Low order modelling and closed-loop thermal control of a ventilated plate subject to a heat source disturbance

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Abstract. A multi-input multi-output (MIMO) thermal control problem in real-time is investigated. An aluminum slab is heated on one side by a radiative heat source and cooled on the other side by a fan panel. Starting from a nominal steady state configuration of heat source power and ventilation level, the objective is to control temperature at 4 chosen locations on the rear side when the thermal system is subject to a perturbation: the heat source power. The 4 actuators are the ventilation levels of 4 fans. The hypothesis of small inputs and temperature responses deviations is made, resulting in the assumption of a linear control problem. The originality of this work is twofold: (i) instead of a (large-sized) classical heat transfer model built from spatial discretization of local partial differential equations governing physics over the system domain, a low order model is identified from experimental data using the Modal Identification Method, (ii) this low order model is used to perform state feedback control in real time through a Linear Quadratic Gaussian (LQG) compensator.

1. Introduction
In the frame of thermal systems, the regulation of temperature at a set of specific locations belongs to the class of closed loop control problems. Although PID has proved to be an efficient tool for thermal control, even for multi-inputs/multi-outputs (MIMO) problems [1], it usually requires a tuning of its parameters. In a different way, optimal control theory appears to be a valuable solution for MIMO systems [2][3]. Assuming small perturbations of a nominal configuration, the optimal closed-loop control theory relies on a state feedback control based on a linearized state space model: the command is computed through a regulator using the state of the system (Linear Quadratic Regulator - LQR). For thermal systems, the state would be a priori the temperature field in the considered domain. If the state is not completely known, which is always the case in practice for 3D thermal systems, one has to use a Kalman Filter to estimate it (Linear Quadratic Estimator - LQE). The combination of both LQR and LQE forms the so-called Linear Quadratic Gaussian (LQG) compensator. Such an approach therefore requires the controller to have a good (linearized) model of the plant. The knowledge of the involved heat transfer modes, geometry, boundary conditions (including heat exchange coefficients), thermophysical parameters, heat sources, is hence needed to correctly model the
system. If all these features are known, then the whole space and time varying temperature field can be calculated. When some of these features are unknown or not known accurately enough, one has to estimate them to calibrate the model. The usual way to model a thermal system is to build a numerical model based on a discrete form of the continuous equations governing heat transfer inside the domain under consideration. This leads to a system of algebraic equations (let us say $N$ equations), which can be large due to zones with sharp temperature gradients and/or when 3D effects can not be neglected. Handling the control problem with a large-sized model leads to two main issues. The first one is to compute matrices of both LQR and LQE, requiring to solve two nonlinear Riccati matrix equations for continuous time formulation. This is possible with models involving up to a hundred degrees of freedom only. The second one is directly related to the computation time needed for the state estimation: a Kalman filter based on a large-sized model would prevent to control the system with a time step as small as desired in a real-time application. Model reduction methods aim to build Reduced Models (RMs), that is, models involving a number of equations $n << N$. RMs are able to reproduce the large-sized model behavior with short computing time while preserving a satisfying accuracy, and are very useful to solve optimization problems such as inverse or control problems. Some model reduction methods also allow to build Low Order Models (LOMs) from experimental data. Although numerous reduction techniques do exist for both linear and nonlinear problems (modal methods [4], balanced truncation [5], POD-Galerkin [3] to cite a few), we focus here on the use of the Modal Identification Method (MIM) to build LOMs from measured data [6][7][8]. In a first step, a LOM linking up a set of temperatures to the independent inputs acting on heat transfer is identified from experimental data with the MIM. This parameter estimation problem uses transient temperature measurements at both points to be controlled and points to be used for state estimation, corresponding to increasing and decreasing steps of each input. Such an experimental modeling allows: to avoid dealing with approximations specific to classical models (boundary conditions, geometric simplifications, knowledge of thermophysical parameters, heat exchange coefficients, emissivities); to get rid of uncertainties in sensors position (the LOM is built from data measured at real sensors location whereas a classical model uses theoretical locations). In a second step, the identified LOM is used to build the LQR and LQE matrices in order to control the temperature at specific points when the nominal configuration is perturbed.

