TWISTED K THEORY INVARIANTS

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Abstract An invariant for twisted K theory classes on a 3-manifold is intro-
duced. The invariant is then applied to the twisted equivariant classes arising from
the supersymmetric Wess-Zumino-Witten model based on the group $SU(2)$. It is
shown that the classes defined by different highest weight representations of the loop
group $LSU(2)$ are inequivalent. The results are compatible with Freed-Hopkins-
Teleman identification of twisted equivariant K theory as the Verlinde algebra.

0. Introduction

Twisted K theory classes arise in a natural way in two dimensional conformal
field theory and can be described in terms of Verlinde algebra, [FHT1-2], [AtSe].
In this paper I shall describe a rather elementary method for a construction of
numerical invariants for the twisted K theory classes in the case of an oriented,
connected, simply connected 3-manifold. The method is then applied to the case
of the group manifold $SU(2)$ and it is shown that indeed the result matches the
prediction in [FHT1-2]. The computations are based on the construction of twisted
K theory classes in terms of the supersymmetric Wess-Zumino-Witten model as
described in [M]. The result, Theorem 2, shows that indeed the equivariant twisted
K theory classes constructed from different highest $SU(2)$ weights are different.

Ordinary complex K theory on a space $X$ can be defined as the abelian group
(with respect to direct sums of Hilbert spaces) of homotopy classes of maps from $X$
to Fredholm operators in a complex Hilbert space $H$. There is a grading mod 2 in
complex $K$ theory. The group $K^0(X)$ is defined by using the space of all Fredholm operators in $H$ whereas $K^1(X)$ is defined with the help of self-adjoint Fredholm operators which have both positive and negative essential spectrum.

To define twisted $K$ theory one needs as an input a principal $PU(H)$ bundle $P$ over $X$. Here $PU(H)$ is the projective unitary group $PU(H) = U(H)/S^1$ in the Hilbert space $H$. These principal bundles are classified by $H^3(X, \mathbb{Z})$; an element $\omega \in H^3(X, \mathbb{Z})$ is called the Dixmier-Douady class of the bundle $P$ and it plays the role of the (first) Chern class for circle bundles. A bundle $P$ is called a gerbe over $X$. Usually a gerbe is equipped with additional structure, the gerbe connection which is a Deligne cohomology class on $X$ with top form $\omega$.

Given $P$ we can define an associated vector bundle

\begin{equation}
Q = P \times_{PU(H)} \mathcal{F},
\end{equation}

where $\mathcal{F}$ denotes the space of (self-adjoint) Fredholm operators in $H$ and the action of $PU(H)$ on $\mathcal{F}$ is defined by conjugation, [BCMMS]. The twisted $K$ theory $K^*(X, \omega)$ is then the set of homotopy classes of sections of the bundle $Q$. It is again an abelian group with respect to direct sums.

As in the case of ordinary $K$ theory, it is sometimes useful to have an alternative equivalent definition. In the case of $K^1(X)$ one can replace self-adjoint Fredholm operators by unitary operators using the trick in [AS]. First one can contract to space of (unbounded) self-adjoint Fredholm operators (with positive and negative essential spectrum) to bounded self-adjoint operators with essential spectrum at the points $\pm 1$. Then one can map these operators to unitaries by $F \mapsto g = - \exp(i\pi F)$. The operator $g$ belongs to the group $U_1(H)$ of unitary operators such that $g - 1$ is a trace-class operator. The advantage with this method is that we can explicitly produce the generators $H^*(U_1(H), \mathbb{Z})$ as differential forms

\begin{equation}
\omega_{2k+1} = \alpha_{2k+1} \text{tr} (g^{-1}dg)^{2k+1}
\end{equation}

where $\alpha_{2k+1}$ is a normalization coefficient. The $K^1$ theory classes on $X$ are then classified, modulo torsion, by the pull-backs of classes $\omega_{2k+1}$ with respect to a mapping $X \to U_1(H)$. 

In the case of twisted K theory we can use the same trick simply by replacing in (0.1) the space \( F \) by \( U_1(H) \); this gives an alternative definition for \( K^1(X, \omega) \). The case \( K^0(X, \omega) \) has to be dealt differently. There is a different unitary group \( U_{\text{res}}(H) \) which has the same homotopy type as the space of all Fredholm operators in \( H \). To define \( U_{\text{res}}(H) \) one needs a polarization \( H = H_+ \oplus H_- \) to a pair of infinite-dimensional subspaces and a grading operator \( \epsilon \), such that \( H_\pm \) has grade \( \pm 1 \). The group \( U_{\text{res}}(H) \) consists the of unitaries \( g \) such that \([\epsilon, g]\) is compact. In fact, instead of compactness one can as well require that \([\epsilon, g]\) belongs to some fixed Schatten ideal \( L_p \) of bounded operators \( A \) such that \(|A|^p \) is trace-class, with \( 1 \leq p < \infty \), [P].

