Divergence form nonlinear nonsmooth elliptic equations with locally arbitrary growth conditions and nonlinear maximal regularity

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Abstract

This is a simplification of our prior work on the existence theory for the Rosseland-type equations. Inspired by the Rosseland equation in the conduction-radiation coupled heat transfer, we use the locally arbitrary growth conditions instead of the common global restricted growth conditions. In the Lebesgue square integrable space, the solution to the linear elliptic equation depends continuously on the coefficients matrix. This is a simple version of the maximal regularity. There exists a fixed point for the linearized map (compact and continuous) in a closed convex set.

Key words: growth conditions; nonlinear maximal regularity; nonlinear elliptic equations; nonsmooth data; Rosseland equation;

1 Introduction

Consider the following elliptic problem: find \( u, (u-u_b) \in H^1_0(\Omega) \), such that
\[
- \text{div}[A(u(x), x) \nabla u] = 0, \quad \text{in } \Omega. \tag{1.1}
\]

For the Rosseland equation: \( A(z, x) = K(x) + z^3 B(x) \), where \( K(x) \) and \( B(x) \) are symmetric and positive definite.

1. \( K(x) + z^3 B(x) \) is positive definite only in an interval for \( z \).
2. it doesn’t satisfy the common growth and smooth conditions and there may be no \( C^{2,\gamma} \) estimate (Theorem 15.11 [1]).

The problem of the existence theory for the Rosseland equation (also named diffusion approximation) was proposed by Laitinen [6] in 2002. It may be useful to keep this equation in mind while reading this paper.

It’s a little technical to prove the existence by the fixed point method in \( L^\infty(\Omega) \) [7, 8]. We will use \( L^2(\Omega) \) in this paper.

Firstly, we make the following assumptions.
\((A1) \ \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain.
(A2) \( A = (a_{ij}). a_{ij} = a_{ji}. T_{\text{min}} \leq T_{\text{max}} \) are two constants.

\[ \lambda|\xi|^2 \leq a_{ij}(z, x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad 0 < \lambda \leq \Lambda, \quad \forall (z, x, \xi) \in [T_{\text{min}}, T_{\text{max}}] \times \Omega \times \mathbb{R}^n. \quad (1.2) \]

(A3)

\[ u_b \in H^1(\Omega). \quad T_{\text{min}} \leq u_b(x) \leq T_{\text{max}}, \quad \text{a.e. in } \partial\Omega. \quad (1.4) \]

(A4) \( A(z, x) \) is uniformly continuous with respect to \( z \) in \( C \), where

\[ C = \{ \varphi \in L^2(\Omega); \ T_{\text{min}} \leq \varphi(x) \leq T_{\text{max}}, \ \text{a.e. in } \Omega \}. \quad (1.5) \]

Remark 1.1 In fact, we had considered a general case: parabolic equations with bounded mixed boundary conditions and nonnegative bounded right-hand term \( f(z, x) \) in \([8]\).

If \( a_{pq} \) is uniformly Hölder continuous with respect to \( z \), (A4) is natural since

\[ \|a_{pq}(z_i(x), x) - a_{pq}(z(x), x)\|_2 \rightarrow 0. \quad (1.6) \]

2 Linearized map and fixed point

Theorem 2.1 (Corollary 11.2 [1]) Let \( \mathcal{C} \) be a closed convex set in a Banach space \( \mathcal{B} \) and let \( \mathcal{L} \) be a continuous mapping of \( \mathcal{C} \) into itself such that the image \( \mathcal{L}\mathcal{C} \) is precompact. Then \( \mathcal{L} \) has a fixed point.

Lemma 2.1 The following set

\[ \mathcal{C} = \{ \varphi \in L^2(\Omega); \ T_{\text{min}} \leq \varphi(x) \leq T_{\text{max}}, \ \text{a.e. in } \Omega \} \quad (2.8) \]

is a closed convex set in the Banach space \( L^2(\Omega) \).

