The Open Quantum Symmetric Simple Exclusion Process

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We introduce and solve a model of fermions hopping between neighbouring sites on a line with random Brownian amplitudes and open boundary conditions driving the system out of equilibrium. The average dynamics reduces to that of the symmetric simple exclusion process. However, the full distribution encodes for a richer behaviour entailing fluctuating quantum coherences which survive in the steady limit. We determine exactly the system state steady distribution. We show that these out of equilibrium quantum fluctuations fulfil a large deviation principle and we present a method to recursively compute exactly the large deviation function. On the way, our approach gives a solution of the classical symmetric simple exclusion process using fermion technology. Our results open the route towards the extension of the macroscopic fluctuation theory to many body quantum systems.

Introduction.– Non-equilibrium phenomena are ubiquitous in Nature. Understanding the fluctuations of the flux of heat or particles through systems is a central question in non equilibrium statistical mechanics. Last decade has witnessed tremendous conceptual and technical progresses in this direction for classical systems, starting from the exact analysis of simple models \cite{1-3}, such as the Symmetric Simple Exclusion Process (SSEP) \cite{4-7}, via the understanding of fluctuation relations \cite{8-10} and their interplay with time reversal \cite{11, 12}, and culminating in the formulation of the macroscopic fluctuation theory (MFT) which is an effective theory adapted to describe transport and its fluctuations in diffusive classical systems \cite{13}. Whether macroscopic fluctuation theory may be extended to the quantum realm is yet unexplored.

In parallel, the study of quantum systems out of equilibrium has received a large amount of attention in recent years \cite{20-23}. On the experimental side, unprecedented experimental control of cold atom gases gave access to the observation of many body quantum systems in inhomogeneous and isolated setups \cite{14-19}. On the theoretical side, results about closed, quantum systems have recently flourished, with a better perception of the roles of integrability, chaos or disorder \cite{24-35}. In critical or integrable models, including the Lieb-Liniger model which is known to describe accurately gases of cold atoms in quasi one dimensional traps, a good understanding has been obtained with a precise description of entanglement dynamics, quenched dynamics, as well as transport \cite{36-47}. These efforts culminated in the development of a hydrodynamic picture adapted to integrable systems, named generalised hydrodynamics \cite{48, 49}. However, these understandings are restricted to closed, predominantly ballistic, systems.

Many important quantum transport processes are diffusive rather than ballistic \cite{50} and, to some extends, physical systems are generically in contact with external environments. It is therefore crucial to extend the previous studies by developing simple models for fluctuations in open, quantum many body, locally diffusive, out of equilibrium systems, opening a perspective on the quantisation of the macroscopic fluctuation theory. Putting aside the quantum nature of the environments leads to consider model systems interacting with classical reservoirs or noisy external fields. In the context of quantum many body systems, and especially quantum spin chains, the study of such models has recently been revitalised \cite{51-57}, partly in connection with random quantum circuit \cite{58-64}, as a way to get a better understanding of entanglement production or information spreading.

In this work, we introduce and solve an iconic example of such models. It is a stochastic variant of the Heisenberg XX spin chain. It describes fermions hopping from site to site on a discretised line, but with Brownian hopping amplitudes, and interacting with reservoirs at the chain boundaries. For reasons explained below, we may call this model the quantum SSEP. Its average dynamics reduces to the classical SSEP, but the model codes for the fluctuations around this mean behaviour. Although decoherence is at play in the mean behaviour, fluctuating quantum coherences survive to the noisy interaction. Their magnitudes typically scale proportionally with the inverse of the square root of the system size. We characterise completely the steady measure on the system state which encodes the fluctuations of the quantum coherences and occupation numbers at large time. We also present a recursive method to compute exactly, order by order, the large deviation function of these fluctuations. These findings open the route towards the extension of the macroscopic fluctuation theory \cite{13} to many body quantum systems.

The open quantum SSEP.– For an open chain in contact with external reservoirs at their boundaries, the quantum SSEP dynamics results from the interplay between unitary, but stochastic, bulk flows with dissipative, but deterministic, boundary couplings. The bulk flows induce unitary evolutions of the system density matrix
\[ \rho_t \text{ onto } e^{-iD_t} \rho_t e^{iD_t} \text{ with Hamiltonian increments} \]

\[ dH_t = \sqrt{D} \sum_{j=0}^{L-1} \left( c_j^\dagger c_j \, dW_t^j + c_j^\dagger c_j + 1 \, d\mathcal{W}_t^j \right), \quad (1) \]

for a chain of length \( L \), where \( c_j \) and \( c_j^\dagger \) are canonical fermionic operators, one pair for each site of the chain, with \( \{c_j, c_k^\dagger\} = \delta_{j,k} \) and \( W_t^j \) and \( \mathcal{W}_t^j \) are pairs of complex conjugated Brownian motions, one pair for each edge along the chain, with quadratic variations \( dW_t^j \mathcal{W}_t^j = \delta_{j,k} \, dt \). This model was shown to describe the effective dynamics of the Heisenberg XX spin chain with dephasing noise in the strong noise limit \cite{65}. It codes for a diffusive evolution of the number operators \( \hat{n}_j = c_j^\dagger c_j \) with the parameter \( D \) being the diffusion constant. This model is one of the simplest model of quantum, stochastic, diffusion. Properties of the closed periodic version of this model were deciphered in \cite{66} via a mapping to random matrix theory. We set \( D = 1 \) in the following.

Assuming the interaction between the chain and the reservoirs to be Markovian, the contacts with the external leads at the boundary sites \( 0 \) and \( L \) can be faithfully estimated by Lindblad terms \cite{67}. The resulting equations of motion read

\[ d\rho_t = -i[D_H, \rho_t] - \frac{1}{2}[D_H, [D_H, \rho_t]] + \mathcal{L}_{\text{bdry}}(\rho_t) \, dt, \quad (2) \]

with \( D_H \) as above and \( \mathcal{L}_{\text{bdry}} \) the boundary Lindbladian. The two first terms result from expanding the unitary increment \( \rho_t \to e^{-iD_t} \rho_t e^{iD_t} \) to second order (because the Brownian increments scale as \( \sqrt{D} \)). The third term codes for the dissipative boundary dynamics in the form of two-point quantum coherences, one pair for each site of the chain, \( \{c_j, c_k^\dagger\} = \delta_{j,k} \) and \( W_t^j \) and \( \mathcal{W}_t^j \) are pairs of complex conjugated Brownian motions, one pair for each edge along the chain, with quadratic variations \( dW_t^j \mathcal{W}_t^j = \delta_{j,k} \, dt \). This model was shown to describe the effective dynamics of the Heisenberg XX spin chain with dephasing noise in the strong noise limit \cite{65}. It codes for a diffusive evolution of the number operators \( \hat{n}_j = c_j^\dagger c_j \) with the parameter \( D \) being the diffusion constant. This model is one of the simplest model of quantum, stochastic, diffusion. Properties of the closed periodic version of this model were deciphered in \cite{66} via a mapping to random matrix theory. We set \( D = 1 \) in the following.

The dynamics being noisy, so is the density matrix and hence the quantum expectations such as the mean quantum occupation numbers \( n_j = \text{Tr} (\hat{n}_j \rho_t) \). Their stochastic averages \( \mathbb{E}[n_j] \) evolve according to

\[ \partial_t \mathbb{E}[n_j] = \Delta_j^{\text{dis}} \mathbb{E}[n_j] + \sum_{k \in \{0, L\}} \delta_{j,k} (\alpha_k (1 - \mathbb{E}[n_k]) - \beta_k \mathbb{E}[n_k]), \]

with \( \Delta_j^{\text{dis}} \) the discrete Laplacian, \( \Delta_j^{\text{dis}} n_j = n_{j+1} - 2n_j + n_{j-1} \), illustrating the diffusive bulk dynamics and the boundary injection/extraction processes. At large time, they reach a linear profile,

\[ n_j^* := \lim_{t \to \infty} \mathbb{E}[n_j] = \frac{n_a (L + b - j) + n_b (j + a)}{L + a + b}, \quad (5) \]

with \( n_a := \frac{\alpha}{\alpha + \beta}, \quad n_b := \frac{\beta}{\alpha + \beta}, \quad a := \frac{1}{n_a - n_b}, \quad b := \frac{1}{n_a + n_b} \). Associated to the steady mean flow from one reservoir to the other. In the large size limit, \( L \to \infty \) at \( x = i/L \) fixed, this mean profile, \( n^*(x) = n_a + x(n_b - n_a) \), interpolates linearly the two boundary mean occupations \( n_a \) and \( n_b \).