2. Experimental device
The experimental device is composed of an aluminum slab ($0.60\text{m} \times 0.38\text{m} \times 0.003\text{m}$), vertically positioned as shown in figure 1. Its front face is heated by a confined radiative heat source (tungsten wire) whose center is located in $x = y = 0$. The other vertical face is cooled by 4 fans (F1, F2, F3 and F4) located at 0.07m from the slab. Hence, the slab loses heat by exchanging with the ambiance, especially the ventilated rear face. Thus, this experiment involves thermal diffusion with convective and radiative boundary conditions. The nonlinear behavior of this thermal system is mainly emphasized by the ventilation. The temperature measurements are given by 13 T-type thermocouples embedded on the rear face of the slab and 2 J-type thermocouples for the ambient temperature. Thanks to preliminary steady-state experiments, the standard deviation of the measurement errors has been estimated at $\sigma_m = 0.044K$ for a mean temperature equal to 307K. In this application, 4 temperatures among 13 have to be controlled by 4 manipulated variables: the ventilation level piloted by the voltage in each of the 4 fans. The disturbance is created by the heat source power. The LabVIEW software is used to control the process in real time via different peripheral devices: the DC power modulator for the 4 fans (the rotation speed of the fan is linear with respect to the supplied voltage (varying from 5V to 12V) contrary to the convective heat exchange coefficient), the DC power modulator for the heat source and the data acquisition system which gives the temperature measurements on the slab and the ambient temperature.
3. Heat transfer equations and experimental modelling

3.1. Heat transfer equations

The local equation governing unsteady heat diffusion in the system can be written as:

$$\rho(M)C_p(M) \frac{\partial T(M,t)}{\partial t} = \nabla \cdot (k(M) \nabla T)$$  \hspace{1cm} (1)

Where $T$ is the temperature at a point $M$ in the domain, $k$ the thermal conductivity, $\rho$ the density and $C_p$ the specific heat. Both convective and radiative boundary conditions do occur on the boundary $\Gamma$ of the slab, except for the area $\Gamma_s$ of the front side covered by the heat source:

$$k \nabla T \cdot \vec{n} = h(M)(T_a - T) + \epsilon(M)\sigma(T_a^4 - T^4) \hspace{1cm} M \in \Gamma \setminus \Gamma_s$$  \hspace{1cm} (2)

Where $h(M)$ and $\epsilon(M)$ are respectively the convective exchange coefficient and emissivity at point $M \in \Gamma \setminus \Gamma_s$, $\sigma = 5.67 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4}$ is the Stefan-Boltzmann constant and $T_a$ is the ambient temperature. The heat flux density provided by the source on $\Gamma_s$ is written as:

$$k \nabla T \cdot \vec{n} = \varphi(M,t) \hspace{1cm} M \in \Gamma_s$$  \hspace{1cm} (3)

Equations (1), (2) and (3) lead to many modelling issues:
- the values of thermophysical parameters ($k$, $C_p$), of density $\rho$, of emissivity $\epsilon$ and the value of natural and forced convective exchange coefficient $h$ are not exactly known;
- the source modelling requires the link between the actual heat flux density $\varphi(M,t)$ prescribed on the front side and the electric power $P(t)$ dissipated through the heating wire;
- the temperatures depend in a nonlinear way on $h$ (ventilation level) and heat power.

