.

$$S(Z, A) = \left\{ \begin{array}{l} z_{\sigma(j,1)} z_{\sigma(j,2)} z_{\sigma(j,3)} = 1 \\ z_s = a_s \\ s = l - m + 1, \ldots, l \end{array} \right. \quad j = 1, \ldots, q - m$$

where $\sigma(j, k) \in \{1, \ldots, l\}$ and $a_i \in \Gamma$. An algorithm is described in [RS95] which, for every $m \in \mathbb{N}$, assigns to each element $g \in \Gamma$ a word $\theta_m(g) \in F$ satisfying

$$\theta_m(g) = g \text{ in } \Gamma$$

called its canonical representative. The representatives $\theta_m(g)$ are not ‘global canonical representatives’, but do satisfy useful properties for certain $m$ and certain finite subsets of $\Gamma$, as follows.

Let $L = q \cdot 2^{5050(\delta + 1)^5(2|A|)^{23}}$. Suppose $\psi : F(Z, A) \to \Gamma$ is a solution of $S(Z, A)$ and denote

$$\psi(z_{\sigma(j, k)}) = g_{\sigma(j, k)}.$$  

Then there exist $h_k^{(j)}, c_k^{(j)} \in F(A)$ (for $j = 1, \ldots, q - m$ and $k = 1, 2, 3$) such that

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1The constant of hyperbolicity $\delta$ may be computed from a presentation of $\Gamma$ using the results of [EH01].
(i) each \( c_k^{(j)} \) has length less than \( 2L \) (as a word in \( F \)),

(ii) \( c_1^{(j)} c_2^{(j)} c_3^{(j)} = 1 \) in \( \Gamma \),

(iii) there exists \( m \leq L \) such that the canonical representatives satisfy the following equations in \( F \):

\[
\theta_m(g_{\sigma(j,1)}) = h_1^{(j)} c_1^{(j)} \left( h_2^{(j)} \right)^{-1} \tag{2}
\]

\[
\theta_m(g_{\sigma(j,2)}) = h_2^{(j)} c_2^{(j)} \left( h_3^{(j)} \right)^{-1} \tag{3}
\]

\[
\theta_m(g_{\sigma(j,3)}) = h_3^{(j)} c_3^{(j)} \left( h_1^{(j)} \right)^{-1}. \tag{4}
\]

In particular, when \( \sigma(j, k) = \sigma(j', k') \) (which corresponds to two occurrences in \( S \) of the variable \( z_{\sigma(j,k)} \)) we have

\[
h_1^{(j')} c_1^{(j')} \left( h_{k+1}^{(j')} \right)^{-1} = h_1^{(j')} c_1^{(j')} \left( h_{k'+1}^{(j')} \right)^{-1}. \tag{5}
\]

Consequently, we construct the systems \( S(X_i, A) \) as follows. For every positive integer \( m \leq L \) and every choice of \( 3(q - m) \) elements \( c_1^{(j)} , c_2^{(j)} , c_3^{(j)} \in F \) \((j = 1, \ldots, q - m)\) satisfying (i) and (ii)\(^2\) we build a system \( S(X_i, A) \) consisting of the equations

\[
x_k^{(j)} c_k^{(j)} \left( x_{k+1}^{(j)} \right)^{-1} = x_k^{(j')} c_k^{(j')} \left( x_{k+1}^{(j')} \right)^{-1} \tag{6}
\]

\[
x_k^{(j)} c_k^{(j)} \left( x_{k+1}^{(j)} \right)^{-1} = \theta_m(a_s) \tag{7}
\]

where an equation of type \((6)\) is included whenever \( \sigma(j, k) = \sigma(j', k') \) and an equation of type \((7)\) is included whenever \( \sigma(j, k) = s \in \{l - m + 1, \ldots, l\} \). To define \( \rho_i \), set

\[
\rho_i(z_s) = \begin{cases} 
  x_k^{(j)} c_k^{(j)} \left( x_{k+1}^{(j)} \right)^{-1}, & 1 \leq s \leq l - m \text{ and } s = \sigma(j, k) \\
  \theta_m(a_s), & l - m + 1 \leq s \leq l 
\end{cases}
\]

where for \( 1 \leq s \leq l - m \) any \( j, k \) with \( \sigma(j, k) = s \) may be used.

If \( \psi : F(Z) \to \Gamma \) is any solution to \( S(Z, A) = 1 \), there is a system \( S(X_i, A) \) such that \( \theta_m(g_{\sigma(j,k)}) \) satisfy \((2) - (4)) \. Then the required solution \( \phi \) is given by

\[
\phi(x_j^{(k)}) = h_j^{(k)}. \]

\(^2\) The bound of \( L \) here, and below, is extremely loose. Somewhat tighter, and more intuitive, bounds are given in [RS95].

\(^3\) The word problem in hyperbolic groups is decidable.
Indeed, (iii) implies that \( \phi \) is a solution to \( S(X_i, A) = 1 \). For \( s = \sigma(j, k) \in \{1, \ldots, l - m\} \),
\[
\sigma_{\pi}^{\rho} = h_k^{(j)} c_k^{(j)} \left( h_{k+1}^{(j)} \right)^{-1} = \theta_m(\sigma(j,k))
\]
and similarly for \( s \in \{l - m + 1, \ldots, l\} \), hence \( \psi = \rho_i \phi \pi \).

Conversely, for any solution \( \phi(x_j^{(k)}) = h_j^{(k)} \) of \( S(X_i) = 1 \) one sees that by (i), hence \( \rho_i \phi \pi \) induces a homomorphism.

2.2 Encoding solutions with the tree \( T \)

An algorithm is described in §5.6 of [KM05b] which constructs, for a given system of equations \( S(X, A) \) over the free group \( F \), a diagram encoding the set of solutions of \( S \). The diagram consists of a directed finite rooted tree \( T \) with the following properties. Let \( G = F_{R(S)} \).

(i) Each vertex \( v \) of \( T \) is labelled by a pair \((G_v, Q_v)\) where \( G_v \) is an \( F \)-quotient of \( G \) and \( Q_v \) a finitely generated subgroup of \( \text{Aut}_F(G_v) \). The root \( v_0 \) is labelled by \((G, 1)\) and every leaf is labelled by \((F(Y) * F, 1)\) where \( Y \) is some finite set (called free variables). Each \( G_v \), except possibly \( G_{v_0} \), is fully residually \( F \).

(ii) Every (directed) edge \( v \to v' \) is labelled by a proper surjective \( F \)-homomorphism \( \pi(v, v') : G_v \to G_{v'} \).

(iii) For every \( \phi \in \text{Hom}_F(G, F) \) there is a path \( p = v_0 v_1 \ldots v_k \) where \( v_k \) is a leaf labelled by \((F(Y) * F, 1)\), elements \( \sigma_i \in Q_{v_i} \), and a \( F \)-homomorphism \( \phi_0 : F(Y) * F \to F \) such that
\[
\phi = \pi(v_0, v_1) \sigma_1 \pi(v_1, v_2) \sigma_2 \cdots \pi(v_{k-2}, v_{k-1}) \sigma_{k-1} \pi(v_{k-1}, v_k) \phi_0.
\]

The algorithm gives for each \( G_v \) a finite presentation \( \langle A_v | R_v \rangle \), and for each \( Q_v \) a finite list of generators in the form of functions \( A_v \to (A_v \cup A_v^{-1})^* \). Note that the choices for \( \phi_0 \) are exactly parametrized by the set of functions from \( Y \) to \( F \).

Let \( S(Z, A) = 1 \) be a system of equations over \( \Gamma \). We will construct a diagram \( T \) to encode the set of solutions of \( S(Z, A) = 1 \), as follows.

Apply Lemma \ref{Lemma: Encoding} to construct the systems \( S_1(X_1, A), \ldots, S_n(X_n, A) \). Create a root vertex \( v_0 \) labelled by \( F(Z, A) \). For each of the systems \( S_i(X_i, A) \), let \( T_i \) be the tree constructed above. Build an edge from \( v_0 \) to the root of \( T_i \) labelled by the homomorphism \( \rho_i \pi S_i \), where \( \pi S_i : F(X_i, A) \to F_{R(S_i)} \) is the canonical projection. For each leaf \( v \) of \( T_i \), labelled by \( F(Y) * F \), build a new vertex \( w \) labelled by \( F(Y) * \Gamma \) and an edge \( v \to w \) labelled by the homomorphism \( \pi_Y : F(Y) * F \to F(Y) * \Gamma \) which is induced from \( \pi : F \to \Gamma \) by acting as the identity on \( F(Y) \).
Define a branch $b$ of $T$ to be a path $b = v_0v_1 \ldots v_k$ from the root $v_0$ to a leaf $v_k$. Let $v_j$ be labelled by $F_{R(S_j)}$ and $v_k$ by $F(Y) \ast \Gamma$. We associate to $b$ the set $\Phi_b$ consisting of all homomorphisms $F(Z) \rightarrow \Gamma$ of the form 

$$\rho_1 \pi_{S_1} \pi_2 \pi(v_1, v_2) \sigma_2 \ldots \pi(v_{k-2}, v_{k-1}) \sigma_{k-1} \pi(v_{k-1}, v_k) \pi_2 \phi$$

(9)

where $\sigma_j \in Q_{v_j}$ and $\phi \in \text{Hom}(F(Y) \ast \Gamma, \Gamma)$. Since $\text{Hom}(F(Y) \ast \Gamma, \Gamma)$ is in bijective correspondence with the set of functions $\Gamma^1$, all elements of $\Phi_b$ can be effectively constructed. We have obtained the following theorem.

