Self-adjoint Dirac type Hamiltonians in one space dimension with a mass jump

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Received 5 July 2014, revised 12 November 2014
Accepted for publication 14 November 2014
Published 5 January 2015

Abstract

Physical self-adjoint extensions and their spectra of the one-dimensional Dirac type Hamiltonian operator, in which both the mass and velocity are constant except for a finite jump at one point of the real axis, are found correctly. Different boundary conditions on envelope wave functions are studied and the limiting case of equal masses (with no mass jump) is reviewed. Transport across one-dimensional heterostructures described by the Dirac equation is considered.

Keywords: self-adjoint extensions, boundary conditions, mass jump, heterostructures, graphene

1. Introduction

The discovery of graphene ended the belief that the Dirac equation was useless in condensed matter physics [1, 2]. The scientific and technological potential for exploiting charge carriers and quasiparticles with relativistic behavior in tunable condensed matter and atomic physics systems is attracting much attention [3–6]. In this regard, an important question, as yet only partly explored, remains: whether quasi-one-dimensional graphene systems exclusively
support Dirac–Weyl massless or constant-mass Dirac fermions, or whether they can induce relativistic quantum field behaviors that require consideration of a position-dependent mass term [7]. The mass jump case was first considered in [8] for spherical defects in III–V semiconductors and then in [9–11] for cylindrically symmetric defects in graphene, where the presence of the mass jump was traced back to the graphene sublattice symmetry violation, which is natural except for short range defects. Using the Dirac–Weyl equation, the breakdown of the sublattice symmetry (the equivalence of the two triangular lattices in graphene) can be described in terms of the effective mass which can be position-dependent. The existence of localized Dirac fermions in graphene with an inhomogeneous effective mass was considered in [12], where the conditions under which the Dirac fermions are confined were found.

Models with an abrupt discontinuity of the mass and velocity at one point can be used to describe the behavior of a quantum particle moving between two different materials, i.e. an electron moving in a medium formed by two different materials. In each material the particle behaves as if it had a different mass and velocity. The discontinuity point represents the junction between these two materials.

We propose a simple model that describes an electron moving in a medium formed by two different materials given by a one-dimensional system in which the mass and velocity are constant except for a finite jump at one point of the real axis, which is chosen to be the origin for simplicity,

$$m(x) = \begin{cases} m_l & \text{if } x < 0 \\ m_r & \text{if } x > 0 \end{cases}$$  \tag{1}

where $m_l$ and $m_r$ ($m_l \neq m_r$) are the masses at rest on the left and right, respectively. In this case, the Hamiltonian operator has the functional form

$$H = \begin{cases} -i\nu_l \sigma_z \frac{d}{dx} + m_l v_l^2 \sigma_z & \text{if } x < 0 \\ -i\nu_r \sigma_z \frac{d}{dx} + m_r v_r^2 \sigma_z & \text{if } x > 0 \end{cases}$$  \tag{2}

where $v_l$, $v_r$ are the Fermi velocities in each medium and

$$\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \tag{3}

are the Pauli matrices.

Finding appropriate boundary conditions in these kinds of models is very important to describe the correct physics. The study of the role of the boundary conditions of quantum systems has became a recent focus of activity in different branches of physics [13, 14]. Some examples of quantum physical phenomena which are intimately related to boundary conditions are the Casimir effect [15], the role of edge states [16] and the quantization of conductivity in the Hall effect [17].

In this paper, we show that the operator (2), in a suitable domain, has infinite self-adjoint extensions. All the self-adjoint extensions have real discrete spectra. Thus, all self-adjoint extensions describe bound states only, but not all the extensions are physically acceptable. We examine which extensions could play an interesting role according to physical arguments.

The paper is organized as follows: in section 2, we find the set of all possible self-adjoint extensions of $H$. In section 3, we calculate the reflection and transmission coefficients for all self-adjoint extensions and we use constraints from physical arguments to reduce the set of all possible self-adjoint extensions. From the equation of the poles of the scattering coefficients, we obtain the spectrum that characterizes each self-adjoint extension.
2. Self-adjoint extensions of $H$

We will follow Reed [18] and Naimark [19] to construct the self-adjoint extensions of the operator $H$. We must begin by defining the smaller domain where the action operator makes sense. In this section we will assume that the operator $H$ is densely defined. The domain of the operator $H$, $\mathcal{D}(H)$, is

$$\mathcal{D}(H) = \{ \psi \in W^{2,1}(\mathbb{R}) \otimes \mathbb{C}^2, \; \psi(0^-) = \psi(0^+) = 0 \},$$

