MORITA EQUIVALENCES OF CYCLOTOMIC HECKE ALGEBRAS OF TYPE $G(r, p, n)$

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ABSTRACT. We prove a Morita reduction theorem for the cyclotomic Hecke algebras $H_{r,p,n}(q, \mathbf{Q})$ of type $G(r, p, n)$ with $p > 1$ and $n \geq 3$. As a consequence, we show that computing the decomposition numbers of $H_{r,p,n}(\mathbf{Q})$ reduces to computing the $p'$-splittable decomposition numbers (see Definition 1.1) of the cyclotomic Hecke algebras $H_{r',p',n'}(\mathbf{Q}')$, where $1 \leq r' \leq r$, $1 \leq n' \leq n$, $p' \mid p$ and where the parameters $\mathbf{Q}'$ are contained in a single $(\epsilon'; q)$-orbit and $\epsilon'$ is a primitive $p'$th root of unity.

Dedicated to Toshiaki Shoji on the occasion of his sixtieth birthday.

1. Introduction

Motivated by “generic features” of the representation theory of finite reductive groups Broué and Malle [3] attached a cyclotomic Hecke algebra to each complex reflection group. These algebras have many good properties and, conjecturally, they arise as the endomorphism algebras of Deligne–Lusztig representations.

This paper is concerned with the cyclotomic Hecke algebras of type $G(r, p, n)$ with $p > 1$ and $n \geq 3$. These algebras were first considered by Broué and Malle [3] and by Ariki [1] in the semisimple case. These algebras have been studied extensively in the non–semisimple case, notably by the first author [9–12] and by Genet and Jacon [20]. In particular, the simple modules of these algebras have been classified over any field of characteristic coprime to $p$ [11].

In the case $p = 1$ the cyclotomic Hecke algebras of type $G(r, 1, n)$ are known as the Ariki–Koike algebras. These algebras are well understood; see [17] and the references therein. The highlight of this theory is Ariki’s celebrated theorem which says that the decomposition numbers of these algebras in characteristic zero can be computed using the canonical bases of the higher level Fock spaces for the quantized affine special linear groups. Another fundamental result for the Ariki–Koike algebras is the Morita equivalence theorem of Dipper and the second author [6] which says that, up to Morita equivalence, these algebras are determined by the $q$–orbits of their parameters.

The first main result in this paper gives an analogue of the Dipper–Mathas Morita equivalence theorem for the Hecke algebras of type $G(r, p, n)$. To state this result explicitly, fix positive integers $r$, $p$ and $n$ with $r = pt$ for some integer $t$, and let $K$ be an algebraically closed field of characteristic coprime to $p$. Fix parameters $q, Q_1, \ldots, Q_t \in K^\times$ and let $\mathbf{Q} = (Q_1, \ldots, Q_t)$. Let $H_{r,n}(\mathbf{Q})$ be the Ariki–Koike algebra and let $H_{r,p,n}(\mathbf{Q})$ be the Hecke algebra of type $G(r, p, n)$ with parameters $q$ and $\mathbf{Q}$. The algebra $H_{r,n}(\mathbf{Q})$ is equipped with an automorphism $\sigma$ of order $p$ and $H_{r,p,n}(\mathbf{Q})$ is the fixed point subalgebra of $H_{r,n}(\mathbf{Q})$ under $\sigma$. There is a second automorphism $\tau$ on $H_{r,n}$ which fixes $H_{r,p,n}$ setwise. For the precise definitions see the third paragraph in Section 2, Definition 2.1 and Definition 3.2.

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Fix a primitive \( p \)th root of unity \( \varepsilon \) in \( K \) and say that \( Q_1 \) and \( Q_2 \) are in the same \((\varepsilon, q)\)-orbit if \( Q_1 = \varepsilon^aq^bQ_2 \), for some integers \( a, b \in \mathbb{Z} \). Given two ordered tuples \( \mathbf{X} = (X_1, \ldots, X_s) \) and \( \mathbf{Y} = (Y_1, \ldots, Y_l) \) of elements of \( K^\times \) set \( \mathbf{X} \lor \mathbf{Y} = (X_1, \ldots, X_s, Y_1, \ldots, Y_l) \).

Suppose that \( A \) is an algebra and that \( \mathbb{Z}_p \) is a group which acts on \( A \) as a group of algebra automorphisms. Let \( A \rtimes \mathbb{Z}_p \) be the algebra with elements \( \{ ag | a \in A \text{ and } g \in \mathbb{Z}_p \} \) and with multiplication
\[
ag \cdot bh = ab^g \cdot gh, \quad \text{for } a, b \in A \text{ and } g, h \in \mathbb{Z}_p.
\]

The first main result of this paper is the following.

**Theorem A.** Suppose that \( Q = Q_1 \lor \cdots \lor Q_\ell \), where \( Q_1 \in Q_\alpha \) and \( Q_2 \in Q_\beta \) are in the same \((\varepsilon, q)\)-orbit only if \( \alpha = \beta \). Let \( t_\alpha = |Q_\alpha| \), for \( 1 \leq \alpha \leq \kappa \). Then \( \mathcal{H}_{r,p,n}(Q) \) is Morita equivalent to the algebra
\[
\bigoplus_{b_1, \ldots, b_n \geq 0 \atop b_1 + \cdots + b_n = n} \left( \mathcal{H}_{pt_1, b_1}(Q_1) \otimes \cdots \otimes \mathcal{H}_{pt_\ell, b_\ell}(Q_\ell) \right) \rtimes \mathbb{Z}_p.
\]

In the theorem, each of the algebras \( \mathcal{H}_{pt_\alpha, b_\alpha}(Q_\alpha) \) has an automorphism \( \sigma_\alpha \) of order \( p \) and, in the direct sum, the automorphism \( \sigma_1 \otimes \cdots \otimes \sigma_\ell \) acts diagonally on the algebra \( \mathcal{H}_{pt_1, b_1}(Q_1) \otimes \cdots \otimes \mathcal{H}_{pt_\ell, b_\ell}(Q_\ell) \). Observe that \( \langle \sigma_1 \otimes \cdots \otimes \sigma_\ell \rangle \cong \mathbb{Z}_p \).

The second result of this paper uses Theorem A to prove a reduction theorem for computing the decomposition numbers of \( \mathcal{H}_{r,p,n}(Q) \). In order to state this result fix a modular system \((F, O, K)\) “with parameters”. That is, we fix an algebraically closed field \( F \) of characteristic zero, a discrete valuation ring \( O \) with maximal ideal \( \pi \) and residue field \( K \cong O/\pi \), together with parameters \( q_0, Q_1, \ldots, Q_\ell \in \mathcal{O}^\times \) such that \( q = q_0 + \pi \) and \( Q_i = Q_0 + \pi \) for each \( i \). Let \( \mathcal{H}_{F}^{r,p,n} = \mathcal{H}_{r,p,n}(Q \otimes \mathcal{O}) \) be the Hecke algebra of type \( G(r, p, n) \) over \( F \) with parameters \( q \) and \( \mathcal{Q} = (Q_1, \ldots, Q_\ell) \) and similarly let \( \mathcal{H}_{\mathcal{O}}^{r,p,n} = \mathcal{H}_{r,p,n}(Q) \) and write \( \mathcal{H}_{r,p,n}^O = \mathcal{H}_{r,p,n}(Q) \). We assume that \( H_{r,p,n}^{F} \) is semisimple. By freeness we have that \( H_{r,p,n}^{F} \cong H_{r,p,n}^{O} \otimes \mathcal{O} \mathcal{F} \) and \( H_{r,p,n}^{K} \cong H_{r,p,n}^{O} \otimes \mathcal{O} K \). Thus, by choosing \( \mathcal{O} \)-lattices we can talk of modular reduction from \( H_{r,p,n}^{F} \)–\text{Mod} to \( H_{r,p,n}^{K} \)–\text{Mod}.

Using the definitions, it is straightforward to check that the automorphisms \( \sigma \) and \( \tau \) commute with modular reduction. Thus, we have compatible automorphisms \( \sigma \) and \( \tau \) on \( H_{r,p,n}^{F} \) and on \( H_{r,p,n}^{K} \).

Let \( R \in \{ F, K \} \) and let \( M \) be an \( H_{r,p,n}^{R} \)–module. Then we define a new \( H_{r,p,n}^{R} \)–module \( M^r \) by “twisting” the action of \( H_{r,p,n}^{R} \) using the automorphism \( \tau \). Explicitly, \( M^r = M \) as a vector space and the \( H_{r,p,n}^{R} \)–action on \( M^r \) is defined by
\[
m \cdot h = m \tau(h), \quad \text{for all } m \in M \text{ and } h \in H_{r,p,n}^{R}.
\]

Since \( \tau^p \) is an inner automorphism of the algebra \( H_{r,p,n}^{R} \), it follows that \( M \cong M^{p} \) for any \( H_{r,p,n}^{R} \)–module \( M \). Therefore, there is a natural action of the cyclic group \( \mathbb{Z}_p \) on the set of isomorphism classes of \( H_{r,p,n}^{R} \)–modules. We define the **inertia group** of \( M \) to be \( G_M = \{ k | 0 \leq k < p, M \cong M^{k} \} \leq \mathbb{Z}_p \).

If \( A \) is any algebra let \( \text{Irr}(A) \) be the complete set of isomorphism classes of irreducible \( A \)–modules. We are interested in the inertia group \( G_S \) and \( G_D \), for \( S \in \text{Irr}(H_{r,p,n}^{R}) \) and \( D \in \text{Irr}(H_{r,p,n}^{K}) \).

1.1. **Definition.** Suppose that \( S \in \text{Irr}(H_{r,p,n}^{F}) \) and \( D \in \text{Irr}(H_{r,p,n}^{K}) \). The decomposition number \( [S : D] \) is a \( p \)–splittable decomposition number of \( H_{r,p,n}(Q) \) if \( G_S = \{ \emptyset \} = G_D \).

The second main result of this paper is the following:
Theorem B. Then the decomposition numbers of the cyclotomic Hecke algebras of type $G(r, p, n)$ are completely determined by the $p'$-splittable decomposition numbers of certain cyclotomic Hecke algebras $H_{r', p', n'}(Q')$, where $p'$ divides $p$, $1 \leq r' \leq r$, $1 \leq n' \leq n$ and where the parameters $Q'$ are contained in a single $(\varepsilon', q)$-orbit and $\varepsilon'$ is a primitive $p'$th root of unity.

The proof of Theorem B explicitly describes the algebras $H_{r', p', n'}(Q')$ and the parameters $Q'$ which appear in this reduction. Thus, once the $p'$-splittable decomposition numbers are known this result gives an algorithm for computing the decomposition matrices of the cyclotomic Hecke algebras of type $G(r, p, n)$.

This paper is organized as follows. In the next section we define the cyclotomic Hecke algebras of type $G(r, p, n)$ and prove the Morita equivalence result for the Hecke algebras of type $G(r, 1, n)$ which underpins all of the results in this paper. In the third section we apply the results for the algebras of type $G(r, 1, n)$ to prove Theorem A. The fourth section of the paper uses Clifford theory to show that if an algebra can be written as a semidirect product then its decomposition numbers are determined by a suitable family of $p'$-splittable decomposition numbers. This result is then applied in section 5 to prove Theorem B.

2. Morita equivalence theorems for Hecke algebras of type $G(r, 1, n)$

In this section we define the cyclotomic Hecke algebras and set our notation. We then recall and generalize the Morita equivalence results that we need for the cyclotomic algebras of type $G(r, 1, n)$.

Throughout this paper we fix positive integers $r$, $p$ and $n$ such that $r = pt$ for some integer $t$. Let $K$ be an algebraically closed field which contains a primitive $p$th root of unity $\varepsilon$. In particular, the characteristic of $K$ is coprime to $p$. Throughout this paper, we assume that $p > 1$ and $n \geq 3$. Fix parameters $q, Q_1, \ldots, Q_t \in K^\times$ and, as in the introduction, let $Q := (Q_1, \ldots, Q_t)$ and write $|Q| = t$.

Let $H_{r,n}(Q)$ be the cyclotomic Hecke algebra of type $G(r, 1, n)$. As a $K$-algebra $H_{r,n}(Q)$ is generated by $T_0, T_1, \ldots, T_{n-1}$ subject to the relations:

\[
\begin{align*}
(T_0^p - Q_1^p)(T_0^p - Q_2^p) \cdots (T_0^p - Q_t^p) &= 0, \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, \\
(T_i + 1)(T_i - q) &= 0, \quad 1 \leq i \leq n - 1, \\
T_1T_{i+1}T_i &= T_{i+1}T_iT_{i+1}, \quad 1 \leq i \leq n - 2, \\
T_iT_j &= T_jT_i, \quad 0 \leq i < j - 1 \leq n - 2.
\end{align*}
\]

That is, $H_{r,n}(Q)$ is the cyclotomic Hecke algebra of type $G(r, 1, n)$ with parameters $\{Q_1', \ldots, Q_t'\}$, where $Q_{i+pj} = \varepsilon^jQ_{i+1}$ for $0 \leq i < p$ and $0 \leq j < t$.

2.1. Definition. The cyclotomic Hecke algebra of type $G(r, p, n)$ is the subalgebra $H_{r,p,n}(Q)$ of $H_{r,n}(Q)$ which is generated by the elements $T_0^p, T_0 = T_0^{-1}T_1T_0$ and $T_1, T_2, \ldots, T_{n-1}$.

When the choice of parameters $q, Q$ is clear we write $H_{r,p,n} = H_{r,p,n}(Q)$ and $H_{r,n} = H_{r,n}(Q)$. When we want to emphasize the coefficient ring we write $H_{r,p,n}^K = H_{r,p,n}(Q)$ and $H_{r,n}^K = H_{r,n}(Q)$, respectively.

