Topological defects, fractals and
the structure of quantum field theory

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In this paper I discuss the formation of topological defects in quantum field theory and the relation between fractals and coherent states. The study of defect formation is particularly useful in the understanding of the same mathematical structure of quantum field theory with particular reference to the processes of non-equilibrium symmetry breaking. The functional realization of fractals in terms of the $q$-deformed algebra of coherent states is also presented. From one side, this sheds some light on the dynamical formation of fractals. From the other side, it also exhibits the fractal nature of coherent states, thus opening new perspectives in the analysis of those phenomena where coherent states play a relevant role. The global nature of fractals appears to emerge from local deformation processes and fractal properties are incorporated in the framework of the theory of entire analytical functions.

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I. INTRODUCTION

In this paper I discuss some specific features of the formation of topological defects during the process of phase transitions and the relation between fractals and coherent states. Fractals and defects are present in an extremely large number of systems and natural phenomena and therefore much attention is devoted to their study.

Examples of defects are magnetic domain walls in ferromagnets, vortices in superconductors and superfluids, dislocations, point defects, etc. in crystals. Even in cosmology cosmic strings [1] may be studied as topological singularities whose role may have been relevant in the phase transition processes in the early Universe. The analogy between defect formation in condensed matter physics and high energy physics and cosmology is quite surprising [2, 3, 4, 5]. Moreover, the study of defect formation may be particularly useful in the understanding of the same mathematical structure of quantum field theory (QFT) with particular reference to the processes of non-equilibrium symmetry breaking, a subject on which my attention in this paper will be mostly focused.

On the other hand, there is no need of many words to express the relevance of fractals in science, from physics to biology, medical sciences, earth science, clustering of galaxies, etc. [6]. It is therefore of great interest to investigate in deep fractal properties. Here I present the functional realization of fractals in terms of coherent states. From one side, this may shed some light on the dynamical formation of fractals. From the other side, it also exhibits the fractal nature of coherent states, thus opening new perspectives in the analysis of those phenomena where coherent states play a relevant role.

In my discussion I will closely follow some previous papers of mine on similar topics, especially Refs. [7]–[11]. I will essentially consider the symmetry properties of the dynamics of the systems under consideration and therefore the conclusions will be mostly model independent. On the contrary, a specific model choice would imply working in some approximation scheme. The general, more formal, approach I will follow is, however, more convenient in discussing some of the general questions which arise in the study of defect formation in the framework of non-equilibrium symmetry breaking phase transitions.

Among the results that will be presented there is the theoretical explanation, based on the microscopic dynamics of quantum fields, of the fact that the formation of topological defects is typically observed during the processes of non-equilibrium symmetry breaking phase transitions when a gauge field is present and an order parameter exists. The fact that the conclusions are, as said, model independent is helpful since those features of defect formation are singled out which are shared by many systems in a wide range of energy scale, independently of specific aspects of the system dynamics.

I will also present the proof that the Nambu-Goldstone (NG) particle acquires an effective non-zero mass due to boundary (finite volume) effects. This is related with the size of the defect and introduces us to the question of the dynamical formation of the boundaries. These can be considered indeed to be themselves like a "defect" of the system, which otherwise, in the absence of boundaries, would extend to infinity. An interesting question is indeed the one of the formation of the system boundaries, which has to be, of course, a dynamical process, not imposed by hand from
the exterior (presumably, such a dynamical formation of boundaries has a lot to do with morphogenesis). As we will see, finite volume effects are related with temperature effects.

Also the discussion of fractals will be of general character, aiming to point out some structural fractal aspect rather than analyzing the features of specific fractals. Indeed, I will focus my attention only on a feature common to all the self-similar fractal structures. The self-similar property of fractals is only one of the many mathematical and phenomenological properties of fractals. It is, however, a characterizing properties of an extremely large class of fractals and therefore I focus my attention on it. The connection will be made with the theory of the entire analytic functions and with the q-deformed algebra of the (Glauber) coherent states. This results in the possibility of incorporating fractal properties in the framework of the theory of entire analytical functions. Conversely, it also allows to recognize, in the specific sense that will be discussed below (cf. Section V), fractal properties of coherent states. The study presented in this paper thus provides a first step in the understanding of the dynamical origin of fractals and of their global nature emerging from local deformation processes. It also provides insights on the geometrical (fractal) properties of coherent states.

The presentation is organized as follows. In Section II I present the notion of spontaneous breakdown of symmetry (SBS) in QFT. As a result of SBS, the invariance properties of the basic field equations manifest themselves in the formation of ordered patterns at the level of the observable fields. We have thus the phenomenon of the dynamical rearrangement of symmetry. In Section III I consider the problem of the formation of ‘extended objects’ (defects) with topological singularities in the processes of phase transitions. It will be then evident the relevant role played by the mathematical structure of QFT characterized by the existence of infinitely many unitarily inequivalent representations of the canonical (anti-)commutation relations. The effects of the non-vanishing mass of the Nambu-Goldstone (NG) particles on the coherent domain size will be discussed in Section IV where temperature effects will be also considered. The functional realization of fractals in the framework of the theory of the entire analytical functions and their relation with the deformed Weyl-Heisenberg algebra of the coherent states is presented in Section V. Section VI is devoted to final remarks.

II. SPONTANEOUS BREAKDOWN OF SYMMETRY AND DYNAMICAL REARRANGEMENT OF SYMMETRY

In an intuitive picture, the formation of a topological defect may be thought to occur when a spatially extended region of the ”normal” (i.e. symmetric) state is surrounded by ”ordered” domains. In a vortex, for example, the normal state constitutes the “core” of the vortex which is “trapped” in an ordered state surrounding it. Therefore, the first notions we need in order to study defect formation are the ones of ordered or non-symmetric state vs the disordered or symmetric state. It is known that the mechanism of spontaneous breakdown of symmetry in QFT [12, 13] is at the origin of the dynamical formation of ordered patterns out of the symmetric state.

In QFT [12, 13] the dynamics, i.e. the Lagrangian and the nonlinear field equations from it derived, is given in terms of the interacting fields, say \( \varphi(x) \), called the Heisenberg fields. The observables measured in experiments are instead described in terms of asymptotic in- (or out-) operator fields (called free or physical fields), say \( \varphi_{\text{in}}(x) \), which satisfy free field equations. Such a dual level structure (Heisenberg or interacting fields vs free or physical or asymptotic fields) is called the Lehmann-Symanzik-Zimmermann (LSZ) standard formalism of QFT [13]. In condensed matter physics and in quark-gluon physics asymptotic in- (or out-) operator fields are not available. In such cases the role of physical fields is played by quasiparticle fields and by fields in the asymptotic freedom regime, respectively. When it happens that the physical vacuum state of the system is not symmetric under the action of one or more generators of the group of transformations, say \( G \), under which the Heisenberg field equations are invariant, we say that the symmetry is spontaneously broken. Here we are assuming that \( G \) is a continuous group of transformations.

Summarizing, we have the dynamics given in terms of Heisenberg fields and the Hilbert space of physical states where asymptotic fields, satisfying free field equations, are realized. The theory is "solved" when the dynamical map between Heisenberg fields and asymptotic fields is found; in other words, when the coefficients of the map are computed by solving the Heisenberg field equations. The dynamical map is a weak equality: it holds between expectation values of the members of the mapping computed in the Hilbert space of the physical states. It is useful to proceed by considering a concrete standard example, the one of the U(1) gauge model where vortices appear. However, our conclusions apply, in their structural features, to more complex models as well, including the non-Abelian case of SO(3) and SU(2) symmetry (as for the monopole and the sphaleron case [13], respectively).

Let \( \mathcal{L}[\phi_H(x), \phi_H^*(x), A_{H\mu}(x)] \) be the Lagrangian density for a complex scalar field \( \phi_H(x) \) interacting with a gauge field \( A_{H\mu}(x) \). For our tasks, there is no need to specify the detailed structure of the Lagrangian. We only require
that $\mathcal{L}$ be invariant under global and local U(1) gauge transformations (the Higgs-Kibble model \[15, 16\]):

\[
\phi_H(x) \rightarrow e^{i\theta} \phi_H(x) , \quad A_{H\mu}(x) \rightarrow A_{H\mu}(x),
\]

\[
\phi_H(x) \rightarrow e^{i\rho_{0}(x)} \phi_H(x) , \quad A_{H\mu}(x) \rightarrow A_{H\mu}(x) + \partial_{\mu}(x),
\]

(1) respectively. Here $\lambda(x) \rightarrow 0$ for $|x_0| \rightarrow \infty$ and/or $|x| \rightarrow \infty$. The Lorentz gauge $\partial_{\mu} A_{H\mu}(x) = 0$ is adopted. Moreover, $\phi_H(x) \equiv \frac{1}{\sqrt{2}} [\psi_H(x) + i\chi_H(x)]$.

