Hadronic contributions to the muon anomalous moment

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Abstract

We discuss the hadronic contributions to the muon anomalous magnetic moment. They are dominated by light quark contributions which are constrained by the mechanism of chiral symmetry breaking. Using the leading order result based on $e^+e^-$ scattering data, we show that the next-to-leading order contributions in the fine structure constant $\alpha$ can be reliably calculated. Extending this idea to the hadronic four-point function we give a prediction for the light-by-light contribution.

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1 Introduction

The prediction of a $g$ factor $g_e = 2$ for the magnetic moment of the electron marked a great success of the relativistic wave equation introduced by Dirac in 1928 [1]. With Schwinger’s pioneering analysis of the electron magnetic moment within perturbation theory in 1948 [2], Quantum Electrodynamics (QED) as the first quantum field theory was established. Presently the lepton anomalous magnetic moments continue to be important observables for precision tests of the Standard Model (SM) [3]. Current data indicate a tension with the theoretical prediction for the anomalous magnetic moment of the muon which is considered to be the most promising observable for a stringent test of the SM and for searches for beyond SM physics (as a review, see e.g. Ref. [4]). The recent experimental value for the anomalous magnetic moment of the muon is [5]

$$g_\mu/2 = 1 + a_\mu = 1.001 165 920 8(6).$$

The Particle Data Group (PDG) gives an updated value for the muon anomaly in the form [6]

$$a_\mu^{\exp} = 116 592 091(54)(33) \times 10^{-11}.$$

The current muon experiment at Fermilab plans to reduce the experimental uncertainty by the factor of four [7], $\sigma_{\text{future}} \approx (1.0 \div 1.5) \times 10^{-10} \approx (10 \div 15) \times 10^{-11}$. This precision clearly is a challenge for the theoretical side to increase the precision of the prediction.
The theoretical results for the muon anomalous magnetic moment in the SM are traditionally represented as a sum of three parts,

\[ a_{\mu}^{\text{SM}} = a_{\mu}^{\text{QED}} + a_{\mu}^{\text{EW}} + a_{\mu}^{\text{had}} \]  \hspace{1cm} (3)

with \( a_{\mu}^{\text{QED}} \), \( a_{\mu}^{\text{EW}} \) being the leptonic and electroweak parts, respectively, and \( a_{\mu}^{\text{had}} \) is the contribution involving the electromagnetic currents of quarks. We note that in eq. (3) \( a_{\mu}^{\text{EW}} \) also contains the quark loops related to the interaction with the heavy weak bosons. Since their contribution is small, we do not worry about a precise description of this hadronic contribution to \( a_{\mu}^{\text{EW}} \).

The leptonic part is computed in perturbation theory and reads \[ a_{\mu}^{\text{QED}} = 116.584.718.95(0.08) \times 10^{-11}. \] \hspace{1cm} (4)

The computation extends up to five-loop level, using both analytical and numerical techniques \[ \text{as a review see e.g. Ref. [9]}. \]

The electroweak part is known to two loops and reads \[ a_{\mu}^{\text{EW}} = 153.6(1.0) \times 10^{-11}. \] \hspace{1cm} (5)

The absolute value of \( a_{\mu}^{\text{EW}} \) is small and the uncertainty of this contribution is negligible, at least for comparison with the present experiments.

The hadronic part \( a_{\mu}^{\text{had}} \) in the SM is related to quark contributions to the electromagnetic currents. To leading order (LO) in the fine structure constant \( \alpha \) it is given by the two-point function of hadronic electromagnetic currents through the vacuum polarization of the photon. In order to match the experimental accuracy of the muon anomaly one has to include next-to-leading order (NLO) contributions in \( \alpha \). At this order the four-point function of hadronic electromagnetic currents starts to contribute.

As we shall discuss below, an accurate calculation of the hadronic contributions due to light quarks is very difficult as they are represented by (almost) massless light quarks and are infrared (IR) singular in perturbation theory. This is the main obstacle for obtaining high precision SM predictions. Instead, the theoretical estimate for the hadronic two-point function utilizes scattering data. The LO hadronic contribution extracted from \( e^+e^- \) data is given by \[ a_{\mu}(\text{LO; had}; e^+e^-) = 6931(33)(7) \times 10^{-11}. \] \hspace{1cm} (6)

Other estimates are based on data from hadronic \( \tau \) lepton decays \[ \text{[4]} \] and yield \[ a_{\mu}(\text{LO; had}; \tau) = (6894.6 \pm 32.5) \times 10^{-11}. \] \hspace{1cm} (7)

In our estimates we stick to the PDG value in Eq. (6) for definiteness, called \( a_{\mu}(\text{LO; had}) \).
The hadronic contribution is rather large and needs to be computed with a precision of one or two per mille. This is a challenge for the theory, since there are no appropriate tools for an analytical theoretical computation. However, presently the lattice is emerging as a promising tool for this task.

In NLO there are further hadronic contributions. They are extensively discussed in the literature and estimated in various approaches. The current total SM prediction reads [6]

\[ a_{\mu}^{\text{SM}} = 116591823(1)(34)(26) \times 10^{-11}. \]  

(8)

The difference

\[ \Delta a_{\mu} = a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = 268(63)(43) \times 10^{-11} \]  

(9)

could be due to physics beyond the SM, but it is not statistically significant yet; however, it is considered to be rather serious for the prospect of discovering new physics.