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1. Invariants for twisted K theory classes over a 3-manifold

Let \( M \) be an oriented compact connected 3-manifold. Fix a triangulation of \( M \) by a finite number of closed sets \( \Delta_\alpha \subset M \), where each \( \Delta_\alpha \) is parametrized by a standard 3-simplex (tetraed). We may assume without restriction that when the closed sets \( \Delta_\alpha \) are slightly extended to open sets \( U_\alpha \) then \( \{U_\alpha\} \) is a good cover of \( M \), i.e., all the multiple intersections of the open sets are contractible. A gerbe over \( M \) is given in terms of transition functions \( \phi_{\alpha\beta} : U_\alpha \cap U_\beta \to PU(H) \). Here \( H \) is a complex (in general, infinite-dimensional) Hilbert space. Since the open sets are contractible, we may lift these functions to maps \( \phi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to U(H) \). The lifts satisfy

\[
\phi_{\alpha\beta}(x)\phi_{\beta\gamma}(x)\phi_{\gamma\alpha}(x) = f_{\alpha\beta\gamma}(x),
\]

where \( f_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to S^1 \). Denote by \( \tau \) the Dixmier-Douady class of the gerbe, given by the above system of local functions satisfying

\[
f_{\alpha\beta\gamma}f_{\alpha\beta\eta}f_{\alpha\gamma\eta}f_{\beta\gamma\eta}^{-1} = 1
\]
on quadruple overlaps. For the logarithms of these functions we get

\[ a_{\alpha\beta\gamma\eta} = \log f_{\alpha\beta\gamma} - \log f_{\alpha\beta\eta} + \log f_{\alpha\gamma\eta} - \log f_{\beta\gamma\eta} = 2\pi in \]

for some integer \( n \). The sum of \( a_{\alpha\beta\gamma\eta} \)'s evaluated at the vertices \( \Delta_{\alpha\beta\gamma\eta} \) is then equal to \( 2\pi i \) times the integral of the Dixmier-Douady class over the 3-manifold \( M \). This can be written as \( 2\pi ik \), where \( k \) is an integer depending only on the Dixmier-Douady class.

A twisted \( K^1 \) theory class with a gerbe as input is then given by a family of functions \( g_\alpha : U_\alpha \to U_1(H) \). Here \( U_1(H) \) is the group of unitaries \( g \) in \( H \) such that \( g - 1 \) is trace-class. On the overlaps \( U_{\alpha\beta} \),

\[ g_\alpha = \phi_{\alpha\beta} g_\beta \phi_{\alpha\beta}^{-1}. \]

We want to determine a homotopy invariant for this class \([\omega] \in K^1(M,\tau)\).

First let us recall that an untwisted \( K \) theory class is the homotopy class of a globally defined function \( g : M \to U_1(H) \). A homotopy invariant for this is

\[ I_M(g) = \frac{1}{24\pi^2} \int_M \text{tr} (g^{-1}dg)^3, \]

i.e., the Witten action. This is an integer depending on the homotopy class of \( g \).

In the twisted case we could try to use the formula

\[ \sum_\alpha \int_{\Delta_\alpha} \text{tr} (g_\alpha^{-1}dg_\alpha)^3. \]

However, this fails to be homotopy invariant due to boundary terms in integration by parts. Instead, we can add correction terms

\[ r_{\alpha\beta} = \int_{\Delta_{\alpha\beta}} \omega_{\alpha\beta} \]

with

\[ \omega_{\alpha\beta} = \frac{1}{8\pi^2} \text{tr} (d\phi_{\alpha\beta} \phi_{\alpha\beta}^{-1})[dg_\alpha g_\alpha^{-1} + g_\alpha^{-1}dg_\alpha + g_\alpha d\phi_{\alpha\beta} \phi_{\alpha\beta}^{-1}g_\alpha^{-1} - d\phi_{\alpha\beta} \phi_{\alpha\beta}^{-1}]. \]
Note that the second and the third term in the brackets are not trace-class operators but their difference is. These correction terms are chosen such that

\[(1.7) \quad d\omega_{\alpha\beta} = \frac{1}{24\pi^2} \text{tr} [(g_{\beta}^{-1}dg_{\beta})^3 - (g_{\alpha}^{-1}dg_{\alpha})^3].\]

Suppose for a moment that all $f_{\alpha\beta\gamma} = 1$. Then

\[(1.8) \quad \omega_{\alpha\beta} + \omega_{\beta\gamma} + \omega_{\gamma\alpha} = 0\]
on triple overlaps of open sets. Define

\[(1.9) \quad I'_M(g) = \sum_{\alpha} I_{\Delta_{\alpha}} + \sum_{\alpha < \beta} r_{\alpha\beta}\]

where we have chosen the finite index set to be $\{1, 2, \ldots, p\}$ so that we have a natural ordering $\alpha < \beta$. Then it is a direct consequence of Stokes’ theorem, the Čech - de Rham cocycle relations (1.7), (1.8), and closedness of the forms $\text{tr} (g^{-1}dg)^3$ that $I'_M(g)$ is a homotopy invariant.

