Distributions in CR-lightlike submanifolds of an indefinite Kaehler statistical manifold

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Abstract
In this paper, the distributions in CR-lightlike submanifolds of an indefinite Kaehler Statistical manifold have been characterized using second fundamental form and the necessary and sufficient conditions for integrability of the same have been obtained. Also, the conditions for the distributions to be totally geodesic with respect to the dual connections in the statistical manifold have been developed.

Keywords
CR-lightlike submanifolds, distribution, indefinite Kaehler Statistical manifold, totally geodesic submanifold, integrability.

AMS Subject Classification
53B05, 53B15, 53B30.

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1. Introduction

A statistical manifold is a contemporary and an interesting branch of manifolds which has developed from the investigation of geometric structures on sets of certain probability distributions. It is a differentiable manifold where each point represents a probability distribution. The set of all probability measures consists of an infinite dimensional statistical manifold which associates each location in parameter space to a probability density function. A parameter space and a manifold together enables us to generalize many concepts from the Euclidean space to the statistical manifold. These manifolds are also geometrically formulated as Riemannian manifolds with a certain affine connection.

Rao [14] was the one to relate geometry with statistics resulting in the formation of the statistical manifold. He used Fisher information matrix to introduce the concept of Riemannian metric. Although various researchers worked in this direction in the subsequent years, yet an appreciable amount of work was done by Amari [20] [1] and Simon [17] when they introduced statistical manifold on the basis of information geometry which is the study of probability and information from the view point of differential geometry having applications in the fields of statistics and applied mathematics. Then Vos [18] developed certain fundamental equations and structural formulae for the statistical manifold. Thereafter, Kurose [11] developed the concept of holomorphic statistical manifold which was further elaborated by Furuhata et.al [7][8][9][16]. Milijevic [22] [12] studied CR-submanifolds of holomorphic statistical manifold which was later on studied extensively by Boyom et.al [3].

The study of the concept of CR- submanifolds being earlier confined to manifolds with positive definite metric and its non applicability to other branches of mathematics motivated Duggal et.al [4][5][21][6][2] to introduce the theory of CR-lightlike submanifolds of Kaehler manifold where the underlying metric is indefinite. This study created widespread interest in the lightlike geometry among the researchers due to its applications in the theory of general relativity. In the recent years, a considerable amount of work has been done in almost Hermitian statistical manifolds.

Keeping in focus the above facts, in this paper, after giv-
ing certain important preliminary concepts, we have established some conditions on the integrability of distributions in CR-lightlike submanifolds of an indefinite Kaehler statistical manifold from their characterisations with the properties on the subbundles of the manifold. Also by using the projections on the distributions, we have developed certain results for the Kaehler statistical manifold with respect to the connection and dual connection.

2. CR-lightlike Submanifolds

Consider \((M, g)\) as an \((m + n)\)-dimensional semi-Riemannian manifold with semi-Riemannian metric \(g\) and of constant index \(q\) such that \(m, n \geq 1, 1 \leq q \leq m + n - 1\).

Let \((M, g)\) be a \(m\)-dimensional lightlike submanifold of \(\bar{M}\). In this case, there exists a smooth distribution \(RadTM\) on \(M\) of rank \(r > 0\), known as radical distribution on \(M\) such that \(RadTM_p = TM_p \cap TM^\perp_p, \forall p \in M\) where \(TM_p\) and \(TM^\perp_p\) are degenerate orthogonal spaces but not complementary. Then \(M\) is called an \(r\)-lightlike submanifold of \(\bar{M}\).

