ON MILNOR AND TJURINA NUMBERS OF FOLIATIONS

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Abstract. We study the relationship between the Milnor and Tjurina numbers of a singular foliation $\mathcal{F}$, in the complex plane, with respect to a balanced divisor of separatrices $\mathcal{B}$ for $\mathcal{F}$. For that, we associated with $\mathcal{F}$ a new number called the $\chi$-number and we prove that it is a $C^1$ invariant for holomorphic foliations. We compute the polar excess number of $\mathcal{F}$ with respect to a balanced divisor of separatrices $\mathcal{B}$ for $\mathcal{F}$, via the Milnor number of the foliation, the multiplicity of some hamiltonian foliations along the separatrices in the support of $\mathcal{B}$ and the $\chi$-number of $\mathcal{F}$. On the other hand, we generalize, in the plane case and the formal context, the well-known result of Gómez-Mont given in the holomorphic context, which establishes the equality between the GSV-index of the foliation and the difference between the Tjurina number of the foliation and the Tjurina number of a set of separatrices of $\mathcal{F}$. Finally, we state numerical relationships between some classic indices, as Baum-Bott, Camacho-Sad, and variational indices of a singular foliation and its Milnor and Tjurina numbers; and we obtain a bound for the sum of Milnor numbers of the local separatrices of a holomorphic foliation on the complex projective plane.

1. Introduction

The Milnor and Tjurina numbers are classical invariants in the theory of complex analytic hypersurfaces with isolated singularity. The Milnor number is the number of spheres in the Milnor fibre of the hypersurface. The notion of the Milnor number for hypersurfaces was introduced in [22, Section 7]. The Tjurina number is the dimension of the base space of a semi-universal deformation of the hypersurface.

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Semi-universal deformations were studied in [29] and as far as our knowledge reaches, the name Tjurina number appears for the first time in [16]. The Milnor number is a topological invariant and the Tjurina number is an analytic invariant of the singularity. There is an abundant and varied bibliography on the Milnor number of singular hypersurfaces. The Tjurina number has not been as well studied, perhaps because it is an analytical invariant, but in recent years new studies have been published. In the context of singular foliations, the notion of Milnor number appears for the first time with that name in [3], although this notion is found in previous works such as that of Van den Essen [30], where a new proof of Seidenberg’s theorem is given. As in the case of hypersurfaces, the Milnor number of a one-dimensional holomorphic foliation is a topological invariant and there is a varied bibliography on it. However, the concept of the Tjurina number of a foliation has been less studied and according to our knowledge, always related to the Gómez-Mont-Seade-Verjovsky index after [14]. We emphasize that Gómez-Mont did not use the terminology of the Tjurina number of a foliation, such a name appears for the first time in [7, Page 159]. In [13, Corollary 2.7] there is a bound of the Tjurina number of a $\mathcal{F}$-invariant curve $C$ as a function of the Tjurina number of $\mathcal{F}$ with respect to $C$. Some verifications on the relationship between the Milnor and Tjurina numbers of non-dicritical foliations and their total union of separatrices are collected in [11].

In this work we deepen into the study of the Tjurina number of a singular foliation along a reduced curve of separatrices and its relationship with the Milnor number and other invariants and indices associated with the foliation such as the polar excess number, Baum-Bott, Camacho-Sad, and variational indices. Taking into account that a singular foliation could admit many infinitely of separatrices – dicritical foliation – the Tjurina number of a foliation will be associated with a balanced divisor of separatrices. The notion of balanced divisor of separatrices for a foliation was introduced by Genzmer in [12]. This is a geometric object formed by a finite set of separatrices, choosing all isolated separatrices and some separatrices from the ones associated to dicritical components, with weights, possibly negative (those that correspond to poles). In the non-dicritical case, this notion coincides with the total union of separatrices of the foliation. A balanced divisor of separatrices provides a control of the algebraic multiplicity of the foliation and of its separatrices (see Proposition 2.3). In Foliation Theory it is not easy to determine whether an invariant associated with a foliation is topological (remember that, for example, it is not yet known if the algebraic multiplicity is). However, it has been shown that
some indices are analytical invariants, such as the Baum-Bott index (see [9, Remark 3.2]), Camacho-Sad and Variational indices (a good exercise for beginners in the world of foliations). We hope that this article contributes for understanding the dicritical foliations and the invariants associated with them.

The paper is organized as follows. In Section 2 we recall some preliminary notions that are necessary in the paper. The first motivation for this work was to understand the relationship between the Milnor and Tjurina numbers of a singular foliation $\mathcal{F}$ and a balanced divisor of separatrices for $\mathcal{F}$. It is for that in Section 3 we associate a new number $\chi_p(\mathcal{F})$ to any (dicritical or non dicritical) singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$. This number is defined in function of the algebraic multiplicities and excess tangency indices of the strict transforms of $\mathcal{F}$ at infinitely near points of $p$. Hence $\chi_p(\mathcal{F})$ is a $C^1$ invariant for holomorphic foliations. We study the properties of the $\chi$-number in Proposition 3.1. In particular, we prove that it is a nonnegative integer number and any foliation of algebraic multiplicity bigger than 1 is of second type if and only if the $\chi$-number equals zero.

Section 4 is devoted to the indices associated with a foliation making use of a generic polar curve of it, as the polar intersection and the polar excess. Proposition 4.2 provides a formula to compute the polar excess number of a foliation with respect to the zero divisor of a reduced balanced divisor of separatrices, and as a consequence, we obtain a characterization of generalized curve foliations, which generalizes [7, Proposition 2] to dicritical context. In Proposition 4.7 we establish the relationship between the Milnor number of a foliation, the multiplicity of the foliation along the separatrices (of a balanced divisor) and the $\chi$-number of the foliation, generalizing [7, Corollary 2] to dicritical foliations. As a consequence we give a new proof, using foliations, of the well-known relationship between the Milnor number of a reduced plane curve and the Milnor numbers of its irreducible components (see Proposition 4.8). Theorem A is the main result in this section and one of the main results in the paper, where we compute the polar excess number of a singular foliation $\mathcal{F}$ with respect to a balanced divisor of separatrices $\mathcal{B}$, $\Delta_p(\mathcal{F}, \mathcal{B})$, via the Milnor number of the foliation, $\mu_p(\mathcal{F})$, the multiplicity of some hamiltonian foliations along the separatrices in the support of $\mathcal{B}$, $\mu_p(dF_B, B)$, and the $\chi$-number of $\mathcal{F}$. More precisely
Theorem A. Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}) - \sum_B a_B \mu_p(dF_B, B) + \deg(\mathcal{B}) - 1 - \chi_p(\mathcal{F}),$$

where $F_B$ is a balanced divisor of separatrices for $\mathcal{F}$ adapted to $\mathcal{B}$. Moreover, if $\mathcal{F}$ is of a second type foliation, then

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}) - \sum_B a_B \mu_p(dF_B, B) + \deg(\mathcal{B}) - 1.$$

As a consequence, in Corollary 4.10 we give a new characterization of generalized curve foliations in the non-dicritical case.

In Section 5, we study the Gómez-Mont-Seade-Verjovsky index (GSV-index). In particular, in Corollary 5.3, we compute the GSV-index of a foliation $\mathcal{F}$ with respect to the zero divisor of a reduced balanced divisor of separatrices for $\mathcal{F}$. In Proposition 5.4, we generalize, to the dicritical case, [7, Proposition 4] which establishes the equality between the GSV-index of a foliation $\mathcal{F}$ (containing perhaps purely formal branches) with respect to an $\mathcal{F}$-invariant curve $C : f(x, y) = 0$ and the intersection multiplicity of $C$ with a divisor which zero divisor is a generic polar curve of $\mathcal{F}$ and its pole divisor is a generic polar curve of $C$. We finish this section establishing, in Proposition 5.7, a relationship between the GSV-index and the multiplicity of a foliation along a fixed separatrix.

In Section 6, we introduce the notion of Tjurina number of a singular foliation $\mathcal{F}$ along a reduced curve of separatrices $C$, denoted by $\tau_p(\mathcal{F}, C)$. Gómez-Mont proved, for a singular foliation with a set of convergent separatrices $C$, that the difference between the Tjurina number of the foliation and the Tjurina number of $C$, $\tau_p(C)$, equals to the GSV-index (see [14, Theorem 1]). In Proposition 6.2, we show that this result also holds, in the formal context, for the Tjurina number of a singular foliation along a reduced curve of separatrices. As a consequence we get the next corollary for non-dicritical foliations:

Corollary B. Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C$ is the total union of separatrices of $\mathcal{F}$. Then

$$\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C) + \chi_p(\mathcal{F}).$$

Moreover, if $\mathcal{F}$ is of second type then $\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C)$. 

The main result in this section, and other of the main results in the paper, is Theorem C where given a balanced divisor of separatrices $\mathcal{B} = \sum_B a_B B$ of the
singular foliation $\mathcal{F}$, we compute the difference of the Milnor of $\mathcal{F}$ and the sum $T_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B \tau_p(\mathcal{F}, B)$ of Tjurina numbers of $\mathcal{F}$ along the components of $\mathcal{B}$:

**Theorem C.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then

$$
\mu_p(\mathcal{F}) - T_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B [\mu_p(d\mathcal{F}_B, B) - \tau_p(B)] - \deg(B) + 1 + \chi_p(\mathcal{F}) - \sum_B a_B [i_p(B, (\mathcal{F}_B)_0 \setminus B) - i_p(B, (\mathcal{F}_B)_\infty)],
$$

where $\mathcal{F}_B$ is a balanced divisor of separatrices for $\mathcal{F}$ adapted to $B$.

As a consequence, we precise this relationship for second type foliations in Corollary 6.6 and for non-dicritical foliations in:

**Corollary D.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C = \bigcup_{j=1}^\ell C_j$ is the total union of separatrices of $\mathcal{F}$. Then

$$
\mu_p(\mathcal{F}) - T_p(\mathcal{F}, C) = \mu_p(C) - \sum_{j=1}^\ell \tau_p(C_j) + \chi_p(\mathcal{F}) - \sum_{j=1}^\ell i_p(C_j, C \setminus C_j),
$$

where $\tau_p(C_j)$ is the Tjurina number of $C_j$.

We complete this section with several examples. In particular, in Example 6.5, we construct a family of dicritical foliations which are not of second type. We finish Section 6 stating numerical relationships between some classic indices, as Baum-Bott, Camacho-Sad, and variational indices, of a singular foliation and the Milnor and Tjurina numbers. Finally, in Section 7 we obtain a bound for the sum of Milnor numbers of the local separatrices of a holomorphic foliation on the complex projective plane.

2. **Basic tools**

In order to fix the terminology and the notation, we recall some basic concepts of local Foliation Theory. Unless we specify otherwise, throughout this text $\mathcal{F}$ denotes a germ of a singular (holomorphic or formal) foliation at $(\mathbb{C}^2, p)$. In local coordinates $(x, y)$ centered at $p$, the foliation is given by a (holomorphic or formal) 1-form

$$
\omega = P(x, y)dx + Q(x, y)dy,
$$

(2.1)
or by its dual vector field

\[ (2.2) \quad v = -Q(x,y) \frac{\partial}{\partial x} + P(x,y) \frac{\partial}{\partial y}, \]

where \( P(x,y), Q(x,y) \in \mathbb{C}[[x,y]] \) are relatively prime, where \( \mathbb{C}[[x,y]] \) is the ring of complex formal power series in two variables. The algebraic multiplicity \( \nu_p(F) \) is the minimum of the orders \( \nu_p(P), \nu_p(Q) \) at \( p \) of the coefficients of a local generator of \( F \).

