When the bispectrum is real-valued

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Abstract

Let \( \{X(t), t \in \mathbb{Z}\} \) be a stationary time series with a.e. positive spectrum. Two consequences of that the bispectrum of \( \{X(t), t \in \mathbb{Z}\} \) is real-valued but nonzero: 1) if \( \{X(t), t \in \mathbb{Z}\} \) is also linear, then it is reversible; 2) \( \{X(t), t \in \mathbb{Z}\} \) can not be causal linear. A corollary of the first statement: if \( \{X(t), t \in \mathbb{Z}\} \) is linear, and the skewness of \( X(0) \) is nonzero, then third order reversibility implies reversibility. In this paper the notion of bispectrum is of a broader scope.

1 Introduction

If a time series is reversible, then all of its polyspectra, if they exist, are real-valued. The frequency-domain test of reversibility in [6] uses this property regarded to the bispectrum, i.e. that real-valuedness of the bispectrum is a necessary condition of reversibility. In this paper we prove that, in essence, when the time series is linear, the real-valuedness of the non-zero bispectrum is a sufficient condition as well, see Theorem 1. This confirms that when linearity is known to hold, then for testing reversibility 1) there are no need for the polyspectra of order higher than three, and 2) the bispectrum-based reversibility test of [6] is consistent (with respect to non-reversibility, and not only with respect to the alternative hypothesis that the bispectrum is not real). There is also another corollary, valid in essence for linear time series with a skewed distribution: third order reversibility (see Definition 2) implies reversibility, see Corollary 1.

Our other theorem, in essence: if the spectrum is positive and the bispectrum is real-valued but nonzero, then the time series can not be causal linear, see Theorem 2.

Let us recall some notions. A time series \( \{X(t), t \in \mathbb{Z}\} \) is called reversible, if

\[
(X_t, X_{t+1}, \ldots, X_{t+k}) \overset{d}{=} (X_{t+k}, X_{t+k-1}, \ldots, X_t)
\]

for all \( k \in \mathbb{N} \) and \( t \in \mathbb{Z} \) (\( \overset{d}{=} \) means equality in distribution). Reversibility implies stationarity, see [8]. A time series \( \{X(t), t \in \mathbb{Z}\} \) is reversible, if and only if it is stationary and

\[
(X_{t_1}, \ldots, X_{t_k}) \overset{d}{=} (X_{-t_1}, \ldots, X_{-t_k})
\]

(1)

for all \( k \in \mathbb{N} \) and \( t_1 < \ldots < t_k \in \mathbb{Z} \). Since Gaussian stationary time series are always reversible, it is enough to deal with the non-Gaussian case.
A time series \( \{X(t), t \in \mathbb{Z}\} \) is called linear, if it has a moving average representation

\[
X(t) = \sum_{k=-\infty}^{\infty} c(k)Z(t-k),
\]

\[
\sum_{k=-\infty}^{\infty} c(k)^2 < \infty, \text{ r.v.s } Z(t), t \in \mathbb{Z}, \text{ are i.i.d. with } EZ(t) = 0, \ E|Z(t)|^2 < \infty.
\]

Obviously, linearity implies stationarity. Because of non-Gaussianity, the representation (2) is called causal, if the summation is over nonnegative indices only, i.e. if the time series does not depend on future \(Z(t)\) values.

The proofs of the main results depend largely on the solution of a particular case of the Cauchy functional equation, see Lemma 4.

The rest of the paper is organized as follows. In Section 2 the notion of the bispectrum is generalized in order to be existing for, among others, linear time series with finite third order absolute moment. The main results of the paper are stated in Section 3. The proofs and the necessary lemmas on the Cauchy functional equation are presented in Section 4.

