A Note on Regularity and Uniqueness of Natural Convection with Effects of Viscous Dissipation in 3D Open Channels

Michal Beneš

Department of Mathematics,
Faculty of Civil Engineering,
Czech Technical University in Prague,
Thákurova 7, 166 29 Prague 6, Czech Republic
email: benes@mat.fsv.cvut.cz

Abstract We prove the existence of unique regular solutions of steady-state buoyancy-driven flows of viscous incompressible heat-conducting fluids in 3D open channels under mixed boundary conditions. The model takes into account the phenomena of the viscous energy dissipation.

1 Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with boundary \( \partial \Omega \), \( \Gamma_D \) and \( \Gamma_N \) are \( C^\infty \)-smooth open disjoint subsets of \( \partial \Omega \) such that \( \partial \Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \), \( \Gamma_D \neq \emptyset \), \( \Gamma_N \neq \emptyset \), \( \mathcal{M} = \partial \Omega - (\Gamma_D \cup \Gamma_N) = \overline{\Gamma_D} \cap \overline{\Gamma_N} = \bigcup_{j \in \mathcal{J}} \mathcal{M}_i \), \( \mathcal{J} = \{1, \ldots, d\} \), and the 2-dimensional measure of \( \mathcal{M} \) is zero and \( \mathcal{M}_i \) are smooth nonintersecting curves (this means that \( \mathcal{M}_i \) are smooth curved nonintersecting edges and vertices (conical points) on \( \partial \Omega \) are excluded). Moreover, all portions of \( \Gamma_N \) are taken to be flat and \( \Gamma_D \) and \( \Gamma_N \) form an angle \( \omega_M = \pi/2 \) at all points of \( \mathcal{M} \) (in the sense of tangential planes). In a physical sense, \( \Omega \) represents a “truncated” region of an unbounded channel system occupied by a moving heat-conducting viscous incompressible fluid. \( \Gamma_D \) will denote the “lateral” surface and \( \Gamma_N \) represents the open parts of the channel \( \Omega \). In addition, we assume that in/outflow channel segments extend as straight pipes (see Figure 1).
The strong formulation of our problem reads as follows:

\[ \rho_0 (u \cdot \nabla) u - \nu \Delta u + \nabla P = \rho(\theta)g \quad \text{in } \Omega, \quad (1) \]
\[ \nabla \cdot (\rho_0 u) = 0 \quad \text{in } \Omega, \quad (2) \]
\[ c_V \rho(\theta) u \cdot \nabla \theta - \lambda \Delta \theta = \alpha_1 \nu e(u) : e(u) \quad \text{in } \Omega, \quad (3) \]
\[ u = 0 \quad \text{on } \Gamma_D, \quad (4) \]
\[ \theta = \theta_D \quad \text{on } \Gamma_D, \quad (5) \]
\[ -Pn + \nu(\nabla u) \cdot n = 0 \quad \text{on } \Gamma_N, \quad (6) \]
\[ \nabla \theta \cdot n = 0 \quad \text{on } \Gamma_N. \quad (7) \]

Equations (1)–(3) represent the balance equations for linear momentum, mass and internal energy of the homogeneous fluid and the system (1)–(7) describes stationary buoyancy-driven flows of viscous incompressible heat-conducting fluids with dissipative heating in the open channel \( \Omega \). In the model, \( u = (u_1, u_2, u_3) \), \( P \) and \( \theta \) denote the unknown velocity, pressure and temperature, respectively. Tensor \( e(u) \) denotes the symmetric part of the velocity gradient \( e(u) = \frac{\nabla u + (\nabla u)^\top}{2} \). \( n \) denotes the unit outward normal with respect to \( \Omega \) along \( \partial \Omega \). Data of the problem are as follows: \( g \) is a body force and \( \theta_D \) is a given function representing the prescribed distribution of the temperature \( \theta \) on \( \Gamma_D \). Positive constant material coefficients represent the kinematic viscosity \( \nu \), reference density \( \rho_0 \), heat conductivity \( \lambda \) and the specific heat at constant volume \( c_V \). Since most of the fluids, especially liquids, are slightly compressible, we consider the fluid to be “mechanically incompressible”, yet “thermally expansible”. Following the well-known Boussinesq approximation, the temperature dependent density is used in the energy equation (3) and to compute the buoyancy force \( \rho(\theta)g \) on the right-hand side of equation (1). Everywhere else in the model, \( \rho \) is replaced by the reference value \( \rho_0 \). Change of density \( \rho \) with temperature is given by strictly positive, nonincreasing and continuous function satisfying

\[ 0 < \rho(\xi) \leq \rho^* < +\infty \quad \forall \xi \in \mathbb{R} \quad (\rho^* = \text{const} > 0). \quad (8) \]
Coefficient $\alpha_1$ reflects the dissipation effect, which is omitted frequently in many mathematical models \cite{19, 24, 25, 26}. Otherwise, taking into account the dissipative term $\alpha_1 \nu e(u) : e(u)$ with quadratic growth of the gradient, the equations (1)–(7) represent the elliptic system with strong nonlinearities without appropriate general existence and regularity theory. In \cite{7}, Frehse presented an example of a discontinuous bounded weak solution of a nonlinear elliptic system of the type $\Delta u = F(u, \nabla u)$, where $F$ is smooth and has quadratic growth in $\nabla u$.

It is matter of discussion which boundary condition should be prescribed on the outlets of the channel system. The boundary condition (6), introduced originally in \cite{10, 11, 12, 27} and which is often called the “do nothing” boundary condition, results from a variational principle and has been proven to be convenient in numerical modeling of parallel flows \cite{12, 18}. The “do nothing” boundary condition is often used in the computational simulation of blood flow in the human vascular system (see e.g. \cite{9}). Let us mention some other interesting problems. Because of the “do nothing” boundary condition (6), some uncontrolled “backward flow” can take place at the outlets of the channel and we are not able to prove an “a priori” estimate for the convective terms in the system (cf. \cite{14}). Consequently, only local solutions can be proven and the question of whether a given solution is unique is, to date, open (even for “small data”). This makes the present problem quite different than in the case of Dirichlet-type boundary conditions for $u$ on the whole boundary frequently studied in the literature (cf. \cite{6, 19, 22, 23, 25, 26}).

In \cite{5}, the authors proved the existence of the local weak solutions of the system (1)–(7) (with constant density $\rho$) in a 3D open cylindrical channel with the prescribed “free surface” boundary condition $-P n + \nu [\nabla u + (\nabla u)^T] n = 0$ on the output of the channel. In \cite{4}, the author proved the $W^{2,8/7}$-regularity for the velocity and temperature of the problem in 2D Lipschitz domains and various types of boundary conditions. In this work we provide an existence proof of $W^{2,s}$-regular solutions of steady-state buoyancy-driven flows in 3D open channels modeled by the problem (1)–(7), which does not require small data for $\theta$ on $\Gamma_D$ described by the function $\theta_D$. Finally, the uniqueness of the solution is discussed in this paper.

