Spherical functions and local densities on the space of $p$-adic quaternion hermitian matrices

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§0 Introduction

Let $G$ be a reductive algebraic group and $X$ a $G$-homogeneous affine algebraic variety, where everything is assumed to be defined over a $p$-adic field $k$. We denote by $G$ and $X$ the sets of $k$-rational points of $G$ and $X$, respectively. Taking a maximal compact subgroup $K$ of $G$, we consider the Hecke algebra $\mathcal{H}(G, K)$, that is the commutative $\mathbb{C}$-algebra generated by the characteristic functions of $KgK$, $g \in G$. Then, a nonzero $K$-invariant function on $X$ is called a spherical function on $X$ if it is an $\mathcal{H}(G, K)$-common eigenfunction.

Spherical functions on homogeneous spaces comprise an interesting topic to investigate and a basic tool to study harmonic analysis on $G$-space $X$. Spherical functions on the spaces of sesquilinear forms are particularly interesting, since they can be regarded as generating functions of local densities of representations of such forms. For the cases of alternating forms of size $2n$ and unramified hermitian forms of size $n$, the main terms of the explicit formulas of spherical functions are related to Hall-Littlewood symmetric polynomials of type $A_n$, which are well studied, hence it is possible to extract local densities of forms (cf. [HS1], [H2]). For the space of unitary hermitian forms of size $m$, the main terms of the explicit formulas are related to Hall-Littlewood polynomials of type $C_n$, where $m = 2n$ or $m = 2n + 1$, according to the parity of $m$, and the unitary group acting on $X$ is of type $C_n$ or $BC_n$, respectively (cf. [HK1], [HK2], [H5]).

In the present paper, we introduce the space $X$ of quaternion hermitian forms of size $n$ on a $p$-adic field $k$ and study spherical functions on it, where we assume $k$ has odd residual characteristic. In §1, we introduce Cartan decomposition of $X$ due to Jacobowitz and define typical spherical functions $\omega(x; s)$ on $X$. In §2, we introduce local densities of representations within quaternion hermitian forms, and give an induction theorem of spherical functions using local densities (Theorem 2.2). By this theorem, we may regard spherical functions as generating functions of local densities, and we give the explicit value of the local density of itself (Theorem 2.3). Then we introduce the normalized Fourier transform $F_0$ on the Schwartz space $\mathcal{S}(K\setminus X)$, which is an injective $\mathcal{H}(G, K)$-module map (Proposition 2.12). In §3, we consider the functional equations and location of possible poles and zeros of $\omega(x; s)$ (Theorem 3.4). Then we introduce the normalized Fourier transform.
transform $F$ by modifying $F_0$, which gives an inclusion of $\mathcal{S}(K \setminus X)$ into the symmetric Laurent polynomial ring $\mathcal{R} = \mathbb{C}[q^{\pm 1}, \ldots, q^{\pm n}]^{S_n}$, where $\mathcal{R}$ is isomorphic to $\mathcal{H}(G, K)$ by Satake transform (Theorem 3.3). In §4, we give the explicit formulas of $\omega(x; s)$ by a general method introduced in [H2], [H4] (Theorem 4.1). In this case, we obtain a different kind of symmetric polynomials as the main terms of explicit formulas from those of other sesquilinear forms (cf. Remark 4.3). In §5, we study $\mathcal{S}(K \setminus X)$ more precisely for small $n$. In §5.1, for size $n \leq 4$, we determine the $\mathcal{H}(G, K)$-module structure of $\mathcal{S}(K \setminus X)$ and show the dimension of spherical functions on $X$ associated to general $z$ is equal to 1. In §5.2, we introduce the Plancherel measure for size 2 proved by Yasushi Komori and give the inversion formula.

§1 The space $X$ and spherical functions on it

Let $k$ be a $p$-adic field, and denote by $\mathfrak{o}$ the ring of integers, $\pi$ a fixed prime element, $\mathfrak{p} = \pi\mathfrak{o}$, and $q$ the cardinality of $\mathfrak{o}/\mathfrak{p}$. Throughout this paper we assume $k$ has odd residual characteristic. Set $D$ be a division quaternion algebra over $k$, $\mathcal{O}$ the maximal order in $D$, and $\mathcal{P}$ the maximal ideal in $\mathcal{O}$. Then there is an unramified quadratic extension $k'$ of $k$ in $D$, for which $k' = k(\epsilon)$, $\epsilon^2 \in \mathfrak{o}^\times$ and we may take a prime element $\Pi$ of $D$ such that $\Pi^2 = \pi$, $\Pi \epsilon = -\epsilon \Pi$. Then the set $\{1, \epsilon, \Pi, \Pi \epsilon\}$ forms a basis for $\mathcal{O}/\mathfrak{o}$ with the involution $* \Delta \Pi$ defined by

$$\alpha = a + b \epsilon + c \Pi + d \Pi \epsilon \mapsto \alpha^* = a - b \epsilon - c \Pi - d \Pi \epsilon, \quad (a, b, c, d \in k), \quad (1.1)$$

and $\alpha \alpha^* \in k$. There is a $k$-algebra inclusion $\varphi : D \rightarrow M_2(k')$ such that

$$\varphi(\alpha) = \begin{pmatrix} a + b \epsilon & (c - de) \pi \\ c + d \epsilon & a - b \epsilon \end{pmatrix} \text{ determined by } \alpha(1 \Pi) = (1 \Pi) \varphi(\alpha), \quad (1.2)$$

$$\det(\varphi(\alpha)) = \alpha \alpha^* = N_{rd}(\alpha) \in k, \quad (1.3)$$

$$\text{trace}(\varphi(\alpha)) = \alpha + \alpha^* = T_{rd}(\alpha) \in k, \quad (1.4)$$

where $\alpha$ is written as in (1.1), $N_{rd}$ is the reduced norm and $T_{rd}$ is the reduced trace on $D$. Based on $\varphi$, we have a $k$-algebra inclusion $\varphi_n : M_n(D) \rightarrow M_{2n}(k')$, and the reduced norm and the reduced trace of an element of $A \in M_n(D)$ give by

$$N_{rd}(A) = \det(\varphi_n(A)), \quad T_{rd}(A) = \text{trace}(\varphi_n(A)) \in k. \quad (1.5)$$

In particular, we see

$$N_{rd}(a) = \det(a)^2, \quad T_{rd}(a) = 2\text{trace}(a), \quad \text{for } \ a \in M_n(k). \quad (1.6)$$

Since $N_{rd}$ and $T_{rd}$ do not depend on the choice of splitting fields of $D$, we may use another $k$-algebra inclusion $\varphi'_n : M_n(D) \rightarrow M_{2n}(k(\Pi))$ based on

$$\varphi'(\alpha) = \begin{pmatrix} a + c \Pi & (b + d \Pi) \epsilon^2 \\ b - d \Pi & a - c \Pi \end{pmatrix} \in M_2(k(\Pi)), \quad \alpha(1 \epsilon) = (1 \epsilon) \varphi'(\alpha), \quad (1.7)$$
One may refer for the above facts to Reiner’s book [Re, §9, 13, 14].

