Repeated restricted Bursts error correcting linear codes Over $GF(q); q > 2$

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Abstract
This paper deals with non binary repeated restricted burst errors. In this paper lower and upper bounds on the number of parity check digits needed for a linear code having the capability to correct the repeated restricted bursts are presented. Restricted bursts are introduced by Tyagi and Lata [11] for non binary case over $GF(3)$. By a restricted burst of length $l$ or less we mean a vector whose all the non zero components are confined to some $l$ consecutive positions, the first and the last of which is nonzero with a restriction that all the non zero consecutive positions contain same field element. For example in non binary case for $q = 3, n = 3$ and $l = 2$, we have the following vectors of length 2 or less $110, 220, 011, 022, 100, 010, 001, 200, 020, 002$.

Keywords
Restricted burst errors, burst correcting codes, burst error, repeated burst error.

AMS Subject Classification
94B20, 94B65, 94B25.

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1. Introduction
During very early stages in the history of coding theory codes were meant for detecting and correcting only random errors. But at a later stage it was observed that in almost all channels errors were more in adjacent positions and quite less in random manner. Adjacent error correcting codes were introduced and developed by Abramson [1]. The generalization of this idea was put in the category of errors that is now known as burst errors. A burst of length $l$ or less may be defined as follows: “A burst of length $l$ is a vector whose only non-zero components are confined to some $l$ consecutive positions, the first and the last of which is non-zero”. This definition is due to P. Fire [6] where he defined such errors as open loop bursts errors. He also defined closed loop bursts errors according to which: “A closed loop bursts of length $l$ is a vector whose only nonzero components are confined to some $l$ consecutive positions the first and the last of which is nonzero and the number of its starting positions is $n”$. (i.e. it is possible to come back cyclically at the first position after the last position for enumeration of the length of the burst. There is yet another burst error due to Chien and Tang [3] according to which: “A burst of length $l$ is a vector whose only non-zero components are confined to some $l$ consecutive positions, the first of which is non-zero”. Among various generalizations of burst errors, Fire’s definition has been found of great importance and a good deal of research has gone into the development of bursts and multiple bursts error correcting codes. See [2, 8, 9, 12, 13] and many more. As there is not any uniform terminology for multiple bursts; repeated bursts errors correcting codes are also put in this category. Dass and Verma [4] introduced the idea of repeated bursts error correcting codes and derived both the bounds on the number of parity check digits needed for correcting repeated burst errors over $GF(q)$. It was also pointed out in the end that they have not been able to construct codes for non binary cases and that it’s a open problem. We in this paper study non-binary repeated restricted burst error correcting codes over $GF(q); q > 2$. While working on the possibility of the existence of 2-burst correcting non-
binary codes, (initially discussed by Tuvi Etzion [5] in binary case) the procedure lead us to the idea of restricted burst errors. Tyagi and Lata [11] have been able to give non-binary optimal restricted 2- burst correcting codes and byte oriented codes over GF(3). We in this paper discuss this new burst defined as ‘restricted burst’ and develop theorems for the existence of restricted burst errors correcting codes. The paper is organized into three sections. Section 2 gives necessary and sufficient conditions for 2 repeated restricted burst of length 2 or less whereas section 3 presents correction of m-repeated restricted burst of length 2 or less. In section 4, we conclude the paper by presenting an examples of (8, 2) codes with an open problem in the end.

Definition 1.1. “An m-repeated restricted burst of length l whose only non-zero components are confined to m-distinct sets of l-consecutive components, the first and last component of each set being non-zero and all the non-zero components contain same field element”.

In particular a 2-repeated restricted burst may be obtained by putting m = 2 in the above definition. The vector (00222200202200) is an example of a 2-repeated restricted burst of length 4 over GF(3).

To prove our theorems, we use the following results:

Result 1.2. (Dass and Verma, [4]) “A q-ary (n, k) linear code correcting m-repeated burst errors of length l or less must satisfy

\[ q^{n-k} \geq q^{m(l-1)} \left[ \left( \frac{n-ml+m}{m} \right) (q-1)^m + \sum_{p=0}^{m-1} \left( \frac{n-ml+p}{p} \right) (q-1)^p q^{m-1-p} \right] \].

Result 1.3. (Dass and Verma, [4]) “A q-ary (n, k) linear code correcting m-repeated burst errors of length l or less (n > 2ml) will always exist

\[ q^{n-k} > q^{2m(l-1)} \left[ \left( \frac{n-2ml+2m-1}{2m-1} \right) (q-1)^{2m-1} + \sum_{p=0}^{2m-2} \left( \frac{n-2ml+p}{p} \right) (q-1)^p q^{2m-2-p} \right] \].

2. Correction of m-Repeated Restricted Bursts

In this section we consider linear codes capable of correcting m-repeated restricted bursts of length l or less and obtain the lower and upper bound for such codes.

