ON THE TWISTED OCTONIONIC EIGENVALUE PROBLEM
AND SOME SEXTICS HYPERSURFACES RELATED TO THE
CARTAN CUBIC

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Abstract

We revisit the octonionic eigenvalue problem from a geometric perspective. In particular, we study a tautological sheaf defined on a sextic related to this problem, the Ogievetskii-Dray-Manogue sextic. We then define and study a twisted version of the octonionic eigenvalue problem. A new sextic arises in this setting and we study the corresponding tautological sheaf supported on it. This twisted version of the octonionic eigenvalue problem is eminently more symmetric than the original one, as reflected by the last result we prove in this paper: the automorphism group of the twisted octonionic eigenvalue problem, though not isomorphic to $E_6$, acts prehomogeneously on the exceptional Jordan algebra $J_3(O)$. This is in sharp contrast with the fact that the generic orbit for the action of the automorphism group of the classical octonionic eigenvalue problem has (at least) codimension 6 in $J_3(O)$.

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1 Introduction

One starting point of our paper is the claim made by Bhargava and Ho that there exists a «less familiar to most people» (as they state it) rank 8 tautological vector bundle on the (complexified) octonionic projective plane $\mathbb{OP}^2 \subset \mathbb{P}^5$ (see Theorem 5.27 of [BH16]). Following their definition 5.21, one should believe that this vector bundle is endowed with a global faithful right action of the (complexified) octonions $\mathbb{O}$. Let us recall the well-known construction of such a tautological bundle in the case of an associative composition algebra instead of $\mathbb{O}$.

We denote by $A$ one of the (complexified) algebras $\mathbb{R} \otimes \mathbb{C}$, $\mathbb{C} \otimes \mathbb{C}$, $\mathbb{H} \otimes \mathbb{C}$ and we let $J_3(A)$ be the space of hermitian matrices with coefficients in $A$:

$$J_3(A) = \left\{ \begin{pmatrix} \lambda_1 & c & \overline{b} \\ \overline{\tau} & \lambda_2 & a \\ b & \overline{\tau} & \lambda_3 \end{pmatrix} \right\}, \quad a, b, c \in A, \quad \lambda_i \in \mathbb{C}.$$

For any $A \in J_3(A)$, we denote by $\det_A(A)$ the number:

$$\det_A(A) = \lambda_1 \lambda_2 \lambda_3 + (ab) + \left( (\overline{b})(\overline{\tau}) \right) \overline{\tau} - \lambda_2 b\overline{b} - \lambda_1 a\overline{\alpha} - \lambda_3 c\overline{c}$$

$$= \lambda_1 \lambda_2 \lambda_3 + 2\text{Re}(cab) - \lambda_2 |b|^2 - \lambda_1 |a|^2 - \lambda_3 |c|^2,$$

where $\text{Re}(a)$ is the real part of $a \in A$ and $|a|^2$ is the square of its norm. This expression corresponds to the Sarrus expansion for a formal determinant of the $3 \times 3$ matrix $A$. We notice the remarkable that $\det_A(A) \in \mathbb{C}$, for all $A \in J_3(A)$. Finally, we denote by $\mathcal{C}_A$ the cubic hypersurface in $\mathbb{P}(J_3(A))$ given by the equation $\det_A = 0$ and $X_A$ the singular locus of this hypersurface. The case by case analysis is well-known ([Rob88] for instance):

- if $A = \mathbb{R} \otimes \mathbb{C}$, then $X_A = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is the Veronese embedding of $\mathbb{P}^2$ and $\mathcal{C}_A$ is the determinantal cubic for $3 \times 3$ symmetric matrices.

- if $A = \mathbb{C} \otimes \mathbb{C}$, then $X_A = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ is the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathcal{C}_A$ is the determinantal cubic for $3 \times 3$ general matrices.

- if $A = \mathbb{H} \otimes \mathbb{C}$, then $X_A = \text{Gr}(2, 6) \subset \mathbb{P}^{14}$ is the Plücker embedding of $\text{Gr}(2, 6)$ and $\mathcal{C}_A$ is the Pfaffian cubic for $6 \times 6$ skew-symmetric matrices.
In each case above, one can provide an easy ad-hoc construction of a rank $n_A = \dim_\mathbb{C} \mathbb{A}$ tautological vector bundle on $X_A$. We shall rather give a uniform construction of this vector bundle which endows it with a global faithful right action of $\mathbb{A}$. For all $A = \begin{pmatrix} \lambda_1 & c & \bar{a} \\ \bar{a} & \lambda_2 & \bar{b} \\ a & b & \lambda_3 \end{pmatrix} \in J_3(A)$, we denote by $\text{Com}(A)$, the element of $J_3(A)$ defined by:

$$\text{Com}(A) = \begin{pmatrix} \lambda_2 \lambda_3 - a \bar{a} & (\bar{b})(\bar{a}) - \lambda_3 c & c a - \lambda_2 \bar{b} \\ ab - \lambda_3 \bar{a} & \lambda_1 \lambda_3 - \bar{b} \bar{a} & (\bar{c})(\bar{b}) - \lambda_1 a \\ (\bar{a})(\bar{c}) - \lambda_2 \bar{b} & b c - \lambda_1 \bar{a} & \lambda_1 \lambda_2 - c \bar{c} \end{pmatrix}.$$ 

Owing to the associativity of $\mathbb{A}$ and to fundamental relation $\text{Re}(cab) = \text{Re}(abc) = \text{Re}(bca)$, for any $c, b, a \in \mathbb{A}$, we get, for any $A \in J_3(A)$, the matrix factorization:

$$\text{Com}(A) \times A = A \times \text{Com}(A) = \text{det}_A(A).I_3,$$

where $I_3$ is the identity $3 \times 3$ matrix. Furthermore, there is a natural embedding of algebras:

$$L : \mathbb{A} \longrightarrow \mathcal{M}_{n_A,n_A}(\mathbb{C}),$$

sending an element to its left multiplication table and which induces an embedding of algebras:

$$L : J_3(A) \longrightarrow \mathcal{M}_{3n_A,3n_A}(\mathbb{C}).$$

We fix $e_1, \ldots, e_{n_A}$ a basis of $\mathbb{A}$ over $\mathbb{C}$ and consider the ring $\mathbb{C}[x_1, \ldots, x_{3n_A+3}]$. We put $c = x_1 e_1 + \ldots + x_n e_n$, $b = x_{n+1} e_1 + \ldots + x_{2n} e_n$, $a = x_{2n+1} e_1 + \ldots + x_{3n_A+2} e_n$, $\lambda_1 = x_{3n_A+1} e_1$, $\lambda_2 = x_{3n_A+2} e_1$ and $\lambda_3 = x_{3n_A+3} e_1$. Consider the $3n_A \times 3n_A$ matrix with entries in $\mathbb{C}[x_1, \ldots, x_{3n_A+3}]$ described in $n_A \times n_A$ blocks by:

$$M_A := L(A) = \begin{pmatrix} \begin{pmatrix} \lambda_1 & \bar{L}_c \\ \bar{L}_c & \lambda_2 \end{pmatrix} & \begin{pmatrix} \bar{L}_b \\ \bar{L}_b \end{pmatrix} \\ \begin{pmatrix} \bar{L}_c & \lambda_2 \end{pmatrix} & \lambda_3 \end{pmatrix}.$$ 

One immediately notices that $M_A$ is a symmetric $3n_A \times 3n_A$ matrix with entries in the ring $\mathbb{C}[x_1, \ldots, x_{3n_A+3}]$. We also consider the matrix:

$$\text{Com}(M_A) := L(\text{Com}(A)) = \begin{pmatrix} \begin{pmatrix} \lambda_2 \lambda_3 - a \bar{a} & (\bar{b})(\bar{a}) - \lambda_3 c & c a - \lambda_2 \bar{b} \\ ab - \lambda_3 \bar{a} & \lambda_1 \lambda_3 - \bar{b} \bar{a} & (\bar{c})(\bar{b}) - \lambda_1 a \\ (\bar{a})(\bar{c}) - \lambda_2 \bar{b} & b c - \lambda_1 \bar{a} & \lambda_1 \lambda_2 - c \bar{c} \end{pmatrix} \\ \begin{pmatrix} \bar{L}_c & \lambda_2 \end{pmatrix} \begin{pmatrix} \bar{L}_b \\ \bar{L}_b \end{pmatrix} \\ \begin{pmatrix} \bar{L}_c & \lambda_2 \end{pmatrix} \begin{pmatrix} \bar{L}_b \\ \bar{L}_b \end{pmatrix} \end{pmatrix}.$$ 

The matrix $\text{Com}(M_A)$ is also a symmetric $3n_A \times 3n_A$ symmetric matrix with coefficients in $\mathbb{C}[x_1, \ldots, x_{3n_A+3}]$. The matrix factorizations in equation (1) and the fact that $L$ is a morphism of algebras show that we have a matrix factorizations in $\mathcal{M}_{3n_A,3n_A}(\mathbb{C}[x_1, \ldots, x_{3n_A+3}]):$

$$\text{Com}(M_A) \times M_A = M_A \times \text{Com}(M_A) = \text{det}_A.I_{3n_A},$$

where $\text{Det}_A$ is the cubic polynomial in $\mathbb{C}[x_1, \ldots, x_{3n_A+3}]$ corresponding to $\text{det}_A$. We deduce from Equation (2) that the degree $3n_A$ polynomial $\text{det}_{\mathbb{A}_{3n_A;3n_A}}(M_A)$ and the cubic polynomial $\text{Det}_A$ define set-theoretically the same hypersurface in $\mathbb{P}^{3n_A+2}$ (that is the locus where $M_A$ is not full-rank). On the other hand, it is known that $\text{Det}_A$ is irreducible (see the appendix of [Rob88]), so that we get the relation:

$$\text{det}_{\mathbb{A}_{3n_A;3n_A}}(M_A) = (\text{Det}_A)^{n_A}.$$
Let us denote by $E_A$ the cokernel of:

$$
\mathbb{A}^{\oplus 3} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(J_3(A))}(-1) \xrightarrow{M_A} \mathbb{A}^{\oplus 3} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(J_3(A))}
$$

The sheaf $E_A$ is supported on $\mathcal{C}_A$, as can be read from equation 2. It is generically of rank $n_A$ as follows from equation 3. Since $A$ is associative, we have:

$$
M_A \times \begin{pmatrix} x \times a \\ y \times a \\ z \times a \end{pmatrix} = \begin{pmatrix} M_A \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} \times a,
$$

for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{A}^{\oplus 3}$ and all $a \in A$. This implies that $E_A$ is induced with a global faithful right action of $A$. The sheaf $E_A$ is a tautological sheaf on $\mathcal{C}_A$. Indeed, for any $A$ in the smooth locus of $\mathcal{C}_A$, the stalk $E_A|_A$ is the cokernel of $A$. In order to transport this sheaf to $X_A$, we recall the conormal construction. We denote by $N_{X_A}$ the normal bundle of $X_A$ in $\mathbb{P}(J_3(A))$. We have the conormal diagram:

$$
\begin{array}{c}
\mathbb{P}(N^*_A(1)) \subset \mathbb{P}(J_3(A)) \times \mathbb{P}(J_3(A)^*) \\
X_A \subset \mathbb{P}(J_3(A)) \xrightarrow{p_A} \mathbb{P}(J_3(A)) \xrightarrow{q_A} X^*_A \subset \mathbb{P}(J_3(A)^*)
\end{array}
$$

where $X^*_A \subset \mathbb{P}(J_3(A)^*)$ is the projective dual of $X_A$. It happens that $X^*_A \cong \mathcal{C}_A$ (Rob88) under an identification $\mathbb{P}(J_3(A)) \cong \mathbb{P}(J_3(A)^*)$ and that $E_A = (q_A)_*(p^*_A F_A)$, where $F_A$ is a rank $n_A$ vector bundle on $X_A$. The bundle $F_A$ is the tautological bundle we were looking for. Indeed, for any $A \in X_A$, the stalk $F_A|_A$ is the orthogonal in $(\mathbb{A}^{\oplus 3})^*$ of the kernel of $A$ (which is a 2$n_A$-dimensional complex vector sub-space of $\mathbb{A}^{\oplus 3}$ as all matrices $A \in X_A$ have rank $n_A$). For the convenience of the reader, we describe explicitly $F_A$ for each $A$:

- if $A = \mathbb{R} \otimes \mathbb{C}$, then $F_A = \mathcal{O}_{\mathbb{P}^2}(1)$,
- if $A = \mathbb{C} \otimes \mathbb{C}$, then $F_A = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 0) \oplus \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, 1)$,
- if $A = \mathbb{H} \otimes \mathbb{C}$, then $F_A = (\mathcal{R}^*)^{\oplus 2}$, where $\mathcal{R}$ is the tautological bundle on $\text{Gr}(2, 6)$.