In mean, the off-diagonal quantum expectations \( G_{ji} := \text{Tr}(c_i^\dagger c_j \rho_t) \) vanish exponentially fast, \( \lim_{t \to \infty} \mathbb{E}[G_{ji}] = 0 \) for \( j \neq i \), hence reflecting decoherence due to destructive interferences induced by the noise. However, this statement is only valid in mean as fluctuating coherences survive at sub-leading orders with a rich statistical structure, with long range correlations.

The steady state : fluctuations and coherences. As exemplified by the above one-point functions, a steady state is attained at large time in the sense that the distribution of quantum expectations reaches a stationary value. Equivalently, the limit \( \lim_{t \to \infty} \mathbb{E}[F(G_{ji})] \) exists for any (sufficiently regular) function \( F \) of the matrix of two-point quantum expectations \( G \) and this defines an invariant measure \( \mathbb{E}_\infty[\cdot] \) of the flow \( (2) \), that we shall denote by \( \square \) to simplify the notation. Diagonal elements \( G_{jj} \) code for occupation numbers while the off-diagonal elements \( G_{ji}, j \neq i \), for coherences, and hence \( \square \) for their steady statistics.

Amongst the two point functions \( \mathbb{E}[G_{ij} G_{kl}] \), only those with \( \{i = j, k = l\} \) and \( \{i = l, j = k\} \) survive at large time, the others decrease exponentially fast. This leaves us with three possible configurations: \( [G_{ii}], [G_{ii} G_{jj}], \) and \( [G_{ij} G_{ji}], j \neq i \), coding respectively for quantum occupation and coherence fluctuations. They are determined by solving the stationarity equations for the invariant measure (see Supplementary Material):

\[ [G_{ij} G_{ji}]^c = \frac{(\Delta n)^2 (i + a)(L - j + b)}{(L + a + b - 1)(a + b + L)(a + b + L + 1)}, \]

\[ [G_{ii} G_{jj}]^c = \frac{-(\Delta n)^2 (i + a)(L - j + b)}{(L + a + b - 1)(a + b + L)(a + b + L + 1)}, \]

\[ [G_{ii}^2]^c = \frac{(\Delta n)^2 (i + a)(L - j + b)}{(2(a + b + L)(a + b + L + 1))}, \]

for \( i < j \) with \( \Delta n = n_b - n_a \) and \( G_{ii} G_{jj}^c = [G_{ii} G_{ij}] - [G_{ii}] [G_{ij}] \). The first lesson is that coherences are present in the large time steady state as their covariances do not vanish exponentially but remains finite. At large size, \( L \to \infty \) with \( x = i/L \), \( y = j/L \) fixed, their second moments behave as

\[ [G_{ij} G_{ji}]^c = \frac{1}{L} (\Delta n)^2 x(1 - y) + O(L^{-2}), \]

\[ [G_{ii} G_{jj}]^c = -\frac{1}{L} (\Delta n)^2 x(1 - y) + O(L^{-3}), \]

for \( x < y \), while \( [G_{ii}^2]^c = \frac{1}{L} (\Delta n)^2 x(1 - x) + O(L^{-2}) \). The second lesson is, on one hand, that these fluctuating coherences scale as \( 1/\sqrt{L} \) in the thermodynamic limit, and on the other hand, that the correlations between the
quantum occupation numbers \( n_i \) and \( n_j \) at distinct sites \( i \neq j \) scale as \( 1/L^2 \) and hence are sub-leading.

These facts hold for higher order cumulants \( [G_{i_1j_1} \cdots G_{i_Nj_N}]^c \) of the matrix of two-point quantum expectations. These cumulants are non vanishing only if the sets \( \{i_1, \ldots, i_N\} \) and \( \{j_1, \ldots, j_N\} \) coincide so that the \( N \)-uplet \( (i_1, \ldots, i_N) \) is a permutation of \( (i_1, \ldots, i_N) \). To such product \( G_{i_1j_1} \cdots G_{i_Nj_N} \) we can associate an oriented graph with a vertex for each point \( i_1, \ldots, i_N \) and an oriented edge from \( i \) to \( j \) for each insertion of \( G_{ij} \). These graphs may be disconnected. The condition that the sets \( \{i_1, \ldots, i_N\} \) and \( \{j_1, \ldots, j_N\} \) coincide translates into the fact that the number of outgoing edges equals that of outgoing edges, at each vertex. For instance, \( [G_{ij}] \) is represented by \( [\square] \), \( [G_{ij}G_{jj}] \) for \( i \neq j \) by \( [\square\square] \), \( [G_{ij}G_{ji}] \) for \( i \neq j \) by \( [\square\square] \) and \( [G_{ij}^2] \) by \( [\square\square] \).

The claim is that expectations of single loop diagrams, corresponding to the expectations of cyclic products \( [G_{i_1i_2} \cdots G_{i_4i_1}]^c \), are the elementary building blocks in the large size limit. They scale proportionally to \( 1/L^{N-1} \) in the thermodynamic limit

\[
[G_{i_1i_2} \cdots G_{i_4i_1}]^c = \frac{1}{L^{N-1}} g_N(x_1, \ldots, x_N) + O(L^{-N}),
\]

with \( x_p = i_p/L \). The expectations \( g_N \) depend on which sector the points \( \mathbf{x} := (x_1, \ldots, x_N) \) belong to, with the sectors indexing how the ordering of the points along the chain match \( \mu \)-match that along the loop graph. Let us choose to fix an ordering of the points along the chain so that \( 0 \leq x_1 < \cdots < x_N \leq 1 \), and let \( \sigma \) be the permutation coding for the ordering of the point vertices around the loop so that by turning around the oriented loop one successively encounters the vertices labeled by \( x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)} \). There are \( (N-1)!/2 \) sectors because the ordering around the loop is defined up to cyclic permutations and because reversing the orientation of the loop preserves the expectations. Let us then set

\[
f_N^\sigma(\mathbf{x}) := g_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}).
\]

The \( f_N^\sigma \)'s are recursively determined by a set of equations which arise from the stationarity conditions of the invariant measure. (See the Supplementary Material).

First, stationarity in the bulk imposes that \( \Delta_x f_N^\sigma(\mathbf{x}) = 0 \) for all \( j \) with \( \Delta_x \) the Laplacian with respect to \( x \), as a consequence of the bulk diffusivity. Second, the couplings at the chain boundaries freeze the fluctuations so that

\[
(f_N^\sigma(\mathbf{x})|_{x_1=0} = f_N^\sigma(\mathbf{x})|_{x_N=1} = 0).
\]

Third, contact interactions associated to the noisy hopping imposes two conditions on expectations at the boundary between the sectors \( \sigma \) and \( \pi_{j,j+1}\sigma \) with \( \pi_{j,j+1} \) the permutation transposing \( j \) and \( j+1 \). The ordering of the point vertices in the sector \( \sigma \) and \( \pi_{j,j+1}\sigma \) differ by the exchange of \( x_j \) and \( x_{j+1} \), so that \( x_{j+1} = x_j \) at these boundaries. The first contact condition is the continuity condition

\[
(f_N^\sigma(\mathbf{x})|_{x_{j+1}=x_j} = f_N^{\pi_{j,j+1}\sigma}(\mathbf{x})|_{x_{j+1}=x_j} \label{eq:contact1}
\]

To write the second contact condition, let us define \( j^+ \) (resp. \( j^- \)) to be the \( \sigma \) pre-image of \( j \) (resp. \( j+1 \)), i.e. \( j = \sigma(j^-) \) and \( j + 1 = \sigma(j^+) \). Since, the vertices \( x_{j+1} \) and \( x_j \) are identified at these sector boundaries, the loop graph splits into two sub-loops, touching at the vertex \( x_j \), one including the circle arc \( x_{\sigma(j^-)}, x_j, x_{\sigma(j^+)} \), and the other containing the circle arc \( x_{\sigma(j^+)} \) and \( x_{\sigma(j^-)} \), respectively denoted \( \ell_j^- \) and \( \ell_j^+ \). The second contact condition is the following Neumann like matching condition

\[
(f_N^\sigma(\mathbf{x})|_{x_{j+1}=x_j} = f_N^{\pi_{j,j+1}\sigma}(\mathbf{x})|_{x_{j+1}=x_j} \label{eq:contact2}
\]

Equations \( \ref{eq:contact1,eq:contact2} \) are the main equations which allow to recursively compute the building block loop expectations \( \ref{eq:loop_growth} \). See FIG. 1 for a graphical representation of \( \ref{eq:loop_growth} \).}

![FIG. 1. Graphical representation of the contact relation (11).](image)

Furthermore, connected expectations of pinched graphs obtained by identifying points in single loop graphs are obtained by continuity from the expectations of the corresponding parent loop graphs, thanks to \( \ref{eq:loop_growth} \). They are of order \( 1/L^{N-1} \) with \( N \) the number of edges in the pinched graph (and hence the number of insertions of matrix elements of \( G \)). All other connected expectations of disconnected graphs are sub-leading in the large size limit.

