Averaging of Random Flows of Linear and Nonlinear Maps

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Abstract. The procedure of generalized averaging of random strongly continuous semigroups of linear operators is investigated by means of Feynman-Chernoff iterations. The main features of this procedure is the fact that the generalized mean value of random semigroup is the semigroup. The generalized mean value of the random generator of a random \( C_0 \)-semigroup is defined as the generator of generalized mean value of a random semigroup. This procedure of averaging of a random semigroup of linear operators is extended onto the averaging procedure for random semiflows of nonlinear maps. For this aim the semiflows of nonlinear mappings of some linear space \( L \) is presented by the semigroup \( U_t, t \geq 0 \), of linear operators in the space \( H \) of functions on the space \( L \) which is integrable with respect to some measure on the space \( L \). The conditions on the measure and on the semiflow sufficient for boundedness of the linear operators \( U_t \) in the space \( H \) are obtained. The sufficient condition of the existence of the generalized mean value of a random semiflow is investigated. The estimates for the probability of the deviation of the values of compositions of independent random semiflows from its mean value similar to the large numbers law are obtained.

1. Introduction

In this paper we investigate a random transformation of a real Hilbert space \( E \) (finite or infinite dimensional) and random maps of the real interval into a space of transformations of Hilbert space \( E \) including random semigroups of linear operators in the space \( E \) and random flows of nonlinear transformations of the space \( E \).

A random operator-function of the real argument is defined as the measurable map \( \xi \) of some space with the probability measure \( (\Omega, \mathcal{A}, \mu) \) into the topological vector space \( (Y, \tau) \) of operator-valued functions of the real argument endowed with Borel \( \sigma \)-algebra. If the values of this map is the one-parametric group or semigroup with respect to the real parameter then this random operator function is called the random one-parameter group (semigroup) of operators in the space \( E \) or the random flow (semiflow) of maps of the space \( E \) into itself.

Let us consider some examples of arising of random semigroups.

1. A random semigroup can be realized as the regularization of singular operator of ill-posed problem by the directed set of regular operators of well-posed problems (see [1, 2, 3]). The sequence \( \{L_\epsilon, \epsilon \in (0, 1), \epsilon \to 0\} \) of generators of semigroups is the approximation of the operator \( L \) of ill-posed Cauchy problem. If the set of regularized generators is endowed by the structure of the space with the measure \( ((0, 1), \mathcal{A}, \mu) \) then the family of semigroups

\[ e^{iL_\epsilon}, t \geq 0; \epsilon \in (0, 1), \]
is the random variable with values in the set of semigroups.

2. A random group or semigroup can arise if the coefficients of a differential operator $L$ or parameters of Hamilton function on the phase space are random variables (see [4, 5]).

3. The choice of the quantization procedure for defining of quantum Hamiltonian by the Hamiltonian of a classical system is not unique. If the set of quantization of classical Hamiltonian systems is endowed with the structure of probability space then, the random quantization procedure defines the random quantum Hamiltonian (see [6]) and the random unitary group.

A random variable with values in the topological vector space or in the topological space is defined by the following way.

Let $(Y, \tau)$ be a Hausdorff topological space and $A_\tau$ be the minimal algebra containing any open set of the topology $\tau$. If $A_\tau$ is the $\sigma$-algebra then the symbol $A_\tau$ denotes Borel $\sigma$-algebra of topological space $(Y, \tau)$.

Let $(\Omega, A, \mu)$ be a probability space by the above. $Y$-valued random variable is the measurable map $\xi : \Omega \rightarrow Y$ of the probability space $(\Omega, A, \mu)$ into the measurable space $(Y, A_\tau)$, where $A_\tau$ is the minimal algebra containing all open sets of the topology $\tau$.

**Definition 1.** The measurable map $\xi : \Omega \rightarrow Y$ of the probability space $(\Omega, A, \mu)$ into the measurable space $(Y, A_\tau)$ is called the random variable in topological space $(Y, \tau)$.

The Hausdorff topological space $(Y, \tau)$ can be realized as the space of scalar functions (random Hamiltonian), as the space of vector valued functions (random vector field) or as the space of operator valued functions (random flows in the phase space).

1a) (The random Hamiltonian). Let $P = Q = R^d$ and $E = Q \oplus P$ be the phase space of some Hamiltonian system. If the Hausdorff topological space $(Y, \tau)$ is the Banach space $C^1(E)$ of continuously differentiable real-valued functions on the phase space $E$ then the random variable $\xi$ with values in the space $C^1(E)$ is the random classical Hamiltonian.

1b) (The random vector field in the phase space). Let the phase space of the ordinary differential equation system be the real Hilbert space $E$ (with finite or infinite dimension).

Let $C^1_b(E, E)$ be the Banach space of continuously differentiable (in Fréchet sense) transformations of the space $E$ endowed with the norm

$$
\|\Phi\|_{C^1_b} = \sup_{\|x\|_E > 0} \|\Phi(x)\|_E + \sup_{x \in E} \sup_{\|h\|_E = 1} \|\Phi'(x)h\|_E.
$$

Then the random vector field on the phase space $E$ is the random variable in the topological space $Y = C^1_b(E, E)$ with the topology of Banach space.

1c) (The random flow of nonlinear transformations of the phase space). If the Hausdorff topological space $(Y, \tau)$ is the Banach space $C(R, C^1_b(E, E))$ of continuous maps of real line into the Banach space $C^1_b(E, E)$ then the random variable $\xi$ with values in the space $C(R, C^1_b(E, E))$ is called random continuous operator-valued function. This random variable $\xi$ is called the random flow if for any $\omega \in \Omega$ the equality $\xi_\omega(t + s, z) = \xi_\omega(t, \xi_\omega(s, z))$ holds for any $t, s \in R$ and any $z \in E$.

The mean value of random Hamiltonian is the function on the phase space. The mean value of random flow is the map $R \rightarrow E$ which can has no group property.

What relationship does exist between this mean values?

The Hausdorff topological space $(Y, \tau)$ can be realized as the space of strongly continuous functions with values in the space of bounded linear operators in Hilbert space (the random semigroup) or as the set of self-adjoint operators in Hilbert space (the random generator).

2a) *The random semigroup of linear operators.* Let $H$ be a complex Hilbert space. Let $Y_s \equiv C_s(R, B(H))$ be the space of strongly continuous maps of the axis $R$ into the Banach space $B(H)$ of bounded linear operators in the Hilbert space $H$. The topology $\tau_s$ in the
space $C_s(R, B(H))$ is generated by the family of the functionals \( \{ \phi_{T,v}, T \in R, v \in H \} \) where \( \phi_{T,v}(f) = \sup_{t \in [-T,T]} \| f(t)v \|_H \forall f \in C_s(R, B(H)) \). Let \((Y_s, \tau_s)\) be the topological vector space $Y_s \equiv C_s(R, B(H))$ endowed with the topology $\tau_s$. Let $A_{\tau_s}$ be the minimal algebra containing the topology $\tau_s$.

The random variable $\xi$ with values in the space $Y_s$ is called random strongly continuous operator valued function. The $Y_s$-valued random variable $\xi$ is called the random strongly continuous unitary group (semigroup) if for any $\omega \in \Omega$ the value $\xi_\omega$ is strongly continuous unitary group (semigroup) in the space $H$.

A random unitary semigroup is a random variable $\xi$ with the values in the measurable space $(Y_s, A_{\tau_s})$ such that every value $\xi_\omega$ of the map $\xi$ is a unitary semigroup.

2b) The random Hamiltonian and the random generator of a random semigroup. According to Stone theorem there is the one-to-one correspondence between the set of strongly continuous unitary groups in the space $H$ and the set $SA(H)$ of self-adjoint operators in the space $H$. This bijection introduces the topology $\tau_{sa}$ of the set $SA(H)$ from the space $C_s(R, B(H))$ with the topology $\tau_s$.

A random variable $\xi$ with values in the topological space $(SA(H), \tau_{sa})$ is called a random quantum Hamiltonian. Thus there is the one-to-one correspondence between the set of random unitary groups and the set of random quantum Hamiltonians.

The plan of investigation is the following. We define the mean value and covariation of the above random variables and obtain the estimates of deviations of random variables from its mean values. Firstly, we realize this plan for the random strongly continuous unitary groups. According to bijection between the strongly continuous unitary groups and self-adjoint operators in Hilbert space the mean value of random Hamiltonian and the deviation of random Hamiltonian from its mean value are defined by means of the same characteristics of the random unitary group.

The aim of the investigation of random semigroups of linear operators and its random generators is to obtain the asymptotic properties of compositions of $n$ independent random linear operators in a Hilbert space for $n \to \infty$. This properties had been investigated in the papers [6, 7, 8, 9]. Note that the usual mean value of the random semigroup of linear operators is the element of the space $Y_s$ without the semigroup properties (see [6, 9]). The generalized mean value of a random semigroup and the generalized mean value of a random generator are defined and studied. This method is based on consideration of the sequence of compositions of $n$ independent random linear operators and its mean value $(T(\frac{1}{n}))^n$, $t \in R$, and passage to the limit for the sequence of iterations $\{(T(\frac{1}{n}))^n, t \in R\}$ (so-called Feynman-Chernoff iterations). Thus, the method of averaging of random semigroups of linear operators is based on the Chernoff theorem ([10, 11]) and its inversion ([6, 7, 12]).

