TORIC DEGENERATIONS AND TROPICAL GEOMETRY OF BRANCHING ALGEBRAS

CHRISTOPHER MANON

Abstract. We construct polyhedral families of valuations on the branching algebra of a morphism of reductive groups. This establishes a connection between the combinatorial rules for studying a branching problem and the tropical geometry of the branching algebra. In the special case when the branching problem comes from the inclusion of a Levi subgroup or a diagonal subgroup, we use the dual canonical basis of Lusztig and Kashiwara to build toric deformations of the branching algebra.

Contents

1. Introduction 1
2. Valuations and tropical geometry 9
3. Algebraic constructions from branching problems 13
4. Degenerations and the dual canonical basis 18
5. Degenerations of $R(\delta_n)$ and $R(i_L,G)$ 23
6. Tensor products in type $A$ 24
References 25

Keywords: branching problem, reductive group, tropical geometry, toric variety, valuation.

1. INTRODUCTION

For every map $\phi : H \to G$ of connected reductive groups over $\mathbb{C}$, we may regard the representations of $G$ as representations of $H$ via its action through $\phi$. For any irreducible representation $V(\lambda)$ of $G$ there is a direct sum decomposition into irreducible $H$ representations.

\begin{equation}
V(\lambda) = \bigoplus_{\beta \in \Delta_H} \text{Hom}_H(V(\beta), V(\lambda)) \otimes V(\beta)
\end{equation}

The vector spaces $\text{Hom}_H(V(\beta), V(\lambda))$ determine the restriction functor $\phi^* : \text{Rep}(G) \to \text{Rep}(H)$ on the categories of finite dimensional representations of $H$ and $G$, and a calculation of the dimensions of these spaces is called a branching rule associated to $\phi$. Recall that equivalence classes of irreducible representations of $G$ are in bijection with the lattice points of convex rational cone $\Delta_G$, a Weyl chamber of $G$.

University of California, Berkeley, chris.manon@math.berkeley.edu, This work was supported by the NSF fellowship DMS-0902710.
Solving the branching problem can be seen as assigning non-negative integers, the dimensions of the above spaces, to the dominant weights in $\Delta_H \times \Delta_G$. This leads naturally to questions about the nature of the sets of points where these numbers are non-zero. One general approach to this problem is to bring in techniques from commutative algebra by integrating the spaces $\text{Hom}_H(V(\beta), V(\lambda))$ into a commutative algebra, $R(\phi)$. This algebra is constructed from the algebra of invariants by a maximal unipotent $U^+ \subset G$ in the coordinate ring $\mathbb{C}[G]$. As a representation of $G$, this algebra is the multiplicity free sum of the irreducible representations of $G$.

This algebra is finitely generated by the representations associated to the fundamental weights in $\Delta_G$. Multiplication can be computed component-wise by the Cartan product, which is projection onto the highest weight component of the tensor product.

**Definition 1.1.** For $\phi : H \to G$ a map of reductive groups, the branching algebra for $\phi$ is defined as the invariant ring of $R_H \otimes R_G$ by $H$, or equivalently as the $U_H$-invariants of $R_G$, where $U_H \subset H$ is a maximal unipotent subgroup.

$$R_G = \mathbb{C}[G]^U = \bigoplus_{\lambda \in \Delta_G} V(\lambda)$$

(2)

It is worth mentioning two special cases of this construction. For $\phi = \text{id}$, the algebra $R(\text{id})$ is the subalgebra of highest weight vectors in $R_G$, in other words it is the toric algebra attached to the semigroup of dominant weights of $G$, $\mathbb{C}[\Delta]$. For the unity morphism $i_G : 1 \to G$, the algebra is just $R_G$. For all $\phi$, $R(\phi)$ is finitely generated and graded by the monoid of dominant weights $\Delta_H \times \Delta_G$, and the non-zero branching numbers for $\phi : H \to G$ correspond to the non-zero graded components of $R(\phi)$, we call this submonoid $C(\phi) \subset \Delta_H \times \Delta_G$. Since $R(\phi)$ is finitely generated, the monoid $C(\phi)$ is finitely generated as well, this implies that its non-negative real saturation is a convex rational cone in $\Delta_H \times \Delta_G$.

The cone $C(\phi)$ itself can be difficult to understand so in order to get more information about the polyhedral and algebraic properties of $C(\phi)$ it helps to find a cone $D$ which can be understood more easily, and a surjection $D \to C(\phi)$. Conceptually, the cone $D$ ”enriches” the branching structure described by $C(\phi)$, and meaningful $D$ correspond to combinatorial rules which solve or simplify the branching problem, such as the Littlewood-Richardson rule or the Pieri rule in type $A$. Throughout this paper we will describe two ways a cone like $D$ can be constructed, how to combine these constructions, and some of the resulting algebraic and geometric implications for $R(\phi)$. The principle driving our constructions is that different meaningful $D$, arising from different combinatorial rules for understanding branching problems, correspond to flat degenerations of $R(\phi)$, and that these degenerations fit together into a polyhedral structure that is described by elements of tropical geometry.

The connection with tropical geometry comes from our use of valuations to construct the degenerations of $R(\phi)$. It is a theorem from tropical geometry (see [P] and Section 2) that the tropical variety $\text{tr}(I)$ of any ideal $I$ from a presentation of
$R(\phi)$ can be constructed as an image of $\mathbb{V}_T(R(\phi))$, the set of all valuations on $R(\phi)$ into the tropical real line $\mathbb{T}$ which are trivial on $\mathbb{C} \subset R(\phi)$. We define a complex of rational, pointed cones $K_\phi$ for each morphism $\phi$ in the category of reductive groups which is informed by the factorizations of $\phi$, and we show that points in this complex define valuations on the branching algebra $R(\phi)$. These valuations do not always have toric associated graded algebras, so we explain how they can be enhanced to toric degenerations of $R(\phi)$ in the special cases when $\phi$ is a diagonal embedding $\delta_n : G \to G^n$ or an inclusion of a Levi subgroup $i_L : L \to G$.

1.1. Construction of the branching complex. For any branching problem $\phi : H \to G$ it is possible to enrich $C(\phi)$ by studying factorizations of $\phi$.

\begin{equation}
\phi = \phi_m \circ \ldots \circ \phi_0
\end{equation}

$H \xrightarrow{\phi_0} K_1 \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_{m-1}} K_m \xrightarrow{\phi_m} G$

By the semisimplicity of the categories $\text{Rep}(K_i)$, the space $\text{Hom}_H(V(\eta), V(\lambda))$ decomposes into a direct sum.

\begin{equation}
\text{Hom}_H(V(\eta), V(\lambda)) = \bigoplus_{\eta, \bar{\eta}, \lambda \in \Delta_{H \times \ldots \times G}} \text{Hom}_H(V(\eta), V(\tau_1)) \otimes \ldots \otimes \text{Hom}_{K_m}(V(\tau_m), V(\lambda))
\end{equation}

We think of elements of these spaces as diagrams of intertwiners.

$V(\eta) \xrightarrow{f_0} V(\tau_1) \ldots V(\tau_m) \xrightarrow{f_m} V(\lambda)$

We abbreviate the summands on the right above as $W(\vec{\phi}, \vec{\tau}) \subset R(\phi)$, and $R(\phi)$ has a direct sum decomposition into these spaces as a vector space. Forgetting all but the dominant weight data results in a fiber product cone $C(\vec{\phi}) = C(\phi_0) \times_{\Delta_{K_1}} \ldots \times_{\Delta_{K_m}} C(\phi_m)$ with a natural map to $C(\phi_m \circ \ldots \circ \phi_0) = C(\phi)$. On the level of commutative algebra the above direct sum decomposition is a filtration of the multiplication operation on $R(\phi)$. The following will be discussed in Section 3.

**Theorem 1.2.** For any factorization of a morphism of reductive groups $\phi = \phi_n \circ \ldots \circ \phi_1$, there is a filtration of $R(\phi)$ defined on the spaces $W(\vec{\phi}, \vec{\lambda})$, by the ordering on the dominant weight labels.

\begin{equation}
W(\vec{\phi}, \vec{\lambda})W(\vec{\phi}, \vec{\eta}) \subset \bigoplus_{\vec{\beta} \leq \vec{\lambda}+\vec{\eta}} W(\vec{\phi}, \vec{\beta})
\end{equation}

The associated graded algebra is isomorphic to $[\otimes_{i=1}^n R(\phi_i)]^{T_1 \times \ldots \times T_{n-1}} \subset [\otimes_{i=1}^n R(\phi_i)]^{T_1 \times \ldots \times T_n}$. Furthermore, $[\otimes_{i=1}^n R(\phi_i)]^{T_1 \times \ldots \times T_{n-1}}$ is a flat degeneration of $R(\phi)$.

The torus $T_1 \times \ldots \times T_{n-1}$ is a product of the maximal tori in the groups $K_i$. A general summand of $[\otimes_{i=1}^n R(\phi_i)]$ is of the form

\begin{equation}
\text{Hom}_H(V(\lambda_0), V(\eta_0)) \otimes \text{Hom}_{K_1}(V(\lambda_1), V(\eta_1)) \otimes \ldots \otimes \text{Hom}_{K_{n-1}}(V(\lambda_{n-1}), V(\eta_{n-1}))
\end{equation}

and this algebra has an action of $T_1^2 \times \ldots \times T_{n-1}^2$, where $T_i^2$ acts with weight $(\eta_{i-1}, \lambda_i)$. We take invariants by the action by the anti-diagonal subtorus $T_1 \times$
\[ \ldots \times T_{n-1} \subset T_1^2 \times \ldots \times T_{n-1}^2, \]

so the components in the invariant algebra satisfy \( \eta_{i-1} = \lambda_i \).

To each such factorization we will then associate a rational polyhedral cone \( B(\tilde{\phi}) \).

We take the product \( \Delta_H^* \times \ldots \times \Delta_G^* \) of the dual Weyl-chambers of the groups, in other words the Weyl chambers of their respective Langlands dual groups. These elements can be realized as linear functionals on the Weyl chambers \( \Delta_H^* \times \ldots \times \Delta_G^* \), so we define \( B(\tilde{\phi}) \) as the quotient of \( \Delta_H^* \times \ldots \times \Delta_G^* \) by the relation which identifies two elements having the same values on \( C(\tilde{\phi}) \).

The following will be proved in Section 3.

**Proposition 1.3.** There is a map \( f_{\tilde{\phi}} : B(\tilde{\phi}) \to V_T(R(\phi)) \).

We construct a complex out of the cones \( B(\tilde{\phi}) \) as \( \tilde{\phi} \) runs over all factorizations of \( \phi \) in the category of reductive groups.