Now, consider \(S(TM)\), known as Screen distribution, as a complementary distribution of radical distribution in \(TM\) i.e.,

\[
TM = RadTM \perp S(TM)
\]

and \(S(TM^\perp)\), called screen transversal vector bundle, as a complementary vector subbundle to \(Rad(TM)\) in \(TM^\perp\) i.e.,

\[
TM^\perp = RadTM \perp S(TM^\perp)
\]

As \(S(TM)\) is non degenerate vector subbundle of \(T\bar{M}|_M\), we have

\[
T\bar{M}|_M = S(TM) \perp S(TM^\perp)
\]

where \(S(TM^\perp)\) is the complementary orthogonal vector subbundle of \(S(TM)\) in \(T\bar{M}|_M\).

Let \(tr(TM)\) and \(ltr(TM)\) be complementary vector bundles to \(TM\) in \(T\bar{M}|_M\) and to \(RadTM\) in \(S(TM^\perp)\). Then we have

\[
tr(TM) = ltr(TM) \perp S(TM^\perp), \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
(ii) There exist vector bundles $S(T^\perp M)$, $S(T^\perp M)^\perp$, $l_{tr}(TM)$, $D_\perp$ and $D'$ over $M$ such that

$$S(TM) = \{ \bar{J}(\text{Rad}TM) \oplus D_\perp \} \perp D_\perp,$$  

where $D_\perp$ is a nondegenerate distribution on $M$ and $L_1$, $L_2$ are vector bundles of $l_{tr}(TM)$ and $S(TM^\perp)$, respectively.

Using the above definition, the tangent bundle $TM$ of $M$ is decomposed as:

$$TM = D \oplus D'$$

where

$$D = \text{Rad}TM \perp \bar{J}\text{Rad}TM \perp D_\perp$$

We denote by $S$ and $Q$, the projections on $D$ and $D'$ respectively. Then we have

$$\bar{J}X = fX + wX$$  \hspace{1cm} (2.2)

for any $X, Y \in \Gamma(TM)$, where $fX = \bar{J}SX$ and $wX = \bar{J}QX$.

Also we set

$$J^V = BV + CV$$  \hspace{1cm} (2.3)

for any $V \in \Gamma(tr(TM))$, where $BV \in \Gamma(TM)$ and $CV \in \Gamma(tr(TM))$.

Unless otherwise stated, $M_1$ and $M_2$ are supposed to be as $\bar{J}L_1$ and $\bar{J}L_2$ where $\bar{J}(L_1) = M_1 \subset D'$ and $\bar{J}(L_2) = M_2 \subset D'$ respectively.

3. The Lightlike approach to an indefinite Statistical Manifold

Following are certain basic known definitions related to the theory of lightlike submanifolds of an indefinite statistical manifold.

Let $\bar{M}$ be a $C^\infty$ manifold of dimension $\bar{m} \geq 2$, $\bar{\nabla}$ be an affine connection on $\bar{M}$ and $\bar{g}$ be a semi-Riemannian metric of constant index $q \geq 1$ on $\bar{M}$. Then

1. $(\bar{M}, \bar{\nabla}, \bar{g})$ is called an indefinite statistical manifold if
   (i) $\bar{\nabla}$ is of torsion free and
   (ii) $\langle \bar{\nabla}_X \bar{g}(Y, Z) - \bar{\nabla}_Y \bar{g}(X, Z) \rangle = 0$ for $X, Y, Z \in \Gamma(\text{TM})$.

2. If $\bar{X}\bar{g}(X, Z) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}_X Z)$; for $X, Y, Z \in \Gamma(\text{TM})$, then $\bar{\nabla}^X$ is called the dual connection of $\bar{\nabla}$ with respect to $\bar{g}$.

If $(\bar{M}, \bar{\nabla}, \bar{g})$ is an indefinite statistical manifold, then $(\bar{M}, \bar{\nabla}^*, \bar{g})$ is also an indefinite statistical manifold. We therefore denote the indefinite statistical manifold by $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$.