The generalized mean value of a random semigroup has the following properties: 1) it is the semigroup and 2) it coincides with the usual mean values of a random semigroup if the last operator function has the semigroup property. The generalized mean value of a random generator of a strong continuous semigroup has the following properties: 1) it is the generator of a strong continuous semigroup and 2) it coincides with the usual mean values of a random generator of a semigroup if the last mean value exists. In this paper we give the short review of the results of the papers [6, 7, 8, 9].

The main interest of this article is the extension of the methods of averaging of random linear operators and random groups of linear operators onto the random nonlinear transformations and flows of nonlinear transformations. For this aim we consider the construction of the representation of random groups of nonlinear transformations of Hilbert space $E$ by means of random groups of linear operators in Hilbert space $\mathcal{H}(E)$. Here $\mathcal{H}(E)$ is the space of square
operators in the space be described by the same probabilistic characteristics of a corresponding random group of linear operators in Hilbert space $H$.

Hence, the probabilistic characteristics of a random flow of nonlinear maps of the space $E$ can be represented as a random group of linear operators in Hilbert space $H$ (E) are obtained.

For investigation of a random flow of nonlinear maps of the space $E$ we endow the space $E$ by the structure of measurable space $(E,\mu)$ with a measure $\mu$. Then (under some additional assumptions) a flow of nonlinear maps of the space $E$ can be represented by the group of linear operators in the space $H(E) = L_2(E,\mu)$. Therefore, a random flow of nonlinear maps of the space $E$ can be represented as a random group of linear operators in the space $H(E)$. Hence, the probabilistic characteristics of a random flow of nonlinear maps of the space $E$ can be described by the same probabilistic characteristics of a corresponding random group of linear operators in the space $H(E)$.

2. Random semigroups of linear operators and its generators

Let $(Y,\tau)$ be the topological vector space $Y_s = C_s(R_d, B(H))$ of strongly continuous maps of semiaxis $R_d$ into the Banach space $B(H)$ of bounded linear maps of a Hilbert space $H$ endowed with corresponding topology (see item 2a).

Definition 2 Let $\xi$ be a random semigroup $\xi : \Omega \to Y_s$. The map $M\xi : R_+ \to B(H)$ is called the mean value of random semigroup $\xi : \Omega \to Y_s$ if the equality

$$M\xi = \int_{\Omega} \xi_{\omega} \, d\mu(\omega),$$

holds in the sense of Pettis integral, i.e. if for any $t \in R_+, u, v \in H$ the equality (1) holds.

$$\langle M\xi(t)u, v \rangle = \int_{\Omega} \langle \xi_{\omega}(t)u, v \rangle \, d\mu(\omega), \quad (1)$$

where the symbol $\langle \cdot, \cdot \rangle$ denotes the scalar product in the space $H$.

Theorem 1. (See [7, 8]) If the measurable map $\xi : \Omega \to Y_s$ is uniformly bounded (A) and densely strongly equicontinuous (B) then $M\xi \in Y_s$.

Conditions (A) and (B) mean respectively:

(A) $\exists C > 0 : \|\xi_{\omega}(t)\|_{B(H)} < C \forall t \geq 0, \omega \in \Omega$

(B) $\exists D \subset H$, where $D$ is dense subspace of $H$, such that $\forall u \in D; \forall \sigma > 0 \exists \delta > 0 : \|\xi_{\omega}(t')u - \xi_{\omega}(t'')u\| < \sigma \forall \omega \in \Omega; |t' - t''| < \delta$.

Give example of the random semigroup such that its mean value has no semigroup property.

In fact, let $\mathbf{A} = \mathbf{A}^*$. Then, for the mean value of two-valued random semigroup $\xi_{\pm}(t) = e^{\pm i\mathbf{A}t}, t \geq 0$ such that $P(\{\xi = e^{i\mathbf{A}t}\}) = \frac{1}{2} = P(\{\xi = e^{-i\mathbf{A}t}\})$ the following holds:

$$M\xi(t) = \cos(\mathbf{A}t) = \frac{1}{2} e^{i\mathbf{A}t} + \frac{1}{2} e^{-i\mathbf{A}t}, t \geq 0.$$
To reconstruct the semigroup property of the mean value of random semigroup and to define the
generalized mean value of random semigroup which is some semigroup we use the following
notion.

**Definition 3.** (See [13, 6]) The functions \( F, G \in C_s(R_+, B(H)) \) such that \( F(0) = G(0) = I \)
are equivalent in Chernoff sense (denote it by \( F \sim G \)) if for any \( T > 0 \) and \( u \in H \) the equality
\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \| ((G(\frac{t}{n}))^n - (F(\frac{t}{n}))^n)u \| = 0
\]
holds.

The function \( F \) is equivalent to the semigroup \( U \) if \( (F(\frac{t}{n}))^n \to U(t) \) uniformly on any segment.
The representation of the semigroup \( U \) by the limit
\[
\lim_{n \to \infty} (F(\frac{t}{n}))^n
\]
is called Feynman formula ([13, 6]).

**Definition 4.** The semigroup \( U \in C_s(R_+, B(H)) \) is the general mean value of the random
semigroup \( \xi \) if \( U \sim M \xi \).

**Theorem 2 ([12, 7]).** Let the function \( F \in C_s(R_+, B(H)) \) satisfy the conditions \( F(0) = I \)
and \( \| F(t) \|_{B(H)} \leq e^{at} \), \( t \in R_+ \) for some \( a \in R \). Let the sequence \( \{ G_n \} \) of operators
\[
G_n(t) = (F(\frac{t}{n}))^n, \quad n \in N, \quad t \in R_+
\]
converge in the strong operator topology uniformly on any segment \([0, T]\). Then the limit function
is \( C_0 \)-semigroup in the space \( H \).

**Theorem 3.** ([7, 6]) Let \( \xi : \Omega \to C_s(R_+, B(H)) \) be a random semigroup such that its
set of values \( \xi_\omega, \omega \in \Omega \), is the strongly continuous contractive semigroup in the space \( H \) with
generators \( L_\omega, \omega \in \Omega \). Let \( D \subset H \) be the essential domain of linear operators \( L_\omega, \omega \in \Omega \) and
\[
\int \| L_\omega x \| d\mu(\omega) < \infty \quad \text{for any} \quad x \in D.
\]
Let the operator \( Su = \int_\Omega L_\omega u d\mu(\omega), \quad u \in D, \) be essentially self-adjoint. Then
\[
e^{tS} \sim M \xi(t),
\]
where \( M \xi(t) \) is the usual mean value of the random semigroup.

Under the assumptions of Theorem 3 the generalized mean value of the random semigroup is
the semigroup generated by the mean value of the random generator. But the result of Theorem 3
give the opportunity to define the mean value of the generator of the random semigroup

**Corollary 1.** Let \( \{ \mu_n \} \) be a sequence of positive numbers satisfying: \( \sum_{n=1}^{\infty} \mu_n = 1 \). Let \( \{ L_j \} \)
be uniformly bounded sequence in the space \( B(X) \). Then
\[
\sum_{j=1}^{\infty} \mu_j e^{tL_j} \sim \sum_{j=1}^{\infty} \mu_j L_j.
\]

Since there is the one-to-one correspondence \( \Psi \) between the set of \( C_0 \)-semigroups in the space
\( H \) as the subset of topological vector space \( Y_s \) and the set \( G(H) \) of generators of \( C_0 \)-semigroups
as the subset of the set of closed linear operators in the space \( H \) (Hille-Yosida theorem) then, we
can endow the set \( G(H) \) with the structure of topological space with the topology inducing by the
bijection \( \Psi \). In particular, according to the Stone theorem there is the one-to-one correspondence
between the set of unitary groups in the space \( H \) and the set \( SA(H) \) of self-adjoint operators
in the space \( H \). Therefore, the topology on the set \( SA(H) \) can be induced by the bijection \( \Psi \)
from the topological space \((Y_s, \tau_s)\). This property, results of Theorem 3 and Corollary 1 give the
opportunity to define the generalized mean value of the generator of the random semigroup in the
space \( H \).

**Definition 5.** The generator \( L \) of some semigroups of linear operators in the space \( H \)
is called the generalized mean value of the random generator \( L_\omega, \omega \in \Omega \), if the function
\[
G(t) = \int_\Omega e^{tL_\omega} d\mu(\omega), \quad t \geq 0
\]
is equivalent (in Chernoff sense) to the semigroup \( e^{tL} \).
Corollary 1 states that Definition 5 is the extension of summation procedure from the space of bounded operators onto the set of generators \(G(H)\). Now we compare the averaging in the sense of Definition 5 and averaging in the sense of quadratic forms.

**Corollary 2.** Let \(L_1, \ldots, L_m\) be self-adjoint operators in Hilbert space \(X\) and \(\sum_{j=1}^{m} p_j = 1\), \(p_j > 0\). Let the quadratic form of the operator \(L_1\) majorize the quadratic forms of operators \(L_k\), \(k = 2, \ldots, m\). Then the domain \(D(L_1)\) is the essential domain of operators \(L_k\), \(k = 2, \ldots, m\) and the equalities \(L = \sum_{j=1}^{m} p_j L_j\) and \(\sum_{j=1}^{m} p_j e^{-itL_j} \sim e^{-it\sum_{j=1}^{m} p_j L_j}\) hold. Here \(L\) is the mean operator.

