Sampling in $\Lambda$-shift-invariant subspaces of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$

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Abstract

The translation of an operator is defined by using conjugation with time-frequency shifts. Thus, one can define $\Lambda$-shift-invariant subspaces of Hilbert-Schmidt operators, finitely generated, with respect to a lattice $\Lambda$ in $\mathbb{R}^{2d}$. These spaces can be seen as a generalization of classical shift-invariant subspaces of square integrable functions. Obtaining sampling results for these subspaces appears as a natural question that can be motivated by the problem of channel estimation in wireless communications. These sampling results are obtained in the light of the frame theory in a separable Hilbert space.

Keywords: Hilbert-Schmidt operators; Weyl transform; Kohn-Nirenberg transform; Translation of operators; $\Lambda$-shift-invariant subspaces; Sampling Hilbert-Schmidt operators.

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1 Introduction

In this paper we obtain sampling results in shift-invariant-like subspaces of the class $\mathcal{HS}(\mathbb{R}^d)$ of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$. To be more precise, these subspaces are obtained by translation in a lattice $\Lambda \subset \mathbb{R}^{2d}$ of a fixed set of Hilbert-Schmidt operators $S_1, S_2, \ldots, S_N$. The translation of an operator $S$ by $z \in \mathbb{R}^{2d}$ is defined by using conjugation with the time-frequency shift $\pi(z)$, where $z = (x, \omega)$ belongs to the phase space $\mathbb{R}^d \times \hat{\mathbb{R}}^d$ (which in the sequel will be identified with $\mathbb{R}^{2d}$) by

$$\alpha_z(S) := \pi(z) S \pi(z)^*, \quad z \in \mathbb{R}^{2d}.$$  

Recall that the time-frequency shift acts on $f \in L^2(\mathbb{R}^d)$ as $\pi(z)f(t) = e^{2\pi i \omega \cdot t} f(t-x)$. The set of translations $\{\alpha_z\}_{z \in \mathbb{R}^{2d}}$ is a unitary representation of the group $\mathbb{R}^{2d}$ on the Hilbert space $\mathcal{HS}(\mathbb{R}^d)$.

If we take a full rank lattice $\Lambda$ in $\mathbb{R}^{2d}$, i.e., $\Lambda = A\mathbb{Z}^d$ where $A$ is a $2d \times 2d$ real invertible matrix, such that the sequence $\{\alpha_{1}(S_{n})\}_{\lambda \in \Lambda}$ is a Riesz sequence in $\mathcal{HS}(\mathbb{R}^d)$ we consider the subspace of $\mathcal{HS}(\mathbb{R}^d)$ given by

$$V^2_S = \left\{ \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda) \alpha_{\lambda}(S_{n}) : \{c_n(\lambda)\}_{\lambda \in \Lambda} \in \ell^2(\Lambda), \, n = 1, 2, \ldots, N \right\}.$$  

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From now on, the subspaces $V^2_S$ obtained in this way will be called \textit{$\Lambda$-shift-invariant subspaces} in $\mathcal{HS}(\mathbb{R}^d)$. These spaces are a generalization of the classical shift-invariant subspaces in $L^2(\mathbb{R}^d)$:

$$V^2_S := \left\{ \sum_{n=1}^{N} \sum_{\alpha \in \mathbb{Z}^d} c_n(\alpha) \varphi_n(t - \alpha) : \{c_n(\alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d), \ n = 1, 2, \ldots, N \right\},$$

where $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ denotes a set of generators of $V^2_S$. Sampling in the shift-invariant subspace $V^2_S$ usually involves, for each $f \in V^2_S$, pointwise samples $\{f(\alpha + \beta_m)\}_{\alpha \in \mathbb{Z}^d}$ and/or average samples $\{(f, \psi_m(\cdot - \alpha))\}_{\alpha \in \mathbb{Z}^d}$, where $\psi_m$ is an \textit{average function} in $L^2(\mathbb{R}^d)$, which not necessarily belong to $V^2_S$. Any stable sampling in $V^2_S$ will involve, necessarily, $M \geq N$ sequences of samples (see, for instance, Refs. \cite{6, 12} and references therein).

A challenge problem here is to choose an appropriate set of samples that should be used for operators in $V^2_S$. Inspired in Ref. \cite{18} and motivated by the problem of channel estimation in wireless communications, in this paper we propose for any $T \in V^2_S$ its \textit{diagonal channel samples} at the lattice $\Lambda \subset \mathbb{R}^{2d}$ defined by

$$s_{T,m}(\lambda) := \langle \alpha_{-\lambda}(T)g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)}, \ \lambda \in \Lambda, \ m = 1, 2, \ldots, M, \ (1)$$

where $g_m, \tilde{g}_m, m = 1, 2, \ldots, M$, are $2M$ fixed functions in $L^2(\mathbb{R}^d)$ (we will see that necessarily $M \geq N$). The name \textit{diagonal channel samples} coined for these samples will become clear later on where a little explanation will be done for both, the choice of Hilbert-Schmidt operators (in $V^2_S$) to be sampled, and the choice of the above samples for any $T \in V^2_S$. As we will see in Section 3.3, the samples defined in (1) are nothing but the \textit{lower symbol} of the operator $T$ with respect $g_m, \tilde{g}_m \in L^2(\mathbb{R}^d)$ and lattice $\Lambda$, i.e., $\langle T \pi(\lambda)g_m, \pi(\lambda)\tilde{g}_m \rangle_{L^2(\mathbb{R}^d)}, \ \lambda \in \Lambda$, or the samples of the Berezin transform $B^{g_m, \tilde{g}_m} T(z) := \langle T \pi(z)g_m, \pi(z)\tilde{g}_m \rangle_{L^2(\mathbb{R}^d)}, \ z \in \mathbb{R}^{2d}$, at the lattice $\Lambda$ (see Ref. \cite{21}). These samples are also a particular case of the \textit{average samples} $\langle T, \alpha_{\lambda}(Q_m) \rangle_{\mathcal{HS}}$ where the \textit{average operator} $Q_m$ is the rank-one operator $g_m \otimes g_m$; average sampling has been used previously in Refs. \cite{6, 12}.

The main aim here is the stable recovery of any $T \in V^2_S$ from its samples (1) by means of a sampling formula in $V^2_S$ having the form

$$T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} s_{T,m}(\lambda) \alpha_{\lambda}(H_m) \text{ in } \mathcal{HS}-\text{norm},$$

for each $T \in V^2_S$. The operators $H_m, m = 1, 2, \ldots, M$, above belong to $V^2_S$ and satisfy that the sequence $\{\alpha_{\lambda}(H_m)\}_{\lambda \in \Lambda; \ m=1, 2, \ldots, M}$ is a \textit{frame} for the Hilbert space $V^2_S$.

For sampling in classical shift-invariant spaces see, for instance, Refs. \cite{1, 13, 14} and references therein. See also Ref. \cite{3} for the case where other unitary representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ is used instead of the classical one given by translations. For the less known topic on sampling operators, see Refs. \cite{6, 12, 18, 20, 22, 23}.

The used techniques in this work are those of the frame theory in a separable Hilbert space. To be precise, the samples used along this paper will be expressed as a discrete convolution system in the product Hilbert space $\ell^2_N(\Lambda) := \ell^2(\Lambda) \times \cdots \times \ell^2(\Lambda)$ ($N$ times),
and then it will be used the close relationship between a discrete convolution system and a sequence of translates in $\ell^2_\Lambda (\Lambda)$ (see, for instance, Ref. [15]). The other involved tools are the Kohn-Nirenberg transform or the Weyl transform for Hilbert-Schmidt operators: both are unitary operators from $L^2(\mathbb{R}^d)$ onto $\mathcal{HS}(\mathbb{R}^d)$ which respect the translations in the sense that, if we denote any of them by $\mathcal{L}$, we have $\mathcal{L}(T_z f) = \alpha_z (\mathcal{L} f)$ for $f \in L^2(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$.

Now we briefly explain a practical motivation for considering the samples defined in Eq. (1) for the elements in $\mathbb{V}^2$. It is a well-known fact in mobile wireless channels that the relative location between transmitter and receiver is varying with time and consequently the input-output relation is modeled by a time-varying system $x \mapsto H x$ that can be expressed as the integral operator

$$H x(t) = \int_{\mathbb{R}^d} h_t(s) x(t-s) \, ds = \int_{\mathbb{R}^d} \sigma(t,\omega) \tilde{x}(\omega) e^{2\pi i \omega t} d\omega,$$

where $\sigma(t,\omega) = \mathcal{F}(h_t)(\omega)$, i.e., the Fourier transform with respect to the last $d$ variables in $h(t, s) := h_t(s)$. In this last formulation, operator $H$ becomes a pseudodifferential operator with Kohn-Nirenberg symbol $\sigma$ (see, for instance, Refs. [16, 25]).