We extend the involution \( * \) on \( A = (a_{ij}) \in M_{nn}(D) \) by \( A^* = (a_{ji}^*) \in M_{nn}(D) \). We define the space \( X_n \) of quaternion hermitian forms and the action of \( G_n = GL_n(D) \) as follows

\[
X_n = \{ A \in G_n \mid A^* = A \},
\]

\[
g \cdot A = gAg^* = A[g^*], \quad \text{for} \quad (g, A) \in G_n \times X_n.
\]

Denote by \( K_n \) the maximal order in \( G_n \), i.e., \( K_n = G_n(\mathcal{O}) \). Then, it is known ([Jac, Theorem 6.2]) that the set \( K_n \setminus X_n \) of \( K_n \)-orbits in \( X_n \) is bijectively correspond to \( \Lambda_n \), where

\[
\tilde{\Lambda}_n = \{ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \},
\]

\[
\Lambda_n = \left\{ \alpha \in \tilde{\Lambda}_n \mid \text{if } \alpha_i \text{ is odd, then } \#\{ j \mid \alpha_j = \alpha_i \} \text{ is even} \right\}.
\]

In fact, writing \( \alpha \in \Lambda_n \) as

\[
\alpha = (\gamma_1, \ldots, \gamma_1, \ldots, \gamma_r, \ldots, \gamma_r), \quad \gamma_1 > \cdots > \gamma_r, \quad m_j > 0, \quad \sum_j m_j = n,
\]

one may take the matrix \( \pi^\alpha = \langle \pi_1^{m_1} \rangle \cdots \langle \pi_r^{m_r} \rangle \in X_n \), where

\[ \langle \pi_1^{m_1}, \ldots, \pi_r^{m_r} \rangle = \left\{ \begin{array}{ll}
\text{Diag}(\pi^{m_1}, \ldots, \pi^{m_r}) & \text{if } \gamma = 2e \\
0 \quad \pi^{m \Pi} & \pi^{m \Pi} 0 & \text{if } \gamma = 2e + 1
\end{array} \right\} \in X_m. \]

Set \( \Lambda_n^+ = \{ \alpha \in \Lambda_n \mid \alpha_n \geq 0 \} \) and \( X_n^+ = X_n \cap M_n(\mathcal{O}) \). Then

\[ X_n^+ = \bigcup_{\alpha \in \Lambda_n^+} K_n \cdot \pi^\alpha. \]

It is easy to see

\[ N_{rd}(\pi^\alpha) = \pi^{[\alpha]}, \quad |\alpha| = \sum_{i=1}^n \alpha_i \in 2\mathbb{Z}, \quad (\alpha \in \Lambda_n) \]

hence we have

\[ N_{rd}(x) \in k^2, \quad \text{for } x \in X_n. \]

For \( g \in G = G_n \), we denote by \( g^{(i)} \) the upper left \( i \times i \)-block of \( g \), \( 1 \leq i \leq n \). We take the Borel subgroup \( B = B_n \) of \( G \) consisting of lower triangular matrices. Then for \( (p, x) \in B \times X_n \), we have

\[ N_{rd}(p^{(i)} \cdot x^{(i)}) = N_{rd}(p^{(i)}, x^{(i)}), \quad \psi_i(p) = N_{rd}(p^{(i)}), \quad 1 \leq i \leq n. \]

Thus, for \( x \in X = X_n \), we may define \( d_i(x) \in k \) by \( d_i(x)^2 = N_{rd}(x^{(i)}) \), \( 1 \leq i \leq n \). Then, \( d_i(x) \) is a \( B \)-relative invariant associated with \( k \)-rational character \( \psi_i \), \( 1 \leq i \leq n \). For \( x \in X \) and \( s \in \mathbb{C}^n \), we consider the integral

\[ \omega(x; s) = \int_{K_n} |d(k \cdot x)|^s dk, \quad |d(y)|^s = \left\{ \begin{array}{ll}
\prod_{i=1}^n |d_i(y)|^{s_i} & \text{if } y \in X^{op} \\
0 & \text{otherwise},
\end{array} \right. \]
where $dk$ is the normalized Haar measure on $K = K_n$, $||$ is the absolute value on $k$ and

$$X_n^\text{op} = \{x \in X_n \mid d_i(x) \neq 0, \text{ for all } 1 \leq i \leq n\}.$$  \hspace{1cm} (1.17)

The integral in (1.16) is absolutely convergent if $\text{Re}(s_i) \geq 0$, $1 \leq i \leq n - 1$, and continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$ (cf. [H2, Remark 1.1]), where $s_n$ is free because $|d_n(k \cdot x)| = |x|$ for $k \in K_n$. Then it becomes an element of

$$C^\infty(K \backslash X) = \left\{\Psi : X \longrightarrow \mathbb{C} \mid \Psi(k \cdot x) = \Psi(x), \ k \in K\right\},$$ \hspace{1cm} (1.18)

and we use the notation $\omega(x; s)$ in such sense. Denote by $\mathcal{H}(G, K)$ the Hecke algebra of $G$ with respect to $K$. We recall the action of $\mathcal{H}(G, K)$ on $C^\infty(K \backslash X)$:

$$f \ast \Psi(x) = \int_G f(g)\Psi(g^{-1} \cdot x)dg, \quad (f \in \mathcal{H}(G, K), \ \Psi \in C^\infty(K \backslash X), \ x \in X),$$ \hspace{1cm} (1.19)

where $dg$ is the normalized Haar measure on $G$. We call $\omega(x; s)$ a spherical function on $X$, since it is a common eigenfunction with respect to the above action of $\mathcal{H}(G, K)$, in fact

$$(f \ast \omega(\ ; s))(x) = \lambda_s(f)\omega(x; s), \quad (f \in \mathcal{H}(G, K)).$$ \hspace{1cm} (1.20)

Here $\lambda_s$ is the $\mathbb{C}$-algebra map

$$\lambda_s : \mathcal{H}(G, K) \longrightarrow \mathbb{C}(q^{s_1}, \ldots, q^{s_n}),$$

$$f \longmapsto \int_B f(p) \prod_{i=1}^n |\psi_i(p)|^{-s_i} \delta(p)dp,$$ \hspace{1cm} (1.21)

where $dp$ is the left invariant measure on $B_n$ with modulus character $\delta$. The Weyl group $S_n$ of $G$ acts on $\{s_1, \ldots, s_n\}$ through its action on the rational characters $\{|\psi_i|^{s_n} \mid 1 \leq i \leq n\}$. It is convenient to introduce a new variable $z \in \mathbb{C}^n$ related to $s \in \mathbb{C}^n$ by

$$s_i = -z_i + z_{i+1} - 2 \ (1 \leq i \leq n-1), \quad s_n = -z_n + n - 1,$$ \hspace{1cm} (1.22)

and denote $\omega(x; s) = \omega(x; z)$ and $\lambda_n = \lambda_z$. Then $S_n$ acts on $\{z_1, \ldots, z_n\}$ by permutation of indices, and the $\mathbb{C}$-algebra map $\lambda_z$ is the Satake isomorphism

$$\lambda_z : \mathcal{H}(G, K) \sim \mathbb{C}[q^{z_1}, \ldots, q^{z_n}]^{S_n}.$$ \hspace{1cm} (1.23)

Because of this isomorphism, all the spherical functions on $X$ are parametrized by eigenvalues $z \in \mathbb{C}^n$ through $\mathcal{H}(G, K) \longrightarrow \mathbb{C}, \ f \longmapsto \lambda_z(f)$, and $\lambda_z$ is determined by the class of $z$ in $\left(\mathbb{C}/\mathbb{Z}\right)^n / S_n$. 

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§2 Local densities and spherical functions

2.1. We will give the induction theorem (Theorem 2.2) of spherical functions by means of local densities, by which we may regard spherical functions as generating functions of local densities of representations. We start with the definition of local densities. For \( A \in X^+_m \) and \( B \in X^+_n \) with \( m \geq n \), we define the local density \( \mu(B, A) \) and primitive local density \( \mu^{pr}(B, A) \) of \( B \) by \( A \) as follows:

\[
\mu(B, A) = \lim_{\ell \to \infty} \frac{N_\ell(B, A)}{q^{\ell n(4m-2n+1)+n(n-1)}}.
\]
\[
\mu^{pr}(B, A) = \lim_{\ell \to \infty} \frac{N^{pr}_\ell(B, A)}{q^{\ell n(4m-2n+1)+n(n-1)}}.
\]

(2.1)

Here

\[
N_\ell(B, A) = \sharp \left\{ u \in M_{mn}(O) / M_{mn}(P^{2\ell}) \mid A[u] - B \in M_n(P^{2\ell-1}) \right\},
\]
\[
N^{pr}_\ell(B, A) = \sharp \left\{ u \in M_{mn}(O/P^{2\ell}) \mid A[u] - B \in M_n(P^{2\ell-1}) \right\},
\]

(2.2)

where we identify \( M_{mn}(O) / M_{mn}(P^{2\ell}) \) with \( M_{mn}(O/P^{2\ell}) \) and denote by \( u \) its element represented by \( u \in M_{mn}(O) \), and an element in \( M_{mn}(O) \) is called primitive if it belongs to the set \( GL_m(O) \left( \begin{smallmatrix} 1_n & \cdot \\ 0 & 0 \end{smallmatrix} \right) \), and we write

\[
M^{pr}_{mn}(O/P^{2\ell}) = \left\{ u \in M_{mn}(O/P^{2\ell}) \mid u \text{ is primitive} \right\}.
\]

(2.3)

Remark 2.1 The above definition is well-defined, since the conditions in (2.2) and (2.3) are independent of the choice of the representative \( u \) of \( \overline{u} \). If \( \ell \) is sufficiently large, then the \( K_n \)-orbit \( K_n \cdot B \) decomposes into a finite union of the set \( B_i + M_n(P^{2\ell-1}) \), and the ratios in the right hand sides of (2.1) becomes stable. This phenomenon is characteristic of local densities for sesquilinear forms (cf. [Ki2], [HS1, §3], [H1, §2]).

