UNIVERSAL BUNDLE FOR GRAVITY, LOCAL INDEX THEOREM, AND COVARIANT GRAVITATIONAL ANOMALIES

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Abstract
Consistent and covariant Lorentz and diffeomorphism anomalies are investigated in terms of the geometry of the universal bundle for gravity. This bundle is explicitly constructed and its geometrical structure will be studied. By means of the local index theorem for families of Bismut and Freed the consistent gravitational anomalies are calculated. Covariant gravitational anomalies are shown to be related with secondary characteristic classes of the universal bundle and a new set of descent equations which also contains the covariant Schwinger terms is derived. The relation between consistent and covariant anomalies is studied. Finally a geometrical realization of the gravitational BRS, anti-BRS transformations is presented which enables the formulation of a kind of covariance condition for covariant gravitational anomalies.

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I. Introduction

It has been shown long ago\textsuperscript{1,2} that certain chiral fermions interacting with external gravitational fields in \(4k + 2\) dimensions may suffer from anomalies which spoil general covariance and local Lorentz symmetry. These gravitational anomalies appear as a non-conservation of the energy momentum tensor and cause its asymmetry.

Like in the Yang-Mills case one can distinguish between a consistent and covariant form of the gravitational anomaly. Actually, the methods used in Ref. 1 give the covariant diffeomorphism anomaly. On the other side, pure covariant Lorentz anomalies have been calculated in Ref. 3 using path-integral methods. Bardeen and Zumino\textsuperscript{4} investigated the relation between the consistent and covariant diffeomorphism anomaly. Moreover they showed that consistent diffeomorphism anomalies can be shifted to consistent Lorentz anomalies by means of a non-polynomial action.

Another manifestation of these anomalies is due to the occurrence of Schwinger terms in the equal time commutator of the energy momentum tensor. The consistent Schwinger terms have been determined in two dimensions and their relation with the gravitational anomaly was investigated.\textsuperscript{5,6}

Recently, an algebraic approach to determine covariant diffeomorphism anomalies has been presented.\textsuperscript{7,8} This method is based on the gauged (anti) BRST symmetry and leads to descent equations for covariant diffeomorphism anomalies and Schwinger terms. Furthermore a kind of consistency condition for these anomalous terms has been formulated in Ref. 8.

The aim of this paper is twofold: In the first place we want to clarify the applicability of the local index theorem of Bismut and Freed\textsuperscript{9–11} to the determination of consistent gravitational anomalies. In the second place we intend to investigate the geometrical structure of covariant anomalies.

Although the geometrical framework and the use of the cohomological Atiyah-Singer index theorem\textsuperscript{12} to the study of the consistent case has been outlined by various authors,\textsuperscript{13–16} a throughout construction has never been worked out explicitly.

Since local cohomology\textsuperscript{16} seems to be the appropriate cohomology theory to detect anomalies, the use of the Atiyah-Singer index theorem needs further clarification. In the Yang-Mills case Atiyah and Singer\textsuperscript{13} introduced a certain universal bundle with connection in order to compute the characteristic classes of the index bundle. Its first Chern class was interpreted as an obstruction to trivializing the associated determinant line bundle. It was argued that a similar situation arises in the gravitational case.

On the other hand the local families index theorem of Bismut and Freed\textsuperscript{10,11} provides a method to calculate the curvature of the determinant line bundle associated with the family of Dirac operators. The relevance of this differential form version of the index theorem in String theory has been stressed by Freed.\textsuperscript{17} He also sketched briefly the gravitational case but he skipped an explicit calculation.

It is one aim of this paper to fill this gap in order to study consistent and covariant gravitational anomalies form a unified geometrical viewpoint. We shall give an explicit construction of the universal bundle for gravity and define a universal connection in the sense of Ref. 13 on it. It is then shown that this bundle with connection provides the geometrical data in order to apply the local index theorem. Finally we calculate the consistent gravitational anomalies.
In view of possible applications in topological gravity a detailed investigation of the geometrical structure of the universal bundle seems to be interesting in itself. A step in this direction has been already made in two dimensions in Ref. 18.

Our other theme is to investigate the geometrical structure of combined covariant diffeomorphism- and Lorentz anomalies of arbitrary ghost degree. Here we shall present a construction extending the formalism which we have introduced in the case of Yang-Mills theory.\(^{19}\) This method is based on the geometry of the universal bundle for gravity and unlike other formalisms\(^{7,14}\) it does not require the use of certain homotopy operators. Moreover our method is appropriate to deal also with non parallelizable manifolds. A further advantage of our formalism is that it provides explicit formulae for consistent and covariant anomalies and their corresponding counter terms.

We shall show that covariant gravitational anomalies are related with secondary characteristic classes of a certain pullback bundle of the universal gravity bundle. Furthermore a new set of descent equations for the covariant gravitational anomalies of arbitrary degree in diffeomorphism- and Lorentz ghost can be derived.

As a by-product of our study we shall see that the covariant diffeomorphism anomaly can be viewed as an equivariant momentum map on the space of all metrics. In the case of covariant Yang-Mills anomalies a similar result has already been obtained in Ref. 20.

Finally we find a geometrical realization of the gravitational BRS, anti-BRS multiplet.\(^ {21}\) The covariant descent equations can be formulated in this extended framework and it is shown that the covariant gravitational anomalies satisfy a kind of covariance condition. For pure diffeomorphism anomalies we recover recent results of Ref. 8.

The paper is organized as follows: In Sec. II the universal bundle for gravity is constructed. Its geometry will be discussed in detail and a natural connection in the sense of Ref. 13 will be constructed. In Sec. III, we present the geometrical framework to discuss both diffeomorphisms and local Lorentz transformations and derive the gravitational BRS multiplet. The family of Dirac operators is constructed and by means of the local families index theorem we calculate the curvature of the corresponding determinant line bundle. Then the consistent gravitational anomalies are determined. Sec. IV is devoted to the determination of covariant gravitational anomalies. A new set of descent equations is derived and finally the relation between consistent and covariant anomalies is investigated. A geometrical realization of the gravitational BRS, anti-BRS multiplet is formulated in Sec. V. It is shown that the covariant anomalies fulfill a certain covariance condition.

II. The universal bundle for gravity

Let \(M\) be a compact, connected, orientable, \(n\)-dimensional spin manifold without boundary. Let \(LM^+\) denote the principal bundle of oriented linear frames over \(M\) with structure group \(GL^+(n)\) and projection \(\pi_{LM^+}\). A frame in \(x \in M\) is an isomorphism \(u: \mathbb{R}^n \rightarrow T_x M\). The right action by \(A \in GL^+(n)\) is denoted by \(r_A(u) := u \cdot A\) and the Lie algebras of \(GL^+(n)\) and \(SO(n)\) will be denoted by \(gl(n)\) and \(so(n)\), respectively.

The group of orientation preserving diffeomorphisms of \(M\) will be denoted by \(\mathcal{D}\). Let us consider the group of bundle automorphisms \(Aut(LM^+)\) of \(LM^+\). The
subgroup of $\text{Aut}(LM^+)$ which induces the identity transformation on $M$ is denoted by $\text{Aut}_0(LM^+)$. There is the following exact sequence of groups

$$1 \to \text{Aut}_0(LM^+) \to \text{Aut}(LM^+) \to \mathcal{D} \to 1. \quad (2.1)$$

Define $l: \mathcal{D} \to \text{Aut}(LM^+)$ by $l(\phi)u := T_{\pi(u)} \phi \circ u$ then $\text{Aut}(LM^+)$ can be regarded as a semidirect product of $\mathcal{D}$ and $\text{Aut}_0(LM^+)$, denoted by $\mathcal{D} \ltimes \text{Aut}_0(LM^+)$, relative to the homomorphism $j: \mathcal{D} \to \text{Aut}(\text{Aut}_0(LM^+))$ which is given by

$$j(\phi)(F) := l(\phi) \circ F \circ l(\phi^{-1}), \quad \phi \in \mathcal{D}, \; F \in \text{Aut}_0(LM^+). \quad (2.2)$$

Correspondingly, any vector field $\zeta \in \mathfrak{X}(M)$ can be lifted to a unique $GL^+(n)$ invariant vector field $Y^\zeta \in \mathfrak{X}(LM^+)$ given by $Y^\zeta_u := \frac{d}{dt} |_{t=0} l(\Phi^\zeta_t) u$, where $\Phi^\zeta_t$ is the flow generated by $\zeta \in \mathfrak{X}(M)$. Notice that $Y^\zeta$ can be considered as infinitesimal generator of the left action $(\phi, u) \mapsto l(\phi)u$ of $\mathcal{D}$ on $LM^+$. Therefore one has $[Y^\zeta_1, Y^\zeta_2] = Y^{[\zeta_1, \zeta_2]}$ for all $\zeta_1, \zeta_2 \in \mathfrak{X}(M)$.

Let $S^2T^*M$ denote the space of symmetric covariant tensors then the space of metrics $\mathfrak{M}$ on $M$ is defined to be the set of all sections $\Gamma(S^2T^*M)$ of $S^2T^*M$ which induce a positive inner product in any tangent space $T_xM$. So $\mathfrak{M}$ is an open cone in $\Gamma(S^2T^*M)$ and therefore it is contractible. See Ref. 22 for a detailed analysis of the structure of $\mathfrak{M}$.

In this paper we shall often regard tensor fields as equivariant functions on $LM^+$. Thereby we follow the calculus developed in Ref. 23. Let $T^{p,q}$ be the tensor algebra of $\mathbb{R}^n$. A representation $GL^+(n) \to GL(T^{p,q})$ is given by

$$(A \cdot f)(a_1, \ldots, a_p, v_1, \ldots, v_q) := f(a_1 \circ A, \ldots, a_p \circ A, A^{-1}(v_1), \ldots, A^{-1}(v_q)), \quad (2.3)$$

where $f \in T^{p,q}$, $a_i \in (\mathbb{R}^n)^*$, $v_i \in \mathbb{R}^n$ and $(\mathbb{R}^n)^*$ is the dual space of $\mathbb{R}^n$. Then the space

$$C(LM^+, T^{p,q}) := \{ f: LM^+ \to T^{p,q} | f(u \cdot A) = (A^{-1} \cdot f)(u) \} \quad (2.4)$$

is isomorphic with the space $\mathfrak{T}^{p,q} = \Gamma(\otimes^p TM \otimes \otimes^q T^*M)$ of tensor fields of type $(p, q)$ on $M$. Let $t \in \mathfrak{T}^{p,q}$ then we denote the corresponding equivariant function by $\hat{t}$. Conversely, an equivariant function $\hat{t}$ determines $t$ by the formula

$$t_{\pi(u)}(\alpha_1, \ldots, \alpha_p, \zeta_1, \ldots, \zeta_q) := \hat{t}(u)(\alpha_1 \circ u, \ldots, \alpha_p \circ u, u^{-1}(\zeta_1), \ldots, u^{-1}(\zeta_q)), \quad (2.5)$$

where $\zeta_i \in T_{\pi(u)}M$ and $\alpha_i \in T^*_{\pi(u)}M$. Let $(e_i)$ be the standard basis of $\mathbb{R}^n$ and $(e^i)$ its dual basis, then the components of $f \in C(LM^+, T^{p,q})$ are the equivariant functions

$$f_{i_1 \ldots i_p}^{j_1 \ldots j_q}(u) := f(u)(e^{i_1}, \ldots, e^{i_p}, e_{j_1}, \ldots, e_{j_q}). \quad (2.6)$$

Let $\Gamma(g) \in \Omega^1(LM^+, gl(n))$ be the Levi-Civita connection in $LM^+ \to M$ associated with the metric $g$. Let us recall that $\Gamma(g)$ is uniquely determined by the equations

$$d_{\Gamma(g)} \hat{g} = 0, \quad d_{\Gamma(g)} \varphi = 0, \quad (2.7)$$

where $\varphi \in \Omega^1(LM^+, \mathbb{R}^n)$ is the soldering form, i.e. $\varphi_u(X_u) = u^{-1}(T_u\pi_{LM^+}(X_u))$. Here $d_{\Gamma(g)}$ denotes the covariant exterior derivative on $LM^+$ with respect to $\Gamma(g)$. For any $v \in \mathbb{R}^n$ there exists a unique horizontal (relative to $\Gamma(g)$) vector field $\bar{v} \in \mathfrak{D}$
\( X(LM^+) \) such that \( \varphi_u((\bar{v})_u) = v [31] \). The components of \( d\Gamma(g)\hat{t} = d_{LM^+}\hat{t} + \Gamma(g)\cdot\hat{t} \),

where \( t \in \mathcal{T}^{p,q} \) are given by

\[
(d\Gamma(g)\hat{t})_{j_1\ldots j_p}^{i_1\ldots i_p} \equiv t_{j_1\ldots j_p}^{i_1\ldots i_p} = i_{\bar{e}_k} (d_{LM^+} t_{j_1\ldots j_p}^{i_1\ldots i_p}),
\]

(2.8)

where \( d_{LM^+} \) is the exterior derivative on \( LM^+ \). For \( \phi \in \mathcal{D}, \) (2.7) implies

\[
\Gamma(\phi^*g) = l(\phi)^*\Gamma(g).
\]

(2.9)

Let \( \Omega_{eq,h}(LM^+, T^{p,q}) \) denote the algebra of equivariant (relative to (2.3)) and horizontal (relative to \( \Gamma(g) \)) differential forms on \( LM^+ \) then there is a natural isomorphism\(^{23}\)

\[
\lambda: \Omega^1_{eq,h}(LM^+, T^{p,q}) \rightarrow C(LM^+, T^{p,q+1})
\]

\[
\lambda(f)(u)(a_1, \ldots, a_p, v_1, \ldots, v_{q+1}) := f_u((\bar{v}_{q+1})_u)(a_1, \ldots, a_p, v_1, \ldots, v_q).
\]

(2.10)

Since \( SO(n) \) is the maximal compact subgroup of \( GL(n)^+ \), \( LM^+ \) admits a reduction to a principal \( SO(n) \) bundle. Each reduction corresponds uniquely to a global section in the associated fibre bundle \( LM^+ \times_{GL(n)^+} (GL(n)^+/SO(n)) \) over \( M \) and is in a one to one correspondence with a Riemann metric on \( M^{24} \) But \( LM^+ \times_{GL(n)^+} (GL(n)^+/SO(n)) \) is isomorphic to the fibre bundle \( LM^+/SO(n) \) over \( M \) where frames differing by an orthonormal transformation are identified. Thus we have the identification \( \mathfrak{M} \cong \Gamma(LM^+/SO(n)) \). Note that \( LM^+/SO(n) \) itself is the base of the principal \( SO(n) \) bundle \( LM^+ \xrightarrow{\pi} LM^+/SO(n) \). In the following we shall write \( \tilde{LM}^+ := LM^+/SO(n) \).

So every \( g \in \mathfrak{M} \) induces the pullback bundle \( SO_g(M) := g^*LM^+ \) over \( M \) which covers all frames of \( LM^+ \) which are orthonormal with respect to \( g \). We introduce the evaluation map \( ev: \mathfrak{M} \times M \rightarrow \tilde{LM}^+ \), \( ev(g, x) = [u, \hat{g}(u)] \in LM^+ \times_{GL(n)^+} GL^+(n)/SO(n) \) with \( \pi(u) = x \), where we have identified the homogenous space \( GL^+(n)/SO(n) \) with the space of symmetric transformations of \( \mathbb{R}^n \). Hence \( ev \) induces the pullback bundle

\[
\tilde{\mathfrak{F}} := ev^*LM^+ \rightarrow \tilde{LM}^+
\]

(2.11)

By definition \( \tilde{\mathfrak{F}} = \{(g, x, u) \in \mathfrak{M} \times M \times LM^+| \) u is orthonormal with respect to \( g_x \} \). Obviously, \( \tilde{\mathfrak{F}} \) restricts to \( SO_g(M) \) on \( \{g\} \times M \), i.e. \( \tilde{\mathfrak{F}}|_{\{g\} \times M} \cong SO_g(M) \).

