ON THE PRESERVATION OF GIBBSIANNESS
UNDER SYMBOL AMALGAMATION

JEAN–RENNÉ CHAZOTTES & EDGARDO UGALDE

Abstract. Starting from the full–shift on a finite alphabet $A$, mingling some symbols of $A$, we obtain a new full shift on a smaller alphabet $B$. This amalgamation defines a factor map from $(A^n, T_A)$ to $(B^n, T_B)$, where $T_A$ and $T_B$ are the respective shift maps. According to the thermodynamic formalism, to each regular function (‘potential’) $\psi : A^n \to \mathbb{R}$, we can associate a unique Gibbs measure $\mu_\psi$. In this article, we prove that, for a large class of potentials, the pushforward measure $\mu_\psi \circ \pi^{-1}$ is still Gibbsian for a potential $\phi : B^n \to \mathbb{R}$ having a ‘bit less’ regularity than $\psi$. In the special case where $\psi$ is a ‘2–symbol’ potential, the Gibbs measure $\mu_\psi$ is nothing but a Markov measure and the amalgamation $\pi$ defines a hidden Markov chain. In this particular case, our theorem can be recast by saying that a hidden Markov chain is a Gibbs measure (for a Hölder potential).

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1. Introduction

From different viewpoints and under different names, the so–called hidden Markov measures have received a lot of attention in the last fifty years [3]. One considers
a (stationary) Markov chain \((X_n)_{n \in \mathbb{N}}\) with finite state space \(A\) and looks at its ‘instantaneous’ image \(Y_n := \pi(X_n)\), where the map \(\pi\) is an amalgamation of the elements of \(A\) yielding a smaller state space, say \(B\). It is well–known that in general the resulting chain, \((Y_n)_{n \in \mathbb{N}}\), has infinite memory. For concrete examples, see e. g. [1] or the more easily accessible reference [3] where they are recalled.

A stationary Markov chain with finite state space \(A\) can be equivalently defined as a shift–invariant Markov measure \(\mu\) on the path space \(A^\mathbb{N}\) (of infinite sequences of ‘symbols’ from the finite ‘alphabet’ \(A\)), where the shift map \(T : A^\mathbb{N} \to A^\mathbb{N}\) is defined by \((Ta)_i = a_{i+1}\). A hidden Markov measure can be therefore seen as the the pushforward measure \(\mu_\psi \circ \pi^{-1}\) on the path space \(B^\mathbb{N}\) formed by the instantaneous image under the amalgamation \(\pi\), of paths in \(A^\mathbb{N}\).

In the present article, instead of focusing on shift–invariant Markov measures, we consider a natural generalization of them. Let \(\psi : A^\mathbb{N} \to \mathbb{R}\) be a ‘potential’, then, under appropriate regularity condition on \(\psi\) (see more details below), there is a unique so–called Gibbs measure \(\mu_\psi\) associated to it. It is a shift–invariant probability measure on \(A^\mathbb{N}\) with remarkably nice properties. Each \(r\)–step Markov measure falls in this category, since an \(r\)–step Markov measure is nothing but a Gibbs measure defined by a \((r+1)\)–symbol potential, i.e., a potential \(\psi\) such that \(\psi(a) = \psi(\hat{a})\) whenever \(a_i = \hat{a}_i, i = 0, \ldots, r\), with \(r\) a strictly positive integer.

On the other hand, given \(\psi\) one can construct a sequence \((\psi_r)\) of \((r+1)\)–symbol potentials (uniformly approximating \(\psi\)) such that the sequence of associated \(r\)–step Markov measures \(\mu_{\psi_r}\) converges to \(\mu_\psi\) (in the vague or weak* topology, at least).

Now let \(B\) be the alphabet obtained from \(A\) by amalgamation of some of the symbols of \(A\).\(^1\) The amalgamation defines a surjective (i.e., onto) map \(\pi : A \to B\) which extends to \(A^\mathbb{N}\) in the obvious way. Given a Gibbs measure \(\mu_\psi\) on \(A^\mathbb{N}\), this map induces a measure \(\mu_\psi \circ \pi^{-1}\) supported on the full shift \(B^\mathbb{N}\). The question we address now reads:

**Question 1.** Under which condition is the measure \(\mu_\psi \circ \pi^{-1}\), supported on the full shift \(B^\mathbb{N}\), still Gibbsian? In other words, under which conditions on \(\psi\) can one build a ‘nice’ potential \(\phi : B^\mathbb{N} \to \mathbb{R}\) such that \(\mu_\psi \circ \pi^{-1} = \mu_\phi\)? In particular, for \(\psi\) a \(2\)–symbol potential, what is the nature of \(\mu_\psi \circ \pi^{-1}\)?

In this article we make the following answer (made precise below, see Theorems 3.1 and 4.1):

Under mild regularity condition on \(\psi\), the pushforward of the Gibbs measure \(\mu_\psi\), namely \(\mu_\psi \circ \pi^{-1}\), is Gibbsian as well, and the associated potential \(\phi\) can be computed from \(\psi\). Furthermore, when \(\psi\) is a \(2\)–symbol potential, the corresponding hidden Markov chain is Gibbsian, and it is associated to a Hölder potential.

A slightly more general problem is the following. Suppose that we do not start with the full shift \(A^\mathbb{N}\) but with a subshift of finite type (henceforth SFT) or a topological Markov chain \(X\) [9]. The image of \(X\) is not in general of finite type but it is a sofic subshift [9]:

**Question 2.** When \(X \subset A^\mathbb{N}\) is a SFT, is the measure \(\mu_\psi \circ \pi^{-1}\) still Gibbsian?

\(^1\)The case \(r = 0\) corresponds to product measures (i.i.d. process).

\(^2\)We assume \(B\) has cardinality at least equal to two.
Question 2 has only received very partial answers up-to-date. We shall comment on that in Section 5.

The present work is motivated, on the one hand, by our previous work in [5] in which we attempted to solve Question 2 and were partially successful. On the other hand, it was motivated by [6] where we were interested in approximating Gibbs measures on sofic subshifts by Markov measures on subshifts of finite type. Here we combine ideas and techniques both from [5] and [6] but we need extra work to get more uniformity than previously obtained.

Let us mention two recent works related to ours. In [11], another kind of transformation of the alphabet is considered, and the method employed to prove Gibbsianity is completely different from ours. In [7], the authors study random functions of Markov chains and obtain results about their loss of memory.

The paper is organized as follows. In the next section we give some notations and definitions. In particular, we present the weak∗ convergence of measures as a projective convergence and we define the notion of Markov approximants of a Gibbs measure. In Section 3, we state Theorem 3.1 which answers Question 1 when the starting potential ψ is Hölder continuous (its modulus of continuity decays exponentially to 0). The proof relies on two lemmas which are proved in Appendices 6.2 and 6.3 respectively. In Section 4, we generalize Theorem 3.1 to a class of potentials with subexponential (strictly subexponential or polynomial) decay of modulus of continuity. We finish (Section 5) by discussing Question 2 and giving a conjecture. Appendix 6.1 is devoted to Birkhoff’s version of Perron–Frobenius theorem for positive matrices, our main tool.

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2. Background Material

2.1. Symbolic dynamics.

Let A be a finite set (‘alphabet’) and A^N be the set of infinite sequences of symbols drawn from A. We define N to be the set {0, 1, 2, ...}, that is, the set of positive integers plus 0. We denote by a, b, etc, elements of A^N and use the notation a^n_m (m ≤ n, m, n ∈ N) for the word a_m a_{m+1} · · · a_{n−1} a_n (of length n − m + 1). We endow A^N with the distance

\[d_\ast(a, b) := \begin{cases} \exp(-\min\{n ≥ 0 : a_0^n ≠ b_0^n\}) & \text{if } a ≠ b \\ 0 & \text{otherwise.} \end{cases}\]

The resulting metric space (A^N, d_\ast) is compact.

The shift transformation T : A^N → A^N is defined by (T a)_n = a_{n+1} for all n ∈ N.

A subshift X of A^N is a closed T–invariant subset of A^N.

Given a set of admissible words \mathcal{L} ⊂ A^\ell for some fixed integer \ell ≥ 2, one defines a subshift of finite type A_\mathcal{L} ⊂ A^N by

\[A_\mathcal{L} := \{a ∈ A^N : a_{n+\ell−1}^n ∈ \mathcal{L}, \ ∀ n ∈ N\}.\]
A subshift of finite type defined by words in \( \mathcal{L} \subset A^2 \) is called a topological Markov chain. It can be equivalently described by the transition matrix \( M : A \times A \to \{0, 1\} \) such that \( M(a, b) = \chi_{\mathcal{L}}(ab) \), where \( \chi_{\mathcal{L}} \) is the indicator function of the set \( \mathcal{L} \). We will use both \( A_{\mathcal{L}} \) and \( A_M \) to denote the corresponding subshift of finite type.

Note that the ‘full shift’ \( (A^N, T) \) can be seen as the subshift of finite type defined by all the words of length \( \ell \), and we have the identification \( A^N \equiv A_{\ell} \).