2 The bispectrum of a linear time series

Assume that the time series \( \{X(t), t \in \mathbb{Z}\} \) is stationary in third order. Let us denote the joint cumulant of the random variables \(X(t_1), X(t_2), X(t_3)\) by \(\text{cum}(X(t_1), X(t_2), X(t_3))\). Because of stationarity we have \(\text{cum}(X(t_1), X(t_2), X(t_3)) = \text{cum}(X(0), X(t_2 - t_1), X(t_3 - t_1)), t_1, t_2, t_3 \in \mathbb{Z}\), thus the third order joint cumulant is, in fact, a function of two variables only. The bispectrum has been defined in [2] as the two variable Fourier-transform of the sequence of third order joint cumulants \(\text{cum}(X(0), X(t_1), X(t_2)), t_1, t_2 \in \mathbb{Z}\). For the Fourier-transform to be meaningful, it has been required that the cumulant series be absolutely summable. Defined in this way, the bispectrum is an integrable function, thus \(\text{cum}(X(0), X(t_1), X(t_2)), t_1, t_2 \in \mathbb{Z}\), is the two variable inverse Fourier-transform of it. The absolute summability condition is, however, too strict, e.g. long range dependent time series generally fail to fulfill it. However, if we define the bispectrum requiring the integrability of the bispectrum only, but not the absolute summability of the cumulant series, then we get a more general concept, what is extensive enough to apply to at least any linear time series with finite absolute moments of third order. On the other hand, the integrability of a function guarantees the one-to-one correspondence between itself and its inverse Fourier transform. Thus, the above mentioned classical definition of the bispectrum can be generalized so that we do not assume absolutely summable cumulants.

**Definition 1** Let \(\{X(t), t \in \mathbb{Z}\}\) be a stationary time series with finite third order absolute moment, and assume that a function \(B(\omega_1, \omega_2)\) defined a.e. on \([0, 2\pi) \times [0, 2\pi)\) is integrable, and its inverse Fourier transform is just the cumulant sequence, i.e.

\[
\text{cum}(X(0), X(t_1), X(t_2)) = \int_0^{2\pi} \int_0^{2\pi} \exp(i(t_1\omega_1 + t_2\omega_2))B(\omega_1, \omega_2)d\omega_1d\omega_2,
\]

\(t_1, t_2 \in \mathbb{Z}\). Then function \(B(\omega_1, \omega_2)\) is called the bispectrum of \(\{X(t), t \in \mathbb{Z}\}\).
In the rest of the paper we use the notion of bispectrum in the sense of Definition 1. As we have already mentioned, there is a one-to-one correspondence between the set of all bispectra and the set of those third order joint cumulant sequences for which the bispectrum exists.

**Lemma 1**

Let \( \{ X(t), t \in \mathbb{Z} \} \) be a linear time series with finite third order absolute moment and moving average representation (2). Then the bispectrum of \( \{ X(t), t \in \mathbb{Z} \} \) exists, and it has the form

\[
B(\omega_1, \omega_2) = \frac{\text{cum}_3(Z(0))}{(2\pi)^2} \varphi(\omega_1) \varphi(\omega_2) \varphi(-\omega_1 - \omega_2)
\]

for a.e. \((\omega_1, \omega_2) \in [0, 2\pi) \times [0, 2\pi)\), where

\[
\varphi(\omega) = \sum_{k=-\infty}^{\infty} c(k)e^{-ik\omega},
\]

\(\omega \in [0, 2\pi)\), is the frequency domain transfer function corresponding to the linear representation (3).

3 Some new relations among reversibility, bispectrum and linearity

**Theorem 1**

Let \( \{ X(t), t \in \mathbb{Z} \} \) be a linear time series with finite third order absolute moment and a.e. positive spectrum. If its bispectrum is real-valued but not a.e. zero, then \( \{ X(t), t \in \mathbb{Z} \} \) is reversible.

Sometimes the 2 and 3 dimensional distributions can be handled more directly than the general finite dimensional ones. Motivated by this, we introduce the following notion, and then state a corollary of the previous theorem.

**Definition 2**

Let \( k \in \{2, 3, \ldots \} \). A stationary time series \( \{ X(t), t \in \mathbb{Z} \} \) is reversible in \( k^{th} \) order, if (1) holds for all \( t_1 < \ldots < t_k \in \mathbb{Z} \).

**Corollary 1**

Let \( \{ X(t), t \in \mathbb{Z} \} \) be a linear time series with finite third order absolute moment, and a.e. positive spectrum. If the skewness of \( X(t) \) is nonzero, then \( \{ X(t), t \in \mathbb{Z} \} \) is reversible in third order, if and only if it is reversible.

