Spatial-symmetry violating electromagnetic fields corrected by nonlinear Lagrangian

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Abstract
We investigate a general nonlinear electromagnetic Lagrangian belonging to a subclass of the Plebanski class. Depending on the form of nonlinear Lagrangian model, in an electrostatic problem, not only the electric field but also the electric flux density can vary from that given by linear classical electromagnetism. The variance is a correction which stems from the nonlinearity of Lagrangian. The nonlinear correction is a divergence-free field and possesses a mathematical vector potential. By considering a two charge system with a non-perturbative approach, we derive a necessary condition on nonlinear Lagrangian such that the nonlinear correction of the electric flux density becomes zero in the entire space. Several frequently considered nonlinear Lagrangian models do not satisfy this condition. As another important result, we show that the vector potential of the nonlinear correction violates mirror symmetry.

Keywords: nonlinear electromagnetism, Plebanski class, nonlinear correction of electric flux density, violation of spatial symmetry

1. Introduction

The symmetries in physics are important because they are directly connected to conserved quantities, as is well known by Noether’s theorem. Studying symmetries is useful for clarifying the mathematical structures and for developing a new physical theory.

In particular, classical electromagnetism is naturally invariant under the Lorentz transformation, and the invariance is crucial in special relativity. In addition, the gauge invariance of classical electromagnetism was a very important step for the Yang–Mills theory [1], the grand unified theory [2], and several other theories. Moreover, the invariance under C, P, and T transformations is a remarkable property of classical electromagnetism.

Nevertheless, the importance of classical electromagnetism owes to the fact that the theory has explained numerous experiments. Recent technological developments [3–5] have resulted in interest in modification or extension of classical electromagnetism. To correspond to a strong field region, several extensions have been considered [6–9]. Furthermore, an extension of classical electromagnetism is taken into account in cosmology [10, 11].

Whether the extended theories satisfy the same symmetries of classical electromagnetism is an interesting problem. For example, several models exist that violate Lorentz invariance [12, 13] and CPT invariance [14–16].

In addition to symmetry, the existence and uniqueness of the solution are fundamental [17–19]. In classical electromagnetism, the existence and uniqueness of the solution are guaranteed, essentially, by Helmholtz’s theorem. However, this is unclear in nonlinear electromagnetism. For example, if a physical system possesses symmetry, a mathematical solution can be irrelevant to the symmetry [20]. However, for the ‘correct’ nonlinear Lagrangian, it would be natural to expect that a unique solution exists and it reflects the physical symmetries.

In this study, we consider a static two charge system for nonlinear electromagnetic Lagrangian belonging to a certain class. This system is axially symmetric with respect to the line that passes through both charges. The main discussion is described in section 3. First, we demonstrate that the magnetic flux density is zero at a sufficiently far region from the...
charges. Then, we derive a necessary condition on nonlinear Lagrangian that the nonlinear correction of electric flux density is zero in the entire space. Subsequently, we introduce a vector potential for the nonlinear correction of electric flux density and examine its property. Throughout the paper, the permittivity of vacuum \( \varepsilon_0 \) and the magnetic permeability \( \mu_0 \) are set to unity.

2. Notations

The electric field \( E \) and magnetic flux density \( B \) are defined by the four potential \( A^\mu \). The electromagnetic tensor [21] is defined by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.
\]

The following four values,

\[
F = |E|^2 - |B|^2, \quad G = E \cdot B, \quad A^\mu A_\mu, \quad F_{\mu\nu} A^\mu A_\nu,
\]

are necessary and sufficient to uniquely determine the electromagnetic fields [22]. The three values except for \( G \) are invariant under the Lorentz transformation. \( G \) varies its sign under \( P \) and \( T \) transformations.