Let \( X \subset A^N \) be a subshift. A point \( a \in X \) is periodic with period \( p \geq 1 \) if \( T^p a = a \), and \( p \) is its minimal period if in addition \( T^k a \neq a \) whenever \( 0 < k < p \). We denote by \( \text{Per}_p(X) \) the collection of all periodic points with period \( p \) in \( X \), and by \( \text{Per}(X) \) the collection of all periodic points in \( X \), i.e., \( \text{Per}(X) = \bigcup_{p \geq 1} \text{Per}_p(X) \).

Given an arbitrary subshift \( X \subset A^N \) and \( m \in \mathbb{N} \), the set of all the \( X \)-admissible words of length \( m + 1 \) is the set \( X_m := \{ w \in A^{m+1} : \exists a \in X, w = a^m_0 \} \).

It is a well known fact that a topological Markov chain \( A_M \) is topologically mixing if its transition matrix \( M \) is primitive, i.e., if and only if there exists an integer \( n \geq 1 \) such that \( M^n > 0 \). In this case, the smallest of such integers is the so called primitivity index of \( M \).

For a subshift \( X \subset A^N \), \( w \in X_m \) and \( m \in \mathbb{N} \), the set \([w] := \{ a \in X : a^m_0 = w \}\) is the cylinder based on \( w \).

We will use boldfaced symbols \( a, b, \) etc, not only for infinite sequences but also for finite ones (i.e., for words). The context will make clear whether we deal with a finite or an infinite sequence.

### 2.2. Thermodynamic formalism.

For a subshift \( X \subset A^N \), cylinders are clopen sets and generate the Borel \( \sigma \)-algebra. We denote by \( \mathcal{M}(X) \) the set of Borel probability measures on \( X \) and by \( \mathcal{M}_T(X) \) the subset of \( T \)-invariant probability measures on \( X \). Both are compact convex sets in weak\(^*\) topology. The weak\(^*\) topology can be metrized \(^2\) by the distance

\[
D(\mu, \nu) := \sum_{m=0}^{\infty} 2^{-(m+1)} \left( \sum_{w \in X_m} |\mu[w] - \nu[w]| \right).
\]

It turns out that the following notion of convergence is very convenient in our later calculations.

**Definition 2.1.** We say that a sequence \( (\mu_n)_{n \in \mathbb{N}} \) of probability measures in \( \mathcal{M}(X) \) converges in the projective sense to a measure \( \mu \in \mathcal{M}(X) \) if for all \( \epsilon > 0 \) and \( N > 1 \) there exists \( N' > 1 \) such that

\[
\exp(-\epsilon) \leq \frac{\mu_n[w]}{\mu[w]} \leq \exp(\epsilon)
\]

for all admissible words \( w \) of length \( k \leq N \), and for all \( n \geq N' \).

\(^2\)On the other hand, if none of the rows or columns of \( M \) is identically zero, \( A_M \) is topologically mixing implies \( M \) is primitive.
It is easy to verify that convergence in the projective sense implies weak$^\ast$ convergence. On the other hand, when all the measures involved share the same support, weak$^\ast$ and projective convergence coincide. Though it is the case in this paper, we will speak of projective convergence.

We make the following definitions.

**Definition 2.2** (($r+1$)-symbol potentials). A function $\psi: \mathbb{A}^N \to \mathbb{R}$ will be called a potential. We say that a potential $\psi: \mathbb{A}^N \to \mathbb{R}$ is an ($r+1$)-symbol potential if there is an $r \in \mathbb{N}$ such that

$$\psi(a) = \psi(b) \text{ whenever } a^r_0 = b^r_0.$$  

Of course, we take $r$ to be the smallest integer with this property.

We will say that $\psi$ is locally constant if it is an ($r+1$)-symbol potential for some $r \in \mathbb{N}$.

A way of quantifying the regularity of a potential $\psi: \mathbb{A}^N \to \mathbb{R}$ is by using its modulus of continuity on cylinders, or variation, defined by

$$\text{var}_n \psi := \sup \{|\psi(a) - \psi(b)| : a, b \in \mathbb{A}^N, a^r_0 = b^r_0\}.$$  

A potential $\psi$ is continuous if and only if $\text{var}_n \psi \to 0$ as $n \to \infty$. An ($r+1$)-symbol potential $\psi$ can be alternatively defined by requiring that $\text{var}_n \psi = 0$ whenever $n \geq r$, and thus it is trivially continuous. If there are $C > 0$ and $\theta \in ]0,1[\text{ such that } \text{var}_n \phi \leq C \theta^n$ for all $n \geq 0$, then $\psi$ is said to be Hölder continuous.

We will use the notation

$$S_n \psi(a) := \sum_{k=0}^{n-1} \psi \circ T^k(a), \ n = 1, 2, \ldots$$  

Throughout we will write

$$x \lesssim y \exp(\pm C) \quad \text{for } x, y \text{ and } C \text{ strictly positive numbers. Accordingly we will use the notation } x \lesssim y \exp(\pm C).$$  

We now define the notion of Gibbs measure we will use in the sequel.

**Definition 2.3** (Gibbs measures). Let $X \subset \mathbb{A}^N$ be a subshift and $\psi: \mathbb{A}^N \to \mathbb{R}$ be a potential such that $\psi|_X$ is continuous. A measure $\mu \in \mathcal{M}_T(X)$ is a Gibbs measure for the potential $\psi$, if there are constants $C = C(\psi, X) \geq 1$ and $P = P(\psi, X) \in \mathbb{R}$ such that

$$\frac{\mu[a^n_0]}{\exp(S_{n+1} \psi(a) - (n+1)P)} \leq C^{\pm 1},$$  

for all $n \in \mathbb{N}$ and $a \in X$. We denote by $\mu_\psi$ such a measure.

The constant $P = P(\psi, X)$ is the topological pressure $\mathfrak{p}$ of $X$ with respect to $\psi$. It can be obtained, for $X$ a subshift of finite type, as follows:

$$P(\psi, X) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{a \in \text{Per}_n(X)} \exp(S_n \psi(a)).$$
We will say that the potential $\psi$ is normalized on $X$ if $P(\psi, X) = 0$. We can always normalize a potential $\psi$ by replacing $\psi$ by $\psi - P(\psi, X)$. This does not affect the associated Gibbs measure $\mu_\psi$.

In the above definition, we allow that $\psi = -\infty$ on $A^N \setminus X$. In other words, $\psi$ is upper semi–continuous on $A^N$.

**Remark 2.1.** If $\mu \in M_T(X)$ is such that the sequence $(\log(\mu[a_0^n]/\mu[a_1^n]))_{n=1}^\infty$ converges uniformly in $a \in X$, then the potential $\psi : X \to \mathbb{R}$ given by

$$\psi(a) = \lim_{n \to \infty} \log \left( \frac{\mu[a_0^n]}{\mu[a_1^n]} \right)$$

is continuous on $X$, and $\mu$ is a Gibbs measure with respect to $\psi$, i.e. $\mu = \mu_\psi$. Furthermore, $\psi$ is such that $P(\psi) = 0$.

Notice that $\mu[a_0^n]/\mu[a_1^n]$ is nothing but the probability under $\mu$ of $a_0$ given $a_{n-1}$. Therefore, by the martingale convergence theorem the sequence $(\log(\mu[a_0^n]/\mu[a_1^n]))_{n=1}^\infty$ converges for $\mu$–a. e. $a \in X$. The uniform convergence is what makes $\mu$ a Gibbs measure.

We have the following classical theorem.

**Theorem 2.1** ([10]). Let $X \subset A^N$ be a topologically mixing subshift of finite type and $\psi : X \to \mathbb{R}$. If

$$\sum_{n=0}^{\infty} \text{var}_n \psi < \infty$$

then there exists a unique Gibbs measure $\mu_\psi$, i.e., a unique $T$-invariant probability measure satisfying (1).

**Remark 2.2.** By this theorem we have a partial converse to (3) in the sense that there the potential is defined by the measure, while in the theorem it is the potential which defines the measure.

Notice that the uniqueness part of the theorem is granted by the Gibbs inequality (1), since two measures satisfying it have to be absolutely continuous with respect to each other. It is the existence part which is nontrivial.

For a proof of Theorem 2.1 see e.g. [8]. This includes the case of Hölder continuous potentials treated in, e.g., [2, 12].