The paper is organized as follows. In Section 2, we introduce basic notations and some appropriate function spaces in order to precisely formulate our problem and prove some auxiliary Lemmas which will be used in the proof of the main result. In Section 3, we formulate the problem in a variational setting under the framework of free divergence functional spaces and establish the main result of our work. The main advantage of the formulation of the Navier-Stokes system in free divergence spaces is the elimination of the pressure $P$ to consider only the couple $u$ and $\theta$ as the primary unknowns of the coupled system. The main result is proved in Section 4. In Subsection 4.1, we prove the existence of the solution introducing iterative scheme to uncouple the system. Let us briefly describe the rough idea of the proof. Introducing the translated function $\vartheta_0 = \theta - \theta_D$ we solve the corresponding problem with homogeneous boundary conditions. For given temperature, say $\vartheta_0$, in the buoyancy term on the right hand side in (1), we find $u$, the solution of the decoupled Navier-Stokes equations (1)–(2) with
mixed boundary conditions (4)–(6) via the Banach contraction principle. Now with \( u \) in hand we modify (3) substituting \( \vartheta_0 \) and \( u \) into convective and dissipative terms and find \( \vartheta \), the solution of the linearized heat (Poisson) equation with the mixed boundary conditions (5) and (7). Finally we show that the map \( \vartheta_0 \rightarrow \vartheta \) is completely continuous and maps some ball into itself. Hence the existence of at least one solution follows from the Leray-Schauder theorem. In Subsection 4.2, the uniqueness of the solution is established under the assumption of Lipschitz continuity of \( \vartheta \).

2 Preliminaries

Throughout the paper, we will always use positive constants \( c, c_1, c_2, \ldots \), which are not specified and which may differ from line to line, however, do not depend on the functions under consideration.

Let

\[
C_{\sigma,D}^\infty := \left\{ u \in C^\infty(\overline{\Omega})^3; \operatorname{div} u = 0, \supp u \cap \Gamma_D = \emptyset \right\},
\]

\[
C_D^\infty := \left\{ \theta \in C^\infty(\overline{\Omega}); \supp \theta \cap \Gamma_D = \emptyset \right\}
\]

and \( V_{\sigma,D}^{k,p} \) be the closure of \( C_{\sigma,D}^\infty \) in the norm of \( W^{k,p}(\Omega)^3, k \geq 0 \) and \( 1 \leq p \leq \infty \).

Similarly, let \( V_D^{k,p} \) be a closure of \( C_D^\infty \) in the norm of \( W^{k,p}(\Omega) \). Then \( V_{\sigma,D}^{k,p} \) and \( V_D^{k,p} \), respectively, are Banach spaces with the norms of the spaces \( W^{k,p}(\Omega)^3 \) and \( W^{k,p}(\Omega) \), respectively.

We suppose that \( r, s \in \mathbb{R} \) are fixed numbers throughout the paper such that \( s \in [4/3, s_0), s_0 = 3 + \varepsilon \) (\( \varepsilon > 0 \) sufficiently small) and

\[
\begin{cases}
  r \in \left[6/5; \frac{3s}{2(3-s)}\right] & \text{for } s \in [4/3, 3), \\
  r \in \left[6/5; +\infty\right) & \text{for } s \in [3, s_0)
\end{cases}
\]  

(9) (the value \( s_0 \) will be clarified later).
To simplify mathematical formulations we introduce the following notations:

\[ a(u, v) := \nu \int_{\Omega} \nabla u : \nabla v \, d\Omega, \quad (10) \]

\[ b(u, v, w) := \int_{\Omega} \varrho_0 (u \cdot \nabla) v \cdot w \, d\Omega, \quad (11) \]

\[ \kappa(\theta, \varphi) := \lambda \int_{\Omega} \nabla \theta \cdot \nabla \varphi \, d\Omega, \quad (12) \]

\[ d(\vartheta, u, \theta, \varphi) := c_v \int_{\Omega} g(\vartheta) u \cdot \nabla \theta \varphi \, d\Omega, \quad (13) \]

\[ e(u, v, \varphi) := \alpha_1 \nu \int_{\Omega} e(u) : e(v) \varphi \, d\Omega, \quad (14) \]

\[ (u, v) := \int_{\Omega} u \cdot v \, d\Omega, \quad (15) \]

\[ (\theta, \varphi)_{\Omega} := \int_{\Omega} \theta \varphi \, d\Omega. \quad (16) \]

In (10)–(16) all functions \( u, v, w, \theta, \varphi \) are smooth enough, such that all integrals on the right-hand sides make sense.

Further, define the spaces

\[ D_s^a := \{ u \mid f \in V^{0,s}_{\sigma,D}, a(u, v) = (f, v) \text{ for all } v \in V^{1,2}_{\sigma,D} \} \quad (17) \]

and

\[ D_r^\kappa := \{ \theta \mid f \in V^{0,r}_D, \kappa(\theta, \varphi) = (f, \varphi)_{\Omega} \text{ for all } \varphi \in V^{1,2}_D \}, \quad (18) \]

equipped with the norms

\[ \| u \|_{D_s^a} := \| f \|_{V^{0,s}_{\sigma,D}} \quad \text{and} \quad \| \theta \|_{D_r^\kappa} := \| f \|_{V^{0,r}_D}, \quad (19) \]

where \( u \) and \( f \) are corresponding functions via (17) and \( \theta \) and \( f \) are corresponding functions via (18).

**Lemma 1.** For given \( s, r \in \mathbb{R} \) satisfying (9) the following embeddings hold:

\[ D_s^a \hookrightarrow W^{2,s}, \quad \| u \|_{W^{2,s}} \leq c(\nu, \Omega) \| u \|_{D_s^a} \quad \forall u \in D_s^a, \quad (20) \]

\[ D_r^\kappa \hookrightarrow W^{2,r}(\Omega), \quad \| \theta \|_{W^{2,r}(\Omega)} \leq c(\lambda, \Omega) \| \theta \|_{D_r^\kappa} \quad \forall \theta \in D_r^\kappa. \quad (21) \]

**Proof.** It is well known that for every \( f \in (V^{1,2}_{\sigma,D})^* \) there exists the uniquely determined \( u \in V^{1,2}_{\sigma,D} \) such that \( a(u, v) = (f, v) \) for every \( v \in V^{1,2}_{\sigma,D} \) and

\[ \| u \|_{V^{1,2}_{\sigma,D}} \leq c(\nu, \Omega) \| f \|_{(V^{1,2}_{\sigma,D})^*}. \]

Similarly, for every \( f \in (V^{1,2}_D)^* \) there exists the uniquely determined \( \theta \in V^{1,2}_D \) such that \( \kappa(\theta, \varphi) = (f, \varphi) \) for every \( \varphi \in V^{1,2}_D \) and

\[ \| \theta \|_{V^{1,2}_D} \leq c(\lambda, \Omega) \| f \|_{(V^{1,2}_D)^*}. \]
Lemma 3. There exist numbers $C_b$, $C_d$ and $C_e$ such that for every $u, v \in D^s_\sigma$ and $\theta \in W^{2-\varepsilon,r}(\Omega)$ we have

\begin{align}
\|b(u, v, \cdot)\|_{V^{0,s}_{\sigma,D}} & \leq g_0(b(\nu, \omega, v))\|u\|_{D^s_\sigma}\|v\|_{D^s_\omega}, \\
\|d(\theta, u, \theta, \cdot)\|_{V^{1,r}_{\sigma,D}} & \leq c_V g^\circ d(\nu, \omega, u)\|u\|_{D^s_\omega}\|\theta\|_{W^{2-\varepsilon,r}(\Omega)}, \\
\|e(u, v, \cdot)\|_{V^{1,r}_{\sigma,D}} & \leq \alpha_1 \nu C_e(\nu, \omega, u)\|u\|_{D^s_\omega}\|v\|_{D^s_\omega}.
\end{align}

Proof. By Hölder inequality, Sobolev embeddings (see e.g. [1][16][17]) and using (9), (20) and (21) we arrive at the estimates

\begin{align}
\|b(u, v, \cdot)\|_{V^{0,s}_{\sigma,D}} & \leq g_0(\|u\|_{V^{0,2}_{\sigma,D}}\|\nabla v\|_{V^{0,2}_{\sigma,D}}) \\
& \leq g_0(\|u\|_{V^{2,\omega}_{\sigma,D}}\|v\|_{V^{2,\omega}_{\sigma,D}}) \\
& \leq g_0(\|u\|_{D^s_\omega}\|v\|_{D^s_\omega}).
\end{align}

