Lazy Abstraction-Based Control for Reachability
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Abstract—We present lazy abstraction-based controller synthesis (ABCS) for continuous-time nonlinear dynamical systems against reach-avoid specifications. State-of-the-art multi-layered ABCS pre-computes multiple finite-state abstractions of different coarseness and applies reactive synthesis to the coarsest abstraction whenever feasible, but adaptively considers finer abstractions when necessary. Our new algorithm improves this technique by constructing abstractions lazily on demand. Our insight is that the abstract transition relation only needs to be locally computed for a small set of frontier states of the coarseness currently required by the synthesis algorithm. We show that lazy ABCS can significantly outperform previous multi-layered ABCS algorithms: on a standard benchmark, lazy ABCS was more than 4 times faster.

I. INTRODUCTION

Abstraction-based controller synthesis (ABCS) is a general three-step procedure for automatic synthesis of controllers for non-linear dynamical systems. First, a time-sampled version of the continuous dynamics of the open-loop system is abstracted by a symbolic finite state model. Second, algorithms from reactive synthesis are used to synthesize a discrete abstract controller on the abstract system. Third, the abstract controller is refined to a controller for the concrete system.

The abstract model can be constructed by fixing a sampling time $\tau$, by partitioning the state space using a a grid size $\eta$, and by adding a transition between two grid cells iff there exists some state in the first cell which can reach some state of the second by following the original dynamics for time $\tau$. This construction establishes a feedback refinement relation (FRR) between the concrete and the abstract system, which establishes soundness of ABCS.

Recent approaches to ABCS use multiple abstraction layers of varying coarseness, i.e., abstract models constructed from progressively larger $\eta$ and $\tau$ [8], [3], [2], [9], [13]. As $\eta$ and $\tau$ grow, the abstract state spaces get smaller, the process gets faster, but the feasible controller domain possibly gets smaller. Therefore the abstract controller synthesis procedure tries to find a controller for the coarsest abstraction whenever feasible, but adaptively considers finer abstractions when necessary. The common bottleneck of these approaches is that the full abstract transition systems for every layer needs to be pre-computed.

This setting seems perfect for lazy abstraction [11], which is a successful paradigm to systematically explore large state spaces through abstraction and refinement, and is the basis for successful model checkers for software, hardware, and timed systems [1], [2], [17], [12].

Contribution. In this paper, we apply lazy abstraction to ABCS by solving the following two technical problems. (I) One key principle of lazy abstraction is that the abstraction is computed in the direction of the search. However, in ABCS, the abstract transition relation can only be computed forward, since it involves solving the ODE of the dynamical system forward up to the sample time $\tau$. This conflicts with reactive synthesis algorithms which work backward by iterating controllable predecessor operators. We solve this problem by keeping a reasonably small frontier, and proving that in the backward controllable predecessor computation, all transitions that need to be considered arise out of these frontier states. Thus, we can construct the abstract transitions lazily by computing the finer abstract transitions only for the frontier.

(II) The proof of correctness for lazy abstractions uses the property that there is a simulation relation (or Galois connection) between any two abstraction layers [6], [10]. This property does not hold in our setting due to different sampling times $\tau$ used in different layers: while there is a FRR between the original dynamics and each abstraction, we may not have FRRs between two arbitrary abstraction layers. Thus, our proofs of soundness and completeness (w.r.t. the finest layer) only use (a) FRRs between any abstraction and the concrete system to argue about the correctness of a controller in a sub-space, and (b) a different, ab initio, argument about the structure of the ranking functions obtained from the fixed point iterations that combines the individual controllers.

Informal Overview. The main steps of our algorithm are summarized in Fig. 1 which we explain by synthesizing a controller to steer a vehicle in a given state space (see Pic. 1 of Fig. 2) to the target (red) while avoiding static obstacles (black). In this example, we use three layers of abstraction $S_1$, $S_2$ and $S_3$ with the parameters $(\eta, \tau)$, $(2\eta, 2\tau)$ and $(4\eta, 4\tau)$.

To begin, we compute the full abstract transition relation for the coarsest abstraction $S_3$ and run the usual reachability fixed-point w.r.t. the target (red) on the obtained game graph, obtaining the winning region marked by blue in Pic. 2. To gain more states, we proceed to $S_2$ with the new target region being the projection of the winning region of $S_1$, marked by red in Pic. 3. Now the problem is that we don’t know the transitions in $S_2$. The naive option is to compute the transition relation of $S_2$ for the entire state space that falls outside the red region. This seems sub-optimal: for the wide region to the right of the obstacles, we could possibly get away with using the transitions of $S_3$, which we have already computed. Hence, we want to efficiently find a realistic over-approximation of the set

1 While an ODE can also be solved backward in time, unlike symbolic model checking for hardware or software, it is not known how to symbolically construct the set of all (controllable) predecessor states of a given set of states just from a set of ODEs.

2 One can design an enumerative forward algorithm for controller synthesis, essentially as a backtracking search of an AND-OR tree [5] but, as we demonstrate empirically, the performance of this algorithm is orders of magnitude worse for our benchmarks.
model, which allows for a forward search based technique to synthesize controllers. While forward search is usually faster than our backward computation à la reactive synthesis, it is not known how to symbolically handle external disturbances and non-determinism in the abstraction in a forward algorithm, and how controllers for ω-regular objectives can be computed over the resulting non-deterministic game graphs efficiently.

Our algorithm is also similar to the the tree-valued abstraction-refinement scheme in [7]. However, due to the use of different sampling times in different abstraction layers, we cannot apply their methodology determining an exact frontier. To balance the over-approximation of the frontier caused by our approach, we use the tuning parameter $m$ to bound the number of calls to the fixed-point algorithm in every iteration, which is not necessary in [7].

II. Preliminaries

**Notation.** We use the symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ to denote the sets of natural numbers, reals, non-negative reals, positive reals, integers, and positive integers, respectively. Given $a, b \in \mathbb{R} \cup \{\pm \infty\}$ with $a \leq b$, we denote by $[a, b]$ a closed interval. Given $a, b \in (\mathbb{R} \cup \{\pm \infty\})^n$, $a < b$, we denote by $a_i$ and $b_i$ their $i$-th element. A cell $[a, b]$ is the closed set $\{x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i\}$. We define the relations $<, \leq, \geq, >$ on $a, b$ component-wise. For a set $A$, we write $A^*$ and $A^\infty$ for the set of finite, and the set of finite or infinite sequences over $A$, respectively. For $w \in A^*$, we write $|w|$ for the length of $w$; the length of an infinite sequence is $\infty$. For $0 \leq k < |w|$ we write $w(k)$ for the $k$-th symbol of $w$.

