Noncompact gauge fields on a lattice: SU(n) theories

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Abstract

Recently it has been found that in a noncompact lattice regularization of the SU(2) gauge theory the physical volume is larger than in the Wilson theory with the same number of sites. In its original formulation the noncompact regularization is directly applicable to U(n) theories for any n and to SU(n) theories for n=2. In this paper we extend it to SU(n) for any n and investigate some of its properties.

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1 Introduction

One of the present problems of lattice gauge theories is how to increase the physical volume where the numerical simulations are performed. The physical size of the lattice is indeed a major limitation in the study of hadronic structure functions and light hadron spectroscopy, and in the evaluation of the ratio $\epsilon'/\epsilon$.

A recent investigation showed that for SU(2) lattice gauge theory a non-compact regularization provides a physical volume larger than the Wilson theory with the same number of sites. It appears therefore interesting to repeat the simulation for the physically relevant SU(3) theory, but in its original formulation this regularization is directly applicable to U(n) theories for any n but to SU(n) theories for n=2 only. It is the purpose of the present paper to extend it to SU(n) for any n and to investigate some of its properties.

As it is well known a formal discretization of gauge theories breaks gauge invariance. To avoid this inconvenient Wilson assumed as dynamical variables elements of the gauge group instead of the gauge fields which live in the group algebra. In this way one gets a theory with an exact symmetry which has the desired formal continuum limit. This theory is said compact because compact are the dynamical variables.

The success of Wilson’s regularization is by now celebrated in textbooks. But one can wonder whether its exact lattice symmetry can also be realized without compactifying the variables, and if this can have some advantages wrt specific issues, reducing the artifacts of the lattice. In addition to the mentioned possibility of having larger volumes, the importance of noncompact gauge fields, especially in their coupling with matter fields, has been advocated in the investigation of a possible fixed point of QED at a finite coupling. Moreover perturbative calculations should be easier since one does not have to expand the link variables of the Wilson theory in terms of the gauge fields. Perturbative calculations are at least necessary to make contact with the continuum formulation, but other applications like the study of renormalons should also be mentioned. Finally in numerical simulations one might expect a faster approach to the scaling, the more so the more important is the summation of the tadpoles generated by the expansion of the link variables.

If one defines the covariant derivative in close analogy to the continuum as an ordinary discrete derivative plus the appropriate element of the algebra of the group, the lattice symmetry is broken, but it can be maintained by introducing compensating auxiliary fields which decouple in the continuum limit. Such a regularization has been studied exhaustively in the case...
of SU(2). Specifically the renormalization group parameter has been evaluated, and the perturbative properties have been shown to agree with Lusher’s calculation in Wilson regularization [11, 12]. Moreover Monte Carlo simulations gave results compatible with Wilson’s theory, but with the interesting difference mentioned at the beginning: the physical volume results larger than in the Wilson theory with the same number of sites [4]. This is what we expect euristically for a regularization closer to the continuum.

In the original formulation of this noncompact regularization, with the exception of the case n=2, invariance wrt SU(n) implies invariance wrt U(n). The aim of this paper is to construct a potential which breaks the U(n) invariance down to SU(n) and to investigate some of its properties. For simplicity, explicit formulae will be given for n=3, but the generalization is obvious. Obvious is also the coupling to matter fields which will therefore not be discussed.

In Section 2 we report, for the convenience of the reader, the regularization for U(n). In Section 3 we show how to construct a SU(n) invariant theory. In Sections 4 and 5 we derive the Ward identities and the formulation in a background gauge, which can be useful in perturbative calculations. In the Appendix we report the explicit expression of the U(3) breaking potential.

2 The noncompact regularization for U(3)

We first consider the regularization of U(n) gauge theories. For n=1 we get a truly noncompact QED, namely noncompact also in the coupling with the matter fields.

