Some remarks on contact manifolds, Monge-Ampère equations and solution singularities

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Abstract. We describe some natural relations connecting contact geometry, classical Monge-Ampère equations and theory of singularities of solutions to nonlinear PDEs. They reveal the hidden meaning of Monge-Ampère equations and sheds new light on some aspects of contact geometry.
1. Introduction

In this note we describe some natural relations among subjects in the title. While contact geometry and Monge-Ampère equations are classical themes, studied in numerous works, their natural ties with the theory of singularities of solutions to nonlinear PDEs, as far as we know, were not yet explicitly established and, as a consequence, duly exploited.

The fact that a (nonlinear) PDE is surrounded by a “cloud” of subsidiary equations, which describe the behavior of singularities of its generalized solutions is also not well-known. These equations were introduced by the author in [12] and since that are waiting to be systematically studied. A detailed exposition of fundamentals of this theory will be published in [14]. It is worth noticing that some particular subsidiary equations are implicitly well-known. For instance, such are equations describing wave front propagation in the theory of hyperbolic equations, or equations of geometrical optics, which, in fact, describe some kind singularities of solutions of Maxwell’s equations. We used “implicitly” to stress that these equations were not identified as a part of solution singularity theory.

A fundamental problem in this theory is whether it is possible to reconstruct the original equation if the subsidiary equations are known. In other words, we are asking whether the laws describing behavior of singularities of a physical field (continuum media, etc), predetermine the equations describing this field, (media, etc). This problem will be called the reconstruction problem. In this note we show that, informally speaking, the classical Monge-Ampère equations are characterized by the fact that the reconstruction problem for them is a tautology. The exact formulation of this assertion requires various facts from geometry of jet spaces, including contact geometry, and theory of generalized solutions of nonlinear PDEs, which are collected and discussed along the paper. When being put in this perspective they bring into the light some structures and their interrelations whose importance was not duly understood even in the case they are formally known. For instance, the intrinsic definition of a contact structure (see below definition 2.1) belongs to this list. These small things are, in fact, rather useful and enrich even
such classical subject as contact geometry. The fact that the intrinsic definition of contact structures immediately leads to the explicit description of contact vector fields (see subsection 2.2) of this point.

For the composition of this note is see the “contents”. Everything (manifolds, vector fields, etc) in it is assumed to be smooth. The $C^\infty(M)$–module of $k$-th order differential forms (resp., vector fields) on a manifold $M$ is denoted by $\Lambda^k(M)$ (resp., $D(M)$).

2. Contact manifolds and generalized solutions

Let $M$ be an $n$–dimensional manifold. Recall that any projective $C^\infty(M)$–module $P$ of finite type is canonically identified with the module $\Gamma(\pi)$ of sections of a vector bundle $\pi$ (see [9]). The fiber $\pi^{-1}(x)$, $x \in M$, of $\pi$ is the quotient module $P_x := P/\mu_x \cdot P$, $\mu_x = \{ f \in C^\infty(M) \mid f(x) = 0 \}$, considered as an $\mathbb{R}$–vector space. The dimension of the vector bundle $\pi$ is the rank of $P$.

Accordingly, below we treat a regular distribution on a manifold $M$ as a projective submodule $D$ of the $C^\infty(M)$–module $D(M)$ of vector fields on $M$. Since $D(M)_x$ is canonically identified with the tangent to $M$ at $x$ space $T_xM$, $D_x$ may be viewed as a subspace of $T_xM$ and this way one recovers the standard definition of a distribution as a family of vector spaces $x \mapsto D_x \subset T_xM$.

Put $\kappa := D(M)/D$. So, $\kappa_x$ is identified with $T_xM/D_x$ and this is the reason to call $\kappa$ normal to $D$ bundle. Obviously, rank $\kappa = n – \text{rank } D$. Recall also that the curvature of $D$ is the $C^\infty(M)$–bilinear form on $D$ with values in $\kappa$ defined as

$$R(X,Y) := [X,Y]( \mod D), \ X, Y \in D.$$ 

If the rank of $\kappa$ is 1, i.e., the distribution $D$ is of codimension 1, then $D$ will be called nondegenerate, if

$$D \ni X \mapsto R(X,\cdot) \in \text{Hom}_{C^\infty(M)}(D, \kappa)$$

is an isomorphism of $C^\infty(M)$–modules. Equivalently, a distribution $D$ is nondegenerate if its curvature is a nondegenerate $\kappa$–valued $C^\infty(M)$–bilinear form on it.

The following is an intrinsic definition of a contact manifold.

**Definition 2.1.** A contact manifold is a manifold $M$ supplied with a nondegenerate distribution $D$ of codimension 1. Such a distribution $D$ is called a contact structure on $M$.

2.1. Comparison with the standard definition. Recall that the fiber at $x \in M$ of the vector bundle associated with the $C^\infty(M)$–module $\text{Hom}_{C^\infty(M)}(P, Q)$ with $P, Q$ being projective $C^\infty(M)$–modules of finite type is $\text{Hom}_\mathbb{R}(P_x, Q_x)$ (see [9]). In particular, this fiber for $\text{Hom}_{C^\infty(M)}(D, \kappa)$ is $\text{Hom}_\mathbb{R}(D_x, \kappa_x)$. But $\mathbb{R}$–vector space $\kappa_x$ is 1-dimensional. So, it is identified with $\mathbb{R}$ by choosing a base vector in it. If $D$ is locally given by the Pfaff equation $\omega = 0$, $\omega \in \Lambda^1(M)$, $X \in D(M)$ is such that $\omega(X) = 1$ and $\nu = X(\mod D)$, then $\nu_x \in D_x$ is a base vector, and $\text{Hom}_{C^\infty(M)}(D, \kappa)$ (resp., $\text{Hom}_\mathbb{R}(D_x, \kappa_x)$) is identified with $D^*_x := \text{Hom}_{C^\infty(M)}(D, C^\infty(M))$ (resp., $D^*_x := \text{Hom}_\mathbb{R}(D_x, \mathbb{R})$).