In the case of the octonions, the determinant $\text{det}_\mathbb{O}(A)$ still makes sense for any $A \in J_3(\mathbb{O})$ and the equation $\text{det}_\mathbb{O} = 0$ defines a cubic in $\mathbb{P}(J_3(\mathbb{O}))$ known as the Cartan cubic. Its singular locus is the octonionic projective plane $\mathbb{O} \mathbb{P}^2 \subset \mathbb{P}(J_3(\mathbb{O}))$, a projective homogeneous space under $E_6$ (see Rob88). There are however many key steps where the above construction of $F_A$ fails for $A = \mathbb{O}$. First of all, the matrix factorization:

$$
\text{Com}(A) \times A = A \times \text{Com}(A) = \text{det}_\mathbb{O}(A) J_3,
$$

does not hold for all $A \in J_3(\mathbb{O})$, due to the lack of associativity of $\mathbb{O}$. Furthermore, the embedding $L : \mathbb{O} \longrightarrow \mathbb{M}_{8,8}(\mathbb{C})$ is not an algebra embedding (because $\mathbb{O}$ is not associative). Hence, even if for all $A \in J_3(\mathbb{O})$ we could find $B_A \in J_3(\mathbb{O})$ such that $B_A \times A = A \times B_A = \text{det}_\mathbb{O}(A) J_3$, there wouldn’t necessarily be any corresponding matrix factorization involving $L(A)$.

Finally, the very notion of a locally free $\mathbb{O} \otimes \mathcal{O}_X$-module on a scheme $X$ is a notoriously challenging one. While alternative modules over $\mathbb{O}$ have been studied by many authors (see for
instance \([\text{Jac}54]\), there are serious difficulties to sheafify these modules. Indeed, let \(X\) be a scheme over \(\mathbb{C}\) and \(F\) be a sheaf of locally free (right) \(\mathcal{O} \otimes \mathcal{O}_X\)-modules of rank \(n\) over \(\mathcal{O} \otimes \mathcal{O}_X\). Let \(U, V\) trivializing opens in \(X\). The transition map \(t_{U,V}\) gives a right \(\mathcal{O}\)-linear isomorphism:

\[
t_{U,V} : \mathcal{O} \otimes \mathcal{O}^n \times \{U \cap V\} \longrightarrow \mathcal{O} \otimes \mathcal{O}^n \times \{U \cap V\}
\]

\[
(a \otimes v, x) \longmapsto (t_{U,V}(x)(a \otimes v), x).
\]

The algebra \(\mathcal{O}\) being non-associative, the only right \(\mathcal{O}\)-linear endomorphisms of \(\mathcal{O}\) are given by multiplication by pure (complexified) real numbers in \(\mathcal{O}\). We thus find that the transition map \(t_{U,V}\) is of the form:

\[
t_{U,V} : \mathcal{O} \otimes \mathcal{O}^n \times \{U \cap V\} \longrightarrow \mathcal{O} \otimes \mathcal{O}^n \times \{U \cap V\}
\]

\[
(a \otimes v, x) \longmapsto (a \otimes r_{U,V}(x)(v), x),
\]

for some cocyle \(r \in H^1(X, \mathbb{GL}_n)\). As a consequence \(F = \mathcal{O} \otimes E\) where \(E\) is a vector bundle on \(X\), and the global \(\mathcal{O}\) structure on \(F\) is trivial. This is in sharp contrast with the fact that the vector bundle \(F_\mathcal{A}\) described above is a locally free \(\mathcal{A} \otimes \mathcal{O}_X\)-module of rank \(n_\mathcal{A}\) (over \(\mathcal{O}_X\)) whose global right \(\mathcal{A}\)-structure is not trivial.

The above analysis suggests that the proof we give that the cokernel of the matrix \(M_{\mathcal{A}}\) is a tautological rank \(n_\mathcal{A}\) bundle on \(\mathcal{C}_\mathcal{A}\) endowed with a right global \(\mathcal{A}\)-action cannot be adapted to the case \(\mathcal{A} = \mathcal{O}\). But it doesn’t actually show that the cokernel of \(M_{\mathcal{O}}\) is the not, by some miracle, the bundle which existence is claimed by Bhargava and Ho. As a matter of fact, it happens that this cokernel doesn’t even live on \(\mathcal{C}_\mathcal{O}\) and our first main result describes some of its geometric and representation theoretic properties. Let \(c = x_1e_1 + \ldots + x_8e_8, b = x_9e_1 + \ldots + x_{16}e_n, a = x_{17}e_1 + \ldots + x_{23}e_n, \lambda_1 = x_{25}e_1, \lambda_2 = x_{26}e_1\) and \(\lambda_3 = x_{27}e_1\). Consider the \(24 \times 24\) matrix with entries in \(\mathbb{C}[x_1, \ldots, x_{27}]\) given in \(8 \times 8\) blocks by:

\[
M_{\mathcal{O}} = \begin{pmatrix}
\lambda_1, I_8 & L_c & L_b \\
L_c & \lambda_2, I_8 & L_a \\
L_b & L_a & \lambda_1, I_8
\end{pmatrix}.
\]

Let \(S_{\text{ODM}}\) be the sextic polynomial in \(\mathbb{C}[x_1, \ldots, x_{27}]\) defined by:

\[
S_{\text{ODM}} = \text{Det}_{\mathcal{O}}^2 - 4\phi(c, b, a) \text{Det}_{\mathcal{O}} - \left| [c, b, a] \right|^2,
\]

where \(\phi(c, b, a) = \frac{1}{2} \text{Re}(c\bar{b} - \bar{c}c)a\), \([c, b, a] = (cb)a - c(ba)\) and \(\text{Det}_{\mathcal{O}}\) is the cubic polynomial corresponding to \(\text{det}_{\mathcal{O}}\).

**Theorem 1**

1. The vanishing locus of \(S_{\text{ODM}}\) is an irreducible sextic (equally denoted by \(S_{\text{ODM}}\)) in \(\mathbb{P}(J_3(\mathcal{O}))\) which singular locus has codimension 4 in \(\mathbb{P}(J_3(\mathcal{O}))\). Furthermore, up to a finite quotient, we have:

\[
\text{Aut}^0(S_{\text{ODM}}) = \text{SL}_3(\mathbb{C}) \times \text{SO}_7(\mathbb{C}).
\]

2. The cokernel of:

\[
\mathcal{O}^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}(J_3(\mathcal{O}))}(-1) \overset{M_{\mathcal{O}}}{\longrightarrow} \mathcal{O}^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}(J_3(\mathcal{O}))}
\]

is supported on \(S_{\text{ODM}}\). It has generically rank 4 and is \(\text{SL}_3(\mathbb{C}) \times \text{SO}_7(\mathbb{C})\)-equivariant.
The sextic in Theorem 1 has been first found out in [Ogi81] and was then rediscovered later in [DM98] with computer aid. We call it the Ogievetskiï-Dray-Manogue sextic and denote it by $N_0$. As far as I know, the objects introduced in Theorem 1 have not been yet investigated from the algebraic geometry perspective and they certainly deserve to be studied in details. They seem however not directly related to the geometry of the Cartan cubic and the tautological bundles that might exist on it. Our second main result relates to this more classical object. We denote by $N_0$ the $24 \times 24$ matrix with entries in $\mathbb{C}[x_1, \ldots, x_{27}]$ given in $8 \times 8$ blocks by:

$$
N_0 = \begin{pmatrix}
\lambda_1 I_8 & R_c & ^tL_b \\
^tR_c & \lambda_2 I_8 & L_a \\
L_b & ^tL_a & \lambda_1 I_8
\end{pmatrix},
$$

where $R_c$ is the matrix of right multiplication by $c \in \mathbb{O}$.

**Theorem 2**  
1. Let $E_0$ be the cokernel of

$$
\mathbb{O}^{\otimes 3} \otimes \mathbb{C} H_{\mathcal{P}(J_3(\mathbb{O}))}(-1) \xrightarrow{N_0} \mathbb{O}^{\otimes 3} \otimes \mathbb{C} \mathcal{H}_{\mathcal{P}(J_3(\mathbb{O}))}.
$$

The sheaf $E_0$ is supported on a degree 9 reducible hypersurface in $\mathbb{P}(J_3(\mathbb{O}))$, which is the union of hypersurface projectively isomorphic to $\mathcal{H}_0$ (the Cartan cubic) and a sextic hypersurface, which we denote $\mathcal{G}$, and whose equation is:

$$
\mathcal{G} = (\lambda_1 |a|^2 + \lambda_2 |b|^2 + \lambda_3 |c|^2 - \lambda_1 \lambda_2 \lambda_3)^2 - 4(\lambda_1 |a|^2 + \lambda_2 |b|^2 + \lambda_3 |c|^2 - \lambda_1 \lambda_2 \lambda_3) \text{Re}(c) \text{Re}(ab)
+ 4((|b|^2 |a|^2 \text{Re}(c))^2 + |c|^2 \text{Re}(ab)^2 - |a|^2 |b|^2 |c|^2),
$$

where $|z|^2 = z \overline{z}$ and $\text{Re}(z) = \frac{z + \overline{z}}{2}$.

2. The cubic hypersurface is $(\text{Spin}_7(\mathbb{C}), \text{SL}_3(\mathbb{C}))$-invariant, where $(\text{Spin}_7(\mathbb{C}), \text{SL}_3(\mathbb{C}))$ is the subgroup of $\text{SL}_{24}$ generated by $\text{Spin}_7(\mathbb{C})$ and $\text{SL}_3(\mathbb{C})$. The restriction of $E_0$ to the cubic has generically rank 4 and is $(\text{Spin}_7(\mathbb{C}), \text{SL}_3(\mathbb{C}))$-equivariant.

3. The hypersurface $\mathcal{G}$ is equally $(\text{Spin}_7(\mathbb{C}), \text{SL}_3(\mathbb{C}))$-invariant. The restriction of $E_0$ to $\mathcal{G}$ has generically rank 2 and is $(\text{Spin}_7(\mathbb{C}), \text{SL}_3(\mathbb{C}))$-equivariant.

It should be noted that the explicit equation of $\mathcal{G}$ has been first found by Jonathan Hauenstein using the **Bertini** software. The embedding of Spin$_7$ in SL$_{24}$ we consider does not factor through E$_6$. As a matter of fact, we haven’t been able to identify the subgroup of SL$_{24}$ generated by Spin$_7(\mathbb{C})$ and SL$_3(\mathbb{C})$. We can however show the:

**Theorem 3** Let $G$ be the subgroup of SL$_{24}$ generated by Spin$_7(\mathbb{C})$ and SL$_3(\mathbb{C})$. The action of $\mathbb{C}^* \times G$ on $J_3(\mathbb{O})$ is prehomogeneous.

Theorem 3 is slightly surprising as the subgroup $G$ of SL$_{24}$ generated by Spin$_7(\mathbb{C})$ and SL$_3(\mathbb{C})$ is not E$_6$ (and is even not a subgroup of E$_6$). We will prove this fact in details in section 3.3, but less us have a glimpse at the argument. Any element of $J_3(\mathbb{O})$ can be diagonalized under the action of $E_6$. Hence, any vector of $J_3(\mathbb{O})$ can be transformed, under the action of $E_6$, to a vector whose corresponding $24 \times 24$ matrix has rank 0, 8, 16 or 24. On the other hand, the action of $G$ on $J_3(\mathbb{O})$ preserves the rank. Since there are vectors in $J_3(\mathbb{O})$ whose corresponding $24 \times 24$ matrix have rank 22, we conclude that $G$ is not E$_6$. The prehomogeneous space $(J_3(\mathbb{O}), \mathbb{C}^* \times G)$ does not appear in the list of Kimura-Sato [KS77]. This is not surprising as it is most likely
not irreducible as a prehomogeneous space. It thus provides an interesting example of a non-
irreducible prehomogeneous vector space which does not split as a sum of lower dimensional
prehomogeneous vector spaces.

We prove Theorems 2 and 3 in the second section of this paper. There, we introduce a variant
of the octonionic eigenvalue problem, which we call the twisted octonionic eigenvalue problem.
We deduce the aforementioned results from both algebraic and geometric features of this twisted
problem. We note that the geometry of the twisted octonionic eigenvalue problem differs strongly
from that of the corresponding untwisted problem. Indeed, one easily notices that the action of
$GL_3 \times SO_7$ on $J_3(\mathbb{O})$ is far from being homogeneous (the generic orbit has codimension at least
6 in $J_3(\mathbb{O})$). Theorem 3 thus shows that the twisted problem is eminently more symmetric than
the untwisted problem.

The sheaf $E_\mathbb{O}$ appearing in Theorem 2 may be considered as a quasi-tautological sheaf on
$\mathcal{C}_\mathbb{O}$ as it is the cokernel of a matrix whose tautological interpretation is clear. However, the
description of $E_\mathbb{O}|_A$ for generic $A \in \mathcal{C}_\mathbb{O}$ is not completely obvious. Furthermore, the restriction
of $E_\mathbb{O}$ to the smooth locus of $\mathcal{C}_\mathbb{O}$ is not a vector bundle, so that $E_\mathbb{O}$ is not the Fourier-Mukai
transform, by the conormal diagram, of a vector bundle on $\mathbb{O}P^2$. One can however expect that
it is the Fourier-Mukai transform of a rank 4 coherent sheaf on $\mathbb{O}P^2$ which could be conceived
as quasi-tautological sheaf on $\mathbb{O}P^2$. The tautological interpretation of this sheaf is again not
completely transparent. Further investigations should reveal many more fascinating features
associated to the twisted octonionic eigenvalue problem. In a forthcoming paper with Jonathan
Hauenstein [AH], we will study in more details the geometry and singularities of $SO_{ODM}$ and
$\mathcal{S}$.