Let $M$ be a submanifold of a statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g})$ and $g$ be the induced metric on $M$. An affine connection $\nabla$ on $M$ is defined by ([12], [9]) as:

$$\nabla_X Y = (\bar{\nabla}_X Y)^\perp$$

where $(\bar{\nabla}_X Y)^\perp$ denotes the orthogonal projection of $\bar{\nabla}_X Y$ on the tangent space with respect to $\bar{g}$, that is $\langle \bar{\nabla}_X Y, Z \rangle = \bar{g}(\bar{\nabla}_X Y, Z)$ for $X, Y, Z \in \Gamma(TM)$. Then $(M, \nabla, g)$ becomes a statistical manifold and $(\nabla, g)$ is called the induced statistical structure on $M$.

$(M, \bar{\nabla}, \bar{g})$ is said to be a statistical submanifold in $(\bar{M}, \bar{\nabla}, \bar{g})$ if $(\nabla, g)$ is induced statistical structure on $M$.

Now $\bar{T}^\perp M$ denote the normal space of $M$ i.e. $\bar{T}^\perp M = \{ v \in T\bar{M} | \bar{g}(v, w) = 0, w \in \bar{T}M \}$ and $g$, the induced metric on $M$. It follows that

$$\bar{\nabla}_X \bar{\nabla}^\perp Y = \bar{\nabla}_X Y + h(X, Y), \bar{\nabla}^\perp \bar{\nabla}_X Y = -A^V_X + \bar{\nabla}^\perp V$$

for $X, Y \in \Gamma(TM), V \in \Gamma(T^\perp M)$.

Then the following hold for $X, Y \in \Gamma(TM), V \in \Gamma(T^\perp M)$:

$$\bar{g}(h(X, Y), V) = g(A^V_X Y, X), \bar{g}(h^*(X, Y), V) = g(A^V_X Y, X)$$

From the concept of structural equations in the lightlike theory available so far, the Gauss and Weingarten formulae for a lightlike submanifold $(M, g)$ of an indefinite statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$ are as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) + h^*(X, Y),$$

$$\bar{\nabla}^\perp \nabla Y = \bar{\nabla}^\perp Y + h(X, Y) + h^*(X, Y)$$

$$\bar{\nabla}_X V = -A^V_X + D^X V + D^\perp \bar{\nabla}^\perp V,$$

$$\bar{\nabla}^\perp \nabla V = -A^V_X + D^X V + D^\perp \bar{\nabla}^\perp V,$$

$$\bar{\nabla}_X N = -A^V_X + D^X N + D^\perp \bar{\nabla}^\perp V,$$

$$\bar{\nabla}^\perp \nabla N = -A^V_X + D^X N + D^\perp \bar{\nabla}^\perp V$$

for any $X, Y \in \Gamma(TM), V \in \Gamma(tr(TM)), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(STM^\perp)$.  

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Considering the corresponding projection morphism \( P \) of tangent bundle \( TM \) to the screen distribution, we have the following decomposition w.r.t \( \nabla \) and \( \nabla^* \):

\[
\nabla_X PY = \nabla_X^P Y + h'(X, PY), \quad \nabla_X^P Y = \nabla_X^P Y + h''(X, PY)
\]

(3.6)

\[
\nabla_X \xi = -A_\xi X + \nabla_X^\ast \xi, \quad \nabla_X^\ast \xi = -A_\xi^\ast X + \nabla_X^\ast \xi
\]

(3.7)

for any \( X, Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM)) \).

Then the following holds:

\[
\bar{g}(h'(X, PY), \xi) = g(A_\xi^0 X, PY),
\]

(3.8)

\[
\bar{g}(h''(X, PY), \xi) = g(A_\xi^0 X, PY)
\]

(3.9)

for any \( X, Y \in \Gamma(TM) \).

3.1 Indefinite Kaehler Statistical Manifold

Let \( \bar{\nabla}^0 \) be the Levi-Civita connection w.r.t \( \bar{g} \). Then, we have \( \bar{\nabla}^0 = \frac{1}{2}(\nabla + \nabla^*) \).

