Non-Existence of Finite Order Solution of Non-homogeneous Second Order Linear Differential Equations

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Abstract. In this paper, we have considered second order non-homogeneous linear differential equations having entire coefficients. We have established conditions ensuring non-existence of finite order solution of such type of differential equations.

1. Introduction

The value distribution theory of meromorphic functions or Nevanlinna theory is a useful tool in the study of complex linear differential equations. We assume readers are familiar with the notations and basic results of Nevanlinna theory. Nevertheless, we have inserted some results for the non-initiated readers. We denote the order of growth, the hyper-order of growth and exponent of convergence of non-zero zeros of an entire function \( f(z) \) by \( \rho(f) \), \( \rho_2(f) \) and \( \lambda(f) \), respectively.
Throughout we consider second order non-homogeneous linear differential equation of the form:

\[
f'' + A(z)f' + B(z)f = H(z)
\]

where \( A(z), B(z) \neq 0 \) and \( H(z) \neq 0 \) are entire functions. The associated homogeneous linear differential equation of the equation (1) is

\[
f'' + A(z)f' + B(z)f = 0
\]

A necessary and sufficient condition for solutions of equation (1) to be of finite order is that \( A(z) \) and \( B(z) \) are polynomials and \( H(z) \) is an entire function of finite order \( \rho \). Therefore, for the existence of infinite order solutions of equation (1), at least one of \( A(z) \) or \( B(z) \) is a transcendental entire function. In [3, 4, 5], authors have established

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conditions on the coefficients of associated homogeneous differential equation \( (2) \) for existence of non-trivial solutions of infinite order.

We now give the outline of our paper. In section 2, for the sake of completeness we give some crucial results of Gundersen, Bank et al., Kwon and Zongxuan which plays a vital role in proofs of our results. The final section 3 contains proofs of our main theorems including results in form of lemmas. These lemmas have been proved for making the steps of the proofs of the theorems easier to read. Final section also contains examples to illustrate validity and wider applicability of the theorems. Before stating our main results, we would like to point out that the hyper-order is crucially used as a measure of growth of entire function of infinite order. Now we state our main results.

In the following result, we have shown that all non-trivial solutions of equation \( (2) \) of infinite order has finite hyper-order of growth.

**Theorem 1.** Suppose \( A(z) \) is an entire function with \( \lambda(A) < \rho(A) \) and \( B(z) \) is a transcendental entire function of finite order satisfying

(a) \( \rho(A) \neq \rho(B) \) or

(b) \( B(z) \) has fabry gap.

Then all non-trivial solutions \( f(z) \) of the equation \( (2) \) satisfies

\[
\rho_2(f) = \max\{\rho(A), \rho(B)\}.
\]

Wang and Laine [8] have established that when \( \rho(A) = \rho(B) \) and \( \rho(H) < \max\{\rho(A), \rho(B)\} \), then all solutions of equation \( (1) \) are of infinite order. In our case we have assumed \( \rho(A) \neq \rho(B) \) and proved the following theorem:

**Theorem 2.** Suppose \( A(z) \) is an entire function with \( \lambda(A) < \rho(A) \) and \( B(z) \) a transcendental entire function satisfying \( \rho(B) \neq \rho(A) \). Furthermore, suppose \( H(z) \) is an entire function satisfying \( \rho(H) < \max\{\rho(A), \rho(B)\} \), then all solutions \( f(z) \) of the equation \( (1) \) are of infinite order.

Our final result gives the hyper-order of solution of equation \( (1) \) with the help of order of its coefficients:

**Theorem 3.** Suppose \( A(z) \) and \( B(z) \) are entire functions of finite order. Then all solutions \( f(z) \) in Theorem 2 satisfies

\[
\rho_2(f) = \max\{\rho(A), \rho(B)\}.
\]

2. preliminary results

In this section we mention the results which have been used to prove our results. For a set \( E \subset (0, \infty) \) we denote linear measure, logarithmic measure, upper logarithmic density and lower logarithmic density of the set \( E \) by \( m(E) \), \( m_1(E) \), \( \log \text{dens}(E) \) and \( \log \text{dens}(E) \), respectively. The following result of Gundersen [2] plays a pivotal role in the proofs of our results:
**Lemma 1.** Let $f(z)$ be a transcendental meromorphic function and let $\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m)\}$ denote finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $i = 1, 2, \ldots, m$. Let $\alpha > 1$ and $\epsilon > 0$ be given real constants. Then there exists a set $E \subset (1, \infty)$ with $m(E)$ is finite and there exists a constant $c > 0$ that depends only on $\alpha$ and $\Gamma$ such that for all $z$ satisfying $|z| = r \notin E \cup [0, 1]$ and for all $(k, j) \in \Gamma$, we have

$$
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq c \left( \frac{T(\alpha r, f)}{r} \right) \log \alpha r \log T(\alpha r, f)^{(k-j)}
$$