**Definition 1.4.** Let \( K_\phi \) be the simplicial set defined as follows. The \( k \)-simplices are defined as length-\( k \) factorizations \( \phi_k \circ \ldots \circ \phi_1 = \phi \) of the morphism \( \phi \). The face and degeneracy maps are defined by the following operations on chains of morphisms.

\[
\begin{align*}
(d^i_k)^* & : B(\phi_1, \ldots, \phi_i \circ \phi_{i-1}, \ldots, \phi_k) \to B(\phi_1, \ldots, \phi_k) \\
(s^i_k)^* & : (\rho_0, \ldots, \rho_k) \to (\rho_0, \ldots, 0, \ldots, \rho_k)
\end{align*}
\]

To make this well-defined, we can take some small category in the category of reductive groups which contains all such factorizations up to equivalence. The cones \( B(\tilde{\phi}) \) carry maps which correspond to the degeneracy maps in the above simplicial set by setting the appropriate coordinate to be 0, and the face maps are realized by adding the two adjacent coordinates which come from the same group.

\[
\begin{align*}
(d_k)^* & : B(\phi_1, \ldots, \phi_i \circ \phi_{i-1}, \ldots, \phi_k) \to B(\phi_1, \ldots, \phi_k) \\
(s_k^i)^* & : (\rho_0, \ldots, \rho_k) \to (\rho_0, \ldots, \rho_i + \rho_{i+1}, \ldots, \rho_k)
\end{align*}
\]

We can glue the \( B(\tilde{\phi}) \) together along these maps by taking a colimit to obtain a polyhedral model of \( K_\phi \), which we call \( K_\tilde{\phi} \). These gluing maps will be shown to be compatible with the maps \( f_{\tilde{\phi}} \).

**Theorem 1.5.** There is a map \( F_{\tilde{\phi}} : K_\tilde{\phi} \to V_T(R(\phi)) \).

In particular, every tropical variety \( \text{tr}(I) \) associated to a presentation of the ring \( R(\phi) \) has a subset which is the image of the polyhedral complex \( K_\tilde{\phi} \). In this way, the structure of the category of reductive groups informs the tropical geometry of branching algebras.
We now describe special factorizations of two types of morphisms, a diagonal \( \delta_n : G \to G^n \) and the inclusion of a Levi subgroup \( i_L : L \to G \). The degenerations defined by these special factorizations can be completed to toric degenerations. As a vector space, the branching algebra \( R(\delta_n) \) is a direct sum of the invariants in all \( n + 1 \)-fold tensor products of representations of \( G \), for this reason we call it the full tensor algebra of \( G \).

\[
R(\delta_n) = \bigoplus_{\lambda \in \Delta^{n+1}} \text{Hom}_G(V(\lambda_0), V(\lambda) \otimes \cdots \otimes V(\lambda_n)) = \bigoplus_{\lambda \in \Delta^{n+1}} [V(\lambda) \otimes \cdots \otimes V(\lambda)]^G
\]

We make a special application of the branching construction described above to reduce the study of \( R(\delta_n) \) to \( R(\delta_2) \) for any \( n \).

**Definition 1.6.** Define an oriented \( n \)-tree to be a \( T \) with \( n + 1 \) leaves labeled \( 0, \ldots, n \), where each edge is directed in such a way that each non-leaf vertex has exactly one in-edge, the 0 leaf is a source, and all other leaves are sinks.

Note that the orientation is entirely defined by the labeling \( \{0, \ldots, n\} \). To each oriented \( n \)-tree we define a factorization of \( \delta_n \) by assigning each vertex \( v \in V(T) \) the morphism \( \delta_{\text{val}(v)}^{-1} : G \to G_{\text{val}(v)}^{-1} \), where \( \text{val}(v) \) is the valence of \( v \).

The branchings defined by this tree construction define a connected subcomplex of \( K_{\delta_n} \), which we call \( D_n(G) \). The factorization diagrams corresponding to \( T \) are labellings of the edges \( E(T) \) of \( T \) by dominant \( G \)-weights, and the vertices \( V(T) \) of \( T \) by \( G \)-tensors. By Theorem 1.2 above, this results in a filtration on \( R(\delta_n) \) with associated graded algebra a subalgebra of torus invariants in \( \bigotimes_{v \in V(T)} R(\delta_{\text{val}(v)}^{-1}) \). If \( T \) is trivalent, we obtain a subalgebra of of \( [R(\delta_2)]^\otimes n-2 \).

**Remark 1.7.** The complex \( D_n(SL_2(\mathbb{C})) \) can be recognized as the space of phylogenetic trees defined by Billera, Holmes and Vogtmann in [BHV] to give a geometric context to phylogenetic algorithms from mathematical biology.

There is a distinguished class of factorizations we can use on the map defined by the inclusion of a Levi subgroup \( i_L : L \to G \) as well, namely factorizations by other Levi subgroups. For any chain of Levi subgroups

\[
L \xrightarrow{i_{L_{L_1}}} L_1 \xrightarrow{i_{L_{L_2}}} \cdots \xrightarrow{i_{L_{L_{m-1}}}} L_m \xrightarrow{i_{L_m,G}} G
\]
we obtain a cone of valuations $B(i_L)$, with a flat degeneration of $R(i_{L,G})$ to $[R(i_{L,G})]^{T_1 \times \ldots \times T_m}$. Each such cone can be represented combinatorially by a nesting of the Dynkin diagrams defining the $L_i$.

![Chain of sub-Dynkin diagrams of type A](image)

**Figure 2.** A chain of sub-Dynkin diagrams of type A

We define $H_{L}(G) \subset K_{i,L,G}$ to be the connected subcomplex defined by all chains of Levi subgroups which begin with $L$ for a fixed choice of maximal torus. Just as exploring $D_n(G)$ for $G = SL_m(C)$ has lead to some interesting combinatorics (see [M3], [SpSt]), we anticipate that the study of $H_L(G)$ for various $L$ and $G$ to yield interesting combinatorial structures as well.

1.2. The dual canonical basis and branching algebras. Factorization diagrams provide a convenient combinatorial filtration of general branching algebras, but the associated graded algebras of these filtrations are not always monoidal. Now we describe how to use the dual canonical basis (crystal basis) of Lusztig to complete the branching degenerations associated to the valuations in $D_n(G)$ and $H_L(G)$ to toric degenerations.

A full solution to a branching problem would be a cone $D \to C(\tilde{\phi})$ where each element of a basis of branching maps $f : V(\beta) \to V(\lambda), (\beta, \lambda) \in \Delta_H \times \Delta_G$ is assigned a unique element of $D$. Ideally, one wants

1. A finitely generated monoid $S(\phi)$

2. A map $\pi : S(\phi) \to \Delta_H \times \Delta_G$ which recovers $C(\phi)$, such that the number of points in the fiber over $(\lambda, \eta)$ is $\dim[Hom_H(V(\lambda), V(\eta))]$.

3. An algebraic relationship between $S(\phi)$ and $R(\phi)$.

We will use the convex polyhedral descriptions of branching for inclusion of Levi subgroup $i_{L,G} : L \subset G$ and the diagonal map $\delta_2 : G \to G^2$ given by Berenstein and Zelevinsky in [BZ1], [BZ2] to give a variety of solutions to 1, 2, and 3 above in these cases.

The dual canonical basis $B \subset R_G$ has a parametrization by $N$–tuples of non-negative integers for each reduced decomposition $i = (i_1, \ldots, i_N)$ of the longest word of the Weyl group $w_0 \in W$ of $G$.

\[ w_0 = s_{\alpha_1} \cdots s_{\alpha_N} \]
This defines an injective map $B \to \Delta \times \mathbb{Z}^N_{\geq 0}$, and a partial ordering on $B$, where $b_{\lambda, \vec{s}} > b_{\eta, \vec{t}}$ if $\lambda > \eta$ as dominant weights, or $\lambda = \eta$ and $\vec{s} > \vec{t}$ lexicographically. The image of this map is the set of integer points in a convex cone $C(i) \subset \Delta \times \mathbb{Z}^N_{\geq 0}$. Multiplication in $R_G$ with respect to the basis $B$ was shown to be lower-triangular by Caldero [C], see also [K].

(17) \[ b_{\lambda, \vec{s}} \times b_{\eta, \vec{t}} = b_{\lambda+\eta, \vec{s}+\vec{t}} + \sum_{\vec{t} < \vec{s}} C_{\lambda+\eta, \vec{t}} b_{\lambda+\eta, \vec{t}} \]

This is similar to the filtration defined by factorization diagrams. We will employ an observation we believe to be essentially due to Zhelobenko [Zh] to realize the branching algebras $R(\delta_2)$ and $R(i_{L,G})$ as subalgebras of $R_G \otimes \mathbb{C}[T]$ and $R_G$ respectively in order to show the following.

**Theorem 1.8.** The algebras $R(\delta_2)$ and $R(i_{L,G})$ inherit the dual canonical basis along with its lower triangular multiplication property from $R_G$.

Using the work of Berenstein and Zelevinsky [BZ3], we observe that the bases of $R(\delta_2)$ and $R(i_{L,G})$ both come with natural labelings by the integer points in convex rational cones $C(\delta_2)$ and $C(i_{L,G})$, respectively, one for each choice of string parameter $i$.

**Corollary 1.9.** The algebras $R(\delta_2)$ and $R(i_{L,G})$ carry flat degenerations $\mathbb{C}[C(\delta_2)]$ and $\mathbb{C}[C(i_{L,G})]$ respectively.

In the examples we explain a particular nice instance of a cone $C_3(i)$ in type $A$. These degenerations allow us to complete the branching degenerations of $R(\delta_n)$ and $R(i_{L,G})$ defined above to toric degenerations.

**Theorem 1.10.** The full tensor algebra $R(\delta_n)$ has a toric degeneration for each choice of the following objects.

1. A trivalent, oriented $n$-tree, $T$.
2. An ordering of the internal vertices $v \in V(T)$.
3. An assignment $i_v$ of strings to internal vertices of $T$.

The resulting rational cone $C_T(i)$ is a fiber product of cones $C_3(i_v)$ over copies of $\Delta_G$ according to the topology of $T$. An element of this cone is a labeling of $T$ by dominant weights on the edges of $T$ and compatible dual canonical basis elements on the trinodes. The map which forgets all data except the labels on the leaves is a map on cones $C_T(i) \to C(\delta_n)$ which enriches the tensor branching cone.

These degenerations are all $T^{n+1}$ invariant with respect to the action which grades $R(\delta_n)$ by the tuple of dominant weights in each tensor product, so we also get toric degenerations of every subalgebra $R(\delta_n, M) \subset R(\delta_n)$ given as the sum of the graded pieces for a submonoid $M \subset \Delta_G^{n+1}$.

(18) \[ R(\delta_n, M) = \bigoplus_{\bar{\lambda} \in M} Hom_G(V(\lambda_0), V(\lambda_1) \otimes \ldots \otimes V(\lambda_n)) \]

In particular, we get a degeneration of the algebra corresponding to the monoid of non-negative integer multiples of a particular tuple of weights, $\mathbb{Z}_{\geq 0} \bar{\lambda} \subset \Delta_G^{n+1}$. This
Figure 3. A member of $C_T(i)$

corresponding to a face of $\Delta_G$ which contains $\lambda_i$.

Corollary 1.11. Let $C_T(i, \vec{\lambda})$ be the polytope obtained from $C_T(i)$ by fixing the leaf edge values at the weights $\lambda_0, \ldots, \lambda_n$. The lattice points in $C_T(i, \vec{\lambda})$ are in bijection with a basis of the space of global sections $H^0(P_{\vec{\lambda}}(G), L(\vec{\lambda}))$.