We assume that spontaneous breakdown of global U(1) symmetry can occur, i.e. that $\langle 0 | \phi_H(x) | 0 \rangle \equiv \tilde{\nu} \neq 0$. $\tilde{\nu}$ is a constant and $\rho_H(x) \equiv \psi_H(x) - \tilde{\nu}$.

A crucial point is that in the presence of SBS the theory contains a massless negative norm field (ghost) $b_{in}(x)$, the Nambu-Goldstone massless mode $\chi_{in}(x)$, and a massive vector field $U_{in}^\mu$, as one can show, e.g. by functional integration techniques \[17\].

We can then exhibit a concrete example of the LSZ mapping (the dynamical map, also called the Haag expansion) between interacting fields and physical fields, mentioned above. It provides the relation between the dynamics and the observable properties of the physical states. The following LSZ maps are indeed found \[17\]:

\[
\phi_H(x) = : \exp \left\{ \frac{Z_{\chi}^2}{\tilde{\nu}} \chi_{in}(x) \right\} \left[ \tilde{\nu} + Z_{\rho}^2 \rho_{in}(x) + F[\rho_{in}, U_{in}^\mu, \partial(\chi_{in} - b_{in})] \right] : ,
\]

\[
A_{H}^\mu(x) = Z_{3}^\mu U_{in}^\mu(x) + Z_{\rho}^\mu \partial^\mu b_{in}(x) + : F^\mu[\rho_{in}, U_{in}^\mu, \partial(\chi_{in} - b_{in})] : .
\]

(3) The colon symbol denotes normal ordering. $Z_{\chi}$, $Z_{\rho}$ and $Z_{3}$ are the wave function renormalization constants. The functionals $F$ and $F^\mu$ are determined when a specific explicit choice for the Lagrangian is assumed. In the present case, the above results are model independent. When considering a specific choice for the Lagrangian they can be also obtained within some approximation, e.g., by using the saddle point expansion.

Another important relation is the one providing the $S$-matrix. This is given by

\[ S = : S[\rho_{in}, U_{in}^\mu, \partial(\chi_{in} - b_{in})] : . \]

(4) I stress that, as already observed above, the dynamical mappings \[3\] and \[4\] are weak equalities, i.e. they are equalities among matrix elements computed in the Fock space for the physical states. The free field equations are

\[
\partial^2 \chi_{in}(x) = 0 , \quad \partial^2 b_{in}(x) = 0 , \quad (\partial^2 + m_{V}^2) \rho_{in}(x) = 0 ,
\]

(5)

\[
(\partial^2 + m_{V}^2) U_{in}^\mu(x) = 0 , \quad \partial_{\mu} U_{in}^\mu(x) = 0 .
\]

(6) with $m_{V}^2 = \frac{Z_{3}}{Z_{\chi}}(e_0 \tilde{\nu})^2$.

I now observe that the global gauge transformations \[11\] of the Heisenberg fields are found to be induced by the in-field transformations (see \[6\] and \[8\]):

\[
\chi_{in}(x) \rightarrow \chi_{in}(x) + \frac{\tilde{\nu}}{Z_{\chi}^2} \partial f(x) ,
\]

(7)

\[
b_{in}(x) \rightarrow b_{in}(x), \quad \rho_{in}(x) \rightarrow \rho_{in}(x), \quad U_{in}^\mu(x) \rightarrow U_{in}^\mu(x).
\]

(8)

The local phase transformations \[2\] are induced by

\[
\chi_{in}(x) \rightarrow \chi_{in}(x) + \frac{e_0 \tilde{\nu}}{Z_{\chi}^2} \lambda(x) , \quad b_{in}(x) \rightarrow b_{in}(x) + \frac{e_0 \tilde{\nu}}{Z_{\chi}^2} \lambda(x) ,
\]

(9)

\[
\rho_{in}(x) \rightarrow \rho_{in}(x) , \quad U_{in}^\mu(x) \rightarrow U_{in}^\mu(x).
\]

(10)

An important remark is that \[7\] with $f(x) = 1$ (translation by a constant c-number) is not unitarily implementable in QFT (i.e. in the limit of infinitely many degrees of freedom): $f(x)$ is introduced in order to make the generator of such a transformation well defined. Mathematical definiteness requires that $f(x)$ be a square integrable function,
solution of the equation for $\chi_{in}(x)$ and $b_{in}(x)$, i.e. $\partial^2 f(x) = 0$. The limit $f(x) \to 1$ (i.e. the infinite volume limit) is to be performed at the end of the computation. The in-field equations and the $S$ matrix are invariant under the above in-field transformations (in the limit $f \to 1$).

Notice that the generator (the Glauber’s displacement operator) of the transformation (7) where $f(x)$ is set to be 1, namely the translation (or displacement or shift) of the operator field $\chi_{in}(x)$ by a constant, is the key ingredient in the theory of coherent states in quantum mechanics (QM). There, since the number of the degrees of freedom is finite (finite volume) the von Neumann theorem guarantees that the representations of the canonical commutation relations are unitarily equivalent. In contrast, in QFT infinitely many unitarily inequivalent representations exist [18]. The possibility of choosing one of them among the many inequivalent ones corresponds to a completely non-trivial condition under which the basic dynamics is realized in terms of observable quantities.

Let me now briefly comment on the physical meaning of the formal structure which has emerged as a result of SBS. We have seen that the global and the local U(1) gauge transformations of the Heisenberg fields are induced, at the level of the physical fields, by the group $G'$ of transformations (8) and (9), respectively. Since $G'$ is a different group of transformation than $G$, one says that $G \to G'$ represents the dynamical rearrangement of symmetry [12] [19]: it is the result of, and expresses the consistency between the invariance of the Lagrangian under the $G$ symmetry transformations and the SBS condition (in the considered model, $G = U(1)$ and the SBS condition is $\langle 0|\phi_H(x)|0 \rangle = \hat{\psi} \neq 0$).

Translations of boson fields (7) and (9) are thus obtained as a consequence of SBS.

Transitions of boson fields (7) and (9) are thus obtained as a consequence of SBS. $G'$ is the group contraction of the U(1) symmetry group of the dynamics; if the dynamics symmetry group is SU(n) or SO(n), the group contraction $G'$ is EU(n-1) or E(n), respectively [20]. Under quite general conditions, the dynamical rearrangement of symmetry is found to lead to the group contraction of the group under which the Lagrangian is invariant. This is an exact result which goes beyond any approximation scheme [20]. I stress that $G'$ is the transformation group relevant to the phase transitions process.

A. Homogeneous and non/homogeneous boson condensation

“Shifts” of the NG fields are thus introduced through the dynamical rearrangement of symmetry. They are controlled by the Abelian subgroup of $G'$. In the global gauge case, the transformation (7), with $f(x) = 1$, describes the NG homogeneous boson condensation. As a result of such a boson condensation in the ground state, coherent long range correlation is established which manifests as the ordered pattern in the system ground state. We thus realize that transitions between the system phases characterized by different ordered patterns in the ground state may be induced by the process of boson condensation. Stated in different words: phase transitions are induced by variations (gradients) of the NG boson condensation. Since in the $f(x) = 1$ limit it does not exist any unitary generator of the boson translation by a constant c-number, vacua with different boson condensation are unitarily inequivalent states and thus we see that phase transitions are transitions among unitarily inequivalent Fock spaces. The essentially non-perturbative nature of the phase transition process is in this way recognized. Moreover, we learn that since phase transitions can only occur when a multiplicity of unitarily inequivalent spaces of states is available, they can only occur in QFT. In quantum mechanics, indeed, all the representations of the canonical commutation relations are unitarily equivalent, as stated by the von Neumann theorem, and therefore no (phase) transition between inequivalent state spaces is conceivable.

When $f(x) \neq 1$ the NG translation (7) describes coherent non-homogeneous boson condensation and (7) is called the boson transformation. As we will see below, extended objects (defects) are described by non-homogeneous boson condensation.

Next step is to consider “macroscopic” manifestations of the boson condensation. In order to do that, I observe that in the framework of the U(1) gauge model discussed above, the Maxwell equations are given by

$$-\partial^2 A_\mu^H(x) = j_\mu^H(x) - \partial^\mu B(x),$$

where

$$B(x) = \frac{e_0 \hat{\psi}}{Z_\chi} [b_{in}(x) - \chi_{in}(x)], \quad \partial^2 B(x) = 0,$$

and $j_\mu^H(x) = \delta \mathcal{L}(x)/\delta A_{\mu H}(x)$. Let the current $j_\mu^H$ be the only source of $A_\mu^H$ in any observable process. This implies

$$\langle p| b| \partial^\mu B(x) |a \rangle_p = 0,$$

i.e.

$$(-\partial^2)_p \langle b| A_\mu^H(x) |a \rangle_p = \langle p| b j_\mu^H(x) |a \rangle_p,$$

$$\text{(13)}$$
where \( A_{\mu}^0(x) \equiv A_{\mu}^0(x) - e_0 \tilde{v} : \partial^\mu b_{in}(x) : \), \(|a\rangle_p\) and \(|b\rangle_p\) are two generic physical states. The condition \( p(b|\partial^\mu B(x)|a\rangle_p = 0 \) leads to the Gupta-Bleuler-like condition
\[
[\chi_{in}^{(-)}(x) - b_{in}^{(-)}(x)]|a\rangle_p = 0 , \tag{14}
\]
where \( \chi_{in}^{(-)} \) and \( b_{in}^{(-)} \) are the positive-frequency parts of the corresponding fields. Eq. \(14\) shows that \( \chi_{in} \) and \( b_{in} \) do not participate to any observable reaction. However, the NG bosons do not disappear from the theory: their condensation in the vacuum can have observable effects.