If $f(z)$ is of finite order then $f(z)$ satisfies:

$$
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\epsilon)}
$$

for all $z$ satisfying $|z| \notin E \cup [0, 1]$ and $|z| \geq R_0$ and for all $(k, j) \in \Gamma$.

The next lemma is used to establish estimates for the transcendental entire function:

**Lemma 2.** Let $A(z) = v(z)e^{P(z)}$ be an entire function, where $P(z)$ is a polynomial of degree $n$ and $v(z)$ be an entire function with order less than $n$. Then for every $\epsilon > 0$ there exists $E \subset [0, 2\pi)$ of linear measure zero such that

(i) for $\theta \in [0, 2\pi)$ such that $\delta(P, \theta) > 0$, there exists $R > 1$ such that

$$
\exp\left((1 - \epsilon)\delta(P, \theta)r^n\right) \leq |A(re^{i\theta})| \leq \exp\left((1 + \epsilon)\delta(P, \theta)r^n\right)
$$

for $r > R$;

(ii) for $\theta \in [0, 2\pi)$ such that $\delta(P, \theta) < 0$, there exists $R > 1$ such that

$$
\exp\left((1 + \epsilon)\delta(P, \theta)r^n\right) \leq |A(re^{i\theta})| \leq \exp\left((1 - \epsilon)\delta(P, \theta)r^n\right)
$$

for $r > R$.

Kwon [6] used Residue theorem to prove the following lemma:

**Lemma 3.** Let $f(z)$ be a non-constant entire function. Then there exist a real number $R > 0$ such that for all $r \geq R$ there exists $z$ with $|z| = r$ satisfying

$$
\left| \frac{f(z)}{f'(z)} \right| \leq r.
$$

The following lemma provides a lower bound for modulus of an entire function in a neighbourhood of a particular $\theta \in [0, 2\pi)$.

**Lemma 4.** Suppose $f(z)$ is an entire function of finite order $\rho$ and $M(r, f) = |f(re^{i\theta})|$ for every $r$. Given $\zeta > 0$ and $0 < C(\rho, \zeta) < 1$ there exists $0 < l_0 < \frac{1}{2}$ and a set $S \subset (1, \infty)$ with $\log\text{dens}(S) \geq 1 - \zeta$ such that

$$
e^{-5\pi}M(r, f)^{1-C} \leq |f(re^{i\theta})|
$$

for all sufficiently large $r \in S$ and for all $\theta$ satisfying $|\theta - \theta_r| \leq l_0$. 
Zongxuan [9] gave an upper bound for the hyper-order of solutions \( f(z) \) of equation (2).

**Lemma 5.** Suppose that \( A(z) \) and \( B(z) \) are entire functions of finite order. Then
\[
\rho_2(f) \leq \max\{\rho(A), \rho(B)\}
\]
for all solutions \( f(z) \) of equation (2).

3. proof of main theorems

This section contains proofs of our main theorems spread over several subsections.

3.1. Lemmas. In this subsection, we have proved some results which have been used in proofs of our main results.

In the following lemma, we give a lower bound on \( T(r, f) \), where \( f(z) \) is a non-constant meromorphic function.

**Lemma 6.** Suppose \( f(z) \) is a meromorphic function with \( \rho(f) \in (0, \infty) \). Then for each \( \epsilon > 0 \), there exists a set \( S \subset (1, \infty) \) that satisfies \( \log \text{dens}(S) > 0 \) and
\[
T(r, f) \geq r^{\rho(f)-\epsilon}
\]
for all \( r \) sufficiently large and \( r \in S \).

**Proof.** On the contrary, suppose there exists an \( \epsilon_0 > 0 \) such that for all sets \( S \subset (1, \infty) \) satisfying \( \log \text{dens}(S) > 0 \) we have
\[
T(r, f) < r^{\rho(f)-\epsilon}
\]
for all \( r \) sufficiently large and \( r \in S \). This will imply that \( \rho(f) < \rho(f) - \epsilon \), which is a contradiction.

