Additive Average Schwarz Method for Elliptic Mortar Finite Element Problems with Highly Heterogeneous Coefficients

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Abstract In this paper, we extend the additive average Schwarz method to solve second order elliptic boundary value problems with heterogeneous coefficients inside the subdomains and across their interfaces by the mortar technique, where the mortar finite element discretization is on nonmatching meshes. In this two-level method, we enrich the coarse space in two different ways, i.e., by adding eigenfunctions of two variants of the generalized eigenvalue problems. We prove that the condition numbers of the systems of algebraic equations resulting from the extended additive average Schwarz method, corresponding to both coarse spaces, are of the order $O(H/h)$ and independent of jumps in the coefficients, where $H$ and $h$ are the mesh parameters.

Keywords Domain decomposition method · Additive Schwarz method · Generalized eigenvalue problem · Mortar finite elements
1 Introduction

Domain decomposition methods are efficient and powerful iterative methods to solve large algebraic systems arising from a finite element discretization of elliptic boundary value problems [32,35,37]. They can also be regarded as a procedure of producing preconditioners for other iterative methods, such as the conjugate gradient method, for achieving fast convergence. In both approaches, to solve an original problem defined on a bounded Lipschitz domain \( \Omega = \bigcup_{i=1}^{N} \Omega_i \), it is equivalent to solve many subproblems defined locally on the subdomains \( \Omega_i \) in parallel. To obtain fast convergence, Dryja and Widlund [12], and Matsonki and Nepomnyaschikh [27] proposed to add one global problem and introduced the additive Schwarz methods to solve the global and local problems in parallel.

Corresponding to the global and local problems, different coarse and fine spaces can be constructed, for instance, see [5,7,11,29]. It is worth mentioning that coarse spaces are more important than fine spaces since they have a key role in the central of scalability for domain decomposition methods. Therefore, in [7] one of the simplest and efficient ways to construct coarse space, called the average coarse space, was proposed. Consequently, the two-level additive average Schwarz method was introduced and developed to solve many kinds of elliptic problems with the continuous and discontinuous coefficients, see [8,14,17,24]. Hence, an extension of the additive average Schwarz method to solve elliptic model problems arising from many applications such as composite materials with highly heterogeneous coefficients is of particular interest in this paper because it has a high-level performance.

In general, in terms of distributions of the coefficients, we can classify elliptic problems with the heterogeneous coefficients into two classes, i.e., elliptic problems with jumps in the coefficients only inside the subdomains and ones with jumps in the coefficients both inside the subdomains and on the subdomain interfaces. For the first class, where the heterogeneous coefficients are piecewise constants for subdomains \( \Omega_i, i = 1, \ldots, N \), the additive Schwarz method was developed and analyzed in [7,9,30,36] and references therein. For the second class with the large variation in the heterogeneous coefficients, the classical coarse spaces lead the condition numbers of the preconditioned systems to blow up. Consequently, the convergence rates of iterative methods will deteriorate. To alleviate this difficulty, the coarse spaces can be enriched by combining their structure with spectral spaces.

The idea of the coarse spectral space was introduced in [6] and extended as the spectral algebraic multigrid method in [15]. The spectral construction of this new space is achieved by solving the generalized eigenvalue problems locally. Due to its crucial role in preventing the impact of large jumps in the coefficients on bounds of the condition numbers of preconditioners, the new coarse space has received considerable attention. As a result, it is extensively developed for overlapping Schwarz methods [16,18,19,34], nonoverlapping additive Schwarz method [26], balancing domain decomposition methods [20].
and nonlinear domain decomposition, where all nodes on the subdomains interfaces for all references are matching grids. Hence, in this paper, we focus on the nonmatching grids, which are common due to heterogeneous materials in real-life problems. In practice, one may use different triangulations for polygonal subdomains independent of other triangulated subdomains. More precisely, there exist situations, where two subdomains with a common interface have fine and coarse (or fine but with different mesh sizes compared to other) triangulations. Therefore, the nonmatching grids on the subdomains interfaces are unavoidable and cause consistency errors in numerical methods, which can be handled by using the mortar techniques. Hence, the additive Schwarz method for the mortar finite element method was introduced, and modified with Crouzeix-Raviart mortar finite elements.

The main aim of this paper is to enrich the coarse space used in the classical nonoverlapping additive average Schwarz method by using the idea of coarse spectral space for elliptic problems with highly heterogeneous coefficients inside the subdomains and across their interfaces where the mortar finite element discretization is on the nonmatching meshes. To achieve this, our new coarse space consists of two subspaces. The first subspace is the common coarse space in the classical additive average Schwarz method used in , i.e., for the fixed $i$, is the range of a linear operator defined on $\Omega_i$ such that it is either the nodal values of a function $u \in V_h$ inside $\Omega_i$ or the average of nodal values of the function $u$ on the mortar and nonmortar sides of $\Omega_i$, where $V_h$ is a finite space of $P_1$ conforming elements defined on a fine triangulation of $\Omega$ and vanishing on $\partial \Omega$. The second subspace has a particular spectral structure. To obtain the basis functions for this subspace, we solve the generalized eigenvalue problems restricted to each subdomain as in .

To define the proper generalized eigenvalue problems, we require determining minimum values of the coefficients over each subdomain’s triangulation to estimate the condition number of additive average Schwarz preconditioners independent of the large eigenvalues caused by the large jumps in the coefficients. Motivated by the ideas from , we consider two different types of the generalized eigenvalue problems based on either minimum of the coefficients over the whole subdomain or just minimum of ones over the layer connected to the boundary of the subdomain with one vertex or with one edge of the triangles inside of the subdomain. Solving these generalized eigenvalue problems lead to finding orthogonal basis functions enriching the coarse spaces. With these new coarse spaces, we prove that the condition numbers of the produced preconditioners are of the order $O(H/h)$ and independent of the number of subdomains. Due to the definition of the second type layer, it has faster performance than the first one through the implementations with numerical-software packages. See also Section for numerical results.

The outline of this paper is as follows. In Section after introducing a discrete problem, we define the mortar condition and the space of basis
functions satisfying that condition. Furthermore, several figures related to those functions are also given there. Section 3 is devoted to introducing the additive average Schwarz method, where the average interpolation operator has two different types. This operator consists of the natural extension of the standard average interpolation operator for mortar case and orthogonal operators defined in the next section. In Section 4, the generalized eigenvalue problems in terms of how the minimum values of jumps in the coefficients over the subdomains’ triangulations can be defined are introduced. In Section 5, following the standard additive Schwarz framework [32], we drive an optimal estimate of the condition number of the produced preconditioners with the aid of removing bad eigenvalues, which are influenced by the large jumps in the coefficients. Finally, to verify our theoretical results’ validity, in Section 6 some numerical experiments are reported.

