Non-Gaussian limit of a tracer motion in an incompressible flow
by
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Abstract. We consider a massless tracer particle moving in a random, stationary, isotropic and divergence free velocity field. We identify a class of fields for which the limit of the laws of appropriately scaled tracer trajectory processes is a non-Gaussian Rosenblatt type process.

1. Introduction. We consider the motion of a massless tracer particle in a random field described by an ordinary differential equation (O.D.E.)

\[
\frac{dx(t)}{dt} = V(t, x(t)), \quad x(0) = x_0.
\]

Here \(V = (V_1, \ldots, V_d) : \mathbb{R}^{1+d} \times \Omega \to \mathbb{R}^d\) is a \(d\)-dimensional random vector field defined over a probability space \((\Omega, \mathcal{V}, \mathbb{P})\). It is one of the most popular models of transport in turbulence considered in statistical hydrodynamics that can be traced back to the works of G. Taylor [40] in the 1920’s and R. Kraichnan [23] in the 1970’s. There exists an extensive literature concerning this model: see, e.g., [13, 8, 29, 32] and the references therein. It is often assumed that the field \(V(t, x)\), thought of as the Eulerian velocity of the fluid, is incompressible, i.e.

\[
\nabla_x \cdot V(t, x) := \sum_{j=1}^{d} \partial_{x_j} V_j(t, x) \equiv 0.
\]

Another assumption frequently made about the velocity flow is its stationarity and statistical isotropy. The above means that the law of the random field \((V(t, x))_{(t, x) \in \mathbb{R}^{1+d}}\) does not depend on either temporal or spatial translations and is also invariant under the action of the group of rotations in space. As

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a consequence of the incompressibility and stationarity of the flow the velocity observed along the trajectory \((V(t, x(t)))_{t \in \mathbb{R}}\), the so called Lagrangian process, is stationary \cite[Theorem 2, p. 500]{35}.

When the field is of zero mean and its fluctuations are small, i.e. \((1.1)\) can be written in the form
\[
\frac{dx(t)}{dt} = \varepsilon V(t, x(t)), \quad x(0) = x_0,
\]
where \(\varepsilon > 0\) is a certain parameter (which turns out to be small), then the scaled trajectory of the tracer, \(x(\varepsilon^{-2} t)\), satisfies the central limit theorem (CLT) as \(\varepsilon \to 0^+\) provided that the field is sufficiently strongly mixing in the temporal variable (see \cite{21} and also \cite{20}). More precisely, one can show the functional CLT, i.e. the continuous trajectory processes \((x(\varepsilon^{-2} t))_{t \geq 0}\) converge in law as \(\varepsilon \to 0^+\) to the process \(\sqrt{D} B_t\), where \((B_t)\) is a standard \(d\)-dimensional Brownian motion and the matrix \(D\) is given by the Taylor–Kubo formula
\[
D_{j,j'} := \int_0^\infty \{ \mathbb{E}[V_j(t,0)V_{j'}(0,0)] + \mathbb{E}[V_{j'}(t,0)V_j(0,0)] \} dt, \quad j,j' = 1, \ldots, d.
\]

Consider now a stationary and isotropic, divergence free field whose energy spectrum satisfies the power law. Its covariance matrix can be written as
\[
R_{j,j'}(t,x) = \mathbb{E}[V_j(t+t',x+x')V_{j'}(t',x')]
= \int_{\mathbb{R}^{d+1}} \exp\{i k \cdot x\} \Gamma_{j,j'}(k) \frac{\hat{\chi}(t,|k|)}{|k|^{d-1}} dk,
\]
for any \(j,j' = 1, \ldots, d\), \(t,t' \in \mathbb{R}\), and \(x,x' \in \mathbb{R}^d\). Here \(i = \sqrt{-1}\) and
\[
\hat{\chi}(t,\xi) = e^{-r(\xi)|t|} a(\xi) \mathcal{E}_0(\xi), \quad (t,\xi) \in \mathbb{R} \times \mathbb{R}_+.
\]
We assume that the cut-off function \(a(\cdot)\) is continuous, non-negative, compactly supported with \(a(0) > 0\). The parameter \(r(\cdot)\), determining the mixing rate in time, and the power-energy density spectrum \(\mathcal{E}_0(\cdot)\) are both non-negative and satisfy \(r(\xi) \sim \xi^{2\beta}\) and \(\mathcal{E}_0(\xi) \sim \xi^{1-2\alpha}\) as \(\xi \ll 1\) for some parameters \(\alpha, \beta\). The factor \(\Gamma_{j,j'}(k) := \delta_{j,j'} - \delta_j k_k k_{j'} / |k|^2\) ensures that the realizations of the field are incompressible. We can see from \((1.4)\) that the spectrum is integrable in \(k\) and the mixing rate decays on large spatial scales provided that \(\alpha < 1\) and \(\beta \geq 0\).

Random fields whose covariance is given by \((1.4)\) play an important role in statistical hydrodynamics. According to the theory of Kolmogorov and Obukhov \cite{22,33}, the Eulerian velocity of a fully turbulent flow of a fluid can be described by a locally isotropic, time-space homogeneous random field whose energy spectrum is of the aforementioned form.
For a random field whose covariance is given by (1.4) the coefficients \( D_{j,j'} \) appearing in the Taylor–Kubo formula (1.3) are finite if and only if

\[
\int_0^\infty \frac{a(\xi)}{\xi^{2\alpha+2\beta-1}} d\xi < \infty,
\]

which leads to the condition \( \alpha + \beta < 1 \). For a tracer motion in a Gaussian field whose covariance satisfies (1.6), the functional CLT has been proved in [10]. One can inquire what would happen when the condition (1.6) is violated. The conjecture that the trajectory evolves on a shorter, superdiffusive scale turns out to be true [11]: it has been shown that if \( \alpha + \beta > 1 \) (and \( \alpha < 1 \)), then the scaled processes \((x(t\varepsilon^{-2\delta}))\) with \( \delta := \beta/(\alpha+2\beta-1) \) converge in law as \( \varepsilon \to 0^+ \) to a zero mean fractional Brownian motion with Hurst exponent \( H = 1/(2\delta) \).

A natural question arises whether a fractional Brownian motion is a universal limit for an anomalous diffusive scaling of the tracer particle. More precisely, take a family of non-Gaussian fields with covariance (1.4) and suppose \( \alpha + \beta > 1 \), so the diffusivity given by the Kubo formula is no longer finite. Does the limit of an appropriately scaled trajectory \((x(t\varepsilon^{-2\delta}))\) tend to a Gaussian as \( \varepsilon \to 0^+ \)? The experimental data in general seem to confirm Gaussianity of the tracer distribution (see, e.g., [16]), but the non-Gaussianity is also observed in some numerical experiments [18], and explicitly solvable models [29, Sec. 5.2.2] and [28, 31, 42]. We are not aware of any rigorous result for isotropic flows that leads to a non-Gaussian limit for the tracer process. We mention here a recent physics paper [17], where a closely related passive scalar model with a non-Gaussian drift has been studied.

In the present paper we consider time stationary and spatially homogeneous random fields \( V(t,x) \) whose components belong to the space \( \mathcal{H}_2 \) in the Fock decomposition over some Gaussian Hilbert space. In addition the fields have an isotropic covariance tensor (see Remark 2.2). We show that the limit of the tracer particle is then a stochastic process that is related to a Rosenblatt process (see Theorem 2.6, Remark 2.10 and Proposition 2.9 below). Rosenblatt processes have been introduced by Taqqu [38]. They are self-similar, with stationary increments and “live” in the second Wiener chaos (see (2.21) for one of possible representations). Unlike in the Gaussian case, in the second chaos there exist many self-similar processes with stationary increments (see [26]), but Rosenblatt processes are the simplest and are considered to be the second chaos counterparts of fractional Brownian motions. These processes, along with other self-similar processes with stationary increments, living in the second and higher order chaos spaces, have attracted quite a lot of interest lately. Some recent relevant literature concerning the
subject includes [39, 41, 26, 27, 1, 7, 4, 5]. Our limit is not strictly speaking a Rosenblatt process, but it is closely related to a process of that type (see (2.20) and Proposition 2.9). To the best of our knowledge, the representation of the Rosenblatt process given in (2.22) seems to be new.

Concerning the proof of our main result (Theorem 2.6), we compute the limits of the moments of the trajectory process (Theorem 4.7). In fact, for convenience we consider the moments of its statistically equivalent version introduced in (4.8) below. These limits coincide with the corresponding moments of the limiting process in question. This in turn implies the convergence of finite-dimensional laws, as the respective moment problem for the limiting law is well posed. The proof of Theorem 4.7 relies on showing that the limits of the moments of the trajectory process coincide with the limits of the respective moments of the approximate process
\[ \varepsilon^{d/\varepsilon^2} \int_0^1 V(s, x_0) \, ds, \]
on obtained by “freezing” the right hand side at the initial position of the tracer. The convergence of the latter is quite straightforward due to Proposition 4.4. To argue that the “true” trajectory can be replaced by its approximate counterpart we use the Taylor expansion (4.31). An important role in the analysis of the arising terms is played by the representation of the products of multiple stochastic integrals in terms of integrals of higher order with the help of appropriate Feynman diagrams (Proposition 3.1) and the formula for their conditional moments (Proposition 3.4).

The organization of the paper is as follows: in Section 2 we introduce basic notions and formulate our main result, Theorem 2.6. In Section 3 we recall some basic facts about multiple stochastic integration in the context of Gaussian stochastic measures. Section 4 is devoted to the proof of the main result. Finally, in Section 5 we prove Proposition 2.9 concerning a representation of Rosenblatt processes.

2. Preliminaries and formulation of the main result

2.1. Some basic notation. We shall use the following notation: for any functions \( f, g : A \to \mathbb{R} \) on some cone \( A \subset \mathbb{R}^d \) we write

\[ f(x) \preceq g(x), \quad x \in A, \]

if there exist \( C, c > 0 \) such that

\[ f(x) \leq Cg(cx), \quad x \in A. \]

We write

\[ f(x) \approx g(x), \quad x \in A, \]

if

\[ f(x) \preceq g(x) \quad \text{and} \quad g(x) \preceq f(x), \quad x \in A. \]
We shall also denote by \( \mathbb{Z}_+ \) and \( \mathbb{N} \) the sets of all non-negative and all positive and integers, respectively, \( \mathbb{Z}_d := \{1, \ldots, d\} \) for some positive integer \( d \), and \( \mathbb{R}_+, \mathbb{R}_+ \) stand for \((0, \infty)\) and \([0, \infty)\), respectively.

Throughout the paper \( C, C_i \) always denote positive constants which may vary from line to line. Possible dependence on parameters is indicated in parentheses.

With boldface we denote vectors, e.g. \( t = (t_1, \ldots, t_n) \); to stress the dimension we also often write \( t_{1,n} \), we also write \( t_{m,n} = (t_m, \ldots, t_n) \) for \( m \leq n \). The differentials \( ds_m \ldots ds_n \) for \( m \leq n \) are abbreviated as \( ds_{m,n} \).

\( \Delta_n \) stands for the \( n \)-dimensional simplex
\[
\Delta_n = \{(s_1, \ldots, s_n) \in \mathbb{R}_+^n : s_1 \geq \cdots \geq s_n \geq 0 \}.
\]

We shall also frequently use sets of the form
\[
\square(t_{1,m}) := [0, t_1] \times \cdots \times [0, t_m], \quad \Delta(t_{1,m}) := \Delta_m \cap \square(t_{1,m}),
\]
\[
\Delta_N(t_{1,m}) := \Delta_{m+N} \cap (\square(t_{1,m}) \times \mathbb{R}^N),
\]
\[
\Delta_N(s', s'') := \{(s_1, \ldots, s_N) : s'' \geq s_1 \geq \cdots \geq s_N \geq s' \}
\]
for \( 0 \leq s' \leq s'' \),
\[
\Delta_m(t_{1,m}, s) = \{(s_1, \ldots, s_m) \in \Delta(t_{1,m}) : s_m \geq s \} \quad \text{for} \ s \geq 0.
\]

### 2.2. Space-time white noise field

Denote by \( L^2(\mathbb{R}^{1+d}) \) the complex Hilbert space of all \( \phi : \mathbb{R}^{1+d} \to \mathbb{C} \) for which
\[
\|\phi\|^2_{L^2(\mathbb{R}^{1+d})} = \int_{\mathbb{R}^{1+d}} |\phi(t, k)|^2 \, dt \, dk < \infty.
\]
The scalar product on \( L^2(\mathbb{R}^{1+d}) \) will be denoted by \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^{1+d})} \).

We let \( L^2_{(s)}(\mathbb{R}^{1+d}) \) be the real Hilbert space consisting of \( \phi \in L^2(\mathbb{R}^{1+d}) \) that satisfy \( \phi^*(t, k) = \phi(t, -k) \). Here \( z^* \) denotes the complex conjugate of \( z \in \mathbb{C} \). The scalar product on \( L^2_{(s)}(\mathbb{R}^{1+d}) \) is given by the formula
\[
\langle \psi, \phi \rangle_{L^2(\mathbb{R}^{1+d})} = \int_{\mathbb{R}^{1+d}} \psi(t, k)\phi(t, -k) \, dt \, dk, \quad \psi, \phi \in L^2_{(s)}(\mathbb{R}^{1+d}).
\]

Fix an orthonormal basis \( (e_m)_{m\geq 1} \) in \( L^2_{(s)}(\mathbb{R}^{1+d}) \) and let \( (\xi_{j,m}), \ j = 1, \ldots, d, \ m = 1, 2, \ldots, \) be i.i.d. one-dimensional, real valued standard normal random variables defined on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). A space-time, \( d \)-dimensional vector valued white noise \( W(dt, dk) = (W_1(dt, dk), \ldots, W_d(dt, dk)) \), \( (t, k) \in \mathbb{R}^{1+d} \), can be defined as an \( \mathbb{R}^d \)-valued stochastic measure, over \( (\Omega, \mathcal{F}, \mathbb{P}) \), in the following way: for any \( \phi \in L^2_{(s)}(\mathbb{R}^{1+d}) \) we let
\[
\langle \phi, W_j \rangle := \sum_{m=1}^{\infty} \xi_{j,m} \langle \phi, e_m \rangle_{L^2(\mathbb{R}^{1+d})}, \quad j = 1, \ldots, d.
\]
The series on the right hand side of (2.2) converges both a.s. and in $L^2$. Observe that, by the Parseval identity,

\[
\mathbb{E}[\langle \psi, W_j \rangle \langle \phi, W_{j'} \rangle] = \delta_{j,j'} \mathbb{E}[\langle \psi, \phi \rangle L^2_{(s)}(\mathbb{R}^{1+d})],
\]

\(j, j' = 1, \ldots, d, \psi, \phi \in L^2_{(s)}(\mathbb{R}^{1+d})\),

where $\delta_{j,j'}$ denotes the Kronecker delta.

2.3. Velocity field. In the present section we introduce the random fields considered throughout the paper. The assumptions made in this section are valid throughout.

Suppose that $a : \mathbb{R}_+ \to \mathbb{R}_+$ is a compactly supported, continuous function satisfying

\[
a_0 := a(0) > 0.
\]

We shall also assume that $r : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function that satisfies

\[
r(\xi) \approx \xi^{2\beta}, \quad \xi \in \mathbb{R}_+, \quad \text{and} \quad \lim_{\xi \to 0} \frac{r(\xi)}{\xi^{2\beta}} = r_0
\]

for some $\beta > 0$, $r_0 > 0$. We also fix a parameter $\alpha \in \mathbb{R}$, $\alpha < 1$.