The following assertion is a direct consequence of the standard formula

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]), \ X, Y \in D(M).$$

**Lemma 2.1.** With the above identifications the curvature $R$ of $D$ is identified with $d\omega |_D$, i.e., $R(X,Y) = -d\omega(X,Y)$, $\forall X, Y \in D$. 

This lemma shows that nondegeneracy of $R_z$ is equivalent to nondegeneracy of $(d\omega)_z$, $\forall x \in M$, i.e., that the bilinear form on the vector space $D_z (d\omega)_z$ is symplectic. This nondegeneracy property may be also expressed by saying that $d\omega$ is nondegenerate on $D$, i.e., that

$$D \ni X \mapsto d\omega(X, \cdot) |_{\mathcal{D}^*}$$

is an isomorphism of $C^\infty(M)$-modules, or, equivalently, that $\omega \wedge d\omega^m$ is a local volume form on $M$ assuming that $n = 2m + 1$. These observations show that definition 2.1 is equivalent to the standard one.

Recall that $\omega \in \Lambda(M)$ is a contact form on $M$ if $\mathcal{D} := \{\omega = 0\}$ is a contact structure on $M$. Contact forms defining the same distribution differ one from another by a nowhere vanishing factor $f \in C^\infty(M)$. Moreover, contact forms associated with a contact distribution are, generally, defined only locally. By these two reasons definition 2.1 is more convenient that the standard one. An illustration of that is in the subsequent subsection.

2.2. Contact transformations and vector fields. A symmetry of a distribution $\mathcal{D}$ on $M$ is a diffeomorphism $F: M \to M$ that preserves $\mathcal{D}$, i.e., $F(X) \in \mathcal{D}$ if $X \in \mathcal{D}$ with $F(X) := (F^*)^{-1} \circ X \circ F^*$ being the image of $X$ via $F$. Traditionally symmetries of a contact structure/manifold are called contact transformations.

An infinitesimal symmetry of $\mathcal{D}$ is a vector field $Z \in \mathcal{D}(M)$ such that $[X, Z] \in \mathcal{D}$, $\forall X \in \mathcal{D}$. Alternatively, infinitesimal symmetries of $\mathcal{D}$ are defined as vector fields whose flows consist of (local) symmetries of $\mathcal{D}$. Infinitesimal symmetries of $\mathcal{D}$ form a Lie subalgebra of $\text{Sym} \mathcal{D}$. If $\mathcal{D}$ is contact, then infinitesimal symmetries of $\mathcal{D}$ are called contact vector fields and some authors use a slightly ambiguous Cont $M$ for $\text{Sym} \mathcal{D}$ if $M$ is a contact manifold (see [4]).

Now we shall show that definition 2.1 directly leads to a natural description of contact vector fields. To this end, we note that with a vector field $Z \in M$ a homomorphism $\phi_Z: \mathcal{D} \to \mathcal{Z}$ of $C^\infty(M)$-modules is associated. Namely, if $X \in \mathcal{D}$, then $\phi_Z(X) = [X, Z] (\mod \mathcal{D})$. Since $[fX, Z] = f[X, Z] - Z(f)X$, one sees that $\phi_Z$ is a $C^\infty(M)$-homomorphism.

**Proposition 2.1.** If $\mathcal{D}$ is contact, then the $\mathbb{R}$-linear map

$$\text{Sym} \mathcal{D} \ni Z \mapsto (Z \mod \mathcal{D}) \in \mathcal{Z}$$

is biunique.

**Proof.** **Injectivity.** If $Z \in \mathcal{D} \cap (\text{Sym} \mathcal{D})$, then

$$R(Z, X) = 0, \forall X \in \mathcal{D} \iff R(Z, \cdot) = 0.$$

Hence, by nondegeneracy of $R$, $Z = 0$.

**Surjectivity.** Let $\mathcal{Z} \ni \nu = Z'(\mod \mathcal{D})$. Since the curvature form $R$ is nondegenerate, there is $Y \in \mathcal{D}$ such that $\phi_{Z'} = R(Y, \cdot)$, i.e., $[X, Z'] = [Y, X] (\mod \mathcal{D})$, $\forall X \in \mathcal{D}$. So, $[X, Y + Z'] \in \mathcal{D}$, $\forall X \in \mathcal{D}$ and, therefore, $Z = Y + Z'$ is contact and $Z (\mod \mathcal{D}) = Z' (\mod \mathcal{D}) = \nu$. $\square$

If the contact vector field $Z$ corresponds via proposition 2.1 to $\nu \in \mathcal{Z}$, then $\nu$ is called the generating function of $Z$ and $Z$ will be denoted by $X_\nu$. In other words, if $Z \in \mathcal{D}(M)$ is contact, then $Z = X_\nu$ with $\nu = Z(\mod \mathcal{D})$. Also, proposition 2.1
allows to transfer the Lie algebra structure in \( \text{Sym} \mathcal{D} \) to \( \mathcal{X} \). Namely, the transferred Lie bracket, denoted by \( \{ \cdot, \cdot \} \), is defined by the relation

\[
X_{(\mu, \nu)} = [X_\mu, X_\nu], \quad \mu, \nu \in \mathcal{X}
\]

The \( \mathbb{R} \)-linear map \( \chi : \mathcal{X} \to D(M) \), \( \chi(\nu) = X_\nu \), is a 1-st order differential operator. To prove it we have to show that

\[
[f, [g, \chi]](\nu) = 0, \quad \forall f, g \in C^\infty(M), \quad \nu \in \mathcal{X}
\]

(see [9]). In view of proposition 2.1 it suffices to show that the vector field

\[
Z = [f, [g, \chi]](\nu) = X_{fg\nu} - fX_{g\nu} - gX_{f\nu} + fgX_\nu
\]

is contact and its generating function is trivial. But, obviously, \( X_{h\nu} - hX_\nu \in \mathcal{D}, \forall h \in C^\infty(M) \). This proves that \( Z \in \mathcal{D} \). Next, if \( X \in \mathcal{D} \), then

\[
[X, Z] = [X, X_{fg\nu}] - f[X, X_{g\nu}] - g[X, X_{f\nu}] + fg[X, X_\nu] - X(f)X_{g\nu} - X(g)X_{f\nu} + X(fg)X_\nu
\]

Each of first 4 terms in this expression, obviously, belongs to \( \mathcal{D} \). The remaining 3 terms of it may be rewritten in the form

\[
X(f)(gX_\nu - X_{g\nu}) + X(g)(fX_\nu - X_{f\nu}),
\]

and, as we have seen before, each of 2 summands in this expression also belongs to \( \mathcal{D} \).

