An MISOCP-Based Solution Approach to the Reactive Optimal Power Flow Problem

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Abstract—In this letter, we present an alternative mixed-integer non-linear programming formulation of the reactive optimal power flow (ROPF) problem. We utilize a mixed-integer second-order cone programming (MISOCP) based approach to find global optimal solutions of the proposed ROPF problem formulation. We strengthen the MISOCP relaxation via the addition of convex envelopes and cutting planes. Computational experiments on challenging test cases show that the MISOCP-based approach yields promising results compared to a semidefinite programming based approach from the literature.

Index Terms—Mixed-integer non-linear programming, reactive optimal power flow, second-order cone programming.

I. INTRODUCTION

The reactive optimal power flow (ROPF) problem is a variant of the well-known optimal power flow (OPF) problem in which additional discrete decisions, such as shunt susceptance and tap ratio, are considered. Due to the presence of these discrete variables in the ROPF problem, it can be formulated as a mixed-integer non-linear programming (MINLP) problem. This letter utilizes the recent developments in the OPF problem to propose an efficient way of solving the ROPF problem.

OPF is one of the most studied problems in the area of power systems and a variety of solution approaches have been proposed in the literature. Local methods such as the interior point method try to solve the OPF problem but they do not provide any assurances of global optimality although empirical evidence suggests that such methods are quite successful in practice. In recent years, convex relaxations of the OPF problem have drawn considerable research interest since the convexity property promises a globally optimal solution under certain conditions. Several approaches have been developed based on convex quadratic [1], semidefinite programming (SDP) [2], second order cone programming (SOCP) [3] and convex-distflow [4] formulations. The ROPF problem has a similar structure with the OPF problem, except the inclusion of shunt susceptance and tap ratio variables, which are typically modelled as discrete variables. The resulting MINLP problem is difficult to solve and the literature has primarily focused on various heuristic methods [5]–[7]. The systematic treatment of the ROPF problem is limited to a partial SDP-based relaxation called tight-and-cheap relaxation (TCR) [8].

The contributions of this letter are as follows. We propose an alternative MINLP formulation for the ROPF problem along with its mixed-integer second-order cone programming (MISOCP) relaxation. In addition, we improve convex envelopes from the literature and use cutting planes to strengthen the MISOCP relaxation. We develop a heuristic approach to obtain a feasible solution for the ROPF problem based on the solution from the MISOCP relaxation. We also test the accuracy and efficiency of our approach with the TCR method from the literature on difficult test cases and obtain promising results especially for the instances with small-angle conditions.

II. MATHEMATICAL MODEL

A. MINLP Formulation

Consider a power network \( \mathcal{N} = (\mathcal{B}, \mathcal{L}) \), where \( \mathcal{B} \) and \( \mathcal{L} \) denote the set of buses and the set of transmission lines respectively. Let \( \mathcal{G} \subseteq \mathcal{B} \) and \( \mathcal{S} \subseteq \mathcal{B} \) and \( \mathcal{T} \subseteq \mathcal{L} \) respectively denote the set of generators connected to the grid, the buses with a variable shunt susceptance and the lines with a variable tap ratio. Let \( \mathcal{G} \) and \( \mathcal{B} \) be the matrices of line conductance and susceptance. Rest of the parameters are given as follows:

- For each bus \( i \in \mathcal{B} \), \( p_i^q \) and \( q_i^q \) are the real and reactive power load, \( \sqrt{p_i^q} \) and \( \sqrt{q_i^q} \) are the bounds on the voltage magnitude, \( \delta(i) \) is the set of its neighbors and \( \{b_{ik}^q : k \in S_i\} \) is the set of allowable shunt susceptances.
- For each generator located at bus \( i \in \mathcal{G} \); active and reactive outputs must be in the intervals \([p_i, p_i]\) and \([q_i, q_i]\), and we have \( p_i = p_i = q_i = q_i = 0 \) for \( i \in \mathcal{B} \setminus \mathcal{G} \).
- For each line \((i, j) \in \mathcal{L}\), \( G_{ij}, G_{ji} \) and \( B_{ij}, B_{ji} \) are the elements of the conductance and susceptance matrices, \( \{\tau_{ij}^l : l \in T_{ij}\} \) is the set of allowable tap ratios, \( \overline{\tau}_{ij} \) is the apparent power flow limit and \( \underline{\tau}_{ij} \) is the bound on the phase angle.