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showing that the subgroup of $E_6$ generated by $\text{Spin}_7$ and $\text{SL}_3$ is $E_6$ itself.

2 The eigenvalue problem for $J_3(\mathbb{O})$ and the Ogievetskiî-Dray-
Manogue sextic

2.1 The octonionic eigenvalue problem

Let $\mathbb{A}$ be $\mathbb{C}$, $\mathbb{C} \otimes \mathbb{R} \mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (the algebra $\mathbb{H}$ and $\mathbb{O}$ being the complexified quaternions and octo-
nions). The left (complex) eigenvalue problem for $J_3(\mathbb{A})$ is the following. Given $A = \begin{pmatrix}
\lambda_1 & c & b \\
\bar{c} & \lambda_2 & a \\
b & \bar{a} & \lambda_3
\end{pmatrix} \in J_3(\mathbb{A})$, what are the $\mu \in \mathbb{C}$ such that there exists a non zero vector
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{A}^3$ with $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 
\mu \begin{pmatrix} x \\ y \\ z \end{pmatrix}$? So that there is no confusion due to the non-commutativity (and possibly non asso-
ciativity) of $A$, the equality $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mu \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is meant to be:

$$\begin{cases} 
\lambda_1 x + cy + \overline{c} z = \mu x \\
\overline{c} x + \lambda_2 y + az = \mu y \\
bx + \overline{a} y + \lambda_2 z = \mu z.
\end{cases}$$

Let us say that $A \in J_3(\mathbb{A})$ is singular or degenerate if 0 is an eigenvalue of $A$. In case $\mathbb{A}$ is associative, the following proposition is well-known. We recall a proof for the convenience of the reader.

**Proposition 2.1.1** Let $\mathbb{A}$ be an associative composition algebra which is finite dimensional over $\mathbb{C}$. Let $\begin{pmatrix} \lambda_1 & c & \overline{c} \\ \overline{c} & \lambda_2 & a \\ b & \pi & \lambda_3 \end{pmatrix} \in J_3(\mathbb{A})$ and $\mu \in \mathbb{C}$. Then, there exists a non-zero vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{A}^3$ if and only if $\det A (A - \mu I_3) = 0$.

**Proof:**

$\triangleright$ If $A = 0$ the result is obvious. We now assume that $A \neq 0$, for instance $c \neq 0$ (the non-vanishing of any other coefficient in $A$ is dealt with exactly in the same way). Assume that $\det A (A - \mu I_3) = 0$. If $\Com(A - \mu I_3) \neq 0$ then, the equation:

$$(A - \mu I_3) \times \Com(A - \mu I_3) = \det A (A - \mu I_3) I_3 = 0$$

shows that any non-zero column vector of $\Com(A - \mu I_3)$ is an eigenvector for $A$ with respect to $\mu$. If $\Com(A - \mu I_3) = 0$, then one easily checks that $\begin{pmatrix} \lambda_2 - \mu \\ -\pi \\ 0 \end{pmatrix}$ is a non-zero eigenvector for $A$ with respect to $\mu$.

Assume on the other side that there exists $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq 0$ such that $(A - \mu I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$. Let $B = \begin{pmatrix} x & x & x \\ y & y & y \\ z & z & z \end{pmatrix}$. We have $(A - \mu I_3) \times B = 0$ and since $\mathbb{A}$ is associative, we also have $\Com(A - \mu I_3) \times (A - \mu I_3) \times B = 0$. But $\Com(A - \mu I_3) \times (A - \mu I_3) = \det A (A - \mu I_3) I_3$, so that we get:

$$\det A (A - \mu I_3) I_3 = 0.$$

The matrix $B$ is non-zero by hypothesis and $\det A (A - \mu I_3) \in \mathbb{C}$ imply that $\det A (A - \mu I_3) = 0$.

$\blacktriangleright$ The algebra $\mathbb{O}$ being not associative, the above result does not apply in this case. Nevertheless, Ogievetskiĭ found out a characteristic equation that determines the complex eigenvalues of $A \in J_3(\mathbb{O})$:..
Theorem 2.1.2 (see [Ogi81, DM98])

1. Let \( A = \begin{pmatrix} \lambda_1 & c & \bar{b} \\ \bar{c} & \lambda_2 & a \\ b & \bar{a} & \lambda_3 \end{pmatrix} \in J_3(\mathbb{O}) \). The matrix \( A \)
is singular if and only if the 24 \( \times \) 24 matrix \( \begin{pmatrix} \lambda_1 I_8 & L_c & \bar{t}L_b \\ \bar{t}L_c & \lambda_2 I_8 & L_a \\ L_b & \bar{t}L_a & \lambda_3 I_8 \end{pmatrix} \) is singular.

2. The degeneracy locus of the matrix \( \begin{pmatrix} \lambda_1 I_8 & L_c & \bar{t}L_b \\ \bar{t}L_c & \lambda_2 I_8 & L_a \\ L_b & \bar{t}L_a & \lambda_3 I_8 \end{pmatrix} \) is given by the equation:

\[
(\det(\mathbb{O}(A))^2 - 4\phi(c, b, a)\det(\mathbb{O}(A)) - |c, b, a|^2 = 0.
\]

As we will build upon it when dealing with the twisted version of the octonionic eigenvalue problem, we recall the main steps of the proof from [Ogi81]:

**Proof:**

The first assertion in the Theorem is obvious and follows from the definitions. We prove the second assertion. For \( a, b, c \in \mathbb{O} \) and \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \) such that \( |a|, |b|, |c|, \lambda_1, \lambda_2, \lambda_3 \neq 0 \) and \( |c|^2 \lambda_1 - \frac{|c|^4}{\lambda_2} \neq 0 \), we define the following 24 \( \times \) 24 matrices in block form:

\[
R_1 = \begin{pmatrix} \frac{1}{|c|^2} & L_c & 0_{8,8} & 0_{8,8} \\ 0_{8,8} & I_8 & 0_{8,8} \\ 0_{8,8} & 0_{8,8} & \frac{1}{|a|^2} & tL_a \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} I_8 & \frac{|c|^2}{\lambda_2} & 0_{8,8} \\ 0_{8,8} & I_8 & 0_{8,8} \\ \frac{1}{\delta B} & \frac{|a|^2}{\lambda_2} & I_8 & I_8 \end{pmatrix}\]

where \( B = L_a L_b L_c - \frac{|a|^2 |c|^2}{\lambda_2} I_8, \delta = |c|^2 \lambda_1 - \frac{|c|^4}{\lambda_2} \) and \( 0_{8,8} \) is the zero 8 \( \times \) 8 matrix and \( I_8 \) the identity matrix of size 8. One easily checks that:

\[
\begin{pmatrix} \lambda_1 I_8 & L_c & \bar{t}L_b \\ \bar{t}L_c & \lambda_2 I_8 & L_a \\ L_b & \bar{t}L_a & \lambda_3 I_8 \end{pmatrix} = R_1 R_2 \begin{pmatrix} \delta I_8 & 0_{8,8} & 0_{8,8} \\ 0_{8,8} & \lambda_2 I_8 & 0_{8,8} \\ 0_{8,8} & 0_{8,8} & \nu I_8 - \frac{1}{\delta B^t B} \end{pmatrix} R_2^t R_1,
\]

where \( \nu = \lambda_3 |a|^2 - \frac{|c|^4}{\lambda_2} \). Since \( \det(R_1) = \frac{1}{|a|^8 |c|^8} \) and \( \det(R_2) = 1 \), we get:

\[
\det \begin{pmatrix} \lambda_1 I_8 & L_c & \bar{t}L_b \\ \bar{t}L_c & \lambda_2 I_8 & L_a \\ L_b & \bar{t}L_a & \lambda_3 I_8 \end{pmatrix} = \frac{1}{|a|^8 |c|^8} \det \left( \delta \nu \lambda_2 I_8 - \lambda_2 (B^t B) \right)
\]

\[
= \det \left( (\lambda_1 \lambda_2 \lambda_3 - |a|^2 - |b|^2 - \lambda_3 |c|^2) I_8 + (L_a L_b L_c + \bar{t}L_c \bar{t}L_b \bar{t}L_a) \right).
\]

where \( A = \begin{pmatrix} \lambda_1 & c & \bar{b} \\ \bar{c} & \lambda_2 & a \\ b & \bar{a} & \lambda_3 \end{pmatrix} \).

The expression \( \det \left( (\lambda_2 I_8 - \lambda_2 B^t B) \right) \) has no denominators so that the above equality holds for all \( a, b, c \in \mathbb{O} \) and \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \). We deduce the following:
The determinant of
\[
\begin{pmatrix}
\lambda_1 I_8 & L_c & \mathbf{i} L_b \\
\mathbf{i} L_c & \lambda_2 I_8 & L_a \\
L_b & \mathbf{i} L_a & \lambda_3 I_8
\end{pmatrix}
\] (complex) eigenvalue of \((L_a L_b L_c + \mathbf{i} L_c L_b L_a)\). We shall now determine the (complex) eigenvalues of \((L_a L_b L_c + \mathbf{i} L_c L_b L_a)\). This means we are looking for the \(\kappa \in \mathbb{C}\) such that:
\[
\exists x \in \mathbb{O}, \; x \neq 0 \quad a.(b.(c.x)) + \overline{c}.(\overline{b}.(\overline{a}.x)) = \kappa.x
\] (4)

Let us consider a four dimensional quaternionic subalgebra of \(\mathbb{O}\) which contains \(a\) and \(c\) (its existence is granted by a Theorem of Hurwitz). Let \(e \in \mathbb{O}\) be a vector orthogonal to \(M\) such that \(\mathbb{O} = M \oplus M.e\). We write \(b = b_0 + b_1.e\) and \(x = x_0 + x_1.e\), with \(b_0, b_1, x_0, x_1 \in M\). Using the relations:
\[
u.(v.e) = (v.u).e, \quad (u.e).(v.e) = -\overline{v}.u, \quad u.e = e.\overline{u} \quad \text{and} \quad (u.e).v = (u.\overline{v}).e,
\]
valid for all \((u,v) \in M^2\), we find that equation (4) is equivalent to the system:
\[
\begin{align*}
2(Re(c.a.b_0)).x_0 + (\overline{a}.\overline{c})x_0.b_1 &= \kappa.x_0 \\
(\overline{b}.c-a.\overline{c}).x_0.b_1 + 2Re(c.b_0.a).x_1 &= \kappa.x_1.
\end{align*}
\]
This system can be written in matrix form:
\[
\begin{pmatrix}
(2(Re(c.a.b_0)) - \kappa).I_4 & (\mathbf{i} L_c L_a - L_a \mathbf{i} L_c) R_{b_1} \\
(\mathbf{i} L_a L_c - L_c \mathbf{i} L_a) R_{b_1} & (2(Re(c.b_0.a) - \kappa))I_4
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1
\end{pmatrix}
= \begin{pmatrix}
0_4,1 \\
0_4,1
\end{pmatrix}.
\]

The existence of such an \(x_0 \neq 0\) is then equivalent to:
\[
\det \begin{pmatrix}
(2(Re(c.a.b_0)) - \kappa).I_4 & (\mathbf{i} L_c L_a - L_a \mathbf{i} L_c) R_{b_1} \\
(\mathbf{i} L_a L_c - L_c \mathbf{i} L_a) R_{b_1} & (2(Re(c.b_0.a) - \kappa))I_4
\end{pmatrix} = 0.
\]

The bottom right entry of the above block-matrix commutes with every matrix, so that the block-computation of the determinant gives:
\[
\det \begin{pmatrix}
(2(Re(c.a.b_0)) - \kappa).I_4 & (\mathbf{i} L_c L_a - L_a \mathbf{i} L_c) R_{b_1} \\
(\mathbf{i} L_a L_c - L_c \mathbf{i} L_a) R_{b_1} & (2(Re(c.b_0.a) - \kappa))I_4
\end{pmatrix}
= \det((2(Re(c.a.b_0)) - \kappa)(2(Re(c.b_0.a) - \kappa) - |\overline{c}.a-a.\overline{c}|^2 |b_1|^2))
= \det((2(Re(c.a.b_0)) - |\overline{c}.a-a.\overline{c}|^2 |b_1|^2) I_4).
\]

We deduce that:
\[
kappa\text{ is an eigenvalue of } (L_a L_b L_c + \mathbf{i} L_c L_b L_a) \iff (2(Re(c.a.b_0)) - \kappa)(2(Re(c.b_0.a) - \kappa) - |\overline{c}.a-a.\overline{c}|^2 |b_1|^2) = 0.
\]

