Conformal measures associated to ends of hyperbolic $n$-manifolds

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Abstract

Let $\Gamma$ be a non-elementary Kleinian group acting on the closed $n$-dimensional unit ball and assume that its Poincaré series converges at the exponent $\alpha$. Let $M_\Gamma$ be the $\Gamma$-quotient of the open unit ball. We consider certain families $\mathcal{E} = \{E_1, ..., E_p\}$ of open subsets of $M_\Gamma$ such that $M_\Gamma \setminus (\bigcup_{E \in \mathcal{E}} E)$ is compact. The sets $E_i$ are called ends of $M_\Gamma$ and $\mathcal{E}$ is called a complete collection of ends for $M_\Gamma$. We show that we can associate to each end $E \in \mathcal{E}$ a conformal measure of dimension $\alpha$ such that the measures corresponding to different ends are mutually singular if non-trivial. Each conformal measure for $\Gamma$ of dimension $\alpha$ on the limit set $\Lambda(\Gamma)$ of $\Gamma$ can be written as a sum of such conformal measures associated to ends $E \in \mathcal{E}$. In dimension 3, our results overlap with some results of Bishop and Jones [7].

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1 Introduction and survey of known results

Consider a non-elementary Kleinian group $\Gamma$ acting on the closed $n$-dimensional unit ball $\mathbb{B}^n \cup \mathbb{S}^{n-1}$ and assume that its Poincaré series converges at the exponent $\alpha$. Let $M_\Gamma$ be the $\Gamma$-quotient of the open unit ball $\mathbb{B}^n$. In this note we study certain families $\mathcal{E} = \{E_1, ..., E_p\}$ of open subsets of $M_\Gamma$ such that $M_\Gamma \setminus (\bigcup_{E \in \mathcal{E}} E)$ is compact. We call the sets $E_i$ ends of $M_\Gamma$, and $\mathcal{E}$ a complete collection of ends for $M_\Gamma$. A point $z$ on the unit sphere $\mathbb{S}^{n-1}$ is called an endpoint for an end $E$ if any geodesic ray $R$ in $\mathbb{B}^n$ towards $z$ contains a subray which projects onto a ray $R'$ in $M_\Gamma$ so that $R' \subset E$ and so that the distance of a point $x$ on $R'$ to the boundary of $E$ tends to infinity as $x$ tends to $z$ along $R'$. The point $z$ is called an end limit point if $z$ is a limit point of $\Gamma$. We construct and investigate conformal measures concentrated on the set of endpoints for an end.

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We sometimes refer to a conformal measure of dimension \( \alpha \) for \( \Gamma \) as an \( \alpha \)-conformal measure for \( \Gamma \). Note that we do not assume that a conformal measure for \( \Gamma \) is necessarily supported by the limit set \( \Lambda(\Gamma) \). The main result of this paper, stated and proved in the text as Theorem 4.6, can be paraphrased as follows.

**Theorem** Let \( \Gamma \) be a non-elementary Kleinian group \( \Gamma \) acting on \( \mathbb{B}^n \) which has a complete collection of ends \( \mathcal{E} = \{E_1, \ldots, E_p\} \) and assume that its Poincaré series converges at \( \alpha \). Let \( \Lambda_i \) be the set of end limit points of \( E_i \). If \( \Lambda_i \neq \emptyset \), there exists a non-trivial \( \alpha \)-conformal measure for \( \Gamma \) supported by the end limit points of \( E_i \), and any two such measures corresponding to different ends are mutually singular. Each \( \alpha \)-conformal measure \( m \) for \( \Gamma \) on \( \Lambda(\Gamma) \) can be written as a sum of \( \alpha \)-conformal measures \( m_i \) for \( \Gamma \) supported by the \( \Lambda_i \).

As a general result, it is known, see Sullivan [26], Roblin [24], that for a Kleinian group \( \Gamma \) acting on \( \mathbb{B}^n \) of divergence type, there exists a unique invariant conformal measure on its limit set, up to multiplication by constants. Specializing to the case of \( n = 3 \), suppose that \( \Gamma \) is a topologically tame Kleinian group acting on \( \mathbb{B}^3 \) for which \( \Lambda(\Gamma) = S^2 \). Then, every \( \Gamma \)-invariant conformal measure on \( S^2 \) is a multiple of Lebesgue measure on \( S^2 \). In this generality, this result can be obtained by combining Theorem 9.1 of Canary [11] with Proposition 3.9 of Culler and Shalen [13]. Note that while Proposition 3.9 of [13] holds for all \( \mathbb{B}^n \), Theorem 9.1 of [11] is specific to \( \mathbb{B}^3 \).

Therefore we need to consider only groups of convergence type. It is known that a topologically tame Kleinian group \( \Gamma \) acting on \( \mathbb{B}^3 \) is of convergence type if and only if \( \Lambda(\Gamma) \neq S^2 \), see Corollary 9.9.3 of Thurston [31]. We note that there are Kleinian groups of convergence type acting on \( \mathbb{B}^3 \) for which the invariant conformal measure on its limit set is unique up to multiplication by constants, such as the examples given by Sullivan [28], which inspired the present work.

Bishop and Jones, see Corollary 1.3 of [7], prove that if \( \Gamma \) is a topologically tame, geometrically infinite Kleinian group acting on \( \mathbb{B}^3 \) for which the injectivity radius of \( M_\Gamma = \mathbb{B}^3/\Gamma \) is bounded away from zero and \( \Lambda(\Gamma) \neq S^2 \), then \( \mathcal{H}^2 \), the Hausdorff measure associated to the gauge function \( \varphi(t) = t^2 \sqrt{\log(\frac{1}{t})} \log \log \log(\frac{1}{t}) \), is a conformal density of dimension 2. As a consequence they obtain that for each geometrically infinite end there is a unique 2-conformal measure (up to multiplicative constants), the action of \( \Gamma \) on the boundary is ergodic with respect to each of these measures, these measures are mutually singular, and finally, any 2-conformal measure for \( \Gamma \) is a linear combination of them. The methods they use are analytic, constructing a positive harmonic function on \( M_\Gamma \) which grows at most linearly in the geometrically infinite ends of \( M_\Gamma \).

If applied to the situation in 3 dimensions, Theorem 4.6 means that one can remove the lower bound on the injectivity radius of \( M_\Gamma \) and replace the condition that the group is of the second kind by the convergence of the Poincaré series in Bishop and Jones’ result. However, our statements are not as strong as theirs, in that we are unable to prove ergodicity. Our arguments employ straightforward topological and dynamical mechanisms, and we feel that the straightforward nature of the proofs makes up for this lack of ergodicity. It is also not clear the extent to which one can expect such strong ergodicity results to hold in all dimensions.

A note on referencing: it is our intention to give due credit to all authors, but we have chosen to sometimes reference standard texts rather than the first statement of a
result. As a standard reference on the basics of Kleinian groups, we use Maskit [16]. As a standard reference on the measure theoretic constructions involving Kleinian groups, we use Nicholls [19].

This paper is structured as follows. In Section 2 we give the basic definitions and fix the notation. In Section 3 we introduce the notions of an end of a hyperbolic \( n \)-manifold and the associated end groups as they are used in this note, and give the topological and geometric properties which are necessary for the measure-theoretical considerations of Section 4. In Section 4 we state and prove our main results and discuss some applications. Finally, in Section 5 we specialize the discussion to the case of a Kleinian group acting on \( B^3 \).

## 2 Basic definitions

Throughout this note, we work in the Poincaré ball model of hyperbolic \( n \)-space, where \( n \geq 2 \). The underlying space is the unit ball \( B^n = \{ x \in \mathbb{R}^n \mid |x| < 1 \} \) in \( \mathbb{R}^n \), with the element of arc-length \( \frac{1}{1-|x|^2} \, dx \). The hyperbolic distance between points \( x, y \) in \( B^n \) is denoted \( d(x, y) \).

The sphere at infinity of \( B^n \) is the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \).

We denote the closure of \( X \) in the Euclidean topology on the closed \( n \)-ball \( B^n \cup S^{n-1} \) by \( \overline{X} \); in particular, \( \overline{B^n} = B^n \cup S^{n-1} \). We denote the boundary of \( X \subset B^n \) in \( B^n \) by \( \partial X \). For a subset \( X \) of \( \overline{B^n} \), let \( \partial_\infty(X) = \overline{X} \cap S^{n-1} \).

A Kleinian group is a discrete subgroup \( \Gamma \) of the group of (possibly orientation-reversing) isometries of hyperbolic \( n \)-space \( B^n \). (We note that Kleinian groups are often assumed to contain only orientation-preserving isometries. This is the case in many of the papers we have referred to. However, this assumption is not relevant to our arguments in Sections 3 and 4 and so we do not make it here.) A Kleinian group is elementary if it contains an abelian subgroup of finite index, and is non-elementary otherwise. Unless otherwise stated, we will assume that a Kleinian group is non-elementary. We denote the induced hyperbolic distance between points \( x \) and \( y \) in the quotient \( B^n/\Gamma \) by \( d(x, y) \).

An orientation-preserving isometry of \( B^n \) extends to a conformal homeomorphism of the sphere at infinity \( S^{n-1} \). The domain of discontinuity \( \Omega(\Gamma) \) of a Kleinian group \( \Gamma \) is the largest open subset of \( S^{n-1} \) on which \( \Gamma \) acts properly discontinuously. The complement of \( \Omega(\Gamma) \) in \( S^{n-1} \) is the limit set \( \Lambda(\Gamma) \) of \( \Gamma \). We equivalently define \( \Lambda(\Gamma) \) to be the set of accumulation points of the orbit \( \Gamma x \) for any point \( x \in \overline{B^n} \). A Kleinian group \( \Gamma \) is of the first kind if \( \Lambda(\Gamma) = S^{n-1} \), and is of the second kind otherwise. In the latter case, \( \Lambda(\Gamma) \) is a closed, nowhere dense subset of \( S^{n-1} \). If \( \Gamma \) is non-elementary, then \( \Lambda(\Gamma) \) is perfect.

Let \( \Gamma \) be a Kleinian group acting on \( B^n \). For a subset \( X \) of \( \overline{B^n} \), the stabilizer \( \Gamma_X \) of \( X \) in \( \Gamma \) is defined to be the subgroup

\[
\Gamma_X := \{ \gamma \in \Gamma \mid \gamma(X) = X \}
\]

of \( \Gamma \).
A limit point $x \in \Lambda(\Gamma)$ of a Kleinian group $\Gamma$ is a conical limit point if the following holds: there exists a hyperbolic ray $R$ in $B^n$ ending at $x$, a number $\varepsilon > 0$, a point $z \in B^n$, and a sequence $\{\gamma_n\}$ of distinct elements of $\Gamma$ so that $\gamma_n(z) \in U_\varepsilon(R)$ for all $n$, and $\gamma_n(z) \to x$ in $B^n$. Here, $U_\varepsilon(R) = \{z \in B^n | d(x, R) < \varepsilon\}$ is the open $\varepsilon$-neighborhood of $R$ in $B^n$. Equivalently, $x$ is a conical limit point of $\Gamma$ if there exists a sequence $\{x_n\}$ of points of $R$ converging to $x$ in $B^n$ so that, if $\pi : B^n \to B^n/\Gamma$ is the covering map, then the $\pi(x_n)$ all lie in a compact subset of $B^n/\Gamma$. The collection of all conical limit points of $\Gamma$ is denoted $\Lambda_c(\Gamma)$.

