Verifying the Movable Elastoplastic Boundary Method by Using Galin’s Problem

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Abstract. Galin’s solution for the problem of biaxial tension of a plate with a hole completely covered by the plastic region appears to be a pearl recognized by the world scientific community. This solution serves as a test for all sorts of approximate approaches to solving elastoplastic problems, including the semi-analytical iterative method being developed by the author, focused on solving more complex problems such as the Kirsch problem in the elastoplastic formulation. The proposed iterative approach for a semi-analytical solution involves an explicit analytical expression for stresses in the plastic region and an iterative numerical solution in the elastic region with a refined border. The paper shows the convergence of the results based on the iterative procedure for the elastoplastic region boundary approaching its analytical position, which follows from the analytical solution of Galin’s elastoplastic problem. Consideration has also been given to obtaining results on the determination of the boundary between the elastic and plastic regions using a competing approximate perturbation method. The advantage of the proposed method lays in not limited modifications in parameters due to the requirement for small differences while formulating a problem from the axisymmetric case as seen in the perturbation method.

Keywords: Galin’s problem; Analytical solution; Numerical solution; Refined elastoplastic boundary method; Perturbation method.

1. Introduction

Solving elastoplastic problems of the perfect plasticity theory is associated with the joint solution for equations of different types, namely elliptic – in the elastic region, hyperbolic – in the plastic one and their conjugation within the previously unknown boundary line between the elastic and plastic material states determined while shaping the solution. L.A. Galin [1] was the first who worked out an analytical solution to a specific problem of this type regarding the distribution of stresses in the vicinity of a circular hole in a plate exposed to the conditions of a plane strain state under biaxial tension at infinity and the Tresca - Saint-Venant plasticity condition. The corresponding mathematical problem for an elastic region with an unknown boundary was reduced to a boundary-value problem for a biharmonic equation, which, when a region with an unknown boundary was conformally mapped to the outside of a circle, was transformed into an analytically solvable problem of the theory of complex variable functions. G.P. Cherepanov applied a similar approach for a plate with a hole influenced by a plane stress state [2]. Expressions for deformations in Galin’s problem were obtained by I.S. Tuba [3] while N.I. Ostrosablin [4] proposed expressions for displacements. B.D. Annin [5] constructed the solution for this problem under another – exponential plasticity condition. It should be noted that the analytical solutions constructed in [1-6] are valid when the plastic region completely
covers the hole. Developing solutions for such problems continues to this day ([7-10] and in many other works), although their area of the load combination is rather narrow [7].

In an incomparably greater number of options for biaxial load of the plate and with uniaxial load exceeding a third of the yield stress of its material occurs incomplete coverage of the hole by the plastic region. As for the elastic formulation, the problem of uniaxial tension of a plate with a circular hole was solved by G. Kirsch [11] at the end of the 19th century. In the elastoplastic formulation, however, the analytical solution to this problem has not been found yet. A. Nadai [12] suggests to qualitatively analyze this problem and informs the results of experimental modeling in his monograph.

An effective numerical method to shape the solution that intends to find the boundary between the elastic and plastic regions is described by I.S. Tuba [13]. I.I. Faerberg [14], P.I. Perlin [15], M. Leitman and P. Villaggio [16] propose approximate approaches. The first of them proposes an approximate solution procedure based on the model of elastoplastic bending of a curved bar; the second one and the third one set out the expected shape of the elastoplastic boundary: in [15] it takes a form of an ellipse section coaxial to the hole intersecting it at the exit points of the plastic region boundary while in [16] it has a form of sections of circles with centers inside the hole on the axis with the maximum concentration of stresses that also cross the hole at the exit points of the plastic region boundary. Both last-mentioned options give an approximate description of the real configuration of the boundary between the elastic and plastic regions only at the initial development stage of the plastic region.

