The Yang monopole in IIA superstring: multi-charge disease and enhançon cure

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Received 4 August 2011, in final form 23 November 2011
Published 23 December 2011
Online at stacks.iop.org/JPhysA/45/045401

Abstract

A brane picture in type IIA superstring for the Yang monopole is reconsidered. It makes use of D2 and D4-branes wrapped on cycles in the K3 surface. When the model was first presented, some problems concerning the charges of the monopoles arose. In this paper, they are shown to be cured by the model itself. Surprisingly, the incompatibility between the multi-charge configuration and the spherical symmetry of the Yang monopole is seen in the brane description as the emergence of the enhançon shell and the fuzzy geometry. This consistency is deep and surprising, and is the point that triggered this work. It nontrivially relates a purely geometrical problem in ordinary spacetime with the emergence of noncommutative geometries. Besides, this paper includes an extended model for SO(4)-monopoles and a T-dual model in type IIB superstring.

PACS numbers: 11.25.−w, 04.60.Cf, 14.80.Hv

1. Introduction

Based on the idea that string theories at low energies must be well approached by supergravity, we have been searching for a brane model in type IIA string theory of the Yang monopole [1]. In the brane picture shown in [1], most of the features of the Yang construction were successfully reproduced, namely the charges, the non-Abelian field and the point-like behavior of Yang configurations. Some corners of the brane model remained unanswered; however, questions whose investigation has revealed some unexpected deep connections between gauge theory and brane physics have finally led to this paper.

The Yang monopole [2] was constructed as a non-Abelian generalization of the Dirac monopole [3]. The Yang configuration is characterized by the flux of the four-form field \( \text{Tr} F \wedge F \), charged under the \( SU(2) \) gauge group, across the four-dimensional sphere that
covers the origin in a \((5 + 1)\)-dimensional spacetime. It corresponds to the conformal mapping into \(S^4\) of the BPST Euclidean instanton solution \([4]\). Again, the origin is singular but now the energy of this solitonic configuration is well behaved in the UV regime, although IR divergences linearly appear. The total energy inside a four-sphere is proportional to its radius. In this realization, the flux is quantized but now the magnetic charge of the Yang monopole can take only two values \(\{+1, -1\}\) \([2]\). This charge, which may correspond to the self-dual and anti-self-dual BPST instanton configurations, respectively, is given by the integral over \(S^4\) of the second Chern class \(\text{Tr}(F \wedge F)/8\pi^2\). The Yang monopole can be easily generalized to higher even-dimensional spacetime where the configuration is characterized by the \(2n\)-form \(F^n\). Explicit solutions can be systematically obtained in \([5–8]\).

We would like to stress at this point the well-known fact that the Yang monopole, due to its spherical symmetry, can carry just two charges \([2]\). This is a purely gauge requirement in the sense that unlike the fuzzy backgrounds which emerge in the brane picture, the charge analysis from the gauge theory point of view concerns topologically nontrivial configurations of gauge fields in an ordinary four-dimensional sphere.

In the brane model presented in \([1]\), the two-charge nature of the Yang monopole seemed to hold nicely as associated with the two ways a single D4-brane can wrap a 2-cycle of the K3 surface. Specifically, the construction given in \([1]\) needs D2- and D4-branes on the K3 surface. In this brane picture, the \(SU(2)\) gauge group of the Yang construction is engineered by means of a D2-brane wrapping shrinking 2-cycles inside the K3 surface. The Yang monopole comes up when a D4-brane wraps the whole K3 and dimensional reduction to six spacetime dimensions is performed. In this way, the Yang properties mentioned above are encoded in the K3 surface features. However, a careful look reveals a mechanism that allows us to obtain a multi-charge spherically symmetric configuration. Basically, the mechanism consists of adding subsequent D4-branes to the setup at no cost of energy since the branes are taken to be BPS states. The addition of \(N\) D4-branes would increase the charge by \(N\) units without losing spherical symmetry since the branes are point-like in 5+1 dimensions. It would lead, as said, to a multi-charge spherically symmetric configuration.

At this stage, the brane setup would be ruined as a model for the Yang monopole given that, as said before, no spherical \(SU(2)\) configuration over \(S^4\) can be multi-charged. However, a deeper investigation of the brane physics reveals the appearance of an enhàçon shell \([10–17]\) and a fuzzy geometry at the core of the monopole for the multi-charge case. Fuzzy geometry explicitly breaks spherical symmetry. So the two-charge property of the spherically symmetric gauge theory configuration is restored. It is surprising because spherical symmetry is broken in the brane picture by a non-ordinary spacetime background: the (noncommutative) fuzzy 4-sphere that the enhàçon mechanism brings aside. In this highly nontrivial way, which involves fuzzy geometries, a gauge requirement for the two allowed charges of the Yang configuration is recovered in the brane picture.

