Topological number and Fermion Green’s function of Strongly Interacting Topological Superconductors

Yi-Zhuang You,1 Zhong Wang,2,3 Jeremy Oon,1 and Cenke Xu1
1Department of physics, University of California, Santa Barbara, CA 93106, USA
2Institute for Advanced Study, Tsinghua University, Beijing, China, 100084
3Collaborative Innovation Center of Quantum Matter, Beijing 100871, China

It has been understood that short range interactions can reduce the classification of topological superconductors in all dimensions. In this paper we demonstrate by explicit calculations that when the topological phase transition between two distinct phases in the noninteracting limit is gapped out by interaction, the bulk fermion Green’s function $G(i\omega)$ at the “transition” approaches zero as $G(i\omega) \sim \omega$ at certain momentum $\vec{k}$ in the Brillouin zone.

Introduction — 
Unlike bosonic systems, fermionic systems can have nontrivial topological insulator (TI) and topological superconductor (TSC) phases without interaction. Noninteracting fermionic TIs and TSCs in all dimensions with various symmetries have been fully classified in Ref. 1–3. However, it remains an open and challenging question that what role does interaction play in the classification of TIs and TSCs. According to a pioneering work Ref. 4, 5 a 1d TSC with time-reversal symmetry, which in the noninteracting limit has a $\mathbb{Z}$ classification 1–3, has only a $\mathbb{Z}_8$ classification in the presence of local interactions. This implies that the boundary state of 8 copies of this 1d TSC (which is 8 flavors of 0d Majorana fermion zero modes) can be gapped out by interaction without ground state degeneracy. The work in Ref. 4, 5 was soon generalized to 2d TSCs with a 1d boundary 6–9, and 3d TSCs with 2d boundary 10, 11. All these TSCs have $\mathbb{Z}$ classification without interaction, namely in all these cases, for arbitrary flavors of the system, the boundary remains gapless without interaction. But with 8 copies of such TSCs in 2d, and 16 flavors of He$^3$B in 3d, a strong enough local interaction that preserves all the symmetries can render the boundary states trivial, namely the boundary states can be gapped out by interaction without degeneracy.

If interaction reduces the classification of a TSC, it not only implies that the boundary becomes trivial under interaction, it also implies that the “TSC” and the trivial state can be adiabatically connected to each other without any phase transition, namely the bulk quantum critical point in the noninteracting limit is gapped out by interaction. A free fermion TI or TSC with $\mathbb{Z}$ classification is usually characterized by a quantized topological number, which can be constructed by fermion Green’s function 12, 17. For example, the so-called TKNN number 12, 13 of integer quantum Hall state can be represented as 18, 19

$$N = \frac{1}{24\pi^2} \int d^3k \epsilon^{\mu\nu\rho} \text{Tr} [G\partial_\mu G^{-1}G\partial_\nu G^{-1}G\partial_\rho G^{-1}], \quad (1)$$

where $G(k)$ is the fermion Green’s function in the frequency and momentum space $k = (i\omega, \vec{k})$, and $i\omega$ is the Matsubara frequency. The formula Eq. (1) can be formally applied to interacting systems too, as long as we use the full interacting fermion Green’s function 12, 17. Recently it was also proved that in the presence of interaction, Eq. (1) is fully determined by zero-frequency Green’s function, and the topological invariant takes a simpler form 16

$$N = \frac{1}{2\pi} \int d^2k F_{xy},$$

where $F_{xy}$ is the Berry curvature calculated from the eigenvectors of $-G^{-1}(\omega = 0, \vec{k})$, as if it is a “noninteracting” Hamiltonian. This TKNN topological number must be quantized, so it can only change through a sharp “transition” in the phase diagram. In the noninteracting limit, this sharp transition is a physical phase transition which corresponds to closing the bulk gap of the fermion, i.e. the fermion Green’s function develops a gapless pole at the transition $G(i\omega) \sim 1/\omega$. With strong interaction, the results in Ref. 4, 5 imply that this quantum critical point can be gapped out by interactions, while this “transition” of topological number must still occur in the phase diagram. Ref. 20, 21 proposed that the quantized topological number such as Eq. (1) not necessarily corresponds to gapless edge states, it can also correspond to zeros of the fermion Green’s function at the edge. In our paper we will study the fermion Green’s function in the bulk, and we will demonstrate with explicit calculations that the fermion Green’s function develops zero at the “transition” of topological numbers, even though there is no transition in bulk spectrum. And the fermion Green’s function vanishes analytically as $G(i\omega) \sim \omega$.