The classical electromagnetic Lagrangian is given by \( F/2 \). Thus, it can be a natural extension to use these four values as arguments of a nonlinear Lagrangian \( L \). Moreover, if we require the argument to be gauge invariant, the arguments are limited to \( F \) and \( G \). A set of such Lagrangians is often called the Plebański class [23]. Let \( \mathcal{L} \) be the nonlinear part of the Lagrangian, i.e.

\[
\mathcal{L} = L - \frac{1}{2} F.
\]

It is zero for the linear classical theory. Because the constant in the Lagrangian is meaningless, we can assume \( \mathcal{L} \rightarrow 0 \) (as \( F, G \rightarrow 0 \)), without loss of generality. We assume \( \mathcal{L} \) is sufficiently smooth. In particular, regarding \( \mathcal{L} \) as a function of \( F \) by substituting \( G = 0 \), we assume \( \mathcal{L}(F, 0) \) to be analytic at \( F = 0 \). Let \( \Lambda > 0 \) be the radius of convergence. In short, if \( |F| < \Lambda \), \( \mathcal{L} \) can be expanded as

\[
\mathcal{L}(F, 0) = \sum_{n=2}^\infty C_n F^n.
\]

Once the Lagrangian \( L \) of the electromagnetic fields is given, the electric flux density \( D \) and magnetic field \( H \) are defined so that the Euler–Lagrange equations correspond to Gauss’s law and Maxwell-Ampère equation. For Lagrangians belonging to the Plebański class, \( D \) and \( H \) are given by

\[
D = \left( 1 + 2 \frac{\partial \mathcal{L}}{\partial F} \right) E + \frac{\partial \mathcal{L}}{\partial G} B,
\]

\[
H = \left( 1 + 2 \frac{\partial \mathcal{L}}{\partial F} \right) B - \frac{\partial \mathcal{L}}{\partial G} E.
\]

For a static charge and current configuration, the distribution of electromagnetic fields is obviously unique, in the physical sense. Therefore, we assume that the nonlinear Lagrangian yields a unique solution for a static problem. As above-mentioned, this assumption of uniqueness is unnecessary for the linear classical electromagnetism because the unique distribution for a static problem is always derived by Helmholtz’s theorem.

3. Two charge system

Let us consider a static two charged point particle system. The boundary condition is given as all the physical fields converge to zero as \( r \to \infty \). A very similar system was considered in [24]. Let \( a > 0 \) be a constant and we fix two charges \( \sigma, \sigma' \neq 0 \) at \( R = (0, 0, a) \) and \( R' = (0, 0, -a) \), respectively. The physical system is mirror-symmetric with respect to a plane containing the \( z \) axis. For a position vector \( r = (x, y, z) \), we define \( r_c = r - R \) and \( r' = r - R' \). Using the three-dimensional Dirac delta, the electric charge density is given by \( \sigma \delta(r_c) + \sigma' \delta(r_c) \). Subsequently, Gauss’s law yields

\[
D = D_c + \tilde{D}, \quad D_c = \frac{\sigma}{4\pi r_c^2} \hat{r} + \frac{\sigma'}{4\pi r_c^2} \hat{r}' \quad \text{and} \quad \nabla \cdot D = 0,
\]

where \( D_c \) is that given in the linear classical electromagnetism and \( \tilde{D} \) is a nonlinear correction which originates from the nonlinearity of considered Lagrangian \( \mathcal{L} \). Note that \( \tilde{D} \) vanishes in classical linear electromagnetism. The remaining Maxwell’s equations are as follows:

\[
\nabla \times E = 0, \quad \nabla \times H = 0, \quad \nabla \cdot B = 0.
\]

We perform calculations in the cylindrical coordinate system in order to clarify the relationship between the physical symmetry and the mathematical solution of the nonlinear Maxwell’s equations. \((\rho, \theta)\) are introduced by \( \rho = (x^2 + y^2)^{1/2} \) and \( x = \rho \cos \theta, y = \rho \sin \theta \). The unit vectors are \( \mathbf{e}_\rho = (\cos \theta, \sin \theta, 0), \mathbf{e}_\theta = (-\sin \theta, \cos \theta, 0) \), and \( \mathbf{e}_z = (0, 0, 1) \).