2.2. Remark. As noted by Malle [18, §4.B], if $n = 2$ then the Hecke algebra of type $G(2m, 2p, 2)$ cannot be identified with a subalgebra of the Hecke algebra of type $G(2m, 1, 2)$. It is for this reason that we assume that $n \geq 3$ in this paper (all of the results in this section are valid for $n \geq 1$).

Let $S_n$ be the symmetric group on $n$ letters. As the type $A$ braid relations hold in $H_{r,n}$ for each $w \in S_n$ there is a well–defined element $T_w \in H_{r,n}$, where
where compositions of $Q$ are indistinguishable.

Notation. Given any sequence $a$ of compositions of $Q$ into parts. If

\[ a = (i_1, \ldots, i_k) \] whenever $k$ is minimal such that $w = (i_1, i_1 + 1) \ldots (i_k, i_k + 1)$. Set $L_1 = T_0$ and $L_{k+1} = q^{-1}T_kL_kT_k$, for $k = 1, \ldots, n - 1$. Then Ariki and Koike [2, Theorem 3.10] showed that

\[ \{ L_{c_1} \ldots L_{c_n} T_w \mid w \in S_n \text{ and } 0 \leq c_i < r \} \]

is a basis of $\mathcal{H}_{r,n}$. The Morita equivalence in Theorem A is a consequence of the following result for the cyclotomic Hecke algebras $\mathcal{H}_{r,n}$.

2.3. Theorem (Dipper–Mathas [6, Theorem 1.1]). Suppose that $Q = Q_1 \vee \cdots \vee Q_\kappa$, such that $\alpha = \beta$ whenever $Q_i \in Q_\alpha$, $Q_j \in Q_\beta$ and $Q_i = q^{\alpha \cdot \beta}Q_j$, for some $a, b \in \mathbb{Z}$. Let $t_\alpha = |Q_\alpha|$. Then $\mathcal{H}_{r,n}(Q)$ is Morita equivalent to the algebra

\[ \bigoplus_{b_1, \ldots, b_\kappa \geq 0} \mathcal{H}_{p_{b_1}, b_1}(Q_1) \otimes \cdots \otimes \mathcal{H}_{p_{b_\kappa}, b_\kappa}(Q_\kappa). \]

As noted in [6] the proof of Theorem 2.3 quickly reduces to the case $\kappa = 2$, so only this case is considered in [6]. Unfortunately, to prove Theorem A we need detailed information about the bimodule which induces the Morita equivalence of Theorem 2.3 for arbitrary $\kappa \geq 1$. Consequently, we need to generalize the results of [6] and construct the bimodule which induces the Morita equivalence of Theorem 2.3 (in the special case when $Q$ is partitioned into a disjoint union of $(\varepsilon, q)$-orbits). In constructing this bimodule we refer the reader back to [6] whenever the details are not substantially different from the case $\kappa = 2$.

First, fix non-negative integers $a$ and $b$ with $a + b \leq n$ and an integer $s$ with $1 \leq s \leq t$. Define

\[ v_{a,b}(s) = \prod_{k=1}^{s} (L_1^a - Q_k^p) \ldots (L_n^a - Q_k^p) \cdot T_{w_{a,b}} \cdot \prod_{k=s+1}^{t} (L_1^b - Q_k^p) \ldots (L_n^b - Q_k^p), \]

where $w_{a,b} = (s_{a+b-1} \ldots s_1)^b$. (So $v_{n-b,b}(s)$ is the element $v_b$ of [6, Definition 3.3].) We write $v_{a,b}(s) = \sum_{k=1}^{s} (L_1^a - Q_k^p) \ldots (L_n^a - Q_k^p) \cdot T_{w_{a,b}}$. It may help the reader to observe that if we write $w_{a,b} \in S_{a+b}$ as a permutation in two-line notation then

\[ w_{a,b} = \left( \begin{array}{cccc} 1 & \cdots & a & a+1 \cdots a+b \\ b+1 & \cdots & a+b & 1 & \cdots & b \end{array} \right). \]

We will use the following notation extensively.

Notation. Given any sequence $a = (a_1, \ldots, a_k)$ and integers $1 \leq i \leq j \leq k$ we set $a_{i..j} = a_i + \cdots + a_j$. If $i < j$ then set $a_{j..i} = 0$.

Until further notice we fix a partition $Q = Q_1 \vee \cdots \vee Q_\kappa$ of $Q$ such that $Q_i \in Q_\alpha$, $Q_j \in Q_\beta$ are in the same $(\varepsilon, q)$-orbit only if $\alpha = \beta$. Set $t = (t_1, \ldots, t_{\kappa})$ where $t_\alpha = |Q_\alpha|$, for $1 \leq \alpha \leq \kappa$. Without loss of generality we assume that $Q_\alpha = (Q_{t_{\kappa}-\alpha+1}, \ldots, Q_{t_{\alpha}-\alpha})$, for $\alpha = 1, \ldots, \kappa$ (set $t_0 = 0$).

Let $\Lambda(n, \kappa) = \{ \mathbf{b} = (b_1, \ldots, b_\kappa) \mid b_{1, \kappa} = n \text{ and } b_\alpha \geq 0 \text{ for } 1 \leq \alpha \leq \kappa \}$ be the set of compositions of $n$ into $\kappa$ parts. If $\mathbf{b} \in \Lambda(n, \kappa)$ then, for convenience, we set $b_{\kappa+1} = 0$.

2.4. Definition. Suppose that $\mathbf{b} \in \Lambda(n, \kappa)$. Define

\[ v_\mathbf{b} = v_{a_{1..1}, b_{1..1}}(t_{1..1}) v_{a_{2..1}, b_{1..2}}(t_{1..2}) \ldots v_{a_{b_1..1}, b_{1..b_1}}(t_{1..b_1}) u_{w_\mathbf{b}}, \]

where

\[ u_{w_\mathbf{b}} := \left( \prod_{k=t_{1..1}+1}^{t_{1..\kappa}} (L_1^b - Q_k^p) \ldots (L_{b_1}^b - Q_k^p) \right) \ldots \left( \prod_{k=t_{1..1}+1}^{t_{1..b_1}} (L_1^b - Q_k^p) \ldots (L_{b_1}^b - Q_k^p) \right). \]
and let \( w_b = w_{b_0, b_1, \ldots, 1} w_{b_{n-1}, b_{n-2}, \ldots, 0} \). Define \( V^b = v_b \mathcal{F}_r, \alpha \).

Note that \( v_b \) depends crucially on our fixed partition \( Q = Q_1 \lor \cdots \lor Q_\alpha \) of \( Q \), so we should really write \( v_b = v_b(Q_1, \ldots, Q_\alpha) \). Our first goal is to understand \( V^b \).

Note that for each \( 2 \leq \alpha \leq \kappa, v^+_w \) has a factor \( \prod_{k=t_1, \alpha-1+1}^t (L^p_k - Q^p_k) \), i.e.,

\[
u^+_w = \prod_{k=t_1, \alpha-1+1}^t (L^p_k - Q^p_k) \prod_{b_1, \alpha-1} (L^p_{b_1, \alpha-1} - Q^p_{b_1, \alpha-1}) B(\alpha),\]

for some polynomial \( B(\alpha) \) in the Murphy operators. Then

\[
v_b = v^+_b(b_1, \alpha) \prod_{k=t_1, \alpha-1} (L^p_{b_1, \alpha-1} - Q^p_{b_1, \alpha-1}) B(\alpha).
\]

This observation will be used in the proof of the following two key properties of the elements \( v_b \).

2.5. Proposition. Suppose that \( b \in \Lambda(n, \kappa) \). Then

(a) \( T_i v_b = v_b T_{s_a(i)} \), whenever \( 1 \leq i < n \) and \( \alpha \neq b_\alpha \) for \( \alpha = 1, \ldots, \kappa \).

(b) \( L_k v_b = v_b L_{s_a(k)} \), whenever \( 1 \leq k \leq n \).

Proof. After translating notation, [6, Prop. 3.4] says that if \( 1 \leq s \leq t \) and \( a \) and \( b \) are non–negative integers with \( a + b \leq n \) then

\[
T_i v_{a,b}(s) = v_{a,b}(s) T_{w_a(i)} \quad \text{and} \quad L_k v_{a,b}(s) = v_{a,b}(s) L_{w_a(b)}(k)
\]

whenever \( 1 \leq i < a + b, \ i \neq a, \ 1 \leq k \leq a + b \). This is precisely the special case of the Proposition when \( \kappa = 2 \). The general case follows from this result, the observation above Proposition 2.5 and the fact that \( T_i v_{a,b}(s) = v_{a,b}(s) T_i \) and \( L_k v_{a,b}(s) = v_{a,b}(s) L_k \) whenever \( a + b < i < n \) and \( a + b < k \leq n \) for non–negative integers \( a \) and \( b \).

Observe that \( v_b T_j = T_{w_b^{-1}(j)} v_b \) and \( v_b L_m = L_{w_b^{-1}(m)} v_b \) by Proposition 2.5, for \( 1 \leq j < n, 1 \leq m \leq n \) with \( j \neq b_\alpha \) for \( \alpha = 1, \ldots, \kappa \).

2.7. Lemma. Suppose that \( b \in \Lambda(n, \kappa), 1 \leq \alpha \leq \kappa \) and \( b_\alpha \neq 0 \). Then

\[
\prod_{Q_\alpha \in \mathcal{Q}_a} (L^p_{1+b_\alpha+1, \alpha} - Q^p_{1}) \cdot v_b = 0 = v_b \cdot \prod_{Q_\alpha \in \mathcal{Q}_a} (L^p_{b_1, \alpha-1+1} - Q^p_{b_1, \alpha-1}).
\]

Proof. Recall that \( \prod_{i=1}^t (L^p_i - Q^p_i) = 0 \) since \( L_1 = T_0 \). Therefore, it follows from the definitions that if \( b_\alpha \neq 0 \) then

\[
\prod_{Q_\alpha \in \mathcal{Q}_a} (L^p_{1+b_\alpha+1, \alpha} - Q^p_{1}) \cdot v_{b, b_{1, \alpha-1}}(t_1, \alpha-1) = 0.
\]

Hence, \( \prod_{Q_\alpha \in \mathcal{Q}_a} (L^p_1 - Q^p_1) \cdot v_b = 0 \). Similarly, if \( b_1 \neq 0 \), then

\[
v_b \cdot \prod_{Q_\alpha \in \mathcal{Q}_a} (L^p_1 - Q^p_1) = 0.
\]

Now suppose that \( 1 \leq \alpha < \kappa, b_\alpha \neq 0 \), and set \( L(\alpha) = \prod_{Q_\alpha \in \mathcal{Q}_a} (L^p_{1+b_\alpha+1, \alpha} - Q^p_{1}) \).

Then, using (2.6), the observation before Proposition 2.5 and the fact that any symmetric polynomial on Murphy operators is central, we deduce that \( L(\alpha) v_b \) has a factor of the form

\[
\prod_{Q_\alpha \in \mathcal{Q}_a} (L^p_1 - Q^p_1) \left( \prod_{k=t_1, \alpha}^t \frac{1}{(L^p_k - Q^p_k)} \right) \left( \prod_{k=t_1, \alpha+1}^t \frac{1}{Q^p_k} \right).
\]
Hence, \( L(\alpha)v_b = 0 \) by combining the relation \( \prod_{i=1}^{n}(L^p_i - Q^p_i) = 0 \) with the last displayed equation. The second statement

\[ v_b \cdot \prod_{Q_i \in Q_s} (L^p_{\lambda_{b,s-1}+1} - Q^p_i) = 0, \]

for \( \alpha = 2, \ldots, \kappa \), is equivalent to what we have just proved because \( v_bL_{\lambda_{b,s-1}+1} = L_{w^{-1}_{b,s-1}+1}v_b = L_{1+b_{s-1}+v}v_b \) by Proposition 2.5.

To proceed we recall the cellular basis of the algebras \( \mathcal{H}_{r,n} \), and the associated combinatorics, introduced in [5]. A \textbf{multipartition of} \( n \) is an ordered \( r \)--tuple of partitions \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) such that \( \mid \lambda^{(1)} \mid + \cdots + \mid \lambda^{(r)} \mid = n \). Let \( \Lambda^{+}_n \) be the set of multipartitions of \( n \). Then \( \Lambda^{+}_n \) is a poset under the \textbf{dominance order}, where \( \lambda \succeq \mu \) if

\[
\sum_{i=1}^{s-1} |\lambda^{(a)}| + \sum_{j=1}^{i} \lambda^{(s)}_j \geq \sum_{a=1}^{s-1} |\mu^{(a)}| + \sum_{j=1}^{i} \mu^{(s)}_j,
\]

for all \( 1 \leq s \leq r \) and all \( i \geq 1 \).

The \textbf{diagram} of \( \lambda \) is the set \( \lambda \) = \{ \( (i,j,s) \mid 1 \leq j \leq \lambda^{(s)}_i \) for \( 1 \leq s \leq r \} \). A \textbf{lambda-tableau} is a bijection \( t: [\lambda] \rightarrow \{1, 2, \ldots, n\} \). The \( \lambda \)--tableau \( t \) is \textbf{standard} if \( t(i,j,s) < t(i',j',s) \) whenever \( i \leq i', j \leq j' \) and \( (i,j,s) \) and \( (i',j',s) \) are distinct elements of \( [\lambda] \). Let \( \text{Std}(\lambda) \) be the set of standard \( \lambda \)--tableaux. Observe that \( \mathcal{S}_n \) acts from the right on the set of \( \lambda \)--tableaux. In particular, if \( w \in \mathcal{S}_n \) and \( t \in \text{Std}(\lambda) \) then \( tw \) is a \( \lambda \)--tableau, however, it is not necessarily standard.