As a consequence of Lemma 6, we have

**Remark 1.** If we take \( f(z) \) to be an entire function in Lemma 6, then the inequality (9) reduces to
\[
M(r, f) > \exp r^{\rho(f)-\epsilon}
\]
for all \( r \) sufficiently large in \( S \).

The next lemma gives a relation between characteristic functions of two meromorphic functions having different orders.

**Lemma 7.** Let \( f(z) \) and \( g(z) \) be two meromorphic functions satisfying \( \rho(g) < \rho(f) \). Then there exists a set \( S \subset (1, \infty) \) with \( \log \text{dens}(S) > 0 \) such that
\[
T(r, g) = o(T(r, f))
\]
for sufficiently large \( r \in S \).
Proof. Using the definition of \( \rho(g) \), we have for \( \epsilon > 0 \), there exists \( R(\epsilon) > 0 \) such that

\[
T(r, g) \leq r^{\rho(g) + \epsilon}
\]

for all \( r > R \). We choose \( 0 < \epsilon < \frac{\rho(f) - \rho(g)}{5} \), use equation (10) and Lemma 6 for function \( f(z) \) to obtain

\[
T(r, g) \leq r^{\rho(g) + \epsilon} \leq r^{\rho(f) - 4\epsilon} \leq r^{-3\epsilon} T(r, f)
\]

for all \( r > R \) and \( r \in S \), where \( S \subset (1, \infty) \) with \( \log \text{dens}(S) > 0 \). This proves the result.

In the following result, we have given a relation between the maximum modulus of two entire functions of different orders.

Proposition 1. Suppose \( f(z) \) and \( g(z) \) be two entire functions satisfying \( \rho(g) < \rho(f) \). Then for \( 0 < \epsilon \leq \min\{ \frac{3\rho(f)}{4}, \frac{\rho(f) - \rho(g)}{2} \} \), there exists \( S \subset (1, \infty) \) with \( \log \text{dens}(S) = 1 \) satisfying

\[
|g(z)| = o(M(|z|, f))
\]

for sufficiently large \( |z| \in S \).

Proof. From the definition of order of \( g(z) \) we get

\[
|g(z)| \leq \exp \left( r^{\rho(g) + \epsilon} \right)
\]

for all sufficiently large \( |z| = r \). Also,

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}
\]

which implies that there exists an increasing sequence \((r_m)\) satisfying

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log \log M(r_m, f)}{\log r_m}
\]

This further implies that for each \( \epsilon > 0 \) we have

\[
\rho(f) - \epsilon \leq \frac{\log \log M(r_m, f)}{\log r_m}
\]

for all large \( m \in \mathbb{N} \). Define \( S = \bigcup_{m=1}^{\infty} [r_m, r_m^2] \), where without loss of generality we may assume that \( r_m \geq 1 \) for all \( m \in \mathbb{N} \). Suppose \( r_m^2 < r < r_{m+t} \) for some \( m, t \in \mathbb{N} \). Then

\[
S \cap [1, r] \supset S \cap [1, r_m^2] \supset S \cap [r_m, r_m^2]
\]

which implies

\[
m_t(S \cap [1, r]) \geq m_t(S \cap [1, r_m^2]) \geq m_t(S \cap [r_m, r_m^2])
\]

and so

\[
\frac{m_t(S \cap [1, r])}{\log r} \geq m_t(S \cap [r_m, r_m^2]) = \frac{\log r_m}{\log r_m + t}
\]
which further implies that
\[
\overline{\log \text{dens}}(S) \geq \limsup_{m \to \infty} \frac{\log r_m}{\log r_{m+1}} \geq \lim_{m \to \infty} \frac{\log r_m}{\log r_{m+1}} \geq 1.
\]
As \(\overline{\log \text{dens}}(S) \leq 1\), therefore, \(\overline{\log \text{dens}}(S) = 1\). Suppose that \(r \in [r_m, r_m^2] \subset S\) for some \(m\). As \(M(r, f)\) is an increasing function of \(r\), we have
\[
\frac{\log \log M(r, f)}{\log r} \geq \frac{\log \log M(r_m, f)}{\log r_m^2}
\]
for some \(m \in \mathbb{N}\). Now using equation (12) we obtain
\[
\frac{\log \log M(r, f)}{\log r} \geq \frac{\rho(f)}{4} \geq \rho(f) - \epsilon
\]
where \(0 < \epsilon \leq \frac{3\rho(f)}{4}\). Thus
\[
\frac{|g(z)|}{M(r, f)} \leq \exp \left( \frac{\rho(g)+\epsilon}{r^\rho} \right) \leq \exp \left( \frac{\rho(f)-\epsilon}{r^\delta} \right) \leq \exp \left( -r^{\delta} \right)
\]
for all sufficiently large \(r \in S\) and \(\delta > 0\). This implies the desired result. \(\Box\)