These spaces have been studied in detail for $G = SL_2(\mathbb{C})$, where $G/P(\lambda)$ is always the projective line, see [HMSV], [HMM], [M1]. We also have a similar statement for the branching algebra $R(i_{L,G})$.

Theorem 1.12. For any choices of the following information we construct a toric degeneration of $R(i_{L,G})$.

1. A chain of Levi subgroups $L \subset L_1 \subset \ldots \subset L_k \subset G$

2. A choice of string $i_j$ for each Levi $L_j$

The result is easiest to describe when the strings $i_j$ are all compatible in the sense that they concatenate to a string $i$ of $G$. For choices of these "adapted" strings, $R(i_{L,G})$ degenerates to the toric algebra $\mathbb{C}[C_T(i)]$. Here $C_T(i)$ is the fiber product $C_L(i_1) \times_{\Delta_{L_1}} \ldots \times_{\Delta_{L_m}} C_L(i_G)$, where $C_L(i_j)$ is the cone for the inclusion of $L_{j-1}$ into $L_j$. There is also a map $C_T(i) \rightarrow C(i_{L,G})$ which enriches the Levi branching cone.

1.3. The case of type $A$ and remarks. In Section 6 we discuss a particular realization of $\mathbb{C}[C_T(i)]$ for $SL_m(\mathbb{C})$ using Berenstein-Zelevinsky triangles [BZ3]. The result is a description of a particular $C_T(i)$ as the non-negative integer points in the intersection of a collection of linear spaces.

In certain cases the two constructions presented here apply to the same algebras. As with $R(\delta_n, \vec{\lambda})$, any subalgebra $R(i_{L,G}, M) \subset R(i_{L,G})$ corresponding to a submonoid $M \subset C(i_{L,G}) \subset \Delta_H \times \Delta_G$ inherits the degenerations defined by a chain of Levi subgroups $i_{L_1}, i_{L_2}, \ldots, i_{L_m,G}$ with selected strings $i_k$. An example of
such a submonoid is given by the multiplies of the \( m - \)th dominant weight \( \omega_m \) of \( GL_n(\mathbb{C}) \). The subalgebra \( R_G(\mathbb{Z}\omega_m) \subset R_G \) is the projective coordinate ring of the Grassmannian \( Gr_m(\mathbb{C}^n) \). This algebra can also be realized as a subalgebra \( R_G(\mathbb{Z}^{\geq 0} \omega_m) \subset R_G \) for \( G = GL_m(\mathbb{C}) \). It would interesting to see how the different degenerations constructed here for \( R_G \) and \( R_G(\delta_n) \) are related under these dualities.

Valuations have been used to connect the combinatorics of branching problems to the commutative algebra of branching algebras before. Howe, Tan and Willenbring [HTW2] arrive at a SAGBI (Sub-algebra Analogue of Gröbner Basis for Ideals), interpretation of the Littlewood-Richardson rule, this is in the same spirit as this paper, as SAGBI degenerations arise from the higher rank valuations implicit in the lexicographic ordering on the dual canonical basis. Also in a similar spirit, Kaveh and Anderson [K], [A], have been connecting the dual canonical basis to a special type of valuation built from a full flag of subspaces in a flag variety or Schubert variety of \( G \). This allows them to realize meaningful polytopes from representation theory as the so-called Okounkov bodies of these valuations. It would be interesting to see a similar interpretation the polytope \( \mathcal{C}_T, \vec{\lambda}(i) \) as an Okounkov body for some flag of subspaces in the projective variety \( P_{\vec{\lambda}}(G) \).

2. Valuations and tropical geometry

In this section we collect a few technical facts about valuations on a commutative algebra \( A \) and their relationship to the tropical geometry of the associated variety \( Spec(A) \). We will define subductive generating sets \( X \subset A \) with respect to a given valuation \( v : A \rightarrow \mathbb{Q} \). Subductive sets are a generalization of SAGBI bases, and are an interesting construction in their own right. While writing this paper we were made aware of the paper [K], where Kaveh has arrived at the same definition. We will use subductive sets to extend some of the results in [C] and [AB] on constructing flat degenerations from term orderings on an algebra, in particular we will construct a cone \( D(X, \Gamma) \) in \( V_T(A) \) with respect to a subductive set \( X \) and a generating set of relations \( \Gamma \). When this construction is applied to the term orders defined by the dual canonical basis on \( R(i_{L,G}) \) and \( R(\delta_n) \) it provides a way to to turn these term orders into flat degenerations.

**Definition 2.1.** We say \( \mathbb{Q} \) is a tropical field, if it is a totally ordered abelian group with a least element \( -\infty \).

For any tropical field \( \mathbb{Q} \) and an index set \( X \), we can form the polynomial semialgebra \( \mathbb{Q}[X] \). An element in \( \mathbb{Q}[X] \) is a tropical polynomial.

\[
\bigoplus q_i \otimes x_1^{\otimes n_1(i)} \otimes \ldots \otimes x_r^{\otimes n_r(i)}
\]

Each tropical polynomial \( F \in \mathbb{Q}[X] \) determines a subspace \( tr(F) \subset \mathbb{Q}^X \), the tropical variety of \( F \). This is the set of points \( \vec{q} \in \mathbb{Q}^X \) where at least two monomials from \( F \) take the maximum value.

**Definition 2.2.** A valuation into \( \mathbb{Q} \) on a commutative algebra \( A \) over \( \mathbb{C} \), is a function \( v : A \rightarrow \mathbb{Q} \) which satisfies the following properties.

1. \( v(ab) = v(a) \otimes v(b) \)
\[(a + b) \leq v(a) \oplus v(b)\]
\[(a) = 0, C \in \mathbb{C}^\times\]

We denote the set of all such functions by \(V_Q(A)\).

Let \(v : K \to Q\) be a valuation on a field \(K\). For a polynomial \(f(X) = \sum C_i x^{\vec{m}_i} \in K[X]\) we define a polynomial in the semiring \(Q[X]\), called the tropicalization of \(f(X)\) as follows.

\[T(f) = \bigoplus v(C_i) x^{\vec{m}_i}\]

For an ideal \(I \subset K[X]\) we can then define the tropical variety \(tr(I) = \cap_{f \in I} tr(T(f))\). Notice that if \(v : K \to Q\) is the trivial valuation, then \(T(f)\) has only the tropically multiplicative identity \(0\) as a coefficient.

When \(Q = \mathbb{T}\), the set \(V_T(A)\) has the structure of a Haussdorf topological space, and it is referred to as the analytification of \(Spec(A)\), see \([P]\), \([B]\). In this case, a tropical variety \(tr(I) \subset \mathbb{T}_X\) is a subfan of the Gröbner fan of \(I\). Higher rank tropical fields appear naturally in combinatorial commutative algebra as well.

**Example 2.3.** In the setting of SAGBI theory, one studies a subalgebra \(A \subset \mathbb{C}[X]\) of a polynomial algebra with the deg-lex term order. This defines a valuation into \(\mathbb{Z}[X]^{+1}\), ordered lexicographically. In particular, a polynomial \(f(x)\) is taken to its degree paired with the exponents of its top monomial.

There is a close relationship between the sets \(tr_Q(I)\) and \(V_Q(A)\).

**Theorem 2.4.** Let \(I\) be a presenting ideal of \(A\).

\[
\begin{array}{cccccc}
0 & \longrightarrow & I & \longrightarrow & \mathbb{C}[X] & \longrightarrow & A & \longrightarrow & 0
\end{array}
\]

Then there is a map,

\[\pi_X : V_Q(A) \to tr_Q(I)\]

furthermore, there is a bijection.

\[V_Q(A) \cong \varprojlim_{X \subset A} tr_Q(I)\]

**Proof.** For this, emulate the proof in \([P]\) of Theorem 1.1. \(\square\)

2.1. The subduction algorithm. In SAGBI theory the subduction algorithm writes an arbitrary polynomial \(p(x) \in A \subset \mathbb{C}[X]\) as a polynomial in a SAGBI basis \(\{f_1, \ldots, f_r\} \subset A\), see for example \([S]\), Chapter 11. We will give a generalization of this algorithm to a general valuation on an algebra \(A\).

**Definition 2.5.** For a valuation \(v : A \to Q\), filter \(A\) by values of \(Q, A = \bigcup A_{\leq v}\). Let \(gr_v(A)\) be the associated graded algebra with respect to this filtration. Let \(X \subset A\) give a homogenous generating set \(X \subset gr_v(A)\). For \(a \in A\) we perform subduction on \(a\) as follows.
(1) Find \( p(\bar{x}) = \bar{a} \in \text{gr}_v(A) \).

(2) Consider \( a - p(x) \in A \), which must have strictly lower filtration value. Repeat (1) on \( a - p(x) \).

If this process always terminates then we say \( X \subset A \) is subductive.

Any subductive set \( X \) generates \( A \), also notice that \( X \) subductive implies that \( X \cup \{ y \} \) for \( y \in A \) subductive.

**Definition 2.6.** For a vector \( \vec{q} \in Q^X \) and a polynomial \( f \in \mathbb{C}[X] \), we define the initial form \( \text{in}_\vec{q}(f) \) to be the sum of the monomials in \( f \) which have highest value on \( \vec{q} \) when tropicalized. For \( \vec{q} \in \text{tr}_Q(I) \) define \( \text{in}_\vec{q}(I) \) to be the ideal generated by the \( \vec{q} \)-initial forms from \( I \).

Each ideal \( \text{in}_\vec{q}(I) \) defines a new algebra \( \mathbb{C}[X]/\text{in}_\vec{q}(I) \). For any \( v \in \mathbb{V}_Q(A) \), both \( v \) and \( \pi_X(v) \) define algebras constructed from \( A \), which leads naturally to the following proposition.

**Proposition 2.7.** If \( X \subset A \) is subductive with respect to \( v \) then we have the following.

\[
\text{gr}_v(A) \cong \mathbb{C}[X]/\text{in}_v(I)
\]

**Proof.** Let \( J \) be the homogeneous ideal which presents \( \text{gr}_v(A) \) as a quotient of \( \mathbb{C}[X] \), we will show \( J = \text{in}_v(I) \). The inclusion \( \text{in}_v(I) \subset J \) is easier, any \( f \in \text{in}_v(I) \) is of the form \( \text{in}_v(f + f') \) for \( f' \) of lower \( v \)-degree in \( \mathbb{C}[X] \), as graded by \( v \). This implies that \( f(\bar{x}) \in A_{<r} \) for \( r \) the \( v \)-degree of the monomials in \( f \), so \( f(\bar{x}) = 0 \) in \( \text{gr}_v(A) \). Now, if \( g \in J \), is homogeneous then \( g(\bar{x}) \in A_{<r} \) for \( r \) the \( v \)-degree of \( g \), which means there is a polynomial in monomial terms of strictly less \( v \)-degree, \( p(\bar{x}) \) such that \( g(\bar{x}) - p(\bar{x}) \in A_{<r} \), for \( r_1 \) the \( v \)-degree of \( p_1 \) and \( r_1 < r \). Continuing this way, we get a polynomial relation \( \hat{g} \) with \( in_v(\hat{g}) = g \) by the termination of the subduction algorithm. \( \square \)

2.2. Partial orderings. We make use of a number of partially ordered cones, specifically Weyl chambers, and products of Weyl chambers with lexicographically ordered products non-negative integers. We will cover some techniques for turning these partial orders into valuations and material for flat degenerations. Consider a commutative algebra \( A = \bigoplus_{c \in C} A_c \) with underlying vector space graded by a commutative monoid \( C \). Suppose there is a partial ordering \( < \) on \( C \) such that \( A_n A_{n_2} \subset \bigoplus_{\eta \leq n_1 + n_2} A_{\eta} \). We can define an associated graded algebra by altering the multiplication operation on the vector space \( \bigoplus_{c \in C} A_c \) by projecting onto the top weighted component.

\[
A_c \times A_d \rightarrow A_{c+d}
\]

We call this new algebra \( \text{gr}_<(A) \). For a tropical field \( Q \) let \( e : C \rightarrow Q \) be a monoidal map such that \( c < d \) implies \( e(c) \oplus e(d) = e(d) \) in \( Q \).

