DISSIPATIVE MARTINGALE SOLUTIONS OF THE STOCHASTICALLY
FORCED NAVIER–STOKES–POISSON SYSTEM ON DOMAINS WITHOUT
BOUNDARY

DONATELLA DONATELLI, PIERANGELO MARCATI, AND PRINCE ROMEO MENSAH

Abstract. We construct solutions to the randomly-forced Navier–Stokes–Poisson system in periodic three-dimensional domains or in the whole three-dimensional Euclidean space. These solutions are weak in the sense of PDEs and also weak in the sense of probability. As such, they satisfy the system in the sense of distributions and the underlying probability space and the stochastic driving force are also unknowns of the problem. Additionally, these solutions dissipate energy, satisfy a relative energy inequality in the sense of [4] and satisfy a renormalized form of the continuity equation in the sense of [5].

1. Introduction

Let \( t \geq 0 \) and \( x \in O \), where \( O = \mathbb{T}^3 \) is the three dimensional torus or \( O = \mathbb{R}^3 \), \( \vartheta, \nu^S > 0 \), \( \nu^B \geq 0 \) be constants and \( f = f(x) \) be a given function. We will construct a class of solution to the following stochastically forced Navier–Stokes–Poisson system

\[
\begin{align*}
\partial_t \varrho + \text{div}(\varrho \mathbf{u}) &= 0, \\
\varrho \partial_t \mathbf{u} + \left[ \text{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) \right] dt &= \left[ \nu^S \Delta \mathbf{u} + (\nu^B + \nu^S) \nabla \text{div} \mathbf{u} \\
&\quad + \vartheta \varrho \nabla V \right] dt + \varrho \mathbf{G}(\varrho, f, \varrho \mathbf{u}) dW(t), \\
\pm \Delta V &= \varrho - f
\end{align*}
\]

(1.1) – (1.3)

which are simultaneously weak in the sense of PDEs and in the sense of probability. The former notion of \textit{weak} means that each individual equation (1.1), (1.2) and (1.3) is satisfied in the sense of distributions, and the latter notion of \textit{weak} means that the underlying probability space and the Wiener process \( W \) are also unknowns of the problem. Furthermore, equation (1.3) is satisfied pointwise almost everywhere in its domain meaning that it indeed satisfied \textit{strongly} in the sense of PDEs.

For simplicity, we assume that the system (1.1)–(1.3) is \textit{isentropic} so that the equation of state is

\[
p(\varrho) = a \varrho^\gamma, \quad a > 0, \quad \gamma > \frac{3}{2}
\]

(1.4)

Remark 1.1. The result of this work also holds for generalized pressure laws. In particular, the result holds for any pressure satisfying

\[
p \in C^1(\mathbb{R}_0), \quad p(0) = 0, \quad \text{and for all } \varrho \geq 0, \quad \frac{1}{a} \varrho^{\gamma - 1} - b \leq p'(\varrho) \leq a \varrho^{\gamma - 1} + b
\]

(1.5)

where \( a > 0 \) and \( b \) are constants. For some physical models covered by the relation (1.5), we refer the reader to [7, Section 1.1].

Remark 1.2. The result also holds in the lower one and two dimensions under an even stronger pressure law, i.e., when the adiabatic exponent satisfies \( \gamma \geq 1 \) and \( \gamma > 1 \) respectively.

\[\text{Remark 1.2.}\]
The system (1.1)–(1.3) under study has various applications including the study of semiconductor devices, nuclear fluids, stellar dynamics, amongst others. In particular, in the momentum equation (1.2), the force term \( \vartheta \rho \nabla V \) may represent the extraneous force acting on the fluid due to the electric field or to self-gravitation. Accordingly, the Poisson equation represents the balance of the related potentials \( V \) (electric or gravitational). The novelty of the above model is that the momentum equations incorporates also the possible influence of external forces (be it random or deterministic), possible noise and errors that the classical purely-deterministic system may fail to capture.

If we set \( G = 0 \) in (1.2), then we obtain a deterministic system (1.1)–(1.3) for which existence of weak solution has been studied by [6] for bounded domains and by [7] on unbounded domains. The former relies primarily on a fixed-point argument whereas the latter uses the now standard multi-layer approximation scheme for the construction of weak solutions to compressible systems. A relevant application of this multi-layer approximation scheme is in the construction of weak martingale solutions for the stochastic compressible Navier–Stokes system (1.1)–(1.2) with \( V = 0 \). This was independently accomplished by [2] (see also, the manuscript [1]) for periodic boundaries and by [20] for bounded domain. By building on [1], the author of [16] extended the result to the whole space. Our result will therefore follow the manuscript [1] to construct solutions to (1.1)–(1.3) on periodic domains and then follow the approach of [16] to complete the picture for the whole space. Refer to Section 1.1 for more details.

1.1. Plan of paper. In the next section, Section 2, we present some notations, conventions and definitions that we will use throughout this paper. Since this paper discusses an existence result, we will in particular define the precise concept of a solution that we are interested in constructing. We will finally complete the section by stating the main results.

Our proof of existence of solution to (1.1)–(1.3) will rely on the energy method. Since this system is stochastic, we will present in Section 3 the formal equivalent derivation of the energy inequality from which regularity of the solution variables are obtained. Unlike deterministic PDEs where such energy estimates are obtained by formally testing the system with the solution, the stochastic energy estimate will be obtained from a formal application of an infinitesimal Itô’s formula.

The main part of the proof of our first main result will commence in Section 4. This is a preliminary approximation layer that uses Cauchy’s collocation method to construct a stochastically strong solution to a finite-dimensional auxiliary system of (1.1)–(1.3) containing an additional artificial pressure term, an artificial viscosity terms and some cut-off of the velocity field in the mass and momentum balance equations (1.1)–(1.2). In the deterministic sense, this solution will be strong in the auxiliary version of the continuity equation (1.1) and the Poisson equation (1.3) but weak in the auxiliary version of the momentum equation (1.2). In order words, the former equations are satisfied pointwise almost everywhere in spacetime whereas the latter is satisfied in the sense of distributions. The construction of the solution to the fluid part (1.1)–(1.2) of the system, albeit the extra electric field term and the data \( f \) input in the noise, follow the approach in [1] by simply freezing these components in time. Given the information about the fluid system and further information on the data \( f \), we are then able to construct a solution to the Poisson equation (1.3) by using standard regularity theorem for the Poisson equation.

The cut-off function of the velocity field mentioned in the previous paragraph is meant to deal with any anticipated vacuum region or potential blowup due to the noise. This cut-off is coupled with a stopping time argument to establish the existence of a unique solution to a finite-dimensional approximation derived from the previous layer. In particular, on each stopping time, we are able to construct a Faedo–Galerkin approximation with no potential blowup once these cut-offs are activated. By passing to the limit in this cut-off parameter (which coincide with passing to the limit in the family of stopping times), we establish the existence of a unique Faedo–Galerkin approximation on the entire time interval in Section 5. The notion of a solution being constructed in this layer is inherited from the previous layer with the obvious exemption of any cut-offs which essentially converge in various norms to the identity operator. Furthermore, this solution is shown to conserve energy so we obtain an energy equality rather than an inequality.
We finally transition from a finite-dimensional system as constructed in the previous layers to an infinitesimal system by passing to the limit in the finite-dimensional projection parameter in Section 6. The notion of a solution in the sense of PDEs is again inherited from the previous layers. However, from a strong solution in the sense of probabilities, we only obtain a weak solution in this infinitesimal approximation layer. Also, energy is dissipated in this layer rather than conserved. As a result of the solution being weak in the sense of probabilities, one is expected to prescribe an initial law and the underlying probability space and stochastic driving force becomes unknowns as well. The steps in the construction of this solution – which will be replicated in subsequent approximation layers – is as follows.

- We approximate a suitable law prescribed on data defined in an infinitesimal space by a family of laws – indexed by the discretization parameter – prescribed on the data from the previous section. This constructed law satisfies the hypothesis of the main result, Theorem 6.2 of this approximation layer.
- By using the finiteness of the ‘limit’ initial law construction above, we obtain a priori bounds for any family of solutions – indexed by the discretization parameter – uniformly in this parameter.
- The laws on these approximate solutions are shown to be tight on the corresponding spaces in which they live and by using the stochastic compactness method by Jakubowski and Skorokhod, we obtain ‘limit’ random variables for these solutions as a result of the finiteness of the ‘limit’ initial law.
- We then show in that these ‘limit’ random variables solves the required system without the discretization parameter.

The bullet points above gives the general structure of the proof which is the same for the corresponding version of the strictly fluid system studied in [1]. The extra detail which needs to be tackled is the establishment of compactness for the family of electric fields due to the additional Poisson equation. As is the case for stochastic fluid systems, this will follow from a tightness argument which requires a further degree of regularity for these fields besides the regularity obtained due to the conservation of energy. Fortunately, since the Possion equation is linear, this is quite straightforward and all we need is information on the data $f$ and the corresponding family of densities. We have the former by way of assumption and the information on the latter is derived from the tightness argument on the family of densities as shown in [1].

Section 7 deals with the vanishing artificial viscosity limit. The main difference in the concept of a solution from the previous sections is that the solution to the continuity equation 1.1 is now weak in the sense of PDEs. However, the construction of this solution will mimic the itemized steps in the previous paragraph. That is, we first construct a suitable law from a family of laws from the previous layer, show that any solution indexed by the current approximation parameter satisfies suitable bounds due to energy dissipation from the previous layer, show compactness of these approximate solutions and finally, identify the limit system. However, an intermediate step between obtaining the usual uniform estimates from the energy inequality and compactness is the requirement to improve the regularity of the density sequence. This problem already exists in the analyses of the deterministic Navier–Stokes–Poisson system and the stochastic Navier–Stokes system and they results from the fact that the continuity equation is only expected to be satisfied weakly in the sense of distributions after the passage to the limit in this artificial viscous term. This improvement in regularity of the density sequence will rely on the stochastic adaptation of the analysis of the effective viscous flux. At the level of the stochastic Navier–Stokes system for fluids, [1], this aforementioned improved regularity is obtained from the application of Itô’s lemma which results in an equation corresponding to formally ‘testing’ the momentum equation with the continuity equation. This crucial step is performed differently in our case. Instead of testing with the continuity equation, we use an equation derived from the combination of the continuity equation and the Poisson equation with the help of the inverse Laplace operator. This gives a much cleaner expression and more importantly, allow us to estimate the term containing the electric field.
The final step in the construction of a solution to \((1.1) - (1.3)\) on the torus is the vanishing pressure limit which is presented in Section 8. The analyses is similar to Section 7 but a further loss of regularity due to the loss of the artificial viscosity means an additional analyses of the oscillation defect measure in order to obtain an improved regularity of the density sequence. Furthermore, unlike the preceding layer where the analyses of the effective viscous flux was performed by testing the momentum equation with a combined equation derived from the two other equations, we are unable to do that in this present layer due to the fact that a variational power of density–which no longer solves the continuity equation–is required. As a result, the analyses of both the effective viscous flux and the oscillation defect measure will have to follow the corresponding arguments in \([1]\) and thus requiring a new treatment of electric field terms in both case. Fortunately, the conservation of mass and its boundedness together with the regularity of the electric field obtained from the energy estimate is enough to resolve this problem.

Section 9 studies a corresponding version of Theorem 2.2 on the whole space. This is stated in Theorem 9.2. This involves approximating the problem on \(\mathbb{R}^3\) by an increasing sequence of periodic problems, establish local-in-space uniform bounds for this family, show tightness of prescribed laws then proceed to apply Jakubowski-Skorokhod compactness to obtain limit variables which are identified in the limit. The crucial term to be identified is the pressure and again, requires on Assumptions on the stochastic force.

2. Assumptions on the stochastic force. We start our analysis in the periodic setting, namely \(\mathcal{O} = \mathbb{T}^3\). Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a stochastic basis endowed with a right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\) and let \(W\) be an \((\mathcal{F}_t)\)-cylindrical Wiener process, i.e.,

\[
W(t) = \sum_{k \in \mathbb{N}} \beta_k(t)e_k, \quad t \in [0, T]
\]

(2.1)

where \((\beta_k)_{k \in \mathbb{N}}\) is a family of mutually independent real-valued Brownian motions and \((e_k)_{k \in \mathbb{N}}\) are orthonormal basis of a separable Hilbert space \(\Omega\). Since the formal sum (2.1) is not expected to
converge in $U$, we can construct a larger space $\mathcal{U}_0 \supset \mathcal{U}$ as follows

$$\mathcal{U}_0 = \left\{ v = \sum_{k \geq 1} c_k e_k : \sum_{k \geq 1} \frac{c_k^2}{k^2} < \infty \right\}$$

(2.2)

and endow it with the norm

$$\|v\|_{\mathcal{U}_0}^2 = \sum_{k \in \mathbb{N}} \frac{c_k^2}{k^2}, \quad v = \sum_{k \in \mathbb{N}} c_k e_k.$$  

One can now check that (2.1) converges in $\mathcal{U}_0$. Furthermore, $W$ has $\mathbb{P}$-a.s. $C([0, T]; \mathcal{U}_0)$ sample paths and the embedding $\mathcal{U} \rightarrow \mathcal{U}_0$ is Hilbert–Schmidt. See [3].

To ensure that the stochastic integral $\int_0^t \varrho \mathbf{G}(\varrho, f, \varrho u) dW$ is a well-defined $(\mathcal{F}_t)$-martingale taking value in a suitable Hilbert space $W^{-1,2}(\mathbb{T}^3)$, $l \geq 0$ say, we set $\mathbf{m} = \varrho \mathbf{u}$ and for $(x, \varrho, f, \mathbf{u}) \in \mathbb{T}^3 \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^3$, we assume that there exists some functions $(g_k)_{k \in \mathbb{N}}$ such that

$$g_k : \mathbb{T}^3 \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \rightarrow \mathbb{R}, \quad g_k \in C^1(\mathbb{T}^3 \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^3)$$

(2.3)

for any $k \in \mathbb{N}$ and in addition, $g_k$ satisfies the following growth conditions:

$$|g_k(x, \varrho, f, \mathbf{m})| \leq c_k (\varrho + f + |\mathbf{m}|),$$

(2.4)

$$|\nabla_{\varrho, \mathbf{m}} g_k(x, \varrho, f, \mathbf{m})| \leq c_k,$$

(2.5)

$$\sum_{k \in \mathbb{N}} c_k^2 \leq 1$$

(2.6)

for some constant $(c_k)_{k \in \mathbb{N}} \in [0, \infty)$. Now if we define the map $\mathbf{G}(\varrho, f, \mathbf{m}) : \mathcal{U} \rightarrow L^1(\mathbb{T}^3)$ by $\mathbf{G}(\varrho, f, \mathbf{m})e_k = g_k(-\varrho(\cdot), \varrho(\cdot), f(\cdot), \mathbf{m}(\cdot))$, then we can conclude that $\varrho \mathbf{G}(\varrho, f, \mathbf{m})$ is uniformly bounded in $L_2(\Omega; W^{-1,2}(\mathbb{T}^3))$ provided that, $\varrho^\gamma, f \rightarrow 1$ and $\varrho|\mathbf{u}|^2$ are integrable in $\mathbb{T}^3$. Indeed, if we let $(\varrho)_{\mathbb{T}^3} < \infty$ represent the mean density on the torus, then it follows from Young’s inequality that

$$\int_{\mathbb{T}^3} \varrho^2 f^2 \, dx \lesssim (\varrho)_{\mathbb{T}^3} \int_{\mathbb{T}^3} \varrho f^2 \, dx \lesssim \int_{\mathbb{T}^3} \varrho^\gamma \, dx + \int_{\mathbb{T}^3} f^{\frac{2}{\gamma-1}} \, dx$$

(2.7)

where $2 < \frac{2\gamma}{\gamma-1} < 6$. Also, by using $\varrho \leq 1 + \varrho^\gamma$, we also gain

$$\int_{\mathbb{T}^3} \varrho^2 (\varrho^2 + |\mathbf{m}|^2) \, dx \lesssim (\varrho)_{\mathbb{T}^3}^3 \int_{\mathbb{T}^3} \varrho^{-1} (\varrho^2 + |\mathbf{m}|^2) \, dx \lesssim \int_{\mathbb{T}^3} (1 + \varrho^\gamma + \varrho|\mathbf{u}|^2) \, dx.$$  

(2.8)

Finally, by using (2.4), (2.6) and (2.7)–(2.8), as well as Sobolev’s embedding, it follows that

$$\|\varrho \mathbf{G}(\varrho, f, \mathbf{m})\|^2_{L_2(U; W^{-1,2})} = \sum_{k \in \mathbb{N}} \|g_k(x, \varrho, f, \mathbf{m})\|^2_{W^{-1,2}} \lesssim \int_{\mathbb{T}^3} \sum_{k \in \mathbb{N}} \varrho^2 \|g_k(x, \varrho, f, \mathbf{m})\|^2 \, dx$$

$$\lesssim \sum_{k \in \mathbb{N}} \int_{\mathbb{T}^3} \varrho^2 (\varrho^2 + |\mathbf{m}|^2) \, dx + \int_{\mathbb{T}^3} \sum_{k \in \mathbb{N}} \varrho^2 f^2 \, dx \lesssim \int_{\mathbb{T}^3} (1 + \varrho^\gamma + \varrho|\mathbf{u}|^2 + f^{\frac{2}{\gamma-1}}) \, dx$$

(2.9)

provided that $l \geq \frac{3}{2}$. Boundedness thus follow if $\varrho^\gamma, f^{\frac{2}{\gamma-1}}$ and $\varrho|\mathbf{u}|^2$ are integrable in $\mathbb{T}^3$.

2.3. Concept of a solution. To continue, let us define the notions of solution that we wish to construct in this paper.

Definition 2.1. Let $\Lambda = \Lambda(\varrho, \mathbf{m}, f)$ be a Borel probability measure on $[L_2^3]$. We say that

$$[[\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}]; \varrho, \mathbf{u}, V, W]$$

(2.10)

is a dissipate martingale solution of (1.1)–(1.3) with initial law $\Lambda$ provided

1. $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
2. $W$ is a $(\mathcal{F}_t)$-cylindrical Wiener process;
(3) the density \( \rho \in C_w([0,T]; L^2_\gamma) \) \( \mathbb{P} \)-a.s., it is \((\mathcal{F}_t)\)-progressively measurable and
\[
\mathbb{E} \left[ \sup_{t \in (0,T)} \|\rho(t)\|_{L^p_\gamma}^p \right] < \infty
\]
for all \( 1 \leq p < \infty \) where \( \gamma = \min\{2, \gamma\} \),
(4) the velocity field \( u \) is an \((\mathcal{F}_t)\)-adapted random distribution and
\[
\mathbb{E} \left[ \int_0^T \|u\|_{W^{1,2}}^p \, dt \right] < \infty
\]
for all \( 1 \leq p < \infty \),
(5) the field \( \nabla V \in C_w([0,T]; L^2_\gamma) \) \( \mathbb{P} \)-a.s., it is \((\mathcal{F}_t)\)-progressively measurable and
\[
\mathbb{E} \left[ \sup_{t \in (0,T)} \|\nabla V(t)\|_{L^p_\gamma}^p \right] + \mathbb{E} \left[ \sup_{t \in (0,T)} \|\nabla V(t)\|_{L^p_\gamma}^p \right] < \infty
\]
for all \( 1 \leq p < \infty \) where \( \gamma = \min\{2, \gamma\} \),
(6) the momentum \( u \rho \in C_w([0,T]; L^2_\gamma) \) \( \mathbb{P} \)-a.s., it is \((\mathcal{F}_t)\)-progressively measurable and
\[
\mathbb{E} \left[ \sup_{t \in (0,T)} \|\rho u(t)\|_{L^p_\gamma}^p \right] + \mathbb{E} \left[ \sup_{t \in (0,T)} \|\rho u(t)\|_{L^p_\gamma}^p \right] < \infty
\]
for all \( 1 \leq p < \infty \),
(7) there exists \( \mathcal{F}_0 \)-measurable random variables \((\rho_0, \rho_0 u_0) = (\rho(0), \rho(0))\) such that \( \Lambda = \mathbb{P} \circ \tau_0^{-1} \),
(8) for all \( \psi \in C^\infty_c([0,T)) \) and \( \phi \in C^\infty(T^3) \), the following
\[
- \int_0^T \partial_t \psi \int_{T^3} \rho(t) \phi \, dx \, dt = \psi(0) \int_{T^3} \rho \phi \, dx + \int_0^T \psi \int_{T^3} \rho u \cdot \nabla \phi \, dx \, dt
\]
hold \( \mathbb{P} \)-a.s.;
(9) for all \( \psi \in C^\infty_c([0,T)) \) and \( \phi \in C^\infty(T^3) \), the following
\[
- \int_0^T \partial_t \psi \int_{T^3} \rho u(t) \cdot \phi \, dx \, dt = \psi(0) \int_{T^3} \rho \phi \, dx + \int_0^T \psi \int_{T^3} \rho u \cdot \nabla \phi \, dx \, dt - \nu^S \int_0^T \psi \int_{T^3} \nabla u \cdot \nabla \phi \, dx \, dt + (\nu^B + \nu^S) \int_0^T \psi \int_{T^3} \text{div} u \, \text{div} \phi \, dx \, dt
\]
hold \( \mathbb{P} \)-a.s.;
(10) equation (1.13) holds \( \mathbb{P} \)-a.s. for a.e. \((t,x) \in (0,T) \times T^3\);
(11) the energy inequality
\[
- \int_0^T \psi \int_{T^3} \left[ \frac{1}{2} \rho |u|^2 + P(\rho) \pm \rho \nabla V^2 \right] \, dx \, dt + (\nu^B + \nu^S) \int_0^T \psi \int_{T^3} \text{div} u^2 \, dx \, dt
\]
\[
+ \nu^S \int_0^T \psi \int_{T^3} |\nabla u|^2 \, dx \, dt \leq \psi(0) \int_{T^3} \left[ \frac{1}{2} \rho_0 |u_0|^2 + P(\rho_0) \pm \rho \nabla V_0^2 \right] \, dx
\]
holds \( \mathbb{P} \)-a.s. for all \( \psi \in C^\infty_c([0,T)) \), \( \psi \geq 0 \) where
\[
P(\rho) = \frac{1}{\gamma - 1} \rho \gamma \]
(12) and (1.1) holds in the renormalized sense, i.e., for any $\phi \in C_c^\infty([0, T) \times \mathbb{T}^3)$ and $b \in C^1_b(R)$ such that $b'(z) = 0$ for all $z \geq M_0$, we have that
\begin{equation}
- \int_0^T \int_{\mathbb{T}^3} b(\phi) \partial_t \phi \, dx \, dt = \int_{\mathbb{T}^3} b(\phi(0)) \phi(0) \, dx \\
+ \int_0^T \int_{\mathbb{T}^3} [b(\phi)u] \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathbb{T}^3} [(b'(\phi) - b(\phi)) \text{div} u] \phi \, dx \, dt
\end{equation}
holds $\mathbb{P}$-a.s.

2.4. Main result. We now state the first main result of this work.

**Theorem 2.2.** Let $\Lambda(\varrho, \mathbf{m}, f)$ be a Borel probability measure on $[L^1_x]^3$ such that
\begin{equation}
\Lambda\left\{ \varrho \geq 0, \ M \leq \varrho \leq M^{-1}, \ f \leq \mathcal{J}, \ \mathbf{m} = 0 \quad \text{when} \ \varrho = 0 \right\} = 1,
\end{equation}
holds for some deterministic constants $M, \mathcal{J} > 0$. Also, assume that
\begin{equation}
\int_{[L^1_x]^3} \left| \int_{\mathbb{T}^3} \left[ \frac{\mathbf{m}^2}{2\varrho} + P(\varrho) \pm \varrho |\nabla V|^2 \right] \, dx \right|^p \, d\Lambda(\varrho, \mathbf{m}, f) \lesssim 1
\end{equation}
holds for some $p \geq 1$ and that
\begin{equation}
f \in L^1_x \cap L^\infty_x \quad \mathbb{P}\text{-a.s.}
\end{equation}
Finally assume that (2.3)–(2.6) are satisfied. Then the exists a dissipative martingale solution of (1.1)–(1.3) in the sense of Definition 2.7.

3. Formal derivation of the energy inequality

Let us first recall the following Itô lemma. See [1] Theorem A.4.1 for the stronger version of this result.

**Theorem 3.1.** Let $W$ be an $(\mathcal{F}_t)$-cylindrical Wiener process on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Let $(r, s)$ be a pair of stochastic processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying
\begin{equation}
dx = [Dr] \, dt + [D\mathcal{V}] \, dW, \quad ds = [Ds] \, dt + [D\mathcal{S}] \, dW
\end{equation}
on the cylinder $(0, T) \times \mathbb{T}^d$. Now suppose that the following
\begin{equation}
r \in C^\infty([0, T] \times \mathbb{T}^d), \quad s \in C^\infty([0, T] \times \mathbb{T}^d)
\end{equation}
holds $\mathbb{P}$-a.s. and that for all $1 \leq q < \infty$
\begin{equation}
\mathbb{E} \left[ \sup_{t \in [0, T]} \|r\|_{W^{1,q}_x}^q \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} \|s\|_{W^{1,q}_x}^q \right] \lesssim_q 1.
\end{equation}
Furthermore, assume that $[Dr], [Ds], [D\mathcal{V}], [D\mathcal{S}]$ are progressively measurable and that
\begin{equation}
[Dr], [Ds] \in L^q(\Omega; L^q(0, T; W^{1,q}_x))
\end{equation}
\begin{equation}
[D\mathcal{V}], [D\mathcal{S}] \in L^2(\Omega; L^2(0, T; L_2(\Omega; L^2_x)))
\end{equation}
and
\begin{equation}
\left( \sum_{k \in \mathbb{N}} [Dr(e_k)]^q \right)^{\frac{1}{q}}, \left( \sum_{k \in \mathbb{N}} [Ds(e_k)]^q \right)^{\frac{1}{q}} \in L^q(\Omega \times (0, T) \times \mathbb{T}^d)
\end{equation}
holds. Finally, for some $\lambda \geq 0$, let $Q$ be $(\lambda + 2)$-continuously differentiable function such that
\begin{equation}
\mathbb{E} \sup_{t \in [0, T]} \|Q'(r)\|_{W^{1,q'}_{x, \varrho} \cap C_x}^2 < \infty, \quad j = 0, 1, 2.
\end{equation}
Then
\[
\int_{T^d} (sQ(r))(t) \, dx = \int_{T^d} s_0 Q(r_0) \, dx + \int_0^t \int_{T^d} \left[ sQ'(r) \, [Dr] + \frac{1}{2} \sum_{k \in \mathbb{N}} s Q''(r) \, ||D^r(e_k)||^2 \right] \, dx \, dt' \\
+ \int_0^t \int_{T^d} sQ(r) |Ds| \, dx \, dt' + \sum_{k \in \mathbb{N}} \int_0^t \int_{T^d} [Ds] |D^r(e_k)| |D^r(e_k)| \, dx \, dt' \\
+ \sum_{k \in \mathbb{N}} \int_0^t \int_{T^d} \left[ sQ'(r) |D^r(e_k)| + Q(r)|Ds| \right] \, dx \, d\beta_k(t').
\]  

(3.7)

In the following argument, we assume that all the unknowns are sufficiently regular and that no vacuum state exists. Subsequently, we rewrite our system as follows
\[
d\varrho + \text{div}(\varrho \mathbf{u}) \, dt = 0, \quad (3.8) \\
d\mathbf{u} + \left[ \mathbf{u} \cdot \nabla \mathbf{u} + \varrho^{-1} \nabla p(\varrho) \right] \, dt = \left[ \nu^S \varrho^{-1} \Delta \mathbf{u} + (\nu^B + \nu^S) \varrho^{-1} \nabla \text{div} \mathbf{u} \right. \\
+ \varrho \nabla \varphi \right] \, dt + \sum_{k \in \mathbb{N}} g_k(x, \varrho, f, \varrho \mathbf{u}) \, d\beta_k, \quad (3.9) \\
\pm \Delta \varphi = \varrho - f. \quad (3.10)
\]

For \( s = \varrho \) and \( Q(r) = \frac{1}{2} |\mathbf{u}|^2 \), applying Itô’s formula, Theorem 3.1 to the functional
\[
F(\varrho, \mathbf{u})(t) = \int_{T^d} \frac{1}{2} \varrho(t) |\mathbf{u}(t)|^2 \, dx
\]

yields
\[
F(\varrho, \mathbf{u})(t) = \int_{T^d} \frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 \, dx - \int_0^t \int_{T^d} \left[ \varrho \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \nabla \varrho \right] \, dx \, ds \\
+ \int_0^t \int_{T^d} \left[ \nu^S \mathbf{u} \cdot \Delta \mathbf{u} + (\nu^B + \nu^S) \mathbf{u} \cdot \nabla \text{div} \mathbf{u} + \varrho \mathbf{u} \cdot \nabla \varphi \right] \, dx \, ds \\
- \int_0^t \int_{T^d} \frac{1}{2} |\mathbf{u}|^2 \, d\varrho(\mathbf{u}) \, dx \, ds + \sum_{k \in \mathbb{N}} \int_0^t \int_{T^d} \frac{1}{2} \varrho |g_k(x, \varrho, f, \varrho \mathbf{u})|^2 \, dx \, ds \\
+ \sum_{k \in \mathbb{N}} \int_0^t \int_{T^d} \varrho \mathbf{u} \cdot g_k(x, \varrho, f, \varrho \mathbf{u}) \, dx \, d\beta_k.
\]  

(3.12)

Now since \( \varrho \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \varrho |\mathbf{u} \cdot \nabla |^2 \), it cancels with the last non-noise term above after integrating by parts. For the pressure term, we use the following elementary identities
\[
\frac{a \nabla \varrho}{\varrho} = a \varrho \gamma^{-2} \nabla \varrho = \frac{a \gamma}{\gamma - 1} \nabla \varrho^{\gamma-1}
\]

so that the use of the continuity equation yields
\[
\int_0^t \int_{T^d} \mathbf{u} \nabla p \, dx \, ds = - \int_0^t \int_{T^d} \text{div}(\varrho \mathbf{u}) \frac{a \gamma}{\gamma - 1} \varrho^{\gamma-1} \, dx \, ds = \int_0^t \int_{T^d} \varrho \partial_s P(\varrho) \, dx \, ds.
\]

(3.13)

Recall (2.14). Finally, by integrating by parts in the \( V \)-term, we can substitute in the following continuity–Poisson equation
\[
-\text{div}(\varrho \mathbf{u}) \, ds = d(\varrho - f) = d(\pm \Delta \varphi)
\]

(3.14)
which holds because $f$ is independent of time. By collecting the above arguments, we obtain
\[
F(\varrho, \mathbf{u})(t) + \int_{T^3} P(\varrho(t)) \, dx \pm \int_{T^3} \varrho |\nabla V(t)|^2 \, dx = \int_{T^3} \frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 \, dx + \int_{T^3} P(\varrho_0) \, dx
\]
\[
\pm \int_{T^3} \varrho |\nabla V(t)|^2 \, dx - \int_{0}^{t} \int_{T^3} \left[ \nu^S |\nabla \mathbf{u}|^2 + (\nu^B + \nu^S)|\text{div} \mathbf{u}|^2 \right] \, dx \, ds
\]
\[
+ \sum_{k \in \mathbb{N}} \int_{0}^{t} \int_{T^3} \frac{1}{2} \varrho |\mathbf{g}_k(x, \varrho, \mathbf{f}, \mathbf{u})|^2 \, dx \, ds + \sum_{k \in \mathbb{N}} \int_{0}^{t} \int_{T^3} \varrho \mathbf{u} \cdot \mathbf{g}_k(x, \varrho, \mathbf{f}, \mathbf{u}) \, dx \, d\beta_k.
\]
Alternative to (3.14), one can use the combination of integration by parts and the continuity equation so that
\[
\int_{0}^{t} \int_{T^3} \varrho \mathbf{u} \nabla V \, dx \, ds = - \int_{0}^{t} \int_{T^3} \text{div}(\varrho \mathbf{u}) V \, dx \, ds = - \int_{0}^{t} \int_{T^3} \varrho \partial_s V \, dx \, ds
\]
holds. The equation (10.10) is therefore equivalent to
\[
F(\varrho, \mathbf{u})(t) + \int_{T^3} P(\varrho(t)) \, dx \mp \int_{T^3} \varrho \partial_t G(t, x) \, dx = \int_{T^3} \frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 \, dx + \int_{T^3} P(\varrho_0) \, dx
\]
\[
\mp \int_{T^3} \varrho_0 \partial_t G(t, x)|_{t=0} \, dx - \int_{0}^{t} \int_{T^3} \left[ \nu^S |\nabla \mathbf{u}|^2 + (\nu^B + \nu^S)|\text{div} \mathbf{u}|^2 \right] \, dx \, ds
\]
\[
+ \sum_{k \in \mathbb{N}} \int_{0}^{t} \int_{T^3} \frac{1}{2} \varrho |\mathbf{g}_k(x, \varrho, \mathbf{f}, \mathbf{u})|^2 \, dx \, ds + \sum_{k \in \mathbb{N}} \int_{0}^{t} \int_{T^3} \varrho \mathbf{u} \cdot \mathbf{g}_k(x, \varrho, \mathbf{f}, \mathbf{u}) \, dx \, d\beta_k
\]
where $G(t, x)$ is the Green’s function of the Poisson equation. For example, in $\mathbb{R}^3$, we can use the Green’s function
\[
G(s, x) = \mp c \int_{\mathbb{R}^3} \frac{(\varrho - f)(s, y)}{|x - y|} \, dy
\]
(3.17)
where $c = c(\pi)$, which solves the Poisson equation so that
\[
\varrho \partial_s G(s, x) = \mp c \int_{\mathbb{R}^3} \frac{\partial_s (\varrho^2 - 2f)(s, y)}{|x - y|} \, dy = \mp \partial_s \frac{c}{2} \int_{\mathbb{R}^3} \frac{(\varrho - 2f)(s, y)}{|x - y|} \, dy = \mp \partial_s \frac{c}{2} \rho \left( (\varrho - 2f) * x \frac{1}{|x|} \right).
\]
It follows that
\[
\int_{0}^{t} \int_{T^3} \varrho \mathbf{u} \nabla V \, dx \, ds = - \int_{0}^{t} \int_{T^3} \frac{c}{2} \varrho \left( (\varrho - 2f) * x \frac{1}{|x|} \right) \, dx \, (t) \, dt
\]
\[
+ \int_{0}^{t} \frac{c}{2} \varrho_0 \left( (\varrho_0 - 2f) * x \frac{1}{|x|} \right) \, dx.
\]
(3.18)
It may be useful to treat the pressure differently when one wants to study singular limits. To do so, we rewrite the continuity equation (3.8) as
\[
d(\varrho - \overline{\varrho}) + \text{div}(\varrho \mathbf{u}) \, dt = 0
\]
(3.19)
and also replace the pressure term in (3.9) by
\[
\varrho^{-1} \nabla [p(\varrho) - p(\overline{\varrho})]
\]
where $\overline{\varrho} \geq 0$, so that if we define
\[
\overline{\mathbf{F}}(\varrho) = \varrho \int_{\overline{\varrho}}^{\varrho} \frac{p(z)}{z^2} \, dz
\]
as the new pressure potential, we have that
\[
\overline{\mathbf{F}}(\overline{\varrho}) = 0, \quad p(\varrho) = \varrho \overline{\mathbf{F}}(\varrho) - \overline{\mathbf{F}}(\varrho).
\]
Furthermore, by applying Gauss’s theorem to (3.19) in order to calculate the flux through $T^3$, then we able to obtain the mass conservation relation from which we gain

$$\frac{d}{dt} \int_{T^3} P'(\varrho)(\varrho - \varrho) \, dx = 0.$$ 

By collecting the above information, we obtain the following versions of the energy inequality

$$F(\varrho, u)(t) + \int_{T^3} H(\varrho(t), \varrho) \, dx \pm \int_{T^3} \vartheta|\nabla V(t)|^2 \, dx = \int_{T^3} \frac{1}{2} \varrho_0|u_0|^2 \, dx + \int_{T^3} H(\varrho_0, \varrho) \, dx$$

$$\pm \int_{T^3} \vartheta|\nabla \varrho_0|^2 \, dx - \int_{0}^{t} \int_{T^3} [\nu^S|\nabla u|^2 + (\nu^B + \nu^S)|\text{div} u|^2] \, dx \, ds$$

$$+ \sum_{k \in \mathbb{N}} \int_{T^3} \frac{1}{2} \vartheta|g_k(x, \varrho, f, \varrho u)|^2 \, dx \, ds + \sum_{k \in \mathbb{N}} \int_{T^3} \varrho u \cdot g_k(x, \varrho, f, \varrho u) \, dx \, d\beta_k$$

and

$$F(\varrho, u)(t) + \int_{T^3} H(\varrho(t), \varrho) \, dx \mp \int_{T^3} \varrho_0 \partial_t G(t, x) \, dx = \int_{T^3} \frac{1}{2} \varrho_0|u_0|^2 \, dx + \int_{T^3} H(\varrho_0, \varrho) \, dx$$

$$\mp \int_{T^3} \varrho_0 \partial_t G(t, x)|_{t=0} \, dx - \int_{0}^{t} \int_{T^3} [\nu^S|\nabla u|^2 + (\nu^B + \nu^S)|\text{div} u|^2] \, dx \, ds$$

$$+ \sum_{k \in \mathbb{N}} \int_{T^3} \frac{1}{2} \vartheta|g_k(x, \varrho, f, \varrho u)|^2 \, dx \, ds + \sum_{k \in \mathbb{N}} \int_{T^3} \varrho u \cdot g_k(x, \varrho, f, \varrho u) \, dx \, d\beta_k$$

where

$$H(\varrho, \varrho) = P'(\varrho) - P'(\varrho)(\varrho - \varrho) - P(\varrho).$$

4. The first approximation layer

Let $H_N$ be a finite dimensional space with an associated $L^2$-orthonormal projection

$$\Pi_N : L^2(T^2) \to H_N.$$ 

Then for any $k \in \mathbb{N}$, $p \in (1, \infty)$ and $v \in W^{k,p}(T^3)$, it follows that

$$\|\Pi_N v\|_{W^{k,p}(T^3)} \lesssim_{k,p} \|v\|_{W^{k,p}(T^3)},$$

$$\Pi_N v \to v \quad \text{in} \quad W^{k,p}(T^3)$$

as $N \to \infty$, c.f. [2] Chapter 3. We now consider the following cut–off

$$\chi \in C^\infty(\mathbb{R}), \quad \chi(f) = \begin{cases} 1 & \text{if } v \leq 0 \\ \chi'(v) & \text{if } 0 < v < 1 \\ \chi(v) & \text{if } v \geq 1 \end{cases}$$

and define the following

$$\chi_{u_R} = \chi(\|u\|_{H_N} - R),$$

$$g_{k,\varepsilon}(x, \varrho, f, \varrho u) = \chi\left(\frac{\varrho - 1}{\varrho} \chi\left(\frac{|u| - \frac{1}{\varrho}}{\varepsilon} \right)g_k(x, \varrho, f, \varrho u).$$

From the definition of the cut–off function (1.2), it follows from (2.5) (2.6) that

$$\text{ess sup}_{(x, \varrho, f, m)} \left(|g_{k,\varepsilon}| + |\nabla \varepsilon, m g_{k,\varepsilon}|\right) \leq c_{k,\varepsilon},$$

$$\sum_{k \in \mathbb{N}} c_{k,\varepsilon}^2 \lesssim 1.$$
holds for some constants \((c_k, \epsilon) \in \mathbb{N} \subset [0, \infty)\). The aim of this section is to now construct a solution to the following auxiliary problem

\[
\begin{align*}
\mathbb{1} + \text{div}(\mathbb{1}(\chi_{u \in \mathbb{R}} u)) \, dt &= \varepsilon \Delta \mathbb{1} \, dt, \\
\mathbb{1} \Pi_N (\mathbb{1}) + \Pi_N [\text{div}(\mathbb{1}(\chi_{u \in \mathbb{R}} u) \otimes u) + \chi_{u \in \mathbb{R}} \nabla p_\mathbb{1}^\mathbb{1}(\mathbb{1})] \, dt &= \Pi_N [\varepsilon \Delta (\mathbb{1}) u + \nu^S \Delta u] \\
+ (\nu^R + \nu^S) \nabla \text{div} u + \varrho \theta \nabla V \, dt + \sum_{k \in \mathbb{N}} \Pi_N [\mathbb{1} \Pi N \mathbb{1}(\mathbb{1}, f, \mathbb{1})] \, d\beta_k,
\end{align*}
\]

(4.7)

\[
\pm \Delta V = \mathbb{1} - f,
\]

(4.9)

where

\[
p_\mathbb{1}^\mathbb{1}(\mathbb{1}) = \mathbb{1}(\mathbb{1}) + \delta(\mathbb{1} + \mathbb{1}^\mathbb{1})
\]

(4.10)

for some \(\delta > 0\) and \(\Gamma \geq \max\{6, \gamma\}\).