Theorem 2.1. An (n, k) linear code over GF(q); q > 2 that corrects all m-repeated restricted bursts of length l or less must satisfy

\[ q^{n-k} \geq (q-1)^{2m(l-1)} \left[ \left( \frac{n-ml+m}{m} \right) + \sum_{p=0}^{m-1} \left( \frac{n-ml+p}{l} \right) 2^{m-1-p} \right] - (q-2). \]

Proof. This theorem can be proved simply by enumerating the total number of correctable error vectors which are m-repeated restricted bursts of length l or less. It has been observed that the total number of m-repeated restricted bursts of length l or less for q = 3 is equal to one less than the double of the total number m-repeated burst errors of length l or less for binary case in Result 1.2 (Theorem 3.1 [4]). For q = 4 the total number of such errors is equal to thrice of the total number of m-repeated burst error of length l or less for binary case in Result 1.2 (Theorem 3.1 [4]) minus two. In this manner we conclude that the total number of m-repeated restricted burst errors of length l or less for q > 2 is

\[ (q-1)^{2m(l-1)} \left[ \left( \frac{n-ml+m}{m} \right) + \sum_{p=0}^{m-1} \left( \frac{n-ml+p}{l} \right) 2^{m-1-p} \right] - (q-2). \]

Since all these error patterns must belong to different cosets for correction and the largest number of cosets available is \( q^{n-k} \), then we have

\[ q^{n-k} \geq (q-1)^{2m(l-1)} \left[ \left( \frac{n-ml+m}{m} \right) + \sum_{p=0}^{m-1} \left( \frac{n-ml+p}{l} \right) 2^{m-1-p} \right] - (q-2). \]

This completes the required proof.

If we put m = 2 in Theorem 2.1, we get a corollary, which gives a necessary condition for the codes having the capacity to correct 2-repeated restricted burst errors of length l or less. We give the corollary as follows:

Corollary 2.2. An (n, k) linear code over GF(q); q > 2 that corrects all 2-repeated restricted bursts of length l or less must satisfy

\[ q^{n-k} \geq (q-1)^{2(l-1)} \left[ \left( \frac{n-2l+2}{2} \right) + \left( \frac{n-2l+1}{1} \right) + 2 \right] - (q-2). \]

We now give a sufficient condition on the number of parity check digits required for the existence of such a code.
Theorem 2.3. The existence an \((n,k)\) linear code over \(GF(q)\); \(q > 2\) that corrects all \(m\)-repeated restricted bursts of length \(l\) or less \((n \geq 2ml)\) provided that

\[
q^{n-k} > (q-1)^2^{\frac{ml-1}{l}} - (q-2) × \end{equation}

\[
(q-1)^2^{\frac{ml+2m-l-1}{l}} + \sum_{p=0}^{2m-2} \left( \frac{j-2ml+p}{p} \right)^{2m^{2}-p} -(q-2).
\]

Proof. This result can be proved by forming an suitable parity check matrix \(H\) by following the method used to prove the Theorem 4.7 [7] also refer Sacks [10] and Theorem 3.2 [4]. Let \(H = [c_1,c_2,c_3,...,c_n]\) for the desired code. First of all we select \(j-1\) columns \(c_1, c_2, c_3, ..., c_{j-1}\) of \(H\) suitably. Now we lay down a restriction to add the \(j^{th}\) column \(c_j\) to the matrix \(H\) as follows: \(c_j\) must not be written in the form of linear sum of just preceding \(l-1\) or lesser columns \(c_{j-1}, c_{j-2}, ..., c_{j-1}\) of \(H\) along with any \((2m-1)\) sets of \(l\) or fewer consecutive columns that are distinct and each of them is from amongst the first \(j-l\) columns \(c_1, c_2, c_3, ..., c_{j-l}\). In different words with same meaning,

\[
c_j \neq (u_1c_{j-1}+u_2c_{j-2}+\cdots+u_{j-1}c_{j-1}) + (v_1c_{i_1}+v_2c_{i_1+1}+\cdots+v_lc_{i_1+l-1}) + (w_1c_{i_2}+w_2c_{i_2+1}+\cdots+w_lc_{i_2+l-1}) + \cdots + (x_1c_{i_{m-1}}+x_2c_{i_{m-1}+1}+\cdots+x_lc_{i_{m-1}+l-1}).
\]

where \(u_1, v_1, w_1, ..., x_l \in GF(q); u_1=v_1=w_1=\cdots=x_l \neq 0\) and \(i_1+i_2+i_3+\cdots+i_{2m-1}+(2m-1)b-(2m-1) \leq j-b\). The total number of coefficients \(u_i\)’s will be equal to \((q-1)×2^{l-1}-(q-2)\). The calculation of the number of the coefficients \(v_i, w_i, x_i\) will be same as the enumeration of \((2m-1)\)-repeated restricted burst errors lying in a vector of length \(j-b\) which can be calculated by using the Theorem 3.1 [4] and given as

\[
(q-1)^2^{\frac{j-2ml+2m-l-1}{l}} + \sum_{p=0}^{2m-2} \left( \frac{j-2ml+p}{p} \right)^{2m^{2}-p} -(q-2).
\]