The main theoretical uncertainties originate from hadronic contributions. These are encoded in the two-point function and the four-point function of the hadronic electromagnetic currents. The four-point function is involved in the topology called light-by-light (LBL) [6]. In the present paper we re-consider these hadronic contributions. The method to deal with the genuinely nonperturbative light-quark contributions makes use of the mechanism of chiral symmetry breaking which is assumed to capture the main physics to describe the hadronic matrix elements involving light quarks. This is described in the next section. We use this idea to update the results of Refs. [11, 12] and calculate the light-by-light contribution in this approach. Some useful formulas are given in the Appendix.

2 Description of the method

One of the key features of massless QCD is spontaneous breaking of chiral symmetry, which determines the low-energy behavior of the lightest states of QCD. Some recent discussion and references can be found in Ref. [13]. This spontaneous symmetry breaking (SSB) results at the level of correlation functions in the generation of mass terms for light fermion propagators in the complex (exact) QCD vacuum, i.e. the light quark is not just the Lagrangian quark anymore but rather a dressed collective excitation.

In perturbation theory the chirality of massless quarks is conserved, which can be read off from the perturbative light quark propagator in Fock space,

\[ S(q) = \frac{1}{\not q} \sim \frac{1}{fx^2} \]  

(10)

However, the (nonperturbative) interaction with soft gluons changes this behavior at large distances x, i.e. for small values of q (cf. e.g. Ref. [14]). Using the operator product expansion (OPE) as proposed by Wilson [15], one finds the expansion (cf. e.g. Ref. [16])

\[ S(q) = \frac{\not q}{q^2} + c_{qq} \frac{\langle \bar{q} q \rangle}{q^4} + c_{GG} \frac{\not q}{q^2} \frac{\langle G^2 \rangle}{q^4} + \cdots \]  

(11)
where $\langle \bar{q}q \rangle$ is the quark condensate, $\langle G^2 \rangle$ is the gluon condensate and $c_{qq}$, $c_{GG}$ are Wilson coefficients which can be computed in perturbation theory.

The physical meaning of the condensate term is that an effective mass term for the light quarks emerges through chiral SSB, which can be taken into account by writing

$$S(q, A) = (q - M(q))^{-1} \text{ with } M(q) = c_{qq} \frac{\langle \bar{q}q \rangle}{q^2}$$  \hspace{1cm} (12)

for the quark two-point point function in the presence of gluons. Here the dynamical mass $M(q)$ describes the effects of chiral symmetry breaking at the level of Green functions (cf. e.g. Ref. [17]), and the relation given in eq. (12) is the asymptotic form of $M(q)$ for large $q^2$. The exact asymptotic behavior at short distances can be obtained in OPE through the quark condensate as order parameter [18], or from the Dyson–Schwinger integral equation in the spirit of self-consistency or gap equations familiar from superconductivity [19].

In fact, this approach becomes more transparent by starting from the functional-integral expression for the two-point function for the hadronic electromagnetic current

$$\Pi_2(x) = \int [DA] \text{Tr}[S(x, A)\gamma^\mu S(-x, A)\gamma_\mu]$$  \hspace{1cm} (13)

where $\int [DA]$ represents the functional integration over the gluon field with a proper weight. Furthermore, $S(x, A)$ is the light-quark propagator in the presence of gluon field in the coordinate space. As a side remark we note, that the expression (13) is the starting point for a lattice calculation of the two-point function for the hadronic electromagnetic current.

The expression (13) is genuinely nonperturbative, so there is no way to perform an actual analytical calculation. In our approach we assume that the major effect of the integration over the gluon fields is the breaking of chiral symmetry which amounts to replace the light-quark propagator by (12) such that

$$\int [DA] \text{Tr}[S(x, A)\gamma^\mu S(-x, A)\gamma_\mu] = \text{Tr}[S(x, \hat{M}(x))\gamma^\mu S(-x, \hat{M}(x))\gamma_\mu]$$  \hspace{1cm} (14)

with the dynamical mass $\hat{M}(x)$ [20 21] which is related to $M(q)$ by Fourier transformation.

The behavior of the dynamical mass as a function of $x$ (or likewise $q$) is only qualitatively known. However, we note that the exact integral of the two-point hadronic function for the muon anomalous magnetic moment as in Eq. (A3) given in Appendix is equal to an integral of the function in Eq. (14) for some constant value $M(q) = m^*$, which is always true for reasonably smooth functions. The entire analysis can be done in Euclidean space-time which contains no particle singularities and where $m^*$ provides a straight infrared cut-off of QCD. Practically, this is a very efficient ansatz as all correlation functions are indeed represented by Feynman diagrams for which the analytical expressions are known.