We note that \(b = b_0 + b_1.e\) and \(e \perp M\), hence:
\[
\text{Re}(c.a.b_0) = \text{Re}(c.a.b), \quad \text{Re}(c.b_0.a) = \text{Re}(c.b.a)\text{ and }|\overline{c}.a-a.\overline{c}|^2 |b_1|^2 = |[a, b_1.e, c]|^2 = ||c, b, a||^2.
\]

We furthermore recall that \(2(Re(c.a.b)) - \det_{\mathbb{O}}(A)\) is an eigenvalue of \((L_a L_b L_c + \mathbf{i} L_c L_b L_a)\) if and only if the matrix
\[
\begin{pmatrix}
\lambda_1 I_8 & L_c & \mathbf{i} L_b \\
\mathbf{i} L_c & \lambda_2 I_8 & L_a \\
L_b & \mathbf{i} L_a & \lambda_3 I_8
\end{pmatrix}
\] is singular. We finally find that the degeneracy locus
of the matrix
\[
\begin{pmatrix}
\lambda_1 I_8 & L_c & ^tL_b \\
^tL_c & \lambda_2 I_8 & L_a \\
L_b & ^tL_a & \lambda_3 I_8
\end{pmatrix}
\] is defined by the equation:
\[
(2\text{Re}(c.a.b) - \kappa)(2\text{Re}(c.b.a) - \kappa) - |[c, b, a]|^2 = 0
\]
\[
\iff \text{Det}_O[2\text{Re}(c.a.b) - 2\text{Re}(c.b.a) + \text{Det}_O] - |[c, b, a]|^2 = 0
\]
\[
\iff \text{Det}_O[2\text{Re}(c(ab - ba))\text{Det}_O - |[c, b, a]|^2 = 0
\]
\[
\iff \text{Det}_O[4\phi(a, b, c)\text{Det}_O - |[c, b, a]|^2 = 0
\]
\[
\iff \text{Det}_O[2 - 4\phi(a, b, c)\text{Det}_O - |[c, b, a]|^2 = 0
\].

2.2 The geometry of the Ogievetskii-Dray-Manogue sextic

We will now describe elementary geometric features of the sextic $S_{ODM} \subset \mathbb{P}^{26}$ defined by the equation:
\[
(\text{det}_O(A))^2 - 4\phi(c, b, a)\text{det}_O(A) - |[c, b, a]|^2 = 0.
\]
Some of them have been obtained with the help of Macaulay2 (GS):

**Theorem 2.2.1**

1. The sextic $S_{ODM} \subset \mathbb{P}^{26}$ is irreducible and its singular locus is of codimension 4 in $\mathbb{P}^{26}$.

2. The neutral component the automorphism group of $S_{ODM}$ is $SO_7 \times SL_3$.

3. The map:
\[
\text{Gr}(6, J_3(\mathbb{O}))/SO_7 \times SL_3 \longrightarrow S^6 \mathbb{C}^6/GL_6
\]
which sends a six dimensional subspace $L \subset J_3(\mathbb{O})$ to the sextic $S_{ODM} \cap L$ is generically étale onto its image.

**Proof**

The codimension of the singular locus of $S_{ODM} \subset \mathbb{P}^{26}$ is obtained computationally with the help of Macaulay2. The only «challenge» is to find a way to write explicitly the equation $S_{ODM}$ as, to the best of my knowledge, there is no obvious package to simulate the octonion algebra in Macaulay2. We refer to the first appendix for a description of our algorithm. The irreducibility of $S_{ODM}$ follows as it is then a normal hypersurface in $\mathbb{P}^{26}$.

We now prove the second and third assertion. We first describe the action of $SO_7 \times SL_3$ on $J_3(\mathbb{O})$.

Let $T_1 \in SO_7$ and $A = \begin{pmatrix} \lambda_1 & c & \bar{b} \\ \bar{c} & \lambda_2 & a \\ b & \bar{a} & \lambda_3 \end{pmatrix} \in J_3(\mathbb{O})$, we put:
\[
T_1.A = \begin{pmatrix} \lambda_1 & T_1(c) & (T_1(b)) \\ (T_1(c)) & \lambda_2 & T_1(a) \\ T_1(b) & (T_1(a)) & \lambda_3 \end{pmatrix} \in J_3(\mathbb{O}),
\]
where $SO_7 \subset SO_8$ is the isotropy group of $1 \in \mathbb{O}$. On the other hand, for $C \in SL_3$ and $A = \begin{pmatrix} \lambda_1 & c & \bar{b} \\ \bar{c} & \lambda_2 & a \\ b & \bar{a} & \lambda_3 \end{pmatrix} \in J_3(\mathbb{O})$, we put:
\[
C.A = ^tC \times A \times C \in J_3(\mathbb{O}),
\]
where $\times$ denotes here the product of $3 \times 3$ matrices (there is no associativity issues since all coefficients of $C$ are complex and thus commute with all coefficients of $A$). Since $T_1(x) = x$, for all $T_1 \in \text{SO}_7$ and for all $x \in \mathcal{O}$ with zero imaginary part, we deduce that:

$$C.(T_1.A) = T_1.(C.A),$$

for all $T_1 \in \text{SO}_7$, $C \in \text{SL}_3$ and $A \in J_3(\mathcal{O})$. Furthermore, we notice that $\text{SO}_7 \cap \text{SL}_3 = \{Id\}$ as subgroups of $\text{SL}_2$. As a consequence, we find that the subgroup of $\text{SL}_2$ generated by $\text{SO}_7$ and $\text{SL}_3$ is $\text{SO}_7 \times \text{SL}_3$. Let $A \in J_3(\mathcal{O})$ be a singular matrix. By definition, there exists $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{O}^3$, with $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq 0_{\mathbb{O}^3}$ such that:

$$A \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0_{\mathbb{O}^3}.$$

For any $T_1 \in \text{SO}_7$, the Triality principle (see section 2 of [DeS01] for instance) insures the existence of $T_2 \in \text{SO}_8$ such that:

$$\forall (u,v) \in \mathbb{O}^2, \; T_2(u,v) = T_1(u).T_2(v).$$

One then easily computes that:

$$(T_1.A) \times \begin{pmatrix} T_2(x) \\ T_2(y) \\ T_2(z) \end{pmatrix} = \begin{pmatrix} T_2 \\ T_2 \\ T_2 \end{pmatrix} \times \begin{pmatrix} A \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = 0_{\mathbb{O}^3}.$$

We have $\begin{pmatrix} T_2(x) \\ T_2(y) \\ T_2(z) \end{pmatrix} \neq 0_{\mathbb{O}^3}$, hence $T_1.A$ is also a singular matrix. Furthermore, we obviously note that for all $C \in \text{SL}_3$, we have:

$$(C.A) \times \begin{pmatrix} C^{-1} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = 0_{\mathbb{O}^3},$$

which shows that $C.A$ is also a singular element of $J_3(\mathcal{O})$. The variety of singular matrices in $J_3(\mathcal{O})$ is thus preserved by the action of $\text{SO}_7 \times \text{SL}_3$. By Theorem 2.1.2 we deduce that $\text{SO}_7 \times \text{SL}_3$ is a subgroup of the neutral component of $\text{Aut}(\text{SODM})$.

In order to prove that $\text{Aut}^0(\text{SODM}) = \text{SO}_7 \times \text{SL}_3$ (up to a finite quotient), we show, using Macaulay2, that the Lie algebra of $\text{Aut}^0(\text{SODM})$ has dimension less or equal to 29. Consider the map:

$$\varphi : M_{6 \times 27} / \text{Aut}^0(\text{SODM}) \longrightarrow S^6 \mathbb{C}^6$$

$$[(m)_{i,j}] \longmapsto \text{SODM}(m_{1,1}.z_1 + \cdots + m_{1,6}.z_6, \cdots, m_{27,1}.z_1 + \cdots + m_{27,6}.z_6),$$

where $z_1, \cdots, z_6$ is a basis for $\mathbb{C}^6$. Let us denote by $\mathfrak{a}(\text{SODM})$ the Lie algebra of $\text{Aut}^0(\text{SODM})$. The differential of $\varphi$ at a point $[(m)_{i,j}] \in M_{6 \times 27}$ is given by:
The sheaf $E_{\dim 4}$ which implies that $29 + 4$ on which has dimension $9$.

The main result of this section is related to the cokernel of $\phi$.

Remark 2.2.2 The generic isotropy group for the action of $SO_7 \times SL_3 \subset Aut^0(S_{ODM})$ and $dim SO_7 \times SL_3 = 29$, we find that:

$$\Aut^0(S_{ODM}) = SO_7 \times SL_3.$$

We can now infer that the map:

$$Gr(6, J_3(\mathbb{O}))/SO_7 \times SL_3 \longrightarrow S^6C^6/GL_6$$

is generically étale onto its image. Indeed, the rank of $d\phi$ being generically 133 and $dim M_{6 \times J_3(\mathbb{O})}/SO_7 \times SL_3 = 162 - 29 = 133$, we find that $\phi$ is generically étale onto its image. We finally notice that $\phi$ is $GL_6$-equivariant, so that the map induces by $\varphi$ on the quotients by $GL_6$:

$$(M_{6 \times J_3(\mathbb{O})}/SO_7 \times SL_3)/GL_6 = Gr(6, J_3(\mathbb{O}))/SO_7 \times SL_3 \longrightarrow S^6C^6/GL_6$$

is equally generically étale onto its image.

2.3 A tautological rank 4 sheaf on the Ogievetskiï-Dray-Manogue sextic

The main result of this section is related to the cokernel of

$$
\begin{pmatrix}
\lambda_1 I_8 & L_c & ^t L_b \\
^t L_c & \lambda_2 I_8 & L_a \\
L_b & ^t L_a & \lambda_3 I_8
\end{pmatrix}
$$

In the following, we work over the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_{27}]$. We let $c = x_1 e_1 + \ldots + x_8 e_8$, $b = x_9 e_1 + \ldots + x_{16} e_n$, $a = x_{17} e_1 + \ldots + x_{24} e_n$, $\lambda_1 = x_{25} e_1$, $\lambda_2 = x_{26} e_1$ and $\lambda_3 = x_{27} e_1$, where $e_1, \ldots, e_8$ is the canonical basis of $\mathbb{C}$.

Theorem 2.3.1 Let $M_\mathbb{O} = \begin{pmatrix}
\lambda_1 I_8 & L_c & ^t L_b \\
^t L_c & \lambda_2 I_8 & L_a \\
L_b & ^t L_a & \lambda_3 I_8
\end{pmatrix}$ in $\mathcal{M}_{24,24}(R)$ and let $E_{ODM}$ be the cokernel of:

$$
\mathbb{C}^{24} \otimes \mathcal{O}_{\mathbb{P}^{26}}(-1) \xrightarrow{M_{\mathbb{O}}} \mathbb{C}^{24} \otimes \mathcal{O}_{\mathbb{P}^{26}}.
$$

The sheaf $E_{ODM}$ is a $SO_7 \times SL_3$-equivariant sheaf supported on $S_{ODM}$ and has generically rank 4 on $S_{ODM}$. 13
Proof: Theorem 2.1.2 insures that the degeneracy of $M_O$ is $S_{OM}$, so that $E_{OM}$ is supported on $S_{OM}$. Note that the determinant of $M_O$ is of degree $24 = 4 \times \deg(S_{OM})$. In particular, we have $\text{rank}(E_{OM}) = 4$. Let us now prove that $M_O$ is $SO_7 \times SL_3$-equivariant. The $SL_3$-equivariance is obvious given the definition of the $SL_3$ action. We are left to prove the $SO_7$-equivariance. Let $T_1 \in SO_7$, the triality principle insures that there exists $T_2 \in SO_8$ such that:

$$\forall (u, v) \in \mathbb{O}^2, \ T_2(uv) = T_1(u)T_2(v).$$

It easily follows that for any $z \in \mathbb{O}$, we have:

$$L_{T_1(z)} = T_2L_zT_2^{-1}.$$ 

As a consequence, for any $A \in J_3(\mathbb{O})$ and any $T_1 \in SO_7$ as above, we have:

$$T_1M_O = \begin{pmatrix} T_2 & T_2^{-1} \\ T_2 & T_2^{-1} \end{pmatrix} \begin{pmatrix} T_2 \\ T_2^{-1} \end{pmatrix}.$$ 

This proves that $M_O$ is also $SO_7$-equivariant. We can conclude that its cokernel, $E_{OM}$ is $SL_3 \times SO_7$-equivariant. ▮

Remark 2.3.2 The complete description of the higher degeneracy loci of $M_O$ seems to be an interesting question. It seems however to be also a difficult one, as the action of $SO_7 \times SL_3$ on $\mathbb{P}^{26}$ has infinitely many orbits (see remark 2.2.2).

