2 + 1 gravity and Doubly Special Relativity

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It is shown that gravity in 2+1 dimensions coupled to point particles provides a nontrivial example of Doubly Special Relativity (DSR). This result is obtained by interpretation of previous results in the field and by exhibiting an explicit transformation between the phase space algebra for one particle in 2+1 gravity found by Matschull and Welling and the corresponding DSR algebra. The identification of 2+1 gravity as a DSR system answers a number of questions concerning the latter, and resolves the ambiguity of the basis of the algebra of observables.

Based on this observation a heuristic argument is made that the algebra of symmetries of ultra high energy particle kinematics in 3+1 dimensions is described by some DSR theory.

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I. INTRODUCTION

Recently a proposal has been much discussed concerning how quantum theories of gravity may be tested experimentally. The doubly or deformed special relativity (DSR) proposal is that quantum gravity effects may lead in the limit of weak fields to modifications in the kinematics of elementary particles characterized by –

1. Preservation of the relativity of inertial frames.

2. Non-linear modifications of the action of Lorentz boosts on energy-momentum vectors, preserving a preferred energy scale, which is naturally taken to be the Planck energy, . In some cases is a maximum mass and/or momentum that a single elementary particle can attain.

3. Non-linear modifications of the energy-momentum relations, because the function of and that is preserved under the exact action of the Lorentz group is no longer quadratic. This could result in Planck scale effects such as an energy-dependent speed of light and modifications of thresholds for scattering, that may be observable in present and near future experiments.

4. Modifications in the commutators of coordinates and momentum and/or non-commutativity of space-time coordinates.

Theories with these characteristics are invariant under modifications of the Poincaré algebra, called generically -Poincaré algebras, where is a dimensional parameter that measures the deformations, usually taken to be proportional to the Planck mass.

In a recent paper , it was argued that quantum gravity in 2+1 dimensions with vanishing cosmological constant must be derivable from the limit of 2+1 quantum gravity with non-zero cosmological constant. The argument is simple and algebraic, the point is that the symmetry which characterizes quantum gravity in 2+1 dimensions with is actually quantum deformed de Sitter to with the quantum deformation parameter given by

\[ z = \ln(q) \approx l_{Planck} \sqrt{\Lambda} \]  (1)
The limit $\Lambda \to 0$ then affects both the scaling of the transformation generators as the De Sitter group is contracted to the Poincaré group, and the limit of $g \to 1$. It is easy to see that because the ratio $\kappa = \hbar \sqrt{\Lambda}/\kappa = G^{-1}_{2+1}$, where $G^{2+1}$ is Newton’s constant in 2 + 1 dimensions, is held fixed, the limit gives the $\kappa$-deformed symmetry group in 2 + 1 dimensions. The conclusion is that the symmetry algebra of 2 + 1 dimensional quantum gravity with $\Lambda = 0$ is not Poincaré, it is a $\kappa$-deformed Poincaré algebra. This means that the theory must be a DSR theory.

Quantum gravity in 2 + 1 dimensions has been the subject of much study in both the classical and quantum domain, beginning with the work of Deser, Jackiw and ’t Hooft $[12]$, $[22]$. If that theory is a DSR theory than the features just listed above must be present, and this could not have been easily missed by investigators.

Indeed, all of the listed features have been seen in the literature on 2 + 1 gravity. In the next section we review some of the long standing results in 2 + 1 gravity and show how they may be understood using the language of DSR theories. To clinch the relation, in section 3 we exhibit an explicit mapping between the phase space of quantum gravity in 2 + 1 dimensions coupled to a single point particle, studied in $[20]$, and the algebra of symmetry generators of a DSR theory.

The observation that 2 + 1 gravity provides examples of DSR theories can help the study of both sides of the relation. The language of DSR theories and their foundations in terms of general principles can unify and explain some results in the literature of 2 + 1 dimensional gravity that, when first discovered, seemed strange and unintuitive. We can now see that some of the features of 2 + 1 gravity are neither strange nor necessarily unique to 2 + 1 dimensions, because they follow only from the general requirement that the transformations between different inertial frames preserve an energy scale.