Apart from the discussion of the multicharge configurations and the broken spherical symmetry, this paper presents a new brane model for \(SO(4)\)-monopoles in six dimensions. The construction is inspired in the model presented in \([1]\). It is, say, its natural extension. The number of charges labeling the topologically different brane configurations in this case is 4. Needless to say that the same argument used for the Yang monopole holds for the \(SO(4)\) construction and also prevents it from having an infinite tower of charges without breaking spherical symmetry.

The organization of this paper is as follows. In section 2, we will review the model proposed in \([1]\) and find a useful T-dual version of the type IIA model in type IIB superstring. In section 3, we will extend the logic of the model for the case of \(SO(4)\)-monopoles. Section 4 deals with the core of the discussion. It is devoted to the discussion of the main
objections concerning the multiple charge configurations the model presented at first sight. We will offer, to these objections, a solution coming from the model itself by a purely stringy effect: the enhanc¸on mechanism and the fuzzy geometry which comes along with it, both intriguing phenomena which are better visualized in the T-dual model. Finally, a brief conclusion sums up the main points of the paper and brings some open questions.

2. Type IIA superstring construction of the Yang monopole

The idea of finding a brane picture for Yang monopoles is not new. Before [1], we suggested the possibility of considering the Yang monopole in M-theory [18]. In this regard, a heterotic M-theory realization was soon proposed [19]. In particular, the authors of [19] show that M5-brane may have boundaries on M9-branes, where the boundary is a D4-brane with an infinite tension so its center of mass is not free to move. The boundaries may be identified with Yang monopoles. Indeed, in this heterotic M-theory picture, there are two Yang monopoles which correspond to the ends of the oriented M5-brane which stretches between two M9-branes. Each monopole (each end) is charged under an $SU(2)$ subgroup of $E_8$ with the topological charges $\{+1, -1\}$, respectively. Using string/string duality and the result of [19], we give a string realization of the Yang monopole for a six-dimensional type IIA superstring obtained from the compactification on a local description of the K3 surface in the presence of wrapped D-branes. The relation between our model and the M-theory model is discussed in detail in [1].

2.1. Superstringy construction of the Yang monopole

Consider a local description of the K3 surface where the manifold develops an $SU(2)$ singularity (known as the $A_1$ singularity). This singularity corresponds to a vanishing two-sphere. Near such a singular point, the K3 surface can be identified with the asymptotically locally Euclidean (ALE) complex space which is algebraically given by

$$f(x, y, z) = xy - z^2 = 0,$$

which is singular at $x = y = z = 0$. In the two-dimensional $N = 2$ linear sigma model with only one $U(1)$ gauge symmetry, the resolution of this singularity is related to the D-term described by the following bosonic potential $V(\phi_1, \phi_2, \phi_3)$:

$$V(\phi_1, \phi_2, \phi_3) = (|\phi_1|^2 - 2|\phi_2|^2 + |\phi_3|^2) - R^2,$$

where $R$ is the $U(1)$ Fayet–Iliopoulos (FI) parameter [21]. Geometrically, this corresponds to replacing the singular point $x = y = z = 0$ by the 2-sphere $S^2$ defined by $V = 0$, which is the only non-trivial 2-cycle on which we can wrap D2-branes. In order to geometrically engineer the $SU(2)$ gauge symmetry, only the compact piece containing the $S^2$ is necessary [22]. Now the system consists of type IIA D2-branes wrapping around $S^2$. This gives a pair of massive vectors $W^\pm$, one for each of the two possible ways of wrapping. The masses of these particles are proportional to the volume of the 2-sphere. They are charged under the $SU(2)$ gauge field obtained by decomposing the type IIA 3-form in terms of the harmonic 2-form on the 2-sphere and the 1-form gauge field in six-dimensional spacetime. In the limit where the 2-sphere shrinks, the $W^\pm$ particles become massless and, together with the 1-form

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6 Independently, a matrix model of the Yang monopole was given in [20].
7 For the ALE geometry $A_n$, the gauge group is $U(1)^n$.
8 In this way, one sees that the $U(1)$ Cartan subgroup of the $SU(2)$ symmetry of the singularity of K3 carries the gauge symmetry of the $N = 2$ supersymmetric linear sigma model.
gauge field, generates the $SU(2)$ adjoint representation. This will be identified with the gauge
symmetry of our Yang monopole. We have obtained the electrically charged sector, associated
with D2-branes wrapping two vanishing cycles in the K3 surface. Lifting consistently to 11
dimensions, the M2-brane is encountered. The magnetic Yang monopoles can be identified
with D4-branes, totally wrapped on the K3 surface. As a consequence, they generate the
magnetic objects in the six-dimensional spacetime. This is expected from the fact that the
D4-brane is the only magnetic object in type IIA superstring theory which can be obtained
from the M5-brane and gives a point-like particle after wrapping the K3 surface. Owing to the
spherical symmetry of the six-dimensional configuration and the fact that, as seen above, the
gauge group origin is linked to the singular limit of the geometry, we strongly believe that all
Yang monopole properties should be derived from the K3 surface data.

