COMPLETE MONOTONICITY OF A FUNCTION INVOLVING THE GAMMA FUNCTION AND APPLICATIONS

FENG QI

ABSTRACT. In the article we present necessary and sufficient conditions for a function involving the logarithm of the gamma function to be completely monotonic and apply these results to bound the gamma function $\Gamma(x)$, the $n$-th harmonic number $\sum_{k=1}^n \frac{1}{k}$, and the factorial $n!$.

1. INTRODUCTION

We recall from [18, Chapter XIII], [29, Chapter 1] and [31, Chapter IV] that a function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and

$$0 \leq (-1)^n f^{(n)}(x) < \infty$$

(1.1)

for $x \in I$ and $n \geq 0$.

We also recall from [1, p. 254, 6.1.1] that the classical Euler gamma function $\Gamma(x)$ may be defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt, \quad x > 0.$$  

(1.2)

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or di-gamma function, and the derivatives $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are respectively called the polygamma functions. They are a series of important special functions and have much extensive applications in many branches such as statistics, probability, number theory, theory of 0-1 matrices, graph theory, combinatorics, physics, engineering, and other mathematical sciences.

Using Hermite-Hadamard’s inequality (see [26, 27, 28]), the double inequality

$$\left( x - \frac{1}{2} \right) \left[ \ln \left( x - \frac{1}{2} \right) - 1 \right] + \ln \sqrt{2\pi} - \frac{1}{24(x-1)} \leq \ln \Gamma(x)$$

$$\leq \left( x - \frac{1}{2} \right) \left[ \ln \left( x - \frac{1}{2} \right) - 1 \right] + \ln \sqrt{2\pi} - \frac{1}{24(\sqrt{x^2 + x + 1/2} - 1/2)}$$

(1.3)

for $x > 1$ was obtained in [4, p. 236, Theorem 1].

In [30, p. 1774, Theorem 2.3], the function

$$H(x) = \ln \Gamma(x+1) - \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) + x + \frac{1}{2} - \frac{1}{2} \ln(2\pi) + \frac{1}{24(x+1/2)}$$

(1.4)

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was proved to be completely monotonic on \((0, \infty)\). From this it was deduced in [30, p. 1775, Corollary 2.4] that the double inequality

\[
\alpha \left( \frac{x + 1/2}{e} \right)^{x+1/2} e^{-1/24(x+1/2)} < \Gamma(x+1) \\
\leq \beta \left( \frac{x + 1/2}{e} \right)^{x+1/2} e^{-1/24(x+1/2)}
\]

holds for \(x > 0\), where \(\alpha = \sqrt{2\pi} = 2.50 \ldots\) and \(\beta = \sqrt{2} e^{7/12} = 2.53 \ldots\) are the best possible constants.

We observe that, by taking the natural exponentials on all sides of (1.3) and replacing \(x\) by \(x + 1\), the inequality (1.3) may be rewritten as

\[
\sqrt{2\pi} \left( \frac{x + 1/2}{e} \right)^{x+1/2} e^{-1/24x} < \Gamma(x+1) \\
\leq \sqrt{2\pi} \left( \frac{x + 1/2}{e} \right)^{x+1/2} e^{-1/24\left(\sqrt{x^2+3x+5/2} - 1/2\right)} , \quad x > 0 .
\]

Hence, it is clear that the left hand side inequality in (1.5) is stronger than the corresponding one in (1.3) or, equivalently, (1.6). But, when \(x \geq 1\), the right hand side inequality in (1.5) is weaker than the corresponding one in (1.3) or, equivalently, (1.6).

For \(\lambda \geq 0\) and \(x \in (0, \infty)\), let

\[ H_\lambda(x) = \ln \Gamma(x+1) - \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) + x + \frac{1}{2} - \ln \sqrt{2\pi} + \frac{1}{24(x + \lambda)}. \]

The first aim of this paper is to find necessary and sufficient conditions for the functions \(\pm H_\lambda(x)\) to be completely monotonic on \((0, \infty)\). The second aim is to apply the complete monotonicity of \(\pm H_\lambda(x)\) to establish inequalities for bounding the gamma function, the \(n\)-th harmonic number \(\sum_{k=1}^{n} \frac{1}{k}\), and the factorial \(n!\).

