Non-existence of standard compact Clifford-Klein forms of homogeneous spaces of exceptional Lie groups

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Abstract

We use a computer-aided approach to prove that there are no standard compact Clifford-Klein forms of homogeneous spaces of exceptional Lie groups. This yields further support for Kobayashi’s conjecture about possible compact Clifford-Klein forms. On one hand, our approach is based on the algorithms developed in this work which eliminate the majority of possibilities. On the other hand, we complete the proof using the algorithmic methods of classifying semisimple subalgebras in simple real Lie algebras developed by Faccin and de Graaf, as well as by calculating invariants like a-hyperbolic rank.

Keywords: proper actions, semisimple algebras, Clifford-Klein forms.

AMS Subject Classification: 57S30, 17B20, 22F30, 22E40, 65-05, 65F15

1 Introduction

Let $G$ be a semisimple linear real Lie group, $H \subset G$ a reductive (non-compact) subgroup such that $G/H$ is non-compact. We say that $G/H$ admits a compact Clifford-Klein form, if there exists a discrete subgroup $\Gamma \subset G$ acting properly and co-compactly on $G/H$. The problem of determining which reductive homogeneous spaces admit such forms goes back to Calabi and Markus (8), Kulkarni (19) and was formulated as a research program by T. Kobayashi. One of the important and challenging problems in the whole area is Kobayashi’s conjecture.
We say that \( G/H \) admits a **standard compact Clifford-Klein form**, if there exists a reductive Lie subgroup \( L \subset G \) such that \( L \) acts properly on \( G/H \) and \( L \cap G/H \) is compact. Note that in this case any co-compact lattice \( \Gamma \subset L \) yields a compact Clifford-Klein form. The Kobayashi conjecture states that for the homogeneous spaces \( G/H \) of reductive type, the existence of a compact Clifford-Klein form on \( G/H \) implies the existence of a standard one. Note that the conjecture does not say that all compact Clifford-Klein forms are standard. There are examples of non-standard ones (see [15], [17]), obtained by a deformation of a lattice \( \Gamma \subset L \).

In fact all known examples of standard Clifford-Klein forms can be obtained as follows. Assume that there exists a reductive Lie subgroup \( L \subset G \) such that \( G = L \cdot H \), and \( L \cap H \) is compact. Under these assumptions we see that

- \( L \) acts transitively on \( G/H \) and there is a diffeomorphism \( G/H \cong L/(L \cap H) \),
- since \( L \cap H \) is compact, any co-compact lattice \( \Gamma \subset L \) acts properly and co-compactly on \( L/(L \cap H) \) and hence on \( G/H \).

Notice that Kobayashi’s conjecture indicates that compact Clifford-Klein forms of non-compact homogeneous spaces \( G/H \) of reductive type are rare and of a special nature. There are many partial results which yield a strong evidence for the conjecture. For instance there are topological obstructions [22, 27], partial results of Margulis [20], Zimmer [28], Hee Oh and Witte [25], Okuda [26] and the authors of this paper [4, 2, 6, 7]. However, Kobayashi’s question (which is challenging and mathematically interesting) is still to be answered. It should be noted that Kobayashi found a criterion for the existence of standard Clifford-Klein forms in terms of the data of \( G/H \): one needs to know the non-compact dimensions of \( G, H, L \) and the action of the little Weyl group of \( G \) on the non-compact parts of the real split Cartan subalgebras of the Lie algebras \( \mathfrak{g}, \mathfrak{h} \) and \( \mathfrak{l} \) of \( G, H, L \), respectively. However, this approach works for the given triples \( (G, H, L) \) and the known embeddings of \( H \) and \( L \) into \( G \) (basically, described in terms of the roots systems).

The aim of this paper is to prove the following.

**Theorem 1.** Let \( G \) be a simple linear real Lie group of exceptional type, \( H \subset G \) a reductive non-compact subgroup such that \( G/H \) is non-compact. Then \( G/H \) does not admit standard compact Clifford-Klein forms.
To prove Theorem 1 we translate the Kobayashi criterion into a computer algorithm to find all possible triples \((G, H, L)\) which may induce compact Clifford-Klein forms. Then we use a method described by Faccin and de Graaf [11] to classify all the equivalence classes of embeddings \(H, L \subset G\). Eventually we eliminate each triple using known criteria of existence of compact Clifford-Klein forms.

