Infrared Non-perturbative Propagators of Gluon and Ghost via Exact Renormalization Group

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Abstract

The recent investigations of pure Landau gauge SU(3) Yang-Mills theories which are based on the truncated Schwinger-Dyson equations (SDE) indicate an infrared power law behavior of the gluon and the ghost propagators. It has been shown that the gluon propagator vanishes (or finite) in the infrared limit, while the ghost propagator is more singular than a massless pole, and also that there exists an infrared fixed point of the running gauge coupling. In this paper we reexamine this picture by means of the exact (non-perturbative) renormalization group (ERG) equations under some approximation scheme, in which we treat not only two point functions but also four point vertices in the effective average action with retaining their momentum dependence. Then it is shown that the gluon and the ghost propagators with the infrared power law behavior are obtained as an attractive solution starting from rather arbitrary ultraviolet bare actions. Here it is found to be crucial to include the momentum dependent four point vertices in the ERG framework, since otherwise the RG flows diverge at finite scales. The features of the ERG analyses in comparison with the SDE are also discussed.

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I. INTRODUCTION

In quantum field theories, the Green functions are the most fundamental objects in order to understand the dynamics. Especially in QCD, the two point functions tell us various informations about non-perturbative phenomena, such as dynamical mass generation of chiral fermions and also color confinement, which are expected to occur in the infrared region. There have been many studies concerning to the dynamical chiral symmetry breaking (D\(\chi\)SB) in QCD and its effective models using SDE and ERG. Though these are successful in describing D\(\chi\)SB, much less has been well known as for color confinement due to complicated gluon self interactions.

The recent SDE studies for the gluon and the ghost propagators in Landau gauge Yang-Mills theories\[1, 2, 3, 4, 5, 6, 7, 8, 9\] initiated by Smekal, Hauck and Alkofer\[1\], indicate the power law behavior: the gluon propagator vanishes (or finite) in the infrared limit, while the ghost propagator is more singular than a massless pole. IR asymptotic forms of the propagators are found to be described by one exponent (ghost anomalous dimension) \(\kappa\) as

\[
D_Z(p^2) \sim (p^2)^{-1+2\kappa}, \quad D_G(p^2) \sim (p^2)^{-1-\kappa},
\]

for the gluon and the ghost propagators respectively. The actual value of \(\kappa\) obtained by solving the SDE is dependent on the approximation scheme. However fit of the data by lattice simulations at some finite Euclidean momentum region\[6\] as well as axiomatic consideration \[7\] indicates \(\kappa \approx 0.5\). It is also interesting that the solutions of the SDE show consistent behavior with the lattice data. Implication of this behavior to confinement is attributed to realization of the so-called Kugo-Ojima confinement scenario via the BRS quartet mechanism of colored particles \[10, 11\]. In the Landau gauge, the confinement criterion is understood as infrared enhancement of the ghost propagator, and has been proven to be fulfilled under general ansatz by using the SDE for ghost in the Landau gauge \[12\]. In addition to the confinement criterion, the IR asymptotic power solutions satisfy also the Gribov-Zwanziger horizon condition \[13, 14\]. The horizon condition is a dynamical consequence obtained in restricting the support of the Faddeev-Popov measure to the interior of the Gribov region to avoid gauge copies. The conditions are represented in terms of the propagators as

\[
\lim_{p^2 \to 0} D_Z(p^2) = 0, \quad \lim_{p^2 \to 0} \left( p^2 D_G(p^2) \right)^{-1} = 0,
\]

which are found to be fulfilled by the SDE solutions. Thus, one may say that the power solutions of the SDE most probably capture an aspect of confinement \[15\], though the string tension has not been derived directly from the two point functions.

Here it would be worthwhile to mention also Zwanziger’s observations that the cutoff prescription at the Gribov horizon resolves an ambiguity in the solutions of SDE \[2, 6\]. First it is noted that cutting off the functional integral at the Gribov horizon does not alter the SDE. Also the boundary contribution does not affect the form of SDE, because the Faddeev-Popov measure vanishes on the boundary. However, it is shown that the cutoff at the Gribov horizon provides supplementary conditions for the solutions of SDE so as to make the Faddeev-Popov measure positive definite. Indeed it is needed to know the infrared asymptotic power law behavior of the solutions in the practical analyses of the SDE. However there are found multiple solutions satisfying the power law ansatz. Then Zwanziger pointed out that infrared asymptotic solution in the SDE analyses done so far has been implicitly chosen to satisfy the supplementary condition \[1, 2, 5\]. Thus, in the practical SDE analyses, it
is important to impose proper infrared boundary conditions in order to find out the physical solution. Contrary to this, however, the exact renormalization group (ERG) approach does not need to use the supplementary conditions for the infrared behaviors, the power law solutions with a certain exponent emerge uniquely as will be shown in this paper.

The ERG describes continuous evolution of the coarse grained effective action in the spirit of Wilson’s renormalization group. In continuum field theories, this evolution is most conveniently expressed in terms of the effective average action $\Gamma_\Lambda$ with the coarse grained (infrared cutoff) scale $\Lambda$. In practice the ERG has been frequently used in various non-perturbative calculations and found to be rather useful. In contrast to the SDE framework, in the ERG, solving the flow equation starting with a ultraviolet bare action $\Gamma_{\Lambda_0}$ gives a unique infrared solution $\Gamma_{\Lambda=0}$ as $\Lambda_0 \rightarrow 0$. Indeed it can be shown schematically that the solutions of ERG satisfy the corresponding SDE on the exact generating functional level. However the correspondence becomes totally unclear, once some approximation or truncation is performed, except for some rare cases. Especially in the cases that the SDE allows multiple solutions, it would be an interesting problem to see whether the physical solution can be obtained by solving the ERG equations. Another good feature of the ERG scheme is the infrared attractive property of the RG flows. In renormalizable quantum field theories, the same infrared effective action is derived from rather arbitrary ultraviolet bare actions owing to the universality argument. Thus it is expected to be rather straightforward to analyze the infrared two point functions of the Yang-Mills theories in the ERG framework.