Furthermore, what one has in the 2 + 1 gravity models, such as those with gravity coupled to $N$ point particles, is a class of non-trivial DSR theories that are completely explicit and solvable, both classically and quantum mechanically. The existence of these examples answers a number of questions and challenges that have been raised concerning DSR theories. Some authors have argued $[22]$ that DSR theories are just ordinary special relativistic theories rewritten in terms of some non-linear combinations of energy and momentum, while, conversely, others have argued that they must be trivial because interactions cannot be consistently included. Both criticisms are shown wrong by the existence of an explicit and solvable class of DSR theories, with interactions, given by quantum gravity in 2 + 1 dimensions coupled to point particles and fields.

Furthermore, we see that in 2 + 1 dimensions the apparent problem of the freedom to choose the basis of the symmetry algebra of a DSR theory is resolved by the fact that the choice of the coupling of matter to the gravitational field picks out the physical energy and momentum. We see in section 3 below that for the case of minimal coupling of gravity to a single point particle the basis picked out is the classical basis.

Finally, one can ask whether the fact that 2 + 1 gravity is a DSR theory has any implications for real physics in $3 = 1$ dimensions. In the final section of the paper we present a heuristic argument that it may.

II. SIGNS OF DSR IN 2 + 1 GRAVITY

In this section we point out where effects characteristic of DSR have been discovered already in the literature on 2 + 1 dimensional gravity. We consider only the case $\Lambda = 0$.

- It is important first to note that Newton’s constant in 2 + 1 dimensions, denoted here by $G$, has dimensions of inverse mass (with only $c = 1$ and no $\hbar$ involved)$^2$. Thus, if the asymptotic symmetry group knows about gravity, it will have to preserve the scale $G^{-1}$. Of course, in theories with sufficiently short range interactions the asymptotic symmetry group does not depend on the coupling constants. But in 2 + 1 gravity the presence of matter causes the geometry of spacetime to become conical and this deforms the asymptotic conditions in a way that depends on $G$. Further, since $\hbar$ is not involved in the definition of the mass scale, $G^{-1}$, the deformation affects also the algebra of the classical phase space. This is the main reason why 2 + 1 gravity is a DSR theory.

- In 2 + 1 gravity coupled to point particles, the hamiltonian, $H$, whose value is equal to the ADM mass, and hence is measured by a surface term, is bounded from both above and below $[21]$, $[22]$.

\[ 0 \leq H \leq \frac{1}{4G} \quad (2) \]

This can be understood in the following way. In 2 + 1 dimensions the spacetime is flat, except where matter is present. A particle, or in fact any compactly supported distribution of matter, is surrounded by an asymptotic region, which is locally flat, and whose geometry is thus characterized by a deficit angle $\alpha$. A standard result is that $[12]$, $[18]$, $[17]$, $[20]$, $[22]$, $[16]$.

\[ \alpha = 8\pi GH \quad (3) \]

But a deficit angle $\alpha$ must be less than or equal to $2\pi$. Hence there is an upper limit on the mass of any system, as measured by the hamiltonian. The upper limit holds for all systems, regardless of how

$^2$ $G$ is identified with inverse of the $\kappa$ deformation parameter of $\kappa$-Poincaré algebra.
many particles there are and what their relative positions or motions are. This upper mass limit must be preserved by the asymptotic symmetry group. Hence the asymptotic symmetry group cannot be the ordinary Poincaré group, it must be a DSR theory with a maximum energy.

- It has further been shown that the spatial components of momentum of a particle in 2+1 gravity are unbounded. This, together, with a bounded energy, implies a modified energy-momentum relation.

- The phase space of a single point particle in 2 + 1 gravity was constructed by Matschull and Welling in 20 and it was found that a classical solution is labelled by a three dimensional position \( Y_\mu, \mu = 0, 1, 2 \) and momentum \( p_\mu \). They find explicitly that the energy momentum relations and the action of the Poincaré symmetry are deformed, in a way that preserves a fixed energy scale. They indeed make explicit reference to the work of Snyder 1, which was an early proposal for DSR.

- Furthermore, Matschull and Welling find that the spacetime coordinates \( Y_\mu \) of a particle are non-commutative under the classical Poisson brackets,

\[
[Y_\mu, Y_\nu] = -2G \epsilon_{\mu\nu\rho} Y_\rho.
\]

This property was found in 22 to extend to systems of \( N \) particles.

