A UNIQUENESS PROPERTY FOR THE QUANTIZATION OF TEICHMÜLLER SPACES

HUA BAI

Abstract. Chekhov, Fock and Kashaev introduced a quantization of the Teichmüller space $T^q(S)$ of a punctured surface $S$, and an exponential version of this construction was developed by Bonahon and Liu. The construction of the quantum Teichmüller space crucially depends on certain coordinate change isomorphisms between the Chekhov-Fock algebras associated to different ideal triangulations of $S$. We show that these coordinate change isomorphisms are essentially unique, once we require them to satisfy a certain number of natural conditions.

Let $S$ be an oriented surface of finite topological type, with at least one puncture. The Teichmüller space $T(S)$ is the space of isotopy classes of complete hyperbolic metrics on $S$. A quantization of the Teichmüller space $T(S)$ of $S$ was introduced by L. Chekhov and V. Fock [4, 5, 7] and, independently, by R. Kashaev [10] (see also [14]) as an approach to quantum gravity in $2 + 1$ dimensions. This is a deformation of the $C^*$-algebra of functions on the usual Teichmüller space $T(S)$ of $S$, depending on a parameter $\hbar$, in such a way that the linearization of this deformation at $\hbar = 0$ corresponds to the Weil-Petersson Poisson structure of $T(S)$. F. Bonahon and X. Liu [2, 12] developed an exponential version of the Chekhov-Fock-Kashaev construction. This exponential version of the quantization can be formulated in terms of non-commutative algebraic geometry, and has the advantage of possessing an interesting finite dimensional representation theory [2], whereas the non-exponential version is defined in terms of self-adjoint operators of Hilbert spaces.

More precisely, let $S$ be an oriented punctured surface of finite topological type, obtained by removing a finite set $\{v_1, v_2, \ldots, v_p\}$ from a closed oriented surface $\overline{S}$. An ideal triangulation is a family $\lambda$ of finitely many disjoint simple arcs $\lambda_1, \lambda_2, \ldots, \lambda_n$ going from puncture to puncture and decomposing $S$ into finitely many triangles with vertices at infinity; in other words, an ideal triangulation consists of the edges of a triangulation of the closed surface $\overline{S}$ whose vertex set consists of the punctures $\{v_1, v_2, \ldots, v_p\}$. Considering $q = e^{\pi i \hbar}$ as an indeterminate over $\mathbb{C}$, the Chekhov-Fock algebra $\mathbb{C}[X_1, X_2, \ldots, X_n]^{\hbar}_{\lambda}$ associated to the ideal triangulation $\lambda$ is the algebra over $\mathbb{C}(q)$ defined by generators $X_i^{\pm 1}, X_j^{\pm 1}, \ldots, X_h^{\pm 1}$ associated to the components of $\lambda$, and by relations $X_i X_j X_i^{-1} X_j^{-1} = q^{2\sigma_{ij}}$, where the $\sigma_{ij}$ are integers determined by the combinatorics of the ideal triangulation and connected to the Weil-Petersson form on Teichmüller space. This algebra has a well-defined fraction division algebra $\mathbb{C}(X_1, X_2, \ldots, X_n)^{\hbar}_{\lambda}$. In practice, the algebra $\mathbb{C}(X_1, X_2, \ldots, X_n)^{\hbar}_{\lambda}$

Date: October 30, 2018.

Key words and phrases. pentagon equation, quantum Teichmüller space, Chekhov-Fock algebra.

This work was partially supported by the grant DMS-0103511 from the National Science Foundation.
consists of all formal rational fractions in variables $X_i$ that skew-commute according to the relations $X_i X_j = q^{2\sigma_{ij}} X_j X_i$.

As one moves from one ideal triangulation $\lambda$ to another $\lambda'$, Chekhov and Fock [7, 14] (as developed in [12, 2]) introduce explicit coordinate change isomorphisms

$$\Phi^q_{\lambda, \lambda'} : \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda'}^q \to \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}^q$$

These are algebra isomorphisms which satisfy the natural property that $\Phi^q_{\lambda', \lambda''} \circ \Phi^q_{\lambda''} = \Phi^q_{\lambda, \lambda''}$ for any ideal triangulations $\lambda, \lambda', \lambda''$. In a triangulation independent way, this associates to the surface $S$ the algebra $\mathcal{T}^q(S)$ defined as the quotient of the family of all $\mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}^q$, with $\lambda$ ranging over all the ideal triangulations of the surface $S$, by the equivalence relation that identifies $\mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}^q$ and $\mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda'}^q$ by the coordinate change isomorphism $\Phi^q_{\lambda, \lambda'}$. By definition, $\mathcal{T}^q(S)$ is the quantum Teichmüller space of the surface $S$.

This construction is motivated by the case where $\hbar = 0$, so that $q = e^{2\pi i \hbar} = 1$. Thurston associated to each ideal triangulation $\lambda$ a set of global coordinates for the (non-quantum) Teichmüller space $\mathcal{T}(S)$, called shear coordinates; see [15, 7]. When $q = 1$, $\mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}$ is the usual algebra $\mathbb{C}(X_1, X_2, \ldots, X_n)$ of rational functions in the commuting variables $X_i$, and the coordinate change isomorphism $\Phi^q_{\lambda, \lambda'} : \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda} \to \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda'}$ exactly corresponds to the coordinate change between the shear coordinates $X_i$ for $\mathcal{T}(S)$ respectively associated to $\lambda$ and $\lambda'$. As a consequence, $\mathcal{T}^1(S)$ is in a natural way the algebra of rational functions on the Teichmüller space $\mathcal{T}(S)$.

We need to be a little more specific in our definitions, by requiring the data of an ideal triangulation $\lambda$ to include an indexing of its components $\lambda_1, \lambda_2, \ldots, \lambda_n$ by the set $\{1, 2, \ldots, n\}$. The permutation group $S_n$ then acts by reindexing on the set $\Lambda(S)$ of isotopy classes of all such indexed ideal triangulations of $S$.

The coordinate change isomorphisms defined by Chekhov-Fock [4], Kashaev [10], and Liu [12] satisfy the following natural conditions:

**Theorem 1** (Chekhov-Fock, Kashaev, Liu). There exists a family of algebra isomorphisms

$$\Phi^q_{\lambda, \lambda'} : \mathbb{C}(X'_1, X'_2, \ldots, X'_n)_{\lambda'}^q \to \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}^q$$

indexed by pairs of ideal triangulations $\lambda, \lambda' \in \Lambda(S)$, which satisfy the following conditions:

1. $\Phi^q_{\lambda', \lambda''} \circ \Phi^q_{\lambda''} = \Phi^q_{\lambda, \lambda''}$ for any $\lambda, \lambda', \lambda'' \in \Lambda(S)$;
2. if $\lambda' = \sigma \lambda$ is obtained by reindexing $\lambda$ by a permutation $\sigma \in S_n$, then $\Phi^q_{\lambda, \lambda'}(X_{\sigma(i)}') = X_{\sigma(i)}$ for any $1 \leq i \leq n$;
3. a Locality Condition precisely described in Section 2.