However, in the case of a nontrivial gerbe the functions $f_{\alpha\beta\gamma} \neq 1$ and the cocycle relation (1.8) does not hold. The correct relation is

\[(1.10) \quad \omega_{\alpha\beta} + \omega_{\beta\gamma} + \omega_{\gamma\alpha} = d\omega_{\alpha\beta\gamma},\]

where $\omega_{\alpha\beta\gamma}$’s are 1-forms on triple overlaps. A solution of (1.10) is given by

\[(1.11) \quad \omega_{\alpha\beta\gamma} = \frac{1}{4\pi^2} h^{-1} dh \log f_{\alpha\beta\gamma},\]

where $h: M \to S^1$ is the globally defined function $h = \det g_{\alpha}$. The choice of the index $\alpha$ is unimportant, since $g_{\alpha} = \phi_{\alpha\beta}g_{\beta}\phi_{\alpha\beta}^{-1}$ so that the determinant is well defined. However, we have to make a choice of the logarithm $\log f_{\alpha\beta\gamma}$. Two different choices differ by the locally constant function $n \cdot 2\pi i$ and give two different solutions to (1.9). In any case, the cocycle property (1.2) shows that

\[(1.12) \quad \omega_{\alpha\beta\gamma} - \omega_{\alpha\beta\eta} + \omega_{\alpha\gamma\eta} - \omega_{\beta\gamma\eta} = a_{\alpha\beta\gamma\eta} h^{-1} dh\]
on quadruple overlaps. Now we make the additional assumption that the function $h : M \to S^1$ is contractible (which would be automatic if $M$ is simply connected). Then we can write

$$a_{\alpha\beta\gamma\eta}h^{-1}dh = d(a_{\alpha\beta\gamma\eta} \log h) \equiv d\omega_{\alpha\beta\gamma\eta}$$

with some choice of logarithm of $h$. Different choices of the logarithm lead to expressions for $\omega_{\alpha\beta\gamma\eta}$ which differ by $(2\pi i)^2$ times an integer.

From this we reduce, by Stokes’ theorem:

**Theorem 1.** Let the determinant function $h$ defined above be contractible. Then the expression

$$I(g) = \sum_\alpha \int_{\Delta_\alpha} \omega_\alpha - \sum_{\alpha < \beta} \int_{\Delta_{\alpha\beta}} \omega_{\alpha\beta} + \sum_{\alpha < \beta < \gamma} \int_{\Delta_{\alpha\beta\gamma}} \omega_{\alpha\beta\gamma} + \sum_{\alpha < \beta < \gamma < \eta} \omega_{\alpha\beta\gamma\eta}$$

is a homotopy invariant; the last term is evaluated at the points $\Delta_{\alpha\beta\gamma\eta}$.

**Remark 1** The quantity $I(g)$ is only well defined modulo $k \times$ an integer. This is because of the arbitrary choice of the branch of the logarithm of $h$. The difference between two choices gives a contribution

$$\delta = 2\pi i \cdot \frac{1}{4\pi^2} \sum a_{\alpha\beta\gamma\eta}.$$

The sum of the numbers $a_{\alpha\beta\gamma\eta}$ is equal to $2\pi ik \times$ an integer, where $k$ is an integer depending only on the Dixmier-Douady class $\tau$ of the gerbe. Thus $\delta$ is equal to $k \times$ an integer and $I(g)$ is well defined mod $k$.

**Remark 2** In the case when $h$ is not contractible we can still use it to define the winding number invariant for the K theory class,

$$w(h) = \frac{1}{2\pi i} \int_{S^1} h^{-1}dh,$$

where $S^1 \subset M$ represents any element of $\pi_1(M)$.

**Example** Take $M = S^3 = SU(2)$. Then $H^3(M, \mathbb{Z})$ is one dimensional, the Dixmier-Douady class $\tau$ is represented as $k$ times the basic 3-form $\frac{1}{24\pi^2} \text{tr} (g^{-1}dg)^3$ on $SU(2)$. The map $I$ takes values in $\mathbb{Z}/k\mathbb{Z}$. 
2. Calculations in the case \( G = SU(2) \)

We study the twisted K theory class over the group \( G = SU(2) \). The Lie algebra of \( G \) is denoted by \( \mathfrak{g} \). Let \( A \) denote the space of smooth \( \mathfrak{g} \) valued vector potentials (1-forms) on the unit circle \( S^1 \). Let \( LG \) be the group of smooth loops in \( G \) and let \( \Omega G \subset LG \) be the group of based loops, i.e., loops \( f \) such that \( f(1) \) is the neutral element in \( G \). Then \( A/\Omega G \) is the group \( G \) of holonomies around the circle. The right action on \( A \) is defined by \( A f = f^{-1} A f + f^{-1} df \). The twisted K theory classes are constructed using the family of hermitean operators \( Q_A \) for \( A \in A \) constructed in [M].