3 The geometry of the twisted eigenvalue problem for $J_3(\mathbb{O})$

The motivation for introducing the twisted eigenvalue problem is the lack of «prehomogeneity» of the space $(\mathbb{P}(J_3(\mathbb{O})), SO_7 \times SL_3 \times \mathbb{C}^*, S_{OM})$. This lack of prehomogeneity is partly explained by the relatively big codimension of the subgroup of $E_6$ generated by $SO_7$ and $SL_3$. On the other hand, it is well-known (see chapter 14 of [Har90]) that $E_6$ is generated by $Spin_8$ and $SL_3$. Thus, one could hope for an eigenvalue problem having a larger group of symmetries. As it happens, the symmetry group of the twisted octonionic eigenvalue problem contains the subgroup of $SL_{27}$ generated by $Spin_7$ and $SL_3$ \footnote{The action of $Spin_7$ on $J_3(\mathbb{O})$ that we exhibit is not the standard one and does not factor through an embedding of $Spin_7$ in $E_6$. As a matter of fact, we haven’t been able to identify the subgroup of $SL_{27}$ generated by $Spin_7$ and $SL_3$.} and we prove that $J_3(\mathbb{O})$ is prehomogeneous for this action. We also show that this twisted problem is singular on the union of the Cartan cubic and a sextic hypersurface. To the best of our knowledge, this geometry has never appeared before in the literature.

3.1 The twisted octonionic eigenvalue problem

Let $A = \begin{pmatrix} \lambda_1 & c & \overline{a} \\ \overline{c} & \lambda_2 & a \\ b & \overline{a} & \lambda_3 \end{pmatrix} \in J_3(\mathbb{O})$. We say that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{O}^3$ is in the «twisted kernel» of $A$ if:

$$\begin{cases} \lambda_1 x + yc + \overline{b}z = 0 \\ \overline{x}c + \lambda_2 y + az = 0 \\ bx + \overline{a}y + \lambda_3 z = 0 \end{cases}$$
We say that $A \in J_3(\mathbb{O})$ is twisted-singular if the twisted kernel of $A$ is not zero. We notice that $(\lambda_1 \ c \ b) \ t$ is twisted-singular if and only if the $24 \times 24$ matrix $N_A = \begin{pmatrix} \lambda_1 I_8 & R_c & tL_b \\ tR_c & \lambda_2 I_8 & L_a \\ L_b & tL_a & \lambda_3 I_8 \end{pmatrix}$ is singular, where $R_c$ is the $8 \times 8$ matrix representing the right multiplication by $c$ in $\mathbb{O}$. Our first result related to the twisted eigenvalue problem is the:

**Theorem 3.1.1** The degeneracy locus of the matrix $N_A$ is given by the equation:

$$\left(\det(\mathcal{O}) - 2\Re(cab) + 2\Re(\overline{c}ba)\right)^4 \times \mathcal{O}^2 = 0,$$

where:

$$\mathcal{O} = (\lambda_1 |a|^2 + \lambda_2 |b|^2 + \lambda_3 |c|^2 - \lambda_1 \lambda_2 \lambda_3)^2 - 4 (\lambda_1 |a|^2 + \lambda_2 |b|^2 + \lambda_3 |c|^2 - \lambda_1 \lambda_2 \lambda_3) \Re(c)\Re(ab) + 4 (|b|^2 |a|^2 \Re(c) - |c|^2 \Re(ab)^2 - |a|^2 |b|^2 |c|^2),$$

with $|z|^2 = z \overline{z}$ and $\Re(z) = \frac{z + \overline{z}}{2}$.

Note that $\det(\mathcal{O}) - 2\Re(cab) + 2\Re(\overline{c}ba) = \det(\mathcal{O}(A))$, where $A' = \begin{pmatrix} \lambda_1 & c & b \\ \bar{c} & \lambda_2 & \bar{b} \\ a & \bar{a} & \lambda_3 \end{pmatrix}$. We refrain from adopting such a notation for the cubic term appearing in the equation of the degeneracy locus of $N_A$ as it may be a source of confusions. We prove Theorem 3.1.1 using a strategy similar to that used for Theorem 2.1.2.

**Proof:**

- For $a, b, c \in \mathbb{O}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that $|a|, |b|, |c|, \lambda_1, \lambda_2, \lambda_3 \neq 0$ and $|c|^2 \lambda_1 - \frac{|c|^4}{\lambda_2} \neq 0$, we define the following $24 \times 24$ matrices in block form:

$$S_1 = \begin{pmatrix} \frac{1}{|c|^2} R_c & 0_{8,8} & 0_{8,8} \\ 0_{8,8} & I_8 & 0_{8,8} \\ 0_{8,8} & 0_{8,8} & 1_{8,8} tL_a \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} I_8 & \frac{|c|^2}{\lambda_2} & 0_{8,8} \\ 0_{8,8} & I_8 & 0_{8,8} \\ \frac{1}{\delta} B & \frac{|a|^2}{\lambda_2} I_8 & I_8 \end{pmatrix},$$

where $B = L_a L_b R_c - \frac{|a|^2 |c|^2}{\lambda_2} I_8$, $\delta = |c|^2 \lambda_1 - \frac{|c|^4}{\lambda_2}$, $0_{8,8}$ is the zero $8 \times 8$ matrix and $I_8$ the identity matrix of size 8. One easily checks that:

$$\begin{pmatrix} \lambda_1 I_8 & R_c & tL_b \\ tR_c & \lambda_2 I_8 & L_a \\ L_b & tL_a & \lambda_3 I_8 \end{pmatrix} = S_1 S_2 \begin{pmatrix} \delta I_8 & 0_{8,8} & 0_{8,8} \\ 0_{8,8} & \lambda_2 I_8 & 0_{8,8} \\ 0_{8,8} & 0_{8,8} & \nu I_8 - \frac{1}{\delta} B^t B \end{pmatrix},$$

where $\nu = \lambda_3 |a|^2 - \frac{|a|^4}{\lambda_2}$. Since $\det(S_1) = \frac{1}{|a|^8 |c|^8}$ and $\det(S_2) = 1$, we get:

$$\det \begin{pmatrix} \lambda_1 I_8 & R_c & tL_b \\ tR_c & \lambda_2 I_8 & L_a \\ L_b & tL_a & \lambda_3 I_8 \end{pmatrix} = \frac{1}{|a|^8 |c|^8} \det \left( \delta \nu \lambda_2 I_8 - \lambda_2 B^t B \right) = \det \left( (\lambda_1 \lambda_2 \lambda_3 - \lambda_1 |a|^2 - \lambda_2 |b|^2 - \lambda_3 |c|^2) I_8 + (L_a L_b R_c + tR_c L_b tL_a) \right) = \det(\det(\mathcal{O}) - 2\Re(cab)) I_8 + (L_a L_b R_c + tR_c L_b tL_a).$$

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The expression \( \det(\det_{\mathbb{O}}(A) - 2\text{Re}(cab)).I_8 + (L_a L_b R_c + t^1 R_c t^L L_a) \) has no denominators so that the above equality holds for all \( a, b, c \in \mathbb{O} \) and \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \). We deduce the following:

\[
\begin{pmatrix}
\lambda_1 I_8 & R_c \\
{^t} R_c & \lambda_2 I_8 \\
L_b & \lambda_3 I_8
\end{pmatrix}
\]

The determinant of \( \begin{pmatrix} {^t} R_c & \lambda_2 I_8 \\ L_b & \lambda_3 I_8 \end{pmatrix} \) vanishes if and only if \( 2\text{Re}(cab) - \det_{\mathbb{O}}(A) \) is a (complex) eigenvalue of \( (L_a L_b R_c + t^1 R_c t^L L_a) \).

We shall now determine the (complex) eigenvalue of \( (L_a L_b R_c + t^1 R_c t^L L_a) \). This means we are looking for the \( \kappa \in \mathbb{C} \) such that:

\[
\exists x \in \mathbb{O}, \ x \neq 0 \ a.(b.(x.c)) + (\overline{\mathcal{F}}(\mathcal{F}.x)).x = \kappa.x
\]  \hspace{1cm} (5)

Let us consider a four dimensional quaternionic subalgebra of \( \mathbb{O} \) which contains \( a \) and \( c \). Let \( e \in \mathbb{O} \) be a vector orthogonal to \( M \) such that \( \mathbb{O} = M \oplus M.e \). We write \( b = b_0 + b_1.e \) and \( x = x_0 + x_1.e \), with \( b_0, b_1, x_0, x_1 \in M \). Using the relations:

\[
u.(v.e) = (v.u).e, \ (u.e).(v.e) = -\overline{v}.u, \ u.e = e.\overline{v} \text{ and } (u.e).v = (u.\overline{v}).e,
\]

valid for all \( (u, v) \in M^2 \), we find that equation (5) is equivalent to the system:

\[
\begin{align*}
\{ & a.b_0.x_0.c + \overline{b_0}.x_0.\overline{c} + a.x_1.b_1.\overline{c} - a.c.x_1.b_1 = \kappa.x_0 \\
& x_1.\overline{b}_0.a + b_1.\overline{x}_0.a + x_1.\overline{b}_0.c - b_1.\overline{x}_0.a.c = \kappa.x_1 
\}
\]

which, by conjugating the second line, is equivalent to:

\[
\begin{align*}
\{ & a.b_0.x_0.c + \overline{b_0}.x_0.\overline{c} + a.x_1.b_1.\overline{c} - a.c.x_1.b_1 = \kappa.x_0 \\
& \overline{a}.b_0.a.x_1 + \overline{c}.x_0.a.b_1 - \overline{c}.\overline{a}.x_0.b_1 = \kappa.x_1 
\}
\]

which, in turn, is equivalent to:

\[
\begin{align*}
\{ & a.b_0.x_0.c + \overline{b_0}.x_0.\overline{c} + a.x_1.b_1.\overline{c} - a.c.x_1.b_1 = \kappa.x_0 \\
& 2\text{Re}(\overline{v} ba)\overline{x}_1 + \overline{a}.x_0.a.b_1 - \overline{c}.\overline{a}.x_0.b_1 = \kappa.x_1 
\}
\]  \hspace{1cm} (6)

as \( \text{Re}(\overline{v} ba) = \text{Re}(\overline{v} ba) \). It happens that the matrix form of this system can not be exploited as easily as in the proof of Theorem 2.1.2. We will nevertheless prove that for all \( a, b, c \in \mathbb{O} \), the complex \( 2\text{Re}(\overline{v} ba) \) is an eigenvalue of multiplicity at least 4 of \( (L_a L_b R_c + t^1 R_c t^L L_a) \). Let \( V_1 \) be the space of vectors \( \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \) such that:

\[
x_0 = 0 \text{ et } \overline{x}_1.b_1 = c.\overline{x}_1.b_1
\]

One directly checks with the help of system (6) that for any such vectors:

\[
(L_a L_b R_c + t^1 R_c t^L L_a) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = 2\text{Re}(\overline{v} ba) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.
\]

The quaternionic structure of \( M \) insures that \( \dim V_1 \geq 2 \). Let \( V_2 \) be the space of vectors \( \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \) such that:

\[
\overline{a}.x_0.c = c.\overline{a}.x_0 \text{ and } a.b_0.x_0.c = a.c.\overline{x}_1.b_1
\]

One directly checks with the help of system (6) that for any such vectors:

\[
(L_a L_b R_c + t^1 R_c t^L L_a) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = 2\text{Re}(\overline{v} ba) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.
\]
The quaternionic structure of $M$ insures that $\dim V_2 \geq 2$. We note that $V_1 \cap V_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which implies $\dim V_1 \oplus V_2 \geq 4$. As a consequence, $2\text{Re}(\overline{c}ba)$ is an eigenvalue of multiplicity at least 4 of $(L_a L_b R_c + ^tR_c ^tL_b ^tL_a)$. In particular, we may write:

$$\det \left( \kappa I_8 - L_a L_b R_c + ^tR_c ^tL_b ^tL_a \right) = (\kappa - 2\text{Re}(\overline{c}ba))^4 T(a, b, c, \kappa),$$

where $T(a, b, c, \kappa)$ is weighted-homogeneous of degree 12 with weights $(1, 1, 1, 3)$.

We can now conclude the proof of the Theorem as we remember that the determinant of

$$\begin{pmatrix} \lambda_1 I_8 & R_c & ^tL_b \\ ^tR_c & \lambda_2 I_8 & L_a \\ L_b & ^tL_a & \lambda_3 I_8 \end{pmatrix}$$

vanishes if and only if $2\text{Re}(cab) - \det O(A)$ is a (complex) eigenvalue of $(L_a L_b R_c + ^tR_c ^tL_b ^tL_a)$. Hence the degeneracy locus of $N_A$ is scheme-theoretically given by the equation:

$$\left( \det O(A) - 2\text{Re}(cab) + 2\text{Re}(\overline{c}ba) \right)^4 \times \Xi = 0,$$

where $\Xi$ is the degree 12 homogeneous polynomial in $a, b, c$ defined by $\Xi = T(a, b, c, 2\text{Re}(cab) - \det O(A))$. One easily proves that $\Xi = \mathcal{G}^2$ is a square and the explicit equation:

$$\mathcal{G} = (\lambda_1 |a|^2 + \lambda_2 |b|^2 + \lambda_3 |c|^2 - \lambda_1 \lambda_2 \lambda_3)^2 - 4 (\lambda_1 |a|^2 + \lambda_2 |b|^2 + \lambda_3 |c|^2 - \lambda_1 \lambda_2 \lambda_3) \text{Re}(c)\text{Re}(ab) + 4 (|b|^2 |a|^2 \text{Re}(c)^2 + |c|^2 \text{Re}(ab)^2 - |a|^2 |b|^2 |c|^2),$$

has been first found by Jonathan Hauenstein using the software Bertini.