**Remark 2.** As \(M(r, f)\) is an increasing function of \(r\) and approaches infinity as \(r\) tends to infinity, we have from Proposition 1, for \(0 < C < 1\),
\[
|g(z)| = o \left( M(r, f)^{(1-C)} \right)
\]
for sufficiently large \(|z| = r \in S\).

We now establish an important relation between the order of solution of equation (1) and its coefficients.

**Proposition 2.** Suppose \(A(z), B(z)\) and \(H(z)\) be entire functions with \(\rho(H) < \max\{\rho(A), \rho(B)\}\) and \(\rho(A) \neq \rho(B)\). Then all finite order solutions \(f(z)\) of equation (1) satisfies \(\rho(f) \geq \max\{\rho(A), \rho(B)\}\).

**Proof.** It is important to note here that under the given hypothesis, all solutions of equation (1) are transcendental entire functions. First suppose that \(\rho(A) < \rho(B)\). We need to show that \(\rho(f) \geq \rho(B)\). Using equation (4), first fundamental theorem of Nevanlinna, lemma of logarithmic derivatives and Lemma 7, we obtain
\[
m(r, B) \leq m \left( r, \frac{f'}{f} \right) + m(r, A) + m \left( r, \frac{f'}{f} \right) + m \left( r, \frac{H}{f} \right)
\]
\[
T(r, B) \leq O(\log r) + T(r, f) + T(r, A) + T(r, H)
\]
\[
\leq O(\log r) + T(r, f) + o(T(r, B))
\]
for all \(r > R\) and \(r \in S\) where \(\overline{\log \text{dens}}(S) > 0\). This implies that \(\rho(B) \leq \rho(f)\). On similar lines, when \(\rho(B) < \rho(A)\), one can show that \(\rho(f) \geq \rho(A)\). This completes the proof. \(\Box\)
However, the given conditions in above proposition is not sufficient. Examples are provided in the next section for its justification. Moreover, we provide some examples to justify that all conditions in Proposition 2 are necessary.

3.2. Examples. In this subsection, we provide some examples to justify that none of conditions in our results can be relaxed.

The following example shows that conclusion of Theorem 2 is not true if $\rho(A) = \rho(B)$.

**Example 1.** The differential equation

$$f'' - e^z f' + (e^z - 1)f = 2$$

has for its solution the function $f(z) = e^{-z}$ which is of finite order.

Example 1 also shows that in Proposition 2, $\rho(f) = \max\{\rho(A), \rho(B)\}$ but $\rho(H) < \rho(A) = \rho(B)$. Moreover, if $\rho(H) > \max\{\rho(A), \rho(B)\}$ and $\rho(A) \neq \rho(B)$, still conclusion of Theorem 2 is not true. The following example establishes this.

**Example 2.** The finite order entire function $f(z) = e^{z^2}$ satisfies the differential equation

$$f'' + e^{-z} f' + \cos(z^2) f = (2 + 4z^2 + 2ze^{-z} + \cos(z^2))e^{z^2}.$$ 

Example 2 also justifies that the condition given in Proposition 2 is not sufficient.

When $\rho(H) = \max\{\rho(A), \rho(B)\}$ and $\rho(A) \neq \rho(B)$, then still conclusion of Theorem 2 does not hold. Following two examples justifies this.

**Example 3.** The function $f(z) = e^z$ satisfies

$$f'' + e^{-z} f' + \cos z^2 f = (1 + \cos z^2)e^z + 1.$$ 

**Example 4.** The function $f(z) = e^{-z^2}$ satisfies the differential equation:

$$f'' + e^{z^2} f' + e^z f = (-2 + 4z^2 + e^z)e^{-z^2} - 2z$$

where $\rho(H) = \rho(A) = 2$ and $\rho(B) < \rho(A)$.