**Continuous-Time Control Systems.** A control system $\Sigma = (X, U, W, f)$ consists of a state space $X = \mathbb{R}^n$, a non-empty input space $U \subseteq \mathbb{R}^m$, a compact cell $W \subset \mathbb{R}^n$, and a non-linear function $f : X \times U \to X$ s.t.

$$\dot{\xi} \in f(\xi(t), u(t)) + W, \quad (1)$$

where $f(\cdot, u)$ is locally Lipschitz in first component for all $u \in U$. Given an initial state $\xi(0) \in X$, a positive parameter $\tau > 0$ and a constant input trajectory $\mu_u : [0, \tau] \to U$ which maps every $t \in [0, \tau]$ to the same $u \in U$, a solution of the inclusion in (1) on $[0, \tau]$ is an absolutely continuous function $\xi : [0, \tau] \to X$ that fulfills (1) for almost every $t \in [0, \tau]$. We collect all such solutions in the set $\text{Sol}_f(\xi(0), \tau, u)$.

**Time-Sampled System.** Given a time sampling parameter $\tau > 0$, we define the time-sampled system $\overline{\Sigma}(\Sigma, \tau) = (X, U, \overline{F})$ associated with $\Sigma$, where $\overline{F} : X \times U \to 2^X$ is the transition function, defined s.t. for all $x \in X$ and for all $u \in U$ it holds that $x' \in \overline{F}(x, u)$ iff there exists a solution $\xi \in \text{Sol}_f(x, \tau, u)$ s.t. $\xi(\tau) = x'$. A trajectory $\overline{\xi}$ of $\overline{\Sigma}(\Sigma, \tau)$ is a finite or infinite sequence $x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \ldots$ such that for each $i \geq 0$, $x_{i+1} \in F(x_i, u_i)$; the collection of all such trajectories defines the behavior $B(\overline{\Sigma}(\Sigma, \tau)) \subseteq X^\infty$.

**Specifications.** We consider specifications given by three parts: a global safety requirement that requires the system to remain within a compact set $Y \subset X$; a reachability objective $T \subseteq X$; and an avoidance objective $O \subseteq X$. Intuitively, we are interested in controllers that keep the state of a system within
Y, and in addition, enforce a visit to $T \cap Y$ while avoiding any states in $O$. In the following, we fix a specification $\Phi = \{Y \mid T \cap O\}$.

**Abstract Systems.** A cover $\hat{X}$ of $X$ is a set of non-empty cells $[a, b]$ with $a, b \in (\mathbb{R} \cup \{\pm \infty\})^n$, s.t. every $x \in X$ belongs to some cell in $\hat{X}$. Fix a grid parameter $\eta \in \mathbb{R}_{>0}$ s.t. the global safety requirement $Y = \{[\alpha, \beta]\}$, where $\beta - \alpha$ is an integer multiple of $\eta$. We define a finite cover of $Y$ using cells with diameter $\eta$. We say a point $c \in Y$ is grid-aligned if there is a $k \in \mathbb{Z}^n$ s.t. for each $i \in \{1, \ldots, n\}$, we have $c_i = \alpha_i + k_i \eta - \frac{1}{2} \eta$. A cell $[a, b]$ is grid-aligned if there is a grid-aligned point $c$ s.t. $a = c - \frac{1}{2} \eta$ and $b = c + \frac{1}{2} \eta$; such cells define sets of diameter $\eta$ whose center-points are grid aligned.

Clearly, the (finite) set of grid-aligned cells is a cover for $Y$.

We define an abstract system $\hat{S}(\Sigma, \tau, \eta, \Phi) = (\hat{X}, \hat{U}, \hat{F})$ associated with the control system $\Sigma$, the time-sampling parameter $\tau$, the grid parameter $\eta$, and a specification $\Phi = \{Y \mid T \cap O\}$ s.t. the following holds: (i) $\hat{X}$ is a finite cover of $X$ and there exists a non-empty subset $\hat{Y} \subseteq \hat{X}$ which is a cover of $Y$ with grid aligned cells, and (a possibly empty) set $\hat{O} \subseteq \hat{Y}$ s.t.

\begin{equation}
\hat{O} = \{\hat{x} \in \hat{Y} \mid \exists x \in \hat{x} \cdot x \in O\},
\end{equation}

(ii) a finite $\hat{U} \subseteq U$, (iii) for all $\hat{x} \in \hat{O} \cup (\hat{X} \setminus \hat{Y})$ and $u \in \hat{U}$ it holds that $\hat{F}(\hat{x}, u) = \emptyset$, and (iv) for all $\hat{x} \in \hat{Y} \setminus \hat{O}$, $\hat{x'} \in \hat{X}$, and $u \in \hat{U}$ it holds that

\begin{equation}
\hat{x'} \in \hat{F}(\hat{x}, u) \Leftrightarrow \{\cup_{x \in x} \text{Sol}(f(x, \tau, \eta))\} \cap \hat{x'} \neq \emptyset.
\end{equation}

Notice that $T$ is not used in this definition, but will be used in our algorithms.

We consider multiple abstract systems obtained in this way. For parameters $\eta_l > 0$ and $\tau_l > 0$, and for $l \in \mathbb{N} \setminus \{1\}$, we define $\eta_l = 2\eta_{l-1}$ and $\tau_l = 2\tau_{l-1}$. With this, we obtain a sequence of $L$ time-sampled systems $\hat{S} := \{\hat{S}(\Sigma, \tau_l, \eta_l, \Phi)\}_{l \in \{1; L\}}$ and a sequence of $L$ abstract systems $S := \{\hat{S}(\Sigma, \tau_l, \eta_l, \Phi)\}_{l \in \{1; L\}}$. For simplicity, we assume that all layers use the same continuous and abstract input spaces $U$ and $\hat{U} \subseteq U$. We use $\hat{O}_l$ to denote the set defined in $\hat{S}$ for layer $l$.