**Definition 4.1.** Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a stochastic basis where the filtration is complete and right-continuous. Let \(W\) be an \((\mathcal{F}_t)\)-cylindrical Wiener process and further let \((\varrho_0, u_0, f)\) be \(\mathcal{F}_0\)-measurable random variables belonging to \(C^{2+\nu}(\mathbb{T}^3) \times H_N \times C^\nu(\mathbb{T}^3)\). Then \([\mathbb{1}, u, V]\) is a pathwise solution of (4.7) with data \((\varrho_0, u_0, f)\) if

1. the density \(\varrho > 0\), \(\varrho \in C([0, \mathbb{T}]; C^{2+\nu}) \) \(\mathbb{P}\)-a.s. and it is \((\mathcal{F}_t)\)-adapted;
2. the velocity field \(u \in C([0, \mathbb{T}]; H_N) \) \(\mathbb{P}\)-a.s. and it is \((\mathcal{F}_t)\)-adapted;
3. the electric field \(V \in C([0, \mathbb{T}]; C^{2+\nu}) \) \(\mathbb{P}\)-a.s. and it is \((\mathcal{F}_t)\)-adapted;
4. the following \((\varrho(0), u(0)) = (\varrho_0, u_0)\) holds \(\mathbb{P}\)-a.s.;
5. equation (4.7) holds \(\mathbb{P}\)-a.s. for a.e. \((t, x) \in (0, \mathbb{T}) \times \mathbb{T}^3\);
6. equation (4.9) holds \(\mathbb{P}\)-a.s. for a.e. \((t, x) \in (0, \mathbb{T}) \times \mathbb{T}^3\);
7. for all \(\varrho \in C^\infty([0, \mathbb{T})]\) and \(\varrho \in H_N\), the following

\[
- \int_0^T \partial_t \varrho(\mathbb{1}) dx dt + \int_0^T \varrho(\mathbb{1}) u \cdot \mathbb{1} dx dt + \int_0^T \mathbb{1}(\mathbb{1}) u \cdot \nabla \mathbb{1} dx dt
\]

\[
- \nu^R \int_0^T \mathbb{1}(\mathbb{1}) \nabla \mathbb{1} dx dt - (\nu^R + \nu^S) \int_0^T \psi \int_0^T \mathbb{1}(\mathbb{1}) u \cdot \nabla \mathbb{1} dx dt
\]

\[
+ \int_0^T \psi \int_0^T \chi_{u \in \mathbb{R}} p_\mathbb{1}^\mathbb{1}(\mathbb{1}) \nabla \mathbb{1} dx dt + \int_0^T \psi \int_0^T \varrho \Delta \mathbb{1} \mathbb{1} dx dt + \int_0^T \psi \int_0^T \varrho \nabla V \cdot \mathbb{1} \mathbb{1} dx dt
\]

\[
+ \int_0^T \psi \int_0^T \sum_{k \in \mathbb{N}} \Pi_N [\mathbb{1} \Pi N \mathbb{1}(\mathbb{1}, f, \mathbb{1})] \mathbb{1} \mathbb{1} dx dt
\]

hold \(\mathbb{P}\)-a.s.

We now state the main result of this section.

**Theorem 4.2.** Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a stochastic basis and let \(\varrho_0 \in C^{2+\nu}(\mathbb{T}^3)\) for \(\nu_0 \in (0, 1)\), \(u_0 \in H_N\), \(f \in C^\nu(\mathbb{T}^3)\) be some \(\mathcal{F}_0\)-measurable random variables such that

\[
\varrho_0 \geq 2 > 0, \quad \|\varrho_0\|_{C^{2+\nu}} \leq \mathbb{F}, \quad \|f\|_{C^\nu} \leq \mathbb{F} \quad \mathbb{P}\text{-a.s.},
\]

(4.12)

\[
\mathbb{E}\|u\|_{H_N}^p \leq \mathbb{F}
\]

(4.13)

for some \(p \geq 1\) and some deterministic constants \(2 > \mathbb{F}, \mathbb{F}, \mathbb{F} > 0\). Then there exists a unique pathwise solution \((\varrho, u, V)\) of (4.7) in the sense of Definition 4.1 such that the estimates

\[
\text{ess sup}_{t \in [0, \mathbb{T}]} \left( \|\varrho(t)\|_{C^{2+\nu}} + \|\partial_t \varrho(t)\|_{C^\nu} + \|\varrho^{-1}(t)\|_{C^\nu} \right) \leq 1, \quad \mathbb{P}\text{-a.s.}
\]

(4.14)

\[
\text{ess sup}_{t \in [0, \mathbb{T}]} \left( \|V(t)\|_{C^{2+\nu}} + \|\partial_t V(t)\|_{C^{2+\nu}} \right) \leq 1, \quad \mathbb{P}\text{-a.s.}
\]

(4.15)

\[
\mathbb{E}\text{ess sup}_{t \in [0, \mathbb{T}]} \|u(t)\|_{H_N}^p \leq 1 + \mathbb{E}\|u_0\|_{H_N}^p
\]

(4.16)

holds for some \(p \geq 1\).
The proof of Theorem 4.2 requires two main ingredients: a stochastically weak solution and pathwise uniqueness. The precise definition of the former is given below.

**Definition 4.3.** If $\Lambda$ is a Borel probability measure on $C^{2+\nu}(\mathbb{T}^3) \times H_N \times C^\nu(\mathbb{T}^3)$. We say that

$$[(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}); \varphi, \mathbf{u}, V, W]$$

is a martingale solution of (4.17) - (4.19) with initial law $\Lambda$ provided

1. $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
2. $W$ is a $(\mathcal{F}_t)$-cylindrical Wiener process;
3. the density $\varphi \in C([0, T]; C^{2+\nu})$, it is $(\mathcal{F}_t)$-adapted and $\varphi > 0$ $\mathbb{P}$-a.s.;
4. the velocity field $\mathbf{u} \in C([0, T]; H_N)$ $\mathbb{P}$-a.s. and it is $(\mathcal{F}_t)$-adapted;
5. the electric field $V \in C([0, T]; C^\nu(\mathbb{T}^3))$ $\mathbb{P}$-a.s. and it is $(\mathcal{F}_t)$-adapted;
6. there exists $\mathcal{F}_0$-measurable random variables $(\varphi_0, \mathbf{u}_0, f) = (\varphi(0), \varphi\mathbf{u}(0), 0)$ such that $\Lambda = \mathbb{P} \circ (\varphi_0, \mathbf{u}_0, f)^{-1}$;
7. we have that $\varphi(0) = \varphi_0$ $\mathbb{P}$-a.s. and holds $\mathbb{P}$-a.s. for a.e. $(t, x) \in (0, T) \times \mathbb{T}^3$;
8. equation (1.19) holds $\mathbb{P}$-a.s. for a.e. $(t, x) \in (0, T) \times \mathbb{T}^3$;
9. for all $\psi \in C^{\infty}_c((0, T))$ and $\phi \in H_N$, the following

$$-\int_0^T \partial_t \psi \int_{\mathbb{T}^3} \varphi \mathbf{u}(t) \cdot \phi \, dx \, dt = \psi(0) \int_{\mathbb{T}^3} \varphi \mathbf{u}_0 \cdot \phi \, dx + \int_0^T \psi \int_{\mathbb{T}^3} \varphi(\chi_{\mathbb{T}^3}) \otimes \mathbf{u} : \nabla \phi \, dx \, dt$$
$$- \nu^S \int_0^T \psi \int_{\mathbb{T}^3} \nabla \phi \cdot \nabla \phi \, dx \, dt - (\nu^B + \nu^S) \int_0^T \psi \int_{\mathbb{T}^3} \text{div} \mathbf{u} \text{div} \phi \, dx \, dt$$
$$+ \int_0^T \psi \int_{\mathbb{T}^3} \varphi(\mathbf{u}_x \cdot \mathbf{v}) \, dx \, dt$$
$$+ \int_0^T \psi \int_{\mathbb{T}^3} \Pi_N [\varphi \Pi_N \mathbf{g}_{k, x}(\varphi, \mathbf{f}, \mathbf{u}_x)] \cdot \phi \, dx \, dt,$$

hold $\mathbb{P}$-a.s.

The following result is reminiscent of [1] Theorem 4.1.3 and it is establishes the existence of a stochastically weak solution in the sense of Definition 4.3, with further uniform bounds on the observables.

**Proposition 4.4.** If $\Lambda$ is a Borel probability measure on $C^{2+\nu}(\mathbb{T}^3) \times H_N \times C^\nu(\mathbb{T}^3)$, $\nu \in (0, 1)$ such that

$$\Lambda \left\{ \varphi \geq \varphi > 0, \quad \|\varphi\|_{C_{2+\nu}} \leq \overline{\varphi}, \quad \|f\|_{C_{1+}} \leq \overline{f} \right\} = 1,$$

and

$$\int \|\mathbf{u}\|_{H_N}^p \, d\Lambda \leq \overline{\Pi}$$

holds for some $p \geq 1$ and some deterministic constants $\overline{\varphi}, \overline{\varphi}, \overline{f}, \overline{\Pi}$. Then there exists a martingale solution of (4.17) - (4.19) in the sense of Definition 4.3 such that the estimates

$$\text{ess sup}_{t \in [0,T]} \left( \|\varphi(t)\|_{C_{2+\nu}} + \|\partial_t \varphi(t)\|_{C_{1+}} + \|\varphi^{-1}(t)\|_{C_{1+}} \right) \lesssim 1, \quad \mathbb{P}$-a.s. \hspace{1cm} (4.21)$$
$$\text{ess sup}_{t \in [0,T]} \left( \|V(t)\|_{C_{2+\nu}} + \|\partial_t V(t)\|_{C_{2+\nu}} \right) \lesssim 1, \quad \mathbb{P}$-a.s. \hspace{1cm} (4.22)$$
$$\mathbb{E} \text{ess sup}_{t \in [0,T]} \|\mathbf{u}(t)\|_{H_N}^p + \mathbb{E} \|\mathbf{u}(t)\|_{C_{3+}(0,T;H_N)}^p \leq 1 + \mathbb{E} \|\mathbf{u}_0\|_{H_N}^p$$

holds whenever $p > 2$ and $\beta \in (0, \frac{1}{2} - \frac{1}{p})$. 

\[ \]
In order to avoid repeating the results in [1] Section 4.1.1–4.1.2.3, we only give the main ideas to solving Proposition 4.4. Just as was done in the cited book, we construct this weak solution via an iterative scheme exactly as was done in [1] Section 4.1.1–4.1.2.3. So we consider the data

\[ \varrho(t) = \varrho_0, \quad u(t) = u_0, \quad t \leq 0 \]  

(4.24)

the following invertible linear map

\[ M[\varrho] : H_N \to H_N \]  

(4.25)

satisfying

\[ M[\varrho](u) = \Pi_N(\varrho u), \quad M^{-1}[\varrho]\Pi_N[\varrho u] = u \]

for any \( u \in H_N \) in the distributional sense, that is, the following

\[ \int_{T^3} M[\varrho](u) \cdot \psi \, dx = \int_{T^3} (\varrho u) \cdot \psi \, dx \]

\[ \int_{T^3} M^{-1}[\varrho](\varrho u) \cdot \psi \, dx = \int_{T^3} u \cdot \psi \, dx \]

holds for all \( \psi \in H_N \). See [10] for further properties of this map but in particular,

\[ \left\| M^{-1}[\varrho] \right\|_{L(H_N, H_N')} \leq \left( \inf_{x \in T^3} \varrho(x) \right)^{-1} \]

and

\[ \left\| M^{-1}[\varrho_1] - M^{-1}[\varrho_2] \right\|_{L(H_N, H_N')} \leq ||\varrho_1 - \varrho_2||_{L^1} \]  

(4.26)

holds for any \( \varrho_1, \varrho_2 \in \mathbb{R}_{>0} \). Then for integers \( n \in \{0, [h^{-1}T]\} \), we examine the following system

\[ \partial_t \varrho + \text{div}(\varrho(\chi_{u_R(nh)} u(nh))) = \varepsilon \Delta \varrho(t), \quad t \in [nh, (n + 1)h) \]  

(4.27)

\[ d\Pi_N(\varrho u) + \Pi_N \left( \text{div}(\partial_t(\chi_{u_R(nh)} u(nh)) \otimes u(nh)) + \chi_{u_{R}(nh)} \nabla p_5(\varrho(t)) \right) \, dt \]

\[ = \Pi_N \left[ \varepsilon \Delta(\varrho(t) u(nh)) + \nu^S \Delta u(nh) + \nu^B + \nu^S \right] \nabla \text{div} u(nh) \]

\[ + \varrho(t) \nabla V(nh) ] \, dt + \sum_{k \in \mathbb{N}} \Pi_N \left[ \varrho(t) \Pi_N g_{k, \varepsilon}(\varrho(nh), f(nh), (\varrho u)(nh)) \right] 
\]

(4.28)

\[ \pm \Delta V = \varrho(t) - f(t), \quad t \in [nh, (n + 1)h) \]  

(4.29)

for \( t \in [nh, (n + 1)h) \) where

\[ \varrho(nh) = \varrho(nh-) := \lim_{s \downarrow nh} \varrho(s), \quad u(nh) = u(nh-) := \lim_{s \downarrow nh} u(s). \]

Since \( u(nh) \in H_N \) is frozen in time and smooth in space, by Proposition 4.1.1 Equation (4.27) has a unique classical solution for any initial data \( \varrho(nh) \) and by [11.1], this solution is strictly positive so long as the initial data \( \varrho(nh) \) is. This unique solution of (4.27), uniquely define a solution \( V \) of (4.29) since \( f \in C^0_{\varepsilon} \) is a given.

Now since the additional terms \( V \) and \( f \) in (4.28) are frozen in time, the analysis in [1] Section 4.1.1 holds true in our case. In particular, by rewriting (4.28) as follows

\[ u(t) = M^{-1}[\varrho(t)](\varrho u)(nh) - M^{-1}[\varrho(t)] \int_{nh}^t \Pi_N \chi_{u_R(nh)} \nabla p_5(\varrho(s)) \, ds \]

\[ \quad - M^{-1}[\varrho(t)] \int_{nh}^t \Pi_N \left[ \text{div}(\varrho(s)(\chi_{u_R(nh)} u(nh)) \otimes u(nh)) \right] \, ds \]

\[ + M^{-1}[\varrho(t)] \int_{nh}^t \Pi_N \left[ \varepsilon \Delta(\varrho(s) u(nh)) + \nu^S \Delta u(nh) + \nu^B + \nu^S \right] \nabla \text{div} u(nh) \]

\[ + \varrho(s) \nabla V(nh) \] \, ds + M^{-1}[\varrho(t)] \int_{nh}^t \sum_{k \in \mathbb{N}} \Pi_N \left[ \varrho(t) \Pi_N g_{k, \varepsilon}(\varrho(nh), f(nh), (\varrho u)(nh)) \right] 
\]

(4.30)
then given that there exist a unique solutions to \((4.27)\) and \((4.29)\), we obtain a unique solution for \((4.30)\) having the data \((4.24)\) and indeed any initial data \(\varphi(nh)\). We can therefore deduce that \((4.24)\) and \((4.27)\) uniquely generates solutions \(\varphi, u\) and \(V\) which are \((\mathcal{F}_t)\)-progressively measurable and for any \(n \in \mathbb{Z}_{>0}\), the following holds \(\mathbb{P}\)-a.s. It goes without saying but since \(V\) solves an elliptic equation, there is natural gain of two spatial derivatives for \(V\) given the regularity of \(f\). Now using the equivalence of norms in the finite-dimensional space \(H_N\) and the fact that \(\varphi_0 \in C^{2+\nu}(\mathbb{T}^3)\), we deduce from \((4.27)\) that

\[
\partial_t \varphi \in C([0,T]; C^\nu_x)
\]

holds \(\mathbb{P}\)-a.s. Hölder continuity in time of \(u\) follows the same argument as in [1] Section 4.1.2.3. Finally, having gained Hölder regularity for the density and its time derivative and given that \(f \in C^\nu(\mathbb{T}^3)\), standard regularity theorem for the Poisson equation, see for instant [11, Chapters 2 and 4], yields \((4.22)\). In order words, given \(f, V\) is uniquely determined by the density.

4.1. Compactness. In order to explore compactness, we denote by \((\varphi_h, u_h, V_h, W)\), the unique solution of \((4.24) - (4.29)\) given in the summary above and define the following spaces

\[
\chi_\varphi = C^{\nu_0}([0,T]; C^{2+\nu}_x), \quad \chi_u = C^{\beta_0}([0,T]; H_N), \quad \chi_V = C^{\nu_0}([0,T]; C^{2+\nu}_x), \quad \chi_W = C([0,T]; \Omega_0),
\]

where \(\nu_0 \in (0, \nu)\) and \(\beta_0 \in (0, \beta)\). Here \(\nu\) and \(\beta\) are the Hölder exponents in Proposition 4.4. We now let \(\mu_{\varphi_h}, \mu_{u_h}, \mu_{V_h}\) and \(\mu_W\) be the respective laws of \(u_h, \varphi_h, V_h\) and \(W\) on the respective spaces \(\chi_\varphi, \chi_u, \chi_V\) and \(\chi_W\). Furthermore, we set \(\mu_h\) as the joint law of \(\varphi_h, u_h, V_h\) and \(W\) on the space \(\chi = \chi_u \times \chi_\varphi \times \chi_V \times \chi_W\).

Lemma 4.5. The set \(\{\mu_h : h \in (0,1)\}\) is tight on \(\chi\).

Proof. To proof the above lemma, we first note the following:

- From \((4.14)\), the set \(\{\mu_{\varphi_h} : h \in (0,1)\}\) is tight on \(\chi_\varphi\).
- From \((4.10)\), the set \(\{\mu_{u_h} : h \in (0,1)\}\) is tight on \(\chi_u\).
- The set \(\{\mu_W\}\) is tight on \(\chi_W\) since its a Radon measure on a Polish space.

For further details, please refer to [1] Proposition 4.1.5.]

For \(V\), we note from \((4.22)\) that since \(V \in W^{1,\infty}(0,T; C^{2+\nu}_x)\), it has a Lipschitz continuous representation (not relabelled) so in particular, it follows from \((4.27)\) that

\[
V \in C^\nu([0,T]; C^{2+\nu}_x) \cap W^{1,\infty}(0,T; C^{2+\nu}_x)
\]

holds \(\mathbb{P}\)-a.s. It therefore follow from the compact embedding

\[
C^\nu([0,T]; C^{2+\nu}_x) \cap W^{1,\infty}(0,T; C^{2+\nu}_x) \hookrightarrow C^{\nu_0}([0,T]; C^{2+\nu}_x)
\]

where \(\nu_0 \in (0, \nu)\) that the set

\[
A_L := \left\{ V \in C^\nu([0,T]; C^{2+\nu}_x) \cap W^{1,\infty}(0,T; C^{2+\nu}_x) \right\}
\]

\[
: \|V(t)\|_{C^\nu([0,T]; C^{2+\nu}_x)} + \|V(t)\|_{W^{1,\infty}(0,T; C^{2+\nu}_x)} \leq L
\]

is relatively compact in \(\chi_V\). Finally, by using Chebyshev’s inequality, we deduce that the measure of the complement of the set above

\[
\mu_{V_h} \left( A_L^C \right) \leq \frac{1}{L} \mathbb{E} \left( \|V(t)\|_{C^\nu([0,T]; C^{2+\nu}_x)} + \|V(t)\|_{W^{1,\infty}(0,T; C^{2}_x)} \right) = 0
\]

as \(L \to \infty\). It follows that \(\{\mu_{V_h} : h \in (0,1)\}\) is tight on \(\chi_V\). \(\square\)

Now since \(\chi\) is a Polish space, we may use the classical Prokhorovs and Skorokhods theorems to obtain the following result.
Lemma 4.6. The exists a subsequence (not relabelled) \( \{ \mu_h : h \in (0, 1) \} \), a complete probability space \((\Omega, \mathcal{F}, P)\) with \( \chi \)-valued random variables

\[
(\tilde{\mu}_h, \tilde{\nu}_h, \tilde{V}_h, \tilde{W}_h) \text{ and } (\tilde{\sigma}, \tilde{\nu}, \tilde{V}, \tilde{W}) , \quad h \in (0, 1)
\]
such that

- the law of \((\tilde{\mu}_h, \tilde{\nu}_h, \tilde{V}_h, \tilde{W}_h)\) on \( \chi \) is \( \mu_h \), \( h \in (0, 1) \),
- the law of \((\tilde{\sigma}, \tilde{\nu}, \tilde{V}, \tilde{W})\) on \( \chi \) is a Radon measure,
- the following convergence
  \[
  \tilde{\mu}_h \to \tilde{\sigma} \quad \text{in } \chi, \\
  \tilde{V}_h \to \tilde{V} \quad \text{in } \chi, \\
  \tilde{W}_h \to \tilde{W} \quad \text{in } \chi
  \]
holds \( \tilde{P} \)-a.s.

We have therefore constructed stochastic processes \((\tilde{\sigma}, \tilde{\nu}, \tilde{V}, \tilde{W})\) which are progressively measurable with respect to the following complete right-continuous canonical filtration

\[
\tilde{\mathcal{F}}_t := \sigma \left( \sigma_t[\tilde{\sigma}], \sigma_t[\tilde{\nu}], \sigma_t[\tilde{V}], \bigcup_{k \in \mathbb{N}} \sigma_t[\tilde{\nu}_k] \right), \quad t \in [0, T].
\]
Furthermore, by [1] Lemma 2.1.35, Corollary 2.1.36], the stochastic process \( \tilde{W} = \sum_{k \in \mathbb{N}} \tilde{\nu}_k(t) e_k \) is an \((\tilde{\mathcal{F}}_t)\)-cylindrical Wiener process.

4.2. Identification of the limit. We now show that the limit random variables derived from Lemma 4.6 satisfies (4.8)–(4.9).

Lemma 4.7. The equation (4.7) is satisfied for \((\tilde{\sigma}, \tilde{\nu}, \tilde{V}, \tilde{W})\) \( \tilde{P} \)-a.s. for a.e. \((t, x) \in (0, T) \times \mathbb{T}^3 \). In addition, (4.8) is satisfied for \((\tilde{\sigma}, \tilde{\nu}, \tilde{V}, \tilde{W})\) for all \( \psi \in C^\infty_c ([0, T]) \) and \( \phi \in H_{\mathbb{P}} \) \( \tilde{P} \)-a.s.

Proof. For the proof of the first part of the above lemma, see [1] Lemma 4.17. For the second part, we follow the argument of [1] Proposition 4.18. First of all, by using the equality of laws given by Lemma 4.6, we get that the estimates

\[
\| \tilde{\nu}_h(\cdot) - \tilde{\nu}(\cdot) \|_{H_N} \lesssim h^{\beta} \| \tilde{\nu}_h \|_{C^\beta([0, T]; H_N)} \quad (4.31)
\]
and

\[
\| \tilde{\nu}_h(\cdot) - \tilde{\nu}(\cdot) \|_{C^{2+\nu}} \lesssim h^{\nu} \| \tilde{\nu}_h \|_{C^{\nu}([0, T]; C^{2+\nu})} \quad (4.32)
\]
holds. In addition,

\[
\| \tilde{V}(\cdot) - \tilde{V}(\cdot) - \tilde{H}(\cdot) \|_{C^{2+\nu}} \lesssim h^{\nu} \| \tilde{V}_h \|_{C^{\nu}([0, T]; C^{2+\nu})} \quad (4.33)
\]
The information above is enough to pass to the limit \( h \to 0 \) in the ‘deterministic’ part of the corresponding momentum equation (4.28) defined on the stochastic basis \((\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{P})\).

For the noise term, we combine (4.31)–(4.33) with Proposition 4.4 and the continuity properties of the operator \( \Pi_N \) and the coefficients \( g_{k, \epsilon} \) to get that for all \( p \in (1, \infty) \), the convergence

\[
\Pi_N \left[ \tilde{\nu} \Pi_N g_{k, \epsilon}(\tilde{\nu}_h(\cdot), f(\cdot), (\tilde{\nu}_h \tilde{\nu}_h)(\cdot)) \right] \to \Pi_N \left[ \tilde{\nu} \Pi_N g_{k, \epsilon}(\tilde{\nu}, f, \tilde{\nu}) \right] \quad (4.34)
\]
holds \( \tilde{P} \)-a.s. in \( L^p((0, T) \times \mathbb{T}^3) \) for any \( k \in \mathbb{N} \). Furthermore, by using the estimates (2.4)–(2.6) (which holds on the new probability space because of the equality of laws established by Lemma 4.6) together with Itô’s isometry, (4.11) and (4.14), we also gain

\[
\tilde{E} \left[ \left\| \int_0^T \Pi_N \left[ \tilde{\nu}_h \Pi_N G_{\epsilon}(\tilde{\nu}_h(\cdot), f(\cdot), (\tilde{\nu}_h \tilde{\nu}_h)(\cdot)) \right] \mathrm{d}\tilde{W}_h \right\|_{L^2(\mathbb{T}^2)}^2 \right] \lesssim 1 \quad (4.35)
\]
just as in the proof of [1] Proposition 4.18. Consequently, in combination with the strong convergence of \( \tilde{W}_h \) in Lemma 4.6, we obtain the following convergence

\[
\Pi_N \left[ \tilde{\nu}_h \Pi_N G_{\epsilon}(\tilde{\nu}_h(\cdot), f(\cdot), (\tilde{\nu}_h \tilde{\nu}_h)(\cdot)) \right] \to \Pi_N \left[ \tilde{\nu} \Pi_N G_{\epsilon}(\tilde{\nu}, f, \tilde{\nu}) \right] \quad (4.36)
\]
\[ \hat{\mathbb{P}}\text{-a.s. in } L^2(0, T; L_2(\mathbb{U}; L_2^2)) \] where
\[ G_\varepsilon(\varrho, f, \varrho u) = \chi\left(\frac{\varepsilon}{\varrho} - 1\right)\chi\left(|\varrho| - \frac{1}{\varepsilon}\right)G(\varrho, f, \varrho u). \]

This finishes the proof. \(\square\)

Finally, the following lemma identifies the Poisson equation.

**Lemma 4.8.** The equation (4.9) is satisfied for \((\hat{\varrho}, \hat{V}, f)\) \(\mathbb{P}\)\(-a.s. for a.e. \((t, x) \in (0, T) \times \mathbb{T}^3\).

**Proof.** As \(f \in C^\infty\) is given, by the equality of laws, i.e. Lemma 4.6, \((\hat{\varrho}, \hat{V})\) satisfies (4.29) as well as the estimates (4.21)–(4.22) on the new probability space. Given that (4.32) holds, we are able to pass to the limit and we get that (4.39) is satisfied for \((\hat{\varrho}, \hat{V}, f)\) \(\mathbb{P}\)\(-a.s. for a.e. \((t, x) \in (0, T) \times \mathbb{T}^3\).

Combining the results we have established above in Section 4.2 completes the proof of Proposition 4.3. In order to solve Theorem 4.2, we require uniqueness.

4.3. **Pathwise uniqueness.** The pathwise uniqueness of the martingale solution constructed in Proposition 4.3 is given in the following result.

**Proposition 4.9.** Consider two martingale solutions
\[ [(\Omega, \mathcal{F}, \langle \mathcal{F}_t \rangle_{t \geq 0}; \varrho_1, \varrho_1, V_1, W) \text{ and } [(\Omega, \mathcal{F}, \langle \mathcal{F}_t \rangle_{t \geq 0}; \varrho_2, \varrho_2, V_2, W] \]
of (4.7)–(4.9) is the sense of Definition 4.3 sharing a data
\[ (\varrho_0, \varrho_0, f) \in C^{2+\nu}(\mathbb{T}^3) \times H_N \times C^\nu(\mathbb{T}^3) \mathbb{P}\)-a.s.
and with both satisfying (4.21)–(4.29). Then
\[ (\varrho_1, \varrho_1, V_1) = (\varrho_2, \varrho_2, V_2) \in C([0, T]; C^{2+\nu}(\mathbb{T}^3) \times H_N \times C^\nu(\mathbb{T}^3)) \mathbb{P}\)-a.s.

**Proof.** The proof of Proposition 4.9 follow the ideas of [1] Proposition 4.1.9, i.e., for \(i = 1, 2\), we introduce the stopping times
\[ \tau_n^i := \inf_{t \in [0, T]} \|\varrho_i(t)\|_{C^{2+\nu}} + \|\varrho_i(t)\|_{L^\infty} + \|\varrho_i(t)\|_{H_N} > n \]
where \(\inf \emptyset = T\) and set \(\tau_n = \tau_n^1 \wedge \tau_n^2\). Note that the collection \((\tau_n)_{n \in \mathbb{N}}\) is increasing and that \(\tau_n \to T\) almost surely by virtue of (4.21)–(4.22).

We now consider the following difference equations
\[ \partial_t (\varrho_1 - \varrho_2) + \text{div} (\varrho_1 (\chi_{u_1, R} - \varrho_2 (\chi_{u_2, R}))) = \varepsilon \Delta (\varrho_1 - \varrho_2), \]
\[ \pm \Delta (V_1 - V_2) = (\varrho_1 - \varrho_2) \]
of two solutions to (4.7) and (4.9) where as in (4.2)–(4.3),
\[ \chi_{u_1, R} = \chi(\|u_1\|_{H_N} - R). \]

By using (4.19), we recall from [1] (4.43)–(4.44) that the estimate
\[ \sup_{\nu \in [0, \tau]} \left[\|\varrho(t)\|_{C^{2+\nu}} + \|\varrho(t)\|_{C^{2+\nu}}\right] \lesssim \sup_{\nu \in [0, \tau]} \|\varrho_1(t) - \varrho_2(t)\|_{H_N} \]
\[ + \left(\int_0^\tau \|\varrho(t)\|_{C^{2+\nu}} dt\right)^{\frac{1}{2}} + \|\varrho(0)\|_{C^{2+\nu}} \]
follow from (4.37). For \(0 \leq l\), we can therefore infer from (4.33) that estimate
\[ \sup_{\nu \in [0, \tau]} \left[\|\nabla^l (V_1 - V_2(t))\|_{C^{2+\nu}} \lesssim \sup_{\nu \in [0, \tau]} \|\Delta (V_1 - V_2(t))\|_{C^{2+\nu}} \lesssim \sup_{\nu \in [0, \tau]} \|\varrho_1(t) - \varrho_2(t)\|_{H_N} \]
\[ + \left(\int_0^\tau \|\varrho(t)\|_{C^{2+\nu}} dt\right)^{\frac{1}{2}} + \|\varrho(0)\|_{C^{2+\nu}} \]

(4.41)
holds true. Also, by using \((4.25)\)–\((4.26)\) and the continuity equation \((4.7)\), we rewrite the momentum equation \((4.8)\) as

\[
d\mathbf{u} + \mathcal{M}^{-1}[\varrho] \Pi_N [\mathbf{u} \partial_t \varrho] + \mathcal{M}^{-1}[\varrho] \Pi_N \text{div} (\varrho \chi_{u_N} \otimes \mathbf{u}) \, dt + \mathcal{M}^{-1}[\varrho] \Pi_N \chi_{u_N} \nabla p^\Gamma_{R} (\varrho) \, dt \\
= \mathcal{M}^{-1}[\varrho] \Pi_N [\varepsilon \Delta (\varrho \mathbf{u}) + \nu^S \Delta \mathbf{u} + (\nu^B + \nu^S) \nabla \text{div} \mathbf{u}] \, dt + \partial \nabla \mathbf{V} \, dt \\
+ \sum_{k \in \mathbb{N}} \Pi_N g_{k,\varepsilon} (\varrho, f, \varrho \mathbf{u}) \, d\beta_k. \tag{4.42}
\]

It follows that \((\mathbf{u}_1 - \mathbf{u}_2)\) satisfies

\[
d(\mathbf{u}_1 - \mathbf{u}_2) = (\mathcal{M}^{-1}[\varrho_2] - \mathcal{M}^{-1}[\varrho_1]) \Pi_N [\mathbf{u}_1 \partial_t \varrho_1] \, dt + \mathcal{M}^{-1}[\varrho_2] \Pi_N [\mathbf{u}_2 \partial_t \varrho_2 - \mathbf{u}_1 \partial_t \varrho_1] \, dt \\
- \mathcal{M}^{-1}[\varrho_1] \Pi_N \left[ \text{div} (\varrho_1 \chi_{u_{N,1}} \otimes \mathbf{u}_1) + \chi_{u_{N,1}} \nabla p^\Gamma_{R} (\varrho_1) \right] \, dt \\
+ \mathcal{M}^{-1}[\varrho_2] \Pi_N \left[ \text{div} (\varrho_2 \chi_{u_{N,2}} \otimes \mathbf{u}_2) + \chi_{u_{N,2}} \nabla p^\Gamma_{R} (\varrho_2) \right] \, dt \\
+ (\mathcal{M}^{-1}[\varrho_1] - \mathcal{M}^{-1}[\varrho_2]) \Pi_N \left[ \varepsilon \Delta (\varrho_1 \mathbf{u}_1) + \nu^S \Delta \mathbf{u}_1 + (\nu^B + \nu^S) \nabla \text{div} \mathbf{u}_1 \right] \, dt \\
+ \mathcal{M}^{-1}[\varrho_2] \Pi_N \left[ \varepsilon \Delta (\varrho_1 \mathbf{u}_1 - \mathbf{u}_2) + \nu^S \Delta (\mathbf{u}_1 - \mathbf{u}_2) + (\nu^B + \nu^S) \nabla \text{div} (\mathbf{u}_1 - \mathbf{u}_2) \right] \, dt \\
+ \partial \nabla (V_1 - V_2) \, dt + \sum_{k \in \mathbb{N}} \Pi_N \left[ g_{k,\varepsilon} (\varrho_1, f, \varrho_1 \mathbf{u}_1) - g_{k,\varepsilon} (\varrho_2, f, \varrho_2 \mathbf{u}_2) \right] \, d\beta_k. \tag{4.43}
\]

By applying Itô’s product rule to \((4.43)\), we obtain

\[
d \int_{\mathbb{T}^3} \frac{1}{2} \| \mathbf{u}_1 - \mathbf{u}_2 \|^2 \, dx = \int_{\mathbb{T}^3} (\mathcal{M}^{-1}[\varrho_2] - \mathcal{M}^{-1}[\varrho_1]) \Pi_N [\mathbf{u}_1 \partial_t \varrho_1] \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, dx \, dt \\
+ \int_{\mathbb{T}^3} \mathcal{M}^{-1}[\varrho_2] \Pi_N [\mathbf{u}_2 \partial_t \varrho_2 - \mathbf{u}_1 \partial_t \varrho_1] \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, dx \, dt \\
- \int_{\mathbb{T}^3} \mathcal{M}^{-1}[\varrho_1] \Pi_N \left[ \text{div} (\varrho_1 \chi_{u_{N,1}} \otimes \mathbf{u}_1) + \chi_{u_{N,1}} \nabla p^\Gamma_{R} (\varrho_1) \right] (\mathbf{u}_1 - \mathbf{u}_2) \, dx \, dt \\
+ \int_{\mathbb{T}^3} \mathcal{M}^{-1}[\varrho_2] \Pi_N \left[ \text{div} (\varrho_2 \chi_{u_{N,2}} \otimes \mathbf{u}_2) + \chi_{u_{N,2}} \nabla p^\Gamma_{R} (\varrho_2) \right] (\mathbf{u}_1 - \mathbf{u}_2) \, dx \, dt \\
+ \int_{\mathbb{T}^3} (\mathcal{M}^{-1}[\varrho_1] - \mathcal{M}^{-1}[\varrho_2]) \Pi_N \left[ \varepsilon \Delta (\varrho_1 \mathbf{u}_1) + \nu^S \Delta \mathbf{u}_1 + (\nu^B + \nu^S) \nabla \text{div} \mathbf{u}_1 \right] (\mathbf{u}_1 - \mathbf{u}_2) \, dx \, dt \\
+ \int_{\mathbb{T}^3} \mathcal{M}^{-1}[\varrho_2] \Pi_N \left[ \varepsilon \Delta (\varrho_1 \mathbf{u}_1 - \mathbf{u}_2) + \nu^S \Delta (\mathbf{u}_1 - \mathbf{u}_2) \right] \, dx \, dt \\
+ (\nu^B + \nu^S) \nabla \text{div} (\mathbf{u}_1 - \mathbf{u}_2) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, dx \, dt + \partial \int_{\mathbb{T}^3} \nabla (V_1 - V_2) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, dx \, dt \\
+ \frac{1}{2} \sum_{k \in \mathbb{N}} \int_{\mathbb{T}^3} \Pi_N \left[ g_{k,\varepsilon} (\varrho_1, f, \varrho_1 \mathbf{u}_1) - g_{k,\varepsilon} (\varrho_2, f, \varrho_2 \mathbf{u}_2) \right] \, dx \, dt \\
+ \sum_{k \in \mathbb{N}} \int_{\mathbb{T}^3} \Pi_N \left[ g_{k,\varepsilon} (\varrho_1, f, \varrho_1 \mathbf{u}_1) - g_{k,\varepsilon} (\varrho_2, f, \varrho_2 \mathbf{u}_2) \right] \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, dx \, d\beta_k. \tag{4.44}
\]

Now just as in \((11)\) \((4.40)\)–\((4.42)\), by using the definition of the stopping time \(\tau_{\varepsilon, \varphi}\) \((4.26)\) \((4.25)\)–\((4.24)\), as well as the Burkholder–Davis–Gundy inequality to tackle the noise term, it follows from \((4.44)\) that for any \(\kappa > 0\), the estimate

\[
E \sup_{t' \in [0, T]} \| (\mathbf{u}_1 - \mathbf{u}_2)(t' \wedge \tau_{\varepsilon, \varphi}) \|^2_{\mathcal{H}^N} \leq \kappa E \sup_{t' \in [0, T]} \| (\varrho_1 - \varrho_2)(t') \|^2_{L^2_{x,t} + \nu^N} \\
+ c_{n,\varepsilon} E \int_0^{T \wedge \tau_{\varepsilon, \varphi}} (\| \mathbf{u}_1 - \mathbf{u}_2 \|^2_{\mathcal{H}^N} + \| \varrho_1 - \varrho_2 \|^2_{L^2_{x,t} + \nu^N} + \| \nabla (V_1 - V_2) \|^2_{L^2_{x,t}}) \, dt + E \| (\mathbf{u}_1 - \mathbf{u}_2)(0) \|^2_{\mathcal{H}^N}. \tag{4.45}
\]
holds true. For \( \tau = T \land \tau_n \), we can combine (4.40)–(4.41) with (4.45) and we get

\[
E \sup_{t' \in [0,T]} \left( \|(u_1 - u_2)(t' \land \tau_n)\|_{H^N}^2 + \|(\varrho_1 - \varrho_2)(t')\|_{C_t^2}^2 + \sum_{i=0}^{\infty} \|\nabla^i (V_1 - V_2)(t')\|_{C_t^2}^2 \right)
\]

\[
\leq E \int_0^{T \land \tau_n} \left( \|(u_1 - u_2)\|_{H^N}^2 + \|(\varrho_1 - \varrho_2)\|_{C_t^2}^2 + \|\nabla (V_1 - V_2)\|_{C_t^2}^2 \right) dt
\]

\[
+ E \|(\varrho_1 - \varrho_2)(0)\|_{C_t^2} + E \|(u_1 - u_2)(0)\|_{H^N}^2.
\]

Gronwall’s lemma, the fact that \( u_1|_{t=0} = u_2|_{t=0} = u_0 \) and \( \varrho_1|_{t=0} = \varrho_2|_{t=0} = \varrho_0 \) finishes the proof.

\[\Box\]

### 4.4. Conclusion

Pathwise uniqueness as shown in Section 4.3 and martingale solution given by Proposition 4.13 combines to give the existence of a pathwise solution Theorem 4.2. This relies on Gyöngy–Krylov characterization of convergence given in [13, Theorem 2.10.3]. In order to show that the hypothesis of the Gyöngy–Krylov result are satisfied, we refer the reader to the short proof of [11, Theorem 4.1.12].

