Describing neutrino oscillations in matter with Magnus expansion

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We present new formalism for description of the neutrino oscillations in matter with varying density. The formalism is based on the Magnus expansion and has a virtue that the unitarity of the S-matrix is maintained in each order of perturbation theory. We show that the Magnus expansion provides better convergence of series: the restoration of unitarity leads to smaller deviations from the exact results especially in the regions of large transition probabilities. Various expansions are obtained depending on a basis of neutrino states and a way one splits the Hamiltonian into the self-commuting and non-commuting parts. In particular, we develop the Magnus expansion for the adiabatic perturbation theory which gives the best approximation. We apply the formalism to the neutrino oscillations in matter of the Earth and show that for the solar oscillation parameters the second order Magnus adiabatic expansion has better than 1% accuracy for all energies and trajectories. For the atmospheric $\Delta m^2$ and small 1-3 mixing the approximation works well (< 3% accuracy for $\sin^2 \theta_{13} = 0.01$) outside the resonance region (2.7 - 8) GeV.

PACS numbers: 14.60.Pq, 95.85.Ry, 14.60.Lm, 26.65.+t

I. INTRODUCTION

Neutrino physics enters the era of precision measurements, studies of the sub-leading oscillation effects and searches for new physics beyond the standard neutrino scenario. The neutrino flavor conversions become a tool of exploration of other particles and objects such as interiors of the Earth and stars. One of the key elements of these studies is neutrino oscillations in matter with varying density, and in particular, the oscillations inside the Earth. The latter is relevant for the solar, supernova and atmospheric neutrinos, as well as for the cosmic and accelerator neutrinos. In this connection it is important to have precise analytical or semi-analytical expressions for oscillation probabilities valid in wide energy ranges. These expressions allow us to simplify numerical computations but also to gain a deeper insight into physics involved. The results can be of special interest in view of discussions of future experiments with the megaton-scale fine structured underwater/underice detectors.

Several analytic and semi-analytic approaches to computing probabilities in matter with non-constant density have been developed recently which use various perturbation theories \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}. In the previous publications \cite{2, 3}, we have proposed a formalism which describes the neutrino oscillations in matter with low density. It make use of smallness of the matter potential $V$ in comparison with the kinetic term: $V \ll \Delta m^2/2E$, where $\Delta m^2$ is the mass squared difference and $E$ is the energy of neutrino. Essentially, the expansion parameter is given by the integral along the trajectory

$$I = \int dx \, V(x) \cos \phi(x),$$

where $\phi(x)$ is the adiabatic phase. The first approximation works very well at low energies $E < 20$ MeV \cite{2}. Validity of the results can be extended to higher energies if the second order term, $\sim I^2$, is taken into account \cite{2}. It can be further improved in certain energy ranges if expansion is performed with respect to the deviation of the potential from some average value.

The problem of this and some other similar approaches is that the unitarity of oscillation amplitudes is not guaranteed, and in fact, is violated at high energies \cite{2}. This violation, in turn, can produce certain problems in numerical computations. In this paper we propose the new type of perturbation theories which maintain the unitarity explicitly in each order of expansion, and therefore at any truncation of the series. The approach is based on the Magnus expansion \cite{12, 13} which was previously used for description of the nonadiabatic neutrino conversion in medium with monotonously varying density \cite{14, 12, 10}. Recently the first order Magnus expansion has been applied to the low energy neutrino oscillations in matter of the Earth \cite{10}. The formula for the regeneration factor in the Earth has been obtained which generalizes our result in \cite{3}. In this paper we develop various perturbation theories using explicitly two orders of the Magnus expansion. Since the Magnus expansion is an expansion in power of commutators, it is the second order that provides non-trivial new results. As a part of the present study we reproduce the formula from \cite{10}.

Essentially, the restoration of unitarity in the Magnus expansion is achieved by an effective re-summation of certain contributions to oscillation amplitudes. This leads to higher accuracy of the semi-analytic results and allows...
us to further extend the range of applications of the approach. Furthermore, it gives better understanding of the previously obtained results and their limits of validity.

We illustrate an accuracy of the approximations computing the transition probabilities for neutrinos crossing the core of the Earth. We find that for the solar oscillation parameters the second order Magnus adiabatic expansion has better than 1% accuracy for all energies and all trajectories. For the atmospheric $\Delta m^2$ and small 1-3 mixing the approximation works very well ($< 3\%$ accuracy for $\sin^2 \theta_{13} = 0.01$) below 2.7 GeV and above 8 GeV for $\sin^2 \theta_{13} = 0.01$. In the region, (2.7 - 8) GeV, where the MSW resonances in the core and in the mantle as well as the parametric resonances take place, a special consideration is required.

The paper is organized as follows. In sec. 2 we present the formalism of Magnus expansion and obtain general expressions for the S-matrix. We calculate the oscillation probabilities using various perturbation approaches based on the Magnus expansion in sec. 3. In particular, we develop the perturbation theory in $I$ and the adiabatic perturbation theory. We compare the results of different semi-analytic approaches in sec. 4. Conclusions follow in sec. 5.

II. MAGNUS EXPANSION

A. S-matrix and Magnus expansion

In what follows we will mainly study the case of 2$\nu$-mixing ($\nu_e, \nu_\alpha$), where $\nu_\alpha$ is, in general, some combination of $\nu_\mu$ and $\nu_\tau$. In a number practical cases the two neutrino results can be immediately embedded in the complete 3$\nu$ mixing scheme. The evolution matrix of neutrinos in matter, $S(x,x_0)$, obeys the first order (operator) differential equation,

$$i \frac{d S(x,x_0)}{d x} = H(x)S(x,x_0),$$

where the Hamiltonian $H(x)$ is given in the flavor basis by

$$H = \frac{MM^\dagger}{2E} + \hat{V} = \frac{1}{2E}U(\theta)M^2_{\Delta}U(\theta)^\dagger + \hat{V}. \quad (2)$$

Here $\hat{V} \equiv diag(V,0)$ is the matrix of potentials,

$$U(\theta) \equiv \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (3)$$

is the mixing matrix, and $M^2_{\Delta} \equiv diag(0, \Delta m^2)$ is the diagonal matrix of mass squared differences.

Formally, the solution of the equation (1) can be written as the chronological product

$$S(x_f,x_0) = Te^{-i \int_{x_0}^{x_f} H(x) \, dx} = \lim_{n \to \infty} e^{-iH(x_n)\Delta x} \cdot e^{-iH(x_{n-1})\Delta x} \cdots e^{-iH(x_1)\Delta x}, \quad (4)$$

$$\Delta x = \frac{x_f - x_0}{n}.$$

In our previous papers, [2], [3], we performed expansion of each exponential factor in eq. (4) and then took limit $n \to \infty$. Such a procedure does not guarantee the unitarity once the series is truncated and finite number of terms of the expansion is taken.

In this paper we will use expansions of powers of exponents and sum up contributions in the power without expansion of exponents themselves. Consequently, the form, $S = e^{-iC}$, of the S-matrix, and therefore, the unitarity are maintained since $C$ is a hermitian matrix. The Magnus expansion [12] has the following form

$$S = e^{-iC[H]} \equiv e^{-i(C_1+C_2+C_3+\cdots)}, \quad (5)$$

where functional $C[H]$ is a series in powers of commutators of the Hamiltonians taken in different points of neutrino
The term $C_k[H]$ contains commutators of order $k - 1$:

$$C_1 = \int_{x_0}^{x_f} dx \ H(x),$$

$$C_2 = -i \int_{x_0}^{x_f} \int_{x_0}^{x} dy \ [H(x), H(y)],$$

$$C_3 = \frac{(-i)^2}{6} \int_{x_0}^{x_f} \int_{x_0}^{x} \int_{x_0}^{y} dz \ \left( [H(x), [H(y), H(z)]] + [[H(x), H(y)], H(z)] \right).$$

The details of derivation of the functionals $C_k[H]$ are given in the Appendix. The representation of the $S$ matrix in eq. (5) with $C_i$ given in (6), (7), (8) is the main tool which we will use for different applications.

