An Integral Kernel
for Weakly Pseudoconvex Domains

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ABSTRACT

A new explicit construction of Cauchy-Fantappié kernels is introduced for an arbitrary weakly pseudoconvex domain with smooth boundary. While not holomorphic in the parameter, the new kernel reflects the complex geometry and the Levi form of the boundary. Some estimates are obtained for the corresponding integral operator, which provide evidence that this kernel and related constructions give useful new tools for complex analysis on this general class of domains.

1 Introduction

The well-known Bochner-Martinelli kernel $K_{BM}(\zeta, z)$ is a natural generalization of the familiar Cauchy kernel to higher dimensions. It leads to a corresponding integral representation formula for holomorphic functions on arbitrary smoothly bounded domains in $\mathbb{C}^n$. However, it lacks critical properties, and this limits its applicability. For once, it is not holomorphic in the parameter $z$ when $n \geq 2$. Furthermore, its singularity is isotropic, and thus it does not at all reflect the non-isotropic complex geometry of boundaries of domains in higher dimension.

In the late 1960s G. Henkin and E. Ramirez, independently, introduced new integral representation formulas on strictly pseudoconvex domains which overcame the shortcomings of the Bochner-Martinelli kernel. These new tools rapidly led to proofs of numerous results concerning the boundary behavior of holomorphic functions and related objects on strictly pseudoconvex domains. (See Ra86 for a systematic exposition.) In particular, they allowed to prove pointwise estimates for solutions of the Cauchy-Riemann equations, such as estimates in supremum norm and in Hölder norms, which were not accessible by the classical $L^2$ - methods of J. J. Kohn, L. Hörmander, and others. Note that in

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Every smoothly bounded domain is trivially strictly pseudoconvex. Such domains therefore provide a natural setting for generalizing results known in dimension one to higher dimensions.

Of course, when \( n \geq 2 \), not every domain of holomorphy with smooth boundary is strictly pseudoconvex. Such more general domains are just (weakly) pseudoconvex, i.e., the Levi form associated to the boundary is only positive semi-definite rather than positive definite as in the strict case. This more general case has been investigated for well over 40 years, and it continues to present major challenges. As conjectured by J. J. Kohn, the right notion of finite type (see Ko72, Ko79, Da82) turned out to be central in the \( L^2 \) theory of the \( \overline{\partial} \)-Neumann problem, providing necessary and sufficient conditions for the existence of subelliptic estimates (Ca87). However, attempts to generalize kernel methods to this setting and obtain, for example, pointwise estimates for solutions of the \( \overline{\partial} \)-equation, have had only limited success (see Ra78, Ra90, Cu97, DF99, DFF99). In fact, there is a fundamental obstruction to extending these methods to this setting, as follows. The Henkin/Ramirez construction made essential use of an explicit holomorphic support function, which in the strictly pseudoconvex case is given locally by the quadratic Levi polynomial. Holomorphic support functions exist also on Euclidean convex domains, but as shown by an example of Kohn and L. Nirenberg [KoNi72], they do not exist in general for pseudoconvex domains of finite type. Consequently the integral kernel methods seemed to have reached their limit, and a more complete understanding of pointwise estimates in the theory of the \( \overline{\partial} \)-Neumann problem in arbitrary dimensions has remained elusive for quite some time.

In this paper we introduce a non-holomorphic local modification \( \Phi(\zeta, z) \) of the Levi polynomial for an arbitrary weakly pseudoconvex domain \( D \subset \subset \mathbb{C}^n \), and a corresponding global kernel generating form

\[
W^S(\zeta, z) = \frac{\sum_{j=1}^n s_j(\zeta, z) d\zeta_j}{S(\zeta, z)} \quad \text{on } bD \times D,
\]

with \( S(\zeta, z) = \Phi(\zeta, z) \) for \( z \) close to \( \zeta \), and where the coefficients \( s_j \) satisfy \( S(\zeta, z) = \sum_{j=1}^n s_j(\zeta_j - z_j) \). While \( \Phi \) and \( W^S \) are not holomorphic in \( z \), they satisfy critical basic properties which open the door to significant applications and new results. In particular, we shall prove the following properties.

a) The form \( \overline{\partial} \Phi(\zeta, z) \) has a zero at \( z = \zeta \) whose order is carefully controlled.

b) \( \Phi \) satisfies precise uniform estimates from below somewhat weaker than those familiar in the strictly pseudoconvex case, and which involve explicitly the eigenvalues of the Levi form of a defining function \( r \) for \( D \).

c) In case the domain is strictly pseudoconvex, \( \Phi \) and \( W^S \)—while not holomorphic—satisfy the classical estimations known in that case.

d) \( \Phi \) satisfies the same symmetry properties that have been successfully used on strictly pseudoconvex domains in earlier work.

The resulting Cauchy-Fantappié kernel \( \Omega_0(W^S) = (2\pi)^{-n} W^S \wedge (\overline{\partial}_\zeta W^S)^{n-1} \) yields a Cauchy-type integral formula for holomorphic functions on a weakly
pseudoconvex domain $D$ which reflects the complex geometry of the boundary. In particular, it has a singularity of order one in the complex normal direction (just like the standard one dimensional Cauchy kernel), and a singularity of order two in each of the complex tangential directions. Furthermore, it will be shown that $\partial_t \Omega_0(W^S)$ is in some sense more regular than the corresponding form for the Bochner-Martinelli kernel. These features suggest that $\Omega_0(W^S)$ might be a useful new tool for complex analysis on weakly pseudoconvex domains. For example, the author has used $\Omega_0(W^S)$ and its higher order versions $\Omega_q(W^S)$, $0 \leq q \leq n$, in the theory of the $\overline{\partial}$–Neumann problem. A major new result he obtained is a pointwise analogon of the classical basic $L^2$ estimate of Morrey and Kohn (see FoKo72). Furthermore, some additional results suggest that it might be possible to develop a suitable version of Kohn’s theory of subelliptic multipliers (see Ko79, Siu10) in the integral representation setting involving the kernels $\Omega_q(W^S)$ and their variants. Such techniques might then allow to prove suitable Hölder estimates on pseudoconvex domains of finite type. A preliminary report discussing these applications has been published on arXiv (Ra11).