**Proposition 2.8.** If \( \text{gr}_<(A) \) is a domain, any \( e \) as above defines a valuation on \( A \).
Proposition 2.9. Let $A$ be filtered by a partially ordered monoid as above, suppose that the down-set $D_c = \{ d \in C \mid d < c \}$ is finite for all $c$, and let $e : C \to Q$ be as above. If $X \subset gr_c(A)$ is a homogeneous generating set, then it lifts to a subductive set $X$ with respect to $e$.

**Proof.** Given $a \in A$ we may assume without loss of generality that $a \in A_c$ for some $c$. Let $p(\hat{x}) = a$ in the associated graded algebra. We have that $a - p(x)$ is composed of terms of weight strictly less than $c$. Since $D_c$ is finite, we only need repeat this process a finite number of times, so the subduction algorithm terminates. \qed

We will prove a strengthening of Proposition 2.8. Let $X \subset A$ give a homogeneous generating set for $gr_c(A)$, and let $J$ be the homogeneous ideal of the corresponding presentation. Using a subduction type argument we can lift an element $g \in J$ to $\hat{g} \in I$ with the property that $\hat{g} = g + \ell$ with all the terms in $\ell$ of weight less than the homogenous degree of $g$. Let $\Gamma$ be a lifting of a generating set of $A$.

**Definition 2.10.** Define $D(X, \Gamma)$ to be the set of monoidal maps $e : C \to Q$ such that $e$ weights the monomials of $g$ higher than any term in $\ell$ for all $\hat{g} \in \Gamma$.

We define a filtration on $A$ by letting $A_{\leq q} = \mathbb{C}\{x_1^{m_1} \ldots x_k^{m_k} \mid \sum m_i e(c_i) \leq q \}$, where $c_i$ is the $C$–degree of $x_i$. This is the span of all monomials in $X$ with $e$-degree less than or equal to $q$.

**Theorem 2.11.** Let $A$ be as above, with $D_c$ finite for all $c$. If $e \in D(X, \Gamma)$, then $e$ defines a valuation from $A$ to $Q$.

**Proof.** The set $X$ is subductive for the filtration defined by $e$, so we have $gr_c(A) = \mathbb{C}X/\text{in}_e(I)$. We will show that $\text{in}_c(I) = J$, which establishes that the associated graded algebra of $e$ is $gr_c(A)$, a domain. One direction is clear by definition. If $f \in J$ then $f = \sum x_i^{\ell_i} g_i$ for $g_i + \ell_i \in G$, by assumption these are all initial forms for $e$. \qed
For a general $f \in I$ we may write $f = \sum f_c$ for $c \in C$. Consider the set $D(f) = \bigcup D_c$, the union of all the finite down-sets of the weights of the terms in the sum above, this is a finite set by assumption. Consider a maximal element $c_m \in D(f)$, and $f_{c_m}(x) \in A$. This must land in the sum $\bigoplus_{c \in D_{c_m}} A_c$. If a term of $f_{c_m}(\bar{x})$ is non-zero in $A_{c_m}$, then it cannot cancel with any terms from $f_c(\bar{x})$ for any other $c \in D(f)$, which contradicts $f \in I$, this implies that $f_{c_m}(\bar{x}) \in \bigoplus_{c < c_m} A_c$, and therefore $f_{c_m} \in J$. We can choose a homogeneous expression $f_{c_m} = \sum x^{M_i}g_i$ and consider $f - \sum x^{M_i}(g_i + \ell_i) \in I$. By construction, this element has the property that the union of all of the down sets of its weights is a strict subset of $D(f)$. After repeating this process a finite number of steps we obtain an expression $f = \sum x^{M_i}(g_i + \ell_i)$ where the terms $x^{M_i}g_i$ for a fixed weight don’t cancel.

This shows two things. First of all, it proves that $G$ generates $I$, and it shows that for any lift $\tilde{f}$ of an initial form $f \in in_c(I)$ we can produce an expression as above, take its initial form, and recover $f$ as a sum of terms of the form $x^{M_i}g_i$, which are all in $J$.

The purpose of these last two propositions is to allow us to start with an algebra that has been filtered by a partially ordered cone, and turn that partial ordering into a valuation. If that valuation takes values in $T$ then we can say more geometrically.

**Proposition 2.12.** If $v : A \to T$ takes non-negative integer values, then $gr_v(A)$ is a flat degeneration of $A$.

This follows from the standard construction of the Reese algebra, see for example Proposition 2.2 of [AB]. For a filtration $F$ on $A$, one constructs a new algebra $R = C[t] \otimes \bigoplus t^n F_{\leq n}$, this algebra is flat over $C[t]$ by construction because $t$ is not a zero divisor. specializing at $t = 0$ produces $gr_v(A)$ and $t \neq 0$ produces $A$.

In this paper, the constructions using the dual canonical basis require us to use Proposition 2.11 to turn a filtration into a flat degeneration. In order to do so, we require an $e \in D(X, \Gamma)$ as above. When the filtration is by a partially ordered rational cone, as is the case with all of our constructions, we can use the argument for Proposition 2.2 in [AB] to create this element.

3. **Algebraic constructions from branching problems**

In this section we construct the filtration by factorization diagrams on $R(\phi)$, and employ Proposition 2.8 to build cones of valuations $B(\tilde{\phi}) \subset \nabla_T(R(\phi))$ for every factorization $\tilde{\phi}$ of $\phi$. These are then glued together to form the complex $K_\phi \subset \nabla_T(R(\phi))$. We describe a set of elements in $K_\phi$ which define flat degenerations of $R(\phi)$, and we will use these constructions on $R(i_{L,G})$ and $R(\delta_n)$ as the first step in constructing toric degenerations of $R(\delta_n)$ and $R(i_{L,G})$.

3.1. **Filtrations and valuations from a factorization.** Recall that the algebra $R(\phi)$ is multigraded by pairs of dominant weights $(\gamma, \lambda) \in \Delta_H \times \Delta_G$. The $(\gamma, \lambda)$ component is isomorphic to $Hom_H(C, V(\gamma) \otimes V(\lambda)) = \hom_H(V(\gamma^*), V(\lambda))$. Recall that $C_* : V(\lambda) \otimes V(\eta) \to V(\lambda + \eta)$ is the Cartan multiplication operation, there is a dual operation $C^* : V(\lambda) \otimes V(\eta) \to V(\lambda + \eta)$ defined by sending the highest weight vector $v_{\lambda + \eta}$ to $v_{\lambda} \otimes v_{\eta}$. The lemma below appears in [M3], and its proof is a straightforward computation.
Lemma 3.1. When the multiplication of two homogeneous elements $\phi_1, \phi_2 \in R(\phi)$ is computed by the following diagram.

\[
V(\gamma_1^1 + \gamma_2^2) \xrightarrow{C^*} V(\gamma_1^1) \otimes V(\gamma_2^2) \xrightarrow{\phi_1 \otimes \phi_2} V(\lambda_1) \otimes V(\lambda_2) \xrightarrow{C^*} V(\lambda_1 + \lambda_2)
\]

Now consider a factorization $\phi = \psi \circ \pi$ through a third reductive group $K$. For each graded piece $\text{Hom}_H(V(\gamma^*), V(\lambda))$ this gives a direct sum decomposition.

\[\text{(26)} \quad \text{Hom}_H(V(\gamma^*), V(\lambda)) = \bigoplus_{\tau \in \Delta_K} \text{Hom}_H(V(\gamma^*), V(\tau^*)) \otimes \text{Hom}_K(V(\tau^*), V(\lambda))\]

Each element in this decomposition can be pictured as a factorization diagram.

\[
V(\gamma^*) \xrightarrow{f} V(\tau^*) \xrightarrow{g} V(\lambda)
\]

Here $f$ is $H$-linear and $g$ is $K$-linear, so we can also regard this as an element of $R(\pi) \otimes R(\psi)$. Now we can write the multiplication in $R(\phi)$ in terms of these diagrams.

\[
\begin{array}{ccc}
V(\gamma_1^1) \otimes V(\gamma_2^2) & \xrightarrow{f_1 \otimes f_2} & V(\tau_1^1) \otimes V(\tau_2^2) \\
& \uparrow{C^*} & \downarrow{C^*} \\
V(\gamma_1^1 + \gamma_2^2) & & V(\lambda_1 + \lambda_2)
\end{array}
\]

The middle component has a direct sum decomposition as a $K$ module,

\[V(\tau_1) \otimes V(\tau_2) = \bigoplus_{\eta} I_\eta \otimes V(\eta).\]

There are projection $P_\eta$ and injection $Q_\eta$ maps for each component in this decomposition. In this way, the product element can be written as follows.

\[\text{(27)} \quad C_\ast \circ [g_1 \otimes g_2] \circ [f_1 \otimes f_2] \circ C^* = \sum C_\ast \circ [g_1 \otimes g_2] \circ P_\eta \circ Q_\eta \circ [f_1 \otimes f_2] \circ C^*
\]

The $\tau_1^1 + \tau_2^2$ component in this sum is exactly $C_\ast \circ [g_1 \otimes g_2] \circ C^* \circ C_\ast \circ [f_1 \otimes f_2] \circ C^*$. By the above observations we have the following.

\[\text{(28)} \quad W(\alpha_1, \eta_1, \beta_1)W(\alpha_2, \eta_2, \beta_2) \subset \bigoplus_{\eta \leq \eta_1 + \eta_2} W(\alpha_1 + \alpha_2, \eta, \beta_1 + \beta_2)
\]

Using the direct sum decomposition above, we can filter $R(\phi) = \bigoplus W(\alpha, \eta, \beta)$ by the ordering on the dominant weights of $K$. Using the filtering index, the $(\alpha_1, \eta_1, \beta_1)$ and $(\alpha_2, \eta_2, \beta_2)$ components of $R(\phi)$ multiply to give elements of weight $(\alpha_1 + \alpha_2, \eta_1 + \eta_2, \beta_1 + \beta_2)$ and lower. Furthermore, if we cut off the components of strictly lower weight, we get the multiplication operation in the ring $R(\pi) \otimes R(\psi)$. This proves the following version of theorem \[28\].