If \( \hat{g} \in C(LM^+, T^{0,2}) \) corresponds to \( g \in \mathfrak{M} \) then we admit an equivalent characterization of \( ev^*LM^+ \) by

\[
\tilde{\mathfrak{F}} = \{(g, u) \in \mathfrak{M} \times LM^+| \) \( \hat{g}(u)(v_1, v_2) = (v_1, v_2)_{\mathbb{R}^n} \) \( \forall v_1, v_2 \in \mathbb{R}^n \},
\]

(2.12)

where \( (,)_\mathbb{R}^n \) denotes the standard scalar product on \( \mathbb{R}^n \). The principal right action is given by \( r_O(\hat{g}, u) := (\hat{g}, u \cdot O) \) with \( O \in SO(n) \). The tangent bundle of \( \tilde{\mathfrak{F}} \) is given by

\[
T\tilde{\mathfrak{F}} = \bigcup_{(g, u) \in \tilde{\mathfrak{F}}} \{(s_g, X_u) \in T_{(g, u)}(\mathfrak{M} \times LM^+)| \) \( s_g(u) = -d_{LM^+} \hat{g}(X_u) \}.
\]

(2.13)
The bundle $\mathfrak{F}$ can also be obtained in the following way: Consider the principal $SO(n)$ bundle $\mathcal{M} \times LM^+ \overset{\bar{\chi}}{\to} \mathcal{M} \times \bar{LM}^+$ with projection $\bar{\pi}(g,u) := (g,[u])$. Here $[u]$ is the equivalence class in $\bar{LM}^+$. Let $\chi: \mathcal{M} \times M \to \mathcal{M} \times \bar{LM}^+$ be defined by $\chi(g,x) := (g,\phi(x))$ then $\chi^*(\mathcal{M} \times LM^+) \cong \mathfrak{F}$.

The vector bundle $\mathcal{M} \times TM \to \mathcal{M} \times M$, which is associated with the principal bundle $\mathcal{M} \times LM^+$, admits a natural metric, given by

$$\nu((0,\zeta^1),(0,\zeta^2)) := g_x((\zeta^1_x,\zeta^2_x)), \quad \zeta^1,\zeta^2 \in TM.$$  \hspace{1cm} (2.14)

Equivalently, $\nu$ can be described by the $GL^+(n)$ equivariant map $\hat{\nu}: \mathcal{M} \times LM^+ \to \mathcal{T}^{0,2}$, defined by $\hat{\nu}(g,u) := \hat{g}(u)$. Hence $\mathfrak{F}$ can be regarded as the bundle of orthonormal frames of $\mathcal{M} \times TM$. Let $i: \mathfrak{F} \hookrightarrow \mathcal{M} \times LM^+$ denote the corresponding embedding.

In the following we restrict to the group $\mathcal{D}_0$ of orientation preserving diffeomorphisms which leave a point and a frame fixed, namely

$$\mathcal{D}_0 := \{ \phi \in \mathcal{D} \mid \phi(x_0) = x_0, \quad T_{x_0}\phi = id_{T_{x_0}M} \}.$$  \hspace{1cm} (2.15)

Then the right action $R^\mathcal{D}_0(g,\phi) := \phi^* g$ of $\mathcal{D}_0$ is free and thus we can consider the principal $\mathcal{D}_0$ bundle $\mathcal{M} \overset{\pi_{\mathcal{D}_0}}{\to} \mathcal{D}_0 \mathcal{M}$ (Notice that $\phi^* g$ corresponds to $l(\phi)^* \hat{g}$ in the equivariant description.)

There is a free right action of $\mathcal{D}_0$ on the two principal bundles $\mathcal{M} \times LM^+ \to \mathcal{M} \times \bar{LM}^+$ and $\mathcal{M} \times LM^+ \to \mathcal{M} \times M$ defined by

$$w_\phi: \mathcal{M} \times LM^+ \to \mathcal{M} \times LM^+, \quad w_\phi(g,u) := (\phi^* g, l(\phi)^{-1} u),$$
$$w'_\phi: \mathcal{M} \times \bar{LM}^+ \to \mathcal{M} \times \bar{LM}^+, \quad w'_\phi(g,[u]) = (\phi^* g, [l(\phi)^{-1} u])$$
$$\bar{w}_\phi: \mathcal{M} \times M \to \mathcal{M} \times M, \quad \bar{w}_\phi(g,x) = (\phi^* g, \phi^{-1}(x)).$$  \hspace{1cm} (2.16)

Finally this action extends to a bundle action on $\mathfrak{F}$. Since

$$w'_\phi \circ \chi(g,x) = (\phi^* g, [l(\phi^{-1} u), \hat{g}(u)]) = (\phi^* g, [l(\phi^{-1} u), l(\phi)^* \hat{g}(l(\phi^{-1}) u)])$$
$$= (\phi^* g, ev(\phi^* g, \phi^{-1}(x))) = \chi \circ \bar{w}_\phi(g,x)$$  \hspace{1cm} (2.17)

the map $\chi$ factorizes to a map $\chi': \mathcal{M} \times_{\mathcal{D}_0} M \to \mathcal{M} \times_{\mathcal{D}_0} \bar{LM}^+$. Taking the quotient by the various $\mathcal{D}_0$ actions we obtain the following commutative diagram:

\[
\begin{array}{ccccccccc}
\mathcal{M} \times LM^+ & \overset{\chi}{\leftarrow} & \mathfrak{F} & \overset{\bar{\pi}}{\longrightarrow} & \mathfrak{F}/\mathcal{D}_0 & \overset{\chi'}{\longrightarrow} & \mathcal{M} \times_{\mathcal{D}_0} LM^+ \\
\downarrow & & \downarrow \pi_{\mathfrak{F}} & & \downarrow \pi_{\mathfrak{F}} & & \downarrow \bar{\pi}' \\
\mathcal{M} \times \bar{LM}^+ & \overset{\chi}{\leftarrow} & \mathcal{M} \times M & \overset{p}{\longrightarrow} & \mathcal{M} \times_{\mathcal{D}_0} M & \overset{\chi'}{\longrightarrow} & \mathcal{M} \times_{\mathcal{D}_0} \bar{LM}^+ & \overset{2.18}{=} & \\
\end{array}
\]

Here $p$, $\bar{p}$ are the projections and $\bar{\chi}$ and $\chi'$ are the canonical bundle maps. Notice that $\mathcal{M} \times_{\mathcal{D}_0} M$, $\mathcal{M} \times_{\mathcal{D}_0} \bar{LM}^+$ and $\mathcal{M} \times_{\mathcal{D}_0} LM^+$ can be considered as fibre bundles.
associated with \( \mathcal{M} \rightarrow \mathcal{M}/\mathcal{D}_0 \), where the fibres are \( M, \widehat{L}M^+ \) and \( LM^+ \), respectively. Obviously one finds the bundle isomorphisms

\[
p^*(\mathfrak{F}/\mathcal{D}_0) \cong \mathfrak{F}, \quad \chi_*(\mathcal{M} \times_{\mathcal{D}_0} LM^+) \cong \mathfrak{F}/\mathcal{D}_0. \tag{2.19}\]

Since \( \mathcal{D}_0 \) acts as an isometry of \( \nu \), this metric factorizes to a metric \( \bar{\nu} \) on \( \mathcal{M} \times_{\mathcal{D}_0} TM \rightarrow \mathcal{M} \times_{\mathcal{D}_0} M \) and hence \( \mathfrak{F}/\mathcal{D}_0 \) is its orthonormal frame bundle. Finally we have the commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{F}/\mathcal{D}_0 & \xrightarrow{i} & \mathcal{M} \times_{\mathcal{D}_0} LM^+\\
\bar{\pi}_{\mathfrak{F}} & \downarrow & \downarrow \pi \\
\mathcal{M} \times_{\mathcal{D}_0} M & \overset{\pi}{\longrightarrow} & \mathcal{M} \times_{\mathcal{D}_0} M
\end{array} \tag{2.20}
\]

where \( i \) is the induced embedding.

In analogy with Ref. 13 we call \( \mathcal{M} \times_{\mathcal{D}_0} LM^+ \) respectively its reduction \( \mathfrak{F}/\mathcal{D}_0 \) the universal bundle for gravity.

In the remainder of this section we want to define a canonical connection on the universal bundle.

Firstly there is a natural Riemannian structure on \( \mathcal{M} \). Each \( g \in \mathcal{M} \) induces a metric \( (\cdot, \cdot)_g \) on \( S^2T^*M \). For \( s^1_g, s^2_g \in T_g\mathcal{M} \cong \Gamma(S^2T^*M) \) we can define an inner product on \( \mathcal{M} \) by

\[
<s^1_g, s^2_g>_g := \int_M (s^1_g, s^2_g)_g \mu_g = \int_M (\hat{g}^{ij} \hat{g}^{kl} (\hat{s}^1_g)_{ik} (\hat{s}^2_g)_{jl}) \mu_g, \tag{2.21}\]

where \( \mu_g \) is the volume form on \( M \) determined by \( \hat{g} \) and the components of \( g \) and \( \hat{s}^1_g, \hat{s}^2_g \) are taken according to (2.6). Notice, that the integrand on the right hand side is \( GL^+(n) \) invariant and thus projects to a well defined function on \( M \).

It was shown \(^{25}\) that \( \mathcal{D}_0 \) acts by isometry, i.e. \( <,> \) is \( \mathcal{D}_0 \) invariant. In order to determine the vertical subbundle in \( \mathcal{M} \rightarrow \mathcal{M}/\mathcal{D}_0 \), let \( R^\mathfrak{M}_g: \mathcal{D}_0 \rightarrow \mathcal{M} \), \( R^\mathfrak{M}_g(\phi) = \phi^*g \) be the orbit map, then

\[
Z^\mathfrak{M}_\zeta(g) = T_idR^\mathfrak{M}_g(\zeta) = \frac{d}{dt}|_{t=0} R^\mathfrak{M}_g(F^\zeta_t) = \frac{d}{dt}|_{t=0} (F^\zeta_t)^*g = L_\zeta g \tag{2.22}\]

is the fundamental vector field generated by \( \zeta = \frac{d}{dt}|_{t=0}F^\zeta_t \in \mathfrak{X}(M) \). Here \( L_\zeta g \) is the Lie derivative of \( g \) with respect to \( \zeta \). Equivalently, one has \( \widetilde{L_\zeta g} = L_Y \hat{g} \) if regarded as element in \( C(LM^+, T^0,2) \). It has been shown in Ref. 25 that the distribution

\[
g \mapsto H_g(\mathcal{M}) = \{ s_g \in \Gamma(S^2T^*M) | < s_g, L_\zeta g >_g = 0, \quad \forall \zeta \in \mathfrak{X}(M) \} \tag{2.23}\]

defines a connection in \( \mathcal{M} \rightarrow \mathcal{M}/\mathcal{D}_0 \). Using (2.8) one finds

\[
<s_g, L_\zeta g >_g = -\int_M \hat{g}^{jk}(\hat{s}_g)_{ij;k} \hat{\zeta}^i \mu_g = < div_g s_g, \zeta >_g, \tag{2.24}\]

where \( \hat{g}^{jk}(\hat{s}_g)_{ij;k} \) are the components of the divergence of \( s_g \) with respect to \( \Gamma(g) \). Thus the horizontal bundle consists of all divergence free, covariant symmetric tensor fields of second degree on \( M \).
The corresponding connection one form $\beta \in \Omega^1(\mathcal{M}, \mathfrak{X}(M))$ is given by

$$\beta_g(s_g) = G_g \circ \text{div}_g(s_g), \quad s_g \in T_g \mathcal{M},$$

(2.25)

where $G_g := (\text{div}_g \circ L_g)^{-1}$. Here $L_g$ denotes the mapping $\mathfrak{X}(M) \to \Gamma(S^2 T^* M)$.

Together with the adjoint action of $\mathcal{D}$ on $\mathfrak{X}(M)$

$$(ad(\phi) \zeta)(x) := T_{\phi^{-1}(x)} \phi (\zeta_{\phi^{-1}(x)}), \quad \zeta \in \mathfrak{X}(M), \ x \in M$$

(2.26)

the transformation property of $\beta$ reads

$$(R^\mathcal{M}_\phi \ast \beta)_g(s_g)(x) = \beta_{\phi^* g}(\phi^* s_g)(x) = T_{\phi(x)} \phi^{-1} (\beta_g(s_g)_{\phi(x)}), \ \phi \in \mathcal{D}_0, \ x \in M.$$  

(2.27)

We now consider the curvature of $\beta$. For vector fields $s^1, s^2$ on $\mathcal{M}$ which are horizontal, i.e. $s^1, s^2 \in \ker(\beta)$, it follows that

$$\mathcal{F}_g(s^1, s^2) = (d\mathfrak{m} + \frac{1}{2}[\beta, \beta])g(s^1, s^2) = -\beta_g([s^1, s^2])_g = -G_g \circ \text{div}_g(s^1, s^2)_g.$$  

(2.28)

For any $\tau \in \Gamma(S^2 T^* M)$ let $\mathcal{F}l^\tau_t(g) = g + t\tau$ denote the flow on $\mathcal{M}$ generated by $\tau$. Then $d_{\Gamma(\mathcal{F}l^\tau_t(g))}\mathcal{F}l^\tau_t(g) = 0, \forall t$, implies

$$d_{\Gamma(g)}\tau = -\Xi_g(\tau) \cdot \dot{g},$$

(2.29)

where the components of $\Xi_g(\tau) \in \Omega^1_{\text{eq}, h}(LM^+, T^{1,1}) \cong C(LM^+, T^{1,2})$ are given by the formula

$$\Xi_g(\tau)^k_{ij} := \left. \frac{d}{dt} \right|_{t=0} \Gamma(g + t\tau)^k_{ij} = \frac{1}{2} \dot{g}^{kl}(\dot{\tau}_{ij} + \ddot{\tau}_{il; j} - \ddot{\tau}_{ij; l}).$$

(2.30)

Notice that an isomorphism $T^{1,1} \cong gl(n)$ is given by the map $A \to A'$, where $A'(a,v) := a(Av)$ with $A \in \text{gl}(n), \ a \in (\mathbb{R}^n)^*$ and $v \in \mathbb{R}^n$. Using the following identity for horizontal vector fields $s^1, s^2$ on $\mathcal{M}$

$$\left. \frac{d}{dt} \right|_{t=0}(\dot{s}^1_{g+ts^2})_{ij;k} = \Xi_g(s^2_g)_{k i}(s^1_g)_{lj} - \Xi_g(s^1_g)_{k j}(s^1_g)_{il},$$

(2.31)

we finally obtain

$$([s^1, s^2]_g)_{i;j} = (s^1_g)_{jk;i}(s^2_g)^{jk} - (s^2_g)_{jk;i}(s^1_g)^{jk} + (s^2_g)^{i k}(s^1_g)_{j k} - (s^1_g)^{i k}(s^2_g)_{j k}. $$

(2.32)

If we regard $[s^1, s^2]_g$ as an element of $\Omega^1_{\text{eq}, h}(LM^+) \cong C(LM^+, T^{0,1})$ then the curvature of $\beta$ reads

$$\mathcal{F}_g(s^1, s^2) = G_g \{ -d_{\Gamma(g)}\dot{s}^2_g, s^1_g > _g - d_{\Gamma(g)}\dot{s}^1_g, s^2_g > _g + s^1_g, d_{LM^+} tr(\dot{s}^2_g) > _g \}

- s^2_g, d_{LM^+} tr(\dot{s}^1_g) > _g \},$$

(2.33)

where $<, > _g$ is the induced inner product in $\Omega(LM^+, T^{p.q})$. 

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Now let $\kappa: gl(n) \times gl(n) \to \mathbb{R}$, $\kappa(A, B) := \text{tr}(A^t B)$ be an inner product on $gl(n)$, where $A^t$ denotes the transpose of $A$. There is a natural Riemann structure on $\mathcal{M} \times LM^+$ given by
\[
\mu((s^1_g, X^1_u), (s^2_g, X^2_u)) := <s^1_g, s^2_g> + (\pi^*(g)u(X^1_u, X^2_u) + \kappa(\Gamma(g)u(X^1_u, \Gamma(g)u(X^2_u))).
\] (2.34)
which induces a Riemannian structure $\mu$ on $\mathfrak{F}$ by $\mu = \bar{\chi}^* \mu$ on $\mathfrak{F}$. This inner product is invariant by $SO(n)$ but not under $GL^+(n)$ transformations in general. Moreover, $\mu$ is $\mathcal{D}$ invariant because for $\phi \in \mathcal{D}$ and $(s^i_g, X^i_u) \in T_{(g,u)}(\mathcal{M} \times LM^+)$, $i = 1, 2$, we find
\[
(w^*_g \mu)((s^1_g, X^1_u), (s^2_g, X^2_u)) = <\phi^*s^1_g, \phi^*s^2_g> + (\pi^*(\phi^*g_u)(T_u l(\phi^{-1}) X^1_u, T_u l(\phi^{-1}) X^2_u)
+ \kappa(\Gamma(\phi^*g_u)(T_u l(\phi^{-1}) X^1_u, \Gamma(\phi^*g_u)(T_u l(\phi^{-1}) X^2_u))
= <s^1_g, s^2_g> + (\pi^*(g_u)(X^1_u, X^2_u) + \kappa(\Gamma(g_u)(X^1_u, \Gamma(g_u)(X^2_u)),-
\] (2.35)
where the invariance of $<,>$ has been used. Thus $\mu$ induces a connection in $\mathcal{M} \times LM^+ \overset{\hat{\alpha}}{\to} \mathcal{M} \times _{\mathcal{D}}^* LM^+$ and moreover gives rise to a Riemannian structure $\bar{\mu}$ on $\mathcal{M} \times _{\mathcal{D}} LM^+$. Then $\bar{\mu} := \bar{\chi}^* \bar{\mu}$ is the corresponding metric on $\mathfrak{F} / \mathcal{D}$.

