Excluding four-edge paths and their complements

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Abstract

We prove that a graph $G$ contains no induced four-edge path and no induced complement of a four-edge path if and only if $G$ is obtained from five-cycles and split graphs by repeatedly applying the following operations: substitution, split graph unification, and split graph unification in the complement (“split graph unification” is a new class-preserving operation that is introduced in this paper).

1 Introduction

All graphs in this paper are finite and simple. We denote by $P_5$ the path on five vertices (and four edges); this path is also called a four-edge path. The complement of a graph $G$ is denoted by $\overline{G}$. The graph $\overline{P_5}$, the complement of the four-edge path, is also called a house. Given graphs $G$ and $H$, we say that $G$ is $H$-free if $G$ does not contains (an isomorphic copy of) $H$ as an induced subgraph. Given a family $\mathcal{H}$ of graphs, we say that a graph $G$ is $\mathcal{H}$-free provided that $G$ is $H$-free for all $H \in \mathcal{H}$. The goal of this paper is to understand the structure of $\{P_5, \overline{P_5}\}$-free graphs.

We begin with a few definitions. The vertex-set of a graph $G$ is denoted by $V_G$. Given $X \subseteq V_G$, we denote by $G[X]$ the subgraph of $G$ induced by $X$; given $v_1, ..., v_n \in V_G$, we often write $G[v_1, ..., v_n]$ instead of $G[\{v_1, ..., v_n\}]$. We denote by $G \setminus X$ the graph $G[V_G \setminus X]$, and for $v \in V_G$, we often write $G \setminus v$ instead of $G \setminus \{v\}$. A clique in $G$ is a set of pairwise adjacent vertices in $G$, and the clique number of $G$, denoted by $\omega(G)$, is the maximum size

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of a clique in \(G\). A stable set in \(G\) is a set of pairwise non-adjacent vertices in \(G\). \(G\) is said to be a split graph if its vertex-set can be partitioned into a (possibly empty) clique and a (possibly empty) empty set. The chromatic number of \(G\) is denoted by \(\chi(G)\). \(G\) is said to be perfect if \(\chi(H) = \omega(H)\) for all induced subgraphs \(H\) of \(G\). Given a graph \(G\), a set \(X \subseteq V_G\), and a vertex \(v \in V_G \setminus X\), we say that \(v\) is complete to \(X\) if \(v\) is adjacent to every vertex of \(X\), and we say that \(v\) is anti-complete to \(X\) if \(v\) is non-adjacent to every vertex of \(X\); \(v\) is said to be mixed on \(X\) if \(v\) is neither complete nor anti-complete to \(X\). \(X\) is said to be a homogeneous set in \(G\) if no vertex in \(V_G \setminus X\) is mixed on \(X\). A homogeneous set \(X\) in a graph \(G\) is said to be proper if \(2 \leq |X| \leq |V_G| - 1\). \(G\) is said to be prime if it does not contain a proper homogeneous set.

We denote by \(C_5\) the cycle on five vertices; this graph is also called a pentagon. The following result about the structure of \(\{P_5, \overline{P_5}\}\)-free graphs was proven by Fouquet in [4].

1.1. [4] For each \(\{P_5, \overline{P_5}\}\)-free graph \(G\) at least one of the following holds:

- \(G\) contains a proper homogeneous set;
- \(G\) is isomorphic to \(C_5\);
- \(G\) is \(C_5\)-free.

1.1 immediately implies that every \(\{P_5, \overline{P_5}, C_5\}\)-free graph can be obtained by “substitution” starting from \(\{P_5, \overline{P_5}, C_5\}\)-free graphs and pentagons (substitution is a well-known operation whose precise definition we give in section 2). Furthermore, it is easy to check that every graph obtained by substitution starting from \(\{P_5, \overline{P_5}, C_5\}\)-free graphs and pentagons is \(\{P_5, \overline{P_5}\}\)-free. We remark that the Strong Perfect Graph Theorem [2] implies that a \(\{P_5, \overline{P_5}\}\)-free graph is perfect if and only if it is \(C_5\)-free. Thus, every \(\{P_5, \overline{P_5}\}\)-free graph can be obtained by substitution starting from \(\{P_5, \overline{P_5}\}\)-free perfect graphs and pentagons. In view of this, the bulk of this paper focuses on \(\{P_5, \overline{P_5}, C_5\}\)-free graphs (equivalently: \(\{P_5, \overline{P_5}\}\)-free perfect graphs).

It is easy to check that all split graphs are \(\{P_5, \overline{P_5}, C_5\}\)-free. Our first result is a decomposition theorem (3.5), which states that every prime \(\{P_5, \overline{P_5}, C_5\}\)-free graph that is not split admits a particular kind of “skew-partition.” Skew-partitions were first introduced by Chvátal [3], and they played an important role in the proof of the Strong Perfect Graph Theorem [2]; we
give the precise definition in section 3. Our second result is another decomposition theorem (4.1), which states that every prime \(\{P_5, \overline{P_5}, C_5\}\)-free graph that is not split admits a new kind of decomposition, which we call a “split graph divide” (see section 4 for the definition). Next, we reverse the split graph divide decomposition to turn it into a composition that preserves the property of being \(\{P_5, \overline{P_5}, C_5\}\)-free. We call this composition “split graph unification” (see section 5 for the definition). Finally, combining our results with 1.1, we prove that every \(\{P_5, \overline{P_5}\}\)-free graph is obtained by repeatedly applying substitution, split graph unification, and split graph unification in the complement starting from split graphs and pentagons (see 6.1 and 6.2).

This paper is organized as follows. Section 2 contains definitions that we use in the remainder of the paper. Section 3 is devoted, first, to proving that every prime \(\{P_5, \overline{P_5}, C_5\}\)-free graph that is not split admits a skew-partition of a certain kind (see 3.5), and then to further analyzing skew-partitions in prime \(\{P_5, \overline{P_5}, C_5\}\)-free graphs. The final result of section 3 (see 3.11) is used in section 4. However, a number of lemmas from section 3 (in particular 3.8, 3.9, and 3.10) that are used to prove 3.11 may also be of independent interest as theorems about skew-partitions in \(\{P_5, \overline{P_5}, C_5\}\)-free graphs. Section 4 deals with split graph divides, and section 5 with split graph unifications. Finally, in section 6, we prove the main theorem of this paper.

2 Definitions

Given a graph \(G\) and a vertex \(v \in V_G\), we denote by \(\Gamma_G(v)\) the set of all neighbors of \(v\) in \(G\). (Thus, \(v \notin \Gamma_G(v)\).) The degree of \(v\) in \(G\) is \(|\Gamma_G(v)|\), that is, the number of neighbors that \(v\) has in \(G\).

A graph is non-trivial if it contains at least two vertices. A graph \(H\) is said to be smaller than a graph \(G\) provided that \(H\) has strictly fewer vertices than \(G\). \(G\) is bigger than \(H\) provided that \(H\) is smaller than \(G\).

Given graphs \(G_1\) and \(G_2\) with disjoint vertex-sets, and a vertex \(u \in V_{G_2}\), we say that a graph \(G\) is obtained by substituting \(G_1\) for \(u\) in \(G_2\) provided that the following hold:

- \(V_G = V_{G_1} \cup (V_{G_2} \setminus \{u\})\);
- \(G[V_{G_1}] = G_1\);
- \(G[V_{G_2} \setminus \{u\}] = G_2 \setminus u\);
• for all $v \in V_{G_2} \setminus \{u\}$, if $v$ is adjacent to $u$ in $G_2$, then $v$ is complete to $V_{G_1}$ in $G$, and if $v$ is non-adjacent to $u$ in $G_2$, then $v$ is anti-complete to $V_{G_1}$ in $G$.

We remark that under these circumstances, $V_{G_1}$ is a homogeneous set in $G$, and the homogeneous set $V_{G_1}$ in $G$ is proper if and only if $G_1$ and $G_2$ both have at least two vertices (equivalently: if $G_1$ and $G_2$ are both smaller than $G$). Thus, a graph $G$ is obtained by substitution from smaller graphs if and only if $G$ contains a proper homogeneous set.

Given a graph $G$ and disjoint sets $A, B \subseteq V_G$, we say that $A$ is complete to $B$ provided that every vertex in $A$ is complete to $B$, and we say that $A$ is anti-complete to $B$ provided every vertex in $A$ is anti-complete to $B$.

We often denote a path by $p_0 - \cdots - p_n$; this means that $p_0, \ldots, p_n$ are the vertices of the path, and that for all distinct $i, j \in \{0, \ldots, n\}$, $p_i$ is adjacent to $p_j$ if and only if $|i - j| = 1$. A path on $n + 1$ vertices and $n$ edges is denoted by $P_{n+1}$; thus, $P_{n+1}$ is an $n$-edge path. The length of a path is the number of edges that it contains; thus, the length of $P_{n+1}$ is $n$. We remind the reader that a house is the complement of a four-edge path. We often denote a house by $p_0 - p_1 - p_2 - p_3 - p_4$; this means that $p_0, p_1, p_2, p_3, p_4$ are the vertices of the house, and that for all distinct $i, j \in \{0, 1, 2, 3, 4\}$, $p_i$ and $p_j$ are non-adjacent if and only if $|i - j| = 1$.

We often denote a cycle by $c_0 - c_1 - \cdots - c_{n-1} - c_0$; this means that $c_0, c_1, \ldots, c_{n-1}$ are the vertices of the cycle, and that for all distinct $i, j \in \{0, \ldots, n-1\}$, $c_i$ and $c_j$ are adjacent if and only if $|i - j| = 1$ or $n - 1$. A cycle on $n$ vertices and $n$ edges is denoted by $C_n$. The length of a cycle is the number of edges (equivalently: the number of vertices) that it contains. A triangle is a cycle of length three, a square is a cycle of length four, and a pentagon is a cycle of length five.

A graph $G$ is connected if $V_G$ cannot be partitioned into two non-empty sets that are anti-complete to each other. A graph $G$ is anti-connected if $\overline{G}$ is connected. A component of a non-null graph $G$ is a maximal connected induced subgraph of $G$, and an anti-component of $G$ is a maximal anti-connected induced subgraph of $G$. A component or an anti-component of a graph is non-trivial if it contains at least two vertices. We remark that the vertex-sets of the components of a graph are anti-complete to each other, and that the vertex-sets of the anti-components of a graph are complete to
each other.