As it was pointed out in Ref. [18], in orthogonal frequency-division multiplexing (OFDM) the digital information, i.e., a sequence of numbers $\{c_\lambda\}$, $\lambda$ in the lattice $\Lambda = a \mathbb{Z}^d \times b \mathbb{Z}^d$ ($a, b > 0$), is used as the coefficients of the input signal $x(t) = \sum_{\mu \in \Lambda} c_\mu \pi(\lambda) g(t)$ of a time-varying system $H$ producing the output $y(t) = H x(t)$. Then, the sequence of numbers

$$d_\lambda = \langle y, \pi(\lambda) \tilde{g} \rangle_{L^2(\mathbb{R}^d)} = \sum_{\mu \in \Lambda} c_\mu \langle H \pi(\mu) g, \pi(\lambda) \tilde{g} \rangle_{L^2(\mathbb{R}^d)}, \quad \lambda \in \Lambda,$$

is considered. The main task of the engineer is to recover the original data $\{c_\lambda\}$ from the received data $\{d_\lambda\}$. The matrix $A = [a_{\lambda,\mu}]$, where $a_{\lambda,\mu} = \langle H \pi(\mu) g, \pi(\lambda) \tilde{g} \rangle_{L^2(\mathbb{R}^d)}$, which appears in Eq. (2), involving $H$ and the time-frequency shifts of a pair of fixed functions $g, \tilde{g} \in L^2(\mathbb{R}^d)$, is the so-called channel matrix associated with $H$ and the functions $g, \tilde{g}$ in $L^2(\mathbb{R}^d)$. As it will be proved in Section 3.3 (see Eq. (5) below), we have that

$$\langle H \pi(\lambda) g, \pi(\lambda) \tilde{g} \rangle_{L^2(\mathbb{R}^d)} = \langle \alpha_{-\lambda}(H) g, \tilde{g} \rangle_{L^2(\mathbb{R}^d)}, \quad \lambda \in \Lambda,$$

i.e., the samples $\langle \alpha_{-\lambda}(H) g, \tilde{g} \rangle_{L^2(\mathbb{R}^d)}$, $\lambda \in \Lambda$, coincide with the diagonal entries of the channel matrix associated with $H$ and windows $g, \tilde{g}$. This is the reason to consider the samples defined in Eq. (1) and to name them as the diagonal channel samples of the operator $H$ with respect to the fixed functions $g, \tilde{g} \in L^2(\mathbb{R}^d)$ and lattice $\Lambda$.

Besides, a simple class of operators $H$ describing time-varying systems, and allowing to live in the Hilbert space setting, is given by the class of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$. A Hilbert-Schmidt operator $H$ on $L^2(\mathbb{R}^d)$ is a compact operator on $L^2(\mathbb{R}^d)$ having the integral representation

$$H x(t) = \int_{\mathbb{R}^d} \kappa(t, s) x(s) \, ds = \int_{\mathbb{R}^d} \kappa(t, t-s) x(t-s) \, ds,$$

with kernel $\kappa \in L^2(\mathbb{R}^{2d})$. Although only Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$ can be described as integral operators with kernel in $L^2(\mathbb{R}^{2d})$, every bounded operator on $L^2(\mathbb{R}^d)$ can
be uniquely described, via the \textit{Schwartz kernel theorem}, by a distributional kernel in \(S' (\mathbb{R}^{2d})\) (see, for instance, Ref. \cite{16}).

The paper is organized as follows: Section \ref{preliminaries} introduces, for the sake of completeness, some preliminaries needed in the sequel; they comprise Hilbert-Schmidt operators and their Kohn-Nirenberg and Weyl transforms, the concept of translation of an operator, and \textit{symplectic Fourier series}. For the theory of bases and frames in a Hilbert space we cite Ref. \cite{3}. Section \ref{sampling results} contains the main sampling results for the multiple generated subspace \(V^2_S\) of \(\mathcal{HS}(\mathbb{R}^d)\). They rely on the expression of the involved samples as the output of a bounded discrete convolution system \(\ell^2_N (\Lambda) \to \ell^2_m (\Lambda)\), and its relationship with a frame of translates for \(\ell^2_N (\Lambda)\).

\section{Some preliminaries}

Next we briefly introduce some mathematical tools used throughout the work. For the needed theory of bases and frames in a Hilbert space we merely make reference to \cite{3}; it mainly comprises Riesz sequences, dual Riesz bases and frames and its duals in a separable Hilbert space. The results for discrete convolution systems and their relationship with frames of translates in \(\ell^2_N (\Lambda)\) can be found, for instance, in Ref. \cite{15}.

The Kohn-Nirenberg and Weyl transforms in the class of Hilbert-Schmidt operators

The class of Hilbert-Schmidt operators in a Hilbert space, \(L^2 (\mathbb{R}^d)\) in our case, can be introduced by using the \textit{Schmidt decomposition} (singular value decomposition) of a compact operator on \(L^2 (\mathbb{R}^d)\) (see, for instance, Ref. \cite{4}). Namely, for a compact operator \(S\) on \(L^2 (\mathbb{R}^d)\) there exist two orthonormal sequences \(\{x_n\}_{n \in \mathbb{N}}\) and \(\{y_n\}_{n \in \mathbb{N}}\) in \(L^2 (\mathbb{R}^d)\) and a bounded sequence of positive numbers \(\{s_n (S)\}_{n \in \mathbb{N}}\) (\textit{singular values of} \(S\)) such that

\[
S = \sum_{n \in \mathbb{N}} s_n (S) x_n \otimes y_n,
\]

with convergence of the series in the operator norm. Here, \(x_n \otimes y_n\) denotes the rank-one operator defined by \((x_n \otimes y_n) (e) = \langle e, y_n \rangle_{L^2} x_n\) for \(e \in L^2 (\mathbb{R}^d)\). For \(1 \leq p < \infty\) we define the \textit{Schatten-p class} \(\mathcal{T}^p\) by

\[
\mathcal{T}^p := \{ S \text{ compact on } L^2 (\mathbb{R}^d) : \{s_n (S)\}_{n \in \mathbb{N}} \in \ell^p (\mathbb{N}) \}.
\]

The Schatten-\(p\) class \(\mathcal{T}^p\) is a Banach space endowed with the norm \(\|S\|_{\mathcal{T}^p}^p = \sum_{n \in \mathbb{N}} s_n^p (S)\).

In particular, for \(p = 1\) we obtain the so-called \textit{trace class operators} \(\mathcal{T}^1\). The \textit{trace} defined by \(\text{tr} (S) = \sum_{n \in \mathbb{N}} \langle S e_n, e_n \rangle_{L^2}\) is a well-defined bounded linear functional on \(\mathcal{T}^1\), and independent of the used orthonormal basis \(\{e_n\}_{n \in \mathbb{N}}\) in \(L^2 (\mathbb{R}^d)\).

For \(p = 2\) we obtain the class of \textit{Hilbert-Schmidt operators} \(\mathcal{HS} (\mathbb{R}^d) := \mathcal{T}^2\). The space \(\mathcal{HS} (\mathbb{R}^d)\) endowed with the inner product \(\langle S, T \rangle_{\mathcal{HS}} = \text{tr} (ST^*)\) becomes a Hilbert space. For the norm of \(S \in \mathcal{HS} (\mathbb{R}^d)\) we have

\[
\|S\|^2_{\mathcal{HS}} = \text{tr} (SS^*) = \sum_{n \in \mathbb{N}} \|S^* (e_n)\|^2_{L^2} = \sum_{n \in \mathbb{N}} \|S (e_n)\|^2_{L^2} = \sum_{n \in \mathbb{N}} s_n^2 (S).
\]
A Hilbert-Schmidt operator $S \in \mathcal{HS}(\mathbb{R}^d)$ can be seen also as a compact operator on $L^2(\mathbb{R}^d)$ defined for each $f \in L^2(\mathbb{R}^d)$ by

$$Sf(t) = \int_{\mathbb{R}^d} \kappa_S(t,x)f(x)dx \quad \text{a.e. } t \in \mathbb{R}^d,$$

with kernel $\kappa_S \in L^2(\mathbb{R}^{2d})$. Besides, $\langle S, T \rangle_{\mathcal{HS}} = \langle \kappa_S, \kappa_T \rangle_{L^2(\mathbb{R}^{2d})}$ for $S, T \in \mathcal{HS}(\mathbb{R}^d)$.

Now, we briefly introduce the Kohn-Nirenberg and Weyl transforms in $L^2(\mathbb{R}^{2d})$, the setting where they will be used in this paper. More information and details about these transforms, also valid in more general settings, can be found in Refs. [7, 9, 16, 24, 26].

The Kohn-Nirenberg transform $L^2(\mathbb{R}^{2d}) \ni \sigma \mapsto K_\sigma \in \mathcal{HS}(\mathbb{R}^d)$ is a unitary operator where $K_\sigma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the Hilbert-Schmidt operator defined in weak sense by

$$\langle K_\sigma \phi, \psi \rangle_{L^2(\mathbb{R}^d)} = \langle \sigma, R(\psi, \phi) \rangle_{L^2(\mathbb{R}^{2d})}, \quad \phi, \psi \in L^2(\mathbb{R}^d);$$

here

$$R(\psi, \phi)(x, \omega) = \psi(x) \overline{\phi(\omega)} e^{-2\pi i x \cdot \omega}, \quad (x, \omega) \in \mathbb{R}^{2d},$$

is the Rihaczek distribution of the functions $\psi, \phi \in L^2(\mathbb{R}^d)$ (see [16, Theorem 14.6.1]).