We define the following decision variables:

- For each bus \( i \in \mathcal{B} \), \( V_i \) and \( \theta_i \) are the voltage magnitude and phase angle, \( b_{ii} \) is the shunt susceptance, \( a_{ii}^k \) is one if \( b_{ii} = b_{ii}^k \) and zero otherwise.
- For each generator located at bus \( i \in \mathcal{G} \), \( p_i^q \) and \( q_i^q \) are the real and reactive power output.
- For each line \((i, j) \in \mathcal{L}\), \( p_{ij}, p_{ji} \) and \( q_{ij}, q_{ji} \) are the real and reactive power flow, \( \tau_{ij} \) is the tap ratio, \( \beta_{ij}^l \) is one if \( \tau_{ij} = \tau_{ij}^l \) and zero otherwise.

Then, the ROPF problem can be modeled as the following MINLP:

\[
\begin{align*}
\min_{i \in \mathcal{B}} \sum f(p_i^q) \\
p_i^q - p_i^q &= g_i |V_i|^2 + \sum_{j \in \delta(i)} p_{ij} \quad i \in \mathcal{B} \\
q_i^q - q_i^q &= -b_{ii} |V_i|^2 + \sum_{j \in \delta(i)} q_{ij} \quad i \in \mathcal{B}
\end{align*}
\]
Then, we define a new variable \( \Gamma^k_i := c_{ii} \alpha^k_i \) to reformulate (14) and include additional constraints as follows:

\[
q_i^q - q_i^d = - \sum_{k \in \mathcal{S}_i} b^k_{ii} \Gamma^k_i + \sum_{j \in \mathcal{B}} q_{ij} \quad i \in \mathcal{B}
\]

\[
c_{ii} \alpha^k_i \leq \Gamma^k_i \leq c_{ii} \alpha^k_i \quad k \in \mathcal{S}_i, \quad c_{ii} := \sum_{k \in \mathcal{S}_i} \Gamma^k_i \quad i \in \mathcal{B}. \tag{15}
\]

We now update power flow constraints using a similar procedure. In particular, we substitute \( 1/\tau_{ij} \) with \( \sum_{l \in \mathcal{T}_{ij}} \beta^l_{ij} / \tau_{ij} \) into constraints (4) and (5). After defining the new variables \( \Phi^l_{ij} := c_{ij} \beta^l_{ij}, \Phi^l_{jj} := c_{jj} \beta^l_{jj} \) and \( \Psi^l_{ij} := s_{ij} \beta^l_{ij} \), we rewrite the real and reactive power flow constraints (4)–(5) together with other equations necessary for the reformulation as follows:

\[
p_{ij} = \sum_{l \in \mathcal{T}_{ij}} \left( G_{ij} \left( \frac{\Phi^l_{ij}}{\tau_{ij}} + \frac{\Phi^l_{ij}}{\tau_{ij}} \right) - B_{ij} \frac{\Psi^l_{ij}}{\tau_{ij}} \right) \quad (i, j) \in \mathcal{L}
\]

\[
p_{ji} = G_{ji} c_{jj} + \sum_{l \in \mathcal{T}_{ij}} \left( G_{ji} \frac{\Phi^l_{ij}}{\tau_{ij}} - B_{ij} \frac{\Psi^l_{ij}}{\tau_{ij}} \right) \quad (i, j) \notin \mathcal{T}
\]

\[
q_{ij} = - \sum_{l \in \mathcal{T}_{ij}} \left( B_{ij} \left( \frac{\Phi^l_{ij}}{\tau_{ij}} + \frac{\Phi^l_{ij}}{\tau_{ij}} \right) + G_{ij} \frac{\Psi^l_{ij}}{\tau_{ij}} \right) \quad (i, j) \in \mathcal{L}
\]

\[
q_{ji} = - B_{ji} c_{jj} - \sum_{l \in \mathcal{T}_{ij}} \left( B_{ji} \frac{\Phi^l_{ij}}{\tau_{ij}} + G_{ji} \frac{\Psi^l_{ij}}{\tau_{ij}} \right) \quad (i, j) \notin \mathcal{T}
\]