Proof Suppose \( u_i \in \mathcal{C}, v \in L^2(\Omega), \|v_i - v\|_2 \rightarrow 0. \) If \( v \notin \mathcal{C} \), there exist two constants \( \delta_0 > 0, \delta_1 > 0 \), such that the Lebesgue measure of the set \( \Omega_0 \equiv \{ x \in \Omega; v(x) \geq T_{\text{max}} + \delta_0 \} \) is bigger than \( \delta_1 > 0 \). Then

\[ \|v_i - v\|^2 = \int_\Omega |v_i - v|^2 \geq \int_{\Omega_0} |v_i - v|^2 \geq \delta_0^2 \delta_1. \quad (2.9) \]

It’s impossible since \( \|v_i - v\|_2 \rightarrow 0. \) Similarly, \( v \geq T_{\text{min}} \) and \( \mathcal{C} \) is closed.

\[ \forall \theta \in [0, 1], \quad \theta v_1 + (1 - \theta)v_2 \leq \theta T_{\text{max}} + (1 - \theta)T_{\text{max}} = T_{\text{max}}. \quad (2.10) \]

So \( \mathcal{C} \) is convex. \( \square \)
**Theorem 2.2** If (A1) – (A4) are satisfied, then

(1) \( \forall z \in \mathcal{C} \), the following equation has a unique solution \( w \in \mathcal{C} \), \( (w - u_0) \in H_0^1(\Omega) \).

\[
\int_{\Omega} A(z(x), x)\nabla w \cdot \nabla \varphi = 0, \quad \forall \varphi \in H_0^1(\Omega). \tag{2.11}
\]

(2) Define a map \( \mathcal{L} : \mathcal{C} \to \mathcal{C} \), \( \mathcal{L} z = w \), then \( \mathcal{L} \mathcal{C} \) is precompact in \( L^2(\Omega) \).

(3) \( \mathcal{L} \) is continuous in \( L^2(\Omega) \). So \( \mathcal{L} \) has a fixed point in \( \mathcal{C} \).

**Proof**

(1) Let \( v = (w - u_0) \in H_0^1(\Omega) \), then we have

\[
\int_{\Omega} A(z(x), x)\nabla v \cdot \nabla \varphi = -\int_{\Omega} A(z(x), x)\nabla u_0 \cdot \nabla \varphi, \quad \forall \varphi \in H_0^1(\Omega). \tag{2.12}
\]

From (A2), if \( z \in \mathcal{C} \), \( A(z(x), x) \in [\lambda, A] \). From (A3), \( |\nabla u_0| \leq C \). Let \( \varphi = v \),

\[
\lambda |v|^2_1 \leq \int_{\Omega} A\nabla v \cdot \nabla v = -\int_{\Omega} A\nabla u_0 \cdot \nabla v
\leq \|A\nabla u_0\|_2 \|\nabla v\|_2 = (\int_{\Omega} A^T A\nabla u_0 \cdot \nabla u_0)^{1/2} \|\nabla v\|_2
\leq \Lambda \|u_0\|_{H^1(\Omega)} |v|_1. \tag{2.13}
\]

Using the well-known Lax-Milgram Lemma, there exists a unique solution \( v \in H_0^1(\Omega) \) to this equation. Using the Poincaré inequality,

\[
\|w\|_{H^1(\Omega)} \leq \|w - u_0\|_{H^1(\Omega)} + \|u_0\|_{H^1(\Omega)}
\leq C(\Omega) \frac{\Lambda |u_0|_{H^1(\Omega)}}{\lambda} + \|u_0\|_{H^1(\Omega)}. \tag{2.14}
\]

Using the maximum principle (Theorem 8.1 [1]), \( u \in \mathcal{C} \). In fact, we can let \( \varphi = (w - T_{\max})_+ \in H_0^1(\Omega) \),

\[
C(\Omega) \lambda \|(w - T_{\max})_+\|_1^2 \leq \lambda \|(w - T_{\max})_+\|_1^2
\leq \int_{\Omega} A(z(x), x) \nabla (w - T_{\max})_+ \cdot \nabla (w - T_{\max})_+
\leq \int_{\Omega} A(z(x), x) \nabla w - \nabla (w - T_{\max})_+ = 0. \tag{2.15}
\]

So \( (w - T_{\max})_+ = 0 \), \( w \leq T_{\max} \). In the same way, \( w \leq T_{\min} \).