Let \( f(x,y) \in \mathbb{C}[[x,y]] \). We say that \( C : f(x,y) = 0 \) is invariant by \( F \) or \( F \)-invariant if

\[ \omega \wedge df = (f.h)dx \wedge dy, \]

for some \( h \in \mathbb{C}[[x,y]] \). If \( C \) is an irreducible \( F \)-invariant curve then we will say that \( C \) is a separatrix of \( F \). The separatrix \( C \) is analytical if \( f \) is convergent. We denote by \( \text{Sep}_p(F) \) the set of all separatrices of \( F \). When \( \text{Sep}_p(F) \) is a finite set we will say that the foliation \( F \) is non-dicritical and we call total union of separatrices of \( F \) to the union of all elements of \( \text{Sep}_p(F) \). Otherwise we will say that \( F \) is a dicritical foliation.

A point \( p \in \mathbb{C}^2 \) is a reduced or simple singularity for \( F \) if the linear part \( Dv(p) \) of the vector field \( v \) in (2.2) is non-zero and has eigenvalues \( \lambda_1, \lambda_2 \in \mathbb{C} \) fitting in one of the two following cases:

1. \( \lambda_1 \lambda_2 \neq 0 \) and \( \lambda_1/\lambda_2 \notin \mathbb{Q}^+ \) (in which case we say that \( p \) is a non-degenerate or complex hyperbolic singularity).

2. \( \lambda_1 \neq 0 \) and \( \lambda_2 = 0 \) (in which case we say that \( p \) is a saddle-node singularity).

The reduction process of the singularities of a codimension one singular foliation over an ambient space of dimension two was achieved by Seidenberg [25]. Van den Essen gave a new proof in [30].

A singular foliation \( F \) at \((\mathbb{C}^2, p)\) is a generalized curve foliation if it has no saddle-nodes in its reduction process of singularities, that is, the case (1). This concept was defined by Camacho-Lins Neto-Sad [5, Page 144]. In this case, there is a system of coordinates \((x, y)\) in which \( F \) is induced by the equation

\[ (2.3) \quad \omega = x(\lambda_1 + a(x,y))dy - y(\lambda_2 + b(x,y))dx, \]

where \( a(x,y), b(x,y) \in \mathbb{C}[[x,y]] \) are non-units, so that \( \text{Sep}_p(F) \) is formed by two transversal analytic branches given by \( \{x = 0\} \) and \( \{y = 0\} \). In the case (2), up to a formal change of coordinates, the saddle-node singularity is given by a 1-form of
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the type

\begin{equation}
\omega = x^{k+1}dy - y(1 + \lambda x^k)dx,
\end{equation}

where \( \lambda \in \mathbb{C} \) and \( k \in \mathbb{Z}_{>0} \) are invariants after formal changes of coordinates (see \cite[Proposition 4.3]{20}). The curve \( \{ x = 0 \} \) is an analytic separatrix, called \textit{strong}, whereas \( \{ y = 0 \} \) corresponds to a possibly formal separatrix, called \textit{weak} or \textit{central}.

Given a foliation \( \mathcal{F} \) at \((\mathbb{C}^2, p)\) we follow \cite[page 1115]{10} to introduce the set \( \mathcal{I}_p(\mathcal{F}) \) of \textit{infinitely near points} of \( \mathcal{F} \) at \( p \). This is defined in a recursive way along the reduction process of the singularities of \( \mathcal{F} \). We do as follows. Given a sequence of blow-ups \( \pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p) \) — an intermediate step in the reduction process, where \( \mathcal{D} = \pi^{-1}(p) \) and \( \tilde{X} \) is the ambient space containing \( \mathcal{D} \) — and a point \( q \in \mathcal{D} \) we set:

- if \( \tilde{\mathcal{F}} \) is \( \mathcal{D} \)-reduced at \( q \), then \( \mathcal{I}_q(\tilde{\mathcal{F}}) = \{ q \} \);
- if \( \tilde{\mathcal{F}} \) is \( \mathcal{D} \)-singular but not \( \mathcal{D} \)-reduced at \( q \), we perform a blow-up \( \sigma : (\tilde{X}, \tilde{\mathcal{D}}) \to (\tilde{X}, \mathcal{D}) \) at \( q \), where \( \tilde{\mathcal{D}} = \sigma^{-1}(\mathcal{D}) = (\sigma^*\mathcal{D}) \cup D \) and \( D = \sigma^{-1}(q) \) (here \( \sigma^*\mathcal{D} \) denotes the strict transform of \( \mathcal{D} \)). If \( q_1, \ldots, q_\ell \) are all \( \tilde{\mathcal{D}} \)-singular points of \( \tilde{\mathcal{F}} = \sigma^*\tilde{\mathcal{F}} \) on \( D \), then

\[ \mathcal{I}_q(\tilde{\mathcal{F}}) = \{ q \} \cup \mathcal{I}_{q_1}(\tilde{\mathcal{F}}) \cup \ldots \cup \mathcal{I}_{q_\ell}(\tilde{\mathcal{F}}). \]

In order to simplify notation, we settle that a numerical invariant for a foliation \( \mathcal{F} \) at \( q \in \mathcal{I}_p(\mathcal{F}) \) actually means the same invariant computed for the transform of \( \mathcal{F} \) at \( q \).

For a fixed reduction process of singularities \( \pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p) \) for \( \mathcal{F} \), a component \( D \subset \mathcal{D} \) can be:

- \textit{non-dicritical}, if \( D \) is \( \tilde{\mathcal{F}} \)-invariant. In this case, \( D \) contains a finite number of simple singularities. Each non-corner singularity of \( D \) carries a separatrix transversal to \( D \), whose projection by \( \pi \) is a curve in \( \text{Sep}_p(\mathcal{F}) \). Remember that a corner singularity of \( D \) is an intersection point of \( D \) with other irreducible component of \( \mathcal{D} \).
- \textit{dicritical}, if \( D \) is not \( \tilde{\mathcal{F}} \)-invariant. The reduction process of singularities gives that \( D \) may intersect only non-dicritical components of \( \mathcal{D} \) and \( \tilde{\mathcal{F}} \) is everywhere transverse to \( D \). The \( \pi \)-image of a local leaf of \( \tilde{\mathcal{F}} \) at each non-corner point of \( D \) belongs to \( \text{Sep}_p(\mathcal{F}) \).

Let \( \sigma \) be the blow-up of the reduction process of singularities \( \pi \) that generated the component \( D \subset \mathcal{D} \). We will say that \( \sigma \) is \textit{non-dicritical} (respectively \textit{dicritical}) if \( D \) is non-dicritical (respectively dicritical).
Denote by $\text{Sep}_p(D) \subset \text{Sep}_p(F)$ the set of separatrices whose strict transforms by $\pi$ intersect the component $D \subset \mathcal{D}$. If $B \in \text{Sep}_p(D)$ with $D$ non-dicritical, $B$ is said to be isolated. Otherwise, it is said to be a dicritical separatrix. This determines the decomposition $\text{Sep}_p(F) = \text{Iso}_p(F) \cup \text{Dic}_p(F)$, where notations are self-evident. The set $\text{Iso}_p(F)$ is finite and contains all purely formal separatrices. It subdivides further in two classes: weak separatrices — those arising from the weak separatrices of saddle-nodes — and strong separatrices — corresponding to strong separatrices of saddle-nodes and separatrices of non-degenerate singularities. On the other hand, if $\text{Dic}_p(F)$ is non-empty then it is an infinite set of analytic separatrices. Observe that a foliation $F$ is dicritical when $\text{Sep}_p(F)$ is infinite, which is equivalent to saying that $\text{Dic}_p(F)$ is non-empty. Otherwise, $F$ is non-dicritical.

Throughout the text, we would rather adopt the language of divisors of formal curves. More specifically, a divisor of separatrices for a foliation $F$ at $(\mathbb{C}^2, p)$ is a formal sum

\begin{equation}
B = \sum_{B \in \text{Sep}_p(F)} a_B \cdot B,
\end{equation}

where the coefficients $a_B \in \mathbb{Z}$ are zero except for finitely many $B \in \text{Sep}_p(F)$. The set of separatrices $\{B : a_B \neq 0\}$ appearing in (2.5) is called the support of the divisor $B$ and it is denoted by $\text{supp}(B)$. The degree of the divisor $B$ is by definition $\deg B = \sum_{B \in \text{supp}(B)} a_B$. We denote by $\text{Div}_p(F)$ the set of all these divisors of separatrices, which turns into a group with the canonical additive structure. We follow the usual terminology and notation:

- $B \geq 0$ denotes an effective divisor, one whose coefficients are all non-negative;
- there is a unique decomposition $B = B_0 - B_\infty$, where $B_0, B_\infty \geq 0$ are respectively the zero and pole divisors of $B$;
- the algebraic multiplicity of $B$ is $\nu_p(B) = \sum_{B \in \text{supp}(B)} \nu_p(B)$.

Given a foliation $F$ and a formal meromorphic equation $F(x, y) = \prod_{i=1}^s f_i(x, y)^{a_i}$, whose irreducible components define separatrices $B_i : f_i(x, y) = 0$ of $F$, we associate the divisor $(F) = \sum_i a_i \cdot B_i$. A curve of separatrices $C$, associated with a reduced equation $F(x, y)$, is identified to the divisor $(F)$ and we write $C = (F)$. Such an effective divisor is named reduced, that is, all coefficients are either 0 or 1. In general, $B \in \text{Div}_p(F)$ is reduced if both $B_0$ and $B_\infty$ are reduced divisors. A divisor $B$ is said to be adapted to a curve of separatrices $C$ if $B_0 - C \geq 0$.

Following [12, page 5] and [13, Definition 3.1], we remember the following notion:
Definition 2.1. A balanced divisor of separatrices for $\mathcal{F}$ is a divisor of the form

$$B = \sum_{B \in \text{Iso}_p(\mathcal{F})} B + \sum_{B \in \text{Dic}_p(\mathcal{F})} a_B \cdot B,$$

where the coefficients $a_B \in \mathbb{Z}$ are non-zero except for finitely many $B \in \text{Dic}_p(\mathcal{F})$, and, for each dicritical component $D \subset \mathcal{D}$, the following equality is respected:

$$\sum_{B \in \text{Sep}_p(D)} a_B = 2 - \text{Val}(D).$$

The integer $\text{Val}(D)$ stands for the valence of a component $D \subset \mathcal{D}$ in the reduction process of singularities, that is, it is the number of components of $\mathcal{D}$ intersecting $D$ other from $D$ itself.

Observe that the notion of balanced divisor of separatrices generalizes, to dicritical foliations, the notion of total union of separatrices for non-dicritical foliations.

A balanced divisor $B = \sum_B a_B B$ of separatrices of $\mathcal{F}$ is called primitive if, $a_B \in \{-1, 1\}$ for any $B \in \text{supp}(B)$. A balanced equation of separatrices is a formal meromorphic function $F(x, y)$ whose associated divisor $C = C_0 - C_\infty$ is a balanced divisor. A balanced equation is reduced, primitive or adapted to a curve $C$ if the same is true for the underlying divisor.

Remember that the intersection number of two formal curves $C$ and $D$ at $(C^2, p)$ is by definition

$$i_p(C, D) = \dim_\mathbb{C} \mathbb{C}[[x, y]]/(g, h),$$

where $C : g(x, y) = 0, D : h(x, y) = 0$, and $(g, h)$ denotes the ideal generated by $g$ and $h$ in $\mathbb{C}[[x, y]]$. The intersection number for formal curves at $(C^2, p)$ is canonically extended in a bilinear way to divisors of curves.

Let $\mathcal{F}$ be a foliation at $(C^2, p)$, given by a 1-form as in (2.1), with reduction process $\pi : (\tilde{X}, \mathcal{D}) \to (C^2, p)$ and let $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$ be the strict transform of $\mathcal{F}$. Denote by $\text{Sing}(\cdot)$ the set of singularities of a foliation. A saddle-node singularity $q \in \text{Sing}(\tilde{\mathcal{F}})$ is said to be a tangent saddle-node if its weak separatrix is contained in the exceptional divisor $\mathcal{D}$, that is, the weak separatrix is an irreducible component of $\mathcal{D}$.

