A generalisation of the deformation variety

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Abstract

Given an ideal tetrahedralisation of a connected 3-manifold with non-empty boundary consisting of a disjoint union of tori, a point of the deformation variety is an assignment of complex numbers to the dihedral angles of the tetrahedra subject to the gluing equations, from which one can recover a representation of the fundamental group of the manifold into the isometries of 3-dimensional hyperbolic space. However, the deformation variety depends crucially on the tetrahedralisation: there may be entire components of the representation variety which can be obtained from the deformation variety with one tetrahedralisation but not another. We introduce a generalisation of the deformation variety, which again consists of assignments of complex variables to certain dihedral angles subject to polynomial equations, but together with some extra combinatorial data concerning degenerate tetrahedra. This “extended deformation variety” deals with many situations that the deformation variety cannot. In particular, we show that given a manifold that admits an ideal tetrahedralisation, we can construct a tetrahedralisation so that we can recover any irreducible representation whose image is not a generalised dihedral group from the associated extended deformation variety.

This paper is organised as follows: in section 1 we recall the definition of the deformation variety and give an example showing how a bad choice of tetrahedralisation can cause problems. In section 2 we introduce Serre’s tree as a better context to work in and prove some lemmas about developing cross ratios in the tree. In section 3 we define the extended deformation variety and its developing map, and show that it is an affine algebraic variety. In section 4 show how to alter a tetrahedralisation so that it has properties needed for the extended deformation variety. In section 5 we define the map from the extended deformation variety to the representation variety and prove (in theorem 5.11) that under good conditions (i.e. a good tetrahedralisation, which we can ensure using the results of section 4), this map hits all of the representations that are irreducible and do not have as image in $\text{PSL}_2(\mathbb{C})$ a generalised dihedral group. In section 6 we return to the example of section 1 and show how the extended deformation variety deals with it, and in section 7 we ask some questions and suggest further directions to explore.

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1 The deformation variety

**Definition 1.1.** Let $M$ be a 3-manifold with non-empty boundary consisting of a disjoint union of tori, and with ideal tetrahedralisation $T$ consisting of $N$ tetrahedra. The **deformation variety** of $M$ with respect to the tetrahedralisation $T$, $\mathcal{D}(M) = \mathcal{D}(M; T)$ is the affine variety in $(\mathbb{C} \setminus \{0, 1\})^{3N}$, defined as the solutions of gluing equations and identities between complex dihedral angles within each tetrahedron, where each of the three complex dihedral angles in each tetrahedron corresponds to a dimension of the ambient space. Specifically, for the 6 dihedral angles within each tetrahedron, angles on opposite edges are the same, as shown in figure 1 (hence the fact that there are three complex variables for each of the $N$ tetrahedra), and $x_1, x_2, x_3$ are related to each other by:

\[
\begin{align*}
x_1x_2x_3 &= -1 \\
x_1x_2 - x_1 + 1 &= 0
\end{align*}
\]

For each edge of $T$, we also require that the product of the complex dihedral angles arranged around an edge of the tetrahedralisation equals 1 (these are the gluing equations).

This definition is essentially first seen in Thurston’s notes [3], chapter 4.

![Figure 1: The 6 dihedral angles in a tetrahedron.](image)

Let $\widetilde{M}$ be the universal cover of $M$ with induced tetrahedralisation $\widetilde{T}$. Let $\tilde{V}$ be the set of vertices of $\widetilde{T}$. Given a point $Z \in \mathcal{D}(M; T)$ the **developing map** $\Phi_Z : \tilde{V} \rightarrow \partial \mathbb{H}^3$ is defined up to conjugation (see Yoshida [6] and later Tillmann [4]: the map is usually defined as from $\widetilde{M}$ to $\mathbb{H}^3$, but our formulation encodes equivalent data and is more natural in the context of this paper). Let $\mathcal{R}(M)$ be the set of representations $\rho : \pi_1 M \rightarrow PSL_2(\mathbb{C}) \cong Isom(\mathbb{H}^3)$. The developing map gives a map $\mathcal{R}_T : \mathcal{D}(M; T) \rightarrow \mathcal{R}(M)$ (again up to conjugation) since seeing where a triple of distinct points on $\partial \mathbb{H}^3$ go under elements of $\pi_1 M$ gives us an element of $PSL_2(\mathbb{C})$.

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1. $\tilde{V}$ is also the set of cusps of $\widetilde{M}$, and so is independent of $T$. 

1.1 Dependence of the deformation variety on the tetrahedralisation

**Theorem 1.2** (Matveev, Theorem 1.2.5 of [1]). *Any two ideal tetrahedralisations of a given manifold $M$ are connected by a sequence of 2-3 and 3-2 moves (see figure 2).*

![2-3 and 3-2 moves](image_url)

Figure 2: 2-3 and 3-2 moves.

We might hope that the deformation varieties for tetrahedralisations $\mathcal{T}^2$ and $\mathcal{T}^3$ of a manifold that differ by a 2-3 move might be equivalent in some sense, and then given the above theorem and induction we would get equivalence for any tetrahedralisation. We would expect to be able to convert between points of $\mathcal{D}(M; \mathcal{T}^2)$ and $\mathcal{D}(M; \mathcal{T}^3)$ as follows:

![2-3 and 3-2 moves with truncated ends](image_url)

Figure 3: The 2-3 and 3-2 moves with truncated ends, showing the dihedral angles.

Figure 3 shows two tetrahedra labelled with complex dihedral angles $x$ and $y$ which share a face, and three tetrahedra labelled by $a$, $b$, and $c$ which all share an edge. We name the angles within one tetrahedron $x_1 = x, x_2 = \frac{x - 1}{x}, x_3 = \frac{1}{x}$, moving clockwise from $x$ on each truncated triangular end and similarly for the
other tetrahedra. The extra edge for the three tetrahedra gives us the equation \( a_1b_1c_1 = 1 \). If we are to have corresponding points of the two deformation varieties, then the dihedral angles outside of the six-sided shape in which the 2-3 move is performed must be the same. Because of the gluing equations, the dihedral angles inside must also be the same and we have the following relations:

\[
\begin{align*}
  a_1 &= x_1y_1 & x_1 &= b_3c_2 & y_1 &= b_2c_3 \\
  b_1 &= x_2y_3 & x_2 &= c_3a_2 & y_2 &= a_2b_3 \\
  c_1 &= x_3y_2 & x_3 &= a_3b_2 & y_3 &= c_2a_3
\end{align*}
\]

In good situations, these equations do allow us to produce a birational map between two deformation varieties of a manifold with tetrahedralisations that differ by a 2-3 move. However, this does not always work, in particular if we happen to have the relation in \( \mathcal{D}(M; T^2) \) that \( xy = 1 \), as we will see in the following example:

### 1.2 Example, part 1

We give an example of two tetrahedralisations of a manifold such that the deformation variety for one tetrahedralisation contains an entire component that does not appear in the deformation variety for the other tetrahedralisation. We consider the punctured torus bundle \( M_{LLR} \) with monodromy given by \( LLR \) (see for example, Guéritaud [5] for the notation), and show two tetrahedralisations of this punctured torus bundle, \( T_4 \) and \( T_5 \) (neither of which is canonical) in figure 4. Vertices of the triangulation of the torus boundary correspond to edges of the tetrahedralisation, and we can read off the gluing equation from the corners of triangles incident to vertices. We see both ends of each edge so each equation appears twice.

For \( \mathcal{D}(M_{LLR}; T_4) \) we obtain the following gluing equations:

\[
\begin{align*}
  hijk &= 1 & (3) \\
  \frac{i-1}{i} j \frac{1}{1-h} &= 1 & (4) \\
  \frac{i}{1-i} \frac{j-1}{j} \frac{1}{1-j} \frac{1}{1-h} \left( \frac{k-1}{k} \right)^2 &= 1 & (5) \\
  \frac{i-1}{i} \frac{1}{1-i} j \frac{1}{1-j} \frac{1}{1-h} \left( \frac{h-1}{h} \right)^2 k \left( \frac{1}{1-k} \right)^2 &= 1 & (6)
\end{align*}
\]

One gluing equation always depends on the others, so we discard the last of these and the others simplify to:

\[
\begin{align*}
  hijk &= 1 & (7) \\
  \frac{i-1}{i} j \frac{1}{1-h} &= 1 & (8) \\
  \frac{i}{1-i} \frac{-1}{j} \frac{1}{1-k} \left( \frac{k-1}{k} \right)^2 &= 1 & (9)
\end{align*}
\]
Figure 4: Two tetrahedralisations, $\mathcal{T}_4$ and $\mathcal{T}_5$, of the punctured torus bundle $M_{LLR}$. In the top left is a fundamental domain for the triangulation on the torus boundary induced by $\mathcal{T}_4$. There are four tetrahedra in this tetrahedralisation, with angles labelled $h, i, j, k$. Each tetrahedron has four ends and we see these four truncated ends as triangles on the torus boundary. To the right is shown the tetrahedra involved in the 2-3 move, and in the bottom left the resulting fundamental domain for $\mathcal{T}_5$, with angles labelled $i, j, p, q, r$. 
The variety consists of two 1-dimensional components, one of which contains the complete structure, and the other of which satisfies the extra condition that $hk = 1$. Then these equations become:

\[
ij = 1 \quad (10)
\]

\[
\frac{i - 1}{i - j} \frac{1}{1 - h} = 1 \quad (11)
\]

\[
\frac{i - 1}{1 - i} \frac{1}{1 - j} (1 - h) = 1 \quad (12)
\]

The latter two equations are redundant and we get a 1-dimensional variety. For $\mathcal{D}(M_{LLR}; \mathcal{T}_S)$ we obtain:

\[
pqr = 1 \quad (13)
\]

\[
ij q - 1 \frac{1}{1 - q} r - 1 \frac{1}{1 - r} = 1 \quad (14)
\]

\[
\frac{i - 1}{i - j} \frac{1}{1 - p} q = 1 \quad (15)
\]

\[
\frac{i - 1}{1 - i} \frac{j - 1}{1 - j} \frac{p - 1}{1 - p} \frac{1 - q}{1 - q} = 1 \quad (16)
\]

\[
\frac{i - 1}{1 - i} \frac{1}{1 - j} \frac{1}{1 - j} \frac{r - 1}{1 - r} = 1 \quad (17)
\]

Equations (14) through (17) correspond to (3) through (6), and we get the extra equation (13). Again discarding the last equation and simplifying we get:

\[
pqr = 1 \quad (18)
\]

\[
ij q - 1 \frac{1}{q} r - 1 \frac{1}{r} = 1 \quad (19)
\]

\[
\frac{i - 1}{i - j} \frac{1}{1 - p} q - 1 \frac{1}{p} = 1 \quad (20)
\]

\[
\frac{i - 1}{1 - i} \frac{j - 1}{1 - j} \frac{p - 1}{1 - p} \frac{1 - q}{1 - q} r - 1 \frac{1}{r} = 1 \quad (21)
\]

This time, if $ij = 1$ then equations (18) and (19) imply that $p = 1$, and there is no solution.

Remark 1.3. By a result of Tillmann [4], if $M$ admits a complete hyperbolic structure of finite volume and a deformation variety for $M$ with some tetrahedralisation is non-empty, then it has a component which corresponds to the Dehn surgery component of the character variety. The above example shows that it may or may not “see” other components.

Remark 1.4. If a particular component of the representation variety has an extra relation on it (in addition to those defining the variety) which implies that certain cusps always appear in the same place under $\Psi_\rho$ (see definition
as we vary $\rho$ within the component, then a tetrahedralisation with an edge between two such cusps will have a deformation variety that will not see this component.

2 Serre’s tree

To solve these problems, we move to a more general setting, replacing dihedral angles in $\mathbb{C}\setminus\{0, 1\}$ with dihedral angles in $\mathbb{C}((\zeta))\setminus\{0, 1\}$, formal Laurent series in a variable $\zeta$, which allow tetrahedra to be degenerate whilst retaining some extra information. The following definition is the same as definition 1.3 in Ohtsuki [2].

Definition 2.1. For an indeterminate variable $\zeta$ we define the tree $T_\zeta$ with the following four conditions:

1. $T_\zeta$ has a special vertex $H$; we call it the home vertex, or just home.

2. We call an infinitely long path from home an end. The set of ends of $T_\zeta$ is identified with the set $\mathbb{C}((\zeta)) \cup \{\infty\}$, to which we give the discrete topology.

3. The part of $T_\zeta$ from $H$ to the set of ends $\mathbb{C}[[\zeta]]$ (power series in $\zeta$) is identified with the series

$$\{H\} \leftarrow \mathbb{C}[\zeta]/(\zeta) \leftarrow \mathbb{C}[\zeta]/(\zeta^2) \leftarrow \mathbb{C}[\zeta]/(\zeta^3) \leftarrow \cdots \leftarrow \mathbb{C}[[\zeta]]$$

where the maps are the natural projections. Here we identify the set of vertices at distance $n$ from $H$ with the set $\mathbb{C}[\zeta]/(\zeta^n)$, the ring of polynomials of degree at most $n-1$. Two vertices are connected by an edge if one of the maps takes one vertex to the other.

4. The part of $T_\zeta$ from $H$ to the set of ends $\mathbb{C}((\zeta)) \cup \{\infty\} \setminus \mathbb{C}[[\zeta]]$ is homeomorphic to $\{\zeta\} \subset \mathbb{C}[[\zeta]]$ by the map $\mathbb{C}((\zeta)) \cup \{\infty\} \rightarrow \mathbb{C}((\zeta)) \cup \{\infty\}$, which takes $x$ to $x^{-1}$.

Definition 2.2. For $x \in \mathbb{C}((\zeta)) \setminus \{0\}$ with coefficients $x_k$, we define

$$\text{order}(x) = \min\{k \in \mathbb{Z} | x_k \neq 0\}$$

$$x_* = x_{\text{order}(x)} \in \mathbb{C} \setminus \{0\}$$

We set $\text{order}(0) = \infty$ and $\text{order}(\infty) = -\infty$.