This paper is devoted to a uniqueness property for these $\Phi^q_{\lambda, \lambda'}$. This will require the property that the surface $S$ is sufficiently large in the sense that its Euler characteristic $\chi(S)$ is less than $-2$. This excludes the sphere with $\leq 4$ punctures, and the torus with $\leq 2$ punctures.

**Theorem 2.** Assume that the surface $S$ is sufficiently large, in the sense that $\chi(S) < -2$. Then the family of coordinate change isomorphisms $\Phi^q_{\lambda, \lambda'}$ in Theorem 1 is unique up to a uniform rescaling and/or inversion of the $X_i$.

Namely, if

$$\Psi^q_{\lambda, \lambda'} : \mathbb{C}(X'_1, X'_2, \ldots, X'_n)_{\lambda'}^q \to \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}^q$$
is another family of isomorphisms satisfying the conditions of Theorem 2, then there exists a non-zero constant $\xi \in \mathbb{C}(q)$ and a sign $\varepsilon = \pm 1$ such that $\Psi^q_{\lambda \lambda'} = \Theta_{\lambda} \circ \Phi^q_{\lambda \lambda'} \circ \Theta^{-1}_{\lambda'}$ for any pair of ideal triangulations $\lambda, \lambda'$, where $\Theta_{\lambda} : \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}^q \to \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}^q$ is the isomorphism defined by $\Theta_{\lambda}(X_i) = \xi X_i^{\varepsilon}$ for every $i$.

Theorem 4 is false when $S$ is the once-punctured torus or the 3–times punctured sphere. The uniqueness property appears to hold for the twice-punctured torus and the 4–times punctured sphere.

The proof of Theorem 2 relies on two crucial ingredients. One is that the $\Psi^q_{\lambda \lambda'}$ are algebra isomorphisms. The other one is that they satisfy the pentagon relation discussed in Section 3.

This uniqueness property has a counterpart in the non-exponential context of [10, 11, 12, 13]. In that context, it is crucial that the coordinate change isomorphisms are defined by transforming self-adjoint operators under meromorphic functions satisfying appropriate conditions. The punch line is then provided by the characterization of the meromorphic function

$$\phi^\hbar(z) = -\frac{\pi \hbar}{2} \int_{-\infty}^{\infty} \frac{e^{-it\pi}}{\sinh \pi t \sinh \pi \hbar t} dt$$

by a certain functional equation. By contrast, our arguments are purely algebraic.

Our proof strongly uses the property that the algebras considered are defined over $\mathbb{C}(q)$, or at least that $q$ is not a root of unity. On the other hand, interesting topological applications occur when we specialize $q$ to a root of unity 2. It would be interesting to investigate what kind of uniqueness property holds when $q^n = 1$, including the classical case $q = 1$.

Acknowledgments: The author would like to thank Francis Bonahon and Xiaobo Liu for many conversations related to this work.

1. The Chekhov-Fock algebra and coordinate change isomorphisms

In the surface $S = \overline{S} - \{v_1, v_2, \ldots, v_g\}$, let $\lambda$ be an ideal triangulation consisting of finitely many disjoint simple arcs $\lambda_1, \lambda_2, \ldots, \lambda_n$ going from puncture to puncture. An easy computation shows that the number $n$ of arcs in $\lambda$ is equal to $-3\chi(S) = 6g + 3p - 6$, where $\chi(S)$ is the Euler characteristic of $S$, $g$ is its genus and $p$ is its number of punctures.

The complement $S - \lambda$ has $2n$ spikes converging towards the punctures, and each spike is delimited by one $\lambda_i$ on one side and one $\lambda_j$ on the other side, with possibly $i = j$. Let $a_{ij}$ denote the number of spikes of $S - \lambda$ that are delimited on the left by $\lambda_i$ and on the right by $\lambda_j$, and define $\sigma_{ij} = a_{ij} - a_{ji}$. It is immediate that $\sigma_{ij} \in \{0, \pm 1, \pm 2\}$ and $\sigma_{ji} = -\sigma_{ij}$. Let $q$ be an indeterminate over $\mathbb{C}$. The Chekhov-Fock algebra $\mathbb{C}[X_1, X_2, \ldots, X_n]_{\lambda}^q$, as defined in [2, 12], is the algebra over $\mathbb{C}(q)$ defined by generators $X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}$ associated to the components of $\lambda$ and by skew-commutativity relations

$$X_i X_j X_i^{-1} X_j^{-1} = q^{2\sigma_{ij}}.$$ 

This is an iterated skew-polynomial algebra. As such, it satisfies the Ore Condition and consequently admits a well-defined fraction division algebra $\mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}^q$; see for instance [3, 6, 11]. In concrete terms, the Chekhov-Fock algebra $\mathbb{C}[X_1, X_2, \ldots, X_n]_{\lambda}^q$ consists of formal Laurent polynomials in variables $X_i$ satisfying the skew-commutativity
relations $X_i X_j X_i^{-1} X_j^{-1} = q^{2a_{ij}}$, while its fraction algebra $\mathbb{C}(X_1, X_2, \ldots, X_n)^\mathcal{L}$ consists of all formal rational fractions in the $X_i$ satisfying the same relations.

2. The Locality Condition

We describe here the Locality Condition mentioned in the introduction.

Define the discrepancy span $D(\lambda, \lambda')$ of two ideal triangulations $\lambda$, $\lambda'$ as the closure of the union of those connected components of $S - \lambda$ which are not isotopic to a component of $S - \lambda'$. For instance, $D(\lambda, \lambda')$ is empty exactly when $\lambda$ and $\lambda'$ coincide after isotopy and reindexing.

The coordinate change isomorphisms $\Phi_{\lambda, \lambda'}^q$ are said to satisfy the Locality Condition if the following holds. Let $\lambda$ and $\lambda'$ be two ideal triangulations indexed in such a way that $\lambda_i \subset D(\lambda, \lambda')$ when $i \leq k$, and $\lambda'_i = \lambda_i$ when $i > k$. Then the Locality Condition requires:

1. $\Phi_{\lambda, \lambda'}^q(X_i^q) = X_i$ for every $i > k$;
2. $\Phi_{\lambda, \lambda'}^q(X_i^q) = f_i(X_1, X_2, \ldots, X_k)$ for every $i \leq k$, where $f_i$ is a multivariable rational function depending only on the combinatorics of $\lambda$ and $\lambda'$ in $D(\lambda, \lambda')$ in the following sense: For any two pairs of ideal triangulation $(\lambda, \lambda'), (\lambda'', \lambda''')$ for which there exists a diffeomorphism $\psi : D(\lambda, \lambda') \to D(\lambda'', \lambda''')$ sending $\lambda_i$ to $\lambda''_i$ and $\lambda'_i$ to $\lambda'''_i$ for every $1 \leq j \leq k$, then
   \[
   \Phi_{\lambda, \lambda'}^q(X_i^q) = f_i(X_1, X_2, \ldots, X_k)
   \]
   for the same rational function $f_i$.