Proposition 3.2. There is a filtration $F_{\pi, \psi}$ on $R(\phi)$ by factorization diagrams associated to each factorization $\phi = \pi \circ \psi$, such that the associated graded algebra $\text{gr}_{\pi, \psi}(R(\phi))$ is isomorphic to $[R(\phi) \otimes R(\pi)]^{T_K}$. 

An element $\rho \in \Delta^*$, the dual Weyl chamber is defined by its non-negativity on positive roots. This implies that the values obtained by applying $\rho$ respect the partial ordering on the dominant weights of $K$. The next corollary follows from Proposition 2.8.

**Corollary 3.3.** To each factorization $\pi \circ \psi = \phi$ in the category of reductive groups there is a map $f_{\pi,\psi} : \Delta_H^* \times \Delta_K^* \times \Delta_G^* \to V_T(R(\phi))$.

**Proposition 3.4.** The image of the above map is the quotient of the cone $\Delta_H^* \times \Delta_K^* \times \Delta_G^*$ by the relation $X \sim Y$ when $X(\omega) = Y(\omega)$ for all $\omega \in C(\pi) \times \Delta_K C(\psi) = C(\pi, \psi)$, and is therefore equal to $B(\psi, \pi)$.

**Proof.** The lattice points in the cone $C(\pi) \times \Delta_K C(\psi)$ give exactly the triples $(\alpha, \beta, \gamma)$ such that the graded component $W(\alpha, \beta, \gamma) \subset R(\pi \circ \psi)$ are non-zero. Since the filtrations defined by $\Delta_H^* \times \Delta_K^* \times \Delta_G^*$ are defined from the weight information from these components, the proposition follows. $\Box$

By construction the filtration by factorization diagrams is $T_H \times T_G$ linear, so it passes to subrings of $R(\phi)$ which are graded by a submonoid $\Delta^*$ of the dominant weights in $\Delta_H \times \Delta_G$. For example, the submonoid $\{0\} \times \Delta_G \subset \Delta_H \times \Delta_G$ corresponds to the invariant ring $R_H^G$, given by the factorization diagrams where the $H$-weight is trivial. Every factorization $\phi = \pi \circ \psi$ gives a filtration on $R_H^G$ with associated graded algebra $[R_H^G \otimes R(\pi)]^{T_K}$. Note also that $[R(\phi) \otimes R(\pi)]^{T_K}$ has an action of $T_K$, so the degeneration adds torus symmetries in accordance with the corresponding factorization. We can extend this construction to any finite factorization of a morphism $\phi : H \to G$.

$\begin{align*}
H \rightarrow \phi_1 &\rightarrow K_1 \ldots K_{n-1} \rightarrow \phi_n \rightarrow K_n
\end{align*}$

We write $R(\phi)$ as a direct sum of spaces of factorization diagrams as above, where $\lambda_i$ is a dominant weight of $K_i$. This gives a filtration on $R(\phi)$ by the same argument as above, we define the spaces $W(\phi_i, \bar{\lambda})$ as above.

(29) $W(\bar{\phi}, \bar{\lambda}) = Hom_H(V(\lambda_0), V(\lambda_1)) \otimes \ldots \otimes Hom_{K_{n-1}}(V(\lambda_{n-1}), V(\lambda_n))$

We can now apply Proposition 2.8 and Proposition 2.12 implies that coweight vectors which give positive integer values define flat degenerations of $R(\phi)$. Furthermore, when the coweight vector $\rho$ is chosen to have strictly positive contributions from all coroots, then $\bar{\tau}_1 < \bar{\tau}_2$ as dominant weights implies $\rho(\bar{\tau}_1) < \rho(\bar{\tau}_2)$, so the flat degeneration is isomorphic to $[R(\phi_1) \otimes \ldots \otimes R(\phi_m)]^{T_1 \times \ldots \times T_m}$.

### 3.2. The complex $K_{\phi}$

We recall the degeneracy and face maps from the introduction.

(30) $(d^*_k) : B(\phi_1, \ldots, \phi_i \circ \phi_{i-1}, \ldots, \phi_k) \rightarrow B(\phi_1, \ldots, \phi_k)$

(31) $(d^*_k) : (\rho_0, \ldots, \rho_k) \rightarrow (\rho_0, \ldots, 0, \ldots, \rho_k)$

(32) $(s^*_k) : B(\phi_1, \ldots, \phi_i, id, \phi_{i+1}, \ldots, \phi_k) \rightarrow B(\phi_1, \ldots, \phi_i, \phi_{i+1}, \ldots, \phi_k)$
In order to prove Theorem 1.5 we must show these maps are well-defined, and commute with the \( f_{\vec{\phi}} : B(\vec{\phi}) \to V_T(R(\vec{\phi})) \). To do this we define corresponding maps on the branching cones \( C(\vec{\phi}) \). We will see shortly that the operations of composing two consecutive morphisms in a diagram corresponds to the following degeneracy map on branching cones. Let \( \Delta_{\vec{\phi}} \) be the product of Weyl chambers for the groups in the factorization defined by \( \vec{\phi} \):

\[
\begin{array}{ccc}
\Delta_{\vec{\phi}} & \xrightarrow{(d^1_{\vec{\phi}})_*} & \Delta_{d^1_{\vec{\phi}}(\vec{\phi})} \\
\uparrow & & \uparrow \\
C(\vec{\phi}) & \xrightarrow{(d^1_{\vec{\phi}})_*} & C(d^1_{\vec{\phi}}(\vec{\phi}))
\end{array}
\]

On branching diagrams, these are the operations of composing two adjacent morphisms and adding in an identity morphism, respectively. Similarly, the operation of inserting an identity into a diagram of morphisms corresponds to the following face map on branching cones.

\[
\begin{array}{ccc}
\Delta_{\vec{\phi}} & \xrightarrow{(s^1_{\vec{\phi}})_*} & \Delta_{s^1_{\vec{\phi}}(\vec{\phi})} \\
\uparrow & & \uparrow \\
C(\vec{\phi}) & \xrightarrow{(s^1_{\vec{\phi}})_*} & C(s^1_{\vec{\phi}}(\vec{\phi}))
\end{array}
\]

Since branching valuations are determined by the branching cones by definition, showing that these maps are well-defined on the \( B(\vec{\phi}) \) and commute with the maps \( f_{\vec{\phi}} \) amounts to proving the following formulas. For any \( \bar{x} \in \Delta_{\vec{\phi}} \) and \( \bar{\rho} \in \Delta_{\vec{\phi}}^* \),

\[
(d^1_{\vec{\phi}})^*(\bar{\rho})(\bar{x}) = \sum_{j \neq i} \rho_j(\lambda_j) + 0\lambda_i = \bar{\rho}(s^1_{\vec{\phi}})(\bar{x})
\]

\[
(s^1_{\vec{\phi}})^*(\bar{\rho})(\bar{x}) = \rho_1(\lambda_1) + \ldots + \rho_i(\lambda_i) + \rho_{i+1}(\lambda_i) + \ldots + \rho_k(\lambda_{k-1}) = \bar{\rho}(s^1_{\vec{\phi}})(\bar{x})
\]

If \( \bar{\rho} \) and \( \bar{\rho}' \) agree on all \( \bar{x} \in C(d^1_{\vec{\phi}}(\vec{\phi})) \) then for any \( \bar{\gamma} \in C(\vec{\phi}) \) we have \( \bar{\rho}(d^1_{\vec{\phi}})(\bar{\gamma}) = \bar{\rho}'(d^1_{\vec{\phi}})(\bar{\gamma}) \) and therefore \( (d^1_{\vec{\phi}})^*(\bar{\rho})(\bar{\gamma}) = (d^1_{\vec{\phi}})^*(\bar{\rho}')(\bar{\gamma}) \). This shows that the map \( (d^1_{\vec{\phi}})^* : B(d^1_{\vec{\phi}}(\vec{\phi})) \to B(\vec{\phi}) \) is well-defined. A similar argument proves that \( (s^1_{\vec{\phi}})^* : B(s^1_{\vec{\phi}}(\vec{\phi})) \to B(\vec{\phi}) \) is well-defined. Now we can check that the maps \( (d^1_{\vec{\phi}})^* \) and \( (s^1_{\vec{\phi}})^* \) commute with the maps \( f_{\vec{\phi}} \) to \( V_T(R(\vec{\phi})) \). We have \( R(\vec{\phi}) = \bigoplus_{\bar{x} \in C(\vec{\phi})} W(\vec{\phi}, \bar{x}) \), as above. The operation \( d^1_{\vec{\phi}}(\vec{\phi}) \) collapses away the weights for \( K_i \), one checks that we get
Proof. Grosshans shows that this is a decomposition as a vector space. Theorem 3.7. For any $g \in G$, $v \in \mathbb{V}_T(A)$ coming from $K_{iG}$, and $g \in G$, the valuation $g \circ v$ is also in the image of $K_{iG}$. Theorem 1.5.

3.3. General $G$–algebras and $G$–actions on valuations. We briefly describe how to construct a map from $K_φ$ to $\mathbb{V}_T(A^H)$ for $φ : H \to G$ and any $G$–algebra $A$. We also discuss the action of $G$ on the complex $K_{iG}$ for the map $i_G : 1 \to G$. First, we look at a generalization of Theorem 1.5.

**Theorem 3.5.** Let $A$ be a $G$–algebra, then there is a map $F_φ : K_φ \to \mathbb{V}_T(A^H)$.

**Corollary 3.6.** There is a map $F_{iG} : K_{iG} \to \mathbb{V}_T(A)$ for any algebra $A$ with a $G$ action.

This is shown with very similar methods employed thus far in this paper. One follows Chapter 3, Section 15 of [G] and notes that $A$ always carries a direct sum decomposition as a vector space.

\begin{equation}
A = \bigoplus_{λ \in Δ} \text{Hom}(V(λ), A) \otimes V(λ)
\end{equation}

Grosshans shows that this is a $G$–stable filtration of algebras, with associated graded algebra $[A^U_+ \otimes R_G]^T$, where $U_+$ is a maximal unipotent. Now the filtrations and valuations on $R_{G,H}^U$ constructed in Section 3 can be extended to $A_{G,H}$ and its $Δ_H \times Δ_G$–multigraded subalgebras, in particular $A^H$. Corollary 3.4 is of particular interest because the valuations $\mathbb{V}_T(A)$ of a $G$–algebra $A$ carry an action by the group $G$ by pre-composition.

\begin{equation}
[g \circ v](f) = v(g^{-1}(f))
\end{equation}

**Theorem 3.7.** For any $v \in \mathbb{V}_T(A)$ coming from $K_{iG}$, and $g \in G$, the valuation $g \circ v$ is also in the image of $K_{iG}$.