2 Preliminaries

We use the basics of Lie theory without explanations, the reader may consult [21]. We consider root systems of complex semisimple Lie algebras with respect to the Cartan subalgebras. If \(\Phi\) is a root system, we write \(\Phi^\vee\) for its dual. We use the notation \(\mathfrak{g}\) for real Lie algebras, while \(\mathfrak{g}^c\) means a complex Lie algebra (or, the complexification of \(\mathfrak{g}\), the context will be always clear). If \(\mathfrak{g} = \mathfrak{k} + \mathfrak{p}\) denotes the Cartan decomposition of a semisimple non-compact real Lie algebra, then there is a maximal \(\mathbb{R}\)-diagonalizable subalgebra \(\mathfrak{a} \subset \mathfrak{p}\), and any two such subalgebras are transformed into each other by an element in \(K\), the maximal compact subgroup of \(G\) (corresponding to \(\mathfrak{k}\)). Moreover, \(\mathfrak{a}\) is a maximal \(\mathbb{R}\)-diagonalizable subalgebra in \(\mathfrak{g}\), and all such subalgebras are conjugate. We set \(\text{rank}_{\mathbb{R}} \mathfrak{g} := \dim \mathfrak{a}\), the real rank of \(\mathfrak{g}\). Let \(N_K(\mathfrak{a})\) and \(Z_K(\mathfrak{a})\) denote the normalizer and the centralizer of \(\mathfrak{a}\) in \(K\). The little Weyl group of \(G\) is, by definition, the group \(W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})\).

We will need the notion of the \(a\)-hyperbolic rank [5, 6] which will be used in the sieving procedure eliminating some possibilities for compact Clifford-Klein forms. Let \(\mathfrak{t}\) be a split Cartan subalgebra in \(\mathfrak{g}\) containing \(\mathfrak{a}\), and \(\Sigma \subset \mathfrak{a}^*\) be a restricted root system of \(\mathfrak{g}\). Choose a subset of positive roots \(\Sigma^+ \subset \Sigma\). We have

\[ \mathfrak{g} = \mathfrak{t} + \mathfrak{a} + \mathfrak{n}, \quad \mathfrak{n} := \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha. \]

The above decomposition is called the Iwasawa decomposition. On the Lie group level we have

\[ G = KAN. \]

where the subgroup \(A\) is simply connected abelian, \(N\) is simply connected unipotent and \(A\) normalizes \(N\). Also \(N\) is the maximal unipotent subgroup of \(G\), any unipotent subgroup of \(G\) is conjugate to a subgroup of \(N\).
Define
\[ a^+ = \{ X \in a \mid \xi(X) \geq 0, \forall \xi \in \Sigma^+ \}. \]
Consider the complexification \( g^c \), the corresponding root system \( \Phi \) with the set of simple roots \( \Pi \). For every \( X \in t^c \) define \( \psi_X : \Pi \to \mathbb{R} \) by the formula \( \alpha \to \alpha(X) \) (the above map is called the weighted Dynkin diagram). We also get the map
\[ \psi : t^c \to \text{Map}(\Pi, \mathbb{R}), X \to \Psi_X. \]
Consider the Weyl group \( W_{g^c} \) of \( g^c \) and choose the longest element \( w_0 \in W_{g^c} \). Define \( -w_0 : t^c \to t^c \) by the formula \( X \to -(w_0X) \). It can be checked that \( \iota = \psi \circ \iota \circ \psi^{-1} \) has the property \( \iota(a^+) \subset a^+ \). Let
\[ b^+ = \{ X \in a^+ \mid \iota(X) = X \}. \]

**Definition 1.** The dimension of \( b^+ \) (as a convex cone) is called the \( a \)-hyperbolic rank of \( g \). It is denoted by \( \text{rank}_{a-hyp} g \).

Here is an example which shows the way of using this invariant.

**Theorem 2** ([6], Theorem 8). Let \( G \) be a connected and semisimple linear Lie group and \( H \) a reductive subgroup with finite number of connected components. Then
- if \( \text{rank}_{a-hyp} g = \text{rank}_{a-hyp} h \), then \( G/H \) does not admit discontinuous actions of non virtually abelian discrete subgroups (and, therefore, compact Clifford-Klein forms),
- if \( \text{rank}_{a-hyp} g > \text{rank}_{a-hyp} h \), then \( G/H \) admits a discontinuous action of a non virtually abelian discrete subgroup.

Note that calculations of \( \text{rank}_{a-hyp} g \) are done in [6].

In this paper we work with semisimple subalgebras in simple real and complex Lie algebras and our approach is based on methods of computer aided classification of embeddings of Lie algebras [11]. This classification is considered up to equivalence or linear equivalence.