2 Local construction of the support function

We assume that $D$ is a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^k$ boundary $bD$ ($k \geq 3$), and we choose a $C^k$ defining function $\varphi$ for $bD$ defined on a neighborhood $U = U(bD)$ of $bD$. In general the level surfaces $M_{\varphi} = \{z : \varphi(z) = -\delta\}$ will not be Levi pseudoconvex for $\delta > 0$.

Proposition 1 There exists $C > 0$ and $U$, such that for all $\zeta \in \overline{D} \cap U$ the Levi form $L$ of the defining function $r(z) = \varphi(z) \exp(-C|z|^2)$ satisfies

$$L(r, \zeta; t) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k} \overline{t_j t_k} \geq 0 \text{ for all } t \in \mathbb{C}^n \text{ with } \sum_{j=1}^{n} \frac{\partial r}{\partial \zeta_j}(\zeta) t_j = 0.$$

As the proof will show, the level surfaces $M_{r(\zeta)}$ of $r$ are actually strictly pseudoconvex for $\zeta \in D \cap U$, but the resulting estimates are not uniform in $\zeta$ as $r(\zeta) \to 0$, unless $bD$ is strictly pseudoconvex to begin with.

Proof. By Theorem 2 in [Ra81], there exist $C > 0$ and $0 < \eta < 1$ such that $\rho = -(-\varphi \exp(-C|z|^2))^\eta$ is strictly plurisubharmonic on $U \cap D$ for $U$ sufficiently small. In particular, the level surfaces of $\rho$ close to $bD$ are strictly pseudoconvex. Let $r = \varphi \exp(-C|z|^2)$. Since $\rho(z) = -\delta^\eta$ if and only if $r(z) = -\delta$, it follows that the level surfaces $M_{-\delta}$ of $r$ are (strictly) pseudoconvex as well for $\delta$ sufficiently small, and the desired result follows.

We now fix this particular global defining function $r$. After shrinking $U$, we may assume that for a fixed $k \geq 3$ the function $r$ has a bounded $C^k$ norm $|r|_k$ over $U$.

For $\zeta \in U(bD)$ let

$$F^{(r)}(\zeta, z) = \sum_{j} \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{j,k} \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k)$$
be the usual Levi polynomial of \( r \), which is a quadratic holomorphic polynomial in \( z \). (See Ra86) The following well known equation is a direct consequence of the 2nd order Taylor expansion of \( r(z) \) at \( \zeta \).

\[
2 \Re \left[ F^{(r)}(\zeta, z) - r(\zeta) \right] = -r(\zeta) - r(z) + \mathcal{L}(r; \zeta; \zeta - z) + O(|\zeta - z|^3).
\]

We now define

\[
\Phi_K(\zeta, z) = F^{(r)}(\zeta, z) - r(\zeta) + K|\zeta - z|^3,
\]

where \( K > 0 \) is a large constant to be suitably chosen later on.

While \(|\zeta - z|^2\) is smooth in \((\zeta, z)\), the term \(|\zeta - z|^3\) which appears in \( \Phi_K \) is of class \( C^2 \) in general and smooth only at points \((\zeta, z)\) with \( \zeta \neq z \). For \( j = 0, 1, 2, \ldots \) we denote by \( \mathcal{E}_j = \mathcal{E}_j(\zeta, z) \) a smooth function or form which satisfies \(|\mathcal{E}_j| \leq \text{const.} |\zeta - z|^l\) for a constant that is independent of \( \zeta \) and \( z \); similarly, we use \( \mathcal{E}^\#_j \) to denote bounded functions or forms which are smooth for \( \zeta, z \) with \( \zeta \neq z \), and which satisfy an estimate \(|\mathcal{E}^\#_j| \leq \text{const.} |\zeta - z|^l\). Then \(|\zeta - z|^3 = \mathcal{E}^\#_3\), and one readily verifies that if \( D^l \) is a partial derivative of order \( l \) with respect to \( \zeta \) and/or \( z \), then \( D^l |\zeta - z|^3 = \mathcal{E}^\#_{3-l} \) for \( l = 1, 2, 3 \). We point out that the precise expressions of both \( \mathcal{E}_j \) and \( \mathcal{E}^\#_j \) may differ from formula to formula, and even within the same formula. The emphasis is on keeping track of the order of vanishing of terms, not of their exact expressions. This will become most relevant in section 5.

Since \( F^{(r)} - r(\zeta) \) is holomorphic in \( z \), it follows that \( \overline{\partial}_z \Phi_K(\zeta, z) = \mathcal{E}^\#_2 \). For the other derivatives, fix \( P \in b\mathcal{D} \) and introduce a \( C^{k-1} \) orthonormal frame \( \{\omega_1, \omega_2, \ldots, \omega_n\} \) for \((1, 0)\) forms on a sufficiently small neighborhood \( V(P) \), with \( \omega_n = \nu \partial r \) for some function \( \nu(\zeta) > 0 \) on \( V \), so that \( ||\omega_n|| = 1 \) on \( V \). Let \( \{L_1, \ldots, L_n\} \) be the corresponding dual frame for \((1, 0)\) vector fields on \( V \). Then \( L_j(r) = 0 \) for \( j < n \), and \( L_n = \gamma(\zeta) \sum_k \partial r / \partial \zeta_k \partial / \partial \zeta_k \) for some \( \gamma(\zeta) > 0 \). The vector fields \( L_j \) act in \( \zeta \); we use the notation \( L_{j, z} \) and \( \overline{\partial}_{j, z} \) if differentiation is taken with respect to \( z \).

**Proposition 2** The following estimates hold for the derivatives of \( \Phi_K \):

i) \( \overline{\partial}_{j, z} \Phi_K(\zeta, z) = \mathcal{E}^\#_2 \) for \( j = 1, \ldots, n \);

ii) \( L_{j, z} \Phi_K(\zeta, z) = \mathcal{E}^\#_j \) for \( j < n \);

iii) \( L_{n, z} \Phi_K(\zeta, z) \neq 0 \).

**Proof.** We already noted i). For ii), note that if \( L_j = \sum_k \omega_j(\zeta) \partial / \partial \zeta_k \), then \( L_j(z) F^{(r)}(z) - r(\zeta) = -\sum_k \omega_j(\zeta) \partial r / \partial \zeta_k + \mathcal{E}_1 = \mathcal{E}_1 \) for \( j < n \). This implies ii). Finally, since \( L_{n, z} F^{(r)}(\zeta, z) = -\gamma(\zeta) \sum_k |\partial r / \partial \zeta_k|^2 + \mathcal{E}_1 \), iii) follows as well.