The numerical value for the parameter $m^*$ of our ansatz, $m^* = 180\text{ MeV}$ [11] (see Sect. 4), has been extracted from data of $e^+e^-$ scattering in LO, i.e. from measured hadronic two-point function.
This value turns out to be rather close to both the pion mass and the usual constituent quark masses. However, this fact is purely accidental, since \( m^* \) in our approach is simply an infrared cutoff parameter in massless QCD and specific for the considered observable, namely the muon anomalous magnetic moment. On the other hand, a value in this ballpark is to be expected, giving us some confidence that the mechanism of chiral SSB is indeed the main input in the light-quark dynamics relevant for the anomalous magnetic moments.

Turning now to the NLO hadronic contributions we first consider the ones which need the hadronic two-point function as an input. NLO in this context means to look at the leptonic corrections and to compute the relevant integration kernel \( K^{(4)} \) which has a very similar shape as the LO integration kernel \( K^{(2)} \), i.e. we have to a very good approximation \[ K^{(4)}(x) \propto \frac{\alpha}{\pi} K^{(2)}(x) \]

and hence we have the same convolution integral with the hadronic two-point function, up to a constant. In the approach discussed above this means that the same value for \( m^* \) has to be used when evaluating the NLO leptonic contributions, and the uncertainty of the input LO contribution then directly translates into that of the NLO contribution.

The extrapolation of this procedure to contributions from the hadronic four-point function is not so obvious, since the integration weight functions (kernels) for the anomalous magnetic moment are now of different form, which would lead to a different value of \( m^* \) for the four-point function. However, if the mechanism of chiral SSB remains to be the main piece, we do not expect \( m^* \) to be grossly different from the one extracted from the two-point function. The assumption, namely that \( m^* \) has the same value in the calculation of the anomalous magnetic moment of the muon, is the main systematic uncertainty of this approach. While this is a point difficult to resolve by analytical methods, the explicit numerical calculations on the lattice can quantitatively test this assumption in the future.

Although we assume that the effect of chiral symmetry breaking and the appearance of the quark condensate is the leading contribution, we may also consider the effect of the gluon condensate. To this end, we can write

\[
S(q, A) = R(q)(q - M(q))^{-1} \text{ with } R(q) = 1 + c_{GG}\frac{G^2}{q^4}
\]  

where the expression for \( R(q) \) is again given in the asymptotic regime. Note that the above expression equals (11) up to terms of order \( 1/q^7 \). The emergence of the gluon condensate in QCD is related rather to the breaking of scale invariance than to the chiral symmetry, and it is not clear whether the gluon condensate is an order parameter of some symmetry breaking phenomenon.

In the same spirit as we discussed the mass \( M(q) \), replacing it effectively by a constant value \( m^* \), we assume that the contributions related to \( R(p) \) can be replaced by a constant value \( r^* \). In this case all quark propagators will be multiplied by the constant \( r^* \), which means that the hadronic contributions of two-point functions will be multiplied by \((r^*)^2\), while the hadronic contributions...
to the four point functions will be multiplied by $(r^*)^4$. This will also modify the extraction of the value of $m^*$, since the value extracted from the data is actually $m^*/r^*$, while the mass entering the four-point function would become $m^*/(r^*)^2$. The total contribution of the four-point function can be rather enhanced compared to a simple picture based on perturbation theory in Fock space. One can draw many topologically different configurations that could be relevant for the lattice, a few examples are found in Fig. 1. There is no a priori reason for them to be small, however, there is also no reason why $r^*$ is very large since the gluon condensate is numerically small. Nevertheless, it is clear that within our approach the main systematic uncertainty in the determination of the hadronic contribution coming from the hadronic four-point function is related to the choice of the numerical value for the effective mass $m^*$.

In the following we give a more detailed account of the calculation and our results.

3 LO hadronic contributions

To LO the hadrons contribute through a two-point function of electromagnetic currents. In the SM this two-point function is the correlator of electromagnetic currents of quarks (see Appendix for details). The top, bottom, and charm quarks are heavy enough for perturbative QCD to apply. The perturbative corrections are given in terms of $\alpha_s(m_Q)$ and are well under control.

For a hadronic scale $m_Q$ the contribution to the muon magnetic moment scales as $(m_\mu/m_Q)^2$. Thus, the top quark contribution is negligible.

The bottom quark ($Q_b = -1/3, m_b = 4.8\,\text{GeV}$) gives

$$a^{\text{had}}_\mu(\text{LO}; b) = 1.9 \times 10^{-11},$$

(16)

where we used the pole mass [22, 23]. This contribution is below the expected experimental uncertainty. The result (16) is stable against the inclusion of higher order QCD corrections which are completely negligible.

The charm quark gives a larger contribution due to its larger electric charge and its smaller mass
\(Q_c = 2/3, \ m_c \sim 1.6 \text{ GeV} \sim m_{J/\psi}/2\)

\[
a_{\mu}^{\text{had}}(\text{LO}; c) = 69.3 \times 10^{-11}.
\]  

(17)

As stated in the introduction, the present requirement for a solid theoretical estimate for the anomalous magnetic moment of the muon is that its uncertainty should be smaller than the benchmark uncertainty of the Fermilab experiment. This is still the case for our estimate of the charm quark contribution. The charm quark mass is given with high precision in [24].

