Radiation Emission In Strong Electromagnetic Fields

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We investigate the capability of the Non-Linear Compton model, the Local Constant Field Approximation and the semi classical radiation integrals (BCK) used in Strong Field Quantum Electrodynamics to simulate the radiation emitted from relativistic particles moving in strong plane wave-like electromagnetic fields. We show that the applicability of the Local Constant Field Approximation model is solely determined by the classical intensity parameter \(\eta\). When \(\eta > 4\), discrepancies between the Local Constant Field approximation and the other two models are found in the peak in the radiation spectrum and increases for lower photon energies. For lower values of \(\eta\), the Non-Linear Compton model and the BCK model start showing distinct features in the spectrum which are not captured by the Local Constant Field Approximation. For all simulations in this paper, agreement between the Non-Linear Compton model and the BCK model is found. The Non-Linear Compton model, when using a plane wave with a varying amplitude, is also shown to be credible for all values of \(\eta > 1\), even for a short (10 fs) laser pulse. An estimate of the suitability of these models can be determined from the formation length, by checking whether the field strength or the plane wave amplitude changes within the formation length. We will show that the formation length can be used as an extremely powerful tool to determine which radiation model to employ and how to implement it.

INTRODUCTION

The radiation from high energy electrons and positrons moving in strong electromagnetic fields has been studied \([1, 2]\) and is being studied \([3, 10]\) intensively. The prevalent theoretical approach is to treat the radiation field as a perturbation while keeping the interaction between external electromagnetic fields inside the electron wave function. The wave function of the radiating particle can, for some specific external fields, be reduced analytically to a point which enables one to evaluate the resulting radiation spectrum. For arbitrary external fields the perturbation method does not work. Instead one evaluates the spectrum numerically as an integral over the particle trajectory through the arbitrary electromagnetic field \([17, 19]\). The numerical method for arbitrary fields has been shown to give reliable results for high energy electrons and positrons penetrating oriented single crystals \([20, 21]\).

Even though the different analytical expressions have been around for decades, the difficulty of actually evaluating the radiation spectra has prevented a detailed comparison. With several upcoming experiments aiming to use these models for theoretical predictions, we believe that a detailed investigation of the different radiation models is useful and important.

In this paper we investigate the strengths and limitations of the different methods developed to evaluate the radiation spectra for electrons and positrons in strong electromagnetic fields. We start by outlining the important points in deriving the theoretical models highlighting where the models differ. We then compare simulated spectra for each model for different relevant experimental cases to see which models can be used given the experimental parameters.

THEORETICAL FORMALISM

The common starting point for all three theoretical models used for radiation emission investigated in this paper is a first order perturbation in the radiated photon field \([17, 22, 23]\) that contains the transition matrix element \(M\). The transition matrix element is given by

\[
M = e \int \bar{\Psi}_{p'}(x') \hat{\gamma}^5 \Psi_p(x) e^{-ik'x} \frac{1}{\sqrt{2k_0}} d^4x, \tag{1}
\]

for an electron that transitions from an initial state \(\Psi_p(x)\) to a final state \(\Psi_{p'}(x)\) from an interaction with a radiated photon. Here \(e\) is the electron charge, \(\hat{c}\) is the photon polarization vector, \(p\) and \(p'\) is the electron four-momentum in its initial and final state respectively, \(k'\) is the photon four-momentum in the final state, and \(\Psi_p(x)\) and \(\Psi_{p'}(x)\) are the electron wave functions. Throughout the paper, ‘\(^\dagger\)’ denotes an operator and ‘\(^\ast\)’ denotes the complex conjugate.

The specific choice of electron wave function is what differentiates the radiation models investigated in this paper. The first two models, the Non-Linear Compton model (NLCM) and the Local Constant Field Approximation (LCFA), are based on the exact solution to the Dirac equation found by Volkov \([22, 23]\). For these two models the vector potential \(A^\mu(\phi)\) of the external electromagnetic field, in which the electron moves, is assumed only to depend on the invariant variable \(\phi = kx\). Here \(k_\mu\) is a zero-length four-vector with \(k_\mu = (|k|, k)\) where \(k\) determines the propagation direction of the external
Quantum effects in the emission process become important when the energy of a single emitted photon is comparable to the energy of the emitting particle. Classically, a relativistic particle can emit photons with frequencies up to the critical frequency \( \omega_c \approx \omega_0 \gamma^3 \) [25]. For a relativistic particle moving perpendicular to a magnetic field, the condition for the emission process to be classical is
\[
1 \gg \frac{\hbar \omega_c}{c} \approx \frac{\gamma H}{H_0} = \chi.
\]