We have the following definition given by Mattei-Salem [21, Définition 3.1.4] to non-dicritical case and used by Genzmer [12] for arbitrary foliations:

Definition 2.2. A foliation is in the second class or is of second type if there are no tangent saddle-nodes in its reduction process of singularities.
Let $B$ be a separatrix of $\mathcal{F}$ at $p$. Suppose that $\{y = 0\}$ is the tangent cone of $B$, then we may choose one of its primitive Puiseux parametrizations $\gamma(t) = (t^n, \phi(t))$ at $p$ such that $n = \nu_p(B)$, where $\nu_p(B)$ denotes the algebraic multiplicity of $B$. The tangency index of $\mathcal{F}$ along $B$ at $p$ (or weak index in [10, page 1114]) is

$$\text{Ind}_p(\mathcal{F}, B) := \text{ord}_t Q(\gamma(t)).$$

The tangency index $\text{Ind}_p(\mathcal{F}, B)$ does not depend on the chosen parametrization of $B$ because by properties of the multiplicity number we get the equality $\text{ord}_t Q(\gamma(t)) = i_p(Q, B)$. The foliation $\mathcal{F}$ given by the 1-form defined in (2.4) verifies $\text{Ind}_p(\mathcal{F}, B) = k + 1 > 1$, where $B : \{y = 0\}$.

The tangency index was defined in [5, page 159], where the authors denominate it multiplicity of $\mathcal{F}$ along $B$ at $p$ and denoted by $\mu_{\mathcal{F}}(B, p)$. In the same paper the authors define a similar notion, the index with respect to a vector field $Z$ (see page 152) and denote it by $\text{Ind}_p(Z/B)$ which coincides with the multiplicity of $\mathcal{F}$ along $B$ at $p$ that we introduce in (4.7) and we denote by $\mu_p(\mathcal{F}, B)$. The reader should pay attention to it to avoid confusion. We adopt the notation given by Genzmer [12], instead of the original given in [5] since $\mu_p(\mathcal{F}, B)$ resembles a Milnor number, which will be studied in Section 4.

Given a component $D \subset \mathcal{D}$, we denote by $\nu(D)$ its multiplicity, which coincides with the algebraic multiplicity of a curve $E$ at $(\mathbb{C}^2, p)$ whose strict transform $\pi^*E$ meets $D$ transversally outside a corner of $\mathcal{D}$. The following invariant is a measure of the existence of tangent saddle-nodes in the reduction process of singularities of a foliation:

**Definition 2.3.** The **tangency excess** of the foliation $\mathcal{F}$ is defined as $\xi_p(\mathcal{F}) = 0$, when $p$ is a reduced singularity, and, in the non-reduced case, as the number

$$\xi_p(\mathcal{F}) = \sum_{q \in \text{SN}(\mathcal{F})} \nu(D_q)(\text{Ind}_q(\tilde{\mathcal{F}}, \tilde{B}) - 1),$$

where $\text{SN}(\mathcal{F})$ stands for the set of tangent saddle-nodes on $\mathcal{D}$, $\tilde{B}$ is the weak separatrix passing by $q \in \text{SN}(\mathcal{F})$, and $D_q$ is the component of $\mathcal{D}$ containing $\tilde{B}$. By (2.4), we observe that $\text{Ind}_q(\tilde{\mathcal{F}}, \tilde{B}) = k + 1 > 1$.

Remark that $\xi_p(\mathcal{F}) \geq 0$ and, by definition, $\xi_p(\mathcal{F}) = 0$ if and only if $\text{SN}(\mathcal{F}) = \emptyset$, that is, if and only if $\mathcal{F}$ is of second type. In several papers (see for example [10], [4]) the tangency excess of $\mathcal{F}$ is denoted by $\tau_p(\mathcal{F})$. In this paper, we denote it by $\xi_p(\mathcal{F})$ since we keep the letter $\tau$ for the Tjurina number of a curve or a foliation.
The following proposition proved by Genzmer (see [12, Proposition 2.4]) will be very useful in this paper:

**Proposition 2.4.** Let $F$ be a singular foliation at $(C^2, p)$ and $B$ a balanced divisor of separatrices for $F$. Denote by $\nu_p(F)$ and $\nu_p(B)$ their algebraic multiplicities respectively. Then

$$\nu_p(F) = \nu_p(B) - 1 + \xi_p(F).$$

Moreover,

$$\nu_p(F) = \nu_p(B) - 1$$

if, and only if, $F$ is a second type foliation.

Take a primitive parametrization $\gamma : (C, 0) \rightarrow (C^2, p)$, $\gamma(t) = (x(t), y(t))$, of a formal irreducible curve $B : f(x, y) = 0$ at $(C^2, p)$. Note that $B$ is a separatrix of the foliation $F : \omega = 0$ if and only if $\gamma^*(\omega) = 0$. If $B$ is not an $F$-invariant curve, we define the **tangency order** of $F$ along $B$ at $p$ as

$$\text{tang}_p(F, B) = \text{ord}_t a(t),$$

where $\gamma^*(\omega) = a(t)dt$. The tangency order does not depend on the chosen parametrization of $B$, since $\text{tang}_p(F, B) = i_p(B, \nu(f))$ where $\nu$ is from (2.2). The tangency order was introduced in this last way in [3, page 22].

The behavior, under blow-up, of the tangency order, in the non-dicritical case, was studied in [7, equality (4)]. The dicritical case is similar. Indeed, if $F : \omega = 0$ is a singular foliation at $(C^2, p)$, $\tilde{F} : \tilde{\omega} = 0$ is its strict transform by the blow-up $\sigma$ at $p$ and $B$ is not an $F$-invariant curve then we have

$$\tilde{\omega} = \begin{cases} x^{-\nu_p(F)}x^{\ast}(\omega) & \text{if } \sigma \text{ is non-dicritical;} \\ x^{-(\nu_p(F)+1)}x^{\ast}(\omega) & \text{if } \sigma \text{ is dicritical.} \end{cases}$$

Evaluating $\tilde{\omega}$ in a parametrization of the strict transform (by $\sigma$) $\tilde{B}$ of $B$ and taking orders we get

$$\text{tang}_p(F, B) = \begin{cases} \nu_p(F)\nu_p(B) + \text{tang}_q(\tilde{F}, \tilde{B}) & \text{if } \sigma \text{ is non-dicritical;} \\ (\nu_p(F) + 1)\nu_p(B) + \text{tang}_q(\tilde{F}, \tilde{B}) & \text{if } \sigma \text{ is dicritical,} \end{cases}$$

where $q \in \tilde{B} \cap \sigma^{-1}(p)$.

In [7, Corollary 1] it was established that $i_p(B, B) \leq \text{tang}_p(F, B) + 1$ and the equality holds if and only if $F$ is of second type. In [4, Lemma 4.2], for complex
analytic foliations, was improved [7, Corollary 1] determining explicitly the difference
\( \text{tang}_p(\mathcal{F}, B) + 1 - i_p(B, B) \), in the next way:
\[
\begin{aligned}
    i_p(B, B) &= \text{tang}_p(\mathcal{F}, B) - \sum_{q \in I_p(\mathcal{F})} \nu_q(B)\xi_q(\mathcal{F}) + 1,
\end{aligned}
\]
where \( \mathcal{F} \) is a singular foliation at \((\mathbb{C}^2, p)\), \( B \) is a balanced divisor of separatrices for \( \mathcal{F} \) and \( B \) is a branch which is not \( \mathcal{F} \)-invariant. A proof, similar to the one given in [4], holds for formal and dicritical foliations.

3. The \( \chi \)-number of a foliation

For a singular foliation \( \mathcal{F} \) at \((\mathbb{C}^2, p)\) we introduce a new number
\[
\chi_p(\mathcal{F}) := \left( \sum_{q \in I_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) \right) - \xi_p(\mathcal{F}).
\]

Observe that
\[
(3.1) \quad \chi_p(\mathcal{F}) = \sum_{q \in I_p(\mathcal{F}) \setminus \{p\}} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) + (\nu_p(\mathcal{F}) - 1)\xi_p(\mathcal{F}).
\]

In [23, Proposition 9.5] the authors prove that the tangency excess is a \( C^\infty \) invariant, and after [24] the algebraic multiplicity of a holomorphic foliation is a \( C^1 \) invariant. Hence the \( \chi \)-number of a holomorphic foliation is a \( C^1 \) invariant.

We enclosed some properties of \( \chi_p(\mathcal{F}) \) in the following proposition:

**Proposition 3.1.** Let \( \mathcal{F} \) be a singular foliation at \((\mathbb{C}^2, p)\), then we get:

1. \( \chi_p(\mathcal{F}) \geq 0 \);
2. if \( \mathcal{F} \) is of second type then \( \chi_p(\mathcal{F}) = 0 \);
3. if \( \chi_p(\mathcal{F}) = 0 \), then either \( \mathcal{F} \) has algebraic multiplicity 1 at \( p \) or \( \mathcal{F} \) is of second type;
4. if \( \nu_p(\mathcal{F}) > 1 \), then \( \chi_p(\mathcal{F}) = 0 \) if and only if \( \mathcal{F} \) is of second type.

**Proof.** By (3.1) we have \( \chi_p(\mathcal{F}) = \beta + (\nu_p(\mathcal{F}) - 1)\xi_p(\mathcal{F}) \), where \( \beta = \sum_{q \in I_p(\mathcal{F}) \setminus \{p\}} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) \).
Clearly, \( \chi_p(\mathcal{F}) \geq 0 \), since it is the sum of two nonnegative numbers. Now, if \( \mathcal{F} \) is of second type, we get \( \xi_q(\mathcal{F}) = 0 \), for all \( q \in I_p(\mathcal{F}) \), which implies that \( \chi_p(\mathcal{F}) = 0 \). On the other hand, if \( \chi_p(\mathcal{F}) = 0 \) then \( \beta = 0 \) and \( (\nu_p(\mathcal{F}) - 1)\xi_p(\mathcal{F}) = 0 \). This finishes the proof of (3). Item (4) is an immediately consequence of (2) and (3).

**Remark 3.2.** Let \( \omega = 4xydx + (y - 2x^2)dy \) be a 1-form. Observe that the foliation \( \mathcal{F} : \omega = 0 \) at \((\mathbb{C}^2, 0)\) is not of second type, its algebraic multiplicity is one but \( \chi_0(\mathcal{F}) = 1 \neq 0 \).
4. Polar intersection, polar excess and Milnor numbers

Let \( \omega = P(x, y)dx + Q(x, y)dy \) be a 1-form, where \( P(x, y), Q(x, y) \in \mathbb{C}[[x, y]] \). If \( \mathcal{F} : \omega = 0 \) is a singular (analytic or formal) foliation then the polar curve of \( \mathcal{F} \) at \((\mathbb{C}^2, p)\) with respect to a point \((a : b)\) of the complex projective line \( \mathbb{P}^1(\mathbb{C}) \) is the (analytic or formal) curve \( \mathcal{P}_{(a:b)}^\omega : aP(x, y) + bQ(x, y) = 0 \). There exists an open Zariski set \( U \) of \( \mathbb{P}^1(\mathbb{C}) \) such that \( \{aP(x, y) + bQ(x, y) = 0 : (a : b) \in U\} \) is an equisingular family of plane curves. Any element of this set is called generic polar curve of the foliation \( \mathcal{F} \) and we will denote it by \( \mathcal{P}_{(a:b)}^\omega \).

We borrow from [13, Section 4] the notion of polar curve of a meromorphic 1-form: let \( \eta = \frac{\omega}{H(x,y)} \) be a meromorphic 1-form, where \( \omega = P(x, y)dx + Q(x, y)dy \) with \( P(x, y), Q(x, y), H(x,y) \in \mathbb{C}[[x, y]] \). The polar curve of \( \eta \) at \((\mathbb{C}^2, p)\) with respect to \((a : b) \in \mathbb{P}^1(\mathbb{C})\) is the divisor \( \mathcal{P}_{(a:b)}^\eta \) with formal meromorphic equation

\[
aP(x, y) + bQ(x, y) \frac{H}{H(x,y)} = 0.
\]

A polar curve \( \frac{aP(x,y) + bQ(x,y)}{H} = 0 \) of a meromorphic 1-form \( \frac{\omega}{H(x,y)} \) will be generic if the polar curve \( aP(x, y) + bQ(x, y) = 0 \) is a generic polar curve of the foliation defined by the 1-form \( \omega \).

