Abstract We investigate convergence properties of generalized Walsh series associated with signals $f \in L^1[0, 1]$. We also show how the dependence of the generalized Walsh bases on $N \times N$ unitary matrices allows for applications in signal encoding and encryption, provided the signals are piece-wise constant on $N$-adic subintervals of $[0, 1]$.

Keywords Conditional expectation · Encoding · Generalized Walsh functions · Martingale · Maple · Public key cryptography

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1 Introduction

The Walsh basis functions form an orthonormal system that can be interpreted roughly as the discrete analog of classic sines and cosines. Unlike these the Walsh functions have several advantages: for example they take only two values $\pm 1$ on sub-intervals defined by dyadic fractions, thus making the computation of coefficients much easier. The Walsh functions are connected to probability, e.g., the Walsh expansion can be seen as conditional expectation, and the partial sums form a Doob martingale. Moreover, partial sums converge a.e. for $L^1$ functions, which is not true of the classic exponential basis.

The Walsh functions have found a wide range of applications: for example in modern communications systems (through the so-called Hadamard matrices, to recover information in the presence of noise and interference), signal processing (reconstruction of signals by...
means of dyadic sampling theorems), and generally in computer science. For detailed accounts of the many areas of the applied sciences where the Walsh functions are used we refer the reader to the books [1, 2, 11], and [12]. Next we describe briefly the classic Walsh system and some of its properties. These can be found in [9, 17] and references therein. Let

$$
\phi_n(t) := \begin{cases} 
1, & \text{if } t \in \left[0, \frac{1}{2^n}\right) \cup \left[\frac{3}{2^n}, \frac{4}{2^n}\right) \cup \cdots \\
-1, & \text{if } t \in \left[\frac{1}{2^n}, \frac{2}{2^n}\right) \cup \left[\frac{4}{2^n}, \frac{5}{2^n}\right) \cup \cdots 
\end{cases}
$$

Usually the dyadic endpoints are not included as above and the value of $\phi_n$ is taken to be zero there (i.e. jump average). Same extension is considered for the Walsh functions. Here we are not affected by this as we are not concerned with convergence questions at such dyadic rationals. Now to define the $n$th Walsh function let $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$ be the base-2 expansion of $n$. Then the classic Walsh functions $(W_n)_{n \geq 0}$ defined by

$$
W_n(t) := \phi_{n_1}(t)\phi_{n_2}(t) \cdots \phi_{n_k}(t)
$$

form an orthonormal basis (ONB) for the Hilbert space $L^2[0, 1]$. There are certain features that make this ONB more desirable to work with than for example the Fourier system. As we will point out below in more detail, Walsh series associated to that make this ONB more desirable to work with than for example the Fourier system. As we will point out below in more detail, Walsh series associated to $f \in L^1[0, 1]$ converge pointwise a.e. to $f$. This is also true for $f$ with bounded variation at a continuity point of $f$.

Various interpretations of the Walsh ONB have been given, in [9, 15], and generalizations in [4, 14, 16]. E.g. for the dyadic group $G$ the Walsh functions can be viewed as characters on $G$, or more generally starting with [16], as characters of a zero-dimensional, separable group. A generalized Walsh system based on $N$-adic numbers and exponentials functions can be found in [4], and has been used to construct algorithms for polynomial lattices (a particular kind of digital net which in turn can be used in sampling methods for multivariate integration), see [6, 7] and their references. However the Walsh-like system in [8] which inspired the present work seems to be new: roughly, this new generalization of the Walsh ONB depends on certain unitary matrices (constant first row) and a simple IFS (iterated function system implemented by map $r$ below). In [8], Theorem 3.1 gives a criteria to obtain ONBs based on Cuntz algebra representations. One byproduct (Proposition 3.10) recovers the classic Walsh ONB, and another (Theorem 3.11) generalizes it as follows: Start with an integer $N \geq 2$, and $A = [a_{ij}]_{i=0,N-1}$ a unitary matrix such that its first row entries are all equal to $\frac{1}{\sqrt{N}}$. For $0 \leq i \leq N - 1$ define

$$
m_i(x) := \sqrt{N} \sum_{j=0}^{N-1} a_{ij} \chi_{[(j/N,(j+1)/N)}(x)
$$

Notice $m_0(x) = 1$, $\forall x \in [0, 1]$. Denote by $r$ the map $r(x) := (Nx) \mod 1 = Nx - l$ if $x \in [l/N,(l+1)/N]$ where $l \in \{0, 1, \ldots, N-1\}$. With $n$ nonnegative integer written in its base-$N$ expansion $n = \sum_{k=0}^{p-1} i_k N^k$, the $n$'th Walsh function associated to matrix $A$ is

$$
W_{n,A}(x) := m_{i_0}(x)m_{i_1}(rx) \cdots m_{i_p}(r^{p-1}x) \tag{1.1}
$$

where $i_0, i_1, \ldots, i_{p-1}$ are the coefficients of the expansion. Notice $W_{0,A} \equiv 1$. When $N = 2$ one obtains the classic Walsh system by picking the unitary matrix to be

$$
A := \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & -1 \end{pmatrix}
$$

and the Rademacher functions are obtained as $\phi_n(t) = m_1(r^n t)$. 

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In the next section we prove convergence results for the generalized Walsh series formed with (1.1) (Theorem 2.1, and Corollary 2.2). In Example 2.5 we have implemented a generalized Walsh system with the aid of the mathematical software Maple to point out issues with the convergence of arbitrary generalized Walsh series. Corollary 2.3 helps define the discrete generalized Walsh transform and is very instrumental in our Maple computations. We end the section with Theorem 2.6, and Corollary 2.8 where we show that the connections with probability are still maintained: the generalized Walsh partial sums (of type \( N^q \)) form martingales, and their series behave as conditional expectations that converge in \( L^p \).

While applications in signal processing could have been investigated, due to the multitude of generalized Walsh systems (each being associated to a unitary matrix) we found it natural to consider data encryption: in the last section of the paper we find a sufficient condition that two unitary matrices should satisfy in order to have secret communication in the spirit of public key cryptography, between two users each possessing a generalized Walsh transform. However as our remarks and examples indicate, a successful protocol depends on whether certain zero-dimensional systems of polynomial (quadratic) equations have infinitely many solutions.

### 2 Pointwise Convergence

We study convergence properties of the new orthonormal bases, and show that some of the convergence results from [13], and [17] extend for any \( N \geq 2 \). For \( N = 2 \) the theorem below was obtained by Walsh (with \( f \) continuous), and Kaczmarz (\( f \in L^1 \)), see also [9] and references therein.

**Theorem 2.1** For \( f \in L^1[0, 1] \) the sequence of partial sums

\[ S_{N^q}(x) = \sum_{n=0}^{N^q-1} \langle f, W_{n,A} \rangle W_{n,A}(x) \]

converges a.e. to \( f(x) \).