2 Discrete problem

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Lipschitz domain with a nonoverlapping partition \( \{ \Omega_i \}_{i=1}^N \) of polygonal subdomains such that \( \overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega_i} \). We consider an elliptic model problem defined on \( \Omega \): Find \( u_\ast \in H^1_0(\Omega) \) such that

\[
a(u_\ast, v) = f(v), \quad v \in H^1_0(\Omega),
\]

where

\[
a(u, v) = \sum_{i=1}^N a_i(u, v) = \sum_{i=1}^N (\alpha_i \nabla u, \nabla v)_{L^2(\Omega_i)}
\]

and

\[
f(v) = \int_{\Omega} fv \, dx = \sum_{i=1}^N \int_{\Omega_i} fv \, dx.
\]

Here, \( f \in L^2(\Omega), \alpha \in L^\infty(\Omega) \) and \( \alpha_i(x) \) is the restriction of \( \alpha(x) \) over \( \Omega_i \). Further, we assume that there exists a positive constant \( \alpha_0 \) such that \( \alpha(x) > \alpha_0 \).

The partition \( \{ \Omega_i \}_{i=1}^N \) forms a coarse triangulation of \( \Omega \) with the mesh parameter \( H = \max\{H_i: i = 1, \ldots, N\} \), where \( H_i \) are diameters of \( \Omega_i \). We assume this partition to be geometrically conforming, i.e., \( \partial \Omega_i \cap \partial \Omega_j \) (\( i \neq j \)) is a vertex or a whole edge of both subdomains \( \Omega_i \) and \( \Omega_j \) or is empty. Further, we denote the set of all vertices of such coarse triangulation, except those belonging to \( \partial \Omega \), by \( \mathcal{V}_H \). We also denote the triangulation of the subdomain \( \Omega_i \) by \( T_i \), which consists of triangles satisfying the shape regular property [10] inside of \( \Omega_i \), and quasi-uniform triangles touching \( \partial \Omega_i \) with the mesh size \( h_i \). Further, we denote the set of all internal nodes of the fine triangulation \( T_i \) by \( \mathcal{N}_i \).

We define the product space \( X_h \) on the computational domain \( \Omega \) by

\[
X_h(\Omega) = X_1(\Omega_1) \times \ldots \times X_N(\Omega_N),
\]
where $X_i(\Omega_i), i = 1, \ldots, N$ are the finite element spaces of the continuous piecewise linear functions defined on $\mathcal{T}_i$ and vanishing on $\partial \Omega \cap \partial \Omega_i$. We denote all nodal points on $\bigcup_{i=1}^{N} \mathcal{T}_i$ except those on $\partial \Omega$ by $\mathcal{N}_\Omega$. Further, we denote the set of basis functions associated with the set of nodal points $\mathcal{N}_\Omega$ by $\{\phi_l\}_{l \in \mathcal{N}_\Omega}$.

Due to independent triangulations inside of each subdomain, on each side of the interface $\Gamma_{ij} = \Omega_i \cap \Omega_j$ we may have different discretization (cf. Figure 1). We select one side of $\Gamma_{ij}$ as the mortar side, and the other side as the nonmortar side denoted by $\gamma_m(i)$ and $\delta_m(j)$, respectively. It is obvious that $\Gamma_{ij} = \gamma_m(i) = \delta_m(j)$. Further, we denote the nodes on the mortar and nonmortar sides by $m_0, m_1, \ldots, m_{n_m+1}$ and $s_0, s_1, \ldots, s_{n_s+1}$, respectively.

Let $W_{h,i}(\Gamma_{ij})$ and $W_{h,j}(\Gamma_{ij})$ denote the restrictions of $X_i(\Omega_i)$ and $X_j(\Omega_j)$ onto $\Gamma_{ij}$, respectively. Now, the nonmatching grids on the subdomain interfaces impose discontinuities for the functions belong to $X_h$. Therefore, we need to define a weak continuity condition. To this end, we first define the projection $P_m(u_i, \text{Tr} v_j) : L^2(\delta_m(j)) \to W_{h,i}(\delta_m(j))$ by

$$
\int_{\delta_m(j)} (u_i|_{\gamma_m(i)} - P_m(u_i, \text{Tr} v_j)) \psi \, dx = 0, \quad \gamma_m(i) = \Gamma_{ij} = \delta_m(j),
$$

for all functions $\psi \in M_{h,j}(\delta_m(j)) = \text{span}\{\phi_l\}_{l=0}^{n_s+1}$ and

$$
P_m(u_i, \text{Tr} v_j)|_{\delta_m(j)} = v_j|_{\delta_m(j)},
$$

where $u_i$, $\text{Tr} v_j$, and $M_{h,j}(\delta_m(j))$ are the restriction of $u$ into each subdomains $\Omega_i, i = 1, \ldots, N$, the trace of $v_j$, and a subspace of $W_{h,j}(\delta_m(j))$ with constant values on the elements touching $\partial \delta_m(j)$, more precisely, at two end points of $\delta_m(j)$, respectively. Now, we say a function $u_h = \{u_i\}_{i=1}^{N} \in X_h$ satisfies the mortar condition on $\delta_m(j)$, if (2) holds. We denote the mortar space by $V_{h}$ in terms of the mortar condition, i.e.,

$$
V_h = \{u_h, v_h \in X_h \mid P_m(u_h, \text{Tr} v_h) = 0, \forall \Gamma_{ij}, i, j = 1, \ldots, N\}.
$$
Fig. 2 The node $x_k$ is represented by the black and thick dot, where the value of the basis function $\phi_k(x)$ associated with $x_k$ is 1. Figures (I)-(IV) illustrate the different positions of $x_k$.

To see the structure of the basis functions spanning the mortar space $V_h$, let $\phi_k^{(i)}(x)$ be a basis function defined on $\Omega_i$ and define $V_h = \text{span}\{\phi_k\}$, where $\phi_k(x)$ is a basis function associated with a node $x_k$. Due to different positions of $x_k$, $\phi_k(x)$ takes different forms as follows (cf. Figure 2).