In this paper, we investigate the infrared behavior of the effective action, specially the gluon and the ghost propagators of Landau gauge Yang-Mills theory by solving the approximated ERG equation. The main purpose of this paper is to see, first whether the infrared power behavior can emerge also in the ERG framework at all, and then which kind of corrections are necessary to be taken into account for that. Previously, Ellwanger, Hirsch and Weber also have studied infrared behavior of the propagators by using ERG, and they concluded that the confining $1/p^4$ behavior for the heavy quark potential which was derived from the effective four quark interactions. However, their RG flows become singular at a finite scale, and therefore the infrared power law behavior has not been seen. The most different point with our analysis is inclusion of the four point vertices among gluons and ghosts, which are necessary ingredient to produce the power law solutions. In their analysis, only the four gluon vertex, which was determined by using the Slavnov-Taylor identities, was included. Our analysis indicates that the effective four point vertices generated by box diagrams play a specially important role, because the large ghost anomalous dimension enhances these four point vertices. Indeed, the infrared solutions with power behavior are found to be obtained by taking these into account, although with a small exponent $\kappa \approx 0.146$.

Application of the ERG to Yang-Mills theories immediately faces problems of the gauge or the BRS invariance, because the infrared cutoff of momentum breaks the local symmetry. Schematically, this problem may be managed by use of the modified Slavnov-Taylor identities (mSTi), which guarantee the broken BRS symmetry of the effective average action $\Gamma_\Lambda$ at an intermediate infrared cutoff scale $\Lambda \neq 0$ to recover the symmetry at infrared limit $\Lambda \rightarrow 0$. There the BRS non-invariant counter terms, such as gluon mass, should be controlled by the mSTi. However, the infrared BRS invariance is not maintained any more once we truncate the effective action. Effectiveness of the mSTi in the practical sense is unclear apart from perturbative analyses. Therefore, in this paper we do not pursuit for the
problems of gauge invariance and simply discard the BRS non-invariant corrections. This is a difficult aspect of non-perturbative analysis of gauge theories not only in the ERG but also in the SDE framework.

Furthermore, for the sake of simplicity, we neglect also corrections with the three gluon vertex in the flow equations, though the resultant flows will no longer be Yang-Mills theory’s ones. However, according to the previous SDE analyses [1, 2, 3, 4, 9], contribution of the three gluon vertex was found to be sub-leading as far as the IR asymptotic solutions are concerned. Therefore we also adopt such a simple approximation as the first trial to apply the ERG equation for the infrared dynamics of Yang-Mills theories.

The paper is organized as follows: In section 2, we explain briefly the exact flow equation for Landau gauge Yang-Mills theory and also introduce some basic notations. In section 3, we present our approximation scheme and give the practical RG equations, which we shall analyze in the followings in detail. Results of the numerical analysis are presented in section 4. There the RG flows of the momentum dependent gluon and ghost form factors and also the running gauge coupling obtained in our scheme are shown. Section 5. is devoted for some discussions and remarks on further issues. Some comparisons between the ERG and the SDE formalisms are also summarized there.

II. EXACT RENORMALIZATION GROUP

The exact renormalization group is a realization of the Wilson renormalization group transformation in continuum field theories, and there have been known various formulations. The most commonly used form is the so-called flow equation written in terms of the effective average action $\Gamma_{\Lambda}$ [17]. The effective average action $\Gamma_{\Lambda}$ is the effective action $\Gamma$ obtained after only quantum fluctuations with momentum scale $p^2 \geq \Lambda^2$ are integrated out. Therefore, it is an efficient tool interpolating the classical bare action $S$ and the quantum effective action $\Gamma$;

$$\lim_{\Lambda \to \Lambda_0} \Gamma[\Phi; \Lambda] = S_{\text{bare}}[\Phi], \quad \lim_{\Lambda \to 0} \Gamma[\Phi; \Lambda] = \Gamma[\Phi].$$

(3)

Here dependence on the ultraviolet cutoff $\Lambda_0$ is suppressed implicitly. Explicitly the effective average action $\Gamma_{\Lambda}$ is defined through the Legendre transformation of the IR regularized generating functional for the connected Green functions $W_{\Lambda}$;

$$W[J; \Lambda] = \log \int \mathcal{D}\Phi \exp \{-S[\Phi] - \Delta S[\Phi; \Lambda] + J \cdot \Phi\},$$

(4)

where $\Delta S[\Phi; \Lambda]$ is the IR cutoff term introduced to suppress the infrared mode of $p^2 < \Lambda^2$, and which is normally quadratic in fields $\Phi$. The Legendre transformation of $W_{\Lambda}$ is defined by

$$\Gamma[\Phi; \Lambda] = -W[J; \Lambda] + J \cdot \Phi - \Delta S[\Phi; \Lambda].$$

(5)

The flow equation is given by infinitesimal variation of $\Gamma_{\Lambda}$ with respect to the IR cutoff scale $\Lambda$ and it can be written down as one-loop exact form. In the case of pure Yang-Mills theory, the flow equation may be given as

$$\partial_{\Lambda} \Gamma[\Phi; \Lambda] = \frac{1}{2} \text{Str} \left[ \partial R \cdot \left( \frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi}[\Phi; \Lambda] + R \right)^{-1} \right],$$

(6)
where we introduced the matrix notation following Ellwanger et al. [24]. The supertrace includes momentum integration as well as summation of color and Lorentz indices. The field $\Phi$ and $\bar{\Phi}$ denote the gluon, ghost and anti-ghost fields in short hand notation as

$$\bar{\Phi} = (A, c, \bar{c}), \quad \Phi = (A, -\bar{c}, c).$$

(7)

The symbol $R$ stands for the diagonal matrix of the cutoff functions, which are explicitly given by

$$R(p) = \text{diag} \left( R^{ab}_{\mu \nu}(p; \Lambda), -\tilde{R}^{ab}_{\mu \nu}(p; \Lambda), -\tilde{R}^{ab}_{\mu \nu}(p; \Lambda) \right),$$

(8)

$$\partial R(p) = \text{diag} \left( \partial_A R^{ab}_{\mu \nu}(p; \Lambda), -\partial_A \tilde{R}^{ab}_{\mu \nu}(p; \Lambda), -\partial_A \tilde{R}^{ab}_{\mu \nu}(p; \Lambda) \right).$$

(9)

The cutoff term added in the definition of the cutoff generating functional given by eq. (4) is written down in terms of these functions as

$$\Delta S[\Phi; \Lambda] = \frac{1}{2} \int_p A^a_\mu (-p) R^{ab}_{\mu \nu}(p; \Lambda) A^b_\nu(p) - \int_p \bar{c}^a(-p) \tilde{R}^{ab}(p; \Lambda) c^b(p).$$

(10)

Here we adopt the cutoff functions of the following form;

$$R^{ab}_{\mu \nu}(p; \Lambda) = \left[ \left( \delta_{\mu \nu} - \frac{p_\mu p_\nu}{p^2} \right) R_\Lambda(p^2) + \frac{1}{\xi(\Lambda)} R_\Lambda(p^2) \frac{p_\mu p_\nu}{p^2} \right] p^2 \delta^{ab},$$

(11)

$$\tilde{R}^{ab}(p; \Lambda) = R_\Lambda(p^2) p^2 \delta^{ab},$$

(12)

for the gluon and the ghost fields respectively. We may choose a general cutoff functions for $R_\Lambda$ here, however in the following analyses, we will take sharp cutoff limit given by,

$$R_\Lambda(p^2) = \lim_{\alpha \to \infty} \left( \frac{p^2}{\Lambda^2} \right)^{-\alpha},$$

(13)

for the calculational simplicity.

