On Vertex Irregular Total $k$-labeling and Total Vertex Irregularity Strength of Lollipop Graphs

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Abstract. Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A vertex irregular total $k$-labeling $\lambda: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$ of a graph $G$ is a labeling of vertices and edges of $G$ in such a way that the weights of any two different vertices $x$ and $y$ are distinct. The weight of a vertex $x$ in $G$, denoted by $wt(x)$, is defined as the sum of the label of $x$ and the labels of all edges incident with the vertex $x$. A vertex irregular total $k$-labeling $\lambda$ is a labeling of vertices and edges of $G$ in such a way that the weights of any two different vertices $x$ and $y$ are distinct. They also defined $tvs(G)$, as the smallest positive integer $k$ for which the graph $G$ has a vertex irregular total $k$-labeling.

Many researchers have investigated the total vertex irregularity strength of some graphs. In 2007, Bača, et al. [1] defined the weight of a vertex $x$ under a total labeling $\lambda$ of graph $G = (V, E)$ as $wt(x) = \lambda(x) + \sum_{xy \in E(G)} \lambda(xy)$, for $x, y \in V(G)$ and $xy \in E(G)$. A vertex irregular total $k$-labeling $\lambda: V \cup E \rightarrow \{1, 2, \ldots, k\}$ of a graph $G$ is a labeling of vertices and edges of $G$ in such a way that the weights of any two different vertices $x$ and $y$ are distinct. They also defined $tvs(G)$, as the smallest positive integer $k$ for which the graph $G$ has a vertex irregular total $k$-labeling.

In 2007 Bača, et al. [1] observed that the total vertex irregularity strength of complete graph is equal to $tvs(K_n) = 2$, $n \geq 2$, while for cycle graph is equal to $tvs(C_n) = \left\lceil \frac{n+2}{3} \right\rceil$, $n \geq 3$. In 2005 Wijaya, et al. [7] obtained $tvs(K_{n,n}) = 3$, $n \geq 3$. Indriati, et al. [3] determined $tvs$ of double star graphs $S_{n,m}$ and caterpillar $S_{n,2,n}$. Nurdin, et al. [4] defined $tvs$ of path $P_n$ and connected graph $G$ having $n_i$ vertices of degree $i$, $\delta \leq i \leq \Delta$, as follows,

$$tvs(G) \geq \max \left\{ \left\lfloor \frac{\delta + n_\delta}{\delta + 1} \right\rfloor, \left\lfloor \frac{\delta + n_\delta + n_{\delta+1}}{\delta + 2} \right\rfloor, \ldots, \left\lfloor \frac{\delta + \sum_{i=\delta}^{\Delta} n_i}{\Delta + 1} \right\rfloor \right\}. \tag{1}$$

In this paper, we determine the total vertex irregularity strength of a lollipop graphs $L_{m,n}$. 

1. Introduction

Let $G(V, E)$ be a simple connected undirected graph with vertex set $V$ and edge set $E$. Wallis [5] defined a labeling of graph as a map that carries graph elements to the number (usually to the positive or non negative integers). The most common choices of domain are the vertex set alone (called a vertex labeling), the edge set alone (called an edge labeling), or the set of vertices and edges (called a total labeling). In the recent development, the graph labeling is also defined as a various functions and one of this is an irregular labeling (see Gallian [2]).

In 2007, Bača, et al. [1] defined the weight of a vertex $x$ under a total labeling $\lambda$ of graph $G = (V, E)$ is $wt(x) = \lambda(x) + \sum_{xy \in E(G)} \lambda(xy)$, for $x, y \in V(G)$ and $xy \in E(G)$. A vertex irregular total $k$-labeling $\lambda: V \cup E \rightarrow \{1, 2, \ldots, k\}$ of a graph $G$ is a labeling of vertices and edges of $G$ in such a way that the weights of any two different vertices $x$ and $y$ are distinct. They also defined $tvs(G)$, as the smallest positive integer $k$ for which the graph $G$ has a vertex irregular total $k$-labeling.