1. $x_k \in \Omega_i$:
   $$\phi_k(x) = \phi_k^{(i)}(x).$$

2. $x_k \in \{m_1, \ldots, m_{m_m}\}$:
   $$\phi_k(x) = \begin{cases} 
   \phi_k^{(i)}(x), & \text{on } \Omega_i, \\
   \Pi_m(\phi_k^{(i)}(x), 0)(x), & \text{on } \delta_{m(j)}, \text{ where } \gamma_m(i) = \delta_m(j), \\
   0, & \text{otherwise}. 
   \end{cases}$$

3. $x_k \in N_H$:

(a) $x_k$ is a point of intersection between two mortar sides $\gamma_{m(i)}$ and $\gamma_{n(i)}$:

$$
\phi_k(x) = \begin{cases} 
\phi_k^{(i)}(x), & \text{on } \gamma_{m(i)} \text{ and } \gamma_{n(i)}, \\
\Pi_m(\phi_k^{(i)}, 0)(x), & \text{on } \delta_{m(j)}, \\
\Pi_n(\phi_k^{(i)}, 0)(x), & \text{on } \delta_{n(j)}, \\
0, & \text{otherwise}.
\end{cases}
$$

(b) $x_k$ is a point of intersection between mortar side $\gamma_{m(i)}$ and nonmortar side $\delta_{n(i)}$:

$$
\phi_k(x) = \begin{cases} 
\phi_k^{(i)}(x), & \text{on } \gamma_{m(i)}, \\
\Pi_m(\phi_k^{(i)}, 0)(x), & \text{on } \delta_{m(j)}, \\
\Pi_n(0, \text{Tr } \phi_k^{(i)})(x), & \text{on } \delta_{n(i)}, \\
0, & \text{otherwise}.
\end{cases}
$$

(c) $x_k$ is a point of intersection between two nonmortar sides $\delta_{m(i)}$ and $\delta_{n(i)}$:

$$
\phi_k(x) = \begin{cases} 
\phi_k^{(i)}(x), & \text{on } \gamma_{m(i)} \text{ and } \gamma_{n(j)}, \\
\Pi_m(0, \text{Tr } \phi_k^{(i)})(x), & \text{on } \delta_{m(i)}, \\
\Pi_n(0, \text{Tr } \phi_k^{(i)})(x), & \text{on } \delta_{n(i)}, \\
0, & \text{otherwise}.
\end{cases}
$$

We now express the main problem as in the following form: Find $u_h^* = \{u_i^h\}_{i=1}^N \in V_h$ such that

$$
a_h(u_h^*, v_h) = f(v_h) \quad \forall v_h \in V_h. \quad (3)
$$

In what follows, we consider the following matrix representation of the linear systems arising from the discretization of $(3)$:

$$
A v = f,
$$

where $v$ is the vector of all unknown coefficients defined on $\overline{\Omega}$. Further, we consider the submatrices $A_{\Omega_i} = R_iAR_i^T$, where $R_i, i = 1, \ldots, N$ are the restriction matrices such that $v_i = R_i v$ are the vectors of coefficients defined on $\Omega_i \setminus \partial \Omega_i$.

Employing the mortar condition, we can compute some coefficients of $v$ in terms of other ones. More precisely, where $\Gamma_{ij} = \gamma_{m(i)} = \delta_{m(j)}$ assume $\nu_\gamma = (v_h(s_i))_{i=1}^n, \nu_\epsilon = (v_h(s_0), v_h(s_{n+1}))^T, \nu_m = (v_h(m_i))_{i=0}^{n+1}$ and consider the following matrix representations as in [13].

$$
M_\gamma := ((\phi_{s_i}, \phi_{m_j})_{L^2(\gamma)})_{i=1; j=0}^{n+1} \in \mathbb{R}^{n_x \times (n_m+2)},
$$

$$
S_\gamma := ((\phi_{s_i}, \phi_{s_j})_{L^2(\gamma)})_{i=1; j=0}^{n_x} \in \mathbb{R}^{n_x \times n_x},
$$

$$
C_\gamma := ((\phi_{s_i}, \phi_{m_j})_{L^2(\gamma)})_{i=1; j=0, n+1}^{n_x} \in \mathbb{R}^{n_x \times 2}.
$$
Hence
\[ \nu_s = S^{-1}_t (M_s \nu_m - C_s \nu_c). \]

Consequently, in the next sections, we will focus only on all mortar, corner, and interior nodes.

3 The additive average Schwarz method

Let \( V^{(i)} \), \( i = 1, \ldots, N \) be decompositions of \( V_h \) that are restrictions of \( V_h \) to \( \Omega_i \) with zero on \( \partial \Omega_i \) and on the subdomains \( \Omega_j \), \( j \neq i \). Further, let \( V_0^{(\text{type})} \) stands for two different types of the coarse spaces, distinguished by two notations I and II, such that
\[ V_h = V_0^{(\text{type})} + \sum_{i=1}^N V^{(i)}, \text{ type } \in \{I, II\}, \]
where
\[ V_0^{(\text{type})} = I_0 V_h + \sum_{i=1}^N W_i^{\text{type}}, \text{ type } \in \{I, II\}. \]

We define the operator \( I_0 \) and the spaces \( W_i^{\text{type}}, \text{ type } \in \{I, II\} \) in the next section. Here, we use the exact bilinear form for all our subproblems. Thus, we have
\[ a(u^h, v^h) = a_i(u^h, v^h), \quad i = 1, \ldots, N, \]
where \( u^h = \{u^h_i\}_{i=1}^N \in V^{(i)} \) and \( v^h = \{v^h_i\}_{i=1}^N \in V^{(i)} \). We define the projection-like operators \( T^{(i)} : V_h \to V^{(i)} \) \( i = 1, \ldots, N \) and \( T_0^{(\text{type})} : V_h \to V_0^{(\text{type})} \) such that for \( u^h \in V_h, T^{(i)} u^h \) and \( T_0^{(\text{type})} u^h \) are the solutions of
\[ a(T^{(i)} u^h, v^h) = a(u^h, v^h), \quad v^h \in V^{(i)}, \quad i = 1, \ldots, N \]
and
\[ a(T_0^{(\text{type})} u^h, v^h) = a(u^h, v^h), \quad v^h \in V_0^{(\text{type})}, \quad \text{ type } \in \{I, II\}. \]

Now, the additive Schwarz operator \( T^{\text{type}} : V_h \to V_h \) is
\[ T^{\text{type}} = T_0^{(\text{type})} + \sum_{i=1}^N T^{(i)}, \quad \text{ type } \in \{I, II\} \]
and the problem (3) can be written as follow.
\[ T^{\text{type}} u^*_h = g^{\text{type}}, \quad \text{ type } \in \{I, II\}, \]
where \( g^{\text{type}} = g_0^{(\text{type})} + \sum_{i=1}^N g^{(i)} \) with \( g_0^{(\text{type})} = T_0^{(\text{type})} u^*_h \) and \( g^{(i)} = T^{(i)} u^*_h \).
4 Enrichment of the coarse space for the mortar discretization

In this section, we design two different coarse spaces for the additive average Schwarz method. To this end, we first denote the sets of nodal nodes of all mortar and nonmortar sides, and also all interior nodes of all subdomains \( \Omega_i \), \( i = 1, \ldots, N \) by \( \mathcal{N}_m, \mathcal{N}_s, \) and \( \mathcal{N}_i \), respectively. Now, the average interpolation operator \( I_0 : V_h \to V_h \) for the mortar discretization as in \[8\] has the following structure.