### 3 Skew-partition decomposition

Given a graph $G$ and sets $X, Y \subseteq V_G$, we say that $(X, Y)$ is a **skew-partition** of $G$ provided that $V_G = X \cup Y$, $X$ and $Y$ are non-empty and disjoint, $G[X]$ is not connected, and $G[Y]$ is not anti-connected. We say that $G$ **admits a skew-partition** provided that there exist sets $X, Y \subseteq V_G$ such that $(X, Y)$ is a skew-partition of $G$. Note that if $(X, Y)$ is a skew-partition of $G$, then $(Y, X)$ is a skew-partition of $\overline{G}$, and consequently, $G$ admits a skew-partition if and only if $\overline{G}$ does.

This section is devoted to analyzing skew-partitions in $\{P_5, \overline{P_5}, C_5\}$-free graphs. It is organized as follows. We consider prime $\{P_5, \overline{P_5}, C_5\}$-free graphs that are not split. A result from \cite{1} (see \S 3.3 below) guarantees that all such graphs contain a certain induced subgraph that “interacts” with the rest of the graph in a certain useful way. We use this result to show that every prime $\{P_5, \overline{P_5}, C_5\}$-free graph $G$ that is not split admits a skew-partition $(X, Y)$ such that either $G[X]$ contains at least two non-trivial components, or $G[Y]$ contains at least two non-trivial anti-components (see \S 3.5). The remainder of the section is devoted to proving a series of lemmas about skew-partitions of this kind in $\{P_5, \overline{P_5}, C_5\}$-free graphs. The final result of this section (\S 3.11) is used in section 4 to construct “split graph divides.”

We begin with a couple of technical lemmas (\S 3.1 and 3.2).

#### 3.1. Let $G$ be a graph, let $X \subseteq V_G$, and let $v \in V_G \setminus X$ be a vertex that is mixed on $X$ in $G$. Then the following hold:

- if $G[X]$ is connected, then there exist adjacent $x, x' \in X$ such that $v$ is adjacent to $x$ and non-adjacent to $x'$;
- if $G[X]$ is anti-connected, then there exist non-adjacent $x, x' \in X$ such that $v$ is adjacent to $x$ and non-adjacent to $x'$.

**Proof.** It suffices to prove the first statement, for then the second will follow by an analogous argument applied to $\overline{G}$. So suppose that $G[X]$ is connected. Since $v$ is mixed on $X$, both $X \cap \Gamma_G(v)$ and $X \setminus \Gamma_G(v)$ are non-empty. As $G[X]$ is connected, $X \cap \Gamma_G(v)$ is not anti-complete to $X \setminus \Gamma_G(v)$ in $G$, and consequently, there exist adjacent vertices $x \in X \cap \Gamma_G(v)$ and $x' \in X \setminus \Gamma_G(v)$. This completes the argument. $\square$
3.2. Let $G$ be a $\{P_5, \overline{P_5}\}$-free graph, and let $(X, Y)$ be a skew-partition of $G$. Then:

- no vertex in $X$ is mixed on more than one anti-component of $G[Y]$;
- no vertex in $Y$ is mixed on more than one component of $G[X]$.

Proof. It suffices to prove the second statement, for the first will then follow by an analogous argument applied to $\overline{G}$. Suppose that $X_1$ and $X_2$ are the vertex-sets of distinct components of $G[X]$, and that some $y \in Y$ is mixed on both $X_1$ and $X_2$ in $G$. By 3.1, there exist adjacent $x_1, x'_1 \in X_1$ such that $y$ is adjacent to $x_1$ and non-adjacent to $x'_1$. Similarly, there exist adjacent $x_2, x'_2 \in X_2$ such that $y$ is adjacent to $x_2$ and non-adjacent to $x'_2$. But now $x'_1 - x_1 - y - x_2 - x'_2$ is an induced four-edge path in $G$, which contradicts the assumption that $G$ is $P_5$-free.  

We now need some definitions. Given a graph $G$ and a vertex $v \in V_G$, we say that $v$ is a simplicial vertex of $G$ provided that $\Gamma_G(v)$ is a clique, and we say that $v$ is an anti-simplicial vertex of $G$ provided that $v$ is a simplicial vertex of $\overline{G}$. In other words, $v$ is simplicial in $G$ provided that $v$’s neighbors in $G$ form a clique, and $v$ is anti-simplicial in $G$ provided that $v$’s non-neighbors in $G$ form a stable set. Last, $H_6$ is a graph with vertex-set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge-set $\{v_1v_2, v_2v_3, v_3v_4, v_2v_5, v_3v_6, v_5v_6\}$.

In what follows, we use a theorem (stated below) proven in [1] by the first two authors of the present paper.

3.3. [1] If $G$ is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph, then either $G$ is a split graph or at least one of $G$ and $\overline{G}$ contains an induced $H_6$ whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is anti-simplicial.

Given a graph $G$, an induced subgraph $H$ of $G$, and vertices $u \in V_H$ and $u' \in V_G \setminus V_H$, we say that $u'$ is a clone of $u$ with respect to $H$ in $G$ provided that for all $v \in V_H \setminus \{u\}$, $u'$ is adjacent to $v$ if and only if $u$ is adjacent to $v$. We now prove a lemma (see 3.4) that describes how vertices in a $\{P_5, \overline{P_5}, C_5\}$-free graph can “attach” to an induced three-edge path in that graph, and then we use this lemma, together with 3.3, to show that every prime $\{P_5, \overline{P_5}, C_5\}$-free graph that is not split admits a certain kind of skew-partition (see 3.5).

3.4. Let $G$ be a $\{P_5, \overline{P_5}, C_5\}$-free graph, and let $a - b - c - d$ be an induced three-edge path in $G$. For each $x \in \{a, b, c, d\}$, let $C_x$ be the set of all
clones of $x$ with respect to $a - b - c - d$ in $G$. Let $A$ be the set of all vertices in $G$ that are anti-complete to $\{a, b, c, d\}$, let $B$ be the set of all vertices in $G$ that are complete to $\{b, c\}$ and anti-complete to $\{a, d\}$, and let $C$ be the set of vertices in $G$ that are complete to $\{a, b, c, d\}$. Then $V_G = \{a, b, c, d\} \cup C_a \cup C_b \cup C_c \cup C_d \cup A \cup B \cup C$.

Proof. Let $x \in V_G$. We need to show that $x \in \{a, b, c, d\} \cup C_a \cup C_b \cup C_c \cup C_d \cup A \cup B \cup C$. If $x \in \{a, b, c, d\}$, then we are done; so assume that $x \in V_G \setminus \{a, b, c, d\}$. We remark that both the premises and the conclusion of 3.4 are complement-invariant (we are using the fact that the complement of a three edge-path is again a three-edge path). Thus, passing to the complement of $G$ if necessary, we may assume that $x$ has at most two neighbors in $\{a, b, c, d\}$. If $x$ is anti-complete to $\{a, b, c, d\}$, then $x \in A$, and we are done. So assume that $x$ has a neighbor in $\{a, b, c, d\}$.

Suppose first that $x$ has a neighbor in $\{a, d\}$; by symmetry, we may assume that $x$ is adjacent to $a$. Since $x - a - b - c - d$ is not an induced four-edge path in $G$, it follows that $x$ has a (unique) neighbor in $\{b, c, d\}$. Since $x - a - b - c - d - x$ is not an induced pentagon in $G$, $x$ is non-adjacent to $d$. But now, if $x$ is adjacent to $b$, then $x \in C_a$; and if $x$ is adjacent to $c$, then $x \in C_b$. In either case, we are done.

Suppose now that $x$ is anti-complete to $\{a, d\}$. If $x$ is adjacent to exactly one of $b$ and $c$, then $x \in C_a \cup C_d$; and if $x$ is complete to $\{b, c\}$, then $x \in B$. This completes the argument.

3.5. Let $G$ be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph that is not split. Then $G$ admits a skew-partition $(X, Y)$ such that either $G[X]$ has at least two non-trivial components, or $G[Y]$ has at least two non-trivial anti-components.

Proof. By 3.3 we know that at least one of $G$ and $\overline{G}$ contains an induced $H_6$ whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is anti-simplicial. Our goal is to prove the following:

(a) if $G$ contains an induced $H_6$ whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is anti-simplicial, then $G$ contains a skew-partition $(X, Y)$ such that $G[Y]$ contains at least two non-trivial anti-components;

(b) if $\overline{G}$ contains an induced $H_6$ whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is anti-simplicial, then $G$ contains a skew-partition $(X, Y)$ such that $G[X]$ contains at least two non-trivial components.
Note that it suffices to prove (b), for (a) will then follow by an analogous argument applied to $\overline{G}$. So assume that $\overline{G}$ contains an induced $H_6$ whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is anti-simplicial. Then there exist pairwise distinct vertices $a, b, c, d, b', c' \in V_G$ such that:

- $a - b - c - d$ is an induced path in $G$;
- $b'c'$ is a non-edge in $G$;
- $b'$ is complete to $\{a, b, c\}$ and non-adjacent to $d$ in $G$;
- $c'$ is complete to $\{b, c, d\}$ and non-adjacent to $a$ in $G$;
- $a$ is simplicial in $G$;
- $b$ and $c$ are anti-simplicial in $G$.

Define sets $C_a, C_b, C_c, C_d, A, B, C$ as in 3.4; by 3.4, we know that $V_G = \{a, b, c, d\} \cup C_a \cup C_b \cup C_c \cup C_d \cup A \cup B \cup C$. We remark that $b' \in C_b$ and $c' \in C_c$.

Since $a$ is simplicial and complete to $C \cup C_b \cup \{b\}$, we know that $C \cup C_b \cup \{b\}$ is a clique. Since $b$ is anti-simplicial and anti-complete to $A \cup C_d \cup \{d\}$, we know that $A \cup C_d \cup \{d\}$ is a stable set; and since $c$ is anti-simplicial and anti-complete to $A \cup C_a \cup \{a\}$, we know that $A \cup C_a \cup \{a\}$ is a stable set.