We note that, for a matrix \( C = C^* \in M_n(D) \), \( C \) belongs to \( M_n(P^{2\ell-1}) \) if and only if \( C \) belongs to \( H_n(P, \ell) \), where

\[
H_n(P, \ell) = \left\{ A = (a_{ij}) \in M_n(O) \mid A = A^*, a_{ii} \in p^\ell, a_{ij} \in P^{2\ell-1}, (i, j) \right\}.
\]

(2.4)

By definition, \( \omega(x; s) \) takes the same value on the \( K_n \)-orbit containing \( x \in X_n \), further we see that

\[
\omega(\pi^r x; s) = q^{-r \sum s_i} \omega(x; s) = q^{r(z_1 + \cdots + z_n)} \omega(x; s), \quad r \in \mathbb{Z}.
\]

Hence it suffices to show the induction theorem for \( \pi^\xi, \xi \in \Lambda^+_m \).
Theorem 2.2 Let $m > n$ and assume that $\Re(s_i) \geq 0$ for any $1 \leq i \leq n$. Then, for any $\xi \in \Lambda^+_m$, one has

$$\omega(\pi, s_1, \ldots, s_n, 0, \ldots, 0) = \frac{w_n(q^{-2})w_{m-n}(q^{-2})}{w_m(q^{-2})} \times \sum_{\alpha \in \Lambda^+_n} \frac{\mu^{pr}(\pi\alpha, \pi\xi)}{\mu(\pi\alpha, \pi\alpha)} \cdot \omega(\pi\alpha; s_1, \ldots, s_n)$$

$$= \frac{w_n(q^{-2})w_{m-n}(q^{-2})}{w_m(q^{-2})} \prod_{i=1}^{n} (1 - q^{-(s_i + \cdots + s_n + 2m-2i+2)}) \times \sum_{\alpha \in \Lambda^+_n} \frac{\mu(\pi\alpha, \pi\xi)}{\mu(\pi\alpha, \pi\alpha)} \cdot \omega(\pi\alpha; s_1, \ldots, s_n),$$

where $w_m(t) = \prod_{i=1}^{m} (1 - t^i)$.

The above theorem can be proved in a similar way to the case for the other sesquilinear forms, i.e. alternating, hermitian and symmetric forms, so we omit the proof (cf. [HS1, Theorem 5], [H1, §2 Theorem]). For the present case the result is proved in the master thesis of Y. Ohtaka ([OY]) in a slightly different definition, and he used it to study the explicit formula of spherical functions of size 2.

The density $\mu(\pi\alpha, \pi\alpha) = \mu^{pr}(\pi\alpha, \pi\alpha)$ is given as follows, which we will prove in §2.2. In §2.3, we will introduce a spherical transform $F_0$ on the Schwartz space on $X$ and show it is injective by using Theorem 2.2 (Proposition 2.12).

Theorem 2.3 Assume $\alpha \in \Lambda_n$ is given as in (1.11). Then one has

$$\mu(\pi\alpha, \pi\alpha) = q^{2n(\alpha) + \frac{1}{2} |\alpha| + \frac{1}{2} s} \{ \begin{array}{ll} w_{m_j}(-q^{-1}) & \text{if } 2 \mid \gamma_j \\ w_{m_j}(q^{-1}) & \text{if } 2 \not\mid \gamma_j \end{array} \}, \quad (2.5)$$

where

$$n(\alpha) = \sum_{i=1}^{n} (i - 1)\alpha_i, \quad |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

2.2. In the following, $(pr)$ means that the identity holds with and without the condition primitive, respectively.

Proposition 2.4 For $A \in X^+_m$ and $B \in X^+_n$ with $m \geq n$ and $e \in \mathbb{N}$, one has

$$\mu^{(pr)}(\pi^e B, \pi^e A) = q^{en(2n-1)} \mu^{(pr)}(B, A). \quad (2.6)$$

Proof. Assume $\ell$ is sufficiently large, and take $X \in M_{mn}(\mathcal{O})$ such that $A[X] - B \in H_{n}(\mathcal{P}, \ell)$. For any $Y \in M_{mn}(\mathcal{O})$, one has

$$(\pi^e A)[X + \pi^\ell Y] - \pi^e B = \pi^e(A[X] - B) + \pi^{e+\ell}(Y^*AX + X^*AY + \pi^\ell YAY^*) \in H_{n}(\mathcal{P}, e + \ell),$$
and \( X + \pi^e Y \) is primitive if \( X \) is. Hence \( N_{e+\ell}^{(pr)}(\pi^e B, \pi^e A) = q^{4emn} N_{\ell}^{(pr)}(B, A) \), and

\[
\mu^{(pr)}(\pi^e B, \pi^e A) = \lim_{\ell \to \infty} \frac{N_{e+\ell}^{(pr)}(\pi^e B, \pi^e A)}{q^{(e+\ell)n(4m-2n+1)+n(n-1)}}
= \lim_{\ell \to \infty} \frac{N_{\ell}^{(pr)}(B, A) q^{4emn}}{q^{(en(4m-2n+1)+n(n-1)+4emn-en(2n-1))}}
= q^{en(2n-1)} \mu^{(pr)}(B, A).
\]

Remark 2.5 Owing to Proposition 2.4 we may define local density and primitive local density for any \( A \in X_m \) and \( B \in X_n \) with \( m \geq n \) as follows: Taking \( e \in \mathbb{N} \) for which \( \pi^e A \in X_m^+ \) and \( \pi^e B \in X_n^+ \),

\[
\mu^{(pr)}(B, A) = q^{-en(2n-1)} \mu^{\text{local}}(\pi^e B, \pi^e A).
\]

Then, we see that Proposition 2.4 is valid for any \( A \in X_m, B \in X_n, \) and \( e \in \mathbb{Z} \).

Proposition 2.6 Assume \( \alpha \in \Lambda^+_m \) is decomposed as \( \alpha = (\gamma, \beta) \) with \( \beta \in \Lambda^+_n \) and \( \gamma \in \Lambda^+_{m-n} \). Then

\[
\mu(\pi^\alpha, \pi^\alpha) = q^{2(m-n)|\beta|} \mu^{\text{pr}}(\pi^\beta, \pi^\alpha) \mu(\pi^\gamma, \pi^\gamma).
\]

In particular, if \( \gamma_{m-n} > \beta_1 \), then \( \mu^{\text{pr}}(\pi^\beta, \pi^\alpha) = \mu(\pi^\beta, \pi^\beta) \) and

\[
\mu(\pi^\alpha, \pi^\alpha) = q^{2(m-n)|\beta|} \mu(\pi^\beta, \pi^\beta) \mu(\pi^\gamma, \pi^\gamma).
\]

Proof. We use the notation \( \bar{\pi}^\alpha = j_m \cdot \pi^\alpha \), where \( j_m \) is the matrix of size \( m \) such that all the anti-diagonal entries are 1 and other entries are 0. Then \( \bar{\pi}^\alpha = \begin{pmatrix} \bar{\pi}^\beta & 0 \\ 0 & \bar{\pi}^\gamma \end{pmatrix} \), where \( \bar{\pi}^\beta \) and \( \bar{\pi}^\gamma \) are defined similarly. Assume \( \ell \) is sufficiently large, and take

\[
X \in M_{mn}^e(O/\mathcal{P}^{2\ell}) \text{ such that } \pi^\alpha [X] - \bar{\pi}^\beta \in H_n(\mathcal{P}, \ell).
\]