Consider a random vector field $V(t, x) = (V_1(t, x), \ldots, V_d(t, x))$ over the space of second degree Hermite polynomials corresponding to the Gaussian Hilbert space for the noise $W(dt, dk)$ (see, e.g., [19, Chapter 2]). More precisely, for any $j = 1, \ldots, d$ we let

\[
V_j(t, x) := \sum_{j' = 1}^d \int_{-\infty}^t \int_{\mathbb{R}^2} \exp \{i(k + k') \cdot x\} E(t - s, t - s', k, k') \Gamma_{j,j'}(k + k')
\]

\[\times W_{j'}(ds, dk) W_{j'}(ds', dk'),\]

where

\[
E(s, s', k, k') := \sqrt{\frac{r(|k|)a(|k|)}{|k|^{(d+\alpha-1)/2}}} \sqrt{\frac{r(|k'|)a(|k'|)}{|k'|^{(d+\alpha-1)/2}}} \exp \left\{ -\frac{1}{2} [r(|k|)|s| + r(|k'|)|s'|] \right\}
\]

and

\[
\Gamma_{j,j'}(k) := \begin{cases} 
\delta_{j,j'} - \frac{k_j k_{j'}}{|k|^2}, & k \neq 0, \\
0, & k = 0.
\end{cases}
\]

For $k \neq 0$ the matrix $[\Gamma_{j,j'}(k)]$ corresponds to the orthogonal projection of $\mathbb{R}^d$ onto the hyperplane orthogonal to $k$, therefore

\[
\sum_{\ell=1}^d \Gamma_{j,\ell}(k) \Gamma_{j',\ell}(k) = \delta_{j,j'}, \quad j, j' = 1, \ldots, d.
\]
We refer to Section 3 for the precise definition of the stochastic integrals appearing in (2.6). To shorten the expressions we will use the notation

\[(2.10) \quad Y_j(ds, dk, ds', dk') := \sum_{j'=1}^{d} \Gamma_{j,j'}(k + k') W_{j'}(ds, dk) W_{j'}(ds', dk').\]

For brevity let us also denote

\[W_j^2(ds, dk, ds', dk') := W_j(ds, dk) W_j(ds', dk'), \quad j = 1, \ldots, d.\]

Note that the structure measure of \(W_j^2(ds, dk, ds', dk')\) equals

\[(2.11) \quad \mathbb{E}[W_j^2(ds_1, dk_1, ds_2, dk_2)] W_j^2(ds_3, dk_3, ds_4, dk_4)\]

\[= \left[\delta(s_1 - s_3)\delta(s_2 - s_4)\delta(k_1 + k_3)\delta(k_2 + k_4) + \delta(s_1 - s_4)\delta(s_2 - s_3)\delta(k_1 + k_4)\delta(k_2 + k_3)\right] \delta_{j,j'} ds_1 ds_2 ds_3 ds_4 dk_1 dk_2 dk_3 dk_4,
\]

while the structure measure of \(Y_j(ds, dk, ds', dk')\), by (2.10) and (2.9), equals

\[(2.12) \quad \mathbb{E}[Y_j(ds_1, dk_1, ds_2, dk_2) Y_j(ds_3, dk_3, ds_4, dk_4)]\]

\[= \left[\delta(s_1 - s_3)\delta(s_2 - s_4)\delta(k_1 + k_3)\delta(k_2 + k_4) + \delta(s_1 - s_4)\delta(s_2 - s_3)\delta(k_1 + k_4)\delta(k_2 + k_3)\right] \Gamma_{j,j'}(k_1 + k_2) ds_1 ds_2 ds_3 ds_4 dk_1 dk_2 dk_3 dk_4.
\]

Since \(W^*(ds, dk) = W(ds, -dk)\), the random vector field \(V(t, x)\) defined by (2.6) is real valued. Using (2.6) and (2.12) we compute its covariance matrix

\[(2.13) \quad R_{j,j'}(t, x) = \mathbb{E}[V_j(t + t', x + x') V_j(t', x')]
\]

\[= \int_{\mathbb{R}^2} \exp\{i(k_1 + k_2) \cdot x\} \delta_{j,j'}(k_1 + k_2) e^{-|r|(|k_1| + |r|(|k_2|))} |a(|k_1|) a(|k_2|)| k_1 d k_2
\]

\[\frac{|r|^d + |\alpha|}{|k_1|} - \frac{|r|^d + |\alpha|}{|k_2|} \approx \frac{1}{|k_2|^{2\alpha + d - 2}}, \quad k \in \mathbb{R}^d
\]

(see, e.g., [36], (6), p. 118), we conclude that \(R_{j,j'}(t, x)\) is of the form (1.4) with

\[\hat{E}(t, |k|) := |k|^{d-1} \int_{\mathbb{R}^d} e^{-|r|(|k - \ell|) + |r|(|\ell|))} \frac{a(|k - \ell|) a(|\ell|)}{|k - \ell|^{d + \alpha - 2}} \ell^{d + \alpha - 1} d \ell, \quad (t, k) \in \mathbb{R}^{1+d}.
\]

Remark 2.1. In light of the assumptions (2.4)–(2.5) we have

\[(2.14) \quad \hat{E}(t, \xi) \approx e^{-\xi^2 |t|} \frac{a(\xi)}{\xi^{2\alpha - 1}}, \quad (t, \xi) \in \mathbb{R} \times \mathbb{R}_+.
\]
Hence the energy spectrum is integrable and the field given by (2.6) is well defined provided that $\alpha < 1$. The diffusivity, defined by the Green–Kubo formula (1.3), becomes infinite if $\alpha + \beta > 1$.

**Remark 2.2.** The random field $V$ is stationary in the strict sense, i.e. for any $(h, y) \in \mathbb{R}^{1+d}$ the fields $V(h + \cdot, y + \cdot)$ and $V(\cdot, \cdot)$ have identical laws. This can be seen from (2.6) as $V(h + \cdot, y + \cdot)$ is given by

$$V_j(t + h, x + y) := \int_{\mathbb{R}^d} \int_{(0, t]^2} \exp\{i(k + k') \cdot x\} E(t - s, t - s', k, k') Y^{h,y}_j(ds, dk, ds', dk'),$$

where $Y^{h,y} = (Y^{h,y}_1, \ldots, Y^{h,y}_d)$ is defined by (2.10) with the noise $W = (W_1, \ldots, W_d)$ replaced by an identically distributed noise $W^{h,y} = (W^{h,y}_1, \ldots, W^{h,y}_d)$ given by

$$\int_{\mathbb{R}^d} f(t, k) W^{h,y}_j(dt, dk) := \int_{\mathbb{R}^d} e^{i y \cdot k} f(t + h, k) W_j(dt, dk),$$

where $f \in L^2_2(\mathbb{R}^{1+d})$, $j = 1, \ldots, d$. In addition, by Corollary A.2 for any $q > 0$ and $T > 0$ there exists a random constant $C(q, T)$ such that

$$\sup_{t \in [0, T]} |V(t, x; \omega)| \leq C(q, T)(1 + |x|^q), \quad x \in \mathbb{R}^d, \mathbb{P}\text{-a.s.}$$

Finally, the field is isotropic (in the wide sense), meaning that its covariance matrix, given by (2.13), behaves as an isotropic tensor under an orthogonal change of spatial coordinates, i.e.

$$R(t, gx) = gR(t, x)g^T, \quad (t, x) \in \mathbb{R}^{1+d},$$

for any orthogonal $d \times d$-matrix $g = [g^{j,j'}]_{j,j'=1,\ldots,d}$. Finally, the factor $\Gamma_{j,j'}$ appearing in (2.10) ensures the incompressibility of the spatial realizations of the field.

**Remark 2.3.** Random velocity fields with covariance matrix of the form (1.4) and power energy spectrum satisfying (2.14) are widely used in the theory of isotropic turbulence (see, e.g., [32, Chapter 7]). Perhaps the simplest examples are furnished by Gaussians $V(t, x) = (V_1(t, x), \ldots, V_d(t, x))$, where

$$V_j(t, x) := \int_{(0, t]^2} \int_{\mathbb{R}^d} \exp\{i k \cdot x\} e^{-r(|k|)(t-s)/2} \sqrt{a(|k|) r(|k|)} |\gamma_{j,j'}|^{\alpha/2-1} \tilde{Y}_j(ds, dk),$$

and

$$\tilde{Y}_j(ds, dk) := \sum_{j'=1}^d \Gamma_{j,j'}(k) W_{j'}(ds, dk), \quad j = 1, \ldots, d,$$
with $\Gamma_{j,j'}(\cdot), a(\cdot), r(\cdot)$ and $\alpha$ as defined above. The fields of this type have been used to model transport phenomena in turbulent flows (see, e.g., [29 Chapter 3], [2, 3, 9]). In the present paper we consider non-Gaussian generalizations of such flows given by (2.6). In our main result (Theorem 2.6), we prove that the long time, large scale limit of the tracer particle trajectory in such a flow is superdiffusive and its limiting statistics is non-Gaussian provided that condition (2.19) holds. To the best of our knowledge it is the first example of a non-Gaussian limit for a tracer in an isotropic flow.

**Remark 2.4.** Let $p \geq 1$ be an integer. Consider the tensor valued field
\[
W(t,x) := \left[ W^{p_1, \ldots, p_d}_j(t,x), \quad j = 1, \ldots, d, \quad p_1, \ldots, p_d \in \mathbb{Z}_+, \quad \sum_{k=1}^d p_k = p \right],
\]
where
\[
W^{p_1, \ldots, p_d}_j(t,x) := i^p \int_{(\mathbb{R}^2 \times \mathbb{R}^d)} \prod_{\ell=1}^d (k + k')^{p_\ell}_\ell \times \exp \{ i(k + k') \cdot x \} E(t-s, t-s', k, k') Y_j(ds, dk, ds', dk').
\]
For $p = 0$ we let $W(t,x) := V(t,x)$.

By the hypercontractivity for double integrals with respect to space-time white noise (cf. Proposition 3.3 below), from (2.7) one can show, after a straightforward calculation, that for any $m \in \mathbb{N}$ there exists $\delta > 0$ such that
\[
\mathbb{E} |W^{p_1, \ldots, p_d}_j(t+h, x+y) - W^{p_1, \ldots, p_d}_j(t,x)|^{2m} \leq (\mathbb{E} |W^{p_1, \ldots, p_d}_j(t+h, x+y) - W^{p_1, \ldots, p_d}_j(t,x)|^2)^m \leq (|h| + |y|)^{d+1+\delta}, \quad (t,x), (h,y) \in \mathbb{R}^{1+d}.
\]
Hence $W(t,x)$ has a continuous trajectory modification (see, e.g., [25 Theorem 1.4.1, p. 31]). This in turn implies that there exists a modification of the field $V(t,x)$ that is $p$ times continuously differentiable in $x$ and $D^p V(t,x) = W(t,x)$, where
\[
D^p V(t,x) := \left[ \partial^{p_1}_{x_1} \ldots \partial^{p_d}_{x_d} V_j(t,x), \quad j = 1, \ldots, d, \quad p_1, \ldots, p_d \in \mathbb{Z}_+, \quad \sum_{k=1}^d p_k = p \right].
\]
We conclude that the field $V(t,x)$ has a modification that is $C^\infty$ smooth in $x$.

**2.4. Statement of the main result.** Suppose that $V$ is given by (2.6) and $x(t)$ is the solution of (1.2) with $x_0 = 0$. Let
\[
\delta = \frac{\beta}{\alpha + 2\beta - 1} \quad \text{and} \quad H := \frac{1}{2\delta}.
\]
Remark 2.5. Note that if $\alpha + \beta > 1$ then $H > 1/2$. On the other hand, since $\alpha < 1$ we have $H < 1$.

Denote
\begin{equation}
(2.17) \quad x_\varepsilon(t) := x\left(\frac{t}{\varepsilon^{2\delta}}\right) = \varepsilon^{1-2\delta} \int_0^t V\left(\frac{s}{\varepsilon^{2\delta}}, x_\varepsilon(s)\right) ds,
\end{equation}
\begin{equation}
(2.18) \quad E_\infty(s,s',k,k') := a_0 r_0 \left(\frac{1}{|k||k'|}\right)^{(\alpha+d-1)/2-\beta} \exp\left\{ -\frac{r_0}{2} [k|2\beta|s + |k'|2\beta|s'|] \right\}.
\end{equation}

It is clear from Remark 2.4 that $V(t,x)$ has a continuous version that is $C^\infty$ smooth in $x$. This guarantees a.s. local existence of trajectories $x_\varepsilon(t)$ given by solutions of (2.17). Their global existence follows from the growth estimate of the velocity field (2.15).

Our main result is the following.

**Theorem 2.6.** Suppose that the velocity field $V(\cdot)$ satisfies the assumptions of Section 2.3. Moreover, assume
\begin{equation}
(2.19) \quad \alpha < 1 \quad \text{and} \quad \alpha + \beta > 1.
\end{equation}
Then the processes $(x_\varepsilon(t))_{t \geq 0}$ converge in law over $C([0,\infty);\mathbb{R}^d)$ as $\varepsilon \to 0$ to the process $(X(t))_{t \geq 0} = (X_1(t), \ldots, X_d(t))_{t \geq 0}$ given by
\begin{equation}
(2.20) \quad X_j(t) := \int_0^t \int_{(-\infty,\sigma]^2} E_\infty(\sigma - s, \sigma - s', k, k') Y_j(ds, dk, ds', dk')
\end{equation}
for $j = 1, \ldots, d$, where $Y_j$ is defined in (2.10).

The proof of the theorem is given in Section 4.

Remark 2.7. It is not difficult to see that the process $(X(t))$ is self-similar with index $H$, i.e. the laws of $(X(ct))_{t \geq 0}$ and $(c^H X(t))_{t \geq 0}$ coincide for any $c > 0$. Moreover, it has stationary increments, that is, its law coincides with that of $(X(t+h) - X(h))_{t \geq 0}$ for any $h \geq 0$. The process is not Gaussian and takes values in the second Wiener chaos corresponding to the time space white noise $(W(dt, dk))$. It is not a Rosenblatt process, but it is yet another example of a self-similar process with stationary increments living in the second Wiener chaos, next to, for example, the non-symmetric Rosenblatt processes (see e.g. [26]). In Proposition 2.9 below we discuss in more detail the relation of $X$ to Rosenblatt processes.

Remark 2.8. By stationarity of increments, self-similarity of $X$ and hypercontractivity for double integrals with respect to space-time white noise (cf. Proposition 3.3 below), for any $m \in \mathbb{N}$ we have
\[ \mathbb{E}|X(t+h) - X(t)|^{2m} \leq C_m h^{2Hm}, \]
hence $X$ has Hölder continuous trajectories with any exponent $\tilde{H} < H$. 
Let us briefly discuss the relation of the process \((X(t))\) to Rosenblatt processes. Recall (see, e.g., \(\text{[41]}\)) that a one-dimensional Rosenblatt process of index \(H \in (1/2, 1)\) is a process of the form

\[
(2.21) \quad \tilde{Z}(t) := c \int_{\mathbb{R}^2} \left\{ \int_0^t (s - y_1)^{-H/2} (s - y_2)^{-H/2} ds \right\} W(dy_1) W(dy_2), \quad t \geq 0,
\]

where \(W(dy)\) is a one-dimensional real valued white noise, \(x^\perp := \max(0, x)\) for \(x \in \mathbb{R}\), and \(c > 0\) is a normalizing constant such that \(E\tilde{Z}(1)^2 = 1\). The process is \(H\)-self-similar, with stationary increments.

A \(d\)-dimensional Rosenblatt process is defined as \(\tilde{Z}(t) := (\tilde{Z}_1(t), \ldots, \tilde{Z}_d(t)), t \geq 0\), where \(\tilde{Z}_1(\cdot), \ldots, \tilde{Z}_d(\cdot)\) are independent one-dimensional Rosenblatt processes. Let us now consider the process analogous to \((2.20)\) but with \(Y_j(ds, dk, ds', dk')\) replaced by \(W_j(ds, dk)W_j(ds'dk')\), which corresponds to taking the Kronecker delta \(\delta_{j,j'}\) instead of \(\Gamma_{j,j'}(k)\) in \((2.10)\), that is, for \(j = 1, \ldots, d\) let

\[
(2.22) \quad Z_j(t) := \int_0^t d\sigma \int_{(\infty, \sigma)^2} \int_{\mathbb{R}^{2d}} E_\infty(\sigma - s, \sigma - s', k, k') W_j(ds, dk) W_j(ds', dk').
\]

**Proposition 2.9.** The process \(Z(t) := (Z_1(t), \ldots, Z_d(t)), t \geq 0\), where \(Z_j(\cdot), j = 1, \ldots, d,\) are given by \((2.22)\), is a \(d\)-dimensional Rosenblatt process, up to a multiplicative constant.

The proof of the proposition is given in Section 5.