Next, we claim that $C_a \cup C_d$ is stable. Suppose otherwise. Since $C_a$ and $C_d$ are stable, there exist adjacent $\hat{a} \in C_a$ and $\hat{d} \in C_d$. Since $b - \hat{d} - b' - \hat{a} - c$ is not an induced house in $G$, $b'$ has a neighbor in $\{\hat{a}, \hat{d}\}$. Similarly, $c'$ has a neighbor in $\{\hat{a}, \hat{d}\}$. Let $P$ be an induced path in $G[b', c', \hat{a}, \hat{d}]$ between $b'$ and $c'$; since $b'c'$ is a non-edge in $G$, we know that $P$ contains at least two edges. But now since $C_a \cup \{a\}$ and $C_d \cup \{d\}$ are stable, it follows that $a - b' - P - c' - d$ is an induced path in $G$ of length at least four, contrary to the fact that $G$ is $P_5$-free. This proves that $C_a \cup C_d$ is stable.

We now know the following:

- $A \cup C_d \cup \{d\}$ is stable;
- $A \cup C_a \cup \{a\}$ is stable;
- $C_a \cup C_d$ is stable;
Now, we claim that there is no path between \( a \) and \( b \) since \( b \) is complete to \( c \). We showed above that \( a, b \) is complete to \( c \), and consequently, to \( C \). Thus, \( a, b \) is complete to \( C \). Further, we showed above that \( a, b, c \) is a stable set; thus, \( a \) is anti-complete to \( C \). It follows that \( a \) is anti-complete to \( X \). Since \( b \) is complete to \( C \), it follows that \( a \) is anti-complete to \( X \). Consequently, \( a, b, c \) is a stable set.

Recall that \( b' \in C_b \) and so \( C_b \neq \emptyset \). Let \( b_1 \) be a vertex in \( C_b \) with as few neighbors as possible in \( C_c \); let \( N \) be the set of all neighbors of \( b_1 \) in \( C_c \). We claim that \( N \) is complete to \( C_b \). Suppose otherwise. Fix \( b_2 \in C_b \) and \( c_1 \in N \) such that \( b_2c_1 \) is a non-edge. By the minimality of \( N \), there exists some \( c_2 \in C_c \setminus N \) such that \( b_2c_2 \) is an edge. Since \( C_b \) is a clique, we know that either \( c_1 - b_1 - b_2 - c_2 \) is an induced three-edge path in \( G \), or \( c_1 - b_1 - b_2 - c_2 - c_1 \) is an induced square in \( G \); in the former case, \( d - c_1 - b_1 - b_2 - c_2 - d \) is an induced pentagon in \( G \), and in the latter case, \( c_2 - b_1 - d - b_2 - c_1 \) is an induced house in \( G \). But neither outcome is possible since \( G \) contains no induced pentagon and no induced house. This proves that \( N \) is complete to \( C_b \).

Set \( Y = C \cup N \cup (C_b \setminus \{b_1\}) \cup \{b, c\} \). By definition, \( b \) is complete to \( C \cup C_c \cup \{c\} \); since \( N \subseteq C_c \), it follows that \( b \) is complete to \( C \cup N \cup \{c\} \). Further, we showed above that \( C_b \cup \{b\} \) is a clique; it follows that \( b \) is complete to \( C_b \), and consequently, to \( C_b \setminus \{b_1\} \). Thus, \( b \) is complete to \( C \cup N \cup (C_b \setminus \{b_1\}) \cup \{c\} = Y \setminus \{b\} \). Since \( Y \setminus \{b\} \neq \emptyset \) (because \( c \in Y \setminus \{b\} \)), it follows that \( Y \) is not anti-connected.

Set \( X = V_G \setminus Y \). Then \( X = A \cup B \cup C_a \cup C_d \cup (C_c \setminus N) \cup \{a, d, b_1\} \). We showed above that \( A \cup C_a \cup \{a\} \) is a stable set; thus, \( a \) is anti-complete to \( A \cup C_a \). Further, by construction, \( a \) is anti-complete to \( B \cup C_d \cup C_c \cup \{d\} \). It follows that \( a \) is anti-complete to \( X \setminus \{a, b_1\} \); since \( b_1 \) is a clone of \( b \) for \( a - b - c - d \) in \( G \), we know that \( ab_1 \) is an edge. Next, we showed above that \( A \cup C_d \cup \{d\} \) is a stable set, and by the definition of \( B \) and \( C_a \), \( d \) is anti-complete to \( B \cup C_a \). Since \( a - b - c - d \) is an induced path in \( G \), and since \( b_1 \in C_b \), we know that \( d \) is anti-complete to \( \{a, b_1\} \). Further, by construction, \( d \) is complete to \( C_c \), and therefore, to \( C_c \setminus N \). It follows that \( d \) is complete to \( C_c \setminus N \) and anti-complete to \( X \setminus ((C_c \setminus N) \cup \{d\}) \) in \( G \).

Now, we claim that there is no path between \( \{a, b_1\} \) and \( (C_c \setminus N) \cup \{d\} \) in \( G[X] \). Suppose otherwise. Let \( P \) be a path of minimum length between \( \{a, b_1\} \) and \( (C_c \setminus N) \cup \{d\} \) in \( G[X] \). Since \( b_1 \) is the only neighbor of \( a \) in \( G[X] \), and since all the neighbors of \( d \) in \( G[X] \) lie in \( C_c \setminus N \), it follows that the endpoints of this path are \( b_1 \) and some vertex \( c \in C_c \setminus N \). Since \( b_1 \) is
anti-complete to $C_c \setminus N$, $P$ has at least two edges. But then $a - b_1 - P - \hat{c} - d$ is an induced path in $G$ of length at least four, which is impossible since $G$ is $P_5$-free. Thus, there is no path between $\{a, b_1\}$ and $(C_c \setminus N) \cup \{d\}$ in $G[X]$. It follows that $G[X]$ is disconnected, and consequently, that $(X, Y)$ is a skew-partition of $G$.

It remains to show that $G[X]$ has at least two non-trivial components. Since $ab_1$ is an edge in $G$, $G[a, b_1]$ is connected, and since $d$ is complete to $C_c \setminus N$ in $G$, $G[(C_c \setminus N) \cup \{d\}]$ is connected. Let $X_1$ be the vertex-set of the component of $G[X]$ that contains $a$ and $b_1$, and let $X_2$ be the vertex-set of the component of $G[X]$ that includes $(C_c \setminus N) \cup \{d\}$. Clearly, $|X_1| \geq 2$, and we just need to show that $|X_2| \geq 2$. It suffices to show that $C_c \setminus N \neq \emptyset$. Suppose otherwise. Then $C_c = N$, and consequently, $b_1$ is complete to $C_c$. But $b' \in C_b$ and $b'$ has a non-neighbor (namely, $c'$) in $C_c$; consequently, $b'$ has fewer neighbors in $C_c$ than $b_1$ does, contrary to the choice of $b_1$. It follows that $|X_2| \geq 2$. This completes the argument.

In the remainder of this section, we study skew-partitions of the kind that appears in 3.5. We start with a few definitions. Given a graph $G$ and a skew-partition $(X, Y)$ of $G$, a decomposition of $(X, Y)$ in $G$ is an ordered six-tuple $(\{X_i\}_{i=1}^m, \{Y_j\}_{j=1}^n, S, K, \{S_j\}_{j=1}^n, \{K_i\}_{i=1}^m)$ such that:

- $X_1, \ldots, X_m$ are the vertex-sets of the non-trivial components of $G[X]$;
- $Y_1, \ldots, Y_n$ are the vertex-sets of the non-trivial anti-components of $G[Y]$;
- $S = X \setminus (X_1 \cup \ldots \cup X_m)$;
- $K = Y \setminus (Y_1 \cup \ldots \cup Y_n)$;
- for each $j \in \{1, \ldots, n\}$, $S_j$ is the set of all vertices in $S$ that are mixed on $Y_j$;
- for each $i \in \{1, \ldots, m\}$, $K_i$ is the set of all vertices in $K$ that are mixed on $X_i$.

Clearly, $X$ is the disjoint union of $X_1, \ldots, X_m, S$; and $Y$ is the disjoint union of $Y_1, \ldots, Y_n, K$. Further, $S$ is a (possibly empty) stable set, and $K$ is a (possibly empty) clique. We note that if $m = 0$, then $X = S$; similarly, if $n = 0$, then $Y = K$.

We say that a skew-partition $(X, Y)$ of $G$ is usable provided that its associated partition $(\{X_i\}_{i=1}^m, \{Y_j\}_{j=1}^n, S, K, \{S_j\}_{j=1}^n, \{K_i\}_{i=1}^m)$ satisfies at least one of the following:
(a) \( m \geq 2; \) the sets \( S_1, \ldots, S_n \) are pairwise disjoint; the sets \( K_1, \ldots, K_m \) are pairwise disjoint, and every vertex of \( Y \) has a neighbor in each of \( X_1, \ldots, X_m \).

(b) \( n \geq 2; \) the sets \( S_1, \ldots, S_n \) are pairwise disjoint; the sets \( K_1, \ldots, K_m \) are pairwise disjoint, and every vertex in \( X \) has a non-neighbor in each of \( Y_1, \ldots, Y_n \).

We say that \( G \) admits a usable skew-partition provided that there exists a usable skew-partition \((X, Y)\) of \( G \). Note that \( G \) admits a usable skew-partition if and only if \( \overline{G} \) does. Our next goal is to prove that every prime \( \{P_5, \overline{P_5}, C_5\} \)-free graph that is not split admits a usable skew-partition (see \( \text{3.8} \)). We first need a couple of lemmas (\( \text{3.6} \) and \( \text{3.7} \)). The first of the two lemmas is used to prove the second, and the second is used in the proof of \( \text{3.8} \).

**3.6.** Let \( G \) be a \( \{P_5, \overline{P_5}, C_5\} \)-free graph, and let \( X, Y \subseteq V_G \) be disjoint sets such that \( G[X] \) is connected and \( G[Y] \) is anti-connected. Let \( v \in V_G \setminus (X \cup Y) \) be complete to \( Y \) and anti-complete to \( X \). Then the following hold:

- if \( y, y' \in Y \) are non-adjacent vertices such that some vertex in \( X \) is adjacent to \( y \) and non-adjacent to \( y' \), then \( y \) is complete to \( X \) and \( y' \) is anti-complete to \( X \);
- if \( x, x' \in X \) are adjacent vertices such that some vertex in \( Y \) is adjacent to \( x \) and non-adjacent to \( x' \), then \( x \) is complete to \( Y \) and \( x' \) is anti-complete to \( Y \).