Considering the all coefficients \(u_i, v_i, w_i, ..., x_i\) simultaneously, we get the total number of linear sums that can not be put to be equal to \(c_j\) is equal to

\[
\begin{align*}
(q-1)^2^{\frac{j-2ml+2m-l-1}{l}} & \times \\
(q-1)^2^{\frac{j-2ml+2m-l-1}{l}} & + \sum_{p=0}^{2m-2} \left( \frac{j-2ml+p}{p} \right)^{2m^{2}-p} -(q-2).
\end{align*}
\]

Therefore in view of the fact that total number of \((n-k)\) tuples is \(q^{n-k}\), addition of the \(j^{th}\) column \(c_j\) to \(H\) can be done provided \(q^{n-k}\) is greater than \((2.6)\). That is

\[
q^{n-k} > (q-1)^2^{\frac{ml-1}{l}} - (q-2) × \end{equation}

\[
(q-1)^2^{\frac{ml+2m-l-1}{l}} + \sum_{p=0}^{2m-2} \left( \frac{j-2ml+p}{p} \right)^{2m^{2}-p} -(q-2).
\]

The proof of the required theorem is completed by replacing \(j\) by \(n\).

If we put \(m = 2\) in Theorem 2.3, we get a corollary, which gives a sufficient condition for the codes having the capacity to correct 2-repeated restricted burst errors of length \(l\) or less. We give the corollary as follows:

**Corollary 2.4.** The existence of an \((n,k)\) linear code over \(GF(q); q > 2\) that corrects all 2-repeated restricted bursts of length \(l\) or less \((n \geq 4l)\) is ensured, provided that

\[
q^{n-k} > (q-1)^2^{\frac{ml-1}{l}} - (q-2) × \end{equation}

\[
(q-1)^2^{\frac{ml+2m-l-1}{l}} + \sum_{p=0}^{2m-2} \left( \frac{j-2ml+p}{p} \right)^{2m^{2}-p} -(q-2).
\]

for the verification of such codes, we are providing an example which is given as follows:

**Example 2.5.** For \(n = 8, m = 2, l = 2\), Consider a \((8,2)\) code over \(GF(3)\) with parity check matrix

\[
H = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 1
\end{bmatrix}
\]

This parity check matrix has been formed by the method used in the proof of Theorem 2.3 for repeated restricted burst errors by taking \(l = 2\) over \(GF(3)\). The error patterns and syndromes table for this parity check matrix is given below as:
### 3. Conclusion

We have shown with the help of one example that 2 repeated restricted burst correcting codes over GF(3) exist. It would be interesting to see a general matrix formation for such codes for any given value of q, l, n and k.

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References

[1] N.M. Abramson, A class of systematic codes for non-independent errors, *IRE Trans. on Information Theory*, IT-5(4)(1959), 150–157.

[2] J.D. Bridwell and J.K. wolf, Burst distance and multiple burst correction, Bell System Tech. J. 99(1970), 889–909.

[3] R.T. Chien and D.T. Tang, On definition of a burst, *IBM Journal of Research and Development*, 9 (4)(1965), 292–293.

[4] B.K. Dass and Rashmi Verma, Repeated burst error correcting linear codes, *Asian-European Journal of Mathematics*, 1(3)(2008), 303–335.

[5] Tuvi Etzion, Construction for perfect 2-burst correcting codes, *IEEE Trans. Inform. Theory*, 47(6)(2001), 2553–2555.

[6] P. Fire, A class of multiple error correcting binary codes for non-independent errors, *Sylvania Rep. RSL-E-2, Sylvania reconnaissance Systems lab, Mountain View, Calif.*, (1959).

[7] W.W. Peterson and E.J. Weldon (Jr.), Error-correcting codes, 2nd edition, *M.I.T. Press, Cambridge, Massachusetts*, 1972.

[8] E.C. Posner, Simultaneous error-correction and burst-error detection using binary linear cyclic codes, *J. Soc. Indust. Appl. Math.*, 13(4)(1965), 1087–1095.

[9] J.J. Stone, Multiple burst error correction, *Information and control*, 4(1961), 324–331.

[10] G.E. Sacks, Multiple error correction by means of parity-checks, *IRE Transactions on Information Theory*, 4(4)(1958), 145–147.

[11] V. Tyagi, and Tarun Lata, Restricted 2-burst correcting non-binary optimal codes, *Journal of Combinatorics and System Sciences*, 42(1-4)(2017), 145–154.

[12] J.K. Wolf, On codes derivable from the tensor product of check matrices, *IEEE Trans. On Information Theory*, IT-11 (2)(1965), 281–284.

[13] A.D. Wyner, Low density burst correcting codes, *IEEE Transactions on Information Theory*, 9(2)(1963), 124.

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