2.3. Jets and generalized solutions of nonlinear PDEs. Fix a manifold \( E^{n+m} \) of dimension \( n + m \), \( n, m \in \mathbb{Z}_+ \). The \( k \)-jet of an \( n \)-dimensional submanifold \( N^n \subset E^{n+m} \) at a point \( a \in N \) is the equivalence class of \( n \)-dimensional submanifolds of \( E \) passing through \( a \), which are tangent to \( N \) with order \( k \). It will be denoted by \( [N]^k_a \). The totality of such jets forms a smooth manifold, denoted by \( J^k(E, n) = \bigcup_{N \subset E, a \in N} [N]^k_a \). There is a natural map

\[
j_k(N) : N \longrightarrow J^k(E, n), \quad a \mapsto [N]^k_a
\]

A function \( f \) on \( J^k(E, n) \) is defined to be smooth if for any \( n \)-dimensional submanifold \( N \subset E \) the function \( j_k(N)^* (f) \) is smooth. The so-defined smooth function algebra, denoted by \( \mathcal{F}_k(E, n) \), supplies \( J^k(E, n) \) with a smooth manifold structure. The minimal distribution on \( J^k(E, n) \) such that any submanifold of the form \( N_{(k)} = \text{Im} j_k(N) \subset J^k(E, n) \) is integral for it is the \( k \)-th order contact structure, or Cartan distribution on \( J^k(E, n) \). Denote it by \( \mathcal{C}_k(E, n) \). If \( m = 1 \), then \( \mathcal{C}_1(E, n) \) is the (classical) contact structure on \( J^1(E, n) \). For \( k \geq l \) the natural projection

\[
J^k(E, n) \xrightarrow{\pi_{k,l}} J^l(E, n), \quad [N]^k_a \mapsto [N]^l_a
\]

sends \( \mathcal{C}_k(E, n) \) to \( \mathcal{C}_l(E, n) \).

An integral submanifold \( L \) of a distribution \( \mathcal{D} \) is called locally maximal if it is not contained even locally in an integral submanifold of greater dimension. Submanifolds \( N_{(k)} \) are locally maximal integral submanifolds of \( \mathcal{C}_k(E, n) \) of dimension \( n \). But except the case \( k = m = 1 \) (contact geometry) there are locally maximal integral submanifolds of other types. More exactly, the type of such a submanifold \( U \) is the dimension of \( \pi_{k,k-1}(U) \), which may vary from 0 to \( n \). For instance, fibers of projection \( \pi_{k,k-1} \) are of type 0. It should be stressed that the notion of type is intrinsic, i.e., can be defined exclusively in terms of distribution \( \mathcal{C}_k(E, n) \). In
this sense the contact geometry on $J^1(E^{n+1}, n)$ is the only exception to the general case.

Recall that a system of (nonlinear) PDEs of order $k$ is geometrically interpreted as a submanifold $E$ of $J^k(E, n)$ for a suitable $E$. In this approach ”usual” solutions of $E$ are interpreted as submanifolds $N \subset E$ such that $N_{(k)} \subset E$. Moreover, this interpretation allows one to define an analogue of the notion of generalized solutions in the theory of linear PDEs for nonlinear equations. This is achieved by enlarging the class of submanifolds of the form $N_{(k)}$ to maximal integral submanifolds of type $n$, called $R$-manifolds. They play the role of Legendrian submanifolds in contact geometry.

While an $R$-manifold $U$ is, by definition, smooth, its singular point are defined to be singular points of the projection $\pi_{k,k-1} |_U$. Their totality, denoted by $U_{\text{sing}}$, is a union of submanifolds with singularities. If $\theta \in U_{\text{sing}}$, then the kernel of the differential $d\theta(\pi_{k,k-1} |_U) : T_{\theta}U \rightarrow T_{\pi_{k,k-1}(\theta)}J^{k-1}(E, n)$ at $\theta \in U$ is not trivial and is called the bend of $U$ at $\theta$. Denote it by $\mathcal{J}_\theta = \mathcal{J}_{\theta,U}$. We shall use the term $s$-bend for a bend of dimension $s$. The notion of a bend is key in the solution singularities theory (see subsection 3.4).

**Definition 2.2.** A (generalized) $s$-bend solution of an equation $E \subset J^k(E, n)$ is an $R$-manifold $U \subset E$ such that for any $\theta \in U_{\text{sing}}$ the dimension of $\mathcal{J}_\theta,U$ is $s$.

PDEs differ each other by types of generalized solutions they admit. Hence the problem of (local) classification of singularities of $R$–manifolds is a central one in geometrical theory of PDEs. In particular, as we shall see below, MA-equations are distinguished by the structure of singularities of their generalized solutions.

### 3. Monge-Ampère equations

#### 3.1. Classical Monge-Ampère equations

Recall that classical Monge-Ampère equations (MAEs) are PDEs in two independent variables of the form

$$N(u_{xx}u_{yy} - u_{xy}^2) + Au_{xx} + Bu_{xy} + Cu_{yy} + D = 0$$

with $N, A, B, C, D$ being some functions of variables $x, y, u, u_x, u_y$. In the current literature the term “Monge-Ampère equation” may refer to one of various generalizations of classical MAEs (see, for instance, [4]).