**Proof.** It suffices to prove the first claim, for then the second will follow by an analogous argument applied to \( \overline{G} \). Suppose that \( y, y' \in Y \) are non-adjacent vertices such that some vertex in \( X \) is adjacent to \( y \) and non-adjacent to \( y' \). Let \( X_0 \) be the set of all vertices in \( X \) that are adjacent to \( y \) and non-adjacent to \( y' \); by construction, \( X_0 \neq \emptyset \). We need to show that \( X_0 = X \). Suppose otherwise. Since \( G[X] \) is connected, there exist adjacent vertices \( x \in X_0 \) and \( x' \in X \setminus X_0 \). Since \( x' \in X \setminus X_0 \), we know that one of the following holds:

- \( x' \) is complete to \( \{y, y'\} \);
- \( x' \) is anti-complete to \( \{y, y'\} \);
- \( x' \) is adjacent to \( y' \) and non-adjacent to \( y \).
But in the first case, \( x' - v - x - y' - y \) is an induced house in \( G \); in the second case, \( x' - x - y - v - y' \) is an induced four-edge path in \( G \); and in the third case, \( x - y - v - y' - x' - x \) is an induced pentagon in \( G \). Since \( G \) is \( \{P_5, \overline{P_5}, C_5\}-free \), none of these three outcomes is possible. Thus, \( X_0 = X \). This completes the argument.

**3.7.** Let \( G \) be a \( \{P_5, \overline{P_5}, C_5\}\)-free graph, and let \( X, Y \subseteq V_G \) be disjoint sets such that \( G[X] \) is connected and \( G[Y] \) is anti-connected. Let \( v \in V_G \setminus (X \cup Y) \) be complete to \( Y \) and anti-complete to \( X \). Then the following hold:

- if \( x_0 \in X \) is mixed on \( Y \), then \( X \) is complete to \( Y \cap \Gamma_G(x_0) \) and anti-complete to \( Y \setminus \Gamma_G(x_0) \);
- if \( y_0 \in Y \) is mixed on \( X \), then \( Y \) is complete to \( X \cap \Gamma_G(y_0) \) and anti-complete to \( X \setminus \Gamma_G(y_0) \).

**Proof.** It suffices to prove the first claim, for then the second will follow by an analogous argument applied to \( \overline{G} \). Suppose that some \( x_0 \in X \) is mixed on \( Y \), and let \( U = Y \cap \Gamma_G(x_0) \) and \( V = Y \setminus \Gamma_G(x_0) \). Then \( U \) and \( V \) are non-empty and disjoint, and \( Y = U \cup V \). We need to show that \( X \) is complete to \( U \) and anti-complete to \( V \). Let \( X_0 \) be the set of all vertices in \( X \) that are complete to \( U \) and anti-complete to \( V \); by construction, \( x_0 \in X_0 \), and consequently, \( X_0 \neq \emptyset \). We need to show that \( X_0 = X \). Suppose otherwise. Then since \( G[X] \) is connected, there exist adjacent \( x \in X_0 \) and \( x' \in X \setminus X_0 \). Since \( x \in X_0 \), we know that \( x \) is mixed on \( Y \). Since \( x' \in X \setminus X_0 \), we know that either \( x' \) has a non-neighbor in \( U \), or \( x' \) has a neighbor in \( V \). In either case, some vertex of \( Y \) is mixed on \( \{x, x'\} \), and so by 3.6 \( x \) is either complete or anti-complete to \( Y \), which is a contradiction.

We now need a definition. Given a graph \( G \), a set \( X \subseteq V_G \), and distinct vertices \( u, v \in V_G \setminus X \), we say that \( u \) dominates \( v \) in \( X \) provided that every neighbor of \( v \) in \( X \) is also a neighbor of \( u \). We are now ready to prove that every prime \( \{P_5, \overline{P_5}, C_5\}\)-free graph that is not split admits a usable skew-partition.

**3.8.** Let \( G \) be a prime \( \{P_5, \overline{P_5}, C_5\}\)-free graph that is not split. Then \( G \) admits a usable skew-partition.

**Proof.** By 3.5, we know that \( G \) admits a skew-partition \( (X', Y') \) such that either \( G[X'] \) has at least two non-trivial components, or \( G[Y'] \) has at least two non-trivial anti-components. Since \( G \) admits a usable skew-partition if and only if \( \overline{G} \) does, we may assume that \( G[X'] \) contains at least two non-trivial components.
Let $X \subseteq V_G$ be an inclusion-wise maximal set such that $(X, V_G \setminus X)$ is a skew-partition of $G$, and $G[X]$ has at least two non-trivial components. Set $Y = V_G \setminus X$. We claim that $(X, Y)$ satisfies (a) from the definition of a usable skew-partition. Let $(\{X_1\}_{i=1}^m, \{Y_1\}_{j=1}^n, S, K, \{S_j\}_{j=1}^n, \{K_i\}_{i=1}^n)$ be a decomposition of $(X, Y)$ in $G$. The fact that the sets $S_1, ..., S_n$ are pairwise disjoint follows from 3.2 as does the fact that the sets $K_1, ..., K_m$ are pairwise disjoint.

It remains to show that every vertex in $Y$ has a neighbor in each of $X_1, ..., X_m$. Suppose otherwise. Fix some $y \in Y$ such that $y$ is anti-complete to at least one of $X_1, ..., X_m$; by symmetry, we may assume that $y$ is anti-complete to $X_1$. Since $G[X_1]$ is a non-trivial component of $G[X]$, since $G[X]$ has at least two non-trivial components, and since $y \notin X$ is anti-complete to $X_1$, we know that $G[X \cup \{y\}]$ has at least two non-trivial components. Now $G[Y \setminus \{y\}]$ must be anti-connected, for otherwise, $X \cup \{y\}$ would contradict the maximality of $X$. Since $G[Y]$ is not anti-connected but $G[Y \setminus \{y\}]$ is anti-connected, we know that $y$ is complete to $Y \setminus \{y\}$.

Since $G$ is prime, $X_1$ is not a homogeneous set in $G$. It follows that some vertex $y' \in V_G \setminus X_1$ is mixed on $X_1$. Clearly, $y' \notin X$, and since $y$ is anti-complete to $X_1$, $y' \neq y$. Thus, $y' \in Y \setminus \{y\}$. Now, $G[X_1]$ is connected, $G[Y \setminus \{y\}]$ is anti-connected, $y$ is anti-complete to $X_1$ and complete to $Y \setminus \{y\}$, and some vertex (namely $y'$) in $Y \setminus \{y\}$ is mixed on $X_1$. By 3.7, we know that $Y \setminus \{y\}$ is complete to $\Gamma_G(y') \cap X_1$ and anti-complete to $X_1 \setminus \Gamma_G(y')$. Since $\Gamma_G(y') \cap X_1$ and $X_1 \setminus \Gamma_G(y')$ are both non-empty (because $y'$ is mixed on $X_1$), it follows that every vertex in $Y \setminus \{y\}$ is mixed on $X_1$. By 3.2, it follows that no vertex in $Y \setminus \{y\}$ is mixed on any one of $X_2, ..., X_m$. Since $m \geq 2$, since none of $X_2, ..., X_m$ is a homogeneous set in $G$, and since (by 3.2) $y$ can be mixed on at most one of them, it follows that $m = 2$, and that $y$ is mixed on $X_2$.

Now, we claim that every vertex in $Y \setminus \{y\}$ dominates $y$ in $X$. Fix $\hat{y} \in Y \setminus \{y\}$, and suppose that $\hat{y}$ does not dominate $y$ in $X$. Fix $\hat{x} \in X$ such that $\hat{y}$ is adjacent to $\hat{x}$ but $\hat{y}$ is non-adjacent to $\hat{x}$. Since $y$ is anti-complete to $X_1$, we know that $\hat{x} \notin X_1$; since $G[X_1]$ is a component of $G[X]$, it follows that $\hat{x}$ is anti-complete to $X_1$. Since $\hat{y} \in Y \setminus \{y\}$, we know that $\hat{y}$ is mixed on $X_1$; since $G[X_1]$ is connected, we know by 3.1 that there exist adjacent vertices $x_1, x_1' \in X$ such that $\hat{y}$ is adjacent to $x_1$ and non-adjacent to $x_1'$. But now $\hat{x} - y - \hat{y} - x_1 - x_1'$ is an induced four-edge path in $G$, which contradicts
the assumption that $G$ is $P_3$-free. This proves that every vertex in $Y \setminus \{y\}$ dominates $y$ in $X$.

Let $Z = \{y\} \cup X_2 \cup (S \cap \Gamma_G(y))$. We claim that $Z$ is a homogeneous set in $G$. Clearly, $X \setminus Z$ is anti-complete to $Z$. Next, we know that $y$ is complete to $Y \setminus \{y\}$, and we proved above that every vertex in $Y \setminus \{y\}$ dominates $y$ in $X$. Thus, $Y \setminus \{y\}$ is complete to $\{y\} \cup (S \cap \Gamma_G(y))$, as well as to $X_2 \cap \Gamma_G(y)$. Now, we know that $y$ is mixed on $X_2$, and so $y$ has a neighbor in $X_2$; consequently, every vertex in $Y \setminus \{y\}$ has a neighbor in $X_2$. Since no vertex in $Y \setminus \{y\}$ is mixed on $X_2$, it follows that $Y \setminus \{y\}$ is complete to $X_2$. Thus, $Y \setminus \{y\}$ is complete to $Z$. This proves that $Z$ is a homogeneous set in $G$. Since $X_2 \subseteq Z$ and $Z \cap X_1 = \emptyset$, it follows that $2 \leq |Z| \leq |V_G| - 2$, and consequently, $Z$ is a proper homogeneous set in $G$. But this contradicts the fact that $G$ is prime. \hfill \Box

The next two lemmas (3.9 and 3.10) examine the behavior of usable skew-partitions in $\{P_5, \overline{P_5}, C_5\}$-free graphs. They will be used in the proof of 3.11, the main result of this section.