From the above discussion, it is evident that the parameter \( \chi \) is decisive in determining if the emission process or the trajectory can be treated classically or has to be treated quantum mechanically. Presently, no experiment has been close to producing fields comparable to the Swinger critical field, so the trajectory for relativistic particles have been treated classically. Due to the extra \( \gamma^3 \) factor on the radiation condition, quantum effects in the radiation process quickly start becoming important as the energy increases. Therefore, it is not surprising that the trajectory of the electrons end up being treated classically, even though a quantum theory is being used. In the following sections we outline the derivations of three models to evaluate the radiation spectrum from an electron moving in electromagnetic fields, with the goal of highlighting the major differences between the methods. This will provide an understanding of the advantages and limitations of each model, that will be investigated later.

Non-Linear Compton Model

Consider an electron moving in a linearly polarized plane wave field with a vector potential of the form \( A^\mu(\phi) = a^\mu \cos(\phi) \), where \( a^\mu \) is a constant four-potential amplitude. Evaluating the classical action integral analytically results in the following wave function [22, 26]
\[
\Psi_p(x) = \left[ 1 - \frac{e k^2 A(\phi)}{2kp} \right] \frac{u(p)}{\sqrt{2p_0}} e^{-iS/\hbar}, \tag{2}
\]
where \( u(p) \) is the free particle plane wave bispinor. The parameter \( S \) is the classical action of an electron moving in an electromagnetic field with vector potential \( A^\mu(\phi) \) given by
\[
S = px + e \int_0^\phi \left[ \frac{pA'(\phi')}{kp} - \frac{eA(\phi')^2}{2kp} \right] d\phi'. \tag{3}
\]
As stated in Ref. [23], the electron wave function is determined by the classical action, meaning that the Volkov wave function is semiclassical. The resulting transition matrix element becomes semiclassical, so the interaction between the electron and the external electromagnetic field is treated classically. In contrast, the interaction between the electron and the radiated photon is treated quantum mechanically. This is true for all radiation models investigated in this paper. It is worth noting that the Volkov wave functions are solutions to the Dirac equation, consequently they are completely quantum mechanical, therefore no semiclassical characteristics are inferred. The semiclassical nature of the wave function could have been predicted by looking at the quantum effects in the photon emission process and in the trajectory of the electron moving in an electromagnetic field. Such considerations are made in Refs. [17, 23]. Quantum effects in the trajectory of the particle becomes important when the energy levels in the electromagnetic field become comparable to the energy of the particle. The quantized energy levels of a particle moving perpendicularly to a magnetic field \( H \), have a spacing of
\[
\hbar \omega_0 = \hbar \frac{|e| |H|}{|p|} \approx |e| H \epsilon, \tag{4}
\]
where \( \omega_0 \) is the classical frequency of revolution of a particle with energy \( \epsilon \), momentum \( p \), and velocity \( v \). The approximation in Eq. (4) holds for \( \gamma \gg 1 \) with \( \gamma \) being the relativistic Lorentz factor. The motion is classical when \( \hbar \omega_0 / \epsilon \ll 1 \). It is convenient to introduce the quantum non-linearity parameter \( \chi \) and the critical Swinger field \( H_0 \) [23]
\[
\chi = \sqrt{(F_{\mu\nu} u^\nu)^2 / H_0}, \quad H_0 = m^2 / e\hbar = 4.4 \cdot 10^{13} \text{ G}, \tag{5}
\]
where \( F_{\mu\nu} \) is the external electromagnetic field tensor, \( u^\mu \) is the four-velocity of the electron. The parameter \( \chi \) can be expressed as the ratio between the Lorentz boosted electromagnetic field and the critical field in the rest frame of the particle. The trajectory is considered to be classical when
\[
1 \gg \frac{\hbar \omega_0}{\epsilon} = \frac{vmH}{\gamma H_0} \approx \frac{H_0}{H_0 \gamma^2} = \frac{\chi}{\gamma^3}. \tag{6}
\]
The functions in Eq. (9) can be written in terms of a discrete Fourier series
\[ \cos^n(\phi)e^{i(\alpha \sin(\phi) - \beta \sin(2\phi))} = \sum_{s=-\infty}^{\infty} A_n(s, \alpha, \beta)e^{is\phi}, \] (11)
where \( s \) is an integer and
\[ A_n(s, \alpha, \beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^n(\phi)e^{i(\alpha \sin(\phi) - \beta \sin(2\phi) - s\phi)} d\phi. \] (12)
The parameters in the above expression are given by
\[ \alpha = z \cos(\theta), \] (13)
\[ z = \frac{\eta^2 \sqrt{1 + \eta^2/2}}{\chi} \sqrt{u(u_s - u)}, \] (14)
\[ u_s = \frac{\eta(1 + \eta^2/2)}{2\chi}, \] (15)
\[ \beta = \frac{\eta^3 u}{8\chi}, \] (16)
where \( u = k k'/(k q') \approx \hbar \omega/(\epsilon - \hbar \omega) \) is related to the ratio of the emitted photon energy to the initial electron energy. The parameter
\[ \eta = e \sqrt{a^2/m_0 \omega}, \] (17)
is an invariant often called the classical non-linearity parameter where \( \omega \) is the frequency of the plane wave in which the particle is moving. The parameter \( \eta \) is equivalent to the work done by the external field on the particle over a wavelength. In some literature the RMS value of the field strength is used instead of \( \sqrt{\sigma} \) [27], then one must include a factor \( 1/\sqrt{2} \) and the resulting cross sections have to be altered accordingly.