Remarkably, Eq. \(15\) is the classical Maxwell equations.

The boson transformation must be also compatible with the physical state condition \(14\). \( B \) changes as
\[
B(x) \to B(x) - \frac{e_0 \tilde{v}^2}{2 \chi} f(x) \tag{15}
\]
under the transformation \( \chi_{in}(x) \to \chi_{in}(x) + \frac{\tilde{v}}{2 \chi} f(x) \). Eq. \(13\) is then violated. In order to restore it, the shift in \( B \) must be compensated by means of the transformation on \( U_{in} \):
\[
U_{in}^\mu(x) \to U_{in}^\mu(x) + Z_3^{-\frac{1}{2}} a^\mu(x) , \quad \partial_\mu a^\mu(x) = 0 , \tag{16}
\]
with a convenient c-number function \( a^\mu(x) \). The dynamical maps of the various Heisenberg operators are not affected by \(16\) provided
\[
(\partial^2 + m_\gamma^2) a^\mu(x) = \frac{m_\gamma^2}{e_0} \partial_\mu f(x) . \tag{17}
\]
Eq. \(17\) is the classical Maxwell equation for the vector potential \( a^\mu \). Thus we see that symmetry breaking phase transitions are characterized by macroscopic ground state effects, such as the vacuum current and field (e.g. in superconductors), originated from the microscopic dynamics.

These results are not confined to the \( U(1) \) model here considered. They are also obtained in more complex models with non-Abelian symmetry groups (see \[14, 19\]), in the relativistic as well as in non-relativistic regime.

### III. DEFECT FORMATION AND PHASE TRANSITIONS

We have seen that non-homogeneous boson condensation is a strict consequence of SBS provided certain conditions are met \((f(x) \neq 1)\). Next question is why defect formation is observed during the processes of non-equilibrium symmetry breaking phase transitions when a gauge field is present and an order parameter exists. To answer to such a question from the perspective of the microscopic dynamics, we need to consider the topological characterization of non-homogeneous boson condensation in SBS theories \[22\].

The boson transformation function \( f(x) \) considered in the previous Section plays the role of a "form factor": the extended object (the defect) appears as the macroscopic envelope of the non-homogeneous boson condensate localized over a finite domain. The topological charge of the defect is thus expected to arise from the topological singularity of the boson condensation function.

The boson condensation has been recognized in Section \[11\] to be formally obtained by translations of boson fields, say \( \chi_{in}(x) \to \chi_{in}(x) + f(x) \), with c-number function \( f(x) \), satisfying the same field equation for \( \chi_{in}(x) \). As already said, these translations are called boson transformations \[12\]. According to its general definition, the boson transformation may be applied to boson fields which need not to be necessarily massless.

We have also seen that transitions between phases characterized by different ordered patterns in the ground state are induced by variations (gradients) of the boson transformation function. Thus, to obtain that in such a transition process the conditions can be met for the formation of topological defects, we need to consider under which constraints the boson transformation function \( f(x) \) can carry a topological singularity. Then we have also to show that these constraints are in fact satisfied in the process of phase transitions.

With respect to the first of these points, one can show that topological singularities of the boson transformation functions are allowed only for for massless bosons \[12, 19\], such as NG bosons of SBS theories where ordered ground states appear.

In the boson transformation \( \chi_{in}(x) \to \chi_{in}(x) + f(x) \), let \( f(x) \) carry a topological singularity. This means that it is path-dependent:
\[
G_{\mu\nu}^\dagger(x) \equiv [\partial_\mu, \partial_\nu] f(x) \neq 0 , \quad \text{for certain} \quad \mu, \nu, x . \tag{18}
\]
\(\partial \mu f\) is related with observables (see below) and therefore it assumed to be single-valued, i.e. \([\partial \mu, \partial \nu] \partial \mu f(x) = 0\). \(f(x)\) is required to be solution of the \(\chi_{in}\) equation. Suppose that in such an equation there is a non-zero mass term: \((\partial^2 + m^2) f(x) = 0\). From the regularity of \(\partial \mu f(x)\) it follows that

\[
\partial \mu f(x) = \frac{1}{\partial^2 + m^2} \partial^\lambda G^\lambda_{\mu}(x),
\]

which leads to \(\partial^2 f(x) = 0\), which in turn implies \(m = 0\). Thus (18) is compatible only with massless \(\chi_{in}\). This explains why topological defects are observed only in systems exhibiting massless modes, such as ordered patterns, namely in the presence of NG bosons sustaining long range ordering correlation.

For the second point, I recall that Eq. (17) is a characterizing equation for the occurrence of the phase transition processes. From such an equation we see that the classical ground state current \(j_\mu\) is given by

\[
j_\mu(x) \equiv \langle 0 | j_{\mu}(x) | 0 \rangle = m^2_V [a_\mu(x) - \frac{1}{\epsilon_0} \partial_\mu f(x)].
\]

Here \(m^2_V a_\mu(x)\) is the Meissner-like current and \(\frac{m^2}{\epsilon_0} \partial_\mu f(x)\) is the boson current.

Eq. (20) shows that the classical field and the classical current do not occur for regular \(f(x)\) (\(G^\lambda_{\mu\nu} = 0\), with \(\partial^2 f(x) = 0\) (which is required when considering NG bosons).

In fact, from (17) \(a_\mu\) is formally given by \(a_\mu(x) = \frac{1}{\partial^2 + m^2_V} \frac{m^2}{\epsilon_0} \partial_\mu f(x)\). Then we have \(\partial^2 a_\mu(x) = \partial^2 \frac{1}{\partial^2 + m^2_V} \frac{m^2}{\epsilon_0} \partial_\mu f(x) = 0\) for regular \(f(x)\), i.e. \(a_\mu(x) = \frac{1}{\epsilon_0} \partial_\mu f(x)\). Thus we see that for regular \(f(x)\) the Meissner-like current and the boson current cancel each other, which implies zero classical current \((j_\mu = 0)\) and zero classical field \((F_{\mu\nu} = \partial_\mu a_\nu - \partial_\mu a_\nu)\).

The gauge potential behaves thus as the “reservoir” compensating the boson transformation gradients [23].

It is interesting to observe that the gauge field potential \(a_\mu(x)\) can be thought as “generated” by the gradient \(\frac{1}{\epsilon_0} \partial_\mu f(x)\) of the boson condensation of the \(\chi_{in}(x)\) field. Its introduction in Eq. (16) was indeed motivated by the stability requirement of the physical state \(|a\rangle_p\) under the shift of the \(\chi_{in}(x)\) field. For regular \(f(x)\), \(a_\mu(x)\) exactly compensates \(\frac{1}{\epsilon_0} \partial_\mu f(x)\). For singular \(f(x)\), the non-vanishing difference \(m^2_V [a_\mu(x) - \frac{1}{\epsilon_0} \partial_\mu f(x)] \equiv j_\mu(x)\) satisfying the continuity equation \(\partial_\mu j_\mu(x) = 0\) behaves as a current (source) in the Maxwell equation (17).

In conclusion, vacuum currents characterizing the processes of phase transition appear only when \(f(x)\) has topological singularities, which, as we have seen above, is only compatible with the condensation of massless bosons, as it happens when SBS occurs.

Summarizing, the same conditions allowing topological singularities in the boson condensation function \(f(x)\) are the ones under which phase transitions may occur in a gauge theory. Therefore, the conditions for the formation of topological defects are met in the phase transition processes, which explains why topological defects are observed in the process of symmetry breaking phase transitions.

I note that the assumption of the regularity of \(\partial_\mu f\) is justified by the (topological) regularity of observable quantities. The classical current (20), which is an observable quantity, is indeed given in terms of gradients \(\partial_\mu f\) of the boson condensation function.

Notice that the appearance of space-time dependent order parameter \(\tilde{\nu}\) is not enough to guarantee that persistent ground state currents (and fields) will exist. Indeed, if \(f\) is a regular function, the space-time dependence of \(\tilde{\nu}\) can be gauged away by an appropriate gauge transformation. Therefore, topological defects cannot be obtained in such case. I also note that in a theory which has only global gauge invariance non-trivial physical effects, like linear flow in superfluidity, may be produced by non-singular boson transformations of the NG fields.

