Flat Semimodules & von Neumann Regular Semirings

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Abstract

Flat modules play an important role in the study of the category of modules over rings and in the characterization of some classes of rings. We study the $e$-flatness for semimodules introduced by the first author using his new notion of exact sequences of semimodules and its relationships with other notions of flatness for semimodules over semirings. We also prove that a subtractive semiring over which every right (left) semimodule is $e$-flat is a von Neumann regular semiring.

Introduction

Semirings are, roughly, rings not necessarily with subtraction. They generalize both rings and distributive bounded lattices and have, along with their semimodules many applications in Computer Science and Mathematics (e.g., [HW1998], [Gla2002], [LM2005]). Many applications can be found in Golan’s book [Gol1999], which is our main reference on this topic.

A systematic study of semimodules over semirings was carried out by M. Takahashi in a series of papers 1981-1990. However, he defined two main notions in a way that turned out to be

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not natural. Takahashi’s tensor products [Tak1982b] did not satisfy the expected Universal Property. On the other hand, Takahashi’s exact sequences of semimodules [Tak1981] were defined as if this category were exact, which is not the case (in general).

By the beginning of the 21st century, several researchers began to use a more natural notion of tensor products of semimodules (cf., [Kat2004]) with which the category of semimodules over a commutative semiring is monoidal rather than semimonoidal [Abu2013]. On the other hand, several notions of exact sequences were introduced (cf., [Pat2003]), each of which with advantages and disadvantages. One of the most recent notions is due to Abuhlail [Abu2014] and is based on an intensive study of the nature of the category of semimodules over a semiring.

In addition to the categorical notions of flat semimodules over a semiring, several other notions were considered in the literature, e.g., the so called m-flat semimodules [Alt2004] (called mono-flat in [Kat2004]). One reason for the interest of such notions is the phenomenon that, a commutative semiring all of whose semimodules are flat is a von Neumann regular ring [Kat2004, Theorem 2.11]. Using a new notion of exact sequences of semimodules over a semiring, Abuhlail introduced ([Abu2014-SF]) a homological notion of exactly flat semimodules, which we call, for short, e-flat semimodules assuming that an appropriate \( \otimes \) functor preserves short exact sequences.

The paper is divided into three sections.

In Section 1, we collect the basic definitions, examples and preliminaries used in this paper. Among others, we include the definitions and basic properties of exact sequences as defined by Abuhlail [Abu2014].

In Section 2, we investigate the e-flat semimodules. A flat semimodule is one which is the direct colimit of finitely presented semimodules [Abu2014-SF]. It was proved by Abuhlail [Abu2014-SF, Theorem 3.6] that flat left \( S \)-semimodules are e-flat. We prove in Lemma 2.13 and Proposition 2.14 that the class of e-flat left \( S \)-semimodules is closed under retracts and direct sums.

In Section 3, we study von Neumann regular semirings. In Theorem 3.11, we show that if \( S \) is a (left and right) subtractive semiring each of its right semimodules is \( S \)-e-flat, then \( S \) is a von Neumann regular semiring. Conversely, we prove that if \( S \) is von Neumann regular, then every normally \( S \)-generated right \( S \)-semimodule is \( S \)-m-flat.

1 Preliminaries

In this section, we provide the basic definitions and preliminaries used in this work. Any notions that are not defined can be found in our main reference [Gol1999]. We refer to [Wis1991] for the foundations of Module and Ring Theory.
1.1. ([Gol1999]) A semiring \((S, +, 0, \cdot, 1)\) consists of a commutative monoid \((S, +, 0)\) and a monoid \((S, \cdot, 1)\) such that \(0 \neq 1\) and

\[
\begin{align*}
a \cdot 0 &= 0 = 0 \cdot a \text{ for all } a \in S; \\
a(b + c) &= ab + ac \text{ and } (a + b)c = ac + bc \text{ for all } a, b, c \in S.
\end{align*}
\]

If, moreover, the monoid \((S, \cdot, 1)\) is commutative, then we say that \(S\) is a commutative semiring. We say that \(S\) is additively idempotent, if \(s + s = s\) for every \(s \in S\).

Examples 1.2. ([Gol1999])

- Every ring is a semiring.
- Any distributive bounded lattice \(\mathcal{L} = (L, \lor, 1, \land, 0)\) is a commutative semiring.
- Let \(R\) be any ring. The set \(\mathcal{I} = (\text{Ideal}(R), +, 0, R)\) of (two-sided) ideals of \(R\) is a semiring.
- The set \((\mathbb{Z}^+, +, 0, \cdot, 1)\) (resp. \((\mathbb{Q}^+, +, 0, \cdot, 1), (\mathbb{Q}^+, +, 0, \cdot, 1)\)) of non-negative integers (resp. non-negative rational numbers, non-negative real numbers) is a commutative semiring (resp. semifield) which is not a ring (not a field).
- \(M_n(S)\), the set of all \(n \times n\) matrices over a semiring \(S\), is a semiring.
- \(\mathbb{B} := \{0, 1\}\) with \(1 + 1 = 1\) is a semiring, called the Boolean semiring.
- The max-min algebra \(\mathbb{R}_{\max, \min} := (\mathbb{R} \cup \{-\infty, \infty\}, \max, -\infty, \min, \infty)\) is an additively idempotent semiring.
- The log algebra \((\mathbb{R} \cup \{-\infty, \infty\}, \oplus, \infty, +, 0)\) is a semiring, where
  \[
  x \oplus y = -\ln(e^{-x} + e^{-y})
  \]

1.3. [Gol1999] Let \(S\) and \(T\) be semirings. The categories \(\mathcal{S} \mathcal{M}\) of left \(S\)-semimodules with arrows the \(S\)-linear maps, \(\mathcal{S} \mathcal{M}_T\) of right \(S\)-semimodules with arrows the \(T\)-linear maps, and \(\mathcal{S} \mathcal{S} \mathcal{M}_T\) of \((S, T)\)-bisemimodules are defined in the usual way (as for modules and bimodules over rings). We write \(L \leq_S M\) to indicate that \(L\) is an \(S\)-subsemimodule of the left (right) \(S\)-semimodule \(M\).

Example 1.4. The category of \(\mathbb{Z}^+\)-semimodules is nothing but the category of commutative monoids.

Definition 1.5. [Gol1999, page 162] Let \(S\) be a semiring. An equivalence relation \(\rho\) on a left \(S\)-semimodule \(M\) is a congruence relation, if it preserves the addition and the scalar multiplication on \(M\), i.e. for all \(s \in S\) and \(m, m', n, n' \in M\):

\[
m \rho m' \text{ and } n \rho n' \implies (m + m') \rho (n + n'),
\]

\[
m \rho m' \implies (sm) \rho (sm').
\]
1.6. ([Gol1999, page 150, 154]) Let $S$ be a semiring and $M$ a left $S$-semimodule.

(1) The **subtractive closure** of $L \leq_S M$ is defined as
\[
    \mathcal{L} := \{ m \in M | \ m + l = l' \text{ for some } l, l' \in L \}.
\]

We say that $L$ is subtractive if $L = \mathcal{L}$. The left $S$-semimodule $M$ is a **subtractive semimodule**, if every $S$-subsemimodule $L \leq_S M$ is subtractive.

(2) The set of **cancellative elements** of $M$ is defined as
\[
    K^+(M) = \{ x \in M | x + y = x + z \implies y = z \text{ for any } y, z \in M \}.
\]

We say that $M$ is a **cancellative semimodule**, if $K^+(M) = M$.

1.7. (cf., [AHS2004]) The category $\mathcal{S}_{SM}$ of left semimodules over a semiring $S$ is a **variety** (i.e. closed under homomorphic images, subobjects and arbitrary products), whence complete (i.e. has all limits, e.g., direct products, equalizers, kernels, pullbacks, inverse limits) and cocomplete (i.e. has all colimits, e.g., direct coproducts, coequalizers, cokernels, pushouts, direct colimits).

1.8. With the tensor product of a right $S$-semimodule $L$ and a left $S$-semimodule $M$, we mean the commutative monoid $L \otimes_S M$ in the sense of [Kat1997, 3.1], and not that in the sense of Takahashi adapted by Golan [Gol1999], which we denote by $M \boxtimes_S N$. Abuhlail [Abu2013] showed that $M \boxtimes_S N = c(M \otimes_S N)$, the **cancellative hull** of $M \otimes_S N$. See also [Abu2014-SF, 2.1].

**Lemma 1.9.** For every right $S$-semimodule $M$, there exists a natural right $S$-isomorphism
\[
    \theta_M : M \otimes_S S \rightarrow M, \ m \otimes_S s \mapsto ms.
\]

**Exact Sequences**

Throughout, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an $S$-module is a left $S$-semimodule.