**Theorem 2.3.** There is an algorithm that, given a system $S(Z, A) = 1$ of equations over $\Gamma$, produces a diagram encoding its set of solutions. Specifically,

$$\text{Hom}(\Gamma_{R(S)}, \Gamma) = \{ \phi \mid \phi \in \Phi_b, \ b \text{ is a branch of } T \}$$

where $T$ is the diagram described above. When the system is coefficient-free, then the diagram encodes $\text{Hom}(G, \Gamma)$ where $G = \langle Z \mid S \rangle$.

Note that in the diagram $T$, the groups $G_v$ appearing at vertices are not quotients of coordinate group $\Gamma_{R(S)}$ and that to obtain a homomorphism from $\Gamma_{R(S)}$ to $\Gamma$ one must compose maps along a complete path ending at a leaf of $T$. In [Gro05] it is shown that for any toral relatively hyperbolic group there exist Hom-diagrams with the property that every group $G_v$ is a quotient of $\Gamma_{R(S)}$ and that every edge map $\pi(v, v')$ is a proper surjective homomorphism. However, we are not aware of an algorithm for constructing these diagrams.

## 3 Embedding into extensions of centralizers

The proof given in [KM09] that any $\Gamma$-limit group $G$ embeds into extensions of centralizers of $\Gamma$ involves two steps: first, $G$ is shown to embed into the coordinate groups of an NTQ system (see §3.1), and second, such groups are shown to embed into extensions of centralizers of $\Gamma$. The first step of this construction relies on the following theorem (Theorem 1.1 of [Gro05]): there exists a finite collection $\{ L_i \}$ of proper quotients of $G$ such that any homomorphism from $G$ to $\Gamma$ factors through one of the $L_i$ (up to a certain equivalence). Algorithmic construction of the set $\{ L_i \}$ is not given, nor are we aware of an algorithm for constructing it.

Instead, we use canonical representatives and results from [KM98b] regarding equations over free groups to construct a collection of NTQ groups $F_{R(S_i)}$ and maps from the generating set of $G$ to each $F_{R(S_i)}$. Regarding each system of equations $S_i$ over $\Gamma$ rather than $F$, at least one of these maps induces an embedding $G \hookrightarrow \Gamma_{R(S_i)}$. The coordinate groups $\Gamma_{R(S_i)}$ can be embedded into extensions of centralizers of $\Gamma$ using the techniques from [KM09].

### 3.1 Quadratic equations and NTQ systems

An equation $s \in G[X]$ over a group $G$ is said to be (strictly) quadratic if every variable appearing in $s$ appears at most (exactly) twice, and a system of
equations $S(X) \subset G[X]$ is (strictly) quadratic if every variable that appears in $S$ appears at most (exactly) twice. Here we count both $x$ and $x^{-1}$ as an appearance of $x$. Constructing NTQ systems involves considerable analysis of quadratic equations, and is aided by considering certain standard forms.

**Definition 3.1.** A standard quadratic equation over a group $G$ is an equation of one of the following forms, where $c_i$ and $d$ are all nontrivial elements of $G$:

\[\prod_{i=1}^{n} [x_i, y_i] = 1, \quad n \geq 1;\] (10)

\[\prod_{i=1}^{n} [x_i, y_i] \prod_{i=1}^{m} z_i^{-1} c_i z d = 1, \quad n, m \geq 0, n + m \geq 1;\] (11)

\[\prod_{i=1}^{n} x_i^2 = 1, \quad n \geq 1;\] (12)

\[\prod_{i=1}^{n} x_i^2 \prod_{i=1}^{m} z_i^{-1} c_i z d = 1, \quad n, m \geq 0, n + m \geq 1.\] (13)

The left-hand sides of the above equations are called the standard quadratic words.

The following result allows us to assume that quadratic equations always appear in standard form.

**Lemma 3.2.** Let $s(X) \in G[X]$ be a strictly quadratic word over a group $G$. Then there is a $G$-automorphism $\phi$ such that $s^\phi$ is a standard quadratic word over $G$.

**Proof.** Follows easily from §I.7 of [LS77].

To each quadratic equation we associate a punctured surface. To (10) we associate the orientable surface of genus $n$ and zero punctures, to (11) the orientable surface of genus $n$ with $m + 1$ punctures, to (12) the non-orientable surface of genus $n$, and to (13) the non-orientable surface of genus $n$ with $m + 1$ punctures. For a standard quadratic equation $S$, denote by $\chi(S)$ the Euler characteristic of the corresponding surface.

Quadratic words of the form $[x, y]$, $x^2$, and $z^{-1} cz$ where $c \in G$, are called atomic quadratic words or simply atoms. An atom $[x, y]$ contributes $-2$ to the Euler characteristic of $S$ while $x^2$ and $z^{-1} cz$ (as well as $d$) each contribute $-1$. A standard quadratic equation $S = 1$ over $G$ has the form

\[r_1 r_2 \ldots r_k d = 1,\]

where $r_i$ are atoms and $d \in G$. We classify solutions to quadratic equations based on the extent to which the images of the atoms commute, as follows.

**Definition 3.3.** Let $S = 1$ be a standard quadratic equation written in the atomic form $r_1 r_2 \ldots r_k d = 1$ with $k \geq 2$. A solution $\phi : G_{R(S)} \rightarrow G$ of $S = 1$ is called
(i) degenerate, if \( r_i^\phi = 1 \) for some \( i \), and non-degenerate otherwise;

(ii) commutative, if \( [r_i^\phi, r_{i+1}^\phi] = 1 \) for all \( i = 1, \ldots, k-1 \), and non-commutative otherwise;

(iii) in general position, if \( [r_i^\phi, r_{i+1}^\phi] \neq 1 \) for all \( i = 1, \ldots, k-1 \).

When the group \( G \) is commutation transitive, a commutative solution satisfies \( [r_i^\phi, r_j^\phi] = 1 \) for all \( i,j \). We will only be interested in the case when \( G \) is toral relatively hyperbolic, hence commutation transitive\(^4\). In this case, solutions also have the following important property.

**Lemma 3.4.** Let \( S \in G[X] \) be a standard quadratic equation over a toral relatively hyperbolic group \( G \) such that \( S \) has at least two atoms and such that \( S = 1 \) has a solution in \( G \). Then either

1. \( S \) has a solution in general position, or
2. every solution of \( S \) is commutative.

Further, there is an algorithm that distinguishes the cases.

**Proof.** The dichotomy is true for all CSA groups, by Proposition 3 of [KM98a]. For the algorithm, let \( S \) have the atomic form \( r_1 r_2 \ldots r_k d \) with variables \( x_1, \ldots, x_n \). Consider the sentences

\[ S_i : \exists x_1 \ldots \exists x_n (S = 1) \land ([r_i, r_{i+1}] \neq 1) \]

for \( i = 1, \ldots, k-1 \). Then all solutions of \( S = 1 \) are commutative if and only if none of the sentences \( S_i \) is true in \( G \). The existential theory of toral relatively hyperbolic groups is decidable ([Dah09]), hence we can decide whether or not each \( S_i \) is true in \( G \).

Now we define NTQ systems. Let \( G \) be a group generated by \( A \) and let \( S(X, A) \) be a system of equations. Suppose \( S \) can be partitioned into subsystems

\[
S_1(X_1, X_2, \ldots, X_n, A) = 1, \\
S_2(X_2, \ldots, X_n, A) = 1, \\
\vdots \\
S_n(X_n, A) = 1
\]

where \( \{X_1, X_2, \ldots, X_n\} \) is a partition of \( X \). Define groups \( G_i \) for \( i = 1, \ldots, n+1 \) by

\[
G_{n+1} = G, \\
G_i = G_R(S_i, \ldots, S_n).
\]

\(^4\)Toral relatively hyperbolic groups are CSA, hence commutation transitive. See [KM92] or [Gro09].
We interpret $S_i$ as a subset of $G_{i-1} \ast F(X_i)$, i.e. letters from $X_i$ are considered variables and letters from $X_{i+1} \cup \ldots \cup X_n \cup A$ are considered as constants from $G_i$.