Let \( B : h(x, y) = 0 \) be a separatrix of a singular foliation \( \mathcal{F} \). The polar intersection number of \( \mathcal{F} \) with respect to \( B \) is the intersection number \( i_p(\mathcal{P}^\omega, B) \).

**Lemma 4.1.** Let \( B : h(x, y) = 0 \) be a separatrix of a singular foliation \( \mathcal{F} \) at \((\mathbb{C}^2, p)\) and consider \( F_B \) and \( G_B \) two balanced divisors of separatrices for \( \mathcal{F} \) adapted to \( B \). Then

\[
i_p(\mathcal{P}^{dF_B}, B) = i_p(\mathcal{P}^{dG_B}, B).
\]

**Proof.** Put \( F_B = \frac{h \cdot f \cdot g_1 \cdots g_m}{\phi_1 \cdots \phi_m} \) and \( G_B = \frac{h \cdot f \cdot h_1 \cdots h_s}{\psi_1 \cdots \psi_r} \), where \( g_i(x, y) = 0, h_i(x, y) = 0, \phi_i(x, y) = 0 \) and \( \psi_i(x, y) = 0 \) are dicritical separatrices of \( \mathcal{F} \), \( \hat{f}(x, y) = 0 \) defines the reduced curve which is the union of all isolated separatrices of \( \mathcal{F} \) except perhaps \( h(x, y) = 0 \) when this is also isolated. We get

\[
dF_B = \frac{\phi(\mathcal{P} \cdot h + \hat{f} \cdot g_1 \cdots g_l \cdot dh) - h \cdot \hat{f} \cdot g_1 \cdots g_l d(\phi)}{\phi^2}
\]

and

\[
dG_B = \frac{\psi(\mathcal{Q} \cdot h + \hat{f} \cdot h_1 \cdots h_s \cdot dh) - h \cdot \hat{f} \cdot h_1 \cdots h_s d(\psi)}{\psi^2}.
\]
where \( \phi = \phi_1 \cdots \phi_m, \psi = \psi_1 \cdots \psi_r, \mathcal{P} = d(\hat{f} \cdot g_1 \cdots g_l) =: \mathcal{P}_1 dx + \mathcal{P}_2 dy \) and \( \mathcal{Q} = d(\hat{f} \cdot h_1 \cdots h_s) =: \mathcal{Q}_1 dx + \mathcal{Q}_2 dy \). Put \( u = \hat{f} \cdot g_1 \cdots g_l \) and \( v = \hat{f} \cdot h_1 \cdots h_s \). Hence
\[
\mathcal{P}^{dF_B} = \frac{[\phi(\mathcal{P}_1 \cdot h + u \cdot \partial_x h) - hu \cdot \partial_x \phi + [\phi(\mathcal{P}_2 \cdot h + u \cdot \partial_y h) - hu \cdot \partial_y \phi]b]}{\phi^2},
\]
and
\[
\mathcal{P}^{dG_B} = \frac{[\psi(\mathcal{Q}_1 \cdot h + v \cdot \partial_x h) - hv \cdot \partial_x \psi + [\psi(\mathcal{Q}_2 \cdot h + v \cdot \partial_y h) - hv \cdot \partial_y \psi]b]}{\psi^2}.
\]
So
\[
i_p(\mathcal{P}^{dF_B}, B) = i_p(\hat{f} \cdot g_1 \cdots g_l \cdot (a \partial_x h + b \partial_y h), h) - i_p(\phi_1 \cdots \phi_m, h)
\]
(4.1)
\[
= i_p(\hat{f} \cdot (a \partial_x h + b \partial_y h), h) + i_p(g_1 \cdots g_l, h) - i_p(\phi_1 \cdots \phi_m, h),
\]
and
\[
i_p(\mathcal{P}^{dG_B}, B) = i_p(\hat{f} \cdot h_1 \cdots h_s \cdot (a \partial_x h + b \partial_y h), h) - i_p(\psi_1 \cdots \psi_r, h)
\]
(4.2)
\[
= i_p(\hat{f} \cdot (a \partial_x h + b \partial_y h), h) + i_p(h_1 \cdots h_s, h) - i_p(\psi_1 \cdots \psi_r, h).
\]
We claim that \( i_p(g_1 \cdots g_l, h) - i_p(\phi_1 \cdots \phi_m, h) = i_p(h_1 \cdots h_s, h) - i_p(\psi_1 \cdots \psi_r, h) \).

Indeed, if every dicritical separatrix of \( \mathcal{F} \) is smooth and transversal to any isolated separatrix, then using properties of the intersection multiplicity we have
\[
i_p(g_1 \cdots g_l, h) - i_p(\phi_1 \cdots \phi_m, h) = \nu_p(h) \left[ \sum_{j=1}^{l} \nu_p(g_j) - \sum_{j=1}^{m} \nu_p(\phi_j) \right]
\]
(4.3)
\[
= \nu_p(h) \left[ \sum_{j=1}^{s} \nu_p(h_j) - \sum_{j=1}^{r} \nu_p(\psi_j) \right]
\]
\[
= i_p(h_1 \cdots h_s, h) - i_p(\psi_1 \cdots \psi_r, h),
\]
where the equality \( \text{[4.3]} \) holds since \( F_B \) and \( G_B \) are two balanced divisors of separatrices for \( \mathcal{F} \).

In the general case, after the reduction of singularities of the foliation we can suppose that the strict transform of every dicritical separatrix of \( \mathcal{F} \) is smooth and transversal to any strict transform of every isolated separatrix. We finish the proof using Noether formula.

Lemma \( \text{[4.1]} \) allows us to define the polar excess number of a singular foliation \( \mathcal{F} \) at \((\mathbb{C}^2, p)\) with respect to a separatrix \( B \) of \( \mathcal{F} \) as
\[
\Delta_p(\mathcal{F}, B) := i_p(\mathcal{P}^\mathcal{F}, B) - i_p(\mathcal{P}^{dF_B}, B),
\]
(4.4)
where $F_B$ is any balanced divisor of separatrices for $F$ adapted to $B$. On the other hand, if the foliation $F$ is non-dicritical then it is enough to consider $F_B$ as the total union of the separatrices of $F$.

Using properties of the intersection number we extend the definitions of polar intersection and polar excess numbers to any divisor $B := \sum_B a_B B$ of separatrices of $F$ in the following way:

$$i_p(P^F, B) = \sum_B a_B i_p(P^F, B)$$

and

$$\Delta_p(F, B) = \sum_B a_B \Delta_p(F, B) = \Delta_p(F, B_0) - \Delta_p(F, B_\infty).$$

If $B$ is a primitive divisor then the difference $\Delta_p(F, B_0) - \Delta_p(F, B_\infty)$ is independent of the chosen primitive balanced divisor of separatrices for $F$ (see [10, Section 3.6, page 1123]).

By [13, Proposition 4.6], $\Delta_p(F, B)$ is a non-negative integer number for any irreducible component $B$ of $B_0$. As a consequence $\Delta_p(F, B)$ is also a non-negative integer number for any effective divisor of separatrices $B$.

The Milnor number $\mu_p(F)$ of the foliation $F$ at $p$ given by the 1-form $\omega = P(x, y)dx + Q(x, y)dy$ is defined by

$$\mu_p(F) = i_p(P, Q).$$

Remember that we consider $P$ and $Q$ coprime, so $\mu_p(F)$ is a non negative integer. In [5, Theorem A] it was proved that the Milnor number of a foliation is a topological invariant.

On the other hand, the Milnor number $\mu_p(C)$ at $p$ of a plane curve $C$ (non necessary irreducible) of equation $f(x, y) = 0$ is

$$\mu_p(C) = i_p(\partial_x f, \partial_y f).$$

Observe that $\mu_p(C)$ is finite if and only if $f$ has not multiple factors, that is, the curve $C$ is reduced.

Generalized curve foliations have a property of minimization of Milnor numbers and are characterized, in the non-dicritical case, by several authors, see for instance [2, Proposition 7] and [8, Théorème 3.3]. Recently, in [13, Theorem A], the authors have characterized singular generalized curve foliations in terms of its polar excess number as follows: let $F$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $B = B_0 - B_\infty$ be
a balanced divisor of separatrices for $\mathcal{F}$, then $\Delta_p(\mathcal{F}, \mathcal{B}_0) = 0$ if and only if $\mathcal{F}$ is a generalized curve foliation.

The following proposition provides a formula to compute the polar excess number of a foliation with respect to the zero divisor of a reduced balanced divisor of separatrices.

**Proposition 4.2.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a reduced balanced divisor of separatrices for $\mathcal{F}$. Then

$$\Delta_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}_\mathcal{F}, \mathcal{B}_0) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_0) + 1.$$ 

Moreover, $\mathcal{F}$ is a generalized curve foliation if and only if

$$i_p(\mathcal{P}_\mathcal{F}, \mathcal{B}_0) = \mu_p(\mathcal{B}_0) + \nu_p(\mathcal{B}_0) - i_p(\mathcal{B}_0, \mathcal{B}_\infty) - 1.$$ 

**Proof.** Let $\omega = P(x, y)dx + Q(x, y)dy$ be a 1-form inducing $\mathcal{F}$ and $f(x, y) = 0$ and $g(x, y) = 0$ be the reduced equations of $\mathcal{B}_0$ and $\mathcal{B}_\infty$ respectively. Since $\mathcal{B}$ is a set of separatrices of $\mathcal{F}$ adapted to $\mathcal{B}_0$, by (4.4) and the definition of the polar curve of $d(f/g)$ we have

$$\Delta_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}_\mathcal{F}, \mathcal{B}_0) - i_p(\mathcal{P}^{d(f/g)}, \mathcal{B}_0)$$

$$= i_p(\mathcal{P}_\mathcal{F}, \mathcal{B}_0) - i_p\left(\frac{(g\partial_x f - f\partial_x g)a + (g\partial_y f - f\partial_y g)b}{g^2}, f\right)$$

$$= i_p(\mathcal{P}_\mathcal{F}, \mathcal{B}_0) - i_p\left(\frac{g(a\partial_x f + b\partial_y f) - f(a\partial_x g + b\partial_y g)}{g^2}, f\right)$$

where $(a : b) \in \mathbb{P}^1$. By applying properties on intersection numbers and Teissier’s Proposition [28, Chapter II, Proposition 1.2], we get

$$\Delta_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}_\mathcal{F}, \mathcal{B}_0) - i_p(a\partial_x f + b\partial_y f, f) + i_p(g, f)$$

$$= i_p(\mathcal{P}_\mathcal{F}, \mathcal{B}_0) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_0) + 1.$$ 

The second part of the proposition follows from the first part and the characterization of generalized curve foliations given in [13, Theorem A].

The second part of Proposition 4.2 generalizes, to dicritical context, [7, Proposition 2] stated in the non-dicritical case. We give a numerical illustration of Proposition 4.2.

**Example 4.3.** Let $\mathcal{F}$ be the foliation at $(\mathbb{C}^2, 0)$ defined by $\omega = xdy - ydx$. Observe that $\mathcal{F}$ is a dicritical generalized curve foliation called the *radial foliation*. It has only
one dicritical component whose valence is 0 in its reduction process of singularities. Thus

$$B = (x) + (y) + (x - y) - (x + y)$$

is a reduced balanced divisor of separatrices for $F$, where $B_0 : xy(x - y) = 0$ and $B_{\infty} : x + y = 0$. We get $\mu_0(B_0) = 4$, $\nu_0(B_0) = 3$, $i_0(B_0, B_{\infty}) = 3$ and $i_0(PF, B_0) = 3$. Observe that if we consider the reduced balanced divisor of separatrices $B = (x) + (y)$ for $F$ then $B_{\infty}$ is a unit, so $i_p(B_0, B_{\infty}) = 0$ and now $\mu_0(B_0) = 1$, $\nu_0(B_0) = 2$. On the other hand, if we consider the foliation $F$ with a saddle-node (so $F$ is not a generalized curve foliation) and equation as in (2.4) we get again $B = (x) + (y)$ but $i_p(PF, B_0) = k + 2 \neq 2$.

