Supplemental Material to “Debt Limits and Credit Bubbles in General Equilibrium”

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Abstract

This supplemental material provides the detailed arguments of some omitted proofs in the example analyzed in Martins-da-Rocha et al. (2019) (henceforth MPV).

Consider a simple deterministic economy with identical agents whose endowments switch between a high and low value and default entails the loss of a fixed amount ℓ > 0. There are two agents I := {i₁, i₂} with a constant relative risk aversion utility function

\[ u(c) = \frac{c^{1-\alpha}}{1-\alpha}, \quad \alpha > 0. \]

At every date t, each agent i’s income \( y^i_t \) alternates between a high value \( y_{h,t} \) and a low value \( y_{l,t} \). Agent \( i_1 \) starts with the high income. Incomes grow at a constant gross rate \( \rho > 1 \):

\[ (y_{h,t}, y_{l,t}) = \rho^t (y_{h}, y_{l}), \quad \text{with} \quad y_{h} > y_{l} > \ell > 0. \]

We focus on symmetric equilibria and denote by \( x_t \) the not-too-tight debt limit at date t, i.e., \( D^i_t = x_t \) for each \( i \). It follows from the general characterization result in Martins-da-Rocha et al. (2019) that there exists \( M_0 \geq 0 \) such that

\[ x_t = \frac{1}{p_t} [\ell(p_t + p_{t+1} + \ldots) + M_0], \]

where \( p_t := q_1 \cdots q_t \). Let \( z_t \) denote the net trade position, i.e.,

\[ z_t := x_t + q_{t+1} x_{t+1}. \]

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1Symmetry means that the debt limit is the same for both agents.

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Since there is growth, we let \( \hat{z}_t := \rho^{-t}z_t \) represent the detrended trade position at period \( t \).

It follows from the arguments in Martins-da-Rocha et al. (2019) that to get the existence of a competitive equilibrium it is sufficient to find a sequence \( (q_{t+1})_{t \geq 0} \) of positive bond prices and a sequence \( (\hat{z}_t)_{t \geq 0} \) of detrended net trades such that

\[
q_{t+1} = \beta \left( \frac{y_h - \hat{z}_t}{\rho(y_h + \hat{z}_{t+1})} \right)^\alpha,
\]

\[
\hat{z}_t = \frac{\ell}{\rho} + \rho q_{t+1} \left( \hat{z}_{t+1} + \frac{\ell}{\rho^{t+1}} \right)
\]

and

\[
\hat{z}_t \in [0, (y_h - y_l)/2].
\]

A sequence of \( (q_t)_{t \geq 1} \) of positive bond prices (or the corresponding sequence \( (p_t)_{t \geq 0} \) of Arrow–Debreu prices) is called an equilibrium when there exits \( M_0 \geq 0 \) such that the sequence \( (\hat{z}_t)_{t \geq 0} \) of detrended net trade defined by

\[
\hat{z}_t := \frac{1}{\rho^t p_t} \left[ \ell (p_t + 2p_{t+1} + 2p_{t+2} + \ldots) + 2M_0 \right]
\]

satisfies (0.1) and (0.3).

We impose the following restrictions on primitives:

\[
1 > \beta \left( \frac{y_h}{\rho y_l} \right)^\alpha > \frac{1}{\rho} > \beta \left( \frac{1}{\rho} \right)^\alpha = \bar{q}.
\]

This condition implies that there exists a unique \( M_0^* \) such that

\[
2M_0^* < (y_h - y_l)/2 \quad \text{and} \quad \beta \left( \frac{y_h - 2M_0^*}{\rho(y_h + 2M_0^*)} \right)^\alpha = \frac{1}{\rho}.
\]

1 Existence of competitive equilibrium

Proposition 1.1. Assume that \( \ell \) is small enough in the sense that

\[
\frac{\ell}{1 - \bar{q}} < M_0^*.
\]

Fix an arbitrary \( M_0(\ell) \geq 0 \) satisfying

\[
\frac{\ell}{1 - \bar{q}} + M_0(\ell) \leq M_0^*.
\]
There exists a competitive equilibrium \((p_t(\ell))_{t \geq 0}\) such that \(p_t(\ell) \geq 1/\rho^t\) and
\[
x_t(\ell) = \frac{1}{p_t(\ell)}[\ell(p_t(\ell) + p_{t+1}(\ell) + \ldots) + M_0(\ell)].
\]
In particular, the not-too-tight debt limit at date \(t\) contains the bubble component \(M_0(\ell)/p_t\).