**Feedback Refinement Relations.** Let $\hat{Q} \subseteq \hat{X} \times \hat{X}$ be a relation s.t. $(x, \hat{x}) \in \hat{Q}$ iff $x \in \hat{x}$ and for each $x$ exists some $\hat{x}$ such that $(x, \hat{x}) \in \hat{Q}$. Then $\hat{Q}$ is a feedback refinement relation (FRR) from $S$ to $\hat{S}$ written $\hat{S} \sqsubseteq \hat{S}$ (see [13], Thm. III.5). That is, for all $(x, \hat{x}) \in \hat{Q}$ we have (i) $U_{\hat{S}}(\hat{x}) \subseteq U_S(x)$, and (ii) $u \in U_{\hat{S}}(\hat{x}) \Rightarrow \hat{Q}(\hat{F}(\hat{x}, u)) \subseteq \hat{F}(\hat{x}, u)$, where

\begin{equation}
U_{\hat{S}}(\hat{x}) := \{u \in U \mid \hat{F}(\hat{x}, u) \neq \emptyset\} \text{ and } U_{\hat{S}}(\hat{x}) := \{u \in U \mid \hat{F}(\hat{x}, u) \neq \emptyset\}.
\end{equation}

For $\hat{S}$ and $S$, we have a sequence $\hat{Q}_l \subseteq \hat{S}$ of FRRs between the corresponding systems. The set of FRRs $\{\hat{Q}_l\}_{l \in \{1; L\}}$ induces transformers $\hat{R}_{l1} \subseteq \hat{X}_1 \times \hat{X}_l$ for $1 \leq l, l' \leq L$ between abstract states of different layers such that

\begin{equation}
\hat{x} \in \hat{R}_{l1}(\hat{x}') \Leftrightarrow \hat{x} \in \hat{Q}_l(\hat{Q}_{l-1}(\hat{x}')).
\end{equation}

However, the relation $\hat{R}_{l1}$ is generally not a FRR between the layers (see [13], Rem.1). This means that $\hat{S}_{l+1}$ cannot be directly constructed from $\hat{S}_l$, unlike in usual abstraction refinement algorithms [6], [11], [17].

**Multi-Layered Controllers and Closed Loops.** Given a multi-layered abstract system $\hat{S}$ and some $P \in \mathbb{N}$, a multi-layered controller $C = \{C^p\}_{p \in \{1; P\}}$ with $C^p \in (B^p, U, G^p)$ and $G^p : B^p \rightarrow 2^{\hat{U}}$ is composable with $\hat{S}$ if for all $p \in \{1; P\}$ there exists a unique $l_p \in \{1; L\}$ s.t. $C^p$ is composable with $\hat{S}_{l_p}$, i.e., $B^p \subseteq \hat{X}_{l_p}$. We denote by dom$(C) = \bigcup_{p \in \{1; P\}} B^p$ the domain of $C$. $p \in P$ are not necessarily related $l \in L$; we allow for multiple controllers to be composable with the same layer or no controller composable for some layers.

The quantizer induced by $C$ is defined as the map $Q : X \rightarrow 2^X \setminus \{\emptyset\}$ with $X = \bigcup_{p \in \{1; P\}} \hat{X}_p$ s.t. for all $x \in X$ holds that $\hat{x} \in Q(x)$ iff either

(i) there exists $p \in \{1; P\}$ s.t. $\hat{x} \in \hat{Q}_{l_p}(x) \cap B^p$ and there exists no $p' \in \{1; P\}$ s.t. $l_p > l_{p'}$ and $\hat{Q}_{l_{p'}}(x) \cap B^{p'} \neq \emptyset$, or

(ii) $\hat{x} \in \hat{Q}_1(x)$ and $\hat{x} \notin \hat{Q}_{l_p}(x) \cap B^p$ for all $p \in \{1; P\}$.

We define $\text{img}(Q) = \{\hat{x} \in X \mid \exists x \in X \cdot \hat{x} \in Q(x)\}$. Intuitively, $Q$ maps states $x \in X$ to the coarsest abstract state $\hat{x}$ that is both related to $x$ and is in the domain of $C$ (condition (i)). If such an abstract state does not exist, $Q$ maps $x$ to its related layer $l = 1$ states (condition (ii)).

The closed loop system formed by interconnecting $\hat{S}$ and $C$ in feedback is defined by the system $\hat{S}^c = (X, F^c)$ with $F^c : \text{img}(Q) \rightarrow 2^{\hat{U}}(Q)$ s.t. $\hat{x} \in F^c(\hat{x})$ iff there exists $p \in \{1; P\}$, $\hat{u} \in G^p(\hat{x})$ and $\hat{x'} \in \hat{R}_{l_p}(\hat{x}, \hat{u})$ s.t. $\hat{x'} \in Q(\hat{Q}_{l_{p'}}(\hat{x'}))$.

As $\hat{S}_l \subseteq \hat{S}_l$ for all $l \in \{1; L\}$, $C$ can be refined into a controller composable with $\hat{S}$ using $Q$ (see [13], Sec. 3.4). This results in the closed loop system $\hat{S}^c = (X, F^c)$ with $F^c : X \rightarrow 2^X$ s.t. $x' \in F^c(x)$ iff there exists $\hat{x} \in Q(x)$, $p \in \{1; P\}$ and $\hat{u} \in G^p(x)$ s.t. $x' \in \hat{F}(\hat{F}_{l_p}(x, \hat{u}))$.

The behaviour of $\hat{S}^c$ and $S^c$ are defined by

\begin{equation}
\hat{B}(\hat{S}^c) := \{\hat{x} \in X \mid \forall 1 \leq k \leq |\hat{x}| \cdot \hat{x}(k) \in F^c(\hat{Q}(\hat{x}, k - 1))\}
\end{equation}

\begin{equation}
\hat{B}(S^c) := \{\hat{x} \in X \mid \forall 1 \leq k \leq |\hat{x}| \cdot \hat{x}(k) \in F^c(\hat{Q}(\hat{x}, k - 1))\}.
\end{equation}

Note that $\hat{B}(\hat{S}^c)$ contains trajectories composed from abstract
III. PROBLEM STATEMENT

The reach-avoid control problem \((\Sigma, \Phi)\) for a control system \(\Sigma\) and a specification \(\Phi = \langle Y \mid O \mid T \rangle\), asks for a controller to be constructed such that every trajectory \(\xi\) of the closed loop system satisfies \(\Phi\) at sampling instances. A multi-layered controller \(C\) therefore enforces \(\Phi = \langle Y \mid O \mid T \rangle\) if for all \(\xi \in B(S)\) holds that \(\xi(k) \notin O\) for all \(k \in dom(\xi)\), and there exists a \(k' \in dom(\xi)\) s.t. \(\xi(k') \in T \cap Y\). Note that in this case the considered sampling instances might be non-uniformly spaced. The set of all multi-layered controllers fulfilling \((\Sigma, \Phi)\) are denoted by \(\mathcal{C}(\Sigma, \Phi)\).