**Remark 4.10.** As in [1, Corollary 4.1.13], we can relax the assumption on the data \((\varrho_0 u_0, f)\) in Theorem 4.2. In particular, this can be replaced by any \(\mathcal{F}_0\)-measurable random variables \((\varrho_0 u_0, f)\) satisfying

\[\varrho_0 \geq \varrho > 0, \quad \varrho_0 \in C_x^{2+\nu}, \quad f \in C_x^\nu, \quad u_0 \in H_N\]

\[\mathbb{P}\text{-a.s.}\]

### 4.5. Energy equality

We now show that the solution constructed in Theorem 4.2 in the sense of Definition 4.1 satisfies an energy equality. This is given the following result.

**Proposition 4.11.** Let \((\varrho, u, V)\) be a pathwise solution of (4.7)–(4.9) in the sense of Definition 4.1. Then the energy equality

\[\begin{align*}
- \int_0^T \partial_t \psi & \int_{T^3} \left[ \frac{1}{2} \varrho |u|^2 + P^\varrho_\mu (\varrho) \pm \varrho |\nabla V|^2 \right] dx dt + \nu^S \int_0^T \psi \int_{T^3} \nabla u^2 dx \, dt \\
+ (\nu^B + \nu^S) \int_0^T \psi \int_{T^3} |\div u|^2 dx \, dt + \varepsilon \int_0^T \psi \int_{T^3} \varrho |\nabla u|^2 dx \, dt \\
+ \varepsilon \int_0^T \psi \int_{T^3} (P^\varrho_\mu)''(\varrho) |\nabla \varrho|^2 dx \, dt = \psi(0) \int_{T^3} \left[ \frac{1}{2} \varrho_0 |u_0|^2 + P^\varrho_\mu (\varrho_0) \pm \int_{T^3} \varrho |\nabla V_0|^2 \right] dx \\
+ \frac{1}{2} \int_0^T \psi \int_{T^3} \varrho \left( \sum_{k \in \mathbb{N}} [\Pi_N g_{k,\varepsilon}(x, \varrho, f, m)]^2 \right) dx \, dt + \int_0^T \psi \int_{T^3} \varrho u \cdot \Pi_N G_\varepsilon (\varrho, f, m) dx \, dW
\end{align*}\]

holds \(\mathbb{P}\text{-a.s.}\) for all \(\varrho \in C_x^\infty ([0, T])\).

In order to show (4.47), we mimic the formal argument of Section 4.3, i.e., we consider

\[d\varrho + \div (\varrho (\chi_{\varrho > 0} u)) \, dt = \varepsilon \Delta \varrho \, dt, \tag{4.48}\]

\[d\Pi_N u + \Pi_N [\varrho^{-1} \Delta \varrho \cdot u + \chi_{\varrho > 0} u \cdot \nabla \varrho] = \varrho^{-1} \chi_{\varrho > 0} \nabla P^\varrho_\mu (\varrho) \, dt = \Pi_N [\varrho \varrho^{-1} \Delta (\varrho u) + \nu^S \varrho^{-1} \Delta u \div \varrho \nabla V] \, dt + \sum_{k \in \mathbb{N}} \Pi_N g_{k,\varepsilon}(x, \varrho, f, \varrho u) \, d\beta_k, \tag{4.49}\]

\[\pm \Delta V = \varrho - f. \tag{4.50}\]

and apply Itô’s lemma, Theorem 4.1, to the functional

\[F(\varrho, u)(t) = \int_{T^3} \frac{1}{2} \varrho(t) |u(t)|^2 \, dx\]
and we obtain

\[
\int_{\mathbb{T}^3} \frac{1}{2} \rho(t)|u(t)|^2 \, dx = \int_{\mathbb{T}^3} \frac{1}{2} \rho_0|u_0|^2 \, dx - \int_0^t \int_{\mathbb{T}^3} [\varepsilon |u|^2 \Delta \rho + \rho u \cdot \chi_{u_R} u \cdot \nabla u \\
+ \varepsilon \chi_{u_R} \nabla \rho^T] \, dx \, ds + \int_0^t \int_{\mathbb{T}^3} [\varepsilon u \cdot (\rho \rho u) + \nu^S u \cdot \Delta u \\
+ (\nu^B + \nu^S) u \cdot \nabla \nabla u + \rho |u| \nabla V] \, dx \, ds - \int_0^t \int_{\mathbb{T}^3} \frac{1}{2} |u|^2 \left[ \operatorname{div}(\rho \chi_{u_R} u) - \varepsilon \Delta \rho \right] \, dx \, ds \\
+ \sum_{k \in \mathbb{N}} \int_0^t \int_{\mathbb{T}^3} \frac{1}{2} \rho N \chi_{k, \epsilon}(x, \rho, f, \rho \rho u)^2 \, dx \, ds + \sum_{k \in \mathbb{N}} \int_0^t \int_{\mathbb{T}^3} \rho u \cdot N \chi_{k, \epsilon}(x, \rho, f, \rho \rho u) \, dx \, d\beta_k.
\]

\[P^\circ \text{-a.s.} \]

Now if we multiply (4.48) by \( P^\circ_3(\rho)' \), then we get the following renormalized continuity equation

\[
\int_{\mathbb{T}^3} P^\circ_3(\rho(t)) \, dx = \int_{\mathbb{T}^3} P^\circ_3(\rho_0) \, dx + \int_0^t \int_{\mathbb{T}^3} [\varepsilon P^\circ_3(\rho)' \Delta \rho - \operatorname{div}(P^\circ_3(\rho) \chi_{u_R} u) \\
- \left[ P^\circ_3(\rho)' \rho - P^\circ_3(\rho) \right] \chi_{u_R} u] \, dx \, ds.
\]

Recall that by Definition 4.48 is satisfied pointwise almost everywhere. The proof is done once we use the identities

\[P^\circ_3(\rho) = P^\circ_3(\rho)' \rho - P^\circ_3(\rho)\]

and

\[
\int_{\mathbb{T}^3} \varepsilon P^\circ_3(\rho)' \Delta \rho \, dx = - \int_{\mathbb{T}^3} \varepsilon P^\circ_3(\rho)' \, \nabla |\rho|^2 \, dx
\]

together with the divergence theorem in (4.56) and substitute the resulting equation into (4.55).

5. The second approximation layer

We now wish to construct a class of solution to the following system

\[
d_0 + \operatorname{div}(\rho u) \, dt = \varepsilon \Delta \rho \, dt,
\]

\[
d N(\rho u) + N \left[ \operatorname{div}(\rho u) \otimes u + \nabla P^\circ_3(\rho) \right] \, dt = N \left[ \varepsilon \Delta (\rho u) + \nu^S \Delta u + (\nu^B + \nu^S) \nabla \nabla u + \phi \nabla V \right] \, dt + \sum_{k \in \mathbb{N}} N \left[ \rho \rho N \chi_{k, \epsilon}(\rho, f, \rho \rho u) \right] \, d\beta_k,
\]

\[\pm \Delta V = \rho - f,\]
which conserves energy by passing to the limit $R \to \infty$ in (4.7)-(4.8). We now give the precise definition of this solution.

**Definition 5.1.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a stochastic basis where the filtration is complete and right-continuous. Let $W$ be an $(\mathcal{F}_t)$-cylindrical Wiener process and further let $(\varrho_0, u_0, f)$ be $\mathcal{F}_0$-measurable random variables belonging to $C^{2+\nu}(\mathbb{T}^3) \times H_N \times C^{\nu}(\mathbb{T}^3)$. Then $[\varrho, u, V]$ is a pathwise solution of (5.1) with data $(\varrho_0, u_0, f)$ if

1. the density $\varrho > 0$, $\varrho \in C([0, T]; C^{2+\nu}_x)$, $\partial_t \varrho \in C([0, T]; C^{\nu}_x)$ $\mathbb{P}$-a.s. and it is $(\mathcal{F}_t)$-progressively measurable;
2. the velocity field $u \in C([0, T]; H_N)$ $\mathbb{P}$-a.s. and it is $(\mathcal{F}_t)$-progressively measurable;
3. the electric field $V \in C([0, T]; C^{2+\nu}_x)$ $\mathbb{P}$-a.s. and it is $(\mathcal{F}_t)$-progressively measurable;
4. the following $(\varrho(0), u(0)) = (\varrho_0, u_0)$ holds $\mathbb{P}$-a.s.;
5. equation (5.3) holds $\mathbb{P}$-a.s. for a.e. $(t, x) \in (0, T) \times \mathbb{T}^3$;
6. equation (5.3) holds $\mathbb{P}$-a.s. for a.e. $(t, x) \in (0, T) \times \mathbb{T}^3$;
7. for all $\psi \in C^\infty_c((0, T))$ and $\phi \in H_N$, the following

$$- \int_0^T \partial_t \psi \int_{\mathbb{T}^3} \varrho u(t) \cdot \phi \, dx \, dt = \psi(0) \int_{\mathbb{T}^3} \varrho_0 u_0 \cdot \phi \, dx + \int_0^T \psi \int_{\mathbb{T}^3} \varrho \otimes u : \nabla \phi \, dx \, dt$$
$$- \nu^S \int_0^T \psi \int_{\mathbb{T}^3} \nabla u : \nabla \phi \, dx \, dt - (\nu^B + \nu^S) \int_0^T \psi \int_{\mathbb{T}^3} \text{div} \, u \, \text{div} \phi \, dx \, dt$$
$$+ \int_0^T \psi \int_{\mathbb{T}^3} P^T_0 \, \text{div} \phi \, dx \, dt + \int_0^T \psi \int_{\mathbb{T}^3} \varepsilon \varrho u \Delta \phi \, dx \, dt$$
$$+ \int_0^T \psi \int_{\mathbb{T}^3} \partial_x \nabla V \cdot \phi \, dx \, dt + \int_0^T \psi \sum_{k \in \mathbb{N}} \int_{\mathbb{T}^3} \Pi_N \{\varrho \Pi_N g_{k, \varepsilon}(\varrho, f, \varrho u)\} \cdot \phi \, dx \beta_k \quad \mathbb{P}\text{-a.s.}$$

hold $\mathbb{P}$-a.s.;
8. the energy equality

$$- \int_0^T \partial_t \psi \int_{\mathbb{T}^3} \left[ \frac{1}{2} \varrho |u|^2 + P^T_0(\varrho) \pm \varrho |\nabla V|^2 \right] \, dx \, dt + \nu^S \int_0^T \psi \int_{\mathbb{T}^3} |\nabla u|^2 \, dx \, dt$$
$$+ (\nu^B + \nu^S) \int_0^T \psi \int_{\mathbb{T}^3} |\text{div} \, u|^2 \, dx \, dt + \varepsilon \int_0^T \psi \int_{\mathbb{T}^3} \varrho |\nabla u|^2 \, dx \, dt$$
$$+ \varepsilon \int_0^T \psi \int_{\mathbb{T}^3} (P^T_0)^\nu(\varrho) |\nabla \varrho|^2 \, dx = \psi(0) \int_{\mathbb{T}^3} \left[ \frac{1}{2} \varrho_0 |u_0|^2 + P^T_0(\varrho_0) \pm \varrho |\nabla V_0|^2 \right] \, dx$$
$$+ \frac{1}{2} \int_0^T \psi \int_{\mathbb{T}^3} \sum_{k \in \mathbb{N}} |\Pi_N g_{k, \varepsilon}(\varrho, f, m)|^2 \, dx \, dt + \int_0^T \psi \int_{\mathbb{T}^3} \varrho u \cdot \Pi_N G_{\varepsilon}(\varrho, f, m) \, dx \, dW;$$

hold $\mathbb{P}$-a.s. for all $\psi \in C^\infty_c((0, T))$.

We now state the main result of this section.

**Theorem 5.2.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. Let $W$ be an $(\mathcal{F}_t)$-cylindrical Wiener process and let $\varrho_0 \in C^{2+\nu}_x$ for $\nu \in (0, 1)$, $u_0 \in H_N$, $f \in C^{\nu}_x$ be some $\mathbb{P}$-a.s. $\mathcal{F}_0$-measurable random variables such that

$$\varpi \geq \varrho_0 \geq \underline{\varrho} > 0 \quad (5.6)$$

holds for some deterministic constants $\varpi$ and $\underline{\varrho}$. Finally, assume that the following moment estimate

$$\mathbb{E} \left[ \int_{\mathbb{T}^3} \varrho_0 |u_0|^2 + P^T_0(\varrho_0) \pm \varrho |\nabla V_0|^2 \right] \leq 1 \quad (5.7)$$

holds for some $p \in (1, \infty)$. Then there exists a unique pathwise solution $[\varrho, u, V]$ of (5.1)-(5.3) in the sense of Definition 5.1.
To proof Theorem 5.2 we first require uniform estimates in $R$ since we intend to pass to the limit $R \to \infty$ in \((4.7) - (4.9)\) in order to gain \((5.1) - (5.3)\). As it turns out, besides uniform-in-$R$ estimates, these bounds are also independent of $N$. To see this, we follow the proof of [1, Proposition 4.2.3] and consider the unique pathwise solution $(\varrho, u, V)$ of \((4.7) - (4.9)\) in the sense of Definition 4.1. By using \((1.1)\) and \((4.5) - (4.6)\) combined with the continuous embedding $L^\infty_x \hookrightarrow L^r_x$, we obtain the following bound
\[
\sum_{k \in \mathbb{N}} \int_{\mathbb{T}^3} \varrho \| \Pi_N g_{k,\varepsilon}(x, \varrho, f, m) \|^2 \, dx \leq \| \varrho \|_{L^r_x} \sum_{k \in \mathbb{N}} \| g_{k,\varepsilon}(x, \varrho, f, m) \|^2_{L^\infty_x} \lesssim \| \varrho \|_{L^r_x}.
\] (5.8)
uniformly in $N$ and $R$. Similarly, for $r = \frac{2^r}{1 - 2^r}$, we gain by using \((1.1), (4.5) - (4.6)\), the embedding $L^\infty_x \hookrightarrow L^2_x$ as well as Young’s inequality
\[
\left( \int_{\mathbb{T}^3} \varrho u \cdot \Pi_N g_{k,\varepsilon}(x, \varrho, f, m) \, dx \right)^2 \leq \left( \| \sqrt{\varrho} \|_{L^2_x} \| \sqrt{\varrho} u \|_{L^2_x} \right) \left( \| \varrho \|_{L^2_x} + \| u \|_{L^2_x} \right)^2.
\] (5.9)
It therefore follow from \((5.9)\) and the Burkholder–Davis–Gundy inequality that the estimate
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left( \int_0^t \int_{\mathbb{T}^3} \varrho u \cdot \Pi_N G_{\varepsilon}(\varrho, f, m) \, dx \, dW \right)^p \right] \lesssim \mathbb{E} \left[ \int_0^T \left( \int_{\mathbb{T}^3} \varrho u \cdot \Pi_N g_{k,\varepsilon}(x, \varrho, f, m) \, dx \right)^2 \, dt \right]^{\frac{p}{2}} \lesssim_{k,\varepsilon,\Gamma} \mathbb{E} \left[ \int_0^T \left( \| \varrho \|_{L^2_x}^2 + \| u \|_{L^2_x}^2 \right) \, dt \right]^{\frac{p}{2}}
\] (5.10)
is true for any $p \in (1, \infty)$. Finally, it follows from Korn’s inequality, [1, Theorem A1.8] and Proposition 4.47 combined with \((5.8), (5.10)\) and Gronwall’s lemma that the estimate
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left( \int_{\mathbb{T}^3} \varrho^2 + P^T_0(\varrho) \pm \varrho \|
abla V\|^2 \right) \, dx \right]^{\frac{p}{2}} + \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^3} \left( \| \nabla u \|^2 + \varrho \| \nabla u \|^2 \right) \, dx \right]^{\frac{p}{2}} + \varepsilon \left( P^T_\delta V''(\varrho) \right) \, dx \right]^{\frac{p}{2}} \lesssim \mathbb{E} \left[ \int_0^T \left( \| \varrho \|_{L^2_x}^2 + P^T_\delta(\varrho) \pm \varrho \|
abla V\|^2 \right) \, dx \right]^{\frac{p}{2}}
\] (5.11)
holds uniformly in $R$ and $N$ for every $p \in (1, \infty)$. Uniform boundedness thus follow from \((5.7)\).

The completion of the proof of Theorem 5.2 will now rely on a stopping time argument. Denote the unique pathwise solution from Theorem 4.2 by $(\varrho_R, u_R, V_R)$ and assume that it has data $(\varrho_0, u_0, f)$ satisfying the assumptions in Theorem 5.2. Define the increasing family $(\tau_R)_{R > 0}$ of $(\mathcal{F}_t)$-stopping times by
\[
\tau_R := \inf_{t \in [0, T]} \left\{ \| u_R(t) \|_{H_N} > R \right\}
\]
where we set $\inf \emptyset = T$. By uniqueness, as given by Theorem 4.2 $\tau_{R_0} \leq \tau_R$ if $R_0 \leq R$ and also, $(\varrho_{R_0}, u_{R_0}, V_{R_0}) = (\varrho_R, u_R, V_R)$ solves \((5.1) - (5.3)\) on $(0, \tau_R)$ in the sense of Definition 5.1. In particular, note that the energy equality remains unchanged from Proposition 4.47 at least on $(0, \tau_R)$. Now since $(\varrho_R, u_R, V_R)$ satisfies an energy estimate uniformly in $R$, the proof is done once
\[
\mathbb{P} \left( \sup_{R} \tau_R = T \right) = 1
\] (5.12)
is shown. The proof of \((5.12)\) is exactly the same as was shown in the proof of [1, Proposition 4.2.3].
In this section, we establish a class of solutions to the following system
\begin{align}
\frac{d\varrho}{dt} + \text{div}(\varrho \mathbf{u}) dt &= \varepsilon \Delta \varrho dt, \\
\frac{d(\varrho \mathbf{u})}{dt} + \left[ \text{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p_\delta^f(\varrho) \right] dt &= \left[ \varepsilon \Delta (\varrho \mathbf{u}) + \nu^S \Delta \mathbf{u} \right. \\
&\quad+ (\nu^B + \nu^S) \nabla \text{div} \mathbf{u} + \partial \varrho \nabla V \big] dt \\
&\quad+ \sum_{k \in \mathbb{N}} \varrho g_{k,\varepsilon}(\varrho, f, \varrho \mathbf{u}) \, d\beta_k, \\
\pm \Delta V &= \varrho - f,
\end{align}
which dissipates energy by passing to the limit \( N \to \infty \) in (5.2). The precise notion of a solution is as follows.

**Definition 6.1.** If \( \Lambda \) is a Borel probability measure on \( C_\varepsilon^{2+\nu} \times L_1^1 \times C_\varepsilon^\nu \), we say that
\[ [(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}); \varrho, \mathbf{u}, V, W] \]
is a dissipative martingale solution of (6.1)–(6.3) with initial law \( \Lambda \) provided
1. \( \Omega, \mathcal{F}, (\mathcal{F}_t) \), \( \mathbb{P} \) is a stochastic basis with a complete right-continuous filtration;
2. \( W \) is a \( (\mathcal{F}_t) \)-cylindrical Wiener process;
3. the density \( \varrho \in C_\varepsilon([0, T]; C_\varepsilon^{2+\nu}) \), it is \( (\mathcal{F}_t) \)-adapted and \( \varrho > 0 \) \( \mathbb{P} \)-a.s.;
4. the velocity field \( \mathbf{u} \in L^2(0, T; W_1^1) \) \( \mathbb{P} \)-a.s. is an \( (\mathcal{F}_t) \)-adapted random distribution;
5. given \( f \in C_\varepsilon^\nu \), there exist \( \mathcal{F}_0 \)-measurable random variables \((\varrho_0, \mathbf{u}_0)\) such that \( \Lambda = \mathbb{P} \circ ((\varrho_0, \varrho_0 \mathbf{u}_0, f)^{-1};
6. we have that \( \varrho(0) = \varrho_0 \) \( \mathbb{P} \)-a.s. and (6.1) holds \( \mathbb{P} \)-a.s. for a.e. \((t, x) \in (0, T) \times \mathbb{T}^3\);
7. equation (6.3) holds \( \mathbb{P} \)-a.s. for a.e. \((t, x) \in (0, T) \times \mathbb{T}^3\);
8. for all \( \psi \in C_\varepsilon^\infty((0, T)) \) and \( \varphi \in C_\varepsilon^\infty \), the following
\begin{align}
- \int_0^T \partial_t \psi \int_{\mathbb{T}^3} \varrho \mathbf{u}(t) \cdot \varphi \, dx dt &= \psi(0) \int_{\mathbb{T}^3} \varrho_0 \mathbf{u}_0 \cdot \varphi \, dx + \int_0^T \psi \int_{\mathbb{T}^3} \varrho \mathbf{u} \cdot \nabla \varphi \, dx dt \\
- \nu^S \int_0^T \psi \int_{\mathbb{T}^3} \nabla \varphi \cdot \nabla \varphi \, dx dt - (\nu^B + \nu^S) \int_0^T \psi \int_{\mathbb{T}^3} \text{div} \mathbf{u} \cdot \nabla \varphi \, dx dt \\
+ \int_0^T \psi \int_{\mathbb{T}^3} p_\delta^f \text{div} \varphi \, dx dt + \int_0^T \int_{\mathbb{T}^3} \varrho \nabla \Delta \varphi \, dx dt \\
+ \int_0^T \psi \int_{\mathbb{T}^3} \nabla V \cdot \varphi \, dx dt + \int_0^T \psi \int_{\mathbb{T}^3} \sum_{k \in \mathbb{N}} \int_{\mathbb{T}^3} \varrho g_{k,\varepsilon}(\varrho, f, \varrho \mathbf{u}) \cdot \varphi \, dx \, d\beta_k
\end{align}
hold \( \mathbb{P} \)-a.s.;
9. the energy inequality
\begin{align}
- \int_0^T \partial_t \psi \int_{\mathbb{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + P_\delta^f(\varrho) \pm \varrho |\nabla V|^2 \right] \, dx \, dt + \nu^S \int_0^T \psi \int_{\mathbb{T}^3} |\nabla \mathbf{u}|^2 \, dx \, dt \\
+ (\nu^B + \nu^S) \int_0^T \psi \int_{\mathbb{T}^3} |\text{div} \mathbf{u}|^2 \, dx \, dt + \varepsilon \int_0^T \psi \int_{\mathbb{T}^3} \varrho |\nabla \mathbf{u}|^2 \, dx \, dt \\
+ \varepsilon \int_0^T \psi \int_{\mathbb{T}^3} (P_\delta^f)^{''}(\varrho) |\nabla \varrho|^2 \, dx \, dt \leq \psi(0) \int_{\mathbb{T}^3} \left[ \frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P_\delta^f(\varrho_0) \pm \varrho |\nabla V_0|^2 \right] \, dx \\
+ \frac{1}{2} \int_0^T \psi \int_{\mathbb{T}^3} \varrho \left( \sum_{k \in \mathbb{N}} |g_{k,\varepsilon}(x, \varrho, f, m)|^2 \right) \, dx \, dt + \int_0^T \psi \int_{\mathbb{T}^3} \varrho \mathbf{u} \cdot \mathbf{G}_{\varepsilon}(\varrho, f, m) \, dx \, dW
\end{align}
holds \( \mathbb{P} \)-a.s. for all \( \psi \in C_\varepsilon^\infty((0, T)) \), \( \psi \geq 0 \).

The main theorem of this section is the following.
Theorem 6.2. Let $\Lambda$ be a Borel probability measure on $C^{2+\nu}_x \times L^1_x \times C^\nu_\gamma$ such that
\[
\Lambda\left\{ M \leq \varrho \leq M^{-1}, \quad f \leq f \right\} = 1,
\] holds for deterministic constants $M, f > 0$. Also assume that
\[
\int_{C^{2+\nu}_x \times L^1_x \times C^\nu_\gamma} \left( \left| \int_{T^3} \left[ \frac{1}{2} \varrho |u|^2 + P_\delta^T(\varrho) \pm \varrho |\nabla V|^2 \right] \, dx \right|^p + \| \varrho \|^p_{C^{2+\nu}} + \| f \|^p_{C^\gamma} \right) \, d\Lambda(\varrho, m, f) \lesssim 1
\] holds for some $p \geq 1$. Then there exists a dissipative martingale solution of $\mathcal{E}_{1}$--(6.3) in the sense of Definition 6.7.

We now devote the rest of this section to the proof of Theorem 6.2.

6.1. Construction of law. Given $f \in C^\nu_\gamma$, consider the $(\mathcal{F}_0)$-measurable random variables $(\varrho_0, m_0)$ ranging in $C^{2+\nu}_x \times L^1_x$ so that the triplet $(\varrho_0, m_0, f)$ has the law $\Lambda$ as prescribed in the assumption of Theorem 6.2. Since $\varrho_0 > 0$, it follows that $u_0 := \frac{m_0}{\varrho_0} \in L^2_x$ $\mathbb{P}$-a.s. Furthermore, by setting $u_{0,N} := \Pi_N u_0$, it follows that $(\varrho_0, u_{0,N}, f)$ satisfies the assumptions of Theorem 6.2 and for $\Lambda_N = \mathbb{P} \circ (\varrho_0, u_{0,N}, f)^{-1}$,
\[
\int_{C^{2+\nu}_x \times L^1_x \times C^\nu_\gamma} \left( \left| \int_{T^3} \left[ \frac{1}{2} \varrho |u|^2 + P_\delta^T(\varrho) \pm \varrho |\nabla V|^2 \right] \, dx \right|^p + \| \varrho \|^p_{C^{2+\nu}} + \| f \|^p_{C^\gamma} \right) \, d\Lambda_N(\varrho, u, f) \lesssim 1
\] holds uniformly in $N$. Furthermore, we can use the bound in (4.1) to obtain
\[
\int_{C^{2+\nu}_x \times L^1_x \times C^\nu_\gamma} \left( \left| \int_{T^3} \left[ \frac{1}{2} \varrho |u|^2 + P_\delta^T(\varrho) \pm \varrho |\nabla V|^2 \right] \, dx \right|^p + \| \varrho \|^p_{C^{2+\nu}} + \| f \|^p_{C^\gamma} \right) \, d\Lambda_N(\varrho, u, f) = \mathbb{E} \left( \left| \int_{T^3} \left[ \frac{1}{2} \varrho |u_N|^2 + P_\delta^T(\varrho) \pm \varrho |\nabla V|^2 \right] \, dx \right|^p + \| \varrho \|^p_{C^{2+\nu}} + \| f \|^p_{C^\gamma} \right) \rightarrow \mathbb{E} \left( \left| \int_{T^3} \left[ \frac{1}{2} \varrho |u|^2 + P_\delta^T(\varrho) \pm \varrho |\nabla V|^2 \right] \, dx \right|^p + \| \varrho \|^p_{C^{2+\nu}} + \| f \|^p_{C^\gamma} \right) \int_{C^{2+\nu}_x \times L^1_x \times C^\nu_\gamma} \left( \left| \int_{T^3} \left[ \frac{1}{2} \varrho |u|^2 + P_\delta^T(\varrho) \pm \varrho |\nabla V|^2 \right] \, dx \right|^p + \| \varrho \|^p_{C^{2+\nu}} + \| f \|^p_{C^\gamma} \right) \, d\Lambda_N(\varrho, u, f).
\]

6.2. Uniform estimates. Now denote by $(\varrho_N, u_N, V_N)$, the pathwise solution constructed in Theorem 6.2 with data $(\varrho, u_{0,N}, f)$. Since (6.11) holds uniformly in $N$ (and clearly of $R$), the argument that the estimate (6.11) holds uniformly in $R$ and $N$ remains valid for $\varrho_N, u_N, V_N$. It follows that
\[
\mathbb{E} \sup_{t \in [0,T]} \| \varrho_N |u_N|^2 \|_{L^1_x} + \mathbb{E} \sup_{t \in [0,T]} \| \varrho_N \|_{L^p_t} + \mathbb{E} \sup_{t \in [0,T]} \| \nabla V_N \|_{L^2_t} + \mathbb{E} \| \nabla u_N \|_{L^2_t}^2 + \mathbb{E} \| \varrho_N V_N \|_{L^2_t}^2 + \mathbb{E} \| \varrho_N u_N \|_{L^2_t}^2 \lesssim_{\epsilon, \delta, \Gamma, p, \varrho_0, N} 1
\] holds uniformly in $N$ for some $q > 2$ and any $p \in (1, \infty)$ where
\[
\mathcal{E}_{0,N} := \frac{1}{2} \varrho_0 |u_{0,N}|^2 + P_\delta^T(\varrho_0) \pm \varrho |\nabla V_0|^2
\]
Lemma 6.3. The set $\chi$ is integrable for any $p$. To see this, we first note that since the given function $f \in C_c(\mathbb{R}^d)$, we now recall again that $\|f\|_{L^p(\mathbb{R}^d)}$ holds $\mu$-a.s. for some $l \in \mathbb{N}$, and $1 < r_1, r_2 < \infty$ (or for $r_1 = 1$ with $r_2 = \infty$) where $X := [W_x^{2,r_2}, W_x^{2,l,r_2}]_{1-r_2^{-1}, r_2}$ is the interpolation space, see [11] for definition. So by combining this with (6.11) and the assumption that the initial density is highly regular, we get in particular that the estimate holds uniformly in $N$ for some $q > 2$.

6.3. Compactness. To derive compactness, we first define the following spaces $N$ bounded in $\mathbb{R}$, and $1 < r_1, r_2 < \infty$ (or for $r_1 = 1$ with $r_2 = \infty$) where $X := [W_x^{2,r_2}, W_x^{2,l,r_2}]_{1-r_2^{-1}, r_2}$ is the interpolation space, see [11] for definition. So by combining this with (6.11) and the assumption that the initial density is highly regular, we get in particular that the estimate holds uniformly in $N$ for some $q > 2$.

$$\mathbb{E}\|\varrho_N\|_{L^p}^q + \mathbb{E}\|\partial_t \varrho_N\|_{L^q_{\gamma}}^q \lesssim_{e,q,p} 1$$

Lemma 6.3. The set $\{\mu_N : N \in \mathbb{N}\}$ is tight on $\chi$.

Proof. First of all, as shown in [1] Corollary 4.3.9, the following holds true.

- The law $\mu_{\varrho_0}$ is tight on $\chi_{\varrho_0}$ since its a Radon measure on a Polish space.
- The set $\{\mu_{\varrho_0,N} : N \in \mathbb{N}\}$ is tight on $\chi_{\varrho_0}$.
- The set $\{\mu_{\varrho_N} : N \in \mathbb{N}\}$ is tight on $\chi_{\varrho}$.
- The set $\{\mu_{\varrho_0,N} : N \in \mathbb{N}\}$ is tight on $\chi_{\varrho_0}$.
- The set $\{\mu_{\varrho_0} : N \in \mathbb{N}\}$ is tight on $\chi_{\varrho_0}$.
- The set $\{\mu_{\varrho_0} : N \in \mathbb{N}\}$ is tight on $\chi_{\varrho}$.

We now recall again that $[\varrho_N, u_N, V_N]$ satisfied (5.5) from which we concluded that it satisfied the estimates (6.11)–(6.13). So in particular, $\varrho_N \in L^\infty(0, T; L^2)$ and $\nabla V_N \in L^\infty(0, T; L^2)$ in moments and as such, $\varrho_N \nabla V_N \in L^\infty(0, T; L^{\frac{2d}{d-2}}_{\infty})$. Since the embedding

$$L^\infty(0, T; L^{\frac{2d}{d-2}}_{\infty}) \hookrightarrow L^r(0, T; W_x^{2-r,2})$$

is continuous for all $r \in (1, \infty)$ and for all $l \geq \frac{2d}{2d-2}$ but $\frac{2d}{2d-2} \in (0, 1)$, indeed, the aforementioned embedding holds for all $r, l > 1$. It follows that

$$\partial_t \varrho_N \nabla V_N \text{ is bounded in } L^p(\Omega; L^r(0, T; W_x^{2-r,2}))$$

for all $p, r, l \in (1, \infty)$. It follows from [1] Proposition 4.3.8 applied to (6.2) that

- The set $\{\mu_{\varrho_0} : N \in \mathbb{N}\}$ is tight on $\chi_{\varrho_0}$.

The proof of Lemma 6.3 is done once we show that

- The set $\{\mu_{\varrho_0} : N \in \mathbb{N}\}$ is tight on $\chi_{\varrho_0}$.

To see this, we first note that since the given function $f(x)$ is continuous in $\mathbb{T}^3$, it is $p$-Lebesgue integrable for any $p \in [1, \infty)$. And since the moment estimates of $\varrho_N$ in $L^\infty L^r_{\gamma}$ are uniformly bounded in $N$, recall the second summand in (6.11), we can deduce from the Poisson equation (6.3) that any moment estimate of $V_N$ in $L^\infty L^2_{\gamma}$ are uniformly bounded in $N$. Additionally, $V_N$ can inherit the following regularity of the density sequence

$$\mathbb{E}\|\varrho_N\|_{C^{0,1}_c L^{\frac{2d}{d-2}}_{\infty}} \lesssim 1$$

(6.14)
uniformly in $N$, see [1] Page 138, through (6.3) and the fact that $f(x)$ is Lebesgue integrable for any finite integrability exponent. Indeed, the spatial regularity in (6.14) can be improved for $V_N$ but this does not take anything away from the analysis. Tightness of $\mu_{V_N}$ will therefore follow from the following compact embedding

$$L^\infty(0, T; W^{2, p}(T^3)) \cap C^{0, 1}_w([0, T]; W^{-2, p}_c(T^3)) \rightarrow C_w([0, T]; W^{2, p}(T^3))$$ (6.15)

established in [19 Corollary B.2.].

We now use the Jakubowski–Skorokhod representation theorem to obtain the following result.

**Lemma 6.4.** The exists a subsequence (not relabelled) $\{\mu_N : N \in \mathbb{N}\}$, a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\chi$-valued random variables

$$\tilde{\varrho}_0, \tilde{\varrho}_0, \tilde{\varrho}, \tilde{\varrho}, \tilde{u}_n, \tilde{u}, \tilde{V}_N, \tilde{V}_N) \quad N \in \mathbb{N}$$

and

$$\tilde{\varrho}_0, \tilde{u}_0, \tilde{\varrho}, \tilde{u}, \tilde{m}, \tilde{V}, \tilde{W}$$

such that

- the law of $(\tilde{\varrho}_0, \tilde{\varrho}_0, \tilde{\varrho}_N, \tilde{\varrho}_N, \tilde{\varrho}_N, \tilde{V}_N, \tilde{V}_N)$ on $\chi$ is $\mu_N$, $N \in \mathbb{N}$,
- the law of $(\tilde{\varrho}_0, \tilde{u}_0, \tilde{\varrho}, \tilde{u}, \tilde{m}, \tilde{V}, \tilde{W})$ on $\chi$ is a Radon measure,
- the following convergence (with each $\rightarrow$ interpreted with respect to the corresponding topology)

$\tilde{\varrho}_0 \rightarrow \tilde{\varrho}$ in $\chi_{\tilde{\varrho}_0}$, \quad $\tilde{\varrho}_0 \rightarrow \tilde{\varrho}$ in $\chi_{\tilde{\varrho}_0}$, 

$\tilde{\varrho}_N \rightarrow \tilde{\varrho}$ in $\chi_{\tilde{\varrho}_N}$, \quad $\tilde{u}_N \rightarrow \tilde{u}$ in $\chi_u$, 

$\tilde{V}_N \rightarrow \tilde{V}$ in $\chi_V$, \quad $\tilde{m}_N \rightarrow \tilde{m}$ in $\chi_m$, 

$\tilde{W}_N \rightarrow \tilde{W}$ in $\chi_W$

holds $\tilde{\mathbb{P}}$-a.s.

### 6.4. Identifying the limit system.

Since our constructed velocity field $\tilde{u}$ lives in a function space endowed with a weak topology, it follows that it is a random distribution in the sense of [1] Definition 2.2.1 rather than the classical notion of a stochastic process as one would usually expects. Loosely speaking, whereas a stochastic process is defined pointwise for all times, our velocity field can only be interpreted in the PDE sense of distributions.

We also have that $\tilde{u}$ is adapted to its complete right-continuous history $(\sigma_t[\tilde{u}])_{t \geq 0}$, see comments on [1] Page 37. It follows from [1] Lemma 2.2.18 that there exist a classical stochastic process $\tilde{u}$ (not relabelled) that coincides with our constructed velocity almost always in $\tilde{\Omega} \times [0, T]$, it is $\sigma_t[\tilde{u}]$-progressively measurable and

$$\tilde{u} \in L^2(0, T; W^{1, 2}_x) \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Subsequently, we can build the following complete right-continuous filtration

$$\sigma\left(\sigma_t[\tilde{\varrho}], \sigma_t[\tilde{u}], \sigma_t[\tilde{V}_N], \bigcup_{k \in \mathbb{N}} \sigma_t[\tilde{\varrho}_N, k]\right), \quad t \in [0, T]$$

which is non-anticipative with respect to $\tilde{W}_N$. This ensures a well-defined stochastic integral, see [1] Remark 2.3.7, and by Lemma 6.4 and [1] Lemma 2.9.3, allows the passage to the limit $N \rightarrow \infty$ to obtain the following filtration

$$\tilde{\mathcal{F}}_t := \sigma\left(\sigma_t[\tilde{\varrho}], \sigma_t[\tilde{u}], \sigma_t[\tilde{V}], \bigcup_{k \in \mathbb{N}} \sigma_t[\tilde{\varrho}_k]\right), \quad t \in [0, T]$$

which is non-anticipative with respect to $\tilde{W}$. 


Lemma 6.5. The following results
\begin{align}
\bar{m} &= \hat{\bar{m}}, \\
\bar{m}_N &= \Pi_N(\hat{\bar{m}}_N), \\
\sqrt{\hat{\bar{m}}}_N \hat{\bar{u}}_N &\to \sqrt{\hat{\bar{m}}} \hat{\bar{u}} \quad \text{in } L^2([0, T) \times T^3), \tag{6.18} \\
\hat{\bar{m}}_N \otimes \hat{\bar{u}}_N &\to \hat{\bar{m}} \otimes \hat{\bar{u}} \quad \text{in } L^1([0, T) \times T^3) \tag{6.19}
\end{align}
holds \( \mathbb{P}\)-a.s.

The proof of the above lemma is shown in [1] Lemma 4.3.11 and Corollary 4.3.12]. Furthermore, due to the high enough regularity enjoyed by \([\hat{\bar{m}}, \hat{\bar{u}}]\), refer to the convergence results in Lemma 6.4, we obtain the following result.

Lemma 6.6. The random distributions \([\hat{\bar{m}}, \hat{\bar{u}}]\) satisfies \(6.1\) a.e. in \((0, T) \times T^3 \mathbb{P}\)-a.s.

Again, the high regularity enjoyed by \([\hat{\bar{V}}, \hat{\bar{V}}]\) and the fact that the Poisson equation is linear results in the following.

Lemma 6.7. The random distributions \([\hat{\bar{V}}, \hat{\bar{V}}]\) satisfies \(6.3\) a.e. in \((0, T) \times T^3 \mathbb{P}\)-a.s.

Also, by relying on [1] Theorem 2.9.1 and Lemma 6.4, we gain analogously to [1] Proposition 4.3.14, the following result.

