Quasi-stationary states and fermion pair creation from a vacuum in supercritical Coulomb field

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Creation of charged fermion pair from a vacuum in the so-called supercritical Coulomb potential is examined for the case when created pair moves in one plane. In which case the quantum dynamics of charged massive or massless fermions can be described by the two-dimensional Dirac Hamiltonians with a Coulomb potential. These Hamiltonians are singular and require the additional definition in order for them to be treated as self-adjoint quantum-mechanical operators. We construct the self-adjoint two-dimensional Dirac Hamiltonians with a Coulomb potential and determine the quantum-mechanical states for such Hamiltonians in the corresponding Hilbert spaces of square-integrable functions. We determine the scattering amplitude in which the self-adjoint extension parameter is incorporated and then obtain the equations implicitly defining the possible discrete energy spectra of the self-adjoint Dirac Hamiltonians with a Coulomb potential. It is shown that the quantum system in the presence of a supercritical Coulomb potential becomes unstable which manifests in the appearance of quasi-stationary states in the lower (negative) energy continuum. The energy spectrum of these states is quasi-discrete, consists of broadened levels whose width is related to the inverse lifetime of the quasi-stationary state as well as the creation probability of charged fermion pair by supercritical Coulomb field. Explicit analytical expressions for the creation probabilities of charged (massive or massless) fermion pair are obtained in a supercritical Coulomb field.

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I. INTRODUCTION

The instability of quantum electrodynamics vacuum in the presence of the so-called supercritical Coulomb potential of a hypothetical atomic nucleus with the charge of a nucleus $Z$ exceeding a certain critical value $Z_{cr} \sim 170$ ($Z_{cr} \sim 1.24$, $\alpha \approx 1/137$ is the fine structure constant) have been studied along time \cite{8}. It has been understood this phenomenon is related to electron-positron pair creation from a vacuum. Because of the absence of such supercharged atomic nuclei the electron-positron pair creation due to such an instability is highly academic problem. Nevertheless, significant efforts to observe the positrons created due to the vacuum instability have been made through colliding heavy-ions with enormously large $Z \sim 170$ but results look ambiguous.

The vacuum instability was found to occur in the spatially two-dimensional quantum systems in which case the Coulomb potential strength $Z_{cr} \alpha \sim 0.65$ ($Z_{cr} \sim 90$) for massive case \cite{8} and $Z_{cr} \alpha_g \approx 0.5$ ($\alpha_g$ is the the effective coupling constant) for massless case \cite{1010}. So the interest in similar phenomena in two-dimensional relativistic quantum systems was revived in connection with the Coulomb impurity problem in graphene \cite{1010,1112,1723}. Indeed, charge carriers in graphene behave as relativistic particles described by the two-dimensional Dirac equation \cite{10,11,2123}, which allows to consider graphene as the condensed matter analog of the quantum electrodynamics in 2+1 dimensions \cite{24,25}. Besides, in graphene, the corresponding “effective fine structure constant” $\alpha_{eff} = e^2/\epsilon_0 \hbar v_F$ (where $e$ is the electron charge, $\epsilon_0$ is the dielectric constant of the medium, $\hbar$ is the action constant and $v_F$ is the Fermi-Dirac velocity) is large ($\alpha_{eff} \sim 1$) \cite{18,21,26} and a cluster of charged impurities can produce the supercritical Coulomb potential, which opens the real possibility of testing the supercritical instability \cite{20}. The electron-hole pair creation (holes in graphene play the role of positrons \cite{1012}) is likely to be now revealed in graphene (see, \cite{27,28}).

The induced electric current due to vacuum polarization of massless fermions in the superposition of Coulomb and Aharonov–Bohm potentials in 2+1 dimensions was addressed in \cite{24}. Vacuum polarization of the massive charged fermions can also be of interest for graphene with Coulomb impurity \cite{31}; the plane density of an induced vacuum charge in a strong Coulomb potential for massless and massive fermions was studied in \cite{32}.

In present paper we study the creation of charged fermion pair from a vacuum by supercritical Coulomb potential for the case when the quantum dynamics of the massive or massless fermions are governed by the two-dimensional Dirac Hamiltonians with a Coulomb potential. These Dirac Hamiltonians are singular and require the supplementary definition in order for them to be treated as self-adjoint quantum-mechanical operators. Self-adjoint Hamiltonians are not unique but each of them can be specified a real "self-adjoint extension" parameter by additional (self-adjoint) boundary conditions. To put it more exactly, a domain, including the singular $r = 0$ region, in the Hilbert space of square-integrable functions must be indicated for each self-adjoint Hamiltonian.

We find the quantum-mechanical states for self-adjoint two-dimensional Hamiltonians with a Coulomb potential by constructing the corresponding Hilbert spaces. We determine the scattering amplitude in which the self-adjoint extension parameter is incorporated and then obtain the equations implicitly defining the energy spectra of these Hamiltonians. It is shown that the quantum system in the presence of supercritical Coulomb potential becomes unstable which manifests in the appearance of quasi-stationary states in the lower (negative) energy continuum.