III. APPROXIMATION SCHEME

In the practical analysis, we need to perform some approximation in order to solve the flow equation schematically given by (6). In the ERG framework it is done typically by truncating interactions in $\Gamma_\Lambda$ at some finite order, which should be compared with truncation of equations in the SDE formalism. We may improve the approximation systematically by increasing the interactions to be taken in. The recent analyses of the SDE treat only the two point functions of the glouon and ghost fields as dynamical ones. However it does not mean that we may truncate the effective action or corrections to the action at the two point function order. Although the solutions of the ERG and the SDE should coincide to each other at the full generating functional level, the truncation ruins the correspondence. As we will show in the subsequent section, it is necessary to include the four point vertices to realize the infrared finite power solution in the ERG formalism.

One of our main interests is to see how the infrared power behavior of the gluon and ghost propagators emerge in the ERG framework. We shall now explain briefly the approximation
scheme which we adopted for this purpose. Our approximation scheme is, in short, to treat all diagrams that contribute to the gluon and ghost two point functions.

Because the flow equation (14) is one loop exact, the vertices contributing to the corrections of the two point functions are not more than four point. Therefore, we truncate $\Gamma_{\Lambda}$ at the forth power of fields and the explicit form is given as follows,

$$
\Gamma[A, \bar{c}, c; \Lambda] = \frac{1}{2} \int p \Gamma_{\mu\nu}(p; \Lambda) A_\mu(-p) A_\nu(p)
+ \int \frac{(2\pi)^4 \delta^4(p + q + r)}{p, q, r} \, V_{\mu\nu\rho}(p, q, r; \Lambda) \, A_\rho(p) A_\mu(q) A_\nu(r)
+ \int \frac{(2\pi)^4 \delta^4(p + q + r + s)}{p, q, r, s} \, V_{\mu\nu\rho\sigma}(p, q, r, s; \Lambda) \, A_\rho(p) A_\nu(q) A_\mu(r) A_\sigma(s)
- \int_p \bar{c}^\mu(p; \Lambda) c^\nu(-p) c^\rho(p)
- \int_p \frac{(2\pi)^4 \delta(p + q + r)}{p, q, r} \, T_{\mu\nu}(p|q, r; \Lambda) \, A_\mu(p) c^\nu(q) c^\rho(r)
+ \int_p \frac{(2\pi)^4 \delta(p + q + r + s)}{p, q, r, s} \, T_{\mu\nu\rho\sigma}(p, q|r, s; \Lambda) \, A_\rho(p) A_\nu(q) c^\nu(r) c^\rho(s)
+ \int_p \frac{(2\pi)^4 \delta(p + q + r + s)}{p, q, r, s} \, T_{\mu\nu\rho\sigma}(p, q, r, s; \Lambda) \, \bar{c}^\rho(p) c^\mu(q) c^\nu(r) c^\rho(s)
+ \mathcal{O}(\Phi^5)
\left( \int_{p, \ldots, q} \equiv \int \frac{d^4 p}{(2\pi)^4} \cdots \int \frac{d^4 q}{(2\pi)^4} \right),
$$

(14)

where the vertices are defined (anti)symmetric, e.g. $V_{\mu\nu\rho}(p, q, r; \Lambda) = V_{\nu\rho\mu}(q, p, r; \Lambda) = \cdots$, and so on. Substituting the action (14) to the flow equation (6) and expanding the RHS of the flow equation in powers of fields, then we obtain the coupled flow equations for the momentum dependent vertex functions.

However we may simplify the equations by using more specific form of the two point functions. For the gluon two point function, we choose,

$$
\Gamma_{\mu\nu}(p; \Lambda) = \left[ \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] f_Z(p^2; \Lambda)
+ \frac{1}{\xi(\Lambda)} \frac{p_\mu p_\nu}{p^2} \right] p^2 \delta^{\mu\nu}. \quad (15)
$$

Although we write also the $\Lambda$-dependent gauge parameter $\xi(\Lambda)$, we will consider only the Landau gauge limit case $\xi(\Lambda) \to 0$ in this paper. It is noted that this limit can be taken consistently, because the Landau gauge is a fixed point $\xi^*(\Lambda) = 0$ under the RG evolution.

To be more precise, the equation (15) is incomplete even in the Landau gauge, because the cutoff term (14) induces the gluon mass term $\sim m(\Lambda)^2 A_\mu A^\mu$ in the averaged effective action due to lack of manifest BRS invariance. Such BRS non-invariant terms must be controlled by the mSTi consistently with the RG evolution. Here, however, we neglect simply such BRS non-invariant terms and also the corrections to them. Then by contracting the flow equation for the gluon two point function with Brown-Pennington tensor $B_{\mu\nu}(p) = (\delta_{\mu\nu} - 4p_\mu p_\nu/p^2)$, we may project out correction to the gluon mass term. For the ghost two point function, we choose,

$$
\bar{\Gamma}_{\mu\nu}(p; \Lambda) = \delta^{\mu\nu} p^2 f_G(p^2; \Lambda).
$$

(16)

Thus the two point functions are reduced to $\Lambda$ dependent one component functions $f_Z$ and $f_G$. Integrating the flow equations toward $\Lambda \to 0$, the gluon and the ghost propagators,
which are of our present interest, are found to be

\[
D_{\mu\nu}(p^2) = \frac{T_{\mu\nu}(p)}{p^2 f_Z(p^2; \Lambda \to 0)}, \quad D_G(p^2) = -\frac{1}{p^2 f_G(p^2; \Lambda \to 0)},
\]

where \(T_{\mu\nu}(p) = (\delta_{\mu\nu} - p_\mu p_\nu/p^2)\) is the transverse projection.