Now recall from Sec. II that the construction of \(\hat{S}(\Sigma, \tau, \eta, \Phi) = (X, \hat{U}, \hat{F})\) ensures every trajectory of the abstract closed loop remains within \(Y\) and avoids \(O\). By adopting the classical result of ABCS using FRR (see [14, Sec.VI.A]) to the multi-layered case (see [13], Sec. 3.4) we therefore know that \(C\) enforces \((\Sigma, \Phi)\), if it solves the corresponding reachability control problem \(\langle \hat{S}, \{\hat{T}_l\}_{l \in [1:L]} \rangle\), where \(\hat{T}_l\) is an under-approximation of \(T\) w.r.t. \(\hat{O}_l\) in (2) s.t.

\[ \hat{T}_l = \{ \hat{x} \in \hat{X}_l \mid \hat{x} \subseteq T \} \setminus \hat{O}_l. \] (5)

That is, \(C \in \mathcal{C}(\Sigma, \Phi)\) if for all \(\hat{\xi} \in B(S)\) holds that

\[ \hat{\xi}(0) \in B \Rightarrow \left( \exists k \in dom(\hat{\xi}) \cdot Q^{-1}(\hat{\xi}(k)) \in T \right). \] (6)

Or equivalently, there must exists a \(k\) s.t. \(\hat{\xi}(k)\) is a layer \(l\) cell which is currently the largest cell in the domain of \(C\) and it is contained in the under-approximation \(\hat{T}_l\) of the target set \(T\). To solve the reach-avoid control problem \((\Sigma, \Phi)\) it therefore suffices to construct a multi-layer controller \(C\) solving the reachability control problem \(\langle \hat{S}, \{\hat{T}_l\}_{l \in [1:L]} \rangle\) in the sense of (6), denoted by \(\mathcal{C} \subseteq \mathcal{C}(\Sigma, \Phi)\).

IV. ABSTRACT REACHABILITY CONTROL

Single-Layered Control We first recall how the abstract reachability control problem \(\langle \hat{S}_l, \hat{T}_l \rangle\) is commonly solved by ABCS for \(L = l = 1\). In this case iteratively computes

\[ W^0 = \hat{T}_l \text{ and } W^{l+1} = \text{CPres}_{\hat{S}_l}(W^l) \cup \hat{T}_l \] (7)

until some iteration \(N \in \mathbb{N}\) is reached where \(W^N = W^{N+1}\). In [2], CPres\(_{\hat{S}_l}\) : \(2^X \rightarrow 2^X\) denotes the controllable predecessor operator, defined for a set \(\hat{Y} \subseteq \hat{X}_l\) by

\[ \text{CPres}_{\hat{S}_l}(\hat{Y}) := \{ \hat{x} \in \hat{X}_l \mid \exists \hat{u} \in \hat{U}. \hat{F}_l(\hat{x}, \hat{u}) \subseteq \hat{Y} \}. \] (8)

The controller \(C = (B, \hat{U}, G)\) with \(B = W^N \setminus \hat{T}\), and

\[ G(\hat{x}) = \{ \hat{u} \in \hat{U} \mid \hat{F}_l(\hat{x}, \hat{u}) \subseteq W^i \} \] (9)

for all \(\hat{x} \in B\), where \(i^* = \min(\{i \mid \hat{x} \in W^l \setminus \hat{T}_l\}) - 1\), is known to be a reachability controller for \(\hat{T}_l\).

The procedure implementing this iterative computation is called Reach\(_{\infty}(\hat{T}_l, l)\). We shall also use a version of Reach which runs \(m > 0\) steps of (7). Formally, given the input \(\Lambda \subseteq \hat{X}_l\), the algorithm Reach\(_m(\Lambda, l)\) returns the set \(W^m\) (the result of the \(m\)-th iteration of (7)) and the corresponding controller \(C\) computed using \(W^1, \ldots, W^m\) in [9]. Note that if the iteration in (7) reaches a fixed point in at most \(m\) iterations, Reach\(_{\infty}\) and Reach\(_m\) return the same results.

Multi-Layered Control We now recall the synthesis of a multi-layered controller \(C\) for the abstract reachability control problem \(\langle \hat{S}, \{\hat{T}_l\}_{l \in [1:L]} \rangle\) presented in [13]. This algorithm is given by a wrapper function EagerReach\(_m(\hat{T}_l, L)\) which calls the iterative algorithm Explore\(_m\) in Alg. 1 with parameters \((\hat{T}_l, L, \emptyset)\) under the assumption that \(S\) is pre-computed for all \(l \in [1; L]\) and ExpandAbstraction\(_m\) is never called (i.e., the boxed lines of Alg. 1 can be ignored in this section).

The recursive procedure Explore\(_m\) in Alg. 1 essentially implements the flow-chart of Fig. 1 by ignoring the green box which corresponds to the subroutine ExpandAbstraction\(_m\) and will be discussed in Sec. VI. Lines 1–11 implement reachability at the coarsest layer (in \(\hat{S}_l\)) and then Explore\(_m\) recursively calls itself to see if the winning states can be extended to a lower abstraction layer. Lines 13–28 implement reachability in a lower layer \((L < L)\). Here, reachability analysis is performed only for \(m\) steps, where \(m\) is a parameter of the algorithm. If the analysis already reaches a fixed point, then, as in the first case, the algorithm checks if further states can be added at a lower layer (lines 16–23). If not, the algorithm attempts to take the new states found by the reachability analysis, compute an under-approximation in a coarser layer, and recursively continues the search at the coarser layer (lines 24–28).

