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A Numerical Study of the Vibration Spectrum for a Double-Walled Carbon Nanotube Model

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1. Introduction

In this paper we offer a series of new results devoted to the numerical analysis of a double-walled carbon nanotube model. This model is given in the form of two coupled Timoshenko beams connected through the distributed Van der Waals force (Gibson et al., 2007; Ru, 2000). Typically, nanotubes can be modeled as quantum systems and studied by a molecular simulations approach, or as classical systems (such as flexible beams, shells membranes (Mahan, 2002; Pantano et al., 2003; 2004; Wang et al., 2004; 2005)), or as specific hybrid models (Wang, 2005). The choice of model in any situation involves a tradeoff in that, while molecular models may yield more accurate results, implementing them is extremely time and labor intensive, which is not the case for models from continuum mechanics.

The scientific and engineering communities have acknowledged the very desirable properties of carbon nanotubes (CNTs) and their potential use in wide-ranging applications. The author of (Jamieson, 2000) argues that nanotechnology, mainly due to CNTs, may impact technology more than did the silicon revolution. Depending on the atomic structure, CNTs have electrical properties that can range from those of metals to those of semiconductors. The mechanical properties of CNTs are also unique. They possess exceptionally high specific stiffness and specific strength; they are extremely elastic, being able to bend through a complete $360^\circ$ without noticeable damage. The application potential for materials with these properties is almost limitless.

Developing mathematical models for CNTs is of critical importance. Such models must be verified and quantified by performing and analyzing experiments. As we have mentioned, two groups of models exist: molecular simulation models and continuum mechanics models. Continuum models are generally based on traditional engineering models such as beams, shells, or membranes. The nanotubes are treated as continuous materials with definite geometries and common material properties such as Young's modulus. In contrast, molecular models consider each atom, and mathematically define the interactions among the atoms. Based on their work on atomic simulations of CNTs, the authors of (Jakobson et al., 1996) provide a justification for incorporating continuum mechanics models into CNTs study,

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stating that “The laws of continuum mechanics are amazingly robust and allow one to treat
even intrinsically discrete objects only a few atoms in diameter.”

The most commonly used models are the following: the Euler–Bernoulli beam model,
Timoshenko beam model, and flexible shell and membrane models. Typically, many models
for multi-walled nanotubes allow for independent wall movement, and the wall interaction is
a function of the local wall separation distance.

Vibration of a double-walled carbon nanotube (DWCNT) generated by a nonlinear interlayer
Van der Waals force is studied in (Xu et al., 2006). The results indicate that the nonlinear
factors of the Van der Waals force, on the one hand, have little effect on the coaxial free
vibrations. On the other hand, these nonlinear factors greatly affect noncoaxial free vibrations.
As is indicated in (Qian et al., 2002), although carbon nanotubes can have diameters only
several times larger than the length between carbon atoms, continuum models have been
found to describe their mechanical behavior very accurately, in many circumstances.

Our analysis of an initial boundary-value problem models small transversal vibrations of a
double-walled carbon nanotube. The system of equations is similar to the ones mentioned in
a number of papers (see references (Gibson et al., 2007; Jakobson et al., 1996; Pantano et al.,
2003; Qian et al., 2002; Ru, 2001; Wang et al., 2006; Xu et al., 2006; Yoon et al., 2003)). The
physical system consists of two nested nanotubes interacting through the distributed Van der
Waals force; each nanotube is modeled as a Timoshenko beam with specific parameters. As
pointed out in (Wang et al., 2006), “Unlike the Euler–Bernoulli beam model, the Timoshenko
beam model allows for the effects of transverse shear deformation and rotary inertia. These
effects become significant for carbon nanotubes with small length-to-diameter ratios that are
normally encountered in applications.”

The model is given in the form of two coupled Timoshenko beams (i.e., in the form of
four coupled hyperbolic partial differential equations). The system is equipped with a set
of nonself-adjoint boundary conditions involving four independent complex parameters.
Indeed, all other articles treating the Timoshenko model consider only the traditional
energy-conserving boundary conditions, thus our treatment is a generalization of their work
(as these latter conditions are just limiting special cases of the nonself-adjoint conditions
treated herein). An asymptotic analysis of the eigenspectrum for this problem was performed
in (Shubov & Rojas-Arenaza, 2010a;b;c), under certain simplifying assumptions. We must
mention that the assumptions are somewhat restrictive—indeed, they cannot be satisfied by
a physical double-walled carbon nanotube system. However, even for this simplified case,
the necessary computations were extremely complex and cumbersome, and it is unclear if the
more general problem even is tractable.

Regardless, this special case is a valid and interesting mathematical problem whose behavior
should be quite similar to the more general physical problem. Thus, we feel that a study of the
vibration spectrum for this case certainly will shed light on the spectrum of the more general
problem, particularly by our choosing values for the physical parameters that are similar to
those for physical carbon nanotubes.

The paper is organized as follows. In Section 2, we introduce the general mathematical
model, perform separation of variables and rewrite the special case of the model treated
in (Shubov & Rojas-Arenaza, 2010a) in dimensionless form. In Section 3, we present the
asymptotic results derived in (Shubov & Rojas-Arenaza, 2010a). The Legendre-tau spectral
method is described in Section 4, and in Section 5 we present our numerical results and
comparison with the asymptotic results predicted by (Shubov & Rojas-Arenaza, 2010a).
2. The mathematical model

We consider the system consisting of two Timoshenko beams coupled through the van der Waals force, as given in (Shubov & Rojas-Arenaza, 2010a;b;c):

\[ \sigma A_1 W_{1x}(x,t) + k_1 G A_1 [\Phi_{1x}(x,t) - W_{1xx}(x,t)] = -C[W_2(x,t) - W_1(x,t)] \]  
(1)

\[ \sigma I_1 \Phi_{1ff}(x,t) - E I_1 \Phi_{1xx}(x,t) + k_1 G A_1 [\Phi_1(x,t) - W_{1x}(x,t)] = 0 \]  
(2)

\[ \sigma A_2 W_{2x}(x,t) + k_2 G A_2 [\Phi_{2x}(x,t) - W_{2xx}(x,t)] = C[W_2(x,t) - W_1(x,t)] \]  
(3)

\[ \sigma I_2 \Phi_{2ff}(x,t) - E I_2 \Phi_{2xx}(x,t) + k_2 G A_2 [\Phi_2(x,t) - W_{2x}(x,t)] = 0. \]  
(4)

For boundary conditions, the left end of each beam is free, while the right end of each is subject to the standard set of two-parameter boundary conditions:

\[ W_{1x}(0,t) - \Phi_1(0,t) = \Phi_{1x}(0,t) = 0 \]  
(5,6)

\[ W_{2x}(0,t) - \Phi_2(0,t) = \Phi_{2x}(0,t) = 0 \]  
(7,8)

\[ k_1 G A_1 [\Phi_1(L,t) - W_{1x}(L,t)] = \sigma I_1 \alpha_1 W_{1f}(L,t) \]  
(9)

\[ E \Phi_{1x}(L,t) = -\sigma \beta_1 \Phi_{1f}(L,t) \]  
(10)

\[ k_2 G A_2 [\Phi_2(L,t) - W_{2x}(L,t)] = \sigma I_2 \alpha_2 W_{2f}(L,t) \]  
(11)

\[ E \Phi_{2x}(L,t) = -\sigma \beta_2 \Phi_{2f}(L,t). \]  
(12)

Here, \( 0 \leq x \leq L \) where \( L \) is the length of each beam, and \( t \geq 0 \). \( W_i(x,t) \) is the transverse displacement of beam \( i \), \( \Phi_i(x,t) \) is the bending angle of beam \( i \), \( i = 1,2 \). The physical and geometrical constants are as follows: \( \sigma \) is the mass per unit volume; \( E \), Young’s modulus; \( G \), the shear modulus; \( A_i \), the uniform cross-sectional area of beam \( i \); \( I_i \) the uniform area moment of inertia of beam \( i \); and \( k_i \); the shear connection factor for beam \( i \). We note that \( E = 2(1 + \nu)G \), where \( \nu \) is the Poisson’s ratio.