2. NECESSARY AND SUFFICIENT CONDITIONS

Our main results are necessary and sufficient conditions for the functions \(\pm H_\lambda(x)\) to be completely monotonic on \((0, \infty)\), which can be stated in the following theorem.

**Theorem 1.** For \(x \in (0, \infty)\) and \(\lambda \geq 0\),

1. if and only if \(0 \leq \lambda \leq \frac{1}{2}\), the function \(H_\lambda(x)\) is completely monotonic;
2. if

\[ \lambda \geq \inf_{t \in (0, \infty)} \left\{ \frac{1}{t} \ln \left[ \frac{24}{t^2} \left( \frac{1}{e^{t/2}} - \frac{t}{e^t - 1} \right) \right] \right\}, \]

the function \(-H_\lambda(x)\) is completely monotonic;
3. if \(\lambda \geq \frac{3}{2}\), a special case of the inequality (2.1), the function \(-H_\lambda(x)\) is completely monotonic on \((0, \infty)\).

**Proof of the necessary condition in (1) of Theorem 1.** If the function \(H_\lambda(x)\) is completely monotonic on \((0, \infty)\) for \(\lambda \geq 0\), then \(H_\lambda(x) \geq 0\), which may be rearranged as

\[ \lambda \leq -x - \frac{1}{24f(x)}, \]
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where
\[ f(x) = \ln \Gamma(x + 1) - \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) + x + \frac{1}{2} - \frac{1}{2} \ln(2\pi), \quad x > 0. \tag{2.2} \]

Since
\[ \ln \Gamma(z) \sim \left( z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \cdots \tag{2.3} \]
as \( z \to \infty \) in \( |\arg z| < \pi \), see [1, p. 257, 6.1.41], we have
\[ f(x) \sim \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) - \frac{1}{2} + \frac{1}{12(x + 1)} + O\left( \frac{1}{x^2} \right) \tag{2.4} \]
as \( x \to \infty \). As a result, we have
\[ -x - \frac{1}{24f(x)} \sim \frac{1}{2} \frac{h_1(x)}{h_2(x)} \]
as \( x \to \infty \), where
\[ h_1(x) = 12x(2x^2 + 3x + 1) \ln \frac{x + 1}{x + 1/2} - 12x^2 - 9x + 1 + O(1) \to \frac{1}{2} \]
and
\[ h_2(x) = 6(2x^2 + 3x + 1) \ln \frac{x + 1}{x + 1/2} - 6x - 5 + O\left( \frac{1}{x} \right) \to -\frac{1}{2} \]
as \( x \to \infty \). So
\[ -x - \frac{1}{24f(x)} \to \frac{1}{2}, \quad x \to \infty. \]

This means that the necessary condition for the function \( H_\lambda(x) \) to be completely monotonic on \((0, \infty)\) is \( \lambda \leq \frac{1}{2} \).

First proof of the sufficient condition in (1) of Theorem 1. When \( \lambda < \frac{1}{2} \), the function \( H_\lambda(x) \) is
\[ H_\lambda(x) = H_{1/2}(x) + \frac{1}{24(x + \lambda)} - \frac{1}{24(x + 1/2)} \]
\[ = H_{1/2}(x) + \frac{1}{24} \int_{-\lambda}^{1/2} \frac{1}{(x + t)^2} \, dt. \]

Since the function \( H_{1/2}(x) = H(x) \) is completely monotonic on \((0, \infty)\), see [30, p. 1774, Theorem 2.3], and the integral \( \int_{-\lambda}^{1/2} \frac{1}{(x + t)^2} \, dt \) is clearly completely monotonic on \((-\lambda, \infty) \supset (0, \infty)\), also since the product and sum of finite completely monotonic functions are completely monotonic on their common domain, it follows immediately that the function \( H_\lambda(x) \) is completely monotonic on \((0, \infty)\) if \( \lambda \leq \frac{1}{2} \).