One also has the following approximate symmetry, which follows directly from the known result in case \( K = 0 \) (see Ra 86).

\[
\Phi_K(\zeta, z) - \overline{\Phi}_K(z, \zeta) = \mathcal{E}_3.
\]
3 Estimations for the support function

Next we prove that $|\Phi_K(\zeta, z)|$ is precisely controlled from below, as follows. We use the convention that $A \gg B$ means that there exists a constant $c > 0$, so that $|A| \geq c|B|$ for all the values of the relevant variables under consideration.

For $\zeta \in U$ consider the level surface $M_{r(\zeta)}$ of $r$ through the point $\zeta$. We introduce the orthogonal projection $\pi_\zeta^t : \mathbb{C}^n \to T^1,0_r(M_{r(\zeta)}) \subset \mathbb{C}^n$, where $T^1,0_r(\mathbb{C}^n)$ is identified with $\mathbb{C}^n$ via the standard basis $\{\frac{\partial}{\partial \zeta_1}, \ldots, \frac{\partial}{\partial \zeta_n}\}$.

**Theorem 3** The neighborhood $U$, the constant $K$, and $\varepsilon > 0$ can be chosen so that for all $\zeta, z \in \overline{D} \cap U$ with $|\zeta - z| < \varepsilon$ one has

$$|\Phi_K(\zeta, z)| \geq \left[\left|\text{Im} F^{(r)}(\zeta, z)\right| + |r(\zeta)| + |r(z)| + \mathcal{L}(r, \zeta; \pi_\zeta^t(\zeta - z)) + K|\zeta - z|^3\right]$$

Note that by pseudoconvexity and by the special choice of the defining function $r$ one has

$$\mathcal{L}(r, \zeta; \pi_\zeta^t(\zeta - z)) \geq 0 \text{ for all } \zeta \in \overline{D} \cap U,$$

so all terms on the right side in the estimation of $|\Phi_K(\zeta, z)|$ are nonnegative!

As in the familiar strictly pseudoconvex case, $r(\zeta)$ and $\text{Im} F^{(r)}(\zeta, z)$ can be used as coordinates in a $C^{k-2}$ real coordinate system in a neighborhood $B(z, \delta)$ of a fixed point $z \in U$ provided $\delta > 0$ is sufficiently small. The crux of the estimate in the Theorem is that $\Phi_K$ is of order 1 in the complex normal direction, while the Levi form completely controls $\Phi_K$ from below in the complex tangential directions.

**Proof.** Decompose $\zeta - z = \pi_\zeta^t(\zeta - z) + \pi_n^t(\zeta - z)$, and notice that $|\pi_n^t(\zeta - z)| = O(|<\partial r(\zeta), \zeta - z>|)$, where $<\partial r(\zeta), \zeta - z> = \sum_j \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j)$.

Since

$$<\partial r(\zeta), \zeta - z> = \Phi_K + r(\zeta) + \mathcal{E}_2 - K|\zeta - z|^3,$$

it follows that

$$\mathcal{L}(r, \zeta; \zeta - z) = \mathcal{L}(r, \zeta; \pi_\zeta^t(\zeta - z)) + \mathcal{E}_1 |<\partial r(\zeta), \zeta - z>| \geq \mathcal{L}(r, \zeta; \pi_\zeta^t(\zeta - z)) - \mathcal{E}_1|\Phi_K| + |r(\zeta)| + K|\zeta - z|^3 - \mathcal{E}_3.$$ 

We choose $\varepsilon > 0$ so small that the above $\mathcal{E}_1$ term satisfies $|\mathcal{E}_1| < 1/2$ for $|\zeta - z| < \varepsilon$. It follows that for $\zeta, z \in \overline{D} \cap U$ with $|\zeta - z| < \varepsilon$ one has

$$2 \text{Re } \Phi_K = 2 \text{Re } \left[F^{(r)}(\zeta, z) - r(\zeta)\right] + 2K|\zeta - z|^3 \geq |r(\zeta)| + |r(z)| + \mathcal{L}(r, \zeta; \pi_\zeta^t(\zeta - z)) + \frac{3}{2}K|\zeta - z|^3 +$$

$$-1/2(\Phi_K + |r(\zeta)|) - A|\zeta - z|^3.$$ 

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for a certain constant $A > 0$. Now choose $K \geq 2A$; it then follows that

$$2 \Re \Phi_K \geq |r(\zeta)| + |r(z)| + \mathcal{L}(r, \zeta; \pi^t_\zeta(\zeta - z)) + K |\zeta - z|^3 + \frac{1}{2}(|\Phi_K| + |r(\zeta)|).$$

After rearranging, it follows that for $\zeta, z \in \overline{\mathcal{D}} \cap \mathcal{U}$ with $|\zeta - z| < \varepsilon$ one has

$$2 \Re \Phi_K + |\Phi_K|/2 \geq |r(\zeta)|/2 + |r(z)| + \mathcal{L}(r, \zeta; \pi^t_\zeta(\zeta - z)) + K |\zeta - z|^3.$$

Since $|\Phi_K| \geq 2 |\Re \Phi_K| + 2 |\Im \Phi_K| \geq 2 \Re \Phi_K + 2 |\Im F^r|$, the preceding inequality readily gives the estimate stated in the Theorem.

**Remark.** If $bD$ is strictly pseudoconvex, then

$$L(r; \zeta; \pi^t_\zeta(\zeta - z)) \geq c |\pi^t_\zeta(\zeta - z)|^2$$

for some $c > 0$.

Note that

$$|\zeta - z|^2 \leq |\pi^t_\zeta(\zeta - z)|^2 + \mathcal{E}_1 |< \partial r(\zeta), \zeta - z>|$$

By estimating $|< \partial r(\zeta), \zeta - z>|$ as before, one obtains

$$|\pi^t_\zeta(\zeta - z)|^2 \geq |\zeta - z|^2 - A_1 |\zeta - z|(|\Phi_K| + |r(\zeta)|) + K |\zeta - z|^3 - A_2 |\zeta - z|^3,$$

where $A_1, A_2$ are certain positive constants. Given $K \geq 0$, one can then choose $\varepsilon = \varepsilon(K)$ so small that

$$|\pi^t_\zeta(\zeta - z)|^2 \geq |\zeta - z|^2 - 1/2(|\Phi_K| + |r(\zeta)|) \text{ for } |\zeta - z| < \varepsilon.$$

This readily implies the stronger estimate

$$|\Phi_K(\zeta, z)| \gtrsim \left| \operatorname{Im} F^r(\zeta, z) \right| + |r(\zeta)| + |r(z)| + c |\zeta - z|^2$$

for any $K \geq 0$ and for $\zeta, z \in \overline{D} \cap U$ with $|\zeta - z| \leq \varepsilon$, provided $\varepsilon$ is chosen sufficiently small. This includes the familiar estimate for the Levi polynomial in the strictly pseudoconvex case, which in the literature has usually been obtained for a strictly plurisubharmonic defining function. The proof here gives the result for an arbitrary defining function.