Further, we note the following:

\[ \alpha_i = \beta_i = 0 \Rightarrow \text{right end of beam } i \text{ is free} \]  
(13)

\[ \alpha_i = \beta_i = \infty \Rightarrow \text{right end of beam } i \text{ is clamped} \]  
(14)

\[ \alpha_i = \infty, \beta_i = 0 \Rightarrow \text{right end of beam } i \text{ is simply-supported} \]  
(15)

\[ \alpha_i = 0, \beta_i = \infty \Rightarrow \text{right end of beam } i \text{ is roller-supported.} \]  
(16)

We separate variables by letting

\[ W_j(x,t) = e^{-i\omega t} w_j(x), \]

\[ \Phi_j(x,t) = e^{-i\omega t} \phi_j(x), \]
\( j = 1, 2, \) and, following the notation in (Shubov & Rojas-Arenaza, 2010a), the system (1)–(12) becomes

\[
\begin{align*}
\omega^2 w_1(x) &= \tilde{k}_1 [\phi'_1(x) - w''_1(x)] + C_1 [w_2(x) - w_1(x)] \quad (17) \\
\omega^2 \phi'_1(x) &= -\frac{E}{\sigma} \phi''_1(x) + \tilde{k}_1 [\phi_1(x) - w'_1(x)] \quad (18) \\
\omega^2 w_2(x) &= \tilde{k}_2 [\phi'_2(x) - w''_2(x)] - C_2 [w_2(x) - w_1(x)] \quad (19) \\
\omega^2 \phi'_2(x) &= -\frac{E}{\sigma} \phi''_2(x) + \tilde{k}_2 [\phi_2(x) - w'_2(x)] \quad (20) \\
w'_1(0) - \phi'_1(0) &= 0 \quad (21) \\
\phi'_1(0) &= 0 \quad (22) \\
w'_2(0) - \phi'_2(0) &= 0 \quad (23) \\
\phi_2(0) &= 0 \quad (24) \\
\tilde{k}_1 [\phi_1(L) - w'_1(L)] &= -i \omega \alpha_1 w_1(L) \quad (25) \\
\frac{E}{\sigma} \phi'_1(L) &= i \omega \beta_1 \phi_2(L) \quad (26) \\
\tilde{k}_2 [\phi_2(L) - w'_2(L)] &= -i \omega \alpha_2 w_2(L) \quad (27) \\
\frac{E}{\sigma} \phi'_2(L) &= i \omega \beta_2 \phi_2(L) \quad (28)
\end{align*}
\]

Here, we have

\[ \tilde{k}_i = \frac{k_i G A_i}{\sigma L}, \quad \tilde{k}_i = \frac{k_i G}{\sigma}, \quad C_i = \frac{C}{\sigma A_i}, \quad i = 1, 2. \]

Again, following (Shubov & Rojas-Arenaza, 2010a), we consider the special case

\[ k_1 = k_2 = \hat{k}, \quad \tilde{k}_1 = \tilde{k}_2 = \hat{k}. \quad (29) \]

We must note that these conditions cannot hold for a physical double-walled carbon nanotube (e.g., the shape factors must be different, \( k_1 \neq k_2 \)). However, without these assumptions, the asymptotic treatment of the problem becomes extremely difficult, and possibly intractable. Thus, at this point in time, this particular special case is the only one for which there are analytical results with which to compare. We now cast the problem in dimensionless form. Following (Trail-Nash & Collar, 1953) and, more appropriately, (Coleman & Schaffer, 2010), we introduce dimensionless quantities as follows:

\[
\begin{align*}
\hat{x} &= \frac{x}{L}, \quad \hat{w}_i(\hat{x}) = \frac{1}{L} w_i(x), \quad \hat{\phi}_i(\hat{x}) = \phi_i(x), \quad i = 1, 2, \\
\lambda &= \sqrt{\frac{\sigma k}{E k} L^2 \omega}, \quad \gamma_1 = \frac{\hat{k}}{kL^2}, \quad \gamma_2 = \frac{E}{\sigma k L^2}, \\
\alpha'_i &= \frac{1}{\sigma A k L} \sqrt{\frac{E k}{\sigma k}} \alpha_i, \quad \beta'_i = \frac{1}{\sigma A k L^3} \sqrt{\frac{E k}{\sigma k}} \beta_i, \quad i = 1, 2, \\
C'_i &= \frac{L^2}{k} C_i = \frac{L^2}{k_i G A_i} C, \quad i = 1, 2.
\end{align*}
\]
We abuse notation, and use \( x, w_i, \phi_i \) instead of \( \hat{x}, \hat{w}_i \) and \( \hat{\phi}_i \), and the resulting dimensionless system is

\[
- \gamma_2 \lambda^2 w_1(x) = -\phi'_1(x) + w''_1(x) - C'_1[w_2(x) - w_1(x)] \tag{31}
\]

\[
- \gamma_1 \gamma_2 \lambda^2 \phi_1(x) = \gamma_2 \phi''_1(x) - \phi_1(x) + w'_1(x) \tag{32}
\]

\[
- \gamma_2 \lambda^2 w_2(x) = -\phi'_2(x) + w''_2(x) + C'_2[w_2(x) - w_1(x)] \tag{33}
\]

\[
- \gamma_1 \gamma_2 \lambda^2 \phi_2(x) = \gamma_2 \phi''_2(x) - \phi_2(x) + w'_2(x), \quad 0 < x < 1, \tag{34}
\]

\[w'_1(0) - \phi_1(0) = 0 \tag{35}\]

\[\phi'_1(0) = 0 \tag{36}\]

\[w'_2(0) - \phi_2(0) = 0 \tag{37}\]

\[\phi'_2(0) = 0 \tag{38}\]

\[\phi_1(1) - w'_1(1) + i \alpha'_1 \lambda w_1(1) = 0 \tag{39}\]

\[\gamma_2 \phi'_1(1) - i \beta'_1 \lambda \phi(1) = 0 \tag{40}\]

\[\phi_2(1) - w'_2(1) + i \alpha'_2 \lambda w_2(1) = 0 \tag{41}\]

\[\gamma_2 \phi'_2(1) + i \beta'_2 \lambda \phi(1) = 0. \tag{42}\]

### 3. Asymptotic estimation of vibration spectrum

The first-order asymptotic estimation of the vibration frequencies for problem (31)–(42) is given in Theorem 2.5, of (Shubov & Rojas-Arenaza, 2010a); we present the results here, but in dimensionless form.