The following theorem is about a relation between real-valuedness of the bispectrum and causal linear representability.

**Theorem 2**

Let \( \{ X(t), t \in \mathbb{Z} \} \) be a time series with a.e. positive spectrum. If the bispectrum of \( X(t) \) exists and is real-valued but not a.e. zero, then \( \{ X(t), t \in \mathbb{Z} \} \) can not have a causal linear representation.
4 Proofs and preliminary lemmas

Proof of Lemma [1] First of all by the Marcinkiewicz–Zygmund inequality r.v.s $Z(t)$ also have finite third order moments. Applying the Schwarz–Cauchy inequality and utilizing the $2\pi$-periodicity of the function $\varphi(\omega)$, one can easily get that $B(\omega_1, \omega_2)$ in (1) is integrable, i.e. $B \in L^1([0, 2\pi] \times [0, 2\pi])$. Let us introduce the notation

$$\varphi^{(K)}(\omega) = \sum_{k=-K}^{K} c(k)e^{-ik\omega},$$

$K \in \mathbb{N}$, $\omega \in [0, 2\pi)$. In the same manner as the integrability of $B(\omega_1, \omega_2)$, one can also get the inequality

$$\int_{0}^{2\pi} \int_{0}^{2\pi} |\psi_1(\omega_1)\psi_2(\omega_2)\psi_3(-\omega_1 - \omega_2)| \, d\omega_1 d\omega_2 \leq \sqrt{2\pi} \|\psi_1\|_2 \|\psi_2\|_2 \|\psi_3\|_2$$

for any $2\pi$-periodic functions $\psi_1, \psi_2, \psi_3$ with the property $\psi_i|_{[0, 2\pi]} \in L^2[0, 2\pi)$, $i = 1, 2, 3$, where $\|\cdot\|_2$ is the $L^2[0, 2\pi)$-norm. Thus we have

$$\lim_{K_1, K_2, K_3 \to \infty} \left( \frac{\text{cum}_3(Z(0))}{(2\pi)^2} \varphi^{(K_1)}(\omega_1)\varphi^{(K_2)}(\omega_2)\varphi^{(K_3)}(-\omega_1 - \omega_2) \right) = B(\omega_1, \omega_2),$$

where l.i.m. denotes limit in $L^1([0, 2\pi] \times [0, 2\pi])$. Hence the Fourier-transform of $B(\omega_1, \omega_2)$ is

$$\int_{0}^{2\pi} \int_{0}^{2\pi} e^{(t_1\omega_1 + t_2\omega_2)} B(\omega_1, \omega_2) d\omega_1 d\omega_2$$

$$= \frac{\text{cum}_3(Z(0))}{(2\pi)^2} \lim_{K_1, K_2, K_3 \to \infty} \int_{0}^{2\pi} \int_{0}^{2\pi} e^{(t_1\omega_1 + t_2\omega_2)} \varphi^{(K_1)}(\omega_1)\varphi^{(K_2)}(\omega_2)\varphi^{(K_3)}(-\omega_1 - \omega_2) d\omega_1 d\omega_2$$

$$= \frac{\text{cum}_3(Z(0))}{(2\pi)^2} \lim_{K_1, K_2, K_3 \to \infty} \sum_{k_1=-K_1}^{K_1} \sum_{k_2=-K_2}^{K_2} \sum_{k_3=-K_3}^{K_3} c(k_1)c(k_2)c(k_3)$$

$$\times \int_{0}^{2\pi} e^{i\omega_1(t_1-k_1+k_3)} d\omega_1 \int_{0}^{2\pi} e^{i\omega_2(t_2-k_2+k_3)} d\omega_2$$

$$= \text{cum}_3(Z(0)) \sum_{k=-\infty}^{\infty} c(t_1 + k)c(t_2 + k)c(k).$$
On the other hand we have
\[
\begin{align*}
\text{cum} \left( X(0), X(t_1), X(t_2) \right) &= \text{cum} \left( \sum_{k_0=-\infty}^{\infty} c(k_0)Z(-k_0), \sum_{k_1=-\infty}^{\infty} c(k_1)Z(t_1 - k_1), \sum_{k_2=-\infty}^{\infty} c(k_2)Z(t_2 - k_2) \right) \\
&= \sum_{k_0=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} c(k_0)c(k_1)c(k_2) \text{cum} \left( Z(-k_0), Z(t_1 - k_1), Z(t_2 - k_2) \right) \\
&= \text{cum}_3(Z(0)) \sum_{k=-\infty}^{\infty} c(k)c(t_1 + k)c(t_2 + k).
\end{align*}
\]