### 2.3. Markov measures and Markov approximants.

Markov measures can be seen as Gibbs measures. Colloquially, an $r$–step Markov measure is defined by the property that the probability that $a_n = a \in A$ given $a_0^{n-1}$ depends only on $a_0^{r-1}$. What is usually called a Markov measure corresponds to 1–step Markov measures. On the full shift, the case $r = 0$ gives product measures. A $T$–invariant probability measure is an $r$–step Markov if and only if it is the Gibbs measure of an $(r + 1)$–symbol potential. Given an $(r + 1)$–symbol potential

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4We assume that $r \geq 1$. The case $r = 0$ corresponds to an i.i.d. process, in which case the Gibbsianity is evident.
\(psi\), which we identify as a function on \(A^{r+1}\), one can define the transition matrix \(M_{\psi} : A^r \times A^r \to \mathbb{R}^+\) such that
\[
M_{\psi}(v, w) := \begin{cases} 
\exp(\psi(vw_{r-1})) & \text{if } v_1^{-1} = w_0^{r-2}, \\
0 & \text{otherwise.}
\end{cases}
\]

By \(vw_{r-1}\) we mean the word obtained by concatenation of \(v\) and \(w_{r-1}\) (the last letter of \(w\)).

By Perron–Frobenius Theorem (cf. Appendix 6.1) there exist a right eigenvector \(\tilde{R}_{\psi} > 0\) such that \(\sum_{a \in A^r} \tilde{R}_{\psi}(a) = 1\), and a left eigenvector \(\tilde{L}_{\psi} > 0\) such that \(\tilde{L}_{\psi}^r \tilde{R}_{\psi} = 1\), associated to the maximal eigenvalue \(0 < \rho_{\psi} := \max \text{spec}(M_{\psi})\). Then the measure \(\mu\) defined by
\[
\mu(a_0^r) := \tilde{L}_{\psi}(a_0^{r-1}) \prod_{j=0}^{n-r} \frac{M_{\psi}(a_j^{j+r-1}, a_{j+1}^{j+r})}{\rho_{\psi}^{n-r+1}} \tilde{R}_{\psi}(a_{n-r+1}^n),
\]
for each \(a \in A^N\) and \(n \in \mathbb{N}\) such that \(n \geq r\), is easily seen to be a \(T\)-invariant probability measure satisfying (1) with
\[
P = \log(\rho_{\psi}) \quad \text{and} \quad C = \rho_{\psi} e^{-r\|\psi\|} \max_{w, w'} \frac{\tilde{L}_{\psi}(w) \tilde{R}_{\psi}(w') : w, w' \in A^r}{\min_{w, w'} \tilde{L}_{\psi}(w) \tilde{R}_{\psi}(w') : w, w' \in A^r},
\]
where \(\|\psi\| := \sup\{|\psi(a)| : a \in A^N\}\). Therefore \(\mu = \mu_{\psi}\) is the unique Gibbs measure associated to the \((r+1)\)-symbol potential \(\psi\).

Markov and locally constant approximants.

Given a continuous \(\psi : A^N \to \mathbb{R}\), one can uniformly approximate it by a sequence of \((r+1)\)-symbol potentials \(\psi_r, r = 1, 2, \ldots\), in such a way that \(\|\psi - \psi_r\| \leq \text{var}_r(\psi)\), which goes to 0 as \(r\) goes to \(\infty\). The \(\psi_r\)'s are not defined in a unique way but this does not matter since the associated \(r\)-step Markov measures \(\mu_{\psi_r}\), approximate the same Gibbs measure \(\mu_{\psi}\). We can choose \(\psi_r(a) := \max\{|\psi(b)| : b \in [a_0^n]\}\) for instance.

The potential \(\psi_r\) will be called the \((r+1)\)-symbol approximant of \(\psi\) and the associated \(r\)-step Markov measure \(\mu_{\psi_r}\) will be the \(r\)th Markov approximant of \(\mu_{\psi}\). It is well known (and not difficult to prove) that \(\mu_{\psi_r}\) converges in the weak* topology to \(\mu_{\psi}\).

3. Main Result

The next theorem answers Question 1 when \(\psi\) is Hölder continuous (Theorem 3.1). For the sake of simplicity we discuss the generalization of that theorem to a class of less regular potentials (\(i.e., \text{var}_n(\psi)\) decreases subexponentially or polynomially) in Section 3.

Amalgamation map. Let \(A, B\) be two finite alphabets, with \(\text{Card}(A) > \text{Card}(B)\), and \(\pi : A \to B\) be a surjective map (‘amalgamation’) which extends to the map \(\pi : A^N \to B^N\) (we use the same letter for both) such that \((\pi a)_n = \pi(a_n)\) for all \(n \in \mathbb{N}\). The map \(\pi\) is continuous and shift–commuting, \(i.e.,\) it is a factor map from \(A^N\) onto \(B^N\).
Theorem 3.1. Let \( \pi : A^N \to B^N \) be the amalgamation map just defined and \( \psi : A^N \to \mathbb{R} \) be a Hölder continuous potential. Then the measure \( \mu_\psi \circ \pi^{-1} \) is a Gibbs measure with support \( B^N \), for a potential \( \phi : B^N \to \mathbb{R} \) such that
\[
\text{var}_n(\phi) \leq D \exp(-cn)
\]
for some \( c, D > 0 \), and all \( n \in \mathbb{N} \).
Furthermore, this potential \( \phi : B^N \to \mathbb{R} \) is normalized and it is given by
\[
\phi(b) = \lim_{r \to \infty} \lim_{n \to \infty} \log \left( \frac{\mu_\psi \circ \pi^{-1}[b_0^{[n]}]}{\mu_\psi \circ \pi^{-1}[b_1^{[n]}]} \right),
\]
where \( \psi_r \) is the \((r+1)\)-symbol approximant of \( \psi \).

If \( \psi \) is locally constant, then for all \( n \)
\[
\text{var}_n(\phi) \leq C \vartheta^n
\]
where \( \vartheta \in ]0, 1[ \), \( C > 0 \).

The case of locally constant potentials in the theorem can be rephrased as follows: When \( \mu_\psi \) is an \( r \)-step Markov measure, with \( r > 0 \), the pushforward measure \( \mu_\psi \circ \pi^{-1} \), i.e. the hidden Markov measure, is a Gibbs measure for a Hölder continuous potential \( \phi \) given by
\[
\phi(b) = \lim_{n \to \infty} \log \left( \frac{\mu_\psi \circ \pi^{-1}[b_0^{[n]}]}{\mu_\psi \circ \pi^{-1}[b_1^{[n]}]} \right).
\]
The case \( r = 0 \) is trivial: the Gibbs measure is simply a product measure and its pushforward is also a product measure.

The proof of Theorem 3.1 relies on the following two lemmas whose proofs are deferred to Appendices 6.2 and 6.3.

Lemma 3.1 (Amalgamation for \((r+1)\)-symbol potentials). The measure \( \mu_{\psi_r} \circ \pi^{-1} \), with \( r > 0 \), is a Gibbs measure for the potential \( \phi_r : B^N \to \mathbb{R} \) obtained as the following limit
\[
\phi_r(b) := \lim_{n \to \infty} \log \left( \frac{\mu_{\psi_r} \circ \pi^{-1}[b_0^{[n]}]}{\mu_{\psi_r} \circ \pi^{-1}[b_1^{[n]}]} \right).
\]
Furthermore, there are constants \( C > 0 \) and \( \theta \in [0, 1[ \) such that, for any positive integer \( n > r \) and for any \( b \in B^N \) we have
\[
\left| \phi_r(b) - \log \left( \frac{\mu_{\psi_r} \circ \pi^{-1}[b_0^{[n]}]}{\mu_{\psi_r} \circ \pi^{-1}[b_1^{[n]}]} \right) \right| \leq C \vartheta^n.
\]

Lemma 3.2 (Projective convergence of Markov approximants). The sequence of measures \( (\mu_{\psi_r}) \) converges in the projective sense to the Gibbs measure \( \mu_\psi \) associated to the potential \( \psi \).
Furthermore, for all \( n, r > 0 \) and \( w \in A^n \), we have \( \mu_{\psi_r}[w] \leq \mu_\psi[w] \exp(\epsilon_{r,n}) \), where
\[
\epsilon_{r,n} := D \sum_{s=r}^{\infty} ((n + (s+1)(s+2))\text{var}_{s} \psi + s \theta^n),
\]
for adequate constants \( D > 0 \) and \( \theta \in [0, 1[ \) (the same \( \theta \) as in Lemma 3.1).
With the two previous lemmas at hand, we can proceed to the proof of Theorem 3.1.

**Proof of Theorem 3.1**

We start by proving that the sequence \((\mu_\psi \circ \pi^{-1})_r\) converges in the projective sense to \(\mu_\psi \circ \pi^{-1}\).

On the one hand, Lemma 3.1 tells us that the measure \(\nu_r := \mu_\psi \circ \pi^{-1}\) is Gibbsian for the potential \(\phi_r : B^N \to \mathbb{R}\) given by

\[
\phi_r(b) = \lim_{n \to \infty} \log \left( \frac{\nu_r(b^n)}{\nu_r(b^1)} \right).
\]

On the other hand, Lemma 3.2 ensures that for each \(n, r > 0\) with \(n \geq r\), and each \(v \in A^n\), we have \(\mu_\psi(v) \leq \mu_\psi(v) \exp(\pm \epsilon_{r,n})\) where \(\epsilon_{r,n}\) is defined as in (10). From this it follows that for each \(w \in B^n\) we have

\[
\nu_r[w] := \sum_{\nu \in A^n : v = w} \mu_\psi(v) \leq \exp(\pm \epsilon_{r,n}) \sum_{\nu \in A^n : v = w} \mu_\psi(v) \leq \exp(\pm \epsilon_{r,n}) \mu_\psi \circ \pi^{-1}(w).
\]

(11)

Otherwise said, the sequence of approximants \((\nu_r = \mu_\psi \circ \pi^{-1})_r\), converges in the projective sense to the induced measure \(\mu_\psi \circ \pi^{-1}\), and the speed of convergence is the same both the factor and the original system.