Definition (1) is descriptive and as such does not reveal the true nature of this class of equations. Our main goal here is to discover the hidden meaning behind analytical expression (1). First of all, it is important to stress that equations (1) are, in their turn, (locally) subdivided into 3 classes, namely, elliptic, parabolic, or hyperbolic ones, according to $AC - B^2 - 4ND < 0$, $= 0$, or $> 0$, respectively.

The first step toward revealing this meaning was due to S. Lie, who observed that the class of elliptic (resp., parabolic, or hyperbolic) MAEs is invariant with respect to contact transformations. Much later some authors and, first of all, Lychagin and his collaborators, interpreted a MA-equation as the condition $\omega|_L = 0$ imposed on Legendrian submanifolds $L$’s of a 5–dimensional contact manifold $M$ for a given effective 2-form $\omega$ (see [6], [8], [4]). The same condition imposed on Legendrian submanifolds of an arbitrary contact manifold is a natural generalization of classical MA-equations to higher dimensions. Nevertheless, though being coordinate-free, this definition is still of a descriptive character.
3.2. Jordan algebras of self adjoint operators. In this subsection $M$ stands for a contact manifold of dimension $2n + 1$. The curvature form allows one to distinguish an important class of operators in $\text{End}_{C^\infty(M)} \mathcal{D}$. Namely, a $C^\infty(M)$–homomorphism $A : \mathcal{D} \to \mathcal{D}$ is called self-adjoint if
\[ R(AX, Y) = R(X, AY), \quad \forall X, Y \in \mathcal{D}. \]

Scalar, i.e., multiplication by a function operators are, obviously, self-adjoint. Self-adjoint operators form a Jordan algebra, denoted by $\text{Sad}(\mathcal{X})$, with respect to the Jordan product
\[ A * B : = \frac{1}{2}(AB + BA). \]

$\text{Sad}(\mathcal{X})$ is, obviously, a $C^\infty(M)$–module, and in what follows we shall concentrate on JS-subalgebras of $\text{Sad}(\mathcal{X})$. These are submodules of $\text{Sad}(\mathcal{X})$ closed with respect to the Jordan product. Such a subalgebra is unital if it contains the identity endomorphism and hence the algebra $C^\infty(M)$ interpreted as a subalgebra of $\text{End}_{C^\infty(M)} \mathcal{D}$. Two JS-algebras are isomorphic if there is an isomorphism of supporting them $C^\infty(M)$–modules preserving the Jordan product.

By literally repeating the above definitions in the situation when an $\mathbb{R}$–vector space $V$ and a skew-symmetric bilinear form $\sigma = \langle \cdot, \cdot \rangle$ on $V$ taking values in a 1-dimensional vector space $W$ substitute $M$, $\mathbb{R}$ and $\mathcal{X}$, respectively, one gets notions of $(\sigma)$-self-adjoint operators in $\text{End} V$, JS-subalgebras in $\text{End} V$, etc.

If $A \subset \text{Sad}(\mathcal{X})$ is a JS-subalgebra, then its fiber $A_x$ at $x \in M$ inherits a structure of a JS-subalgebra in $\text{End}_\mathbb{R} \mathcal{D}_x$ with respect to the $\mathcal{X}_x$–valued form $R_x$. By choosing a base vector in $\mathcal{D}_x$ one may identify it with a symplectic (=skew-symmetric and nondegenerate) bilinear form on the vector space $V = \mathcal{D}_x$. Recall that a symplectic vector space (over $\mathbb{R}$) is an $\mathbb{R}$–vector space supplied with a non-degenerate skew-symmetric form. A unital JS-subalgebra of dimension $n$ with respect to the structure symplectic form on a $2n$–dimensional symplectic vector space will be called basic.

Importance of basic algebras is that they classify geometric solution singularities of (nonlinear) PDEs (see [12]). So, the problem of describing basic algebras is fundamental in the solution singularities theory. This problem is not trivial just because it includes the problem of describing finite-dimensional commutative algebras over $\mathbb{R}$.

By slightly abusing the language we shall call basic also a JP-subalgebra $\mathcal{A}$ of $\text{Sad}(\mathcal{X})$ such that $\mathcal{A}_x$ is basic in $\text{End}_\mathbb{R}(\mathcal{D}_x)$ for almost all $x \in M$, i.e., for all $x \in N$ where $N$ is an everywhere dense open in $M$. Almost all fibers $\mathcal{A}_x$ of $\mathcal{A}$ are of dimension $n$. By this reason we say that the dimension of $\mathcal{A}$ is $n$.

3.3. 2-dimensional unital JS-algebras. Here we shall illustrate the above-said in the simplest nontrivial case $n = 2$. But before we shall list some elementary properties of symplectic self-adjoint operators for arbitrary $n$.

Lemma 3.1. Let $A$ be an symplectic self-adjoint operator on a $2n$–dimensional symplectic vector space $V$. Then

1. The symplectic form $\langle \cdot, \cdot \rangle$ vanishes on any cyclic subspace
\[ C_v : = \text{Span}\{A^k(v)\}_{k \geq 0}, \quad v \in V. \]
(2) Root subspaces of $A$ corresponding to different eigenvalues of $A$ are symplectic orthogonal.

(3) $<\text{Ker } A, \text{Im } A> = 0$.

Proof. (1) This is an obvious consequence of $< Aw, w > = 0$ for any $w \in V$. But $< Aw, w > = < w, Aw > = -< Aw, w >$.