Second proof of the sufficient condition in (1) of Theorem 1. The famous Binet’s first formula of \( \ln \Gamma(x) \) for \( x > 0 \) is given by
\[ \ln \Gamma(x) = \left( x - \frac{1}{2} \right) \ln x - x + \ln \sqrt{2\pi} + \theta(x), \tag{2.5} \]
where
\[ \theta(x) = \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{e^{-xt}}{t} \, dt \tag{2.6} \]
for $x > 0$ is called the remainder of Binet’s first formula for the logarithm of the gamma function, see [17, p. 11] or [25, p. 462]. The formulas
\[ \Gamma(z) = k^z \int_0^\infty t^{z-1}e^{-kt} dt \] (2.7)
and
\[ \ln \frac{b}{a} = \int_0^\infty \frac{e^{-au} - e^{-bu}}{u} du \] (2.8)
for $\Re z > 0$, $\Re k > 0$, $a > 0$ and $b > 0$ can be found in [1, p. 255, 6.1.1 and p. 230, 5.1.32]. Utilizing these formulas yields
\[ f(x) = \left( x + 1/2 \right) \ln \frac{x}{x + 1/2} + 1/2 + \theta(x), \] (2.9)
\[ f'(x) = \frac{1}{2x} + \ln \frac{x}{x + 1/2} + \theta'(x) \]
\[ = \int_0^\infty \left( \frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} \right) e^{-xt} dt, \] (2.10)
\[ H'_\lambda(x) = f'(x) - \frac{1}{24(x + \lambda)^2} \]
\[ = \int_0^\infty \left( \frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} - \frac{te^{-\lambda t}}{24} \right) e^{-xt} dt. \] (2.11)
From (1.7), (2.2) and (2.9), it is easy to see that
\[ \lim_{x \to \infty} H_\lambda(x) = \lim_{x \to \infty} f(x) + \lim_{x \to \infty} \frac{1}{24(x + \lambda)} = 0. \] (2.12)
Therefore, in order to prove the complete monotonicity of $\mp H_\lambda(x)$, it suffices to show $\pm H'_\lambda(x)$ is completely monotonic on $(0, \infty)$. For this, it is sufficient to have
\[ \frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} - \frac{te^{-\lambda t}}{24} \geq 0 \]
for all $t \in (0, \infty)$, which is equivalent to
\[ \lambda \geq \frac{-1}{t} \ln \left[ \frac{24}{t} \left( \frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} \right) \right] \] (2.13)
for all $t \in (0, \infty)$.

We claim that
\[ \frac{-1}{t} \ln \left[ \frac{24}{t} \left( \frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} \right) \right] \geq \frac{1}{2} \]
for $t \in (0, \infty)$. In fact, this inequality can be reduced to
\[ \frac{24}{t} \left( \frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} \right) \leq e^{-t/2}, \] (2.14)
equivalently,
\[
(t^2 - 24)e^t + 24te^{t/2} - t^2 + 24 = \sum_{k=5}^{\infty} \left( [k(k-1) - 24]2^k + 48k \right) \frac{t^k}{k!2^k}
\]
Thus, when \( 0 \leq \lambda \leq \frac{1}{2} \), the function \( H_\lambda(x) \) is completely monotonic on \((0, \infty)\).

