Quadratic Multiform Separation: A New Classification Model in Machine Learning

Ko-Hui Michael Fan
AEM Technology Taiwan Corporation
Taipei, Taiwan, R.O.C.

Chih-Chung Chang
Department of Mathematics
National Taiwan University
Taipei, Taiwan, R.O.C.

Kuang-Hsiao-Yin Kongguoluo
AEM Technology Taiwan Corporation
Taipei, Taiwan, R.O.C.

Abstract

In this paper we present a new classification model in machine learning. Our result is threefold: 1) The model produces comparable predictive accuracy to that of most common classification models. 2) It runs significantly faster than most common classification models. 3) It has the ability to identify a portion of unseen samples for which class labels can be found with much higher predictive accuracy. Currently there are several patents pending on the proposed model (Fan et al., 2021a,b).

Keywords: quadratic multiform separation, loyalty extraction machine, machine learning, supervised learning, statistical classification

1. Introduction

Machine learning has become one of the most important topics within many scientific disciplines. What makes it attractive is that it builds a model of a system solely based on observed input-output pairs without referring to the system itself. Once the model “template” is selected, an algorithm is applied to modify model parameters to match between the observed and predicted input-output pairs. In this paper we consider the classification problem and propose a novel model for its solution. To our knowledge the proposed model does not resemble any existing ones. Some of common classification models are logistic regression, support vector machine, naive bayes, decision tree, random forest, and neural network (see, for example, Mitchell (1997), Nasteski (2017)).

The classification problem is defined as follows. Let $\Omega \subset \mathbb{R}^p$ be a training set containing instances of $x$ from input-output pair $(x, y)$ generated by the unknown membership function $\mu : \Omega \rightarrow \mathcal{M}$. Here $x \in \mathbb{R}^p$ is a vector, $y$ is a class label selected from the set

$$\mathcal{M} = \{1, \ldots, m\}$$

and $m$ is the number of possible classes. The problem is to find a classifier $\hat{\mu}(\cdot)$, serves as an approximation of $\mu(\cdot)$, by equating the behavior of $\hat{\mu}(\cdot)$ to the samples in the training
set. After a classifier is constructed, a test set of unseen samples is used to evaluate its performance. For a classification problem to be considered properly formulated, the test set is assumed to possess a similar sample distribution to that of the training set.

For $i \in M$, let $\Omega_i$ denote the member set defined by

$$\Omega_i = \{ x \in \Omega : \mu(x) = i \}$$

Thus the collection of $\Omega_1, \ldots, \Omega_m$ forms a partition of the training set $\Omega$. The essence of the proposed model is quadratic multiform separation (QMS), in which we seek pairwise separation of the member sets using quadratic polynomials that satisfy a certain cyclic conditions. We show that QMS is equivalent to finding nonnegative quadratic polynomials $f_1, \ldots, f_m$ and defining the classifier $\hat{\mu}$ by

$$\hat{\mu}(x) \in \{ k : f_k(x) \leq f_i(x) \text{ for all } i \}$$  \hspace{1cm} (1)

To this end, a QMS-specific loss function is introduced and a corresponding optimization problem is formulated. We then propose a simple yet very effective and efficient algorithm, called coordinate perturbation method (CPM), to minimize the loss function. CPM is superior in speed because the method does not require full evaluation of the loss function (not even just for once), nor it requires any gradient computation or line search of that sort. With careful implementation, CPM appears to run significantly faster than most of common classification models.

We also move further into the notion of loyalty extraction machine (LEM). The primary functionality of LEM is to identify a specific portion of the test set for which we will be able to find class labels with much higher predictive accuracy. LEM is related to the field of probabilistic classification and has many advantages over non-probabilistic classifiers.

Incidentally, in his master’s thesis entitled "Gradient-based quadratic multiform separation" (Chang, 2021), W.T. Chang studies the topic of designing QMS classifiers using the Adam optimization algorithm (Kingma and Ba, 2017). The Adam optimization algorithm is an extension to stochastic gradient descent that has recently seen broader adoption for deep learning applications in computer vision and natural language processing. Chang’s empirical numerical result shows that, in terms of predictive accuracy, QMS performs at least as good as most of common classification models. Moreover, QMS’s superior performance is almost comparable to that of gradient boosting algorithms which have won numerous machine learning competitions.

The remainder of the paper is organized as follows: in Section 2 we present the theory of QMS, in Section 3 we discuss LEM, in Section 4 we provide numerical results of running QMS and LEM using the Fashion MNIST dataset (Xiao et al., 2017), and finally, Appendix A contains all the proofs.

2. Quadratic Multiform Separation

In this section we present the theory of QMS. We start by introducing the concept of multiform separation.
Definition 1. Multiform separation is a mathematical model for classification. It consists of $m$ piecewise continuous functions $f_1, \ldots, f_m$, which are constructed based on the training set $\Omega$. Accordingly, a classifier $\hat{\mu} : \Omega \to M$ in the form of (1) can be realized. The functions $f_1, \ldots, f_m$ are called separators.