We also update the constraint on voltage magnitude bounds (6) as follows:

\[
\n_i^2 \leq v_i^2 \leq \overline{v}_i^2 \quad i \in \mathcal{B}. \tag{17}
\]

Finally, we define the following consistency constraints:

\[
c_{ij}^2 + s_{ij}^2 = c_{ij} c_{jj} \quad (i, j) \in \mathcal{L} \tag{18}
\]

\[
\left( \Phi^l_{ij} \right)^2 + \left( \Psi^l_{ij} \right)^2 = \Phi^l_{ij} c_{jj} \quad l \in \mathcal{T}_{ij}, (i, j) \in \mathcal{L} \tag{19}
\]

\[
\theta_j - \theta_i = \arctan(s_{ij} / c_{ij}) \quad (i, j) \in \mathcal{L}. \tag{20}
\]

Equation (18) preserves the trigonometric relation between the variables \( c_{ij}, c_{jj} \). If we multiply (18) by \( \beta^l_{ij} \), we can get a similar condition for the variables \( \Phi^l_{ij}, \Phi^l_{jj} \) and \( \Psi^l_{ij} \).

The alternative formulation minimizes (1) subject to constraints (8)–(13) and (15)–(20).

### C. MISOCP Relaxation

The feasible region of the alternative MINLP formulation is non-convex due to constraints (18)–(20). Let us relax these constraints as follows:

\[
c_{ij}^2 + s_{ij}^2 \leq c_{ij} c_{jj} \quad (i, j) \in \mathcal{L} \tag{21}
\]

\[
\left( \Phi^l_{ij} \right)^2 + \left( \Psi^l_{ij} \right)^2 \leq \Phi^l_{ij} c_{jj} \quad (i, j) \in \mathcal{L}. \tag{22}
\]
Then, an MISOCP relaxation is obtained as (1), (8)–(13), (15)–(17) and (21).

### D. Tightened MISOCP Relaxation

To tighten the MISOCP relaxation, we also consider an outer-approximation of constraints (18) and (20), which is an improved version of a similar approach proposed in [9]. Let us define the set $\mathcal{P} = \{(c, s, \theta) \in \mathcal{P} : \theta = \arctan(s/c), c^2 \leq s^2 \leq \pi^2\}$, where $\theta_i - \theta_j$ is denoted by $\theta$ and the other subscripts are omitted. The four points of interest are given as follows:

- $\zeta^1 = (\tau, \pi, \arctan(\pi/\tau))$
- $\zeta^2 = (\tau, \pi, \arctan(\pi/\tau))$
- $\zeta^3 = (\tau, \pi, \arctan(\pi/\tau))$
- $\zeta^4 = (\tau, \pi, \arctan(\pi/\tau))$

The following proposition provides two upper envelopes for $A$:

**Proposition 1:** Let $\theta = \gamma^1 + \mu^1 c + v^1 s$ and $\theta = \gamma^2 + \mu^2 c + v^2 s$ be the planes passing through points $\{\zeta^1, \zeta^2, \zeta^3, \zeta^4\}$, respectively. Then, two valid inequalities for $A$ can be obtained as

$$\tau^m + \mu^m c + v^m s \geq \arctan(s/c),$$

with $\tau^m = \gamma^m + \Delta m$, $m = 1, 2$, where

$$\Delta m = \max_{(c, s, \theta) \in A} \{\arctan(s/c) - (\gamma^m + \mu^m c + v^m s)\} \quad (22)$$

We will omit the proof of Proposition 1 since the statement holds true by construction. However, the interesting property related to the optimization problem (22) is that although both its objective function and feasible region are non-convex, it can still be solved globally. The key idea is to re-state this optimization problem in the polar coordinates as

$$\Delta m = -\tau^m + \max_{r \in [\tau, \pi], \theta \in [\theta]} \{\theta - r(\mu^m \cos(\theta) + v^m \sin(\theta))\},$$

where $r := \sqrt{c^2 + s^2}$. Since problem (23) is linear in $\tau$, it can be solved for the two end-points of the interval $[\tau, \pi]$ separately. Finally, the remaining one-dimensional optimization problems in $\theta$ can be solved by checking the KKT points. We note that Proposition 1 is an improvement over Proposition 3.8 from [9] since the feasible region of problem (22) is a smaller subset of the corresponding optimization problem in [9].