In order to map abstract states between different layers of abstraction, we use the operator

\[ \Gamma_{l'}(Y_{l'}) = \left\{ \begin{array}{ll} \hat{R}_{l'l'}(Y_{l'}), & l \leq l' \\ \{ \hat{x} \in \hat{X}_l \mid \hat{R}_{l'l'}(\hat{x}) \subseteq Y_{l'} \}, & l > l' \end{array} \right. \] (10)

where \(l, l' \in [1; L]\) and \(Y_{l'} \subseteq \hat{X}_{l'}\). The operation \(\Gamma_{l'l'}(Y_{l'})\) is defined as \(\hat{S}_l\) under-approximates a set \(Y_{l'} \subseteq \hat{X}_{l'}\) from layer \(l'\) to \(l\).

It is proven in [13] that EagerReach\(_m\) is sound and relatively complete w.r.t. single-layer controller synthesis on layer \(l = 1\).

Theorem 1 ([13]). Let \((\Sigma, \Phi)\) be a reach-avoid control problem and \(\hat{S} = \{\hat{S}_l\}_{l \in [1:L]}\) a sequence of abstractions. Let \((C) = \text{EagerReach}_m(\hat{T}_l, L)\). Further, let \(C = (B, \hat{U}, G)\) be returned by Reach\(_{\infty}(\hat{T}_l, 1)\). Then \(C \in \mathcal{C}(\Sigma, \Phi)\) and \(B = \hat{Q}_1(\hat{Q}^{-1}(\text{dom}(C)))\), i.e. \(C\) is sound and relatively complete w.r.t. single-layer control for layer \(l = 1\).

V. LAZY ABSTRACTION-REFINEMENT

We now describe the new lazy abstraction algorithm which implements the procedure ExpandAbstraction\(_m\) in Alg. 1 and is the main contribution of this paper.

Let LazyReach\(_m(\hat{T}_l, L)\) be a wrapper function which calls Explore\(_m\) in Alg. 1 with parameters \((\hat{T}_l, L, \emptyset)\), but without assuming that \(\hat{S}_l\) is pre-computed for \(l < L\). Hence, whenever the reachability algorithm at layer \(l < L\) of Explore\(_m\) needs
Algorithm 1 Procedure Explore\(_m\)

Require: \(\Upsilon \subseteq \hat{X}_1, l, C\)
1: if \(l = L\) then
2: \((W, C) \leftarrow \text{Reach}_\infty(\Gamma^L_1(\Upsilon), l)\)
3: \(C \leftarrow C \cup \{C\}\)
4: \(\Upsilon \leftarrow \Upsilon \cup \Gamma^L_1(W)\) // Save W to \(\Upsilon\)
5: if \(L = 1\) then // Single-layered reachability
6: \(\text{return} \,(\Upsilon, C)\)
7: else // Go finer
8: \(\text{ExpandAbstraction}_m(\Upsilon, l - 1)\)
9: \((\Upsilon, C) \leftarrow \text{Explore}_m(\Upsilon, l - 1, C)\)
10: \(\text{return} \,(\Upsilon, C)\)
11: end if
12: else
13: \((W, C) \leftarrow \text{Reach}_m(\Gamma^{l+1}_1(\Upsilon), l)\)
14: \(C \leftarrow C \cup \{C\}\)
15: \(\Upsilon \leftarrow \Upsilon \cup \Gamma^L_1(W)\) // Save W to \(\Upsilon\)
16: if Fixed-point is reached in line 13 then
17: if \(l = 1\) then // Finest layer reached
18: \(\text{return} \,(\Upsilon, C)\)
19: else // Go finer
20: \(\text{ExpandAbstraction}_m(\Upsilon, l - 1)\)
21: \((\Upsilon, C) \leftarrow \text{Explore}_m(\Upsilon, l - 1, C)\)
22: \(\text{return} \,(\Upsilon, C)\)
23: end if
24: else // Go coarser
25: \(\text{ExpandAbstraction}_m(\Upsilon, l + 1)\)
26: \((\Upsilon, C) \leftarrow \text{Explore}_m(\Upsilon, l + 1, C)\)
27: \(\text{return} \,(\Upsilon, C)\)
28: end if
29: end if

Algorithm 2 ExpandAbstraction\(_m\)

Require: \(\Upsilon \subseteq \hat{X}_1, l\)
1: \(W' \leftarrow \text{Pre}_{A^l_1}(\Gamma^L_1(\Upsilon)) \setminus \Gamma^L_1(\Upsilon)\)
2: \(W'' \leftarrow \text{Pre}_{L}^{K_L}(W')\)
3: Compute \(F^{\hat{L}}_1\) as in 3 for all \(\hat{x} \in W'' \setminus \hat{O}_1\) and \(\tau = \tau_1\).

to compute the abstract transition relation, the procedure ExpandAbstraction\(_m\) is used to compute just the necessary portion of the abstract transition relation.

Constructing the Frontier Intuitively, ExpandAbstraction\(_m\) computes the frontier by computing the predecessors of the already obtained set \(\Upsilon\) optimistically (i) by using (coarse) auxiliary abstractions for this computation, and (ii) by applying a cooperative predecessor operator.

This requires a set of auxiliary systems, given by \(\hat{A} = \{A^l_j\}_{l=1}^m\), where \(A^l_j := \mathcal{S}(\Sigma, \tau_l, \eta_L, \{Y\}) T \{\emptyset\} = (\hat{X}_L, \hat{U}, \hat{F}^L_j)\). Recall from Sec. 3 that this implies that \(F^{\hat{L}}_1\) is defined for all \(\hat{x} \in Y_L, \hat{x}' \in \hat{X}_L\), and \(u \in \hat{U}\) by 3 with \(\tau = \tau_1\). Intuitively, \(A^l_j\) is the abstract system induced by \(\Sigma\), which ignores the avoid part of the specification \(O\), and computes the set of layer \(L\) cells which are reachable by applying a constant control input for duration \(\tau_1\), where \(l \leq L\). Ignoring \(O\) is important, as usually \(\hat{O}_1 \subset \tilde{O}_L\) for \(l < L\). Hence, projection of the “good states” in layer \(l\) might be covered by an obstacle in layer \(L\) (see Fig. 2). Using \(\tau_1\) instead of \(\tau_L\) is important, as \(\tau_L\) might cause “holes” between the computed frontier and the current target \(\Upsilon\) which cannot be bridged by control actions in layer \(l\). This would render LazyReach\(_m\) unsound.