**Proof** We show that if \( x \in [0, 1] \) is a Lebesque point of \( f \) then the generalized Walsh series \( S_{N^q} \) converges to \( f(x) \). The calculations are based on the orthogonality conditions that the columns/rows of matrix \( A \) satisfy. We will also write \( W_n \) instead of \( W_{n,A} \) (as long as we deal with a fixed \( A \)). To not clutter our expressions we will consider the case \( N = 3 \) and point out how/why the arbitrary dimensional analogue carries through. We set out to prove a couple of properties of the Dirichlet kernel \( D_q(x, t) \). Let us recall that in general \( S_q = f(t)D_q(x, t)dt \) where \( D_q(x, t) = \sum_{n=0}^{q-1} W_n(x)W_n(t) \). Notice first

\[ D_{3^q}(x, t) = \prod_{j=0}^{q-1} [1 + m_1(r^jx)m_1(r^jt) + m_2(r^jx)m_2(r^jt)] \quad (2.1) \]

The relationship is easily checked by multiplying through all parentheses in the righthand side and using (1.1). If \( N \) is arbitrary then the generic factor in the product above is of the form \( \sum_{n=0}^{m_{N-1}(r^jx)m_{N-1}(r^jt)} \).

For \( x \in [0, 1] \) and \( q \in \mathbb{N} \) there exists a unique \( m = m(q, x) \in \{0, 1, \ldots, 3^q - 1\} \) such that

\[ \frac{m}{3^q} \leq x < \frac{m + 1}{3^q} \].

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We claim the following formula holds (with obvious \(N^q\) replacement in the general case):

\[
D_{3^q}(x, t) = \begin{cases} 
3^q & \text{if } t \in (\alpha_{q,x}, \beta_{q,x}) \\
0 & \text{otherwise}
\end{cases} \tag{2.2}
\]

Let \(t \in (\alpha_{q,x}, \beta_{q,x})\). With respect to base-3 expansion \(m = k_{q-1}3^0 + \cdots + k_{q-1-j}3^j + \cdots + k_03^{q-1}\), \(k_j \in \{0, 1, 2\}\). Then for all \(j \in \{0, 1, \ldots, 3^q - 1\}\) we have \(m_1(r^j x) = m_1(r^j t) = \sqrt{3}a_{1,k_j}\), and \(m_2(r^j x) = m_2(r^j t) = \sqrt{3}a_{2,k_j}\). To see this notice that \(r^j(x) \in [k_j/3, (k_j+1)/3]\) for all \(0 \leq j \leq q - 1\). (indeed, for \(j = 0\) apply the inequalities

\[
\frac{k_0}{3} = \frac{k_03^{q-1}}{3^q} \leq \frac{m}{3^q} \leq \frac{1 + m}{3^q} \leq \frac{1 + \sum_{j=0}^{q-2} 2 \cdot 3^j + k_03^{q-1}}{3^q} = \frac{1 + k_0}{3}
\]

then \(x \in [k_0/3, (k_0+1)/3]\) and \(\frac{m-k_03^{q-1}}{3^q} < r(x) = 3x - k_0 < \frac{m-k_03^{q-1}+1}{3^q}\). One can continue with same lower-upper inequalities to get \(r(x) \in [k_1/3, (k_1+1)/3]\), and so on.

From (2.1) we obtain

\[
D_{3^q}(x, t) = \prod_{j=0}^{q-1} \left[ 1 + 3|a_{1,k_j}|^2 + 3|a_{2,k_j}|^2 \right] = \prod_{j=0}^{q-1} 3 \left[ \left( \frac{1}{\sqrt{3}} \right)^2 + |a_{1,k_j}|^2 + |a_{2,k_j}|^2 \right] = \prod_{j=0}^{q-1} 3 = 3^q
\]

In the second product above the norm of the \(k_j\)th column of matrix \(A\) appears. Because \(A\) is unitary this norm equals to 1.

Assume now \(t \notin [\alpha_{q,x}, \beta_{q,x}]\). It follows that there exists a \(j \in \{0, 1, \ldots, q - 1\}\) such that \(r^j(t) \notin [k_j/3, (k_j+1)/3]\). Then a factor in \(D_{3^q}(x, t)\) from (2.1) must be of the form \(1 + 3a_{1,k_j} \vec{a}_{1,k_j'} + 3a_{2,k_j} \vec{a}_{2,k_j'}\) with \(k_j \neq k_j'\). Rewriting it as \(3((1/\sqrt{3})^2 + a_{1,k_j} \vec{a}_{1,k_j'} + a_{2,k_j} \vec{a}_{2,k_j'})\) we recognize the inner product between the \(k_j\)th and \(k_j'\)th columns of \(A\). This last inner product of course vanishes as \(k_j \neq k_j'\) and \(A\) is unitary. In conclusion (2.2) holds. Next we estimate \(|f(x) - S_{3^q}(x)|\). For \(x \in [0, 1]\) we have

\[
|f(x) - S_{3^q}(x)| = |f(x) - \int_0^1 f(t) D_{3^q}(x, t) dt| = |f(x) - 3^q \int_{\alpha_{q,x}}^{\beta_{q,x}} f(t) dt| \tag{2.3}
\]

The last equality follows from (2.2). Now when \(q \to \infty\) the last term above converges to 0 when \(x\) is Lebesgue point for \(f\), because \(\beta_{q,x} - \alpha_{q,x} = 1/3^q\). Thus for \(f \in L^1[0, 1]\) the convergence holds pointwise a.e.

The corollary below was shown by Walsh for \(N = 2\). Because the Rademacher functions values are “jump-averaged” at points of discontinuity the \(2^q\)-type of Walsh sums converge at dyadic points. In our case this would mean that if \(x = a\) is a \(N\)-adic rational then \(S_{3^q}(a) = \sum_{n=0}^{N^q-1} \langle f, W_n,A \rangle W_n(a)\) converges to \(\frac{f(a)+f(a-)}{2}\). We did not average out the discontinuities of the Rademacher-like functions \(m_1(r^j x)\) above, and we will not emphasize here the convergence of \(S_{3^q}\) at \(N\)-adic rationals. Actually, what happens at a finite-jump discontinuity can be analyzed with the aid of the corollary below applied to \(f_1\) and \(f_2\) where

\[
f_1(x) = \begin{cases} 
f(x), & \text{if } x < a \\
f(a-), & \text{if } x \geq a
\end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 
f(x), & \text{if } x > a \\
f(a+), & \text{if } x \leq a
\end{cases}
\]

Whether or not full generalized Walsh series (i.e. \(\lim_{k \to \infty} \sum_{n=0}^{k} \langle f, W_n,A \rangle W_n,A\)) converge to \(f\) at a continuity point when \(f\) is of bounded variation seems to be quite a different
Corollary 2.2 If \( f \in L^1[0, 1] \) is continuous in a neighborhood of \( x = a \) then the convergence in Theorem 2.1 is uniform inside an interval centered at \( a \).