The Magnus expansion is an integral version of the Baker-Campbell-Hausdorff (BCH) equality. Recall that according to the BCH-equation, the summation of powers in exponents leads to

$$e^a \cdot e^b = e^{a+b+\frac{1}{2}[a,b]+\frac{1}{12}([a-b][a,b])+...},$$

that is, to appearance of commutator of the operators. In fact, in matter with varying density the Hamiltonians taken in different spatial points do not commute $[H(x_i), H(x_j)] \neq 0$.

The calculation of the $S$-matrix (4) requires an extension of the BCH-equality to a product of many exponential factors, and eventually, a transition to the continuous limit.

B. Properties of Magnus expansion.

Let us consider general properties of the Magnus expansion given in eqs. (5, 6, 7, 8).

1). If $H(x) = \text{constant}$, then $C_i = 0$ for $i > 1$, and therefore in the uniform medium the $S$-matrix is given by

$$S = e^{-i \int_{x_0}^{x_f} dx \ H(x)} = e^{-iH(x_f-x_0)},$$

This reproduces immediately the standard oscillation results. All corrections (due to the non-constant Hamiltonian) are given by the commutators. Essentially, the Magnus expansion is the expansion in the number of commutators.

2). The terms of the Magnus expansion contain factorials in denominator, therefore a convergence of the series is better than a convergence of the usual expansion (see eq. (89) in the Appendix). The Magnus series has good convergence even if $H$ is not small.

3). The commutators themselves may contain an additional smallness. The weaker dependence of $H$ on distance the smaller the commutators. So, in a sense, we deal here with a kind of adiabatic expansion.

4). If $H(x)$ is a symmetric function with respect to the middle point of a neutrino trajectory,

$$\bar{x} = \frac{x_f + x_0}{2},$$

that is,

$$H(x) = H(2\bar{x} - x),$$

one can show that $C_{2n} = 0$ ($n = 1, 2, ...$) [13], and only the odd terms in the expansion are non-zero. Let us prove that $C_2 = 0$ (general proof is given in [13]). According to eq. (7), the integration region $(y = x_0 \div x, \ x = x_0 \div x_f)$ is symmetric with respect to the diagonal line $y = 2\bar{x} - x$, that is, symmetric under reflection:

$$(x, y) \rightarrow (2\bar{x} - y, 2\bar{x} - x)$$

(x > y). Taking into account the symmetry of Hamiltonian it is easy to show that under the reflection the commutator $[H(x), H(y)]$ changes the sign. Therefore the integration of this commutator gives zero.
C. Magnus expansion in the “interaction” representation

Let us split the total Hamiltonian into two parts

$$H(x) = H_0(x) + \Upsilon(x)$$  \hspace{1cm} (11)

in such a way that $H_0(x)$ is self-commuting along a trajectory. That is, for any two points of the trajectory $x_i, x_j$: $[H_0(x_i), H_0(x_j)] = 0$. The rest of the Hamiltonian, $\Upsilon(x)$, is not self-commuting, in general, and if small can be treated as a perturbation. In this case it is convenient to solve the problem in the basis of new states, $\psi_I$, related to the initial basis by

$$\psi = U_I(x)\psi_I = e^{-i\int_{x_0}^x dt H_0(t)}\psi_I.$$  \hspace{1cm} (12)

Inserting this relation into the evolution equation we find that $\psi_I$, and the corresponding $S$–matrix, satisfy the evolution equation with the Hamiltonian $H_I \equiv \Upsilon_I$, where

$$\Upsilon_I(x, x_0) = U_I^\dagger(x)\Upsilon(x)U_I(x) = e^{i\int_{x_0}^x dt H_0(t)} e^{i\int_{x_0}^x dt \Upsilon(x)} e^{-i\int_{x_0}^x dt H_0(t)}.$$  \hspace{1cm} (13)

The transformation to new basis \cite{12} is equivalent to transition to a “interaction representation” if $H_0$ is interpreted as the Hamiltonian of free propagation. $\Upsilon_I$ can be considered as an operator in the interaction representation.

The evolution matrix in the interaction representation is given by

$$S_I(x_f, x_0) = e^{-iC[\Upsilon_I(x, x_0)]},$$  \hspace{1cm} (14)

that is, in the formulas \cite{5, 11, 5} one should substitute $H(x) \rightarrow \Upsilon_I(x, x_0)$. Then, according to eq. \cite{12} the $S$–matrix in the original basis equals

$$S(x_f, x_0) = U_I(x_f)S_I(x_f, x_0)U_I(x_0)^\dagger,$$  \hspace{1cm} (15)

or explicitly,

$$S(x_f, x_0) = e^{-i\int_{x_0}^{x_f} dt H_0(t)} e^{-iC[\Upsilon_I(x, x_0)]}.$$  \hspace{1cm} (16)

(The exponent on the RH side of this equality disappears because of the integration limits.) If $\Upsilon(x) \ll H_0(x)$, so that it can be considered as a small perturbation, a convergence of the series will be fast.

The Hamiltonian is self-commuting if its dependence on distance can be factorized:

$$H_0(x) = f(x) \cdot M, $$  \hspace{1cm} (17)

here $f(x)$ is an arbitrary function of $x$ and $M$ is an arbitrary constant matrix. Specific realizations of \cite{17} include constant ($x$-independent) Hamiltonians as well as the diagonal Hamiltonians $H_0(x) = \text{diag}[f_1(x), f_2(x)]$. In the latter case subtracting a matrix proportional to the unit matrix: $0.5(f_1 + f_2)\text{diag}(1, 1)$, one can reduce the Hamiltonian to the form \cite{17}.

In the case of small mixing (which can be achieved selecting certain basis of neutrino states) we can split the Hamiltonian as

$$H(x) = H^{\text{diag}}(x) + H^{\text{off-diag}}(x)$$

and identify $H^{\text{off-diag}}(x)$ with $\Upsilon$.

D. Evolution in symmetric potential

Let us consider a symmetric density profile so that the Hamiltonian satisfies the equality \cite{9}. In this case it is convenient to perform the integration in $C_i$ from the middle point of neutrino trajectory, $\bar{x}$, and to choose the evolution basis $\psi_I$, such that $\psi = U_I\psi_I$ with

$$U_I(x) = e^{-i\int_{\bar{x}}^x dt H_0(t)}.$$  \hspace{1cm} (18)
Essentially here we have substituted $x_0$ by $\bar{x}$. Now (similarly to the consideration in the previous section) the evolution matrix can be written as

$$S_I(x_f, x_0) = e^{-iC[I(x_f, \bar{x})]}, \quad (19)$$

where

$$\Upsilon_I(x, \bar{x}) = e^{i \int^x_{\bar{x}} H_0(t) dt} \Upsilon(x) e^{-i \int^x_{\bar{x}} H_0(t) dt}. \quad (20)$$

Then, the evolution matrix in the original basis equals

$$S(x, x_0) = \bar{U}_I(x) S_I(x, x_0) \bar{U}_I(x_0)^\dagger, \quad (21)$$

or explicitly, for an evolution from $x_0$ to $x_f$ we obtain

$$S(x_f, x_0) = e^{-i \int^x_{x_0} H_0(t) dt} e^{-i C[I(x_f, \bar{x})]} e^{-i \int^x_{\bar{x}} H_0(t) dt}. \quad (22)$$

Notice that in contrast to $\Upsilon(x)$ the operator $\Upsilon_I(x, \bar{x})$ has no definite symmetry with respect to the middle of a trajectory even for a constant density profile. Therefore the even coefficients $C_{2k}$, are non-zero:

$$\bar{C}_1 \equiv C_1[I(x_f, \bar{x})] = \int^x_{x_0} d \bar{x} \Upsilon_I(x),$$

$$\bar{C}_2 \equiv C_2[I(x_f, \bar{x})] = -\frac{i}{2} \int^x_{x_0} d \bar{x} \int^r_{x_0} dy [\Upsilon_I(x), \Upsilon_I(y)], \quad (23)$$

etc.. Here “bar” indicates that $\bar{C}_i$ have been calculated in the interaction representation with the $\bar{U}_I$-matrix integrated from the middle point of trajectory.

