Composite Operators in QCD*

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We give a formula for the derivatives of a correlation function of composite operators with respect to the parameters (i.e., the strong fine structure constant and the quark mass) of QCD in four-dimensional euclidean space. The formula is given as spatial integration of the operator conjugate to a parameter. The operator product of a composite operator and a conjugate operator has an unintegrable part, and the formula requires divergent subtractions. By imposing consistency conditions we derive a relation between the anomalous dimensions of the composite operators and the unintegrable part of the operator product coefficients.

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1. Introduction

The dimensional regularization with the minimal subtraction has become everyone’s favorite method for perturbative calculations of renormalizable field theories. The method has three advantages. First, calculations are simple. Secondly, it is mass independent. Thirdly, the singularities higher than $1/\epsilon$ are determined completely by the simple $1/\epsilon$ poles.

Consider the $(\phi^4)_4$ theory as an example. The theory has three parameters $g_1$, $m^2$, and $\lambda$ with scaling dimensions 4, 2, and 0, respectively. These parameters are conjugate to the operators $1$, $\phi^2/2$, and $\phi^4/4!$ whose scaling dimensions are 0, 2, and 4, respectively. Under the renormalization group (RG), the coordinate distance $r$ transforms into $re^{-l}$, while the renormalization point is always fixed at $r = 1$. Hence, the RG acts toward the infrared limit, and it differs from the standard definition by rescaling. Then, under the RG the parameters transform as follows:

$$\frac{d}{dl} g_1 = 4g_1 + \frac{(m^2)^2}{2} \beta_1(\lambda)$$

$$\frac{d}{dl} m^2 = (2 + \beta_m(\lambda))m^2$$

$$\frac{d}{dl} \lambda = \beta_\lambda(\lambda).$$

As we can see, the scaling dimensions of the parameters are additively preserved under the RG; only terms of scaling dimension 4 are allowed in the first equation, while only terms of scaling dimension 2, 0 are allowed in the second and third, respectively. In fact in ref. it was shown that we can always choose parameters such that the scaling dimensions of the parameters are additively preserved in any field theory with an ultraviolet fixed point. These parameters were also shown to be appropriate to describe the short distance physics. Locality of the theory implies that the RG equations involve only integral powers of these parameters. Hence, the RG equations become finite polynomials of dimensionful parameters whose coefficients are power series in dimensionless parameters. Within perturbation theory the results of ref. applies also to the $\phi^4$ theory. The minimal subtraction with dimensional regularization provides an example of such a choice of parameters. The structure of (1.1) is called mass independent, since the beta functions depend only on the dimensionless parameter $\lambda$.

Now, in the minimal subtraction scheme in the dimensional regularization, the beta functions and the anomalous dimensions of composite operators are directly related to the
simple $1/\epsilon$ poles of the unrenormalized correlation functions. In other words, nontrivial renormalization properties of the theory, i.e., nonvanishing beta functions and anomalous dimensions, demand that the bare correlation functions have simple $1/\epsilon$ poles. Since the higher order poles in $1/\epsilon$ are related to the simple $1/\epsilon$ poles, we can say that all divergences in $\epsilon$ are expected consequences of nonvanishing beta functions and anomalous dimensions.

So far we have explained the advantages of the dimensional regularization with the minimal subtraction. We should not be totally happy with this method, however, since the physical meaning of the divergences in $\epsilon$ is unclear. The purpose of this paper is to consider physical singularities of correlation functions at short distances and relate the short distance singularities, rather than the unphysical singularities in $\epsilon$, to the anomalous dimensions. Our results are valid beyond perturbation theory, and we will use QCD in four dimensional euclidean space as an example.

The paper is organized as follows. We summarize the relevant facts on the RG equations in sect. 2 and those on the operator product expansions in sect. 3. Then, we introduce the main formula in this paper in sect. 4 that describes the change of correlation functions of composite operators under an infinitesimal change of parameters. In sect. 5 we derive a relation between the operator product coefficients and anomalous dimensions of composite operators by considering the consistency between the main formula of sect. 4 and the RG eqs. In sect. 6 we make yet another consistency check of the main formula of sect. 4, coming from commutativity of derivatives. In sect. 7 we introduce a convention for composite operators. We give concluding remarks in sect. 8.

2. Renormalization group equations

In this section we introduce relevant facts on the RG equations in QCD with massive quarks in four dimensional euclidean space. The theory is characterized by three parameters $g_1$, $m$, and $g_E$. The parameter $g_1$ is an additive constant to the lagrangian density, and its scaling dimension is four. The parameter $m$ is the quark mass parameter with scaling dimension one, and $g_E$ is the strong fine structure constant with scaling dimension zero.\footnote{We do not consider a gauge fixing parameter, since we will only consider gauge invariant operators in this paper.} We denote the operators conjugate to these parameters by $1$, $O_m$, and $O_E$, respectively. The operator $1$ is the identity operator. The operator $O_m$ is the mass density operator
$\bar{\psi}\psi$, and $O_E$ corresponds to the energy density operator $F_{\mu\nu}F_{\mu\nu}$. These three operators are the only gauge invariant scalar operators with scaling dimension less than or equal to four that conserve C and P. There is no other independent operator with these properties; $\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$ and $\bar{\psi}\gamma^\mu D_\mu\psi = m\bar{\psi}\psi$ by the eqs. of motion.

We can write down the RG eqs. in the following form:

$$
\frac{d}{dl} g_1 = 4g_1 + \frac{m^4}{4!} \beta_1(g_E)
$$

$$
\frac{d}{dl} m = (1 + \beta_m(g_E))m
$$

$$
\frac{d}{dl} g_E = \beta_c(g_E).
$$

The above form is determined by the requirement that the scaling dimensions of the parameters are additively conserved under the RG. In ref. [3] it was shown that we can always choose such parameters. In the first eq. in (2.1) the coefficient of $g_1$ is simply 4, since the parameter $g_1$, being an additive constant in the lagrangian, has nothing to do with interactions. Locality of the theory implies that the beta functions can be expanded in powers of $g_E$ near $g_E = 0$:

$$
\beta_1(g_E) = \beta_{1,0} + \beta_{1,1}g_E + ...
$$

$$
\beta_m(g_E) = \beta_{m,1}g_E + \frac{\beta_{m,2}}{2} g_E^2 + ...
$$

$$
\beta_E(g_E) = \frac{\beta_{E,1}}{2} g_E^2 + \frac{\beta_{E,2}}{3!} g_E^3 + ... .
$$

Let $F(g_1, m, g_E)$ be the free energy density. Then, by definition of the RG, we find

$$
\frac{d}{dl} F = 4F.
$$

Since

$$
\langle 1 \rangle_{m,g_E} = 1 = \frac{\partial}{\partial g_1} F, \quad \langle O_m \rangle_{m,g_E} = \frac{\partial}{\partial m} F, \quad \langle O_E \rangle_{m,g_E} = \frac{\partial}{\partial g_E} F,
$$

we find, from (2.1), (2.3), and (2.4), that the composite operators $O_m$ and $O_E$ satisfy the RG eqs.

$$
\frac{d}{dl} O_m = (3 - \beta_m(g_E))O_m - \frac{m^3}{3!} \beta_1(g_E) 1
$$

$$
\frac{d}{dl} O_E = (4 - \beta_E'(g_E))O_E - m\beta_m'(g_E)O_m - \frac{m^4}{4!} \beta_1'(g_E) 1
$$

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In principle there can be additional scalar operators whose expectation values vanish, for example total derivative operators. But there is no other scalar operators of dimension less than or equal to four, and the above equations are exact.

We now introduce RG eqs. for general composite operators.