- Matschull and Welling also find that the components of the energy-momentum vector for a point particle in 2 + 1 gravity live on a curved manifold, which is 2+1 dimensional Anti-de Sitter spacetime. This was shown in 28, 30 to be a feature of DSR theories.

- Ashtekar and Varadarajan 16 found a relationship between two definitions of energy relevant for 2 + 1 gravity, which is reminiscent of non-linear redefinitions of the energy used in changing bases between different realizations of DSR theories. The case they studied has to do with 3 + 1 gravity, with two Killing fields, one rotational and one axial. One first dimensionally reduces to 2 + 1 dimensions, in which case the dynamics of GR in 3+1 is expressed as a scalar field coupled to 2 + 1 dimensional GR. The ADM Hamiltonian \( H \) still exists and still is bounded from above as in 2. But in the presence of the additional, rotational Killing field, the theory can be represented by a scalar field evolving in a flat reference Minkowski spacetime, with the ordinary hamiltonian

\[
H_{flat} = \frac{1}{2} \int_0^\infty drr \left[ \dot{\phi}^2 + (\partial_r \phi)^2 \right]
\]

\( H_{flat} \) is of course unbounded above. They find the relationship between them is

\[
H = \frac{1}{4G} (1 - e^{-4GH_{flat}})
\]

This exact relation is in fact present in the literature on DSR 28. It holds in the a presentation of the \( \kappa \)-Poincaré algebra known as the “bi-crossproduct” basis. In that case \( H_{flat} = E \) is, as in the present case, the zeroth component of an energy momentum vector and \( H \) is the “physical rest mass,\( m_0 \) defined by,

\[
\frac{1}{m_0} = \lim_{p \to 0} \frac{1}{p} \frac{dE}{dp} \bigg|_{E=p_0}
\]

It is intriguing that this is the inertial mass, while, for the solutions with rotational symmetry, the ADM energy is the active gravitational mass. Since they are both expressed in terms of the zeroth component of the energy-momentum vector by the same equation, they coincide on the subset of solutions on which they are both defined, which are the rotationally invariant solutions. This appears to be a direct demonstration of the equality of inertial and gravitational mass, within this context. Indeed, this observation suggests that the Ashtekar-Varadarajan form of the ADM mass is more general than their calculation shows. Indeed it is not hard to see that this is the case. Let us study the free scalar field in 2 + 1 dimensional Minkowski spacetime, with no condition of rotational symmetry. This system is not a dimensional reduction of general relativity, only a subspace of solutions, those with rotational symmetry, are related to general relativity. But it still may serve as a useful example of a DSR theory. Of course the theory has full Poincaré invariance, with momentum generators \( P_i \) and boost generators \( K_i \) satisfying the usual Poincaré algebra. But eq. 9 implies that they form with \( H \) a DSR algebra

\[
\{ K_i, H \} = (1 - 4GH) P_i
\]

\[
\{ K_i, P_j \} = -\frac{1}{4G} \delta_{ij} \ln|1 - 4GH|
\]

with the other commutators undeformed. The physical energy momentum relations are deformed to

\[
P_i^2 + m^2 = \frac{1}{16G^2} [\ln(1 - 4GH)]^2
\]

\[\text{References:} \]

3 Although in reference 28, 30 the momentum space for a class of DSR theories was shown to be de Sitter spacetime. We discuss below the difference between positively and negatively curved momentum spaces.
Recent calculations indicate that quantum deformations of symmetries play a role in gravitational scattering of particles in 2+1 dimensions.

All of these pieces of evidence show that 2+1 gravity coupled to matter can be understood as a DSR system. Of course, the 2+1 dimensional model system is not completely analogous to real physics in 3+1 dimensions. But this result answers cleanly several queries and criticisms that have been levied against the DSR proposal.

First, some authors have suggested that DSR theories are physically indistinguishable from ordinary special relativity. They argue that in some cases, one can arrive at a DSR system from a non-linear mapping of energy-momentum space to itself. These results show that argument fails, for there is no doubt that the model system of point particles in 2+1 gravity is physically distinguishable from the model system of free particles in flat 2+1 dimensional spacetime. This is here a clean result, with no quantization ambiguities, because the deformation parameter $\kappa = 1/4G$ is entirely classical and the modification is of the structure of the classical phase space. The two phase spaces are not isomorphic, when gravity is turned on, the phase space is curved, but when $G = 0$ the phase space is flat.