In practice, the functions $f_i$'s are chosen in a manner so that $\hat{\mu} (\cdot)$ serves as a good representation of $\mu (\cdot)$. Just like every mathematical model in machine learning, selecting separators in multiform separation involves a trade-off between simplicity and fidelity of the model. There is no doubt that the set of piecewise continuous functions would be too large to handle. We would like to add sufficient complexity into the model to improve its realism, but maintain the simplicity of the model to make it easier to understand and analyze. With that in mind, we propose to select separators from the set of nonnegative quadratic polynomials, which we shall call member functions.

Definition 2. A polynomial $f : \mathbb{R}^p \to \mathbb{R}$ is said to be nonnegative if $f(x) \geq 0$ for all $x \in \mathbb{R}^p$.

Definition 3. Let $q$ be a positive integer. A function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be a member function (or $q$-dimensional member function) if it can be written as $f(x) = \|Ax - b\|^2$ for some $A \in \mathbb{R}^{q \times p}$ and $b \in \mathbb{R}^q$, where $\| \cdot \|$ denotes the Euclidean norm.

Lemma 1. A quadratic polynomial $f : \mathbb{R}^p \to \mathbb{R}$ is nonnegative if and only if it is a $q$-dimensional member function for some $q \leq p + 1$.

Proof. See Appendix A.

Definition 4. Quadratic multiform separation is a scenario of multiform separation where the separators are selected among member functions.

2.1 Geometric Interpretation

We provide below a geometric interpretation for QMS. In this interpretation, we make use of the notion of having Property A, and a result for the converse of a trivial statement: the difference of two member functions is a quadratic polynomial.

Definition 5. Given sets $S_1, S_2 \subset \mathbb{R}^p$ and a quadratic polynomial $h : \mathbb{R}^p \to \mathbb{R}$, the triplex $\{S_1, S_2, h\}$ is said to have Property A if $h(x) < 0$ for all $x \in S_1$ and $h(x) > 0$ for all $x \in S_2$.

In other words, Property A asserts that the sets $S_1$ and $S_2$ reside on two different sides of the manifold defined by $h(x) = 0$, or the function $h$ separates the sets $S_1$ and $S_2$.

Lemma 2. Every quadratic polynomial can be written as the difference of two member functions.

Proof. See Appendix A.

Let us first consider the case when only two classes in the classification problem, that is, $m = 2$. It is easy to see that in order to have a classifier $\hat{\mu} (\cdot)$ match perfectly the membership function $\mu (\cdot)$, the task of QMS is essentially tailored to look for member functions $f_1$ and $f_2$ in such a way that

\[ f_1(x) < f_2(x) \text{ for all } x \in \Omega_1, \quad \text{and} \]
\[ f_2(x) < f_1(x) \text{ for all } x \in \Omega_2. \]
Suppose the conditions (2) and (3) hold for some member functions \( f_1 \) and \( f_2 \). It is obvious that \( f_1 - f_2 \) is a quadratic polynomial and the triplex \( \{\Omega_1, \Omega_2, f_1 - f_2\} \) has Property A. On the other hand, suppose that the triplex \( \{\Omega_1, \Omega_2, h\} \) has Property A for some quadratic polynomial \( h \). In view of Lemma 2, \( h \) can be written as \( h = f_1 - f_2 \) for some member functions \( f_1 \) and \( f_2 \). Then it is also obvious that the conditions (2) and (3) hold for \( f_1 \) and \( f_2 \). This leads to an interesting interpretation of QMS: for the case when \( m = 2 \), QMS amounts to separating \( \Omega_1 \) and \( \Omega_2 \) by a quadratic polynomial. We assume that this quadratic polynomial \( h \) exists and it has been found by some method. Then the following simple procedure leads to a perfect classifier: obtain member functions \( f_1 \) and \( f_2 \) so that \( h = f_1 - f_2 \), and define \( \hat{\mu}(\cdot) \) by (1). The existence of a separating quadratic polynomial ensures the existence of a perfect classifier. In reality, however, such an ideal situation may not be true and thus one might alternatively seek an approximation of separation defined in terms of some optimal sense.

Are we able to extend the above interpretation for the general \( m \)? The answer is yes but it is more nuanced. Consider the case when \( m = 3 \). The task of QMS to achieve a perfect classifier in this case is essentially tailored to look for member functions \( f_1, f_2, \) and \( f_3 \) to satisfy the conditions

\[
\begin{align*}
&f_1(x) < f_2(x), \ f_1(x) < f_3(x) \text{ for all } x \in \Omega_1, \text{ and} \\
&f_2(x) < f_1(x), \ f_2(x) < f_3(x) \text{ for all } x \in \Omega_2, \text{ and} \\
&f_3(x) < f_1(x), \ f_3(x) < f_2(x) \text{ for all } x \in \Omega_3
\end{align*}
\]

We assume the conditions (4)-(6) hold for some member functions \( f_1, f_2, \) and \( f_3 \), and define the quadratic polynomials as \( h_{12} = f_1 - f_2, \ h_{23} = f_2 - f_3 \) and \( h_{31} = f_3 - f_1 \). Then it is easy to see that the following conditions hold.

\[
\begin{align*}
&\{\Omega_1, \Omega_2, h_{12}\} \text{ has Property A} \\
&\{\Omega_2, \Omega_3, h_{23}\} \text{ has Property A} \\
&\{\Omega_3, \Omega_1, h_{31}\} \text{ has Property A} \\
&h_{12} + h_{23} + h_{31} = 0
\end{align*}
\]

Therefore it is very straightforward to derive from member functions \( f_i \)'s to the corresponding quadratic polynomial \( h_{ij} \)'s. The derivation for going the other direction is less obvious. Lemma 3 asserts its feasibility. We will omit the proof here since it is a special case of Theorem 1 (see below).