Schematically, the ten-dimensional spacetime where type IIA lives is occupied as follows.
If we consider that the K3 surface extends along dimensions 6789, and the vanishing 2-cycle
of K3 at 67 positions, then the D2-brane is at 067 and the D4-brane at 06789.

We will show that the charges $\{+1, -1\}$ can have different compatible K3 surface
interpretations. First, there are different ways in which D4-branes are wrapped on the K3
surface. They are classified by their fourth homotopy group. As seen before, in order to
construct the $SU(2)$ gauge group, it is necessary to work with a local model of K3 with a
singularity $A_1$. The deformed geometry is given by the product of the complex $C$ plane and
a two-sphere $S^2$. Since $\Pi_q(X \times Y) = \Pi_q(X) \times \Pi_q(Y)$, we have the following remarkable
relation:

$$\pi_4(A_1) \sim \pi_4(S^2) = Z_2. \quad (2.3)$$

The two charges of the Yang monopole are related to the two ways the geometry allows a
D4-brane to wrap on it.\(^9\)

It is known that the energy of the Yang monopole diverges linearly in spacetime. This fact
is not manifest in this geometric construction. However, the divergence in the energy can be
intuitively seen in the T-dual model (see section 2.3), where the D4-brane turns into an infinite
effective D1-brane which pulls at a pair of coincident NS5-branes.

2.2. Six-dimensional effective field theory

The six-dimensional field theory that remains after compactification of the IIA supergravity
and ignoring all massive Kaluza–Klein modes can be consistently truncated to

$$S = \int \sqrt{-g}(R - (\partial \sigma)^2 - e^{-\sigma} \text{tr} |F|^2), \quad (2.4)$$

where $\sigma$ is the 6D dilaton and the trace is taken over the color indices of the $SU(2)$ gauge
field. Monopole solutions from (2.4) are obtained with a spherically symmetric ansatz for the
metric:

$$ds^2 = -e^{\lambda r} \Delta(r) \, dt^2 + dr^2 / \Delta(r) + r^2 d\Omega^2_4, \quad (2.5)$$
in terms of functions $\lambda$ and $\Delta$, and $d\Omega^2_4$ which is the invariant metric over the 4-sphere, together
with the Yang monopole field strength and radial ansatz $\sigma(r)$ for the dilaton.

The dilaton cannot be consistently set to a constant and then eliminated from the action.
So the solutions which minimize action (2.4) are in principle different from those of a pure

\(^9\) This last statement has left some room for controversy. After reading our paper [1], David Tong suggested that there
should be five charges corresponding to the two ways the D4-brane and the D4-antibrane wrap the $A_1$ manifold plus
the trivial one (zero charge). We believe that David’s idea is right for the case of extended Yang monopoles, the ones
with gauge group $SO(4)$; we will go further on this point in section 3.
Yang–Mills theory to which the Yang monopole belongs. However, because the Yang–Mills field strength 2-form for the Yang monopole solution has components only on the 4-sphere, it continues to minimize action (2.4) and solve the Yang–Mills equations as modified by the dilaton. This is the reason why we still keep the name.

2.3. Dual model in type IIB superstring

As we will see in the following, the appearance of the enhancion mechanism and the fuzzy geometry in the realization of the Yang monopole we describe in this paper is better visualized in a dual model on the type IIB side. In [1] we related our D2–D4 system on the K3 setup with the M-theory (and heterotic string theory) model proposed in [7] by the S-duality. As already noted in [10], a configuration with $N$ $D(p + 1)$-branes stretched between two NS5-branes is $T$-dual to the same number of $D(p + 2)$-branes wrapped on the two-cycle of an $A_1$ ALE space, giving rise to $N$ effective $D_p$-branes. This means that the D2–D4 brane on a vanishing 2-cycle of the K3 setup is dual to a D1-brane and a D3-brane stretched between two approaching NS5-branes, the vanishing limit of the IIA picture corresponding to the coalescence of the NS5-branes. The spatial position of the branes in the IIB picture can be described as follows. Let us state that both NS5-branes occupy dimensions 012345. The D1-brane extends along a transversal dimension (say, 6) and has an end on each NS5-brane. The D3-brane, which is actually supporting the magnetic charge of the configuration, shares the last direction plus two transversal dimensions more, which will be named 78. These two new directions correspond to the two remaining compact dimensions of K3 that do not belong to the vanishing 2-cycle.