The scattering amplitude in a supercritical Coulomb potential becomes ambiguous function; it has a discontinuity in the complex plane of energy and additional poles on the negative energy axis of the second (nonphysical) sheet of the Riemann surface. The quasi-discrete spectrum of quasi-stationary states consists of broadened levels whose width (defined by the imaginary part of energy $E$) is related to the inverse lifetime (the decay rate) of the quasi-stationary state. We derive equations for the energy spectra and quasi-stationary state lifetimes and analyze (solve) these equations in physically important cases. The quasi-stationary states are directly associated with the fermion pair creation in the quantum electrodynamics and the modulus of the imaginary part $E$ determines the doubled probability of the creation of charged fermion pair by Coulomb potential.

We shall adopt the units where $c = \hbar = 1$.

II. SPECTRA OF THE SELF-ADJOINT RADIAL DIRAC HAMILTONIANS

We are only interested on the planar dynamics of a charged fermion in a Coulomb potential what implies that the fermion moves in the $xy$ plane and the fermion momentum projection $p_z = 0$, together
with the imposition of the Coulomb field should be now intrinsically two-dimensional (see, for instance, [33]). For which reason we shall assume in what follows that the space of fermion quantum states is the two-dimensional Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^2) \) of square-integrable functions \( \Psi(\mathbf{r}), \mathbf{r} = (x, y) \) with the scalar product

\[
(\Psi_1, \Psi_2) = \int \Psi_1^\dagger(\mathbf{r})\Psi_2(\mathbf{r})d\mathbf{r}, \quad d\mathbf{r} = dx dy.
\] (1)

One also knows [34] that the two-dimensional Dirac matrices can be represented in terms of the Pauli matrices, namely, \( \gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2 \), where the parameter \( s = \pm 1 \) can be introduced to label two types of fermions in accordance with the signature of the two-dimensional Dirac matrices [34]; for the case of massive fermions it can be applied to characterize two states of the fermion spin (spin "up" and "down") [35].

Then, the Dirac Hamiltonian for a fermion of the mass \( m \) and charge \( e = -e_0 < 0 \) in a Coulomb potential \( A_0(r) = a/e_0 r, A_r = 0, A_\varphi = 0, a > 0 \) in polar coordinates \( r = \sqrt{x^2 + y^2}, \varphi = \arctan(y/x) \), is

\[
H_D = \sigma_1 P_2 - s\sigma_2 P_1 + \sigma_3 m - e_0 A_0(r),
\] (2)

where \( P_\mu = -i\partial_\mu - e A_\mu \) is the generalized fermion momentum operator (a three-vector). The Hamiltonian (2) should be defined as a self-adjoint operator in the Hilbert space of square-integrable functions \( \Psi(\mathbf{r}), \mathbf{r} = (x, y) \) with the scalar product (1).

Eigenfunctions of the Hamiltonian (2) can be represented as (see, for instance [36,38])

\[
\Psi(t, \mathbf{r}) = \frac{1}{\sqrt{2\pi r}} \begin{pmatrix} f_1(r) \\ f_2(r) e^{is\varphi} \end{pmatrix} \exp(-iEt + il\varphi),
\] (3)

where \( E \) is the fermion energy, \( l \) is an integer. The wave function \( \Psi \) is an eigenfunction of the total angular momentum \( J \equiv L_z + s\sigma_3/2 \), where \( L_z \equiv -i\partial/\partial\varphi \), with eigenvalue \( j = l + s/2 \) and

\[
hF = EF, \quad F = \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix},
\] (4)

where

\[
h = is\sigma_2 \frac{d}{dr} + \sigma_1 \frac{l + \mu + s/2}{r} + \sigma_3 m - \frac{a}{r}.
\] (5)

The radial Hamiltonian \( h \) is singular and so the supplementary definition is required in order for it to be treated as a self-adjoint quantum-mechanical operator, therefore, we must indicate the Hamiltonian domain in the Hilbert space of square-integrable functions on the half-line, including the \( r = 0 \) region. One knows that self-adjoint Dirac Hamiltonians can be constructed for a symmetrical operator \( h \), if

\[
\int_0^\infty G_\dagger(r)hF(r)dr = \int_0^\infty [hG(r)]^\dagger F(r)dr
\] (6)

for any doublets \( F(r) \) and \( G(r) \).

Let us define the operator \( h^0 \) in the Hilbert space \( \Sigma^2(0, \infty) \) as

\[
h^0 : \begin{cases} 
D(h^0) = D(0, \infty), \\
h^0 F(r) = hF(r),
\end{cases}
\]

where \( D(0, \infty) \) is the standard space of smooth functions on \( (0, \infty) \) vanishing at \( r \to \infty \). It is evident that \( h^0 \) is the symmetrical operator.