For \(\Upsilon \subseteq \hat{X}_L\) and \(l \in [1; L]\), we define the co-operative predecessor operator
\[
\text{Pre}_{A^l_1}(\Upsilon) = \{\hat{x} \in \hat{X}_L \mid \exists u \in \hat{U} . \hat{F}^{\hat{L}}_1(\hat{x}, \hat{u}) \cap \Upsilon \neq \emptyset\}.
\]

This operator is applied \(m\) times in ExpandAbstraction\(_m\), i.e.,
\[
\text{Pre}_{A^L_1}(\Upsilon) = \text{Pre}_{A^L_1}(\Upsilon) \cap \text{Pre}_{A^L_1}(\text{Pre}_{A^L_1}(\text{Pre}_{A^L_1}(\Upsilon))).
\]

When calling ExpandAbstraction\(_m\) with parameters \(\Upsilon \subseteq \hat{X}_1\) and \(l < L\), it applies \(\text{Pre}_{A^L_1}\) to the over-approximation of \(\Upsilon\) by \(L\)-states. This over-approximation is defined as the dual operator of the under-approximation operator \(\Gamma^L_\uparrow\).
\[
\Gamma^L_\uparrow(\Upsilon) := \left\{ \begin{array}{ll}
\hat{R}^L(\Upsilon), & l \leq l' \\
\{\hat{x} \in \hat{X}_L \mid \hat{R}^L(\hat{x}) \cap \Upsilon \neq \emptyset\}, & l > l'.
\end{array} \right.
\]

where \(l, l' \in [1; L]\) and \(\Upsilon \subseteq \hat{X}_L\).

Soundness and Relative Completeness The soundness and completeness of LazyReach\(_m\) follows from Thm. 1 if we can ensure that in every iteration of LazyReach\(_m\), the set of states returned by ExpandAbstraction\(_m\), for which the abstract transition relation is computed, is not smaller than the set of states subsequently added to \(\Upsilon\) by Reach\(_m\) in the next iteration. We obtain this result by the following series of lemmata, proven in the appendix, leading to our main result stated in Thm. 2.

We first observe that computing the under-approximation of the \(m\)-step cooperative predecessor w.r.t. the auxiliary system \(A^l_j\) of a set \(\Upsilon\) (as used in ExpandAbstraction\(_m\)) over-approximates the set obtained by computing the \(m\)-step cooperative predecessor w.r.t. the abstract system \(\hat{S}_1\) for a set \(\Upsilon\) (as used in Reach\(_m\)) if \(\Upsilon\) over-approximates \(\Upsilon\).

Lemma 1. Let \(\hat{S}\) be a multi-layered system and \(\Upsilon, l \subseteq \hat{X}_L\) and \(\Upsilon \subseteq \hat{X}_L\) for some \(l < L\) s.t. \(\Upsilon \subseteq \Gamma^L_1(\Upsilon)\).
Then \(\Gamma^L_1\left(\text{Pre}_{A^L_1}(\Upsilon)\right) \supseteq \text{Pre}_{\hat{S}_1}(\Upsilon)\). Furthermore, for all \(m > 0\), it holds that \(\Gamma^L_1\left(\text{Pre}_{A^L_1}(\Upsilon)\right) \supseteq \text{Pre}_{\hat{S}_1}^m(\Upsilon)\), where \(\text{Pre}_{\hat{S}_1}\) and \(\text{Pre}_{\hat{S}_1}^m\) for \(\hat{S}_1\) are defined analogously to (11) and (12) respectively.

Lem. 1 can be used to show that ExpandAbstraction\(_m\) constructs the transition function \(F^L_1(\hat{x}, \hat{u})\) for all \(\hat{x}\) which would be in the winning state set computed by mathsfEagerReach\(_m\).

Lemma 2. For all \(l < L\), \(\Upsilon \subseteq \hat{X}_L\) and \(C = (B, \hat{U}, \hat{G})\) returned by Reach\(_m\)(\(\Gamma^L_1(\Upsilon), l\)) on pre-computed \(\hat{S}_1\), it holds that \(\hat{x} \in B \setminus \Gamma^L_1(\Upsilon)\) implies \(\hat{x} \in W''\), where \(W''\) is returned by the second line of ExpandAbstraction\(_m\)(\(\Upsilon, l\)).

With Lem. 2 soundness and completeness of LazyReach\(_m\) directly follows from Thm. 1 as proven in the appendix.

Theorem 2. Let \((\Sigma, \Phi)\) be a reach-avoid control problem and \(S = \{\hat{S}_1\}_{l \in [1; L]}\) a sequence of abstractions. Let
VI. IMPLEMENTATION AND CASE STUDY

Our algorithm is implemented in C++ as an extension to SCOTS [15]. We use binary decision diagrams (BDDs) to symbolically represent and manipulate states and transitions; for this we leverage the CUDD package [16]. All the experiments presented in this section were performed on an Intel Core i5 3.40 GHz processor.

For the efficient implementation of the $\Gamma^\uparrow_{W_l}$ and $\Gamma^\downarrow_{W_l}$ operators, we use similar ideas to those presented in [13]. Since the ratio of $n$-s of the adjacent layers of $S$ is always 2, the BDD variables can be arranged in a particular pattern, as shown in Fig. 3 for the simple case of $|X_L| = 2$. Let $W_l \subseteq X_l$ be a set of states in layer $l$ represented by the BDD $\mathcal{W}_l$ over a set of boolean variables $\{b_1, b_2, \ldots, b_k\}$. We compute the approximations as follows:

- $\Gamma^\uparrow_{W_{l+1}}(W_l) = \forall \{b_1, W_l\}[\{b_1, \ldots, b_k\} / \{b_{2j-k}, \ldots, b_{j-1}\}]$,
- $\Gamma^\downarrow_{W_{l+1}}(W_l) = \exists \{b_1, W_l\}[\{b_1, \ldots, b_k\} / \{b_{2j-k}, \ldots, b_{j-1}\}]$,
- $\Gamma^\downarrow_{W_{l-1}}(W_l) = \Gamma^\uparrow_{W_l}(W)$

where $[\mathcal{B}]/\{x_1, x_2, \ldots, x_p\} / \{y_1, \ldots, y_p\}$ represents the renaming of the variables $\{x_1, \ldots, x_p\}$ of the BDD $\mathcal{B}$ to the variables $\{y_1, \ldots, y_p\}$. Recall that renaming of BDD variables can be done in linear time in the size of the BDD.