Proof Let \([c, d]\) be an interval around \( x = a \) on which \( f \) is uniformly continuous such that \([c, d] = [k/N^q, (k + 3)/N^q] \), and \( a \in [(k + 1)/N^q, (k + 2)/N^q] \). We show that \( S_{Nq} \) converges uniformly to \( f \) on the \( N \)-adic sub-interval \([ (k + 1)/N^q, (k + 2)/N^q] \). For \( \varepsilon > 0 \) there exists \( \delta_\varepsilon > 0 \) such that \( |f(t) - f(t')| < \varepsilon \) whenever \( t, t' \in [c, d] \) and \( |t - t'| < \delta_\varepsilon \). For \( q_\varepsilon \in \mathbb{N} \) large enough we have \( 1/N^q < \min\{\delta_\varepsilon, 1/N^{q_\varepsilon}\} \) for all \( q \geq q_\varepsilon \). Then, with the notations in the proof of Theorem 2.1, the interval \([a_q, \beta_q, i] \) is contained in the interval \([c, d]\) for any \( t \in [(k + 1)/N^q, (k + 2)/N^q] \), and \( q \geq q_\varepsilon \). Using (2.3) we have

\[
|S_{Nq}(t) - f(t)| = N^q \left| \int_{a_q, t}^{\beta_q, t} [f(\tau) - f(t)] d\tau \right| \leq \varepsilon \tag*{□}
\]

Next corollary is easy to prove with the aid of (2.2) and (2.3), and can be used for data encoding/encrypting.

Corollary 2.3 Assume \( f : [0, 1] \to \mathbb{C} \) is constant on the interval \( I := [i/N^q, (i + 1)/N^q] \) for some \( i \in \{0, 1, \ldots, N^q - 1\} \), and \( A \) is a unitary \( N \times N \) matrix with constant \( 1/\sqrt{N} \) first row. Then for all \( x \in I \):

\[
f(x) = \sum_{n=0}^{N^q-1} (f, W_{n,A})W_{n,A}(x) \tag{2.4}
\]

Remark 2.4 Given positive integers \( N \geq 2 \) and \( q \), and a matrix \( A \in \mathcal{M}_{N \times N}(\mathbb{C}) \) as above, one can define the discrete generalized Walsh transform \( DTWA : \mathbb{C}^{N^q} \to \mathbb{C}^{N^q} \) as follows

\[
DTWA(v) := \left[ \sum_{j=0}^{N^q-1} v_j W_{i,A} \left( \frac{j + 1}{2N^q} \right) \right]_{i=0}^{N^q-1} \tag{2.5}
\]

where \( v = [v_j]_{j=0}^{N^q-1} \). Relation (2.5) represents the sequence \([ (f, W_{i,A}) ]_{i=0, \ldots, N^q-1} \) where \( f \) is the function constant \( v_j \) on each interval \([i/N^q, (i + 1)/N^q] \). The integration in each inner product translates into a finite sum because for \( 0 \leq i \leq N^q - 1 \) the Walsh function \( W_{i,A} \) is constant on intervals \([i/N^q, (i + 1)/N^q]) \), \( \forall j = 1, \ldots, N^q - 1 \). Hence \( (f, W_{i,A}) = \sum_{j=0}^{N^q-1} v_j W_{i,A}(t_j) \) where for \( t_j \) we picked the midpoint of the interval \([i/N^q, (i + 1)/N^q]) \). Now with \( x = k/N^q \), \( 0 \leq k < N^q \) substituted in (2.4) we obtain \( v_k = \sum_{i=0}^{N^q-1} [DTWA(v)]_i W_{i,A}(x) \forall 0 \leq k < N^q \), i.e. \( DTWA \) is invertible.

Example 2.5 Consider the step function

\[
f(x) = \begin{cases} 0, & x \in [0, 1/16) \cup [1/8, 3/16) \cup [1/4, 1/2) \\ 1, & x \in [1/16, 1/8) \cup [3/16, 1/4) \cup [1/2, 1] \\
\end{cases}
\]
We have implemented the generalized Walsh system associated to the unitary matrix
\[
A = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} \\
-\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6}
\end{pmatrix}
\]
By Theorem 2.1 the partial sums \( \sum_{n=0}^{k} \langle f, W_{n,A} \rangle W_{n,A}(x) \) converge to \( f(x) \) for \( k = N^q - 1 \). This is clearly the behaviour pictured in Figs. 1 and 2 for \( k = 27 \) and \( k = 81 \) (notice that Corollary 2.3 is not applicable here and therefore the graphs for the partial sums having \( 3^q \) terms do not coincide with \( f \)'s, which is piecewise constant on a subdivision of \([0, 1]\) coarser than one with triadic points). However even for a high number of terms (\( k = 300 \)) it seems that the partial sums do not settle at \( f(x) \). Notice also that in our example \( f \) has bounded variation.

Next we show that similarly to the classic Walsh functions the generalized Walsh expansions also can be interpreted as conditional expectations and Doob martingales. This enables us to conclude the convergence of the Walsh series in \( L^p[0, 1] \) with \( 1 < p < \infty \).

Let \( \mathcal{F}_q \) be the \( \sigma \)-algebra generated by the intervals \( [j/N^q, (j + 1)/N^q) \), \( j = 0, \ldots, N^q - 1 \). Obviously \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_q \subset \mathcal{F}_{q+1} \subset \cdots \subset \mathcal{B} \) where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra on \([0, 1]\). Given a unitary matrix \( A \) as before and a function \( f \) we will denote by \( S_k(f) \) the generalized Walsh series \( \sum_{n=0}^{k} \langle f, W_{n,A} \rangle W_{n,A} \).

**Theorem 2.6** For \( f \in L^1[0, 1] \) the sequence \( (S_{N^q}(f))_{q=1}^{\infty} \) is a martingale i.e.
\[
\mathbb{E}[f|\mathcal{F}_q] = S_{N^q}(f), \quad \forall q \in \mathbb{N}
\]
\[
\mathbb{E}[S_{N^q+1}(f)|\mathcal{F}_q] = S_{N^q}(f), \quad \forall q \in \mathbb{N}
\]
Proof Let \( x \in [0, 1) \) and \( m \) the unique number in \( \{0, \ldots, N^q - 1\} \) such that \( x \in [m/N^q, (m + 1)/N^q) \). As in the proof of Theorem 2.1 we have

\[
S_{N^q}(f)(x) = N^q \int_{m/N^q}^{(m+1)/N^q} f(t) \, dt
\]

Thus \( S_{N^q}(f) \) is a piece-wise constant function, constant on each interval \([j/N^q, (j + 1)/N^q)\), equal to the average of \( f \) on that interval. Then one can see that, for \( j = 0, \ldots, N^q - 1 \):

\[
\int_0^1 f(t) \cdot \chi_{[j/N^q, j+1/N^q)}(t) = \int_0^1 S_{N^q}(f)(t) \cdot \chi_{[j/N^q, j+1/N^q)}(t) \, dt
\]

This proves (2.6). We get that \( (S_{N^q}(f))_{q=1}^{\infty} \) is a martingale.