Thus, for each operator $S \in \mathcal{HS}(\mathbb{R}^d)$ there exists a unique function $\sigma_S \in L^2(\mathbb{R}^{2d})$, called its Kohn-Nirenberg symbol, i.e., $S = K_{\sigma_S}$, and such that

$$\langle S, T \rangle_{\mathcal{HS}} = \langle \sigma_S, \sigma_T \rangle_{L^2(\mathbb{R}^{2d})} \quad \text{for each } S, T \in \mathcal{HS}(\mathbb{R}^d).$$

The Weyl transform $L^2(\mathbb{R}^{2d}) \ni f \mapsto L_f \in \mathcal{HS}(\mathbb{R}^d)$ is also a unitary operator where $L_f : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the Hilbert-Schmidt operator defined in weak sense by

$$\langle L_f \phi, \psi \rangle_{L^2(\mathbb{R}^d)} = \langle f, W(\psi, \phi) \rangle_{L^2(\mathbb{R}^{2d})}, \quad \phi, \psi \in L^2(\mathbb{R}^d);$$

here

$$W(\psi, \phi)(x, \omega) = \int_{\mathbb{R}^d} \psi(x + \frac{t}{2}) \overline{\phi(x - \frac{t}{2})} e^{-2\pi i \omega \cdot t} dt, \quad (x, \omega) \in \mathbb{R}^{2d},$$

is the cross-Wigner distribution of the functions $\psi, \phi \in L^2(\mathbb{R}^d)$ (see Ref. [16, Theorem 14.6.1]).

Thus, for each operator $S \in \mathcal{HS}(\mathbb{R}^d)$ there exists a unique function $a_S \in L^2(\mathbb{R}^{2d})$, called its Weyl symbol, i.e., $S = L_{a_S}$, and such that

$$\langle S, T \rangle_{\mathcal{HS}} = \langle a_S, a_T \rangle_{L^2(\mathbb{R}^{2d})} \quad \text{for each } S, T \in \mathcal{HS}(\mathbb{R}^d).$$

If $a_S$ denotes the Weyl symbol of $S$, its Kohn-Nirenberg symbol $\sigma_S$ is given by $U a_S$ where $U$ is the unitary operator on $L^2(\mathbb{R}^{2d})$ such that $\widehat{U a_S}(\xi, u) = e^{\pi i u \cdot \xi} \hat{a}_S(\xi, u), \quad (\xi, u) \in \mathbb{R}^{2d}$ (see the details in Ref. [16]).

The Kohn-Nirenberg (or Weyl) transform can be defined for $\sigma$ (or $f$) in $S' (\mathbb{R}^{2d})$, i.e., for tempered distributions by using the dualities $(S(\mathbb{R}^d), S'(\mathbb{R}^d))$ and $(S(\mathbb{R}^{2d}), S'(\mathbb{R}^{2d}))$ in Eq. (3) (or Eq. (4)); see, for instance, Refs. [16, 24].
Translation of operators

For $z = (x, \omega) \in \mathbb{R}^{2d}$, the time-frequency shift operator $\pi(z) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined as

$$\pi(z) \varphi(t) = e^{2\pi i t \cdot z} \varphi(t - x) \quad \text{for} \quad \varphi \in L^2(\mathbb{R}^d).$$

It is used to define the short-time Fourier transform (Gabor transform) $V_{\psi} \varphi$ of $\varphi$ with window $\psi$, both in $L^2(\mathbb{R}^d)$, by

$$V_{\psi} \varphi(z) = \langle \varphi, \pi(z) \psi \rangle_{L^2(\mathbb{R}^d)}, \quad z \in \mathbb{R}^{2d}.$$

Its adjoint operator is $\pi(z)^* \equiv e^{-2\pi i x \cdot \omega} \pi(-z)$ for $z = (x, \omega) \in \mathbb{R}^{2d}$. By using conjugation with $\pi(z)$ one can define the translation by $z \in \mathbb{R}^{2d}$ of an operator $S \in \mathcal{HS}(\mathbb{R}^d)$. Namely,

$$\alpha_z(S) := \pi(z) S \pi(z)^*, \quad z \in \mathbb{R}^{2d}.$$

For instance, for $\varphi, \psi \in L^2(\mathbb{R}^d)$ we get $\alpha_z(\varphi \otimes \psi) = [\pi(z) \varphi] \otimes [\pi(z) \psi]$, $z \in \mathbb{R}^{2d}$.

Since $\alpha_z$ defines a unitary operator on $\mathcal{HS}(\mathbb{R}^d)$, $\alpha_z \alpha_{z'} = \alpha_{z+z'}$ for $z, z' \in \mathbb{R}^{2d}$, and the map $z \mapsto \alpha_z(S)$ is continuous for each $S \in \mathcal{HS}(\mathbb{R}^d)$ we have that $\{\alpha_z\}_{z \in \mathbb{R}^{2d}}$ is a unitary representation of the group $\mathbb{R}^{2d}$ on the Hilbert space $\mathcal{HS}(\mathbb{R}^d)$. More properties and applications can be found, for instance, in Refs. [21, 24, 26].

Symplectic Fourier series

Let $\Lambda$ be a full rank lattice in $\mathbb{R}^{2d}$, i.e., $\Lambda = A \mathbb{Z}^{2d}$ with $A \in GL(2d, \mathbb{R})$ and volume $|\Lambda| = \det A$. Its dual group $\Lambda^\circ$ is identified with $\mathbb{R}^{2d}/\Lambda^\circ$, where $\Lambda^\circ$ is the annihilator group

$$\Lambda^\circ = \{ \lambda^\circ \in \mathbb{R}^{2d} : e^{2\pi i \sigma(\lambda^\circ, \lambda)} = 1 \; \text{for all} \; \lambda \in \Lambda \},$$

where $\sigma$ denotes here the standard symplectic form $\sigma(z, z') = \omega \cdot x' - \omega' \cdot x$ for $z = (x, \omega)$ and $z' = (x', \omega')$ in $\mathbb{R}^{2d}$. Notice that, since $\Lambda$ is discrete its dual group $\Lambda^\circ$ is compact. The group $\Lambda^\circ$ is itself a lattice: the so-called adjoint lattice of $\Lambda$. The symplectic characters $\chi_z(z') := e^{2\pi i \sigma(z, z')}$ are the natural way of identifying the group $\mathbb{R}^{2d}$ with its dual group via the bijection $z \mapsto \chi_z$.

The Fourier transform of $c \in \ell^1(\Lambda)$ is the symplectic Fourier series

$$\mathcal{F}^\Lambda_s(c)(\hat{z}) := \sum_{\lambda \in \Lambda} c(\lambda) e^{2\pi i \sigma(\lambda, \hat{z})}, \quad \hat{z} \in \mathbb{R}^{2d}/\Lambda^\circ,$$

where $\hat{z}$ denotes the image of $z$ under the natural quotient map $\mathbb{R}^{2d} \to \mathbb{R}^{2d}/\Lambda^\circ$.

Since $\mathcal{F}^\Lambda_s$ is a Fourier transform it extends to a unitary mapping $\mathcal{F}^\Lambda_s : \ell^2(\Lambda) \to L^2(\hat{\Lambda})$. It satisfies $\mathcal{F}^\Lambda_s(c \ast_\Lambda d) = \mathcal{F}^\Lambda_s(c) \mathcal{F}^\Lambda_s(d)$ for $c \in \ell^1(\Lambda)$ and $d \in \ell^2(\Lambda)$. Moreover, if $c, d \in \ell^2(\Lambda)$ with $c \ast_\Lambda d \in \ell^2(\Lambda)$, then $\mathcal{F}^\Lambda_s(c \ast_\Lambda d) = \mathcal{F}^\Lambda_s(c) \mathcal{F}^\Lambda_s(d)$. As usual, the convolution $\ast_\Lambda$ of two sequences $c, d$ is defined by

$$(c \ast_\Lambda d)(\lambda) = \sum_{\mu \in \Lambda} c(\mu) d(\lambda - \mu), \quad \lambda \in \Lambda.$$

For more details, see, for instance, Refs. [5, 10, 11, 24].
3 Sampling in the case of multiple generators

For a fixed set $S = \{S_1, S_2, \ldots, S_N\} \subset \mathcal{H}(\mathbb{R}^d)$, we are interested that the sequence of translates $\{\alpha_\lambda(S_n)\}_{\lambda \in \Lambda; n=1,2,\ldots,N}$ forms a Riesz sequence for $\mathcal{H}(\mathbb{R}^d)$ where $\Lambda \subset \mathbb{R}^{2d}$ is a full rank lattice with dual group $\tilde{\Lambda}$.

3.1 Riesz sequences of translated operators in $\mathcal{H}(\mathbb{R}^d)$

As it was said before, the Weyl transform $f \mapsto L_f$ is a unitary operator $L^2(\mathbb{R}^{2d}) \to \mathcal{H}(\mathbb{R}^d)$ which respects translations in the sense that

$$L_{T_z f} = \alpha_z(L_f) \quad \text{for } f \in L^2(\mathbb{R}^{2d}) \text{ and } z \in \mathbb{R}^{2d}. $$

These two properties are very important throughout this work. In particular, as it was pointed out in Refs. [6, 24], for fixed $S \in \mathcal{H}(\mathbb{R}^d)$ with Weyl symbol $a_s \in L^2(\mathbb{R}^{2d})$ and lattice $\Lambda$ in $\mathbb{R}^{2d}$, the sequence $\{\alpha_\lambda(S)\}_{\lambda \in \Lambda}$ is a Riesz sequence in $\mathcal{H}(\mathbb{R}^d)$, i.e., a Riesz basis for $V^2_S := \varphi_0 \mathcal{H}(\mathbb{R}^d)\{\alpha_\lambda(S)\}_{\lambda \in \Lambda}$, if and only if the sequence $\{T_\lambda(a_s)\}_{\lambda \in \Lambda}$ is a Riesz sequence in $L^2(\mathbb{R}^{2d})$, i.e., a Riesz basis for the shift-invariant subspace $V^2_{\alpha_s}$ in $L^2(\mathbb{R}^{2d})$ generated by $a_s$.