(4)

where $W^{2,1}(\mathbb{R})$ is the corresponding Sobolev space and $\psi$ is a two-component spinor wave function

$$\psi(x) = \begin{pmatrix} \psi_a(x) \\ \psi_b(x) \end{pmatrix}. \quad (5)$$

The operator $H$ is symmetric and closed. Let $H^\dagger$ be the adjoint of $H$, with domain

$$\mathcal{D}(H^\dagger) = \{ \psi \in W^{2,1}(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2 \}.$$  

(6)

Note that $\mathcal{D}(H) \subset \mathcal{D}(H^\dagger)$. The deficiency subspaces of $H$ are given by

$$\mathcal{N}_\pm = \left\{ \psi_\pm \in \mathcal{D}(H^\dagger), \; H^\dagger \psi_\pm = \pm i \psi_\pm \right\},$$  

(7)

with the respective dimensions $n_+$ and $n_-$. These are called the deficiency indices of the operator $H$ and will be denoted by the ordered pair $(n_+, n_-)$. The normalized solutions of $H^\dagger \psi_\pm = \pm i \psi_\pm$ are

$$\psi_+(\pm)(x) = \left( \frac{1 + m_\psi v_\psi}{\sqrt{v_\psi}} \right)^{1/4} \left( \frac{1}{\sqrt{\frac{1 + m_\psi v_\psi}{i + m_\psi v_\psi}}} \right) \theta(x) e^{-i \frac{\pm \sqrt{v_\psi^2 - m_\psi^2}}{m_\psi}} \theta(x) e^{i \frac{\pm \sqrt{v_\psi^2 - m_\psi^2}}{m_\psi}},$$  

(8a)

$$\psi_-(\pm)(x) = \left( \frac{1 + m_\psi v_\psi}{\sqrt{v_\psi}} \right)^{1/4} \left( \frac{1}{\sqrt{\frac{1 + m_\psi v_\psi}{i + m_\psi v_\psi}}} \right) (-x) \theta(x) e^{-i \frac{\pm \sqrt{v_\psi^2 - m_\psi^2}}{m_\psi}} \theta(x) e^{i \frac{\pm \sqrt{v_\psi^2 - m_\psi^2}}{m_\psi}},$$  

(8b)

$$\psi_+(\pm)(x) = \left( \frac{1 + m_\psi v_\psi}{\sqrt{v_\psi}} \right)^{1/4} \left( \frac{1}{\sqrt{\frac{1 + m_\psi v_\psi}{m_\psi v_\psi - i}}} \right) \theta(x) e^{-i \frac{\pm \sqrt{v_\psi^2 - m_\psi^2}}{m_\psi}} \theta(x) e^{i \frac{\pm \sqrt{v_\psi^2 - m_\psi^2}}{m_\psi}},$$  

(8c)

$$\psi_-(\pm)(x) = \left( \frac{1 + m_\psi v_\psi}{\sqrt{v_\psi}} \right)^{1/4} \left( \frac{1}{\sqrt{\frac{1 + m_\psi v_\psi}{m_\psi v_\psi - i}}} \right) (-x) \theta(x) e^{-i \frac{\pm \sqrt{v_\psi^2 - m_\psi^2}}{m_\psi}} \theta(x) e^{i \frac{\pm \sqrt{v_\psi^2 - m_\psi^2}}{m_\psi}},$$  

(8d)

where $\theta(x)$ represents the Heaviside step function. Since all the solutions of equations $H^\dagger \psi_\pm = \pm i \psi_\pm$ belong to $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, the deficiency indices are $(2, 2)$ and, according to Naimark [19], every self-adjoint extension is parametrized by a $U(2)$ matrix. This matrix defines a unique self-adjoint extension, $H_U$, of $H$ with the domain characterized by the set of all functions $\phi \in \mathcal{D}(H^\dagger)$ which satisfy the conditions

$$\begin{pmatrix} \psi_{a2}(0^-) & \psi_{a1}(0^-) \\ \psi_{b2}(0^-) & \psi_{b1}(0^-) \end{pmatrix} \begin{pmatrix} \phi(0^-) \\ \phi(0^-) \end{pmatrix} = \frac{v_\psi}{\sqrt{v_\psi}} \begin{pmatrix} \psi_{a2}(0^+) & \psi_{a1}(0^+) \\ \psi_{b2}(0^+) & \psi_{b1}(0^+) \end{pmatrix} \begin{pmatrix} \phi(0^+) \\ \phi(0^+) \end{pmatrix} \quad (9)$$
where $\psi(0^\pm) \equiv \lim_{x \to 0^\pm} \psi(x)$ and $\phi(0^\pm) \equiv \lim_{x \to 0^\pm} \phi(x)$, and

\[
\psi_1(x) = \psi_1^{(+)}(x) + U_{11}\psi_1^{(-)}(x) + U_{21}\psi_2^{(-)}(x)
\]

\[
\psi_2(x) = \psi_2^{(+)}(x) + U_{12}\psi_2^{(+)}(x) + U_{22}\psi_2^{(-)}(x)
\]

where $\psi_1(x), \psi_2(x) \in D(H_U)$, and $U_{11}, U_{12}, U_{21}, U_{22}$ are complex numbers that determine $U$.