\section*{3.2 A quasi-tautological rank 4 sheaf on the Cartan cubic}

We shall first explain how Spin$_7$ and SL$_3$ act on $N_A = \begin{pmatrix} \lambda_1 I_8 & R_c & ^tL_b \\ ^tR_c & \lambda_2 I_8 & L_a \\ L_b & ^tL_a & \lambda_3 I_8 \end{pmatrix}$. The action of SL$_3$ is again by transpose-conjugation and is the same as in section 2.2. To describe the action of Spin$_7$, we shall need a few more facts on triality (we refer to [DeS01] for more details on the triality principle). We recall that there is an embedding (the triality embedding) of Spin$_8$ in SO$_8 \times$ SO$_8 \times$ SO$_8$ defined as follows:

$$\text{Spin}_8 = \{(T_1, T_2, T_3) \in (\text{SO}_8)^3, T_1(x)T_2(y) = T_3(xy), \text{ for all } (x, y) \in \mathbb{O}^2\}.$$ 

There are three copies of Spin$_7$ inside Spin$_8$ which have remarkable group-theoretic properties with respect to the triality embedding. We are interested in one of them:

$$\text{Spin}_7 = \{(T_1, T_2, T_1) \in (\text{SO}_8)^3, T_1(x)T_2(y) = T_1(xy), \text{ for all } (x, y) \in \mathbb{O}^2\}.$$ 

The projection map:

$$\text{Spin}_7 \longrightarrow \text{SO}_8$$

$$(T_1, T_2, T_1) \longrightarrow T_2$$

is a double cover of SO$_7$ embedded in SO$_8$ as the stabilizer of $1 \in \mathbb{O}$. For any $T \in$ SO$_8$, we denote by $K_T$ the element of SO$_8$ defined by $K_T(x) = \overline{T(x)}$. It is easily checked that if $(T_1, T_2, T_1)$ is
a triality, then $(K_{T_1}, T_1, T_2)$ and $(T_2, K_{T_1}, K_{T_1})$ are equally trialities (but, in general, they lie in different copies $\text{Spin}_7 \subset \text{Spin}_8$).

We now come back to the twisted eigenvalue problem. Let $A = \begin{pmatrix} \lambda_1 & c & b \\ \tau & \lambda_2 & a \\ b & \overline{\tau} & \lambda_3 \end{pmatrix} \in J_3(\mathcal{O})$ and let $(x, y, z) \in \mathcal{O}^3$ is in the twisted kernel of $A$, namely:

$$\begin{cases} 
\lambda_1 x + yc + b z = 0 \\
x \tau + \lambda_2 y + az = 0 \\
b x + \overline{\tau} y + \lambda_3 z = 0
\end{cases}$$

A quick computation shows that for any $(T_1, T_2, T_1) \in \text{Spin}_7$, the vector $(T_1(x), T_1(y), T_1(z)) \in \mathcal{O}^3$ is in the twisted kernel of

$$\begin{pmatrix} 
\lambda_1 & T_2(c) & K_{T_1}(b) \\
T_2(c) & \lambda_2 & T_1(a) \\
K_{T_1}(b) & T_1(a) & \lambda_3
\end{pmatrix},$$

that is:

$$\begin{cases} 
\lambda_1 T_1(x) + T_1(y)T_2(c) + T_1(b)T_2(z) = 0 \\
T_1(x)T_2(c) + \lambda_2 T_1(y) + T_1(a)T_2(z) = 0 \\
K_{T_1}(b)T_1(x) + \overline{T_1(a)}y + \lambda_3 T_2(z) = 0
\end{cases}$$

For any $(T_1, T_2, T_1) \in \text{Spin}_7$ and any $A = \begin{pmatrix} \lambda_1 & c & b \\ \tau & \lambda_2 & a \\ b & \overline{\tau} & \lambda_3 \end{pmatrix} \in J_3(\mathcal{O})$, we then define:

$$(T_1, T_2, T_1).A = \begin{pmatrix} \lambda_1 & T_2(c) & \overline{K_{T_1}(b)} \\
T_2(c) & \lambda_2 & T_1(a) \\
K_{T_1}(b) & T_1(a) & \lambda_3
\end{pmatrix}.$$ 

This defines an action of $\text{Spin}_7$ on $J_3(\mathcal{O})$ which preserves the twisted eigenvalue problem. The main result of this section is related to the cokernel of

$$\begin{pmatrix} \lambda_1 I_8 & R_c & t^1 L_b \\
R_c & \lambda_2 I_8 & L_a \\
t^1 L_b & L_a & \lambda_3 I_8
\end{pmatrix}.$$ 

As in section 2.3, we work over the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_{27}]$. We let $c = x_1e_1 + \ldots + x_8e_8$, $b = x_9e_1 + \ldots + x_8e_{16}$, $a = x_{17}e_1 + \ldots + x_{24}e_8$, $\lambda_1 = x_{25}e_1$, $\lambda_2 = x_{26}e_1$ and $\lambda_3 = x_{27}e_1$, where $e_1, \ldots, e_8$ is the canonical basis of $\mathcal{O}$.

**Theorem 3.2.1** Let $N_0 = \begin{pmatrix} \lambda_1 I_8 & R_c & t^1 L_b \\
R_c & \lambda_2 I_8 & L_a \\
t^1 L_b & L_a & \lambda_3 I_8
\end{pmatrix}$ in $M_{24,24}(R)$ and let $E_0$ be the cokernel of:

$$\mathbb{C}^{24} \otimes \mathcal{O}_{\mathbb{P}^{26}}(-1) \xrightarrow{N_0} \mathbb{C}^{24} \otimes \mathcal{O}_{\mathbb{P}^{26}}.$$

1. The sheaf $E_0$ is $(\text{Spin}_7, \text{SL}_3)$-equivariant (where $(\text{Spin}_7, \text{SL}_3)$ is the subgroup of $\text{SL}_{24}$ generated by $\text{SL}_3$ and $\text{Spin}_7$ as above).
2. It is supported on the union of a cubic hypersurface isomorphic to \( \mathcal{C}_\mathfrak{O} \) and the degree 6 hypersurface given by the equation \( \mathcal{S} = 0 \). Both hypersurfaces are \( (\text{Spin}_7, \text{SL}_3) \)-invariant.

3. The restriction of \( E_\mathfrak{O} \) to the cubic hypersurface has generically rank 4 while its restriction to the sextic \( \mathcal{S} = 0 \) has generically rank 2.

In the following, we shall also denote by \( \mathcal{S} \) the hypersurface defined by \( \mathcal{S} = 0 \).

**Proof:**

Theorem 3.1.1 insures that the degeneracy locus of \( N_\mathfrak{O} \) is scheme-theoretically given by

\[
(D_\mathfrak{O} - \text{Re}(cab) + \text{Re}(\overline{cba}))^4 \times \mathcal{S}^2 = 0.
\]

We already noted that \( (\det(\mathfrak{A}) - \text{Re}(cab) + \text{Re}(\overline{cab})) = \det_\mathfrak{O}A' \), where \( A' = \begin{pmatrix} \lambda_1 & \sigma & \pi \\ c & \lambda_2 & b \\ a & \overline{b} & \lambda_3 \end{pmatrix} \). As a consequence, the equation defines a cubic isomorphic to the Cartan cubic, which we denote by \( \mathcal{C}_\mathfrak{C} \). The sheaf \( E_\mathfrak{O} \) is then supported on the union of \( \mathcal{C}_\mathfrak{C} \) and the degree 6 hypersurface \( \mathcal{S} \). The rank of the restriction of \( E_\mathfrak{O} \) to \( \mathcal{C}_\mathfrak{C} \) is exactly 4 as \( (D_\mathfrak{O} - \text{Re}(cab) + \text{Re}(\overline{cab})) \) does not divide \( \mathcal{S} \). The same argument shows that the rank of \( E_\mathfrak{O} \) restricted to \( \mathcal{S} = 2 \).

The \( \text{SL}_3 \)-equivariance of \( N_\mathfrak{O} \) follows from the definition of the \( \text{SL}_3 \) action (by transpose-conjugation). Let us prove that \( N_\mathfrak{O} \) is also \( \text{Spin}_7 \)-equivariant. For any \( (T_1, T_2, T_1) \in \text{Spin}_7 \), we know that \( (K_{T_1}, T_1, T_2) \) is also in \( \text{Spin}_8 \). Hence, for any \( (x, y, z, w) \in \mathcal{O}^4 \) and any \( (T_1, T_2, T_1) \in \text{Spin}_7 \), we have:

\[
R_{T_2(x)} = T_1^1 R_{T_1} T_1^{-1}, \quad L_{K_{T_1}(y)} = T_1^1 L_y T_1^{-1}, \quad L_{T_1(z)} = T_1 L_z T_1^{-1} \quad \text{and} \quad L_{T_1(w)} = T_2^1 L_w T_1^{-1}.
\]

As a consequence, for any \( (T_1, T_2, T_1) \in \text{Spin}_7 \), we find:

\[
(T_1, T_2, T_1).N_\mathfrak{O} = \begin{pmatrix} T_1 & T_1 & T_1^{-1} \\ T_1 & T_1 & T_1^{-1} \\ T_2 & T_1^{-1} & T_2^{-1} \end{pmatrix} N_\mathfrak{O} \begin{pmatrix} T_1^{-1} \\ T_1^{-1} \\ T_2^{-1} \end{pmatrix},
\]

which proves the \( \text{Spin}_7 \)-equivariance of \( N_\mathfrak{O} \). In fine, we can conclude that \( E_\mathfrak{O} \), the cokernel of \( N_\mathfrak{O} \), is \( (\text{Spin}_7, \text{SL}_3) \)-equivariant, where \( (\text{Spin}_7, \text{SL}_3) \) is the subgroup of \( \text{SL}_{24} \) generated by \( \text{SL}_3 \) and \( \text{Spin}_7 \). Since \( E_\mathfrak{O} \) is \( (\text{Spin}_7, \text{SL}_3) \)-equivariant, its support must be \( (\text{Spin}_7, \text{SL}_3) \)-invariant. This proves that the sextic \( \mathcal{S} \) is \( (\text{Spin}_7, \text{SL}_3) \)-invariant.

**Remark 3.2.2** On easily checks that the jumping locus of the restriction of \( E_\mathfrak{O} \) to the cubic \( \mathcal{C}_\mathfrak{O} \) is not the singular locus of \( \mathcal{C}_\mathfrak{O} \) (that is a copy of \( \mathcal{O} \mathbb{P}^2 \)) : it has strictly smaller codimension. It would be interesting to know if we can give an interpretation of \( E_\mathfrak{O}|_A \) for any \( A \) in the smooth locus of \( \mathcal{C}_\mathfrak{O} \) using only the octonionic geometry in \( \mathcal{O} \) (as we can do for the tautological sheaves described in the introduction for the associative exceptional algebras \( \mathbb{R} \otimes \mathbb{C}, \mathbb{C} \otimes \mathbb{C} \) and \( \mathbb{H} \otimes \mathbb{C} \)).

### 3.3 On the automorphism group of twisted eigenvalue problem

It is well known (see the section 14 of [Har90]) that the subgroup of \( \text{E}_6 \) generated by \( \text{SL}_3 \) and \( \text{Spin}_8 \) is \( \text{E}_6 \) itself. In fact, one can prove that the subgroup of \( \text{E}_6 \) generated by \( \text{SL}_3 \) and \( \text{Spin}_7 \) is also \( \text{E}_6 \). The following argument has been communicated to me by Robert Bryant. Let \( h \in \text{SL}_3 \) defined by:

\[
h = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
Let us denote by $K_2$ the copy of $\text{Spin}_7$ embedded in $\text{Spin}_8$ we considered in the previous section:

$$K_2 = \{(T_1, T_2, T_3) \in (\text{SO}_8)^3, \; T_1(x)T_2(y) = T_1(xy), \text{ for all } (x, y) \in \mathbb{O}^2\}.$$ 

It is easily checked that we have $hK_2h^{-1} = K_1$ as subgroups of $E_6$, where $K_1$ is the copy of $\text{Spin}_7$ embedded in $\text{Spin}_8$ as:

$$K_1 = \{(T_2, T_1, T_3) \in (\text{SO}_8)^3, \; T_2(x)T_1(y) = T_1(xy), \text{ for all } (x, y) \in \mathbb{O}^2\}.$$ 

It is known that these two copies of $\text{Spin}_7$ are sufficient to generate $\text{Spin}_8$ (see the section 14 of [Har90]) and we are done.