Now we prove that the pushforward measure \(\nu := \mu_\psi \circ \pi^{-1}\) is a Gibbs measure. According to Lemma 3.1 and Eq. (11), for any \(b \in B^N\), and \(n, r > 0\) with \(n \geq r\), we have

\[
(12) \quad \left| \phi_r(b) - \log \left( \frac{\nu(b^n)}{\nu(b^1)} \right) \right| \leq 2 \epsilon_{r,n} + C r^2 \theta^r.
\]

Let us take, for each \(r > 0\), \(n = n(r) := r^2\), and let \(r^* > 0\) be such that both \(s \mapsto s^2 \theta^s\) and \(s \mapsto \epsilon_{s,s^*}\) define decreasing functions in \([r^*, \infty)\). Hence, using the triangle inequality we obtain

\[
\left| \phi_r(b) - \phi_{r^*}(b) \right| \leq 2 \left( 2 \epsilon_{r,r^2} + C r^2 \theta^{r^*} \right)
\]

for all \(r^* \leq r < r'\), and for any \(b \in B^N\). This proves uniform convergence of the sequence of potentials \((\phi_r)_r\). The limit is the continuous function \(\phi : B^N \to \mathbb{R}\) defined by

\[
\phi(b) := \lim_{n \to \infty} \log \left( \frac{\nu(b)^n}{\nu(b)^1} \right).
\]

If we verify that \(\phi\) satisfies condition (1), then, according to the observation following Theorem 2.1 this will prove that \(\nu \equiv \mu_\psi \circ \pi^{-1}\) is the unique Gibbs measure for \(\phi\). From Ineq. (12) it follows that

\[
\left| \phi(b) - \phi(\tilde{b}) \right| \leq \left| \phi(b) - \phi_r(b) \right| + \left| \phi_r(b) - \log \left( \frac{\nu(b^n)}{\nu(b^1)} \right) \right| + \left| \phi_r(b) - \log \left( \frac{\nu(\tilde{b}^n)}{\nu(\tilde{b}^1)} \right) \right| + \left| \phi_r(\tilde{b}) - \phi(\tilde{b}) \right|
\]

\[
\leq 4 \left( 2 \epsilon_{r,r^2} + C r^2 \theta^{r^*} \right) + 2 \left( 2 \epsilon_{r,n} + C r^2 \theta^{r^*} \right),
\]

for all \(b, \tilde{b} \in B^N\) such that \(\tilde{b} \in [b^n]\), and every \(n > r \geq r^*\).
Since $\psi$ is Hölder continuous and
\[
\epsilon_{r,r^2} := D \sum_{s=r}^{\infty} (r^2 + (s + 1)(s + 2)) \var r s \psi + s \theta^s,
\]
then there exist $C > 0$ and $\rho \in [\theta, 1]$ (remember that $\theta \in [0, 1]$) such that $\max(\epsilon_{r,r^2}, r^2 \theta^r) \leq C \rho^r$. We take again $n = n(r) = r^2$ and obtain, for all $n \in \mathbb{N}$,
\[
\var n \phi \leq D \exp(-c \sqrt{n})
\]
with a convenient $D \geq 6C(2 + C)$, and $c = -\log(\rho)$.

The case of a locally constant $\psi$ is the immediate consequence of Lemma 3.1 and one has $\var \eta = 0 \theta$. The theorem is now proved □

Remark 3.1. The competition between the terms $\epsilon_{r,n}$ and $\theta^{n/r}$ in the upper bound of $\var n \phi$ leads to a subexponential bound, namely $\var n \phi \leq D \exp(-c n^{\delta})$, for any $\delta > 0$. We made the choice $\delta = 1$.

4. Generalization to less regular potentials

In this section we go beyond Hölder continuous potentials and look at potentials $\psi$ such that $\var r \psi$ decreases slower than exponentially. Besides the fact that $\sum_r \var r \psi < \infty$ is always assumed, the only place where a finer control in the decrease of $\var r \psi$ is required, is inside the proof of Lemma 3.2. There, the projective convergence of the Markov approximants depends on the fact that
\[
\epsilon_{r,n} := D \sum_{s=r}^{\infty} ((n + (s + 1)(s + 2)) \var r s \psi + s \theta^s) \to 0, \text{ when } r \to \infty,
\]
for each $n > 0$. Furthermore, the variation of the induced potential, $\var n \phi$, is upper bounded by a linear combination of $\epsilon_{r,n}$ and $n \theta^{n/r}$. After this consideration, we can generalize Theorem 3.1 as follows.

Theorem 4.1. Let $\pi : A^\mathbb{N} \to B^\mathbb{N}$ be the amalgamation map just defined and $\psi : A^\mathbb{N} \to \mathbb{R}$ be such that $\sum_{s=0}^\infty s^2 \var s \psi < \infty$. Then the measure $\mu_{\psi} \circ \pi^{-1}$ is a Gibbs measure with support $B^\mathbb{N}$ for a normalized potential $\phi : B^\mathbb{N} \to \mathbb{R}$ defined by the limit
\[
\phi(b) = \lim_{r \to \infty} \lim_{n \to \infty} \log \left( \frac{\mu_{\psi_r} \circ \pi^{-1}[b^n_0]}{\mu_{\psi_r} \circ \pi^{-1}[b^n_1]} \right),
\]
where $\psi_r$ is the $(r + 1)$-symbol approximant of $\psi$.

If $\var n \psi$ has subexponential decreasing, i.e., if $\var n \psi \leq C \exp(-c n^{\gamma})$ for some $C > 0$ and $\gamma \in [0, 1]$, then there are constants $D > C$ and $0 < d < c$ such that
\[
\var n \phi \leq D \exp(-d n^{\gamma})
\]
for all $n \in \mathbb{N}$.

If $\var n \psi$ is polynomially decreasing, i.e., if $\var n \psi \leq C n^{-q}$, for some $C > 0$ and $q > 3$, then for all $\epsilon \in (0, q - 3)$ there is a constant $D > C$ such that
\[
\var n \phi \leq D \frac{1}{n^{q-2-\epsilon}}
\]
for all $n \in \mathbb{N}$.

**Remark 4.1.** As mentioned above, the $n$–variation of the induced potential is upper bounded by linear combination of $\epsilon_{r,n}$ and $r^2 \theta_{n/r}$. We have to optimize the choice of the function $r \mapsto n(r)$ in such a way that $n/r \to \infty$ when $r \to \infty$, and that the resulting $n$–variation of $\psi$ has the fastest possible decreasing. In the subexponential case, $\var_n \psi \leq C \exp(-cn^\gamma)$, the optimal choice turns to be $n(r) = r^{1+\gamma}$, while in the polynomially decreasing case, $\var_n \psi \leq C n^{-q}$, the optimal choice is $n(r) = r^{(q-1)/(q-1-\epsilon)}$. This gives a bound in $n^{-q+2+\epsilon}$.

5. Comments and Open Questions

In our previous work [5] we made two restrictive assumptions, namely that $\psi$ is a locally constant potential and the image of the starting SFT under the amalgamation map $\pi$ is still a SFT (in general it is a sofic subshift). In that setting, we could prove, under sufficient conditions, that $\mu_{\psi} \circ \pi^{-1}$ is a Gibbs measure for a Hölder continuous potential $\phi$. We also exhibited an example showing that one of our sufficient conditions turns out to be necessary in that otherwise the induced potential $\phi$ is not defined everywhere.

We conjecture the following: Let $\pi : A \to B$ be an amalgamation map as above, $X \subset A^\mathbb{N}$ a SFT and $Y \subset B^\mathbb{N}$ the resulting sofic subshift. Then the pushforward measure of a Gibbs measure for a Hölder continuous potential is a “weak” Gibbs measure $\mu_{\phi}$ in that (1) does not hold for every $a$ but for almost all $a$ (w.r.t. $\mu_{\phi}$).