The vertical subbundle $V^\bar{q}(\mathcal{M} \times LM^+)$ of $\mathcal{M} \times LM^+ \overset{\hat{\alpha}}{\to} \mathcal{M} \times _{\mathcal{D}}^* LM^+$ is generated by fundamental vector fields with respect to the action $w$. In fact, let $\bar{w}_{(g,u)}: \mathcal{D} \to \mathcal{M} \times LM^+$ denote its orbit map, then
\[
T_{id} \bar{w}_{(g,u)}(\zeta) = \frac{d}{dt} \bigg|_{t=0} w_{F\mathcal{L}_t}^\bar{q}(g, u) = (Z^\mathcal{M}_\zeta(g), -Y^\mathcal{M}_\zeta).
\] (2.36)
So we have
\[
V^\bar{q}(\mathcal{M} \times LM^+) = \bigcup_{(g,u) \in \mathcal{M} \times LM^+} \left\{ (L_{Y^\mathcal{M} \zeta} \Tilde{g}, -Y^\mathcal{M}_\zeta) \big| \zeta \in \mathfrak{X}(M) \right\}
\] (2.37)
and furthermore $V^\bar{q}(\mathcal{M} \times LM^+) \cong V^p(\mathfrak{F})$, where the bundle $V^p(\mathfrak{F})$ is the vertical subbundle of the principal $\mathcal{D}$ bundle $\mathfrak{F} \overset{\bar{p}}{\to} \mathfrak{F} / \mathcal{D}$.

The vertical bundle corresponding to $\mathcal{M} \times \bar{L}M^+ \overset{\bar{\alpha}}{\to} \mathcal{M} \times _{\mathcal{D}} \bar{L}M^+$ is given by
\[
V^\bar{q}(\mathcal{M} \times \bar{L}M^+) = \bigcup_{(g,u) \in \mathcal{M} \times \bar{L}M^+} \left\{ (L_{Y^\mathcal{M} \zeta} \Tilde{g}, -T_u \bar{\pi} Y^\mathcal{M}_\zeta) \big| \zeta \in \mathfrak{X}(M) \right\}
\] (2.38)
and finally the vertical bundle of $\mathcal{M} \times M \overset{\Tilde{p}}{\to} \mathcal{M} \times _{\mathcal{D}} M$ reads
\[
V^p(\mathcal{M} \times M) = \bigcup_{(g,x) \in \mathcal{M} \times M} \left\{ (L_{Y^\mathcal{M} \zeta} \Tilde{g}, -\zeta_x) \big| \zeta \in \mathfrak{X}(M) \right\}.
\] (2.39)
Let us denote by $s^h_g$ the horizontal component of $s_g \in T_g \mathcal{M}$ with respect to the connection $\beta$ in (2.25). Then any $(s_g, X_u) \in T_{(g,u)}(\mathcal{M} \times LM^+)$ can be written in the form
\[
(s_g, X_u) = (s^h_g, X_u + Y^\beta_{u(s_g)}) + (L_{Y^\beta_{u(s_g)}} \Tilde{g}, -Y^\beta_{u(s_g)}).
\] (2.40)
Proposition 1. The projector $S: T(\mathcal{M} \times \mathbb{LM}^+) \to T(\mathcal{M} \times \mathbb{LM}^+)$ given by

$$S_{(g,u)}(s_g, X_u) := (s^h_g, X_u + Y^\beta_g(s_g))$$  \hspace{1cm} (2.41)

defines a connection in $\mathcal{M} \times \mathbb{LM}^+ \xrightarrow{\bar{\gamma}} \mathcal{M} \times \mathcal{D}_0 \mathbb{LM}^+$.

Proof. Let us define $H_{(g,u)}(\mathcal{M} \times \mathbb{LM}^+) := \text{im}(S_{(g,u)})$, then (2.40) corresponds to the splitting

$$T_{(g,u)}(\mathcal{M} \times \mathbb{LM}^+) = H_{(g,u)}(\mathcal{M} \times \mathbb{LM}^+) \oplus V_{(g,u)}(\mathcal{M} \times \mathbb{LM}^+).$$  \hspace{1cm} (2.42)

Since

$$T_u l(\phi^{-1}) Y^\beta_g(s_g) = \frac{d}{dt} \big|_{t=0} l(F^a d(\phi^{-1}) Y^\beta_g(s_g)) l(\phi^{-1}) u = Y^{\beta \phi^*}_g(\phi^* s_g)$$

we finally get

$$T_{(g,u)} w_{\phi} \circ S_{(g,u)}(s_g, X_u) = (\phi^* (s^h_g), T_u l(\phi^{-1})(X_u + Y^\beta_g(s_g))$$

$$= ((\phi^* s_g)^h, T_u l(\phi^{-1}) X_u + Y^{\beta \phi^*}_g(\phi^* s_g))$$

$$= S_{w_{\phi}(g,u)} \circ T_{(g,u)} w_{\phi}(s_g, X_u).$$  \hspace{1cm} (2.44)

□

We should remark that $S$ also gives a connection in $\mathcal{G} \to \mathcal{G}/\mathcal{D}_0$ because

$$s^h_g = s_g - L_{Y^\beta_g(s_g)} \hat{\gamma} = -(d_{\mathbb{LM}^+} \hat{\gamma})(X_u + Y^\beta_g(s_g))$$  \hspace{1cm} (2.45)

if $(s_g, X_u) \in T_{(g,u)} \mathcal{G}$ and hence $S_{(g,u)}(s_g, X_u) \in T_{(g,u)} \mathcal{G}$.

The induced metric $\bar{\mu}$ on $\mathcal{M} \times \mathcal{D}_0 \mathbb{LM}^+$ can now be calculated by the formula

$$\bar{\mu}_{\gamma}(T_{(g,u)} q(s^1_g, X^1_u), T_{(g,u)} q(s^2_g, X^2_u)) := \mu_{(g,u)}(S_{(g,u)}(s^1_g, X^1_u), S_{(g,u)}(s^1_g, X^1_u)).$$  \hspace{1cm} (2.46)

Finally the induced $SO(n)$ action on $T(\mathcal{M} \times \mathcal{D}_0 \mathbb{LM}^+)$ is given by

$$r^O_{\gamma}(T_{(g,u)} q(s_g, X_u)) = T_{(g,u)} q \circ S_{(g,u \cdot O)}(s_g, T_u \gamma O(X_u)), \hspace{1cm} O \in SO(n).$$  \hspace{1cm} (2.47)

Proposition 2. The metric $\bar{\mu}$ is $SO(n)$ invariant. Thus it induces a connection in the bundle $\mathcal{M} \times \mathcal{D}_0 \mathbb{LM}^+ \xrightarrow{\bar{\gamma}'} \mathcal{M} \times \mathcal{D}_0 \mathbb{LM}^+$.

Proof. Since $\mu$ is $SO(n)$ invariant we get

$$(r^O_{\gamma} \bar{\mu})_{\gamma}(T_{(g,u)} q(s^1_g, X^1_u), T_{(g,u)} q(s^2_g, X^2_u))$$

$$= \mu_{(g,u \cdot O)}(S_{(g,u \cdot O)}(s^1_g, T_u \gamma O(X^1_u)), T_{(g,u)} q \circ S_{(g,u \cdot O)}(s^2_g, T_u \gamma O(X^2_u)))$$

$$= \mu_{(g,u \cdot O)}(S_{(g,u \cdot O)}(s^1_g, T_u \gamma O(X^1_u)), S_{(g,u \cdot O)}(s^2_g, T_u \gamma O(X^2_u)))$$

$$= (r^O_{\gamma} \mu)_{(g,u)}(S_{(g,u)}(s^1_g, X^1_u), S_{(g,u)}(s^2_g, X^2_u))$$

$$= \mu_{(g,u)}(T_{(g,u)} q(s^1_g, X^1_u), T_{(g,u)} q(s^2_g, X^2_u)).$$  \hspace{1cm} (2.48)

□
As a consequence, $\bar{\mu}_0$ induces a connection in the universal bundle. It is our aim to determine this connection explicitly.

Since $q^* T(\mathcal{M} \times \mathcal{D}_0 \times LM^+) \cong \bigcup_{(g,u) \in \mathcal{M} \times LM^+} H_{(g,u)} (\mathcal{M} \times LM^+)$ we can identify any tangent vector $\tau_{[g,u]}$ of $\mathcal{M} \times \mathcal{D}_0 \times LM^+$ with a vector $(s_g, X_u)$ which is horizontal with respect to $S$.

Let $\tau_{[g,u]} = T_{(g,u)} \bar{q}(s_g, X_u)$ where $(s_g, X_u) \in H_{(g,u)} (\mathcal{M} \times LM^+)$. For $\tau$ being vertical in $\mathcal{M} \times \mathcal{D}_0 \times LM^+ \to \mathcal{M} \times \mathcal{D}_0 \times \tilde{LM}^+$ it follows that

$$T_{(g,u)} \bar{q} (s_g, T_u \tilde{\pi}(X_u)) = 0,$$

(2.49)

where $q : \mathcal{M} \times \tilde{LM}^+ \to \mathcal{M} \times \mathcal{D}_0 \times \tilde{LM}^+$ is the corresponding projection.

This implies that $(s_g, T_u \tilde{\pi}(X_u)) \in V^q \tau_{[g,u]} (\mathcal{M} \times \tilde{LM}^+)$. Therefore $s_g = T_u \tilde{\pi}(X_u) = 0$.

So there exists $\xi \in \mathfrak{so}(n)$ such that $X_u = Z_\xi(u)$, where $Z_\xi$ is the fundamental vector field on $LM^+$ generated by $\xi$. As a result, the vertical bundle is

$$V^{\tilde{\pi}}(\mathcal{M} \times \mathcal{D}_0 \times LM^+) = \bigcup_{[g,u] \in \mathcal{M} \times \mathcal{D}_0 \times LM^+} \{ T_{(g,u)} \bar{q}(0_g, Z_\xi(u)) \mid \xi \in \mathfrak{so}(n) \}.$$  

(2.50)

Now we are able to determine the horizontal bundle $H^{\bar{\mu}}(\mathcal{M} \times \mathcal{D}_0 \times LM^+)$ of the connection which is induced by the metric $\bar{\mu}$. If $\tau_{[g,u]} \in H^{\bar{\mu}}_{[g,u]} (\mathcal{M} \times \mathcal{D}_0 \times LM^+)$, then it must fulfill the following equation

$$\bar{\mu}_{[g,u]} (T_{(g,u)} \bar{q}(0_g, Z_\xi(u)), T_{(g,u)} \bar{q}(s_g, X_u)) = \kappa(\xi, \Gamma(g)u(X_u)) = 0$$

(2.51)

for all $\xi \in \mathfrak{so}(n)$. Here we have represented $\tau_{[g,u]} = T_{(g,u)} \bar{q}(s_g, X_u)$, with $(s_g, X_u) \in H_{(g,u)} (\mathcal{M} \times LM^+)$. Since $\Gamma(g)$ admits a natural decomposition

$$\Gamma(g) = \Gamma^a(g) + \Gamma^s(g) := \frac{1}{2}(\Gamma(g) - \Gamma(g)^t) + \frac{1}{2}(\Gamma(g) + \Gamma(g)^t)$$

(2.52)

into an antisymmetric and a symmetric part with respect to the standard scalar product in $\mathbb{R}^n$, (2.51) implies that

$$\kappa(\xi, \Gamma^a(g)u(X_u)) = 0, \quad \forall \xi \in \mathfrak{so}(n).$$

(2.53)

But $\Gamma^a(g)$ gives a connection in $LM^+ \to \tilde{LM}^+$ whose horizontal bundle will be denoted by $H^\Gamma^a(LM^+)$. So the horizontal bundle we are looking for is given by

$$H^\bar{\mu}(\mathcal{M} \times \mathcal{D}_0 \times LM^+) = \bigcup_{[g,u] \in \mathcal{M} \times \mathcal{D}_0 \times LM^+} \{ T_{(g,u)} q(s_g, X_u) \mid s_g \in H_g(\mathcal{M}), X_u \in H^a_{(g)} u(LM^+) \}.$$  

(2.54)

Let us now define the following connection one form on $\mathcal{M} \times LM^+ \to \mathcal{M} \times \tilde{LM}^+$

$$\epsilon_{(g,u)}(s_g, X_u) = \epsilon_{(g,u)}^{(1,0)}(s_g) + \epsilon_{(g,u)}^{(0,1)}(X_u) = (i_{Y_\xi^s(s_g)} \Gamma^a(g))(u) + \Gamma^a(g)u(X_u).$$

(2.55)
Proposition 3. The connection $\epsilon$ induces a well defined connection $\bar{\epsilon}$ in $\mathcal{M} \times \mathcal{D}_0$ $LM^+ \to \mathcal{M} \times \mathcal{D}_0 \mathcal{M}^+$. In fact, $\bar{\epsilon}$ coincides with the connection induced by $\bar{\mu}$.

Proof. $\epsilon$ is $\mathcal{D}_0$ invariant since
\[
(w^s_{(g,u)}(s_g, X_u) = \epsilon(\phi^*g, \iota(\phi^{-1})u)(T_g R^\mathcal{M}_{\phi}(s_g, T_u \iota(\phi^{-1})X_u)
\]
\[
= \Gamma^s_{\phi^*g}(T_u \iota(\phi^{-1})X_u) + \Gamma^a(\phi^*g)(Y_{(\phi^{-1})u}^\beta \phi^*s_g)
\]
\[
= \epsilon(g, u)(s_g, X_u). \tag{2.56}
\]
Furthermore we have $\epsilon(g, u)(Z^\mathcal{M}(g) - Y^\mathcal{M}(g)) = 0$ implying that $\epsilon$ is basic. Thus there exists a one form $\bar{\epsilon} \in \Omega^1(\mathcal{M} \times \mathcal{D}_0, \mathcal{M}^+, \mathcal{D}(\mathcal{D}_0))$ such that $\bar{\epsilon} \epsilon = \epsilon$. Let $\tau_{[g, u]} = T_{(g, u)} q(s_g, X_u)$ be horizontal with respect to $\bar{\epsilon}$. Because $(s_g, X_u) \in H(g, u)(\mathcal{M} \times \mathcal{M}^+)$ we conclude from
\[
\epsilon_{[g, u]}(\tau_{[g, u]}) = \epsilon_{(g, u)}(s_g, X_u) = \Gamma^a(g, u)(X_u) = 0 \tag{2.57}
\]
that $X_u$ must be horizontal with respect to $\Gamma^a(g)$. □

In fact, $\bar{\chi}^* \epsilon$ agrees with the connection induced by the metric $\bar{\chi}^* \bar{\mu}$ on $\mathfrak{F}/\mathcal{D}_0$. In analogy with ref. 13 this connection will be called the universal connection.

For our purpose it is, however, necessary to have a connection on $\mathcal{M} \times \mathcal{D}_0 \mathcal{M}^+$ which is reducible to a connection on $\mathfrak{F}/\mathcal{D}_0$, i.e. a connection which is compatible with the vertical metric $\bar{v}$.

Therefore let us define the following connection on $\mathcal{M} \times \mathcal{M}^+ \to \mathcal{M} \times \mathcal{M}$ by
\[
\omega(g, u)(s_g, X_u) = \omega^{[1,0]}(g, u)(s_g) + \omega^{[0,1]}(g, u)(X_u) = \Gamma^a(g, u)(Y^\mathcal{D}(s_g) + \rho_g(s_g)(u) + \Gamma^a(g, u)(X_u). \tag{2.58}
\]
Here $\rho$ is a $C(LM^+, \mathcal{T})$ valued one form on $\mathcal{M}$ defined by the components according to (2.6)
\[
\rho_g(s_g)^{(i)}(u) = \frac{1}{2} \hat{g}^{ij}(u)(\hat{s}_g)_{i\beta}(u), \tag{2.59}
\]
where it is summed over the index $j$. Notice, that $\hat{g}^{ij}$ are the components of $\hat{g}^{-1}$.