The conditions \( \ref{eq:contact1,eq:contact2} \) allow to determine all leading expectations recursively. For \( N = 3 \), there is only one sector and \( g_N(x,y,z) = (\Delta n)^3 x(1-2y)(1-z) \) for \( x < y < z \), so that

\[
[G_{ik}G_{kj}G_{ij}]^c = \frac{1}{L^2}(\Delta n)^3 x(1-2y)(1-z) + O(L^{-3}),
\]

with \( x = i/L, \ y = j/L \) and \( z = k/L \ (i < j < k) \). For \( N = 4 \), there are 3 sectors respectively associated to the identity and the transpositions \( \pi_{1;2} \) and \( \pi_{2;3} \):
For \( x_1 < x_2 < x_3 < x_4 \), their expectations are respectively:
\[
\begin{align*}
\frac{1}{L^3} (\Delta n)^4 x_1 (1 - 3x_2 - 2x_3 + 5x_2 x_3) (1 - x_4), \\
\frac{1}{L^3} (\Delta n)^4 x_1 (1 - 3x_2 - 2x_3 + 5x_2 x_3) (1 - x_4), \\
\frac{1}{L^3} (\Delta n)^4 x_1 (1 - 4x_2 - 3x_3 + 5x_2 x_3) (1 - x_4),
\end{align*}
\]
up to \( O(L^{-4}) \) contributions.

The scaling behaviour of the single loop expectations (8) ensures that the fluctuations of the matrix of quantum two-point expectations \( G \) satisfy a large deviation principle, in the sense that their generating function is such that \( [e^{\text{Tr}(AG)}] \geq \frac{1}{L} \log \left( e^{L \mathfrak{F}(A)} \right) \) for some function \( \mathfrak{F}(A) \), called the large deviation function,
\[
\mathfrak{F}(A) = \lim_{L \to \infty} \frac{1}{L} \log \left( e^{\text{Tr}(AG)} \right).
\]
This function admits a series expansion, \( \mathfrak{F}(A) = \sum_{N} \frac{\mathfrak{F}(N)}{N!} \), with the \( \mathfrak{F}(N) \)'s given by the multiple sums \( L^{-N} \sum_{j_1, \ldots, j_N} [G_{i_1j_1} \cdots G_{i_Nj_N}] \) which converge to multiple integrals. To lowest order
\[
\mathfrak{F}(A) = \int_{0}^{1} dx \, n^*(x) A(x, x) + (\Delta n)^2 \int_{0}^{x} dy \, (1 - y) A(x, y) A(y, x) + \cdots.
\]
Higher orders can be recursively computed by using equations (9,10,11).

**Sketch of proof.**— Since both the Hamiltonian increments (1) and the Lindbladians (3) are quadratic in the fermionic creation and annihilation operators, the stochastic evolution (2) preserves Gaussian states of the form \( \rho_t = Z_t^{-1} e^{\text{Tr}(MCt)} \), with \( M \) a \( L \times L \) matrix and \( Z_t = \text{Tr}(e^{\text{MCt}}) \). These density matrices are parametrised by \( M \) or, equivalently, by the matrix of quantum two-point expectations \( G_{ij} = \text{Tr}(\rho_t c_i^j c_i) \). \( G_t \) can be obtained from the Hamiltonian (2) then becomes a stochastic equation for \( M_t \) or \( G_t \). For instance, for \( 0 < i < j \neq L \),
\[
dG_{ij} = -2G_{ij} dt + i (G_{i,j-1} dW_{i-1}^j + G_{i,j+1} dW_{i}^j) - i (G_{i-1,j} dW_{i-1}^{j} + G_{i+1,j} dW_{i}^{j}),
\]
with similar equations for \( G_{ii} \) and at the two chain boundaries. Imposing the stationarity of the measure amounts to demand that the statistical expectations \( [F(G_t)] \) are time independent for any function \( F \). Since the Itô derivatives of polynomials \( G_t \) are polynomials in \( G_t \) of the same degrees, the stationarity conditions are sets of linear equations on moments of given order. There are two types of contributions arising from the Itô derivatives of polynomials: one completing the drift term in (15) to produce discrete Laplacians acting on products of \( G_t \)'s, the other producing contact interactions. For instance, \( dG_{ij} dG_{i,j+1} \mid_{\text{contact}} = -(G_{i,j+1} dW^j) (G_{ij} dW^i) = -G_{ij+1} G_{ij} dt \) which implements the transposition of the adjacent points \( j \) and \( j+1 \). As a consequence, the Itô derivatives of graphs coding for products of \( G_t \)'s with adjacent indices induce a reshuffling of the connections of these graphs. See FIG. 2 for an illustration. Thus, the stationarity conditions yield relations between expectations of reshuffled graphs from which the relations (9,10,11) can be deduced. (See Supplementary Material). More details will be described elsewhere [68].

**Connecting to the classical SSEP.**— The mean density matrix \( \bar{\rho}_t := \mathbb{E} [\rho_t] \) evolves according to the Lindblad type equation \( \partial_t \bar{\rho}_t = \mathcal{L}_{\text{bulk}}(\bar{\rho}_t) + \mathcal{L}_{\text{bdry}}(\bar{\rho}_t) \) with \( \mathcal{L}_{\text{bdry}} \) defined in (3) and bulk Lindbladian
\[
\mathcal{L}_{\text{bulk}}(\bar{\rho}_t) = -\frac{1}{2} \sum_{j=0}^{L-1} (|c^l_{j+1}, c^l_{j+1}, \bar{\rho}_t| + \text{h.c.})
\]
For density matrices diagonal in the occupation number basis, this codes for the time evolution of SSEP. Indeed, asymptotically in time, decoherence is effective and the mean density matrix is diagonal, \( \bar{\rho}_t = \sum_{|n|} \bar{Q}_t |m| \otimes P_m \) where the \( P_m \)'s are the projectors on the occupation number eigen-states \( |n| \) and \( \bar{Q}_t |m| \) the mean populations. The \( P_m \)'s are products of projectors \( P_{n_j} \) on each site of the chain, with \( n_j = 0 \) (resp. \( n_j = 1 \)) for empty (resp. full). On adjacent pairs of projectors, the bulk Lindbladian acts as
\[
\mathcal{L}_{\text{bulk}}(\mathbb{P}_{|n_j|^1}) = -\mathbb{P}_{|n_j|^0}^{1} + \mathbb{P}_{|n_j|^1}^{0},
\]
whereas \( \mathcal{L}_{\text{bulk}}(\mathbb{P}_{|n_j|^0}^{0}) = 0 \) and \( \mathcal{L}_{\text{bulk}}(\mathbb{P}_{|n_j|^1}^{1}) = 0 \). This is equivalent to the SSEP transition matrix.