We want to construct a covariant derivative $D_\mu$ which transforms according to

$$D_\mu'(x) = g(x)D_\mu(x)g^\dagger(x + \mu)$$

when $g(x)$ is an element of U(n). A simple discretization of the continuum would give

$$D_\mu = \nabla_\mu + i \left( \chi_\mu \mathbb{I} + A_\mu^a T_a \right),$$

where $\nabla_\mu$ is the ordinary right discrete derivative, and $\chi_\mu$ and $A_\mu^a$ are the abelian and nonabelian gauge fields respectively. We adopt for the generators of SU(3) in the fundamental representation the normalizations

$$\{T_a, T_b\} = \frac{4}{3} \delta_{ab} \mathbb{I} + 2f_{abc} T_c, \quad [T_a, T_b] = 2i f_{abc} T_c.$$  

These normalizations are slightly different from those used in [10].
As it is well known with such a definition of the covariant derivative it is impossible to fulfill the transformation rule of Eq. (1). The way out that we reconsider here is based on the use of auxiliary compensating fields. It turns out that the lattice covariant derivative transforms in the right way if it acts on a field $\psi$ in the fundamental representation according to

$$
(D_\mu \psi)(x) = D_\mu(x)\psi(x + \mu) - \frac{1}{a}\psi(x),
$$

where $D_\mu$ has the following form

$$
D_\mu(x) = \left[ \frac{1}{a} - \sigma_\mu(x) + i\chi_\mu(x) \right] I + \left[ iA^a_\mu - \alpha^a_\mu(x) \right] T_a.
$$

In the above equation "a" is the lattice spacing, and $\sigma_\mu$ and $\alpha_\mu$ are the additional fields necessary to enforce the lattice gauge invariance. With a little abuse of language we will call also $D_\mu$ covariant derivative. The action of $U(3)$ on the fields, for $g(x) \simeq I - iT_a \theta^a(x) - iT_0 \theta^0(x)$, is

$$
\left( A^a_\mu(x) \right)' = A^a_\mu(x) + \Delta_\mu \theta^a(x) + 2f^a_{bc} \theta^b(x) A^c_\mu(x) - a\sigma_\mu(x) \Delta_\mu \theta^a(x) - a f^a_{bc} \alpha^b_\mu(x) \Delta_\mu \theta^c(x) - a \alpha^a_\mu(x) \Delta_\mu \theta^0(x)
$$

$$
\left( \alpha^a_\mu(x) \right)' = \alpha^a_\mu(x) + 2f^a_{bc} \theta^b(x) \alpha^c_\mu(x) - a f^a_{bc} \alpha^b_\mu(x) \Delta_\mu \theta^c(x) + a \chi_\mu(x) \Delta_\mu \theta^a(x) + a A^a_\mu(x) \Delta_\mu \theta^0(x) + a A^a_\mu(x) \Delta_\mu \theta^0(x)
$$

$$
\left( \chi_\mu(x) \right)' = \chi_\mu(x) + \Delta_\mu \theta^0(x) - \frac{2}{3} a \alpha^a_\mu(x) \Delta_\mu \theta^a(x) - a \sigma_\mu(x) \Delta_\mu \theta^0(x)
$$

$$
\left( \sigma_\mu(x) \right)' = \sigma_\mu(x) + \frac{2}{3} a A^a_\mu(x) \Delta_\mu \theta^a(x) + a \chi_\mu(x) \Delta_\mu \theta^0(x).
$$

Since all the fields are mixed by the gauge transformations, we cannot say at this point which are the physical fields. They are selected by the action as we will see by studying the Ward identities.

A lattice action invariant under the above transformations is

$$
\mathcal{L}_{YM}(x) = \frac{1}{4} \beta \text{Tr} F^+_{\mu\nu} F_{\mu\nu},
$$

where $F_{\mu\nu}$ is the stress tensor

$$
F_{\mu\nu}(x) = D_\mu(x)D_\nu(x + \mu) - D_\nu(x)D_\mu(x + \nu).
$$
We emphasize that in such a formulation the measure in the partition function is flat.

In the formal continuum limit the field $\sigma_{\mu}$ becomes invariant and decouples together with $\alpha_{\mu}$, so that these seem to be the auxiliary fields. But the situation can be different at the quantum level. To control the decoupling of the redundant fields in the presence of quantum effects we use the fact that in a noncompact regularization, besides $L_{YM}$, there are other local invariants, which can be used to construct an appropriate potential and to give divergent masses to the fields which must stay decoupled. One such potential is

$$L_1 = \beta_1 \sum_{\mu} Tr \left[ D_{\mu}^1(x) D_{\mu}(x) - \frac{1}{a^2} \right]^2$$