For an extension \( Y = (XZ) \in GL_m(O) \) of \( X \), we have

\[
\pi^\alpha [Y] = \begin{pmatrix} \pi^\alpha [X] & X^* \pi^\alpha Z \\ Z^* \pi^\alpha X & \pi^\alpha [Z] \end{pmatrix}, \quad \pi^\alpha [X] - \bar{\pi}^\beta \in H_n(\mathcal{P}, \ell),
\]

and we may assume that \( X^* \pi^\alpha Z \equiv 0 \pmod{\mathcal{P}^{2\ell}} \) after changing the extension (since \( \beta_1 \leq \gamma_{m-n} \)), then \( \pi^\alpha [Z] \) is \( K_{m-n} \)-equivalent to \( \bar{\pi}^\gamma \). Hence there is an extension \( Y \) of \( X \) such that

\[
\pi^\alpha [Y] - \bar{\pi}^\alpha \in H_m(\mathcal{P}, \ell),
\]

or equivalently

\[
\bar{\pi}^\alpha [Y^{-1}] - \pi^\alpha \in H_m(\mathcal{P}, \ell).
\]
For such extensions $Y_1$ and $Y_2$ of $X$, we see

$$
\tilde{\pi}^\alpha[Y_1^{-1}Y_2] - \tilde{\pi}^\alpha \in H_m(\mathcal{P}, \ell), \quad Y_2 = Y_1 \begin{pmatrix} 1_n & W \\ 0 & V \end{pmatrix} (\text{ in } GL_m(\mathcal{O})).
$$

(2.14)

Since

$$
\tilde{\pi}^\alpha \left[ \begin{pmatrix} 1_n & W \\ 0 & V \end{pmatrix} \right] = \begin{pmatrix} \tilde{\pi}^\beta & \tilde{\pi}^\beta W \\ W^* \tilde{\pi}^\beta & \tilde{\pi}^\beta W + \tilde{\pi}^\gamma [V] \end{pmatrix},
$$

and $\ell$ is large enough, we see the number of extensions $Y$ of type (2.12) for the fixed $X$ as in (2.10) is equal to

$$
\# \left\{ \overline{W} \in M_{n,m-n}(\mathcal{O})/M_{n,m-n}(\mathcal{P}^{2\ell}) \mid \tilde{\pi}^\beta W \equiv 0 \ (\text{mod } \mathcal{P}^{2\ell-1}) \right\} \times N_\ell(\pi^\gamma, \pi^\gamma)
$$

On the other hand, since the number of $\overline{Y} \in M^pr_{\ell}(\mathcal{O}/\mathcal{P}^{2\ell}) \cong GL_m(\mathcal{O}/\mathcal{P}^{2\ell})$ satisfying (2.12) is equal to $N^pr_{\ell}(\pi^\alpha, \pi^\alpha) = N_\ell(\pi^\gamma, \pi^\gamma)$, we see

$$
\mu(\pi^\alpha, \pi^\alpha) = q^{-\ell m(2m+1) - m(m-1)} N_\ell(\pi^\alpha, \pi^\alpha)
$$

$$
= q^{-\ell m(2m+1) - m(m-1) + 2(m-n)(\beta + 2n(m-n))} N^pr_{\ell}(\pi^\beta, \pi^\alpha) N_\ell(\pi^\gamma, \pi^\gamma)
$$

$$
= q^{2(m-n)/|\beta|} \cdot N^pr_{\ell}(\pi^\beta, \pi^\alpha) \times N_\ell(\pi^\gamma, \pi^\gamma)
$$

$$
= q^{2(m-n)/|\beta|} \cdot N^pr_{\ell}(\pi^\beta, \pi^\alpha) \cdot N_\ell(\pi^\gamma, \pi^\gamma).
$$

Next, assume $\beta_1 < \gamma_{m-n}$. For any $V \in M_{m-n,n}(\mathcal{O})$, there is $W \in K_n = GL_n(\mathcal{O})$ such that $\tilde{\pi}^\beta [W] = \pi^\beta - \pi^\gamma [V]$, since $\pi^\beta - \pi^\gamma [V]$ is $K_n$-equivalent to $\pi^\beta$. Then

$$
\begin{pmatrix} V \\ W \end{pmatrix} \in M^pr_{n,m}(\mathcal{O}) \text{ and } \pi^\alpha \left[ \begin{pmatrix} V \\ W \end{pmatrix} \right] \equiv \pi^\beta \pmod{H_n(\mathcal{P}, \ell)},
$$

and the number of choice of such $\overline{W} \in M_n(\mathcal{O}/\mathcal{P}^{2\ell})$ is equal to $N_\ell(\pi^\beta, \pi^\beta)$. Hence, if $\beta_1 < \gamma_{m-n}$, one has

$$
\mu^pr_{\ell}(\pi^\beta, \pi^\alpha) = q^{-\ell n(2m-n+1) - n(n-1)} N^pr_{\ell}(\pi^\beta, \pi^\alpha)
$$

$$
= q^{-\ell n(2m-n+1) - n(n-1) + 4\ell n(m-n)} \cdot N_\ell(\pi^\beta, \pi^\beta)
$$

$$
= \mu(\pi^\beta, \pi^\beta),
$$

which yields (2.9) together with (2.15).

By Proposition 2.4 and Proposition 2.6 in order to prove Theorem 2.3, it is enough to calculate $\mu(1_n, 1_n)$ and $\mu(h_t, h_t)$, where

$$
h_t = \pi^{12t} = \begin{pmatrix} 0 & IT \\ -IT & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & IT \\ -IT & 0 \end{pmatrix} \in X_{2t}.
$$

(2.16)

We define a $k$-bilinear pairing on the set $\{ X \in M_n(D) \mid X^* = X \}$ as follows: For $B = (b_{ij}), C = (c_{ij})$, set

$$
\langle B, C \rangle = \sum_{i=1}^n b_{ii}c_{ii} + \sum_{1 \leq i < j \leq n} T_{rd}(b_{ij}c_{ij}) \in k,
$$

(2.17)
then we have character sum expressions for $N^{(pr)}(B, A)$ as follows.

**Proposition 2.7** Let $\ell \geq 1$ and take a character $\chi = \chi_\ell$ of $\mathfrak{o}/p^\ell$ such that $\chi$ is nontrivial on $p^{\ell-1}/p^\ell$. For $A \in X^+_m$ and $B \in X^+_n$ with $m \geq n$, one has

$$N^{(pr)}(B, A) = q^{-\ell n(2n-1)} \sum_{\overline{Y} \equiv Y^* \pmod{p^{2\ell}}} \sum_{\overline{X} \in M_{nn}(\mathcal{O})/M_{nn}(p^{2\ell})} \chi((A[X] - B, Y)).$$

(2.18)

where $\overline{X}$ and $\overline{Y}$ determine the element $(A[X] - B, Y)$ in $\mathfrak{o}$ modulo $p^\ell$.

**Proof.** We write $A[X] - B = (c_{ij})$, and we understand $c_{ij}$'s and entries of $Y$ as elements in $\mathcal{O}/p^{2\ell}$. We calculate the right hand side of the above identity.

$$= \sum_{\overline{X}} \prod_{i=1}^n \sum_{y \in \mathfrak{o}/p^\ell} \chi(c_{ii} y) \cdot \prod_{i<j} \sum_{y \in \mathfrak{o}/p^{2\ell}} \chi(T_{rd}(c_{ij} y))$$

$$= \sum_{\overline{X}} \prod_{i=1}^n \left( q^\ell \text{ if } c_{ii} \equiv 0 \pmod{p^\ell} \right) \times \prod_{i<j} \left( q^4 \text{ if } c_{ij} \in p^{2\ell-1} \right)$$

$$= \sum_{\overline{X}} \prod_{i=1}^n \left( q^\ell \text{ if } c_{ii} \equiv 0 \pmod{p^\ell} \right) \times \prod_{i<j} \left( q^4 \text{ if } c_{ij} \in p^{2\ell-1} \right)$$

$$= q^{n(2n-1)} N^{(pr)}(B, A).$$

For the convenience of later calculation, we note the following.