**Theorem** (Shubov, Rojas-Arenaza). Assume that the boundary parameters \( \alpha'_i \) and \( \beta'_i \), \( i \geq 1, 2 \), satisfy the following conditions

\[
\alpha'_1 \neq \alpha'_2, \quad \beta'_1 \neq \beta'_2, \quad \alpha'_i \neq \sqrt{\gamma_2}, \quad \beta'_i \neq \sqrt{\gamma_1 \gamma_2}, \quad \text{and}
\]

\[
\left| \frac{\alpha'_1 - \sqrt{\gamma_2}}{\alpha'_1 + \sqrt{\gamma_2}} \right| \neq \left| \frac{\beta'_1 - \sqrt{\gamma_1 \gamma_2}}{\beta'_1 + \sqrt{\gamma_1 \gamma_2}} \right|.
\]

Then, the set of frequencies \(-i\lambda\) of system (31)–(42) splits into the following four separate branches:

\[
- i \lambda_n^{(1)} = \frac{1}{2 \sqrt{\gamma_2}} \left[ \log \left( \frac{1 - \alpha''_1}{1 + \alpha''_1} + 2 n \pi i \right) + O \left( \frac{1}{n} \right) \right], \tag{43}
\]

\[
- i \lambda_n^{(2)} = \frac{1}{2 \sqrt{\gamma_1}} \left[ \log \left( \frac{1 - \beta''_1}{1 + \beta''_1} + 2 n \pi i \right) + O \left( \frac{1}{n} \right) \right], \tag{44}
\]

\[
- i \lambda_n^{(3)} = \frac{1}{2 \sqrt{\gamma_2}} \left[ \log \left( \frac{1 - \alpha''_2}{1 + \alpha''_2} + 2 n \pi i \right) + O \left( \frac{1}{n} \right) \right], \tag{45}
\]

\[
- i \lambda_n^{(4)} = \frac{1}{2 \sqrt{\gamma_1}} \left[ \log \left( \frac{1 - \beta''_2}{1 + \beta''_2} + 2 n \pi i \right) + O \left( \frac{1}{n} \right) \right], \quad n = \pm 1, 2, 3, \ldots \tag{46}
\]

where

\[
\alpha''_i = \frac{1}{\sqrt{\gamma_2}} \alpha'_i, \quad \beta''_i = \frac{1}{\sqrt{\gamma_1 \gamma_2}} \beta'_i, \quad i = 1, 2. \quad \square
\]
We note that log represents the complex logarithm, \( \log z = \ln |z| + i \arg z \). We note also the important fact that the Van der Waals force between the two tubes does not appear in the first-order approximation.

4. The legendre-tau spectral method

We compare the asymptotic results of the Theorem with a numerical approximation of the spectrum using the Legendre-tau spectral method (Gottlieb et al., 1984). This entails transforming problem (1)–(12) to one on the interval \(-1 \leq x \leq 1\) by letting \( x = \frac{2}{L} x - 1 \). Assuming there will be no confusion, we keep the original variables \( x, w_i, \phi_i, i = 1, 2 \), and the resulting system is

\[
\omega^2 w_1(x) = \frac{2 \tilde{k}}{L} \phi_1'(x) - \frac{4 \tilde{k}}{L^2} w_1''(x) + C_1 [w_2(x) - w_1(x)] \quad (47)
\]

\[
\omega^2 \phi_1(x) = -\frac{4 E}{\sigma L^2} \phi_1''(x) + \tilde{k} \phi_1(x) - \frac{2 \tilde{k}}{L} w_1'(x) \quad (48)
\]

\[
\omega^2 w_2(x) = \frac{2 \tilde{k}}{L} \phi_2'(x) - \frac{4 \tilde{k}}{L^2} w_2''(x) - C_2 [w_2(x) - w_1(x)] \quad (49)
\]

\[
\omega^2 \phi_2(x) = -\frac{4 E}{\sigma L^2} \phi_2''(x) + \tilde{k} \phi_2(x) - \frac{2 \tilde{k}}{L} w_2'(x), \quad -1 \leq x \leq 1, \quad (50)
\]

\[
\frac{2}{L} w_1'(-1) - \phi_1(-1) = 0 \quad (51)
\]

\[
\phi_1'(-1) = 0 \quad (52)
\]

\[
\frac{2}{L} w_2'(1) - \phi_2(-1) = 0 \quad (53)
\]

\[
\phi_2'(1) = 0 \quad (54)
\]

\[
\tilde{k} \phi_1(1) - \frac{2 \tilde{k}}{L} w_1'(1) = -i \omega \alpha_1 w_1(1) \quad (55)
\]

\[
\frac{\partial E}{\sigma L} \phi_1'(1) = i \omega \beta_1 \phi_1(1) \quad (56)
\]

\[
\tilde{k} \phi_2(1) - \frac{2 \tilde{k}}{L} w_2'(1) = -i \omega \alpha_2 w_2(1) \quad (57)
\]

\[
\frac{\partial E}{\sigma L} \phi_2'(1) = i \omega \beta_2 \phi_2(1). \quad (58)
\]

We let

\[
w_1(x) = \sum_{n=0}^{N} a_n P_n(x), \quad \phi_1(x) = \sum_{n=0}^{N} b_n P_n(x),
\]

\[
w_2(x) = \sum_{n=0}^{N} c_n P_n(x), \quad \phi_2(x) = \sum_{n=0}^{N} d_n P_n(x),
\]

where \( P_n \) is the Legendre polynomial of degree \( n \).

We then compare coefficients of \( x^n \), for \( n = 0, 1, \ldots, N - 2 \), in each of the equations resulting from (47)–(50) and, including the 8 equations resulting from boundary conditions (51)–(58),

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the result is a system of $4N + 4$ equations in the $4N + 4$ unknowns $a_n, b_n, c_n, d_n, n = 0, 1, \ldots, N,$ and the parameter $\omega$. We may rewrite the system in the form

$$(\omega^2 A + \omega B + C)(a_0, \ldots, a_N, b_0, \ldots, b_N, c_0, \ldots, c_N, d_0, \ldots, d_N)^T = 0,$$  

(59)

where $A, B$ and $C$ are $(4N + 4) \times (4N + 4)$ matrices. Then, the vibration spectrum consists of those numbers $-i\omega$, where $\omega$ is a latent value of (59), i.e., where $\omega$ satisfies

$$\det(\omega^2 A + \omega B + C) = 0.$$  

(60)

It is easy to show that $\omega$ satisfies (60) if and only if $\omega$ is an eigenvalue of the $(8N + 8) \times (8N + 8)$ matrix

$$\begin{pmatrix} -A^{-1}B - A^{-1}C & I \\ I & 0 \end{pmatrix},$$

where $I$ is the $(4N + 4) \times (4N + 4)$ identity matrix and 0 the $(4N + 4) \times (4N + 4)$ 0-matrix. In practice, $A$ is often singular—indeed, that is the case here. We remedy the situation by letting

$$\omega = \frac{\zeta - 1}{\zeta + 1},$$

yielding the equation

$$\det(\zeta^2 X + \zeta Y + Z),$$

where $X, Y, Z$, of course, are $(4N + 4) \times (4N + 4)$ matrices $X$ is nonsingular, so we may proceed by finding the eigenvalues of

$$\begin{pmatrix} -X^{-1}Y - X^{-1}Z & I \\ I & 0 \end{pmatrix}$$

and transforming back.

5. Comparison of numerical and asymptotic results

Assumptions (29) imply that $k_1 = k_2$ and $A_1/L_1 = A_2/L_2$. While, as mentioned above, this means that we are not looking at a double-walled tube, these assumptions have the advantage of allowing us better to see the effect that the damping parameters and Van der Waals force have on the imaginary parts—i.e., the actual “frequency” parts—of the eigenfrequencies, as we shall see below.

Form our physical and geometrical parameters, we choose the carbon nanotube data given in (Wang et al., 2006). Thus, we have $E = 1$ TPa, $G = 4$ TPa, $A = 2.3090706$ nm$^2$, $I = 0.59649366$ nm$^4$ and $\rho = 2.3$ g/cm$^3$, and with a Van der Waals constant of $C = 0.06943$ TPa. Further, from our previous work, we have seen that, as the value of the slenderness ratio $L/d$ increases, one must go further out along the spectrum in order to find agreement with the asymptotic results. Thus we choose $L = 2.5$ nm, resulting in $L/d = 2.85714286$.