Most importantly, the third order correction term allows to prove the following more delicate estimate which is critical for the estimations of Cauchy-Fantappié kernels involving the support function $\Phi_K$.

**Proposition 4** Fix $z \in \overline{D} \cap U$ and let $\lambda_1(z), ..., \lambda_{n-1}(z)$ be the eigenvalues of the Levi form of the chosen defining function $r$ at the point $z$. There exists a unitary change of coordinates in the $\zeta$ variables (in dependence of $z$), so that in the new coordinates one has the estimate

$$|\Phi_K(\zeta, z)| \gtrsim \left| \operatorname{Im} F^r(\zeta, z) \right| + |r(\zeta)| + |r(z)| + \sum_{j=1}^{n-1} \lambda_j(z) |\zeta_j - z_j|^2 + K/2 |\zeta - z|^3.$$
for \( \zeta, z \in \overline{D} \cap U \) with \( |\zeta - z| < \delta \), provided \( K \) is sufficiently large and \( \delta \) is sufficiently small. All relevant constants can be chosen to be independent of \( \zeta \) and \( z \in \overline{D} \cap U \).

**Proof.** Fix \( z \), and choose the orthonormal frame \( L_1, \ldots, L_n \) so that \( L_1, \ldots, L_{n-1} \) form an orthonormal basis for \( T^1,0_{\zeta}(M_r(\zeta)) \) for \( |\zeta - z| < \delta \leq \varepsilon \) which diagonalizes the Levi form restricted to \( T^1,0_{\zeta}(M_r(z)) \) at the point \( \zeta = z \). Note that this is a condition at the single point \( \zeta = z \); in general, there is no smooth frame which diagonalizes the Levi form in a neighborhood of \( z \). By pseudoconvexity and the choice of the defining function the eigenvalues \( \lambda_j(z), j = 1, \ldots, n-1 \), are nonnegative. After a unitary change of coordinates in \( \zeta_1, \ldots, \zeta_n \), one can assume that \( L_j|_z = \sqrt{2} \frac{\partial}{\partial \zeta_j}|_z \), and hence \( L_j|_\zeta = \sqrt{2} \frac{\partial}{\partial \zeta_j}|_\zeta + V_j \), where the coefficients of \( V_j \) are of type \( E_1 \). We call such coordinates \( z \)-diagonalizing. It then follows that with respect to these particular coordinates one has

\[
\mathcal{L}(r, z; \pi^i(\zeta - z)) = \sum_{j=1}^{n-1} \lambda_j(z) |\zeta_j - z_j|^2.
\]

Since the coefficients of the Levi form are smooth in \( \zeta \), one obtains

\[
\mathcal{L}(r, \zeta; \pi^i(\zeta - z)) = \sum_{j=1}^{n-1} \lambda_j(z) |\zeta_j - z_j|^2 + \mathcal{R}(\zeta, z),
\]

where the error term \( \mathcal{R}(\zeta, z) \) is of type \( E_3 \). Now choose \( K \) in the definition of \( \Phi_K \) so large that this error term satisfies \( |\mathcal{R}(\zeta, z)| \leq \frac{K}{2} |\zeta - z|^3 \). The desired estimate then follows from the estimate in Theorem 3.
for a (1, 1)-form $\Omega$ with $\mathcal{E}_1$ coefficients. The proposition then follows by wedging the last equation with $\partial r \wedge \overline{\partial r} = \gamma(\zeta) \omega_n \wedge \overline{\omega}_n$. 

**Corollary 6** With $z \in \overline{D} \cap U$ fixed and $\zeta$ the corresponding $z$-diagonalizing coordinates as above, one has the following representation for $\zeta \in \overline{D} \cap U$ and $0 < |\zeta - z| < \varepsilon$:

$$
\frac{\partial r(\zeta) \wedge \overline{\partial r}(\zeta) \wedge \overline{\partial r}(\zeta)}{\Phi_K(\zeta, z)} = \omega_n \wedge \overline{\omega}_n \wedge \left[ \sum_{j=1}^{n-1} A_j d\zeta_j \wedge \overline{d\zeta_j} + \sum_{j,l} B_{jl} d\zeta_j \wedge \overline{d\zeta_l} \right],
$$

where

$$
|A_j(\zeta, z)| \lesssim \frac{1}{|\text{Im} F^{(r)}(\zeta, z)| + |r(\zeta)| + |r(z)| + |\zeta_j - z_j|^2 + \frac{C}{2} |\zeta - z|^3}
$$

and

$$
|B_{jl}(\zeta, z)| \lesssim \frac{1}{|\text{Im} F^{(r)}(\zeta, z)| + |r(\zeta)| + |r(z)| + \frac{C}{2} |\zeta - z|^2}.
$$

**Proof.** From the preceding propositions one obtains

$$
\frac{\partial r \wedge \overline{\partial r} \wedge \overline{\partial r}(\zeta)}{\Phi_K(\zeta, z)} = \partial r \wedge \overline{\partial r} \wedge \left[ \frac{1/2 \sum_{j=1}^{n-1} \lambda_j(z) d\zeta_j \wedge \overline{d\zeta_j}}{\Phi_K(\zeta, z)} + \frac{\Omega_1}{\Phi_K(\zeta, z)} \right].
$$

By using the estimate for $|\Phi_K|$ in Theorem 3 and after cancelling $|\zeta - z|$, one readily sees that the coefficients $B_{jl}$ of the form $\Omega_1/\Phi_K$ satisfy the required estimate. For the leading terms, estimate

$$
\left| \frac{\lambda_j(z)}{\Phi_K(\zeta, z)} \right| \lesssim \frac{\lambda_j(z)}{|\text{Im} F^{(r)}(\zeta, z)| + |r(\zeta)| + |r(z)| + \lambda_j(z) |\zeta_j - z_j|^2 + \frac{C}{2} |\zeta - z|^3}.
$$