For a statistical manifold \( (\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*) \), the difference (1,2) tensor \( \bar{K} \) of a torsion free affine connection \( \bar{\nabla} \) and Levi-civita connection \( \bar{\nabla}^0 \) is defined as

\[
\bar{K}(X, Y) = \bar{K}_XY = \bar{\nabla}XY - \bar{\nabla}^*XY
\]

(3.10)

Since \( \bar{\nabla} \) and \( \bar{\nabla}^0 \) are torsion free, we have

\[
\bar{K}(X, Y) = \bar{K}(Y, X), \quad \bar{g}(\bar{K}_XY, Z) = \bar{g}(\bar{Y}, \bar{K}_XZ)
\]

for any \( X, Y, Z \in \Gamma(TM) \).

Also we have

\[
\bar{K}(X, Y) = \bar{\nabla}^0 XY - \bar{\nabla}^*XY
\]

From the above equations, we have

\[
\bar{K}(X, Y) = \frac{1}{2}(\bar{\nabla}XY - \bar{\nabla}^*XY).
\]

Also, from (3.10), we have

\[
\bar{g}(\bar{\nabla}XY, Z) = \bar{g}(\bar{K}(X, Y), Z) + \bar{g}(\bar{\nabla}^*XY, Z)
\]

We have the following result from [13]:

\[
\bar{g}(\bar{K}_XY, Z) = -\bar{g}(\bar{Y}, (\bar{\nabla}^*)_XZ)
\]

(3.11)

holds for any \( X, Y, Z \in \Gamma(TM) \) for an almost Hermitian manifold \( (\bar{M}, \bar{g}, \bar{J}, \bar{\nabla}, \bar{\nabla}^*) \). Now, from [19], we have the following equations for the almost Hermitian manifold:

\[
(\bar{\nabla}XY) = (\bar{\nabla}^*XY) + (K_XY)\bar{J}Y
\]

\[
(\bar{\nabla}^*XY) = (\bar{\nabla}^*XY) - (K_XY)\bar{J}Y
\]

4. Characteristics of Distributions

Definition 4.1. A CR-lightlike submanifold of an indefinite Kaehler Statistical manifold is called D-totally geodesic with respect to \( \bar{\nabla} \) (respectively \( \bar{\nabla}^* \)) if \( h(X, Y) = 0 \) (respectively \( h^*(X, Y) = 0 \)) for all \( X, Y \in D \).

Definition 4.2. A CR-lightlike submanifold of an indefinite Kaehler statistical manifold is called mixed totally geodesic with respect to \( \bar{\nabla} \) (resp. \( \bar{\nabla}^* \)) if \( h(X, Y) = 0 \) (resp. \( h^*(X, Y) = 0 \)) for \( X \in D \) and \( Y \in D' \).

Theorem 4.3. Let \( (\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*) \) be an indefinite Kaehler statistical manifold and \( \bar{M} \) be CR-lightlike submanifold of \( \bar{M} \). If \( \bar{M} \) is D-totally geodesic with respect to \( \bar{\nabla} \) and \( \bar{\nabla}^* \), then \( w\bar{\nabla}XY + w\bar{\nabla}^*XY = 0 \).
Then we have

\[ h(X, Y) + h^*(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y + \tilde{\nabla}_Y X - \nabla_Y X \]

Using the relation of connections \( \tilde{\nabla} \) and \( \nabla^* \) with Levi Civita connection \( \nabla^0 \) in statistical manifold \( M \), we get

\[
\begin{align*}
  h(X, Y) + h^*(X, Y) &= 2(\tilde{\nabla}^0_X Y - \nabla^0_X Y) \\
  &= 2(\tilde{\nabla}^0_X JY + J\tilde{\nabla}^0_X Y + \nabla^0_X Y)
\end{align*}
\]