Many researchers have investigated the total vertex irregularity strength of some graphs. In 2007 Bača, et al. [1] observed that the total vertex irregularity strength of complete graph is equal to $tvs(K_n) = 2$, $n \geq 2$, while for cycle graph is equal to $tvs(C_n) = \left\lceil \frac{n+2}{3} \right\rceil$, $n \geq 3$. In 2005 Wijaya, et al. [7] obtained $tvs(K_{n,n}) = 3$, $n \geq 3$. Indriati, et al. [3] determined $tvs$ of double star graphs $S_{n,m}$ and caterpillar $S_{n,2,n}$. Nurdin, et al. [4] defined $tvs$ of path $P_n$ and connected graph $G$ having $n_i$ vertices of degree $i$, $\delta \leq i \leq \Delta$, as follows,

$$tvs(G) \geq \max \left\{ \left\lfloor \frac{\delta + n_\delta}{\delta + 1} \right\rfloor, \left\lfloor \frac{\delta + n_\delta + n_{\delta+1}}{\delta + 2} \right\rfloor, \ldots, \left\lfloor \frac{\delta + \sum_{i=\delta}^{\Delta} n_i}{\Delta + 1} \right\rfloor \right\}. \tag{1}$$

In this paper, we determine the total vertex irregularity strength of a lollipop graphs $L_{m,n}$. 

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2. Main Result

Weisstein [6] defined the \((m, n)\)-lollipop graphs, denoted by \(L_{m,n}\), as a graph obtained by joining a complete graph \(K_m\) to a path graph \(P_n\) with a bridge. Lollipop graphs \(L_{m,n}\) has vertex set \(V(L_{m,n}) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}\), where \(v_1, v_2, \ldots, v_m\) are the vertices of the complete graph \(K_m\) and \(u_1, u_2, \ldots, u_n\) are the vertices of the path graph \(P_n\). Moreover, vertex \(v_i, 1 \leq i \leq m\), adjacent with vertex \(v_j, 1 \leq j \leq m, i \neq j\). Vertex \(u_i, 1 \leq i \leq n-1\), adjacent with vertex \(u_{i+1}\). Vertex \(v_m\) adjacent with vertex \(u_n\). Lollipop graphs \(L_{m,n}\) can be see at Figure 1.

![Figure 1. Lollipop graph \(L_{m,n}\)](image)

Theorem 2.1 For \(m \geq 3\) and \(n \geq 1\), the total vertex irregularity strength of lollipop graphs \(L_{m,n}\) is equal to \(\text{tvs}(L_{m,n}) = \max\{\lceil \frac{n+1}{3}\rceil, \lceil \frac{m+n}{m}\rceil\}\).

Proof. Lollipop graphs \(L_{m,n}\) has 1 vertex of degree 1, \((n-1)\) vertices of degree 2, \((m-1)\) vertices of degree \((m-1)\), and 1 vertex of degree \(m\). According (1), a lower bound of lollipop graph for \(m \geq 3\) and \(n \geq 1\) is,

\[
\text{tvs}(L_{m,n}) \geq \max\left\{\frac{1+1}{1+1}, \frac{1+1+(n-1)}{1+2}, \frac{1+1+(n-1)+(m-1)}{1+(m-1)}, \frac{1+1+(n-1)+(m-1)+1}{m+1}\right\}
\]

\[
= \max\left\{\left\lceil \frac{2}{2}\right\rceil, \left\lceil \frac{n+1}{3}\right\rceil, \left\lceil \frac{m+n}{m}\right\rceil, \left\lceil \frac{m+n+1}{m+1}\right\rceil\right\}
\]

\[
= \max\left\{\left\lceil \frac{n+1}{3}\right\rceil, \left\lceil \frac{m+n}{m}\right\rceil\right\}
\]

We now prove that \(\text{tvs}(L_{m,n}) \leq \max\{\lceil \frac{n+1}{3}\rceil, \lceil \frac{m+n}{m}\rceil\}\) for \(m \geq 3\) and \(n \geq 1\). Let \(k = \max\{\lceil \frac{n+1}{3}\rceil, \lceil \frac{m+n}{m}\rceil\}\). Then we define \(\lambda\) as follows.

\[
\lambda(v_m) = 1, \quad \text{for } m \geq 3.
\]

\[
\lambda(u_i) = \begin{cases} 
1, & \text{for } 1 \leq i \leq 4; \\
\left\lceil \frac{i+1}{3}\right\rceil, & \text{for } 5 \leq i \leq n.
\end{cases}
\]

\[
\lambda(u_iu_{i+1}) = \left\lceil \frac{i+1}{3}\right\rceil, \quad \text{for } 1 \leq i \leq n-1.
\]

\[
\lambda(v_mu_n) = \left\lceil \frac{n+1}{3}\right\rceil.
\]
For vertex labeling of $\lambda(v_i)$, $i \neq m$, and $\lambda(v_iv_j)$, $1 \leq i, j \leq m$, there are 3 cases.