Let \( h \) be the self-adjoint extension of \( h^0 \) in \( \Sigma^2(0, \infty) \) and let us consider the adjoint operator \( h^* \) given by Eq. (5) but defined as follows

\[
h^* : \begin{cases} 
D(h^*) = \{ F(r) : F(r) \text{ are absolutely continuous in } (0, \infty), \\
h^* F(r) = hF(r),
\end{cases}
\] (7)

i.e. \( D(h^0) \subset D(h^*) \). A symmetric operator \( h \) is self-adjoint, if its domain \( D(h) \) coincides with that of its adjoint operator \( D(h^*) \equiv D^* \).
As far as \( \lim_{r \to \infty} F(r) = 0 \) for any \( F(r) \) of \( D(h^*) \), integrating (6) by parts, we reduce Eq. (6) to boundary conditions at \( r = 0 \)

\[
\lim_{r \to 0} G^I(r) i\sigma_2 F(r) = 0. \tag{8}
\]

If (5) is satisfied for any doublet \( F(r) \) of \( D^* \) then the operator \( h^* \) is symmetric and, so, self-adjoint. This means that the operator \( h \) is essentially self-adjoint, i.e., its unique self-adjoint extension is its closure \( h = h^\dagger \), which coincides with the adjoint operator \( h = h^* \). If (5) is not satisfied then the self-adjoint operator \( h = h^\dagger \) can be found as the narrowing of \( h^* \) on the so-called maximum domain \( D(h) \subset D^* \).

Thus, any doublet \( F(r, E) \) of \( D^* \) must satisfy boundary condition \( 39 \)

\[
(F^I(r, E)i\sigma_2 F(r, E))|_{r=0} = (f_1^I f_2 - f_2^I f_1)|_{r=0} = 0, \tag{9}
\]

where \( f^* \) is the complex conjugate function. Physically, Eq. (9) shows that the probability current density is equal to zero at the origin.

The needed regular (at \( r = 0 \)) doublet can be conveniently represented as

\[
F(r, E) = \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} = Ar^\gamma e^{ipr} \left( \frac{\sqrt{E + m}}{i\sqrt{E - m}} \right) \left[ \Phi(b, c; x) \pm \frac{\gamma - iaE/p}{\nu - i\sigma m/p} \Phi(b + 1, c; x) \right]. \tag{10}
\]

Here

\[
\gamma = \sqrt{(l + 1/2)^2 - a^2}, \nu = l + 1/2, b = \gamma - iaE/p, c = 2\gamma + 1, x = -2ipr, p = \sqrt{E^2 - m^2}, \tag{11}
\]

\( A \) is a constant, \( \Phi(b, c; x) \) is the confluent hypergeometric function \( 40 \) and because the fermion states in a Coulomb potential are doubly degenerate with respect to \( s \) we put \( s = 1 \).

The asymptotic behavior of doublets (wave functions) at \( r \to 0 \) is determined with quantity \( \gamma \), which is real for \( a^2 \leq \nu^2 \) and is imaginary \( \gamma = i\sqrt{a^2 - \nu^2} = i\sigma \) for \( a^2 > \nu^2 \). Applying known formula

\[
\Phi(b, c; x) = e^x \Phi(c - b, c; -x),
\]

we have for \( \Phi(b + 1, c; x) \)

\[
\Phi(b + 1, c; x) = e^x \Phi^*(b, c; x), \tag{12}
\]

and so Eq. (10) takes the form

\[
\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} = \begin{pmatrix} \Phi(b, c; x) \\ e^{ipr} \Phi(b, c; x) \end{pmatrix} = Y(r, \gamma, E), \tag{13}
\]

where

\[
e^{-2i\sigma} = \frac{\gamma - iaE/p}{\nu - i\sigma m/p} \tag{14}
\]

and \( A^\prime \) is the normalization constant.

One can show that for \( \gamma \neq n/2, n = 1, 2, \ldots \) two doublets

\[
U_1(r, \gamma, E) = Y(r, \gamma, E), \quad U_2(r, \gamma, E) = Y(r, -\gamma, E) \tag{15}
\]

are linear independent; their asymptotic behavior as \( r \to 0 \) is given by

\[
U_1(r, \gamma, E) = (r)^\gamma u_+ + O(r^{\gamma + 1}), \quad U_2(r, \gamma, E) = (r)^{-\gamma} u_- + O(r^{-\gamma + 1}), \tag{16}
\]

where

\[
u_{\pm} = \begin{pmatrix} u(\pm \gamma, a) \\ 1 \end{pmatrix}. \tag{17}
\]

We see that the Hamiltonian domain in the Hilbert space of square-integrable functions is specified by the two doublets \( U_{1,2}(r, \gamma, E) \) and therefore essentially depends on \( \gamma \) as well as on \( a \). One can separate out three regions of the values of \( \gamma \).

In the region \( \gamma \geq 1/2 \), only the wave function \( \sim U_1(r, \gamma, E)/\sqrt{r} \) is square integrable at \( r = 0 \), but \( \sim U_2(r, \gamma, E)/\sqrt{r} \) is not. For \( \gamma \geq 1/2 \), the wave function \( \sim U_1(r, \gamma, E)/\sqrt{r} \) is regular at \( r = 0 \). The generalized eigenfunctions \( F(r, \gamma, E) \) of the radial self-adjoint Hamiltonian are \( U_1(r, \gamma, E) \). Its energy
spectrum is continuous in the region $E \geq m$ and discrete levels in the region $m > E > 0$ have to exist in a Coulomb potential in addition to continuous part of the energy spectrum.