3.2. Low Order Model (LOM)

Rather than using classical spatial discretization methods (finite differences, finite volumes, finite elements, ...) of the local heat diffusion equation (1) and associated boundary conditions (2) and (3), in order to build a (large-sized) model of heat transfer in the system, we propose an experimental modelling approach: a linear Low Order Model (LOM) is going to be built from data measured on the experiment. The inputs of this model are the perturbation of heat source power $\delta P$ and the variations of actuators $\delta V_1$, $\delta V_2$, $\delta V_3$ and $\delta V_4$ of fans F1, F2, F3 and

![Figure 1. Experimental device: sensors and heat source position (in mm).](image1)

![Figure 2. State feedback control diagram.](image2)
The outputs are a set of temperatures at specific chosen locations. The LOM equations write as follows [9]:

\[ \dot{\delta X} = A\delta X + B_U\delta U + B_P\delta P \]

(4)

\[ \delta Y = [\delta T_1 \ \cdots \ \delta T_{13}]^T = Y - Y_{nom} = C\delta X \]

(5)

Where \( \delta U = [\delta V_1 \ \delta V_2 \ \delta V_3 \ \delta V_4]^T \) is the command vector, \( \delta X \in \mathbb{R}^n \) is the low order state vector and \( A \in \mathbb{R}^{n \times n} \) is the associated state matrix. \( B_U \in \mathbb{R}^{n \times 4} \) and \( B_P \in \mathbb{R}^{n \times 1} \) are respectively the input matrix and vector relative to the command vector \( \delta U \) and the perturbation \( \delta P \). \( C \in \mathbb{R}^{q \times n} \) is the output matrix. The output vector \( \delta Y \) contains temperature deviations \( \delta T_i \), \( i=1,...,13 \), at points 1 to 13 shown in figure 1 (\( q=13 \) in our case). We call \( \delta Z \) the part of \( \delta Y \) relative to the temperatures at points 1, 3, 5 and 6 to be controlled:

\[ \delta Z = [\delta T_1 \ \delta T_3 \ \delta T_5 \ \delta T_6]^T = C_Z\delta X \]

(6)

In order to build this LOM, the Modal Identification Method (MIM) is used. This technique, presented and used in [8] for instance, aims at identifying low order models through the use of optimization techniques, from the knowledge of input-output data, coming from either numerical simulations or experimental recordings. In the present work, data measured on the experiment are used to build the LOMs. Practically, data \( \delta Y^m(t) \) used for the LOM identification are recorded for a discrete number of time steps \( N_t \). For a given model order \( n \), the objective functional \( F^{(n)} \) therefore writes:

\[ F^{(n)} = \sum_{i=1}^{i=q=13} \sum_{j=1}^{N_t} (\delta Y_i(t_j) - \delta Y^m_i(t_j))^2 \]

(7)

The minimization of \( F^{(n)} \) is performed starting from \( n=1 \) and increasing the LOM order by 1 until a satisfying accuracy is obtained or until \( F^{(n)} \) can not be decreased anymore. A home-made code using the stochastic order 0 Particle Swarm Optimization (PSO) method [10] has been used to perform the computation of matrices \( A, B_U \) and vector \( B_P \) of equation (4). As for matrix \( C \) of equation (5), it is computed in a single step at each iteration of the PSO using ordinary least squares [8]. Table 1 summarizes the mean quadratic discrepancy (i.e. root mean square of the residues) between data \( \delta Y^m \) to be fitted and LOM outputs \( \delta Y \), defined by \( \sigma^{(n)}_Y = \sqrt{\frac{F^{(n)}}{q \times N_t}} \).

### Table 1. Evolution of \( \sigma^{(n)}_Y \) function of the LOM order \( n \).  

| \( n \) | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \sigma^{(n)}_Y \) (K) | 0.238 | 0.185 | 0.155 | 0.127 | 0.115 | 0.111 | 0.103 | 0.101 | 0.100 | 0.099 |

4. Thermal regulation problem

4.1. State feedback thermal control

4.1.1. Linear Quadratic Regulator (LQR). Making \( Z \) tend towards \( Z_0 \), i.e. making \( \delta Z \) tend towards 0 (null vector), can be recast in the minimization of the objective functional \( J \), for large time horizon problems:

\[ J = \frac{1}{2} \int_0^\infty (\delta Z^T Q \delta Z + \delta U^T R \delta U) dt \]

(8)
Figure 3. Input signals (a) and 4 among 13 corresponding output temperatures (b,c,d,e) used for the LOM identification. Comparison between temperatures measured on the experiment (solid lines) and those computed by the order 7 LOM (dashed lines).