Thus, we obtain the additive analog of Trotter formula \(e^{tA}e^{tB} \sim e^{(A+B)\lambda} \sim \frac{e^{2itA}+e^{2itB}}{2}\).

In the paper [6] we give the example of the random unitary semigroup such that the mean value of the random self-adjoint generator in the sense of Definition 5 exists, but there is no the mean value of the random generator in the sense of quadratic form. In this example the random generator takes values in the set \(SA(H)\), in addition, its generalized mean value is the maximal dissipative, but not self-adjoint operator. In this example the common domain of the random generator values is smaller than the essential domain of the mean value of the random generator.

This approach to the approximation of the semigroup generated by the sum of generators of other semigroups is studied in the works [14, 15]. In the paper [3] we give the example of the random semigroup such that it has no generalized mean value in the following sense: the sequence \(\{(G(x))^{n}, t \in R_+\}\) of iteration of the mean value \(G(t) = \int_{\Omega} e^{itA}d\mu(\omega)\), \(t \in R_+\), does not converge (it has different partial limits).

### 3. The linear representation of the nonlinear flow

Now we describe the representation of the one-parameter family of nonlinear maps by the one-parameter family of linear operators and the representation of the flow of nonlinear maps by the group of linear operators.

Let \(E\) be a set, and \(F\) be some ring of subsets of the set \(E\). Let \(A_F\) be the minimal algebra containing the ring \(F\) (see [16]).

**Definition 6.** The map \(X : E \rightarrow E\) of the phase space \(E\) into itself is called acceptable by the measure \(\lambda\) on the measurable space \((E, F)\) if the following conditions hold:

1) \(X\) is the measurable map of the measurable space \((E, F)\) into itself;

2) the measure \(\lambda \circ X\) on the space \(E\) is absolutely continuous with respect to the measure \(\lambda\) (where \(\lambda \circ X(A) = \lambda(X^{-1}(A)) \forall A \in F\));

3) the density \(\rho\) of the measure \(\lambda \circ X\) with respect to the measure \(\lambda\) satisfies the condition \(\rho \in L_\infty(E, F, \lambda, R)\).

In this case the map \(X : E \rightarrow E\) of phase space \(E\) into itself and the measure \(\lambda\) on the measurable space \((E, F)\) are called coordinated.

The conditions of invariance and quasi-invariance of measures on infinitely dimensional spaces with respect to the dynamical mappings are studied in the papers [17, 16].

Let us consider the examples of the mappings \(X : E \rightarrow E\) of the space \(E\) into itself with the measures \(\lambda\) of the space \(E\) which are accepted by this mappings.

1. Let the measure \(\lambda\) on a space \(E\) be invariant with respect to a transformation \(X : E \rightarrow E\). Then this transformation \(X\) is accepted by this invariant measure \(\lambda\).

2. Let \(E\) be a domain in Euclidean space \(R^d\), and \(X\) be reversible continuously differentiable map \(X : E \rightarrow E\) such that \(\det X'(x) \in [\delta, \frac{1}{\delta}] \forall x \in E\) with some \(\delta \in (0, 1)\). Then the transformation \(X\) is accepted by Lebesgue measure on the space \(R^d\).
Therefore the flows of continuously differentiable maps \( X_t, t \in \mathbb{R} \), of the phase space \( E = \mathbb{R}^d \) into itself with uniformly bounded Jacobians \( |\det(X_t)'(x)| \leq M \forall (x, t) \in E \times \mathbb{R} \) with some \( M > 0 \) are accepted by Lebesgue measure \( \lambda \) on the space \( \mathbb{R}^d \) (since \( (X_t)^{-1} = X_{-t} \) and
\[
|\det(X_t)'(x)||\det(X_{-t})'(X_t(x))| = 1 \forall (x, t) \in E \times \mathbb{R}.
\]

In the case of finitely dimensional phase space \( E \) solutions representation of nonlinear ODE by means of its first integrals and solutions of linear first order PDE is well-known theory. In particular, the conditions of coordinating of the transformation of the space \( E \) with the measure on this space is well studied in the case of smooth transformations. But in the case of infinitely dimensional space \( E \) this approach is not fully investigated. The Hamiltonian approach to infinitely dimensional quantum dynamics is constructed in the work [18]. In particular, the problems of Liouville equation approach to the infinitely dimensional nonlinear ODE arises due to the absence of Lebesgue measure on the infinite dimensional space. To construct this Liouville approach to the ODE in infinitely dimensional phase space the analog of Lebesgue measure on a Hilbert space (Banach space) had been constructed in the papers [19, 20, 21]. This analog is the translation invariant, nonnegative, nontrivial, complete measure on some ring of subsets of Hilbert space. Let us consider this construction.

3. Let \( E \) be a real separable Hilbert space. Now we describe the measure on the space \( E \) which is invariant under any shift and any orthogonal mapping of the space \( E \) (see [20, 21]).

The set \( \Pi \subset E \) is called rectangle if there are the orthonormal basis \( E = \{e_j\} \) and the elements \( a, b \in l_\infty \) such that
\[
\Pi = \{x \in E : (x, e_j) \in [a_j, b_j] \forall j \in \mathbb{N}\}.
\]
This set can be empty. The rectangle (2) is called measurable if either \( \Pi = \emptyset \) or the following condition holds:
\[
\sum_{j=1}^{\infty} \max \{0, \ln(b_j - a_j)\} < \infty. \tag{3}
\]

Let \( \mathcal{K} \) be the class of measurable rectangles in the space \( E \) and \( \mathcal{R} \) be the minimal ring of subsets containing the class \( \mathcal{K} \). Let \( \lambda \) be the function of the set on the class \( \mathcal{K} \) which is defined by the condition: \( \lambda(\emptyset) = 0 \) and
\[
\lambda(\Pi) = \exp\left[\sum_{j=1}^{\infty} \ln(b_j - a_j)\right] \tag{4}
\]
for any measurable nonempty rectangle (2). According to condition (3) \( \lambda(\Pi) \in [0, +\infty) \) for any \( \Pi \in \mathcal{K} \).

According to the papers [20, 21] the function of the set \( \lambda \) is additive on the class of sets \( \mathcal{K} \) and has the unique additive extension onto the measure \( \lambda \) on the ring \( \mathcal{R} \).

A set \( A \subset E \) is called \( \lambda \)-measurable if for any \( \epsilon > 0 \) there are the sets \( A_\epsilon, A^* \in \mathcal{R} \) such that \( A_\epsilon \subset A \subset A^* \) and \( \lambda(A^* \setminus A_\epsilon) < \epsilon \). The class \( \Lambda \) of \( \lambda \)-measurable sets in the space \( E \) is the ring and the measure \( \lambda \) has the unique extension from the ring \( \mathcal{R} \) onto the ring \( \Lambda \) by the equality \( \lambda(A) = \inf_{A^* \in \mathcal{R}, A^* \supset A} \lambda(A^*) \forall A \in \Lambda \). The measure \( \lambda \) is invariant with respect to the shift on any vector of the space \( E \) and with respect to any orthogonal transformation of the space \( E \) (see [20, 21]).

The function \( \lambda : \Lambda \rightarrow [0, +\infty) \) is the finite additive measure on the ring \( \Lambda \) of subsets of the space \( E \) which are invariant with respect to the shifts and orthogonal mappings. This measure is locally finite, complete, but it is not \( \sigma \)-additive; moreover, the algebra \( \Lambda \) can not contain some open balls in the space \( E \) (see [20, 21]).

Let \( V \) be a linear bijective operator from the space \( E \) onto itself with bounded inverse operator. Then there is the unique positive self-adjoint operator \( S \) and the orthogonal linear
operator $\mathbf{U}$ such that $\mathbf{V} = \mathbf{US}$. This representation of a bounded linear bijective operator is called its polar decomposition.

In fact, the operator $\mathbf{V}^*\mathbf{V}$ is the bounded positive linear operator with bounded inverse operator. Then there is the unique positive linear operator $\mathbf{S} = (\mathbf{V}^*\mathbf{V})^{\frac{1}{2}}$, and the operator $\mathbf{S}$ has bounded inverse operator. Therefore, the bounded linear operator $\mathbf{U} = \mathbf{VS}^{-1}$ has the bounded linear inverse operator, and for any $x \in E$ the equality $\|\mathbf{U}x\|_E^2 = (\mathbf{VS}^{-1}x, \mathbf{VS}^{-1}x) = (x, \mathbf{S}^{-1}\mathbf{V}^*\mathbf{V}\mathbf{S}^{-1}x) = \|x\|_E^2$ holds. Hence $\mathbf{U}$ is unitary operator.

**Theorem 4.** Let $\mathbf{V}$ be a linear isomorphism of the space $E$ onto itself, and the equality $\mathbf{V} = \mathbf{US}$ be its polar decomposition. Then the linear operator $\mathbf{V}$ is accepted by the measure $\lambda$ if and only if the self-adjoint operator $\mathbf{S} - I$ is the trace class operator.

The statement of Theorem 4 is the consequence of following three lemmas. Firstly, the property of acceptability of the map $\mathbf{V} = \mathbf{US}$ for the measure $\lambda$ will be investigated under the assumption that the operator $\mathbf{S}$ has the basis of eigenvectors. Further, we prove that this property of the operator $\mathbf{S}$ is the necessary condition for the acceptability.