**Definition 2** ([11],[12]). Let \( g \) and \( \tilde{g} \) be two Lie algebras. Two embeddings \( \varepsilon : g^c \to \tilde{g}^c, \varepsilon' : g^c \to \tilde{g}^c \) are called equivalent if there is an inner automorphism \( \phi \in \text{Aut}(\tilde{g}^c) \) such that \( \varepsilon = \phi \varepsilon' \). They are called linearly equivalent if for all representations \( \rho : \tilde{g}^c \to \mathfrak{gl}(V^c) \) the induced representations \( \rho \circ \varepsilon \) and \( \rho \circ \varepsilon' \) are equivalent.
The equivalence implies linear equivalence, but the converse is false (see, for example, [21]). However, it is mentioned in [11] that the coincidence of these classes is ubiquitous if \( \tilde{g}^c \) is an exceptional simple Lie algebra and \( g^c \) is semisimple.

In Section 5 we consider symmetric pairs \((g, h)\), that is, when \( h \) is the subalgebra of all fixed points of an involutive automorphism of \( g \). They correspond to pseudo-Riemannian symmetric spaces. We use only some classification tables from [26], so we don’t discuss this topic referring to [16].

3 Kobayashi’s criterion for proper actions and sieving algorithms

3.1 A criterion for properness

Let \( G \) be a simple linear real Lie group and \( H, L \subset G \) reductive subgroups of \( G \). Choose a Cartan decomposition

\[
g = \mathfrak{k} + \mathfrak{p},
\]

where \( \mathfrak{k} \) is the Lie algebra of the maximal compact subgroup \( K \) of \( G \). By assumption, \( \mathfrak{h} \) and \( \mathfrak{l} \) are reductive subalgebras of \( \mathfrak{g} \), therefore, they admit Cartan decompositions

\[
\mathfrak{h} = \mathfrak{k}_h + \mathfrak{p}_h \text{ and } \mathfrak{l} = \mathfrak{k}_l + \mathfrak{p}_l,
\]

such that \( \mathfrak{p}_h, \mathfrak{p}_l \subset \mathfrak{p} \) (see [18] for more details). One can choose maximal abelian subalgebras

\[
\mathfrak{a} \subset \mathfrak{p}, \quad \mathfrak{a}_h \subset \mathfrak{p}_h, \text{ and } \mathfrak{a}_l \subset \mathfrak{p}_l
\]

such that \( \mathfrak{a}_h, \mathfrak{a}_l \subset \mathfrak{a} \). Let us recall the definition of the proper group action. Let \( S \) be a locally compact topological group acting continuously on a locally compact Hausdorff topological space \( X \). This action is \textit{proper} if for every compact subset \( S \subset X \) the set

\[
L_S := \{ g \in L \mid g \cdot S \cap S \neq \emptyset \}
\]

is compact. Kobayashi proved the following criterion for the properness of a Lie group action.
Theorem 3 ([18], Theorem 4.1). The following three conditions are equivalent

- $L$ acts on $G/H$ properly.
- $H$ acts on $G/L$ properly.
- For any $w \in W$ (where $W$ denotes the little Weyl group of $\mathfrak{g}$)
  \[(w \cdot a_I) \cap a_h = \{0\}.\]

Let
\[d(G) := \dim \mathfrak{p}, \ d(H) := \dim \mathfrak{p}_h, \ d(L) := \dim \mathfrak{p}_l.\]

Corollary 1 ([18], Theorem 4.7). If a triple $(G, H, L)$ induces a standard compact Clifford-Klein form then
\[d(G) = d(H) + d(L).\]  \hspace{1cm} (2)

Conversely, if $L$ acts properly and co-compactly on $G/H$ and (2) is satisfied, then $(G, H, L)$ determines a standard compact Clifford-Klein form. Moreover for any $g \in G$ we have
\[gHg^{-1} \cap L \subset K',\]
where $K' \subset G$ is a conjugate of $K$.

We also need the following result.

Theorem 4 ([1], Corollary 3). If $G/H$ admits a compact Clifford-Klein form then the center of $H$ is compact.

We will need also the following easy corollary to the previous results.

Proposition 1. Assume that for any non-compact semisimple subgroup $H' \subset G$ the non-compact homogeneous space $G/H'$ does not admit a standard compact Clifford-Klein form. Then for any noncompact reductive subgroup $H \subset G$ the non-compact homogeneous space $G/H$ does not admit standard compact Clifford-Klein forms.