A widespread approximate method for solving elastoplastic problems for small elastoplastic deformations is the method of elastic solutions suggested by A.A. Ilyushin [17], which, substantially, represents a sequence of linear problems of the elasticity theory, and their solutions, with an increase in the serial number, converge to solving the problems of the plasticity theory. The method has modifications that differ in the form of additional loads and in the form of variable elasticity parameters. In the first case, the appearance of plastic deformations is taken into account by the introduction of some fictitious additional loads, in the second one, by a modification in the elastic modulus and Poisson’s ratio, which are the functions of spatial coordinates in each approximation. The modification in the position of the plastic region boundary in both modifications of the elastic solution method is not monitored. In the case of a perfect elastoplastic body, the iterations by this method increase in number unlimitedly.

The closest to the proposed method for the corrected position of the elastoplastic boundary is the perturbation method (small parameter), first used to solve elastoplastic problems by A.P. Sokolov [18] and then systematically applied by D.D. Ivlev and L.V. Ershov [6] in their monograph for a number of problems of elastoplastic strain of bodies. This method is used to consider the problems of biaxial tension of thin and thick plates with circular and elliptical holes, a space with a spherical groove and an ellipsoidal cavity, a cylindrical tube with an eccentrically located hole and a conical tube at constant internal pressure. In the described range of problems, the perturbation method is limited, mainly, by the requirement for a small difference between the studied stress state and the axisymmetric one. The iterative analytical method used for testing on Galin’s elastoplastic problem below does not contain restrictions due to the presence of a small parameter. It was previously tested on a simpler elastoplastic Lame problem [19].

2. Methods

2.1. Analytical Solution for Galin’s Elastoplastic Problem

Let’s consider Galin’s problem of biaxial tension of a plate with a hole of radius $R$ and external tensile forces $P_1$ and $P_2$ (figure 1).
From the elastic analytical solution for this problem, expressions for stresses in polar coordinates \((r, \theta)\) have the following form [20]:

\[
\sigma_r = \frac{p_1 + p_2}{2} \left( 1 - \frac{R^2}{r^2} \right) + \frac{p_2 - p_1}{2} \left( 1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4} \right) \cos 2\theta
\]

\[
\sigma_\theta = \frac{p_1 + p_2}{2} \left( 1 + \frac{R^2}{r^2} \right) - \frac{p_2 - p_1}{2} \left( 1 + \frac{3R^4}{r^4} \right) \cos 2\theta
\]

\[
\tau_r = -\frac{P_2 - P_1}{2} \left( 1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4} \right) \sin 2\theta
\]  
(1)

By applying the Tresca yield criterion for plane strain [1]:

\[
\left( \sigma_r - \sigma_\theta \right)^2 + 4\tau_r^2 = 4k^2
\]  
(2)

where \(k\) is half of the yield stress \(\sigma_y\), taking into account (1), we will find the parametric equation to determine the plasticity region boundary:

\[
(P_2 - P_1) \left( 3R^4 - 2R^2 r^2 + r^4 \right) \cos 2\theta + R^2 r^2 (P_1 + P_2) = 2kr^4
\]  
(3)

As is seen from (3), it reduces to the biquadratic equation:

\[
A_1 r^4 + A_2 r^2 + A_3 = 0
\]  
(4)

where \(A_1 = (P_2 - P_1) \cos 2\theta - 2k\), \(A_2 = R^2 \left[(P_2 + P_1) - 2(P_2 - P_1) \cos 2\theta] \right\), \(A_3 = 3R^4 (P_2 - P_1) \cos 2\theta\).

The solution to this equation, which satisfies the condition of complete coverage of the hole by the plastic region, has the following form:

\[
r = -\frac{\sqrt{-2A_1 (A_2 + \sqrt{4A_1 A_3 - A_1^2})}}{2A_1}
\]  
(5)
Let us take a closer look at the example used in [1] to determine the elastoplastic boundary. In this case, the values of external efforts are taken as equal to $P_1 = 3k$ and $P_2 = 2.4k$. Substitution of these values into expression (5) taking into account (4) shows that the plastic region boundary in the first approximation is close to an ellipse with semiaxes $1.84R$ along $X$ and $1.49R$ along $Y$.