Let $\Gamma$ be a Kleinian group acting on $B^n$. Fix a point $y \in B^n$. For $x \in B^n$, define the Poincaré series $P_\Gamma(x, y, s)$ of $\Gamma$ based at $x$ to be

$$P_\Gamma(x, y, s) = \sum_{\gamma \in \Gamma} \exp(-s d(x, \gamma(y))).$$

By the triangle inequality, if $P_\Gamma(x, y, s)$ converges at $s$ for some $x \in B^n$, then it converges at $s$ for all $x \in B^n$. The same holds for $y$.

The critical exponent $\delta(\Gamma)$ of $\Gamma$ is defined to be

$$\delta = \delta(\Gamma) := \inf\{s > 0 \mid P_\Gamma(x, y, s) \text{ converges}\}.$$

$\Gamma$ is of $\delta$-convergence type, or simply of convergence type, if $P_\Gamma(x, y, \delta)$ converges, and of divergence type if $P_\Gamma(x, y, \delta)$ diverges.

Let $\Gamma$ be a Kleinian group acting on $B^n$. Then, we have that $\delta(\Gamma) \leq n - 1$, see Nicholls [19], Theorem 1.6.1. If $\Gamma$ is of the second kind, then $\Gamma$ is of $(n - 1)$-convergence type, see Nicholls [19], Theorem 1.6.2. If $\Gamma$ has finite volume quotient $B^n/\Gamma$, then $\delta(\Gamma) = n - 1$ and $\Gamma$ is of $(n - 1)$-divergence type. If $\Gamma$ is geometrically finite and of the second kind, then $\delta(\Gamma) < n - 1$, see Sullivan [27], Tukia [32]. We have that $\delta(\Gamma) > 0$ for any non-elementary Kleinian group $\Gamma$, see Beardon [5].

There exist finitely generated, geometrically infinite Kleinian groups of the first kind acting on $B^3$ which are of $2$-divergence type (see e.g. Thurston [31], Sullivan [30], Rees [22], [23], or Aaronson and Sullivan [1]).

Throughout we shall work with real-valued, non-negative, finite measures on $\overline{B^n}$. We shall say that a support of such a measure is a measurable, not necessarily uniquely determined subset of $\overline{B^n}$ whose complement has measure zero.

### 3 Ends and endgroups of Kleinian groups acting on $B^n$

Let $\Gamma$ be a Kleinian group acting on $B^n$. We associate to $\Gamma$ the following orbit spaces:

$$M_\Gamma = B^n/\Gamma,$$

$$\overline{M}_\Gamma = (B^n \cup \Omega(\Gamma))/\Gamma,$$

$$\partial_\infty M_\Gamma = \Omega(\Gamma)/\Gamma.$$
For a subset $X$ of $\overline{M_Γ}$, let $X$ denote the closure of $X$ in the Euclidean topology on $\overline{M_Γ}$, and let $\partial_∞(X) = X \cap \partial_∞(M_Γ)$.

We now give the basic definition of this paper. A connected open subset $E$ of $M_Γ$ is an end of $M_Γ$ if $\partial E$ is compact and non-empty and if $E$ has non-compact closure in $M_Γ$. The bulk of the paper is devoted to exploring this definition of end, and to constructing measures associated to ends. Note that the word end has been used in several different senses for hyperbolic manifolds. For instance, the end in Bonahon [3] can be described as a certain type of restricted equivalence class of ends in our sense.

One of the cases we consider in detail, in Section 3, is that $n = 3$ and $\partial E$ is a separating compact surface, which we will often take to be a boundary component of a compact core of $M_Γ$. While it is sometimes the case that $\partial E$ is incompressible, meaning that $π_1(\partial E)$ is infinite and the inclusion of $\partial E$ into $M_Γ$ induces an injection on fundamental groups, the more complicated case occurs when $\partial E$ is not incompressible.

Let $π : B^n → M_Γ$ be the covering projection, let $E$ be an end of $M_Γ$, and let $E^0_i, i ∈ I$, be the components of $π^{-1}(E)$. Let $Γ_i := Γ|E^0_i$ be the stabilizer of $E^0_i$ in $Γ$. We call $E^0_i$ an end of $Γ$ and $Γ_i$ the corresponding end group of $Γ$. By construction, the groups $Γ_i$ are conjugate subgroups of $Γ$.

If we say that $\tilde{E}$ is an end of $Γ$, we mean that $\tilde{E}$ is obtained as above, i.e. $\tilde{E}$ is a component of $π^{-1}(E)$ for an end $E$ of $M_Γ$. Thus ends of $Γ$ are subsets of $B^n$, while the ends of $M_Γ$ are subsets of $M_Γ$.

Let $F^0_i = (\overline{B^n} \setminus \overline{E^0_i}) \cup \partial E^0_i$. Since the sets $F^0_i/Γ_i$ are canonically homeomorphic, we set $F = F^0_i/Γ_i$; in some loose sense, $F$ captures the behavior of an end group on the complement of its corresponding end. Note that $F^0_i$ is invariant under the action of $Γ_i$, since $\overline{B^n}, \overline{E^0_i}$, and $\partial E^0_i$ are all invariant under the action of $Γ_i$. Also, since $\overline{E^0_i} \cap S^{n-1}$ is a non-empty closed subset of $S^{n-1}$ invariant under $Γ_i$, we see that $Λ(Γ_i) \subset \overline{F^0_i} \cap S^{n-1}$, and so $F^0_i \subset B^n \cup Ω(Γ_i)$. If $F$ is compact, we say that $E$ is a bounded end. Additionally, in this case we refer to the $E^0_i$ as bounded ends of $Γ$ and the $Γ_i$ as bounded end groups.

Thus, if we refer to $\tilde{E}$ as a bounded end of $Γ$, $\tilde{E}$ is obtained in this manner.

We can equivalently characterize a bounded end as follows: Since each $E^0_i$ is precisely invariant under its stabilizer $Γ_i$ in $Γ$, we can identify the quotient $E^0_i/Γ_i$ with the end $E$ of $M_Γ$, and so we can regard $E$ as a subset of $\overline{M_i} = \overline{M_Γ}$. The end $E$ of $M_Γ$ (or equivalently the end $E^0_i$ of $Γ$) is a bounded end if $\overline{M_i \setminus (E \cup \partial_∞ E)}$ is compact.

Let $Γ$ be a Kleinian group acting on $B^n$. We say that a point $z ∈ S^{n-1}$ is an endpoint of an end $E^0$ of $Γ$ if, whenever $R$ is a hyperbolic ray with endpoint at infinity $z$, there exists a subray $R'$ of $R$ that is contained in $E^0$, for which the hyperbolic distance $d(x, ∂E^0) → ∞$ as $x \to z$ on $R'$. Note that if $S'$ is the projection of $R'$ to the quotient $M_Γ$, then we still have that $d(x, ∂E^0) → ∞$ as $x$ tends towards infinity on $S'$ and where $E^0$ projects to $E$. Note that by definition, an endpoint of an end $E^0$ of $Γ$ is never a conical limit point of $Γ$.

If $Φ$ is the end group associated to $E^0$, that is $Φ = Γ|E^0$, if $z$ is an endpoint of $E^0$, and if in addition $z ∈ Λ(Φ)$, we say that $z$ is an end limit point of $E^0$. We will see (Lemma 3.1) that $z ∈ Λ(Φ)$ as soon as $z ∈ Λ(Γ)$. We denote the set of end limit points of $Φ$ and $E^0$ by $Λ_c(E^0, Φ) = Λ_c(E^0, Φ)$. Note that we have the inclusion $Λ_c(E^0, Φ) ⊂ Λ(Φ) \setminus Λ_c(Φ)$.
If the end is bounded, this is an equality (cf. Lemma 3.1) but this need not be so in general.

Example 1. To give a concrete example of an end, consider the following example in the case \( n = 2 \). Let \( S \) be a Riemann surface of genus 0 and infinite analytic type; for instance, let \( S \) be the domain of discontinuity of a 2-generator Schottky group \( \Phi \). We can uniformize \( S \) by a Fuchsian group \( \Gamma \), so that \( S = \mathbb{B}^2 / \Gamma \). Let \( c \) be a simple closed geodesic on \( \Omega(\Phi) / \Phi \) which lifts to a simple closed geodesic \( C \) on \( \Omega(\Phi) = S \) (such a curve always exists), and consider the curves \( \{ \varphi(C) \mid \varphi \in \Phi \} \) on \( S \). Each of these curves is compact and separating, and so each determines a pair of ends, namely the two components of \( S \setminus \varphi(C) \). Now, let \( C_1 \) and \( C_2 \) be two disjoint lifts of the curve \( c \) on \( \Omega(\Phi) / \Phi \) to \( S = \Omega(\Phi) \). For \( C_1 \), we can choose the end \( E_1 \) containing \( C_2 \), and for \( C_2 \), we can choose the end \( E_2 \) contained in \( E_1 \). Obviously, we can continue this and obtain an infinite sequence of ends \( E_1 \supset E_2 \supset \ldots \) and there is no smallest end in the sequence. The nested sequence of ends induces an inclusion on the sets of end limit points, that is, \( \Lambda_c(E_1) \supset \Lambda_c(E_2) \supset \ldots \).