Integrating the wave function in Eq. (8) over \( d^4x \) in Eq. (1) leads to a delta function \( \delta(s k' - q_i) \) for each term in the sum over \( s \). Since the energy is being delivered in quanta of \( k \), one can relate each term in the sum to a specific number of absorbed photons \( s \) from the background field. This is surprising since the interaction between the background field and the electron is determined by the classical action, and the quantum nature of the background field (interpreting it as a density of photons) is not included.

The result for an electron moving in a linearly polarized plane wave to emit a photon with energy ratio \( x_\omega = \hbar \omega/\epsilon \) per unit time becomes
\[ \frac{dP}{dx_\omega} = \frac{e^2 m^2}{8\pi q_0} \sum_{s=1}^{\infty} \int_0^{2\pi} \left( A_0^2 + \frac{\eta^2}{2(1 + u)} \left( A_1^2 - A_0 A_2 \right) \right) d\theta, \] (18)
where the ratio \( x_\omega \) relates to \( u \) by \( u = x_\omega/(1 - x_\omega) \).

**Local Constant Field Approximation**

A special case of the NLCM is when a plane wave field is approximated by letting the frequency \( \omega \) become very small, letting the classical non-linearity parameter \( \eta \to \infty \), which corresponds to the particle being in a constant crossed field. This approximation is called the LCFA. Analytically this is achieved by expressing the vector potential to be linear in the phase \( \phi = k x \) and on the form \( A^\mu(\phi) = u^\mu \phi \) [23]. The corresponding electron wave function then becomes
\[ \Psi_p(x) = \left[ 1 - \frac{e k k \phi}{2k p} \frac{u(p)}{\sqrt{2q_0}} \exp \left( i \frac{\epsilon a^2}{6k p} \phi^3 + ip x \right) \right], \] (19)
and the transition matrix elements will contain functions of the type
\[ \phi^n \exp \left[ i \left( \frac{\alpha \phi^2}{2} - \frac{4\beta \phi^3}{3} \right) \right], \quad n = 0, 1, 2, \] (20)
where \( \alpha \) and \( \beta \) are defined in Eq. (10). The functions can be written in terms of a continuous Fourier integral
\[ \phi^n e^{i[(\alpha \phi^2/2 - 4\beta \phi^3/3)]} = i^n \int_{-\infty}^{\infty} e^{i\theta} \frac{\partial C(s, \alpha, \beta)}{\partial s^n} ds, \] (21)
where
\[ C(s, \alpha, \beta) = (4\beta)^{-1/3} \exp \left( -is \frac{\alpha}{8\beta} + i \frac{8\beta}{3} \left( \frac{\alpha}{8\beta} \right)^3 \right) Ai(y), \] (22)
depends on the Airy function \( Ai(y) \) and its derivative \( Ai'(y) \) where their arguments are given by \( y = (4\beta)^{2/3}[s/4\beta - (\alpha/8\beta)^3] \). The resulting probability for an electron to emit a photon with energy ratio \( x_\omega \) per unit time becomes [17 22 23]
\[ \frac{dP}{dx_\omega} dt(\chi, u) = -\frac{\alpha m^2}{\epsilon} \left\{ \int_0^{\infty} dt Ai(t) \right\} + \frac{Ai'(z)}{z} \left[ 2 + \frac{u^2}{1 + u} \right], \] (23)
where \( z = [u/\chi]^{2/3} \).

