Statistical Entropy of Magnetic Black Holes from Near-Horizon Geometry

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Abstract

Four-dimensional magnetic black holes including dilaton and abelian gauge fields which are solutions of supergravity can also be obtained by dimensional reduction of the Einstein-Maxwell gravity in five dimensions. In the extremal case the five-dimensional solutions have horizon and their near-horizon geometry is $AdS_3 \times S^2$. In the non-extremal case the near-horizon geometry is shown to be the product of the three-dimensional Bañados-Teitelboim-Zanelli black hole and $S^2$. This allows to perform microscopic counting of statistical entropy of magnetic black holes. Exact agreement with the geometric entropy is found. The microstates responsible for statistical entropy are located in the near-horizon region.

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1 Introduction

Recently considerable attention has been paid to the problem of the microscopic calculation of the entropy of black holes by reducing the problem to counting the number of microstates of the Bañados-Teitelboim-Zanelli (BTZ) black hole [1]. The counting [2] is based on the fact that diffeomorphisms which preserve the asymptotic of the $AdS_3$ vacuum solution at spatial infinity are generated by two copies of the Virasoro algebra with a certain central charge [3]. It appears that some non-extremal solutions of the type-IIA supergravity can be reduced via the U-duality transformations to different forms, including those which under dimensional reduction yield non-extremal 4D (5D) black holes and the others which reduce to the product of the non-extremal BTZ black hole and 2 (3)-dimensional sphere parts [4]. Using these transformations it is possible to connect the geometric entropy to statistical entropy calculated by counting the number of states in a conformal theory associated with the BTZ black hole [5]. Characteristic features of these solutions are: (i) they are exact; (ii) the BTZ and the sphere parts completely decouple and are exact classical solutions.

A generalization of these results was proposed in [6] where it was argued that calculation of statistical entropy can be carried out equally well for any black hole whose near-horizon geometry is locally $AdS_3$. A realization of this idea was presented in [6, 7], where it was shown that there exist compactifications of heterotic string theory which yield supersymmetric solutions containing $AdS_3$ as a factor. An explicit example of such construction was presented also in [7] where, starting from a configuration of 5M-branes, the geometric entropy of 4D black holes was related to the entropy of the BTZ black hole by noting that by dimensional reduction the near-horizon region of the solution transforms to the product $BTZ \times S^2$. In this case, in contrast to the solutions obtained by means of the U-duality transformations or the heterotic theory compactifications [9], the $BTZ \times S^2$ form of the solution is only approximate in a sense that it is valid only in the near-horizon region ($Q/r << 1$).

In the present paper we discuss the well-studied example of 4D dilaton gravity coupled to an abelian gauge field. The equations of motion admit a magnetically charged black hole; both extremal and non-extremal solutions are available [8]. In the extremal case, the metric and dilaton are singular at the horizon. However, a remarkable property of the solution is that, for certain values of the dilaton coupling, the singularity is resolved by reinterpreting the solution as an object in a higher-dimensional spacetime [8]. Moreover, it appears that in a neighborhood of the horizon, the higher-dimensional metric is of the form $AdS_3 \times S^2$. In the non-extremal case, similar reinterpretation of the four-dimensional solution in terms of a higher-dimensional one shows that in a neighborhood of the horizon the geometry is of the form $BTZ \times S^2$. This suggests a possibility to compare the Bekenstein-Hawking entropy of the (non-extremal) 4D magnetic black hole with statistical entropy obtained by counting states in a conformal theory associated with the BTZ black hole.
2 Magnetic black hole solution

A class of magnetic black hole solutions is obtained by solving the equations of motion of the action
\[ S_4 = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left( R^{(4)} - 2(\partial \phi)^2 - e^{-2\alpha} F_2^2 \right), \]  
where \( a = \sqrt{\frac{p}{p+2}} \), \( p = 0, 1, \ldots \).