Lemma 2.7 The operator \( f \to S_{N^q}(f) \) is bounded from \( L^p[0, 1] \) to \( L^p[0, 1] \) for all \( 1 \leq p \leq \infty \).

Proof We have

\[
\|S_{N^q}(f)\|_1 \leq \|f\|_1
\]

(2.8)

\[
\|S_{N^q}(f)\|_{\infty} \leq \|f\|_{\infty}
\]

(2.9)

Indeed, because \( S_{N^q}(f) \) is piece-wise constant (the average of \( f \) on \( N \)-adic intervals we have

\[
\int_0^1 |S_{N^q}(f)(t)| \, dt = \sum_{j=0}^{N^q-1} \left| \int_{j/N^q}^{(j+1)/N^q} f(t) \, dt \right| \leq \int_0^1 |f(t)| \, dt
\]

This proves (2.8). Also (2.9) follows from:

\[
|S_{N^q}(f)(t)| \leq N^q \int_{j/N^q}^{(j+1)/N^q} |f(t)| \, dt \leq \|f\|_{\infty}
\]

The two inequalities imply that the operator \( f \to S_{N^q}(f) \) is bounded between \( L^\infty[0, 1] \to L^\infty[0, 1] \) and \( L^1[0, 1] \to L^1[0, 1] \). Then, by the Riesz-Thorin interpolation theorem, the operator \( S_{N^q} \) is bounded from \( L^p[0, 1] \) to \( L^p[0, 1] \) for all \( 1 \leq p \leq \infty \).

Corollary 2.8 Let \( 1 \leq p \leq \infty \), and \( f \in L^p[0, 1] \). Then \( S_{N^q}(f) \to f \) a.e. in \([0, 1]\). For \( 1 < p < \infty \) we have \( S_{N^q}(f) \to f \) in \( L^p[0, 1] \).

Proof Using Theorem 2.6 and Lemma 2.7 we have

\[
\|E[f|F_q]\|_p = \|S_{N^q}(f)\|_p \leq \|f\|_p \quad \text{for all } q \in \mathbb{N} \text{ and } p = 1, \ldots, \infty
\]

By Doob’s martingale Convergence theorem we have \( S_{N^q}(f) \to f \) a.e. in \([0, 1]\) and \( S_{N^q}(f) \to f \) in \( L^p[0, 1] \) for \( 1 < p < 1 \).
3 An Encryption Protocol à la Diffie-Hellman

Remark 3.1 Corollary 2.3 suggests the following encoding or encryption scheme: Given data recorded by a function $f$ which is piecewise constant on intervals of length $1/N^q$, compute the generalized Walsh coefficients $(\langle f, W_{n,A} \rangle)_{n=0}^{N^q-1}$, for a choice of unitary matrix $A$. One can generate unitary matrices $A \in M_{3 \times 3}(\mathbb{R})$ with constant first row $1/\sqrt{3}$ by randomly choosing an entry $a \in [-\sqrt{2}/3, \sqrt{2}/3]$ in the second or third row and then solving for the remaining ones (we show how to implement such an algorithm using Maple software later in this section). The restriction $|a| \leq \sqrt{2}/3$ comes from asking certain quadratic equations have solutions, and it can be easily observed by requiring the matrix be unitary.

One should hedge against brute force attacks to “guessing” the value $a$, which would act as secret key. For example if the range of $f$ is known (e.g. the alphabet letters are indexed from 1 to 26 and $f$ represents a message of length $3^q$) then one can estimate within a certain margin $|\hat{a} - a| < \epsilon$ an approximate message

$$\hat{f}(x) = \sum \langle f, W_{n,A} \rangle W_{n,A}(x)$$

assuming that the finite sequence $(\langle f, W_{n,A} \rangle)_n$ representing $f$ has been intercepted, e.g. through an unsecure network. Hence it may be safer to consider the process $f \rightarrow (\langle f, W_{n,A} \rangle)_{n=0}^{N^q-1}$ just an encoding step, which due to its complexity is suitable to further encryption (e.g. using classical bit operations). For example one could add a perturbation $h(a, x)$ to the sequence encoding $f$, depending on the entry $a$ and other variables which may be part of the secret key.

Of course security can be increased by allowing complex entries in $A$ (even though the data to be represented is made of real numbers). We note here that a scheme as above pertains to the area of symmetric key cryptography, i.e. both sender and receiver have knowledge of the matrix $A$ which generates the Walsh system. Generating such unitary $A$ is easily done with the aid of mathematical software, and the scheme described above can also be thought of as one time pad encryption.

In what follows we will study the theoretical feasibility of a protocol that shares similarities with both Diffie-Hellman key-exchange protocol and public key cryptography (RSA), based on generalized Walsh systems. More precisely we ask whether communication between Alice and Bob without sharing of the matrices $A$ and $B$ is possible. Our results indicate that some information about $A$ or $B$ has to be shared prior to message transmission (this theoretical “weakness” will be discussed later in the section). Hence this protocol is not “pure” Diffie-Hellman; the information to be shared (a system of quadratic polynomial equations) may be considered as public key, which the sender possesses (as opposed to common public key cryptography protocols where the receiver makes his public key known to anyone). For theoretical details regarding RSA, Diffie-Hellmann key exchange protocols, and other public key cryptosystems and algorithms we refer the reader to [10] and [5].

Remark 3.2 We describe first a more general set up: let $H_1$ be a non empty set (the space of messages) and $H_2$ another set (the space of encrypted messages). Then one can set up communication through an unsecure channel (e.g. Alice wants to send Bob a secret message $v \in H_1$ and Eve is capable to intercept all communications) without prior contact provided Alice and Bob are each in possession of operators $A : H_1 \rightarrow H_2$ and $B : H_1 \rightarrow H_2$ such that $B^{-1} \circ A \circ B^{-1} \circ A = I_{H_1}$, where $I_{H_1}$ is the identity operator (one might consider a slightly different approach e.g. require that both $A$ and $B$ are defined on $H_1$ and adjust the above...
identity, and/or that their inverses exists and are defined on a smaller subspace). One should take care of a few requirements: for example the computation of $B^{-1}$ should be reliable (and easy); also there should be plenty of operators $A$ and $B$ Alice and Bob could choose from without revealing their choice to each other or anyone else. Ideally there should be infinitely many $A$’s each of which admits infinitely many (or a large number of) $B$’s with $B^{-1}AB^{-1}A = I_{H_1}$. Such a family of operators is public and any pair (Alice, Bob) using the protocol will freely choose a pair $(A, B)$ with which they can start communicating. The situation where both choose the same operator pertains to the realm of symmetric key cryptography and consists only of the first two steps below (i.e. not all four); moreover such an occurrence would be improbable if there are infinitely many pairs $A$ and $B$ as above (see also remark 3.6 below). If all these are satisfied then one can start the exchange as follows:

1. Alice to Bob: $w_1 = A(v) \in H_2$
2. Bob to Alice: $w_2 = B^{-1}A(v) \in H_1$
3. Alice to Bob: $w_3 = AB^{-1}A(v) \in H_2$
4. Bob applies $B^{-1}$ to $w_3$.