**Proof of (2) in Theorem 1.** This follows from the inequality with the sign \( \geq \) in (2.13).

**Proof of (3) in Theorem 1.** Suppose that

\[
- \frac{1}{t} \ln \left[ \frac{24}{t} \left( \frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} \right) \right] \leq \lambda
\]

for \( t \in (0, \infty) \). Then

\[
\frac{24}{t} \left( \frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} \right) \geq e^{-\lambda t},
\]

which can be rewritten as

\[
24e^{\lambda t} (e^t - te^{t/2} - 1) \geq t^2 e^{t/2} (e^t - 1).
\]

Expanding at \( t = 0 \) the functions on both sides of the above inequality into power series yields

\[
24 \sum_{k=3}^{\infty} \left[ (\lambda+1)^k - \lambda^k - k \left( \lambda + \frac{1}{2} \right)^{k-1} \right] \frac{k^k}{k!} \geq \sum_{k=3}^{\infty} \left[ \frac{3}{2} \right]^{k-2} - \left( \frac{1}{2} \right)^{k-2} \frac{1}{(k-2)!}.
\]

Let \( h_{k;\lambda}(u) = (\lambda + u)^{k-1} \). Then, by the left hand side inequality in the double integral inequality

\[
(b - a)^2 \leq \frac{1}{b - a} \int_a^b f(t) \, dt - f \left( \frac{a + b}{2} \right) \leq \frac{(b - a)^2}{24} M,
\]

where \( f : [a, b] \to \mathbb{R} \) be a twice differentiable mapping and \( m \leq f''(t) \leq M \) for all \( t \in (a, b) \), see [5, 6] and [26, p. 236, Theorem A], we obtain

\[
(\lambda + 1)^k - \lambda^k - k \left( \lambda + \frac{1}{2} \right)^{k-1} = k \left[ \frac{1}{1 - 0} \int_0^1 h_{k;\lambda}(u) \, du - h_{k;\lambda} \left( \frac{0 + 1}{2} \right) \right]
\]

\[
\geq k \cdot \frac{(1-0)^2}{24} \inf_{u \in (0,1)} h_{k;\lambda}'(u)
\]

\[
= \frac{1}{24} k(k-1)(k-2)\lambda^{k-3}.
\]

Therefore, in order to have the inequality (2.15) hold, it is sufficient to make

\[
(k - 2)\lambda^{k-3} \geq \left( \frac{3}{2} \right)^{k-2} - \left( \frac{1}{2} \right)^{k-2},
\]
which is equivalent to

$$\lambda \geq k^{-3} \sqrt{\frac{1}{k-2} \left( \frac{3}{2} \right)^{k-2} - \left( \frac{1}{2} \right)^{k-2}}$$

for all \( k \geq 3 \). Since

$$k^{-3} \sqrt{\frac{1}{k-2} \left( \frac{3}{2} \right)^{k-2} - \left( \frac{1}{2} \right)^{k-2}} \leq k^{-3} \sqrt{\frac{3}{2^{k-2}} - \left( \frac{1}{2} \right)^{k-2}} \to \frac{3}{2}$$

as \( k \to \infty \), it follows that when \( \lambda \geq \frac{3}{2} \), the inequality (2.15) holds. This means that when \( \lambda \geq \frac{3}{2} \), the negative of the function (2.11) is completely monotonic on \((0, \infty)\). The proof of (2) in Theorem 1 is complete.

3. Applications

In this section, we apply the complete monotonicity of \( \pm H_\lambda(x) \) to establish inequalities for bounding the gamma function, the \( n \)-th harmonic number \( \sum_{k=1}^{n} \frac{1}{k} \), and the factorial \( n! \).

**Theorem 2.** For \( x \in (0, \infty) \), the gamma function \( \Gamma(x+1) \) can be bounded by

$$\sqrt{2\pi} \left( \frac{x+1/2}{e} \right)^{x+1/2} \exp \left( -\frac{1}{24(x+1/2)} \right) < \Gamma(x+1) < \sqrt{2\pi} \left( \frac{x+3/2}{e} \right)^{x+1/2} \exp \left( \frac{1}{24} \left( \frac{2x}{x+1/2} - 12(\ln \pi - 1) \right) \right)$$