Observe that there exists a constant $C > 0$ independent of $z$, such that $0 \leq \lambda_j(z) \leq C$ for $j = 1, \ldots, n - 1$. Thus, if $\lambda_j(z) > 0$, one has $1/\lambda_j(z) \geq 1/C > 0$, and one may cancel the factor $\lambda_j(z)$ from numerator and denominator to obtain (with a new constant in $\lesssim$)

$$
\left| \frac{\lambda_j(z)}{\Phi_K(\zeta, z)} \right| \lesssim \frac{1}{|\text{Im} F^{(r)}(\zeta, z)| + |r(\zeta)| + |r(z)| + |\zeta_j - z_j|^2 + \frac{C}{2} |\zeta - z|^3}.
$$

Trivially this estimate holds also $\lambda_j(z) = 0$. 

**Remark.** The special $z$-diagonalizing coordinates in dependence of $z$ introduced above are used only for the estimations in the preceding propositions and corollary, and for relevant estimates in section 5. Unless something different is explicitly mentioned, it is assumed that in all other places the coordinates are the fixed standard coordinates of $\mathbb{C}^n$ introduced at the beginning. In particular, the function $\Phi_K$ is defined with respect to these standard coordinates of $\mathbb{C}^n$, which will continue to be used in the following sections.
4 Factorization and the generating form

Following standard procedure, in order to build a kernel generating form on $bD \times (D \cap U)$ from the support function $\Phi_K$, one needs a decomposition

$$\Phi_K(\zeta, z) = \sum_{j=1}^{n} g_j(\zeta, z)(\zeta_j - z_j).$$

Note that for $\zeta \in bD$ one has $\Phi_K(\zeta, z) = \Phi_K(\zeta, z) + r(\zeta) = F^R(\zeta, z) + K|\zeta - z|^3$. Clearly there is such a decomposition for the Levi polynomial $F^R$, with the corresponding coefficients holomorphic in $z$.

Since

$$|\zeta - z|^3 = \sum_{j=1}^{n} |\zeta - z|(\zeta_j - z_j)(\zeta_j - z_j),$$

there is a decomposition for $\Phi_K + r(\zeta)$ with $g_j$ equal to a sum of a holomorphic term and a term of type $E^#_1$. This implies the following lemma.

**Lemma 7** There is a decomposition

$$\Phi_K(\zeta, z) + r(\zeta) = \sum_{j=1}^{n} g_j(\zeta, z)(\zeta_j - z_j) \text{ for } \zeta \in U,$$

where $\overline{\partial}_z g_j$ is of type $E^#_1$ for $j = 1, \ldots, n$.

We now define the $(1, 0)$ form $g = \sum_{j=1}^{n} g_j d\zeta_j$ and set

$$W^K(\zeta, z) = \frac{g}{\Phi_K(\zeta, z)}$$

for $(\zeta, z) \in (\overline{D} \cap U) \times (\overline{D} \cap U)$ with $0 < |\zeta - z| < \varepsilon$. Note that the $(1, 0)$ form $W^K(\zeta, z)$ has smooth coefficients for $\zeta \neq z$ and that it satisfies

$$< W^K(\zeta, z), \zeta - z > = \sum_{j=1}^{n} \frac{g_j}{\Phi_K}(\zeta_j - z_j) = 1$$

for $\zeta \in bD$ and $0 < |\zeta - z| < \varepsilon$; it therefore is a (local) generating form in the terminology of Ra86.

In order to globalize $\Phi_K$ and $W^K$ in $z$, we patch with the corresponding terms from the Bochner-Martinelli kernel, as was done in Ra86 in case of strictly pseudoconvex domains. We choose a $C^\infty$ function $\chi(t)$ such that $0 \leq \chi(t) \leq 1$, $\chi(t) = 1$ for $t \leq \varepsilon/2$, and $\chi(t) = 0$ for $t \geq 3/4 \varepsilon$. Define $S(\zeta, z)$ on $U \times \overline{D}$ by

$$S(\zeta, z) = \chi(|\zeta - z|)\Phi_K(\zeta, z) + [1 - \chi(|\zeta - z|)]|\zeta - z|^2.$$

We need to ensure that $S$ does not have any new zeroes.
Lemma 8 $S(\zeta, z) \neq 0$ for $\zeta \in \overline{D} \cap U$ with $|r(\zeta)| \leq \varepsilon$ and all $z \in \overline{D}$ with $z \neq \zeta$.

Proof. Since

$$|S| \geq |\Re S| + |\Im S| \geq \Re S + |\Im S| \geq \chi \Re \Phi_K(\zeta, z) + (1 - \chi) |\zeta - z|^2,$$

and since $\chi(|\zeta - z|) \equiv 0$ for $|\zeta - z| \geq 3/4 \varepsilon$, we can estimate the first term by utilizing the estimate for $\Re \Phi_K(\zeta, z)$ from section 3 to obtain

$$|S| \geq \chi(|r(\zeta)|/2 + |r(z)| + \mathcal{L}(\varepsilon, \zeta; \pi^t \Phi_K) + K |\zeta - z|^3 - \chi |\Phi_K|/2 + (1 - \chi) |\zeta - z|^2. $$

By estimating $\chi |\Phi_K| \leq |S| + (1 - \chi) |\zeta - z|^2$ it then follows that

$$|S| \geq \chi K |\zeta - z|^3 + 1/2 |S| + (1 - \chi) |\zeta - z|^2 + (1 - \chi) |\zeta - z|^2.$$ 

Therefore

$$|S| \geq \chi K |\zeta - z|^3 + 1/2(1 - \chi) |\zeta - z|^2 \geq |\zeta - z|^3,$$

and the lemma is proved.