Note that the D3-brane is effectively a D1-brane with an infinite mass, given that the two remaining directions, parallel to the NS5-brane, are infinite. So in the limit where the two NS5 coincide, the D3-brane must be interpreted as an infinite D1-brane pulling at both NS5-branes. Thus, the D3-brane is seen as a point from the NS5-brane, in a way that the nontrivial magnetic configuration it carries is point-like in the (5+1)-dimensional brane world. The fact that the effective D1-brane has infinite mass is the reason why the Yang monopole in our brane picture has infinite energy, in agreement with the well-known gauge computation. The other transverse D1-brane, whose size goes to zero, produces enhancement of the gauge symmetry from $U(1)$ to $SU(2)$.

3. $SO(4)$-monopoles

Another D-brane construction can be achieved if we consider an $SO(4)$ gauge group instead of $SU(2)$ on $S^4 \subset R^{5,1}$. This object will be called the extended-Yang monopole [9]. The procedure is similar to the one shown in section 2.1. Before entering into details, let us first note that the existence of two charges in the usual Yang construction is not obvious from the brane picture. D4-branes and D4-antibranes can each wrap a 2-cycle in two different ways. The number of charges would in principle be 4, each one labeling a possible realization. This is not the case as we will see. Let us call

$$\lambda_i : S^2 \rightarrow S^4 \quad i = 1, 2$$

the two homotopically inequivalent maps of (2.3), corresponding to the two ways a D4-brane wraps a 2-sphere, and $\tilde{\lambda}_i$ the homologous map for a D4-antibrane. The point is that there exists a homotopic deformation which makes $\lambda_1 = \tilde{\lambda}_2$ and $\lambda_2 = \tilde{\lambda}_1$, leaving us with just two (plus the trivial) homotopically inequivalent maps and, consequently, two charges. Roughly speaking, this means that the ‘one-way’ wrap of the D4-brane is actually identified with the ‘other-way’ wrap of the D4-antibrane.
Figure 1. The four ways a D4-brane can wrap two distinguishable vanishing 2-cycles of the K3 surface. The configurations are homotopically different and so the number of charges is 4. Note that at the singular limit (right) the two-dimensional part of the D4-brane connecting the two shrinking cycles vanishes into a string and just the two wrapped cycles remain.

Things are different if we consider an extended-Yang monopole. As shown in [9], imposing spherical symmetry on an $SO(4)$ bundle over $R^7 - 0$ leaves four homotopically different possibilities. So the number of charges is 4 in this case. This is intimately related to the fact that the algebra of $SO(4)$ is isomorphic to the Cartesian product of two copies of $su(2)$, as can be visualized in its Dynkin diagram. In the brane picture, the isomorphism of the algebras together with the geometric engineering mechanism we have used in subsection 2.1, suggests that the construction of the $SO(4)$-monopole involves two vanishing 2-cycles on K3. Recall that in the ALE space, each vanishing cycle is an $A_1$ singularity where the D2-brane is wrapped. Now, two D2-branes wrap a shrinking cycle each and geometrically engineer an $su(2)$ factor. The singularities are well separated and disconnected. They are distinguishable. Now, as in the Yang case, we wrap them with a D4-brane (see figure 1), which also wraps the rest of the K3 surface. As before, the ‘one-way’ wrap of the D4-brane is identified with the ‘other-way’ wrap of the D4-antibrane, so only branes are considered. Now, the fact that $A_1$ singularities are distinguishable raises the number of possible inequivalent configurations to 4. The homotopy group which labels the homotopically inequivalent maps is now

$$\pi_4(A_1 \times A_1) = \pi_4(A_1) \times \pi_4(A_1) = \mathbb{Z}_2 \times \mathbb{Z}_2,$$

corresponding to the four charges for the extended-Yang monopole, in agreement with [9].

4. The multi-charge disease and the enhançon cure

As mentioned in the introduction, there is an apparent inconsistency in the D2–D4 system on the K3 model we have just considered and which we claim as the Yang monopole. It precisely concerns its charges. In this regard, it has been objected that the model was not by itself capable of explaining the two-charge quality of the Yang monopole as opposed to the $Z$-tower of charges that characterizes Dirac monopoles.