Example 2. The following is a simple example of a bounded end. Let \( \Gamma \) be a Kleinian group acting on \( \mathbb{B}^n \). Let \( v \) be a parabolic fixed point of \( \Gamma \) such that \( \Gamma_v \) is free abelian of rank \( n - 1 \). There then exists an open horoball \( B \) at \( v \) that is precisely invariant under \( \Gamma_v \) in \( \Gamma \), so that \( \gamma(B) = B \) for all \( \gamma \in \Gamma_v \) and \( \gamma(B) \cap B = \emptyset \) for all \( \gamma \in \Gamma \setminus \Gamma_v \). (As usual, an open horoball is a Euclidean ball contained in \( \mathbb{B}^n \) whose boundary sphere is tangential to \( S^{n-1} \).) Hence, the stabilizers \( \Gamma_B \) and \( \Gamma_v \) of \( B \) and \( v \), respectively, coincide. Let \( S^0 = \partial B \), and note that \( \partial B \) is the disjoint union of \( B \), \( \partial B \), and \( \partial_\infty(B) = \{ v \} \). Then \( S = S^0 / \Gamma = S^0 / \Gamma_v \) is compact since \( \Gamma_v \) has full rank. The same is true of \( (\mathbb{B}^n \setminus (B \cup \{ v \})) / \Gamma \), which is homeomorphic to \( S \times [0, 1] \). Thus \( B \) is a bounded end of \( \Gamma \) with end group \( \Gamma_v \), and \( B / \Gamma \) is a bounded end of \( M_\Gamma \). Let \( R \) be any geodesic ray in \( \mathbb{B}^n \) ending at \( v \). Then, it follows from basic properties of hyperbolic space that there exists a subray \( R' \) of \( R \) that is contained in \( B \), and for which the hyperbolic distance \( d(x, \partial B) \to \infty \) as \( x \to z \) on \( R' \). Hence, \( v \) is an end limit point of \( B \). In fact, in this case we have that \( \{ v \} = \Lambda_v(B) = \Lambda(\Gamma_v) / \Lambda_c(\Gamma_v) \), since any hyperbolic ray ending at any point \( z \neq v \) in \( S^{n-1} \) must eventually exit any given horoball based at \( z \).

The crucial fact about bounded ends is the following tripartite division of points of the sphere at infinity \( S^{n-1} \) of \( \mathbb{B}^n \).

**Lemma 3.1** Let \( \Gamma \) be a Kleinian group acting on \( \mathbb{B}^n \). Let \( \Phi \) be a bounded end group of \( \Gamma \) associated to the bounded end \( E^0 \) of \( \Gamma \). Let \( z \in S^{n-1} \). Then either \( z \in \Omega(\Phi) \), \( z \in \Lambda_c(\Phi) \), or \( z \) is an end limit point of \( E^0 \).

If \( z \in \Omega(\Gamma) \) and \( z \) is an endpoint of \( E^0 \), then \( z \in \Lambda(\Phi) \) and this is true even if the end \( E^0 \) is not bounded.

**Proof** Let \( F^0 = (\mathbb{B}^n \setminus E^0) \cup \partial E^0 \), and let \( F = F^0 / \Phi \). Let \( D = F^0 \cap S^{n-1} = S^{n-1} \setminus E^0 \). By definition of a bounded end, \( F^0 / \Phi \) is compact, and hence \( \partial_\infty(F^0 / \Phi) = D / \Phi \) is compact as well.

Since \( D / \Phi \) is compact, there are finitely many open hyperbolic half-spaces \( H_j \), \( 1 \leq j \leq p \), contained in \( \mathbb{B}^n \setminus E^0 \) so that \( W = \cup_{1 \leq j \leq p} \varphi(H_j) \) is a neighborhood of \( D \) in \( \mathbb{B}^n \). The quotient \( (F^0 \setminus W) / \Phi \) is compact, as it is a closed subset of the compact set \( F^0 / \Phi \).
Let \( z \in S^{n-1} \) and let \( R \) be a hyperbolic ray with endpoint at infinity \( z \). If \( R \) has a subray \( R' \) with endpoint at infinity \( z \) such that \( R' \subset E^0 \), then as we move \( x \) along \( R' \) towards \( z \), there are two possibilities. One is that \( d(x, \partial E^0) \to \infty \) as \( x \to z \), and hence \( z \) is an endpoint of \( E^0 \); in this case, either \( z \in \Omega(\Phi) \) or \( z \in \Lambda_c(E^0) \). The other case is that there is a positive number \( r \) so that \( d(x, \partial E^0) < r \) for \( x \in R' \) arbitrarily close to \( z \). Since \( \partial E^0 / \Gamma \) is compact, it follows that in this latter case \( z \in \Lambda_c(\Phi) \).

If there is no subray of \( R \) contained in \( E^0 \), then let \( X = F^0 \setminus W \) (with \( W \) constructed as above) so that \( X/\Phi \) is compact. In this case, either \( R \) contains points \( x_i \in R \cap X \) such that \( x_i \to z \), in which case \( z \in \Lambda_c(\Phi) \), or else \( R \) meets some \( \gamma H_i, \gamma \in \Phi \). But then \( R \) contains a subray \( R' \) with endpoint at infinity \( z \) such that \( R' \subset \gamma H_i \) and it follows that \( z \in \Omega(\Phi) \).

Finally, suppose that \( z \in \Lambda(\Gamma) \) is an endpoint of \( E_0 \). We claim that \( z \in \Lambda(\Phi) \). To prove this, choose \( x \in B^n \) which is outside all \( \gamma E^0, \gamma \in \Gamma \). Thus there exist \( \gamma_i \in \Gamma \) such that \( \gamma_i(x) \to z \). The fact that \( x \notin \bigcup_{\gamma \in \Gamma} \gamma E^0 \) implies \( \gamma_i(x) \notin E^0 \). Hence, if \( R_i \) is the hyperbolic ray with endpoints \( z \) and \( \gamma_i(x) \), then there is \( y_i \in R_i \cap \partial E^0 \), since \( z \) is an endpoint of \( E_0 \). Finally, since \( \gamma_i(x) \to z \), we see that \( y_i \to z \). Now fix an \( x_0 \in \partial E^0 \). Since \( \partial E^0 / \Gamma = \partial E^0 / \Phi \) is compact (because \( E^0 \) and \( \partial E^0 \) are precisely invariant under \( \Phi \) in \( \Gamma \)), there exist \( M > 0 \) and \( \varphi_i \in \Phi \) such that \( d(\varphi_i(x_0), y_i) \leq M \). It follows that \( \varphi_i(x_0) \to z \) and hence \( z \in \Lambda(\Phi) \). Note that this argument does not require that \( E \) is bounded.

We now show that the set of endpoints of disjoint ends are disjoint.

**Lemma 3.2** Let \( \Gamma \) be a Kleinian group acting on \( B^n \). If \( E_1^0 \) and \( E_2^0 \) are disjoint ends of \( \Gamma \), then their sets of endpoints are disjoint. Furthermore, if \( E_2^0 \) is bounded, then the set of endpoints of \( E_1^0 \) and the limit set \( \Lambda(\Phi_2), \Phi_2 := \Gamma_{E_2^0} \), are also disjoint.

**Proof** If \( x \) is an endpoint of \( E_1^0 \), then \( x \) is the endpoint of a ray \( R \) contained in \( E_1^0 \) such that \( d(z, \partial E_1^0) \to \infty \) as \( z \to x \) on \( R \). This can be true for at most one end. Therefore, the sets of endpoints of \( E_1^0 \) and \( E_2^0 \) are disjoint, and thus \( \Lambda_c(E_1^0) \cap \Lambda_c(E_2^0) = \emptyset \). Now, if \( E_2^0 \) is bounded, then by Lemma 3.1 we know that \( \Lambda(E_2^0) = \Lambda_c(\Phi_2) \cup \Lambda_c(E_2^0) \). Clearly, an endpoint of \( E_2^0 \) cannot be in \( \Lambda_c(\Phi_2) \) and the lemma follows.

Let \( \mathcal{E} = \{ E_1, \ldots, E_n \} \) be a finite collection of ends of \( M_\Gamma \). For any end \( E \) of \( M_\Gamma \), let \( \bar{E} = E \cup \partial_\infty E \). We say that the collection \( \mathcal{E} \) forms a complete collection of ends of \( M_\Gamma \) if \( E_i \) and \( E_j \) are disjoint for \( i \neq j \) and if \( \overline{M_\Gamma \setminus \bigcup_{E \in \mathcal{E}} \bar{E}} \) is compact. If \( \mathcal{E} \) forms a complete collection of ends of \( M_\Gamma \) and if in addition each \( E_i \) is a bounded end of \( M_\Gamma \), we say that \( \mathcal{E} \) forms a complete collection of bounded ends of \( M_\Gamma \).

Let \( \mathcal{E} \) be a complete collection of ends for \( M_\Gamma \). Let

\[ F = \{ F^0 | F^0 \text{ is a component of } \pi^{-1}(E) \text{ for some } E \in \mathcal{E} \}; \]

we say that \( F \) is a complete collection of ends for \( \Gamma \). If in addition each \( F^0 \in F \) is a bounded end of \( \Gamma \), we say that \( F \) is a complete collection of bounded ends of \( \Gamma \). If the \( \{ \text{bounded} \} \) (pairwise disjoint) ends \( F_1^0, \ldots, F_k^0 \) of \( \Gamma \) generate \( F \), so that \( F = \bigcup_i \Gamma F_i^0 \), we also say that \( \{ F_1^0, \ldots, F_k^0 \} \) forms a complete collection of [bounded] ends for \( \Gamma \).
Lemma 3.3 Let $\Gamma$ be a Kleinian group acting on $B^n$. If $\mathcal{F}$ is a complete collection of ends for $\Gamma$, then $\Lambda(\Gamma)$ is the disjoint union of $\Lambda_c(\Gamma)$ and of the end limit point sets $\Lambda_e(F^0)$, $F^0 \in \mathcal{F}$.

The proof Lemma 3.3 is similar to that of Lemma 3.1.

4 Conformal measures associated to ends

Let $\Gamma$ be a Kleinian group. A measure $m$ on a $\Gamma$-invariant subset $E$ of $B^n$ is called an $\alpha$-conformal measure for $\Gamma$ (or alternatively a conformal measure of dimension $\alpha$ for $\Gamma$) if all Borel subsets of $E$ are measurable and if for any measurable subset $A$ of $E$ and for every $\gamma \in \Gamma$, we have

$$m(\gamma(A)) = \int_A |\gamma'|^\alpha dm. \tag{1}$$

Here, $|\gamma'(\xi)|$ is the operator norm of the derivative of $\gamma$ at $\xi$, and $\alpha$ is some non-negative number. Sometimes, when it is clear from the context what group $\Gamma$ and what number $\alpha$ are meant, we shall just call a measure conformal whenever the condition above is satisfied. Note that we explicitly allow conformal measures to be supported by $B^n$, which stands in contrast to the usual understanding that such measures are defined only in the limit set of a group. Patterson [20], [21] has given the construction of a probability measure $m$ on $\Lambda(\Gamma)$ which is $\delta(\Gamma)$-conformal, where $\delta(\Gamma)$ is the critical exponent of $\Gamma$. In general, a conformal measure $m$ of dimension $\alpha$ is not unique, which is one of the motivations for this work. However, such an $m$ is unique for groups of divergence type when $\alpha = \delta(\Gamma)$, see Sullivan [20], Roblin [24]. There are also other cases when such an $m$ is unique, see for instance Sullivan [28].