The expression (9) can be written in the form

\[
\begin{pmatrix}
\psi_\alpha(0^+)
\psi_\beta(0^+)
\phi_\alpha(0^+)
\phi_\beta(0^+)
\end{pmatrix}
= \begin{pmatrix}
\phi_a(0^-)
\phi_b(0^-)
\phi_a(0^-)
\phi_b(0^-)
\end{pmatrix} \begin{pmatrix}
\psi_{a2}(0^+)
\psi_{b2}(0^+)
\psi_{a1}(0^+)
\psi_{b1}(0^+)
\end{pmatrix}^{-1}
\begin{pmatrix}
\psi_{a2}(0^-)
\psi_{b2}(0^-)
\psi_{a1}(0^-)
\psi_{b1}(0^-)
\end{pmatrix},
\]

where the $n_+ \times n_-$ matrix $\mathbb{I}$ is given by

\[
\mathbb{I} = \frac{v_1}{v_i} \begin{pmatrix}
\psi_{a2}(0^+)
\psi_{b2}(0^+)
\psi_{a1}(0^+)
\psi_{b1}(0^+)
\end{pmatrix}^{-1}
\begin{pmatrix}
\psi_{a2}(0^-)
\psi_{b2}(0^-)
\psi_{a1}(0^-)
\psi_{b1}(0^-)
\end{pmatrix},
\]

whose determinant is given by

\[
|\det \mathbb{I}| = \frac{v_1}{v_i}.
\]

The matrix $\mathbb{I}$ gives the matching conditions at the origin. From (10), we can rewrite the matrix $\mathbb{I}$ in the form

\[
\mathbb{I} = \frac{\sqrt{v_1}}{2U_{12}\sqrt{v_i}} \begin{pmatrix}
T_{11} & T_{12}
T_{21} & T_{22}
\end{pmatrix}
\]

with

\[
T_{11} = \frac{(1 + m_i^2 v_i^4)^\frac{1}{2} \left( (1 - im_1 v_1^2)(1 + U_{11}) - (1 + im_1 v_1^2)(U_{22} + \text{det}(U)) \right)}{(1 + m_i^2 v_i^4)^\frac{1}{2}}
\]

(16a)

\[
T_{12} = -(1 + \text{det}(U) + U_{11} + U_{22}) \left( (1 + m_i^2 v_i^4)(1 + m_i^2 v_i^4)^\frac{1}{2} \right)^\frac{1}{2}
\]

(16b)

\[
T_{21} = \frac{(i + m_1 v_1^2 + U_{22}(m_1 v_1^2 - i))(U_{11}(m_1 v_1^2 - i) + m_i v_i^2 + i)}{(1 + m_i^2 v_i^4)(1 + m_i^2 v_i^4)^\frac{1}{2} - \frac{U_{12} U_{21}(m_1 v_1^2 - i)(m_i v_i^2 - i)}{(1 + m_i^2 v_i^4)(1 + m_i^2 v_i^4)^\frac{1}{2}}}
\]

(16c)

\[
T_{22} = \frac{(1 + m_i^2 v_i^4)^\frac{1}{2} \left( (1 - im_1 v_1^2)(1 + U_{22}) - (1 + im_1 v_1^2)(U_{11} + \text{det}(U)) \right)}{(1 + m_i^2 v_i^4)^\frac{1}{2}}
\]

(16d)
The determinant of (15) is given by

$$\det \mathcal{T} = \frac{v_1 \tilde{U}_{21}}{v_1 \tilde{U}_{12}}. \quad (17)$$

By comparing (17) with (14), we have that $|U_{12}| = |U_{21}|$.

3. Scattering coefficients and the spectra of $H$

In this section we will derive the spectra for the self-adjoint extensions $H_U$ from the poles of the scattering amplitudes. For this, let us parametrize the unitary matrix $U$ as

$$U = e^{ia}A, \quad \det(A) = 1, \quad (18)$$

where

$$A = \begin{pmatrix} a_0 - ia_3 & -a_2 - ia_1 \\ a_2 - ia_1 & a_0 + ia_3 \end{pmatrix}. \quad (19)$$

with $a_0, a_1, a_2, a_3 \in \mathbb{R}$, $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$ and $\alpha \in [0, \pi]$. Notice that the points $\alpha = 0$ and $\alpha = \pi$ have to be identified. Substituting (18) and (19) in (16a), we obtain the components of matrix $\mathcal{T}$:

$$T_{11} = 2ie^{-ia} \left( 1 + m_1^2 v_1^4 \right)^{\frac{1}{2}} \left( a_3 + \sin \alpha - m_1 v_1^2 (a_0 + \cos \alpha) \right) \left( 1 + m_1^2 v_1^4 \right)^{\frac{1}{2}}. \quad (20a)$$

$$T_{12} = 2e^{-ia} (a_0 + \cos \alpha) \left( 1 + m_1^2 v_1^4 \right)^{\frac{1}{2}} \left( 1 + m_1^2 v_1^4 \right)^{\frac{1}{2}}. \quad (20b)$$

$$T_{21} = 2e^{-ia} \left( a_0 - \cos \alpha + m_1 m_2 v_1^2 v_2^3 (a_0 + \cos \alpha) + m_1 v_1^2 (a_3 - \sin \alpha) - m_1 v_1^2 (a_3 + \sin \alpha) \right) \left( 1 + m_1^2 v_1^4 \right)^{\frac{1}{2}} \left( 1 + m_1^2 v_1^4 \right)^{\frac{1}{2}}. \quad (20c)$$

$$T_{22} = -2ie^{-ia} \left( 1 + m_1^2 v_1^4 \right)^{\frac{1}{2}} \left( a_3 - \sin \alpha + m_1 v_1^2 (a_0 + \cos \alpha) \right) \left( 1 + m_1^2 v_1^4 \right)^{\frac{1}{2}}. \quad (20d)$$

In terms of (20), the matching conditions (12) are

$$\begin{pmatrix} \phi_b(0^+) \\ \phi_b(0^-) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \phi_b(0^+) \\ \phi_b(0^+) \end{pmatrix}. \quad (21)$$

Let us assume that an incoming monochromatic wave $\left( \begin{pmatrix} 1 \\ \sqrt{E - m_1 v_1^2} \end{pmatrix} e^{ik_1 x} \right)$, $k_1 = \frac{\sqrt{E^2 - m_1^2 v_1^2}}{v_1}$, $E > \max (m_1 v_1^2, m_1 v_1^2)$, comes from the left, so that the wave function for
\[ x < 0 \text{ is } \left( \frac{1}{E - m_1 v_1^2} \right) e^{ik_1 x} + \eta \left( \frac{1}{\sqrt{E + m_1 v_1^2}} \right) e^{-ik_1 x}, \] and the wave function for \( x > 0 \) is

\[ t_1 \left( \frac{1}{\sqrt{E - m_2 v_1^2}} \right) e^{ik_1 x}, \quad k_1 = \frac{\sqrt{E^2 - m_2^2 v_1^2}}{v_i}, \quad E > \max (m_1 v_1^2, m_2 v_1^2), \] where \( \eta \) and \( t_1 \) are the reflection and transmission amplitudes, respectively, for an incoming wave coming from the left. Then, the matching conditions (21) at the origin give

\[
\left( \begin{array}{c} 1 + \eta \\ E - m_1 v_1^2 \\ E + m_1 v_1^2 
\end{array} \right) = \mathbb{I} \left( \begin{array}{c} t_1 \\ \sqrt{E - m_2 v_1^2} \\ \sqrt{E + m_2 v_1^2} 
\end{array} \right) \tag{22}
\]

and then finally one obtains the expressions of \( \eta \) and \( t_1 \) as

\[
\eta = \frac{\mathcal{N}}{\mathcal{D}} \tag{23}
\]

\[
t_1 = 2 \frac{\sqrt{\frac{a_1^2 + a_2^2}{v_i}}} {\mathcal{D}} \mathcal{I} \quad \tag{24}
\]

with

\[
\mathcal{N} = -\sqrt{m_1^2 v_1^4 + 1} \sqrt{E - m_1 v_1^2} \left( \sqrt{E + m_1 v_1^2} \left( a_3 + \sin \alpha - m_1 v_1^2 \left( a_0 + \cos \alpha \right) \right) + i (a_0 + \cos \alpha) \sqrt{\left( m_1^2 v_1^4 + 1 \right) \left( E - m_1 v_1^2 \right)} \right) - \sqrt{m_1^2 v_1^4 + 1} \sqrt{E - m_1 v_1^2} \sqrt{E + m_1 v_1^2} \left( a_3 - \sin \alpha + m_1 v_1^2 \left( a_0 + \cos \alpha \right) \right) - i \sqrt{E + m_1 v_1^2} \sqrt{E + m_1 v_1^2} \left( m_1 m_2 v_1^2 v_2^2 \left( a_0 + \cos \alpha \right) + a_0 \right) - \cos \alpha + \left( m_1 v_1^2 \left( a_3 - \sin \alpha \right) - m_2 v_2^2 \left( a_3 + \sin \alpha \right) \right), \tag{25}
\]