For $(\zeta, z) \in bD \times \overline{D}$ one has the decomposition

$$S(\zeta, z) = \sum_{j=1}^{n} s_j(\zeta, z)(\zeta_j - z_j),$$

with $s_j(\zeta, z) = \chi(|\zeta - z|)g_j(\zeta, z) + (1 - \chi)(\zeta_j - z_j)$. Finally, by introducing the (1, 0) form $s = \sum_j s_j(\zeta, z) d\zeta_j$, one obtains the global generating form

$$W^S(\zeta, z) = \frac{s(\zeta, z)}{S(\zeta, z)}$$

on $bD \times \overline{D} - \{(\zeta, \zeta) : \zeta \in bD\}$. Note that for $\zeta \in bD$ and $0 < |\zeta - z| \leq \varepsilon/2$ one has $W^S(\zeta, z) = g/\Phi_K$.

We now introduce the corresponding Cauchy-Fantappié kernel (of order 0)

$$\Omega_0(W^S) = (2\pi i)^{-n} W^S \wedge (\overline{\partial \zeta} W^S)^{n-1}$$

on $bD \times \overline{D} - \{(\zeta, \zeta) : \zeta \in bD\}$.

We recall the following standard properties of Cauchy-Fantappié kernels, that is, for their pull-backs to $bD$:

a) $\overline{\partial \zeta} \Omega_0(W^S) = 0,$

b) $\Omega_0(W^S) = (2\pi i)^{-n} W^S \wedge (\overline{\partial \zeta} W^S)^{n-1},$

c) $f(z) = \int_{bD} f(\zeta) \Omega_0(W^S)$ for $f \in \mathcal{O}(D) \cap C(\overline{D})$ and $z \in D$.

(See Ra86, for example.)
5 Some regularity properties

As we pointed out at the beginning, for a given weakly pseudoconvex domain it is in general not possible to find an explicit Cauchy-Fantappié kernel which is holomorphic near the singularity \( z = \zeta \in \partial D \). The kernel \( \Omega_0(W^S) \) we constructed in the preceding section is—in some way—optimal in this general setting. On the one hand, just like the universal Bochner-Martinelli kernel, if \( z \in \partial D \), the singularity of \( \Omega_0(W^S) \) at \( \zeta = z \) is not integrable over \( \partial D \), but it is right at the border line: as we will show, for any \( \alpha > 0 \),

\[
|\zeta - z|^{-\alpha} \Omega_0(W^S)
\]

is indeed integrable over \( \partial D \). This implies that the operator \( T^S : L^1(\partial D) \rightarrow C(D) \) defined by

\[
T^S(f) = \int_{\partial D} f(\zeta) \Omega_0(W^S)
\]

preserves a variety of function spaces, at least if one allows for an arbitrarily small loss in regularity. Such results, however, require detailed new proofs, as the singularity of the operator \( T^S \) does not appear to fit into any of the classical theories. Much more significant is the fact that \( \Omega_0(W^S) \) reflects the complex geometry of the boundary \( \partial D \), and hence should be much more useful in complex analysis than the Bochner-Martinelli kernel. For example, while \( T^S(f) \) is not holomorphic in general, one has the following result, which makes it explicit that \( T^S(f) \) enjoys some special complex analytic properties.

**Theorem 9** For any \( \delta < 2/3 \) there exists a constant \( C_\delta \) such that for all functions \( f \) continuous on \( \partial D \) one has \( T^S(f) \in C^\infty(D) \) and

\[
|\overline{\partial} T^S(f)(z)| \leq C_\delta |f|_0 \text{dist}(z, \partial D)^{\delta - 1} \text{ for } z \in D.
\]

In contrast, the Bochner-Martinelli kernel satisfies the analogous estimate only for \( \delta = 0 \); it does not give preference to partial derivatives with respect to the conjugate variables \( \overline{\zeta} \).

We note that given the structure of the coefficients of \( \Omega_0(W^S) \) it follows by standard arguments that \( T^S(f) \in C^\infty(D) \) for any \( f \) that is integrable over \( \partial D \). For the proof of the desired estimate we may differentiate with respect to \( \zeta_j \) under the integral sign, and we shall utilize variations of the estimates for \( \partial \overline{\partial} r/\Phi_K \) that were established in section 3. Since \( |S| \geq \gamma > 0 \) for \( |\zeta - z| \geq \varepsilon/2 \), for the purposes of estimations it is enough to consider the case where \( \zeta \in \partial D \) and \( |\zeta - z| \leq \varepsilon/2 \), so that

\[
s_j = g_j = \partial r/\partial \zeta_j - 1/2 \sum_k \partial^2 r/\partial \zeta_j \partial \zeta_k (\zeta_k - z_k) + \mathcal{E}_2^#.
\]

By making use of the specific form of the second term of \( s_j = g_j \), it follows that

\[
\begin{align*}
s &= \partial r(\zeta) + \mathcal{E}_1^# , \quad \text{and} \\
\overline{\partial} s &= \overline{\partial} \partial r(\zeta) + \mathcal{E}_1^#.
\end{align*}
\]

\(^4|f|_0 \) denotes the supremum norm of \( f \) over \( \partial D \).
We now fix \( z \in U \cap D \) and introduce the frame \( L_1, ..., L_n \) and the \( z \)-diagonalizing coordinates for \( \zeta \), as in the proof of Proposition 4. The proof of Proposition 5 shows that

\[
\overline{\partial_s} \zeta = \left[ \mathcal{L}(z) + \mathcal{E}_1^\# \right] + N,
\]

where \( \mathcal{L}(z) = 1/2 \sum_{j=1}^{n-1} \lambda_j(z) d\zeta_j \wedge d\overline{\zeta}_j \), and \( N \) represents a "normal" 2-form of the type \( \Lambda_1 \wedge \omega_n + \Lambda_2 \wedge \overline{\omega}_n \), where \( \Lambda_1 \) and \( \Lambda_2 \) are suitable 1-forms which may change from formula to formula. Note that the pull-back of \( N \wedge N \) to \( bD \) is zero; we shall therefore ignore such terms in the following.

**Lemma 10** If \( \iota_{bD} : bD \to U \) is the inclusion, for any \( 1 \leq q \leq n-1 \) the form \( \iota_{bD}^*(d\zeta \wedge (\overline{\partial_s} \zeta))^q \) is a linear combination with \( \mathcal{E}_1^\# \) coefficients of \((q + 1, q)\)-forms which contain factors of the type

\[
\sum_{|J|=l} \lambda_J(z) (d\zeta \wedge d\overline{\zeta})^J \wedge [\mathcal{E}_1^\#]^{q-l} \text{ with } 0 \leq l \leq q,
\]

where the summation is over strictly increasing \( l \)-tuples \( J = [j_1, ..., j_l] \subset [1, 2, ..., n-1] \),

\[
\lambda_J(z) (d\zeta \wedge d\overline{\zeta})^J = \prod_{j=1}^l \lambda_{j_j}(z) d\zeta_{j_j} \wedge d\overline{\zeta}_{j_j},
\]

and \([\mathcal{E}_1^\#]^{q-l}\) is a form of degree \( 2(q-l) \) whose coefficients are products of \( q-l \) factors of type \( \mathcal{E}_1^\# \).

