Distribution of the Entanglement Entropy in Disordered One-Dimensional Fermions

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Abstract. The entanglement entropy is related to quantum phase transitions. It behaves differently when a quantum system is in a phase with a finite correlation length (such as an insulator phase), compared to a system in a critical phase in which the correlation length is infinite such as a metallic phase. The entanglement entropy of disordered systems does not depend only on the length of the relevant segment and the size of the system but also on the specific realization of disorder. Thus, the entanglement entropy of a disordered system exhibits mesoscopic fluctuations. Here we shall show numerical results that exhibit a crossover in the distribution of the entanglement entropy from a normal distribution at the weak localization regime to a log-normal distribution at the strong localization regime.

1. Introduction
Entanglement entropy (EE) has garnered much recent interest [1]. For condensed matter physics, perhaps the most interesting property related to the EE is its behavior in the vicinity of a quantum phase transition (QPT) [1, 2, 3, 4]. As usual the EE for a system described by a pure state $|\Psi\rangle$ measures the entanglement between two regions A and B composing the system. Specifically, The EE $S_A$ of a region A is defined by using the eigenvalues $\lambda_i$ of its reduced density operator $\hat{\rho}_A = \text{Tr}_B |\Psi\rangle\langle\Psi|$ (where $\text{Tr}_B$ is a trace over all degrees of freedom belonging to region B), as

$$S_A = -\sum \lambda_i \ln(\lambda_i). \quad (1)$$

For an interacting one-dimensional electronic system $S_A$ will grow logarithmic with the size of region A, $L_A$, if the system is in the metallic regime (i.e., has no disorder), while for insulators (disordered systems) it will saturate once the size of region A will exceed the localization length $\xi$ [5]. Other aspects of the crossover in the EE of one-dimensional electronic systems can be utilized to identify and study various QPT, such as the phase transition predicted for the population switching in quantum dots coupled to Luttinger liquid leads [6].

Using the concept of EE in the framework of disordered systems raises, as usual in this context, the question of fluctuations and averaging. As with the conductance through a disordered system, one would expect also the EE to show mesoscopic fluctuation. This immediately leads to the question what is the form of the distribution of the EE, and does it depend of the regime of the system, i.e., whether the system length $L$ or the segment length $L_A$ is much smaller than localization length $\xi$ or vise-versa.
In this paper we will show some numerical evidence that the EE distribution crosses over from a normal distribution in the weak localization regime \((\xi \gg L, L_A)\) to a log-normal distribution in the strongly localized regime \((\xi \ll L, L_A)\). This crossover is reminiscent of the behavior of the distribution of the conductance in one- and higher dimensional disordered systems [12].

2. Model
The Hamiltonian of the one-dimensional spinless electronic system of size \(L\) with repulsive nearest neighbor interactions and with on-site disorder is given by the Anderson model:

\[
H = \sum_{j=1}^{L} \epsilon_j \hat{c}_j^\dagger \hat{c}_j - t \sum_{j=1}^{L-1} (\hat{c}_j^\dagger \hat{c}_{j+1} + h.c.) + U \sum_{j=1}^{L-1} (\hat{c}_j^\dagger \hat{c}_j - \frac{1}{2})(\hat{c}_{j+1}^\dagger \hat{c}_{j+1} - \frac{1}{2}),
\]

for which \(\hat{c}_j^\dagger (\hat{c}_j)\) is the creation (annihilation) operator of a spinless electron at site \(j\). The random on-site energies are given by \(\epsilon_j\) drawn from a uniform distribution \([-W/2, W/2]\). \(U\) is the nearest-neighbour interaction strength where a positive background is included, and the hopping matrix element \(t\) is taken as unity. For the non-interacting case the localization length is known to follow \(\xi(W, U = 0) \approx 105/W^2\). When interactions are considered \(\xi(U) = (\xi(U = 0))^{1/(3-2g(U))}\) [7] where the Luttinger liquid parameter \(g(U) = \pi/[2 \cos^{-1}(-U/2)]\) [8]. Since for repulsive interactions \(g < 1\) decreases as a function of the interaction strength, the localization length decreases as a function of the interaction strength [9].

DMRG (Density matrix renormalization group) [10, 11] will be used in order to numerically calculate the ground state EE of the Hamiltonian in Eq. (2) for different realizations of disorder, (600 realizations for the shortest samples, 400 for the longer ones), different length \(L = 300, 500, 700\) and different interaction strength \(U = 0, 0.6\) (corresponding to \(g = 1, 0.836\)).

3. Results

![Figure 1](image1.png)

**Figure 1.** The distribution \(P(S)\) of the EE \(S\) at the midpoint of the sample. The disorder was chosen as \(W = 0.7\) (\(\xi = 214\)) and there are no electron-electron interactions \((U = 0)\). Symbols depict the numerical results, while the curves correspond to a fit to the normal distribution with an average and variance given in the legend.

![Figure 2](image2.png)

**Figure 2.** As in Fig 1, but in this figure the strength of the electron-electron interactions \((U = 0.6)\) corresponds to a many particle localization length of \(\xi = 57\). Thus, for all length of the sample we are in the strong localized regime. It can be clearly seen that the normal distribution gives a rather poor fit to the numerical results.
First we examine the distribution of the EE for a non-interacting case \((U = 0, g = 1)\). In all cases we plot the distribution of the EE at the middle of the sample, i.e., \(L_A = L/2\). The results are depicted in Fig. 1. For the shortest samples, \(L = 300\) (for which \(\xi \approx L\)) the fit to the normal distribution is not bad. For longer sample length \(L = 500, 700\) the fit becomes worse. When the interacting case is considered (See Fig. 2), for which \(\xi(U = 0.6) = 57 \ll L\) for all length considered here, the fit badly fails, especially in the tail of the distribution.

Alternatively we fit the numerical data to a log-normal distribution as is shown in Figs. 3,4. While for the non-interacting case (Fig. 3) the log-normal fit is not great (indeed, it is quite bad for the shortest samples \(L = 300\)), it fits pretty well for the interacting case (Fig. 4). Thus it seems that the numerical data supports a crossover from the normal distribution at the weak localization regime \((L \leq \xi)\) to a log-normal distribution at the strong localization limit \((L \gg \xi)\).

4. Discussion
This crossover of the EE distribution could have some implications to the study of the EE in disordered systems. The most obvious is related to using the EE as an indicator of the position and properties of the localization transition. As we mentioned the EE is expected to saturate once \(L_A > \xi\). The most natural approach would be to study the average value of the EE over different realizations of disorder. Nevertheless, once the distribution is not a normal one must be careful in performing the average. Since one would like to span the whole regime from \(L_A \ll \xi\) to \(L_A \gg \xi\) without changing the averaging method, probably the best approach would be to treat the median of the distribution as the representative EE for the system [5].

It is also interesting to try to extrapolate from the one-dimensional behavior of the EE to the general behavior of the EE at any dimension. Leaning on the similarity of the distribution of conductance and EE for one-dimensional systems it is tempting to speculate that also for higher dimensions the EE distribution will follow the normal distribution for the metallic regime (with a fixed width like the UCF?), while it should follow a log-normal distribution in the localized regime. These speculations need to be substantiated by further studies.

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