Let \( \{ \mathcal{O}_i \} \) be a basis of gauge invariant scalar operators, where \( \mathcal{O}_i \) has scaling dimension \( x_i \). Then the RG eqs. take the form

\[
\frac{d}{dl} \mathcal{O}_i = x_i \mathcal{O}_i + \sum_j \Gamma_{i,j}(m, g_E) \mathcal{O}_j,
\]

where the anomalous dimensions \( \Gamma_{i,j} \) can be nonvanishing only if \( x_i - x_j \) is a non-negative integer. More precisely, the anomalous dimensions are finite polynomials of \( m \) with coefficients as power series in \( g_E \):

\[
\Gamma_{i,j}(m, g_E) = \gamma_{i,j}(g_E)m^{x_i - x_j},
\]

where \( \gamma_{i,j}(g_E) \) is a power series. We assume that

\[
\Gamma_{i,j}(0, 0) = 0
\]

so that at the UV fixed point \( m = g_E = 0 \) there is no mixing of operators. In the matrix notation, in which \( \mathcal{O}_i \) is an infinite dimensional column vector \( \mathcal{O} \), we can write eqs. (2.6) as

\[
\frac{d}{dl} \mathcal{O} = X \mathcal{O} + \Gamma(m, g_E) \mathcal{O},
\]

where \( X_{i,j} = x_i \delta_{ij} \) is a diagonal matrix.

Before closing this section, we will examine the RG eqs. (2.1) further for later convenience. We define the running parameters \( \bar{g}_E(l; g_E), \bar{m}(l; m, g_E) \), and \( \bar{g}_1(l; g_1, m, g_E) \) as the solutions of the RG eqs.

\[
\frac{\partial}{\partial l} \bar{g}_E(l; g_E) = \beta_E(\bar{g}_E(l; g_E)),
\]

\[
\frac{\partial}{\partial l} \bar{m}(l; m, g_E) = (1 + \beta_m(\bar{g}_E(l; g_E))) \bar{m}(l; m, g_E)
\]

\[
\frac{\partial}{\partial l} \bar{g}_1(l; g_1, m, g_E) = 4 \bar{g}_1(l; g_1, m, g_E) + \frac{\bar{m}(l; m, g_E)^4}{4!} \beta_1(\bar{g}_E(l; g_E)),
\]

with the initial conditions

\[
\bar{g}_E(0; g_E) = g_E, \bar{m}(0; m, g_E) = m, \bar{g}_1(0; g_1, m, g_E) = g_1.
\]
We will suppress the initial conditions from now on, and we will denote, for example, simply $\bar{g}_E(l)$ instead of $\bar{g}_E(l; g_E)$. Then we note that

$$\bar{g}_E(\ln r), \quad \bar{m}(\ln r), \quad \bar{g}_1(\ln r)$$

are all RG invariants, since the dependence on the change of the coordinate $r$ under the RG is canceled by the dependence on the change of the initial parameters $g_1, m, g_E$ under the RG. Using the beta functions we can write the running parameter $\bar{m}(l)$ as follows:

$$\bar{m}(l) = m e^l E(\bar{g}_E(l), g_E), \quad (2.12)$$

where $E(y, x)$ is defined by

$$E(y, x) \equiv \exp \left[ \int_x^y dz \frac{\beta_m(z)}{\beta_E(z)} \right]. \quad (2.13)$$

Note that the function $E$ satisfies

$$\beta_E(y) \frac{\partial}{\partial y} E(y, x) = \beta_m(y) E(y, x), \quad \beta_E(x) \frac{\partial}{\partial x} E(y, x) = -E(y, x) \beta_m(x). \quad (2.14)$$

3. Operator product expansions

In this section we summarize relevant facts on the operator product expansions (OPE).\[^{[4]}\] We will be interested in the two types of operator products:

$$\mathcal{O}_m(r) \mathcal{O}(0) = C_m(r; m, g_E) \mathcal{O}(0) + o \left( \frac{1}{r^4} \right)$$

$$\mathcal{O}_E(r) \mathcal{O}(0) = C_E(r; m, g_E) \mathcal{O}(0) + o \left( \frac{1}{r^4} \right). \quad (3.1)$$

Here, both $C_m$ and $C_E$ are part of the operator coefficient functions that are at least as singular as $1/r^4$. In taking the operator products we have taken the average over the orientation of the coordinate vector $r_\mu$; without the average the right-hand sides of (3.1) would include non-scalar composite operators.

One of the most fundamental properties of OPE is that for a fixed coordinate $r$ the coefficient functions can be expanded in powers of the small parameters $m, g_E$. We will
find that this analyticity, together with the RG, implies that the coefficient functions are 
finite polynomials in \( m \) whose coefficients are infinite power series in \( g_E \), i.e.,

\[
C_{m,i,j}(r; m, g_E) = \sum_{n=0}^{x_i-x_j-1} \frac{m^n}{n!} \frac{\partial^n}{\partial m^n} C_{m,i,j}(r; 0, g_E)
\]

\[
C_{E,i,j}(r; m, g_E) = \sum_{n=0}^{x_i-x_j} \frac{m^n}{n!} \frac{\partial^n}{\partial m^n} C_{E,i,j}(r; 0, g_E)
\]

We note that this implies that \( C_{m,i,j} \) can be nonvanishing only for \( x_i \geq x_j + 1 \), while \( C_{E,i,j} \) can be nonvanishing only for \( x_i \geq x_j \). Let us derive the above results.

The coefficient functions are closed under the RG, and we find

\[
\frac{d}{dl} C_m(r; m, g_E) = (3 - \beta_m(g_E))C_m(r; m, g_E) + [X + \Gamma(m, g_E), C_m(r; m, g_E)]
\]

\[
\frac{d}{dl} C_E(r; m, g_E) = (4 - \beta'_E(g_E))C_E(r; m, g_E) - m\beta_m'(g_E)C_m(r; m, g_E)
\]

\[
+ [X + \Gamma(m, g_E), C_E(r; m, g_E)].
\]

In order to solve these RG eqs. we introduce a matrix \( G(r; m, g_E) \) that satisfies

\[
\frac{d}{dl} G(r; m, g_E) = (X + \Gamma(m, g_E))G(r; m, g_E)
\]

and the initial condition \( G(1; m, g_E) = 1 \). The solution is given by

\[
G(r; m, g_E) = \frac{1}{r^x} \mathcal{T} \exp \left[ \int^{g_E}_{\bar{g}_E(\ln r)} dx \frac{1}{\beta_E(x)} \Gamma(\bar{m}(\ln r)E(x, \bar{g}_E(\ln r)), x) \right],
\]

where \( \mathcal{T} \) denotes the increasing ordering of \( x \) from right to left. Due to eq. (2.7) \( \Gamma_{i,j}(m, g_E) = m^{x_i-x_j}\gamma_{i,j}(g_E) \), the matrix element \( G_{i,j}(r; m, g_E) \) can be nonvanishing only if \( x_i - x_j \) is a non-negative integer. As far as the power of \( r \) is concerned (i.e., we ignore logarithmic corrections), \( \bar{m}(\ln r) \) is proportional to \( r \) from eq. (2.12). Hence the dependence of \( G_{i,j}(r; m, g_E) \) on the powers of \( m \) and \( r \) is given by

\[
G_{i,j}(r; m, g_E) \propto \frac{m^{x_i-x_j}}{r^{x_j}}.
\]

Similarly, we find

\[
G_{i,j}^{-1}(r; m, g_E) \propto m^{x_i-x_j} r^{x_i}.
\]
It is also helpful to note that the matrix \( G(r; m, g_E) \) satisfies
\[
\frac{\partial}{\partial \ln r} G(r; m, g_E) = -G(r; m, g_E) \left( X + \Gamma(\bar{m}(\ln r), \bar{g}_E(\ln r)) \right).
\] (3.8)

Using the matrix \( G \) we can solve the RG eqs. (3.3) as follows:
\[
C_m(r; m, g_E) = \frac{1}{r^3} E(\bar{g}_E(\ln r), g_E) G(r; m, g_E) H_m(\bar{m}(\ln r), \bar{g}_E(\ln r)) G^{-1}(r; m, g_E)
\]
\[
C_E(r; m, g_E) = \frac{1}{r^4} \frac{\beta_E(\bar{g}_E(\ln r))}{\beta_E(g_E)} G(r; m, g_E) H_E(\bar{m}(\ln r), \bar{g}_E(\ln r)) G^{-1}(r; m, g_E)
\] (3.9)
\[
+ \frac{m}{\beta_E(g_E)} (\beta_m(\bar{g}_E(\ln r)) - \beta_m(g_E)) C_m(r; m, g_E),
\]
where we define
\[
H_m(m, g_E) \equiv C_m(1; m, g_E), \quad H_E(m, g_E) \equiv C_E(1; m, g_E).
\] (3.10)