(2) The root subspace of $A$ corresponding to a real (resp., complex) eigenvalue $\lambda$ of it is of the form $\text{Ker } f(A)$ where $f(t) = (t-\lambda)^k$ (resp., $f(t) = (t^2 - (\lambda+\bar{\lambda})t + \lambda \bar{\lambda})^m$). If eigenvalues $\lambda_1$ and $\lambda_2$ are different, then the corresponding to them polynomials $f_1(t)$ and $f_2(t)$ are relatively prime and hence there are polynomials $g_1(t)$ and $g_2(t)$ such that

$$f_1(t)g_1(t) + f_2(t)g_2(t) = 1 \implies f_1(A)g_1(A) + f_2(A)g_2(A) = \text{id}_V.$$ 

I follows from the last relation that $f_1(A)$ is invertible on $\text{Ker } f_2(A)$ and vice versa. So,

$$0 = < f_1(A)(\text{Ker } f_1(A)), \text{Ker } f_2(A) > = < \text{Ker } f_1(A), f_1(A)(\text{Ker } f_2(A)) > = < \text{Ker } f_1(A), \text{Ker } f_2(A) > .$$

(3) If $v \in \text{Ker } A$, then $< v, Aw > = < Av, w > = 0$. □

Proposition 3.1. Let $V$ be a symplectic vector space of dimension 4 and $A$ be a non scalar, symplectic self-adjoint operator on $V$. Then the minimal polynomial $f(t)$ of $A$ is of second order and

(1) if $f(t)$ has complex roots, then $A$ supplies $V$ with a complex structure, all its proper subspaces are complex lines and these lines understood as 2-dimensional real planes are Lagrangian (elliptic case);

(2) if $f(t)$ has different real roots $\lambda_1, \lambda_2$, then eigenspaces $\text{Ker } (A - \lambda_i \text{id}_V)$ are symplectic orthogonal and non-Lagrangian planes (hyperbolic case);

(3) if roots of $f(t)$ coincide, i.e., $f(t) = (t-\lambda)^2$, then the unique eigenspace $W = \text{Ker } (A - \lambda \text{id}_V)$ of $A$ is a Lagrangian plane, which coincides with $\text{Im } (A - \lambda \text{id}_V)$. Lagrangian planes intersecting $W$ by a line are cyclic subspaces of $A$ of dimension 2 and vice versa (parabolic case).

Proof. First, note that a linear operator possesses a cyclic subspace whose dimension equals to the degree of its minimal polynomial. By lemma 3.1, (1), the symplectic form vanishes on a cyclic subspace and hence its dimension cannot be greater than 2. It cannot be 1-dimensional, since $A$ is not scalar. Hence the minimal polynomial of $A$ is of second order.

(1) If $\lambda_1 = a + bi$ and $B = b^{-1}(A - a \text{id}_V)$, then $B^2 = -\text{id}_V$. So, $B$ supplies $V$ with a structure of $\mathbb{C}$-vector space and $A$ is $\mathbb{C}$-linear. In view of lemma 3.1, (1), other assertions directly follows from this fact.

(2) In this case $V$ is the direct sum of eigenspaces. By lemma 3.1, (2), they are symplectic orthogonal, and, so, none of them could be of dimension 1. Indeed, the symplectic orthogonal to a line is a 3-dimensional subspace containing this line. So, the eigenspaces are 2-dimensional, and none of them can be Lagrangian, since they are symplectic orthogonal.

(3) If $B = A - \lambda \text{id}_V$, then $B^2 = 0$, i.e., $\text{Im } B \subset \text{Ker } B, W = \text{Ker } B$ and $B$ is symplectic self-adjoint. If $\text{Im } B$ is 1-dimensional, then $\dim(\text{Ker } B) = \dim W = 3$. On the other hand, by lemma 3.1, (3), $W$ is symplectic orthogonal to $\text{Im } B$ and, therefore, coincides with the symplectic orthogonal complement of $\text{Im } B$. If $0 \neq
v ∈ Im B and v = Bu, then u ̸∈ W and hence is not symplectic orthogonal to v. But this contradicts to the fact that <u, Bu> = 0 (lemma 3.1, (1)). Hence dim W = 2, W = Im B. Moreover, W is Lagrangian ss a symplectic orthogonal to Im B subspace.

Finally, let L be a Lagrangian plane that intersects W by a line ℓ and u ∈ L \ ℓ. Then 0 ̸= Bu ∈ W. But the span L' of u and Bu is a Lagrangian plane. It is symplectic orthogonal to ℓ, since such are u and Bu. But any Lagrangian plane, which is symplectic orthogonal to a line, contains this line. In particular, L' contains ℓ and, therefore, L' ∩ W = ℓ. So, Bu ∈ ℓ and hence L = L' and is cyclic. The converse is obvious.

□

Remark 3.1. It is curious to observe that a Lagrangian plane W in a symplectic V^2 defines a symplectic self-adjoint operator B such that B^2 = 0 and Ker B = W. Such an operator is unique up to a scalar factor. Indeed, if u ∈ V \ W, then Bu should belong to W and be symplectic orthogonal to u. Since the symplectic orthogonal complement of u intersects W by a line, say, ℓ_u, Bu should belong to ℓ_u and hence is unique up to a factor. In order to construct one such operator choose a non-Lagrangian and complementary to W plane U and an isomorphism h : U → W such that h(u) ∈ ℓ_u. Then any v ∈ V is uniquely presented in the form v = u + w, u ∈ U, w ∈ W, and we put Bu := h(u).

Recall that a 2-dimensional unital associative algebra is isomorphic to the algebra of ζ-complex numbers whose elements are of the form x + yζ with ζ^2 = −1, 0 or 1 and called double, dual or complex, numbers, respectively. Accordingly, denote these algebras by C−(= ℂ), C_0 and C_+.

Corollary 3.1. There are 3 isomorphism classes of basic algebras for n = 2 represented by algebras C_+ and C_0.

Proof. Any basic algebra for n = 2 is generated by id_V and a non-scalar symplectic self-adjoint operator A. A linear combination of these operators is an operator B such that B^2 = ±id_V or 0. It follows from proposition 3.1 that two such operators of the same type are symplectically isomorphic.

□

Remark 3.2. There is a simple approach to description of symplectic self-adjoint operators based on following observations.