A necessary and sufficient condition for $\{\alpha_\lambda(S)\}_{\lambda \in \Lambda}$ to be a Riesz sequence in $\mathcal{H}(\mathbb{R}^d)$ is given in Ref. [24]. There, it is assumed that $S \in \mathcal{B}$, a Banach space of continuous operators with Weyl symbol $a_s$ in the Feichtinger’s algebra $\mathcal{S}_0(\mathbb{R}^{2d})$; in essence, $\mathcal{B}$ consists of trace class operators on $L^2(\mathbb{R}^d)$ with a norm-continuous inclusion $\iota : \mathcal{B} \hookrightarrow \mathcal{T}^1$ (see the details in Refs. [17, 24]).

Recall that the Feichtinger’s algebra $\mathcal{S}_0(\mathbb{R}^d)$ is the space of all tempered distributions $\psi$ in $\mathbb{R}^d$ such that

$$\|\psi\|_{\mathcal{S}_0} := \int_{\mathbb{R}^d} |V_{\varphi_0}\psi(z)| dz < \infty,$$

where $\varphi_0$ denotes the $L^2$-normalized gaussian $\varphi_0(x) = 2^{d/4} e^{-\pi x \cdot x}$ for $x \in \mathbb{R}^d$. With this norm, $\mathcal{S}_0(\mathbb{R}^d)$ is a Banach space of continuous functions and an algebra under multiplication and convolution; see the details in Refs. [16, 19, 24].

**Theorem 1.** ([24, Theorem 6.1]) Let $\Lambda$ be a lattice and $S \in \mathcal{B}$. The sequence $\{\alpha_\lambda(S)\}_{\lambda \in \Lambda}$ is a Riesz sequence in $\mathcal{H}(\mathbb{R}^d)$ if and only if the function

$$P_{\Lambda^\circ}(|\mathcal{F}_W(S)|^2)(\hat{z}) := \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} |\mathcal{F}_W(S)(z + \lambda^\circ)|^2, \quad z \in \mathbb{R}^{2d},$$

has no zeros in $\hat{\Lambda}$.

It involves the periodization operator $P_{\Lambda^\circ}$ in $\Lambda^\circ$ and the Fourier-Wigner transform $\mathcal{F}_W$ of an operator $S$. In this case, we have that $\mathcal{F}_W(S) = \mathcal{F}_s(a_s)$, where $\mathcal{F}_s$ denotes the symplectic Fourier transform of $a_s$ defined by

$$\mathcal{F}_s(a_s)(z) := \int_{\mathbb{R}^{2d}} a_s(z') e^{-2\pi i \sigma(z,z')} dz', \quad z \in \mathbb{R}^{2d},$$

where $\sigma$ denotes here the standard symplectic form in $\mathbb{R}^{2d}$. The Fourier-Wigner transform of an operator $S$ is defined as the function

$$\mathcal{F}_W(S)(z) := e^{-\pi i x \cdot \omega} \text{tr}[\pi(-z)S], \quad z = (x, \omega) \in \mathbb{R}^{2d}.$$
See the details in Ref. [24]. A similar result to that in the above theorem for a rank-one operator $S = \psi \otimes \phi$, where $\psi, \phi \in L^2(\mathbb{R}^d)$, can be found in Refs. [2, 6].

In case $\{\alpha_\lambda(S)\}_{\lambda \in \Lambda}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$, the operator $S$ is the generator of the $\Lambda$-shift-invariant subspace $V^2_S$ which can be described by

$$V^2_S := \text{span}_{\mathcal{HS}} \{\alpha_\lambda(S)\}_{\lambda \in \Lambda} = \left\{ \sum_{\lambda \in \Lambda} c(\lambda) \alpha_\lambda(S) : \{c(\lambda)\}_{\lambda \in \Lambda} \in \ell^2(\Lambda) \right\}.$$ 

Observe that operators in $V^2_S$ are nothing but Gabor multipliers in case $S = \varphi \otimes \psi$. Indeed, for $\eta \in L^2(\mathbb{R}^d)$ we have

$$\sum_{\lambda \in \Lambda} c(\lambda) \alpha_\lambda(S)(\eta) = \sum_{\lambda \in \Lambda} c(\lambda) (\pi(\lambda) \varphi \otimes \pi(\lambda) \psi)(\eta) = \sum_{\lambda \in \Lambda} c(\lambda) V_S \eta(\lambda) \pi(\lambda) \varphi,$$

that is, $\sum_{\lambda \in \Lambda} c(\lambda) \alpha_\lambda(S) = G^\psi_\varphi$, the Gabor multiplier with windows $\psi, \varphi$ and mask $c$ in $\ell^2(\Lambda)$ used in time-frequency analysis (see, for instance, Ref. [24]).

Analogously, a necessary and sufficient condition can be obtained for the multiply generated case. Indeed, let $S = \{S_1, S_2, \ldots, S_N\}$ be a fixed subset of $\mathcal{HS}(\mathbb{R}^d)$ and let $\Lambda$ be a lattice in $\mathbb{R}^{2d}$. We are searching for a necessary and sufficient condition such that $\{\alpha_\lambda(S_n)\}_{\lambda \in \Lambda; n=1,2,\ldots,N}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$, i.e., a Riesz basis for the closed subspace

$$V^2_S := \text{span}_{\mathcal{HS}} \{\alpha_\lambda(S_n)\}_{\lambda \in \Lambda; n=1,2,\ldots,N} \subset \mathcal{HS}(\mathbb{R}^d).$$

For the multiply generated case we have the following result:

**Theorem 2.** Let $\Lambda$ be a lattice and $S_n \in B$, $n = 1, 2, \ldots, N$. Then, $\{\alpha_\lambda(S_n)\}_{\lambda \in \Lambda; n=1,2,\ldots,N}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$ if and only if there exist two constants $0 < m \leq M$ such that

$$m I_N \leq G^W_S(z) \leq M I_N \quad \text{for any } z \in \mathbb{R}^{2d},$$

where $G^W_S(z)$ denotes the $N \times N$ matrix-valued function

$$G^W_S(z) := \sum_{\lambda^0 \in \Lambda^0} F_W(S)(z + \lambda^0) F_W(S)(z + \lambda^0)^\top, \quad z \in \mathbb{R}^{2d},$$

and $F_W(S) = (F_W(S_1), F_W(S_2), \ldots, F_W(S_N))^\top$.

**Proof.** As indicated above, it will be a Riesz sequence in $\mathcal{HS}(\mathbb{R}^d)$ if and only if the sequence $\{T_\lambda(a_{s_n})\}_{\lambda \in \Lambda; n=1,2,\ldots,N}$ is a Riesz sequence in $L^2(\mathbb{R}^{2d})$. To this end, we introduce the $N \times N$ matrix-valued function

$$G^\sigma_S(z) := \sum_{\lambda^0 \in \Lambda^0} F_s(a_s)(z + \lambda^0) F_s(a_s)(z + \lambda^0)^\top, \quad z \in \mathbb{R}^{2d},$$

where $F_s(a_s) = (F_s(a_{s_1}), F_s(a_{s_2}), \ldots, F_s(a_{s_N}))^\top$. It is known (see, for instance, Ref. [1]) that the sequence $\{T_\lambda(a_{s_n})\}_{\lambda \in \Lambda; n=1,2,\ldots,N}$ is a Riesz sequence in $L^2(\mathbb{R}^{2d})$ if and only if there exist two constants $0 < m \leq M$ such that $m I_N \leq G^\sigma_S(z) \leq M I_N, \text{ a.e. } z \in \mathbb{R}^{2d}$, where $I_N$ denotes the $N \times N$ identity matrix. Assuming as before that $S_n \in B$, $n = 1, 2, \ldots, N$, the functions $F_s(a_{s_n})$ are continuous and $F_W(S_n) = F_s(a_{s_n})$ for $n = 1, 2, \ldots, N$. Hence, the above necessary and sufficient condition can be expressed in terms of the hermitian matrix $G^W_S(z)$ as in the statement of the theorem.
In this case, \( S = \{S_1, S_2, \ldots, S_N\} \) is a set of generators for the \( \Lambda \)-shift-invariant subspace \( V_S^2 := \text{span}_{HS} \{\alpha_\lambda(S_n)\} \) \( \lambda \in \Lambda; n=1,2,\ldots,N \) which can be described by

\[
V_S^2 = \left\{ \sum_{n=1}^N \sum_{\lambda \in \Lambda} c_n(\lambda) \alpha_\lambda(S_n) : \{c_n(\lambda)\}_{\lambda \in \Lambda} \in \ell^2(\Lambda), \ n = 1, 2, \ldots, N \right\}.
\]