The dimensionless parameters then become

$$\gamma_1 = 0.03185$$

$$\gamma_2 = 0.0652925$$

$$C' = 0.5729492131.$$
For the damping constants, there is nothing in the literature to guide our choices. However, we can see that, if each \( \alpha''_i < 1 \) and each \( \beta''_i < 1 \) in (43)–(46), the asymptotic behavior of the imaginary parts of the eigenfrequencies will behave as though both right ends are free; similarly, if the arguments in the logs all are negative, the behavior will be as if both right ends are clamped. (Of course, there are many more possibilities; however, “clamped” and “free” are the most common types, so, due to space limitations, we restrict ourselves to these two cases. Also, we mention that the critical cases \( \alpha'' = 1 \) and \( \beta'' = 1 \) are studied in (Coleman & Schaffer, preprint), for the single Timoshenko beam.) Further, our choices are guided by the wish to see clearly the separation of the spectrum into branches.

To study the case where the right ends are free-like, we choose our dimensionless damping parameters to be

\[
\alpha'_1 = .2, \quad \beta'_1 = .01, \quad \alpha'_2 = .1, \quad \beta'_2 = .001. \tag{61}
\]

For clamped-like, we choose:

\[
\alpha'_1 = .3, \quad \beta'_1 = .013, \quad \alpha'_2 = 2, \quad \beta'_2 = .02. \tag{62}
\]

For all of our numerical examples, we have performed computations at \( N = 180, 200 \) and \( 220 \) Legendre polynomials, and we see that all results have converged to at least 10 decimal places.

1) For our first example, we consider the case with damping parameters given by (61) and with no Van der Waals force. This will give us a baseline for later examples, and will allow us to see how the spectrum separates into four branches. The results can be seen in Tables 1A and 1B, where we actually separate the frequencies into their four branches. First, however, we must note that the branching is an asymptotic phenomenon, thus one needs to go out along the spectrum before it can be seen. As mentioned earlier, for larger values of \( L/d \), one must go very far out before one sees the branching starting to occur. Here, we begin to see the branching and agreement with the asymptotic results pretty clearly after about the 4th or 5th eigenfrequency of each branch. For the first few, however, it may not even make sense to assign them to a branch; thus, while we do so by making our best guess, we mark them with * to denote the fact that this assignment is problematic.

Table 1A, then, lists the first 40 eigenfrequencies, and the 50th, 60th, 70th, 80th, 90th and 100th eigenfrequencies, of each \( \alpha \)-branch. The final column lists the asymptotic approximations for the imaginary parts, and the line at the bottom gives the asymptotic approximations for the real parts. Table 1B does the same, but for the \( \beta \)-branches.

As mentioned, in both tables the frequencies seem clearly to have split into branches, based on the real parts, well before the 10th frequency. By the 100th frequency in each branch, we have at least a three-decimal place match between the numerical and asymptotic real parts, and a four-decimal place match between the numerical and asymptotic imaginary parts.

One item of note: we see that the first frequency of the \( \alpha \)-branch predicted by the asymptotic results does not appear. As we shall see, it appears that this frequency may have been “damped out” by the boundary damping.

2) For Example 2, we use the damping parameters given in (62), and Tables 2A and 2B are analogous to Tables 1A and 1B, respectively. Here, it is not clear how to deal with the first few entries in each table. However, they separate into branches very quickly. In Table 2A we see that, by the 100th frequency, we have at least a three-decimal place match between the numerical and asymptotic real parts, and a three-decimal-place match between the numerical and asymptotic imaginary parts. In Table 2B, by the 100th frequency we see a four-decimal place match between the numerical and asymptotic frequencies. Meanwhile, for
the $\beta_2$ branch, the numerical and asymptotic real parts match to three decimal places. For the $\beta_1$ branch, the match is not as good (two decimal places), though they still clearly seem to be converging.

For the remaining examples we introduce the Van der Waals force. Specifically, we wish to see what happens to the spectrum as the Van der Waals constant increases from 0 to about twice the value of the physically realistic value of $C' = .5729492131$. Thus, we consider what happens for the values

$$C' = 0, \quad .25, \quad .5, \quad .75 \quad \text{and} \quad 1.$$  

3) For Examples 3 and 4, we look at two cases without boundary damping. Example 3 considers the case where the right ends are free, that is, for which

$$a'_1 = \beta'_1 = a'_2 = \beta'_2 = 0;$$

while Example 4 considers the right ends to be clamped, i.e.,

$$a'_1 = \beta'_1 = a'_2 = \beta'_2 = \infty.$$ 

We note that, in Examples 3 and 4, all numerical real parts are of absolute value $< 1.0E - 10$. The results for Example 3 can be found in Tables 3A and 3B. In Table 3A, we list the imaginary parts of the first 40 frequencies. The first column represents the double $\alpha$- and $\beta$-branches, identical for $C' = 0$. Introducing $C' > 0$ leads to the splitting of these pairs. What is striking is that, for each pair of frequencies, one decreases as the value of $C'$ increases, while the other is unaffected. (Indeed, it turns out that each of the even-numbered frequencies is unchanged to 13 decimal places!) Secondly, as we go out along the spectrum, the first member of each pair is less affected by the Van der Waals force, so that, when we get to the 39th–40th pair, they agree to three decimal places. (We look more closely at this phenomenon in Table 3B.)

Further, in comparing these results with those of Example 1, we see that the first predicted frequencies, missing in Table 1A, do appear here. Thus, as mentioned, it appears that the first pair was damped out via the boundary damping in Example 1, and that only one of these seems to be damped out by the inclusion of the Van der Waals force. Further, by comparing the first column of Table 3A with the results of Example 1, it is clear that the damping also affects the imaginary or “frequency” parts of the eigenfrequencies.

In Table 3B, we list the 49th–50th, 99th–100th, 149th–150th, 199th–200th, 249th–250th, 299th–300th, 349th–350th and 399th–400th eigenfrequencies, both numerical and asymptotic, for the case $C' = 1$ (i.e., corresponding to the last column in Table 3A). We see still closer agreement between the entries in each pair, and very close agreement with the asymptotics, as well. (Note that we list the branch for each eigenfrequency.) (Of course, the numbering here is very different from the numbering in Examples 1 and 2; e.g., the 40th entry in Table 3A corresponds to the 12th entry in Table 1A.)

4) The results of Example 4 are given in Tables 4A and 4B, in the same format as Tables 3A and 3B, respectively. In Table 4A, we see that the matching between the members of each pair is quite similar to that occurring in Table 3A. And again here, we see in Table 4B still closer agreement in each pair, and with the asymptotic results.

5) Example 5 is combination of Examples 1 and 3, and Example 6 is a combination of Examples 2 and 4. Example 5 looks at the damped system with the free-like parameters in (61), for the Van der Waals constant with values $C' = 0, .5$ and 1. The results are given in Tables 5A and 5B. In Table 5A, we proceed as in Table 3A, by listing the first 40 eigenfrequencies, although here we consider only the three values of $C'$. We see here that, for each pair, both
imaginary parts are affected by the Van der Waals force. However, we still see the closer matching of each pair as we go out along the spectrum. Meanwhile the real parts (damping rates) also are affected by the Van der Waals force, although there does not seem to be a noticeable pattern in that, in some cases it increases, while for others it decreases; in particular, there seems to be no branch-related pattern. Table 5B, then, is analogous to Table 3B, again using only the Van der Waals constant $C' = 1$. For the imaginary parts, the results are quite similar to those given in Table 3B. Meanwhile, the effect of the van der Waals on the real parts is diminished, as well, with the exception of the $\beta_2$-branch. However, this must be due to the fact that the $\beta_2$ damping rates are an order of magnitude smaller than the other damping rates.