A simple observation can support this conclusion above: in Eq. (1), $G$ and $G^{-1}$ are almost equivalent. Thus the topological number should either change through a pole of $G$ (the noninteracting case), or change through a zero of $G$, which corresponds to a pole of $G^{-1}$. This implies that when the “transition” of topological number is gapped out by interactions, the Green’s function should approach zero as $G(i\omega) \sim \omega^\alpha$ with positive $\alpha$. Since at the “transition” the fermion spectrum is fully gapped, the fermion correlation function in real space-time must be short ranged, thus $\alpha$ must be an integer, and the simplest scenario is $\alpha = 1$. 
1d Example —

Before providing a general argument of the existence of zeros in the fermion Green’s function for arbitrary dimension, we will first demonstrate this behavior explicitly in a 1d example using 8 copies of Kitaev’s chain [22], also known as the 1d Fidkowski-Kitaev model [4]. Consider a 1d lattice, on each site $i$, we introduce 8 Majorana fermions denoted by $\chi_{i\alpha}$ with $\alpha = 1, \cdots, 8$ labeling the fermion flavors (species), which satisfy the Majorana fermion anti-commutation relations $\{\chi_{i\alpha}, \chi_{i'\alpha'}\} = 2\delta_{i'i}\delta_{\alpha\alpha'}$. The model Hamiltonian reads

$$H = H_u + H_w,$$

$$H_u = \frac{1}{2} \sum_{i,j,\alpha} i \nu_{ij} \chi_{i\alpha} \chi_{j\alpha},$$

$$H_w = -w \sum_{i,\{\alpha_k\}} X_{\alpha_1\alpha_2\alpha_3\alpha_4} \chi_{i\alpha_1} \chi_{i\alpha_2} \chi_{i\alpha_3} \chi_{i\alpha_4}.$$  \hspace{1cm} (2)

$H_u$ is the inter-site coupling term with $u_{ij} \in \mathbb{R}$ and $u_{ij} = -u_{ji}$. $H_w$ is the on-site Fidkowski-Kitaev interaction, in which the coefficient $X_{\alpha_1\alpha_2\alpha_3\alpha_4} \equiv \langle e_1 | \chi_{i\alpha_1} \chi_{i\alpha_2} \chi_{i\alpha_3} \chi_{i\alpha_4} | e_1 \rangle$ is given as the expectation value of the four-fermion operator on a chosen on-site ground state $|e_1\rangle$ (to be specified later). Here we have omitted the site index $i$, and the summation of flavor $\sum_{\alpha}$ in $H_w$ is to sum over all the possible quartets $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ for $\alpha_k = 1, \cdots, 8$. To specify the state $|e_1\rangle$, we may choose a representation of the Majorana operators as $\chi_{2n-1} = \sigma_1^n \chi_{1} \chi_{2n} = \sigma_2^{n_2} \prod_{m<n} \sigma_m^3 \chi_n$ for $n = 1, \cdots, 4$, with $\sigma_\mu$ being the $\mu$-th Pauli matrix acting on the $n$-th tensor factor. Then the chosen ground state $|e_1\rangle$ can be written as $|e_1\rangle = ((0000) - (1111))/\sqrt{2}$ (where $|0\rangle$ and $|1\rangle$ are the eigen states of $\sigma^2$). The choice of $|e_1\rangle$ is not unique, but we will stick to this convention. It turns out that there are 14 non-zero $X_{\alpha_1\alpha_2\alpha_3\alpha_4}$’s, corresponding to the 14 four-fermion terms in $H_w$ on each site. The explicit form of $H_w$ was given in Ref. [4] and its physical meaning has been discussed in Ref. [7] [21].