Where $Q'$ and $R$ are symmetric positive matrices. $Q'$ is the identity matrix $I_4$ of rank 4 if no weighting on controlled outputs is applied. $R$ is usually written as $R = l^2 I_4$ where $l$ allows to limit the command magnitude: the larger $l$ is, the lower $\delta U$ is. Using equation (6) and letting $Q = C'_T Q' C_Z$, $J$ is written as:

$$J = \frac{1}{2} \int_0^\infty (\delta X^T Q \delta X + \delta U^T R \delta U) dt$$  \hspace{1cm} (9)

The command vector $\delta U$ of the LQR allowing to make $\delta Z$ tend towards 0, is given by [11][12]:

$$\delta U = -K_r \delta X$$  \hspace{1cm} (10)
Where the gain matrix $K_r$ of the LQR is:

$$K_r = R^{-1}B_U^T\mathcal{P} = \left(1/l^2\right)B_U^T\mathcal{P}$$  (11)

Where matrix $\mathcal{P}$ is solution of the algebraic Riccati equation:

$$0 = A^T\mathcal{P} + \mathcal{P}A - \mathcal{P}B_UR^{-1}B_U^T\mathcal{P} + Q$$  (12)

The command correction vector defined by equation (10) requires the knowledge of the state deviation vector $\delta X$ at each time step of the control process. If the state vector was the temperature variation vector for the whole field, it would be impossible to know it, because in practice only temperatures at local positions are available. In our case, $\delta X$ is of small size but is not a temperature vector and hence cannot be measured directly. In both situations, there is a need to use a state estimator to compute an approximation of the state vector $\delta X$ from the vector $\delta Y^m$ gathering measured temperature deviations from the nominal values.

4.1.2. Linear Quadratic Estimator (LQE). The Kalman filter or LQE is defined by [11]:

$$\dot{\hat{\delta X}} = A\hat{\delta X} + B_U\delta U + K_e(\delta Y^m - C\hat{\delta X})$$  (13)

The Kalman filter is a deterministic dynamical system allowing to compute an estimate $\hat{\delta X}$ of the state vector $\delta X$ from noisy data $\delta Y^m$ which may be seen as the output vector $\delta Y$ defined by equation (5) corrupted by measurement errors $w_m$:

$$\delta Y^m = Y^m - Y_{nom} = \delta Y + w_m = C\delta X + w_m$$  (14)

Using equation (14), one gets from equation (13):

$$\dot{\hat{\delta X}} = A\hat{\delta X} + B_U\delta U + K_e(\delta X - \hat{\delta X}) + \hat{\delta X}w_m$$  (15)

$K_eC(\delta X - \hat{\delta X})$ in equation (15) is a forcing term which makes the estimated state $\hat{\delta X}$ tend to the real state $\delta X$. The gain matrix $K_e$ of the LQE allowing to minimize the variance of the state estimation error is given by:

$$K_e = SC^TW_m^{-1}$$  (16)

Where matrix $S$ is solution of the algebraic Riccati equation:

$$0 = A\bar{S} + \bar{S}A^T - \bar{S}C^TW_m^{-1}C\bar{S} + B_PW_PB_P^T$$  (17)

$W_m$ and $W_P$ are the spectral densities of (a priori) gaussian noises $w_m$ and $\delta P$.

4.1.3. Linear Quadratic Gaussian (LQG) Compensator. The Linear Quadratic Gaussian (LQG) compensator is formed by the LQR and the LQE [13]. The principle is to replace equation (10) of the LQR giving the command correction $\delta U$ function of state deviation $\delta X$ by:

$$\delta U = -K_r\hat{\delta X}$$  (18)

Where $\hat{\delta X}$ is the estimate of the state deviation $\delta X$ and is computed by equation (13). Injecting equation (18) in equation (13), we get:

$$\hat{\delta X} = A\hat{\delta X} - B_UK_r\hat{\delta X} + K_e(\delta Y^m - C\hat{\delta X})$$  (19)