Proof. Let $v$ be defined by the coweights $(ρ_1, \ldots, ρ_m)$ on a chain of reductive groups.
We define a new valuation $v_g$ by taking the same co-weights, and conjugating $\phi_m$ by $g$.

\[ 1 \xrightarrow{i_{k_1}} K_1 \xrightarrow{\phi_1} \ldots \xrightarrow{g\phi_m g^{-1}} G \]

We show that $g \circ v = v_g$. For $K_i$ in this chain, the map $\phi_1 \circ \ldots \circ g\phi_m g^{-1}$ is equal to $g[\phi_1 \circ \ldots \circ \phi_m]g^{-1}$. We consider the branching of an irreducible representation viewed as a subspace of $A$ with respect to this map.

\[
V(\lambda) = \bigoplus \text{Hom}_{K_i}(V(\eta), V(\lambda)) \otimes V(\eta)
\]

The element $g$ defines an isomorphism of $K_i$ modules $i_g : A \to gA$, where $K_i$ acts on $A$ through $\phi_1 \circ \ldots \circ \phi_m$, and on $gA$ through $g\phi_1 \circ \ldots \circ g\phi_m g^{-1}$. Under this isomorphism, $gV(\lambda)$ decomposes as $\bigoplus g\text{Hom}_{K_i}(V(\eta), V(\lambda)) \otimes V(\eta)$. This implies the following equation.

\[
v_g(g \circ f) = v(f)
\]

This implies that $[g^{-1} \circ v_g](f) = v(f)$, so $v_g(f) = [g \circ v](f)$. □

A consequence of this proof is that the isomorphism type of each part of the chain is preserved by the $G$-action as well. In this way, the space $K_{iG}$ decomposes into chambers depending on these isomorphism types.

**Corollary 3.8.** Let $v : A \to \mathbb{T}$ be a valuation from by the branching cone defined by the chain

\[ 1 \to T \to G, \]

then $g \circ v$ is a valuation from the branching cone defined by the chain

\[ 1 \to gTg^{-1} \to G. \]

A valuation $v$ as above is a choice $\chi \in \text{Hom}(\mathbb{C}^*, T) \otimes \mathbb{R}$ which is then evaluated on the weights of $A$ via its eigen-decomposition. Since any two maximal tori of $G$ are conjugate, this shows that the set of all branching valuations coming from maximal tori is covered by $G$-translates of those coming from a fixed maximal torus. Similarly, the set of all branching valuations coming from a chain of Levi subgroups is covered by $G$-translates of the set of branching valuations coming from chains of Levi subgroups which share a common maximal torus, $H_T(G)$.

**Remark 3.9.** It is straightforward to show that $H_T(G)$ is a cone over the spherical building of $G$.

### 4. Degenerations and the dual canonical basis

In this section we will give a brief overview of the labeling of the dual canonical basis $B \subset R_G$, by the string parameters associated to a decomposition $i \in R(w_0)$ of the longest word in the Weyl group of $G$. To begin, we assume that $G$ is simple. Lusztig [Lu] (also Kashiwara [Ka]) constructed a canonically defined basis $\mathcal{B}$ of a certain subalgebra $U^+$ of the quantized universal enveloping algebra of the Lie algebra of $G$, $\mathcal{U}_q(g)$. Specialization this basis at $q = 1$ yields a basis for each irreducible representation $V(\lambda)$ of $G$, we denote the dual of this basis in $V(\lambda^*)$ by $B(\lambda^*)$. 
4.1. String parameters. In [BZ1], Berenstein and Zelevinsky studied the parameterization of this basis by the string parameters associated to \( i \), which identifies \( B(\lambda) \) with a subset of \( \mathbb{Z}_{\geq 0}^N \subset \mathbb{R}^N \). After summing over all \( \lambda \in \Delta \), there is a bijection of \( B = \bigsqcup_{\lambda \in \Delta} B(\lambda) \) with the lattice points in a rational cone \( C(i) \subset \Delta \times \mathbb{Z}_{\geq 0}^N \). In order to describe this cone we must first define the system for indexing weights in representations invented by Berenstein and Zelevinsky [BZ1] called \( \iota \)-trails. An \( \iota \)-trail from a weight \( \gamma \) to a weight \( \eta \) in the weight polytope of a representation \( V \) is a sequence of weights \( (\gamma, \gamma_1, \ldots, \gamma_{\ell-1}, \eta) \), such that consecutive differences of weights are integer multiples of simple roots from \( i \), \( \gamma_i - \gamma_i+1 = c_{ik} \alpha_{ik} \), and the application of the raising operators \( e_{\alpha_i}^{c_{ik}} \circ \cdots \circ e_{\alpha_j}^{c_{jk}} : V_\gamma \to V_\gamma \) is non-zero. For any \( \iota \)-trail \( \pi \), Berenstein and Zelevinsky define \( b_{k}(\pi) = \frac{1}{2}H_{\alpha_{ik}}(\gamma_{k-1} + \gamma_k) \). In what follows, the entries of the Cartan matrix \( A \) are denoted \( a_{ij} \).

**Definition 4.1.** The cone \( C(i) \subset \Delta \times \mathbb{Z}_{\geq 0}^N \) is the set of \((\lambda, \bar{i})\) defined by the following inequalities.

1. \( \sum_k b_k(\pi) \tau_k \geq 0 \) for any \( \iota \)-trail \( \omega_i \to v_0 s_{i} \omega_i \) in \( V(\omega_i) \), for all fundamental weights \( \omega_j \) of the dual Langlands group.

2. \( \tau_k \leq H_{\alpha_{ik}}(\lambda) - \sum_{\ell=k+1}^{N} a_{ii,k} \tau_\ell \) for \( k = 1, \ldots, N \).

After specializing the \( \Delta \) component at a dominant weight, one obtains the string parameters which index the elements of the dual canonical basis in \( V(\lambda) \). For the algebra \( R_G = \bigoplus_{\lambda \in \Delta} V(\lambda) \) the \( \Delta \) parameter on the dual canonical basis corresponds to the grading by the right \( T \)-action. The left \( T \)-action gives a grading on \( R_G \) where the dual canonical basis element \( b_{\lambda, \bar{s}} \) has weight \((\sum a_i \alpha_i) - \lambda \). Elements of \( B \) can then be ordered first by dominant weight, with ties broken by the lexicographic order on the string parameters. Recall that Caldero [C] has shown multiplication in \( R_G \) to be lower triangular with respect to this ordering, see also [AB].

\[
(46) \quad b_{\lambda, \bar{s}} \times b_{\eta, \bar{t}} = b_{\lambda+\eta, \bar{s+t}} + \sum_{\bar{r} \leq \bar{s} + \bar{t}} C_{\lambda + \eta, \bar{r}} b_{\lambda + \eta, \bar{r}}
\]

As a result Caldero obtains the following theorem, the associated graded algebra is defined in the sense of Proposition 2.8.

**Theorem 4.2.** The algebra \( R_G \) has a filtration with associated graded ring isomorphic to \( \mathbb{C}[C(i)] \).

4.2. Bases for \( R(\delta_2) \) and \( R(i_{L,G}) \). The dual canonical basis \( B(\lambda) \subset V(\lambda) \) is known to be a good basis, see [BZ2], [Mat], [Lu]. This means that its restriction to certain subspaces \( V_{\chi, \beta}(\lambda) \subset V(\lambda) \) and \( V_{i,\eta}(\lambda) \subset V(\lambda) \) is still a basis. Here all \( \chi, \beta, \lambda, \eta \) are weights, and \( I \) is a collection of simple roots. Let \( e_i \) be the raising operator in the lie algebra \( \mathfrak{g} \) of \( G \).

\[
(47) \quad V_{\chi, \alpha}(\lambda) = \{ v \mid t \circ v = \chi(t)v, e_i^{H_i(\alpha)+1}v = 0 \} \subset V(\lambda).
\]

\[
(48) \quad V_{i,\eta}(\lambda) = \{ v \mid t \circ v = \eta(t)v, e_i v = 0, i \in I \} \subset V(\lambda).
\]
Notice that the $T$-weight spaces of $V(\lambda)$ are realized as a special case $I = \emptyset$ of the second definition. The space $V_{I, \eta}(\lambda)$ is by definition the $U_L$-fixed points of weight $\eta$ for $U_L \subset L$ the Levi subgroup corresponding to $I$ the chosen subset of roots, which means we have an identification.

\[(49) \quad V_{I, \eta}(\lambda) \cong \text{Hom}_L(V(\eta), V(\lambda))\]

Here $\eta$ is viewed as a dominant weight for $L$. When we fix $\chi = \mu - \beta$ in the first definition, for $\mu$ and $\beta$ dominant weights there is an identification,

\[(50) \quad V_{\mu - \beta, \beta}(\lambda) \cong \text{Hom}_G(V(\mu), V(\beta) \otimes V(\gamma)).\]

The string parameterizations allow Berenstein and Zelevinsky to produce the following polyhedral descriptions of the elements of the dual canonical basis in these multiplicity spaces.

**Theorem 4.3** (Berenstein, Zelevinsky, [BZ1]). The set of dual canonical basis members which span $V_{\mu - \beta, \beta}(\lambda) \subset V(\lambda)$ are indexed by the points in $\mathbb{Z}_{\geq 0}^N$ such that the following hold.

1. $\sum_k d_k(\pi)t_k \geq 0$ for any $i$–trail from $\omega_j$ to $w_0s_j\omega_j$ in $V(\omega_j)$, for all fundamental weights $\omega_j$ of the dual Langlands group.

2. $-\sum_k t_k \alpha_k + \lambda + \beta = \mu$

3. $\sum_k d_k(\pi)t_k \geq H_{\alpha_i(\beta)}$ for any $i$–trail from $s_j\omega_i$ to $w_0\omega_j$ in $V(\omega_j)$, for all fundamental weights $\omega_j$ of the dual Langlands group.

4. $t_k + \sum_{j > k} a_{ik,i_j} t_j \geq H_{\alpha_{ik}}(\lambda)$

The first and last conditions say that $(\lambda, \vec{t})$ is a member of $C(\vec{i})$ in the fiber over the weight $\lambda$, the second condition says that the basis members lie in the weight $\mu - \beta$ subspace of $V(\lambda)$, and the third condition says that the appropriate raising operators annihilate the basis members. Berenstein and Zelevinsky state this for a general semi-simple group, but it extends to a general reductive group as follows. The weights that define a non-zero $\text{Hom}_G(V(\mu), V(\lambda) \otimes V(\beta))$ are of the form $\mu' + \tau_1$, $\beta' + \tau_2$ and $\lambda' + \tau_3$ where $\tau_i$ are characters of the center of $G$ with $\tau_1 = \tau_2 + \tau_3$ and $\mu', \beta', \lambda'$ are dominant weights of the semisimple part of $G$. The subspace $V_{\mu - \beta, \beta}(\lambda)$ is the same as the subspace $V_{\mu' - \beta' + (\tau_1 - \tau_2), \beta'}(\lambda' + \tau_3) = V_{\mu' - \beta' + \tau_2, \beta'}(\lambda' + \tau_3) = V_{\mu' - \beta', \beta'}(\lambda') \otimes C\tau_3$. So this space inherits the subset of the dual canonical basis of the semi-simple part of $G$ coming from $V_{\mu' - \mu', \beta'}(\lambda')$ tensored with the character $\tau_3$. From now on we label the dual canonical basis members in this space with the triple of dominant weights $b_{\lambda, \mu, \beta, \beta} \in V_{\mu - \beta, \beta}(\lambda)$.