As a consequence, the SSEP generating function for the occupation number fluctuations can be identified with the statistical average of the generating function of quantum expectations of the number operators,
\[
\langle e^{\sum_i a_i n_i} \rangle_{\text{SSEP}} = \text{Tr}(\bar{\rho} e^{\Sigma_i a_i \hat{n}_i}) = \left[ \text{Tr}(\rho e^{\Sigma_i a_i \hat{n}_i}) \right],
\]
with \( \hat{n}_i = c^l_i c_i^\dagger \). It can be computed using Wick’s theorem, so that the SSEP cumulants read
\[
\langle n_{j_1} \cdots n_{j_N} \rangle_{\text{SSEP}} = \left( \frac{N!}{L^{N-1}} \sum \mathfrak{f}_N(x) + \mathcal{O}(L^{-N}) \right).
\]
with \( x_k = j_k/L \) all distinct. The sum is over permutations \( \sigma \) modulo cyclic permutations. (See Supplementary Material). The expectations (8) of the matrix of quantum two-point expectations cannot be reconstructed from the SSEP expectations (for \( N \geq 4 \)) because the latter are symmetric under permutations and hence only involve the sum of the sectors.

**Discussion.**— We have introduced a quantum extension of the symmetric simple exclusion process (SSEP) and outlined how to solve it exactly by characterising its invariant measure and computing the large deviation function of the matrix of quantum two-point expectations. The quantum SSEP is a simple, if not the simplest, model coding for diffusive behaviour of quantum operators, including fluctuations, in a many body fermionic systems. In mean, it reduces to the classical SSEP in the sense that the statistical averages of the quantum expectations of the number operators coincide with those of the classical SSEP. In passing, the approach we have outlined provides an alternative way of solving the classical SSEP, based on fermionic techniques, without using matrix product ansatz, and hence gives an explanation of the, yet unexplained, determinant representation of the classical SSEP solution [6, 7, 69].

The quantum SSEP is strictly finer than its classical counterpart, and contains much more information, including fluctuations of quantum coherences. Although decoherence is at work on the mean steady state, we have observed and quantified sub-leading fluctuating coherences. In the thermodynamic large size limit, the system state approaches a self averaging non equilibrium state dressed by occupancy and coherence fluctuations whose amplitudes scale proportionally to the inverse of the square root of the system size. We have described how to compute the large deviation function for these fluctuations, order by order.

As an example of quantum out of equilibrium exclusion processes and of fluctuating quantum discrete hydrodynamics, our findings open several new research directions. The first concerns the integrable structure underlying the exact solution we have presented and its connection with the existing solution methods for classical exclusion processes [70–73]. The second concerns the extension of our work to deal with the quantum analogue of the asymmetric simple exclusion process (ASEP). We have already noticed that the appropriate generalisation amounts to couple the fermionic system to quantum noise [74, 75] and we plane to report soon on this problem [77]. But the most important ones deal with using the present model, and its generalisations, to formalise the extension of the macroscopic fluctuation theory [13] to many body quantum systems. How to extend the additivity principle [76], which holds classically with some degree of generality, to keep track of the fluctuating quantum coherence? How to extend this study to more general interacting spin or fermionic systems, such as the noisy dephasing Heisenberg spin chain? How to take the continuum limit of those models to provide a quantisation of the macroscopic fluctuation theory? We plane to report on these questions in the near future [77].

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Supplemental Material

The Open Quantum Symmetric Simple Exclusion Process

Denis Bernard and Tony Jin

Low order cumulants

From eq.(2) in the main text, we derive the stochastic equations satisfied by the matrix $G_t$:

$$dG_{i;i} = (G_{i+1;i+1} - 2G_{i;i} + G_{i-1;i-1})dt + \sum_{\beta \in \{0,L\}} \delta_{i,\beta}(\alpha_{i} - \beta_{i})dG_{i;i}dt$$

$$+ i\{G_{i-1}dW_{i-1}^{i-1} + G_{i+1}dW_{i+1}^{i+1} - G_{i-1;i}dW_i^i - G_{i+1;i}dW_i^i\}, \quad (j = i)$$

$$dG_{i;j} = -2G_{i;j}dt - \frac{1}{2} \sum_{\beta \in \{0,L\}} (\alpha_{i} + \beta_{i})(\delta_{i,\beta} + \delta_{j,\beta})G_{i;j}dt$$

$$+ i\{G_{i-1}dW_{i-1}^{j-1} + G_{i+1}dW_{i+1}^{j+1} - G_{i-1;j}dW_i^j - G_{i+1;j}dW_i^j\}, \quad (j \neq i)$$

Here and in the following, it will always be implicitly assumed that we have to truncate the equations keeping only the appropriate terms when evaluating them at boundaries, i.e. either taking $i$ or $j$ equal to 0 or $L$. For instance

$$dG_{0;j} = -G_{0;j}dt - \frac{1}{2} \sum_{\beta \in \{0,L\}} (\alpha_{i} + \beta_{i})(\delta_{0,\beta} + \delta_{j,\beta})G_{0;j}dt$$

$$+ i\{G_{0}dW_{0}^{j-1} + G_{0+1}dW_{0+1}^{j+1} - G_{0;j}dW_i^j\}, \quad (j \neq 0),$$

and similarly at the other end of the chain.

Cumulants of order 2.– Let us see how to compute the cumulants of order 2. To each product of $G$’s not vanishing in the long-time limit, there is an associated stationary equation. Let’s illustrate how this works for instance for $E[G_{i;j}G_{j;i}]$ with $i$ and $j \neq 0$ or $L$. Working in the Itô convention, the differential of $E[G_{i;j}G_{j;i}]$ is given by:

$$dE[G_{i;j}G_{j;i}] = E[dG_{i;j}G_{j;i} + dG_{i;j}G_{j;i} + dG_{i;j}G_{j;i}]$$

$$= E[(\Delta_{i}^{\text{dis}} + \Delta_{j}^{\text{dis}})G_{i;j}G_{j;i} + 2(\delta_{i,j} - \delta_{i,j})G_{i-1;i}G_{i;i} + 2(\delta_{i,j} - \delta_{i+1,j})G_{i;i}G_{i+1;i+1}]$$

$$- \sum_{\beta \in \{0,L\}} \gamma_{p}(\delta_{i,\beta} + \delta_{j,\beta})(G_{i;j}G_{j;i})dt,$$

(21)

where $\Delta^{\text{dis}}_i$ is the discrete Laplacian as in the main text. To go from the first line to the second line, we made use of the statistical properties of the complex noises, i.e.

$$E[dW^{i}_{f}] = E[dW^{j}_{f}] = E[(dW^{i}_{f})^{2}] = E[(dW^{j}_{f})^{2}] = 0$$

and $E[dW^{i}_{f}dW^{j}_{f}] = E[dW^{i}_{f}dW^{j}_{f}] = \delta^{i;j}dt$. In the steady state $E_{\infty}[G_{i;j}G_{j;i}] = f(i,j)$ and $df(i,j) = 0$ which leads to:

$$0 = (\Delta^{\text{dis}}_{i} + \Delta^{\text{dis}}_{j})f(i,j) + 2(\delta_{i,j} - \delta_{i-1,j})n(i - 1, i) + 2(\delta_{i,j} - \delta_{i+1,j})n(i, i + 1) - \sum_{\beta \in \{0,L\}} \gamma_{p}(\delta_{i,\beta} + \delta_{j,\beta})f(i,j).$$

with $\gamma_k := \alpha_k + \beta_k$ (i.e. $\gamma_0 = 1/a$ and $\gamma_L = 1/b$). The terms weighted by the Kronecker delta’s are what we will refer to in the following as the contact terms. A stationarity equation is always composed of three parts as above: the Laplacian terms, the contact terms and the boundary terms.