Now we turn to the contribution from the light quarks. However, here a perturbative calculation is not possible, since the scale for QCD corrections is \(\mu \sim m_q \ll \Lambda_{\text{QCD}}\), and thus we use the method described in Sect. 2. We estimate the contribution of light quarks using \(e^+e^-\) data as

\[
a_{\mu}^{\text{had}}(\text{LO}; \text{uds} - \text{data}) = a_{\mu}^{\text{had}}(\text{LO}; \text{all}) - a_{\mu}^{\text{had}}(\text{LO}; b) - a_{\mu}^{\text{had}}(\text{LO}; c) = (6931(34) - 1.9 - 69.3) \times 10^{-11} = 6860(34) \times 10^{-11}.
\]

(18)

From this result the numerical value for \(m^*\) is extracted as \(m^* = 180.0 \pm 0.5\) MeV.

In fact, the data-based result for the LO contribution (18) includes implicitly some of the NLO hadronic corrections. These are, for instance, additional leptonic and hadronic contributions to the vacuum polarization diagrams, or vertex corrections which are found in both the \(e^+e^-\) data and the muon anomalous magnetic moment. This is a well known problem of potential double counting, which is intensively discussed in the literature. We do not consider this problem here and take the value from Eq. (18) as our input for the LO part.

### 4 NLO hadronic contributions

As has been discussed above, we fix the nonperturbative effective IR mass of the light quarks using the LO value for the hadronic contribution to the muon magnetic moment to obtain \(m^* = 180\) MeV as proposed in [11]. To be precise, computing the LO value within our approach with \(m^* = 180.0 \pm 0.5\) MeV we obtain \(a_{\mu}^{\text{had}}(\text{LO}; \text{uds}) = (6852 \pm 38) \times 10^{-11}\), i.e. we reproduce the value given in (18). One should not take the high precision of the determination of the effective mass \(m^*\) too seriously, since the main uncertainty of our approach is systematic, see the discussion of the method in Sect. 2. However, the structure for hadronic correlators is completely fixed in our approach and, with the value of the infrared mass \(m^*\) known from LO, we have an explicit model for the NLO calculations. This model can be easily applied, since all necessary formulas are well known in the literature; the main source for the analytical results results used here is [25].

As has been pointed out before, we have at NLO contributions involving the hadronic two-point as well as the hadronic four-point function, which will be considered in the subsequent subsections.

#### 4.1 Two-point function and photon–muon corrections

The first contribution is given by the vertex of the type \(K^{(4)}\) in Ref. [25] (cf. also Ref. [26]). Using the expression for the kernel \(K^{(4)}\) from Ref. [25] and our approximation for the quark propagators,
we find for the NLO contribution
\[ a^\text{had}_\mu \text{(ver; NLO; }uds) = -188 \times 10^{-11}. \]

We note that this contribution can be also obtained by an expansion in inverse powers of the quark mass, which in our case will be the effective mass \( m_q = m^* \). The relevant ratio is in this case \( m^2/4m^{*2} \) since the threshold is actually at \( 2m^* \). This expansion yields (without the QCD color factors)
\[ a^\text{had}_\mu \text{(ver; NLO, }q) = -\frac{8}{3} \left( \frac{m_\mu}{2m_q} \right)^2 \left( -\frac{2689}{5400} + \frac{\pi^2}{15} + \frac{23}{90} \ln \frac{m_q}{m_\mu} \right) \left( \frac{\alpha}{\pi} \right)^3 \] (19)

Inserting the QCD color factors and \( m_q = m^* \) we get (cf. [27])
\[ a^\text{had}_\mu \text{(ver; NLO; }uds) = -171 \times 10^{-11}. \] (20)

From this results we conclude that the expansion in inverse powers of \( m^* \) yields already a pretty precise prediction.

The charm quark and bottom quark contributions can be calculated perturbatively; for the charm quark we get the result
\[ a^\text{had}_\mu \text{(ver; NLO; }c) = -4 \times 10^{-11}. \] (21)
while the bottom quark contributes a tiny amount
\[ a^\text{had}_\mu \text{(ver; NLO; }b) = -0.1 \times 10^{-11}. \] (22)

Both contributions are smaller than the expected uncertainty of a new experimental value and can be neglected. Obviously, the leading-order mass expansion gives sufficient accuracy for heavy quark contributions. One can also use the expansion for the \( K^{(4)} \) kernel of Ref. [25] given in Ref. [28], even though the exact result is easy to handle as well.

The total vertex-type contribution reads
\[ a^\text{had}_\mu \text{(ver; NLO)} = -192 \times 10^{-11}. \] (23)

The second contribution is of the double bubble (db) vacuum polarization type where the second 1PI block is given by leptons different from the muon, as the muon has been already included in the vertex type contribution as defined in [25]. The electron loop gives
\[ a^\text{had}_\mu \text{(db; NLO; }e&uds) = 104 \times 10^{-11} \] (24)
that should be compared to Ref. [11, 27, 28]. The \( \tau \) lepton loop is negligible,
\[ a^\text{had}_\mu \text{(db; NLO; }\tau&uds) = 0.05 \times 10^{-11}. \] (25)