Equations (19) and (18) are the final equations to be implemented in our control command algorithm. The real-time control algorithm has been implemented in LabVIEW. Figure 2 summarizes the diagram of the state feedback control.
Figure 4. Control results for a regulation case using the LOM of order 7 with $\Delta t = 2s$ and $l = 1.4 \cdot 10^{-3} \text{K.V}^{-1}$. From top to bottom: perturbed heat power, actuators, temperatures $T_1$, $T_3$, $T_5$ and $T_6$ to be controlled. First hour with control, second hour without control.

4.2. Regulation results

The nominal steady state is set to $P_{nom} = 200\text{W}$ and $V_1 = V_2 = V_3 = V_4 = V_{nom} = 8.5\text{V}$. A pre-control transient regime is done during 4 hours to reach this steady state. Temperatures $T_{1nom}$ to $T_{13nom}$ are measured and stored in $Y_{nom}$. A two-hour control experiment then begins. It is split into two parts. A one-hour control sequence is first performed using the LOM of order 7. Our objective is to maintain temperatures $T_1$, $T_3$, $T_5$ and $T_6$ equal to their nominal values $T_{1nom}$, $T_{3nom}$, $T_{5nom}$ and $T_{6nom}$ measured in steady state. The second hour corresponds to an uncontrolled case during which the actuators are set to their nominal values $V_{nom}$. The disturbance $\delta P$ in the heat power is composed of successive steps of random magnitude, as shown in figure 4. The
duration of each perturbation step is chosen equal to 300s in order to allow the system to reach large temperature deviations in the uncontrolled phase. The random sequence of the control phase is repeated for the uncontrolled phase in order to assess the benefit of the control. The time step of control is $\Delta t = 2$s. At each one of the $N_t=1800$ time steps, the following actions are successively performed:

- heat power perturbation: $P = P_{\text{nom}} + \delta P$;
- voltage change: $V_i = V_{\text{nom}} + \delta V_i$ $i = 1 \ldots 4$ (computed from previous time step);
- measurement of $T^m_i$ to $T^m_{13}$ stored in $Y^m$;
- update of $\delta Y^m = Y^m - Y_{\text{nom}}^m$;
- computation of $\hat{\delta}X$ by solving equation (19) using an Euler implicit scheme for time integration;
- update of $\delta U = [\delta V_1 \ \delta V_2 \ \delta V_3 \ \delta V_4]^T$ using equation (18).

It can be pointed out in figure 4 that when heat power is increased, temperatures tend to grow and the control algorithm increases fan voltages. A zoom on the 4 controlled temperatures is given for both the controlled and uncontrolled sequences. Temperature deviations are clearly damped by the control algorithm. The mean quadratic discrepancy for the controlled temperatures $\sigma_Z = \sqrt{\sum_{j=1}^{N_t=1800} \sum_{i=\{1,3,5,6\}} (\delta T^m_i(t_j))^2/(4 \times N_t)}$ is equal to 0.12K whereas it is equal to 0.87K for the uncontrolled case. It can be noted that this configuration corresponds to a delicate case because the controlled temperatures are affected by many fans. The regulation of $T_4$, $T_6$, $T_9$ and $T_{11}$ is for example an easier case.

5. Conclusion
A state feedback control strategy using a low order model has been investigated for the real-time regulation of temperature at some locations on a metallic slab. The plate was heated on its front side by a heat source and cooled on the rear side by fans. The fan voltages were used as actuators to control temperature at four chosen locations, when a random heat power was supplied to the source. It has been shown that it was possible to control four outputs using four actuators. The low order model was built from experimental data measured on the set-up. Its small number of equations allowed us to solve easily Riccati equations whose solutions are needed to compute the gain matrices of both the state estimator and the regulator. Moreover, real-time control was achieved thanks to the low size of the model. In future works, we plan to add new disturbances by making the source move around its nominal position. We will also increase the number of actuators in order to control additional temperatures.

6. References
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