\[
I_0 u^h(x) = \begin{cases} u^h_i(x) & x \in \mathcal{N}_m \cup \mathcal{N}_s, \\ \bar{u}^h_i & x \in \mathcal{N}_i, \\ \end{cases} \quad i = 1, \ldots, N,
\]

where \( \bar{u}^h_i \) is the average value of \( u^h_i \) over \( \Omega_i \), i.e.,

\[
\bar{u}^h_i = \frac{1}{\mu_i(\delta, \gamma)} \left( \sum_{\gamma_m(i) \subset \partial \Omega_i} \bar{u}^h_{m(i)} + \sum_{\delta_m(i) \subset \partial \Omega_i, \gamma_m(j) = \delta_m(i)} \bar{u}^h_{m(j)} \right), \quad (4)
\]

where \( \mu_i(\delta, \gamma) \) is the number of all mortar and nonmortar sides of \( \Omega_i \), and \( |\gamma_m(i)| \) is the length of \( \gamma_m(i) \). To express the interpolation operator \( I_0 \) in terms of the matrix form denoted by \( R_0 \), let \( N_c \) be a set of all nodal nodes at the end of all nonmortar sides of \( \Omega \) and let also \( I_{(c)} \in \mathbb{R}^{N_c \times N_c} \) and \( I_{(m)} \in \mathbb{R}^{N_m \times N_m} \) be identity matrices, where \( N_c = \dim(\mathcal{N}_c) \) and \( N_m = \dim(\mathcal{N}_m) \). Further, consider \( H = \text{diag}(H_1, \ldots, H_N) \in \mathbb{R}^{(N_c + N_m) \times N_i} \), where \( H_i = (\bar{u}^h_i)^{k_{n_m} n_i} \), \( N_i = \dim(\mathcal{N}_i) \), and \( n_i = \dim(\mathcal{N}_i) \). Hence

\[
R_0 = \begin{bmatrix} \text{diag}(I_{(c)}, I_{(m)}) \end{bmatrix} H \in \mathbb{R}^{(N_c + N_m) \times N_i}, \quad N_i = N_c + N_m + N_i.
\]

Fig. 3 The boundary layer \( \Omega_i^B \subset \Omega_i \) is highlighted by the colorful triangles.

To define two different coarse spaces, we first introduce the boundary layers \( \Omega_i^B \), \( i = 1, \ldots, N \), where \( \Omega_i^B \subset \Omega_i \) is the sum of all triangles such as \( \tau \in T_i \) such that \( \partial \tau \cap \partial \Omega_i \neq \phi \) (cf. Figure 3). We then define the following local
minimums of the coefficients over the subdomains $\Omega_i$:

$$\alpha_i := \min_{x \in \Omega_i} \alpha(x), \quad \alpha_{i,B} := \min_{x \in \Omega_{i,B}} \alpha(x), \quad i = 1, \ldots, N.$$  

We now proceed to construct the second part of $V_0^{(type)}$, i.e., $W_i^{type}$, $type \in \{I, II\}, i = 1, \ldots, N$ which are the spaces of adaptively chosen eigenfunctions of specially constructed generalized eigenvalue problems defined locally in each subdomain and extended by zero to the rest of the domain. Hence, the generalized eigenvalue problems is to find all eigen pairs $(\lambda_j^{i,type}, \psi_j^{i,type}) \in (\mathbb{R}_+, V_{i,h})$ such that

$$A_{\Omega_i} x = \lambda^{type} (B_{\text{type}}^{i} \psi_j^{i,type}) x, \quad i = 1, \ldots, N, \quad (5)$$

where $B^{type}(\cdot, \cdot)$, $type \in \{I, II\}$ are the matrix representations of the following bilinear forms.

$$b^I(u, v) := \sum_{i=1}^{N} \int_{\Omega_i} \alpha_i \nabla u \cdot \nabla v \, dx, \quad u, v \in V_{i,h},$$

$$b^II(u, v) := \sum_{i=1}^{N} \left( \int_{\Omega_i} \alpha_i \nabla u \cdot \nabla v \, dx + \int_{\Omega_i \setminus \Omega_{i,B}} \alpha \nabla u \cdot \nabla v \, dx \right), \quad u, v \in V_{i,h}.$$  

We also denote all eigenvalues of (5) by

$$D^{type} = \text{diag}(D_1^{type}, D_2^{type}, \ldots, D_N^{type}) \in \mathbb{R}^{N \times N}, \quad \text{type} \in \{I, II\},$$

where

$$D_i^{type} = \text{diag}(\lambda_1^{i,type}, \lambda_2^{i,type}, \ldots, \lambda_{n_i}^{i,type}) \in \mathbb{R}^{n_i \times n_i}, \quad i = 1, \ldots, N,$$

such that

$$\lambda_1^{i,type} \geq \lambda_2^{i,type} \geq \ldots \geq \lambda_{n_i}^{i,type} > 0.$$  

We define

$$W_i^{type} := \text{span}\{\psi_k^{i,type}\}_{k=1}^{n_i}, \quad \text{type} \in \{I, II\}, \quad i = 1, \ldots, N$$

as the space of the eigenfunctions associated with the eigenvalues $\lambda_j^{i,type}$. We also correspond to these spectral spaces the matrix forms denoted by

$$W^{type} = \text{diag}(W_1^{type}, W_2^{type}, \ldots, W_N^{type}), \quad \text{type} \in \{I, II\}.$$  

Further, we use the notation $R_0^{type}$ as the matrix representation of the operator $T_0^{(type)}$ which has the following structure.

$$R_0^{type} = \begin{bmatrix} R_0 & O \\ O & W^{type} \end{bmatrix}, \quad R_0^{type} \in \mathbb{R}^{N^2 \times (N^2 + N_i)}, \quad \text{type} \in \{I, II\}.$$
4.1 The additive average Schwarz preconditioner

The main aim of this section is to introduce new enrichment preconditioners denoted by \( B_{E,\text{type}} \), \( \text{type} \in \{I, II\} \). The idea is based on expressing these preconditioners in terms of a combination of \( B_{C,\text{type}} \), \( \text{type} \in \{I, II\} \) and \( \sum_{i=1}^{N} R_i^T A_i^{-1} R_i \), where

\[
B_{C,\text{type}} = (R_0^{\text{type}})^T \left( R_0^{\text{type}} A_N (R_0^{\text{type}})^T \right)^{-1} R_0^{\text{type}}
\]

and

\[
A_N = \begin{bmatrix}
A_N^{(11)} & A_N^{(12)} \\
A_N^{(21)} & A_N^{(22)}
\end{bmatrix}, \quad A_N \in \mathbb{R}^{(N_D+N_i)(N_D+N_i)}
\]

such that

\[
A_N^{(11)} := \text{square submatrix of } A \text{ whose rows and columns are corresponding to the nodes belong to } N_c \cup N_m \cup N_i,
\]

\[
A_N^{(12)} := \text{rectangular submatrix of } A \text{ whose rows are corresponding to all nodes belong to } N_c \cup N_m \cup N_i, \text{ meanwhile whose columns are corresponding to the nodes belong to } N_i,
\]

\[
A_N^{(21)} = (A_N^{(12)})^T,
\]

\[
A_N^{(22)} := \text{square submatrix of } A \text{ whose rows and columns are corresponding to the nodes belong to } N_i. \text{ Indeed, } A_N^{(22)} = \text{diag}(A_{\Omega_1}, A_{\Omega_2}, \ldots, A_{\Omega_N}).
\]