The LCFA derivation is similar to the NLCM derivation where the major difference arises from taking the limit of \( \eta \to \infty \). As a result, the discrete series in \( s \) that appeared in the NLCM is replaced by a continuous integral over \( s \) in the LCFA. This is not so surprising since the discrete series in the limit of \( \eta \to \infty \) can be approximated as a continuous integral. Another consequence of the LCFA is that the electron quasimomentum \( q^\mu \) is replaced by its "regular" momentum \( p^\mu \) and that the only relevant parameter in describing the emission
process is the parameter \( \chi \). This reduction of dimensionality by one is of significant practical importance when calculating the radiation spectrum in a simulation. This makes it easier to approximate the function, which is a difficult integral to evaluate numerically, by a function that is much faster to evaluate.

**Radiation Integral-BCK model**

For the two previous models, a specific vector potential has been assumed, which limits the versatility of the models to specific field configurations. This final method provides a general expression to calculate the resulting radiation emitted from a relativistic charged particle moving along a classical trajectory in an arbitrary electromagnetic field. Starting with the semiclasical wave function of a particle moving in an external electromagnetic field

\[
\Psi = \frac{1}{\sqrt{2\hbar}} u(p) e^{-i\hat{H}t} \phi(x),
\]

where \( \hat{H} \) is the Hamiltonian operator. The function \( \phi(x) \propto e^{-iS/\hbar} \) is the spatial wave function and depends on the classical action \( S \). In perturbation theory, the probability to emit a photon with frequency \( \omega \) and wave vector \( k \) is usually evaluated by squaring the transition matrix element, multiplying by the density of states and summing over the final states of the electron

\[
dP = \sum f |M|^2 \frac{d^3k}{(2\pi)^3} = \sum f \left| \int V(t) dt \right|^2 \frac{d^3k}{(2\pi)^3}.
\]

The transition matrix element in Eq. (1) is the same as in the above expression, but now we separate the integral \( dx \) into \( d^3x dt \) where the element \( V(t) \) now becomes a 3-dimensional integral over space,

\[
V(t) = \sum f \langle \hat{P} \hat{Q}(t) \rangle, \tag{26}
\]

where \( \alpha \) is the regular Dirac matrix. We can write the new matrix element in terms of the operator \( \hat{Q}(t) \)

\[
\hat{V} = \frac{e\sqrt{2\pi}}{\sqrt{\hbar \omega}} \langle f | \hat{Q}(t) | i \rangle e^{\omega t}, \tag{27}
\]

which is defined as

\[
\hat{Q}(t) = \frac{u(t) \langle \hat{p}(t) \rangle}{\sqrt{2\hat{H}(t)}} e^{i\omega t} \cdot \alpha e^{-i\hat{K} \times u(t) \hat{p}(t)} \frac{1}{\sqrt{2\hat{H}(t)}}, \tag{28}
\]

where factors of the form \( e^{-i\hat{H}t} \) have been absorbed, converting the operators into a time-dependent Heisenberg operator. Since the spatial wave functions form a complete basis, we can write \( \sum f |f| \langle f | = \delta(x_2 - x_1) \), which reduces the emission probability to an expectation value given by

\[
dP = \frac{e^2 d^3k}{4\pi^2 \hbar \omega} \int \langle i | \hat{Q}^\dagger(t_2) \hat{Q}(t_1) | i \rangle e^{i\omega(t_1 - t_2)} dt_1 dt_2. \tag{29}
\]

