Entropy of extremal black holes

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Abstract

After summarizing the development of black hole thermodynamics in the seventies, we describe a recent microscopic model. This model suggests that the Bekenstein-Hawking area formula holds for extremal black holes as well as for ordinary (non-extremal) ones. On the other hand, semiclassical studies have suggested a discontinuity between non-extremal and extremal cases. We indicate how a reconciliation has been brought about by summing over topologies.

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1 Introduction

In Einstein’s theory of gravitation, the gravitational field due to a point mass is described by a metric which has many interesting properties. Its black hole
features have been known for a very long time, but in the seventies it began to appear that thermodynamic concepts like temperature and entropy were also associated with it. Gradually it was realized that these were quantum effects. But the degrees of freedom associated with the entropy could not be clearly identified. Many suggestions have been made. A recent one made in 1996 itself involves the embedding of some black holes in string theory. It has been possible to identify the quantum states contributing to the black hole entropy, which naturally has the standard value \([1]\). However, these embeddings involve supersymmetry and are not quite universal. One would like to identify the relevant states in the original theory instead of the embedded one. It is to be hoped that the coming years will bring further progress.

Meanwhile, in this talk a more conventional framework will be used to discuss the entropy of what are known as extremal black holes. Over the past couple of years it has been unclear whether the so-called Bekenstein-Hawking formula is applicable to these black holes. There is no evidence that the originators of this formula supposed it to continue to hold for the special class; indeed, some of the arguments which were originally used to derive this formula indicate that the formula may fail in the extremal case. On the other hand several non-critical authors have used the formula loosely without pausing to think whether all black holes have to obey it. Fortunately, a clear picture does seem to have emerged on this issue now, as we hope to explain.

After summarizing the history of black hole entropy as it developed in the seventies, we first refer to a model which may be said to have indicated that extremal black holes are related to other black holes in a not particularly discontinuous way (this model came after the string theory developments mentioned above.) Thereafter we go back to the usual semiclassical approach to extremal black holes and recall the glaring indications of discontinuity in that approach. As we understand the situation now, this discontinuity arises in one way of quantization of the classical theory. An alternative way which leads to the Bekenstein-Hawking formula even for extremal black holes is reviewed in some detail.

2 Black hole entropy in the seventies

A precursor of the idea of entropy in the context of black holes was the so-called area theorem \([2]\). According to this theorem, the area of the horizon
of a system of black holes always increases in a class of spacetimes. The
asymmetry in time is built into the definition of this class: these spacetimes
are predictable from partial Cauchy hypersurfaces. This result is certainly
reminiscent of thermodynamical entropy.

Some other observations made around that time were collected together
into a set of laws of black hole mechanics analogous to the laws of thermo-
dynamics [3].

- The zeroth law states that the surface gravity $\kappa$ remains constant on
  the horizon of a black hole.

- The first law states that

$$\frac{\kappa dA}{8\pi} = dM - \phi dQ,$$

where $A$ represents the area of the horizon and $\phi$ the potential at the
horizon. For the Reissner-Nordström black hole, with horizons at

$$r_\pm = M \pm \sqrt{M^2 - Q^2},$$

$$\kappa = \frac{r_+ - r_-}{2r_+^2}, \quad \phi = Q/r_+, \quad A = 4\pi r_+^2.$$

- The second law is just the area theorem already stated.

When these observations were made, there was no obvious connection
with thermodynamics, it was only a matter of analogy. But it was soon
realized [4] that the existence of a horizon imposes a limitation on the amount
of information available and hence may lead to an entropy, which should then
be measured by the geometric quantity associated with the horizon, namely
its area. Thus, upto a factor, $A$ should represent the entropy and $\frac{\kappa}{8\pi}$ the
temperature.

This interpretation of the laws of black hole mechanics was not fully con-
vincing, and in any case the undetermined factor left a question mark. Fortu-
nately, the problem was solved very soon. It was discovered that quantum
theory causes dramatic changes in the behaviour of black hole spacetimes. A
scalar field theory in the background of a Schwarzschild black hole indicates the occurrence of radiation of particles \[\bar{\text{h}}\] at a temperature

\[T = \frac{\bar{\text{h}}}{8\pi M} = \frac{\hbar \kappa}{2\pi},\]  

(4)

This demonstrated the connection of the laws of black hole mechanics with thermodynamics and fixed the scale factor. It involves Planck’s constant and is a quantum effect.