Lemma 6.8. The random distributions \([\hat{\bar{u}}, \hat{\bar{u}}, \hat{\bar{V}}]\) satisfies \(6.5\) for all \(\psi \in C^\infty_c([0, T))\) and \(\phi \in C^\infty(T^3) \mathbb{P}\)-a.s.

Note that since \(f\) is a given, the proof of [1] Proposition 4.3.14] which emphasis convergence in the stochastic forcing term can be carried over, almost verbatim, to the proof of Lemma 6.8. Finally, note that these random variables are still regular enough to pass to the limit in the energy inequality for the sequence variables and that the noise term can be treated in the same vain as [1] Proposition 4.3.15.

Lemma 6.9. The random distributions \([\hat{\bar{u}}, \hat{\bar{u}}, \hat{\bar{V}}]\) satisfies \(6.6\) for all \(\psi \in C^\infty_c([0, T))\), \(\psi \geq 0 \mathbb{P}\)-a.s.

7. The fourth approximation layer

We devote this section to the establishment of a class of solution to the following system
\begin{align}
d\varrho + \text{div}(\varrho \mathbf{u}) \, dt &= 0, \quad \tag{7.1} \\
d(\varrho \mathbf{u}) + \left[\text{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p^\varrho_\delta(\varrho)\right] \, dt &= \left[\varrho^S \Delta \mathbf{u} + (\nu^B + \nu^S) \nabla \text{div} \mathbf{u} + \varrho \varrho \nabla \varphi\right] \, dt + \sum_{k \in \mathbb{N}} \varrho g_k(\varrho, \varrho, \varrho, \mathbf{u}) \, d\beta_k, \tag{7.2} \\
\pm \Delta \varphi &= \varrho - f, \tag{7.3}
\end{align}
which dissipates energy by passing to the limit \(\varepsilon \to 0\) in \((6.1) - (6.2)\). The precise definition of this solution is as follows.

Definition 7.1. Let \(\Lambda\) be a Borel probability measure on \(L^1_\mathbb{P}\). We say that
\[
\left[\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right); \varrho, \mathbf{u}, V, W\right]
\]
is a dissipative martingale solution of \(7.1 - 7.3\) with initial law \(\Lambda\) provided
\begin{enumerate}
  \item \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) is a stochastic basis with a complete right-continuous filtration;
  \item \(W\) is a \((\mathcal{F}_t)\)-cylindrical Wiener process;
  \item the density \(\varrho \in C_\infty([0, T]; L^2_\mathbb{P})\) is a random distribution, it is \((\mathcal{F}_t)\)-adapted and \(\varrho > 0 \mathbb{P}\)-a.s.;
  \item the velocity field \(\mathbf{u} \in L^2([0, T]; W^{1,2}_\mathbb{P}) \mathbb{P}\)-a.s. is an \((\mathcal{F}_t)\)-adapted random distribution;
  \item given \(f \in L^\infty_\mathbb{P}\), there exists \(\mathcal{F}_0\)-measurable random variables \((\varrho_0, \mathbf{u}_0)\) such that \(\Lambda = \mathbb{P} \circ (\varrho_0, \varrho_0 \mathbf{u}_0, f)^{-1}\).
\end{enumerate}
(6) for all \( \psi \in C_c^\infty((0,T)) \) and \( \phi \in C^\infty(\mathbb{T}^3) \), the following
\[
- \int_0^T \partial_t \psi \int_{\mathbb{T}^3} \phi(t) \phi \, dx \, dt = \psi(0) \int_{\mathbb{T}^3} \phi_0 \phi \, dx + \int_0^T \psi \int_{\mathbb{T}^3} \phi \, \nabla \phi \, dx \, dt,
\]
\( \mathbb{P}\)-a.s.;

(7) equation (5.3) holds \( \mathbb{P}\)-a.s. for a.e. \((t,x) \in (0,T) \times \mathbb{T}^3\);

(8) for all \( \psi \in C_c^\infty((0,T)) \) and \( \phi \in C^\infty(\mathbb{T}^3) \), the following
\[
- \int_0^T \partial_t \psi \int_{\mathbb{T}^3} \phi(t) \phi \, dx \, dt = \psi(0) \int_{\mathbb{T}^3} \phi_0 \phi \, dx + \int_0^T \psi \int_{\mathbb{T}^3} \phi \, \nabla \phi \, dx \, dt
\]
\[
- \nu^S \int_0^T \psi \int_{\mathbb{T}^3} \nabla \phi \, dx \, dt - (\nu^B + \nu^S) \int_0^T \psi \int_{\mathbb{T}^3} \text{div} \phi \, dx \, dt + \int_0^T \psi \int_{\mathbb{T}^3} p \, \text{div} \phi \, dx \, dt \quad (7.6)
\]
holds \( \mathbb{P}\)-a.s.;

(9) equation (7.1) holds in the renormalized sense, i.e., for any \( \phi \in C_c^\infty([0,T) \times \mathbb{T}^3) \) such that \( b'(z) = 0 \) for all \( z \geq M_b \), we have that
\[
- \int_0^T \int_{\mathbb{T}^3} b(\phi) \partial_t \phi \, dx \, dt = \int_{\mathbb{T}^3} b(\phi(0)) \phi(0) \, dx + \int_0^T \int_{\mathbb{T}^3} [b(\phi) \mathbf{u}] \cdot \nabla \phi \, dx \, dt
\]
\[
- \int_0^T \int_{\mathbb{T}^3} (b'(\phi) - b(\phi)) \text{div} \mathbf{u} \, dx \, dt \quad (7.7)
\]
holds \( \mathbb{P}\)-a.s.;

(10) the energy inequality
\[
- \int_0^T \partial_t \psi \int_{\mathbb{T}^3} \left[ \frac{1}{2} \phi |\mathbf{u}|^2 + P_0^T(\phi) \pm \phi |\nabla \phi|^2 \right] \, dx \, dt + \nu^S \int_0^T \psi \int_{\mathbb{T}^3} |\nabla \phi|^2 \, dx \, dt
\]
\[
+ (\nu^B + \nu^S) \int_0^T \psi \int_{\mathbb{T}^3} |\text{div} \mathbf{u}|^2 \, dx \, dt \leq \psi(0) \int_{\mathbb{T}^3} \left[ \frac{1}{2} \phi |\mathbf{u}_0|^2 + P_0^T(\phi_0) \pm \phi |\nabla \phi_0|^2 \right] \, dx
\]
\[
+ \frac{1}{2} \int_0^T \psi \int_{\mathbb{T}^3} \sum_{k \in \mathbb{N}} |\mathbf{g}_k(x, \phi, f, m)|^2 \, dx \, dt + \int_0^T \psi \int_{\mathbb{T}^3} \phi \cdot \mathbf{G}(\phi, f, m) \, dx \, d\mathcal{W};
\]
holds \( \mathbb{P}\)-a.s. for all \( \psi \in C_c^\infty([0,T)) \), \( \psi \geq 0 \).

The main result in this section is the following.

**Theorem 7.2.** Let \( \Gamma \geq 6 \) and \( \Lambda \) be a Borel probability measure on \([L^1_2]^3\) such that
\[
\Lambda \left\{ M \leq \phi \leq M^{-1} < \infty, \quad f \leq \mathcal{F} \right\} = 1,
\]
holds for deterministic constants \( M, \mathcal{F} > 0 \). Also assume that
\[
\int \left| \left| \frac{\mathbf{m}}{2\phi} \right| \right|^p \, dx \, d\Lambda(\phi, m, f) \leq 1 \quad (7.9)
\]
holds for some \( p \geq 1 \). Then the exists a dissipative martingale solution of (7.1)–(7.3) in the sense of Definition 7.7.

7.1. **Construction of law.** We now construct the law \( \Lambda \) as described in Theorem 7.2 through an approximation of laws \( \Lambda_\varepsilon \) constructed from the previous section in Theorem 6.2. In order words, given \( \Lambda_\varepsilon \) satisfying (6.7)–(6.8), we obtain (7.9)–(7.10) uniformly in \( \varepsilon \).

Now since \([L^1_2]^3\) is a Polish space, by [11] Corollary 2.6.4, there exists a stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\)
containing \( \mathcal{F}_0 \)-measurable random variables \((\varrho_0, m_0)\) so that given \( f \in L_q^\infty \), the triplet \((\varrho_0, m_0, f)\) has values in \([L_1^2]_3\) as well as having \( \Lambda \) as their law. It follows in particular that

\[ f \leq \overline{f} \]

holds \( \mathbb{P} \)-a.s. for a deterministic constant \( \overline{f} > 0 \). Also, one can find random variables \( \varrho_{0, \varepsilon} \in C^{2+\nu}_2 \) for some \( \nu \in (0, 1) \) such that

\[ M \leq \varrho_{0, \varepsilon} \leq M^{-1}, \]

hold \( \mathbb{P} \)-a.s. for a deterministic constant \( M > 0 \) and that

\[ \varrho_{0, \varepsilon} \to \varrho_0 \quad \text{in} \quad L^{p_0}(\Omega; L_1^p) \quad \forall p_0 \in [1, p \Gamma]. \]  

(7.11)

If we also set \( V_{0, \varepsilon} = V_0 = \pm \Delta_{\varepsilon}^{-1}(\varrho_0 - f) \), then we have that

\[ \mathbb{E} \left[ \int_{T^3} \varrho |\nabla V_{0, \varepsilon}|^2 \, dx \right]^{p_0} \lesssim 1 \]

uniformly in \( \delta \) for all \( p_0 \in [1, p] \) and that

\[ \nabla V_{0, \varepsilon} \to \nabla V_0 \quad \text{in} \quad L^{p_0}(\Omega; L_1^p) \quad \text{for all} \quad p_0 \in [1, 2p]. \]  

(7.12)

Furthermore, as in [17, Page 154], one can find a \( C^2_2 \)-valued random variable \( h_\varepsilon \), such that for \( m_{0, \varepsilon} = h_\varepsilon \sqrt{\varrho_{0, \varepsilon}} \), we have

\[ \frac{|m_{0, \varepsilon}|^2}{\varrho_{0, \varepsilon}} \in L^{p_0}(\Omega; L_1^1) \quad \forall p_0 \in [1, p] \]

uniformly in \( \varepsilon \) and with the help of (7.11),

\[ m_{0, \varepsilon} \to m_0 \quad \text{in} \quad L^{p_0}(\Omega; L_1^1) \quad \forall p_0 \in [1, p], \]  

(7.14)

\[ \frac{m_{0, \varepsilon}}{\sqrt{\varrho_{0, \varepsilon}}} \to \frac{m_0}{\sqrt{\varrho_0}} \quad \text{in} \quad L^{p_0}(\Omega; L_1^p) \quad \forall p_0 \in [1, 2p]. \]  

(7.15)

We now set \( \Lambda_{\varepsilon} = \mathbb{P} \circ (\varrho_{0, \varepsilon}, m_{0, \varepsilon}, f)^{-1} \) where \( f \leq \overline{f} \) holds \( \mathbb{P} \)-a.s. for a deterministic constant \( \overline{f} > 0 \) and from (7.11) and (7.14), it follows that \( \Lambda_{\varepsilon} \to \Lambda \) in the sense of measures in \([L_1^1]_3\). Note that \( f \in L^p_2 \) for all \( p \in [1, \infty] \).

7.2. Uniform estimates. From Section 7.1 it follows from Theorem 6.2 that for any \( \varepsilon \in (0, 1) \), there exists a dissipative martingale solution

\[ ([\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}); \varrho_\varepsilon, u_\varepsilon, V_\varepsilon, W] \]

of (5.1)-6.3 with law \( \Lambda_{\varepsilon} \) in the sense of Definition 4.1. The justification of the choice of a single stochastic basis and a single Wiener process for these solutions is standard, see for instance, [17, Remark 5.3.1] or [11, Remark 4.0.4].

Since the solution satisfies the energy estimate (6.6), by approximating the characteristic function \( \chi_{[0,s]} \) for any \( s \in (0, T) \), by a sequence of test functions \( \psi \in C_c^\infty([0, T]) \), it follows that

\[ \int_{T^3} \left[ \frac{1}{2} \partial_t u_\varepsilon |u_\varepsilon|^2 + P_\varepsilon^\Gamma(\varrho_\varepsilon) \pm \varrho |\nabla V_\varepsilon|^2 \right](s) \, dx + \nu S \int_0^s \int_{T^3} |\nabla u_\varepsilon|^2 \, dx \, dt \\
+ \nu B + \nu S \int_0^s \int_{T^3} |\text{div} u_\varepsilon|^2 \, dx \, dt + \nu \int_0^s \int_{T^3} \varrho_\varepsilon |\nabla u_\varepsilon|^2 \, dx \, dt \\
+ \varepsilon \int_0^s \int_{T^3} (P_\varepsilon^\Gamma)''(\varrho_\varepsilon) |\varrho_\varepsilon|^2 \, dx \, dt \leq \int_{T^3} \left[ \frac{1}{2} |\varrho_{0, \varepsilon} u_{0, \varepsilon}|^2 + P_\varepsilon^\Gamma(\varrho_{0, \varepsilon}) \pm \varrho |\nabla V_{0, \varepsilon}|^2 \right] \, dx \\
+ \frac{1}{2} \int_0^s \int_{T^3} \varrho_\varepsilon \sum_{k \in \mathbb{N}} |g_{k, \varepsilon}(x, \varrho_\varepsilon, f, m_\varepsilon)|^2 \, dx \, dt + \int_0^s \int_{T^3} \varrho_\varepsilon u_\varepsilon \cdot G_\varepsilon(\varrho_\varepsilon, f, m_\varepsilon) \, dx \, dW \]

(7.16)
holds $\mathbb{P}$-a.s. for a.e. $s \in (0, T)$. Similar to \([5.8] - [5.10]\), we can use \([2.5] - [2.6]\), the fact that the cut-off function \([4.2]\) is uniformly bounded by one to obtain the bound

$$
\mathbb{E} \left[ \sup_{s \in [0, T]} \left| \int_0^s \sum_{k \in \mathbb{N}} \int_{\mathbb{T}^3} \phi_k(x, \varepsilon, f, m_\varepsilon)^2 \, dx \, dt \right|^p \right] \lesssim_{p, k, \Gamma} \mathbb{E} \left[ \int_0^T \| \phi_\varepsilon \|^p_{L^p_s} \, dt \right] \tag{7.17}
$$

and together with the Burkholder–Davis–Gundy inequality, also the bound

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \int_{\mathbb{T}^3} \phi_\varepsilon \cdot G_\varepsilon(\varepsilon, f, m_\varepsilon) \, dx \, dW_t \right|^p \right] \lesssim_{k, \Gamma} \mathbb{E} \left[ \int_0^T \left( \| \phi_\varepsilon \|^2_{L^2_s} + \| \phi_\varepsilon \|^2_{L^2_s} \right) \, dt \right] \tag{7.18}
$$

uniformly in $\varepsilon$ for any $p \in (1, \infty)$. By taking moments in \([7.10]\), it therefore follow from Gronwall’s lemma that the exact same estimate \([5.11]\) still holds true except that additionally, it now holds uniformly in $\varepsilon$ for any $p \in (1, \infty)$. So as in \([6.11]\), we gain from \([7.16]\), the following crucial estimates

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \phi_\varepsilon \|u_\varepsilon\|^2_{L^2_s} \right|^p \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} \| \phi_\varepsilon \|^p_{L^2_s} \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} \| \nabla \phi_\varepsilon \|^2_{L^2_s} \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} \| \Delta \phi_\varepsilon \|^p_{L^2_s} \right] \lesssim_{k, \Gamma, p, \varepsilon_0, s} 1
$$

which holds uniformly in $\varepsilon$ where

$$
\mathcal{E}_{\varepsilon_0, s} := \frac{1}{2} \| \phi_0 \|_{L^6_s}^2 + P^T_\varepsilon(\phi_0) \pm |\nabla \phi_0|^2
$$

is $\varepsilon$-uniformly bounded in $L^p(\Omega; L^2_s)$, c.f. Section \([7.1]\). In addition, by Hölder inequality and Young’s inequality, we have that

$$
\left( \int_0^T \| \phi_\varepsilon \|^2_{L^2_s} \, dt \right)^{\frac{1}{2}} \lesssim \sup_{t \in [0, T]} \| \phi_\varepsilon \|_{L^2_s} \left( \int_0^T \| u_\varepsilon \|^2_{L^2_s} \, dt \right)^{\frac{1}{2}} \tag{7.20}
$$

where $1 < \Gamma' \leq \frac{6}{5}$. Therefore, it follows from \([7.19]\) that

$$
\mathbb{E} \left\| \phi_\varepsilon \right\|_{L^2_s}^{2p} \lesssim_{k, \Gamma, p, \varepsilon_0, s} 1
$$

holds uniformly in $\varepsilon$ for all $p \in (1, \infty)$ where $3 \leq \frac{6 \Gamma'}{1 + 6 \Gamma'} < 6$. At this point, we note that the only information we have on the pressure is given by the second summand in \([7.19]\). However, unlike the previous sections where this regularity and the fact that the continuity equations were satisfied in the strong PDE sense combined to enable us perform our analysis, this aforementioned summand is not enough at this stage. Indeed, any family of pressures enjoying this regularity may only converge to a measure which is not useful in identifying the limit pressure term. We therefore improve the pressure (or density) regularity in the following.

7.3. **Improved pressure estimate.** Let $\Delta_{\varepsilon, s}^{-1}(\phi_\varepsilon - f)$ be a solution of the Poisson equation \([6.3]\). Notice that since $f$ is independent of time, we can combine the continuity equation \([6.1]\) and Poisson equation \([6.3]\) and then obtain the following

$$
\pm d \Delta_{\varepsilon, s}^{-1}(\phi_\varepsilon - f) = [\varepsilon \phi_\varepsilon - \text{div}(\phi_\varepsilon u_\varepsilon)] \, dt.
$$

So by applying the operator $\nabla \Delta_{\varepsilon, s}^{-1}$ to the resulting combined equation, we obtain

$$
\pm d \nabla V_\varepsilon = [\varepsilon \phi_\varepsilon - \nabla \Delta_{\varepsilon, s}^{-1} \text{div}(\phi_\varepsilon u_\varepsilon)] \, dt. \tag{7.22}
$$
Also, as the momentum equation (7.24) is only satisfied weakly in the sense of (6.5), a suitable approximation of the characteristic function \(\chi_{[0,s]}\) for any \(s \in (0,T]\), by a sequence of test functions \(\psi \in C_{c}^{\infty}([0,T])\) results in the following
\[
\int_{T^{3}} \varrho_{\varepsilon}u_{\varepsilon}(t) \cdot \phi \, dx = \int_{T^{3}} \varrho_{0,\varepsilon}u_{0,\varepsilon} \cdot \phi \, dx + \int_{0}^{s} \int_{T^{3}} \varrho_{\varepsilon}u_{\varepsilon} \otimes u_{\varepsilon} : \nabla \phi \, dxdt
\]
\[
- \nu' \int_{0}^{s} \int_{T^{3}} \nabla u_{\varepsilon} : \nabla \phi \, dxdt - (\nu' + \nu'') \int_{0}^{s} \int_{T^{3}} \text{div} \, u_{\varepsilon} \, \text{div} \, \phi \, dxdt
\]
\[
+ \int_{0}^{s} \int_{T^{3}} \varrho_{\varepsilon} \nabla \varepsilon \cdot \phi \, dxdt + \int_{0}^{s} \int_{T^{3}} \varepsilon \varrho_{\varepsilon} \Delta \phi \, dxdt
\]
\[
+ \int_{0}^{s} \int_{T^{3}} \partial_{\varepsilon} \varrho_{\varepsilon} \nabla \varepsilon \cdot \phi \, dxdt + \int_{0}^{s} \sum_{k \in \mathbb{N}} \int_{T^{3}} \partial_{\varepsilon} g_{k,\varepsilon}(\varrho_{\varepsilon}, f, \varphi_{\varepsilon} u_{\varepsilon}) \cdot \phi \, dx \, \beta_{k}
\]
(7.23)

\(\mathbb{P}\)-a.s. Notice that (7.23) can easily be rewritten in differential form. By applying the generalized Itô formula\(^{1}\) Theorem A.4.1\(^{1}\) to (7.22) and (7.23), we obtain the following
\[
\int_{0}^{s} \int_{T^{3}} \varrho_{\varepsilon}^{\Gamma}(\varrho_{\varepsilon}) D_{\varepsilon} \varphi \, dxdt = \int_{T^{3}} \varrho_{\varepsilon} u_{\varepsilon}(t) \cdot \nabla \varphi \, dx - \int_{T^{3}} \varrho_{0,\varepsilon} u_{0,\varepsilon} \cdot \nabla \varphi_{0,\varepsilon} \, dx
\]
\[
- \int_{0}^{s} \int_{T^{3}} \varrho_{\varepsilon} u_{\varepsilon} \otimes u_{\varepsilon} : \nabla^{2} \varphi \, dxdt - \int_{0}^{s} \int_{T^{3}} \text{div} \, u_{\varepsilon} \Delta \varphi \, dxdt
\]
\[
+ \nu' \int_{0}^{s} \int_{T^{3}} \nabla u_{\varepsilon} : \nabla^{2} \varphi \, dxdt - \int_{0}^{s} \int_{T^{3}} \epsilon \nabla(\varrho_{\varepsilon} u_{\varepsilon}) : \nabla^{2} \varphi \, dxdt
\]
\[
- \int_{0}^{s} \int_{T^{3}} \partial_{\varepsilon} \varrho_{\varepsilon} [\nabla \varphi_{\varepsilon}]^{2} \, dxdt - \int_{0}^{s} \int_{T^{3}} \varrho_{\varepsilon} u_{\varepsilon} [\epsilon \nabla \varrho_{\varepsilon} - \nabla \Delta_{\varepsilon}^{-1} \text{div}(\varrho_{\varepsilon} u_{\varepsilon})] \, dxdt
\]
\[
- \int_{0}^{s} \int_{T^{3}} \sum_{k \in \mathbb{N}} \int_{T^{3}} \partial_{\varepsilon} g_{k,\varepsilon}(\varrho_{\varepsilon}, f, \varphi_{\varepsilon} u_{\varepsilon}) \cdot \nabla \varphi \, dx \, \beta_{k} =: I_{1} + \ldots + I_{9}
\]
(7.24)

Without loss of generality (as we intend to take norms anyways), the equation (7.24) was deduced for the + sign in (7.22).

Now if we denote the left-hand side of (7.24) as \(I_{0}\), then from (6.10) and (6.13), we have that
\[
I_{0} = \int_{0}^{s} \int_{T^{3}} [p(\varrho_{\varepsilon}) + \delta(\varrho_{\varepsilon} + \varrho_{\varepsilon}^{0})] \varphi \, dx \, dt - \int_{0}^{s} \int_{T^{3}} p_{\varepsilon}^{\Gamma}(\varrho_{\varepsilon}) f \, dx \, dt =: I_{1}^{1} + I_{0}^{1}
\]
where from (7.21) and (7.19),
\[
\mathbb{E}[I_{0}^{1}] \leq \delta_{\gamma} \mathbb{E} \left[ \sup_{t \in [0,T]} \|\varrho_{\varepsilon}\|_{L_{\varepsilon}^{1}} \|f\|_{L_{\varepsilon}^{\infty}} \right]^{p} \leq \delta_{\gamma,\varepsilon_{0},s,T} \mathbb{E} \left[ \sup_{t \in [0,T]} \|\varrho_{\varepsilon}\|_{L_{\varepsilon}^{1}} \right]^{p} \leq \delta_{\gamma,\varepsilon_{0},s,T} 1
\]
holds uniformly in \(\varepsilon\).

Notice that by using the Poisson equation and the continuity of the operator \(\nabla^{2} \Delta_{\varepsilon}^{-1}\), we have that
\[
\|\nabla^{2} \varphi\|_{L_{\varepsilon}^{p}} \approx \|\nabla^{2} \Delta_{\varepsilon}^{-1} \varphi\|_{L_{\varepsilon}^{p}} \lesssim \|\Delta \varphi\|_{L_{\varepsilon}^{p}}.
\]
(7.26)

To estimate \(I_{1}\), we use the continuous embedding \(W_{x,\Gamma}^{1} \hookrightarrow L_{\varepsilon}^{\infty}\) which holds for \(\Gamma \geq 3\), (7.26) and (7.19) so that
\[
\mathbb{E}[I_{1}]^{p} \leq \mathbb{E}[\|\varrho_{\varepsilon} u_{\varepsilon}(t)\|_{L_{\varepsilon}^{1}} \|\nabla \varphi_{\varepsilon}(t)\|_{L_{\varepsilon}^{\infty}}]^{p} \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \|\varrho_{\varepsilon} u_{\varepsilon}\|_{L_{\varepsilon}^{1}}^{\frac{p}{\Gamma}} \right]^{p} + \mathbb{E} \left[ \sup_{t \in [0,T]} \|\nabla \varphi_{\varepsilon}\|_{L_{\varepsilon}^{1}}^{\frac{p}{\Gamma}} \right]^{p}
\]
\[
\lesssim \delta_{\Gamma,p,\varepsilon_{0},s} 1
\]
(7.27)
holds uniformly in $\varepsilon$ since $\frac{3}{2} \leq \frac{2\Gamma}{1 - \Gamma} < 2$ and $\Gamma \geq 6$. An analogous estimate holds for $I_2$. Again, since for $\Gamma \geq 6$, we have that $\frac{3}{2} < r \leq 2$ where $\frac{1}{r} = 1 - \frac{3}{6} - \frac{1}{\Gamma}$, it follows from (7.26) and (7.19) that

$$
E|I_3|^p \leq E \left( \int_0^T \|\varrho_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \|_{L^r_T}^p \right) dt \leq E \left( \|\varrho_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \|_{L^r_T}^p \right) \leq E \left( \|\varrho_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \|_{L^r_T}^p \right)
$$

(7.28)

uniformly in $\varepsilon$. We obtain from Hölder inequality and $\Gamma \geq 2$ that the estimate for $E \left( \|\nabla \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \|_{L^r_T}^p \right)$ holds uniformly in $\varepsilon$ as a result of (7.19). A similar estimate holds for $I_5$. We now note that since $\varepsilon \in (0, 1)$,

$$
\varepsilon \nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \leq \sqrt{\varepsilon} (\sqrt{\varepsilon} \nabla \varphi_{\varepsilon} \varphi_{\varepsilon} + \varphi_{\varepsilon} \nabla \varphi_{\varepsilon} \varphi_{\varepsilon})
$$

And since $2 < \frac{2\Gamma}{1 - \Gamma} \leq 3$, it therefore follows from (7.26) and (7.19) that

$$
E|I_4|^p \leq E \left( \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right) dt \leq E \left( \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right) \leq E \left( \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right)
$$

(7.29)

uniformly in $\varepsilon$. We obtain from Hölder inequality and $\Gamma \geq 6$ that $\Gamma \geq 2$ holds uniformly in $\varepsilon$ (note that $\varepsilon \frac{3}{2} < 1$). For $I_7$, we use the continuous embedding $W^{1,\Gamma}_x \hookrightarrow L^\infty_x$, (7.20) and (7.19) to obtain

$$
E|I_7|^p \leq E \left( \|\varrho_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right) \leq E \left( \|\varrho_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right) \leq E \left( \|\varrho_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right)
$$

(7.30)

uniformly in $\varepsilon$. The estimate for $I_8$ is as follows:

$$
E|I_8|^p \leq E \left( \|\varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right) \leq E \left( \|\varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right) \leq E \left( \|\varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right)
$$

(7.31)

which holds uniformly in $\varepsilon$ for all $p \geq 1$ because of (7.19) and (7.21). To estimate $I_9$ we first note

$$
\left( \int_{\mathbb{R}^3} \varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} (\varrho_{\varepsilon}, f, \varrho_{\varepsilon} \varphi_{\varepsilon}) \cdot \nabla \varphi_{\varepsilon} \right) \leq c_k \|\varrho_{\varepsilon} \varphi_{\varepsilon} \|_{L^\infty}^2 \left( \|\varrho_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p + \|\nabla \varphi_{\varepsilon} \varphi_{\varepsilon} \|_{L^r_T}^p \right)
$$

(7.32)

and so similar to the argument in (7.17) - (7.18) we can invoke the Burkholder–Davis–Gundy inequality and use (7.26), (7.20), and the fact that the cut-off function $|\varphi_{\varepsilon}|$ is uniformly bounded by one to obtain from (7.19), the following estimate

$$
E|I_9|^p \leq E \left( \int_0^T \sum_{k=0}^{4} \int_{\mathbb{R}^3} \varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} (\varrho_{\varepsilon}, f, \varrho_{\varepsilon} \varphi_{\varepsilon}) \cdot \nabla \varphi_{\varepsilon} \right) dt \leq E \left( \int_0^T \sum_{k=0}^{4} \int_{\mathbb{R}^3} \varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} (\varrho_{\varepsilon}, f, \varrho_{\varepsilon} \varphi_{\varepsilon}) \cdot \nabla \varphi_{\varepsilon} \right) dt \leq E \left( \int_0^T \sum_{k=0}^{4} \int_{\mathbb{R}^3} \varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} (\varrho_{\varepsilon}, f, \varrho_{\varepsilon} \varphi_{\varepsilon}) \cdot \nabla \varphi_{\varepsilon} \right) dt
$$

(7.33)

and so similar to the argument in (7.17) - (7.18) we can invoke the Burkholder–Davis–Gundy inequality and use (7.26), (7.20), and the fact that the cut-off function $|\varphi_{\varepsilon}|$ is uniformly bounded by one to obtain from (7.19), the following estimate

$$
E|I_9|^p \leq E \left( \int_0^T \sum_{k=0}^{4} \int_{\mathbb{R}^3} \varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} (\varrho_{\varepsilon}, f, \varrho_{\varepsilon} \varphi_{\varepsilon}) \cdot \nabla \varphi_{\varepsilon} \right) dt \leq E \left( \int_0^T \sum_{k=0}^{4} \int_{\mathbb{R}^3} \varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} (\varrho_{\varepsilon}, f, \varrho_{\varepsilon} \varphi_{\varepsilon}) \cdot \nabla \varphi_{\varepsilon} \right) dt \leq E \left( \int_0^T \sum_{k=0}^{4} \int_{\mathbb{R}^3} \varrho_{\varepsilon} \varphi_{\varepsilon} \varphi_{\varepsilon} (\varrho_{\varepsilon}, f, \varrho_{\varepsilon} \varphi_{\varepsilon}) \cdot \nabla \varphi_{\varepsilon} \right) dt
$$

(7.34)
uniformly in $\varepsilon$ for all $p \geq 2$. The fact that $p \geq 2$ follow from the fact that $|\cdot|^{p/2}$ is convex only if $p/2 \geq 1$.

By collecting the various estimates above, we have shown that

$$
\mathbb{E} \left| \int_0^t \int_{\mathbb{T}^3} [q_\varepsilon p(q_\varepsilon) + \delta(q_\varepsilon^2 + q_\varepsilon^{p+1})] \, dx \, dt \right|^{p} \lesssim_{k, \Gamma, \delta, \gamma, \xi_0, \varepsilon, T} \varepsilon \tag{7.35}
$$

holds uniformly in $\varepsilon$ for all $p \geq 2$ and in particular, for $\Gamma \geq 6$, the estimate

$$
\mathbb{E} \left| \int_0^t \int_{\mathbb{T}^3} q_\varepsilon^{p+1} \, dx \, dt \right|^{p} \lesssim_{p, k, \Gamma, \delta, \gamma, \xi_0, \varepsilon, T} \varepsilon \tag{7.36}
$$

holds uniformly in $\varepsilon$ for all $p \geq 2$.

7.4. **Compactness.** In order to establish compactness, we first need some preparation. We denote the energy by

$$
\mathcal{E}_\varepsilon := \frac{1}{2} q_\varepsilon|u_\varepsilon|^2 + P^p_0(q_\varepsilon) \pm \nu \|\nabla V_\varepsilon\|^2 \tag{7.37}
$$

and let the weakly-* measurable mapping

$$
\nu_\varepsilon: [0, T] \times \mathbb{T}^3 \rightarrow \mathcal{P}(\mathbb{R}^{20})
$$

defined by

$$
\nu_{\varepsilon, t, x}(\cdot) = \delta_{(q_\varepsilon, u_\varepsilon, \nabla u_\varepsilon, q_\varepsilon, u_\varepsilon, f, \nabla V_\varepsilon)(t, x)}(\cdot)
$$

be the canonical Young measure associated to $[q_\varepsilon, u_\varepsilon, \nabla u_\varepsilon, q_\varepsilon, u_\varepsilon, f, \nabla V_\varepsilon]$. See the discussion in [11, Section 2.8, Section 4.4.3.1] [15, Section 2.8] on how this allows us to interpret $\nu_\varepsilon$ as a random variable taking values in the non-Polish space $(L^\infty([0, T] \times \mathbb{T}^3); \mathcal{P}(\mathbb{R}^{20}), w^*)$ endowed with the weak-* topology which is determined by

$$
L^\infty([0, T] \times \mathbb{T}^3; \mathcal{P}(\mathbb{R}^{20})) \rightarrow \mathbb{R}, \quad \nu \mapsto \int_0^T \int_{\mathbb{T}^3} \psi(t, x) \int_{\mathbb{R}^{20}} \phi(\xi) \, d\nu_{\omega, t, x}(\xi) \, dx \, dt
$$

for all $\psi \in L^1([0, T] \times \mathbb{T}^3)$, for all $\phi \in C_b(\mathbb{R}^{20})$. We now define the following spaces

$$
\chi_{\theta_0} = L^1_\varepsilon, \quad \chi_{m_0} = L^1_\varepsilon, \quad \chi_{\frac{m_0}{\sqrt{\eta_0}}} = L^2_\varepsilon, \quad \chi_u = (L^2(0, T; W^{1, 2}_x, w), \quad \chi_{\nu} = C_w \left( [0, T]; W^{2, r}_x \right), \]

$$
$$
\chi_{W} = C \left( [0, T]; U_0 \right), \quad \chi_{\nu_{\varepsilon, u}} = C_w \left( [0, T]; L^{p\Phi}_x \right) \cap C \left( [0, T]; W^{-k, 2}_x \right), \]

$$
$$
\chi_{\theta} = C_w \left( [0, T]; L^1_\varepsilon \right) \cap \left( L^2_{\varepsilon, x, t} \right), \quad \chi_{\varepsilon, x} = (L^2_{\varepsilon, x}, M_0(\mathbb{T}^3), w^*), \quad \chi_{\nu} = L^\infty([0, T] \times \mathbb{T}^3; \mathcal{P}(\mathbb{R}^{20}), w^*)
$$

for $\Gamma \geq 6$ and $k > \frac{3}{2}$. We now let $\mu_{\theta_0, \varepsilon}, \mu_{m_0, \varepsilon}, \mu_{\frac{m_0}{\sqrt{\eta_0}}, \varepsilon}, \mu_\theta, \mu_u, \mu_{\theta_0, \varepsilon}, \mu_{\varepsilon, x}, \mu_{\nu_{\varepsilon, u}}, \mu_{\nu_{\varepsilon, x}}, \mu_{\nu_{\varepsilon, u}}$ and $\mu_{\nu_{\varepsilon, x}}$ be the respective laws of $\theta_{0, \varepsilon}, m_{0, \varepsilon}, \frac{m_{0, \varepsilon}}{\sqrt{\eta_0, \varepsilon}}, \theta, u, \theta_0, u, \varepsilon, u, \varepsilon, \nu_{\varepsilon, x}, \nu_{\varepsilon, u}$ and $W$ on the respective spaces $\chi_{\theta_0}, \chi_{m_0}, \chi_u, \chi_{\nu_{\varepsilon, u}}, \chi_{\varepsilon, x}, \chi_{\nu_{\varepsilon, x}}, \chi_{\nu_{\varepsilon, u}}$ and $\chi_{\nu_{\varepsilon, x}}$. Furthermore, we set $\mu_\varepsilon$ as their joint law on the space $\chi = \chi_{\theta_0} \times \chi_{m_0} \times \chi_{\frac{m_0}{\sqrt{\eta_0, \varepsilon}}} \times \chi_\theta \times \chi_u \times \chi_{\nu_{\varepsilon, u}} \times \chi_{\varepsilon, x} \times \chi_{\nu_{\varepsilon, x}} \times \chi_{\nu_{\varepsilon, u}} \times \chi_{\nu_{\varepsilon, x}}$.

Now since the following spaces are Polish, it follows from Prokhorov's theorem theorem that:

- The set $\{\mu_{\theta_0, \varepsilon} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\theta_0}$.
- The set $\{\mu_{m_0, \varepsilon} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{m_0}$.
- The set $\{\mu_{\frac{m_0}{\sqrt{\eta_0, \varepsilon}}} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\frac{m_0}{\sqrt{\eta_0, \varepsilon}}}$.
- The set $\{\mu_{\theta} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\theta}$.
- The set $\{\mu_u : \varepsilon \in (0, 1)\}$ is tight on $\chi_u$.
- The set $\{\mu_{\varepsilon, x} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\varepsilon, x}$.
- The set $\{\mu_{\nu_{\varepsilon, u}} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\nu_{\varepsilon, u}}$.
- The set $\{\mu_{\nu_{\varepsilon, x}} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\nu_{\varepsilon, x}}$.
- The set $\{\mu_{\nu_{\varepsilon, u}} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\nu_{\varepsilon, u}}$.
- The set $\{\mu_{\nu_{\varepsilon, x}} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\nu_{\varepsilon, x}}$.

Additionally, in analogy with the corresponding result in Lemma 5.3 we have that:

- The set $\{\mu_\theta : \varepsilon \in (0, 1)\}$ is tight on $\chi_\theta$.
- The set $\{\mu_u : \varepsilon \in (0, 1)\}$ is tight on $\chi_u$.
- The set $\{\mu_{\varepsilon, x} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\varepsilon, x}$.
- The set $\{\mu_{\nu_{\varepsilon, u}} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\nu_{\varepsilon, u}}$.
- The set $\{\mu_{\nu_{\varepsilon, x}} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\nu_{\varepsilon, x}}$.
- The set $\{\mu_{\nu_{\varepsilon, u}} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\nu_{\varepsilon, u}}$.
- The set $\{\mu_{\nu_{\varepsilon, x}} : \varepsilon \in (0, 1)\}$ is tight on $\chi_{\nu_{\varepsilon, x}}$.

Also, analogous to [11, Proposition 4.4.6, Proposition 4.4.7],...
• The set \( \{ \mu_\varepsilon : \varepsilon \in (0, 1) \} \) is tight on \( \chi_\varepsilon \).
• The set \( \{ \mu_\nu : \varepsilon \in (0, 1) \} \) is tight on \( \chi_\nu \).

The following lemma thus hold.

**Lemma 7.3.** The set \( \{ \mu_\varepsilon : \varepsilon \in (0, 1) \} \) is tight on \( \chi \).

We can now apply the Jakubowski–Skorokhod theorem \([15]\) and we obtain the following result.