The electron loop together with a charm quark loop is marginal,
\[ a^\text{mod}_\mu \text{(db; NLO; }e&c) = 1.1 \times 10^{-11}, \] (26)
while the electron loop together with a bottom quark loop is completely negligible.
4.2 Four-point function contributions

The contributions of the four-point function $\Pi_4$ to the muon magnetic moment is difficult to estimate since it is completely nonperturbative by nature. Therefore it is clearly dangerous to identify the corresponding hadronic matrix element with perturbative diagrams. We nevertheless find it convenient to characterize different contributions at NLO by a decomposition of the four-point function as

$$\langle 0 | T[j(x)j(y)j(z)j(w)] | 0 \rangle = \langle 0 | T[j(x)j(y)j(z)j(w)] | 0 \rangle_{\text{conn}} \tag{27}$$

$$+ \langle 0 | T[j(x)j(y)] | 0 \rangle \langle 0 | T[j(z)j(w)] | 0 \rangle + \text{permutations } x, y, z, w$$

where $j(x)$ is a hadronic electromagnetic current (we suppress the Lorenz index for simplicity) and the matrix elements are to be computed with the QCD interaction only, e.g. on the lattice. This expression is similar to the definition of the contributions which are one-particle irreducible with regards to the photon lines. In perturbation theory the second term generates double insertions of the hadronic two-point functions while the first term is a genuine “nonfactorizable” contribution to the four-point function of electromagnetic currents. In the data-based analysis the two-point functions are taken from data and the QED corrections that are formally given by the four-point function are already partly included in the parametrization. In Fig. (2) we illustrate this situation by contributions of the leading order perturbative diagrams. The QED corrections to the connected part which is given by the light-by-light configuration at the leading order will generate terms beyond NLO in the fine structure constant and should be dropped.

While our modeling of hadronic contributions may look a bit naïve and certainly has not yet been rigorously justified in our analysis as a systematic effective theory the method is physically attractive and very efficient computationally. Indeed, theoretically the four-point function contributions can be uniquely identified in perturbation theory and, practically, the numerical results can be obtained by using directly the formulas for NLO leptonic contributions to the muon magnetic moment that are readily available in the literature.

We have the following contributions:
i) The double vacuum polarization in perturbation theory with different quarks is mainly analogous to the mixed lepton–quark vacuum polarization as it does not require internal corrections to the quark loop (Källén–Sabry correction [29], cf. the next item).

The charm quark together with light modes gives

\[ a_{\mu}^{\text{had}}(\text{db}; \text{NLO}; c\&u\&d\&s) = 0.1 \times 10^{-11}. \] (28)

The reiteration of light modes with different quarks reads

\[ a_{\mu}^{\text{had}}(\text{db}; \text{NLO}; u\&d\&s\&u'd's') = 3 \times 10^{-11}. \] (29)

ii) In addition to double bubbles of the same fermion we have diagrams with an internal structure, the Källén–Sabry correction. In our approach the new contribution, i.e. an internal structure of an effective quark loop in \( \Pi_4 \), is computable. The general formula for the contributions of a fermion (without symmetry and group factors) is given by

\[ a_{\mu}^{\text{ferm}}(4; \text{NLO}; \text{ferm}) = \frac{41}{486} \left( \frac{m_{\mu}}{m_{\text{ferm}}} \right)^2 \left( \frac{\alpha}{\pi} \right)^3. \] (30)

The contribution of charm quarks is negligible,

\[ a_{\mu}^{\text{had}}(4; \text{NLO}; c) = 0.3 \times 10^{-11}. \] (31)

iii) The result for the double bubble with light quarks, where the same quark is running in the loop is taken together with the Källén–Sabry type correction to the single bubble and yields

\[ a_{\mu}^{\text{had}}(4; \text{NLO}; u\&d\&s) = 25 \times 10^{-11}. \] (32)

In fact, one has to add terms with the color structure \( N_c^2 \) not separable in a lepton-type calculation of Ref. [25]. However, these terms can be computed explicitly. An additional group factor is \( N_c(Q_u^4 + Q_d^4 + Q_s^4) = 2/3 \), and the result is small, \( 0.5 \times 10^{-11} \).

The NLO estimates of this type based on data are named “dispersive NLO” and are defined to be any contribution except the genuine light-by-light piece. The result is [6]

\[ a_{\mu}(\text{disp}; \text{NLO}; \text{had}; e^+e^-) = -98.7(0.9) \times 10^{-11}. \] (33)

This numerical value corresponds to the contribution of the two-point hadronic function and should be compared to the sum of our results above.

iv) The light-by-light contribution is the genuine \( \Pi_4 \) contribution that is most unknown and controversial. Unlike for the hadronic two-point function there is no way at present to extract sufficient helpful information about this contribution from experiment and thus this topology contributes a large part of the theoretical uncertainty.
We use our approach described in Sect. 2 and compute the LBL in terms of a “dressed” quark with the effective mass \(m^\ast\). The LBL contribution for fermions then reads\[30\]

\[
a_{\mu}^{\text{ferm}}(\text{LBL; NLO}; m_q) = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \left(\frac{m_\mu}{m_q}\right)^2 \left(\frac{3}{2}\zeta(3) - \frac{19}{16}\right) + \left(\frac{m_\mu}{m_q}\right)^4 \left(-\frac{161}{810}\ln^2\left(\frac{m_q}{m_\mu}\right) - \frac{16189}{48600}\ln\left(\frac{m_q}{m_\mu}\right) + \frac{13}{18}\zeta(3) - \frac{161}{9720}\pi^2 - \frac{831931}{972000}\right) \right\}.
\]