**Proposition 2.8**

$$\mu(1, 1_n) = \mu^{pr}(1, 1_n) = 1 - (-q^{-1})^n,$$

(2.19)

$$\mu(1_n, 1_n) = \prod_{i=1}^n (1 - (-q^{-1})^i) = w_n(-q^{-1}).$$

(2.20)

**Proof.** Take $\ell$ to be sufficiently large and $\chi = \chi_\ell$ as in Proposition 2.7. For $0 \leq e < \ell$, we set $\chi_{\ell-e}(x) = \chi(p^e x)$. Then we may regard $\chi_{\ell-e}$ as a character of $\mathfrak{o}/p^{\ell-e}$ that is nontrivial on $p^{\ell-e-1}/p^{\ell-e}$. We may take the representatives of $\mathfrak{o}/p^{\ell}$ as

$$\{0\} \cup \bigcup_{e=0}^{\ell-1} \left\{ \pi^e u \mid \pi \in (\mathfrak{o}/p^{\ell-e})^\times \right\}.$$  

(2.21)
Then, by (2.18), we have
\[ q^\ell N^{pr}_\ell (1, 1_n) = q^\ell N_\ell (1, 1_n) \]
\[ = \sum_{y \in \mathbb{O} / p^\ell} \sum_{x_i \in \mathbb{O} / p^{2\ell}, 1 \leq i \leq n} \chi((\sum_{i=1}^n N_{rd}(x_i) - 1)y) \]
\[ = q^{4n\ell} + \sum_{e=0}^{\ell-1} \sum_{x \in \mathbb{O} / p^{2(\ell-e)}} \left( \sum_{\pi \in (\mathbb{O} / p^{\ell-e})^\times} \chi(\pi \pi u N_{rd}(x)) \right) \chi(-\pi^e u) \] (2.22)
\[ = q^{4n\ell} + \sum_{e=0}^{\ell-1} \left\{ \left( q^{4\ell} \sum_{x \in \mathbb{O} / p^{2(\ell-e)}} \chi_{\ell-e}(N_{rd}(x)) \right)^n \sum_{\pi \in (\mathbb{O} / p^{\ell-e})^\times} \chi_{\ell-e}(u) \right\}, \]
where, since \( N_{rd}(\mathbb{O}^\times) = o^\times \), one may erase \( u \) in the sum with respect to \( x \) in (2.22), and obtain the last expression. Since
\[ \sum_{\pi \in (\mathbb{O} / p^m)^\times} \chi(u) = \begin{cases} -1 & \text{if } m = 1 \\ 0 & \text{if } m > 1 \end{cases}, \] (2.23)
we have, as continuation of the above calculation
\[ q^\ell N^{pr}_\ell (1, 1_n) = q^{4n\ell} \left( \sum_{x \in \mathbb{O} / p^{2\ell}} \chi_1(N_{rd}(x)) \right)^n, \] (2.24)
It is easy to see
\[ \sum_{x \in \mathbb{O} / p^2} \chi_1(N_{rd}(x)) = q^2 + \sum_{u \in (\mathbb{O} / p)^\times} \frac{q^2(q^2-1)}{q-1} \chi_1(u) = -q^3, \] (2.25)
hence, we obtain by (2.24) and (2.25)
\[ N^{pr}_\ell (1, 1_n) = q^{-\ell} \left( q^{4n\ell} - (-q^{4\ell-1})^n \right) = q^{(4n-1)\ell}(1 - (-q^{-1})^n), \]
\[ \mu^{pr}_\ell (1, 1_n) = (1 - (-q^{-1})^n). \]
Finally, by Proposition 2.6 we have
\[ \mu(1_n, 1_n) = \prod_{r=1}^n \mu^{pr}_\ell (1, 1_r) = \prod_{r=1}^n (1 - (-q^{-1})^r) = w_n(-q^{-1}). \]

Next we consider about \( \alpha = (1, \ldots, 1) \in \Lambda_n \), and set \( n = 2t \) and \( \pi^\alpha = h_t \), where \( h_t \) is defined in (2.16). It is convenient to consider the following density
\[ N^{pr}_\ell (0, h_t) = \# \left\{ \pi \in M_{wh^t}^{pr}(\mathbb{O} / p^{2\ell}) \mid h_t[x] \equiv 0 \pmod{p^\ell} \right\}, \]
\[ \mu^{pr}_\ell (0, h_t) = \lim_{\ell \to \infty} \frac{N^{pr}_\ell (0, h_t)}{q^{(4n-1)t}}. \] (2.26)
where \( M_{n_1}^{pr}(O/P^{2\ell}) \) is defined in (2.3), and in this case
\[
M_{n_1}^{pr}(O/P^{2\ell}) = \{ \pi \in M_{n_1}(O/P^{2\ell}) \mid x \notin (P)^n \}.
\]

**Lemma 2.9** Let \( n = 2t \). Then
\[
\mu(h_1, h_1) = q^{3} \mu^{pr}(0, h_1), \tag{2.27}
\]
\[
\mu(h_t, h_t) = q^{4n-5} \mu^{pr}(0, h_t) \cdot \mu(h_{t-1}, h_{t-1}), \quad (t \geq 2). \tag{2.28}
\]

**Proof.** Take \( \ell \) to be sufficiently large. For any \( \pi \in M_{n_1}^{pr}(O/P^{2\ell}) \) satisfying \( h_t[x] \equiv 0 \) (mod \( p^{\ell} \)), \( x \) can be extended to an element \( U \in K_n = GL_\ell(O) \) such that \( h_t[U] - h_t \in H_n(P, \ell) \). For two such extensions \( U \) and \( V \) of \( x \), we see \( h_t[U^{-1}V] - h_t \in H_n(P, \ell) \). Hence the number of such extensions \( \pi \in M_{n_1}(O/P^{2\ell}) \) of \( \pi \) is equal to the number of \( W \in GL_\ell(O/P^{2\ell}) \) such that, when \( t \geq 2 \),
\[
W = \begin{pmatrix}
1 & b \\
0 & c \\
b^* & c^*
\end{pmatrix}
\begin{pmatrix}
\tilde{b} \\
\tilde{c} \\
D
\end{pmatrix} \in GL_\ell(O), \quad h_t[W] \equiv h_t \pmod{H_n(P, \ell)}, \tag{2.29}
\]
where upper left 1, \( b, c \in O \) and other entries are taken with suitable size. When \( t = 1 \), only upper left 2 \( \times \) 2-block of \( W \) in (2.29) appears, and we may ignore other entries. We continue the case \( t \geq 2 \). Since
\[
h_t[W] = \begin{pmatrix}
1 & 0 \\
0 & c^* \\
b^* & c^*
\end{pmatrix}
\begin{pmatrix}
0 & IIc \\
-II & -IIb \\
0 & h_{t-1}d
\end{pmatrix}
\begin{pmatrix}
\Pi \tilde{c} \\
-\Pi \tilde{b} \\
h_{t-1}[D]
\end{pmatrix} \equiv h_t \pmod{H_n(P, \ell)}, \tag{2.30}
\]
it is easy to see
\[
c \equiv 1 \pmod{P^{2\ell-2}}, \quad \tilde{c} \equiv 0 \pmod{P^{2\ell-2}}, \tag{2.31}
\]
and the choice of \( (c, \tilde{c}) \pmod{P^{2\ell}} \) is \( q^{4(n-1)} \) in \( M_{1,n-1}(O) \). Then (2.30) becomes
\[
h_t[W] \equiv \begin{pmatrix}
0 & II \\
-II & -T_{rd}(\Pi b) + h_{t-1}[d] \\
0 & b^* II + D^*h_{t-1}D
\end{pmatrix}
\begin{pmatrix}
\Pi \tilde{b} + d^*h_{t-1}D \\
h_{t-1}[D]
\end{pmatrix} \pmod{H_n(P, \ell)}, \tag{2.32}
\]
hence we see
\[
D \in GL_{n-2}(O), \quad h_{t-1}[D] \equiv h_{t-1} \pmod{H_{n-2}(P, \ell)}. \tag{2.33}
\]
For any \( D \) as in (2.33) and \( d \in M_{n-2,1}(O) \), we may take \( b \) and \( \tilde{b} \) satisfying \( T_{rd}(\Pi b) \equiv h_{t-1}[d] \pmod{P^{\ell}} \) and \( \Pi \tilde{b} \equiv d^* h_{t-1}D \pmod{P^{2\ell-1}} \), actually the choice of \( b \pmod{P^{2\ell}} \) is \( q^{3\ell+1} \) in \( O/P^{2\ell} \) and that of \( \tilde{b} \pmod{P^{2\ell}} \) is \( q^{4(n-2)} \) in \( M_{1,n-2}(O/P^{2\ell}) \). If we take \( W \) in this
way, $W$ becomes an element of $GL_n(O/P^{2\ell})$ since $h_t[W] \equiv h_t \pmod{H_n(O, \ell)}$. Hence we see, for $t \geq 2$