Proposition 4. The connection $\omega$ induces a connection $\bar{\omega}$ on $\mathcal{M} \times \mathcal{D}_0 \mathcal{M}^+ \to \mathcal{M} \times \mathcal{D}_0 \mathcal{M}$. Furthermore $\omega$ and $\bar{\omega}$ satisfy $i^* \omega = \chi^* \epsilon$ and $i^* \bar{\omega} = \chi^* \epsilon$.

Proof. It is easy to verify that $\omega$ is basic, thus inducing a connection $\bar{\omega}$. For $(s_g, X_u) \in T_{(g, u)} \mathfrak{F}$ one finds from (2.7)
\[
(\hat{s}_g)_{i\beta}(u) = -d^\mathcal{M} g \hat{g}^{ij}(X_u) = \Gamma(g, u)(X_u) \hat{g}^{ik}(u) - \Gamma^a(g, u)(X_u)^{ki} \hat{g}_{kj}(u). \tag{2.60}
\]
Together with $s_g = s_g^h + Lg_{\beta}(s_g) \hat{g}$ and $\hat{g}^{ij}(u) = \delta^{ij}$ one obtains
\[
(i^* \omega)(g, u)(s_g, X_u)^{i\beta}(u) = \Gamma^a(g, u)(X_u) + \Gamma^a(g, u)(Y^\mathcal{D}(s_g)) = (\bar{\chi}^* \epsilon)(g, u)(s_g, X_u), \tag{2.61}
\]
and hence $i^* \bar{\omega} = \chi^* \epsilon$. □

Let $(f^{ts})^{ij}_k := \hat{g}^{ij} f^{ij}_k \hat{g}_{lk}$ denote the transpose of $f \in C(LM^+, \mathcal{T})$ with respect to the metric $g$ and let $f^{as} := 1/2(f - f^{ts})$ be its antisymmetric part. Furthermore let $R(g)$ be the curvature of the Levi Civita connection. According to the product structure of $\mathfrak{M} \times \mathcal{M}^+$ there is a bigrading of the space of differential forms $\Omega(\mathfrak{M} \times \mathcal{M}^+) = \oplus_{i,j \geq 0} \Omega^{(i,j)}$ on it. The exterior derivative splits into $d = d_{\mathfrak{M}} + d_{\mathcal{M}^+}$ where
\[
d_{\mathfrak{M}}: \Omega^{(i,j)} \to \Omega^{(i+1,j)}
\]
\[
d_{\mathcal{M}^+}: \Omega^{(i,j)} \to \Omega^{(i,j+1)},
\]
\[
d_{\mathcal{M}^+} := (-1)^i d_{\mathcal{M}^+}. \tag{2.62}
\]
Proposition 5. The components of the curvature $\Omega_\omega$ of $\omega$ are given by

$$\begin{align*}
\Omega^{(2,0)}_{\omega(g,u)}(s^1_g, s^2_g) &= (i_{Y^g(s^1_g), s^2_g}) \Gamma^a g(g) + i_{Y^g(s^1_g), s^2_g} \Xi^g(s^1_g) - i_{Y^g(s^1_g), s^2_g} \Xi^g(s^2_g) \\
&+ [i_{Y^g(s^1_g), s^2_g}, \Gamma^a g(g)] - i_{Y^g(s^1_g), s^2_g} \Xi^g(s^1_g) - \Xi^g(s^{2}_g) \\
&- [\rho_g(s^1_g), \rho_g(s^2_g)] + [\rho_g(s^1_g), i_{Y^g(s^2_g), \Gamma^a g(g)}] \\
&+ [\rho_g(s^2_g), i_{Y^g(s^1_g), \Gamma^a g(g)}(u)] \\
\Omega^{(1,1)}_{\omega(g,u)}(s_g, X_u) &= (\Xi^g(s_g) - d \Gamma(g)(i_{Y^g(s_g), \Gamma^a g(g)}(u))(X_u) \\
\Omega^{(0,2)}_{\omega(g,u)}(X^1_u, X^2_u) &= R(g)(X^1_u, X^2_u) \quad (2.63)
\end{align*}$$

Proof. Since $s^h_g = s_g - L_{Y^g(s_g)} \hat{g}$, one can write (2.58) in the form

$$\omega_{(g,u)}(s_g, X_u) = \Gamma^a g(g)(Y^g(s_g) + \rho_g(s_g)(u) + \Gamma(g)(u)(X_u). \quad (2.64)$$

The result for the $(0,2)$ component is obvious. The $(1,1)$ component can be calculated from the formula

$$\Omega^{(1,1)}_\omega = d\hat{\omega}^{(0,1)} - dL_{M^+} \omega^{(1,0)} - [\omega^{(0,1)}, \omega^{(1,0)}]. \quad (2.65)$$

Using (2.29) we get

$$d \Gamma(g) \rho_g(s_g)^i_k = \frac{1}{2} (\Xi^i(s_g)^i_k + \Xi^j(s_g)^i_k) \quad (2.66)$$

and hence inserting into (2.65) gives the final result. In order to find the $(2,0)$ component we observe that the following identity holds for any vector fields $s^1, s^2$ on $\mathcal{M}$

$$d\hat{\omega} (i_{Y^g(s^2), \Gamma^a g(g)})(s^1_g) = \Xi^a g(s^1_g)u(Y^g(s^2_g)) + \Gamma^a g(g)(u)Y^g(s^1_g(s^2_g)) + [\rho_g(s^1_g), i_{Y^g(s^2), \Gamma^a g(g)}(u)]. \quad (2.67)$$

Using (2.28) the result follows. $\square$

Now the curvature $\Omega_\omega$ of $\omega$ can be easily calculated by noticing that

$$\Omega_{\omega[g,u]}(\tau^1_{[g,u]}, \tau^2_{[g,u]}) = \Omega_{\omega[g,u]}((s^1_g, X^1_u), (s^2_g, X^2_u)) \quad (2.68)$$

where $\tau^i_{[g,u]} = T_{(g,u)} \hat{g}(s^i_g, X^i_u)$ for $i = 1, 2$ and $(s^i_g, X^i_u) \in H_{(g,u)}(\mathcal{M} \times L M^+)$. Hence the relevant components are given by

$$\begin{align*}
\Omega^{(2,0)}_{\omega(g,u)}(s^1_g, s^2_g) &= (i_{Y^g(s^1_g), s^2_g}) \Gamma^a g(g) - [\rho_g(s^1_g), \rho_g(s^2_g)](u) \\
\Omega^{(1,1)}_{\omega(g,u)}(s_g, X_u) &= \Xi^a g(s_g)u(X_u) \\
\Omega^{(0,2)}_{\omega(g,u)}(X^1_u, X^2_u) &= R(g)(X^1_u, X^2_u). \quad (2.69)
\end{align*}$$

III. Consistent gravitational anomalies

13
A. The gravitational BRS multiplet

In this section we shall construct an appropriate geometrical framework to discuss both the consistent Lorentz and diffeomorphism anomalies. It will be explicitly proved that the consistent gravitational anomalies can be derived using the local index formula of Bismut and Freed.\textsuperscript{11}

Let $\mathcal{G}$ denote the group defined by

$$\mathcal{G} = \{ F \in Aut_0(LM^+) \mid F(u) = u \cdot \hat{F}(u), \hat{F}(u) \in SO(n), \forall u \in LM^+ \}. \quad (3.1)$$

For a fixed metric $g$, the group $\mathcal{G}$ can be identified with the gauge group of $SO_g(M)$.

The mappings $\bar{ev}(F, g, u) = (g, F(u))$ and $\varrho(F, g, x) = (F, ev(g, x))$ make the following diagram commute

$$
\begin{array}{ccc}
\mathcal{G} \times \mathcal{F} & \xrightarrow{\tilde{\varrho}} & \mathcal{G} \times \mathcal{M} \times LM^+ \\
\downarrow \text{id}_G \times \pi_\mathcal{F} & & \downarrow \text{id}_G \times \pi \\
\mathcal{G} \times \mathcal{M} \times M & \xrightarrow{\varrho} & \mathcal{G} \times \mathcal{M} \times \widetilde{LM}^+ \\
\downarrow \pi & & \downarrow \text{pr} \\
\mathcal{M} \times LM^+ & \xrightarrow{\text{pr}} & \mathcal{M} \times \widetilde{LM}^+
\end{array}
$$

where $\tilde{\varrho}$ is the relevant bundle map and $\text{pr}$ is the canonical projection. Thus we find the bundle isomorphism

$$\mathcal{G} \times \mathcal{F} \cong \varrho^*(\mathcal{G} \times \mathcal{M} \times LM^+). \quad (3.3)$$

Let $\Theta \in \Omega^1(\mathcal{G}, \text{Lie}\mathcal{G})$ denote the Maurer Cartan form on $\mathcal{G}$. Any $\mathcal{Y}_F \in T_F\mathcal{G}$ can be written in the form $\mathcal{Y}_F = \mathcal{Y}_{\Theta F}(\mathcal{Y}_F)(F)$, where $\mathcal{Y}_{\Theta F}(\mathcal{Y}_F)$ denotes the left invariant vector field on $\mathcal{G}$ induced by $\Theta_F(\mathcal{Y}_F) \in \text{Lie}\mathcal{G}$. Hence $\mathcal{Y}$ generates the flow $F\mathcal{Y}_F(u) = F(u) \cdot \text{exp} \ t\Theta_F(\mathcal{Y}_F)(u)$.

Let us consider the connection $\tilde{\omega} := \bar{ev}^*\omega$ on $\mathcal{G} \times \mathcal{M} \times LM^+ \rightarrow \mathcal{G} \times \mathcal{M} \times M$. Since

$$T_{(F,g,u)}\bar{ev}(\mathcal{Y}_F, s_g, X_u)) = (s_g, T_u F (X_u + Z\Theta_F(\mathcal{Y}_F)(u))), \quad (3.4)$$

where $Z\Theta_F(\mathcal{Y}_F)$ is the fundamental vector field on $LM^+$ the components of $\tilde{\omega}$ are given by

$$\tilde{\omega}_{(F,g,u)}(\mathcal{Y}_F, s_g, X_u) = \tilde{\omega}^{(1,0,0)}_{(F,g,u)}(\mathcal{Y}_F) + \tilde{\omega}^{(0,1,0)}_{(F,g,u)}(s_g) + \tilde{\omega}^{(0,0,1)}_{(F,g,u)}(X_u)$$

$$= \Theta_F(\mathcal{Y}_F)(u) + F^*(i_{\mathcal{Y}_F}(s_g))\Gamma(g) + \rho(s_g^h)(u) + (F^\ast\Gamma(g))u(X_u). \quad (3.5)$$

It is evident that $\tilde{\omega}$ restricted to $\mathcal{G} \times \mathcal{F}$ coincides with $(\bar{ev} \circ \tilde{\varrho})^*\epsilon$.

We define the semidirect product $\mathfrak{G} := \mathcal{G} \times \mathfrak{D}_0$ with group structure

$$(F_1, \phi_1) \cdot (F_2, \phi_2) := (l(\phi_2^{-1}) \circ F_1 \circ l(\phi_2) \circ F_2, \phi_1 \circ \phi_2). \quad (3.6)$$

Its Lie algebra $\text{Lie}\mathfrak{G} = \text{Lie}\mathcal{G} \ltimes \mathfrak{X}(M)$ is the semidirect product of $\mathfrak{X}(M)$ and $\text{Lie}\mathcal{G}$, where we have the identification

$$\text{Lie}\mathcal{G} = \{ \xi : LM^+ \rightarrow so(n) \mid \xi(u \cdot O) = \text{ad}(O^{-1}) \xi(u), O \in SO(n) \}. \quad (3.7)$$
We define a right action of $\mathfrak{g}$ on $G \times M \times LM^+$ by
\[
\alpha_{(\phi,F')}(F,g,u) := (l(\phi^{-1}) \circ F \circ l(\phi) \circ F', \phi^*g, (l(\phi) \circ F')^{-1}(u)). \tag{3.8}
\]
This action is free and naturally extends to a free action on $G \times \mathfrak{g}$. Furthermore, $\alpha$ commutes with the principal $GL^+(n)$ action and thus induces a free action on $G \times M \times M$ by
\[
\bar{\alpha}_{(\phi,F')} := (l(\phi^{-1}) \circ F \circ l(\phi) \circ F', \phi^*g, \phi^{-1}(x)). \tag{3.9}
\]
Now we can identify $(G \times M \times LM^+)/\mathfrak{g} \cong M \times \mathcal{D}_0 LM^+$ by the map
\[
h: (G \times M \times LM^+)/\mathfrak{g} \to M \times \mathcal{D}_0 LM^+, \quad [F,g,u]_\mathfrak{g} \mapsto [g,F(u)]_{\mathcal{D}_0}, \tag{3.10}
\]
and furthermore we have $(G \times M \times M)/\mathfrak{g} \cong M \times \mathcal{D}_0 M$ provided by
\[
h': (G \times M \times M)/\mathfrak{g} \to M \times \mathcal{D}_0 M, \quad [F,g,u]_\mathfrak{g} \mapsto [g,u]_{\mathcal{D}_0}, \tag{3.11}
\]
where the brackets are the equivalence classes with respect to the actions of $\mathfrak{g}$ and $\mathcal{D}_0$ respectively. We can summarize our constructions in the following commutative diagram:
\[
\begin{array}{ccc}
G \times \mathfrak{g} & \xrightarrow{\bar{f}} & \mathfrak{g}/\mathcal{D}_0 \\
\downarrow{id \times \pi_{\mathfrak{g}}} & & \downarrow{\bar{\pi}} \\
G \times M \times M & \xrightarrow{f} & M \times \mathcal{D}_0 M \\
\downarrow{id \times \pi_{\mathcal{D}_0}} & & \downarrow{id \times \pi_{\mathcal{D}_0}} \\
G \times M & \xrightarrow{id \times \pi_{\mathcal{D}_0}} & M/\mathcal{D}_0 \\
\end{array}
\]
where $f(F,g,x) = [g,x]_{\mathcal{D}_0}$, $\bar{f}(F,g,u) = [g,F(u)]_{\mathcal{D}_0}$ and $\bar{f}' = \bar{f} \circ i$. Obviously, one has the bundle isomorphisms
\[
f^*(\mathfrak{g}/\mathcal{D}_0) \cong G \times \mathfrak{g}, \quad f^*(M \times \mathcal{D}_0 LM^+) \cong G \times M \times LM^+. \tag{3.12}
\]

**Proposition 6.** The connection $\tilde{\omega}$ satisfies $\bar{f}^*\tilde{\omega} = \tilde{\omega}$ and descends to a well defined connection $\bar{\omega}'$ on $(G \times M \times LM^+)/\mathfrak{g} \to (G \times M \times M)/\mathfrak{g}$.