**Lemma 7.4.** The exists a subsequence (not relabelled) \( \{ \mu_\varepsilon : \varepsilon \in (0, 1) \} \), a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with \( \chi \)-valued random variables 
\[
(\tilde{\varrho}_{0,\varepsilon}, \tilde{m}_{0,\varepsilon}, \tilde{n}_{0,\varepsilon}, \tilde{\varrho}_\varepsilon, \tilde{u}_\varepsilon, \tilde{m}_\varepsilon, \tilde{V}_\varepsilon, \tilde{E}_\varepsilon, \tilde{\nu}_\varepsilon, \tilde{W}_\varepsilon) \quad \varepsilon \in (0, 1)
\]

and
\[
(\tilde{\varrho}_0, \tilde{m}_0, \tilde{n}_0, \tilde{\varrho}_1, \tilde{\varrho}_2, \tilde{u}, \tilde{m}, \tilde{V}, \tilde{E}, \tilde{\nu}, \tilde{W})
\]

such that
• the law of \((\tilde{\varrho}_{0,\varepsilon}, \tilde{m}_{0,\varepsilon}, \tilde{n}_{0,\varepsilon}, \tilde{\varrho}_\varepsilon, \tilde{u}_\varepsilon, \tilde{m}_\varepsilon, \tilde{V}_\varepsilon, \tilde{E}_\varepsilon, \tilde{\nu}_\varepsilon, \tilde{W}_\varepsilon)\) on \( \chi \) coincide with \( \mu_\varepsilon, \varepsilon \in (0, 1) \),
• the law of \((\tilde{\varrho}_0, \tilde{m}_0, \tilde{n}_0, \tilde{\varrho}_1, \tilde{\varrho}_2, \tilde{u}, \tilde{m}, \tilde{V}, \tilde{E}, \tilde{\nu}, \tilde{W})\) on \( \chi \) is a Radon measure,
• the following convergence (with each \( \to \) interpreted with respect to the corresponding topology)
  \[
  \tilde{\varrho}_{0,\varepsilon} \to \tilde{\varrho}_0 \quad \text{in} \quad \chi_{\tilde{\varrho}_0}, \quad \tilde{m}_{0,\varepsilon} \to \tilde{m}_0 \quad \text{in} \quad \chi_{\tilde{m}_0}, \\
  \tilde{n}_{0,\varepsilon} \to \tilde{n}_0 \quad \text{in} \quad \chi_{\tilde{n}_0}, \quad \tilde{\varrho}_\varepsilon \to \tilde{\varrho}_1 \quad \text{in} \quad \chi_{\tilde{\varrho}_1}, \\
  \tilde{u}_\varepsilon \to \tilde{u} \quad \text{in} \quad \chi_{\tilde{u}}, \quad \tilde{m}_\varepsilon \to \tilde{m} \quad \text{in} \quad \chi_{\tilde{m}}, \\
  \tilde{V}_\varepsilon \to \tilde{V} \quad \text{in} \quad \chi_{\tilde{V}}, \quad \tilde{E}_\varepsilon \to \tilde{E} \quad \text{in} \quad \chi_{\tilde{E}}, \\
  \tilde{\nu}_\varepsilon \to \tilde{\nu} \quad \text{in} \quad \chi_{\tilde{\nu}}, \quad \tilde{W}_\varepsilon \to \tilde{W} \quad \text{in} \quad \chi_{\tilde{W}}
\]
holds \( \mathbb{P} \)-a.s.;
• consider any Carathéodory function

\[
C = C(t, x, \varrho, u, m, f, V) \quad \text{with}
\]
\[
(t, x, \varrho, u, m, f, V) \in [0, T] \times \mathbb{T}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3,
\]

and where the following estimate
\[
|C| \lesssim 1 + |\varrho|^4 + |u|^2 + |U|^3 + |m|^4 + |f|^6 + |V|^6
\]
holds uniformly in \((t, x)\) for some \( r_i > 0, i = 1, \ldots, 6 \). Then as \( \varepsilon \to 0 \), it follows that
\[
\tilde{C}(\tilde{\varrho}_\varepsilon, \tilde{u}_\varepsilon, \nabla \tilde{u}_\varepsilon, \tilde{m}_\varepsilon, f, \nabla V_\varepsilon) \to \tilde{C}(\tilde{\varrho}_1, \tilde{u}, \nabla \tilde{u}, \tilde{m}, f, \nabla V)
\]
holds in \( L^r((0, T) \times \mathbb{T}^3) \) for all
\[
1 < r \leq \frac{\Gamma + 1}{r_1} \wedge \frac{2}{r_2} \wedge \frac{2\Gamma}{r_4(\Gamma + 1)} \wedge \frac{2}{r_6}
\]
\( \mathbb{P} \)-a.s.

As a consequence of Lemma 7.4, we obtain the following result.

**Corollary 7.5.** The following holds \( \mathbb{P} \)-a.s.
\[
\tilde{\varrho}_{0,\varepsilon} = \tilde{\varrho}_0(0), \quad \tilde{m}_{0,\varepsilon} = \tilde{\varrho}_1(0), \quad \tilde{n}_{0,\varepsilon} = \tilde{\varrho}_1(0), \quad \tilde{u}_\varepsilon = \tilde{\varrho}_1(0), \quad \tilde{m}_\varepsilon = \tilde{\varrho}_1(0),
\]
\[
\tilde{E}_\varepsilon = \tilde{E}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{u}_\varepsilon, \tilde{V}_\varepsilon), \quad \tilde{\nu}_\varepsilon = \delta_{\tilde{\varrho}_\varepsilon, \tilde{u}_\varepsilon, \nabla \tilde{u}_\varepsilon, \tilde{m}_\varepsilon},
\]
and that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \int_{\mathbb{T}^3} \tilde{E}_\varepsilon \, dx \right]^p = \mathbb{E} \left[ \sup_{t \in [0, T]} \int_{\mathbb{T}^3} \left( \frac{1}{2} \tilde{\varrho}_\varepsilon |\tilde{u}_\varepsilon|^2 + P^\varepsilon_\delta(\tilde{\varrho}_\varepsilon) \pm \varrho |\nabla \tilde{V}_\varepsilon|^2 \right) \, dx \right]^p
\leq \delta_{T, p, \varepsilon_{0,\varepsilon}}
\]
holds uniformly in \( \varepsilon \) just as in 7.39.
Furthermore, a consequence of the last item of Lemma 7.4 is the following.

**Corollary 7.6.** There exists \( p_0^f(\bar{\varrho}) \), \( |\nabla V|^2 \), \( \bar{\varrho} \nabla V \) and \( \bar{\varrho} g_k(\bar{\varrho}, f, \ddot{u}) \) such that
\[
\begin{align*}
p_0^f(\bar{\varrho}_e) &\to p_0^f(\bar{\varrho}) \quad (7.40) \\
|\nabla \bar{V}_e|^2 &\to |\nabla V|^2 \quad (7.41) \\
\bar{\varrho}_e \nabla \bar{V}_e &\to \bar{\varrho} \nabla V \quad (7.42) \\
\bar{\varrho}_e g_k(\bar{\varrho}_e, f, \ddot{u}_e) &\to \bar{\varrho} g_k(\bar{\varrho}, f, \ddot{u}) \quad (7.43)
\end{align*}
\]

\( \hat{\mathbb{P}} \)-a.s. in \( L^r((0, T) \times \mathbb{T}^3) \) for some \( r > 1 \).

**7.5. Identifying the limit system.** Just as was done in Section 6.4 we can now construct and endow the family of sequences \( \hat{\mathcal{F}}_t := \sigma(\sigma_i[\bar{\varrho}], \sigma_i[\ddot{u}], \sigma_i[\bar{V}], \bigcup_{k \in \mathbb{N}} \sigma_i[\bar{\varrho}_e, k]) \), \( t \in [0, T] \) and
\[
\hat{\mathcal{F}}_t := \sigma(\sigma_i[\bar{\varrho}], \sigma_i[\ddot{u}], \sigma_i[\bar{V}], \bigcup_{k \in \mathbb{N}} \sigma_i[\bar{\varrho}_e, k]) \quad t \in [0, T]
\]
on the family of sequences \( (\bar{\varrho}_0, \bar{\varrho}_e, \bar{\varrho}_0, \bar{\varrho}_e, \bar{\varrho}_e, \bar{\varrho}_e, \bar{\varrho}_e, \bar{\varrho}_e, \bar{\varrho}_e) \) and the random variables \( (\hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}) \) respectively. The following can be found in \([1]\) Lemma 4.4.11.

**Lemma 7.7.** The random distributions \( [\hat{\varrho}, \hat{\varrho}] \) satisfies \( (7.45) \) for all \( \psi \in C_c^\infty((0, T)) \) and \( \phi \in C^\infty(\mathbb{T}^3) \) \( \hat{\mathbb{P}} \)-a.s.

Similar to Lemma 6.7 we obtain the following result.

**Lemma 7.8.** The random variables \( [\hat{\varrho}, \hat{\varrho}] \) satisfies \( (7.3) \) a.e. in \( (0, T) \times \mathbb{T}^3 \) \( \hat{\mathbb{P}} \)-a.s.

By using Corollary 7.6 we obtain the following result which is analogous to \([1]\) Proposition 4.4.12.]

**Lemma 7.9.** The following convergence
\[
\hat{\varrho}_e \ddot{u}_e \otimes \ddot{u}_e \to \hat{\varrho} \ddot{u} \otimes \ddot{u} \quad \text{in} \quad L^1(0, T; L^1(\mathbb{T}^3))
\]

holds \( \hat{\mathbb{P}} \)-a.s. and the random distributions \( [\hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}, \hat{\varrho}] \) satisfies
\[
\begin{align*}
- \int_0^T \partial_t \psi \int_{\mathbb{T}^3} \hat{\varrho}(t) \cdot \phi \, dx \, dt &= \psi(0) \int_{\mathbb{T}^3} \bar{\varrho}_0 \bar{u}_0 \cdot \phi \, dx + \int_0^T \psi \int_{\mathbb{T}^3} \bar{\varrho} \ddot{u} \otimes \ddot{u} : \nabla \phi \, dx \, dt \\
- \nu^S \int_0^T \psi \int_{\mathbb{T}^3} \nabla \ddot{u} : \nabla \phi \, dx \, dt &= -(\nu_B + \nu^S) \int_0^T \psi \int_{\mathbb{T}^3} \text{div} \ddot{u} \, \text{div} \phi \, dx \, dt \\
+ \int_0^T \psi \int_{\mathbb{T}^3} p_0^f(\bar{\varrho}) \, \text{div} \phi \, dx \, dt &= \int_0^T \psi \int_{\mathbb{T}^3} \bar{\varrho} \bar{\varrho} \nabla \ddot{u} : \phi \, dx \, dt \\
+ \int_0^T \psi \sum_{k \in \mathbb{N}} \int_{\mathbb{T}^3} \hat{\varrho} g_k(\bar{\varrho}, f, \ddot{u}) \cdot \phi \, dx \, dt &= \int_0^T \psi \int_{\mathbb{T}^3} \nu^S \bar{\varrho} \ddot{u} \otimes \ddot{u} : \nabla \phi \, dx \, dt \\
\end{align*}
\]

for all \( \psi \in C_c^\infty((0, T)) \) and \( \phi \in C^\infty(\mathbb{T}^3) \) \( \hat{\mathbb{P}} \)-a.s.

**Lemma 7.9** does not completely identify the momentum equation since the nonlinear pressure, the term containing the electric field and the noise term are only expressed in terms of arbitrary limits which we do not know to be exactly of the form \( (7.2) \). We shall require strong convergence of the density in order to establish that these arbitrary 'product limit' terms actually coincide with their corresponding form in \( (7.2) \). The following lemma will help us in this direction.
Lemma 7.10. The following convergence

$$\lim_{\varepsilon \to 0} \int_{T_1}^{T_2} \left[ p_{\Gamma}^{\varepsilon}(\tilde{\varrho}_e) - (\nu^B + 2\nu^S) \text{div} \, \tilde{\mathbf{u}}_e \right] \tilde{\varrho}_e \, dx \, dt = \int_{0}^{s} \int_{T_3} \left[ p_{\Gamma}^{\varepsilon}(\varrho) - (\nu^B + 2\nu^S) \text{div} \, \tilde{\mathbf{u}} \right] \varrho \, dx \, dt$$

holds $\tilde{\mathbb{P}}$-a.s. for a.e. $s \in (0, T)$.

Proof. First of all, as a result of the equality of laws established by Lemma 7.4, it follows from (7.24) that

$$\mathbb{E} \left[ \int_{T_1}^{T_2} \tilde{\varrho}_e \tilde{\mathbf{u}}_e \cdot \nabla \tilde{V}_e \, dx \right] + \mathbb{E} \left[ \int_{T_1}^{T_2} \tilde{\varrho}_0 \tilde{\mathbf{u}}_0 \cdot \nabla \tilde{V}_0 \, dx \right]$$

holds $\tilde{\mathbb{P}}$-a.s. Notice that the noise term in now driven by several Brownian motions $\tilde{\beta}_{k,e}$ for each $k \in \mathbb{N}$ which is a result of the existence of $\tilde{W}_e$ as given by Lemma 7.4. Also notice that the commutativity of the differential operators $\partial_{x_i}$ and $\partial_{x_j}$ mean that after integrating by parts twice,

$$J_4 = \nu^S \int_{T_1}^{T_2} \text{div} \, \tilde{\mathbf{u}}_e \cdot \tilde{V}_e \, dx$$

provided that $\tilde{\mathbf{u}}_e$ and $\tilde{V}_e$ are regular enough to allow this many integration by parts. Since (7.24) and thus (7.46) was derived through the application of Itô’s lemma after a preliminary regularization step (7.47) holds true indeed. Now similar to (7.46), we use use the generalized Itô’s formula to obtain from Lemma 7.9

$$\mathbb{E} \left[ \int_{T_1}^{T_2} \tilde{\varrho} \tilde{\mathbf{u}} \cdot \nabla \tilde{V} \, dx \right] + \mathbb{E} \left[ \int_{T_1}^{T_2} \tilde{\varrho}_0 \tilde{\mathbf{u}}_0 \cdot \nabla \tilde{V}_0 \, dx \right]$$

We now show that $I_9, I_{10} \to \tilde{\mathbb{P}}$-a.s. as $\varepsilon \to 0$. This is because since $2 < \frac{2\Gamma}{\nu^S} \leq 3$ (recall that $\Gamma \geq 6$), it follows from Hölder inequality that $\tilde{\mathbb{P}}$-a.s., the estimate

$$J_9 = \int_{T_1}^{T_2} \left[ \varepsilon \tilde{\varrho}_e \nabla \tilde{\mathbf{u}}_e + \varepsilon \sqrt{\varepsilon} \nabla \tilde{\varrho}_e \cdot \tilde{\mathbf{u}}_e \right] \cdot \tilde{\nabla} \tilde{V}_e \, dx \, dt \leq \varepsilon \int_{T_1}^{T_2} \| \tilde{\varrho}_e \|_{L_\infty} \| \nabla \tilde{\mathbf{u}}_e \|_{L_2} \| \tilde{\nabla} \tilde{V}_e \|_{L_\infty} \, dx \, dt$$

holds uniformly in $\varepsilon$. Now since the embedding $L_x^\infty L_x^{\frac{2\Gamma}{\nu^S}} \to L_x^2 L_x^{\frac{2\Gamma}{\nu^S}}$ is continuous, it follows from the estimates (7.19), (7.20) and Lemma 7.4 that

$$J_9 \lesssim \sqrt{\varepsilon} \to 0$$
\( \hat{P} \)-a.s. as \( \varepsilon \to 0 \). Again, since \( 2 < \frac{3\tau}{\tau - 3} \leq 3 \), it follows from Lemma 7.4 that

\[
J_{10} \leq \sqrt{\varepsilon} \int_0^T \| \hat{\varrho}_e \|_{L^p} \| \hat{u}_e \|_{L^q} \| \nabla \hat{\varrho}_e \|_{L^2} \, dt \lesssim \sqrt{\varepsilon} \| \hat{\varrho}_e \|_{L^p} \| \nabla \hat{u}_e \|_{L^q} \| \nabla \hat{\varrho}_e \|_{L^p} \lesssim \sqrt{\varepsilon} \, \to \, 0
\]  

(7.51)

\( \hat{P} \)-a.s. as \( \varepsilon \to 0 \).

Now notice that since the given function \( f \in L^p_x \) for all \( p \in [1, \infty] \), it follows from Lemma 7.4 and (7.40) that

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^3} [p_0^f(\hat{\varrho}_e) - (\nu^B + 2\nu^S) \text{div} \, \hat{u}_e] \Delta \hat{\varrho}_e \, dx \, dt \\
- \int_0^T \int_{\mathbb{T}^3} [p_0^f(\hat{\varrho}_e) - (\nu^B + 2\nu^S) \text{div} \, \hat{u}_e] \Delta \hat{\varrho}_e \, dx \, dt \\
= \lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^3} [p_0^f(\hat{\varrho}_e) - (\nu^B + 2\nu^S) \text{div} \, \hat{u}_e] \hat{\varrho}_e \, dx \, dt \\
- \int_0^T \int_{\mathbb{T}^3} [p_0^f(\hat{\varrho}_e) - (\nu^B + 2\nu^S) \text{div} \, \hat{u}_e] \hat{\varrho}_e \, dx \, dt
\]  

(7.52)

Also notice that since \( \Gamma > 3 \), it follows from Lemma 7.4 and the compact embedding \( W^{1, \Gamma}_x \rightarrow C \) that

\[ \nabla \hat{\varrho}_e \rightarrow \nabla \hat{\varrho} \quad \text{in} \quad C([0, T] \times \mathbb{T}^3) \]  

(7.53)

\( \hat{P} \)-a.s. It follows from Lemma 2.6.6, (7.33) and (7.33) that

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^3} \hat{\varrho}_e \varrho \, g_{\lambda, \varepsilon}(\hat{\varrho}_e, f, \hat{\varrho}_e \, \hat{u}_e) \cdot \nabla \hat{\varrho}_e \, dx \, dt = \int_0^T \int_{\mathbb{T}^3} \varrho \varrho \, g(\hat{\varrho}, \hat{\varrho} \, \hat{u}) \cdot \nabla \hat{\varrho} \, dx \, dt
\]  

(7.54)

\( \hat{P} \)-a.s. and from Lemma (7.4), and (7.63) that

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^3} \hat{\varrho}_e \hat{u}_e(t) \cdot \nabla \hat{\varrho}_e \, dx \, dt = \int_0^T \int_{\mathbb{T}^3} \hat{\varrho}_0 \hat{u}_0 \cdot \nabla \hat{\varrho}_0 \, dx \, dt
\]  

(7.55)

Also, (7.42) and (7.53) yields

\[ \lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^3} \varrho \, \delta \nabla \hat{\varrho}_e \, dx \, dt = \int_0^T \int_{\mathbb{T}^3} \varrho \, \delta \nabla \hat{\varrho} \, dx \, dt
\]  

(7.56)

\( \hat{P} \)-a.s. By combining (7.46) – (7.48) with (7.50) – (7.52) and (7.54) – (7.56) it follows that

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^3} \hat{\varrho}_e \varrho_0 \, g_{\lambda, \varepsilon}(\hat{\varrho}_e, f, \hat{\varrho}_e \, \hat{u}_e) \cdot \nabla \hat{\varrho}_e \, dx \, dt \\
- \int_0^T \int_{\mathbb{T}^3} \varrho \, g(\hat{\varrho}, \hat{\varrho} \, \hat{u}) \cdot \nabla \hat{\varrho} \, dx \, dt \\
= \lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^3} \hat{\varrho}_e \hat{u}_e \varrho \Delta_{\mathbb{T}^3} \, div(\delta \hat{\varrho}_e \hat{u}_e) - \delta \hat{\varrho}_e \hat{u}_e \otimes \hat{u}_e : \nabla^2 \hat{\varrho}_e \, dx \, dt \\
+ \int_0^T \int_{\mathbb{T}^3} \hat{\varrho} \, \delta \nabla \Delta_{\mathbb{T}^3}^{-1} \text{div}(\delta \hat{\varrho} \hat{u} - \delta \hat{\varrho} \otimes \hat{u} : \nabla^2 \hat{\varrho}) \, dx \, dt
\]  

(7.57)

\( \hat{P} \)-a.s. We now want to show that the right-hand side of (7.57) is zero. Since the periodic inverse Laplacian commutes with spatial derivatives and

\[ \pm \nabla^2 V = \nabla^2 \Delta_{\mathbb{T}^3}^{-1} (\varrho - f) \]
in coordinates form, the first integral term of the right-hand side of (7.57) is
\[
\int_0^s \int_{T^3} \tilde{u}_x^i \left[ \tilde{\theta}_x \partial_{x_i} \Delta_{T^3}^{-1} \partial_{x_j} (\tilde{\theta}_x \tilde{u}_x^j) - \tilde{\theta}_x \tilde{u}_x^i \partial_{x_j} \Delta_{T^3}^{-1} \partial_{x_j} (\tilde{\theta}_x - f) \right] \, dx \, dt
\]
\[
= \int_0^s \int_{T^3} \tilde{u}_x^i \left[ \tilde{\theta}_x \partial_{x_i} \Delta_{T^3}^{-1} \partial_{x_j} (\tilde{\theta}_x \tilde{u}_x^j) - \tilde{\theta}_x \tilde{u}_x^i \partial_{x_j} \Delta_{T^3}^{-1} \partial_{x_j} (\tilde{\theta}_x - f) \right] \, dx \, dt
\]
\[+ \int_0^s \int_{T^3} \tilde{\theta}_x \tilde{u}_x^i \tilde{u}_x^j \partial_{x_j} \Delta_{T^3}^{-1} f \, dx \, dt =: I_1^s + I_2^s
\]
\[\tilde{P}\text{-a.s.} \]
The same result holds for the second integral term of the right-hand side of (7.57), i.e., an obvious comparison of the form
\[
\int_0^s \int_{T^3} \tilde{u}_x^i \left[ \tilde{\theta}_x \partial_{x_i} \Delta_{T^3}^{-1} \partial_{x_j} (\tilde{\theta}_x \tilde{u}_x^j) - \tilde{\theta}_x \tilde{u}_x^i \partial_{x_j} \Delta_{T^3}^{-1} \partial_{x_j} (\tilde{\theta}_x - f) \right] \, dx \, dt = I_1 + I_2
\]
\[\tilde{P}\text{-a.s.} \] provided that we choose \( \Gamma \geq 5 \). By using this convergence, the right-hand side of equation (7.57) is zero and the proof is done. \( \square \)

Now since \( \tilde{\theta} \in L^\infty_t L^1_x \tilde{P}\text{-a.s.} \) with \( \Gamma \geq 2 \) and \( \tilde{u} \in L^2_t W^{1,2}_x \tilde{P}\text{-a.s.} \), we may use the renormalized theory of DiPerna-Lions \[5\] to get that \((\tilde{\theta}, \tilde{u})\) satisfies (7.1) in the renormalized sense. Since the map \( \varrho \mapsto \varrho \log \varrho \) is strictly convex, it follows that
\[
\tilde{\theta}_x \rightarrow \tilde{\theta}, \quad \text{in} \quad L^r_t L^1_x \quad (7.61)
\]
and hence,
\[
\Delta \tilde{V}_x \rightarrow \Delta \tilde{V}, \quad \text{in} \quad L^r_t L^1_x \quad (7.62)
\]
\(\tilde{P}\text{-a.s.} \) for any \( r \in [1, \infty) \). Furthermore, as in \([1\text{, }4.177]\), we can use the Lipschitz continuity of \( g_k \) and (7.61) to that
\[
\int_{T^3} \tilde{\theta}_x g_k(\tilde{\theta}_x, f, \tilde{\theta}_x \tilde{u}_x) \cdot \phi \, dx \rightarrow \int_{T^3} \tilde{\theta} g_k(\tilde{\theta}(f, \tilde{\theta}_x \tilde{u}_x) \cdot \phi \, dx \quad \text{a.e. in} \ (0,T)
\]
holds \( \mathbb{P}\text{-a.s.} \) for any \( \phi \in C^\infty(T^3) \) from which we infer that
\[
\overline{\tilde{\theta}} g_k(\tilde{\theta}(f, \tilde{\theta}_x \tilde{u}_x) = \overline{\tilde{\theta}} g_k(\tilde{\theta}(f, \tilde{\theta}_x \tilde{u}_x) \quad \text{a.e. in} \ \tilde{\Omega} \times (0,T) \times T^3.
\]
We can therefore conclude with the following lemma

**Lemma 7.11.** The random distributions \([\tilde{\theta}, \tilde{u}, \tilde{V}, \tilde{W}]\) which has been shown to satisfy (7.45) in Lemma 7.9 further satisfy (7.2) for all \( \psi \in C^\infty_c([0,T)) \) and \( \phi \in C^\infty(T^3) \) \( \mathbb{P}\text{-a.s.} \).

To complete the proof of Theorem 7.2 we identify the limit energy inequality.

**Lemma 7.12.** The random distributions \([\tilde{\theta}, \tilde{u}, \tilde{V}]\) satisfies (7.8) for all \( \psi \in C^\infty_c([0,T)), \psi \geq 0 \ \mathbb{P}\text{-a.s.} \).

Since we have strong convergence of the density sequence (4.61), the only issue is the identification of the noise term (last term) in (7.8). Since \( f \in L^\infty_c \) is given, the argument follow exactly as in [1 Proposition 4.4.13].
8. The Fifth Approximation Layer

Under the assumptions of Theorem 2.2, we establish the existence of a solution in the sense of Definition 2.1 to the following system

\[ \begin{align*}
\frac{d\varrho}{dt} + \text{div}(\varrho u) &= 0, \\
\frac{d(\varrho u)}{dt} + \left[ \text{div}(\varrho u \otimes u) + \nabla p(\varrho) \right] &= \left[ \nu^g \Delta u + (\nu^B + \nu^S) \nabla \varrho \nabla V \right] dt + \sum_{k \in \mathbb{N}} \varrho g_k(\varrho, u) d\beta_k, \\
 \pm \Delta V &= \varrho - f,
\end{align*} \]

by passing to the limit \( \delta \to 0 \) in (7.1)–(7.2).

8.1. Construction of law. Before we start the main arguments, we first construct a law \( \Lambda \) satisfying the assumptions of Theorem 2.2. This will be done from the approximation laws \( \Lambda_\delta \) satisfying Theorem 7.2 such that \( \Lambda_\delta \overset{\text{w}}{\to} \Lambda \) in the sense of measures on \( [L^1]^3 \). To see this, let \( \Lambda \) be a Borel probability measure on \( [L^1]^3 \). Since \( [L^1]^3 \) is a Polish space, by [1, Corollary 2.6.4], there exists a stochastic basis \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) containing some \( [L^1]^3 \)-valued \( \mathcal{F}_0 \)-measurable random variables \( (\varrho_0, m_0, f) \) having the law \( \Lambda \). First of all, since these random variables have law \( \Lambda \),

\[ f \leq \overline{f} \]

holds \( \mathbb{P} \)-a.s. for a deterministic constant \( \overline{f} > 0 \). It follows from this boundedness and the fact that \( \mathbb{T}^3 \) is of finite measure that \( f \in L^1_x \) for all \( r \in [1, \infty) \). Now set \( \varrho_{0, \delta} = \varrho_0 \) for all \( \delta > 0 \) so that

\[ 0 < M \leq \varrho_{0, \delta} \leq M^{-1} \]

holds \( \mathbb{P} \)-a.s. for a \( \delta \)-uniform deterministic constant \( M > 0 \). Since \( \varrho_0 \) is bounded, it follows that \( \mathbb{P} \)-a.s. \( \varrho_{0, \delta} \in L^1_x \) and in particular for \( \gamma \leq \Gamma \), we have that

\[ \varrho_{0, \delta} \to \varrho_0 \quad \text{in} \quad L^1_x \quad \mathbb{P} \text{-a.s.} \]

If we also set \( V_{0, \delta} = V_0 = \pm \Delta^{-1}_{\mathbb{T}^3} (\varrho_0 - f) \), then we have that

\[ \mathbb{E} \left[ \int_{\mathbb{T}^3} \varrho |\nabla V_{0, \delta}|^2 \, dx \right]_{\varrho_0} \lesssim 1 \quad (8.4) \]

uniformly in \( \delta \) for all \( p_0 \in [1, p] \) and that

\[ \nabla V_{0, \delta} \to \nabla V_0 \quad \text{in} \quad L^{p_0}(\Omega; L^2_x) \quad \text{for all} \quad p_0 \in [1, 2p]. \quad (8.5) \]

Now set

\[ \tilde{m}_{0, \delta} = \begin{cases} m_0 & \text{if} \ \varrho > 0 \\
0 & \text{if} \ \varrho = 0 \end{cases} \]

so that

\[ \mathbb{E} \left[ \int_{\mathbb{T}^3} \frac{1}{2} \frac{\tilde{m}_{0, \delta}}{\varrho_{0, \delta}}^2 \, dx \right]_{\varrho_0} \lesssim 1 \quad (8.7) \]

uniformly in \( \delta \) for all \( p_0 \in [1, p] \). Now if we represent the smooth version (after mollification say,)

\[ \left( \frac{\tilde{m}_{0, \delta}}{\sqrt{\varrho_{0, \delta}}} - s_\delta \right) \to 0 \quad \text{in} \quad L^{p_0}(\Omega; L^2_x) \]

for all \( p_0 \in [1, 2p] \), then for \( m_{0, \delta} = s_\delta \sqrt{\varrho_{0, \delta}} \), we have that

\[ \mathbb{E} \left[ \int_{\mathbb{T}^3} \frac{1}{2} \frac{m_{0, \delta}}{\varrho_{0, \delta}}^2 \, dx \right]_{\varrho_0} \lesssim 1 \]

(8.8)
uniformly in \( \delta \) for all \( p_0 \in [1, p] \). In addition (recall again that \( \varrho_{0, \delta} = \varrho_0 \)),

\[
\begin{align*}
\text{m}_{0, \delta} &\rightarrow \text{m}_0 & \text{in } L^{p_0}(\Omega; L^1_2) & \text{ for all } p_0 \in [1, p], \\
\text{m}_{0, \delta} &\rightarrow \text{m}_0 & \text{in } L^{p_0}(\Omega; L^2_2) & \text{ for all } p_0 \in [1, 2p].
\end{align*}
\]

By using the inequality \((a + b)^r \leq a^r + b^r\), we obtain the estimate

\[
\mathbb{E} \left[ \int_{\mathbb{T}^3} \left( \frac{1}{2} \frac{\text{m}_{0, \delta}^2}{\varrho_{0, \delta}} + P^F_{\delta}(\varrho_{0, \delta}) + \vartheta |\nabla V_0^\delta|^2 \right) dx \right]^{p_0}_{\varrho_0, 1} \leq_{p_0, 1}
\]

uniformly in \( \delta \) by collecting the suitable information above. Now if we set \( \Lambda_{\delta} = \mathbb{P} \circ (\varrho_{0, \delta}, \text{m}_{0, \delta}, f)^{-1} \), then we observe that \( \Lambda_{\delta} \) satisfies the corresponding assumptions of Theorem 7.2 uniformly in \( \delta \) and in addition, \( \Lambda_{\delta} \xrightarrow[\delta \to 0]{} \Lambda \) in the sense of measures on \([L^1_2]^3\).

8.2. Uniform estimates. Given that the data \((\varrho_{0, \delta}, \text{m}_{0, \delta}, f)\) just constructed above satisfies the assumptions of Theorem 7.2 there exists a dissipative martingale solution \([\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}); \varrho_{\delta}, \text{u}_{\delta}, V_{\delta}, W\]

of (7.1)–(7.3) in the sense of Definition 7.1 with \( \Lambda_{\delta} \) as its law. As usual, without loss of generality, we consider a single stochastic basis and a single Wiener process for each \( \delta > 0 \). Since the solution satisfies the energy estimate (7.8), by approximating the characteristic function \( \chi_{[0, s]} \) for any \( s \in (0, T) \), by a sequence of test functions \( \psi \in C_\infty(\mathbb{R}) \), it follows that

\[
\begin{align*}
\int_{\mathbb{T}^3} \left[ \frac{1}{2} \varrho_{0} |\text{u}_{\delta}|^2 + P^F_{\delta}(\varrho_{0}) + \vartheta |\nabla V_0^\delta|^2 \right](S) dx + \nu^S \int_0^s \int_{\mathbb{T}^3} |\nabla \text{u}_{\delta}|^2 dx dt \\
+ (\nu^B + \nu^S) \int_0^s \int_{\mathbb{T}^3} |\text{div} \text{u}_{\delta}|^2 dx dt \leq \int_{\mathbb{T}^3} \frac{1}{2} \frac{\text{m}_{0, \delta}^2}{\varrho_{0, \delta}} + P^F_{\delta}(\varrho_{0, \delta}) + \vartheta |\nabla V_0^\delta|^2 \right] dx \\
+ \frac{1}{2} \int_0^s \int_{\mathbb{T}^3} \varrho_{\delta} \sum_{k \in \mathbb{N}} |g_{0, \varepsilon}(x, \varrho_{\delta}, f, \text{m}_{\delta})|^2 dx dt + \int_0^s \varrho_{0} \text{u}_{\delta} \cdot \text{G}_{\varepsilon}(\varrho_{\delta}, f, \text{m}_{\delta}) dx dW;
\end{align*}
\]

holds \( \mathbb{P} \)-a.s. for a.e. \( s \in (0, T) \). Given that

\[
E_{0, \delta} := \frac{1}{2} \frac{\text{m}_{0, \delta}^2}{\varrho_{0, \delta}} + P^F_{\delta}(\varrho_{0, \delta}) + \vartheta |\nabla V_0^\delta|^2
\]

is \( \delta \)-uniformly bounded in \( L^p(\Omega; L^1_2) \), just as in Section 7.2 we obtain the following estimates

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0, T]} \| \varrho_{\delta} |\text{u}_{\delta}|^2 \|_{L^2_2} \right]^p + \mathbb{E} \left[ \sup_{t \in [0, T]} \| \varrho_{\delta} \|_{L^2_2} \right]^p + \mathbb{E} \left[ \sup_{t \in [0, T]} \| \varrho_{\delta} \|_{L^4_2} \right]^p + \mathbb{E} \left[ \sup_{t \in [0, T]} \| \nabla V_0^\delta \|_{L^2_2} \right]^p \\
+ \mathbb{E} \left[ \sup_{t \in [0, T]} \| \Delta V_0^\delta \|_{L^2_2} \right]^p + \mathbb{E} \left[ \| \text{u}_{\delta} \|_{L^2_2 W^{1, 2} \Omega} \right]^p \leq \mathbb{E} \left[ \sup_{t \in [0, T]} \| \varrho_{\delta} |\text{u}_{\delta}|^2 \|_{L^2_2} \right]^p \mathbb{E} \left[ \sup_{t \in [0, T]} \| \nabla V_0^\delta \|_{L^2_2} \right]^p \mathbb{E} \left[ \sup_{t \in [0, T]} \| \Delta V_0^\delta \|_{L^2_2} \right]^p \mathbb{E} \left[ \sup_{t \in [0, T]} \| \text{u}_{\delta} \|_{L^2_2 W^{1, 2} \Omega} \right]^p \leq_{p, \vartheta, \epsilon_0, \delta, 1} 1
\end{align*}
\]

which holds uniformly in \( \delta \). Furthermore, it follows from the continuity equation (8.11) and (8.10) that

\[
\| \varrho_{\delta}(t) \|_{L^1_2} = \| \varrho_{0, \delta} \|_{L^1_2} \leq M^{-1}
\]

for any \( t \in [0, T] \).

8.3. Improved pressure estimate. Now since \((\varrho_{\delta}, \text{u}_{\delta})\) satisfies the renormalized continuity equation

\[
0 = \int_0^T \int_{\mathbb{T}^3} b(\varrho_{\delta}) \partial_t \phi dx dt + \int_0^T \int_{\mathbb{T}^3} b(\varrho_{0}(0)) \phi(0) dx \\
+ \int_0^T \int_{\mathbb{T}^3} [b(\varrho_{\delta}) \text{u}_{\delta}] \cdot \nabla \phi dx dt - \int_0^T \int_{\mathbb{T}^3} [(b' (\varrho_{\delta}) \varrho_{\delta} - b(\varrho_{\delta})) \text{div} \text{u}_{\delta}] \phi dx dt
\]

(8.16)
\(\mathbb{P}\)-a.s. or any \(\phi \in C_c^\infty([0,T) \times T^3)\) and \(b \in C_1^0(\mathbb{R})\) such that \(b'(z) = 0\) for all \(z \geq M_b\), it follows that for \(b(\varrho) = \varrho_0^\delta\) where \(0 < \Theta \leq \frac{1}{3}\), we have that

\[
\begin{align*}
&- \int_{T^3} \varrho_0^\delta(0) \phi(0) \, dx - \int_0^s \int_{T^3} \varrho_0^\delta \partial_t \phi \, dx \, dt \\
&\quad = \int_0^s \int_{T^3} \left[ \varrho_0^\delta u_\delta \right] \cdot \nabla \phi \, dx \, dt - (\Theta - 1) \int_0^s \int_{T^3} \left[ \varrho_0^\delta \operatorname{div} u_\delta \right] \phi \, dx \, dt
\end{align*}
\]

holds \(\mathbb{P}\)-a.s. for any \(\phi \in C_c^\infty([0,T) \times T^3)\). Now for \(\phi = \Delta_\varrho^{-1} \operatorname{div} \psi\) where \(\psi \in C_c^\infty([0,T) \times T^3)\), we obtain

\[
\int_{T^3} \nabla \Delta_\varrho^{-1} \varrho_0^\delta (0) \psi(0) \, dx + \int_0^s \int_{T^3} \nabla \Delta_\varrho^{-1} \varrho_0^\delta \partial_t \psi \, dx \, dt = \int_0^s \int_{T^3} \nabla \Delta_\varrho^{-1} \operatorname{div} \left[ \varrho_0^\delta u_\delta \right] \psi \, dx \, dt
\]

and

\[
(\Theta - 1) \int_0^s \int_{T^3} \nabla \Delta_\varrho^{-1} \left[ \varrho_0^\delta \operatorname{div} u_\delta \right] \psi \, dx \, dt
\]

\(\mathbb{P}\)-a.s. Also, as the momentum equation (7.22) is only satisfied weakly in the sense of (7.6), a suitable approximation of the characteristic function \(\chi_{[0,s]}\) for any \(s \in (0, T]\), by a sequence of test functions \(\psi \in C_c^\infty([0,T))\) yields

\[
\int_{T^3} \varrho_0 u_\delta(t) \cdot \phi \, dx = \int_{T^3} \varrho_0 u_\delta(0) \cdot \phi \, dx + \int_0^s \int_{T^3} \varrho_0 u_\delta \otimes u_\delta : \nabla \phi \, dx \, dt \\
- \nu^s \int_0^s \int_{T^3} \nabla u_\delta : \nabla \phi \, dx \, dt - (\nu^B + \nu^S) \int_0^s \int_{T^3} \operatorname{div} u_\delta \operatorname{div} \phi \, dx \, dt
\]

and

\[
\begin{align*}
&+ \int_0^s \int_{T^3} \nu (\varrho_0) \operatorname{div} \phi \, dx \, dt + \int_0^s \int_{T^3} \psi (\varrho_0) \nabla \psi \cdot \phi \, dx \, dt \\
&+ \int_0^s \int_{T^3} \nu (\varrho_0) \operatorname{div} \phi \, dx \, dt
\end{align*}
\]

\(\mathbb{P}\)-a.s. Since both (8.18) and (8.19) are in their distributional form, after a long and tedious regularization procedure (which we do not show), we obtain from Iō’s lemma,

\[
\begin{align*}
&\int_0^s \int_{T^3} \nu (\varrho_0) \varrho_0^\delta \, dx \, dt = \int_{T^3} \varrho_0 u_\delta(t) \cdot \nabla \Delta_\varrho^{-1} \varrho_0^\delta \, dx - \int_{T^3} \varrho_0 u_\delta(0) \cdot \nabla \Delta_\varrho^{-1} \varrho_0^\delta \, dx \\
&\quad - \int_0^s \int_{T^3} \varrho_0 u_\delta \otimes u_\delta : \nabla \Delta_\varrho^{-1} \varrho_0^\delta \, dx - \int_0^s \int_{T^3} \varrho_0 u_\delta(0) \cdot \nabla \Delta_\varrho^{-1} \varrho_0^\delta(0) \, dx \\
&\quad - \nu^s \int_0^s \int_{T^3} \nabla u_\delta : \nabla \Delta_\varrho^{-1} \varrho_0^\delta \, dx + (\nu^B + \nu^S) \int_0^s \int_{T^3} \operatorname{div} u_\delta \operatorname{div} \varrho_0^\delta \, dx \, dt
\end{align*}
\]

Now since the embedding \(W_2^{1,r} \hookrightarrow L_2^{\infty}\) is continuous for \(r \geq 3\), it follows from (8.15) and the fact that \(0 < \Theta \leq \frac{1}{3}\),

\[
\begin{align*}
\sup_{t \in [0,T]} \left\| \nabla \Delta_\varrho^{-1} \varrho_0^\delta(t) \right\|_{L_2^{\infty}} &\leq \left\| \varrho_0 \right\|_{L_2} \leq M^{-\Theta}, \quad \mathbb{P}\text{-a.s. where } q = \frac{1}{\Theta}
\end{align*}
\]

uniform in \(\delta\). By using (8.21), we can replicate the estimates in (7.3) so that we obtain from (8.20),

\[
\mathbb{E} \left[ \int_0^s \int_{T^3} \nu (\varrho_0) \varrho_0^\delta \, dx \, dt \right] \leq \left\| p_{\ell} \right\|_{p, k, \gamma, \ell, \Theta, M}
\]

uniformly in \(\delta\) for all \(p \geq 2\). We remind the reader of (4.10), i.e.,

\[
p_{\ell} (\varrho_0) \varrho_0^\delta = [p(\varrho_0) + \delta(\varrho_0 + \varrho_0^\delta)] \varrho_0^\delta
\]
and also of \cite{8, 13} which states that $E_{0, \delta}$ is uniformly bounded in $\delta$.