**Lemma 4.4.** Let $F$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $B$ be a balanced divisor of separatrices for $F$. Then

$$i_p(PF, B) = \mu_p(F) + \nu_p(F) - \sum_{q \in I_p(F)} \nu_q(F) \xi_q(F),$$

where the summation runs over all infinitely near points of $F$ at $p$.

**Proof.** Let $PF$ be a generic polar. Denote by $\Gamma(PF)$ the set of irreducible components of $PF$. It follows from [4, Lemma 4.2] that

$$i_p(PF, B) = \sum_{A \in \Gamma(PF)} i_p(A, B)$$

$$= \sum_{A \in \Gamma(PF)} \left( \text{tang}_p(F, A) - \sum_{q \in I_p(F)} \nu_q(A) \xi_q(F) + 1 \right)$$

$$= \sum_{A \in \Gamma(PF)} \left( \text{tang}_p(F, A) + 1 \right) - \sum_{A \in \Gamma(PF)} \left( \sum_{q \in I_p(F)} \nu_q(A) \xi_q(F) \right)$$

$$= \sum_{A \in \Gamma(PF)} \left( \text{tang}_p(F, A) + 1 \right) - \sum_{q \in I_p(F)} \left( \sum_{A \in \Gamma(PF)} \nu_q(A) \right) \xi_q(F)$$

(4.6)

$$= \sum_{A \in \Gamma(PF)} \left( \text{tang}_p(F, A) + 1 \right) - \sum_{q \in I_p(F)} \nu_q(PF) \xi_q(F).$$

According to the proof of [7, Proposition 2], we have

$$\sum_{A \in \Gamma(PF)} \left( \text{tang}_p(F, A) + 1 \right) = \mu_p(F) + \nu_p(F),$$
and by [7, Remark 1] $\nu_q(PF) = \nu_q(F)$. Substituting these terms in the equation (5.5), we obtain

$$i_p(PF, B) = \mu_p(F) + \nu_p(F) - \sum_{q \in I_p(F)} \nu_q(F)\xi_q(F).$$

□

Lemma 4.4 improves [7, Proposition 2] determining explicitly the difference between the polar intersection number with respect to a balanced divisor of separatrices $B$ and the sum of the Milnor number and the algebraic multiplicity of the foliation $F$ and generalizing the result to dicritical foliations. On the other hand comparing Lemma 4.4 and [4, Proposition 4.3] (proved for complex analytic foliations, but it also holds for formal foliations) we conclude that

$$\sum_{q \in I_p(F)} \nu_q(F)\xi_q(F) = \sum_{q \in I_p(F)} \nu_q(B)\xi_q(F),$$

for any $B$ which is not an $F$-invariant curve. Hence the sum $\sum_{q \in I_p(F)} \nu_q(F)\xi_q(F)$ coincides with the tangency excess of $F$ along any irreducible curve which is not an $F$-invariant curve, introduced in [4, equality (8)]. In particular, after the definition of the $\chi$-number, this tangency excess equals $\chi_p(F) + \xi_p(F)$.

Let $F$ be a singular foliation at $(\mathbb{C}^2, p)$ induced by the vector field $v$ and $B$ be a separatrix of $F$. Let $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, p)$ be a primitive parametrization of $B$, we can consider the multiplicity of $F$ along $B$ at $p$ defined by

$$(4.7) \quad \mu_p(F, B) = \text{ord}_t \theta(t),$$

where $\theta(t)$ is the unique vector field at $(\mathbb{C}, 0)$ such that $\gamma_\ast \theta(t) = v \circ \gamma(t)$, see for instance [5, page 159]. If $\omega = P(x, y)dx + Q(x, y)dy$ is a 1-form inducing $F$ and $\gamma(t) = (x(t), y(t))$, we get

$$(4.8) \quad \theta(t) = \begin{cases} -\frac{Q(\gamma(t))}{x'(t)} & \text{if } x(t) \neq 0 \\ \frac{P(\gamma(t))}{y'(t)} & \text{if } y(t) \neq 0. \end{cases}$$

Hence, by taking order, we obtain

$$(4.9) \quad \mu_p(F, B) = \begin{cases} \text{ord}_t Q(\gamma(t)) - \text{ord}_t x(t) + 1 & \text{if } x(t) \neq 0; \\ \text{ord}_t P(\gamma(t)) - \text{ord}_t y(t) + 1 & \text{if } y(t) \neq 0. \end{cases}$$
The following proposition has been proved in [7, Proposition 1] for non-dicritical foliations, but we may check that it is valid to dicritical foliations.

**Proposition 4.5.** Consider a separatrix $B$ of a singular (holomorphic or formal) foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$. We have

$$i_p(\mathcal{P}^\mathcal{F}, B) = \mu_p(\mathcal{F}, B) + \nu_p(B) - 1.$$ 

**Remark 4.6.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C = \bigcup_{j=1}^\ell C_j$ is the total union of separatrices of $\mathcal{F}$. Applying Proposition 4.5 to $\mathcal{F}$ and $df$, where $C : f(x, y) = 0$, we get $\Delta(\mathcal{F}, C_j) = \mu_p(\mathcal{F}, C_j) - \mu_p(df, C_j)$, for $j = 1, \ldots, \ell$. Since $\Delta(\mathcal{F}, C_j) \geq 0$ we have $\mu_p(\mathcal{F}, C_j) \geq \mu_p(df, C_j)$ for any separatrix $C_j$ of $\mathcal{F}$.

As consequence of Lemma 4.4 and Proposition 4.5 we obtain a generalization of [7, Corollary 2].

**Proposition 4.7.** Let $\mathcal{F}$ be a singular (holomorphic or formal) foliation at $(\mathbb{C}^2, p)$ and let $B = \sum_B a_B B$ be a balanced divisor for separatrices of $\mathcal{F}$. We have

$$\mu_p(\mathcal{F}) = \sum_B a_B \mu_p(\mathcal{F}, B) + \chi_p(\mathcal{F}) - \deg(B) + 1.$$ 

**Proof.** By summing up polar intersection numbers over all irreducible components of $B$ and applying Proposition 4.5 we get

$$i_p(\mathcal{P}^\mathcal{F}, \mathcal{B}) = \sum_B a_B i_p(\mathcal{P}^\mathcal{F}, B) = \sum_B a_B (\mu_p(\mathcal{F}, B) + \nu_p(B) - 1)$$

$$= \sum_B a_B \mu_p(\mathcal{F}, B) + \left(\sum_B a_B \nu_p(B)\right) - \deg(B)$$

$$= \sum_B a_B \mu_p(\mathcal{F}, B) + \nu_p(B) - \deg(B).$$ 

From Lemma 4.4, Proposition 2.4 and the definition of the $\chi$-number of $\mathcal{F}$ we get

$$\mu_p(\mathcal{F}) = \sum_B a_B \mu_p(\mathcal{F}, B) + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F}) \xi_q(\mathcal{F}) - \nu_p(\mathcal{F}) + \nu_p(B) - \deg(B)$$

$$= \sum_B a_B \mu_p(\mathcal{F}, B) + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F}) \xi_q(\mathcal{F}) - \xi_p(\mathcal{F}) - \deg(B) + 1$$

$$= \sum_B a_B \mu_p(\mathcal{F}, B) + \chi_p(\mathcal{F}) - \deg(B) + 1.$$

□
Let $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial, where the origin is an isolated singular point of the hypersurface $f^{-1}(0)$. The notion of the Milnor number $\mu(f)$ was introduced in [22, Section 7] as the degree of the mapping $z \rightarrow \frac{\nabla f}{||\nabla f||}$, where $\nabla$ denotes the gradient function. In particular, for complex plane curves, Milnor proved, using topological tools, the purely algebraic equality $\mu(f) = 2\delta(f) - r(f) + 1$, where $\delta(f)$ is the number of double points and $r(f)$ is the number of irreducible factors of $f$ (see [22, Theorem 10.5]). The reader can find further formulae for the Milnor number of a plane curve in [31, Section 6.5]). In particular in [31, Theorem 6.5.1]) it was established the relationship between the Milnor number of a reduced plane curve and the Milnor numbers of its irreducible components. The ingredients of the proof of Wall are Milnor fibrations and the Euler characteristic. We give another proof of this relationship, using foliations:

**Proposition 4.8.** Let $C : f(x, y) = 0$ be a germ of reduced singular curve at $(\mathbb{C}^2, p)$. Assume that $C = \bigcup_{j=1}^\ell C_j$ is the decomposition of $C$ in irreducible components $C_j : f_j(x, y) = 0$, where $f(x, y) = f_1(x, y) \cdots f_\ell(x, y)$. Then

$$\mu_p(C) + \ell - 1 = \sum_{j=1}^\ell \mu_p(C_j) + 2 \sum_{1 \leq i < j \leq \ell} i_p(C_i, C_j).$$

**Proof.** Applying Proposition 4.7 to the foliation defined by $\omega = df$ and to the balanced divisor of separatrices $C = \sum_{j=1}^\ell C_j$ we have

$$\mu_p(C) + \ell - 1 = \sum_{j=1}^\ell \mu_p(df, C_j).$$

It follows from Proposition 4.5 that

$$\mu_p(df, C_j) = i_p(P^{df}, C_j) - \nu_p(C_j) + 1, \quad \text{for } j = 1, \ldots, \ell.$$

Using properties on intersection numbers, we have

$$i_p(P^{df}, C_j) = i_p(P^{df}, C_j) + \sum_{i \neq j} i_p(C_i, C_j).$$

From Teissier’s Proposition [28, Chapter II, Proposition 1.2], we get

$$i_p(P^{df}, C_j) = \mu_p(C_j) + \nu_p(C_j) - 1.$$

Thus

$$\mu_p(df, C_j) = \mu_p(C_j) + \sum_{i \neq j} i_p(C_i, C_j).$$

(4.11)
The proof ends, by substituting (4.11) into (4.10).

**Theorem A.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}) - \sum_B a_B \mu_p(dF_B, B) + \deg(\mathcal{B}) - 1 - \chi_p(\mathcal{F}),$$

where $F_B$ is a balanced divisor of separatrices for $\mathcal{F}$ adapted to $B$. Moreover, if $\mathcal{F}$ is of a second type foliation, then

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}) - \sum_B a_B \mu_p(dF_B, B) + \deg(\mathcal{B}) - 1.$$

**Proof.** By (4.5) and (4.4)

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B \Delta_p(\mathcal{F}, B) = \sum_B a_B \left( i_p(\mathcal{P}^\mathcal{F}, B) - i_p(\mathcal{P}^{dF_B}, B) \right)$$

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B i_p(\mathcal{P}^{dF_B}, B).$$

Hence, after Lemma 4.4 and Proposition 4.5 we have

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F}) \xi_q(\mathcal{F}) - \sum_B a_B \left( \mu_p(dF_B, B) + \nu_p(B) - 1 \right)$$

$$= \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F}) \xi_q(\mathcal{F}) - \left( \sum_B a_B \mu_p(dF_B, B) \right) - \nu_p(\mathcal{B}) + \deg(\mathcal{B}).$$

We finish the proof after Proposition 2.4 and the definition of the $\chi$-number of $\mathcal{F}$. On the other hand if $\mathcal{F}$ is a second type foliation then $\chi_p(\mathcal{F}) = 0$ and the second part of the theorem follows.

From Theorem A and Proposition 4.7 we get:

**Corollary 4.9.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B\left( \mu_p(\mathcal{F}, B) - \mu_p(dF_B, B) \right),$$

where $F_B$ is a balanced divisor of separatrices for $\mathcal{F}$ adapted to $B$.