From the analyticity assumption, both \( H_m(m, g_E) \) and \( H_E(m, g_E) \) are power series in \( m \) and \( g_E \). Suppose \( H_{mi,j}(m, g_E) \) has a term proportional to \( m^n \). Then, from eq. (3.9) we find that \( C_{mi,j}(r; m, g_E) \) has a term proportional to \( m^n/r^{3+x_i-x_j-n} \). Since \( C_m(r; m, g_E) \) must be at least as singular as \( 1/r^4 \) by definition, we must have \( n \leq x_i - x_j - 1 \). Hence, \( H_{mi,j}(m, g_E) \) is a polynomial of degree \( x_i - x_j - 1 \) as far as \( m \) is concerned. The \( m^n \) term in \( H_{mi,j} \) can also contribute to \( C_{mi',j'}(r; m, g_E) \), where \( x'_i \geq x_i > x_j \geq x'_j \). But from (3.6) and (3.7) we find again that \( n \leq x_i - x_j - 1 \). Similarly, we can conclude that \( H_{Ei,j}(m, g_E) \) is a polynomial of degree \( x_i - x_j \) with respect to \( m \). To summarize, we find the structure
\[
H_{mi,j}(m, g_E) = \sum_{n=0}^{x_i-x_j-1} \frac{m^n}{n!} H^{(n)}_{mi,j}(g_E),
\]
\[
H_{Ei,j}(m, g_E) = \sum_{n=0}^{x_i-x_j} \frac{m^n}{n!} H^{(n)}_{Ei,j}(g_E).
\] (3.11)

4. Variational formulas

We will introduce a main formula that gives operator realization of the derivatives with respect to \( m \) and \( g_E \) in this section. We will see that the consistency of this formula with the RG eqs. gives a relation between the operator coefficients and anomalous dimensions.
So far we have been saying casually that the parameter $m$ and the operator $\mathcal{O}_m$ are conjugate to each other. (The following discussion applies to $g_E$ and $\mathcal{O}_E$ as well.) The precise meaning of the conjugacy relation is that the derivative $-\frac{\partial}{\partial m}$ is realized by the operator $\mathcal{O}_m$. One example is eq. (2.4). This conjugacy relation must be valid not only for the simple expectation value as (2.4) but also for any correlation functions of composite operators. So, we expect a formula of the type

$$- \frac{\partial}{\partial m} \langle \mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n) \rangle_{m,g_E} = \int d^4r \ (\mathcal{O}_m(r)\mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n))_{m,g_E}^c, \quad (4.1)$$

where

$$\langle \mathcal{O}_m(r)\mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n) \rangle_{m,g_E}^c \equiv \langle (\mathcal{O}_m(r) - \langle \mathcal{O}_m \rangle_{g_E})\mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n) \rangle_{m,g_E}. \quad (4.2)$$

We immediately notice, however, that the above formula is not well defined due to the short distance singularities in the product of $\mathcal{O}_m$ and $\mathcal{O}_i$. We emphasize that these short distance singularities are physical singularities that exist in QCD even after renormalization. (We will make a brief remark on the relation between these short distance singularities and the singularities we encounter in perturbative calculations in the concluding section.) There must be a way of fixing this problem in a local way, since the singularities result from the UV physics rather than the IR physics. Thus, we postulate the validity of the following variational formulas in QCD (an analogous formula was mentioned in ref. [4]):

$$- \frac{\partial}{\partial m} \langle \mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n) \rangle_{m,g_E} = \langle \mathcal{O}_m^*\mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n) \rangle_{m,g_E}$$

$$\equiv \lim_{\epsilon \to 0} \left[ \int_{|r-r_i| \geq \epsilon} d^4r \ (\mathcal{O}_m(r)\mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n))_{m,g_E}^c \right]$$

$$+ \sum_{k=1}^n A_{m i k, j}(\epsilon; m, g_E) \langle \mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{j}(r_k)\ldots\mathcal{O}_{i_n}(r_n) \rangle_{m,g_E} \quad (4.3)$$

$$- \frac{\partial}{\partial g_E} \langle \mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n) \rangle_{m,g_E} = \langle \mathcal{O}_E^*\mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n) \rangle_{m,g_E}$$

$$\equiv \lim_{\epsilon \to 0} \left[ \int_{|r-r_i| \geq \epsilon} d^4r \ (\mathcal{O}_E(r)\mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{i_n}(r_n))_{m,g_E}^c \right]$$

$$+ \sum_{k=1}^n A_{E i k, j}(\epsilon; m, g_E) \langle \mathcal{O}_{i_1}(r_1)\ldots\mathcal{O}_{j}(r_k)\ldots\mathcal{O}_{i_n}(r_n) \rangle_{m,g_E},$$

where we define

$$A_{m i j}(\epsilon; m, g_E) \equiv - \int_{1 \geq r \geq \epsilon} d^4r \ C_{m i j}(r; m, g_E) + c_{m i j}(m, g_E) \quad (4.4)$$

$$A_{E i j}(\epsilon; m, g_E) \equiv - \int_{1 \geq r \geq \epsilon} d^4r \ C_{E i j}(r; m, g_E) + c_{E i j}(m, g_E).$$
The variational formulas \((4.3)\) explain why it is important to treat the operator coefficients at least as singular as \(1/r^4\); less singular coefficients are irrelevant for the subtractions. The finite counterterms \(c_m, c_E\) are necessary in order to compensate the arbitrariness involved in the subtracting procedure. The finite counterterms must have the same structure as the operator coefficients \(C_m, C_E\) (for a fixed \(r\)), and they can be expanded as

\[
\begin{align*}
 c_{mi,j}(m, g_E) &= \sum_{n=0}^{x_i-x_j-1} \frac{m^n}{n!} c^{(n)}_{mi,j}(g_E) \\
 c_{Ei,j}(m, g_E) &= \sum_{n=0}^{x_i-x_j} \frac{m^n}{n!} c^{(n)}_{Ei,j}(g_E).
\end{align*}
\]

(4.5)

For later convenience we define the maximal parts of the counterterms by

\[
\begin{align*}
 &\bar{c}_{mi,j}(m, g_E) \equiv m^{x_i-x_j-1}/(x_i-x_j-1)! c^{(x_i-x_j-1)}_{mi,j}(g_E) \\
 &\bar{c}_{Ei,j}(m, g_E) \equiv m^{x_i-x_j}/(x_i-x_j)! c^{(x_i-x_j)}_{Ei,j}(g_E).
\end{align*}
\]

(4.6)

5. Consistency condition

It is very important to note that the variational formulas introduced in the previous section are assumed to be valid for finite parameters \(m, g_E\), i.e., they are assumed to be valid beyond perturbation theory. Though it is difficult to derive the variational formulas from first principles, it is easy to check their consistency. Especially we will check their consistency with the RG eqs.