### 3.2 The isomorphism \( T_S \)

Our sampling results rely on the following isomorphism \( T_S \) which involves the spaces \( \ell^2(\Lambda) \), the shift-invariant subspace \( V_S^2 \) in \( L^2(\mathbb{R}^d) \) generated by the Kohn-Nirenberg symbols \( \sigma_{S_n} \) of \( S_n, \ n = 1, 2, \ldots, N \), and the \( \Lambda \)-shift-invariant subspace \( V_S^2 \). Namely,

\[
T_S : \ell^2(\Lambda) \quad \longrightarrow \quad V_S^2 \subset L^2(\mathbb{R}^d) \quad \longrightarrow \quad V_S^2 \subset HS(\mathbb{R}^d)
\]

\[
(c_1, c_2, \ldots, c_N)^T \quad \longmapsto \quad \sum_{n=1}^N \sum_{\lambda \in \Lambda} c_n(\lambda) T_\lambda \sigma_{S_n} \quad \longmapsto \quad \sum_{n=1}^N \sum_{\lambda \in \Lambda} c_n(\lambda) \alpha_\lambda(S_n).
\]  

(5)

The isomorphism \( T_S \) is the composition of the isomorphism \( T_{\sigma_S} : \ell^2(\Lambda) \rightarrow V_{\sigma_S}^2 \) which maps the standard orthonormal basis \( \{\delta_\lambda\}_{\lambda \in \Lambda} \) for \( \ell^2(\Lambda) \) onto the Riesz basis \( \{T_\lambda \sigma_{S_n}\}_{\lambda \in \Lambda; n=1,2,\ldots,N} \) for \( V_{\sigma_S}^2 \), and the Kohn-Nirenberg transform transform between \( V_{\sigma_S}^2 \) and \( V_S^2 \).

Recall that the Kohn-Nirenberg transform \( L^2(\mathbb{R}^d) \ni f \mapsto K_f \in HS(\mathbb{R}^d) \) is a unitary operator which respects translations in the sense that \( K_{T_\lambda f} = \alpha_\lambda(K_f) \) for \( f \in L^2(\mathbb{R}^d) \) and \( \lambda \in \mathbb{R}^d \). See, for instance, Ref. [7][16].

### 3.3 An expression for the samples

For each \( T = \sum_{n=1}^N \sum_{\mu \in \Lambda} c_n(\mu) \alpha_\mu(S_n) \) in \( V_S^2 \), we define a set of diagonal channel samples as

\[
s_T(\lambda) := \langle \alpha_{-\lambda}(T) g_m, \tilde{g}_m \rangle, \langle \alpha_{-\lambda}(T) g_2, \tilde{g}_2 \rangle, \ldots, \langle \alpha_{-\lambda}(T) g_M, \tilde{g}_M \rangle^T, \ \lambda \in \Lambda,
\]  

(6)

where \( g_m, \tilde{g}_m, \ m = 1, 2, \ldots, M \), denote \( 2M \) fixed functions in \( L^2(\mathbb{R}^d) \). For \( m = 1, 2, \ldots, M \) the above samples can be expressed by

\[
s_{T,m}(\lambda) := \langle \alpha_{-\lambda}(T) g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \sum_{n=1}^N \sum_{\mu \in \Lambda} c_n(\mu) \alpha_{-\mu-\lambda}(S_n) g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)}
\]

\[
= \sum_{n=1}^N \sum_{\mu \in \Lambda} c_n(\mu) \langle \alpha_{-\mu-\lambda}(S_n) g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \sum_{n=1}^N (a_{m,n} * \lambda c_n)(\lambda), \ \lambda \in \Lambda,
\]  

(7)

where \( a_{m,n}(\mu) := \langle \alpha_{-\mu}(S_n) g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} \), \( \mu \in \Lambda \). Observe that \( a_{m,n}(\lambda), \ \lambda \in \Lambda \), are precisely the samples \( s_{S_n}(\lambda), \ \lambda \in \Lambda \), of the generator \( S_n \).

**Lemma 3.** Concerning the samples defined in Eq. (7) we have:

1. For \( m = 1, 2, \ldots, M \) these samples can be written as

\[
\langle \alpha_{-\lambda}(T) g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \langle T \pi(\lambda) g_m, \pi(\lambda) \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \langle T, \alpha_{\lambda}(\tilde{g}_m \otimes g_m) \rangle_{HS}, \ \lambda \in \Lambda.
\]  

(8)
2. The sequences \( \{a_{m,n}(\lambda)\}_{\lambda \in \Lambda} \) appearing in Eq. (7) belong to \( \ell^2(\Lambda) \) for \( m = 1,2,\ldots,M \) and \( n = 1,2,\ldots,N \).

Proof. For the first equality in (8) we have that

\[
s_{T,m}(\lambda) = \langle \alpha_\lambda(T) g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \langle \sigma(T) R(\pi(-\lambda) g_m, \pi(-\lambda) g_m) \rangle_{L^2(\mathbb{R}^d)}, \quad \lambda \in \Lambda.
\]

On the other hand, it is easy to check that for the Rihaczek distribution one gets

\[
R(\pi(-\lambda) g_m, \pi(-\lambda) g_m)(z) = R(\pi(\lambda) \tilde{g}_m, \pi(\lambda) g_m)(z), \quad z \in \mathbb{R}^d.
\]

Hence, for each \( \lambda \in \Lambda \) we obtain

\[
\langle \alpha_\lambda(T) g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \langle \sigma(T) R(\pi(\lambda) \tilde{g}_m, \pi(\lambda) g_m) \rangle_{L^2(\mathbb{R}^d)} = \langle T \pi(\lambda) g_m, \pi(\lambda) \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)}.
\]

For the second equality we get

\[
\langle T, \alpha_\lambda(\tilde{g}_m \otimes g_m) \rangle_{HS} = \langle \sigma(T) \pi(\lambda) \tilde{g}_m \otimes \pi(\lambda) g_m \rangle_{HS} = \langle \sigma(T) R(\pi(\lambda) \tilde{g}_m, \pi(\lambda) g_m) \rangle_{L^2(\mathbb{R}^d)} = \langle T \pi(\lambda) g_m, \pi(\lambda) \tilde{g}_m \rangle_{L^2(\mathbb{R}^2d)}.
\]

We have used that the Kohn-Nirenberg symbol of \( \pi(\lambda) \tilde{g}_m \otimes \pi(\lambda) g_m \) coincides with the Rihaczek distribution of the pair of functions \( \pi(\lambda) \tilde{g}_m \) and \( \pi(\lambda) g_m \) in \( L^2(\mathbb{R}^d) \).

In particular we have proved that

\[
a_{m,n}(\lambda) = \langle \alpha_\lambda(S_n) g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \langle S_n, \alpha_\lambda(\tilde{g}_m \otimes g_m) \rangle_{HS} = \langle \alpha_\lambda(S_n), \tilde{g}_m \otimes g_m \rangle_{HS}, \quad \lambda \in \Lambda.
\]

Since \( \{\alpha_\lambda(S_n)\}_{\lambda \in \Lambda, n=1,2,\ldots,N} \) is a Riesz sequence for \( \mathcal{HS}(\mathbb{R}^d) \), it is in particular a Bessel sequence in \( \mathcal{HS}(\mathbb{R}^d) \). Hence, the sequences \( \{\langle \alpha_\lambda(S_n), \tilde{g}_m \otimes g_m \rangle_{HS}\}_{\lambda \in \Lambda} \) belongs to \( \ell^2(\Lambda) \) for \( m = 1,2,\ldots,M \) and \( n = 1,2,\ldots,N \).

Once we have that \( a_{m,n} \in \ell^2(\Lambda) \) for each \( m = 1,2,\ldots,M \) and \( n = 1,2,\ldots,N \), and denoting \( A = [a_{m,n}] \) the corresponding \( M \times N \) matrix with entries in \( \ell^2(\Lambda) \), the sampling process in (6) is described by means of the discrete convolution system

\[
T = \sum_{n=1}^N \sum_{\lambda \in \Lambda} c_n(\lambda) \alpha_\lambda(S_n) \in V_S^2 \mapsto s_T(\lambda) = (A * A c)(\lambda) = \sum_{\mu \in \Lambda} A(\lambda - \mu) c(\mu), \quad \lambda \in \Lambda,
\]

where \( c = (c_1, c_2, \ldots, c_N) \top \in \ell^2_{\Lambda}(\Lambda) := \ell^2(\Lambda) \times \cdots \times \ell^2(\Lambda) \) (\( N \) times). Note that the \( m \)-th entry of \( A * A c \) is \( \sum_{n=1}^N (a_{m,n} * A c)_n \).