6) In Table 6A, we proceed as in Table 4A, by listing the first 40 eigenfrequencies, but again only considering the three values of $C'$. We see again that, for each pair, both imaginary parts are affected by the Van der Waals force. Again we see the closer matching of each pair as we go out along the spectrum. Indeed, the last few pairs match more closely than the undamped pairs in Table 4A. The real parts behave quite the same as in Table 5A. Table 6B, then, is analogous to Table 4B, once more using only the Van der Waals constant $C' = 1$. Again, the imaginary parts behave quite similarly to those in Table 4B, and the real parts behave quite similarly to those in Table 5B.

In closing, we should mention that, although the results in (Shubov & Rojas-Arenaza, 2010b) show that the system is nonconservative, we have been unable to find any unstable eigenfrequencies in our numerical investigations.

| Numerical | Asymptotic (Im) |
|-----------|-----------------|
| $\alpha_1$ Branch | $\alpha_2$ Branch |
| Re | Im | Re | Im | Re | Im |
| 1. | — | — | — | — | 6.14735 |
| 2.* | -2.746 | 9.44918 | -3.783 | 9.93037 | 18.4421 |
| 3.* | -3.529 | 21.4099 | -9.628 | 21.7742 | 30.7368 |
| 4.* | -3.658 | 39.8147 | -1.490 | 38.6929 | 43.0315 |
| 5. | -3.823 | 53.2242 | -9.151 | 51.9313 | 55.3262 |
| 6. | -4.613 | 64.3926 | -7.899 | 65.1048 | 67.6209 |
| 7. | -4.754 | 77.6284 | -8.924 | 78.0511 | 79.9156 |
| 8. | -4.696 | 90.6127 | -1.224 | 90.7904 | 92.2103 |
| 9. | -4.357 | 103.890 | -1.951 | 102.998 | 104.505 |
| 10. | -4.743 | 114.106 | -1.357 | 114.409 | 116.800 |
| 11. | -4.902 | 127.122 | -7.845 | 127.016 | 129.094 |
| 12. | -4.907 | 139.799 | -5.884 | 139.671 | 141.389 |
| 13. | -4.690 | 152.112 | -5.443 | 152.252 | 153.684 |
| 14. | -4.920 | 164.680 | -5.754 | 164.776 | 165.979 |
| 15. | -4.899 | 177.203 | -6.898 | 177.241 | 178.273 |
| 16. | -4.712 | 189.886 | -8.915 | 189.528 | 190.568 |
| 17. | -4.831 | 201.836 | -8.153 | 201.610 | 202.863 |
| 18. | -4.929 | 214.000 | -6.139 | 213.968 | 215.157 |
| 19. | -4.932 | 226.449 | -5.353 | 226.399 | 227.452 |
| 20. | -4.867 | 238.754 | -5.199 | 238.814 | 239.747 |
| 21. | -4.934 | 251.173 | -5.397 | 251.213 | 252.042 |
| 22. | -4.919 | 263.582 | -6.054 | 263.584 | 264.336 |
| 23. | -4.715 | 276.142 | -7.003 | 275.851 | 276.631 |
A Numerical Study of the Vibration Spectrum for a Double-Walled Carbon Nanotube Model

Table 1A. Numerical eigenfrequencies 1–40, 50, 60, 70, 80, 90 and 100 for the $\alpha_1$ and $\alpha_2$ branches from Example 1. The asymptotic imaginary parts are given in the last column, while the asymptotic real parts appear at the bottom.

| $\beta_1$ Branch | $\beta_2$ Branch | Asymptotic (Im) |
|------------------|------------------|-----------------|
| Re   | Im   | Re   | Im   | Re   | Im   |
| 1.*  | -2.335 | 27.3749 | -1.321 | 26.9029 | 8.80167 |
| 2.*  | -3.537 | 35.5415 | -3.172 | 36.5570 | 26.4050 |
| 3.*  | -5.139 | 51.3999 | -4.007 | 50.2733 | 44.0084 |
| 4.*  | -6.082 | 67.0480 | -3.538 | 67.8770 | 61.6117 |
| 5.   | -6.339 | 83.3545 | -3.632 | 83.7187 | 79.2150 |
| 6.   | -5.939 | 100.464 | -4.061 | 99.7040 | 96.8184 |
| 7.   | -6.716 | 118.544 | -3.738 | 118.957 | 114.422 |
| 8.   | -7.386 | 135.540 | -3.642 | 135.511 | 132.025 |
| 9.   | -7.623 | 152.566 | -3.874 | 152.736 | 149.628 |
| 10.  | -7.641 | 169.729 | -3.682 | 169.794 | 167.232 |
| 11.  | -7.510 | 187.131 | -3.879 | 186.797 | 184.835 |
| 12.  | -7.625 | 204.784 | -3.775 | 205.036 | 202.438 |
| 13.  | -7.843 | 222.194 | -3.695 | 222.184 | 220.042 |
| 14.  | -7.919 | 239.557 | -3.762 | 239.624 | 237.645 |
| 15.  | -7.900 | 256.965 | -3.713 | 256.973 | 255.248 |
| 16.  | -7.830 | 274.497 | -3.920 | 274.214 | 272.852 |
| 17.  | -7.903 | 292.103 | -3.738 | 292.187 | 290.455 |
|   | \( \text{Re} \) | \( \text{Im} \) | \( \text{Re} \) | \( \text{Im} \) |   |
|---|---|---|---|---|---|
| 18. | -7.995 | 309.592 | -3.715 | 309.577 | 308.059 |
| 19. | -8.018 | 327.064 | -3.725 | 327.096 | 325.662 |
| 20. | -7.990 | 344.566 | -3.728 | 344.549 | 343.265 |
| 21. | -7.957 | 362.152 | -3.730 | 362.244 | 360.869 |
| 22. | -8.015 | 379.737 | -3.726 | 379.758 | 378.472 |
| 23. | -8.057 | 397.260 | -3.730 | 397.244 | 396.075 |
| 24. | -8.060 | 414.783 | -3.723 | 414.796 | 413.679 |
| 25. | -8.032 | 432.332 | -3.743 | 432.293 | 431.282 |
| 26. | -8.027 | 449.937 | -3.743 | 449.937 | 448.885 |
| 27. | -8.068 | 467.510 | -3.726 | 467.512 | 466.489 |
| 28. | -8.087 | 485.056 | -3.749 | 485.055 | 484.092 |
| 29. | -8.081 | 502.606 | -3.727 | 502.609 | 501.695 |
| 30. | -8.058 | 520.184 | -3.772 | 520.119 | 519.299 |
| 31. | -8.069 | 537.788 | -3.740 | 537.822 | 536.902 |
| 32. | -8.095 | 555.359 | -3.728 | 555.355 | 554.505 |
| 33. | -8.102 | 572.920 | -3.732 | 572.931 | 572.109 |
| 34. | -8.091 | 590.488 | -3.731 | 590.482 | 589.712 |
| 35. | -8.077 | 608.084 | -3.875 | 608.067 | 607.315 |
| 36. | -8.095 | 625.681 | -3.732 | 625.690 | 624.919 |
| 37. | -8.110 | 643.255 | -3.732 | 643.248 | 642.522 |
| 38. | -8.110 | 660.826 | -3.730 | 660.831 | 660.125 |
| 39. | -8.098 | 678.408 | -3.738 | 678.391 | 677.729 |
| 40. | -8.094 | 696.011 | -3.760 | 696.054 | 695.332 |
| 41. | -8.119 | 713.710 | -3.733 | 713.914 | 713.656 |
| 42. | -8.130 | 731.413 | -3.732 | 731.852 | 731.650 |
| 43. | -8.134 | 749.116 | -3.735 | 749.232 | 749.213 |
| 44. | -8.135 | 766.820 | -3.733 | 766.880 | 766.866 |
| 45. | -8.138 | 784.524 | -3.743 | 784.674 | 784.703 |
| 46. | -8.141 | 802.228 | -3.734 | 802.378 | 802.413 |