From (5) and (6) we have
\[
\text{cum} \left( X(0), X(t_1), X(t_2) \right) = \int_0^{2\pi} \int_0^{2\pi} e^{i(t_1 \omega_1 + t_2 \omega_2)} B(\omega_1, \omega_2) d\omega_1 d\omega_2,
\]
meaning that function $B(\omega_1, \omega_2)$ in (4) is the bispectrum of $\{X(t), t \in \mathbb{Z}\}$.

As a preliminary to the proof of Theorem 1 we solve the Cauchy functional equation modulo $\pi$, defined a.e.. It has been solved, separately, both when it is defined a.e. and when it holds modulo $\pi$. At first we quote these results, both in simplified form.

**Lemma 2** ([7], [3]) Let $(Y, +)$ be a commutative group, and let the function $f : \mathbb{R} \to Y$ satisfy the Cauchy functional equation a.e., i.e.
\[
f(x + y) = f(x) + f(y)
\]
for almost all pairs $(x, y) \in \mathbb{R}^2$ (in the sense of Lebesgue measure on $\mathbb{R}^2$). Then there exists a function $\phi : \mathbb{R} \to Y$ satisfying (7) everywhere in $\mathbb{R}^2$ and being a.e. (in the sense of Lebesgue measure on $\mathbb{R}$) equal to $f$.

**Lemma 3** ([1]) Let $F$ be a real topological vector space and assume that $L : F \to \mathbb{R}$ is a continuous linear functional. Suppose $\phi : \mathbb{R} \to F$ satisfies
\[
\phi(x + y) - \phi(x) - \phi(y) \in L^{-1}(\mathbb{Z})
\]
for all $x, y \in \mathbb{R}$. If $\phi$ is measurable, then there exists a continuous linear operator $M : \mathbb{R} \to F$ such that
\[
\phi(x) - M(x) \in L^{-1}(\mathbb{Z})
\]
for all $x \in \mathbb{R}$.

Now, consider the Cauchy functional equation when it holds modulo $\pi$ and a.e., simultaneously.
Lemma 4 Let the measurable function \( f : \mathbb{R} \to \mathbb{R} \) fulfil the congruence
\[
f(x + y) = (f(x) + f(y)) \mod \pi
\]
for a.e. \((x, y) \in \mathbb{R}^2\), i.e.,
\[
f(x + y) - f(x) - f(y) \in \pi \mathbb{Z}
\]
for a.e. \((x, y) \in \mathbb{R}^2\). Then \( f \) must be of the form
\[
f(x) = cx + k(x)
\]
for a.e. \( x \in \mathbb{R} \), where \( c \in \mathbb{R} \) and \( k : \mathbb{R} \to \pi \mathbb{Z} \).

