Nonlinear self-adjointness and conservation laws for some equation systems of two-phase media

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Abstract. The T B Anderson — R Jackson system of equations with the Stokes regime of wrapping around is considered, which describes in the isothermal case the dynamics of a mixture of gas and fine solid particles, taking into account the difference in velocities and the presence of the own particle phase pressure. The system includes the equations for the conservation of mass and momentum, written for each phase and supplemented by equations of state. The aim of the paper is the search of conservation laws for the system by means of N Kh Ibragimov’s nonlinear self-adjointness method. For this purpose it was found a substitutions family, which corresponds to the nonlinear self-adjointness of equations of the T B Anderson — R Jackson system. Using this result and known symmetries of the equations system, the series of conservation laws were derived. It is noted that the analogous results are valid for the Kh A Rakhmatulin system of equations of a mixture dynamics with one pressure.

1. Introduction
Consider the T B Anderson — R Jackson system of equations [1] with the Stokes regime of wrapping around [2]

\[
\begin{align*}
\frac{\partial \rho_1}{\partial t} + \frac{\partial (\rho_1 u_1)}{\partial x} &= 0, \\
\frac{\partial \rho_2}{\partial t} + \frac{\partial (\rho_2 u_2)}{\partial x} &= 0, \\
\frac{\partial (\rho_1 u_1)}{\partial t} + \frac{\partial (\rho_1 u_1^2)}{\partial x} + m_1 \frac{\partial P_1}{\partial x} &= -\frac{\rho_2 (u_1 - u_2)}{\tau}, \\
\frac{\partial (\rho_2 u_2)}{\partial t} + \frac{\partial (\rho_2 u_2^2)}{\partial x} + m_2 \frac{\partial P_2}{\partial x} &= \frac{\rho_2 (u_1 - u_2)}{\tau},
\end{align*}
\]

(1)

which describes in the isothermal case the dynamics of a mixture of gas and fine solid particles, taking into account the difference in velocities and the presence of the own particle phase pressure. The system includes the equations for the conservation of mass and momentum, written for each phase and supplemented by equations of state. Here \(\rho_i = m_i \rho_{ii}\) is the average density of \(i\)-th phase, \(m_i, \rho_{ii}, u_i, P_i\) are the volume concentration, the genuine density, the velocity and the pressure respectively for \(i\)-th phase, \(i = 1, 2\), the constant \(\tau\) is the Stokes velocity relaxation time.
The three conservation laws of system (2) are obvious: the last of them corresponds to the equation \( \rho F_{x,t} = 0 \) on \( R \), adjointness method of N Kh Nigmatullin [3], we write the equations for determining the adjoint \( P \) into account that 

\[
\begin{align*}
F_1 &:= \rho_t + \rho_x u + \rho u_x = 0, \\
F_2 &:= \sigma_t + \sigma_x v + \sigma v_x = 0, \\
F_3 &:= u_t + u u_x + \rho^{-1}(1 - c\sigma)(P_\rho \rho_x + P_\sigma \sigma_x) + b \rho^{-2}(u - v) = 0, \\
F_4 &:= v_t + v v_x + c(P_\rho \rho_x + P_\sigma \sigma_x) + \sigma^{-1}(Q_\rho \rho_x + Q_\sigma \sigma_x) - b(u - v) = 0.
\end{align*}
\]

The three conservation laws of system (2) are obvious:

\[
\begin{align*}
C^t_1 &= \rho, \quad C^x_1 = \rho u; \\
C^t_2 &= \sigma, \quad C^x_2 = \sigma v; \\
C^t_3 &= \rho u + \sigma v, \quad C^x_3 = \rho u^2 + \sigma v^2 + P(\rho, \sigma) + Q(\rho, \sigma).
\end{align*}
\]

The last of them corresponds to the equation \( \rho F_3 + \sigma F_4 = 0 \).