The operator \( Q_A \) is a sum of a 'free' supercharge \( Q \) and an interaction term \( \hat{A} \). The Hilbert space \( H \) is a tensor product of a 'fermionic' Fock space \( H_f \) and a 'bosonic' Hilbert space \( H_b \). The space \( H_b \) carries an irreducible representation of the loop algebra \( Lg \) of level \( k \) where The highest weight representations of level \( k \) are classified by the \( SU(2) \) representation of dimension \( 2j_0 + 1 \) on the 'vacuum sector'. We denote the generators of the loop algebra by \( T^a_n \), where \( n \in \mathbb{Z} \) is the Fourier index and \( a = 1, 2, 3 \) labels a basis of \( \mathfrak{g} \). The commutation relations are

\[
[T^a_n, T^b_m] = \lambda_{abc} T^c_{n+m} + \frac{k}{4} \delta_{ab} \delta_{n,-m},
\]

where \( a, b, c = 1, 2, 3 \) are the structure constant of \( \mathfrak{g} \); in this case when \( \mathfrak{g} \) is the Lie algebra of \( SU(2) \) the nonzero structure constants are completely antisymmetric and we use the normalization \( \lambda_{123} = \frac{1}{\sqrt{2}} \). (This comes from a normalization of the basis vectors \( T^a_0 \in \mathfrak{g} \) with respect to the Killing form.) In addition, we have the hermiticity relations \( (T^a_n)^* = -T^a_{-n} \). With this normalization of the basis, \( k \) is a nonnegative integer and \( 2j_0 = 0, 1, 2 \ldots k \). The case \( k = 0 \) corresponds to a trivial representation and we shall assume in the following that \( k \) is strictly positive.

The Fock space \( H_f \) carries an irreducible representations of the canonical anticommutation relations (CAR),

\[
\psi^a_n \psi^b_m + \psi^b_m \psi^a_n = 2 \delta_{ab} \delta_{n,-m},
\]
and \((\psi_{n}^{a})^{*} = \psi_{-n}^{a}\). The representation is fixed by the requirement that there is an irreducible representation of the Clifford algebra \(\{\psi_{0}^{a}\}\) in a subspace \(H_{f,vac}\) such that \(\psi_{n}^{a} v = 0\) for \(n < 0\) and \(v \in H_{f,vac}\).

The central extension of the loop algebra at level 2 is represented in \(H_{f}\) through the operators

\[
K_{n}^{a} = -\frac{1}{4} \sum_{b,c=1,2,3;m \in \mathbb{Z}} \lambda_{abc} \psi_{n-m}^{b} \psi_{m}^{c},
\]

that is,

\[
[K_{n}^{a}, K_{m}^{b}] = \lambda_{abc} K_{n+m}^{c} + \frac{1}{2} n \delta_{ab} \delta_{n,-m}.
\]

We set \(S_{n}^{a} = T_{n}^{a} + K_{n}^{a}\). This gives a representation of the loop algebra at level \(k + 2\) in the tensor product \(H = H_{f} \otimes H_{b}\).

Next we define

\[
Q = i \psi_{n}^{a} T_{-n}^{a} + \frac{i}{3} \psi_{n}^{a} K_{-n}^{a}.
\]

This operator satisfies \(Q^{2} = h\), where \(h\) is the hamiltonian of the supersymmetric Wess-Zumino-Witten model,

\[
h = -\sum_{a,n} : T_{n}^{a} T_{-n}^{a} : + \frac{k+2}{8} \sum_{a,n} : n \psi_{n}^{a} \psi_{-n}^{a} : + \frac{1}{8},
\]

where the normal ordering :: means that the operators with negative Fourier index are placed to the right of the operators with positive index, \(: \psi_{-n}^{a} \psi_{n}^{b} : = -\psi_{n}^{b} \psi_{-n}^{a}\) if \(n > 0\) and \(AB := AB\) otherwise. In the case of the bosonic currents \(T_{n}^{a}\) the sign is + on the right-hand-side of the equation. See [KT] for details on the supersymmetric current algebra.

Finally, \(Q_{A}\) is defined as

\[
Q_{A} = Q + i \tilde{k} \psi_{n}^{a} A_{-n}^{a}
\]

where the \(A_{n}^{a}\)'s are the Fourier components of the \(g\)-valued function \(A\) in the basis \(T_{n}^{a}\) and \(\tilde{k} = \frac{k+2}{4}\). All the formulas above can be generalized in a straight-forward
way to arbitrary simple Lie algebras, with the modification that the last term $1/8$ in (2.6) is replaced by $\dim g/24$ and the level $k$ is quantized as integer times twice the length squared of the longest root with respect to the dual Killing form.