6. Proofs

6.1. Preliminary result: Birkhoff’s refinement of Perron–Frobenius Theorem.

Let $E, E'$ be finite sets and $M : E \times E' \to \mathbb{R}^+$ be a row allowable non–negative matrix, i.e., a matrix such that $Mx > 0$ whenever $x > 0$. Let us define the set

$$\Delta_E := \left\{ x \in [0,1]^E : |x|_1 := \sum_{e \in E} x(e) = 1 \right\},$$

and similarly $\Delta_{E'}$. We supply $\Delta_E$ with the distance

$$\delta_E(x,y) := \max_{e,f \in E} \log \frac{x(e)y(f)}{x(f)y(e)}.$$

On $\Delta_{E'}$ we define $\delta_{E'}$ accordingly. Let us now define

$$\tau(M) := \frac{1 - \sqrt{\Phi(M)}}{1 + \sqrt{\Phi(M)}}$$

where

$$\Phi(M) := \begin{cases} \min_{e,f \in E, e',f' \in E'} \frac{M(e,e')M(f,f')}{M(e,f')M(e,e')} & \text{if } M > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here $M > 0$ means that all entries of $M$ are strictly positive.
Theorem 6.1 (After Garrett Birkhoff). Let $M : E \times E' \to \mathbb{R}^+$ be row allowable, and $F_M : \Delta_{E'} \to \Delta_E$ be such that

$$F_M \mathbf{x} := \frac{M \mathbf{x}}{||M \mathbf{x}||} \quad \text{for each } \mathbf{x} \in \Delta_{E'}.$$

Then, for all $\mathbf{x}, \mathbf{y} \in \Delta_{E'}$, we have

$$\delta_E(F_M \mathbf{x}, F_M \mathbf{y}) \leq \tau(M) \delta_{E'}(\mathbf{x}, \mathbf{y}).$$

We have $\tau(M) < 1$ if and only if $M > 0$.

For a proof of this important result, see [4] for instance. It can also be deduced from the proof of a similar theorem concerning square matrices which can be found in [13]. As a corollary of this result we obtain the following form of the Perron-Frobenius Theorem.

Corollary 6.1 (Enhanced Perron-Frobenius Theorem). Suppose that $M : E \times E \to \mathbb{R}^+$ is primitive i.e., there exists $\ell \in \mathbb{N}$ such that $M^\ell > 0$. Then its maximal eigenvalue $\rho_M$ is simple and it has a unique right eigenvector $\bar{R}_M \in \Delta_E$, and a unique left eigenvector $L_M$ satisfying $L_M^\dagger \bar{R}_M = 1$. Furthermore, for each $\mathbf{x} \in \Delta_E$ and each $n \in \mathbb{N}$ we have

$$M^n \mathbf{x} \leq (\bar{L}_M \mathbf{x}) \rho_M^n \bar{R}_M \exp \left( \pm \frac{\ell \delta_E(\mathbf{x}, F_M \mathbf{x})}{1 - \tau} \tau |n/\ell| \right),$$

with $\tau := \tau(M^\ell) < 1$.

Proof. Let us first remark that $F_M^\ell = F_M^\dagger$. Since $M^\ell > 0$, then Theorem 6.1 and the Contraction Mapping Theorem imply the existence of a unique fixed point $\mathbf{x}_M = F_M \mathbf{x}_M \in \Delta_E$ such that

$$\delta_E(F_M^n \mathbf{x}_M, \mathbf{x}_M) \leq \sum_{k=0}^{\infty} \delta_E \left( F_M^{n+k\ell} \mathbf{x}_M, F_M^{n+(k+1)\ell} \mathbf{x}_M \right) \leq \frac{\delta_E(\mathbf{x}, F_M^\ell \mathbf{x}) \tau |n/\ell|}{1 - \tau} \leq \frac{\ell \delta_E(\mathbf{x}, F_M \mathbf{x}) \tau |n/\ell|}{1 - \tau},$$

for each $n \in \mathbb{N}$ and $\mathbf{x} \in \Delta_E$. From the definition of projective distance it follows that, for each $\mathbf{x} \in \Delta_E$ and $n \in \mathbb{N}$ there is a constant $C = C(\mathbf{x}, n)$ such that

$$M^n \mathbf{x} \leq C(\mathbf{x}, n) \mathbf{x}_M \exp \left( \pm \frac{\ell \delta_E(\mathbf{x}, F_M \mathbf{x})}{1 - \tau} \tau |n/\ell| \right). \quad (13)$$

Let us now prove that $\mathbf{x}_M \equiv \bar{R}_M \in \Delta_E$ is the unique positive right eigenvector associated to the maximum eigenvalue $\rho_M := \max \text{spec}(M)$. Indeed, since $F_M \mathbf{x}_M = \mathbf{x}_M$, then $M \mathbf{x}_M = \lambda \mathbf{x}_M$ for some $\lambda > 0$. Now, if $M \mathbf{y} = \lambda \mathbf{y}$ for some $\mathbf{y} \in \mathbb{C}^E$, and taking into account that $M$ is a real matrix, then $\mathbf{y} = a \mathbf{z}$ for some $a \in \mathbb{C}$ and $\mathbf{z} \in \Delta_E$. Therefore $\lambda$ is a simple eigenvalue. It follows from Theorem 6.1 and the contraction mapping theorem that $\mathbf{z} = \bar{R}_M$ is the associated eigenvector. Consider the map $\mathbf{x} \mapsto \min_{e \in E}(M \mathbf{x})(e)/\mathbf{x}(e)$ on $\Delta_E$, and extend it to $\text{clos}(\Delta_E)$ (the closure is taken with respect to the euclidean distance), by allowing values in the extended reals $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. The resulting transformation is upper

\footnote{Here we are following a standard argument which can be found in [13] for instance.}
semicontinuous, and therefore there exists $x_0 \in \text{clos}(\Delta_E)$ attaining the supremum, i.e., such that

$$\rho := \sup_{x \in \Delta_E} \min_{e \in E} \frac{(Mx)(e)}{x(e)} = \min_{e \in E} \frac{(Mx_0)(e)}{x_0(e)}.$$ 

This supremum is an eigenvalue, and the point where it is attained is its corresponding positive eigenvector. Indeed, if $Mx_0 \neq \rho x_0$, i.e. if $(Mx_0)(e) > \rho x_0(e)$ for some $e \in E$, then $M^{\ell+1}x_0 > \rho M^\ell x_0$ which implies that $\rho < \sup_{x \in \Delta_E} \min_{e \in E} (Mx)(e)/x(e)$.

Therefore $x_0$ is a non-negative eigenvector for $M$, but since $M^\ell x_0 = \rho^\ell x_0 > 0$, then necessarily $x_0 = \tilde{R}_M$ and $\lambda = \rho$.

Finally, if $0 \neq y \in \mathbb{C}^E$ is a right eigenvector of $M$, associated to another eigenvalue $\lambda' \in \mathbb{C}$, then

$$|\lambda'| |y| = |My| \leq |M| |y|,$$

where $|z|$ denotes the coordinatewise absolute value of the vector $z \in \mathbb{C}^E$, and the inequality holds at each coordinate. If $|\lambda'| < \min_{e \in E}|((My)(e))/|y(e)|$, we can find a vector $y^+ \in \Delta_E$ by slightly changing $|y|$ at coordinates $e \in E$ where $y(e) = 0$ and then normalizing, so that $|\lambda'| \leq \min_{e \in E}|(My^+)(e)/y^+(e)|$. If on the contrary $|\lambda'| = \min_{e \in E}|(My)(e)/|y(e)|$, then $M^{\ell+1}y \geq |\lambda'| |My|$, and normalizing $M^\ell |y|$ we obtain $y^+ \in \Delta_E$ such a way that $|\lambda'| \leq \min_{e \in E}|(My^+)(e)/y^+(e)|$. We conclude that

$$|\lambda'| \leq \sup_{x \in \Delta_E} \min_{e \in E} \frac{(Mx)(e)}{x(e)} := \rho$$

for each $\lambda' \in \text{spec}(M)$, therefore $\rho \equiv \rho_M := \max \text{spec}(M)$.

It remains to prove that in Ineq. (13), we have $C(x, n) = (\tilde{L}_M^\dagger x) \rho_M^{n\ell}$, where $\tilde{L}_M > 0$ is the left eigenvector associated to $\rho_M$, normalized so that $\tilde{L}_M^\dagger \tilde{R}_M = 1$. For this note that, by multiplying Ineq. (13) at left by $\tilde{L}_M$, we obtain

$$\rho_M^n(\tilde{L}_M^\dagger x) \leq (\tilde{L}_M^\dagger \tilde{R}_M) C(x, n) \exp \left( \pm \frac{\ell \delta_E(x, F_M x)}{1 - \tau} n/\ell \right),$$

hence $C(x, n) = \rho_M^n(\tilde{L}_M^\dagger x)/(\tilde{L}_M^\dagger \tilde{R}_M)$, and the proof is finished. \hfill \Box

### 6.2. Proof of Lemma 3.1

#### 6.2.1. The right eigenvector.

Notice that the transition matrix $M_r := M_{\psi_r}$ is primitive with primitivity index $r$, hence, according to Corollary 6.1,

$$M^n_r x \leq (\tilde{L}_r^\dagger x) \rho_r^n \tilde{R}_r \exp \left( \pm \frac{r \delta_{M_r} (x, F_r x)}{1 - \tau} n/\ell \right),$$

for each $x \in \Delta_E$ and $n \in \mathbb{N}$. Here $\rho_r$ denotes the maximal eigenvalue of $M_r$, $\tilde{R}_r \in \Delta_{M_r}$ its unique right eigenvector in the simplex, $\tilde{L}_r$ its unique associated left eigenvector satisfying $\tilde{L}_r^\dagger \tilde{R}_r = 1$, and $\tau := \tau(M^n_r)$ denotes the contraction coefficient associated to the positive matrix $M^n_r$. 