First of all, the mapping \( A : \ell^2_{\Lambda}(\Lambda) \to \ell^2_{\Lambda}(\Lambda) \) which maps \( c \mapsto A * A c \) is a well-defined bounded operator if and only if the \( M \times N \) matrix-valued function \( \hat{A} : \hat{\Lambda} \to \mathbb{F}^M_N \) is a bounded operator, where \( \hat{\Lambda} \) is the dual space of \( \Lambda \) and \( \mathbb{F}^M_N \) is the space of \( M \times N \) matrices. Note that the \( m \)-th component of \( A * A c \) is

\[
[a * c]_m(\lambda) = \sum_{n=1}^N (a_{m,n} * A c)_n(\lambda) = \langle c, T^*_\lambda a_m^\top \rangle_{\ell^2_N(\Lambda)}, \quad \lambda \in \Lambda.
\]
where \( a_{m,n}^* \) denotes the involution \( a_{m,n}^*(\lambda) := a_{m,n}(-\lambda) \), \( \lambda \in \Lambda \). As a consequence, the operator \( A \) is the analysis operator of the positive semidefinite matrix \( \hat{A}(\xi)^* \hat{A}(\xi) \) (see Ref. [15]).

Concerning the duals of \( \{ T_\lambda a_m^* \}_{\lambda \in \Lambda; m=1,2,...,M} \) having its same structure, consider two matrices \( \hat{A} \in \mathcal{M}_{M \times N}(L^\infty(\hat{\Lambda})) \) and \( \hat{B} \in \mathcal{M}_{N \times M}(L^\infty(\hat{\Lambda})) \), and let \( b_m \) denote the \( m \)-th column of the matrix \( B \) associated to \( \hat{B} \). Then, the sequences \( \{ T_\lambda a_m^* \}_{\lambda \in \Lambda; m=1,2,...,M} \) and \( \{ T_\lambda b_m \}_{\lambda \in \Lambda; m=1,2,...,M} \) form a pair of dual frames for \( \ell^2_\Lambda(\Lambda) \) if and only if \( \hat{B}(\xi) \hat{A}(\xi) = I_N \), a.e. \( \xi \in \hat{\Lambda} \); equivalently, if and only if \( \mathcal{B} \mathcal{A} = I_{\ell^2_\Lambda(\Lambda)} \), i.e., the convolution system \( \mathcal{B} \) with matrix \( B \) is a left-inverse of the convolution system \( \mathcal{A} \) with matrix \( A \). Thus, we have the frame expansion

\[
\mathbf{c} = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} \langle \mathbf{c}, T_\lambda a_m^* \rangle_{\ell^2_\Lambda(\Lambda)} \mathcal{T}_\lambda b_m \quad \text{for each } \mathbf{c} \in \ell^2_\Lambda(\Lambda).
\]

Observe that a possible left-inverse \( \hat{B}(\xi) \) of the matrix \( \hat{A}(\xi) \) is given by its Moore-Penrose pseudo-inverse \( \hat{A}(\xi)^+=\left[(\hat{A}(\xi)^*\hat{A}(\xi))^{-1}\hat{A}(\xi)^*\right] \), a.e. \( \xi \in \hat{\Lambda} \).

### 3.4 The sampling results

Next we prove the main sampling result in this paper:

**Theorem 4.** Suppose that for each \( T \in V^2_S \) we consider the samples defined by (\ref{eq:samples}), and such that the matrix \( A = [a_{m,n}] \), where \( a_{m,n}(\lambda) = \langle \alpha_\lambda(S_n)g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} \), \( \lambda \in \Lambda \), satisfies conditions in Eq. (\ref{eq:condition}). Then, there exist \( M \geq N \) elements \( H_m \in V^2_S \), \( m = 1, 2, \ldots, M \), such that the sampling formula

\[
T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} s_{T,m}(\lambda) \alpha_\lambda(H_m) \quad \text{in } \mathcal{H}\mathcal{S}-\text{norm}
\]

holds for each \( T \in V^2_S \) where \( \{ \alpha_\lambda(H_m) \}_{\lambda \in \Lambda; m=1,2,...,M} \) is a frame for \( V^2_S \). The convergence of the series is unconditional in Hilbert-Schmidt norm.

Moreover, the \( \ell^2 \)-norm of the samples \( \| s_T \|_{\ell^2_M} \) defines an equivalent norm to \( \| T \|_{\mathcal{H}\mathcal{S}} \) in \( V^2_S \), and for each \( f \in L^2(\mathbb{R}^d) \) we have the pointwise expansion

\[
Tf = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} s_{T,m}(\lambda) \alpha_\lambda(H_m)f \quad \text{in } L^2(\mathbb{R}^d).
\]
Proof. Under the hypotheses of the theorem the sequence \( \{ T_\lambda a_m^* \}_{\lambda \in \Lambda; m=1,2,\ldots,M} \) is a frame for \( \ell^2_N(\Lambda) \), and we can consider a dual frame \( \{ T_\lambda b_m \}_{\lambda \in \Lambda; m=1,2,\ldots,M} \) with the same structure. As a consequence, for each \( T = \sum_{n=1}^N \sum_{\lambda \in \Lambda} c_n(\lambda) \alpha_\lambda(S_n) \) in \( V_\mathcal{S}^2 \) we have

\[
c = \sum_{m=1}^M \sum_{\lambda \in \Lambda} \langle c, T_\lambda a_m^* \rangle_{\ell^2_N(\Lambda)} T_\lambda b_m = \sum_{m=1}^M \sum_{\lambda \in \Lambda} s_{T,m}(\lambda) T_\lambda b_m \quad \text{in} \quad \ell^2_N(\Lambda),
\]

where \( c = (c_1, c_2, \ldots, c_N)^\top \in \ell^2_N(\Lambda) \). Notice that the fact that \( \{ T_\lambda a_m^* \}_{\lambda \in \Lambda; m=1,2,\ldots,M} \) is a frame for \( \ell^2_N(\Lambda) \) and the isomorphism \( \mathcal{T}_\mathcal{S} \) in Eq. (5) give the equivalence of the norms.

The isomorphism \( \mathcal{T}_\mathcal{S} \) defined by Eq. (5) applied in Eq. (11) gives the sampling expansion (10), where \( H_m = K_{h_m} \in V_\mathcal{S}^2 \) with Kohn-Nirenberg symbol \( h_m = \mathcal{T}_\mathcal{S}(b_m) \in V_\mathcal{S}^2 \), \( m = 1,2,\ldots,M \). Furthermore, since \( \{ \alpha_\lambda(H_m) \}_{\lambda \in \Lambda; m=1,2,\ldots,M} \) is a frame for \( V_\mathcal{S}^2 \) the convergence of the series in the Hilbert-Schmidt norm is unconditional. Notice that \( \mathcal{T}_\mathcal{S}(T_\lambda b_m) = T_\lambda(T_{\mathcal{S}}b_m) = T_\lambda(h_m), \) where the same symbol \( T_\lambda \) denotes both the translation by \( \lambda \) in \( \ell^2_N(\Lambda) \) and in \( L^2(\mathbb{R}^{2d}) \) respectively. Notice that if \( b_m = (b_{1,m}(\lambda), b_{2,m}(\lambda), \ldots, b_{N,m}(\lambda))^\top \), then

\[
H_m = \sum_{n=1}^N \sum_{\lambda \in \Lambda} b_{n,m}(\lambda) \alpha_\lambda(S_n), \quad m = 1,2,\ldots,M.
\]

Since convergence in \( \mathcal{HS} \)-norm implies convergence in operator norm we deduce the pointwise expansion for each \( f \in L^2(\mathbb{R}^d) \).

Observe that, due to conditions (9) in Theorem 4 we have necessarily \( M \geq N \). Whenever \( M > N \), there are infinite dual frames \( \{ T_\lambda b_m \}_{\lambda \in \Lambda; m=1,2,\ldots,M} \) of \( \{ T_\lambda a_m^* \}_{\lambda \in \Lambda; m=1,2,\ldots,M} \) given by the samples (7). They are obtained from the left-inverses \( \hat{B}(\xi) \) of \( \hat{A}(\xi) \) which are deduced, from the Moore-Penrose pseudo-inverse \( \hat{A}(\xi)^\dagger \), as the \( N \times M \) matrices

\[
\hat{B}(\xi) := \hat{A}(\xi)^\dagger + C(\xi) [I_M - \hat{A}(\xi)\hat{A}(\xi)^\dagger], \quad \text{a.e.} \quad \xi \in \hat{\Lambda},
\]

where \( C \) denotes any \( N \times M \) matrix with entries in \( L^\infty(\hat{\Lambda}) \).

More can be said in case \( M = N \):

**Corollary 5.** In case \( M = N \), assume that the conditions

\[
0 < \text{ess inf}_{\xi \in \hat{\Lambda}} |\text{det}[\hat{A}(\xi)]| \leq \text{ess sup}_{\xi \in \hat{\Lambda}} |\text{det}[\hat{A}(\xi)]| < +\infty
\]

hold. Then, there exist \( N \) unique elements \( H_n, n = 1,2,\ldots,N, \) in \( V_\mathcal{S}^2 \) such that the associated sequence \( \{ \alpha_\lambda(H_n) \}_{\lambda \in \Lambda; n=1,2,\ldots,N} \) is a Riesz basis for \( V_\mathcal{S}^2 \) and the sampling formula

\[
T = \sum_{n=1}^N \sum_{\lambda \in \Lambda} s_{T,n}(\lambda) \alpha_\lambda(H_n) \quad \text{in} \quad \mathcal{HS}-\text{norm}
\]

holds for each \( T \in V_\mathcal{S}^2 \). Moreover, the interpolation property \( \langle \alpha_\lambda(H_m) g_n, \tilde{g}_n \rangle = \delta_{m,n} \delta_{\lambda,0} \), where \( \lambda \in \Lambda \) and \( m, n = 1,2,\ldots,N \), holds.
Proof. In this case, the square matrix \( \hat{A}(\xi) \) is invertible and the statement (12) in corollary is equivalent to condition \( 0 < \alpha_A \leq \beta_A < +\infty \) in (9); besides, any Riesz basis has a unique dual basis. The uniqueness of the coefficients in a Riesz basis expansion gives the interpolation property.