From the fact that \( M \) is a Kaehler statistical manifold and from the equations (2.1), (2.2) (2.3), we have

\[
\begin{align*}
  h(X, Y) + h^*(X, Y) &= -2\nabla^0_X JY - 2\nabla^0_X Y \\
  &= -(2[f\nabla^0_X Y + w\nabla^0_X Y + \frac{1}{2}(Bh(X, Y) + Bh^*(X, Y)) + \frac{1}{2}(Ch(X, JY) + Ch^*(X, JY)) - \nabla^0_X Y]
\end{align*}
\]

On equating normal parts, we obtain

\[
\begin{align*}
  h(X, Y) + h^*(X, Y) &= -2\nabla^0_X JY - 2\nabla^0_X Y - Ch(X, JY) - Ch^*(X, JY)
\end{align*}
\]

Using the given hypothesis that \( M \) is \( D \)-totally geodesic with respect to \( \tilde{\nabla} \) and \( \nabla^* \), we get the desired result.

**Lemma 4.4.** Let \( (\tilde{\nabla}, \tilde{\nabla}^0, \tilde{\nabla}^* \) be an indefinite Kaehler statistical manifold and \( M \) be a CR-lightlike submanifold of \( \tilde{\nabla} \). Then we have

\[ f[X, Y] = \nabla^0_X JY - \nabla^0_Y JX \]

for any \( X, Y \in \Gamma(D) \).

**Proof:** For any \( X, Y \in \Gamma(D) \), we have

\[
\begin{align*}
  h(X, JY) + h^*(X, JY) &= \tilde{\nabla}_X JY - \nabla_X JY + \tilde{\nabla}_Y X - \nabla_Y X \\
  &= 2(\tilde{\nabla}^0_X JY - \nabla^0_X JY) \\
  &= 2(\tilde{\nabla}^0_X JY - \nabla^0_X JY)
\end{align*}
\]

Similarly

\[
\begin{align*}
  h(JX, Y) + h^*(JX, Y) &= 2(\tilde{\nabla}^0_JX JY - \nabla^0_JX JY)
\end{align*}
\]

From above equations, we have

\[
\begin{align*}
  h(X, JY) + h^*(X, JY) - h(JX, Y) - h^*(JX, Y) &= 2(\tilde{\nabla}^0_X JY - \nabla^0_X JY) \\
  &= 2(\tilde{\nabla}^0_JX JY - \nabla^0_JX JY - \nabla^0_X JY)
\end{align*}
\]

Hence the proof.

**Theorem 4.5.** Let \( (\tilde{\nabla}, \tilde{\nabla}^0, \tilde{\nabla}^* \) be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold of \( M \). Then \( D' \) is integrable with respect to \( \tilde{\nabla} \) and \( \tilde{\nabla}^* \) if and only if

\[ -A_wZ + A_wZ = f[Z, W] \]

\[ \nabla^0_Z wW - \nabla^0_W wZ = w[Z, W] \]

for any \( Z, W \in \Gamma(D') \).

**Proof:** The given hypothesis implies that

\[ \nabla^0_Z JW = J\nabla^0_Z W \]

for any \( Z, W \in \Gamma(D') \). Thus from the relation of Levi-civita connection with dual connections \( \tilde{\nabla}, \tilde{\nabla}^* \), we obtain

\[
\begin{align*}
  2\nabla^0_Z JW &= 2J\nabla^0_Z W + J(h(Z, W) + h^*(Z, W)) \\
  2\nabla^0_Z (fW + wW) &= 2J\nabla^0_Z W + J(h(Z, W) + h^*(Z, W)) \\
  2(-A_wZ + \nabla^0_w Z) &= 2(f\nabla^0_W Z + w\nabla^0_W Z) \\
  + Bh(Z, W) + Ch(Z, W) + Bh^*(Z, W) + Ch^*(Z, W)
\end{align*}
\]

Similarly, we get

\[
\begin{align*}
  2(-A_wZ + \nabla^0_w Z) &= 2(f\nabla^0_W Z + w\nabla^0_W Z) \\
  + Bh(Z, W) + Ch(W, Z) + Bh^*(W, Z) + Ch^*(W, Z)
\end{align*}
\]

Thus we have

\[
\begin{align*}
  2(-A_wZ + A_wZ + \nabla^0_w Z - \nabla^0_w Z) &= 2(f[Z, W] + w[Z, W])
\end{align*}
\]

By taking tangential and normal components of this equation, we get the desired result.