(1) If W is an n-dimensional ℝ-vector space, then V = W ⊕ W* is symplectic with respect to the form [(w_1, φ_1), (w_2, φ_2)] = φ_1(w_2) − φ_2(w_1).

(2) Let W and W' be Lagrangian subspaces of a symplectic vector space V, which are complementary to each other. Then i : w' →< w', · >, w' ∈ W', is an isomorphism between W' and W*, which extends to the symplectic isomorphism v = w + w' → w ⊕ i(w') between V and W ⊕ W*.

(3) If F ∈ EndW, then F ⊕ F* ∈ EndW ⊕ W* is symplectic self-adjoint.

Now it is easily follows from (1)-(3) and lemma 3.1, (1), that any symplectic self-adjoint operator is of the form F ⊕ F*.

The approach we have chosen above is motivated by the fact that it is better adapted to the solution singularities theory.

3.4. Geometric singularities of solutions of PDEs. Here we shall assemble some facts concerning classification of singularities of R-manifolds that are necessary to encode the meaning of classical MA-equations (see [12], [14]). The classification
group is that of symmetries of the Cartan distribution $C_k(E, n)$. According to the Lie-Bäcklund theorem this group consists of diffeomorphisms of $E$ lifted to $J^k(E, n)$ if $m > 1$ and of lifted contact transformations of $J^1(E, n)$ if $m = 1$.

The simplest question here is the first order classification, i.e., classification of tangent spaces to $R$-manifolds at singular points. The first result in this direction is that this classifications is equivalent to classification of bends (see subsection 2.3). The second important fact is that the classification of bends depends only on the dimension, say, $s$, of bends, i.e., does not depends on $m \geq 1, k > 1$ and $n \geq s$. This reduces the problem to the case $m = 1, k = 2, n = s$.

As it follows from the definition, the bend at $\theta$ of an $R$–manifold $U$ is a subspace of $T_\theta \Phi$ where $\Phi$ is the fiber of $\pi_{k,k-1}$ passing through $\theta$. If $m = 1$, then $T_\theta \Phi$ is identified with the space $P_{k,n}^k$ of homogeneous polynomials of degree $k$ in $n$ variables over $\mathbb{R}$ (see [15]). In these terms, the $s$–bend of an $R$–manifold passing through $\theta$ can be characterized as a special subspace $\mathfrak{J}$ in $P_{k,s}$, which, by abusing the language, we shall also call a bend.

It holds the following key assertion (see [14])

**Proposition 3.2.** If $\mathfrak{J} \subset P_{k,s}$ is an $s$–bend, then the span of polynomials in $P_{k+1,s}$ whose derivatives belong to $\mathfrak{J}$ is also an $s$–bend in $P_{k+1,s}$.

In the case of MA–equations we have $m = 1, k = n = 2$. Peculiarity of this case is twofold. First, it is easy to see that any 2-dimensional subspace of $P_{2,2}$ in the condition of proposition 3.2, and hence any 2-dimensional subspace tangent to a fiber of $\pi_{2,1}$ is a 2-bend. Then, a second order PDE $E \subset J^2(E^3, 2)$, generally, intersects fibers of $\pi_{2,1}$ by 2-dimensional submanifolds. So, tangent spaces to these submanifolds are 2-bends.

### 3.5. 2-bends, basic algebras and $\zeta$–holomorphic functions.

Proposition 3.2 establishes a natural relation between bends and basic algebras, which we shall make explicit in the case $n = 2, m = 1$ (see [14] for the general case).

Let $\mathfrak{J} \subset P_{k,2}, k \geq 2$, be a bend. Then, according to proposition 3.2, there is a polynomial $f(x, y) \in P_{k+1,2}$ whose derivatives $f_x$ and $f_y$ form a basis of $\mathfrak{J}$. By the same proposition there should be a non proportional to $f$ polynomial $g$ whose derivatives belong to $\mathfrak{J}$. Hence $g_x = \alpha f_x + \beta f_y, g_y = \gamma f_x + \delta f_y$, $\alpha, \ldots, \delta \in \mathbb{R}$, and $(\alpha f_x + \beta f_y)_y = (\gamma f_x + \delta f_y)_x$, or, equivalently,

$$
\gamma f_{xx} + (\delta - \alpha) f_{xy} + \beta f_{yy} = 0.
$$

(2)

Since $g$ is not proportional to $f$, coefficients of equation 2 do not vanish simultaneously, and this equation can be brought to one of the forms $u_{xx} \pm u_{yy} = 0$ or $u_{xx} = 0$. Accordingly, one of basic basic algebras $\mathbb{R}_x$ or $\mathbb{R}_y$ is associated with $\mathfrak{J}$.

More exactly, this basic algebra is generated by the operator whose matrix is

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}.
$$

Normal forms of 2-bends are easily deduced from this fact and they are

$$
\text{Span}(\text{Re } z^k, \text{Im } z^k), \quad z = x + \zeta y,
$$

where Re and Im refer to the “real” and “imaginary” parts of a $\zeta$-complex number (see subsection 3.3), respectively.

Examples of R-manifolds with one singular point, in which these normal 2–bends are realized, are easily constructed in terms of $\zeta$-holomorphic functions. Namely,
let $N$ be a 2-dimensional manifold. An $C^\infty(N)$-homomorphism $A: D(N) \to D(N)$ such that $A^2 = \zeta^2 \text{id}_{D(N)}$ supplies $N$ with a structure of a $\zeta$-complex curve. A function $u \in C^\infty(N)$ is $\zeta$-harmonic if $A^*(du) = dv$ for a function $v \in C^\infty(N)$. Here $A^*: \Lambda^1(N) \to \Lambda^1(N)$ is the dual to $A$ homomorphism. The function $v$ is unique up to a constant and is called $(\zeta)$-conjugate to $u$. It is also $\zeta$-harmonic and together with $u$ form a $\mathbb{C}_\zeta$-valued function $f = u + \zeta v$. Such a function is called $\zeta$-holomorphic.