For a Schwarzschild black hole, the first law of thermodynamics can be written as

\[T dS = dM\]  

(5)

and can be integrated, because of (4), to yield

\[S = \frac{4\pi M^2}{\hbar} = \frac{A}{4\hbar}.\]  

(6)

Although the expression for \(T\) given above is specific to the case of Schwarzschild black holes, the relation between the temperature and the surface gravity given in (4) is more generally valid in the case of black holes having \(g_{tt} \sim (1 - \frac{r}{M})\). The first law of black hole mechanics then becomes

\[Td \frac{A}{4\hbar} = dM - \phi dQ.\]  

(7)

Comparison with the first law of thermodynamics

\[TdS = dM - \tilde{\phi}dQ\]  

(8)

is not straightforward because the chemical potential \(\tilde{\phi}\) is not clearly known. However, one way of satisfying these two equations involves the identification

\[S = \frac{A}{4\hbar}, \quad \tilde{\phi} = \phi.\]  

(9)

In another approach, the grand partition function is used. For charged black holes \(\bar{\text{h}}\) it can be related to the classical action by

\[Z_{\text{grand}} = e^{-\frac{M-TS-\phi Q}{\bar{\text{h}}}} \approx e^{-T/\hbar},\]  

(10)
where the functional integral over all configurations consistent with the appropriate boundary conditions is semiclassically approximated by the exponential weight factor for the classical action $I$ of the black hole. This action (see below) is given by a quarter of the area of the horizon when the Euclidean time goes over one period, i.e., from zero to $\hbar/T$. Consequently,

$$M = T(S + \frac{A}{4\hbar}) + \phi Q.$$  \hspace{1cm} (11)

Now there is a standard formula named after Smarr [7],

$$M = \frac{\kappa A}{4\pi} + \phi Q,$$  \hspace{1cm} (12)

which can be rewritten as

$$M = T \frac{A}{2\hbar} + \phi Q.$$  \hspace{1cm} (13)

Comparison with (11) suggests once again the relations (9). Although the result is the same, it should be noted that there is a new input: the functional integral. There is a hope that corrections to the above formulas may be obtained by improving the approximation used in the calculation of the functional integral.

### 2.1 On-shell action

To see that the action equals a quarter of the area, let us consider a euclidean Reissner-Nordström black hole in a manifold $\mathcal{M}$ with a boundary which is subsequently taken to infinity. The action has the expression

$$I = -\frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{g}R + \frac{1}{8\pi} \int_{\partial \mathcal{M}} d^3x \sqrt{\gamma}(K - K_0) + \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{g}F_{\mu\nu}F^{\mu\nu}. \hspace{1cm} (14)$$

Here $\gamma$ is the induced metric on the boundary $\partial \mathcal{M}$ and $K$ the extrinsic curvature, from which a subtraction has to be made to make the action finite.

The first term of the action vanishes because Einstein’s equations lead to $R = 0$. 

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To evaluate the second term, we take the boundary of the manifold at \( r = r_B \to \infty \). Then

\[
K = -\frac{1}{\sqrt{g_{tt}r^2}} \frac{1}{\sqrt{g_{rr}}} \frac{d}{dr}(\sqrt{g_{tt}r^2}) = -\frac{1}{r^2 \frac{d}{dr}}[(1 - \frac{M}{r} + \cdots)r^2] = -\frac{1}{r^2 \frac{d}{dr}}(r^2 - Mr),
\]

and

\[
\int d^3x \sqrt{\gamma} = \int dt (1 - \frac{M}{r} + \cdots)4\pi r^2.
\]