Table 1B. Numerical eigenfrequencies 1–40, 50, 60, 70, 80, 90 and 100 for the \( \beta_1 \) and \( \beta_2 \) branches from Example 1. The asymptotic imaginary parts are given in the last column, while the asymptotic real parts appear at the bottom.
|   | α₁ Branch |   | α₂ Branch |   |
|---|-----------|---|-----------|---|
|   | Re       | Im       | Re       | Im       |
| 0.* | 2.767     | 20.7386  | .3306    | 20.7810  |
| 1.* | 3.365     | 32.4318  | .3818    | 31.1907  |
| 2.* | 4.405     | 43.3261  | .4055    | 44.5802  |
| 3.  | .889      | 59.0805  | .4281    | 59.3359  |
| 4.  | 5.844     | 74.6045  | .5117    | 74.9216  |

Table 2A. Numerical eigenfrequencies 1–40, 50, 60, 70, 80, 90 and 100 for the α₁ and α₂ branches from Example 2. The asymptotic imaginary parts are given in the last column, while the asymptotic real parts appear at the bottom.
Table 2B. Numerical eigenfrequencies 1–40, 50, 60, 70, 80, 90 and 100 for the $\beta_1$ and $\beta_2$ branches from Example 2. The asymptotic imaginary parts are given in the last column, while the asymptotic real parts appear at the bottom.
A Numerical Study of the Vibration Spectrum for a Double-Walled Carbon Nanotube Model

| $C' = 0$ | $C' = .25$ | $C' = .5$ | $C' = .75$ | $C' = 1$ |
| --- | --- | --- | --- | --- |
| 2.9458212164 | 1.2833048444 | — | — | — |
| 10.678276269 | 10.173172662 | 9.9095407887 | 9.6376145423 | |
| 22.186791607 | 21.956728997 | 21.840975799 | 21.724712242 | |
| 26.977657335 | 26.82366881 | 26.819700315 | 26.767023640 | |
| 29.77657335 | 29.77657335 | 29.77657335 | 29.77657335 | |
| 36.35895938 | 36.35895938 | 36.385895938 | 36.385895938 | |
| 39.346020018 | 39.234922527 | 39.179434389 | 39.123994694 | |
| 50.846627673 | 50.846627673 | 50.846627673 | 50.846627673 | |
| 52.832231307 | 52.752514490 | 52.713137027 | 52.674092535 | |
| 64.810103855 | 64.757222238 | 64.668361902 | 64.620894071 | |
| 67.511260081 | 67.482874126 | 67.446934409 | 67.4092535 | |
| 77.945061900 | 77.86047284 | 77.785081589 | 77.709174702 | |
| 83.358682455 | 83.331747254 | 83.323650666 | 83.315549557 | |
| 83.358682455 | 83.358682455 | 83.358682455 | 83.358682455 | |
| 90.953565995 | 90.938453817 | 90.913453919 | 90.879634490 | |
| 90.953565995 | 90.938453817 | 90.913453919 | 90.879634490 | |
| 99.219305787 | 99.189697002 | 99.174574327 | 99.159714381 | |
| 99.219305787 | 99.189697002 | 99.174574327 | 99.159714381 | |
| 104.36226852 | 104.33845569 | 104.3145536 | 104.29067265 | |
| 104.36226852 | 104.33845569 | 104.3145536 | 104.29067265 | |
| 113.79274626 | 113.76971861 | 113.74302537 | 113.71805740 | |
| 113.79274626 | 113.76971861 | 113.74302537 | 113.71805740 | |
| 119.25906418 | 119.24969905 | 119.23105072 | 119.22184083 | |
| 119.25906418 | 119.24969905 | 119.23105072 | 119.22184083 | |
| 126.98609223 | 126.95009682 | 126.9240897 | 126.87578972 | |
| 126.98609223 | 126.95009682 | 126.9240897 | 126.87578972 | |
| 135.64225489 | 135.62835685 | 135.61433864 | 135.58247482 | |
| 135.64225489 | 135.62835685 | 135.61433864 | 135.58247482 | |
| 139.71595073 | 139.69029336 | 139.6642947 | 139.63895904 | |
| 139.71595073 | 139.69029336 | 139.6642947 | 139.63895904 | |

Table 3A. The first 40 imaginary parts of the numerical eigenfrequencies from Example 3, computed for five different values of the Van der Waals constant $C'$. The “real-life” value of the constant is approximately .57.

| Numerical Asymptotic |
| --- | --- |
| 177.211 | 178.283 (α-branch) |
Table 3B. Numerical and asymptotic eigenfrequencies (imaginary parts) 49, 50, 99, 100, 149, 150, 199, 200, 249, 250, 299, 300, 349, 350, 399, 400 from Example 3, computed for the Van der Waals constant $C' = 1$.

| $C' = 0$ | $C' = .25$ | $C' = .5$ | $C' = .75$ | $C' = 1$ |
|---------|----------|----------|----------|----------|
| 50. 177.293 | 178.283  " |          |          |          |
| 99. 361.308 | 360.869 (β-branch) |          |          |          |
| 100. 361.331 | 360.869  " |          |          |          |
| 149. 537.847 | 536.902 (β-branch) |          |          |          |
| 150. 537.848 | 536.902  " |          |          |          |
| 199. 718.895 | 719.240 (α-branch) |          |          |          |
| 200. 718.916 | 719.240  " |          |          |          |
| 249. 903.393 | 903.661 (α-branch) |          |          |          |
| 250. 903.410 | 903.661  " |          |          |          |
| 299. 1083.03 | 1082.61 (β-branch) |          |          |          |
| 300. 1083.06 | 1082.61  " |          |          |          |
| 349. 1260.06 | 1260.21 (α-branch) |          |          |          |
| 350. 1260.07 | 1260.21  " |          |          |          |
| 399. 1444.46 | 1444.62 (α-branch) |          |          |          |
| 400. 1444.47 | 1444.62  " |          |          |          |
Table 4A. The first 40 imaginary parts of the numerical eigenfrequencies from Example 4, computed for five different values of the Van der Waals constant $C'$.

| Numerical  | Asymptotic |
|------------|------------|
| 49.        | 183.705    | 184.421 ($\alpha$-branch) |
| 50.        | 183.779    | 184.421 $''$ |
| 99.        | 367.917    | 368.841 ($\alpha$-branch) |
| 100.       | 367.955    | 368.841 $''$ |
| 149.       | 546.609    | 545.704 ($\beta$-branch) |
| 150.       | 546.609    | 545.704 $''$ |
| 199.       | 725.047    | 725.388 ($\alpha$-branch) |
| 200.       | 725.068    | 725.388 $''$ |
| 249.       | 909.548    | 909.808 ($\alpha$-branch) |
| 250.       | 909.565    | 909.808 $''$ |
| 299.       | 1091.80    | 1091.41 ($\beta$-branch) |
| 300.       | 1091.80    | 1091.41 $''$ |
| 349.       | 1267.85    | 1267.44 ($\beta$-branch) |
| 350.       | 1267.85    | 1267.44 $''$ |
| 399.       | 1450.61    | 1450.78 ($\alpha$-branch) |
| 400.       | 1450.62    | 1450.78 $''$ |
Table 4B. Numerical and asymptotic eigenfrequencies (imaginary parts) 49, 50, 99, 100, 149, 150, 199, 200, 249, 250, 299, 300, 349, 350, 399, 400 from Example 4, computed for the Van der Waals constant $C' = 1$.