Now the semiclassical operator method developed by Baier and Katkov [17] is employed, where the product of commuting operators are replaced by their classical non-operator value. Through approximations, the product \( \hat{Q}^\dagger \hat{Q} \) is rewritten in terms of commuting operators which are subsequently replaced by their classical non-operator value. The accuracy of this approximation is 1/\( \gamma \) making it a valid approximation for the relativistic particles simulated in this paper. A detailed derivation of the approximation is shown in Refs. [17, 23, 28]. The result is reduced to a single integral over time (first seen in [29] and derivations shown in Ref. [19 and 30]) given by,

\[
\frac{d^2P}{dh\omega dt} = \frac{e^2}{4\pi^3} \left( \frac{e^2 + e^2}{2h\omega^2} |I|^2 + \frac{\hbar \omega}{2E^2\gamma^2} |J|^2 \right), \tag{30}
\]

where \( e^* = e - h\omega, (e^* \omega^* = \omega \) and \( I \) and \( J \) are given by

\[
I = \int_{-\infty}^{\infty} n \times |(n - \beta) \times \dot{\beta}| e^{i\omega(t-n r)} dt, \tag{31}
\]

\[
J = \int_{-\infty}^{\infty} \frac{n \cdot \dot{\beta}}{1 - n \cdot \beta} e^{i\omega(t-n r)} dt. \tag{32}
\]

We call this model the Belkacem, Cuce, Kimbal (BCK) model. Here \( n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \) is the direction of emission with polar and azimuthal angles \( \theta \) and \( \varphi \) defined relative to a suitable axis and \( dt = \sin \theta d\theta d\varphi \). The classical trajectory is characterized by the instantaneous position \( r(t) \), the instantaneous velocity \( \beta(t) \), and the instantaneous acceleration \( \dot{\beta}(t) \).

As evident from the derivation, no details of the external field or the trajectory has been inferred, only that the trajectory should be described classically, which holds for a \( 1 \gg \chi/\gamma^3 \). From the discussion in Ref. [17], the angle of the emitted radiation has been assumed to be of the order 1/\( \gamma \) when determining the comutativity of the operators, and we can therefore expect errors of the order 1/\( \gamma^2 \) when using this method.

When applying the BCK model, it is necessary to determine the trajectory first and then evaluate the spectrum numerically for arbitrary external fields. An implementation of the BCK model is shown in Ref. [24]. Classically the radiation from an electron can be determined by the Liemand-Wiechart radiation integrals [25] which is very similar to the BCK model. So the fact that a semi classical version, integrating over the classical trajectory, exists might not be surprising. Lindhard
showed [31] that if the only non-negligible quantum effect is the photon recoil, when neglecting spin, a simple substitution of the frequency variable in the classical photon number spectrum ($\omega \rightarrow \omega^*$) regardless of the details of the motion of the particle, reproduce the exact quantum mechanical expression for a spin-0 particle [18].

Formation Length

A relevant concept to keep in mind for these three models is the formation/coherence length. Each model considered here are single photon emission processes so the trajectory over which you integrate should amount to a probability of emitting a photon below 1. When considering the oscillating phase term in the exponent of Eqs. (31) and (32), the particle can only travel a certain length before the oscillating phase becomes large and starts canceling out contributions of the radiation probability. In practice this means that if one integrates over long trajectories, the initial and final part of the trajectory do not add coherently to the spectrum. The initial and final part of the trajectory could be considered as two separate trajectories where the radiation in each part would just add incoherently to the combined spectrum of the total trajectory. The formation length can be determined by expanding the phase term into small quantities (see [21, 30]) and neglecting terms that vary with the particle position, to give

$$l_f = 2\gamma^2 \frac{E - \hbar \omega}{E \omega}.$$  \hspace{1cm} (33)

For our simulations it is important that the trajectory over which one integrates is several formations lengths long. This is especially true when applying the BCK model, where you need a suitable time over which to integrate the trajectory. If the trajectory is divided into smaller sections, to keep the probability of photon emission below one for each section, the error in the spectrum is on the order of a formation length. From the classical point of view, the formation length can be considered as the distance an electron and the emitted photon travel before they are separated by more than one reduced wavelength. Therefore the formation length can also be expressed as

$$l_f = 2\gamma^2 / \omega.$$  \hspace{1cm} (34)
The quantum version of the formation length in Eq. (33) can be retrieved by substituting $\omega \rightarrow \omega^*$ as discussed in the previous section. A detailed discussion of the concept of the formation lengths is presented in Ref. [32].