For the branching multiplicities over $L$, any string $i$ of the dual canonical basis gives a parametrization of the points in $V_{i, I}(\lambda)$, however in special cases there is a polyhedral description. Following [BZ1], we choose $i$ to be a concatenation of factorizations $i_k$ of $w_0(I)$, the longest word in the parabolic subgroup of $W$ corresponding to $I$, and $i_2$, a factorization of $w_0(I)^{-1}w_0$. In this case, the projection
of $C(i)$ onto the $\mathbb{Z}_{\geq 0}^N$, the factor splits as a product of string cones, see Theorem 3.11 of [BZ1].

**Theorem 4.4** (Berenstein, Zelevinsky, [BZ1]). The set of dual canonical basis members which span $V_\eta(\lambda) \subset V(\lambda)$ are $i_1 \circ i_2 = i_2$-parametrized by the points in $\mathbb{Z}_{\geq 0}^N$ with the first $N - \ell(w_0(I)^{-1}w_0)$ coordinates equal to zero, such that the following hold.

1. $\sum_{k > N - \ell(w_0(I)^{-1}w_0)} d_k(\pi)t_k \geq 0$ for any $i_2$-trail from $w_0(I)\omega_j$ to $w_0s_j\omega_j$ in $V(\omega_j)$, for all fundamental weights $\omega_j$ of the dual Langlands group.

2. $\sum_{k > N - \ell(w_0(I)^{-1}w_0)} t_k \alpha_k = \lambda - \eta$

3. $t_k + \sum_{j \geq k} a_{i_2, i_j} t_j \geq H_{\alpha_{i_2}}(\lambda)$

We label the basis members in this subspace $b_{\eta, \lambda, \rho}$. The purpose of the rest of this section is to define the two cones $C_W$ and $C_L$ which have cross-sections equal to the above polytopes and index bases $B_3(i)$ and $B_2(i)$ in the two branching algebras $R(\delta_2)$ and $R(i_L, G)$. In order to do so, we will relate both algebras to $R_G$. Let $U_-$ and $U_+$ be the maximal unipotent subgroups of $G$ associated to the negative and positive simple roots. Following Zhelobenko, Chapter XVIII, page 383 of [ Zh], we have the following commutative diagram.

$$
\begin{array}{cccc}
T \times G/U_+ & \longrightarrow & G/U_+ \times G/U_+ & \longrightarrow & U_- \setminus [G/U_+ \times G/U_+] \\
\uparrow & & \uparrow & & \uparrow \\
T \times U_- \times T & \longrightarrow & U_- \times T \times U_- \times T & \longrightarrow & U_- \setminus [U_- \times T \times U_- \times T]
\end{array}
$$

The bottom row of this diagram is dense in the top row. An invariant function $f \in C[G/U_+ \times G/U_+]$ satisfies

$$
(51) \quad f(u_1t_1, u_2t_2) = f(t_1, u_1^{-1}u_2t_2),
$$

so we can realize $C[G/U_+ \times G/U_+]^{\mathbb{U}_-}$ as a subalgebra of $R(G) \otimes C[T]$. The algebra $C[G/U_+ \times G/U_+]^{\mathbb{U}_-}$ is isomorphic to $C[G/U_+ \times G/U_+ \times G/U_+]^G \cong R(\delta_2)$. From [ Zh], we get that the map on the graded pieces is

$$
(52) \quad Hom_{U_-}(C_{\mu}, V(\lambda) \otimes V(\beta)) \rightarrow V_{\mu - \beta, \beta}(\lambda) \otimes C_{\nu_\beta} \subset V(\lambda) \otimes C_{\nu_\beta}.
$$

In this way, $R(\delta_2)$ inherits the dual canonical basis $B_3 = \coprod_{\mu, \beta, \lambda} B(\lambda) \otimes \nu_\beta \cap V_{\mu - \beta, \beta}(\lambda) \otimes \nu_\beta$, along with the multiplication operation in $R_G \otimes C[T]$. This implies that the basis in $R(\delta_2)$ has the lower-triangularity property with respect to multiplication in $R(\delta_2)$.

The algebra $R_G \otimes C[T]$ has 4 torus actions by $T \subset G$, namely the left and right action on both components. We label these $T_1 \times T_2 \times T_3 \times T_4$, from left to right. A graded component of this action is the space $V_\mu(\lambda) \otimes \nu_\beta$. Here $\mu$ is the character of $T_1$, and $\lambda$ is the character of $T_2$. The character $\beta$ and $-\beta$ are associated to $T_3$ and $T_4$. A graded component of this action is the space $V_\mu(\lambda) \otimes \nu_\beta$. Here $\mu$ is the character of $T_1$, and $\lambda$ is the character of $T_2$. The character $\beta$ and $-\beta$ are associated to $T_3$ and $T_4$.

The algebra $R(\delta_2)$ has 3 distinguished torus actions, one for each representation parameter, the character spaces of $T_a \times T_b \times T_c$ are the spaces $Hom_G(V(\mu), V(\lambda) \otimes V(\beta))$, where $T_a$ has character $\mu$, $T_b$ has character $\lambda$ and $T_c$ has character $\beta$. Under
the map above, $T_a$ is the diagonal in $T_1 \times T_3$, $T_b = T_2$ and $T_c = T_4$. In this way, the subspace $V_{\mu - \beta, \beta}(\lambda)$ has $T_1$-character equal to $\mu - \beta$, $T_2$-character equal to $\lambda$ and $T_3 \times T_1$ character $(-\beta, \beta)$, so this space corresponds to the $(\mu - \beta + \beta, \lambda, \beta) = (\mu, \lambda, \beta)$ character space of $T_a \times T_b \times T_c$.

**Definition 4.5.** For a string parameterization $i$, the cone $C_3(i)$ is defined by the following inequalities on $(\lambda, \tilde{t}, \beta) \in \Delta \times C(i) \times \Delta \subset \Delta \times \mathbb{Z}_{\geq 0}^N \times \Delta$.

1. $\sum_k d_k(\pi) t_k \geq 0$ for any $i$-trail from $\omega_j$ to $w_0 s_j \omega_j$ in $V(\omega_j)$, for all fundamental weights $\omega_j$ of the dual Langlands group.

2. $- \sum_k t_k \alpha_k + \lambda + \beta \in \Delta$

3. $\sum_k d_k(\pi) t_k \geq H_{\alpha_k}(\beta)$ for any $i$-trail from $s_j \omega_j$ to $w_0 \omega_j$ in $V(\omega_j)$, for all fundamental weights $\omega_j$ of the dual Langlands group.

4. $t_k + \sum_{j > k} a_{i, j, t} j \geq H_{\alpha_j}(\lambda)$

We can prove Theorem 4.2 as follows. If $b_{\lambda, \tilde{t}} \otimes v_\beta \in R_G \otimes \mathbb{C}[T]$ satisfies the above conditions, then we can recover a third weight $\mu = \sum_k t_k \alpha_k - \lambda + \beta$, and by the theorem of Berenstein and Zelevinsky, $b_{\lambda, \tilde{t}} \otimes v_\beta \in V_{\mu - \beta, \beta}(\lambda) \otimes \mathbb{C} v_\beta \subset R(\delta_2) \subset R_G \otimes \mathbb{C}[T]$. By construction, if $b_{\lambda, \tilde{t}} \otimes v_\beta$ is in $R(\delta_2)$, then it must be in some $V_{\mu - \beta, \beta}(\lambda) \otimes \mathbb{C} v_\beta$, and must have string parameters in $C_3(i)$.

Note that $C_3(i)$ is a rational cone, and specializing the parameters $\lambda, \beta, \mu$ yields the conditions from [BZ1] which index the members of the dual canonical basis in the space $V_{\mu - \beta, \beta}(\lambda)$. Just as the elements of $C(i)$ have an ordering, we place an order on $C_3(i)$ where $(\lambda_1, \lambda_2, \lambda_3, \tilde{s}) < (\eta_1, \eta_2, \eta_3, \tilde{t})$ if $(\lambda_1, \lambda_2, \lambda_3) < (\eta_1, \eta_2, \eta_3)$ as dominant weights of $G^\imath$, or $(\lambda_1, \lambda_2, \lambda_3) = (\eta_1, \eta_2, \eta_3)$ and $\tilde{s} \prec \tilde{t}$ lexicographically. Multiplication in $R(\delta_2)$ is lower triangular because the same holds for $R_G \otimes \mathbb{C}[T]$.

We can now carry out the same construction for the branching algebra $R(i_{L,G})$. First we recall the two ways to see a branching algebra.

\begin{equation}
R(i_{L,G}) = L[R_L \otimes R_G] \cong R_G^{UL}
\end{equation}

The algebra $R_G^{UL}$ sits inside $R_G$ as the direct sum of the invariant spaces $V(\lambda)^{UL}$. Each of these spaces in turn decomposes over the dominant weights of $L$.

\begin{equation}
V(\lambda)^{UL} = \bigoplus_{\eta \in \Delta_L} V_{i, \eta}(\lambda)
\end{equation}

We have $V_{i, \eta}(\lambda) \cong Hom_L(V(\eta), V(\lambda))$. The algebra $R(i_{L,G})$ comes with a natural bigrading by $\Delta_L \times \Delta_G$, which matches the bigrading by pairs of dominant weights on the spaces $V_{i, \eta}(\lambda)$. By Theorem 4.3 above, each space $V_{i, \eta}(\lambda)$ has a basis obtained by $B(I, \eta, \lambda) = B \cap V_{i, \eta}(\lambda)$, note that this is independent of the string parameterization. We choose a $i$ that is adapted to $L$ as above, with the first part of the string a reduced factorization $w_0(I)^{-1}w_0$ and the second part a factorization of $w_0(I)$. 
Definition 4.6. We define the cone $C_L(i) \subset \Delta \times \mathbb{Z}_{\geq 0}^N$ for $i = i_1 \circ i_2$ to be the subcone of $C(i)$ of strings with first $N - \ell(w_0(I^{-1}w_0))$ entries 0, which satisfy the following inequalities.

1. $\sum_{k > N - \ell(w_0(I^{-1}w_0))} d_k(\pi)t_k \geq 0$ for any $i_2$-trail from $w_0(I)\omega_j$ to $w_0s_j\omega_j$ in $V(\omega_j)$, for all fundamental weights $\omega_j$ of the dual Langlands group.

2. $\sum_{k > N - \ell(w_0(I^{-1}w_0))} t_k \alpha_k - \lambda \in \Delta_L$

3. $t_k + \sum_{j > k} a_{i_k,i_j}t_j \geq H_{i_k}(\lambda)$

The cone $C_L(i)$ also has a partial ordering given by $(\eta_1, \lambda_1, \vec{s}) < (\eta_2, \lambda_2, \vec{t})$ if $(\eta_1, \lambda_1) < (\eta_2, \lambda_2)$ as dominant weights of $L \times G$, or $(\eta_1, \lambda_1) = (\eta_2, \lambda_2)$ and $\vec{s} < \vec{t}$ lexicographically. Both $R(\delta_2)$ and $R(i_{L,G})$ are then filtered by partially ordered monoids, as in Proposition 2.8. The weight spaces in this filtration are all 1 dimensional, and the associated graded rings are $\mathbb{C}[C_3(i)]$ and $\mathbb{C}[C_L(i)]$ respectively. We can apply a tuple of coweights to the dominant weight components above to obtain a valuation into $\mathbb{Z}^{N+1}$ with the lexicographic ordering by Proposition 2.8. Since the basis is indexed by the points of a rational cone, we can also use, Proposition 2.11 and Proposition 2.12 to prove Theorem 1.8.