**Definition 1.10.** A morphism of left $S$-semimodules $f : L \rightarrow M$ is

- **$k$-normal**, if whenever $f(m) = f(m')$ for some $m, m' \in M$, we have $m + k = m' + k'$ for some $k, k' \in \text{Ker}(f)$;

- **$i$-normal**, if $\text{Im}(f) = \overline{f(L)} := \{ m \in M | m + l \in L \text{ for some } l \in L \}$.

- **normal**, if $f$ is both $k$-normal and $i$-normal.

**Remark 1.11.** Among others, Takahashi ([Tak1981]) and Golan [Gol1999] called $k$-normal (resp., $i$-normal, normal) $S$-linear maps $k$-regular (resp., $i$-regular, regular) morphisms. Our terminology is consistent with Category Theory noting that the **normal epimorphisms** are exactly the normal surjective $S$-linear maps, and the **normal monomorphisms** are exactly the normal injective $S$-linear maps (see [Abu2014]).
The following technical lemma is easy to prove.

**Lemma 1.12.** Let $L \overset{f}{\to} M \overset{g}{\to} N$ be a sequence of semimodules.

(1) Let $g$ be injective.

(a) $f$ is $k$-normal if and only if $g \circ f$ is $k$-normal.

(b) If $g \circ f$ is $i$-normal (normal), then $f$ is $i$-normal (normal).

(c) Assume that $g$ is $i$-normal. Then $f$ is $i$-normal (normal) if and only if $g \circ f$ is $i$-normal (normal).

(2) Let $f$ be surjective.

(a) $g$ is $i$-normal if and only if $g \circ f$ is $i$-normal.

(b) If $g \circ f$ is $k$-normal (normal), then $g$ is $k$-normal (normal).

(c) Assume that $f$ is $k$-normal. Then $g$ is $k$-normal (normal) if and only if $g \circ f$ is $k$-normal (normal).

The proof of the following lemma is straightforward:

**Lemma 1.13.** (1) Let $\{f_\lambda : L_\lambda \to M_\lambda\}_{\lambda \in \Lambda}$ be a family of left $S$-semimodule morphisms and consider the induced $S$-linear map $f : \bigoplus\limits_{\lambda \in \Lambda} L_\lambda \to \bigoplus\limits_{\lambda \in \Lambda} M_\lambda$. Then $f$ is normal (resp. $k$-normal, $i$-normal) if and only if $f_\lambda$ is normal (resp. $k$-normal, $i$-normal) for every $\lambda \in \Lambda$.

(2) A morphism $\varphi : L \to M$ of left $S$-semimodules is normal (resp. $k$-normal, $i$-normal) if and only if $\text{id}_F \otimes_S \varphi : F \otimes_S L \to F \otimes_S M$ is normal (resp. $k$-normal, $i$-normal) for every non-zero free right $S$-semimodule $F$.

(3) If $P_S$ is projective and $\varphi : L \to M$ is a normal (resp. $k$-normal, $i$-normal) morphism of left $S$-semimodules, then $\text{id}_F \otimes_S \varphi : P \otimes_S L \to P \otimes_S M$ is normal (resp. $k$-normal, $i$-normal).

There are several notions of exactness for sequences of semimodules. In this paper, we use the relatively new notion introduced by Abuhlail:

**Definition 1.14.** ([Abu2014, 2.4]) A sequence

$$ L \overset{f}{\to} M \overset{g}{\to} N $$

(2) of left $S$-semimodules is **exact**, if $g$ is $k$-normal and $f(L) = \text{Ker}(g)$.

**1.15.** We call a sequence of $S$-semimodules $L \overset{f}{\to} M \overset{g}{\to} N$

- **proper-exact** if $f(L) = \text{Ker}(g)$ (exact in the sense of Patchkoria [Pat2003]);
- **semi-exact** if $f(L) = \text{Ker}(g)$ (exact in the sense of Takahashi [Tak1982a]).
1.16. We call a (possibly infinite) sequence of $S$-semimodules
\[ \cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \to \cdots \] (3)

chain complex if $f_{j+1} \circ f_j = 0$ for every $j$;
exact (resp., proper-exact, semi-exact, quasi-exact) if each partial sequence with three terms $M_j \xrightarrow{f_j} M_{j+1} \xrightarrow{f_{j+1}} M_{j+2}$ is exact (resp., proper-exact, semi-exact, quasi-exact).

A short exact sequence (or a Takahashi extension [Tak1982b]) of $S$-semimodules is an exact sequence of the form
\[ 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \]

The following examples show some of the advantages of the new definition of exact sequences over the old ones:

**Lemma 1.17.** Let $L, M$ and $N$ be $S$-semimodules.

(1) $0 \to L \xrightarrow{f} M$ is exact if and only if $f$ is injective.

(2) $M \xrightarrow{g} N \to 0$ is exact if and only if $g$ is surjective.

(3) $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ is proper-exact and $f$ is normal (semi-exact and $f$ is normal) if and only if $L \simeq \text{Ker}(g)$.

(4) $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ is exact if and only if $L \simeq \text{Ker}(g)$ and $g$ is $k$-normal.

(5) $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is semi-exact and $g$ is normal if and only if $N \simeq M/f(L)$.

(6) $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact if and only if $N \simeq M/f(L)$ and $f$ is $i$-normal.

(7) $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact if and only if $L \simeq \text{Ker}(g)$ and $N \simeq M/L$.

**Corollary 1.18.** The following assertions are equivalent:

(1) $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an exact sequence of $S$-semimodules;

(2) $L \simeq \text{Ker}(g)$ and $N \simeq M/f(L)$;

(3) $f$ is injective, $f(L) = \text{Ker}(g)$, $g$ is surjective and (k-)normal.

In this case, $f$ and $g$ are normal morphisms.

**Remark 1.19.** An $S$-linear map is a monomorphism if and only if it is injective. Every surjective $S$-linear map is an epimorphism. The converse is not true in general.
Lemma 1.20. Let

\[
\begin{array}{c}
A' \xrightarrow{i} A \xrightarrow{p} A'' \\
\downarrow f \quad \downarrow g \quad \downarrow h \\
B' \xrightarrow{j} B \xrightarrow{q} B''
\end{array}
\]

be a commutative diagram of left S-semimodules and semi-exact rows. If \(p\) is normal epimorphism, then there exists a unique S-linear map \(h : A'' \to B''\) making the augmented diagram commute.

1. If, moreover, \(q\) is a normal epimorphism, \(f\) is surjective and \(g\) is injective (an isomorphism), then \(h\) is injective (an isomorphism).

2. If, moreover, \(A\) and \(B\) are cancellative, \(j\), \(f\) and \(h\) are injective, then \(g\) is injective.

Proof. Since \(p\) is normal, \(A'' \cong Cok(i)\) by Lemma (1.17) (5). Since \((q \circ g) \circ i = q \circ j \circ i = 0\), the existence and uniqueness of \(h\) follows directly from the Universal Property of Cokernels. However, we give an elementary proof that \(h\) is well-defined using diagram chasing. Let \(a'' \in A''\). Since \(p\) is surjective, exists \(a \in A\) such that \(p(a) = a''\). Consider

\[h : A'' \to B'', \; a'' \mapsto q(g(a)).\]

Claim: \(h\) is well defined.

Suppose there exist \(p(a_1) = a'' = p(a_2)\). Since \(p\) is \(k\)-normal, \(a_1 + k_1 = a_2 + k_2\) for some \(k_1, k_2 \in Ker(p) = \text{Im}(i)\). Let \(a'_1, \tilde{a}_1, a'_2, \tilde{a}_2 \in A'\) be such that \(k_1 + i(a'_1) = i(\tilde{a}_1)\) and \(k_2 + i(a'_2) = i(\tilde{a}_2)\). It follows that

\[a_1 + k_1 + i(a'_1) = a_1 + i(\tilde{a}_1)\quad \text{and} \quad a_2 + k_2 + i(a'_2) = a_2 + i(\tilde{a}_2)\]

and so

\[(q \circ g \circ i)(\tilde{a}_1) = (q \circ j \circ f)(\tilde{a}_1) = 0 = (q \circ j \circ f)(\tilde{a}_2) = (q \circ g \circ i)(\tilde{a}_2).\]

So

\[
(q \circ g)(a_1) = (q \circ g)(a_1) + (q \circ j \circ f)(\tilde{a}_1) \\
= (q \circ g)(a_1) + (q \circ g \circ i)(\tilde{a}_1) \\
= (q \circ g)(a_1 + i(\tilde{a}_1)) \\
= (q \circ g)(a_1 + k_1 + i(a'_1)) \\
= (q \circ g)(a_1 + k_1) + (q \circ j \circ f)(a'_1) \\
= (q \circ g)(a_2 + k_2) \\
= (q \circ g)(a_2 + k_2) + (q \circ j \circ f)(a'_2) \\
= (q \circ g)(a_2 + k_2 + i(a'_2)) \\
= (q \circ g)(a_2 + i(\tilde{a}_2)) \\
= (q \circ g)(a_2) + (q \circ j \circ f)(\tilde{a}_2) \\
= (q \circ g)(a_2).
\]