Let us introduce the variable

$$r \equiv x - \bar{x} = x - \frac{x_f + x_0}{2} \quad (24)$$

which is the distance from the middle of trajectory. Then

$$\bar{C}_1 = \int_{-L}^{L} dr \Upsilon_I(r),$$

$$\bar{C}_2 = -\frac{i}{2} \int_{-L}^{L} dr \int_{-L}^{r} dp [\Upsilon_I(r), \Upsilon_I(p)]. \quad (25)$$

Here

$$L \equiv \frac{x_f - x_0}{2} \quad (27)$$

and

$$\Upsilon_I(r) = e^{i \int^r_{x_0} H_0(t) dt} \Upsilon(r) e^{-i \int^r_{x_0} H_0(t) dt}. \quad (28)$$

Notice that the expressions $^{(25, 26, 28)}$ are valid for any density profile and we have not used yet any symmetry of the Hamiltonian.

Let us now assume that $V(x)$, and consequently, the Hamiltonian, are symmetric functions with respect to the middle point of a trajectory, $r = 0$, (as for neutrinos crossing the Earth). In this case $H_0$ and $\Upsilon$ are the even functions of $r$:

$$H_0(-r) = H_0(r), \quad \Upsilon(-r) = \Upsilon(r). \quad (29)$$

Denoting

$$\Phi_0 \equiv \int_0^r H_0(t) dt \quad (30)$$
we have
\[ \Phi_0(-r) = -\Phi_0(r) \]  \hspace{1cm} (31)
provided that \( H_0 \) is real. Let us show that in this case \( \hat{C}_1 \) and \( \hat{C}_2 \) are the real symmetric matrices. The proof is straightforward in the case of real \( \Upsilon \). The function \( \Upsilon_I(r) \) is not symmetric with respect to \( r = 0 \). Indeed, rewriting (28) as
\[ \Upsilon_I(r) = e^{i\Phi_0(r)} \Upsilon(r) e^{-i\Phi_0(r)}, \]  \hspace{1cm} (32)
one can see immediately that under \( r \rightarrow -r \)
\[ \Upsilon_I(-r) = \Upsilon_I(r)^*. \]  \hspace{1cm} (33)
Using this relation and the definition (25) we obtain
\[ \hat{C}_1^* = \int_{-L}^L dr \Upsilon_I(r)^* = \int_{-L}^L dr \Upsilon_I(-r) = \int_{-L}^L dr \Upsilon_I(r) = \hat{C}_1, \]
where in the last equality we made a substitution \( r \rightarrow -r \). Furthermore, since \( \hat{C}_1 \) is Hermitian, \( \hat{C}_1 = \hat{C}_1^\dagger \), i.e., the matrix is symmetric.

Similarly we can show that \( \hat{C}_2^* = \hat{C}_2 \). Here in addition to the property (33) and the change of the signs of variables, we use that
\[ \int_{-L}^L dr \int_{-L}^L dp [\Upsilon_I(r), \Upsilon_I(p)] = - \int_{-L}^L dr \int_{-L}^L dp [\Upsilon_I(r), \Upsilon_I(p)]. \]
Again, since \( \hat{C}_2 \) is Hermitian, the matrix \( \hat{C}_2 \) should be symmetric.

Performing integration in the expressions for \( \hat{C}_i \) (25) from the middle point of a trajectory we obtain
\[ \hat{C}_1 = 2 \int_0^L dr \text{Re} \Upsilon_I(r), \]
\[ \hat{C}_2 = 2 \int_0^L dr \int_0^r dp [\text{Im} \Upsilon_I(r), \text{Re} \Upsilon_I(p)] \]  \hspace{1cm} (34)
from which we immediately conclude that \( \hat{C}_i \) are real.

As we will see in sect. III.C, in the adiabatic perturbation theory \( \Upsilon \) is purely imaginary matrix. Moreover, since \( \Upsilon \propto dV/dx \), for a symmetric potential we have the antisymmetric \( \Upsilon \). So,
\[ \Upsilon(r)^* = -\Upsilon(r), \quad \Upsilon(-r) = -\Upsilon(r), \]  \hspace{1cm} (35)
and therefore \( \Upsilon(-r) = \Upsilon(r)^* \). Using the equalities (35) one can show that in this case \( \Upsilon_I(r) \) also satisfies the equality (33), and consequently the matrices \( \hat{C}_i \) can be calculated as in eq. (34).

For a symmetric potential using the property (31) we can write the S-matrix in the original basis (22) as
\[ S(x_f, x_0) = e^{-i\Phi_0(L)} e^{-iC[\Upsilon_I(x, \bar{x})]} e^{-i\Phi_0(L)}. \]  \hspace{1cm} (36)

### III. OSCILLATION PROBABILITIES

In applications of the Magnus expansion, adjusting the formalism to a specific physical situation we can select
- propagation basis, that is, the basis of neutrino states in which we consider evolution;
- split of the Hamiltonian into self-commuting and non-commuting parts;
- perturbation terms.

In what follows we will consider a symmetric density profile keeping in mind applications to the neutrino propagation inside the Earth.
A. Low energy and low density limit

In the low energy or/and low density case it is convenient to consider the neutrino evolution in the mass eigenstates basis, $\nu_{\text{mass}} = (\nu_1, \nu_2)$. In this basis the Hamiltonian can be written as

$$H(x) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta m^2 / 2E \end{pmatrix} + U^\dagger \begin{pmatrix} V(x) & 0 \\ 0 & 0 \end{pmatrix} U,$$

(37)

where $U$ is the vacuum mixing matrix (3). We split the Hamiltonian, according to (11), in the following way. The self-commuting part can be chosen as

$$H_0(x) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta^m(x) \end{pmatrix},$$

(38)

where $\Delta^m(x)$ is the difference of the instantaneous eigenvalues of the Hamiltonian (37):

$$\Delta^m(x) \equiv \Delta m^2 / 2E \sqrt{\left(\cos 2\theta - \frac{2EV(x)}{\Delta m^2}\right)^2 + \sin^2 2\theta}.$$  

(39)

Then, according to (37), the perturbation part equals

$$\Upsilon(x) = A(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B(x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(40)

where

$$A(x) = \frac{1}{2} \sin 2\theta \ V(x),$$

$$B(x) = \frac{1}{2} \left[ \Delta^m(x) - \Delta m^2 / 2E + V(x) \cos 2\theta \right].$$

(41)

For a weak potential $V$: $V \ll \Delta m^2 / 2E$, we have

$$B(x) = \frac{1}{4} (V \sin 2\theta)^2 \frac{2E}{\Delta m^2} + O(V^3) \approx A^2(x) \frac{2E}{\Delta m^2}.$$  

(42)

According to (38) the matrix of transition to the interaction representation equals

$$\bar{U}_I(x) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi(x)} \end{pmatrix},$$

(43)

where

$$\phi(x) \equiv \int_0^x \Delta^m(r) \ dr$$

(44)

is the adiabatic phase (here the integration runs from the middle point of a trajectory). Then the Hamiltonian in the interaction representation, $\Upsilon_I(x) = \bar{U}^\dagger(x) \Upsilon(x) \bar{U}(x)$ can be written as

$$\Upsilon_I(x) = A(x) \begin{pmatrix} 0 & e^{-i\phi(x)} \\ e^{i\phi(x)} & 0 \end{pmatrix} + B(x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(45)