The scheme is safe provided Eve cannot decipher $v$ even when she is in possession of $w_i$, $i = 1, 2, 3$. In other words Eve should not be able to figure out neither $A$ nor $B$ (or their inverses). Of course it is assumed that Eve is only eavesdropping without other interaction in the process (such as impersonating Alice and/or Bob). The question is where to look for such operators? One could encode the data to be transmitted in a finite dimensional vector $v$, hence naturally the operators $A$ and $B$ may be thought of as matrices. In this case we would have to find a infinite or very large number of matrices $C$ such that $C^2 = I$, and decompositions of such matrices $C = AB$. Computationally we would have to deal with finding inverses of some of the matrices involved, which is unreliable when dealing with matrices of relatively large dimension. There are other ideas to consider, e.g. when the operators arise from the faithful representation of a group $G$ on a Hilbert space (such as $L^2(\mathbb{R}^n)$, which encodes the signals or the messages). Both encryption and security are then a consequence of the intrinsic properties of the group (generators and relations) and its representation (for example acting by ‘shifting’ should be considered unsecure).

**Example 3.3** Let $A$ and $B$ two unitary $N \times N$ matrices having constant $1/\sqrt{N}$ first row. We allow greater generality by working with complex numbers entries in both $A$ and $B$. We will denote by $W_A$ the generalized Walsh transform associated to matrix $A$ i.e. $W_A : L^2[0, 1] \to l^2(\mathbb{N})$, $W_A(f) = \langle f, W_{n,A} \rangle_{n \geq 0}$, and similarly for the Walsh transform of $B$. Where well-defined (e.g. for finite sequences) the inverse transform operates as follows: if $(a_n)_{n \geq 0} \in l^2(\mathbb{N})$ then $W_A^{-1}((a_n)) = \sum_n a_n W_{n,A}$. Also note that we have kept the same value $N$ in both systems, thus the same map $r(x) = (Nx) \mod 1$ enters in the definition of both Walsh ONBs. Then for a given $f \in L^2([0, 1])$ the requirement

$$W_B^{-1} \circ W_A \circ W_B^{-1} \circ W_A(f) = f$$

has the following interpretation:

- Alice encrypts “message” $f$ using her matrix $A$ as the sequence $W_A(f) = \langle f, W_{n,A} \rangle_{n \geq 0}$ which she sends to Bob.
- Using his own matrix $B$, Bob constructs a new message $W_B^{-1} \circ W_A(f) = \sum_n \langle f, W_{n,A} \rangle \times W_{n,B}(x)$ which he sends back to Alice. Notice that Bob sends a “whole” function whereas Alice sends a sequence, however in practice both $f$ and $W_B^{-1} \circ W_A(f)$ are piece-wise constant thus easy to record as finite length sequences.
Alice sends the coefficients $W_A \circ W_B^{-1} \circ W_A(f)$ back to Bob.

Bob finally applies the $W_B^{-1}$ transform to the previous sequence and recovers the original $f$.

Of course one should not expect that the intertwining relation (3.1) just holds for any pair of matrices $A$ and $B$. We are interested in finding plenty of cases when it does. We would actually like to have infinitely many such pairs $(A, B)$ and to make it impossible to detect $A$ given $B$ or vice versa. We will work under the assumption that for a fixed positive integer $q$, the “message” function $f$ is real valued, piecewise constant on $N$-adic intervals as in Corollary 2.3. We have:

$$W_B^{-1} \circ W_A \circ W_B^{-1} \circ W_A(f) = \sum_{k=0}^{Nq-1} \left( \sum_{l=0}^{Nq-1} \langle f, W_l, A \rangle \langle W_l, B, W_k, A \rangle \right) W_k, B$$

Next we assume the “commutation” relation $\langle W_l, B, W_k, A \rangle = \langle W_l, A, W_k, B \rangle$. Hence we can continue the last equality with

$$W_B^{-1} \circ W_A \circ W_B^{-1} \circ W_A(f) = \sum_{l=0}^{Nq-1} \langle f, W_l, A \rangle \sum_{k=0}^{Nq-1} \langle W_l, A, W_k, B \rangle W_k, B$$

Notice that all generalized Walsh functions $x \rightarrow W_{l,A}(x)$ are piecewise constant on $N$-adic intervals so that Corollary 2.3 can be applied:

$$\sum_{k=0}^{Nq-1} \langle W_l, A, W_k, B \rangle W_k, B(x) = W_{l,A}(x)$$

We therefore obtain

$$W_B^{-1} \circ W_A \circ W_B^{-1} \circ W_A(f) = \sum_{l=0}^{Nq-1} \langle f, W_l, A \rangle W_l, A = f$$

The last equality follows from Corollary 2.3 applied to $f$. We record these computations in the following

**Proposition 3.4** Let $q \in \mathbb{N}$, $N > 1$ an integer. Then relation (3.1) holds for any $f$ piecewise constant on each interval $[i/N^q, (i+1)/N^q]$, $i \in \{0, 1, \ldots, N^q - 1\}$, provided

$$\langle W_l, B, W_k, A \rangle = \langle W_l, A, W_k, B \rangle, \quad \forall k, l = 0, \ldots, N^q - 1$$

where $A$, and $B$ are unitary in $\mathcal{M}_{N \times N}(\mathbb{C})$, having constant $1/\sqrt{N}$ first row.

We would like to find a condition that is easier to implement in an algorithm than the above (3.2). Actually at this point it is not obvious that there should exist unitary matrices $A$ and $B$ satisfying (3.2). Note that the inner product in (3.2) is taken in the Hilbert space $L^2[0, 1]$ whereas the one below in (3.3) is the usual $\mathbb{C}^N$ inner product.
Theorem 3.5 Let \( A = [a_{ij}]_{i,j=0,N-1} \) and \( B = [b_{ij}]_{i,j=0,N-1} \) be unitary matrices in \( M_{N \times N}(\mathbb{R}) \) with constant \( 1/\sqrt{N} \) first row. Using notation row\(_{i,A} \) for the \( i^{th} \) row in matrix \( A \), condition (3.2) is equivalent to

\[
\langle \text{row}_{i,B}, \text{row}_{j,A} \rangle = \langle \text{row}_{i,A}, \text{row}_{j,B} \rangle \quad \text{for all } k, l \text{ in } \{1, 2, 3, \ldots, N\} \tag{3.3}
\]

Proof The implication (3.2) \( \Rightarrow \) (3.3) follows faster. First we will denote by \( m_i^A \) the functions which define the Walsh system associated with unitary matrix \( A \), and similarly for \( B \). Notice that the first rows of both \( A \) and \( B \) are written \( (a_{0,i})_{i=0}^{N-1} \) and \( (b_{0,j})_{j=0}^{N-1} \) (as in Introduction).

