SUBCLASSES OF MEROMORPHIC STARLIKE FUNCTIONS CONNECTED TO MULTIPLIER FAMILY

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Abstract. The object of this paper is studying some properties of meromorphic functions which satisfy in the condition
\[ \text{Re}(zf(z)) > \alpha |z^2 f'(z) + zf(z)|. \]
Parallel results for some related classes are also obtained.

1. Introduction and Definitions

Denote by \( \sum \) the family of functions
\[ f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \quad (1) \]
which are analytic in the punctured disc \( E = \{ z : 0 < |z| < 1 \} \) with simple pole at \( z = 0 \). A function \( f \in \sum \) is said to be in the class \( \sum^*(\alpha) \) of meromorphic starlike functions of order \( \alpha \) if and only if
\[ \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < -\alpha \quad (z \in E; 0 \leq \alpha < 1). \]
Set \( \sum^*(0) = \sum^* \).

We further let \( ME(\alpha), 0 \leq \alpha, \) be the subclasses of \( \sum \) consisting of functions of the form (1) which satisfy the condition
\[ \text{Re}(zf(z)) > \alpha |z^2 f'(z) + zf(z)|. \]
Set \( ME = ME(1) \). Also for, \( 0 \leq \alpha < 1 \), let \( MF(\alpha) \) be the subclasses of \( \sum \) consisting of functions of the form (1) which satisfy the condition
\[ \left| \frac{zf'(z)}{f(z)} + 1 \right| < 1 - \alpha, \]
for \( z \in E \). Set \( MF = MF(0) \).

Many important properties and characteristics of various interesting subclasses of the class \( \sum \), including (for example) the class \( \sum^*(\alpha) \), were investigated by (among others) Liu and Srivastava \cite{5}, \cite{6} and Aouf \cite{1}. Also some interesting properties of analytic functions related to multiplier family were studied by Fournier et al in \cite{3} and Ahuja et al in \cite{2} and Rosy et al in \cite{9}. In this paper we aim to obtain several properties of functions belong to the classes \( ME(\alpha), MF(\alpha) \) and \( MTE(\alpha) \).

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2. Main Results

We begin by proving inclusion relation between classes which are defined in the above.

**Theorem 2.1**

$$ME(\alpha) \subset MF(1 - \frac{1}{\alpha}) \subset \sum^*(1 - \frac{1}{\alpha}) \quad 1 \leq \alpha.$$ 

If $\alpha = 1$ all inclusions are proper and for $\alpha > 1$ the result is sharp.

**Proof.** If $f \in ME(\alpha)$, then

$$|zf(z)| > \alpha |z^2f'(z) + zf(z)| \quad \text{or} \quad \left| \frac{zf'(z)}{f(z)} + 1 \right| < \frac{1}{\alpha}. \quad (2)$$

Hence

$$Re \left( \frac{zf'(z)}{f(z)} \right) < -1 + \frac{1}{\alpha} \quad \text{or} \quad -Re \left( \frac{zf'(z)}{f(z)} \right) > 1 - \frac{1}{\alpha}. \quad (3)$$

By making use of (2) and (3) we get our result. But for $\alpha = 1$ it is easy to see that $e^z \in MF - ME$ and $(1-z^2)^{\frac{1}{2}} \in \sum^* - MF$. Now for sharpness set $f(z) = \frac{1+cz}{z-cz^2}$, $c = (1 + \alpha^2)^{\frac{1}{2}} - \alpha$. Then $f \in ME(\alpha)$ because for $|z| = r < 1$

$$Re f(z) = 1 - \frac{c^2 z^2}{1-cz^2} \geq \alpha \frac{2cr}{1-cz^2} = \alpha |z^2f'(z) + zf(z)|.$$

Note that

$$\frac{-zf'(z)}{f(z)} = 1 - \frac{2c}{1-cz^2},$$

for $z = -r, r \to 1$, this last expression approaches to $1 - \frac{2c}{1-cr^2} = 1 - \frac{1}{\alpha}$. Thus $f \notin MF(\beta)$ and $f \notin \sum^*(\beta)$ for $\beta > 1 - \frac{1}{\alpha}$.

Next we determine a sufficient condition for a function of the form (1) to be in the class $ME(\alpha)$.