In particular, for the case \( N = M = 1 \) we have:

Corollary 6. Assume that the sequence \( a = \{a(\lambda)\}_{\lambda \in \Lambda} \), where \( a(\lambda) = \langle \alpha_{-\lambda}(S)g, \bar{g}\rangle_{L^2(\mathbb{R}^d)} \), \( \lambda \in \Lambda \), for a fixed pair of functions \( g, \bar{g} \in L^2(\mathbb{R}^d) \), satisfies the conditions

\[
0 < \text{ess inf}_{\xi \in \hat{\Lambda}} |\mathcal{F}^A_{\alpha}(a)(\xi)| \leq \text{ess sup}_{\xi \in \hat{\Lambda}} |\mathcal{F}^A_{\alpha}(a)(\xi)| < \infty.
\]

Then, there exists a unique \( H \in \mathcal{V}^2_s \) such that the sequence \( \{\alpha_{\lambda}(H)\}_{\lambda \in \Lambda} \) is a Riesz basis for \( \mathcal{V}^2_s \) and the sampling formula

\[
T = \sum_{\lambda \in \Lambda} \langle \alpha_{-\lambda}(T)g, \bar{g}\rangle_{L^2(\mathbb{R}^d)} \alpha_{\lambda}(H) \quad \text{in } \mathcal{HS}\text{-norm}
\]

holds for each \( T \in \mathcal{V}^2_s \). Moreover, the interpolation property \( \langle \alpha_{-\lambda}(H)g, \bar{g}\rangle = \delta_{\lambda,0}, \lambda \in \Lambda, \) holds; in particular, \( \langle Hg, \bar{g}\rangle = 1 \).

It is worth to remark that in the above sampling result is not necessary that the operators in \( \mathcal{V}^2_s \) have a bandlimited Kohn-Nirenberg symbol as in Ref. [18, Theorem 2].

The bandlimited case is obtained as a particular case. Let \( \Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d \) be a lattice in \( \mathbb{R}^{2d} \) with \( a, b > 0 \). Assume that the generator \( S \) of \( \mathcal{V}^2_s \) is a bandlimited operator to \( Q := [\frac{1}{2a}, \frac{1}{2a}]^d \times [\frac{1}{2b}, \frac{1}{2b}]^d \), i.e., it belongs to \( \text{OPW}^2(Q) := \{T \in \mathcal{HS}(\mathbb{R}^d) : \text{supp } \hat{T} \subseteq Q\} \). Then any \( T \in \mathcal{V}^2_s \) also belongs to \( \text{OPW}^2(Q) \). In case conditions (13) are satisfied, any \( T \in \mathcal{V}^2_s \) can be recovered from its diagonal channel samples as

\[
T = \sum_{\lambda \in \Lambda} \langle T\pi(\lambda)g, \pi(\lambda)\bar{g}\rangle_{L^2(\mathbb{R}^d)} \alpha_{\lambda}(H) \quad \text{in } \mathcal{HS}\text{-norm},
\]

where \( H = \sum_{\lambda \in \Lambda} b(\lambda) \alpha_{\lambda}(S) \) in \( \mathcal{V}^2_s \) is obtained from the sequence \( b = \{b(\lambda)\}_{\lambda \in \Lambda} \) in \( \ell^2(\Lambda) \) such that \( \mathcal{F}^A_{\alpha}(b)(\xi) \mathcal{F}^A_{\alpha}(a)(\xi) = 1 \), a.e. \( \xi \in \hat{\Lambda} \).

In Ref. [18] the reconstruction of pseudodifferential operators with a bandlimited Kohn-Nirenberg symbol is considered. In particular, Theorem 2 of the same reference proves that, under some appropriate assumptions, for any \( T \in \text{OPW}^2(Q) \) we have

\[
\sigma_T = \frac{1}{(ab)^d} \sum_{\lambda \in \Lambda} \langle T\pi(\lambda)g, \pi(\lambda)g\rangle_{L^2(\mathbb{R}^d)} T\lambda(\text{sinc}_{a,b} * k) \quad \text{in } L^2(\mathbb{R}^{2d}),
\]

where the function \( k \), independent of \( T \), belongs to \( L^1(\mathbb{R}^{2d}) \) and \( \text{sinc}_{a,b} \) denotes the sinc function adapted to the lattice \( \Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d \), namely

\[
\text{sinc}_{a,b}(x) = \prod_{j=1}^{d} \frac{\sin \pi ax_j}{\pi ax_j} \prod_{j=d+1}^{2d} \frac{\sin \pi bx_j}{\pi bx_j}, \quad x \in \mathbb{R}^{2d}.
\]
Using the Kohn-Nirenberg transform, the above sampling formula for $\sigma_T$ can be written as

$$T = \frac{1}{(ab)^d} \sum_{\lambda \in \Lambda} \langle T \pi(\lambda) g, \pi(\lambda) g \rangle_{L^2(\mathbb{R}^d)} \alpha_\lambda(k * K_{\text{sinc}_{a,b}})$$ in $\mathcal{HS}$-norm,

where $k * K_{\text{sinc}_{a,b}}$ denotes the Hilbert-Schmidt operator obtained from the convolution of the function $k$ and the operator $K_{\text{sinc}_{a,b}}$; we have also used the following result:

**Lemma 7.** Let $K_f$ be an operator in $\mathcal{HS}(\mathbb{R}^d)$ with Kohn-Nirenberg symbol $f \in L^2(\mathbb{R}^{2d})$, and let $g$ a function in $L^1(\mathbb{R}^{2d})$. Then we have that $K_{g*f} = g * K_f$.

**Proof.** Recall that the convolution $g * K_f$ is the operator in $\mathcal{HS}(\mathbb{R}^d)$ defined by the operator-valued integral (in weak sense)

$$g * K_f = \int_{\mathbb{R}^{2d}} g(z) \alpha_z(K_f) dz,$$

i.e.,

$$\left\langle \left( \int_{\mathbb{R}^{2d}} g(z) \alpha_z(K_f) dz \right) \varphi, \psi \right\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} g(z) \left\langle \alpha_z(K_f) \varphi, \psi \right\rangle dz, \quad \varphi, \psi \in L^2(\mathbb{R}^d).$$

See the details in Refs. [21, 24, 26]. Since the map $K : \mathcal{HS}(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ such that $K(K_f) = f$ is a unitary operator and bounded operators commute with convergent integrals [21, Proposition 2.4] we get

$$K(g * K_f) = \int_{\mathbb{R}^{2d}} g(z) K(\alpha_z(K_f)) dz = \int_{\mathbb{R}^{2d}} g(z) K(K_{T_f}) dz = \int_{\mathbb{R}^{2d}} g(z) f(\cdot - z) dz = g * f,$$

that is, $K_{g*f} = g * K_f$. \qed

In the same manner we can consider average sampling in $V^2_{\mathcal{S}}$. Namely, for any $T \in V^2_{\mathcal{S}}$, its average samples at $\Lambda$ are defined by

$$\left\langle T, \alpha_\lambda(Q_m) \right\rangle_{\mathcal{HS}}, \quad \lambda \in \Lambda, \quad m = 1, 2, \ldots, M,$$

from $M$ fixed operators $Q_1, Q_2, \ldots, Q_M$ in $\mathcal{HS}(\mathbb{R}^d)$, not necessarily in $V^2_{\mathcal{S}}$. Observe that, having in mind Eq. (3) in Lemma 3 the diagonal channel samples defined in Eq. (6) are a particular case of average sampling where $Q_m = \tilde{g}_m \otimes g_m$, $m = 1, 2, \ldots, M$. The average samples of any $T = \sum_{n=1}^N \sum_{\mu \in \Lambda} c_n(\mu) \alpha_\mu(S_n)$ can be also expressed as a discrete convolution system in $l^2(\Lambda)$. Indeed, for $m = 1, 2, \ldots, M$ we have

$$\left\langle T, \alpha_\lambda(Q_m) \right\rangle_{\mathcal{HS}} = \left\langle \sigma_T, T_\lambda \sigma_{Q_m} \right\rangle_{L^2(\mathbb{R}^{2d})} = \left\langle \sum_{n=1}^N \sum_{\mu \in \Lambda} c_n(\mu) T_\mu \sigma_{S_n} , T_\lambda \sigma_{Q_m} \right\rangle_{L^2(\mathbb{R}^{2d})}$$

$$= \sum_{n=1}^N \sum_{\mu \in \Lambda} c_n(\mu) \left\langle T_\mu \sigma_{S_n} , T_\lambda \sigma_{Q_m} \right\rangle_{L^2(\mathbb{R}^{2d})} = \sum_{n=1}^N \sum_{\mu \in \Lambda} c_n(\mu) \left\langle \sigma_{S_n} , T_{\lambda - \mu} \sigma_{Q_m} \right\rangle_{L^2(\mathbb{R}^{2d})}$$

$$= \sum_{n=1}^N \left( a_{m,n} *_\Lambda c_n \right)(\lambda), \quad \lambda \in \Lambda,$$
where \( a_{m,n}(\mu) := \langle \sigma_{S_n}, T_\mu \sigma_{Q_m} \rangle_{L^2(\mathbb{R}^{2d})} = \langle S_n, \alpha_\mu(Q_m) \rangle_{\mathcal{HS}}, \mu \in \Lambda, \) and \( \sigma_{S_n}, \sigma_{Q_m} \) are the Kohn-Nirenberg symbols of \( S_n, Q_m \) respectively.