We first examine the transformation of the left-hand sides of the variational formulas \((4.3)\) under the RG. From eqs. \((2.1)\), \((2.5)\), and \((2.9)\) we find

\[
\begin{align*}
 \frac{d}{dl} \left( -\frac{\partial}{\partial m} \langle O_{i_1}(r_1) \ldots O_{i_n}(r_n) \rangle_{m, g_E} \right) \\
 &= \sum_{k=1}^{n} (X_{i_k,j} + \Gamma_{i_k,j}(m, g_E)) \left( -\frac{\partial}{\partial m} \langle O_{i_1}(r_1) \ldots O_{i_k-j}(r_k) \ldots O_{i_n}(r_n) \rangle_{m, g_E} \right) \\
 &\quad - (1 + \beta_m(g_E)) \left( -\frac{\partial}{\partial m} \langle O_{i_1}(r_1) \ldots O_{i_n}(r_n) \rangle_{m, g_E} \right) \\
 &\quad - \sum_{k=1}^{n} \partial_m \Gamma_{i_k,j}(m, g_E) \langle O_{i_1}(r_1) \ldots O_{i_k-j}(r_k) \ldots O_{i_n}(r_n) \rangle_{m, g_E}
\end{align*}
\]

(5.1)
and
\[
\frac{d}{dl} \left( -\frac{\partial}{\partial g_E}(O_{i_1}(r_1)\ldots O_{i_n}(r_n))_{m,E} \right)
\]
\[
= \sum_{k=1}^{n} (X_{ik,j} + \Gamma_{ik,j}(m, g_E)) \left( -\frac{\partial}{\partial g_E}(O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n))_{m,E} \right)
\]
\[
- \beta_{E'}(g_E) \left( -\frac{\partial}{\partial g_E}(O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n))_{m,E} \right)
\]
\[
- m\beta_m'(g_E) \left( -\frac{\partial}{\partial m}(O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n))_{m,E} \right)
\]
\[
- \sum_{k=1}^{n} \frac{\partial g_E}{g_E} \Gamma_{ik,j}(m, g_E)(O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n))_{m,E}.
\]

Next we consider the transformation of the right-hand sides of (4.3). From eqs. (2.3) and (2.9) we obtain
\[
\frac{d}{dl} \langle O^*_m O_{i_1}(r_1)\ldots O_{i_n}(r_n) \rangle_{m,E}
\]
\[
= \sum_{k=1}^{n} (X_{ik,j} + \Gamma_{ik,j}(m, g_E)) \langle O^*_m O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n) \rangle_{m,E}
\]
\[
- \langle 1 + \beta_m(g_E) \rangle (O^*_m O_{i_1}(r_1)\ldots O_{i_n}(r_n))_{m,E}
\]
\[
- 2\pi^2 \sum_{k=1}^{n} H_{mi_{k,j}}(m, g_E) \langle O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n) \rangle_{m,E}
\]
\[
+ \sum_{k=1}^{n} \left( \frac{d}{dl} c_{mi_{k,j}} + (1 + \beta_m(g_E)) c_{mi_{k,j}} + [c_m, X + \Gamma(m, g_E)]_{ik,j} \right)
\]
\[
\cdot \langle O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n) \rangle_{m,E}
\]
and
\[
\frac{d}{dl} \langle O^*_E O_{i_1}(r_1)\ldots O_{i_n}(r_n) \rangle_{m,E}
\]
\[
= \sum_{k=1}^{n} (X_{ik,j} + \Gamma_{ik,j}(m, g_E)) \langle O^*_E O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n) \rangle_{m,E}
\]
\[
- \beta_{E'}(g_E) \langle O^*_E O_{i_1}(r_1)\ldots O_{i_n}(r_n) \rangle_{m,E} - m\beta_m'(g_E) \langle O^*_m O_{i_1}(r_1)\ldots O_{i_n}(r_n) \rangle_{m,E}
\]
\[
- 2\pi^2 \sum_{k=1}^{n} H_{Ei_{k,j}}(m, g_E) \langle O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n) \rangle_{m,E}
\]
\[
+ \sum_{k=1}^{n} \left( \frac{d}{dl} c_{Ei_{k,j}} + \beta_{E'}(g_E) c_{Ei_{k,j}} + m\beta_m'(g_E) c_{mi_{k,j}} + [c_E, X + \Gamma(m, g_E)]_{ik,j} \right)
\]
\[
\cdot \langle O_{i_1}(r_1)\ldots O_{i_k}(r_k)\ldots O_{i_n}(r_n) \rangle_{m,E}
\]
Here we get the $H_m, H_E$ terms from the boundary integrals at $r = 1$, and $2\pi^2$ is the volume integral of a unit 3-sphere.

By comparing eq. (5.1) and eq. (5.3), and eq. (5.2) and eq. (5.4) we obtain

$$2\pi^2 H_m(m, g_E) = \partial_m \Gamma(m, g_E) + \frac{d}{dl} c_m(m, g_E)$$
$$+ (1 + \beta_m(g_E)) c_m(m, g_E) + [c_m(m, g_E), X + \Gamma(m, g_E)]$$

$$2\pi^2 H_E(m, g_E) = \partial_{g_E} \Gamma(m, g_E) + \frac{d}{dl} c_E(m, g_E)$$
$$+ \beta_E'(g_E)c_E(m, g_E) + m\beta_m'(g_E)c_m(m, g_E)$$
$$+ [c_E(m, g_E), X + \Gamma(m, g_E)].$$

These formulas give relations between the operator coefficients and anomalous dimensions. We can rewrite these in a more suggestive way as follows:

$$2\pi^2 H_m = \partial_m \Phi + [c_m, \Phi] + \beta_E(\partial_{g_E} c_m - \partial_m c_E + [c_E, c_m])$$
$$2\pi^2 H_E = \partial_{g_E} \Phi + [c_E, \Phi] - m(1 + \beta_m)(\partial_{g_E} c_m - \partial_m c_E + [c_E, c_m]),$$

where

$$\Phi(m, g_E) \equiv X + \Gamma + \beta_E c_E + m(1 + \beta_m)c_m.$$ (5.7)

Eqs. (5.6) constitute a main result of this paper, and they are analogous to the relation between the anomalous dimensions and $1/\epsilon$ poles in the dimensional regularization with the minimal subtraction.

We will come back to the geometric meaning of the expressions (5.6) in sect. 7. Note that the anomalous dimensions $\Gamma$ contribute only to the maximal parts of the coefficient functions.

Now, by substituting eqs. (5.3) into eqs. (3.9) we can express the coefficient functions for an arbitrary $r$ as total derivatives:

$$C_m(r; m, g_E)$$
$$= \frac{1}{2\pi^2 mr^4} \frac{\partial}{\partial \ln r} \left( G(r; m, g_E) \left( X + \bar{m}(\ln r)c_m(\bar{m}(\ln r), \bar{g}_E(\ln r)) \right) G^{-1}(r; m, g_E) \right)$$

$$C_E(r; m, g_E)$$
$$= \frac{1}{2\pi^2 \beta_E(g_E) r^4} \frac{\partial}{\partial \ln r} \left( G(r; m, g_E) \left( \Gamma(\bar{m}(\ln r), \bar{g}_E(\ln r)) - \beta_m(g_E)X ight. 
+ \beta_E(g_E(\ln r))c_E(\bar{m}(\ln r), \bar{g}_E(\ln r)) 
\left. + \bar{m}(\ln r)(\beta_m(\bar{g}_E(\ln r)) - \beta_m(g_E))c_m(\bar{m}(\ln r), \bar{g}_E(\ln r)) \right) G^{-1}(r; m, g_E) \right).$$ (5.8)
To derive this result we need to use eqs. (2.12), (3.8), and

\[ [X, \Gamma(m, g_E)] = m\partial_m \Gamma(m, g_E), \] (5.9)

where the last equation is a consequence of eq. (2.7).

We can now integrate the above coefficient functions to obtain an expression for the subtractions \(A_m\) and \(A_E\), defined by (4.4), in terms of the anomalous dimensions \(\Gamma\) and the counterterms \(c_m, c_E\). We find

\[ A_m(\epsilon; m, g_E) = \frac{1}{m} \left( -X + G(\epsilon; m, g_E)XG^{-1}(\epsilon; m, g_E) + G(\epsilon; m, g_E)\bar{m}(\ln \epsilon)c_m(\bar{m}(\ln \epsilon), \bar{g}_E(\ln \epsilon)) - \Gamma(m, g_E) + G(\bar{m}(\ln \epsilon), \bar{g}_E(\ln \epsilon)) \right), \]

\[ A_E(\epsilon; m, g_E) = \frac{1}{\beta(\epsilon)} \left( \beta_m(\epsilon)X - G(\epsilon; m, g_E)XG^{-1}(\epsilon; m, g_E) \right) + G(\bar{g}_E(\ln \epsilon)c_E(\bar{m}(\ln \epsilon), \bar{g}_E(\ln \epsilon)) + (\beta_m(\epsilon(\ln \epsilon) - \beta_m(\epsilon))\bar{m}(\ln \epsilon)c_m(\bar{m}(\ln \epsilon), g_E(\ln \epsilon)) - \Gamma(m, g_E) + G(\bar{m}(\ln \epsilon), \bar{g}_E(\ln \epsilon)) \right) \right) \] (5.10)