Using this expression and eqs. (34) we obtain

$$\bar{C}_1 + \bar{C}_2 = Z(L) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + Y(L) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(46)

with

$$Z(L) = 2 \int_0^L \ dr A(r) \cos \phi(r) + 4 \int_0^L \ dr \int_0^r \ dp \ A(r) B(p) \sin \phi(r),$$

$$Y(L) = 2 \int_0^L \ dr B(r) - 4 \int_0^L \ dr \int_0^r \ dp \ A(r) A(p) \sin \phi(r) \cos \phi(p).$$

(47)
Let us estimate these quantities with accuracy \( \sim V^2 \). Since \( A \sim V \) and \( B \sim V^2 \), the last term in \( Z \), being of the order \( V^3 \), can be neglected. For the function \( Y(x) \) performing integration by parts in the second integral we have

\[
Y(L) = 2 \int_0^L dr \left[ B(r) - 2 \frac{A(r)}{\Delta m^2(r)} \right] + 4 \int_0^L dr \int_0^r dp \frac{d}{dr} \left[ \frac{A(r)}{\Delta m^2(r)} \right] \frac{d}{dp} \left[ \frac{A(p)}{\Delta m^2(p)} \right] \sin \phi(p) \cos \phi(r)
\]

\[
= \sin^2 2\theta \int_0^L dr \int_0^r dp \frac{d}{dr} \left[ \frac{V(r)}{\Delta m^2(r)} \right] \frac{d}{dp} \left[ \frac{V(p)}{\Delta m^2(p)} \right] \sin \phi(p) \cos \phi(r) + O(V^3),
\]

where in the last equality we used expression (42).

Neglecting \( Y(x) \) we find

\[
\tilde{C} \simeq \tilde{C}_1 + \tilde{C}_2 \approx I_V \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and

\[
I_V \equiv \sin 2\theta \int_0^L dr V(r) \cos \phi(r).
\]

Using eqs. (50), (49) and (48) we obtain the \( S \)-matrix (16) in the mass-eigenstates basis

\[
S = \begin{pmatrix} 1 & 0 \\ 0 & \cos I_V \cos 2\theta \end{pmatrix} \begin{pmatrix} \cos I_V & -i \sin I_V \\ -i \sin I_V & \cos I_V \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \cos I_V \end{pmatrix}.
\]

Here \( \phi \) is the half of the oscillation phase:

\[
\phi = \phi_{\tilde{x} \rightarrow x} = \phi_{x_0 \rightarrow \tilde{x}} = \phi(L).
\]

Notice that both the matrix that originates from the self-commuting part and the perturbation, \( I_V \), depend on the same adiabatic phase.

For the transition between the mass states we have immediately from (51):

\[
P_{\nu_2 \rightarrow \nu_1} = |S_{21}|^2 = \sin^2 I_V.
\]

The \( S \)-matrix for the mass-to-flavor transitions equals

\[
S_{\text{mass-flavor}} = U(\theta) \cdot S,
\]

and the \( \nu_i \rightarrow \nu_\alpha \) probability is

\[
P_{\nu_i \rightarrow \nu_\alpha} = |(U \cdot S)_{i\alpha}|^2.
\]

From (52), (51) and (53) we obtain

\[
P_{\nu_2 \rightarrow \nu_\alpha} = \sin^2 \theta + \frac{1}{2} \sin 2\theta \sin 2I_V \sin \phi + \cos 2\theta \sin^2 I_V,
\]

where the first term is simply projection squared of \( \nu_2 \) state onto \( \nu_\alpha \). Eq. (55) reproduces the formula given in (10). If \( |I_V| \ll 1 \), we find making expansion in powers of \( I_V \)

\[
P_{\nu_2 \rightarrow \nu_\alpha} = \sin^2 \theta + I_V \sin 2\theta \sin \phi + I_V^2 \cos 2\theta
\]

which exactly coincides with our result in (3) (see eq. (15)). In a sense, the result (55) corresponds to a re-summation of certain contributions to the probability. It is the substitution \( I_V \rightarrow \sin I_V \) that restores the unitarity. Notice that \( I_V = \sin 2\theta \cdot I \), where \( I \) is the integral used as the expansion parameter in (3). According to the present result (55) the expansion parameter includes also \( \sin 2\theta \) which makes convergence even better in the case of small vacuum mixing. Our present consideration explains also the reason why the second order effect in ref. (3) depends on the same integral \( I \).

The \( S \)-matrix for transitions between the flavor states equals

\[
S_{\text{flavor-flavor}} = U \cdot S \cdot U^\dagger.
\]
In particular, for the $\nu_e \rightarrow \nu_\alpha$ channel we obtain

$$P_{\nu_e \rightarrow \nu_\alpha} = \cos^2 I_V \sin^2 2\theta \sin^2 \phi + \frac{1}{2} \sin 2I_V \sin 4\theta \sin \phi + \sin^2 I_V \cos^2 2\theta$$

$$= (\cos I_V \sin 2\theta \sin \phi + \sin I_V \cos 2\theta)^2. \quad (57)$$

In the limit $V \rightarrow 0$, we have $I_V \rightarrow 0$ and the first term reproduces the standard vacuum oscillation probability. For small $I_V$ the following form of the probability can be useful:

$$P_{\nu_e \rightarrow \nu_\alpha} = \sin^2 2\theta \sin^2 \phi + \frac{1}{2} \sin 2I_V \sin 4\theta \sin \phi + \sin^2 I_V (\cos^2 2\theta - \sin^2 2\theta \sin^2 \phi). \quad (58)$$

The result in the second order of the Magnus expansion can be obtained keeping term proportional to $Y(x)$ in $\tilde{C}$ \cite{46}. Straightforward calculations give

$$S = \left( \begin{array}{cc} \cos X - i\sqrt{2} \sin X & -ie^{-i\phi} e^{-i\phi} \sin X \\ -ie^{-i\phi} e^{-i\phi} \sin X & \cos X + i\sqrt{2} \sin X \end{array} \right), \quad (59)$$

where $X = \sqrt{Z^2 + Y^2}$. Apparently, the result \cite{51} follows from this expression in the limit $Y \rightarrow 0$, $Z \rightarrow I_V$.

**B. Perturbation around average potential $V_0$**

Let us consider the same situation as in the previous section but perform the expansion with respect to an average potential $V_0$. This means that we use the basis of neutrino eigenstates in matter with constant potential $V_0$, as the propagation basis. These eigenstates are related to the flavor states by the mixing matrix in matter

$$\nu_f = U(\theta_0^m)\nu_0^m, \quad (60)$$

where $U$ is defined in \cite{3} and $\theta_0^m = \theta^m(V_0)$ is the mixing angle in matter with the potential $V_0$, the angle $\theta^m(V)$ is given by

$$\sin 2\theta^m(V) = \frac{\sin 2\theta}{\sqrt{(\cos 2\theta - 2EV/\Delta m^2)^2 + \sin^2 2\theta}}. \quad (61)$$

In the $\nu_0^m$-basis the Hamiltonian equals

$$H(x) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Delta_0^m \end{array} \right) + U^\dagger(\theta_0^m) \left( \Delta V(x) \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) U(\theta_0^m), \quad (62)$$

where $\Delta_0^m$ is the difference of the eigenvalues in matter with the potential $V_0$, and

$$\Delta V(x) = V(x) - V_0.$$

We split the Hamiltonian into the self-commuting part and the perturbation using the same $H_0$ as in the previous case \cite{38}. Then the perturbation equals

$$\Upsilon(x) = \frac{1}{2} \sin 2\theta \Delta V(x) \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right) + \frac{1}{2} \Delta m^2 + \Delta V(x) \cos 2\theta_0^m \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right). \quad (63)$$

Consequently, for the matrix $\tilde{C}$ we obtain the same expression as in Eq. \cite{49} with substitution $I_V \rightarrow I'_V$, where

$$I'_V = \sin 2\theta_0^m \int_{0}^{L} \Delta V(x) \cos \phi(x) \, dx. \quad (64)$$

In turn, $I'_V$ differs from $I_V$ by the substitutions $V \rightarrow \Delta V$ and $\theta \rightarrow \theta_0^m$.