We shall be interested in the following solution
\[ ds_4^2 = -(1 - \frac{r_+}{r})(1 - \frac{r_-}{r})^{1/2}dt^2 + (1 - \frac{r_+}{r})^{-1}(1 - \frac{r_-}{r})^{-1/2}dr^2 + r^2(1 - \frac{r_-}{r})^{1/2}d\Omega_2^2 \]
\[ e^{\alpha \phi} = (1 - \frac{r_-}{r})^{-1/4} \]
\[ F_2 = Q \epsilon_2 \]
\[ Q^2 = \frac{3}{4} r_- r_+ . \]

Here \( r_+ \) and \( r_- \) are the outer and inner horizons, the extremal limit being \( r_+ = r_- = \mu \) and \( \epsilon_2 \) is the volume two-form on the unit 2-sphere, \( a = \frac{1}{\sqrt{3}} \).

For \( a = \frac{1}{\sqrt{3}} \) the solution (3) can be also obtained by dimensional reduction of a solution of the pure Einstein-Maxwell action in five dimensions.

\[ S_5 = \frac{1}{16\pi G_5} \int d^5\sqrt{-g} \left( R^{(5)} - F_2^{(5)2} \right) \]

This action admits a solution of the form
\[ ds_5^2 = -(1 - \frac{r_+}{r})dt^2 + (1 - \frac{r_-}{r})dy^2 + (1 - \frac{r_+}{r})^{-1}(1 - \frac{r_-}{r})^{-1}dr^2 + r^2d\Omega_2^2 \]
\[ F_2^{(5)} = Q \epsilon_2, \]
where \( Q \) and \( \epsilon_2 \) are the same as in (2). Introducing a new variable \( \omega \) according to \( 1 - \frac{r_+}{r} = \omega^2 \), we rewrite the metric (4) as
\[ ds_5^2 = -\omega^2 dt^2 + (\delta + \omega^2)\frac{r_-}{r_+} dy^2 + \frac{dw^2}{\delta + \omega^2} \frac{4r_+}{r_- (1 - \omega^2)^4} \frac{1}{r_+^2 (1 - \omega^2)^2} d\Omega_2^2, \]
where \( \delta = \frac{r_+}{r_-} - 1 \). In the extremal limit, \( r_+ = r_- \) and \( \delta = 0 \). In this case the near-horizon geometry is that of the \textit{AdS}_3 space (5). Analytically continuing \( \omega \) to complex values, i.e., changing \( \omega \to i\omega \), we transform (5) to
\[ ds_5^2 = -(\omega^2 - \delta)\frac{r_+}{r_-} dy^2 + \omega^2 dt^2 + \frac{dw^2}{\omega^2 - \delta} \frac{4r_+}{r_- (1 + \omega^2)^4} \frac{1}{r_+^2 (1 + \omega^2)^2} d\Omega_2^2. \]

It is seen that near the horizon \( r \approx r_+ \), i.e. at small \( \omega \), the metric (6) describes the product of the BTZ black hole with \( S^2 \), provided we interpret \( y \) as the non-compact and \( t \) as the compact coordinates. To make the correspondence with the BTZ black hole more transparent we identify
\[ \delta = M, \quad \omega^2 = \frac{r_+^2}{l^2}, \quad dy = \frac{r_+}{r_-}^{1/2} dr, \quad \omega^2 dt^2 = r_2 d\phi^2, \quad \Omega_2^2 = \frac{4r_+}{r_-} \]
and finally obtain
\[ ds_5^2 = -\frac{r_+ - ML^2}{l^2} d\tau^2 + r_2 d\phi^2 + \frac{l^2 d\tau^2}{r_+ - ML^2 (1 + \frac{r_-}{r_+})^2} + \frac{r_+^2 d\Omega_2^2}{(1 + \frac{r_-}{r_+})^2}. \]
3 Bekenstein-Hawking entropies.