$$N_t(h_t, h_t) = N_{t}^{pr}(0, h_t) \cdot q^{4(n-1)} \cdot N_{t}(h_{t-1}, h_{t-1}) \cdot q^{4t(n-2)} \cdot q^{3t+1} \cdot q^{4(t-2)}$$

$$= q^{(4n-5)+8n-11} \cdot N_{t}^{pr}(0, h_t) \cdot N_{t}(h_{t-1}, h_{t-1}) \quad (n = 2t)$$

$$= q^{(4n-5)+8n-11} \cdot q^{4t(n-1)} \cdot \mu_{pr}(0, h_t) \cdot q^{\ell(n-2)(2n-3)+(n-2)(n-3)} \mu(h_{t-1}, h_{t-1})$$

$$= q^{2n(2n+1)+n^2+3n-5} \mu_{pr}(0, h_t) \cdot \mu(h_{t-1}, h_{t-1}),$$

which yields

$$\mu(h_t, h_t) = q^{4n-5} \mu_{pr}(0, h_t) \cdot \mu(h_{t-1}, h_{t-1}).$$

(2.35)

As for the case $t = 1$, we see the condition of $W$ to be $h_1[W] \equiv h_1 \pmod{H_2(P, \ell)}$ is

$c \equiv 1 \pmod{p^{2\ell-2}}$ and $T_{rd}(IIb) \equiv 0 \pmod{p^\ell}$, by (2.30) and (2.32). Hence

$$N_t(h_1, h_1) = N_{t}^{pr}(0, h_1) \cdot q^{4} \cdot q^{3t+1},$$

(2.36)

$$\mu(h_1, h_1) = \lim_{\ell \to \infty} \frac{N_t(h_1, h_1)}{q^{10\ell+2}} = \lim_{\ell \to \infty} \frac{N_{t}^{pr}(0, h_1)}{q^{3\ell+1}} \cdot q^{3},$$

(2.37)

Lemma 2.10 For each $t \geq 1$, it holds

$$\mu_{pr}(0, h_t) = q(1 - q^{-4t}).$$

Proof. We may check the identity (2.18) holds even when $B = 0$, and we have

$$q^\ell N_t(0, h_t) = \sum_{\pi \in O/P^\ell} \sum_{\pi \in (O/P^{2\ell})^{2\ell}} \chi_{t}(h_t[w]z)$$

$$= \sum_{\pi \in O/P^\ell} \left( \sum_{\pi \in (O/P^{2\ell})^{2\ell}} \chi(zT_{rd}(xIIy)) \right)^t$$

$$= q^{1+8\ell} + \sum_{e=0}^{\ell-2} q^{\ell-e} (1 - q^{-1}) \left( \sum_{\pi \in O/P^{2\ell}} \chi(\pi^eT_{rd}(xIIy)) \right)^t,$$

(2.38)

where we use the representatives of $O/P^\ell$ written in (2.21) and the property $T_{rd}$ is $k$-linear. To take the sum for $\pi$ in (2.38), we take the representatives of $O/P^{2\ell}$ for each $e$ with $0 \leq e \leq \ell - 2$, as follows:

$$\{ \pi^{\ell-e-1} x \mid \pi \in O/P^{2(\ell+1)} \} \cup \bigcup_{r=0}^{2\ell-2e-3} \{ \pi^ru \mid \pi u \in (O/P^{2\ell-r})^x \}.$$

(2.39)

Then, we have

$$\sum_{\pi \in O/P^{2\ell}} \chi(\pi^eT_{rd}(xIIy))$$

$$= q^{4(\ell+e+1)} + \sum_{r=0}^{2\ell-2e-3} q^{4\ell-2e-3r} (1 - q^{-2}) \sum_{\pi \in O/P^{2\ell}} \chi(\pi^eT_{rd}(xII^{r+1}))$$

$$= q^{4(\ell+e+1)},$$

(2.40)
where we used the fact
\[ \sum_{x \in O/P^2} \chi(\pi^e T_{rd}(x\Pi^{r+1})) = 0, \tag{2.41} \]
which holds since \(2e + r + 1 < 2\ell - 1\) and \(\chi\) is nontrivial on \(p^{\ell-1}/p\ell\). Hence we obtain
\[ q^\ell N_{\ell}(0, h_t) = q^{1+8t\ell} + \sum_{e=0}^{\ell-2} q^{\ell-e}(1 - q^{-1})q^{4t(\ell+e+1)}. \tag{2.42} \]

Next, we calculate the number \(N_{\ell}^{imp}(0, h_t)\) of imprimitive solutions for \(h_t[x] \equiv 0 \pmod{p^\ell}\) by character sum as follows
\[ q^\ell N_{\ell}^{imp}(0, h_t) = q^{\ell} \# \{ \pi \in M_{n,1}(P/P^{2\ell}) \mid h_t[x] \equiv 0 \pmod{p^\ell} \} \]
\[ = \sum_{x \in \mathbb{F}} \sum_{\pi \in (P/P^{2\ell})^2} \chi(h_t[w]z) \]
\[ = \sum_{x \in \mathbb{F}} \left( \sum_{\pi, y \in (P/P^{2\ell})^2} \chi(z T_{rd}(x\Pi y)) \right)^t \]
\[ = q^{2+4t(2\ell-1)} + \sum_{e=0}^{\ell-3} q^{\ell-e}(1 - q^{-1}) \left( \sum_{\pi, y \in (P/P^{2\ell})^2} \chi(\pi^e T_{rd}(x\Pi y)) \right)^t \tag{2.43} \]
where we use (2.21) and the fact \(N_{rd}(x\Pi y) \in p^{2}\) for \(x, y \in P\). In the similar way to calculate (2.40), we have for each \(e\) with \(0 \leq e \leq \ell - 3\),
\[ \sum_{\pi, y \in (P/P^{2\ell})^2} \chi(\pi^e T_{rd}(x\Pi y)) \]
\[ = q^{4\ell-2+4e+6} + \sum_{r=1}^{2\ell-2e-4} q^{4\ell-2r}(1 - q^{-2}) \sum_{\pi \in (P/P^{2\ell})^2} \chi(\pi^e T_{rd}(x\Pi^{r+1})) \]
\[ = q^{4(\ell+e+1)} + \sum_{r=1}^{2\ell-2e-4} q^{4\ell-2r}(1 - q^{-2}) \cdot 0 \]
\[ = q^{4(\ell+e+1)}. \tag{2.44} \]

Hence we have
\[ q^\ell N_{\ell}^{imp}(0, h_t) = q^{2+4t(2\ell-1)} + \sum_{e=0}^{\ell-3} q^{\ell-e}(1 - q^{-1})q^{4t(\ell+e+1)} \tag{2.45} \]

By (2.42) and (2.45), we obtain
\[ N_{\ell}^{pr}(0, h_t) = q^{8t-1}\ell q(1 - q^{-4t}), \quad (n = 2t) \tag{2.46} \]
which yields \(\mu^{pr}(0, h_t) = q(1 - q^{-4t})\).