(3.1)

and

$$\sqrt{2\pi} \left( \frac{x+1/2}{e} \right)^{x+1/2} \exp \left( \frac{1}{24} \left( \frac{2x}{3(x+3/2)} - 12(\ln \pi - 1) \right) \right) < \Gamma(x+1) < \sqrt{2\pi} \left( \frac{x+1/2}{e} \right)^{x+1/2} \exp \left( -\frac{1}{24(x+3/2)} \right).$$

(3.2)

**Proof.** By (2.4), it is easy to see that

$$\lim_{x \to \infty} H_\lambda(x) = 0.$$  

(3.3)

Moreover, it is immediate that

$$\lim_{x \to 0^+} H_\lambda(x) = \frac{1}{24} \left( \frac{1}{\lambda} + 12 - 12 \ln \pi \right).$$

By Theorem 1, it readily follows that when and only when \( 0 \leq \lambda \leq \frac{1}{2} \), the function \( H_\lambda(x) \) is decreasing on \((0, \infty)\). So, we have

$$0 < H_\lambda(x) < \frac{1}{24} \left( \frac{1}{\lambda} + 12 - 12 \ln \pi \right),$$

that is,

$$\sqrt{2\pi} \left( \frac{x+1/2}{e} \right)^{x+1/2} \exp \left( -\frac{1}{24(x+\lambda)} \right) < \Gamma(x+1) < \sqrt{2\pi} \left( \frac{x+1/2}{e} \right)^{x+1/2} \exp \left( \frac{1}{24} \left( \frac{1}{\lambda} + 12 - 12 \ln \pi - \frac{1}{x+\lambda} \right) \right)$$

(3.4)
for \(0 \leq \lambda \leq \frac{1}{2}\) and \(x \in (0, \infty)\). The inequality (3.1) is proved.

Similarly, when \(\lambda \geq \frac{3}{2}\), the function \(H_\lambda(x)\) is increasing on \((0, \infty)\), and the inequality (3.4) is reversed on \((0, \infty)\) for \(\lambda \geq \frac{3}{2}\). The inequality (3.2) follows. □

**Theorem 3.** For \(n \in \mathbb{N}\), the \(n\)-th harmonic number \(\sum_{k=1}^{n} \frac{1}{k}\) can be bounded by

\[
\ln\left(n + \frac{1}{2}\right) + \frac{1}{24(n + 1/2)^2} + 1 - \frac{3}{2} - \frac{1}{54} \leq \sum_{k=1}^{n} \frac{1}{k} < \ln\left(n + \frac{1}{2}\right) + \frac{1}{24(n + 1/2)^2} + 1 - \frac{3}{2} - \frac{1}{90} \quad (3.5)
\]

and

\[
\ln\left(n + \frac{1}{2}\right) + \frac{1}{24(n + 3/2)^2} + \gamma < \sum_{k=1}^{n} \frac{1}{k} \leq \ln\left(n + \frac{1}{2}\right) + \frac{1}{24(n + 3/2)^2} + 1 - \frac{3}{2} - \frac{1}{90} \quad (3.6)
\]

where \(\gamma = 0.577 \ldots\) stands for Euler-Mascheroni’s constant.

**Proof.** By Theorem 1 and the definition of completely monotonic functions, it follows that

1. when \(0 \leq \lambda \leq \frac{1}{2}\), the function \(H'_\lambda(x)\) is increasing on \((0, \infty)\),
2. when \(\lambda \geq \frac{3}{2}\), the function \(H'_\lambda(x)\) is decreasing on \((0, \infty)\).