Further,

\begin{align}
\|d(\theta, u, \theta, \cdot)\|_{V^{1,r}_{\sigma,D}} & \leq c_V g^\circ(\|u\|_{V^{0,2}_{\sigma,D}}\|\nabla \theta\|_{L^{2r}(\Omega)}) \\
& \leq c_V g^\circ(\|u\|_{V^{2,\omega}_{\sigma,D}}\|\theta\|_{W^{2-\varepsilon,r}(\Omega)}) \\
& \leq c_V g^\circ C_d(\nu, \omega, u)\|u\|_{D^s_\omega}\|\theta\|_{W^{2-\varepsilon,r}(\Omega)}
\end{align}

and finally

\begin{align}
\|e(u, v, \cdot)\|_{V^{1,r}_{\sigma,D}} & \leq \alpha_1 \nu \|\nabla u\|_{V^{0,2}_{\sigma,D}}\|\nabla v\|_{V^{0,2}_{\sigma,D}} \\
& \leq \alpha_1 \nu c(\|u\|_{V^{2,\omega}_{\sigma,D}}\|v\|_{V^{2,\omega}_{\sigma,D}}) \\
& \leq \alpha_1 \nu C_e(\nu, \omega, u)\|u\|_{D^s_\omega}\|v\|_{D^s_\omega}.
\end{align}

Remark 2. Throughout the paper $\varepsilon$ denotes a sufficiently small positive real number.

Lemma 4. Let $u \in D^s_\sigma$, $\theta \in W^{2-\varepsilon,r}(\Omega)$ and $\theta_n$ be a sequence in $W^{2-\varepsilon,r}(\Omega)$ such that $\theta_n \to \theta$ in $W^{2-\varepsilon,r}(\Omega)$. Then

\begin{equation}
\|d(\theta_n, u, \theta, \cdot) - d(\theta, u, \theta, \cdot)\|_{V^{1,r}_{\sigma,D}} \to 0.
\end{equation}
Proof. Let \( u \in D^s_a \) and \( \theta \in W^{2-\varepsilon, r}(\Omega) \). Then
\[
\| u \cdot \nabla \theta \|_{L^r(\Omega)} \leq \| u \|_{\mathcal{V}^{0,\sigma}_r D^s_a} \| \nabla \theta \|_{L^r(\Omega)^3} \\
\leq c(\Omega) \| u \|_{\mathcal{V}^{2,\sigma}_r D^s_a} \| \theta \|_{W^{2-\varepsilon, r}(\Omega)} \\
\leq C_d(\nu, \Omega) \| u \|_{\mathcal{D}^s_a} \| \theta \|_{W^{2-\varepsilon, r}(\Omega)},
\]
which yields \( u \cdot \nabla \theta \in L^r(\Omega) \). Hence, the function \( f \) defined by the formula (recall (8))
\[
f(x, \xi) = c_V \rho(\xi) u(x) \cdot \nabla \theta(x)
\]
has the Carathéodory property (see [8, Definition 12.2]), i.e. (i) for all \( \xi \in \mathbb{R} \), the function \( f_{\xi}(x) = f(x, \xi) \) (as function of the variable \( x \)) is measurable on \( \Omega \) and (ii) for almost all \( x \in \Omega \), the function \( f_x(\xi) = f(x, \xi) \) (as function of the variable \( \xi \)) continuous on \( \mathbb{R} \) (by continuity of \( \rho \)).

Define the so-called Němystkiů operator, \( N : L^r(\Omega) \rightarrow L^r(\Omega) \), by the formula
\[
N(\theta)(x) = f(x, \theta).
\]
By [8, Theorem 12.10] we deduce that the Němystkiů operator \( N \) is a continuous operator from \( L^r(\Omega) \) into \( L^r(\Omega) \) (recall (8)). Since we assume \( \theta_n \rightarrow \theta \) in \( W^{2-\varepsilon, r}(\Omega) \) and \( W^{2-\varepsilon, r}(\Omega) \hookrightarrow L^r(\Omega) \), we get (25).

Remark 5. By Lemma 1 and the standard compact embedding of Sobolev spaces the embedding \( D^s_\sigma \rightarrow V^{2-\varepsilon, r}_{\sigma,D} \) is compact and we denote by \( C_\varepsilon \) the embedding constant such that
\[
\| \theta \|_{V^{2-\varepsilon, r}_{\sigma,D}} \leq C_\varepsilon \| \theta \|_{D^s_\sigma} \quad \forall \theta \in D^s_\sigma.
\]

3 The main result

Our problem reads as follows: for given \( g \in \mathcal{V}^{0,\sigma}_r \) and \( \theta_D \in W^{2,r}(\Omega) \) find a couple \([u, \theta]\) such that \( u \in D^s_a \), \( \theta \in \theta_D + D^s_\sigma \) and the following system
\[
a(u, v) + b(u, u, v) = (\rho(\theta) g, v), \quad (26)
\]
\[
\kappa(\theta, \varphi) + d(\theta, u, \theta, \varphi) = e(u, u, \varphi) \quad (27)
\]
holds for every \([v, \varphi] \in V^{1,2}_{\sigma,D} \times V^{1,2}_{\sigma,D} \). The couple \([u, \theta]\) will be called the strong solution to the system (1)–(7).

Theorem 1 (Main result). (i) Let \( g \in \mathcal{V}^{0,\sigma}_r \) and \( \theta_D \in W^{2,r}(\Omega) \) and assume that
\[
\| g \|_{\mathcal{V}^{0,\sigma}_r} \leq \frac{\beta}{4C_{\rho}^2 \rho_0^2} < \frac{1}{2C_{\kappa}^2 C_{\omega} C_{\nu} (\rho_0^2)^2} \quad (28)
\]
with some \( \beta \in (0, 1) \). Then there exists the strong solution to the system (1)–(7).

(ii) Let \( r > 3/2 \) and, in addition, \( \rho \) be Lipschitz continuous, i.e.
\[
|\rho(\zeta_1) - \rho(\zeta_2)| \leq C_{\rho} |\zeta_1 - \zeta_2| \quad \forall \zeta_1, \zeta_2 \in \mathbb{R} \quad (C_{\rho} = \text{const} > 0).
\]
Then there exists \( \gamma > 0 \) such that, if \( [u, \theta] \) is the strong solution of the system (1)–(7), \( u \in D^s_a, \theta \in \theta_D + D^r_\kappa \) and satisfying \( (\|\theta\|_{W^{2,r}(\Omega)} + \|u\|_{D^s_a}) < \gamma \), then it is unique.

**Remark 6.** Let us note that the assumption \( r > 3/2 \) in Theorem 1(ii) ensures \( \theta \in L^{\infty}(\Omega) \), which is required in the proof of uniqueness.

### 4 Proof of the main result

#### 4.1 Existence of strong solutions

For an arbitrary fixed \([u_0, \vartheta_0] \in D^s_a \times V^2_{\theta_D} \) we now consider the following auxiliary problem: to find a couple \([w, \vartheta] \in D^s_a \times D^r_\kappa \), such that

\[
a(w, v) = (\vartheta(\vartheta + \theta_D)g, v) - b(u_0, u_0, v),
\]

\[
\kappa(\vartheta, \varphi) = e(w, w, \varphi) - d(\vartheta + \theta_D, w, \vartheta + \theta_D, \varphi) - \kappa(\theta_D, \varphi)
\]

for every \([v, \varphi] \in V^1_{\sigma_D} \times V^1_{\theta_D} \).