If \( \lambda \in \Lambda^{+}_n \) let \( \mathcal{S}_\lambda = \mathcal{S}_{\lambda^{(1)}} \times \cdots \times \mathcal{S}_{\lambda^{(r)}} \) be the corresponding Young (or parabolic) subgroup of \( \mathcal{S}_n \). We set

\[
x_\lambda = \sum_{w \in \mathcal{S}_\lambda} T_w \quad \text{and} \quad u^+_\lambda = \prod_{s=2}^{r} \prod_{k=1}^{\lambda^{(s)} - 1} (L_k - Q^p_s) \cdot (L^p_{\lambda^{(s-1)} - 1} - Q^p_s).
\]

Then \( x_\lambda \) and \( u^+_\lambda \) are commuting elements of \( \mathcal{H}_{r,n} \). Next, if \( s \) is a standard \( \lambda \)--tableau let \( d(s) \) be the corresponding distinguished right coset representative of \( \mathcal{S}_\lambda \) in \( \mathcal{S}_n \). Finally, given a pair \( (s, t) \) of standard \( \lambda \)--tableaux define \( m_{st} = T^*_{d(s)}x_\lambda u^+_\lambda T_{d(t)} \), where \( * \) is the unique anti--isomorphism of \( \mathcal{H}_{r,n} \) which fixes \( T_0, \ldots, T_{n-1} \). Then

\[
\{ m_{st} \mid s, t \in \text{Std}(\lambda) \text{ for some } \lambda \in \Lambda^{+}_n \}
\]

is a cellular basis of \( \mathcal{H}_{r,n} \) by [5, Theorem 3.26].

We can relate \( V^b \) to the combinatorics of the cellular basis \( \{m_{st}\} \) by defining \( \omega_b = (\omega^{(1)}_b, \ldots, \omega^{(r)}_b) \) to be the multipartition with

\[
\omega^{(s)}_b = \begin{cases} (1_{b^{(s)}}), & \text{if } s = pt_{1:..} \text{ for some } \alpha, \\ (0), & \text{otherwise}. \end{cases}
\]

From the definitions, \( u^+_w = \prod_{s=1}^{r-1} \prod_{Q_i \in Q_{s+1}} (L^p_i - Q^p_i) \cdots (L^p_{\lambda^{(s-1)} - 1} - Q^p_{\lambda^{(s)} - 1}) \). Hence, using (2.6) we obtain the following.

2.8. \textbf{Lemma.} Suppose that \( b \in \Lambda(n, \kappa) \). Then \( v_b = v_b^{-1}u^+_w \), where

\[
v_b^{-1} = \prod_{\alpha=2}^{\kappa} \prod_{i=1}^{t_{\alpha-1}} (L^p_i - Q^p_i) \cdots (L^p_{\lambda^{(s-1)} - 1} - Q^p_{\lambda^{(s)} - 1}) \cdot T_{w_{\lambda^{(s-1)} - 1}}
\]

and in the product \( \alpha \) decreases in order from left to right.

Following [5] define \( M^\beta = u^+_w \mathcal{H}_{r,n} \). By the Lemma, there is a surjective \( \mathcal{H}_{r,n} \)--module homomorphism \( \theta_b: M^\beta \rightarrow V^b \) given by \( \theta_b(h) = v_b^{-1}h \), for all \( h \in M^\beta \).
Suppose that \( \lambda \) is a multipartition and that \( t \) is a standard \( \lambda \)-tableau. For each integer \( k \), with \( 1 \leq k \leq n \), define \( \text{comp}_t(k) = s \) if \((i, j, s)\) is the unique node in \([\lambda]\) such that \( t(i, j, s) = k \).

2.9. Definition. Suppose that \( \lambda \) is a multipartition of \( n \). Define

\[
\text{Std}_b(\lambda) = \{ t \in \text{Std}(\lambda) \mid \text{comp}_t(k) \leq pt_{1..a} \text{ if } 1 \leq k \leq b_{1..a} \}
\]

and

\[
\text{Std}^+_b(\lambda) = \{ t \in \text{Std}(\lambda) \mid pt_{1..a-1} < \text{comp}_t(k) \leq pt_{1..a} \text{ if } b_{1..a-1} < k \leq b_{1..a} \}
\]

Then \( \text{Std}_b(\lambda) \neq \emptyset \) if only if \( \sum_{i=1}^{pt_{1..a}} |\lambda(s)| \geq b_{1..a} \) for \( 1 \leq \alpha \leq \kappa \) and \( \text{Std}^+_b(\lambda) \neq 0 \) if and only if \( \sum_{i=1}^{pt_{1..a-1}+1} |\lambda(s)| = b_{1..a} \) for \( 1 \leq \alpha \leq \kappa \). Hence, \( \text{Std}^+_b(\lambda) \subseteq \text{Std}_b(\lambda) \).

2.10. Lemma. Suppose that \( b \in \Lambda(n, \kappa) \).

a) \( M^{ab} \) has basis \( \{ m_{st} \mid s \in \text{Std}_b(\lambda), t \in \text{Std}(\lambda) \text{ for some } \lambda \in \Lambda^+_n \} \).

b) Suppose that \( s \in \text{Std}_b(\lambda) \setminus \text{Std}^+_b(\lambda) \) and \( t \in \text{Std}(\lambda) \). Then \( \theta_b(m_{st}) = 0 \).

Proof. Part (a) is a translation of [5, Theorem 4.14] into the current notation. See the proof of [6, Lemma 3.9] for more details.

For part (b) we follow the proof of [6, Lemma 3.10]. Let \( c = (c_1, \ldots, c_\kappa) \), where \( c_\alpha = |\lambda(\alpha^{pt_{1..a-1}+1})| + \cdots + |\lambda(\alpha^{pt_{1..a}})| \), for \( 1 \leq \alpha \leq \kappa \). Then \( c_{1..a} \geq b_{1..a} \), for \( 1 \leq \alpha \leq \kappa \), since \( s \in \text{Std}_b(\lambda) \) and \( c \neq b \) since \( s \notin \text{Std}^+_b(\lambda) \). Choose \( \beta \) to be minimal such that \( c_\beta > b_\beta \). Then \( 1 \leq \beta < \kappa \) and if \( 1 \leq \alpha \leq \beta \) then \( pt_{1..a-1} < \text{comp}_s(k) \leq pt_{1..a} \) if \( b_{1..a-1} < k \leq b_{1..a} \) since \( c_1 = b_1, \ldots, c_{\beta-1} = b_{\beta-1} \) and \( s \in \text{Std}_b(\lambda) \). Choose \( \gamma \geq \beta \) to be minimal such that \( b_\gamma \neq 0 \). Let \( w \) be a permutation of \( \{b_{1..\gamma} + 1, b_{1..\gamma} + 2, \ldots, n\} \) of minimal length such that \( s' = sw \) is a standard \( \lambda \)-tableau with \( pt_{1..a-1} < \text{comp}_s(k) \leq pt_{1..a} \) if \( c_{1..a-1} < k \leq c_{1..a} \), for \( \gamma + 1 \leq \alpha \leq \kappa \). (Such a permutation exists because we can first swap integers \( k \), with \( c_{1..\kappa-1} < k \leq n \) and \( \text{comp}_s(k) \leq pt_{1..\kappa-1} \), with the integers \( l \), where \( b_{1..\kappa-1} < l \leq n \) and \( pt_{1..\kappa-1} < \text{comp}_s(l) \leq pt_{1..\kappa} \); let \( s_1 \) be the resulting \( \lambda \)-tableau. Then we swap integers \( k \), with \( c_{1..\kappa-2} < k \leq c_{1..\kappa-1} \) and \( \text{comp}_s(k) \leq pt_{1..\kappa-2} \), with the integers \( l \), where \( b_{1..\kappa-2} < l \leq n \) and \( pt_{1..\kappa-2} < \text{comp}_s(l) \leq pt_{1..\kappa-1} \); and so on; compare [6, Lemma 3.10].)

As a result, \( \text{comp}_s(k) \leq pt_{1..\gamma} \) if \( k \leq c_{1..\gamma} \). Then \( d(s) = d(s')w \), with the lengths adding, so that \( m_{st} = T^s_w m_{s't} \). Furthermore, by construction, there is a composition \( c' \in \Lambda(n, \kappa) \) such that \( s' \in \text{Std}_{c'}(\lambda) \), \( c'_\alpha = c_\alpha \) for \( 1 \leq \alpha < \beta \) or \( \gamma \leq \alpha \leq \kappa \), and \( c'_{1..a} \geq b_{1..a} \) for \( 1 \leq \alpha \leq \kappa \). Hence, \( m_{s't} \in M^{c'} \) by part (a), so that \( m_{st} = T^s_w u^+_w h \) for some \( h \in M^n \). Therefore, \( \theta_b(m_{st}) = T^s_w u^+_w v^+_w h \).

To simplify the notation, for the remainder of the proof set \( w(\alpha) = w_{b_{1..a}, b_{1..a-1}} \) and \( u^-(m, s) = \prod_{i=1}^{t} (L^P_q - Q^P_q) \cdots (L^P_m - Q^P_m) \), for \( 1 \leq \alpha \leq \kappa \), \( 1 \leq m \leq n \) and \( 1 \leq s \leq t \). Similarly, set \( u^+(m, s) = \prod_{i=1}^{t} (L^P_q - Q^P_q) \cdots (L^P_m - Q^P_m) \). Then

\[
\theta_b(m_{st}) = v^+_w T^s_w u^+_w h = \prod_{\alpha=2}^{t} u^-(b_\alpha, t_{1..a-1}) T^{*}_{w(\alpha)} \cdot T^s_w u^+_w h;
\]

where the product is taken in order with \( \alpha \) decreasing from left to right. Now, \( u^+(m, s) \) commutes with \( T_i \) if \( i \neq m \). Therefore, since \( w \) is a permutation of \( \{b_{1..\gamma} + 1, b_{1..\gamma} + 2, \ldots, n\} \), we have

\[
\theta_b(m_{st}) = \prod_{\alpha=\gamma+1}^{t} u^-(b_\alpha, t_{1..a-1}) T^{*}_{w(\alpha)} \cdot T^s_w \prod_{\alpha=2}^{\gamma} u^-(b_\alpha, t_{1..a-1}) T^{*}_{w(\alpha)} \cdot u^+_w h;
\]

for some \( w' \in \mathfrak{S}_n \), where again both products are ordered with \( \alpha \) decreasing from left to right. By definition, \( u^+_w = \prod_{\alpha=1}^{t} u^+(c'_{1..a}, t_{1..a}+1) \), where this product can
be taken in any order. So,
\[ \prod_{\alpha=2}^{\gamma} u^{-}(b_\alpha, t_1..\alpha-1)T_w(\alpha) \cdot u_{\mu'}^{\alpha} = \prod_{\alpha=2}^{\gamma} u^{-}(b_\alpha, t_1..\alpha-1)T_w(\alpha) \cdot \prod_{\alpha=1}^{\kappa-1} u^{\alpha}(c_{1..\alpha}', t_1..\alpha + 1). \]
Now, \( w(\alpha) = w_{b_\alpha, b_1..\alpha-1} \in \mathcal{S}_{b_1..\alpha}. \) So if \( 1 \leq \alpha < \gamma \) then \( T_w(\alpha) \) commutes with \( u^{\alpha}(c_{1..\alpha}', t_1..\alpha + 1) \) since \( c_{1..\alpha}' = c_1..\alpha > b_1..\alpha. \) Consequently, the last displayed equation contains \( u^{-}(b_{\gamma}, t_1..\gamma-1)T_w(\gamma)u^{\alpha}(c_{1..\gamma}, t_1..\gamma + 1) \) as a factor. Since \( c_1..\gamma > b_1..\gamma-1, \) this element is equal to
\[ v_{b_{\gamma}, b_1..\gamma-1}(t_1..\gamma-1) \prod_{s=t_1..\gamma+1}^{t} (L_{b_{1..\gamma-1}+1}^p - Q_s^p) \cdots (L_{c_1..\gamma}^p - Q_s^p) = 0, \]
where the last equality comes from applying the right hand equation of Lemma 2.7 in the special case when \( \kappa = 2. \) Putting all of these equations together, we have shown that \( \theta_b(m_{st}) = 0, \) as required.

\( \square \)

Suppose that \( t \) is a standard \( \lambda \)-tableau and that \( 1 \leq k \leq n. \) Let \( \text{Shape}_k(t) \) be the multipartition with diagram \( t^{-1}(\{1, \ldots, k\}) \); that is, \( \text{Shape}_k(t) \) is the multipartition given by the positions of \( \{1, \ldots, k\} \) in \( t. \) If \( t \in \text{Std}(\lambda) \) and \( v \in \text{Std}(\mu) \) then we write \( t \triangleright v \) if \( \lambda \triangleright \mu \) or if \( \lambda = \mu \) and \( \text{Shape}_k(t) \triangleright \text{Shape}_k(v) \) for \( 1 \leq k \leq n. \) We extend this partial order to pairs of standard tableaux in the obvious way.

2.11. Lemma. Suppose that \( \lambda \) is a multipartition of \( n \) and that \( s \in \text{Std}_b^+(\lambda) \) and \( t \in \text{Std}(\lambda). \) Let \( s' = sw_{b}^{-1}. \) Then there exists an invertible element \( u \in R \) such that
\[ \theta_b(m_{st}) = um_{s't} + \sum_{(u,v) \triangleright (s',t)} r_{uv}m_{uv}, \]
for some \( r_{uv} \in R. \)

Proof. By [13, Prop. 3.7], if \( 1 \leq k \leq n \) and \( p(a - 1) < \text{comp}_1(k) \leq pa \) then
\[ m_{st}L_k^p = q^{p(j-1)}Q_{\alpha}^p m_{st} + \sum_{(u,v) \triangleright (s',t)} r_{uv}m_{uv}, \]
for some \( r_{uv} \in R. \) Using this formula we can compute \( \theta(m_{st}) = v_{s'}m_{st} \) directly, which shows that \( m_{st} \) appears with non-zero coefficient in \( \theta_b(m_{st}) \) only if \( (u,v) \triangleright (s',t). \) Finally, \( m_{s't} \) appears with non-zero coefficient in \( \theta_b(m_{st}) \) because \( Q = Q_1 \lor \cdots \lor Q_k \) is a partition of \( Q \) into \((\varepsilon, q)\)-orbits. (Compare with the proof of [6, Lemma 3.11].)