The $S$-matrix in the $\nu_0^m$-basis equals

$$S_0^m = \left( \begin{array}{cc} \cos I'_V & -ie^{-i\phi} \sin I'_V \\ -ie^{i\phi} \sin I'_V & e^{-2i\phi} \cos I'_V \end{array} \right).$$
and the phase $\phi$ is defined in (52).

Since $\nu = U^\dagger (\theta) U(\theta^m) \nu^m = U(\theta^m - \theta) \nu^m$, the $S$-matrix of the mass-to-flavor transitions equals

$$S_{\text{mass-flavor}} = U(\theta^m) \cdot S_0^m \cdot U(\theta^m - \theta) \dagger.$$ 

Then the $\nu_2 \rightarrow \nu_e$ probability is

$$P_{\nu_2 \rightarrow \nu_e} = \cos^2 I' \sin^2 \theta + \sin 2 \theta^m \sin 2(\theta^m - \theta) \sin^2 \phi + \frac{1}{2} \sin 2 I' \sin 2(\theta^m - \theta) \sin \phi + \sin^2 I' \cos^2 2(\theta^m - \theta).$$

(65)

Apparently this expression is reduced to the one in eq. (55), if $\theta^m_0 = \theta$.

For the mass-to-mass transitions the $S$-matrix equals

$$S_{\text{mass-mass}} = U(\theta^m_0 - \theta) \cdot S_0^m \cdot U(\theta^m_0 - \theta) \dagger,$$

and therefore the probabilities are given by the same expressions as for the flavor-to-flavor transitions in the previous section with the substitutions $\theta \rightarrow (\theta^m_0 - \theta)$ and $I_V \rightarrow I'_V$:

$$P_{\nu_2 \rightarrow \nu_e} = \cos^2 I' \sin^2 \theta + \sin 2 \theta^m \sin 2(\theta^m_0 - \theta) \sin^2 \phi + \frac{1}{2} \sin 2 I' \sin 2(\theta^m_0 - \theta) \sin \phi + \sin^2 I' \cos^2 2(\theta^m_0 - \theta)$$

$$= [\cos I' \sin 2(\theta^m_0 - \theta) \sin \phi + \sin I' \cos 2(\theta^m_0 - \theta)]^2.$$ 

(66)

For the flavor-to-flavor transition we have

$$S_{\text{flavor-flavor}} = U(\theta^m_0) \cdot S_0^m \cdot U(\theta^m_0) \dagger.$$ 

Consequently, the probability follows immediately from (66) substituting $(\theta^m_0 - \theta) \rightarrow \theta^m_0$:

$$P_{\nu_2 \rightarrow \nu_e} = \cos^2 I' \sin^2 \theta + \sin 2 \theta^m \sin 2(\theta^m_0 - \theta) \sin^2 \phi + \frac{1}{2} \sin 2 I' \sin 2(\theta^m_0 - \theta) \sin \phi + \sin^2 I' \cos^2 2(\theta^m_0 - \theta)$$

$$= (\cos I' \sin 2 \theta^m_0 \sin \phi + \sin I' \cos 2 \theta^m_0)^2.$$ 

(67)

An interesting feature of the obtained results is that the probabilities for symmetric transitions: the flavor-to-flavor and mass-to-mass ones can be written as a square of the sum of two terms proportional to $\cos I_V$ and $\sin I_V$.

C. Adiabatic perturbation theory in Magnus expansion

Let us again consider symmetric density profile. As the propagation basis, we take the basis of the eigenstates of instantaneous Hamiltonian, $\nu^m = (\nu_1^m, \nu_2^m)$:

$$\nu_I = U(\theta^m(x)) \nu^m.$$ 

Here $\theta^m(x)$ is the instantaneous mixing angle in matter (61). The Hamiltonian for the eigenstates equals $H(x) = H_0 + \Upsilon_\theta(x)$, where

$$H_0(x) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta^m(x) \end{pmatrix}, \quad \Upsilon_\theta(x) = \dot{\theta}^m(x) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$ 

(68)

and

$$\dot{\theta}^m(x) \equiv \frac{d \theta^m(x)}{dx} = \frac{\sin 2 \theta^m(x)}{2 \Delta^m(x)} \frac{dV(x)}{dx}.$$ 

(69)

In what follows we will use $H_0$ and $\Upsilon_\theta(x)$ as the self-commuting and perturbation parts correspondingly. Notice that the self-commuting part is the same as before, but the perturbation is different since the basis of states differs from the one we used before. Now $\Upsilon_\theta(x)$ is a complex and non-symmetric matrix with respect to the middle of trajectory. Straightforward calculations give according to (23) or (54)

$$\bar{C}_1 = I_\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{C}_2 = I_{\theta \theta} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$ 

(70)
where

\[ I_{\theta} = -2 \int_{x}^{x_f} \hat{\theta}^m(x) \sin \phi_{x-x} dx = \]

\[ = 2 \int_{x}^{x_f} [\theta^m(x) - \theta^m_0] \Delta^m(x) \cos \phi_{x-x} dx, \quad (71) \]

\[ I_{\theta\theta} = - \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} \hat{\theta}^m(x) \hat{\theta}^m(y) \sin \phi_{y-x} dy = \]

\[ = 4 \int_{x}^{x_f} dx \int_{x}^{x_f} \hat{\theta}^m(x) \hat{\theta}^m(y) \sin \phi_{x-y} \cos \phi_{x-x} dy. \quad (72) \]

Here \( \theta^m = \theta^m(x_0) = \theta^m(x_f) \) is the mixing angle at the surface of the Earth. Taking into account (69) one sees that \( I_{\theta\theta} \) has the same structure as the integral in \( Y(x) \) (15).

Neglecting the second order term \( \propto I_{\theta\theta} \), we have \( \bar{C} = \bar{C}_1 \) which coincides, according to (70), with total \( \bar{C} \) in eq. (49) up to the change \( I_{V'} \rightarrow I_{\theta} \). Therefore the adiabatic probabilities equal to those in the previous subsection with the substitutions \( I_{V'} \rightarrow I_{\theta} \), \( \theta^m \rightarrow \theta^m_s \):

\[ P_{\nu_2 \rightarrow \nu_e} = \cos^2 I_{\theta} \sin^2 \theta + \cos^2 I_{\theta} \sin 2\theta^m_s \sin 2(\theta^m_s - \theta) \sin^2 \phi + \]

\[ + \frac{1}{2} \sin 2I_{\theta} \sin 2(\theta^m_s - \theta) \sin \phi + \sin^2 I_{\theta} \cos^2 (2\theta^m_s - \theta), \quad (73) \]

\[ P_{\nu_2 \rightarrow \nu_1} = \cos^2 I_{\theta} \sin^2 2(\theta^m_s - \theta) \sin^2 \phi + \frac{1}{2} \sin 2I_{\theta} \sin 4(\theta^m_s - \theta) \sin \phi \]

\[ + \sin^2 I_{\theta} \cos^2 (2\theta^m_s - \theta), \quad (74) \]

\[ P_{\nu_e \rightarrow \nu_\alpha} = \cos^2 I_{\theta} \cos^2 2\theta^m_s \sin^2 \phi + \frac{1}{2} \sin 2I_{\theta} \sin 4\theta^m_s \sin \phi + \]

\[ + \sin^2 I_{\theta} \cos^2 2\theta^m_s. \quad (75) \]

Notice that \( I_{\theta} \approx I_{V'} \), when \( \theta^m - \theta^m_s \ll 1 \).