8.4. **Compactness.** We now define the following spaces

$$\chi_{\theta_0} = L^1, \quad \chi_{m_0} = L^1, \quad \chi_{\frac{m_{0,m}}{\nu_{0,m}}} = L^2_w, \quad \chi_u = (L^2(0, T; W^{1,2}_x), \omega), \quad \chi_V = C_{w}([0, T]; W^2_{x,\gamma}),$$

$$\chi_W = C([0, T]; \Omega_0), \quad \chi_{\nu m} = C_{w}([0, T]; L^{\frac{2}{k-2}}) \cap C([0, T]; W^{-k,2}),$$

$$\chi_{\phi} = C_{w}([0, T]; L^2_{\gamma}) \cap (L^{2}_{\gamma + \Theta}, w) \quad \chi_x = (L^{2}_{\gamma}; M^0(T^3), w), \quad \chi_{\nu} = (L^{\infty}(0, T) \times T^3; \mathcal{P}(\mathbb{R}^{20})), w^*)$$

for $\gamma > \frac{1}{2}$ and $k > \frac{3}{2}$. We now let $\mu_{\theta_{0, \delta}}, \mu_{m_{0, \delta}}, \mu_{\frac{m_{0,m}}{\nu_{0,m}}}, \mu_{\phi}, \mu_{U}, \mu_{e}, \nu_{\phi}, \nu_{U}, \nu_{e}$ and $W$ be the respective laws of $\theta_{0, \delta}, m_{0, \delta}, \frac{m_{0,m}}{\nu_{0,m}}, \phi, u, U, \epsilon, \nu, \delta$ and $W$ on the respective spaces $\chi_{\theta_0}, \chi_{m_0}, \chi_{\frac{m_{0,m}}{\nu_{0,m}}}, \chi_{\phi}, \chi_{\nu m}, \chi_U, \chi_{\nu U}, \chi_{\nu e}, \chi_{\nu}$ and $\chi_{W}$. Furthermore, we set $\mu_{\phi}$ as their joint law on the space $\chi = \chi_{\theta_0} \times \chi_{m_0} \times \chi_{\frac{m_{0,m}}{\nu_{0,m}}} \times \chi_{\phi} \times \chi_U \times \chi_{\nu U} \times \chi_{\nu e} \times \chi_{\nu} \times \chi_{\nu} \times \chi_{W}$.

In analogy with \cite{3}, we have the following result.

**Lemma 8.1.** The set $\{\mu_{\phi}: \delta \in (0, 1)\}$ is tight on $\chi$.

By applying the Jakubowski–Skorokhod theorem \cite{15}, we obtain the following result.

**Lemma 8.2.** The exists a subsequence (not relabelled) $\{\mu_{\phi}: \delta \in (0, 1)\}$, a complete probability space $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\chi$-valued random variables

$$(\tilde{\theta}_{0, \delta}, \tilde{m}_{0, \delta}, \tilde{u}_{0, \delta}, \tilde{\phi}, \tilde{u}, \tilde{m}, \tilde{V}, \tilde{E}, \tilde{\nu}, \tilde{W}) \quad \delta \in (0, 1)$$

such that

- the law of $(\tilde{\theta}_{0, \delta}, \tilde{m}_{0, \delta}, \tilde{u}_{0, \delta}, \tilde{\phi}, \tilde{u}, \tilde{m}, \tilde{V}, \tilde{E}, \tilde{\nu}, \tilde{W})$ on $\chi$ coincide with $\mu_{\phi}, \delta \in (0, 1)$,
- the law of $(\tilde{\theta}_{0}, \tilde{m}_{0}, \tilde{u}, \tilde{\phi}, \tilde{u}, \tilde{m}, \tilde{V}, \tilde{E}, \tilde{\nu}, \tilde{W})$ on $\chi$ is a Radon measure,
- the following convergence (with each $\rightarrow$ interpreted with respect to the corresponding topology)

$$\tilde{\theta}_{0, \delta} \rightarrow \tilde{\theta}_0 \quad \text{in} \quad \chi_{\theta_0}, \quad \tilde{m}_{0, \delta} \rightarrow \tilde{m}_0 \quad \text{in} \quad \chi_{m_0},$$

$$\tilde{u}_{0, \delta} \rightarrow \tilde{u}_0 \quad \text{in} \quad \chi_{\nu m}, \quad \tilde{\phi} \rightarrow \tilde{\phi} \quad \text{in} \quad \chi_{\phi},$$

$$\tilde{u} \rightarrow \tilde{u} \quad \text{in} \quad \chi_U, \quad \tilde{m} \rightarrow \tilde{m} \quad \text{in} \quad \chi_m,$$

$$\tilde{V} \rightarrow \tilde{V} \quad \text{in} \quad \chi_V, \quad \tilde{E} \rightarrow \tilde{E} \quad \text{in} \quad \chi_E,$$

$$\tilde{\nu} \rightarrow \tilde{\nu} \quad \text{in} \quad \chi_{\nu}, \quad \tilde{W} \rightarrow \tilde{W} \quad \text{in} \quad \chi_W$$

holds $\tilde{\mathbb{P}}$-a.s.;

- consider any Carathéodory function $C = C(t, x, \phi, u, U, m, f, V)$ with

$$(t, x, \phi, u, U, m, f, V) \in [0, T] \times T^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3,$$

and where the following estimate

$$|C| \lesssim 1 + |\phi|^r + |u|^r + |U|^r + |m|^r + |f|^r + |V|^r$$

holds uniformly in $(t, x)$ for some $r_i > 0$, $i = 1, \ldots, 6$. Then as $\delta \rightarrow 0$, it follows that

$$C(\tilde{\theta}, \tilde{u}, \tilde{\nu} \tilde{u}, \tilde{m}, f, \nabla \tilde{V}) \rightarrow C(\tilde{\theta}, \tilde{u}, \nabla \tilde{u}, \tilde{m}, f, \nabla V)$$

holds in $L^r([0, T) \times T^3)$ for all

$$1 < r \leq \frac{2 + \Theta}{r_1} \wedge \frac{2}{r_2} \wedge \frac{2\gamma}{r_4(\gamma + 1)} \wedge \frac{2}{r_6}$$

$\tilde{\mathbb{P}}$-a.s.

The following corollary follow from Lemma 8.2.
Corollary 8.3. The following holds $\tilde{P}$-a.s.
\[
\tilde{\varrho}_{0,\delta} = \tilde{\varrho}_0(0), \quad \tilde{m}_{0,\delta} = \tilde{\varrho}_0 \tilde{u}_3(0), \quad \tilde{n}_{0,\delta} = \frac{\tilde{m}_{0,\delta}}{\sqrt{\tilde{\varrho}_{0,\delta}}}, \quad \tilde{m}_4 = \tilde{\varrho}_0 \tilde{u}_4,
\]
and that
\[
\tilde{\varepsilon}_\delta = \varepsilon_\delta(\tilde{\varrho}_0, \tilde{u}_3, \tilde{V}_\delta), \quad \tilde{\nu}_\delta = \delta_{\tilde{\varrho}_0, \tilde{u}_3, \nabla \tilde{u}_3, \tilde{\varrho}_0 \nabla \tilde{V}_\delta}
\]
for any $\delta > 0$.

The following result thus holds true.

Corollary 8.4. For any $\phi \in C^c_c([0, T) \times \mathbb{T}^3)$ and $b \in C^1_b(\mathbb{R})$ such that $b'(z) = 0$ for all $z \geq M_b$, we have that
\[
0 = \int_0^T \int_{\mathbb{T}^3} b(\tilde{\varrho}_0) \partial_t \phi \, dx \, dt + \int_{\mathbb{T}^3} b(\tilde{\varrho}_0) \phi(0) \, dx
\]
\[
+ \int_0^T \int_{\mathbb{T}^3} [b(\tilde{\varrho}_0) \tilde{u}_3] \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathbb{T}^3} [(\nu(\tilde{\varrho}_0) \tilde{\varrho}_0 - b(\tilde{\varrho}_0)) \div \tilde{u}_3] \phi \, dx \, dt
\]
holds $\tilde{P}$-a.s.

Now analogous to Corollary 8.3, we also have the following result.

Corollary 8.5. There exists $p^\delta_f(\tilde{\varrho}_0), |\nabla \tilde{V}|^2, \tilde{\varrho}_0 \nabla \tilde{V}$ and $\tilde{\varrho}_0 \tilde{g}_k(\tilde{\varrho}_0, \tilde{f}, \tilde{u}_3)$ such that
\[
p^\delta_f(\tilde{\varrho}_0) \rightarrow p^\delta_f(\varrho)
\]
\[
|\nabla \tilde{V}|^2 \rightarrow |\nabla \tilde{V}|^2
\]
\[
\tilde{\varrho}_0 \nabla \tilde{V}_\delta \rightarrow \tilde{\varrho}_0 \nabla \tilde{V}
\]
\[
\tilde{\varrho}_0 \tilde{g}_k(\tilde{\varrho}_0, \tilde{f}, \tilde{u}_3) \rightarrow \tilde{\varrho}_0 g_k(\tilde{\varrho}_0, \tilde{f}, \tilde{u}_3)
\]
$\tilde{P}$-a.s. in $L^r((0, T) \times \mathbb{T}^3)$ for some $r > 1$.

The corresponding versions of Lemma 7.7–Lemma 7.45 on this approximation layer holds true.

Lemma 8.6. The random distributions $[\tilde{\varrho}, \tilde{u}]$ satisfies (8.12) for all $\psi \in C^c_c([0, T))$ and $\phi \in C^\infty(\mathbb{T}^3)$ $\tilde{P}$-a.s.

Lemma 8.7. The random variables $[\tilde{\varrho}, \tilde{V}]$ satisfies (8.3) a.e. in $(0, T) \times \mathbb{T}^3$ $\tilde{P}$-a.s.
Lemma 8.8. The random distributions \([\mathcal{G}, \mathcal{V}, \mathcal{W}]\) satisfies

\[
- \int_0^T \partial_t \psi \int_{\mathcal{T}^3} \mathcal{G}(t) \cdot \phi \, dx \, dt = \psi(0) \int_{\mathcal{T}^3} \mathcal{G}(0) \cdot \phi \, dx + \int_0^T \psi \int_{\mathcal{T}^3} \mathcal{G}(t) \cdot \nabla \phi \, dx \, dt
\]

\[
- \nu^S \int_0^T \psi \int_{\mathcal{T}^3} \nabla \mathcal{V} : \nabla \phi \, dx \, dt - (\nu^B + \nu^S) \int_0^T \psi \int_{\mathcal{T}^3} \left[ \mathcal{V} \cdot \nabla \phi \right] \, dx \, dt
\]

\[
+ \int_0^T \psi \int_{\mathcal{T}^3} \mathcal{U} \cdot \phi \, dx \, dt + \int_0^T \psi \int_{\mathcal{T}^3} \left[ \mathcal{U} \cdot \nabla \phi \right] \, dx \, dt
\]

\[
+ \int_0^T \psi \sum_{k \in \mathbb{N}} \int_{\mathcal{T}^3} \mathcal{W}(\theta) \cdot \phi \, dx \, dt
\]

for all \(\psi \in C^\infty_c([0, T))\) and \(\phi \in C^\infty(\mathcal{T}^3)\) \(\mathbb{P}\)-a.s.

8.5. Strong convergence of Density. Notice the strong convergence \([7,61]\) of the density sequence in the previous approximation layer relied crucially on the DiPerna-Lions renormalization theorem \([5]\). In order to apply this theorem, one requires the density to be at least squared-integrable in time and space, which happened to be the case in Section 7 (recall that \(\Gamma \geq 6\)). Since the fluid limit density in this current approximation layer is only \((\gamma + \Theta)\)-integrable in time and space, we can no longer apply the theorem of DiPerna-Lions since \(\gamma > \frac{4}{3}\) and \(0 < \Theta \leq \frac{1}{3}\). To remedy this problem, we rely on the oscillation defect measure introduced by Feireisl \([8]\) in order to establish the smallness of the amplitude of oscillations in the density sequence. In the present stochastic setting and approximation layer, the measure is given by

\[
\text{osc}_{\gamma+1}[\mathcal{G} \rightarrow \mathcal{G}](\Omega \times Q) = \sup_{k \geq 1} \left( \limsup_{\delta \rightarrow 0} \mathbb{E} \int_Q \left| T_k(\mathcal{G}) - T_k(\mathcal{G}) \right|^{\gamma+1} \, dx \, dt \right)
\]

where \(Q = (0, T) \times \mathcal{T}^3\) and for any \(k \in \mathbb{N}\), \(T_k\) is a cut-off function defined as

\[
T_k(s) = kT \left( \frac{s}{k} \right), \quad T \in C^\infty([0, \infty)), \quad T(s) = \begin{cases} s & \text{if } 0 \leq s \leq 1 \\ T''(s) & \text{if } 1 < s < 3 \\ 2 & \text{if } s \geq 3. \end{cases}
\]

8.5.1. The effective viscous flux. By using the weak compactness result of Carathéodory functions established in Lemma 8.2, we obtain the following

\[
T_k(\mathcal{G}) \rightarrow \overline{T_k(\mathcal{G})} \quad \text{in} \quad L^p((0, T) \times \mathcal{T}^3),
\]

\[
T_k(\mathcal{G}) \rightarrow \overline{T_k(\mathcal{G})} \quad \text{in} \quad C_w((0, T); L^p(\mathcal{T}^3)),
\]

\[
[T_k(\mathcal{G}) - T_k(\mathcal{G})] \mathcal{V} \rightarrow \overline{[T_k(\mathcal{G}) - T_k(\mathcal{G})] \mathcal{V}} \quad \text{in} \quad L^q((0, T) \times \mathcal{T}^3)
\]

\(\mathbb{P}\)-a.s. for any \(p \in (1, \infty)\) and for some \(q > 1\). We can then use \([8.29] + [8.37]\) to pass to limit \(\delta \rightarrow 0\) in \([8.27]\) with \(b = T_k\) to obtain

\[
0 = \int_0^T \int_{\mathcal{T}^3} T_k(\mathcal{G}) \partial_t \phi \, dx \, dt + \int_{\mathcal{T}^3} T_k(\mathcal{G}) \phi(0) \, dx
\]

\[
+ \int_0^T \int_{\mathcal{T}^3} \mathcal{V} \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathcal{T}^3} \left[ (T_k(\mathcal{G}) - T_k(\mathcal{G})) \mathcal{V} \phi \right] \, dx \, dt
\]

\[\text{(8.38)}\]
As in (8.47), we can rewrite the viscous terms in (8.39) and (8.40) appropriately and in analogy with

\[ \int_0^s \int_{T^3} p(\tilde{\rho}_k) T_k(\tilde{\rho}) \, dx \, dt = \int_0^s \int_{T^3} \tilde{\rho}_k(t) \cdot \nabla \Delta_{T^3}^{-1} T_k(\tilde{\rho}_k) \, dx \, dt - \int_0^s \int_{T^3} \tilde{\rho}_0,\delta \tilde{u}_{0,\delta} \cdot \nabla \Delta_{T^3}^{-1} T_k(\tilde{\rho}_k(0)) \, dx \, dt \\
- \int_0^s \int_{T^3} \tilde{\rho}_k \nabla \tilde{u}_k \cdot \nabla \Delta_{T^3}^{-1} T_k(\tilde{\rho}) \, dx \, dt + \int_0^s \int_{T^3} \tilde{\rho}_k \nabla \Delta_{T^3}^{-1} \tilde{u}_k \, dx \, dt + \int_0^s \int_{T^3} \tilde{\rho}_k \nabla \Delta_{T^3}^{-1} \tilde{u}_k \, dx \, dt \]

(8.40)

As in (8.39), we can rewrite the viscous terms in (8.40) appropriately and in analogy with the derivation of Lemma 7.6, we obtain in (8.40),

\[ \lim_{\delta \to 0} \int_0^s \int_{T^3} p(\tilde{\rho}_k) T_k(\tilde{\rho}) \, dx \, dt = \int_0^s \int_{T^3} \left[ p(\tilde{\rho}_k) - (\nu^B + 2\nu^S) \nabla \tilde{u}_k \right] T_k(\tilde{\rho}_k) \, dx \, dt \]

(8.41)

holds \( \mathbb{P} \)-a.s. for a.e. \( s \in (0, T) \) with the help of (8.39) – (8.40), (8.37) – (8.38) and [1] Lemma 2.6.6.

8.5.2. Oscillation defect measure. We now show that the estimate

\[ \text{osc}_{s+1}[\tilde{\rho}_k \to \tilde{\rho}](\bar{Q} \times Q) \lesssim 1. \]

(8.42)

holds uniformly in \( k \). To see this, we first note the identity

\[ \lim_{\delta \to 0} \int_Q \left( p(\tilde{\rho}_k) T_k(\tilde{\rho}) - p(\tilde{\rho}) T_k(\tilde{\rho}) \right) \, dx \, dt = \int_Q \left[ p(\tilde{\rho}_k) T_k(\tilde{\rho}) - p(\tilde{\rho}) T_k(\tilde{\rho}) \right] \, dx \, dt \]

(8.43)

from (8.41) where by use of the inequality

\[ |T_k(t) - T_k(s)|^{\gamma+1} \leq (t^\gamma - s^\gamma) (T_k(t) - T_k(s)), \]

it follows that

\[ \lim_{\delta \to 0} \int_Q \left( p(\tilde{\rho}_k) T_k(\tilde{\rho}) - p(\tilde{\rho}) T_k(\tilde{\rho}) \right) \, dx \, dt \geq \lim_{\delta \to 0} \int_Q \left| T_k(\tilde{\rho}_k) - T_k(\tilde{\rho}) \right|^{\gamma+1} \, dx \, dt \]

\[ + \int_Q \left( p(\tilde{\rho}_k) - p(\tilde{\rho}) \right) \left( T_k(\tilde{\rho}) - T_k(\tilde{\rho}) \right) \, dx \, dt. \]

(8.44)
By combining (8.43) - (8.44) with the negativity of the lest term in (8.44) above (this negativity follows from the convexity of $t \mapsto t^\gamma$ and the concavity of $t \mapsto T_k(t))$, we gain

$$
\limsup_{\delta \to 0} \mathbb{E} \int_Q |T_k(\tilde{\varphi}_\delta) - T_k(\varphi)|^{\gamma+1} \, dx \, dt \\
\leq (\nu^B + 2\nu^S) \limsup_{\delta \to 0} \mathbb{E} \int_Q [\text{div} \, \tilde{u}_\delta T_k(\tilde{\varphi}_\delta) - \text{div} \, \tilde{u} T_k(\varphi)] \, dx \, dt.
$$

(8.45)

However, the use of Hölder and triangle inequalities further yield

$$
\limsup_{n \to \infty} \mathbb{E} \int_Q [\text{div} \, \tilde{u}_n T_k(\tilde{\varphi}_\delta) - \text{div} \, \tilde{u} T_k(\varphi)] \, dx \, dt \\
\leq \limsup_{n \to \infty} \mathbb{E} \|\text{div} \, \tilde{u}_n\|_{L^2(Q)} \mathbb{E} \left(\|T_k(\tilde{\varphi}_\delta) - T_k(\varphi)\|_{L^2(Q)} + \|T_k(\varphi) - T_k(\varphi)\|_{L^2(Q)}\right).
$$

(8.46)

And by lower semi-continuity of norms,

$$
\mathbb{E} \left(\|T_k(\varphi) - T_k(\varphi)\|_{L^2(Q)}\right) \leq \liminf_{\delta \to 0} \mathbb{E} \left(\|T_k(\tilde{\varphi}_\delta) - T_k(\varphi)\|_{L^2(Q)}\right) \\
\leq \limsup_{\delta \to 0} \mathbb{E} \left(\|T_k(\tilde{\varphi}_\delta) - T_k(\varphi)\|_{L^2(Q)}\right).
$$

(8.47)

By using the embedding $L^{\gamma+1} \to L^2$, we can substitute (8.47) into (8.46) to get

$$
\limsup_{n \to \infty} \mathbb{E} \int_Q [\text{div} \, \tilde{u}_n T_k(\tilde{\varphi}_\delta) - \text{div} \, \tilde{u} T_k(\varphi)] \, dx \, dt \\
\leq 2 \limsup_{n \to \infty} \left(\mathbb{E} \|\text{div} \, \tilde{u}_n\|_{L^2(Q)} \mathbb{E} \left(\|T_k(\tilde{\varphi}_\delta) - T_k(\varphi)\|_{L^{\gamma+1}(Q)}\right)\right).
$$

(8.48)

Finally, we substitute (8.48) into (8.45) and apply Young’s inequality to obtain

$$
\limsup_{\delta \to 0} \mathbb{E} \int_Q |T_k(\tilde{\varphi}_\delta) - T_k(\varphi)|^{\gamma+1} \, dx \, dt \\
\leq \limsup_{n \to \infty} \left(\mathbb{E} \|\text{div} \, \tilde{u}_n\|_{L^2(Q)} \mathbb{E} \left(\|T_k(\tilde{\varphi}_\delta) - T_k(\varphi)\|_{L^{\gamma+1}(Q)}\right)\right) \\
\leq \frac{\gamma}{\gamma + 1} \sup_n \left(\mathbb{E} \|\text{div} \, \tilde{u}_n\|_{L^2(Q)}\right)^{\gamma+1} + \frac{1}{\gamma + 1} \limsup_{\delta \to 0} \mathbb{E} \int_Q |T_k(\tilde{\varphi}_\delta) - T_k(\varphi)|^{\gamma+1} \, dx \, dt
$$

uniformly in $k$. The estimate (8.42) follow by absolving the last term above into the left-hand side (note that $\frac{1}{\gamma + 1} < \frac{2}{\gamma}$ is small enough) and keep in mind that $\frac{\gamma+1}{\gamma} < 2$.

8.5.3. The renormalized solution for the limit process. If we now regularize $\mathbb{S}_{m}$ with some regularizing operator $S_m$, multiply the resulting deterministically strong equation by $\tilde{b}'(S_m(T_k(\varphi)))$ and pass to the limit $m \to \infty$, we obtain

$$
0 = \int_0^T \int_{\mathbb{T}^3} \tilde{b}'(T_k(\varphi)) \partial_t \phi \, dx \, dt + \int_{\mathbb{T}^3} \tilde{b}(T_k(\varphi)) \phi(0) \, dx + \int_0^T \int_{\mathbb{T}^3} \tilde{b}(T_k(\varphi)) \tilde{u} \cdot \nabla \phi \, dx \, dt \\
+ \int_0^T \int_{\mathbb{T}^3} \tilde{b}'(T_k(\varphi)) [T_k(\varphi) - T_k(\varphi)] \text{div} \, \tilde{u} \phi \, dx \, dt \\
- \int_0^T \int_{\mathbb{T}^3} [\tilde{b}'(T_k(\varphi)) T_k(\varphi) - \tilde{b}(T_k(\varphi))] \text{div} \, \tilde{u} \phi \, dx \, dt
$$

(8.50)
\( \mathbb{P} \)-a.s. for any \( \phi \in C^\infty_c((0, T) \times \mathbb{T}^3) \). Now if we let \( Q = (0, T) \times \mathbb{T}^3 \) and set \( Q_{k,M} = \{(\omega, t, x) \in \Omega \times Q : |T_k(\bar{\theta})| \leq M\} \), then it follows from Lemma \[ \text{(5.53)} \] that

\[
\left\| b'(T_k(\bar{\theta}))(T_k(\bar{\theta}) \bar{\theta} - T_k(\bar{\theta})) \operatorname{div} \tilde{u} \right\|_{L^1(\Omega \times Q)} \\
\leq \left( \sup_{0 \leq z \leq M} |b'(z)| \right) \liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^1(\Omega \times Q)} \\
\leq \left( \sup_{0 \leq z \leq M} |b'(z)| \right) \left( \sup_{\delta > 0} \left\| \operatorname{div} \tilde{u} \right\|_{L^2(\Omega \times Q)} \right) \liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(\Omega \times Q)} \\
\lesssim_M \liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(Q_{k,M})} \\
 \leq \left( \sup_{0 \leq z \leq M} |b'(z)| \right) \liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(\Omega \times Q)} \\
\lesssim M \liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(Q_{k,M})}
\]}

where by interpolation, we also have that

\[
\left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(Q_{k,M})} \\
\leq \left( \sup_{0 \leq z \leq M} |b'(z)| \right) \liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(\Omega \times Q)} \\
\lesssim M \liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(Q_{k,M})} \\
\leq \left( \sup_{0 \leq z \leq M} |b'(z)| \right) \liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(\Omega \times Q)} \\
\lesssim M \liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(Q_{k,M})}
\]

It follows that

\[
\liminf_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(Q_{k,M})} \\
\lesssim \limsup_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(Q_{k,M})} \lesssim \limsup_{\delta \to 0} \left\| \bar{\delta} \right\|_{L^2(\Omega \times Q)}
\]

where

\[
\limsup_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^2(\Omega \times Q)} \lesssim \limsup_{\delta \to 0} \left\| \bar{\delta} \right\|_{L^2(\Omega \times Q)} \\
\lesssim \left( \frac{1}{k} \right)^{\frac{1}{2}} \frac{1}{2} \limsup_{\delta \to 0} \left\| \bar{\delta} \right\|_{L^2(\Omega \times Q)} \to 0
\]

as \( k \to \infty \) and by triangle inequality,

\[
\left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \bar{\delta} \right\|_{L^{\gamma+1}(Q_{k,M})} \lesssim \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \right\|_{L^{\gamma+1}(Q_{k,M})} + \frac{1}{M}.
\]

Now since

\[
\left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \right\|_{L^{\gamma+1}(Q_{k,M})} \leq \limsup_{\delta \to 0} \left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \right\|_{L^{\gamma+1}(\Omega \times Q)}
\]

and \( \frac{1}{k^2} < 1 \), we obtain from \( \text{(8.54)} \),

\[
\left\| (T_k(\bar{\theta}) - T_k(\bar{\theta})) \right\|_{L^{\gamma+1}(Q_{k,M})} \lesssim \limsup_{\delta \to 0} \left\| \bar{\delta} \right\|_{L^{\gamma+1}(\Omega \times Q)} + \frac{1}{M}.
\]

for a constant that is independent of \( k \).

Now substituting \( \text{(8.53)}, \text{(8.51)}, \text{and (8.56)} \) into \( \text{(8.51)} \), we obtain

\[
\left\| b'(T_k(\bar{\theta}))(T_k(\bar{\theta}) \bar{\theta} - T_k(\bar{\theta})) \operatorname{div} \tilde{u} \right\|_{L^1(\Omega \times Q)} \\
\lesssim_M \left( \frac{1}{k} \right)^{\frac{1}{2}} \frac{1}{2} \limsup_{\delta \to 0} \left\| \bar{\delta} \right\|_{L^{\gamma+1}(\Omega \times Q)} \left( \frac{1}{M} \right) + \frac{1}{M}.
\]

where the right-hand side converges to zero as \( k \to \infty \) since

\[
\text{osc}_{\gamma+1}[\bar{\delta}] \to 0 \quad 0 < \gamma < 1
\]
uniformly in $k$, recall \[8.42\]. We have therefore shown that
\[
b'(T_k(\hat{\varrho}))(T_k(\hat{\varrho}) - T_k(\hat{\varrho})) \text{ div } \mathbf{u} \to 0 \text{ in } L^1(\Omega \times (0,T) \times \mathbb{T}^3)
\] (8.59)
as $k \to \infty$. The convergence \[8.59\] together with
\[
T_k(\hat{\varrho}) \to \hat{\varrho} \text{ in } L'(\Omega \times (0,T) \times \mathbb{T}^3)
\] (8.60)
for all $r \in (1, \gamma)$ as $k \to \infty$ (see \[1\] Eq. (4.232)) allows us to pass to the limit $k \to \infty$ in \[8.50\] and we obtain
\[
0 = \int_0^T \int_{\mathbb{T}^3} b(\hat{\varrho}) \partial_t \phi \, dx \, dt + \int_{\mathbb{T}^3} b(\hat{\varrho}_0) \phi(0) \, dx
+ \int_0^T \int_{\mathbb{T}^3} [b(\hat{\varrho}) \mathbf{u}] \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathbb{T}^3} \left[ (b'(\hat{\varrho}) - b(\hat{\varrho})) \text{ div } \mathbf{u} \right] \phi \, dx \, dt
\] (8.61)
$\mathbb{P}$-a.s. for any $\phi \in C_c^\infty([0,T) \times \mathbb{T}^3)$. If we now introduce
\[
L_k(z) = \begin{cases} z \log(z) & \text{if } z \in [0,k), \\ z \log(k) + z \int_k^z \frac{T_k(s)}{s^2} \, ds & \text{if } z \in [k, \infty) \end{cases}
\] (8.62)
which satisfies
\[
zL_k'(z) - L_k(z) = T_k(z)
\] (8.63)
in place of $b$ in \[8.27\] and \[8.61\], then $\hat{\mathbb{P}}$-a.s., we obtain
\[
0 = \int_0^T \int_{\mathbb{T}^3} L_k(\hat{\varrho}_0) \partial_t \phi \, dx \, dt + \int_{\mathbb{T}^3} L_k(\hat{\varrho}_0,0) \phi(0) \, dx
+ \int_0^T \int_{\mathbb{T}^3} \left[ L_k(\hat{\varrho}_0) \mathbf{u}_0 \right] \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathbb{T}^3} T_k(\hat{\varrho}_0) \text{ div } \mathbf{u}_0 \phi \, dx \, dt
\] (8.64)
and
\[
0 = \int_0^T \int_{\mathbb{T}^3} L_k(\hat{\varrho}) \partial_t \phi \, dx \, dt + \int_{\mathbb{T}^3} L_k(\hat{\varrho}_0) \phi(0) \, dx
+ \int_0^T \int_{\mathbb{T}^3} \left[ L_k(\hat{\varrho}) \mathbf{u} \right] \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathbb{T}^3} T_k(\hat{\varrho}) \text{ div } \mathbf{u} \phi \, dx \, dt
\] (8.65)
respectively and where
\[
L_k(\hat{\varrho}_0) \to \overline{L_k(\hat{\varrho})} \quad \text{in} \quad C_w([0,T];L^p(\mathbb{T}^3))
\] (8.66)
\[
\hat{\varrho}_n \log(\hat{\varrho}_0) \to \overline{\hat{\varrho} \log(\hat{\varrho})} \quad \text{in} \quad C_w([0,T];L^p(\mathbb{T}^3))
\] (8.67)
\[
L_k(\hat{\varrho}_0) \to \overline{L_k(\hat{\varrho})} \quad \text{in} \quad C([0,T];W^{-1,2}(\mathbb{T}^3))
\] (8.68)
\[
T_k(\hat{\varrho}_0) \text{ div } \mathbf{u}_0 \to \overline{T_k(\hat{\varrho}) \text{ div } \mathbf{u}} \quad \text{in} \quad L^p(\mathbb{T}^3)
\] (8.69)
holds $\hat{\mathbb{P}}$-a.s. for any $p \in (1, \gamma)$ and some $q > 1$. We can now use \[8.66\]–\[8.69\] to pass to the limit in \[8.64\] and we obtain
\[
0 = \int_0^T \int_{\mathbb{T}^3} \overline{L_k(\hat{\varrho})} \partial_t \phi \, dx \, dt + \int_{\mathbb{T}^3} L_k(\hat{\varrho}_0) \phi(0) \, dx
+ \int_0^T \int_{\mathbb{T}^3} \left[ L_k(\hat{\varrho}) \mathbf{u} \right] \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathbb{T}^3} \overline{T_k(\hat{\varrho})} \text{ div } \mathbf{u} \phi \, dx \, dt.
\] (8.70)
If we now take the difference of \[8.65\] and \[8.70\] and consider $\phi(t,x) = \phi_m(t)\phi_n(x)$ where $\phi_m$ and $\phi_n$ are approximation sequences of the characteristic functions $\chi_{[0,s]}$, $s \in [0,T]$ and $\chi_{\mathbb{T}^3}$ respectively,
then we obtain
\[
\int_0^T \int_{\mathbb{T}^3} \left[ L_k(\tilde{\varrho}) - L_k(\tilde{\varrho}) \right] \, dx \, dt = \int_0^T \int_{\mathbb{T}^3} \left[ T_k(\tilde{\varrho}) \text{div} \tilde{\mathbf{u}} - T_k(\tilde{\varrho}) \text{div} \mathbf{u} \right] \, dx \, dt
\]
\[
= \int_0^T \int_{\mathbb{T}^3} \left[ T_k(\tilde{\varrho}) \text{div} \tilde{\mathbf{u}} - T_k(\tilde{\varrho}) \text{div} \mathbf{u} \right] \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} \left[ T_k(\tilde{\varrho}) - T_k(\tilde{\varrho}) \right] \text{div} \mathbf{u} \, dx \, dt
\]
\[
=: I_1 + I_2
\]
where by a similar argument as in (8.49), we have that
\[
I_1 \geq \frac{1}{(\nu^B + 2\nu^S)} \lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} \left| T_k(\tilde{\varrho}) - T_k(\tilde{\varrho}) \right|^{7+1} \, dx \, dt.
\]
By the bounded of \( \tilde{\mathbf{u}} \) in \( L^2_{t}W^{1,2}_x \), the smallness of the oscillation defect measure (8.42) and properties of \( T_k \), we also have that
\[
I_2 \to 0
\]
as \( k \to \infty \). Also, one can verify that the left-hand side of (8.71) is bounded by using (8.63). The collection of the above information implies that
\[
\int_{\mathbb{T}^3} \left[ \frac{\partial}{\partial t} \log \varrho - \tilde{\varrho} \log \tilde{\varrho} \right] (t) \, dx = 0 \quad \text{for all} \quad t \in [0, T]
\]
as \( k \to \infty \) in (8.71). This implies that
\[
\tilde{\varrho}(t) \to \tilde{\varrho}(t) \quad \text{in} \quad L^1(\mathbb{T}^3) \quad \text{for any} \quad t \in [0, T]
\]
since \( \tilde{\varrho} \to \tilde{\varrho} \log \tilde{\varrho} \) is strictly convex. It also follows from (8.75) that
\[
\Delta \tilde{V}_k(t) \to \Delta \tilde{V}(t) \quad \text{in} \quad L^1(\mathbb{T}^3) \quad \text{for any} \quad t \in [0, T].
\]
We can now use the Lipschitz continuity of \( g_k \) and (8.76) to obtain
\[
\int_{\mathbb{T}^3} \tilde{\varrho} g_k(\tilde{\varrho}, f, \tilde{\varrho} \tilde{\mathbf{u}}) \cdot \phi \, dx \to \int_{\mathbb{T}^3} \tilde{\varrho} g_k(\tilde{\varrho}, f, \tilde{\varrho} \tilde{\mathbf{u}}) \cdot \phi \, dx \quad \text{a.e. in} \quad (0, T)
\]
\( \mathbb{P} \)-a.s. for any \( \phi \in C^\infty(\mathbb{T}^3) \) from which we infer that
\[
\tilde{\varrho} g_k(\tilde{\varrho}, f, \tilde{\varrho} \tilde{\mathbf{u}}) = \tilde{\varrho} g_k(\tilde{\varrho}, f, \tilde{\varrho} \tilde{\mathbf{u}}) \quad \text{a.e. in} \quad \tilde{\Omega} \times (0, T) \times \mathbb{T}^3.
\]
We are therefore able to conclude that the nonlinear pressure term in (8.32) is indeed isentropic by the use of (8.70). Furthermore, the nonlinear noise term in (8.32) is also of the require form since we have (8.78). Finally as in Lemma 7.12 we also have the following result.

**Lemma 8.9.** The random distributions \( \tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{V} \) satisfies (2.13) for all \( \psi \in C^\infty_c((0, T)) \), \( \psi \geq 0 \) \( \mathbb{P} \)-a.s.

### 9. Existence on the whole space

In this section we deal with the existence in the whole domain case \( \mathcal{O} = \mathbb{R}^3 \). For \( (x, \varrho, f, \mathbf{u}) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \), we assume that there exists some functions \( (g_k)_{k \in \mathbb{N}} \) such that for any \( k \in \mathbb{N} \),
\[
g_k : \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}, \quad g_k \in C^1 \left( \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \right)
\]
and the support of each \( g_k \) in \( \mathbb{R}^3 \) is \( K \) where \( K \subset \mathbb{R}^3 \), i.e.,
\[
\exists K \subset \mathbb{R}^3 \quad \text{such that} \quad g_k = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus K.
\]
In addition, \( g_k \) satisfies the growth conditions (2.41) - (2.46).

We study the problem under the far-field condition
\[
\mathbf{u} \to 0, \quad \varrho \to \tilde{\varrho} > 0
\]
as \( |x| \to \infty \). By the above conditions, with the same lines of arguments of Section 2.2, we have that the stochastic integral \( \int_0^T \varrho G(\tilde{\varrho}, f, \mathbf{u}) \, dW \) is well-defined.
9.1. Concept of a solution.