Next we restrict the three gluon vertex and the three ghost-gluon vertex to the following forms:

\[
V_{\mu\nu\rho}^{abc}(p, q, r; \Lambda) = 0,
\]

\[
T_{\mu}^{abc}(p|q, r; \Lambda) = ig(\Lambda) f^{abc} q_\mu,
\]

where the three gluon vertex is just neglected. It is because the gluon loop corrections give sub-leading contributions concerning to the infrared momentum region where the power behavior is seen in the SDE \[1, 2, 3, 4, 9\]. Of course the three gluon vertex is not negligible where the three gluon vertex is just neglected. It is because the gluon loop corrections give sub-leading contributions concerning to the infrared momentum region where the power behavior is seen in the SDE \[1, 2, 3, 4, 9\]. Of course the three gluon vertex is not negligible at ultraviolet and intermediate momentum region, and therefore resultant RG flows cannot be regarded as those for the Yang-Mills theories. However we examine the ERG by imposing this condition just for simplicity as the first step of the analyses. While the three ghost-gluon vertex is given by the bare form \[15\]. Owing to the non-renormalization theorem for the three ghost-gluon vertex in Landau gauge \[25\], the coupling constant is not modified \(g(\Lambda) = g(\Lambda_0)\). In fact, if we truncate the bare form \[15\], the coupling non-renormalization \(\partial_\Lambda g(\Lambda) = 0\) is also shown by using the flow equation similarly to the perturbative one-loop argument.

Now let us derive the flow equations for the two point functions. From now on we will consider SU(3) Yang-Mills theory, although the extension to general SU(N) gauge group can be done without difficulty. Contracting the flow equation for the gluon two point function with BP tensor \(B_{\mu\nu}(p)\), we can easily obtain the following flow equation,

\[
\frac{1}{2} \delta^{ab} p^2 \partial_\Lambda f_Z(p^2; \Lambda) = -\frac{1}{2} \delta^{ab} g^2(\Lambda_0) \int k^2 \partial_\Lambda R_A(k^2) P_G^2 (k^2; \Lambda) P_G (k + p)^2; \Lambda) \left[k_\alpha(k + p)_\beta B_{\alpha\beta}(p)\right] \]

\[-\frac{1}{2} \delta^{ab} g^2(\Lambda_0) \int (k + p)^2 \partial_\Lambda R_A ((k + p)^2) P_G ((k + p)^2; \Lambda) P_G (k^2; \Lambda) \left[k_\alpha(k + p)_\beta B_{\alpha\beta}(p)\right] \]

\[-\frac{1}{3} \int k^2 \partial_\Lambda R_A (k^2) P_G (k^2; \Lambda) \left[B_{\alpha\beta}(p) T^{abcd}_{\alpha\beta} (p, -p|k, -k; \Lambda) \delta^{cd}\right] \]

\[-\frac{1}{6} \int k^2 \partial_\Lambda R_A (k^2) P_Z^2 (k^2; \Lambda) \left[B_{\alpha\beta}(p) V^{abcd}_{\alpha\beta\mu\nu} (p, -p, k, -k; \Lambda) T_{\mu\nu}(k) \delta^{cd}\right] \times 12, \tag{20}
\]

for the ghost two point function. The flow equation for the gluon two point function is also found to be

\[-\delta^{ab} p^2 \partial_\Lambda f_G(p^2; \Lambda) = -3 \delta^{ab} g^2(\Lambda_0) \int k^2 \partial_\Lambda R_A (k^2) P_G (k^2; \Lambda) P_Z ((k + p)^2; \Lambda) \left[k_\mu k_\nu T_{\mu\nu}(k + p)\right] \]

\[-3 \delta^{ab} g^2(\Lambda_0) \int (k + p)^2 \partial_\Lambda R_A ((k + p)^2) P_Z^2 ((k + p)^2; \Lambda) P_G (k^2; \Lambda) \left[k_\mu k_\nu T_{\mu\nu}(k + p)\right] \]

\[-\int k^2 \partial_\Lambda R_A (k^2) P_G (k^2; \Lambda) \left[T^{abcd}(p, -p, k, -k; \Lambda) \delta^{cd}\right] \times 4 \]

\[-\frac{1}{2} \int k^2 \partial_\Lambda R_A (k^2) P_Z^2 (k^2; \Lambda) \left[T_{\mu\nu}(k) T^{cd\beta}_{\mu\nu} (k, -k|p, -p; \Lambda) \delta^{cd}\right] \times 2, \tag{21}
\]
FIG. 1: The diagrammatic representation of the beta functions for the gluon and the ghost two point functions; $f_Z(p^2; \Lambda)$ (upper diagram) and $f_G(p; \Lambda)$ (lower diagram). The crossed circle implies insertion of the cutoff function $\partial_\Lambda R_\Lambda$.

where we introduced short hand notations for the cutoff propagators:

$$P_Z(p^2; \Lambda) = \frac{1}{k^2[f_Z(k^2; \Lambda) + R_\Lambda(k^2)]}, \quad (22)$$

$$P_G(p^2; \Lambda) = \frac{1}{k^2[f_G(k^2; \Lambda) + R_\Lambda(k^2)]}. \quad (23)$$

The integrals in the beta functions may be interpreted as specific one loop corrections and their diagrammatic representations are presented in Fig.[11]

Even though the three point vertices are simple in the Landau gauge, it is still difficult to solve the flow equations. Because the two point functions and the four point functions appearing in the beta functions are fully momentum dependent. In order to integrate the flow equations toward infrared numerically, it is quite preferable to perform the shell momentum integrals in the RHS of the flow equations analytically. Then the numerical analyses are much simplified. For this purpose, we adopt one dimensional (y-max) approximation and take sharp cutoff limit as follows.

The one dimensional approximation is often used also in SDE analyses, which replaces the each form factor as,

$$f_i((k+p)^2; \Lambda) \rightarrow f_i(\max\{k^2, p^2\}; \Lambda), \quad (i = G, Z), \quad (24)$$

in the above flow equations [20] and [21]. Next we consider to perform the angular integrals of internal momentum. It is noted that the cutoff propagators $P_i((k+p)^2; \Lambda)$ are reduced to the factorized form as

$$P_i((k+p)^2; \Lambda) \rightarrow \frac{1}{(k+p)^2 f_i(\max\{k^2, p^2\}; \Lambda)} \theta^\epsilon ((k+p)^2 - \Lambda^2), \quad (25)$$

in the sharp cutoff limit given by [13]. The superscript $\epsilon$ indicates the sharpness of the cutoff function ($\epsilon \sim \alpha^{-1}$ for [13]). However, naive sharp cutoff limit induces non-analyticity at the origin of momentum space, reflecting the non-local behavior in position space. One way out this problem is to use momentum scale $p \sim \sqrt{p_\mu p^\mu}$ instead of the momentum components $p_\mu$ [27]. Here, however, we avoid this problem by simply replacing the all momenta associated
with the cutoff propagators \( k + p_1 + \cdots + p_n \) (\( k \) is inner-loop momentum, and \( p_i \) is external momentum) with the inner-loop momentum \( k \). Resultantly the above manipulations correspond to replacement of the cutoff propagators as

\[
P_i \left( (k + p)^2 ; \Lambda \right) \rightarrow \frac{1}{(k + p)^2 f_i (\max \{k^2, p^2\}; \Lambda)} \theta^\epsilon \left( k^2 - \Lambda^2 \right).
\] (26)