First we prove some estimates for terms on the right hand sides in (29)–(30). For an arbitrary \([u_0, \vartheta_0] \in D^s_a \times V^2_{\theta_D} \) we arrive at

\[
\|((\vartheta(\vartheta + \theta_D)g, \cdot))_{V^0_{\sigma_D}} \| \leq \vartheta^2\|g\|_{V^0_{\sigma_D}},
\]

\[
\|b(u_0, u_0, \cdot)\|_{V^0_{\sigma_D}} \leq \vartheta_0 C_b \|u_0\|_{D^s_a}^2,
\]

where \( C_b = C_b(\nu, \Omega) \) (see (22)). The inequalities (31)–(32) together with (19) yield the estimate for the unique solution \( w \in D^s_a \) of the problem (29)

\[
\|w\|_{D^s_a} \leq \vartheta^2\|g\|_{V^0_{\sigma_D}} + \vartheta_0 C_b \|u_0\|_{D^s_a}^2.
\]

Now having \( w \in D^s_a \) and \( \vartheta_0 \in V^2_{\theta_D} \), for all terms on the right hand side of (30) we arrive at the estimates

\[
\|\kappa(\theta_D, \cdot)\|_{V^0_{\theta_D}} \leq \vartheta \|\theta_D\|_{W^{2,r}(\Omega)},
\]

\[
\|e(w, w, \cdot)\|_{V^0_{\theta_D}} \leq \vartheta \|w\|_{D^s_a}^2
\]

and

\[
\|d(\vartheta + \theta_D, w, \vartheta + \theta_D, \cdot)\|_{V^0_{\theta_D}} \leq C_d \vartheta^2 c_V \|w\|_{D^s_a} \left(\|\vartheta\|_{V^2_{\theta_D}} + \|\theta_D\|_{W^{2,r}}\right).
\]

Now the inequalities (34)–(36) together with (19) yield the estimate for the unique solution \( \vartheta \in D^r_\kappa \) of the problem (30)

\[
\|\vartheta\|_{V^2_{\theta_D}} \leq C_r \|\vartheta\|_{D^r_\kappa} \leq C_r C_d \vartheta^2 c_V \|w\|_{D^s_a} \|\vartheta\|_{V^0_{\theta_D}} + C_1 C_r \|\theta_D\|_{W^{2,r}(\Omega)} + C_2 \vartheta \|w\|_{D^s_a}^2 + C_3 C_d \vartheta^2 c_V \|w\|_{D^s_a} \|\theta_D\|_{W^{2,r}(\Omega)}.
\]

For a given couple \([u_0, \vartheta_0] \in D^s_a \times V^2_{\theta_D} \) let \( w \in D^s_a \) be the unique solution of the equation (29). Fix \( \vartheta_0 \in V^2_{\theta_D} \) and consider the map
\begin{align*}
\mathcal{K}_{\phi_0} : D^*_a \to D^*_a \quad \text{with} \quad \mathcal{K}_{\phi_0}(u_0) = w.

\text{Lemma 7.} \quad \text{Operator } \mathcal{K}_{\phi_0} \text{ realizes contraction in the closed ball (} \beta \text{ is the constant from (28))}
\end{align*}

\begin{align*}
M = \left\{ v \in D^*_a ; \|v\|_{D^*_a} \leq \frac{\beta}{2C_{b,b_0}} \right\}.
\end{align*}

\text{Proof.} \quad \text{Using the estimate (33) and the assumption (28) we get } \mathcal{K}_{\phi_0}(M) \subset M. \quad \text{Let } u_0 \text{ and } \bar{u}_0 \in M. \quad \text{Then by (22) we arrive at}
\begin{align*}
\|\mathcal{K}_{\phi_0}(u_0) - \mathcal{K}_{\phi_0}(\bar{u}_0)\|_{D^*_a} &= \|b(u_0, u_0, \cdot) - b(\bar{u}_0, \bar{u}_0, \cdot)\|_{V_{\sigma,D}^{0,s}} \\
&\leq \varrho_0 C_b(\|u_0\|_{D^*_a} + \|\bar{u}_0\|_{D^*_a})\|u_0 - \bar{u}_0\|_{D^*_a} \\
&< \beta\|u_0 - \bar{u}_0\|_{D^*_a}
\end{align*}

with \( \beta \in (0, 1) \) (cf. (28)). \quad \text{The proof is complete.} \quad \square

As a consequence of Lemma 7 and the Banach fixed point theorem there exists the unique \( w \in M \) such that \( \mathcal{K}_{\phi_0}(w) = w \). \quad \text{Define the operator } \mathcal{T}_1 : V_{D}^{2-r, r} \to M \quad \text{by } \mathcal{T}_1(\bar{\vartheta}) = w. \quad \text{Let } \vartheta = \mathcal{T}_2(\mathcal{T}_1(\bar{\vartheta}), \bar{\vartheta}), \vartheta \in D^*_c \to V_{D}^{2-r, r}, \quad \text{be the solution of the problem (30). If there exists a fixed point of } \mathcal{T}_2 \quad \text{and } \mathcal{T} = \mathcal{T}_1(\bar{\vartheta}) \text{ solve the system (26)–(27).}

\text{Let us note that the ball } M \text{ is independent of the choice of } \bar{\vartheta}_0 \in V_{D}^{2-r, r} \text{ and the right hand side of the inequality (37) depends linearly on } \|\bar{\vartheta}_0\|_{V_{D}^{2-r, r}}. \quad \text{Moreover, for } w \in M \text{ and taking into account the assumption (28) we obtain}
\begin{align*}
C_d C_d g^\sigma c_V \|w\|_{D^*_a} \leq C_d C_d g^\sigma c_V \frac{\beta}{2C_{b,b_0}} < 1.
\end{align*}

Consequently, there exists sufficiently large \( R \) such that \( \mathcal{T}_2 \) maps the ball
\begin{align*}
B = \left\{ \bar{\vartheta} \in V_{D}^{2-r, r} ; \|\bar{\vartheta}\|_{V_{D}^{2-r, r}} \leq R \right\}
\end{align*}

into itself. By the compact embedding \( D^*_c \hookrightarrow V_{D}^{2-r, r} \) the operator \( \mathcal{T}_2 \) is completely continuous if we prove that \( \mathcal{T}_2 \) is continuous.

\text{Let } \bar{\vartheta}_0 \in V_{D}^{2-r, r} \text{ and } (\bar{\vartheta}_0)_n \text{ be a sequence in } V_{D}^{2-r, r} \text{ such that } (\bar{\vartheta}_0)_n \to \bar{\vartheta}_0. \quad \text{Let } w = \mathcal{T}_1(\bar{\vartheta}_0) \text{ and } w_n = \mathcal{T}_1((\bar{\vartheta}_0)_n). \quad \text{By noting (19) we arrive at the estimate}
\begin{align*}
\|w - w_n\|_{D^*_a} &\leq \|b(w, w, \cdot) - b(w_n, w_n, \cdot)\|_{V_{\sigma,D}^{0,s}} \\
&\quad + \|(g(\bar{\vartheta}_0 + \bar{\vartheta}_D)g, \cdot) - (g((\bar{\vartheta}_0)_n + \bar{\vartheta}_D)g, \cdot)\|_{V_{\sigma,D}^{0,s}} \\
&\leq \varrho_0 C_b(\|w\|_{D^*_a} + \|w_n\|_{D^*_a})\|w - w_n\|_{D^*_a} \\
&\quad + \|(g(\bar{\vartheta}_0 + \bar{\vartheta}_D)g, \cdot) - (g((\bar{\vartheta}_0)_n + \bar{\vartheta}_D)g, \cdot)\|_{V_{\sigma,D}^{0,s}} \\
&\leq \beta\|w - w_n\|_{D^*_a} + \|(g(\bar{\vartheta}_0 + \bar{\vartheta}_D)g, \cdot) - (g((\bar{\vartheta}_0)_n + \bar{\vartheta}_D)g, \cdot)\|_{V_{\sigma,D}^{0,s}}
\end{align*}