While Patterson’s construction gives $\delta(\Gamma)$-conformal measures on $\Lambda(\Gamma)$, it is also possible under certain circumstances to construct $\alpha$-conformal measures on $\Lambda(\Gamma)$ for $\alpha \geq \delta(\Gamma)$. In fact, an $\alpha$-conformal measure for $\Gamma$ can only exist if $\alpha \geq \delta$. See Sullivan (Theorem (2.19) of [20]), and also the discussion in Nicholls [19], Chapter 4. Thus, the critical exponent of $\Gamma$ can be defined as the infimum of all numbers $\alpha$ for which there exists an $\alpha$-conformal measure on $\Lambda(\Gamma)$. It is well-known that if the Poincaré series converges at $\alpha$, then any $\alpha$-conformal measure gives zero measure to the conical limit set (see for instance [19], Theorem 4.4.1). A construction of such $\alpha$-conformal measures when the Poincaré series converges at $\alpha$ is given in Theorem 4.1 below. Note that this construction is different in nature to Patterson’s construction [20] and in some cases [14] it gives different measures than Patterson’s method. Both methods work if $\alpha$ is the exponent of convergence and the Poincaré series converges at this exponent; in this case the non-compactness condition of the theorem is automatically met since otherwise all limit points are conical limit points and hence the Poincaré series diverges at the exponent of convergence.

Theorem 4.1 Let $\Gamma$ be a Kleinian group acting on $B^n$. If $(B^n \cup \Omega(\Gamma))/\Gamma$ is not compact, and if the Poincaré series for $\Gamma$ converges at the exponent $\alpha$, then there exists a non-trivial conformal measure of dimension $\alpha$ for $\Gamma$ on $\Lambda(\Gamma)$. 
Proof Since \((B^n \cup \Omega(\Gamma))/\Gamma\) is not compact, there exists a sequence of points \(\{z_i\}\) of \(B^n\) such that the orbits \(\Gamma z_i\) converge to \(\Lambda(\Gamma)\) in the Hausdorff metric on closed subsets of \(B^n\). (In order to see this, one only needs to choose the \(z_i\) so that for any set of the form \(\Gamma C\), where \(C \subset B^n \cup \Omega(\Gamma)\) is compact, there exists \(I = I_C\) so that \(z_i \not\in \Gamma C\) for \(i > I\). The existence of such \(z_i\) is guaranteed by the assumption that \((B^n \cup \Omega(\Gamma))/\Gamma\) is not compact.) Since the Poincaré series converges, there exists an atomic \(\alpha\)-conformal measure \(\mu\) for \(\Gamma\) on \(\Gamma z_i\) of total mass 1. A subsequence has a weak limit and this limit is the desired non-trivial conformal measure on \(\Lambda(\Gamma)\). QED

Let now \(E\) be an end of \(\Gamma\) in \(B^n\). Choose the points \(z_i\) in the proof of Theorem 4.1 to be points of \(E\). We will show that we obtain a conformal measure supported by the endpoints of \(E\). In order to prove this, we need some estimates on measures of shadows of hyperbolic balls when viewed from the origin \(0 \in B^n\). The next lemma is basically one half of Sullivan’s shadow lemma (we only need the estimate in one direction), but we sharpen the statement in the sense that the constants in the lemma can be chosen not to depend on the measure \(m\) if \(m\) has total mass 1. We also show that the constant does not change under conjugation.

We need only to adapt an argument of Tukia [34] to the present situation. The half-space model was considered in [34], as calculations were easier due to the fact that Euclidean similarities preserving the half-space are hyperbolic isometries.

If \(\gamma\) is a Möbius transformation and \(m\) is a conformal measure of dimension \(\alpha\) on the closed ball \(\overline{B^n}\), then we can define a measure \(m_\gamma\), the image measure of \(\gamma\), by

\[
m_\gamma(A) = \int_A |\gamma'|^\alpha dm. \quad (2)
\]

Thus \(m\) is a conformal measure of dimension \(\alpha\) for \(\Phi\) if and only if \(m_\gamma = m\) for every \(\gamma \in \Phi\). Considerations involving the Radon-Nikodym derivative show that \(m_{\gamma_1 \gamma_2} = (m_{\gamma_2})_{\gamma_1}\) for any two Möbius transformations \(\gamma_1\) and \(\gamma_2\). It follows that if \(m\) is a conformal measure of dimension \(\alpha\) for \(\Phi\), then \(m_\gamma\) is a conformal measure of dimension \(\alpha\) for \(\gamma \Phi \gamma^{-1}\) and is supported by the set \(\gamma A\) if \(m\) is supported by \(A\).

We let \(B(z, r)\) be the Euclidean \(n\)-ball of radius \(r\) centred at \(z\).

Lemma 4.2 Let \(\Phi\) be a Kleinian group. Let \(C \subset B^n\) be compact and fix \(k > 0\). Then there exists \(M > 1\) such that the following holds: Let \(m\) be a conformal measure of dimension \(\alpha\) for \(\Phi\) on \(\overline{B^n}\) of total mass 1. Let \(\gamma\) be a Möbius transformation and define \(m_\gamma\) as in (2). Consider a point \(z \in S^{n-1}\) such that for some \(0 \leq t < 1\) we have \((1-t)z \in \gamma(\Phi C)\). Then

\[
m_\gamma(B(z, kt)) \leq Mt^\alpha.
\]

Proof This is basically Lemma 2C of [34]. In the formulation of [34], we did not claim that the lemma was valid for any measure of total mass 1 but rather fixed the conformal measure and then found the constants. However, we need the lemma only to have the upper estimate and we need only to set \(\nu(\overline{B^n}) = 1\) on the second line of p. 247 of the proof of Lemma 2C in [34] in order to see that the constant in the upper estimate does not depend on the measure if the total mass is 1.
After this observation we transform Lemma 2C of [34] to $B^n$ by means of the stereographic projection. We easily obtain that Lemma 4.2 is true if $z = -e_n = (0, ..., 0, -1)$ which corresponds to 0 under the stereographic projection. Other points are obtained by means of an auxiliary rotation which transforms the point to 0. Since $|\gamma'| = 1$ for a rotation we obtain our claim in view of the conjugacy invariance of Lemma 2C of [34].

QED

The following lemma is a direct consequence of Lemma 4.2 and is in fact the statement which shall be used in the proof of Theorem 4.6. Except for conjugacy by a M"obius transformation and independence of the measure, assumed to be a probability measure, it is a direct consequence of Sullivan’s shadow lemma. Here, $S_r(y) \subset B^n$ denotes the shadow in $B^n$ from the origin of the open hyperbolic ball $D(y, r)$ of radius $r > 0$ and center $y \in B^n$. Thus a point $w \in B^n$ is in $S_r(y)$ if and only if the hyperbolic ray from 0 to $w$ intersects $D(y, r)$. Note that our definition of the shadow of a hyperbolic ball is slightly different from the usual one, where only the part contained in $S^{n-1}$ is considered. We apply Lemma 4.2 in the case that the compact set $C$ is a one-point set \{y\}.

**Lemma 4.3** Let $\Phi$ be a non-elementary Kleinian group, let $y \in B^n$, and let $r > 0$ and $\alpha > 0$ be positive constants. Then there is $c > 0$ such that if $m$ is a conformal measure of dimension $\alpha$ for $\Phi$ on $B^n$ of total mass 1, then the following is true. Let $\gamma$ be an arbitrary M"obius transformation, and define the conformal measure $m_\gamma$ for the Kleinian group $\gamma\Phi\gamma^{-1}$ as in (2). Then,

$$m_\gamma(S_r(\gamma(z))) \leq c \exp(-\alpha d(0, \gamma(z)))$$

for all $z$ in the orbit $\Gamma y$.

Armed with these estimates, we can derive some results on the distribution of mass and extension of conformal measures. We start with the refinement of Theorem 4.1 which says that we can find a measure supported by the end limit points.

**Theorem 4.4** Let the situation be as in Theorem 4.1 and let $E$ be an end of $\Gamma$ in $B^n$ such that the end limit point set $\Lambda_e(E) \neq \emptyset$. Then there is an $\alpha$-conformal measure $m$ for $\Gamma$ such that $m$ is supported by the end limit points of $E$ and of the ends of $\Gamma$ equivalent to $E$ under $\Gamma$.

In particular, if the Poincaré series for $\Gamma_E$ converges at exponent $\alpha$ (even if the Poincaré series for $\Gamma$ diverges), there is a non-trivial conformal measure of dimension $\alpha$ for $\Gamma_E$ supported by $\Lambda_e(E)$.

**Remark.** If $\Lambda_e(E) = \emptyset$, then our method still constructs a measure on the endpoint set of $E$ and of the ends equivalent to $E$ under $\Gamma$. However, the set of endpoints which are not end limit points is open and hence the $(n-1)$-dimensional Hausdorff measure is a conformal measure on this set.

**Proof** Pick $y \in \Lambda_e(E)$ and let $R$ be a hyperbolic ray with endpoint $y$. Pick points $z_i \in R$ such that $z_i \to y$ as $i \to \infty$. Since $y$ is a limit point of $\Gamma$, it follows that the $z_i$ exit any $\Gamma C$, where $C \subset B^n \cup \Omega(\Gamma)$ is compact. Since the Poincaré series converges,
there is a conformal measure $m_i$ on $\Gamma Z_i$ of total mass 1. We can assume that the sequence formed by the $m_i$ has a weak limit $m$. Since the $\Gamma Z_i$ exit any compact subset of $B^n \cup \Omega(\Gamma)$, it follows that $m$ is supported by $\Lambda(\Gamma)$. We show that $m$ is supported by $\Lambda_\varepsilon(E)$ and the endpoints of the ends equivalent to $E$ under $\Gamma$.

Let $E_\Gamma = E/\Gamma$ and let $X = \partial E_\Gamma$. We can assume that $M_\Gamma \setminus X$ has only finitely many components. For instance, we can cover $X$ by a finite number of hyperbolic balls $B_i$ and replace $E_\Gamma$ by the component of $M_\Gamma \setminus (\cup_i B_i)$ containing the original $E_\Gamma$. Let $F_1, ..., F_q$ be the components of $M_\Gamma \setminus X$ whose closures in $M_\Gamma$ are non-compact and which are distinct from $E_\Gamma$. Then each $F_i$ is an end of $M_\Gamma$ and $\{E_\Gamma, F_1, ..., F_q\}$ is a complete collection of ends for $M_\Gamma$. In view of Lemma 3.3 it suffices to show that if $F$ is a lift of some $F_i$ to $B^n$, then $m(\Lambda_\varepsilon(F)) = 0$.