We may also discuss the effects of topological singularity in the \(S\) matrix. Since boson transformations with regular \(f\) do not affect observable quantities, the \(S\) matrix must be actually given by

\[
S = S [\rho_{in} U^\mu_{in} - \frac{1}{m_V} \partial (\chi_{in} - b_{in})];
\]

which is in fact independent of the boson transformation with regular \(f\):

\[
S \rightarrow S' = S [\rho_{in} U^\mu_{in} - \frac{1}{m_V} \partial (\chi_{in} - b_{in})] + Z_3^{\frac{1}{2}} (a^\mu - \frac{1}{\epsilon_0} \partial^\mu f) ;
\]

since \(a_\mu(x) = \frac{1}{\epsilon_0} \partial_\mu f(x)\) for regular \(f\). However, \(S' \neq S\) for singular \(f\): in such a case (22) shows that \(S'\) includes the interaction of the quanta \(U^\mu_{in}\) and \(\rho_{in}\) with the classical field and current. This shows how it may happens that quanta interact and have effects on classically behaving macroscopic extended objects.

The above conclusions are not limited by dimensional considerations or by the Abelian or non-Abelian nature of the symmetry group. They apply to a full set of topologically non-trivial extended objects, such as topological line
singularity, surface singularity, grain boundaries and dislocation defects in crystals, SU(2)-triplet model and monopole
singularity. The topological singularity and the topological charge of the related extended object can be completely
characterized. A detailed account can be found in Refs. 19 [22]. The general character of our conclusions also
shows why the features of the defect formation are shared by quite different systems, from condensed matter to
cosmology. They account for the macroscopic behavior of extended objects and their interaction with quanta in a
unified theoretical scheme. In the following Section I consider finite volume and temperature effects in such a scheme.

IV. DEFECT FORMATION AND NON-VANISHING EFFECTIVE MASS OF THE
NAMBU-GOLDSTONE BOSONS

The two-point function of the \( \chi(x) \) field in the considered U(1) model can be computed in full generality by using
the Ward-Takahashi identities. It has \[17\] the following pole structure for:

\[
\langle \chi(x)\chi(y) \rangle = \lim_{p \to 0} \left\{ \frac{i}{(2\pi)^4} \int d^4p \frac{Z_\chi e^{-ip(x-y)}}{p^2 - m_\chi^2 + i\epsilon a_\chi + \text{cont. c.}} \right\} ,
\]

where \( Z_\chi \) and \( a_\chi \) are renormalization constants and 'cont. c.' denote continuum contributions. The space integration
of \( \langle \chi(x)\chi(y) \rangle \) picks up the pole contribution at \( p^2 = 0 \), and leads to \[24\] \[25\]

\[
\tilde{v} = \frac{Z_\chi}{a_\chi} v \iff m_\chi = 0 , \quad \text{or} \quad \tilde{v} = 0 \iff m_\chi \neq 0 ,
\]

where \( v \) denotes a convenient c-number \[17\]. Eq. \[24\] proves the existence of a massless particle corresponding to the
pole singularity. It expresses the well known Goldstone theorem: if the symmetry is spontaneously broken (\( \tilde{v} \neq 0 \)),
the NG massless mode exists, whose interpolating Heisenberg field is \( \chi_H(x) \). It spans the whole system since it is
massless and manifests as a long range correlation mode. Thus it is responsible for the vacuum ordering.

I restrict now the space integration of Eq. \[23\] over the finite (but large) volume \( V \equiv \eta^{-3} \). For each space
component of \( p \) we have:

\[
\delta_\eta(p) = \frac{1}{2\pi} \int \frac{dx e^{ixp}}{x} = \frac{1}{\pi p} \sin \frac{p}{\eta} .
\]

As well known, \( \lim_{\eta \to 0} \delta_\eta(p) = \delta(p) \) and

\[
\lim_{\eta \to 0} \int dp \delta_\eta(p) f(p) = f(0) = \lim_{\eta \to 0} \int dp \delta(p - \eta) f(p) .
\]

Using \( \delta_\eta(p) \approx \delta(p - \eta) \) for small \( \eta \), one obtains

\[
\tilde{v}(y, \epsilon, \eta) = i\epsilon v e^{-i\eta y} \Delta_\chi(\epsilon, \eta, p_0 = 0) ,
\]

where

\[
\Delta_\chi(\epsilon, \eta, p_0 = 0) = \left[ \frac{Z_\chi}{-\omega^2_{\eta} + i\epsilon a_\chi + \text{continuum contributions}} \right] ,
\]

and \( \omega^2_{\eta} = \eta^2 + m_\chi^2 \). Thus, \( \lim_{\epsilon \to 0} \lim_{\eta \to 0} \tilde{v}(y, \epsilon, \eta) \neq 0 \) only if \( m_\chi = 0 \), otherwise \( \tilde{v} = 0 \). The Goldstone theorem
is of course recovered in the infinite volume limit (\( \eta \to 0 \)) (the QFT limit).

Note that if \( m_\chi = 0 \) and \( \eta \) is given a non-zero value (i.e. by reducing to a finite volume, i.e. in the presence of
boundaries), then \( \omega^2_{\eta} \neq 0 \) and it acts as an "effective mass" for the \( \chi \) bosons. Then, in order to have the order
parameter \( \tilde{v} \) different from zero \( \epsilon \) must be kept non-zero. I remark that an impurity embedded in the system always
generates "boundaries" around it, thus producing finite volume effects.

In conclusion, near the boundaries (\( \eta \neq 0 \)) the NG bosons acquire an effective mass \( m_{\epsilon,ff} \equiv \omega_{\eta} \). Then they
propagate over a range of the order of \( \xi \equiv \frac{1}{\eta} \), which is the linear size of the condensation domain, or, in the presence
of topological singularity, the size of the topologically non-trivial condensation, namely of the extended object (the
defect). It must be observed that the topological singularity tends to be washed out since, according to the conclusion
of the previous Section, it is not compatible with the non-vanishing value of the NG boson effective mass. Near the
boundaries we thus expect (topologically) regular boson condensation. Far from the boundaries, topological
singularities might survive.
If $\eta \neq 0$ then $\epsilon$ must be non-zero in order to have the order parameter different from zero, $\tilde{v} \neq 0$ (at least locally). In such a case the symmetry breakdown is maintained because $\epsilon \neq 0$: $\epsilon$ acts as the coupling with an external field (the pump) providing energy. Energy supply is required in order to condensate modes of non-zero lowest energy $\omega_{\mu=0}$. Boundary effects are thus in competition with the breakdown of symmetry \cite{22}. They may preclude its occurrence or, if symmetry is already broken, they may reduce to zero the order parameter.

The above discussion fits with the intuitive picture: for large but finite volume one expects that the order parameter is constant “inside the bulk” far from the boundaries. However, “near” the boundaries, one might expect “distortions” in the order parameter: “near” the system boundaries we may have non-homogeneous order parameter, $\tilde{v} = \tilde{v}(x)$ (or even $\tilde{v} \to 0$). Such non-homogeneities in the boson condensation “smoothes out” in the $V \to \infty$ limit.

Remarkably, the roles of the boundaries and of the NG effective mass in the above discussion may be exchanged, i.e. if for some kinematical or dynamical reasons the NG modes acquire non-vanishing effective mass, say of the order of $\eta$, then the ordered domains will have linear dimension of the order of $\frac{1}{\eta}$, which means that the domain boundaries are dynamically generated.

I observe that by use of a model system where two level atoms are considered in interaction with their radiative field, the analysis of stability of the solutions of field equations shows \cite{20} that the e.m. field, as an effect of the spontaneous breakdown of the phase symmetry, gets a massive component (the amplitude field), as indeed expected in the Anderson-Higgs-Kibble mechanism \cite{17}, there is a (surviving) massless component (the phase field) playing the role of the NG mode and the stability regime is reached provided the phase locking of the e.m. and matter fields is attained. The physical meaning of the phase locking can be stated as follows. The gauge arbitrariness of the field $a_{\mu}(x)$ is meant to compensate exactly the arbitrariness of the phase of the matter field in the covariant derivative $D_{\mu} = \partial_{\mu} - ig a_{\mu}(x)$. Should one of the two arbitrariness be removed by the dynamics, the invariance of the theory requires the other arbitrariness, too, must be simultaneously removed, namely the appearance of a well defined phase of the matter field implies that a specific gauge function must be selected. The above link between the phase of the matter field and the gauge of $a_{\mu}(x)$ is stated by the equation $a_{\mu}(x) \propto x (\partial_{\mu} f(x)) (a_{\mu}(x)$ is a pure gauge field) and the analysis above reported in connection with the (topological) regularity and singularity of $f(x)$ is then recovered.

A. Remark on temperature effects and critical regime

Let me briefly comment now on the temperature effects on the order parameter (symmetry may be restored at or above a critical temperature $T_C$). See \cite{14, 27} for further details.

Since the order parameter goes to zero when NG modes acquire non-zero effective mass (unless, as observed above, external energy is supplied), the effect of thermalization may be represented in terms of finite volume effects by putting, e.g., $\eta \propto \sqrt{\frac{T-C}{T_C}}$, so that temperature fluctuations around $T_C$ may produce fluctuations in the size $\xi$ of the condensed domain.