For all experiments, we use a non-linear kinematic system model commonly known as the unicycle model as presented in [13] and is given in the following:

- $\dot{x}_1 = u_1 \cos(x_3) + W_1$
- $\dot{x}_2 = u_1 \sin(x_3) + W_2$
- $\dot{x}_3 = u_2$;

where $x_1$, $x_2$, $x_3$ are the state variables representing the horizontal, vertical and angular displacement respectively, $u_1$, $u_2$ are the control input variables representing the linear and angular velocities respectively, and $W_1$, $W_2$ are the perturbation bounds in the respective dimension given by $W_1 = W_2 = [-0.05, 0.05]$. We run controller synthesis experiments for the unicycle inside a two dimensional state space with obstacles and a designated target area, as can be seen in Fig. 4. We use three layers for the multi-layered algorithms EagerReach and LazyReach.

Algorithm Comparison We use the synthesis problem as presented in [13] for a comparison of four algorithms: Reach, EagerReach, LazyReach, and a multi-layered baseline performance.

![Figure 4. Solution of the unicycle reach-avoid problem by LazyReach](image-url)

Table I

| Algorithm | Reach | EagerReach | LazyReach | Forward |
|-----------|-------|------------|-----------|---------|
| A         | 2590  | 2628       | 588       | n/a     |
| S         | 818   | 73         | 21        | n/a     |
| T         | 3408  | 2701       | 609       | > 36000 |

- (126%) (100%) (22.5%)

Hyperparameter Investigation We investigate the performance of LazyReach as we vary $m > 0$. The results are shown in Fig. 5(a). The monotonic behavior suggests that the cost of switching (i.e. projecting sets of states) between layers is negligible compared to the savings from the minimization of the exploration done in the finest layer.

Varying State Space Complexity We investigate how our lazy algorithm and the multi-layered baseline perform with respect to the incremental variation of the difficulty of the problem, which is achieved by varying the number of identical obstacles.
obstacles, \( o \), placed in the open area of the state space. This can also be seen as varying the non-uniformity (decreasing with increasing \( o \)) of the state space. The runtimes for \( \text{EagerReach}_2 \) and \( \text{LazyReach}_2 \) are plotted in Fig. 5(b). We observe that \( \text{LazyReach}_2 \) is able to keep its runtime low when there are fewer obstacles by only constructing the abstraction in the finest layer for the immediate surroundings of those obstacles. For \( o = 20 \), \( \text{LazyReach}_2 \) explores the entire state space in the finest layer, resulting in its performance being slightly worse than that of \( \text{EagerReach}_2 \). The general decreasing trend in the abstraction construction runtime for \( \text{EagerReach}_2 \) is because transitions outgoing from obstacle states are not computed.

VII. CONCLUSION

In this paper we extend the paradigm of lazy abstractions from the area of program verification to the area of abstraction-based control for continuous dynamical systems with reach-avoid specifications. We show how one can lazily compute the abstractions of different granularity as per the need, while prioritizing the usage of coarser abstractions to save computational effort. By empirically comparing our algorithm with the ones available in the literature using a benchmark example, we show that our algorithm is more than 4 times faster than the state-of-the art.

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APPENDIX

First we state some properties of \( \Gamma_{l'}^t(\cdot) \) and \( \Gamma_{l}^t(\cdot) \). Let \( A_{l'}, B_{l'} \subseteq X_{l'} \) be any two sets. Then

(a) \( \Gamma_{l'}^t(\cdot) = \Gamma_{l'}^{t,k}(\Gamma_{l'}^{k,t}(\cdot)) \) and \( \Gamma_{l}^t(\cdot) = \Gamma_{l}^{t,k}(\Gamma_{l}^{k,t}(\cdot)) \) for all \( k \) s.t. \( \min\{l,l'\} \leq k \leq \max\{l,l'\} \).

(b) \( \Gamma_{l'}^t(\cdot) \) and \( \Gamma_{l}^t(\cdot) \) are monotonic, i.e. \( A_{l'} \subseteq B_{l} \Rightarrow \Gamma_{l'}^t(A_{l'}) \subseteq \Gamma_{l}^t(B_{l'}) \) and \( A_{l} \subseteq B_{l'} \Rightarrow \Gamma_{l}^t(A_{l}) \subseteq \Gamma_{l'}^t(B_{l'}) \).

(c) For \( l > l' \), both \( \Gamma_{l'}^t(\cdot) \) and \( \Gamma_{l}^t(\cdot) \) are closed under union and intersection.

(d) \( l \leq l' \Rightarrow \Gamma_{l'}^t(\cdot) \subseteq \Gamma_{l}^t(\cdot) \).

(e) \( l \leq l' \Rightarrow \Gamma_{l}^t(\Gamma_{l}^t(A_{l})) = \Gamma_{l}^t(\Gamma_{l'}^t(A_{l'})) = A_{l'} \).

Using (d), we additionally have \( l \leq l' \Rightarrow \Gamma_{l}^t(\Gamma_{l}^t(A_{l})) = A_{l'} \), i.e., when \( l \leq l' \), the composition \( \Gamma_{l}^t \circ \Gamma_{l}^t \) for \( \ast \in \{↑, ↓\} \) is the identity function.

(f) For all \( x \in X, Q_{l}^t(x) \in A_{l'} \Rightarrow Q_{l}^t(x) = \Gamma_{l}^t(A_{l'}) \). Equivalently, for all \( x' \in X_{l'}, x' \in A_{l'} \Rightarrow \tilde{R}_{l}^t(\tilde{x}') \in \Gamma_{l}^t(A_{l'}) \).

(g) For all \( x \in X, Q_{l}^t(x) \in A_{l'} \Rightarrow Q_{l}^t(x) \in A_{l'} \). Equivalently, for all \( \tilde{x} \in X_{l}, \tilde{x} \in A_{l'} \Rightarrow \tilde{R}_{l}^t(\tilde{x}) \in A_{l'} \).
where (16) follows from (c) and (17) follows by applying the necessary pre-condition, and (ii) for the right side of the

Proof of Lem. 2 Let \( \hat{x} \in \text{Pre}_{S_1}(\Upsilon_1) \) be an abstract state, which implies that there exists \( x \in X \) s.t. \( Q_l(x) = \hat{x}, \hat{x} \in \hat{U} \) and \( \hat{x}' \in \Upsilon_1 \) s.t. \( \text{Sol}_f(x, \tau, u) \cap Q_l^{-1}(\hat{x}') \neq \emptyset \) (from (3)).