**Proof.** Denote by \(P\) the set of all sequences \((p_t)_{t \geq 0}\) such that \(p_0 = 1\) and
\[
\frac{1}{\rho^t} \leq p_t \leq (\bar{q})^t.
\]
Observe that \(P\) is non-empty, convex and compact for the product topology.

For every \(p \in P\) and \(t \geq 0\), we let
\[
\check{z}_t(p) := \frac{1}{\rho^t p_t} [\ell(p_t + 2p_{t+1} + 2p_{t+2} + \ldots) + 2M_0(\ell)].
\]
Since \(p_t \leq (\bar{q})^t\) and \(\bar{q} < 1\), we can apply the Lebesgue Dominated Convergence to show that \(p \mapsto \check{z}_t(p)\) is continuous for the product topology. Moreover, for every \(p \in P\), we have
\[
\check{z}_t(p) \leq \frac{2\ell}{1 - \bar{q}} + 2M_0(\ell) \leq 2M_0^* < \frac{y_h - y_l}{2}.
\]
For every \(p \in P\) and \(t \geq 1\), we let
\[
\varphi_t(p) := \beta \left( \frac{y_h - \check{z}_t(p)}{\rho(y_l + \check{z}_{t+1}(p))} \right)^\alpha.
\]
Since \(0 < \check{z}_t(p) \leq 2M_0^*\) and \(0 < \check{z}_{t+1}(p) \leq 2M_0^*\), we deduce that
\[
\bar{q} = \beta \left( \frac{y_h}{\rho y_l} \right)^\alpha \geq \varphi_t(p) \geq \beta \left( \frac{y_h - 2M_0^*}{\rho(y_l + 2M_0^*)} \right)^\alpha = \frac{1}{\rho}.
\]
For every \(t \geq 0\), we pose
\[
\psi_0(p) := 1 \quad \text{and} \quad \psi_t(p) = \varphi_t(p)\psi_{t-1}(p), \quad \text{for all } t \geq 1.
\]
We then deduce that \((\psi_t(p))_{t \geq 0}\) belongs to \(P\). We have then constructed a continuous mapping \(\Psi : P \to P\) defined by
\[
\Psi(p) := (\psi_t(p))_{t \geq 0}.
\]
Applying Brouwer’s Fixed Point Theorem, we get the existence of a sequence \((p_t)_{t \geq 0} \in P\) satisfying
\[
p_{t+1}/p_t = \beta \left( \frac{y_h - \check{z}_t(p)}{\rho(y_l + \check{z}_{t+1}(p))} \right)^\alpha.
\]
This corresponds to a competitive equilibrium. \(\square\)
Few observations deserve attention. First, there exists a *continuum* of bubbly equilibria as the size $M_0(\ell)$ can be arbitrarily chosen in the interval
\[ 0, M_0^* - \frac{\ell}{1 - \bar{q}}. \]
Second, a *non-bubbly* equilibrium can also be supported for any $\ell > 0$ satisfying (1.1) by simply choosing $M_0(\ell) = 0$. That is, a non-bubbly equilibrium always co-exists with bubbly equilibria. Third, $M_0(\ell)$ can be arbitrarily close to $M_0^*$ when $\ell$ is arbitrarily close to 0. In particular, if $\ell = 0$, there exists a competitive equilibrium with time-invariant interest rate characterized by
\[ p_t = \frac{1}{\rho^t} \quad \text{and} \quad x_t = \rho^t M_0^*. \]

## 2 Additional Results

Recall that when $\ell = 0$ we can set the bubble component equal to its maximal value (i.e., $M_0 = M_0^*$) and support an equilibrium where $p_t = 1/\rho^t$ and $z_t = \rho^t(2M_0^*)$, i.e., trade grows at the constant rate $\rho$. In this section, we focus on the equilibria described in Proposition 1.1. We show below that for $\ell > 0$ we can assure a lower bound on the rate at which trade grows over time. To formalize this, let $\rho_0$ be the gross rate associated to the autarkic price $\bar{q}$, i.e.,
\[ \rho_0 := \frac{1}{\bar{q}}. \]
Restriction (4.5) in MPV implies that $\rho_0 > 1$.