\[
\mathcal{D} = \sqrt{m_1^2 v_1^4 + 1} \sqrt{E - m_1 v_1^2} \left( \sqrt{E + m_1 v_1^2} \left( a_3 + \sin \alpha - m_1 v_1^2 \left( a_0 - \cos \alpha \right) \right) - i (a_0 + \cos \alpha) \sqrt{\left( m_1^2 v_1^4 + 1 \right) \left( E - m_1 v_1^2 \right)} \right) - \sqrt{m_1^2 v_1^4 + 1} \sqrt{E - m_1 v_1^2} \sqrt{E + m_1 v_1^2} \left( a_3 - \sin \alpha + m_2 v_2^2 \left( a_0 + \cos \alpha \right) \right) + i \sqrt{E + m_1 v_1^2} \sqrt{E + m_1 v_1^2} \left( m_1 m_2 v_1^2 v_2^2 \left( a_0 + \cos \alpha \right) + a_0 \right) - \cos \alpha + \left( m_1 v_1^2 \left( a_3 - \sin \alpha \right) - m_2 v_2^2 \left( a_3 + \sin \alpha \right) \right), \tag{26}
\]

\[
\mathcal{I} = \sqrt{\left( m_1^2 v_1^4 + 1 \right) \left( m_2^2 v_2^4 + 1 \right) \left( E - m_1 v_1^2 \right) \left( E + m_2 v_2^2 \right)}. \tag{27}
\]

Since the matrix \( \mathbb{I} \) is not real, the transmission amplitudes are different and the self-adjoint extensions are not explicitly time reversal invariant [20, 21].
Physically, the term \(e^{-i \arctan \left( \frac{E}{m} \right)}\) in (24) does not add new information to the phase shift, since \(a_0, a_1, a_2\) and \(a_3\) are independent of the energy, so we can put \(a_2 = 0\) without any loss of information. The matrix \(\Pi\) coincides with the corresponding one in the nonrelativistic case [22] precisely when \(a_2 = 0\). In this situation, we have that the matching conditions (21) are

\[
\begin{pmatrix}
\phi_b(0^+)
\phi_b(0^-)
\end{pmatrix}
= \Pi
\begin{pmatrix}
\phi_b(0^+)
\phi_b(0^-)
\end{pmatrix}
\]

(28)

the determinant of which is

\[
\det \Pi \bigg|_{a_2=0} = \frac{\eta_1}{\eta_2}.
\]

(29)

Making use of \(a_0^2 + a_1^2 + a_2^2 = 1\), we have \(\left|\eta_1\right|^2 + \left|\eta_2\right|^2 + \frac{E + m_F^2}{\sqrt{E - m_F^2}} \frac{\eta_1}{\eta_2} = 1\). The poles of \(\eta_1\) and \(\eta_2\) satisfy the following equation

\[
\mathfrak{D} = 0.
\]

(30)

The poles of \(\eta_1\) and \(\eta_2\) (\(r_l\) and \(t_r\)) are the reflection and transmission amplitudes, respectively, for an incoming wave coming from the right) also satisfy (30). The zero values of (23) correspond to transmission resonances [23, 24]. The zero values of (24) are called zero momentum resonances [25] and they occur at \(E = \pm m_F v_F^2\) and \(E = \pm m_F v_F^2\) \(v_F\) is the Fermi velocity). The anti-particle is described by the hole wave function corresponding to the absence of the state with \(E = \pm m_F v_F^2\) [25].

In the following sub-sections, we discuss the spectra of some self-adjoint extensions of (2) corresponding to a one-dimensional spatial Dirac Hamiltonian: (a) with an equally mixed point interaction potential (PIP) at the origin plus a mass jump at the same point, (b) with an inverted mixed PIP at the origin plus a mass jump at the same point, (c) with a vector PIP at the origin plus a mass jump at the same point and (d) with a scalar PIP at the origin plus a mass jump at the same point. The one-dimensional Dirac Hamiltonian with PIPs without a mass jump is analyzed in [26]. In this paper, the self-adjoint extensions of the one-dimensional Dirac operator with point interactions can also be obtained (heuristically) starting from the operator \(H = -i\sigma \frac{d}{dx} + m\sigma + U(x)\), with \(U(x) = \left( g_v + g_s \right) u(x)\), where \(u(x)\) is any peaked function at \(x = 0\) satisfying \(\int_{-\infty}^{\infty} u(x)dx = 1\). \(g_v\) and \(g_s\) are the strengths of the vector and scalar components of the potential, respectively. When \(g_v = 0, g_s = 0\), \(g_v = g_s = -g_s\), we say that we have a scalar PIP, vector PIP, equally mixed PIP and inverted mixed PIP, respectively. For simplicity and comparison, in the following sections we will impose that \(\eta_1 = \eta_2 = \eta_F\). Thus, (29) equals one, similarly to the case of equal masses.