Proof. Assume that $G/H$ admits a standard compact Clifford-Klein form for some reductive subgroup $H \subset G$ so there exists a triple $(G, H, L)$. We see that $G/L$ and $G/H$ admit (standard) compact Clifford-Klein forms. Thus the centers of $H$ and $L$ are compact by Theorem 4. Let $H' \subset H$ and $L' \subset L$ be the semisimple parts of $H$ and $L$, respectively. We have
\[d(H) = d(H'), \quad d(L) = d(L')\]
and so $(G, H', L')$ induces a standard compact Clifford-Klein form. \qed
3.2 Algorithms

The algorithms proposed in this work use the numerical characteristics of the triple \((g, h, l)\) which are a consequence of Proposition 1, Theorem 3 and Corollary 1.

Proposition 2. Assume that \((G, H, L)\) determines a standard compact Clifford-Klein form. Then the following restrictions on the triple \((g, h, l)\) hold:

1. By Proposition 2 we may assume that \(h\), and \(l\) are semisimple,
2. \(\text{rank}_\mathbb{R}(g) = \text{rank}_\mathbb{R}(h) + \text{rank}_\mathbb{R}(l)\),
3. \(\dim p = \dim p_h + \dim p_l\),
4. \(\dim k > \dim k_h, \dim k > \dim k_l, \text{ since } h, l \subset g\).

Proof. The condition 2 follows from Theorem 3. We have \(\text{rank}_\mathbb{R}(g) \geq \text{rank}_\mathbb{R}(h) + \text{rank}_\mathbb{R}(l)\), so assume that \(L\) acts properly and co-compactly on \(G/H\) and

\[
\text{rank}_\mathbb{R}(g) > \text{rank}_\mathbb{R}(h) + \text{rank}_\mathbb{R}(l).
\]

Let \(g = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}, h = \mathfrak{k}_h + \mathfrak{a}_h + \mathfrak{n}_h, l = \mathfrak{k}_l + \mathfrak{a}_l + \mathfrak{n}_l\) be the Iwasawa decompositions of \(g, h, l\), respectively. Since \(d(G) = \dim \mathfrak{a} + \dim \mathfrak{n}\) we obtain

\[
\dim \mathfrak{n} - \dim \mathfrak{n}_h - \dim \mathfrak{n}_l < 0.
\]

After conjugating \(H\) and \(L\) we have \(\mathfrak{n}_h, \mathfrak{n}_l \subset \mathfrak{n}\). Thus there exists a non-trivial subalgebra \(\mathfrak{n}_0 := \mathfrak{n}_h \cap \mathfrak{n}_l\). Let \(N_0 \subset H, L\) be a connected (non-compact) subgroup corresponding to \(\mathfrak{n}_0\). Since \(N_0 \subset H, L\) is non-compact, \(L\) can not act properly on \(G/H\). A contradiction. 

Thus, the first step of our computer aided analysis is based on the following plan.

1. If \((G, H, L)\) is a triple which yields a standard Clifford-Klein form, then the restrictions 1)-4) of Proposition 2 hold.
2. Ideally we should begin with describing all possible triples \((g, h, l)\), where \(g\) is a simple real Lie algebra, and \(h, l\) are semisimple subalgebras. It is too complicated to do it in full generality, however, one can begin with an easier task of finding all triples \((\mathfrak{g'}, \mathfrak{h'}, \mathfrak{l'})\). This is known since the work of Dynkin, and one can use the data base [12].
3. Thus, one proceeds as follows: writes down all possible triples 
\((g, h, l)\), where each of the real algebras is a real form of the 
corresponding complex Lie algebra. Note that it may happen, 
that there is no true embedding of \(h \hookrightarrow g\), or \(l \hookrightarrow g\). This is 
because in general, if \(\varepsilon : g^c \hookrightarrow \tilde{g}^c\) is a complex embedding of 
arbitrary Lie algebras \(g^c\) and \(\tilde{g}^c\), it may happen that \(\varepsilon(g) \not\subset \tilde{g}\).