Using the above formulas we can count the powers of \(1/\epsilon\) in the subtractions. First we note, from (3.6) and (3.7), that

\[ (X - GXG^{-1})_{i,j} = 0, \quad \text{if } x_i = x_j \]

\[ \propto m^{|x_i - x_j|}, \quad \text{if } x_i > x_j. \] (5.11)

Since \(G(1; m, g_E) = 1\), the term \((X - GXG^{-1})_{i,j}\) corresponds to purely logarithmic contributions in \(\epsilon\) that vanish at \(\ln \epsilon = 0\). Similarly, we find that

\[ \left( \Gamma(m, g_E) - G(\bar{m}(\ln \epsilon), \bar{g}_E(\ln \epsilon))G^{-1} \right)_{i,j} \propto m^{x_i - x_j} \] (5.12)

are purely logarithmic in \(\epsilon\) without powers of \(\epsilon\). Secondly we note, from (4.5), that

\[ \left( G\bar{m}(\ln \epsilon)c_m(\bar{m}(\ln \epsilon), g_E(\ln \epsilon)) \right)_{i,j} \sim c_{m', j'}(\bar{g}_E(\ln \epsilon)) \left( m\epsilon \right)^{n+1} \] (5.13)
where \( x_i \geq x_{i'} \geq x_j \geq x_{j'} \). Especially we find

\[
\left( G\bar{m}(\ln \epsilon)\bar{c}_m(\bar{m}(\ln \epsilon), \bar{g}_E(\ln \epsilon))G^{-1}\right)_{i,j} \sim \bar{c}_{mi',j'}(\bar{g}_E(\ln \epsilon))m^{x_{i'} - x_j} \\
\left( G\bar{c}_E(\bar{m}(\ln \epsilon), \bar{g}_E(\ln \epsilon))G^{-1}\right)_{i,j} \sim \bar{c}_{Ei',j'}(\bar{g}_E(\ln \epsilon))m^{x_{i'} - x_j}.
\]

These are not necessarily zero at \( \epsilon = 1 \) and involve terms that are finite as \( \epsilon \to 0 \). In conclusion the anomalous dimensions \( \Gamma \) give only logarithmic contributions to the subtractions \( A_m, A_E \), while the non-maximal parts of the counterterms \( c_m, c_E \) give contributions that are proportional to integral powers of \( 1/\epsilon \), and the maximal parts \( \bar{c}_m, \bar{c}_E \) give both logarithmic and finite contributions in \( \epsilon \). In the absence of maximal counterterms there is no finite subtraction. Thus, the necessary subtractions are minimal when the maximal parts of the counterterms vanish.

6. Commutativity

In the previous section we checked the consistency of the variational formulas with the RG equations. In this section we will perform another consistency check. The variational formulas give single derivatives of correlation functions with respect to \( m, g_E \),

and we can use the formulas recursively to realize multiple derivatives in terms of multiple integrals over operators.

We define two insertions \( \langle O^*_E O^*_m \ldots \rangle_{m, g_E} \) (in this particular order) such that it equals \(-\partial_{g_E} \langle O^*_m \ldots \rangle_{m, g_E} \); namely we apply the definition of \( O^*_E \) to the correlation function in the definition of \( \langle O^*_m \ldots \rangle_{m, g_E} \):

\[
\langle O^*_E O^*_m O_{i_1}(r_1) \ldots O_{i_n}(r_n) \rangle_{m, g_E} \equiv \lim_{\epsilon \to 0} \left[ \int_{|r - r_i| \geq \epsilon} d^4 r \langle O^*_E O^*_m(r) O_{i_1}(r_1) \ldots O_{i_n}(r_n) \rangle_{m, g_E} \right] \\
+ \sum_{k=1}^{n} A_{mi_k,j}(\epsilon; m, g_E)\langle O^*_E O_{i_1}(r_1) \ldots O_{j}(r_k) \ldots O_{i_n}(r_n) \rangle_{m, g_E} \\
- \sum_{k=1}^{n} \partial_{g_E} A_{mi_k,j}(\epsilon; m, g_E)\langle O_{i_1}(r_1) \ldots O_{j}(r_k) \ldots O_{i_n}(r_n) \rangle_{m, g_E}.
\]

(6.1)
Similarly, we define $\langle O_m^* O_E^* O_1 \ldots O_n \rangle_{m,g_E}$ such that it equals $-\partial_m \langle O_E^* O_1 \ldots O_n \rangle_{m,g_E}$:

$$
\langle O_m^* O_E^* O_{i_1} (r_1) \ldots O_{i_n} (r_n) \rangle_{m,g_E} \equiv \lim_{\epsilon \to 0} \left[ \int_{|r_i'-r_i| \geq \epsilon} d^4 r' \langle O_m^* O_E (r') O_{i_1} (r_1) \ldots O_{i_n} (r_n) \rangle_{m,g_E}^c 
+ \sum_{k=1}^n A_{Ei,k,j} (\epsilon; m, g_E) \langle O_m^* O_{i_1} (r_1) \ldots O_j (r_k) \ldots O_{i_n} (r_n) \rangle_{m,g_E}
- \sum_{k=1}^n \partial_m A_{Ei,k,j} (\epsilon; m, g_E) \langle O_{i_1} (r_1) \ldots O_j (r_k) \ldots O_{i_n} (r_n) \rangle_{m,g_E} \right].
$$

(6.2)

In a similar way we can define $\langle O_m^* O_m^* \ldots \rangle_{m,g_E}$ and $\langle O_E^* O_E^* \ldots \rangle_{m,g_E}$.

The insertions $O_m^*$ and $O_E^*$ are operator realizations of the partial derivatives $-\partial_m$ and $-\partial_{g_E}$. Since the partial derivatives commute with one another, we must satisfy

$$
\langle O_E^* O_m^* O_{i_1} (r_1) \ldots O_{i_n} (r_n) \rangle_{m,g_E} = \langle O_m^* O_E^* O_{i_1} (r_1) \ldots O_{i_n} (r_n) \rangle_{m,g_E}.
$$

(6.3)

The evaluation of the difference $O_E^* O_m^* - O_m^* O_E^*$ is straightforward but tedious,

and we will sketch the calculation in Appendix A. For simplicity we give the result only for the case $n = 1$ (For the case of a general $n$, see Appendix A.):

$$
\langle (O_E^* O_m^* - O_m^* O_E^*) O_1 (0) \rangle_{m,g_E}
= (\Omega_{Em})_{i,j} \langle O_j (0) \rangle_{m,g_E} - (\partial_{g_E} c_m - \partial_m c_E + [c_E, c_m])_{i,j} \langle O_j (0) \rangle_{m,g_E}.
$$

(6.4)

Here we define the curvature

$$(\Omega_{Em})_{i,j} \langle O_j (0) \rangle_{m,g_E}\equiv \int_{1 \geq r} d^4 r \ F.P. \int_{1 \geq r'} d^4 r' \langle O_m (r) (O_E (r') O_1 (0) - C_{Ei,j} (r'; m, g_E) O_j (0)) 
- O_E (r) (O_m (r') O_1 (0) - C_{mi,j} (r'; m, g_E) O_j (0)) \rangle_{m,g_E},$$

(6.5)

where F.P. denotes an integrable part with respect to $r$.