If $n = \text{order}(x)$ then $x = x_n \zeta^n + x_{n+1} \zeta^{n+1} + \cdots$

Given four distinct ends $a, b, c$ and $d$ of $T_\zeta$ we define the cross ratio

$$z = \frac{(a - c)(b - d)}{(a - d)(b - c)} \in \mathbb{C}((\zeta)) \setminus \{0, 1\} \quad (22)$$
Just as in $\mathbb{C} \cup \{\infty\}$, if $a = \infty$, for example, we interpret this as $z = \frac{b-d}{b-c}$, using the rules $\infty + x = \infty$ for $x \in \mathbb{C}((\zeta))$ and $\infty/\infty = 1$. There are six possible cross ratios to take, but only three if we preserve orientation. They are related to each other as $z, \frac{z-1}{z}, \frac{1}{1-z}$. At least one of these three is a preferred cross ratio $z \in \mathbb{C}[[\zeta]]$ such that $z_0 \neq 1$ (if $z \in \mathbb{C}((\zeta)) \setminus \mathbb{C}[[\zeta]]$ then $\frac{1}{1-z} \in \mathbb{C}[[\zeta]]$ and $(\frac{1}{1-z})_0 = 0$, and if $z_0 = 1$ then $(\frac{z-1}{z})_0 = 0$). If $z \in \mathbb{C}[[\zeta]], z_0 \notin \{0, 1\}$ then all of the three cross ratios are preferred. Note that the order of the preferred cross ratio determines the length of the spine determined by the four ends, as in figure 6.

Figure 5: Part of the Serre tree, with the home vertex $H$ marked.

Figure 6: On the left, a spine in $T_\zeta$ determined by four ends $a, b, c, d$. The order of the preferred cross ratio is the same as the length of the spine (given by the number of edges in the midsection), which is 2 in this example. On the right, a spine isometric to the first viewed as part of the dual tree to a normal surface within a tetrahedron, this time $a, b, c, d$ are vertices of the tetrahedron.

Given three distinct ends $a, b, c \in \mathbb{C}((\zeta)) \cup \{\infty\}$ and cross ratio $z$, then solving
for the fourth gives

\[ d = \frac{(b - c)z - (a - c)b}{(b - c)z - (a - c)} \]  

(23)

We deal with \( \infty \) in this equation in the same way we did for the cross ratio.

Given a pair of triangles which share an edge, we may assign a **dihedral angle** between them, which is a cross ratio in \( \mathbb{C}(\zeta) \setminus \{0,1\} \). This tells us, given specified ends \( e, e', e'' \in \mathbb{C}(\zeta) \cup \{\infty\} \) for the locations of the vertices of one triangle, the location of the fourth vertex using equation (23). See figure 7. We call this process of determining the position of a vertex from a dihedral angle and the positions of three vertices **developing**. We specify a choice of cross ratio for the dihedral angle from the three possibilities so that the labelling on the left hand diagram of figure 7 matches equation (22). The right hand diagram of figure 7 and the independence of the cross ratio under swapping \( a \) with \( b \) and \( c \) with \( d \) shows us that it doesn’t matter which way up the picture is when we choose which cross ratio is the dihedral angle.

**Definition 2.3.** A chain of triangles in \( \mathcal{T} \) (or \( \tilde{\mathcal{T}} \)) is a sequence of triangular faces \( \triangle^{(1)}, \triangle^{(2)}, \ldots, \triangle^{(n)} \in \mathcal{T} \) (or \( \tilde{\mathcal{T}} \)) such that neighbouring triangles are two faces of a tetrahedron of \( \mathcal{T} \) (or \( \tilde{\mathcal{T}} \)) (and therefore share an edge).

We can similarly develop positions of vertices for chains of triangles with dihedral angles between neighbours.

![Figure 7: Developing across a dihedral angle between two neighbouring triangles.](image)

**Remark 2.4.** For any field \( \mathbb{F} \), \( \text{PSL}_2(\mathbb{F}) \) acts freely and transitively on \( \mathbb{P}^{1} \) and preserves cross ratios, and so this is true for the field \( \mathbb{C}(\zeta) \). Therefore choosing different initial points \( e, e', e'' \) for the first triangle moves the set of developed
vertex positions consistently. In particular, for any construction in this paper there will be only countably many developed vertex positions but a CP^1's worth of vertices next to H so we may choose e, e', e'' so that all vertex positions are in \( \mathbb{C}[[\zeta]] \), avoiding the one edge leading away from H in the direction of \( \infty \).

**Lemma 2.5.** If \( z \) is the preferred cross ratio of \( a, b, c, d \in \mathbb{C}[[\zeta]] \) then

\[
z_\star = \frac{(a - c)_\star(b - d)_\star}{(a - d)_\star(b - c)_\star}
\]

**Proof.** Let \( (a - c) = p, (b - d) = q, (a - d) = r, (b - c) = s \), all in \( \mathbb{C}[[\zeta]] \). Then

\[
z = \frac{pq}{rs} = \zeta^k \frac{(p_\star + \zeta p')(q_\star + \zeta q')}{(r_\star + \zeta r')(s_\star + \zeta s')} \quad p', q', r', s' \in \mathbb{C}[[\zeta]], k \in \mathbb{N}
\]

where \( k \geq 0 \) since this is the preferred cross ratio. Then

\[
z = \zeta^k \frac{p_\star q_\star + \zeta e}{r_\star s_\star + \zeta f} = \left( \frac{\zeta^k}{r_\star s_\star} \right) \frac{p_\star q_\star + \zeta e}{1 - \zeta\frac{f}{r_\star s_\star}} \quad e, f \in \mathbb{C}[[\zeta]]
\]

so

\[
z = \left( \frac{\zeta^k}{r_\star s_\star} \right) (p_\star q_\star + \zeta e)(1 + \zeta g) = \zeta^k \left( \frac{p_\star q_\star}{r_\star s_\star} + \zeta h \right) \quad g, h \in \mathbb{C}[[\zeta]]
\]

hence

\[
z_\star = \frac{p_\star q_\star}{r_\star s_\star}
\]

\[\square\]

This provides motivation that under good conditions we should be able to ignore all higher order information about cross ratios and developed positions if we only care about lowest order information about those objects. This is the subject of the next few lemmas.

**Lemma 2.6.** Suppose that \( a, b, c \in \mathbb{C}[[\zeta]] \), that at most one of order \((a - b), (a - c), (b - c)\) is greater than 0, \( z \) is the preferred cross ratio for the four ends \( a, b, c, d \) and that \( d \in \mathbb{C}[[\zeta]] \). Then \((d - a)_\star, (d - b)_\star, (d - c)_\star, z_\star\) are determined by \((a - b)_\star, (a - c)_\star, (b - c)_\star, z_\star\) and the orders of those terms.

**Proof.** Note the following relations:

\[
(d - a) = \frac{1}{(b - c)z - (a - c)}(a - b)(a - c) \quad (24)
\]

\[
(d - b) = \frac{1}{(b - c)z - (a - c)}(b - c)(a - b)z \quad (25)
\]

\[
(d - c) = \frac{1}{(b - c)z - (a - c)}(a - c)(b - c)(z - 1) \quad (26)
\]
Let \( q = (b - c)z - (a - c) \). If \( \text{order}(q) = n \geq 0 \) then \( q_\ast = q_n \neq 0 \) and

\[
\frac{1}{q} = \frac{1}{\zeta^n q_n + \zeta^{n+1} q'} = \frac{1}{\zeta^n q_n} \frac{1}{1 + \zeta(q/q_n)} = \frac{1}{\zeta^n q_n} \sum_{m=0}^{\infty} \left( -\frac{q'}{q_n} \right)^m \zeta^m
\]

where \( q' \in \mathbb{C}[[\zeta]] \). Thus \( \left( \frac{1}{q} \right)_\ast = \frac{1}{q_n} = \frac{1}{q_\ast} \).

First, \( z \) is the preferred cross ratio, so \( \text{order}(z) \geq 0 \). \( \text{order}(q) \geq \min\{\text{order}(b-c) + \text{order}(z), \text{order}(a-c)\} \), with equality unless \( \text{order}(b-c) + \text{order}(z) = \text{order}(a-c) \) and \( (b-c)_\ast z_\ast - (a-c)_\ast = 0 \). We rule this out by considering the possible cases for this given that at most one of \( \text{order}(a-b), \text{order}(a-c), \text{order}(b-c) \) is greater than 0:

1. \( \text{order}(b-c) > 0 \) and so \( \text{order}(a-c) = 0 \). If this is true, then since \( \text{order}(z) \geq 0 \), then \((b-c)z - (a-c))_0 = -(a-c)_0 \neq 0 \), so this case is impossible.

2. \( \text{order}(a-b) > 0 \) and so \( \text{order}(b-c) = \text{order}(a-c) = 0 \). \( \text{order}(b-c) + \text{order}(z) = \text{order}(a-c) \) so \( \text{order}(z) = 0 \). Then we have that \( (b-c)_0 z_0 - (a-c)_0 = 0 \), so \( (b_0 - c_0) z_0 - (a_0 - c_0) = 0 \). Since \( \text{order}(a-b) > 0 \), \( a_0 = b_0 \) and therefore \( z_0 = 1 \), which is ruled out by the choice of preferred cross ratio.

3. \( \text{order}(a-c) = n \geq 0 \), \( \text{order}(a-b) = \text{order}(b-c) = 0 \). \( \text{order}(b-c) + \text{order}(z) = \text{order}(a-c) \) so \( \text{order}(z) = n \). Then the numerator of \( d \) in equation (23) is

\[
(b-c)za - (a-c) = (b-c)_0 z_n \zeta^n a - (a-c)_n \zeta^n b + \text{h.o.t.}
\]

\[
= (a-c)_n \zeta^n (a-b) + \text{h.o.t.}
\]

\[
= (a-c)_n (a-b)_0 \zeta^n + \text{h.o.t.}
\]

Here h.o.t. stands for “higher order terms”. Since \( (a-c)_n (a-b)_0 \neq 0 \), the lowest power of \( \zeta \) in the numerator is less than that in the denominator, which is strictly greater than \( n \) because of the cancellation, and so \( d \notin \mathbb{C}[[\zeta]] \), contradicting the hypothesis.

So we have equality: \( \text{order}(q) = \min\{\text{order}(b-c) + \text{order}(z), \text{order}(a-c)\} \), and \( q_\ast \) is one of \((b-c)_\ast z_\ast, (a-c)_\ast \) or \((b-c)_\ast z_\ast - (a-c)_\ast \), none of which is zero. We then have that

\[
(d - a)_\ast = \left( \frac{1}{q_\ast} \right) (a-b)_\ast (a-c)_\ast \tag{27}
\]

\[
(d - b)_\ast = \left( \frac{1}{q_\ast} \right) (b-c)_\ast (a-b)_\ast z_\ast \tag{28}
\]

\[
(d - c)_\ast = \left( \frac{1}{q_\ast} \right) (a-c)_\ast (b-c)_\ast (z - 1)_\ast \tag{29}
\]
For \((d - c)_s\), since \(z_0 \neq 1\), \((z - 1)_s = (z - 1)_0 = z_0 - 1\), which is either \(z_s - 1\) or just \(-1\) depending on order\((z)\). In all cases therefore, \((d - a)_s\), \((d - b)_s\) and \((d - c)_s\) and their orders are determined by \((a - b)_s\), \((a - c)_s\), \((b - c)_s\), \(z_s\) and their orders.

Similar analysis shows that case 3 in the lemma is the only case in which \(d\) can fail to be in \(C[\zeta]\).

**Definition 2.7.** For each triple \(\{a, b, c\}\) of distinct ends of \(T_\zeta\) we define the **tripod** of \(\{a, b, c\}\) to be the smallest connected subtree of \(T_\zeta\) that contains an infinite tail of each end (recall that an end is an infinitely long path from \(H\)). The unique trivalent vertex of the tripod is the **center** of the tripod.

**Definition 2.8.** A triple \(\{a, b, c\}\) of distinct ends of \(T_\zeta\) is **domestic** if the tripod of \(\{a, b, c\}\) contains \(H\) and **foreign** otherwise.

We can thus rephrase the condition in lemma 2.6 on the orders of \((a - b)\), \((a - c)\) and \((b - c)\) as the triple \(\{a, b, c\}\) being domestic.

**Lemma 2.9.** Let \(\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(n)}\) be a chain of triangles with specified dihedral angles \(w^{(i)}\) between \(\Delta^{(i)}\) and \(\Delta^{(i+1)}\). Suppose initial points \(e, e', e''\) for the locations of the 3 vertices of \(\Delta^{(1)}\) are chosen so that order\((e - e') = order(e' - e'') = order(e'' - e) = 0\), all developed vertices are in \(C[\zeta]\) and suppose that each triple of developed vertices given by those triangular faces is domestic. Then for every developed vertex \(f \in C[\zeta]\) corresponding to a vertex of one of the \(\Delta^{(i)}\), \(f_0\) depends only on \(e_0, e'_0, e''_0\) and the \(z^{(i)}_s\), order\((z^{(i)})\) where \(z^{(i)}\) is the preferred cross ratio corresponding to the dihedral angle \(w^{(i)}\).

**Proof.** This is essentially an induction using lemma 2.6. Each triangular face with vertex positions \(\{a, b, c\} \subset C[\zeta]\) we develop from is domestic. This implies that at most one of the orders order\((a - b)\), order\((a - c)\), order\((b - c)\) is greater than 0. If \(a\) and \(b\) are positions for vertices at opposite ends of an edge of a \(\Delta^{(i)}\), then by induction, \(\{a - b\}\) and order\((a - b)\) are determined by \((e - e')_s\), \((e' - e'')_s\), and the \(z^{(i)}_s\) and their orders. All of those orders are 0 by our choice of \(e, e', e''\), and so \((e - e')_s = (e - e')_0 = e_0 - e'_0\) and similarly for the others, so in fact \((a - b)_s\) and order\((a - b)\) are determined by \(e_0, e'_0, e''_0, z^{(i)}_s\) and order\((z^{(i)})\).