**Proof.** Since $\bar{f} = \tilde{g} \circ \phi \circ \bar{\mathcal{O}}$, the relation between $\tilde{\omega}$ and $\bar{\omega}$ is evident. Because $\bar{\mathcal{O}} \circ \alpha_{(\phi,F')} = \omega \circ \bar{\mathcal{O}}$ we find that $\alpha_{(\phi,F')}^*\omega = \tilde{\omega}$, $\forall (F',\phi) \in \mathfrak{g}$. Let $\bar{\alpha}_{(F,g,u)}(F',\phi) := \alpha_{(\phi,F')}((F,g,u)$ be the orbit map corresponding to $\alpha$. Then the fundamental vector field generated by $(\xi,\zeta) \in Lie\mathfrak{g}$ reads
\[
\bar{\xi}_{(F,g,u)}(\xi,\zeta) := \frac{d}{dt}|_{t=0} \bar{\alpha}_{(F,g,u)}(exp(t\xi),FL_t^\xi). \tag{3.13}
\]
Hence $\tilde{\omega}_{(F,g,u)}((\bar{\xi}_{(F,g,u)}(\xi,\zeta))) = \omega_{(F,u)}(Z^{\mathfrak{m}}_\xi(g)) - Y_F^\zeta = 0$, proving that $\tilde{\omega}$ is basic. $\square$
According to the triple grading of
\[
\Omega(\mathcal{G} \times \mathfrak{M} \times LM^+) = \bigoplus_{i,j,k} \Omega^{(i,j,k)}(\mathcal{G} \times \mathfrak{M} \times LM^+) \tag{3.15}
\]
the exterior derivative can be written in the form \(d = d_{\mathcal{G}} + d_{\mathfrak{M}} + d_{LM^+}\), where
\[
d_{\mathcal{G}} : \Omega^{(i,j,k)} \rightarrow \Omega^{(i+1,j,k)}
\]
\[
d_{\mathfrak{M}} : \Omega^{(i,j,k)} \rightarrow \Omega^{(i,j+1,k)}, \quad d_{\mathfrak{M}} := (-1)^id_{\mathfrak{M}}
\]
\[
d_{LM^+} : \Omega^{(i,j,k)} \rightarrow \Omega^{(i,j,k+1)}, \quad d_{LM^+} := (-1)^{i+j}d_{LM^+}. \tag{3.16}
\]

Using (3.4) we can calculate the curvature \(\tilde{\Omega}\) of \(\tilde{\omega}\)
\[
\Omega_{\tilde{\omega}}(F,g,u)((Y^1_F, s^1_g, X^1_u), (Y^2_F, s^2_g, X^2_u)) = (\tilde{e} \tilde{u}^* \Omega_{\omega})(F,g,u)((Y^1_F, s^1_g, X^1_u), (Y^2_F, s^2_g, X^2_u))
\]
\[
= \Omega_{\omega_{(F,g,u)}}((s^1_g, T_uF(X^1_u), (s^2_g, T_uF(X^2_u))), \tag{3.17}
\]
since \(T_uF(Z_\xi(u))\) is vertical \(\forall \xi \in \text{Lie} \mathcal{G}\). Together with (2.63) one can prove

**Proposition 7.** The components of \(\Omega_{\tilde{\omega}}\) are given by
\[
\Omega_{\tilde{\omega}}^{(2,0,0)} = \Omega_{\tilde{\omega}}^{(1,1,0)} = \Omega_{\tilde{\omega}}^{(1,0,1)} = 0
\]
\[
\Omega_{\tilde{\omega}}^{(0,2,0)}(F,g,u)(s^1_g, s^2_g) = \Omega_{\omega_{(F,g,u)}}(s^1_g, s^2_g)
\]
\[
\Omega_{\tilde{\omega}}^{(0,0,2)}(F,g,u)(X^1_u, X^2_u) = (F^*R(g))(u)(X^1_u, X^2_u)
\]
\[
\Omega_{\tilde{\omega}}^{(0,1,1)}(F,g,u)(s_g, X_u) = F^*(\mathbb{E}^a_s(s_g) - d_{\Gamma(g)}(i_{Y_{s_g}}(s_g)\Gamma^a_g))))u(X_u). \tag{3.18}
\]

Now we shall display the gravitational BRS multiplet for combined Lorentz transformations and diffeomorphisms. For fixed \(g \in \mathfrak{M}\) let us define the map
\[
i_g : \mathcal{G} \times \mathcal{D}_0 \times LM^+ \hookrightarrow \mathcal{G} \times \mathfrak{M} \times LM^+
\]
\[
i_g(F, \phi, u) := (F, \phi^*g, u). \tag{3.19}
\]
From (2.30) one obtains
\[
\Xi_g(Z^\mathfrak{M}_\xi(g)) = \frac{d}{dt}\bigg|_{t=0} \Gamma(Fl^\xi g) = L_{Y_\xi} \Gamma(g). \tag{3.20}
\]
Using the formula
\[
L_X \Gamma(g) = i_X R(g) + d_{\Gamma(g)}(i_X \Gamma(g)) \tag{3.21}
\]
which holds for any \(X \in \mathfrak{X}(LM^+)\) the following result can be established.

**Corollary 8.** The components of \(i^*_g \Omega_{\tilde{\omega}}\) are given by
\[
(i^*_g \Omega_{\tilde{\omega}})^{(2,0,0)} = (i^*_g \Omega_{\tilde{\omega}})^{(1,1,0)} = (i^*_g \Omega_{\tilde{\omega}})^{(1,0,1)} = 0
\]
\[
(i^*_g \Omega_{\tilde{\omega}})^{(0,2,0)}_{(F, \phi, u)}(\xi^1_\phi, \xi^2_\phi) = R(g)(u)(Y^\phi_{\xi^1_\phi}, Y^\phi_{\xi^2_\phi})
\]
\[
(i^*_g \Omega_{\tilde{\omega}})^{(0,0,2)}_{(F, \phi, u)}(X^1_u, X^2_u) = ((l(\phi) \circ F)^*R(g))(u)(X^1_u, X^2_u)
\]
\[
(i^*_g \Omega_{\tilde{\omega}})^{(0,1,1)}_{(F, \phi, u)}(\xi_\phi, X_u) = (i_{Y_{\phi}}(\xi_\phi)R(g))(u)(X_u), \tag{3.22}
\]
where \( c \in \Omega^1(D_0, \mathfrak{X}(M)) \) is the Maurer Cartan form on \( D_0 \) and \( \zeta^i_{\phi} \in T_0 D_0 \) with \( i = 1, 2 \).

In order to derive the BRS structure in our formalism we shall use the following abbreviations:

\[
(i_g^* \bar{\omega})^{(1,0,0)} = \Theta \\
(i_g^* \bar{\omega})^{(0,0,1)} = \Gamma \\
(i_g^* \bar{\omega})^{(0,1,0)} = \delta = d_{\mathcal{G}} + \tilde{d}_{D_0}.
\]

Using (3.22) we finally obtain the system of equations

\[
\begin{align*}
\delta\Gamma &= d\Gamma(\Theta + i_c \Gamma) + i_c R \\
\delta(\Theta + i_c \Gamma) &= i_c i_c R - \frac{1}{2}[\Theta + i_c \Gamma, \Theta + i_c \Gamma] \\
\delta c &= \frac{1}{2} [c, c],
\end{align*}
\]

where the last equation in (3.24) is the Maurer Cartan equation in \( D_0 \). In Ref. 26 these equations have been derived in a different way. Thus we have explicitly verified that the bundle \( \mathfrak{G} \times LM_+ \to \mathfrak{G} \times M \) provides an appropriate geometrical framework to formulate the gravitational BRS relations.

B. Local index formula and consistent descent equations

The quantum version of a classical action which describes the kinematics of fields in an external bosonic background is usually represented by a determinant of a certain family of Dirac operators. Geometrically, the determinant can be viewed as a section of the determinant line bundle associated with the given operator family. In the gravitational case the configuration space and the corresponding Dirac family can be constructed as follows:

The spin structure on \( M \) determines a lift of \( \mathcal{G} \times \tilde{\mathfrak{F}} \) to a principal \( Spin(n) \) bundle \( \mathcal{G} \times \tilde{\mathfrak{F}} \) over \( \mathcal{G} \times \mathfrak{M} \times M \), where \( \tilde{\mathfrak{F}} \) is the double cover of \( \mathfrak{F} \). For fixed metric \( g \), \( \tilde{\mathfrak{F}} \) restricts to the bundle of spin frames \( Spin_g(M) \) on \( M \). Let \( \Delta^\pm \) denote the half-spin representations. For fixed \( (F, g) \in \mathcal{G} \times \mathfrak{M} \), the associated bundles \( \Lambda^\pm = (\mathcal{G} \times \tilde{\mathfrak{F}}) \times_{Spin(n)} \Delta^\pm \) restrict to the bundles of positive and negative spinors over \( M \) respectively. Consider the metric compatible connection \( F^* \Gamma(g) \) on \( LM_+ \) where \( F \in \mathcal{G} \). For fixed \( F \) and metric \( g \), it reduces to a \( SO(n) \) connection on \( SO_g(M) \) and can be uniquely lifted to a connection on \( Spin_g(M) \). The chiral Dirac operators can now be constructed in the usual way

\[
D_{(F,g)}: \Gamma(\Lambda^\pm_{\{(F,g)\} \times M}) \to \Gamma(\Lambda^\mp_{\{(F,g)\} \times M}).
\]

They fit together to form a family of elliptic operators parametrized by \( \mathcal{G} \times \mathfrak{M} \). If we consider only those diffeomorphisms from \( D_0 \) which preserves the spin structure\(^{27}\) then there exists a lifting to an action of \( \mathfrak{G} \) on \( \mathcal{G} \times \tilde{\mathfrak{F}} \). Since the operators (3.25) transform under \( \mathfrak{G} \) transformations according to

\[
(l(\phi) \circ F')^* \circ D_{(F,g)} \circ (l(\phi) \circ F')^{-1} = D_{(l(\phi^{-1}) \circ F \circ l(\phi) \circ F', \phi^* g)},
\]

for \( \phi \in \mathfrak{G} \) and \( F \in \mathcal{G} \), the consistent descent equations hold.

\[3.25\]

\[3.26\]
\{D_{(F,g)}\} is an equivariant family. The action of \(\mathfrak{G}\) is free and therefore one can equivalently work on the quotient family of operators parametrized by \((\mathcal{G} \times \mathcal{M})/\mathfrak{G} \cong \mathcal{M}/\mathcal{D}_0\).

In the following we shall specify the geometrical data according to Ref. 10 in order to calculate the curvature of the determinant line bundle associated with (3.25).

Let us define a connection in the trivial fibre bundle \(\mathcal{M} \times M \rightarrow \mathcal{M}\) by

\[
P: T(\mathcal{M} \times M) \rightarrow T(\mathcal{M} \times M), \quad P_{(g,x)}(s_g, \zeta_x) := (0_g, \zeta_x + \beta_g(s_g)_x). \tag{3.27}
\]

The metric \(\langle \cdot, \cdot \rangle\) lifted to the horizontal subspaces (determined by \(P\)) together with \(\nu\) combine to form a metric \(\nu'\) on \(\mathcal{M} \times M\). Let \(\nabla^{\nu'}\) denote the corresponding Levi-Civita connection, then \(\nabla^{\text{ver}} := P \circ \nabla^{\nu'}\) is a connection on \(\mathcal{M} \times TM\). Explicitly, \(\nabla^{\text{ver}}\) is determined by

\[
2\nu(\nabla^{\text{ver}}_UV, W) = \nu(P(U, V), W) - \nu(P(U), [V, W]) + \nu(P(W, U), V) + U(\nu(V, W)) + V(\nu(P(U), W)) - W(\nu(P(U), V)),
\]

where \(U \in \mathfrak{X}(\mathcal{M} \times M)\) and \(V, W \in \Gamma(\mathcal{M} \times TM)\). With respect to the identification \(\mathcal{M} \times TM \cong (\mathcal{M} \times LM^+) \times_{GL(n)} \mathbb{R}^n\) every section \(V \in \Gamma(\mathcal{M} \times TM)\) can be written in the following form

\[
V(g, x) = (0_g, \bar{V}(g, x)) = (0_g, [u, \bar{V}(g, u)]). \tag{3.29}
\]

**Proposition 9.** The connection \(P\) projects to a well defined connection \(\bar{P}\) on \(\mathcal{M} \times _{\mathcal{D}_0} M \rightarrow \mathcal{M}/\mathcal{D}_0\).

**Proof.** Since \(T(\mathcal{M} \times _{\mathcal{D}_0} M) \cong TM \times_{\mathcal{D}_0} TM\), \(T_{(g,x)}\bar{q}(s_g, \zeta_x) = T_{(g,x)}q(s'_g, \zeta'_x)\) implies that there exists \(\phi \in \mathcal{D}_0\) and \(\bar{\zeta}_\phi \in T_\phi \mathcal{D}_0\) such that

\[
(s'_g, \zeta'_x) = (T_g P^\mathcal{M}_\phi(s_g) + Z^{\mathcal{M}}_{c_\phi(\bar{\zeta}_\phi)}, T_g \phi^{-1}(\zeta_x) - c_\phi(\bar{\zeta}_\phi)\phi^{-1}(x)) \tag{3.30}
\]

and thus

\[
\bar{P}_{(g,x)}(T_{(g,x)}\bar{q}(s_g, \zeta_x)) := T_{(g,x)}q \circ P_{(g,x)}(s_g, \zeta_x) \tag{3.31}
\]

gives a well defined connection on \(\mathcal{M} \times _{\mathcal{D}_0} M \rightarrow \mathcal{M}/\mathcal{D}_0\). \(\Box\)

There is, however, another connection on \(\mathcal{M} \times TM\) which is induced from \(\omega\). In fact, let \(\bar{U}\) denote the horizontal lift of \(U \in \mathfrak{X}(\mathcal{M} \times M)\) with respect to \(\omega\). Then the induced covariant derivative is given by the formula

\[
(\nabla_{\bar{U}}V)(g, x) = (0_g, [u, (i_\bar{U} d_{\mathfrak{M} \times LM^+} \bar{V})(g, u)]). \tag{3.32}
\]

**Proposition 10.** Let \(P\) be the connection defined in (3.27). Then the covariant derivatives \(\nabla^{\text{ver}}\) and \(\nabla\) coincide.

**Proof.** Any \(U \in \mathfrak{X}(\mathcal{M} \times M)\) can be split into the following summands

\[
U_{(g,x)} = (s_{(g,x)}, \zeta_{(g,x)}) = (s^h_{(g,x)}, 0_x) + (L_{(g,s_{(g,x)})} \bar{g}, -\beta_g(s_{(g,x)})) + P_{(g,x)}U_{(g,x)}. \tag{3.33}
\]
If $U \in \Gamma(\mathfrak{M} \times TM)$ then it is obvious from (3.28) and (3.32) that $\nabla^\text{ver}$ and $\nabla$ reduce to the Levi Civita connection on the fibres for the metric $\nu$.

For $U_{(g,x)} = (s_{(g,x)}, 0_x)$, where $s_{(g,x)} \in T_g \mathfrak{M}$ is horizontal $\forall x$, and for $V \in \Gamma(\mathfrak{M} \times TM)$ the combination of the metric on $\mathfrak{M}$ and that the horizontal lift of $U$ is $\tilde{U}_{(g,x)} = (\hat{s}_{(g,x)}, Z\xi(u))$, where $\xi^i_k(u) = -\frac{1}{2}\hat{g}^{ij}(u)(\hat{s}_{(g,x)})_{jk}(u)$ and $\pi(u) = x$. We derive from (3.32)

$$\left< \nabla_U V, g, x \right> = \left( \nu_{(g,x)} \left[ \left[ \frac{d}{dt} \right]_{t=0} V(g + ts_{(g,x)}, x) \right] \right).$$

Since $(L_{\nu} \tilde{V}(g,.))_{ij}(u) = \frac{1}{2}\hat{g}^{ij}(u)(\hat{s}_{(g,x)})_{jk}(u)\tilde{V}^k(g, u)$, where the components are taken according to (2.6), the scalar product of (3.36) and $W$ finally gives (3.35).

For $U_{(g,x)} = Z^{\mathfrak{M} \times M}_{\nu} (s_{(g,x)}), (g, x)$, where $Z^{\mathfrak{M} \times M}$ is the fundamental vector on $\mathfrak{M} \times M$, it is not difficult to see that $\left[ U, V \right]$ is vertical. So (3.28) implies

$$2\nu(\nabla^\text{ver}_U V, W)_{(g,x)} = \nu_{(g,x)} \left[ \left[ \frac{d}{dt} \right]_{t=0} \tilde{V}(g + ts_{(g,x)}, x) \right] \nu_{(g,x)} \left[ \left[ \frac{d}{dt} \right]_{t=0} \tilde{W}(g + ts_{(g,x)}, x) \right] - \nu_{(g,x)} \left[ \left[ \frac{d}{dt} \right]_{t=0} \tilde{V}(g + ts_{(g,x)}, x) \right] \nu_{(g,x)} \left[ \left[ \frac{d}{dt} \right]_{t=0} \tilde{W}(g + ts_{(g,x)}, x) \right]$$

where we have used the fact that $Z^{\mathfrak{M} \times M}_{\nu}$ is a Killing vector field of $\nu$ for arbitrary $\xi \in \mathfrak{X}(M)$.

On the other hand, using the identity

$$\left( L_{Z^{\mathfrak{M} \times M}_{\nu}} \tilde{V} \right)_{(g,u)} = \left( L_{Z^{\mathfrak{M} \times LM+}_{\nu}} \tilde{V} \right)_{(g,u)}$$

and that the horizontal lift of $U$ is $\tilde{U}_{(g,u)} = Z^{\mathfrak{M} \times LM+}_{\nu} (s_{(g,x)})$, where $Z^{\mathfrak{M} \times LM+}$ is the fundamental vector field on $\mathfrak{M} \times LM^+$, one can derive from (3.32)

$$\left( \nabla_U V \right)_{(g,x)} = \left( 0_g, [u, L_{Z^{\mathfrak{M} \times LM+}_{\nu}} \tilde{V}]_{(g,u)} \right) = \left( 0_g, L_{Z^{\mathfrak{M} \times LM+}_{\nu}} \tilde{V} \right)_{(g,x)}.$$  

Taking the scalar product with $W$ finally gives the result. In summary we can conclude $\nabla^\text{ver} = \nabla$. \qed

The metric $\nu'$ projects to a metric $\tilde{\nu}'$ on $\mathfrak{M} \times \mathfrak{D}_0 M$, which, in fact, is given by the combination of the metric on $\mathfrak{M}/\mathfrak{D}_0$ induced by $<,>$ together with $\tilde{\nu}$. Let $\nabla^{\tilde{\nu}'}$ denote its Levi Civita connection and define $\nabla^{\text{ver}} := P \circ \nabla^{\tilde{\nu}'}$. Together with proposition 10 this leads to
Corollary 11. Let $\bar{\nabla}$ be the covariant derivative on $\mathfrak{M} \times \mathfrak{D}_0 TM$ induced by the connection $\tilde{\omega}$ on $\mathfrak{M} \times \mathfrak{D}_0 LM^+ \rightarrow \mathfrak{M} \times \mathfrak{D}_0 M$. Then $\nabla$ and $\bar{\nabla}^{\text{ver}}$ coincide.