**Remark 2.10.** The process \((X(t))\) that appears in the statement of Theorem 2.6 is subordinated to the one defined in \((2.22)\) in the following sense. From representation \((2.22)\) one concludes that \(\tilde{Z}(t) = X(t) + \tilde{X}(t)\), where \(\tilde{X}(t) = (\tilde{X}_1(t), \ldots, \tilde{X}_d(t))\) and

\[
\tilde{X}_j(t) := \sum_{j'=1}^d \int_0^t d\sigma \int_{(\infty, \sigma)^2} \int_{\mathbb{R}^{2d}} E_\infty(\sigma - s, \sigma - s', k, k') \times |k + k'|^{-2} (k + k')_j (k + k')_{j'} W_j(ds, dk) W_{j'}(ds', dk'), \quad j = 1, \ldots, d.
\]

The process \((\tilde{X}(t))\) is also self-similar with index \(H\) and has stationary increments.

Moreover \((X(t))\) and \((\tilde{X}(t))\) are uncorrelated, i.e. \(\mathbb{E}[X_j(t)\tilde{X}_{j'}(t')] = 0\) for all \(t, t' \geq 0\) and \(j, j' = 1, \ldots, d\).

3. **An interlude on multiple stochastic integrals.** In this section we recall the notion of multiple stochastic integrals used in the definition of the field \(V(\cdot, \cdot)\) and process \(X(\cdot)\). We also discuss some of their properties, which
will be used in the proofs of Theorem 2.6 and Proposition 2.9. We take the approach of [19, Section 7.2] employing Wick products. It is equivalent to the construction presented in [33] (see [30, Theorem 4.7, p. 41]). We start by describing Feynman diagrams, which provide a useful tool in our subsequent analysis.

Given positive integers \( n, r \) we let
\[
Z_{n,r} := \{ (\ell, m) : \ell = 1, \ldots, n, m = 1, \ldots r \}.
\]
The first element of a pair \((\ell, m)\) will be called a node, and the second a hand. We refer to the pair as the \( m \)th hand of node \( \ell \). We consider diagrams in which each hand can be either free (not linked to any other), or linked to exactly one hand of a different node (for \( r = 2 \) imagine a group of \( n \) people holding hands or not). We will write \(((\ell, m), (\ell', m'))\) to denote a link between hand \( m \) of node \( \ell \) and hand \( m' \) of node \( \ell' \). We always require that \( \ell \neq \ell' \) and each \((\ell, m)\) can belong to at most one link. The elements of the link are not ordered but in some calculations it is useful to order them in such a way that \( \ell < \ell' \). Such diagrams will be called Feynman diagrams. They can be equivalently represented as graphs with \( Z_{n,r} \) being the set of vertices and the edges corresponding to the links. The vertices with hands that are not paired will be called free. In what follows we will represent diagrams described above as a set of links and free vertices. They will usually be denoted by the letter \( G \). The set of all possible diagrams formed over \( n \) nodes with \( r \) hands will be denoted by \( \mathcal{D}_{r,n} \). For \( G \in \mathcal{D}_{r,n} \), we denote by \( G_{\text{free}} \), \( G_{\text{links}} \) the sets of free elements and links, respectively, while \( f(G) \), \( \ell(G) \) stand for the cardinalities of these sets. Of special importance are those diagrams for which \( G_{\text{free}} = \emptyset \); we call them complete. Following [37] we denote them by \( \mathcal{G}_{n} := \{ G \in \mathcal{D}_{r,n} : G_{\text{free}} = \emptyset \} \). Clearly, if \( nr \) is odd, then \( \mathcal{G}_{n}^r = \emptyset \).

Suppose that \( G \in \mathcal{G}_{n}^r \). We wish to define a notion of a connected component of \( G \), that is, a maximal set consisting of those links that are connected via common nodes. More precisely, we say that two nodes \( \ell, \ell' \) are directly connected if there exists a link \(((\ell, m), (\ell', m'))\) for some \( m, m' \). We say that nodes \( \ell \) and \( \ell' \) are connected, and write \( \ell \sim \ell' \), if there exists a positive integer \( k \) and a sequence of nodes \( \ell_1, \ldots, \ell_k \) such that \( \ell = \ell_1, \ell' = \ell_k \) and the node \( \ell_{i-1} \) is directly connected to \( \ell_i \) for each \( i = 2, \ldots, k \). It is clear that \( \sim \) is an equivalence relation on the set of nodes. We say that \( G' \subset G \) is a connected component of \( G \) if the nodes of its vertices form an equivalence class of the relation \( \sim \). We denote by \( \mathcal{G}_{n}^r \) the subclass of \( \mathcal{G}_{n}^r \) made up of those complete Feynman diagrams that have only one connected component.

Feynman diagrams in \( \mathcal{D}^1 \) can be used to describe Wick products. Diagrams of the class \( \mathcal{G}_{n}^r \) are useful in computing moments of Wick products, or of multiple stochastic integrals (see, e.g., [15], [12], [37] or [19, (3.6)]).
For example consider the case when \( r = 1 \). We then omit writing hands in our notation of vertices. Let \( f_1, \ldots, f_n \in L^2(\mathbb{R}^{d+1}) \). Suppose that \((W_1(ds, dk), \ldots, W_d(ds, dk))\) is a white noise as introduced in Section 2.2 Denote by \( \mathcal{J}_{n,d}^1 \) the set of all multi-indices \( j := (j_1, \ldots, j_n) \in \mathbb{Z}_d^n \). For each \( j \in \mathcal{J}_{n,d}^1 \) and \( f := f_1 \otimes \cdots \otimes f_n \) with \( f_j \in L^2(\mathbb{R}^{1+d}) \) for \( j = 1, \ldots, n \) we define the multiple stochastic integral (cf. [19, (7.22)–(7.23)]),

\[
I_n^j(f) = \int_{\mathbb{R}^{(1+d)n}} f_1(s_1, k_1) \cdots f_n(s_n, k_n) W_{j_1}(ds_1, dk_1) \cdots W_{j_n}(ds_n, dk_n)
\]

\[
:= \sum_{G \in \mathcal{G}^1_n} (-1)^{\ell(G)} \prod_{(\ell, \ell')} \delta_{j_\ell,j_{\ell'}} \left( \int_{\mathbb{R}^{1+d}} f_\ell(s, k) f_{\ell'}(s, -k) \, ds \, dk \right) \prod_{\ell \in G_{\text{free}}} \int_{\mathbb{R}^{d+1}} f_\ell(s, k) W_{j_\ell}(ds, dk).
\]

Note that the right hand side of (3.1) is simply the Wick product of the stochastic integrals \( \int_{\mathbb{R}^{1+d}} f_1 \, dW_{j_1}, \ldots, \int_{\mathbb{R}^{1+d}} f_n \, dW_{j_n} \). Then

\[
\mathbb{E}|I_n^j(f)|^2 \leq n! \int_{\mathbb{R}^{n(1+d)}} |f(s_1, k_1, \ldots, s_n, k_n)|^2 \, ds_1 \, dk_1 \cdots \, ds_n \, dk_n.
\]

The definition of \( I_n^j \) is extended by linearity to \( L^2(\mathbb{R}^{n(1+d)}) \). We always have \( \mathbb{E}I_n^j(f) = 0 \) for \( n \geq 1 \) and \( j \in \mathcal{J}_{n,d}^1 \). Moreover \( I_n^j(f) \) is invariant with respect to permutations of those arguments of \( f \) that correspond to the same noises \( W_j(\cdot) \).

Next, we wish to formulate an analogue of (3.1) for products of double stochastic integrals. Denote by \( \mathcal{J}_{2,d}^2 \) the set of all multi-indices \( j := (j_\ell, m) \), where \( j_\ell, m \in \mathbb{Z}_d, (\ell, m) \in \mathbb{Z}_{n,2} \). Suppose that \( f \in L^2(\mathbb{R}^{2n(1+d)}) \) and \( G \in \mathcal{G}_n^2 \). For \( j = (j_\ell, m) \in \mathcal{J}_{2,d}^2 \) we define

\[
I_{G,J}(f) := \int_{\mathbb{R}^{(1+d)2n}} f(s_{1,1}, k_{1,1}, s_{1,2}, k_{1,2}, \ldots, s_{n,1}, k_{n,1}, s_{n,2}, k_{n,2}) \prod_{((\ell, m), (\ell', m'))} \delta_{j_\ell,j_{\ell'}} \delta(s_{\ell, m} - s_{\ell', m'}) \delta(k_{\ell, m} + k_{\ell', m'}) \, ds_{\ell, m} \, dk_{\ell, m} \sum_{((\ell, m), (\ell', m'))} \delta_{j_\ell,j_{\ell'}} \delta(s_{\ell, m} - s_{\ell', m'}) \delta(k_{\ell, m} + k_{\ell', m'}) \, ds_{\ell', m'} \, dk_{\ell', m'} \prod_{(\ell, m) \in G_{\text{free}}} W_{j_\ell, m}(ds_{\ell, m}, dk_{\ell, m}).
\]

The above formula can be interpreted as follows: to compute \( I_{G,J}(f) \) we identify the variables corresponding to a link \(((\ell, m), (\ell', m'))\) and integrate out according to (2.3). The remaining variables, corresponding to free vertices \((\ell, m)\) in \( G \), are integrated with respect to the corresponding noises \( W_{j_\ell, m}(\cdot) \). The resulting multiple stochastic integral of order \( f(G) \) is interpreted via (3.1). The following result is [19, Theorem 7.33].
Proposition 3.1. Suppose that \( f_1, \ldots, f_n \in L^2(\mathbb{R}^{2(1+d)}) \). Let \( \mathbf{j} = (j_\ell,m) \) be a multi-index as above. Then
\[
I^{j_1,1,j_2,2}_2(f_1) \cdots I^{j_n,1,j_n,2}_2(f_n) = \sum_{G \in \mathcal{G}_n^2} I_G(f_1 \otimes \cdots \otimes f_n).
\]

Using the fact that the mean of any multiple stochastic integral vanishes, one immediately obtains, as a corollary from Proposition 3.1, a well-known formula for the moment of the (Wick) product of \( n \) double integrals (see, e.g., [37, Lemma 2.1]).

Corollary 3.2. Under the assumptions of Proposition 3.1 we have
\[
\mathbb{E}[I^{j_1,1,j_2,2}_2(f_1) \cdots I^{j_n,1,j_n,2}_2(f_n)] = \sum_{G \in \mathcal{G}_n^2} I_G(f_1 \otimes \cdots \otimes f_n).
\]

The moments of multiple integrals can be estimated using the hypercontractivity property of Wiener chaos [19, Theorem 3.50, p. 39].

Proposition 3.3. For any \( p \geq 1 \) there exists \( C_p > 0 \) such that for any \( f \in L^2(\mathbb{R}^{(d+1)n}) \) and \( \mathbf{j} \in J_{n,d}^2 \) we have
\[
(\mathbb{E}|I_1^n(f)|^p)^{1/p} \leq C_p(\mathbb{E}|I_1^n(f)|^2)^{1/2}.
\]

Finally, we formulate a result on conditioning, which is derived in exactly the same way as [33, Lemma 1.2.5]. Denote
\[
\mathcal{F}_s = \sigma\{\langle \phi, W_j \rangle : \phi \in L^2(\mathbb{R}^{1+d}), \text{ supp } \phi \in (-\infty, s] \times \mathbb{R}^d, j = 1, \ldots, d\}.
\]

Proposition 3.4. Let \( f \in L^2(\mathbb{R}^{(d+1)n}) \). Then
\[
\mathbb{E}(I_1^n(f)|\mathcal{F}_s) = I_1^n(f 1^\otimes_{(-\infty,s] \times \mathbb{R}^d}) \quad \text{for any } \mathbf{j} \in J_{n,d}^2.
\]

4. Proof of Theorem 2.6

4.1. Outline of the proof. In this section we describe the main steps of the proof. The proofs of some technical lemmas are presented in the following sections.

4.1.1. Reformulation of the problem. The first step is to reformulate our problem. Let \( T := \varepsilon^{-\delta/\beta} \) and \( \tilde{x}_T(t) := x_\varepsilon(t) \). By (2.16) and (2.17),
\[
\tilde{x}_T(t) = T^{2\beta(1-H)} \int_0^t V(sT^{2\beta}, \tilde{x}_T(s)) \, ds.
\]

Our goal is to prove the convergence in law of \( \tilde{x}_T(\cdot) \) as \( T \to \infty \).
Let $V_T = (V_{T,1}, \ldots, V_{T,d})$, where

$$V_{T,j}(t, x) := \sum_{j'=1}^{d} V_{T,j,j'}(t, x),$$

$$V_{T,j,j'}(t, x) := \int_{\mathbb{R}^{2d}} \int_{(-\infty,t]^2} \exp \{ i(k+k') \cdot x \} E_T(t-s, t-s', k, k')$$

$$\times \Gamma_{j,j'}(k+k') W_{j'}(ds, dk) W_{j'}(ds', dk'), \quad (t, x) \in \mathbb{R}^{1+d},$$

with

$$E_T(s, s', k, k') := \left( \frac{r_T(|k|)a(|k/T|)r_T(|k'|)a(|k'/T|)}{|k'|^{d+\alpha-1}|k|^{d+\alpha-1}} \right)^{1/2}$$

$$\times \exp \left\{ -\frac{1}{2} [r_T(|k|)|s| + r_T(|k'|)|s'|] \right\},$$

$$r_T(\xi) := T^{2\beta}(1-H) V_{T,2\beta} T x.$$  \hspace{1cm} (4.5)

Recalling (2.4)–(2.8) and (2.18), and using the fact that the stochastic processes

$$W_T(f) := \int_{\mathbb{R}^{1+d}} f\left( \frac{s}{T^{2\beta}}, kT \right) W(ds, dk),$$

defined for $f$ belonging to the Schwartz class $S(\mathbb{R}^{1+d})$ and

$$W'_T(f) := T^{\beta-d/2} \int_{\mathbb{R}^{1+d}} f(s, k) W(ds, dk), \quad f \in S(\mathbb{R}^{1+d}),$$

have the same laws, we conclude the following.

**Lemma 4.1.** Assume that $V(\cdot)$ is the random field defined by (2.6) and satisfies the assumptions made in Section 2.3. Let $H$ be given by (2.16). Then the laws of the random vector fields $(T^{2\beta}(1-H) V(T^{2\beta} t, T x))_{(t, x) \in \mathbb{R}^{1+d}}$ and $(V_T(t, x))_{(t, x) \in \mathbb{R}^{1+d}}$, over the space of $d$-dimensional continuous trajectory vector fields, are identical.

The proof of this lemma is straightforward, using (4.6), so we omit it.

Directly from (4.1) and Lemma 4.1 we obtain the following.

**Corollary 4.2.** Under the assumptions of Theorem 2.6 and for $T = \varepsilon^{-\delta/\beta}$ the process $x_\varepsilon(\cdot)$ has the same law as $z_T(\cdot)$, where

$$z_T(t) = \int_{0}^{t} V_T\left( s, \frac{z_T(s)}{T} \right) ds, \quad t \in \mathbb{R},$$

with $V_T$ defined by (4.2)–(4.5).

Hence, to prove Theorem 2.6 it suffices to show that, as $T \to \infty$, the processes $z_T(\cdot)$ converge in law in $C([0, \infty), \mathbb{R}^{d})$ to the process $X(\cdot)$ defined in (2.20). The general approach is standard: we prove the convergence of
finite-dimensional distributions (in fact, we show their moment convergence) and then we establish tightness.

4.1.2. A result on stationarity. Note that $V_T(\cdot)$ given by (4.2)–(4.5) is again stationary and divergence free. It is well known that in this case the process $(V_T(s, z_T(s)/T)_{s \in \mathbb{R}}$ with $z_T(\cdot)$ given by (4.8) is stationary (see [35]). In the course of the proof we will need a somewhat stronger result.

Given $p \in \mathbb{Z}_+$ consider the family of fields

(4.9)
$$D^p(V_T) := \left\{ \frac{1}{T^p} \frac{\partial^{p_1}}{\partial z_1} \ldots \frac{\partial^{p_d}}{\partial z_d} V_{T,j} : j = 1, \ldots, d, p_1, \ldots, p_d \in \mathbb{Z}_+, \sum_{k=1}^d p_k = p \right\}.$$  

We allow $p = 0$ with the convention $D^0(V_T) = \{V_{T,j} : j = 1, \ldots, d\}$.

**Proposition 4.3.** Let $z_T$ be given by (4.8). For any $T > 0$, $p_j \in \mathbb{Z}_+$, $s_j, s'_j \in \mathbb{R}$, $H_j \in D^p_j(V_T)$, $j = 1, \ldots, m$, the process

$$(H_1(s_1 + t, z_T(s_1 + t)/T), \ldots, H_m(s_m + t, z_T(s'_m + t)/T))_{t \in \mathbb{R}}$$

is stationary.

