The Gauss product and Raabe’s integral for \(k\)-gamma functions

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Abstract: We obtain an extension of the famous Gauss product formula to the case of \(k\)-gamma functions. The Sándor–Tóth short product formula [16] is also attended to these functions. An asymptotic formula and Raabe’s integral analogue are also considered.

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1 Introduction

As a generalization of the classical Euler gamma function \(\Gamma(x)\), in 2007 R. Diaz and E. Pariguan [6] have introduced and studied the notion of \(k\)-gamma function.

For \(k > 0\), the \(\Gamma_k\)-function is defined by

\[
\Gamma_k(x) = \lim_{n \to \infty} \frac{n!h^n(nk)^{\frac{x}{k} - 1}}{(x)_{n,k}},
\]

for \(x \in \mathbb{C} \setminus k\mathbb{Z}^-\), where \(\mathbb{C}\) is the set of complex numbers, \(\mathbb{Z}^-\) is the set of negative integers, \((x)_{n,k}\) denotes the classical Pochhammer symbol \((x)_{n,k} = x(x + k)(x + 2k)\ldots(x + (n - 1)k)\).

For \(x \in \mathbb{C}\), with \(\text{Re}(x) > 0\), it can be proved the integral representation [6]

\[
\Gamma_k(x) = \int_{0}^{\infty} t^{x-1}e^{-\frac{t}{k}} dt.
\]
Also, it satisfies the following properties [6]:

\[(i)\] \(\Gamma_k(x + k) = x\Gamma_k(x),\)

\[(ii)\] \(\frac{\Gamma_k(x + nk)}{\Gamma_k(x)} = (x)_{n,k},\)

\[(iii)\] \(\Gamma_k(k) = 1,\)

\[(iv)\] \(\frac{1}{\Gamma_k(x)} = x^{-\frac{2}{k}}e^{\frac{2}{k}\gamma} \prod_{n=1}^{\infty} \left( 1 + \frac{x}{nk} \right) e^{-\frac{2}{k}\pi}.\) (3)

It is obvious also that \(\Gamma_1(x) \equiv \Gamma(x).\)

One of the motivations of introduction of the \(\Gamma_k(x)\)-function is in its connection with the symbol \((x)_{n,k}\) which appears in a variety of contexts (see [5] and the references). In the recent years, there is an increasing interest about the \(k\)-gamma function (see, e.g., [5, 6, 8–11]).

The famous short product formula of Gauss for the Euler gamma function states that one has the identity

\[\Gamma \left( \frac{1}{n} \right) \Gamma \left( \frac{2}{n} \right) \ldots \Gamma \left( \frac{n-1}{n} \right) = \frac{(2\pi)^{n-1}}{\sqrt{n}}.\] (4)

In 1989, J. Sándor and L. Tóth [16] studied the short product

\[
\prod_{l=1, (l,n)=1}^{n} \Gamma \left( \frac{l}{n} \right) = \frac{(2\pi)^{\varphi(n)}}{e^{\Lambda(n)/2}},
\] (5)

where \(\varphi(n)\) is the Euler totient function, and \(\Lambda(n)\) is the von Mangoldt function. This paper has evoked large interest, see, e.g. [1–4, 12–15]. Particularly, the recent paper by M. E. Bachraoui and J. Sándor [2] offers an extension of (5) to the \(\Gamma_q\)-function, which is a classical extension of gamma function, due to F. H. Jackson (see the references from [2]).

The aim of this paper is to extend (4) and (5) to the case of \(k\)-gamma functions.

### 2 Main results

The main results are contained in the following.

**Theorem 2.1.** One has the identity

\[
\prod_{l=1}^{n-1} \Gamma_k \left( \frac{kl}{n} \right) = \left( \frac{2\pi}{k} \right)^{\frac{n-1}{2}} \frac{1}{\sqrt{n}}. \] (6)

**Theorem 2.2.** One has the identity

\[
P_k(n) = \prod_{l=1, (l,n)=1}^{n} \Gamma_k \left( \frac{kl}{n} \right) = \frac{(2\pi)^{\varphi(n)}}{e^{\Lambda(n)/2}} \begin{cases} \frac{(2\pi)^{\varphi(n)}}{\sqrt{p}} & \text{for } n = p^m, \\ \frac{(2\pi)^{\varphi(n)}}{\sqrt{p}} & \text{for } n \neq p^m \end{cases}, \] (7)

where \(p\) is an arbitrary prime, and \(m\) is an arbitrary positive integer.
Theorem 2.3. One has the following Raabe type integral formula
\[
\int_0^1 \log \Gamma_k(kx) \, dx = \log \sqrt{\frac{2\pi}{k}}.
\]
(8)

Theorem 2.4. One has the following asymptotic formula
\[
\sum_{n \leq x} \log P_k(n) = 3 \log \left( \frac{2\pi}{k} \right) \frac{x^2}{2\pi^2} + O(x \log x),
\]
(9)
where \( P_k(n) \) is defined in Theorem 2.2.

First, one needs the following auxiliary result.

Lemma 2.1. The following extension of the Euler reflexion formula holds true:
\[
\Gamma_k(x) \Gamma_k(k - x) = \frac{\pi}{k \sin \left( \frac{\pi x}{k} \right)}.
\]
(10)

Proof. By using the fundamental identity (i) of (3) one can write that \( \Gamma_k(k - x) = -x \Gamma_k(-x) \).
By the Weierstrass type relation (iv) of (3) one gets
\[
\frac{1}{\Gamma_k(x) \Gamma_k(k - x)} = x \prod_{n=1}^\infty \left( 1 - \frac{x^2}{n^2 k^2} \right)
\]
(where we have omitted some obvious computations). Now, by the classical Euler formula
\[
\frac{\sin \pi x}{\pi x} = \prod_{n=1}^\infty \left( 1 - \frac{x^2}{n^2} \right)
\]
(11)
with the application for \( x := \frac{x}{k} \), identity (10) follows.