and hence
\begin{align*}
(1 - \beta)\|w - w_n\|_{D^*_a} \leq \|(g(\bar{\vartheta}_0 + \bar{\vartheta}_D)g, \cdot) - (g((\bar{\vartheta}_0)_n + \bar{\vartheta}_D)g, \cdot)\|_{V_{\sigma,D}^{0,s}}, \quad (38)
\end{align*}
where $\beta \in (0,1)$ (given by (28)). Consequently, the operator $T_1 : V^{2-\varepsilon,r}_D \to M$ is continuous. To prove the continuity of $\mathcal{T}_2$ let us estimate

$$
\| T_2(\vartheta_0) - T_2((\vartheta_0)_n) \|_{V^{2-\varepsilon,r}_D} \leq C_\varepsilon \| T_2(\vartheta_0) - T_2((\vartheta_0)_n) \|_{D^*_\varepsilon} \\
\leq C_\varepsilon \left( \| e(\vartheta_0, w - w_n, \cdot) \|_{V^{0,r}_D} + \| e(w - w_n, w_n, \cdot) \|_{V^{0,r}_D} \\
+ \| d(\vartheta_0 + \theta_D, w, \vartheta_0 + \theta_D, \cdot) - d((\vartheta_0)_n + \theta_D, w, (\vartheta_0)_n + \theta_D, \cdot) \|_{V^{0,r}_D} \\
+ \| d((\vartheta_0)_n + \theta_D, w_n, (\vartheta_0)_n - \vartheta_0, \cdot) \|_{V^{0,r}_D} + \| d((\vartheta_0)_n + \theta_D, w_n - w, (\vartheta_0)_n + \theta_D, \cdot) \|_{V^{0,r}_D} \\
+ \| (\vartheta_0, (\vartheta_0)_n + \theta_D)g \cdot (w - w_n, \cdot) \|_{V^{0,r}_D} \right).
$$

By Lemma 4 estimates (23), (24) and (38) and continuity of $g$ we conclude that

$$
\| T_2(\vartheta_0) - T_2((\vartheta_0)_n) \|_{V^{2-\varepsilon,r}_D} \to 0
$$

whenever $\| (\vartheta_0) - ((\vartheta_0)_n) \|_{V^{2-\varepsilon,r}_D} \to 0$. Consequently, $\mathcal{T}_2$ is completely continuous and $\mathcal{T}_2(B) \subset B$. The existence of at least one fixed point $\vartheta = T_2(\vartheta)$ follows from the Leray-Schauder theorem. Now the couple $[u, \vartheta]$, $u = T_1(\vartheta)$ and $\vartheta = \theta_D + \vartheta$, is the solution of the problem (26)–(27).

4.2 Uniqueness

Here we prove the uniqueness of the strong solution stated in the main result. Suppose that all assumptions of Theorem 1 are satisfied and there are two strong solutions $[u_1, \vartheta_1], [u_2, \vartheta_2]$ of the system (11)–(17) such that $u_1, u_2 \in D^*_\sigma, \vartheta_1, \vartheta_2 \in \theta_D + D^*_\kappa$. Denote $z = u_1 - u_2$ and $\sigma = \vartheta_1 - \vartheta_2$. Then $z$ and $\sigma$ satisfy the equations

$$
a(z, v) + b(z, u_2, v) + b(u_1, z, v) - (\vartheta_1 - \vartheta_2)g \cdot v = 0,
$$

$$
\kappa(\sigma, \varphi) + d(\vartheta_1, u_1, \vartheta_2, \varphi) - d(\vartheta_2, u_1, \vartheta_2, \varphi) + d(\vartheta_2, z, \vartheta_1, \varphi) + d(\vartheta_2, u_2, \varphi) - e(z, u_1, \varphi) - e(u_2, \varphi) = 0
$$

for every $[v, \varphi] \in V^{1,2}_{\sigma,D} \times V^{1,2}_{\sigma,D}$. By (19) we arrive at the estimates

$$
\| z \|_{D^*_\varepsilon} \leq \| b(z, u_2, \cdot) \|_{D^*_\varepsilon} + \| b(u_1, z, \cdot) \|_{D^*_\varepsilon} + \| (\vartheta_1 - \vartheta_2)g \|_{V^{0,r}_{\sigma,D}} \tag{39}
$$

and

$$
\| \sigma \|_{D^*_\varepsilon} \leq \| d(\vartheta_1, u_1, \vartheta_2, \cdot) - d(\vartheta_2, u_1, \vartheta_2, \cdot) \|_{V^{0,r}_D} + \| d(\vartheta_2, z, \vartheta_1, \cdot) \|_{V^{0,r}_D} \tag{40}
\| d(\vartheta_2, u_2, \sigma, \cdot) \|_{V^{0,r}_D} + \| e(z, u_1, \cdot) \|_{V^{0,r}_D} + \| e(u_2, z, \cdot) \|_{V^{0,r}_D}.
$$

To estimate term by term on the right-hand sides of (39) and (40) we use Lemma 3 to obtain

$$
\| b(z, u_2, \cdot) \|_{V^{0,r}_{\sigma,D}} \leq \vartheta_0 C_b \| z \|_{D^*_\varepsilon} \| u_2 \|_{D^*_\varepsilon},
$$

$$
\| b(u_1, z, \cdot) \|_{V^{0,r}_{\sigma,D}} \leq \vartheta_0 C_b \| u_1 \|_{D^*_\varepsilon} \| z \|_{D^*_\varepsilon},
$$