We also obtain two envelopes for $A$.

**Proposition 2:** Let $\theta = \gamma^3 + \mu^3 c + v^3 s$ and $\theta = \gamma^4 + \mu^4 c + v^4 s$ be the planes passing through points $\{\zeta^3, \zeta^4\}$, respectively. Then, two valid inequalities for $A$ are defined as $\gamma^m + \mu^m c + v^m s \leq \arctan(s/c)$ with $\gamma^m = \gamma^m + \Delta n$, $n = 3, 4$, where

$$\Delta n = \max_{(c, s, \theta) \in A} \{(\gamma^m + \mu^m c + v^m s) - \arctan(s/c)\}.$$  

Finally, we add the following valid inequalities to the MISOCP relaxation:

$$\tau_{ij}^m + \mu_{ij}^m s_j + v_{ij}^m s_j \geq \theta_j - \theta_i, \quad m = 1, 2, \quad (i, j) \in \mathcal{L}$$

and

$$\tau_{ij}^m + \mu_{ij}^m s_j + v_{ij}^m s_j \leq \theta_j - \theta_i, \quad n = 3, 4, \quad (i, j) \in \mathcal{L}.$$  

We will use the abbreviation MISOCPA+ to refer to this stronger relaxation. Additionally, we generate cutting planes for each cycle in the cycle basis using a method called SDP Separation, more details can be found in [3]. We denote this further improved relaxation as MISOCPA+.

### III. COMPUTATIONAL EXPERIMENTS

#### A. Algorithm

We first solve the continuous relaxation of the MISOCPA formulation by relaxing the integrality of $a_k^b$ and $b_j^k$ variables. Then, for each cycle in the cycle basis, we use the SDP separation method to generate cutting planes to separate this continuous relaxation solution from the feasible region of the SDP relaxation of the cycle. The separation process is parallelized over cycles. We repeat this procedure five times consecutively. Then, we solve the final MISOCPA+ relaxation to obtain a lower (LB) bound, and then fix the binary variables in the MINLP formulation to obtain an upper bound (UB) from the remaining non-linear program (NLP) using a local solver, which provides a feasible solution to the ROPF problem. In case of infeasible NLP, we eliminate the fixed binary variable combination from the feasible region and resolve the MISOCPA to potentially find other combinations of binary variables. The optimality gap is computed as $\% Gap = 100 \times (1 - LB/UB)$.

#### B. Results

We compare the percentage optimality gap and the computational time of the MISOCPA+ approach with the publicly available implementation of TCR relaxation of Type 2 (TCR2) from [8] under default settings. All computational experiments have been carried out on a 64-bit desktop with Intel Core i7 CPU with 3.20 GHz processor and 64 GB RAM. Our code is written in Python language using Spyder environment. The solvers Gurobi, IPOPT and MOSEK are used to solve the MISOCPA+ relaxation, NLP and separation problems, respectively. We run Gurobi with the default settings except for changing the time limit as 30 seconds.

For the computational experiments, we use the OPF instances from the NESTA library; typical operating conditions (TYP), congested operating conditions (API) and small angle difference conditions (SAD). We only consider difficult instances in which the SOCP optimality gap is more than 1% [3]. We include one shunt element and one transformer for those instances which do not have them.

The sets of the discrete values are determined as $b^i_h \in \{0, 1\}$ for $i \in \mathcal{S}$ and $t^i_j \in \{1 \pm 0.0125 \times h : h \in \{0, 1, \ldots, 8\}\}$ for $(i, j) \in \mathcal{T}$, which represent the on/off status of the shunt susceptance and values of the tap ratio, respectively.