$$= \beta_1 \sum_{\mu} \left\{ \frac{12}{a^2} \sigma_{\mu}^2(x) + 8 a^2 \alpha_{\mu}^a(x) \alpha_{\mu}^a(x) - \frac{12}{a} \sigma_{\mu}(x) \left( \sigma_{\mu}^2(x) + \chi^2_{\mu}(x) \right) \\
- \frac{8}{a} \left( 3 \sigma_{\mu}(x) \alpha_{\mu}^a(x) \alpha_{\mu}^a(x) + 2 \sigma_{\mu}(x) A_{\mu}^a(x) A_{\mu}^a(x) + 2 \chi_{\mu}(x) A_{\mu}^a(x) \alpha_{\mu}^a(x) \right) \\
+ 3 \left( \sigma_{\mu}^2(x) + \chi^2_{\mu}(x) \right)^2 + \frac{4}{3} \left( A_{\mu}^a(x) + \alpha_{\mu}^a(x) \right)^2 + 4 A_{\mu}^a(x) A_{\mu}^a(x) \left( \sigma_{\mu}^2(x) + 3 \chi^2_{\mu}(x) \right) \\
+ 4 \alpha_{\mu}^a(x) \alpha_{\mu}^a(x) \left( 3 \sigma_{\mu}^2(x) + \chi^2_{\mu}(x) \right) + 16 \sigma_{\mu}(x) \chi_{\mu}(x) A_{\mu}^a(x) \alpha_{\mu}^a(x) \right\}$$

$$+ 8 d_b c \left[ (A_{\mu}^a(x) A_{\mu}^b(x) + \alpha_{\mu}^a(x) \alpha_{\mu}^b(x)) \left( \sigma_{\mu}(x) \alpha_{\mu}^c(x) + \chi_{\mu}(x) A_{\mu}^c(x) - \frac{1}{a} \alpha_{\mu}^c(x) \right) \right]$$

$$+ 8 f_{ab} f_{cd} A_{\mu}^a(x) \alpha_{\mu}^b(x) A_{\mu}^c(x) \alpha_{\mu}^d(x) + 2 d_{ab} d_{cd} \left( A_{\mu}^a(x) A_{\mu}^b(x) A_{\mu}^c(x) A_{\mu}^d(x) \right)$$

$$+ \alpha_{\mu}^a(x) \alpha_{\mu}^b(x) \alpha_{\mu}^c(x) \alpha_{\mu}^d(x) + 2 A_{\mu}^a(x) A_{\mu}^b(x) A_{\mu}^c(x) A_{\mu}^d(x)$$

$$+ 8 d_b c \left[ (A_{\mu}^a(x) A_{\mu}^b(x) + \alpha_{\mu}^a(x) \alpha_{\mu}^b(x)) \left( A_{\mu}^c(x) A_{\mu}^d(x) + \alpha_{\mu}^c(x) \alpha_{\mu}^d(x) \right) \right].$$

(9)

We see that $L_1$ provides the desired divergent masses to the auxiliary fields in the trivial vacuum. A more general analysis of the mass spectrum will be given in Section 4. There are other invariant terms, which can be used for instance to make the propagator of some of the auxiliary fields strictly local on the lattice \[11\], but we will ignore them for simplicity.

The effect of $L_1$ can be well understood by adopting a definition of the covariant derivative where the abelian fields are in a polar representation

$$D_{\mu}(x) = \hat{D}_{\mu}(x) \exp i \phi_{\mu}(x),$$

(10)

where

$$\hat{D}_{\mu} = \rho_{\mu} \mathbb{I} + \left[ i \left( A' \right)^a_{\mu} - \left( \alpha' \right)^a_{\mu} \right] T_a.$$  

(11)

Due to $L_1$, the $\rho$-field acquires a non vanishing expectation value $< \rho_{\mu} > = 1/a$. The U(3) symmetry is "spontaneously" broken, and the components of
φ_μ are the Goldston bosons\(^2\). As we will see by studying the Ward identities, the physical fields are φ_μ and A'_μ.

It is worth while noticing that in the absence of the spontaneous symmetry breaking there is not even a discrete derivative, the term 1/a being absent in the definition of D_μ. The present definition of gauge theories on a lattice can then be regarded as a matrix model where the space-time dynamics is generated by a spontaneous breaking of the gauge symmetry.

3 The noncompact regularization for SU(n)

A derivative covariant wrt SU(n) transformations only, must in general contain all the fields of the U(n) theory, the only difference being that the field χ_μ becomes another auxiliary field. So we cannot restrict ourselves to the SU(n) symmetry by changing the covariant derivative, and at the same time the potential \( L_1 \) does not generate a mass for the χ-field. Moreover, as it will be confirmed in the next Section by the Ward identities, no U(n) invariant potential can generate a mass for both abelian fields. We must therefore break explicitly the U(3) symmetry in order to give to the would be Goldstone bosons a mass, actually a divergent mass.