Example 3 and Example 4 justifies that conditions given in Proposition 2 cannot be relaxed. When $\rho(A) = \rho(B)$ and $\rho(H) \geq \rho(A)$, then still conclusion of Theorem 2 does not hold. Following two examples establishes this:

**Example 5.** The differential equation

$$f'' - e^z f' + (e^z + 1)f = 2e^z$$

has finite order solution $f(z) = e^z$. 

The following example shows that conclusion of Theorem 2 is not true if $\rho(A)$ equals $\rho(B)$.
Example 6. The finite order function \( f(z) = e^{z^2} \) satisfies
\[
f'' - e^z f' + 2(z e^z - 1) f = 4z^2 e^{z^2}.
\]

Next three examples justifies the fact that the hypothesis in Proposition 2 are necessary. If in the hypothesis of Proposition 2 we take \( \rho(A) = \rho(B) \), then conclusion may not hold. Following two examples justifies this.

Example 7. The non-homogeneous linear differential equation
\[
f'' + z e^z f' - (e^z - 1) f = z
\]
is satisfied by the function \( f(z) = z \).

Example 8. The differential equation
\[
f'' + e^{z^2} f' + e^{z^2} f = e^{-z}
\]
has finite order solution \( f(z) = e^{-z} \).

If in hypothesis of Proposition 2 we take \( \rho(H) = \max\{\rho(A), \rho(B)\} \) and \( \rho(A) \neq \rho(B) \), then conclusion may not hold. The following example illustrates this.

Example 9. The differential equation
\[
f'' + e^{z^2} f' + e^{-z} f = e^{z^2 + z} + e^z + 1
\]
possesses a finite order solution \( f(z) = e^z \).

3.3. Proof of Theorem 1.

Proof. (a) We know that all solutions \( f(\neq 0) \) of the equation (2) are of infinite order, when \( \rho(B) \neq \rho(A) \) by [Theorem 4, 4]. Then using Lemma 1 for \( \epsilon > 0 \), there exists a set \( E \subset [1, \infty) \) that has finite logarithmic measure such that for all \( z \) satisfying \( |z| = r \notin E \cup [0, 1] \) we have
\[
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq c [T(2r, f)]^{2(k-j)}
\]
where \( c > 0 \) is a constant.

If \( \rho(A) < \rho(B) \) then from [Theorem 1, 6] and Theorem 5 we get that \( \rho_2(f) = \max\{\rho(A), \rho(B)\} \).

If \( n = \rho(A) > \rho(B), n \in \mathbb{N} \), one can choose \( \beta \) such that \( \rho(B) < \beta < \rho(A) \). Now choose \( \theta \in [0, 2\pi) \) such that \( \delta(P, \theta) > 0 \) and a sequence \( (r_m) \) outside \( E \cup [0, 1] \) satisfying \( r_m \to \infty \) as \( m \to \infty \) such that equations (5), (7) and (14) are satisfied for \( z_m = r_m e^{i\theta} \).
Using equation (2), (5), (7) and (14) for \( z_m = r_m e^{i\theta} \) we obtain

\[
\exp \left\{ (1 - \epsilon) \delta(P, \theta) r_m^n \right\} \leq |A(r_m e^{i\theta})| \\
\leq \left| \frac{f''(r_m e^{i\theta})}{f'(r_m e^{i\theta})} \right| + |B(r_m e^{i\theta})| \left| \frac{f(r_m e^{i\theta})}{f'(r_m e^{i\theta})} \right| \\
\leq c\left[ T(2r_m, f) \right]^2 + \exp \left\{ r_m^\beta r_m \right\}.
\]

As \( \beta < n \), this implies that

\[
\limsup_{m \to \infty} \frac{\log^+ \log^+ T(r_m, f)}{\log r_m} \geq \rho(A). \tag{15}
\]

Using Theorem 5 and equation (15) we have

\[
\rho_2(f) = \max\{ \rho(A), \rho(B) \}
\]

(b) It has been already proved that all non-trivial solutions \( f(z) \) of equation (2), with \( A(z) \) and \( B(z) \) satisfying the hypothesis of the theorem, are of infinite order, [Theorem 4, [4]]. Also if \( \rho(A) \neq \rho(B) \) then using case (a) above,

\[
\rho_2(f) = \max\{ \rho(A), \rho(B) \}.
\]