At $T > T_C$, but near to $T_C$, and in the presence of an external driving field ($\epsilon \neq 0$), one may have the formation of ordered domains of size $\xi \propto (\sqrt{\frac{T-T_C}{T_C}})^{-1}$ even before transition to fully ordered phase is achieved as $T \to T_C$. As far as $\eta \neq 0$, the ordered domains (and the topological defects) are unstable. They disappear as the external field coupling $\epsilon \to 0$. If ordered domains are still present at $T < T_C$, they also disappear as $\epsilon \to 0$. The surviving possibility of such ordered domains below $T_C$ depends on the speed at which $T$ is lowered (which is related to the speed at which $\eta \to 0$), compared to the speed at which the system is able to get homogeneously ordered. The system is said to be in the critical or Ginzburg regime during the lapse of time in which a maximally stable new configuration is attained since the transition has started. In many cases, information on the critical regime behavior is provided by using the harmonic approximation for the evolution of the order parameter $\tilde{v}(x)$ (non-homogeneous condensate) \cite{4, 8, 28}. The reality condition on the ‘mass parameter’ $M_k(t)$ turns out to be a condition on the $k$-modes propagation. The “effective causal horizon” \cite{23, 30} can happen to be inside the system (possible formation of more than a domain) or outside (single domain formation) according to whether the time occurring for reaching the boundaries of the system is longer or shorter than the allowed propagation time. This determines the dimensions to which the domains can expand. The number of defects (of vortices) $n_{\text{def}}$ possibly appearing during the critical regime and the evolution of the size of the domain can be computed \cite{23, 29, 30}.

It can be shown that higher momentum modes survive longer, which implies that smaller size domains are more stable than larger size domains \cite{23, 30}. Correlation modes with non-vanishing effective mass thus generate domains which tend to break down into smaller, more stable domains. It might be worth to investigate deeper such an occurrence since it might shed some light also on macroscopic phenomena in biology, finance, etc., in view of the fact that coherent condensed states are nearest to classical states (we have seen in Sections \cite{11} and \cite{11} how classical motion equations and classical observables are obtained from the microscopic analysis). As an example, the above analysis might suggest, within proper conditions and under convenient extrapolations, that a global market may be
maintained only provided an (enormous) amount of energies is pumped in; its “natural” destiny being, otherwise, its breakdown into separate “pieces”.

V. FRACTALS AND THE ALGEBRA OF COHERENT STATES

In this Section I show that a relation between fractals and the algebra of coherent states exists (I have conjectured the existence of such a relation in Ref. [11]).

In the following I consider the case of fractals which are generated iteratively according to a prescribed recipe, the so-called deterministic fractals (fractals generated by means of a random process, called “random fractals” [6], will be considered in a future work).

I will focus my discussion on the self-similarity property which is in some sense the most important property of fractals (p. 150 in Ref. [31]).

To be specific, I consider the Koch curve (Fig. 1). One starts with a one-dimensional, \( d = 1 \), segment \( u_0 \) of unit length \( L_0 \), sometimes called the initiator [6]. I call this, as usually done, the step, or stage, of order \( n = 0 \). The length \( L_0 \) is then divided by the reducing factor \( s = 3 \), and the rescaled unit length \( L_1 = \frac{1}{3}L_0 \) is adopted to construct the new “deformed segment” \( u_1 \) made of \( \alpha = 4 \) units \( L_1 \) (step of order \( n = 1 \)). \( u_1 \) is called the generator [6]. Of course, such a “deformation” of the \( u_0 \) segment is only possible provided one “gets out” of the one dimensional straight line \( r \) to which the \( u_0 \) segment belongs: this suggests that in order to construct the \( u_1 \) segment “shape” the one dimensional constraint \( d = 1 \) is relaxed. We thus see that such a shape, made of \( \alpha = 4 \) units \( L_1 \), lives in some \( d \neq 1 \) dimensions and thus we write \( u_{1,q}(\alpha) \equiv q^\alpha u_0, \; q = \frac{1}{3}, \; d \neq 1 \), where \( d \) has to be determined and the index \( q \) has been introduced in the notation of the deformed segment \( u_1 \).

![Fig. 1. The first five stages of Koch curve.](image)

Usually one considers the familiar scaling laws of lengths, surfaces and volumes when lengths are (homogeneously) scaled. Denoting by \( \mathcal{H}(L_0) \) lengths, surfaces or volumes one has

\[
\mathcal{H}(\lambda L_0) = \lambda^d \mathcal{H}(L_0),
\]

under the scale transformation: \( L_0 \rightarrow \lambda L_0 \). A square \( S \) whose side is \( L_0 \) scales to \( \frac{1}{\sqrt{\lambda}}S \) when \( L_0 \rightarrow \lambda L_0 \) with \( \lambda = \frac{1}{2} \). A cube \( V \) of same side with same rescaling of \( L_0 \) scales to \( \frac{1}{\sqrt[3]{\lambda}}V \). Thus \( d = 2 \) and \( d = 3 \) for surfaces and volumes, respectively. Note that \( \frac{S(\frac{1}{2}L_0)}{S(L_0)} = p = \frac{1}{4} \) and \( \frac{V(\frac{1}{2}L_0)}{V(L_0)} = p = \frac{1}{8} \), respectively, so that in both cases \( p = \lambda^d \). Similarly, for
the length $L_0$ it is $p = \frac{1}{3} = \frac{1}{d_\lambda} = \lambda^d$ and of course it is $d = 1$.

By generalizing and extending this to the case of any other “ipervolume” $\mathcal{H}$ one considers thus the ratio

$$\frac{\mathcal{H}(\lambda L_0)}{\mathcal{H}(L_0)} = p,$$

and assuming that Eq. (29) is still valid “by definition”, one obtains

$$p \mathcal{H}(L_0) = \lambda^d \mathcal{H}(L_0),$$

i.e. $p = \lambda^d$. Going back then to the discussion of the Koch curve and setting $\alpha = \frac{1}{p} = 4$ and $q = \lambda^d = \frac{4}{3}$, the relation $p = \lambda^d$ gives

$$q \alpha = 1, \quad \text{where} \quad \alpha = 4, \quad q = \frac{1}{3^d},$$

i.e.

$$d = \frac{\ln 4}{\ln 3} \approx 1.2619.$$  \hspace{2cm} (33)

The non-integer $d$ is called the fractal dimension, or the self-similarity dimension [31].

With reference to the Koch curve, I observe that the meaning of Eq. (33) is that in the “deformed space”, to which $u_{1,q}$ belongs, the set of four segments of which $u_{1,q}$ is made “equals” (is equivalent to) the three segments of which $u_0$ is made in the original “undeformed space”. The (fractal) dimension $d$ is the dimension of the deformed space which allows the possibility of such an “equivalence”, i.e. that ensures the existence of a solution of the relation $\frac{1}{\alpha} = \frac{4}{3} = \frac{1}{d_\lambda} = q$, which for $d = 1$ would be trivially wrong. In this sense $d$ is a measure of the “deformation” of the $u_{1,q}$-space with respect to the $u_0$-space. In other words, we require that the measure of the deformed segment $u_{1,q}$ with respect to the undeformed segment $u_0$ be 1: $\frac{u_{1,q}}{u_0} = 1$, namely $\alpha q = \frac{4}{3} = 1$. In the following, for brevity I will thus set $u_0 = 1$, whenever no misunderstanding arises.

After having partitioned $u_0$ in three equal segments, since the deformation of $u_0$ into $u_{1,q}$ is performed by varying the number $\alpha$ of such segments from 3 to 4, we expect that $\alpha$ and its derivative $\frac{d}{\alpha}$ play a relevant role in the fractal structure. We will see indeed that $(\alpha, \frac{d}{\alpha})$ play the role of conjugate variables (cf. Eq. (38)).

Steps of higher order $n$, $n = 2, 3, 4, \ldots \infty$, can be obtained by iteration of the deformation process keeping $q = \frac{4}{3}$ and $\alpha = 4$. For example, in the step $n = 2$, we rescale $L_1$ by a further factor 3, $L_2 = \frac{1}{3}L_1$, and construct the deformed segment $u_{2,q}(\alpha) \equiv q\alpha u_{1,q}(\alpha) = (q\alpha)^2 u_0$, and so on. For the $n$th order deformation we have

$$u_{n,q}(\alpha) = (q\alpha) u_{n-1,q}(\alpha), \quad n = 1, 2, 3, \ldots$$

i.e., for any $n$

$$u_{n,q}(\alpha) = (q\alpha)^n u_0.$$  \hspace{2cm} (35)

By proceeding by iteration, or, equivalently, by requiring that $\frac{u_{n,q}(\alpha)}{u_0}$ be 1 for any $n$, gives $(q\alpha)^n = 1$ and Eq. (33) is again obtained. Notice that the fractal is mathematically defined in the limit of infinite iterations of the deformation process, $n \to \infty$: in this sense, the fractal is the limit of the deformation process for $n \to \infty$. As a matter of facts, the definition of fractal dimension is given starting from $(q\alpha)^n = 1$ in the $n \to \infty$ limit. Since $L_n \to 0$ for $n \to \infty$, the Koch fractal is a curve which is everywhere non-differentiable [31].