A formal Lagrangian has the form \( L = RF_1 + SF_2 + UF_3 + VF_4 \). Using the nonlinear self-adjointness method of N Kh Ibragimov [3], we write the equations for determining the adjoint functions \( R, S, U, V \) and functions \( \alpha_i, \beta_i, \gamma_i, \delta_i, i = 1, 2, 3, 4 \), which will be assumed to depend only on \( x, t \):

\[
\begin{align*}
\frac{\delta L}{\delta \rho} &= -D_t R - u D_x R - \rho^{-2}(1 - c\sigma)(P_\rho \rho_x + P_\sigma \sigma_x) U + \\
&\quad + \rho^{-1}(1 - c\sigma)(P_{\rho \rho} \rho_x + P_{\rho \sigma} \sigma_x) U - D_x(\rho^{-1}(1 - c\sigma)P_\rho U) - b \rho^{-2} \sigma (u - v) U + \\
&\quad + c(P_{\rho \rho} \rho_x + P_{\rho \sigma} \sigma_x) V - c D_x(P_\rho V) + \sigma^{-1}(Q_{\rho \rho} \rho_x + Q_{\rho \sigma} \sigma_x) V - D_x(\sigma^{-1} Q_\rho V) = \\
&\quad = -D_t R - u D_x R - \rho^{-2}(1 - c\sigma)P_\sigma \sigma_x U + c \rho^{-1} P_\rho \rho_x U - \\
&\quad - \rho^{-1}(1 - c\sigma) P_\rho D_x U - b \rho^{-2} \sigma (u - v) U - (c P_\rho + \sigma^{-1} Q_\rho) D_x V + \sigma^{-2} Q_\rho \sigma_x V = \\
&\quad = \alpha_1 F_1 + \alpha_2 F_2 + \alpha_3 F_3 + \alpha_4 F_4,
\end{align*}
\]

\[
\frac{\delta L}{\delta \sigma} = -D_t S - v D_x S - c \rho^{-1}(P_\rho \rho_x + P_\sigma \sigma_x) U + \rho^{-1}(1 - c\sigma)(P_{\rho \sigma} \rho_x + P_{\sigma \sigma} \sigma_x) U - \\
- D_x(\rho^{-1}(1 - c\sigma)P_\sigma U) + b \rho^{-1} (u - v) U + c(P_{\rho \rho} \rho_x + P_{\rho \sigma} \sigma_x) V - c D_x(P_\sigma V) - \\
- \sigma^{-2}(Q_{\rho \rho} \rho_x + Q_{\rho \sigma} \sigma_x) V + \sigma^{-1}(Q_{\rho \sigma} \rho_x + Q_{\sigma \sigma} \sigma_x) V - D_x(\sigma^{-1} Q_\sigma V) = 
\]

2. Nonlinear self-adjointness

To reduce the number of indexes and for convenience of computation, we denote \( \rho = \rho_1, \sigma = \rho_2, u = u_1, v = u_2, \tau^{-1} = b, \rho^{-2} = c, m = m_2 = \sigma \), \( P = P_1, Q = P_2 \). Further, using the basic identity of heterogeneous media mechanics \( m_1 + m_2 = 1 \), we get \( m_1 = 1 - m = 1 - \sigma \). Taking into account that \( P = P(\rho, \sigma), Q = Q(\rho, \sigma) \), we rewrite system of equations (1) in the form

\[
\begin{align*}
A_1 &= \rho_1 + \rho_2 u + \rho_2 u_x = 0, \\
A_2 &= \sigma_1 + \sigma_2 v + \sigma_2 v_x = 0, \\
A_3 &= u_1 + u_2 u_x + \rho^{-1}(1 - c\sigma)(P_\rho \rho_x + P_\sigma \sigma_x) + b \rho^{-2} \sigma (u - v) = 0, \\
A_4 &= v_1 + v_2 v_x + c(P_\rho \rho_x + P_\sigma \sigma_x) + \sigma^{-1}(Q_{\rho \rho} \rho_x + Q_{\rho \sigma} \sigma_x) - b(u - v) = 0.
\end{align*}
\]