The \( \Lambda \) derivative of the cutoff propagators are also replaced by

\[
(k + p)^2 \partial_\Lambda R_\Lambda \left( (k + p)^2 \right) P_i \left( (k + p)^2 ; \Lambda \right) = -\partial_\Lambda \left\{ \frac{1}{(k + p)^2 [f_i ((k + p)^2; \Lambda) + R_\Lambda ((k + p)^2)]} \right\} \big|_{\Lambda' = \Lambda} \\
\rightarrow -\frac{1}{(k + p)^2 f_i (\max \{k^2, p^2\}; \Lambda)} \partial_\Lambda \theta^\epsilon \left( k^2 - \Lambda^2 \right).
\] (27)

After this replacement, the shell integrals may be carried out analytically by using the formula [27];

\[
\delta^\epsilon \left( k^2 - \Lambda^2 \right) h \left( \theta^\epsilon (k^2 - \Lambda^2) \right) \rightarrow \delta \left( k^2 - \Lambda^2 \right) \int_0^1 d\theta (t) \quad \text{as} \quad \epsilon \rightarrow 0,
\] (28)

where \( h \left( \theta^\epsilon (k^2 - \Lambda^2) \right) \) is an arbitrary function of the step function \( \theta^\epsilon \). Indeed it is far from obvious whether the one dimensional approximation and the sharp cutoff limit introduced in our approximation scheme is effective well or not. However it would be inevitable to use of these approximations in order to solve the flow equations with retaining the full momentum dependence of the two and four point vertex functions. Needless to say this approximation may be too brute, although the analysis of the flow equations becomes easy. Another approach base on derivative expansion of the effective average action may be also possible, however, we would like to leave it for future work[28].

Lastly let us consider the four point vertices. By substituting the truncated action (14) into the flow equation (6), we may obtain the beta functions for the four point vertices: \( T^{abcd}_{\mu\nu} (p, q | r, s; \Lambda) \), \( V^{abcd}_{\mu\nu\rho\sigma} (p, q, r, s; \Lambda) \), \( T^{abcd} (p, q, r, s; \Lambda) \), \( T^{cdab}_{\mu\nu} (k, -k | p, -p; \Lambda) \). However it is still difficult to solve these flow equations with retaining all the informations of the three independent momenta, color and Lorentz indices. Therefore we shall further restrict their functional forms below. As stated at the beginning of this section, our truncation scheme deals with only the diagrams which contribute to the two point functions. Looking at the flow equations (20) and (21), we observe that the four types of four point vertices appear in the flow equations with two independent momenta (the external momentum \( p \) and the internal momentum \( k \)) associated with the following momentum channels;

\[
T^{abcd}_{\alpha\beta} (p, -p | k, -k; \Lambda), \quad V^{abcd}_{\mu\nu\alpha\beta} (p, -p, k, -k; \Lambda), \\
T^{abcd} (p, -p, k, -k; \Lambda), \quad T^{cdab}_{\mu\nu} (k, -k | p, -p; \Lambda).
\] (29)

Here, we have distinguished the internal momentum \( k \) from the external momentum \( p \), because these momenta are treated asymmetrically in the followings. Then we shall approximate the beta functions of the four point vertices by restricting to the above types of vertices; we neglect differences in the other momentum channels. Also we restrict the color
and Lorentz index structures with the following forms;

\[ T_{\alpha \beta}^{abcd}(p, -p|k, -k; \Lambda) = \left[ \delta^{ab} \delta^{cd} / 8 \right] \left[ T_{\alpha \beta}(p)/3 \right] W_1 \left( p^2, k^2; \Lambda \right), \]  

(31)

\[ V_{\mu \nu \alpha \beta}^{abcd}(p, -p, k, -k; \Lambda) = \left[ \delta^{ab} \delta^{cd} / 8 \right] \left[ T_{\alpha \beta}(p)/3 \times T_{\mu \nu}(k)/3 \right] W_2 \left( p^2, k^2; \Lambda \right), \]  

(32)

\[ T_{\mu \nu}^{abcd}(p, -p, k, -k; \Lambda) = \left[ \delta^{ab} \delta^{cd} / 8 \right] W_3 \left( p^2, k^2; \Lambda \right), \]  

(33)

\[ T_{\mu \nu}^{abcd}(k, -k|p, -p; \Lambda) = \left[ \delta^{ab} \delta^{cd} / 8 \right] \left[ T_{\mu \nu}(k)/3 \right] W_4 \left( p^2, k^2; \Lambda \right), \]  

(34)

where \( T_{\mu \nu}(p) \) denotes the transverse projector and the normalization factors are chosen for the later conveniences. Here we introduced the four types of four point vertices \( W_1, W_2, W_3, W_4 \) as functions dependent on two momentum variables \( p^2 \) and \( k^2 \) as well as the cutoff scale \( \Lambda \). Thus we have reduced the four point vertices to these scalar functions by restricting structures of the four point vertices, and also by using the one dimensional approximation in the sharp cutoff limit. Then the flow equations for the four point vertices are reduced to a set of coupled partial differential equations with respect to the momentum dependent functions: \( f_G, f_Z, W_1, W_2, W_3, W_4 \). One may wonder if the functions \( W_i \) and \( W_4 \) are not independent mutually, as is seen from their definition given by (31) and (34). However we will treat them as independent functions because of the following reason in approximation. In order to write down the beta functions for \( W_i \), it is necessary to evaluate the box diagrams which consist of the three ghost-gluon vertices (see Fig.2). Then one encounters the two angle integrations [31], which are induced by the propagators associated with the momenta \((k' + p)^2\) and \((k' + k)^2\), where \( k' \) is the internal loop momentum in the box diagrams. This angular integrals in the box diagrams cannot be performed analytically. Here, therefore, we will approximate the integrand by expanding the inner products of momenta \( k_\mu \), that is, \((k'k)\) and \((pk)\) to the second order, and then average over the direction of \( k_\mu \). Thus \( W_1 \) and \( W_4 \) receive different radiative corrections in this evaluation of beta functions, so we treat them as independent functions.