Hence

\[
R_0^{\text{type}} A_N (R_0^{\text{type}})^T = \begin{bmatrix}
R_0 A_N^{(11)} R_0^T & R_0 A_N^{(12)} (W_0)^T \\
(W_0 A_N^{(12)} (W_0)^T)^T & W_0 A_N^{(22)} (W_0)^T
\end{bmatrix}.
\]

Since \( W_0 A_N^{(22)} (W_0)^T = D^{\text{type}}, \text{type} \in \{I, II\} \) and it is the nonsingular matrix, we can easily compute \( B_{C,\text{type}} \). To this end and for the sake of simplicity, we first assume

\[
G^{\text{type}} = R_0 A_N^{(12)} (W_0)^T,
\]

\[
S^{\text{type}} = R_0 A_N^{(11)} R_0^T - G^{\text{type}} (D^{\text{type}})^{-1} (G^{\text{type}})^T.
\]

Hence, the inverse of \( 2 \times 2 \) block matrix \( R_0^{\text{type}} A_N (R_0^{\text{type}})^T \) implies

\[
B_{C(11)}^{\text{type}} = R_0^T (S^{\text{type}})^{-1} R_0,
\]

\[
B_{C(12)}^{\text{type}} = -R_0^T (S^{\text{type}})^{-1} G^{\text{type}} (D^{\text{type}})^{-1} W_0^{\text{type}},
\]

\[
B_{C(21)}^{\text{type}} = -(B_{C(12)}^{\text{type}})^T,
\]

\[
B_{C(22)}^{\text{type}} = (W^{\text{type}})^T \left( (D^{\text{type}})^{-1} + (G^{\text{type}} (D^{\text{type}})^{-1})^T (S^{\text{type}})^{-1} G^{\text{type}} (D^{\text{type}})^{-1} \right) W^{\text{type}}.
\]
Due to different sizes of the submatrices of $B_{\text{type}}$, we can use the restriction matrices $R_i$, $i = 1, \ldots, N$ to obtain new matrices not only with identical sizes, but also with their previous performances. Let $R_i = \text{diag}(R_{i1}, R_{i2}, \ldots, R_{iN})$, $B_{\text{type}}^0 = B_{\text{type}}^{(11)} + R_T^T R_{\text{type}}^{(21)}$, and $B_{\text{type}}^{00} = B_{\text{type}}^{(12)} R_r + R_T^T B_{\text{type}}^{(22)} R_r$. Hence, $B_{\text{type}}^E$, $\text{type} \in \{I, II\}$ as the enrichment preconditioners can be written explicitly in the following matrix form.

$$B_{\text{type}}^E = B_{\text{type}}^0 + B_{\text{type}}^{00} + \sum_{i=1}^{N} R_i^T A^{-1}_i R_i$$

To estimate an upper bound for the condition number of $B_{\text{type}}^E$ in the next section, we define the $b_{\text{type}}$-orthogonal projection operator $\pi_{\text{type}}^i : V(i) \rightarrow V(i)$ as

$$\pi_{\text{type}}^i v^h = \sum_{k=1}^{N_i} b_{\text{type}}^i(v^h, \psi_{k,i,\text{type}}) \psi_{k,i,\text{type}}, \quad v^h \in V_h, \quad \text{type} \in \{I, II\}.$$  

For any $u^h \in V_h$, we consider the function $w^h = u^h - I_0 u^h \in V_h$ with zero value both on $\partial \Omega_i$ and on the rest of the domain $\Omega$. In addition, we define $I_{0,\text{type}}^i : V_h \rightarrow V_{0,\text{type}}$ as

$$I_{0,\text{type}}^i u^h = I_0 u^h + \sum_{i=1}^{N} \pi_{\text{type}}^i u^h, \quad u^h \in V_h, \quad \text{type} \in \{I, II\}.$$  

5 On the estimation of the condition number bound

This section is devoted to obtaining condition number estimates of the additive average Schwarz preconditioners, which are defined in the previous section for $\text{type} \in \{I, II\}$. The proof is based on the standard additive Schwarz framework, where three assumptions to be held, see [32, p. 155]. Here, we need only to show that there exists the stable splitting for all $u \in V_h$ as in Theorem. Note that the other assumptions hold since there is no overlapping, and we also use the exact bilinear forms. For simplicity, throughout this section, we use $C$ as a positive constant, which is independent of the mesh sizes. We add a notation or an index to $C$ if we want to emphasize a special constant. Furthermore, we use the notation $\lesssim$ to remove any constants except the mesh sizes in the inequalities.

Let $M_{\text{type}}^i$ be a given number such that $0 \leq M_{\text{type}}^i < n_i$ and $\lambda_{M_{\text{type}}^i+1} \leq \lambda_{M_{\text{type}}^i}$. We define

$$\tilde{W}_{\text{type}}^i := \text{span}\{\psi_{k,i,\text{type}}\}_{k=1}^{M_{\text{type}}^i}, \quad \text{type} \in \{I, II\}$$
and
\[
\widetilde{V}_0^{\text{type}} = I_0 V_h + \sum_{i=1}^N \tilde{W}_i^{\text{type}}, \quad \text{type} \in \{I, II\}.
\]

For the analysis of the additive average Schwarz method, we define \( \tilde{I}_0^{\text{type}} : V_h \to \widetilde{V}_0^{\text{type}} \) as
\[
\tilde{I}_0^{\text{type}} u^h = I_0 u^h + \sum_{i=1}^N \tilde{\pi}^{\text{type}}_i u^h, \quad u^h \in V_h, \quad \text{type} \in \{I, II\},
\]
where
\[
\tilde{\pi}^{\text{type}}_i u^h = \sum_{k=1}^{M_i^{\text{type}}} b^{\text{type}}(u^h, \psi^{i,\text{type}}_k) \psi^{i,\text{type}}_k, \quad \text{type} \in \{I, II\}.
\]

To prove our main result, we need the following lemma.