We would like to introduce an explicit set of eigen basis for $H_u$. Let us focus on a single site, and omit the site index $i$. In the 16-dimensional Hilbert space of the 8 Majorana fermions on each site, the Hamiltonian $H_u$ can be diagonalized. All its on-site many-body eigenstates can be constructed from $|e_1\rangle$ by applying Majorana fermion operators. We can divide these states into even and odd sectors according to their fermion parity. Given the ground state $|e_1\rangle$ is an even parity state, the odd parity states $|\epsilon_{\alpha}\rangle = \chi_{\alpha} |e_1\rangle$ are obtained by acting a single fermion operator; and the even parity states $|\eta_{\alpha}\rangle = \chi_{\alpha} \chi_{\alpha} |e_1\rangle$ can be constructed by acting the two-fermion operator with the first operator fixed as $\chi_{\alpha}$. These states $|p_{\alpha}\rangle (p = e, \epsilon, \eta = \alpha = 1, \cdots, 8)$ form a set of orthonormal basis, i.e., $\langle p_{\alpha} | p_{\alpha}' \rangle = \delta_{pp'} \delta_{\alpha\alpha'}$. These states can be defined on each site as $|p_{\alpha}\rangle_i$. In this set of basis, the interaction Hamiltonian $H_w$ is diagonalized,

$$H_w = \sum_i \left( -14w |e_1\rangle_i \langle e_1\rangle_i + 2w \sum_{\alpha=2}^{8} |e_\alpha\rangle_i \langle e_\alpha\rangle_i \right). \hspace{1cm} (3)$$

Odd parity states are annihilated by $H_w$, i.e. of zero eigenvalue, so they do not appear in Eq. (3). The many-body ground state of $H_w$ is simply the $|e_1\rangle$ product state, denoted as $|0\rangle = \bigotimes_i |e_1\rangle_i$, with the ground state energy $E_0$ scaling with the system size.

Now we turn on the bilinear fermion coupling term $H_u$ perturbatively, assuming $u_{ij} \ll w$. In this section we will calculate the Green’s function up to the first order perturbation of $u_{ij}/w$. In the next section we will prove that our qualitatively valid result is valid after summing over the entire infinite perturbation series. $H_u$ will take the unperturbed ground state $|0\rangle$ to a group of two-fermion excited states $\chi_{i\alpha} \chi_{j\alpha} |0\rangle$, all of which have approximately the same energy $E_0 + 28w$. So according to the perturbation theory, the perturbed ground state (to the first order in $u_{ij}$) reads

$$|g\rangle = \left( 1 - \frac{H_u}{28w} \right) |0\rangle. \hspace{1cm} (4)$$

The Majorana fermion Green’s function in the real space can be calculated from [16] [17]

$$G(i\omega)_{i\alpha,i'\alpha'} = \sum_m \left( \frac{\langle g | \chi_{i\alpha} | m \rangle \langle m | \chi_{i'\alpha'} | g \rangle}{i\omega - (E_m - E_0)} \right)$$

$$+ \sum_{\mu} \left( \frac{\langle m | \chi_{i\alpha} | g \rangle \langle g | \chi_{i'\alpha'} | m \rangle}{i\omega + (E_m - E_0)} \right). \hspace{1cm} (5)$$

Since $H_u \ll H_w$, we have $(E_m - E_0) \simeq 14w$, therefore

$$G(i\omega)_{i\alpha,i'\alpha'} = \frac{\langle g | \chi_{i\alpha} \chi_{i'\alpha'} | g \rangle}{i\omega - 14w} + \frac{\langle g | \chi_{i'\alpha'} \chi_{i\alpha} | g \rangle}{i\omega + 14w}$$

$$= \frac{i\omega}{(i\omega)^2 - (14w)^2} \langle g | \chi_{i\alpha} \chi_{i'\alpha'} | g \rangle$$

$$+ \frac{14w}{(i\omega)^2 - (14w)^2} \langle g | \chi_{i\alpha} \chi_{i'\alpha'} | g \rangle. \hspace{1cm} (6)$$