Thus, the consistency condition becomes

$$
\partial_{g_E} c_m - \partial_m c_E + [c_E, c_m] = \Omega_{Em}.
$$

(6.6)

The left-hand side is reminiscent of a gauge field strength with $c_m, c_E$ as the gauge fields. We will elaborate on this viewpoint in the next section. It is important to emphasize that

the curvature $\Omega_{Em}$ is related essentially to a product of three operators close together, and it cannot be deduced from the OPE coefficients which have to do with the products of only two operators close together.
Before we close this section we must ask if we get further consistency conditions from more number of insertions of \( \mathcal{O}_m^* \) and \( \mathcal{O}_E^* \). Let us consider insertion of \( K \mathcal{O}_m^* \)'s and \( L \mathcal{O}_E^* \)'s. These are defined using (4.3) recursively. We can classify all possible insertions into two classes, those of type 1
\[
\langle \mathcal{O}_m^* \mathcal{O}^\ast \ldots \mathcal{O}_{i_1} (r_1) \ldots \mathcal{O}_{i_n} (r_n) \rangle_{m, g_E}
\]
and those of type 2
\[
\langle \mathcal{O}_E^* \mathcal{O}^\ast \ldots \mathcal{O}_{i_1} (r_1) \ldots \mathcal{O}_{i_n} (r_n) \rangle_{m, g_E},
\]
where \( \mathcal{O}^\ast \ldots \) corresponds to the remaining \( K + L - 1 \) insertions. Suppose that the insertion of \( K' \mathcal{O}_m^* \)'s and \( L' \mathcal{O}_E^* \)'s is independent of ordering if \( K' + L' < K + L \). Now, using the variational formulas (4.3) \textit{only once}, we get
\[
\langle \mathcal{O}_m^* \mathcal{O}^\ast \ldots \mathcal{O}_{i_1} (r_1) \ldots \mathcal{O}_{i_n} (r_n) \rangle_{m, g_E} = -\partial_m \langle \mathcal{O}^\ast \ldots \mathcal{O}_{i_1} (r_1) \ldots \mathcal{O}_{i_n} (r_n) \rangle_{m, g_E}
\]
\[
\langle \mathcal{O}_E^* \mathcal{O}^\ast \ldots \mathcal{O}_{i_1} (r_1) \ldots \mathcal{O}_{i_n} (r_n) \rangle_{m, g_E} = -\partial_{g_E} \langle \mathcal{O}^\ast \ldots \mathcal{O}_{i_1} (r_1) \ldots \mathcal{O}_{i_n} (r_n) \rangle_{m, g_E}.
\]
Hence, by the assumption all correlation functions are equal in each class. The equality between the two classes amounts to show
\[
\langle \mathcal{O}_m^* \mathcal{O}_E^\ast \ldots \mathcal{O}_{i_1} (r_1) \ldots \mathcal{O}_{i_n} (r_n) \rangle_{m, g_E} = \langle \mathcal{O}_E^* \mathcal{O}_m^* \ldots \mathcal{O}_{i_1} (r_1) \ldots \mathcal{O}_{i_n} (r_n) \rangle_{m, g_E} \quad (6.7)
\]
where \( \mathcal{O}^\ast \ldots \) stands for the remaining \( K + L - 2 \) insertions. Since \( \langle \mathcal{O}^\ast \ldots \mathcal{O}_{i_1} (r_1) \ldots \mathcal{O}_{i_n} (r_n) \rangle_{m, g_E} \) is given as an integral over \( K + L - 2 + n \)-point correlation function (ignoring counterterms for simplicity), we can prove the equality by going through the same proof as given in Appendix A for the case of two insertions. Therefore, we will not get any further consistency condition from multiple insertions.

7. Unique convention for composite operators

We would like to redefine the composite operators in such a way that the corresponding counterterms have vanishing maximal parts. The reason is that this convention is the simplest in which finite subtractions are unnecessary.

The most general redefinition of the operators \( \{ \mathcal{O}_i \} \) that we can think of is given by
\[
\tilde{\mathcal{O}}_i = \sum_j N_{i,j}(m, g_E) \mathcal{O}_j, \quad (7.1)
\]
where the matrix $N$ is invertible. The element $N_{i,j}$ can be nonvanishing only if $x_i - x_j$ is a non-negative integer, and $N_{i,j}$ must be proportional to $m^{x_i - x_j}$:

$$\partial_m N = \frac{1}{m} [X, N]. \quad (7.2)$$

This condition is required in order to preserve the scaling dimensions of the redefined operators additively under the RG. With respect to $g_E$, the element $N_{i,j}$ must be a power series as required by locality of the theory.

For the redefined operators $\tilde{O}$, the counterterms for the variational formulas (4.3) are given by

$$\tilde{c}_m = N c_m N^{-1} - \partial_m N \cdot N^{-1}$$
$$\tilde{c}_E = N c_E N^{-1} - \partial_{g_E} N \cdot N^{-1}. \quad (7.3)$$

Hence, the finite counterterms transform as gauge fields in the space of parameters $m, g_E$. The inhomogeneous terms are maximal, and for the maximal parts alone we find

$$\tilde{c}_m = N \tilde{c}_m N^{-1} - \partial_m N \cdot N^{-1}$$
$$\tilde{c}_E = N \tilde{c}_E N^{-1} - \partial_{g_E} N \cdot N^{-1}. \quad (7.4)$$

The transformation properties (7.3) imply that we can interpret $(c_m, c_E)$ as a gauge field on the two-dimensional theory space whose coordinates are $m$ and $g_E$. The operators $\{O_i\}$ form a basis of an infinite dimensional vector bundle. The commutativity condition (6.6) gives the curvature of the gauge field in terms of an integral over a correlation function (6.5). Since the integral has no reason to vanish, the curvature is nonvanishing in general.

The field $\Phi$ defined by (5.7) can be seen to transform as

$$\tilde{\Phi} = N \Phi N^{-1}. \quad (7.5)$$

Thus, $\Phi$ is a scalar field in the adjoint representation on the theory space. Eqs. (5.6) and (6.6) give the vector field $(H_m, H_E)$ in terms of a covariant derivative of $\Phi$ and the curvature $\Omega_{Em}$.

Hence, $(H_m, H_E)$ is in the adjoint representation as expected. So much for geometric interpretations of our results.
The simplest convention for the finite counterterms is the vanishing maximal counterterms:

\[ \tilde{c}_m = 0, \quad \tilde{c}_E = 0. \tag{7.6} \]

But in general this is impossible, since the curvature \( \Omega_{Em} \), defined by (6.5), is nonvanishing. We remove the ambiguity of the convention by imposing

\[ \tilde{c}_E(m, g_E) = 0 \tag{7.7} \]

and

\[ \tilde{c}_m(m, g_E = 0) = 0. \tag{7.8} \]

This is the analogue of the temporal gauge used in non-abelian gauge theories. For completeness we will derive the matrix of transformation \( N \) in Appendix B. The gauge conditions (7.7) and (7.8) fix \( N \) up to a left multiplication by a constant matrix. Hence, the choice of the composite operators is uniquely determined up to a

left multiplication by a constant matrix.

Finally we ask whether the unique convention we obtained depends on the choice of the renormalized parameters \( m, g_E \). The most general redefinition of the parameters are given by

\[ \tilde{m} = m f(g_E) \]
\[ \tilde{g}_E = g(g_E) \tag{7.9} \]

where

\[ f(g_E) = 1 + f_1 g_E + \frac{f_2}{2} g_E^2 + \ldots \]
\[ g(g_E) = g_E + \frac{g_1}{2} g_E^2 + \frac{g_2}{3!} g_E^3 + \ldots . \tag{7.10} \]

The corresponding changes in the partial derivatives are

\[ \partial_{\tilde{m}} = \frac{1}{f(g_E)} \partial_m \]
\[ \partial_{\tilde{g}_E} = \frac{1}{g'(g_E)} \partial_{g_E} - m \frac{f'(g_E)}{g'(g_E)f(g_E)} \partial_m. \tag{7.11} \]

The operators \( \mathcal{O}_m \) and \( \mathcal{O}_E \) transform in the same way as a vector:

\[ \tilde{\mathcal{O}}_m = \frac{1}{f(g_E)} \mathcal{O}_m \]
\[ \tilde{\mathcal{O}}_E = \frac{1}{g'(g_E)} \mathcal{O}_E - m \frac{f'(g_E)}{g'(g_E)f(g_E)} \mathcal{O}_m. \tag{7.12} \]
This induces homogeneous transformations of the operator coefficients:

\[
\tilde{C}_m(r; \tilde{m}, \tilde{g}_E) = \frac{1}{f(g_E)} C_m(r; m, g_E)
\]
\[
\tilde{C}_E(r; \tilde{m}, \tilde{g}_E) = \frac{1}{g'(g_E)} C_E(r; m, g_E) - m \frac{f'(g_E)}{g'(g_E)f(g_E)} C_m(r; m, g_E).
\]

From eqs. (7.11) and (7.13) we find that the counterterms also transform homogeneously as a vector:

\[
\tilde{c}_m(\tilde{m}, \tilde{g}_E) = \frac{1}{f(g_E)} c_m(m, g_E)
\]
\[
\tilde{c}_E(\tilde{m}, \tilde{g}_E) = \frac{1}{g'(g_E)} c_E(m, g_E) - m \frac{f'(g_E)}{g'(g_E)f(g_E)} c_m(m, g_E).
\]

Therefore, under the change of parameters the gauge condition (7.7) is not preserved, while the condition (7.8) is preserved. Thus, in order to remove the ambiguity in the choice of composite operators by the gauge conditions (7.7) and (7.8), we need to introduce a convention for the choice of \(m\) and \(g_E\). This can be done by applying conditions analogous to (7.7), (7.8) to the conjugate operators \(O_m, O_E\).