3.1. Magnetic flux density

We first consider the magnetic flux density. \( B_\theta = 0 \) for a sufficiently large \( r \) is demonstrated as follows. Because of the assumption of uniqueness, all components of the electromagnetic fields must be independent of \( \theta \). Thus, \( \nabla \times H = 0 \) leads to

\[
\frac{\partial H_\theta}{\partial \zeta} = 0, \quad \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\theta) = 0.
\]

With a constant \( H_0 \), we obtain \( H_\theta = H_0/\rho \). Taking \( r \to \infty \) with a fixed \( \rho, H_0/\rho \) is unchanged but should converge to zero, i.e. \( H_0 = 0 \) is necessary. Consequently, we obtain \( H_0 = 0 \). Similarly, we obtain \( E_\theta = 0 \). Therefore, from
equation (6), we obtain

\[ 0 = \left( 1 + 2 \frac{\partial f}{\partial F} \right) B_0. \]  

(10)

If \( r \gg a \), it holds that \( |F|, |G| \ll 1 \) and the absolute value of the second term in the parenthesis is much smaller than unity. Therefore, we obtain

\[ B_0 = 0. \]  

(11)

It is difficult to prove the remaining components similarly. Therefore, we consider the electromagnetic vector potential \( A \), instead of \( B \). As is well known, we cannot determine \( A \) uniquely without gauge fixing. Owing to its arbitrariness, \( A \) does not have to reflect the physical symmetry. For example, its component can depend on \( \theta \) and it does not have to be mirrorsymmetric. However, due to the fact that \( B = \nabla \times A \) and each component of \( B \) does not depend on \( \theta \), the electromagnetic vector potential can be expressed in the form of

\[ A = A_{\theta \phi}(\rho, z) \epsilon_\phi + A_{\theta \phi}(\rho, z) \epsilon_\theta + A_{\rho \phi}(\rho, z) \epsilon_\rho + \nabla \chi(\rho, \theta, z). \]  

(12)

The \( \theta \) dependence appears only in the unit vectors \( \epsilon_\phi, \epsilon_\theta \) and in the gradient term \( \nabla \chi(\rho, \theta, z) \) which does not affect the magnetic flux density. Because any gauge transformations can be included in the gradient term, we can think that \( A_{\theta \phi} \) reflects mirror symmetry, i.e. \( A_{\theta \phi} = 0 \). Therefore, we obtain \( B_\rho = 0 \) and \( B_\theta = 0 \), and we can conclude that

\[ B = 0 \]  

(13)

for a sufficiently large \( r \). Note that contrary to the linear classical theory, this result is not clear in a nonlinear Lagrangian case.

3.2. In the case of \( \Phi = 0 \)

In this subsection, we consider a necessary condition on nonlinear Lagrangian that the nonlinear correction of electric flux density \( \Phi \) is zero in the entire space, i.e. \( \Phi \equiv 0 \).

For \( r \gg a \), the electric field produced by two charges is sufficiently weak and \( |F| < \Lambda \) must hold. Therefore, the relationship between the electric field and electric flux density is given by

\[ D = D_0 = \left( 1 + 2 \sum_{p=2} pC_p f^{p-1} \right) E. \]  

(14)

Note that \( |D| \neq 0 \) holds almost everywhere. Let \( D_0 = |D| \) and \( e = D/D_0 \). The electric field is parallel to \( D \) and it can be expressed as \( E = fe \). The function \( f \) satisfies the following almost everywhere,

\[ D_0 = \left( 1 + 2 \sum_{p=2} pC_p f^{p-1} \right) f. \]  

(15)

The electric field is expressed as

\[ E = \frac{f}{4\pi D_0} \times \left[ \left( \frac{\sigma}{r_+^3} + \frac{\sigma'}{r_-^3} \right) \epsilon_p + \left( \frac{\sigma(z-a)}{r_+^3} + \frac{\sigma'(z+a)}{r_-^3} \right) \epsilon_e \right]. \]  

(16)

In cylindrical coordinates, the \( \rho \) and \( z \) components of \( \nabla \times E \) are clearly zero. Subsequently, let us calculate the \( \theta \) component. We define two absolutely convergent series by

\[ \alpha = \sum_{p=2} p^2 C_p f^{2(p-1)}, \quad \gamma = \sum_{p=2} p^2 C_p f^{2(p-1)}. \]  

(17)

Equation (15) can be expressed as \( D_0 = (1 + 2\alpha)f \). Finally, we obtain

\[ \left( \nabla \times E \right)_\theta = \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = \frac{3(\alpha - \gamma)\sigma\sigma' a^2 \rho [\sigma(z + a) r_+ + \sigma'(z - a) r_-]}{4\pi D_0^2 (1 + 4\gamma - 2\alpha)(1 + 2\alpha) r_+^8 r_-^8}. \]  