Let us now display our geometrical data:

1. The fibre bundle $\mathfrak{M} \times \mathfrak{D}_0 M \rightarrow \mathfrak{M}/\mathfrak{D}_0$ whose tangent bundle along its fibres is $\mathfrak{M} \times \mathfrak{D}_0 TM \rightarrow \mathfrak{M} \times \mathfrak{D}_0 M$ with curvature $\Omega_{\tilde{\omega}}$.
2. The vertical metric $\bar{\nu}$ on $\mathfrak{M} \times \mathfrak{D}_0 TM$.
3. The connection $\bar{\nu}$ on $\mathfrak{M} \times \mathfrak{D}_0 M \rightarrow \mathfrak{M}/\mathfrak{D}_0$ which induces $\nabla$ on $\mathfrak{M} \times \mathfrak{D}_0 TM$.
4. Additionally, one may consider a representation $\tau$ of Spin($n$) in order to construct vector bundles associated to $\mathfrak{M} \times \mathfrak{D}_0 LM^+$ to which the Dirac operators (3.25) couple. Examples are the Rarita Schwinger operator and the signature operator which corresponds to chiral spin $3/2$ fermions and self dual antisymmetric tensor fields respectively.

The curvature $\Omega^{(L)}$ of the determinant line bundle $L \rightarrow \mathfrak{M}/\mathfrak{D}_0$ associated with the equivariant family $\{ D_{(F,g)} | (F,g) \in G \times \mathfrak{M} \}$ can now be calculated by means of the local index formula

$$\Omega^{(L)} = \left[ 2\pi i \int_M \hat{A}(\Omega_{\tilde{\omega}}) \ ch(\tau(\Omega_{\tilde{\omega}})) \right] \in \Omega^2(\mathfrak{M}/\mathfrak{D}_0, \mathbb{R}) \quad (3.40)$$

where the 2-form component of the differential form on $\mathfrak{M}/\mathfrak{D}_0$ is taken. Here $\hat{A}$ and $ch$ are the usual polynomials

$$\hat{A}(\Omega_{\tilde{\omega}}) = \left( \text{det} \left( \frac{\Omega_{\tilde{\omega}}/4\pi}{\sinh(\Omega_{\tilde{\omega}}/4\pi)} \right) \right)^{1/2}, \quad ch(\Omega_{\tilde{\omega}}) = \text{tr}(e^{i\tau(\Omega_{\tilde{\omega}})}/2\pi). \quad (3.41)$$

Let $I^m(GL(n))$ denote the space of ad invariant, symmetric multilinear real valued functions on $gl(n)$ of degree $m$ where we have chosen $2m - 2 = n$. Let $Q \in I^m(GL(n))$ be given in such a way that the integrand of (3.40) can be written in the form

$$\overline{Q(\Omega_{\tilde{\omega}})} := \hat{A}(\Omega_{\tilde{\omega}}) \ ch(\tau(\Omega_{\tilde{\omega}})) \in \Omega^{2m}(\mathfrak{M} \times \mathfrak{D}_0 M, \mathbb{R}^n), \quad (3.42)$$

where the $2m$ form component of the integrand has been taken. The form $\overline{Q(\Omega_{\tilde{\omega}})}$ is uniquely determined by $Q(\Omega_{\tilde{\omega}}) \in \Omega^{2m}(\mathfrak{M} \times \mathfrak{D}_0 LM^+)$ with $\pi^* \overline{Q(\Omega_{\tilde{\omega}})} = Q(\Omega_{\tilde{\omega}})$.

In the next step we want to determine the descent equations for combined consistent Lorentz- and diffeomorphism anomalies within our formalism. We choose a fixed connection $B$ on $LM^+ \rightarrow M$ and extend it to a connection on $G \times \mathfrak{M} \times LM^+$ where we denote it with the same symbol. Let $\Omega_B \in \Omega^{(0,0,2)}$ be its curvature. Since $Q(\Omega_B) = 0$ because of dimensional reasons the descent equations for consistent gravitational anomalies can be derived from the transgression formula

$$Q(\Omega_{\tilde{\omega}}) = Q(\Omega_{\tilde{\omega}}) - Q(\Omega_B) = d_{G \times \mathfrak{M} \times LM^+} TQ(\tilde{\omega}, B)$$

$$= d_{G \times \mathfrak{M} \times LM^+} \left( m \int_0^1 dt \ Q(\tilde{\omega} - B, \tilde{\Omega}_t, \ldots, \tilde{\Omega}_t) \right), \quad (3.43)$$

where $TQ(\tilde{\omega}, B) \in \Omega^{2m-1}(G \times \mathfrak{M} \times LM^+)$ is the secondary characteristic form and $\tilde{\Omega}_t$ is the curvature of the interpolating connection $\tilde{\omega}_t = t\tilde{\gamma} + (1-t)B$. For $\alpha$ being
For fixed $(n, \ldots, R)$ component of $\tilde{\Omega}$ we define $\tilde{\Omega}_t = t\Omega + \frac{1}{2}(t^2 - t)[\alpha, \alpha]$. Explicitly, the components of $\tilde{\Omega}_t$ read

\[
\tilde{\Omega}_t^{(2,0)}_{(F,g,u)}(Y_F, Y_F) = \frac{t^2 - t}{2}[\Theta_F, \Theta_F](Y_F, Y_F)(u)
\]

\[
\tilde{\Omega}_t^{(0,2)}_{(F,g,u)}(s_g, s_g) = t\Omega^{(2,0)}_{\omega(g,F(u))}(s_g, s_g) + (t^2 - t)
\times F^*[i_{Y^\rho(g(s_g))}^\rho g + \rho_g(s_g)^i, i_{Y^\rho(g(s_g))}^\rho g](u)
\]

\[
\tilde{\Omega}_t^{(0,1)}_{(F,g,u)}(X_u, s_g) = (t^2 - t)[\Theta_F Y_F](u), (F^*\Gamma(g) - B)u(X_u)
\]

\[
\tilde{\Omega}_t^{(0,1)}_{(F,g,u)}(s_g, X_u) = t\Omega^{(1,1)}_{\omega(g,F,u)}(s_g, X_u) + (t^2 - t)[\omega^{(1,0)}(s_g), (F^*\Gamma(g) - B)u(X_u)].
\]

(3.44)

Notice that $TQ(\tilde{\omega}, B)$ is a basic form, i.e. it descends to a form $TQ(\tilde{\omega}, B)$ on $G \times \mathfrak{m} \times M$. With respect to the grading of $\Omega(G \times \mathfrak{m} \times LM^+)$ (3.43) can be written in the form

\[
(Q(\omega) - Q(\Omega_B))^{(i,j,2m-i-j)} = d_g TQ(\tilde{\omega}, B)^{(i-1,j,2m-i-j)} + \hat{d}_{LM} TQ(\tilde{\omega}, B)^{(i,j-1,2m-i-j)}
\]

\[+ \hat{d}_{LM} TQ(\tilde{\omega}, B)^{(i,j,2m-i-j-1)}.\]

(3.45)

Now we can apply the transgression procedure.\(^{14}\) Let $Q(\Omega_\omega)$, $Q(\Omega_B)$ denote the projections of $Q(\Omega_\omega)$ respectively $Q(\Omega_B)$ on $G \times \mathfrak{m} \times M$. Using proposition 6 and (3.40) the transgression of $\Omega(\mathcal{L})$ gives

\[
\pi^*_G \times \mathfrak{m} \Omega(\mathcal{L}) = 2\pi i \int_M f^*Q(\Omega_\omega) = 2\pi i \int_M Q(\Omega_B)
\]

\[= 2\pi i \int_M [Q(\Omega_\omega) - Q(\Omega_B)] = d_g \times \mathfrak{m} \left(2\pi i \int_M TQ(\tilde{\omega}, B)\right).
\]

(3.46)

For fixed $g \in \mathfrak{m}$ the pullback of (3.46) through $i_g$ yields

\[
d_\mathcal{L} \left(2\pi i \int_M i_g^* TQ(\tilde{\omega}, B)\right) = 0,
\]

(3.47)

which can be regarded as the consistency condition for the consistent anomaly. Actually, the fibre integral in (3.47) projects onto the $(1, n)$ component of $i_g^* TQ(\tilde{\omega}, B)$ with respect to the bigrading of $\Omega^{n+1}(G \times LM^+)$. Using (3.44), (3.20) and (3.21), we obtain for the combined consistent gravitational anomaly

\[
i_g^* TQ(\tilde{\omega}, B)^{(1,2m-2)}(\tilde{\xi}, \tilde{\zeta}) = m \int_0^1 dt (Q(\xi + t\gamma, \Gamma(g), R(g)t, \ldots, R(g)t)
\]

\[+ (m-1)(t^2 - t)Q(\Gamma(g) - B, [\xi + t\gamma, \Gamma(g), \Gamma(g) - B], R(g)t, \ldots, R(g)t)
\]

\[+ (m-1)tQ(\Gamma(g), i_Y R(g), R(g)t, \ldots, R(g)t))(u),
\]

(3.48)
where \((\xi, \zeta) \in \text{Lie}\mathfrak{G}\). This result agrees in form with previous calculations done in Refs. 14,29.

In summary we have seen that the local index theorem determines the expression for the consistent gravitational anomaly.

**IV. Determination of covariant gravitational anomalies**

In this section we want to derive descent equations for covariant Lorentz- and diffeomorphism anomalies. In particular, we intend to examine their geometrical structure.

We define the following connection on \(\mathcal{G} \times \mathfrak{M} \times LM^+\) by

\[
\eta_{(F,g,u)}(\mathcal{Y}_F, s_g, X_u) = (F^*\Gamma(g))_u(X_u).
\]  

(4.1)

This connection is \(\mathfrak{G}\) invariant but it does not induce a connection on the universal bundle.

**Proposition 12.** The curvature \(\Omega_{\eta}\) of \(\eta\) has the following components:

\[
\Omega_{\eta}^{(2,0,0)} = \Omega_{\eta}^{(0,2,0)} = \Omega_{\eta}^{(1,1,0)} = 0
\]

\[
\Omega_{\eta}^{(0,0,2)}(X_u^1, X_u^2) = (F^*R(g))_u(X_u^1, X_u^2)
\]

\[
\Omega_{\eta}^{(1,0,1)}(\mathcal{Y}_F, X_u) = (d_{F*}\Gamma(g)\Theta_F(\mathcal{Y}_F))_u(X_u)
\]

\[
\Omega_{\eta}^{(0,1,1)}(s_g, X_u) = (F^*\Xi_g(s_g))_u(X_u).
\]  

(4.2)

**Proof.** Only the \((1,0,1)\) component of \(\Omega_{\eta}\) remains to be calculated. Using (3.21) this follows from

\[
(d_{\mathcal{G}} \eta_{(\mathcal{G},g,u)})_F(\mathcal{Y}_F)(X_u) = \frac{d}{dt}|_{t=0}((\exp t\Theta_F(\mathcal{Y}_F))^*F^*\Gamma(g))_u(X_u)
\]

\[
= (L_{Z\Theta_F(\mathcal{Y}_F)}F^*\Gamma(g))_u(X_u) = (d_{F*}\Gamma(g)\Theta_F(\mathcal{Y}_F))_u(X_u).
\]  

(4.3)

\[
\square
\]

Let \(Q \in \mathcal{I}^m(GL^+(n))\), then the covariant descent equations can be obtained from the transgression formula

\[
Q(\Omega_{\mathcal{G}}) - Q(\Omega_{\eta}) = d_{\mathcal{G} \times \mathfrak{M} \times LM^+} TQ(\tilde{\omega}, \eta).
\]  

(4.4)

The covariant anomaly enters the calculation concerning consistent gravitational anomalies if in the transgression formula (3.46) \(\pi_{\mathcal{G} \times \mathfrak{M}}^\ast \Omega^{(\mathcal{L})}\) is replaced with \(\pi_{\mathcal{G} \times \mathfrak{M}}^\ast \Omega^{(\mathcal{L})} - \int_M Q(\Omega_{\eta})\).

With respect to the triple grading of \(\Omega(\mathcal{G} \times \mathfrak{M} \times LM^+)\) we obtain from (4.4)

\[
(Q(\bar{\Omega}) - Q(\Omega_{\eta}))^{(i,j,2m-i-j)} = d_{\mathcal{G} \times \mathfrak{M} \times LM^+} TQ(\tilde{\omega}, \eta)^{(i-1,j,2m-i-j)} + d_{\mathfrak{M}} TQ(\tilde{\omega}, \eta)^{(i,j-1,2m-i-j)} + d_{LM^+} TQ(\tilde{\omega}, \eta)^{(i,j,2m-i-j-1)}.
\]  

(4.5)
Using (4.3) and

\[ (dG \tilde{\omega}^{(0,1,0)}_{(g,u)})(\Omega_F)(s_g) = \frac{d}{dt}|_{t=0} \omega^{(1,0)}_{(g,F,l_t(F),u)}(s_g) \]

\[ = -[\Theta_F(\Omega_F)(u), (i_{\gamma}(s_g) \Gamma^a(g) + \rho_g(s_g))(u)] \]  

(4.6)

the components of the curvature of the interpolating connection \( \gamma_t = t\tilde{\omega} + (1-t)\eta \) are given by

\[
\begin{align*}
\Omega_t^{(2,0,0)}(\Omega_F, \Omega_F) &= \frac{t(t-1)}{2} [\Theta, \Theta](\Omega_F, \Omega_F)(u) \\
\Omega_t^{(0,2,0)}(s_g, s_g) &= \tilde{\Omega}_t^{(0,2,0)}(s_g, s_g) \\
\Omega_t^{(0,0,2)}(X_u, X_u) &= (F^* R(g))_u(X_u, X_u) \\
\Omega_t^{(1,1,0)}(\Omega_F, s_g) &= t(t-1)[\Theta_F(\Omega_F)(u), \omega^{(1,0)}_{(g,F(u))}(s_g)] \\
\Omega_t^{(1,0,1)}(\Omega_F, X_u) &= (1-t)(d_{F^*} \Gamma(g) \Theta_F(\Omega_F))_u(X_u) \\
\Omega_t^{(0,1,1)}(s_g, X_u) &= F^* (\Xi(s_g) - td_{\Gamma(g)}(\omega^{(1,0)}_{(g,\cdot)}(s_g)))_u(X_u).
\end{align*}
\]  

(4.7)

If we restrict (4.4) to a \( \mathfrak{g} \) orbit through \((id_G, g)\) and consider the decomposition according to the bigrading of \( \Omega(\mathfrak{g} \times LM^+) \) we find the following system of descent equations

\[
(Q(i_g \Omega) - Q(i_g \Omega))^{(k,2m-k)} = d_{\mathfrak{g}} TQ(i_g^* \tilde{\omega}, i_g^* \eta)^{(k-1,2m-k)} + \hat{d}_{LM^+} TQ(i_g^* \tilde{\omega}, i_g^* \eta)^{(k,2m-k-1)}
\]  

(4.8)

Explicitly, we find for the non-integrated combined covariant anomaly

\[
TQ(i_g^* \tilde{\omega}, i_g^* \eta)^{(1,2m-2)}(id_{\mathfrak{g}}, u) = m Q(\Theta + i_Y \Gamma(g), R(g), \ldots, R(g))(u).
\]  

(4.9)

For \( k = 1 \) the descent system (4.8) gives

\[
0 = i_g^*(Q(\tilde{\Omega}) - Q(\Omega^c))^{(1,n+1)} = \hat{d}_{LM^+} TQ(i_g^* \tilde{\omega}, i_g^* \eta)^{(1,n)}.
\]  

(4.10)

So the covariant anomaly induces a class \([TQ(i_g^* \tilde{\omega}, i_g^* \eta)]^{(1,n)}\) in \( H^{(1,n)}(\mathfrak{g} \times M, \mathbb{R}) \). This result is similar to that for the covariant Yang-Mills anomaly.\^{19,30} In fact, the form \( TQ(i_g^* \tilde{\omega}, i_g^* \eta)^{(1,n)} \) is local in the sense of Ref. 30, i.e. it is polynomial in \( \Gamma(g) \), \( R(g) \) and linear in the ghost \( \Theta + i_Y \Gamma(g) \). Thus it induces a class in the local de Rham cohomology of \( LM^+ \) with one ghost.