In the first term, $\langle g | \chi_{i\alpha} \chi_{i'\alpha'} | g \rangle = 2\delta_{ii'}\delta_{\alpha\alpha'}$, according to the Majorana fermion anti-commutation relations. The second term is not vanishing only for $i \neq i'$, and under such condition, $\langle g | \chi_{i\alpha} \chi_{i'\alpha'} | g \rangle = 2g | \chi_{i\alpha} \chi_{i'\alpha'} | g \rangle = 4|0\rangle \chi_{i\alpha} \chi_{i'\alpha'} - \frac{H_u}{28w} |0\rangle = 2iu_{ii'}\delta_{\alpha\alpha'}/(14w)$. Hence

$$G(i\omega)_{i\alpha,i'\alpha'} = \frac{2\delta_{ii'}\delta_{\alpha\alpha'}}{(i\omega)^2 - (14w)^2} (i\omega u_{ii'} + iu_{ii'}). \hspace{1cm} (7)$$

Fourier transform to the momentum space, assuming the coupling $u_{ij}$ is alternating between $u$ and $v$ along the 1d chain, s.t. $u_k = u - ve^{ik}$, then on the sublattice basis,

$$G(i\omega, k)_{\alpha,\alpha'} = \frac{2\delta_{\alpha\alpha'}}{(i\omega)^2 - (14w)^2} \begin{bmatrix} i\omega & -iu_{k} \\ iu_{k} & i\omega \end{bmatrix}. \hspace{1cm} (8)$$
At the topological “transition” point \( u = v, \ u_0 \sim -i0k \) vanishes as \( k \to 0 \), so the Green’s function approaches zero analytically at the zero frequency and momentum \( i\omega, k \to 0 \) as expected. In contrast, in the free fermion case, such a topological transition happens through a gapless critical point characterized by the pole in the Green’s function instead.

**Zero in Green’s Function in general dimension —**

The above perturbative calculation is not limited to the 1d model, but can be immediately generalized to higher spatial dimensions. It is found that for a series of \( d \)-dimensional models with arbitrary \( d \), the topological number defined with full fermion Green’s function can change via the zero in the Green’s function at some moment in the Brillouin zone. This statement actually holds to all orders of perturbation.

We will consider lattice models for a \( d \)-dimensional TSC (\( d \geq 2 \)) like the follows:

\[
H = 3 \sum_{\alpha = 1}^{8} \chi_{\alpha, k} \left( \sum_{\mu = 1}^{d} \sin k_{\mu} \right) \chi_{\alpha, k} + m \chi_{\alpha, -k} \left( \sum_{\mu = 1}^{d} \cos k_{\mu} - d + 1 \right) \Gamma^{d+1} \chi_{\alpha, k};
\]

where \( \Gamma^{1\ldots d} \) are symmetric matrices, while \( \Gamma^{d+1} \) is an antisymmetric matrix. When \( d = 2 \), \( \Gamma^{1} = \sigma^{x} \), \( \Gamma^{2} = \sigma^{y} \), \( \Gamma^{3} = -\sigma^{z} \); \( m = 0 \) is the quantum critical point between 8 copies of \( p + ip \) TSC (\( m > 0 \)) and \( p - ip \) TSC (\( m < 0 \)), the topological number defined in Eq. [1] changes by 16 at this transition. When \( d = 3 \), \( \Gamma^{1}, 2, 3 \) are 4 x 4 symmetric matrices, \( \Gamma^{4} \) and \( \Gamma^{5} = \prod_{\alpha = 1}^{4} \Gamma^{\alpha} \) are antisymmetric matrices. \( m = 0 \) is the quantum critical point between 8 copies of He\(^{3}B \) TSC with topological number \(-1 \) and \(+1 \). The time-reversal symmetry acts on the Majorana fermions as \( Z_{2}^{T} : \chi_{\alpha, k} \to i^{d} K \chi_{\alpha, -k} \), where \( K \) stands for complex conjugation. The topological phase for He\(^{3}B \) phase can be represented with Fermion Green’s function:

\[
N = \frac{1}{48\pi^{2}} \int d^{3}k \epsilon^{abc} \text{Tr} [\Gamma^{5} G \partial_{\mu} G^{-1} G \partial_{\nu} G^{-1} G \partial_{\sigma} G^{-1}] (10)
\]

where \( G = G(0, k) \) is the zero frequency Green’s function. The momentum integral is carried out in the Brillouin zone.