(2) \( \|w\|_{H^1(\Omega)} \leq C \). \( \mathcal{L} \mathcal{C} \) is bounded in \( H^1(\Omega) \). From the Rellich Theorem, \( \mathcal{L} \mathcal{C} \) is precompact in \( L^2(\Omega) \).

(3) Suppose

\[
z_i, z \in \mathcal{C}, \quad \|z_i - z\|_2 \to 0, \quad \mathcal{L} z_i = w_i, \quad \mathcal{L} z = w. \tag{2.16}
\]

\( H_0^1(\Omega) \) is a Hilbert and then a reflexive space, so there exists a subsequence \( \{i_k\} \) and \( v_0 = (w_0 - u_0) \in H_0^1(\Omega) \) such that

\[
(w_{i_k} - u_k) \to (w_0 - u_0), \quad \text{weakly in } H_0^1(\Omega). \tag{2.17}
\]

\[
H_0^1(\Omega) \subset L^2(\Omega), \quad (L^2(\Omega))^\prime \subset (H_0^1(\Omega))^\prime. \tag{2.18}
\]
\( (w_{ik} - w_0) \to (w_0 - u_b), \) weakly in \( L^2(\Omega). \) \hfill (2.19)
\{w_{ik} - u_b\} is bounded in \( H^1(\Omega), \) so there exists a subsequence \( \{i_m\} \subset \{i_k\} \) and \( v_* \in L^2(\Omega) \) such that
\( (w_{im} - u_b) \to v_*, \) strongly in \( L^2(\Omega). \) \hfill (2.20)
\( (w_{im} - u_b) \to v_0, \) weakly in \( L^2(\Omega). \) \hfill (2.21)
So \( v_* = v_0. \) Since each subsequence of \( \{\|w_{ik} - u_b - v_0\|_2\} \) has a subsequence which converges to 0, \( \|w_{ik} - u_b - v_0\|_2 \to 0, \|w_{ik} - w_0\|_2 \to 0. \)

Since
\( (w_{ik} - u_b) \to (w_0 - u_b), \) weakly in \( H^1_0(\Omega). \) \hfill (2.22)
\( \forall \bar{\psi} \in L^2(\Omega; \mathbb{R}^n), \) \( \langle \bar{\psi}, h \rangle_{H^1_0(\Omega)} \equiv \int_\Omega \nabla h \cdot \bar{\psi}, \) \( \forall h \in H^1_0(\Omega), \) \hfill (2.23)
is a bounded linear functional.
\[ \int_\Omega \nabla (w_{ik} - u_b) \cdot \bar{\psi} \to \int_\Omega \nabla (w_0 - u_b) \cdot \bar{\psi}. \] \hfill (2.24)
\[ \forall \bar{\psi} \in L^2(\Omega; \mathbb{R}^n), \int_\Omega \nabla w_{ik} \cdot \bar{\psi} \to \int_\Omega \nabla w_0 \cdot \bar{\psi}. \] \hfill (2.25)

From the Riesz representation theorem, the dual space \( (L^2(\Omega; \mathbb{R}^n))' \approx L^2(\Omega; \mathbb{R}^n), \)
\[ \nabla w_{ik} \to \nabla w_0, \] weakly in \( L^2(\Omega; \mathbb{R}^n). \) \hfill (2.26)
From (A4) and \( \|z_i - z\|_2 \to 0, \)
\[ \sup_{1 \leq p, q \leq n} \|a_{pq}(z_{ik}(x), x) - a_{pq}(z(x), x)\|_2 \to 0. \] \hfill (2.28)