4. For each of the triples \((g, h, l)\) obtained in step 3) one checks 
conditions 1)-4) of Proposition 2.

Using this list we create the following algorithms to obtain the list of 
possible triples \((g, h, l)\) (note again that at this stage we do not check 
if \(h, l\) can be realized as subalgebras of \(g\)).
Using these algorithms we obtain the following table of possible triples \((g, h, l)\) which might yield standard compact Clifford-Klein forms, because they do not violate the restrictions of Proposition 2.
|   | \(g\)     | \(h\)         | \(l\)         |
|---|-----------|---------------|---------------|
| 1 | \(e_6(6)\) | \(\text{su}^*(6)\) | \(\hat{f}_4(4)\) |
| 2 | \(e_6(6)\) | \(\text{so}(2, 7)\) | \(\hat{f}_4(4)\) |
| 3 | \(e_6(6)\) | \(\text{so}(3, 7)\) | \(\text{so}(3, 7)\) |
| 4 | \(e_6(6)\) | \(\text{su}(2) + \text{su}^*(6)\) | \(\hat{f}_4(4)\) |
| 5 | \(e_6(2)\) | \(\text{so}^*(10)\) | \(\text{so}^*(10)\) |
| 6 | \(e_6(-14)\) | \(\hat{f}_4(-20)\) | \(\hat{f}_4(-20)\) |
| 7 | \(e_6(-26)\) | \(\text{su}(1, 5)\) | \(\hat{f}_4(-20)\) |
| 8 | \(e_6(-26)\) | \(\text{su}(2) + \text{su}(1, 5)\) | \(\hat{f}_4(-20)\) |
| 9 | \(e_7(7)\) | \(\text{su}(3, 5)\) | \(e_6(2)\) |
| 10 | \(e_7(7)\) | \(\text{so}^*(12)\) | \(e_6(2)\) |
| 11 | \(e_7(7)\) | \(\text{su}(2) + \text{so}^*(12)\) | \(e_6(2)\) |
| 12 | \(e_7(-5)\) | \(e_6(-14)\) | \(e_6(-14)\) |
| 13 | \(e_8(8)\) | \(e_7(-5)\) | \(e_7(-5)\) |
| 14 | \(e_8(8)\) | \(e_7(-5)\) | \(\text{su}(2) + e_7(-5)\) |
| 15 | \(e_8(8)\) | \(\text{su}(2) + e_7(-5)\) | \(e_7(-5)\) |
| 16 | \(e_8(8)\) | \(\text{su}(2) + e_7(-5)\) | \(\text{su}(2) + e_7(-5)\) |
| 17 | \(\hat{f}_4(4)\) | \(\text{so}(2, 7)\) | \(\text{so}(2, 7)\) |
| 18 | \(g_2(2)\) | \(\text{su}(1, 2)\) | \(\text{su}(1, 2)\) |

Table 1: The list of triples \((g, h, l)\) from Algorithm 2

### 3.3 Implementation of algorithms

We have implemented Algorithm 1 and 2 in the computer algebra system GAP [14]. We have also created a special plugin CKForms [3] which uses the following plugins: SLA [13], CoReLG [9].

### 4 Analyzing Table 1 and the second step of proof of Theorem 1

#### 4.1 The problem

Now we describe the second ingredient of proof of Theorem 1. We see that one has to deal with two issues:

- is a triple \((g, h, l)\) represented by true monomorphisms of real Lie
algebras $\mathfrak{h} \hookrightarrow \mathfrak{g}$ and $\mathfrak{l} \hookrightarrow \mathfrak{g}$?

- If yes, does it yield a compact Clifford-Klein form?

We begin with the first issue. Note that our description is heavily based on the methods borrowed from the work of Faccin and de Graaf [11]. It is also in order to stress that we use methods rather than their computer implementation. For that reason we need to describe this work in greater detail. The description of semisimple subalgebras in complex simple Lie algebras was done by Dynkin [10]. The problem of embeddings of semisimple real Lie algebra into a simple real Lie algebra is more subtle. One may notice the following. If $\mathfrak{g}$ is a real form of $\mathfrak{g}^c$, then for any $\varphi \in \text{Aut}(\mathfrak{g}^c)$ the subalgebra $\varphi(\mathfrak{g})$ is also a real form. Therefore the problem of describing of embeddings $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ can be formulated as follows.

**Problem 1.** Let $\varepsilon : \mathfrak{g}^c \hookrightarrow \tilde{\mathfrak{g}}^c$ be an embedding of complex Lie algebras $\mathfrak{g}^c$ and $\tilde{\mathfrak{g}}^c$, and $\tilde{\mathfrak{g}}_1, ..., \tilde{\mathfrak{g}}_l$ be the (isomorphism classes of) real forms of $\tilde{\mathfrak{g}}^c$. Let $\mathfrak{g}$ be a real form of $\mathfrak{g}^c$. Find all $i$ such that $\phi(\varepsilon(\mathfrak{g})) \subset \tilde{\mathfrak{g}}_i$ for some $\phi \in \text{Aut}(\mathfrak{g}^c)$.