Provided one specializes the values of the “deformation” parameter $q$ and of the $\alpha$ parameter, Eqs. (34) and (35) fully characterize the fractal. They express in analytic form, in the $n \to \infty$ limit, the self-similarity property of a large class of fractals (the Sierpinski gasket and carpet, the Cantor set, etc.): “cutting a piece of a fractal and magnifying it isotropically to the size of the original, both the original and the magnification look the same” [6]. In this sense one also says that fractals are “scale free”, namely viewing a picture of part of a fractal one cannot deduce its actual size if the unit of measure is not given in the same picture [32]. It has to be stressed that only in the $n \to \infty$ limit self-similarity is defined (self-similarity does not hold when considering only a finite number $n$ of iterations). I also recall that invariance (always in the limit of $n \to \infty$ iterations) only under anisotropic magnification is called self-affinity. The discussion below can be extended to self-affine fractals. I will not discuss here the measure of lengths in fractals, the Hausdorff measure, the fractal “mass” and other fractal properties. The reader is referred to the existing literature.

My main observation is now that Eq. (35) expresses the deep formal connection with the theory of coherent states and the related algebraic structure. I discuss this in the next subsection.
A. Fractals and deformed coherent states

In order to make the connection between fractals and the algebra of coherent states explicit, I observe that, by considering in full generality the complex $\alpha$-plane, the functions

$$u_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}}, \quad u_0(\alpha) = 1, \quad n \in \mathbb{N}_+, \quad \alpha \in \mathbb{C},$$

form in the space $\mathcal{F}$ of the entire analytic functions a basis which is orthonormal under the gaussian measure $d\mu(\alpha) = \frac{1}{\sqrt{\pi}} e^{-|\alpha|^2} d\alpha d\bar{\alpha}$. In Eq. (36) the factor $\frac{1}{\sqrt{n!}}$ has been introduced to ensure the normalization condition with respect to the gaussian measure.

The functions $u_n(a^\dagger a)^i|q\rightarrow 1$ in Eq. (36) (for the factor $q \neq 1$ see the discussion below), are thus immediately recognized to be nothing but the restriction to real $\alpha$ of the functions in Eq. (36), apart the normalization factor $\frac{1}{\sqrt{n!}}$. The study of the fractal properties may be thus carried on in the space $\mathcal{F}$ of the entire analytic functions, by restricting, at the end, the conclusions to real $\alpha$, $\alpha \rightarrow \Re(\alpha)$. Furthermore, since actually in Eq. (35) it is $q \neq 1$ ($q < 1$), one also needs to consider the “$q$-deformed” algebraic structure of which the space $\mathcal{F}$ provides a representation. This is indeed the plan I will follow below.

Let me start by observing that the space $\mathcal{F}$ is a vector space which provides the so called Fock–Bargmann representation (FBR) [33] of the Weyl–Heisenberg algebra generated by the set of operators $\{a, a^\dagger, 1\}$:

$$[a, a^\dagger] = 1, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a,$$

where $N \equiv a^\dagger a$, with the identification:

$$N \rightarrow a \frac{d}{d\alpha}, \quad a^\dagger \rightarrow \alpha, \quad a \rightarrow \frac{d}{d\alpha}.$$ (38)

The $u_n(\alpha)$ (Eq. (36)) are easily seen to be eigenkets of $N$ with integer (positive and zero) eigenvalues. The FBR is the Hilbert space $\mathcal{K}$ generated by the $u_n(\alpha)$, i.e. the whole space $\mathcal{F}$ of entire analytic functions. Any vector $|\psi\rangle$ in $\mathcal{K}$ is associated, in a one-to-one correspondence, with a function $\psi(\alpha) \in \mathcal{F}$ and is thus described by the set $\{c_n; c_n \in \mathbb{C}, \sum_{n=0}^{\infty} |c_n|^2 = 1\}$ defined by its expansion in the complete orthonormal set of eigenkets $\{|n\rangle\}$ of $N$:

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \rightarrow \psi(\alpha) = \sum_{n=0}^{\infty} c_n u_n(\alpha),$$ (39)

$$\langle \psi|\psi\rangle = \sum_{n=0}^{\infty} |c_n|^2 = \int |\psi(\alpha)|^2 d\mu(\alpha) = ||\psi||^2 = 1,$$ (40)

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle,$$ (41)

where $|0\rangle$ denotes the vacuum vector, $a|0\rangle = 0$, $\langle 0|0\rangle = 1$. The series expressing $\psi(\alpha)$ in Eq. (38) converges uniformly in any compact domain of the $\alpha$-plane due to the condition $\sum_{n=0}^{\infty} |c_n|^2 = 1$ (cf. Eq. (40)), confirming that $\psi(\alpha)$ is an entire analytic function. Note that, as expected in view of the correspondence $\mathcal{K} \rightarrow \mathcal{F}$ ($|n\rangle \rightarrow u_n(\alpha)$),

$$a^\dagger u_n(\alpha) = \sqrt{n+1} u_{n+1}(\alpha), \quad a u_n(\alpha) = \sqrt{n} u_{n-1}(\alpha),$$ (42)

$$N u_n(\alpha) = a^\dagger a u_n(\alpha) = \alpha \frac{d}{d\alpha} u_n(\alpha) = n u_n(\alpha).$$ (43)

Equations (42) and (43) establish the mutual conjugation of $a$ and $a^\dagger$ in the FBR, with respect to the measure $d\mu(z)$.

Note that, upon introducing $H \equiv N + \frac{1}{2}$, the three operators $\{a, a^\dagger, H\}$ close, on $\mathcal{K}$, the relations

$$\{a, a^\dagger\} = 2H, \quad [H, a] = -a, \quad [H, a^\dagger] = a^\dagger,$$ (44)

that are equivalent to (37) on $\mathcal{K}$ and show the intrinsic nature of superalgebra of such scheme.

The Fock–Bargmann representation provides a simple frame to describe the usual coherent states (CS) [33, 34] $|\alpha\rangle$:

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad a|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C},$$ (45)
The unitary displacement operator $D(\alpha)$ in (45) (mentioned in Section II) is given by:

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha a) = \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(\alpha a^\dagger) \exp(-\bar{\alpha} a).$$

(47)

The explicit relation between the CS and the entire analytic function basis $\{u_n(\alpha)\}$ (Eq. (36)) is:

$$u_n(\alpha) = e^{\frac{i}{2}|\alpha|^2} \langle n | \alpha \rangle .$$

(48)

The following relations hold

$$D^{-1}(\alpha) a D(\alpha) = a + \alpha ,$$

(49)

$$D(\alpha)D(\beta) = \exp(i \text{Im}(\alpha \bar{\beta}))D(\alpha + \beta) ,$$

(50)

$$D(\alpha)D(\beta) = \exp(2i \text{Im}(\alpha \bar{\beta}))D(\beta)D(\alpha) .$$

(51)

The operator $D(\alpha)$ is a bounded operator defined on the whole $\mathcal{K}$. Eq. (50) is nothing but a representation of the Weyl–Heisenberg group usually denoted by $W_1[33]$. It must be remarked that the set $\{\alpha\}$ is an overcomplete set of states, from which, however, a complete set can be extracted. Is well known that in order to extract a complete set of CS from the overcomplete set it is necessary to introduce in the $\alpha$-complex plane a regular lattice $L$, called the von Neumann lattice $[33, 35]$. For shortness, I will not discuss this point here, see [33] for a general discussion and original references (see also [32] where the von Neumann lattice is discussed also in connection with the deformation of the Weyl–Heisenberg algebra introduced below).