Note that \((a - b)_0\) is either 0 or \((a - b)_s\), depending on if order\((a - b) > 0\) or not, so \((a - b)_0 = a_0 - b_0\) is also determined by the same data. Now consider a path of edges walking along a sequence of vertex positions \(f^{(1)}, f^{(2)}, \ldots, f^{(m)}\) where \(f^{(1)} \in \{e, e', e''\}\) and \(f^{(m)} = f\). Then for each neighbouring pair \((f^{(j)}, f^{(j+1)})\), \(f^{(j)} - f^{(j+1)}\) is determined by the data, we start from one of \(e_0, e'_0, e''_0\) and so again by induction \(f^{(m)}_0 = f_0\) is also determined by the data.

Now suppose we have an ideal tetrahedralisation \(T\) of an orientable 3-manifold with boundary \(M\) and an embedded surface \(S \subset M\) in (spin-)normal form relative to \(T\). We lift \(T\) to \(\tilde{T}\), a tetrahedralisation of \(\tilde{M}\), the universal cover of \(M\)
and lift $S$ to $\tilde{S} \subset \tilde{M}$ a surface in (spun-)normal form relative to $\tilde{T}$. We allow the possibility that $S$ is boundary parallel. Dual to $\tilde{S}$ we have a tree denoted $T_{\tilde{S}}$, which we can view as made by gluing together spines, with the tree dual to the intersection of $\tilde{S}$ with a tetrahedron of $\tilde{T}$ being such a spine, as in figure 6. Vertices of $T_{\tilde{S}}$ correspond to connected components of $\tilde{M} \setminus \tilde{S}$.

We will assign dihedral angles (elements of $\mathbb{C}(\{\zeta\}) \setminus \{0, 1, \infty\}$) to the six edges between pairs of triangles in each tetrahedron of $\tilde{T}$. Given this information and any chain of triangles in $\tilde{T}$ together with the positions of the vertices of the first triangle we can develop the positions of all vertices along the chain.

**Definition 2.10.** For $f \in \mathbb{C}(\{\zeta\}) \cup \{\infty\}$, the direction of $f$ from $H$ is

\[
    f_H := \begin{cases} 
        f_0 & \text{if } f \in \mathbb{C}[\zeta] \\
        \infty & \text{if } f \in (\mathbb{C}(\{\zeta\}) \cup \{\infty\}) \setminus \mathbb{C}[\zeta]
    \end{cases}
\]

**Condition 2.11.** The degeneration order condition on dihedral angles assigned to a tetrahedron $t$ states that if $t$ contains no quadrilateral of $S$ then all dihedral angles $w$ must have $w_H \neq 0, 1, \infty$ (and hence order($w$) = 0), and if $t$ has $k$ quadrilaterals then $w_H$ is as in figure 8 and the order of the preferred cross ratio corresponding to $w$ is $k$.

This is consistent with the connection between ideal points of the deformation variety and spun-normal surfaces as in Tillmann [4]. This also means that for a tetrahedron $t \in \tilde{T}$, the spine dual to $S \cap t$ is the same shape as the corresponding spine in $T_\zeta$ we get when developing through any pair of triangles of $t$. Spines corresponding to neighbouring tetrahedra glue to each other in the same way in both contexts.

![Figure 8: The $w_H$ for dihedral angles $w$ in a tetrahedron with a quadrilateral.](image)

**Lemma 2.12.** Suppose we assign dihedral angles to each tetrahedron of $\tilde{T}$ that satisfy the degeneration order condition with respect to the surface $S$. Let $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(n)}$ be a chain of triangles in $\tilde{T}$ and $R_0$ be the component of $\tilde{M} \setminus \tilde{S}$ that contains the central region of $\Delta^{(1)}$. Suppose we can trace a path following $R_0$ continuously through the $\Delta^{(i)}$ and we develop positions of the vertices starting from $e, e', e''$ for the locations of the 3 vertices of $\Delta^{(1)}$ as in lemma...
and all developed positions are in $\mathbb{C}[[\zeta]]$. Then the positions for the triple of vertices for each $\triangle^{(i)}$ is domestic.

Proof. The triple of images of the vertices of $\triangle^{(1)}$ is domestic by assumption. When we develop to $\triangle^{(2)}$, the image of the new vertex ($f$ say) is arranged with respect to $e, e', e''$ in $\mathbb{C}[[\zeta]]$ in the same arrangement as the corresponding vertices in $T^{\tilde{S}}$, by the degeneration order condition (2.11). The tripod in $T^{\tilde{S}}$ corresponding to $\triangle^{(1)}$ has center the vertex dual to $R_0$ and the tripod corresponding to $\triangle^{(2)}$ together with the tripod corresponding to $\triangle^{(1)}$ form a spine as in the right diagram of figure 6, as a subtree of $T^{\tilde{S}}$. See also figure 9.

As we continue to develop to further triangles, the condition that $R_0$ has non-empty intersection with each $\triangle^{(i)}$ corresponds to the condition that each new tripod in $T^{\tilde{S}}$ contains the central vertex of the tripod for $\triangle^{(1)}$. The tripods we add in $\mathbb{C}[[\zeta]]$ satisfy the corresponding condition, which is that they contain the home vertex, and so are all domestic. \hfill $\square$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{Two neighbouring triangles and their intersections with $\tilde{S}$. The corresponding spine in $T^{\tilde{S}}$. The central region of the first triangle is marked $R_0$ and the corresponding vertex of the spine is marked $V_0$.}
\end{figure}

**Theorem 2.13.** Suppose we assign dihedral angles $w$ to each tetrahedron of $\mathcal{T}$ that satisfy the degeneration order condition with respect to the surface $S$. Let $\triangle^{(1)}, \triangle^{(2)}, \ldots, \triangle^{(n)}$ be a chain of triangles in $\mathcal{T}$ (so neighbouring triangles are both part of a single tetrahedron) and $R_0$ be the component of $M \setminus \tilde{S}$ that contains the central region of $\triangle^{(1)}$. Suppose we can trace a path following $R_0$ continuously through the $\triangle^{(i)}$ and we develop positions of the vertices starting from $e, e', e''$ for the locations of the three vertices of $\triangle^{(1)}$ where the tripod of $\{e, e', e''\}$ has center $H$. Then for every developed position $f$ corresponding to a vertex of one of the $\triangle^{(i)}$, $f_H$ depends only on $e_H, e'_H, e''_H$ and the $z, \text{order}(z)$ where $z$ is the preferred cross ratio for the dihedral angle $w$.

Proof. This is a combination of lemmas 2.9, 2.12 and the observation that we can drop the condition that we develop only into $\mathbb{C}[[\zeta]]$ by noting that the result is true when we do restrict to $\mathbb{C}[[\zeta]]$, and we can always conjugate the developed vertex positions so that they are all in $\mathbb{C}[[\zeta]]$ without altering the cross ratios, and hence the developing map. \hfill $\square$
We will use the same trick again later, assuming without loss of generality that we develop only into $C[[\zeta]]$ but then dropping this requirement for the result.

Remark 2.14. If we identify $C[\zeta]/(\zeta) \cup \{\infty\}$ with $\partial H^3$, then this gives us a way to define a developing map from chains of triangles that contiguously contain $R_0$ into $H^3$. Some of the triangles may be degenerate, having two vertices at the same location on $\partial H^3$, although due to the construction we never have any triangles with all three vertices at the same location (this would correspond to a foreign triple of vertices).

We have yet to put any conditions on the assigned dihedral angles other than the degeneration order condition. In the construction following we will add some more conditions, in particular appropriate versions of the relations from equations (1) and (2), although in some cases only for certain tetrahedra. We will also require a condition playing the role of the gluing equations.

3 The extended deformation variety

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be the edge sets of $\mathcal{T}$ and $\tilde{\mathcal{T}}$. We consider disjoint subsets $\mathcal{E}_0, \mathcal{E}_+ \subset \mathcal{E}$ whose union is $\mathcal{E}$, and define $\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_+$ as their lifts to $\tilde{\mathcal{T}}$. The idea is that $\tilde{\mathcal{E}}_0$ will be the set of developed edges of zero length in $H^3$, and $\tilde{\mathcal{E}}_+$ the set of developed edges of positive (i.e. non-zero) length.

Definition 3.1. Given a choice of edges $\mathcal{E}_0 \subset \mathcal{E}$ we say that a loop of edges in $\mathcal{E}$ is a bad loop if exactly one edge is in $\mathcal{E}_+$ and all others are in $\mathcal{E}_0$. A choice of edges $\mathcal{E}_0 \subset \mathcal{E}$ has no bad loops if there is no such loop.

We will only allow choices of $\mathcal{E}_0$ with no bad loops. In particular, if two of the three edges of a face of $\mathcal{T}$ are in $\mathcal{E}_0$ then the third must also be to avoid a bad loop:

Definition 3.2. We refer to the three possibilities for the arrangement of edges of $\mathcal{E}_0$ around a triangle of $\mathcal{T}$ as types 111 (no edges in $\mathcal{E}_0$), 21 (one edge in $\mathcal{E}_0$), and 3 (all edges in $\mathcal{E}_0$). We make the same definition for triangles of $\mathcal{\tilde{T}}$.

Definition 3.3. There are five possibilities for the arrangement of edges in $\mathcal{E}_0$ around each tetrahedron of $\mathcal{T}$, as in figure 10. We refer to these five possibilities as types 1111, 211, 22, 31 and 4 respectively. Again we make the same definition for tetrahedra of $\mathcal{\tilde{T}}$.

The naming convention in these two definitions is derived from the size and number of equivalence classes of vertices, where two vertices are equivalent if they are connected by an edge of $\mathcal{E}_0$. See also figure 11.

Definition 3.4. Given a choice of edges $\mathcal{E}_0 \subset \mathcal{E}$ with no bad loops, we construct a corresponding normal surface as follows: each tetrahedron $t \in \mathcal{T}$ has one of the five arrangements of edges in $\mathcal{E}_0$ as in figure 10. We also place normal quadrilaterals and triangles in these tetrahedra as in the figure. Each face of $\mathcal{T}$ is of type
Figure 10: The 5 possible arrangements of edges in $\mathcal{E}_0$ (the dotted tetrahedron edges) around a tetrahedron with corresponding parts of normal surfaces. The barycenters of faces of the tetrahedra are shown with large dots, and a valid chain of triangles corresponds to a path of these dots along the graph formed by the dashed lines.

Figure 11: From left to right, tetrahedra in $\widetilde{T}$ of type 1111, 211, 31, 22. The dual tree.
111, 21 or 3. These faces have either three normal arcs cutting off each vertex, two normal arcs parallel to the edge in in $E_0$, or no normal arcs, respectively. Therefore these normal quadrilaterals and triangles match up across the faces of $T$ to form a normal surface.

If $S$ is a normal surface corresponding to a choice of edges with no bad loops, then we say $S$ is in horo-normal form relative to $T$, and that $S$ is a horo-normal surface.

Note that we can construct $E_0$ from a horo-normal surface by reading off from figure [10] and so horo-normal surfaces and choices of $E_0$ with no bad loops are in one to one correspondence.

**Lemma 3.5.** A horo-normal surface $S$ is orientable, closed, and cuts $M$ into a compact inside region $R_{in}$ and an outside region $R_{out}$ which contains $\partial M$.

**Proof.** The quadrilateral and triangle parts of $S$ are prescribed and finite in each of the finitely many tetrahedra of $T$, and so $S$ is made from finitely many parts and is therefore finite and closed. $S$ is orientable because there is a consistent choice of the side of the parts of $S$ which face the vertices of the tetrahedra. It follows that $S$ cuts $M$ into the two regions (not necessarily connected). $\square$

**Definition 3.6.** A chain of triangles $(\triangle = \triangle^{(0)}, \triangle^{(1)}, \ldots, \triangle^{(n)})$ in $T$ (or $\tilde{T}$) is valid if $\tilde{R}_m$ (the lift of $R_m$, as in lemma 3.5) intersects $\triangle$ as its central region (in other words all edges of $\triangle$ are in $\tilde{E}_+$), and the chain of triangles contiguously intersects $\tilde{R}_m$ (in other words, the edge between neighbouring triangles is in $\tilde{E}_+$).

**Definition 3.7.** The data for a point of the extended deformation variety of $M$ with tetrahedralisation $T$ and horo-normal surface $S$ consists of:

- A cross ratio $z \in \mathbb{C} \setminus \{0, 1\}$ for each tetrahedron of type 1111
- A cross ratio $z\zeta \in \mathbb{C}[\zeta], z \in \mathbb{C} \setminus \{0\}$ for each tetrahedron of type 211
- A cross ratio $z\zeta^2 \in \mathbb{C}[\zeta], z \in \mathbb{C} \setminus \{0\}$ for each tetrahedron of type 22
- Two complex angles $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ for each tetrahedron of type 31

This determines dihedral angles for each edge of a tetrahedron that is in $\tilde{E}_+$, as in figure [12] (only lowest orders are shown; the dihedral angles for type 211 for example are $z\zeta, \frac{z\zeta - 1}{z\zeta}$, and $\frac{1}{1 - z\zeta}$, so the preferred cross ratio is always $z\zeta$).