**Proof.** By the above representations for \( s \) and \( \overline{\partial_s} \zeta \), one obtains

\[
s \wedge (\overline{\partial_s} \zeta)^q = N \wedge (\overline{\partial_s} \zeta)^q + \mathcal{E}_1^\# \wedge (\overline{\partial_s} \zeta)^q
\]

and

\[
(\overline{\partial_s} \zeta)^q = \left[ \mathcal{L}(z) + \mathcal{E}_1^\# \right]^q + \left[ \mathcal{L}(z) + \mathcal{E}_1^\# \right]^{q-1} \wedge N = \sum_{i=0}^q [\mathcal{L}(z)]^i \wedge [\mathcal{E}_1^\#]^{q-i} + \sum_{i=0}^{q-1} [\mathcal{L}(z)]^i \wedge [\mathcal{E}_1^\#]^{q-1-i} \wedge N,
\]

where we have ignored terms with \( N \wedge N \). It follows that

\[
s \wedge (\overline{\partial_s} \zeta)^q = (N + \mathcal{E}_1^\#) \wedge \sum_{i=0}^q [\mathcal{L}(z)]^i \wedge [\mathcal{E}_1^\#]^{q-i} +
\]

\[
+ \mathcal{E}_1^\# \wedge \sum_{i=0}^{q-1} [\mathcal{L}(z)]^i \wedge [\mathcal{E}_1^\#]^{q-1-i} \wedge N = \sum_{i=0}^q [\mathcal{L}(z)]^i \wedge [\mathcal{E}_1^\#]^{q-i} + ...,\]

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where we have retained just the leading terms, i.e., those with the lowest order of vanishing. The lemma then follows by expanding $|\mathcal{L}(z)|^l = \left[1/2 \sum_{j=1}^{n-1} \lambda_j(z) d\zeta_j \wedge d\bar{\zeta}_j \right]^l$ and absorbing all numerical factors in the $N$ and/or $E^\#_1$ terms.

Before coming to the proof of theorem 9, we use the preceding results to prove the following statement that we had anticipated earlier.

**Proposition 11** For any $\alpha > 0$ there exists a constant $C_\alpha$ such that for any $z \in \overline{D}$ with $\text{dist}(z, bD) < \varepsilon/2$ one has

$$
\int_{bD} |\zeta - z|^\alpha |\Omega_0(W^S)(\zeta, z)| \leq C_\alpha.
$$

**Proof.** We fix $z$ as required and introduce the $z$-diagonalizing coordinates on a ball $B(z, \delta)$, with $\delta \leq \varepsilon/2$. It is clearly enough to prove the estimate locally, i.e., for $\zeta \in bD \cap B(z, \delta)$. From Lemma 10, with $q = n - 1$, one sees that the pull-back to $bD$ of $|\zeta - z|^\alpha \Omega_0(W^S) = |\zeta - z|^\alpha c_n s \wedge (\overline{d\zeta}s)^{n-1}/\Phi_K^n$ decomposes into a linear combination of terms

$$
|\zeta - z|^\alpha \frac{\omega_n \wedge \sum_{|J|=l} \lambda_J(z)(d\zeta \wedge \overline{d\zeta})^J \wedge (\mathcal{E}^\#)^{n-1-l}}{\Phi_K^n}
$$

for $l = 0, \ldots, n - 1$, and of other terms of this type with higher order of vanishing in the numerator. In order to integrate over $bD \cap B(z, \delta)$ we introduce a $C^{k-2}$ coordinate system $(\zeta_1 - z_1, \ldots, \zeta_{n-1} - z_{n-1}, \text{Im} F^r(\zeta, z), r(\zeta))$ on $B(z, \delta)$, where $\delta$ may have to be chosen smaller, but can in any case be fixed independently of the point $z$. After renumbering, and since $r(\zeta) = 0$ on $bD$, one is left with estimating

$$
I_l(z) = \int_{bD \cap B(z, \delta)} |\zeta - z|^\alpha \frac{\lambda_1(z) \ldots \lambda_l(z) |\zeta - z|^{n-1-l}}{|\Phi_K|^l} \cdot d\zeta_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_{n-1} \wedge d\zeta_{n-1} \wedge d\text{Im} F^r(\zeta, z)
$$

for any $l = 0, \ldots, n - 1$. Choose $\gamma > 0$ so that $\gamma^\# = \alpha - 3\gamma(l+1) > 0$, and set $\zeta' = (\zeta_{l+1}, \ldots, \zeta_{n-1}) \in \mathbb{C}^{n-1-l}$. Since $|\Phi_K| \geq |\zeta - z|^3$ one has $|\zeta - z|^\alpha / |\Phi_K|^\gamma(l+1) \leq |\zeta - z|^\gamma^\#$, and it follows from Proposition 4 that

$$
\lambda_j(z)/|\Phi_K|^{1-\gamma} \leq 1/|r(z)| + |\zeta_j - z_j|^2(1-\gamma).
$$

By using these estimates one obtains

$$
I_l(z) \lesssim \int_{bD \cap B(z, \delta)} \prod_{j=1}^{l} \frac{|d\zeta_j \wedge d\overline{\zeta}_j|}{|r(z)| + |\zeta_j - z_j|^2(1-\gamma)} \cdot \frac{d\text{Im} F^r}{|\text{Im} F^r|^{\gamma^\#}} \cdot \frac{|\zeta - z|^\gamma^\# + n-1-l}{|\Phi_K|^{(n-1-l)}} dV(\zeta').
$$
Note that in case $l = n - 1$ the last factor is uniformly bounded, while if $l < n - 1$, one has
\[
\frac{|\zeta - z|^{\gamma + n-1-l}}{|\Phi_K|^{(n-1-l)}} \leq \frac{1}{|\zeta' - z'|^{2(n-1-l)-\gamma}}
\]
which is integrable in $\zeta'$. Consequently all factors in the resulting iterated integral for $I_l(z)$ are bounded independently of $z$.