**Proof.** To simplify the notation we assume that $T = 1$ and $H_j = V_{\ell_j}$ for $j = 1, \ldots, m$, $\ell_j = 1, \ldots, d$. The general case can be treated using the same argument. Assume also that $\mathbb{P}$ is the law of the field $V(\cdot, \cdot)$ on $\Omega = C^{0,1}_\text{div}(\mathbb{R}^{1+d}; \mathbb{R}^d)$, the space of all $d$-dimensional vector fields $\omega : \mathbb{R}^{1+d} \to \mathbb{R}^d$ that are continuous and such that $\partial_x \omega(t, x)$, $\ell = 1, \ldots, d$, are continuous and $\sum_{\ell = 1}^d \partial_{x_{\ell}} \omega(t, x) \equiv 0$. This space is Polish, when equipped with the standard Fréchet metric, and $\mathbb{P}$ is defined on its Borel $\sigma$-algebra $B(\Omega)$. Let $\tau_{t,x} : \Omega \to \Omega$ be the shift operator, given by $(\tau_{t,x}\omega)(s, y) = \omega(t + s, x + y)$ for $(t, x), (s, y) \in \mathbb{R}^{1+d}$. The stationarity of the field implies that for each $(t, x) \in \mathbb{R}^{1+d}$ the push-forward measure $\mathbb{P} \circ \tau_{t,x}$ is $\mathbb{P}$. We can assume that

$V(t, x; \omega) = \mathbb{V}(\tau_{t,x}\omega)$,

where $\mathbb{V}(\omega) := \omega(0,0)$. Let $x(t; \omega)$ be the solution of the O.D.E. (1.1) corresponding to $x_0 = 0$. Let $S_t : \Omega \to \Omega$ be given by $S_t(\omega) := \tau_{t,x(t;\omega)}$. Thanks to the uniqueness of the solution of the O.D.E. we conclude that $(S_t)_{t \in \mathbb{R}}$ forms a dynamical system, i.e. $S_{t+s} = S_t S_s$, $t, s \in \mathbb{R}$. Using [17] Theorem 10(a) (see also [35] Corollary 1, p. 499]), we conclude that it preserves the measure $\mathbb{P}$, i.e. the process $(Z(S_t(\omega)))_{t \in \mathbb{R}}$ is strictly stationary for any random vector $Z$. The conclusion of the proposition then follows, once we consider the random vector

$$Z(\omega) := \left( \mathbb{V}_{\ell_1}(\tau_{s_1-s'_1,0}S_{s'_1}(\omega)), \ldots, \mathbb{V}_{\ell_m}(\tau_{s_m-s'_m,0}S_{s'_m}(\omega)) \right).$$

4.1.3. Convergence of finite-dimensional distributions. Thanks to the reformulation in Corollary 4.2 it is quite easy to guess what the limit should be. Since $z_T(s)/T$ is expected to become small as $T \to \infty$, formula (4.8) suggests
that \( z_T \) should be close to the process

\[
y_T(t) := \int_0^t V_T(s, 0) \, ds, \quad t \geq 0,
\]

at least in the sense of the proximity of the respective laws. In some special cases (when \( \alpha + \beta/2 > 1 \)) we can also argue that \( y_T(t) \) and \( z_T(t) \) are close in \( L^q \) (see Remark 4.6 below).

Note that by (4.5) and (2.5) we have

\[
\lim_{T \to \infty} r_T(\xi) = r_0 |\xi|^{2\beta} \quad \text{and} \quad r_T(\xi) \leq C |\xi|^{2\beta}, \quad \xi \in \mathbb{R}^d.
\]

Hence, using the continuity of \( a(\cdot) \) at 0 and (2.5), we conclude that

\[
\lim_{T \to \infty} E_T(s, s', k, k') = E_\infty(s, s', k, k')
\]

pointwise (cf. (4.4) and (2.18)). By (2.4), boundedness of \( a \) and (4.11),

\[
E_T(s, s', k, k') \preceq E_\infty(s, s', k, k'), \quad (s, s', k, k') \in \mathbb{R}_+^2 \times \mathbb{R}^{2d}, \quad T \geq 1.
\]

Hence \( y_T(\cdot) \) should converge to \( X(\cdot) \), defined in (2.20), which indeed is the case.

**Proposition 4.4.** Let \( y_T \) be defined by (4.10) and (4.2)–(4.5). Then under the assumptions of Theorem 2.6, for any \( q > 0 \) we have

\[
\lim_{T \to \infty} E|y_T(t) - X(t)|^q = 0, \quad t > 0.
\]

The proof of the above proposition is given in Section 4.2.

The above argument justifies heuristically the claim of Theorem 2.6. Note, however, that the field \( V_T(\cdot) \) itself does not converge as \( T \to \infty \). We have the following lemma, which will be useful later on.

**Lemma 4.5.** Suppose that the assumptions of Theorem 2.6 are satisfied. Then the following are true.

(a) There exists \( C > 0 \) such that

\[
E|V_T(0, 0)|^2 = \sum_{j=1}^d E V_{T,j}^2(0, 0) = CT^{2(1-\alpha)}, \quad T > 0.
\]

(b) For any \( n \in \mathbb{N}, \ q > 0 \) and \( j, j_1, \ldots, j_n \in \{1, \ldots, d\} \) there exists \( C_q > 0 \) such that

\[
\left( E \left| \frac{1}{T^n} \partial_{x_{j_1}, \ldots, x_{j_n}} V_T(0, 0) \right|^q \right)^{1/q} \leq C_q T^{1-\alpha}, \quad T \geq 1.
\]
(c) For any \( n \in \mathbb{N} \) and \( j' \in \{1, \ldots, d\} \) there exists \( C > 0 \) such that
\[
E\left( \frac{1}{T} \int_0^t \partial_{x_j} V_{T,j}(s,0) \, ds \right)^2 \leq Ct(t \lor 1) \begin{cases} 
T^{-2(\alpha+\beta-1)} & \text{if } \alpha + \beta < 2, \\
T^{-2} & \text{if } \alpha + \beta = 2, \\
T^{-2} & \text{if } \alpha + \beta > 2.
\end{cases}
\]

The proof of this lemma is given in Section 4.3.

**Remark 4.6.** Using Lemma 4.5 we can easily deduce the convergence of finite-dimensional distributions in Theorem 2.6 for a narrower range of parameters when \( \alpha < 1 \) and \( \alpha + \beta / 2 > 1 \). By (4.8) and (4.10) we can write
\[
z_{T,\ell}(t) - y_{T,\ell}(t) = \sum_{j=1}^d \int_0^t ds \int_0^s \frac{1}{T} \partial_{x_j} V_{T,\ell}(s,0) \, ds \partial_{x_j} V_{T,j}(u, \frac{z_T(u)}{T}) \, du.
\]
Changing the order of integration and applying the Cauchy–Schwarz inequality we have
\[
E|z_{T,\ell}(t) - y_{T,\ell}(t)| \leq \sum_{j=1}^d \left\{ \mathbb{E} \left[ \int_0^t ds \left( \frac{1}{T} \partial_{x_j} V_{T,\ell}(s,0) \right)^2 du \right] \mathbb{E} \left[ \int_0^t du \partial_{x_j} V_{T,j}(u, \frac{z_T(u)}{T})^2 \right] \right\}^{1/2}.
\]
By Proposition 4.3, for any \( u \geq 0 \) and \( s_1, s_2 \geq u \) we have the following equality in law:
\[
V_T(u, \frac{z_T(u)}{T}) \overset{d}{=} V_T(0,0)
\]
and
\[
(\partial_{x_j} V_{T,\ell}(s_1, z_T(u)/T), \partial_{x_j} V_{T,\ell}(s_2, z_T(u)/T)) \overset{d}{=} (\partial_{x_j} V_{T,\ell}(s_1 - u, 0), \partial_{x_j} V_{T,\ell}(s_2 - u, 0)).
\]
Hence
\[
E|z_T(t) - y_T(t)| \leq \sqrt{t} \sum_{j,\ell=1}^d \left\{ \mathbb{E} \left[ \int_0^t ds \left( \frac{1}{T} \partial_{x_j} V_{T,\ell}(s,0) \right)^2 \right]^2 du \right\}^{1/2} \{EV_{T,j}^2(0,0)\}^{1/2}.
\]
By Lemma 4.5(a,c) we conclude that the right hand side of (4.18) converges to 0 as \( T \to \infty \) provided that \( \alpha + \beta / 2 > 1 \) and \( \alpha < 1 \). This together with (4.14) yields in particular the weak convergence of finite-dimensional distributions of \( z_T(\cdot) \) to \( X(\cdot) \), in this case. By hypercontractivity (see Proposition 3.3) one can also argue that \( z_T(t) \) converges \( X(t) \) in \( L^q \) for any \( q > 0 \) and \( t \geq 0 \).
Quite remarkably, extension of the above argument to the case $\alpha + \beta > 1$, $\alpha < 1$ (considered in Theorem 2.6) eludes us, although we believe that the finite-dimensional distributions of $z_T(\cdot)$ should then converge in $L^q$ for any $q > 0$ as well. However, the natural idea of using a longer Taylor expansion and applying the Cauchy–Schwarz (or Hölder) inequality to estimate the remainder does not seem to work. Therefore, to show the convergence of finite-dimensional distributions we use the method of moments. There we also use the Taylor expansion and the Cauchy–Schwarz inequality but in a more subtle way.

It is known (see, e.g., [12, p. 113]) that the law of a random variable belonging to the second Wiener chaos is determined by its moments (this is no longer true in chaoses of higher order). Therefore, to prove convergence of finite-dimensional distributions it suffices to show that for any $m, n \in \mathbb{N}$, $0 < t_n \leq \cdots \leq t_1$ and $a_{r,j} \in \mathbb{R}$ for $r = 1, \ldots, n$, $j = 1, \ldots, d$ we have

$$
\lim_{T \to \infty} \mathbb{E} \left( \sum_{r=1}^n \sum_{j=1}^d a_{r,j} z_T(t_r) \right)^m = \mathbb{E} \left( \sum_{r=1}^n \sum_{j=1}^d a_{r,j} X_j(t_r) \right)^m.
$$

The latter is a direct consequence of the following.

**Theorem 4.7 (Convergence of moments).** Suppose that the assumptions of Theorem 2.6 are satisfied and $z_T$ is defined by (4.8). Then for any $t_1 \geq \cdots \geq t_m \geq 0$ and non-negative integers $j_1, \ldots, j_m \in \mathbb{Z}_d$ we have

$$
\lim_{T \to +\infty} \mathbb{E} \left[ \prod_{n=1}^m z_{T,n}(t_n) \right] = \mathbb{E} \left[ \prod_{n=1}^m X_{j_n}(t_n) \right].
$$

The proof of the above result is the most difficult part of the argument. Its main steps are presented in Section 4.4, leaving the demonstration of some technical lemmas to Sections 4.5 and 4.6. The key ingredients of the proof of Theorem 4.7 are: inductive Taylor expansions (using (4.8)), applications of conditioning and stationarity, resulting from the incompressibility of $V(\cdot)$ (following from Proposition 4.3) and Propositions 3.1–3.4 applied to calculate the expectations (both conditional and unconditional) of the expressions resulting from the Taylor expansion.

As mentioned above, from Theorem 4.7 we obtain in particular

**Corollary 4.8 (Convergence of finite-dimensional distributions).** For any $t_1 \geq \cdots \geq t_m \geq 0$ the laws of $(z_T(t_1), \ldots, z_T(t_m))$ converge weakly to the law of $(X(t_1), \ldots, X(t_m))$.

**4.1.4. Tightness.** Finally, the last step is to prove tightness. We show

**Proposition 4.9.** Under the assumptions of Theorem 2.6 there exists $C > 0$ such that

$$
\mathbb{E} |z_T(s + t) - z_T(s)|^2 \leq Ct^{2H} \quad \text{for any } s, t \geq 0, T > 1.
$$
The above result implies tightness of the family of laws of \( z_T(\cdot) \) for \( T > 1 \) in \( C([0, \infty); \mathbb{R}^d) \) provided that \( H > 1/2 \) (see, e.g., [6, Theorem 12.3, p. 95]). The proof of Proposition 4.9 is given in Section 4.7. It uses the techniques developed in the proof of Theorem 4.7.

Combining Corollaries 4.8 and 4.2 with Proposition 4.9, recalling that \( H > 1/2 \) (see Remark 2.5), concludes the proof of Theorem 2.6.

4.2. Proof of Proposition 4.4. Using the definitions of the processes \( X(\cdot) \), \( z_T(\cdot) \) and the field \( V_T(\cdot) \) (see (2.20) and (4.2)–(4.8)), and the \( L^2 \) isometry for multiple integrals, we have

\[
\sum_{j=1}^{d} \mathbb{E}[y_{T,j}(t) - X_j(t)]^2 = 2(d - 1) \int_{\mathbb{R}^{2d+2}} dk \, dk' \, ds \, ds' \\
\times \left[ \int_{0}^{t} \mathbb{1}_{\{s \leq r, s' \leq r\}} \left( E_T(r - s, r - s', k, k') - E_\infty(r - s, r - s', k, k') \right) \, dr \right]^2
\]

(cf. (4.4) and (2.18)). By (4.12) and (4.13) we can use the Lebesgue dominated convergence theorem to infer (4.14) for \( q = 2 \), provided we can show that

\[
J_c := \int_{\mathbb{R}^{2d+2}} dk \, dk' \, ds \, ds' \left[ \int_{0}^{t} \mathbb{1}_{\{s \leq r, s' \leq r\}} E_\infty(c(r - s), c(r - s'), k, k') \, dr \right]^2 < \infty
\]

for \( c > 0 \). Integrating with respect \( s \) and \( s' \) and using

\[
\int_{-\infty}^{r \wedge r'} |k|^{2\beta} e^{-cr_0 |r - r'|} |k|^{2\beta} ds = \frac{1}{cr_0} e^{-cr_0 |r - r'|} |k|^{2\beta}
\]

we conclude that \( J_c \) equals, up to a constant,

\[
\int_{\mathbb{R}^{2d}[0,t]^2} \frac{1}{\langle |k| |k'| \rangle^{\alpha+d-1}} \exp \left\{-\frac{cr_0}{2} |r - r'|(|k|^{2\beta} + |k'|^{2\beta}) \right\} \, dr \, dr' \, dk \, dk'.
\]

We have the elementary inequality

\[
\frac{1}{\gamma} (1 - e^{-\gamma t})(1 + \gamma) \leq C(t) := 2(t \vee 1), \quad \gamma, t > 0.
\]

Note that for any \( \gamma, t > 0 \),

\[
\int_{[0,t]^2} e^{-\gamma |r - r'|} \, dr \, dr' = \frac{2}{\gamma} \int_{0}^{t} (1 - e^{-\gamma r}) \, dr \leq 2t \frac{1 - e^{-\gamma t}}{\gamma} \leq 2tC(t) \frac{1}{1 + \gamma}.
\]

Using the elementary inequality

\[
\frac{1}{1 + a^2 + b^2} \leq \frac{2}{(1 + a)(1 + b)}, \quad a, b > -1,
\]
we conclude that \((4.22)\) is bounded from above, modulo some absolute constant, by

\[
(4.24) \quad c(t) \int_{\mathbb{R}^{2d}} \frac{dk \, dk'}{(|k|/|k'|)^{\alpha + d - 1}(1 + |k|^{2\beta} + |k'|^{2\beta})} \leq c(t) \left[ \int_{\mathbb{R}^d} \frac{dk}{|k|^{\alpha + d - 1}(1 + |k|^{\beta})} \right]^2 < \infty,
\]
due to the second inequality in \((2.19)\). Here

\[
(4.25) \quad c(t) := t(t \vee 1), \quad t \in \bar{\mathbb{R}}_+.
\]

This finishes the proof of \((4.20)\), and thus \((4.14)\) follows for \(q = 2\). The result for an arbitrary \(q \geq 1\) then follows from the fact that \(y_T(t) - X(t)\) belongs to the second Gaussian chaos space, corresponding to \((W(\cdot))\), where all the \(L^q\) norms are equivalent (see Lemma 3.3). ■

4.3. Proof of Lemma 4.5. (a) In the same way as in the proof of Proposition 4.4 we have

\[
\mathbb{E} V^2_T(0, 0) = 2(d - 1) \int_{\mathbb{R}^{2d+2}} 1_{|s| \leq 0, |s'| \leq 0} E^2_T(-s, -s', k, k') \, dk \, dk' \, ds \, ds',
\]

with \(E_T\) defined by \((4.4)\). Integrating with respect to \(s\) and \(s'\), similarly to \((4.21)\), we obtain

\[
\mathbb{E} V^2_T(0, 0) = 2(d - 1) \left( \int_{\mathbb{R}^d} \frac{a(|kT|)}{|k|^{\alpha + d - 1}} \, dk \right)^2
\]

\[
= 2(d - 1) T^{2(1 - \alpha)} \left( \int_{\mathbb{R}^d} \frac{a(|k|)}{|k|^{\alpha + d - 1}} \, dk \right)^2.
\]

Note that the integral on the right hand side is finite, since \(\alpha < 1\) and \(a(\cdot)\) has compact support. This ends the proof of part (a) of the lemma.