Now let \( \rho(A) = \rho(B) = n, n \in \mathbb{N} \). Using [Lemma 5, [4]], for \( \epsilon > 0 \), there exist \( H \subset (1, \infty) \) satisfying \( \log \dens(H) \geq 0 \) such that for all \( |z| = r \in H \) we have

\[
|B(z)| > \exp \left( r_m^{n-\epsilon} \right). \tag{16}
\]

Now we choose \( \theta \in [0, 2\pi) \) with \( \delta(P, \theta) < 0 \) and a sequence \( (r_m) \subset H \setminus \{ E \cup [0, 1] \} \) satisfying \( r_m \to \infty \). Using equations (2), (6), (14) and (16) we obtain

\[
\exp \left( r_m^{n-\epsilon} \right) < |B(r_m e^{i\theta})| \leq \left| \frac{f''(r_m e^{i\theta})}{f'(r_m e^{i\theta})} \right| + |A(r_m e^{i\theta})| \left| \frac{f(r_m e^{i\theta})}{f'(r_m e^{i\theta})} \right| \\
\leq cT(2r_m, f)^4 + \exp \left\{ (1 - \epsilon) \delta(P, \theta) r_m^n \right\} \exp \left\{ (1 - \epsilon) \delta(P, \theta) r_m^n \right\} cT(2r_m, f)^2 \\
\leq cT(2r_m, f)^4(1 + o(1)).
\]

Thus we conclude that

\[
\limsup_{m \to \infty} \frac{\log^+ \log^+ T(r_m, f)}{\log r_m} \geq n. \tag{17}
\]

Using Theorem 5 and equation (17) we get

\[
\rho_2(f) = \{ \rho(A), \rho(B) \}.
\]

\( \square \)
3.4. Proof of Theorem [2]

**Proof.** As $\lambda(A) < \rho(A)$, we have $A(z) = v(z)e^{P(z)}$, where $P(z)$ is a polynomial of degree $n$ and $v(z)$ is an entire function with $\rho(v) < n$. Suppose there exists a solution $f(z)$ of equation (11) having finite order. Then using Proposition [2], we have $\rho(f) \geq \max\{\rho(A), \rho(B)\} > \rho(H)$. Using Lemma [1], there exists a set $S \subseteq \mathbb{C}$ such that for all sufficiently large $r$, we have

$$M(r, B) \geq \exp r^{\rho(B) - \epsilon}$$

for all $r \in S$ and $r > R$. We suppose that $|f(re^{i\theta})| = M(r, f)$ for each $r$. Using Lemma [4], for $\delta > 0$ and $0 < C < 1$, there exists $0 < \theta_0 < \frac{1}{2}$ and $S_2$ with $\log \text{dens}(S_2) \geq 1 - \frac{\epsilon}{2}$ such that

$$e^{-5\pi M(r, f)^{1-C}} \leq |f(re^{i\theta})|$$

for all sufficiently large $r \in S_2$ and $\theta$ such that $|\theta - \theta_0| \leq \theta_0$. Also using Proposition [1] we obtain

$$\frac{|H(z)|}{M(r, f)} \to 0$$

as $r \to \infty$ where $r \in S_3$ and $\log \text{dens}(S_3) = 1$. We know that

$$\chi_{S_1 \cap S_2} = \chi_{S_1} + \chi_{S_2} - \chi_{S_1 \cup S_2}$$

and

$$\log \text{dens}(S_1 \cap S_2) \leq 1$$

therefore,

$$\log \text{dens}(S_1 \cap S_2) \geq \log \text{dens}(S_1) + \log \text{dens}(S_2) - \log \text{dens}(S_1 \cup S_2)$$

$$\geq \zeta + 1 - \frac{\zeta}{2} - 1 = \frac{\zeta}{2}.$$

Also,

$$\log \text{dens}(S_1 \cap S_2 \cap S_3) \geq \log \text{dens}(S_1 \cap S_2) + \log \text{dens}(S_3) - \log \text{dens}(S_1 \cup S_2 \cup S_3)$$

$$\geq \frac{\zeta}{2} + 1 - 1 = \frac{\zeta}{2} > 0.$$  

As $m_l(E) < \infty$, this gives $\log \text{dens}(S_1 \cap S_2 \cap S_3 \setminus E) > 0$. Hence we can choose $z_m = r_m e^{i\theta_m}$ with $r_m \to \infty$ such that

$$r_m \in (S_1 \cap S_2 \cap S_3 \setminus E), \quad |f(r_m e^{i\theta_m})| = M(r_m, f).$$