**Lemma 1.** Let $\mathbf{V}$ be a linear isomorphism of the space $E$ onto itself and the equality $\mathbf{V} = \mathbf{US}$ be its polar decomposition. Let the operator $\mathbf{S}$ have the orthonormal basis of eigenvectors and spectrum $\sigma = \{s_j, j \in \mathbb{N}\}$. If $\{\alpha_j\} = \{s_j - 1\} \in \ell_1$. Then the linear map $\mathbf{V}$ is accepted by the measure $\lambda$.

Let us consider the measurable rectangle $\Pi \in \mathcal{K}$ with the lengths of edges $\{d_j, j \in \mathbb{N}\}$. The image $\Pi' = \mathbf{V}(\Pi)$ is the rectangle in the class $\mathcal{K}$ with the lengths of edges $\{d'_j = sjd_j, j \in \mathbb{N}\}$. Hence

$$
\Pi_{j=1}^{\infty}d'_j = (\Pi_{j=1}^{\infty} s_j)\lambda(\Pi) = (\lambda(\Pi) \exp[\sum_{j=1}^{\infty} \ln(1 + \alpha_j)]) \leq \lambda(\Pi) \exp[\|\alpha\|_{\ell_1}]
$$

and thus, $\lambda(\Pi') \leq \lambda(\Pi) \exp[\|\{s_j - 1\}\|_{\ell_1}]$. For the inverse mapping $\mathbf{V}^{-1}$ we obtain analogously $\lambda(\Pi) \leq \lambda(\Pi') \exp[\|\{\frac{1}{s_j - 1}\}\|_{\ell_1}].$

Therefore, if $\{s_j - 1\} \in \ell_1$ then there is the number $\Pi_{j=1}^{\infty} s_j = \Delta \in (0, +\infty)$ such that $\lambda(\mathbf{V}(\Pi)) = \Delta(\Pi)$ for any $\Pi \in \mathcal{K}$. Since any set $A \in \Lambda$ can be approximated by a sequence of the sets from the ring $\mathcal{R}$ then $\lambda(\mathbf{V}(A)) = \Delta\lambda(A)$ for any $A \in \Lambda$.

**Lemma 2.** Let $\mathbf{V}$ be a linear isomorphism of the space $E$ onto itself, and the equality $\mathbf{V} = \mathbf{US}$ be its polar decomposition. Let the operator $\mathbf{S}$ have the orthonormal basis of eigenvectors and spectrum $\sigma = \{s_j, j \in \mathbb{N}\}$. If the linear operator $\mathbf{V}$ is accepted by the measure $\lambda$ then $\{\alpha_j\} = \{s_j - 1\} \in \ell_1$.

Let us assume the contrary: $\{\alpha_j\} = \{s_j - 1\} \notin \ell_1$. Then either there is the subsequence of negative numbers $\{\alpha_{j_k}\}$ such that $\sum_{k=1}^{\infty} \alpha_{j_k} = -\infty$, or there is the subsequence of positive numbers $\{\alpha_{j_m}\}$ such that $\sum_{m=1}^{\infty} \alpha_{j_m} = \infty$. In the second case the rectangle $\Pi' = \mathbf{V}(\Pi)$ (where $\Pi$ is some rectangle with the unit edges) is not measurable since the condition (3) does not hold for $\Pi'$: $\sum_{m=1}^{\infty} \ln(s_{j_m}) = \sum_{m=1}^{\infty} \ln(1 + \alpha_{j_m}) = +\infty$. In the first case rectangle $\Pi' = \mathbf{V}^{-1}(\Pi)$ is not measurable. In both case we obtain the contradiction since the map $\mathbf{V}$ is not admitted by the measure $\lambda$.

**Lemma 3.** Let $\mathbf{V}$ be a linear isomorphism of the space $E$ onto itself, and the equality $\mathbf{V} = \mathbf{US}$ be its polar decomposition. If the linear operator $\mathbf{V}$ is accepted by the measure $\lambda$ then the self-adjoint operator $\mathbf{S} - I$ is the trace class operator.

Let us assume the contrary: the self-adjoint operator $\mathbf{S} - I$ is not the trace class operator. If it has the nontrivial component of continuous spectrum in the interval $(0, +\infty)$ or $(-\infty, 0)$ then there are the infinite orthonormal system $\{e_j\}$ and the number $\delta > 0$ such that either $\|\mathbf{Se}_j\|_E \geq (1 + \delta)\|e_j\|$ or $\|\mathbf{Se}_j\|_E \leq \frac{1}{1+\delta}\|e_j\|$ for all $j \in \mathbb{N}$. Therefore, according to the proof
of lemma 2 the map \( V \) is not accepted by the measure \( \lambda \). Hence, the spectrum of self-adjoint operator \( S - I \) has only discrete component and only eigenvalue 0 can have infinite multiplicity. Then \( S - I \) is trace class operator according to the lemma 2.

Any flow of acceptable by the measure \( \lambda \) maps \( X_t, t \in R \), in the phase space \( E \) is associated with the group \( U_t, t \in R \), of the linear maps of the Hilbert space \( \mathcal{H} = L_2(E, \mathcal{F}, \lambda, C) \) acting by the equality

\[
U_t \psi(x) = \psi(X_{-t}(x)), x \in E. \tag{5}
\]

*The flows of nonlinear maps of the phase space \( E \) which is acceptable by the measure \( \lambda \) on the space \( E \) can be studying as the groups of linear operators in the space \( L_2(E, \mathcal{F}, \lambda, C) \).*

4. **Mean values of random flows and random vector fields**

Let \( E \) be a real Hilbert space and \( \Lambda \) be a ring of subsets of Hilbert space \( E \).

Let \( \lambda \) be the nonnegative finite additive measure on the space \( E \) which is invariant with respect to shifts and rotations. Set \( \mathcal{H} = L_2(E, \mathcal{F}, \lambda, C) \).

Let \( M(E, \Lambda) \) be the space of measurable maps of measurable space \( (E, \Lambda) \) into itself and \( M(E, \Lambda, \lambda) \) be the subset of maps of the space \( M(E, \Lambda) \) which are accepted by the measure \( \lambda \).

Let \( C^1(E, E) \) be the Banach space of continuously differentiable maps of the space \( E \) into itself.

Let \( F(E) = C(R, C^1(E, E)) \) be the Banach space of continuous maps of the segment \([0, +\infty)\) into the Banach space \( C^1(E, E) \) endowed with the norm \( \|f\|_{F(E)} = \sup_{t \in R} \|f(t)\|_{C^1(E, E)} \). Let \( F^1(E) = C^1(R, C^1(E, E)) = \{f \in F(E) : f' \in F(E)\} \) be the Banach space endowed with the norm \( \|f\|_{F^1(E)} = \|f\|_{F(E)} + \|f'\|_{F(E)} \).

A map \( X \in F(E) \) is called a flow if \( X_{t+s} = X_t \circ X_s \forall t, s \in R \).

Let \( \mathcal{A}_B \) be Borel sigma-algebra of subsets in the space \( F(E) \).

A random flow is defined as a measurable map of the space with a measure \((\Omega, \mathcal{A}, \mu)\) into the measurable space \((F(E), \mathcal{A}_B)\) such that the values of this map are flows.

Let \( F_\lambda(E) \) be the subset of the Banach space \( F(E) \) which is defined by the conditions

\[
F_\lambda(E) = \{X \in F(E) : \forall t \in R \ X_t \in M(E, \Lambda, \lambda)\}. \tag{6}
\]

If \( X \in F_\lambda(E) \) then the operator-function \( U : R \to \mathcal{B}(\mathcal{H}) \) acting by the equality (5) is defined. The symbol \( \Phi_\lambda \) means the map which is defined on the set \( F_\lambda(E) \), takes values in the space of maps \( R \to \mathcal{B}(\mathcal{H}) \) and is defined by the equality (5).

4.1. **Chernoff equivalence and mean values of nonlinear maps**

According to O.G. Smolyanov the function \( F \in C_s(R_+, B(H)) \) is equivalent in the space \( C_s(R_+, B(H)) \) to the function \( G \in C_s(R_+, B(H)) \) if \( F(0) = G(0) = I \), and

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \|(F(\frac{t}{n})^n - (G(\frac{t}{n})^n))u\|_H = 0 \quad \text{for all } T > 0, u \in H \quad \text{(see [13, 7, 6])}.
\]

**Definition 7.** Operator valued functions \( X, Y \in F_\lambda(E) \) are called equivalent in the space \( F_\lambda(E) \) to each other in nonlinear Chernoff sense if and only if the functions \( \Phi_\lambda(X), \Phi_\lambda(Y) \in C_s(R, B(\mathcal{H})) \) are equivalent to each other in Chernoff sense.

**Definition 8.** The flow \( Y \in F_\lambda(E) \) is called the generalized mean value of the random flow \( X : \Omega \to F_\lambda(E) \) if \( M\Phi_\lambda(X) \sim \Phi_\lambda(Y) \).

Does the relation \( M\Phi_\lambda(X) \sim \Phi_\lambda(MX) \) holds if the both mean values exist?
Firstly, we present the sufficient conditions for the validity of relation \( \Phi(X) \sim \Phi(MX) \).