Generally, non-constant $\zeta$-holomorphic functions on $N$ exist only locally. A pair of $\zeta$-conjugate local functions $x$ and $y$ on $N$ forms a local $\zeta$-complex chart on $N$, and, locally, $\zeta$-holomorphic functions may be viewed as functions of $\zeta$-complex variable $z = x + \zeta y$. In particular, the transition function between two $\zeta$-complex charts is $\zeta$-holomorphic. Standard complex curves are, obviously, $\zeta$-complex ones for $\zeta^2 = -1$. If $\zeta^2 = 1$ (resp., $\zeta^2 = 0$), then, as it is easy to see, a $\zeta$-complex curve is a 2-dimensional manifold supplied with two transversal to each other 1-dimensional distributions (resp., one 1-dimensional distribution).

Locally, the condition $A^*(du) = dv$ is equivalent to $(d \circ A \circ d)(u) = 0$. The operator $\Delta_\zeta = d \circ A \circ d$ will be called the $\zeta$-Laplace operator, since in a $\zeta$-complex chart it reads $\Delta_\zeta = \partial^2/\partial x^2 - \zeta^2 \partial^2/\partial y^2$. Accordingly, $u_{xx} - \zeta^2 u_{yy} = 0$ is the $\zeta$-Laplace equation. So, in these terms, $\zeta$-harmonic functions are solutions of the $\zeta$-Laplace equation. This equation express the compatibility condition for the $\zeta$-Cauchy-Riemann equations $A^*(du) = dv$, which in a $\zeta$-complex chart read $u_x = -\zeta^2 v_y$, $u_y = \zeta^2 v_x$. In other words, a function $f = u + \zeta v$ is $\zeta$-holomorphic iff its real and imaginary parts $u = \text{Re} f$ and $v = \text{Im} f$, respectively, satisfy the $\zeta$-Cauchy-Riemann equations.

Now consider the $(k − 2)$-th prolongation $\mathcal{E}_{(k-2)} \subset J^k(E, n)$ of the $\zeta$-Laplace equation $\mathcal{E} \subset J^2(E, n)$. It is given by equations

$$u_{2+r,s} - \zeta^2 u_{r,s+2} = 0, \quad r + s \leq k - 2$$

with $u_{p,q}$ being the function on $J^{p+q}(E, n)$ representing the operator $\partial^{p+q}/\partial x^p \partial y^q$ in a standard local jet-chart. Intersections of $\mathcal{E}_{(k-2)}$ with fibers of $\pi_{k,k-1}$ are given by equations

$$u_{p,q} = \text{const}, \quad p + q \leq k - 1, \quad u_{2+r,s} - \zeta^2 u_{r,s+2} = 0, \quad r + s = k - 2$$

and hence are 2-dimensional. Vectors

$$\nu_1 = \sum_{r=0}^{[k/2]} \zeta^{2r} \partial/\partial u_{2r,k-2r}, \quad \nu_2 = \sum_{r=0}^{[k-1/2]} \zeta^{2r} \partial/\partial u_{1+2r,k-1-2r}$$

form a basis of the tangent space to such an intersection. In coordinates the identification of tangent to fibers of $\pi_{k,k-1}$ vectors with homogeneous polynomials of degree $k$ (see subsection 3.4) looks as

$$\frac{\partial}{\partial u_{r,k-r}} \leftrightarrow \frac{1}{r!(k-r)!} x^r y^{k-r}$$

So, the vectors $\nu_1$ and $\nu_2$ are identified with $\frac{1}{k!} \text{Re}(x + \zeta y)^k$ and $\frac{1}{k!} \text{Im}(x + \zeta y)^k$, respectively.

Let now $1 < l \in \mathbb{Z}$ and consider the $\mathbb{R}$-manifold $L^\zeta_{k,l}$ in $J^k(E, n)$ given by equations (3) and equations:
where \((s + \frac{1}{2})! := (1 + \frac{1}{2})(2 + \frac{1}{2}) \cdots (s + \frac{1}{2}), \ s \in \mathbb{Z}_+.\) This R-manifold has the unique singular point \(u_{(r,s)} = 0, \ r + s \leq k,\) in which the bend is \(\text{Span}(\text{Re}z^k, \text{Im}z^k).\) It may be viewed as the real part of the \(k\)-th jet of the multivalued \(\zeta\)-holomorphic function \(f(z) = z^k + \zeta.\)

It is not difficult to show that R-manifolds \(L^\zeta_{k,l}\) corresponding to different \(l\)’s are not equivalent, while they have the common bend.

Remark 3.3. The problem of finding \(s\)-bend solutions for relevant equations of mathematical physics and differential geometry was not yet systematically studied even for \(s = 2.\) Some 2-bend solutions of the vacuum Einstein equations were constructed in [10]. Among them are foam-like solutions describing “parallel universes” separated by singularities. The square root of the Schwarzschild metric is the simplest of that kind.

3.6. Intrinsic definition of classical Monge-Ampère equations. Now we are ready to formulate a conceptual definition of classical MA-equations. We say that a Legendrian submanifold \(L \subset M\) is invariant with respect to a JS-subalgebra \(A \subset \text{Sad}(\kappa)\) if \(T_xL\) is \(A_x\)-invariant for all \(x \in M.\) \(A\)-invariant Legendrian submanifolds are called solutions of this problem.

In order to connect definitions 1 and 3.1, recall that according to the classical Darboux lemma, a given contact form \(\omega\) on a \(2n + 1\)-dimensional manifold locally admits a Darboux chart \((x_i, p_i, u),\ i = 1, \ldots, n,\) in which it takes the canonical form \(\omega = du - \sum_{i=1}^n p_i dx_i.\) A regular with respect to this chart Legendrian submanifold is given by equations

\[ u = f(x), \ p_i = \frac{\partial f(x)}{\partial x_i}, \ i = 1, \ldots, n, \ \text{with} \ x = (x_1, \ldots, x_n). \]

To underline that such a submanifold is described by a function \(f(x)\) we denote it by \(L_f.\)

Proposition 3.3. Solutions of equation (1) are solutions of a classical Monge-Ampère problem for Legendrian submanifolds of the form \(L_f\) and vice versa.