By Lemma 2.9 and Lemma 2.10, we have the following.
Proposition 2.11  Let \( n = 2t \) and \( \alpha = (1, \ldots, 1) \in \Lambda_n^+ \). Then \( \pi^\alpha = h_t \), and it holds

\[
\mu(h_t, h_t) = q^{4t^2} \cdot \prod_{i=1}^{t} (1-q^{-4i}) = q^{n^2} \cdot w_n(q^{-4}). \tag{2.47}
\]

Proof of Theorem 2.3  Take \( \alpha \in \Lambda_n^+ \) as in (1.11). Then, by Propositions 2.4, 2.6, 2.8 and 2.11 we see

\[
\mu(\pi^\alpha, \pi^\alpha) = q^{\sum_{j=2}^{r}(m_1 + \cdots + m_{j-1})m_j \gamma_j + \sum_{1 \leq j \leq r, 2 | \gamma_j} \frac{\gamma_j}{2}m_j(2m_j - 1) + \sum_{1 \leq j \leq r, 2 \nmid \gamma_j} (\frac{\gamma_j - 1}{2}m_j(2m_j - 1) + m_j^2)} \times
\]

\[
\prod_{1 \leq j \leq r} w_{m_j}(-q^{-1}) \times \prod_{1 \leq j \leq r} w_{m_j}(q^{-4}),
\]

where

\[
m_\alpha = \sum_{i=2}^{r} 2(m_1 + \cdots + m_{j-1})m_j \gamma_j + \sum_{1 \leq j \leq r, 2 | \gamma_j} \frac{\gamma_j}{2}m_j(2m_j - 1)
\]

\[
+ \sum_{1 \leq j \leq r, 2 \nmid \gamma_j} (\frac{\gamma_j - 1}{2}m_j(2m_j - 1) + m_j^2)
\]

\[
= 2 \sum_{j=2}^{r} (m_1 + \cdots + m_{j-1})m_j \gamma_j + \sum_{j=1}^{r} (m_j(m_j - 1) + \frac{1}{2}m_j \gamma_j + \frac{1}{2}) \sum_{1 \leq i \leq r, 2 | \gamma_i} m_j
\]

\[
= 2n(\alpha) + \frac{1}{2} |\alpha| + \frac{1}{2} \# \{ i \mid \alpha_i \text{ is odd} \}.
\]

For \( \bar{\alpha} = \alpha + (2e, \ldots, 2e) \in \Lambda_n \), we have \( \mu(\pi^{\bar{\alpha}}, \mu^{\bar{\alpha}}) = q^{en(2n-1)} \mu(\pi^\alpha, \mu^\alpha) \) by Remark 2.5. On the other hand, we have \( 2n(\bar{\alpha}) + \frac{1}{2} |\bar{\alpha}| = 2n(\alpha) + \frac{1}{2} |\alpha| + en(2n-1) \), hence we see (2.5) holds for any \( \alpha \in \Lambda_n \).

2.3. We introduce the Schwartz space \( S(K\backslash X) \) by

\[
S(K\backslash X) = \{ \varphi : X \to \mathbb{C} \mid \text{left } K\text{-invariant, compactly supported} \},
\]

that is spanned by the characteristic functions of \( K \cdot \pi^\alpha, \alpha \in \Lambda_n \) over \( \mathbb{C} \). It is an \( \mathcal{H}(G,K) \)-submodule of \( C^\infty(K\backslash X) \) (cf. (1.18), (1.19)). We define an integral transform \( F_0 \) on \( S(K\backslash X) \) as follows:

\[
F_0 : S(K\backslash X) \to \mathbb{C}(q^{s_1}, \ldots, q^{s_n}),
\]

\[
\varphi \mapsto \int_X \varphi(x) \omega(x^{-1}; s) dx,
\]

where \( dx \) is a \( G \)-invariant measure on \( X \). We call \( F_0 \) a spherical Fourier transform on \( S(K\backslash X) \), and we will normalize \( F_0 \) suitably and define the spherical transform \( F \) in §3.
Proposition 2.12  The spherical Fourier transform $F_0$ defined in (2.48) is injective and compatible with the action of $\mathcal{H}(G, K)$:

$$F_0(f * \varphi)(s) = \lambda_s(f)F_0(\varphi), \quad f \in \mathcal{H}(G, K), \varphi \in \mathcal{S}(K \setminus X),$$

where $\lambda_s$ is defined in (2.21).

The injectivity of $F_0$ is proved in a similar way to the cases of other sesquilinear forms by using Lemma 2.13 below and Theorem 2.2 (cf. [H1, Lemma 2.13]). We define a binary relation $\succ$ on $\Lambda_n$ by

$$\lambda \succ \mu \iff \lambda = \mu, \text{ or there is some } t \text{ with } 1 \leq t \leq n - 1 \text{ satisfying } \lambda_{n-t} > \mu_{n-t}, \lambda_{n-t+1} = \mu_{n-t+1}, \ldots, \lambda_n = \mu_n.$$

Lemma 2.13  Let $n \geq 2$. For any $\alpha \in \Lambda_n^+$, there exists $\beta \in \Lambda_{n-1}^+$ such that

(i) $\mu^{pr}(\pi^\beta, \pi^\alpha) \neq 0$, and

(ii) if $\gamma \in \Lambda_n^+$ satisfies $|\gamma| = |\alpha|$, $\gamma \succ \alpha$ and $\mu^{pr}(\pi^\beta, \pi^\gamma) \neq 0$, then $\gamma = \alpha$.

Similar lemma was introduced first by Kitaoka ([K11]) for symmetric forms and by the author for hermitian forms ([H1 §3]), and the above lemma can be proved similarly, so we note here that one may take $\beta \in \Lambda_{n-1}$ as $\beta = (\alpha_2, \alpha_3, \ldots, \alpha_n)$ (resp. $(\alpha_2 + 1, \alpha_3, \ldots, \alpha_n)$ if $\alpha_1$ is even (resp. odd).

Proof of Proposition 2.12  Let $f \in \mathcal{H}(G, K)$ and $\varphi \in \mathcal{S}(K \setminus X)$. Then we have

$$F_0(f * \varphi)(s) = \int_X \int_G f(g)\varphi(g^{-1} \cdot x)\omega(x^{-1}; s) dg dx$$

$$= \int_G \int_X f(g)\varphi(y)\omega(g^{-1} \cdot y^{-1}; s) dy dg \quad (y = g^{-1} \cdot x)$$

$$= \int_X \int_G f(g^\ast)\omega(g^{-1} \cdot y^{-1}; s) dg \varphi(y) dy$$

$$= \lambda_s(f) \int_X \varphi(y)\omega(y^{-1}; s) = \lambda_s(f)F_0(\varphi).$$

We prove the injectivity of $F_0$ by induction on $n$. For $\alpha \in \Lambda_n$, we denote by $\varphi_\alpha \in \mathcal{S}(K \setminus X)$ the characteristic function of $K \cdot (\pi^\alpha)^{-1}$. Then we have $F_0(\varphi_\alpha) = \text{volume}(K \cdot (\pi^\alpha)^{-1}) \times \omega(\pi^\alpha; s) \neq 0$, hence the injectivity of $F_0$ is equivalent to the linear independence of $\omega(\pi^\alpha; s)$ for $\alpha \in \Lambda_n$. It is clear that $F_0$ is injective for $n = 1$. We assume that $F_0$ is injective for $n - 1$ and not injective for $n$, and take $0 \neq \varphi \in \text{Ker}(F_0)$. We may assume $\varphi$ is of the following shape:

$$\varphi = \sum_{i=1}^\ell c_i \varphi_{\alpha(i)}, \quad \ell \geq 2, \ c_i \neq 0, \ \alpha(i) \in \Lambda_n^+ (1 \leq i \leq \ell), \ \alpha(i) \neq \alpha(j) \text{ if } i \neq j. \quad (2.49)$$

Since we have, for any $\alpha \in \Lambda_n$,

$$\omega(\pi^\alpha; s) = q_0^{\alpha_0} s^0 \omega(\pi^\alpha; s_1, \ldots, s_{n-1}, 0), \quad \omega(\pi^\alpha; s_1, \ldots, s_{n-1}, 0) \in \mathbb{C}(q^{s_1}, \ldots, q^{s_{n-1}}),$$

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looking at the exponent of \( q^n \) in \( \omega(\pi^\alpha; s) \), we may assume that \(|\alpha^{(i)}| = |\alpha^{(1)}|\) for any \( i \). Assume that \( \alpha^{(1)} \) is the smallest with respect to the order \( \succ \) within \( \{ \alpha^{(i)} | 1 \leq i \leq \ell \} \). Since \( F_0(\varphi) = 0 \), we obtain by Theorem 2.2

\[
\sum_{i=1}^{\ell} c_i d_i \sum_{\gamma \in \Lambda_{n-1}^+} \frac{\mu^{pr}(\pi^\gamma, \pi^\alpha^{(i)})}{\mu(\pi^\gamma, \pi^\gamma)} \omega(\pi^\gamma; s_1, \ldots, s_{n-1}) = 0,
\]