Let us go a little deeper into the problem and explain how the model cures itself. Consider a D4-brane wrapped on the K3 surface and located at $r = 0$ in a polar coordination of the six-dimensional spacetime. Being BPS states, there is no reason against the addition of a second, a third or subsequent D4-brane superpositions. It results in a pile of an arbitrary

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10 Strictly speaking, they would actually be at infinite distance in the ALE space.
11 Thank, for example, that K3 occupies dimensions 6789. The 2-cycles may both be placed at 67 and the D4-brane wraps it all.
12 We thank P K Townsend for this remark.
number of D4-branes wrapped on the K3 surface at the same location, say \( r = 0 \). Each brane accounts for a unit of magnetic charge, so the total charge is basically the number of D4-branes which is, in principle, arbitrary. Besides, the configuration manifestly preserves spherical symmetry from the six-dimensional spacetime point of view since all the branes are located at the same point. In essence, this simple mechanism allows us to construct a monopole with arbitrary charge from our model which, for this reason, could no longer be claimed as the Yang monopole. Is there any way out?

Before answering the question, let us analyze the argument a little deeper. In the original paper \[2\], Yang did not merely construct the field strengths for the monopoles but also, and by means of the construction itself, proved that (up to isomorphism) there are only two nontrivial static spherically symmetric bundles of \( SU(2) \) over \( R^{5,1} - \{0\} \). They are respectively labeled by \( \{+1, -1\} \) charges\(^{13}\). Now, it seems inappropriate to claim the model as the Yang monopole when its internal logic (the brane superposition) allows us to add arbitrary charge. The field configurations they correspond to are either not \( SU(2) \)-bundles or they are simply not spherically symmetric. However, both properties have been explicitly and intentionally imposed in the construction of our brane model. That is the problem.

The enhançon mechanism (or more precisely, the fuzzy geometry it involves for multicharge configurations), by means of which the second and subsequent branes feel a repulsive interaction with respect to the first one and cannot reach the origin but smear onto the enhançon locus\(^{14}\) at radius \( r_e \), comes as a surprising solution to our problem. It is also a nontrivial proof of consistency of the brane model we propose for the Yang monopole, a pure stringy mechanism that comes for the brane picture to agree with a purely gauge theory requirement.

4.1. The enhançon

In the picture shown in section 2 of the D2–D4 system on the K3 model we did not make any assumption about the energy of the brane states. It was not necessary since our discussion was purely geometrical and did not need the description of an explicit six-dimensional effective field theory. In the following discussion, however, it becomes essential for the branes to be BPS states.

When originally proposed \[10\], the authors realized that their new mechanism (the enhançon) could resolve a naked timelike singularity produced by a Dp-brane wrapped in a 2-cycle of the K3 surface, which was being called a repulson\(^{15}\). In order to investigate it, they used the ten-dimensional supergravity of the system D2–D6 on the K3 model although they showed that the same conclusions hold for any Dp–D(p + 4) on the K3 model\(^{16}\). The geometric locus of the enhançon is independent of the model. Let us see how the enhançon comes up in our model for the case \( p = 0 \).

We will use the D0/D4-branes on the K3 surface, that is, without including D2-branes, so that the gauge theory is Abelian in the low energy approximation that follows. D2-branes are not relevant for the phenomenon we are describing; they do not ‘see’ the enhançon shell. This fact allows us to simplify the computation but still trust the result as extended for the full model.

\(^{13}\) A systematic analysis of the possible static spherically symmetric \( SO(2n) \)-bundles over \( R^{5+1,1} - \{0\} \), and consequently of the charges that label them, is carried out in \[9\].

\(^{14}\) The enhançon is, by definition, the locus of points where a probe brane is tensionless in certain brane configurations as it tries to reach the origin.

\(^{15}\) In the geometry of a repulson, a massive particle would naively feel a repulsive gravitational force by a potential which becomes infinite as the particle approaches a point at finite distance from the physical location of the branes.

\(^{16}\) Dp-branes are in the solution for consistency since even if one does not put them in by hand they virtually appear.
The simplest static supergravity solution consistently truncated to its bosonic part can be written as
\[ ds^2 = -Z_0^{-1/2} Z_4^{-1/2} dr^2 + Z_0^{1/2} Z_4^{1/2} (dr^2 + r^2 d\Omega_4^2) + V^{1/2} Z_0^{-1/2} Z_4^{-1/2} d\Omega_{K3}^2, \]
\[ e^{2\phi} = g_s Z_0^{3/2} Z_4^{-1/2}, \]
\[ C_1 = (Z_0 g_s)^{-1} dr, \]
\[ C_5 = (Z_0 g_s)^{-1} dr \wedge \epsilon_{K3}. \]
(4.1)