**Theorem 5.** Let \( E = R^d, \mathcal{H} = L_2(R^d) \), \( \lambda \) be the Lebesque measure on \( R^d \). Let \( X \in F^1(E) \) be the operator valued function such that for any \( t \in R \), \( X_t : E \rightarrow E \) is the reversible map and \( X^{-1} \in F^1(E) \), where \((X^{-1})_t = (X_t)^{-1} \forall t \in R \).

Then \( X \sim S_a \) where \( X'((0)) = a \in C^1(R^d, R^d) \) and \( S_a \) is the flow of transformation of the space \( R^d \) generated by differential equation \( \dot{x}(t) = a(x(t)), t \in R \).

The proof of the Theorem 5 is given in the following three lemmas.

**Lemma 4.** Let the assumptions of Theorem 5 hold. Then for any \( t \in R \) the map \( X_t \) is acceptable by the Lebesgue measure \( \lambda \) on the space \( E \).

*Proof.* In fact, \( X_t^{-1} \in F^1(E) = C^1(R, C^1(R, R^d)) \) by the assumption of Theorem 5. According to the “replacement theorem in the integral” for any function \( \psi \in L_2(R^d) \) the following equality
\[
\int_{R^d} |\psi(X_t^{-1}(x))|^2 dx = \int_{R^d} |\psi(y)|^2 |(X_t^{-1})'(y)| dy = \int_{R^d} |\psi(y)|^2 |(X_t')^{-1}(y)| dy
\]
holds, where \(|(X_t^{-1})'(x)| = \det|\frac{\partial (X_t^{-1}(x))}{\partial x}|\) and \(|(X_t')(x)| = \det|\frac{\partial (X_t(x))}{\partial x}|\), \( (t, x) \in R \times R^d \).

Let \( \frac{d}{dt}(X_t(x)) = b(t, x) = a(t, X_t(x)), t \in R, x \in R^d \) then \( a(t, x) = b(t, X_t^{-1}(x)) \in F(E) \).

Since the maps \( X_t, t \in R \), of the flow \( X \) are continuously differentiable, the equality \( \frac{d}{dt}(X_t(x)) = a(t, X_t(x)), t \in R, x \in R^d \) holds. Then \( \frac{d}{dt}\det|\frac{\partial (X_t(x))}{\partial x}| = \det|\frac{\partial (X_t(x))}{\partial x}| \), where the function \( \det(a(t, x)) = \sum_{k=1}^d \frac{\partial a(t, x)}{\partial x_k} \) is bounded by the condition (4).

Therefore, there is the constant \( L = L(\|X\|_{F^1(E)}, \|X^{-1}\|_{F^1(E)}) \) such that Jacobians of the mappings \( X_t, t \in R \) admit the estimates
\[
\sup_{(t, x) \in [-T, T] \times R^d} |(X_t)'(x)| \leq e^{LT}.
\]

Thus the norms of values of the operator function \( T_t, t \geq 0 \), admit the estimates \( \|T_t\|_{B(H)} \leq e^{L_0 t}, t \geq 0 \). Therefore, the maps \( X_t, t \in R \), of the flow \( X \) are accepted by the Lebesgue measure on the space \( R^d \).

According to Lemma 4 the one-parameter family of maps \( X(t), t \in R \), of the space \( R^d \) generates the operator-valued function \( T_t = \Phi(X(t)), t \in R \), with values in the Banach space of bounded linear operators \( B(H) \) in the space \( H = L_2(R^d) \) acting by the equality
\[
T_t \psi(x) = \Phi(X(t)) \psi(x) = \psi((X_t)^{-1}(x)), t \in R, x \in R^d; \psi \in H.
\]

**Lemma 5.** Let the assumptions of Theorem 5 hold. Then \( T = \Phi(X) \in C_t(R, B(H)) \).

Since \( X \in C^1(R, C^1(R, R^d)) \) then Cauchy problem for the system of ODE \( \dot{x}(t) = a(t, x(t)), t \in R; \ x(0) = x_0 \) for any \( x_0 \in R^d \) has the unique solution \( X_t(x_0) \in C^1(R, R^d) \).

Thus, according to Lemma 4 the one-parameter family \( T_t, t \in R \), of linear operators of the space \( H = L_2(R^d) \) is defined. This family of linear operators has the following properties:

I) the space \( W_2^1(R^d) \) is invariant with respect to operators \( T_t, t \in R \);

II) for any \( u \in W_2^1(R^d) \) the equality \( \frac{d}{dt}T_t u(x) = (\nabla T_t u(x), a(t, X_t(x))) \) holds.

Hence,

1) for any element \( u \in W_2^1(R^d) \) and number \( T > 0 \) there is a number \( L > 0 \) such that
\[
\|T_{t+\tau}u - T_t u\|_H \leq L|\tau|\|u\|_{W_2^1}
\]
for all \( t, t + \tau \in [-T, T] \);
2) for any $\psi \in L_2(\mathbb{R}^d)$ and $t \in R$ the estimate holds: $\|T_t\psi\|_{L_2(\mathbb{R}^d)} \leq e^{\lambda t} \|\psi\|_{L_2(\mathbb{R}^d)}$ where $A = \sup_{x \in \mathbb{R}^d} |\text{div}(x)|$.

Since the space $W_2^1(\mathbb{R}^d)$ is the dense subspace of the space $L_2(\mathbb{R}^d)$ then the equality $\lim_{t \to 0} \|T_{t+\epsilon}u - T_{t}u\|_H = 0$ is the consequence of the properties I), II) of the operator-function $T$.

If, in fact, $\psi \in L_2(\mathbb{R}^d)$ and $\epsilon > 0$ then there is the vector $u \in W_2^1(\mathbb{R}^d)$ such that $\|\psi - u\|_{L_2} < \frac{\epsilon}{2}$. Then $\|T_t\psi - \psi\|_H \leq \|T_t(\psi - u)\|_H + \|T_tu - u\|_H + \|u - \psi\|_H$. According to the condition 2) the first part of the sum admits the estimate by the value $\frac{\epsilon}{4}$ for sufficiently small $|t|$, the second term of the sum admits the estimate by the value $\frac{\epsilon}{4}$ for sufficiently small $|t|$ according to (8).

Since the vector field $a(0, \cdot) \in C^1(E, E)$ is continuously differentiable then Cauchy problem for the autonomic system of differential equation $\dot{x} = a(0, x)$; $x(0) = x_0$, has the unique solution $x(t, x_0)$, $t \in R$. Therefore, the one-parameter family of maps defined by the equality $(S^t)_{0}(x_0) = x(t, x_0)$, $x_0 \in E$, $t \in R$, is the flow of continuously differentiable maps, moreover, $S^t \in F^1(E)$.

According to Lemma 4 the flow of continuously differentiable maps $S^t$ is acceptable by the Lebesgue measure $\lambda$ on the space $\mathbb{R}^d$. Therefore, the one-parameter family $F_t$, $t \in R$, of linear operators in Hilbert space $H = L_2(\mathbb{R}^d)$ is defined by the equality $F = \Phi_\lambda(S^t)$. Then the equalities $F_t\cdot u(x) = u((S^t)_{-1}(x))$, $t \in R$, $x \in \mathbb{R}^d$, $u \in H$ define the one-parameter family $F_t$, $t \in R$, of linear operators in Hilbert space $H = L_2(\mathbb{R}^d)$. The one-parameter family $F_t$, $t \in R$, is one-parameter group since $F_t\cdot u(x) = u((S^t)_{-1}(x))$, $t \in R$, $x \in \mathbb{R}^d$, $u \in H$, and family of maps $S^t$, $t \in R$, is the flow. According to Lemma 5 the one-parameter group $F = \Phi_\lambda(S^t)$ of linear operators is continuous in the strong operator topology.

Lemma 6. Let the assumptions of Theorem 5 hold. Then the operator valued functions $\Phi_\lambda(X), \Phi_\lambda(S^t) \in C_s(R, B(H))$ are equivalent to each other in Chernoff sense.

Let us verify the condition of Chernoff theorem about the differentiability of the operator-function $T = \Phi_\lambda(X)$. For any $u \in W_2^1(\mathbb{R}^d)$ there is the derivative $\frac{d}{dt}(T_t(\psi))(t=0) = (a(0, \cdot), \nabla)u$.

The operator $(a(0, \cdot), \nabla)$ is densely defined on the space $W_2^1(\mathbb{R}^d)$, and it is the generator of the group $F_t$, $t \in R$. According to Lemma 5 $T \in C_s(R, B(H))$, and according to (7) the estimate $\|T(\psi)(t)\|_{B(H)} \leq e^{\lambda t}$, $t \geq 0$ holds. Then according to Chernoff Theorem for operator function with values in the Banach space $B(H)$ the relation $T \sim F$ holds. Therefore, $X \sim S^t$ according to Definition 7. Theorem 5 is proved.

Corollary 5. Let the operator valued functions $X, Y \in F^1(E)$ satisfy the conditions $X^{-1}, Y^{-1} \in F^1(E)$. If $\frac{d}{dt}(X_t)(t=0) = \frac{d}{dt}(Y_t)(t=0)$ then the operator valued functions $X, Y$ are equivalent to each other in nonlinear Chernoff sense.

Let $\frac{d}{dt}(X_t)(t=0) = \frac{d}{dt}(Y_t)(t=0) = a \in C^1(E, E)$. Since $T_1 = \Phi_\lambda(X)$, $T_2 = \Phi_\lambda(Y)$ and $F = \Phi_\lambda(S^t)$ then according to Theorem 5 $X \sim S^t \sim Y$ in the sense of Definition 7.