**Lemma 3.** Suppose the conditions (7)-(10) hold for some quadratic polynomials \( h_{12}, h_{23} \) and \( h_{31} \). Then there exist member functions \( f_1, f_2, \) and \( f_3 \) so that \( h_{12} = f_1 - f_2, \ h_{23} = f_2 - f_3, \ h_{31} = f_3 - f_1 \) and the conditions (4)-(6) hold.

Lemma 3 reveals a somewhat surprising result. For the case when \( m = 3 \), QMS is not equivalent to independently solving pairwise separation of the sets \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) by quadratic polynomials. Instead, Lemma 3 says those quadratic polynomials have to satisfy one additional constraint which is sum of them is equal to zero.

Now we turn the discussion to the general case. Theorem 1 below, which extends the result in Lemma 3, grants a geometric interpretation of QMS. In addition, with the quadratic
polynomials in place, explicit expressions for constructing member functions are provided in its proof. The geometric interpretation is stated as follows: QMS is a mathematical model for classification in which one seeks pairwise separation of the member sets using quadratic polynomials that satisfy a certain cyclic conditions.

**Theorem 1.** Suppose there exist quadratic polynomials \( h_{ij}, \ i, j \in \mathcal{M} \) such that (i) the triplex \( \{\Omega_i, \Omega_j, h_{ij}\} \) has Property A for all \( i, j \in \mathcal{M}, i \neq j \) and (ii) the cyclic condition

\[
h_{ij} + h_{jk} + h_{ki} = 0
\]

holds for all \( i, j, k \in \mathcal{M} \). Then there exist member functions \( f_1, \ldots, f_m \) such that, for all \( i \in \mathcal{M} \), the following condition holds

\[
f_i(x) < f_j(x) \text{ for all } x \in \Omega_i \text{ and for all } j \in \mathcal{M}, j \neq i
\]

Proof. See Appendix A.

**2.2 Loss Function**

At this point we have completed the analysis stage of the classification problem. In that stage, we employ a mathematical model and scientific principles to help us predict the design results. The next step is the optimization stage where a systematic process using design constraints and criteria takes place to locate the optimal member functions \( f_1, \ldots, f_m \). To facilitate the process, a loss function must be defined.

A loss function maps the member functions \( f_1, \ldots, f_m \) onto a real number intuitively representing some “cost”. Let us first consider part of the loss function that is associated with a particular sample, say, \( x \in \Omega_1 \). This part of the loss function should resemble a degree of violation for the inequalities

\[
f_1(x) < f_j(x) \text{ for all } j = 2, \ldots, m
\]

Therefore a simple candidate for its choice could be

\[
\sum_{j=2}^{m} \max \{-\epsilon, f_1(x) - f_j(x)\}
\]

(11)

where \( \epsilon \) is a small positive number. However, (11) is not numerically sound since it will depend on the problem scaling. To be specific, we see that multiplying a positive constant to all \( f_i \)'s will change (11) but it should not be doing so. Instead, the following choice seems to be a better one.

\[
\sum_{j=2}^{m} \max \left\{ \alpha, \frac{f_1(x)}{f_j(x)} \right\}
\]

(12)

where \( \alpha \in [0, 1) \). The use of \( \alpha \) in (12) is to remove the constraint \( f_1(x) < f_j(x) \) during optimization as long as \( f_1(x) \) stays sufficiently smaller than \( f_j(x) \).

Now we include all \( x \) from the training set \( \Omega \) and formally present the definition. Select \( \alpha \in [0, 1) \) and define the loss function \( \phi \) by

\[
\phi = \sum_{i \in \mathcal{M}} \phi_i
\]

(13)
where, for $i \in \mathcal{M}$, $\phi_i$ denotes

$$
\phi_i = \sum_{x \in \Omega_i} \sum_{j \in \mathcal{M}, j \neq i} \max \left\{ \alpha \frac{f_i(x)}{f_j(x)} \right\}
$$

(14)

### 2.3 Coordinate Perturbation Method

In this section, we propose a simple yet very effective and efficient algorithm for minimizing the loss function defined in (13) and (14). The algorithm works as follows. It successively minimizes along each and every coordinate, one at a time, to find a descenting point. Along each coordinate, the algorithm considers at most two nearby points, one along each of two opposite directions. It moves to a nearby descending point or otherwise does nothing if both nearby points do not produce a lower value for the loss function. The algorithm then switches to the next coordinate and continues. We shall call this algorithm the coordinate perturbation method (CPM).

We point out that CPM does not require full evaluation of the loss function $\phi$ (not even just for once), nor it requires any gradient computation or line search of that sort. With careful implementation, CPM appears to run significantly faster than most of common classification models. In Section 4, among other things, we will report a timing result of CPM using the Fashion MNIST dataset.