The aim of the paper is the search of conservation laws for system (1) by means of N Kh Ibragimov’s nonlinear self-adjointness method [3]. For this purpose it was found a substitutions family, which corresponds to the nonlinear self-adjointness of equations system (1). Using this result and known symmetries of system (1), the series of conservation laws were derived. It is noted that the analogous results are valid for Kh A Rakhmatulin’s system of equations [4] of a mixture dynamics with one pressure.
\[
\begin{align*}
\rho(t) = \rho = \rho - \rho v S
c = \rho D_x U + b \rho^{-1}(u-v)U - \sigma^2 Q \rho_x V - (c \rho \sigma + \sigma^{-1} Q) D_x V = \\
= \beta_1 F_1 + \beta_2 F_2 + \beta_3 F_3 + \beta_4 F_4,
\end{align*}
\]

Here, in accordance to [3],

\[
\begin{align*}
\frac{\delta L}{\delta u} &= \frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial \rho_t} - D_x \frac{\partial L}{\partial \rho_x}, \\
\frac{\delta L}{\delta \sigma} &= \frac{\partial L}{\partial \sigma} - D_t \frac{\partial L}{\partial \sigma_t} - D_x \frac{\partial L}{\partial \sigma_x}, \\
\frac{\delta L}{\delta v} &= \frac{\partial L}{\partial v} - D_t \frac{\partial L}{\partial \sigma_t} - D_x \frac{\partial L}{\partial \sigma_x}.
\end{align*}
\]

Equating to zero the coefficients at the free variables \( \rho_t, \sigma_t, u_t, v_t \) in system (3)-(6), we get that

\[
\begin{align*}
R &= -\alpha_1 \rho - \alpha_2 \sigma - \alpha_3 u - \alpha_4 v - \alpha_5, \\
S &= -\beta_1 \rho - \beta_2 \sigma - \beta_3 u - \beta_4 v - \beta_5, \\
U &= -\gamma_1 \rho - \gamma_2 \sigma - \gamma_3 u - \gamma_4 v - \gamma_5, \\
V &= -\delta_1 \rho - \delta_2 \sigma - \delta_3 u - \delta_4 v - \delta_5.
\end{align*}
\]

Now splitting (3)-(6) with respect to the free variables \( \rho_x, \sigma_x, u_x, v_x \), we obtain the system of equations

\[
\begin{align*}
R_t + u R_x + \rho^{-1}(1 - \sigma) P \rho U_x + b \rho^{-2} \sigma (u-v) U + \\
+ (c \rho + \sigma^{-1} Q) V_x + \alpha_3 b \rho^{-1} \sigma (u-v) - \alpha_4 b (u-v) = 0, \\
u R_t + \rho^{-1}(1 - \sigma) P \rho U_t + (c \rho + \sigma^{-1} Q) V_t + \\
+ \alpha_1 u + \alpha_3 \rho^{-1}(1 - \sigma) \rho + \alpha_4 (c \rho + \sigma^{-1} Q) = 0, \\
u R_x + (1 - \sigma) P \rho U_x + (c \rho + \sigma^{-1} Q) V_x + \\
+ \alpha_4 b u + \alpha_3 \rho^{-1}(1 - \sigma) \rho + \alpha_4 (c \rho + \sigma^{-1} Q) = 0, \\
u R_v + \rho^{-1}(1 - \sigma) P \rho U_v + (c \rho + \sigma^{-1} Q) V_v + + \alpha_4 b u + \alpha_3 \rho^{-1}(1 - \sigma) \rho + \alpha_4 (c \rho + \sigma^{-1} Q) = 0, \\
S_t + v S_x + \rho^{-1}(1 - \sigma) P \rho U_x - b \rho^{-1}(u-v) U + (c \rho \sigma + \sigma^{-1} Q) V_x + \\
+ \beta_3 \rho^{-1} \sigma (u-v) - \beta_4 b (u-v) = 0, \\
v S_t + \alpha_3 \rho^{-1}(1 - \sigma) \rho + \alpha_4 (c \rho + \sigma^{-1} Q) V_t + \\
+ \beta_3 \rho^{-1}(1 - \sigma) \rho + \alpha_4 (c \rho + \sigma^{-1} Q) V_t + \\
+ \beta_2 v + \beta_3 \rho^{-1}(1 - \sigma) \rho + \beta_4 (c \rho + \sigma^{-1} Q) = 0, \\
v S_u + \rho^{-1}(1 - \sigma) P \rho U_u + (c \rho + \sigma^{-1} Q) V_u + + \beta_1 u + \beta_3 \rho^{-1}(1 - \sigma) \rho + \beta_4 (c \rho + \sigma^{-1} Q) = 0, \\
v S_v + \rho^{-1}(1 - \sigma) P \rho U_v + (c \rho + \sigma^{-1} Q) V_v + + \beta_2 v + \beta_3 \rho^{-1}(1 - \sigma) \rho + \beta_4 (c \rho + \sigma^{-1} Q) = 0, \\
U_t + \rho R_x + u U_x - b \rho^{-1} \sigma U + b V + \gamma_3 \rho^{-1} \sigma (u-v) - \gamma_4 b (u-v) = 0, \\
\end{align*}
\]
corresponds. Here