Thus \(h\) is well defined and \(h \circ p = q \circ g\) by the definition of \(h\). Clearly, \(h\) is unique.
(1) Suppose that $h(x_1) = h(x_2)$ for some $x_1, x_2 \in A''$. Since $p$ is surjective, $x_1 = p(a_1)$ and $x_2 = p(a_2)$ for some $a_1, a_2 \in A$. So,

$$q(g(a_1)) = h(p(a_1)) = h(x_1) = h(x_2) = h(p(a_2)) = q(g(a_2)).$$

Since the second row is semi-exact, there exists $y_1, y_2 \in \text{Im}(j)$ such that $g(a_1) + y_1 = g(a_2) + y_2$. Let $z_1, \bar{z}_1, z_2, \bar{z}_2 \in B'$ be such that $y_1 + j(z_1) = j(\bar{z}_1)$ and $y_2 + j(z_2) = j(\bar{z}_2)$. It follows that

$$g(a_1) + y_1 + j(z_1) + j(z_2) = g(a_2) + y_2 + j(z_2) + j(z_1)$$

$$g(a_1) + j(\bar{z}_1) + j(z_2) = g(a_2) + j(\bar{z}_2) + j(z_1)$$

$$g(a_1) + j(\bar{z}_1 + z_2) = g(a_2) + j(\bar{z}_2 + z_1)$$

Since $f$ is surjective, there exists $w_1, w_2 \in A'$ such that $f(w_1) = \bar{z}_1 + z_2$ and $f(w_2) = \bar{z}_2 + z_1$. So, we have

$$g(a_1 + i(w_1)) = g(a_1) + (g \circ i)(w_1) = g(a_1) + (j \circ f)(w_1)$$

$$= g(a_1) + j(\tilde{z}_1 + z_2) = g(a_2) + j(\tilde{z}_2 + z_1)$$

$$= g(a_2) + (g \circ i)(w_2) = g(a_2 + i(w_2))$$

Since $g$ is injective, we have $a_1 + i(w_1) = a_2 + i(w_2)$, whence

$$x_1 = p(a_1) = p(a_1 + i(w_1)) = p(a_2 + i(w_2)) = p(a_2) = x_2.$$

It follows that $h$ is injective. If $g$ is surjective, then $h \circ p = q \circ g$ is surjective, whence $h$ is surjective.

(2) Suppose that $g(a_1) = g(a_2)$ for some $a_1, a_2 \in A$. It follows that

$$(h \circ p)(a_1) = (q \circ g)(a_1) = (q \circ g)(a_2) = (h \circ p)(a_2),$$

whence $p(a_1) = p(a_2)$ ($h$ is injective, by assumption). Since the first row is semi-exact, there exist $y_1, y_2 \in \text{Im}(i) = \text{Ker}(p)$ such that $a_1 + y_1 = a_2 + y_2$. Let $w_1, \tilde{w}_1, w_2, \tilde{w}_2 \in A'$ be such that $y_1 + i(w_1) = i(\tilde{w}_1)$ and $y_2 + i(w_2) = i(\tilde{w}_2)$. It follows that

$$a_1 + y_1 + i(w_1) = a_1 + i(\tilde{w}_1)$$

and

$$a_2 + y_2 + i(w_2) = a_2 + i(\tilde{w}_2).$$

Consequently, we have

$$a_1 + y_1 + i(w_1) + i(w_2) = a_2 + y_2 + i(w_2) + i(w_1)$$

$$a_1 + i(\tilde{w}_1) + i(w_2) = a_2 + i(\tilde{w}_2) + i(w_1)$$

$$a_1 + i(\tilde{w}_1 + w_2)) = a_2 + i(\tilde{w}_2 + w_1)$$

It follows that

$$g(a_1) + (g \circ i)(\tilde{w}_1 + w_2) = g(a_2) + (g \circ i)(\tilde{w}_2 + w_1)$$

$$g(a_1) + (j \circ f)(\tilde{w}_1 + w_2) = g(a_2) + (j \circ f)(\tilde{w}_2 + w_1).$$

Since $B$ is cancellative and both $f$ and $j$ are injective, we conclude that $\tilde{w}_1 + w_2 = \tilde{w}_2 + w_1$. Since $A$ is cancellative, we conclude $a_1 = a_2$.
Lemma 1.21. Consider an exact sequence

\[ 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \]

of left $S$-semimodules. If $U \leq_S N$ is a (subtractive) subsemimodule, then in the pullback $(P, t', g')$ of $t : U \leftrightarrow N$ and $g : M \to N$ the $S$-linear map $g' : P \to M$ is a (normal) monomorphism.

**Proof.** Let $U \leq_S N$. By [Tak1982b, 1.7], the pullback of $t : U \to N$ and $g : M \to N$ is of $(P, t', g')$, where

\[ P : = \{ (u, m) \in U \times M \mid t(u) = g(m) \} \]

\[ t' : P \to U, \ (u, m) \mapsto u; \]

\[ g' : P \to M, \ (u, m) \mapsto m. \]

Consider the following diagram of left $S$-semimodules

\[ \begin{array}{ccc}
0 & \to & 0 \\
0 \downarrow & & \downarrow 0 \\
L \to P & \xrightarrow{f'} & U \to 0 \\
\downarrow \text{id} & & \downarrow 1 \\
0 \to M & \xrightarrow{g} & N \to 0 \\
\end{array} \]

**Claim I:** $g'$ is injective.

Suppose that $g'(u_1, m_1) = g'(u_2, m_2)$ for some $u_1, u_2 \in U$ and $m_1, m_2 \in M$. Then $m_1 = m_2$ and moreover,

\[ t(u_1) = g(m_1) = g(m_2) = t(u_2), \]

whence $u_1 = u_2$ since $t$ is injective. Consequently, $g'$ is injective.

**Claim II:** If $U \leq_S N$ is subtractive, then $P \leq_S M$ is subtractive.

Let $m \in \text{Im}(g')$, i.e. $m + g'(u_1, m_1) = g'(u_2, m_2)$ for some $u_1, u_2 \in U$ and $m_1, m_2 \in M$. It follows that $m + m_1 = m_2$, whence $g(m) + g(m_1) = g(m_2)$ and $g(m) + t(u_1) = t(u_2)$. Since $t$ is a normal monomorphism, it follows that $g(m) = t(u)$ for some $u \in U$ and so $m = g'(m, u)$. Consequently, $g'(P) = \text{Im}(g')$, i.e. $g'$ is a normal monomorphism. \[ \square \]

Proposition 1.22. (cf. [Bor1994, Proposition 3.2.2]) Let $\mathcal{C}, \mathcal{D}$ be arbitrary categories and $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}$ be covariant functors such that $(F, G)$ is an adjoint pair.

1. $F$ preserves all colimits which turn out to exist in $\mathcal{C}$.
2. $G$ preserves all limits which turn out to exist in $\mathcal{D}$.

Corollary 1.23. Let $S, T$ be semirings and $\tau F_S$ a $(T, S)$-bisemimodule.

1. $F \otimes_S : S\text{SM} \to \tau \text{SM}$ preserves all colimits.
(2) For every family of left $S$-semimodules $\{X_\lambda\}_\Lambda$, we have a canonical isomorphism of left $T$-semimodules

$$F \otimes_S \bigoplus_{\lambda \in \Lambda} X_\lambda \simeq \bigoplus_{\lambda \in \Lambda} (F \otimes_S X_\lambda).$$

(3) For any directed system of left $S$-semimodules $(X_j, \{f_{jj}\})_J$, we have an isomorphism of left $T$-semimodules

$$F \otimes_S \lim_{\rightarrow} X_j \simeq \lim_{\rightarrow} (F \otimes_S X_j).$$

(4) $F \otimes_S -$ preserves coequalizers.

(5) $F \otimes_S -$ preserves cokernels.

**Proof.** The proof can be obtained as a direct consequence of Proposition 1.22 and the fact that $(F \otimes_S -, \text{Hom}_T(F, -))$ is an adjoint pair of covariant functors [KN2011].