We define the connected two-point expectations (for $i < j$):

$$n^{c}(i,j) := [G_{ii}G_{jj} - [G_{ii}][G_{jj}], \quad m^{c}(i) := [G_{ii}^{2} - [G_{ii}]_{2}, \quad f(i,j) = [G_{ij}G_{ji}].$$

(22)
The bulk stationarity equations impose that \((\Delta_i^{\text{dis}} + \Delta_j^{\text{dis}}) n^c(i, j) = 0\) and \((\Delta_i^{\text{dis}} + \Delta_j^{\text{dis}}) f(i, j) = 0\), with \(\Delta_i^{\text{dis}}\) is the discrete Laplacian as in the main text, which we enforce by demanding that
\[
\Delta_i^{\text{dis}} n^c(i, j) = \Delta_j^{\text{dis}} n^c(i, j) = 0, \quad \Delta_i^{\text{dis}} f(i, j) = \Delta_j^{\text{dis}} f(i, j) = 0.
\] (23)

The bulk/boundary stationarity equations demand that (with \(j \neq 0, L\) and \(i \neq 0, L\)):
\[
n^c(1, j) - n^c(0, j) + \Delta_j^{\text{dis}} n^c(0, j) = \gamma_a n^c(0, j),
\] (24)
\[
f(1, j) - f(0, j) + \Delta_j^{\text{dis}} f(0, j) = \gamma_a f(0, j),
\] (25)
\[
n^c(i, L - 1) - n^c(i, L) + \Delta_i^{\text{dis}} n^c(i, L) = \gamma_b n^c(i, L),
\] (26)
\[
f(i, L - 1) - f(i, L) + \Delta_i^{\text{dis}} f(i, L) = \gamma_b f(i, L),
\] (27)

The two first equations give \(n^c(1, j) = (\gamma_a + 1)n^c(0, j)\) and \(f(1, j) = (\gamma_a + 1)f(0, j)\). Using \(\Delta_i^{\text{dis}} n^c(i, j) = 0\) and \(\Delta_i^{\text{dis}} f(i, j) = 0\), this then recursively yields (for \(i < j\)):
\[
n^c(i, j) = (i\gamma_a + 1)n^c(0, j), \quad f(i, j) = (i\gamma_a + 1)f(0, j).
\] (28)

Similarly, starting from the other hand of the chain gives (for \(i < j\)):
\[
n^c(i, j) = ((L - j)\gamma_b + 1)n^c(i, L), \quad f(i, j) = ((L - j)\gamma_b + 1)f(i, L).
\] (29)

Hence, for \(i < j\),
\[
n^c(i, j) = N \cdot (i + a)(L - j + b), \quad f(i, j) = F \cdot (i + a)(L - j + b),
\] (30)
for some constants \(N\) and \(F\) (with \(a = 1/\gamma_a = 1/(\alpha_0 + \beta_0)\) and \(b = 1/\gamma_b = 1/(\alpha_L + \beta_L)\).
The constants \(N\) and \(F\) are then determined by the stationarity conditions at the contact points \(j = i\) and \(j = i + 1\).
The equation fixes \(m^c(i)\) as a function of \(f(i, j)\) and \(n^c(i, j)\). Using again the bulk relations \(\Delta_i n^c(i, j) = 0\) and \(\Delta_i f(i, j) = 0\), the stationarity conditions for \(f(i, i + 1)\) and for \(n^c(i, i + 1)\) give
\[
m^c(i + 1) + m^c(i) = 2f(i, i + 1) + n^c(i + 1, i + 1) + n^c(i, i),
\] (32)
\[
m^c(i + 1) + m^c(i) = 2n^c(i, i + 1) + f(i, i + 1) + f(i, i) - (n^*(i + 1) - n^*(i))^2,
\] (33)
with \(n^*(i)\) the mean profile. Eliminating \(m^c(i)\) from the above three equations give two equations for the constants \(N\) and \(F\):
\[
(F - N)(b + a + L - 1) = (F + N)(a + b + L + 1) = (n^*(i + 1) - n^*(i))^2.
\] (34)

Since \(n^*(i + 1) - n^*(i) = (\Delta n)/(b + a + L)\), with \(\Delta n = n_b - n_a\), this yields
\[
F = \frac{(\Delta n)^2}{(a + b + L - 1)(a + b + L)(a + b + L + 1)},
\] (35)
\[
N = -\frac{(\Delta n)^2}{(a + b + L - 1)(a + b + L)^2(a + b + L + 1)}.
\] (36)

Hence
\[
n^c(i, j) = -\frac{(\Delta n)^2((L - j) + b)}{(a + b + L - 1)(a + b + L)^2(a + b + L + 1)},
\] (37)
\[
f(i, j) = \frac{(\Delta n)^2((L - j) + b)}{(a + b + L - 1)(a + b + L)(a + b + L + 1)}.
\] (38)

The cumulant \(m^c(i)\) are then determined using (31):
\[
m^c(i) = \frac{\Delta n^2(2(i + a)(L - i + b) - (L + b + a))}{2(a + b + L)^2(a + b + L + 1)}.
\] (39)
Cumulants of order 3.– For $N = 3$, with $0 \leq i < j < k \leq L$, the non zero terms in the stationary state are:

\[
\begin{align*}
[\begin{array}{ccc}
  \cdot & \cdot & \cdot \\
  j & j & k
\end{array}] &= [G_{iij}G_{jkk}], \\
[\begin{array}{ccc}
  \cdot & \cdot & \cdot \\
  i & j & k
\end{array}] &= [G_{iij}G_{jik}], \\
[\begin{array}{ccc}
  \cdot & \cdot & \cdot \\
  i & j & k
\end{array}] &= [G_{ijj}G_{jkk}], \\
[\begin{array}{ccc}
  \cdot & \cdot & \cdot \\
  i & j & k
\end{array}] &= [G_{ijj}G_{jkk}], \\
[\begin{array}{ccc}
  \cdot & \cdot & \cdot \\
  i & j & k
\end{array}] &= [G_{ijj}G_{jkk}], \\
[\begin{array}{ccc}
  \cdot & \cdot & \cdot \\
  i & j & k
\end{array}] &= [G_{iij}G_{jkk}], \\
[\begin{array}{ccc}
  \cdot & \cdot & \cdot \\
  i & j & k
\end{array}] &= [G_{iij}G_{jkk}],
\end{align*}
\]

(40)

where the convention is that two indices written with different letters are evaluated at different sites. We define the connected expectations of the five first terms as:

\[
\begin{align*}
[\begin{array}{ccc}
  \cdot & \cdot & \cdot \\
  i & j & k
\end{array}]^c &= [G_{iij}G_{jkk}] - [G_{iij}G_{jkk}] - [G_{ijj}G_{jkk}] + 2[G_{ijj}G_{jkk}]
\end{align*}
\]

(45)

We name these terms $g^c_m(i, j, k)$ where the index $m$ runs from 1 to 5 and labels the diagrams. The stationarity conditions for the $g^c_m$ follow all the same pattern:

\[
\begin{align*}
0 &= (\Delta_{i}^{\text{dis}} + \Delta_{j}^{\text{dis}} + \Delta_{k}^{\text{dis}})(g^c_m(i, j, k)) + \text{“contact terms”} \\
&+ (\gamma_{0}\delta_{i,0} + \gamma_{L}\delta_{k,L})g^c_m(i, j, k)
\end{align*}
\]

(50)

where the contact terms are non zero only when two indices are close to each other: $j = i + 1$, $k = j + 1$ and depends on $m$. As with the $N = 2$ case, using bulk and boundary stationarity, we find that all the $g^c_m$'s must be of the following polynomial form:

\[
g^c_m(i, j, k) = N_m(a + i)(1 + C_m)(L + b - k)
\]

We make use of the contact terms to determine the constant $N_m$ and $C_m$. We will not show the explicit derivation as it does not entail any difficulties but is a bit cumbersome. Knowing the $g^c_m$'s, it is then easy to retrieve the remaining
terms by again making use of the various contact terms. The result is:

\[
\begin{align*}
[\bigotimes_{j}^{k}]^c &= -\frac{4(\Delta n)^3(a + i)(a - b + 2j - L)(L + b - k)}{(L + a + b - 2)(L + a + b - 1)(L + a + b)^3(L + a + b + 1)(L + a + b + 2)} \\
[\bigotimes_{j}^{k}]^c &= \frac{2(\Delta n)^3(a + i)(a - b + 2j - L)(L + b - k)}{(L + a + b - 2)(L + a + b - 1)(L + a + b + 1)(L + a + b + 2)}
\end{align*}
\]

(51) (52)

\[
\begin{align*}
[\bigotimes_{j}^{k}]^c &= \left[\bigotimes_{j}^{k}\right]^c = [\bigotimes_{j}^{k}]^c
\end{align*}
\]

(53)

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= -\frac{(\Delta n)^3(a + i)(-L - b + a + 2j)(L + b - k)}{(L + a + b - 2)(L + a + b - 1)(L + a + b)(L + a + b + 1)(L + a + b + 2)}
\end{align*}
\]

(54)

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= \frac{(\Delta n)^3(2(a + i)(-L - b + a + 2i) + L + a + b)(L + b - j)}{(L + a + b - 1)(L + a + b^3(L + a + b + 1)(L + a + b + 2)}
\end{align*}
\]

(55)