3. Diagonal Exchanges

The permutation group $\mathcal{S}_n$ acts on the set $\Lambda(S)$ of isotopy classes of (indexed) ideal triangulations of $S$ by reindexing of their components. The set $\Lambda(S)$ admits another natural operation.

The $i$-th diagonal exchange $\Delta_i : \Lambda(S) \to \Lambda(S)$ is defined as follows. The $i$-th component $\lambda_i$ of the ideal triangulation $\lambda$ separates two triangle components $T_1$ and $T_2$ of $S_\lambda$. If these two triangles are distinct, the union $T_1 \cup T_2 \cup \lambda_i$ is an open square $Q$ with diagonal $\lambda_i$. Then the ideal triangulation $\Delta_i(\lambda) \in \Lambda(S)$ is obtained from $\lambda$ by replacing $\lambda_i$ by the other diagonal $\lambda'_i$ of the square $Q$. In the remaining case where $T_1 = T_2$, then $\Delta_i(\lambda) = \lambda$ by convention; note that this case occurs exactly when $\lambda_i$ is the only component of $\lambda$ converging towards a certain puncture of $S$.

We say that the ideal triangulation $\lambda' = \Delta_i(\lambda)$ is obtained from $\lambda$ by an embedded diagonal exchange if, with the above notation, the four sides of the square $Q = T_1 \cup T_2 \cup \lambda_i$ correspond to distinct components of $\lambda$ (and of $\lambda'$).

![Figure 1. A diagonal exchange](image-url)
The coordinate change isomorphisms $\Phi_{\lambda,\lambda'}^q$ constructed by Chekhov and Fock have the property that, if $\lambda' = \Delta_i(\lambda)$ is obtained from $\lambda$ by an embedded $i$-th diagonal exchange with indices as shown in Figure 1,

$$
\Phi_{\lambda,\lambda'}^q(X_a') = (1 + qX_i)X_a \\
\Phi_{\lambda,\lambda'}^q(X_b') = (1 + qX_i^{-1})^{-1}X_b \\
\Phi_{\lambda,\lambda'}^q(X_c') = (1 + qX_i)X_c \\
\Phi_{\lambda,\lambda'}^q(X_d') = (1 + qX_i^{-1})^{-1}X_d, \\
\Phi_{\lambda,\lambda'}^q(X_i') = X_i^{-1}
$$
while $\Phi_{\lambda,\lambda'}^q(X_j') = X_j$ for every $j \neq a, b, c, d, i$.

One needs different formulas for non-embedded diagonal exchanges, which are given in [12].

There is an issue that we have somewhat neglected when stating Theorem 1, which is that it is not completely trivial that the coordinate change isomorphisms $\Phi_{\lambda,\lambda'}^q$ of Chekhov and Fock satisfy the Locality Condition of Section 2. Now is probably a good time to quickly address this issue.

Lemma 3. The Chekhov-Fock coordinate change isomorphisms $\Phi_{\lambda,\lambda'}^q$ of [7] [5] [4] [12] satisfy the Locality Condition of Section 2.

Proof. Having defined the coordinate change isomorphisms when $\lambda$ and $\lambda'$ are related by a diagonal exchange or a reindexing, Chekhov and Fock define $\Phi_{\lambda,\lambda'}^q$ for general $\lambda$, $\lambda'$ as follows. They connect $\lambda$ to $\lambda'$ by a sequence $\lambda = \lambda^{(0)}$, $\lambda^{(1)}$, $\ldots$, $\lambda^{(k)} = \lambda'$ such that each $\lambda^{(i)}$ is obtained from $\lambda^{(i-1)}$ by a diagonal exchange or a reindexing; the existence of such a sequence is for instance guaranteed by [13] [5] [9]. Then they define $\Phi_{\lambda,\lambda'}^q$ as

$$
\Phi_{\lambda,\lambda'}^q = \Phi_{\lambda,\lambda^{(1)}}^q \circ \Phi_{\lambda^{(1)},\lambda^{(2)}}^q \circ \cdots \circ \Phi_{\lambda^{(k-1)},\lambda'}^q
$$
and show that this is independent of the sequence of $\lambda^{(i)}$ connecting $\lambda$ to $\lambda'$. (The possibility of non-embedded diagonal exchanges in this sequence is neglected by Chekhov and Fock, and is rigorously dealt with in [12]).

Hatcher [9] proves a stronger result, namely that one can choose the sequence of reindexing and diagonal exchanges in which all the $\lambda^{(i)}$ are contained in the discrepancy span $D(\lambda, \lambda')$. If one uses this sequence to define $\Phi_{\lambda,\lambda'}^q$, it becomes evident that $\Phi_{\lambda,\lambda'}^q$ satisfies the Locality Condition.

The main step in the proof of Theorem 2 is the following, which is its specialization to the case of embedded diagonal exchanges.

Proposition 4 (Main step). Assume that $\chi(S) < -2$. Suppose that we have a family of algebraic isomorphisms

$$
\Psi_{\lambda,\lambda'}^q : C(X_1', X_2', \ldots, X_n')_{\lambda'} \rightarrow C(X_1, X_2, \ldots, X_n)_{\lambda},
$$
indexed by pairs of ideal triangulations $\lambda, \lambda' \in \Lambda(S)$ such that:

(a) $\Psi_{\lambda,\lambda'}^q = \Psi_{\lambda',\lambda''}^q \circ \Psi_{\lambda''}^q$ for any $\lambda$, $\lambda'$ and $\lambda'' \in \Lambda(S)$ ;

(b) if $\lambda' = \sigma\lambda$ is obtained by re-indexing $\lambda$ by $\sigma \in S_n$, then $\Psi_{\lambda,\lambda'}^q(X_i') = X_{\sigma(i)}$ for every $i$;