Suppose that \( \lambda \) is a multipartition of \( n. \) Let \( \mathcal{H}^\lambda_{r,n} \) be the module with basis \( \{m_{uv}\} \) where \( u \) and \( v \) range over the standard \( \mu \)-tableaux with \( \mu \triangleright \lambda. \) It follows from the general theory of cellular algebras [16, Lemma 2.3] that \( \mathcal{H}^\lambda_{r,n} \) is a two-sided ideal of \( \mathcal{H}_{r,n}. \)

Fix \( s \in \text{Std}(\lambda). \) Then, as a vector space, the **Specht module** (or cell module) \( S(\lambda) \) is the module with basis \( \{m_{st} + \mathcal{H}^\lambda_{r,n} \ | \ t \in \text{Std}(\lambda)\}. \) The theory of cellular algebras [16, 2.4] shows that \( S(\lambda) \) is an \( \mathcal{H}_{r,n} \)-module and that, up to isomorphism, \( S(\lambda) \) does not depend on the choice of \( s. \)

Finally, we need the classification of the blocks for \( \mathcal{H}_{r,n}. \) For each \( \lambda \in \Lambda^+_n \) define a “content function” \( c_{\lambda} : R \rightarrow \mathbb{N} \) by
\[ c_{\lambda}(x) = \# \{(i,j,a+pb) \in [\lambda] \ | \ 0 \leq a < p \text{ and } x = q^{j-1}e^aQ_b\}, \]
for \( x \in R. \) Then the Specht modules \( S(\lambda) \) and \( S(\mu) \) are in the same block only if \( c_{\lambda}(x) = c_{\mu}(x), \) for all \( x \in R, \) by [8, Prop. 5.9(ii)]. (Although we will not need this we note that the converse is also true by [14, Theorem A].)
With the results that we have now proved we can complete the proof of Theorem 2.3 with only minor modifications of the arguments of [6]. Consequently, we sketch the rest of the proof and give references to [6] for those readers who require more detail.

2.12. **Definition.** Suppose that \( b \in \Lambda(n, \kappa) \) and that \( s \in \text{Std}_b^+(\lambda) \), \( t \in \text{Std}(\lambda) \) for some \( \lambda \in \Lambda_n^+ \).

- **a)** Set \( v_{st} = \theta_b(m_{st}) \in V^b \).
- **b)** If, in addition, \( t \in \text{Std}_b^+(\lambda) \), then let \( \theta_{st} \in \text{End}_R(V^b) \) be the endomorphism \( \theta_{st}(v_b h) = v_{st} h \), for all \( h \in \mathcal{H}_{r,n} \).

We remark that it is not clear from the definition that the maps \( \theta_{st} \) are well-defined.

For \( b \in \Lambda(n, \kappa) \) let

\[
\Lambda_b = \left\{ \lambda \in \Lambda_n^+ \mid b_\lambda = |\lambda^{(pt_1, \ldots, pt_{\alpha+1})}| + \cdots + |\lambda^{(pt_1, \alpha)}|, \quad \text{for } \alpha = 1, \ldots, k \right\}.
\]

Note that \( \text{Std}_b^+(\lambda) \neq \emptyset \) if and only if \( \lambda \in \Lambda_b^+ \).

2.13. **Proposition.** Suppose that \( b \in \Lambda(n, \kappa) \). Then:

- **a)** \( V^b \) has basis

\[
\left\{ v_{st} \mid s, t \in \text{Std}_b^+(\lambda) \text{ for some } \lambda \in \Lambda_b^+ \right\}.
\]

- **b)** If \( b \neq c \in \Lambda(n, \kappa) \) then \( \text{Hom}_{\mathcal{H}_{r,n}}(V^b, V^c) = 0 \).

- **c)** \( \text{End}_{\mathcal{H}_{r,n}}(V^b) \) is a vector space with basis

\[
\left\{ \theta_{st} \mid s, t \in \text{Std}_b^+(\lambda) \text{ for some } \lambda \in \Lambda_b^+ \right\}.
\]

**Proof.** (a) This follows directly from Lemma 2.10(b) and Lemma 2.11.

(b) As in [6, Theorem 3.16] it follows from part (a) and the construction of the Specht modules that \( V^b \) has a filtration \( V^b = V_1 \supset V_2 \supset \cdots \supset V_k = 0 \) such that (1) \( V_i/V_{i+1} \cong S(\lambda_i) \), for some \( \lambda_i \in \Lambda_b^+ \); and (2) if \( \mu \in \Lambda_b^+ \) then

\[
\# \text{Std}_b^+(\mu) = \# \left\{ 1 \leq i < k \mid V_i/V_{i+1} \cong S(\mu) \right\}.
\]

Now, if \( b \neq c \) and \( \lambda \) and \( \mu \) are two multipartitions such that \( \text{Std}_b^+(\lambda) \neq \emptyset \) and \( \text{Std}_c^+(\mu) \neq \emptyset \) then it is easy to see (cf. the proof of [6, Cor. 3.17]) that \( c_\lambda \neq c_\mu \). Consequently, by the remarks before the Theorem, the Specht modules \( S(\lambda) \) and \( S(\mu) \) are in different blocks. Therefore, all of the composition factors of \( V^b \) and \( V^c \) belong to different blocks, so \( \text{Hom}_{\mathcal{H}_{r,n}}(V^b, V^c) = 0 \).

(c) The proof is identical to that of [6, Theorem 3.19]. In outline, the argument is as follows. By [5, Theorem 6.16] and Lemma 2.10(a), \( \text{End}_{\mathcal{H}_{r,n}}(M^{\omega_b}) \) has basis \( \{ \varphi_{st} \mid s, t \in \text{Std}_b^+(\lambda) \text{ for some } \lambda \in \Lambda_b^+ \} \), where \( \varphi_{st}(m_{st} h) = m_{st} h \) for all \( h \in \mathcal{H}_{r,n} \). As in the proof of part (b), the filtration \( 0 \subseteq \ker \theta_{st} \subseteq M^{\omega_b} \) can be extended to a Specht filtration which is compatible with the Specht filtration of \( V^b \cong M^{\omega_b}/\ker \theta_b \). Using this Specht filtration and the classification of the blocks of \( \mathcal{H}_{r,n} \) given above it follows that all of the irreducible constituents of \( V_b \) and \( \ker \theta_b \) belong to different blocks. Therefore, the map \( \theta_{st} : M^{\omega_b} \rightarrow V^b \) splits. Let \( \theta_{st}^{-1} \) be a right inverse to \( \theta_{st} \). Then a straightforward calculation shows that

\[
\theta_{st} \varphi_{st} \theta_{st}^{-1} = \begin{cases} \theta_{st}, & \text{if } s, t \in \text{Std}_b^+(\lambda), \\ 0, & \text{otherwise.} \end{cases}
\]

As the maps \( \{ \theta_{st} \varphi_{st} \theta_{st}^{-1} \mid s, t \in \text{Std}_b^+(\lambda) \} \) span \( \text{End}_{\mathcal{H}_{r,n}}(V^b) \), this proves part (c). \( \square \)
Let $\mathcal{H}_b = \mathcal{H}_{pt,b_1}(Q_1) \otimes \cdots \otimes \mathcal{H}_{pt,b_n}(Q_n)$. Let $T_i^{(\alpha)} = 1 \otimes \cdots \otimes T_i \otimes \cdots \otimes 1$ be a generator of $\mathcal{H}_b$, where the $T_i$ occurs in the $\alpha$th tensor factor, for $1 \leq \alpha \leq \kappa$. Then, as an algebra, $\mathcal{H}_b$ is generated by the elements $\{ T_i^{(\alpha)} | 1 \leq \alpha \leq \kappa \}$.

We need some combinatorial machinery to describe the $\mathcal{H}_b$–modules. For $\lambda \in \Lambda_\kappa^+$ let $A_\lambda = (\lambda^{(1)}_\kappa, \ldots, \lambda^{(\kappa)}_\kappa)$, where $\lambda^{(\alpha)}_a = (\lambda^{pt_1 \cdot a-1+i}_1, \ldots, \lambda^{pt_1 \cdot a-1})$. Then the Specht modules of $\mathcal{H}_b$ are all of the form $S(\lambda^{(1)}_\kappa) \otimes \cdots \otimes S(\lambda^{(\kappa)}_\kappa)$, for $\lambda \in \Lambda_\kappa^+$, and there is a natural bijection $\text{Std}^+_\kappa(\lambda) \cong \text{Std}(\lambda^{(1)}_\kappa) \times \cdots \times \text{Std}(\lambda^{(\kappa)}_\kappa)$.

Let $\mathfrak{S}_b = \mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_n}$, which we consider as a subgroup of $\mathfrak{S}_n$ via the natural embedding. Let $D_b$ be the set of distinguished (minimal length) right coset representatives for $\mathfrak{S}_b$ in $\mathfrak{S}_n$. Observe that if $\lambda \in \Lambda_\kappa^+$ then
\[
(2.14) \quad \text{Std}(\lambda) = \prod_{d \in D_b} \text{Std}^+_d(\lambda) d.
\]

Recall that a **progenerator**, or projective generator, for an algebra $A$ is a projective $A$–module $V$ which contains every projective indecomposable $A$–module as a direct summand. The algebras $A$ and $\text{End}_A(V)$ are Morita equivalent and, moreover, every Morita equivalence arises in this way.

2.15. **Proposition.**

a) Let $V = \bigoplus_{b \in \Lambda(n, \kappa)} V^b$. Then $V$ is a progenerator for $\mathfrak{H}_{r,n}$.

b) Suppose that $b \in \Lambda(n, \kappa)$. Then:

i) $V^b$ is a projective $\mathfrak{H}_{r,n}$–module;

ii) $\text{End}_{\mathfrak{H}_{r,n}}(V^b) \cong \mathcal{H}_b$; and

iii) as left $\mathcal{H}_b$–modules, $\mathcal{H}_b \cong V^0_0$ and $V^b = \bigoplus_{d \in D_b} V^d_T d$, where $V^d_T$ is the subspace of $V^b$ with basis $\{ v_{st} | s, t \in \text{Std}^+_d(\lambda) \text{ for some } \lambda \in \Lambda_\kappa^+ \}$.

**Sketch of proof.** Using Proposition 2.13 and (2.14) it is straightforward to show that $\mathfrak{H}_{r,n} \cong \bigoplus_{b \in \Lambda(n, \kappa)} \bigoplus_{d \in D_b} T^d V^b$ (see the proof of [6, Theorem 3.20]). Hence, $V^b$ is a projective $\mathfrak{H}_{r,n}$–module, proving b(i). Part (a) now follows because $V^b \cong T^d V^b$, for all $d \in D_b$.

Now consider the remaining statements of (b). By Proposition 2.5 and Lemma 2.7 there is an action of $\mathcal{H}_b$ on $V^b$ by left multiplication which is uniquely determined by letting the generator $T_i^{(\alpha)}$ of $\mathcal{H}_b$ act as left multiplication by $L_{1+\alpha b_{a+1}}$, if $i = 0$ and by $T_{i+1+b_{a+1}}$, if $1 \leq i < b_{a}$. Thus, there is a map from $\mathcal{H}_b$ into $\text{End}_{\mathfrak{H}_{r,n}}(V^b)$. The argument used to prove [6, Theorem 4.7] now shows that if $\lambda \in \Lambda_\kappa^+$ and $s, t \in \text{Std}^+_d(\lambda)$ then the map $\theta_{st} \in \text{End}_{\mathfrak{H}_{r,n}}(V^b)$ corresponds to left multiplication by the corresponding Murphy basis element of $\mathcal{H}_b$, where we use the bijection $\text{Std}^+_d(\lambda) \cong \text{Std}(\lambda^{(1)}_\kappa) \times \cdots \times \text{Std}(\lambda^{(\kappa)}_\kappa)$. That is if $h \in \mathfrak{H}_{r,p,n}$ then $\theta_{st}(v_b h) = (m_s^{(1)} u^{(1)} \otimes \cdots \otimes m_t^{(\kappa)} u^{(\kappa)}) v_b h$, where $u \in \text{Std}^+_d(\lambda)$ maps to $(u^{(1)}, \ldots, u^{(\kappa)})$ under the bijection above; see the proof of [6, Lemma 4.6]. This shows that $\text{End}_{\mathfrak{H}_{r,n}}(V^b) \cong \mathcal{H}_b$. Finally, part b(iii) follows from b(ii) and the observation that $V^b = \bigoplus_{d \in D_b} V^d_T d$, as a left $\mathcal{H}_b$–module. \(\square\)

Notice, in particular, that Theorem 2.3 is an immediate Corollary of the Proposition. Later we need the following result which follows directly from Proposition 2.15b(iii); compare [6, Remark 3.15].

2.16. **Corollary.** Suppose that $b \in \Lambda(n, \kappa)$. Then
\[
\{ v_b L_1^{c_1} \cdots L_n^{c_n} T_w | w \in \mathfrak{S}_n \text{ and } 0 \leq c_i < pt_{a} \text{ whenever } b_{1 \cdot a-1} < i \leq b_{1 \cdot a} \}
\]
is a basis of $v_b \mathfrak{H}_{r,n}$.