This is clear also for the multiparticle system, where there are non-trivial interactions, depending on $G$, which make the system measurably distinct from the free particle case with $G = 0$.

The multiparticle system in 2+1 gravity also serves as an example of a counterintuitive property of some DSR models in 3+1 gravity. This is that the upper mass limit $M_{upper} = 1/4G$ is independent of the number of particles in the system. This of course cannot be the case in the real world, so it is good to know that there are implementations of DSR in 3+1 dimensions that do not have an upper mass limit for systems of many particles, or where the upper mass limit grows with the number of particles or the mass of the total system, in such a way as to not violate experience.

However, it is also good to know that there is a model system, which is sensible physically, in which this non-intuitive feature is completely realized. Moreover it suggests the start of a physical answer to one of the puzzling questions about DSR models. This is that the addition of energy and momentum in DSR theories is non-linear. This can be understood as a consequence of the non-linear action of the Lorentz group, for example it follows from the fact that the energy-momentum space has non-zero curvature. It appears to remain even in realizations of DSR that remove the mass limit for composite systems.

Some physicists have criticized the DSR proposal by pointing out that the non-linear corrections to addition of energy-momentum vectors for a system of two particles can be interpreted by saying that there is a binding energy between pairs of particles that does not depend on the distance between them, but depends only on the individual energies and momenta.

This may be counter-intuitive, but it is precisely the what happens in 2+1 dimensions. Because spacetime is locally flat, each particle contributes a deficit angle to the overall geometry that affects all the other particles’ motions, no matter how far away. The result is that there is a binding energy that is independent of distance.

This suggests a speculative remark: might there be even in 3+1 dimensions a small component of the binding energy of pairs of particles, of order $M_1M_2$, which is independent of distance? Might this be interpreted as a kind of quantum gravity effect?

In the last section we make some speculative remarks concerning the question of whether these results have any bearing on real physics in 3+1 dimensions.

III. PHASE SPACE OF DSR IN 2+1 DIMENSIONS

In this section we will compare the phase space of 2+1 dimensional DSR with that of 2+1 dimensional gravity with one particle. Let us start with the former.

A. Phase spaces of DSR

As in 3+1 dimensions, the starting point to find the phase space of DSR theory in the 2+1 dimensional case is the 2+1 dimensional $\kappa$-Poincaré algebra, the quantum algebra whose generators are momenta $p_\mu = (p_0, p_i)$ and Lorentz algebra generators $J_\mu = (M, N_\mu)$ boosts. Taking the co-algebra of the $\kappa$-Poincaré quantum algebra and using the so-called “Heisenberg-double construction” it is possible to derive the position variables, conjugate to momenta, $x_\mu$, as well as the brackets between them and the $\kappa$-Poincaré algebra generators.

This (quantum) algebraic construction has a geometrical counterpart, described in [24, 27]. Here the manifold on which momenta live is de Sitter space (in the case at hand the 3 dimensional one). The positions and the Lorentz transformations are symmetries acting on the space of momenta. Thus they form the three dimensional de Sitter algebra $SO(3,1)$. It is convenient to define the de Sitter space of momenta as a three dimensional surface

$$-\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 = \kappa^2$$

in the four dimensional Minkowski space with coordinates $(\eta_0, \ldots, \eta_3)$. The physical momenta $p_\mu$ are then the coordinates on the surface [10]. This means that we can think of $\eta_A = \eta_A(p_\mu)$ as of the given functions of momenta, for which the equation [10] is identically satisfied. In the DSR terminology, the choice of a particular coordinate system on de Sitter space corresponds to a

\[4\] The Greek indices run from 1 to 3, the Latin ones from 1 to 2, while the capital ones from 0 to 3.
choice of the so called DSR basis (see \[28, 30\]). It turns out that in order to relate DSR to the 2+1 gravity one has to choose the so called classical basis, characterized by $\eta_A = p_{\mu}$. This choice will be implicit below, however we find it more convenient to write down the formulas below in terms of the variables $\eta_A$.