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= \frac{(\Delta n)^3(a + i)(a(2(L + b - j) - 1) - 2(L + b - 2j)(L + b - j) - L - b)}{(L + a + b - 1)(L + a + b^3(L + a + b + 1)(L + a + b + 2)}
\end{align*}
\]

(56)

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= -\frac{(\Delta n)^3(2(a + i)(-L - b + a + 2i) + L + a + b)(L + b - j)}{2(L + a + b - 1)(L + a + b)(L + a + b + 1)(L + a + b + 2)}
\end{align*}
\]

(57)

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= -\frac{(\Delta n)^3(a + i)(a(2(L + b - j) - 1) - 2(L + b - 2j)(L + b - j) - L - b)}{2(L + a + b - 1)(L + a + b^3(L + a + b + 1)(L + a + b + 2)}
\end{align*}
\]

(58)

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= -\frac{(\Delta n)^3(a + i)(a(4(L + b - i) - 3) + 4i(L + b - i) - 3(b + L))}{4(L + a + b - 1)(L + a + b^3(L + a + b + 1)(L + a + b + 2)}
\end{align*}
\]

(59)

where the connected expectations are defined according to eqs.(45) with appropriate identification of the indices. For \(N = 3\) the leading order in the thermodynamic limit is \(O(L^{-2})\). These include \([\bigotimes_{i}^{k}]^c\), \([\bigotimes_{i}^{k}]^c\), \([\bigotimes_{i}^{k}]^c\) and \([\bigotimes_{i}^{k}]^c\). In the thermodynamic limit, with \(x = i/L, y = j/L, z = k/L, \) they read:

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= \frac{(\Delta n)^3 x(1 - 2x)(1 - y)}{L^2}
\end{align*}
\]

(60)

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= \frac{(\Delta n)^3 x(1 - 2y)(1 - y)}{L^2}
\end{align*}
\]

(61)

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= \frac{(\Delta n)^3 x(1 - 2x)(1 - x)}{L^2}
\end{align*}
\]

(62)

\[
\begin{align*}
[\bigotimes_{i}^{k}]^c &= \frac{(\Delta n)^3 x(1 - 2y)(1 - z)}{L^2}
\end{align*}
\]

(63)

As stated in the main text we see that in the thermodynamic limit, the knowledge of the single loop \([\bigotimes_{i}^{k}]^c\) is enough to get the other terms by continuity.

**Conditions for stationarity, blow-ups and higher order cumulants**

We recall the stochastic equations (18,19,20).

**Conditions for stationarity and blow-ups.** – We look at expectations of the following form

\[
\begin{align*}
\left[\bigotimes_{i}^{k}\right], \left[\bigotimes_{i}^{k}\right], \left[\bigotimes_{i}^{k}\right], \left[\bigotimes_{i}^{k}\right],
\end{align*}
\]

(64)

where \(A\) and \(B\) are subgraphs (We could also add another subgraph linking \(A\) and \(B\)).
We write the stationarity conditions, using (18,19,20). The first set of bulk relations, for \( i \) and \( j \) far away from the boundaries, are satisfied if

\[
\Delta^\text{dis}_i \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] = \Delta^\text{dis}_j \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] = 0, \quad \Delta^\text{dis}_i \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] = \Delta^\text{dis}_j \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] = 0, \tag{65}
\]

with \( \Delta^\text{dis} \) the discrete Laplacian as in the main text. Notice that \( \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] \) is not discrete harmonic.

As for the two- and three- point functions, it is easy to see that the stationarity conditions imposes the connected components of \( \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] \) and \( \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] \) to vanish at the boundaries in the large size limit.

We then write the contact conditions which come from the stationarity conditions for \( j = i \) or \( j = i \pm 1 \). We start with \( j = i - 1 \). The case \( j = i + 1 \) is recovered from the case \( j = i + 1 \) up to the exchange of \( A \) and \( B \). Using (65), stationarity of \( \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] \) gives

\[
\left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow \bullet \end{array} \right] = \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j \rightarrow j + 1 \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow j \rightarrow \bullet \end{array} \right] \tag{66}
\]

Similarly, stationarity of \( \left[ \begin{array}{c} \circ \rightarrow j \rightarrow j + 1 \rightarrow \bullet \end{array} \right] \) gives

\[
\left[ \begin{array}{c} \circ \rightarrow j \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow j + 1 \rightarrow \bullet \end{array} \right] = \left[ \begin{array}{c} \circ \rightarrow j \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow j + 1 \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j \rightarrow j + 1 \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow j \rightarrow \bullet \end{array} \right] \tag{67}
\]

The stationarity conditions for \( \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] \) yields

\[
4 \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] = \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j \rightarrow j + 1 \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j \rightarrow j + 1 \rightarrow \bullet \end{array} \right] \tag{68}
\]

**Stationarity conditions for higher order cumulants.** – Eliminating \( \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow \bullet \end{array} \right] \) from the two equations (66,67) yields

\[
\left[ \begin{array}{c} \circ \rightarrow j \rightarrow j + 1 \rightarrow \bullet \end{array} \right] - \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow j + 1 \rightarrow \bullet \end{array} \right] - \left[ \begin{array}{c} \circ \rightarrow j \rightarrow j + 1 \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow j \rightarrow \bullet \end{array} \right] \tag{69}
\]

The l.h.s. is the difference of discrete derivatives. In the large size limit, the r.h.s. is dominated by the disconnected contributions to the expectations because the approximation

\[
\left[ \begin{array}{c} \circ \rightarrow j \rightarrow j \rightarrow \bullet \end{array} \right] = \left[ \begin{array}{c} \circ \rightarrow j \rightarrow \bullet \end{array} \right] + \left[ \begin{array}{c} \circ \rightarrow j + 1 \rightarrow \bullet \end{array} \right] + \cdots, \tag{70}
\]

where the dots refer to sub-leading terms in \( 1/L \), is valid in the large size limit. Hence, equation (69) can be written as

\[
\nabla_x^\text{dis} \left[ \begin{array}{c} \circ \rightarrow y \rightarrow \bullet \end{array} \right] \bigg|_{y=x+} - \nabla_y^\text{dis} \left[ \begin{array}{c} \circ \rightarrow x \rightarrow \bullet \end{array} \right] \bigg|_{y=x+} = \left( \nabla_x^\text{dis} \left[ \begin{array}{c} \circ \rightarrow y \rightarrow \bullet \end{array} \right] \right) \cdot \left( \nabla_y^\text{dis} \left[ \begin{array}{c} \circ \rightarrow x \rightarrow \bullet \end{array} \right] \right) \bigg|_{y=x} \tag{71}
\]

up to sub-leading terms in \( 1/L \), where we adopt a more explicit continuous indexation (i.e. \( x = i/L \), etc.), and with \( \nabla_x^\text{dis} f(x) = f(x + 1/L) - f(x) \simeq L^{-1} \nabla_x f(x) \) in the large \( L \) limit.
Exchanging the role of $A$ and $B$ (which amounts to exchange $x$ and $y$) gives two relations. Exchanging and taking the sum yields (in the large $L$ limit)

$$
(\nabla_x \text{dis} - \nabla_y \text{dis}) \left( [\begin{array}{c} x \\ y \\ z \\ u \\ v \\ w \end{array}] + [\begin{array}{c} y \\ x \\ z \\ u \\ v \\ w \end{array}] \right) \bigg|_{y=x} = 2 \left( \nabla_x \text{dis} \left[ \begin{array}{c} x \\ y \\ z \end{array} \right] \right) \cdot \left( \nabla_y \text{dis} \left[ \begin{array}{c} y \\ x \\ z \end{array} \right] \right) \bigg|_{y=x} \quad (72)
$$

Exchanging and taking the difference yields (in the large $L$ limit)

$$
(\nabla_x \text{dis} + \nabla_y \text{dis}) \left( [\begin{array}{c} x \\ y \\ z \\ u \\ v \\ w \end{array}] - [\begin{array}{c} y \\ x \\ z \\ u \\ v \\ w \end{array}] \right) \bigg|_{y=x} = 0. \quad (73)
$$