In closing this section, we would like to mention one more thing. Our implementation of CPM uses only single-thread/no-GPU computer resource. In spite of its exceptional speed performance, it is not clear how to take full advantage of the multi-cpu/multi-thread/GPU features that exist in today’s computer hardware. This is currently under investigation.

### 3. Loyalty Extraction Machine

In machine learning, an ensemble method uses multiple classifiers to obtain better predictive performance than could be obtained from any of the constituent classifiers alone. In this section, we also consider multiple classifiers created by altering the definition of the loss function. Even though our method improves overall predictive performance, it is not our goal to do so. Instead, among other things, we would like to identify a specific portion of the test set for which class labels can be found with much higher predictive accuracy. In order to achieve this goal, a scheme (or a machine) is developed to divide the test set according to the so-called loyalty type of its samples. There are three loyalty types defined here: strong, weak, and normal. A finer partition of the test set is readily applicable with the expenses of complexity and computation time. As a result, the test set turns into the union of three disjoint subsets, each corresponds to a different loyalty type. We shall call this process the loyalty extraction machine (LEM). The loyalty type is an estimate of how confident a sample adheres to its predicted class. This estimate of course is based on the statistical information learned from the training set.

LEM is related to the field of probabilistic classification. A probabilistic algorithm computes a probability of the sample being a member of each of the possible classes. The best class is normally then selected as the one with the highest probability. Similar to the probabilistic algorithms, LEM has the following advantages over non-probabilistic classifiers:
- Samples with strong loyal type typically have much higher predictive accuracy than the overall predictive accuracy.

- Samples with weak loyal type typically have much lower predictive accuracy than the overall predictive accuracy. Therefore, one could refrain from using the predicted class and switch to other means of classification instead.

- LEM can be incorporated into the medical diagnosis problem to minimize the total cost incurred in the diagnosis (see Section 3.3 below).

- LEM can be more effectively incorporated into larger machine learning tasks, in a way that partially or completely avoids the problem of error propagation.

### 3.1 Loyalty Type

We will first generate $2^m$ new classifiers in designing the loyalty extraction machine. To keep the explanation simple, we focus on the case when $m = 3$. Therefore, there are total of six new classifiers. It is easy to see that extension to the general $m$ is quite straightforward.

Recall that the loss function $\phi$ is defined by

$$\phi = \phi_1 + \phi_2 + \phi_3$$

where $\phi_i$’s are defined by (14). Let $\beta < 1$ and $\gamma > 1$ be positive numbers. Consider the following six altered loss functions.

\[
\begin{align*}
\phi &= \beta \phi_1 + \phi_2 + \phi_3 \implies \hat{\mu}(1, \beta, \cdot) \quad (15) \\
\phi &= \gamma \phi_1 + \phi_2 + \phi_3 \implies \hat{\mu}(1, \gamma, \cdot) \quad (16) \\
\phi &= \phi_1 + \beta \phi_2 + \phi_3 \implies \hat{\mu}(2, \beta, \cdot) \quad (17) \\
\phi &= \phi_1 + \gamma \phi_2 + \phi_3 \implies \hat{\mu}(2, \gamma, \cdot) \quad (18) \\
\phi &= \phi_1 + \phi_2 + \beta \phi_3 \implies \hat{\mu}(3, \beta, \cdot) \quad (19) \\
\phi &= \phi_1 + \phi_2 + \gamma \phi_3 \implies \hat{\mu}(3, \gamma, \cdot) \quad (20)
\end{align*}
\]

For each of the altered loss functions, we carry out the classification task in the similar fashion as given in Section 2 and obtain a corresponding classifier ($\hat{\mu}(\cdot, \cdot, \cdot)$’s in (15)-(20)). Hence six new classifiers are hatched. These classifiers usually differ from the nominal classifier (that is, the one with the unaltered loss function). They also differ from each other in a way should become less ambiguous in a moment. The rationale for doing so centers on differentiating the adherence of samples pertaining to one member set from those pertaining to all other member sets.

We provide further explanation by using the loss function defined by (15). Since $\beta < 1$, it raises a situation that, during the course of optimization, violation of the inequalities

$$f_1(x) < f_2(x), \ f_1(x) < f_3(x)$$

for any $x \in \Omega_1$ will be partially ignored. Therefore, the optimization algorithm will place emphasis in keeping other inequalities stay away from being violated. As a result, this will
decrease the chance for 1 to be selected as the predicted class for every training sample. Given the circumstances, suppose that we have some \( x \in \Omega \) whose predicted class is still 1. Then it is an indication that \( x \) has strong adherence to the class 1.

Let us look at a different case when the loss function is defined by (16). Since \( \gamma > 1 \), it raises a situation that, during the course of optimization, violation of the inequalities (21) for any \( x \in \Omega \) will be more recognized. Therefore, the optimization algorithm will place emphasis in keeping (21) stay away from being violated. As a result, this will increase the chance for 1 to be selected as the predicted class for every training sample. Similar statements can be made if the loss function is defined by (18) or by (20). Now suppose for some \( x \in \Omega \) whose predicted class is 1 when the loss function is defined by (16), is 2 when the loss function is defined by (18), and is 3 when the loss function is defined by (20), that is, \( x \) has a foot in both camps. Given the circumstances, it indicates that \( x \) has weak adherence to every class.