Corollary 6. Let $E = \mathbb{R}^d$, $H = L_2(\mathbb{R}^d)$ and $X$ be the random flow with a finite set of values $X^{(1)}, \ldots, X^{(m)} \in C^1(R, C^1(\mathbb{R}^d, \mathbb{R}^d))$.

If $MX = p_1X^{(1)} + \ldots + p_mX^{(m)}$, where $p_1, \ldots, p_m \geq 0$, $p_1 + \ldots + p_m = 1$, then $MX \sim S^{Ma}$, where $Ma = p_1a_1 + \ldots + p_m a_m$, $a_k = \frac{d}{dt}X^{(k)}(t)(t=0)$, $k = 1, \ldots, m$.

Thus, under the assumption of Theorem 6 the mean value of random vector field generates the generalized mean value of random flow: $M\Phi_\lambda(X) \sim \Phi_\lambda(MX) \sim \Phi_\lambda(S^{Ma}(0))$.

4.2. The examples of violation of the property $M\Phi_\lambda(X) \sim \Phi_\lambda(MX)$

Example 1. Let $E = R$, $H = L_2(R)$, and

$X_t^{(\omega)}(x) = x + \sqrt{t}\omega$, $x \in E$, $t \geq 0$,

where $\omega$ is $E$-valued random variable such that its three first moments satisfy the conditions

$M\omega = 0$; $M(\omega)^2 = \sigma^2$; $M(|\omega|^3) < \infty$.  \hfill (9)
Any map $X_t^ω$ is accepted by the Lebesgue measure on the space $E$ and therefore,

$$\Phi_λ(X_t^ω)u(x) = u(x - \sqrt{t}ω) = e^{-\sqrt{t}ω D}u(x);$$

where $Du = u' \forall u \in W^1_0(R)$. Thus, according to the assumptions (9)

$$MX_t = I; \quad ΦMX_t = I;$$

$$MΦ(X_t) = ch(σ\sqrt{t}D) \sim \exp(\frac{σ^2}{2}D^2t).$$

Thus, $Φ(MX)$ and $MΦ(X)$ are not equivalent in Chernoff sense.

**Example 2.** Let $E = H$ be a real separable Hilbert space. According to the paper [20] the space $H$ can be endowed with the complete measure $λ$ on the space $H$ which is translation, rotationally invariant and defined on the minimal ring $R$ of subsets of $H$ containing the set $K$ of rectangles with absolutely converging products of edges lengths. Hilbert space $\mathcal{H} = L_2(H, R, λ, C)$ of square integrable with respect to the measure $λ$ functions $f: H \to C$ is constructed and its properties are investigated in the paper [20].

Let $E = \{e_j\}$ be some orthonormal basis in the space $E$. Let $K_E \subset K$ be the class of measurable rectangles in the space $E$ with edges collinear to the vectors of the system $E$. Let $\mathcal{R}_E$ be the minimal ring of subsets containing the class $K_E$.

Let $λ$ be the function of the set on the class $K$ defined by condition (4).

The function $λ: K \to [0, +∞)$ is additive and has the unique continuation onto the measure $λ$ on the ring $R$. Then the function $λ_E \equiv λ|_R \subset \mathcal{R}_E$ is the measure on the ring $\mathcal{R}_E$. A set $A \subset E$ is called $λ_E$-measurable if for any $ε > 0$ there are the sets $A_1, A_2 \subset A \subset A_2$ and $λ(A_2 \setminus A_1) < ε$. The set $Λ_E$ of $λ_E$-measurable subsets of the space $E$ is the ring and the function $λ_E(A) = \inf_{B\subset A} λ_E(B)$ is the complete measure on the ring $Λ_E$. Then $\mathcal{H} = L_2(E, Λ_E, λ_E, C)$ is the Hilbert space and $\mathcal{H}_E = L_2(E, Λ_E, λ_E, C)$ is the closed subspace of the space $\mathcal{H}$.

Let

$$X_t^h(x) = x + \sqrt{t}h, x \in H, t \geq 0, \quad (10)$$

where $h$ is $H$-valued random variable with distribution given by the Gaussian measure $ν_D$ with trace class covariation operator $D$. Therefore, the mean value $MX_t = \int H X_t^h dν_D(h)$ of the random map $X_t^h$ exists and $MX_t = I, t \geq 0$.

Since $MX_t = I, t \geq 0$, then $ΦMX_t = I, t \geq 0$.

Since the measure $λ$ is invariant with respect to shifts then any map $X_t^h, t \geq 0, h \in H$, is acceptable by the measure $λ$. Therefore, the linear operators $Φ_λ(X_t^h), t \geq 0, h \in H$, are defined on the space $\mathcal{H}$ and are given by the equality

$$Φ(X_t^h)u = S_{\sqrt{t}h}u;$$

where $S_h, h \in H$, is the operator of the argument shift on the vector $h$: $S_h u(x) = u(x+h), x \in H$.

Since $h$ is $H$-valued random variable with distribution given by Gaussian measure $ν_D$ with covariation operator $D$ then the mean value $M[Φ(X_t)] = \int H X_t^h dν_D(h)$ of the random linear operators $Φ(X_t^h), t \geq 0, h \in H$, exists. If the measure $ν_D$ is countable additive (this condition is equivalent to the condition: $D ∈ σ_{1}(H)$ is the trace class operator) then by the papers [20, 21] the one-parameter family of mean values $M[Φ(X_t)], t \geq 0$, is the one-parametric semigroup $U_D(t), t \geq 0$, of nonnegative contractions in the space $\mathcal{H}$. This fact is the consequence of the semigroup property with respect to the convolution for the Gaussian measures $(ν_{D} * ν_{sD}) = ν_{t+sD}$ for all $t, s \geq 0$. 

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Let $E$ be the orthonormal basis of eigenvectors of the correlation operator $D$ of Gaussian measure $\nu_{D}$. By the paper [21] the space $H_{\xi}$ is the invariant subspace of the operators of the semigroup $M[\Phi(X)]$, $t \geq 0$. If in addition, the operator $D^{\frac{1}{2}}$ is the trace-class operator then by the papers [20, 21] the restriction of the semigroup $M[\Phi(X)]$ on the invariant subspace $H_{\xi}$ is strongly continuous semigroup of the self-adjoint contractions of the space $H_{\xi}$. Thus, the operator-valued functions $\Phi(MX) = I$ and $M[\Phi(X) = U_{D}]$ are not equivalent in Chernoff sense.

5. The law of large numbers

Now we investigate the properties of the sequence of compositions of independent identically distributed random semigroups of linear operators. Using the concept of Chernoff equivalence and Chernoff Theorem we obtain the analog of the low of large numbers for the products of independent identically distributed random semigroups of linear operators.

The random linear operator $U$ is the measurable map of the set $\Omega$ of the probability space $(\Omega, A, \mu)$ into the topological vector space $(B(H), \tau_{s})$ endowed with the $\sigma$-algebra $A_{s}$ of Borel subsets. The random operators $U_{1}, U_{2}$ are called independent random operators if for any $A_{1}, A_{2} \in A_{s}$ the equality $\mu(\{U_{1} \in A_{1}, U_{2} \in A_{2}\}) = \mu(\{U_{1} \in A_{1}\}) \mu(\{U_{2} \in A_{2}\})$ holds.

Let $\{(\Omega_{n}, A_{n}, \mu_{n})\}$ be the sequence of probability spaces. Let $U_{k}: \omega_{k} \rightarrow Y_{k} \subset C_{s}(R_{+}, B(H))$ be the random semigroup of bounded linear operators in the space $H$ for any $k \in N$. Let $\{U_{n}\}$ be the sequence of independent random semigroups of bounded linear operators in the space $H$.

This sequence $\{U_{n}\}$ can be realized by the following way. For any $n \in N$ the composition of $n$ independent random semigroups of bounded linear operators $U_{1},..., U_{n}$ is defined as the map $U_{n} \circ ... \circ U_{1}$ of the space with the measure $(\Omega_{1} \times ... \times \Omega_{n}, A_{1} \otimes ... \otimes A_{n}, \mu_{1} \otimes ... \otimes \mu_{n})$ into the measurable space $(Y_{n}(H), A_{n})$. This map is defined by the equality $U_{n} \circ ... \circ U_{1}(\omega_{1},..., \omega_{n}) = U_{n}(\omega_{n}) \circ ... \circ U_{1}(\omega_{1})$, $(\omega_{1},..., \omega_{n}) \in \Omega_{1} \times ... \times \Omega_{n}$. Then for any $n \in N$ the composition of $n$ independent random semigroups of linear operators is the random operator valued function with values in the measurable space $(Y_{n}(H), A_{n})$.

**Theorem 7.** Let for any $n \in N$ the random operators $U_{1}, U_{2},..., U_{n}$ be independent. Then

$$M[U_{n} \circ ... \circ U_{1}] = M[U_{n}] \circ ... \circ M[U_{1}] \quad \forall \ n \in \mathbb{N}.$$ 

This statement is the consequence of the Fubini Theorem (see [22], Theorem 3.4.4) in its application to the function

$$f_{u,v,t}(\omega_{1},..., \omega_{n}) = (U_{n}(\omega_{n}) \circ ... \circ U_{1}(\omega_{1})u,v)_{H}, \ (\omega_{1},..., \omega_{n}) \in \Omega_{1} \times ... \times \Omega_{n},$$

for an arbitrary $u, v \in H$.