(b) It is clear that it suffices to show the estimate for each component \(V_{T,j,j'}\) of \(V_{t,j}\) (cf. \((4.2)\)–\((4.3)\)). By \((4.3)\) we have

\[
(4.26) \quad \frac{1}{T^n} \sum_{j_1, \ldots, j_n} V_{T,j,j'}(t, 0) = i^n \int_{\mathbb{R}^{2d}} \int_{(-\infty, t]^2} \left[ \prod_{t=1}^n \left( \frac{(k + k')_{j_t}}{T} \right) \right] E_T(t - s, t - s', k, k') \Gamma_{j,j'}(k + k')
\]

\[
\times W_{j'}(ds, dk) W_{j'}(ds', dk')
\]

Note that \(|\Gamma_{j,j'}(k + k')| \leq 1\) and the function \(k \mapsto |k|^n a(|k|)\) is bounded and compactly supported. Hence, the second moment can be estimated as in part (a). Thus, (b) holds for \(q = 2\). The general case follows by hypercontractivity (see Proposition 3.3).
(c) By stationarity we have
\[
\mathbb{E}\left[ \frac{1}{T} \int_{0}^{t} \partial_{x_{m}} V_{T,j}(s,0) \, ds \right]^{2} = \frac{2}{T^{2}} \sum_{j=1}^{d} \int_{0}^{u} \mathbb{E}[\partial_{x_{m}} V_{T,j}(s,0) \partial_{x_{m}} V_{T,j}(0,0)] \, ds.
\]
Applying (4.26), (4.4), (4.11) and (4.21) we obtain
\[
\mathbb{E}\left[ \frac{1}{T} \int_{0}^{t} \partial_{x_{m}} V_{T,j}(s,0) \, ds \right]^{2} \leq 2 \int_{0}^{u} \frac{a(|kT^{-1}|)a(|k'T^{-1}|)}{|k|^{d+\alpha-1} |k'|^{d+\alpha-1}(1 + |k|^{2\beta} + |k'|^{2\beta})} \, dk \, dk' \, ds \, du
\]
for some constant \(c > 0\). Here we have also used the fact that \(|\Gamma_{j,j'}(k+k')| \leq 1\) and the elementary inequality
\[
\left| \frac{k + k'}{T} \right|^{2} \leq 2 \left( \left| \frac{k}{T} \right|^{2} + \left| \frac{k'}{T} \right|^{2} \right).
\]
Thanks to the fact that \(a(\cdot)\) is bounded and (4.23) we conclude that
\[
\mathbb{E}\left[ \frac{1}{T} \int_{0}^{t} \partial_{x_{m}} V_{T,j}(s,0) \, ds \right]^{2} \leq c(t) \int_{\mathbb{R}^{2d}} \frac{|kT^{-1}|^2 a(|kT^{-1}|)}{|k|^{d+\alpha-1} |k'|^{d+\alpha-1}(1 + |k|^{2\beta} + |k'|^{2\beta})} \, dk \, dk',
\]
where \(c(t)\) is given by (4.25). Substituting \(k' \mapsto k'(1 + |k|^{2\beta})^{1/2}\beta\) we get
\[
\int_{\mathbb{R}^{d}} \frac{dk'}{|k'|^{d+\alpha-1}(1 + |k|^{2\beta} + |k'|^{2\beta})} = C (1 + |k|^{2\beta})^{-\frac{\alpha-1+2\beta}{2\beta}}
\]
for some constant \(C > 0\). Summarizing, we have shown so far that
\[
\sum_{m=1}^{d} \mathbb{E}\left[ \frac{1}{T} \int_{0}^{t} \partial_{x_{m}} V_{T}(s,0) \, ds \right]^{2} \leq c(t)I(T), \quad T, t > 0,
\]
where
\[
I(T) := \int_{\mathbb{R}^{d}} \frac{|kT^{-1}|^2 a(|kT^{-1}|)}{|k|^{d+\alpha-1}(1 + |k|^{2\beta})^{-\frac{\alpha-1+2\beta}{2\beta}}} \, dk.
\]
If \(\alpha + \beta > 2\) we use the fact that \(a(\cdot)\) is bounded and estimate
\[
I(T) \lesssim \frac{1}{T^{2}} \int_{\mathbb{R}^{d}} \frac{dk}{|k|^{d+\alpha-3}(1 + |k|^{2\beta})^{-\frac{\alpha-1+2\beta}{2\beta}}} = CT^{-2}
\]
A tracer motion in an incompressible flow

for some constant $C > 0$. For $\alpha + \beta < 2$, we use the fact that $a(\cdot)$ is compactly supported, and therefore

$$I(T) \leq \frac{1}{T^2} \int_{\mathbb{R}^d} \frac{a(|kT^{-1}|)}{|k|^{d+2\alpha+2\beta-4}} dk = C T^{-(2\alpha+2\beta-2)}.$$ 

Finally, if $\alpha + \beta = 2$ then, assuming that $\text{supp} \ a(\cdot) \subset [0, K]$ for some $K > 0$, we can write

$$I(T) \leq T^{-2} \left( \int_{|k| \leq 1} \frac{dk}{|k|^{d+\alpha-3}} + \int_{1 < |k| < KT} \frac{dk}{|k|^d} \right) \leq T^{-2}(1 + \log T) \leq CT^{-2} \log T$$

for $T \geq 1$. This concludes the proof of the lemma. ■

**4.4. Proof of Theorem 4.7.** Recall that $z_T(\cdot)$ is defined by (4.8) and $y_T$ by (4.10). By Proposition 4.4 and by the triangle and Hölder inequalities, to prove (4.19) it suffices to show that for any $t_1 \geq \cdots \geq t_m \geq 0$ and $j_1, \ldots, j_m \in \mathbb{Z}_d$,

$$\lim_{T \to \infty} \left( \mathbb{E} \left[ \prod_{n=1}^m z_{T,j_n}(t_n) \right] - \mathbb{E} \left[ \prod_{n=1}^m y_{T,j_n}(t_n) \right] \right) = 0.$$ 

By (4.8) and (4.10) and recalling the notation $t_{1,m} = (t_1, \ldots, t_m)$ and (2.1), the above is equivalent to

(4.27) $$\lim_{T \to \infty} \left\{ \mathbb{E} \left[ \int_{\square(t_{1,m})} \prod_{n=1}^m V_{T,j_n} \left( s_n, \frac{z_T(s_n)}{T} \right) ds_{1,m} \right] - \mathbb{E} \left[ \int_{\square(t_{1,m})} \prod_{n=1}^m V_{T,j_n} \left( s_n, 0 \right) ds_{1,m} \right] \right\} = 0.$$ 

If we split the multiple integrals in (4.27) depending on the order of the variables $s_n$, and note that for any permutation $(\pi_n)_{1 \leq n \leq m}$ of $\mathbb{Z}_m$ the set

$$\{ s_{1,m} \in \mathbb{R}_+^m : s_{\pi_1} \leq \cdots \leq s_{\pi_m}, s_{\pi_1} \leq t_{\pi_n}, n = 1, \ldots, m \}$$

is equal to

$$\{ s_{1,m} \in \mathbb{R}_+^m : s_{\pi_1} \leq \cdots \leq s_{\pi_m}, s_{\pi_1} \leq t_{\pi_n} \wedge t_{\pi_{n+1}} \wedge \cdots \wedge t_{\pi_m}, n = 1, \ldots, m \}$$

then it is clear that to prove (4.27) it suffices to show that

(4.28) $$\lim_{T \to +\infty} \left\{ \int_{\Delta(t_{1,m})} \mathbb{E} \left[ \prod_{n=1}^m V_{T,j_n} \left( s_n, \frac{z_T(s_n)}{T} \right) \right] ds_{1,m} - \int_{\Delta(t_{1,m})} \mathbb{E} \left[ \prod_{n=1}^m V_{T,j_n} \left( s_n, 0 \right) \right] ds_{1,m} \right\} = 0,$$

where $\Delta(t_{1,m})$ is defined in (2.1).
Let us consider the first integral in (4.28). The idea is to write first the Taylor expansion of $V_{T,j_1}(s_1, z_T(s_1)/T)$ in the second variable at $z_T(s_2)/T$ using (4.8). It turns out that if the expansion is long enough, then we can show that the remainder term converges to 0. To calculate the limit of the other terms we take the Taylor expansion with respect to $s_2$ at $z_T(s_3)/T$ and repeat the procedure until we reach the variable $s_m$, expanding around 0. Eventually, discarding the remainder terms we arrive at expressions without a random spatial argument. After some explicit calculation we deduce (4.28).

To make this precise we need several lemmas and some additional notation. Given $s_1 \geq \cdots \geq s_m$ and $H_j \in \mathcal{D}^{p_j}(V_T)$, $p_j \in \mathbb{Z}_+$, $j = 1, \ldots, m$ (see (4.9)) let

$$H^{(0)}(s_{1,m}, z) := \prod_{j=1}^m H_j(s_j, z).$$

(4.29)

Recall that $s_{1,m} := (s_1, \ldots, s_m)$. Suppose that $H^{(N)}(s_{1,m+N}, z)$ has been defined for some $N \geq 0$. For $s_1 \geq \cdots \geq s_{m+N+1}$ we let

$$H^{(N+1)}(s_{1,m+N+1}, z) := \frac{1}{T} \sum_{j'=1}^d \frac{\partial}{\partial z_j} H^{(N)}(s_{1,m+N}, z) V_{T,j}(s_{m+N+1}, z).$$

(4.30)

Suppose that $s_1 \geq \cdots \geq s_m \geq s'$. Using the above notation we can write the Taylor expansion of $H^{(0)}(s_{1,m}, z_T(s_m)/T)$ around $z_T(s'/T)$ as follows:

$$H^{(0)}\left(s_{1,m}, \frac{z_T(s_m)}{T}\right) = \sum_{k=0}^{N-1} S^{(k)}\left(s_{1,m}, s', \frac{z_T(s')}{T}\right) + R^{(N)}(s_{1,m}, s'),$$

(4.31)

where

$$S^{(0)}(s_{1,m}, s', z) := H^{(0)}(s_{1,m}, z),$$

(4.32)

$$S^{(k)}(s_{1,m}, s', z) := \int_{\Delta_k(s', s_m)} H^{(k)}(s_{1,m}, s_{1,k}, z) \, ds''_{1,k}, \quad k = 1, 2, \ldots,$$

(4.33)

$$R^{(N)}(s_{1,m}, s'_1) := \int_{\Delta_N(s', s_m)} H^{(N)}\left(s_{1,m}, s''_{1,N}, \frac{z_T(s''_N)}{T}\right) \, ds''_{1,N}.$$
for any \( N \geq 1 \) we can find a constant \( C_N > 0 \) for which (cf. (2.1))

\[
\text{(4.35)} \quad \sup_{0 \leq s'_N \leq t_m} \mathbb{E} \left\{ \mathbb{E}_{s'_N} \left[ \int_{\Delta_N(t_{1,m})} H^{(N)} \left( s_{1,m}, s'_{1,N-1}, \frac{z_T(s'_N)}{T} \right) \, ds_{1,m} \, ds'_{1,N-1} \right]^2 \right. \\
\quad \leq C_N \left[ t_1^p C^{1-\rho}(t_1) \right]^{2(m+N-1)} T^{2(1-\alpha)-\gamma N}
\]

for all \( t_1 \geq \cdots \geq t_m \geq 0 \) and \( T > 0 \). Here \( C(\cdot) \) is defined by (4.23).

Let \( \epsilon := (1, \ldots, 1) \). It follows from Proposition 4.3 that the conditional expectation on the left hand side of (4.35) has the same distribution as

\[
\mathbb{E}_0 \left[ \int_{\Delta_N(t_{1,m}-s'_N \epsilon)} H^{(N)}(s_{1,m}, s'_{1,N-1}, 0,0) \, ds_{1,m} \, ds'_{1,N-1} \right].
\]

Moreover, since \( \mathcal{V}_0 \subset \mathcal{F}_0 \), where \( \mathcal{F}_s \) is defined by (3), by Jensen’s inequality for any random variable \( \xi \) we have

\[
\mathbb{E}(\mathbb{E}(\xi|\mathcal{V}_0))^2 \leq \mathbb{E}(\mathbb{E}(\xi|\mathcal{F}_0))^2.
\]

Hence, to prove Lemma 4.10 it suffices to show the following.

**LEMMA 4.11.** Under the assumptions of Lemma 4.10 there exist \( \gamma > 0 \) and \( \rho \in (0,1) \) such that for any \( N \geq 1 \) we can find a constant \( C_N > 0 \) for which

\[
\text{(4.36)} \quad \mathbb{E} \left\{ \int_{\Delta_{N-1}(t_{1,m})} \mathbb{E}[H^{(N)}(s_{1,m}, s_{m+1,m+N-1}, 0,0)|\mathcal{F}_0] \, ds_{1,m} \, ds'_{1,N-1} \right\}^2 \\
\quad \leq C_N \left[ t_1^p C^{1-\rho}(t_1) \right]^{2(m+N-1)} T^{2(1-\alpha)-\gamma N}
\]

for all \( t_1 \geq \cdots \geq t_m \geq 0 \) and \( T > 0 \). Here \( C(\cdot) \) is defined by (4.23).

The proof of this lemma is given in Section 4.5.

Our next result deals with the case when at least one of the factors appearing in the product in (4.29) contains a derivative, that is, belongs to \( \mathcal{D}^p(V_T) \) for some \( p > 0 \).

**LEMMA 4.12.** Suppose that \( m \geq 1 \) and \( p_j \in \mathbb{Z}_+, \, H_j \in \mathcal{D}^{p_j}(V_T) \) (see (4.9)) and at least one of the \( p_j \)'s is positive. Then

\[
\text{(4.37)} \quad \lim_{T \to \infty} \int_{\Delta(t_{1,m})} \mathbb{E} \left[ \prod_{j=1}^m H_j(s_j,0) \right] \, ds_{1,m} = 0.
\]

The proof of the lemma is given in Section 4.6.

To prove (4.28) we will need a more general result, which is a consequence of Lemmas 4.10 and 4.12. For fixed \( m \geq 1, \, n \geq 0, \, p_\ell \in \mathbb{Z}_+, \, H_\ell \in \mathcal{D}^{p_\ell}(V_T), \)
\[ \ell \in \mathbb{Z}_{m+n} \text{ and } t_1 \geq \cdots \geq t_{m+n} \geq 0 \text{ denote} \]
\[ I_T^{m,n}(t_{1,m+n};(H_{\ell})) := \int_{\Delta(t_{1,m+n})} \mathbb{E} \left[ \prod_{\ell=1}^{m} H_{\ell} \left( s_\ell, \frac{z_T(s_m)}{T} \right) \prod_{\ell=m+1}^{m+n} H_{\ell} \left( s_\ell, \frac{z_T(s_\ell)}{T} \right) \right] ds_{1,m+n}, \]

with the convention that the product over an empty set is 1. Recall that \( t_{1,m+n} := (t_1, \ldots, t_{m+n}) \) and \( \Delta(t_{1,m+n}) \) is defined in (2.1).