Besides the Majorana fermion hopping terms, we will also turn on the on-site Fidkowski-Kitaev interaction \( H_{w} \) to gap out the fermions,

\[
H_{w} = -w \sum_{j} \sum_{\sigma, \{\alpha_{k}\}} X_{\{\alpha_{k}\} \chi_{\alpha_{1}, j} \chi_{\alpha_{2}, j} \chi_{\alpha_{3}, j} \chi_{\alpha_{4}, j}} \text{,}
\]

\[
H_{w} = -w \sum_{j} \sum_{\sigma, \{\alpha_{k}\}} X_{\{\alpha_{k}\} \chi_{\alpha_{1}, j} \chi_{\alpha_{2}, j} \chi_{\alpha_{3}, j} \chi_{\alpha_{4}, j}} \text{,}
\]

where \( \sigma \) labels the orbital degrees of freedom, \( j \) labels the lattice sites. The on-site interaction in Eq. [11] has a nondegenerate ground state on every site, thus with strong interaction Eq. [11], the quantum critical point \( m = 0 \) in Eq. [9] is indeed gapped, and the system is in a trivial direct product state.

Now we will demonstrate that in the two dimensional phase diagram of \( w \) and \( m \), \( m = 0 \) is always the transition line of topological number defined with the Green’s function (Fig. [1]). With weak interaction \( w \), because the topological number defined in Eq. [1] and Eq. [10] are always quantized, a weak interaction will not change the topological numbers. With strong interaction, the fermion Green’s function can be computed by perturbation theory as the previous section. With first order perturbation, the Green’s function in any dimension would take the same form as Eq. [8] It is straightforward to verify that the strongly interacting Green’s function from first order perturbation theory always leads to the same 2d TKNN number (Eq. [1]) as 8 copies of noninteracting \( p \pm ip \) superconductors with the same \( m \). Also, for strongly interacting He\(^{3}B \), according to Eq. [8] the zero frequency Greens function reads \( G(0, k) \sim 1/G(0, k)_{\mu} \), where \( G(i\omega, k)_{\mu} \) is the free fermion Green’s function without interaction at all. In the definition of topological number Eq. [10] \( G(0, k) \) and \( G(0, k)^{-1} \) are totally interchangable since \( G(0, k) \) anticommutes with \( \Gamma_{5} \), thus the topological numbers in the noninteracting limit and strong coupling limit are identical. This means that \( m = 0 \) is always the “transition” line of topological number, even though the qualitative spectrum of the system might not change across this line. The argument above was based on the explicit form of the fermion Green’s function from the first order perturbation theory. But since the topological numbers are always quantized, we expect the higher order perturbations will not change the topological numbers, as long as \( w \) is strong enough.