We see that \( \int d^3x \sqrt{\gamma}K \) diverges as \( r \to \infty \), but this can be cured by subtracting from \( K \) the flat space contribution \( K_0 = -\frac{1}{r^2} \frac{d}{dr}r^2 \). The second piece of the action becomes

\[
-\frac{1}{8\pi} \int dt (1 - \frac{M}{r} + \cdots)4\pi r^2 \frac{1}{r^2} \frac{d}{dr}(-Mr)|_{r=r_B \to \infty} = -\frac{1}{2} \int dt (-M) = \frac{1}{2} \beta M.
\]

Finally, the third term of the action becomes

\[
-\frac{1}{16\pi} \int dt A_4 \int dr^2 2. \frac{Q^2}{r^4} = -\frac{1}{2} \int dt \frac{Q^2}{r_+} = -\frac{1}{2} \beta Q \phi,
\]

where \( \phi \) is the electrostatic potential at the horizon. The sign is negative here because in the euclidean solution the electric field is purely imaginary.

Putting all pieces of the action together, we find the numerical value of the action to be

\[
I = \frac{1}{2} \beta (M - Q \phi) = \frac{A}{4}.
\]

As indicated above, this leads to an entropy of the same value (in natural units).
3 Microscopic model for near-extremal black holes

It has recently been observed (cf. [8]) that a one-dimensional gas of massless particles can be used as a model for black holes in any number of dimensions. The particles can be either left-moving or right-moving – there is no mixing between the two types. Both bosons and fermions can be present. If the total length of the one-dimensional space is \( L \), the entropy and the energy are given by

\[
S = \frac{\pi L}{6\hbar}[n_L T_L + n_R T_R], \quad E = \frac{\pi L}{12\hbar}[n_L T_L^2 + n_R T_R^2],
\]

(20)

where \( n_L (n_R) \) is the number of left(right)-moving bosons plus half the corresponding number of fermions. In the absence of interactions, the left and right degrees of freedom are independent, and the corresponding temperatures can be different. The effective temperature may be defined by \( (\frac{dS}{dE})^{-1} \), the differentiation being carried out at constant momentum, \( i.e., \) constant difference between \( E_L \) and \( E_R \). This leads to a temperature

\[
T = \frac{2T_L T_R}{T_L + T_R}
\]

(21)

equal to the harmonic mean of \( T_L \) and \( T_R \). If \( n_L = n_R = n \), these equations get somewhat simplified and one has

\[
E = \frac{\pi n L}{12\hbar} \left( \frac{6\hbar S}{\pi n L} \right)^2 - \frac{6\hbar ST}{\pi n L},
\]

(22)

whence,

\[
\frac{12\hbar S}{\pi n L} = T + \sqrt{T^2 + \frac{48E}{\pi n L}}.
\]

(23)

To compare these quantities with those for a near-extremal charged black hole in four dimensions, we put

\[
E = E_0 + \epsilon, \quad T = T(M, Q) = T(Q + \epsilon, Q)
\]

(24)

to get

\[
S = \sqrt{\frac{\pi n L E_0}{3}} + \frac{n L}{24} \sqrt{\frac{2\epsilon}{Q^3}} + \epsilon \left[ \sqrt{\frac{\pi n L}{12 E_0}} - \frac{n L}{6Q^2} + \frac{(\pi n L)^{3/2}}{192\sqrt{3}\pi^2 Q^3\sqrt{E_0}} \right] + \cdots
\]

(25)
Comparison with the area formula

\[ S = \pi(Q^2 + 2Q\sqrt{2Q\epsilon} + 2Q\epsilon) + \cdots \]  

(26)

for the black hole shows that agreement occurs for a continuous range of values of \( \epsilon \) provided that

\[ nL = 48\pi Q^3, \quad E_0 = \frac{Q}{16}. \]  

(27)

The first equality here relates the number \( n \) to the parameter \( Q \) characterizing the family of black holes being considered; the second equality fixes a zero-point shift. If these conditions are satisfied, the gas of massless particles can be regarded as a model for the family of near-extremal black holes. The one-dimensional particles can be modes of a string. In this sense, the model may be embedded into string theory [8]. In any case, the model indicates the entropy of the black hole family to be continuous in the limit \( \epsilon \to 0 \).