Now when \( 0 \leq k, l \leq N - 1 \) their base \( N \) expansion are simply \( k = k_1 \cdot N^0 \) and \( l = l_1 \cdot N^0 \), hence \( W_{i,C}(x) = m_i^C(x) \) for \( i = k \), or \( i = l \), and matrix \( C = A \), or \( C = B \). We have

\[
\langle W_{k,A}, W_{l,B} \rangle = \int_0^1 m_k^A(x)m_l^B(x) dx = N \int_0^1 \sum_{j=0}^{N-1} a_{kj} b_{lj} \chi_{I_j \cap I_l}(x)
\]

where \( I_j \cap I_l = [j/N, (j + 1)/N] \cap [i/N, (i + 1)/N] \). In conclusion

\[
\langle W_{k,A}, W_{l,B} \rangle = N \int_0^1 \sum_{j=0}^{N-1} a_{kj} b_{lj} \chi_{I_j}(x) = \langle \text{row}_{l+1,A}, \text{row}_{l+1,B} \rangle
\]

We thus obtain (3.2) \( \Rightarrow \) (3.3). We will show the converse with \( N = 3 \), however the reader can easily replace the calculations for general \( N \) as the pattern does not change much. When \( k = 0 \) or \( l = 0 \) or \( k = l \) the relation (3.2) is true because of either orthogonality and \( W_{0,A} = 1 = W_{0,B} \), or symmetry of the inner product over \( \mathbb{R}^N \). Also, when \( k = 1 \) or \( l = 1 \) (3.3) is true because of unitary requirements on \( A \) and \( B \). Hence the main assumption becomes

\[
\langle \text{row}_{2,A}, \text{row}_{3,B} \rangle = \langle \text{row}_{3,A}, \text{row}_{2,B} \rangle \tag{3.4}
\]

We want to deduce

\[
\int_0^1 m_{i_0}^A(x)m_{i_1}^A(rx) \cdots m_{i_p}^A(r^p x) m_{j_0}^B(x)m_{j_1}^B(rx) \cdots m_{j_p}^B(r^p x) dx
\]

\[
= \int_0^1 m_{j_0}^B(x)m_{j_1}^B(rx) \cdots m_{j_p}^B(r^p x) m_{i_0}^A(x)m_{i_1}^A(rx) \cdots m_{i_p}^A(r^p x) \tag{3.5}
\]

for all non negative integers \( p \) and \( l \), and for all \( i_0, i_1, \ldots, i_p, j_0, j_1, \ldots, j_l \) in \( \{0, 1, 2\} \). Following up on the discussion above, (3.5) is true when \( (l = 0 = p) \) and \( (i_0 = 0 = j_0 = 0 = i_0 = j_0) \). When \( i_0 = j_0 \) and both non zero, these subscripts represent the 2nd and 3rd row of either \( A \) or \( B \), and (3.5) follows from (3.4). To better digest the proof of (3.5) we will go through one more particular case, e.g. we will show

\[
\int_0^1 m_{i_0}^A(x)m_{i_1}^A(rx)m_{j_0}^B(x) dx = \int_0^1 m_{j_0}^A(x)m_{i_0}^B(x)m_{i_1}^B(rx) \tag{3.6}
\]

Starting with the left-hand side we have

\[
\int_0^1 m_{i_0}^A(x)m_{i_1}^A(rx)m_{j_0}^B(x) dx = \sum_{k,l,t=0}^2 a_{i_0,k} \chi_{I_{\frac{k}{2}, \frac{k+1}{2}}}(x) \cdot a_{i_1,l} \chi_{I_{\frac{l}{2}, \frac{l+1}{2}}}(rx) \cdot b_{j_0,t} \chi_{I_{\frac{t}{2}, \frac{t+1}{2}}}(x)
\]
Replacing the

Without loss of generality assume

Using notation

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In the calculations above we have used \( \lambda((\frac{k}{3}, \frac{k+1}{3}] \cap \frac{l}{3}, \frac{l+1}{3}) = 0 \) whenever \( k \neq l \), and \( \lambda([\frac{k}{3}, \frac{k+1}{3}] \cap r^{-1}([\frac{l}{3}, \frac{l+1}{3}])) = \frac{1}{3} \), for all \( k, l \in \{0, 1, 2\} \) (this follows by inspecting the action of \( r \) on \([0, 1] \), for example \( rx \in [0, 1/3] \) iff \( x \in [0, 1/9] \cup [1/3, 4/9] \cup [2/3, 7/9] \) etc). By applying similar arguments to the right-hand side of (3.6) we obtain

\[
\int_0^1 m_{l_0}^A(x)m_{l_0}^B(x)m_{l_1}^B(x) = \frac{1}{3^2} \sum_{k=0}^{2} a_{l_0,k} b_{l_0,k} \cdot \sum_{l=0}^{2} b_{l_1,l}
\]

Because \( A, B \) are unitary with constant first row we have \( \sum_{l=0}^{2} a_{l_1,l} = \sum_{l=0}^{2} b_{l_1,l} (= \) either 0 or \( 3/\sqrt{3} \), replaced by \( N/\sqrt{N} \) in the general setting), and thus (3.6) follows due to (3.3). Now to prove (3.5) for any \( l \) and \( p \) we highlight the following property of \( r \) which we mentioned in the particular case above. The reader can check it easily based on the observation that each set \( r^{-1}([\frac{k}{3}, \frac{k+1}{3}] \cap \frac{l}{3}, \frac{l+1}{3}) \) contains precisely one component out of three of measure \( 1/9 \) inside any interval \([\frac{k}{3}, \frac{k+1}{3}] \), where \( k, l \in \{0, 1, 2\} \). Hence, if \( l \leq p \) are nonnegative integers, and \( t_0, t_1, \ldots, t_l, q_0, q_1, \ldots, q_p \) are \([0, 1, 2] \)-digits, the Lebesgue measure of the set

\[
S_{t_0,t_1,\ldots,t_l,q_0,q_1,\ldots,q_p} := \left[ \frac{t_0}{3}, \frac{t_0+1}{3}\right] \cap r^{-1}\left[ \frac{t_1}{3}, \frac{t_1+1}{3}\right] \cap \cdots \cap r^{-l+1}\left[ \frac{t_l}{3}, \frac{t_l+1}{3}\right]
\]

\[
\cap \left[ \frac{q_0}{3}, \frac{q_0+1}{3}\right] \cap r^{-l}\left[ \frac{q_1}{3}, \frac{q_1+1}{3}\right] \cap \cdots \cap r^{-p+1}\left[ \frac{q_p}{3}, \frac{q_p+1}{3}\right]
\]

is obtained

\[
\lambda(S_{t_0,t_1,\ldots,t_l,q_0,q_1,\ldots,q_p}) = \begin{cases} 
\frac{1}{3^{p+1}} & \text{if } t_0 = q_0 \text{ and } t_1 = q_1 \text{ and } \ldots \text{ and } t_l = q_l \\
0 & \text{otherwise}
\end{cases}
\]