**Theorem 2.2.** A sufficient condition for a function of the form (1) to be in the $ME(\alpha)$ is that

$$\sum_{n=0}^{\infty} [1 + \alpha(n+1)] |a_n| \leq 1.$$ 

**Proof.** Let $f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n$, then

$$Re(zf(z)) = Re \left( 1 + \sum_{n=0}^{\infty} a_n z^{n+1} \right) \geq 1 - \sum_{n=0}^{\infty} |a_n| \quad (4)$$

and also

$$\alpha |z^2f'(z) + zf(z)| \leq \alpha \sum_{n=0}^{\infty} (n+1) a_n z^{n+1}. \quad (5)$$

By making use of (4) and (5) we get our result.

Remark 1. By Theorem 2.1 it follows that $ME \subset \sum^*$, also we note that Theorem 2.2 implies that $g(z) = z^{-1} + \frac{z}{n+2} z^n \in ME$ for any $n \geq 1$, but if $n > \frac{2-\alpha}{\alpha}$ then $g$ is not in $\sum^*(\alpha)$, hence $ME$ is not subset of $\sum^*(\alpha)$ for any $\alpha > 0$.

We shall need the following lemma, which is due to Miller and Mocanu [7] to prove the coefficient estimates for functions belonging to the class $ME(\alpha)$.
Lemma 1. Let a function \( w(z) = a + w_m z^m + \cdots \) be analytic in the unit disc with \( w(z) \neq a \) and \( m \geq 1 \). If \( z_0 = r_0 e^{i\theta} \) (\( 0 < r_0 < 1 \)) and \( |w(z_0)| = \max_{|z| \leq r_0} |w(z)| \). Then \( z_0 w'(z_0) = kw(z_0) \) and \( \Re \left( 1 + \frac{z_0 w'(z_0)}{w(z_0)} \right) \geq k \), where \( k \) is real and \( k \geq m \).

Theorem 2.3. If the function \( f \) given by (1) belongs to the class \( ME(\alpha) \), then

\[
|a_n| \leq \frac{2}{\sqrt{\alpha^2(n + 1)^2 + 1 + \alpha(n + 1)}}, \quad n \geq 0.
\]

The result is sharp for function \( z f(z) = \frac{1 + d_n z^n}{1 - d_n z^n} \) where \( d_n = \sqrt{\alpha^2 n^2 + 1 + \alpha n} \).

Proof. Let \( f \in ME(\alpha) \) and \( z f(z) = 1 + Az^n + \ldots \). It is sufficient to show that \( |A| \leq 2d_n \). For this let \( z f(z) = \frac{1 + d_n w(z)}{1 - d_n w(z)} \). It is easy to see that \( w \) is analytic in the unit disc and \( w(0) = 0 \). We wish to show that \( |w(z)| < 1 \), for all \( z \) in the unit disc. For, if not, by Lemma 1 there exists \( z_0 \) in the unit disc such that \( |w(z_0)| = 1 \) and \( z_0 w'(z_0) = kw(z_0), k \geq n \) and hence

\[
Re(z_0 f(z_0)) - \alpha|z_0 f(z_0)|^2 = \frac{1 - d_n^2 - 2kd_n}{|1 - d_n - nw(z_0)|^2} \leq \frac{1 - d_n^2 - 2nd_n}{|1 - d_n - nw(z_0)|^2} = 0,
\]

which contradicts \( f \in ME(\alpha) \). Now the result follows from the well known result of Robertson [10].

Remark 2. Also we note that if \( f \in ME(\alpha), \alpha \geq 1, \) then \( Re(z^2 f'(z)) < 0, z \in E \). Since if \( f(z) = \frac{g'(z)}{z} \in ME(\alpha) \) where \( g(z) \) is an analytic function in the unit disk, then \( f'(z) = \frac{z g''(z) - g(z)}{z^2} \). Hence

\[
Re(-z^2 f'(z)) = Re(g'(z) - Re zg''(z) > Re g'(z) - \alpha|zg''(z)| > 0,
\]

which yields result.

3. Neighborhoods And Partial Sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [4] and Ruscheweyh [8], we begin by introducing here the \( \delta \)-neighborhood of a function of the form (1) by the means of the definition

\[
N_\delta(f) = \{ g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k \mid g \text{ is analytic in } E \text{ and } \sum_{k=1}^{\infty} k|a_k - b_k| \leq \delta \}.
\]

From Theorem 2.2 for the function \( f(z) = \frac{1}{z} \), we immediately have \( N_{\frac{1}{1 + \gamma} f(1)} \subset ME(\alpha) \).