The basic property of the family of self-adjoint Fredholm operators $Q_A$ is that it is equivariant with respect to the action of the central extension of the loop group $LG$. Any element $f \in LG$ is represented by a unitary operator $S(f)$ in $H$ but the phase of $S(f)$ is not uniquely determined. The equivariantness property is

$$S(f^{-1})Q_A S(f) = Q_{Af}$$

with $Af = f^{-1}Af + f^{-1}df$. The infinitesimal version of this is

$$[S_n^a, Q_A] = i\tilde{k}(n\psi^n_a + \sum_{b,c;m} \lambda_{abc} \psi^b_m A^c_{n-m})$$

which can be checked directly from (2.1), (2.2), and (2.4).

The group $LG$ can be viewed as a subgroup of the group $PU(H)$ through the projective representation $S$. The space $A$ of smooth vector potentials on the circle is the total space for a principal bundle with fiber $\Omega G \subset LG$. Since now $\Omega G \subset PU(H)$, $A$ may be viewed as a reduction of a $PU(H)$ principal bundle over $G$. The $\Omega G$ action by conjugation on the Fredholm operators in $H$ defines an associated vector bundle $Q$ over $G$ and the family of operators $Q_A$ defines a section of this vector bundle. Thus $\{Q_A\}$ is a twisted $K^1$ theory class over $G$ where the twist is determined by the level $k+2$ projective representation of $LG$.

Using the method in [AS] we replace the family of unbounded hermitean operators by a family of bounded operators $F_A = Q_A/(|Q_A| + e^{-Q_A^2})$ which represent the same $K$ theoretic class. The perturbation in the denominators is introduced to avoid singularities with zero modes of $Q_A$. The operator $F_A$ differs from the sign operator $Q_A/|Q_A|$ by a trace-class perturbation. For this reason the unitary operators $g_A = -e^{i\pi F_A}$ differ from the unit by a trace-class operator.

We shall now study the twisted $K$ theory class represented by the family $g_A$ of unitary operators. Note that this family is still gauge equivariant,

$$S(f)^{-1}g_A S(f) = g_{Af}$$
where \( f \in LG \).

Since \( S^3 = A/\Omega G \), we write the K theory class as a function from the three dimensional unit disk \( D^3 \) to unitaries of the form \( 1 + \text{trace-class operators} \) such that on the boundary \( S^2 \) the operators are gauge conjugate. Concretely, this is achieved as follows. For each point \( \mathbf{n} \in S^2 \) we define a constant \( SU(2) \) vector potential \( A(\mathbf{n}) = \frac{1}{2i} \mathbf{n} \cdot \sigma \). Pauli matrices satisfy \( \sigma_1 \sigma_2 = i \sigma_3 \) (and cyclic permutations) and \( \sigma_j^2 = 1 \). The holonomy around the circle \( S^1 \) is equal to \(-1\) for each of the potentials \( A(\mathbf{n}) \), thus they belong to the same \( \Omega G \) orbit in \( A \). Next we define a disk \( D^3 \) of potentials \( A(t, \mathbf{n}) = tA(\mathbf{n}) \) where \( 0 \leq t \leq 1 \) is the radial variable in the disk \( D^3 \) and \( \mathbf{n} \) are the angular coordinates. This disk projects to a closed sphere \( S^3 \) in \( G = A/\Omega G \). For each \( A \in D^3 \) we have the corresponding supercharge
\[
Q_A = Q + \frac{k+2}{4} t \cdot \sqrt{2} \psi_0^a n^a
\]
where the factor \( \sqrt{2} \) comes from the normalization of the basis \( T^a \) of \( g \) relative to the Pauli matrix basis.

Now we have a family of unitaries \( g(t, \mathbf{n}) = g_{A(t,\mathbf{n})} \) which are gauge conjugate on the boundary through the projective unitary representation of \( LG \) of level \( k + 2 \). This means that the homotopy class of the functions \( g(t, \mathbf{n}) \) gives an element in \( K^1_G(S^3, k + 2) \). In the language of section 1, we may replace the triangulation \( \{ \Delta_\alpha \} \) by two sets: the disk \( D^3 \) as the southern hemisphere of \( S^3 \) and a second disk \( D^3 \) as the northern hemisphere. On the southern hemisphere we have the unitary matrix valued function \( g(t, \mathbf{n}) \) whereas on the northern hemisphere we have a constant function \( g_0 = -\exp(\pi i F_0) \). On the equator parametrized by \( \mathbf{n} \in S^2 \) they are all gauge conjugate.

The \( G \) equivariantness follows from the fact that the family \( Q_A \) is gauge equivariant with respect to the full group \( LG \) of gauge transformations and not only with respect to the based gauge transformations \( \Omega G \).