3.9. Let $G$ be a $\{P_5, \overline{P_5}, C_5\}$-free graph, and let $(X, Y)$ be a usable skew-partition of $G$ with associated decomposition $(\{X_i\}_{i=1}^m, \{Y_j\}_{j=1}^n, S, K, \{S_j\}_{j=1}^n, \{K_i\}_{i=1}^m)$. Then:

- if $(X, Y)$ satisfies (a) from the definition of a usable skew-partition, then no vertex in $X_1 \cup \ldots \cup X_m$ is mixed on any one of $Y_1, \ldots, Y_n$;

- if $(X, Y)$ satisfies (b) from the definition of a usable skew-partition, then no vertex in $Y_1 \cup \ldots \cup Y_n$ is mixed on any one of $X_1, \ldots, X_m$.

\textbf{Proof.} It suffices to prove the first claim, for then the second will follow by an analogous argument applied to $\overline{G}$. So assume that $(X, Y)$ satisfies (a) from the definition of a usable skew-partition. We need to show that no vertex in $X_1 \cup \ldots \cup X_m$ is mixed on any one of $Y_1, \ldots, Y_n$. Suppose otherwise. By symmetry, we may assume that some vertex $x_1 \in X_1$ is mixed on $Y_1$. By 3.1 there exist non-adjacent vertices $y_1, y'_1 \in Y_1$ such that $x_1$ is adjacent to $y_1$ and non-adjacent to $y'_1$. Since $(X, Y)$ satisfies (a) from the definition of a usable skew-partition, it follows that for each $i \in \{1, 2\}$, $y'_1$ has a neighbor $x'_i \in X_i$. Then $y'_1$ is mixed on $X_1$ (because $y'_1$ is adjacent to $x'_1$ and non-adjacent to $x_1$), and so by 3.2 $y'_1$ is not mixed on $X_2$. Since $y'_1$ has a neighbor (namely $x'_2$) in $X_2$, it follows that $y'_1$ is complete to $X_2$. Next, since $(X, Y)$ satisfies (a) from the definition of a usable skew-partition, $y_1$ has a neighbor...
$x_2 \in X_2$. Now $G[X_1]$ is connected, $G[y_1, y_1']$ is anti-connected, $x_2$ is anti-complete to $X_1$ and complete to $\{y_1, y_1'\}$; $y_1$ is non-adjacent to $y_1'$, and $x_1$ is adjacent to $y_1$ and non-adjacent to $y_1'$; thus, by 3.6 $y_1'$ is anti-complete to $X_1$. But this is impossible because $y_1'$ has a neighbor (namely $x_1'$) in $X_1$. This completes the argument.

3.10. Let $G$ be a $\{P_5, \overline{P_5}, C_5\}$-free graph, and let $(X, Y)$ be a usable skew-partition of $G$ with associated decomposition $(\{X_i\}_{i=1}^m, \{Y_j\}_{j=1}^n, S, K, \{S_j\}_{j=1}^n)$.

• If the skew-partition $(X, Y)$ satisfies (a) from the definition of a usable skew-partition, then either $n = 0$, or the following hold:
  - $S_1, ..., S_n$ are non-empty, and
  - there exists some $j \in \{1, ..., n\}$ such that $S_j$ is anti-complete to $Y \setminus (Y_j \cup K)$;

• If the skew-partition $(X, Y)$ satisfies (b) from the definition of a usable skew-partition, then either $m = 0$, or the following hold:
  - $K_1, ..., K_m$ are non-empty, and
  - there exists some $i \in \{1, ..., m\}$ such that $K_i$ is complete to $X \setminus (X_i \cup S)$.

Proof. It suffices to prove the first statement, for then the second will follow by an analogous argument applied to $\overline{G}$. So suppose that the skew-partition $(X, Y)$ satisfies (a) from the definition of a usable skew-partition. If $n = 0$, then we are done; so suppose that $n \geq 1$.

We first show that the sets $S_1, ..., S_n$ are non-empty. By symmetry, it suffices to show that $S_1 \neq \emptyset$. By 3.9 no vertex of $X_1 \cup ... \cup X_m$ is mixed on $Y_1$. By construction, no vertex in $Y \setminus Y_1$ is mixed on $Y_1$. Since $2 \leq |Y_1| \leq |V_G| - 1$, and $Y_1$ is not a proper homogeneous set in $G$, it follows that some vertex in $S$ is mixed on $Y_1$, and therefore $S_1 \neq \emptyset$.

It remains to show that there exists some $i \in \{1, ..., n\}$ such that $S_j$ is anti-complete to $Y \setminus (Y_j \cup K)$. By what we just showed, the sets $S_1, ..., S_n$ are non-empty, and since the skew-partition $(X, Y)$ is usable, the sets $S_1, ..., S_n$ are pairwise disjoint. Now, let $\tilde{H}$ be the directed graph with vertex-set $\{S_1, ..., S_n\}$, and in which for all distinct $i, j \in \{1, ..., n\}$, $(S_i, S_j)$ is an arc in $\tilde{H}$ provided that some vertex in $S_i$ is complete to $Y_j$. 

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We first prove the following: for all \( t \geq 2 \) and all pairwise distinct indices \( i_1, \ldots, i_t \in \{1, \ldots, n\} \), if \( S_{i_1} - \ldots - S_{i_t} \) is a directed path in \( \vec{H} \), then \( S_{i_t} \) is anti-complete to \( Y_{i_1} \cup \ldots \cup Y_{i_{t-1}} \) in \( G \). We proceed by induction on \( t \).

For the base case, fix distinct \( i_1, i_2 \in \{1, \ldots, n\} \), and suppose that \( S_{i_1} - S_{i_2} \) is a directed path in \( \vec{H} \); then \((S_{i_1}, S_{i_2}) \) is an arc in \( \vec{H} \), and consequently, some vertex \( s_1 \in S_{i_1} \) is complete to \( Y_{i_2} \). Now, suppose that \( S_{i_1} \) is not anti-complete to \( Y_{i_1} \); fix \( s_2 \in S_{i_2} \) such that \( s_2 \) has a neighbor in \( Y_{i_1} \). Then since no vertex in \( S_{i_2} \) is mixed on \( Y_{i_1} \), we know that \( s_2 \) is complete to \( Y_{i_1} \). By definition, we know that \( s_2 \) is mixed on \( Y_{i_2} \); by 3.1 there exist non-adjacent vertices \( y_2, y_2' \in Y_{i_2} \) such that \( s_2 \) is adjacent to \( y_2 \) and non-adjacent to \( y_2' \).

Next, by definition, \( s_1 \) is mixed on \( Y_{i_1} \), and so there exists some \( y_1' \in Y_{i_1} \) such that \( s_1 \) is non-adjacent to \( y_1' \). Since \( Y_{i_1} \) is complete to \( Y_{i_2} \), we know that \( y_1' \) is complete to \( \{y_2, y_2'\} \), and since \( S_{i_1} \cup S_{i_2} \) is a stable set, we know that \( s_1 \) is non-adjacent to \( s_2 \). But now \( y_1' - s_1 - s_2 - y_2' - y_2 \) is an induced house in \( G \), which is a contradiction. This completes the base case.

For the induction step, suppose that the claim holds for some \( t \geq 2 \); we need to show that it holds for \( t + 1 \). Suppose that \( i_1, \ldots, i_{t+1} \in \{1, \ldots, n\} \) are pairwise distinct and that \( S_{i_1} - \ldots - S_{i_{t+1}} \) is a directed path in \( \vec{H} \). We need to show that \( S_{i_{t+1}} \) is anti-complete to \( Y_{i_1} \cup \ldots \cup Y_{i_{t+1}} \). By the induction hypothesis applied to the directed path \( S_{i_2} - \ldots - S_{i_{t+1}} \), we know that \( S_{i_{t+1}} \) is anti-complete to \( Y_{i_2} \cup \ldots \cup Y_{i_t} \), and so we just need to show that \( S_{i_{t+1}} \) is anti-complete to \( Y_{i_1} \). Suppose otherwise. Then there exists some \( s_{t+1} \in S_{i_{t+1}} \) such that \( s_{t+1} \) has a neighbor in \( Y_{i_1} \); since no vertex in \( S_{i_{t+1}} \) is mixed on \( Y_{i_1} \), this means that \( s_{t+1} \) is complete to \( Y_{i_1} \). Next, since \((S_{i_1}, S_{i_2}) \) is an arc in \( \vec{H} \), we know that some vertex \( s_1 \in S_{i_1} \) is complete to \( Y_{i_2} \). By construction, \( s_1 \) is mixed on \( Y_{i_1} \), and so by 3.1 we know that there exist non-adjacent \( y_1, y_1' \in Y_{i_2} \) such that \( s_1 \) is adjacent to \( y_1 \) and non-adjacent to \( y_1' \). Now, fix \( y_2 \in Y_{i_2} \). Then \( s_{t+1} \) is non-adjacent to \( y_2 \) and \( s_1 \) is adjacent to \( y_2 \). But now \( y_1 - y_1' - s_1 - s_{t+1} - y_2 \) is an induced house in \( G \), which is a contradiction. This completes the induction.

Now, let \( S_{i_1} - \ldots - S_{i_t} \) be a maximal directed path in \( \vec{H} \). Set \( j = i_t \), and fix \( k \in \{1, \ldots, n\} \setminus \{j\} \); we need to show that \( S_j \) is anti-complete to \( Y_k \). If \( k \in \{i_1, \ldots, i_{t-1}\} \), then the result follows from what we just showed. So assume that \( k \notin \{i_1, \ldots, i_{t-1}\} \). Suppose that \( S_j \) is not anti-complete to \( Y_k \). Then some vertex \( s_j \in S_j \) has a neighbor in \( Y_k \); since \( s_j \) is not mixed on \( Y_k \), it follows that \( s_j \) is complete to \( Y_k \), and consequently, \((S_j, S_k) \) is an arc
in $\tilde{H}$. But now $S_{i_1} - ... - S_{i_q} - S_k$ is a directed path in $\tilde{H}$, contrary to the maximality of $S_{i_1} - ... - S_{i_t}$. Thus, $Y_j$ is anti-complete to $Y_k$, and the result is immediate. \hfill $\square$

We are finally ready to prove the main result of this section.