Since \(H'_\lambda(x) = \psi(x + 1) - \ln\left(x + \frac{1}{2}\right) - \frac{1}{24(x + \lambda)^2}\) and \(\lim_{x \to \infty} H'_\lambda(x) = 0\), it follows readily that

\[
\ln\left(x + \frac{1}{2}\right) + \frac{1}{24(x + 1/2)^2} + 1 - \gamma - \frac{3}{2} - \frac{1}{54} \leq \psi(x + 1)
\]

\[
< \ln\left(x + \frac{1}{2}\right) + \frac{1}{24(x + 1/2)^2} \quad (3.7)
\]

and

\[
\ln\left(x + \frac{1}{2}\right) + \frac{1}{24(x + 3/2)^2} < \psi(x + 1)
\]

\[
\leq \ln\left(x + \frac{1}{2}\right) + \frac{1}{24(x + 3/2)^2} + 1 - \gamma - \frac{3}{2} - \frac{1}{90} \quad (3.8)
\]

for \(x \in [1, \infty)\). Taking \(x = n\) and using

\[
\psi(n + 1) = \sum_{k=1}^{n} \frac{1}{k} - \gamma, \quad (3.9)
\]

see [1, p. 258, 6.3.2], in (3.7) and (3.8) give inequalities (3.5) and (3.6). □

We recall from [2, 22] that a function \(f\) is said to be logarithmically completely monotonic on an interval \(I \subseteq \mathbb{R}\) if it has derivatives of all orders on \(I\) and its logarithm \(\ln f\) satisfies

\[
(-1)^k[\ln f(x)]^{(k)} \geq 0 \quad (3.10)
\]
for \( k \in \mathbb{N} \) on \( I \).

**Theorem 4.** For \( x \in (0, \infty) \) and \( \lambda \geq 0 \), let

\[
G_\lambda(x) = \frac{e^\lambda \Gamma(x + 1)}{(x + 1/2)^{x+1/2}} \exp \frac{1}{24(x + \lambda)}.
\]  
(3.11)

Then the function \( G_\lambda(x) \) has the following properties:

1. if and only if \( 0 \leq \lambda \leq \frac{1}{2} \), the function \( G_\lambda(x) \) is logarithmically completely monotonic on \((0, \infty)\);
2. if the inequality (2.1) is valid, the reciprocal of the function \( G_\lambda(x) \) is logarithmically completely monotonic on \((0, \infty)\);
3. if \( \lambda \geq \frac{3}{2} \), a special case of the inequality (2.1), the reciprocal of the function \( G_\lambda(x) \) is logarithmically completely monotonic on \((0, \infty)\).

**Proof.** This follows from the obvious fact that

\[
\ln G_\lambda(x) = H_\lambda(x) - \frac{1 - \ln(2\pi)}{2}
\]

and the definition of logarithmically completely monotonic functions. \(\square\)

**Theorem 5.** For \( n \in \mathbb{N} \), the factorial \( n! \) can be bounded by

\[
\sqrt{2\pi} \left( \frac{n + 1/2}{e} \right)^{n+1/2} \exp \left( -\frac{1}{24(n + 1/2)} \right) < n! 
\leq \sqrt{2\pi} \left( \frac{n + 1/2}{e} \right)^{n+1/2} \exp \left( \frac{1}{24} \left[ 12 \left( 3 - \ln \pi + \ln \frac{4}{27} \right) - \frac{1}{3(n+1/2)} \right] \right) 
\]  
(3.12)

and

\[
\sqrt{2\pi} \left( \frac{n + 1/2}{e} \right)^{n+1/2} \exp \left( \frac{1}{24} \left[ 12 \left( 3 - \ln \pi + \ln \frac{4}{27} \right) + \frac{1}{5(n + 3/2)} \right] \right) \leq n! 
< \sqrt{2\pi} \left( \frac{n + 1/2}{e} \right)^{n+1/2} \exp \left( -\frac{1}{24(n + 3/2)} \right). 
\]  
(3.13)