We want to compute the quantum invariant for the class \([g]\) by evaluating the Witten functional
\[
I(g) = \frac{1}{24\pi^2} \int_{D^3} \text{tr}(g^{-1}dg)^3.
\]
Note that in the present setting the correction terms are absent since $g_0$ is constant (which can be deformed to the unit matrix since the group of unitaries $U_1(H)$ is connected). A direct computation of the integral of the trace in an infinite-dimensional Hilbert space $H$ is difficult. Instead, we shall apply first various homotopy deformations to $g$ to bring the trace into more manageable form.

**First deformation.** We need first a Lemma:

**Lemma 1.** The spectral projections $P_\Lambda$ of $|Q|$ commute with $Q_A$ when $A = \frac{1}{2i} t n \cdot \sigma$.

**Proof.** Now $Q_A$ is given by (2.11). Using the canonical anticommutation relations for $\psi^a_n$'s we observe that

$$[Q_A - Q, Q]_+ = -2\tilde{k} S^a_0 A^a_0.$$

On the other hand, $[S^a_0, Q] = 0$ so that

$$[Q_A, Q^2] = [Q_A - Q, Q^2] = 2\tilde{k} (-S^a_0 A^a_0 Q + Q S^a_0 A^a_0) = 0$$

from which follows $[Q_A, |Q|] = 0$ and thus also $[Q_A, P_\Lambda] = 0$ where $P_\Lambda$ is the spectral projection $|Q| \leq \Lambda$.

The Lemma implies that the spectral subspaces $H_\Lambda = P_\Lambda H$ and $H^\perp_\Lambda$ are invariant under $Q_A, F_A, \text{and } g_A$.

Since $(Q_A - Q)^2 = 2t^2 \tilde{k}^2$ we see that the restriction of $Q_A$ to the subspace $H^\perp_\Lambda$ is invertible if we choose $\Lambda > \sqrt{2}\tilde{k}$.

Let us deform the denominator $|Q_A| + e^{-Q_A^2}$ in $F_A$. Define $D(s) = |Q(t, n)| + st(1 - t)e^{-Q^2} + (1 - s)e^{-Q(t,n)^2}$ for $0 \leq s \leq 1$. For any fixed $s$ these operators are gauge conjugate at the boundary $t = 1$ because at $t = 1$ we have $D(s) = |Q(1, n)| + (1 - s)e^{-Q(t,n)^2}$. At $s = 0$ this is the original family of denominators whereas for $s = 1$ we get $D(1) = |Q(t, n)| + t(1 - t)e^{-Q^2}$. This is our first deformation: We replace the original $g(t, n)$ by the homotopy equivalent family

$$g(t, n) = -e^{i\pi F(t, n)} \text{ with } F(t, n) = \frac{Q(t, n)}{|Q(t, n)| + t(1 - t)e^{-Q^2}}.$$
Second deformation By a similar $s$ dependent family as above we can replace the denominator $D(n)$ by $|Q(t, n)| + t(1 - t)P_{\Lambda}$ for any $\Lambda > \tilde{k}$. This is because $Q(t, n)$ is invertible in the complement of $H_{\Lambda}$ and is invertible in the whole space $H$ for $t = 0, 1$. (For $t = 0$ this is clear since $Q^2 \geq 1/8$ and for $t = 1$ one observes that the spectrum of $Q(1, n)^2 = Q^2 + \frac{1}{8}(k + 2)^2 + i\sqrt{2}(k + 2) \mathbf{n} \cdot \mathbf{S}_0$ is of the form $\frac{1}{8}[1 + (k + 2)p]$ where $p$ is an integer.) For the intermediate values $0 < t < 1$ both $t(1 - t)e^{-Q^2}$ and $t(1 - t)P_{\Lambda}$ are strictly positive in $H_{\Lambda}$. So after the second deformation

\begin{equation}
(2.13) \quad g(t, n) = -e^{i\pi F(t, n)} \text{ with } F(t, n) = \frac{Q(t, n)}{|Q(t, n)| + t(1 - t)P_{\Lambda}}.
\end{equation}

In particular, since the eigenvalues of $Q/|Q|$ are $\pm 1$, the restriction of $g$ to $H_{\Lambda}^\perp$ is equal to the unit operator. Thus

\begin{equation}
(2.14) \quad I(g) = \frac{1}{24\pi^2} \int_{D^3} \text{tr}_{H_{\Lambda}}(g^{-1}dg)^3.
\end{equation}

We need now to compute the trace of $(g^{-1}dg)^3$ only in the finite-dimensional subspace $H_{\Lambda}$. We use the formula

\begin{equation}
(2.15) \quad \text{tr}(g^{-1}dg)^3 = d\text{tr} \, dX \eta(ad_X) dX,
\end{equation}

where $X = \log(g)$ and $\eta(x) = \frac{\sinh(x) - x}{x^2}$. By Stokes’ theorem the integral defining $I(g)$ is then equal to the integral of the 2-form $dX \eta(ad_X) dX$ over $S^2 = \partial D^3$. But on the boundary $t = 1$ we have $F(1, n) = Q(1, n)/|Q(1, n)|$. This simplifies $\eta(ad_X)$ so that the 2-form becomes

\[\frac{i}{16\pi} \text{tr} FdFdF \text{ for } F = F(1, n).\]