**Lemma 1.** For all \( u \in V^{(i)}, i = 1, \ldots, N, \)
\[
|u - \tilde{\pi}^i u|^2_{H^1(\Omega), a} \leq C \lambda_i^{1,1} \| \nabla^{1/2} u \|_{L^2(\Omega)}^2,
\]
\[
|u - \tilde{\pi}^i u|^2_{H^1(\Omega), a} \leq C \lambda_i^{1,\text{II}} \| \Omega B \nabla u \|_{L^2(\Omega)}^2 + \| \alpha_i^{1/2} \nabla u \|_{L^2(\Omega \setminus \Omega_i^\text{II})}^2,
\]
where \( | \cdot |^2_{H^1(\Omega), a} = a_i(\cdot, \cdot). \)

**Proof.** We first express any \( u \in V^{(i)} \) uniquely in terms of the eigenfunctions, i.e.,
\[
u = \sum_{i=1}^{n_i} b^{\text{type}}(u, \psi^{i,\text{type}}_k) \psi^{i,\text{type}}_k.
\]
Hence
\[
u - \tilde{\pi}^{\text{type}} u = \sum_{i=M_i^{\text{type}}+1}^{n_i} b^{\text{type}}(u, \psi^{i,\text{type}}_k) \psi^{i,\text{type}}_k, \quad \text{type} \in \{I, II\}
\]
and we have
\[
u - \tilde{\pi}^{\text{type}} u = \sum_{i=M_i^{\text{type}}+1}^{n_i} \left( b^{\text{type}}(u, \psi^{i,\text{type}}_k) \right)^2 \| \psi^{i,\text{type}} \|_{H^1(\Omega), a}^2.
\]

Furthermore, \( (W^{\text{type}}_i)^T A_i W^{\text{type}} = D^{\text{type}}_i \) is equivalent to
\[
| \nabla \psi^{i,\text{type}}_k |^2_{H^1(\Omega), a} = \lambda^{i,\text{type}}_k, \quad k = 1, \ldots, n_i, \quad i = 1, \ldots, N, \quad \text{type} \in \{I, II\}.
\]
Therefore
\[
u - \tilde{\pi}^{\text{type}} u = \sum_{i=M_i^{\text{type}}+1}^{n_i} \left( b^{\text{type}}(u, \psi^{i,\text{type}}_k) \right)^2 \lambda^{i,\text{type}}_k, \quad \text{type} \in \{I, II\}.
\]
(6)

Using the Schwarz inequality for \( \text{type} = I \), yields
\[
u - \tilde{\pi}^{\text{type}} u = \sum_{i=M_i^{\text{type}}+1}^{n_i} \| a_i^{1/2} \nabla \psi^{i,\text{type}}_k \|_{L^2(\Omega_i)} | \psi^{i,\text{type}}_k |_{L^2(\Omega_i)} \lambda^{i,1}_k.
\]

Since \( (W^{\text{type}}_i)^T (R_i B^{\text{type}} R_i^T) W^{\text{type}}_i = I, \quad \text{type} \in \{I, II\} \) and \( i = 1, \ldots, N \) we have equivalently
\[
\| a_i^{1/2} \nabla \psi^{i,\text{type}}_k \|_{L^2(\Omega_i)} = 1, \quad \| a_i^{1/2} \nabla \psi^{i,\text{type}}_k \|_{L^2(\Omega_i)} + \| a_i^{1/2} \nabla \psi^{i,\text{type}}_k \|_{L^2(\Omega_i \setminus \Omega_i^\text{II})} = 1.
\]
(7)
Proof. We now proceed to prove a similar result for type II. Consequently, we get

\[ u \leq C \| \nabla u \|_{L^2(\Omega)} + \sum_{i=1}^{N} u^i, \]

where \( u^i = (0,\ldots,0,w_i,0,\ldots,0) \in V(i) \) and \( w_i = (u^h - \tilde{I}_0 u^h) |_{\Omega_i} \). Consequently, we get

\[ \alpha_i(w - \tilde{\pi}_i w, w - \tilde{\pi}_i w) \leq a(\tilde{I}_0 u^h, \tilde{I}_0 u^h, w - \tilde{\pi}_i w), \]

where \( a(\tilde{I}_0 u^h, \tilde{I}_0 u^h, w - \tilde{\pi}_i w) = \sum_{i=1}^{N} \alpha_i(w^i, w^i) = \sum_{i=1}^{N} \alpha_i(u^h - \tilde{I}_0 u^h, u^h - \tilde{\pi}_i u^h) \)

\[ = \sum_{i=1}^{N} \alpha_i(w - \tilde{\pi}_i w, w - \tilde{\pi}_i w), \quad \text{type } \in \{I, II\}. \]

We first use Lemma [3] for the type I. Then, we have

\[ a_i(w - \tilde{\pi}_i w, w - \tilde{\pi}_i w) \leq C \| \nabla (u^h - I_0 u^h) \|_{L^2(\Omega_i)}^2 \]

\[ \leq C \| \nabla u^h \|_{L^2(\Omega_i)}^2 + |\nabla I_0 u^h|_{L^2(\Omega_i)}^2. \]

Further, from [3] pp. 8-9 we get similarly

\[ \alpha_i |\nabla I_0 u^h|_{L^2(\Omega_i)}^2 \leq C \frac{H}{h} a_i(u^h, u^h). \]  (8)

For type I, the proof is completed by summing (8) over \( i = 1,\ldots,N \). We now proceed to prove a similar result for type II. Due to definition of \( I_0 u^h \), we use this fact that \( \nabla I_0 u^h \) is equal to zero on each triangle \( \tau \notin \Omega_i^B \), \( i = 1,\ldots,N \). Hence, we use Lemma [3] to get the first inequality as follow.

Theorem 1. For all \( u^h \in V_h \) the following results hold:

\[ a(\tilde{I}_0 u^h, \tilde{I}_0 u^h) \leq \max \lambda^{type}_{M_1, H} \frac{H}{h} a(u^h, u^h), \quad \text{type } \in \{I, II\}, \quad i = 1,\ldots,N. \]
\[ a_i(w - \frac{\partial}{\partial x_i} w, w - \frac{\partial}{\partial x_i} w) \]

\[ \leq C \lambda_i^{\frac{1}{\nu_i+1}} \left( \| \alpha_i^{1/2} \nabla (u^h - I_0 u^h) \|_{L^2(\Omega_i \setminus \Omega^h_i)}^2 + \omega_i \| \nabla (u^h - I_0 u^h) \|_{L^2(\Omega^h_i)}^2 \right) \]

\[ \leq C \lambda_i^{\frac{1}{\nu_i+1}} \left( \| \alpha_i^{1/2} \nabla u^h \|_{L^2(\Omega_i \setminus \Omega^h_i)}^2 + \omega_i \| \nabla (u^h - I_0 u^h) \|_{L^2(\Omega^h_i)}^2 \right) \]

\[ \leq C \lambda_i^{\frac{1}{\nu_i+1}} \left( \| \alpha_i^{1/2} \nabla (u^h - I_0 u^h) \|_{L^2(\Omega_i \setminus \Omega^h_i)}^2 + \omega_i \| \nabla u^h \|_{L^2(\Omega_i \setminus \Omega^h_i)}^2 + \| \nabla I_0 u^h \|_{L^2(\Omega^h_i)}^2 \right) \]

\[ \leq C \lambda_i^{\frac{1}{\nu_i+1}} \left( \| \alpha_i^{1/2} \nabla (u^h - I_0 u^h) \|_{L^2(\Omega_i \setminus \Omega^h_i)}^2 + \omega_i \| \nabla u^h \|_{L^2(\Omega_i \setminus \Omega^h_i)}^2 + \| \nabla I_0 u^h \|_{L^2(\Omega^h_i)}^2 \right) \]