We will define for each $0 < r < 1$ a set $U_r$ so that $U_r$ will be a neighborhood of $\Lambda_\varepsilon(F)$ in $B^n$ and that $m_i(U_r) \leq c_r$ where $c_r \to 0$. This is the basic reason why $m(\Lambda_\varepsilon(F)) = 0$, and the precise argument is given below.

Let $H_r = \{ z : 1 - r \leq |z| \leq 1 \} \subset B^n$. We can assume that $0 \in E$. Let $U_r$ be the union of all geodesic rays $R_a = \{ ta : 1 - r < t < 1 \}$, $a \in S^{n-1}$, with the property that there exists $1 - r < t < 1$ such that $ta \in F$. Clearly, $U_r$ is a neighborhood of $\Lambda_\varepsilon(F)$ in $B^n$. We fix a point $z_0 \in \partial F$ and a number $R > 0$ such that $\bigcup_{\gamma \in \Gamma} D(\gamma(z_0), R) \supset \bigcup_{\gamma \in \Gamma} \gamma(\partial F)$ where $D(z, R)$ is the open hyperbolic ball with center $z$ and radius $R$. Let $S_\gamma$ be the shadow of $D(\gamma(z_0), R)$ from 0, so that $S_\gamma$ contains all the points $w \in B^n$ such that the hyperbolic line segment or ray with endpoints 0 and $w$ intersects $D(\gamma(z_0), R)$. Let $V_r$ be the union of all shadows $S_\gamma$, $\gamma \in \Gamma$, such that $D(\gamma(z_0), R)$ intersects $H_r$. It is not difficult to see that $\Gamma z_0 \cap U_r \subset V_r$.

Next, we apply Lemma 4.3 to the measures $m_i$ whose limit is $m$. Thus there exists a constant $c$ such that $m_i(S_\gamma) \leq c \exp(-\alpha d(0, \gamma(z_0)))$ regardless of $i$ and therefore

$$\sum m_i(S_\gamma) \leq c \sum \exp(-\alpha d(0, \gamma(z_0))) =: c_r$$

where both sums are restricted to elements $\gamma \in \Gamma$ such that $\gamma(z_0) \in H_r$. If $r \to 1$, the right hand side tends to zero and thus $c_r \to 0$ as claimed. Since the mass of $m_i$ in $U_r$ is contained in the shadows $S_\gamma$, it follows that $c_r$ is indeed an upper bound for $m(U_r)$.

To see that $m(\Lambda_\varepsilon(F)) = 0$, let $\Lambda_p$ be the set of points $z \in S^{n-1}$ such that the line segment $tz$, $t \in [1 - 1/p, 1)$, is contained in $F \cup \partial F$. Note that each $\Lambda_p$ is a closed set and $U_r$ is a neighborhood of $\Lambda_p$ for every $0 < r < 1$. Since $\Lambda_p$ is closed, the inequalities $m_i(U_r) \leq c_r$ imply that $m(\Lambda_p) \leq c_r$ for all $r$ and hence $m(\Lambda_p) = 0$. Since $\Lambda_\varepsilon(E)$ is contained in the union of the $\Lambda_p$, it follows that $m(\Lambda_\varepsilon(E)) = 0$.

To see the last paragraph, we only need to observe that $E$ is an end for $\Gamma_E$ as well, and that the set of end limit points is the same whether we regard $E$ as an end of $\Gamma$ or of $\Gamma_E$ (see Lemma 3.1).

QED

The above theorem (and the remark following it) asserts that we always have a conformal measure for $\Gamma$ supported by the endpoints of an end $E$ and the ends equivalent to $E$ under $\Gamma$. Conversely, suppose that we have a conformal measure $m$ for $\Gamma_E$ supported by $\Lambda_\varepsilon(E)$. We might ask whether it is possible to extend $m$ to a conformal measure for the whole group $\Gamma$. That this is possible is shown in the next proposition.
Proposition 4.5 Let \( \Gamma \) be a Kleinian group whose Poincaré series converges at the exponent \( \alpha \), and let \( E \) be an end of \( \Gamma \). Let \( m \) be an \( \alpha \)-conformal measure for \( \Gamma_E \) supported by the set of endpoints of \( E \). Then there is a unique extension of \( m \) to a conformal measure of \( \Gamma \) supported by the endpoints of \( E \) and of the ends equivalent to \( E \) under \( \Gamma \).

Proof Choose representatives \( \gamma_i, i \in \mathbb{N} \), from the cosets \( \Gamma/\Gamma_E \). Let \( \gamma_0 \) be the identity. Thus \( E_i = \gamma_iE \) is an end distinct from \( E = E_0 \) if \( i \neq 0 \). Let \( \Lambda_i \) be the set of endpoints of \( E_i \). Thus, the sets \( \Lambda_i, i \in \mathbb{N} \), form a family of pairwise disjoint sets. If we can extend \( m \) to a conformal measure for \( \Gamma \) on \( \bigcup \Lambda_i \), then the restriction of \( m \) to \( \Lambda_i \) must be the image measure \( m_{\gamma_i} \) of \( \Gamma \). This proves the uniqueness of the extension. Now, the rule 

\[
\mu_{\gamma_1\gamma_2} = (m_{\gamma_1})_{\gamma_2},
\]

and the property that \( m_\gamma = m \) if \( \gamma \in \Gamma_E \) together imply that \( m_{\gamma_i} \) is independent of the choice of the representative, and that it satisfies the transformation rule for conformal measures, or equivalently, that \( m_\gamma = m \) for \( \gamma \in \Gamma \). These properties also imply that setting \( m := m_{\gamma_i} \) on \( \Lambda_i \), we obtain a measure on \( \bigcup_i \Lambda_i \) which satisfies the transformation rule for conformal measures. Thus it is a conformal measure for \( \Gamma \) if it is finite.

We will now prove the finiteness of \( m \) by relating \( m(\bigcup \Lambda_i) \) to the sum of the Poincaré series at \( \alpha \), which is assumed to be finite. It suffices to prove that \( \sum_{i \neq 0} m(\Lambda_i) < \infty \).

By conjugation with suitable Möbius transformations we can assume that \( 0 \in E \). Next, we pick \( y \in \partial E \). Since \( \partial E/\Gamma \) is compact, there is a number \( r > 0 \) such that the shadows from 0 of the hyperbolic balls \( D(\gamma(y), r), \gamma \in \Gamma \), cover \( \bigcup_i \partial E_i \). Suppose that \( z \in \Lambda_i \) for some \( i \neq 0 \). Then the hyperbolic ray with endpoints 0 and \( z \) intersects \( \partial E_i \) and hence this ray intersects also some \( D(\gamma(y), r) \) so that \( z \in S_r(\gamma_i(y)) \) for some \( g \in \Gamma = \Gamma_E \). Lemma \ref{lem:bounded} then implies the existence of a constant \( C > 0 \) such that

\[
m(\bigcup_{i \neq 0} \Lambda_i) \leq m\left( \bigcup_{\gamma \in \Gamma} S_r(\gamma(y)) \right) \leq \sum_{\gamma \in \Gamma} C \exp(-\alpha d(0, \gamma(y))) < \infty.
\]

QED

Remark. We have formulated Theorem \ref{thm:fin} and Proposition \ref{prop:fin} for the case at hand so that we take the weak limit of atomic measures supported by an orbit or extend measures supported by the end limit point set. A more general formulation would be as follows. Let \( E \) be an end of \( \Gamma \) in \( \mathbb{B}^n \) and set \( \widetilde{E} = (E \cap (\mathbb{B}^n \cup \Omega(\Gamma))) \) \( \cup \Lambda(E) \). The formulation of Proposition \ref{prop:fin} would be that an \( \alpha \)-conformal measure \( \mu \) for \( E_\Gamma \) which is supported by \( \widetilde{E} \) can be extended to an \( \alpha \)-conformal measure for \( \Gamma \) supported by \( \bigcup_{\gamma \in \Gamma} \widetilde{E} \). Theorem \ref{thm:fin} is formulated in the clearest way for conformal measures for \( \Gamma_E \) (thus \( \Gamma = \Gamma_E \)). In this case, if the measures \( \mu_i \) are supported by \( \widetilde{E} \) and have total mass 1, then also their weak limit \( \mu \) is supported by \( \widetilde{E} \).

Suppose that \( \mathcal{E} = \{E_1, ..., E_q\} \) is a complete collection of ends for \( \Gamma \) where each \( E_i \) is an end of \( \Gamma \) in \( \mathbb{B}^n \) and where the \( E_i \) are pairwise disjoint. Thus, setting \( E_i^{\Gamma} = E_i/\Gamma \), it follows that \( M_{\Gamma} \setminus (\bigcup_{i \leq q} E_i^{\Gamma}) \) is compact. Let \( \Gamma_i := \Gamma_E \). Fix \( \alpha > 0 \) such that the Poincaré series for \( \Gamma \) converges with exponent \( \alpha \). We suppose that \( E_i, i \leq p \), are the ends such that \( \Lambda(E_i) \neq \emptyset \) and denote

\[
\mathcal{M} = \text{the family of } \alpha \text{-conformal measures for } \Gamma \text{ on } \Lambda(\Gamma).
\]
\[ \mathcal{M}_i = \text{the family of } \alpha\text{-conformal measures for } \Gamma_i \text{ on } \Lambda_e(E_i), \ i \leq p. \]

We can now prove our main theorem. The natural situation for us is that the measures live on \( \Lambda(\Gamma) \) but everything remains valid if \( \mathcal{M} \) is the set of all conformal measures on \( S^{n-1} \) and \( \mathcal{M}_i \) is the set of conformal measures on endpoints of \( E_i \) as \( i \) varies from 1 to \( q \).

**Theorem 4.6** Let \( \mu \in \mathcal{M} \) be a conformal measure for \( \Gamma \) on \( \Lambda(\Gamma) \), and let \( \mu_i \in \mathcal{M}_i \) be the restriction of \( \mu \) to \( \Lambda_e(E_i) \). Then

\[ \mu = \mu_1^* + ... + \mu_p^* \]

where \( \mu_i^* \) is the unique extension of \( \mu_i \) to a conformal measure on \( \bigcup_{\gamma \in \Gamma} \Lambda_e(E_i) \) given by Proposition 4.5. The measures \( \mu_i^* \) and \( \mu_j^* \), \( i \neq j \), are mutually singular if non-trivial.

If \( \mu_i \in \mathcal{M}_i \), then (4) defines a conformal measure \( \mu \) for \( \Gamma \) on \( \Lambda(\Gamma) \). For each \( i \leq p \) there is a non-trivial measure \( \mu_i \in \mathcal{M} \) and thus, if there are \( p \) ends in \( \mathcal{E} \) such that \( \Lambda_e(E_i) \neq \emptyset \), then there are at least \( p \) mutually singular non-trivial conformal measures of dimension \( \alpha \) for \( \Gamma \) on \( \Lambda(\Gamma) \).