We may suppose that there exists a subsequence $(\theta_m)$ such that

$$\lim_{m \to \infty} \theta_m = \theta_0.$$

We consider the following cases:
(a) Firstly, suppose that $\delta(P, \theta_0) > 0$. As $\delta(P, \theta)$ is a continuous function, we get

$$\frac{1}{2} \delta(P, \theta_0) < \delta(P, \theta_m) < \frac{3}{2} \delta(P, \theta_0)$$

for all sufficiently large $m \in \mathbb{N}$. Using part (i) of Lemma 2 we obtain

$$\exp\left(\frac{1}{2} \delta(P, \theta_0) r_m^n\right) \leq |A(z_m)| \leq \exp\left(\frac{3}{2} \delta(P, \theta_0) r_m^n\right)$$

for all sufficiently large $m \in \mathbb{N}$. Using continuity of $\delta(P, \theta)$, we deduce that

$$\frac{3}{2} \delta(P, \theta_0) < \delta(P, \theta_m) < \frac{1}{2} \delta(P, \theta_0)$$

for sufficiently large $m \in \mathbb{N}$. Using part (ii) of Lemma 2 we obtain

$$\exp\left((1 + \epsilon) \frac{3}{2} \delta(P, \theta_0) r_m^n\right) \leq |A(z_m)| \leq \exp\left((1 - \epsilon) \frac{1}{2} \delta(P, \theta_0) r_m^n\right)$$

for all sufficiently large $m \in \mathbb{N}$. Using equations (1), (18), (19), (21) and (22) we get

$$\exp r_m^{\rho(B)} \leq M(r_m, B) \leq \left|f''(z_m)\right| + |A(z_m)| \left|f'(z_m)\right| + \frac{|H(z_m)|}{|f(z_m)|}$$

$$\leq r_m^{2\rho(f)} \left(1 + \exp\left((1 + \epsilon) \frac{3}{2} \delta(P, \theta_0) r_m^n\right)\right) + \frac{|H(z_m)|}{M(r_m, f)}$$

$$\leq r_m^{2\rho(f)} \left(1 + \exp\left((1 + \epsilon) \frac{3}{2} \delta(P, \theta_0) r_m^n\right)\right) + o(1)$$

for all large $m \in \mathbb{N}$. This is a contradiction to the fact that $\rho(B) = n$.

(b) We now let $\delta(P, \theta_0) < 0$. Using continuity of $\delta(P, \theta)$, we deduce that

$$\frac{3}{2} \delta(P, \theta_0) < \delta(P, \theta_m) < \frac{1}{2} \delta(P, \theta_0)$$

for sufficiently large $m \in \mathbb{N}$. Using part (ii) of Lemma 2 we obtain

$$\exp\left((1 + \epsilon) \frac{1}{2} \delta(P, \theta_0) r_m^n\right) \leq |A(z_m)| \leq \exp\left((1 - \epsilon) \frac{3}{2} \delta(P, \theta_0) r_m^n\right)$$

for all sufficiently large $m \in \mathbb{N}$. Using equations (1), (18), (19), (21) and (22) we get

$$\exp r_m^{\rho(B)-\epsilon} \leq M(r_m, B) \leq \left|f''(z_m)\right| + |A(z_m)| \left|f'(z_m)\right| + \frac{|H(z_m)|}{|f(z_m)|}$$

$$\leq r_m^{2\rho(f)} \left(1 + \exp\left((1 - \epsilon) \frac{1}{2} \delta(P, \theta_0) r_m^n\right)\right) + \frac{|H(z_m)|}{M(r_m, f)}$$

$$\leq r_m^{2\rho(f)} \left(1 + o(1)\right) + o(1)$$

for sufficiently large $m \in \mathbb{N}$. This is a contradiction to the fact that $\rho(B) > 1$.