The algebra of symmetries of the de Sitter space of momenta \[11\] can be most easily read off by writing down the action of these symmetries on the four-dimensional Minkowski space with coordinates $\eta_A$ and then pulling them down to the surface \[10\]. Let us note however that while it is easy to identify the Lorentz generators $J_{\mu} = (M, N_i)$ as the elements of the $SO(2,1)$ subalgebra of the $SO(3,1)$, it is a matter of convenience which linearly independent combination of generators is to be identified with positions (i.e. the generators of translation in momentum space.) Technically speaking we are free to choose the decomposition of $SO(3,1)$ into the sum of $SO(2,1)$ and its remainder.

In the case of the DSR phase space, the action of the symmetries is given by

$$[M, \eta_i] = \epsilon_{ij} \eta_j, \quad [N_i, \eta_j] = \delta_{ij} \eta_0, \quad [N_i, \eta_0] = \eta_i,$$  \hspace{1cm} (11)

$$[J_\mu, \eta_3] = 0,$$  \hspace{1cm} (12)

with $J_\mu$ satisfying the algebra

$$[M, N_i] = \epsilon_{ij} N_j, \quad [N_i, N_j] = -\epsilon_{ij} M,$$  \hspace{1cm} (13)

$$[x_0, \eta_3] = \frac{1}{\kappa} \eta_0, \quad [x_0, \eta_0] = \frac{1}{\kappa} \eta_3, \quad [x_0, \eta_i] = 0,$$  \hspace{1cm} (14)

$$[x_i, \eta_3] = [x_i, \eta_0] = \frac{1}{\kappa} \eta_i, \quad [x_i, \eta_j] = \frac{1}{\kappa} \delta_{ij}(\eta_0 - \eta_3),$$  \hspace{1cm} (15)

Note that it follows from these equations that

$$[x_0, x_i] = -\frac{1}{\kappa} x_i, \quad [x_i, x_j] = 0.$$  \hspace{1cm} (16)

It is worth mentioning also that such a decomposition is possible in any dimension. In particular in the 3+1 case the bracket \[10\] describes the so-called $\kappa$-Minkowskian type of non-commutativity.

One can repeat this geometric construction in the case when the momenta manifold is the anti de Sitter space

$$-\eta_0^2 + \eta_1^2 + \eta_2^2 - \eta_3^2 = \kappa^2.$$  \hspace{1cm} (17)

Now the symmetry algebra is $SO(2,2)$, having again the three dimensional Lorentz algebra $SO(2,1)$ described by \[11, 12\] as its subalgebra. The algebra of positions, which we denote $y_{\mu}$ (i.e. translations of momenta) changes only slightly and now reads

$$[y_0, \eta_3] = -\frac{1}{\kappa} \eta_0, \quad [y_0, \eta_0] = \frac{1}{\kappa} \eta_3, \quad [y_0, \eta_i] = 0,$$  \hspace{1cm} (18)

$$[y_i, \eta_3] = \frac{1}{\kappa} \delta_{ij}(\eta_0 - \eta_3),$$  \hspace{1cm} (19)

From \[18, 19\] it follows that

$$[y_0, y_i] = -\frac{1}{\kappa} y_i + \frac{1}{\kappa^2} N_i, \quad [y_i, y_j] = -2 \kappa \epsilon_{ij} M.$$  \hspace{1cm} (20)

We see that the bracket \[20\] does not describe the $\kappa$-Minkowskian type of non-commutativity. Since the non-commutativity type is related to the co-algebra structure of the quantum Poincaré algebra, this result indicates that along with the $\kappa$-Poincaré algebra there exists another quantum Poincaré algebra with the same algebra, but different co-algebra, which we expect to be related to the former by a twist$^5$.