Thus, we see that if we know how to solve Problem 1, we can apply this method to each of the pair coming from the triples in Table 1, and get the table of triples which indeed may yield Clifford-Klein forms.

### 4.2 Notation

In this subsection $\mathfrak{g}^c$ is a complex semisimple Lie algebra with a Cartan subalgebra $\mathfrak{t}^c$. The root system determined by the pair $(\mathfrak{g}^c, \mathfrak{t}^c)$ is denoted by $\Phi$, the set of simple roots of $\Phi$ is $\Delta = \{\alpha_1, ..., \alpha_l\}$, $l = \text{rank } \mathfrak{g}^c$. The set of positive roots is denoted by $\Phi^+$, and $\mathfrak{g}_\alpha$ stands for the root space of $\alpha \in \Phi$. The *Chevalley basis* consists of vectors

$$h_{\alpha_i} = h_i, i = 1, ..., l, \ x_{\alpha_i}, \alpha \in \Phi$$

such that

$$[x_{\alpha_i}, x_{-\alpha_i}] = h_{\alpha_i}, \ [x_{\alpha_i}, x_{\beta}] = N_{\alpha, \beta} x_{\alpha + \beta}, \ [h_{\alpha}, x_{\alpha}] = 2x_{\alpha},$$

where $x_\gamma = 0$, if $\gamma \notin \Phi$. Here

$$N_{\alpha, \beta} = \pm (r + 1), \ r = \max_p \{p \mid \beta - p\alpha \in \Phi\}, \ q = \max_p \{p \mid \beta + p\alpha \in \Phi\}.$$
One also considers the set of generators of $\mathfrak{g}$ as a Lie algebra:

$$\{h_i, x_i, y_i, \ i = 1, \ldots, l\}, \ h_i := h_{\alpha_i}, \ x_i := x_{\alpha_i}, \ y_i := y_{\alpha_i}$$

with the relations

$$[x_i, y_j] = \delta_{ij} h_i, [h_i, h_j] = 0, [h_i, x_j] = \langle \alpha_j, \alpha_i^\vee \rangle x_j, [h_i, y_j] = -\langle \alpha_j, \alpha_i^\vee \rangle y_j.$$

Given a Chevalley basis, put

$$u = \text{Span}_R(\sqrt{-1}h_1, \ldots, \sqrt{-1}h_l, x_\alpha - x_{-\alpha}, \sqrt{-1}(x_\alpha + x_{-\alpha}) | \alpha \in \Phi^+ ) \ (3)$$

This is the compact real form of $\mathfrak{g}$, so $\mathfrak{g} = u + \sqrt{-1}u$. The complex conjugation with respect to $u$ will be denoted by $\tau$. Thus, in the sequel we will always consider the pair $(u, \tau)$. Let $\theta \in \text{Aut}(\mathfrak{g})$ be an involutive automorphism of $\mathfrak{g}$. If $\theta$ commutes with $\tau$, then $\theta(u) \subset u$. Since $\theta$ is involutive, one can decompose $u$ into the $(+1)$- and $(-1)$-eigenspaces of $\theta$ and write

$$u = u_1 + u_{-1}. \ (4)$$

4.3 A procedure for constructing all real forms of $\mathfrak{g}$

**Theorem 5** ([23]). Every real form $\mathfrak{g}$ of $\mathfrak{g}$ can be constructed according to the following procedure.

1. Fix the Chevalley basis;
2. construct the pair $(u, \tau)$ according to (3);
3. take an involutive $\theta \in \text{Aut}(\mathfrak{g})$ such that $\tau \theta = \theta \tau$ and construct the decomposition (4);
4. set $\mathfrak{k} = u_1, \mathfrak{p} = \sqrt{-1}u_{-1}, \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

The decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition.

**Definition 3.** We will say that the real form $\mathfrak{g}$ constructed in Theorem 5 is determined by the triple $(u, \tau, \theta)$.

Note that all constructions are done up to an automorphism in $\text{Aut}(\mathfrak{g})$. 12
4.4 A characterization of complex embeddings 

\( \varepsilon : g^c \hookrightarrow \tilde{g}^c \) with the property \( \varepsilon(g) \subset \tilde{g} \)

**Theorem 6** ([11]). Let \( g \) and \( \tilde{g} \) be real forms of \( g^c \) and \( \tilde{g}^c \), and \( \varepsilon : g^c \hookrightarrow \tilde{g}^c \) be an embedding of complex Lie algebras. Assume that \( \theta \) is an involutive automorphism of \( g^c \) determining the real form \( g \) according to Theorem 5.