The case n=2 is exceptional, because for SU(2) transformations, namely for \( \theta = 0 \), Eqs. 6 do not mix the multiplet A_μ, σ_μ with the multiplet a_μ, χ_μ. Therefore we can break U(2) by omitting the latter fields to get an SU(2) invariant theory. This case has already been exhaustively studied [10, 11, 12].

There are two terms ( whose expression will be spelled out in the Appendix) which break the U(3) invariance of the action, explicitly

\[
L_2 = \beta_2 \phi \sum_\mu \left[ \det D_\mu(x) + \det D_\mu^\dagger(x) \right]
\]

\[
L_2' = \beta_2' \phi \sum_\mu \left[ \det D_\mu(x) - \det D_\mu^\dagger(x) \right].
\]

But we can always get rid of one of them by the global trasformation

\[
D_\mu = D_\mu' \exp i\alpha_\mu.
\]

For instance, we can get rid of \( L_2' \) by setting in the above equation \( \alpha = 1/3 \arctg (\beta_2/\beta_2') \). We assume this to be the case.

We now determine the minima of the action at constant fields in the presence of \( L_2 \). We assume that the color symmetry is not spontaneously

\(^2\)Needless to say, the U(3) symmetry remains exact. While for \( \langle \rho_\mu \rangle = 0 \) it is realized linearly, for \( \langle \rho_\mu \rangle \neq 0 \) it is realized nonlinearly.
broken. As a consequence the colored fields cannot develop a nonvanishing expectation value, neither can they mix with the auxiliary abelian fields. By adopting the abelian polar representation of Eq. (10) we minimize \( L_2 \) wrt \( \phi_\mu \) at fixed \( \rho_\mu \), and then minimize the resulting action wrt \( \rho_\mu \).

By noticing that
\[
L_2 = \beta_2 \frac{2}{a} \rho_\mu^3 \cos \left( 3\phi_\mu \right)
\]
we obtain the stationarity condition
\[
\sin 3\phi_\mu = 0.
\]

Assuming \( \beta_2 < 0 \), the minimum of \( L_2 \) occurs at \( \phi_\mu = 0, 2\pi/3, 4\pi/3 \), namely the covariant derivative at the minimum belongs to the center of SU(3)\(^3\).

Next we require, as a normalization condition, that the total action have one and only one minimum at \( \rho_\mu = 1 \). To achieve this result we find it necessary to add another potential term
\[
L_3 = \beta_3 \frac{1}{a^2} \sum_\mu \left[ -\frac{6}{a} \sigma_\mu(x) + 3\sigma_\mu^2(x) + 3\chi_\mu^2(x) + 2a_\mu(x)\alpha_\mu(x) + 2A_\mu^a(x)A_\mu^a(x) \right].
\]

This term seems to give a mass also to all the colored fields, but it has already been shown that this is not the case for SU(2), and the proof will be generalized in the next Section.

Taking into account that at the minimum
\[
L_2 = -2\frac{|\beta_2|}{a} \rho_\mu^3,
\]
we then have, omitting some constant terms
\[
\mathcal{L} = \sum_\mu \left\{ 3\beta_1 \left( \rho_\mu^2 - \frac{1}{a^2} \right)^2 - 2\frac{|\beta_2|}{a} \rho_\mu^2 \right\} + \frac{3\beta_3}{a^2} \rho_\mu^2.
\]

This lagrangian density is stationary for
\[
\rho_\mu^{(0)} = 0, \quad \rho_\mu^{(\pm)} = \frac{1}{4a\beta_1} \left[ |\beta_2| \pm \sqrt{\beta_2^2 + 8\beta_1 (2\beta_1 - \beta_3)} \right].
\]

\(^3\)All the minima are therefore in one-to-one correspondence with those of the Wilson theory, and the difficulty raised in ref. [13] in connection with this degeneracy can then be overcome as in the compact regularization.
We have to choose the couplings so that $\rho_{\mu}^{(+)}$ be a minimum and equal to $1/\alpha$; this requirement gives

$$|\beta_2| = \beta_3, \quad 4\beta_1 > \beta_3. \quad (21)$$

Under these conditions $\rho_{\mu}^{(0)}$ is another minimum, which we must require to lie higher than the minimum at $\rho_{\mu}^{(+)}$. This further strengthens the above inequality to $3\beta_1 > \beta_3$.