We now have the information that we need to start proving Theorem A from the introduction.
3. Morita equivalence theorems for algebras of type $G(r, p, n)$

In this section we prove Theorem A, the Morita reduction theorem for the Hecke algebras of type $G(r, p, n)$, by analyzing the structure of $V^b = v_b \mathcal{H}_{r,n}$ as an $\mathcal{H}_{r,p,n}$-module. We maintain our notation from the previous section. In particular, we fix a partitioning $Q = Q_1 \cup \ldots \cup Q_s$ of $Q$ such that $Q_i$ and $Q_j$ are in different $(\varepsilon, q)$-orbits whenever $Q_i \in Q_1$, $Q_j \in Q_3$ and $\alpha \neq \beta$.

Following Ariki [1], for each integer $m$ with $1 \leq m \leq n$, define:

$$S_m = \begin{cases} T^p_0, & \text{if } m = 1; \\ T_0^{-1}L_m, & \text{if } 2 \leq m \leq n. \end{cases}$$

The elements $S_1, S_2, \ldots, S_n$ are the Murphy operators of $\mathcal{H}_{r,p,n}$. We need these elements to prove the following fundamental fact.

3.1. Lemma. The algebra $\mathcal{H}_{r,p,n}$ has basis

$$\{ L_1^{c_1} \ldots L_n^{c_n}T_w \mid w \in \mathcal{S}_n, 0 \leq c_i < r \text{ and } c_1 + \cdots + c_n \equiv 0 \mod p \}.$$ 

Proof. Ariki [1, Prop. 1.6] showed that $\mathcal{H}_{r,p,n}$ is the submodule of $\mathcal{H}_{r,n}$ with basis

$$\{ S_1^{c_1} \cdots S_n^{c_n}T_w \mid w \in \mathcal{S}_n, 0 \leq c_i < r \text{ for } 2 \leq i \leq n, \text{ and } 0 \leq pc_1 - c_2 - \cdots - c_n < r \}.$$ 

Applying the definitions $S_1^{c_1} \cdots S_n^{c_n}T_w = L_1^{p_1 c_1 - \cdots - c_n}L_2^{c_2 - \cdots - c_n} \cdots L_n^{c_n}T_w$, hence, the Lemma is just a reformulation of Ariki’s result. \hfill \square

Recall from the introduction that there are two algebra automorphisms $\sigma$ and $\tau$ of $\mathcal{H}_{r,n}$.

3.2. Definition. The automorphism $\tau$ is the $K$-algebra automorphism of $\mathcal{H}_{r,n}$ which is given by $\tau(h) = T_0^{-1}hT_0$, for all $h \in \mathcal{H}_{r,n}$. The map $\sigma$ is the $K$-algebra automorphism of $\mathcal{H}_{r,n}$ which is determined by

$$\sigma(T_0) = \varepsilon T_0 \quad \text{and} \quad \sigma(T_i) = T_i, \quad \text{for } i = 1, \ldots, n - 1.$$ 

The reader can check that $\mathcal{H}_{r,n}$ is the fixed point subalgebra of $\mathcal{H}_{r,n}$ under $\sigma$.

Suppose that $A$ is an algebra with an automorphism $\theta$ of order $p$. Define $A \times_\theta \mathbb{Z}_p$ to be the $K$-algebra with elements

$$\{ a\theta^k \mid a \in A \text{ and } 0 \leq k < p \}$$

and with multiplication $a\theta^k \cdot b\theta^l = a\theta^{k+l}$, for $a, b \in A$ and $0 \leq k, l < p$. As in the introduction, if $M$ is an $A$–module then we can define a new $A$–module $M^\theta$ which is isomorphic to $M$ as a vector space but with the $A$–action twisted by $\theta$. Informally, it is convenient to think of $M^\theta$ as the set of elements $\{ m\theta \mid m \in M \}$ with $A$–action $m\theta \cdot a = (m\theta(a))\theta$, for $m \in M$ and $a \in A$.

3.3. Lemma. Suppose that $0 \leq b \leq n$. Then $\sigma(v_b) = v_b$ and $\tau(v_b) \in v_b \mathcal{H}_{r,p,n}$. Consequently, $(v_b \mathcal{H}_{r,n})^\sigma = v_b \mathcal{H}_{r,n}$ and $(v_b \mathcal{H}_{r,p,n})^\tau = v_b \mathcal{H}_{r,p,n}$ as $\mathcal{H}_{r,p,n}$-modules.

Proof. Since $\sigma(T_i) = T_i$, for $1 \leq i < n$, and $\sigma(L^b_k) = L^b_k$, for $1 \leq k \leq n$, we see that $\sigma(v_b) = v_b$. Furthermore, $T_0v_b = v_bL_{b_1 \cdot \ldots \cdot b_{s-1} + 1}$ and $v_bT_0 = L_{b_2 \cdot \ldots \cdot b_1 + 1}v_b$, by Proposition 2.5, so $\tau(v_b) = T_0^{-1}v_bT_0 = v_bL_{b_1 \cdot \ldots \cdot b_{s-1} + 1}L_1 = v_bS_{b_1 \cdot \ldots \cdot b_{s-1} + 1} \in v_b \mathcal{H}_{r,p,n}$. From what we have proved, $\sigma(v_b \mathcal{H}_{r,n}) = v_b \mathcal{H}_{r,n}$. Consequently, the map $v_b h \mapsto \sigma(v_b)h = (v_b)\sigma(h)$ defines a module isomorphism $v_b \mathcal{H}_{r,n} \cong (v_b \mathcal{H}_{r,n})^\sigma$, for $h \in \mathcal{H}_{r,n}$. Similarly, $(v_b \mathcal{H}_{r,p,n})^\tau \cong v_b \mathcal{H}_{r,p,n}$ as $\mathcal{H}_{r,p,n}$-modules. \hfill \square
3.4. Proposition. Suppose that $0 \leq b \leq n$. Then
\[
\left\{ v_b L_1^c \cdots L_n^c T_w \left| w \in S_n \text{ and } 0 \leq c_i < pt_a \text{ whenever } b_{1..a-1} < i \leq b_{1..a} \right. \right. \\
\left. \right. \left. \text{ and } c_1 + \cdots + c_n \equiv 0 \pmod p \right\}
\]
is a basis of $v_b \mathcal{H}_{r,p,n}$. In particular, $\dim v_b \mathcal{H}_{r,p,n} = \frac{1}{p} \dim v_b \mathcal{H}_{r,n}$.

Proof. First, observe that $\dim v_b \mathcal{H}_{r,p,n} \geq \frac{1}{p} \dim v_b \mathcal{H}_{r,n}$ since $\mathcal{H}_{r,n}$ is a free $\mathcal{H}_{r,p,n}$-module of rank $p$. By Corollary 2.16 the number of the elements given in the statement of the Proposition is exactly $\frac{1}{p} \dim v_b \mathcal{H}_{r,n}$. Therefore, it suffices to show that the elements in the statement in the Proposition span $v_b \mathcal{H}_{r,p,n}$.

By Lemma 3.1 the module $v_b \mathcal{H}_{r,p,n}$ is spanned by the elements
\[
\left\{ v_b L_1^c \cdots L_n^c T_w \left| w \in S_n, 0 \leq c_i < r \text{ for } 1 \leq i \leq n \right. \right. \\
\left. \right. \left. \text{ and } c_1 + \cdots + c_n \equiv 0 \pmod p \right\}.
\]
The elements $L_1, \ldots, L_n$ commute by [2, Lemma 3.3]. Therefore, to prove the Proposition it is enough to show for $\alpha = 1, \ldots, \kappa$ that if $b_{1..a-1} < i \leq b_{1..a}$ then $v_b L_i^\alpha$ is a linear combination of terms of the form $v_b L_{b_{1..a-1}+1}^{a_{b_{1..a-1}+1}} \cdots L_i^{a_i-1} T_w$, where $0 \leq a_j < pt_a$ for all $j$ and $w$ is an element of the symmetric group on the letters $\{b_{1..a-1} + 1, \ldots, i\}$. We prove this by induction on $i$.

Fix $\alpha$ such that $b_{\alpha} \neq 0$ and $1 \leq \alpha \leq \kappa$. Suppose first that $i = i_0$, where $i_0 = b_{1..a-1} + 1 \leq b_{1..a}$. Recall from Lemma 2.7 that
\[
v_b \cdot \prod_{Q \in \mathcal{Q}_a} (L_{i_0}^p - Q^p) = 0.
\]
Therefore, $v_b L_{i_0}^{pt_a}$ can be written as a linear combination of the elements $v_b L_{i_0}^{pt}$, for $0 \leq k < t_a$. Note that, modulo $p$, we have not changed the exponent of $L_{i_0}$. Hence, we may assume that $0 \leq c_i < pt_a$ when $i = i_0$. Now suppose that $i_0 < i \leq b_{1..a}$.

Arguing by induction (see [2, Lemma 3.3]), it follows easily that
\[
(3.5) \quad L_i^\alpha = q^{-1} T_{i-1} L_i^{\alpha} T_{i-1} + (1 - q^{-1}) \sum_{d=1}^{c-1} L_{i-1}^{d} L_i^{\alpha-d} T_{i-1}.
\]
Therefore, using Proposition 2.6,
\[
v_b L_i^\alpha = q^{-1} v_b T_{i-1} L_i^{\alpha} T_{i-1} + (1 - q^{-1}) \sum_{d=1}^{c-1} v_b L_{i-1}^{\alpha-d} L_i^{d} T_{i-1}
\]
\[
= q^{-1} T_{w^{-1}(i-1)} v_b L_{i-1}^{\alpha} T_{i-1} + (1 - q^{-1}) \sum_{d=1}^{c-1} v_b L_{i-1}^{\alpha-d} L_i^{d} T_{i-1}.
\]
If $c \geq pt_a$ then, by induction on $i$, we can rewrite $v_b L_{i-1}^{\alpha}$ as a linear combination of terms of the form $v_b L_{i_0}^{a_{i_0}} \cdots L_{i-1}^{a_{i-1}} T_w$, where $0 \leq a_j < pt_a$ for all $j$ and $w$ is an element of the symmetric group on the letters $\{i_0, \ldots, i-1\}$. Now, $L_1, \ldots, L_n$ commute with each other, and $T_{i-1}$ commutes with $L_j$ if $j \neq i-1, i$, so
\[
T_{w^{-1}(i-1)} v_b L_{i_0}^{a_{i_0}} \cdots L_{i-1}^{a_{i-1}} T_w T_{i-1} = v_b T_{i-1} L_{i_0}^{a_{i_0}} \cdots L_{i-1}^{a_{i-1}} T_w T_{i-1}
\]
\[
= v_b L_{i_0}^{a_{i_0}} \cdots L_{i-2}^{a_{i-2}} T_{i-1} L_{i-1}^{a_{i-1}} T_w T_{i-1}.
\]
Hence, using (3.5) once again, we can rewrite $v_b L_i^\alpha$ as a linear combination of terms of the form $v_b L_{i_0}^{a_{i_0}} \cdots L_i^{a_i} T_w$, where $0 \leq a_j < pt_a$ for all $j$ and $w$ is an element of the symmetric group on the letters $\{i_0, \ldots, i\}$. This proves our claim. Moreover, this completes the proof of the Proposition because, modulo $p$, the sums of the exponents of $L_1, \ldots, L_n$ are unchanged in all of the formulae above. \qed
If \( M \) is an \( H_{r,p,n} \)-module let \( M^+_{r,p,n} = M \otimes_{H_{r,p,n}} H_{r,n} \) be the corresponding induced \( H_{r,n} \)-module. Similarly, if \( N \) is an \( H_{r,n} \)-module let \( N_{r,p,n} \) be the restriction of \( N \) to \( H_{r,p,n} \). Since \( H_{r,n} \) is free as an \( H_{r,p,n} \)-module both induction and restriction are exact functors.

3.6. **Corollary.**

a) \( v_b H_{r,n} \cong (v_b H_{r,p,n})^+ H_{r,p,n} \),

b) \( (v_b H_{r,n})^- H_{r,p,n} \cong (v_b H_{r,p,n})^{\otimes p} \),

c) \( (v_b H_{r,n})^+ H_{r,p,n} \cong (v_b H_{r,n})^{\otimes p} \).

**Proof.** Since \( H_{r,n} = \bigoplus_{b=0}^{b-1} T^b_{H_{r,p,n}} \), there is a surjective homomorphism from \( (v_b H_{r,n})^+ H_{r,p,n} \) to \( v_b H_{r,n} \). By Corollary 2.16 and Proposition 3.4 both modules have the same dimension so this map must be an isomorphism, proving (a). Part (b) now follows from Proposition 2.15(b(iii)); alternatively, use part (a) and Lemma 3.3. Part (c) follows from parts (a) and (b).

3.7. **Corollary.** Suppose that \( 0 \leq b \leq n \) as above. Then

\[
\frac{1}{p} \dim \text{End}_{H_{r,p,n}} (v_b H_{r,n}) = \dim \text{End}_{H_{r,n}} (v_b H_{r,n}) = p \dim \text{End}_{H_{r,p,n}} (v_b H_{r,p,n}).
\]

**Proof.** The left and right hand equalities follow using Corollary 3.6 and Frobenius reciprocity.

We can now prove Theorem A from the introduction.