Multiplying with the necessary group factors we can compute the light-by-light contribution for the different fermions. The light quarks give the contribution

\[
a_{\mu}^{\text{mod}}(\text{LBL; NLO}; uds) = 139 \times 10^{-11}
\]

while the result for the \(c\) quark is

\[
a_{\mu}^{\text{mod}}(\text{LBL; NLO}; c) = 2 \times 10^{-11}.
\]

We point out that this estimate is model dependent, so we also cannot assign a reliable estimate of the corresponding theoretical uncertainty.

### 5 Results

We are now ready to collect the different contributions. The NLO result related to the two-point function is the sum of Eqs. (23), (24) and (26),

\[
a_{\mu}^{\text{had}}(\text{NLO}; \Pi_2) = (-192 + 104 + 1) \times 10^{-11} = -87 \times 10^{-11}.
\]

This result is based on LO from Eq. (18) and is very stable. Our model calculation reproduces the integration of the LO data with appropriate kernels. This is because the two kernels \(K^{(2)}\) and \(K^{(4)}\) behave similarly in the important region of integration (cf. e.g. Ref. [12]).

The result related to the four-point function without LBL is given by Eqs. (29) and (32) and by some other small terms \((1 \times 10^{-11})\),

\[
a_{\mu}^{\text{had}}(\text{NLO}; \Pi_4\text{pol}) = (3 + 25 + 1) \times 10^{-11} = 29 \times 10^{-11}.
\]

The main contribution comes from the Källén-Sabry term \(32\). In fact, one would perhaps had to subtract this term from the LO contribution before fitting \(m^\ast\). However, as we have already discussed before, the problem of double counting is too complicated to be considered here.

The LBL term from Eqs. (35) and (36) gives

\[
a_{\mu}^{\text{had}}(\text{NLO}; \Pi_4\text{LBL}) = (139 + 2) \times 10^{-11} = 141 \times 10^{-11}.
\]

Therefore, the NLO contribution related to the four-point function is given by

\[
a_{\mu}^{\text{had}}(\text{NLO}; \Pi_4\text{tot}) = (141 + 29) \times 10^{-11} = 170 \times 10^{-11}.
\]
A difficult question is to estimate the accuracy of the obtained results. It is clear that the statistical uncertainty due to the error of the only parameter of our model \( m^* = 180 \pm 0.5 \text{MeV} \) is very small and is basically irrelevant while the main uncertainty of the predictions is a systematic one, i.e. the uncertainty of the model itself. As we have discussed in detail in Sects. 2 and 6, we think that the actual contribution of the four-point function can be up to a factor two larger. We take a conservative point of view and include a 50% uncertainty in our result as a systematics to get the prediction for the NLO contribution related to the four-point function in the form

\[
a_{\mu}^{\text{had}}(\text{NLO}; \Pi_4 \text{fin}) = (170 \div 255) \times 10^{-11} = (213 \pm 43) \times 10^{-11}.
\]

Unfortunately, the conservative uncertainty of the prediction in Eq. (41) is larger than an allowed uncertainty of \( \sigma_{\text{future}} \approx (10 \div 15) \times 10^{-11} \), but we believe that based on the current techniques it is a realistic one.

Our prediction for the total hadronic NLO now reads

\[
a_{\mu}^{\text{had}}(\text{had}; \text{NLO}) = [-87 + (213 \pm 43)] \times 10^{-11} = (126 \pm 43) \times 10^{-11}.
\]

The data-based result is

\[
a_{\mu}^{\text{had}}(\text{N(N)LO}; e^+ e^-) = 19(26) \times 10^{-11}.
\]

In both cases the error is dominated by the LBL contribution, or, more generally, by the contribution of the four-point hadronic function.

Using the hadronic LO contribution from Eq. (6) we obtain the total hadronic contribution

\[
a_{\mu}^{\text{had}}(\text{had}) = [(6931 \pm 34) + (126 \pm 43)] \times 10^{-11} = (7057 \pm 55) \times 10^{-11}.
\]

Adding in the leptonic contributions we obtain the SM value

\[
a_{\mu}^{\text{SM}} = 116591929(55) \times 10^{-11}.
\]

and the comparison with the data yields

\[
a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = \Delta a_{\mu}(\text{SM}) = 162(54)(33)(55) \times 10^{-11}.
\]

showing the well known tension at the level of 2\( \sigma \)'s. A future measurement with a significantly reduced uncertainty will certainly shed some light on this tension, but the theoretical uncertainty seems to persist to stay larger than the experimental one. The lattice calculations can become crucial for the analysis of this observable in the SM with such a high accuracy.