Definition 9.1. Let $\Lambda = \Lambda(\rho, m, f)$ be a Borel probability measure on $[L^1(\mathbb{R}^3)]^3$. We say that
\[
([\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}); \rho, u, V, W]
\] is a \textit{dissipate martingale solution} of \eqref{1.1}–\eqref{1.3} with initial law $\Lambda$ provided

1. $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
2. $W$ is a $(\mathcal{F}_t)$-cylindrical Wiener process;
3. the density $\rho \in C_w([0, T]; L^\infty(\mathbb{R}^3))$ $\mathbb{P}$-a.s., it is $(\mathcal{F}_t)$-progressively measurable and
\[
\mathbb{E} \left[ \sup_{t \in (0, T)} \| \rho(t) \|_{L^p(\mathbb{R}^3)}^p \right] < \infty
\]
for all $1 \leq p < \infty$ where $\gamma = \min\{2, \gamma\}$;
4. the velocity field $u$ is an $(\mathcal{F}_t)$-adapted random distribution and
\[
\mathbb{E} \left[ \int_0^T \| u(t) \|_{W^{1,2}(\mathbb{R}^3)}^2 \, dt \right] < \infty
\]
for all $1 \leq p < \infty$;
5. the momentum $\rho u \in C_w([0, T]; L^{\frac{2p}{p+1}}(\mathbb{R}^3))$ $\mathbb{P}$-a.s., it is $(\mathcal{F}_t)$-progressively measurable and
\[
\mathbb{E} \left[ \sup_{t \in (0, T)} \| \rho u(t) \|_{L^p(\mathbb{R}^3)}^p \right] + \mathbb{E} \left[ \sup_{t \in (0, T)} \| \rho u(t) \|_{L^{\frac{2p}{p+1}}(\mathbb{R}^3)}^p \right] < \infty
\]
for all $1 \leq p < \infty$;
6. the field $\Delta V \in C_w([0, T]; L^2(\mathbb{R}^3))$ $\mathbb{P}$-a.s., it is $(\mathcal{F}_t)$-progressively measurable and
\[
\mathbb{E} \left[ \sup_{t \in (0, T)} \| \Delta V(t) \|_{L^p(\mathbb{R}^3)}^p \right] + \mathbb{E} \left[ \sup_{t \in (0, T)} \| \nabla V(t) \|_{L^2(\mathbb{R}^3)}^{2p} \right] < \infty
\]
for all $1 \leq p < \infty$ where $\gamma = \min\{2, \gamma\}$;
7. given $f \in L_1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, there exists $\mathcal{F}_0$-measurable random variables $(\rho_0, \rho_0 u_0) = (\rho(0), \rho u(0))$ such that $\Lambda = \mathbb{P} \circ (\rho_0, \rho_0 u_0, f)^{-1}$;
8. for all $\psi \in C_c^\infty([0, T))$ and $\phi \in C_c^\infty(\mathbb{R}^3)$, the following
\[
- \int_0^T \partial_t \psi \int_{\mathbb{R}^3} \rho(t) \phi \, dx \, dt = \psi(0) \int_{\mathbb{R}^3} \rho_0 \phi \, dx + \int_0^T \psi \int_{\mathbb{R}^3} \rho u \cdot \nabla \phi \, dx \, dt,
\] hold $\mathbb{P}$-a.s.;
9. for all $\psi \in C_c^\infty([0, T))$ and $\phi \in C_c^\infty(\mathbb{R}^3)$, the following
\[
- \int_0^T \partial_t \psi \int_{\mathbb{R}^3} \rho u(t) \cdot \phi \, dx \, dt = \psi(0) \int_{\mathbb{R}^3} \rho_0 u_0 \cdot \phi \, dx + \int_0^T \psi \int_{\mathbb{R}^3} \rho u \otimes u : \nabla \phi \, dx \, dt
\] \[= - \nu^S \int_0^T \psi \int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi \, dx \, dt - (\nu^B + \nu^S) \int_0^T \psi \int_{\mathbb{R}^3} \text{div} u \text{div} \phi \, dx \, dt
\] \[+ \int_0^T \psi \int_{\mathbb{R}^3} \rho(t) \phi \, dx \, dt + \int_0^T \psi \int_{\mathbb{R}^3} \partial \rho \nabla V \cdot \phi \, dx \, dt
\] \[+ \int_0^T \psi \int_{\mathbb{R}^3} \phi \, G(\rho, f, m) \cdot \phi \, dx \, dW,
\] hold $\mathbb{P}$-a.s.;
10. equation \eqref{1.3} holds $\mathbb{P}$-a.s. for a.e. $(t, x) \in (0, T) \times \mathbb{R}^3$;
we first consider the following family of cut-off functions

\( \psi \)

Finally assume that

\( \nu^B + \nu^S \)

Theorem 9.2.

\[ \Lambda \left( \varrho > 0, \ M < \varrho \leq M^{-1}, \ m = 0 \quad \text{when} \ \varrho = 0 \right) = 1, \]

holds for some deterministic constants \( M, \tilde{f} > 0 \). Also assume that

\[ \int_{[L^1(\mathbb{R}^3)]^3} \left| \int_{\mathbb{R}^3} \left[ \frac{m^2}{2 \varrho} + H(\varrho, \tilde{\varphi}) \pm \varrho |\nabla V|^2 \right] dx \right|^p \ \rho \ \varrho \ = \ 0 \quad \text{for all} \ \varrho \in C^\infty_c([0,T]), \ \varrho \geq 0 \quad \text{where} \]

and \( P \) is given by (2.14);

(12) and (1.1) holds in the renormalized sense, i.e., for any \( \phi \in C^\infty_c([0,T] \times \mathbb{R}^3) \) and \( b \in C^1_b(\mathbb{R}) \)

such that \( b'(z) = 0 \) for all \( z \geq M_b \), we have that

\[ - \int_0^T \int_{\mathbb{R}^3} b(\varrho) \partial_t \phi \ dx \ dt = \int_0^T \int_{\mathbb{R}^3} b(\varrho(0)) \phi(0) \ dx \]

\[ + \int_0^T \int_{\mathbb{R}^3} [b(\varrho)u] \cdot \nabla \phi \ dx \ dt - \int_0^T \int_{\mathbb{R}^3} [(b'(\varrho) - b(\varrho)) \div u] \phi \ dx \ dt \]

holds \( \mathbb{P} \)-a.s.

9.2. Main result. The main result is the following.

**Theorem 9.2.** Let \( \tilde{\varphi} \geq 1 \) and let \( \Lambda = \Lambda(\varrho, m, f) \) be a Borel probability measure on \([L^1(\mathbb{R}^3)]^3\) such that

\[ \Lambda \left\{ \varrho \geq 0, \ M \leq \varrho \leq M^{-1}, \ f \leq \tilde{f}, \ m = 0 \quad \text{when} \ \varrho = 0 \right\} = 1, \]

holds for some deterministic constants \( M, \tilde{f} > 0 \). Also assume that

\[ \int_{[L^1(\mathbb{R}^3)]^3} \left| \int_{\mathbb{R}^3} \left[ \frac{m^2}{2 \varrho} + H(\varrho, \tilde{\varphi}) \pm \varrho |\nabla V|^2 \right] dx \right|^p \ \rho \ \varrho \ = \ 0 \quad \text{for all} \ \varrho \in C^\infty_c([0,T]), \ \varrho \geq 0 \quad \text{where} \]

Finally assume that (2.3) – (2.4) as well as (9.2) are satisfied. Then the exists a dissipative martingale solution of (1.1) – (1.3) in the sense of Definition 9.4.

9.3. Construction of law. To obtain a suitable initial law satisfying the preamble of Theorem 9.2 we first consider the following family of cut-off functions

\[ \eta_L \in C^\infty_c([-L,L]^3), \quad 0 \leq \eta_L \leq 1, \]

\[ \eta_L \equiv 1 \quad \text{in} \quad \left( \frac{-L}{2}, \frac{L}{2} \right)^3 \]

defined for \( L \geq 1 \) and let \( \tilde{\varphi} > 0 \) be the anticipated far-field condition in (2.3). Since the law \( \Lambda \) in Theorem 9.2 is a measure on a Polish space, it follows from Skorokhod’s theorem that there exists some \( \mathcal{F}_0 \)-measurable random variables \((\varrho_0, m_0, f)\) defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and having \( \Lambda \) as its law. With (9.11) in hand, we construct the following family

\[ \varrho_{0,L} = \eta_L \varrho + \left( 1 - \eta_L \right) \tilde{\varphi}, \quad m_{0,L} := \varrho_{0,L} u_{0,L} = \eta_L \sqrt{\varrho_{0,L}} m_0, \]

\[ V_{0,L} = \pm \Delta^{-1}_{[-L,L]^3}(\varrho_{0,L} - f) \]

of periodic functions having the property that

\[ \varrho_{0,L}|_{[-L,L]^3} = \tilde{\varphi}, \quad m_{0,L}|_{[-L,L]^3} = 0 \]
and hence satisfies the far-field condition \([9.3]\). We also have that
\[
(\varrho_0, L, m_{0,L}) \to (\varrho_0, m_0) \text{ a.e. in } \mathbb{R}^3
\]  
(9.14)
as \(L \to \infty\). Now for an arbitrary \(K \in \mathbb{R}^3\) and a choice of \(L \gg 1\) such that \(K \subseteq [-L, L]^3\), we have that \(|\varrho_0, L| \lesssim 0 + \mathbb{T}\) and \(|m_{0,L}| \lesssim 1 + m_0\) holds uniformly in \(L\). Furthermore, per the assumptions on \(\Lambda\), \(\varrho_0 + \mathbb{T} \in L^1(K)\) and \(1 + m_0 \in L^1(K)\) so that
\[
(\varrho_0, L, m_{0,L}, f) \to (\varrho_0, m_0, f) \text{ in } [L^1(K)]^3
\]
a.s. Subsequently, we gain
\[
\Lambda_L = \mathbb{P} \circ (\varrho_0, L, m_{0,L}, f)^{-1} \Rightarrow \mathbb{P} \circ (\varrho_0, m_0, f)^{-1} = \Lambda
\]
in the sense of measures on \([L_{loc}^1(\mathbb{R}^3)]^3\) by the arbitrariness of \(K \subseteq \mathbb{R}^3\).

9.4. A priori bounds. By periodicity and invariance, it follows from (2.2) that there exists a family of dissipative martingale solutions
\[
[(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) ; \varrho_L, u_L, V_L, W]
\]
in the sense of Definition (2.1) which are defined for \(k \geq 0\) for all \(L \in \mathbb{R}^3\) such that \(0 \leq \varrho(\varrho + P(\varrho) + |\nabla V|^2) dx = 0\), \(d\Lambda_L(\varrho, m, f) < \infty\),
\[
(9.15)
\]
for some \(q \geq 1\).

Furthermore, in analogy with (2.3)–(2.6), for \(m_L := \varrho_L u_L\), the noise term \(G(\varrho_L, m_L, f) : \mathcal{X} \to L^1(T^1_L)\) is defined as
\[
G(\varrho_L, m_L, f)e_k = g_k(\cdot, \varrho_L(\cdot, m_L(\cdot), f(\cdot))
\]
for all \(k \in \mathbb{N}\) such that
\[
g_k : T^3_L \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times T^3_L \to \mathbb{R}^3, \quad g_k \in C^1 \left( T^3_L \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times T^3_L \right)
\]
(9.20)
for any \(k \in \mathbb{N}\) and in addition, \(g_k\) satisfies the following growth conditions:
\[
|g_k(x, \varrho_L, f, m_L)| \leq c_k (\varrho_L + f + |m_L|),
\]
(9.21)
\[
|\nabla_{\varrho,f,m} g_k(x, \varrho_L, f, m_L)| \leq c_k,
\]
(9.22)
\[
\sum_{k \in \mathbb{N}} c^2_k < 1
\]
(9.23)
for some constant \((c_k)_{k \in \mathbb{N}} \in [0, \infty)\).

If we now approximate the characteristic map \(t \mapsto \chi_{[0,t]}\) by a sequence of nonnegative compactly supported smooth functions \((\psi_n)_{m \in \mathbb{N}}\) and observe that the resulting inequality is preserved under an affine perturbation of \(P(z)\), then for
\[
H(z, \overline{v}) = P(z) - P'(\overline{v})(z - \overline{v}) - P(\overline{v})
\]
(9.24)
where \( \overline{\eta} > 0 \), it follows from (2.13) that
\[
\int_{\mathbb{T}_L^1} \left[ \frac{1}{2} \varrho_L |u_L|^2 + H(\varrho_L, \overline{\eta}) \pm \vartheta |\nabla V_L|^2 \right] (t) \, dx + \nu^S \int_0^T \int_{\mathbb{T}_L^1} |\nabla u_L|^2 \, dx \, ds
+ (\nu^B + \nu^S) \int_0^T \int_{\mathbb{T}_L^1} |\text{div} \, u_L|^2 \, dx \, ds \leq \int_{\mathbb{T}_L^1} \left[ \frac{1}{2} \varrho_0 |u_0, L|^2 + H(\varrho_0, L, \overline{\eta}) \pm \vartheta |\nabla V_0|^2 \right] \, dx
\]
holds uniformly in \( P \) and also yields
\[
-\text{a.s., c.f. [17, Page 57]. In comparison with (2.9), it follows from (9.21) that for any } E \in \mathbb{R}^3, \text{ if we now take the } p\text{-th-moment of the supremum in (9.25) and use Gronwall's lemma, then it follows from (9.25) and (9.28) that}
\]
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left( \int_0^T \int_{\mathbb{T}_L^1} \varrho_L u_L \cdot G(\varrho_L, f, m_L) \, dx \, dW \right)^p \right] \lesssim_p \mathbb{E} \left( \sup_{t \in [0, T]} \int_{\mathbb{T}_L^1} \varrho_L |u_L|^2 \, dx \right)^p
+ \mathbb{E} \int_0^T \left( \int_K (1 + \varrho_L^2 + \varrho_L |u_L|^2 + f \varrho_L) \, dx \right)^p \, ds
\]
holds uniformly in \( L \) where \( K \) is given in (9.2). The use of the Burkholder–Davis–Gundy inequality also yields
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left( \int_0^T \int_{\mathbb{T}_L^1} \varrho_L u_L \cdot G(\varrho_L, f, m_L) \, dx \, dW \right)^p \right] \lesssim_p \mathbb{E} \left( \sup_{t \in [0, T]} \int_{\mathbb{T}_L^1} \varrho_L |u_L|^2 \, dx \right)^p
+ \nu^S \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}_L^1} |\nabla u_L|^2 \, dx \, ds \right]^p
\]
holds uniformly in \( L \) where the right-hand side is finite as a result of (9.10). In particular, it therefore follow from (9.29) that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|\varrho_L |u_L|^2\|_{L^1(\mathbb{T}_L^1)}^p \right] \lesssim 1, \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|H(\varrho_L, \overline{\eta})\|_{L^1(\mathbb{T}_L^1)}^p \right] \lesssim 1,
\]
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|\nabla V_L\|_{L^2(\mathbb{T}_L^1)}^p \right] \lesssim 1, \quad \mathbb{E} \left[ \left( \int_0^T \|\nabla u_L\|_{L^2(\mathbb{T}_L^1)}^2 \, dt \right)^{\frac{p}{2}} \right] \lesssim 1, \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|\varrho_L - \overline{\eta}\|_{L^2(\mathbb{T}_L^1)}^p \right] \lesssim 1
\]
holds uniformly in \( L \) where \( \overline{\eta} = \min\{\gamma, 2\} \).
Lemma 9.3. For all \( p \in [1, \infty) \), the following estimate

\[
E \left[ \left( \int_0^T \| u_L \|^2_{L^2(T^3_L)} \, dt \right)^{\frac{p}{2}} \right] \lesssim_p 1
\]

holds uniformly in \( L \).

Proof. First of all, if we decompose \( T^3_L \) into

\[
\{ x \in T^3_L : 2|\varrho_L - \overline{\varrho}| \leq 1 \} \quad \text{and} \quad \{ x \in T^3_L : 2|\varrho_L - \overline{\varrho}| \geq 1 \},
\]

then since \( \overline{\varrho} \geq 1 \), we obtain

\[
E \left[ \int_0^T \| u_L \|^2_{L^2(T^3_L)} \, dt \right] \leq E \left[ 2 \sup_{t \in [0,T]} \| \varrho_L - \overline{\varrho} \|_{L^\infty(T^3_L)} \int_0^T \| u_L \|^2_{L^2(T^3_L)} \, dt \right] + E \left[ 2 \sup_{t \in [0,T]} \| \varrho_L \|_{L^1(T^3_L)} \right]^p
\]

for any \( p \in [1, \infty) \). Next, since we have \( 0 < \frac{3}{2\overline{\varrho}} < 1 \), \( 0 < 1 - \frac{3}{2\overline{\varrho}} < 1 \) and

\[
\frac{\overline{\varrho} - 1}{2\overline{\varrho}} = \left( 1 - \frac{3}{2\overline{\varrho}} \right) \times \frac{1}{2} + \frac{3}{2\overline{\varrho}} \times \frac{1}{6}
\]

we can interpolate \( u_L \) between \( L^2(T^3_L) \) and \( L^6(T^3_L) \) which yields for any \( \delta, c_3 > 0 \),

\[
E \left[ 2 \sup_{t \in [0,T]} \| \varrho_L - \overline{\varrho} \|_{L^\infty(T^3_L)} \int_0^T \| u_L \|^2_{L^2(T^3_L)} \, dt \right] \leq \delta \left( E \left[ \int_0^T \| u_L \|^2_{L^2(T^3_L)} \, dt \right] \right)^p \left( \int_0^T \| u_L \|^2_{L^2(T^3_L)} \, dt \right)^{\frac{p}{2}}
\]

\[
+ c_3 \left( E \left[ \sup_{t \in [0,T]} \| \varrho_L - \overline{\varrho} \|_{L^\infty(T^3_L)} \right] \right)^p \left( \int_0^T \| u_L \|^2_{L^6(T^3_L)} \, dt \right)^{\frac{p}{2}}
\]

uniformly in \( L \) for all \( p_1, p_2, p_3 \in [1, \infty) \) such that \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p} \). Thus our result follow from 9.30.

We can now conclude from velocity and gradient velocity estimates in 9.30 as well as Lemma 9.3 that

\[
E \left[ \left( \int_0^T \| u_L \|^2_{W^{1,2}(T^3_L)} \, dt \right)^{\frac{p}{2}} \right] \lesssim_p 1
\]

holds uniformly in \( L \). Also,

\[
E \left[ \sup_{t \in [0,T]} \| \varrho_L u_L \|_{L^{\frac{2p}{p+2}}(T^3_L)} \right]^p \lesssim_{p,\gamma} E \left[ \sup_{t \in [0,T]} \| \varrho_L \|^\gamma_{L^{\gamma}(T^3_L)} \right]^p
\]

\[
+ E \left[ \sup_{t \in [0,T]} \| \varrho_L u_L \|^2_{L^1(T^3_L)} \right]^p \lesssim 1
\]

and

\[
E \left[ \left( \int_0^T \| \varrho_L u_L \|_{L^{\frac{2p}{p+2}}(T^3_L)} \, dt \right)^{\frac{p}{2}} \right] \lesssim E \left[ \sup_{t \in [0,T]} \| \varrho_L u_L \|_{L^{\gamma}(T^3_L)} \right]^p
\]

\[
\times E \left[ \left( \int_0^T \| u_L \|^2_{L^3(T^3_L)} \, dt \right)^{\frac{p}{2}} \right] \lesssim 1
\]

holds uniformly in \( L \).

Lemma 9.4. Let \( L \gg 1 \) be large enough so that \( B \cap T^3_L = B \) for a ball \( B \) of arbitrary radius. Then for all \( \Theta < \frac{\pi}{4} \gamma - 1 \), we have

\[
E \left[ \int_0^T \int_B \varrho_L^{\gamma+\Theta} \, dx \, dt \right]^p \lesssim_{p,\gamma,\Theta} 1
\]
uniformly in $L$ for any $p \in [2, \infty)$

Proof. For $B \subseteq \tilde{B}$, let $\eta \in C_0^\infty(\tilde{B})$ be such that $\eta = 1$ in $B$. Since $(\varrho_L, u_L)$ is a solution in the sense of Definition 2.1, it satisfies the renormalized continuity equation (2.11) for a sequence of compactly supported smooth functions $b_n$ that approximate $\varrho \mapsto \varrho^p$. Also, $(\varrho_L, u_L, V_L)$ satisfies the momentum balance equation (2.12). Since the aforementioned equations (2.11) and (2.12) are weak in the PDE sense, we may regularize them by mollification with the usual mollifier $\varrho_n$ to get them to be satisfied strongly in the sense of PDEs. We can then apply the operator $\eta \nabla \Delta^{-1}_B$ to the (strong regularized) renormalized continuity equation, multiply the resulting equation with the (strong regularized) momentum balance equation by the use of Itô’s product rule. If we subsequently pass to the limit $\kappa \to 0$ in the product, then just as in [17, Lemma 3.3.4], we obtain

$$E \left[ \int_0^t \int_{\mathbb{R}^3} \eta p(\varrho_L) \varrho_L^{\Theta} \, dx \, ds \right]^p \lesssim_p E \sum_{i=1}^{14} |J_i|^p \quad (9.37)$$

where

$$E \sum_{i=1}^{14} |J_i|^p := E \left[ \int_{\mathbb{R}^3} \eta(p(\varrho_L) u_L(t) \cdot \nabla \Delta^{-1}_B (\varrho_L^\Theta)(t) \, dx \right]^p + E \left[ \int_{\mathbb{R}^3} \eta \varrho_{0,L} \cdot \nabla \Delta^{-1}_B [\varrho_{0,L}] \, dx \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \varrho_L \cdot \nabla \Delta^{-1}_B [\text{div}(\varrho_L^\Theta u_L)] \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \varrho_L \cdot \nabla \Delta^{-1}_B [\Theta \varrho_L^\Theta \text{div} u_L] \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \varrho_L \cdot \nabla \Delta^{-1}_B [\varrho_L^\Theta \text{div} u_L] \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \varrho_L \otimes u_L : \nabla^2 \Delta^{-1}_B (\varrho_L^\Theta) \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta (\nu^B + \nu^S) \varrho_L^\Theta \text{div} u_L \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta (\nu^B + \nu^S) \varrho_L^\Theta \text{div} u_L \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \nabla \Theta \cdot \nabla \Delta^{-1}_B (\varrho_L^\Theta) \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \nabla \Theta \cdot \nabla \Delta^{-1}_B (\varrho_L^\Theta) \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \nu^B \nabla u_L \cdot \nabla^2 \Delta^{-1}_B (\varrho_L^\Theta) \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \nu^S \nabla u_L \cdot \nabla \Theta \cdot \nabla \Delta^{-1}_B (\varrho_L^\Theta) \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \varrho_L \nabla V_L \cdot \nabla \Delta^{-1}_B (\varrho_L^\Theta) \, dx \, ds \right]^p + E \left[ \int_0^t \int_{\mathbb{R}^3} \eta \varrho_L \nabla \Delta^{-1}_B (\varrho_L^\Theta) G(\varrho_L, f, \varrho_L u_L) \, dx \, dW \right]^p.$$

Now if we use the notation

$$\| \cdot \|_p := \| \cdot \|_{L^p(\tilde{B})}$$
and consider positive real numbers $q_1, q_2, q_3$ satisfying

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1,$$

then analogous to [17] (3.66)-(3.67) we have that

$$E[J_1]^p \lesssim \left( E \| \theta_L \|_{L^\gamma}^{2 \Theta} \right)^{\frac{p}{2 \Theta}} \left( E \| \theta_L \|_{L^1} \| u_L \|_{L^2}^{2} \right)^{\frac{p}{2}} \left( E \| \theta_L \|_{L^q \Theta} \right)^{\frac{p}{q}} \lesssim 1 \quad (9.39)$$

uniformly in $L$ provided $2 \Theta < \gamma$. A similar estimate holds for $J_2$. In comparison with [17] (3.79)-(3.80),

$$E[J_3]^p \lesssim \left( E \sup_{t \in [0,T]} \| \theta_L \|_{L^\gamma}^{2 \Theta} \right)^{\frac{p}{2 \Theta}} \left( E \left[ \int_0^T \left( \| u_L \|_{L^2} + \| \nabla u_L \|_{L^2} \right) ^{q} \right]^{\frac{p}{q}} \right)^{\frac{p}{q}}$$

$$\times \left( E \sup_{t \in [0,T]} \| \theta_L \|_{L^q \Theta} \right)^{\frac{p}{q}} \lesssim 1 \quad (9.40)$$

uniformly in $L$ provided $\Theta < \frac{3p}{2q - 3}$ where $r = \frac{3p}{2q - 3}$. Also

$$E[J_4 + J_5]^p \lesssim \left( E \sup_{t \in [0,T]} \| \theta_L \|_{L^\gamma}^{3 \Theta} \right)^{\frac{p}{2 \Theta}} \left[ E \left( \int_0^T \| \nabla u_L \|_{L^2} \right) \right]^{\frac{p}{q}}$$

$$\times \left( E \left[ \sup_{t \in [0,T]} \| \theta_L \|_{L^q \Theta} \right] \right) \left[ E \left( \int_0^T \| \nabla u_L \|_{L^2} \right) \right] \lesssim 1 \quad (9.41)$$

uniformly in $L$ provided $k \Theta = \frac{3p}{2q - 3} \Theta \leq \gamma$.

$$E[J_6]^p \lesssim \left( E \sup_{t \in [0,T]} \| \theta_L \|_{L^\gamma}^{q_1} \right)^{\frac{p}{q_1}} \left( E \left[ \int_0^T \| u_L \|_{L^2} \right]^{q_2} \right)^{\frac{p}{q_2}}$$

$$\times \left( E \sup_{t \in [0,T]} \| \theta_L \|_{L^q \Theta} \right)^{\frac{p}{q}} \lesssim 1 \quad (9.42)$$

is bounded uniformly in $L$ provided $r \Theta = \frac{3p + 6 \Theta}{2q - 3} < \gamma$. The estimate for $J_9$ follows is similar to $J_6$.

$$E[J_7]^p \lesssim \left( E \sup_{t \in [0,T]} \| \theta_L \|_{L^\gamma}^{2 \Theta} \right)^{\frac{p}{2}} \left( E \left[ \int_0^T \| \nabla u_L \|_{L^2} \right]^{\frac{p}{2}} \right) \lesssim 1 \quad (9.43)$$

uniformly in $L$ if $2 \Theta < \gamma$. The estimate for $J_{10}$ follow similarly to $J_7$. Also,

$$E[J_8]^p \lesssim E \left( \int_0^T \| \nabla u_L \|_{L^2}^2 dt + \sup_{t \in [0,T]} \| \theta_L \|_{L^q \Theta}^p \right) \lesssim 1. \quad (9.44)$$

The estimate for $J_{11}$ follows is similar to $J_8$. Now

$$E[J_{12}]^p \lesssim E \left\{ \sup_{t \in [0,T]} \left( \| \theta_L \|_{L^q \Theta}^2 \right) \right\} \lesssim 1. \quad (9.45)$$

In order to estimate $J_{13}$, we can use the Poisson equation and several integration by parts to rewrite it as

$$J_{13} = \int_0^t \int_{\mathbb{R}^d} \eta \vartheta (f + \Delta V_L) \nabla V_L \cdot \nabla \Delta^{-1}_B (\theta_L) dx ds = \sum_{j=1}^5 J_{13}^j$$
Now since by \[17\] (3.58), we have that
\[
L \leq \sup_{x \in [0,T]} \|\nabla V_L\|_{W}^{q_3} \left( E \sup_{t \in [0,T]} \|\varrho_{L\theta}\|_{L_{q_3}^{\Theta}} \right)^{\frac{q_3}{q_3 - 2}} \lesssim 1
\]
is bounded uniformly in \(L\) provided \(2\Theta < \gamma\). Also, since for a.e. \((\omega, t) \in \Omega \times [0, T]\), we have that
\[
\int_{\mathbb{R}^3} \|\nabla V_L \otimes \nabla V_L : \nabla \Delta^{-1}_B (\varrho^\omega_L)\| \, dx \lesssim \|\nabla V_L \otimes \nabla V_L\|_{W^{-1, r'}(\mathbb{R}^3)} \|\nabla \Delta^{-1}_B (\varrho^\omega_L)\|_{W^{1, r}(\mathbb{R}^3)},
\]
for any \(r > 3\) and its Hölder conjugate \(r'\) such that \(r\Theta < \gamma\), we gain from \[17\] (3.59) and the continuous embedding \(L^1(\mathbb{R}^3) \hookrightarrow W^{-1, r'}(\mathbb{R}^3)\), the estimate
\[
E[J_{13}^2]^p \lesssim \left( E \sup_{t \in [0,T]} \|\nabla V_L\|_{L}^{q_3} \right)^{\frac{p}{q}} \left( E \sup_{t \in [0,T]} \|\varrho_{L\theta}\|_{L_{q_3}^{\Theta}} \right)^{\frac{p}{q_3} + \frac{p}{q' \Theta}} \lesssim 1
\]
uniformly in \(L\). The same estimate as \(J_{13}^2\) holds for \(J_{13}^3\) and \(J_{13}^4\). In order to estimate \(J_{13}^5\), we first use the Poisson equation to rewrite it as follows
\[
J_{13}^5 = \pm \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \varrho \eta |\nabla \Delta^{-1}_B (\varrho_{L\theta} - f)|^2 \, dx \, ds.
\]
Now since \(f\) is Lebesgue integrable for any finite or infinite Lebesgue power, it follows from \[17\] (3.57) that
\[
|J_{13}^5| \lesssim \int_0^T \|\nabla \Delta^{-1}_B (\varrho_{L\theta} - f)\|_{L^2}^2 \|\varrho^\omega_L\|_{L^2} \, ds \lesssim \int_0^T \left( 1 + \|\varrho_{L\theta}\|_{L^2}^2 \right) \|\varrho_{L\theta}\|_{L_{q_3}^{\Theta}} \, ds.
\]
for \( \gamma' = \frac{2 \gamma}{\gamma + 1} \) where by (3.58),
\[
\| \nabla \Delta_B^{-1}(\vartheta_L) \|_{L^4_B}^{2 \gamma'} \lesssim_B \| \vartheta_L \|_2^{2 \gamma'} = \| q_L \|_{2 \Theta}^{2 \gamma'}.
\]
(9.51)
Since the noise coefficients are essentially bounded, it follows from the Burkholder–Davis–Gundy inequality that
\[
E[J_{14}]^p \leq E \left[ \int_0^T \int_{|x| < R} \left( \sum_{k \in \mathbb{N}} | \int_{\mathbb{R}^3} \eta_{pl \vartheta} \nabla \Delta_B^{-1}(\vartheta_L)^k g_k(\vartheta_L, f, \vartheta_L u_L) \, dx \right)^2 \, ds \right] \lesssim_{k, \gamma, \Theta} E \left[ \sup_{t \in [0,T]} \| \vartheta_L \|_{L^2/(\gamma + 1)}^{2 \gamma} \right] + E \left[ \sup_{t \in [0,T]} \| q_L \|_{L^2/(\gamma + 1)}^{2 \gamma} \right] \lesssim_{k, \gamma, \Theta} 1
\]
(9.52)
uniformly in \( L \) provided \( 2 \Theta < \gamma \).
Summing up all the estimates above finishes the proof.

9.5. **Compactness.** As in Sections 7.3 and 8.3 we denote the energy by
\[
\mathcal{E}_L := \frac{1}{2} \vartheta_L |u_L|^2 + P(\vartheta_L) \pm \vartheta |\nabla V_L|^2
\]
and let the weakly-* measurable mapping
\[
\nu_L : [0,T] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R}^2)
\]
defined by
\[
\nu_{L,t,x}(\cdot) = \vartheta_L(\vartheta_L, \nabla u_L, \vartheta_L u_L, f, \nabla V_L[t,x](\cdot))
\]
be the canonical Young measure associated to \([\vartheta_L, u_L, \nabla u_L, \vartheta_L u_L, f, \nabla V_L]\) where \( \nu_L \) is interpreted as a random variable taking values in \((L^\infty(0,T;L^\infty_{\text{loc}}(\mathbb{R}^3));\mathcal{P}(\mathbb{R}^2), w^*)\) endowed with the weak-* topology determined by
\[
L^\infty(0,T;L^\infty_{\text{loc}}(\mathbb{R}^3);\mathcal{P}(\mathbb{R}^2)) \to \mathbb{R}, \quad \nu \mapsto \int_0^T \int_{\mathbb{R}^3} \psi(t,x) \int_{\mathbb{R}^2} \phi(\xi) \, d\nu_{L,t,x}(\xi) \, dx \, dt
\]
for all \( \psi \in L^1(0,T;L^1_{\text{loc}}(\mathbb{R}^3)) \) and for all \( \phi \in C_b(\mathbb{R}^2) \). We now define the following spaces
\[
\chi_{\vartheta_0} = L^\gamma_{\text{loc}}(\mathbb{R}^3), \quad \chi_{m_0} = L^1_{\text{loc}}(\mathbb{R}^3), \quad \chi_{m_0}^{\mu_{m_0}} = L^2_{\text{loc}}(\mathbb{R}^3), \quad \chi_u = (L^2(0,T;W^{1,2}(\mathbb{R}^3)), w), \quad \chi_V = C_w \left( \left[0,T\right];W^{2,\gamma}(\mathbb{R}^3) \right), \quad \chi_W = C \left( \left[0,T\right];\mathcal{M}_{\mathbb{R}^2} \right), \quad \chi_{\vartheta u} = C_w \left( \left[0,T\right];L^\infty_{\text{loc}}(\mathbb{R}^3) \right) \cap C \left( \left[0,T\right];W^{-k,2}(\mathbb{R}^3) \right), \quad \chi_\varphi = C_w \left( \left[0,T\right];L^\infty(\mathbb{R}^3) \right) \cap (L^\gamma+\Theta(0,T;L^\infty_{\text{loc}}(\mathbb{R}^3)), w), \quad \chi_e = (L^\infty(0,T;\mathcal{M}_{\mathbb{R}^2}(\mathbb{R}^3)), w^*), \quad \chi_\nu = (L^\infty(0,T;L^\infty_{\text{loc}}(\mathbb{R}^3));\mathcal{P}(\mathbb{R}^2), w^*)
\]
for \( \Gamma \geq 6 \) and \( k > \frac{2}{\gamma} \). We now let \( \mu_{\vartheta_0,L}, \mu_{m_0,L}, \mu_{m_0,L}^{\mu_{m_0,L}}, \mu_{\vartheta,L}, \mu_{u,L}, \mu_{\vartheta u,L}, \mu_{V,L}, \mu_{\vartheta u,L} \) and \( \mu_W \) be the respective laws of \( \vartheta_0,L, m_0,L, m_0,L^{\mu_{m_0,L}}, \vartheta_L, u_L, \vartheta_L u_L, \nabla V_L, \chi_\varphi, \chi_{m_0}, \chi_{m_0}^{\mu_{m_0,L}}, \chi_u, \chi_{\vartheta u}, \chi_V, \chi_e, \chi_\nu, \chi_W \). Furthermore, we set \( \mu_L \) as their joint law on the space
\[
\chi = \chi_{\vartheta_0} \times \chi_{m_0} \times \chi_{m_0}^{\mu_{m_0,L}} \times \chi_\varphi \times \chi_u \times \chi_{\vartheta u} \times \chi_V \times \chi_e \times \chi_\nu \times \chi_W.
\]
**Lemma 9.5.** The set \( \{\mu_L : L \geq 1\} \) is tight on \( \chi \).

The application of the Jakubowski–Skorokhod theorem yields the following.

**Lemma 9.6.** The exists a subsequence (not relabelled) \( \{\mu_L : L \geq 1\} \), a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with \( \chi \)-valued random variables
\[
(\tilde{\vartheta}_L, \tilde{m}_0,L, \tilde{n}_0,L, \tilde{\vartheta}_L, \tilde{u}_L, \tilde{m}_L, \tilde{V}_L, \tilde{E}_L, \tilde{\nu}_L, \tilde{W}_L) \quad L \in (0,1)
\]
and

$$(\hat{\varrho}_0, \hat{m}_0, \hat{n}_0, \hat{\varrho}, \hat{u}, \hat{m}, V, \hat{\varv}, \hat{W})$$

such that

- the law of $(\hat{\varrho}_{0,L}, \hat{m}_{0,L}, \hat{n}_{0,L}, \hat{\varrho}_L, \hat{u}_L, \hat{m}_L, \hat{V}_L, \hat{\varv}_L, \hat{W}_L)$ on $\chi$ coincide with $\mu_L, \ L \geq 1$,
- the law of $(\varrho_0, m_0, \varrho, u, m, V, \varv, W)$ on $\chi$ is a Radon measure,
- the following convergence (with each $r$ interpreted with respect to the corresponding topology)
  $$\begin{align*}
  \hat{\varrho}_{0,L} & \to \hat{\varrho}_0 \text{ in } \chi_{\varrho_0}, \\
  \hat{m}_{0,L} & \to \hat{m}_0 \text{ in } \chi_{m_0}, \\
  \hat{n}_{0,L} & \to \hat{n}_0 \text{ in } \chi_{n_0}, \\
  \hat{\varrho}_L & \to \hat{\varrho} \text{ in } \chi_{\varrho}, \\
  \hat{u}_L & \to \hat{u} \text{ in } \chi_u, \\
  \hat{m}_L & \to \hat{m} \text{ in } \chi_m, \\
  \hat{V}_L & \to \hat{V} \text{ in } \chi_V, \\
  \hat{\varv}_L & \to \hat{\varv} \text{ in } \chi_{\varv}, \\
  \hat{W}_L & \to \hat{W} \text{ in } \chi_W
  \end{align*}$$

holds $\hat{\mathbb{P}}$-a.s.;
- consider any Carathéodory function $C = C(t, x, \varrho, \mathbf{u}, \mathbf{m}, f, V)$ with
  $$(t, x, \varrho, \mathbf{u}, \mathbf{m}, f, V) \in [0, T] \times K \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \times \mathbb{R}^3 \times \mathbb{R}^3,$$
  $K \subseteq \mathbb{R}^3,$
  and where the following estimate
  $$|C| \lesssim 1 + |f|^r + |\mathbf{u}|^r + |\mathbf{m}|^r + |f|^r + |V|^r$$
  holds uniformly in $(t, x)$ for some $r_i > 0$, $i = 1, \ldots, 6$. Then as $L \to 0$, it follows that

$$C(\hat{\varrho}_L, \hat{u}_L, \nabla \hat{u}_L, \hat{m}_L, f, \nabla \hat{V}_L) \to C(\hat{\varrho}, \hat{u}, \nabla \hat{u}, \hat{m}, f, \nabla \hat{V})$$

holds in $L^t((0, T) \times K)$ for all

$$1 < r \leq \frac{\gamma + \Theta}{r_1} \wedge \frac{2\gamma}{r_2} \wedge \frac{2\gamma}{r_4(\gamma + 1)} \wedge \frac{2}{r_6}$$

$\hat{\mathbb{P}}$-a.s.

And we have the following corollary.

**Corollary 9.7.** The following holds $\hat{\mathbb{P}}$-a.s.

$$\hat{\varrho}_{0,L} = \hat{\varrho}_L(0), \quad \hat{m}_{0,L} = \hat{m}_L \hat{u}_L(0), \quad \hat{n}_{0,L} = \frac{\hat{m}_{0,L}}{\hat{\varrho}_{0,L}}, \quad \hat{m}_L = \hat{\varrho}_L \hat{u}_L,$$

$$\hat{\varv}_L = \mathcal{E}_L(\hat{\varrho}_L, \hat{u}_L, \hat{V}_L), \quad \hat{\varv}_L = L_{[\hat{\varrho}_L, \hat{u}_L, \nabla \hat{u}_L, \hat{m}_L, f, \nabla \hat{V}_L]}$$

and that

$$\hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \int_{\mathbb{R}^3} \hat{\mathcal{E}}_L \, dx \right]^p = \hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \int_{\mathbb{R}^3} \left( \frac{1}{2} \hat{\varrho}_L |\hat{u}_L|^2 + P(\hat{\varrho}_L) \pm |\nabla \hat{V}_L|^2 \right) \, dx \right]^p \lesssim_{\gamma, p, \hat{\varrho}_{0,L}} 1$$

(9.56)

and for any ball $B \subset \mathbb{R}^3$, the estimate

$$\hat{\mathbb{E}} \left[ \int_B (\hat{\varrho}_L + \Theta) \, dx \, dt \right]^p \lesssim_{\gamma, p, \hat{\varrho}_{0,L}} 1$$

(9.57)

holds uniformly in $L$.

Furthermore, the random variables can be endowed with the following filtrations

$$\hat{\mathcal{F}}^L_t := \sigma_t(\hat{\varrho}_L, \hat{u}_L, \hat{V}_L) \cup \bigcup_{k \in \mathbb{N}} \sigma_t(\hat{\varrho}_{L,k}), \quad t \in [0, T]$$
and
\[ \mathcal{F}_t := \sigma\left( \{ \sigma_t[^\varphi], \sigma_t[^\hat{u}], \sigma_t[^\tilde{V}], \bigcup_{k \in \mathbb{N}} \sigma_t[^\tilde{\varphi}_k] \} \right), \quad t \in [0, T] \]
for the sequence and limit random variables respectively. Now as in Lemma 9.27, we have the following result.

**Lemma 9.8.** For any \( \phi \in C^\infty_c([0, T) \times \mathbb{R}^3) \) and \( b \in C^1_b(\mathbb{R}) \) such that \( b'(z) = 0 \) for all \( z \geq M_b \), we have that
\[
0 = \int_0^T \int_{\mathbb{R}^3} b(\tilde{\varphi}_L) \partial_t \phi \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} b(\hat{\varphi}_0, L) \phi(0) \, dx \\
+ \int_0^T \int_{\mathbb{R}^3} \left[ b(\tilde{\varphi}_L) \hat{u}_L \right] \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} \left[ (b'(\tilde{\varphi}_L) \tilde{\varphi}_L - b(\tilde{\varphi}_L)) \text{div} \hat{u}_L \right] \phi \, dx \, dt
\]
holds \( \tilde{\mathbb{P}} \)-a.s.