Corollary 4.9 restricted to non-dicritical singular foliations provides us a new characterization of non-dicritical generalized curve foliations.
Corollary 4.10. Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C = \bigcup_{j=1}^{\ell} C_j$ is the total union of separatrices of $\mathcal{F}$. Then $\mathcal{F}$ is a generalized curve foliation if and only if
\[ \mu_p(\mathcal{F}, C_j) = \mu_p(df, C_j), \quad \text{for all } j = 1, \ldots, \ell, \]
where $f(x, y) = 0$ is a reduced equation of $C$ at $p$.

Proof. According to [13, Theorem A], $\mathcal{F}$ is a generalized curve foliation at $(\mathbb{C}^2, p)$ if and only if $\Delta_p(\mathcal{F}, C) = 0$, where $C$ is the total union of separatrices of $\mathcal{F}$. It follows from Corollary 4.9 that $\Delta_p(\mathcal{F}, C) = \sum_{j=1}^{\ell} (\mu_p(\mathcal{F}, C_j) - \mu_p(df, C_j))$, where $f(x, y) = 0$ is a reduced equation of $C$ at $p$. Thus, by Remark 4.6, $\Delta_p(\mathcal{F}, C) = 0$ if and only if $\mu_p(\mathcal{F}, C_j) = \mu_p(df, C_j)$ for all $j = 1, \ldots, \ell$. □

Remark 4.11. Observe that if, in Corollary 4.10, the curve $C : f(x, y) = 0$ is irreducible then we rediscover the classic characterization of generalized curve foliations, that is, $\mu_p(\mathcal{F}) = \mu_p(df) = \mu_p(C)$.

5. The Gómez-Mont-Seade-Verjovsky index

Let $\mathcal{F} : \omega = 0$ be a singular foliation at $(\mathbb{C}^2, p)$. Let $C : f(x, y) = 0$ be an $\mathcal{F}$-invariant curve, where $f(x, y) \in \mathbb{C}[[x, y]]$ is reduced. Then, as in the convergent case, there are $g, h \in \mathbb{C}[[x, y]]$ (depending on $f$ and $\omega$), with $f$ and $g$ and $h$ relatively prime and a 1-form $\eta$ (see [26, Lemma 1.1 and its proof]) such that
\[ g \omega = h df + f \eta. \]

The Gómez-Mont-Seade-Verjovsky index of the foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ (GSV-index) with respect to an analytic $\mathcal{F}$-invariant curve $C$ is
\[ \text{GSV}_p(\mathcal{F}, C) = \frac{1}{2\pi i} \int_{\partial C} \frac{g}{h} d \left( \frac{h}{g} \right), \]
where $g, h \in \mathbb{C}\{x, y\}$ are from (5.1). This index was introduced in [15] but here we follow the presentation of [2]. If $C$ is irreducible then equality (5.2) becomes
\[ \text{GSV}_p(\mathcal{F}, C) = \text{ord}_t \left( \frac{h}{g} \circ \gamma \right)(t) = i_p(f, h) - i_p(f, g), \]
where $\gamma(t)$ is a Puiseux parametrization of $C$. By [2, page 532], we get the adjunction formula
\[ \text{GSV}_p(\mathcal{F}, C_1 \cup C_2) = \text{GSV}_p(\mathcal{F}, C_1) + \text{GSV}_p(\mathcal{F}, C_2) - 2i_p(C_1, C_2), \]
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for any two analytic $F$-invariant curves, $C_1$ and $C_2$, without common irreducible components.

The equality (5.3) allows us to extend the definition of the GSV-index to a purely formal (non-analytic) irreducible $F$-invariant curve; and the equality (5.4) allows us to extend the definition of the GSV-index to $F$-invariant curves containing purely formal branches.

The following lemma generalizes the equality (5.3) to any reduced $F$-invariant curve (containing perhaps purely formal branches):

Lemma 5.1. Let $C : f(x, y) = 0$ be any reduced invariant curve of a singular foliation $F$ at $(\mathbb{C}^2, p)$. Then

$$GSV_p(F, C) = i_p(f, h) - i_p(f, g),$$

where $g, h \in \mathbb{C}[[x, y]]$ are from (5.1).

Proof. Suppose, without lost of generality, that $f(x, y) = f_1(x, y)f_2(x, y)$, where $f_1, f_2 \in \mathbb{C}[[x, y]]$ are irreducible and put $C_i : f_i(x, y) = 0$ for $1 \leq i \leq 2$. By (5.1) we get

$$g \omega = hf_2df_1 + f_1(hdf_2 + f_2 \eta),$$

for some $g, h \in \mathbb{C}[[x, y]]$ relative prime with $f$ and a 1-form $\eta$. Hence, if $\gamma_1(t)$ is a Puiseux parametrization of $C_1$, then after (5.3) we have

$$GSV_p(F, C_1) = \text{ord}_t \left( \frac{hf_2}{g} \circ \gamma_1 \right)(t) = \text{ord}_t \left( \frac{h}{g} \circ \gamma_1 \right)(t) + \text{ord}_t (f_2 \circ \gamma_1)(t)$$

$$= \text{ord}_t \left( \frac{h}{g} \circ \gamma_1 \right)(t) + i_p(C_1, C_2).$$

Similarly, if $\gamma_2(t)$ denotes a Puiseux parametrization of $C_2$ then we have

$$GSV_p(F, C_2) = \text{ord}_t \left( \frac{h}{g} \circ \gamma_2 \right)(t) + i_p(C_1, C_2).$$

The proof follows after equality (5.4) and properties of the intersection number. □

In this section, we will use the following result due to Genzmer-Mol [12 Theorem B] that establishes a relationship between the GSV-index and the polar excess number of a foliation with respect to a set of separatrices.
**Theorem 5.2.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Let $C$ be a reduced curve of separatrices and $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a balanced divisor of separatrices for $\mathcal{F}$ adapted to $C$. Then

$$GSV_p(\mathcal{F}, C) = \Delta_p(\mathcal{F}, C) + i_p(C, \mathcal{B}_0 \setminus C) - i_p(C, \mathcal{B}_\infty).$$

We note that the above theorem implies that if $\mathcal{B}$ is an effective balanced divisor of separatrices for $\mathcal{F}$, then $GSV_p(\mathcal{F}, \mathcal{B}) = \Delta_p(\mathcal{F}, \mathcal{B})$.

On the other hand, as a consequence of Proposition 4.2 and Theorem 5.2 we get

**Corollary 5.3.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Let $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a reduced balanced divisor of separatrices for $\mathcal{F}$. Then

$$GSV_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}^\mathcal{F}, \mathcal{B}_0) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_0) + 1.$$

**Proof.** By applying Theorem 5.2 to $C = \mathcal{B}_0$, we have

$$(5.5) \quad GSV_p(\mathcal{F}, \mathcal{B}_0) = \Delta_p(\mathcal{F}, \mathcal{B}_0) - i_p(\mathcal{B}_0, \mathcal{B}_\infty),$$

and it follows from Proposition 4.2 that

$$(5.6) \quad \Delta_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}^\mathcal{F}, \mathcal{B}_0) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_0) + 1.$$ 

The proof ends, by substituting (5.6) in (5.5). □

**Proposition 5.4.** Let $C : f(x, y) = 0$ be any reduced invariant curve of a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$. Then

$$GSV_p(\mathcal{F}, C) = i_p(\mathcal{P}^\mathcal{F}, C) - i_p(\mathcal{P}^df, C).$$

**Proof.** Let $\omega = P(x, y)dx + Q(x, y)dy$ be a 1-form inducing $\mathcal{F}$. By equality (5.1) we get $g\omega = hdf + f\eta$, where $\eta$ is a formal 1-form and $g, h \in \mathbb{C}[[x, y]]$ with $g$ and $f$ relatively prime and $h$ and $f$ also relatively prime. Let $(a : b) \in \mathbb{P}^1$ such that the polar curves $aP(x, y) + bQ(x, y) = 0$ and $a\partial_x f + b\partial_y f = 0$ of $\mathcal{F}$ and $df$ respectively, are generic. We have $g \cdot (aP + bQ) = h \cdot (a\partial_x f + b\partial_y f) + fk$, for some $k \in \mathbb{C}[[x, y]]$. Then $i_p(f, g \cdot (aP + aQ)) = i_p(f, h \cdot (a\partial_x f + b\partial_y f) + fk) = i_p(f, h \cdot (a\partial_x f + b\partial_y f))$. So $i_p(f, g \cdot (aP + aQ)) = i_p(f, h) + i_p(f, a\partial_x f + b\partial_y f)$.

On the other hand $i_p(f, g \cdot (aP + bQ)) = i_p(f, g) + i_p(f, aP + bQ)$, hence

$$(5.7) \quad i_p(f, g) + i_p(f, aP + bQ) = i_p(f, h) + i_p(f, a\partial_x f + b\partial_y f).$$

We finish the proof using Lemma 5.1 and equality (5.7). □
Remark 5.5. If $\mathcal{F}$ is a non-dicritical singular foliation at $(\mathbb{C}^2, p)$, where $C$ is the total union of separatrices of $\mathcal{F}$ then $i_p(\mathcal{P}^F, f) - i_p(\mathcal{P}^{df}, f) = \Delta_p(\mathcal{F}, C)$, but in general this two values are different as the following example shows: consider the foliation $\mathcal{F}$ defined by $\omega = 2xdy - 3ydx$ and the curve $C : y^2 - x^3 = 0$. Note that $\mathcal{F}$ admits the meromorphic first integral $y^2/x^3$ and so that $C$ is $F$-invariant. We get $i_0(\mathcal{P}^F, C) - i_0(\mathcal{P}^{df}, C) = -1$, and $\Delta_0(\mathcal{F}, C) = 0$, since $\mathcal{F}$ is a generalized curve foliation.

We obtain the following corollary.

**Corollary 5.6.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C$ is the total union of separatrices of $\mathcal{F}$. Then

$$\text{GSV}_p(\mathcal{F}, C) = \mu_p(\mathcal{F}) - \mu_p(C) - \chi_p(\mathcal{F}).$$

**Proof.** By Proposition 5.4 we have $\text{GSV}_p(\mathcal{F}, C) = i_p(\mathcal{P}^F, C) - i_p(\mathcal{P}^{df}, C)$ and applying Lemma 4.4 to $\mathcal{F}$ and $C$, we get $i_p(\mathcal{P}^F, C) = \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q \in I_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F})$. Teissier’s Proposition [28, Chapter II, Proposition 1.2] implies that

$$i_p(\mathcal{P}^{df}, C) = \mu_p(C) + \nu_p(C) - 1.$$

Thus

$$\text{GSV}_p(\mathcal{F}, C) = i_p(\mathcal{P}^F, C) - i_p(\mathcal{P}^{df}, C) = \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q \in I_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) - (\mu_p(C) + \nu_p(C) - 1) = \mu_p(\mathcal{F}) - \mu_p(C) + \underbrace{(\nu_p(\mathcal{F}) - \nu_p(C) + 1)}_{\xi_p(\mathcal{F})} - \sum_{q \in I_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) = \mu_p(\mathcal{F}) - \mu_p(C) - \chi_p(\mathcal{F}).$$

To finish this section we state a relationship between the GSV-index and the multiplicity of $\mathcal{F}$ along a fixed separatrix.
Proposition 5.7. Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and $B : f(x, y) = 0$ be a separatrix of $\mathcal{F}$. Then

\begin{equation}
\mu_p(\mathcal{F}, B) = \begin{cases} 
\text{GSV}_p(\mathcal{F}, B) + \text{ord}_t \partial_y f(\gamma(t)) - \text{ord}_t x(t) + 1 & \text{if } x(t) \neq 0; \\
\text{GSV}_p(\mathcal{F}, B) + \text{ord}_t \partial_x f(\gamma(t)) - \text{ord}_t y(t) + 1 & \text{if } y(t) \neq 0,
\end{cases}
\end{equation}

where $\gamma(t) = (x(t), y(t))$ is a Puiseux parametrization of $B$. In particular, if $B$ is a non-singular separatrix, then $\mu_p(\mathcal{F}, B) = \text{GSV}_p(\mathcal{F}, B)$.