Let \( \hat{y} = \Gamma_{L_L}^{-1}(\{\hat{x}\}) \) which implies \( Q_L(x) = \hat{y} \) (from (1)). Also note that since \( \hat{x}' \in \Upsilon_1 \), hence \( \Gamma_{L_L}^{-1}(\{\hat{x}'\}) \subseteq \Upsilon_L \) holds by assumption, implying that for all \( x' \in X \) s.t. \( Q_l(x') = \hat{x}' \), \( Q_L(x') \in \Gamma_{L_L}^{-1}(\{\hat{x}'\}) \subseteq \Upsilon_L \) (from (1)). Consequently we have that \( \text{Sol}_f(x, \tau, u) \cap \Upsilon_L \neq \emptyset \), and hence \( \hat{y} \in \text{Pre}_{A_1}(\Upsilon_L) \).

Moreover, using (14c) we have that \( \hat{x} \in \Gamma_{L_L}^{-1}(\{\hat{y}\}) \) which leads to \( \hat{x} \in \text{Pre}_{A_1}^{m}(\Upsilon_L) \).

The second claim is proven by induction on \( m \). The base case for \( m = 1 \) is given by the first claim proven above. Now assume that \( \Gamma_{L_L}^{-1}(\text{Pre}_{m}(\Upsilon_L)) \supseteq \text{Pre}_{S_1}(\Upsilon_1) \) holds for some \( m > 0 \). This together with (14c) implies:

\[
\text{Pre}_{A_1}^{m}(\Upsilon_L) \supseteq \Gamma_{L_L}^{-1}(\text{Pre}_{S_1}(\Upsilon_1)).
\]

Now note that by (12), we have \( \text{Pre}_{A_1}^{m}(\cdot) = \text{Pre}_{A_1}^{m-1}(\cdot) \cup \text{Pre}_{A_1}^{m}(\cdot) \) and it holds that

\[
\Gamma_{L_L}^{-1}(\text{Pre}_{A_1}^{m}(\Upsilon_L)) \supseteq \text{Pre}_{S_1}(\Upsilon_1).
\]

Proof of Thm. 2 For the proof, we leverage Thm. 1. First, we prove that both algorithms terminate after the same depth of recursion \( D \). Second, we show that the overall controller domain that we get from \( \text{EagerReach}_m \) is same as the one that we get from \( \text{LazyReach}_m \), i.e. \( \cup_{d \in [1; D]} B_d = \cup_{d \in [1; D]} B_d \), where \( B_d \) and \( B_d \) are the controller domains obtained in depth \( d \) of the algorithms \( \text{EagerReach}_m \) and \( \text{LazyReach}_m \) respectively. (We actually prove a stronger statement: for all \( d \in [1; D] \), \( B_d = B_d \).) Then, since \( \text{EagerReach}_m \) is sound and complete w.r.t. Reach\_\infty, hence \( \text{LazyReach}_m \) will also be sound and complete w.r.t. Reach\_\infty.

The “\( \geq \)” direction of the proof is trivial and is based on two simple observations: (a) the amount of information of the abstract transition systems \( S \) which is available to \( \text{LazyReach}_m \) is never greater than the same available to \( \text{EagerReach}_m \); (b) whenever \( \text{LazyReach}_m \) invokes ExpandAbstraction\_m for computing transitions for some set of abstract states, ExpandAbstraction\_m returns the full information of the outgoing transitions for those states to \( \text{LazyReach}_m \).

The second part is crucial, as partial information of outgoing transitions might possibly lead to false positive states in the controller domain. Combining these two arguments, we have that for all \( d \in [1; D] \), \( B_d \supseteq B_d \). (We are yet to show that the maximum recursion depth is \( D \) for both the algorithms \( \text{EagerReach}_m \) and \( \text{LazyReach}_m \).)

The other direction will be proven by induction on the depth of the recursive calls of the two algorithms. Let \( L_l \) and \( l_l \) be the corresponding layers in depth \( d \) of algorithm \( \text{LazyReach}_m \) and \( \text{EagerReach}_m \) respectively. It is clear that \( B_1 = B_1 \) and \( l_1 = l_1 \) (induction base) since we start with full abstract transition system for layer \( L \) in both cases. Let us assume that for some depth \( d \), \( B_d = B_d \) and \( l_d = l_d \) holds true for all \( d \leq d \) (induction hypothesis). Now in \( \text{LazyReach}_m \), the check in Line 16 of \( \text{LazyReach}_m \) is fully iff the corresponding check in Line 15 of \( \text{EagerReach}_m \) (i.e. Alg. 1) is fulfilled. This shows by induction that (a) the maximum depth of recursion in \( \text{LazyReach}_m \) and \( \text{EagerReach}_m \) are the same (call it \( D \)), and (b) the concerned layer in each recursive call is the same for both algorithms.

Now in the beginning of depth \( d + 1 \), we have that \( \Upsilon = \cup_{d \in [1; D]} \text{Pre}_{A_1}^{d}(\Gamma_{L_L}(\Upsilon)) \). In the rest of the proof, let’s call \( l_{d+1} = l_l \) for simpler notation. Suppose \( \hat{x} \in \hat{X}_l \) be a state which was added in depth \( d + 1 \) in the controller domain \( B_{d+1} \) for the first time, i.e. (a) \( \hat{x} \in B_{d+1} \), and (b) \( \hat{x} \in \Gamma_{L_L}(\Upsilon) \). Then by Lem. 2 we have that \( \hat{x} \in W_m \).

Since ExpandAbstraction\_m also computes all the outgoing transitions from the states in \( W_m \) (Line 3 in Alg. 2), hence full information of the outgoing transitions of all the states which are added in \( B_{d+1} \) will be available to the \( \text{LazyReach}_m \) algorithm in depth \( d + 1 \). In other words given \( \hat{x} \in \hat{X}_l \), if there is an \( m \)-step controllable path from \( \hat{x} \) to \( \Upsilon \) in \( \text{EagerReach}_m \), there will be an \( m \)-step controllable path in \( \text{LazyReach}_m \) as well. Hence \( \hat{x} \) will be added in \( B_{d+1} \) as well. This proves that for all \( d \in [1; D] \), \( B_d \supseteq B_d \).