The masses of the auxiliary fields turn out to be

$$m_\rho^2 = \frac{6}{a^2} (4\beta_1 - \beta_3), \quad m_\phi^2 = \frac{18}{a^2} \beta_3, \quad m_\alpha^2 = \frac{8}{a^2} (2\beta_1 + \beta_3). \quad (22)$$

In conclusion the full classical lagrangian is

$$\mathcal{L}_G = \mathcal{L}_{YM} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3. \quad (23)$$

4 Ward identities

To determine the mass spectrum and identify the physical fields we investigate the Ward identities.

We start with $U(3)$ invariance and we assume that the color symmetry is not spontaneously broken. Therefore the effective action $\Gamma$ must be stationary

$$\frac{\partial \Gamma}{\partial A_\mu^a(x)} = \frac{\partial \Gamma}{\partial \alpha_\mu^a(x)} = \frac{\partial \Gamma}{\partial \chi_\mu(x)} = \frac{\partial \Gamma}{\partial \sigma_\mu(x)} = 0 \quad (24)$$

for

$$A_\mu^a(x) = \alpha_\mu^a(x) = 0, \quad \chi_\mu = \chi_\mu, \quad \sigma_\mu = \sigma_\mu. \quad (25)$$

Because of gauge invariance we have

$$\delta \Gamma = \sum_{\mu,x} \left[ \delta \chi_\mu(x) \frac{\partial \Gamma}{\partial \chi_\mu(x)} + \delta \sigma_\mu(x) \frac{\partial \Gamma}{\partial \sigma_\mu(x)} \right]$$

$$+ \sum_{a,\mu,x} \left[ \delta A_\mu^a(x) \frac{\partial \Gamma}{\partial A_\mu^a(x)} + \delta \alpha_\mu^a(x) \frac{\partial \Gamma}{\partial \alpha_\mu^a(x)} \right] = 0. \quad (26)$$

Introducing the explicit expressions for the variations and integrating by parts we obtain

$$\delta \Gamma = \sum_{\mu,x} \theta^a(x) \left\{ \frac{2}{3} a \Delta_\mu^{(-)} \left[ \alpha_\mu^a(x) \frac{\partial \Gamma}{\partial A_\mu^a(x)} - A_\mu^a(x) \frac{\partial \Gamma}{\partial \sigma_\mu(x)} \right] \right. \right.$$

$$- \Delta_\mu^{(-)} \frac{\partial \Gamma}{\partial A_\mu^a(x)} + 2 f_{bc}^a A_\mu^b(x) \frac{\partial \Gamma}{\partial A_\mu^c(x)} + a \Delta_\mu^{(-)} \left[ \sigma_\mu(x) \frac{\partial \Gamma}{\partial A_\mu^a(x)} \right]$$
\[-f^a_{bc} A^b_\mu(x) \frac{\partial \Gamma}{\partial A^c_\mu(x)} + d^a_{bc} \alpha^b_\mu(x) \frac{\partial \Gamma}{\partial A^c_\mu(x)} \] 
\[+ a \Delta^{-}(x) \left\{ -\chi(x) \frac{\partial \Gamma}{\partial \alpha^a_\mu(x)} - f^a_{bc} \alpha^b_\mu(x) \frac{\partial \Gamma}{\partial \alpha^c_\mu(x)} - d^a_{bc} A^b_\mu(x) \frac{\partial \Gamma}{\partial \alpha^c_\mu(x)} \right\} \] 
\[- \sum_{\mu,x} \theta^a(x) \alpha^a_\mu(x) \left\{ (1 - a \sigma_\mu(x)) \frac{\partial \Gamma}{\partial \chi_\mu(x)} + a \chi_\mu(x) \frac{\partial \Gamma}{\partial \sigma_\mu(x)} - \alpha^a_\mu(x) \frac{\partial \Gamma}{\partial A^a_\mu(x)} + A^a_\mu(x) \frac{\partial \Gamma}{\partial \alpha^a_\mu(x)} \right\} = 0. \] 
(27)