For function \( f \in \sum \) given by (1) and \( g \in \sum \) given by

\[
g(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n,
\]

we define the Hadamard product (or convolution) of \( f \) and \( g \) by

\[
(f \ast g)(z) = z^{-1} + \sum_{n=0}^{\infty} a_n b_n z^n = (g \ast f)(z).
\]

We next give a multiplier convolution characterization for \( ME(\alpha) \).

Theorem 3.1 \( f \in ME(\alpha) \Leftrightarrow Re f(z) * \frac{1 + z(\alpha e^{i\gamma} - 1)}{z(1 - z^2)} > 0, \gamma \in (-\pi, \pi], z \in E \).

Proof. We have
\[ zf(z) + e^{i\gamma }az(zf)' = zf(z) \ast \left[ \frac{1}{1 - z} + e^{i\gamma } \frac{z}{(1 - z)^2} \right] \]

\[ = zf(z) \ast \left( \frac{1 + z(e^{i\gamma } - 1)}{z(1 - z)^2} \right). \]

(2.1)

Hence we get our result.

**Theorem 3.2.** If \( \frac{L(z) - \epsilon z^{-1}}{1 - \epsilon} \in ME(\alpha) \), for \( \delta < \epsilon < 1 \), then \( N_\gamma(f) \subset ME(\alpha) \) where \( \gamma = \frac{1}{1 + 2\alpha} \).

**Proof.** Let \( h(z) = z^{-1} + \sum_{k=0}^{\infty} c_k z^k = \frac{1 + z(e^{i\gamma } - 1)}{z(1 - z)^2} \). It is not difficult to verify that \( |c_k| \leq (1 + \alpha(k + 1)), k = 0, 1, 2, 3, \ldots \). Let \( g \in N_\gamma(f) \) and \( g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k \). Then

\[ \text{Re}(z(g * h)) = \text{Re}(z((g - f) * h) + z(f * h)) = \text{Re}(z((g - f) * h)) + \text{Re}(z(f * h)) \] (8)

But

\[ \text{Re}(z((g - f) * h)) \geq -|z((g - f) * h)| = -\sum_{k=0}^{\infty} (b_k - a_k)c_k z^k| > -\delta, \] (9)

since \( g \in N_\gamma(f) \). Again \( \frac{L(z) - \epsilon z^{-1}}{1 - \epsilon} \in ME(\alpha) \), for \( \delta < \epsilon < 1 \), implies that \( \text{Re}(z(\frac{L(z) - \epsilon z^{-1}}{1 - \epsilon} * h)) > 0 \) by Theorem 3.1. That is,

\[ \text{Re}(z(f * h)) > \epsilon \quad \text{for} \quad \delta < \epsilon < 1. \] (10)

Using (9),(10) in (8) we see that \( \text{Re}(z(g * h)) > 0 \) for all \( z \in E \). Hence Theorem 3.1 show that \( g \in ME(\alpha) \).

**4. Negative Coefficients.**

In this section at first we introduce the subclass \( TME(\alpha) \) consisting of all functions \( f \in ME(\alpha) \) which are in the form

\[ f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geq 0), \]

and then we obtain several properties of functions belong to \( TME(\alpha) \).

**Theorem 4.1.** A function \( f \) of the form \( f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^n \) is in \( TME(\alpha) \) if and only if

\[ \sum_{n=1}^{\infty} (1 + \alpha(n + 1))a_n \leq 1. \]

The result is sharp for the function \( f(z) \) given by

\[ f(z) = z^{-1} - \left( \frac{1}{1 + \alpha(n + 1)} \right) z^n, \quad n = 1, 2, 3, \ldots. \]

**Proof.** In view of Theorem 2.2, we need only show that \( f \in TME(\alpha) \) satisfies the coefficient condition. For \( z = re^{i\theta}, 0 \leq r < 1 \) and \( 0 \leq \theta < 2\pi \) we have

\[ rf(r) = 1 - \sum_{n=1}^{\infty} a_n r^{n+1} \quad \text{and} \quad \alpha |r^2 f'(r) + r f(r)| = \alpha \sum_{n=1}^{\infty} (n + 1)a_n r^{n+1}. \]

The result follows upon letting \( r \to 1 \).