Additionally, the non-integrated covariant Schwinger term is given by

\[
TQ(i_g^* \tilde{\omega}, i_g^* \eta)^{(2,2m-1)}(id_{\mathfrak{g}}, u) = m(m-1) Q(\Theta + i_Y \Gamma(g), i_Y R(g), R(g)) \ldots, R(g))
\]

\[
+ m(m-1) \frac{2}{2} Q(\Theta(i_Y \Gamma(g), d_{\Gamma(g)}(\Theta + i_Y \Gamma(g)), R(g)) \ldots, R(g)).
\]  

(4.11)

Notice that the first term on the right hand side of (4.11) can be regarded as a basic \( n \) form on \( LM^+ \) and thus it disappears if it is restricted to a \( n - 1 \) dimensional submanifold of \( M \).
Evidently, pure covariant diffeomorphism anomalies are obtained by considering only forms of the type $TQ(i_g^*\tilde{\omega}, i_g^*\eta)^{(0,j,2m-j-1)}$. In order to make contact with previous calculations for $j = 1$, we notice that $L_{Y^C}\varphi = 0$ implies

\begin{equation}
(i_{Y^C}\Gamma(g))\varphi = d_{\Gamma(g)}(i_{Y^C}\varphi).
\end{equation}

Let $\hat{\zeta} \in C(LM^+, T^{1,0})$ be the equivariant function corresponding to $\zeta \in \mathfrak{X}(M)$. Its components are given by

\begin{equation}
\hat{\zeta}^i(u) = e^i(u^{-1}(\zeta_{\pi_{LM^+}}(u))) = e^i(i_{Y^C}\varphi)(u) = (i_{Y^C}\varphi)^i(u).
\end{equation}

Application of $\lambda$ (2.10) on both sides of (4.12) and using (4.13) give

\begin{equation}
(i_{Y^C}\Gamma(g))^i_k = \hat{\zeta}^i_{;k}
\end{equation}

Hence we obtain

\begin{equation}
TQ(i_g^*\tilde{\omega}, i_g^*\eta)^{(0,1,n)}(i_{\id_{\varphi},u}) = m Q(d_{\Gamma(g)}\hat{\zeta}, R(g), \ldots, R(g)).
\end{equation}

This result has been found in Refs. 4,8,14 by other methods.

Let us now briefly comment on covariant Lorentz anomalies. Let us fix a metric $g$. Consider the map $\tilde{\gamma}_g: \mathcal{G} \times SO_g(M) \to \mathcal{G} \times \mathfrak{m} \times LM^+$, defined by $\tilde{\gamma}_g(F, u) = (F, g, u)$. Since $Q(\tilde{\gamma})(i,0,2m-i) = 0$, if $i \neq 0$, (4.5) reduces to

\begin{equation}
-Q(\tilde{\gamma}_g^*\Omega)(i,2m-i) = d_g TQ(\tilde{\gamma}_g^*\tilde{\omega}, \tilde{\gamma}_g^*\eta)^{(i-1,2m-i)} + \tilde{\gamma}_g^*SO_g(M) TQ(\tilde{\gamma}_g^*\tilde{\omega}, \tilde{\gamma}_g^*\eta)^{(i,2m-i-1)},
\end{equation}

which gives the set of descent equations for pure covariant Lorentz anomalies of arbitrary ghost degree. The Levi Civita connection becomes $so(n)$ valued if restricted to $SO_g(M)$. In fact, $\Gamma(g)$ coincides with $\Gamma^a(g)$ on the subbundle $SO_g(M)$ for fixed $g$. Explicitly, the covariant Lorentz anomaly is given by

\begin{equation}
TQ(\tilde{\gamma}_g^*\tilde{\omega}, \tilde{\gamma}_g^*\eta)^{(1,2m-2)}(\xi) = m Q(\xi, R_a(g), \ldots, R_a(g))(u),
\end{equation}

where $R_a(g)$ is the curvature of $\Gamma^a$ restricted to $SO_g(M)$. This result has also be derived by path integral methods.\(^3\)

The statement that pure consistent and covariant Lorentz anomalies can also be obtained on the same ground as gauge anomalies in Yang-Mills theory is now easy to understand. In fact, under the assumption from above, $\mathcal{G} = Aut_0(SO_g(M))$ and (3.2) can be reduced to the commutative diagram

\begin{equation}
\begin{array}{ccc}
Aut_0(SO_g(M)) \times SO_g(M) & \overset{ev_g}{\longrightarrow} & LM^+\\
\downarrow & & \downarrow \\
Aut_0(SO_g(M)) \times M & \overset{\bar{ev}_g}{\longrightarrow} & \bar{LM}^+
\end{array}
\end{equation}

with $ev_g(F, u) = F(u)$ and $\bar{ev}_g(F, x) = g(x)$.

The connections

\begin{equation}
\omega^g_{(F,u)}(\mathcal{Y}_F, X_u) := (ev_g^*\Gamma^a(g))(\mathcal{Y}_F, X_u) = \Theta_F(\mathcal{Y}_F)(u) + (F^*\Gamma^a(g))_u(X_u)
\end{equation}

\begin{equation}
\eta^g_{(F,u)}(\mathcal{Y}_F, X_u) := (F^*\Gamma^a(g))_u(X_u)
\end{equation}
on $\text{Aut}_0(\text{SO}_g(M)) \times \text{SO}_g(M) \rightarrow \text{Aut}_0(\text{SO}_g(M)) \times M$, where $\Gamma^a(g)$ is restricted to $\text{SO}_g(M)$, can be used to derive the consistent and covariant descent equations for pure Lorentz anomalies. So they can be seen as gauge anomalies for $\text{SO}_g(M)$. For the consistent case this viewpoint has been adopted in Ref. 16.

Let us now determine the counter terms which relate consistent and covariant gravitational anomalies. These counter terms for anomalies of arbitrary ghost degree can be found by considering the identity \(^{31}\)

\[
TQ(\tilde{\omega}, \eta) = TQ(\tilde{\omega}, B) - TQ(\eta, B) + d_G \otimes \mathfrak{g} \otimes \mathfrak{LM}^+ \otimes S_Q(B, \eta, \tilde{\omega}), \tag{4.20}
\]

where $S_Q(B, \eta, \tilde{\omega}) = m(m-1) \int_{t_1 + t_2 \leq 1} dt_1 dt_2 Q(\eta - B, \tilde{\omega} - B, \Omega_{t_1}^{(1)}, \ldots, \Omega_{t_2}^{(1)})$. Here $\Omega_{t_j}^{(1)}$ is the curvature of $B + t_1(\eta - B) + t_2(\tilde{\omega} - B)$. One should remark that $S_Q$ is a basic form and thus projects to a form on $\mathfrak{G} \otimes \mathfrak{M} \otimes M$. Finally we identify the counter terms with

\[
\lambda(\tilde{\omega}, \eta, B)^{(i,j,2m-(i+j+1)} = TQ(\eta, B)^{(i,j,2m-(i+j+1)} - d_G S_Q(B, \eta, \tilde{\omega})^{(i-1,j,2m-(i+j+1))} - d_{\mathfrak{M}} S_Q(B, \eta, \tilde{\omega})^{(i,j-1,2m-(i+j+1))} - d_{\mathfrak{LM}} S_Q(B, \eta, \tilde{\omega})^{(i,j,2m-(i+j+2)).} \tag{4.21}
\]

To be explicit we calculate the counter term for $k = 1$. The components of the curvature $(\Omega_{t})_t$ of $t\eta + (1-t)B$ are given by

\[
(\Omega_{t})_t^{(2,0,0)} = (\Omega_{t})_t^{(0,2,0)} = (\Omega_{t})_t^{(1,1,0)} = 0
\]

\[
(\Omega_{t})_t^{(0,0,2)}(X^1_u, X^2_u) = (F^* R(g)_t + (\Omega_B)_{(1-t)} + t(1-t)[F^* \Gamma(g), B])_u(X^1_u, X^2_u)
\]

\[
(\Omega_{t})_t^{(1,0,1)}(F, g, u)(\Psi_F, X_u) = t(d_{\mathfrak{F}^*} (\Psi_F)_{(1-t)} + t(1-t)[F^* \Xi(g), B])_u(X_u)
\]

\[
(\Omega_{t})_t^{(0,1,1)}(s, g, u) = t(F^* \Xi(g)_t)_u(X_u).
\]

So up to an exact form, the counter term for $k = 1$ reads

\[
TQ(\tilde{\eta}, B)^{(1,2m-2)} = m \int_0^1 dt tQ(F^* \Gamma(g) - B, F^* \Xi(s)_t + d_{\mathfrak{F}^*} (\Psi_F)_{(1-t)} + t(1-t)[F^* \Xi(g), B])_u(X_u).
\]

This term is the gravitational analogue of the Bardeen Zeeino counter term in Yang-Mills case. \(^4\)

Before closing this section we want to give another geometrical interpretation of the covariant diffeomorphism anomaly. This will be an extension of a study of covariant Yang-Mills anomalies in terms of presymplectic geometry. \(^{32}\) We begin with an investigation of the properties of covariant gravitational anomalies under $\mathfrak{G}$ transformations. For $\vartheta \in \Omega^1(\mathfrak{G} \otimes \mathfrak{M} \otimes \mathfrak{LM}^+) \otimes \mathfrak{G}$ transformations. For $\vartheta \in \Omega^1(\mathfrak{G} \otimes \mathfrak{M} \otimes \mathfrak{LM}^+)$ we define

\[
\kappa(\vartheta)(F, g, u)(\xi, \zeta, X_u) := \vartheta(F, g, u)(\mathfrak{Y}^\text{left}_\xi(F), Z^\text{LM}_\zeta(g), X_u).
\]

\[
4.24
\]
Lemma 13. Let \( \hat{\alpha}(\phi, F')(F, g) := (l(\phi^{-1}) \circ F \circ l(\phi) \circ F', \phi^*g) \) then

\[
\kappa(\hat{\omega})(\hat{\alpha}(\phi, F') (F, g), u)(\xi, \zeta, X_u)) = \kappa(\hat{\omega})(F, g, l(\phi) \circ F') u((l(\phi) \circ F')^{-1} \ast \xi, \text{ad}(\phi) \zeta, T_u(l(\phi) \circ F') X_u) \quad (4.25)
\]

and the formula is also true for \( \eta \).

Let us define the integrated covariant gravitational anomaly of ghost degree \( k \) by

\[
\mathcal{A}^{(k)}_{(F, g)}(\xi, \zeta) := \int_N \left( i_g \overrightarrow{TQ}(\omega, \eta)^{(k, 2m - k)} \right)_{(F, \text{id}_{\mathcal{D}_0})} (\mathcal{Q}^{\text{left}}_{\xi}(F), \zeta), \quad (4.26)
\]

where \( N \) is an appropriate \( 2m - k \) dimensional submanifold of \( M \). Then it is obvious from the above Lemma that the following formula holds:

\[
\mathcal{A}^{(k)}_{(\hat{\alpha}(\phi, F') (F, g))}(\xi, \zeta) = \mathcal{A}^{(k)}_{(F, g)}((l(\phi) \circ F')^{-1} \ast \xi, \text{ad}(\phi) \zeta), \quad (4.27)
\]

Let us write \( \tilde{\mathcal{A}}(g, \zeta) = \mathcal{A}^{(1)}_{(\text{id}_{\mathcal{D}_0}, g)}(0, \zeta) \) for the pure covariant diffeomorphism anomaly. The closed 2-form on \( \mathcal{M} \)

\[
\kappa_g(s^1_g, s^2_g) = \int_M Q(\Omega_g)^{(0, 2, n)}(s^1_g, s^2_g) = m(m - 1) \int_M Q(\Xi_g(s^1_g), \Xi_g(s^2_g), R(g), \ldots, R(g)) \quad (4.28)
\]

is \( \mathcal{D}_0 \) invariant and defines a presymplectic structure on \( \mathcal{M} \). Notice that \( \kappa \) is also exact. This property is just a consequence of the transgression formula \( Q(\Omega_g) = Q(\Omega_B) = \partial_{g \times \mathcal{M} \times LM^+} TQ(\eta, B) \) and does not depend on the topology of \( \mathcal{M} \). However, \( \kappa \) is the differential of the gravitational Bardeen Zumino term \( (4.23) \) for pure diffeomorphism anomalies.

A direct computation, using \( d_{\mathcal{M}} R(g) = d_{\Gamma(g)} \Xi \), finally gives

\[
i_{\mathcal{K}} \kappa = -d_{\mathcal{M}} \tilde{\mathcal{A}}(\zeta). \quad (4.29)
\]

According to \( (4.27) \), \( \tilde{\mathcal{A}} \) transforms equivariantly under the adjoint action of \( \mathcal{D}_0 \). Hence the covariant diffeomorphism anomaly is an equivariant momentum map\(^{32}\) for the \( \mathcal{D}_0 \) action on the presymplectic manifold \( \mathcal{M} \). The form \( \kappa \) also represents the obstruction for \( \tilde{\mathcal{A}} \) to fullfil a gravitational analogue of the Wess Zumino consistency condition. This is the content of the descent system \( (4.5) \) for the values \( i = 0 \), \( j = 1 \).

V. Gravitational BRS, anti-BRS algebra and the covariance condition

In this section we want to derive a covariance condition for the combined covariant Lorentz- and diffeomorphism anomalies. The first step will be the construction of a geometrical realization of the gravitational BRS, anti-BRS algebra.\(^{21}\)

Let us consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{G} \times \mathcal{D}_0 \times \mathcal{G} \times \mathcal{M} \times LM^+ & \xrightarrow{\psi} & \mathcal{M} \times LM^+ \\
\hat{\pi} & \downarrow & \pi & \downarrow & \pi \\
\mathcal{G} \times \mathcal{D}_0 \times \mathcal{G} \times M \times M & \xrightarrow{\tilde{\psi}} & \mathcal{M} \times M
\end{array}
\]

(5.1)
where $\psi(F_1, \phi, F_2, g, u) := (g, (F_2 \circ F_1 \circ l(\phi))(u))$ and $\bar{\psi}$ is the induced map. Let $\mathcal{B}$ denote the bundle on the left hand side of (5.1). This bundle is a certain pull-back bundle of the universal bundle. In fact, let us consider a right $\mathfrak{G} \times \mathfrak{G}$ action on $\mathcal{B}$, given by

$$
(F_1, \phi, F_2, g, u) \mapsto ((F_1, \phi) \cdot (F', \phi'), \alpha(\bar{F}, \bar{\phi})(F_2, g, u)),
$$

(5.2)

where $((F', \phi'), (\bar{F}, \bar{\phi})) \in \mathfrak{G} \times \mathfrak{G}$. Then $\mathcal{B} / (\mathfrak{G} \times \mathfrak{G})$ is isomorphic with the universal bundle.