1. If \( \varepsilon(g) \subset \tilde{g} \), then there exists a pair \( (\tilde{u}, \tilde{\tau}) \) determined by a compact real form \( \tilde{u} \) of \( \tilde{g}^c \) satisfying the following:

\[
\varepsilon(u) \subset \tilde{u}, \quad \varepsilon\theta = \tilde{\theta} \varepsilon, \quad \tilde{\theta} \tilde{\tau} = \tilde{\tau} \tilde{\theta}, \quad \tag{5}
\]

and the real form \( \tilde{g} \) is determined by the pair \( (\tilde{u}, \tilde{\tau}) \).

2. Conversely, given a compact pair \( (\tilde{u}, \tilde{\tau}) \) and an involutive automorphism \( \theta \) such that (5) holds, the real form \( \tilde{g} \) constructed in Theorem 5 has the property \( \varepsilon(g) \subset \tilde{g} \).

We see that in order to find all embeddings \( \varepsilon \) with the property \( \varepsilon(g) \subset \tilde{g} \), one needs to find all pairs of triples

\( (u, \tau, \theta), (\tilde{u}, \tilde{\tau}, \tilde{\theta}) \)

satisfying (5).

4.5 A procedure for finding \( \varepsilon \) with \( \varepsilon(u) \subset \tilde{u} \)

We will denote the Cartan subalgebra of \( \tilde{g}^c \) by \( \tilde{t}^c \), and the root system determined by \( (\tilde{g}^c, \tilde{t}^c) \) will be denoted by \( \Psi \). Assume that \( \varepsilon(u) \subset \tilde{u} \). Then for \( \alpha \in \Phi \) there is a subset \( A_\alpha \subset \Psi \) such that

\[
\varepsilon(x_\alpha) = \sum_{\beta \in A_\alpha} a_{\alpha,\beta} y_\beta
\]

\[
\varepsilon(x_{-\alpha}) = \sum_{\beta \in A_\alpha} b_{\alpha,\beta} y_{-\beta},
\]

where \( a_{\alpha,\beta}, b_{\alpha,\beta} \in \mathbb{C} \).

**Definition 4.** We say that \( \varepsilon \) is balanced if \( \varepsilon(t^c) \subset \tilde{t}^c \) and

\[
b_{\alpha,\beta} = \tilde{a}_{\alpha,\beta}, \quad \forall \alpha \in \Phi, \beta \in A_\alpha.
\]

**Theorem 7** ([11]). An embedding \( \varepsilon : g^c \hookrightarrow \tilde{g}^c \) has the property \( \varepsilon(u) \subset \tilde{u} \) for some compact real forms \( u \) and \( \tilde{u} \) if and only if it is balanced.
A procedure for finding balanced $\varepsilon$

1. Fix Chevalley bases in $\mathfrak{g}^c$ and $\tilde{\mathfrak{g}}^c$. Let $y_{\beta}$ denote the root vectors in $\tilde{\mathfrak{g}}^c$.

2. Put

$$X_i = \sum_{\beta \in A_{\alpha_i}} (s_{\alpha_i, \beta} + \sqrt{-1} t_{\alpha_i, \beta}) y_{\beta}$$

$$Y_i = \sum_{\beta \in A_{\alpha_i}} (s_{\alpha_i, \beta} - \sqrt{-1} t_{\alpha_i, \beta}) y_{-\beta}$$

with unknowns $s_{\alpha_i, \beta}, t_{\alpha_i, \beta}$.

3. Write down the conditions

$$[\varepsilon(h_i), \varepsilon(h_j)] = 0,$$

$$[\varepsilon(h_i), X_j] = (\alpha_j, \alpha_i^\vee) X_j,$$

$$[\varepsilon(h_i), Y_j] = - (\alpha_j, \alpha_i^\vee) Y_j,$$

$$[X_i, Y_j] = \delta_{ij}\varepsilon(h_i).$$

4. These conditions yield a set of polynomial equations in unknown variables $s_{\alpha_i, \beta}, t_{\alpha_i, \beta}$. Solve it and get numbers $s^*_{\alpha_i, \beta}, t^*_{\alpha_i, \beta}$ (we assume that there is a solution).