\( \gamma \) by virtue of (23)

(19) it follows that

\( \beta \) Splitting this equation with respect to

In [3] it is shown that to a symmetry

self-adjoint with substitution

Let

Now equalities (13), (17) are satisfied. Thus, the following statement is proved.

\( \gamma \) which must satisfy equalities (11), (13), (16), (17), (21), (26). Substituting these functions in

\( \alpha \)

We substitute the functions (7)–(10) into (11)–(30), then by virtue of (24) \( \alpha_3 = \gamma_1 \), according to (30) \( \beta_4 = \delta_2 \), and from (25) and (29) it follows that \( \alpha_4 = \beta_3 = \gamma_2 = \gamma_4 = \delta_1 = \delta_3 = 0 \).

We shall assume that \( P \) is not constant. Integrating equation (22) with respect to \( \rho \), under assumption \( \gamma_3 \neq 0 \) we obtain

\[ P = \frac{\alpha_1 \rho^3}{3(1-\sigma)} + A(\sigma), \]

where \( A \) is a single variable function. Then by virtue of (23)

\[ -\alpha_2 \rho^2 + \frac{\alpha_1 \rho^3}{3(1-\sigma)} + \gamma_3(1-\sigma)A'(\sigma) = 0. \]

Splitting this equation with respect to \( \rho \), obtain that \( \alpha_1 = \alpha_2 = 0 \), hence \( P = A \) is a constant. Contradiction means that \( \gamma_3 = 0 \). Therefore, due to (22), (23) \( \alpha_1 = \alpha_2 = 0 \). From equation (19) it follows that \( \beta_1 = 0 \). Hence \( \delta_1 = 0 \) as coefficient at \( u \) in (26), and due to (28) \( \beta_2 = 0 \).

Now we have the functions \( R = -\alpha_3 u - \alpha_5, S = -\beta_4 v - \beta_5, U = -\gamma_1 \rho - \gamma_5, V = -\delta_2 \sigma - \delta_5, \)

which must satisfy equalities (11), (13), (16), (17), (21), (26). Substituting these functions in equation (21), we get \( \gamma_5 = \delta_5 = 0, \alpha_3 = \beta_4 = \gamma_1 = \delta_2, \alpha_5 = -\gamma_1 x, \)

therefore, \( \alpha_3 \) does not depend on \( x, \gamma_1 u = -\alpha_5 x \). Then from equation (26) it follows that \( \delta_2 u = -\beta_5 x, \) equation (11) implies the equality \( \alpha_5 t = 0, \) and due to equation (16) we have \( \beta_5 t = 0, \beta_5 u = -\beta_5 x. \) Consequently, \( \alpha_5 x = \beta_5 x = 0. \) Finally, \( \alpha_3 = \beta_4 = \gamma_1 = \delta_2 = -At - B, \alpha_5 = Ax - C, \beta_5 = Ax - D. \)

\[ R = A(tu - x) + Bu + C, \quad S = A(tv - x) + Bv + D, \quad U = At \rho + B \rho, \quad V = At \sigma + B \sigma. \]

Now equalities (13), (17) are satisfied. Thus, the following statement is proved.