B. Phase space of 2+1 gravity

The phase space algebra of one particle in 2+1 dimensional gravity is the algebra of asymptotic charges. This algebra has been carefully analyzed by Matschull and Welling in \[20\]. They find that the physical momentum manifold is anti de Sitter space and that $\eta_i = p_i$, as stated above. This means that 2+1 gravity seems to pick the classical basis of DSR as the one having physical relevance. Further, Matschull and Welling employ a particular decomposition of the $SO(3,1)$ algebra, in which the positions $\mathcal{Y}_A$ act on momenta as right multiplication and have the following brackets with $\eta_A$.

$$[\mathcal{Y}_3, \eta_3] = -\frac{1}{\kappa} \eta_0, \quad [\mathcal{Y}_0, \eta_0] = \frac{1}{\kappa} \eta_3, \quad [\mathcal{Y}_0, \eta_i] = \frac{1}{\kappa} \epsilon_{ij} \eta_j,$$  \hspace{1cm} (21)

$$[\mathcal{Y}_i, \eta_3] = \frac{1}{\kappa} \eta_i, \quad [\mathcal{Y}_i, \eta_0] = \frac{1}{\kappa} \epsilon_{ij} \eta_j,$$  \hspace{1cm} (22)

$$[\mathcal{Y}_i, \eta_j] = \frac{1}{\kappa} \left( \epsilon_{ij} \eta_0 - \delta_{ij} \eta_3 \right).$$  \hspace{1cm} (23)

Comparing the expressions \[18, 19\] with \[21, 22\] we easily find that these decompositions are related by

$$\mathcal{Y}_0 = y_0 - \frac{1}{\kappa} M, \quad \mathcal{Y}_i = y_i - \frac{1}{\kappa} (N_i - \epsilon_{ij} N_j).$$  \hspace{1cm} (24)

It can be also easily checked that

$$[\mathcal{Y}_A, \mathcal{Y}_B] = -\frac{2}{\kappa} \epsilon_{\alpha\beta\mu} \mathcal{Y}^\mu.$$  \hspace{1cm} (25)

$^5$ This expectation is based on the classification of Poisson structures on Poincaré group presented in \[21\].
Thus the DSR anti de Sitter phase space is (up to a trivial reshuffling of the generators) equivalent to the phase space of a single particle in 2+1 gravity.

It is an open problem whether one can get de Sitter space as a manifold of momenta from 2+1 quantum gravity. It would be interesting to see if this is the case. If so, there exist two kinds of phase spaces of a particle in a 2+1 gravitational field corresponding to two DSR phase space algebras presented above.

IV. IMPLICATIONS FOR PHYSICS IN 3 + 1 DIMENSIONS

We present here an argument that suggests that the results of this paper, and of those we reference, concerning 2+1 dimensional quantum gravity coupled to point particles may have implications for real physics in 3 + 1 dimensions.

The main idea is to construct an experimental situation that forces a dimensional reduction to the 2 + 1 dimensional theory. It is interesting that this can be done in quantum theory, using the uncertainty principle as an essential element of the argument.

Let us consider a system of two relativistic interacting elementary particles in 3 + 1 dimensions, whose masses are less than $G^{-1}$. In the center of mass frame the motion will be planar. Let us consider the system as described by an inertial observer who travels perpendicular to the plane of the system’s motion, which we will call the z direction. From the point of view of that observer, the plane of the system’s motion, which we will call the 1−z plane, by an equivalent 2 + 1 dimensional problem in which the gravitational field is dimensionally reduced along the z direction so that the two “cosmic strings” which are the sources of the gravitational field, are replaced by two punctures.

The dimensional reduction is governed by a length $d$, which is the extent in z that the system extends. We cannot take $d < L$ without violating the uncertainty principle. It is then convenient to take $d = L$. Further, since the system consists of elementary particles, they have no intrinsic extent, so there is no other scale associated with their extent in the z direction. We can then identify $z = 0$ and $z = L$ to make an equivalent toroidal system, and then dimensionally reduce along z. The relationship between the four dimensional Newton’s constant $G^{3+1}$ and the three dimensional Newton’s constant $G^{2+1} = G$, which played a role so far in this paper is given by

$$G^{2+1} = \frac{G^{3+1}}{2L} = \frac{G^{3+1} P^{tot}_z}{2\hbar}$$

Thus, in the analogous 2+1 dimensional system, which is equivalent to the original system as seen from the point of view of the boosted observer, the Newton’s constant depends on the longitudinal momenta.

Of course, in general there will be an additional scalar field, corresponding to the dynamical degrees of freedom of the gravitational field. We will for the moment assume that these are unexcited, but exciting them will not affect the analysis so long as the gravitational excitations are invariant also under the Killing field and are of compact support.