One knows that bound states are identified as the poles of the scattering amplitude at the analytic continuation in the region $E < m$ [3]; these poles are located on the first (physical) sheet of the Riemann surface $\text{Re} \sqrt{m^2 - E^2} > 0, \text{Im} \sqrt{m^2 - E^2} > 0$ in the complex $E$-plane. The scattering amplitude relates incoming and outgoing wave functions of a quantum system undergoing a scattering process. In this manner, the poles are determined at the zeros of the coefficient in the ingoing wave at the analytic continuation in the region $E < m$ (see, for instance [41]). It will be recalled that the asymptotic form of radial doublets at $r \to \infty$ in the case under study is

$$
\left( \frac{f_1(r)}{f_2(r)} \right) = D \sqrt{E + m} \cos \left( \frac{\pi}{2} \frac{r}{|\nu|} - \frac{\pi}{4} + \delta_1 \right),
$$

where $D$ is a constant and the phase shifts $\delta_1$ are determined by the potential at small $r$.

The asymptotic behavior of the wave function at $r \to \infty$ one can find by means of the formula for the confluent hypergeometric function at $z \to \infty$

$$
\Phi(b; c; z) = \frac{\Gamma(c)}{\Gamma(c - b)} (-z)^{-b} + \frac{\Gamma(c)}{\Gamma(b)} e^z z^{b-c}
$$

in which the first (leading asymptotic) term only is significant. Here $\Gamma(z)$ is the Gamma function [40]. After simple calculations, we obtain for the ingoing wave at $r \to \infty$

$$
B_l(\gamma, E) e^{-i(\nu r - \pi/2)/2 - \pi/4 + C \ln pr},
$$

where

$$
B_l(\gamma, E) = \frac{\Gamma(\gamma + 1 + iaE/p)}{\Gamma(\gamma + 1 - iaE/p)} e^{-2i\eta(\gamma) - i\pi(|l+1/2| - \gamma)}
$$

and $C = aE/p$.

Analytic continuation in the region $E < m$ on the first sheet is performing by means of replacements

$$
\sqrt{E - m} \to i\sqrt{m - E}, p \to i\lambda, \lambda = \sqrt{m^2 - E^2}, aE/p \to -iaE/\lambda,
$$

and as a result, we obtain

$$
B_l(\gamma, \lambda, E) = \frac{\Gamma(\gamma + 1 + aE/\lambda)}{\Gamma(\gamma + 1 - aE/\lambda)} \frac{\gamma - aE/\lambda}{\nu - \lambda} e^{-i\pi(|l+1/2| - \gamma)}.
$$

Discrete energy levels of bound states are defined as roots of the equation $B_l(\gamma, \lambda, E) = 0$ (as they say, the zeros of $B_l(\gamma, \lambda, E)$), i.e., either

$$
\gamma + 1 - aE/\lambda = -n, \quad n = 0, 1, 2, \ldots
$$

(at these points $\Gamma(\gamma + 1 - aE/\lambda)$ has the poles including a pole at $l = 0, 1/2 > \gamma > 0$) or

$$
\gamma - aE/\lambda = 0
$$

(in which case $\nu < 0, l = -1$). Therefore, the discrete energy spectrum of bound states at $\lambda \leq 1/2$ has the form (see, also [3])

$$
E_{n,l} = m - \frac{n + \sqrt{(l + 1/2)^2 - a^2}}{\sqrt{[n + \sqrt{(l + 1/2)^2 - a^2}^2 + a^2]}}, \quad n = 0, 1, 2, \ldots, -\infty < l < \infty,
$$

It can be easily shown that the discrete spectrum accumulates at the point $E = m$, and its asymptotic form as $n \gg 1$ is given by the formula

$$
E_{n,l} = m - \frac{ma^2}{2n^2}.
$$

In the region $1/2 > \gamma > 0$ the wave function $\sim U_2(r, E)/\sqrt{r}$ is singular but square-integrable at $r \to 0$ with respect to the measure $rdr$ and the generalized eigenfunctions $F(r, E)$ of the radial Hamiltonian should be chosen in the form

$$
F(r, \gamma, E) = U_1(r, \gamma, E) + \xi U_2(r, \gamma, E),
$$
where $-\infty \leq \xi \leq \infty$ is the real parameter. For each $l = -1, 0$ there exist self-adjoint Dirac Hamiltonians $h_\xi$ parameterized by $\xi$ (the values $\xi = \pm \infty$ are equivalent) and specified by the asymptotic self-adjoint boundary conditions at the origin.