Let us now obtain explicit an upper bound for \( \tau \) and for the distance \( \delta_{A^r}(x, F_r x) \) for particular values of \( x \in \Delta_e \). First,

\[
\Phi(M_r^e) \geq \min_{u, v, u', v', v'' \in A^r} M_r(u, u') M_r(v, v'') \geq \exp \left( -2 \sum_{k=0}^{r} \var_k \psi \right) > 0.
\]

Therefore \( \tau \leq 1 - e^{-\sum_{k=0}^{r} \var_k \psi} \) and \( (1 - \tau)^{-1} \leq e^{\sum_{k=0}^{r} \var_k \psi} \).

Let \( s_\psi := \sum_{k=0}^{\infty} \var_k \psi \), and \( \theta := 1 - e^{-s_\psi} \). With this, and taking into account the upper bound for \( \tau \) and \( (1 - \tau)^{-1} \), we obtain

\[
M_r^e x \leq (L_r^1 x) \rho_r^e R_r \exp \left( \pm r \delta_{A^r}(x, F_r x) e^{s_\psi - \theta |x|} \right).
\]

On the other hand, for \( \bar{\mu} := (1/(\text{Card}(A^r)), \ldots, 1/(\text{Card}(A^r)))^{\top} \in \Delta_{A^r} \), we have

\[
\delta_{A^r}(\bar{\mu}, F_r \bar{\mu}) := \max_{u, w \in A^r} \log \left( \frac{\bar{\mu}(u')(M_r \bar{\mu})(w)}{\bar{\mu}(u)(M_r \bar{\mu})(w')} \right) \leq r \log(\text{Card}(A)) + 2 \|\psi\| < r C_0,
\]

where \( C_0 := 2 (\log(\text{Card}(A)) + \|\psi\|) \). Therefore, by taking \( x = \bar{u} \) and \( n = r^2 \) in (13), we finally obtain

\[
R_r(u) \leq \sum_{a \in \text{Ver, } a(\Delta^0 \cup \{w\})} e^{S_{a,-r-1} \psi \theta(a)} \rho_r^e |L_r|^{1/2} e^{C_0 r^2 \exp(s_\psi) \theta^r}.
\]

\[6.2.2.
\]

Ansatz for the induced potential.

To each word \( w \in B^r \) we associate the simplex

\[
\Delta_w := \left\{ x \in (0, 1)^{E_w} : |x|_1 := \sum_{v \in E_w} x_v = 1 \right\},
\]

where \( E_w := \{v \in A^r : \pi v = w\} \).

Let \( M_r, \rho_r, \bar{L}_r, \bar{R}_r \) and \( \bar{L}_r, \bar{R}_r \) be as above, and define, for each \( w \in B^r \), the restriction \( \bar{L}_{r,w} := \bar{L}_r|_{E_w} \in (0, \infty)^{E_w} \). Define \( \bar{R}_{r,w} \) in the analogous way, and for each \( w \in B^{r+1} \) let \( M_{r,w} \) be the restriction of \( M_r \) to the coordinates in \( E_{w_0} \times E_{w_1} \). Using this, and taking into account Eq. (3), which applies to our \((r + 1)\)-symbol potential \( \psi_r \), we derive the matrix expression

\[
\nu_r[b^n_0] \equiv \sum_{a^n_0 = b^n_0} \mu_{\psi_r}[a^n_0] = \bar{L}_{r,b_0^{-1}}^{\dagger} \left( \prod_{j=0}^{n-r} M_r b_j^{r+r} \rho_r^{n-r+1} \right) \bar{R}_{r,b_0^{n-r+1}},
\]

for the induced measure \( \nu_r := \mu_{\psi_r} \circ \pi^{-1} \). It follows from this that

\[
\log \frac{\nu_r[b^n_0]}{\nu_r[b_0^0]} = \log \left( \frac{\bar{L}_{r,b_0^{-1}}^{\dagger} \prod_{j=0}^{n-r} M_r b_j^{r+r} \bar{R}_{r,b_0^{n-r+1}}}{\bar{L}_{r,b_0^{-1}}^{\dagger} \prod_{j=1}^{n-r} M_r b_j^{r+r} \bar{R}_{r,b_0^{n-r+1}}} \right) - \log(\rho_r).
\]

For each \( w \in A^{r+s} \), with \( s \geq 1 \), let \( M_{r,w} := \prod_{j=0}^{s-1} M_{r,w_j^{r+s}} \), and define the transformation \( F_{r,w} : \Delta_{w_s^{r+s-1}} \rightarrow \Delta_{w_0^{r-1}} \) such that

\[
F_{r,w} x = \frac{M_{r,w} x}{|M_{r,w}|_1}.
\]
For each $b \in B^N$ and $s, t \in \mathbb{N}$, let

$$x_{r,b_{s+1}^r} := F_{r,b_{s+1}^r} \circ \cdots \circ F_{r,b_{s+1}^r} \left( \frac{\tilde{R}_{r,b_{s+1}^r}}{\tilde{R}_{r,b_{s+1}^r}} \right) \in \Delta_b^{r+1}.$$ 

By convention, $x_{r,b_{s+1}^r} := \tilde{R}_{r,b_{s+1}^r}/|\tilde{R}_{r,b_{s+1}^r}| \in \Delta_b^{r+1}$. Using this notation, and after the adequate renormalization, Eq. (16) becomes

$$\log \left( \frac{\nu_r[b^n]}{\nu_r[b^n]} \right) = \log \left( \frac{\bar{L}_{r,b^n}^{-1}}{(\bar{L}_{r,b^n})^{-1}} \right) - \log(\mu_r). \quad (17)$$

### 6.2.3. Convergence of the inhomogeneous product.

Let us now prove the convergence of the sequence $(x_{b}^n)_{n \geq r}$. For this notice that

$$x_{r,b_{s}^r} := F_{r,b_{s+1}^r} \circ \cdots \circ F_{r,b_{s+1}^r} x_{r,b_{s+1}^r} = F_{r,b_{s}^r} x_{r,b_{s}^r} F_{r,b_{s+1}^r} \circ \cdots \circ F_{r,b_{s+1}^r} x_{r,b_{s+1}^r},$$

where $k := \left[ \frac{n}{r} \right] - 1$. Now, since $\mathcal{M}_{r,w} > 0$ for each $w \in B^2$, then Theorem 6.1 ensures that the associated transformation $F_{r,w} : \Delta^{2r-1} \to \Delta^{-1}$, is a contraction with contraction coefficient $\tau_w = (1 - \sqrt{\Phi_w})/(1 + \sqrt{\Phi_w})$, where

$$\Phi_w := \min_{v, u \in E, v, u' \in E} \mathcal{M}_{r,w}(v, u')/\mathcal{M}_{r,w}(u, u') \geq \min_{v, u, v', u' \in A} \mathcal{M}_{r,w}(v, u')/\mathcal{M}_{r,w}(u, u') \geq \exp \left( -2 \sum_{k=0}^{r} \text{var}_k \psi \right) \geq e^{-2s\psi} > 0. \quad (18)$$

Recall that $s\psi = \sum_{k=0}^{\infty} \text{var}_k \psi$. From Ineq. (18) we obtain a uniform upper bound for the contraction coefficients, $\tau_w \leq \theta := 1 - \exp(-s\psi) < 1$, which allows us to establish the uniform convergence of the sequence $(x_{b}^n)_{n \geq r}$ with respect to $b \in B^N$.

Indeed, for $b \in B^N$ fixed and $m > n$, we have

$$\delta_{E_{b}^{n}}(x_{b_{s}^r}, x_{b_{s}^n}) \leq \theta^k \delta_{E_{b_{s}^{r+1}}^{(k+1)r}}(x_{r,b_{s}^{r+1}}, x_{r,b_{s}^{n+1}}) \quad (19)$$

where $k := \left[ \frac{n}{r} \right] - 1$. On the other hand,

$$\delta_{E_{b_{s}^{r+1}}^{(k+1)r}}(x_{r,b_{s}^{r+1}}, x_{r,b_{s}^{n+1}}) \leq \sum_{j=0}^{k'} \delta_{E_{b_{s}^{r+1}}^{(k+1)r}}(x_{r,b_{s}^{r+1}}, x_{r,b_{s}^{n+1}}) + \delta_{E_{b_{s}^{r+1}}^{(k+1)r}}(x_{r,b_{s}^{n+1}}, x_{r,b_{s}^{n+1}}),$$

where $k' := \left[ (m-n)/r \right] - 1$. By convention, when $k' = -1$, the summation in the right–hand side is zero. Then, since all the matrices $\mathcal{M}_{r,w}$ are row allowable and positive for $w \in B^2$, then we have

$$\delta_{E_{b_{s}^{r+1}}^{(k+1)r}}(x_{r,b_{s}^{r+1}}, x_{r,b_{s}^{n+1}}) \leq T_1 + T_2 + T_3 \quad (20)$$
where

\begin{equation}
T_1 := \delta E_{n-r+1} \left( x_r b_n^{n-r} + F_r b_{n-r+1}^{n-r} \right),
\end{equation}

\begin{equation}
T_2 := \sum_{j=1}^{k'} \theta^j \delta E_{n+j} \left( x_r b_{n+(j-1)+1}^{n+j} + F_r b_{n+(j-1)+1}^{n+j} \right)
\end{equation}

and

\begin{equation}
T_3 := \theta^{k'} \delta E_{n+(k'+1)} \left( x_r b_{n+k'+1}^{n+(k'+1)} + F_r b_{n+k'+1}^{n+(k'+1)} \right).
\end{equation}

Once again, we convene that \( T_2 = 0 \) if \( k' = -1 \).