We record no data for tetrahedra of type 4. What this data amounts to is the following:

1. Choose dihedral angles from $\mathbb{C}((\zeta)) \setminus \{0, 1\}$ for each tetrahedron of types 1111, 211, and 22 for each of the six edges, subject to:

\[\text{The surface is similar to a horosphere, cutting the vertices that are all in the same place on } \partial \mathbb{H}^3 \text{ away from all of the other vertices in other locations.}\]
(a) The degeneration order condition
(b) Opposite dihedral angles being equal
(c) Equations (1) and (2) holding as equations in $C((\zeta))$

2. Choose dihedral angles from $C((\zeta)) \setminus \{0, 1\}$ for each tetrahedron of type 31 for the 3 edges not in $E_0$ subject to:
(a) The degeneration order condition
(b) Equation (1) holding as an equation in $C((\zeta))$

Figure 12: Lowest order coefficients for dihedral angles associated to edges of tetrahedra in $\tilde{E}_+$ as in definition 3.7. When we develop we only ever use the preferred cross ratios.

For types 1111, 211, and 22 the two equations and opposite dihedral angles being equal reduce the choice for each tetrahedron to a single cross ratio, and the degeneration order condition implies the power on the cross ratios as in definition 3.7. In light of theorem 2.13 we only need to record the lowest order information for the preferred cross ratio in order to be able to develop positions of vertices along valid chains of triangles. For type 31 we similarly only record data for dihedral angles that a valid chain of triangles in $\tilde{T}$ could turn through. We likewise never need to record data for type 4, since we will never develop through any of these dihedral angles.

The data of definition 3.7 then allows us develop through any valid chain of triangles, starting from three ends of $T_0$ whose tripod center is $H$. We require one further condition on this data:

**Condition 3.8.** The consistent development condition on the data for a point of the extended deformation variety states that given any two valid chains of triangles that start from $\Delta \in \tilde{T}$, if a vertex $v \in \tilde{V}$ is a vertex of triangles in both chains then the developed positions $f$ and $f'$ of $v$ under the two chains satisfy $f_H = f'_H$.

**Remark 3.9.** Conditions 1(b), 1(c) and 2(b) in definition 3.7 ensure that valid chains contained within a single tetrahedron satisfy the consistent development condition.
Example 3.10. If \( E_0 = \phi \) then \( E_+ = \mathcal{E} \), the horo-normal surface \( S \) is boundary parallel and \( \widetilde{R}_m \) is isotopic to \( \widetilde{M} \), all tetrahedra are of type 1111, the consistent development condition is equivalent to the gluing equations holding and we get the standard deformation variety for \( M \) with tetrahedralisation \( \mathcal{T} \).

Definition 3.11. Let \( M \) be a 3-manifold with boundary a disjoint union of tori and with ideal tetrahedralisation \( \mathcal{T} \) and \( S \) a surface in horo-normal form relative to \( \mathcal{T} \) such that:

1. \( \widetilde{R}_m \) is connected.
2. For each component of \( \partial M \) there exists an \( e \in \mathcal{E}_+ \) that has at least one endpoint on that component.
3. There exists a triangle \( \triangle \in \mathcal{T} \) of type 111 (i.e. all three edges are in \( \mathcal{E}_+ \), or equivalently \( \widetilde{R}_m \) intersects \( \triangle \) as its central region).

Then the extended deformation variety of \( M \) with tetrahedralisation \( \mathcal{T} \) and horo-normal surface \( S \), \( \hat{D}(M; \mathcal{T}; S) \) consists of points of the extended deformation variety (as in definition 3.7) subject to the consistent development condition. We will also consider the disjoint union of all such varieties with a fixed tetrahedralisation but ranging over all horo-normal surfaces satisfying these conditions. We call this set the extended deformation variety of \( M \) with tetrahedralisation \( \mathcal{T} \), and write it as \( \hat{D}(M; \mathcal{T}) \).

Note that the second condition is automatic for manifolds with only one boundary component and the third holds as long as there are any tetrahedra of type 1111 or 211.

Definition 3.12. A surface in horo-normal form relative to \( \mathcal{T} \) that satisfies conditions 1 and 2 of definition 3.11 is called omnipresent.

Definition 3.13. Given an element \( Z \in \hat{D}(M; \mathcal{T}; S) \) we define the developing map

\[ \Phi_Z : \mathcal{V} \to \partial \mathbb{H}^3 \]

up to conjugation, determined by the distinct images (positions on \( \partial \mathbb{H}^3 \)) we choose for the vertices of \( \Delta \). (In fact we choose ends of \( T_\zeta \) for those positions, but by theorem 2.13 only the lowest order information of these matters.) To determine the image of any other vertex \( v \) of \( \mathcal{V} \), take any valid chain of triangles from \( \Delta \) to a triangle containing \( v \), and develop along that chain. Such a chain exists by the omnipresence conditions: each \( v \) corresponds to a lift of some component of \( \partial M \), let \( e \in \mathcal{E}_+ \) be as in condition 2 of omnipresence, then some lift \( \widetilde{e} \) of \( e \) has \( v \) as an endpoint and intersects \( \widetilde{R}_m \). Traverse triangles with non-empty intersection with \( \widetilde{R}_m \) from \( \Delta \) to \( \widetilde{e} \) (which we can always do by condition 1) and we get a position for \( v \) on \( \partial \mathbb{H}^3 \) by forgetting all but the lowest order information, which is independent of the chain we took by the consistent development condition, 3.8.
Lemma 3.14. Suppose $\bar{e} \in \bar{E}$ has endpoints $v, v'$ and we can develop through a valid chain of triangles to a triangle containing $\bar{e}$. Then $\bar{e} \in \bar{E}_0$ if and only if $\Phi_Z(v) = \Phi_Z(v')$.

Proof. We prove a slightly stronger result: that if $a, a' \in C[\zeta]$ are the positions of $v, v'$, developed along some valid chain of triangles then

$$\text{order}(a - a') = \begin{cases} 1 & \text{if } e \in \bar{E}_0 \\ 0 & \text{if } e \in \bar{E}_+ \end{cases}$$

We prove this by induction along chains of triangles starting with $\triangle$. By definition, no edges of $\triangle$ are in $\bar{E}_0$ so we start out developing from three distinct points of $\partial \mathbb{H}^3$, by which we mean three ends $e, e', e'' \in T_\zeta$ such that $e_H, e_H', e_H''$ are all different. This shows that the base case holds. The inductive step follows from the degeneration order condition: for every new triangle we develop to, the order of the difference between the developed positions of the endpoints is determined by the orders for the edges in the triangle we develop from and the dihedral angle we develop through. The degeneration order condition ensures that we get the correct order for the new edges.

Lemma 3.15. If $v, v' \in \bar{V}$ are in the same component $(\bar{R}_\text{out})_0$ of $\bar{R}_\text{out}$, the lift of $R_\text{out}$ to $\bar{M}$, then $\Phi_Z(v) = \Phi_Z(v')$.

Proof. We first develop from $\triangle$ out to $v$, and then note that one valid chain of triangles from $\triangle$ out to $v'$ is to go via a triangle containing $v$ and then traverse along the lift of $S$ that separates $\left(\bar{R}_\text{out}\right)_0$ from $\bar{R}_\text{in}$. This gives us a path of edges on the $\left(\bar{R}_\text{out}\right)_0$ side, all of which are in $\bar{E}_0$. The vertex at the start of the path is $v$, the vertex at the end is $v'$. Now apply lemma 3.14 at each step of the path.

Remark 3.16. If we allowed in definition 3.11 a choice of $S$ and hence $\bar{E}_0$ with a bad loop consisting of all but one edge in $\bar{E}_0$ and one edge in $\bar{E}_+$, then the two endpoints of the path of edges in $\bar{E}_0$, $v$ and $v'$, would necessarily be in the same component of $\bar{R}_\text{out}$. By the above lemma, their developed positions would be the same on $\partial \mathbb{H}^3$, and then the tetrahedra that contain the single edges of $\bar{E}_+$ would not be able to have dihedral angles that satisfied both the degeneration order and consistent development conditions.

Remark 3.17. If there is no $\triangle \in \bar{T}$ of type 111 then all tetrahedra are of types 22, 31 or 4. $\bar{R}_\text{in}$ is then sandwiched between only two components of $\bar{R}_\text{out}$, and given lemma 3.15 the only sensible developing map we might define (see footnote [1] in the proof of theorem 3.18 on how we can develop starting from a triangle of type 21) would have all vertices appear at one of only two positions on $\partial \mathbb{H}^3$, and there would be no hope of constructing a representation from this data. If all tetrahedra are of type 4 then $\bar{R}_\text{in} = \bar{M}$ and the developing map would have to have all vertices appear at only one position on $\partial \mathbb{H}^3$. See also the proof of theorem 5.9.
Theorem 3.18. \( \mathcal{O}(M; T; S) \) is an affine algebraic variety.

Proof. We may view \( \mathcal{O}(M; T; S) \) as a subset of \( \mathbb{C}^{N_1+2N_2} \), where \( N_1 \) is the number of tetrahedra of \( T \) of types 1111, 211 and 22, and \( N_2 \) the number of type 31, as in definition 3.7. We now need to show that the consistent development condition 3.8 gives a finite number of polynomial conditions on these variables.

First consider all the dihedral angles as elements of \( \mathbb{C}((\zeta)) \setminus \{0,1\} \) and assume without loss of generality that all developed positions are in \( \mathbb{C}[[\zeta]] \). By repeated application of equation (23), every developed position \( f \) is a rational function of the preferred cross ratios \( z^{(i)} \in \mathbb{C}[[\zeta]] \) corresponding to the dihedral angles, and the initial three points \( e, e', e'' \in \mathbb{C}[[\zeta]] \), so

\[
f = f(z^{(i)}, e, e', e'') = \frac{p(z^{(i)}, e, e', e'')}{q(z^{(i)}, e, e', e'')}
\]

where \( p \) and \( q \) are polynomials. We can then write \( p = \zeta^k (p_0 + \zeta p_1) \) where \( p_0 \) is a polynomial in the \( z^{(i)} \) and \( e_0, e'_0, e''_0 \) (but not \( \zeta \)), and \( p_1 \) is a polynomial. We can similarly write \( q = q_0 + \zeta q_1 \) (where there is no factor of \( \zeta \) since \( f \in \mathbb{C}[[\zeta]] \)). Then

\[
f = \zeta^k \frac{p_0 + \zeta p_1}{q_0 + \zeta q_1} = \zeta^k \frac{p_0 + \zeta p_1}{q_0 (1 - \zeta \frac{q_1}{q_0})} = \zeta^k \frac{p_0 + \zeta p_1}{q_0} \left( 1 + \zeta \frac{q_1}{q_0} + \left( \frac{q_1}{q_0} \right)^2 + \ldots \right)
\]

Thus \( f_H = \frac{p_0}{q_0} \), or 0 if \( k > 0 \), and the equation \( f_H = f'_H \) can be rearranged into the form of a polynomial in the \( z^{(i)} \) together with \( e_0, e'_0, e''_0 \) being equal to 0. Whether or not the two developed points coincide does not depend on the starting positions \( e_0, e'_0, e''_0 \), since moving these points consists of conjugating, which likewise conjugates all of the developed vertices of triangles in the chains, since cross ratios are preserved. Thus if we fix the values of \( e_0, e''_0 \) and the \( z^{(i)} \) we get a polynomial in only \( e_0 \), but which is constantly zero and so the polynomial cannot depend on \( e_0 \) at all. Similarly for \( e'_0 \) and \( e''_0 \), and the polynomial equation depends only on the \( z^{(i)} \).

We require that there are a finite number of such polynomial equations needed to ensure consistent development. Again we assume without loss of generality that all developed positions are in \( \mathbb{C}[[\zeta]] \). Given a triangle \( \Delta' \in \bar{T} \) with vertices \( u', v', w' \) to which we have developed positions in \( \mathbb{C}[[\zeta]] \), then due to lemma 2.6, \( \Phi_Z(u'), \Phi_Z(v'), \Phi_Z(w') \) together with the first order coefficient of the difference between developed positions of a pair of these vertices with the same image under \( \Phi_Z \) (if such a pair exists) are all the information we need in order to entirely determine \( \Phi_Z(v) \) for further vertices. Moreover, the consistent development condition also means that this first order coefficient of the difference is also independent of the chain we take. If it were not, we could continue.

\footnote{This is the sense in which we can develop starting from a triangle of type 21.}
developing from two chains that produced different answers for the first order coefficient of the difference along some valid chain that connects $\triangle'$ to a triangle of type 111. Then the different answers from the two chains would produce different positions of the images under $\Phi_Z$, contradicting consistent development. (If we cannot reach a triangle of type 111 then there was no choice of $\triangle$ to start developing from in the first place.)

Let $C_1$ and $C_2$ be two valid chains of triangles that start at $\triangle$ and end at $\triangle'$:

$$C_1 = (\triangle = \triangle^{(0)}_1, \triangle^{(1)}_1, \ldots, \triangle^{(n_1)}_1 = \triangle')$$

$$C_2 = (\triangle = \triangle^{(0)}_2, \triangle^{(1)}_2, \ldots, \triangle^{(n_2)}_2 = \triangle')$$

Now suppose $C_1$ and $C_2$ agree at some triangle in the middle, at $\triangle'' := \triangle^{(j_1)}_1 = \triangle^{(j_2)}_2$. We can then split these chains in two, getting:

$$C_{1,A} = (\triangle = \triangle^{(0)}_1, \triangle^{(1)}_1, \ldots, \triangle^{(j_1)}_1 = \triangle'')$$

$$C_{2,A} = (\triangle = \triangle^{(0)}_2, \triangle^{(1)}_2, \ldots, \triangle^{(j_2)}_2 = \triangle'')$$

$$C_{1,B} = (\triangle'' = \triangle^{(j_1)}_1, \triangle^{(j_1+1)}_1, \ldots, \triangle^{(n_1)}_1 = \triangle')$$

$$C_{2,B} = (\triangle'' = \triangle^{(j_2)}_2, \triangle^{(j_2+1)}_2, \ldots, \triangle^{(n_2)}_2 = \triangle')$$

Then by the above discussion, consistency of development (possibly including consistency of first order coefficient of differences) between $C_1$ and $C_2$ is implied by consistency between $C_{1,A}$ and $C_{2,A}$, and between $C_{1,B}$ and $C_{2,B}$.