**Proof.** Combining (3.3) and

\[
H_\lambda(1) = \frac{1}{24} \left[ \frac{1}{\lambda + 1} - 12 \ln(2\pi) - 36 \ln \frac{3}{2} \right],
\]

with Theorem 1 and the proof of Theorem 2 reveals

\[
\sqrt{2\pi} \left( \frac{x + 1/2}{e} \right)^{x+1/2} \exp \left( -\frac{1}{24(x + 1/2)} \right) < \Gamma(x + 1) 
\]

\[
\leq \sqrt{2\pi} \left( \frac{x + 1/2}{e} \right)^{x+1/2} \exp \left( \frac{1}{24} \left[ 12 \left( 3 - \ln \pi + \ln \frac{4}{27} \right) - \frac{1}{3(x+1/2)} \right] \right) 
\]  
(3.14)

and

\[
\sqrt{2\pi} \left( \frac{x + 1/2}{e} \right)^{x+1/2} \exp \left( \frac{1}{24} \left[ 12 \left( 3 - \ln \pi + \ln \frac{4}{27} \right) + \frac{1}{5(x + 3/2)} \right] \right) \leq 
\Gamma(x + 1) < \sqrt{2\pi} \left( \frac{x + 1/2}{e} \right)^{x+1/2} \exp \left( -\frac{1}{24(x + 3/2)} \right). 
\]  
(3.15)
Letting $x = n$ and using $\Gamma(n+1) = n!$ in (3.14) and (3.15) leads to inequalities (3.12) and (3.13). The proof is complete. □

4. Remarks

In this section, we would like to comment some results above-presented.

Remark 1. The inequality (2.14) and the inequality (2.15) for $\lambda \geq \frac{3}{2}$ may be rearranged as the double inequality

$$\frac{1}{e^{x/2}} - \frac{x}{24e^{\beta x}} \leq \frac{x}{e^x - 1} \leq \frac{1}{e^{x/2}} - \frac{x}{24e^{\alpha x}}$$

(4.1)

on $(0, \infty)$, where $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. This improves the right hand side and partially improves the left hand side of the double inequality

$$\frac{1}{e^x} < \frac{x}{e^x - 1} < \frac{1}{e^{x/2}}, \quad x > 0$$

(4.2)

in [23, p. 2550, Proposition 4.1].

We guess that the scalar $\alpha = \frac{3}{2}$ in (4.1) can be replaced by a smaller number, for example, 1, but the constant $\beta = \frac{1}{2}$ in (4.1) is the best possible.

In [19], some related inequalities for the exponential function $e^x$ were constructed. In a subsequent paper, we will refine the right hand side of the double inequality (4.1) and employ it to strengthen double inequalities for bounding Mathieu's series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2}, \quad r > 0$$

(4.3)

and the like. For more information on bounding Mathieu type series, please refer to [16, 20, 23] and closely related references therein.

Remark 2. There have been plenty of references devoted to bounding the $n$-th harmonic number $\sum_{k=1}^{n} \frac{1}{k}$ for $n \in \mathbb{N}$, for example, [7, 10, 11, 24] and closely related references therein.

Remark 3. Several inequalities for bounding the gamma function were also established and collected in [9, 12, 13]. See also [21, pp. 52–57] and lots of references cited therein.

Remark 4. It was proved once again in [3, 8, 22] that the set of logarithmically completely monotonic functions is a subset of the completely monotonic functions. This implies that Theorem 4 is not trivial.

Remark 5. In [14, 15], it was shown that the function

$$g_\beta(x) = \frac{e^x \Gamma(x + 1)}{(x + \beta)^{x+\beta}}$$

(4.4)

on the interval $(\max\{0, -\beta\}, \infty)$ for $\beta \in \mathbb{R}$ is logarithmically completely monotonic if and only if $\beta \geq 1$ and that the function $[g_{\alpha, \beta}(x)]^{-1}$ is logarithmically completely monotonic if and only if $\beta \leq \frac{1}{2}$. See also [21, pp. 53–54, Section 5.6]. Motivated by this and Theorem 4, we would like to ask a question: How about the logarithmically complete monotonicity of the function

$$G_{\lambda, \mu}(x) = \frac{e^x \Gamma(x + 1)}{(x + \mu)^{x+\mu}} \exp \frac{1}{24(x + \lambda)}$$

(4.5)

on the interval $(\max\{0, -\lambda, -\mu\}, \infty)$? where $\lambda$ and $\mu$ are given real numbers.
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