Now we note that, if there are no other particles or excited degrees of freedom, the energy of the system can to a good approximation be described by the hamiltonian $H$ of the two dimensional dimensionally reduced system. This is described by a boundary integral, which may be taken over any circle that encloses the two particles. But this is bounded from above, by $G^{3+1}$. This may seem strange, but it is easy to see that it has a natural four dimensional interpretation.

The bound is given by

$$M < \frac{1}{4G^{2+1}} = \frac{2L}{4G^{3+1}}$$

where $M$ is the value of the ADM hamiltonian, $H$. But this just implies that

$$L > 2G^{3+1}M = R_{Sch}$$
Let us call this subspace of Hilbert space $P$ large. It involves the subgroup of $P$ that contains two particles, and is an eigenstate of $\hat{P}$ described by the system we have just constructed. It is to good approximation described by a two-particle system in 2 + 1 gravity. However, we know from the results cited in the previous sections that the symmetry algebra acting there is not by the ordinary 2 + 1 dimensional Poincaré algebra, but by the $\kappa$-Poincaré algebra in 2 + 1 dimensions, with

$$\kappa^{-1} = \frac{4G^{3+1}P_z^{\text{tot}}}{\hbar} \tag{32}$$

In particular, there is a maximum energy given by

$$M_{\text{max}}(P_z^{\text{tot}}) = \frac{M_{\text{Planck}}^2}{4P_z^{\text{total}}} \tag{33}$$

This gives us a last condition,

$$MP_z^{\text{total}} < \frac{M_{\text{Planck}}^2}{4} \tag{34}$$

which is compatible with the previous conditions. Thus, when all the conditions are satisfied, the deformed symmetry algebra must be identified with $P_G^{3+1}$.

Now we can note the following. Whatever $P_G^{3+1}$ is, it must have the following properties:

- It depends on $G^{3+1}$ and $\hbar$, so that it’s action on each subspace $H_{P_z}$ for each choice of $P_z$ is the $\kappa$ deformed 2 + 1 Poincaré algebra, with $\kappa$ as above.
- It does not satisfy the rule that momenta and energy add, on all states in $H$, since they are not satisfied in these subspaces.
- Therefore, whatever $P_G^{3+1}$ is, it is not the classical Poincaré group.

Thus the theory of particle kinematics at ultra high energies is not Special Relativity, and the arguments presented above suggest that it might be Doubly Special Relativity.

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[1] H. S. Snyder, “Quantized Space-Time,” Phys. Rev. 71 (1947) 38.
[2] V. Fock, The theory of space-time and gravitation, Pergamon Press, 1964.
[3] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, “Q deformation of Poincaré algebra,” Phys. Lett. B 264 (1991) 331; J. Lukierski, A. Nowicki and H. Ruegg, “New quantum Poincare algebra and k deformed field theory,” Phys. Lett. B 293 (1992) 344.
[4] S. Majid and H. Ruegg, “Bicrossproduct structure of kappa Poincaré group and noncommutative geometry,” Phys. Lett. B 334 (1994) 348 arXiv:hep-th/9405107;
[5] J. Lukierski, H. Ruegg and W. J. Zakrzewski, “Classical quantum mechanics of free kappa relativistic systems,” Annals Phys. 243 (1995) 90 arXiv:hep-th/9312153.
[6] G. Amelino-Camelia, “Testable scenario for relativity with minimum-length,” Phys. Lett. B 510, 255 (2001) arXiv:hep-th/0012238.
[7] G. Amelino-Camelia, “Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale,” Int. J. Mod. Phys. D 11, 35 (2002) arXiv:gr-qc/0012051.
[8] N. R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, “Deformed boost transformations that saturate at the Planck scale,” Phys. Lett. B 522, 133 (2001) arXiv:hep-th/0107039.

[9] J. Magueijo and L. Smolin, “Lorentz invariance with an invariant energy scale,” Phys. Rev. Lett. 88 (2002) 190403 arXiv:hep-th/0112000.

[10] G. Amelino-Camelia, L. Smolin and A. Starodubtsev, arXiv:hep-th/0306134.