Let me now introduce the finite difference operator $D_q$, also called the $q$-derivative operator [37], defined by:

$$D_q f(\alpha) = \frac{f(q\alpha) - f(\alpha)}{(q - 1)\alpha} ,$$

(52)

with $f(\alpha) \in \mathcal{F}$, $q = e^\zeta$, $\zeta \in \mathbb{C}$. $D_q$ reduces to the standard derivative for $q \rightarrow 1$ ($\zeta \rightarrow 0$). In the space $\mathcal{F}$, $D_q$ satisfies, together with $\alpha$ and $\frac{d}{d\alpha}$, the commutation relations:

$$[D_q, \alpha] = q^\alpha \frac{d}{d\alpha} , \quad \left[\alpha \frac{d}{d\alpha}, D_q\right] = -D_q , \quad \left[\alpha \frac{d}{d\alpha}, \alpha\right] = \alpha ,$$

(53)

which, as for Eq. (33), lead us to the identification

$$N \rightarrow \alpha \frac{d}{d\alpha} , \quad \hat{a}_q \rightarrow \alpha , \quad a_q \rightarrow D_q ,$$

(54)

with $\hat{a}_q = \hat{a}_{q=1} = a^\dagger$ and $\lim_{q \rightarrow 1} a_q = a$ on $\mathcal{F}$. With such an identification the algebra [37] can be seen as the $q$-deformation of the algebra [37]. For shortness I omit to discuss further the properties of $D_q$ and the $q$-deformed algebra [37]. More details can be found in Ref. [36]. Here I only recall that the relations analogous to (42) for the $q$-deformed case are

$$\hat{a}_q u_n(\alpha) = \sqrt{n+1} u_{n+1}(\alpha) , \quad a_q u_n(\alpha) = q^{n-1} \frac{|n|_q}{\sqrt{n}} u_{n-1}(\alpha) ,$$

(55)

where $|n|_q = \frac{q^{n+1} - q^{-1}}{q^2 - q^{-2}}$, and that the operator $q^N$ acts on the whole $\mathcal{F}$ as

$$q^N f(\alpha) = f(q\alpha) , \quad f(\alpha) \in \mathcal{F} .$$

(56)

This result was originally obtained in Ref. [36]. There it has been remarked that since the $q$-deformation of the algebra has been essentially obtained by replacing the customary derivative with the finite difference operator, then a deformation of the operator algebra acting in $\mathcal{F}$ should arise whenever one deals with some finite scale, e.g. with some discrete structure, lattice or periodic system (periodicity is but a special invariance under finite difference operators),
which cannot be reduced to the continuum by a limiting procedure. The $q$-deformation parameter is related with the finite scale. A finite scale occurs indeed also in the present case of fractals and therefore also in this case we expect and in fact have a deformation of the algebra.

Eq. (60) applied to the coherent state functional (16) gives

$$ q^N(\alpha) = |q\alpha\rangle = \exp \left(-\frac{|q\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(q\alpha)^n}{\sqrt{n!}} |n\rangle, \quad (57) $$

and, since $q\alpha \in \mathcal{C}$, from Eq. (65),

$$ a |q\alpha\rangle = q\alpha |q\alpha\rangle, \quad q\alpha \in \mathcal{C}. \quad (58) $$

By recalling that we have set $u_0 \equiv 1$, the $n$th fractal iteration, Eq. (35), is obtained by projecting out the $n$th component of $|q\alpha\rangle$ and restricting to real $q\alpha$, $q\alpha \to Re(q\alpha)$:

$$ u_{n,q}(\alpha) = (q\alpha)^n = \sqrt{n!} \exp \left(\frac{|q\alpha|^2}{2}\right) \langle n | q\alpha \rangle, \quad \text{for any } n, \quad q\alpha \to Re(q\alpha). \quad (59) $$

Taking into account that $\langle n \rangle = \langle 0 \rangle^n \frac{\langle a \rangle^n}{\sqrt{n!}}$, Eq. (59) gives

$$ u_{n,q}(\alpha) = (q\alpha)^n = \exp \left(\frac{|q\alpha|^2}{2}\right) \langle 0 | (a)^n | q\alpha \rangle, \quad \text{for any } n, \quad q\alpha \to Re(q\alpha), \quad (60) $$

which shows that the operator $(a)^n$ acts as a “magnifying” lens [6]: the $n$th iteration of the fractal can be “seen” by applying $(a)^n$ to $|q\alpha\rangle$ and restricting to real $q\alpha$:

$$ \langle q\alpha | (a)^n | q\alpha \rangle = (q\alpha)^n = u_{n,q}(\alpha), \quad q\alpha \to Re(q\alpha). \quad (61) $$

Note that the equivalence between Eq. (60) and Eq. (61) stems from the coherent state properties [33].

Eq. (58) expresses the invariance of the coherent state representing the fractal under the action of the operator $\frac{1}{q\alpha} a$. This reminds us of the fixed point equation $W(A) = A$, where $W$ is the Hutchinson operator [6], characterizing the iteration process for the fractal $A$ in the $n \to \infty$ limit. Such an invariance property allows to consider the coherent functional $\psi(q\alpha)$ as an “attractor” in $\mathcal{C}$.

In conclusion, the meaning of Eq. (57) is that the operator $q^N$ applied to $|\alpha\rangle$ “produces” the fractal in the functional form of the coherent state $|q\alpha\rangle$. The $n$th fractal stage of iteration, $n = 0, 1, 2, ..., \infty$, is represented, in a one-to-one correspondence, by the $n$th term in the coherent state series in Eq. (57). I call $q^N$ the fractal operator.

Eqs. (59), (60) and (61) formally establish the searched connection between fractals and the ($q$-deformed) algebra of the coherent states.

B. Fractals, squeezed coherent states and noncommutative geometry in the phase space

Let me now look at the fractal operator $q^N$ from a different perspective. I consider the identity

$$ 2\alpha \frac{d}{d\alpha} \psi(\alpha) = \left\{ \frac{1}{2} \left[ \left( \alpha + \frac{d}{d\alpha} \right)^2 - \left( \alpha - \frac{d}{d\alpha} \right)^2 \right] - 1 \right\} \psi(\alpha), \quad (62) $$

which holds in the Hilbert space identified with the space $\mathcal{F}$ of entire analytic functions $\psi(\alpha)$. It is convenient to set $\alpha \equiv x + iy$, $x$ and $y$ denoting the real and the imaginary part of $\alpha$, respectively. I then introduce the operators

$$ c = \frac{1}{\sqrt{2}} \left( \alpha + \frac{d}{d\alpha} \right), \quad c^\dagger = \frac{1}{\sqrt{2}} \left( \alpha - \frac{d}{d\alpha} \right), \quad [c, c^\dagger] = 1. \quad (63) $$

Their relation with the FBR operators $a$ and $a^\dagger$ is

$$ \alpha = \frac{1}{\sqrt{2}} (c + c^\dagger) \to a^\dagger, \quad \frac{d}{d\alpha} = \frac{1}{\sqrt{2}} (c - c^\dagger) \to a. \quad (64) $$
In \( \mathcal{F} \), \( c^\dagger \) is indeed the conjugate of \( c \). In the limit \( \alpha \rightarrow Re(\alpha) \), i.e. \( y \rightarrow 0 \), \( c \) and \( c^\dagger \) turn into the conventional annihilation and creator operators associated with an infinitesimal translation parameter label “different”, i.e. unitarily inequivalent representations: different values of the \( q \)-deformation parameter label “different”, i.e. unitarily inequivalent representations in the limit of infinitely many degrees of freedom (infinite volume limit, infinite fractal dimension). In the scheme here presented, these fractals are then described by the matrix elements \( S(\alpha) \) of the squeezing transformation. Note that the right hand side of Eq. \( (65) \) is an SU(1,1) group element. In fact, by defining \( K_- = \frac{1}{2}c^2, K_+ = \frac{1}{2}c^\dagger 2, K_\alpha = \frac{1}{2}(c^\dagger c + \frac{1}{2}) \), one easily checks they close the algebra \( su(1,1) \). We indeed obtain the SU(1,1) (Bogoliubov) transformations for the \( c \)'s operators:

\[
\hat{S}^{-1}(\alpha) c \hat{S}(\alpha) = c \cos \alpha - c^\dagger \sin \alpha \quad (66)
\]

\[
\hat{S}^{-1}(\alpha) c^\dagger \hat{S}(\alpha) = c^\dagger \cos \alpha + c \sin \alpha \quad (67)
\]

and in the \( y \rightarrow 0 \) limit (still in \( \mathcal{F} \))

\[
\hat{S}^{-1}(\alpha) c \hat{S}(\alpha) \rightarrow \hat{S}^{-1}(\alpha) a \hat{S}(\alpha) = a \cos \alpha - a^\dagger \sin \alpha \quad (68)
\]

\[
\hat{S}^{-1}(\alpha) c^\dagger \hat{S}(\alpha) \rightarrow \hat{S}^{-1}(\alpha) a^\dagger \hat{S}(\alpha) = a^\dagger \cos \alpha - a \sin \alpha \quad (69)
\]

Moreover, in the \( y \rightarrow 0 \) limit,

\[
\hat{S}^{-1}(\alpha) \alpha \hat{S}(\alpha) = \frac{1}{q} \alpha \rightarrow \frac{1}{q} x \quad (70)
\]

\[
\hat{S}^{-1}(\alpha) p_\alpha \hat{S}(\alpha) = q p_\alpha \rightarrow q p_x \quad (71)
\]

where \( p_\alpha \equiv -i \frac{\partial}{\partial \alpha} \), and

\[
\int d\mu(\alpha) \hat{\psi}(\alpha)\hat{S}^{-1}(\alpha) \alpha \hat{S}(\alpha) \psi(\alpha) \rightarrow \frac{1}{q} \langle x \rangle > (72)
\]

\[
\int d\mu(\alpha) \hat{\psi}(\alpha)\hat{S}^{-1}(\alpha) p_\alpha \hat{S}(\alpha) \psi(\alpha) \rightarrow q \langle p_x \rangle > (73)
\]

so that the root mean square deviations \( \Delta x \) and \( \Delta p_x \) satisfy

\[
\Delta x \Delta p_x = \frac{1}{2} \quad , \quad \Delta x = \frac{1}{q} \sqrt{\frac{T}{2}} \quad , \quad \Delta p_x = q \sqrt{\frac{T}{2}} \quad (74)
\]

This confirms that the \( q \)-deformation plays the role of squeezing transformation. Note that the action variable \( \int p_x \, dx \) is invariant under the squeezing transformation.