Resultantly, our ERG equations are reduced to a set of coupled partial differential equations for the two propagator functions \( f_G(p^2; \Lambda) \), \( f_Z(p^2; \Lambda) \) and four types of the scalar functions \( W_i(p^2, k^2; \Lambda), (i = 1, \cdots, 4) \). Diagrammatic representations of the beta functions for the two and four point functions are depicted in Fig[1] and Fig[2] respectively. Concrete expressions of the flow equations are summarized in Appendix A. We will solve this set of ERG equations numerically in the next section.

Before closing this section it is also helpful to define more primitive approximation scheme for later discussions. We define the scheme in which the four point vertices are neglected by putting \( W_i(p^2, k^2; \Lambda) = 0 \) under the RG evolution. This scheme is equivalent to take into account only the first diagrams of each flow equations for \( f_G(p^2; \Lambda) \) and \( f_G(p^2; \Lambda) \) given in Fig[1] At first sight this scheme is more similar to the SDE approximation scheme, which has the infrared power solution. However, as we will show in the next section, this scheme will fail to reach deep infrared region. The reason will be considered in the last section.

IV. RESULTS OF NUMERICAL ANALYSIS

In this section we will present the results obtained by numerically integrating the flow equations, specially with paying much attention on infrared behavior of the gluon and the ghost propagator functions, \( f_Z(p^2; \Lambda) \) and \( f_G(p^2; \Lambda) \). First of all, we need to specify the
FIG. 2: Diagrammatic representations of the beta functions for the four point vertex functions: the 4-ghost-gluon vertices $W_1(p^2, k^2; \Lambda)$ and $W_4(p^2, k^2; \Lambda)$ (upper line), the 4-gluon vertex $W_2(p^2, k^2; \Lambda)$ (center line) and the 4-ghost vertex $W_3(p^2, k^2; \Lambda)$ (lower line). Here $p$ denotes the external and $k$ denotes the internal momenta when they are used in the flow equations for the two point functions shown in Fig.1.

initial condition for the effective action in order to proceed the numerical integration. If one choose a very large scale as the initial cutoff scale $\Lambda_0$, then the boundary condition for the flow equations may well be given by the bare action. This is because the arbitrary irrelevant terms do not affect the infrared behavior according to the discussion given in section III. Obviously the large initial cutoff scale $\Lambda_0$ is characterized by the small coupling constant in asymptotically free theories. We thus define the initial conditions at the initial cutoff scale $\Lambda_0$ simply by

$$g(\Lambda_0) = 2.0,$$  \hspace{1cm} (35)
$$f_Z(p^2; \Lambda_0) = f_G(p^2; \Lambda_0) = 1,$$  \hspace{1cm} (36)
$$W_i(p^2, k^2; \Lambda_0) = 0 \hspace{0.5cm} (i = 1, \cdots, 4).$$  \hspace{1cm} (37)

Here it should be noted also that the bare four gluon vertex does not appear in the initial conditions (37). The presence of such a momentum independent four point vertex is irrelevant in the flow equations, because the corrections are contracted by the B-P tensor $B_{\mu\nu}(p)$.

It would be better to use the initial conditions defined with including quantum fluctuation from larger momentum region, $p^2 > \Lambda_0^2$. This is easily carried out by solving the perturbative flow equations under suitable renormalization conditions[20, 26]. However such corrections are relevant for flows only at the ultraviolet region. Our main concern is to see whether the infrared power solution is realized as an infrared attractive solution in the ERG framework. Therefore we employ the above simple initial conditions (35)~(37).

Now let us present the results of numerical analysis of the flow equations. In Figs.3 and 4 we show flows of the gluon function $f_Z(p^2; \Lambda)$ and the ghost function $f_G(p^2; \Lambda)$ respectively with varying the infrared cutoff scale $\Lambda = \exp(-t)\Lambda_0$. In the figures their snapshots at $t = 0.00, 3.12, 4.16, 4.94, 8.06, 13.00$ are presented in one panel. From these flows one may
observe that the propagator functions $f_Z(p^2; \Lambda)$ and $f_C(p^2; \Lambda)$ freeze immediately once the cutoff scale is lowered than the momentum scale, namely, at the region of $\Lambda^2 < p^2 < \Lambda_0^2$. The graphs are strongly bended around $p^2 \sim \Lambda^2$ and become almost flat at lower momentum region. It may be supposed that the bends are due to the crude approximation. Because the integrand of these beta functions are infrared cutoff, and therefore the corrections around the $p^2 \sim \Lambda^2$ may be overestimated due to the one dimensional approximation in the sharp cutoff limit employed in the previous section. Anyway, as lowering the cutoff scale $\Lambda$, it is found that the frozen solutions drastically change around $p^2/\Lambda_0^2 \sim 0.0001$ and eventually show the power scaling. This means also that power behavior is certainly realized in the infrared propagators defined by (17).

In Fig. 3, the result for these propagator functions obtained at $t = 13.00$ are shown. There we also show these functions evaluated with neglecting the four point vertices $W_i(p^2, k^2; \Lambda)$ in the flow equations for comparison. It is seen clearly that these functions enjoy power behavior in the infrared without singular poles. On the other hand if the four point vertices are neglected, we observe that the ghost propagator function diverges at a finite scale, and our numerical calculation is broken down. In the present case, the singularity appears at $t \approx 5.00$. The snapshots in Fig. 3 is given adjacent to the singularity. It is seen that the two schemes present qualitatively the same behavior at the ultraviolet region, however the evolutions in the infrared become drastically different. Thus we may conclude that contributions through the four point vertices are significant in the infrared region to achieve power behavior with avoiding singularity.

Interestingly these infrared power solutions are related with the running gauge coupling constant, which may be defined naively in the RG framework by

$$\alpha(t) = \frac{g^2(\Lambda_0)}{4\pi f_Z(p^2 = 0; \Lambda(t)) f_C^2(p^2 = 0; \Lambda(t))}. \quad (38)$$
FIG. 4: The flows (Λ dependence) of the ghost propagator function \( f_{G}(p^{2};\Lambda) \) at various cutoff scales \( t = 0.00, 3.12, 4.16, 4.94, 8.06, 13.00 \).

The running coupling obtained by the present analysis is shown in Fig. 4 compared with that in the scheme neglecting the four vertices. We see that the running coupling converges to an infrared fixed point \( \alpha^{*} \approx 4.70 \). In contrast to this the running coupling without four vertex contributions is found to hit the Landau pole. In the SDE approach, the running gauge coupling is sometimes defined as a function of momentum, which is regarded as the renormalization scale [1, 2]. In our ERG framework, the coupling constant would correspond to

\[
\alpha(p^{2}) = \frac{g^{2}(\Lambda_{0})}{4\pi f_{Z}(p^{2};\Lambda \rightarrow 0)f_{G}^{2}(p^{2};\Lambda \rightarrow 0)}. \tag{39}
\]

It is seen also that this coupling constant shows the qualitatively same behavior with \( \alpha(t) \) and that an infrared fixed point exists.