A system $S(X, A) = 1$ is called triangular quasi-quadratic (TQ) if it can be partitioned as above such that for each $i$ one of the following holds:

(I) $S_i$ is quadratic in variables $X_i$;

(II) $S_i = \{ [x, y] = 1, \ [x, u] = 1 \mid x, y \in X_i, \ u \in U_i \}$ where $U_i$ is a finite subset of $G_{i+1}$ such that $(U_i) = C_{G_{i+1}}(g)$ for some $g \in G_{i+1}$;

(III) $S_i = \{ [x, y] = 1 \mid x, y \in X_i \}$;

(IV) $S_i$ is empty.

The system is called non-degenerate triangular quasi-quadratic (NTQ) if for every $i$ the system $S_i(X_i, \ldots, X_n, A)$ has a solution in the coordinate group $G_{R(S_i, \ldots, S_n)}$.

**Definition 3.5.** A group $H$ is called a $G$-NTQ group if there is a NTQ system $S$ over $G$ such that $H \cong G_{R(S)}$.

For any quadratic system $S$ over $G$ one can, by eliminating linear variables, find a strictly quadratic system $S'$ over $G$ such that every variable occurs in exactly one equation and $G_S \cong G_{S'}$. Consequently, if $H$ is an NTQ group with $H \cong G_{R(S)}$ then we may assume that every system $S_i$ of $S$ that has the form consists of a single quadratic equation in standard form.

In order to study NTQ groups by induction on the height $n$ of the NTQ system, we will need the following lemma.

**Lemma 3.6.** Let $S(X, A)$ and $T(Y, A)$ be systems of equations over a group $G$ with $X \cap Y = \emptyset$ and let $G_1 = G[X]/R_G(S)$. Then

$$G_{R(S \cup T)} \cong G_1[Y]/R_{G_1}(T).$$

**Proof.** Let $X = \{ x_1, \ldots, x_n \}$, $Y = \{ y_1, \ldots, y_m \}$, $u = u(x_1, \ldots, x_n, y_1, \ldots, y_m) \in G[X \cup Y]$. We will show that the natural map, which sends $u$ to the element represented by $u$ in $G_1[Y]/R_{G_1}(T)$, is an isomorphism.

To see that the map is well-defined, suppose $u \in R_G(S \cup T)$. It suffices to show that $u \in R_{G_1}(T)$. Let $\varphi : Y \to G_1$ be any solution of $T$ over $G_1$ and denote $y^\varphi_j = w_j(x_1, \ldots, x_n)$. We need to show that $u^\varphi = 1$ in $G_1$, i.e. $u^\varphi \in R_G(S)$. Let $\psi : X \to G$ be any solution of $S$ over $G$, and denote $x^\psi_i = g_i$. Consider the map $\alpha : X \cup Y \to G$ defined by

$$\begin{align*}
x_i & \to g_i, \\
y_j & \to w_j(g_1, \ldots, g_n),
\end{align*}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The map $\alpha$ is a solution to $S \cup T$. Indeed, if $s \in S$ then $s^\alpha = s^\psi$, and $\psi$ is a solution to $S$ so $s^\psi = 1$. If $t \in T$ then

$$t^\alpha = t(w_1(g_1, \ldots, g_n), \ldots, w_m(g_1, \ldots, g_n)) = (t^\varphi)^\psi.$$
Since \( \varphi \) is a solution to \( T \) over \( G_1 \), we have that \( t^\varphi \in R_G(S) \) and since \( \psi \) is a solution to \( S \) over \( G \) we have that \( (t^\varphi)^\psi = 1 \) in \( G \), proving that \( \alpha \) is a solution to \( S \cup T \). Since \( u \in R_G(S \cup T) \), \( u^\alpha = 1 \) hence

\[
1 = u^\alpha = (u^\varphi)^\psi
\]

so \( u^\varphi \in R_G(S) \) as required.

The fact that the natural map is surjective is trivial, so it remains to prove injectivity. Let \( u \in G[X \cup Y] \) with \( u \notin R_G(S \cup T) \). We must show that \( u \notin R_G(T) \). Since \( u \notin R_G(S \cup T) \), there exists a solution \( \alpha : X \cup Y \to G \) of \( S \cup T \) such that \( u^\alpha \neq 1 \). The restriction \( \alpha|_Y \) of \( \alpha \) to \( Y \) is a solution to \( T \) over \( G_1 \). Indeed, if \( t \in T \) then variables of \( X \) do not occur in \( t \), so

\[
t^\alpha|_Y = t^\alpha = 1
\]

in \( G \), hence \( t^\alpha|_Y = 1 \) in \( G_1 \) as well. Since \( \alpha|_X \) is a solution to \( S \) over \( G \) and

\[
\left( u^{\alpha|_Y} \right)^{\alpha|_X} = u^\alpha \neq 1
\]

we conclude that \( u^\alpha|_Y \) is non-trivial in \( G_1 \) hence \( u \) is not in \( R_G(T) \), as required. \( \square \)

It follows from the lemma that for every \( i = 1, \ldots, n \),

\[
G_i \simeq G_{i+1}[X_i]/R_{G_{i+1}}(S_i).
\]

(14)

Note that this isomorphism holds for any system of equations that can be partitioned in triangular form, not just for NTQ systems. It is essential to observe that when \( R_{G_{i+1}}(S_i) = ncl_{G_{i+1}}(S_i) \), \( G_i \) admits the presentation

\[
G_i = \langle G_{i+1}, X_i \mid S_i \rangle.
\]

In this case, \( G_i \) has a graph of groups decomposition of one of the following four types, according to the form of \( S_i \):

(I) as a graph of groups with vertices \( v_1, v_2 \) where \( G_{v_1} = G_{i-1} \) and \( G_{v_2} \) is a QH-subgroup;

(II) as a graph of groups with vertices \( v_1, v_2 \) where \( G_{v_1} = G_{i-1} \), \( G_{v_2} \) is a free abelian group of rank \( m \) and the edge groups generate a maximal abelian subgroup of \( G_{v_1} \) (‘rank \( m \) extension of centralizer’);

(III) as a free product with a finite rank free abelian group;

(IV) as a free product with a finitely generated free group.

A frequently used method of proving that \( R_{G_{i+1}}(S_i) = ncl_{G_{i+1}}(S_i) \) is the following well-known fact.
Lemma 3.7. Let $S(X)$ be a system of equations over a group $G$. If $G_S$ is residually $G$, then $R_G(S) = \text{ncl}_G(S)$ and hence $G_{R(S)} = G_S$.

Proof. It is always the case that $\text{ncl}_G(S) \subset R_G(S)$, so assume for contradiction that there exists $w \in R_G(S) \setminus \text{ncl}_G(S)$. Then $w \neq 1$ in $G_S$, so there exists a homomorphism $\phi : G_S \to G$ such that $w^\phi \neq 1$. But $\phi$ is a solution to $S$ and $w \in R_G(S)$ so $w^\phi = 1$, a contradiction. \qed

For NTQ systems over toral relatively hyperbolic groups, [KM09] has shown that the condition $R_{G_{i+1}}(S_i) = \text{ncl}_{G_{i+1}}(S_i)$ holds except in some exceptional cases. We recall the relevant definitions from [KM09].

Definition 3.8. A standard quadratic equation $S = 1$ over a group $G$ is said to be regular if either $\chi(S) \leq -2$ and $S$ has a non-commutative solution over $G$, or $S = 1$ is an equation of the form $[x, y]d = 1$ or $[x_1, y_1][x_2, y_2] = 1$. An NTQ system is called regular if every quadratic equation appearing in case (I) is regular.

Proposition 3.9 ([KM09]). Let $G$ be a toral relatively hyperbolic group and $S = S_1 \cup \ldots \cup S_n$ a regular NTQ system over $G$. Then for all $i = 1, \ldots, n$,

$$R_{G_{i+1}}(S_i) = \text{ncl}_{G_{i+1}}(S_i).$$

The condition $R_{G_{i+1}}(S_i) = \text{ncl}_{G_{i+1}}(S_i)$ allows us to use the graph of groups decomposition of $G_i$ to derive properties of NTQ groups inductively. In particular, we have the following.

Lemma 3.10. Let $\Gamma = \langle A | R \rangle$ be a toral relatively hyperbolic group and $G$ a $\Gamma$-NTQ group such that $R_{G_{i+1}}(S_i) = \text{ncl}_{G_{i+1}}(S_i)$ for all $i = 1, \ldots, n$. Then $G$ is toral relatively hyperbolic and fully residually $\Gamma$.