(c) the $\Psi_{\lambda,\lambda'}^q$ satisfy the Locality Condition of Section 2.
Then, there exists an invertible element \( \xi \in \mathbb{C}(q) \) and \( \varepsilon = \pm 1 \), defining for each ideal triangulation \( \lambda \) an isomorphism
\[
\Theta_{\lambda} : \mathbb{C}(X_1, X_2, \ldots, X_n)^{q}_{\lambda} \to \mathbb{C}(X_1, X_2, \ldots, X_n)^{q}_{\lambda}
\]
by the property that \( \Theta_{\lambda}(X_k) = \xi X_k^{\varepsilon} \) for every \( k \), such that
\[
\Theta_{\lambda}^{-1} \circ \Psi_{\lambda^i} \circ \Theta_{\lambda}(X_k) = (1 + qX_i)X_a
\]
\[
\Theta_{\lambda}^{-1} \circ \Psi_{\lambda^i} \circ \Theta_{\lambda}(X_b) = (1 + qX_i)^{-1}X_b
\]
\[
\Theta_{\lambda}^{-1} \circ \Psi_{\lambda^i} \circ \Theta_{\lambda}(X_c) = (1 + qX_i)X_c
\]
\[
\Theta_{\lambda}^{-1} \circ \Psi_{\lambda^i} \circ \Theta_{\lambda}(X_d) = (1 + qX_i)^{-1}X_d
\]
\[
\Theta_{\lambda}^{-1} \circ \Psi_{\lambda^i} \circ \Theta_{\lambda}(X_i') = X_i^{-1}
\]
whenever \( \lambda' = \Delta_i(\lambda) \) is obtained from \( \lambda \) by an embedded \( i \)-th diagonal exchange with indices as shown in Figure 7.

The next section is devoted to the proof of Proposition 4.

4. Proof of Proposition 4

We begin with a few algebraic preliminaries which will be regularly used throughout the proof.

A division algebra \( A \) and an algebra isomorphism \( \alpha : A \to A \) determine a skew polynomial algebra \( A[X]_{\alpha} \) consisting of all formal polynomials \( \sum_{i=1}^{n} a_iX_i \) in a variable \( X \), but where the multiplication is defined in such a way that \( Xa = \alpha(a)X \). This is a special case of Ore extension [6, 11, 3]. A fundamental property of these skew polynomial algebras is that they satisfy the Ore condition, so that they can be enlarged to a fraction division algebra \( A(X)_{\alpha} \). The elements of \( A(X)_{\alpha} \) are skew rational fractions, which can be expressed as rational fractions in \( X \) with coefficients in \( A \) but multiply according to the relation \( Xa = \alpha(a)X \).

In particular, the Chekhov-Fock division algebras \( \mathbb{C}(X_1, X_2, \ldots, X_n)^{q}_{\lambda} \) are obtained by iteration of such constructions.

Lemma 5. Let \( A \) be a division algebra over the field \( \mathbb{C}(q) \), and consider the skew rational function algebra \( A(X)_{\alpha} \) associated to an isomorphism \( \alpha : A \to A \). If the skew rational fraction \( f(X) \in A(X)_{\alpha} \) satisfies \( f(X) = f(q^2X) \), then \( f(X) \) is a constant function.

Proof. As in the commutative case, \( f(X) \) admits a unique Laurent series expansion \( f(X) = \sum_{k=0}^{\infty} a_k X^k \) with \( n \in \mathbb{Z} \) and \( a_k \in A \) for every \( k \). Then \( f(X) = f(q^2X) \) implies that \( a_k = a_k q^{2k} \) for every \( k \). Hence \( a_k = 0 \) if \( k \neq 0 \), and \( f(X) \) is a constant function.

In the commutative set-up where \( \alpha \) is the identity, we will frequently use the following elementary property, which we consequently state as a lemma.

Lemma 6. If \( A \) is a division algebra over the field \( \mathbb{C}(q) \) and if \( f(X) \in A(X) \) is a (commutative) rational function in the variable \( X \), then
\[
Xf(Y)X^{-1} = f(XYX^{-1})
\]
in the division algebra \( A[X,Y] \) consisting of all rational fractions in the non-commuting variables \( X \) and \( Y \) (but where \( X \) and \( Y \) commute with the elements of \( A \)).
We are now ready to begin the proof of Proposition \ref{lem}. Suppose that we are given a family of algebraic isomorphisms
\[ \Psi_{\lambda,\lambda'}^q : \mathbb{C}(X'_1, X'_2, \ldots, X'_n)^q \rightarrow \mathbb{C}(X_1, X_2, \ldots, X_n)^q \]
indexed by pairs of ideal triangulations \( \lambda, \lambda' \in \Lambda(S) \) which satisfy the hypotheses of Proposition \ref{lem}.

Throughout this section, we will assume that \( \chi(S) < -2 \).

To avoid repetition, whenever we encounter a diagonal exchange, we will also implicitly assume that the components of the corresponding ideal triangulations are indexed as in Figure 1.

The next four lemmas use only the Locality Condition and the fact that the \( \Psi_{\lambda,\lambda'}^q \) are algebra isomorphisms.

**Lemma 7.** There exists rational fractions \( f, g \) and \( h \in \mathbb{C}(q)(X) \) such that, for every embedded diagonal exchange indexed as in Figure 1,
\[ \Psi_{\lambda,\lambda'}^q(X'_a) = f(X_i)X_a \]
\[ \Psi_{\lambda,\lambda'}^q(X'_b) = g(X_i)X_b \]
\[ \Psi_{\lambda,\lambda'}^q(X'_c) = f(X_i)X_c \]
\[ \Psi_{\lambda,\lambda'}^q(X'_d) = g(X_i)X_d, \]
\[ \Psi_{\lambda,\lambda'}^q(X'_i) = h(X_i). \]

**Proof.** By the Locality Condition, we know there is a multi-variable rational function \( f_1 \) such that \( \Psi_{\lambda,\lambda'}^q(X'_a) = f_1(X_a, X_b, X_c, X_d, X_i) \) for every embedded diagonal exchange. We are going to take advantage of the fact that \( f_1 \) is independent of the global properties of \( \lambda \) and \( \lambda' \), and depends only on the indexing of the components of \( \lambda \) on the square \( D(\lambda, \lambda') \) where the diagonal exchange takes place.

Because of our hypothesis that \( \chi(S) < -2 \), we can find by inspection a pair of ideal triangulations \( \lambda, \lambda' \) which are deduced from each other by an embedded diagonal exchange and for which, in addition, there is a component \( \lambda_e \) of \( \lambda \) such that the corresponding generator \( X_e \) commutes with \( X_a, X_b, X_c, X_i \) and skew-commutes with \( X_d \) in the sense that \( X_eX_d = q^2X_dX_e \). Figure 2 provides examples of such \( \lambda \) for the smallest surfaces allowed by our hypothesis that \( \chi(S) < -2 \); these actually are the hardest cases and the reader will easily extend these examples to surfaces with more punctures.