From the definition of the running coupling [38], it is clear that appearance of the infrared fixed point can be realized by exact cancellation of the corrections to the gluon and ghost propagators. The infrared power behavior implies that the exponents must satisfy the specific ratio as shown in (1). The exponents, that is the anomalous dimensions, may be defined also as Λ-dependent quantity,

\[
\eta_{Z}(t) = -\frac{1}{2}\frac{\partial}{\partial t}\log f_{Z}(p^{2} = 0;\Lambda(t)), \quad \eta_{G}(t) = -\frac{1}{2}\frac{\partial}{\partial t}\log f_{G}(p^{2} = 0;\Lambda(t)). \tag{40}
\]

In the infrared limit \( t = 13 \) these exponents are found to converge to stable constants given by

\[
\kappa \equiv \eta_{G}^{*} = -\frac{1}{2}\eta_{Z}^{*} \approx 0.146, \tag{41}
\]

where \( \kappa \) denotes the exponent in Eq. (1) used frequently in the SDE studies.
FIG. 5: The propagator functions $f_Z(p^2; \Lambda)$ (full line) and $f_G(p^2; \Lambda)$ (long-dashed line) obtained at $t = 13.00$, which show power behavior in the infrared region. For comparison these functions evaluated with neglecting the four point vertices are also shown for $f_Z(p^2; \Lambda)$ (dashed-dotted line) and $f_G(p^2; \Lambda)$ (dashed-dashed-dotted line) just before reaching singularity ($t=4.93$).

V. CONCLUSION AND DISCUSSION

We have performed numerical analysis of the approximated ERG equations for Landau gauge SU(3) Yang-Mills theory. There flows of the momentum dependent four point vertices as well as the momentum dependent gluon and ghost propagators were examined. In order to setup the approximated flow equations with retaining momentum dependence of vertices in a minimal way, we restricted the diagrams to those which contribute to the two point functions. Further we employed one-dimensional approximation and sharp cutoff limit to make numerical analysis easier. The resultant flow equations are given in the form of coupled partial differential equations which have been solved numerically. We solved the flow equations with an ordinary bare action as the initial conditions for the effective action. However infrared behavior of the flows are not affected by the arbitrary irrelevant terms in the effective action at ultraviolet.

Our numerical solution exhibits infrared power behavior for the propagators $D_Z(p^2) \sim (p^2)^{-1+2\kappa}$, $D_G(p^2) \sim (p^2)^{-1-\kappa}$ with the exponent $\kappa \approx 0.146$, which is similar to the result obtained by SDE studies. Also the running gauge coupling is found to approach an infrared fixed point $\alpha^* \approx 4.70$. Compared with the SDE analyses, the most different point is that inclusion of the four point vertices generated during flow is crucial to obtain the power solutions with avoiding infrared divergence. In the SDE approach, only the dressed four gluon vertex appears through the two loop diagram in the equation for gluon propagator and existence of the power solution has been confirmed irrespectively to the vertex contributions 1, 2, 3, 4, 8.

This difference may be considered as follows. The SDEs are a set of integral equations among the Green functions, and there all momentum modes are coupled to each other.
we solve the SDEs for the propagators iteratively, then the propagator may be represented by an infinite sum of diagrams whose internal momenta are not restricted. On the other hand the ERG equation is reduced to a set of the beta functions for the Green functions, which are given by one-loop exact forms diagrammatically. Therefore if we could solve the flow equations iteratively, then the Green functions are given by an infinite sum of diagrams involving all loops, which must be identical to the solution of the SDE. However this is not the case after truncation. Note that internal momenta of the one-loop integrals of the beta functions are restricted around the cutoff scale $\Lambda$, and contributions from higher momentum loops are already taken in the effective vertices in the ERG framework. Also the beta functions for the effective vertices involve the higher point vertices as well as the lower ones. Therefore, in our present concern, some part of the corrections to the two point functions comes through four point effective vertices. If we truncate the effective action and discard the four and higher point functions, the corrections generated through higher point vertices are lost in the resultant propagators. Speaking more explicitly, the diagrams in which the inner loops carry lower momenta than the outer loops are suppressed in the diagrammatic terminology. Inclusion of the four point functions improve this situation drastically indeed. In the case of Landau gauge Yang-Mills theory, the ghost propagator is considered to be enhanced at infrared. Therefore, particularly couplings of the soft ghost modes with other high energy modes are expected to give non-negligible contributions.

Finally we shall discuss residual problems and the advantageous features of the ERG approach. Although we found the power solution in the ERG framework, the obtained exponent $\kappa \approx 0.146$ is fairly smaller than the currently most reliable value $\kappa \approx 0.5$ expected from the SDE analysis and also from axiomatic considerations. Furthermore transient from the logarithmic to the power behavior of the propagators is not smooth. Probably these are due to the fact that our approximation scheme is still poor in the following points. Firstly the corrections for the two point functions with $p^2 \sim \Lambda^2$ are fairly influenced by use of the
one-dimensional approximation and the sharp cutoff limit. Then the four point vertices
which have other channels and also higher order vertices may contribute to flows of the
two point functions. These issues are currently investigated with employing another type of
approximation scheme[28]. Further, inclusion of three gluon interactions, which have been
omitted in this paper just for the simplicity, and also dynamical quarks are remained as
obvious problems.

The advantageous features of ERG may be considered as follows. We could obtain the
infrared power behavior for the gluon and the ghost propagators without any ansatz for their
infrared behaviors, and by starting from rather arbitrary ultraviolet bare actions. On the
contrary all the SDE analyses so far were based on the infrared asymptotic analysis, which
needs an appropriate ansatz. In the case of Landau gauge Yang-Mills theories, the power law
ansatz was found to suit nicely. However in the other situations such as in the deconfinement
phase the appropriate ansatz is unknown a priori. It would be another advantage of the
ERG to obtain some informations of the infrared four point functions at the same time. In
these respects the ERG formalism seems to be suitable to explore the infrared dynamics in
more general cases. Especially exploration of dynamics near phase transitions would be quite
interesting and challenging problem to be considered, and the ERG approach is expected to
be useful also in such studies.

Note added : After completion of this paper, Ref.[29] appeared which addresses related
issues.