8. Concluding remarks

In this paper we obtained two main results. First we obtained the variational formulas (4.3) that realize the partial derivatives with respect to \(m, g_E\) in terms of the operator insertions \(O^*_m, O^*_E\). Secondly we found a relation between the anomalous dimensions of composite operators and their operator product coefficients, given by eqs. (5.6).

Using the second result we can compute the subtractions \(A_m, A_E\), which are necessary in the variational formulas (4.3), in terms of the beta functions, anomalous dimensions \(\Gamma\), and finite counterterms \(c_m, c_E\).

The finite counterterms are not unique, since the composite operators are susceptible to redefinitions (7.1). Once we remove this ambiguity by imposing conditions (7.7) and (7.8), the finite counterterms become unique, and they are determined by eqs. (5.6) and the commutativity condition (6.6).

The non-maximal part of the finite counterterms corresponds to subtractions divergent like powers of the cutoff distance \(\epsilon\). (See the last paragraph of sect. 5.) The dimensional regularization only sees logarithmic divergences and has no counterpart to the non-maximal finite counterterms. The maximal part of the finite counterterms gives finite subtractions.
in the variational formulas. This results from our asymmetric treatment of conjugate operators when we evaluate higher order derivatives of correlation functions with respect to parameters. In the dimensional regularization each operator conjugate to a parameter is treated symmetrically.

We would like to emphasize the conceptual aspects of our results rather than their potential usefulness for perturbative calculations. In fact some modifications will be necessary to apply the variational formulas (4.3) for perturbative calculations of the correlation functions of composite operators. The reason is that the operator $O_E$, conjugate to $g_E$, roughly corresponds to

$$O_E \sim \frac{1}{g_E^2} \text{tr} (F_{\mu\nu})^2,$$

where the field strength is defined by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

in terms of the gauge field, since the lagrangian is given by

$$L = \frac{1}{g_E} \text{tr} (F_{\mu\nu})^2.$$

Note that this is not an artifact of normalization. Even if we redefine a new $A$ by $\sqrt{g_E} A$, we still get $O_E \sim F^2/g_E$. The form of the operator $O_E$ given by (8.1) implies that the correlation functions involving $O_E$ in fact diverge in the limit $g_E \to 0$. (But this does not upset our assumption of the analyticity of the operator coefficients $C_m, C_E$ with respect to $g_E$. In eqs. (8.1) we have taken the angular average, which eliminates these singularities.) This kind of complication is absent in the $(\phi^4)^4$ theory. Hence, we can apply the variational formulas (4.3) to evaluate multiple derivatives of the correlation functions of composite operators at $\lambda = 0$. In this way we can relate the short distance singularities in the renormalized theory to the singularities we encounter in perturbative calculations. Then, we can interpret eqs. (5.6) (or (5.10)) as a relation between the perturbative singularities and anomalous dimensions. Calculations of OPE coefficients for the conjugate operators have been done up to first order in $\lambda$.

I would like to thank Orlando Alvarez for discussions.
Appendix A. Calculation of $O^*_E O^*_m - O^*_m O^*_E$

In this appendix we will sketch the derivation of the commutator (6.4). From the definition (6.1) we find, using (4.3),

$$
(\partial_m \partial_{g_E} - \partial_{g_E} \partial_m) \langle \mathcal{O}_i(0) \rangle_{m,g_E} = \lim_{\epsilon \to 0} \left( \int_{r \geq \epsilon} d^4r \lim_{\eta \to 0} \left( \int_{r' \geq \eta} d^4r' \left[ \langle \mathcal{O}_m(r')\mathcal{O}_E(r) - \mathcal{O}_E(r')\mathcal{O}_m(r) \rangle_{0,i} \middle( \mathcal{O}_i(0) \rangle_{m,g_E} \ight] 
+ A_{m,i,j}(\eta)\langle \mathcal{O}_E(r)\mathcal{O}_j(0) \rangle_{m,g_E} - A_{E,i,j}(\eta)\langle \mathcal{O}_m(r)\mathcal{O}_j(0) \rangle_{m,g_E} \right) 
- \int_{1 \geq r' \geq \epsilon} d^4r' Sg \lim_{\eta \to 0} \left( \text{ditto} \right) 
- \partial_m c_{E,i,j} \langle \mathcal{O}_j \rangle_{m,g_E} + \partial_{g_E} c_{m,i,j} \langle \mathcal{O}_j \rangle_{m,g_E} 
+ c_{E,i,j} \lim_{\epsilon \to 0} \left( \int_{r' \geq \epsilon} d^4r' \langle \mathcal{O}_m(r')\mathcal{O}_j(0) \rangle_{m,g_E} + A_{m,j,k}(\eta)\langle \mathcal{O}_k \rangle_{m,g_E} \right) 
- c_{m,i,j} \lim_{\epsilon \to 0} \left( \int_{r' \geq \epsilon} d^4r' \langle \mathcal{O}_E(r')\mathcal{O}_j(0) \rangle_{m,g_E} + A_{E,j,k}(\eta)\langle \mathcal{O}_k \rangle_{m,g_E} \right) \right) \right).$$

(A.1)

Here $Sg$ denotes the singular part with respect to the coordinate $r$.

Due to the antisymmetry of the integrand under interchange of $m$ and $g_E$, the first integral over $r'$ can be restricted to $r' \leq \epsilon$. In the second integral over $r'$, we must keep the entire range of integration, since the operation of taking the singular part spoils this antisymmetry. Hence, we obtain

$$
(\partial_m \partial_{g_E} - \partial_{g_E} \partial_m) \langle \mathcal{O}_i \rangle_{m,g_E} = \lim_{\epsilon \to 0} \left( \int_{r \geq \epsilon} d^4r \left( \lim_{\eta \to 0} \left( \int_{r' \geq \eta} d^4r' \langle \mathcal{O}_E(r)\mathcal{O}_m(r') \rangle_{0,i} - C_{m,i,j}(r')\langle \mathcal{O}_j(0) \rangle_{m,g_E} \right) 
- \mathcal{O}_m(r)\langle \mathcal{O}_E(r')\mathcal{O}_i(0) - C_{E,i,j}(r')\mathcal{O}_j(0) \rangle_{m,g_E} \right) + \int_{1 \geq r' \geq \epsilon} d^4r' \left( - C_{m,i,j}(r')\langle \mathcal{O}_E(r)\mathcal{O}_j(0) \rangle_{m,g_E} + C_{E,i,j}(r')\langle \mathcal{O}_m(r)\mathcal{O}_j(0) \rangle_{m,g_E} \right) \right) 
- \int_{r \geq \epsilon} d^4r Sg \lim_{\eta \to 0} \left( \int_{r' \geq \eta} d^4r' \langle \mathcal{O}_E(r)\mathcal{O}_m(r') \rangle_{0,i} - \mathcal{O}_m(r)\langle \mathcal{O}_E(r')\mathcal{O}_i(0) \rangle_{m,g_E} \right) 
+ \int_{1 \geq r' \geq \epsilon} d^4r' \left( - C_{m,i,j}(r')\langle \mathcal{O}_E(r)\mathcal{O}_j(0) \rangle_{m,g_E} + C_{E,i,j}(r')\langle \mathcal{O}_m(r)\mathcal{O}_j(0) \rangle_{m,g_E} \right) \right) 
+ (\partial_{g_E} c_m - \partial_m c_E + [c_E, c_m])_{i,j} \langle \mathcal{O}_j \rangle_{m,g_E} \right) \right) \right).$$