![Figure 2. Triangulations of sufficiently large surfaces](image-url)
In this situation, we have $X'_aX'_aX'_a^{-1} = X'_a$. Therefore
\[ \Psi^q_{\lambda,\lambda'}(X'_a)\Psi^q_{\lambda,\lambda'}(X'_a)\Psi^q_{\lambda,\lambda'}(X'_a)^{-1} = \Psi^q_{\lambda,\lambda'}(X'_a) \]
since $\Phi^q_{\lambda,\lambda'}$ is an algebra isomorphism. Because the components $\lambda$ and $\lambda'$ are not in $D(\lambda, \lambda')$, the Locality Condition implies that $\Psi^q_{\lambda,\lambda'}(X'_a) = X_c$. It follows that $X_c f_1(X_a, X_b, X_c, X_d, X_i) X_c^{-1} = f_1(X_a, X_b, X_c, X_d, X_i)$. Applying Lemma 6, we conclude that $f_1(X_a, X_b, X_c, X_d, X_i) = f_1(X_a, X_b, X_c, X_d, X_i)$.

Lemma 5 then shows that $f_1$ must be independent of the variable $X_d$.

Similarly, one can find another pair of ideal triangulations $\lambda$ and $\lambda'$, obtained from each other by a diagonal exchange, for which there is now a generator $X_c$ which skew-commutes with $X_c$ and commutes with the generators $X_a, X_b, X_d$ associated to the other sides of the square $D(\lambda, \lambda')$. The same argument as above then shows that $f_1$ is independent of $X_c$.

Another application of the same argument shows that $f_1$ is independent of $X_b$.

Therefore, $\Phi(X'_a) = f_1(X_a, X_b, X_c, X_d, X_i) = f_2(X_a, X_i)$ for some rational function in two variables.

Now, we again find two ideal triangulations $\lambda$ and $\lambda'$, obtained from each other by a diagonal exchange, for which there is now a generator $X_c$ which commutes with $X_b, X_c, X_d$ and for which $X_c X_a = q^2 X_a X_c$. The argument is here slightly different. From $X'_cX'_aX'_a^{-1} = q^2 X'_a$, we deduce $f_2(q^2 X_a, X_i) = q^2 f_2(X_a, X_i)$. This is equivalent to say $f_2(q^{-2} X_a, X_i)(q^2 X_a)^{-1} = f_2(X_a, X_i)X_a^{-1}$, which forces $f_2(X_a, X_i)X_a^{-1}$ to be independent of variable $X_a$ by Lemma 5. In other words, $\Psi^q_{\lambda,\lambda'}(X'_a) = f_2(X_a, X_i) = f(X_i)X_a$ for some rational function $f \in \mathbb{C}(q)$, as required.

Combining the Locality Condition with the symmetries of the square, $\Psi^q_{\lambda,\lambda'}(X'_a) = g(X_i)X_b$,
\[ \Psi^q_{\lambda,\lambda'}(X'_b) = f(X_i)X_c, \]
and $\Psi^q_{\lambda,\lambda'}(X'_d) = g(X_i)X_d$ for another rational function $g \in \mathbb{C}(q)$.

Finally, we know by the Locality Condition that $\Psi^q_{\lambda,\lambda'}(X'_i) = h_1(X_a, X_b, X_c, X_d, X_i)$ for some multi-variable rational function $h_1$. Finding again a pair $\lambda, \lambda'$ where there is a generator $X_c$ which commutes with $X_b, X_c, X_d, X_i$ but skew-commutes with $X_a$, the same method as above shows that $\Psi^q_{\lambda,\lambda'}(X'_i)$ is independent of $X_a$. Similarly $\Psi^q_{\lambda,\lambda'}(X'_i)$ is also independent of $X_b, X_c$ and $X_d$, hence $\Psi^q_{\lambda,\lambda'}(X'_i) = h(X)$ for some rational function $h \in \mathbb{C}(q)$.

Lemma 8. The rational function $h$ of Lemma 5 is of the form $h(X) = \eta X^{-1}$ for some $\eta \in \mathbb{C}(q) - \{0\}$.

Proof. Applying $\Psi^q_{\lambda,\lambda'}$ to both sides of the relation $X'_aX'_aX'_a^{-1} = q^2 X'_aX'_a$, we obtain $h(X_i)f(X_i)X_a = q^2f(X_i)X_a h(X_i)$. The terms $f(X_i)$ cancel out. Using the relation $X_aXX_a^{-1} = q^2 X$ and Lemma 5 we get $h(q^{-2}X_i) = q^2 h(X_i)$. Rewriting this as $(q^{-2}X_i)h(q^{-2}X_i) = X_i h(X_i)$, we see that $X_i h(X_i) = \eta$ for some constant $\eta \in \mathbb{C}(q)$ by Lemma 5. Because the subalgebra of $\mathbb{C}(X_1, X_2, \ldots, X_n)$ consisting of all rational functions in the variable $X_i$ is abstractly isomorphic to $\mathbb{C}(q)(X_i)$, we conclude that $h(X) = \eta X^{-1}$ in $\mathbb{C}(q)(X_i)$.

The next lemma uses the fact that the diagonal exchange map $\Delta_i : \Lambda(S) \to \Lambda(S)$ is an involution.

Lemma 9. The rational functions $f, g \in \mathbb{C}(q)(X)$ of Lemma 5 are such that $g(X) = f(\eta X^{-1})^{-1}$, where $\eta \in \mathbb{C}(q)$ is the constant of Lemma 5.
Proof. Because \( \lambda = \Delta_i(X) = \Delta^2(\lambda) \),

\[
X'_a = \Psi_{\lambda',\lambda} \circ \Psi_{\lambda}^q(X'_a) = \Psi_{\lambda',\lambda}^q(f(X_i)X_a) = f(\eta X_i^{-1})g(X'_i)X'_a
\]
since \( \Psi_{\lambda',\lambda}(X_i) = \eta X_i^{-1} \) and \( \Psi_{\lambda'}^q(X_a) = g(X'_i)X'_a \). The result follows. \( \square \)

**Lemma 10.** The rational function \( f \) in Lemma 7 satisfies \( f(X) = \xi q X f(q^{-2}\xi^{-2}X^{-1}) \) for some \( \xi \in \mathbb{C}(q) \) with \( \xi^2 = \eta^{-1} \).

Proof. The product \( X'_a X'_b X'_c X'_d X'_i \) commutes with each of \( X'_a, X'_b, X'_c, X'_d \) and \( X'_i \). Since \( \Psi_{\lambda'}^q \) is an algebra isomorphism, \( \Psi_{\lambda'}^q(X'_a X'_b X'_c X'_d X'_i) \) therefore commutes with \( X_a, X_b, X_c, X_d \) and \( X_i \). Using Lemma 3 and the skew-commutativity relations, this element is equal to

\[
\Psi_{\lambda',\lambda}^q(X'_a X'_b X'_c X'_d X'_i) = f(X_i)X_a \cdot f(\eta X_i^{-1})^{-1} X_b \cdot f(X_i)X_c \cdot f(\eta X_i^{-1})^{-1} X_d \cdot \eta X_i^{-1}
\]

Since \( X_a X_b X_c X_d X_i \) commutes with \( X_a, X_b, X_c, X_d \) and \( X_i \), it follows that \( [f(X_i)(qX_i f(q^{-2} \eta X_i^{-1}))]^{-1} \) also commutes with \( X_a \). An application of Lemmas 3 and 4 then shows that this rational function of \( X_i \) is constant. As a consequence, \( f(X_i)(qX_i f(q^{-2} \eta X_i^{-1}))^{-1} = \xi \) for some \( \xi \in \mathbb{C}(q) \).