We firstly assume \( \theta_a = 0 \). By taking the derivative wr to \( \chi_\nu \) and to \( \sigma_\nu \) we get at the minimum
\[(1 - a \sigma_\mu(x)) \frac{\partial^2 \Gamma}{\partial \chi_\nu(y) \partial \chi_\mu(x)} + a \chi_\mu(x) \frac{\partial^2 \Gamma}{\partial \sigma_\nu(y) \partial \sigma_\mu(x)} = 0 \] 
(28)

Analogously we assume \( \theta^0 = 0 \) and take the derivatives with respect to \( A_\mu, \alpha_\mu \)
\[(1 - a \sigma_\mu(x)) \frac{\partial^2 \Gamma}{\partial A_\nu(y) \partial A_\mu(x)} + a \chi_\mu(x) \frac{\partial^2 \Gamma}{\partial \alpha_\nu(y) \partial \alpha_\mu(x)} = 0. \] 
(29)

These equations show that in general there is a combination of the fields \( \chi_\mu, \sigma_\mu \)
\[\chi'_\mu(x) = \frac{1}{a} [ -s_\mu (1 - a \sigma_\mu(x)) + a c_\mu \chi_\mu(x)] \] 
(30)

and a combination of the fields \( A_\mu, \alpha_\mu \)
\[A'_\mu(x) = -s_\mu \alpha_\mu(x) + c_\mu A_\mu(x), \] 
(31)

with
\[c_\mu = \frac{1 - a \sigma_\mu}{\left( (1 - a \sigma_\mu)^2 + a^2 \chi_\mu^2 \right)^{\frac{1}{2}}}, \quad s_\mu = \frac{a \chi_\mu}{\left( (1 - a \sigma_\mu)^2 + a^2 \chi_\mu^2 \right)^{\frac{1}{2}}}, \] 
(32)

which are massless. These are the physical fields. The actual auxiliary fields are the orthogonal combinations
\[
\begin{align*}
\sigma'_\mu(x) & = \frac{1}{a} \left[ 1 - c_\mu (1 - a \sigma_\mu(x)) - a s_\mu \chi_\mu(x) \right] \\
\alpha'_\mu(x) & = c_\mu \alpha_\mu(x) + s_\mu A_\mu(x).
\end{align*}
\]

The rotation to the primed fields is obtained by multiplying \( D_\mu \) by an element of the center of SU(3).

In SU(3) invariant theories we have only Eq. (29), so that a mass for both abelian fields is no longer forbidden. In this case both abelian fields are auxiliary.

5 The background gauge and the BRS symmetry

In the present regularization a background field can be introduced in close analogy with the continuum (see for example [13] and references therein) by performing a shift of the gauge fields. We define a background covariant derivative, which depends solely on the background fields, and the quantum fluctuation wrt these fields

\[
D_\mu(x) = D_{B,\mu}(x) + Q_\mu(x)
\]

where

\[
D_{B,\mu}(x) = \left[ \frac{1}{a} - \sigma_{B,\mu}(x) + i \chi_{B,\mu}(x) \right] \mathbb{1} + \left[ i A^a_{B,\mu}(x) - \alpha^a_{B,\mu}(x) \right] T_a,
\]

\[
Q_\mu(x) = \left[ - \sigma_{Q,\mu}(x) + i \chi_{Q,\mu}(x) \right] \mathbb{1} + \left[ i A^a_{Q,\mu}(x) - \alpha^a_{Q,\mu}(x) \right] T_a.
\]

A gauge transformation of the covariant derivative \( D_\mu \)

\[
D'_\mu(x) = [D_{B,\mu}(x) + Q_\mu(x)]' = g(x) [D_{B,\mu}(x) + Q_\mu(x)] g^\dagger(x + \mu).
\]

can be interpreted, among the others, in the two following ways I interpretation

\[
\begin{align*}
(D_{B,\mu}(x))' & = D_{B,\mu}(x), \\
(Q_\mu(x))' & = g(x) [D_{B,\mu}(x) + Q_\mu(x)] g^\dagger(x + \mu) - D_{B,\mu}(x)
\end{align*}
\]

II interpretation

\[
\begin{align*}
(D_{B,\mu}(x))' & = g(x) D_{B,\mu}(x) g^\dagger(x + \mu), \\
(Q_\mu(x))' & = g(x) Q_\mu(x) g^\dagger(x + \mu)
\end{align*}
\]
According to the first interpretation the background derivative is invariant, while following the second interpretation it transforms as the full covariant derivative. In the second case the quantum fluctuation undergoes a rotation like a matter field in the adjoint representation.