**Proposition 1.24.** Let $\tau G S$ be a $(T, S)$-bisemimodule and consider the functor $G \otimes_S - : \text{SM} \to \tau\text{SM}$. Let

$$L \xrightarrow{f} M \xrightarrow{g} N \to 0 \quad (6)$$

be a sequence of left $S$-semimodules and consider the sequence of left $T$-semimodules

$$G \otimes_S L \xrightarrow{G \otimes f} G \otimes_S M \xrightarrow{G \otimes g} G \otimes_S N \to 0 \quad (7)$$

(1) If $M \xrightarrow{g} N \to 0$ is exact and $g$ is normal, then $G \otimes_S M \xrightarrow{G \otimes g} G \otimes_S N \to 0$ is exact and $G \otimes g$ is normal.

(2) If (6) is semi-exact and $g$ is normal, then (7) is semi-exact and $G \otimes g$ is normal.

(3) If (6) is exact and $G \otimes_S f$ is $i$-normal, then (7) is exact.

**Proof.** The following implications are obvious: $M \xrightarrow{g} N \to 0$ is exact $\implies$ $g$ is surjective $\implies$ $G \otimes g$ is surjective $\implies$ $G \otimes_S M \xrightarrow{G \otimes g} G \otimes_S N \to 0$ is exact.

(1) Assume that $g$ is normal and consider the exact sequence of $S$-semimodules

$$0 \to \text{Ker}(g) \xrightarrow{1} M \xrightarrow{g} N \to 0.$$

Then $N \simeq \text{Coker}(t)$. By Corollary 1.23 (1), $G \otimes_S -$ preserves cokernels and so $G \otimes g = \text{coker}(G \otimes t)$ whence normal.

(2) Apply Lemma 1.17: The assumptions on (6) are equivalent to $N = \text{Coker}(f)$. Since $G \otimes_S -$ preserves cokernels, we conclude that $G \otimes_S N = \text{Coker}(G \otimes f)$, i.e. (7) is semi-exact and $G \otimes g$ is normal.

(3) This follows directly from (2) and the assumption on $G \otimes f$. ■
2 Flat Semimodules

The notion of *exactly flat semimodules* was introduced by Abuhlail [Abu2014-SF, 3.3] where it was called *normally flat*. The terminology *e-flat* was first used in [AIKN2018].

2.1. Let \( F_S \) be a right \( S \)-semimodule. Following Abuhlail [Abu2014-SF], we say that \( F_S \) is a *flat* right \( S \)-semimodule, if \( F \) is the directed colimit of *finitely presented* projective right \( S \)-semimodules.

2.2. Let \( M \) be a left \( S \)-semimodule. A right \( S \)-semimodule \( F \) is called

- **normally \( M \)-flat**, if for every *subtractive* \( S \)-subsemimodule \( L \leq_S M \), we have a *subtractive* submonoid \( F \otimes_S L \leq F \otimes_S M \);

- **\( M \)-i-flat**, if for every *subtractive* \( S \)-subsemimodule \( L \leq_S M \), we have a submonoid \( F \otimes_S L \leq F \otimes_S M \).

- **\( M \)-m-flat**, if for every \( S \)-subsemimodule \( L \leq_S M \), we have a submonoid \( F \otimes_S L \leq F \otimes_S M \).

We say that \( F_S \) is *normally flat* (resp., *i-flat*, *m-flat*), iff \( F \) is normally \( M \)-flat (resp., \( M \)-i-flat, \( M \)-m-flat) for every left \( S \)-semimodule \( M \).

**Definition 2.3.** A right \( S \)-semimodule \( F \) is called *\( M \)-e-flat*, where \( M \) is a left \( S \)-semimodule, iff for every short exact sequence

\[
0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0
\]  

\( 8 \)

of left \( S \)-semimodules the induced sequence

\[
0 \rightarrow F \otimes_S L \xrightarrow{F \otimes f} F \otimes_S M \xrightarrow{F \otimes g} F \otimes_S N \rightarrow 0
\]  

\( 9 \)

of commutative monoids is exact. We say that \( F_S \) is *\( e \)-flat*, iff the covariant functor \( F \otimes_S - : SSM \rightarrow Z_{++}SM \) repasts short exact sequences.

**Remark 2.4.** The prefix in “\( m \)-flat” stems from **mono-flat semimodules** introduced by Katsov [Kat2004], and is different from that of *\( k \)-flat semimodules* in the sense of Al-Thani [Alt2004], since the tensor *product* we adopt here is in the sense of Katsov which is different from that in the sense of Al-Thani (see [Abu2013] for more details).

**Proposition 2.5.** Let \( F \) be a right \( S \)-semimodule and \( M \) a left \( S \)-semimodule.

1. \( F_S \) is normally \( M \)-flat if and only if for every short exact sequence of the form \( 8 \), Sequence \( 9 \) is exact.

2. \( F_S \) is \( M \)-i-flat if and only if for every short exact sequence of the form \( 8 \), Sequence \( 9 \) is semi-exact, \( F \otimes f \) is \( k \)-normal and \( F \otimes g \) is normal.

3. \( F_S \) is \( M \)-m-flat if and only if for every semi-exact sequence of the form \( 8 \) in which \( f \) is \( k \)-normal and \( g \) is normal, Sequence \( 9 \) is semi-exact, \( F \otimes f \) is \( k \)-normal and \( F \otimes g \) is normal.
Proof. (1) and (2).

(\Rightarrow) Notice that \( L = \text{Ker}(g) \) is a subtractive \( S \)-subsemimodule of \( M \). Since \( F_S \) is \( M \)-i-flat (normally \( M \)-flat), we know that \( F \otimes f \) is a (normal) monomorphism. It follows by Proposition 1.24 (2) and (3) that (9) is semi-exact (exact) and \( F \otimes g \) is a normal epimorphism.

(\Leftarrow) Let \( L \leq S M \) be a subtractive \( S \)-subsemimodule. Then

\[
0 \rightarrow L \xrightarrow{\iota} M \xrightarrow{\pi_L} M/L \rightarrow 0
\]  

is a short exact sequence of left \( S \)-semimodules, where \( \iota \) is the canonical injection and \( \pi_L : M \rightarrow M/L \) is the canonical projection. By our assumptions, the induced sequence of commutative monoids

\[
0 \rightarrow F \otimes_S L \xrightarrow{F \otimes \iota} F \otimes_S M \xrightarrow{F \otimes \pi_L} F \otimes_S M/L \rightarrow 0
\]

is semi-exact (exact) and \( F \otimes \iota \) is \( k \)-normal, whence a (normal) monomorphism.

(3) (\Rightarrow) Since \( F_S \) is \( M \)-m-flat, we know that \( F \otimes f \) is a monomorphism, whence \( k \)-normal. Moreover, it follows by Proposition 1.24 (2) that (9) is semi-exact and \( F \otimes g \) is a normal epimorphism.

(\Leftarrow) Let \( L \leq S M \) be an \( S \)-subsemimodule. Then (10) is a semi-exact sequence of left \( S \)-semimodules in which \( \iota \) is \( k \)-normal and \( \pi_L \) is normal. By our assumption, Sequence (11) is semi-exact and \( F \otimes \iota \) is \( k \)-normal, whence \( F \otimes f \) is injective. \( \blacksquare \)

Proposition 2.6. Let

\[
L \xrightarrow{f} M \xrightarrow{g} N
\]

be a sequence of left \( S \)-semimodules, \( F \) a right \( S \)-semimodule and consider the sequence

\[
F \otimes_S L \xrightarrow{F \otimes f} F \otimes_S M \xrightarrow{F \otimes g} F \otimes_S N
\]  

of commutative monoids.

(1) If (12) is exact with \( g \) normal and \( F_S \) is e-flat, then (13) is exact and \( F \otimes g \) is normal.

(2) If (12) is exact with \( g \) normal and \( F_S \) is i-flat, then (13) is semi-exact and \( F \otimes g \) is \( k \)-normal.

(3) If (12) is exact and \( F_S \) is m-flat, then (13) is semi-exact and \( F \otimes g \) is \( k \)-normal.