**Proposition 3.3.1** The action of $\text{Spin}_7$ on $J_3(\mathbb{O})$ we considered in Theorem 3.2.1 does not factor through the action $E_6$ on $J_3(\mathbb{O})$.

**Proof:**

Assume by absurd that it does. Then the remark above shows that $(\text{Spin}_7, \text{SL}_3) = E_6$. One immediately checks that the action of $(\text{Spin}_7, \text{SL}_3)$ we defined on $J_3(\mathbb{O})$ preserves the rank, when we see elements of $J_3(\mathbb{O})$ as $24 \times 24$ matrices. Since any element of $J_3(\mathbb{O})$ can be diagonalized by the action of $E_6$ (see the chapter 14 of [Har90]), this would prove that any element of $J_3(\mathbb{O})$ has rank $0, 8, 16$ or $24$ as a $24 \times 24$ matrix. But we know that a generic point on $\mathcal{G} = 0$ has rank $22$ (see Theorem 3.2.1), a contradiction.

We now state the main result of this section:

**Theorem 3.3.2** The action of $\text{Aut}^0(\mathcal{G}) \times \mathbb{C}^*$ on $J_3(\mathbb{O})$ is prehomogeneous.

This result is slightly surprising as the subgroup of $\text{SL}_{27}$ generated by $\text{Spin}_7(\mathbb{C})$ and $\text{SL}_3(\mathbb{C})$ is not $E_6$ (see the above proposition). One also notices that the prehomogeneous space $(J_3(\mathbb{O}), \text{Aut}^0(\mathcal{G})) \times \mathbb{C}^*$ does not appear in the list of Kimura-Sato [KS77]. This is not surprising as it is unlikely to be an irreducible prehomogeneous space. It thus provides an interesting example of a non-irreducible prehomogeneous vector space which does not split as a sum of lower dimensional prehomogeneous vector spaces. Theorem 3.3.2 also shows that the geometry of the twisted octonionic eigenvalue problem differs eminently from that of the untwisted eigenvalue problem: the former is much more «symmetric» than the latter. Indeed, we noticed in section 2.2 that $\text{Aut}^0(\text{SO}_{27}) \times \mathbb{C}^* = \text{SO}_7 \times \text{SL}_3 \times \mathbb{C}^*$ does not act prehomogeneously on $J_3(\mathbb{O})$.

**Proof:**

We have shown in the previous section that the subgroup $G$ of $\text{SL}_{27}$ generated by $\text{SL}_3$ and $\text{Spin}_7$ is included in $\text{Aut}^0(\mathcal{G})$. We are going to prove that the action of $\mathbb{C}^* \times G$ on $J_3(\mathbb{O})$ is prehomogeneous. We first start with:

**Lemma 3.3.3** Let $\{1, i, j, k, l, m, n, o\}$ be a basis of $\mathbb{O}$ over $\mathbb{C}$. We fix a splitting $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l$, with $\mathbb{H} = \text{Vect}(1, i, j, k)$. The group $\text{Spin}_7$ acts transitively on the following varieties:

$$S^6 \times \{(x + y, l) \in \mathbb{H} \oplus \mathbb{H}l, \; (x, y) = 0, x \neq 0, y \neq 0\} \subset \text{Im}(\mathbb{O}) \times \mathbb{O},$$

$$S^6 \times (\mathbb{H} \backslash \{0\} \oplus \{0\}) \subset \text{Im}(\mathbb{O}) \times \mathbb{O},$$

$$S^6 \times (\{0\} \oplus \mathbb{H}l \backslash \{0\}) \subset \text{Im}(\mathbb{O}) \times \mathbb{O},$$

where $S^6$ denotes the subvariety of $\text{Im}(\mathbb{O})$ defined by $|z|^2 = 1$. 

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This lemma is well-known, we refer to corollary 5.4 in [Ada96]. We recall a proof for the convenience of the reader since the results of [Ada96] are stated over \( \mathbb{R} \) and there are some minor modifications to be made over \( \mathbb{C} \).

**Proof:**

The projection map:

\[
\begin{array}{c}
\text{Spin}_7 \longrightarrow \text{SO}_8 \\
(T_1, T_2, T_1) \longrightarrow T_2
\end{array}
\]

is a double cover of \( \text{SO}_7 \) embedded in \( \text{SO}_8 \) as the stabilizer of 1 ∈ \( \mathbb{O} \). Hence, \( \text{Spin}_7 \) acts transitively on \( S^6 \) and the stabilizer of any point in \( S^6 \) is \( \text{Spin}_6 \). We know that \( \text{Spin}_6 \simeq \text{SL}_4 \) when we make the identification \( \mathbb{H} \oplus \mathbb{H}.l = \mathbb{C}^4 \oplus (\mathbb{C}^4)^* \). Furthermore, for any \( (x + y.l) \in \mathbb{H} \oplus \mathbb{H}.l \), the scalar product \( \langle x, y \rangle \) is the perfect pairing between \( \mathbb{C}^4 \) and \( (\mathbb{C}^4)^* \). This pairing is obviously invariant for the product of the natural and the contragedient representation of \( \text{SL}_4 \). The action of \( \text{SL}_4 \) on \( \mathbb{C}^4 \setminus \{0\} \) is transitive and the stabilizer in \( \text{SL}_4 \) of any non-zero \( x \in \mathbb{C}^4 \) contains \( \text{SL}_3 \), which acts on \( x^\perp \subset (\mathbb{C}^4)^* \). The transitive action of \( \text{SL}_3 \) on \( \mathbb{C}^3 \setminus \{0\} \) allows us to conclude the proof of the lemma.

We can now proceed to the proof of Theorem 3.3.2. Recall that the action of \( \text{Spin}_7 \) on \( J_3(\mathbb{O}) \) is given as follows. For any \( (T_1, T_2, T_1) \in \text{Spin}_7 \) and any \( A = \begin{pmatrix} \lambda_1 & c & b \\ \overline{c} & \lambda_2 & a \\ b & \overline{a} & \lambda_3 \end{pmatrix} \in J_3(\mathbb{O}) \), we have:

\[
(T_1, T_2, T_1).A = \begin{pmatrix} T_2(c) & \overline{K_{T_1}(b)} \\ \overline{T_2(c)} & T_1(a) \\ K_{T_1}(b) & \overline{T_1(a)} \end{pmatrix}.
\]

We start with \( A = \begin{pmatrix} \lambda_1 & c & b \\ \overline{c} & \lambda_2 & a \\ b & \overline{a} & \lambda_3 \end{pmatrix} \) generic in \( J_3(\mathbb{O}) \). By lemma 3.3.3, we can find \( (T_1, T_2, T_1) \in \text{Spin}_7 \) such that \( T_2(c) = r_1 + r_2.i \), with \( r_1, r_2 \in \mathbb{C} \) non zeros (by genericity of \( A \)). Hence, we have:

\[
A_1 = (T_1, T_2, T_1).A = \begin{pmatrix} \lambda_1 & r_1 + r_2.i & b_1 \\ r_1 - r_2.i & \lambda_2 & a' \\ b_1 & \overline{a'} & \lambda_3 \end{pmatrix},
\]

with \( a' = T_1(a) \) and \( b_1 = K_{T_1}(b) \). Let us write \( a' = a_0' + a_1'.l \) with \( a_0', a_1' \in \mathbb{H} \). Since \( A \) is assumed generic, we can arrange \( T_1 \) in the previous transformation so that \( (r_1 - r_2.i, a_1') \neq 0 \).

We plan to make the transformation \( a' \leftrightarrow a'' = \frac{\langle a_0', a_1' \rangle}{\langle r_1 - r_2.i, a_1' \rangle}(r_1 - r_2.i) \). We thus consider

\[
H = \begin{pmatrix} 1 & 0 & -\langle a_0', a_1' \rangle \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3 \text{ and we have:}
\]

\[
A_2 = \overline{H}A_1H = \begin{pmatrix} \lambda_1 & r_1 + r_2.i & b_2 \\ r_1 - r_2.i & \lambda_2 & a'' \\ b_2 & \overline{a''} & \lambda_3' \end{pmatrix},
\]

with \( a'' = a' - \frac{\langle a_0', a_1' \rangle}{\langle r_1 - r_2.i, a_1' \rangle}(r_1 - r_2.i), b_2 \in \mathbb{O} \) and \( \lambda_3' \in \mathbb{C} \). If we put \( a'' = a_0'' + a_1'.l \) with \( a_0'' \in \mathbb{H} \), then we have \( \langle a_0'', a_1' \rangle = 0 \) by construction. By lemma 3.3.3 we can find \( (T_1, T_2, T_1) \in \text{Spin}_7 \) such that
that $T_2(r_1 + r_2.i) = r_1 + r_2.n$ and $T_1(a''') = r_3.i + r_4.n$, with $r_3, r_4 \in \mathbb{C}$ non zeros (by genericity of $A$). Hence, we have:

$$A_3 = (T_1, T_2, T_1).A_2 = \begin{pmatrix} \lambda_1 & r_1 + r_2.n & \overline{b_3} \\ r_1 - r_2.n & \lambda_2 & r_3.i + r_4.n \\ b_3 & -r_3.i - r_4.n & \lambda_3'' \end{pmatrix},$$

with $b_3 \in \mathbb{O}$. We now plan to make the transformation $r_3.i + r_4.n \leftarrow r_3.i + r_4.n + \frac{r_4}{r_2}(r_1 - r_2.n)$.

We thus consider $H = \begin{pmatrix} 1 & 0 & \frac{r_4}{r_2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL_3$ and we have:

$$A_4 = {}^t H A_3 H = \begin{pmatrix} \lambda_1 & r_1 + r_2.n & \overline{b_4} \\ r_1 - r_2.n & \lambda_2 & r_5 + r_3.i \\ b_4 & r_5 - r_3.i & \lambda_3'' \end{pmatrix},$$

with $r_5 = \frac{r_4}{r_2}r_1 \in \mathbb{C}$, $\lambda_3'' \in \mathbb{C}$ and $b_4 \in \mathbb{O}$. By lemma 3.3.3 we can find $(T_1, T_2, T_1) \in Spin_7$ such that $T_2(r_1 + r_2.n) = r_1 + r_2.i$ and $T_1(r_5 + r_3.i) = r_5 + r_3.i$. Hence, we have:

$$A_5 = (T_1, T_2, T_1).A_4 = \begin{pmatrix} \lambda_1 & r_1 + r_2.i & \overline{b_5} \\ r_1 - r_2.i & \lambda_2 & r_5 + r_3.i \\ b_5 & r_5 - r_3.i & \lambda_3'' \end{pmatrix},$$

with $b_5 \in \mathbb{O}$. We now plan to make the transformation $r_1 + r_2.i \leftarrow r_1 + r_2.i + \frac{r_2}{r_3}(r_5 - r_3.i)$. We thus consider $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{r_2}{r_3} & 0 & 1 \end{pmatrix} \in SL_3$ and we have:

$$A_6 = {}^t H A_5 H = \begin{pmatrix} \lambda'_1 & r_6 & \overline{b_6} \\ r_6 & \lambda_2 & r_5 + r_3.i \\ b_6 & r_5 - r_3.i & \lambda_3'' \end{pmatrix},$$

with $r_6 = r_1 + \frac{r_2}{r_3}r_5 \in \mathbb{C}$, $\lambda'_1 \in \mathbb{C}$ and $b_6 \in \mathbb{O}$. By lemma 3.3.3 we can find $(T_1, T_2, T_1) \in Spin_7$ such that $T_1(r_5 + r_3.i) = r_7 \in \mathbb{C}$ (and $T_2(r_6) = r_6$, automatically). Hence, we have:

$$A_7 = (T_1, T_2, T_1).A_6 = \begin{pmatrix} \lambda'_1 & r_6 & \overline{b_7} \\ r_6 & \lambda_2 & r_7 \\ b_7 & r_7 & \lambda_3'' \end{pmatrix}.$$

We now consider:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{r_6}{r_7} & 0 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{r_7}{\lambda_2} \\ 0 & 0 & \lambda_2 \end{pmatrix}, H_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We have:

$$A_8 = {}^t H_3^t H_2^t H_1 A_7 H_1 H_2 H_3 = \begin{pmatrix} \mu_1 & \overline{b_8} & 0 \\ b_8 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix},$$

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with $b_8 \in \mathbb{O}$ and $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$. By lemma 3.3.3 we can find $(T_1, T_2, T_1) \in \text{Spin}_7$ such that $T_2(b_8) = r_8 + r_9 \cdot i$, with $r_8, r_9 \in \mathbb{C}$. Hence, we have:

$$A_9 = (T_1, T_2, T_1).A_8 = \begin{pmatrix} \mu_1 & r_8 + r_9 \cdot i & 0 \\ r_8 - r_9 \cdot i & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}.$$  

We let $H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \text{SL}_3$. We have:

$$A_{10} = {}^t H.A_9.H = \begin{pmatrix} \mu_3 & 0 & 0 \\ 0 & \mu_2 & r_8 + r_9 \cdot i \\ 0 & r_8 - r_9 \cdot i & \mu_1 \end{pmatrix}.$$  

By lemma 3.3.3, we can find $(T_1, T_2, T_1) \in \text{Spin}_7$ such that $T_1(r_8 + r_9 \cdot i) = r_{10}$, with $r_{10} \in \mathbb{C}$. Hence, we have:

$$A_{11} = (T_1, T_2, T_1).A_{10} = \begin{pmatrix} \mu_3 & 0 & 0 \\ 0 & \mu_2 & r_{10} \\ 0 & r_{10} & \mu_1 \end{pmatrix}.$$  

Notice that $A_{11}$ is symmetric with complex coefficients. Since $A \in J_3(\mathbb{O})$ was chosen generic, we have $\det(A_{11}) = \det_\mathbb{O}(A) \neq 0$ and we deduce that there exists $H \in \text{GL}_3$ such that:

$$^t H A_{11} H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

We thus have proved that a generic element $A \in J_3(\mathbb{O})$ is in the same orbit as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for the action of the subgroup of $\text{GL}_{24}$ generated by $\text{Spin}_7$ and $\text{GL}_3$. As a consequence, the action of $\mathbb{C}^* \times \text{Aut}^0(\mathfrak{G})$ on $J_3(\mathbb{O})$ is prehomogeneous. \hfill \blacksquare
A On the dimension the Lie-algebra $\text{aut}(\text{SO}_\text{DM})$

In this section, we provide a Macaulay2 code to get an upper bound on the dimension of $\text{aut}(\text{SO}_\text{DM})$. We refer to the proof of Theorem 2.2.1 in section 2, where this computation is used to determine the neutral component of $\text{Aut}(\text{SO}_\text{DM})$.