$$
\| (\vartheta_1 - \vartheta_2)g \|_{V^{0,r}_{\sigma,D}} \leq C_\varepsilon \| \sigma \|_{V^{0,\infty}_\sigma} \| g \|_{V^{0,\infty}_{\sigma,D}}.
$$
Since \( r > 3/2 \) we have, using known embedding for Sobolev spaces, \( \| \sigma \|_{V^{0,\infty}} \leq C_1 \| \sigma \|_{D^s} \). Further
\[
\|d(\theta_1, u_1, \theta_2, \cdot) - d(\theta_2, u_1, \theta_2, \cdot)\|_{V^{p,r}} \leq c_V C_1 C_d C_\sigma \| \sigma \|_{D^s} \| u_1 \|_{D_s^\alpha} \| \theta_2 \|_{W^{2,r}(\Omega)}
\]
and finally
\[
\|d(\theta_2, z, \theta_1, \cdot)\|_{V^{p,r}} \leq c_V C_d \| \sigma \|_{D^s} \| \theta_1 \|_{W^{2,r}(\Omega)},
\]
\[
\|d(\theta_2, u_2, \sigma, \cdot)\|_{V^{p,r}} \leq c_V C_d \| \sigma \|_{D^s},
\]
\[
\|e(z, u_1, \cdot)\|_{V^{p,r}} \leq \alpha_1 \nu C_c \| z \|_{D^s} \| u_1 \|_{D_s^\alpha},
\]
\[
\|e(u_2, z, \cdot)\|_{V^{p,r}} \leq \alpha_1 \nu C_c \| u_2 \|_{D_s^\alpha} \| z \|_{D^s}.
\]
Hence
\[
\| \sigma \|_{D^s} \leq (c_V C_1 C_d C_\sigma \| u_1 \|_{D_s^\alpha} \| \theta_2 \|_{W^{2,r}(\Omega)} + c_V C_d \| \sigma \|_{D^s} \| u_2 \|_{D_s^\alpha}) \| \sigma \|_{D^s}
+ (c_V C_d \| \sigma \|_{D^s} \| \theta_1 \|_{W^{2,r}(\Omega)} + \alpha_1 \nu C_c \| u_1 \|_{D_s^\alpha} + \alpha_1 \nu C_c \| u_2 \|_{D_s^\alpha}) \| z \|_{D^s} \] (41)
and
\[
\| z \|_{D_s^\alpha} \leq C_1 C_d \| g \|_{V^{0,\infty}} \| \sigma \|_{D^s} + (\theta_0 C_b(\| u_1 \|_{D_s^\alpha} + \| u_2 \|_{D_s^\alpha})) \| z \|_{D^s}
\leq C_1 C_d \| g \|_{V^{0,\infty}} (c_V C_1 C_d C_\sigma \| u_1 \|_{D_s^\alpha} \| \theta_2 \|_{W^{2,r}(\Omega)} + c_V C_d \| \sigma \|_{D^s} \| u_2 \|_{D_s^\alpha}) \| \sigma \|_{D^s}
+ C_1 C_d \| g \|_{V^{0,\infty}} (c_V C_d \| \sigma \|_{D^s} \| \theta_1 \|_{W^{2,r}(\Omega)} + \alpha_1 \nu C_c \| u_1 \|_{D_s^\alpha} + \alpha_1 \nu C_c \| u_2 \|_{D_s^\alpha})
\theta_0 C_b(\| u_1 \|_{D_s^\alpha} + \| u_2 \|_{D_s^\alpha})) \| z \|_{D^s}. \] (42)

Adding (41) and (42) together we get the inequality of the form
\[
\| \sigma \|_{D^s} + \| z \|_{D_s^\alpha} \leq R_1(u_1, u_2, \theta_1, \theta_2) \| \sigma \|_{D^s} + R_2(u_1, u_2, \theta_1, \theta_2) \| z \|_{D^s}.
\]

Thus, if \( R_1(u_1, u_2, \theta_1, \theta_2) < 1 \) and \( R_2(u_1, u_2, \theta_1, \theta_2) < 1 \), we have \( \| \sigma \|_{D^s} = 0 \) and \( \| z \|_{D_s^\alpha} = 0 \). Therefore \( \theta_1 = \theta_2 \) and \( u_1 = u_2 \).

**A Regularity of solutions to appropriate linear elliptic problems**

In this Appendix, we discuss appropriate linear boundary value problems for the Poisson equation and the Stokes system in the channel \( \Omega \) with the mixed boundary conditions. We establish regularity results for weak solutions based on the assumptions on the geometry of the channel \( \Omega \) and provided the data of the problems are sufficiently smooth. Recall that \( \Gamma_D \) and \( \Gamma_N \) belong to the class \( C^\infty \) and form an angle \( \omega_M = \pi/2 \) (in the sense of tangential planes) at all points of \( M \) (the set in which boundary conditions change their type).
A.1 The mixed problem for the Stokes system

We consider the problem

\[-\Delta u + \nabla P = f \quad \text{in } \Omega, \tag{43}\]
\[\nabla \cdot u = 0 \quad \text{in } \Omega, \tag{44}\]
\[u = 0 \quad \text{on } \Gamma_D, \tag{45}\]
\[P n + \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N. \tag{46}\]

Without loss of generality we suppose that the viscosity of the fluid is normalized to one ($\nu = 1$).

For arbitrary real $p \in (1, \infty)$ and $\delta = (\delta_1, \ldots, \delta_d)$, $\delta_i > -2/p$, $i = 1, \ldots, d$, we denote by $W^{k,p}_\delta(\Omega)$ the weighted Sobolev space with the norm (see e.g. [21, Chapter 8.3.1.])

\[\|\varphi\|_{W^{k,p}_\delta(\Omega)} = \left( \int_{\Omega} \prod_{i=1}^{d} r_i(x)^{p\delta_i} \left( \sum_{|\alpha| \leq k} |\partial^{\alpha}_x \varphi(x)|^p \, d\Omega \right)^{1/p} \right),\]

where $r_i(x) = \text{dist}(x, M_i)$.

Let $x_0$ be an arbitrary point of $M$. Define a dihedral angle $D$, $x_0 \in D$, with faces $\gamma_D$ and $\gamma_N$, such that $\gamma_D$ and $\gamma_N$ are tangential planes to $\Gamma_D$ and $\Gamma_N$, respectively, at the point $x_0$ (see Figure 2).

There is a ball-neighborhood $U(x_0)$ of the point $x_0$ with radius $r_0$ such that the
domain $\Omega$ is diffeomorphic to the dihedral angle $D$ in the neighborhood $U(x_0)$.

Consider a cut-off function $\chi(|x|) \in C^\infty(\mathbb{R}^3)$, $0 \leq \chi(|x|) \leq 1$,

$$\chi(|x|) = \begin{cases} 1 & \text{for } |x| < r_0/2, \\
0 & \text{for } |x| > r_0. \end{cases}$$

Multiplying the system of equations (43)–(46) by $\chi$ we get the boundary value problem for $\tilde{u} = \chi u$ and $\tilde{P} = \chi P$ in the dihedral angle $D$

$$-\Delta \tilde{u} + \nabla \tilde{P} = \tilde{f} \quad \text{in } D, \quad \tilde{\nabla} \cdot \tilde{u} = \tilde{g} \quad \text{in } D, \quad \tilde{\nabla} \tilde{u} = 0 \quad \text{on } \gamma_D, \quad -\tilde{P} n + \frac{\partial \tilde{u}}{\partial n} = 0 \quad \text{on } \gamma_N,$$

where

$$\tilde{f} = -u \Delta \chi - 2 \frac{\partial u}{\partial x_1} \frac{\partial \chi}{\partial x_1} - 2 \frac{\partial u}{\partial x_2} \frac{\partial \chi}{\partial x_2} - 2 \frac{\partial u}{\partial x_3} \frac{\partial \chi}{\partial x_3} + f \chi + P(\nabla \chi) \quad (51)$$

and

$$\tilde{\nabla} \cdot (\tilde{u}_1, \tilde{u}_2) = \tilde{g}.$$  

After the application of the Fourier transform with respect to $x_3$, $x_3 \to \eta$, and letting $\eta = 0$ we get the corresponding two-dimensional problem in the plane angle $K$ with the sides $\Gamma_{KD}$ and $\Gamma_{KN}$

$$-\Delta \hat{u}_1 + \nabla \hat{P} = (\hat{f}_1, \hat{f}_2) \quad \text{in } K, \quad -\Delta \hat{u}_3 = \hat{f}_3 \quad \text{in } K, \quad \nabla \cdot (\hat{u}_1, \hat{u}_2) = \hat{g} \quad \text{in } K, \quad (\hat{u}_1, \hat{u}_2, \hat{u}_3) = (0, 0, 0) \quad \text{on } \Gamma_{KD},$$

$$-\hat{P} n_x + \frac{\partial (\hat{u}_1, \hat{u}_2)}{\partial n_x} = (0, 0) \quad \text{on } \Gamma_{KN}, \quad \frac{\partial \hat{u}_3}{\partial n_x} = 0 \quad \text{on } \Gamma_{KN}.$$  