We can conclude that, \( \forall \phi \in C_0^\infty(\Omega), \)
\[ |\int_\Omega [A(z_{ik}(x), x)\nabla w_{ik} - A(z(x), x)\nabla w_0] \cdot \nabla \phi| \leq |\int_\Omega [A(z_{ik}(x), x)\nabla w_{ik} - A(z(x), x)\nabla w_{ik}] \cdot \nabla \phi| \]
\[ + |\int_\Omega [A(z(x), x)\nabla w_{ik} - A(z(x), x)\nabla w_0] \cdot \nabla \phi| \]
\[ = |\int_\Omega [A(z_{ik}(x), x) - A(z(x), x)]\nabla w_{ik} \cdot \nabla \phi| \]
\[ + |\int_\Omega [\nabla w_{ik} - \nabla w_0] \cdot A(z(x), x) \nabla \phi| \]
\[ \leq C \sup_{1 \leq p, q \leq n} \|a_{pq}(z_{ik}(x), x) - a_{pq}(z(x), x)\|_2 + \epsilon(i_k) \to 0. \] \hfill (2.29)
\[ \int_\Omega A(z_{ik}(x), x)\nabla w_{ik} \cdot \nabla \phi = 0, \]
\[ \int_\Omega A(z(x), x)\nabla w_0 \cdot \nabla \phi = 0. \] \hfill (2.30)
\[ \int_\Omega A(z(x), x)\nabla w_0 \cdot \nabla \varphi = 0, \] \( \forall \varphi \in H^1_0(\Omega). \) \hfill (2.31)

Since the solution is unique from the step (1), \( w_0 = \mathcal{L}z = w. \) So \( \|w_{ik} - w\|_2 \to 0. \) Each subsequence of \( \{\|w_i - w\|_2\} \) has a sub-subsequence which converges to 0, so \( \|w_i - w\|_2 \to 0. \) We have obtain the continuity of \( \mathcal{L}. \)

From Theorem 2.1, there exists a fixed point. \( \square \)
Remark 2.1 For the continuity of $\mathcal{L}$ in $C^0(\bar{\Omega})$, we can use the well-known De Giorgi-Nash estimate: $\{w_i\}$ is bounded in $C^{0,\alpha}(\bar{\Omega})$ if $u_b \in C^{0,\alpha}(\partial \Omega)$ and $\Omega$ satisfies a uniform exterior cone condition (Theorem 8.29 [1]).

Then from the Arzel`a-Ascoli Lemma, $\|w_{ik} - w_0\|_{C^0(\bar{\Omega})} \to 0$. By the same method, $w_0 = w$ and $\|w_i - w\|_{C^0(\bar{\Omega})} \to 0$.

From the linear maximal regularity [4], a natural conjecture is: $\mathcal{L}$ is continuous in $C^{0,\alpha}(\bar{\Omega})$ and $H^1(\Omega)$.

Definition 2.1

$$\mathcal{C}_\infty = \{ \varphi \in L^\infty(\Omega); \ T_{\min} \leq \varphi(x) \leq T_{\max}, \ a.e. \ in \ \Omega \}$$

is a closed convex set in the Banach space $L^\infty(\Omega)$.

We replace (A4) in $L^2$ with (A4') in $L^\infty$:

$$\|A(z_i(x), x) - A(z(x), x)\|_{C^0(\bar{\Omega})} \to 0, \ if \ |z_i(x) - z(x)|_{\infty} \to 0. \quad (2.33)$$

Corollary 2.1 If (A1) - (A3), (A4') are satisfied, define a map $\mathcal{L} : \mathcal{C}_\infty \to \mathcal{C}_\infty$, $\mathcal{L}z = w :$ such that $w \in \mathcal{C}_\infty$, $(w - u_b) \in H^1_0(\Omega)$ and

$$\int_{\Omega} A(z(x), x) \nabla w \cdot \nabla \varphi = 0, \ \forall \varphi \in H^1_0(\Omega). \quad (2.34)$$

Then $\mathcal{L}$ is continuous in $H^1(\Omega)$.