In light of these observations, we can provide the definition for the loyalty type. We reiterate that assigning a loyalty type to an unseen sample is solely based on the training set. It has nothing to do with other unseen samples.

**Definition 6.** A sample \( x \) is said to be of strong loyalty type if the set

\[
\{ i \in \mathcal{M} : \hat{\mu}(i, \beta, x) = i \}
\]

is a singleton. A sample \( x \) is said to be of weak loyalty type if it is not of strong loyalty type and the set

\[
\{ i \in \mathcal{M} : \hat{\mu}(i, \gamma, x) = i \}
\]

has at least three elements. A sample \( x \) is said to be of normal loyalty type if it is neither of strong nor of weak loyalty type.

It is easy to see that Definition 6 implies every sample can only be either of strong or of normal loyalty type for the case when \( m = 2 \).

### 3.2 Confusion Tensor

A confusion tensor is a generalization of the confusion matrix to adopt the loyalty type. It is a three-dimensional matrix of size \( 3 \times m \times m \). Such an object has three layers, each layer is a confusion matrix constructed for samples corresponding to a loyalty type. Many performance measures such as accuracy, precision and recall can be similarly defined with respect to the confusion matrix in each layer.

### 3.3 Medical Diagnosis Problem

In this section, we introduce a mathematical problem which is called the medical diagnosis problem. Its formulation is made possible by using the loyalty extraction machine.

This paragraph uses excerpts from Wikipedia for wording. Medical diagnosis here is referred to the process of determining whether a particular disease or condition explains a person’s symptoms and medical signs. A symptom is a subjective or objective departure from normal function or feeling which is apparent to a patient, reflecting the presence of an
unusual state, or of a disease. A medical sign is an objective indication of some medical fact or characteristic that may be detected by a patient or anyone, especially a physician, before or during a physical examination of a patient. The information required for diagnosis is typically collected from a history and physical examination of the person seeking medical care. Often, one or more diagnostic procedure, such as medical tests, are also done during the process. A medical test is a medical procedure performed to detect, diagnose, or monitor diseases, disease processes, susceptibility, or to determine a course of treatment. Medical tests relate to clinical chemistry and molecular diagnostics, and are typically performed in a medical laboratory.

Symptoms and medical signs is collectively called evidence. The acquisition of an evidence may or may not involve a cost, which may be either of monetary nature such as lab fees, or intangible such as waiting time or physical/psychological impact to the patient. Let $\theta$ be the predictive accuracy of the medical diagnosis derived from the full list of evidence using the existing medical records (namely, the training set in the context of machine learning). Obviously, $\theta$ is the maximal predictive accuracy since, intuitively, a medical diagnosis is likely to get worse using only a partial list of evidence. When a new medical diagnosis process starts, in order to reach predictive accuracy as high as possible, it seems inevitable to acquire all evidence, and thus results in a cost which may be unnecessarily high. Remarkably and counterintuitively, loyalty extraction machine demonstrates the ability of maintaining the maximal predictive accuracy, while acquiring merely a partial list of evidence in performing the medical diagnosis. This observation makes the following problem a meaningful challenge.

**Medical Diagnosis Problem.** The problem is to determine the optimal acquisition order of evidence that satisfies the following two properties:

1. The acquisition process terminates as soon as the evidence collected so far are sufficient to determine whether or not the patient has the disease or condition with the predictive accuracy no less than $\theta$.

2. The expected total diagnosis cost is the lowest among all acquisition orders.

### 4. Numerical Results

In this section we give numerical results of running QMS and LEM. The experiment runs on a notebook computer with specifications given in Table 1. Our code is implemented in

| Model          | ThinkPad T14 Gen 1 |
|----------------|--------------------|
| Memory         | 46.7 GiB           |
| Processor      | Intel Core i7 10510U @1.80GHz×4 |
| OS Name        | Fedora 32         |
| OS Type        | 64-bit            |

**Table 1: Computer specifications**

the C language using single precision arithmetic and SIMD instructions. The code only invokes a single user thread and does not use any GPU.
4.1 Dataset
We use the Fashion MNIST dataset. It is a dataset of Zalando’s article images, consisting of a training set of 60,000 samples and a test set of 10,000 samples. Each sample is a 28x28 grayscale image, associated with a label from 10 classes. Fashion MNIST is intended to serve as a direct drop-in replacement for the original MNIST dataset (LeCun and Cortes, 2010). The MNIST dataset is often used as the “Hello, World” of benchmarking machine learning algorithms for computer vision. It contains images of handwritten digits in a format identical to that of the articles of clothing in the Fashion MNIST dataset. Both datasets are relatively small and are used to verify that an algorithm works as expected. They are good starting points to test and debug code.