The space of differential forms $\Omega$ on $\mathcal{G} \times \mathcal{D}_0 \times \mathcal{G} \times \mathcal{M} \times \mathcal{M}^+$ admits a fivefold grading. We define a connection on $\mathcal{B}$ by $\hat{\omega} := \psi^* \omega$. Since

$$
T(\psi(F_1, \zeta, \phi, Y F_2, s g, X u) = \bigg( s g, T_u (F_2 \circ F_1) \{ T_u l(\phi)(X u + Y u^c(\zeta_\phi)) + Z_{\Theta F_1}(Y F_1) + ad(F_1^{-1})\Theta F_2(Y F_2)(u) \} \bigg)
$$

the components of $\hat{\omega}$ are given by

$$
\hat{\omega}_{(F_1, \phi, F_2, g, u)}(\zeta_\phi) = \big( (F_2 \circ F_1 \circ l(\phi))^* (g)(u) \big)(\zeta_\phi)$$

(5.5)

Using (5.4) and (2.62) the components of the curvature $\hat{\Omega}$ of $\hat{\omega}$ read

$$
\hat{\Omega}_{(2,0,0,0,0)} = \hat{\Omega}_{(1,1,0,0,0)} = \hat{\Omega}_{(1,0,1,0,0)} = \hat{\Omega}_{(1,0,0,1,0)} = \hat{\Omega}_{(1,0,0,0,1)}
$$

$$
= \hat{\Omega}_{(0,1,1,0,0)} = \hat{\Omega}_{(0,0,1,1,0)} = \hat{\Omega}_{(0,0,0,1,1)} = 0
$$

$$
\hat{\Omega}_{(0,2,0,0,0)}(\zeta_\phi) = \big( (F_2 \circ F_1 \circ l(\phi))^* R(g) \big)(u) \big( Y u^c(\zeta_\phi), Y u^c(\zeta_\phi) \big)
$$

$$
\hat{\Omega}_{(0,0,2,0)}(s g, s g) = \hat{\Omega}_{(0,0,0,2)}(s g, s g) = \Omega_{(g, (F_2 \circ F_1 \circ l(\phi))(u))}(s g, s g)
$$

$$
\hat{\Omega}_{(1,0,0,1,1)}(s g, X u) = \big( (F_2 \circ F_1 \circ l(\phi))^* \Xi_{s g}(s g) - d_{\Gamma(g)}(i_{Y s g} \Gamma_{s g}) \big)(u) \big( Y u^c(\zeta_\phi), X u \big)
$$

$$
\hat{\Omega}_{(0,1,0,1,1)}(s g, X u) = \big( (F_2 \circ F_1 \circ l(\phi))^* \Xi_{s g}(s g) - d_{\Gamma(g)}(i_{Y s g} \Gamma_{s g}) \big)(u) \big( Y u^c(\zeta_\phi), X u \big)
$$

$$
\hat{\Omega}_{(0,0,0,1,1)}(s g, X u) = \big( (F_2 \circ F_1 \circ l(\phi))^* \Xi_{s g}(s g) - d_{\Gamma(g)}(i_{Y s g} \Gamma_{s g}) \big)(u) \big( Y u^c(\zeta_\phi), X u \big)
$$

(5.6)
Now we can derive the BRS, anti-BRS relations for Lorentz transformations and diffeomorphisms. Let \( j_g : \mathcal{G} \times \mathcal{D}_0 \times \mathcal{G} \times \mathcal{D}_0 \times \mathcal{M} \times \mathcal{M}^+ \rightarrow \mathcal{G} \times \mathcal{D}_0 \times \mathcal{G} \times \mathcal{M} \times \mathcal{M}^+ \), defined by \( j_g(F_1, \phi_1, F_2, \phi_2, u) = (F_1, \phi_1, F_2, \phi_2 g, u) \) denote the embedding through \( g \in \mathcal{M} \). We shall use the following abbreviations

\[
\begin{align*}
(j_g^* \omega)^{(1,0,0,0,0)} & = \Theta \\
(j_g^* \omega)^{(0,1,0,0)} & = \bar{\Theta} \\
(j_g^* \omega)^{(0,0,1,0)} & = i_c \bar{\epsilon} \\
(j_g^* \omega)^{(0,0,0,1)} & = i_c \epsilon \\
(j_g^* \omega)^{(0,0,0,0)} & = i_c \epsilon \\
\delta := d_{\mathcal{G}}^{(1)} + d_{\mathcal{D}_0}^{(1)} & = d_{\mathcal{G}}^{(2)} + \bar{d}_{\mathcal{D}_0}^{(2)},
\end{align*}
\]  

where \( d_{\mathcal{D}_0}^{(1)} \) and \( d_{\mathcal{D}_0}^{(2)} \) are the exterior derivatives on \( \mathcal{D}_0 \) with respect to the first and second copy of \( \mathcal{D}_0 \) respectively. By considering \( j_g^* \hat{\Omega} \) the following relations can be obtained.

\[
\begin{align*}
d_{\mathcal{G}}^{(2)} \Theta & = 0, \quad d_{\mathcal{G}}^{(1)} \bar{\Theta} + [\Theta, \bar{\Theta}] = 0 \\
\delta(\Theta + i_c \epsilon) & = -\frac{1}{2}[\Theta + i_c \epsilon, \Theta + i_c \epsilon] + i_c i_c R \\
\bar{\delta}(\bar{\Theta} + i_c \bar{\epsilon}) & = -\frac{1}{2}[\bar{\Theta} + i_c \bar{\epsilon}, \bar{\Theta} + i_c \bar{\epsilon}] + i_c i_c R \\
\delta(\Theta + i_c \epsilon) + \bar{\delta}(\bar{\Theta} + i_c \bar{\epsilon}) & = -[\Theta + i_c \epsilon, \bar{\Theta} + i_c \bar{\epsilon}] + i_c i_c R,
\end{align*}
\]  

The equations in the first line of (5.8) reflect the BRS, anti-BRS structure in Yang-Mills theory. However, if the combined transformations (Lorentz and diffeomorphism) are considered the derivatives \( \delta \) and \( \bar{\delta} \) act symmetrically. Thus we have seen that \( \mathcal{B} \) provides an appropriate geometrical framework to realize the gravitational BRS, anti-BRS multiplet.

In this extended framework the calculation of consistent anomalies can be carried out if in the relevant formulas of Sec. III, \( \hat{\omega} \) is replaced with \( \hat{\omega} \).

In order to study the covariant case we define another connection on \( \mathcal{B} \) by

\[
\begin{align*}
\tilde{\eta}^{(0,0,1,0,0)} & = \tilde{\eta}^{(0,0,0,1,0)} = 0 \\
\tilde{\eta}^{(1,0,0,0,0)}_{(F_1, \phi, F_2, g, u)} (Y_{F_1}) = (l(\phi)^* \Theta_{F_1}(Y_{F_1}))(u) \\
\tilde{\eta}^{(0,1,0,0,0)}_{(F_1, \phi, F_2, g, u)} (\zeta_\phi) = ((F_2 \circ F_1 \circ l(\phi))^* \Gamma(g))_u(Y_u^{c_\phi}(\zeta_\phi)) \\
\tilde{\eta}^{(0,0,0,0,1)}_{(F_1, \phi, F_2, g, u)} (X_u) = ((F_2 \circ F_1 \circ l(\phi))^* \Gamma(g))_u(X_u).
\end{align*}
\]  

The components of the corresponding curvature can now been easily calculated.
They read
\[
\Omega_{t}^{(2,0,0,0,0)} = \Omega_{t}^{(0,0,2,0,0)} = \Omega_{t}^{(0,0,0,2,0)} = \Omega_{t}^{(1,0,1,0,0)} = \Omega_{t}^{(1,0,0,1,0)} = \Omega_{t}^{(1,0,0,0,1)} = \Omega_{t}^{(0,1,0,0,1)} = \Omega_{t}^{(0,1,0,1,0)} = \Omega_{t}^{(0,0,1,0,1)} = \Omega_{t}^{(0,0,0,1,1)} = 0
\]
\[
\Omega_{t}^{(0,2,0,0,0)}(\zeta_{\phi}^{1}, \zeta_{\phi}^{2}) = ((F_{2} \circ F_{1} \circ l(\phi))^* R(g))_{u}(Y_{u}^{c_{\phi}^{1}}(\zeta_{\phi}^{1}), Y_{u}^{c_{\phi}^{2}}(\zeta_{\phi}^{2}))
\]
\[
\Omega_{t}^{(0,0,2,0,0)}(\zeta_{\phi}, Y_{F_{2}}) = ((F_{1} \circ l(\phi))^* d_{F_{2}}^{*} T_{2}(g) \Theta_{F_{2}}(Y_{F_{2}}))_{u}(Y_{u}^{c_{\phi}}(\zeta_{\phi}))
\]
\[
\Omega_{t}^{(0,1,1,0,0)}(\zeta_{\phi}, Y_{F_{2}}, X_{u}) = ((F_{1} \circ l(\phi))^* d_{F_{2}}^{*} T_{2}(g) \Theta_{F_{2}}(Y_{F_{2}}))_{u}(X_{u})
\]
\[
\Omega_{t}^{(0,1,0,0,0)}(\zeta_{\phi}, s_{g}) = ((F_{2} \circ F_{1} \circ l(\phi))^* \Xi_{g}(s_{g}))_{u}(Y_{u}^{c_{\phi}}(\zeta_{\phi}))
\]
\[
\Omega_{t}^{(0,0,1,0,0)}(s_{g}, X_{u}) = ((F_{2} \circ F_{1} \circ l(\phi))^* \Xi_{g}(s_{g}))_{u}(X_{u})
\]
\[
\Omega_{t}^{(0,1,0,0,0)}(\zeta_{\phi}, X_{u}) = ((F_{2} \circ F_{1} \circ l(\phi))^* R(g))_{u}(Y_{u}^{c_{\phi}}(\zeta_{\phi}), X_{u}).
\]

We proceed like in Sec. IV. Let \( \gamma_{t} = t \hat{\omega} + (1 - t) \hat{\eta} \) be the interpolating connection then its curvature admits the following components
\[
\Omega_{t}^{(2,0,0,0,0)} = \Omega_{t}^{(1,0,1,0,0)} = \Omega_{t}^{(1,0,0,1,0)} = \Omega_{t}^{(1,0,0,0,1)} = \Omega_{t}^{(1,0,0,0,0)} = 0
\]
\[
\Omega_{t}^{(0,2,0,0,0)}(\nu_{F_{1}}, \nu_{F_{2}}) = \frac{t(t - 1)}{2}((F_{1} \circ l(\phi))^* ([\Theta, \Theta](\nu_{F_{1}}, \nu_{F_{2}}))(u)
\]
\[
\Omega_{t}^{(0,0,2,0,0)}(s_{g}, s_{g}) = \hat{\Omega}_{t}^{(2,0,0)}(s_{g}, s_{g})
\]
\[
\Omega_{t}^{(0,2,0,0,0)}(\zeta_{\phi}, \zeta_{\phi}^{2}) = ((F_{2} \circ F_{1} \circ l(\phi))^* R(g))_{u}(Y_{u}^{c_{\phi}}(\zeta_{\phi}^{1}), Y_{u}^{c_{\phi}}(\zeta_{\phi}^{2}))
\]
\[
\Omega_{t}^{(0,0,0,0,0)}(s_{g}, X_{u}) = ((F_{2} \circ F_{1} \circ l(\phi))^* R(g))_{u}(X_{u})
\]
\[
\Omega_{t}^{(0,1,1,0,0)}(s_{g}, Y_{F_{2}}) = t(t - 1)((F_{1} \circ l(\phi))^* \Theta_{F_{2}}(Y_{F_{2}}), F_{2}^{*}\omega_{(g_{\nu})}^{(1,0)}(s_{g}))(u)
\]
\[
\Omega_{t}^{(0,1,0,0,0)}(\zeta_{\phi}, Y_{F_{2}}) = (1 - t)((F_{1} \circ l(\phi))^* (d_{F_{2}}^{*} T_{2}(g) \Theta_{F_{2}}(Y_{F_{2}})))_{u}(Y_{u}^{c_{\phi}}(\zeta_{\phi}))
\]
\[
\Omega_{t}^{(0,1,0,0,0)}(s_{g}, X_{u}) = ((F_{2} \circ F_{1} \circ l(\phi))^* \Xi_{g}(s_{g}))_{u}(X_{u})
\]
\[
\Omega_{t}^{(0,1,0,1,0)}(\zeta_{\phi}, s_{g}) = ((F_{2} \circ F_{1} \circ l(\phi))^* \Xi_{g}(s_{g}))_{u}(Y_{u}^{c_{\phi}}(\zeta_{\phi}))
\]
\[
\Omega_{t}^{(0,0,1,1,0)}(s_{g}, X_{u}) = ((F_{2} \circ F_{1} \circ l(\phi))^* \Xi_{g}(s_{g}))_{u}(X_{u})
\]
\[
\Omega_{t}^{(0,1,0,0,0)}(\zeta_{\phi}, X_{u}) = ((F_{2} \circ F_{1} \circ l(\phi))^* R(g))_{u}(Y_{u}^{c_{\phi}}(\zeta_{\phi}), X_{u}).
\]

Let us now consider the corresponding transgression formula
\[
Q(\hat{\Omega}) - Q(\Omega_{t}) = d_{G} \times D_{0} \times G \times \mathfrak{M} \times LM^{+} TQ(\hat{\omega}, \hat{\eta}),
\]
which according to the product structure of \( G \times D_{0} \times G \times \mathfrak{M} \times LM^{+} \) admits a decomposition into the following system of descent equations
\[
(Q(\hat{\Omega}) - Q(\Omega_{t}))^{(i,j,k,l,s)} = d_{G}^{(1)} TQ(\hat{\omega}, \hat{\eta})^{(i-1,j,k,l,s)} + d_{D_{0}} TQ(\hat{\omega}, \hat{\eta})^{(i,j-1,k,l,s)}
\]
\[
+ d_{G}^{(2)} TQ(\hat{\omega}, \hat{\eta})^{(i,j,k-1,l,s)} + d_{\mathfrak{M}} TQ(\hat{\omega}, \hat{\eta})^{(i,j,k,l-1,s)}
\]
\[
+ d_{LM^{+}} TQ(\hat{\omega}, \hat{\eta})^{(i,j,k,l,s-1)},
\]
where \( s = 2m - (i + j + k + l) \). From (5.6), (5.10) and (5.11) it is obvious that
\[
Q(\hat{\Omega})(i,j,k,l,2m-(i+j+k+l)) = Q(\Omega_\eta)(i,j,k,l,2m-(i+j+k+l))
\]
\[
= TQ(\hat{\omega}, \hat{\eta})(i,j,k,l,2m-(i+j+k+l+1)) = 0, \quad i \neq 0. \quad (5.14)
\]

A comparison of (5.11) with (4.7) gives
\[
TQ(j_g^* \hat{\omega}, j_g^* \hat{\eta})^{(0,0,i,j,2m-i-j-1)}(\phi, \eta)_{(id_g, iD_0, F, \phi, u)} = TQ(i_g^* \hat{\omega}, i_g^* \eta)^{(i,j,2m-i-j-1)}(F, \phi, u).
\]

Thus we have recovered the usual expression for the covariant anomalies. Setting \( i = 1 \) we obtain from (5.13)
\[
d_g^{(1)} \delta TQ(\hat{\omega}, \hat{\eta})(0,j,k,l,2m-(j+k+l+1)) = 0. \quad (5.16)
\]

If the pull-back by \( j_g \) is taken, (5.16) can be viewed as strong covariance condition for combined Lorentz- and diffeomorphism anomalies under Lorentz transformations.

Using (5.6), (5.10) and (5.11) a lengthy calculation gives
\[
i_{Y_{\phi}(\zeta_\phi)} TQ(\hat{\omega}, \hat{\eta})^{(0,0,k,l,2m-(k+l+1))} = i_{\zeta_\phi} TQ(\hat{\omega}, \hat{\eta})^{(0,1,k,l,2m-(k+l+2))}
\]
\[
i_{Y_{\phi}(\zeta_\phi)} Q(\hat{\Omega})^{(0,0,k,l,2m-(k+l))} = i_{\zeta_\phi} Q(\hat{\Omega})^{(0,1,k,l,2m-(k+l+1))}
\]
\[
i_{Y_{\phi}(\zeta_\phi)} Q(\Omega_\eta)^{(0,0,k,l,2m-(k+l))} = i_{\zeta_\phi} Q(\Omega_\eta)^{(0,1,k,l,2m-(k+l+1))}. \quad (5.17)
\]

Here \( i_{Y_{\phi}(\zeta_\phi)} \) is the substitution operator on \( LM^+ \) whereas \( i_{\zeta_\phi} \) denotes the substitution operator on \( D_0 \). If we apply \( i_{Y_{\phi}(\zeta_\phi)} \) on (5.13) for index values \( i = 0, j = 0 \) and compare the result with the result obtained by applying \( i_{\zeta_\phi} \) on (5.13) for index values \( i = 0, j = 1 \) we find
\[
i_{\zeta_\phi} d_{D_0} TQ(\hat{\omega}, \hat{\eta})^{(0,0,k,l,2m-(k+l+1))} = (-1)^{k+l} L_{Y_{\phi}(\zeta_\phi)} TQ(\hat{\omega}, \hat{\eta})^{(0,0,k,l,2m-(k+l+1))}. \quad (5.18)
\]

If we combine (5.16) and (5.18) and take the pull-back by \( j_g \) we obtain
\[
\delta TQ(j_g^* \hat{\omega}, j_g^* \hat{\eta})^{(0,0,k,l,2m-(k+l+1))} = (-1)^{k+l} L_{Y_{\phi}(\zeta_\phi)} TQ(j_g^* \hat{\omega}, j_g^* \hat{\eta})^{(0,0,k,l,2m-(k+l+1))}. \quad (5.19)
\]

Eq. (5.19) is the strong covariance condition for combined covariant diffeomorphism- and Lorentz anomalies. For pure diffeomorphism anomalies this condition agrees with a corresponding one which has previously been derived in Ref. 8. Hence we have found a geometrical interpretation of this condition in terms of the geometry of the bundle \( B \).

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