### 3.1. A one-dimensional spatial Dirac Hamiltonian with a equally mixed PIP at the origin plus a mass jump at the same point

The boundary conditions corresponding to a one-dimensional spatial Dirac Hamiltonian with a vector PIP at the origin plus a mass jump at the same point are obtained by the following equations:

\[
a_0 = -\cos \alpha, \quad (31a)
\]

\[
a_1 = \sin \alpha, \quad (31b)
\]

\[
a_3 = 0, \quad (31c)
\]
\[\cot \alpha = -\frac{\delta + (m_1 + m_r)v_F^3}{2v_F}, \quad \delta < 0, \tag{31d}\]

where \(\delta\) is the strength of the PIP. By inserting (31) in (28), we obtain the matching conditions for this self-adjoint extension:

\[
\begin{pmatrix}
\phi_a(0^-) \\
\phi_b(0^-)
\end{pmatrix} =
\begin{pmatrix}
\frac{\sqrt[4]{m_r^2v_F^4 + 1}}{\sqrt[4]{m_l^2v_F^4 + 1}} & 0 \\
\frac{i\delta}{v_F\sqrt[4]{m_l^2v_F^4 + 1}} & \frac{i\delta}{v_F\sqrt[4]{m_r^2v_F^4 + 1}}
\end{pmatrix}
\begin{pmatrix}
\phi_a(0^+) \\
\phi_b(0^+)
\end{pmatrix} \tag{32}
\]

By inserting (31) in (30), we obtain the spectral equation (bound states energy equation)

\[
\frac{\delta}{v_F}\sqrt{(E + v_F^2m_1)(E + v_F^2m_r) + \sqrt{(v_F^2m_l^2 + 1)(v_F^2m_1 - E)(E + v_F^2m_r)}} + \sqrt{(v_F^4m_r^2 + 1)(E + v_F^2m_1)(v_F^2m_r - E)} = 0. \tag{33}
\]

The energy of bound states lies between \(-\min{(m_1v_F^2, m_rv_F^2)}\) and \(\min{(m_1v_F^2, m_rv_F^2)}\). Note that equation (33) is invariant under the change of \(m_1\) by \(m_r\).

For \(m_1 = 1, m_r = 2\) and \(v_F = 1\), the solution curve of (33) as a function of \(\delta\) is represented in figure 1. The solution curves intersect at a point, which means that they have the same energy for a given value of \(\delta\). The value of the energy is insensitive to the ratio of the masses.

As stated in [26], the boundary of the lower continuum is never reached for finite values of \(\delta\) due to the presence of the scalar potential term, but the energy level crosses zero because of the vector potential term (see figure 1).
For \( m_1 \approx m_r \equiv m \), (33) becomes
\[
2\sqrt{v_F} \left( \gamma^0 \left( v_F^2 m^2 + 1 \right) \right) + \delta \sqrt{\left( E + v_F^2 m \right)^2} = 0,
\]
which gives the value of the energy for the bound state
\[
E = m v_F^2 + 4 m^2 v_F^2 \delta^2.
\]
Defining \( \delta = \frac{\delta}{v_F \sqrt{v^2 m^2 + 1 + m^2 v_F^2 + \delta^2}} \), the above expression can be rewritten as
\[
E = m v_F^2 + 4 \frac{\delta^2}{\delta^2 + \delta^2}.
\]
The energy (36) coincides with that found in [26] and [27] for the self-adjoint extension called the equally mixed potential.

At high energies, we have
\[
|\psi|^2 \sim \frac{4 v_F^2 \left( \gamma^0 \left( v_F^2 m^2 + 1 \right) \right)}{v_F^2 \left( \gamma^0 \left( m^2 + 1 + m^2 v_F^2 + \delta^2 \right) \right)^2} + \delta^2
\]
so the transmission does not occur as the potential becomes sufficiently strong. Therefore, the interaction of equally mixed PIP at the origin plus a mass jump at the same point does confine particles. The same conclusion is reported in [26].

3.2. A one-dimensional spatial Dirac Hamiltonian with a inverted mixed PIP at the origin plus a mass jump at the same point