6 Discussion

We have revisited and discussed the NLO hadronic contributions to the muon anomalous magnetic moment. The contribution originating from the hadronic two-point function is reasonably well under
control since the bulk part of it can be extracted from the data on $e^+e^-$ scattering to hadrons. On the other hand, for the theory an important observation is that this contribution can be computed using the two-point function of quark electro-magnetic currents in Euclidean domain. While this route is more reliable for performing theoretical calculations than using data the main obstacle persists – the contribution of light quarks in perturbation theory is infrared divergent. In our model we cure the deficiency of the perturbation theory approach by the direct method of regularizing the emerging infrared divergence by introducing an effective mass for a light quark. It is not a formal regularization parameter though but a quantity with solid physical meaning. It is well known that generation of a mass parameter at small momenta for the originally massless Lagrangian quark is a general feature of QCD where the chiral symmetry is spontaneously broken. As a technical implementation of this physical feature in our method we choose the effective mass to be a constant as the relevant observable is an integral over the momenta that is saturated at scales around one GeV. The numerical value for this constant is fixed from the results of LO analysis based on data. However, using data for computing the LO contribution requires a careful analysis of which pieces are included in the data input and which need to be subtracted. A naïve analysis based on Feynman diagrams bears the danger of double counting, e.g. the contributions from the double-bubble diagrams.

The important problem is also related to the input of the fine-structure constant $\alpha$ at the NLO analysis. At the current level of precision one can use the accurately measured value of $\alpha$ from the electron anomalous magnetic moment which eventually means that we compare the anomalous magnetic moments of the muon and the electron. A lot of the uncertainties are common to both which also means that the uncertainties in both quantities are correlated even though the mass dependence of the results is much more important in the muon case. A similar remark may apply to other precise sources for extracting $\alpha$.

The main theoretical problem in the context of hadronic contributions to the muon magnetic moment is the reliable predictions for the hadronic four-point function. In the paper we discussed a model based on considerations of chiral symmetry breaking. As an extension of the simple version of the model of ref. [11] we now argue that the contributions related to the hadronic four-point function can be essentially enhanced. This conclusion emerges from the analysis of the large number of different topologies that appear in higher orders of perturbation theory and can play an important role in lattice computations. It is possible that the expectations based on perturbation theory in Fock space with a small number of low mass resonances give an oversimplified picture of a genuine four-point function contribution. Indeed, in the analysis inspired by data the main contribution to LBL comes from an exchange by the neutral pion due to the anomalous dimension-five interaction $\pi^0\tilde{G}/f_{\pi}$ [31] (see also Refs. [32, 33, 34]). However, within the chiral perturbation theory ($\chi$PT) approach this contribution is subleading in power counting and, therefore, not unique. It is suppressed by a natural $\chi$PT scale $\Lambda_\chi = 2\pi f_{\pi} \approx m_\rho$. The leading contribution given by the
Goldstone modes is very small. Thus, the charged pions give \[ a_{\mu}(\text{LBL}) = \left( \frac{\alpha}{\pi} \right)^3 a_{\mu}(\gamma\gamma); \]
\[ a_{\mu}(\gamma\gamma; \text{SQED}) = \frac{m^2_{\mu}}{m^2_\pi} \left( \frac{1}{4} \zeta(3) - \frac{37}{96} \right) = -0.0849 \frac{m^2_{\mu}}{m^2_\pi}, \] (47)
and the contribution of charged kaons is totally negligible. Note also that the sign is negative compared to fermions,
\[ a_{\mu}(\gamma\gamma; \text{QED}) = \frac{m^2_{\mu}}{m^2_f} \left( \frac{3}{2} \zeta(3) - \frac{19}{16} \right) = 0.6156 \frac{m^2_{\mu}}{m^2_f}. \] (48)
The smallness of the pions contribution is related to the fact that pions are spinless particles with no own magnetic moments. In view of \( \chi \)PT power counting, however, the contribution of vector mesons like \( \rho \)-mesons, or even of baryons like protons, are of the same order as the neutral pion contribution, since the scales are close \( (m_\rho = 777 \text{ MeV} \text{ and } \Lambda_{\chi} = 600 \div 800 \text{ MeV} \approx 2\pi f_\pi) \). Within various effective theory schemes, even neutrons can contribute as they interact with photons via their own magnetic moment. Therefore, there are many contributions that are formally of the same order in power counting as the neutral pion one. The contribution of \( \pi^0 \) in its local form is ambiguous as it depends strongly on the ultraviolet cuts used and the usual cut is provided by the \( \rho \)-meson mass.

Therefore, the contributions related to the four-point function can be enhanced even in the resonance-based approach. It supports the conclusion obtained by looking at the number of different topologies that emerge at higher orders of perturbation theory. And even though the perturbative QCD is not applicable for their quantitative evaluation, they all appear in the analysis within the lattice approach.

To conclude, we think that there is still some room in the SM to accommodate for the current experimental value of the muon anomalous magnetic moment, and we are looking forward to results of new measurements.