**Lemma 4.13.** For fixed \( m \geq 1, n \geq 0, p_{\ell} \in \mathbb{Z}_+ \) and \( H_{\ell} \in \mathcal{D}^{p_{\ell}}(V_T), \ell \in \mathbb{Z}_{m+n}, \)
\[ (4.38) \lim_{T \to +\infty} \left\{ I_T^{m,n}(t_{1,m+n};(H_{\ell})) - \int_{\Delta(t_{1,m+n})} \mathbb{E} \left[ \prod_{\ell=1}^{m+n} H_{\ell}(s_\ell,0) \right] ds_{1,m+n} \right\} = 0 \]

for any \( t_1 \geq \cdots \geq t_{m+n} \geq 0. \)

**Proof.** The proof is by induction on \( n. \) If \( n = 0, \) then (4.38) follows for all \( m \in \mathbb{Z}_+ \) by Proposition 4.3 and stationarity of \( H_1, \ldots, H_m \) in the time variable, which in turn follows from stationarity of \( V_T. \)

Now suppose that (4.38) holds for some \( n \geq 0 \) and all \( m \in \mathbb{Z}_+. \) Suppose that \( t_1 \geq \cdots \geq t_{m+n+1} \geq 0 \) and \( H_{\ell} \in \mathcal{D}^{p_{\ell}}(V_T) \) for some \( p_{\ell} \in \mathbb{Z}_+, \) where \( \ell \in \mathbb{Z}_{m+n+1}. \) Thanks to (4.31) we can write
\[ I_T^{m,n+1}(t_{1,m+n+1};(H_{\ell})) = \sum_{k=0}^{N-1} S_k(T) + R_N(T) \]

with \( N \geq 2 \) to be chosen later and
\[ S_k(T) := \int_{\Delta(t_{1,m+n+1})} \mathbb{E} \left[ S^{(k)}(s_{1,m}, s_{m+1}, \frac{z_T(s_{m+1})}{T}) \prod_{\ell=m+1}^{m+n+1} H_{\ell} \left( s_\ell, \frac{z_T(s_\ell)}{T} \right) \right] ds_{1,m+n+1}, \]
\[ R_N(T) := \int_{\Delta(t_{1,m+n+1})} \mathbb{E} \left[ \prod_{\ell=m+1}^{m+n+1} H_{\ell} \left( s_\ell, \frac{z_T(s_\ell)}{T} \right) \right] ds_{1,m+n+1}. \]

By (4.33) and (4.30), \( S_k(T) \) can be written as a sum of terms of the form \( I_T^{m_{k,n}}(t_{m_k+n};(H_{\ell})) \) corresponding to some \( m_k \geq 1, \) and \( \bar{H}_{\ell} \in \mathcal{D}^{\bar{p}_{\ell}}(V_T) \) for some \( \bar{p}_{\ell} \in \mathbb{Z}_+ \) and all \( \ell \in \mathbb{Z}_{m_k+n}. \) When \( k \geq 1 \) we need to have \( \bar{p}_{\ell} > 0 \) for at least one \( \ell. \) Using the induction hypothesis and Lemma 4.12 we conclude that
\[ \lim_{T \to +\infty} S_k(T) = 0 \quad \text{for } k = 1, \ldots, N-1. \]
The only term that may possibly have non-zero limit is
\[ S_0(T) = \int_{\Delta(t_1,m+n+1)} \mathbb{E} \left[ \prod_{\ell=1}^{m+1} H_\ell \left( s_\ell, \frac{z_T(s_{m+1})}{T} \right) \prod_{\ell=m+2}^{m+n+1} H_\ell \left( s_\ell, \frac{z_T(s_\ell)}{T} \right) \right] ds_{1,m+n+1}. \]

We can again use the induction hypothesis to conclude that
\[ \lim_{T \to \infty} \left\{ S_0(T) - \int_{\Delta(t_1,m+n+1)} \mathbb{E} \left[ \prod_{\ell=1}^{m+n+1} H_\ell(s_\ell,0) \right] ds_{1,m+n+1} \right\} = 0. \]

Finally, we claim that
\[ \lim_{T \to \infty} \mathcal{R}_N(T) = 0 \]
provided that \( N \) is sufficiently large. Indeed, by Proposition 4.3 the laws of \( H_\ell(s_\ell,z_T(s_\ell)/T) \) and of \( H_\ell(0,0) \) coincide. Therefore, using the Hölder inequality, we can write (cf. (2.1))
\[ |\mathcal{R}_N(T)| \leq \int_{\Delta(t_{m+1},m+n+1)} \int_{\Delta_m(t_1,m+s_{m+1})} \mathbb{E} \left[ \sum_{s_{m+1}} R^{(N)}(s_1,m,s_{m+1}) ds_{1,m+n+1} \right] \mathbb{E} \left[ \frac{1}{2} \prod_{\ell=m+1}^{m+n+1} [\mathbb{E} H_\ell^{2(n+1)}(0,0)]^{1/(2n+2)} \right]. \]

By (4.16) we can estimate \( [\mathbb{E} H_\ell^{2(n+1)}(0,0)]^{1/(2n+2)} \leq T^{1-\alpha} \) for all \( T > 0 \). Combining this observation with (4.35) we infer that there exist \( \gamma > 0 \) and \( \rho \in (0,1) \) such that for a given \( N \geq 1 \) we can find \( C_N > 0 \) for which
\[ |\mathcal{R}_N(T)| \leq C_N t_1^{n+1} \left[ t_1 C^{-1-\rho}(t_1) \right]^{N+m-1} T^{-\gamma N/2} T^{(1-\alpha)(n+2)}, \]
\[ t_1 \geq \cdots \geq t_m \geq 0, T > 0. \]

Choosing \( N \) so large that \( N\gamma > 2(1-\alpha)(n+2) \) we get (4.39).

This finishes the proof of the conclusion of Lemma 4.13 for \( n+1 \) and any \( m \in \mathbb{Z}_+ \). ■

The identity (4.28) is a particular case of Lemma 4.13 hence it follows immediately, thus finishing the proof of convergence of moments. ■

4.5. Proof of Lemma 4.11

By (4.30) we have
\[ H^{(N)}(s_1,m,s_{m+1,m+N-1},0,0) = \sum_{\ell=1}^d \frac{1}{T} \partial_{x_\ell} H^{(N-1)}(s_1,m,s_{m+1,m+N-1},0)V_T,\ell(0,0). \]
Using the fact that $V_{T,\ell}(0,0)$ is $\mathcal{F}_0$-measurable and applying the Cauchy–Schwarz inequality we have

$$
\mathbb{E}\left\{ \int_{\Delta_{N-1}(t_1,m)} \mathbb{E}[H^{(N)}(s_{1,m+N-1},0,0)|\mathcal{F}_0] \, ds_{1,m+N-1} \right\}^2
\leq d \sum_{\ell=1}^d \left\{ \mathbb{E}\left\{ \int_{\Delta_{N-1}(t_1,m)} \mathbb{E}\left[ \frac{1}{T} \partial_{x_\ell} H^{(N-1)}(s_{1,m+N-1},0) \big| \mathcal{F}_0 \right] \, ds_{1,m} \, ds'_{1,N-1} \right\}^4 \right\}^{1/2}
\times \{ \mathbb{E}V_{T,\ell}^4(0,0) \}^{1/2}.
$$

Since both the random variables $V_{T,\ell}(0,0)$ and the conditional expectation appearing above have a finite Wiener chaos expansion (see Proposition 3.4) their $L^4$ norms are equivalent to the $L^2$ norms [19, Theorem 3.50, p. 39]. In view of (4.15) and writing $N$ instead of $N-1$, to prove the lemma it suffices to show that there exist $\gamma > 0$ and $\rho \in (0,1)$ such that

$$
\mathbb{E}\left\{ \int_{\Delta_{N}(t_1,m)} \mathbb{E}\left[ \frac{1}{T} \partial_{x_\ell} H^{(N)}(s_{1,m+N},0) \big| \mathcal{F}_0 \right] \, ds_{1,m+N} \right\}^2
\leq \left[ t_1^\rho C^{1-\rho}(t_1) \right]^{2(m+N)} T^{-\gamma(N+1)},
$$

for $t_1 \geq \cdots \geq t_m \geq 0$ and $T > 0$.

To simplify the notation we let $H^{(0)}(s_{1,m},z)$, defined in (4.29), be of the form

$$
H^{(0)}(s_{1,m},z) := \prod_{j=1}^m V_{T,\ell_j}(s_j,z)
$$

with $\ell_j \in \mathbb{Z}_d$ for $j = 1, \ldots, m$. To obtain the estimate of the lemma we will use the fact that $a(|k|)$ appearing in the kernel defining $V_T(\cdot)$ is bounded. In the general case of $H_j \in D^p_j(V_T)$, recalling (4.26), we can instead apply the fact that $|k|^n a(|k|)$ is bounded for any $n$, since $a$ is continuous and has a compact support. Otherwise the relevant estimates are obtained in the same way.

Clearly, $T^{-1}\partial_{x_\ell} H^{(N)}$ is again a product of terms from $\bigcup_{p \in \mathbb{Z}_d} D^p(V_T)$. We can decompose each $V_{T,j}(\cdot)$ appearing in $T^{-1}\partial_{x_\ell} H^{(N)}$ into a sum of $V_{t,j,j'}(\cdot)$, according to (4.2), and represent $H^{(N)}$ as a sum of corresponding products and then estimate each of the terms in this sum separately.

More precisely, for any $u = (u_1, \ldots, u_m) \in \mathbb{Z}_d^m$, let

$$
H_u^{(0)}(s_{1,m},z) := \prod_{i=1}^m V_{T,\ell_i,u_i}(s_i,z).
$$
Obviously,
\[ H^{(0)}(s_{1,m}, z) = \sum_{\mathbf{u}} H^{(0)}_{\mathbf{u}}(s_{1,m}, z) := \prod_{i=1}^{m} V_{T,i,u_i}(s_i, z), \]
where the summation extends over all indices \( \mathbf{u} \). Moreover, for \( \mathbf{j} := (j, j', j'') \in \mathbb{Z}_d^3 \) and \( \mathbf{u} \) as above we let
\[ U^0_{\mathbf{u}\mathbf{j}}(s_{1,m}, z) := \frac{1}{T} \partial_{z_{j''}} H^{(0)}_{\mathbf{u}}(s_{1,m}, z). \]
The indices \( j, j' \) are redundant but we have added them to maintain consistency with the ensuing notation. Given \( N + 2 \) multi-indices \( \mathbf{j}_i := (j_i, j'_i, j''_i) \in \mathbb{Z}_d^3, i = 0, \ldots, N + 1 \), and a multi-index \( \mathbf{u} \) as above we let
\[ U^{\mathbf{u}\mathbf{j}_0,\ldots,\mathbf{j}_{N+1}}_{N+1,T}(s_{1,m+N+1}, z) := \frac{1}{T} \partial_{z_{j''_{N+1}}} \{ U^{\mathbf{u}\mathbf{j}_0,\ldots,\mathbf{j}_{N}}_{N+1,T}(s_{1,m+N+1}, z) V_{T,j_{N+1},j'_{N+1}}(s_{m+N+1}, z) \}. \]
By (4.41) we have
\[ \sum_{\mathbf{u},j''_0,j_1,j'_1} U^0_{1,T}^{\mathbf{u}_1\mathbf{j}_1}(s_{1,m+1}, z) \delta_{j_1,j'_1} = \frac{1}{T} \partial_{z_{j''_1}} \{ \sum_{\mathbf{u},j''_0,j'_1} U^0_{0,T}^{\mathbf{u}_0,j'_1}(s_{1,m}, z) V_{T,j'_0,j'_1}(s_{m+1}, z) \} = \frac{1}{T} \partial_{z_{j''_1}} H^{(1)}(s_{1,m+1}, z). \]
By induction we can extend the above formula to all \( N \):
\[ \sum_{\mathbf{u},j''_0,j_1,\ldots,j_{N-1},j_N,j'_N} U^{\mathbf{u}\mathbf{j}_0,\ldots,\mathbf{j}_N}_{N,T}(s_{1,m+N}, z) \prod_{i=1}^{N} \delta_{j_i,j''_{i-1}} = \frac{1}{T} \partial_{z_{j''_N}} H^{(N)}(s_{1,m+N}, z). \]
By (4.43), to prove (4.36) it suffices to show that (cf. (2.1)) there exist \( \gamma > 0 \) and \( \rho \in (0,1) \) such that for each \( N \geq 1 \),
\[ \mathbb{E} \left\{ \int_{\Delta_N(t_{1,m})} \mathbb{E}[U^{\mathbf{u}\mathbf{j}_0,\ldots,\mathbf{j}_N}_{N,T}(s_{1,m+N}, 0)|\mathcal{F}_0] \, ds_{1,m+N} \right\}^2 \leq [t_{1}^{\rho} C^{1-\rho}(t_{1})]^{2(m+N-1)}T^{-\gamma(N+1)} \]
for any \( \mathbf{j}_1, \ldots, \mathbf{j}_N \in \mathbb{Z}_d^3, t_1 \geq \cdots \geq t_m \geq 0, \) and \( T > 0 \).
To simplify the notation, we consider only the case when \( j'_i = u_\ell = j''_0 \) for all \( 0 \leq i \leq N \) and \( 1 \leq \ell \leq m \). Then the fields appearing in the definition of \( U^{\mathbf{u}\mathbf{j}_0,\ldots,\mathbf{j}_N}_{N,T}(\cdot) \) are based on the same noise. The case when noises may be independent can be treated in a similar way. In fact, this leads to better estimates, due to the fact that terms corresponding to covariances of inde-
pendent noises vanish. To simplify the notation even further we will suppress $\mathbf{u}, j_0, \ldots, j_N$ and simply write $U_{N,T}$ instead of $U_{N,T}^{u,j_0,\ldots,j_N}$.

Using (4.42) and Proposition 3.1 we obtain
\[
U_{N,T}(s_{1,m+N}, z) = \frac{1}{T^{N+1}} \sum_{G \in \mathcal{D}_N^{2}} I_G(f(\cdot; s_{1,m+N}, z)),
\]
where $z \in \mathbb{R}^d$ and
\[
f(s', k; s_{1,m+N}, z) := i^{N+1} \prod_{n=0}^{N} \left( \sum_{n'=1}^{m+n} (k_{n',1} + k_{n',2}) \right) j_n' \exp \left\{ iz \cdot \left( \sum_{n=1}^{m+N} (k_{n,1} + k_{n,2}) \right) \right\} \times \prod_{n=1}^{m+N} \left[ E_T(s_n - s_{n,1}', s_n - s_{n,2}', k_{n,1}, k_{n,2}) \mathbb{1}_{\{s_{n,1}' < s_{n,2} \leq s_n\}} \right] \times \prod_{n=1}^{m+N} I_{\Gamma_n,j_n}(k_{n,1} + k_{n,2}).
\]
We have used the shorthand notation $s', k$ for the ensembles of the respective variables $s_{n,j}'$ and $k_{n,j}$. The symbol $I_G(\cdot)$ denotes the multiple stochastic integral introduced in (3.3); we omit writing the multi-index in this case, as all the noises in our situation are identical.