As an application, one obtains the following result for the space $\Lambda_\alpha(bD)$ of functions which are Hölder continuous of order $\alpha$, with norm $|.|_\alpha$.

**Corollary 12** For any $\alpha > 0$ the integral operator $T^S : L^1(bD) \to C^\infty(D)$ satisfies
\[
|T^S(f)|_0 \leq C_\alpha |f|_\alpha
\]
for all $f \in \Lambda_\alpha(bD)$.

More generally, it is quite likely that for any $0 < \alpha' < \alpha < 1$ the operator $T^S$ is bounded from $\Lambda_\alpha(bD)$ to $\Lambda_{\alpha'}(D)$. Such variations will be discussed at some other time.

**Proof of the theorem.** Since $|r(z)| \approx \text{dist}(z, bD)$, it is enough to prove the estimate in the theorem with $|r(z)|$ instead of $\text{dist}(z, bD)$. We apply $\frac{\partial}{\partial z}$ to $T^S(f)$ under the integral sign. Note that
\[
\overline{\omega} \Omega_0(W^S) = c_n \frac{\partial s \wedge (\partial \zeta s)^{n-1} + (n-1) s \wedge \partial \zeta s \wedge (\partial \zeta s)^{n-2}}{\Phi_K^n} +
\]
\[
- c_n \frac{s \wedge (\partial \zeta s)^{n-1} \wedge \partial \zeta \Phi_K}{\Phi_K^{n+1}}.
\]

As in the proof of the proposition, we fix $z$ and choose the frame and unitary coordinate change adapted to that point $z$. By analyzing the terms above as before, and estimating $\frac{\partial s \Phi_K}{\Phi_K^n}$ by proposition 2, one sees that the critical integrals that one needs to estimate are those with factors of type

\[\text{(I) } \omega_n \wedge \sum_{|j| = l} \lambda_j(z) (d\zeta \wedge d\zeta')^j \wedge |s|^n 2^{-l} \]

for $0 \leq l \leq n - 2$ and

\[\text{(II) } \omega_n \wedge \sum_{|j| = l} \lambda_j(z) (d\zeta \wedge d\zeta')^j \wedge |s|^n 2^{-l} \frac{s \#}{\Phi_K^n} \]

for $0 \leq l \leq n - 1$. Note that $|s \# / \Phi_K|$ is bounded by a constant. Therefore, if $l \leq n - 2$, the terms in (II) are estimated by this constant multiplied with terms of type (I). We thus need to estimate terms of type (I), and those of type (II) with $l = n - 1$. 

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Consider (I) first. For a fixed \( l \leq n - 2 \)—after renumbering indices—it is enough to consider

\[
I_l(z) = \int_{bD \cap B(z \delta/2)} \frac{\lambda_1(z) ... \lambda_l(z) |\zeta - z|^{n-2-l}}{|\Phi_K|^n} \cdot |d\zeta_1 \wedge d\zeta_1 \wedge ... \wedge d\zeta_{n-1} \wedge d\zeta_{n-1} \wedge d\Im F^r(\zeta, z)|
\]

for \( z \in D \cap U \). Fix \( 0 < \delta < 2/3 \), and choose \( \gamma > 0 \) so small that \( 3(l + 1)\gamma < 2 - 3\delta \). We factor \( |\Phi_K|^{l+1} = |\Phi_K|^{(l+1)(1-\gamma)} |\Phi_K|^{(l+1)\gamma} \) as before and consider the remaining factor

\[
J_l(z) = \frac{|\zeta - z|^{n-2-l}}{|\Phi_K|^{n-1-l+(l+1)\gamma}}.
\]

Let us use the estimate \( |\Phi_K| \geq |\Phi_K|^{\delta} |r(z)|^{1-\delta} \geq |\zeta - z|^{3\delta} |r(z)|^{1-\delta} \) for one of the factors in the denominator, and the estimate \( |\Phi_K| \geq |\zeta - z|^\delta \) for the other factors. It follows that

\[
J_l(z) \leq |r(z)|^{\delta-1} \frac{|\zeta - z|^{n-2-l}}{|\zeta - z|^3 |r(z)|^{3(l+1)(1-\gamma)}} \leq |r(z)|^{\delta-1} \frac{1}{|\zeta' - z'|^{2(n-2-l)+3\delta(l+1)\gamma}},
\]

where \( \zeta' = (\zeta_{l+1}, ..., \zeta_{n-1}) \in \mathbb{C}^{n-1-l} \). Since by the choice of \( \gamma \) one has \( 3[\delta + (l + 1)\gamma] < 2 \), this last expression is integrable in \( \zeta' \in B(z', \delta/2) \). It follows that the integral \( I_l(z) \) is bounded by \( |r(z)|^{\delta-1} \) multiplied with a uniformly bounded iterated integral. Finally, we must consider the term (II) for \( l = n - 1 \). Choose \( \gamma > 0 \) so that \( 3n\gamma \leq 2 - 3\delta \), and factor \( |\Phi_K|^n = |\Phi_K|^{n(1-\gamma)} |\Phi_K|^\gamma \). By factoring the integrand as before, and by estimating the remaining factor

\[
\frac{|\zeta - z|^2}{|\Phi_K|^{1+n\gamma}} \leq \frac{|\zeta - z|^2}{|\Phi_K|^{3+n\gamma} |r(z)|^{1-\gamma}} \leq \frac{|\zeta - z|^2}{|\zeta - z|^{3(\delta+n\gamma)} |r(z)|^{1-\delta}} \leq |r(z)|^{\delta-1},
\]

the desired estimate follows. \( \blacksquare \)

6 Concluding remarks

The estimates we proved in the preceding section show that the support function \( \Phi_K \), the generating form \( W^S = s/S \), and the associated Cauchy-Fantappié kernel \( \Omega_0(W^S) \) provide a partial replacement for the missing holomorphic Cauchy kernel on arbitrary weakly pseudoconvex domains. Similarly, the corresponding higher order forms \( \Omega_q(W^S) \), \( 0 \leq q \leq n - 1 \), have proved useful in applications to pointwise estimates in the \( \overline{\partial} \)-Neumann theory (see Ra11), and as indicated there, might provide critical basic ingredients to prove deep new results in case the domain is assumed to be of finite type. Such applications, along with others, will be the subject of forthcoming articles.
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