It should be noted that the neglected phase term is not always negligible and can, in some cases, cause a significant increase or decrease the formation length. If the particle’s trajectory bends considerably outside the $1/\gamma$ light cone, the coherence is broken and the effective formation length is smaller. For the NLCM, a similar oscillating phase appears in the derivation (see eq (50) in [33] where we neglect the $\theta^2$ term) which can be written as

$$l_f = \frac{2\gamma^2}{E\omega(1+\eta^2/2)} - 4\gamma^2 \omega \chi,$$  \hspace{1cm} (35)

In this case the position varying part of the phase is non-negligible since the formation length diverges for particular photon frequencies. In reality, the formation length diverges at the exact harmonic frequencies in the Non-Linear Compton spectrum. That the parameter $\eta$ suppresses the formation length is no surprise, since it is equivalent to the amount of transverse momentum the particle receives during one plane wave cycle. This means that for large $\eta$, particles will be bent outside the light cone breaking the coherence.

An interesting phenomena occurs when the formation length diverges for a plane wave. For specific photon frequencies, the electron has not "decided" what photon to emit, even after an extremely long trajectory in the plane wave.

For the BCK model, the phase terms inside the radiation integrals has rapidly and slowly varying terms, which when integrated, can cancel out. When starting and ending the integration in a region of non-zero field at random times and positions, the first and last cycles of the slowly varying envelopes are not guaranteed to cancel out. This gives a small non-physical contribution to the spectrum on the order of the formation length, and results in a small underlying Bethe-Heitler type bremsstrahlung that extends towards higher photon energies. When using the BCK model, breaking up the trajectory into smaller sections is necessary to keep the probability of emitting a photon during a section less than one. If the trajectory becomes too short, these non-physical effects occur which is important to keep in mind when analyzing the resulting spectra, as it is unavoidable because the probability of emitting a photon during a section has to be less than 1.

For the LCFA and NLCM where the trajectory is analytically integrated to infinity, the above mentioned effect is not present. At high fields (large $\eta$), the single-photon emission theories are no longer sufficient and employing a multi-photon emission theory becomes necessary.

SIMULATIONS

Theoretical predictions of SFQED experiments are almost always based on simulations where it is necessary to evaluate the radiation spectra multiple times and for different parameters. As discussed in the previous section, all three radiation models originate from the same initial matrix element but they deviate when approximations or assumptions are made to simplify the calculation. The difficulty of numerically evaluating the radiation spectrum using each model varies a lot, with the LCFA model being the easiest and fastest to evaluate. Depending on how many terms in the sum over absorbed photons one needs, the BCK model and NLCM can take several orders of magnitude longer to evaluate than the LCFA model. It is therefore important to understand when the approximations for the three models are valid and when they fail under various experimental conditions.

In the laboratory, there are in principle two different sources of strong fields, tightly focused pulsed lasers, and oriented crystals using high energy particles. It is clear that the underlying assumption of the external field used in the NLCM does not apply for crystals, so the primary focus in this paper will be laser fields. First we will investigate a regular plane wave for different values of $\eta$ and $\chi$ where the NLCM should give the exact solution to the emitted radiation. Then we will investigate the case of a tightly focused laser pulse, which corresponds to putting a gradient on the plane wave amplitude.

For the LCFA model, the photon spectrum is evaluated at each particle time step based on the local value of $\chi$. Whereas the NLCM divides the trajectory into bigger sections where the average $\eta$ and $\chi$ values are used to evaluate the spectrum for each section. The total NLCM spectrum is the sum of the individual spectra from each section. For simplicity, the electrons’ energy stays at 13 GeV during the entire simulation for each model even though they emit photons. As a result, the BCK model can integrate over the entire trajectory which leads to probabilities of emitting a photon being larger than one. We do this to correctly capture the coherent parts of the trajectory in the radiation spectrum which results in an overall scaling of the intensity.

Plane wave Simulation

In Fig. 1 we show the radiation spectra from a head on collision between electrons propagating 100 fs through a counter propagating plane wave. The radiation spectra have been evaluated using all three models, for four different configurations of $\eta$ and $\chi$. The value of $\eta$ and $\chi$ are determined by the wavelength $\lambda$, electron energy, and peak value of the electric field. The electrons energy for all simulations is 13 GeV, while the wavelength is either 800 or 8000 nm, and the plane wave amplitude is fixed.
Figure 2. Formation length for a 13 GeV electron with different values of $\eta$ and $\chi$ moving through a plane wave with 8000 nm wavelength according to Eq. (33) (red), and Eq. (35) (black) for increasing values of absorbed photons from the external field $s$ throughout the spectrum. The blue line marks a length of 0.1 $\mu$m. The values of $s$ are not equidistant and are chosen to represent the entire spectrum.