**9.6. Identifying the limit system.** To begin with, we have the following result which is analogous to Corollary 9.9

**Corollary 9.9.** There exists \( p(\varphi), |\nabla \varphi|^2, \tilde{\varphi}_L \) and \( \tilde{\varphi} \) such that
\[
p(\tilde{\varphi}_L) \to \tilde{p}(\varphi) \\
|\nabla \tilde{V}_L|^2 \to |\nabla \varphi|^2 \\
\tilde{\varphi}_L \nabla \tilde{V}_L \to \tilde{\varphi}_L \nabla \varphi \\
\tilde{\varphi}_L \tilde{\varphi}_L \tilde{V}_L \to \tilde{\varphi}_L \tilde{\varphi}_L \tilde{\varphi}
\]
\( \tilde{\mathbb{P}} \)-a.s. in \( L^r(0, T; L^2_{\text{loc}}(\mathbb{R}^3)) \) for some \( r > 1 \). Furthermore, for any \( \phi \in C^\infty_c(\mathbb{R}^3) \), the following \( \tilde{\mathbb{P}} \)-a.s. convergence holds
\[
\phi \tilde{\varphi}_L \hat{u}_L \to \phi \tilde{\varphi} \hat{u} \quad \text{in} \quad L^2(0, T; W^{-1,2}(\mathbb{R}^3)),
\]
as \( L \to \infty \).

**Lemma 9.10.** The random distributions \( [\tilde{\varphi}, \tilde{u}] \) satisfies 9.5 for all \( \psi \in C^\infty_c([0, T)) \) and \( \phi \in C^\infty_c(\mathbb{R}^3) \) \( \tilde{\mathbb{P}} \)-a.s.

**Lemma 9.11.** The random variables \( [\tilde{\varphi}, \tilde{V}] \) satisfies 1.3 a.e. in \( (0, T) \times \mathbb{R}^3 \) \( \tilde{\mathbb{P}} \)-a.s.

**Lemma 9.12.** For any \( \psi \in C^\infty_c([0, T) \times \mathbb{R}^3) \), we have that
\[
0 = \int_0^T \int_{\mathbb{R}^3} T_k(\tilde{\varphi}_L) \partial_t \psi \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} T_k(\hat{\varphi}_0, L) \psi(0) \, dx + \int_0^T \int_{\mathbb{R}^3} [T_k(\tilde{\varphi}_L) \hat{u}_L] \cdot \nabla \psi \, dx \, dt \\
- \int_0^T \int_{\mathbb{R}^3} \left[ (T_k(\tilde{\varphi}_L) \tilde{\varphi}_L - T_k(\hat{\varphi}_0, L)) \text{div} \hat{u}_L \right] \psi \, dx \, dt
\]
holds \( \tilde{\mathbb{P}} \)-a.s. Furthermore, if for \( j = 1, 2, 3 \), we set \( \partial_j := \partial_{x_j} \) and the define \( A_j = \partial_j \Delta^{-1} \mathbb{R}^3 \) and \( R_{ij} = \partial_i A_j \), then for any \( \phi \in C^\infty_c(\mathbb{R}^3) \), any \( p \in (1, \infty) \), any \( q_1 \in (1, \infty) \) and any \( q_2 \in [1, \infty) \), we have that
\[
\phi T_k(\tilde{\varphi}_L) \to \phi T_k(\tilde{\varphi}) \quad \text{in} \quad C_w([0, T]; L^p(\mathbb{R}^3)),
\]
\[
\phi T_k(\hat{\varphi}_0, L) \to \phi T_k(\tilde{\varphi}) \quad \text{in} \quad L^p((0, T) \times \mathbb{R}^3),
\]
\[
\phi T_k(\tilde{\varphi}_L) \to \phi T_k(\tilde{\varphi}) \quad \text{in} \quad L^2([0, T]; W^{-1,2}(\mathbb{R}^3)),
\]
\[
A_i[\phi T_k(\tilde{\varphi}_L)] \to A_i[\phi T_k(\tilde{\varphi})] \quad \text{in} \quad L^p((0, T); L^q(\mathbb{R}^3))
\]
holds \( \tilde{P} \)-a.s as \( L \to \infty \). Finally, for some \( q_3 \in (1, \infty) \) and any \( \phi \in C^\infty_c(\mathbb{R}^3) \),

\[
\begin{align*}
\phi(T_k'(\bar{\theta}_L) \bar{\theta}_L - T_k(\bar{\theta}_L)) & \text{div} \bar{u}_L \\
+ \phi(T_k'(\bar{\theta}) \bar{\theta} - T_k(\bar{\theta})) & \text{div} \bar{u} \\
A_i[\phi(T_k'(\bar{\theta}_L) \bar{\theta}_L - T_k(\bar{\theta}_L)) & \text{div} \bar{u}_L] \\
+ \phi(T_k'(\bar{\theta}) \bar{\theta} - T_k(\bar{\theta})) & \text{div} \bar{u} \\
\end{align*}
\]

\( L^q \) \((0, T) \times \mathbb{R}^3 \), \hspace{1cm} (9.69)

\[
\begin{align*}
R_{ij}[\phi(\bar{\theta}_L \bar{u}_L^i) \phi(T_k(\bar{\theta}_L)) - \phi(\bar{\theta}_L \bar{u}_L^i) R_{ij}[\phi T_k(\bar{\theta}_L)]) \\
+ \phi(\bar{\theta}_L \bar{u}_L^i) T_k(\bar{\theta}_L) \partial_j \phi - \bar{\theta}_L \bar{u}_L A_i[\phi^2 T_k(\bar{\theta}_L)] \partial_j \phi \\
+ \phi(\bar{\theta} \bar{u}_L^i) T_k(\bar{\theta}) \partial_j \phi - \bar{\theta} \bar{u}_L A_i[\phi^2 T_k(\bar{\theta})] \partial_j \phi \\
\end{align*}
\]

\( L^2(0, T; W^{1,2}(\mathbb{R}^3)) \), \hspace{1cm} (9.70)

\[
\begin{align*}
\rho(T_k'(\bar{\theta}) \bar{\theta} - T_k(\bar{\theta})) & \phi \partial \psi \text{div} \bar{u}_L \\
+ \rho(T_k'(\bar{\theta}) \bar{\theta} - T_k(\bar{\theta})) & \phi \partial \psi \text{div} \bar{u} \\
A_i[\phi(\bar{\theta}_L \bar{u}_L^i) T_k(\bar{\theta}_L) \partial_j \phi - \bar{\theta}_L \bar{u}_L A_i[\phi^2 T_k(\bar{\theta}_L)] \partial_j \phi \\
+ \phi(\bar{\theta}_L \bar{u}_L^i) T_k(\bar{\theta}_L) \partial_j \phi - \bar{\theta}_L \bar{u}_L A_i[\phi^2 T_k(\bar{\theta}_L)] \partial_j \phi \\
\end{align*}
\]

\( L^2 \) \((0, T) \times \mathbb{R}^3 \), \hspace{1cm} (9.71)

also holds \( \tilde{P} \)-a.s. as \( L \to \infty \).

We can use (9.62) and (9.60) to pass to the limit \( L \to \infty \) in (9.64) to obtain

\[
0 = \int_0^T \int_{\mathbb{R}^3} \frac{T_k(\bar{\theta})}{\phi} \psi \, dx \, dt + \int_{\mathbb{R}^3} \frac{T_k(\bar{\theta})}{\phi} \psi(0) \, dx \hspace{1cm} (9.73)
\]

\( \mathbb{P} \)-a.s. for any \( \psi \in C^\infty_c(0, T) \times \mathbb{R}^3 \).

If we now apply Itô's formula to the function \( f(g_L, \tilde{m}_L^i) = \int_{\mathbb{R}^3} \tilde{m}_L^i \cdot \phi A_i[\phi g_L] \, dx \) where \( g_L = T_k(\bar{\theta}_L) \) satisfies (9.61) and \( \tilde{m}_L^i = \bar{\theta}_L \bar{u}_L^i \), then we obtain after a regularization argument,

\[
\hat{E} \int_0^T \int_{\mathbb{R}^3} (p(\bar{\theta}_L) - (\nu\beta + 2\nu) \text{div} \bar{u}_L) \phi T_k(\bar{\theta}_L) \, dx \, ds = \hat{E} \sum_{k=1}^9 I_k \hspace{1cm} (9.74)
\]

where for \( i = 1, 2, 3 \),

\[
\hat{E} \sum_{k=1}^9 I_k := \hat{E} \int_{\mathbb{R}^3} \phi \bar{\theta}_L \bar{u}_L A_i[\phi T_k(\bar{\theta}_L)] \, dx - \hat{E} \int_{\mathbb{R}^3} \phi \bar{\theta}_L \bar{u}_L A_i[\phi T_k(\bar{\theta}_L(0))] \, dx
\]

\[
\begin{align*}
+ \nu^S \hat{E} \int_0^T \int_{\mathbb{R}^3} \phi \bar{\theta}_L T_k(\bar{\theta}_L) \partial_i \phi \, dx \, ds \\
- \hat{E} \int_0^T \int_{\mathbb{R}^3} [p(\bar{\theta}_L) - (\nu\beta + 2\nu) \text{div} \bar{u}_L] A_i[\phi T_k(\bar{\theta}_L)] \partial_i \phi \, dx \, ds \\
+ \hat{E} \int_0^T \int_{\mathbb{R}^3} \phi \bar{\theta}_L \bar{u}_L A_i[\phi (T_k'(\bar{\theta}_L) \bar{\theta}_L - T_k(\bar{\theta}_L))] \, dx \, ds \\
+ \hat{E} \int_0^T \int_{\mathbb{R}^3} \bar{u}_L \left( R_{ij}[\phi \bar{\theta}_L \bar{u}_L^i] \phi T_k(\bar{\theta}_L) - \phi \bar{\theta}_L \bar{u}_L R_{ij}[\phi T_k(\bar{\theta}_L)] \right) \, dx \, ds \\
+ \hat{E} \int_0^T \int_{\mathbb{R}^3} \bar{u}_L^i \left( A_i[\phi \bar{\theta}_L \bar{u}_L^i] T_k(\bar{\theta}_L) \partial_j \phi - \bar{\theta}_L \bar{u}_L A_i[\phi T_k(\bar{\theta}_L)] \partial_j \phi \right) \, dx \, ds \\
- \nu^S \hat{E} \int_0^T \int_{\mathbb{R}^3} (\partial_j \partial_j \phi) \bar{u}_L A_i[\phi T_k(\bar{\theta}_L)] \, dx \, ds - \hat{E} \int_0^T \int_{\mathbb{R}^3} \phi \bar{\theta}_L \partial_i \bar{V}_L A_i[\phi T_k(\bar{\theta}_L)] \, dx \, ds
\end{align*}
\]

and for the limit variables,

\[
\hat{E} \int_0^T \int_{\mathbb{R}^3} \phi \phi T_k(\bar{\theta}) \, dx \, ds = \hat{E} \sum_{k=1}^9 K_k
\hspace{1cm} (9.76)
where for $i = 1, 2, 3$,
\begin{equation}
\begin{aligned}
\int \sum_{k=1} \mathcal{K}_k = \mathbb{E} \int_{\mathbb{R}^3} \phi \hat{\partial}_t^i \mathcal{A}_i \left[ \phi T_k(\hat{\varrho}) \right] dx - \mathbb{E} \int_{\mathbb{R}^3} \phi \hat{\partial}_t^i(0) \mathcal{A}_i \left[ \phi T_k(\hat{\varrho}(0)) \right] dx \\
\quad + \nu \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \phi \hat{\partial}_t^i T_k(\hat{\varrho}) \partial_t \phi dx ds - \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \left[ \nabla - (\lambda + \nu) \text{div} \hat{\mathbf{u}} \right] \mathcal{A}_i \left[ \phi T_k(\hat{\varrho}) \right] \partial_t \phi dx ds \\
\quad + \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \phi \hat{\partial}_t^i \mathcal{A}_i \left[ \phi (T_k(\hat{\varrho}) - T_k(\hat{\varrho})) \right] \text{div} \hat{\mathbf{u}} dx ds \\
\quad + \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \hat{u}^i \left( R_{ij} \phi \hat{\partial}_t^j \right) \mathcal{A}_i \left[ \phi T_k(\hat{\varrho}) - \phi \hat{\partial}_t^i \phi T_k(\hat{\varrho}) \right] dx ds \\
\quad + \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \hat{u}^i \left( \mathcal{A}_i \phi \hat{\partial}_t^i \hat{\partial}_t^j \left( T_k(\hat{\varrho}) \right) \phi - \hat{\partial}_t^i \mathcal{A}_i \phi T_k(\hat{\varrho}) \phi T_k(\hat{\varrho}) \hat{\partial}_t^j \phi \right) dx ds \\
\quad - \nu \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \left( \partial_t \partial_t \phi \right) \hat{u}^i \mathcal{A}_i \left[ \phi \hat{\partial}_t^i \phi T_k(\hat{\varrho}) \right] dx ds - \theta \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \phi \hat{\partial}_t \partial_t \mathcal{A}_i \left[ \phi \hat{\partial}_t^i \phi T_k(\hat{\varrho}) \right] dx ds
\end{aligned}
\end{equation}

Firstly, it follow from uniform integrability, see [17, (3.165)], that
\[
\mathbb{E} (I_1 + I_2) \to \mathbb{E} (K_1 + K_2).
\]
Now if let \(\{9.54, 9.68\}\) be the convergence result in [9.54] with respect to the velocity, then the following weak-strong duality pairings
\[
\{9.54, 9.68\}, \{9.70, 9.63\}, \{9.54, 9.71\}, \{9.54, 9.72\}, \{9.54, 9.65\}, \{9.61, 9.68\},
\]
yields \(\mathbb{E} I_3 \to \mathbb{E} K_3, \mathbb{E} I_4 \to \mathbb{E} K_4, \mathbb{E} I_5 \to \mathbb{E} K_5, \mathbb{E} I_6 \to \mathbb{E} K_6, \mathbb{E} I_7 \to \mathbb{E} K_7, \mathbb{E} I_8 \to \mathbb{E} K_8\) and \(\mathbb{E} I_9 \to \mathbb{E} K_9\) respectively. It follows that
\begin{equation}
\begin{aligned}
\lim_{L \to \infty} \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \left[ p(\hat{\varrho}_L) - (\nu B + 2\nu S) \text{div} \hat{\mathbf{u}} \right] \mathcal{A}_i \left[ \phi T_k(\hat{\varrho}_L) \right] dx ds \\
= \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \left[ \nabla - (\nu B + 2\nu S) \text{div} \hat{\mathbf{u}} \right] \mathcal{A}_i \left[ \phi T_k(\hat{\varrho}) \right] dx ds
\end{aligned}
\end{equation}
holds for any \(\phi \in C^1_c(\mathbb{R}^3)\) and for any \(t \in [0, T]\). We can now use [9.78] to obtain
\begin{equation}
\limsup_{L \to \infty} \mathbb{E} \int_0^t \int_{\mathbb{R}^3} \phi \left| T_k(\hat{\varrho}_L) - T_k(\hat{\varrho}) \right|^{\gamma + 1} dx ds \lesssim 1
\end{equation}
uniformly in \(k\) for any \(\phi \in C^1_c(\mathbb{R}^3)\) and any \(t \in [0, T]\), see [17, Lemma 3.4.14.]. If we set \(Q := (0, T) \times \mathbb{R}^3\), then follows from [9.79] that
\begin{equation}
\left| T_k(\hat{\varrho}) - \hat{\varrho} \right|_{L^p(\hat{\Omega} \times Q)}^p \leq \limsup_{L \to \infty} \left| T_k(\hat{\varrho}_L) - \hat{\varrho}_L \right|_{L^p(\hat{\Omega} \times Q)}^p \lesssim k^{\frac{1}{p} - \frac{1}{p}} \to 0
\end{equation}
for any \(p \in [1, \gamma)\) as \(k \to \infty\). Note that \(\frac{1}{\gamma} - \frac{1}{p} < 0\). As such,
\begin{equation}
T_k(\hat{\varrho}) \to \hat{\varrho} \quad \text{in} \quad L^p(\hat{\Omega} \times (0, T) \times \mathbb{R}^3)
\end{equation}
for all \(p \in [1, \gamma)\). Furthermore, if we regularize [9.73], with some regularizing operator \(S_m\), multiply the resulting equation by \(b'(S_m[T_k(\hat{\varrho})])\) and pass to the limit \(m \to \infty\), we obtain
\begin{equation}
0 = \int_0^T \int_{\mathbb{R}^3} b' \left( T_k(\hat{\varrho}) \right) \partial_t \psi dx dt + \int_{\mathbb{R}^3} b \left( T_k(\hat{\varrho}(0)) \right) \psi(0) dx + \int_0^T \int_{\mathbb{R}^3} \left[ b \left( T_k(\hat{\varrho}) \right) \hat{\mathbf{u}} \cdot \nabla \psi \right] dx dt \\
+ \int_0^T \int_{\mathbb{R}^3} b' \left( T_k(\hat{\varrho}) \right) \left( T_k(\hat{\varrho}) - b \left( T_k(\hat{\varrho}) \right) \right) \text{div} \hat{\mathbf{u}} \psi dx dt \\
- \int_0^T \int_{\mathbb{R}^3} b' \left( T_k(\hat{\varrho}) \right) T_k(\hat{\varrho}) - b \left( T_k(\hat{\varrho}) \right) \text{div} \hat{\mathbf{u}} \psi dx dt
\end{equation}
\(\hat{\mathbb{P}}\)-a.s for any \(\psi \in C_\infty_c([0, T) \times \mathbb{R}^3)\) where just as in \([17]\) Lemma 3.4.17, it follows from \([9.79]\) that
\[
b'(T_k(\hat{\varrho}))\left[(T_k'(\hat{\varrho})\hat{\varrho} - T_k(\hat{\varrho})) \text{div} \hat{u}\right] \psi \to 0
\]
\[(9.83)\]
in \(L^1(\Omega \times (0, T) \times \mathbb{R}^3)\) as \(k \to \infty\).

By combining \([9.81]\) with \([9.83]\), we may pass to the limit \(k \to \infty\) in \((9.84)\) and obtain up to the taking of possible subsequence,
\[
0 = \int_0^T \int_{\mathbb{R}^3} b(\hat{\varrho}) \partial_t \psi \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} b(\hat{\varrho}_0) \psi(0) \, dx \, dt
+ \int_0^T \int_{\mathbb{R}^3} [b(\hat{\varrho}) \hat{u}] \cdot \nabla \psi \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} [b'(\hat{\varrho}) \hat{\varrho} - b(\hat{\varrho})] \text{div} \hat{u} \psi \, dx \, dt
\]
\[(9.84)\]
\(\hat{\mathbb{P}}\)-a.s for any \(\psi \in C_\infty_c([0, T) \times \mathbb{R}^3)\). If we now take \(b = \tilde{L}_k\) in \((9.58)\) and \((9.84)\) and pass to the limit \(L \to \infty\), then we obtain
\[
\int_{\mathbb{R}^3} [\tilde{L}_k(\hat{\varrho}) - L_k(\hat{\varrho})] \varphi \, dx + \int_0^t \int_{\mathbb{R}^3} [\tilde{T}_k(\hat{\varrho}) \text{div} \hat{u} - T_k(\hat{\varrho}) \text{div} \tilde{u}] \varphi \, dx \, d\tau
= \int_0^t \int_{\mathbb{R}^3} [\tilde{L}_k(\hat{\varrho}) - L_k(\hat{\varrho})] \hat{u} \cdot \nabla \varphi \, dx \, d\tau
\]
\[(9.85)\]
\(\hat{\mathbb{P}}\)-a.s. for any \(t \in [0, T]\) and \(\varphi(x) \in C_\infty_c(\mathbb{R}^3)\). If we now consider \(\varphi = \phi_m\) for \(\phi_m\) as given in \([13]\) Eq. 4.14.12, Eq. 7.11.43, then by using convexity of \(z \mapsto L_k(z)\), triangle inequality and semi-continuity, it follows from \((9.85)\) and \((9.73)\) that
\[
\frac{1}{\nu B + 2\nu^8} \limsup_{L \to \infty} \hat{E} \int_0^t \int_{\mathbb{R}^3} [T_k(\hat{\varrho}_L) - T_k(\hat{\varrho})]^{\gamma+1} \phi_m \, dx \, d\tau
+ \hat{E} \int_0^t \int_{\mathbb{R}^3} [T_k(\hat{\varrho}) - T_k(\hat{\varrho})] \text{div} \hat{u} \phi_m \, dx \, d\tau
\leq \hat{E} \int_0^t \int_{\mathbb{R}^3} [\tilde{L}_k(\hat{\varrho}) - L_k(\hat{\varrho})] \hat{u} \cdot \nabla \phi_m \, dx \, d\tau
\]
\[(9.86)\]
after the taking of possible subsequence where
\[
\hat{E} \int_0^t \int_{\mathbb{R}^3} [T_k(\hat{\varrho}_L) - T_k(\hat{\varrho})] \text{div} \hat{u} \phi_m \, dx \, d\tau \to 0
\]
\[(9.87)\]
as \(k \to \infty\) and
\[
\hat{E} \int_0^t \int_{\mathbb{R}^3} [L_k(\hat{\varrho}) - L_k(\hat{\varrho})] \hat{u} \cdot \nabla \phi_m \, dx \, d\tau \to 0
\]
\[(9.88)\]
as \(m \to \infty\). We subsequently obtain from
\[
\lim_{k \to \infty} \limsup_{L \to \infty} \hat{E} \int_0^t \int_K [T_k(\hat{\varrho}_L) - T_k(\hat{\varrho})]^{\gamma+1} \, dx \, d\tau \leq 0
\]
\[(9.89)\]
for any \(K \subset \mathbb{R}^3\). If we now use the following convergence
\[
\lim_{k \to \infty} \left( \limsup_{L \to \infty} \|T_k(f_L) - f_L\|_{L^p(K)} + \|T_k(f) - f\|_{L^p(K)} \right) \to 0
\]
\[(9.90)\]
which holds for any \(p \in (1, q)\) provided that \(f \in L^q(K)\), refer to \([17]\) (2.33)-(2.34)], then we can use triangle inequality and the continuous embedding \(L^{\gamma+1} \hookrightarrow L^1\) to obtain from \((9.89)\) that
\[
\hat{\varrho}_L \to \hat{\varrho} \quad \text{in} \quad L^1(\tilde{\Omega} \times (0, T) \times K).
\]
\[(9.91)\]
It also follows from \((9.91)\) that
\[
\Delta \hat{V}_L \to \Delta \hat{V} \quad \text{in} \quad L^1(\tilde{\Omega} \times (0, T) \times K).
\]
\[(9.92)\]
We can now use the Lipschitz continuity of \(\varrho_k\) and \((9.01)\) to obtain
\[
\int_{\mathbb{R}^3} \hat{\varrho}_L \varrho_k(\hat{\varrho}_L, \hat{\varrho} \hat{u}_L) \cdot \phi \, dx \to \int_{\mathbb{R}^3} \hat{\varrho} \varrho_k(\hat{\varrho}, \hat{\varrho} \hat{u}) \cdot \phi \, dx \quad \text{a.e. in} \quad (0, T)
\]
\[(9.93)\]
Martingale solutions of the stochastic Navier–Stokes–Poisson system

\( \mathbb{P}\)-a.s. for any \( \phi \in C_\infty^\infty(\mathbb{R}^3) \) from which we infer that
\[
\bar{\varrho} \underline{g}_k(\bar{\varrho}, \bar{f}, \bar{\varrho} \bar{u}) = \underline{\varrho} g_k(\varrho, f, \varrho u) \quad \text{a.e. in } \bar{\Omega} \times (0, T) \times K
\]
where \( K \subseteq \mathbb{R}^3 \).

We are now able to properly identify all the nonlinear terms the momentum equation \( (2.12) \) and the energy inequality \( (2.13) \). The following results therefore holds.

**Lemma 9.13.** The random distributions \( [\hat{\varrho}, \hat{u}, \hat{V}, W] \) satisfies \( (2.12) \) for all \( \psi \in C_\infty(\mathbb{R}^3) \) and \( \phi \in C_\infty^\infty(\mathbb{R}^3) \) \( \mathbb{P}\)-a.s.

**Lemma 9.14.** The random distributions \( [\tilde{\varrho}, \tilde{u}, \tilde{V}] \) satisfies \( (2.13) \) for all \( \psi \in C_\infty(\mathbb{R}^3) \), \( \psi \geq 0 \) \( \mathbb{P}\)-a.s.

### 10. The relative energy estimate

Before we start, we remark that the result presented in this section also hold in \( \mathbb{T}^3 \). In this case the far field density \( \varrho = 0 \) and we do not require functions to be compactly supported.

For the purposes of studying singular limits of our system \( (1.1)–(1.3) \), amongst other reasons, it is sometimes useful to have an estimate that ‘measures the distance’ between the functions that solves \( (1.1)–(1.3) \) and some test function of some limit system which mimics the behaviour of \( (1.1)–(1.3) \). In this regard, we let
\[
E(\varrho, u, V | r, U, W) = \int_{\mathbb{R}^3} \left[ \frac{1}{2} \varrho |u - U|^2 + H(\varrho, r) \pm \varrho |\nabla (V - W)|^2 \right] \, dr,
\]
be the relative energy functional where the test function \( (r, U, W) \) solves the system
\[
\begin{align*}
\frac{d}{dt} r &= D_t^4 r \, dt + D_t^4 r \, dW, \\
\frac{d}{dt} U &= D_t^4 U \, dt + D_t^4 U \, dW, \\
0 &= \mp \Delta W + r - f
\end{align*}
\]
in strong sense of PDEs. Depending on what we wish to study, this solution of \( (10.1) \) can further be weak or strong in the sense of Probabilities. In the above, \( f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) is a given function depending only in space, \( D_t^4 r \) and \( D_t^4 U \) are smooth functions of \( (\omega, t, x) \) whereas \( D_t^4 r \) and \( D_t^4 U \) belongs to the function space \( L^2(\Omega; L^2(\mathbb{R}^3)) \) for a.e \( (\omega, t) \in \Omega \times [0, T] \). For simplicity, we assume that
\[
\begin{align*}
(r - \bar{r}) &\in C_\infty^\infty([0, T] \times \mathbb{R}^3), \\
U &\in C_\infty^\infty([0, T] \times \mathbb{R}^3) \\
W &\in C_\infty^\infty([0, T] \times \mathbb{R}^3)
\end{align*}
\]
\( \mathbb{P}\)-a.s. and for all \( 1 \leq q < \infty \),
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|r\|_{W^{1,q}(\mathbb{R}^3)}^q \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} \|U\|_{W^{1,q}(\mathbb{R}^3)}^q \right] \leq c(q),
\]
\[
0 < r \leq r(t, x) \leq \bar{r} \quad \mathbb{P}\text{-a.s.} \quad (10.5)
\]
Moreover, \( r \) and \( U \) satisfy
\[
D_t^4 r, D_t^4 U \in L^q(\Omega; L^q(0, T; W^{1,q}(\mathbb{R}^3))),
\]
\[
D_t^4 r, D_t^4 U \in L^2(\Omega; L^2(0, T; L^2(\mathbb{R}^3))),
\]
as well as
\[
\left( \sum_{k \in \mathbb{N}} \|D_t^4 r(e_k)\|^q \right)^{\frac{1}{q}} \in L^q(\Omega; L^q(0, T; L^q(\mathbb{R}^3))),
\]
\[
\left( \sum_{k \in \mathbb{N}} \|D_t^4 U(e_k)\|^q \right)^{\frac{1}{q}} \in L^q(\Omega; L^q(0, T; L^q(\mathbb{R}^3))).
\]
(10.6)
The remainder term is
\[ P \]
holds \( \mathbb{P} \)-a.s. where the tensor \( S \) is such that
\[ \int_0^t \int_{\mathbb{R}^3} S(\nabla v) : \nabla v \, dx \, ds = \int_0^t \int_{\mathbb{R}^3} (\nu^3 |\nabla v|^2 + (\mu^B + \nu^S) |\text{div} v|^2) \, dx \, ds \]
the remainder term is
\[ \mathcal{R}(\varrho, u, V | r, U, W) = \int_{\mathbb{R}^3} S(\nabla U) : (\nabla U - \nabla u) \, dx \]
+ \int_{\mathbb{R}^3} \varrho(D_t^s U + u \cdot \nabla U) \cdot (U - u) \, dx 
+ \int_{\mathbb{R}^3} [(r - \varrho)P''(r)D_t^s r + \nabla P'(r) \cdot (rU - \varrho u)] \, dx 
+ \int_{\mathbb{R}^3} [p(r) - p(\varrho)] \text{div}(U) \, dx - \int_{\mathbb{R}^3} \varrho \partial_t U \cdot \nabla V \, dx 
+ \frac{1}{2} \int_0^t \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^3} \left| p''(r) - \varrho P''(r) \right| D_t^s r(e_k) \, dx \, ds 
+ \frac{1}{2} \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^3} \varrho |g_k(\varrho, f, \varrho u) - D_t^s U(e_k)|^2 \, dx 
\]
and \( M_{RE} \) is a real valued square integrable martingale given by
\[ M_{RE}(t) = \int_0^t \int_{\mathbb{R}^3} \varrho (u - U) \cdot G(\varrho, f, \varrho u) \, dx \, dW 
- \int_0^t \int_{\mathbb{R}^3} \varrho (u - U) \cdot D_t^s U \, dx \, dW + \int_0^t \int_{\mathbb{R}^3} (p'(r) - \varrho P'(r)) D_t^s r \, dx \, dW. \]

Remark 10.2. As opposed to \([17, (3.250)]\), notice the extra density in front of the first right-hand term of \((10.9)\). This is because of how we represented the noise in \((1.2)\).

In order to obtain \((10.7)\), we first note that just as in \([17\, \text{Section } 3.6]\), we gain the following
\[ \int_{\mathbb{R}^3} \frac{1}{2} \varrho |U|^2 \, dx = \int_{\mathbb{R}^3} \frac{1}{2} \left| \left( \varrho U(0) \right) \right|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} \varrho u \cdot \nabla U \cdot U \, dx \, ds \]
+ \int_0^t \int_{\mathbb{R}^3} \varrho U \cdot D_t^s U \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} \varrho U \cdot D_t^s U \, dx \, dW + \frac{1}{2} \int_0^t \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^3} \varrho |D_t^s U(e_k)|^2 \, dx \, ds. \]
\( \mathbb{P} \)-a.s. by applying Itô’s lemma to \((11)\) and \((10.12)\). Also, the identity \( rP'(r) - P(r) = p(r) \) helps us obtain from \((10.11)\), the following
\[ \int_{\mathbb{R}^3} [rP'(r) - P(r)](t) \, dx = \int_{\mathbb{R}^3} [rP'(r) - P(r)](0) \, dx + \int_0^t \int_{\mathbb{R}^3} rP''(r) D_t^s r \, dx \, ds \]
+ \int_0^t \int_{\mathbb{R}^3} p'(r) D_t^s r \, dx \, dW + \frac{1}{2} \int_0^t \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^3} p''(r) |D_t^s r(e_k)|^2 \, dx \, ds. \]
Furthermore, the following identity holds
\[
\int_{\mathbb{R}^3} \rho P'(r) dx = \int_{\mathbb{R}^3} \rho(0) P'(r(0)) dx + \int_0^t \int_{\mathbb{R}^3} \rho \nabla P'(r) \cdot \mathbf{u} dx ds + \int_0^t \int_{\mathbb{R}^3} \rho P''(r) \cdot \mathbf{D}_t^e \mathbf{u} dx ds \\
+ \int_0^t \int_{\mathbb{R}^3} \rho P''(r) \cdot \mathbf{D}_t^e \mathbf{u} dx dW + \frac{1}{2} \int_0^t \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^3} \rho P''(r) \left| \mathbf{D}_t^e \mathbf{u}(r_k) \right|^2 dx ds
\]
(10.12)
P-a.s. However, due to the presence of the electric field in the momentum equation \[122\], the corresponding version of \[17\] (3.245) in our present case is now
\[
\int_{\mathbb{R}^3} \rho \mathbf{u} \cdot \mathbf{U} dx = \int_{\mathbb{R}^3} (\rho \mathbf{u})(0) \cdot \mathbf{U}(0) dx + \int_0^t \int_{\mathbb{R}^3} \rho \mathbf{u} \cdot \mathbf{D}_t^q \mathbf{U} dx ds + \int_0^t \int_{\mathbb{R}^3} \rho \mathbf{u} \cdot \mathbf{D}_t^e \mathbf{U} dx dW \\
- \int_0^t \int_{\mathbb{R}^3} \mathbf{S}(\nabla \mathbf{u}) : \nabla \mathbf{U} dx ds + \int_0^t \int_{\mathbb{R}^3} \rho(\mathbf{u}) \text{div}(\mathbf{U}) dx ds + \int_0^t \int_{\mathbb{R}^3} \rho \mathbf{u} \cdot \nabla \mathbf{U} \cdot \mathbf{u} dx ds \\
+ \int_0^t \int_{\mathbb{R}^3} \rho \mathbf{u} \cdot \nabla \mathbf{V} dx ds + \int_0^t \int_{\mathbb{R}^3} \rho \mathbf{U} \cdot \mathbf{G}(\rho, f, \rho \mathbf{u}) dx dW + \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^3} \rho \mathbf{D}_t^e \mathbf{U}(r_k) \cdot \mathbf{g}_k(\rho, f, \rho \mathbf{u}) dx ds.
\]
(10.13)
P-a.s. which follow from the application of Itô’s formula to the function \( f^2(\mathbf{m}, \mathbf{U}) = \int \mathbf{m} \mathbf{U} dx \) where \( \mathbf{m} = \rho \mathbf{u} \) is the momentum. Now since \((V, W)\) solve stationary equations, we can infer that
\[
\pm ([\nabla W]^2 - \nabla V \cdot \nabla W)(t) = \pm ([\nabla W]^2 - \nabla V \cdot \nabla W)(0)
\]
(10.14)
for any \( t \in (0, T) \). Finally we recall that the energy inequality
\[
\int_{\mathbb{R}^3} \left[ \frac{1}{2} \rho |\mathbf{u}|^2 + P(\rho) \pm \rho |\nabla V|^2 \right](t) dx + \int_0^t \int_{\mathbb{R}^3} \mathbf{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} dx ds \\
\leq \int_{\mathbb{R}^3} \left[ \frac{1}{2} \rho |\mathbf{u}|^2 + P(\rho) \pm \rho |\nabla V|^2 \right](0) dx \\
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \rho \sum_{k \in \mathbb{N}} |\mathbf{g}_k(x, \rho, f, \mathbf{m})|^2 dx ds + \int_0^t \int_{\mathbb{R}^3} \rho \mathbf{u} \cdot \mathbf{G}(\rho, f, \mathbf{m}) dx dW;
\]
(10.15)
hold \( \mathbb{P}\)-a.s. because \((\rho, \mathbf{u}, V)\) is a solution of \[1.1\] in the sense of Definition \[9.1\]. Now since the following identities
\[
\frac{1}{2} \rho |\mathbf{u} - \mathbf{U}|^2 = \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{2} \rho |\mathbf{U}|^2 - \rho \mathbf{u} \cdot \mathbf{U} \\
H(\rho, r) = P(\rho) - \rho P'(r) + [P'(r)r - P(r)] \\
|\nabla (V - W)|^2 = |\nabla V|^2 + |\nabla W|^2 - 2 \nabla V \cdot \nabla W,
\]
hold, we can collect \[10.10\]–\[10.15\] to obtain our result.

11. APPENDIX

**Proposition 11.1.** Let \( \mathbf{u} \in C([0, T]; C^2(\mathbb{T}^3)) \) be given and assume that for some constants \( \nu > 0 \) and \( \underline{\theta}, \overline{\theta} > 0 \), the following
\[
\theta(0) = \theta_0 \in C^{2+\nu}(\mathbb{T}^3), \quad \underline{\theta} \leq \theta_0(x) \leq \overline{\theta}
\]
holds. Then the equation
\[
\partial_t \theta + \text{div}(\rho \mathbf{u}) = \varepsilon \Delta \theta
\]
where \( \varepsilon > 0 \) has a unique classical solution \( \theta \in C([0, T]; C^{2+\nu}(\mathbb{T}^3)) \) satisfying the bounds
\[
\underline{\theta} \exp \left( - \int_0^t \| \text{div} \mathbf{u} \|_{L^\infty(\mathbb{T}^3)} ds \right) \leq \theta(t, x) \leq \overline{\theta} \exp \left( \int_0^t \| \text{div} \mathbf{u} \|_{L^\infty(\mathbb{T}^3)} ds \right)
\]
(11.1)
for all \((t, x) \in [0, T] \times T^3\). Furthermore, if \(\varphi_1\) and \(\varphi_2\) are two of any such classical solutions with velocities \(u_1\) and \(u_2\) respectively and that additionally,

\[
\|u_1\|_{L^\infty([0,T];W^{1,\infty}(T^3))} + \|u_2\|_{L^\infty([0,T];W^{1,\infty}(T^3))} \lesssim 1,
\]

then we have that

\[
\|\varphi_1 - \varphi_2\|_{C([0,T];W^{1,2}(T^3))} \lesssim T\|u_1 - u_2\|_{C([0,T];W^{1,2}(T^3))}.
\]

References

[1] Breit, D., Feireisl, E., Hofmanová, M.: Stochastically Forced Compressible Fluid Flows. De Gruyter Series in Applied and Numerical Mathematics. De Gruyter (2018).

[2] Breit, D., Hofmanová, M.: Stochastic Navier–Stokes equations for compressible fluids. Indiana Univ. Math. J. 65(4), 1183–1250 (2016).

[3] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions, vol. 152. Cambridge university press (2014).

[4] Dafermos, C.M.: The second law of thermodynamics and stability. Arch. Rational Mech. Anal. 70(2), 167–179 (1979).

[5] DiPerna, R.J., Lions, P.L.: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. 98(3), 511–547 (1989).

[6] Donatelli, D.: Local and global existence for the coupled Navier-Stokes-Poisson problem. Quart. Appl. Math. 61(2), 345–361 (2003).

[7] Ducomet, B., Feireisl, E., Petzeltová, H., Straškraba, I.: Global in time weak solutions for compressible barotropic self-gravitating fluids. Discrete Cont. Dyn. Syst. 11(1), 113–130 (2004).

[8] Feireisl, E.: On compactness of solutions to the compressible isentropic Navier–Stokes equations when the density is not square integrable. Comment. Math. Univ. Carol. 42(1), 83–98 (2001).

[9] Feireisl, E., Novotný, A.: Singular limits in thermodynamics of viscous fluids. Springer Science & Business Media (2009).

[10] Feireisl, E., Novotný, A., Petzeltová, H.: On the existence of globally defined weak solutions to the Navier–Stokes equations. J. Math. Fluid Mech. 11(4), 358–392 (2009).

[11] Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Classics in Mathematics. Springer-Verlag, Berlin (2001).

[12] Grafakos, L.: Classical Fourier analysis, Graduate Texts in Mathematics, vol. 249, second edn. Springer, New York (2008).

[13] Gyöngy, I., Krylov, N.: Existence of strong solutions for Itô’s stochastic equations via approximations. Probab. Theory Related Fields 105(2), 143–158 (1996).

[14] Hieber, M., Prüss, J.: Heat kernels and maximal \(L^p\)-\(L^q\) estimates for parabolic evolution equations. Comm. Partial Differential Equations 22(9-10), 1647–1669 (1997).

[15] Jakubowski, A.: Short communication: The almost sure skorokhod representation for subsequences in nonmetric spaces. Theory Probab. Appl. 42(1), 167–174 (1998).

[16] Mensah, P.R.: Existence of martingale solutions and the incompressible limit for stochastic compressible flows on the whole space. Ann. Mat. Pura Appl. (4) 196(6), 2105–2133 (2017).

[17] Mensah, P.R.: The stochastic compressible Navier–Stokes system on the whole space and some singular limits. Ph.D. thesis, Heriot–Watt University (2019). https://www.researchgate.net/publication/337682116.

[18] Novotný, A., Straškraba, I.: Introduction to the mathematical theory of compressible flow. Oxford University Press, New York (2004).

[19] Ondreját, M.: Stochastic nonlinear wave equations in local Sobolev spaces. Electron. J. Probab. 15, no. 33, 1041–1091 (2010).

[20] Smith, S.A.: Random perturbations of viscous, compressible fluids: global existence of weak solutions. SIAM J. Math. Anal. 49(6), 4521–4578 (2017).

[21] Tartar, L.: An introduction to Sobolev spaces and interpolation spaces, Lecture Notes of the Unione Matematica Italiana, vol. 3. Springer, Berlin; UMI, Bologna (2007).