The matching conditions for this self-adjoint extension are
\[
\left( \begin{array}{c}
\phi_0^+(0^+) \\
\phi_0^-(0^-)
\end{array} \right) =
\left( \begin{array}{cc}
\left( 1 + m_r^2 v_F^2 \right)^{1/2} & -i v_F \left( 1 + m_r^2 v_F^2 \right)^{1/2} \\
\left( 1 + m_r^2 v_F^2 \right)^{1/2} & 0 \\
0 & \left( 1 + m_r^2 v_F^2 \right)^{1/2}
\end{array} \right)
\left( \begin{array}{c}
\phi_0^+(0^-) \\
\phi_0^-(0^-)
\end{array} \right)
\]
where \( \lambda \) is the strength of the PIP, \( \lambda > 0 \). The spectral equation is
\[
\sqrt{\left( v_F^2 m^2 + 1 \right) \left( v_F^2 m_1 - E \right) \left( E + v_F^2 m_r \right) + \left( v_F^2 m_r + 1 \right) \left( E + v_F^2 m_1 \right) \left( v_F^2 m_r - E \right) + \lambda \sqrt{\left( v_F^2 m_1 - E \right) \left( E + v_F^2 m_r \right) + \left( v_F^2 m_r + 1 \right) \left( E + v_F^2 m_1 \right) \left( v_F^2 m_r - E \right)}} = 0,
\]
where \( -\min \left( m_1 v_F^2, m_r v_F^2 \right) < E < \min \left( m_1 v_F^2, m_r v_F^2 \right) \). This equation is invariant under the change of \( m_1 \) by \( m_r \).

For \( m_1 \approx m_r \equiv m \), (39) becomes
\[
2 v_F \sqrt{\left( v_F^2 m^2 - E^2 \right) + \lambda \sqrt{\left( v_F^2 m^2 + 1 \right) \left( v_F^2 m - E \right)}} = 0,
\]
which gives the value of the energy for the bound state

$$E = -mv_F^2 \frac{4 - \frac{\lambda^2}{2} + \frac{\lambda}{2}}{4 + \frac{\lambda}{2}},$$

with \( \lambda \equiv v_F \sqrt{1 + m^2 v_F^4} \). The energy (41) coincides with the one found in [26] and [27] for the self-adjoint extension called the inverted mixed potential.

For \( m_1 = 1, m_r = 2 \) and \( v_F = 1 \), the solution curve of (39) as a function of \( \lambda \) is represented in figure 2. The solution curves intersect at a point, which means that they have the same energy for \( \lambda = 0.6324 \). For weak coupling, the bound state’s energy reaches the lower negative continuum, as distinct from that shown in [26].

At high energies, the transmission coefficient becomes

$$|t|^2 \sim \frac{4v_F^2m_1^2 + 1}{(v_F^2m_1^2 + 1)(v_F^2m_r^2 + 1)\lambda^2}$$

so that the transmission does not occur as the potential becomes sufficiently strong. As in the previous subsection, the interaction inverted mixed PIP at the origin plus a mass jump at the same point does confine particles. This same conclusion is reported in [26] for this self-adjoint extension.

### 3.3. A one-dimensional spatial Dirac Hamiltonian with a pure scalar PIP at the origin plus a mass jump at the same point

The boundary conditions corresponding to a one-dimensional spatial Dirac Hamiltonian with a pure scalar PIP at the origin plus a mass jump at the same point are

$$\begin{pmatrix}
\phi_0(0^+) \\
\phi_0(0^-)
\end{pmatrix} = \begin{pmatrix}
\left(1 + m_1^2v_F^4\right)^{1/2} \cosh \left(\frac{a}{v_F}\right) & i \sinh \left(\frac{a}{v_F}\right) \\
-i \sinh \left(\frac{a}{v_F}\right) & \left(1 + m_r^2v_F^4\right)^{1/2} \cosh \left(\frac{a}{v_F}\right)
\end{pmatrix} \begin{pmatrix}
\phi_0(0^-) \\
\phi_0(0^+)\end{pmatrix}$$

where \( a \) is the strength of the PIP, \( a < 0 \). The spectral equation is

$$4v_F^2m_1^2 + 1 + \frac{1}{2} \left(v_F^2m_1 - E \right) \left(v_F^2m_r - E\right)$$

$$+ \left(v_F^2m_1 - E \right) \left(v_F^2m_r + E\right) \sinh \left(\frac{a}{v_F}\right)$$

$$+ \left(\frac{1}{4} + \frac{1}{2} \left(v_F^2m_1 - E\right) \left(v_F^2m_r + E\right) \cosh \left(\frac{a}{v_F}\right) = 0,$$

where \( -\min(m_1, m_r) < E < \min(m_1, m_r) \). This equation is invariant under the change of \( m_1 \) by \( m_r \).
Pairs of permitted energy values appear, which is a common feature of the other scalar-type potentials [28]. As seen in figure 3, the same strength $a$ of the scalar potential can bind particles and antiparticles alike. As stated in [26], the energy level never reaches zero. The positive and negative energy states remain well separated even if the potential becomes strong.

For $m_1 \approx m_r \equiv m$, (44) becomes

$$\sqrt{v_F^2 m^2 - E^2} \cosh \left( \frac{a}{v_F} \right) + v_F^2 m \sinh \left( \frac{a}{v_F} \right) = 0,$$

(45)

Figure 2. Solution curves of (39) (solid line) and (41) (dashed line) as a function of $\lambda$.