Consider the graph $\tilde{G}$ with vertex set the triangles of $\tilde{T}$ and edges between two vertices if the corresponding triangles are in the same tetrahedron and the edge between them is in $\tilde{E}_+$ (see the large dots and dashed lines in figure 10). Any valid chain of triangles corresponds to a path in $\tilde{G}$. Showing that any pair of valid chains of triangles develop to the same result is equivalent to showing that any valid chain that is a loop in $\tilde{G}$ develops back to the starting position of the first triangle (for any choice of first triangle).

![Figure 13: Chains of triangles, combining a large loop from two smaller ones.](image)

See figure [13]. Given a loop $A \circ B$ (i.e. following the chain $A$ then $B$), $A \circ B$ develops consistently if and only if $A \circ C \circ C^{-1} \circ B$ does (since all of
our developing steps are reversible), which by the above discussion is implied by the loops $A \circ C$ and $C^{-1} \circ B$ developing consistently. Thus, all we need is a generating set of loops of triangles to develop consistently in order to ensure that every loop develops consistently. A finite set of such loops can be found, using a set of generating loops for $H_1(G)$, where $G$ is the graph defined analogously to $\widetilde{G}$, but with vertices corresponding to the triangles of $\mathcal{T}$ rather than $\widetilde{T}$. The consistency of developing around each loop is a polynomial condition in the dihedral angles, and the result follows.

4 Retetrahedralising

Given a 3-manifold $M$ with tetrahedralisation $\mathcal{T}$ we can consider the (finite) set of horo-normal surfaces in $M$ relative to $\mathcal{T}$. We would like to have horo-normal surfaces that are omnipresent, so that we can use them as the surface in the definition of the extended deformation variety. In this section we show how to alter the tetrahedralisation in order to turn horo-normal surfaces that are not omnipresent into ones that are. The tool we will use to do this involves the following object:

Definition 4.1. A pillow is a pair of tetrahedra that share an edge and are the only two tetrahedra incident to that edge. A pillow folded over on itself along an edge $e$ is a pillow with two faces glued as in figure 14. The edge $e$ is incident to only one tetrahedron.

![Figure 14: A pillow and a pillow folded over on itself.](image)

Definition 4.2. The move of inserting a pillow alters a tetrahedralisation $\mathcal{T}$ as follows: take a pair of triangles of $\mathcal{T}$ which share an edge $e$, open up a gap between the two pairs of faces of tetrahedra that meet at the pair of triangles and insert a pillow between them. There is an analogous pillow insertion if the pair of triangles is the same triangle twice, with the edge $e$ being one of its edges: we open up a gap between the pair of faces of tetrahedra that meet at the triangle, and insert a pillow folded over on itself along $e$.

Remark 4.3. Inserting a pillow corresponds to Matveev’s lune move (definition 1.2.9 of [1]).
Definition 4.4. In the new tetrahedralisation, $T'$, in place of the edge $e$ there are two corresponding edges $e^1$ and $e^2$, and in place of each triangle involved are two corresponding triangles, each of which shares the same vertices as the edge or triangle it came from. We refer to these edges and triangles as splits of $e$ and the original triangle respectively. If we go on to make further pillow insertions we will also recursively refer to splits of those splits as splits.

Let $E$ be the edge set of the tetrahedralisation $T$ and $T'$ the tetrahedralisation obtained by inserting a pillow across an edge $e$. $T'$ has edge set $E' = (E \setminus \{e\}) \cup \{e^1, e^2, f\}$ where $f$ is the edge joining opposite vertices of the pair of triangles. If we have a choice of edges $E_0 \subset E$, we can ask what possible choices of edges $E'_0 \subset E'$ are compatible with $E_0$, meaning that $E_0 \subset E'_0$ if $e \notin E_0$ and $(E_0 \setminus \{e\}) \cup \{e^1, e^2\} \subset E'_0$ if $e \in E_0$. We analyse the possibilities in figure 15, which shows all possible configurations without bad loops in the original pair of triangles. Note that the splits of $e$ must be either both in $E'_0$ or both in $E'_0$ in order to avoid a bad loop.

![Figure 15](image)

Figure 15: The seven possible configurations (up to symmetry) of edges in $E_0$ or $E'_0$ for two triangles that meet at an edge, and the 9 possible configurations after inserting a pillow. Edges in $E_0$ are shown with dotted lines. The numerical labels refer to the number of edges of the two triangles that are in $E_0$, and where there is more than one configuration with that many we have assigned subscripts to the labels arbitrarily. For each of the five configurations 1, 2, 3, and 5 there is only one possibility for whether the added edge is in $E_0$ or $E_+$ that avoids bad loops. For the two configurations 0 and 1 the local picture does not force this, although edges outside of the two triangles may do so.

Lemma 4.5. Suppose for a tetrahedralisation $T$ we have a choice of $E_0$ with no bad loops. Let $T'$ be the tetrahedralisation obtained from $T$ by inserting a pillow,
so $E' = (E \setminus \{e\}) \cup \{e^\uparrow, e^\downarrow, f\}$. Let $E'_0 = (E_0 \setminus \{e\}) \cup \{e^\uparrow, e^\downarrow\}$ if $e \in E_0$, and $E'_0 = E_0$ if not. Then at least one of the choices $E'_0 = E'_0$ or $E'_0 = E'_0 \cup \{f\}$ has no bad loops in $E'$.

Proof. Let the endpoints of $f$ be $v$ and $w$. Suppose that either choice for $f$ would result in a bad loop. Then there would be a path of edges in $E'$ connecting $v$ to $w$ entirely contained in $E'_0$ (so that $f \in E'_+$ would give a bad loop), and another path that would have all but one edge in $E'_0$ (so that $f \in E'_0$ would give a bad loop). Then joining these two paths together and replacing any splits $e^\uparrow$ or $e^\downarrow$ in the path with $e$ would result in a bad loop for the choice of $E_0 \subset E$, contradicting the hypothesis.

Definition 4.6. Because of lemma 4.5, given a horo-normal surface relative to a tetrahedralisation $T$, there are either one or two corresponding horo-normal surfaces relative to the tetrahedralisation $T'$ where $T'$ is obtained from $T$ by inserting a pillow. We call such a derived horo-normal surface a child of the original surface, and the original surface the parent of the derived surface. If we go on to insert further pillows, we call the children of children (with any number of generations) of a surface the descendants of that surface. We extend these definitions to the choices of $E_0$ with no bad loops corresponding to the horo-normal surfaces in the obvious way.

Lemma 4.7. Suppose that $T'$ is obtained from $T$ by inserting a pillow and that $S$ is a horo-normal surface relative to $T'$. Then $S$ has exactly one parent.

Proof. Let $E'_0$ be the choice corresponding to $S$, and let $e^\uparrow, e^\downarrow, f$ be the edges added by inserting the pillow. The choice of $E'_0$ has no bad loops, and collapsing $e^\uparrow, e^\downarrow$ to $e$ and removing $f$ cannot create a bad loop, so taking $E_0 = (E_0 \setminus \{e^\uparrow, e^\downarrow, f\}) \cup \{e\}$ if $e^\uparrow, e^\downarrow \in E'_0$, or $E_0 = E'_0 \setminus \{f\}$ if not produces a parent of $S$. Any parent of the choice of $E'_0$ must agree with $E'_0$ everywhere but at $f$, and contains $e$ if and only if $E'_0$ contains $e^\uparrow$ and $e^\downarrow$, so the parent is unique.

Given a sequence of pillow insertions starting from a tetrahedralisation $T$, the horo-normal surfaces of the various tetrahedralisations thus form a graded forest (a disjoint union of trees, one tree for each horo-normal surface relative to $T$), with all roots of trees at the level of $T$ and a non-decreasing number of nodes at successive levels.

Lemma 4.8. Suppose that $S$ is a surface in horo-normal form relative to a tetrahedralisation $T$ which is omnipresent. Then all descendants of $S$ are also omnipresent.

Proof. Let $S'$ be a descendant of $S$. Condition 2 of definition 3.11 is clear since inserting a pillow only removes edges when replacing them with splits (which have the same endpoints and are in $E_+$ if the original edge is). Therefore the choice of $E'_+$ corresponding to $S'$ contains an edge or a split of it if the choice of $E_+$ corresponding to $S$ does.
To show condition 1, first note that we can see regions of $\tilde{R}_m$ from diagrams of the edges of tetrahedra marked as being in $\tilde{E}_0$ or $\tilde{E}_+$ by looking at the midpoints of edges in $\tilde{E}_+$. Observe that two midpoints are part of the same component of $\tilde{R}_m$ if and only if they are connected by a path of midpoints of edges in $\tilde{E}_+$ where neighbouring midpoints are midpoints of edges of triangles of $\tilde{T}$ (see figure 10).

Now consider the possible moves in figure 15. In all possible configurations of pairs of triangles and pillows only the pair of triangles in configuration 1 has the midpoints of edges in $\tilde{E}_+$ corresponding to potentially distinct components of $\tilde{R}_m$. Those potentially distinct components get connected together when we insert the pillow, no matter which choice we make for the added edge. Thus connectivity can only increase for the descendants of a horo-normal surface.

Remark 4.9. Notice in particular that the midpoints of splits of an edge in $\tilde{E}_+$ are always connected to each other, and so this will be true for all splits of that edge produced by further pillow insertions.

Definition 4.10. A strip of triangles in $\tilde{T}$ (or $\tilde{T}$) is a sequence of triangular faces $(\triangle_1, \triangle_2, \ldots, \triangle_n)$ of $\tilde{T}$ (or $\tilde{T}$) alternating with a sequence of edges $(e_1, e_2, \ldots, e_{n-1})$ such that neighbouring triangles $\triangle_k, \triangle_{k+1}$ share the edge $e_k$ (the internal edge between triangles). For each $\triangle_k$, $k = 2, 3, \ldots, n - 1$, the internal edges either side, $e_{k-1}$ and $e_k$, must be distinct. We allow repeated triangles and edges in the strip, even consecutive repeated triangles.

We will sometimes write a strip as $(\triangle_1, e_1, \triangle_2, e_2, \ldots, e_{n-1}, \triangle_n)$. Compare with definition 2.3.

In order to organise the positions of strips of triangles we will use the following structure derived from a tetrahedralisation:

Definition 4.11. If $\tilde{T}$ is an ideal tetrahedralisation of a manifold $M$, we construct a corresponding handle decomposition $\tilde{T}^\circ$ of $M$ by “thickening up” the edges and triangles of $\tilde{T}$. Each edge $e$ is replaced by a polygonal tubular neighbourhood (or thick edge) $e^\circ$, where the number of sides of the polygon is equal to the number of triangles incident to the edge. Each triangle $\triangle$ is replaced by a triangular prism (or thick triangle) $\triangle^\circ$ with the rectangular faces coinciding with the rectangular faces of the thick edges. Each tetrahedron $t$ remains combinatorially the same but shrinks a little to become $t^\circ$ so that its faces coincide with the triangular faces of the thick triangles. See figure 16.

Inserting a pillow into $\tilde{T}$ has a corresponding effect on $\tilde{T}^\circ$ and splits of thick edges and triangles are defined in an analogous way.

Definition 4.12. A strip of triangles in $\tilde{T}^\circ$ (or $\tilde{T}^\circ$) is a sequence of $n$ triangles embedded in and respecting the product structure of the thick triangles, alternating with $n - 1$ rectangular pieces embedded in and respecting the product structure of the thick edges such that the pieces connect together analogously to

\footnote{As in, internal to the strip.}
as in definition 4.10. See figure 17. If we collapse all of the product structures to recover $T$ then a strip of triangles in $T^\ominus$ becomes a strip of triangles in $T$.

We can think of a strip of triangles in $T^\ominus$ as having all of the data of a strip of triangles in $T$ together with ordering information in the case when the strip passes through a triangle or edge multiple times.

**Theorem 4.13.** Suppose $S$ is a horo-normal surface relative to an ideal tetrahedralisation $T$ of a connected 3-manifold $M$. Then there exists a tetrahedralisation $T^*$ of $M$ obtained from $T$ by a finite number of pillow insertions such that all descendants of $S$ in horo-normal form relative to $T^*$ are omnipresent.

**Proof.** We deal with condition 1 of omnipresence first. $\tilde{M}$ is connected, so there exists a strip of triangles of $\tilde{T}$ from any edge of $\tilde{E}$ to any other. If all of the internal edges of a strip are in $\tilde{E}_+$ then the midpoints of those edges are all in the same component of $\tilde{R}_m$. We split $\tilde{E}_+$ into equivalence classes, where two edges are equivalent if there is a strip in $\tilde{T}$ from one to the other with all internal
edges in $\tilde{E}_+$. 

If $\tilde{R}_{in}$ is not connected then there is more than one equivalence class, and a strip of triangles of $\tilde{T}$ connecting edges of two equivalence classes must pass across an internal edge in $\tilde{E}_0$. We consider a minimal strip $\sigma$ with these properties, which must be of the form $\sigma = (\triangle^1, \triangle^2, \ldots, \triangle^n)$, where $\triangle^1, \triangle^n$ are type 21 with the edge of $\tilde{E}_0$ shared with $\triangle^2, \triangle^{n-1}$ respectively, and $\triangle^2, \ldots, \triangle^{n-1}$ are type 3. Let $e^i$ be the internal edge between $\triangle^i$ and $\triangle^{i+1}$.