5. Substitute them instead of $s_{\alpha_i, \beta}, t_{\alpha_i, \beta}$ and get vectors $\hat{X}_i, \hat{Y}_i$.

**Theorem 8** ([11]). The map

$$h_i \rightarrow \varepsilon(h_i), \quad x_{\alpha_i} \rightarrow \hat{X}_i, \quad x_{-\alpha_i} \rightarrow \hat{Y}_i$$

determines an embedding $\hat{\varepsilon} : \mathfrak{g}^c \hookrightarrow \tilde{\mathfrak{g}}^c$, which is balanced. The embeddings $\varepsilon$ and $\hat{\varepsilon}$ are linearly equivalent.

**Corollary 2.** Assume that all linearly equivalent embeddings of $\mathfrak{g}^c$ to $\tilde{\mathfrak{g}}^c$ are actually equivalent. Then there is not more than one (up to equivalence) embedding $\varepsilon$ such that $\varepsilon(\mathfrak{g}) \subset \tilde{\mathfrak{g}}$.

5  Proof of Theorem [1]

We eliminate each case in Table [I] using the results from the previous sections and the relations between linear equivalence classes and equivalence classes of embeddings of subalgebras into exceptional Lie subalgebras found in [21]. In greater detail, we use the following.
Theorem 9 ([21], Theorem 7). Let $\tilde{g}$ be a complex exceptional Lie algebra, and $g$ a semisimple Lie algebra. If the pair $(g, \tilde{g})$ is not listed below then the linear equivalence class of an arbitrary embedding $\varepsilon(g) \subset \tilde{g}$ consists of a single class of equivalent embeddings.

$$\tilde{g} = e_{6}, \ g = sl_{3}^{c}, so(5, C), g_{2}' \begin{array}{l}
\tilde{g} = e_{7}, \ g = sl_{3}^{c} + sl_{2}^{c}, sl_{2}^{c} + sl_{2}^{c} + sl_{2}^{c}, \\
\tilde{g} = e_{8}, \ g = e_{4}^{c} + sl_{3}^{c}, so(5, C) + sl_{4}^{c}, sl_{3}^{c},
\end{array}$$

where $h_{1}^{c} = sl_{3}^{c}, so(5, C), g_{2}', so(8, C), sl_{2}^{c} + sl_{2}^{c} + sl_{2}^{c} + sl_{2}^{c} + sl_{2}^{c} + sl_{2}^{c}$. 

Now we perform a case-by-case analysis.

1. The cases 1-4, 7, 8 can be eliminated using Theorem 10 below.

Theorem 10 ([5], Theorem 2). If $L$ acts properly on $G/H$ then

$$\text{rank}_{a-hyp}(l) + \text{rank}_{a-hyp}(h) \leq \text{rank}_{a-hyp}(g).$$

But $\text{rank}_{a-hyp}(e_{6(6)}) = 4$, $\text{rank}_{a-hyp}(e_{6(-26)}) = 1$ and the $a$-hyperbolic ranks of the subalgebras in cases 1-4,7,8 are equal to their real ranks. One easily checks that in each of these cases:

$$\text{rank}_{a-hyp}(l) + \text{rank}_{a-hyp}(h) > \text{rank}_{a-hyp}(g).$$

2. The cases 5,6,12,13,16-18 can be eliminated using Corollary 2 and Theorem 9. Since $h$ and $l$ are equivalent, $H$ and $L$ are conjugate inside $G$ and thus $L$ cannot act properly on $G/H$.

3. The cases 9-11,14,15 can be eliminated using Table 3 in [26]. Table 3 lists examples of irreducible simple symmetric pairs $(g, h)$ such that the corresponding symmetric space $G/H$ does not admit compact Clifford-Klein forms. Since there is only one equivalence class of embedding $su(2) + \mathfrak{e}_{7(-5)} \subset \mathfrak{e}_{8(8)}$ it follows that $(e_{8(8)}, su(2) + \mathfrak{e}_{7(-5)})$ is a symmetric pair. Since it is listed in Table 3 in [26], the corresponding symmetric space cannot admit (any) compact Clifford-Klein forms. Analogously $e_{6(2)} \subset \mathfrak{e}_{6(2)} + so(2) \subset \mathfrak{e}_{7(7)}.$ Let $(G, H, L)$ be a triple so that $g = \mathfrak{e}_{7(7)}$ and $h = e_{6(2)}$. Let $H' \subset G$ be a Lie group corresponding to $e_{6(2)} + so(2)$. It follows from Theorem 3 that $L$ acts properly on $G/H$ if and only if it acts properly on $G/H'$. Also $d(H) = d(H')$. Thus $G/H$ admits a (standard) compact Clifford-Klein form if and only if $G/H'$ does. But $G/H'$ does not admit compact Clifford-Klein forms because it is contained in Table 3 in [26].
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