4.2 Running QMS
We follow the design procedure outlined in Section 2 and choose $\alpha = 0$. The classifier uses $q$-dimensional member functions with $q = 18$ for all separators. Table 2 summarizes the result. The resulting confusion matrices are depicted in Table 3 and Table 4.

| Training accuracy | 92.69% |
|-------------------|--------|
| Test accuracy     | 88.63% |
| Execution time    | 58 sec |

Table 2: QMS result

4.3 Running LEM
We follow the design procedure outlined in Section 3 and choose $\beta = 0.05$ and $\gamma = 20$. Twenty additional classifiers are constructed. Table 5 summarizes the result for the training set. In Table 5, $n$ denotes the total number of samples (which is 60000). For each loyalty

![Figure 1: Fashion MNIST samples (by Zalando, MIT License)](image-url)
Table 3: Confusion matrix (training set)

|      | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|---|---|---|---|---|---|---|---|---|
| 0    | 5931 | 1 | 55 | 4 | 0 | 6 | 0 | 2 | 1 |
| 1    | 5198 | 44 | 399 | 0 | 290 | 0 | 10 | 0 |
| 2    | 21 | 5679 | 124 | 0 | 77 | 0 | 3 | 0 |
| 3    | 311 | 127 | 5290 | 0 | 249 | 0 | 7 | 1 |
| 4    | 0 | 0 | 0 | 0 | 5885 | 0 | 69 | 3 | 43 |
| 5    | 2 | 368 | 95 | 318 | 0 | 4646 | 0 | 27 | 1 |
| 6    | 0 | 0 | 0 | 0 | 53 | 0 | 5830 | 0 | 117 |
| 7    | 0 | 8 | 17 | 10 | 4 | 19 | 6 | 5922 | 5 |
| 8    | 0 | 0 | 0 | 1 | 35 | 0 | 148 | 3 | 5813 |

Table 4: Confusion matrix (test set)

|      | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|---|---|---|---|---|---|---|---|---|
| 0    | 858 | 0 | 14 | 26 | 4 | 1 | 91 | 0 | 6 | 0 |
| 1    | 963 | 5 | 19 | 7 | 0 | 2 | 0 | 0 | 0 | 0 |
| 2    | 1814 | 10 | 88 | 0 | 63 | 0 | 3 | 0 |
| 3    | 4 | 12 | 897 | 31 | 0 | 25 | 0 | 6 | 0 |
| 4    | 1 | 79 | 27 | 830 | 0 | 62 | 0 | 0 | 0 |
| 5    | 0 | 0 | 0 | 1 | 949 | 0 | 19 | 2 | 29 |
| 6    | 141 | 83 | 27 | 66 | 0 | 674 | 0 | 8 | 0 |
| 7    | 0 | 0 | 0 | 0 | 0 | 18 | 0 | 954 | 0 | 28 |
| 8    | 3 | 8 | 6 | 6 | 3 | 11 | 3 | 960 | 0 |
| 9    | 0 | 0 | 0 | 0 | 5 | 1 | 30 | 0 | 964 |

Table 5: LEM result (training set)

| loyalty type | \(n_1\) | \(n_1/n\) | \(n_2\) | \(n_2/n_1\) |
|--------------|--------|----------|--------|-------------|
| strong       | 43,249 | 72.08%   | 43,173 | 99.82%      |
| normal       | 13,118 | 21.86%   | 10,473 | 79.84%      |
| weak         | 3,633  | 6.06%    | 1,970  | 54.23%      |

Table 6: LEM result (test set)

Let us give the highlights:

- For the test set, 69.67 percent of samples are of strong loyalty type.
- For any samples of strong loyalty type, the estimated predictive accuracy is 99.82 percent.
- For samples of strong loyalty type in the test set, the actual predictive accuracy is 97.45 percent.
| loyalty type | $n_1$ | $n_1/n$ | $n_2$ | $n_2/n_1$ |
|--------------|-------|---------|-------|-----------|
| strong       | 6,967 | 69.67%  | 6,789 | 97.45%    |
| normal       | 2,299 | 22.99%  | 1,699 | 73.90%    |
| weak         | 734   | 7.34%   | 375   | 51.09%    |

Table 6: LEM result (test set)

Finally, layers of the confusion tensor are depicted in Table 7, Table 8, and Table 9, respectively. Notice that sum of those layers is equal to the confusion matrix given in Table 4.

| 530 0 2 4 0 0 9 0 2 0 |
| 0 943 2 11 1 0 0 0 0 0 |
| 1 0 456 5 5 0 1 0 0 0 |
| 3 1 2 691 2 0 3 0 2 0 |
| 0 1 8 3 406 0 8 0 0 0 |
| 0 0 0 0 0 0 882 0 7 0 14 |
| 21 0 12 5 10 0 256 0 5 0 |
| 0 0 0 0 0 0 2 0 824 0 8 |
| 0 0 2 2 2 1 2 0 925 0 |
| 0 0 0 0 0 2 1 6 0 876 |

Table 7: Confusion tensor, strong loyalty type layer (test set)

| 286 0 5 12 1 0 61 0 3 0 |
| 4 14 3 6 2 0 2 0 0 0 |
| 5 0 297 1 58 0 30 0 0 0 |
| 9 2 3 154 19 0 9 0 3 0 |
| 1 0 47 18 329 0 25 0 0 0 |
| 0 0 0 1 0 64 0 11 2 14 |
| 93 0 39 8 29 0 319 0 2 0 |
| 0 0 0 0 0 0 16 0 129 0 18 |
| 1 0 3 2 0 2 3 3 21 0 |
| 0 0 0 0 0 3 0 21 0 86 |