Proof. The second statement is proved in [KM09]. For the first, we proceed by induction on the height of the NTQ system. The base $\Gamma$ is toral relatively hyperbolic. Now assume that $G_{n-1}$ is toral relatively hyperbolic. We will show that $G_n$ is toral relatively hyperbolic by applying Theorem 0.1 of [Dah03] (‘Combination theorem’) to the four possible decompositions of $G_i$ described above.

Cases (IV) and (III) follow from Theorem 0.1 parts (3) and (2), respectively, by amalgamating over the trivial subgroup. Note that to use Theorem 0.1 (2) we need the fact that if $G$ is hyperbolic relative to the collection of subgroups $\mathcal{H}$ then it is also hyperbolic relative to $\mathcal{H} \cup \{1\}$. Case (II) follows from Theorem 0.1 (2) by amalgamating over $P = \langle U_i \rangle$, which is maximal abelian in $G_{i-1}$.

For case (I), consider first the case when the surface corresponding to the quadratic equation has punctures. In this case we form $G_i$ by amalgamating $G_{i-1}$ with a free group over a $\mathbb{Z}$ subgroup, followed HNN-extensions over $\mathbb{Z}$ subgroups. It follows from the results of [Osi06b] that these $\mathbb{Z}$ subgroups are maximal parabolic subgroups, hence we may apply Theorem 0.1 (3), (3'). \qed
Remark 3.11. From the Combination Theorem it follows that $G$ has finitely many maximal non-cyclic abelian subgroups up to conjugation, and we can construct, by induction, the list of them along with a finite generating set for each. In the base group $\Gamma$ this is possible using the results of [Dah08].

NTQ groups over free groups played a central role in the solution to Tarski’s problems by Kharlampovich-Miasnikov and Sela. In Sela’s work, they are called $\omega$-residually free towers [Sel01].

3.2 Embedding into extensions of centralizers

Let $G = \langle Z \mid S \rangle$ be a finitely presented group. We consider $S$ as a (coefficient-free) system of equations over $\Gamma$. Let $\Gamma$ be presented by $\langle A \mid R \rangle$.

For a system of equations over free groups, Kharlampovich and Miasnikov proved that every solution factors through one of finitely many NTQ groups, which can be effectively constructed.

Proposition 3.12. [KM98b] There is an algorithm that, given a system of equations $T(X, A) = 1$ over a free group, produces finitely many $F$-NTQ systems $T_1(X_1, A) = 1, T_2(X_2, A) = 1, \ldots, T_n(X_n, A) = 1$ and homomorphisms $\mu_i : F(X) \to F_{R(T_i)}$ such that for every homomorphism $\psi : F_{R(T)} \to F$ there is an integer $i$ and a homomorphism $\phi : F_{R(T_i)} \to F$ such that $\psi = \mu_i \phi$.

Given this result, we may assume that the systems $S_1(X_1, A), \ldots, S_n(X_n, A)$ constructed in Lemma 2.2 are in fact NTQ systems. For each of these systems $S_i$ we consider the system $S_i^\pi$ over $\Gamma$.

In the following lemma, we construct homomorphisms from $G$ to the coordinate groups $\Gamma_{R(S_i^\pi)}$, one of which must be an embedding if $G$ is fully residually $\Gamma$.

Lemma 3.13. There is an algorithm that, given a finitely presented group $G = \langle Z \mid S \rangle$, produces

(i) finitely many $F$-NTQ systems $S_1(X_1, A), \ldots, S_m(X_m, A)$, and

(ii) homomorphisms $\alpha_i : G \to \Gamma_{R(S_i)}$

such that

(1) if $G$ is fully residually $\Gamma$, then there exists $i \in \{1, \ldots, m\}$ such that $\alpha_i$ is injective, and

(2) if $G$ is residually $\Gamma$, then for every $g \in G$ there exists $i \in \{1, \ldots, m\}$ such that $g^{\alpha_i} \neq 1$. 

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Proof. Refer to Figure 2 for a diagram of the maps constructed in this proof. Construct the $F$-NTQ systems $S_1(X_1, A), \ldots, S_n(X_n, A)$ and the homomorphisms $\rho_i : F(Z) \to F_{R(S_i)}$ from Lemma 2.2. Let $\gamma_i : F_{R(S_i)} \to \Gamma_{R(S_i)}$ be the canonical epimorphism and set $\alpha_i = \frac{\rho_i\phi\pi}{\gamma_i}$. That is, for any $\underline{u} \in G$,

$$\underline{u}^{\alpha_i} = u^{\rho_i\gamma_i}.$$ 

Since $\rho_i$ is given as a word mapping, so is $\alpha_i$.

To check that $\alpha_i$ is well-defined, let $u \in F(Z)$ with $\underline{u} = 1$ (in $G$). Since $u \in \text{ncl}_{F(Z)}(S)$, there exist $s_j \in S$ and $w_j \in F(Z)$ such that $u = \prod_{j=1}^m s_j^{w_j}$ hence

$$u^{\rho_i\gamma_i} = \prod_{j=1}^m (s_j^{\rho_i\gamma_i})^{w_j^{\gamma_i}}.$$ 

Recall from the description of canonical representatives in Lemma 2.2 that $s_j$ has the form $s_j = z_1z_2z_3$ and hence $s_j^{\rho_i}$ has the form

$$s_j^{\rho_i} = (x_1c_1x_2^{-1})(x_2c_2x_3^{-1})(x_3c_3x_1^{-1})$$

where $c_1c_2c_3 = 1$ in $\Gamma$ and $x_1, x_2, x_3 \in X_i$. Hence

$$s_j^{\rho_i} = (c_1c_2c_3)^{x_1}.$$ 

Since the relators of $\Gamma$ are elements of $R_{\Gamma}(S_i)$ we have that $s_j^{\rho_i\gamma_i} = 1$ in $\Gamma_{R(S_i)}$ hence $u^{\rho_i\gamma_i} = 1$ and $\alpha_i$ is well-defined.

Suppose now that $G$ is fully residually $\Gamma$. For each $i \in \{1, \ldots, n\}$ set

$$\Phi_i = \{\rho_i\phi\pi \mid \phi \in \text{Hom}(F_{R(S_i)}, F)\}.$$ 

From Lemma 2.2 we know that

$$\text{Hom}(G, \Gamma) = \bigcup_{i=1}^n \Phi_i.$$ 

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Since \( G \) is fully residually \( \Gamma \), there exists \( i \) such that \( \Phi_i \) is a discriminating family of homomorphisms. Indeed, if no \( \Phi_i \) discriminates \( G \), then for each \( i \) there is a finite subset \( W_i \subset G \) such that every \( \phi \in \Phi_i \) is not injective on \( W_i \). Then \( \bigcup_{i=1}^m W_i \) is a finite subset that is not discriminated by \( \bigcup_{i=1}^m \Phi_i \).

Let \( i \) be such that \( \Phi_i \) is a discriminating family and let \( \pi \) be any non-trivial element of \( G \). Then there exists \( \rho_i \phi \pi \in \Phi_i \) with \( u^{\rho_i \phi \pi} \neq 1 \). The homomorphism \( \phi \pi : F_{R(S_i)} \to \Gamma \) is a solution to \( S_i \) over \( \Gamma \) which does not send \( u^{\rho_i \phi \pi} \) to 1, hence \( u^{\rho_i \phi \pi} \) is not in \( R_{\Gamma}(S_i) \). Consequently,

\[
\pi^{\alpha_i} = (u^{\rho_i})^{\gamma_i} \neq 1
\]

so \( \alpha_i \) is injective.

Now suppose that \( G \) is residually \( \Gamma \) and let \( \pi \in G \). Since \( \text{Hom}(G, \Gamma) = \bigcup_{i=1}^n \Phi_i \), there exists \( i \) and \( \rho_i \phi \pi \in \Phi_i \) such that \( u^{\rho_i \phi \pi} \neq 1 \). As above, this implies \( \pi^{\alpha_i} \neq 1 \). \( \square \)

**Remark 3.14.** Though at least one of the homomorphisms \( \alpha_i \) must be injective when \( G \) is fully residually \( \Gamma \), we are not aware of a method for determining which one (there may be several).

Our objective now is to construct an effective embedding of each coordinate group \( \Gamma_{R(S_i)} \) into a group obtained from \( \Gamma \) by a series of extensions of centralizers. We will need the following lemma in order to argue by induction.

**Lemma 3.15.** Let \( H \leq G \) be any torsion-free groups and let \( S \) be a system of equations over \( H \) such that \( S \) has one of the NTQ forms (I)–(IV). Then the canonical homomorphism \( H_S \to G_S \) is an embedding.