5. Degenerations of $R(\delta_n)$ and $R(i_{L,G})$

In this section we combine valuations constructed from factorization diagrams with those defined by the dual canonical basis to obtain toric degenerations of $R(\delta_n)$ and $R(i_{L,G})$. We go through the proof for $\delta_n$ only, as the steps are identical for $i_{L,G}$.

After choosing a $T$ to give a filtration of $R(\delta_n)$, we assign a string $i(v)$ to each internal vertex $v \in V(T)$, we call this a $T$-string. Each space $W(T, \vec{\lambda})$ is a tensor product of spaces $\text{Hom}_G(V(\lambda_1(v)), V(\lambda_2(v)) \otimes V(\lambda_3(v)))$, each with a partially ordered basis of dual canonical basis members $b_{\lambda_1(v),\lambda_2(v),\lambda_3(v)}$. The space $W(T, \vec{\lambda})$ therefore has a basis of factorization diagrams labeled by string parameters at each internal vertex.

$$b_{T,\vec{s},\vec{x}} = \bigotimes_{v \in V(T)} b_{\lambda_2(v),\lambda_1(v),\lambda_3(v),\vec{s}(v)}$$

The labels form a cone $C_T(i)$, which is a fiber product of $|V(T)|$ copies of $C_3(i)$ over copies of $\Delta_G$ assigned to the edges of $T$. This cone has a partial ordering, where $(T, \vec{x}, \vec{s}) < (T, \vec{y}, \vec{t})$ if $\vec{x} < \vec{y}$ as dominant weights of $G^{|E(T)|}$, or $\vec{x} = \vec{y}$, and $\vec{s} < \vec{t}$ lexicographically, by the ordering induced by choosing an ordering on the internal vertices of $T$.

Proposition 5.1. Multiplication in $R(\delta_n)$ with respect to the basis $b_{T,\vec{s},\vec{x}}$ is lower triangular.

$$b_{T,\vec{s},\vec{x}} \times b_{T,\vec{s},\vec{x}} = b_{T,\vec{s} + \vec{t},\vec{t} + \vec{s}} + \sum C_{T,\vec{s},\vec{r}} b_{T,\vec{s},\vec{r}}$$

where $(T, \vec{r}, \vec{r}) < (T, \vec{s} + \vec{t}, \vec{s} + \vec{t})$.

Proof. This follows from the lower triangularity of the multiplication rules for branching filtrations and the dual canonical bases. \qed
We can now apply Propositions 2.11 and 2.12 as we did on the algebras $R(\delta_2)$ and $R(i_L,G)$ to obtain Theorems 1.10 and 1.12.

6. Tensor products in type $A$

The cones $C_3(i)$ are useful in that they give toric degenerations of the full tensor algebra, but they are a challenge to write down in practice. For type $A$ partial results were obtained by Berenstein and Zelevinsky in [BZ2], with a full answer given by Gleizer and Postnikov in [GP] in terms of a device they call "web functions." These are equivalent to the honeycomb polytopes of Knutson - Tao used in the proof of the saturation conjecture for $GL_n(\mathbb{C})$, [KT], and the Berenstein Zelevinsky triangles, first described in [BZ2]. These are assignments of non-negative integers to the intersection points of diagrams like figure 4 below, with the condition that pairs on opposite sides of the same hexagon add to the same number.

![Figure 4](image-url)

Note that the sides of this triangle are lined with small triangles, $K_3(SL_m(\mathbb{C}))$ is all such assignments to the diagram with $m - 1$ small triangles to a side. We orient the triangle counter-clockwise. For a particular weighting $w$ of this diagram, let $\lambda_1(w)$, $\lambda_2(w)$, and $\lambda_3(w)$ be the vectors of numbers obtained from sides 1 2 and 3 of the diagram by adding the pairs of numbers assigned to the vertices of the little triangles, so the $j$-th entry of $\lambda_i(w)$ is the sum of the two numbers assigned to the two vertices on the $j$-th little triangle bordering the $i$-th side. These numbers define dominant weights of $SL_m(\mathbb{C})$ by $\sum \lambda_i(w)_j\omega_j$, where $\omega_j$ is the $j$-th fundamental weight. A weighting $w \in K_3(SL_m(\mathbb{C}))$ represents an invariant in the triple tensor product $V(\lambda_1(w)) \otimes V(\lambda_2(w)) \otimes V(\lambda_3(w))$. The following is essentially proved in [BZ3] and [BZ2].

**Proposition 6.1.** For a particular choice of $i$, there is a commutative diagram of linear maps of cones.

$$
\begin{array}{ccc}
K_3 & \longrightarrow & C_3(i) \\
\downarrow & & \downarrow \\
\mathbb{R}^3 & \overset{d_2}{\longrightarrow} & \mathbb{R}^3
\end{array}
$$
Here the vertical maps are projection onto triples of dominant weights, the bottom horizontal map is duality on the middle dominant weight, and the top map is a lattice isomorphism of cones.

The string cone $C_3(i)$ in this theorem is the cone of "partitions," defined in [BZ2], which realize the Littlewood-Richardson rule. The full tensor product algebra $R(\delta_n)$ therefor degenerates to the toric algebra of the fiber product of $K_3(SL_m(C))$ over a trivalent tree $T$. We represent elements of this monoid with labellings of "quilt diagrams" of BZ triangles, shown in figure 5.

![Figure 5. A quilt diagram for $R(\delta_8)$](image)

Each triangle is a BZ triangle, triangles which border each other must have the same dominant weights along their edges, and $T$ can be recovered as the dual tree to the arrangement of triangles. We refer to this semigroup as $K_T(SL_m(C))$.

We give $K_T(SL_m(C))$ as a particular product of Theorem 1.10 which appears to be amenable to computational methods. Deciphering the structure of this semigroup for small $m$ should not be difficult. We also present BZ quilts as a possible tool for studying other interesting algebras. The subsemigroup $K_T(\vec{r},SL_m(C))$ of quilts which have a multiple of $(r_1\omega_1,\ldots,r_n\omega_1)$ for boundary weights is a toric degeneration of the projective coordinate ring $\mathbb{C}[Gr_m(\mathbb{C}^n)/\vec{r}T]$ the weight variety at $\vec{r}$ of the Grassmannian. This is a classical algebra from invariant theory, and not much is known about it outside the case $m = 2$, see [HMSV]. We also observe that a similar analysis could be carried out using a Levi branching degeneration using Theorem 1.12.

References

[A] D. Anderson, Okounkov bodies and toric degenerations [arXiv:1001.4566v1].
[BSS] A. Berenstein, R. Sjamaar, Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion, J. Amer. Math. Soc. 13 (2000), no. 2, 433–466
[AB] V. Alexeev and M. Brion, Toric degenerations of spherical varieties, Selecta Mathematica 10, no. 4, (2005), 459–478.
[B] V. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Vol. 33 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI (1990).

[BHV] L. J. Billera, S. Holmes, and K. Vogtman, *The Geometry of the Space of Phylogenetic Trees*, Adv. in Appl. Math, 1999, 7333-767.

[BZ1] A. Berenstein and A. Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties*, Invent. Math, 2001, 143, 77–128.

[BZ2] A. Berenstein and A. Zelevinsky, *Canonical bases for the quantum group of type A* and *piecewise-linear combinatorics*, Duke Math. J. 82 (1996), 473.

[BZ3] A. Berenstein and A. Zelevinsky, *Triple Multiplicities for sl(r + 1) and the Spectrum of the Algebra of the Adjoint Representation*, Journal of Algebraic Combinatorics 1 (1992), 7-22.

[C] P. Caldero, *Toric degenerations of Schubert varieties*, Transform. Groups 7 (2002), no. 1, 5160.

[D] I. Dolgachev, *Lectures on Invariant Theory*, London Math. Soc. Lecture Note Series 296 (2003).

[E] D. Eisenbud *Commutative Algebra With A View Toward Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer-Verlag, 1995.

[F] W. Fulton, *Introduction to Toric Varieties* Princeton University Press, Princeton, NJ, 1993.

[1] FH W. Fulton, J. Harris, *Representation Theory* GTM, Vol. 129, Springer, Berlin, 1991.

[G] F. D. Grosshans, *Algebraic homogeneous spaces and invariant theory*, Springer Lecture Notes, vol. 1673, Springer, Berlin, 1997.

[GP] O. Gleizer and E. Postnikov, *Littlewood-Richardson coefficients via Yang-Baxter equation*, Internat. Math. Res. Notices, 1999, 741–774.

[HMM] B. J. Howard, C. A. Manon, J. J. Millson, *The toric geometry of triangulated polygons in Euclidean space*, Canadian Journal of Mathematics, to appear.

[HMSV] B. J. Howard, J. J. Millson, A. Snowden, and R. Vakil, *The projective invariants of ordered points on the line*, Duke Math. J., 146 No. 2 (2009), 175-226.

[HTW] Roger E. Howe, Eng-Chye Tan, and Jeb F. Willenbring, *Reciprocity Algebras and branching for classical symmetric pairs*, [arXiv:math/0407467v2 [math.RT]].

[HTW2] Roger Howe, Eng-Chye Tan, and Jeb Willenbring, *A Basis for the GLn Tensor Product Algebra*, Adv. Math., vol 196, issue 2, 531-564.

[K] K. Kaveh, *Crystal bases and Newton-Okounkov bodies*, [arXiv:1101.1687v1 [math.AG]].

[Ka] M. Kashiwara, *The crystal base and Littelmanns refined Demazure character formula*, Duke Math. J. 71 (1993), 839858.

[KT] A. Knutson, T. Tao, and C. Woodward, *The honeycomb model of GL(n) tensor products II: Puzzles determine facets of the Littlewood-Richardson cone*, Journal of the AMS, 17 (2004) 19-48.

[Lu] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. 4 (1991) 356-421.

[M1] C. Manon, *Presentations of semigroup algebras of weighted trees*, J. Alg. Comb. 2010 31: 467-489.

[M2] C. Manon, *The algebra of conformal blocks*, submitted, [arXiv:0910.0577v4 [math.AG]]

[M3] C. Manon, *Dissimilarity vectors and the representation theory of SL_m(C)*, J. Alg. Comb. 2011 33: 199-213.

[Mat] O. Mathieu, *Good bases for G-modules*, Geometriae Dedicata, Volume 36, Number 1, 51-66.

[Mu] D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik und Ihrer Grenzgebiete 34 (1965), Springer.

[P] S. Payne, *Analytification is the limit of all tropicalizations* Math. Res. Lett. 16 (2009), no. 3, 543556.

[SpSt] D. Speyer and B. Sturmfels, *The tropical Grassmannian*, Adv. Geom. 4, no. 3, (2004), 389-411.

[St] B. Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996.

[Zh] D. P. Zhelobenko, *Compact Lie Groups and Their Representations*, American Mathematical Society, Providence, 1973.