For $0 < \gamma < 1/2$, the coefficient $B_l(\gamma, E, \xi)$ takes the form

$$B_l(\gamma, E, \xi) = \frac{\Gamma(\gamma + 1 + i aE/p)}{\Gamma(\gamma + 1 - i aE/p)} e^{-2i\eta(\gamma) - i\pi(l+1/2) - \gamma} +$$

$$+ \frac{2\nu}{m} \left[ \frac{\Gamma(\gamma + 1 + i aE/p)}{\Gamma(\gamma + 1 - i aE/p)} e^{-2i\eta(\gamma + \tau) + i\pi(l+1/2) + \gamma} \right].$$

Performing the analytic continuation in the region $E < m$ by formula (22), we obtain the equation that determines discrete energy levels of bound states in the form $B_l(\gamma, E, \xi) = 0$. For $\xi = 0$, the entire analysis is similar to the one above, and all the formulas obtained there remain applicable in the case $0 < \gamma < 1/2$.

We fail to derive an explicit formula for the discrete energy spectrum in this region, but we can show: 1) the discrete energy levels are in the region $-m \leq E < m$ for $\xi \neq 0$, i.e. there exist values $\xi$ at which the lowest energy level can reach the boundary of the lower energy continuum $E = -m$, 2) the spectrum accumulates at the point $E = m$ and is described by formula (27), independent of $\xi$.

For $\gamma = i\sigma$ the behavior of the two functions $\sim U_{1,2}(r, i\sigma, E)/\sqrt{\eta}$ essentially differ from the one for real $\gamma$: both these functions oscillate near $r \to 0$. So, the most correct representation for the generalized eigenfunctions $F(r, \gamma, E)$ has to be

$$F(r, \gamma, E) = U_1(r, i\sigma, E)e^{i\theta} + U_2(r, i\sigma, E)e^{-i\theta},$$

where $0 \leq \theta \leq \pi$ is the real parameter in which case $\theta = 0, \pi$ are equivalent. For $\gamma = i\sigma$, there exists one-parameter family of self-adjoint Hamiltonians $h_\theta$ parameterized by the $\theta$ and specified by the asymptotic self-adjoint boundary conditions at the origin. For $\gamma = i\sigma$ discrete energy levels are also identified as the poles of the scattering amplitude at the analytic continuation in the region $-m \leq E < m$ located on the first sheet of the Riemann surface or as the zeros of the $B_l(\sigma, E, \xi)$ in the region $-m \leq E < m$. Having made the analytic continuation of $B_l(\sigma, E, \theta)$ in the region $-m \leq E < m$ by formula (22), we obtain

$$B_l(\sigma, E, \theta) = \frac{\Gamma(i\sigma + 1 + aE/\lambda)}{\Gamma(i\sigma + 1 - aE/\lambda)} e^{-2i\eta(i\sigma) - i\pi[l+1/2]} +$$

$$+e^{-2i\eta} \left[ \frac{2\nu}{m} \right]^{1/2} \frac{\Gamma(i\sigma + 1 + aE/\lambda)}{\Gamma(i\sigma + 1 - aE/\lambda)} e^{-2i\eta(i\sigma) + i\pi[l+1/2]}$$

and then derive the equation determining discrete energy levels ($B_l(\sigma, E, \xi) = 0$) on the first sheet of the Riemann surface in the form

$$\frac{\Gamma(2i\sigma)}{\Gamma(-2i\sigma)} \frac{\Gamma(-i\sigma - aE/\lambda)}{\Gamma(i\sigma - aE/\lambda)} e^{-2i\sigma \ln(2\eta/m) + i\pi[l+1/2]} = e^{2i\theta},$$

or

$$-\sigma \ln(2\eta/m) + \arg \Gamma(2i\sigma) - \arg \Gamma(i\sigma - aE/\lambda) - \pi[l+1/2] - \theta = k\pi, \quad k = 0, \pm 1, \ldots$$

We emphasize that Eq. (33) determines the fermion energy levels in the region $m > E \geq -m$ implicitly. Analysis of Eq. (33) shows that the number of discrete energy levels is finite in the interval $-m \leq E < 0$, and the spectrum for $k \gg 1$ is described by the right-hand side of asymptotic formula (27). We also note that near the boundary of the lower energy continuum, i.e., at $E = -m + \epsilon, \epsilon > 0$, Eq. (33) closely resembles the formula for the electron energy spectrum in a strong cutoff Coulomb potential in $2+1$ dimensions near $E = -m$ (see [19]).

One can derive an explicit formula for energy levels near the boundary of the lower energy continuum $E = -m + \epsilon, \epsilon > 0$, in the limit $\sigma \ll 1$. Using formulas

$$\frac{\Gamma(2i\sigma)}{\Gamma(-2i\sigma)} \approx -(1 + 4i\sigma \psi(1)) = e^{i\pi - 4i\sigma \psi}, \quad \frac{\Gamma(-i\sigma - aE/\lambda)}{\Gamma(i\sigma - aE/\lambda)} \approx$$

$$\approx 1 - 2i\sigma \psi \left( \sqrt{\frac{a^2 m}{2\epsilon}} \right) = e^{-i\sigma \ln(a^2 m/2\epsilon) + i\pi \sqrt{2\epsilon/ma^2}},$$

where $\psi$ is the digamma function.
where \( \psi(z) \) is the logarithmic derivative of Gamma function \( \psi(z) \), \( C = -\psi(1) = 0.57721 \) is the Euler constant, as well as formula

\[
\psi(z)|_{z \to \infty} \approx \ln z - \frac{1}{2z} - \frac{1}{12z^2},
\]

we obtain

\[
\sqrt{\epsilon_{n,l}/2ma^2} \approx \theta - \pi/2 + \sigma(\ln 2a + 2C) + \pi n, \quad n = 0, \pm 1, \ldots
\]