|     | $C' = 0$ |     | $C' = .5$ |     | $C' = 1$ |
|-----|----------|-----|-----------|-----|----------|
|     | Re       | Im  | Re        | Im  | Re       |
| 1.  | —        | —   | —         | —   | —        |
| 2.  | —        | —   | —         | —   | —        |
| 3.  | -2.746   | 9.449183995 | -2.877 | 9.172040478 | -2.972 | 8.827341448 |
| 4.  | -6.783   | 9.930366988 | -7.344 | 9.678929479 | -8.576 | 9.477841762 |
| 5.  | -3.529   | 21.40990132 | -3.575 | 21.29275588 | -3.611 | 21.16999982 |
| 6.  | -9.628   | 21.77418316 | -9.720 | 21.65493676 | -9.945 | 21.54041229 |
| 7.  | -1.321   | 26.90229372 | -1.312 | 26.85129821 | -1.308 | 26.79542808 |
| 8.  | -2.335   | 27.37492665 | -2.342 | 27.32751386 | -2.343 | 27.28372611 |
| 9.  | -3.537   | 35.54153378 | -3.556 | 35.47898352 | -3.569 | 35.40700136 |
| 10. | -3.172   | 36.55704093 | -3.177 | 36.50955538 | -3.190 | 36.43763732 |
| 11. | -1.490   | 38.69290303 | -1.496 | 38.63128423 | -1.504 | 38.56573542 |
| 12. | -3.658   | 39.81469654 | -3.664 | 39.76185775 | -3.665 | 39.71047687 |
| 13. | -4.007   | 50.27733227 | -4.028 | 50.23454838 | -4.048 | 50.18747340 |
| 14. | -5.139   | 51.39985683 | -5.143 | 51.37684915 | -5.148 | 51.35549893 |
| 15. | -0.915   | 51.93127919 | -0.918 | 51.87409462 | -0.9225 | 51.81686142 |
| 16. | -3.823   | 53.22422671 | -3.821 | 53.18663028 | -3.816 | 53.15176710 |
| 17. | -4.613   | 64.39259632 | -4.626 | 64.34564444 | -4.637 | 64.29753890 |
| 18. | -7.899   | 65.10483274 | -7.935 | 65.05333361 | -7.986 | 65.00630011 |
| 19. | -6.082   | 67.04799021 | -6.082 | 67.03545255 | -6.081 | 67.02281832 |
| 20. | -3.538   | 67.87701409 | -3.539 | 67.86205923 | -3.539 | 67.84780896 |
| 21. | -4.754   | 77.62836514 | -4.762 | 77.58540148 | -4.768 | 77.54197597 |
| 22. | -8.924   | 78.05109828 | -8.968 | 78.00762757 | -9.026 | 77.96494901 |
| 23. | -6.339   | 83.35453188 | -6.335 | 83.34816855 | -6.331 | 83.34180146 |
| 24. | -3.632   | 83.71868280 | -3.635 | 83.70997652 | -3.638 | 83.70136960 |
| 25. | -4.696   | 90.61268633 | -4.700 | 90.57580895 | -4.703 | 90.53866701 |
| 26. | -1.224   | 90.79035136 | -1.230 | 90.75906706 | -1.237 | 90.71182474 |
| 27. | -4.061   | 99.70402490 | -4.068 | 99.69160776 | -4.076 | 99.67839224 |
| 28. | -5.939   | 100.4640803 | -5.931 | 100.4692419 | -5.924 | 100.4751321 |
| 29. | -1.951   | 102.9837356 | -1.960 | 102.9543524 | -1.970 | 102.9099689 |
| 30. | -4.352   | 103.8895026 | -4.350 | 103.8631919 | -4.346 | 103.8377247 |
| 31. | -4.743   | 114.1061665 | -4.749 | 114.0804654 | -4.754 | 114.0546217 |
| 32. | -1.357   | 114.4090908 | -1.352 | 114.3757397 | -1.348 | 114.3424705 |
| 33. | -6.716   | 118.5438854 | -6.722 | 118.5429195 | -6.727 | 118.5418301 |
| 34. | -3.738   | 118.9568732 | -3.736 | 118.9487280 | -3.735 | 118.9408092 |
| 35. | -7.845   | 127.0161580 | -7.832 | 126.9880193 | -7.825 | 126.9598517 |
| 36. | -4.902   | 127.1222585 | -4.906 | 127.0951053 | -4.909 | 127.0679252 |
| 37. | -3.642   | 135.5114464 | -3.642 | 135.5080511 | -3.642 | 135.5046704 |
| 38. | -7.386   | 135.5399041 | -7.388 | 135.5375232 | -7.390 | 135.5351432 |
| 39. | -5.884   | 139.6706111 | -5.884 | 139.6447863 | -5.888 | 139.6189351 |
| 40. | -4.907   | 139.7988314 | -4.910 | 139.7733338 | -4.912 | 139.7478086 |

Table 5A. The first 40 numerical eigenfrequencies from Example 5, computed for three different values of the Van der Waals constant C.
Table 5B. Numerical and asymptotic eigenfrequencies (imaginary parts) 49, 50, 99, 100, 149, 150, 199, 200, 249, 250, 299, 300, 349, 350, 399, 400 from Example 5, computed for the Van der Waals constant $C' = 1$.

| $C' = 0$ | $C' = 5$ | $C' = 1$ |
|----------|----------|----------|
| Re | Im | Re | Im | Re | Im |
| 1 | -2.961 | 9.936871385 | -3.079 | 9.795166383 | -3.169 | 9.644727421 |
| 2 | -1.610 | 12.43254005 | -1.618 | 12.28811944 | -1.632 | 12.14607344 |
| 3 | -2.767 | 20.73861326 | -2.794 | 20.64458863 | -2.815 | 20.55053147 |
| 4 | -3.306 | 20.78104522 | -3.395 | 20.70597402 | -0.355 | 20.63295359 |
| 5 | -3.190 | 29.37005728 | -3.187 | 29.32574131 | -3.182 | 29.28098877 |
| 6 | -1.397 | 31.18963235 | -3.840 | 31.12726000 | -3.882 | 31.06474018 |
| 7 | -3.818 | 31.19073440 | -1.392 | 31.12875704 | -1.390 | 31.06625299 |
| 8 | -3.365 | 32.43182622 | -3.369 | 32.36004659 | -3.369 | 32.28961412 |
| 9 | -4.405 | 43.32612319 | -4.402 | 43.28601387 | -4.398 | 43.24625757 |
| 10 | -4.055 | 44.58018737 | -4.065 | 44.54246627 | -4.083 | 44.50537030 |
| 11 | -1.634 | 45.08538379 | -1.632 | 45.02792865 | -1.634 | 44.96962880 |
| 12 | -4.064 | 46.00707085 | -4.068 | 45.95274391 | -4.070 | 45.89830745 |
| 13 | -4.495 | 57.84423832 | -4.479 | 57.81269810 | -4.461 | 57.78159556 |
| 14 | -1.657 | 58.59889870 | -1.656 | 58.54816779 | -1.659 | 58.49709714 |
| 15 | -4.281 | 59.33578065 | -4.286 | 59.31422388 | -4.295 | 59.29779828 |
| 16 | -4.889 | 59.80848155 | -4.905 | 59.76734039 | -4.920 | 59.72601742 |
| 17 | -1.621 | 71.79615374 | -1.620 | 71.75209563 | -1.621 | 71.70774465 |
Table 6A. The first 40 numerical eigenfrequencies from Example 6, computed for three different values of the Van der Waals constant $C'$.