Proof. By Corollary 1.18, we have a short exact sequence of left \( S \)-semimodules

\[
0 \rightarrow \text{Ker}(g) \xrightarrow{\iota} M \xrightarrow{\pi} M/\text{Ker}(g) \rightarrow 0
\]

where \( \iota \) and \( \pi \) are the canonical \( S \)-linear maps. Since (12) is proper exact, \( f(M) = \text{Ker}(g) \) and \( M/\text{Ker}(g) = M/f(M) \cong \text{Coker}(f) \). By the Universal Property of Kernels, there exists a unique \( S \)-linear map \( \tilde{f} : L \rightarrow \text{Ker}(g) \) such that \( \iota \circ \tilde{f} = f \). On the other hand, by the Universal Property
of Cokernels, there exists a unique $S$-linear map $\tilde{g} : M / \text{Ker}(g) \to N$ such that $\tilde{g} \circ \pi = g$. So, we have a commutative diagram of left $S$-semimodules

$$
\begin{array}{c}
0 \\
\downarrow \\
0 \\
F \otimes S \text{Ker}(g) \\
\downarrow \\
F \otimes S M \\
\downarrow \\
F \otimes S M / \text{Ker}(g) \\
\downarrow \\
0
\end{array}
$$

Applying the contravariant functor $F \otimes S -$ , we get the sequence

$$
0 \to F \otimes S \text{Ker}(g) \xrightarrow{F \otimes 1} F \otimes S M \xrightarrow{F \otimes \pi} F \otimes S M / \text{Ker}(g) \to 0
$$

and we obtain the commutative diagram

$$
\begin{array}{c}
F \otimes S \text{L} \\
\downarrow \\
F \otimes S \text{Ker}(g) \\
\downarrow \\
F \otimes S M \\
\downarrow \\
F \otimes S M / \text{Ker}(g) \\
\downarrow \\
F \otimes S N
\end{array}
$$

of commutative monoids.

Notice that $\tilde{g}$ is injective since $g$ is $k$-normal. On the other hand, $\tilde{f}$ is surjective since $f(M) = \text{Ker}(g)$.

(1) Let $F_S$ be $e$-flat and $g = \tilde{g} \circ \pi$ be normal. Then $\tilde{g}$ is a normal monomorphism by Lemma 1.12 (2-b), $F \otimes \tilde{g}$ is a normal monomorphism and Sequence (16) is exact.

**Step I:** We have

$$
\begin{align*}
\text{Ker}(F \otimes g) & = \text{Ker}((F \otimes \tilde{g}) \circ (F \otimes \pi)) \\
& = \text{Ker}(F \otimes \pi) \quad (F \otimes \tilde{g} \text{ is injective}) \\
& = \text{im}(F \otimes \iota) \quad (16) \text{ is (proper-)exact} \\
& = \text{im}((F \otimes \iota) \circ (F \otimes \tilde{f})) \quad (F \otimes \tilde{f} \text{ is surjective}) \\
& = \text{im}(F \otimes f).
\end{align*}
$$

Since $F \otimes \tilde{g}$ is injective and $F \otimes \pi$ is normal (by Proposition 1.24 (1)), it follows by Lemma 1.12 that $F \otimes g = (F \otimes \tilde{g}) \circ (F \otimes \pi)$ is $k$-normal.
Step II: Since $F_S$ be $e$-flat, $F \otimes \tilde{g}$ is a normal monomorphism. Moreover, $F \otimes \pi$ is a normal epimorphism, it follows by Lemma 1.12 (1-c or 2-c) that $F \otimes g = (F \otimes \tilde{g}) \circ (F \otimes \pi)$ is normal.

(2) Let $F_S$ be $i$-flat and $g = \tilde{g} \circ \pi$ be normal. Then $\tilde{g}$ is a normal monomorphism by Lemma 1.12 (2-b), whence $F \otimes g$ is a monomorphism. By Proposition 1.24 (2), Sequence (16) is semi-exact, whence $\text{im}(F \otimes f) = \text{Ker}(F \otimes g)$. Calculations similar to those in Step I of (1), show that $\text{im}(F \otimes f) = \text{Ker}(F \otimes g)$.

(3) Let $F_S$ be $m$-flat. Since $g$ is $k$-normal, $\tilde{g}$ is injective whence $F \otimes \tilde{g}$ is a monomorphism and, as shown in (1), $F \otimes g$ is $k$-normal. As clarified in (2), $\text{im}(F \otimes f) = \text{Ker}(F \otimes g)$.

Theorem 2.7. Let $M$ be a left $S$-Semimodule. The following are equivalent for a right $S$-semimodule $F$ :

(1) $F_S$ is normally $M$-flat;

(2) $F_S$ is $M$-e-flat;

(3) For every exact sequence of left $S$-semimodules (12) with $g$ normal, the induced sequence of commutative monoids (13) is exact and $F \otimes g$ is normal.

Proof. (1) $\iff$ (2) This follows by Proposition 2.5 (1).

(1) $\Rightarrow$ (3) This follows by Proposition 2.6 (1).

(3) $\Rightarrow$ (1) This follows directly by applying the assumption to the exact sequences of left $S$-semimodules of the form $0 \to M \to N$ with $g$ normal.

Theorem 2.8. Let $M$ be a left $S$-Semimodule. The following are equivalent for a right $S$-semimodule $F$ :

(1) $F_S$ is $M$-i-flat;

(2) For every short exact sequence (8) of left $S$-semimodules, Sequence (9) is semi-exact, $F \otimes f$ is $k$-normal and $F \otimes g$ is normal.

(3) For every exact sequence of left $S$-semimodules (12) with $g$ normal, the induced sequence of commutative monoids (13) is semi-exact and $F \otimes g$ is $k$-normal.

Proof. (1) $\iff$ (2) This follows by Proposition 2.5 (2).

(1) $\Rightarrow$ (3) This follows by Proposition 2.6 (2).

(3) $\Rightarrow$ (1) This follows directly by applying the assumption to the exact sequences of left $S$-semimodules the form $0 \to M \to N$ with $g$ normal.

Theorem 2.9. Let $M$ be a left $S$-Semimodule. The following are equivalent for a right $S$-semimodule $F$ :
(1) \( F_S \) is \( M\)-\( m\)-flat;

(2) For every semi-exact sequence of the form \((8)\) in which \( f \) is \( k\)-normal and \( g \) is normal, Sequence \((9)\) is semi-exact, \( F \otimes f \) is \( k\)-normal and \( F \otimes g \) is normal.

(3) For every exact sequence \((12)\), Sequence \((13)\) is semi-exact and \( F \otimes g \) is \( k\)-normal.

**Proof.** (1) \( \iff \) (2) This follows by Proposition 2.5 (3).

(1) \( \implies \) (3) This follows by Proposition 2.6 (3).

(3) \( \implies \) (1) This follows directly by applying the assumption to the exact sequences of left \( S \)-semimodules the form \( 0 \rightarrow M \xrightarrow{g} N \).

**Corollary 2.10.** Let \( S \) and \( T \) be semirings, \( F \) a \((T, S)\)-bisemimodule and \( \tilde{F} \) a right \( T \)-semimodule. If \( F_S \) is \( e\)-flat (\( m\)-flat) and \( \tilde{F}_T \) is \( e\)-flat (\( m\)-flat), then \( (\tilde{F} \otimes_T F)_S \) is a normally flat (\( m\)-flat).

**Proof.** Let \( F_S \) \( e\)-flat (\( m\)-flat) and \( \tilde{F}_T \) be \( e\)-flat (\( m\)-flat). By our assumptions and Proposition 2.5, the two functors

\[
F \otimes_S - : S^\tau \mathcal{M} \rightarrow \tau S^\tau \mathcal{M} \quad \text{and} \quad \tilde{F} \otimes_S - : \tau S^\tau \mathcal{M} \rightarrow Z^+ S^\tau \mathcal{M}
\]

respect short exact sequences (monomorphisms), whence the functor

\[
\tilde{F} \otimes_T F = (\tilde{F} \otimes_S -) \circ (F \otimes_S -) : S^\tau \mathcal{M} \rightarrow Z^+ S^\tau \mathcal{M}
\]

respects short exact sequences (monomorphisms). Consequently, \( (\tilde{F} \otimes_T F)_S \) is \( e\)-flat (\( m\)-flat).

**Proposition 2.11.** ([Abu2014-SF, Theorem 3.6]) Let \( S \) be any semiring. If \( F \) is a flat left (right) \( S \)-semimodule, then \( F \) is \( e\)-flat (resp. \( m\)-flat).

**Remark 2.12.** Let \( M \) be a right \( S \)-semimodule and denote with \( \mathcal{F}_S^e(M) \) (resp. \( \mathcal{F}_S^i(M) \), \( \mathcal{F}_S^m(M) \)) the class of \( M\)-\( e\)-flat (resp. \( M\)-\( i\)-flat, \( M\)-\( m\)-flat) left \( S \)-semimodules. Dropping \( M \) means that we take the union over all left \( S \)-semimodules \( M \). Moreover, we denote with \( \mathcal{F}_S \) the class of all flat right \( S \)-semimodules. It follows directly from the definitions that \( M\)-\( e\)-flat and \( M\)-\( m\)-flat right semimodules are \( M\)-\( i\)-flat. Moreover, flat semimodules are \( m\)-flat by [Kat2004, Proposition 2.1] \( e\)-flat by Proposition 2.11. Summarizing, we have the following inclusions:

\[
\mathcal{F}_S^e(M) \cup \mathcal{F}_S^m(M) \subseteq \mathcal{F}_S^i(M) \quad \text{and} \quad \mathcal{F}_S \subseteq \mathcal{F}_S^e \cap \mathcal{F}_S^m \subseteq \mathcal{F}_S^i.
\]

(18)

**Lemma 2.13.** (1) Let \( M \) be a left \( S \)-semimodule. Any retract of an \( M\)-\( i\)-flat (resp. \( M\)-\( e\)-flat, \( m\)-flat) right \( S \)-semimodule is \( M\)-\( i\)-flat (resp. \( M\)-\( e\)-flat, \( m\)-flat).