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A Appendix

A.1 The LO hadronic contribution phenomenology

The two-point correlator is given by

\[ i \int \langle T j_\mu^{\text{had}}(x) j_\nu^{\text{had}}(0) \rangle e^{iqx} dx = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi^{\text{had}}(q^2). \]  

(A1)

The definition of the fine structure constant \( \alpha \) requires a normalization \( \Pi^{\text{had}}(0) = 0 \). The dispersion representation for \( \Pi^{\text{had}}(q^2) \) can then be written with one subtraction, leading to

\[ \Pi^{\text{had}}(q^2) = \frac{q^2}{4m^2} \int_0^\infty \frac{ds \text{ Im } \Pi^{\text{had}}(s)}{s - q^2}. \]  

(A2)

The LO expression for the muon anomaly is

\[ a^{\text{had}}_\mu(\text{LO}) = 4\pi \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty \frac{ds}{4m^2} K^{(2)}(s) \text{ Im } \Pi^{\text{had}}(s) \]  

(A3)

with a one-loop kernel of the form

\[ K^{(2)}(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)s/m^2}. \]  

(A4)

This expression is useful for the analysis based on the hadronic cross section of \( e^+e^- \) annihilation.

For the theory analysis, Eq. (A3) can be rewritten as an integral over Euclidean values \( t = -q^2 \) for \( \Pi^{\text{had}}(q^2) \),

\[ a^{\text{had}}_\mu(\text{LO}) = 4\pi^2 \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty \left\{ -\Pi^{\text{had}}(-t) \right\} W(t)dt \]  

(A5)

with

\[ W(t) = \frac{4m^4}{\sqrt{t^2 + 4m^2 t} \left( t + 2m^2 + \sqrt{t^2 + 4m^2 t} \right)^2}. \]  

(A6)

This form is well known in the form of a parametric integral [26, 36].

Eq. (A5) is more suitable for a theoretical study as the theory is preferably applicable in Euclidean domain and on the lattice in particular. One can further write

\[ \frac{1}{\pi} \int_0^\infty \frac{ds}{s} K^{(2)}(s) \text{ Im } \Pi^{\text{had}}(s) = \int_0^\infty \left( -\frac{d\Pi^{\text{had}}(-t)}{dt} \right) F(t)dt, \quad F(t) = \int_0^t W(\zeta)d\zeta \]  

(A7)

with

\[ F(t) = \frac{1}{2} \left( \frac{t + 2m^2 - \sqrt{t^2 + 4m^2 t}}{t + 2m^2 + \sqrt{t^2 + 4m^2 t}} \right) = \frac{2m^4}{\left( t + 2m^2 + \sqrt{t^2 + 4m^2 t} \right)^2}. \]  

(A8)
Therefore, the analysis of the anomaly is based on the derivative of the hadron vacuum polarization function \( d\Pi_{\text{had}}(-t)/dt \) which is closely related to the famous Adler function \([37]\)

\[
D(t) = -t \frac{d\Pi_{\text{had}}(-t)}{dt}.
\]  

The Adler function can be computed in perturbative QCD with massless quarks for large \( t \),

\[
D(t) = e^2 q N_c \frac{1}{12\pi^2} \left( 1 + \frac{\alpha_s(t)}{\pi} \right). \tag{A10}
\]

### A.2 The LO: technical formulas

The LO technical formulas given here are used to fix \( m_q \) from the LO hadronic contribution. A fermion with mass \( m_q \) without QCD group factors (as a lepton) gives a LO contribution to the muon anomaly of the form

\[
a_{\mu}^{\text{ferm}}(\text{LO}) = I(m_q) \left( \frac{\alpha}{\pi} \right)^2 \tag{A11}
\]

with

\[
I(m_q) = \int_0^\infty \frac{\rho_q(s) K^{(2)}(s)}{s} ds \tag{A12}
\]

and

\[
\rho_q(s) = \frac{1}{3} \sqrt{1 - \frac{4m_q^2}{s}} \left( 1 + \frac{2m_q^2}{s} \right). \tag{A13}
\]

The explicit integration over \( s \) with the kernel \( K^{(2)}(s) \) from Eq. (A4) gives

\[
I(m_q) = \frac{1}{3} \int_0^1 dx (1 - x) \left[ -\pi(x, m_q) \right] \tag{A14}
\]

where

\[
\pi(x, m_q) = \left( \frac{1}{3z} - 1 \right) \varphi(z) - \frac{1}{9} \tag{A15}
\]

and

\[
\varphi(z) = \frac{1}{\sqrt{z}} \operatorname{artanh}(\sqrt{z}) - 1, \quad z = \frac{m_q^2 x^2}{4m_q^2 (1 - x) + m_\mu^2 x^2} = \frac{t}{4m_q^2 + t}, \quad t = \frac{m_\mu^2 x^2}{(1 - x)} \tag{A16}
\]

An analytical expression for the function \( I(m_q) \) is known. However, the integral representation given in Eq. (A14) is sufficient for practical applications.

The iterated contribution for two fermions (double bubble generalization of Eq. (A14)) is given by \([26]\)

\[
a_{\mu}^{\text{ferm}}(\text{db}; f_1 & f_2) = \left( \frac{\alpha}{\pi} \right)^3 \int_0^1 dx (1 - x) \left( -\pi(x, m_{f_1}) \right) \left( -\pi(x, m_{f_2}) \right). \tag{A17}
\]

The actual application of this formula in QCD should account for symmetry factors (a factor 2 if the fermions are different) and for group factors.
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