The analytical solution to Galin’s elastoplastic problem [1] leads to the elliptical boundary of the plastic region

$$
\frac{x^2}{(1+\delta)^2} + \frac{y^2}{(1-\delta)^2} = 1
$$

where $\delta = \frac{(P_1 - P_2)}{2k}$, $X = x / r_s$, $Y = y / r_s$, $r_s = R\exp\left[\left(q - 1\right)/2\right]$, $q = \frac{(P_1 + P_2)}{2k}$.

At the selected load values, it intersects the positive axes of the Cartesian coordinates at points $3.054R$ (along $X$) and $1.64R$ – along $Y$. It is seen that the elastic solution leads to significant errors in determining the plastic region boundaries.

Let us now consider the following approximations for solving Galin’s elastoplastic problem by the iterative method. Here, the solutions in the elastic region will be constructed numerically, and analytical representations will be used for the solutions in the plastic region.

2.2. Iterative Solution of Galin’s Elastoplastic Problem by the Semi-analytical Method

Examine the solution to Galin’s elastoplastic problem by the proposed iterative method. Figure 2a. shows the initial design model for a plate with sides $a$ and $b$, hole radius $R$ and tensile external forces $P_1$ and $P_2$. Due to symmetry, it is sufficient to consider a $1/4$ part of the plate.

The numerical solution for this problem in the elastic formulation makes it possible to determine the distribution for the stress components $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ depending on the radial coordinate $r$ with the starting at the plate hole center.

By using the Tresca yield criterion (2), we will obtain the plastic region boundary in the first approximation in the form of a certain curve $f_1(X,Y)$ (in figure 2b, it is shown by a black dashed line).

For the second iteration, we examine an elastic region with an inner boundary $f_1(X,Y)$, which passes along the outer plasticity region boundary determined in the first iteration. The expressions for the stress components along this boundary can be considered known since the inner boundary of the plastic region is a hole, near which the stress state can be considered axisymmetric; for stresses in it, the following formulas are valid [21]:

\[ \text{Figure 2.} \text{ Design schemes for the iterative procedure stages.} \]
\[ \sigma_r^p = 2k \ln \left( \frac{r}{R} \right), \quad \sigma_\theta^p = 2k \left[ 1 + \ln \left( \frac{r}{R} \right) \right], \quad \tau_{r\theta}^p = 0 \quad (7) \]

Here: \( \sigma_r^p \), \( \sigma_\theta^p \), and \( \tau_{r\theta}^p \) are stresses in polar coordinates \((r, \theta)\), and \( p \) is an index applied for expressing stresses in the plastic region.

By determining stress projection (7) on the \( X, Y \) axis of the Cartesian coordinate system with the help of the transition formulas [22]:

\[ \begin{align*}
\sigma_x^p &= \sigma_r^p \cos^2 \theta + \sigma_\theta^p \sin^2 \theta \\
\sigma_y^p &= \sigma_r^p \sin^2 \theta + \sigma_\theta^p \cos^2 \theta \\
\tau_{xy}^p &= 0.5 \left( \sigma_r^p - \sigma_\theta^p \right) \sin 2\theta
\end{align*} \quad (8) \]

we will find the functions of variation for the normal \( \sigma_n \) and tangential \( \tau_n \) stress components to the elastoplastic boundary of the first approximation [22]:

\[ \begin{align*}
\sigma_n &= \sigma_x^p \cos^2 \phi + \sigma_y^p \sin^2 \phi + \tau_{xy}^p \sin 2\phi \\
\tau_n &= 0.5 \left( \sigma_x^p - \sigma_y^p \right) \sin 2\phi + \tau_{xy}^p \cos 2\phi
\end{align*} \quad (9) \]

here \( \phi \) is the angle between the normal to the \( f_1(X, Y) \) boundary and the \( X \) axis. Figure 2b presents a design model of the elastic problem for the second iteration, which, taking into account criterion (2), leads to a new boundary of the plastic region \( f_2(X, Y) \) shown in figure 2b with a red dashed line. The subsequent iterations are performed similarly to the one described until the positions of the plastic region boundary for the last two iterations coincide.

3. Results and Discussion

3.1. Implementing the Numerical and Semi-analytical Solutions of Galin’s Elastoplastic Problem

Let’s get the solution for Galin’s problem in the elastoplastic formulation by using the numerical complex ANSYS [23].