If $[\begin{array}{c} a \\ b \\ c \end{array}]$ is a single loop diagram, then $A$ and $B$ are circle arcs and $[\begin{array}{c} a \\ b \end{array}]$ and $[\begin{array}{c} b \\ a \end{array}]$ are single loops with respectively $N_a$ and $N_b$ edges (so that the parent single loop has $N = N_a + N_b$ edges). By recursion, the expectations $[\begin{array}{c} a \\ b \end{array}]$ and $[\begin{array}{c} b \\ a \end{array}]$ scale respectively as $1/L^{N_a-1}$ and $1/L^{N_b-1}$, and the l.h.s. of (72) scales as $1/L^{N_a-1+N_b-1+2} = 1/L^N$. Hence, by (72), $[\begin{array}{c} a \\ b \end{array}]$ scales as $1/L^{N_a-1}$. Equation (72) then reads

$$
(\nabla_x - \nabla_y) \left( [\begin{array}{c} x \\ y \\ z \\ u \\ v \\ w \end{array}] + [\begin{array}{c} y \\ x \\ z \\ u \\ v \\ w \end{array}] \right) \bigg|_{y=x} = \frac{2}{L} \left( \nabla_x \left[ \begin{array}{c} x \\ y \\ z \end{array} \right] \right) \cdot \left( \nabla_y \left[ \begin{array}{c} y \\ x \\ z \end{array} \right] \right) \bigg|_{y=x} \quad (74)
$$

which coincides with the main contact relation of the main text.

Similarly, in the large size limit, equation (73) becomes the continuity equation

$$
[\begin{array}{c} a \\ b \end{array}] \bigg|_{y=x} = [\begin{array}{c} b \\ a \end{array}] \bigg|_{y=x} \quad (75)
$$

Finally, equation (68) in the large size limit gives

$$
[\begin{array}{c} a \\ b \\ c \end{array}] = [\begin{array}{c} b \\ c \end{array}], \quad (76)
$$

which says that connected expectations of pinched diagrammes are obtained by continuity from the parent diagramme. Furthermore, we can use the stationarity equations (66,67,68) to prove that $[\begin{array}{c} a \\ b \\ c \end{array}]$ is sub-leading compare to $[\begin{array}{c} a \\ b \\ y \end{array}]$ by a factor $1/L$.

**Explicit solutions for the first cases.** Here, we solve the stationarity conditions for the first cumulants. Recall that $n^*(x) = n_a + x(n_b - n_a)$. The case $N = 2$ was done in the main text (from the discrete solution) with output

$$
[\begin{array}{c} x \\ y \end{array}] = \frac{1}{L} f_2(x, y), \quad f_2(x, y) = (\Delta n)^2 x(1 - y). \quad (77)
$$

For $N = 3$, there are only one diagram

$$
[\begin{array}{c} x \\ y \end{array}] = \frac{1}{L^2} f_3(x, y, z), \quad (78)
$$

with $0 \leq x < y < z \leq 1$ by convention. Because there is only one sector in the case $N = 3$, the equations for $f_3(x, y, z)$
simplify to
\[\Delta_x \left[ \begin{array}{c} z \\ x \end{array} \right] y = \Delta_y \left[ \begin{array}{c} z \\ x \end{array} \right] y = \Delta_z \left[ \begin{array}{c} z \\ x \end{array} \right] y = 0 \quad (79)\]

\[\begin{array}{c}
\left[ \begin{array}{c} z \\
0 \end{array} \right] \left[ \begin{array}{c} y \\
x \end{array} \right] = \left[ \begin{array}{c} 1 \\
x \end{array} \right] y = 0
\end{array} \quad (80)\]

\[\left( \nabla_x - \nabla_y \right) \left[ \begin{array}{c} z \\ x \end{array} \right] y \bigg|_{x=y} = \frac{1}{L} \left( \nabla_x \left[ \begin{array}{c} z \\ x \end{array} \right] y \right) \cdot \left( \nabla_y \left[ \begin{array}{c} y \\ x \end{array} \right] z \right) \bigg|_{x=y} \quad (81)\]

\[\left( \nabla_y - \nabla_z \right) \left[ \begin{array}{c} z \\ y \end{array} \right] y \bigg|_{y=z} = \frac{1}{L} \left( \nabla_y \left[ \begin{array}{c} y \\ y \end{array} \right] y \right) \cdot \left( \nabla_z \left[ \begin{array}{c} x \\ x \end{array} \right] z \right) \bigg|_{y=z} \quad (82)\]

The solution is of the form \( f_3(x, y, z) = x Q(y) (1 - z) \) for some polynomial \( Q(y) \) of degree one. Solving for it using the above equations gives
\[\begin{array}{c}
\left[ \begin{array}{c} z \\
x \end{array} \right] y = \frac{(\Delta_n)^3}{L^2} x(1 - 2y)(1 - z).
\end{array} \]

For \( N = 4 \) there are a priori 3 different types of one-loop diagrams: \( \left[ \begin{array}{cc} x_4 & x_3 \\
x_2 & x_1 \end{array} \right] \), \( \left[ \begin{array}{cc} x_3 & x_4 \\
x_1 & x_2 \end{array} \right] \), and \( \left[ \begin{array}{cc} x_1 & x_3 \\
x_2 & x_4 \end{array} \right] \) with \( 0 \leq x_1 < x_2 < x_3 < x_4 \leq 1 \). They fulfill the following equations:
\[\Delta_{x_j} \left[ \begin{array}{c} x_j \\
x_j \end{array} \right] x_j = \Delta_{x_k} \left[ \begin{array}{c} x_k \\
x_k \end{array} \right] x_k = \Delta_{x_l} \left[ \begin{array}{c} x_l \\
x_l \end{array} \right] x_l = 0, \quad \forall j \in [1, 4] \quad (83)\]

\[\left[ \begin{array}{c} x_j \\
x_j \end{array} \right] x_j = \left[ \begin{array}{c} x_j \\
x_j \end{array} \right] x_j = \left[ \begin{array}{c} x_j \\
x_j \end{array} \right] x_j = 0 \quad (84)\]

\[\left[ \begin{array}{c} x_j \\
x_j \end{array} \right] x_j = \left[ \begin{array}{c} x_j \\
x_j \end{array} \right] x_j = \left[ \begin{array}{c} x_j \\
x_j \end{array} \right] x_j = 0 \quad (85)\]

\[\left( \nabla_{x_1} - \nabla_{x_2} \right) \left[ \begin{array}{c} x_1 \\
x_1 \end{array} \right] x_1 + \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 \bigg|_{x_1=x_2} = \frac{2}{L} \left( \nabla_{x_1} \left[ \begin{array}{c} x_1 \\
x_1 \end{array} \right] x_1 \right) \cdot \left( \nabla_{x_2} \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 \right) \bigg|_{x_1=x_2} \quad (86)\]

\[\lim_{x_1 \to x_2} \left( \left[ \begin{array}{c} x_1 \\
x_1 \end{array} \right] x_1 \bigg|_{x_1=x_2} - \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 \bigg|_{x_2=x_1} \right) = 0 \quad (87)\]

\[\left( \nabla_{x_2} - \nabla_{x_3} \right) \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 + \left[ \begin{array}{c} x_3 \\
x_3 \end{array} \right] x_3 \bigg|_{x_2=x_3} = \frac{2}{L} \left( \nabla_{x_2} \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 \right) \cdot \left( \nabla_{x_3} \left[ \begin{array}{c} x_3 \\
x_3 \end{array} \right] x_3 \right) \bigg|_{x_2=x_3} \quad (88)\]

\[\lim_{x_2 \to x_3} \left( \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 \bigg|_{x_2=x_3} - \left[ \begin{array}{c} x_3 \\
x_3 \end{array} \right] x_3 \bigg|_{x_3=x_2} \right) = 0 \quad (89)\]

\[\left( \nabla_{x_3} - \nabla_{x_4} \right) \left( \left[ \begin{array}{c} x_4 \\
x_4 \end{array} \right] x_4 + \left[ \begin{array}{c} x_3 \\
x_3 \end{array} \right] x_3 \bigg|_{x_3=x_4} = \frac{2}{L} \left( \nabla_{x_3} \left[ \begin{array}{c} x_3 \\
x_3 \end{array} \right] x_3 \right) \cdot \left( \nabla_{x_4} \left[ \begin{array}{c} x_4 \\
x_4 \end{array} \right] x_4 \right) \bigg|_{x_3=x_4} \quad (90)\]