Proof Suppose

$$z_i, z \in \mathcal{C}_\infty, \ |z_i - z|_{\infty} \to 0, \ \mathcal{L}z_i = w_i, \ \mathcal{L}z = w. \quad (2.35)$$

1. $\{w_i\}$ is bounded in $H^1(\Omega)$. For any $\tilde{\varphi} \in L^2(\Omega; \mathbb{R}^n)$, each subsequence of $\int_{\Omega} \nabla(w_i - w) \cdot \tilde{\varphi}$ has a sub-subsequence converges to 0. So $\nabla w_i \to \nabla w$ weakly in $L^2(\Omega; \mathbb{R}^n)$. $\|A(z_i(x), x) - A(z(x), x)\|_{\infty} \to 0$,

$$\nabla w_0 \in L^2(\Omega; \mathbb{R}^n), \ \int_{\Omega} A(z_i) \nabla w_i \cdot \nabla w_0 \to \int_{\Omega} A(z) \nabla w \cdot \nabla w_0. \quad (2.36)$$

$$\int_{\Omega} A(z_i) \nabla w_i \cdot \nabla w_0 = \int_{\Omega} A(z_i) \nabla w_i \cdot \nabla w_i, \quad (2.37)$$

$$\int_{\Omega} A(z) \nabla w \cdot \nabla w_0 = \int_{\Omega} A(z) \nabla w \cdot \nabla w. \quad (2.38)$$

$$\int_{\Omega} A(z_i) \nabla w_i \cdot \nabla w_i \to \int_{\Omega} A(z) \nabla w \cdot \nabla w. \quad (2.39)$$

$$\int_{\Omega} [A(z_i) - A(z)] \nabla w_i \cdot \nabla w_i \leq C \|A(z_i(x), x) - A(z(x), x)\|_{\infty} \to 0. \quad (2.40)$$

$$\int_{\Omega} A(z) [\nabla w \cdot \nabla w - \nabla w_i \cdot \nabla w_i] \to 0. \quad (2.41)$$

$$\int_{\Omega} |\nabla w_i - \nabla w|^2$$

$$\leq \frac{1}{\lambda} \int_{\Omega} A(z) [\nabla w_i - \nabla w] \cdot [\nabla w_i - \nabla w]$$

$$= \frac{1}{\lambda} \int_{\Omega} A(z) [\nabla w \cdot \nabla w + \nabla w_i \cdot \nabla w_i - 2 \nabla w \cdot \nabla w_i$$

$$\to 0. \quad (2.42)$$

$(w_i - w) \in H^1_0(\Omega)$, then use the Poincaré inequality, $\|w_i - w\|_{H^1} \to 0$. □
3 Nonlinear maximal regularity

For the linear parabolic/elliptic equations with nonsmooth data, the theory of maximal regularity has been established [2, 3, 4, 5]. In brief, maximal regularity is about the smoothness of the data-to-solution-map [5]. This smooth dependence has its physical meaning: many physical processes are stable with respect to the parameters (except the chaos and critical theory). For the mathematicians, "the door is open to apply the powerful theorems of differential calculus" ([5], e.g. the Implicit Function Theorem).

In the following, we will discuss the continuous dependence (between the solutions and the data) for the following kind of nonlinear equations: find \( u, (u - u_b) \in H^1_0(\Omega) \), such that

\[
\int_{\Omega} A(u(x), x) \nabla u \cdot \nabla \varphi = 0, \quad \forall \varphi \in H^1_0(\Omega). \tag{3.43}
\]

In this section, we strengthen (A4) with the following equicontinuous conditions.

(A4.1) Each element of \( \mathcal{A}(z, x) \) satisfies (A2):

\[
A(z, x)|_{x \in \Omega} \in [\lambda, \Lambda], \quad \forall A(z, x) \in \mathcal{A}(z, x). \tag{3.44}
\]

\[
\mathcal{C} = \{ \varphi \in L^2(\Omega); T_{\text{min}} \leq \varphi(x) \leq T_{\text{max}}, \text{ a.e. in } \Omega \}. \tag{3.45}
\]

(A4.2) \( \mathcal{A}(z, x) \) is equicontinuous with respect to \( z \) in \( \mathcal{C} \) (uniformly for \( x \)). In other words, for any \( z_i, z \in \mathcal{C} \), if \( \|z_i - z\|_2 \to 0 \),

\[
\sup_{A = (a_{pq}) \in \mathcal{A}(z, x)} \|a_{pq}(z_i(x), x) - a_{pq}(z(x), x)\|_2 \leq \epsilon(i). \tag{3.46}
\]

\( \epsilon(i) \) denotes a higher-order infinitesimal which depends only on \( i \).