3.8. **Theorem.** Suppose that \( Q = Q_1 \vee \cdots \vee Q_\kappa \), where \( Q_1 \in Q_\alpha \) and \( Q_j \in Q_\beta \) are in the same \((\varepsilon,q)\)-orbit only if \( \alpha = \beta \). Then \( H_{r,p,n} \) is Morita equivalent to the algebra

\[
\bigoplus_{b \in \Lambda(n,\kappa)} H_b \rtimes \mathbb{Z}_p.
\]

**Proof.** By Proposition 2.15, \( \bigoplus_{b \in \Lambda(n,\kappa)} V^b \) is a progenerator for \( H_{r,n} \). Hence, by restriction, it is also a progenerator for \( H_{r,p,n} \). By Proposition 2.13(b), Frobenius reciprocity and Corollary 3.6(c) if \( b \neq c \) then \( \text{Hom}_{H_{r,p,n}} (V^b, V^c) = 0 \). Therefore, \( H_{r,p,n} \) is Morita equivalent to

\[
\text{End}_{H_{r,p,n}} \left( \bigoplus_{b \in \Lambda(n,\kappa)} V^b \right) = \bigoplus_{b \in \Lambda(n,\kappa)} \text{End}_{H_{r,p,n}} (V^b).
\]

Hence, to prove the Proposition it suffices to show that

\[
\text{End}_{H_{r,p,n}} (v_b H_{r,n}) \cong H_b \rtimes \mathbb{Z}_p.
\]

for all \( b \in \Lambda(n,\kappa) \).

Recall that each of the algebras \( H_{\alpha} \otimes_{H_{\alpha}} (Q_{\alpha}) \), for \( 1 \leq \alpha \leq \kappa \), has an automorphism \( \sigma_{\alpha} \) of order \( p \). The automorphism \( \sigma_1 \otimes \cdots \otimes \sigma_\kappa \) acts diagonally on the algebra \( H_b \). Note that \( \langle \sigma_1 \otimes \cdots \otimes \sigma_\kappa \rangle \cong \mathbb{Z}_p \). By Proposition 2.15, we have that

\[
H_b \cong \text{End}_{H_{r,n}} (v_b H_{r,n}) \cong \text{End}_{H_{r,p,n}} (v_b H_{r,n}).
\]

On the other hand, \( \sigma(v_b) = v_b \) by Lemma 3.3. So, \( \sigma \) induces an automorphism of \( v_b H_{r,p,n} \) which is given by \( v_b h \mapsto v_b \sigma(h) \), for all \( h \in H_{r,p,n} \). Hence, we have an injective map \( \mathbb{Z}_p \hookrightarrow \text{End}_{H_{r,p,n}} (v_b H_{r,n}) \). Since \( \sigma \) is an outer automorphism of \( H_{r,p,n} \), it follows that we have an embedding

\[
H_b \rtimes \mathbb{Z}_p \hookrightarrow \text{End}_{H_{r,p,n}} (v_b H_{r,n}).
\]

Using Corollary 3.7 to compare the dimensions on both sides of this equation, we conclude that \( \text{End}_{H_{r,p,n}} (v_b H_{r,n}) \cong H_b \rtimes \mathbb{Z}_p \).
Therefore, dim $H$ and that dim $H = n^\kappa$ dim $H_b$. Next, let $H_{p,b}$ be the subalgebra of $H_b$ generated by $H_{p,b}$ and the elements $\{ T_0^{(\alpha)}(T_0^{(\alpha)})^{-1} \mid 1 < \alpha \leq \kappa \}$.

3.9. Proposition. Suppose that $Q = Q_1 \lor \cdots \lor Q_\kappa$, where $Q_1$ and $Q_j$ are in different $(\varepsilon, q)$-orbits whenever $Q_1 \in Q_\alpha$ and $Q_j \in Q_\beta$ for some $\alpha \neq \beta$. Let $b \in \Lambda(n, \kappa)$. Then $H_{r,p,n}$ is Morita equivalent to the algebra

$$\bigoplus_{b \in \Lambda(n, \kappa)} H_{p,b}.$$

Proof. By Proposition 2.15(a), $\bigoplus_b v_b H_{r,p,n}$ is a progenerator for $H_{r,n}$. Therefore, $\bigoplus_b v_b H_{r,p,n}$ is a progenerator for $H_{r,p,n}$ by Corollary 3.6(b). Furthermore, if $b \neq c \in \Lambda(n, \kappa)$ then Hom$_{H_{r,n}}(v_b H_{r,p,n}, v_c H_{r,p,n}) = 0$ by Proposition 2.13(b). By Corollary 3.6 and Frobenius reciprocity, Hom$_{H_{r,n}}(v_b H_{r,p,n}, v_c H_{r,p,n}) = 0$. Combining these results we see that $H_{r,p,n}$ and $H_{p,b}$ are Morita equivalent.

Fix $b \in \Lambda(n, \kappa)$ and let $E_b = $ End$_{H_{r,p,n}}(v_b H_{r,p,n})$. To complete the proof it is enough to show that $E_b \cong H_{p,b}$. As a left $H_b$-module, $v_b H_{r,p,n}$ is isomorphic to a direct sum of $[S_n]_b [S_n]$ copies of the regular representation of $H_b$ by Corollary 2.16. By Proposition 2.15(b(iii)), $H_{p,b}$ acts faithfully on $v_b H_{r,p,n}$ by restriction and this action commutes with the action of $H_{r,p,n}$ from the right. Hence, we can identify $H_{p,b}$ with a subalgebra of $E_b$.

By Lemma 3.3 and Proposition 2.15(b(iii)), $\tau(v_b) = L_{b_{2,1}^{-1}} L_{b_{1,1}^{-1}} v_b \in v_b H_{r,p,n}$ acts on $v_b H_{r,p,n}$ in the same way that $T_0^{(1)}(T_0^{(\kappa)})^{-1} = T_0 \otimes 1 \otimes \cdots \otimes T_0^{-1}$ acts on $v_b H_{r,p,n}$. More generally, for $\alpha = 1, \ldots, \kappa - 1$ let $\rho_\alpha$ be the automorphism of $v_b H_{r,p,n}$ given by left multiplication by $L_{b_{2,0}} L_{b_{1,0}}^{-1} L_{b_{0,0}}^{-1}$ in $H_b$. By Proposition 2.5,

$$L_{b_{2,0}} L_{b_{1,0}}^{-1} L_{b_{0,0}}^{-1} v_b = v_b L_{1} L_{b_{1,0}}^{-1} = v_b S_{1}^{-1} v_b \in v_b H_{r,p,n}.$$

Therefore, $\rho_\alpha \in E_b$, for $2 \leq \alpha \leq \kappa$, since $\rho_\alpha$ commutes with the action of $H_{r,p,n}$. Thus, $\rho_\alpha$ coincides with the action of $T_0^{(1)}(T_0^{(\alpha)})^{-1} \in H_b$ on $V_b$ in Proposition 2.15. Consequently, $\rho_\alpha \not\in H_{p,b}$ and the automorphisms $\rho_2, \ldots, \rho_\alpha$ commute. By definition, $H_{p,b}$ is isomorphic to the subalgebra of $E_b$ generated by $H_{p,b}$ and $\rho_2, \ldots, \rho_\kappa$.

Hence, $H_{p,b}$ is a subalgebra of $E_b$.

For each $\alpha$, the map $\rho_\alpha^p$ acts as left multiplication by $(T_0^{(1)})^p T_0^{(\alpha)}$ in $H_b$, so $\rho_\alpha^p \in H_{p,b}$. Note that the extensions of the endomorphisms $\rho_0, \ldots, \rho_{\alpha - 1}$ to End$_{H_{r,n}}(v_b H_{r,n})$, for $2 \leq \alpha \leq \kappa$, are all linearly independent since End$_{H_{r,n}}(V_b) \cong H_b$ by Proposition 2.15. As these maps act on different components of $H_{p,b}$ it follows that dim $H_{p,b} = n^\kappa$ dim $H_{p,b}$. However, by Lemma 3.7 and Proposition 2.15(ii),

$$\dim E_b = \frac{1}{p} \dim \text{End}_{H_{r,n}}(V_b) = \frac{1}{p} \dim H_{p,b} = n^\kappa \dim H_{p,b}.$$

Therefore, dim $H_{p,b} = n^\kappa \dim H_{p,b} = \dim E_b$. So $E_b \cong H_{p,b}$, as required.
4. Splittable decomposition numbers

In this section we use Clifford theory to show that if an algebra can be written as a semidirect product then its decomposition numbers are determined by the corresponding "$p'$-splittable" decomposition numbers of a related family of algebras. First, we recall some general results about the representation theory of semidirect product algebras. The basic references for this topic are [4, 15, 19].

Let $A$ be a finite dimensional algebra over an algebraically closed field $K$ and suppose that $θ$ is an algebra automorphism of $A$ of order $p$. We identify $\mathbb{Z}_p$ with the group generated by $θ$ and consider the algebra $A \rtimes \mathbb{Z}_p$. Then, as a set,

$$A \rtimes \mathbb{Z}_p = \{ aθ^k \mid a \in A \text{ and } 0 \leq k \leq p \}$$

and the multiplication in $A \rtimes \mathbb{Z}_p$ is defined by

$$(aθ^k) \cdot (bθ^m) = aθ^k(b) \cdot θ^{k+m},$$

for $a, b \in A$ and $0 \leq k, m < p$. If $H$ is a subgroup of $\mathbb{Z}_p$ we identify $A \rtimes H$ with a subalgebra of $A \rtimes \mathbb{Z}_p$ in the natural way. In particular, by taking $H = 1$ we can view $A$ as a subalgebra of $A \rtimes \mathbb{Z}_p$. Moreover, there are natural induction and restriction functors between the module categories of all of these algebras.

Suppose that $L$ is an $A$-module. Then we can twist $L$ by $θ$ to get a new $A$-module $L^θ$. As a vector space we set $L^θ = L$, and we define the action of $A$ on $L^θ$ by

$$v \cdot a := vθ(a), \quad \text{for all } v \in L \text{ and } a \in A.$$ 

It is straightforward to check that $L$ is irreducible if and only if $L^θ$ is irreducible.

The inertial group of $L$ is the group

$$G_L := \{ θ^k \in Z_p \mid L \cong L^{θ^k} \}.$$ 

Then $G_L$ is a subgroup of $Z_p$ and, in particular, it is cyclic. Let $l = |G_L|$. Then $p = lk$ and $G_L$ is generated by $θ^k$. Recall that we have fixed a primitive $p$th root of unity $ε \in K$. Since $K$ is algebraically closed we can choose an $A$-module isomorphism $φ : L \rightarrow L^{θ^k}$ such that $φ^l = 1_L$, the identity map on $L$. For each integer $i \in Z$ define the $(A \rtimes G_L)$-module $L_{i,i}$ as follows: as a vector space $L_{i,i} := L$ and the action of $(A \rtimes G_L)$ on $L_{i,i}$ is given by:

$$v \cdot (aθ^m)^k := ε^{mk} φ^m(va), \quad \text{for all } m \in Z, v \in L_{i,i} \text{ and } a \in A.$$ 

It is easy to check that $L_{i,i}$ is an $(A \rtimes G_L)$-module and, by definition, that $L_{i,i+1} \cong L_{i,i}$, for all $i \in Z$.

Recall that $\text{Irr}(A)$ is the complete set of isomorphism classes of simple $A$-modules. Let $L \in \text{Irr}(A)$. Then, since $L_{i,i} \downarrow A \cong L_i$, it follows that $L_{i,i}$ is a simple $(A \rtimes G_L)$-module. In fact, it is shown in [4, 15, 19] that

$$\left\{ L_{i,i} \uparrow_{A \times G_L} \mid i \in \mathbb{Z} \right\} \subseteq \text{Irr}(A) \uparrow_{A \rtimes G_L} \text{ and } l = |G_L|,$$

where $L' \sim L$ if and only if $L' \cong L^{θ^i}$ for some integer $i$, is a complete set of pairwise non-isomorphic simple $(A \rtimes Z_p)$-modules.

4.1. Lemma. Suppose that $L$ is a simple $A$-module and that $H$ is a subgroup of $G_L$. Let $l = |G_L|$, $h = |H|$ and fix $i$ with $1 \leq i \leq l$. Then $L_{i,i} \uparrow_{A \rtimes H}$ is an irreducible $(A \rtimes H)$-module and

$$L_{i,i} \uparrow_{A \rtimes H} \cong \bigoplus_{j=1}^{|G_L:H|} L_{i,i+j}.$$
Proof. As remarked above, we can view $A$ and $A \times H$ as subalgebras of $A \times G_L$. Since $L_{i+1}A_{xG_L} \cong L_i$ is irreducible we see that $L_{i+1}A_{xG_L} \cong L_i$ is irreducible. Next, observe that $g^{p/h}$ is a generator of $H$. Hence, by the argument above if $1 \leq j \leq l = |G_L|$ then $L_{i+1}A_{xH} \cong L_iA_{xH}$ is irreducible. Therefore, by Frobenius reciprocity and Schur’s Lemma,

$$\text{Hom}_{A \times G_L}(L_{i+1}A_{xH}, L_iA_{xH} \uparrow_{A \times H}^{A \times G_L}) \cong \text{Hom}_{A \times H}(L_{i+1}A_{xH}, L_iA_{xH} \uparrow_{A \times H}^{A \times G_L}) \cong K.$$ 

The lemma now follows by comparing dimensions. \qed

Now suppose that we have a modular system $(F, O, K)$ such that $A = A_K$ has an $O$-lattice $A_0$ which is an $O$-algebra, $\theta$ can be lifted to an automorphism of $A_0$ of order $p$, $F$ is an algebraically closed field of characteristic zero, and $A_F := A_0 \otimes_O F$ is a (split) semisimple $F$-algebra. We abuse notation and write $\theta$ for the corresponding automorphism of $A_F$. Note that if $H$ is a subgroup of $Z_p$ then $A_K \times H \cong (A_0 \times H) \otimes_O K$, so that we also have a modular system for the algebras $A_F \times H$ and $A_K \times H$.