The line element corresponds to the string frame, \( d\Omega_{K3}^2 \) is the metric of the K3 surface of unit volume and \( \epsilon_{K3} \) is its corresponding volume form. Providing that the solution is asymptotically flat, the harmonic functions are
\[ Z_0 = 1 - \frac{V}{r} f(r), \]
\[ Z_4 = 1 + f(r), \]
(4.2)

where \( V \) is the volume of K3 at \( r = \infty \) and \( V_0 = (4\pi l_s)^4 = \frac{\mu_0}{\mu_4} \). The volume of K3 at arbitrary \( r \) can be read off from (4.1):
\[ V(r) = \frac{V Z_0(r)}{Z_4(r)} \]
(4.3)

and the function \( f(r) \) is, for a solution with \( N \) branes,
\[ f(r) = \frac{1}{4} \sum_{\mu} \frac{\mu g_s r_\mu^2}{r^3} = \frac{V r_0^3}{V_0 r^3}, \]
(4.4)

where \( r_0 \) is the radius where a naked singularity (repulson) is placed, as confirmed by inserting (4.4) in (4.1) and computing the Kretschmann curvature scalar. As argued in [10], this singularity is unphysical. This may be seen by probing the geometry (4.1) with the other D4-brane. The action of the probe may be written as
\[ S_{\text{probe}} = \int dt e^{-\Phi} (\mu_4 V(r) - \mu_0) \sqrt{-g} + \mu_4 \int C_5 - \mu_0 \int C_1, \]
(4.5)

where the function \( m(r) \) of the DBI term is the mass of the probe. The probe becomes massless at \( m(r) = 0 \), that is, when \( V(r) = V_0 \). We will define \( r_e \) as the point where the K3 volume becomes \( V_0 \), so \( V(r_e) = V_0 \). Now, \( r_e \) defines the enhançon locus. Its value can be easily computed as
\[ f(r) = \frac{1}{2} \left( \frac{V}{V_0} - 1 \right) \rightarrow r_e = \frac{2V}{V - V_0} |r_0| > |r_0|. \]
(4.6)

The last inequality of (4.6) shows that the enhançon radius is always bigger than \( r_0 \), the radius where the naked singularity is placed, provided that \( V > V_0 \).\(^{17}\)

For smaller values of \( r \) than the enhançon radius, the tension of the probe becomes negative and the solution is considered unphysical from the supergravity perspective. The probe cannot approach the point \( r = 0 \). Moreover, as ‘seen’ by the probe, the \( N \) D4-branes are not at \( r = 0 \) but smeared over the enhançon locus as well.

If we now try to start building the model, say, from empty space, the first D4-brane does not present any enhançon difficulties to reach \( r = 0 \), so the 1-(anti)brane solution is spherically symmetric. However, as we try to add a second and subsequent branes to the former, the enhançon mechanism prevents them reaching the first brane at the origin. Even if it is not straightforward to see that this breaks spherical symmetry (the enhançon shell is, in principle, a sphere), it is not difficult to believe that the lack of point-like behavior will trigger multipole

\(^{17}\) The supergravity equations imply that \( V(r) \) is an increasing function. So \( V = V(r \to \infty) > V_0 \) is always true.
contributions. It happens, for example, that the homogeneous distribution of $N$ branes over the spherical enhancer shell is, as noticed in [14, 24], just a particular supergravity solution. The enhancer, in general, has arbitrary shape. Think for example of an oblate shape. One can then define a brane density on this enhancer surface whose integral gives back the $N$ constituent branes. This density is not homogeneous. Moreover, as ruled by supergravity for static solutions the brane density on the enhancer behaves very much like an electrostatic distribution of charges on a metal [14], that is, growing in regions where the curvature of the enhancer surface is higher. Indeed, it seems a little naive to think of such sharp geometries at a region where the branes are blown in new (noncompact) dimensions and spacetime seems not to behave ordinarily. Indeed, it is believed that the correct description of the geometry near the enhancer locus is a fuzzy sphere [25]. Fuzzy spheres and more generally non-commutative geometries break unavoidably spherical symmetry. Moreover, Riemannian geometry and the concept of manifold are no longer valid in this context. It is remarkable, besides, that the fuzzy geometry appears for $N > 1$ and so the charge-1 monopole does not get affected and recovers spherical symmetry as expected.