(2) Any retract of an \( i\)-flat (resp. \( e\)-flat, \( m\)-flat) right \( S \)-semimodule is \( i\)-flat (resp. \( e\)-flat, \( m\)-flat).

**Proof.** We only need to prove “1” for relative \( i\)-flatness (resp. relative \( e\)-flatness); the proof for relative \( m\)-flatness is similar.
Let $M$ be a left $S$-semimodule, $U \leq_S M$ a subtractive subsemimodule, $F_S$ an $M$-e-flat right $S$-semimodule and $\tilde{F}$ a retract of $F$. Then there exist $S$-linear maps $\tilde{F} \xrightarrow{\psi} F \xrightarrow{\theta} \tilde{F}$ such that $\theta \circ \psi = \text{id}_{\tilde{F}}$. Consider the commutative diagram

\begin{equation*}
\begin{array}{c}
0 \to \tilde{F} \otimes_S U \to \tilde{F} \otimes M \\
\downarrow \psi \otimes U \quad \quad \downarrow \psi \otimes M \\
0 \to F \otimes_S U \to F \otimes S M \\
\downarrow \theta \otimes U \quad \quad \downarrow \theta \otimes M \\
\tilde{F} \otimes_S U \to \tilde{F} \otimes S M
\end{array}
\end{equation*}

Indeed, $(\theta \otimes_S \text{id}_U) \circ (\psi \otimes_S \text{id}_U) = \text{id}_{\tilde{F} \otimes_S U}$ and $(\theta \otimes_S \text{id}_M) \circ (\psi \otimes_S \text{id}_M) = \text{id}_{\tilde{F} \otimes S M}$; i.e. $\tilde{F} \otimes S U$ is a retract of $F \otimes S U$ and $\tilde{F} \otimes S M$ is a retract of $F \otimes S M$. In particular, $\psi \otimes U$ and $\psi \otimes M$ are monomorphisms.

If $F_S$ is $M$-i-flat (resp. $M$-e-flat), then $\text{id}_{\tilde{F}} \otimes_S \iota_U : F \otimes S U \to F \otimes S M$ is (normal) monomorphism. It follows that $\text{id}_{\tilde{F}} \otimes_S \iota_U$ is a (normal) monomorphism by Lemma 1.12 “1”, i.e. $\tilde{F} \otimes S U \leq_S F \otimes S M$ is a (subtractive) $S$-semimodule. Consequently, $\tilde{F}$ is $M$-i-flat ($M$-e-flat).

**Proposition 2.14.** Let $\{F_\lambda\}_\Lambda$ be a family of right $S$-semimodules.

1. Let $M$ be a left $S$-semimodule. Then $\bigoplus_{\lambda \in \Lambda} F_\lambda$ is $M$-i-flat (resp. $M$-e-flat, $M$-m-flat) if and only if $F_\lambda$ is $M$-i-flat (resp. $M$-e-flat, $M$-m-flat) for every $\lambda \in \Lambda$.

2. $\bigoplus_{\lambda \in \Lambda} F_\lambda$ is $i$-flat (resp. $e$-flat, $m$-flat) if and only if $F_\lambda$ is $i$-flat (resp. $e$-flat, $m$-flat) for every $\lambda \in \Lambda$.

**Proof.** We only need to prove “1” for relative $i$-flatness (resp. relative $e$-flatness); the proof for relative $m$-flatness is similar (cf. [Alt2004, Proposition 2.3] for $k$-flat semimodules).

(\(\Longrightarrow\)) For every $\lambda \in \Lambda$, $F_\lambda$ is a retract of $\bigoplus_{\lambda \in \Lambda} F_\lambda$, whence $M$-i-flat (resp. $M$-e-flat, $M$-m-flat) by Lemma 2.13.

(\(\Longleftarrow\)) Let $F := \bigoplus_{\lambda \in \Lambda} F_\lambda$ and consider the projections $\pi_\lambda : F \to F_\lambda$, $(f_\lambda)_\Lambda \mapsto f_\lambda$ for $\lambda \in \Lambda$. Let $U \leq_S M$ be a subtractive $S$-subsemimodule. Assume that $F_\lambda$ is $M$-i-flat ($M$-e-flat) for every $\lambda \in \Lambda$. Then $F_\lambda \otimes_S U \leq_S F_\lambda \otimes_S M$ is a (subtractive) subsemimodule for every $\lambda \in \Lambda$, whence $\bigoplus_{\lambda \in \Lambda} (F_\lambda \otimes_S U) \leq_S \bigoplus_{\lambda \in \Lambda} (F_\lambda \otimes_S M)$ is a (subtractive) subsemimodule by Lemma 1.13 (1).

Since $\bigoplus_{\lambda \in \Lambda} (F_\lambda \otimes_S U) \simeq \bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S U$ and $\bigoplus_{\lambda \in \Lambda} (F_\lambda \otimes_S M) \simeq \bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S M$, we conclude that $\bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S U \leq_S \bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S M$ is a (subtractive) subsemimodule. It follows that $\bigoplus_{\lambda \in \Lambda} F_\lambda$ is $M$-i-flat ($M$-e-flat).
Proposition 2.15. Let $F$ be a right $S$-semimodule and 

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$$

be an exact sequence of left $S$-semimodules.

(1) If $F$ is $M$-i-flat (resp. $M$-e-flat, $M$-m-flat), then $F$ is $L$-i-flat (resp. $L$-e-flat, $L$-m-flat).

(2) If $F$ is $M$-m-flat (resp. $M$-i-flat), then $F$ is $N$-m-flat (resp. $N$-i-flat).

Proof. (1) Let $F_S$ be $M$-i-flat (resp. $M$-e-flat) and $U \leq_S L$ be a subtractive $S$-subsemimodule. Then $U \leq_S M$ is a subtractive $S$-semimodule (notice that $L \leq_S M$ is subtractive). Since $F$ is $M$-i-flat ($M$-e-flat), $F \otimes_S U \leq F \otimes_S M$ and $F \otimes_S L \leq F \otimes_S M$ are (subtractive) submonoids, whence $F \otimes_S U \leq F \otimes_S L$ is a (subtractive) submonoid (by Lemma 1.12 (1-b)). Consequently, $F$ is $L$-i-flat ($L$-e-flat). Similarly, one can prove that if $F_S$ is $M$-m-flat, then $F_S$ is $L$-m-flat.

(2) Consider the pullback $(P, t', g')$ of $t : U \hookrightarrow N$ and $g : M \rightarrow N$ given by Lemma 1.21. Applying $F \otimes_S -$ to Diagram (5) yields the following commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & 0 \\
F \otimes_S L & \xrightarrow{F \otimes f'} & F \otimes_S P & \xrightarrow{F \otimes t'} & F \otimes_S U & \rightarrow & 0 \\
F \otimes_S L & \xrightarrow{F \otimes g'} & F \otimes_S M & \xrightarrow{F \otimes 1} & F \otimes_S N & \rightarrow & 0
\end{array}
$$

of commutative monoids. Assume that $F_S$ is $M$-m-flat (resp. $M$-i-flat), so that $F \otimes g'$ is injective. Since the rows are semi-exact, $F \otimes t'$ is surjective and $F \otimes g$ is a normal epimorphism (by Proposition 1.24), it follows by Lemma 1.20 (1) that $F \otimes t$ is injective.

Lemma 2.16. Let $F$ be a right $S$-semimodule.

(1) Let $M$ be a left $S$-semimodule. Then $F$ is $M$-m-flat if and only if for every finitely generated $S$-subsemimodule and exact sequence $0 \rightarrow K \xrightarrow{1_K} M$ the sequence of commutative monoid

$$0 \rightarrow F \otimes_S K \xrightarrow{F \otimes 1_K} F \otimes_S M$$

is exact.

(2) Let $L$ and $N$ be cancellative left $S$-semimodules. If $F$ is cancellative, $L$-m-flat and $N$-m-flat, then $F$ is $L \oplus N$-m-flat.

(3) Let $\{M_\lambda\}_\Lambda$ be a collection of cancellative left $S$-semimodules. If $F$ is cancellative and $M_\lambda$-$m$-flat for every $\lambda \in \Lambda$, then $F$ is $\bigoplus_{\lambda \in \Lambda} M_\lambda$-$m$-flat.
Proof. (1) (\(\Longrightarrow\)) Obvious.