We insert a pillow between $\triangle^1$ and $\triangle^2$ (as an operation on $\tilde{T}$ rather than $\tilde{\tilde{T}}$). If $n = 2$, and the strip $\sigma$ has only two triangles, then we are in configuration 11 of figure 15. When we insert the pillow the previously disconnected components of $\tilde{R}_{in}$ become connected for both children of $S$. Otherwise we are in configuration 3, the added edge $f \in E'_+ \tilde{E}_0$ and we have two new triangles which have $f$ as an edge. Both of these triangles are of type 21, and one of them is arranged together with $\triangle^3$ in configuration 3 of figure 15 (if $n > 3$) or configuration 11 (if $n = 3$). We repeat, inserting pillows until we reach the same situation as when $n = 2$, and the previously disconnected components of $\tilde{R}_{in}$ become connected. See figure 18.

![Figure 18: Inserting pillows along a strip joining two components of $\tilde{R}_{in}$. Edges in $\tilde{E}_+$ are solid, edges in $\tilde{E}_0$ are dotted, and the last added edge (which could be in either) is dashed.](image)

It is possible that some early pillow insertions could affect our ability to perform later pillow insertions, if some edge or triangle of our strip $\sigma$ has been replaced by splits. By remark 4.9 however, we only care about connecting along the strip to either the edges of the last triangle in $\tilde{E}_+$, or some split of them. We proceed as follows:

Suppose that we have inserted pillows along the strip up to and including $\triangle^k$. Then we have a triangle $\triangle^k$ of type 21 with one vertex shared with $\triangle^1$, and the edge opposite that vertex, $e^k$, is either $e^k$ or a split of it. Call the current tetrahedralisation $\tilde{T}^k$. The next triangle of the strip, $\triangle^{k+1} \in \tilde{T}$ may have been replaced by splits of it in $\tilde{T}^k$, and there may even be no split of $\triangle^{k+1}$ which has
as an edge. See the left diagram of figure 19 for an example.

If there does exist a split of $\triangle^{k+1}$ which has $e^*_k$ as an edge then we choose one arbitrarily and insert a pillow between it and $\triangle^*_k$ to continue. If not then we will need to find a detour strip: a strip in $\tilde{T}^k$ with $\triangle^*_k$ and $e^*_k$ as first triangle and first internal edge and $e^*_m$ and $\triangle^{k+1}_*$ as last internal edge and last triangle, where $\triangle^{k+1}_*$ is a split of $\triangle^{k+1}$ and $e^*_m$ a split of $e^*$ incident to $\triangle^{k+1}_*$.

We then insert pillows along this detour strip starting with the triangle $\triangle^*_k$, going through internal edges of the detour strip in either $\tilde{E}_0$ or $\tilde{E}_+$ (it will not matter which). We eventually get a triangle $(\triangle^*_k)'$ as in the rightmost diagram of figure 19 which has edges between the same endpoints as $\triangle^*_k$, and so the new edges must be in $E_+$ to avoid a bad loop. Thus this new triangle $(\triangle^*_k)'$ is arranged next to $\triangle^{k+1}_*$ so that we can insert a pillow as if we hadn’t needed to take a detour at all. See the middle diagram of figure 19 for an example of such a detour strip.

Figure 19: An obstruction in the way of a strip with first two triangles having vertices $A, B, C$ and $A, B, D$, a detour around the obstruction and the general picture of inserting pillows along a detour. Solid lines are in $\tilde{E}_+$, dotted in $\tilde{E}_0$ and dashed in either. Note that in the rightmost diagram of the general picture, the strip could be “twisted” so that the top edge of the strip goes from $A$ to $B$ rather than $A$ to $A$ and the bottom from $B$ to $A$ rather than $B$ to $B$. This does not affect the argument.

We need to make sure that in inserting pillows along the detour strip we do not find ourselves in the position that required a detour strip in the first place, i.e. that pillow insertions early in the strip could split apart an internal edge later in the strip and there might not be a split of the next triangle in the strip to continue into. This can happen for example if the detour strip in $\tilde{T}^k$ has two internal edges $e'_i, e'_j$ that are lifts of the same edge in $T^k$, and the triangles of the strip on either side of those edges are lifts of triangles of $T^k$ that alternate around the edge.

To construct a detour strip that avoids this problem, start with any strip
\( \tilde{\sigma} \) in \( \tilde{\mathcal{F}}^k \) starting \( \Delta_a^k, e_a^k \) and ending \( e_{a_*}^k, \Delta_{a_*}^{k+1} \) (say, a minimal length such strip). Let \( \sigma \) in \( \mathcal{F}^k \) be the image of the strip under the covering map, \( \sigma = (\Delta_1, e_1, \Delta_2, e_2, \ldots, e_{j-1}, \Delta_j) \). We build a strip \( \sigma' \) embedded in \( \mathcal{F}^{k\odot} \) as follows: start with the first triangle of \( \sigma \) and embed a triangle in the corresponding thick triangle of \( \mathcal{F}^{k\odot} \). Now suppose we have constructed the strip up to a triangle embedded in the thick triangle \( \Delta_i^{\odot} \) corresponding to \( \Delta_i \), and we next want to put in a rectangle and triangle corresponding to \( e_{i+1} \) and \( \Delta_{i+1} \). See figure [20].

![Figure 20: A rectangle in the way of extending the strip and the result of pushing it aside. The left thick triangle is \( \Delta_i^{\odot} \), the right thick triangle is \( \Delta_{i+1}^{\odot} \) and the thick edge in the middle is \( e_{i+1}\odot \). For clarity, the “corner” parts of the strip are not shown in the second diagram. In these diagrams we do not specify how many thick triangles are incident above and below \( e_{i+1}^{\odot} \), and the number does not alter the argument.](image)

As we see in the figure, there may be rectangles of the already constructed strip in the way. We “push aside” any such rectangles, removing them and re-connecting the strip by adding two triangles and three rectangles each as shown, one of the rectangles pushing into one of the thick edges incident to \( \Delta_{i+1}^{\odot} \) other than \( e_{i+1}^{\odot} \). It doesn’t matter if we choose to push the rectangle to the far or the near side of \( \Delta_{i+1}^{\odot} \). This process adds a finite number of pieces to the strip and the result is once again embedded. Note that there could be other triangles of
the strip already in $\Delta^{i+1}_i$, including triangles that connect to a rectangle in $e^{i+1}_i$ as shown in the first diagram of figure 20 or that connect out to both of the other two thick edges incident to $\Delta^{i+1}_i$ (not shown). We push our triangle in between these layered triangles arbitrarily, although the picture is cleanest when we push aside as few rectangles as possible, i.e. only those rectangles in $e^{i+1}_i$ whose incident thick triangles are arranged in an alternating fashion with $\Delta^{i+1}_i$ and $\Delta^{i+1}_i$ around $e^{i+1}_i$. We continue in this manner, and eventually get a strip $\sigma'$ embedded in $T^{k,1}_i$ that starts and ends at the corresponding thick triangles and thick edges to the triangles and edges of $\sigma$. Note that pushing rectangles aside will never change the first triangle or which thick edge contains the first rectangle.

Now take a lift of $\sigma'$ to a strip $\tilde{\sigma}'$ embedded in $\tilde{T}^{k,1}_i$ (and the union of all of its $\pi_1M$-translates is also embedded). By construction $\tilde{\sigma}'$ starts with a triangle and rectangle in $\Delta^{k,1}_i, e^{k,1}_i$ and ends with a rectangle and triangle in $e^{k,1}_{a_1}, \Delta^{k+1,1}_{a_1}$. We now insert pillows along the detour strip. The embedding of the strip in the handle structure of $T^{k,1}_i$ will tell us how to move around parts of the strip we have yet to get to, when we insert pillows along triangles that the strip passes through more than once. See figure 21, which shows a thick pillow, the set of handles corresponding to a pillow.

When we insert a pillow at a pair of triangles sharing an edge, in the corresponding thick picture we replace a pair of thick triangles both incident to a thick edge with a thick pillow as in figure 22. If we have two triangles and a rectangle embedded in the left diagram of figure 22 then any other parts of a strip within the left diagram are either above or below the two triangles and rectangle. We put those parts of strip into either $\Delta^{1,1}_1, e^{1,1}_1, \Delta^{1,1}_2$ or $\Delta^{1,1}_1, e^{1,1}_1, \Delta^{1,1}_2$ depending on if they are above or below the two triangles and rectangle. It is clear that the strip remains embedded and parts of the strip we moved connect up with parts we did not.

This move removes two triangles and a rectangle from the start of a strip. To continue into the next rectangle and triangle, we insert a triangle into $\Delta^{1,1}_L$ or $\Delta^{1,1}_R$ (depending on which direction the strip continues in, see figure 21) and connect it to the next rectangle of the strip. We then repeat the process. The embedding thus makes it clear that we can insert pillows along the whole of the detour strip without requiring any further detours.

It is possible that at some points of the process, for instance at the first pillow insertion of a detour strip, we may want to insert a pillow along two triangles that are in fact the same triangle twice. We insert a folded over pillow instead and the same argument goes through without change.

We call the tetrahedralisation we obtain after inserting pillows along the
Figure 21: Top right: a pillow in $\mathcal{T}$. Bottom left: the triangles inside the pillow, redrawn for clarity. Diagonally from top left to bottom right: the corresponding thick pillow in $\mathcal{T}^\odot$ in exploded view. Top left are two thick triangles and a thick edge, next a tetrahedron, next two thick triangles and a thick edge. The lower half is the mirror image of the upper half.

Figure 22: Thick versions of a pair of triangles and the corresponding pillow, only the boundary faces are shown. The versions shown here correspond to the situation in figure [16] with one triangle incident to the central edge above and one below. If there are more above then the rectangular face is cut into more strips, in both the pair of triangles and pillow diagrams, and similarly below. There are analogous pictures for if there is no triangle on one side of the pair, or if the two triangles are actually the same and the pillow is folded over on itself.
strip \( \mathcal{T}_1 \). We now have a different set of equivalence classes of \( \tilde{E}_+ \) as some previously disjoint classes have merged. Note that we are still considering the equivalence classes of edges of the original tetrahedralisation \( \mathcal{T} \) but with the equivalence relation on those edges given by strips in \( \mathcal{T}_1 \). If \( \tilde{E}_+ \) now has only one equivalence class then we are done. If not then we take another minimal strip \( \sigma_1 = (\Delta_{1_1}^1, \Delta_{1_2}^1, \ldots, \Delta_{1_{n_1}}^1) \) of types \( 21, 3, \ldots, 3, 21 \) in \( \mathcal{T} \) which connects two of the new equivalence classes. Just as before we can insert pillows along this strip, taking detour strips as necessary (there was nothing special about it being the first strip we took in the detouring process or argument) and will eventually connect together the components at either end of the strip.

To show that connecting a finite number of times along strips of this type will produce a connected \( \tilde{R}_0 \), choose some connected fundamental domain \( D_0 \subset M \) for \( M \) made from some union of tetrahedra of the original tetrahedralisation \( \mathcal{T} \) and consider the connected components of \( \tilde{R}_0 \) that intersect \( D_0 \). There will be some finite number of these since \( D_0 \) is made from finitely many tetrahedra, each of which can intersect at most one component of \( \tilde{R}_0 \). \( D_0 \) has some finite number of \( \pi_1 M \) translates that touch it along faces of \( \mathcal{T} \). Connecting all of the components that intersect \( D_0 \) together, and connecting the resulting single component for \( D_0 \) to each of the components intersecting the touching \( \pi_1 M \) translates (if they are not already the same component as the one intersecting \( D_0 \)) results in a connected \( \tilde{R}_0 \). Finally, the only way that two components of \( \tilde{R}_0 \) are not separated by a strip of triangles of types \( 21, 3, 3, \ldots, 3, 21 \) is if any path from one to the other would have to pass through some intermediary component and by connecting to intermediaries we eventually connect the two original components.

Now that we have \( \tilde{R}_0 \), connected, satisfying condition 2 is similar but easier. If it fails then a vertex \( v \in \mathcal{V} \) has only edges in \( \mathcal{E}_0 \) incident to it, and so all triangles incident to it are of type 3. All we have to do is take a strip from a type 21 triangle through type 3 triangles out to \( v \). Choosing a minimal such strip in \( \mathcal{T} \) and inserting pillows along it as before, taking detours when necessary, will suffice.

**Theorem 4.14.** Suppose \( M \) is a connected 3-manifold with ideal tetrahedralisation \( \mathcal{T} \). Then there exists a tetrahedralisation \( \mathcal{T}_* \) of \( M \) obtained from \( \mathcal{T} \) by a finite number of pillow insertions such that all surfaces in horo-normal form relative to \( \mathcal{T}_* \) are omnipresent.

**Proof.** \( \mathcal{T} \) has a finite number of horo-normal surfaces \( S_1, S_2, \ldots, S_n \). For each such surface \( S_j \) there are a finite number of strips \( s_{j_1}^1, s_{j_2}^2, \ldots, s_{j_{m_j}}^m \) in \( \mathcal{T} \) which if pillows are inserted along (taking detours as necessary), will make all descendants of \( S_j \) omnipresent. We start with \( S_1 \) and make all of its descendants omnipresent as in theorem 4.13. In the resulting tetrahedralisation \( \mathcal{T}_1 \), there may be many different descendants of \( S_2, \ldots, S_n \). However, by the same argument as in lemma 4.8 all of the descendants of \( S_j, j \neq 1 \) have at least as
much connectivity (in the sense of the number of connections between components that need to be made and the number of components of ∂M without an edge of E incident) as $S_j$. The descendants of $S_2$ are all identical outside of the inserted pillows, and so inserting pillows along $s_2^1, s_2^2, \ldots, s_2^m$, starting from $T_2^*$ and taking detours as necessary will connect all of the components of each of the descendants of $S_2$ (and give edges of $E + \text{incident}$ to each component of $\partial M$). This makes all of the descendants of $S_2$ in the new tetrahedralisation $T_2^*$ omnipresent. We will produce yet more descendants of $S_2$ as we do this, but by lemma 4.15 all of these will be omnipresent. Repeat for $S_3$ through $S_n$ to obtain the desired tetrahedralisation $T^* = T_1^* = T_2^* = \ldots = T_n^*$.