**Proposition 2.1.** Consider a competitive equilibrium as described in Proposition 1.1. If $\ell > 0$, then the sequence $(\rho_0^t p_t)$ is non-increasing and we have
\[ \lim_{t \to 0} \frac{z_t}{\rho_0^t} = \frac{2M_0(\ell)}{\zeta} > 0, \tag{2.1} \]
where $\zeta := \lim_{t \to \infty} \rho_0^t p_t$. In particular, the sequence $(z_t)$ grows at least at the rate $\rho_0$.

**Proof.** First, observe that
\[ \frac{p_{t+1}}{p_t} = \beta \left( \frac{y_h - \hat{z}_t}{p(y_h + \hat{z}_{t+1})} \right)^\alpha \leq \bar{q} = \frac{1}{\rho_0} \]
since $\hat{z}_t \geq 0$ and $\hat{z}_{t+1} \geq 0$. 

4
We then deduce that the sequence \((\rho_t^0 p_t)\) is non-increasing and we denote by \(\zeta \geq 0\) its limit. Recall that
\[
\frac{z_t}{\rho_t^0} = \frac{1}{\rho_t^0 p_t} [\ell (p_t + 2p_{t+1} + \ldots) + 2M_0(\ell)].
\]
Using the fact that \(q_s \leq \bar{q}\) for any \(s > t\), we get that
\[
\frac{2M_0(\ell)}{\rho_t^0 p_t} \leq \frac{z_t}{\rho_t^0} \leq \frac{1}{\rho_t^0} \times \frac{2\ell}{1 - \bar{q}} + \frac{2M_0(\ell)}{\rho_t^0 p_t}.
\]
Passing to the limit, we get the desired result.

Net trade grows at a rate at least as large as \(\rho_0\). However, detrended trade vanishes.

**Proposition 2.2.** Consider a competitive equilibrium as described in Proposition 1.1. If the size \(M_0(\ell)\) of the bubble component is strictly lower than the maximum \(M^*_0\), then the normalized trade asymptotically vanishes, i.e.,
\[
\lim_{t \to \infty} \frac{z_t}{\rho^t} = 0.
\]
In particular, if \(\ell > 0\), then the normalized trade necessarily vanishes. However, if \(\ell = 0\) and the size of the bubble component is maximal (i.e., \(M_0(0) = M^*_0\)), then we can choose \(p_t = 1/\rho^t\) and the normalized trade is constant:
\[
\frac{z_t}{\rho^t} = 2M^*_0.
\]

**Proof.** First, observe that
\[
\frac{p_{t+1}}{p_t} = \beta \left( \frac{y_t - \hat{z}_t}{\rho (y_t + \hat{z}_{t+1})} \right)^{\alpha} \leq \bar{q}
\]
since \(\hat{z}_t \geq 0\) and \(\hat{z}_{t+1} \geq 0\). Using the fact that \(p_t \leq 1\) and \(p_t \leq (\bar{q})^t\), we deduce that
\[
\hat{z}_t = \frac{1}{\rho^t p_t} [\ell (p_t + 2p_{t+1} + \ldots) + 2M_0(\ell)] \leq 2\ell/(1 - \bar{q}) + 2M_0(\ell) \leq 2M^*_0.
\]
This implies that
\[
\frac{p_{t+1}}{p_t} \geq \beta \left( \frac{y_t - 2M^*_0}{\rho (y_t + 2M^*_0)} \right)^{\alpha} = \frac{1}{\rho}.
\]
Consequently, the sequence \((\rho_t^t p_t)_{t \geq 0}\) is non-decreasing and converges to some limit \(\chi \in [1, \infty]\). Since
\[
p_t + 2p_{t+1} + \ldots \leq 2\bar{q}^t \frac{1}{1 - \bar{q}}
\]
we deduce that
\[ \lim_{t \to \infty} \hat{z}_t = \frac{2M_0(\ell)}{\chi}. \]
Observe that if \( M_0 < M_0^* \), then we must have \( \chi = \infty \). This is because
\[ \lim_{t \to \infty} \frac{p_{t+1}}{p_t} = \beta \left( \frac{y_h - 2M_0(\ell)/\chi}{\rho(y_l + 2M_0(\ell)/\chi)} \right)^\alpha > \frac{1}{\rho^2} \]
If \( \ell > 0 \), then we must have \( M_0(\ell) < M_0^* \) and the normalized trade vanishes. Assume now that \( \ell = 0 \) and \( M_0(0) = M_0^* \). The price process \( p_t := 1/\rho^t \) satisfies the sufficient conditions for a competitive equilibrium.