Therefore, \( f(X)(qX f(q^{-2} \eta X^{-1}))^{-1} = \xi \) in \( \mathbb{C}(q)(X) \). Substituting \( q^{-2} \eta X^{-1} \) for \( X \) in this equation and combining with the original relation, we obtain that \( \xi^2 = \eta^{-1} \).

This proves the relation \( f(X) = \xi q X f(q^{-2} \xi^{-2} X^{-1}) \). \( \square \)

Without loss of generality, we can replace all isomorphisms

\[
\Psi_{\lambda,\lambda'}^q : \mathbb{C}(X'_1, X'_2, \ldots, X'_n)_{\lambda'} \rightarrow \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}
\]

by \( \Theta_{\lambda}^{-1} \circ \Theta_{\lambda'}^{q} \circ \Theta_{\lambda'} \), where, for each ideal triangulation \( \lambda \),

\[
\Theta_{\lambda} : \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda} \rightarrow \mathbb{C}(X_1, X_2, \ldots, X_n)_{\lambda}
\]

is the isomorphism defined by \( \Theta_{\lambda}(X_k) = \xi X_k \) for every \( k \). This replaces the rational function \( f(X) \) by \( f'(X) = f(\xi^{-1}X) \) which satisfies the relation \( f'(X) = qX f'(q^{-2}X^{-1}) \), the function \( g(X) = f(\xi^{-2}X^{-1})^{-1} \) by \( g'(X) = f'(X^{-1})^{-1} \), and \( h(X) = \xi^{-2}X^{-1} \) by \( h'(X) = X^{-1} \). Therefore, we can henceforth assume that \( \xi = 1 \).

So far, we have only used the Locality Condition and the skew commutativity properties of the generators of the Chekhov-Fock algebra. We now take advantage of a subtler property, namely the Pentagon Relation illustrated in Figure 3. Because of our assumption that \( \chi(S) < -2 \), the surface \( S \) admits an ideal triangulation \( \lambda \) for which the closure of three components of \( S - \lambda \) forms an embedded pentagon. We then have a sequence of 5 diagonal exchanges which returns to the original ideal triangulation. (Actually, the indexing is modified by a transposition of the two diagonals of the pentagon, but this will be irrelevant here).

For each ideal triangulation \( \lambda \) occurring in the pentagon move, it is convenient to index the components \( \lambda_i \) and \( \lambda_j \) of \( \lambda \) that are diagonals of the pentagon as indicated on Figure 3. This convention has the advantage that the corresponding generators \( X_i \) and \( X_j \) of the Chekhov-Fock algebra of \( \lambda \) satisfy the same relation \( X_j X_i = q^2 X_i X_j \). Also, as one moves from one such ideal triangulation \( \lambda \) to the next one \( \lambda' \), \( \Psi_{\lambda,\lambda'}^q(X'_i) = f(X_i)X_j \) and \( \Psi_{\lambda',\lambda}^q(X'_j) = X_i^{-1} \).
This leads us to introduce the algebra $\mathcal{W}$ defined by generators $U^{\pm 1}$, $V^{\pm 1}$ and by the relation $VU = q^2UV$, and the algebra isomorphism $\Psi : \mathcal{W} \to \mathcal{W}$ defined by $\Psi(U) = f(U)V$ and $\Psi(V) = U^{-1}$. From the above observation, a consequence of the Pentagon Relation is that the composition $\Psi^5$ is the identity.

**Lemma 11.** The isomorphism $\Psi : \mathcal{W} \to \mathcal{W}$ defined by $\Psi(U) = f(U)V$ and $\Psi(V) = U^{-1}$ has period 5 if and only if $f(f(U)V) = VF(U^{-1}f(V^{-1}))U$.

**Proof.** Write $U_k = \Psi^k(U)$ and $V_k = \Psi^k(V)$. Because $V_{k+1} = U_k^{-1}$, $\Psi$ is periodic with period 5 if and only if $U_{k+5} = U_k$ for some $k$, or, equivalently, or for all $k$.

Noting that $U_{k+1} = f(U_k)V_k = f(U_k)U_k^{-1}$ and $U_{k-1} = U_{k+1}^{-1}f(U_k)$, an immediate computation gives $U_2 = f(f(U)V)U^{-1}$ and $U_{-3} = VF(U^{-1}f(V^{-1}))$. The result follows. □

**Lemma 12.** If a rational function $f$ is such that $f(X) = qXf(q^{-2}X^{-1})$ in $\mathbb{C}(q)(X)$ and $f(f(U)V) = VF(U^{-1}f(V^{-1}))U$ in the algebra $\mathcal{W}$, then $f(X) = 1 + qX$ or $f(X) = (1 + q^{-1}X^{-1})^{-1}$.

**Proof.** Let us transform the relation $f(f(U)V) = VF(U^{-1}f(V^{-1}))U$.

\[ f(f(U)V) = VF(U^{-1}f(V^{-1}))U \\
= Vf(qU^{-1}V^{-1}f(q^{-2}V))U \\
= Vq^2U^{-1}V^{-1}f(q^{-2}V)f(q^{-2}q^{-1}f(q^{-2}V)^{-1}VU))U \\
= (U^{-1}f(q^{-2}V)U) (U^{-1}f(q^{-3}f(q^{-2}V)^{-1}VU))U \\
= f(V)f(q^{-1}f(V)^{-1}VU) \]

Therefore the equation $f(f(U)V) = VF(U^{-1}f(V^{-1}))U$ is equivalent to the relation $f(f(U)V) = f(V)f(q^{-1}f(V)^{-1}VU)$.

Consider the Laurent expansion $f(X) = X^s(a_0 + a_1X + a_2X^2 + \ldots)$ of $f$, with $a_0 \neq 0$. We will distinguish cases.

**Case 1:** $s < 0$.

If we expand $q^{-1}f(V)^{-1}VU$ as a Laurent series in $V$ with coefficients in $\mathbb{C}(q)(U)$, its lowest degree term in $V$ has degree $1 - s \geq 2$. We can therefore use again the
Laurent series expansion of $f(X)$ to expand both sides of the equation $f(f(U)V) = f(V)f(q^{-1}f(V)^{-1}VU)$ as a Laurent series in $V$ with coefficients in $\mathbb{C}(q)(U)$.