Remark 4.15. The tetrahedralisations obtained in the above theorems are unlikely to be very efficient in the numbers of tetrahedra used. If we only have to fix a single horo-normal surface then a similar tunnelling procedure using 2-3 moves will also work and likely require fewer tetrahedra. However, this gives less control on what happens to descendants of other surfaces.

5 Representations

Definition 5.1. We define a map

$$\mathcal{R}(\mathcal{T}; S) : \hat{D}(M; \mathcal{T}; S) \to \mathcal{R}(M)$$

up to conjugation as follows: Given $Z \in \hat{D}(M; \mathcal{T}; S)$, the developing map $\Phi_Z$ gives us for each vertex in $\hat{V}$ a position on $\partial \mathbb{H}^3$. To construct a representation $\mathcal{R}(\mathcal{T}; S)(Z) = \rho_Z : \pi_1 M \to \text{PSL}_2(\mathbb{C})$, for each $\gamma \in \pi_1 M$ we consider the translate $\gamma \Delta$ of $\Delta$ (the triangle we start developing from), which is another triangle of $\mathcal{T}$. No edges of $\Delta$ are in $\hat{E}$, and so the same is true for the edges of $\gamma \Delta$, since both are lifts of the same triangle of $\mathcal{T}$. The positions of the vertices $v, v', v''$ of $\Delta$ on $\partial \mathbb{H}^3$ are distinct by definition, and the developed positions of the vertices $\gamma v, \gamma v', \gamma v''$ of $\gamma \Delta$ are distinct by lemma 3.14. We define $\rho_Z(\gamma)$ to be the unique element of $\text{PSL}_2(\mathbb{C})$ that takes the positions of $v, v', v''$ on $\partial \mathbb{H}^3$ to the positions of $\gamma v, \gamma v', \gamma v''$. We also define

$$\mathcal{R} \mathcal{T} : \hat{D}(M; \mathcal{T}) \to \mathcal{R}(M)$$

as $\mathcal{R}(\mathcal{T}; S)$ on each $\hat{D}(M; \mathcal{T}; S)$ in the disjoint union that makes up $\hat{D}(M; \mathcal{T})$.

To show that the map results in a representation, we need the following:

Lemma 5.2. $\mathcal{R}(\mathcal{T}; S)(Z) = \rho_Z : \pi_1 M \to \text{PSL}_2(\mathbb{C})$ is a homomorphism.

Proof. Suppose we have valid chains of triangles

$$C_1 = \left( \Delta = \Delta_1^{(0)}, \Delta_1^{(1)}, \ldots, \Delta_1^{(n_1)} = \gamma_1 \Delta \right)$$

$$C_2 = \left( \Delta = \Delta_2^{(0)}, \Delta_2^{(1)}, \ldots, \Delta_2^{(n_2)} = \gamma_2 \Delta \right)$$

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an example of a chain from $\triangle$ to $\gamma_2\gamma_1\triangle$ is given by

$$C_3 = \left( \triangle = \triangle_2^{(0)}, \triangle_2^{(1)}, \ldots, \triangle_2^{(n_2)} = \gamma_2\triangle = \gamma_2\triangle_1^{(0)}, \gamma_2\triangle_1^{(1)}, \ldots, \gamma_2\triangle_1^{(n_1)} = \gamma_2\gamma_1\triangle \right)$$

Also let $C_0 = (\triangle)$ be the trivial chain. Let $C^{(j)}$ denote the subchain of $C$ up to the $j$th triangle. Let $\Phi_{(C,\gamma)}(v)$ be the developed position of $v$ into the ends of $T_\gamma$ along a chain of triangles $C$ (so $\Phi_{(C,\gamma)}(v)_H = \Phi_{\gamma}(v)$, but the higher order information can depend on $C$). Let $I_{(C,\gamma)} \in \text{PSL}_2(\mathbb{C}((\zeta)))$ be the unique element that takes the developed position of the first triangle of $C$ to the developed position of the last.

By definition,

$$\Phi_{(C,\gamma)}(\gamma_2\triangle) = I_{(C,\gamma)} \Phi_{(C,\gamma)}(\triangle)$$

Here we are abusing notation in having $\Phi$ take input the ordered triple of vertices forming a triangle rather than just one vertex, and $\Phi_{(C,\gamma)}(\triangle)$ is a complicated way to write the position of $\triangle$ that we start developing from.

We would like to show that for each $j = 0, \ldots, n_1$:

$$\Phi_{(C,\gamma)}(\gamma_2\triangle_1^{(j)}) = I_{(C,\gamma)} \Phi_{(C,\gamma)}(\triangle_1^{(j)})$$

(30)

When $j = 0$ this is the previous equality, and when $j = n_1$ this says that:

$$\Phi_{(C,\gamma)}(\gamma_2\gamma_1\triangle) = I_{(C,\gamma)} \Phi_{(C,\gamma)}(\gamma_1\triangle)$$

and so

$$\Phi_{(C,\gamma)}(\gamma_2\gamma_1\triangle) = I_{(C,\gamma)} \Phi_{(C,\gamma)}(\gamma_1\triangle) = I_{(C,\gamma)} I_{(C,\gamma)} \Phi_{(C,\gamma)}(\triangle)$$

which is what we need. We show equation (30) by induction. As noted before, we have the case when $j = 0$. When we developing from $\gamma_2\triangle_1^{(j)}$ to $\gamma_2\triangle_1^{(j+1)}$, and from $\triangle_1^{(j)}$ to $\triangle_1^{(j+1)}$, in both cases we are developing out one further vertex position, the position determined by the positions of the triangles we are developing from and the dihedral angle. The dihedral angle is the same in both cases because the corresponding dihedral angle in $\mathcal{F}$ is the same. Since elements of $\text{PSL}_2(\mathbb{C}((\zeta)))$ preserve cross ratios, the positions of the two new vertices are related in the appropriate way by $I_{(C,\gamma)}$.

A similar argument shows that the representation we get is independent (up to conjugation) of the triangle $\triangle$ we choose to start developing from.

The following definition closely follows an argument of Tillmann [4].

**Definition 5.3.** Let $T_1, \ldots, T_h$ be the torus boundary components of $M$, and so $\partial M = \bigcup T_i$. A vertex $v \in \mathcal{V}$ is stabilised by a unique subgroup $P_v \subset \pi_1 M$
which is conjugate to \( \text{im}(\pi_1T_i \to \pi_1M) \) for some \( i \). Let \( v_1, \ldots, v_h \) be a choice of such vertices, one for each of the tori. Let \( \rho: \pi_1M \to \text{PSL}_2(\mathbb{C}) \) be any representation. The subgroup \( \rho(P_{v_i}) \subset \text{PSL}_2(\mathbb{C}) \) fixes either one or two points of \( \partial \mathbb{H}^3 \), or the whole of \( \partial \mathbb{H}^3 \) if \( \rho(P_{v_i}) = \{1\} \). For each \( i \), choose a \( w_i \in \partial \mathbb{H}^3 \) which is fixed by \( \rho(P_{v_i}) \). We define a map

\[
\Psi_\rho: \tilde{V} \to \partial \mathbb{H}^3
\]

by extending to all other vertices equivariantly.

Compare with definition 5.13. \( \Psi_\rho \) depends on the choices of fixed point \( w_i \), but the maps are otherwise well defined up to conjugation of \( \mathbb{H}^3 \). There are at most \( 2^h \) choices of \( \Psi_\rho \) unless \( \rho(P_{v_i}) = \{1\} \) for some \( v_i \), in which case there are infinitely many choices.

**Definition 5.4.** Let \( \rho \in \mathcal{R}(M) \) and fix a choice of \( \Psi_\rho \). We associate a horo-normal surface

\[
S(\Psi_\rho)
\]

to \( \Psi_\rho \) as follows: Let \( \mathcal{E} \) be the edge set of \( \mathcal{I} \). Consider the images of the endpoints of edges in \( \tilde{\mathcal{E}} \) under \( \Psi_\rho \). Let \( \tilde{\mathcal{E}}_0 \) be the set of such edges whose endpoints map to the same point of \( \partial \mathbb{H}^3 \). By the equivariance of \( \Psi_\rho \), \( \tilde{\mathcal{E}}_0 \) descends to \( \mathcal{E}_0 \subset \mathcal{E} \). Let \( \mathcal{E}_+ = \mathcal{E} \setminus \mathcal{E}_0 \). Let \( S(\Psi_\rho) \) be the horo-normal surface associated with this choice of \( \mathcal{E}_0 \) and \( \mathcal{E}_+ \) as in definition 3.4.

**Lemma 5.5.** The choice of \( \mathcal{E}_0 \) in the above definition has no bad loops.

**Proof.** Suppose for contradiction that this choice of \( \mathcal{E}_0 \) has a bad loop. Then by definition of a bad loop (definition 3.4) all but one edge of the loop is in \( \mathcal{E}_0 \). Let the loop be given by edges \( e_1, e_2, \ldots, e_n \in \tilde{\mathcal{E}}_0 \), where \( e_i \) has endpoints \( v_i, v_{i+1} \in \tilde{V} \) for \( i = 1, 2, \ldots, n-1 \) and \( e_n \) has endpoints \( v_n, v_1 \). Suppose that \( e_1, e_2, \ldots, e_{n-1} \in \tilde{\mathcal{E}}_0 \) and \( e_n \in \tilde{\mathcal{E}}_+ \). Then \( \Psi_\rho(v_1) = \Psi_\rho(v_2) = \cdots = \Psi_\rho(v_n) \), but \( \Psi_\rho(v_n) \neq \Psi_\rho(v_1) \), which is our contradiction. \( \square \)

**Definition 5.6.** If \( A \) is an abelian group, the **generalised dihedral group** of \( A \) is the semidirect product \( A \rtimes \mathbb{Z}_2 \) with \( \mathbb{Z}_2 \) acting on \( A \) by inverting elements.

**Lemma 5.7.** If \( \rho \in \mathcal{R}(M) \) is irreducible then either \( \rho(\pi_1M) \) is a generalised dihedral group or every \( \Psi_\rho \) is such that \( |\Psi_\rho(\tilde{V})| \geq 3 \).

**Proof.** We use the same notation as in definition 5.3. Let \( w_i \) be a point fixed by \( \rho(P_{v_i}) \), so \( w_i \in \Psi_\rho(\tilde{V}) \). Let \( \text{fix}(w_i) \subset G = \rho(\pi_1M) \) be the set of isometries that have \( w_i \) as a fixed point. \( \rho \) is irreducible, which means that no point of \( \partial \mathbb{H}^3 \) is fixed by all of \( G \), and in particular \( \text{fix}(w_i) \) is a proper subset of \( G \).

Either there are at least three translates of \( w_i \) (and we have \( |\Psi_\rho(\tilde{V})| \geq 3 \)) or every element of \( G \setminus \text{fix}(w_i) \) is of order 2, taking \( w_i \) to some \( w_i' \neq w_i \) and back. In this case we must also have \( \text{fix}(w_i') = \text{fix}(w_i) \), otherwise if we could move one
without moving the other, we would obtain a third point. Thus the subgroup \(\text{fix}(w_i)\) is abelian, since it fixes two distinct points on \(\partial \mathbb{H}^3\).

If we arrange the fixed points at 0 and \(\infty\) on \(\partial \mathbb{H}^3\) then every element \(a \in \text{fix}(w_i)\) is diagonal and every element \(r \in G \setminus \text{fix}(w_i)\) is anti-diagonal. One can verify that \(rar = a^{-1}\), and so \(G = \text{fix}(w_i) \rtimes \mathbb{Z}_2\) is a generalised dihedral group.

**Example 5.8.** If \(M\) is a knot complement in \(S^3\) then for any \(\rho \in \mathcal{R}(M)\), \(\rho(\pi_1 M)\) is not a generalised dihedral group, because the abelianisation of a knot group is always \(\mathbb{Z}\), whereas every element in the abelianisation of a generalised dihedral group is of order 2 and the abelianisation has at least two factors of \(\mathbb{Z}_2\). Therefore the knot group could not have surjected onto the generalised dihedral group.

**Theorem 5.9.** Let \(M\) be a connected 3-manifold with non-empty boundary consisting of a disjoint union of tori and \(\mathcal{T}\) an ideal tetrahedralisation of \(M\). Let \(\rho \in \mathcal{R}(M)\) such that there is a choice of \(\Psi_\rho\) with \(|\Psi_\rho(\tilde{V})| \geq 3\) and \(S(\Psi_\rho)\) omnipresent. Then there exists \(Z_\rho \in \hat{\mathcal{D}}(M; \mathcal{T}; S(\Psi_\rho))\) such that \(R(\mathcal{T}; S(\Psi_\rho))(Z_\rho) = \rho\) up to conjugation.

**Proof.** By assumption \(S(\Psi_\rho)\) is omnipresent. Now suppose for contradiction that condition 3 of definition 3.11 fails. Then all tetrahedra are of types 22, 31 or 4. They cannot all be of type 4 since then \(|\Psi_\rho(\tilde{V})| = 1\). \(R_{in}\) is connected by condition 1 of omnipresence and intersects edges of every vertex in \(\tilde{V}\) by condition 2, so we can follow chains of triangles that contiguously intersect \(R_{in}\) starting from some triangle \(\triangle\) of type 21, and going to each vertex. Every triangle we move through is of type 21, and we see that the vertices of \(\tilde{V}\) fall into two sets, those that are connected by paths of edges in \(\tilde{E}_0\) to either the pair of vertices of \(\triangle\) connected by an edge of \(\tilde{E}_0\), or the other vertex of \(\triangle\). Vertices from these two sets are never connected by an edge of \(\tilde{E}_0\), since that would give a bad loop. Thus in this case \(|\Psi_\rho(\tilde{V})| = 2\).