For \( 0 < \sigma \ll 1 \) Eq. \( \psi(z) \) has real solution \( E = -m, \epsilon_{n,l} = 0 \) for \( n = 0 \) and \( \sigma_{cr}(a_{cr}, l_{cr}) \) as a function of \( a_{cr}, l_{cr} \) related to \( \theta_{cr} = \pi/2 - \sigma_{cr}(\ln 2a_{cr} + 2C) \). It should be noted the various values of self-adjoint extension parameter lead to inequivalent physical cases (see, for example, \[42–44\] and \[45\]). It should be emphasized that in the supercritical Coulomb potential with \( \gamma = i\sigma \), when solving the problem by the physical regularization procedure, which is applied in the conventional quantum mechanics, the stronger singularity of the Coulomb potential at the origin has to be regularized by a cutoff radius \( R \) of a Coulomb potential at small distances \( r \). Therefore, physically, in such a supercritical potential the self-adjoint extension parameter can be interpreted in terms of product \( mR \). So, the nonzero value \( \theta_{cr} \) at which the lowest energy level reaches the boundary \( E = -m \) obviously means that when solving the problem by the physical regularization, the the lowest energy level can become equal \(-m\) only in a supercritical (but cutoff) Coulomb potential.

### III. FERMIONS PAIR PRODUCTION

**Massive case.** Now we show that if \( a, \sigma \) are further increased, the lowest energy level dives into the lower continuum \( (E < -m) \) and becomes a quasi-stationary state (a resonance) at some \( a > a_{cr}, \sigma > \sigma_{cr} \). Hence, as we allow a small change in \( \sigma \) such that \( \sigma > \sigma_{cr} \), a sudden change in spectrum has to occur. For which reason the scattering amplitude has a discontinuity associated with the disappearance of its bound state pole from the physical sheet: for \( E < -m \) only the continuous spectrum exists, but below \( \Re E > 0, \Im E > 0 \) there is a second (nonphysical), \( (\Re E < 0, \Im E < 0, \Re \sqrt{m^2 - E^2} < 0, \Im \sqrt{m^2 - E^2} < 0) \) sheet on which now the former bound state resides at \( \sigma > \sigma_{cr} \). The key difference of the case \( \sigma > \sigma_{cr} \) from \( \sigma < \sigma_{cr} \) is that the former bound states at \( \sigma > \sigma_{cr} \) become quasi-stationary ones; they have “complex energies” \( E = |E|e^{i\tau} \). It follows from the equation \( \psi(z) \) that the diving point \( (\epsilon = 0) \) defines a critical coupling, \( a_{cr}, \sigma_{cr} \), and strongly depends on the self-adjoint extension parameter (i.e., physically, on the cutoff radius \( R \) of a Coulomb potential).

In order to make the needed analytic continuation we must determine \( B_i(\sigma,E,\theta) \) on the upper edge of the cut chosen to run along the negative real energy axis from the branch point \( -m \to -\infty \) and then have to go across the cut on the second sheet (see, for example, \[46\]). As a result, we derive the transcendental complex equation that determine implicitly the “complex energies” of the quasi-stationary states in the form

\[
\frac{\Gamma(2i\sigma)\Gamma(-i\sigma - iaE/p)}{\Gamma(-2i\sigma)\Gamma(i\sigma - iaE/p)} e^{-2i(\sigma \ln(-2ip/m) + \pi|l+1/2|)} = -e^{2i\theta}. \tag{37}
\]

For \( \sigma \ll 1 \) it is only natural to look for solutions of this equation in the \( E = |m + \epsilon| \exp(i\tau), 1 \gg \epsilon > 0 \). Then, using formulas

\[
\frac{\Gamma(-i\sigma - iaE/p)}{\Gamma(i\sigma - iaE/p)} \bigg|_{\sigma \ll 1} \approx e^{-2i\sigma(\Re \psi(z) + \Im \psi(z))}, z = -iaE/p \equiv -ia|E| \exp(i\tau)/p, \tag{38}
\]

\[
\Im \psi(ix) = \frac{1}{2x} + \frac{\pi}{2} \coth \pi x, \text{ for } x \gg 1,
\]

as well as Eq. \( \psi(z) \), we find \( \tau \approx \pi + (\pi/2)e^{-\sqrt{2m(\pi a)^2}/c} \) and

\[
\sigma \epsilon_{n,l}/6ma^2 \approx \theta - \pi/2 + \pi n + \sigma(2C + \ln 2a). \tag{39}
\]

We see that the rights of equations \( \psi(z) \) and \( \psi(z) \) do not hallucinate.
It is seen that when $\sigma > \sigma_{cr}$, the lowest state dives into the negative energy continuum and becomes a quasi-stationary state (a quasi-localized resonance) described with the quasi-discrete spectrum $E = \text{Re}E - i|\text{Im}E|$.