**Theorem 4.6.** Let \( (\tilde{\nabla}, \tilde{\nabla}^0, \tilde{\nabla}^* \) be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold of \( M \). Then \( D' \) is integrable if and only if

\[
\begin{align*}
  (i) \tilde{g}(h'(X, Y), N) + \tilde{g}(h'^*(X, Y), N) &= \tilde{g}(h'(X, Y), N) \\
  + \tilde{g}(h'^*(Y, X), N) \\
  (ii) \tilde{g}(h'(X, JY), N) + \tilde{g}(h'^*(X, JY), N) &= \tilde{g}(h'(Y, JX), N) \\
  + \tilde{g}(h'^*(Y, JX), N) \\
  (iii) \tilde{g}(h'(X, JY), W) + \tilde{g}(h'^*(X, JY), W) &= \tilde{g}(h'(Y, JX), W) \\
  + \tilde{g}(h'^*(Y, JX), W) \\
  (iv) \tilde{g}(\nabla^0_X JY J\xi) + \tilde{g}(\nabla_X Y J\xi) &= \tilde{g}(\nabla^0_Y JX J\xi) + \tilde{g}(\nabla_X Y J\xi)
\end{align*}
\]

for any \( X, Y \in \Gamma(D_0) \), \( N \in \Gamma(Tr(TM)) \), \( \xi \in \Gamma(Rad(TM)) \) and \( W \in \Gamma(S(TM^1)) \).

**Proof:** The definition of CR-lightlike submanifold \( M \) of \( M \) implies that \( D_0 \) is integrable if and only if

\[
\begin{align*}
  \tilde{g}([X, Y], JN) &= \tilde{g}([X, Y], JW) \]
\]

(4.1)
for any \(X, Y \in \Gamma(D_{\chi}), N \in \Gamma(\text{Rad}(TM)), \xi \in \Gamma(\text{Rad}(TM)), W \in \Gamma(S(TM))\).

For connections \(\tilde{\nabla}\) and \(\tilde{\nabla}^*\), we have
\[
\tilde{g}([X, Y], N) = \tilde{g}(\tilde{\nabla}_X Y, N) - \tilde{g}(\tilde{\nabla}_Y X, N) = -\tilde{g}(Y, \tilde{\nabla}_X N) + \tilde{g}(X, \tilde{\nabla}_Y N).
\]

Using the equation (3.9), we obtain
\[
\tilde{g}([X, Y], N) = \frac{1}{2}[\tilde{g}(h''(X, Y), N) + \tilde{g}(h'(X, Y), N) - \tilde{g}(h''(X, Y), N) + \tilde{g}(h'(X, Y), N)]
\]

and using the statistical character of the manifold and equations (3.4), (3.9) we have
\[
\tilde{g}([X, Y], \tilde{J}N) = -\tilde{g}(\tilde{J}\tilde{\nabla}_X Y, N) + \tilde{g}(\tilde{J}\tilde{\nabla}_Y X, N).
\]

Also
\[
\tilde{g}([X, Y], \tilde{J}W) = \tilde{g}(\tilde{\nabla}_X Y, \tilde{J}W) - \tilde{g}(\tilde{\nabla}_Y X, \tilde{J}W).
\]