We first need to write explicitly the equation of $\text{SO}_\text{DM}$ in Macaulay2. This is somehow the main challenge, as the octonion algebra cannot be simulated in Macaulay2.

$$kk = \mathbb{Z}/313$$

$$V = kk[x_1..x_8,y_1..y_27]$$

$$\text{L} = \text{mutableMatrix}(\text{M})$$

for k from 0 to 7 do (for i from 0 to 7 do
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_j=> y_(j+8))))

$$\text{S2} = \text{matrix}(\text{L})$$

$$\text{L} = \text{mutableMatrix}(\text{N})$$

for k from 0 to 7 do (for i from 0 to 7 do
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_j=> y_(j+8))))

$$\text{R2} = \text{matrix}(\text{L})$$

for k from 0 to 7 do (for i from 0 to 7 do
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_j=> y_(j+16))))
S3 = matrix(L)
L = mutableMatrix(N)
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_j=> y_(j+16))))
R3 = matrix(L)
L = mutableMatrix(M)
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_j=> y_j)))
S1 = matrix(L)
L = mutableMatrix(N)
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_j=> y_j)))
R1 = matrix(L)
L = mutableMatrix(M)
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_1 =>y_1*y_17-y_2*y_18-y_3*y_19-y_4*y_20-y_5*y_21-y_6*y_22-y_7*y_23-y_8*y_24)))
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_2 =>y_1*y_18+ y_2*y_17+y_3*y_20-y_4*y_19+y_5*y_22-y_6*y_21-y_7*y_24+y_8*y_23)))
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_3 =>y_1*y_19-y_2*y_20+y_3*y_17+y_4*y_18+y_5*y_23+y_6*y_24-y_7*y_21-y_8*y_22)))
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_4 =>y_1*y_20+y_2*y_19-y_3*y_18+y_4*y_17+y_5*y_24-y_6*y_23+y_7*y_22-y_8*y_21)))
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_5 =>y_1*y_21-y_2*y_22-y_3*y_23-y_4*y_24+y_5*y_17+y_6*y_18+y_7*y_19+y_8*y_20)))
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_6 =>y_1*y_22+y_2*y_21-y_3*y_24+y_4*y_23-y_5*y_18+y_6*y_17-y_7*y_20+y_8*y_19)))
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_7 => y_1*y_23+y_2*y_24+y_3*y_21-y_4*y_22+y_5*y_19+y_6*y_20+y_7*y_17-y_8*y_18)))
for k from 0 to 7 do (for i from 0 to 7 do 
  (for j from 1 to 8 do L_(k,i)= sub(L_(k,i), x_8 => y_1*y_24-y_2*y_23+y_3*y_22+y_4*y_21-y_5*y_20-y_6*y_19+y_7*y_18+y_8*y_17)))
S13 = matrix(L)
for k from 0 to 7 do (for i from 0 to 7 do (for j from 2 to 8 do 
L_(k,i)= sub(L_(k,i), y_j=>-y_j)))

for k from 0 to 7 do (for i from 0 to 7 do (for j from 17 to 24 do 
L_(k,i)= sub(L_(k,i), y_j=> y_(j-8))))

Sb12 = matrix(L)
L = mutableMatrix(Sb12)

for k from 0 to 7 do (for i from 0 to 7 do (for j from 1 to 8 do 
L_(k,i) = sub(L_(k,i), y_j=> y_(16+j)))

for k from 0 to 7 do (for i from 0 to 7 do (for j from 9 to 16 do 
L_(k,i)= sub(L_(k,i), y_j=> y_(j-8))))

for k from 0 to 7 do (for i from 0 to 7 do (for j from 2 to 8 do 
L_(k,i)= sub(L_(k,i), y_j=> -y_j)))

Sb3b1 = matrix(L)
for k from 0 to 7 do (for i from 0 to 7 do (for j from 17 to 24 do 
L_(k,i) = sub(L_(k,i), y_j=> y_(j-8)))

for k from 0 to 7 do (for i from 0 to 7 do (for j from 2 to 8 do 
L_(k,i)= sub(L_(k,i), y_j=> -y_j)))

S2b1 = matrix(L)

L = mutableMatrix(Sb21)
for k from 0 to 7 do (for i from 0 to 7 do (for j from 2 to 8 do 
L_(k,i) = sub(L_(k,i), y_j=> -y_j)))

S12 = matrix(L)
for k from 0 to 7 do (for i from 0 to 7 do (for j from 10 to 16 do 
L_(k,i)= sub(L_(k,i), y_j=> -y_j)))

S1b2 = matrix(L)
L = mutableMatrix(Sb21)
for k from 0 to 7 do (for i from 0 to 7 do (for j from 10 to 16 do 
L_(k,i) = sub(L_(k,i), y_j=> -y_j)))

S21 = matrix(L)
AA = (S13)*S2 + transpose(S2)*(Sb3b1)

P0 = AA_(0,0)

P1 = -y_27*((S1*transpose(S1))_(0,0))

P2 = -y_26*((S2*transpose(S2))_(0,0))

P3 = -y_25*((S3*transpose(S3))_(0,0))

P = y_25*y_26*y_27 + P0 + P1 + P2 + P3

PHI123 = (1/2)*((S12-S21)*transpose(S3))_(0,0)

L = mutableMatrix(S1)

for k from 0 to 7 do (for i from 0 to 7 do
L_(k,i) = sub(L_(k,i), y_1=> 0))

M1 = matrix(L)

L = mutableMatrix(S2)

for k from 0 to 7 do (for i from 0 to 7 do
L_(k,i) = sub(L_(k,i), y_9=> 0))

M2 = matrix(L)

L = mutableMatrix(S3)

for k from 0 to 7 do (for i from 0 to 7 do
L_(k,i) = sub(L_(k,i), y_17=> 0))

M3 = matrix(L)

G = matrix{{(M1*transpose(M1))_(0,0),(M1*M2)_ (0,0),(M1*transpose(M3))_ (0,0)},
{(transpose(M2)*transpose(M1))_(0,0),(M2*transpose(M2))_ (0,0),
(transpose(M2)*transpose(M3))_ (0,0)},
{(M3*transpose(M1))_ (0,0),(M3*M2)_ (0,0),(M3*transpose(M3))_ (0,0)}}

ASS123square = 4*(det(G) - (PHI123)*(PHI123))

SODM = P^2 - 4*(PHI123)*P - ASS123square

Here SODM is the equation of the sextic S_{ODM}, P is Det_O, PHI123 is ϕ(c, b, a) and ASS123square
is \( |c, b, a|^2 \). We have used the formula (see lemma 6.56 and 6.61 in [Har90]):

\[ |c, b, a|^2 = 4 \left( \text{Gram}(\text{Im}(c), \text{Im}(b), \text{Im}(a)) - \phi(c, b, a)^2 \right), \]

which allows to get the expression of \( |c, b, a|^2 \) without computing triple products explicitly in \( \mathbb{O} \). We recall that differential of the map \( \varphi \) introduced in the proof of Theorem 2.2.1 at a point \([(m)_{i,j}] \in M_{6 \times 27} \) is given by:

\[ d\varphi_{[m_{i,j}]} : M_{6 \times 27}/\text{null}(\text{SODM}) \longrightarrow S^6\mathbb{C}^6 \]

\[ [q_{i,j}] \longrightarrow \sum_{i=1}^{27} \left( \sum_{j=1}^{6} q_{i,j} \cdot z_j \right) \times \frac{\text{SODM}}{\partial x_i} (m_{1,1}z_1 + \cdots + m_{1,6}z_6, \cdots, m_{27,1}z_1 + \cdots + m_{27,6}z_6), \]

We produce a matrix representing \( d\varphi \) at a random point \([(m)_{i,j}] \in M_{6 \times 27} \).

\[
D = \text{mutableMatrix}(V, 1, 27)
\]

\[
\text{for } i \text{ from } 1 \text{ to } 27 \text{ do } D_-(0, i-1) = \text{diff}(y_i, \text{SODM})
\]

\[
M = \text{random}(\mathbb{Z}[z_{-27}, \mathbb{Z}^6, \text{Height}=>50)
\]

\[
M^2 = M**V
\]

\[
m = M^2 \ast \text{matrix}\{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}\}
\]

\[
\text{for } i \text{ from } 0 \text{ to } 26 \text{ do } \text{for } j \text{ from } 1 \text{ to } 27 \text{ do } D_-(0, i) = \text{sub}(D_-(0, i), y_j => m_(j-1,0))
\]

\[
W = \mathbb{K}[x_1..x_6]
\]

\[
D\varphi = \text{mutableMatrix}(W, 1, 27)
\]

\[
\text{for } j \text{ from } 0 \text{ to } 26 \text{ do } D\varphi_-(0, j) = \text{sub}(D_-(0, j), W)
\]

The matrix \( D\varphi \) represents \( d\varphi \) at the random \([(m)_{i,j}] \in M_{6 \times 27} \) chosen. We are left to compute the rank of the image of \( D\varphi \) in \( S^6\mathbb{C}^6 \).

\[
I = (0)
\]

\[
\text{for } j \text{ from } 0 \text{ to } 26 \text{ do } \text{for } i \text{ from } 1 \text{ to } 6 \text{ do } I = I + x_i*\text{ideal}(D\varphi W_-(0, j))
\]

\[
\text{II} = \text{module } I
\]

\[
\text{III} = \text{II}**\mathbb{K}
\]

\[
\text{rank } III
\]

The answer (=133) is obtained on a portable workstation in less than 15 minutes. The semi-continuity Theorem insures that the rank of \( d\varphi \) at a generic point \([(m)_{i,j}] \in M_{6 \times 27} \) and in characteristic 0 is bigger or equal to 133.
References

[Ada96] J.F. Adams. Lectures on exceptional Lie groups. Chicago lectures in Mathematics series. University of Chicago Press, 1996.

[AH] R. Abuaf and J. D. Hauenstein. The geometry and singularities of a few octonionic sextics. In preparation.

[BH16] M. Bhargava and W. Ho. Coregular spaces and genus one curves. Cambridge J. Math., 4(1):1–119, 2016.

[DeS01] R. DeSapio. On spin(8) and triality: A topological approach. Exp. Math., 19:143–161, 2001.

[DM98] T. Dray and C.A. Manogue. The octonionic eigenvalue problem. Adv. Appl. Clifford Algebras, 8(2):341–364, 1998.

[GS] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[Har90] F.R. Harvey. Spinors and Calibrations. Perspectives in Math (vol 9). Academic Press, 1990.

[Jac54] N. Jacobson. Structure of Alternative and Jordan Bimodule. Osaka Math. Journal, 6(1):1–71, 1954.

[KS77] T. Kimura and M. Sato. A classification of irreducible prehomogeneous vector spaces and their relative invariants. Nagoya Math. J., 65:1–155, 1977.

[Ogi81] O.V. Ogievetskiî. A Characteristic Equation for 3 × 3 Matrices over the Octonions. Uspekhi Mat. Nauk, 36:197–198, 1981.

[Rob88] J. Roberts. Projective embeddings of Algebraic Varieties. Monografías del Instituto de Matemáticas. Universidad Nacional Autónoma de México, 1988.