**Case 1.** For $1 \leq n \leq m-2$ and $m \geq 3$

\[
\lambda(v_i) = \begin{cases} 
1, & \text{for } 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor; \\
2, & \text{for } \left\lceil \frac{m}{2} \right\rceil < i < m.
\end{cases}
\]

$\forall i, 1 \leq i \leq m, i \neq j$

\[
\lambda(v_iv_j) = \begin{cases} 
1, & \text{for } 1 \leq j \leq m - i + 1; \\
2, & \text{for } m - i + 2 \leq j \leq m.
\end{cases}
\]

**Case 2.** For $n \geq m - 1$ and $m \geq 3$ and not for $m = 4$ and $n \equiv 2 \pmod{3}$, $n \geq 8$

\[
\lambda(v_i) = \begin{cases} 
\left\lceil \frac{n-m+2}{3} \right\rceil, & \text{for } 1 \leq i \leq \left\lfloor \frac{m-3+(m-n+1) \mod 3}{2} \right\rfloor; \\
1 + \left\lceil \frac{n-m+2}{3} \right\rceil, & \text{for } \left\lfloor \frac{m-3+(m-n+1) \mod 3}{2} \right\rfloor < i < m.
\end{cases}
\]

$\forall i, 1 \leq i, j \leq m, i \neq j$

\[
\lambda(v_iv_j) = \begin{cases} 
\left\lceil \frac{n-m+2}{3} \right\rceil, & \text{for } 1 \leq j \leq m - i - (n - m + 4) \mod 3 \text{ and } 1 \leq i \leq m - (n - m + 4) \mod 3; \\
1 + \left\lceil \frac{n-m+2}{3} \right\rceil, & \text{for } m - i - (n - m + 4) \mod 3 < j < m \text{ and } 1 \leq i \leq m - (n - m + 4) \mod 3.
\end{cases}
\]

\[
\lambda(v_iv_m) = \begin{cases} 
\left\lceil \frac{n-m+2}{3} \right\rceil, & \forall i \text{ and } (n - m + 4) \equiv 0, 1 \pmod{3}; \\
k, & \text{for } i = m - 1 \text{ and } (n - m + 4) \equiv 2 \pmod{3}; \\
1 + \left\lceil \frac{n-m+2}{3} \right\rceil, & \text{for } i \neq m - 1 \text{ and } (n - m + 4) \equiv 2 \pmod{3}.
\end{cases}
\]

**Case 3.** For $m = 4$ and $n \equiv 2 \pmod{3}$, $n \geq 8$

\[
\lambda(v_i) = \begin{cases} 
\left\lceil \frac{n-2}{3} \right\rceil, & \text{for } i = 1; \\
1 + \left\lceil \frac{n-2}{3} \right\rceil, & \text{for } i \neq 1.
\end{cases}
\]

$1 \leq j \leq 4, i \neq j$

\[
\lambda(v_iv_j) = \begin{cases} 
\left\lceil \frac{n-2}{3} \right\rceil, & \text{for } 1 \leq j \leq 3 - i \text{ and } i = 1, 2; \\
1 + \left\lceil \frac{n-2}{3} \right\rceil, & \text{for } 4 - i \leq j \leq 4 \text{ and } 1 \leq i \leq 3.
\end{cases}
\]
We can see that \( \lambda \) is a map that carries \( V(L_{m,n}) \cup E(L_{m,n}) \) to \( 1, 2, \ldots, k \). Therefore, \( \lambda \) is a total \( k \)-labeling with \( k = \max\{\lceil \frac{n+1}{3} \rceil, \lceil \frac{m+n}{m} \rceil \} \). Then, the weight of vertices are.
\[
wt(u_i) = i + 1, \quad \text{for } 1 \leq i \leq n.
\]