Here $\hat{f} = \mathcal{F}_{x_3 \to \eta}[f]$, $\hat{\nabla} = \mathcal{F}_{x_3 \to \eta}[\hat{f}]$, $\hat{u} = \mathcal{F}_{x_3 \to \eta}[\hat{u}]$, $\hat{P} = \mathcal{F}_{x_3 \to \eta}[\hat{P}]$. Under the polar
coordinates \((r, \omega)\) the problem \(52\)–\(57\) becomes

\[
- \left( \frac{\partial^2 \bar{u}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{u}_1}{\partial \omega^2} \right) + \frac{\partial \bar{P}}{\partial r} \cos \omega - \frac{1}{r} \frac{\partial \bar{P}}{\partial \omega} \sin \omega = \bar{f}_1 \quad \text{in } \bar{S}, (58)
\]

\[
- \left( \frac{\partial^2 \bar{u}_2}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{u}_2}{\partial \omega^2} \right) + \frac{\partial \bar{P}}{\partial r} \sin \omega + \frac{1}{r} \frac{\partial \bar{P}}{\partial \omega} \cos \omega = \bar{f}_2 \quad \text{in } \bar{S}, (59)
\]

\[
- \left( \frac{\partial^2 \bar{u}_3}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_3}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{u}_3}{\partial \omega^2} \right) = \bar{f}_3 \quad \text{in } \bar{S}, (60)
\]

\[
\frac{\partial \bar{u}_1}{\partial r} \cos \omega - \frac{1}{r} \frac{\partial \bar{u}_1}{\partial \omega} \sin \omega + \frac{\partial \bar{u}_2}{\partial r} \sin \omega + \frac{1}{r} \frac{\partial \bar{u}_2}{\partial \omega} \cos \omega = \bar{g} \quad \text{in } \bar{S}, (61)
\]

\[
\bar{u}_1(r, \pi/2) = 0, \quad \bar{u}_2(r, \pi/2) = 0, \quad \bar{u}_3(r, \pi/2) = 0, (62-64)
\]

\[
\frac{\partial \bar{u}_1}{\partial \omega}(r, 0) = 0, \quad \frac{\partial \bar{u}_2}{\partial \omega}(r, 0) = 0, \quad \frac{\partial \bar{u}_3}{\partial \omega}(r, 0) = 0, (65-67)
\]

where \(\bar{S} = \{(r, \omega) : 0 < r < \infty, 0 < \omega < \pi/2\}\) is the infinite plane angle described in polar coordinates \((r, \omega)\), \(\bar{u}(r, \omega) = \hat{u}(x_1, x_2)\), \(\bar{P}(r, \omega) = \sqrt{x_1^2 + x_2^2}\hat{P}(x_1, x_2)\), \(\bar{f}(r, \omega) = \hat{f}(x_1, x_2)\), \(\bar{g}(r, \omega) = \hat{g}(x_1, x_2)\).

Applying the Mellin transform

\[
\mathcal{M}[\phi] = \int_0^\infty r^{-z-1} \phi(r) \, dr = \hat{\phi}(z)
\]

we get the following system of equations depending on a parameter \(z \in \mathbb{C}\) in the
\[ -\frac{\partial^2 \tilde{u}_1}{\partial \omega^2} - z^2 \tilde{u}_1 + (z - 1) \tilde{P} \cos \omega - \frac{\partial \tilde{P}}{\partial \omega} \sin \omega = \tilde{f}_1, \quad (68) \]
\[ -\frac{\partial^2 \tilde{u}_2}{\partial \omega^2} - z^2 \tilde{u}_2 + (z - 1) \tilde{P} \sin \omega + \frac{\partial \tilde{P}}{\partial \omega} \cos \omega = \tilde{f}_2, \quad (69) \]
\[ -\frac{\partial^2 \tilde{u}_3}{\partial \omega^2} - z^2 \tilde{u}_3 = \tilde{f}_3, \quad (70) \]
\[ z \tilde{u}_1 \cos \omega - \frac{\partial \tilde{u}_1}{\partial \omega} \sin \omega + z \tilde{u}_2 \sin \omega + \frac{\partial \tilde{u}_2}{\partial \omega} \cos \omega = \tilde{g}, \quad (71) \]
\[ \tilde{u}_1(z, \pi/2) = 0, \quad (72) \]
\[ \tilde{u}_2(z, \pi/2) = 0, \quad (73) \]
\[ \tilde{u}_3(z, \pi/2) = 0, \quad (74) \]
\[ \frac{\partial \tilde{u}_1}{\partial \omega}(z, 0) = 0, \quad (75) \]
\[ -\tilde{P}(z, 0) + \frac{\partial \tilde{u}_2}{\partial \omega}(z, 0) = 0, \quad (76) \]
\[ \frac{\partial \tilde{u}_3}{\partial \omega}(z, 0) = 0, \quad (77) \]

where \( \tilde{f}_1 = \mathcal{M}_{r \rightarrow z}[\tilde{f}_1], \tilde{f}_2 = \mathcal{M}_{r \rightarrow z}[\tilde{f}_2], \tilde{f}_3 = \mathcal{M}_{r \rightarrow z}[\tilde{f}_3], \tilde{g} = \mathcal{M}_{r \rightarrow z}[\tilde{g}], \tilde{u}_1 = \mathcal{M}_{r \rightarrow z}[\tilde{u}_1], \tilde{u}_2 = \mathcal{M}_{r \rightarrow z}[\tilde{u}_2], \tilde{u}_3 = \mathcal{M}_{r \rightarrow z}[\tilde{u}_3], \tilde{P} = \mathcal{M}_{r \rightarrow z}[\tilde{P}]. \)

Now the problem (68)–(77) can be treated as the operator equation
\[ \mathcal{A}(z)[\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{P}] = [\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{g}, 0, 0, 0, 0, 0, 0]. \]

Every complex number \( z_0 \) such that \( \ker \mathcal{A}(z_0) \neq 0 \) is said to be an eigenvalue of \( \mathcal{A}(z) \) and the set of all such eigenvalues is called the spectrum of \( \mathcal{A}(z) \). Note that the problem (68)–(77) with \( [\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{g}] = [0, 0, 0, 0] \) consists of two decoupled boundary value problems with parameter \( z \) which can be handled separately. The spectrum of \( \mathcal{A}(z) \) consists of the numbers \( z_k = 2k + 1, \) where \( k = 0, 1, 2, \ldots \), corresponding to the problem (70),(67) and (77) with \( \tilde{f}_3 = 0 \) and the only unknown \( \tilde{u}_3 \) (see [21, Section 8.3.1]). In addition, the spectrum of \( \mathcal{A}(z) \) includes the solutions of the transcendental equation (we refer to [2, eq. (2.9)])
\[ z^2 - 4 \cos^2 \left( \frac{z\pi}{2} \right) - \sin^2 \left( \frac{z\pi}{2} \right) = 0. \quad (78) \]

Note that for the roots \( z \) of (78) there exists a nontrivial solution of the problem (68)–(69), (71)–(76), (72)–(73) with \( [\tilde{f}_1, \tilde{f}_2, \tilde{g}] = [0, 0, 0] \) and the unknowns \( \tilde{u}_1, \tilde{u}_2 \) and \( \tilde{P} \) (see [2]).

Denote by \( \mu_M \) the greatest real number such that the strip
\[ 0 < \Re z < \mu_M, \; z \in \mathbb{C}, \]
contains only the eigenvalue \( z = 1 \) of the operator \( \mathcal{A}(z) \).

The following result is a consequence of [20, Theorem 5.5].