2.1 Pareto Improving Bubbles.

In this section we set \( \ell = 0 \). We know that, starting with no financial endowments, autarky is always a competitive equilibrium. In addition, we know that we can also support a bubbly equilibrium where the bubble component equal its maximal value \( M_0^* \). This requires though initial financial endowments.

Recall that the equilibrium consumption is given by
\[ c_{h,t} = \rho^t(y_h - 2M_0^*) \quad \text{and} \quad c_{l,t} = \rho^t(y_l + 2M_0^*). \]

Normalizing subsequently by the growth rate \( \rho^t \) we obtain a standard two-period steady-state. Define next the following objects
\[ V_h(\hat{z}) = u(y_h - \hat{z}) + \beta \rho^{1-\alpha} u(y_h + \hat{z}) + (\beta \rho^{1-\alpha}) u(y_h - \hat{z}) + \ldots \]
and
\[ V_l(\hat{z}) = u(y_l + \hat{z}) + \beta \rho^{1-\alpha} u(y_l - \hat{z}) + (\beta \rho^{1-\alpha}) u(y_l + \hat{z}) + \ldots \]
Observe that
\[ V_h(\hat{z}) = \frac{1}{1 - \beta} \left[ \frac{u(y_h - \hat{z}) + \hat{\beta} u(y_h + \hat{z})}{1 + \hat{\beta}} \right] \quad \text{and} \quad V_l(\hat{z}) = \frac{1}{1 - \beta} \left[ \frac{u(y_l + \hat{z}) + \hat{\beta} u(y_l - \hat{z})}{1 + \hat{\beta}} \right] \]
where \( \hat{\beta} := \beta \rho^{1-\alpha} \).

\footnote{Indeed, there must exist \( \varepsilon > 0 \) and \( \tau \geq 1 \) such that \( q_{t+1} \geq (1 + \varepsilon)/\rho \) for every \( t \geq \tau \). This implies that \( p_t/p_\tau \geq (1 + \varepsilon)/\rho )^{t-\tau} \) for every \( t \geq \tau + 1 \). We then get that \( \rho^t p_t \geq A(1 + \varepsilon)^t \) with \( A := p_\tau/(1 + \varepsilon)^\tau \). This is sufficient to deduce that \( \chi = \infty \).}
Our purpose is to compare the autarkic allocation to the allocation obtained at the bubbly equilibrium. Since the normalized two-period steady-state requires non-zero initial financial endowments, we need to introduce a one-period transition phase before the equilibrium reaches the steady-state. Formally, let \( \kappa_1 \) be the unique price in \((0, 2/\rho)\) satisfying

\[
\kappa_1 = \beta \left( \frac{y_h - \kappa_1 \rho M_0^*}{\rho (y_l + 2M_0^*)} \right)^\alpha.
\]

Consider the sequence \((q_{t+1}, (c_{u,t}, c_{l,t}))_{t \geq 0}\) defined as follows:

- at \( t = 0 \),
  
  \[
  q_1 := \kappa_1,\quad c_{u,0} := y_h - \kappa_1 \rho M_0^* \quad \text{and} \quad c_{l} := y_l + \kappa_1 \rho M_0^*;
  \]

- at \( t \geq 1 \),
  
  \[
  q_{t+1} := \frac{1}{\rho}, \quad c_{u} := \rho^t (y_h - 2M_0^*) \quad \text{and} \quad c_{l,t} := \rho^t(y_l + 2M_0^*).
  \]