So \(S(\Psi_\rho)\) is a horo-normal surface that satisfies all of the conditions of definition 3.11. We now need to construct \(Z_\rho \in \hat{\mathcal{D}}(M; \mathcal{T}; S(\Psi_\rho))\) such that \(R(\mathcal{T}; S(\Psi_\rho))(Z_\rho) = \rho\) up to conjugation. Because of the definition of \(\mathcal{R}(\mathcal{T}; S(\Psi_\rho))\) (definition 5.1) and the fact that elements of \(\text{PSL}_2(\mathbb{C})\) are determined by their action on 3 distinct points of \(\partial \mathbb{H}^3\), it is enough to construct \(Z_\rho\) so that \(\Psi_\rho = \Phi_{Z_\rho}\) as maps \(\tilde{V} \rightarrow \partial \mathbb{H}^3\).

Fix a conjugation of \(\mathbb{H}^3\) so that \(\infty \notin \Psi_\rho(\tilde{V})\). \(\Psi_\rho\) determines the position on \(\partial \mathbb{H}^3\) of every vertex of \(\tilde{V}\), and each edge of \(\tilde{E}_0\) has both endpoints in the same position. To determine the data (see definition 3.7) for a tetrahedra of type 1111 in \(\mathcal{T}\) we simply read off the cross ratio given by the positions of the vertices of one of its lifts in \(\mathcal{T}\). The answer we get is independent of the choice.
of lift since elements of $\text{PSL}_2(\mathbb{C})$ preserve cross ratios and lifts are taken to each other by deck transformations $\gamma \in \pi_1 M$, and their images in $\partial \mathbb{H}^3$ taken to each other by $\rho(\gamma) \in \text{PSL}_2(\mathbb{C})$. We will use a similar construction to deal with the degenerate tetrahedra:

For each $e_i \in E_0$, arbitrarily choose a lift $\tilde{e}_i \in \tilde{E}_0$. We also arbitrarily choose an **offset** $\delta_i \in \mathbb{C} \setminus \{0\}$ to $\tilde{e}_i$, viewing it as a directed edge between its endpoints $u_i, v_i \in \tilde{V}$. The idea is that although $\Psi_\rho(u_i) = \Psi_\rho(v_i)$, we want to introduce some extra information to talk about the difference between the two positions, namely that the difference will be $\delta_i \zeta \in \mathbb{C}[\zeta]$. We extend the choice of offset to all other lifts of edges in $E_0$ by conjugation: if $\tilde{e}_i$ and $\tilde{e}_i'$ are lifts of $e_i \in E_0$ with endpoints $u_i, v_i, u_i', v_i'$ then there is some deck transformation $\gamma \in \pi_1 M$ such that $\gamma \tilde{e}_i = \tilde{e}_i'$. If $\Psi_\rho(u_i) = \Psi_\rho(v_i) = x \in \mathbb{C}$ then $\rho(\gamma)(x) = \Psi_\rho(u_i') = \Psi_\rho(v_i')$. If

$$\rho(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant 1 then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x + \delta_i \zeta \\ 1 \end{pmatrix} = \begin{pmatrix} ax + a \delta_i \zeta + b \\ cx + c \delta_i \zeta + d \end{pmatrix}$$

and

$$\frac{a x + a \delta_i \zeta + b}{c x + c \delta_i \zeta + d} = \frac{a x + b}{c x + d} + \frac{\delta_i \zeta}{(c x + d)^2} + (\text{h.o.t. in } \zeta)$$

$$\frac{a x + b}{c x + d} = \Psi_\rho(u_i') = \Psi_\rho(v_i')$$

and we take the offset for $\tilde{e}_i'$ to be $\frac{a x + b}{c x + d} \zeta \in \mathbb{C} \setminus \{0\}$. One can verify that this choice is consistent in that we get the same answer under action by products of elements of $\text{PSL}_2(\mathbb{C})$. Note also that the only element of $\pi_1 M$ that fixes an edge of $\tilde{E}$ is the identity element, which of course fixes the offset.

We can also see the consistency as follows: consider four points $x + \delta_i \zeta, y, x, w \in \mathbb{C}[\zeta]$, where $x, y, w \in \mathbb{C}$ are distinct. The cross ratio $z$ of these four points is preserved under elements of $\text{PSL}_2(\mathbb{C}) \subset \text{PSL}_2(\mathbb{C}(\zeta))$, and by lemma 2.5

$$z_* = \frac{\delta_i (y - w)}{(x - w)(y - x)}$$

So $\delta_i$ is determined by $z_*$ and $x, y, w$. If we then apply some combination of elements of $\text{PSL}_2(\mathbb{C})$ then $z_*$ stays fixed and our new $\delta_i$ is determined by the new positions for $x, y, w$, which are independent of the combination of elements of $\text{PSL}_2(\mathbb{C})$ that take us here.

So we have a $\delta_i$ assigned to each $\tilde{e}_i \in \tilde{E}_0$. We use these and lemma 2.5 to read off the lowest order information of the preferred cross ratios for tetrahedra of types 211, and 22. See figure 23. Again the answer we get is independent of the choice of lift of tetrahedron. For tetrahedra of type 31 we only care about
the dihedral angle between pairs of triangles that meet at an edge in $\tilde{E}_+$. We can read this cross ratio off as

$$z = \frac{(x - (x + \delta_j \zeta))(w - (x + \delta_j \zeta))}{(x - (x + \delta_i \zeta))(w - (x + \delta_j \zeta))} = \frac{\delta_j (w - (x + \delta_j \zeta))}{\delta_i (w - (x + \delta_j \zeta))}$$

so $z_* = \delta_j / \delta_i$.

Notice that we do not require that the offset of the third edge of the 31 tetrahedron together with the first two link up to form a triangle. We track the first order offset (difference) between two points with the same position on $\partial \mathbb{H}^3$, but not the absolute first order positions.

We now have the data for a point of the extended deformation variety, we need to show that these choices satisfy the consistent development condition. Suppose that we have two triangles $\triangle_1$ and $\triangle_2$ which share an edge in $E_+$, the positions on $\partial \mathbb{H}^3$ of the vertices of $\triangle_1$ as given by $\Psi_\rho$, and the offset for any edge of $\triangle_1$ in $\tilde{E}_0$, together with the cross ratio data for the dihedral angle. Then lemma 2.6 tells us that we can recover the position on $\partial \mathbb{H}^3$ of the vertex of $\triangle_2$ not shared with $\triangle_1$, and any offsets for edges of $\triangle_2$ in $\tilde{E}_0$. As we develop through valid chains we always get the correct answer (agreeing with $\Psi_\rho(\mathring{V})$, and with our offsets) no matter which chain of triangle we develop along, so we get consistent development. So if we start developing from a triangle with vertex positions agreeing with $\Psi_\rho$, we get $\Psi_\rho = \Phi_{Z_\rho}$ as maps $\mathring{V} \rightarrow \partial \mathbb{H}^3$. □

**Theorem 5.10.** Let $M$ be a connected 3-manifold with non-empty boundary consisting of a disjoint union of tori and $\mathcal{T}$ an ideal tetrahedralisation of $M$. Let $\rho \in \mathcal{R}(M)$ be irreducible such that $\rho(\pi_1 M)$ is not a generalised dihedral group. Then there exists a tetrahedralisation $\mathcal{T}^*$, a surface $S$ in horo-normal...
form relative to $T^*$ and $Z_\rho \in \widehat{D}(M; T^*; S)$ such that $R_T(Z_\rho) = \rho$, up to conjugation.

**Proof.** Use theorem 4.13 to modify $T$ to $T'$, such that the descendants of $S(\Psi_\rho)$ are omnipresent. One of the descendants of $S(\Psi_\rho)$ will correspond to $\Psi_\rho$ relative to $T'$. By lemma 5.7 $|\Psi_\rho(\tilde{V})| \geq 3$. Now apply theorem 5.9. □

**Theorem 5.11.** Let $M$ be a connected 3-manifold with non-empty boundary consisting of a disjoint union of tori and suppose that $M$ admits an ideal tetrahedralisation. Then there exists an ideal tetrahedralisation $T_\ast$ of $M$ such that for every irreducible $\rho \in \mathcal{R}(M)$ such that $\rho(\pi_1 M)$ is not a generalised dihedral group, $\rho$ is in the image under $\mathcal{R}_T$ of $\widehat{D}(M; T^*)$, up to conjugation.

**Proof.** By theorem 4.14 we obtain $T_\ast$, in which every horo-normal surface is omnipresent. Given $\rho$ with the above properties, we obtain $S(\Psi_\rho)$, which must be omnipresent and by lemma 5.7 $|\Psi_\rho(\tilde{V})| \geq 3$. Now apply theorem 5.9. □

6 Example, part 2

We return to the example of section 1.2. The component satisfying $ij = 1$ in $\mathcal{D}(M_{LLR}; T_4)$ has $hk = 1$, so the top right diagram of figure 2 has all of the “equatorial” dihedral complex angles being 1. (The back dihedral angle is obviously 1, the other two dihedral angles turn out to be 1 via the equations internal to each tetrahedron, (1) and (2).) Then the north and south vertices of the two tetrahedra are in the same place on $\partial \mathbb{H}^3$, and so the added edge $e$ in $T_5$ is the single edge in $E_0$. The corresponding horo-normal surface $S$ is shown in figures 24 and 25.

As in figure 25, all tetrahedra are of types 1111 or 211, and so $\widetilde{R}_m$ is connected. (If there are no tetrahedra of type 31 or 4 then all triangles are of type 111 or 21, and so $\widetilde{R}_m$ connects through the center of each triangular face of $T_\ast$.) In this example, if we perform one compression move to the surface $S$ we obtain a boundary parallel torus, and consistent development for $\mathcal{D}(M_{LLR}; T_5; S)$ is achieved if we have the gluing equations (or rather, lowest order versions, using angles as in figure 12) for each edge apart from $e \in E_0$, together with consistency for a chain of triangles going around $e$. The gluing equation (13) is gone, the gluing equations (14) through (17) become:

\begin{align*}
ijq\zeta(-q^{-1}\zeta^{-1})r\zeta(-r^{-1}\zeta^{-1}) &= 1 \\
\frac{i-1}{i}j(-p^{-1}\zeta^{-1})q\zeta &= 1 \\
\frac{i-1}{i-j}j(-p^{-1}\zeta^{-1})q\zeta &= 1 \\
\frac{i-1}{1-i}j(-p^{-1}\zeta^{-1})j1 &= 1 \\
j(-p^{-1}\zeta^{-1})1r\zeta(-r^{-1}\zeta^{-1}) &= 1
\end{align*}

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Figure 24: Tetrahedra incident to one fundamental domain of the boundary torus, as seen in perspective. Only some of the vertical edges are shown (each vertex below has an edge above it going towards the vertex at infinity). The dotted lines are all translates of the one edge \( e \in E_0 \). Shown are some of the pieces of the horo-normal surface: tubes made from quadrilaterals around the vertical lifts of \( e \) and triangles nearest the vertex at infinity.

which simplify (caring only about lowest order) to:

\[
\begin{align*}
ij &= 1 \\
\frac{i - 1}{i} \left(-\frac{q}{p}\right) &= 1 \\
\frac{i}{i - 1} \left(-\frac{p}{q}\right) &= 1 \\
\frac{1}{ij} &= 1
\end{align*}
\]

(35)  
(36)  
(37)  
(38)

There are obvious redundancies here. To see the consistency for a chain of triangles going around \( e \), consider developing around the left hand vertical tube in figure 24 There are three triangles, all of which share the vertex at infinity. The
Figure 25: The view from above. Shown here are all parts of the horo-normal surface. Each edge of $E_+$ intersects the surface twice, the edge $e \in E_0$ (dotted) is disjoint from the surface. Tetrahedra labelled with dihedral angles $i$ and $j$ are of type 1111, $p\zeta, q\zeta$ and $r\zeta$ are type 211.

Dihedral angles between those triangles are $p\zeta(-r^{-1}\zeta^{-1}) = -\frac{p}{r}, q\zeta(-p^{-1}\zeta^{-1}) = -\frac{q}{p}, r\zeta(q^{-1}\zeta^{-1}) = -\frac{r}{q}$. The three triangles are three faces of a tetrahedron (in fact the tetrahedron labelled $h$ in $\mathcal{T}_4$, from before the 2-3 move, see figure 4), and consistent development around these triangles is the same as equations (1) and (2) for that tetrahedron. The first is satisfied automatically, and the second simplifies to the last equation we need for $\hat{D}(M_{LLR}; \mathcal{T}_5; S)$:

$$p + q + r = 0 \quad (39)$$

Remark 6.1. We have three independent equations in five variables, and so this variety is 2-dimensional, whereas the corresponding component of the deformation variety with triangulation $\mathcal{T}_4$ is 1-dimensional. The extra dimension comes from the choices of $p, q$ and $r$, all of which can be scaled by some constant at once to give another point of $\hat{D}(M_{LLR}; \mathcal{T}_5; S)$, and the scaled and original points map to the same representation of $\mathcal{R}(M)$.

7 Further questions and directions

1. How should we compactify the extended deformation variety, similarly to Tillmann’s compactification of the standard deformation variety in $\mathbb{H}^2$?
2. If we can solve the previous question, how much of the Culler-Shalen machinery can we reproduce in the context of tetrahedralisations? The set of ideal points of the extended deformation variety for a 3-manifold with a tetrahedralisation with all horo-normal surfaces omnipresent should contain ideal points corresponding to each ideal point of the character variety (for components not made up of reducible representations, or made up of representations with image a generalised dihedral group).

3. Are there manifolds for which the standard deformation variety for every tetrahedralisation “misses” some component seen by the extended deformation variety? It may not generally be easy to find a tetrahedralisation for which the standard deformation variety sees everything that the extended deformation variety does, but we have yet to find an example for which there is no such tetrahedralisation.

4. Can we make an analogous generalisation of an angle structure, replacing complex dihedral angles in the extended deformation variety with real dihedral angles?

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