**Proof** By the uniqueness of the extension of Proposition 4.5, \( \mu_i^* \) and \( \mu \) coincide on \( E_i^* = \bigcup_{\gamma \in \Gamma} \gamma \Lambda_e(E_i) \). Every limit point of \( \Gamma \) is either in some \( E_i^* \) or is a conical limit point of \( \Gamma \) (Lemma 3.3). Since the Poincaré series converges, conical limit points have zero measure and hence (4) is true. Since \( \mu_i^* \) is supported by \( E_i^* \), and the \( E_i^* \) are disjoint, it follows that \( \mu_i \) and \( \mu_j \) are mutually singular if non-trivial.

To obtain the last paragraph, we note that Theorem 4.4 gives the non-trivial measure \( \mu_i \) on \( \Lambda_e(E_i) \) which can be extended to the conformal measure \( \mu_i^* \) by Proposition 4.5. Other points of the last paragraph are obvious.

**Bounded ends.** Theorem 4.6 says that it is possible to write a conformal measure \( \mu \) for \( \Gamma \) on \( \Lambda(\Gamma) \) as a sum of extended measures, which are obtained from measures supported by the end limit point sets of ends. This is a way to decompose \( \mu \) to simpler measures. The decomposition of (4) is not ideal, since \( \mu_i \) is supported by the end limit set of \( \Gamma_i = \Gamma_{E_i} \). It would be better if we could allow \( \mu_i \) to be supported by \( \Lambda(\Gamma_i) \) since, for instance, the Patterson-Sullivan construction of a conformal measure gives measures on \( \Lambda(\Gamma_i) \) and it seems that these measures are not necessarily supported by the end limit points.

This problem disappears if the end is bounded since then every \( z \in \Lambda(\Gamma_i) \) is either an end limit point or a conical limit point by Lemma 3.1 and the measure of the conical limit set vanishes in the case of convergence of the Poincaré series. Thus, in this case we can replace \( \Lambda_e(E_i) \) by \( \Lambda(\Gamma_i) \).

We combine our main theorems applied to bounded ends to

**Theorem 4.7** Let \( E \) be a bounded end of \( \Gamma \) and suppose that the Poincaré series for \( \Gamma \) converges at the exponent \( \alpha \). If \( \mu \) is a \( \alpha \)-conformal measure for \( \Gamma_E \) on \( \Lambda(\Gamma_E) \), then \( \mu \) has a unique extension, denoted by \( \mu^* \), to a conformal measure for \( \Gamma \) such that the extension is supported by the subset \( \bigcup_{\gamma \in \Gamma} \gamma \Lambda(\Gamma) \) of \( \Lambda(\Gamma) \).
Let \( \{ E_1, ..., E_p \} \) be a complete collection of ends for \( \Gamma \) such that each \( E_i \) is a bounded end in \( B^n \). If \( \mu \) is a conformal measure of dimension \( \alpha \) on \( \Lambda(\Gamma) \), then \( \mu = \mu_1^* + ... + \mu_p^* \) where \( \mu_i \) is the restriction of \( \mu \) to \( \Lambda(\Gamma_{E_i}) \), and the \( \mu_i^* \), \( i \neq j \), are mutually singular if they are non-trivial.

**Applications.** We have the following consequences of the measure extension Proposition 4.5. The first theorem is proved using the fact, discussed in Example 2 in Section 3 that if \( v \) is a parabolic fixed point of full rank, then its stabilizer \( \Gamma_v \) is a bounded end group. Thus the atomic measure which gives mass 1 to \( v \) is conformal for \( \Gamma_v \) and hence can be extended to a conformal measure for \( \Gamma \).

**Theorem 4.8** Let the Kleinian group \( \Gamma \) be as in Theorem 4.6 and let \( v \in S^{n-1} \) be a parabolic fixed point such that \( \Gamma_v \) has rank \( n - 1 \). Then there is a non-trivial atomic conformal measure for \( \Gamma \) supported by the orbit \( \Gamma v \) of \( v \). If \( g_i, i \in I, \) are representatives of the cosets of \( \Gamma / \Gamma_v \), then

\[
\sum_{i \in I} |g_i(v)|^\alpha < \infty.
\]

**Theorem 4.9** Let \( \Gamma \) be a Kleinian group and let \( E \) be an end of \( \Gamma \) in \( B^n \). Let \( m \) be a conformal measure for \( \Gamma_E \) on \( \Lambda(\Gamma_E) \). If \( g_i, i \in I, \) are representatives of the cosets of \( \Gamma / \Gamma_E \), then

\[
\sum_{i \in I} |g_i(x)|^\alpha < \infty \tag{5}
\]

for \( m \)-almost every \( x \in \Lambda_E(E) \). If the end \( E \) is bounded, then (3) is true for \( m \)-almost every \( x \in \Lambda(\Gamma_E) \).

**Ends with boundary.** We have defined ends as subsets \( E \) of \( M_\Gamma \) (or the lift to \( B^n \) of such a set) such that \( E \) is a non-compact component of \( M_\Gamma \setminus \partial E \) where the boundary \( \partial E \) of \( E \) in \( M_\Gamma \) is compact. We could have done this in \( \overline{M}_\Gamma = (B^n \cup \Omega(\Gamma))/\Gamma \) so that \( E \) would now be a component of \( \overline{M}_\Gamma \setminus \partial E \) where now the boundary \( \partial \) is taken in \( \overline{M}_\Gamma \). We call such a set \( E \) an end of \( M_\Gamma \) with boundary (of course, such an end does not necessarily intersect the boundary of \( M_\Gamma \)). Like before, we also call a lift \( \tilde{E} \) of \( E \) to \( B^n \cup \Omega(\Gamma) \) an end of \( \Gamma \) with boundary. Such an end is bounded if \( (B^n \cup \Omega(\Gamma_E)) \setminus \tilde{E} \) has compact \( \Gamma_E \)-quotient. The definition of the endpoints and end limit points of an end with boundary can be given as before.

All the earlier results are valid with this definition. Lemma 3.3 and its tripartite division of points of \( S^{n-1} \) into ordinary, conical limit points or end limit points is valid. Similarly, if we define a complete collection of ends with boundary to be a collection \( \mathcal{E} = \{ E_1, ..., E_p \} \) such that \( \overline{M}_\Gamma \setminus (E_1 \cup ... \cup E_p) \) is compact, or the lift of such a collection, then Lemma 3.3 is valid, and every \( z \in \Lambda(\Gamma) \) is either a conical limit point or an endpoint of a lift to \( B^n \) of some \( E_i \). If an endpoint \( z \) of \( E_i \) is in \( \Lambda(\Gamma) \), then \( z \in \Lambda(\Gamma_{E_i}) \).

Finally, as in Theorem 4.4 we can find a non-trivial conformal measure supported by the end limit point set of an end \( E \) in \( B^n \) and Proposition 4.5 is still valid, allowing to extend a conformal measure for \( \Gamma_E \) supported by the end limit points to a conformal measure for \( \Gamma \). Note that in the proof of these statements we now should choose that
z₀ and the point 0, from which we take shadows, are in the hyperbolic convex hull $H_\Gamma$ of $\Lambda(\Gamma)$. (The convex hull is discussed in Section 5.) Thus the hyperbolic line or segment joining 0 and a point of some $E_i$ or a point of $\Gamma z₀$ lies in $H_\Gamma$. Therefore, in the proofs of Proposition 4.5 and Theorem 4.4 we can replace $F$ by $F \cap H_\Gamma$ and $E_i$ by $E_i \cap H_\Gamma$, both of which can be covered by the shadows of the balls $D(\gamma(z₀), R)$ and $D(\gamma(y), R)$, $\gamma \in \Gamma$.

In analogy to Theorem 4.6 we could state that if $E = \{E_1, ..., E_p\}$ is a complete collection of ends with boundary, then a conformal measure $\mu$ of $\Gamma$ on $\Lambda(\Gamma)$ of dimension $\alpha$ such that the Poincaré series converges at $\alpha$, admits a unique decomposition as a sum $\mu = \mu_1^* + ... + \mu_p^*$, where $\mu_i$ is a conformal measure supported by the end limit points of $E_i$ and $\mu_i^*$ is the extension of $\mu_i$ to a conformal measure for $\Gamma$ which is given by the analogue of Proposition 4.5 for ends with boundary. Also, other parts of Theorem 4.6 are valid for ends with boundary and so is the analogue of Theorem 4.7 for bounded ends with boundary.

5 Ends and end groups of Kleinian groups acting on $B^3$

We now specialize to the case of a Kleinian group acting on $B^3$, and discuss the behavior of ends of hyperbolic 3-manifolds in the context of the terminology of this paper. Throughout this section, let $\Gamma$ be a non-elementary, finitely generated, purely loxodromic, torsion-free Kleinian group acting on $B^3$. We further assume that the Kleinian groups considered in this section contain only orientation-preserving isometries of $B^3$. The assumptions that $\Gamma$ is purely loxodromic and torsion-free are not essential, but are made for ease of exposition.

A Kleinian group $\Gamma$ acting on $B^3$ is topologically tame if $M_\Gamma$ is homeomorphic to the interior of a compact 3-manifold with (possibly empty) boundary. In particular, topologically tame Kleinian groups are finitely generated. It is conjectured that all finitely generated Kleinian groups are topologically tame. Agol [2] and Calegari and Gabai [9] have recently and independently announced proofs of this conjecture. We note that in the case that $\Gamma$ is topologically tame and geometrically infinite, it is known that $\delta(\Gamma) = 2$, see Canary [12], [11]. Again for ease of exposition, we restrict our attention in this section to topologically tame Kleinian groups.

Let $E^0$ be an end of $\Gamma$ with associated end group $\Gamma_{E^0}$. Say that $E^0$ is finite if $E^0 \cap S^2 = \emptyset$, where $D$ is a component of $\Omega(\Gamma)$. Say that $E^0$ is infinite if $E^0 \cap S^2 \subset \Lambda(\Gamma)$. As we will see below, an end of $\Gamma$ is not necessarily either finite or infinite. By Lemma 3.4 the end $E^0$ of $\Gamma$ being infinite is equivalent to the fact that $E^0 \cap S^2 \subset \Lambda(\Gamma_{E^0})$. We note that the ends of $M_\Gamma$ that are finite by this definition correspond to the ends of $M_\Gamma$ that are geometrically finite ends by the more standard definition.