(18)

From Maxwell’s equations, the left-hand side is zero. The right-hand side becomes zero if and only if

\[ (\alpha - \gamma)\sigma\sigma' a^2 \rho [\sigma(z + a) r_+ + \sigma'(z - a) r_-] = 0. \]  

(19)

From the presupposition, \( \sigma, \sigma' \neq 0, a \neq 0 \) are clear. In addition, the square bracket and \( \rho \) never become zero almost everywhere in the considered area. That is, for almost everywhere in the area,

\[ \alpha - \gamma = \sum_{p=2} p(1 - p) C_p f^{2(p-1)} = 0 \]  

(20)

must hold. Because \( f \) varies continuously, it is necessary that all the coefficients are to be zero. This result can be expressed as follows,

\[ \Phi \equiv 0 \iff \forall p \geq 2, \left[ C_p = 0 \right]. \]  

(21)

3.3. Vector potential of \( \Phi \)

In this subsection, we consider a nonlinear Lagrangian model which does not satisfy the necessary condition shown in equation (21). In this case, \( \Phi \) should be taken into account. For example, consider a region \( \Omega \) for a sufficiently large \( r \) and suppose \( \Phi = 0 \) holds almost everywhere in \( \Omega \). Then, similar to the preceding calculation, the same necessary condition is derived. Therefore, \( \Phi \) is different from zero almost everywhere in \( \Omega \). Because \( \Phi \) is a divergence-free vector field, it can be expressed as

\[ \Phi = \nabla \times \mathbf{P}. \]  

(22)

The mathematical vector potential \( \mathbf{P} \) is introduced. Let us temporarily call it a corrective vector potential and see its property. Similar to the discussion on the magnetic flux density, each component of \( \Phi \) is independent of \( \theta \) and \( D_0 = 0 \) holds for a sufficiently large \( r \). Therefore, the corrective
vector potential $\mathbf{P}$ can be expressed as

$$
P = P_0(\rho, z)\mathbf{e}_\rho + P_0(\rho, z)\mathbf{e}_\phi + P_0(\rho, z)\mathbf{e}_z + \nabla \chi_D(\rho, \theta, z).
$$

(23)

Substituting equations (23) into (22) and using $\nabla \times \nabla \chi_D = 0$, we obtain

$$
\tilde{\mathbf{D}} = \left(\frac{1}{\rho} \frac{\partial P_\phi}{\partial \rho} - \frac{\partial P_\rho}{\partial \rho} \right)\mathbf{e}_\rho + \left(\frac{\partial P_\phi}{\partial \rho} - \frac{\partial P_\rho}{\partial \rho}\right)\mathbf{e}_\phi
$$

$$
+ \left(\frac{1}{\rho} \frac{\partial P_\rho}{\partial \phi} - \frac{1}{\rho} \frac{\partial P_\phi}{\partial \phi}\right)\mathbf{e}_z,
$$

(24)

where the arguments $(\rho, z)$ are omitted. We have clarified that $P_\phi$ and $P_\rho$ do not depend on $\theta$ and $\tilde{\mathbf{D}}_\theta = 0$ holds for a sufficiently large $r$. Therefore, we obtain for a sufficiently large $r$,

$$
\tilde{\mathbf{D}} = \frac{\partial P_\phi}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial P_\rho}{\partial \rho} \mathbf{e}_\phi.
$$

(25)

Clearly, $P_\phi \neq 0$ is necessary for $\tilde{\mathbf{D}} \neq 0$ for a sufficiently large $r$. This result shows that the corrective vector potential violates mirror symmetry with respect to a plane containing the $z$ axis.