For a given $G \in \mathcal{D}_N^{2}$ we let
\[
(4.44) \quad J_G(t_1,m) := \frac{1}{T^{N+1}} \int_{\Delta_N(t_1,m)} \mathbb{E}[I_G(f(\cdot; s_{1,m+N}, 0))| \mathcal{F}_0] \, ds_{1,m+N}.
\]
Then obviously
\[
(4.45) \quad \int_{\Delta_N(t_1,m)} \mathbb{E}[U_{N,T}(s_{1,m+N}, 0)| \mathcal{F}_0] \, ds_{1,m+N} = \sum_{G \in \mathcal{D}_N^{2}} J_G(t_1,m).
\]
Observe that if $((\ell, j), (\ell', j')) \in \mathcal{G}_{\text{links}}$ with $\ell < \ell'$, then $s_\ell > s_{\ell'}$ and
\[
(4.46) \quad \left[ r_T \mathcal{E}_T(|k_{\ell,j}|) r_T \mathcal{E}_T(|k_{\ell',j'}|) \right]^{1/2} \times \int_{\mathbb{R}^2} \exp \left\{ -\frac{1}{2} r_T(|k_{\ell,j}|)(s_\ell - s_{\ell,j}') - \frac{1}{2} r_T(|k_{\ell',j'}|)(s_{\ell'} - s_{\ell',j'}) \right\} \times \mathbb{1}_{[s_{\ell,j} \leq s_\ell]} \mathbb{1}_{[s_{\ell',j'} \leq s_{\ell'}]} \delta(s_{\ell,j} - s_{\ell',j'}) \delta(k_{\ell,j} + k_{\ell',j'}) \, ds_{\ell,j} \, ds_{\ell',j'}
\]

\[= \hat{c}_T(s_\ell - s_{\ell'}, k_{\ell,j}) \delta(k_{\ell,j} + k_{\ell',j}),\]
with
\[ \tilde{e}_T(s, k) := \frac{e^{-\frac{1}{2}r_T(|k|)|s|a(|k|/T)}}{|k|^{d-\alpha-1}}, \quad (s, k) \in \mathbb{R}^{1+d}, T > 0. \]

From (3.3), Proposition 3.4 and (4.46) we obtain
\[ \frac{1}{T^{N/(2\beta)}} \int_{\Delta_N(t_{1,m})} \mathbb{E}[I_G(f(\cdot; s_{1,m+N}, 0))|F_0] \, ds_{1,m+N} \]
\[ = \int_{\mathbb{R}^{(1+d)f(G)}} \left\{ \int_{\Delta_N(t_{1,m})} H_G(s_{1,m+N}, s'_{G_{\text{free}}}, k_{G_{\text{free}}}) \, ds_{1,m+N} \right\} \]
\[ \times \prod_{(\ell,j) \in G_{\text{free}}} W_{j'}(ds'_{\ell,j}, dk_{\ell,j}), \]

where
\[ s'_{G_{\text{free}}} := \{ s'_{\ell,j} : (\ell, j) \in G_{\text{free}} \} \quad \text{and} \quad k_{G_{\text{free}}} = \{ k_{\ell,j} : (\ell, j) \in G_{\text{free}} \} \]

and
\[ H_G(s_{1,m+N}, s'_{G_{\text{free}}}, k_{G_{\text{free}}}) \]
\[ := \frac{i^{N+1}}{T^{N+1}} \int_{\mathbb{R}^{(1+d)f(G)}} \prod_{n=0}^N \left( \sum_{\ell=1}^{m+n} (k_{\ell,1} + k_{\ell,2}) \right) j''_n \]
\[ \times \prod_{(\ell,j) \in G_{\text{free}}} \tilde{e}_T(s_{\ell} - s'_{\ell,j}, k_{\ell,j}) \]
\[ \times \prod_{((\ell,j),(\ell',j')) \in G_{\text{links}}} \delta(k_{\ell,j} + k_{\ell',j'}) \, dk_{\ell,j} \, dk_{\ell',j'}, \]

and
\[ e_T(s, k) := \left( r_T(|k|)\tilde{e}_T(s, k) \right)^{1/2}, \quad (s, k) \in \mathbb{R}^{1+d}, T > 0. \]

In what follows we adopt the convention \( s_{m+N+1} := 0 \) and use the notation \( \tau_{\ell} := s_{\ell} - s_{\ell+1}, \ell = 1, \ldots, m + N. \)

If \( (\ell, j) \in G_{\text{free}} \), then we can write
\[ e^{-\frac{1}{2}s_{\ell}r_T(|k_{\ell,j}|)} = \exp \left\{ -\frac{1}{2} \sum_{p=\ell}^{m+N} \tau_p r_T(|k_{\ell,j}|) \right\}. \]
If, on the other hand, \( \{(\ell, j), (\ell', j')\} \in G_{\text{links}} \), then

\[
(4.49) \quad e^{-\frac{1}{2}(s_{\ell} - s_{\ell'})r_T(|k_{\ell,j}|)} = \exp \left\{ -\frac{1}{2} \sum_{p=\ell}^{\ell'-1} \tau_p r_T(|k_{\ell,j}|) \right\}.
\]

Using (4.48), (4.49) we can write

\[
\prod_{(\ell,j) \in G_{\text{free}}} \left[ \exp \left\{ -\frac{1}{2}(s_{\ell} - s_{\ell'})r_T(|k_{\ell,j}|) \right\} \mathbb{1}_{\{s_{\ell,j} \leq 0\}} \right] \\
\times \prod_{((\ell,j),(\ell',j')) \in G_{\text{links}}} \exp \left\{ -\frac{1}{2}(s_{\ell} - s_{\ell'})r_T(|k_{\ell,j}|) \right\} \\
= \exp \left\{ -\frac{1}{2} \sum_{m+N \geq i} \sum_{j=1}^{2} \tau_i \sigma_{\ell,j}^i r_T(|k_{\ell,j}|) \right\} \\
\times \prod_{(\ell,j) \in G_{\text{free}}} \left[ \exp \left\{ \frac{1}{2}s_{\ell,j}r_T(|k_{\ell,j}|) \right\} \mathbb{1}_{\{s_{\ell,j} \leq 0\}} \right],
\]

where

\[
\sigma_{\ell,j}^i = \begin{cases} 
1 & \text{if } (\ell, j) \in G_{\text{free}}, \\
1 & \text{if } ((\ell, j), (\ell', j')) \in G_{\text{links}} \text{ and } \ell \leq i \leq \ell' - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Fix \( i \in \{1, \ldots, N\} \). Note that

\[
\prod_{i=0}^{N} \sum_{\ell=1}^{m+i} \sum_{j=1}^{2} k_{\ell,j} \prod_{\ell,j \in G_{\text{links}}} \delta(k_{\ell,j} + k_{\ell',j'}) \\
= \prod_{i=0}^{N} \sum_{\ell=1}^{m+i} \sum_{j=1}^{2} \sigma_{\ell,j}^i k_{\ell,j} \prod_{\ell,j \in G_{\text{links}}} \delta(k_{\ell,j} + k_{\ell',j'}) \\
\leq \prod_{i=0}^{N} \left( \sum_{\ell=1}^{m+i} \sum_{j=1}^{2} \sigma_{\ell,j}^i |k_{\ell,j}| \right) \prod_{\ell,j \in G_{\text{links}}} \delta(k_{\ell,j} + k_{\ell',j'}).
\]

Using (4.5), (2.4) and (2.5) we conclude that for some \( c_* > 0 \) we have

\[
\tilde{e}_T(s, k) \leq \frac{e^{-c_*|k|^{2\beta}|s|}}{|k|^{d+\alpha-1}} a(|k|/T)
\]

and similarly

\[
e_T(s, k) \leq \frac{e^{-c_*|k|^{2\beta}|s|}}{|k|^{(d+\alpha-1-2\beta)/2}}, \quad (s, k) \in \mathbb{R}^{1+d}, \ T > 0.
\]
From the above and \((4.47)\) we conclude that
\[
(4.50) \quad \left| \int_{\Delta_N(t_{1,m})} H_G(s_{1,m+N}, s'_{\text{free}}, k_{\text{free}}) \, ds_{1,m+N} \right|
\]
\[
\leq T^{-(N+1)} \int_{\mathbb{R}^{2(1+d)(G)}} \left\{ \prod_{i=0}^{N} \left( \sum_{\ell=1}^{m+i} \sum_{j=1}^{2} \sigma_{\ell,j}^i \right) \left| k_{\ell,j} \right| \right\} \times \left\{ \exp \left\{ -c_* \sum_{m+N \geq i \geq \ell \geq 1} \sum_{j=1}^{2} \tau_i \sigma_{\ell,j}^i \left| k_{\ell,j} \right|^{2\beta} \right\} \right\} \, d\tau_{1,m+N}
\]
\[
\times \prod_{(\ell,j) \in G_{\text{free}}} \left[ \exp \left\{ \frac{1}{2} s'_{\ell,j} r_T(\left| k_{\ell,j} \right|) \right\} \frac{1 \{ s'_{\ell,j} \leq 0 \}}{\left| k_{\ell,j} \right|^{(\alpha-2\beta+d-1)/2}} \right]
\]
\[
\times \prod_{(\ell,j), (\ell', j') \in G_{\text{links}}} \delta(k_{\ell,j} + k_{\ell', j'}) \frac{a \left( \left| k_{\ell,j} \right| / T \right) \, dk_{\ell,j} \, dk_{\ell', j'}}{\left| k_{\ell,j} \right|^{\alpha+d-1}},
\]
where \(d\tau_{m,n} := d\tau_\ell \ldots d\tau_n\) for \(0 \leq m \leq n\), and
\[
\tilde{\Delta}_n(t) := \left\{ (\tau_1, \ldots, \tau_n) : \tau_1, \ldots, \tau_n \geq 0, \sum_{j=1}^{n} \tau_j \leq t \right\}.
\]

We can estimate the integral over the simplex appearing on the right hand side of \((4.50)\) by
\[
(4.51) \quad \prod_{i=1}^{m+N} t_1 \int_{0}^{t_1} \exp \left\{ -c_* \tau_i \sum_{\ell=1}^{i} \sum_{j=1}^{2} \sigma_{\ell,j}^i \left| k_{\ell,j} \right|^{2\beta} \right\} \, d\tau_i.
\]

Using \((4.23)\) and the elementary estimate \((1 - e^{-\gamma t}) / \gamma \leq t\) for \(\gamma, t > 0\), we conclude that for any \(\rho \in (0, 1)\) we have
\[
(4.52) \quad \frac{1 - e^{-\gamma t}}{\gamma} \leq t^\rho C^{1-\rho}(t) \quad \frac{1}{(1 + \gamma)^{1-\rho}}, \quad \gamma, t > 0,
\]
with \(C(t)\) given by \((4.23)\). Integrating over \(\tau_i\) in \((4.51)\) and using \((4.52)\) we conclude that the expression in \((4.51)\) can be estimated by
\[
C[t_1^\rho C^{1-\rho}(t_1)]^{m+N} \prod_{i=1}^{m+N} \left( 1 + \sum_{\ell=1}^{i} \sum_{j=1}^{2} \sigma_{\ell,j}^i \left| k_{\ell,j} \right| \right)^{-2\beta(1-\rho)}
\]
with \(C\) some constant independent of \(t_1, T\).

Let \(\gamma \in (0, 1)\) be arbitrary. Since \(a(\cdot)\) is compactly supported we have
\[
K_{m,N} := \sup_{\xi_{\ell,j} > 0} \prod_{i=0}^{N} \left( \sum_{\ell=1}^{m+i} \sum_{j=1}^{2} \sigma_{\ell,j}^i \left| \xi_{\ell,j} \right| \right)^{-1-\gamma} \left\{ \prod_{\ell=1}^{m+N} \left[ a(\left| \xi_{\ell,1} \right|) a(\left| \xi_{\ell,2} \right|) \right] \right\}^{1/2} < \infty
\]
for \( m, N \geq 1 \). Using the above we get

\[
\left| \int_{\Delta_{m+N}(t_1)} H_G(s_{1,m+N}, s'_{G_{\text{free}}}, k_{G_{\text{free}}}) \; ds_{1,m+N} \right| \\
\leq K_{m,N} [t_1^\rho C^{1-\rho}(t_1)]^{m+N} T^{-\gamma(N+1)} \\
\times \prod_{(\ell,j) \in G_{\text{free}}} \left[ \exp\{c_\ast s'_{\ell,j} |k_{\ell,j}|^{2\beta}\} \frac{\mathbb{1}\{s'_{\ell,j} \leq 0\}}{|k_{\ell,j}|^{(\alpha-2\beta+d-1)/2}} \right] \\
\times \prod_{((\ell,j),(\ell',j')) \in G_{\text{links}}} \delta(k_{\ell,j} + k_{\ell',j'}) \frac{dk_{\ell,j} dk_{\ell',j'}}{|k_{\ell,j}|^{\alpha+d-1}}.
\]

Now suppose that \( \gamma < 2\beta(1-\rho) \). Using the fact that \( \sigma_{\ell,j}^\ell \) equals 1 for all \((\ell, j) \in G_{\text{free}}\) and all left vertices \((\ell, j)\) we obtain

\[
\left\{ \prod_{i=1}^{m+N} \left( 1 + \sum_{\ell=1}^i \sum_{j=1}^2 \sigma_{\ell,j}^i |k_{\ell,j}| \right) \right\}^{1-2\beta(1-\rho)} \prod_{((\ell,j),(\ell',j')) \in G_{\text{links}}} \delta(k_{\ell,j} + k_{\ell',j'}) \\
\leq \left\{ \prod_{\ell=1}^{m+N} \left( 1 + \sum_{j=1}^2 \sigma_{\ell,j}^\ell |k_{\ell,j}| \right) \right\}^{1-2\beta(1-\rho)} \prod_{((\ell,j),(\ell',j')) \in G_{\text{links}}} \delta(k_{\ell,j} + k_{\ell',j'}) \\
\leq \prod_{(\ell,j) \in G_{\text{free}}} \left( 1 + |k_{\ell,j}| \right)^{1/2-\beta(1-\rho)} \prod_{((\ell,j),(\ell',j')) \in G_{\text{links}}} \left[ (1 + |k_{\ell,j}|)^{1/2-\beta(1-\rho)} \delta(k_{\ell,j} + k_{\ell',j'}) \right].
\]

Therefore,

\[
(4.53) \quad \left| \int_{\Delta_{N}(t_{1,m})} H_G(s_{1,m+N}, s'_{G_{\text{free}}}, k_{G_{\text{free}}}) \; ds_{1,m+N} \right| \\
\leq [t_1^\rho C^{1-\rho}(t_1)]^{m+N} T^{-(N+1)\gamma} \int_{[0,\infty)^d} \frac{dk}{k^{\alpha+d-1}(1+|k|)^{\beta(1-\rho)-\gamma/2}} \left\{ \prod_{(\ell,j) \in G_{\text{free}}} \left[ \exp\{c_\ast s'_{\ell,j} |k_{\ell,j}|^{2\beta}\} \frac{\mathbb{1}\{s'_{\ell,j} \leq 0\}}{|k_{\ell,j}|^{(\alpha-2\beta+d-1)/2}} \right] \right\}^{\ell(G)}.
\]

From the definition of \( J_G(t_{1,m}) \) (see (4.44)) and (4.53) we conclude that
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\[ E J^2_G(t_{1,m}) = \int_{\mathbb{R}^{f(G)(1+d)}} \left[ \int_{\Delta_N(t_{1,m})} H_G(s_{1,m+N},s'_{G_{\text{free}},k_{G_{\text{free}}}}) \, ds_{1,m+N} \right]^2 \]

\[ \times \prod_{(\ell,j) \in G_{\text{free}}} ds'_{\ell,j} \, dk_{\ell,j} \]

\[ \leq \left[ t_1^\rho C^{1-\rho(t_1)} \right]^{2(m+N)} T^{2(N+1)\gamma} \int_{\mathbb{R}^d} \frac{dk}{|k|^{\alpha+d-1} (1 + |k|)^2(1-\rho)-\gamma} \]

\[ \times \left\{ \int_{\mathbb{R}^d} \frac{dk}{|k|^{\alpha+d-1} (1 + |k|)^2(1-\rho)-\gamma} \right\}^{2f(G)}. \]

Since \( \alpha + 2\beta \geq \alpha + \beta > 1 \), we can choose sufficiently small \( \gamma > 0 \) and \( \rho \in (0,1) \) so that the integrals appearing on the right hand side converge. In light of (4.43) and (4.45) this concludes the proof of Lemma 4.11.

4.6. Proof of Lemma 4.12. To simplify the notation we shall prove the lemma for

\[ H_1(s,z) = \frac{1}{T} (\partial_{z_{j''}} V_{j''}) \left( s, \frac{z}{T} \right), \quad H_\ell(s,z) = V_{j'} \left( s, \frac{z}{T} \right), \quad \ell = 2, \ldots, m, \]

for some \( j'',j'_1,\ldots,j'_m \in \mathbb{Z}_d \). The general case can be dealt with in the same fashion; see the end of the section for some additional remarks.