It is easy to see that the above forms a competitive a bubbly equilibrium with zero initial financial endowments. We claim that the consumption allocation Pareto dominates the autarkic allocation. Indeed, let \( U_u \) (\( U_l \)) be the lifetime utility at \( t = 0 \) of the agent starting with high (low) income. We have

\[
U_u = u(y_h - \kappa_1 \rho M_0^*) + \beta V_l(2M_0^*) \quad \text{and} \quad U_l = u(y_l + \kappa_1 \rho M_0^*) + \beta V_h(2M_0^*).
\]

Since \( \kappa_1 \leq 2/\rho \), we have

\[
U_u > u(y_h - 2M_0^*) + \beta V_h(2M_0^*) = V_h(2M_0^*).
\]

Moreover, we also have

\[
U_l \geq u(y_l) + \beta V_u(2M_0^*).
\]

To prove that the bubbly equilibrium allocation improves upon the autarkic one, it is sufficient to show that \( V_u(2M_0^*) > V_u(0) \). This follows from the property that the function

\[
\hat{z} \mapsto u(y_h - \hat{z}) + \beta u(y_l + \hat{z})
\]

is strictly increasing on \([0, (y_h - y_l)/2] \).
2.2 Logarithmic Utility ($\alpha = 1$)

We analyze in detail the computation of a competitive equilibrium when $\alpha = 1$. A competitive equilibrium where the low income agent always issues debt up to the limit and the not-too-tight debt limits are symmetric is characterized by a sequence $(z_t, q_t)_{t \geq 0}$ of net trades and prices (where $q_0 = 1$ by convention) satisfying the following conditions:

- the Euler equation associated to the high-income-agent’s decision:

  \[ q_{t+1} \rho (y_l + \hat{z}_{t+1}) = \beta (y_h - \hat{z}_t) \]  
  \[ \text{(EE)} \]

  where $\hat{z}_t := \rho^{-t} z_t$ is the detrended value of net trade;

- the characterization of debt limits:

  \[ z_t + \ell = 2 \ell + q_{t+1} (z_{t+1} + \ell) \]  
  \[ \text{(NTT)} \]

- the Euler equation associated to the low-income-agent’s decision:

  \[ \hat{z}_t \leq \frac{y_h - y_l}{2}. \]  
  \[ \text{(EE2)} \]

To recover the sequence of debt limits, recall that $D_t = x_t$ where

\[ x_t = \frac{1}{p_t} [\ell (p_t + p_{t+1} + \ldots) + M_0] \]

and $z_t = x_t + q_{t+1} x_{t+1}$. This implies that

\[ x_t = \frac{z_t + \ell}{2}. \]

Fix a time period $t$ and an initial value $\hat{z}_t$. Combining equations (EE) and (NTT), we get the following system of two equations and two unknowns $(q_{t+1}, \hat{z}_{t+1})$

\[ q_{t+1} \rho (y_l + \hat{z}_{t+1}) = \beta (y_h - \hat{z}_t) \quad \text{and} \quad \rho q_{t+1} \left( \hat{z}_{t+1} + \frac{\ell}{\rho^{t+1}} \right) = \hat{z}_t - \frac{\ell}{\rho^t}. \]

If we let $v_{t+1} := q_{t+1} \hat{z}_{t+1}$, we get the linear system on the unknowns $(q_{t+1}, v_{t+1})$ where $\hat{z}_t$ is a parameter:

\[ \rho q_{t+1} y_l + \rho v_{t+1} = \beta y_h - \beta \hat{z}_t \quad \text{and} \quad \rho v_{t+1} + q_{t+1} \frac{\ell}{\rho^t} = \hat{z}_t - \frac{\ell}{\rho^t}. \]
The solution of the above system is

\[ q_{t+1} = \frac{\beta(y_h - \hat{z}_t)}{\rho y_h} - \frac{\hat{z}_t}{\rho y_h} + \frac{\ell}{\rho^{t+1} y_L} =: g(t, \hat{z}_t) \]  