Proof. A natural basis of \(\mathcal{D}\) in the Darboux chart \((x_i, p_i, u)\) is

\[ \partial_{x_1} + p_1 \partial u, \ \partial_{x_2} + p_2 \partial u, \ \partial_{p_1}, \ \partial_{p_2}, \]

and the operator $\mathfrak{A}$ whose matrix in this basis is
\[
\begin{pmatrix}
  B & -2A & 0 & -2N \\
  2C & -B & 2N & 0 \\
  0 & 2D & B & 2C \\
  -2D & 0 & -2A & -B
\end{pmatrix}
\]
is such that $\mathfrak{A}^2 = \Delta I$ where $I$ is the unit matrix and $\Delta = B^2 - 4AC + 4ND$. Recall that equation (1) is elliptic (resp., parabolic, or hyperbolic) if $\Delta < 0$ (resp., = 0, or $> 0$). $\mathfrak{A}$ and $id_D$ span a basic algebra $A$. Obviously, solutions of the corresponding Monge-Ampère problem are $\mathfrak{A}$-invariant Legendrian submanifolds.

Vector fields
\[
Z_1 = \partial_{x_1} + p_1 \partial u + f_{x_1 x_1} \partial_{p_1} + f_{x_1 x_2} \partial_{p_2}, \quad Z_2 = \partial_{x_2} + p_2 \partial u + f_{x_2 x_1} \partial_{p_1} + f_{x_2 x_2} \partial_{p_2}
\]
are tangent to $L$, while
\[
\mathfrak{A}(Z_1) = (B - 2f_{x_1 x_2}N)Z_1 + 2(C + f_{x_1 x_1}N)Z_2 - 2E\partial p_2
\]
where $E = N(f_{x_1 x_1}f_{x_2 x_2} - f_{x_1 x_2}^2) + Af_{x_1 x_1} + Bf_{x_1 x_2} + Cf_{x_2 x_2} + D$, and similarly for $\mathfrak{A}(Z_2)$. This shows that $L$ is $\mathfrak{A}$-invariant iff $E = 0$.

Thus this proposition establishes a one-to-one correspondence between MA-equations and 2-dimensional basic algebras on contact 5-folds. This correspondence put in evidence the fact that any solution of an MA-equation $\mathcal{E}$ is a bi-dimensional manifold $L$ supplied with an algebra of endomorphisms of $D(L)$. Namely, this algebra, denoted by $\mathcal{A}_L$, is composed from restricted to $L$ elements of the associated with $\mathcal{E}$ basic algebra $A$. If $\mathcal{E}$ is elliptic (resp., parabolic or hyperbolic), then $\mathcal{A}_L$ is of type $\mathbb{C}_-$ (resp., $\mathbb{C}_0$ or $\mathbb{C}_+$), i.e., $L$ is a $\zeta$-complex curve for the corresponding $\zeta$. In fact, solutions of arbitrary second order PDE in two independent variables are naturally supplied with a such an algebra of endomorphisms. This is due to the fact that any 2-dimensional subspace tangent to a fiber of $\pi_{2,1}$ is a 2-bend.

**Remark 3.4.** One of central questions in solution singularity theory is to what extent a given PDE $\mathcal{E}$ is predetermined by behavior of singularities of its solutions. More exactly, a series of subsidiary equations $\mathcal{E}_\Sigma$ where $\Sigma$ is a singularity type admitted by solutions of $\mathcal{E}$ is associated with $\mathcal{E}$. See [5] for some examples. The reconstruction problem is: whether $\mathcal{E}$ can be reconstructed if all equations $\mathcal{E}_\Sigma$ are known? A spectacular, though implicit, historical example of solution of this problem is Maxwell’s deduction of his famous equations from previously found elementary (Coulomb, ..., Faraday) laws of electricity and magnetism, which, according to the modern point of view, describe behavior of singularities of electromagnetic fields. In this context classical MA-equations are distinguished by the fact that the reconstruction problem for them is a tautology, namely, by definition 3.1.

3.7. **Concluding remarks.** The above interpretation of classical MA-equations was enlightening in the development fundamentals of solution singularities theory as one of key examples. Indeed, the discussed in this section constructions and results can be generalized to dimensions $n > 2$. For instance, definition 3.1 in an obvious manner generalizes to higher dimensions. It is not difficult to see that $n$-dimensional Monge-Ampère problem is described by a system of $\frac{n(n-1)}{2}$ second order PDEs, which which may be viewed as another natural generalization of classical MA-equations.
This interpretation is useful in the study of classical MA-equations themselves. In particular, it suggests more exact techniques for integrating concrete MA-equations, finding their classical symmetries, conservation laws, etc. For instance, classical infinitesimal symmetries of the MA-equation associated with a basic algebra $A$ may be defined as contact fields $Z$ such that $[Z,A] \subset A$ and this interpretation directly leads to a simple computational procedure.

Another example we would like to mention here is the classical problem of contact classification of MA-equations, which, essentially, is equivalent to the problem of finding a sufficient number of their scalar differential invariants. Most systematically this problem was studied by V. Lychagin and his collaborators (V. Rubtsov, B. Kruglikov, A. Kushner and others) in last three decades. These authors exploited Lychagin's idea to represent MA-equations in terms of effective 2-forms. Most complete results in this approach were obtained by A. Kushner (see short note [3]). Full details of his work are reproduced in monograph [4] together with earlier results of these authors. In these works scalar differential invariants of hyperbolic and elliptic MA-equations are constructed indirectly as differential invariants of the associated e–structures. On the contrary, definition 3.1 allows a direct construction of scalar differential invariants in terms of operators of the corresponding basic algebra, which leads to more complete and exact results (see [7], [2], [13], [1]).

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