Given \( \ell \in \mathbb{Z}_d \) we can write \( H_\ell = \sum_{j=1}^{d} H_\ell^{(j)} \), where \( H_\ell^{(j)} \) corresponds to the \( Y_j \) component of the noise in (2.6). For any \( j := (j_1,\ldots,j_m) \in \mathbb{Z}_d^m \) set

\[ L^j(s_{1,m}) := E \left[ \prod_{\ell=1}^{m} H_\ell^{(j_\ell)}(s_j,0) \right] . \]

Then

\[ E \left[ \prod_{\ell=1}^{m} H_\ell(s_j,0) \right] = \sum_j L^j(s_{1,m}). \]

The proof of the lemma will be completed as soon as we show that

\[ \lim_{T \to \infty} \int_{\Delta(t_{1,m})} E[L^j(s_{1,m})] \, ds_{1,m} = 0 \]

for each \( j \in \mathbb{Z}_d^m \). We consider only the case when \( j_0 = \cdots = j_m \), as the other cases can be dealt with similarly. As before we drop the multi-indices. According to (3.5) we have

\[ E[L(s_{1,m})] = \sum_{G \in \mathbb{G}_m^o} I_G(f(\cdot;s_{1,m})). \]
where $I_G$ is given by (3.3) and
\begin{equation}
(4.57) \quad \tilde{f}(s', k, s_{1,m}) = \mathcal{i}^m T^{-1}(k_{1,1} + k_{1,2})j''
\times \prod_{\ell=1}^m \left[ E_T(s_{\ell} - s'_{\ell,1}, s_{\ell} - s'_{\ell,2}, k_{\ell,1}, k_{\ell,2}) \mathbb{I}_{(s'_{\ell,1}, s'_{\ell,2} \leq s_{\ell})} \Gamma_{j'_{\ell},j_0} (k_{\ell,1} + k_{\ell,2}) \right].
\end{equation}

Here the summation $\sum_{G \in \mathcal{G}_m^2}$ extends over all complete Feynman diagrams with $m$ nodes and two hands, $E_T$ is defined in (4.4), and $s'$ and $k$ denote the ensembles of variables ($s'_{\ell,j}$) and ($k_{\ell,j}$), respectively. We adopt the convention that $s_{m+1} = 0$. Invoking the definition of $E_T(s_{\ell} - s'_{\ell,1}, s_{\ell} - s'_{\ell,2}, k_{\ell,1}, k_{\ell,2})$ and assumptions (2.4)–(2.5) it is clear that $\tilde{f}$ converges pointwise to 0 as $T \to \infty$. Moreover, since $a$ has bounded support we have
\begin{equation}
(4.58) \quad |\tilde{f}(s', k, s_{1,m})| \leq g(s', k, s_{1,m}),
\end{equation}
where
\[ g(s', k, s_{1,m}) = \prod_{\ell=1}^m \prod_{j=1}^{2} \mathcal{P}_{\ell,j} \frac{e^{-c_{s}|k_{\ell,j}|^{2\beta}(s_{\ell}-s'_{\ell,j})} \mathbb{I}_{(s'_{\ell,j} \leq s_{\ell})}}{|k_{\ell,j}|^{(\alpha+d-1)/2-\beta}}, \]
for some positive constant $c_s$. Hence, by (4.58), (4.56) and the dominated convergence theorem, to prove (4.55) it is enough to show that
\[ J_G := \int_{0}^{t} \int_{\Delta_m(t_{1,m})} I_G(g(\cdot, s_{1,m})) \, ds_{1,m} < \infty \quad \text{for any } G \in \mathcal{G}_m^2. \]

Recalling the definition (3.3) and integrating out with respect to variables $s'_{\ell,j}$ we find that for each $m$ there exists a constant $C > 0$, independent of $t_1$, such that for $T, t_1 > 0$,
\[ J_G \leq C \int_{\Delta_m(t_1)} \mathcal{P}_{s_{1,m}} \mathcal{P}_{s_{1,m}} \prod_{\ell < \ell'} \frac{e^{-c_s |k_{\ell,j}|^{2\beta}(s_{\ell}-s'_{\ell'})} \mathbb{I}_{(s'_{\ell,j} \leq s_{\ell})}}{|k_{\ell,j}|^{d-1}} \delta(k_{\ell,j} + k_{\ell',j'}) \, dk.
\]
Substituting $\tau_{\ell} := s_{\ell} - s_{\ell+1}$, estimating $e^{-c_s |k_{\ell,j}|^{2\beta}(s_{\ell}-s'_{\ell'})} \leq e^{-c_s |k_{\ell,j}|^{2\beta} \tau_{\ell}}$ and enlarging the domain of integration from the simplex $\{ \sum_{\ell=1}^{m} \tau_{\ell} \leq t_1, \tau_{\ell} \geq 0, \ell = 1, \ldots, m \}$ to $[0, t_1]^m$ we get
\begin{equation}
(4.59) \quad J_G \leq C \int_{[0,t_1]^m} \mathcal{P}_{s_{1,m}} \mathcal{P}_{s_{1,m}} \prod_{\ell < \ell'} \frac{e^{-c_s |k_{\ell,j}|^{2\beta} \tau_{\ell}} \mathbb{I}_{(s'_{\ell,j} \leq s_{\ell})}}{|k_{\ell,j}|^{\alpha+d-1}} \delta(k_{\ell,j} + k_{\ell',j'}) \, dk_{\ell,j} \, dk_{\ell',j'} \, d\tau_{1,m}.
\end{equation}
When we integrate with respect to $\tau_{\ell}$ on the right hand side, there are three possibilities:
Then, since \( (4.24) \) we conclude that all integrals on the right hand side of \((4.63)\) converge, \(Ct \) can be rewritten as vertices of some links, and using \((4.62)\), we see that the right hand side of \(\tau \) to set of those \(\ell\)'s obviously equals \(l_1(G)\). In this case the integral with respect to \(\tau_\ell\) equals \(t_1\).

Note the obvious identity
\[
(4.62) \quad 2l_1(G) + l_2(G) = m. 
\]

Changing variables \(t_1^{1/(2\beta)}k_{\ell,j} \mapsto k_{\ell,j}\), when \((\ell,j) \in L(G)\), the set of all left vertices of some links, and using \((4.62)\), we see that the right hand side of \((4.59)\) can be rewritten as

\[
(4.63) \quad \frac{C t_1^{mH}}{c_1 l_1(G) + l_2(G)} \int_{\mathbb{R}^d} \prod_{\ell \in L_1(G)} \frac{1 - e^{-c_1(k_{\ell,1}|2\beta + |k_{\ell,2}|2\beta)}}{|k_{\ell,1}|2\beta + |k_{\ell,2}|2\beta} \prod_{(\ell,j) \in L_1(G)} \frac{dk_{\ell,j}}{|k_{\ell,j}|^{\alpha+d-1}} \\
\times \prod_{\ell \in L_2(G)} \prod_{(\ell,j) \in L(G)} \frac{1 - e^{-c_1(k_{\ell,1}|2\beta + |k_{\ell,2}|2\beta)}}{|k_{\ell,1}|2\beta + |k_{\ell,2}|2\beta} \frac{dk_{\ell,j}}{|k_{\ell,j}|^{\alpha+d-1}}.
\]

Invoking the elementary fact that \((1 - e^{-x})/x \approx (1 + x)^{-1}\) for \(x > 0\) and \((4.24)\) we conclude that all integrals on the right hand side of \((4.63)\) converge, since \(\alpha + \beta > 1\) and \(\alpha < 1\). The proof of the lemma is therefore finished for \(H_\ell\) of the form \((4.54)\).

In the general case the proof is essentially the same. The corresponding function \(\tilde{f}\) in \((4.57)\) will contain some additional factors of the form \(T^{-1}(k_{j,1} + k_{j,2})\) arising after each differentiation. To show the desired con-
vergence it is important that $\tilde{f}$ contains at least one of them, so it converges pointwise to 0 under the integral. The presence of the additional factors does not affect the dominated convergence theorem argument, because in the majorization we make use of the fact that the expressions $|k/T|^p a(|k/T|)$ are uniformly bounded for any $p \geq 0$, so (4.58) is still in force.

**Remark 4.14.** Note that, in light of (4.62), the above argument shows that

$$J_G \preceq t_1^{mH}, \quad T, t_1 > 0.$$  

This estimate will be useful in the proof of tightness.

**4.7. Tightness: end of proof of Theorem 2.6.** In light of Theorem 4.7 it remains to show the tightness of the laws of $(z_T(t))_{t \geq 0}$, $T > 0$ over $C([0, +\infty); \mathbb{R}^d)$. The processes $(z_T(t))_{t \geq 0}$, $T > 0$, have stationary increments, so to demonstrate tightness as $T \to \infty$ it suffices to show that for any $S > 0$ there exist $C, \kappa > 0$ such that

$$E|z_T(t)|^2 \leq Ct^{1+\kappa}, \quad t \in [0, S], \; T > 1.$$  

Thanks to the stationarity of $(V_T(s, z_T(s)/T))_{s \in \mathbb{R}}$ (cf. Proposition 4.3) we have

$$E|z_T(t)|^2 = 2 \int_0^t \int_0^u E\left[V_T \left( s, \frac{z_T(s)}{T} \right) \cdot V_T \left( u, \frac{z_T(u)}{T} \right) \right] \, du \, ds \quad = 2 \int_0^t \int_0^u E\left[V_T \left( s, \frac{z_T(s)}{T} \right) \cdot V_T(0, 0) \right] \, ds.$$

We expand $V_T(s, z_T(s)/T)$ around 0, according to (4.31). We can easily see by a direct calculation, similar to those in Section 4.2, that

$$\int_0^t \int_0^u E[V_T(s, 0) \cdot V_T(0, 0)] \, ds \leq t^{2H}, \quad t > 0, \; T > 1.$$  

By Remark 4.14 (see estimate (4.64)), the terms resulting from $S^{(k)}$, $k = 1, \ldots, N-1$, (cf. (4.31)) can be bounded from above by $Ct^{(k+1)H}$ for $t \in [0, S]$, where the constant $C > 0$ is independent of $T > 0$. According to (4.40), the remainder term can be estimated by $Ct^{\rho (N+1)}$ for some $\rho \in (0, 1)$, independent of $N$. Choosing the latter so large that $\rho (N+1) > 1$ and remembering that $(k+1)H \geq 2H > 1$ for $k = 1, \ldots, N-1$ we obtain (4.65), finishing the proof of tightness and of Theorem 2.6.

**5. Proof of Proposition 2.9.** To simplify we assume that $a_0, r_0$ in (2.18) both equal 1. Since a Rosenblatt process is determined by its moments we can use the moment method to establish the result.
Fix any \( N \in \mathbb{N}, b_1, \ldots, b_N \in \mathbb{R} \) and \( r_1, \ldots, r_N \in \mathbb{R}_+ \) and denote

\[
\psi(t) = \sum_{m=1}^{N} b_m \mathbb{1}_{[0,r_m]}(t).
\]

For a given \( j = 1, \ldots, d \) we can write

\[
\sum_{m=1}^{N} b_m Z_j(r_m) = \int_{\mathbb{R}^{2d+2}} F(s, k, s', k') W_j(ds, dk) W_j(ds', dk'),
\]

where

\[
F(s, k, s', k') := \int_{\mathbb{R}} \psi(t) \mathbb{1}_{(-\infty,t]}^2(s, s')(|k||k'|)^{-(\alpha+d-1)/2-\beta} \times \exp\{-\frac{1}{2}|k|^2(t-s)\} \exp\{-\frac{1}{2}|k'|^2(t-s')\} \, dt.
\]

By Lemma 3.2 we have

\[
\mathbb{E}\left(\sum_{m=1}^{N} b_m Z_j(r_m)\right)^n = \sum_{G \in \mathcal{G}^2_n} I_G,
\]

where

\[
I_G = \int_{\mathbb{R}^{2nd+2n}} F(s_{1,1}, k_{1,1}, s_{1,2}, k_{1,2}) \ldots F(s_{n,1}, k_{n,1}, s_{n,2}, k_{n,2})
\]

\[
\times \prod_{((\ell,m),(\ell',m')) \in G} \delta(s_{\ell,m} - s_{\ell',m'}) \delta(k_{\ell,m} + k_{\ell',m'}) \, ds_{\ell,m} ds_{\ell',m'} \, dk_{\ell,m} dk_{\ell',m'}
\]

Clearly if \( G = G_1 \cup \cdots \cup G_m \), where the \( G_i \) are the connected components of \( G \), then \( I_G = I_{G_1} \ldots I_{G_m} \). Following [7, proof of Theorem 3.2], it suffices to show that there exists \( C > 0 \) such that for any \( G \in \mathcal{G}^2_n \) (the set of complete diagrams with a single connected component),

\[
I_G = C^n \int_{\mathbb{R}^n} \prod_{j=1}^{n} [\psi(t_j)|t_j - t_{j+1}|^{H-1}] \, dt_{1,n}, \quad n \geq 1,
\]

with \( H \) given by (2.16) and \( t_{n+1} := t_1 \). If \( G \in \mathcal{G}^2_n \) then (5.3) can be written as

\[
I_G = \int_{\mathbb{R}^{n(d+1)}} F(s_1, k_1, s_2, -k_2) F(s_2, k_2, s_3, -k_3) \ldots F(s_n, k_n, s_1, -k_1) \, dk_{1,n} ds_{1,n}.
\]

Note that

\[
\int_{s \vee s'}^{+\infty} \exp\{-|k|^{2\beta}[t - (s + s')/2]\} \, dt = \frac{e^{-\frac{1}{2}|s-s'||k|^{2\beta}}}{|k|^{2\beta}}.
\]
We integrate over $s_{1,n}$ in (5.5). Using (5.6) we obtain
\[ I_G = \int_{\mathbb{R}^{n(d+1)}} \prod_{j=1}^{n} \left[ \psi(t_j) \frac{e^{-\frac{1}{2} |k_j|^2 |t_j-t_{j+1}|}}{|k_j|^{\alpha+d-1}} \right] dk_1,n \ dt_{1,n}. \]

Substituting $k_i' := k_i |t_i - t_{i-1}|^{\frac{1}{2\beta}}$, $i = 1, \ldots, n$, we get
\[ I_G = \left( \int_{\mathbb{R}^d} \frac{e^{-\frac{1}{2} |k|^2 |t_i - t_{i+1}|^{H-1}}}{|k|^{\alpha+d-1}} dk \right)^n \int_{\mathbb{R}^n} \prod_{j=1}^{n} \left[ \psi(t_j) |t_i - t_{i+1}|^{H-1} \right] dt_{1,n}. \]

(cf. (2.16)) with the convention $t_{n+1} := t_1$. Hence (5.4) follows. ■

Appendix A. Growth of a stationary and regular random field

**Proposition A.1.** Suppose that $(F(t, x))_{(t, x) \in \mathbb{R}^{1+d}}$ is a stationary $\mathbb{R}^d$-valued random field over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that
\[ (A.1) \quad \mathbb{E} \left[ \sup_{(t, x) \in [0, T] \times [0, 1]^d} |F(t, x)|^q \right] < \infty \]
for some $q, T > 0$. Then
\[ (A.2) \quad \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \frac{|F(t, x)|}{1 + |x|^{d/q}} < \infty \quad \mathbb{P}\text{-a.s.} \]

**Proof.** Define a stationary field by
\[ F_m := \sup_{(t, x) \in [0, T] \times [0, 1]^d} |F(t, x + m)|^q, \quad m = (m_1, \ldots, m_d) \in \mathbb{Z}^d. \]

We have $\mathbb{E}F_m < \infty$. Let $|x| := \max_{1 \leq \ell \leq d} |x_\ell|$ for $x \in \mathbb{R}^d$ and let $m := |m|_{\infty}$. Obviously
\[ \frac{F_m}{(2m)^d} \leq \frac{1}{(2m)^d} \sum_{|m'|_{\infty} \leq m} F_{m'}. \]

The right hand side converges to a finite limit a.s. by the multiparameter ergodic theorem [24, Theorem 6.2.8, p. 205]. The above implies (A.2). ■

**Corollary A.2.** Suppose that $(V(t, x))_{(t, x) \in \mathbb{R}^{1+d}}$ is a continuous trajectory version of the field defined by (2.6). Then, for any $T, q > 0$,
\[ (A.3) \quad \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \frac{|V(t, x)|}{1 + |x|^{d/q}} < \infty \quad \mathbb{P}\text{-a.s.} \]

**Proof.** Using the hypercontractivity for double integrals with respect to space-time white noise [19, Theorem 3.50, p. 39], we conclude that for any
integer $n > 1 + d/2$,
\[
\mathbb{E}|V(t, x) - V(s, y)|^{2n} \leq (\mathbb{E}|V(t, x) - V(s, y)|^2)^n \\
\leq \sum_{\ell=1}^{d} [R_{\ell,\ell}(0, 0) - R_{\ell,\ell}(t-s, x-y)]^n \leq |t-s|^n + |x-y|^{2n},
\]
for $(s, y), (t, x) \in [0,T] \times [0,1]^d$. The last estimate follows from (2.13) and the fact that $\nabla_x R_{\ell,\ell}(0, 0) = 0$.

According to [25, Theorem 1.4.1, pp. 31–32] we then have
\[
(A.4) \quad \mathbb{E} \left[ \sup_{(t,x) \in [0,T] \times [0,1]^d} |V(t, x)|^n \right] < \infty.
\]
The conclusion of the corollary is then a consequence of Proposition [A.1].

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