Let us take into account the second order of the Magnus expansion. Now \( \bar{C} \) contains the term proportional to the diagonal matrix. Apparently, \( \bar{C} \) has the same form as in (46) with the substitutions \( Z \rightarrow I_{\theta} \) and \( Y \rightarrow I_{\theta\theta} \). So, using the results (58), (71), (18) and (22), we find the \( S \)-matrix in the basis of eigenstates of the Hamiltonian,

\[ S_m = \left( \begin{array}{ccc}
\cos I_t - i \frac{I_{\theta} \theta^m_s}{I_t} \sin I_t & -i \frac{I_{\theta}}{I_t} \sin I_t e^{-i\phi} \\
-i \frac{I_{\theta} t}{I_t} \sin I_t e^{-i\phi} & \cos I_t + i \frac{I_{\theta} t}{I_t} \sin I_t e^{-2i\phi} \end{array} \right). \quad (76) \]

Here

\[ I_t = \sqrt{I_{\theta}^2 + I_{\theta\theta}^2}, \quad (77) \]

and the adiabatic phase \( \phi \) is defined in (52). The \( S \)-matrix for the flavor-to-flavor transitions is then given by

\[ S_{\text{flavor-flavor}} = U(\theta^m_s) \cdot S_m \cdot U^\dagger(\theta^m_s). \quad (78) \]

For the probability of \( \nu_e \rightarrow \nu_\alpha \) oscillations, \( P_{\nu_e \rightarrow \nu_\alpha} = |(S_{\text{flavor-flavor}})_{ee}|^2 \), we obtain explicitly

\[ P_{\nu_e \rightarrow \nu_\alpha} = \left[ \sin 2\theta^m_s \cos I_t \sin \phi + \sin I_t(I_{\theta} \cos 2\theta^m_s - I_{\theta\theta} \sin 2\theta^m_s \cos \phi) \right]^2. \quad (79) \]

The \( S \)-matrix for the mass-to-flavor transitions equals

\[ S_{\text{mass-flavor}} = U(\theta^m_s) \cdot S_m \cdot U^\dagger(\theta^m_s - \theta). \quad (80) \]
In particular, the $\nu_2 \rightarrow \nu_e$ - probability can be calculated as $P_{\nu_2 \rightarrow \nu_e} = |(S_{\text{mass-flavor}})_{e2}|^2$; and explicitly we obtain

$$P_{\nu_2 \rightarrow \nu_e} = \left[ \sin(2\theta^m - \theta) \cos I_t \sin \phi + \frac{\sin I_t I_{\theta}}{I_t} (I_{\theta} \cos(2\theta^m - \theta) - I_{\theta\theta} \sin(2\theta^m - \theta) \cos \phi) \right]^2$$

$$+ \sin^2 \theta \left[ \cos I_t \cos \phi + \frac{\sin I_t I_{\theta\theta}}{I_t} \sin \phi \right]^2. \quad (81)$$

Notice that the adiabatic perturbation theory is essentially a series in

$$\frac{\dot{\theta}_m}{\Delta m} = V \frac{\sin 2\theta \Delta m^2}{2(\Delta m)^3}, \quad (82)$$

i.e., in gradient of the potential rather that in $2VE/\Delta m^2$. Therefore this theory is applied also for $2VE/\Delta m^2 > 1$. The largest value of the parameter $[82]$, at least for small vacuum mixing, is achieved in the MSW-resonance, where

$$\frac{\dot{\theta}_m}{\Delta m} = \frac{1}{2\pi} \frac{V}{V} \frac{l_\nu}{2 \sin 2\theta \tan 2\theta} = \frac{1}{2\pi} \frac{l^{R\nu}_{res}}{\Delta r_R}. \quad (83)$$

Here $l_\nu \equiv 4\pi E/\Delta m^2$ and $l^{R\nu}_{res} \equiv l_\nu/\sin 2\theta$ are the oscillation lengths in vacuum and in matter with the resonance density, $\Delta r_R \equiv 2 \tan 2\theta (V/V)$ is the spatial width of the resonance layer. So, the approximation is not expected to work well in resonance for small mixing.

**IV. ACCURACY OF SEMI-ANALYTIC APPROXIMATIONS**

To illustrate an accuracy of the obtained semi-analytical results we consider neutrino oscillations along the trajectory which crosses the center of the Earth (the central trajectory). We take the 5-layer approximation for the Earth density profile [9]. We compute $P_{\text{exact}}$ using exact numerical method, and $P_{\text{analytic}}$ - the approximate probabilities, using different semi-analytic formulas obtained in this paper. The Table I lists approximations we use to produce the figures with indication of abbreviations and references to the corresponding formulas in the text.
FIG. 2: The deviation of the approximate value of the $\nu_2 \rightarrow \nu_e$ probability from the exact value as a function of neutrino energy. The lines correspond to the second order of usual (non-unitary) perturbation theory (dot-dashed); the second order Magnus expansion with shifted potential (dashed); the first order adiabatic Magnus expansion (dotted); the second order adiabatic Magnus expansion (solid).

**TABLE I: Approximations**

| Notation | Approximation                                      | Equation |
|----------|---------------------------------------------------|----------|
| 1MA      | first order Magnus adiabatic expansion            | (73)     |
| 2US      | second order usual expansion with shifted potential | (85)     |
| 2MS      | second order Magnus expansion with shifted potential | (65)     |
| 2MA      | second order Magnus adiabatic assumption          | (81)     |

In Fig. 1 we show the probabilities of $\nu_2 \rightarrow \nu_e$ oscillations driven by the parameters $\Delta m^2 = 7 \cdot 10^{-5}$ eV$^2$ and $\sin^2 \theta_{12} = 1/3$. Fig. 2 presents the differences of the semi-analytic and exact results,

$$
\Delta P \equiv P_{\text{analytic}} - P_{\text{exact}},
$$

as functions of the neutrino energy. The solid line in Fig. 1 shows $P_{\text{exact}}$. Apparently, the probabilities and the differences of probabilities increase with energy; the probabilities become of the order 1 in the resonance region $E \sim 100$ MeV.

Let us discuss the quality of different approximations.

- The dot-dashed lines show $P_{\text{analytic}}$ (Fig. 1) and $\Delta P$ (Fig. 2) computed in the second order of the usual perturbation theory in (practically in $I'_V$) with a shifted potential, 2US. The probability is given by an expansion of the expression (65) in powers of $I'_V$:

$$
P_{\nu_2 \rightarrow \nu_e} = \sin^2 \theta + \sin 2\theta_0^m \sin 2(\theta_0^m - \theta) \sin^2 \phi + I'_V \sin 2(\theta_0^m - \theta) \sin \phi + (I'_V)^2 [\cos^2(2\theta_0^m - \theta) - \sin^2 \theta - \sin 2\theta_0^m \sin 2(\theta_0^m - \theta) \sin^2 \phi].
$$

This probability coincides with our result in [3]. We use the average value of potential, $V_0$, that corresponds to the electron density $n_e = 1.92 \times 10^6$ $N_A$ $mol/cm^3$, where $N_A$ is the Avogadro number. For the central trajectory the probability [53], satisfies inequality $P \leq 1$. However, for some other trajectories, e.g., with the nadir angle $\Theta \sim 10^0$, the unitarity is violated.
In the first order of usual perturbation theory, the probability is given by eq. (85) without last term (the line is not shown in the figure). It becomes $P > 1$ for the central trajectory in the region (80 - 90) MeV reflecting the violation of unitarity.

The dashed line in fig. 1 shows the probability (65) computed in the second order Magnus expansion, $2MS$, with $\Delta V (I')$ as the perturbation. The unitarity is restored and $P \leq 1$ for all energies and for all the trajectories. The difference of probabilities $\Delta P$ is shown in fig. 2. At high energies (large $I'$) this probability gives substantially better approximation than the non-unitary one: the deviation is below 5%. At low energies, $E < 45$ MeV, (small $I'$) both approximations have similar accuracy. As follows from the figure the deviation $\Delta P$ at high energies becomes even smaller: below 2% in the range 80 - 100 MeV.