The convex core $C_\Gamma$ of the hyperbolic 3-manifold $M_\Gamma = B^3/\Gamma$ is the smallest closed, convex subset of $M_\Gamma$ whose inclusion into $M_\Gamma$ is a homotopy equivalence. Equivalently, $C_\Gamma$ is the quotient under $\Gamma$ of the convex hull $H_\Gamma$ of $\Lambda(\Gamma)$ in $B^3$, where the convex hull $H_\Gamma$ of $\Lambda(\Gamma)$ is the smallest closed, convex subset of $B^3$ containing all the lines in $B^3$ both of whose endpoints at infinity lie in $\Lambda(\Gamma)$. A Kleinian group $\Gamma$ is geometrically finite if
a unit neighborhood of $C_\Gamma$ has finite volume, and is \textit{geometrically infinite} otherwise. A Kleinian group acting on $B^2$ is geometrically finite if and only if it is finitely generated. A geometrically finite Kleinian group acting on $B^3$ is necessarily finitely generated, but the converse fails.

A \textit{compact core} for $M_\Gamma$ is a compact submanifold $Y$ of $M_\Gamma$ for which the inclusion map $Y \hookrightarrow M_\Gamma$ induces a homotopy equivalence. It is a theorem of Scott [25] that for any finitely generated Kleinian group $\Gamma$ acting on $B^3$ (in fact, for any irreducible, orientable 3-manifold with finitely generated fundamental group), there exists a compact core for $M_\Gamma$. Note that if $\Gamma$ is topologically tame, then $M_\Gamma$ is homeomorphic to the interior of a compact 3-manifold $Z$ with boundary, and a compact core for $M_\Gamma$ can be obtained by removing a collar neighborhood of $\partial Z$ from $Z$. In this case, the connected components of $M_\Gamma \setminus Y$ form a complete collection of ends and so, in the case that $\Gamma$ is topologically tame, there exists a complete collection of ends $E$ whose elements are in one-to-one correspondence with the boundary components of $Z$.

Let $E$ be an end of $M_\Gamma$. We say that $E$ is a \textit{peripheral end} of $M_\Gamma$ if there exists a compact core $Y$ of $M_\Gamma$ so that $E$ is contained in a component of $M_\Gamma \setminus Y$. Note that there is a great deal of flexibility in the definition of a peripheral end, as we have a great deal of flexibility in choosing the compact core. (We acknowledge that this is not the most general definition of peripheral end that can be given. However, given the generality with which we are treating ends, it seems an appropriate definition.) With this definition, a peripheral end is a neighborhood of an end in the sense of Bonahon [8]. A peripheral end is finite if it faces a component of $\Omega(\Gamma)/\Gamma$. If $E$ is a peripheral end of $M_\Gamma$, then we call any component $E^0$ of the lift of $E$ to $B^3$ a peripheral end of $\Gamma$.

There is one basic class of Kleinian groups that are of interest to us here. Let $\Gamma$ be a purely loxodromic Kleinian group isomorphic to the fundamental group of a closed orientable surface $S$ of negative Euler characteristic. By Bonahon’s criterion for tameness [8], each such $M_\Gamma$ is homeomorphic to the interior of a compact 3-manifold with boundary, and basic 3-manifold topology, see e.g. Hempel [15], implies that $M_\Gamma = S \times (0,1)$.

For any $t$ in $(0,1)$, the surface $S_t := S \times \{t\}$ is compact and separating, and so the two components $E_{(0,t)} := S \times (0,t)$ and $E_{(t,1)} := S \times (t,1)$ of $M_\Gamma \setminus S_t$ are peripheral ends of $M_\Gamma$. In fact, if $E$ is any peripheral end of $M_\Gamma$, then, for any $t \in (0,1)$, the symmetric difference of $E$ with one of $E_{(0,t)}$ or $E_{(t,1)}$ has compact closure, and so to describe the peripheral ends of $M_\Gamma$, it suffices to consider $E_{(0,t)}$ and $E_{(t,1)}$. (To see this, note that if $E$ is any peripheral end of $M_\Gamma$, then $\partial E \cup \partial E_{(0,t)} \cup \partial E_{(t,1)}$ is compact, and so is contained in $S \times [a,b]$ for some closed interval $[a,b]$. Then, we see that $E \setminus S \times [a,b]$ must coincide with either $E_{(0,t)} \setminus S \times [a,b]$ or $E_{(t,1)} \setminus S \times [a,b]$, say $E_{(0,t)} \setminus S \times [a,b]$. In particular, the symmetric difference of $E$ and $E_{(0,t)}$ is then contained in $S \times [a,b]$, which is compact.) In fact, since $M_\Gamma$ is the product $S \times (0,1)$, we see that for any peripheral end $E$ of $M_\Gamma$, the other component of $M_\Gamma \setminus \partial E$ is itself a peripheral end of $M_\Gamma$, which we refer to as the complementary end to $E$ in $M_\Gamma$.

Since $S_t$ is incompressible, it lifts to a properly embedded open topological disc $S^0_t$ in $B^3$, which is necessarily invariant under $\Gamma$. The two components $E^0_{(0,t)}$ and $E^0_{(t,1)}$ of $B^3 \setminus S^0_t$ are connected, invariant under $\Gamma$, and not equivalent under $\Gamma$. Again up to taking the symmetric difference with the $\Gamma$-translates of a compact set, these are the
only two peripheral ends of \( \Gamma \). Note that since both \( E^0_{(0,t)} \) and \( E^0_{(t,1)} \) are invariant under \( \Gamma \), we have that \( \Lambda(\Gamma) \subset \overline{E^0_{(0,t)}} \) and \( \Lambda(\Gamma) \subset \overline{E^0_{(t,1)}} \).

The domain of discontinuity \( \Omega(\Gamma) \) of \( \Gamma \) contains at most two components, and all components of \( \Omega(\Gamma) \) are invariant under \( \Gamma \). If the peripheral end \( E^0 \) of \( \Gamma \) does not face a component of \( \Omega(\Gamma) \), then (up to taking its symmetric difference with the \( \Gamma \)-translate of a compact set) \( E^0 \) is one of \( E^0_{(0,t)} \) or \( E^0_{(t,1)} \) and is contained in the convex hull \( H_\Gamma \) of \( \Lambda(\Gamma) \) in \( B^3 \); in particular, \( E^0 \) is disjoint from \( \Omega(\Gamma) \), as any geodesic ray in \( B^3 \) ending at a point of \( \Omega(\Gamma) \) must exit the convex hull \( H_\Gamma \) of \( \Lambda(\Gamma) \) in \( B^3 \). Hence, if \( E^0 \) does not face a component of \( \Omega(\Gamma) \), then \( E^0 \) is an infinite peripheral end.

To summarize:

**Proposition 5.1** Let \( \Gamma \) be a purely loxodromic Kleinian group isomorphic to the fundamental group of a closed orientable surface \( S \) of negative Euler characteristic. Then, every peripheral end of \( \Gamma \) is either finite or infinite.

A peripheral end of \( \Gamma \) is finite if and only if it faces a component of \( \Omega(\Gamma) \), and a peripheral end of \( \Gamma \) is infinite if and only if it is contained in the convex hull \( H_\Gamma \) of \( \Lambda(\Gamma) \) (up to having symmetric difference contained in the \( \Gamma \)-translates of a compact set).

If the peripheral end \( E \) of \( M_\Gamma \) is finite, then every endpoint of \( E \) is a point of \( \Omega(\Gamma) \), and if \( E \) is infinite, then every endpoint of \( E \) is an end limit point of \( E \).

Consider now the peripheral end \( E^0_{(0,t)} \) of \( \Gamma \). Suppose that its complementary end \( E^0_{(t,1)} \) faces a component \( \Delta_1 \) of \( \Omega(\Gamma) \). Set

\[
F^0_{(0,t)} = \overline{B^3 \setminus \overline{E^0_{(0,t)}}} \cup \partial E^0_{(0,1)}.
\]

Since

\[
\overline{B^3 \setminus \overline{E^0_{(0,t)}}} = E^0_{(t,1)} \cup \Delta_1
\]

and \( \partial E^0_{(0,t)} = \partial E^0_{(t,1)} \), we have that

\[
F^0_{(0,t)} = E^0_{(t,1)} \cup \partial E^0_{(t,1)} \cup \Delta_1,
\]

and so \( F^0_{(0,t)}/\Gamma \) is compact. That is, if \( E \) is a peripheral end of \( M_\Gamma \) and if its complementary end faces a component of \( \Omega(\Gamma) \), then \( E \) is bounded. If its complementary end does not face a component of \( \Omega(\Gamma) \), then \( E \) is not bounded, as the boundary at infinity of its complementary end is empty.

To summarize:

**Proposition 5.2** Let \( \Gamma \) be a purely loxodromic Kleinian group isomorphic to the fundamental group of a closed orientable surface \( S \) of negative Euler characteristic. Then, a peripheral end of \( \Gamma \) is bounded if and only if its complementary end is finite.

Say that \( \Gamma \) is *quasifuchsian* if \( \Lambda(\Gamma) \) is a Jordan curve and no element of \( \Gamma \) interchanges the two components of \( S^2 \setminus \Lambda(\Gamma) \). In particular, \( \Omega(\Gamma) \) has two components, each
invariant under \( \Gamma \). In this case, both peripheral ends of \( \Gamma \) are finite and both peripheral ends of \( \Gamma \) are bounded.

Say that \( \Gamma \) is **degenerate** if \( \Omega(\Gamma) \) is non-empty and consists of one simply connected component. In this case, \( \Gamma \) has one finite peripheral end and one infinite peripheral end. The peripheral end not facing the one component of \( \Omega(\Gamma) \) is infinite, while the peripheral end facing the component of \( \Omega(\Gamma) \) is finite. Moreover, the infinite peripheral end is bounded, while the finite peripheral end is not bounded. A degenerate group \( \Gamma \) for which there is a global positive lower bound on the injectivity radius over all of \( M_\Gamma \) is sometimes known as a **hyperbolic half-cylinder**; these are the hyperbolic 3-manifolds considered by Sullivan in [28].

Say that \( \Gamma \) is **doubly degenerate** if \( \Lambda(\Gamma) = S^2 \). Examples of doubly degenerate groups include the fiber covering of a closed hyperbolic 3-manifold fibering over \( S^1 \), though there are examples of doubly degenerate groups that are not associated to fibered 3-manifolds. In this case, both peripheral ends of \( \Gamma \) are infinite, and neither peripheral end of \( \Gamma \) is bounded.

In general, we have the following basic construction of the complete collection of ends of any non-elementary, finitely generated, purely loxodromic Kleinian group acting on \( B^3 \). (We note that the following proposition also holds for elementary Kleinian groups, but we leave the proof to the interested reader.)

**Proposition 5.3** Let \( \Gamma \) be a purely loxodromic, topologically tame Kleinian group acting on \( B^3 \). Then \( M_\Gamma \) has a complete collection of ends such that every end \( E \in \mathcal{E} \) is peripheral and every end \( E \in \mathcal{E} \) is either finite or infinite.