Using the relation of connections \(\tilde{\nabla}, \tilde{\nabla}^*\) with Levi-Civita Connection and the equation 3.3, we get
\[
\tilde{g}([X, Y], \tilde{J} \tilde{W}) = \frac{1}{2}[\tilde{g}(h''(X, Y), W) - \tilde{g}(h''(X, Y), W) + \tilde{g}(h'(X, Y), W) + \tilde{g}(h''(X, Y), W)].
\]

Now from equations (3.6), (3.7) we obtain
\[
\tilde{g}([X, Y], \tilde{J}\tilde{\xi}) = \frac{1}{2}[\tilde{g}(\tilde{\nabla}_X Y, \tilde{J}\tilde{\xi}) + \tilde{g}(\tilde{\nabla}_Y X, \tilde{J}\tilde{\xi}) - \tilde{g}(\tilde{\nabla}_Y X, \tilde{J}\tilde{\xi}) - \tilde{g}(\tilde{\nabla}_X Y, \tilde{J}\tilde{\xi})].
\]

Therefore using the given hypothesis, we have the desired result.

**Theorem 4.7.** Let \((\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)\) be an indefinite Kaehler statistical manifold and \(M\) be a CR-lightlike submanifold of \(\bar{M}\). Then \(\text{Rad}(TM)\) is integrable if and only if
(i) \(\bar{g}(h'(\xi, J\tilde{\xi}'), W) + \bar{g}(h''(\xi, J\tilde{\xi}''), W) = \bar{g}(h''(\xi, J\tilde{\xi}''), W) + \bar{g}(h''(\xi, J\tilde{\xi}''), W)\)
(ii) \(\bar{g}(h'(\xi, J\tilde{\xi}'), \tilde{\xi}') + \bar{g}(h''(\xi, J\tilde{\xi}''), \tilde{\xi}') = \bar{g}(h''(\xi, J\tilde{\xi}''), \tilde{\xi}') + \bar{g}(h''(\xi, J\tilde{\xi}''), \tilde{\xi}')\)

Proof: We know that for a CR-lightlike submanifold \(M\) of \(\text{Rad}(TM)\) is integrable if and only if
\[
\bar{g}([\bar{\xi}', \bar{\xi}'', J\bar{\xi}''], W) = 0.
\]

Thus for connections \(\tilde{\nabla}\) and \(\tilde{\nabla}^*\), we have
\[
\tilde{g}([\xi, \xi'], \bar{\xi}) = \tilde{g}(\bar{\xi}, [\xi, \xi'], W) = 0.
\]

On the other hand, we get
\[
\tilde{g}([\xi, \xi'], \bar{\xi}) = \tilde{g}(\bar{\xi}, [\xi, \xi'], W) = 0.
\]

Thus the proof is complete by using given hypothesis from (4.2).
Theorem 4.8. Let \([\mathcal{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*]\) be an indefinite Kaehler statistical manifold and \(M\) be a CR-lightlike submanifold of \(\mathcal{M}\). Then each maximal integrable manifold of the Radical distribution is totally geodesic in \(M\) if and only if

(i) \(\bar{g}(\bar{\nabla}_\bar{Y}^A_\xi X) + \bar{g}(\nabla^*_\bar{Y} X) = 0\)

(ii) \(\bar{g}(\bar{\nabla}_\bar{Y}^A_\xi X) + \bar{g}(\nabla^*_\bar{Y} X) = 0\)

(iii) \(\bar{g}(\bar{\nabla}_\bar{Y}^A_\xi X) = 0\)

for any \(X, Y, \xi \in \Gamma(\text{Rad}(TM)), N \in \Gamma(\text{tr}(TM)), W \in \Gamma(S(TM^+))\) where \(M_1 = J(L_1)\) and \(\bar{\nabla}^*\) is the dual connection of \(\bar{\nabla}^*\) w.r.t. \(\bar{g}\).