3.11. Let $G$ be a $\{P_5, \overline{P_5}, C_5\}$-free graph. Then at least one of the following holds:

1. $G$ is a split graph;
2. $G$ contains a proper homogeneous set;
3. $G$ admits a skew-partition $(X, Y)$ with associated decomposition $(\{X_i\}_{i=1}^m, \{Y_j\}_{j=1}^n, S, K, \{S_j\}_{j=1}^n, \{K_i\}_{i=1}^m)$ such that:
   1. $m \geq 1$,
   2. the sets $K_1, ..., K_m$ are pairwise disjoint and non-empty,
   3. for all $i \in \{1, ..., m\}$, no vertex in $Y \setminus K_i$ is mixed on $X_i$,
   4. for all $i \in \{1, ..., m\}$, at least two vertices in $V_G \setminus (X_i \cup S)$ are anti-complete to $X_i$,
   5. there exists some $i \in \{1, ..., m\}$ such that $K_i$ is complete to $X \setminus (X_i \cup S)$;
4. $G$ admits a skew-partition $(X, Y)$ with associated decomposition $(\{X_i\}_{i=1}^m, \{Y_j\}_{j=1}^n, S, K, \{S_j\}_{j=1}^n, \{K_i\}_{i=1}^m)$ such that:
   1. $n \geq 1$,
   2. the sets $S_1, ..., S_n$ are pairwise disjoint and non-empty,
   3. for all $j \in \{1, ..., n\}$, no vertex in $X \setminus S_j$ is mixed on $Y_j$,
   4. for all $j \in \{1, ..., n\}$, at least two vertices in $V_G \setminus (Y_j \cup K)$ are complete to $Y_j$,
   5. there exists some $j \in \{1, ..., m\}$ such that $S_j$ is anti-complete to $Y \setminus (Y_j \cup K)$.

Proof. If $G$ is a split graph, or if $G$ contains a proper homogeneous set, then we are done. So assume that $G$ is a prime graph, and that $G$ is not split. Then by 3.8 $G$ admits a usable skew-partition. Let $(X, Y)$ be a usable skew-partition of $G$, and let $(\{X_i\}_{i=1}^m, \{Y_j\}_{j=1}^n, S, K, \{S_j\}_{j=1}^n, \{K_i\}_{i=1}^m)$ be its associated decomposition. By definition, if $(X, Y)$ satisfies (a) from the definition of a usable skew-partition, then $m \geq 2$, and if $(X, Y)$ satisfies (b)
from the definition of a usable skew-partition, then \( n \geq 2 \). Thus, there are four cases to consider:

- \((X, Y)\) satisfies (a) from the definition of a skew-partition, \( m \geq 2 \), and \( n = 0 \);
- \((X, Y)\) satisfies (a) from the definition of a skew-partition, \( m \geq 2 \), and \( n \geq 1 \);
- \((X, Y)\) satisfies (b) from the definition of a skew-partition, \( n \geq 2 \), and \( m = 0 \);
- \((X, Y)\) satisfies (b) from the definition of a skew-partition, \( n \geq 2 \), and \( m \geq 1 \).

We claim that in the first and fourth case, (3) holds, and in the second and third case, (4) holds. But note that given the symmetry between \( G \) and \( \overline{G} \), we need only consider the first and fourth case.

Suppose first that \((X, Y)\) satisfies (a) from the definition of a skew-partition, \( m \geq 2 \), and \( n = 0 \). Then (3.1) is immediate. We next prove (3.2). Since each of \( X_1, \ldots, X_m \) contains at least two vertices, and since \( G \) is prime, we know that for all \( i \in \{1, \ldots, m\} \), some vertex in \( V_G \setminus X_i \) is mixed on \( X_i \); clearly, this vertex cannot lie in \( X \), and so it must lie in \( Y \). Since \( Y = K \), it follows that for all \( i \in \{1, \ldots, m\} \), some vertex in \( K \) is mixed on \( X_i \), and by definition, this vertex lies in \( K_i \). This proves that each of \( K_1, \ldots, K_m \) is non-empty. The fact that \( K_1, \ldots, K_m \) are pairwise disjoint follows from the definition of a usable skew-partition. This proves (3.2). For (3.3), we first observe that since \( n = 0 \), we have that \( Y = K \). By the definition of \( K_1, \ldots, K_m \), we know that for all \( i \in \{1, \ldots, m\} \), no vertex in \( K \setminus K_i \) is mixed on \( X_i \). This proves (3.3). Next, \( X_2 \) is anti-complete to \( X_1 \) and \( |X_2| \geq 2 \), and (3.4) follows. For (3.5), we note that since \((X, Y)\) satisfies (a) from the definition of a usable skew-partition, every vertex in \( Y \) has a neighbor in each of \( X_1, \ldots, X_m \). Now, fix an arbitrary \( i \in \{1, \ldots, m\} \). Then each vertex in \( K_i \) has a neighbor in each of \( X_1, \ldots, X_m \). Since the sets \( K_1, \ldots, K_m \) are pairwise disjoint, it follows that no vertex in \( K_i \) is mixed on any one of \( X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m \), and consequently, every vertex in \( K_i \) is complete to each of \( X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m \). Thus, \( K_i \) is complete to \( X \setminus (X_i \cup S) \), and (3.5) follows.

Suppose now that \((X, Y)\) satisfies (b) from the definition of a skew-partition, \( n \geq 2 \), and \( m \geq 1 \). Then (3.1) is immediate. The fact that the sets
$K_1, ..., K_m$ are non-empty follows from 3.10 and the fact that they are pairwise disjoint follows from the definition of a usable skew-partition; this proves (3.2). (3.3) follows from 3.9. For (3.4), fix $i \in \{1, ..., m\}$, and fix $x_i \in X_i$. Since $(X, Y)$ satisfies (a) from the definition of a usable skew-partition, we know that $x_i$ has a non-neighbor in each of $Y_1, ..., Y_n$; since $n \geq 2$, there exist $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $x_i$ is non-adjacent to both $y_1$ and $y_2$. Thus, $y_1$ and $y_2$ both have a non-neighbor (namely $x_i$) in $X_i$. By 3.9 neither $y_1$ nor $y_2$ is mixed on $X_i$; thus, $y_1$ and $y_2$ are both anti-complete to $X_1$. This proves (3.4). Finally, (3.5) follows from 3.10.

4 Split graph divide

Given a graph $G$ and pairwise disjoint (possibly empty) sets $A, B, C, L, T \subseteq V_G$, we say that $(A, B, C, L, T)$ is a split graph divide of $G$ provided that the following hold:

- $V_G = A \cup B \cup C \cup L \cup T$;
- $|A| \geq 2$;
- $|C| \geq 2$;
- $L$ is a non-empty clique;
- $T$ is a (possibly empty) stable set;
- every vertex of $L$ is mixed on $A$;
- $A$ is complete to $B$ and anti-complete to $C \cup T$;
- $L$ is complete to $B \cup C$;
- $T$ is anti-complete to $C$.

We say that a graph $G$ admits a split graph divide provided that there exist sets $A, B, C, L, T \subseteq V_G$ such that $(A, B, C, L, T)$ is a split graph divide of $G$.

Split graph divide can be thought of as a relaxation of the homogeneous set decomposition. A set $X \subseteq V_G$ is a homogeneous set in $G$ if no vertex in $V_G \setminus X$ is mixed on $X$. In the case of the split graph divide, there are vertices that are mixed on the set $A$, but they all lie in the clique $L$, and adjacency between $L$ and the rest of the graph is heavily restricted.
We now use \(3.11\) to prove another decomposition theorem for \(\{P_5, \overline{P_5}, C_5\}\)-free graphs, which is the main result of this section.

4.1. Let \(G\) be a \(\{P_5, \overline{P_5}, C_5\}\)-free graph. Then at least one of the following holds:

- \(G\) is a split graph;
- \(G\) contains a proper homogeneous set;
- at least one of \(G\) and \(\overline{G}\) admits a split graph divide.

Proof. We may assume that \(G\) is prime and that it is not a split graph, for otherwise we are done. Then by \(3.11\) \(G\) admits a skew-partition \((X, Y)\) with associated decomposition \((\{X_i\}_{i=1}^m, \{Y_j\}_{j=1}^n, S, K, \{S_j\}_{j=1}^n, \{K_i\}_{i=1}^m)\) that satisfies either (3.1)-(3.5) or (4.1)-(4.5) from \(3.11\). We claim that if \((X, Y)\) satisfies (3.1)-(3.5) from \(3.11\) then \(G\) admits a split graph divide, and if \((X, Y)\) satisfies (4.1)-(4.5) from \(3.11\), then \(\overline{G}\) admits a split graph divide. But it suffices to prove the first claim, for then the second will follow by an analogous argument applied to \(\overline{G}\).

So suppose that \((X, Y)\) satisfies (3.1)-(3.5) from \(3.11\). By (3.5) and by symmetry, we may assume that \(K_1\) is complete to \(X \setminus (X_1 \cup S)\), that is, that \(K_1\) is complete to \(X_2 \cup \ldots \cup X_m\). We now construct sets \(A, B, C, L, T\) as follows:

- let \(A = X_1\);
- let \(B\) be the set of all vertices in \(Y\) that are complete to \(X_1\);
- let \(C\) be the union of the following three sets:
  - \(X_2 \cup \ldots \cup X_m\),
  - the set of all vertices in \(Y\) that are anti-complete to \(X_1\),
  - the set of all vertices in \(S\) that are complete to \(K_1\);
- let \(L = K_1\);
- let \(T\) be the set of all vertices in \(S\) that have a non-neighbor in \(K_1\).

First, it is clear that the sets \(A, B, C, L, T\) are pairwise disjoint. Next, it is clear that \(X \cup K_1 \subseteq A \cup B \cup C \cup L \cup T\), and it is also clear that all vertices in \(Y\) that are not mixed on \(X_1\) lie in \(A \cup B \cup C \cup L \cup T\). But since by (3.3), no vertex in \(Y \setminus K_1\) is mixed on \(X_1\), it follows that \(Y \setminus K_1 \subseteq A \cup B \cup C \cup L \cup T\).
This proves that $V_G = A \cup B \cup C \cup L \cup T$. It is immediate by construction that $G[A]$ is connected and that $|A| \geq 2$. The fact that $|C| \geq 2$ follows from (3.4). By construction, $L$ is a clique, and by (3.2), $L$ is non-empty. By construction, $T$ is a stable set, every vertex of $L$ is mixed on $A$, and $A$ is complete to $B$ and anti-complete to $C \cup T$. It remains to prove that $L$ is complete to $B \cup C$, and that $T$ is anti-complete to $C$.