**Proof.** The case when \( S \) is a standard quadratic equation is Proposition 2 of [KM98a], the case when \( S \) is an extension of a centralizer follows immediately from the theory of normal forms for HNN-extensions, and the cases of a free product with a free group or free abelian group are obvious. \( \square \)

We now prove the main technical lemma.

**Lemma 3.16.** Let \( \Gamma = \langle A \mid R \rangle \) a finitely presented torsion-free hyperbolic group. There exists an algorithm that, given an NTQ system \( S(X, A) \) over the free group \( F \), constructs a group \( H \) obtained from \( \Gamma \) by a series of extensions of centralizers and an embedding

\[
\beta : \Gamma_{R(S)} \hookrightarrow H.
\]

Further, both groups \( \Gamma_{R(S)} \) and \( H \) are toral relatively hyperbolic and a generating set for any maximal abelian subgroup can be effectively constructed.

**Proof.** Let \( S(X, A) \) be partitioned as an NTQ system as \( S_1, \ldots, S_n \). Consider \( S \) as a system of equations over \( \Gamma \), with \( G_{n+1} = \Gamma \) and

\[
G_i = G_{i+1}[X_i]/R_{G_{i+1}}(S_i).
\]

Note that \( \Gamma_{R(S)} = G_1 \).
We proceed by induction on \( n \). For the base case \( n = 0 \) there are no equations or variables in \( S \) so \( \Gamma_{|\ell(S)|} = \Gamma \) so we may take \( H = \Gamma \) and \( \beta \) the identity.

Now assume the theorem holds up to \( n - 1 \). That is, assume we have constructed a group \( H' \) obtained by extensions of centralizers of \( \Gamma \) and an embedding \( \beta' : G_2 \to H' \). We argue based on the form (\text{IV}) of the system of equations \( S_1(X_1, A) \). In the following we will frequently use without mention Lemma 3.7 to obtain a presentation of \( G_1 \), and Lemma 3.10 and Remark 3.11 to show that \( G_1 \) is toral relatively hyperbolic with a finite collection of maximal abelian subgroups (up to conjugation), generating sets of which can be effectively constructed.

**Form (\text{IV}): Free product with a free group.** Suppose \( S_1 \) has the form (\text{IV}), that is, \( S_1 \) is empty. We will show that the group \( \langle G_2, X_1 \mid - \rangle \simeq G_2 \ast F(X_1) \) embeds in a group obtained from \( G_2 \) by extensions of centralizers. It will suffice to consider the case of two variables, \( X_1 = \{ x, y \} \). Let \( u, v \in G_2 \) such that \( C(u) \cap C(v) = 1 \), and consider the series of extensions of centralizers

\[
\begin{align*}
G_2' &= \langle G_2, t \mid [C(u), t] \rangle, \\
G_2'' &= \langle G_2', s \mid [C(v), s] \rangle, \\
G_2''' &= \langle G_2'', r \mid [C(u), s] \rangle.
\end{align*}
\]

One checks that \( C_{G_2'}(v) = C_{G_2'}(v) \), that \( t \) and \( s \) generate a rank two free subgroup in \( G_2'' \), that \( C_{G_2''}(ust) \cap G_2 = 1 \), and that \( C_{G_2''}(ust) \cap \langle t, s \rangle = 1 \). Define \( \phi : G_2 \ast F(x, y) \to G_2'' \) by \( x^\phi = t^r \), \( y^\phi = s^r \), and \( g^\phi = g \) for \( g \in G_2 \). A non-trivial element \( w \in G_2 \ast F(x, y) \) has reduced form

\[
w = g_1 w_1(x, y) g_2 w_2(x, y) \ldots g_m w_m(x, y) g_{m+1}
\]

and is sent under \( \phi \) to

\[
w^\phi = g_1 r^{-1} w_1(t, s) r g_2 r^{-1} w_2(t, s) r \ldots g_m r^{-1} w_m(t, s) r g_{m+1}.
\]

This word has no reduction of the form \( r g_i r^{-1} \to g_i \), since \( C_{G_2''}(ust) \cap G_2 = 1 \), and no reduction of the form \( r^{-1} w_i(t, s) r \to w_i(t, s) \), since \( C_{G_2''}(ust) \cap \langle t, s \rangle = 1 \) and \( \langle t, s \rangle \) is free of rank two. Hence \( w^\phi \) is reduced and therefore non-trivial by Britton’s Lemma, so \( \phi \) is injective.

We conclude that \( \langle G_2, X_1 \mid - \rangle \) is residually \( G_2 \), hence \( G_1 = G_1 \ast F(x, y) \) and \( G_1 \) is toral relatively hyperbolic. By Lemma 3.15 \( G_1 \) embeds canonically in \( H' \ast F(x, y) \). Repeating the construction above with \( H' \) in place of \( G_2 \) we may construct an embedding of \( H' \ast F(x, y) \) into a group \( H \) obtained by extensions of centralizers from \( H' \).

**Form (\text{III}): Free product with a free abelian group.** Suppose \( S_1 \) has the form (\text{III}). First, suppose that \( |X_1| = 2 \), and so \( \langle G_2, X_1 \mid S_1 \rangle \simeq G_2 \ast \mathbb{Z}^2 \). Lemma 16 of [KM98a] shows that \( G_2 \ast \mathbb{Z}^2 \) embeds in every non-trivial extension of a centralizer of \( G_2 \). Consequently, \( G_2 \ast \mathbb{Z}^2 \) is residually \( G_2 \) so \( G_1 \simeq G_2 \ast \mathbb{Z}^2 \) and is toral relatively hyperbolic.
From Lemma 3.15, $G_1$ embeds canonically in $H' \ast \mathbb{Z}^2$. Apply Lemma 16 of [KM09a] again to embed $H' \ast \mathbb{Z}^2$ in an extension of centralizers $H$ of $H'$. It follows immediately from the proof that the embedding is effective, provided we can produce two non-commuting elements of $H'$, which we may do by Lemma 1.1.

If $|X_1| > 2$, we partition $S_1$ into two subsystems

$$S_{1,a} = \{ [x_i, x_j] = 1, [x_i, u] = 1, \mid i, j \in \{3, \ldots, m\}, u \in U_1, a \}$$
$$S_{1,b} = \{ [x_1, x_2] = 1 \}$$

where $X_1 = \{ x_1, \ldots, x_n \}$ and $U_1 = \{ x_1, x_2 \}$. The system $S_{1,b}$ has the form (III) with two variables, which we have dealt with above, and $S_{1,a}$ is an extension of the centralizer $C_{G_1}(x_1) = \langle x_1, x_2 \rangle$ in $G_1 \simeq G_2 \ast \langle x_1, x_2 \rangle$, which we deal with in form (II) below.

**Form (II): Extension of a centralizer.** Suppose $S_1$ has the form (II). If $U_1$ generates the trivial subgroup in $G_2$, which we may check since the word problem in $G_2$ is decidable, then we have the form (III) and we may argue as above.

Otherwise, let $U'$ be the centralizer of $U_1$ in $G_2$. In general, $\langle U_1 \rangle$ is a proper subgroup of $U'$. We must construct a generating set $u_1, \ldots, u_m$ for $U'$. By induction, $G_2$ has, up to conjugation, finitely many parabolic (i.e. abelian) subgroups $P_1, \ldots, P_l$ and we have constructed a generating set for each one. The centralizer $U'$ is a maximal abelian subgroup of $G_2$, hence is either conjugate to one of the $P_i$ or is cyclic.

It follows from [Bum04] and the fact that conjugacy in the abelian groups $P_i$ is decidable (see also Theorem 5.6 of [Osi06a]), that for any element $g \in G_2$ and parabolic subgroup $P_i$, we can decide whether or not $g$ is conjugate to an element of $P_i$, and if so find a conjugating element. Applying this to any non-trivial element $g$ of $U_1$, we either identify $U'$ as a conjugate of one of the $P_i$ and construct a generating set by conjugating the generating set of $P_i$, or we determine that $U'$ is in fact cyclic and we find a generator using Lemma 1.1.

Now consider the system of equations

$$S'_1 = \{ [x, u], [x, y] \mid x, y \in X_1, i \in \{1, \ldots, m\} \}$$

over $G_2$. Since $G_2$ is commutation-transitive, we know that if $\phi : X_1 \to G_2$ is any solution to the system $S_1$ then $[x^\phi, u_i] = 1$ for all $x \in X_1$ and $i = 1, \ldots, m$. Consequently, $[x, u_i] \in R_{G_2}(S_1)$ for all $x \in X_1$ and $i = 1, \ldots, m$ so $S'_1 \subset R_{G_2}(S_1)$. The group $\langle G_2, X \mid S'_1 \rangle$ is an extension of a centralizer of $G_2$. It follows from §5 of [BMR02] and Proposition 1.1 of [KM09a] that any extension of a centralizer of a toral relatively hyperbolic group $K$ is (fully) residually $K$, so $\langle G_2, X \mid S'_1 \rangle$ is residually $G_2$. Then by Lemma 3.7

$$R_{G_2}(S_1) = R_{G_1}(S'_1) = ncl_{G_2}(S'_1)$$

hence $G_1 = \langle G_2, X_1 \mid S'_1 \rangle$ and is toral relatively hyperbolic.