Now we analyze the symmetry of the two-fermion Green’s function \( G(k)_{\alpha, \alpha'} = -\langle \chi_{\alpha, k} \chi_{\alpha', k} \rangle \) where \( \chi_{\alpha, k} \) denotes the \( \alpha \)-flavor Majorana fermion of the frequency-momentum \( k = (i\omega, k) \). In the flavor space (indexed by \( \alpha, \alpha' \), \( H_{u} \) has the full SO(8) symmetry and \( H_{w} \) has a smaller SO(7) symmetry\[^4\], so the total Hamiltonian \( H = H_{u} + H_{w} \) is SO(7) symmetric, and \( \chi_{\alpha} \) carries a 8-dimensional spinor representation of the SO(7) symmetry. Since with strong interaction Eq. [11] the system is gapped and nondegenerate, the ground state must be SO(7) invariant. Thus the Green’s function must be diagonal in the flavor space: \( G(k)_{\alpha, \alpha'} = g(k) \delta_{\alpha, \alpha'} \) where \( g(k) \) is the Green’s function in the orbital space (see Appendix A for a mathematical proof). Physically this can be understood since \( H_{w} \) only breaks the SO(8) symmetry by driving a four-fermion ordering without generating any fermion-bilinear order, so the broken symmetry will not be revealed at the two-fermion level. Then we turn to the orbital space Green’s function \( g(k) \). Because the Fidkowski-Kitaev interaction \( H_{w} \) has the full orbital space symmetry, in the strong interaction limit, the orbital degrees of freedom must be degenerate at each en-
energy level. So the Green’s function can be written as a polynomial of $H_n/(i\omega - H_w)$ where $H_w$ will just behave like a constant in the orbital space. Given that at the “transition” point $H_n$ is constructed using symmetric matrices $\Gamma^\mu$ only, so the Green’s function $g(k)$ must also be a symmetric matrix (in the momentum space), i.e. $g(k) = g(k)^T$. However, by definition $g(k) = -\langle \chi_k \chi_k^\dagger \rangle$, and for the off-diagonal part of $g(k)$, we have $g(-k) = -\langle \chi_{-k} \chi_k \rangle = \langle \chi_k \chi_{-k}^\dagger \rangle = -g(k)^\dagger = -g(k)$, meaning that the off-diagonal part of $g(k)$ must be odd in $k$. Since in the large $w$ limit the whole system is gapped, the real space fermion Green’s function must decay exponentially, thus the off-diagonal components of $g(k)$ must be analytic when $k \to 0$. Thus we conclude that the off-diagonal components of $g(k)$ must scale as $g(k) \sim k^\alpha$, where $\alpha$ must be a positive odd integer.

So now we are left with the diagonal part of $g(k)$. Using the spectral representation of the Green’s function, we can show that the diagonal part of $g(k)$ at zero-momentum must take the following form

$$g(i\omega, k = 0)_{\sigma, \sigma} = \sum_m \frac{2i\omega |\langle m | \chi_0 | g \rangle|^2}{(i\omega)^2 - (E_m - E_g)^2}, \quad (12)$$

where $|g\rangle$ is the ground state, and $|m\rangle$ denotes a generic excited state. We can see $g(i\omega, k = 0)_{\sigma, \sigma}$ must be odd in $i\omega$ at $k = 0$. As long as we are in the fully gapped phase (under strong enough interaction), the excitation energy $(E_m - E_g)$ will never vanish, so there will be no singularity at zero frequency, then it follows that diagonal term of the Green’s function must also vanish with $k \to 0$. So in conclusion, the fermion Green’s function at the topological “transition” in the strong interaction phase must develop a zero at zero frequency around the free-fermion Dirac points in the Brillouin zone.

Summary and Discussion —

Let us summarize our results in Fig. 1. In all dimensions we have considered in this paper, the $m = 0$ line in the phase diagrams are always the transition lines for topological numbers. And in all dimensions a short range interaction can gap out this transition line, but the fermion Green’s function on this line develops zero at zero frequency.

In 1d, the system we consider is a 8 copies of Kitaev’s chain with interactions. In 1d, the quantum critical point at the noninteracting limit will be immediately gapped out by infinitesimal interactions, but in 2d and 3d a short range interaction is irrelevant for gapless Dirac and Majorana fermions, thus the interaction can only gap out the critical line $m = 0$ beyond certain value $w_c$. The 2d example of our paper is a transition between eight $p + ip$ to eight $p - ip$ TSCs, and since these two phases have opposite chiral edge states (opposite thermal Hall effects), thus they cannot be adiabatically connected to each other in the phase diagram. Thus although the $m = 0$ line is gapped out by strong enough interactions, there must be another phase transition line in the phase diagram (the horizontal line in Fig. 1) that separates the strongly interacting phase (which is a trivial direct product state) from the $p + ip$ and $p - ip$ TSCs. In 1d and 3d cases discussed in our paper, the two sides across the line $m = 0$ can be adiabatically connected to each other without any physical phase transition, i.e. the entire phase diagram has only one phase.