Theorem 3.1 If (A1) – (A3), (A4.1) – (A4.2) are satisfied and the solution to the nonlinear equation is unique, then

1. \( u \) depends continuously (in \( L^2 \) or \( C^0(\overline{\Omega}) \)) on the coefficients matrix \( A(\cdot, x) \) in \( \mathcal{A}(z, x) \).
2. \( u \) depends continuously (in \( L^2 \) or \( C^0(\overline{\Omega}) \)) on the boundary value \( u_b \).
3. \( A(z, x) \nabla z \) is a continuous (with respect to \( z \) in \( \mathcal{C} \cap H^1(\Omega) \)) functional in \( (C^{0,1}(\overline{\Omega}))' \).

Proof (1) Suppose \( A_i(\cdot, x) \in \mathcal{A}(\cdot, x) \), then there exists a \( u_i, (u_i - u_b) \in H^1_0(\Omega) \), such that

\[
\int_{\Omega} A_i(u_i(x), x) \nabla u_i \cdot \nabla \varphi = 0, \quad \forall \varphi \in H^1_0(\Omega). \tag{3.47}
\]

We will prove that \( \|u - u_i\|_2 \to 0 \) (or \( \|u - u_i\|_{C^0(\overline{\Omega})} \to 0 \)) if \( \|A(z(x), x) - A_i(z(x), x)\|_2 \to 0 \) for any \( z(x) \in \mathcal{C} \). \{\( u_i \)\} is bounded in \( H^1(\Omega) \) (or in \( C^{0,1}(\overline{\Omega}) \)). So \( u_{i_k} \to u_0 \), strongly in \( L^2(\Omega) \) (or in \( C^{0,1}(\overline{\Omega}) \)); \( \nabla u_{i_k} \to \nabla u \), weakly in \( L^2(\Omega; \mathbb{R}^n) \).

\[
\|A_{i_k}(u_{i_k}(x), x) - A(u_0(x), x)\|_2 \\
\leq \|A_{i_k}(u_{i_k}(x), x) - A_{i_k}(u_0(x), x)\|_2 + \|A_{i_k}(u_0(x), x) - A(u_0(x), x)\|_2 \\
\to 0. \tag{3.48}
\]
∀ \phi \in C^{0,\infty}(\Omega),

\begin{equation}
0 = \int_{\Omega} A_{i_k}(u_{i_k}(x), x) \nabla u_{i_k} \cdot \nabla \phi \rightarrow \int_{\Omega} A(u_0(x), x) \nabla u_0 \cdot \nabla \phi. \tag{3.49}
\end{equation}

Because of the density,

\begin{equation}
\int_{\Omega} A(u_0(x), x) \nabla u_0 \cdot \nabla \varphi = 0, \quad \forall \varphi \in H^1_0(\Omega). \tag{3.50}
\end{equation}

Since the solution is unique, so \(w_i \rightarrow w_0\), strongly in \(L^2(\Omega)\) (or in \(C_0(\Omega)\)).