(2.2)

and

\[ v_{t+1} = \frac{\hat{z}_t}{\rho} - \frac{\ell}{\rho^{t+1}} \left[ 1 + \frac{\beta(y_h - \hat{z}_t)}{\rho y_h} \right]. \]

Since \( \hat{z}_{t+1} = v_{t+1}/q_{t+1} \), we then get

\[ \hat{z}_{t+1} = \frac{\hat{z}_t}{\rho} - \frac{\ell}{\rho^{t+1}} \left[ 1 + \frac{\beta(y_h - \hat{z}_t)}{\rho y_h} \right] =: f(t, \hat{z}_t). \]  

(2.3)

Therefore, for every initial value \( \hat{z}_0 \), we can compute the sequence \((\hat{z}_{t+1}(\hat{z}_0))_{t \geq 0}\) recursively by posing

\[ \hat{z}_{t+1}(\hat{z}_0) = f(t, \hat{z}_t(\hat{z}_0)). \]

We recover the corresponding sequence of prices \((q_{t+1}(\hat{z}_0))_{t \geq 0}\) by posing

\[ q_{t+1}(\hat{z}_0) = g(t, \hat{z}_t(\hat{z}_0)). \]

The sequence \((q_{t+1}(\hat{z}_0), \hat{z}_{t+1}(\hat{z}_0))_{t \geq 0}\) implements a competitive equilibrium if, and only if,

\[ \hat{z}_{t+1}(\hat{z}_0) \leq \frac{y_h - y_l}{2}, \quad \text{for all } t \geq 0. \]  

(2.4)

Moreover, the bubble component associated to this equilibrium is

\[ 2M_0(\hat{z}_0) := \lim_{t \to \infty} p_{t+1}(\hat{z}_0) \hat{z}_{t+1}(\hat{z}_0). \]

The sequence \((f(t, \cdot))_{t \geq 0}\) of functions plays a crucial role to prove existence of an equilibrium.
For each $t \geq 0$, the function $f(t, \cdot) : [0, \infty) \rightarrow \mathbb{R}$ is strictly increasing. When $\ell > 0$, there exists $\xi_t > M_0^\star$ such that $f(t, \xi_t) = \xi_t$ and satisfying

$$
\begin{cases}
    f(t, \hat{z}) > \hat{z} & \text{if } \xi_t < \hat{z} \leq (y_h - y_L)/2, \\
    f(t, \hat{z}) < \hat{z} & \text{if } 0 \leq \hat{z} < \xi_t.
\end{cases}
$$

Fix an arbitrary $\hat{z}_0 \geq 0$. If there exists some $\tau \geq 0$ such that $\hat{z}_\tau(\hat{z}_0) > \xi_t$, then we get

$$
\lim_{t \to \infty} \hat{z}_t(\hat{z}_0) = \infty
$$

and the condition $\hat{z}_t(\hat{z}_0) \leq (y_h - y_L)/2$ will eventually be violated. We should then restrict attention to initial values $\hat{z}_0$ satisfying

$$
\hat{z}_0 \leq \xi_0, \quad f(0, \hat{z}_0) \leq \xi_1, \quad f(1, f(0, \hat{z}_0)) \leq \xi_2, \quad f(2, f(1, f(0, \hat{z}_0))) \leq \xi_3, \ldots
$$

Denote by $h_t : [f(t, 0), \infty) \rightarrow [0, \infty)$ be the inverse of $f(t, \cdot)$. We then look for an initial value $\hat{z}_0$ such that

$$
\hat{z}_0 \leq \xi_0, \quad \hat{z}_0 \leq h_0(\xi_1), \quad \hat{z}_0 \leq h_0(h_1(\xi_2)), \quad \hat{z}_0 \leq h_0(h_1(h_2(\xi_3))), \ldots
$$

Since we want the largest initial value $\hat{z}_0$, we pose

$$
\hat{z}_0^\star := \inf\{\{\xi_0\} \cup \{h_0 \circ \ldots \circ h_t(\xi_{t+1}) : t \geq 0\}).
$$
The sequence \((\xi_t)_{t \geq 0}\) is strictly decreasing and converges to \(2M_0^*\) (see Figure 1).