We need to show that $G_1$ embeds in an extension of centralizer of $H'$. By induction, we may construct a finite generating set $v_1, \ldots, v_l$ for the maximal
abelian subgroup of $H'$ that contains $U'$. Consider the system of equations

$$T = \{ [x, u_i], [x, y] \mid x, y \in X_1, \ i \in \{1, \ldots, l\}\}$$

and the group $H = \langle H', X_1 | T \rangle$, which is an extension of centralizer of $H'$.

Define the map $\beta : G_1 \to H$ by $x^\beta = x$ for $x \in X_1$ and $g^\beta = g^\delta$ for $g \in G_2$. One easily checks that $\beta$ is a (well-defined) homomorphism. To show that $\beta$ is injective, let $w \in G_1$ be non-trivial. Since $G_1$ is residually $G_2$, there is a function $\phi : X_1 \to G_2$ which is a solution to $S'_1$ and such that $w^\phi$ is a non-trivial element of $G_2$. We claim that $\phi \beta' : X_1 \to H'$ is a solution to $T$. For $x, y \in X_1$ we have

$$[x^{\phi \beta'}, y^{\phi \beta'}] = [x^\phi, y^\phi]^\beta' = 1^\beta' = 1.$$

For $x \in X$ and any $u_i$ we have that

$$[x^{\phi \beta'}, u_i^\beta] = [x^\phi, u_i]^\beta' = 1^\beta' = 1$$

so by commutation-transitivity $[x^{\phi \beta'}, v_j] = 1$ for all $j$. Hence $\phi \beta'$ is a solution as required, and induces a homomorphism $\phi \beta' : G_1 \to H'$. The image of $w^\beta$ under this homomorphism is

$$(w^\beta)\phi \beta' = w^{\phi \beta'}$$

and is non-trivial since $w^\phi \neq 1$ and $\beta'$ is injective. Consequently, $w^\beta \neq 1$ in $H$ as required.

**Form (I): Quadratic equation.** Suppose that $S_1$ is a quadratic equation. Then $S_1$ has one of the standard forms (I)–(IV). The words $c_i$ and $d$ in the standard form are non-trivial in $F_{R(S_2 \cup \ldots \cup S_n)}$, but may be trivial in $G_2$. We can check which are trivial by solving the word problem in $G_2$. Form an equation $S_{1,a}$ by

(i) erasing from $S_1$ each atom $c_i^{z_i}$ such that $c_i = 1$ in $G_2$, and

(ii) if $d = 1$ in $G_2$, by erasing $d$ and replacing the rightmost atom of the form $c_i^{z_i}$ by $c_i$.

Let $Z$ be the set of variables of $X_1$ not appearing in $S_{1,a}$ (i.e. the $z_i$ from the erased atoms, as well as the rightmost $z_i$ if $d = 1$). Partition $X$ into $X \setminus Z$ and $Z$. The system of equations $S_1(X_1, A)$ is equivalent over $G_2$ to the union of the systems $S_{1,b} = \emptyset$ in variables $Z$ and $S_{1,a}$ in variables $X_1 \setminus Z$, so we replace $S_1(X_1, A)$ with these two systems and apply case (IV) to $S_{1,b}$.

The equation $S_{1,a}$ is a quadratic equation in standard form over $G_2$. To simplify notation, we rename $S_{1,a}$ to $S_1$ and $X_1 \setminus Z$ to $X_1$. We study cases based on the Euler characteristic $\chi(S_1)$ of the surface associated with $S_1$.

**Case $\chi(S_1) \leq -2$.** Assume that $\chi(S_1) \leq -2$. First, check using Lemma 3.4 whether or not $S_1$ has a solution in general position over $G_2$. If so, then $S_1$ is regular. Whenever $S_1$ is regular and $G_2$ is toral relatively hyperbolic, Theorem 4.1 of [KM09] proves that the group $(G_2, X_1 | S_1)$ embeds into a group
$H$ obtained from $G_2$ by a series of extensions of centralizers. Consequently, this group is residually $G_2$ hence $G_1 = \langle G_2, X_1 \mid S_1 \rangle$ and $G_1$ is toral relatively hyperbolic. Embed $G_1$ canonically into $\langle H', X_1 \mid S_1 \rangle$, using Lemma 3.16.

The equation $S_1$ is regular over $H'$, and $H'$ is toral relatively hyperbolic, so again applying Theorem 4.1 of [KM09] we obtain that $\langle H', X_1 \mid S_1 \rangle$ embeds into a group obtained from $H'$ by a series of extensions of centralizers. It suffices to show that this embedding is effective. The reader may verify that in order to obtain an effective embedding from the proof given in [KM09], one must be able to solve the following three problems: (a) solve the word problem in $H'$, (b) decide whether or not a quadratic equation over $H'$ has a non-commutative solution, and (c) find such a solution. We can solve (a) by Lemma 1.1 since $H'$ is toral relatively hyperbolic, (b) by Lemma 3.4 and (c) by enumerating all possible solutions until we find a non-commutative one.

Now suppose that $S_1$ does not have a solution in general position over $G_2$. By Lemma 3.3 all solutions are commutative. We consider cases based on the from of $S_1$.

**Orientable forms.** Suppose $S_1$ contains a commutator. If $S_1 = [x_1, y_1][x_2, y_2]$, then $S_1$ is regular by definition and we may proceed as above. Otherwise, by Proposition 4.3 of [KM09], $S_1$ has a solution in general position in a group $K$ obtained from $G_2 * F$, where $F$ is a finite-rank free group, by a series of centralizer extensions. Since $K$ is discriminated by $G_2$ (see form (IV)), it follows that $S_1$ has a solution in general position in $G_2$, which contradicts the fact that all solutions are commutative.

**Genus zero forms.** Suppose that $S_1$ has the form

$$c_1^{z_1} \cdots c_k^{z_k} d.$$  

Although $\chi(S_1) \leq -2$ implies that $k \geq 3$, we will assume only $k \geq 2$. Since $G_2$ has the CSA property, we may apply Corollary 3 of [KM98a] to obtain that

$$R_{G_2}(S_1) = \text{ncl} \left( \{ [a_i^{-1} z_i, C], [a_j^{-1} z_j, C] \mid i, j = 1 \ldots k \} \right)$$

where $C = C_{G_2}(c_1^{a_1}, \ldots, c_m^{a_m})$ and $z_j \rightarrow a_j$ is a solution to $S_1$. A solution must exist since $S_1$ has a solution over $F_{R(S_2, \ldots, \cup S_n)}$, and $G_2$ is a quotient of $F_{R(S_2, \ldots, \cup S_n)}$. We may find such a solution by enumerating all possible solutions.

Since $G_2$ is CSA, the group $C$ is precisely the maximal abelian subgroup which is the centralizer of $c_1^{a_1}$. By assumption, we may compute a generating set $\{u_1, \ldots, u_m\}$ for $C$. Then

$$G_1 \cong \langle G_2, t_1, \ldots, t_k \mid [t_i, u_l], [t_i, t_j], 1 \leq i, j \leq k, 1 \leq l \leq m \rangle$$

via the isomorphism $t_i \rightarrow a_i^{-1} z_i$. Since this is an extension of a centralizer, we complete the argument by reasoning as in Case [II].

**Non-orientable forms.** Suppose that $S_1$ corresponds to a non-orientable surface. Suppose $S_1$ has the form

$$x_1^2 \cdots x_p^2$$

where, by assumption, $p \geq 4$. Then any two non-commuting elements $g, h \in G_2$ yield the non-commutative solution $x_1 \rightarrow g$, $x_2 \rightarrow g^{-1}$, $x_3 \rightarrow h$, $x_4 \rightarrow h^{-1}$, and
Suppose $S_1$ has the form
\[ x_1^2 \cdots x_p^2 d \]
with $d \neq 1$ and $p \geq 3$. For any commutative solution $x_i \to s_i$ and any $g \not\in C_{G_2}(s_1)$, the function $x_1 \to g$, $x_2 \to g^{-1}$, $x_3 \to s_1 \cdots s_p$, and $x_i \to 1$ for $i > 3$ is a non-commutative solution, which is a contradiction.