$P_{analytic}$ and $\Delta P$ calculated in the adiabatic perturbation theory in the first order of the Magnus expansion $1MA$ (73) are shown by the dotted lines. According to the figures a quality of the first order adiabatic approximation with restored unitarity is similar to that of the second order in the $\Delta V-$ perturbation theory (the previous case). This means that the adiabatic perturbation theory is more relevant in combination with the Magnus expansion than the usual perturbation theory (practically in $VE/\Delta m^2$). The adiabatic perturbation theory gives better re-summation of the series. The comparable qualities of these approximations are related also to the fact that in both cases we have taken the same values of the electron density: the surface density in the adiabatic case and the average density in the $\Delta V-$ perturbation, so that $\theta_s = \theta_0$. Furthermore, as we have mentioned in the sec. IIIC. the true expansion parameter is $\dot{\theta}_m \propto V$, and the perturbation theory works also for $VE/\Delta m^2 \geq 1$.

The solid line in fig. 2 shows the difference of probabilities for $P_{analytic}$ computed in the second order of the adiabatic Magnus expansion (81) $2MA$. The second order expansion further improves approximation for all energies and especially in the range (50 -65) MeV and at $E > 70$ MeV. In fact, the approximation works well in whole energy range: below the resonance, in the resonance and above. The adiabaticity is well satisfied due to large value of the vacuum mixing angle. To illustrate this in fig. 3 we show the probabilities computed in different approximations at high energies - above the resonance.

Similar picture appears for other neutrino trajectories. In fig. 4 we show dependence of the integral error of the approximations defined as

\[
\sigma \equiv \frac{1}{E_{max} - E_{min}} \int_{E_{min}}^{E_{max}} (P_{analytic} - P_{exact})^2 dE
\]  

(86)
FIG. 4: Dependence of the energy integrated errors of various approximations on the nadir angle (in radians) of neutrino trajectory. The errors (in units $10^{-4}$) are computed for the $\nu_2 \to \nu_e$ oscillation channel with parameters $\Delta m^2 = 7 \cdot 10^{-5} \text{ eV}^2$ and $\sin^2 \theta_{12} = 1/3$. The lines correspond to different approximations as in fig. 2.

on the nadir angle $\Theta$ (in radians). We show the range of the angles which corresponds to the core crossing trajectories; $\Theta = 0$ determines the central trajectory considered above. We take $E_{\text{min}} = 40 \text{ MeV}$ and $E_{\text{max}} = 90 \text{ MeV}$. As follows from the figure, the Magnus expansion gives much better approximation than the usual (non-unitary) perturbation theory. Again, the accuracy of the first order adiabatic Magnus expansion $1MA$ and the second order Magnus expansion in $\Delta V$, $2MS$, are comparable. The second order adiabatic Magnus expansion, $2MA$, gives much better approximation for all the energies.

According to fig. 4 the errors become very small for $\Theta \to 0.58$ which corresponds to the only-mantle crossing trajectories. This means that the proposed approximations have even higher accuracy for neutrinos propagating only in the mantle.

In fig. 5 we compare the $\nu_e \to \nu_\alpha$ probabilities due to $\Delta m^2 = 2 \cdot 10^{-3} \text{ eV}^2$ and $\sin^2 \theta = 10^{-2}$. The solid line is the result of exact computations. Comments on the accuracy of different approximations follow.

- The dot-dashed line is a result of the second order of the usual non-unitary $\Delta V$ - perturbation theory. It corresponds to expansion of the probability (67):

$$P_{\nu_e \to \nu_\alpha} = \sin^2 2\theta^m_0 \sin^2 \phi + I'_V \sin 4\theta^m_0 \sin \phi + (I'_V)^2 (\cos^2 2\theta^m_0 - \sin^2 2\theta^m_0 \sin^2 \phi).$$

The approximation work well at $E < 1.7 \text{ GeV}$, where the probability is small $P < 0.3$. For higher energies it fails completely. According to the figure at $E > 3 \text{ GeV}$ this probability becomes negative indicating a violation of the unitarity.

- The green dotted line shows the probability in the first order of the adiabatic Magnus expansion $1MA$. The Magnus expansion allows us to expand the region up to 2.7 GeV, i.e. practically up to the resonance in the core of the Earth.

- The dash-dotted line represents the probability in the second order of the adiabatic Magnus expansion $2MA$. It has even better accuracy: For $E = 2.3 \text{ GeV}$ we obtain $\Delta P \sim 0.05$ for the first order and $\Delta P \sim 0.02$ for the second one. The approximation becomes invalid for $E > 2.8 \text{ GeV}$ because the adiabaticity is broken in the resonance.
FIG. 5: The probabilities of the $\nu_e \to \nu_\alpha$ transition as functions of neutrino energies computed in various approximations. The lines correspond to the exact numerical calculations (solid), the 2nd order of usual non-unitary expansion (dot-dashed), the first order adiabatic Magnus expansion (dotted) and the second order adiabatic Magnus expansion (dashed). The values of oscillation parameters are $\Delta m^2 = 2 \cdot 10^{-3}$ eV$^2$ and $\sin^2 \theta_{13} = 0.01$.

Let us underline that the Magnus expansion allows one to extend the application of approximation to the region where the probabilities are large. The semianalytic result does not work in the resonance region. It gives good approximation above 8 GeV, that is, above the resonance in the mantle.

Let us compare an accuracy of our semi-analytic results with the exact results of calculations for the widely used two-layer density approximation of the Earth profile [17]. In this approximation the densities of the mantle and the core of the Earth are taken to be constant and equal to the mean densities in the mantle and the core along a given neutrino trajectory. Fig. 6 shows the $(\nu_2 \to \nu_e)$ probabilities for the central trajectory for the exact (5 layers) density profile (solid line) and the two-layer density approximation (dotted line). At high energies, $E = 60 - 70$ MeV, the accuracy of approximation is about 4 - 5%. The accuracy becomes worse with a decrease of energy: at $E \sim 45$ MeV and below, it is about (20 - 30)%. Partly the loss of accuracy is related to the fact that at low energies the propagation becomes more adiabatic and therefore the result of propagation in the mantle is determined by the density at the surface of mantle, rather than the average density.

Comparing fig. 6 and fig. 1 we conclude that the semianalytic results based on the Magnus expansion give better approximation outside the resonance regions than the exact results obtained for the two-layer model of the Earth density profile.

Let us finally comment on embedding of our 2$\nu$-results in the complete 3$\nu$-mixing framework. In certain limits relevant for applications the dynamics of 3$\nu-$system is reduced to the dynamics of 2$\nu-$system. These include the limits of low energies (substantially below the 1-3 resonance energy), and high energies (substantially larger than the 1-2 resonance energy).