### 4 Extremal black holes

In the recent past there has been special interest in extremal black holes. First it was pointed out [9] that the entanglement entropy, which is usually proportional to the area of a black hole, ceases to be so for extremal black holes. Thereafter, [10] noticed that euclidean topology is discontinuous in the passage from non-extremal to extremal black holes and argued that the entropy of extremal black holes might actually vanish. In [11] it was shown that this argument could be relaxed and an extremal black hole could be allowed to have an entropy proportional to the mass. Model calculations analogous to [8] developed formulations that are continuous in the limit of non-extremal black holes going into extremal ones. There is then something of a contradiction. Is there a discontinuity, or is there none?

#### 4.1 Vanishing Action

To see that the action vanishes in the extremal case, and is thus discontinuous, let us once again consider the euclidean Reissner - Nordström black hole in the manifold with boundary.
The first term of the action (14) vanishes again because Einstein’s equations lead to \( R = 0 \). As before, the second piece of the action is \( \frac{1}{2} \beta M \). The third term of the action is

\[
-\frac{1}{16\pi} \int dt.4\pi \int dr^2.2\frac{Q^2}{r^4}
= -\frac{1}{2} \int \frac{Q^2}{r_+} dr
= -\frac{1}{2} \beta M^2
= -\frac{1}{2} \beta M
\]  

as \( Q = M = r_+ \) here.

Putting all pieces of the action together, we find the value of the action to be

\[
I = \frac{1}{2} \beta (M - M) = 0.
\]  

(29)

In doing this calculation, \( \beta \) has been assumed finite. If the extremal limit of a non-extremal black hole is taken, this quantity actually goes to infinity. However, as pointed out in [10], there is no conical singularity in the extremal case, so that there is no reason to fix the temperature in this case, and the temperature should be regarded as arbitrary.

### 4.2 Entropy proportional to mass

The laws of black hole physics suggest that nonextremal black holes possess an entropy proportional to the area of the horizon. When the scale is fixed by comparing the temperature thus suggested with that given by the semiclassical calculations of [3], the entropy turns out to be a quarter of the area. If one is interested in an extremal black hole, one may be tempted to regard it as a special limiting case of a sequence of nonextremal black holes and thus infer that the same formula should hold for the entropy. But it was pointed out in the context of Reissner - Nordström black holes [10] that the extremal and nonextremal cases of the euclidean version are topologically different, so that continuity need not hold. Moreover, it was shown in [11] that the derivation of an expression for the thermodynamic entropy of an extremal
black hole following [6] allows an extra term proportional to the mass of the black hole. It will be instructive to elaborate a little on the discussion of the Reissner - Nordström black hole in [11].

For a charged black hole, the first law of thermodynamics

\[ T dS = dM - \Phi dQ, \]  

involves two “intensive” variables, viz. \( T \), the temperature, conjugate to \( M \), and \( \Phi \) the chemical potential, conjugate to \( Q \). We are interested in the extremal case \( Q = M \). Then there is only one independent thermodynamical variable, \( Q \) or \( M \), so the first law should involve only one conjugate variable and can be written as

\[ dS = \gamma dM. \]  

If this equation is sought to be understood in terms of the previous one, \( \gamma \) must be interpreted as \( \frac{1 - \Phi}{T} \) (see below).

To understand the meaning of \( \gamma \), one has to imagine a thermodynamic system of mass \( M \) and charge \( Q \) in contact with a reservoir of energy and charge such that exchanges of energy and charge with the system are always constrained to be equal. In this situation, the total change of entropy of the system and the reservoir is given by

\[ dS_{\text{tot}} = \gamma dM - \frac{dM}{T_{\text{reservoir}}} + \frac{\Phi_{\text{reservoir}} dM}{T_{\text{reservoir}}}. \]  

The condition for equilibrium is then

\[ \gamma = \frac{1 - \Phi_{\text{reservoir}}}{T_{\text{reservoir}}}. \]  

Thus, instead of the usual equality of temperatures and chemical potentials, there is only one condition, with \( \gamma \) equalling a certain combination of the temperature and the chemical potential of the reservoir. In other words, one cannot even talk separately of a temperature and a chemical potential for the system: there is only this combination \( \gamma \). Correspondingly, the ensemble is not quite grand canonical, but a reduced grand canonical one.