(c) When $\delta(P, \theta_0) = 0$, for large $m \in \mathbb{N}$, we have $|\theta_m - \theta_0| \leq l_0$. Choosing $\theta_m^*$ satisfying $\frac{0}{3} \leq \theta_m^* - \theta_m \leq l_0$ and letting $\theta_m^* \to \theta_0^*$ as $m \to \infty$, we obtain

$$\theta_m + \frac{l_0}{3} \leq \theta_m^* \leq \theta_m + l_0$$
which further gives
\[
\theta_0 + \frac{l_0}{3} \leq \theta_0^* \leq \theta_0 + l_0
\]
as \( m \to \infty \). Without loss of generality, we may assume \( \delta(P, \theta_0^*) > 0 \).

As done for case (a), we obtain on similar lines,
\[
\exp \left( (1 - \epsilon) \frac{1}{2} \delta(P, \theta_0^*) r_m^n \right) \leq |A(z_m^*)| \leq \exp \left( (1 + \epsilon) \frac{3}{2} \delta(P, \theta_0^*) r_m^n \right).
\]

Again, as done for case (a), using equations (1), (18), (19), (20) (21) and (26), we get a contradiction.

We now consider the case when \( \rho(B) < \rho(A) \). Similar to Theorem 2, we consider the following three cases:

(a) When \( \delta(P, \theta_0) > 0 \), using equations (1), (7), (18), (21) and (22) we obtain
\[
\exp \left( (1 - \epsilon) \frac{1}{2} \delta(P, \theta_0) r_m^n \right) \leq |A(z_m)| \leq r_m^2 \rho(f)\left(1 + |B(z_m)|\right) + |H(z_m)| M(r_m, f') \leq r_m^2 \rho(f) \left(1 + \exp \left(r \rho(B) + \epsilon\right)\right) + o(1)
\]
for all large \( m \in \mathbb{N} \). This leads to a contradiction.

(b) As done before, when \( \delta(P, \theta_0) < 0 \) or \( \delta(P, \theta_0) = 0 \), we arrive at a contradiction.

\[
3.5. \text{Proof of Theorem 3.}
\]

**Proof.** Suppose \( f(z) \) is a solution of the equation (11). Then
\[
f(z) = C_1(z)f_1(z) + C_2(z)f_2(z)
\]
where \( f_1 \) and \( f_2 \) are linearly independent solutions of the associated homogeneous linear differential equation (2) and
\[
C'_1(z) = -\frac{Hf_2}{f_1f'_2 - f'_1f_2}, \quad C'_2(z) = -\frac{Hf_1}{f_1f'_2 - f'_1f_2}.
\]

In the case when \( \rho(A) < \rho(B) \), we have \( \rho_2(f_i) = \rho(B) \), \( i = 1, 2 \) and \( \rho(H) < \infty \). Using equation (27), we obtain
\[
T(r, f) \leq 2T(r, f_1) + 2T(r, f_2) + 2T(r, H) + O(1) \leq 4 \exp \left(r \rho(B) + \epsilon\right) + O(1)
\]
for all sufficiently large \( r \). This will imply that \( \rho_2(f) \leq \rho(B) \). As done in proof of Theorem 2 using equations (14) and (23) or equations (14) and (25), we get that \( \rho_2(f) \geq \rho(B) \).
Again, when \( \rho(B) < \rho(A) \), we have \( \rho_2(f_i) = \rho(A) \) for \( i = 1, 2 \). Hence, we get \( \rho_2(f) \leq \rho(A) \). Also, as done in proof of Theorem 1, we deduce that \( \rho_2(f) \geq \rho(A) \). □

Note 1. It is important to mention that all infinite order solutions \( f(z) \) of equation (7) also satisfies \( \lambda(f) = \infty \). This can be seen as follows:

Using equation (7) and lemma of logarithmic derivatives we have

\[
\frac{1}{f} = -\frac{1}{H}\left( \frac{f''}{f} + A(z) \frac{f'}{f} + B(z) \right).
\]

As a result, one obtains

\[
m(r, \frac{1}{f}) \leq m\left(r, \frac{f''}{f}\right) + m\left(r, \frac{f'}{f}\right) + m(r, A) + m(r, B) + m\left(r, \frac{1}{H}\right)
\leq S(r, f) + o(T(r, f)) + m(r, H) + O(1)
= S(r, f) + o(T(r, f)) + O(1).
\]

Hence, we have

\[
T(r, f) + O(1) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right)
\leq N\left(r, \frac{1}{f}\right) + S(r, f) + o(T(r, f)) + O(1).
\]

From this, we conclude that if \( \rho(f) = \infty \) then \( \lambda(f) = \infty \).

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