Physically, the appearance of new fermion states with negative energies implies a rearrangement of the vacuum. The additional distortion of the negative energy continuum (due to the quasi-stationary states) leads to a negative charge density due to the “real vacuum polarization” \cite{4}. The energies of the quasi-stationary states defines and depends upon the parameter $\theta$.

It follows from the equation (39) that at $1 \gg \sigma > \sigma_{cr}$

$$\text{Re}E \approx -m - \epsilon_s, \quad \epsilon_s \approx 6ma_{cr}^{3/2} \frac{\sigma - \sigma_{cr}}{\sigma_{cr}} (2C + \ln 2a_{cr}). \quad (40)$$

The modulus of the imaginary part $|E_w| = 2|\text{Im}E|$ is the doubled probability of the production of the fermion pair by supercritical Coulomb field. It is exponentially small in this case and the lifetime of the quasi-stationary supercritical level is diverging $\Delta t \sim 1/w$.

If $a$ is further increased, other levels will sequentially dive into the lower energy continuum at higher $a$ (see, for instance, \cite{4}).

**Massless case.** In the massless case the spectrum is continuous everywhere and the bound states are absent, nevertheless, the quasi-stationary states emerge in a supercritical Coulomb potential. In solving the problem with massless fermions, it is evident, we can use the some needed formulas (putting in them $m = 0$) which we have derived above for the massive case. In particular, the main equation (13) obviously is valid for the massless case if we put in it: $m = 0, p = |E|, x = -2i|E|r, E/p = e' \equiv \text{sign}E$.

In the region $|E| > 0$ the energy spectrum is continuous for real $\gamma$ and a charged massless fermion in a Coulomb potential does not have bound states. Nevertheless, it is helpful to analyze the neighborhood of zeros of the coefficient in the ingoing wave; they, for example, may characterize some kind of accumulation points of fermion states. It is rewarding to study directly the case $1/2 > \gamma > 0$. This case is described by the equation (29) with taking into account the above replacements. Then, solving the equation $B_i(\gamma, E, \xi) = 0$, we obtain the following complex equation

$$\left(2\frac{|E|}{E_0}\right)^{-2\gamma} = \xi \left[\frac{\Gamma(1 - 2\gamma)\Gamma(\gamma - ie'a)}{\Gamma(1 + 2\gamma)\Gamma(\gamma - ie'a)}\right] e^{i\pi(\gamma' - 1/2)} \quad (41)$$

and using the known representation

$$\text{arg}\Gamma(x + iy) = y \left[-C + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{y} \text{arctan} \frac{y}{x + n - 1}\right)\right], \quad (42)$$

we rewrite (41) in the form of two real equations:

$$E(a, \gamma, \xi) = \frac{e'}{2} E_0 \left[\frac{\Gamma(1 + 2\gamma)\Gamma(-\gamma - ie'a)}{\xi\Gamma(1 - 2\gamma)\Gamma(\gamma - ie'a)}\right]^{1/2\gamma} \quad (43)$$

and

$$\pi \left(e'\gamma - \frac{1}{2}\right) + \text{arctan} \frac{e'a}{\gamma} - \sum_{n=1}^{\infty} \text{arctan} \frac{2e'a\gamma}{n^2 + a^2 - \gamma^2} = (s - 1) \frac{\pi}{2}, \quad (44)$$

where $s = \xi/|\xi| = \pm 1$, $s = 1(-1)$ for $\infty > \xi \geq 0 (0 \geq \xi > -\infty)$ and we introduce the positive constant $E_0$ with the dimension of mass. We see that an energy scale in the massless case set explicitly by the parameter $E_0$. The values $a, \gamma, s, e', \xi$ have to be determined by these two equations.

For $\gamma \to 1/2$ near $|E| = 0$, we find

$$E = e' \frac{1 - 2\gamma}{2|\xi|} \frac{|\Gamma(-1/2 - ie'a)|}{|\Gamma(1/2 - ie'a)|}. \quad (45)$$

whence it follows that $|E| = 0$ and equality (44) is satisfied by the value $\gamma = 1/2$ for $e' = 1$ and $s = 1$ and for $e' = -1$ and $s = -1$ only if $a^2 = (l + 1/2)^2 = 1/4$, i.e. only in free case $a = 0$. In the free case, the quantum system exhibits the particle-antiparticle symmetry.

For $\gamma \to 0$, $|E|$ tends to 0 as $2E \approx e'(|\xi|)^{1/2\gamma}$ and equality (44) is satisfied by $e' = \pm 1, \gamma = 0$ only if $s = -1(0 \geq \xi > -\infty)$. This implies that particle states with $E > 0$ and, on the other hand
antiparticle states with \( E < 0 \) accumulate near the point \( E = 0 \) if \(|\xi| > 1\) but separate as long as \( a < a_{cr} = 1/2 \) (see \[43\]).

As \( a > a_c = 1/2 \) the energy spectrum of massless fermions has to be changed; the scattering amplitude has a discontinuity associated now with the appearance of quasi-stationary states with negative “complex” energies \( (E = |E|e^{i\beta}) \), which are located on the second sheet. It will be recalled that only the stationary states with the continuous spectrum exists in the region \( E < -m \) on the physical sheet and the point \( E = 0 \) is the branch point of the scattering amplitude in the complex plane of \( E \).