Our result implies that when the topological number such as Eq. 12 is nonzero, the system can still remain trivial. Ref. 24 proposed a different method to diagnose TIs and TSCs in 1d and 2d: if $|\Psi\rangle$ is a nontrivial TI or TSC, and $|\Omega\rangle$ is a trivial state, then the following quantity, so called strange correlator, must either saturate to a constant or decay as power-law in the limit $|r - r'| \to +\infty$:

$$C(r, r') = \frac{\langle \Omega | c(r) c^\dagger(r') | \Psi \rangle}{\langle \Omega | \Psi \rangle}, \quad (13)$$

even though $|\Omega\rangle$ and $|\Psi\rangle$ both only have short range correlation between fermion operators. In another work we will demonstrate that the strange correlator proposed in Ref. 24 can still correctly distinguish trivial states from nontrivial TSCs under interaction. In particular, we calculate $C(r, r')$ for eight copies of 1d Kitaev’s chain[22]. In the TSC phase, $C(r, r')$ is a constant in the noninteracting limit, while $C(r, r')$ immediately becomes short range with infinitesimal short range interactions [23].

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[1] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, AIP Conf. Proc. 1134, 10 (2009).
[2] S. Ryu, A. Schnyder, A. Furusaki, and A. Ludwig, New J. Phys. 12, 065010 (2010).
[3] K. A. Yu, AIP Conf. Proc 1134, 22 (2009).
Explicit Verification of the Emergent SO(8) Symmetry

In the fermion flavor space, the Hamiltonian $H = H_u + H_w$ only has an SO(7) symmetry, but the resulting two-fermion Green’s function actually has an emergent SO(8) symmetry, which is larger than the symmetry of the Hamiltonian. The smaller SO(7) symmetry will be manifested in the four-fermion (and higher order) Green’s function, while at the two-fermion level, the Green’s function still processes the full SO(8) symmetry.

To verify this statement, we note that the 8 flavors of the Majorana fermions form the 8-dimensional spinor representation of the SO(7) group. We may write down the 21 generators of the SO(7) group in this spinor representation explicitly. Let $T_{ij}$ be the generator that performs the SO(7) rotation in the $ij$-plane. Its spinor representation is an $8 \times 8$ matrix, which can be written as a direct product of three Pauli matrices, denoted as $\sigma^{\mu\nu\lambda} = \sigma^\mu \otimes \sigma^\nu \otimes \sigma^\lambda$,

$$
\begin{align*}
T_{12} &= i\sigma^{21} \\
T_{24} &= -i\sigma^{12} \\
T_{37} &= -i\sigma^{30} \\
T_{13} &= -i\sigma^{021} \\
T_{25} &= i\sigma^{112} \\
T_{45} &= i\sigma^{032} \\
T_{14} &= i\sigma^{201} \\
T_{26} &= i\sigma^{200} \\
T_{46} &= -i\sigma^{320} \\
T_{15} &= -i\sigma^{123} \\
T_{27} &= -i\sigma^{132} \\
T_{47} &= i\sigma^{012} \\
T_{16} &= i\sigma^{121} \\
T_{28} &= -i\sigma^{122} \\
T_{48} &= i\sigma^{010} \\
T_{17} &= -i\sigma^{213} \\
T_{29} &= i\sigma^{210} \\
T_{49} &= i\sigma^{020} \\
T_{18} &= i\sigma^{202} \\
T_{210} &= -i\sigma^{102} \\
T_{410} &= i\sigma^{332}.
\end{align*}
$$

(14)

To respect the SO(7) symmetry, the two-fermion Green’s function $G$ must commute with all the 21 generators listed above. We search over the space of all $8 \times 8$ matrices by a Mathematica program, and found that $\sigma^{000}$ (the $8 \times 8$ identity matrix) is the only matrix that commute with all the SO(7) generators. So we must have $G \propto \sigma^{000}$ (or $G_{\alpha\alpha'} \propto \delta_{\alpha\alpha'}$) in the fermion flavor space, which indeed has the full SO(8) symmetry.