Suppose $S_1$ has the form
\[ x_1^2 \cdots x_p^2 c_1 \cdots c_k d. \]
with $p \geq 2$. Though $\chi(S_1) \leq -2$ implies $k \neq 0$, the following argument applies for all $k \geq 0$. Construct any (commutative) solution $x_i \to s_i$, $z_j \to a_j$. From transitivity of commutation, it follows that
\[ [c_i^{a_i}, c_j^{a_j}] = [c_i^{a_i}, s_1 \cdots s_p] = 1 \]
for all $i, j = 1, \ldots, k$. Let $U = C_{G_2}(c_1^{a_1}, \ldots, c_k^{a_k}, s_1 \cdots s_p)$ and construct a generating set $\{u_1, \ldots, u_m\}$ for $U$. From the proof of Proposition 8 of [KM98], which needs only the fact that $G_2$ is commutation-transitive and torsion-free, we see that $G_1$ is isomorphic to the group
\[ \langle G_2, t_1, \ldots, t_{p+k-1} | [u_l, t_j], [t_i, t_j], 1 \leq i, j \leq p + k - 1, 1 \leq l \leq m \rangle * \langle x_p \rangle \]
via the isomorphism $t_i \to a_i^{-1} z_i$ for $i = 1, \ldots, k$ and $t_i \to x_i$ for $i = k+1, \ldots, k+p-1$. This group is an extension of a centralizer followed by free product with $\mathbb{Z}$, so we proceed as in Case (III) and Case (IV).

Finally, suppose $S_1$ has the form
\[ x_1^2 c_1^{z_1} \cdots c_k^{z_k} d. \]
It is shown in the proof of Proposition 8 of [KM98] that there exists $s \in G_2$ such that every solution of $S_1$ sends $x_1$ to $s$. Consequently, $s^{-1} x_1$ is in the radical of $S_1$ over $G_2$, hence
\[ G_1 \cong G_2[z_1, \ldots, z_k]/R_{G_2}(c_1^{z_1} \cdots c_k^{z_k} d) \]
and we may argue as in the genus zero case above. Note that we may find $s$ by finding any solution.

Case $\chi(S_1) > -2$. Assume that $\chi(S_1) > -2$. We consider cases based on the form of $S_1$.

Orientable forms. There are two possible forms, $[x, y]d$ and $[x, y]$. The form $[x, y]d$ is a regular quadratic equation (by definition), and the argument for regular equations given at the beginning of the case $\chi(S_1) \leq -2$ applies. For the form $[x, y]$, we apply Case (III).

Non-orientable forms. The possible forms are $x^2$, $x^2 d$, $x^2 y^2$, $x^2 y^2 d$, and $x^2 y^2 z^2$. For the form $x^2$, $x \to 1$ is the unique solution since $G_2$ is torsion-free. Hence $x \in R_{G_2}(S_1)$ and $G_1 \cong G_2$, so there is nothing further to prove.
For the form \(x^2d\), find a solution \(x \to a\). Note that \(d = a^{-2}\). Suppose, for contradiction, that there exists a second solution \(x \to b\). Then since \([a, a^{-2}] = 1\), \([b, b^{-2}] = 1\), and \(a^{-2} = b^{-2}\) we conclude \([a, b] = 1\) by transitivity of commutation. Then
\[
(ab^{-1})^2 = a^2b^{-2} = d^{-1}d = 1
\]
which implies \(a = b\) since \(G_2\) is torsion-free. Consequently, \(x \to a\) is the unique solution and \(xa^{-1}\) is in the radical of \(x^2d\) over \(G_2\). Then \(\langle G_2, x | xa^{-1}, x^2d \rangle \simeq G_2\) hence \(R_{G_2}([x^2d]) = ncl_{G_2}(xa^{-1})\) and \(G_1 \simeq G_2\).

For the form \(x^2y^2\), the analysis is similar. First, we may check for the existence of a non-trivial solution using the fact that the existential theory of toral relatively hyperbolic groups is decidable ([Sel09], [Dah09]). If all solutions are trivial, then \(G_1 \simeq G_2\). Otherwise, let \(x \to a\), \(y \to b\) be a non-trivial solution. Since \([a, a^2] = 1\) and \([b, b^{-2}] = 1\) we obtain \([a, b] = 1\) by transitivity of commutation. As above, \((ab)^2 = 1\) implies \(ab = 1\) hence \(xy\) is in the radical of \(x^2y^2\). The group \(\langle G_2, x, y | xy, x^2y^2 \rangle \simeq G_2 * \langle x \rangle\) is fully residually \(G_2\) hence \(R_{G_2}(x^2y^2) = ncl_{G_2}(xy)\) so
\[G_1 \simeq G_2 * \mathbb{Z}\]
and we may argue as in Case [IV].

For the form \(x^2y^2d\), first we determine whether or not all solutions are commutative, using Lemma 3.3. If all solutions are commutative, the proof given for the case \(S_1 \leq -2\) and \(S_1 = x_1^\alpha \ldots x_p^\alpha c_1^\pm \ldots c_k^\pm d\) with \(p \geq 2\) applies, since there we allowed \(k = 0\). Otherwise, find any (non-commutative) solution \(x \to a\), \(y \to b\). Consider the series of extensions of centralizers
\[
\begin{align*}
G_4' &= \langle G_2, t | [C(ab), t] \rangle, \\
G_4'' &= \langle G_2, s | [C(atat), s] \rangle, \\
G_4''' &= \langle G_2, r | [C(s^{-1}atst^{-1}b), r] \rangle,
\end{align*}
\]
and the map \(\psi : \langle G_2, x, y | x^2y^2d \rangle \to G_4'''\) given by
\[
\begin{align*}
x &\to (at)^sr \\
y &\to r^{-1}t^{-1}b.
\end{align*}
\]
Since \((x^2y^2d)^\psi = 1\), hence \(\psi\) defines a homomorphism. Using normal forms for elements of HNN-extensions, we can show that \(\psi\) is injective (see for example the proofs in §5 of [KM98a]). Consequently, \(\langle G_2, x, y | S_1 \rangle\) is residually \(G_2\) hence \(G_1 = \langle G_2, x, y | S_1 \rangle\). By Lemma 3.15, \(G_1\) embeds canonically into \(\langle H', x, y | S_1 \rangle\). We then apply the construction above to \(\langle H', x, y | S_1 \rangle\) to embed this group into a group \(H\) obtained from \(H'\), hence from \(\Gamma\), by extensions of centralizers.

For the form \(x^2e^d\), first we determine whether or not all solutions are commutative, using Lemma 3.3. Suppose all solutions are commutative. Find any (commutative) solution \(x \to a\), \(z \to b\). Let \(x \to a_1\), \(z \to b_1\) be any other solution. We have that \(d = (a^2b)^{-1} = (a_2^2b_1)^{-1}\) and \([e^b, d] = [c^{b_1}, d] = 1\) since both solutions are commutative. By transitivity of commutation, \([e^b, c^{b_1}] = 1\), and from the CSA property it follows that \([b_1b^{-1}, e] = 1\). This equation may be
rewritten as $c^h = c^{h_1}$, and consequently $a_1^2 = a^2$. If $a = 1$, then $a_1 = 1$ since $G_2$ is torsion-free. If $a \neq 1$, then by transitivity of commutation $[a_1, a] = 1$ hence $(a_1a)^2 = 1$ so $a_1 = a$. In either case, $xa^{-1} \in R_{G_2}(S_1)$. Since

$$\langle G_2, x, z \mid xa^{-1}, x^2c^2 d \rangle \simeq \langle G_2, z \mid c^2 da^2 \rangle$$

we may apply the argument for the case $S_1 = c^2 d$, given below.

If not all solutions are commutative, find any (non-commutative) solution $x \to a$, $z \to b$. As was done in [KM98a], consider the sequence of extensions of centralizers

\begin{align*}
G'_2 &= \langle G_2, t \mid [C(d), t] \rangle, \\
G''_2 &= \langle G'_2, s \mid [C(c^h), s] \rangle, \\
G'''_2 &= \langle G''_2, r \mid [C(c^h t), r] \rangle,
\end{align*}

and the map $\psi : \langle G_2, x, y \mid x^2 c^2 d \rangle \to G'''_2$ given by

\begin{align*}
x &\to a^t \\
y &\to b \text{str}.
\end{align*}

As in the previous case, $(x^2 c^2 d)\psi = 1$ and we may prove using normal forms that $\psi$ is injective and complete the argument as above.