Now, for each \( w, w' \in B^r \), and \( v \in B^s \) with \( r < s < 2r \), and such that \( v_0 = w, v_{s-r+1} = w' \), we have

\[ \delta_w(x_r, w, F_r, v, x_r, w') = \max_{u, u' \in E_w} \log \left( \frac{x_r w(u) (F_r u x_r w')(u)}{x_r w(u') (F_r u x_r w')(u)} \right) \]

\[ \leq \max_{u, u' \in E_w} \log \left( \frac{R_r(u) (M_r u R_r w')(u')}{{R_r(u')} (M_r u R_r w')(u')} \right). \]

Hence, using the estimate for the right eigenvectors given in Eq. (15), it follows that

\[ \delta_w(x_r, w, F_r, v, x_r, w') \leq \max_{u, u' \in A^r} \log \left( \frac{\sum_{a \in Per_2(A^0)} e^{S_r x-r-1 \psi(a)} \sum_{a \in Per_2 (A^0)} e^{S_r x-r-1 \psi(a)}}{\min_{a \in Per_2(A^0)} e^{S_r x-r-1 \psi(a)} \min_{a \in Per_2 (A^0)} e^{S_r x-r-1 \psi(a)}} \right) \]

\[ + 2 r^2 C_0 \left( e^\theta r - 1 \right) C_0 (e^\theta r + 1), \]

with \( C_0 = 2(\log(\text{Card}(A)) + ||\psi||) \) and \( \theta = 1 - \exp(-s) \) as in Eq. (15). Using this upper bound in (21), (22) and (23), we obtain from (20)

\[ \delta_{E_{b_n}^{n-r+1}} (x_r b_{n-r+1}^n, x_r b_{n-r+1}^n) \leq 2 r(r + 1) C_0 (e^\theta r + 1) \left( \theta^{1 - \theta} + \frac{1}{\theta} \right), \]

and with this, Ineq. (19) becomes

\begin{equation}
\delta_{E_{b_n}^{n-r+1}} (x_r b_{n-r+1}^n, x_r b_{n-r+1}^n) \leq 2 r(r + 1) C_0 (e^\theta r + 1 + \theta^{1 - \theta} + \frac{1}{\theta} \theta^{1 - \theta} + \frac{1}{\theta}),
\end{equation}

which holds for all \( b \in B^N \) and \( r < n < m \). Hence, \((x_{b_n}^{n})_{n \geq r} \) is a Cauchy sequence in complete space \( \Delta_{b_n} \), and the existence of the limit \( x_{b_n}^{n} := \lim_{m \rightarrow \infty} x_{b_n}^{m} \) is ensured for each \( b \in B^N \). Furthermore, from Eq. (21) it follows that

\begin{equation}
\delta_{E_{b_n}^{n}} (x_{b_n}^{n}, x_{b_n}^{n}) \leq 2 r(r + 1) C_0 (e^\theta r + 1 + \theta^{1 - \theta} + \frac{1}{\theta} \theta^{1 - \theta} + \frac{1}{\theta} \theta^{1 - \theta}) \leq C_1 r^2 \theta^r,
\end{equation}

with \( C_1 := 4C_0(1 + e^\theta)/(\theta^2(1 - \theta)). \)
6.2.4. The induced potential and the Gibbs inequality.

Taking the Eq. (25), it follows that the limit

\[ \phi_r (b) = \lim_{n \to \infty} \log \left( \frac{\nu_r [b_0^n]}{\nu_r [b_1^n]} \right) = \log \left( \frac{(\bar{L}_r b_r^{-1})^t \cdot M_r b_r x_r b_0^n}{(\bar{L}_r b_1^{-1})^t \cdot x_r b_0^n} \right) - \log (\rho_r), \]

exists for each \( b \in B^N \), and defines a continuous function \( b \mapsto \phi_r (b) \). This proves that the limit \( \phi \) in the statement of the lemma does exist. It remains to find an upperbound to its modulus of continuity.

Inequality (23), and the fact that \( |x_{b_1^n}| = |x_{b_0^n}| = 1 \), imply that

\[ x_{b_1^n} \subseteq x_{b_0^n} \exp \left( \pm C_1 r^2 \theta^2 \right) \]

for all \( b \in B^N \) and \( n > r \). With this, and taking into account Eqs. (17) and (26), it follows that

\[ \phi_r (b) - \log \left( \frac{\nu_r [b_0^n]}{\nu_r [b_1^n]} \right) \leq \log \left( \frac{(\bar{L}_r b_r^{-1})^t \cdot M_r b_r x_r b_0^n}{(\bar{L}_r b_1^{-1})^t \cdot x_r b_0^n} \right) - \log (\rho_r) \]

\[ \leq C r^2 \theta^2, \]

for all \( b \in B^N \), \( n > r \) and \( C := 2C_1 = 8C_0 (1 + e^{s_x} \theta)/(\theta^2 (1 - \theta)). \) This proves (23) in the statement of the lemma.

From this it can be easily deduced that \( \nu_r = \mu_{\psi_r} \circ \pi^{-1} \) satisfies the Gibbs Inequality (1) with potential \( \phi_r \) and constants \( P(\phi_r, B^N) = 0 \) and

\[ C(\phi_r, B^N) = \max_{b \in B^N} \left( \frac{\exp(S_{r, \psi} (b))}{\nu_r [b_0^n]} \cdot \frac{\nu_r [b_0^n]}{\exp(S_{r, \psi} (b))} \right) \exp \left( \frac{C r^2 \theta^2}{1 - \theta^2} \right). \]

This proves the first statement of the lemma the proof of which is now complete. □

Remark 6.1. As mentioned above (see (2)), the topological pressure of \( \psi \) is given by

\[ P(\psi) = P(\psi, A^N) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{a \in \text{Per}_n (A^N)} e^{S_n \psi(a)} \right). \]

Since \( \psi \subseteq \psi_r \pm \text{var} \psi \), we get

\[ \log (\rho_r) = \lim_{n \to \infty} \frac{1}{n} \log (\text{Trace} (M^n)) \]

\[ \leq \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{a \in \text{Per}_n (A^N)} e^{S_n \psi(a)} \right) \pm \text{var} \psi \]

\[ \leq P(\psi) \pm \text{var} \psi, \]

for each \( r \in \mathbb{N} \).

6.3. Proof of Lemma 3.2
6.3.1. Periodic approximations.

Each Markov approximant $\mu_\psi$ can be seen as the limit of measures supported on periodic points as follows. Fix $n, r \in \mathbb{N}$ with $n \geq r$, and $w \in A^n$. Then, for each $p > r + n$ we have,

$$\mathcal{P}_{r}[w] := \sum_{a \in \text{Per}_p(A^n) \cap [w]} e^{S_{\mu_\psi}(a)} \mu(w).$$

We can rewrite the above equation as

$$\mathcal{P}_{r}[w] = \begin{cases} \prod_{i=0}^{n-1} \mathcal{M}_r(w^n_{s+1}, w_{s+1}) e^{\frac{1}{r+1}} \mathcal{M}_r^{-n-r} e^{\frac{1}{r} \theta} \mu(w) & \\
\sum_{i \in \mathbb{N}} e^{\frac{1}{r} \mathcal{M}_r^{-n-r} (\mathcal{M}_r^{-n-r})} \end{cases}$$

with $\mathcal{M}_r$ as above, and $\bar{e}_\xi \in \{0, 1\}^A$ the vector with 1 at coordinate $\xi$ and zeros everywhere else. Now, since $\mathcal{M}_r^n \bar{e}_\xi > 0$ for each $k \geq r$ and $\xi \in A^r$, then Corollary 6.1 applies, and using (3) we obtain

$$\mathcal{P}_{r}[w] \leq \frac{\mathcal{L}_{\mathcal{M}_r^n \bar{e}_\xi}}{\mathcal{L}_{\mathcal{M}_r^n \bar{e}_\xi}} \left( \prod_{i=0}^{n-1} \mathcal{M}_r(w^n_{s+1}, w_{s+1}) e^{\frac{1}{r+1}} \mathcal{M}_r^{-n-r} e^{\frac{1}{r} \theta} \mu(w) \right),$$

with $\mathcal{L}_{\mathcal{M}_r^n \bar{e}_\xi}$, $\rho_r$ and $\tau := \tau(\mathcal{M}_r)$ as before, and

$$D_0 := 2 \max_{\xi \in \mathbb{N}} \delta_{\mathcal{M}_r^n \bar{e}_\xi, \mathcal{M}_r^{n+1} \bar{e}_\xi}.$$