By definition, $\text{Irr}(A_F)$ is the complete set of isomorphism classes of simple $A_F$-modules — the “semisimple” $A_F$-modules. As the automorphism $\theta$ lifts to $A_F$ for each simple $A_F$-module $S \in \text{Irr}(A_F)$ we have an inertia group $G_S \leq Z_p$ and, as above, we can define $(A_F \times G_S)$-modules $S_{s,j}$ for $j \in \mathbb{Z}$ where $s = |G_S|$. Consequently,

$$\{S_{s,1}A_{F \times Z_p} : D_{d,1}A_{xZ_p} \}$$

is a complete set of pairwise non-isomorphic simple $(A_F \times Z_p)$-modules.

Suppose that $S \in \text{Irr}(A_F)$ and that $D \in \text{Irr}(A_K)$ and let $s = |G_S|$ and $d = |G_D|$. Given $i$ and $j$ with $1 \leq i \leq s$ and $1 \leq j \leq d$, we want to determine the decomposition number

$$[S_{s,i}A_{F \times G_S} : D_{d,j}A_{xG_D}],$$

which gives the multiplicity of $D_{d,j}A_{xG_D}$ as an irreducible composition factor of a modular reduction of $S_{s,i}A_{xG_D}$.

4.2. **Definition.** Suppose that $S \in \text{Irr}(A_F)$ and $D \in \text{Irr}(A_K)$ and set $s = |G_S|$ and $d = |G_D|$. Then the pair $(S, D)$ has cyclic decomposition numbers if

$$[S_{s,i}A_{F \times G_S} : D_{d,j}A_{xG_D}] = [S_{s,i+1}A_{F \times G_S} : D_{d,j+1}A_{xG_D}]$$

for all $i, j \in \mathbb{Z}$.

In the next section we show that all pairs of irreducible $\mathcal{H}_{r, n}$-modules have cyclic decomposition numbers.

4.3. **Proposition.** Let $S \in \text{Irr}(A_F)$ and $D \in \text{Irr}(A_K)$ and suppose that $(S, D)$ has cyclic decomposition numbers. Set $s = |G_S|$, $d = |G_D|$ and let $d_0 = \gcd(s, d)$. Then

$$[S_{s,i+d_0}A_{F \times G_S} : D_{d,j+d_0}A_{xG_D}] = [S_{s,i}A_{F \times G_S} : D_{d,j}A_{xG_D}]$$

for all $i, j, l, l' \in \mathbb{Z}$.

Proof. Since the groups $G_S, G_D$ and $G_0 = G_S \cap G_D$ are all cyclic subgroups of $Z_p$, we have that $|G_0| = \gcd(|G_S|, |G_D|) = \gcd(s, d) = d_0$, so that there exist integers $u$ and $v$ such that $d_0 = us + vd$. Suppose that $l, l' \in \mathbb{Z}$. Recall that $S_{s,i}A_{F \times G_S} \cong S_{s,i}$
and $D_{d,j+md} \cong D_{d,j}$ for all $m \in \mathbb{Z}$. Then, using the assumption that $(S, D)$ has cyclic decomposition numbers for the third equality, we find that

$$[S_{s,i+q}, D_{d,j+q}] \cong D_{d,j} \cong [S_{s,i+q}, D_{d,j+q}]$$

as required. 

4.4. **Definition.** Suppose that $S \in \text{Irr}(A_F)$ and $D \in \text{Irr}(A_K)$ and let $s = |G_S|$ and $d = |G_D|$ as above. Then the decomposition number $[S_{s,i}, D_{d,j}] \cong D_{d,j}$ is $p$-splittable if $G_S \cong G_D \cong G$. 

We will see in Corollary 4.5 below that this definition of $p$-splittable decomposition number agrees with Definition 1.1 when applied to the algebras $\mathcal{H}_{r,p,n}$. 

4.5. **Corollary.** Suppose that $S \in \text{Irr}(A_F)$ and $D \in \text{Irr}(A_K)$ have cyclic decomposition numbers and let $G_0 = G_S \cap G_D$, $s = |G_S|$ and $d = |G_D|$. Then

$$[S_{s,i} : D_{d,j}] \cong D_{d,j}$$

for all $1 \leq i \leq s$ and all $1 \leq j \leq d$. In particular, all of the decomposition numbers appearing in the summation are $p'$-splittable in the sense of Definition 4.4, where $p' = |G_0|$. 

**Proof.** Observe that if $L \not\cong L'$ are simple $(A \times G_0)$-modules then the induced modules $L \uparrow_{A \times G_0}$ and $L' \uparrow_{A \times G_0}$ have no common irreducible composition factors. Next observe that if $1 \leq i \leq s$ then $S_{s,i} \uparrow_{A \times G_0}$ is an irreducible $(A \times G_0)$-module by Lemma 4.1. Similarly, if $1 \leq j \leq d$ then $D_{d,j} \uparrow_{A \times G_0}$ is an irreducible $(A \times G_0)$-module. The first claim now follows by combining these observations with Lemma 4.1 and Proposition 4.3. Finally, all of these decomposition numbers are $p'$-splittable because the inertia groups of the modules $S_{s,i}$ and $D_{d,j}$ inside $G_0$ are both equal to $G_0$. 

Corollary 4.5 reduces the calculation of decomposition numbers of the algebra $A \times \mathbb{Z}_p$ to determining the $p'$-splittable decomposition numbers of the subalgebras $A \times \mathbb{Z}_{p'}$, where $p'$ divides $p$.

We now apply Corollary 4.5 in the special case where $A$ has a tensor product decomposition. That is, we suppose that $A = A^{(1)} \otimes \cdots \otimes A^{(r)}$, for some finite dimensional $K$-algebras $A^{(1)}, \ldots, A^{(r)}$ which are equipped with automorphisms $\theta_1, \ldots, \theta_r$, respectively, of order $p$. Set $\theta = \theta_1 \otimes \cdots \otimes \theta_r$. We assume that the tensor product decomposition of $A$ is compatible with the modular system $(F, \mathcal{O}, K)$ so that $A_F = A_F^{(1)} \otimes \cdots \otimes A_F^{(r)}$.

Fix an integer $f \in \{1, \ldots, r\}$ and let $S_f \in \text{Irr}(A_F)$ and $D_f \in \text{Irr}(A_K)$ be irreducible modules. Then $S = S(1) \otimes \cdots \otimes S(r) \in \text{Irr}(A_F)$ and $D = D(1) \otimes \cdots \otimes D(r) \in \text{Irr}(A_K)$. Further, $G_S = G_S(1) \cap \cdots \cap G_S(r)$, and $G_D = G_D(1) \cap \cdots \cap G_D(r)$. Set $s = |G_S|$ and $d = |G_D|$. Then $s = |G_S| = \gcd(|G_S(1)|, \ldots, |G_S(r)|)$ and $d = |G_D| = \gcd(|G_D(1)|, \ldots, |G_D(r)|)$.

Suppose that $(S, D)$ has cyclic decomposition numbers and let $G_0 = G_S \cap G_D$. Then, by Corollary 4.5, the decomposition numbers of $A \times \mathbb{Z}_p$ are completely determined by the decomposition numbers of the form $[S_{s,i}, D_{d,j}] \cong D_{d,j}$, for $1 \leq a \leq p/|G_D|, 1 \leq i \leq s$ and $1 \leq j \leq d$. 


4.6. **Theorem.** Suppose that \( S = S^{(1)} \otimes \cdots \otimes S^{(s)} \in \text{Irr}(A_F) \) and \( D = D^{(1)} \otimes \cdots \otimes D^{(k)} \in \text{Irr}(A_K) \). Let \( s = |G_S| \), \( d = |G_D| \) and \( G_0 = G_S \cap G_D \) and set \( d_0 = |G_0| \). Then

\[
[S_{s,i}^{A_F \times G_S} : D_{d,j}^{A_K \times G_D}] = \sum_{0 \leq j_1, \ldots, j_s < d_0} \prod_{\alpha=1}^{k} [S_{d_0,\alpha}^{(s)} : D_{d_0,\alpha}^{(s)}] \quad \text{for all } i \text{ and } j \text{ with } 1 \leq i \leq s \text{ and } 1 \leq j \leq d.
\]

**Proof.** Suppose that \( R \in \{F, K\} \). Let \( k = \frac{p}{d_0} = |\mathbb{Z}_p/G_0| \), so that \( G_0 = \langle \theta^k \rangle \). Consider the algebra

\[
\hat{A}_R = R G_0 \otimes (A_R^{(1)} \times \langle \theta^k \rangle) \otimes \cdots \otimes (A_R^{(k)} \times \langle \theta^k \rangle).
\]

Then it is straightforward to check that there is an embedding of algebras \( A_R \times G_0 \hookrightarrow \hat{A}_R \) given by

\[
(a_1 \otimes \cdots \otimes a_{s}) \theta^{mk} \mapsto \theta^{mk} \otimes (a_1 \theta_i^{mk}) \otimes \cdots \otimes (a_{s} \theta_{s}^{mk}),
\]

for \( a_1, \ldots, a_{s} \in A_R \) and \( m \in \mathbb{Z} \).

Let \( L = L^{(1)} \otimes \cdots \otimes L^{(k)} \) be a simple \( A_R \)-module, where \( L = S \) if \( R = F \), or \( L = D \) if \( R = K \). Let \( l = |G_L| \) and suppose that \( 1 \leq i \leq l \). Then \( L_{i,i} \) is a simple \((A_R \times G_L)\)-module. Recall that the modules \( L_{i,i} \) are defined using a fixed isomorphism \( \phi : L \rightarrow L^{(j)} \) satisfying \( \phi^j = 1_L \). We may assume that \( \phi \) is compatible with the tensor decomposition of \( L \); that is, \( \phi = \hat{\phi}_1 \otimes \cdots \otimes \hat{\phi}_k \), where for \( f = 1, \ldots, k \), the order of the inertia group \( G_L(f) \) of \( L^{(f)} \) in \( \mathbb{Z}_p \) is \( l_f \), \( \phi_f = (\phi_f)^{l_f} \).

\[
\phi_f : L^{(f)} \rightarrow (L^{(f)})^{\theta_j^{i_f}}
\]

is an isomorphism defining the module \( L_{i_f}^{(f)} \). Note that

\[
L_{i_f}^{(f)} \xrightarrow{\alpha} A_R^{(f)} \otimes \xi_{A_R^{(f)}} \cong L_{d_0,i_f}^{(f)}
\]

Applying the definitions, given any integers \( i, i_0, i_1, \ldots, i_k \) with \( i \equiv i_1 + \cdots + i_k \) (mod \( d_0 \)) there is a natural isomorphism of \((A_R \times G_0)\)-modules

\[
L_{i_0, i_1}^{A_R \times G_L} \cong \left( \varepsilon^{-i_0} \otimes L_{d_0, i_1}^{(1)} \otimes \cdots \otimes L_{d_0, i_k}^{(k)} \right)_{A_R \times G_0}, \tag{\dag}
\]

where \( \varepsilon^{-i_0} \) is the one dimensional representation of \( G_0 = \langle \theta^k \rangle \) upon which \( \theta^k \) acts as multiplication by \( \varepsilon^{-i_0} \).

By (\dag), if \( 1 \leq i \leq s \) then \( S_{s,i}^{A_F \times G_S} \cong \left( \varepsilon^{-(k-1)i} \otimes S_{d_0, i_1}^{(1)} \otimes \cdots \otimes S_{d_0, i_k}^{(k)} \right)^{A_F \times G_0} \).

Therefore, we can find the \((A_F \times G_S)\)-module composition factors of \( S_{s,i}^{A_F \times G_S} \) by first finding the \( \tilde{A}_K \)-module composition factors of \( \varepsilon^{-(k-1)i} \otimes S_{d_0, i_1}^{(1)} \otimes \cdots \otimes S_{d_0, i_k}^{(k)} \) and then restricting to \( A_K \times G_0 \). The \( \tilde{A}_K \)-module composition factors of \( \varepsilon^{-(k-1)i} \otimes S_{d_0, i_1}^{(1)} \otimes \cdots \otimes S_{d_0, i_k}^{(k)} \) are all of the form \( \varepsilon^{-(k-1)i} \otimes D_{d_0, j_1}^{(1)} \otimes \cdots \otimes D_{d_0, j_k}^{(k)} \), where \( 0 \leq j_1, \ldots, j_k < d_0 \). Further, by (\dag), if \( j \equiv j_1 + \cdots + j_k \) (mod \( d_0 \)), then

\[
D_{d_0, j}^{A_K \times G_0} \cong \left( \varepsilon^{-(k-1)i} \otimes D_{d_0, j_1}^{(1)} \otimes \cdots \otimes D_{d_0, j_k}^{(k)} \right)^{A_K \times G_0}.
\]

The theorem now follows. \( \Box \)

4.7. **Corollary.** Suppose that \( S = S^{(1)} \otimes \cdots \otimes S^{(s)} \in \text{Irr}(A_F) \) and \( D = D^{(1)} \otimes \cdots \otimes D^{(k)} \in \text{Irr}(A_K) \) have cyclic decomposition numbers. Let \( s = |G_S| \), \( d = |G_D| \) and \( G_0 = G_S \cap G_D \) and set \( d_0 = |G_0| \). Then

\[
[S_{s,i}^{A_F \times Z_p} \times D_{d,j}^{A_K \times Z_p}] = \sum_{0 \leq j_1, \ldots, j_s < d_0} \prod_{\alpha=1}^{k} [S_{d_0,\alpha}^{(s)} : D_{d_0,\alpha}^{(s)}] \quad \text{for all } i \text{ and } j \text{ with } 1 \leq i \leq s \text{ and } 1 \leq j \leq d.
\]
Proof. This follows from Corollary 4.5 and Theorem 4.6. □

5. Reduction theorems for the decomposition numbers of $\mathcal{H}_{r,p,n}$

We now combine the results of last two sections to show that the decomposition numbers of the cyclotomic Hecke algebras $\mathcal{H}_{r,p,n}$ are completely determined by the $p'$-splittable decomposition numbers of an explicitly determined family of cyclotomic Hecke algebras. We first show that the simple $\mathcal{H}_{r,p,n}$-modules always have cyclic decomposition numbers.