(A.2)
Now using operator product expansions we find

\[
\lim_{\epsilon \to 0} \int_{r \geq 1} d^4r \int_{r' \geq \epsilon} d^4r' \langle \mathcal{O}_E(r)(\mathcal{O}_m(r')\mathcal{O}_i(0) - C_{mi,j}(r')\mathcal{O}_j(0)) \rangle_{m,g_E} = 0 \quad (A.3)
\]

and

\[
S_g \int_{r' \geq 1} d^4r' \langle \mathcal{O}_m(r')\mathcal{O}_E(r)\mathcal{O}_i(0) \rangle_{m,g_E} = \int_{r' \geq 1} d^4r' \langle \mathcal{O}_m(r')C_{Ei,j}(r)\mathcal{O}_j(0) \rangle_{m,g_E} \quad (A.4)
\]

and similar equations obtained by interchanging \( m \) and \( g_E \).

Substituting the above into \( (A.2) \), we obtain

\[
\left( \partial_m \partial_{g_E} - \partial_{g_E} \partial_m \right) \langle \mathcal{O}_i \rangle_{m,g_E}
\]

\[
= \lim_{\epsilon \to 0} \left[ \int_{1 \geq r \geq \epsilon} d^4r \left( \int_{1 \geq r' \geq \epsilon} d^4r' \langle \mathcal{O}_E(r)(\mathcal{O}_m(r')\mathcal{O}_i(0) - C_{mi,j}(r')\mathcal{O}_j(0)) \rangle_{m,g_E} \right.ight.
\]

\[
- \mathcal{O}_m(r)(\mathcal{O}_E(r')\mathcal{O}_i(0) - C_{Ei,j}(r')\mathcal{O}_j(0)) \rangle_{m,g_E} 
\]

\[
- \int_{1 \geq r \geq \epsilon} d^4r S_g \left( \text{ditto} \right) 
\]

\[
+ \left( \partial_{g_E} c_m - \partial_m c_E + [c_E, c_m] \right)_{i,j} \langle \mathcal{O}_j \rangle_{m,g_E}. \quad (A.5)
\]

This gives \( (6.4) \).

In the above result the range of the double integral is restricted to \( r \leq 1 \), since we used \( r = 1 \) as the renormalization point when we introduced divergent subtractions \( A_E, A_m \) in \( (4.4) \). If we use a different renormalization point \( r = r_0 < 1 \) to define the subtractions \( A_E, A_m \), then we must change the finite counterterms \( c_E, c_m \) to

\[
c'_E = c_E - \int_{1 \geq r \geq r_0} d^4r \ C_E(r), \quad c'_m = c_m - \int_{1 \geq r \geq r_0} d^4r \ C_m(r). \quad (A.6)
\]

Then eq. \( (A.3) \) can be rewritten as

\[
\left( \partial_m \partial_{g_E} - \partial_{g_E} \partial_m \right) \langle \mathcal{O}_i \rangle_{m,g_E}
\]

\[
= \int_{r_0 \geq r} d^4r \ \text{F.P.} \int_{r_0 \geq r'} d^4r' \langle \mathcal{O}_E(r)(\mathcal{O}_m(r')\mathcal{O}_i(0) - C_{mi,j}(r')\mathcal{O}_j(0)) \rangle_{m,g_E} 
\]

\[
- \mathcal{O}_m(r)(\mathcal{O}_E(r')\mathcal{O}_i(0) - C_{Ei,j}(r')\mathcal{O}_j(0)) \rangle_{m,g_E} 
\]

\[
+ (\partial_{g_E} c'_m - \partial_m c'_E + [c'_E, c'_m] \right)_{i,j} \langle \mathcal{O}_j \rangle_{m,g_E}, \quad (A.7)
\]
where F.P. denotes a finite part with respect to $r$. We can take the radius $r_0$ as small as we want. This guarantees the locality of the double integral.

In the presence of other composite operators the consistency condition (A.3) is modified to

$$
(\partial_m \partial_{g_E} - \partial_{g_E} \partial_m) \langle O_{i_1}(r_1) \ldots O_{i_n}(r_n) \rangle_{m,g_E} = \sum_{k=1}^{n} \left[ \int_{r_0 \geq |r-r_k|} d^4r \text{ F.P.} \int_{r_0 \geq |r'-r_k|} d^4r' \langle O_E(r)(O_m(r')O_{i_k}(r_k) - C_{m_{ik,j}}(r'-r_k)O_j(r_k)) - O_{i_1}(r_1) \ldots O_{i_n}(r_n) \rangle_{m,g_E} \right],
$$

where we can take $r_0$ smaller than the smallest distance between any two of the $n$ points $r_1, \ldots, r_n$.

**Appendix B. Explicit calculation of $N(m,g_E)$**

We have found that the counterterms $c_m, c_E$ must satisfy the consistency condition (6.6)

$$
\partial_{g_E} c_m - \partial_m c_{E} + [c_E, c_m] = \Omega_{Em}, \quad \text{(B.1)}
$$

where the curvature matrix $\Omega_{Em}$ is defined by eq. (6.5). By taking the maximal part we obtain

$$
\partial_{g_E} \bar{c}_m - \partial_m \bar{c}_E + [\bar{c}_E, \bar{c}_m] = \bar{\Omega}_{Em}, \quad \text{(B.2)}
$$

where the bars denote the maximal parts.

Under a redefinition of the operators

$$
O'_i = N_{i,j}(m,g_E)O_j, \quad \text{(B.3)}
$$

where

$$
N_{i,j}(m,g_E) \propto m^{x_i-x_j}, \quad \text{(B.4)}
$$

the finite counterterms transform as

$$
c'_m = Nc_m N^{-1} - \partial_m N \cdot N^{-1}
$$

$$
c'_E = Nc_E N^{-1} - \partial_{g_E} N \cdot N^{-1}, \quad \text{(B.5)}
$$

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The non-maximal terms transform homogeneously, while the maximal terms transform inhomogeneously:
\[
\tilde{c}_m' = N\tilde{c}_m N^{-1} - \partial_m N \cdot N^{-1} \\
\tilde{c}'_E = N\tilde{c}_E N^{-1} - \partial_{gE} N \cdot N^{-1}.
\] (B.6)

We wish to choose the matrix \(N(m, g_E)\) such that
\[
\tilde{c}'_E(m, g_E) = 0 \quad \text{(B.7)}
\]
\[
\tilde{c}'_m(m, 0) = 0. \quad \text{(B.8)}
\]

The gauge condition (B.7) is solved by
\[
N(m, g_E) = M(m) \cdot R \exp \left[ \int_0^{g_E} dx \, \tilde{c}_E(m, x) \right], \quad \text{(B.9)}
\]
where \(R\) implies the increasing ordering of \(x\) toward right, and \(M(m)\) is an arbitrary invertible matrix dependent only on \(m\). The exponential is analytic in both \(m\) and \(g_E\).

Since
\[
N(m, 0) = M(m), \quad \text{(B.10)}
\]
we find
\[
\tilde{c}_m'(m, 0) = M c_m(m, 0) M^{-1} - \partial_m M \cdot M^{-1}.
\] (B.11)

Hence, to satisfy the other gauge condition (B.8), we must choose
\[
M(m) = L \cdot R \exp \left[ \int_0^m dy \, c_m(y, 0) \right], \quad \text{(B.12)}
\]
where \(L\) is an arbitrary invertible constant matrix. Note that \(M(m)\) is analytic in \(m\).

Finally we note that under the gauge conditions (B.7), (B.8), we get a relation
\[
c'_m(m, g_E) = \int_0^{g_E} dx \, \Omega'_{Em}(m, x). \quad \text{(B.13)}
\]
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