The presence of the background field enables us to introduce a gauge fixing term which breaks the symmetry wrt the first interpretation, while preserving the symmetry according to the second one. So doing we shall obtain an effective action which is a gauge invariant functional of the background field.

To define the gauge fixed theory we follow, for example, ref. \[15\], therefore we build a quantum lagrangian renormalizable by power counting, BRS invariant and with zero ghost number. The fundamental fields of the quantum theory are

\[ D_{B,\mu}(x), Q_\mu(x), c(x), \bar{c}(x), b(x) \]  \hspace{1cm} (39)

where \( c(x), \bar{c}(x) \) are scalar Grassmann fields with, respectively, positive and negative unit ghost number and canonical dimension equal to 1 while \( b(x) \) is a scalar c-number field with vanishing ghost number and canonical dimension equal to 2; the gauge quantum and background fields obviously have vanishing ghost number.

We now determine the equations for a BRS transformation of the various fields. It is worthwhile noticing that the BRS symmetry corresponds to the gauge symmetry broken by the gauge fixing term, therefore we determine the BRS equations starting from those for an infinitesimal gauge transformation according to the first interpretation which are, for \( g(x) \simeq 1 - i\theta^a(x)T_a \)

\[
\delta D_{B,\mu}(x) = 0 \\
\delta Q_\mu(x) = -i\theta^a(x)T_aD_\mu(x) + iD_\mu(x)\theta^a(x + \mu)T_a 
\] \hspace{1cm} (40)

A BRS transformation is obtained by means of the \( s \) operator, whose action on the various fields is specified by the following equations

\[
s D_{B,\mu} = 0,
\]

\[
s Q_\mu(x) = -ic(x)Q_\mu(x) + iQ_\mu c(x + \mu)
\]

\[
-ic(x)D_{B,\mu}(x) + iD_{B,\mu} c(x + \mu),
\]

\[
s c(x) = -iK(x),
\]

\[
s \bar{c}(x) = b(x),
\]

\[
s b(x) = 0,
\] \hspace{1cm} (41)

and the quantity \( K(x) \) is determined so as to obtain the nilpotency of the \( s \) operator, namely

\[ K(x) = c(x)c(x). \] \hspace{1cm} (42)
The quantum theory is defined by the path integral

\[
Z \left[ D_{B,\mu}(x) \right] = \int DQ_{\mu}(x) Dc(x) D\tau(x) Db(x) \exp \left\{ - \sum_{x,\mu} \left[ \mathcal{L}_G(x) + \mathcal{L}_{BRS}(x) \right] \right\}
\]

where

\[
\mathcal{L}_{BRS}(x) = -\lambda \beta \mathrm{Tr} \left\{ \tau(x) [G(x) - b(x)] \right\} = -\lambda \beta \mathrm{Tr} \left\{ b(x) G(x) - b(x) b(x) \right\} + \lambda \beta \mathrm{Tr} \left\{ \tau(x) s G(x) \right\} = \mathcal{L}_{gf}(x) + \mathcal{L}_{ghost}(x).
\]

The quantity \( G(x) = i \mathcal{G}^0(x) \mathbb{1} + \mathcal{G}^a(x) T_a \) is the gauge fixing constraint and \( \lambda \) is a real positive parameter. We can get rid off the \( b(x) \) field with a gaussian integration, so obtaining

\[
\mathcal{L}_{gf}(x) = -\frac{\lambda \beta}{2} \mathcal{G}^a(x) \mathcal{G}^a(x).
\]

A gauge fixing term which preserves the exact gauge symmetry for transformations of the background field is

\[
\mathcal{G}(x) = -i \sum_{\mu} \left[ D_{B,\mu}^\dagger(x - \mu) Q_{\mu}(x - \mu) - Q_{\mu}(x) D_{B,\mu}^\dagger(x) \right].
\]

Following the second interpretation \( \mathcal{G}(x) \) varies according to

\[
(\mathcal{G}(x))^\prime = g(x) \mathcal{G}(x) g^\dagger(x).
\]

As a consequence the gauge fixing term is invariant under gauge transformations of the background field and the effective action is a gauge invariant functional of the latter.