**Proof** Let \( Y \) be a compact core for \( M_\Gamma \). Note that \( \partial Y \) is empty if and only if \( M_\Gamma \) is itself compact. In this case, the collection of ends of \( M_\Gamma \) is empty.

Suppose now that \( M_\Gamma \) has infinite volume. The ends will be the components of the complement of \( Y \). Enumerate the components of \( \partial Y \) as \( \partial Y = \bigcup_{j=1}^{p} S_j \), where \( S_j \) faces a component of \( \Omega(\Gamma)/\Gamma \) for \( 1 \leq j \leq \ell \), where \( \ell \leq p \). Let \( E_j \) be the component of \( M_\Gamma \setminus Y \) facing \( S_j \). Note that \( S_j = \partial E_j \) is a connected component of \( \partial Y \), and hence is a separating surface in \( M_\Gamma \), and that \( E_j \) is a non-compact component of \( M_\Gamma \setminus \partial E_j \), and so \( \mathcal{E} = \{E_1, \ldots, E_p\} \) forms a complete collection of ends for \( M_\Gamma \). Moreover, by construction, each end in \( \mathcal{E} \) is peripheral.

It remains only to determine the types of the ends in \( \mathcal{E} \). The natural retraction of \( \overline{M_\Gamma} \) onto \( C_\Gamma \) yields that the ends \( E_1, \ldots, E_\ell \) facing the components of \( \Omega(\Gamma)/\Gamma \) are finite. The ends \( E_{\ell+1}, \ldots, E_p \) are essentially contained in the convex core \( C_\Gamma \) of \( M_\Gamma \), and hence are infinite. (By essentially here, we mean that for each \( \ell + 1 \leq j \leq p \), there is a peripheral end \( E'_j \) of \( M_\Gamma \) contained in the convex core \( C_\Gamma \) of \( M_\Gamma \) for which the symmetric difference of \( E_j \) and \( E'_j \) has compact closure.)

**QED**

We note here that with the definition given here, there are many ends that are not peripheral, as they are not components of the complement of a compact core. The following example contains an example of such a non-peripheral end. Consider a non-elementary, finitely generated, purely loxodromic Kleinian group \( \Gamma \) whose quotient \( M_\Gamma \) has the following structure: Let \( N \) be a compact hyperbolizable acylindrical 3-manifold with 3 boundary components \( S_1, S_2, \) and \( S_3 \). Put a hyperbolic structure on
the interior $\text{int}(N) = M_\Gamma$ in such a way that the inclusion of $\pi_1(S_1)$ into $\Gamma$ is Fuchsian, the inclusion of $\pi_1(S_2)$ into $\Gamma$ is quasifuchsian, and the inclusion of $\pi_1(S_3)$ into $\Gamma$ is a degenerate group. (Such a manifold $N$ can be constructed by taking the complement of a sufficiently complicated 3 component graph in $S^3$, and such a hyperbolic structure can be constructed by taking an appropriate limit of geometrically finite hyperbolic structures on $\text{int}(N)$.) Doubling across $S_1$, we obtain a hyperbolic 3-manifold $P$ so that $P$ contains a separating totally geodesic surface $S$ (corresponding to $S_1$). The two components of $P \setminus S$, namely the two ends $E_0$ and $E_1$ associated to $S$, are isomorphic (by reflection across $S$), and $S = \partial E_k$ is incompressible in $P$. Neither end is finite nor infinite, since it is not contained in the convex core of $P$, and neither end faces a component of the domain of discontinuity of the Kleinian group uniformizing $P$. (In fact, each end has subends that do both of these.) In this case, though, both ends are bounded: since the subgroup $\Phi$ of the Kleinian group uniformizing $P$ corresponding to $S$ is Fuchsian, the complementary end of each $E_k$ is half of the Fuchsian manifold $\overline{M_\Phi}$.

Now, let $\Gamma$ be a purely loxodromic, topologically tame Kleinian group, let $E$ be an end of $M_\Gamma$, and suppose that $\partial E$ is a separating surface in $M_\Gamma$. (We restrict to the case that $\partial E$ is a surface for ease of exposition.) Say that $E$ is an incompressible end if $\partial E$ is incompressible and if the inclusion of $\partial E$ into $E \cup \partial E$ induces an isomorphism of fundamental groups. We can characterize the incompressible ends of $M_\Gamma$.

**Proposition 5.4** Let $\Gamma$ be a purely loxodromic, topologically tame Kleinian group which is not isomorphic to the fundamental group of a closed, orientable surface of negative Euler characteristic, and let $E$ be an incompressible end of $M_\Gamma$ with end group $\Phi$. Then, either $\Phi$ is quasifuchsian and $E$ is finite, or $\Phi$ is degenerate and $E$ is infinite; in either case, $E$ is bounded.

**Proof** Since $\partial E$ is an incompressible surface, we have that $\Phi$ is a purely loxodromic Kleinian group that is isomorphic to the fundamental group of a closed, orientable surface of negative Euler characteristic, and hence is either quasifuchsian, degenerate, or simply degenerate. (This fact is standard. See e.g. Anderson [4].) Moreover, by the assumption made on $\Gamma$, it must be that $\Phi$ has infinite index in $\Gamma$.

This immediately implies that $\Phi$ cannot be doubly degenerate: if $\Phi$ is doubly degenerate, then by the end covering theorem of Canary [10], the covering $M_\Phi \to M_\Gamma$ is one-to-one, a contradiction. So, $\Phi$ is either quasifuchsian or degenerate. Note that $E$ can then also be considered an end of $M_\Phi$. In particular, $E$ is either finite or infinite, since this dichotomy holds for ends of quasifuchsian and degenerate groups, see Proposition 5.1.

It remains only to show that $E$ cannot be the finite end of a degenerate group. So, suppose that $E$ is a finite end of $M_\Gamma$ and that $\Phi$ is degenerate. We can view $E$ as an end of $M_\Phi$ as well. Let $E'$ be the complementary end of $E$ in $M_\Phi$. Since $\Phi$ is degenerate, $E'$ is infinite, and so the end covering theorem implies that the restriction of the covering map $\pi : M_\Phi \to M_\Gamma$ to $E'$ is finite-to-one. Since the image of $E$ (viewed as an end of $M_\Phi$) under $\pi$ is just $E$ (viewed as an end of $M_\Gamma$), the image of $E'$ in $M_\Gamma$ is the complementary end of $E$ in $M_\Gamma$. Since the restriction of $\pi$ to $E'$ is finite-to-one,
this implies that $\Phi$ has finite index in $\Gamma$, and so $\Gamma$ is isomorphic to the fundamental group of a surface, a contradiction.

Since $E$ is either the infinite end of a degenerate group, or is a finite end of a quasifuchsian group, we see by Proposition 5.2 that $E$ is bounded. \textbf{QED}

In general, it is not possible to come up with a crisp statement of when an end is bounded. We present here a few examples to show what sorts of things can go wrong. There are some trivial situations in which an end is always bounded. For example, let $\Gamma$ be any non-elementary Kleinian group, and let $H$ be a small closed 3-ball embedded in $M_\Gamma$. Then, under the definition we have given here, the complement $E = M_\Gamma \setminus H$ is an end of $M_\Gamma$. By definition, $E$ is bounded as $H$ is compact.

This example also illustrates why we have restricted our attention to peripheral ends. Suppose that $M_\Gamma$ has several peripheral ends. By taking the complement of a small compact set in $M_\Gamma$, we can construct an end of $M_\Gamma$ that contains several peripheral ends of $M_\Gamma$. The behavior of such an end then becomes very complicated from the interaction of the several peripheral ends it contains.

Now, consider the following example of a peripheral end with compressible boundary. Let $\Phi_1$ be a purely loxodromic quasifuchsian group, and let $\Phi_2$ be a purely loxodromic degenerate group. Let $\Gamma$ be the Klein combination of $\Phi_1$ and $\Phi_2$ (see Maskit [16]). Then, there is a component $\Delta$ of $\Omega(\Gamma)$ that is invariant under all of $\Gamma$. The peripheral end of $M_\Gamma$ corresponding to $\Delta/\Gamma$ is then a finite end of $M_\Gamma$. This end is not bounded, since its complementary piece contains the peripheral end corresponding to the degenerate group. Note that the end group corresponding to this end is the whole group $\Gamma$. However, using the Klein-Maskit combination theorems, it is possible to realize this example in more general Kleinian groups.

Finally, we can form a Kleinian group $\Gamma$ that is the free product of the quasifuchsian group $\Phi_1$ and the degenerate group $\Phi_2$ in such a way that $\Gamma$ contains no parabolic elements and so that every component of $\Omega(\Gamma)$ is invariant under a conjugate of $\Phi_1$. (This Kleinian group is similar to the one constructed in the previous paragraph, except that the invariant component has degenerated and is no longer visible in the domain of discontinuity. This construction is adapted from a construction due to Maskit; see Section 5 of [17].) In this case, $M_\Gamma$ has three peripheral ends: one finite end with incompressible boundary, corresponding to the single surface $\Omega(\Gamma)$; one infinite end with incompressible boundary, coming from the degenerate group $\Phi_2$, and one infinite end with compressible boundary. Note that the end group corresponding to the infinite end with compressible boundary is the whole group $\Gamma$, and this infinite end with compressible boundary is not bounded. As in the previous paragraph, it is possible to realize this example in more general Kleinian groups using the Klein-Maskit combination theorems.

We close this section by stating a mild refinement of the tripartite division of points of the sphere at infinity $S^2$ of $B^3$ as given in Lemma 5.1. The proof of Lemma 5.5 is essentially contained in the proofs of Lemma 3.1 and Proposition 5.4. The main distinction is that for topologically tame Kleinian groups, we are able to remove the assumption of boundedness of the ends.
Lemma 5.5 Let $\Gamma$ be a purely loxodromic, topologically tame Kleinian group acting on $B^3$.

If $\Gamma$ is doubly degenerate, let $E_0^0$ and $E_0^1$ be peripheral ends of $\Gamma$, and note that the associated end groups of $E_0^0$ and $E_0^1$ are both $\Gamma$. Let $z \in S^2$. Then either $z \in \Lambda_c(\Gamma)$, $z \in \Lambda_c(E_0^0)$, or $z \in \Lambda_c(E_0^1)$.

Suppose that $\Gamma$ is not doubly degenerate, and let $E^0$ be an incompressible infinite peripheral end of $\Gamma$. Let $\Phi$ be the end group of $\Gamma$ associated to $E^0$, so that $\Phi$ is degenerate. Let $z \in S^2$. Then either $z \in \Omega(\Phi)$, $z \in \Lambda_c(\Phi)$ or $z$ is an end limit point of $E^0$. If $z \in \Lambda(\Gamma)$ and $z$ is an endpoint of $E^0$, then $z \in \Lambda_c(E^0)$.

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