In this expansion, the lowest degree term in $V$ has degree $s + s(1 - s)$ for the right hand side, and $s$ of the left hand side. Since $s = s + s(1 - s)$ has no negative solution, it follows that this case actually cannot occur.

Case 2: $s = 0$.

In this case the lowest degree term of $q^{-1}f(V)^{-1}VU$ has degree 1, so we can expand again $f(q^{-1}f(V)^{-1}VU)$ term by term.

Comparing the constant terms (with respect to $V$) on both sides of $f(f(U)V) = f(V)f(q^{-1}f(V)^{-1}VU)$, we get $a_0 = a_0^2$. So $a_0 = 1$.

The next term in the expansion has degree $t = \min\{T : T > 0, a_T \neq 0\}$. The coefficient of $V^t$ on the left hand side of the equation comes from $a_t(f(U)V)^t$, and is equal to $a_t f(U)f(q^2U) \cdots f(q^{2(t-1)}U)$. On the right hand side, remembering that $a_0 = 1$, the coefficient of $V^t$ comes from $a_t V^t + a_t(q^{-1}VU)^t$ and is equal to $a_t + a_t q^{t^2} U^t$. This gives

$$f(U)f(q^2U) \cdots f(q^{2(t-1)}U) = 1 + q^{t^2} U^t.$$  

Expanding the left hand side as a Laurent series in $U$ and looking at the coefficient of $U^t$, we conclude that $t = 1$ and $f(X) = 1 + qU$.

Case 3: $s > 0$.

In this case, the Laurent expansion of $q^{-1}f(V)^{-1}VU$ in $V$ begins with a term of non-positive degree, so that we cannot expand $f(q^{-1}f(V)^{-1}VU)$ term by term any more. We will use an internal symmetry of the problem.

Set $\hat{f}(X) = f(q^{-2}X^{-1})^{-1} = qXf(X)^{-1}$. We claim that this rational function $\hat{f}(X)$ also satisfies the hypotheses of the Lemma. Let us check this. First,

$$qX \hat{f}(q^{-2}X^{-1}) = qX f(X)^{-1} f(q^{-2}X^{-1})^{-1} = \hat{f}(X).$$

Then

$$\hat{f}(f(U)V) = f(q^{-2}V^{-1} \hat{f}(U)^{-1})^{-1} = f(q^{-2}V^{-1} f(q^{-2}U^{-1}))^{-1} = f(f(U^{-1})q^{-2}V^{-1})^{-1},$$

while

$$\hat{f}(V) \hat{f}(q^{-1} \hat{f}(V)^{-1}VU) = f(q^{-2}V^{-1})^{-1} f(q^{-2}qU^{-1}V^{-1} \hat{f}(V))^{-1} = f(q^{-2}V^{-1})^{-1} f(q^{-1}U^{-1}V^{-1} f(q^{-2}V^{-1})^{-1})^{-1} = f(q^{-1} f(q^{-2}V^{-1})^{-1} q^{-2}V^{-1} U^{-1})^{-1} f(q^{-2}V^{-1})^{-1}.$$  

Therefore we need to check that

$$f(f(U^{-1})q^{-2}V^{-1}) = f(q^{-2}V^{-1}) f(q^{-1} f(q^{-2}V^{-1})^{-1} q^{-2}V^{-1} U^{-1}).$$

But this property holds by applying to the identity $f(f(U)V) = f(V)f(q^{-1}f(V)^{-1}VU)$ the homomorphism $\Theta : \mathcal{W} \to \mathcal{W}$ defined by $\Theta(U) = U^{-1}$ and $\Theta(V) = q^{-2}V^{-1}$.

Therefore, the rational function $\hat{f}(X) \in \mathbb{C}(q)(X)$ satisfies the hypotheses of the Lemma. The Laurent series expansion of $\hat{f}(X) = qXf(X)^{-1}$ begins with a term of degree $1 - s \geq 0$. It follows from Cases 1 and 2 that $\hat{f}(X) = 1 + qX$. Therefore, $f(X) = qX \hat{f}(X)^{-1} = (1 + q^{-1}X^{-1})^{-1}$.  

$\square$
Note that, when \( f(X) = 1 + qX \), the function \( g(X) \) of Lemmas 7 and 10 is such that \( g(X) = (1 + q^{-1}X)^{-1} \). When \( f(X) = (1 + q^{-1}X)^{-1} \), then \( g(X) = 1 + q^{-1}X \).

At this point we have shown that there exists \( \xi \in \mathbb{C} \setminus \{0\} \) such that, if isomorphisms

\[
\Theta_\lambda : \mathbb{C}(X_1, X_2, \ldots, X_n)^q \to \mathbb{C}(X_1, X_2, \ldots, X_n)^q
\]

are defined by \( \Theta_\lambda(X_k) = \xi X_k \) for every \( k \), either we have

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_a) = (1 + qX_i)X_a
\]

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_b) = (1 + qX_i^{-1})^{-1}X_b
\]

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_c) = (1 + qX_i)X_c
\]

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_d) = (1 + qX_i^{-1})^{-1}X_d
\]

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_i) = X_i^{-1}
\]

whenever \( \lambda' \) is obtained from \( \lambda \) by an embedded diagonal exchange (with the edge indexing conventions of Figure 1), or we have

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_a) = (1 + q^{-1}X_i^{-1})^{-1}X_a
\]

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_b) = (1 + q^{-1}X_i)X_b
\]

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_c) = (1 + q^{-1}X_i^{-1})^{-1}X_c
\]

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_d) = (1 + q^{-1}X_i)X_d
\]

\[
\Theta_\lambda^{-1} \circ \Psi^q_{\lambda \lambda'} \circ \Theta_{\lambda'}(X'_i) = X_i^{-1}
\]

again whenever \( \lambda' \) is obtained from \( \lambda \) by an embedded diagonal exchange.

In the first case, we have reached the conclusion desired for Proposition 1.