Proof: The given hypothesis of \(M\) being a CR-lightlike submanifold of \(\mathcal{M}\) implies that each maximal integrable manifold is totally geodesic if and only if

\[
\bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\xi) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, Z_\xi) = 0
\]

where \(X, Y \in \Gamma(\text{Rad}(TM))\)

Since \(\mathcal{M}\) is an indefinite Kaehler statistical manifold and (3.7) holds, we have

\[
\bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\xi) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, Z_\xi) = 0
\]

and from (3.4) and (3.9), we obtain

\[
\bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, Z_\xi) = 0
\]

and

\[
\bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W}) = \frac{1}{2}[\bar{g}(\bar{h}(X, J\bar{Y}), N) - \bar{g}(\bar{h}(X, J\bar{Y}), N)]
\]

Thus the result holds using (4.3).

Theorem 4.9. Let \([\mathcal{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*]\) be an indefinite Kaehler statistical manifold and \(M\) be a CR-lightlike submanifold of \(\mathcal{M}\). Then \(\text{JRad}(TM)\) is integrable if and only if

(i) \(\bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\xi) + \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W}) + \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W})\)

(ii) \(\bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) + \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) + \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W})\)

(iii) \(\bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) + \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W})\)

(iv) \(\bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) + \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W}) = \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{N}) + \bar{g}(\bar{\nabla}_\bar{X}^A_\xi Y, J\bar{W})\)

for any \(\xi, \xi', \xi'' \in \Gamma(\text{Rad}(TM)), N \in \Gamma(\text{tr}(TM)), W \in \Gamma(S(TM^+))\).

Proof: We have

\[
\bar{g}([\bar{J}\xi', \bar{J}\xi], N) = \bar{g}(\bar{\nabla}_\bar{J}\xi' \bar{J}\xi, N) - \bar{g}(\bar{\nabla}_\bar{J}\xi' \bar{J}\xi, N) = -\bar{g}(\bar{J}\xi', \bar{J}\xi, N) + \bar{g}(\bar{J}\xi', \bar{J}\xi, N) = \frac{1}{2}[\bar{g}(\bar{J}\xi', \bar{J}\xi, N) - \bar{g}(\bar{J}\xi', \bar{J}\xi, N)]
\]

and

\[
\bar{g}([\bar{J}\xi', \bar{J}\xi], J\bar{W}) = -\bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi, W) + \bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi, W) = -\bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi, W) + \bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi, W)
\]

Also

\[
\bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}\xi'') = \bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi'', \bar{J}\xi') - \bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi'', \bar{J}\xi') = \frac{1}{2}[\bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi'', \bar{J}\xi') - \bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi', \bar{J}\xi'')]
\]

and

\[
\bar{g}([\bar{J}\xi', \bar{J}\xi], X) = \bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi, X) - \bar{g}(\bar{J}\bar{V}_\bar{J}\xi', \bar{J}\xi, X)
\]

Now \(M\) being a CR-lightlike submanifold of the manifold \(\mathcal{M}\) concludes that \(\text{JRad}(TM)\) is integrable if and only if

\[
\bar{g}([\bar{J}\xi', \bar{J}\xi], N) = \bar{g}([\bar{J}\xi', \bar{J}\xi], J\bar{W}) = \bar{g}([\bar{J}\xi', \bar{J}\xi], J\bar{W}) = \bar{g}([\bar{J}\xi', \bar{J}\xi], X) = 0.
\]

Hence using the above assertion, we get the desired result.

5. Conclusion and Scope

In this paper, the concept of lightlike geometry in the indefinite Kaehler Statistical manifold has been discussed and the characterisation of distributions in the CR lightlike submanifolds of the same with respect to their geodesicity and integrability has been done. The statistical manifold is based on the information geometry having applications in the field of neural networks and when endowed with the lightlike geometry, becomes applicable in various branches of mathematics and physics. So this branch of study can imbibe great interest in the contemporary geometers to work upon the further properties of distributions of CR lightlike submanifolds in the indefinite Kaehler statistical manifold.
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