5.1. Lemma. The algebras $\mathcal{H}_{r,p,n}$ and $\mathcal{H}_{r,n} \rtimes \mathbb{Z}_p$ are Morita equivalent.

Proof. As a right $\mathcal{H}_{r,p,n}$-module $\mathcal{H}_{r,n} = \bigoplus_{k=0}^{p-1} \mathcal{H}_{r,p,n}^k$ by Lemma 3.1. Consequently, $\mathcal{H}_{r,n}$ is a progenerator for $\mathcal{H}_{r,p,n}$, so $\mathcal{H}_{r,p,n}$ is Morita equivalent to $\text{End}_{\mathcal{H}_{r,n}}(\mathcal{H}_{r,n})$. Observe that $\sigma \in \text{End}_{\mathcal{H}_{r,n}}(\mathcal{H}_{r,n})$ since $\sigma$ is trivial on $\mathcal{H}_{r,p,n}$. Furthermore, as vector spaces, $\text{End}_{\mathcal{H}_{r,p,n}}(\mathcal{H}_{r,n}) \cong \text{Hom}_{\mathcal{H}_{r,n}}(\mathcal{H}_{r,n}^\oplus, \mathcal{H}_{r,p,n}^\oplus)$

$\cong \text{Hom}_{\mathcal{H}_{r,p,n}}(\mathcal{H}_{r,n}^\oplus, \mathcal{H}_{r,n}^\oplus)^\oplus_p$

$\cong \text{Hom}_{\mathcal{H}_{r,n}}(\mathcal{H}_{r,n}, \mathcal{H}_{r,n})^\oplus \cong \mathcal{H}_{r,n}$

where the third isomorphism comes from Frobenius reciprocity. Hence, by counting dimensions, $\text{End}_{\mathcal{H}_{r,p,n}}(\mathcal{H}_{r,n}) \cong \mathcal{H}_{r,n} \rtimes \mathbb{Z}_p$. (Alternatively, apply Theorem A with $\kappa = 1$.) □

The proof of this Lemma gives a Morita equivalence from the category of (finite dimensional right) $\mathcal{H}_{r,n}^R \rtimes \mathbb{Z}_p$-modules to the category of $\mathcal{H}_{r,p,n}^R$-modules. This functor sends the finite dimensional $(\mathcal{H}_{r,n} \rtimes \mathbb{Z}_p)$-module $M$ to the $\mathcal{H}_{r,p,n}$-module $F(M) = M \otimes_{\mathcal{H}_{r,n}^R \rtimes \mathbb{Z}_p} \mathcal{H}_{r,n}^R$.

5.2. Lemma. Suppose that $L$ is a simple $\mathcal{H}_{r,n}^R$-module. Then

$L \downarrow_{\mathcal{H}_{r,p,n}^R} \cong F(L \downarrow_{\mathcal{H}_{r,n}^R \rtimes \mathbb{Z}_p})$.

Proof. Applying the definitions and standard properties of tensor products,

$F(L \downarrow_{\mathcal{H}_{r,n}^R \rtimes \mathbb{Z}_p}) = (L \otimes_{\mathcal{H}_{r,n}^R} \mathcal{H}_{r,n}^R \rtimes \mathbb{Z}_p) \otimes_{\mathcal{H}_{r,n}^R \rtimes \mathbb{Z}_p} \mathcal{H}_{r,n}^R \cong L \downarrow_{\mathcal{H}_{r,p,n}^R}$.

□

5.3. Proposition ([7, 2.2], [20, 2.2]) and [9, (5.4), (5.5), (5.6)]). Suppose that $L$ is a simple $\mathcal{H}_{r,n}^R$-module and let $k > 0$ be minimal such that $L \cong L^\phi^k$. Then $1 \leq k \leq p$ and $l := \frac{n}{k}$ is the smallest positive integer such that $L' \cong L'^{l_1}$ whenever $L'$ is a simple $\mathcal{H}_{r,p,n}$-submodule of $L$.

Now fix an isomorphism $\phi : L \to L^\phi$ such that $\phi^2 = 1$ and for $i \in \mathbb{Z}$ define

$L_i := \{ v \in L \mid \phi(v) = e^{-ik}v \}.$

Then $L \downarrow_{\mathcal{H}_{r,p,n}^R} = L_0 \oplus \cdots \oplus L_{l-1}$. Moreover, $L_i = L_{i+1}$ and $L_{i+1} \cong L_i^\phi$, for any $i \in \mathbb{Z}$. Consequently, $L \downarrow_{\mathcal{H}_{r,p,n}^R} \cong L_0 \oplus L_0^\phi \oplus \cdots \oplus L_0^{l-1}$.

For each simple $\mathcal{H}_{r,p,n}$-module $L$ we henceforth fix an isomorphism $\phi : L \to L^\phi$ such that $\phi^2 = 1$, where $k$ and $l = \frac{n}{k}$ as in the Lemma. Observe that $l = [G_L]$, where $G_L$ is the inertia group of $L$. For each integer $i$ we have defined $\mathcal{H}_{r,p,n}$-modules $L_i$ and $L_{i,i}$. The next result gives the connection between these two modules.
5.4. Lemma. Suppose that $L$ is a simple $\mathcal{H}_{r,n}^R$-module with inertia group $G_L$ and let $l = |G_L|$. Then, for each $i \in \mathbb{Z}$, we have
\[ L_i \cong F(L_{i,1} \uparrow_{\mathcal{H}_{r,n}^R \times \mathbb{Z}_p}^{\mathcal{H}_{r,n}^R \times G_L}). \]

Proof. As in the proof of Lemma 5.2, $F(L_{i,1} \uparrow_{\mathcal{H}_{r,n}^R \times \mathbb{Z}_p}^{\mathcal{H}_{r,n}^R \times G_L}) \cong L_{i,1} \otimes_{\mathcal{H}_{r,n}^R \times G_L} \mathcal{H}_{r,n}^R$.

Therefore, there is a natural map $\psi : F(L_{i,1} \uparrow_{\mathcal{H}_{r,n}^R \times \mathbb{Z}_p}^{\mathcal{H}_{r,n}^R \times G_L}) \rightarrow L$ given by $\psi(v \otimes \mathcal{H}_{r,n} \times \mathbb{Z}_p) = \psi(h) = vh$, for $v \in L_{i,1}$ and $h \in \mathcal{H}_{r,n}^R$. Clearly, $\psi \neq 0$ so it suffices to show that the image of $\psi$ is contained in $L_i$. Now, if $v \in L_{i,1}$ and $h \in \mathcal{H}_{r,n}^R$ as above then, using the definition of $L_{i,1}$, we see that
\[ \varepsilon^{ik} \phi(vh) = v \cdot (\sigma^k) = \psi(v \cdot (\sigma^k) \otimes \mathcal{H}_{r,n} \times \mathbb{Z}_p 1) = \psi(v \otimes \mathcal{H}_{r,n} \times \mathbb{Z}_p h) = vh. \]

Hence, $\psi(vh) \in L_i$ as required. \qed

5.5. Corollary. Suppose that $S \in \text{Irr}(\mathcal{H}_{r,n})$ and $D \in \text{Irr}(\mathcal{H}_{r,n}^K)$ and let $s = |G_S|$ and $d = |G_D|$. Then
\[ [S_{s,i} \uparrow_{\mathcal{H}_{r,n} \times \mathbb{Z}_p}^{\mathcal{H}_{r,n} \times G_S} : D_{d,j} \uparrow_{\mathcal{H}_{r,n} \times G_D}^{\mathcal{H}_{r,n} \times G_D}] = [S_{s,i} \vdash_{\mathcal{H}_{r,n} \times \mathbb{Z}_p, G_S} : D_{d,j} \vdash_{\mathcal{H}_{r,n} \times \mathbb{Z}_p, G_D}], \]
for all $i, j \in \mathbb{Z}$. That is, $(S, D)$ has cyclic decomposition numbers.

Proof. Using Lemma 5.4 and Proposition 5.3 we have
\[
[S_{s,i} \uparrow_{\mathcal{H}_{r,n} \times \mathbb{Z}_p}^{\mathcal{H}_{r,n} \times G_S} : D_{d,j} \uparrow_{\mathcal{H}_{r,n} \times G_D}^{\mathcal{H}_{r,n} \times G_D}] = [S_{s,i} \vdash_{\mathcal{H}_{r,n} \times \mathbb{Z}_p, G_S} : D_{d,j} \vdash_{\mathcal{H}_{r,n} \times \mathbb{Z}_p, G_D}],
\]
as required. \qed

Note that Proposition 5.3 and Lemma 5.4 also imply that the two notions of $p$-splittable decomposition numbers for the algebras $\mathcal{H}_{r,p,n}$ coincide.

5.6. Corollary. Suppose that $S \in \text{Irr}(\mathcal{H}_{r,n}^F)$ and $D \in \text{Irr}(\mathcal{H}_{r,n}^K)$ and let $s = |G_S|$ and $d = |G_D|$. Fix integers $i, j \in \mathbb{Z}$. Then the decomposition number
\[ [S_{s,i} \uparrow_{\mathcal{H}_{r,n} \times \mathbb{Z}_p}^{\mathcal{H}_{r,n} \times G_S} : D_{d,j} \uparrow_{\mathcal{H}_{r,n} \times G_D}^{\mathcal{H}_{r,n} \times G_D}] \]
is $p$-splittable in the sense of Definition 4.4 if and only if the decomposition number
\[ [F(S_{s,i} \uparrow_{\mathcal{H}_{r,n} \times \mathbb{Z}_p}^{\mathcal{H}_{r,n} \times G_S} : F(D_{d,j} \uparrow_{\mathcal{H}_{r,n} \times \mathbb{Z}_p}^{\mathcal{H}_{r,n} \times G_D})] \]
is $p$-splittable in the sense of Definition 1.1.

5.7. Theorem. Then every decomposition number of $\mathcal{H}_{r,p,n}(Q)$ is equal to a sum of $p'$-splittable decomposition number of some cyclotomic Hecke algebra $\mathcal{H}_{r,p',n}(q, Q')$, where $p = kp'$ and
\[ Q' = (Q_1, \varepsilon Q_1, \ldots, \varepsilon^{k-1} Q_1, Q_2, \ldots, \varepsilon^{k-1} Q_2, \ldots, Q_t, \ldots, \varepsilon^{k-1} Q_t). \]

Proof. By Proposition 5.3 every irreducible $\mathcal{H}_{r,p,n}^F$-module is equal to $S_i$ for some $S \in \text{Irr}(\mathcal{H}_{r,n}^F)$ and with $1 \leq i \leq s = |G_S|$. Similarly, every irreducible $\mathcal{H}_{r,p,n}^K$-module is equal to $D_j$ for some $D \in \text{Irr}(\mathcal{H}_{r,n}^K)$ and with $1 \leq j \leq s = |G_D|$. Therefore, by Lemma 5.4, Corollary 5.5 and Corollary 4.5,
\[ [S_i : D_j] = \sum_{1 \leq a \leq \frac{s}{p}} [S_{s,i} \uparrow_{\mathcal{H}_{r,n} \times \mathbb{Z}_p}^{\mathcal{H}_{r,n} \times G_0} : D_{d,j} \uparrow_{\mathcal{H}_{r,n} \times G_0}^{\mathcal{H}_{r,n} \times G_0}], \]
where $G_0 = G_S \cap G_D$. Suppose that $G_0 = \langle \sigma^k \rangle$ and write $p = kp'$. Then we have shown that $[S_i : D_j]$ is a sum of $p'$-splittable decomposition number of $\mathcal{H}_{r,n} \times G_0$. 

As at the beginning of section 2, write \( r = pt \). Then \( r = p'^k t \) and in \( \mathcal{H}_{r,p,n}(\mathbb{Q}) \) the ‘order relation’ for \( T_0 \) is
\[
0 = \prod_{b=1}^{t} (T_0^p - Q_b^p) = \prod_{b=1}^{t} \prod_{a=0}^{k-1} (T_0^{p'_a} - (\varepsilon^a Q_b)^{p'_a}).
\]
Observe that the right hand side is the ‘order relation’ for \( T_0^{p'_a} \) in \( \mathcal{H}_{r,p',n'}(\mathbb{Q'}) \). It now follows using Lemma 5.1 that \( \mathcal{H}_{r,\mathbb{Z}(\varepsilon),n}(\mathbb{Q}) \) is Morita equivalent to \( \mathcal{H}_{r,p',n}(q',\mathbb{Q'}) \), where the parameters \( Q' \) are as given in the statement of the theorem. This completes the proof of the theorem.

\[\square\]

**Proof of Theorem B.** This follows from an recursive application of Theorem A, Corollary 4.7, Corollary 5.5 and Theorem 5.7.

Theorem B gives a recursive algorithm for computing all of the decomposition numbers of a cyclotomic Hecke algebra \( \mathcal{H}_{r,p,n}(\mathbb{Q}) \) in terms of the \( p' \)-splittable decomposition numbers of a family of “smaller” cyclotomic Hecke algebras \( \mathcal{H}_{r',p',n'}(\mathbb{Q'}) \), where \( 1 \leq r' \leq r, 1 \leq n' \leq n, 1 \leq p' \mid p \), and the parameters \( Q' \) are contained in a single \((\varepsilon',q')\)-orbit of \( Q \), where \( \varepsilon' \) is a primitive \( p' \)-th root of unity. Therefore, the \( p' \)-splittable decomposition numbers of the cyclotomic Hecke algebras of type \( G(r',p',n') \) completely determine the decomposition numbers of all cyclotomic Hecke algebras \( \mathcal{H}_{r,p,n}(\mathbb{Q}) \).

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