Eq. \( (70) \) shows that \( \alpha \rightarrow \frac{1}{q} \alpha \) under squeezing transformation, which, in view of the fact that \( q^{-1} = \alpha \) (cf. Eq. \( (32) \)), means that \( \alpha \rightarrow \alpha^2 \), i.e. under squeezing we proceed further in the fractal iteration process. Thus, the fractal iteration process can be described in terms of the coherent state squeezing transformation.

I recall that by means of the squeezing transformations the Weyl-Heisenberg representations are labeled by the \( q \)-parameter and that in the infinite volume limit (infinite degrees of freedom) they are unitarily inequivalent representations; different values of the \( q \)-deformation parameter label “different”, i.e. unitarily inequivalent representations in QFT \( (3, 32) \). By changing the value of the \( q \)-parameter one thus moves from a given representation to another one, unitarily inequivalent to the former one. Besides the scale parameter one might also consider, phase parameters and translation parameters characterizing (generalized) coherent states (such as SU(2), SU(1,1), etc. coherent states). For example, by changing the parameters in a deterministic iterated function process, also referred to as multiple reproduction copy machine process, (such as phases, translations, etc.) the Koch curve may be transformed into another fractal (e.g. into Barnsley’s fern \( (31) \)). In the scheme here presented, these fractals are then described by corresponding unitarily inequivalent representations in the limit of infinitely many degrees of freedom (infinite volume limit, infinite fractal dimension).
limit). The trajectories induced by the changes of the parameters over the space of the representations can be shown to be, under quite general conditions, chaotic trajectories [39]. This might shed some light on the richness of the variety of “different” fractal shapes obtainable by changing the parameters of the fractal one starts with [31]. Work is in progress on such a subject.

Due to the holomorphy conditions holding for $f(\alpha) \in \mathcal{F}$

$$\frac{d}{d\alpha} f(\alpha) = \frac{d}{dx} f(\alpha) = -i \frac{d}{dy} f(\alpha) , \quad (75)$$

in the $y \rightarrow 0$ limit we get form [68]

$$c \rightarrow \frac{1}{\sqrt{2}} (x + ip_x) \equiv \hat{z} , \quad c^\dagger \rightarrow \frac{1}{\sqrt{2}} (x - ip_x) \equiv \hat{z}^\dagger , \quad [\hat{z}, \hat{z}^\dagger] = 1 , \quad (76)$$

where $p_x = -i \frac{d}{dx}$. $\hat{z}$ and $\hat{z}^\dagger$ are the usual creation and annihilation operators in the configuration representation. Under the action of the squeezing transformation, use of (70) and (71) leads to

$$\hat{z}_q = \frac{1}{q \sqrt{2}} (x + iq^2 p_x) , \quad \hat{z}_q^\dagger = \frac{1}{q \sqrt{2}} (x - iq^2 p_x) , \quad [\hat{z}_q, \hat{z}_q^\dagger] = 1 . \quad (77)$$

Notice that the $\hat{z}$ and $\hat{z}^\dagger$ algebra is preserved under the squeezing transformation, which is indeed a canonical transformation. In the “deformed” phase space let us denote the coordinates $(x, q^2 p_x)$ as $(x_1, x_2)$, where $x_1 \equiv x$ and $x_2 \equiv q^2 p_x$. Coordinates do not commute in this (deformed) space:

$$[x_1, x_2] = iq^2 . \quad (78)$$

We thus recognize that $q$-deformation introduces non-commutative geometry in the $(x_1, x_2)$-space. In such a space the noncommutative Pythagoras’s theorem gives the distance $D$:

$$D^2 = x_1^2 + x_2^2 = 2q^2 (\hat{z}_q^\dagger \hat{z}_q + \frac{1}{2}) . \quad (79)$$

In $\mathcal{F}$, in the $y \rightarrow 0$ limit, form the known properties of creation and annihilation operators we then get

$$D_n^2 = 2q^2 (n + \frac{1}{2}) , \quad n = 0, 1, 2, 3... , \quad (80)$$

i.e. in the space $(x_1, x_2)$ associated to the coherent state fractal representation, the $(x_1, x_2)$-distance is quantized according to the unit scale set by $q$. Eq. (80) also shows that in the space $(x_1, x_2)$ we have quantized “disks” of squared radius vector $D_n^2$. It is interesting to observe that the “smallest” of such disks has non-zero radius given by the deformation parameter $q$ (recall that $q = \frac{1}{2}$ when Koch fractal is considered). Recalling the expression of the energy spectrum of the harmonic oscillator, one could write Eq. (80) as $D_n^2 = 2q^2 (n + \frac{1}{2}) \equiv E_n$, $n = 0, 1, 2, 3...$, where $E_n$ might be thought as the “energy” associated to the fractal $n$-stage.

VI. FINAL REMARKS

Let me close the paper by observing that in the case of topologically non-trivial condensation at finite temperature the order parameter $v(x, \beta)$ provides a mapping between the domains of variation of $(x, \beta)$ and the space of the unitarily inequivalent representations of the canonical commutation relations. As well known, this is the homotopy mapping between the $(x, \beta)$ variability domain and the group manifold. In the vortex case one has the mapping $\pi$ of $S^1$, surrounding the $r = 0$ singularity, to the group manifold of $U(1)$ which is topologically characterized by the winding number $n \in Z \in \pi_1(S^1)$. Such a singularity at $r = 0$ is carried by the boson condensation function of the NG modes. In the monopole case [14], the mapping $\pi$ is the one of the sphere $S^2$, surrounding the singularity $r = 0$, to $SO(3)/SO(2)$ group manifold, with homotopy classes of $\pi_2(S^2) = Z$. Same situation occurs in the sphaleron case [14], provided one replaces $SO(3)$ and $SO(2)$ with $SU(2)$ and $U(1)$, respectively.

As discussed in the previous sections, transitions between phases characterized by an order parameter imply “moving” over unitarily inequivalent representations, and this is achieved by gradients in NG boson condensation function. In the presence of a gauge field, macroscopic ground state field and currents can only be obtained by non-homogeneous NG boson condensation with topological singularities. The occurrence of such topologically non-trivial condensation
allows the formation of topological defects. This explains why topological defect formation is observed in symmetry breaking phase transition processes.

Finite volume effects, effective mass of the NG bosons and temperature effects have been briefly discussed and related to the ordered domain size. Correlation modes with non-vanishing effective mass generate domains which tend to break down into smaller, more stable domains. An interesting development which can be pursued in a future research is the one referring to the stability of macroscopic correlated domains occurring in biology, finance, etc., to which our conclusions might extend by exploiting the remarkable interplay shown to emerge between the microscopic dynamics of the system components and the macroscopic features of the system. As I have observed, the analysis presented in the previous sections might suggest, for example, that, within proper conditions, the natural evolution of the global market is its breakdown into separate parts, unless, of course, an (enormous) amount of energies is pumped in.

Finally, I have discussed the functional realization of fractal properties, with particular reference to self-similarity property, in terms of the \( q \)-deformed algebra of coherent states. The relation of fractals with the squeezing operator and noncommutative geometry in phase space has been also exhibited. Fractal study can be thus incorporated in the theory of entire analytical functions.

On the other hand, the discussion presented above also shows that the reverse is also true: under convenient choice of the \( q \)-deformation parameter and by a suitable restriction to real \( \alpha \), coherent states exhibit fractal properties in the \( q \)-deformed space of the entire analytical functions.

Since both, fractal structures and coherent states are recognized to appear in an enormous number of systems and natural phenomena, the above conclusions may be of large interest in many applications. Moreover, the relation here established between fractals and coherent states introduces dynamical considerations in the study of fractals and of their origin, as well as geometrical insight in the coherent states properties.

I also remark that fractals are global systems arising from local deformation processes. Therefore they cannot be purely geometric objects. Their deep connection with coherent states is therefore not only expected, but, I would say, necessary, since coherence is the only available tool existing in our knowledge of physical phenomena able to provide long range (macroscopic) correlations out of the microscopic dynamics of elementary components.

In this paper I have not considered "random fractals", i.e. those fractals obtained by randomization processes introduced in their iterative generation. Their characteristics suggest that they must be related with dissipative systems and since self-similarity is still a characterizing property in many of such random fractals, my conjecture is that also in such cases there must exist a deep connection with the coherent state algebraic structure. This will be the subject of future work.

VII. ACKNOWLEDGEMENTS

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