Proof. Let $\omega = P(x, y)dx + Q(x, y)dy$ be a 1-form inducing $\mathcal{F}$ and $f(x, y) = 0$ be a reduced equation of $B$. By equality (5.1), we get

\[ g\omega = h\partial_y f + f\eta, \]

where $\eta$ is a formal 1-form and $g, h \in \mathbb{C}[[x, y]]$, where $g$ and $f$ are relatively prime and $h$ and $f$ are relatively prime. From equality (4.8), we have that the unique vector field $\theta(t)$ such that $\gamma^* \theta(t) = v(\gamma(t))$, where $v = -Q(x, y)\partial_x + P(x, y)\partial_y$, is given by

\begin{equation}
\theta(t) = \begin{cases} 
-\left(\frac{h}{g}\right) (\gamma(t))\partial_y f(\gamma(t)) & \text{if } x(t) \neq 0; \\
\frac{h}{g} (\gamma(t))\partial_x f(\gamma(t)) & \text{if } y(t) \neq 0.
\end{cases}
\end{equation}

Therefore, the proof follows taking orders.

6. **Tjurina number**

Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ defined by the 1-form $\omega = P(x, y)dx + Q(x, y)dy$ and $C : f(x, y) = 0$ be a $\mathcal{F}$-invariant reduced curve. The **Tjurina number** of $\mathcal{F}$ with respect to $C$ is

\[ \tau_p(\mathcal{F}, C) = \dim_\mathbb{C} \mathbb{C}[[x, y]]/(f, P, Q), \]

The **Tjurina number** of any germ of reduced curve $C : f(x, y) = 0$, with $f(x, y) \in \mathbb{C}[[x, y]]$ is by definition

\[ \tau_p(C) = \dim_\mathbb{C} \mathbb{C}[[x, y]]/(f, \partial_x f, \partial_y f). \]

In this section we will study the Tjurina number of a foliation with respect to a balanced divisor of separatrices. First of all we present a lemma on Commutative Algebra which we need on the next, but we do not know a precise reference:
Lemma 6.1. Let \( f, g, p, q \in \mathbb{C}[x, y] \), where \( f \) and \( g \) are relatively prime. Then
\[
\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, gp, gq) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, p, q) + \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, g).
\]

Proof. Observe that \( \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, r_1, \ldots, r_n) = \dim_{\mathbb{C}} \mathcal{O}/(r_1^i, \ldots, r_n^i) \), where \( \mathcal{O} = \mathbb{C}[[x, y]]/(f) \) and \( r_i^i = r_i + (f) \) for any \( i \in \{1, \ldots, n\} \) and any \( r_i \in \mathbb{C}[[x, y]] \). We finish the proof using the following exact sequence:
\[
0 \to \mathcal{O}/(p', q') \xrightarrow{\sigma} \mathcal{O}/(g'p', g'q') \xrightarrow{\delta} \mathcal{O}/(g') \to 0,
\]
where \( \sigma(z' + (p', q')) = g'z' + (g'p', g'q') \) and \( \delta(z' + (g'p', g'q')) = z' + (g') \), for any \( z' \in \mathcal{O} \). \( \square \)

The following proposition has been proved by X. Gómez-Mont \cite{[1]} Theorem 1] for a foliation with a set of convergent separatrices. We show that the same result holds in the formal context for effective reduced balanced divisor of separatrices.

Proposition 6.2. Let \( \mathcal{F} \) be a singular foliation at \((\mathbb{C}^2, p)\) and \( C \) be a reduced curve of separatrices of \( \mathcal{F} \). Then
\[
\tau_p(\mathcal{F}, C) - \tau_p(C) = GSV_p(\mathcal{F}, C).
\]

Proof. Let \( \omega = P(x, y)dx + Q(x, y)dy \) be a 1-form inducing \( \mathcal{F} \) and \( f(x, y) = 0 \) be the reduced equation of \( C \). By equality (5.1) we get \( g\omega = hdf + f\eta \), where \( \eta \) is a formal 1-form and \( g, h \in \mathbb{C}[[x, y]] \) with \( g \) and \( f \) relatively prime and \( h \) and \( f \) also relatively prime.

Hence \( gPdx + gQdy = (h\partial_x f + f\eta_x)dx + (h\partial_y f + f\eta_y)dy \), where \( \eta = \eta_x dx + \eta_y dy \). We get
\[
(6.1) \quad gP = h\partial_x f + f\eta_x, \quad \text{and} \quad gQ = h\partial_y f + f\eta_y.
\]

After equalities (6.1), properties of the intersection number and Lemma 6.1, we have
\[
\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, gP, gQ) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h\partial_x f + f\eta_x, h\partial_y f + f\eta_y)
= \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h\partial_x f, h\partial_y f)
= \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h) + \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, \partial_x f, \partial_y f)
= \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h) + \tau_p(C).
\]

Again, by Lemma 6.1 we get
\[
\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, P, Q) + \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, g) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h) + \tau_p(C).
\]

Hence \( \tau_p(\mathcal{F}, C) - \tau_p(C) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h) - \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, g) = i_p(f, h) - i_p(f, g) \). The proof follows from Lemma 5.1. \( \square \)
Example 6.3. Let $\mathcal{F}$ be the foliation defined by the formal normal form of a saddle-node (see (2.4)) $\omega = x^{k+1} dy - y(1 + \lambda x^k) dx, \quad k \geq 1, \quad \lambda \in \mathbb{C}$. The total union of separatrices of $\mathcal{F}$ is $C = C_1 \cup C_2$, where $C_1 : x = 0$ (strong separatrix) and $C_2 : y = 0$ (weak separatrix). An equality (5.1) for $C_1$ is given for $g = 1$, $h = -y(1 + \lambda x^k)$ and $\eta = x^k dy$, hence by Lemma 5.3 we get $\operatorname{GSV}_0(\mathcal{F}, C_1) = i_0(x, h) - i_0(x, g) = 1$.

Similarly, an equality (5.1) for $C_2$ is given for $g = 1$, $h = x^{k+1}$ and $\eta = -(1 + \lambda x^k) dx$, thus $\operatorname{GSV}_0(\mathcal{F}, C_2) = i_0(y, h) - i_0(y, g) = k + 1$. Therefore, one finds

$$\operatorname{GSV}_0(\mathcal{F}, C) = \operatorname{GSV}_0(\mathcal{F}, C_1) + \operatorname{GSV}_0(\mathcal{F}, C_2) - 2 i_0(C_1, C_2) = 1 + (k + 1) - 2 = k.$$

On the other hand, we get $\tau_0(\mathcal{F}, C) - \tau_0(C) = (k + 1) - 1 = k = \operatorname{GSV}_0(\mathcal{F}, C)$.

Corollary B. Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C$ is the total union of separatrices of $\mathcal{F}$. Then

$$\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C) + \chi_p(\mathcal{F}).$$

Moreover, if $\mathcal{F}$ is of second type then $\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C)$.

Proof. It is a consequence of Proposition 6.2 and Corollary 5.6. □

Now, we characterize generalized curve foliations at $(\mathbb{C}^2, p)$ in terms of its Tjurina numbers.

Corollary 6.4. Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and $B = B_0 - B_{\infty}$ be a reduced balanced divisor of separatrices for $\mathcal{F}$. Then $\mathcal{F}$ is a generalized curve if and only if $\tau_p(B_0) - \tau_p(\mathcal{F}, B_0) = i_p(B_0, B_{\infty})$.

Proof. It follows from [13] Theorem A] that $\mathcal{F}$ is a generalized curve foliation if and only if $\Delta_p(\mathcal{F}, B_0) = 0$. Applying Theorem 5.2 to $C = B_0$, we get $\operatorname{GSV}_p(\mathcal{F}, B_0) = \Delta_p(\mathcal{F}, B_0) - i_p(B_0, B_{\infty})$. Hence $\mathcal{F}$ is a generalized curve foliation if and only if $\operatorname{GSV}_p(\mathcal{F}, B_0) = -i_p(B_0, B_{\infty})$. The proof ends, by applying Proposition 6.2 to $C = B_0$. □

If $B = \sum_B a_B B$ is a divisor of separatrices for $\mathcal{F}$ then we put

$$T_p(\mathcal{F}, B) = \sum_B a_B \tau_p(\mathcal{F}, B).$$

The following theorem gives a relationship between the Milnor and Tjurina numbers and the $\chi$-number, studied in Section 3.
**Theorem C.** Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2,p)$ and let $B = \sum B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then

$$
\mu_p(\mathcal{F}) - T_p(\mathcal{F}, B) = \sum_B a_B [\mu_p(dF_B, B) - \tau_p(B)] - \deg(B) + 1 + \chi_p(\mathcal{F}) - \sum_B a_B [i_p(B, (F_B)_0 \setminus B) - i_p(B, (F_B)_\infty)],
$$

where $F_B$ is a balanced divisor of separatrices for $\mathcal{F}$ adapted to $B$.

**Proof.** By Proposition 6.2 we get $T_p(\mathcal{F}, B) = \sum_B a_B (GSV_p(\mathcal{F}, B) + \tau_p(B))$. Then $T_p(\mathcal{F}, B) - \sum_B a_B \tau_p(B) = \sum_B a_B GSV_p(\mathcal{F}, B)$. From Theorem 5.2 we have

$$
T_p(\mathcal{F}, B) - \sum_B a_B \tau_p(B) = \sum_B a_B [\Delta_p(\mathcal{F}, B) + i_p(B, (F_B)_0 \setminus B) - i_p(B, (F_B)_\infty)]
$$

$$
= \Delta_p(\mathcal{F}, B) + \sum_B a_B [i_p(B, (F_B)_0 \setminus B) - i_p(B, (F_B)_\infty)],
$$

where $F_B$ is a balanced divisor of separatrices for $\mathcal{F}$ adapted to $B$. We finish the proof using Theorem A. □

In order to illustrate Theorem C we present a family of dicritical foliations that are not of second type.

**Example 6.5.** Let $\lambda \in \mathbb{C}$ and $k \geq 3$ integer. Let $\mathcal{F}_k$ be the singular foliation at $(\mathbb{C}^2, 0)$ defined by

$$
\omega_k = y(2x^{2k-2} + 2(\lambda + 1)x^2 y^{k-2} - y^{k-1})dx + x(y^{k-1} - (\lambda + 1)x^2 y^{k-2} - x^{2k-2})dy.
$$

The foliation $\mathcal{F}_k$ is dicritical.
After one blow-up, the foliation has a unique non-reduced singularity \( q \). A further blow-up applied to \( q \) produces a reduction of singularities of the foliation with a dicritical component and a tangent saddle-node with strong separatrix transversal to exceptional divisor. Therefore \( \mathcal{F}_k \) is not of second type. Let \( B_1 : y = 0 \) and \( B_2 : x = 0 \), then \( B = B_1 + B_2 \) is an effective balanced divisor of separatrices for \( \mathcal{F}_k \).

A simple calculation leads to:

\[
\nu_0(\mathcal{F}_k) = k, \quad \nu_q(\mathcal{F}_k) = k - 1, \quad \xi_0(\mathcal{F}_k) = k - 1, \quad \xi_q(\mathcal{F}_k) = k - 1,
\]

thus \( \chi_p(\mathcal{F}_k) = 2(k - 1)^2 \). Moreover \( \mu_0(\mathcal{F}_k) = (k - 2)(2k - 2) + 5k - 4 \), \( T_0(\mathcal{F}_k, B) = 3k - 1 \), \( \tau_0(B_1) = \tau_0(B_2) = 0 \). Since \( F(x, y) = xy \) defines a balanced divisor of separatrices for \( \mathcal{F}_k \) adapted to \( B_1 \) and \( B_2 \), we get \( \mu_0(dF, B_1) + \mu_0(dF, B_2) = 2 \).

Hence we have \( T_0(\mathcal{F}_k, B) = 3k - 1 \) and Theorem [C] is verified.