Proof. First we prove that there exists a measurable function \( h : \mathbb{R} \to \mathbb{R} \), such that
\[
h(x) = f(x) \mod \pi
\]
for a.e. \( x \in \mathbb{R} \), and
\[
h(x + y) = (h(x) + h(y)) \mod \pi
\]
for all \((x, y) \in \mathbb{R}^2\). (Notice the difference between “for a.e. \((x, y) \in \mathbb{R}^2\)” and “for all \((x, y) \in \mathbb{R}^2\)”.) Consider the factor group \((\mathbb{R} / (\pi \mathbb{Z}) , \circledast) = ((0, \pi), \circledast)\), where \( \circledast \) is the modulo \( \pi \) addition. This is a commutative group. There exists a measurable function \( g : \mathbb{R} \to [0, \pi) \), such that
\[
g(x) = f(x) \mod \pi
\]
for all \( x \in \mathbb{R} \), and
\[
g(x + y) = (g(x) + g(y)) \mod \pi,
\]
i.e.
\[
g(x + y) = g(x) \circledast g(y)
\]
for a.e. \((x, y) \in \mathbb{R}^2\). (To see this take the function \( g \) to be \( f \mod \pi \).) Thus by Lemma 2 there exists a function \( h : \mathbb{R} \to [0, \pi) \), such that
\[
h(x) = g(x)
\]
for a.e. \( x \in \mathbb{R} \), and
\[
h(x + y) = h(x) \circledast h(y)
\]
for all \((x, y) \in \mathbb{R}^2\). Considering now that (3) holds for all \( x \in \mathbb{R} \), and (11) holds for a.e. \( x \in \mathbb{R} \), it follows that (8) holds for a.e. \( x \in \mathbb{R} \).

Next we prove that the function \( h \) must be of the form
\[
h(x) = cx + \ell(x)
\]
for all \( x \in \mathbb{R} \), where \( c \in \mathbb{R} \) and \( \ell : \mathbb{R} \to \pi \mathbb{Z} \). The conditions of Lemma 3 are fulfilled, since:
• by (12) we have
\[
h(x + y) - h(x) - h(y) \in \pi \mathbb{Z}
\]
for all \( x, y \in \mathbb{R} \);
• \( h \) is measurable, since so is \( g \), and (11) holds.
Thus by Lemma 3 there exist a constant \( c \in \mathbb{R} \) and a function \( \ell : \mathbb{R} \to \pi \mathbb{Z} \) such that (13) holds for all \( x \in \mathbb{R} \).

Combining the results of the previous two parts we get the statement of the lemma. ■

**Proof of Theorem 1.** By Lemma 1 \( \{X(t), t \in \mathbb{Z}\} \) has a bispectrum \( B(\omega_1, \omega_2) \) of the form (4).

Thus, using the notations of Lemma 1, we have

\[
\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_1 + \omega_2) = \frac{4\pi^2 \text{cum}_3(Z(0))}{B(\omega_1, \omega_2)} B(\omega_1, \omega_2)
\]

(14)

for a.e. \((\omega_1, \omega_2) \in [0, 2\pi) \times [0, 2\pi)\), where \( \text{cum}_3(Z(0)) \neq 0 \) because the bispectrum is not identically zero. Since the spectrum of \( \{X(t), t \in \mathbb{Z}\} \), denoted by \( S(\omega) \), is a.e. positive, and \( S(\omega) = |\varphi(\omega)|^2 \) for a.e. \( \omega \in [0, 2\pi) \), it follows that \( \varphi(\omega) \neq 0 \) for a.e. \( \omega \in [0, 2\pi) \). Hence \( B(\omega_1, \omega_2) \neq 0 \) for a.e. \((\omega_1, \omega_2) \in [0, 2\pi) \times [0, 2\pi)\) (this can be seen by dividing the set where the bispectrum is zero into three sets corresponding to the three factors on the left hand side of (14), and observing that each of these three sets is of zero measure). Taking the logarithm and then the imaginary part in (14), we have

\[
\psi(\omega_1) + \psi(\omega_2) - \psi(\omega_1 + \omega_2) \in \pi \mathbb{Z},
\]

(15)

for a.e. \((\omega_1, \omega_2) \in [0, 2\pi) \times [0, 2\pi)\), where

\[
\psi(\omega) \equiv \text{Im} \log(\varphi(\omega)),
\]

and the principal branch \( \log : \mathbb{C} \setminus \{0\} \to \mathbb{R} + i[-\pi, \pi) \) of the complex logarithm function is used.