The results of our computational experiments are reported in Table I. The computational time is measured in seconds, and includes the time spent for solving the separation problems. If we compare the averages of optimality gap, MISOCPA+ outperforms TCR2 in all types of NESTA instances. MISOCPA+ has the best performance on SAD instances and dominates TCR2 in all of them. Although not reported in this letter, we also observe that MISOCPA+ has better accuracy than the full SDP-based approach SDR2 from [8] for this type of test cases. Overall, we note that MISOCPA+ relaxation has more accurate solutions with 6.87% optimality gap, on average, than TCR with 8.68%. In terms of computational time, MISOCPA+ is slower with 8.14 seconds, on average, than TCR2 with 1.77. We also point out the even better performance MISOCPA+ in terms of accuracy on more difficult instances with the SOC gap more than 3%.

1) **The Effect of Tap Ratio Discretization:** We also analyze the effect of tap ratio discretization on the ROPF problem. The algorithm is repeated with a different discrete set $t^i_j \in \{1 \pm 0.05 \times h : h \in \{0, 1, 2\}\}$ for $(i, j) \in \mathcal{T}$. We observe that UBs and optimality gaps do not change significantly with this coarser discretization. In fact, the average absolute percentage change of UBs is only 0.02%. Since the computational effort increases with the number of discrete steps, we
conclude it may be more practical to use a small number of discrete steps, especially for small-size test cases.

2) A Test Case for a Larger Instance: In order to solve a large scale instance within acceptable time limits, we modify our algorithm as follows: The final MISOCPA+ relaxation is also solved by relaxing the integrality of $\alpha_{ki}$ and $\beta_{lj}$ variables. The convex combinations of discrete values $\{b_{ki} : k \in S_i\}$ and $\{\tau_{lj} : l \in T_ij\}$ with coefficients $\alpha_{ki}$ and $\beta_{lj}$ are rounded off to the nearest discrete value in these sets. Then, we fix the binary variables with respect to the rounded-off values in the MINLP formulation. Once tested on 1354-bus PEGASE SAD test case, the modified algorithm produces a feasible solution with a cost of $1,220,718 with an optimality gap of 3.82% in approximately 6 minutes. We note that the TCR2 relaxation experiences numerical difficulties for this test case.

3) The Comparison With the Global Optimal Solution: In general, our solution approach does not guarantee to provide global optimal solutions due to the computational difficulty of the nonconvex MINLP problem. In order to validate the accuracy of our solution approach, we provide illustrative examples for three small test cases: 3lmbd_sad, 4gs_sad, and 5pjim_sad. We first enumerate all possible combinations of the binary variables. Then, we fix the binary variables and solve the rectangular formulation of the remaining OPF instance by Gurobi (version 9.0) to global optimality. The results show that our solution approach reaches the global optimum solution for these test cases although the proven optimality gap is positive.

IV. CONCLUSION

In this letter, we propose an MISOCP-based approach, namely MISOCPA+, to approximate globally optimal solutions of the ROPF problem. The accuracy and efficiency of this approach are compared with TCR2 using difficult OPF instances from the NESTA library. The computational results indicate that MISOCPA+ is quite promising to solve any type of instances accurately, especially the ones with small angle conditions.

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TABLE I

| Case      | TCR2 | MISOCPA+ |
|-----------|------|----------|
| 3lmbd_typ | 0.35 | 0.64     |
| 5pjim_typ*| 0.73 | 0.40     |
| 30tee_tpy*| 1.13 | 0.45     |
| 118tee_tpy| 4.66 | 34.45    |
| Average   | 1.72 | 10.02    |
| 3lmbd_api*| 0.35 | 0.15     |
| 6wgg_api*  | 0.98 | 0.87     |
| 14jgg_api* | 0.93 | 1.34     |
| 30aapi*   | 2.38 | 0.92     |
| 30srr_api* | 2.16 | 0.89     |
| 39epri_api| 2.59 | 31.33    |
| 118tee_epri*| 4.47 | 34.47    |
| Average   | 1.98 | 9.99     |

| Case      | TCR2 | MISOCPA+ |
|-----------|------|----------|
| 3lmbd_sad*| 0.35 | 0.40     |
| 4gs_sad*  | 0.40 | 0.40     |
| 5pjim_sad*| 0.45 | 0.45     |
| 9wssc_sad | 0.48 | 0.48     |
| 30aas_sad*| 3.19 | 3.19     |
| 50ermin_sad*| 3.19 | 3.19     |
| Overall   | 1.77 | 3.8    |
| Overall*  | 1.75 | 3.8    |