Figure 3. Radiation spectra for 13 GeV electrons colliding head-on with a pulsed laser having a 40 fs FWHM pulse duration and a transverse spot size of $w_0 = 3 \mu$m. Legends indicate the maximum $\eta$ and $\chi$ parameter at the focal point of the laser pulse. The black curve is based on the LCFA, the red curve is based on the BCK model and the blue curve is based on the NLCM.

to either $\eta = 1$ and $\eta = 7$.

When $\eta = 1$ for both wavelengths, the structure of the spectra clearly display the number of photons absorbed (harmonics) by the electron. While this spectra is fundamental to the NLCM, it is also captured perfectly by the BCK model, where the harmonics originate from the frequency of the oscillatory motion the particle makes in its rest frame. In addition, the LCFA model fails to reproduce the NLCM/BCK models for both wavelengths, which is no surprise since the LCFA model is identical to the NLCM in the limit $\eta \rightarrow \infty$.

As discussed earlier, the formation length (see Eq. (35)) diverges for the harmonic frequencies of the spectrum. This is evident in the spectrum and is why the harmonics in the BCK model are not as sharp as the NLCM, where the particle trajectory is integrated analytically to infinity.

In both $\eta = 7$ cases, there is good agreement between the NLCM and BCK spectra where now the strong harmonic peaks appear as noise. Distinguishing these spectra experimentally would be extremely difficult because there is no clear difference even at low energy.

Although the LCFA in principle is the same as the NLCM in the limit $\eta \rightarrow \infty$, it is somewhat surprising that the NLCM spectrum has such prominent features for the 8000 nm plane wave. The features in the spectrum can be understood by examining the formation length. According to Eq. (33), a 13 GeV electron emitting a 2 GeV photon, has a value of $l_f \approx 0.1 \mu$m. For a plane wave with 8000 nm wavelength, the field can safely be considered a constant within 0.1 $\mu$m, but evidently the formation length extends far beyond what Eq. (33) predicts.

In Fig. 2 we show the formation length as a function of emitted photon frequency according to Eqs. (33) and (35) for both $\eta = 7$ and $\eta = 1$ in the 8000 nm case. The divergence of the formation length clearly has a large effect in the $\eta = 1$ case. The diverging peaks, that occur at the number of absorbed of photons $s$ from the external field
field, spans a large region of the spectrum. Increasing the value of $\eta$, increases the transverse momentum a particle receives during a plane wave cycle. If the particle is deflected outside the light cone, which is of the order $1/\gamma$, the coherence in the motion of the particle is lost. As a result, the effective coherence length becomes smaller than the formation length for larger $\eta$. This $\eta$ suppression of the formation length is clearly visible in the $\eta = 7$ case, where only a small fraction of the spectrum extends above the region where the field is not constant.

The difference in spectra for low energy photons between the LCFA model and NLC/BCK models can be understood in terms of the formation length. The formation length becomes much longer for low energy photons for all cases investigated in this paper.

**Short Pulsed Laser Simulation**

In Fig. 3 we show the radiation spectra for a head-on collision between 13 GeV electrons with a pulsed laser having a 40 fs FWHM pulse duration, transverse spot size of $w_0 = 3 \, \mu m$, and 800nm wavelength for three different laser intensities. When $\eta \approx 2$, harmonics in the spectrum are still significant for the NLC/BCK models but not for the LCFA. When the laser intensity is increased the harmonics disappear and we see agreement between all three models except near the peak of the spectrum. The large discrepancy between the LCFA and the NLC/BCK models at small $\eta$ is expected, but the pulsed laser enhances this discrepancy. Even for $\eta \approx 11$, discrepancies in the spectrum are present up to the energy of 3 GeV.

Even when the laser pulse has a large gradient in the electromagnetic fields, the spectrum from the BCK and NLC models agree. This can be explained by examining the formation length based on Eqs. (33) and (35), which is shown in Fig. 4 and compare it to three wavelengths of the laser cycle where the laser field can be approximated by a monochromatic plane wave. The formation length based on the NLCM remains below the three wavelength line for all cases of $\eta$. Decreasing $\eta$ increases the formation length but also decrease the gradient of the field. As a result, the NLCM is applicable for all values of $\eta$ as long as the multi-photon emission processes remains negligible, which is the case for all the laser parameters simulated in this paper.