4. Final remarks

We have studied nonlinear electromagnetic Lagrangian $L(F, G)$ belonging to the Plebański class and $L(F, 0)$ is analytic at $F = 0$. We have shown that the electric flux density can be different from that given in the linear classical electromagnetism by considering a static two charges system. We derived a necessary condition on nonlinear Lagrangian such that the nonlinear correction of electric flux density $\tilde{\mathbf{D}}$ to be zero in the entire space. In the case that a nonlinear Lagrangian model does not satisfy this necessary condition, the nonlinear correction never becomes identically zero. For example, the Born–Infeld model and a frequently considered model of the form of $L = F^2 / 2 + c_{2,0}F^2 + c_{2,2}G^2$ (where $c_{2,0}$ and $c_{2,2}$ are constants and $c_{2,0} \neq 0$) are nonlinear models that the nonzero nonlinear correction $\tilde{\mathbf{D}}$ must be taken into account. Moreover, we would like to refer to the important Heisenberg–Euler (HE) model. The original HE model is not analytic at $F = G = 0$ and the present study is not directly applicable. On the other hand, the HE model is frequently approximated and a truncated polynomial is treated. Such a truncated polynomial is considered to be effective, provided that the electromagnetic fields are sufficiently weaker than the Schwinger limit. For the truncated polynomial, the present study is applicable and nonzero nonlinear correction $\tilde{\mathbf{D}}$ appears. If the truncated polynomial is a good approximation, it can be expected that the nonzero $\tilde{\mathbf{D}}$ will also appear for the original HE model.

In the case of the nonzero nonlinear correction, it is a divergence-free vector field and possesses a mathematical vector potential $\mathbf{P}$. It is found that the corrective vector potential $\mathbf{P}$ violates mirror symmetry in the two charge system. In the case of the two charge system we have considered here, it can be expected that the physical fields $\mathbf{E}, \mathbf{D}$ satisfy mirror symmetry and other spatial symmetries, e.g. they may not depend on $\theta$ and may not possess $\theta$ component. However, for a more general charge distribution, the corrective vector potential $\mathbf{P}$ may violate a spatial symmetry of the physical system. As a result, there would be a solution of $\mathbf{E}, \mathbf{D}$ that the spatial symmetry is violated. This possibility may affect the discussion on the uniqueness of the solution. In fact, in many problems of nonlinear electromagnetism, the direction and arguments of electromagnetic fields are frequently restricted by physical considerations and assumptions. However, in the case of a nonlinear Lagrangian model where the nonlinear correction needs to be considered, such considerations would prevent from finding the correct solution.

If nonlinear Maxwell’s equations are solved in the entire space, the nonlinear correction will be uniquely determined. However, it is difficult in most cases. If one tries to construct an approximate solution locally, the nonlinear correction should be accurately taken into account. However, it is apparently not easy, e.g. let $\mathbf{D}$ be a nonlinear correction around a point where $\mathbf{D}_0 = 0$, then $-\mathbf{D}$ would be another candidate of the nonlinear correction. Both $\mathbf{D}$ and $-\mathbf{D}$ satisfy Maxwell’s equations locally. Construction of the local and precise nonlinear correction may be an important problem in nonlinear electromagnetism.

In many preceding studies, the electric flux density is assumed to be that given in the classical theory, i.e. $\mathbf{D} = \mathbf{D}_0$, and the nonlinear correction on the electric field is considered. It seems that only a few studies [24] partially takes into account the nonlinear correction of electric flux density $\tilde{\mathbf{D}}$.

Instead of a static charge distribution, a static current distribution will require a nonlinear correction of magnetic field $\tilde{\mathbf{H}}$. Similar attention will be necessary for $\tilde{\mathbf{H}}$ and its mathematical scalar potential.

Both $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{H}}$ mathematically depend on the form of nonlinear Lagrangian. On the other hand, in a physical sense, both of them are uniquely determined only by the distribution of charge and current. Therefore, a detailed study of the nonlinear corrections of electric flux density and magnetic field shall be useful to specify the correct form of nonlinear electromagnetic Lagrangian. This study is a step for the understanding of these nonlinear correction fields.

In the last of this study, we would like to make a speculative discussion. The existence of nonzero nonlinear corrections $\mathbf{D}, \mathbf{H}$ depend on the nonlinear Lagrangian model. If the physical nature of electromagnetic fields is nonlinear and these corrections exist, there are mathematical vector and scalar potentials. One unclear problem is that whether these mathematical potentials are physically existing fields or not. Recalling that the electromagnetic vector potential $\mathbf{A}$ was originally introduced by the mathematical manner, an experimental test will be necessary for these mathematical potentials. For example, an experiment like Aharanov–Bohm effect [25, 26] will be highly interesting.
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