**Corollary 6.** Let $U$ be a random operator, $n \in \mathbb{N}$ and $U_{1}, U_{2},..., U_{n}$ be independent identically distributed random operators such that for any $k \in \{1,..., n\}$ the random operator $U_{k}$ has the same distribution as the random operator $U$. Then

$$M[U_{n} \circ ... \circ U_{1}] = (M[U])^{n} \quad \forall \ n \in \mathbb{N}. \quad (11)$$

5.1. The law of large numbers for compositions of linear maps

For the sequence $\{\xi_{k}\}$ of independent real-valued random variables with finite mean values the law of large numbers states that

$$\lim_{n \to \infty} P(|\frac{\xi_{n}}{n} + ... + \frac{\xi_{1}}{n} - M(\frac{\xi_{n}}{n} + ... + \frac{\xi_{1}}{n})| > \epsilon) = 0.$$
We extend this law onto the sequence \( \{U_k\} \) of independent random semigroups of bounded linear operators considering the averaging composition \((U_n)^{\frac{1}{n}} \circ \ldots \circ (U_1)^{\frac{1}{n}}\) instead of the averaged sum \( \frac{\xi_0}{n} + \ldots + \frac{\xi_n}{n} \). The especial property of the semigroup of linear operators in a Hilbert space gives the opportunity to define the operator valued function \((U(t))^{\frac{1}{n}}, t \geq 0, \) by the equality \((U(t))^{\frac{1}{n}} = U(t)^{\frac{1}{n}}, t \geq 0\). This is not the unique of root of the equation \((X(t))^n = U(t), t \geq 0,\) but the choice of the solution \(X(t) = U(t)^{\frac{1}{n}}, t \geq 0,\) gives the dynamical definition of the unique root.

Therefore the mean value of the averaged composition \((U_n(t))^{\frac{1}{n}} \circ \ldots \circ (U_1(t))^{\frac{1}{n}}, t \geq 0,\) of independent random semigroups is the iteration \((F(t))^{n}, t \geq 0,\) of operator function \(F = M[U]\). Thus, the Chernoff Theorem describes the asymptotic properties of the mean value of the composition of \(n\) independent random semigroups for \(n \to \infty\).

**Definition 9.**

1. The sequence \( \{U_n\} \) of the independent random semigroups of bounded linear operators with the common mean value \(U\) is said to be satisfying the large numbers law in the strong operator topology if the following condition

\[
\lim_{n \to \infty} P\{|(U_n(t))^{\frac{1}{n}} \circ \ldots \circ (U_1(t))^{\frac{1}{n}} - U(t)|_H > \epsilon\} = 0
\]

is fulfilled for every \(x \in H, x \neq 0, t > 0\) and \(\epsilon > 0\).

2. The sequence \( \{U_n\} \) of the independent random semigroups of bounded linear operators with the common mean value \(U\) is said to be satisfying the large numbers law in the topology of operators norm if the following condition

\[
\lim_{n \to \infty} P\{|(U_n(t))^{\frac{1}{n}} \circ \ldots \circ (U_1(t))^{\frac{1}{n}} - U(t)|_{B(H)} > \epsilon\} = 0
\]

is fulfilled for every \(t > 0\) and \(\epsilon > 0\).

To obtain estimates for compositions of independent random maps and the sufficient condition for the law of large numbers for random maps we introduce here the concept of the dispersion of random bounded linear operators and random semigroups.

The second moment of a random bounded linear operator \(A\) is the operator \(M[A^*A]\).

The second moment of a random semigroups \(U\) is the operator valued function \(M[U^*U]\).

Thus, the second moment of a random bounded linear operator is the nonnegative bounded linear operator. The dispersion of a random semigroup \(U\) is the nonnegative operator-valued function

\[
D(U) = M[U - M[U]] = M[U^*U] - M[U]^*M[U].
\]

We investigate the second moments of the values of the sequence of iterations of independent random linear operators.

Let \(t \geq 0\) and \(n \in \mathbb{N}\). Then according to (11) the following equality for the composition of independent identically distributed random semigroups \(U_1, \ldots, U_n\) holds:

\[
D(U_n(t) \circ \ldots \circ U_1(t)) = M[U_1^*(t) \circ \ldots \circ U_n^*(t) \circ U_1(t) \circ \ldots \circ U_1(t)] - (M[(U(t))]^n)^*M[(U(t))]^n.
\]

Let \(\{U_n(t), t \geq 0,\}\) be the sequence of independent identically distributed random semigroup of unitary operators. Then

\[
D(U_n(t) \circ \ldots \circ U_1(t)) = I - ((M[U(t)])^n)^*(M[U(t)])^n.
\]

If, in addition, the mean value \(M[U]\) of the random semigroup \(U(t), t \geq 0,\) is the semigroup then

\[
D(U_n(t) \circ \ldots \circ U_1(t)) = I - (M[U(nt)])^*M[U(nt)] = D(U(nt)).
\]
Lemma 7. (Chebyshev inequality). If $A$ is a random variable with values in the Banach space $B(H)$ such that its dispersion is the operator $D(A) \in B(H)$ then for any vector $x \in H : \|x\|_H = 1$ the following Chebyshev inequality is valid:

$$P(\{\|(A - MA)x\|_H > \varepsilon\}) \leq \frac{1}{\varepsilon^2}\|D(A)x\|_H. \quad (14)$$

Proof. Since $D(A) = \int (\langle A - MA\rangle x) d\mu$ then for any vector $x \in H$ such that $\|x\|_H = 1$ the following relations hold:

$$\|D(A)x\|_H = \sup_{\|u\|_H = 1} (u, \int (\langle A - MA\rangle x) d\mu) \geq (x, \int (\langle A - MA\rangle x) d\mu) = \int (x, (A - MA)x) d\mu = \int \|A - MA\|_H^2 d\mu \geq \int \|A - MA\|_H^2 d\mu \geq \varepsilon^2 \mu(\{\omega \in \Omega : \|A - MA\|_H \geq \varepsilon\}).$$

Therefore, Chebyshev inequality (14) holds.

Corollary 7. Let $\{U_n\}$ be the sequence of independent random semigroups with the common mean value $\bar{U}$ and the bounded in the norm of the space $B(H)$ sequence of dispersions $\{D(A_n)\}$. If the sequence $\{U_n\}$ satisfies the condition

$$\lim_{n \to \infty} \|D((A_n)^{\frac{1}{n}} \circ \cdots \circ (A_1)^{\frac{1}{n}})x\|_H = 0 \ \forall \ x \in H,$$

then the law of large numbers in the strong operator topology (12) for the sequence $\{U_n\}$ holds.

Theorem 8. Let $\xi : \Omega \to SA(H)$ be a random variable with the values in the set $SA(H)$ of self-adjoint operators in the space $H$, and

$$U_\omega(t) = \exp(it\xi(\omega)t), \ t \geq 0, \ \omega \in \Omega \quad (15)$$

be the corresponding random semigroup. Let $D$ be a dense linear manifold in the space $H$ such that for any $u \in D$ the condition $\int_\Omega \|\xi(\omega) u\|_H d\mu(\omega) < \infty$ holds.

Let $\{U_k\}$ be a sequence of independent identically distributed random semigroups such that any random semigroup $U_k$ has the same distribution as the random semigroup (15).

Let the linear operator $\xi : D \to H$ defined by the equality $\xi u = \int_\Omega (\xi(\omega) u) d\mu(\omega), \ u \in H$, be essentially self-adjoint. Then the sequence $\{U_0 \circ \cdots \circ U_1\}$ of compositions of independent identically distributed random semigroups satisfies the equality

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \|D(U_n(t)^{\frac{1}{n}} \circ \cdots \circ U_1(t)^{\frac{1}{n}})x\|_H = 0 \ \forall \ T > 0, \ \forall \ x \in H. \quad (16)$$

Proof. For any $n \in N$ the composition $U_n \circ \cdots \circ U_1$ of $n$ independent identically distributed semigroups is the map of the type $\{U_{\omega_n} \circ \cdots \circ U_{\omega_1}, \{\omega_1, \ldots, \omega_n\} \in \Omega \times \ldots \times \Omega\}$. Therefore, the mean value of the composition $U_n \circ \cdots \circ U_1$ satisfies (according to the theorem 7 and (11)) the equality

$$M[U_n \circ \cdots \circ U_1] = M[U_n] \circ \cdots \circ M[U_1] = (M[U])^n \ \forall \ n \in N.$$
Therefore, for any \( t \geq 0 \) and \( n \in \mathbb{N} \) the following equality holds:

\[
D(U_n(t) \circ \ldots \circ U_1(t)) = M[U_1^*(t) \circ \ldots \circ U_n^*(t) \circ U_n(t) \circ \ldots \circ U_1(t)] - M((U^*(t))^n)M((U(t))^n).
\]

Hence the equality

\[
D(U_n(t) \circ \ldots \circ U_1(t)) = I - ((M(U(t))^n)^*([M(U(t))])^n).
\]

is fulfilled for every sequence of compositions of independent random unitary semigroups \( \{U_n \circ \ldots \circ U_1\} \), \( \{\omega_1, \ldots, \omega_n\} \in \Omega \times \ldots \times \Omega \).