**Case 1.** For \( 1 \leq n \leq m - 2 \) and \( m \geq 3 \)
\[
wt(v_i) = \begin{cases} 
  m + i + \left\lceil \frac{n-m+2}{3} \right\rceil + (n-m) \mod 3 + (m-1) \left( \frac{(n-m)-(n-m) \mod 3}{3} \right), & \text{for } 1 \leq i \leq m - 1; \\
  2m - 1 + \left\lceil \frac{n+1}{3} \right\rceil + (m-1) \left( \frac{(n-m)-(n-m) \mod 3}{3} \right), & \text{for } i = m \\
\end{cases}
\]

**Case 2.** For \( n \geq m - 1 \) and \( m \geq 3 \) and not for \( m = 4 \) and \( n \equiv 2 \pmod{3} \), \( n \geq 8 \)
\[
wt(v_i) = \begin{cases} 
  m + i + \left\lceil \frac{n-m+2}{3} \right\rceil + (n-m) \mod 3 + (m-1) \left( \frac{(n-m)-(n-m) \mod 3}{3} \right), & \text{for } 1 \leq i \leq m - 1; \\
  2m - 1 + \left\lceil \frac{n+1}{3} \right\rceil + (m-1) \left( \frac{(n-m)-(n-m) \mod 3}{3} \right), & \text{for } i = m \\
\end{cases}
\]

**Case 3.** For \( m = 4 \) and \( n \equiv 2 \pmod{3} \), \( n \geq 8 \)
\[
wt(v_i) = \left\lceil \frac{n-2}{3} \right\rceil + n + i - 1, \quad \text{for } 1 \leq i \leq 4.
\]

We can see that the weight of all vertices are distinct. Then, we can obtain a vertex irregular total \( k \)-labeling of lollipop graphs \( L_{m,n} \). That means, \( tvs(L_{m,n}) = k \). So, it is proven that \( tvs(L_{m,n}) \leq \max\{\lceil \frac{n+1}{3} \rceil, \lceil \frac{m+n}{m} \rceil \} \) for \( m \geq 3 \) and \( n \geq 1 \).

If \( m = 4, n > 5 \) and \( m \geq 5, n > 2 \), then the total vertex irregularity strength of lollipop graphs \( L_{m,n} \) is \( tvs(L_{m,n}) = \lceil \frac{n+1}{3} \rceil \). If \( m = 3, n \geq 1, m = 4, n \leq 5 \), and \( m \geq 5, n \leq 2 \), then the total vertex irregularity strength of lollipop graphs \( L_{m,n} \) is \( tvs(L_{m,n}) = \lceil \frac{m+n}{m} \rceil \).
Example for case 1 and case 2.

**Figure 2.** Vertex irregular total 2-labeling of $L_{4,2}$

The vertex labeling of $L_{4,2}$, is as follows.

\[
\begin{align*}
\lambda(u_1) &= 1, & \lambda(v_1) &= 2, & \lambda(v_3) &= 2, \\
\lambda(u_2) &= 1, & \lambda(v_2) &= 1, & \lambda(v_4) &= 1.
\end{align*}
\]

The edge labeling of $L_{4,2}$, is as follows.

\[
\begin{align*}
\lambda(u_1u_2) &= 1, & \lambda(v_1v_2) &= 1, & \lambda(v_1v_4) &= 1, & \lambda(v_2u_4) &= 2, \\
\lambda(u_2v_4) &= 1, & \lambda(v_1v_3) &= 1, & \lambda(v_2v_3) &= 1, & \lambda(v_3v_4) &= 2.
\end{align*}
\]

The weight of all vertices of $L_{4,2}$, is as follows.

\[
\begin{align*}
wt(u_1) &= 2, & wt(v_1) &= 4, & wt(v_3) &= 6, \\
wt(u_2) &= 3, & wt(v_2) &= 5, & wt(v_4) &= 7.
\end{align*}
\]

Based on Figure 2, we can see that the biggest label is 2 and the weight of all vertices are distinct, so that we obtain a vertex irregular total 2-labeling of $L_{4,2}$. That means, $tvs(L_{4,2}) = 2$.