(\(\Longleftarrow\)) Let \(U \leq_S M\). Suppose that

\[
(F \otimes t_U) \left( \sum_{i=1}^{m} f_i \otimes_S u_i \right) = (F \otimes t_U) \left( \sum_{j=1}^{n} \tilde{f}_j \otimes_S \tilde{u}_j \right),
\]

and consider \(K \leq_S M\) generated by \(\{u_1, \ldots, u_m, \tilde{u}_1, \ldots, \tilde{u}_n\}\). Notice that

\[
(F \otimes t_K) \left( \sum_{i=1}^{m} f_i \otimes_S u_i \right) = (F \otimes t_U) \left( \sum_{i=1}^{m} f_i \otimes_S u_i \right) = (F \otimes t_U) \left( \sum_{j=1}^{n} \tilde{f}_j \otimes_S \tilde{u}_j \right) = (F \otimes t_K) \left( \sum_{j=1}^{n} \tilde{f}_j \otimes_S \tilde{u}_j \right),
\]

whence \(\sum_{i=1}^{m} f_i \otimes_S u_i = \sum_{j=1}^{n} \tilde{f}_j \otimes_S \tilde{u}_j\) since \(F \otimes t_K\) is injective. It follows that \(F \otimes t_U\) is injective. Consequently, \(F\) is \(M\)-\(m\)-flat.

(2) Let \(U \leq_S L \oplus N\) and consider the short exact sequence

\[
0 \rightarrow L \xrightarrow{i} L \oplus N \xrightarrow{\pi} N \rightarrow 0
\]

of cancellative left \(S\)-semimodules. Consider the pullback \((P, \lambda', t')\) of \(\lambda : U \xrightarrow{} L \oplus N\) and \(t : L \xleftarrow{} L \oplus N\) given by

\[
\begin{align*}
P &= \{(u, l) \in U \times L \mid \lambda(u) = \pi(l)\}; \\
\lambda' : P &\xrightarrow{} U, (u, l) \mapsto u; \\
t' : P &\xrightarrow{} L, (u, l) \mapsto l
\end{align*}
\]

and the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{} & P \\
\downarrow \lambda' & \quad & \downarrow \pi' \\
0 & \xrightarrow{} & U \\
\downarrow \lambda & \quad & \downarrow \pi \\
0 & \xrightarrow{} & U/P
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{} & L \\
\downarrow t' & \quad & \downarrow \lambda \\
0 & \xrightarrow{} & L \oplus N \\
\downarrow \pi & \quad & \downarrow \pi \\
0 & \xrightarrow{} & N
\end{array}
\]

of cancellative \(S\)-semimodules. Applying \(F \otimes_S -\) to Diagram \((20)\) yields the following
of cancellative commutative monoids in which the second row is exact. By Proposition 1.24, the first row is semi-exact and \( F \otimes \pi' \) is a normal epimorphism. Since \( F \) is \( L \)-flat and \( N \)-flat, both \( F \otimes t' \) and \( F \otimes h \) are injective. It follows by Lemma 1.20 (2) that \( F \otimes \lambda \) is injective. Consequently, \( F \) is \( L \oplus N \)-flat.

(3) Let \( U \leq_S \bigoplus_{\lambda \in \Lambda} M_{\lambda} \). In light of (1), we can assume that \( U \) is finitely generated, whence contained in a finite number of direct sums. So, we are done by (2).

3 Von Neumann Regular Rings

In this section, we study the so called von Neumann regular semirings that are not necessarily rings.

Definition 3.1. A semiring \( S \) is a von Neumann regular semiring if for every \( a \in S \) there exists some \( s \in S \) such that \( a = asa \).

Assuming all semimodules of a given commutative semiring \( S \) to be (mono-)flat forces the semiring to be a von Neumann regular ring (cf., [Kat2004, Theorem 2.11]. This suggests other notions of flatness, e.g. \( e \)-flatness and \( i \)-flatness.

Definition 3.2. [Gol1999, page 71] Let \( S \) be a semiring. We say that \( S \) is a left subtractive semiring (right subtractive semiring) if every left (right) ideal of \( S \) is subtractive. We say that \( S \) is a subtractive semiring if \( S \) is both left and right subtractive.

Remark 3.3. Whether a left subtractive semiring is necessarily right subtractive was an open problem till a counterexample was given in [KNT2011, Fact 2.1].

Homological Lemmata

The proofs of the following lemmata are adapted by diagram chasing, with appropriate modifications, which is a well-known tool in the classical proofs which can be found in standard book of Homological Algebra (cf., [Rot2009, Proposition 2.70, Corollary 3.59, Proposition 3.60]).
Lemma 3.4. Let $A$ be a right $S$-semimodule and consider for every left ideal $I \leq S$ the canonical surjective map of commutative monoids

$$\theta_I : A \otimes_S I \to AI, \ a \otimes_S i \mapsto ai.$$  \hfill (22)

(1) $A_S$ is $S$-$m$-flat if and only if $A \otimes_S I \cong AI$ for every (finitely generated) left ideal $I$ of $S$.

(2) $A_S$ is $S$-$i$-flat if and only if $A \otimes_S I \cong AI$ for every subtractive left ideal $I$ of $S$.

(3) $A_S$ is $S$-$e$-flat if and only if $A \otimes_S I \cong AI$ and $AI \leq A$ is subtractive for every subtractive left ideal $I$ of $S$.

Proof. Let $I$ be a left ideal of $S$. Consider the embeddings $t_I : I \hookrightarrow S$, $\zeta_I : AI \hookrightarrow A$ and recall the canonical isomorphism $A \otimes_S S \cong A$ (Lemma 1.9). Then we have the commutative diagram of commutative monoids

$$\begin{array}{c}
A \otimes_S I & \xrightarrow{A \otimes t_I} & A \otimes_S S \\
\theta_I \downarrow & & \varphi_A \downarrow \\
0 & \xrightarrow{0} & AI \\
& \xrightarrow{\zeta_I} & A
\end{array}$$

The result follows now directly from the definitions noticing that for every left ideal of $A$ we have $A \otimes t_I$ is injective if and only if $\theta_I$ is injective. In light of Lemma 2.16, it is sufficient in (1) to consider only the finitely generated left ideals of $S$.  

Remark 3.5. Part ($\Rightarrow$) of (3) in Lemma 3.4 was proved in [NG2019, Theorem 2, Corollary 1] assuming the semiring $S$ is commutative. Notice that (2) provides a complete characterization of right $S$-semimodules $A_S$ for which $A \otimes_S I \cong AI$ is an isomorphism for every subtractive left ideal $I$ of $S$.

Definition 3.6. We say that a left $S$-semimodule $M$ is normally $S$-generated, if there exists a normal epimorphism $S^{(A)} \xrightarrow{\pi} M \to 0$. We say that $S$ is a normal generator iff every left $S$-semimodule is normally $S$-generated.

Proposition 3.7. Let $S$ be a cancellative semiring and $F$ a cancellative right $S$-semimodule. The following are equivalent:

(1) $F_S$ is $S$-$m$-flat;

(2) The canonical map $\theta_I : F \otimes_S I \to FI$ of commutative monoids is injective, whence an isomorphism, for every (finitely generated) left ideal $I$ of $S$;

(3) $F_S$ is $N$-$m$-flat for every normally $S$-generated left $S$-semimodule $N$.  

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Proof. The equivalence (1) $\iff$ (2) follows by Lemma 3.4 (without assuming that $S$ is cancellative). The implication (3) $\Rightarrow$ (1) is trivial. Assume (1). Let $N$ be normally $S$-generated so that there exists a normal epimorphism $\pi : S^{(\Lambda)} \to N$ for some index set $\Lambda$. Consider the short exact sequence

$$0 \to \text{Ker}(\pi) \to S^{(\Lambda)} \xrightarrow{\pi} N \to 0.$$ 

Since $F_S$ is $S$-$m$-flat by (1), it follows by Lemma 2.16 that $F_S$ is $S^{(\Lambda)}$-$m$-flat. Then $F_S$ is $N$-flat by Proposition 2.15. ■

The assumptions of the following result hold in particular when $S$ is a ring, whence it recovers the classical result (e.g., [Wis1991, 12.6]).

Corollary 3.8. Let $S$ be a cancellative semiring such that $S$ is a normal generator. A cancellative right $S$-semimodule $F$ is $m$-flat if and only if $F_S$ $S$-$m$-flat.

Lemma 3.9. Let $F$ be a $m$-flat ($i$-$S$-flat) right $S$-semimodule and $K \hookrightarrow F$ a subtractive $S$-subsemimodule.