Let us calculate and compare the geometric entropy of the magnetic black hole (2) with that of the BTZ black hole. Using the metric (2), we obtain for the geometric entropy the following expression:

$$S^{(4)} = \frac{4\pi r^2 (r_+^2)^{1/2} \delta^{1/2}}{4G_4}. \quad (9)$$

Here $G_4$ is the four-dimensional Newton constant. The BTZ black hole is a solution to the equations of motion of the three-dimensional gravity described by the action

$$S_3 = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} (R^{(3)} + \frac{2}{\hat{l}^2}) \quad (10)$$
yielding the entropy

$$S^{(3)} = \frac{2\pi \hat{l} M^{1/2}}{4G_3}. \quad (11)$$

For a special class of the five-dimensional metrics, to which belongs the metric (4), the Einstein-Maxwell action (3) reduces to the action (1). The correspondence requires the identification $\frac{1}{G_4} = \frac{1}{G_5} \int dy$. (12)

For the geometries which are approximately the product of a three-dimensional part and $S^2$, the five-dimensional Ricci scalar $R^{(5)}$ is the sum of $R^{(3)} + R^{(2)}$, where $R^{(2)} = 2/r^2_+$. The term $F^2_2 = (Q\epsilon_2)^2$ is equal to $\frac{3}{2} \frac{r_+ r_-}{r^2_+}$. Thus, the action (3) reduces to

$$S_5 = \frac{1}{16\pi G_5} 4\pi r^2_+ \int d^3x \sqrt{-g} \left(\frac{2}{r^2_+} - \frac{3}{2} \frac{r_+ r_-}{r^2_+}\right) \quad (13)$$

which should be compared with (10) giving

$$\frac{1}{G_3} = \frac{1}{G_5} 4\pi r^2_+. \quad (14)$$

Now let us consider the near-extremal case for which $r_+ \rightarrow 1$ and $\delta << 1$. From the identification (7) we have $\hat{l}^2 \approx 4r^2_+$. On the other hand, independent of this identification, the scale $\hat{l}$ is defined by the cosmological constant which enters the three-dimensional gravity action. From (13) in the near-extremal case we obtain

$$\frac{2}{r^2_+} \left(1 - \frac{3r_+ r_-}{4r^2_+}\right) \approx \frac{2}{\hat{l}^2}. \quad (15)$$

and thus $\hat{l}^2 \approx 4r^2_+$. The correspondence between expressions for $l^2$ and $\hat{l}^2$ further supports the self-consistency of our approach.

Finally, taking $y$ to be a compact variable on the circle of the radius $l$, we find that the geometric entropies $S^{(3)}$ and $S^{(4)}$ are equal to each other.\footnote{Although there are no a priori limitations on the radius of the circle, it would be desirable to understand the reason of this choice.}
4 Counting the black-hole microstates.

Having established the correspondence between the three and four-dimensional Bekenstein-Hawking entropies, now we address to the question of the microscopic calculation of the entropy. As above, we discuss the near-extremal case. Let us consider the near-horizon region $\sqrt{M} << \frac{r}{l} << 1$, where the configuration is approximately the product of the BTZ black hole and $S^2$. The condition for validity of the semi-classical description of the black hole $l >> G_3$ is consistent with the restriction to the near-horizon region if $\frac{l}{G_3} >> \frac{1}{\sqrt{M}} >> 1$.

A puzzle that arises in understanding the origin of the entropy of the BTZ black hole is that, although the entropy is physically associated with the near-horizon degrees of freedom, it can be calculated from the Virasoro algebra connected with the diffeomorphisms that preserve the asymptotic behavior of the metric at large $r$ [6]. This could have seriously damaged our discussion because on the one hand, the initial metric (5) was defined for $\omega^2 < 1$, and on the other hand, the metric (8) can be interpreted as the product of the BTZ part and $S^2$ only for small $r$.

However, the process of counting of states can be performed using a general result concerning the Chern-Simons action defined on a manifold with topology $\Sigma_2 \times R$ [11, 12]. In the three-dimensional Chern-Simons theory there are no local degrees of freedom, the only relevant degrees of freedom being global charges which generate residual gauge transformations. For a special, but general enough to include most of interesting cases, set of boundary conditions and assumptions on the form of the parameters of residual gauge transformations at the boundary, the global charges form an algebra. For a particular class of the boundary conditions this algebra contains the Virasoro algebra.

In the three-dimensional gravity with a negative cosmological constant represented as the Chern-Simons theory for the $AdS_3$ space, there appear two chiral copies of the Virasoro algebra. The Virasoro algebra written in the standard form has the classical central charge $c = \frac{3l^2}{2G_3}$. The “mass” of the BTZ black hole is related to the zero modes of the Virasoro generators $L_0$ and $\bar{L}_0$ as $M = \frac{8G_3}{l^2}(L_0 + \bar{L}_0)$. This provides the microscopic entropy which was found to be equal to the Bekenstein-Hawking entropy [2] calculated using the Cardy’s formula for the asymptotic density of states.