Indeed, the partition function also has to be written as

\[ Z = e^{S - \gamma M} = e^{-I}. \]  

10
Here, $I$ is the effective action, which is set equal to the classical on-shell action in the lowest approximation, and vanishes in the extremal case, as seen above. This implies

$$S = \gamma M.$$  \hspace{1cm} (35)

Comparison with the first law (31) then shows that

$$d\gamma = 0,$$  \hspace{1cm} (36)

so that $\gamma$ is a constant, hence the entropy is a constant times the mass. This constant may of course vanish, but that is a special case.

### 4.3 The area law again

While the possibility of microscopic models is interesting, the suggestion that the entropy is continuous in the extremal limit is intriguing in view of the developing belief that the area formula applies only to non-extremal black holes. It is true that the borderline between non-extremal and extremal cases is very thin and if one takes the extremal limit of non-extremal black holes instead of an extremal black hole directly, one obtains the area answer. But, as mentioned above, the euclidean topologies are different, so one should consider not the limit but the extreme black hole by itself; and then the semiclassical approach does not yield the area law. A simple way out of this mismatch would be to say that the model is wrong, but it would certainly be more positive to look for a way of obtaining the area answer directly for an extremal black hole.

Usually, when one quantizes a classical theory, one tries to preserve the classical topology. In this spirit, one usually seeks to have a quantum theory of extremal black holes based exclusively on extremal topologies. As an alternative, we shall try out a quantization where a sum over topologies is carried out. Thus, in our consideration of the functional integral, classical configurations corresponding to both topologies will be included. The extremality condition will then be imposed on the averages that result from the functional integration. We shall, following \[3\] and \[12\], use a grand canonical ensemble. Here the temperature and the chemical potential are supposed to be specified as inputs, and the average mass $M$ and charge $Q$ of the black hole are outputs. So the actual definition of extremality that we have in
mind for a Reissner-Nordström black hole is $Q = M$. This may be described as *extremalization after quantization*, as opposed to the usual approach of *quantization after extremalization*.

The action for the euclidean version of a Reissner-Nordström black hole on a four dimensional manifold $\mathcal{M}$ with a boundary has been given in (14). A class of spherically symmetric metrics [12] is considered on $\mathcal{M}$:

$$ds^2 = b^2 d\tau^2 + \alpha^2 dr^2 + r^2 d\Omega^2,$$

(37)

with the variable $r$ ranging between $r_+$ (the horizon) and $r_B$ (the boundary), and $b, \alpha$ functions of $r$ only. There are boundary conditions as usual:

$$b(r_+) = 0, \quad 2\pi b(r_B) = \beta.$$

(38)

Here $\beta$ is the inverse temperature and $r_B$ the radius of the boundary which will be taken to infinity. There is another boundary condition involving $b'(r_+)$: It reflects the extremal/non-extremal nature of the black hole and is therefore different for the two conditions:

$$\frac{b'(r_+)}{\alpha(r_+)} = 1 \ldots \text{in non-extremal case},$$

$$\text{but} \quad = 0 \ldots \text{in extremal case}.$$

(39)

The vector potential is taken to be zero and the scalar potential satisfies the boundary conditions

$$A_\tau(r_+) = 0, \quad A_\tau(r_B) = \frac{\beta \Phi}{2\pi i}.$$

(40)

The action (14) with this form of the metric depends on the functions $b(r), \alpha(r)$ and $A_\tau(r)$: this may be regarded as a reduced action. Variation of these functions with proper boundary conditions leads to reduced versions of the Einstein-Maxwell equations. The solution of a subset of these equations, namely the Gauss law and the Hamiltonian constraint, is given by [12]

$$\frac{1}{\alpha} = [1 - \frac{2m}{r} + \frac{q^2}{r^2}]^{1/2}, \quad A'_\tau = -\frac{iqb\alpha}{r^2},$$

(41)

with the mass parameter $m$ and the charge $q$ arbitrary. The reason why these parameters have not been expressed as functions of $\beta, \Phi$ is that some of the
equations of motion and the corresponding boundary conditions have not
been imposed on the solution. Instead of that, the action may be expressed
in terms of \( m, q \) and then extremized with respect to \( m, q \) \cite{12}.