There are two remarks that are worth pointing out. First, it should be noticed that the branes considered in seminal papers [10–12] were wrapping a nonvanishing 2-cycle of the K3 surface. Our model, by contrast, is built by the so-called fractional branes, that is, D$p$-branes which wrap vanishing 2-cycles of the K3 surface18. This difference could raise some doubts about the above arguments. However, fractional brane solutions have also been proved to show enhancer behavior [13]. A second remark concerns the gauge symmetry. As said before, although it is true that the explicit calculations were carried out only for an Abelian gauge field, the extension for a $SU(2)$ Yang–Mills field (as geometrically engineered in our model) is not expected to show any obstruction for the enhancer given that the D2-brane is not affected by the enhancer.

We will indicate within the next section the relation between the brane setup, the monopoles and the fuzzy geometry.

5. Connection with fuzzy spheres

Fuzzy spheres are examples of noncommutative geometry [25]. The main idea that underlies the construction of these spaces is the one-to-one correspondence between the differential geometry of manifolds and the commutative algebra of functions defined on them. The coordinates are the generators of the algebra and the vector fields are the derivations. A fuzzy sphere differs from an ordinary sphere because the algebra of functions on it is not commutative. Taking spherical harmonics as a basis, fuzzy spheres are generated by harmonics whose spin $l$ is not greater than a given $j$. The terms in the product of two spherical harmonics that involves harmonics with exceeding $j$ are just omitted. This truncation makes the algebra of functions noncommutative. So fuzzy spheres are labeled by an integer number $j$. For $j = 2$, the algebra describes an $S^2$-sphere poorly; in fact, only the north and the south poles are distinguishable. For $j = 3$, one can make out the equator as well, so the geometry gets less fuzzy. The ordinary sphere is recovered in the limit $j \to \infty$. In what follows, we will turn $j$ into $N$, the number of branes, because as we will see, the number of branes will actually rule the degree of fuzziness. The radius of the fuzzy sphere is given by

$$R^2 = \kappa (N^2 - 1),$$

(5.1)

where $\kappa$ must be proportional to $\alpha'$ for dimensions.

18 D$p$-branes on the K3 surface are equivalent to D($p - 2$)-branes on $T^4/\mathbb{Z}_2$, the orbifold limit of the K3 surface. So D4-branes become D2, and D2 become D0 fractional branes.
In what sense are these fuzzy geometries related to the multiple charge configurations? Unfortunately, we do not know the proper mathematical framework to show how fuzzy geometry comes up in the world volume of the NS5-branes in our model, as nicely shown in [17] (by means of exploiting the DBI action) for non-Abelian and non-BPS setups where a pile of $N$ D1-branes stretches between two D5-branes. We lack a low energy action for the worldvolume of NS5-branes. However, it is commonly believed that the enhanc¸on mechanism comes along with fuzzy geometry, and in the case of fuzzy spheres, (5.1) holds for the fuzzy radius, where $N$ is the number of transversal branes. It is clear from (5.1) that $N = 1$ do not lead to fuzziness, but for any $N > 1$ it does. So multicharge solutions are fuzzy.

There is a remarkably nice explanation of how fuzzy geometry comes out from a similar setup [10] to our model. It is worth reviewing in this section.

The model they used was built with the D6–D2 brane system wrapped on the K3 surface. In previous works [11], the authors already noticed that such a system would make a t’Hooft–Polyakov monopole when dimensionally reduced to four dimensions. Specifically, they consider a D6-brane wrapping the K3 surface and two more compact flat dimensions, where the D2-brane lives. The enhanc¸on mechanism produces an enhancement of the gauge symmetry $U(1) \to SU(2)$ in the four-dimensional region $r < r_c$, which will be the core of the monopole. The configuration gets Higgsed as one moves far from $r_c$, recovering the $U(1)$-magnetic charge.

In order to better visualize it, the brane setup that was actually used [15, 16] to show how fuzzy spacetime geometry enters the enhanc¸on picture was not the D6–D2 system on the K3 model already mentioned but a dual one in type IIB instead, which consists of a pair of parallel NS5-branes with $N$ D3-branes stretched between them. The separation of the NS5-branes is parameterized by $\sigma \in [-1, +1]$ at which extremes the branes are initially located. The presence of the D3-branes deforms the geometry of the NS5-branes into a double-trumpet shape [15]. The enhanc¸on locus is precisely a transversal section of the bunch of D3-branes at the point $\sigma = 0$ where the two NS5-branes make contact. As a consequence of the two-brane connection, the gauge symmetry is enhanced to $SU(2)$ at the enhanc¸on locus. That was expected. It is also expected that $N$ BPS monopoles enter this picture. They are placed at the ends of the D3-branes as seen by the part of the NS5-brane worldvolume transverse to the D3-brane. The positions of those ends can be coordinated by $\Phi^i (i = 1, 2, 3)$. These ‘coordinates’ fulfill Nahm equations in the BPS case:

$$\frac{d\Phi^i}{d\sigma} = \frac{1}{2} \epsilon_{ijk} [\Phi^j, \Phi^k].$$

The appropriate solutions (BPS monopoles) are those for which the $\Phi^i (\sigma)$ have a single pole at the end of the interval $\sigma \in [-1, +1]$ and the residues $\Sigma^i$ form the $N \times N$ irreducible representation of $SU(2)$: $[\Sigma^i, \Sigma^j] = 2i\epsilon_{ijk} \Sigma^k$. The general solution of this kind takes one $SU(2)$ representation and twists it to another as it crosses the interval. The enhanc¸on (recall the section on the stack of D3-branes) is a fuzzy sphere for finite $N \neq 1$ [25]. For example, for a two-D3-brane setup, a section will capture only two clear points of $S^2$, the north and the south poles. For a single D3-brane, however, the radius of the fuzzy sphere vanishes (5.1). For large $N$, the geometry becomes less fuzzy, recovering the usual $S^2$ as $N \to \infty$.

Their construction is different from the D4–D2 system on the K3 surface that we propose for the six-dimensional Yang monopole. Differences include, for instance, the origin of the enhanced $SU(2)$ symmetry. We did not make use of the enhanc¸on mechanism but we geometrically engineered the gauge group instead, as explained in section 2. We did so in order to account for an $SU(2)$ gauge group all over the six-dimensional space (and not only within the enhanc¸on region), as needed for the Yang monopole configuration. Unfortunately, for the case of D1-branes stretched between a pair NS5-brane (our T-dual model), we lack the
Nahm equations and an analogous procedure as just shown for the emergence of fuzzy spheres cannot be set. It would be very interesting to study the analogous algebra in our model for the emergence of the fuzzy 4-sphere at the enhançon locus. We leave it for a future work.

Despite this and other obvious differences, the essence of the problem concerning spherical symmetry still remains. It is because the enhançon shell they considered was spherical and supported homogeneously a melting $N$ D6-branes on it. An inconsistency with field theory is again encountered (and cured by the fuzzy geometry) in their case. That is because spherically symmetric ‘t’Hooft–Polyakov monopoles with multiple magnetic charge do not exist [26].

Needless to say the same enhançon-fuzzy mechanism applies for the $SO(4)$ monopole and cures it of the multicharge disease which it could undergo in the brane picture had the geometry not become fuzzy.

6. Discussion and open questions

In this paper, a type IIA geometric realization of the Yang monopole in six dimensions given in [1] is revisited and its apparent contradictions are clarified. In the construction of the magnetic object, the result of the duality between type IIA superstring compactified on the K3 surface and heterotic superstring on $T^4$ has been used. The $SU(2)$ gauge symmetry of the Yang monopole is considered as the enhanced gauge symmetry corresponding to shrinking 2-cycles inside the K3 surface, and the Yang monopole comes up by wrapping D-branes on the K3 non-trivial cycles. In this way, the properties of the Yang monopole are encoded in the K3 surface data.

With respect to the charges of the configuration, suggestions and objections that came up during the presentation of [1] have been taken into full consideration. In our opinion, this work brings light to the main objections strengthening and completing the brane picture of the Yang monopole. Firstly, it was claimed that the number of charges of one D4-brane setup should be four, accounting for the two ways the brane and the antibrane can wrap a 2-cycle. The answer is given at the end of section 2. There it is explained that the four configurations are actually identified in pairs, so it results in just two homotopically different configurations. Indeed, as explained in the same section, the $SO(4)$ extended-Yang monopole is the one that carries four charges and a brane picture for it is proposed.

The second objection has been taken into analysis in section 4. When more than one D4-brane is added to the model, its interpretation as a Yang monopole gets into trouble since an infinite tower of charges seems to appear. This is what we have called the multicharge disease. The multi-charge problem of this construction is satisfactorily solved by the dynamics of the enhançon mechanism which, as explained in section 4, ruins spherical symmetry in the multi-brane setup and then saves the model from contradiction.

The question of how such an involved concept as the enhançon locus and its correlated fuzzy geometry comes on stage to make the brane configuration non-spherically symmetric as required, for different reasons, by the gauge field theory is a point that in our opinion requires further analysis. More on this will be reported in future works.

Acknowledgments

We thank M Asorey, L J Boya, E H Saidi, P K Townsend, K Goldstein and C Hoyos-Badajoz for discussions and enlightening comments. This work has been partially supported by CICYT (grant FPA-2009-09638) and DGIID-DGA (grant 2007-E24/2). We thank also the support by grants A9335/07 and A9335/10 (Fisica de alta energia: Particulas, cuerdas y cosmologia).
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