First, we know by construction that $K_1$ is complete to $Y \setminus K_1$. Thus, to show that $L$ is complete to $B \cup C$, we just need to show that $K_1$ is complete to $X_2 \cup \ldots \cup X_m$. But this follows from the choice of $K_1$.

It remains to show that $T$ is anti-complete to $C$. This means that we have to show that $T$ is anti-complete to each of the following three sets:

- $X_2 \cup \ldots \cup X_m$,
- the set of all vertices in $Y$ that are anti-complete to $X_1$,
- the set of all vertices in $S$ that are complete to $K_1$;

It is clear that $T$ is anti-complete to the first and the third set, and we just need to prove that $T$ is anti-complete to the second of the three sets above. Suppose otherwise. Fix adjacent $s \in T$ and $y \in Y$ such that $y$ is anti-complete to $X_1$. Since $s \in T$, we know that $s$ has a non-neighbor $k_1 \in K_1$. Since $k_1 \in K_1$, we know that $k_1$ is mixed on $X_1$. Since $G[X_1]$ is connected, we know by 3.1 that there exist adjacent vertices $x_1, x'_1 \in X_1$ such that $k_1$ is adjacent to $x_1$ and non-adjacent to $x'_1$. But now $s - y - k_1 - x_1 - x'_1$ is an induced four-edge path in $G$, which is impossible. Thus, $T$ is anti-complete to $C$. This completes the argument.

\[\Box\]

5 Split graph unification

In this section we define a composition operation that “reverses” the split graph divide decomposition. Let $A, B, C, L, T$ be pairwise disjoint sets, and assume that $A$ and $C$ are non-empty. Let $a, c$ be distinct vertices such that $a, c \notin A \cup B \cup C \cup L \cup T$. Let $G_1$ be a graph with vertex-set $V_{G_1} = A \cup L \cup T \cup \{c\}$, and adjacency as follows:

- $L$ is a (possibly empty) clique;
- $T$ is a (possibly empty) stable set;
• A is anti-complete to T;
• c is complete to L and anti-complete to $A \cup T$.

Let $G_2$ be a graph with vertex-set $V_{G_2} = B \cup C \cup L \cup T \cup \{a\}$, and adjacency as follows:
• $G_2[L \cup T] = G_1[L \cup T]$;
• T is anti-complete to C;
• $L$ is complete to $B \cup C$;
• $a$ is complete to $B$ and anti-complete to $C \cup L \cup T$.

Under these circumstances, we say that $(G_1, G_2)$ is a composable pair. The split graph unification of a composable pair $(G_1, G_2)$ is the graph $G$ with vertex-set $A \cup B \cup C \cup L \cup T$ such that:
• $G[A \cup L \cup T] = G_1 \setminus c$;
• $G[B \cup C \cup L \cup T] = G_2 \setminus a$;
• $A$ is complete to $B$ and anti-complete to $C$ in $G$.

Thus to obtain $G$ from $G_1$ and $G_2$, we “glue” $G_1$ and $G_2$ along their common induced subgraph $G_1[L \cup T] = G_2[L \cup T]$, and this induced subgraph is a split graph (hence the name of the operation).

We say that a graph $G$ is obtained by split graph unification from graphs with property $P$ provided that there exists a composable pair $(G_1, G_2)$ such that $G_1$ and $G_2$ both have property $P$, and $G$ is the split graph unification of $(G_1, G_2)$. We say that $G$ is obtained by split graph unification in the complement from graphs with property $P$ provided that $\overline{G}$ is obtained by split graph unification from graphs with property $P$.

We now prove that every graph that admits a split graph divide is obtained by split graph unification from smaller graphs.

5.1. If a graph admits a split graph divide, then it is obtained from a composable pair of smaller graphs by split graph unification.

Proof. Let $G$ be a graph that admits a split graph divide, and let $(A, B, C, L, T)$ be a split graph divide of $G$. Let $G_1$ be the graph obtained from $G[A \cup L \cup T]$
by adding a new vertex \( c \) complete to \( L \) and anticomplete to \( A \cup T \). Since \( |C| \geq 2 \), we know that \( |V_{G_1}| < |V_G| \). Let \( G_2 \) be obtained from \( G[B \cup C \cup L \cup T] \) by adding a new vertex \( a \) complete to \( B \) and anticomplete to \( C \cup L \cup T \). Since \( |A| \geq 2 \), we know that \( |V_{G_2}| < |V_G| \). Now \( (G_1, G_2) \) is a composable pair, and \( G \) is obtained from it by split graph unification. This completes the argument.

Note that in the proof of\(^{5.1} \) \( G_1 \) is obtained from \( G \) by first deleting \( B \), and then “shrinking” \( C \) to a vertex \( c \). On the other hand, \( G_2 \) is obtained from \( G \) by first deleting all the edges between \( A \) and \( L \) and then “shrinking” \( A \) to a vertex \( a \). Thus, \( G_1 \) is (isomorphic to) an induced subgraph of \( G \), but \( G_2 \) need not be.

Split graph unification can be thought of as generalized substitution. Indeed, we obtain the graph \( G \) from \( G_1 \) and \( G_2 \) by first substituting \( G_1[A] \) for \( a \) in \( G_2 \), and then “reconstructing” the adjacency between \( A \) and \( L \) in \( G \) using the adjacency between \( A \) and \( L \) in \( G_1 \). We include \( T \) and \( c \) in \( G_1 \) in order to ensure that split graph unification preserves the property of being \( \{P_5, \overline{P_5}, C_5\} \)-free. In fact, we prove something stronger than this: split graph unification preserves the (individual) properties of being \( P_5 \)-free, \( \overline{P_5} \)-free, and \( C_5 \)-free.

5.2. Let \( (G_1, G_2) \) be a composable pair, and let \( G \) be the split graph unification of \( (G_1, G_2) \). Then:

- if \( G_1 \) and \( G_2 \) are \( P_5 \)-free, then \( G \) is \( P_5 \)-free;
- if \( G_1 \) and \( G_2 \) are \( \overline{P_5} \)-free, then \( G \) is \( \overline{P_5} \)-free;
- if \( G_1 \) and \( G_2 \) are \( C_5 \)-free, then \( G \) is \( C_5 \)-free.

Proof. Let \( H \in \{P_5, \overline{P_5}, C_5\} \), and suppose that \( G_1 \) and \( G_2 \) are \( H \)-free. We need to show that \( G \) is \( H \)-free. Suppose otherwise. Fix \( W \subseteq V_G \) such that \( G[W] \cong H \). Let \( A, B, C, L, T, a, c \) be as in the definition of a composable pair. Since \( H \) is prime, the class of \( H \)-free graphs is closed under substitution. Now, since \( G_1 \) and \( G_2 \) are \( H \)-free, and since \( G \setminus B \) is obtained by substituting \( G_2[C] \) for \( c \) in \( G_1 \), we know that \( G \setminus B \) is \( H \)-free. Thus, \( W \cap B \neq \emptyset \). Next, since \( G \setminus A \) is an induced subgraph of \( G_2 \), and \( G_2 \) is \( H \)-free, we know that \( W \cap A \neq \emptyset \). Since \( G_1 \) and \( G_2 \) are \( H \)-free, and \( G \setminus L \) is obtained by substituting \( G_1[A] \) for \( a \) in \( G_2 \setminus L \), we know that \( W \cap L \neq \emptyset \). Since \( B \) is complete to \( A \cup L \) in \( G \), and since \( W \) intersects both \( B \) and \( A \cup L \) in \( G \), we know that \( G[W \cap (A \cup B \cup L)] \) is not anti-connected. Since \( H \) is
anti-connected, we know that $W \not\subseteq A \cup B \cup L$; thus, $W \cap (C \cup T) \neq \emptyset$. Since $W$ intersects each of $A$, $B$, $L$, and $C \cup T$, and since $|W| = 5$, we know that $1 \leq |W \cap A| \leq 2$.

Suppose first that $|W \cap A| = 2$; set $W \cap A = \{a_1, a_2\}$. Since $W$ intersects each of $B$, $L$, and $C \cup T$, and since $|W| = 5$, it follows that $|W \cap B| = |W \cap L| = |W \cap (C \cup T)| = 1$. Set $W \cap B = \{b\}$, $W \cap L = \{l\}$, and $W \cap (C \cup T) = \{w\}$. Then $W = \{a_1, a_2, b, l, w\}$. Since $G[W]$ is prime, $\{a_1, a_2\}$ cannot be a proper homogeneous set in $G[W]$; since $b$ is complete to $A$ and $w$ is anti-complete to $A$ in $G$, we know that $l$ is mixed on $\{a_1, a_2\}$. By symmetry, we may assume that $l$ is adjacent to $a_1$ and non-adjacent to $a_2$. Then $\{a_1, b, l\}$ is a triangle in $G[W]$, and consequently, $H$ is a house. Thus, $a_2$ must have at least two neighbors in $G[W]$. Since $a_2$ is non-adjacent to $l$ (by assumption) and to $w$ (because $A$ is anti-complete to $C \cup T$), it follows that $a_2$ is adjacent to $a_1$ and $b$. But now $\{a_1, a_2, b\}$ and $\{a_1, b, l\}$ are two distinct triangles in the house $G[W]$, contrary to the fact that a house has only one triangle.

It remains to consider the case when $|W \cap A| = 1$. Set $W \cap A = \{\hat{a}\}$. If $\hat{a}$ is anti-complete to $W \cap L$, then $G[W]$ is an induced subgraph of $G_2$, contrary to the fact that $G_2$ is $H$-free. Thus, $\hat{a}$ has a neighbor $l \in L$. Since $W \cap B \neq \emptyset$, there exists some $b \in W \cap B$. Since $B$ is complete to $A \cup L$ in $G$, we know that $\{\hat{a}, b, l\}$ is a triangle in $G[W]$, and consequently, $H$ is a house. Now, suppose that $|W \cap B| \geq 2$, and fix some $b' \in (W \cap B) \setminus \{b\}$. But then $\{\hat{a}, b, l\}$ and $\{\hat{a}, b', l\}$ are distinct triangles in the house $G[W]$, contrary to the fact that a house contains only one triangle. Thus, $W \cap B = \{b\}$. Next, suppose that $|W \cap L| \geq 2$, and fix some $l' \in W \cap L$. But then since $L$ is a clique in $G$, and since $B$ is complete to $L$ in $G$, we know that $\{b, l, l'\}$ is a triangle in $G[W]$, and so $G[W]$ contains at least two triangles (namely $\{\hat{a}, b, l\}$ and $\{b, l, l'\}$), contrary to the fact that $G[W]$ is a house. Thus, $W \cap L = \{l\}$. It now follows that $|W \cap (C \cup T)| = 2$; set $W \cap (C \cup T) = \{c_1, c_2\}$. Since $\{\hat{a}, b, l\}$ is the unique triangle of the house $G[W]$, we know that $c_1, c_2$ is an edge; since $T$ is a stable set in $G$, and since $C$ is anti-complete to $T$ in $G$, this implies that $c_1, c_2 \in C$. Since $c_1c_2$ is an edge in $G[W]$, and since $C$ is complete to $L$ in $G$, we know that $\{c_1, c_2, l\}$ is a triangle in $G[W]$. But now the house $G[W]$ contains two distinct triangles (namely $\{\hat{a}, b, l\}$ and $\{c_1, c_2, l\}$), which is impossible. This completes the argument.