|    |    |    |    |    |
|----|----|----|----|----|
| 49 | $-1.516$ | 183.750 | $-1.618$ | $184.421$ (α₂-branch) |
| 50 | $-4.408$ | 183.941 | $-4.118$ | $184.421$ (α₁-branch) |
| 99 | $-1.538$ | 367.949 | $-1.618$ | $368.841$ (α₂-branch) |
| 100 | $-4.344$ | 367.965 | $-4.118$ | $368.841$ (α₁-branch) |
| 149 | $-0.4874$ | 546.610 | $-0.4821$ | $545.704$ (β₂-branch) |
| 150 | $-7.142$ | 546.620 | $-7.208$ | $545.704$ (β₁-branch) |
| 199 | $-1.619$ | 725.057 | $-1.618$ | $725.388$ (α₂-branch) |
| 200 | $-4.122$ | 725.058 | $-4.118$ | $725.388$ (α₁-branch) |
| 249 | $-1.618$ | 909.557 | $-1.618$ | $909.808$ (α₂-branch) |
| 250 | $-4.121$ | 909.559 | $-4.118$ | $909.808$ (α₁-branch) |
| 299 | $-0.4987$ | 1091.80 | $-0.4821$ | $1091.41$ (β₂-branch) |
| 300 | $-7.168$ | 1091.81 | $-7.208$ | $1091.41$ (β₁-branch) |
| 349 | $-7.173$ | 1267.83 | $-7.208$ | $1267.44$ (β₁-branch) |
| 350 | $-0.5013$ | 1267.84 | $-0.4821$ | $1267.44$ (β₂-branch) |
A Numerical Study of the Vibration Spectrum for a Double-Walled Carbon Nanotube Model

Table 6B. Numerical and asymptotic eigenfrequencies (imaginary parts) 49, 50, 99, 100, 149, 150, 199, 200, 249, 250, 299, 300, 349, 350, 399, 400 from Example 6, computed for the Van der Waals constant $C^\prime = 1$.

6. References

Coleman, M.P. & Schaffer, L. (2010). Asymptotic analysis of the vibration spectrum of coupled Timoshenko beams with a dissipative joint, *Eur. J. Mech. A Solids*, Vol. 29, No. 4, 629-636.

Coleman, M.P. & Schaffer, L. The single Timoshenko beam with general boundary damping, an asymptotic and numerical study, preprint.

Gibson, R.F.; Ayorinde, E.O. & Wen, Y.-F. (2007). Vibrations of carbon nanotubes and their composites: A review, *Comp. Sci. Tech.*, Vol. 67, 1-27.

Gottlieb, D.; Hussaini, M.Y. & Orszag, S.A. (1984). Theory and applications of spectral methods, *Spectral methods for partial differential equations*, pp. 1-54, Hampton, VA, 1982, SIAM, Philadelphia, PA.

Jakobson, B.I.; Brabec, C.J. & Berhold, J. (1996). Nanomechanics of carbon nanotubes: Instabilities beyond linear response, *Phys. Rev. Lett.*, Vol. 76, No. 14, 2511-2514.

Jameson, V. (2000). Carbon nanotubes roll on, *Phys. World*, Vol. 13, No. 6, 29-30.

Mahan, G.D. (2002). Oscillations of a thin hollow cylinder: Carbon nanotubes, *Phys. Rev. B*, Vol. 65, No. 23, 235402.1-235402.7.

Pantano, A.; Boyce, M.C. & Parks, D.M. (2003). Nonlinear structural mechanics based modeling of carbon nanotube deformation, *Phys. Rev. Lett.*, Vol. 91, No. 14, 145504.1-145504.4.

Pantano, A.; Boyce, M.C. & Parks, D.M. (2004). Mechanics of deformation of single and multi-wall carbon nanotubes, *J. Mech. Phys. Solids*, Vol. 52, No. 4, 789-821.

Qian, D.; Wagner, G.J.; Liu, W.K.; Yu, M.-F. & Ruoff, R.S. (2002). Mechanics of carbon nanotubes, *Appl. Mech. Rev.*, Vol. 55, No. 6, 495-533.

Ru, C.Q. (2000). Effect of Van der Waals forces on axial buckling of a double-walled carbon nanotube, *J. Appl. Phys.*, Vol. 87, 1712-1715.

Ru, C.Q. (2001). Degraded axial buckling strain of multiwalled carbon nanotubes due to interlayer slips, *J. Appl. Phys.*, Vol. 89, No. 6, 3426-3433.

Shubov, M.A. & Rojas-Arenaza, M. (2010a). Vibrational frequency distribution for nonconservative model of double-walled carbon nanotube, *Appl. Math. Comput.*, Vol. 217, No. 3, 1246-1252.

Shubov, M.A. & Rojas-Arenaza, M. (2010b). Mathematical analysis of carbon nanotube model, *J. Comput. Appl. Math.*, Vol. 234, No. 6, 1631-1636.

Shubov, M.A. & Rojas-Arenaza, M. (2010c). Asymptotic distribution of eigenvalues of dynamics generator governing vibrations of double-walled carbon nanotube model, *Asymptotic Anal.*, Vol. 68, No. 1–2, 89-124.

Traill-Nash, R.W. & Collar, A.R. (1953). The effects of shear flexibility and rotatory inertia on the bending vibrations of beams, *Quart. J. Mech. Appl. Math.*, Vol. 6, 186-222.

Wang, C.M.; Tan, V.B.C. & Zhang, Y.Y. (2006). Timoshenko beam model for vibration analysis of multi-walled carbon nanotubes, *J. Sound Vibrations*, Vol. 294, 1060-1072.
Wang, C.Y.; Ru, C.Q. & Mioduchowski, A. (2004). Applicability and limitations of simplified elastic shell equations for carbon nanotubes, *J. Appl. Mech.*, Vol. 71, 622-631.

Wang, C.Y.; Ru, C.Q. & Mioduchowski, A. (2005). Free vibrations of multiwall carbon nanotubes, *J. Appl. Phys.*, Vol. 97, 114323.1-114323.10.

Wang, Q. (2005). Wave propagation in carbon nanotubes via nonlocal continuum mechanics, *J. Appl. Phys.*, Vol. 98, 124301.

Xu, K.Y.; Guo, X.N. & Ru, C.Q. (2006). Vibration of a double-walled carbon nanotube aroused by nonlinear intertube Van der Waals forces, *J. Appl. Phys.*, Vol. 99, 064303.1-064303.7.

Yoon, J.; Ru, C.Q. & Mioduchowski, A. (2003). Vibration of an embedded multiwall carbon nanotube, *Comp. Sci. Tech.*, Vol. 63, 1533-1542.
Carbon nanotubes (CNTs), discovered in 1991, have been a subject of intensive research for a wide range of applications. These one-dimensional (1D) graphene sheets rolled into a tubular form have been the target of many researchers around the world. This book concentrates on the semiconductor physics of carbon nanotubes, it brings unique insight into the phenomena encountered in the electronic structure when operating with carbon nanotubes. This book also presents to reader useful information on the fabrication and applications of these outstanding materials. The main objective of this book is to give in-depth understanding of the physics and electronic structure of carbon nanotubes. Readers of this book should have a strong background on physical electronics and semiconductor device physics. This book first discusses fabrication techniques followed by an analysis on the physical properties of carbon nanotubes, including density of states and electronic structures. Ultimately, the book pursues a significant amount of work in the industry applications of carbon nanotubes.

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