The value of the action corresponding to the solution depends on the
boundary condition:

\[
I = \beta(m - q\Phi) - \pi(m + \sqrt{m^2 - q^2})^2 \text{ for non-extremal bc},
\]

\[
I = \beta(m - q\Phi) \text{ for extremal bc.} \tag{42}
\]

The first line is taken from \cite{12}, where the non-extremal boundary condition
was used in connection with a semiclassically quantized non-extremal black
hole. The second line corresponds to the extremal boundary condition used in
connection with a semiclassically quantized extremal black hole \cite{14}. As the
euclidean topologies of non-extremal and extremal black holes are different,
quantization was done separately for the two cases in \cite{12, 14}. The topology
was selected before quantization.

As indicated above, a different approach has to be used here. The two
topologies are to be summed over in the functional integral \cite{13} and the
extremality condition imposed afterwards.

Thus the partition function is of the form

\[
\sum_{\text{topologies}} \int d\mu(m) \int d\mu(q) e^{-I(q,m)}, \tag{43}
\]

with \( I \) given by (42) as appropriate for non-extremal/extremal \( q \).

The semiclassical approximation involves replacing the double integral by
the maximum value of the integrand, \( i.e., \) by the exponential of the negative
of the minimum \( I \). We consider the variation of \( I \) as \( q, m \) vary in both
topologies. It is clear from \cite{12} that the non-extremal action is lower than
the extremal one for each set of values of \( q, m \). Consequently, the partition
function is to be approximated by \( e^{-I_{\min}} \), where \( I_{\min} \) is the classical action
for the \textit{non-extremal} case, \textit{minimized} with respect to \( q, m \). The result, which
should be a function of \( \beta, \Phi \), can be read off \cite{12}. It leads to an entropy
equal to a quarter of the area for all values of \( \beta, \Phi \). The averages \( Q, M \), as
opposed to the parameters \( q, m \), are calculated from \( \beta, \Phi \). We are interested
in \( |Q| = M, i.e., \) the extremal black hole. This is obtained for limiting values

\[
\beta \rightarrow \infty, \ |\Phi| \rightarrow 1, \ \text{with} \ \beta(1 - |\Phi|) \ \text{finite} \tag{44}
\]
for the ensemble parameters and is described by the effective action
\[ I = \pi M^2 = \frac{(\beta(1 - |\Phi|))^2}{4\pi}. \] (45)

It is worth emphasizing again that for extremal black holes, the parameters \( \beta, \Phi \) necessarily enter in the combination \( \gamma \equiv \beta(1 - |\Phi|) \). This combination does occur here as it also does in the case with purely extremal topology [14]. Thus in the limit the partition function takes the form
\[ Z = e^{-\frac{\gamma^2}{4\pi}} = e^{-\pi M^2} = e^{-\frac{A}{4}}. \] (46)

This continues to correspond to an entropy of a quarter of the area of the horizon, which is the value of the entropy consistent with the microscopic model.

To reach this goal, we defined extremality not by equating the classical parameters \( q, m \) before quantization, but in terms of the averages \( Q, M \) which are calculated from the ensemble characteristics \( \beta, \Phi \) and which reduce to \( q, m \) for the configuration with the minimum action in the semiclassical approximation. It is because of this altered definition, and the use of the sum over topologies, that non-extremal configurations have entered and we have obtained the area law for the entropy instead of the smaller values obtained in [10, 11]. This suggests that the microscopic model discussed above implicitly involves a quantization procedure where the classical euclidean topology is ignored and the condition of extremality imposed only after quantization.

It may be clarified here that this need not be the only correct way of quantization. In other areas of physics, there are different, often inequivalent, ways of quantization, many of them equally acceptable. The results referred to in the previous section correspond to quantization with fixed euclidean topology, while the new models agree with, but do not explicitly involve, a sum over topologies.

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