The coefficient characterization of Theorem 4.1 enables us to determine extreme points and distortion theorems.

**Corollary 1.** The extreme points of \( TME(\alpha) \) are \( f_1(z) = z^{-1} \) and

\[ f_n(z) = z^{-1} - \frac{z^n}{1 + \alpha(n + 1)}, n = 1, 2, 3, \ldots. \]
And $f \in TME(\alpha)$ if and only if $f$ can be written in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f(z), \quad \text{where} \quad \lambda_k \geq 0, \quad \sum_{k=1}^{\infty} \lambda_k = 1.$$ 

**Corollary 2.** If $f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0$ is in $TME(\alpha)$, then

$$\frac{1}{r} - \frac{1}{1+2\alpha} r \leq |f(z)| \leq \frac{1}{r} + \frac{1}{1+2\alpha} r,$$

with equality for $f(z) = \frac{1}{r} - \frac{1}{1+2\alpha} z$ at $z = r, i r$.

Finally we prove

**Theorem 4.2.** Let $f \in \sum$ be given by (1) and define the partial sums $S_1(z)$ and $S_n(z)$ by $S_1(z) = z^{-1}$ and $S_n(z) = z^{-1} + \sum_{k=1}^{n-1} a_k z^k$.

Suppose also that

$$\sum_{k=1}^{\infty} d_k |a_k| \leq 1 \quad (d_k = 1 + \alpha(n + 1)).$$  \hspace{1cm} (11)

Then we have

$$\text{Re} \left( \frac{f(z)}{S_n(z)} \right) > 1 - \frac{1}{d_n} \quad \text{and} \quad \text{Re} \left( \frac{S_n(z)}{f(z)} \right) > \frac{d_n}{1 + d_n} \quad (z \in E; n \in N = \{1, 2, 3, \ldots \}).$$  \hspace{1cm} (12)

Each of the bounds in (12) is the best possible for $n \in N$.

**Proof.** For the coefficients $d_k$ given by (11), it is not difficult to verify that $d_{k+1} > d_k > 1, k = 1, 2, 3, \ldots$. Therefore, by using the hypothesis (11), we have

$$\sum_{k=1}^{n-1} |a_k| + d_n \sum_{k=n}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} d_k |a_k| \leq 1.$$  \hspace{1cm} (13)

By setting

$$g_1(z) = d_n \left( \frac{f(z)}{S_n(z)} - (1 - \frac{1}{d_n}) \right) = 1 + \frac{d_n \sum_{k=n}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^{n-1} a_k z^{k+1}}$$  \hspace{1cm} (14)

and applying (13), we find that

$$\frac{g_1(z) - 1}{g_1(z) + 1} \leq \frac{d_n \sum_{k=n}^{\infty} |a_k|}{2 - 2 \sum_{k=1}^{n-1} |a_k| - d_n \sum_{k=n}^{\infty} |a_k|} \leq 1 \quad (z \in E),$$  \hspace{1cm} (15)

which readily yields the left assertion (12) of Theorem 4.2. If we take

$$f(z) = z^{-1} - \frac{z^n}{d_n},$$  \hspace{1cm} (16)

then

$$\frac{f(z)}{S_n(z)} = 1 - \frac{z^{n+1}}{d_n} \rightarrow 1 - \frac{1}{d_n} \quad \text{as} \quad z \rightarrow 1^-,$$

which shows that the bound in (12) is the best possible for each $n \in N$. Similarly, if we put

$$g_2(z) = (1 + d_n) \left( \frac{S_n(z)}{f(z)} - \frac{d_n}{1 + d_n} \right) = 1 - \frac{(1 + d_n) \sum_{k=n}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^{\infty} a_k z^{k+1}}$$  \hspace{1cm} (17)

and make use of (13) we obtain

$$\frac{g_2(z) - 1}{g_2(z) + 1} \leq \frac{(1 + d_n) \sum_{k=n}^{\infty} |a_k|}{2 - 2 \sum_{k=1}^{n-1} |a_k| + (1 - d_n) \sum_{k=n}^{\infty} |a_k|} \leq 1 \quad (z \in E).$$
which leads us to the assertion (12) of Theorem 4.2. The bounds given in the right of (12) is sharp with the function given by (16). The proof of Theorem 4.2 is thus complete.

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