Figure 3. Solution curves of (44) (solid line) and (46) (dashed line) as a function of $a$ for $m_1 = 1$, $m_r = 2$ and $v_F = 1$. 
which gives the value of the energies for bound states

$$E = \pm m v_F \sech \left( \frac{a}{v_F} \right).$$

The energy (46) coincides with the one found in [26] for the self-adjoint extension called the pure scalar potential. For this potential, at high energy, the transmission coefficient becomes

$$|\kappa|^2 \sim 4 \left( \sqrt{1 + v_F^2 m_1^2} \right) \left( \sqrt{1 + v_F^2 m_2^2} \right) \left( \sqrt{1 + v_F^2 m_1^2} \right) \left( \sqrt{1 + v_F^2 m_2^2} \right) \sech^2 \left( \frac{a}{v_F} \right).$$

Thus, the pure scalar potential leads to particle confinement when $|a| \to \infty$.

### 3.4. A one-dimensional spatial Dirac Hamiltonian with a pure vector PIP at the origin plus a mass jump at the same point

The boundary conditions corresponding to a one-dimensional spatial Dirac Hamiltonian with a pure vector PIP at the origin plus a mass jump at the same point are

$$
\begin{pmatrix}
\phi_b(0^-) \\
\phi_b(0^+)
\end{pmatrix} =

\begin{pmatrix}
m_r \cos \left( \frac{a}{v_F} \right) & -i \sin \left( \frac{a}{v_F} \right) \\
-i \sin \left( \frac{a}{v_F} \right) & m_r \cos \left( \frac{a}{v_F} \right)
\end{pmatrix}
\begin{pmatrix}
\phi_b(0^-) \\
\phi_b(0^+)
\end{pmatrix}
$$

with $a > 0$, contrary to the assertion in [26], where the sign of the strength $a$ is immaterial as far as the existence of bound states is concerned.
The spectral equation is
\[
\left( m_t^2 \sqrt{v_F^2 m_t + E \sqrt{v_F^2 m_t - E} + m_r^2 \sqrt{v_F^2 m_t - E} + m_l^2 \sqrt{v_F^2 m_t - E}} \right) \cos \left( \frac{a}{v_F} \right) \\
+ m_t m_l \left( \sqrt{v_F^2 m_t + E \sqrt{v_F^2 m_t + E} - \sqrt{v_F^2 m_t - E} \sqrt{v_F^2 m_t - E}} \right) \sin \left( \frac{a}{v_F} \right) = 0,
\] (49)

where \(-\min(m_l v_F^2, m_t v_F^2) < E < \min(m_l v_F^2, m_t v_F^2)\). Unlike the previous cases, (49) is not invariant under the change of \(m_l\) by \(m_r\). For \(m_l = 1, m_t = 2\) and \(v_F = 1\), the solution curve of (49) as a function of \(a\) is represented in figure 4.

For \(m_l \approx m_r \equiv m\), (49) becomes
\[
\sqrt{v_F^2 m^2 - E^2} \cos \left( \frac{a}{v_F} \right) + E \sin \left( \frac{a}{v_F} \right) = 0,
\] (50)
which gives the value of the energies for the bound states
\[
E = \pm m v_F^2 \cos \left( \frac{a}{v_F} \right).
\] (51)

The transmission coefficient is bounded from below,
\[
|t_1|^2 \geq \frac{8m_l^2 m_t^2 \sqrt{E - v_F^2 m_l \sqrt{E + v_F^2 m_t}} \sqrt{E - v_F^2 m_t} \sqrt{E + v_F^2 m_t} \sqrt{E + v_F^2 m_t}}{4m_l^2 m_t^2 \sqrt{E - v_F^2 m_l \sqrt{E + v_F^2 m_t}} \sqrt{E - v_F^2 m_t} \sqrt{E + v_F^2 m_t} + 8 \left( m_l^2 + m_t^2 \right)^2}
\] (52)

so the transmission always occurs. There are transmission resonances or virtual bound states [29] for values \(\frac{a}{v_F} = n\pi, n \in \mathbb{Z}\) (see figures 5 and 6).
4. Concluding remarks

Using von Neumann’s theory of self-adjoint extensions and eliminating spurious phases of the transmission amplitude, we found the general matching conditions (28) that describe each one of the different domains of the various self-adjoint extensions of (2). Using the scattering theory, we obtained the spectrum of each one of the extensions, where each corresponds to a different Hamiltonian operator with interaction.

Finally, we found that of the four different self-adjoint extensions of (2), the first three are confining self-adjoint extensions, while the last one is not. For three of the self-adjoint extensions, there is a value of the strength of the interaction point at which the energy of the particle is the same for the mass jump case and without it.

Acknowledgments

This work was supported by IVIC under Project No. 1089.

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