(3.7)

Without loss of generality assume \( l \leq p \) and start with the left-hand side (LHS) of (3.5). Replacing the \( m \)'s and integrating the characteristic functions we obtain

\[
\text{LHS} = \sum_{l_0,t_1,\ldots,t_l=0}^{2} a_{l_0,t_0} a_{l_1,t_1} \cdots a_{l_l,t_l} \cdot b_{j_0,q_0} b_{j_1,q_1} \cdots b_{j_p,q_p} \cdot \lambda(S_{l_0,t_1,\ldots,t_l,q_0,q_1,\ldots,q_p})
\]
Using (3.7) we continue with

\[
LHS = \frac{1}{3^{p+1}} \sum_{t_0, t_1, \ldots, t_l=0}^2 a_{l_0, t_0} b_{j_{l+1}, q_{l+1}} \cdots a_{l_t, t_l} b_{j_{l+1}, q_{l+1}} \cdots b_{j_{l_t}, q_{l_t}}
\]

\[
= \frac{1}{3^{p+1}} \prod_{x=0}^l \langle \text{row}_{i_x, A}, \text{row}_{j_x, B} \rangle \cdot \prod_{y=14}^p (b_{j_y, 0} + b_{j_y, 1} + b_{j_y, 2})
\]

Now due to (3.3) we may switch \( A \) with \( B \) in the first product above. As for the second product, notice the sum of the \( j_y \)th row of matrix \( B \): each such sum is equal to the sum of the \( j_y \)th row of matrix \( A \) (both being equal to either 0 or \( 3/\sqrt{3} \)), according to perpendicularity requirements. Therefore

\[
LHS = \frac{1}{3^{p+1}} \prod_{x=0}^l \langle \text{row}_{i_x, B}, \text{row}_{j_x, A} \rangle \cdot \prod_{y=14}^p (a_{j_y, 0} + a_{j_y, 1} + a_{j_y, 2})
\]

\[
= \frac{1}{3^{p+1}} \sum_{t_0, t_1, \ldots, t_l=0}^2 b_{l_0, t_0} a_{j_{l+1}, q_{l+1}} \cdots b_{l_t, t_l} a_{j_{l+1}, q_{l+1}} \cdots a_{j_{l_t}, q_{l_t}}
\]

The last term we have obtained is equal to the right-hand side RHS of (3.5), as we can express it using (3.7) precisely in the same way we started with LHS. In conclusion (3.5) follows from (3.3) and we are done. \( \square \)

**Remark 3.6** We describe next the theoretical framework underlying a possible cryptographic protocol based on the ideas above.

(i) To obtain a generalized Walsh ONB, Alice sets up the following equations:

\[
\begin{align*}
(1) & \quad a_{1,j} = 1/\sqrt{N}, \quad \forall j = 1, \ldots, N \\
(2) & \quad \sum_{j=1}^N |a_{i,j}|^2 = 1, \quad \forall i = 2, \ldots, N \\
(3) & \quad \sum_{j=1}^N a_{i,j} = 0, \quad \forall i = 2, \ldots, N \\
(4) & \quad \sum_{i=1}^N a_{i,k} \overline{a_{j,k}} = 0, \quad \forall 1 < i < j \leq N
\end{align*}
\]

Discarding the first item we are left with \( \frac{N^2-N}{2} + N - 1 \) equations with \( N(N-1) \) unknowns \( a_{i,j}, i = 2, \ldots, N, j = 1, \ldots, N \). If \( a_{i,j} \in \mathbb{R} \) then one obtains a system of \( \frac{N^2-N}{2} + N - 1 \) polynomial (quadratic) equations with infinitely many solutions as long as \( N \geq 3 \). All Alice has to do is pick a few prescribed entries (with some care so as to maintain norm 1 on the row the entry comes from) and solve for the remaining entries (see example below). Of course one can allow for complex unknowns \( a_{ij} \) with non zero imaginary parts. In this case the system above can again be thought of as a system of polynomial equations with real value unknowns by splitting each equation.
into real and imaginary parts. Actually in this case the number of unknowns doubles (each $a_{i,j}$ contributes two more unknowns, real and imaginary) while the equations in item (ii) above do not. Of course it was obvious that there are infinitely many unitary matrices but what we spelled out here was the precise requirements we need in order to implement in a computer.

(ii) Next comes the “sharing” part: obviously Alice must not reveal $A$ but she will have to “help” Bob choose the right matrix $B$, i.e. such that (3.3) holds. We will assume all entries are real numbers although one can adjust to complex ones as well. Hence in relationship (3.3) the symmetries can be discarded and only $\frac{N^2-N}{2}$ equations will be relevant: it means that Alice sends Bob the following system of equations in unknowns $b_{i,j}$, $i = 2, \ldots, N$, $j = 1, \ldots, N$:

$$
(5) \quad \sum_{j=1}^{N} a_{kj}b_{ij} = \sum_{j=1}^{N} a_{ij}b_{kj}, \quad \forall \ 1 < l < k \leq N
$$

In the above equations Alice’s secret key, $a_{ij}$ seems to be revealed: of course this would be very damaging but Alice can simply multiply each equation by a random number thus masking her matrix.

(iii) Bob considers a system of equations similar to Alice’s above to which he adds item (5). When working with real values Bob deals with a system of $N^2 - N$ polynomial equations and $N^2 - N$ unknowns ($b_{ij}$, $i = 2, \ldots, N$, $j = 1, \ldots, N$). This system of equations clearly has solutions (i.e. Alice’s own $a_{ij}$) but it is important to get a large number of (possibly infinitely many) solutions. This will hedge against an eavesdropper detecting matrix $B$. In an example below we display such a situation using Maple (infinitely many matrices $B$ corresponding to a given $A$); however we would like to obtain infinitely many $A$ for which there are infinitely many $B$ satisfying all items (1) through (5). It is not our scope here to go into a thorough study of the system of equations above, nevertheless we feel it is a very interesting question to settle the existence of infinitely many examples of matrices $A$ and $B$ as above. E.g. Maple is capable to calculate Gröbner bases (theoretical tool that among other things tells whether a zero-dimensional system of polynomial equations has finitely many solutions) and find approximate solutions for the systems above. Finding Gröbner bases is based on Buchberger algorithm and the process is time consuming even for $N = 4$ (see [3], also the Help section in Maple which contains practical details on these bases and more efficient algorithms).