Summarizing we obtain

\begin{equation}
(2.16) \quad I(g) = \frac{i}{16\pi} \int_{S^2} \text{tr}_{H_{\Lambda}} FdFdF.
\end{equation}
**Third deformation** We use the fact that the parameter $\Lambda$ is free except for the constraint $\Lambda > \sqrt{2k}$. Since the spectrum of $Q$ is discrete (the eigenvalues of $Q^2$ are quantized in units $(k + 2)/2$), we can choose $\Lambda - \sqrt{2k}$ so small that the eigenvalues of $|Q|^2$ which are smaller or equal to $\Lambda^2$ are also strictly smaller than $2k^2$. With this choice $Q_A$ becomes invertible in $H_A$. Furthermore, also $Q_s(n) = sQ + \sqrt{2k}n \cdot \psi_0$ is invertible in $H_A$ for all $0 \leq s \leq 1$. We use the homotopy $Q_s$ to replace $F = Q(1, n)/|Q(1, n)|$ in the integral $I(g)$ by the operator $F = n \cdot \psi_0$. Now

$$\text{(2.17)} \quad \text{tr}_{H_A} FdFdF = n \cdot dn \times dn \text{tr}_{H_A} \psi_0^1 \psi_0^2 \psi_0^3.$$  

The integral of $n \cdot dn \times dn$ over $S^2$ is equal to twice the volume of $S^2$ and so

$$\text{(2.18)} \quad I(g) = -\frac{i}{2} \text{tr}_{H_A} \Gamma \text{ with } \Gamma = \psi_0^1 \psi_0^2 \psi_0^3.$$  

The trace is essentially the Witten index. The operator $\Gamma$ almost anticommutes with the supercharge $Q$. Define

$$\text{(2.19)} \quad Q_+ = i \sum_{n \neq 0} \psi_n^a T_{-n}^a + \frac{i}{12} \sum_{n, m, n + m \neq 0} \lambda_{abc} \psi_n^a \psi_m^b \psi_{-n-m}^c.$$  

We have $Q_+ \Gamma = -\Gamma Q_+$ since $\psi_0^a$ anticommutes with $\psi_n^b$ for every $n \neq 0$.

**Lemma 2.** The operator $Q_+$ commutes with the spectral projections $P_\Lambda$.

**Proof.** Write $Q = Q_0 + Q_+$. Then $Q_0$ commutes with $\Gamma$. Now

$$Q^2 = Q_0^2 + Q_+^2 + [Q_0, Q_+] = h$$

is even with respect to $\Gamma$. The first two terms on the right are even, so the third term which is odd has to vanish and so $h = Q_0^2 + Q_+^2$. This implies $[h, Q_+] = [Q_0^2, Q_+] = Q_0[Q_0, Q_+] + [Q_0, Q_+] Q_0 = 0$ and so the spectral projections of $h$ commute with $Q_+$. Since $|Q|^2 = h$, the same is true for the spectral projections $P_\Lambda$ of $|Q|$.

**Lemma 3.** $Q_+^2 = Q_0^2 - \sum_a (T_{0}^a + K^a_{0}^2) - \frac{N}{2}$ where $K^a_{0} = -\frac{1}{4} \sum_{n \neq 0; b, c} \lambda_{abc} \psi_n^b \psi_{-n}^c$. The $K^a_{0}$’s satisfy the same commutation relations as the $T^a$’s.

**Proof.** By a direct computation.
Lemma 4. Let $G = SU(2)$. Then the kernel of $Q_+$ in $H_\Lambda$ is equal to the vacuum sector $H_0 \subset H$ consisting of eigenvectors of $h$ associated to the minimal eigenvalue $1/8$.

Proof. Clearly $H_0 \subset \text{ker} Q_+$ since $\psi^a_n v = T^a_n v = 0$ for any $v \in H_0$ for $n < 0$. We have to show that $|Q_+|$ is strictly positive in the orthogonal complement $H_0^\perp$ in $H_\Lambda$.

Let $d$ be the derivation in the affine Lie algebra based on $SU(2)$. By definition, $[d, T^a_n] = nT^a_n$ and $[d, \psi^a_n] = n\psi^a_n$. From the weight inequalities for lowest weight representations of affine Lie algebras, [K], Prop. 11.4, follows that in a $SU(2)$ subrepresentation with angular momentum $\ell$,

$$\ell_0(\ell_0 + 1) - d_0(k + 2) \geq \ell(\ell + 1) - d(k + 2),$$

where $\ell_0$ is the angular momentum of the lowest weight vector and $d_0$ is the eigenvalue of $d$ for the lowest weight vector.