## 6 Summary

We have reconsidered a lattice regularization of gauge theories which makes use of auxiliary fields in order to enforce exact gauge invariance with non-compact fields. The form of the covariant derivative, for \( n > 2 \), is the same for U(n) and SU(n) theories. This means that the physical abelian field of the U(n) theory must become an additional auxiliary field in the SU(n) theory. This can be guaranteed at the quantum level by breaking explicitly the U(n) symmetry in such a way as to generate a divergent mass for this field. The terms of the lagrangian which realize this condition have been exhibited and
their effect investigated. The regularization can now be used on essentially the same footing for every n.

We have also investigated the Ward identities of the effective action, confirming that the mass spectrum has the desired properties. Finally we have formulated the theory in the background gauge and written the BRS identities, showing that a perturbative treatment can be done in close analogy with the continuum, avoiding the cumbersome expansion of the link variables.

7 Appendix A

In this Appendix we report the explicit expression of $\mathcal{L}_2$

$$\mathcal{L}_2 = \beta_2 \frac{1}{a} \sum_{\mu} \left[ -\text{det} \, D_{\mu}(x) - \text{det} \, D_{\mu}^1(x) + \frac{2}{a^2} \right]$$

$$= \beta_2 \frac{1}{a} \sum_{\mu} \left\{ \frac{6}{a^2} \sigma_{\mu}(x) - \frac{6}{a} \left( \frac{1}{a} \right) \sigma_{\mu}^2(x) \right\} - \frac{2}{a} \left( A_{\mu}^a(x) A_{\mu}^a(x) - \alpha_{\mu}^a(x) \alpha_{\mu}^a(x) \right)$$

$$+ 2 \sigma_{\mu}^3(x) - 6 \sigma_{\mu}(x) \chi_{\mu}(x) + 4 \chi_{\mu}(x) A_{\mu}^a(x) \alpha_{\mu}^a(x)$$

$$+ 2 \sigma_{\mu}(x) \left( A_{\mu}^a(x) A_{\mu}^a(x) - \alpha_{\mu}^a(x) \alpha_{\mu}^a(x) \right)$$

$$- 4 \sum_{a=1}^{8} d_{aa}^8 \left[ 2 A_{\mu}^a(x) A_{\mu}^a(x) \alpha_{\mu}^a(x) + \alpha_{\mu}^a(x) \left( A_{\mu}^a(x) A_{\mu}^a(x) - \alpha_{\mu}^a(x) \alpha_{\mu}^a(x) \right) \right]$$

$$- 4 \sum_{a=1}^{8} d_{aa}^4 \left[ 2 A_{\mu}^a(x) A_{\mu}^a(x) \alpha_{\mu}^a(x) + \alpha_{\mu}^a(x) \left( A_{\mu}^a(x) A_{\mu}^a(x) - \alpha_{\mu}^a(x) \alpha_{\mu}^a(x) \right) \right]$$

$$- 8 d_{17} \left[ A_{\mu}^1(x) \left( A_{\mu}^7(x) \alpha_{\mu}^7(x) + A_{\mu}^7(x) \alpha_{\mu}^5(x) \right) + \alpha_{\mu}^1(x) \left( A_{\mu}^5(x) A_{\mu}^7(x) + \alpha_{\mu}^5(x) \alpha_{\mu}^7(x) \right) \right]$$

$$- 8 d_{46} \left[ A_{\mu}^1(x) \left( A_{\mu}^6(x) \alpha_{\mu}^6(x) + A_{\mu}^6(x) \alpha_{\mu}^4(x) \right) + \alpha_{\mu}^1(x) \left( A_{\mu}^4(x) A_{\mu}^6(x) + \alpha_{\mu}^4(x) \alpha_{\mu}^6(x) \right) \right]$$

$$- 8 d_{17} \left[ A_{\mu}^2(x) \left( A_{\mu}^7(x) \alpha_{\mu}^7(x) + A_{\mu}^7(x) \alpha_{\mu}^5(x) \right) + \alpha_{\mu}^2(x) \left( A_{\mu}^5(x) A_{\mu}^7(x) + \alpha_{\mu}^5(x) \alpha_{\mu}^7(x) \right) \right]$$

$$- 8 d_{56} \left[ A_{\mu}^2(x) \left( A_{\mu}^6(x) \alpha_{\mu}^6(x) + A_{\mu}^6(x) \alpha_{\mu}^5(x) \right) + \alpha_{\mu}^2(x) \left( A_{\mu}^5(x) A_{\mu}^6(x) + \alpha_{\mu}^5(x) \alpha_{\mu}^6(x) \right) \right] \right\} \right\}$$

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