(iv) Suppose $f = (a_1, a_2, \ldots, a_{N^2})$ represents the secret message to be transmitted by Alice to Bob. From the previous steps we obtained $A$ and $B$ that satisfy (3.3). By Theorem 3.5 Eq. (3.1) holds. Now the communication continues as in Example 3.3.

**Example 3.7** In this example the matrix $A$ allows for infinitely many matrices $B$ for which the above relation (3.3) holds. We have experimented with other $3 \times 3$ and $4 \times 4$ matrices $A$ for which Maple did not find infinitely many $B$ satisfying equation (5) in Remark 3.6. At this point we do not know if such occurrences are rare, and we do not have yet a theoretical tool to characterize all such matrices.

- Alice has matrix

$$
A = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{\sqrt{7}}{2} & 0 & -\frac{\sqrt{7}}{2} \\
-\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6}
\end{pmatrix}
$$
Equation (3.3) in this case becomes $p^{2^{-1/2}} - r^{2^{-1/2}} + x 6^{-1/2} - y (1/3) 6^{1/2} + z 6^{-1/2} = 0$, and is sent to Bob, after multiplication by a random number (to mask it from possible eavesdroppers).

- Replacing Bob’s unknowns $[b_{i,j}]_{i,j=1,2,3}$ by $x, y, z, p, q, r$ the following system must be solved:
  $$
  \begin{align*}
  x^2 + y^2 + z^2 - 1 &= 0, \\
  p^2 + q^2 + r^2 - 1 &= 0 \\
  x + y + z &= 0, \\
  p + q + r &= 0 \\
  xp + yq + zr &= 0, \\
  p2^{-1/2} - r2^{-1/2} + x6^{-1/2} - y(1/3)6^{1/2} + z6^{-1/2} &= 0
  \end{align*}
  $$

In this case (i.e. for $A$ above) Maple solve command finds infinitely many solutions indexed by (the free) variable $r$ below (indeterminate $Z$ is a place-holder for the unknown in the quadratic equations):

$$
\begin{align*}
  p &= r + (1/2)\sqrt{2}\text{RootOf}(6Z^2 + 6r^2 + 3\sqrt{2}Z\sqrt{6}r - 1)\sqrt{6} \\
  q &= -2r - (1/2)\sqrt{2}\text{RootOf}(6Z^2 + 6r^2 + 3\sqrt{2}Z\sqrt{6}r - 1)\sqrt{6} \\
  r &= r, \\
  x &= \text{RootOf}(6Z^2 + 6r^2 + 3\sqrt{2}Z\sqrt{6}r - 1) + (1/2)\sqrt{2}\sqrt{6}r \\
  y &= \text{RootOf}(6Z^2 + 6r^2 + 3\sqrt{2}Z\sqrt{6}r - 1) \\
  z &= -(1/2)\sqrt{2}\sqrt{6}r - 2\text{RootOf}(6Z^2 + 6r^2 + 3\sqrt{2}Z\sqrt{6}r - 1)
\end{align*}
$$

- Bob picks a value $r$ such that the quadratic equation $6Z^2 + 6r^2 + 3\sqrt{2}Z\sqrt{6}r - 1 = 0$ has real solutions in indeterminate $Z$. Notice that such values can be chosen randomly in a subinterval of $[-1, 1]$. E.g. for $r = 0.2$ Bob sets up a matrix

$$
B = \begin{pmatrix}
3^{-1/2} & 3^{-1/2} & 3^{-1/2} \\
-0.2226063221 & -0.5690164837 & 0.7916228058 \\
-0.7855654600 & 0.5855654600 & 0.2
\end{pmatrix}
$$

Maple finds a numeric approximation when solving the systems of polynomial equations (it considers them with rational coefficients). Thus matrix $B$ is “almost” unitary. E.g. Maple gives the following computation:

$$
B^*B = \begin{pmatrix}
0.99999999998 & 0.00000000001 & 0.0 \\
0.00000000001 & 0.99999999999 & -0.00000000001 \\
0.0 & -0.00000000001 & 1.0
\end{pmatrix}
$$

- The signals/messages to be transmitted must be of length $3^q$. For example let $f = 00011000001111110002222222$ be a signal of length 27 which is encoded as a step function

$$
f(x) = \begin{cases}
0, & \text{if } 0 \leq x < 1/9 \\
1, & \text{if } 1/9 \leq x < 5/27 \\
0, & \text{if } 5/27 \leq x < 10/27 \\
1, & \text{if } 10/27 \leq x < 17/27 \\
0, & \text{if } 7/27 \leq x < 20/27 \\
2, & \text{if } 20/27 \leq x < 1
\end{cases}
$$
(i) The sequence $W_A f$ (without its first Walsh coefficient) which Alice sends to Bob:

$\begin{align*}
-0.5443310539, & \quad -0.05237828008, & \quad -0.1814436847, & \quad 0.2222222222, \\
0.1283000598, & \quad 0.2618914004, & \quad 0., & \quad -0.07407407407, & \quad -0.04536092117, \\
0.1666666667, & \quad 0.03207501497, & \quad -0.2222222222, & \quad 0.1360827635, \\
-0.07856742012, & \quad 0.1283000598, & \quad 0., & \quad -0.09072184234, & \quad 0.2618914004, \\
0.09622504490, & \quad 0.09259259259, & \quad -0.06415002993, & \quad 0.07856742012, \\
0.04536092117, & \quad 0.03703703704, & \quad 0., & \quad -0.1047565601
\end{align*}$

(ii) Maple display of the graph of $W_B^{-1}W_A f$, which is sent to Alice:

(iii) Alice applies her Walsh transform to Bob’s function and sends him the sequence $W_A W_B^{-1} W_A f$:

$\begin{align*}
0.4268793977, & \quad 0.3417802807, & \quad -0.05238646443, & \quad 0.1424437841,
\end{align*}$
0.08209867227, 0.3142683164, 0.2103987320, 0.005704364048,
0.01428020223, 0.1948148148, 0.06725825079, −0.06424603320,
0.001060076263, −0.1578695927, −0.2267207154, −0.133155569,
−0.01534027859, 0.05039404776, 0.003108220861, 0.06444444442
−0.03427062570, 0.01804658637, −0.05818088527, −0.1209391520,
0.02910416584, −0.06844063412

(iv) Maple graphs of $W_B^{-1}W_AW_B^{-1}W_Af$ and $f$ coincide. This is recovered by Bob by applying $W_B^{-1}$ to the sequence in iii):

One should add that Maple displays both graphs as “equal” which from the point of view of reading off values in the range $\{0, 1, 2\}$ is quite sufficient. Computationally the functions are almost equal. For example, in our Maple program we evaluated the value

$$W_B^{-1}W_AW_B^{-1}W_Af(0.4) = 1.185185185 - 0.1069167165\sqrt{3} \sim 0.9999999998 \sim 1 = f(0.4)$$

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