Now we need to determine the coefficient \( B_l(\sigma, E, m = 0, \theta) \) given on the upper edge of the cut \((-\infty, 0)\) in going across the cut on the second sheet. As a result, we derive the following equations (\( \gamma = i\sigma \))

\[
\frac{\Gamma(i(e'/a - \sigma))}{\Gamma(i(e'/a + \sigma))} e^{-\pi\sigma + 2\sigma\beta} = 1, \quad e' = -1
\]

(46)

and

\[
2\sigma \ln(2|E|/E_0) - 2 \arg \Gamma(2i\sigma) + \arg \Gamma(-ia + i\sigma) - \arg \Gamma(-ia - i\sigma) = 2(\theta - \pi n - \pi/2).
\]

(47)

Here \( n = 1, 2, \ldots, \pi \geq \theta \geq 0 \) and now \( e' = -1 \) corresponds to the nonphysical sheet. Increasing \( a \) (\( \sigma \)) will increase \( n \) and decrease the energy. Using the equation \[42\], we write Eq. \[47\] for \( \sigma \ll 1 \) in the form

\[
\sigma \ln(2|E|/E_0) = (\theta - \pi n - \pi/2) - \sigma C - \sigma \ln \sqrt{1 + a^2}.
\]

(48)

For small \( \sigma \ll 1 \), Eq. \[46\] has approximate solution \( \beta \approx -1/2a + \text{Im} \psi(i\sigma) + \pi/2 \) and \( \beta \approx [1 + \coth(\pi/2)]\pi/2 \approx (1 + 0.04)\pi \) for \( a = 1/2 \). Eqs. \[40\] and \[45\] are approximately satisfied near \(|E| = 0\) only in the region \( E < 0 \). Then, for \( \sigma \ll 1 \) we find the spectrum of supercritical resonances in the form

\[
E_{n,\beta,\sigma,0} = (E_0/2) \cos \frac{\beta}{\sqrt{1 + a^2}} e^{-(\pi n - \theta/2 + \sigma C + \sigma \ln \sqrt{1 + \sigma^2})/\sigma}.
\]

(49)

This spectrum, as function of \( a \), has an essential singularity at \( a = a_c \), \( \sigma = 0 \) and the infinite number of quasi-discrete levels occurs \[11, 13, 48\]. In the massless case there is no natural length scale to characterize the localization region of the infinite number of emerging quasi-stationary states. The stronger singularity of the Coulomb potential at the origin has to be regularized in the supercritical regime, by a finite size \( R \) of the Coulomb impurity and the dimensionless self-adjoint extension parameter \( \theta \) can now be interpreted in terms of product \( E_0R \).

It should be emphasized that in the massless case there is no sequential diving into the lower energy continuum but the infinite number of quasi-stationary states occurs. It is seen that the energy spectrum of these states is quasi-discrete, consists of a number of broadened levels whose width (defined by the imaginary part \( E \)) is related to the inverse lifetime (decay rate) of a \( n \)-quasi-stationary state. These quasi-localized states have negative energies and are directly associated with the positron creation in the quantum electrodynamics \[49\]. Again the modulus of the imaginary part \( E \)

\[
w = 2|\text{Im} E| = E_0|\sin \beta| e^{-(\pi n - \theta/2 + \sigma C + \sigma \ln \sqrt{1 + \sigma^2})/\sigma}
\]

(50)

is the doubled probability of the creation of the massless fermion pair by supercritical Coulomb field.

IV. SUMMARY

In this paper we study the creation of charged (massive and massless) fermion pair by supercritical Coulomb field. We construct the self-adjoint two-dimensional Dirac Hamiltonians with a singular Coulomb potential and determine the quantum-mechanical states for self-adjoint Hamiltonians in the corresponding Hilbert spaces of square-integrable functions. The domain (including the singular \( r = 0 \) region) in these spaces, parameterized by extension parameters and classified by boundary conditions at \( r = 0 \), is found for each self-adjoint Hamiltonian.

We determine the scattering amplitude as a function of the “complex energy” in which the dimensionless self-adjoint extension parameter is incorporated and then obtain the equations implicitly defining the possible discrete energy spectra of the self-adjoint Dirac Hamiltonians with a Coulomb potential. We establish that the quantum system in the presence of supercritical Coulomb potential becomes unstable which manifests in the appearance of quasi-stationary states in the lower (negative) energy continuum.

The above scattering amplitude in the presence of a supercritical Coulomb potential is shown to become ambiguous function; it has a discontinuity in the complex plane of energy and additional singularities on
the negative energy axis of the nonphysical sheet of the Riemann surface. The imaginary part of energy is related to the inverse lifetime of the quasi-stationary state as well as the creation probability of charged fermion pair by supercritical Coulomb field. Explicit analytical expressions for the creation probabilities of charged (massive or massless) fermion pair are obtained in a supercritical Coulomb field.

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