Using the inverse inequality and the definition of the operator \( I_0 \), implies

\[ \| \nabla I_0 u^h \|_{L^2(\Omega^h_i)}^2 = \sum_{\tau \in \mathcal{T}_h^i} \| \nabla I_0 u^h \|_{L^2(\tau)}^2 = \sum_{\tau \in \mathcal{T}_h^i} \| \nabla (I_0 u^h - \overline{u}^h) \|_{L^2(\tau)}^2 \leq C \sum_{\tau \in \mathcal{T}_h^i} h_\tau^{-2} \| I_0 u^h - \overline{u}^h \|_{L^2(\tau)}^2 \leq C \sum_{x \in \partial \Omega_i} (u^h_i(x) - \overline{u}_i^h)^2. \] (9)

In what follow, we need to use the following Poincaré inequality [36].

\[ \| g \|_{L^2(\Omega_i)}^2 \leq C_1 \| g \|_{H^1(\Omega_i)}^2 + C_2 \left( \int_{\Omega_i} g \, dx \right)^2, \quad g \in H^1(\Omega_i), \] (10)

where \( C_1 \) and \( C_2 \) are two positive constants independent of the mesh size of \( \Omega \). Consider the function \( g = u^h_i - c \), where \( c = \frac{1}{\text{meas}(\Omega_i)} \int_{\Omega_i} u^h_i \, dx \). Now, (10) implies

\[ \| u^h_i - c \|_{L^2(\Omega_i)}^2 \leq C_1 \| u^h_i \|_{H^1(\Omega_i)}^2. \] (11)

Furthermore, we use (4) and this property that all boundary elements of subdomains \( \Omega_i, i = 1, \ldots, N \) are quasi-uniform which implies that the number of all nodes belonging to \( \partial \Omega_i \), denoted by \( M_i \), is of the order \( H_i/h_i \), to get

\[ M_i \left( \frac{u^h_i - c}{\frac{1}{2}} \right)^2 \leq CM_i \frac{1}{\rho(\delta, \gamma)} \sum_{\tau \subset \partial \Omega_i} \frac{1}{\tau} \| u^h_i - c \|_{L^2(\tau)}^2 \leq C h_i^{-1} \| u^h_i - c \|_{L^2(\partial \Omega_i)}^2, \] (12)

where \( \Gamma \) is \( \gamma_m(i) \) and \( \delta_m(i) \).
Now, we use $g$ and (12) to estimate an upper bound for (9) as follows.

$$
||\nabla I_0 u_h||^2_{L^2(\Omega^h)} \leq C \sum_{x \in \partial \Omega_h} (g(x) - \bar{g})^2 \\
\leq Ch_i^{-1}||g||^2_{L^2(\partial \Omega_i)} \\
\leq Ch_i^{-1}H_i||\hat{g}||^2_{L^2(\partial \hat{\Omega})},
$$

where $\hat{\Omega}$ is the reference element of unit diameter. From this inequality we deduce

$$
||\nabla I_0 u_h||^2_{L^2(\Omega^h)} \leq Ch_i^{-1}H_{i+1}||\hat{g}||^2_{L^2(\partial \hat{\Omega})} + H_i^{-1}||g||^2_{L^2(\partial \Omega_i)} \\
\leq C \frac{H_i}{h_i} a_i(u^h, u^b),
$$

by the trace inequality, Theorem 3.1.2 in [10] and (11). To complete the proof, it suffices to take a summation over $i = 1, \ldots, N$.

To estimate upper bounds of the condition numbers of $B_{EA}^{type}$, $type \in \{I, II\}$ it suffices to use Lemma 3 in [32, pp. 156-158] and Theorem [11].

**Theorem 2.** The condition numbers of the enriched additive average Schwarz preconditioners are bounded by

$$
\kappa(B_{EA}^{type}) \leq C \left( \frac{H}{h} \right) \max_i \lambda_i^{type} \lambda_{H_i^{type}+1}^{type}, \quad type \in \{I, II\}, \quad i = 1, \ldots, N,
$$

where $C$ is a positive constant.

6 Numerical results

In this section, numerical results confirm the additive average Schwarz method’s validity and efficiency with adaptive enrichment, where the jumps in the coefficient $\alpha(x)$ in (1) are very large and even change rapidly. Those jumps might be occurred inside of the subdomains, or even across the subdomain boundaries. To have a complicated distribution of jumps, we use the following pattern [26], depicted in Figure 4, which is a periodic pattern when the number of subdomains are increased (cf. Figure 5). We also consider background channels, crossing channels and corner channels denoted by $\alpha_b$, $\alpha_i$ and $\alpha_c$, respectively. For different values of $\alpha_b$, $\alpha_c$ and $\alpha_i$, our test problem has the right-hand side function $f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y)$ defined in $\Omega = [0, 1] \times [0, 1]$.

All presented numerical results are based on the nonmatching triangulations across the subdomains interfaces, where $H \in \{1/6, 1/9\}$, $h \in \{1/36, 1/54\}$, and $h^* \in \{1/54, 1/81\}$ (cf. Figure 5). We use the similar
Fig. 4 The decomposition of $\Omega = [0,1] \times [0,1]$ into $3 \times 3$ subdomains (numbered from 1 to 9) and the locations of $\alpha(x)$ consisting of $\alpha_b$, $\alpha_c$ and $\alpha_i$ in the white, red and green areas, respectively. Further, all mortar sides are denoted by the thick and black line segments.