\[\lim_{x_3 \to x_4} \left( \left[ \begin{array}{c} x_3 \\
x_3 \end{array} \right] x_3 \bigg|_{x_3=x_4} - \left[ \begin{array}{c} x_4 \\
x_4 \end{array} \right] x_4 \bigg|_{x_4=x_3} \right) = 0 \quad (91)\]

\[\left( \nabla_{x_2} - \nabla_{x_3} \right) \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 \bigg|_{x_2=x_3} = \frac{1}{L} \nabla_{x_2} \left( \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 \right) \cdot \nabla_{x_3} \left( \left[ \begin{array}{c} x_3 \\
x_3 \end{array} \right] x_4 \right) \bigg|_{x_2=x_3} \quad (92)\]

\[\left( \nabla_{x_1} - \nabla_{x_2} \right) \left[ \begin{array}{c} x_1 \\
x_1 \end{array} \right] x_1 \bigg|_{x_1=x_2} = \frac{1}{L} \nabla_{x_1} \left( \left[ \begin{array}{c} x_1 \\
x_1 \end{array} \right] x_1 \right) \cdot \nabla_{x_2} \left( \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_4 \right) \bigg|_{x_1=x_2} \quad (93)\]

\[\left( \nabla_{x_2} - \nabla_{x_4} \right) \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 \bigg|_{x_2=x_4} = \frac{1}{L} \nabla_{x_2} \left( \left[ \begin{array}{c} x_2 \\
x_2 \end{array} \right] x_2 \right) \cdot \nabla_{x_4} \left( \left[ \begin{array}{c} x_4 \\
x_4 \end{array} \right] x_4 \right) \bigg|_{x_2=x_4} \quad (94)\]
The bulk/boundary conditions impose that all loops must be polynomial of the form \( x_1Q(x_2, x_3)(1 - x_4) \). Using the remaining conditions, one gets:

\[
\begin{align*}
\left[ \begin{array}{c}
  x_1 \\
  x_2 \\
  x_3 \\
  x_1
\end{array} \right] &= \left( \frac{\Delta n}{L^3} \right)^4 x_1(1 - 3x_2 - 2x_3 + 5x_2x_3)(1 - x_4) \\
\left[ \begin{array}{c}
  x_1 \\
  x_2 \\
  x_3 \\
  x_1
\end{array} \right] &= \left( \frac{\Delta n}{L^3} \right)^4 x_1(1 - 3x_2 - 2x_3 + 5x_2x_3)(1 - x_4) \\
\left[ \begin{array}{c}
  x_1 \\
  x_2 \\
  x_3 \\
  x_1
\end{array} \right] &= \left( \frac{\Delta n}{L^3} \right)^4 x_1(1 - 4x_2 - 3x_3 + 5x_2x_3)(1 - x_4)
\end{align*}
\]

(95) (96) (97)

**Classical SSEP**

Let us first recall the connection between the quantum and classical SSEP models:

\[
\langle \epsilon \sum_i \hat{a}_i \hat{n}_i \rangle_{\text{ssep}} = \text{Tr}(\hat{\rho} e^{\sum_i \hat{a}_i \hat{n}_i}) = \left[ \text{Tr}(\rho e^{\sum_i \hat{a}_i \hat{n}_i}) \right],
\]

(98)

with \( \hat{n}_i = c_i^\dagger c_i \) the quantum number operators and \( n_i \) the classical SSEP occupation variables.

In this part, we prove eq.(17) of the main text as a consequence of eq.(8) in the main text. First, using Wick’s theorem we have:

\[
\left[ \begin{array}{c}
  \text{Tr}(\rho \hat{n}_i \hat{n}_j \cdots \hat{n}_{i_N}) \end{array} \right] = \sum_{\mathcal{P} = \{u_1, \cdots, u_m\}} \sum_{\{\sigma_1, \cdots, \sigma_m\}} (-1)^{N+m} \left[ \begin{array}{c}
  i_{\sigma_u(u)} \cdots i_{\sigma_u(N)} \\
  i_{\sigma_1(1)} \cdots i_{\sigma_1(N)} \\
  \cdots \\
  i_{\sigma_m(u_m)} \cdots i_{\sigma_m(N)}
\end{array} \right]
\]

(99)

where \( \mathcal{P} = \{u_1, \cdots, u_m\} \) are partitions of \( N \), i.e. \( \sum_{j=1}^m u_j = N \) and the \( \sigma_u \)'s denote all permutations of \( p \) indices up to cyclic permutations (there are \( (p - 1)! \) of them). This formula is for \( i_1 \neq i_2 \neq \cdots \neq i_N \), and taking two indices to be equal amounts to consider a term of order \( N - 1 \) since \( \hat{n}^2 = \hat{n} \). The graph \( \begin{array}{c}
  \includegraphics[width=0.5\textwidth]{fig}
\end{array} \) designates a one-loop diagram connecting \( p \) points. One can directly check that \( \left[ \begin{array}{c}
  \text{Tr}(\rho \hat{n}_i \hat{n}_j \cdots \hat{n}_{i_N}) \end{array} \right] \) equals the terms of order \( a_{i_1} \cdots a_{i_N} \), in

\[
\exp \left( \sum_{p=1}^{\infty} \sum_{i_1, \cdots, i_p} \frac{(-1)^{p-1}}{p!} a_{i_1} \cdots a_{i_p} \sum_{\sigma_p} i_{\sigma_p(u)} \cdots i_{\sigma_p(N)} \right).
\]

(100)

The two last formula can also be retrieved from the well known formula for fermionic expectation values:

\[
\text{Tr}(\rho \exp(\sum_i a_i \hat{n}_i)) = \text{Det}(1 + G(e^A - 1)).
\]

(101)

with \( A \) the diagonal matrix with entries \( a_i \). Expanding the determinant as a sum over permutations and decomposing these permutations into product of cycles yields (99).

Recall the definition of the connected correlation function \( [X_1, \cdots, X_q]^c \) of random variables \( X_k \):

\[
[X_1, \cdots, X_q]^c = \frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_q} \log \left( \exp(\sum_{i=1}^q a_i X_i) \right) \big|_{a_1 = a_2 = \cdots = a_q = 0}
\]

Combining this definition with (100) and the correspondence between the quantum and classical SSEP correlations, it is clear the the classical SSEP expectations are given by:

\[
\langle n_{i_1} \cdots n_{i_N} \rangle_{\text{ssep}}^c = \sum_{\mathcal{P} = \{u_1, \cdots, u_m\}} \sum_{\{\sigma_1, \cdots, \sigma_m\}} (-1)^{N+m} \left[ \begin{array}{c}
  i_{\sigma_u(u)} \cdots i_{\sigma_u(N)} \\
  i_{\sigma_1(1)} \cdots i_{\sigma_1(N)} \\
  \cdots \\
  i_{\sigma_m(u_m)} \cdots i_{\sigma_m(N)}
\end{array} \right]^c,
\]

(102)
for all $i_k$ distinct. The only remaining terms in the thermodynamic limit among the connected diagrams are the one-loop diagrams, thus proving eq.(17) of the main text:

$$\langle n_{j_1} \cdots n_{j_N} \rangle^c_{\text{sep}} = \frac{(-)^{N-1}}{L^{N-1}} \sum_{\sigma} f_{\sigma}^N(x) + O(L^{-N}).$$

(103)

Up to order four, we checked that this formula indeed agreed with known results for SSEP (see references [6,7,76] of the main text):

$$\langle n_{j_1} \rangle_{\text{sep}} = [ x_1 \text{ } ] = n_a + (\Delta n) x_1$$

$$\langle n_{j_1} n_{j_2} \rangle_{\text{sep}}^c = \frac{\Delta n}{L} x_1 (1 - x_2)$$

$$\langle n_{j_1} n_{j_2} n_{j_3} \rangle_{\text{sep}}^c = \frac{\Delta n}{L^2} 2x_1 (1 - 2x_2)(1 - x_3)$$

$$\langle n_{j_1} n_{j_2} n_{j_3} n_{j_4} \rangle_{\text{sep}}^c = -\frac{\Delta n}{L^3} 2x_1 (3 - 10x_2 - 5x_3 + 15x_2x_3)(1 - x_4)$$

Notice again that the classical SSEP cumulants are given by the sum over the different sectors of the quantum SSEP single loops.