Appendix

Each measurement in a non-computational basis is equivalent to applying a unitary matrix and then measuring in the computational basis (Nielsen and Chuang, 2002). It follows that the circuit in Figure 2 is equivalent to Figure A.1, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ are unitary matrices determined based on the given non-computational basis. This Appendix shows how this one-qubit circuit is equivalent to the two-qubit quantum circuit depicted in Figure 3 for the general case where $A$ and $B$ are unitary matrices.

For the one-qubit circuit in Figure A.1, the qubit after gate $A$ will be in the state $A|0\rangle = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$.

The first measurement collapses the state to $|0\rangle$ with probability equal to $|a_{11}|^2$ (EVENT $A^+$) and to $|1\rangle$ with probability $|a_{21}|^2$ (EVENT $A^-$). For the order effect, $A^+$ would correspond to answering “Yes” to the first question, and $A^-$ would correspond to answering “No”.

If EVENT $A^+$ happens, then the output state after applying $B$ is $B|0\rangle = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$. The second measurement collapses the state to $|0\rangle$ with probability equal to $|b_{11}|^2$ (EVENT $B^+$) and collapses to $|1\rangle$ with probability equal to $|b_{21}|^2$ (EVENT $B^-$).
With the same argument for the other path we have:

| EVENTS | $A^+B^+$ | $A^+B^-$ | $A^-B^+$ | $A^-B^-$ |
|--------|---------|---------|---------|---------|
| Result State | $|00\rangle$ | $|01\rangle$ | $|10\rangle$ | $|11\rangle$ |
| Probability | $|a_{11}b_{11}|^2$ | $|a_{11}b_{21}|^2$ | $|a_{21}b_{12}|^2$ | $|a_{21}b_{22}|^2$ |

On the other hand the output of the two-qubit circuit, when the control qubit is at the top of the circuit, is $a_{11}b_{11}|00\rangle + a_{11}b_{21}|01\rangle + a_{21}b_{21}|10\rangle + a_{21}b_{11}|11\rangle$ with associated probabilities:

| Result State | $|00\rangle$ | $|01\rangle$ | $|10\rangle$ | $|11\rangle$ |
| Probability | $|a_{11}b_{11}|^2$ | $|a_{11}b_{21}|^2$ | $|a_{21}b_{21}|^2$ | $|a_{21}b_{11}|^2$ |

Since $B$ is unitary, its columns (or its rows) form an orthonormal basis (Steeb, 2006). In the case of a 2-by-2 unitary matrix, $|b_{11}|^2 + |b_{12}|^2 = |b_{11}|^2 + |b_{21}|^2 = 1$ which implies $|b_{12}|^2 = |b_{21}|^2$, and $|b_{11}|^2 + |b_{12}|^2 = |b_{21}|^2 + |b_{22}|^2 = 1$ which implies $|b_{11}|^2 = |b_{22}|^2$. It therefore follows that the final probabilities for the two circuits are the same. (Note that the intermediate probabilities are not the same.)