Let us consider the low energy case, $E < 100$ MeV. At low energies one can neglect the matter effect on the 1-3 mixing, and furthermore, the oscillations related to the third mass eigenstate (separated by the atmospheric $\Delta m^2$) are averaged out. This state essentially decouples from the dynamics and evolves independently. In this case it is straightforward to show that, e.g., the 3$\nu$-probability of the $\nu_2 \to \nu_e$ transition, $P^{(3\nu)}_{\nu_2 \to \nu_e}$, is given by

$$P^{(3\nu)}_{\nu_2 \to \nu_e} = \cos^2 \theta_{13} P_{\nu_2 \to \nu_e} (\theta_{12}, \Delta m^2_{21}, V \cos^2 \theta_{13}).$$

Here $P_{\nu_2 \to \nu_e}$ is the two neutrino probability derived in this paper, (see eqs. [55] [65] [66]) which should be computed using the reduced value of the potential: $V \cos^2 \theta_{13}$. 

\[\text{FIG. 6: The probabilities of the } \nu_e \to \nu_\alpha \text{ transition as functions of neutrino energies computed in various approximations. The lines correspond to the exact numerical calculations (solid), the 2nd order of usual non-unitary expansion (dot-dashed), the first order adiabatic Magnus expansion (dotted) and the second order adiabatic Magnus expansion (dashed). The values of oscillation parameters are } \Delta m^2 = 2 \cdot 10^{-3} \text{ eV}^2 \text{ and } \sin^2 \theta_{13} = 0.01.\]
V. CONCLUSIONS

We have developed new formalism of computations of the oscillation probabilities in matter with varying density. It is based on the Magnus expansion and has a virtue to be unitary in each order of the expansion. The formalism can be adjusted to a specific physical situation by choosing a neutrino evolution basis and a split of the Hamiltonian into the self-commuting and non-commuting parts. The latter can be used as a perturbation. Using the Magnus expansion one can develop different perturbation theories, and in particular, the improved adiabatic perturbation theory. The evolution due to self-commuting part can be accounted for in a way which is equivalent to a transition to the “interaction representation” in quantum mechanics.

We have obtained the semi-analytical formulas for various oscillation probabilities in the second order of the Magnus expansion. The Magnus expansion (apart from being unitary) leads also to better convergence of series. We show that the Magnus expansion corresponds to certain re-summation of contributions in the usual perturbation theory, and it is this re-summation that leads to restoration of unitarity. The developed unitary formalism gives new insight into the previously obtained results and their limitations.

Using several explicit examples we show that the restoration of unitarity gives better approximation to the results of exact numerical calculations, especially in the region where the transition probabilities are large. We find that the best approximation (among the considered examples) is provided by the adiabatic Magnus expansion.

The results in sec. II and III have a general character valid for wide class of potentials not necessarily related to the Earth density profile. Using the proposed method one can develop other perturbation approaches adjusting to particular physical conditions the evolution basis and split of the Hamiltonian. For instance, at high energies one can use the matter part of the Hamiltonian as the self-commuting part: $H_0 = \text{diag}(V, 0)$, and the vacuum (kinetic) part as a perturbation. This theory will give good approximation at high energies, where $V > \Delta m^2/2E$.

We have illustrated our results computing the oscillation probabilities for neutrinos crossing the core of the Earth (actually most of the figures are produced for the central trajectory). We find that for the solar oscillation parameters, $\Delta m^2_{21}$ and $\theta_{12}$, the second order of the Magnus adiabatic expansion gives a very good precision ($< 1\%$) for all energies. For the mantle-only trajectories the precision is even higher. For the atmospheric parameters $\Delta m^2_{31}$ and small 1-3 mixing the approximation works well ($< 3\%$ accuracy for $\sin^2 \theta_{13} = 0.01$) below ($E < 2.7$ GeV) and
above \((E > 8 \text{ GeV})\) the resonance region. In the region \((2.7 - 8 \text{ GeV})\) the MSW-resonances in the core and in the mantle as well as the parametric resonances take place and the Magnus adiabatic approximation fails since the adiabaticity is broken. In this region one should use some other approach. For the mantle-only crossing trajectory the approximation fails in the region \((5 - 8 \text{ GeV})\) for \(\sin^2 \theta_{13} = 0.01\).

The results obtained here can be used for description of propagation of the solar and supernova neutrinos inside the Earth. They also can be used to describe the flavor oscillations of the atmospheric and accelerator neutrinos. For solar neutrinos, \(E < 18 \text{ MeV}\), the transition probability is small, so that already usual perturbation theory gives very good approximation. The Magnus expansion adds little, as far as accuracy is concerned. For the galactic supernova the detectable tail of the energy spectrum extends up to \(50\text{ - 70 MeV} \) (depending on a distance to supernova and a size of detector). The range of energies \(E > 40 \text{ MeV}\), where the Earth matter effect is enhanced, is of special interest both for measurements of the neutrino parameters and for physics of gravitational collapse and mechanism of star explosion. It is this range where the Magnus expansion gives substantial improvement of accuracy. For the atmospheric neutrinos, the Magnus adiabatic approximation can be used to describe oscillations driven by the \(1\text{-}2\) mass split and \(1\text{-}2\) mixing for all neutrino energies and all trajectories. It is especially relevant for low energies: the atmospheric neutrinos, the Magnus adiabatic approximation can be used to describe oscillations driven by the \(1\text{-}2\) mass split and \(1\text{-}2\) mixing for all neutrino energies and all trajectories. It is especially relevant for low energies: the sub-GeV events as well as events below \(100 \text{ MeV}\). The results can be applied for oscillations induced by the \(1\text{-}3\) mass split and \(1\text{-}3\) mixing outside the resonance regions. They can be used for long baseline experiments with neutrino energies below \(3 \text{ GeV}\) (thus covering the range of proposed superbeams) and for high energy beams from neutrino factories \((E > 8 \text{ GeV})\). The results can be applied for neutrinos of cosmic origin.

VI. APPENDIX

The functionals \(C_k[H]\) can be derived in the following way. The standard expansion of the chronological product

\[
T \ e^{-i\int_{x_0}^{x_f} H(x) \ dx} = 1 - i \int_{x_0}^{x_f} dx \ H(x) + (-i)^2 \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \ H(x) H(y) \\
+ (-i)^3 \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \int_{x_0}^{x_f} dz \ H(x) H(y) H(z) + \cdots \tag{89}
\]

can be rewritten in terms of the commutators of the Hamiltonian using the following identities

\[
\int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \ H(x) H(y) = \frac{1}{2} \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \ [H(x), \ H(y)] + \frac{1}{2} \left( \int_{x_0}^{x_f} dx \ H(x) \right)^2, \tag{90}
\]

\[
\int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \int_{x_0}^{x_f} dz \ H(x) H(y) H(z) =
\]

\[
\frac{1}{6} \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \int_{x_0}^{x_f} dz \ \{ [H(x), [H(y), H(z)] + [H(x), [H(y), H(z)]] + [H(x), [H(y), [H(z), H(z)]]] + [H(x), [H(y), H(z)]] + [H(x), [H(y), H(z)]] + [H(x), [H(y), H(z)]]
\]

\[
+ \frac{1}{2} \left\{ \int_{x_0}^{x_f} dx \ H(x) \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \ [H(x), H(y)] + \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \ [H(x), H(y)] \int_{x_0}^{x_f} dx \ H(x) \right\}
\]

\[
+ \frac{1}{6} \left( \int_{x_0}^{x_f} dx \ H(x) \right)^3, \tag{91}
\]

etc.. These identities follow from an extension of all the integrations over whole range from \(x_0\) to \(x_f\). For instance, eq. (90) can be derived taking into account that in the double integral over \(x\) and \(y\)

\[
I(y = x_0 \div x) + I(y = x \div x_f) = \left[ \int_{x_0}^{x_f} dx H(x) \right]^2,
\]

and on the other hand

\[
I(y = x \div x_f) = I(y = x_0 \div x) - \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy [H(x), H(y)].
\]

Inserting (90) and (91) into (89) we obtain

\[
S = 1 - i \int_{x_0}^{x_f} dx \ H(x) + (-i)^2 \frac{1}{2} \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \ [H(x), H(y)] + (-i)^3 \frac{1}{2} \left( \int_{x_0}^{x_f} dx \ H(x) \right)^2 + \cdots, \tag{92}
\]
where we have written explicitly the commutators up to the third order. On the other hand expanding (5) we have

\[ S = 1 - i(C_1 + C_2 + C_3) + \frac{(-i)^2}{2}(C_1^2 + 2C_1C_2) + \frac{(-i)^3}{6}C_1^3 + \cdots \] (93)

Comparing (92) and (93) we obtain immediately the results (6, 7, 8).

The work of A.I. was supported by the NFSAT grant No. ARP2-3234-Ye-04.

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