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APPENDIX A: APPROXIMATED FLOW EQUATIONS

The flow equations derived by using the approximation scheme explained in section [II] are
summarized here. The all momentum variables and dimensionful functions are normalized
by the cutoff scale Λ, and we use \( x = p^2 / Λ^2 \), \( y = k^2 / Λ^2 \). By contracting the external gluon
momentum with B-P tensor \( B_{\mu\nu}(p) \), all vertex functions \( W_i(x, y; Λ) \) are factorized with the
momentum variables. So we define the factorized functions by

\[
W_i(x, y; Λ) = \frac{xy}{16\pi^2} \bar{W}_i(x, 1; Λ)
\]

and use them below. The flow equations, which are somewhat lengthy though, are found to
be in the followings.

\[
\partial_t f_Z(x; Λ) = -2π\partial_x f_Z(x; Λ)
+ \frac{g^2(Λ_0)}{16\pi^2} f_Z^{-1}(1; Λ) f_G^{-1}(\max(x, 1); Λ) \times \begin{cases} x^{-2}(3 - 2x^{-1}) & (x > 1) \\ 1 & (x \leq 1) \end{cases}
+ \frac{1}{12\pi^2} f_G^{-1}(1; Λ) \bar{W}_1(x, 1; Λ) + \frac{1}{2\pi^2} f_Z^{-1}(1; Λ) \bar{W}_2(x, 1; Λ),
\]

\[
\partial_t f_G(x; Λ) = -2π\partial_x f_G(x; Λ)
- \frac{g^2(Λ_0)}{32\pi^2} f_G^{-1}(1; Λ) f_Z^{-1}(\max(x, 1); Λ) \times \begin{cases} x^{-2} & (x > 1) \\ 1 & (x \leq 1) \end{cases}
\]
\[
\frac{1}{2\pi^2} f_G^{-1}(1; \Lambda) \tilde{W}_3(x, 1; \Lambda) - \frac{1}{8\pi^2} f_{Z}^{-1}(1; \Lambda) \tilde{W}_4(x, 1; \Lambda),
\]  
(A2)

\[
\frac{\partial_t \tilde{W}_1(x, y; \Lambda)}{y - \lambda} = -4 \sum_{\lambda} \left[ 5 \frac{u}{y} \tilde{W}_1(x, 1; \Lambda) \tilde{W}_2(y, 1; \Lambda) + \frac{1}{2\pi^2} f_{Z}^{-1}(1; \Lambda) \tilde{W}_3(x, 1; \Lambda) \tilde{W}_4(y, 1; \Lambda) \right]
\]

\[
\times \left[ \frac{\partial_t \tilde{W}_2(x, y; \Lambda)}{y - \lambda} = -4 \sum_{\lambda} \left[ 5 \frac{u}{y} \tilde{W}_1(x, 1; \Lambda) \tilde{W}_2(y, 1; \Lambda) + \frac{1}{2\pi^2} f_{Z}^{-1}(1; \Lambda) \tilde{W}_3(x, 1; \Lambda) \tilde{W}_4(y, 1; \Lambda) \right]
\]  
(A3)

\[
\frac{\partial_t \tilde{W}_3(x, y; \Lambda)}{y - \lambda} = -4 \sum_{\lambda} \left[ 5 \frac{u}{y} \tilde{W}_1(x, 1; \Lambda) \tilde{W}_2(y, 1; \Lambda) + \frac{1}{2\pi^2} f_{Z}^{-1}(1; \Lambda) \tilde{W}_3(x, 1; \Lambda) \tilde{W}_4(y, 1; \Lambda) \right]
\]  
(A4)

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\[ -g^2(\Lambda_0) \frac{f_G^{-2}(1; \Lambda) f_G^{-1}(\max(x, 1); \Lambda)}{64\pi^2} W_4(y, 1; \Lambda) \times \begin{cases} (3y^{-1} - y^{-2}) & (y > 1) \\ (3 - y) & (y \leq 1) \end{cases} \\
+ \frac{9g^4(\Lambda_0)}{64\pi^2} f_G^{-2}(1; \Lambda) f_G^{-1}(\max(x, 1); \Lambda) f_G^{-1}(\max(y, 1); \Lambda) \times \begin{cases} x^{-2} & (x > 1) \\ 1 & (x \leq 1) \end{cases} \]

\[ \frac{g^4(\Lambda_0)}{16\pi^2} f_G^{-2}(1; \Lambda) f_G^{-1}(\max(x, 1); \Lambda) W_4(x, y; \Lambda) \times \begin{cases} (3x^{-1} - x^{-2}) & (x > 1) \\ (3 - x) & (x \leq 1) \end{cases} \]  

(A5)

\[
\partial_t W_4(x, y; \Lambda) = -4W_4(x, y; \Lambda) - 2x\partial_x W_4(x, y; \Lambda) - 2y\partial_y W_4(x, y; \Lambda) 
- \frac{1}{2\pi^2} f_G^{-2}(1; \Lambda) W_4(x, 1; \Lambda) W_2(y, 1; \Lambda) + \frac{1}{2\pi^2} f_G^{-2}(1; \Lambda) W_3(x, 1; \Lambda) W_1(y, 1; \Lambda) 
- \frac{3g^2(\Lambda_0)}{8\pi^2} f_G^{-2}(1; \Lambda) f_G^{-1}(\max(x, 1); \Lambda) W_2(y, 1; \Lambda) \times \begin{cases} (3x^{-1} - x^{-2}) & (x > 1) \\ (3 - x) & (x \leq 1) \end{cases} 
+ \frac{9g^2(\Lambda_0)}{32\pi^2} f_G^{-2}(1; \Lambda) f_G^{-1}(\max(x, 1); \Lambda) W_1(y, 1; \Lambda) \times \begin{cases} x^{-2} & (x > 1) \\ 1 & (x \leq 1) \end{cases} 
+ \frac{3g^2(\Lambda_0)}{4\pi^2} f_G^{-2}(1; \Lambda) f_G^{-1}(\max(y, 1); \Lambda) W_3(x, 1; \Lambda) \times \begin{cases} (3y^{-2} - 2y^{-3}) & (y > 1) \\ 1 & (y \leq 1) \end{cases} 
+ \frac{3g^4(\Lambda_0)}{8\pi^2} f_G^{-2}(1; \Lambda) f_G^{-1}(\max(y, 1); \Lambda) f_G^{-1}(\max(x, 1); \Lambda) \times \begin{cases} (5 + 3y) & (x > 1) \\ (1 + y)^3 & (x \leq 1) \end{cases} \]  

(A6)

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