We now prove a partial converse of 5.2.
5.3. Let $(G_1, G_2)$ be a composable pair, let $A, B, C, L, T, a, c$ be as in the definition of a composable pair, and let $G$ be the split graph unification of $(G_1, G_2)$. Then:

- if $G$ is $P_5$-free, then $G_1$ and $G_2$ are $P_5$-free;
- if $G$ is $\overline{P_5}$-free, and every vertex of $L$ has a non-neighbor in $A$ in $G$, then $G_1$ and $G_2$ are $\overline{P_5}$-free;
- if $G$ is $C_5$-free, then $G_1$ and $G_2$ are $C_5$-free.

Proof. Let $H \in \{P_5, \overline{P_5}, C_5\}$, and suppose that $G$ is $H$-free. If $H = \overline{P_5}$, we additionally assume that every vertex of $L$ has a non-neighbor in $A$ in $G$ (and consequently, in $G_1$ as well). We need to show that $G_1$ and $G_2$ are both $H$-free. Clearly, $G_1$ is an induced subgraph of $G$, and consequently, $G_1$ is $H$-free. It remains to show that $G_2$ is $H$-free. Suppose otherwise. Fix some $W \subseteq V_{G_2}$ such that $G[W] \cong H$.

First, we claim that $a \in W$, and that $W$ intersects each of $B$, $L$, and $C \cup T$. Since $G_2 \setminus a$ is an induced subgraph of $G$, and $G$ is $H$-free, we know that $a \in W$. Next, since $|W| = 5$, since $G[W]$ is connected, since $a \in W$, and since all neighbors of $a$ in $G_2$ lie in $B$, we know that $W \cap B \neq \emptyset$. Since $G_2 \setminus L$ is (isomorphic to) an induced subgraph of $G$, we know that $W \cap L \neq \emptyset$. Since $\{a\} \cup L$ is complete to $B$ in $G_2$, and since $W$ intersects both $\{a\} \cup L$ and $B$, we know that $G[W \cap (\{a\} \cup B \cup L)]$ is not anti-connected. Since $G[W]$ is anti-connected, it follows that $W \not\subseteq \{a\} \cup B \cup L$. Consequently, $W \cap (C \cup T) \neq \emptyset$. This proves our claim.

Now, we deal with the following two cases separately: when $H = \overline{P_5}$, and when $H \in \{P_5, C_5\}$.

Suppose first that $H = \overline{P_5}$. Then by assumption, every vertex in $L$ has a non-neighbor in $A$ in $G$, and it follows that for all $l \in L$, $G_2 \setminus (L \setminus \{l\})$ is (isomorphic to) an induced subgraph of $G$. Thus, $|W \cap L| \geq 2$. Since $|W| = 5$, it follows that $a \in W$, $|W \cap L| = 2$, and $|W \cap B| = |W \cap (C \cup T)| = 1$. Since all neighbors of $a$ in $G_2$ lie in $B$, and since $|W \cap B| = 1$, it follows that $a$ has at most one neighbor in $G[W]$. But this is impossible because $G[W]$ is a house, and every vertex of a house is of degree at least two.

It remains to consider the case when $H \in \{P_5, C_5\}$. Fix $b \in W \cap B$ and $l \in W \cap L$. Since $a$ is complete to $B$ and anti-complete to $L$ in $G_2$, and since
$B$ is complete to $L$ in $G_2$, we know that $a - b - l$ is an induced path in $G_2[W]$.

We claim that $W \cap B = \{b\}$. Suppose otherwise; fix $b' \in (W \cap B) \setminus \{b\}$. Then $a - b - l - b' - a$ is a (not necessarily induced) square in $G[W]$, which is impossible because $G[W]$ is either a four-edge path or a pentagon. Thus, $W \cap B = \{b\}$. Next, we claim that $W \cap L = \{l\}$. Suppose otherwise; fix $l' \in (W \cap L) \setminus \{l\}$. Since $L$ is a clique in $G_2$, and since $B$ is complete to $L$ in $G_2$, it follows that $\{b, l, l'\}$ is a triangle in $G[W]$. But this is impossible since $G[W]$ is either a four-edge path or a pentagon. This proves that $W \cap L = \{l\}$.

Since all neighbors of $a$ in $G_2$ lie in $B$, and since $W \cap B = \{b\}$, we know that $a$ is of degree at most one in $G[W]$. Consequently, $G[W]$ is a four-edge path; in fact, the four-edge path $G[W]$ must be of the form $a - b - l - c_1 - c_2$, where $c_1, c_2 \in W \cap (C \cup T)$. Since $c_1c_2$ is an edge, and since $T$ is a stable set that is anti-complete to $C$ in $G_2$, it follows that $c_1, c_2 \in C$. But $C$ is complete to $L$ in $G_2$, and so $\{c_1, c_2, l\}$ is a triangle in $G_2[W]$, which is impossible since $G_2[W]$ is a four-edge path. This completes the argument.

We remark that the additional assumption in the second statement of 5.3 is needed because of the following example. Let $G_1$ be a path $a_1 - l - c$, and let $G_2$ be a house $b_1 - b_2 - c_1 - a - l$. Set $A = \{a_1\}$, $B = \{b_1, b_2\}$, $C = \{c_1\}$, $L = \{l\}$, and $T = \emptyset$. With this setup, $(G_1, G_2)$ is easily seen to be a composable pair. Let $G$ be the split graph unification of $(G_1, G_2)$. It is easy to check that $G$ is $P_5$-free, even though $G_2$ is a house.

We complete this section with a strengthening of 5.1 which we will need in section 6.

5.4. Let $H \in \{P_5, \overline{P_5}, C_5\}$, and let $G$ be an $H$-free graph that admits a split graph divide. Then $G$ is obtained from a composable pair of smaller $H$-free graphs by split graph unification.

Proof. Let $(A, B, C, L, T)$ be a split graph divide of $G$, and let $G_1$ and $G_2$ be constructed as in the proof of 5.1. As shown in the proof of 5.1, $G_1$ and $G_2$ are both smaller than $G$. $(G_1, G_2)$ is a composable pair, and $G$ is the split graph unification of $(G_1, G_2)$. Further, since $(A, B, C, D, L, T)$ is a split graph divide of $G$, we know that every vertex in $L$ is mixed on $A$ in $G$, and in particular, that every vertex of $L$ has a non-neighbor in $A$ in $G$. By 5.3 then, $G_1$ and $G_2$ are both $H$-free. \qed
6 The main theorem

In this section, we use 1.1 and the results of sections 4 and 5 to prove 6.1, the main theorem of this paper.

6.1. A graph $G$ is $\{P_5, \overline{P}_5\}$-free if and only if at least one of the following holds:

- $G$ is a split graph;
- $G$ is a pentagon;
- $G$ is obtained by substitution from smaller $\{P_5, \overline{P}_5\}$-free graphs;
- $G$ or $\overline{G}$ is obtained by split graph unification from smaller $\{P_5, \overline{P}_5\}$-free graphs.

Proof. We first prove the “if” part. If $G$ is a split graph or a pentagon, then it is clear that $G$ is $\{P_5, \overline{P}_5\}$-free. Since both $P_5$ and $\overline{P}_5$ are prime, we know that the class of $\{P_5, \overline{P}_5\}$-free graphs is closed under substitution, and consequently, any graph obtained by substitution from smaller $\{P_5, \overline{P}_5\}$-free graphs is $\{P_5, \overline{P}_5\}$-free. Finally, if $G$ or $\overline{G}$ is obtained by split graph unification from smaller $\{P_5, \overline{P}_5\}$-free graphs, then the fact that $G$ is $\{P_5, \overline{P}_5\}$-free follows from 5.2 and from the fact that the complement of a $\{P_5, \overline{P}_5\}$-free graph is again $\{P_5, \overline{P}_5\}$-free.

For the “only if” part, suppose that $G$ is a $\{P_5, \overline{P}_5\}$-free graph. We may assume that $G$ is prime, for otherwise, $G$ is obtained by substitution from smaller $\{P_5, \overline{P}_5\}$-free graphs, and we are done. If some induced subgraph of $G$ is isomorphic to the pentagon, then by 1.1 $G$ is a pentagon, and again we are done. Thus we may assume that $G$ is $\{P_5, \overline{P}_5, C_5\}$-free. By 4.1, we know that either $G$ is a split graph, or one of $G$ and $\overline{G}$ admits a split graph divide. In the former case, we are done. In the latter case, 5.4 implies that $G$ or $\overline{G}$ is the split graph unification of a composable pair of smaller $\{P_5, \overline{P}_5, C_5\}$-free graphs, and again we are done. \qed

As an immediate corollary of 6.1, we have the following.

6.2. A graph is $\{P_5, \overline{P}_5\}$-free if and only if it is obtained from pentagons and split graphs by repeated substitutions, split graph unifications, and split graph unifications in the complement.

Finally, a proof analogous to the proof of 6.1 (but without the use of 1.1) yields the following result for $\{P_5, \overline{P}_5, C_5\}$-free graphs.
6.3. A graph $G$ is $\{P_5, \overline{P_5}, C_5\}$-free if and only if at least one of the following holds:

- $G$ is a split graph;
- $G$ is obtained by substitution from smaller $\{P_5, \overline{P_5}, C_5\}$-free graphs;
- $G$ or $\overline{G}$ is obtained by split graph unification from smaller $\{P_5, \overline{P_5}, C_5\}$-free graphs.

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