Fig. 5 The distribution of all jumps in $\alpha(x)$ following the extended pattern in Figure 4, where $\Omega = [0,1] \times [0,1]$ is divided into $6 \times 6$ subdomains. The nonmatching discretization parameters are $H = 1/6$, $h = 1/36$ and $h^* = 1/54$. Further, all mortar sides are the coarse mortar.

locations for $\alpha_c$ and $\alpha_i$ as the periodic patterns and the similar mortar sides, as depicted in Figure 4 for $N \geq 9$. For instance, see Figure 5 where $N = 36$. Further, in what follows, we use two terms, i.e., coarse mortar and fine mortar to distinguish between the coarse and fine triangulations connected
to the mortar sides. To withdraw numerical results, we use the additive average Schwarz method to produce the enrichment preconditioners. We estimate the condition number of such preconditioners for $type \in \{I, II\}$. Moreover, the iteration numbers in all tables come from the preconditioned conjugate gradient method, with the tolerance $5e^{-6}$, based on the produced preconditioners.

| ADD | Different Values for $\alpha_b$, $\alpha_c$ and $\alpha_i$ |  |  |
|-----|----------------------------------------------------------|---|---|
|     |                                                          | Coarse Mortar | Fine Mortar | Coarse Mortar | Fine Mortar |
|     |                                                          |               |             |               |             |
| $6 \times 6$ | 6.07e1 (52) | 4.25e1 (45) | 5.99e1 (56) | 4.24e1 (49) |
| $9 \times 9$ | 7.99e1 (55) | 5.56e1 (50) | 7.69e1 (58) | 5.51e1 (52) |

Table 1 The condition number of $B^{type}_{E}A$ and the number of iterations of the preconditioned conjugate gradient method (in parentheses) for $type = II$ with different values for $\alpha_b$, $\alpha_c$, and $\alpha_i$. For $6 \times 6$ and $9 \times 9$ subdomains, $h$ and $h^*$ belong to $\{1/36, 1/54\}$ and $\{1/54, 1/81\}$, respectively. In addition, selecting the number of eigenfunctions for each subdomain to construct the enrichment coarse space is based on the adaptive enrichment, where the threshold is 50.

In Table 1 we set different values for $\alpha_b$, $\alpha_c$, and $\alpha_i$ for different number of subdomains, for instance, see Figure 5 including $6 \times 6$ subdomains and the coarse mortar case. We implement the additive average Schwarz method with the adaptive enrichment coarse space $type = II$, where the threshold is 50, i.e., all eigenfunctions associated with all eigenvalues larger than 50 are considered. Note that the condition number of the non-enrichment preconditioner is very large. For instance, in the case $6 \times 6$ subdomains and coarse mortar, it is $1.32e7$. We first observe that the condition numbers of the enrichment preconditioners are proportional to the ratio $H/h$ and independent of the number of subdomains. We also observe that if we consider the same number of eigenfunctions for each subdomain based on a proper threshold, consequently, the condition number estimates are independent of values of $\alpha_b$, $\alpha_c$, and $\alpha_i$.

Table 2 demonstrates only the condition number estimates and iteration numbers of the preconditioned conjugate gradient method (in parentheses) in the cases $type = II$, coarse mortar, $6 \times 6$ and $9 \times 9$ subdomains, where $\alpha_b = 1$, $\alpha_c = 1e4$, and $\alpha_i = 1e6$. For the other cases and also different values of jumps in the coefficient $\alpha(x)$, we have similar results. Also, the construction of the enrichment coarse space is relied on the fixed number of eigenfunctions for each subdomain varying from 0 to 7.

It is conclusive that both approaches to enrich the standard coarse space, i.e., using the given threshold and imposing the fixed numbers of
Number of Subdomains

| ADD | 0 | 1 | 2  | 3  | 4  | 5  | 6  | 7  |
|-----|---|---|----|----|----|----|----|----|
| 6 × 6 | 1.32e7 | 3.25e6 | 2.65e5 | 4.10e4 | 4.12e3 | 5.7e1  | 5.6e1 | 5.58e1 |
|      | (1756) | (1168) | (584) | (295) | (112) | (50) | (48) | (47) |
| 9 × 9 | 2.29e7 | 7.54e6 | 1.41e6 | 3.82e5 | 1.23e5 | 6.21e3 | 5.80e2 | 7.38e1 |
|      | (4981) | (3889) | (2368) | (1180) | (353) | (118) | (63) | (50) |

Table 2 The implementation of the additive average Schwarz method, type = II, with the fixed number of eigenfunctions for each subdomain to estimate the condition number of \( B_{\text{type}}^T A \) and the number of iterations of the preconditioned conjugate gradient method (in parentheses) for 6 × 6 \((h = 1/36, h^* = 1/54)\) and 9 × 9 \((h = 1/54, h^* = 1/81)\) subdomains and the coarse mortar case. Further, the distribution of jumps in \( \alpha(x) \) are \( \alpha_b = 1, \alpha_c = 1e4 \) and \( \alpha_i = 1e6 \).

| ADD | 6 × 6 | 9 × 9 |
|-----|-------|-------|
| I   | 625   | 1988  |
| II  | 75    | 240   |
| I   | 525   | 1680  |
| II  | 80    | 233   |

Table 3 The total numbers of the eigenfunctions associated with the eigenvalues greater than 50 to enrich the coarse spaces used in the additive average Schwarz method for both types I and II.

eigenfunctions for each subdomain, lead to similar results. This fact can be viewed by comparing, for instance, the third column of Table 1 with the last column of Table 2.

Table 3 gives the total number of required eigenfunctions for the adaptive enrichment coarse spaces in the cases 6 × 6 and 9 × 9 subdomains, type = I and II and different values for \( \alpha(x) \). As we can see from this table, solving the second type of the generalized eigenvalue problem \(^5\) is more efficient than solving the first type. To analyze the distribution of eigenfunctions, Figure 6 contains the polar histograms for both coarse and fine mortars in the case 6 × 6 subdomains, and type \( \in \{I, II\} \). It clearly shows that we need only consider a few eigenfunctions for each subdomain in the case type = II compared to type = I.
Fig. 6 The histograms of the number of eigenfunctions associated with the eigenvalues greater than 50 corresponding to the partition of $\Omega = [0, 1] \times [0, 1]$ into 6 x 6 subdomains, where $\alpha_b = 1$, $\alpha_c = 1e4$ and $\alpha_i = 1e6$. The number of subdomains are ordered anticlockwise around the polar histograms from 1 to 36. For type = I and II, the number of eigenfunctions are grouped into the ranges 0 to 70 and 0 to 10, respectively. The largest numbers of the eigenfunctions in (a) – (d) are 39, 6, 63 and 5, respectively.
7 Conclusion

In this paper, we have employed the additive average Schwarz method with the enrichment coarse spaces to solve second order elliptic boundary problem coupled with very large jumps in the function \( \alpha(x) \), where the finite element discretization of that problem has been based on the nonmatching triangulations across the subdomains interfaces. We have proved that the condition number estimates for the produced preconditioners by the additive average Schwarz methods for \( \text{type} \in \{I, II\} \) are proportional to the ratio \( H/h \) and independent of the number of subdomains. Besides, we have compared the numerical results to conclude that, in practice, \( \text{type} = II \) has much better performance than \( \text{type} = I \).

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