In the second case, we conjugate by an additional family of isomorphisms

\[
\Theta'_\lambda : \mathbb{C}(X_1, X_2, \ldots, X_n)^q \to \mathbb{C}(X_1, X_2, \ldots, X_n)^q
\]

defined by \( \Theta_\lambda(X_k) = X_k^{-1} \) for every \( k \). Then

\[
(\Theta_\lambda \circ \Theta'_\lambda)^{-1} \circ \Psi^q_{\lambda \lambda'} \circ (\Theta_\lambda \circ \Theta'_\lambda)(X'_a) = X_a(1 + q^{-1}X_i) = (1 + qX_i)X_a
\]

\[
(\Theta_\lambda \circ \Theta'_\lambda)^{-1} \circ \Psi^q_{\lambda \lambda'} \circ (\Theta_\lambda \circ \Theta'_\lambda)(X'_b) = X_b(1 + q^{-1}X_i^{-1})^{-1} = (1 + qX_i^{-1})^{-1}X_b
\]

\[
(\Theta_\lambda \circ \Theta'_\lambda)^{-1} \circ \Psi^q_{\lambda \lambda'} \circ (\Theta_\lambda \circ \Theta'_\lambda)(X'_c) = X_c(1 + q^{-1}X_i) = (1 + qX_i)X_c
\]

\[
(\Theta_\lambda \circ \Theta'_\lambda)^{-1} \circ \Psi^q_{\lambda \lambda'} \circ (\Theta_\lambda \circ \Theta'_\lambda)(X'_d) = X_d(1 + q^{-1}X_i^{-1})^{-1} = (1 + qX_i^{-1})^{-1}X_d
\]

\[
(\Theta_\lambda \circ \Theta'_\lambda)^{-1} \circ \Psi^q_{\lambda \lambda'} \circ (\Theta_\lambda \circ \Theta'_\lambda)(X'_i) = X_i^{-1}
\]

We again have reached the conclusions of Proposition 1.

This completes the proof of this statement. \( \square \)

5. Proof of the main Theorem 2

Let \( \Phi^q_{\lambda \lambda'} : \mathbb{C}(X'_1, X'_2, \ldots, X'_n)^q \to \mathbb{C}(X_1, X_2, \ldots, X_n)^q \) be the Chekhov-Fock family of isomorphisms provided by Theorem 1, and let \( \Psi^q_{\lambda \lambda'} : \mathbb{C}(X'_1, X'_2, \ldots, X'_n)^q \to \mathbb{C}(X_1, X_2, \ldots, X_n)^q \) be another family of isomorphisms satisfying the same properties.
Proposition \[4\] shows that, after conjugating the \(\Psi_{\lambda\lambda'}^q\), with appropriate rescaling/inversion isomorphisms \(\Theta_{\chi} : \mathbb{C}\langle X_1, X_2, \ldots, X_n\rangle^q X_k \rightarrow \mathbb{C}\langle X_1, X_2, \ldots, X_n\rangle^q X_k\) defined by \(\Theta_{\lambda}(X_k) = \xi X_k^{\pm 1}\), we can arrange that \(\Psi_{\lambda\lambda'}^q = \Phi_{\lambda\lambda'}^q\), whenever \(\lambda\) and \(\lambda'\) differ only by an embedded diagonal exchange.

Because of our hypothesis that \(\chi(S) < -2\), there are only two possible types of non-embedded diagonal exchanges:

1. one type where, in the square where the diagonal exchange takes, two opposite sides are identified in \(S\);
2. another type where two adjacent sides of this square are identified; this is possible only when \(S\) has at least two punctures.

Explicit formulas for the \(\Phi_{\lambda\lambda'}^q\) associated to non-embedded diagonal exchanges are given in [12].

By inspection on the surface \(S\) and using the fact that \(\chi(S) < -2\), one can construct an ideal triangulation of \(S\) where three faces form a pentagon \(P\), in such a way that the only two sides of the pentagon \(P\) that are identified in \(S\) are two non-adjacent sides. This gives rise to a sequence of diagonal exchanges

\[
\lambda(0) \rightarrow \lambda(1) \rightarrow \lambda(2) \rightarrow \lambda(3) \rightarrow \lambda(4) \rightarrow \lambda(5) = \lambda(0),
\]

all taking place in the pentagon \(P\) as in the pentagon relation, where \(\lambda(0) \rightarrow \lambda(1)\) is a non-embedded diagonal exchange of Type (1), and where all other diagonal exchanges \(\lambda(i) \rightarrow \lambda(i+1)\) are embedded.

From the properties that \(\Phi_{\lambda\lambda'}^q = \Phi_{\lambda\lambda'}^q \circ \Phi_{\chi\chi'}^q\) and \(\Psi_{\lambda\lambda'}^q = \Psi_{\lambda\lambda'}^q \circ \Psi_{\chi\chi'}^q\), we conclude that

\[
\Phi_{\lambda(0)\lambda(1)}^q = \left(\Phi_{\lambda(1)\lambda(2)}^q \circ \Phi_{\lambda(2)\lambda(3)}^q \circ \Phi_{\lambda(3)\lambda(4)}^q \circ \Phi_{\lambda(4)\lambda(5)}^q\right)^{-1}
\]

and

\[
\Psi_{\lambda(0)\lambda(1)}^q = \left(\Psi_{\lambda(1)\lambda(2)}^q \circ \Psi_{\lambda(2)\lambda(3)}^q \circ \Psi_{\lambda(3)\lambda(4)}^q \circ \Psi_{\lambda(4)\lambda(5)}^q\right)^{-1}.
\]

Since \(\Phi_{\lambda\lambda'}^q\) and \(\Psi_{\lambda\lambda'}^q\) coincide on embedded diagonal exchanges, we conclude that \(\Phi_{\lambda(0)\lambda(1)}^q = \Psi_{\lambda(0)\lambda(1)}^q\). By the Locality Condition, it follows that \(\Phi_{\lambda\lambda'}^q\) and \(\Psi_{\lambda\lambda'}^q\) coincide on all non-embedded diagonal exchanges of Type (1).

Similarly, when \(S\) has at least two punctures, one can find an ideal triangulations with a pentagon whose only side identifications are between two adjacent sides. The same argument as above then shows that \(\Phi_{\lambda\lambda'}^q\) and \(\Psi_{\lambda\lambda'}^q\) coincide on all non-embedded diagonal exchanges of Type (2).

Therefore, \(\Phi_{\lambda\lambda'}^q\) and \(\Psi_{\lambda\lambda'}^q\) coincide on all diagonal exchanges, embedded or non-embedded.

Finally, any two ideal triangulations \(\lambda\) and \(\lambda'\) can be joined by a sequence of diagonal exchanges

\[
\lambda = \lambda(0) \rightarrow \lambda(1) \rightarrow \cdots \rightarrow \lambda(n) = \lambda'.
\]

See for instance [13, 9]. This decomposes \(\Phi_{\lambda\lambda'}^q\) and \(\Psi_{\lambda\lambda'}^q\) as a product of \(\Phi_{\lambda(i)\lambda(i+1)}^q = \Psi_{\lambda(i)\lambda(i+1)}^q\) associated to diagonal exchanges, thereby showing that \(\Phi_{\lambda\lambda'}^q = \Psi_{\lambda\lambda'}^q\) for any ideal triangulations \(\lambda\) and \(\lambda'\).

This completes the proof of Theorem 2.
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E-mail address: huabai@usc.edu

Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, U.S.A.