For the form $x^2 y^2 z^2$, first we determine whether or not all solutions are commutative, using Lemma 3.4. Suppose all solutions are commutative. It follows from commutation-transitivity of $G_2$ that $[x, y], [x, z], [y, z] \in R_{G_2}(S_1)$, and then from the fact that $G_2$ is torsion-free that $xyz \in R_{G_2}(S_1)$. Let $S'_1$ be the system of equations $\{x^2, y^2, z^2, [x, y], [x, z], [y, z], xyz\}$. Then

$$\langle G_2, x, y, z \mid S'_1 \rangle \simeq G_2 \ast Z^2.$$ 

It follows from Case (III) that this group is fully residually $G_2$ and hence

$$\text{ncl}_{G_2}(S'_1) = R_{G_2}(S'_1) = R_{G_2}(S_1).$$

Then $G_1 = G_2 \ast Z^2$ and we may argue as in Case (III).

Now find any solution $x \to a$, $y \to b$, $z \to c$ of $S_1$ in general position. Consider the series of six extensions of centralizers

\begin{align*}
G^{(1)}_2 &= \langle G_2, s \mid [s, C(ab)] \rangle, \\
G^{(2)}_2 &= \langle G^{(1)}_2, r \mid [r, C(s^{-1}bc)] \rangle, \\
G^{(3)}_2 &= \langle G^{(2)}_2, v \mid [v, C(abrs^{-1}bc)] \rangle, \\
G^{(4)}_2 &= \langle G^{(3)}_2, t \mid [t, C(vasvas)] \rangle, \\
G^{(5)}_2 &= \langle G^{(4)}_2, u \mid [u, C(s^{-1}brs^{-1}br)] \rangle, \\
G^{(6)}_2 &= \langle G^{(5)}_2, w \mid [w, C(r^{-1}rv^{-1}r^{-1}rv^{-1})] \rangle,
\end{align*}

\[24\]
and the map $\psi : \langle G_2, x, y, z \mid x^2y^2z^2 \rangle \to G_2^{(6)}$ given by
\[
\begin{align*}
x &\rightarrow (vas)^t \\
y &\rightarrow (s^{-1}br)^u \\
z &\rightarrow (r^{-1}cv^{-1})^w.
\end{align*}
\]
As in the previous case, $(x^2y^2z^2)^\psi = 1$ and we may prove, with a lengthy argument using normal forms, that $\psi$ is injective and complete the argument as before.

*Genus zero forms.* The possible forms are $c^2d$ and $c_1^1c_2^2d$. The form $c_1^1c_2^2d$ was covered under genus zero forms for $\chi(S_1) \leq -2$, since the proof there needed only $k \geq 2$.

For the form $c^2d$, find a solution $z \rightarrow a$ and a generating set $\{u_1, \ldots, u_m\}$ for $C_{G_2}(c)$. We claim that $[za^{-1}, u_i]$ is in the radical of $c^2d$, for all $i$. Indeed, if $z \rightarrow b$ is any solution to $c^2d = 1$ over $G_2$ then
\[
[ba^{-1}, c] = ab^{-1}c^{-1}ba^{-1}c = ada^{-1}c = c^{-1}c = 1
\]
and by transitivity of commutation we have $[ba^{-1}, u_i] = 1$, hence $[za^{-1}, u_i]$ is in the radical. Then
\[
\langle G_2, z \mid [za^{-1}, u_i], i = 1, \ldots, m \rangle \simeq \langle G_2, t \mid [t, u_i], i = 1, \ldots, m \rangle
\]
is an extension of the centralizer of $c$, hence is residually $G_2$. Consequently, $G_1$ is isomorphic to the extension of centralizer
\[
G_1 \simeq \langle G_2, t \mid [t, u_i], i = 1, \ldots, m \rangle
\]
and we may argue as in Case (II).

All possible forms of $S_1$ have been covered, so the proof is complete. \(\square\)

We may now prove the main result of the paper.

**Theorem 3.17.** Let $\Gamma$ be any torsion-free hyperbolic group. There is an algorithm that, given a finitely presented group $G$, constructs

(i) finitely many groups $H_1, \ldots, H_n$, each given as a series of extensions of centralizers of $\Gamma$, and

(ii) homomorphisms $\phi_i : G \to H_i$,

such that

(1) if $G$ is fully residually $\Gamma$, then at least one of the $\phi_i$ is injective, and

(2) if $G$ is residually $\Gamma$, the map $\phi_1 \times \ldots \times \phi_n : G \to H_1 \times \ldots \times H_n$ is injective.

This also holds for $G$ in the category of $\Gamma$-groups.
Proof. For each system of equations $S_i$ constructed in Lemma 3.13, let $\beta_i : \Gamma R(S_i) \rightarrow H_i$ be as constructed in Lemma 3.16 and set $\phi_i = \alpha_i\beta_i$. The result then follows from Lemma 3.13 and the fact the each $\beta_i$ is injective.

As a corollary, we obtain a polynomial-time solution to the word problem in any finitely presented residually $\Gamma$ group.

Corollary 3.18. Let $\Gamma$ be a torsion-free hyperbolic group and $G = \langle Z \mid S \rangle$ any finitely presented group that is known to be residually $\Gamma$. There is an algorithm that, given a word $w$ over the alphabet $Z^\pm$, decides whether or not $w = 1$ in $G$ in time polynomial in $|w|$.

Proof. We compute in advance the embedding $\phi : G \rightarrow H_1 \times \ldots \times H_n$, i.e. we compute $z^\phi$ for each $z \in Z$. Given the input word $w$, we need only compute $w^\phi$ and solve the word problem in $H_1 \times \ldots \times H_n$. There is a fixed constant $L$ such that $|\pi_{H_i}(w^\phi)| \leq L|w|$, where $\pi_{H_i}$ is projection onto $H_i$, so we have a polynomial reduction to $n$ word problems in the groups $H_1, \ldots, H_n$. It then suffices to show that each $H_i$ has a polynomial time word problem.

Let $H_i$ be formed by a sequence of $m$ extensions of centralizers and proceed by induction. If $m = 0$, then $H_i = \Gamma$ so the word problem in $H_i$ is decidable in polynomial time. Now assume that

$$H_i = \langle H'_i, t \mid [t, C(u)] \rangle$$

(15)

where $u \in H'_i$ and $H'_i$ is formed from $\Gamma$ by a sequence of $m - 1$ extensions of centralizers and has a polynomial time word problem. Let $w$ be a word in $H_i$. It suffices to produce a reduced form for $w$ as an element of the HNN-extension (15), if any $t^{\pm 1}$ appears in the reduced form then $w \neq 1$, and if no $t^{\pm 1}$ appears then $w \in H'_i$ and we check whether or not $w = 1$ using the word problem algorithm for $H'_i$.

We produce a reduced form for $w$ by examining all subwords of the form $tvt^{-1}$ and $t^{-1}vt$ where no $t^{\pm 1}$ appears in $v$, and making reductions

$$tvt^{-1} \rightarrow v, \quad t^{-1}vt \rightarrow v$$

whenever $v \in C_{H'_i}(u)$. The element $v$ is in $C_{H'_i}(u)$ if and only if $[v, u] = 1$ in $H'_i$, which is an instance of the word problem in $H'_i$ and so may be checked in polynomial time. It is clear that we need only examine a polynomial number of subwords $tvt^{-1}$ and $t^{-1}vt$ before reaching a reduced form.

The result below follows from the proof of Lemma 3.16.

Proposition 3.19. Let $\Gamma = \langle A \mid R \rangle$ be a torsion-free hyperbolic group. There exists an algorithm that, given a system of equations $U(X, A) = 1$ over $\Gamma$, constructs a finite number of $\Gamma$-NTQ systems $S_i(X, A) = 1$ over $\Gamma$ that correspond to the fundamental sequences of solutions of $U(X, A) = 1$ that satisfy the second restriction on fundamental sequences as in Section 7.9 in [KM06]. Namely,

a) edge groups in the decompositions on each level are not mapped along this sequence into trivial elements,
b) images of $QH$ subgroups on each level are non-cyclic,
c) images of rigid subgroups are non-cyclic.

Each homomorphism $\Gamma_{R(U)} \to \Gamma$ factors through one of these fundamental sequences. (Such fundamental sequences correspond to strict resolutions in Sela’s terminology [Sel09].)

**Corollary 3.20.** The universal theory of a torsion-free hyperbolic group is decidable.

**Proof.** To show that the universal theory of $\Gamma$ is decidable we have to show that there is an algorithm to decide whether the conjunction of a system of equations $U(X,A) = 1$ and a system of inequalities $V(X,A) \neq 1$ has a solution in $\Gamma$. The conjunction has a solution if and only if there exists an index $i$ such that the images of all elements from $V(X,A)$ are non-trivial in $\Gamma_{R(S_i(X,A))}$. This we can check because the word problem in $\Gamma_{R(S_i(X,A))}$ is solvable.

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