**Theorem 2 (Regularity in weighted Sobolev spaces).** Let \( f \in (V_{\sigma,D}^{1,2})^* \) and \( u \in V_{\sigma,D}^{1,2} \) be the weak solution of the problem (23)–(26), i.e. satisfying the equation

\[
a(u, v) = (f, v)
\]

for all \( v \in V_{\sigma,D}^{1,2} \). Suppose that \( f \in W_0^{0,p}(\Omega)^3 \) and the components \( \delta_1, \ldots, \delta_d \) satisfy the inequalities

\[
\max(0, 2 - \mu_M) < \delta_i + 2/p < 2, \quad i = 1, \ldots, d.
\]

Then \( u \in W_\delta^{2,p}(\Omega)^3 \) and

\[
\|u\|_{W_\delta^{2,p}(\Omega)^3} \leq c(\Omega) \|f\|_{W_\delta^{0,p}(\Omega)^3}.
\]

**Remark 8.** Let us note (see [2, Remark 2.2]) that it can be shown numerically that there are only two roots of the equation (78), \( z_0 \approx 1.352317 \) and \( z_00 = 1 \), in the strip \( \Re z \in (0, 2) \). Hence \( \mu_M = \Re z_0 \) and we set \( s_0 = \frac{2}{\Re z_0 + 2} (\approx 3.087930) \). Since we consider \( \frac{4}{3} < s < s_0 \) (recall (12)), we have \( \max(0, 2 - \mu_M) < 2/s < 2 \).

For \( \delta = 0 \), we obtain the regularity results in nonweighted Sobolev spaces. The following assertion holds as a consequence of Theorem 2 and Remark 8.

**Corollary 3.** Let \( u \in V_{\sigma,D}^{1,2} \) be the weak solution of the problem (23)–(26) and \( f \in L^s(\Omega)^3 \), \( 4/3 < s < s_0 \). Then \( u \in W^{2,s}(\Omega)^3 \) and

\[
\|u\|_{W^{2,s}(\Omega)^3} \leq c(\Omega) \|f\|_{L^s(\Omega)^3}.
\]

### A.2 The mixed problem for the Poisson equation

We consider the problem

\[
-\Delta \varphi = h \quad \text{in } \Omega, \quad (79)
\]

\[
\varphi = 0 \quad \text{on } \Gamma_D, \quad (80)
\]

\[
\nabla \varphi \cdot n = 0 \quad \text{on } \Gamma_N. \quad (81)
\]

By the Lax-Milgram theorem, for every \( h \in (V_D^{1,2})^* \) there exists a uniquely determined \( \varphi \in V_D^{1,2} \) (the weak solution to the problem (79)–(81)) such that \( \kappa(\varphi, \varphi) = \langle h, \varphi \rangle \) for every \( \varphi \in V_D^{1,2} \). The following regularity result is a consequence of [15, Section 33.5 and Section 26.3] (see also [21, Corollary 8.3.2]).

**Proposition 4.** Let \( h \in L^p(\Omega) \cap (V_D^{1,2})^* \), \( p > 1 \), and \( \varphi \in V_D^{1,2} \) be the weak solution to the problem (79)–(81). Then \( \varphi \in W^{2,p}(\Omega) \) and

\[
\|\varphi\|_{W^{2,p}(\Omega)^3} \leq c(\Omega) \|h\|_{L^p(\Omega)}.
\]
Acknowledgment

This research was supported by the project GAČR 13-18652S.

References

[1] R.A. Adams, J.J.F. Fournier, Sobolev spaces. Elsevier, 2003.

[2] M. Beneš, Solutions to the mixed problem of viscous incompressible flows in a channel. Arch. Math. 93 (2009), 287–297.

[3] M. Beneš, Strong solutions to non-stationary channel flows of heat-conducting viscous incompressible fluids with dissipative heating, Acta Appl. Math. 116 (2011), 237–254.

[4] M. Beneš, A note on the regularity of thermally coupled viscous flows with critical growth in non-smooth domains. Math. Meth. Appl. Sci. 36 (2013), 1290–1300.

[5] M. Beneš, P. Kučera, On the Navier-Stokes flows for heat-conducting fluids with mixed boundary conditions. J. Math. Anal. Appl. 389 (2012), 769–780.

[6] Z. Charki, Existence and Uniqueness of Solutions for the Steady Deep Bénard Convection Problem. Z. Angew. Math. Mech. 75 (1995), 909–915.

[7] J. Frehse, A discontinuous solution of mildly nonlinear elliptic systems. Math. Z. 134 (1973), 229–230.

[8] S. Fučík, A. Kufner, Nonlinear Differential Equations. Elsevier, 1980.

[9] G.P. Galdi, R. Rannacher, A.M. Robertson, S. Turek, Hemodynamical Flows. Modeling, Analysis and Simulation. Birkhäuser, 2008.

[10] R. Glowinski, Numerical Methods for Nonlinear Variational Problems. SpringerVerlag, New York-Berlin-Heidelberg-Tokyo, 1984.

[11] P.M. Gresho, Incompressible fluid dynamics: some fundamental formulation issues. Ann. Rev. Fluid Mech. 23 (1991), 413–453.

[12] J.G. Heywood, R. Rannacher and S. Turek, Artificial boundaries and flux and pressure conditions for the incompressible Navier–Stokes equations. Int. J. Numer. Meth. Fl. 22 (1996),325-352.

[13] P. Kučera, Solution of the Stationary Navier–Stokes Equations with Mixed Boundary Conditions in a Bounded Domain. Pitman Res. Notes Math. Ser. 379 (1998), 127–131.
[14] S. Kračmar, J. Neustupa, *A weak solvability of a steady variational inequality of the Navier-Stokes type with mixed boundary conditions*. Nonlinear Anal. **47** (2001), 4169–4180.

[15] A. Kufner, A.M. Sändig, *Some Applications of Weighted Sobolev Spaces*. B.G. Teubner, 1987.

[16] A. Kufner, O. John, S. Fučík, *Function Spaces*. Academia, 1977.

[17] J. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*. Dunod, 1968.

[18] J. Liu, *Open and traction boundary conditions for the incompressible Navier-Stokes equations*. J. Comput. Phys. **228** (2009), 7250–7267.

[19] S.A. Lorca, J.L. Boldrini, *Stationary solutions for generalized Boussinesq models*. J. Differential Equations **124** (1996), 389–406.

[20] V.G. Mazya, J. Rossmann, *$L_p$ estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains*. Math. Nachr. **280** (2007), 751–793.

[21] V.G. Mazya, J. Rossmann, *Elliptic equations in polyhedral domains*, Mathematical Surveys and Monographs Vol. 162, 2010.

[22] E. Marušić-Paloka, I. Pažanin, *Modelling of heat transfer in a laminar flow through a helical pipe*. Math. Comput. Modelling **50** (2009), 1571–1582.

[23] E. Marušić-Paloka, I. Pažanin, *Non-isothermal fluid flow through a thin pipe with cooling*. Appl. Anal. **88** (2009), 495–515.

[24] K.R. Rajagopal, M. Ruzicka, A.R. Srinivasa, *On the Oberbeck-Boussinesq Approximation*. Math. Models Methods Appl. Sci. **6** (1996), 1157–1167.

[25] M.S. da Rocha, M.A. Rojas-Medar, M.D. Rojas-Medar, *On the existence and uniqueness of the stationary solution to equations of natural convection with boundary data in $L^2$*. Proc. R. Soc. Lond. A **459** (2003), 609–621.

[26] E.J. Villamizar-Roa, M.A. Rodriguez-Bellido, M.A. Rojas-Medar, *The Boussinesq system with mixed nonsmooth boundary data*. C. R. Acad. Sci. Paris, Ser. I **343** (2006), 191–196.

[27] R.L. Sani, P.M. Gresho, *Résumé and remarks on the open boundary condition minisymposium*, Int. J. Numer. Meth. Fl. **18** (1994), 983-1008.