(2) Let \(u_{bi}, u_{b0} \in H^1(\Omega), \|u_{bi} - u_{b0}\|_{H^1(\Omega)} \rightarrow 0\). \(w_i = u_i - u_{bi} \in H^1_0(\Omega)\),

\begin{equation}
\int_{\Omega} A(w_i + u_{bi}) \nabla w_i \cdot \nabla \varphi = -\int_{\Omega} A(w_i + u_{bi}) \nabla u_{bi} \cdot \nabla \varphi. \tag{3.51}
\end{equation}

\(\{w_i\}\) is bounded in \(H^1(\Omega)\) (or in \(C^{0,\infty}(\Omega)\)). So \(w_i \rightarrow w_0\), strongly in \(L^2(\Omega)\) (or in \(C^0(\Omega)\)); \(\nabla w_i \rightarrow \nabla w_0\), weakly in \(L^2(\Omega; \mathbb{R}^n)\). By the same method in (1), \(\forall \phi \in C^{0,\infty}(\Omega), \forall \varphi \in H^1_0(\Omega), \forall \eta \in C^{0,1}(\Omega)\),

\begin{equation}
\int_{\Omega} A(w_0 + u_{b0}) \nabla w_0 \cdot \nabla \varphi = -\int_{\Omega} A(w_0 + u_{b0}) \nabla u_{b0} \cdot \nabla \phi. \tag{3.52}
\end{equation}

So \(w_i \rightarrow w_0\), strongly in \(L^2(\Omega)\).

(3) For any \(\eta \in C^{0,1}(\Omega), \forall \varphi \in H^1_0(\Omega)\),

\begin{equation}
|\langle A(z, x) \nabla z, \eta \rangle| = |\int_{\Omega} A(z, x) \nabla z \cdot \nabla \eta| \leq \|\eta\|_{C^{0,1}(\Omega)} \int_{\Omega} |A(z, x) \nabla z| \leq C\|\eta\|_{C^{0,1}(\Omega)} \tag{3.53}
\end{equation}

For any \(z \in C \cap H^1(\Omega), \langle A(z, x) \nabla z, \eta \rangle \) is a linear continuous functional in \((C^{0,1}(\Omega))'\).

Suppose \(z_i \rightarrow z \in H^1(\Omega), \forall \eta \in C^{0,1}(\Omega)\),

\begin{equation}
|\langle A(z_i, x) \nabla z_i - A(z, x) \nabla z, \eta \rangle| \leq \|\eta\|_{C^{0,1}(\Omega)} \int_{\Omega} |A(z_i, x) \nabla z_i - A(z, x) \nabla z| \rightarrow 0. \tag{3.54}
\end{equation}

\(\|A(z_i, x) \nabla z_i - A(z, x) \nabla z\|_{C^{0,1}(\Omega)'} \rightarrow 0. \square
\)

Remark 3.1 We can consider the continuous dependence in \(H^1(\Omega)\).

4 Existence for Galerkin method

Let \(h \in (0, 1)\) be the step size, \(\{\phi_{i,h}\}\) is a kind of finite element basis in \(H^1_0(\Omega)\).
∀ \( z_h = (\sum_i z_{i,h} \phi_{i,h} + u_b) \in C \), the following equation has a unique solution \( w_h = \sum_i w_{i,h} \phi_{i,h}, (w_h + u_b) \in C \cap H^1_0(\Omega) \).

\[
\int_{\Omega} A(\sum_i z_{i,h} \phi_{i,h} + u_b) \nabla(\sum_i w_{i,h} \phi_{i,h}) \cdot \nabla \phi_{j,h} = - \int_{\Omega} A(\sum_i z_{i,h} \phi_{i,h} + u_b) \nabla u_b \cdot \nabla \phi_{j,h}, \quad \forall \phi_{j,h}. \quad (4.55)
\]

Define \( Lz_h = (w_h + u_b) \), then

1. \( L \subset C \).
2. \( L \) is compact.
3. \( L \) is continuous.
4. There exists a fixed point \( u_h = \sum_i w_{i,h} \phi_{i,h} \) and \( u_h \to u \).

5 Acknowledge

This work is supported by the National Nature Science Foundation of China (No. 90916027). This is a part of my PhD thesis [8] in AMSS, Chinese Academy of Sciences, and a simplification of our prior paper [7]. So I will thank my advisor Professor Jun-zhi Cui (he is also a member of the Chinese Academy of Engineering) and the referees for their careful reading and helpful comments. My E-mail is: zhangqf@lsec.cc.ac.cn.

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