When describing the behavior of a material, we will set the following properties: yield stress \( \sigma_y = 250 \) MPa, Young’s modulus \( E = 210 \) GPa. As for the plate parameters, let us set out \( R = 1 \) mm, and \( a = b = 40 \) mm. The base sides of the numerical model adjacent to the hole (figure 2) can move freely only along their axes.

A quadrangular grid with quadratic approximation of the variables was taken to perform computation. The average length of the rib element is 0.1 mm. For external loads, the values were as follows: \( P_1 = 3k \) and \( P_2 = 2.4k \), where \( k = \sigma_y / 2 \).

Through the numerical solution of the original problem, it was found that the plastic region completely covers the hole, and its boundary practically coincides with the boundary obtained from the analytical solution while crossing the coordinate axes at points \( 3.045R \) along the X axis and \( 1.64R \) along the Y axis.

When implementing a semi-analytical iterative procedure, a gradual convergence of the approximate boundary of the plastic region with the analytical boundary was obtained. Figure 3 provides the calculation results.
Figure 3. Results of calculations to determine plasticity regions for Galin’s problem from analytical, numerical, and semi-analytical iterative solutions.

As can be seen from figure 3, the plasticity region boundary determined from the numerical solution of the elastic problem (shown by the dashed line) practically coincides with the boundary found from the analytical elastic solution (7), the first iteration of the semi-analytical method (shown by the blue solid line). Subsequently, with an increase in the iteration number, the approximate contour of the plastic region boundary gradually approaches the boundary obtained from the analytical elastoplastic solution (6) (shown by the green solid line), and already at the fourth iteration, almost complete coincidence with this boundary is observed. Its position does not vary with the successive approximations. Earlier, a similar result was obtained when testing the method on solving the elastoplastic Lame problem [19].

As a comparison, let us also consider the solution for the same problem by using the perturbation method [6]. Figure 4 suggests the results of 4 approximations for the plastic region boundary with the values $P_1 = 3k$ and $P_2 = 2.4k$ obtained by this method.
Figure 4. Calculation results obtained while determining the plasticity regions for Galin’s problem by the perturbation method.

In figure 4, the solid lines show the positions of the hole and the plastic region boundary based on the analytical solution, and the dashed lines reflect the positions of the plastic region boundary for each iteration. Despite a generally good convergence of the results based on the approximate solution by the perturbation method with the analytical ones, it can be seen that, beginning from the 3rd iteration, the position of the plastic region boundary does not improve. In such a case, the obtained curves for the boundaries at each iteration stage are described by curves that differ from ellipses and can go beyond the outer limits of the plasticity region from the analytical solution. It is also worth mentioning that the perturbation method works only for small differences while formulating the problem from the axisymmetric case and, therefore, cannot be used, for instance, to solve the Kirsch problem in the elastoplastic formulation.

In the proposed iterative method, the position of the plasticity region boundary gradually approaches the plasticity boundary found from the analytical solution and does not go beyond it. No limits on the ratio of external loads also speak of the advantage of the method, and its further application can be considered for problems in a non-axisymmetric formulation where plastic regions do not completely cover the hole.

4. Conclusions
Based on Galin’s model problem in an elastoplastic formulation, this work suggests an iterative method of a semi-analytical solution, when explicit analytical expression for the stress components are applied in the plastic region, and a numerical solution is found in the elastic region. The convergence of the iterative procedure is numerically estimated. The elastoplastic region boundary and the values of the normal and tangential stress components on it are determined at each stage from the solution of the elastic problem. It is shown that the convergence of the method for finding the plastic region boundary to the analytical solution requires no more than 4 iterations; in this case, the boundary of the obtained plasticity region completely fits into the analytical elliptical boundary at each iteration.
A similar result in this problem is achieved using alternative perturbation method. In this case, at the 4th iteration, the plasticity zone has a maximum deviation from the plasticity boundary found from the analytical solution of about 1.4 percent. And its applicability is limited by the requirement of small
difference while formulating the problem from the axisymmetric case and, therefore, cannot be used to solve such problems as the elastoplastic Kirsch problem.

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