The vertex labeling of $L_{4,5}$, is as follows.

\[
\begin{align*}
\lambda(u_1) &= 1, & \lambda(u_4) &= 1, & \lambda(v_2) &= 2, \\
\lambda(u_2) &= 1, & \lambda(u_5) &= 2, & \lambda(v_3) &= 2, \\
\lambda(u_3) &= 1, & \lambda(v_1) &= 1, & \lambda(v_4) &= 1.
\end{align*}
\]

The edge labeling of $L_{4,5}$, is as follows.

\[
\begin{align*}
\lambda(u_1u_2) &= 1, & \lambda(u_4u_5) &= 2, & \lambda(v_1v_2) &= 2, & \lambda(v_2v_3) &= 2, \\
\lambda(u_2u_3) &= 1, & \lambda(u_5v_4) &= 2, & \lambda(v_1v_3) &= 2, & \lambda(v_2v_4) &= 2, \\
\lambda(u_3u_4) &= 2, & \lambda(v_1v_4) &= 2, & \lambda(v_3v_4) &= 3.
\end{align*}
\]

The weight of all vertices of $L_{4,5}$, is as follows.

\[
\begin{align*}
wt(u_1) &= 2, & wt(u_4) &= 5, & wt(v_2) &= 8, \\
wt(u_2) &= 3, & wt(u_5) &= 6, & wt(v_3) &= 9, \\
wt(u_3) &= 4, & wt(v_1) &= 7, & wt(v_4) &= 10.
\end{align*}
\]

Based on Figure 3, we can see that the biggest label is 3 and the weight of all vertices are distinct, so that we obtain a vertex irregular total 3-labeling of $L_{4,5}$. That means, $tvs(L_{4,5}) = 3$. 

---

**Figure 3.** Vertex irregular total 3-labeling of $L_{4,5}$
Example for case 3.

\[ \begin{align*}
  v_1 \quad &v_4 \\
  v_2 \quad &v_3 \\
  u_1 \quad &u_2 \quad &u_3 \quad &u_4 \quad &u_5 \quad &u_6 \quad &u_7 \quad &u_8 \\
  11111 \quad &1223 \quad &21 \quad &222333 \\
  2 \quad &3 \quad &3 \quad &3 \quad &3 \quad &3 \quad &3 \quad &3 \quad &3 \quad &11 \quad &9 \quad &5 \quad &10 \quad &6 \quad &3 \quad &2 \quad &8 \quad &12 \quad &4 \quad &13 \quad &7
\end{align*} \]

Figure 4. Vertex irregular total 3-labeling of \( L_{4,8} \)

The vertex labeling of \( L_{4,8} \), is as follows.

\[
\begin{align*}
  \lambda(u_1) &= 1, & \lambda(u_5) &= 2, & \lambda(v_1) &= 2, \\
  \lambda(u_2) &= 1, & \lambda(u_6) &= 2, & \lambda(v_2) &= 3, \\
  \lambda(u_3) &= 1, & \lambda(u_7) &= 2, & \lambda(v_3) &= 3, \\
  \lambda(u_4) &= 1, & \lambda(u_8) &= 3, & \lambda(v_4) &= 1.
\end{align*}
\]

The edge labeling of \( L_{4,8} \), is as follows.

\[
\begin{align*}
  \lambda(u_1u_2) &= 1, & \lambda(u_5u_6) &= 2, & \lambda(v_1v_2) &= 2, & \lambda(v_2v_4) &= 3, \\
  \lambda(u_2u_3) &= 1, & \lambda(u_6u_7) &= 3, & \lambda(v_1v_3) &= 3, & \lambda(v_3v_4) &= 3, \\
  \lambda(u_3u_4) &= 2, & \lambda(u_7u_8) &= 3, & \lambda(v_1v_4) &= 3, \\
  \lambda(u_4u_5) &= 2, & \lambda(u_8v_4) &= 3.
\end{align*}
\]

The weight of all vertices of \( L_{4,8} \), is as follows.

\[
\begin{align*}
  wt(u_1) &= 2, & wt(u_5) &= 6, & wt(v_1) &= 10, \\
  wt(u_2) &= 3, & wt(u_6) &= 7, & wt(v_2) &= 11, \\
  wt(u_3) &= 4, & wt(u_7) &= 8, & wt(v_3) &= 12, \\
  wt(u_4) &= 5, & wt(u_8) &= 9, & wt(v_4) &= 13.
\end{align*}
\]

Based on Figure 4, we can see that the biggest label is 3 and the weight of all vertices are distinct, so that we obtain a vertex irregular total 3-labeling of \( L_{4,8} \). That means, \( tvs(L_{4,8}) = 3 \).

3. Conclusion

According to the discussion above it can be concluded that total vertex irregularity strength of a lollipop graphs \( L_{m,n} \) is as stated in Theorem 2.1.

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