**Theorem 1.** Let \( P = P(\rho, \sigma) \) be a non-constant function. Then system (2) is nonlinearly self-adjoint with substitution (31).

Note that this statement does not depend on a form of the function \( Q \).

3. Conservation laws

In [3] it is shown that to a symmetry

\[ X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta_\rho \frac{\partial}{\partial \rho} + \eta_\sigma \frac{\partial}{\partial \sigma} + \eta_u \frac{\partial}{\partial u} + \eta_v \frac{\partial}{\partial v} \]

of a nonlinearly self-adjoint system of equations the conservation law

\[ C^t = W^\rho \frac{\partial L}{\partial \rho_t} + W^\sigma \frac{\partial L}{\partial \sigma_t} + W^u \frac{\partial L}{\partial u_t} + W^v \frac{\partial L}{\partial v_t}, \quad C^x = W^\rho \frac{\partial L}{\partial \rho_x} + W^\sigma \frac{\partial L}{\partial \sigma_x} + W^u \frac{\partial L}{\partial u_x} + W^v \frac{\partial L}{\partial v_x}. \]

corresponds. Here

\[ W^\rho = \tau \rho_t + \xi \rho_x - \eta_\rho, \quad W^\sigma = \tau \sigma_t + \xi \sigma_x - \eta_\sigma, \quad W^u = \tau u_t + \xi u_x - \eta_u, \quad W^v = \tau v_t + \xi v_x - \eta_v. \]
It is easy to check, that the kernel of the principal Lie algebras \([5]\) of system (2) with all \(P, Q\) is generated by the operators

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v}.
\]

Now we shall use these symmetries for obtaining of conservation laws for system (2).

Take \(C = 1\), \(A = B = D = 0\) in (31), then the formal Lagrangian \(L = F_1\). For the symmetry \(X_1\) due to formula (33) we have \(W^\nu = \rho_t, W^\sigma = \sigma_t, W^u = u_t, W^v = v_t\). Then by formula (32)

\[
C^t_4 = \rho_t, \quad C^x_4 = u\rho_t + \rho u_t.
\]

For \(X_2\) analogously we obtain the conservation law

\[
C^{t}_5 = \rho_x, \quad C^{x}_5 = u\rho_x + \rho u_x,
\]

and for \(X_3\) we have \(W^\nu = t\rho_x, W^\sigma = t\sigma_x, W^u = tu_x - 1, W^v = tv_x - 1\), and the conservation law has the form

\[
C^t_6 = t\rho_x, \quad C^x_6 = tu\rho_x + t\rho u_x - \rho.
\]

If \(D = 1\), \(A = B = C = 0\), then \(L = F_2\), and the conservation laws corresponding to the symmetries are

\[
C^t_7 = \sigma_t, \quad C^x_7 = v\sigma_t + \sigma v_t;
\]

\[
C^t_8 = \sigma_x, \quad C^x_8 = v\sigma_x + \sigma v_x;
\]

\[
C^t_9 = t\sigma_x, \quad C^x_9 = tv\sigma_x + t\sigma v_x - \sigma.
\]