In the chain of transformations of the metric $ds_5^2$ which lead to the metric (8), we performed the transformation $\omega \to i\omega$. Let us discuss this issue first from the point of view of the $AdS_3$ theory [4]. Starting from the $SL(2, R)$ theory in which the group element is parameterized as

$$g = \begin{pmatrix}
  x_1 + x_2 & x_3 + x_0 \\
  x_3 - x_0 & x_1 - x_2
\end{pmatrix},$$

where

$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = 1,$$  \hspace{1cm} (16)

one can consider (16) as the embedding equation of $AdS_3$ in a flat space. The universal covering space of $AdS_3$ can be covered by three patches. Since the discussion is the same for any of them, let us, for definiteness, consider the region parameterized as

$$
  x_1 = r \cosh \phi \\
  x_2 = r \sinh \phi \\
  x_0 = \sqrt{r^2 - 1} \sinh t \\
  x_3 = \sqrt{r^2 - 1} \cosh t,
$$  \hspace{1cm} (17)
where \( r^2 > 1, -\infty < t, \phi < \infty \). In every patch the group metric is

\[
ds^2 = -(r^2 - 1)dt^2 + r^2d\phi^2 + (r^2 - 1)^{-1}dr^2.
\]  (18)

Changing \( r \to r' = ir \) and introducing a new variable \( \rho \) as \( \rho^2 = r'^2 + 1 \) we see from (17) that, supplemented with the transformation \( \phi \leftrightarrow t \) the change leaves the metric (18) invariant. Thus, we conclude that this transformation is a symmetry of the theory.

The BTZ black hole can be generated from the \( AdS_3 \) solution by the change of variables

\[
r^2 = \frac{\hat{r}^2 - r_-^2}{r_+^2 - r_-^2}, \quad t = r_+ \hat{t} - r_- \hat{\phi}, \quad \phi = -r_- \hat{t} + r_+ \hat{\phi}, \quad M = r_+^2 + r_-^2, \quad J = 2r_+r_- \tag{19}
\]

yielding

\[
ds^2 = -(\hat{t}^2 - M)\hat{d}t^2 - J\hat{d}\hat{\phi}^2 + \hat{r}^2d\hat{\phi}^2 + (\hat{r}^2 - M + \frac{J^2}{4\hat{r}^2})^{-1}d\hat{r}^2 \tag{20}
\]

supplemented with the identification

\[
\hat{\phi} \sim \hat{\phi} + 2\pi \tag{21}
\]

Since the \( AdS_3 \) solution was argued to be invariant under the transformation \( r \to i\sqrt{\rho^2 - 1} \), one can interchange \( t \) and \( \phi \) in the change of variables leading to the BTZ solution and simultaneously regard \( t \) as a compact variable.

## 5 Discussion

Microscopic calculations of the entropy of the black hole usually start from construction of a configuration of the black hole in terms of the exact solutions of 10D or 11D supergravities. In this paper our starting point was a simple 4D dilatonic supergravity interacting with the abelian gauge field which can be lifted to a 5D Einstein-Maxwell gravity. No construction in terms of intersecting M-branes was used.

In examples in which the BTZ black hole appears as a factor in 10D (11D) configuration decoupled from the remaining part of solution obtained by application of the U-duality transformations there is no natural scale for a definition of a near-horizon region. Separation of the near-horizon region appears if there is no exact decoupling of the 3D black hole from the remaining part of the configuration; the decoupling parameter being also the scale which defines the near-horizon region; this refers both to the examples of intersecting branes [7] and to the present case. Moreover, it appears that microscopic degrees of freedom responsible for the entropy of the black hole are located in the near-horizon region.

The result of papers [11, 12], quoted in the preceding section in connection with the problem of counting of the microstates concerning the appearance of the Virasoro algebra, do not require consideration of the asymptotic region \( r \to \infty \) and is valid for an arbitrary boundary surface located at a finite distance from the horizon.

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