According to the Theorem 3 (see also [6]) the operator-function \( M[U](t) \), \( t \geq 0 \), is equivalent to the semigroup \( \exp(it)) \), \( t \geq 0 \) in Chernoff sense (see [13, 6]). It means that the sequence \( \{M[U(t)]^n, t \geq 0, \} \) converges to the semigroup \( \exp(it) \), \( t \geq 0, \) in the strong operator topology uniformly on any segment of the semiaxis \( R_+ \). Therefore, the equality \( \lim_{n \to \infty} \sup_{t \in [0,T]} ||((M(U(t))^n)^*([M(U(t))])^n - I||_H = 0 \) holds for every \( T > 0 \) and \( x \in H \).

**Corollary 8.** Let the conditions of Theorem 8 are fulfilled. Then the law of large numbers in the strong operator topology is valid for the sequence \( \{U_k\} \).

**Remark.** According to the estimates of Chernoff Lemma (see [11, 10]) the deviation of the mean square of compositions of the independent random semigroup \( M[U_n(t) \circ \ldots \circ U_1(t)] \) from the limit semigroup \( e^{itL}, t \geq 0 \) in any seminorm of the strong operator topology of the space \( B(H) \) has the order \( O(1/n) \) as \( n \to \infty \) uniformly on any segment \( [0,T] \).

We give the example of the sequence of compositions of independent identically distributed random semigroups satisfying the law of large numbers in the topology of the operator norm.

**Example 3.** Let us consider the example of the random semigroup of linear operators in the finitely dimensional Euclidean space \( H \) which has the random generator with uniformly bounded values. Let \( \{U_1, \ldots, U_n\} \) be the random semigroup of linear operators in the finitely dimensional Euclidean space \( H \) such that \( \|U_1(t)\| \leq 1 \) for all \( t \geq 0 \) and \( \omega \in \Omega \). Let \( \Omega = \{\omega_1, \ldots, \omega_m\} \) be a finite set and \( L_{\omega_j}, \omega_j \in \Omega \) be the bounded linear operators in \( H \) generating the semigroup \( U_{\omega_j} \), respectively. Then the mean value of the random semigroup \( F(t) = M[U_{\omega_j}(t)], t \in R_+ \), is the operator function which is equivalent in Chernoff sense to the semigroup \( U(t) = e^{tL}, t \geq 0 \), with the mean generator \( L = \sum_{j=1}^{m} p_j L_{\omega_j} \) (see [6]).

Let us consider the sequence of compositions \( \{U_n \circ \ldots \circ U_1\} \) of independent identically distributed random semigroups such that the distribution of every component \( U_k, k \in \mathbb{N} \) of this composition is described in the example 3. Then the sequence \( \{U_n \circ \ldots \circ U_1\} \) of compositions of random semigroups satisfies all conditions of Theorem 8. Hence, the estimate (16) is fulfilled for this sequence. Therefore, the law of large numbers in the strong operator topology is valid for the sequence \( \{U_n \circ \ldots \circ U_1\} \). Since the norm of the finitely dimensional space \( B(H) \) can be majorized by a sum of a finite set of seminorms of the strong operator topology of \( B(H) \) then, the law of large numbers in the topology of operators norm is valid for the sequence \( \{U_n\} \).

5.2. The examples of violation of the law of large numbers for the sequence of random semigroups

Firstly, we consider the example of the sequence \( \{U_n \circ \ldots \circ U_1\} \) of compositions of the independent identically distributed random semigroups which satisfies the law of large numbers in the strong operator topology but does not satisfy this law in the topology of the operator norm.

**Example 4.** (See [9]). (Violation of the large numbers law in the topology of the norm) Let \( \Omega = \{1, 2\} \) and measure \( \mu \) on the \( \sigma \)-algebra \( 2^\Omega \) be given by the equality \( \mu(\{1\}) = \mu(\{2\}) = \frac{1}{2} \). Let
be the self-adjoint operator with the basis of eigenvectors \( \{ e_k \} \) such that \( A e_k = k e_k \forall k \in \mathbb{N} \). Let \( \xi \) be the random generator of the random semigroup \( U(t) = \exp(it\xi), \) where \( \xi(1) = -A \) and \( \xi(2) = A \). Then according to Theorem 3, the law of large numbers in the strong operator topology is fulfilled. But \( M(U(t)) = \cos(A t), t \geq 0, M(U^n(t)U(t)) = I, t \geq 0. \) Then the large numbers low in the topology of operator norm is not satisfied since for any \( t > 0 \) the following estimates hold:

\[
P(\|U(\frac{t}{n})\|^n - M(U(\frac{t}{n}))^n)_{B(H)} > \frac{1}{2}) = P(\|U(\frac{t}{n})\|^n - (\cos(A \frac{t}{n}))^n)_{B(H)} > \frac{1}{2}) \geq
\]

\[
P(\sup_{k \in \mathbb{N}}\|U(\frac{t}{n})\|^n - (\cos(A \frac{t}{n}))^n e_k\|_H > \frac{1}{2}) = P(\sup_{k \in \mathbb{N}}|1 - (\cos(\frac{k t}{n}))^n| > \frac{1}{2}).
\]

Therefore, for any \( \delta > 0 \) there is a \( t \in (0, \delta) \) such that \( P(\|U(\frac{t}{n})\|^n - M(U(\frac{t}{n}))^n)_{B(H)} > \frac{1}{2}) = 1 \) for any \( n \in \mathbb{N}. \) It means that the law of large numbers in the strong operator topology is violated.

5.3. The law of large numbers for the compositions of random flows of nonlinear operators.

Let \( E \) be a real separable Hilbert space. Let \( \lambda \) be a measure on the space \( E. \) We investigate here the operator valued functions and the flows from the set \( F(\lambda) \) in Banach space \( F(E) \) (see (6)).

**Definition 10.** The sequence \( \{ X_n \} \) of independent random flows with values in the space \( F(\lambda) \) is called satisfying the law of large numbers in the topology generated by the strong operator topology of the space \( B(\mathcal{H}_\lambda) \) on the set \( F(\lambda) \) if \( E(\lambda) \) holds for any \( h \in \mathcal{H} \) and \( \epsilon > 0. \)

**Example 5.** Let \( \{ X_n \} \) be the sequence of independent identically distributed random flows in the space \( E = R^d \) such that the distribution of any \( n \in \mathbb{N} \) random flow \( X_n \) satisfies the assumptions of Corollary 6. Let \( \lambda \) be Lebesgue measure on the space \( E \) and \( F(\lambda) \) be the set (6) in the space \( F(E) \) consisting of the operator-functions \( \lambda \)-acceptable values. Then, according to Corollary 6 the sequence \( \{ X_n \} \) satisfies the law of large numbers in the topology generated by the strong operator topology of the space \( B(\mathcal{H}_\lambda) \) on the set \( F(\lambda). \)

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**References**

[1] Sakbaev V Zh 2010 *Infinite Dimensional Analysis Quantum Probability and Related Topics* 13 65–85
[2] Sakbaev V Zh 2016 *Journal of Mathematical Sciences* 213 287–459
[3] Sakbaev V Zh and Smolyanov O G 2016 *Doklady Mathematics* 94 654–58
[4] Volovich I V 2015 *P-Adic Numbers Ultrametric Analysis and Applications* 7 56–70
[5] Sakbaev V Zh and Volovich I V 2017 *P-Adic Numbers Ultrametric Anal. Appl.* 9 39–52
[6] Orlov Yu N, Sakbaev V Zh and Smolyanov O G 2016 *Izvestiya: Mathematics* 80 1131–58
[7] Orlov Yu N, Sakbaev V Zh and Smolyanov O G 2014 *Proceedings of the Steklov Institute of Mathematics* 285 222–32
[8] Efremova L S and Sakbaev V Zh 2015 *Theoret. and Math. Phys.* 185 1582–98
[9] Sakbaev V Zh 2016 *Russian Mathematics (Izvestiya VUZ. Matematika)* 60 72–6
[10] Bogachev V I and Smolyanov O G 2009 Real and functional analysis: a university course (Moscow: Regular and Chaotic Dynamics)
[11] Chernoff P 1968 J. Funct. Anal. 84 238–42
[12] Neklyudov A Yu 2008 Mathematical Notes 83 530–8
[13] Smolyanov O G, Weizsacker H and Wittih O 2007 Potential Anal. 26 1–29
[14] Butko Y A, Schilling R L and Smolyanov O G 2012 Infin. Dimens. Anal. Quantum Probab. Relat. Top. 15 26–39
[15] Remizov I D 2017 Dokl. Akad. Nauk 476 17-21
[16] Orlov Yu N, Sakbaev V Zh and Smolyanov O G 2017 Doklady Mathematics 96(3) 574–7
[17] Kozlov V V and Smolyanov O G 2015 Dokl. Akad. Nauk 465 527–31
[18] Kozlov V V and Smolyanov O G 2012 Dokl. Akad. Nauk 444 607–11
[19] Baker R 1991 Proc. Am. Math. Soc. 113 1023–29.
[20] Sakbaev V Zh 2017 Theoret. and Math. Phys. 191 886–909.
[21] Sakbaev V Zh 2017 Itogi Nauki i Tekhniki. Seriya ” Sovr. Mat. i ee pril.” 140 88-118 (Russian)
[22] Bogachev V I 2007 Measure theory vol 1 (Berlin: Springer)