Observe that, for each \( m = 1, 2, \ldots, M \) and \( n = 1, 2, \ldots, N, \) the sequence \( \{a_{m,n}(\lambda)\}_{\lambda \in \Lambda} \) belongs to \( \ell^2(\Lambda) \) since, in particular, \( \{T_\lambda \sigma_{S_n}\}_{\lambda \in \Lambda; n=1,2,\ldots,N} \) is a Bessel sequence in \( L^2(\mathbb{R}^{2d}) \).

**Corollary 8.** Assume that the matrix \( A = [a_{m,n}] \) with entries \( a_{m,n}(\lambda) = \langle S_n, \alpha_\mu(Q_m) \rangle_{\mathcal{HS}}, \lambda \in \Lambda, \) satisfies conditions in [13]. Then, there exist \( M \geq N \) operators \( H_m \in \mathcal{V}_S^2, m = 1, 2, \ldots, M, \) such that the sampling formula

\[
T = \sum_{m=1}^M \sum_{\lambda \in \Lambda} \langle T, \alpha_\lambda(Q_m) \rangle_{\mathcal{HS}} \alpha_\lambda(H_m) \quad \text{in } \mathcal{HS}-\text{norm}
\]

holds for each \( T \in \mathcal{V}_S^2 \) where \( \{\alpha_\lambda(H_m)\}_{\lambda \in \Lambda; m=1,2,\ldots,M} \) is a frame for \( \mathcal{V}_S^2 \). The convergence of the series is unconditional in Hilbert-Schmidt norm.

The above sampling formula was obtained in Ref. [12] by using the Weyl symbols of \( S_n \) and \( Q_m \) instead of their Kohn-Nirenberg symbols. Finally, it is worth to mention that each sampling result in this section admit a kind of converse result; see the details in Theorems 1-2 and Corollary 3 of Ref. [12].

### An illustrative example

Assume that in \( \mathcal{V}_S^2 \) we have \( N \) stable generators of the form \( S_n = \varphi_n \otimes \bar{\varphi}_n \) with \( \varphi_n, \bar{\varphi}_n \in S_0(\mathbb{R}^d), n = 1, 2, \ldots, N. \) In this regard, note that in order to apply Theorem 2 we have that

\[
\mathcal{F}_W(\varphi_n \otimes \bar{\varphi}_n)(z) = e^{\pi i x \cdot \omega} V_{\bar{\varphi}_n} \varphi_n(z), \quad z = (x, \omega) \in \mathbb{R}^{2d} \quad \text{(see Ref. [21]).}
\]

Next, for each \( T \in \mathcal{V}_S^2 \) we consider the diagonal channel samples \( \langle T \pi(\lambda)g_m, \pi(\lambda)\bar{g}_m \rangle_{L^2(\mathbb{R}^d)} \), \( \lambda \in \Lambda \) and \( m = 1, 2, \ldots, M. \) with \( g_m, \bar{g}_m \in S_0(\mathbb{R}^d). \) In this case, for \( m = 1, 2, \ldots, M \) and \( n = 1, 2, \ldots, N, \) we get

\[
\begin{align*}
  a_{m,n}(\lambda) &= \langle \alpha_{\lambda}(\varphi_n \otimes \bar{\varphi}_n), \bar{g}_m \rangle_{L^2(\mathbb{R}^d)} = \langle (\varphi_n \otimes \bar{\varphi}_n)\pi(\lambda)g_m, \pi(\lambda)\bar{g}_m \rangle_{L^2(\mathbb{R}^d)} \\
  &= \langle (\pi(\lambda)g_m, \bar{\varphi}_n)\varphi_n, \pi(\lambda)\bar{g}_m \rangle_{L^2(\mathbb{R}^d)} = \mathcal{V}_{g_m} \varphi_n(\lambda) \bar{g}_m(\lambda), \quad \lambda \in \Lambda.
\end{align*}
\]

From Proposition 4.1 in Ref. [24] we deduce that the sequences \( \{a_{m,n}(\lambda)\}_{\lambda \in \Lambda} \) belong to \( \ell^1(\Lambda) \) and, as a consequence, the entries in the transfer matrix \( \hat{A} \) are continuous functions on the compact \( \hat{\Lambda}. \) In order to apply Theorem 3 conditions in Eq. (9) reduce to

\[
\det[\hat{A}(\xi)^* \hat{A}(\xi)] \neq 0 \quad \text{for all } \xi \in \hat{\Lambda}.
\]

Under the above circumstances, any \( T \in \mathcal{V}_S^2 \) which is nothing but \( T = \sum_{n=1}^N G_{\varphi_n} \varphi_n \) a finite sum of Gabor multipliers, can be recovered, in a stable way, from its diagonal channel samples \( \langle T \pi(\lambda)g_m, \pi(\lambda)\bar{g}_m \rangle_{L^2(\mathbb{R}^d)} \), \( \lambda \in \Lambda \) and \( m = 1, 2, \ldots, M. \)

### 3.5 Sampling in a sub-lattice of \( \Lambda \)

Let \( \Lambda' \) be a sub-lattice of \( \Lambda \) with finite index \( L, \) i.e., the quotient group \( \Lambda/\Lambda' \) has finite order \( L. \) We consider \( \{\lambda_1, \lambda_2, \ldots, \lambda_L\} \) a set of representatives of the cosets of \( \Lambda'. \) That is, the
lattice $\Lambda$ be decomposed as

$$\Lambda = \bigcup_{l=1}^{L} (\lambda_l + \Lambda') \quad \text{with} \quad (\lambda_l + \Lambda') \cap (\lambda_{l'} + \Lambda') = \emptyset \quad \text{for} \quad l \neq l'.$$

Thus, the space $V^2_S$ can be written as

$$V^2_S = \left\{ \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda) \alpha_\lambda(S_n) : c_n \in \ell^2(\Lambda) \right\} = \left\{ \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{\mu \in \Lambda'} c_n(\lambda_l + \mu) \alpha_{\lambda_l + \mu}(S_n) \right\} = \left\{ \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{\mu \in \Lambda'} c_{nl}(\mu) \alpha_\mu(S_{nl}) : c_{nl} \in \ell^2(\Lambda') \right\},$$

where $c_{nl}(\mu) := c_n(\lambda_l + \mu)$ and $S_{nl} := \alpha_\lambda(S_n)$, and the new index $nl$ goes from 11 to $NL$. As a consequence, the subspace $V^2_S$ can be rewritten as $V^2_{\tilde{S}}$ with $NL$ generators $\tilde{S} = \{S_{nl}\}$ and coefficients $c_{nl}$ in $\ell^2(\Lambda')$.

Let $T = \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{\nu \in \Lambda'} c_{nl}(\nu) \alpha_\nu(S_{nl})$ be in $V^2_S$; its samples $\langle \alpha_{-\mu}(T)g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)}$, $\mu \in \Lambda'$, can be expressed by

$$s_{T,m}(\mu) := \langle \alpha_{-\mu}(T)g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \left\langle \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{\nu \in \Lambda'} c_{nl}(\mu) \alpha_{-\nu}(S_{nl})g_m, \tilde{g}_m \right\rangle_{L^2(\mathbb{R}^d)} = \sum_{n=1}^{N} \sum_{l=1}^{L} \left( a_{m,\nu, *} \alpha_{\nu}(S_{nl})g_m, \tilde{g}_m \right)_{L^2(\mathbb{R}^d)}, \quad \mu \in \Lambda',$$

where $a_{m,\nu}(\nu) := \langle \alpha_{-\nu}(S_{nl})g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)}$, $\nu \in \Lambda'$. Hence, Theorem 4 gives:

**Corollary 9.** Let $A = [a_{m,\nu}]$ be the $M \times NL$ matrix with entries

$$a_{m,\nu}(\nu) = \langle \alpha_{-\nu}(S_{nl})g_m, \tilde{g}_m \rangle_{L^2(\mathbb{R}^d)}, \quad \nu \in \Lambda',$$

for $m = 1, 2, \ldots, M$ and $nl = 11, 12, \ldots, NL$. Assume that $A$ satisfies conditions in (9) with respect to the dual $\Lambda'$. Then, there exist $M \geq NL$ operators $H_m \in V^2_S$, $m = 1, 2, \ldots, M$, such that the sampling formula

$$T = \sum_{m=1}^{M} \sum_{\mu \in \Lambda'} s_{T,m}(\mu) \alpha_\mu(H_m) \quad \text{in } \mathcal{HS}-\text{norm}$$

holds for each $T \in V^2_S$ where $\{\alpha_\mu(H_m)\}_{\mu \in \Lambda'; m = 1, 2, \ldots, M}$ is a frame for $V^2_S$. The convergence of the series is unconditional in Hilbert-Schmidt norm.

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