Table 8: Confusion tensor, normal loyalty type layer (test set)

Appendix A. Proofs

Proof of Lemma 1. A member function is certainly nonnegative. Let

$$f(x) = \sum_{1 \leq i \leq j \leq p} a_{ij}x_i x_j + \sum_{i=1}^{p} a_i x_i + a_0$$

be a real, nonnegative quadratic polynomial. The way to express it as a member function is not unique. We describe one possible way below.
Lc be nonnegative. Thus, without loss of generality, we may simply consider the case 
for some $g$ indexing the variables. Let $x$ in variables $a$
If all $x$ becomes a polynomial without variable $f$
not, then $f$ linear equation $L$
clearly, $g$
since $f$
the
\[
\begin{array}{cccccccc}
42 & 0 & 7 & 10 & 3 & 1 & 21 & 0 & 1 & 0 \\
0 & 6 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \\
15 & 1 & 61 & 4 & 25 & 0 & 32 & 0 & 3 & 0 \\
13 & 1 & 7 & 52 & 10 & 0 & 13 & 0 & 1 & 0 \\
0 & 0 & 24 & 6 & 95 & 0 & 29 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 & 0 & 1 \\
27 & 1 & 32 & 14 & 27 & 0 & 99 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 2 \\
\end{array}
\]
Table 9: Confusion tensor, weak loyalty type layer (test set)

Clearly, $a_0 \geq 0$ and $a_{ii} \geq 0$ for all $i$ since $f$ is nonnegative. Moreover, if $a_{ii} = 0$
for some $i$, then $a_i = a_{ij} = 0, j \neq i$, since otherwise one could choose suitable constants 
c_j, j \neq i, to obtain a nontrival linear function $f(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_p)$ which cannot
be nonnegative. Thus, without loss of generality, we may simply consider the case $a_{ii} > 0, i = 1, \ldots, p$.

Define $f_j, g_j, j = 1, \ldots, p$, inductively as follows. For $j = 1$, define $f_1 = f - g_1$, and
$g_1(x) = L_1(x)^2$,
where $L_1(x)$ is a linear polynomial in variables $x_1, \ldots, x_p$:
$L_1(x) = \sqrt{a_{11}} x_1 + \frac{a_{12}}{2\sqrt{a_{11}}} x_2 + \cdots + \frac{a_{1p}}{2\sqrt{a_{11}}} x_p + \frac{a_1}{2\sqrt{a_{11}}}$.

Since $g_1$ collects all the terms in $f$ containing variable $x_1$, the function
\[
f_1(x) = f(x) - g_1(x) = \left( a_{22} - \frac{a_{12}^2}{4a_{11}} \right) x_2^2 + \cdots + \left( a_0 - \frac{a_1^2}{4a_{11}} \right)
\]
becomes a polynomial without variable $x_1$. We claim that $f_1$ is also nonnegative. Suppose not, then $f_1(\vec{t}) < 0$ for some $\vec{t} = (t_2, \ldots, t_p) \in \mathbb{R}^{p-1}$. Now solve $\vec{t} \in \mathbb{R}$ such that the linear equation $L_1(\vec{t}) = 0$, where $\vec{t} = (t_1, \ldots, t_p)$. Given $\vec{t} = (t_2, \ldots, t_p)$, $t_1$ can be solved since $a_{11} > 0$. Plugging $\vec{t}$ into $f$ then yields a contradiction $f(\vec{t}) = f_1(\vec{t}) + g_1(\vec{t}) = f_1(\vec{t}) + 0 < 0$.

The nonnegativeness of $f_1$ implies
\[
a_{22}^{(1)} = a_{22} - \frac{a_{12}^2}{4a_{11}} \geq 0, \ldots, a_{pp}^{(1)} \geq 0, a_0^{(1)} = a_0 - \frac{a_1^2}{4a_{11}} \geq 0.
\]
If all $a_{ii}^{(1)} = 0, i = 2, \ldots, p$, we are done. Otherwise, assume $a_{22}^{(1)} > 0$ with possibly re-indexing the variables. Let $g_2 = L_2(x)^2$ and $f_2 = f_1 - g_2$, where $L_2$ is a linear polynomial in variables $x_2, \ldots, x_p$:
$L_2(x) = \sqrt{a_{22}^{(1)}} x_2 + \frac{a_{23}^{(1)}}{2\sqrt{a_{22}^{(1)}}} x_3 + \cdots + \frac{a_{2p}^{(1)}}{2\sqrt{a_{22}^{(1)}}} x_p + \frac{a_2^{(1)}}{2\sqrt{a_{22}^{(1)}}}$.
Since \(g_2\) collects all the terms in \(f_1\) containing variable \(x_2\), \(f_2 = f_1 - g_2\) is a polynomial in variables \(x_3, \ldots, x_p\). The nonnegativeness of \(f_2\) can be seen easily. However, for simplicity, below we give a more general observation.