1. If $F/K$ is $m$-flat ($S$-$i$-flat) and $KI \leq S$ $K$ is subtractive, then $K \cap FI = KI$ for every (subtractive) ideal $I$ of $S$.

2. If $K \cap FI = KI$ for every finitely generated left ideal of $S$, then $F$ is $S$-$m$-flat.

3. If $K \cap FI = KI$ (and $FI \leq F$ is subtractive) for every subtractive ideal $I$ of $S$, then $F/K$ is $S$-$i$-flat ($S$-$e$-flat).

Proof. Consider the right $S$-semimodule $A := F/K$ and recall, by Lemma 1.17 (7), that we have a short exact sequence of right $S$-semimodules

$$0 \to K \xrightarrow{i} F \xrightarrow{\phi} A \to 0. \quad (23)$$

Let $I \leq S$ $S$ be an arbitrary (subtractive) left ideal. Applying $- \otimes_S I$ to the exact sequence (23), it follows by Lemma 1.24 (3) that the following sequence

$$K \otimes_S I \xrightarrow{i \otimes I} F \otimes_S I \xrightarrow{\phi \otimes I} A \otimes_S I \to 0$$

of commutative monoids is semi-exact and $\phi \otimes I$ is a normal epimorphism. Consider the following commutative diagram

$$
\begin{array}{ccc}
K \otimes_S I & \xrightarrow{i \otimes I} & F \otimes_S I \\
\downarrow \theta_K & & \downarrow \theta_F \\
K I & \xrightarrow{i'} & F I \\
\end{array}
\begin{array}{ccc}
& & \xrightarrow{\phi \otimes I} \\
& & \downarrow \gamma \\
& & A \otimes_S I \\
\end{array}
\begin{array}{ccc}
& & \xrightarrow{i} \\
& & \downarrow \gamma \\
& & F I / K I \\
\end{array}
\begin{array}{ccc}
& & \xrightarrow{0} \\
\end{array}
\quad (24)
$$

of commutative monoids with semi-exact rows.
Notice that $\theta_F$ is injective, whence an isomorphism, since $F_S$ is $S$-$m$-flat ($S$-$i$-flat). Since $\varphi \otimes I$ and $\pi$ are normal epimorphisms, $\theta_K$ is surjective and $\theta_F$ is injective, there exists by Lemma 1.20 a unique isomorphism

$$\gamma : A \otimes_S I \longrightarrow FI/KI$$

of commutative monoids that makes Diagram (24) commute.

Since $\varphi : F \longrightarrow A$ is surjective, $\varphi(FI) = AI$. Consider the restriction $\varphi|_{FI} : FI \rightarrow AI$ and notice that $\text{Ker}(\varphi|_{FI}) = FI \cap K$. Consider

$$\beta : AI \rightarrow FI/(FI \cap K), \; ai \mapsto [fi] \text{ where } \varphi(f) = a.$$

**Claim I:** $\beta$ is well-defined.

Suppose that $\varphi(f) = a = \varphi(f')$ for some $f, f' \in F$. Since $\varphi$ is $k$-normal, there exist $k, k' \in K$ such that $f + k = f' + k'$, whence $fi + ki = f'i + k'i$ for every $i \in I$. Since $ki, k'i \in FI \cap K$, we get $[fi] = [f'i]$. So, $\beta$ is well defined as it is well defined on a generating set of $AI$.

**Claim II:** $\beta$ is injective.

Suppose that $\beta(\sum a_{ij} j) = \beta(\sum a'_{ij} j)$ for some $\sum a_{ij} j, \sum a'_{ij} j \in AI$. Then $[\sum f_{ij} j] = [\sum f'_{ij} j]$ for some $f_{ij}, f'_{ij} \in F$ satisfying $\varphi(f_{ij}) = a_{ij}$ and $\varphi(f'_{ij}) = a'_{ij}$. It follows that $\sum f_{ij} j + z = \sum f'_{ij} j + z'$ for some $z, z' \in FI \cap K$ and so

$$\sum a_{ij} j = \sum \varphi(f_{ij}) i_{ij} = \varphi(\sum f_{ij} j + z) = \varphi(\sum f'_{ij} j + z') = \sum a'_{ij} j.$$

Consider the commutative diagram

$$\begin{array}{ccc}
A \otimes_S I & \xrightarrow{\gamma} & FI/KI \\
\downarrow{\theta_A} & & \downarrow{\sigma} \\
0 & \rightarrow & AI \\
\downarrow{\beta} & & \downarrow{\beta} \\
0 & \rightarrow & FI/(FI \cap K)
\end{array}$$

where

$$\sigma : FI/KI \longrightarrow FI/(FI \cap K), \; [fi]_{KI} \mapsto [fi]_{FI \cap K}.$$

Since $\gamma$ is an isomorphism, we conclude that $\sigma$ is injective (whence an isomorphism) if and only if $\theta_A$ is injective (an isomorphism).
Let $A$ be $S$-flat ($S$-i-flat). In this case, $\theta_A$ is an isomorphism for every (subtractive) left ideal $I \leq_S S$ by Lemma 3.4 and it follows that $\sigma$ is injective. In particular, $(FI \cap K) / KI = Ker(\sigma) = 0$. Since $KI \leq_S K$ is subtractive (by assumption), we conclude that $KI = FI \cap K$.

(2) If $FI \cap K = KI$ for any finitely generated ideal $I$ of $S$, then $\sigma$ is injective, whence $\theta_A$ is injective. The result follows now by Lemma 3.4.

(3) The proof is similar to that of (2). ■

The proof of the following technical lemma is similar to that in the case of von Neumann regular rings (e.g. [Wis1991, 2.3, 3.10]).

**Lemma 3.10.** Let $S$ be a von Neumann regular semiring. The following are equivalent for a left ideal $I$ of $S$.

1. $SI$ is finitely generated;
2. $SI$ is principal;
3. $I = Se$ for some idempotent $e \in S$;
4. $I \leq_S S$ (a direct summand);
5. $S = I \oplus Se'$ for some idempotent $e' \in S$.

The assumption that all left $S$-semimodules of a (left and right) subtractive semiring are $S$-e-flat is sufficient for $S$ to be a von Neumann semiring.

**Theorem 3.11.** Let $S$ be a semiring.

1. If $S$ is subtractive and every right $S$-semimodule is $S$-e-flat, then $S$ is a von Neumann regular semiring.
2. If $S$ is von Neumann regular, then every normally $S$-generated right $S$-semimodule is $S$-m-flat.

**Proof.** (1) Let $a \in S$. By our assumption, $S$ is right subtractive, whence the right $S$-semimodule $K : = aS$ is a subtractive right ideal of $S$ and

$$0 \rightarrow aS \rightarrow S \rightarrow S/aS \rightarrow 0$$

is an exact sequence of right $S$-semimodules by Lemma 1.17 (7). Indeed, $F := S_S$ is $(S)$-e-flat. By our assumptions, the right $S$-semimodules $aS$ and $S/aS$ are both $S$-e-flat and so it follows, by Lemma 3.9, that for every subtractive left ideal $I$ of $S$:

$$aS \cap I = aS \cap SI = K \cap FI = KI = (aS)I.$$ 

By our assumption, $S$ is left subtractive and so the left ideal $I := Sa \leq_S S$ is subtractive, whence

$$aSa = (aS)(Sa) = aS \cap Sa.$$

It follows that $a \in aSa$, i.e. exists some $s \in S$ such that $a = asa$. 

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(2) Let $S$ be von Neumann regular. Let $A$ be a normally $S$-generated right $S$-semimodule. Then there exists an exact sequence of left $S$-semimodules

$$0 \rightarrow K \xrightarrow{i} F \xrightarrow{\pi} A \rightarrow 0$$

where $F \simeq S^{(\Lambda)}$ for some index set $\Lambda$, and $K := \text{Ker}(\pi)$. Since $F_S$ is free, it is flat and in particular $m$-flat. Let $f$ be a finitely generated left ideal of $S$. By Lemma 3.10, $I = Se$ for some idempotent $e$ of $S$. Since $S$ is von Neumann regular, there exists some $e' \in S$ such that $e = ee'e$. Let $k = fe \in FI \cap K$ for some $k \in K$ and $f \in F$. Then

$$k = fe = f(ee'e) = (fe)(e'e) = (ke'e)e \in KI.$$

The result follows now by Lemma 3.9. ■

**Corollary 3.12.** If $S$ is subtractive commutative semiring such that every $S$-semimodule is $S$-e-flat, then $S$ is a von Neumann regular semiring.

In light of Theorem 3.11 and the fact that a commutative semiring over which all semimodules are flat is a von Neumann regular ring, we raise the following question:

**Question:** Does the $e$-flatness of all right (left) semimodules characterize subtractive von Neumann regular semirings?

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