At \(B = 1, A = C = D = 0\) we have \(L = uF_1 + vF_2 + \rho F_3 + \sigma F_4\). Then to the symmetry \(X_1\) the conservation law

\[
C^t_{10} = u\rho_t + v\sigma_t + \rho u_t + \sigma v_t, \quad C^x_{10} = (u^2 + P_\rho + Q_\rho)\rho_t + (v^2 + P_\sigma + Q_\sigma)\sigma_t + 2\rho uu_t + 2\sigma vv_t
\]

corresponds, for the symmetry \(X_2\) we have

\[
C^t_{11} = u\rho_x + v\sigma_x + \rho u_x + \sigma v_x, \quad C^x_{11} = (u^2 + P_\rho + Q_\rho)\rho_x + (v^2 + P_\sigma + Q_\sigma)\sigma_x + 2\rho uu_x + 2\sigma vv_x,
\]

and for \(X_3\) the conservation law is

\[
C^t_{12} = tu\rho_x + tv\sigma_x + t\rho u_x - \rho + t\sigma v_x - \sigma,
\]

\[
C^x_{12} = t(u^2 + P_\rho + Q_\rho)\rho_x + t(v^2 + P_\sigma + Q_\sigma)\sigma_x + 2\rho uu_x(tu_x - 1) + 2\sigma v(tv_x - 1).
\]

Finally, at \(A = 1, B = C = D = 0\) in (31) we have \(L = (tu - x)F_1 + (tv - x)F_2 + t\rho F_3 + t\sigma F_4\). Then three conservation laws corresponding to the symmetries have the form

\[
C^t_{13} = (tu - x)\rho_t + (tv - x)\sigma_t + t\rho u_t + t\sigma v_t;
\]

\[
C^x_{13} = (tu^2 + P_\rho + Q_\rho)\rho_x + (tv^2 + P_\sigma + Q_\sigma)\sigma_x + (tu - x)\rho u_x + (tv - x)\sigma v_x;
\]

\[
C^t_{14} = (tu^2 + P_\rho + Q_\rho)\rho_x + (tv^2 + P_\sigma + Q_\sigma)\sigma_x + (2tu - x)\rho u_x + (2tv - x)\sigma v_x;
\]

\[
C^x_{14} = (tu^2 + P_\rho + Q_\rho)\rho_x + (tv^2 + P_\sigma + Q_\sigma)\sigma_x + (2tu - x)\rho u_x + (2tv - x)\sigma v_x;
\]

\[
C^t_{15} = (tu - x)\rho_x + t(2u - x)\sigma_x + t\rho(tu_x - 1) + t\sigma(tv_x - 1),
\]

\[
C^x_{15} = (tu^2 + P_\rho + Q_\rho)\rho_x + (tv^2 + P_\sigma + Q_\sigma)\sigma_x + (tu - x)\rho(tu_x - 1) + (2tv - x)\sigma(tv_x - 1).
\]
3.1. The system of Kh A Rakhmatulin
In contrast to the system (2) of T B Anderson — R Jackson, the equations system

\[
\begin{align*}
    \rho_t + \rho_x u + \rho u_x &= 0, \\
    \sigma_t + \sigma_x v + \sigma v_x &= 0, \\
    u_t + uu_x + \rho^{-1}(1 - c\sigma)(P_\rho \rho_x + P_\sigma \sigma_x) + b\rho^{-1}\sigma(u - v) &= 0, \\
    v_t + vv_x - c(P_\rho \rho_x + P_\sigma \sigma_x) - b(u - v) &= 0,
\end{align*}
\]

(34)
of Kh A Rakhmatulin [4] of a mixture dynamics depends on one pressure function only. As you can see from Theorem 1, the form of the pressure \( Q \) does not affect on the fact of nonlinear self-adjointness of system (2). Taking \( Q \equiv 0 \) in all previous reasoning we will obtain the same results for Kh.A.Rakhmatulin’s system without any changes.

**Theorem 1.** Let \( P = P(\rho, \sigma) \) be a non-constant function. Then system (34) is nonlinearly self-adjoint with substitution (31).

Conservation laws for (34) can be derived from the obtained above conservation laws for the T B Anderson — R Jackson system by taking \( Q \equiv 0 \).

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