Suppose that the decomposition process described above has been carried out \(k, k \leq p\) times and we reach the expression \(f = g_1 + g_2 + \cdots + g_k + f_k\), where \(g_j(x) = L_j(x)^2\) and \(L_j(x) = \sqrt{a_{jj}^{(j-1)}} x_j + \cdots + a_{jj}^{(0)} > 0, j = 1, \ldots, k\), is a linear polynomial in variables \(x_j, x_{j+1}, \ldots, x_p\). This process ensures that \(f_k\) is a quadratic polynomial independent of \(x_1, \ldots, x_k\). We claim that \(f_k\) is nonnegative so that the induction argument can proceed. If not, there exists \((\tilde{t}_{k+1}, \ldots, \tilde{t}_p) \in \mathbb{R}^{p-k}\) such that \(f_k(x, \tilde{t}_{k+1}, \ldots, \tilde{t}_p) < 0\). By the fact \(k \leq p\) and the upper triangular nature of the system of \(k\) linear equations, \(k\) unknowns \(\tilde{t}_1, \ldots, \tilde{t}_k \in \mathbb{R}\) can be found so that \(L_1(\tilde{t}) = \cdots = L_k(\tilde{t}) = 0\), where \(\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_p)\). Once more this gives a contradiction \(f(\tilde{t}) = 0 + f_k(\tilde{t}) < 0\). Hence \(f_k\) must be nonnegative.

In the end the decomposition process stops at some expression \(f_{\ell-1} = g_\ell + f_\ell\), where \(f_\ell\) is nonnegative and independent of all variables \(x_1, \ldots, x_p\). Consequently, \(f_\ell\) must be a nonnegative constant \(a_0^{(\ell)} \geq 0\). Obviously, the number \(\ell\) of decompositions must be less than or equal to \(p\), the number of variables. The decomposition
\[
 f = g_1 + \cdots + g_\ell + a_0^{(\ell)}
\]
indicates how the entries of both \(A \in \mathbb{R}^{q \times p}\) and \(b \in \mathbb{R}^q\) can be chosen with \(q \leq \ell + 1 \leq p + 1\). This completes the proof.

**Proof of Lemma 2.** In view of Lemma 1, it suffices to prove that every quadratic polynomial can be written as a difference of two nonnegative quadratic polynomials. The case of monomials can be easily checked. For example, \(x_i x_j = \frac{1}{4} [(x_i + x_j)^2 - (x_i - x_j)^2]\), \(x_i = \frac{1}{4} [(x_i + 1)^2 - (x_i - 1)^2]\). The general case follows trivially.

**Proof of Theorem 1.** By construction. It suffices to show that there exist member functions \(f_1, \ldots, f_m\) such that
\[
 h_{ij} = f_i - f_j \quad \text{for all } i, j \in \mathcal{M}
\]  
(22)

First it is easy to see that the cyclic conditions imply \(h_{ii} = 0\) for all \(i \in \mathcal{M}\), and \(h_{ji} = -h_{ij}\) for all \(i, j \in \mathcal{M}\). In view of Lemma 2, for \(i \in \mathcal{M}\), there exist member functions \(u_i\) and \(v_i\) such that
\[
 h_{1i} = u_i - v_i
\]
Notice that \(u_1 = v_1\). For \(i \in \mathcal{M}\), define the member function \(f_i\) by
\[
 f_i = v_i - u_i + \sum_{k=1}^{m} u_k
\]
Consequently we have \(f_1 = \sum_{k=1}^{m} u_k\) and \(f_i = v_i - u_i + f_1\) for all \(i \in \mathcal{M}\). Finally, the validity of (22) follows from direct verification as follows.
\[
 h_{ij} = h_{1j} - h_{1i} = (u_j - v_j) - (u_i - v_i) = (f_1 - f_j) - (f_1 - f_i) = f_i - f_j
\]
The proof is then complete.
References

Wen-Teng Chang. Gradient-based quadratic multiform separation. Master’s thesis, National Tsing Hua University, 2021.

Ko-Hui Michael Fan, Chih-Chung Chang, Ye-Hwa Chen, and Kuang-Hsiao-Yin Kongguoluo. Classification algorithm based on multiform separation. US. Patent 17/148,860, filed January 14, 2021a.

Ko-Hui Michael Fan, Chih-Chung Chang, and Kuang-Hsiao-Yin Kongguoluo. Loyalty extraction machine. US. Patent 17/230,283, filed April 14, 2021b.

Diederik P. Kingma and Jimmy Ba. Adam: A method for stochastic optimization, 2017.

Yann LeCun and Corinna Cortes. MNIST handwritten digit database. 2010. URL http://yann.lecun.com/exdb/mnist/.

Tom M. Mitchell. Machine Learning. McGraw-Hill series in computer science. McGraw-Hill, 1st edition, 1997. ISBN 9780070428072,0070428077.

Vladimir Nasteski. An overview of the supervised machine learning methods. HORIZONS.B, 4:51–62, 12 2017. doi: 10.20544/HORIZONS.B.04.1.17.P05.

Han Xiao, Kashif Rasul, and Roland Vollgraf. Fashion-mnist: a novel image dataset for benchmarking machine learning algorithms, 2017.