**Corollary 6.6.** Let \( \mathcal{F} \) be a singular foliation at \((\mathbb{C}^2, p)\) and let \( B = \sum_B a_B B \) be a balanced divisor of separatrices for \( \mathcal{F} \). If \( \mathcal{F} \) is of second type, then

\[
\mu_p(\mathcal{F}) - T_p(\mathcal{F}, B) = \sum_B a_B [\mu_p(dF_B, B) - \tau_p(B)] - \deg(B) + 1 \]

\[
- \sum_B a_B [i_p(B, (F_B)_0 \setminus B) - i_p(B, (F_B)_\infty)],
\]

where \( F_B \) is a balanced divisor of separatrices for \( \mathcal{F} \) adapted to \( B \).
Proof. By Lemma 3.1 item (2), $\chi_p(F) = 0$ and the proof follows from Theorem C.

The following example shows that, in general, the reciprocal of Corollary 6.6 is not true.

**Example 6.7 (Dulac’s foliation).** The foliation $F$ defined by the 1-form $\omega = (ny + x^n)dx - xdy$, with $n \geq 2$, admits a unique separatrix $C : x = 0$. Since $\nu_0(F) = 1 \neq 0 = \nu_0(C) - 1$, so $F$ is not of second type. Moreover $T_0(F, C) = 1$, $\tau_0(C) = 0$, $\mu_0(F) = 1$, $\mu_0(dx, C) = 0$. Hence, $F$ verifies the equality of Corollary 6.6 but it is not a foliation of second type at $0 \in \mathbb{C}^2$.

![Figure 2. Dulac foliation $F : \omega = (ny + x^n)dx - xdy$](image)

Now, we apply Theorem C to non-dicritical singular foliations at $(\mathbb{C}^2, p)$.

**Corollary D.** Let $F$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $F$ is non-dicritical and $C = \bigcup_{j=1}^\ell C_j$ is the total union of separatrices of $F$. Then

$$\mu_p(F) - T_p(F, C) = \mu_p(C) - \sum_{j=1}^\ell \tau_p(C_j) + \chi_p(F) - \sum_{j=1}^\ell i_p(C_j, C \setminus C_j).$$

Proof. By taking the effective divisor $B = (f)$, where $f(x, y) = 0$ is an equation of $C$, and applying Proposition 4.7 to the foliation $df$, we get

$$\mu_p(C) = \sum_{j=1}^\ell \mu_p(df, C_j) - \ell + 1. \quad (6.2)$$

The proof follows from Theorem C. □
Corollary 6.8. Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C = \bigcup_{j=1}^{\ell} C_j$ is the total union of separatrices of $\mathcal{F}$. Then

$$T_p(\mathcal{F}, C) - \tau_p(\mathcal{F}, C) = \sum_{j=1}^{\ell} \tau(C_j) - \tau(C) + 2 \sum_{1 \leq i < j \leq \ell} i_p(C_i, C_j).$$

Here is an example to illustrate Corollary 6.8.

Example 6.9. Let $\omega = 4xydx + (y - 2x^2)dy$ be a 1-form defining a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$. The unique separatrix of $\mathcal{F}$ is the curve $C : y = 0$. Since $\nu_0(\mathcal{F}) = 1$ and $\nu_0(C) = 1$, the foliation $\mathcal{F}$ is not of second type at the origin (see Proposition 2.4). Moreover, we get $T_0(\mathcal{F}, C) = 2$, $\tau_0(C) = 0$, $\mu_0(\mathcal{F}) = 3$, $\mu_0(C) = 0$, and $\chi_0(\mathcal{F}) = 1$.

![Foliation $\mathcal{F}$: $\omega = 4xydx + (y - 2x^2)dy$](image-url)

**Figure 3.** Foliation $\mathcal{F}$: $\omega = 4xydx + (y - 2x^2)dy$

6.1. Milnor and Tjurina numbers and some residue-type indices. We finish this section stating numerical relationships between some classic indices, as Baum-Bott, Camacho-Sad and variational indices, of a singular foliation at $(\mathbb{C}^2, p)$ and the Milnor and Tjurina numbers.

Let $\mathcal{F}$ a singular foliation defined by a 1-form $\omega$ as in (2.2). Let $J(x,y)$ be the Jacobian matrix of $(Q(x,y), -P(x,y))$. The Baum-Bott index (see [2]) of $\mathcal{F}$ at $p$ is

$$BB_p(\mathcal{F}) = \text{Res}_p \left\{ \frac{(\text{tr}J)^2}{-P \cdot Q} \right\} dx \wedge dy,$$

where $\text{tr}J$ denotes the trace of $J$. 
The Camacho-Sad index of $\mathcal{F}$ (CS index) with respect to an analytic $\mathcal{F}$-invariant curve $C$ is

\begin{equation}
\text{CS}_p(\mathcal{F}, C) = -\frac{1}{2\pi i} \int_{\partial C} \frac{1}{h} \eta,
\end{equation}

where $g, h$ are from (5.1). The Camacho-Sad index was introduced by these authors in [6] for a non-singular $\mathcal{F}$-invariant curve $C$. Later, Lins Neto in [19] and Suwa in [26] generalize this index to singular $\mathcal{F}$-invariant curves. If $C$ is irreducible then equality (6.3) becomes

\begin{equation}
\text{CS}_p(\mathcal{F}, C) = -\text{Res}_{t=0} \left( \gamma^* \frac{1}{h} \eta \right),
\end{equation}

where $\gamma(t)$ is a Puiseux parametrization of $C$. By [3, page 38] (see also [27]) we get the adjunction formula

\begin{equation}
\text{CS}_p(\mathcal{F}, C_1 \cup C_2) = \text{CS}_p(\mathcal{F}, C_1) + \text{CS}_p(\mathcal{F}, C_2) + 2i_p(C_1, C_2),
\end{equation}

for any two analytic $\mathcal{F}$-invariant curves, $C_1$ and $C_2$, without common irreducible components. The equality (6.4) allows us to extend the definition of the CS index to a purely formal (non-analytic) irreducible $\mathcal{F}$-invariant curve; and the equality (6.5) allows us to extend the definition of the CS index to $\mathcal{F}$-invariant curves containing purely formal branches.

In a neighborhood of a non-singular point of the foliation $\mathcal{F}$, there is a 1-form $\alpha$ such that $d\omega = \alpha \wedge \omega$. If $\alpha'$ is other such 1-form, then $\alpha$ and $\alpha'$ coincide over every leaf of $\mathcal{F}$. Hence, in a neighborhood of 0 (away 0) there exists a holomorphic multi-valued 1-form $\alpha$ such that $d\omega = \alpha \wedge \omega$ and that its restriction to each leaf of $\mathcal{F}$ is single-valued. We say that $\alpha$ is an exponent form for $\omega$. The variational index or variation of $\mathcal{F}$ relative to $C$ at $p$ is

\begin{equation}
\text{Var}_p(\mathcal{F}, C) = \text{Res}_{t=0} \left( \alpha \bigg|_C \right) = \frac{1}{2\pi i} \int_{\partial C} \alpha.
\end{equation}

The variational index was introduced in [17]. It is additive:

\begin{equation}
\text{Var}_p(\mathcal{F}, C_1 \cup C_2) = \text{Var}_p(\mathcal{F}, C_1) + \text{Var}_p(\mathcal{F}, C_2),
\end{equation}

where $C_1$ and $C_2$ are $\mathcal{F}$-invariant curves without common factors. For any divisor $B = \sum_B a_B B$ of separatrices for $\mathcal{F}$ we put

\begin{equation}
\text{Var}_p(\mathcal{F}, B) = \sum_B a_B \text{Var}_p(\mathcal{F}, B).
\end{equation}
For any analytic $F$-invariant curve $C$ we have, after \cite[Proposition 5]{2},

\begin{equation}
\text{Var}_p(F,C) = \text{CS}_p(F,C) + \text{GSV}_p(F,C).
\end{equation}

**Proposition 6.10.** Let $F$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $B = \sum_B a_B B$ be a balanced divisor of separatrices for $F$. Then

\[ BB_p(F) = \text{Var}_p(F, B) + \mu_p(F) - \sum_B a_B \mu_p(dF_B, B) + \deg(B) - 1 - \chi_p(F) + \sum_{q \in I_p(F)} \xi^2_q(F), \]

where $F_B$ is a balanced divisor of separatrices for $F$ adapted to $B$.

Moreover, if $F$ is non-dicritical and $C$ is the total union of separatrices of $F$ then

\[ BB_p(F) = \text{CS}_p(F, C) + \tau_p(F, C) - \tau_p(C) + \mu_p(F) - \mu_p(C) - \chi_p(F) + \sum_{q \in I_p(F)} \xi^2_q(F). \]

**Proof.** Suppose that $F$ is a singular foliation. By \cite[Theorem 5.2]{10} we get

\[ BB_p(F) = \text{Var}_p(F, B) + \Delta_p(F, B) + \sum_{q \in I_p(F)} \xi^2_q(F), \]

where the summation runs over all infinitely near points of $F$ at $p$. The proof of the first equality follows applying Theorem A to $\Delta_p(F, B)$.

The proof of the non-dicritical case follows from the first part of the proposition (for $B = C$ and $F_{C_i} = f$, for any $1 \leq i \leq \ell$, where $f(x, y) = 0$ is an equation of $C$), equality (6.6), Proposition 6.2 and equality (6.2). \qed

### 7. Bound for the Milnor sum of an algebraic curve

Let $F$ be a holomorphic foliation on the complex projective plane $\mathbb{P}^2$. The degree of $F$ is the number of tangencies between $F$ and a generic line. Let $C$ be an algebraic $F$-invariant curve in $\mathbb{P}^2$. We say that $C$ is *non-dicritical* if every singular point of $F$ on $C$ is non-dicritical.

As an application of our previous results, we propose the following theorem:

**Theorem 7.1.** Let $F$ be a holomorphic foliation on $\mathbb{P}^2$ of degree $d$ leaving invariant a non-dicritical algebraic curve $C$ of degree $d_0$ such that for each $p \in \text{Sing}(F) \cap C$, all local branches of $\text{Sep}_p(F)$ are contained in $C$. Then

\[ \sum_{p \in C} \mu_p(C) \leq \left( \sum_{p \in \text{Sing}(F) \cap C} \mu_p(F) \right) + d_0^2 - (d + 2)d_0. \]
Moreover, if $\text{Sing}(\mathcal{F}) \subset C$, then
\[ \sum_{p \in C} \mu_p(C) \leq d^2 - d(d_0 - 1) + (d_0 - 1)^2. \]

**Proof.** Fix an arbitrary point $p \in \text{Sing}(\mathcal{F}) \cap C$. Then, by Corollary 5.6, we have
\[ \mu_p(C) = \mu_p(\mathcal{F}) - \text{GSV}_p(\mathcal{F}, C) - \chi_p(\mathcal{F}). \]
Since $\chi_p(\mathcal{F}) \geq 0$ by Proposition 3.1 item (1), we get
\[ \mu_p(C) \leq \mu_p(\mathcal{F}) - \text{GSV}_p(\mathcal{F}, C). \]
The equality \[ \sum_{p \in \text{Sing}(\mathcal{F}) \cap C} \text{GSV}_p(\mathcal{F}, C) = (d + 2)d_0 - d_0^2 \] (cf. [2, Proposition 4]) implies the inequality
\[ \sum_{p \in C} \mu_p(C) \leq \left( \sum_{p \in \text{Sing}(\mathcal{F}) \cap C} \mu_p(\mathcal{F}) \right) + d_0^2 - (d + 2)d_0. \]
On the other hand, if $\text{Sing}(\mathcal{F}) \subset C$, we use the equality \[ \sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = d^2 + d + 1 \] (cf. [3, page 28]) to finish the proof. \qed

**Remark 7.2.** Observe that the hypothesis *all local branches of Sep$_p(\mathcal{F})$ and all singularities of $\mathcal{F}$ are contained in $C$* in Theorem 7.1 implies, by [10, Proposition 6.1], that $\deg C = \deg \mathcal{F} + 2$ and $\mathcal{F}$ is a logarithmic foliation.

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