We first apply the inequality to the bosonic representation in $H_b$. The bosonic hamiltonian is

$$h_b = -\sum_{a,n} : T^a_n T^a_{-n} := \frac{k + 2}{2} d_b.$$ 

Now the lowest eigenvalue of $d_b$ is equal to the eigenvalue of the Casimir operator $-\sum_a T^a_0 T^a_0$ which is equal to $\frac{1}{2} j_0(j_0 + 1)$ where $j_0 = 0, \frac{1}{2}, 1, \ldots k/2$ labels the vacuum representation of $SU(2)$. Thus we obtain

$$h_b - \frac{1}{2} j(j + 1) \geq 0$$

for any $SU(2)$ representation $j$ contained in $H_b$.

Similarly, on the fermionic sector $H_f$ we have

$$d_f \geq \frac{1}{2} \ell(\ell + 1)$$

since the vacuum eigenvalue of $d_f = \frac{k + 2}{2} h_f$ is zero; here $h_f = \frac{k + 2}{8} n : \psi^a_n \psi^a_{-n} :$ and $\frac{1}{2} \ell(\ell + 1)$ is the eigenvalue of the invariant $-\sum_a K'^a_0 K'^a_0$ in a given irreducible.
representation. This inequality follows from the anticommutation relations of the
fermion operators $\psi^a_n$: In order to increase the value of $\ell$ from zero (in the vacuum)
to a given value $\ell$ one must apply the fermion operators at least for energies $n = 1, 2, \ldots, \ell$ which leads to the eigenvalue $\frac{1}{2}\ell(\ell + 1)$ for $d_f$.

Thus we have

$$h_f = \frac{k+2}{2}d_f \geq \frac{k+2}{4}\ell(\ell + 1).$$

Now by Lemma 3, in a given $(j, \ell)$ subrepresentation of the commuting algebras
$(T^a_0)$ and $(K'^a_0)$,

$$Q_+^2 = h_b + h_f - \frac{1}{2}(j + \ell)(j + \ell + 1) \geq h_f - \frac{1}{2}\ell(\ell + 1) - j\ell$$

(2.21)

$$\geq \frac{k+2}{4}\ell(\ell + 1) - \frac{1}{2}\ell(\ell + 1) - j\ell \geq \ell\left(\frac{k}{4}(\ell + 1) - j\right).$$

In the subspace $H_\Lambda$ we have $\frac{(k+2)^2}{8} \geq h \geq h_b \geq \frac{1}{2}j(j+1)$ so that $j \leq (k+1)/2$.

For $\ell \geq 2$ the right-hand-side of (2.21) is strictly greater than zero. In the case
$\ell = 0$ we have $Q_+^2 = h_f + h_b - \frac{1}{2}j(j+1) - \frac{1}{8}$ and the claim follows from the fact
that $h_b - \frac{1}{2}j(j+1)$ vanishes only on the vacuum sector. The remaining case $\ell = 1$
is clear from (2.21) if $j < k/2$. But since we restrict to the subspace $H_\Lambda$ where
$h \leq (k+2)^2/8$ the cases $j \geq k/2$ are excluded by the energy inequalities

$$h = h_b + h_f + \frac{1}{8} \geq \frac{1}{2}j(j+1) + \frac{k+2}{4}\ell(\ell + 1) + \frac{1}{8} = \frac{1}{2}j(j+1) + \frac{k+2}{2} + \frac{1}{8}$$

for $\ell = 1$.

**Theorem 2.** The family of operators $Q_A$ defined by the weight $(k,j_0)$ of a highest
weight representation of $LG$ defines an element in $K^1_G(G, k + 2)$ for $G = SU(2)$.
The value of the invariant $I \mod k + 2$ for this $K$ theory class is equal to $2j_0 + 1$
and therefore they are inequivalent for the allowed values $2j_0 = 0, 1, 2, \ldots, k$.

**Proof.** By (2.18) the value of the invariant $I(g)$ is given as

(2.22)$$I(g) = -\frac{i}{2}\text{tr}_\ker Q_+\Gamma$$

since $Q_+$ anticommutes with $\Gamma$. But since the kernel of $Q_+$ is equal to $H_0$ and
$\Gamma = \psi_0^1\psi_0^2\psi_0^3 = \sigma_1\sigma_2\sigma_3 = i$ on the vacuum sector, we get $I(g) = \frac{1}{2}\dim H_0 = 2j_0 + 1$,
where we have taken into account that the dimension of $H_{f,vac}$ is two. In particular, it follows that the trivial one dimensional representation $j_0 = 0$ gives the generator in $\mathbb{Z}/(k+2)\mathbb{Z}$.

**Remark** The construction of the operators $Q_A$ works for any semisimple compact group $G$, [M]. However, the twisted K theory classes are not parametrized by a single invariant $I(g)$. Instead, one should study reductions of the K theory classes to various $SU(2)$ subgroups corresponding to a choice of simple roots of $G$.

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