Since $\tau = (1 - \sqrt{\Phi})/(1 + \sqrt{\Phi})$, with

$$\Phi := \begin{cases} \min_{v, u, v', u' \in A^r} \mathcal{M}_r(v, v') \mathcal{M}_r(u, u') & \\
\mathcal{M}_r(v, v') \mathcal{M}_r(u, u'), \mathcal{M}_r(v, v'), \mathcal{M}_r(u, u') \end{cases}$$

then $\tau \leq \theta := 1 - \exp(-s_\psi)$, and $(1 - \tau)^{-1} \leq \exp(s_\psi)$, with $s_\psi := \sum_{k=0}^{\infty} \var_k \psi$ as in Lemma 3.1. On the other hand,

$$D_0 \leq 2 \max_{w', u, w' \in A^r} \log \left( \frac{\mathcal{M}_r^{n+1}(u, w')^{n+1}}{\mathcal{M}_r^{n+1}(u, w')^{n+1}(u, w')} \right)$$

$$\leq 4 \left( \log(\text{Card}(A)) + \sum_{k=1}^{\infty} \var_k \psi + ||\psi|| \right)$$

$$\leq 4 (\log(\text{Card}(A)) + s_\psi + ||\psi||) =: D_1.$$

Using this explicit bound just obtained, we deduce the inequalities

$$\mu_\psi[w] \leq \frac{\sum_{a \in \text{Per}_p(A^n) \cap [w]} e^{S_{\mu_\psi}(a)}}{\sum_{a \in \text{Per}_p(A^n) \cap [w]} e^{S_{\mu_\psi}(a)}} \exp \left( \pm D_1 \right).$$
for each \( w \in A^n \), and all \( p > n + r \). It is easy to check that these inequalities extend to each \( w \in \bigcup_{k=1}^n A^k \), and we finally obtain

\[
(27) \quad \mu_{\psi_r}[w] \leq \frac{\sum_{a \in \text{Per}_p(A^n) \cap [w]} e^{S_p \psi_r(a)}}{\sum_{a \in \text{Per}_p(A^n)} e^{S_p \psi_r(a)}} \exp \left( \pm D_1 r e^{s \psi \theta^{p-m(n, r)-2}} \right)
\]

for all \( r, n \in \mathbb{N} \), and \( w \in A^n \).

### 6.3.2. Telescopic product.

Let us now compare two consecutive Markov approximants. Fix \( n, r > 0 \), and \( w \in A^n \). Then, for each \( p > n + r + 1 \), Inequalities (27) ensure that

\[
\frac{\mu_{\psi_r}[w]}{\mu_{\psi_{r+1}}[w]} \leq \frac{\sum_{a \in \text{Per}_p(A^n) \cap [w]} e^{S_p \psi_r(a)}}{\sum_{a \in \text{Per}_p(A^n)} e^{S_p \psi_r(a)}} \exp \left( \pm C r \theta^{\varphi(r,n) - 2} \right),
\]

with \( q = \max(r + 1, n) \) and \( C := 2 e^{s \psi D_1} \). Since \( \psi_{r+1} \leq \psi_r \pm \varphi_{r+1, \psi} \), then we have

\[
\frac{\mu_{\psi_r}[w]}{\mu_{\psi_{r+1}}[w]} \leq \exp \left( \pm \left( 2 p \varphi_{r+1, \psi} + C r \theta^{\varphi(r,n) - 2} \right) \right) \leq \exp \left( \pm \left( 2 p \varphi_{r+1, \psi} + C r \theta^{\varphi(r,n) - 2} \right) \right)
\]

for all \( r \in \mathbb{N} \), \( w \in \bigcup_{k=1}^n A^k \), and \( p > n + r + 1 \). Let \( p = (r + 1)(r + 2) + n - 1 \), then for each \( r' > r \in \mathbb{N} \) and \( w \in \bigcup_{k=1}^n A^k \) we have

\[
\frac{\mu_{\psi_r}[w]}{\mu_{\psi_{r+1}}[w]} \leq \exp \left( \pm D \sum_{s=r}^{\infty} ((n + (s + 1)(s + 2)) \varphi_{s, \psi} + s \theta^r) \right),
\]

with \( D := \max(2, C) \). Since \( \psi \) is Hölder continuous and \( \theta \in (0, 1) \), then

\[
\epsilon_{r,n} := D \sum_{s=r}^{\infty} ((n + (s + 1)(s + 2)) \varphi_{s, \psi} + s \theta^r) \rightarrow 0 \quad \text{when} \quad r \rightarrow \infty,
\]

for each \( n, r \in \mathbb{N} \). We conclude that, \( \mu[w] := \lim_{r \to \infty} \mu_{\psi_r}[w] \) exists for each \( w \in \bigcup_{k=0}^\infty A^k \), and we have

\[
\frac{\mu_{\psi_r}[w]}{\mu[w]} \leq \exp \left( \pm D \sum_{s=r}^{\infty} ((|w| + (s + 1)(s + 2)) \varphi_{s, \psi} + s \theta^r) \right),
\]

for every \( r \in \mathbb{N} \) and \( w \in \bigcup_{k=1}^\infty A^k \).

### 6.3.3. The limit \( \lim_{r \to \infty} \mu_{\psi_r} \) is the Gibbs measure \( \mu_{\psi} \).

It only remains to prove that \( \mu \) such that \( \mu[w] := \lim_{r \to \infty} \mu_{\psi_r} \) coincides with the original Gibbs measure \( \mu_{\psi} \). Note first that \( \mu \) so defined is \( T \)-invariant. Indeed, it is the weak* limit of the sequence \( (\mu_{\psi_r})_{r \geq 1} \) of \( T \)-invariant Markov approximants, it is a \( T \)-invariant probability measure as well.

Now, replacing \( \psi_r \) by \( \psi \pm \varphi_{S, \psi} \), and making \( p = (r + 1)(r + 2) + n - 1 \) in Ineq. (27), it follows that

\[
\mu[w] \leq \mu_{\psi_r}[w] \exp (\pm \epsilon_{r,n}) \leq \frac{\sum_{a \in \text{Per}_p(A^n) \cap [w]} e^{S_p \psi(a)}}{\sum_{a \in \text{Per}_p(A^n)} e^{S_p \psi(a)}} \exp (\pm 2 \epsilon_{r,n})
\]
for every \( w \in \bigcup_{k=1}^{r^2} A^k \). By taking \( n = r^2 \), we obtain

\[
\mu[w] \leq \frac{\sum_{\alpha \in \text{Per}(2r+1)(r+1)(A^t) \cap [w]} e^{S(2r+1)(r+1)\psi(\alpha)}}{\sum_{\alpha \in \text{Per}(2r+1)(r+1)(A^t)} e^{S(2r+1)(r+1)\psi(\alpha)}} \exp \left( \pm 2\epsilon_{r,r^2} \right)
\]

for each \( r \in \mathbb{N} \) and \( w \in \bigcup_{k=1}^{r^2} A^k \). On the other hand, the Gibbs measure \( \mu_\psi \), whose existence is ensured by the fact that \( \sum_r \text{var}_r \psi < \infty \), is such that

\[
\mu_\psi[w] \leq C^{\pm 1} \frac{\sum_{\alpha \in \text{Per}(2r+1)(A^t) \cap [w]} e^{S_r \psi(\alpha)}}{\sum_{\alpha \in \text{Per}(2r+1)(A^t)} e^{S_r \psi(\alpha)}},
\]

for each \( w \in A^k \) with \( k \leq p \). Since \( \epsilon_{r,r^2} \to 0 \) when \( r \to \infty \), it follows from this and Ineq. (28) that \( \mu_\psi \) is absolutely continuous with respect to \( \mu_\psi \). The Ergodic Decomposition Theorem implies that \( \mu_\psi \) is the only ergodic measure entering in the decomposition of the invariant measure \( \mu \), therefore \( \mu = \mu_\psi \).

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J.-R. Chazottes: Centre de Physique Théorique, CNRS–École Polytechnique, 91128 Palaiseau Cedex, France.

E-mail address: jeanrene@cpt.polytechnique.fr

E. Ugalde: INSTITUTO DE FÍSICA, UNIVERSIDAD AUTONOMA DE SAN LUIS POTOSÍ, SAN LUIS DE POTOSÍ, S.L.P., 78290 MÉXICO.

E-mail address: ugalde@ifisica.uaslp.mx