With longer laser pulse lengths, the probability of several interactions occurring during a collision between the laser and an electron increases, making it difficult to experimentally distinguish if an electron has emitted one hard photon or two softer photons. To avoid this situation, we investigate the spectrum produced from an ultra short laser pulse. In Fig. 5 we show the radiation spectrum from electrons colliding head on with a pulsed laser having a 10 fs FWHM pulse duration, a transverse spot size of $w_0 = 3 \, \mu m$, and a wavelength of 800 nm for two different laser intensities. As evident from the $\eta = 1.25$ case, the difference between the LCFA and the NLC/BCK models has increased as the laser pulse length has decreased. In addition, we now start seeing a difference between the NLCM and the BCK model. While the average spectra agrees fairly well, the harmonics are now missing from the NLC spectrum. The missing harmonics occurs because the plane wave amplitude used to calculate the Non-Linear Compton spectrum is no longer a constant within the formation length. Unfortunately these features that are prominent in simulated spectra will be difficult to measure experimentally due to required energy resolution necessary to distinguish these differences.

For the $\eta = 3.95$ case, we see the same trend as in previous simulations. The $\eta$ suppression of the formation length ensures that the plane wave amplitude does not change significantly within the reduced formation length, as a consequence the NLCM agree well with the BCK model. We still see disagreement with the LCFA model starting from the peak moving to lower energy.

**DISCUSSION**

In our investigation we have shown that even though all three radiation models (NLCM, LCFA and BCK) originate from the same matrix element in time dependent perturbation theory, there applicability varies greatly with regard to which external field the radiating particle traverses. Comparing radiation spectrum from an electron beam and perfect plane waves with 800 and 8000 nm wavelength showed that the BCK model in excellent agreement with the NLC even considering that the NLC gives the exact solution for a perfect plane wave. The discrepancy between the two spectra is evident in the sharpness of the Compton edges which depends on how long you integrate over the trajectory through the plane wave. Achieving complete agreement would become an academic exercise in how long one can numerically integrate the trajectory.

We showed that the formation length is an extremely powerful tool in predicting which model is applicable for a given operational situation. According to the formation length version (Eq. (33)), the LCFA model should be applicable for the 8000 nm wavelength plane wave cases. But in reality, the formation length diverges at the harmonics in the Non-Linear Compton spectrum, and as a consequence, the relevant parameter for the applicability of the LCFA model is the $\eta$ parameter. The discrepancy between the LCFA model and the NLC/BCK models is highly dependent on $\eta$ and not on the gradient of the plane wave which changes in e.g. a pulsed laser. As $\eta$ increases, the BCK and NLC spectra becoming smoothly varying, and start to agree with the LCFA model for high photon energies. The largest discrepancy between
Figure 4. Formation length for a 13 GeV electron with different values of $\eta$ and $\chi$ moving through a plane wave with 800 nm wavelength according to Eq. (33) (red), and according to Eq. (35) (black) for increasing values of $s$ throughout the spectrum. The blue line marks a length of 2.4 $\mu$m (3 wavelengths). The values of $s$ are not equidistant and are chosen to represent the entire spectrum.

Figure 5. Radiation spectra for 13 GeV electrons colliding head-on with a pulsed laser having a 10 fs FWHM pulse duration and a spot size of $w_0 = 3 \mu$m. Legends indicate the maximum $\eta$ and $\chi$ parameter at the focal point of the laser pulse. The black curve is based on the LCFA, the red curve is based on the BCK model and the blue curve is based on the NLCM.

the LCFA and BCK/NLC models occurs below the peak in the spectrum at low photon energies.

We also investigated the applicability of the NLCM in the case of a pulsed laser, where the plane wave amplitude changes over a number of cycles. For a 40 fs long pulse duration, the field amplitude changes slow enough within the formation length that the NLCM agrees with the BCK model. When increasing $\eta$, and thereby the gradient of the field, we found that the $\eta$ suppression of the formation length drops faster than the field changes. Therefore if a regime exists where the NLCM would not be applicable in a head on collision with a laser pulse, it would be for $\eta$ values around 1 and below. Even for a 10 fs long laser pulse, where the pulse duration is only a couple of plane wave cycles, the $\eta$ suppression kicks in and makes the NLCM pertinent. For $\eta = 1.25$ in the 10 fs pulse duration case, we start seeing a notable difference in the underlying features of the BCK and NLCMs.

The average spectrum is still in agreement and the difference in spectrum would not be measurable in current experiments due to the high energy resolution required.

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