![Figure 1: The sequence \((\xi_t)_{t \geq 0}\)](https://ssrn.com/abstract=3463757)

This implies that \(f(t, \xi_t) = \xi_t \geq \xi_{t+1}\) and we deduce that \(\xi_t \geq h_t(\xi_{t+1})\). By monotonicity of \(h_0 \circ \ldots \circ h_{t-1}\), we get that

\[
h_0 \circ \ldots \circ h_{t-1}(\xi_t) \geq h_0 \circ \ldots \circ h_t(\xi_{t+1}).
\]

This implies that

\[
\hat{z}_0^* = \lim_{t \to \infty} h_0 \circ \ldots \circ h_t(\xi_{t+1}).
\]

Recall that for every \(t \geq 0\), we have \(2M_0^* < \xi_t\). This implies that \(f(t, 2M_0^*) < 2M_0^*\) and, consequently, \(2M_0^* < h_t(2M_0^*) < h_t(\xi_{t+1})\) where the last inequality follows from the strict monotonicity of \(h_t\). Repeating this argument for \(t - 1\) and using strict monotonicity of \(h_{t-1}\), we deduce that

\[
2M_0^* < h_{t-1}(2M_0^*) < h_{t-1}(h_t(2M_0^*)) < h_{t-1}(h_t(\xi_{t+1})).
\]

We then deduce that \(2M_0^* < h_0 \circ \ldots \circ h_t(\xi_{t+1})\) for every \(t \geq 0\). In particular, we have that \(\hat{z}_0^* \geq 2M_0^*\). The above results are summarized below.

**Lemma 2.1.** The initial (detrended) net trade \(\hat{z}_0^*\) defined by

\[
\hat{z}_0^* = \inf\{h_0 \circ \ldots \circ h_t(\xi_{t+1}) : t \geq 0\} = \lim_{t \to \infty} h_0 \circ \ldots \circ h_t(\xi_{t+1})
\]
satisfies $\hat{z}_0^* \geq 2M_0^*$ and is such that the associated sequence $(q_{t+1}(\hat{z}_0^*), \hat{z}_{t+1}(\hat{z}_0^*))_{t \geq 0}$ implements a competitive equilibrium. Moreover, it is the largest initial value for net trade that satisfies this last property.

By construction, we have $\hat{z}_0^* \leq \xi_0$. The definition of $\xi_0$ implies that

$$\hat{z}_1(\hat{z}_0^*) = f(0, \hat{z}_0^*) \leq \hat{z}_0^*.$$ 

By construction, we also have $\hat{z}_0^* \leq h_0(\xi_1)$. This means that $\hat{z}_1(\hat{z}_0^*) = f(0, \hat{z}_0^*) \leq \xi_1$. The definition of $\xi_1$ implies that

$$\hat{z}_2(\hat{z}_0^*) = f(1, \hat{z}_1(\hat{z}_0^*)) \leq \hat{z}_1(\hat{z}_0^*).$$

Repeating the above arguments, we can show that $\hat{z}_t(\hat{z}_0^*) \leq \xi_t$ a

$$\hat{z}_{t+1}(\hat{z}_0^*) \leq \hat{z}_t(\hat{z}_0^*) \leq \xi_t.$$

Therefore, the sequence $(\hat{z}_t(\hat{z}_0^*))_{t \geq 0}$ is decreasing and bounded from above by the sequence $(\xi_t)_{t \geq 0}$. Since the sequence $(\xi_t)_{t \geq 0}$ converges to $2M_0^*$ and $\hat{z}_t(\hat{z}_0^*) \geq 2M_0^*$, we deduce the following property.

**Lemma 2.2.** The sequence of equilibrium (detrended) net trades associated to the largest initial net trade value $\hat{z}_0^*$ is decreasing and converges to $2M_0^*$, i.e.,

$$\lim_{t \to \infty} \hat{z}_t(\hat{z}_0^*) = 2M_0^*.$$

**References**

Martins-da-Rocha, V. F., Phan, T., and Vailakis, Y. (2019). Debt limits and credit bubbles in general equilibrium. Available at SSRN: [https://papers.ssrn.com/abstract_id=3463753](https://papers.ssrn.com/abstract_id=3463753)