[11] A. Staruszkiewicz, “Gravitation theory in three-dimensional space”, Acta Phys. Polon. 24 (1963) 734.

[12] S. Deser, R. Jackiw and G. ‘t Hooft, “Three-Dimensional Einstein Gravity: Dynamics Of Flat Space,” Annals Phys. 152 (1984) 220.

[13] A. Achucarro and P. K. Townsend, “A Chern-Simons Action For Three-Dimensional Anti-De Sitter Supergravity Theories,” Phys. Lett. B 180, 89 (1986).

[14] E. Witten, “(2+1)-Dimensional Gravity As An Exactly Soluble System,” Nucl. Phys. B 311, 46 (1988).

[15] J.E. Nelson, T. Regge, F. Zertuche, Nucl. Phys. B339, 316 (1990).

[16] A. Ashtekar and M. Varadarajan, “A Striking property of the gravitational Hamiltonian,” Phys. Rev. D 50 (1994) 4944 arXiv:gr-qc/9406040.

[17] E. Buffenoir, K. Noui and P. Roche, “Hamiltonian quantization of Chern-Simons theory with SL(2,C) group,” Class. Quant. Grav. 19, 4953 (2002) arXiv:hep-th/0202121.

[18] G. ‘t Hooft, “Canonical quantization of gravitating point particles in (2+1)-dimensions,” Class. Quant. Grav. 10 (1993) 1653 gr-qc/9305008. G. ‘t Hooft, “Quantization of Point Particles in 2+1 Dimensional Gravity and Space-Time Discreteness;” Class. Quant. Grav. 13 (1996) 1023 gr-qc/9601014.

[19] A. Ashtekar, V. Husain, C. Rovelli, L. Smolin, J. Samuel, “2+1 quantum gravity as a toy model for the 3+1 theory”, Class. and Quant. Grav. 6 (1989) L185.

[20] H. J. Matschull and M. Welling, “Quantum mechanics of a point particle in 2+1 dimensional gravity,” Class. Quant. Grav. 15 (1998) 2981 arXiv:gr-qc/9708054.

[21] J. Louko and H. J. Matschull, “The 2+1 Kepler problem and its quantization,” Class. Quant. Grav. 18 (2001) 2731 arXiv:gr-qc/0103085.

[22] H. J. Matschull, “The phase space structure of multi particle models in 2+1 gravity,” Class. Quant. Grav. 18 (2001) 3497 arXiv:gr-qc/0103084.

[23] D.V. Ahluwalia, M. Kirchbach, and N. Dadhich, “Operational insistinguishability of doubly special relativities from special relativity,” arXiv:gr-qc/0212128.

[24] J. Magueijo and L. Smolin, “Generalized Lorentz invariance with an invariant energy scale,” Phys. Rev. D 67 (2003) 044017 arXiv:gr-qc/0207085; Giovanni Amelino-Camelia, “Kinematical solution of the UHE-cosmic-ray puzzle without a preferred class of inertial observers”, arXiv:astro-ph/0209232.

[25] F. A. Bais, N. M. Muller and B. J. Schroers, “Quantum group symmetry and particle scattering in (2+1)-dimensional quantum gravity,” Nucl. Phys. B 640 (2002) 3 arXiv:hep-th/0205021.

[26] A. Nowicki, math.QA/9803064.

[27] J. Lukierski and A. Nowicki, Proceedings of Quantum Group Symposium at Group 21, (July 1996, Goslar) Eds. H.-D. Doebner and V.K. Dobrev, Heron Press, Sofia, 1997, p. 186.

[28] J. Kowalski-Glikman and S. Nowak, “Non-commutative space-time of doubly special relativity theories,” Int. J. Mod. Phys. D 12 (2003) 299 arXiv:hep-th/0204245.

[29] J. Kowalski-Glikman, “De Sitter space as an arena for doubly special relativity,” Phys. Lett. B 547 (2002) 291 arXiv:hep-th/0207279.

[30] J. Kowalski-Glikman and S. Nowak, “Doubly special relativity and de Sitter space,” arXiv:hep-th/0304101.

[31] S. Zakrzewski, “Poisson Structures on Poincare Group,” Commun. Math. Phys. 185 (1997) 285.