The following auxiliary result was stated first by A. Hurwitz ([7, 16]).

Lemma 2.2 Let \( s : [0, 1] \to \mathbb{C} \) be an arbitrary function, and put
\[
f(n) = \sum_{k \in A(n)} s \left( \frac{k}{n} \right), \quad g(n) = \sum_{k=1}^n s \left( \frac{k}{n} \right),
\]
where \( A(n) = \{ l : 1 \leq l \leq n, (l, n) = 1 \} \). Then one has
\[
f(n) = \sum_{d|n} \mu(d) g \left( \frac{n}{d} \right),
\]
(12)
where \( \mu \) is the classical Möbius function.

Corollary 2.1 If \( F(n) = \prod_{k \in A(n)} s \left( \frac{k}{n} \right) \) and \( G(n) = \prod_{k=1}^n s \left( \frac{k}{n} \right) \), then
\[
F(n) = \prod_{d|n} \left( G \left( \frac{n}{d} \right) \right)^{\mu(d)}.
\]
(13)

Proof. This follows by letting \( f = \ln F \) and \( g = \ln G \) in Lemma 2.2.

□
3 Proofs of the theorems

Proof of Theorem 2.1. Letting \( x = \frac{kl}{n} \) in identity (10), we get

\[
\Gamma_k \left( \frac{kl}{n} \right) \Gamma_k \left( k \left( 1 - \frac{l}{n} \right) \right) = \frac{\pi}{k} \frac{1}{\sin \frac{\pi l}{n}}.
\]  

(14)

By remarking that, when \( l = 1, 2, \ldots, n - 1 \) one has

\[
\prod_{l=1}^{n-1} \Gamma_k \left( k \left( 1 - \frac{l}{n} \right) \right) = \prod_{l=1}^{n-1} \Gamma_k \left( k \frac{l}{n} \right),
\]

as \( 1 - \frac{l}{n} = \frac{n-l}{n} \), and applying identity (14) to \( l = 1, 2, \ldots, n-1 \), by term-by-term multiplication of the of the obtained relation, we get

\[
\left( \prod_{l=1}^{n-1} \Gamma_k \left( \frac{kl}{n} \right) \right)^2 = \left( \frac{\pi}{k} \right)^{n-1} \frac{1}{\prod_{l=1}^{n-1} \sin \frac{\pi l}{n}} = \left( \frac{\pi}{k} \right)^{n-1} \frac{2^{n-1}}{n}
\]

by the well-known trigonometric identity \( \prod_{l=1}^{n-1} \sin \frac{\pi l}{n} = \frac{2^{n-1}}{n} \).

Now, relation (6) follows at once from the above. □

Proof of Theorem 2.2. By Theorem 2.1 and Corollary 2.1, the left-hand side of (7) can be written as

\[
\left( \frac{2\pi}{k} \right)^{\frac{1}{2} \sum_{d|n} \mu(d) \left( \frac{n}{d} \right) - \frac{1}{2} \sum_{d|n} \mu \left( \frac{n}{d} \right)} \right)^{\frac{1}{2}} \frac{1}{\sqrt{h(n)}},
\]

where \( h(n) = \prod_{d|n} d \mu \left( \frac{n}{d} \right) \).

Now, it is well-known that (see, e.g., [7]) \( \sum_{d|n} \mu \left( \frac{n}{d} \right) = \sum_{d|n} \mu(d) = \varphi(n) \) and \( \sum_{d|n} \mu \left( \frac{n}{d} \right) = \sum_{d|n} \mu(d) = 0 \).

Also, \( \log h(n) = \Lambda(n) = \log p \) if \( n = p^m \), and it is equal to 0, if \( n \neq p^m \). Thus, identity (7) follows. □

Proof of Theorem 2.3. We will use the classical Riemann sum approach, based on the limit formula

\[
\int_0^1 f(x)dx = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{i}{n} \right).
\]

(15)

Let \( f(x) = \log \Gamma_k(kx) \). By Theorem 2.1, and relation (15) one has

\[
\int_0^1 \log \Gamma_k(kx)dx = \lim_{n \to \infty} \left( \frac{n-1}{2n} \log \left( \frac{2\pi}{k} \right) - \frac{1}{2n} \log n \right) = \frac{1}{2} \log \left( \frac{2\pi}{k} \right).
\]

Thus gives relation (8). □
Proof of Theorem 2.4. By Theorem 2.2 one can write

\[
\sum_{n \leq x} \log P_k(n) = \sum_{n \leq x} \left( \frac{\varphi(n)}{2} \log \left( \frac{2\pi}{k} \right) - \frac{1}{2} \Lambda(n) \right)
\]

\[
= \frac{1}{2} \log \frac{2\pi}{k} \varphi(n) - \frac{1}{2} \sum_{n \leq x} \Lambda(n)
\]

\[
= \frac{1}{2} \log \frac{2\pi}{k} \left( \frac{3}{\pi^2} x^2 + O(x \log x) \right) - \frac{1}{2} O(x)
\]

\[
= \frac{3}{2}\frac{\log \frac{2\pi}{k} x^2 + O(x \log x),
\]

where we have used the classical asymptotic relations (see, e.g., [7]):

\[
\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x),
\]

and

\[
\sum_{n \leq x} \Lambda(n) = O(x).
\]

This completes the proof.

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