Denote the periodic continuation of \( \psi \) by the same letter, i.e.

\[
\psi : \mathbb{R} \to \mathbb{R},
\]

\[
\psi(\omega + 2\pi) = \psi(\omega),
\]

for all \( \omega \in \mathbb{R} \). Thus we have (15) for a.e. \((\omega_1, \omega_2) \in \mathbb{R}^2\). Now, by Lemma 4 we have

\[
\psi(\omega) = c\omega + k(\omega)
\]

(16)

for a.e. \( \omega \in \mathbb{R} \), where \( c \in \mathbb{R} \) and \( k : \mathbb{R} \to \pi \mathbb{Z} \). Using the \( 2\pi \)-periodicity of \( \psi \) it follows from (16) that

\[
c = \frac{n}{2},
\]

for some \( n \in \mathbb{Z} \). Thus we have

\[
\psi(\omega) = \frac{n}{2}\omega + k(\omega)
\]

for a.e. \( \omega \in \mathbb{R} \), where \( n \in \mathbb{Z} \) and \( k : \mathbb{R} \to \pi \mathbb{Z} \). Substituting this form of \( \psi(\omega) \) into the argument of the transfer function \( \varphi(\omega) \) we have

\[
\varphi(\omega) = r(\omega)e^{i\psi(\omega)} = r(\omega)e^{i\frac{n}{2}\omega}e^{ik(\omega)},
\]

(17)

where \( r(\omega) = |\varphi(\omega)| \).
Let us calculate the coefficients in the moving average representation (2) of $X(t)$. We have

$$c(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\ell \omega} \varphi(\omega) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} r(\omega) e^{ik(\omega)} e^{i(t-\frac{n}{2})\omega} d\omega $$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} r(\omega) e^{ik(\omega)} e^{i(t-\frac{n}{2})\omega} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\omega) e^{-i(n-\ell)\omega} d\omega = c(n-\ell),$$

for each $\ell \in \mathbb{Z}$, where in the second equation we used (17), while the third equation follows from the fact that both the coefficient $c(\ell)$ and $e^{ik(\omega)}$ are real. Now, (18) means that the sequence of coefficients in (2) is symmetric to the index $n$. Hence, using the necessary and sufficient condition of reversibility of $\mathbb{X}$ (what states, that a linear time series with a.e. positive spectrum is reversible if and only if either the series $\langle c \rangle$ in (2) is symmetric to some index, or it is skew-symmetric and the r.v. $Z(0)$ has symmetric distribution) it follows the statement of our theorem. 

**Proof of Corollary 1.** If $\{X(t), t \in \mathbb{Z}\}$ is reversible in third order, then it is reversible also in second order. Thus for $k = 3$ relation (11) holds even if the indices are not all different. Thus

$$\text{cum}(X(t_1), X(t_2), X(t_3)) = \text{cum}(X(-t_1), X(-t_2), X(-t_3))$$

for all $t_1, t_2, t_3 \in \mathbb{Z}$, particularly

$$\text{cum}(X(0), X(t_1), X(t_2)) = \text{cum}(X(0), X(-t_1), X(-t_2))$$

for all $t_1, t_2 \in \mathbb{Z}$. Hence, using (2) and the one-to-one correspondence between the bispectra and the cumulants, it follows that the bispectrum is real-valued. Moreover, from (11) we have $\text{cum}_3(X(0)) = \text{constant} \times \text{cum}_3(Z(0))$, implying that $\text{cum}_3(X(0)) \neq 0$, since $\text{cum}_3(X(0)) = E(X(0))^3 \neq 0$. On the other hand, by By Lemma (1) $\{X(t), t \in \mathbb{Z}\}$ has a bispectrum $B(\omega_1, \omega_2)$ of the form (4). $B(\omega_1, \omega_2)$ is not a.e. zero, because otherwise either $\text{cum}_3(Z(0)) = 0$ or the transfer function $\varphi(\omega)$ would be zero on a set of positive Lebesgue measure, and then the spectrum $S(\omega) = |\varphi(\omega)|^2$ would also be zero on a set of positive measure, what is a contradiction. Hence by Theorem (1) follows the reversibility of $\{X(t), t \in \mathbb{Z}\}$.

**Proof of Theorem 2.** Let us assume linearity and repeat the proof of Theorem (1) up to the conclusion that the sequence of coefficients in the linear representation is symmetric. We are ready, because a linear representation with symmetric coefficients are necessarily two-sided, while a causal representation would be one-sided.

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**Proof.**

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