(2.50)

where \( d_i (> 0) \) is the volume of \( K \cdot (\pi^{\alpha^{(i)}})^{-1} \). By induction hypothesis that \( F_0 \) is injective for \( n - 1 \), we see by (2.50)

\[
\sum_{i=1}^{\ell} c_i d_i \mu(\pi^\gamma, \pi^{\alpha^{(i)}}) = 0, \quad \text{for every } \gamma \in \Lambda_{n-1}^+.
\]

(2.51)

For \( \beta \in \Lambda_{n-1}^+ \) associated with \( \alpha^{(1)} \) in Lemma 2.13, we see that \( \mu(\pi^{\beta}, \pi^{\alpha^{(1)}}) \neq 0 \) and \( \mu(\pi^\beta, \pi^{\alpha^{(i)}}) = 0 \) for \( i \neq 1 \), and we obtain \( c_1 = 0 \), which contradicts (2.49). \( \blacksquare \)
§3 Functional equations of spherical functions

First we note the result for size 2, which follows from Theorem 2.2 with some calculation of local densities (cf. [Oh]).

Proposition 3.1 For any \( \alpha \in \Lambda_2 \), one has

\[
\omega(\pi^\alpha; z) = \begin{cases} 
\frac{q^{(\lambda, z_0)}}{1 - q^{-2}} \cdot \frac{1}{q^{z_2} - q^{z_1 + 1}} \sum_{\sigma \in S_2} \sigma \left( q^{(\lambda, z)} \frac{(q^{z_1} - q^{z_2 - 1})(q^{z_1} - q^{z_2 + 1})}{q^{z_1} - q^{z_2}} \right) & \text{if } \alpha = 2\lambda, \\
q(1 - q^{-1}) \frac{q^{(\lambda_1 + \lambda_2)}}{q^{z_2} - q^{z_1 + 1}} & \text{if } \alpha = (2e - 1, 2e - 1),
\end{cases}
\]

where \( z_0 = (-1, 1) \) corresponds to the s-variable \( \mathbf{0} = (0, 0) \), \( \langle \lambda, z \rangle = \lambda_1 z_1 + \lambda_2 z_2 \) and \( S_2 \) acts on \( \{ z_1, z_2 \} \) by permutation. Especially, for any \( x \in X_2 \), one has

\[
(q^{z_2} - q^{z_1 + 1}) \cdot \omega(x; z) \in \mathbb{C}[q^{\pm z_1}, q^{\pm z_2}] S_2. \tag{3.1}
\]

The property (3.1) follows from the explicit formula for \( \omega(\pi^\alpha; z) \), since any \( x \in X_2 \) belongs to some orbit \( K_2 \cdot \pi^\alpha, \alpha \in \Lambda_2 \) and \( \omega(x; z) = \omega(\pi^\alpha; z) \).

For the study of the functional equations and holomorphy of \( \omega(x; s) \) for general \( n \), we use the same strategy used in the case of unramified hermitian forms. We introduce the following integral for \( \xi \in \mathcal{S}(K \backslash X) \)

\[
\Phi(s, \xi) = \int_X |d(x)|^s \xi(x) dx, \quad |d(x)|^s = \begin{cases} \prod_{i=1}^n |d_i(x)|^{s_i} & \text{if } x \in X \text{ op} \\
0 & \text{otherwise}, \end{cases} \tag{3.2}
\]

where \( dx \) is a \( G \)-invariant measure on \( X \). The above integral is a finite linear sum of spherical functions \( \omega(x; s) \), hence it is absolutely convergent if \( \text{Re}(s_i) \geq 0, 1 \leq i \leq n - 1 \), and continued to a rational function of \( q^{s_1}, \ldots, q^{s_n} \). Keeping the relation (1.22) between \( s \) and \( z \), we denote \( \Phi(z, \xi) \).

Lemma 3.2 Let \( n \geq 2 \) and take \( \alpha \) with \( 1 \leq \alpha \leq n - 1 \). Assume that \( \text{Re}(s_i) \geq 0, 1 \leq i \leq n - 1 \). Then for any \( \xi \in \mathcal{S}(K \backslash X) \), the following identity holds

\[
\Phi(s, \xi) = \int_{X \text{ op}} \prod_{\alpha \neq \alpha + 1} |d_i(x)|^{s_i} \cdot \prod_{j = \alpha \pm 1} |d_j(x)|^{s_j} \cdot \omega(x; s_i) \cdot \omega(\bar{x}; s, -\frac{s_i}{2}) dx, \tag{3.3}
\]

where \( \bar{x} \) is the lower right \( (2 \times 2) \)-block of \( (x^{(\alpha + 1)})^{-1} \) and \( \omega^{(2)}(y; s) \) indicates the spherical function of size 2.

Proof. Take any \( \alpha \) with \( 1 \leq \alpha \leq n - 1 \) and \( \xi \in \mathcal{S}(K \backslash X) \). We assume that \( \text{Re}(s_i) \geq 0, 1 \leq i \leq n - 1 \). We define an embedding \( \iota = \iota_\alpha \) from \( K_2 = GL_2(O) \) into \( K = K_n \) by

\[
\iota : K_2 \rightarrow K, \quad k \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \\ \end{pmatrix},
\]

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and consider the integral

\[
\Phi(s, \xi) = \int_X |d(x)|^s \int_{K_2} \xi(\iota(k)^{-1} \cdot x) dk dx = \int_{K_2} \int_X |d(\iota(k) \cdot x)|^s \xi(x) dx dk.
\]

(3.4)

Here we recall a well known fact on minor determinants of matrices over a field \(F\):

\[
\det(A(i)) = \det(A) \det(A^{-1(n-i)}), \quad \text{for } A \in GL_n(F),
\]

(3.5)

where \(A(i)\) (resp. \(A(j)\)) indicates the upper left \((i \times i)\)-block (resp. the lower right \((j \times j)\)-block) of \(A\). In the present case, we consider the reduced norm \(N_{rd}: D \rightarrow k\) and relative invariants \(d_i(x)\), which satisfy \(d_i(x)^2 = N_{rd}(x^{(i)})\) on \(X\), as introduced in §1. Hence we have

\[
d_i(\iota(k) \cdot x) = d_i(x) \quad \text{unless } i = \alpha,
\]

\[
N_{rd}((\iota(k) \cdot x)^{(\alpha)}) = N_{rd}(x^{(\alpha + 1)})N_{rd}((\iota(k) \cdot x)^{(\alpha + 1)})^{-1}(1),
\]

\[
d_\alpha(\iota(k) \cdot x) = d_{\alpha+1}(x)d_1(j_2 k^{s-1} \cdot \bar{x}).
\]

We continue the calculation (3.4) as follows

\[
\Phi(s, \xi) = \int_{X^{op}} \xi(x) \prod_{i \neq \alpha, \alpha + 1} |d_i(x)|^{s_i} \cdot |d_{\alpha+1}(x)|^{s_{\alpha+1}} \int_{K_2} |d_1(j_2 k^{s-1} \cdot \bar{x})|^{s_\alpha} dk dx
\]

\[
= \int_{X^{op}} \xi(x) \prod_{i \neq \alpha, \alpha + 1} |d_i(x)|^{s_i} \cdot |d_{\alpha+1}(x)|^{s_{\alpha+1}} \cdot \omega^{(2)}(\bar{x} ; s_\alpha, 0) dx.
\]

By the definition of \(\bar{x}\), we see

\[
d_2(\bar{x}) = d_{\alpha-1}(x)d_{\alpha+1}(x)^{-1},
\]

and we obtain

\[
\Phi(s, \xi) = \int_{X^{op}} \xi(x) \prod_{i \neq \alpha, \alpha \pm 1} |d_i(s)|^{s_i} \cdot \prod_{j = \alpha \pm 1} |d_j(x)|^{s_j + \frac{s_\alpha}{2}} \cdot \omega^{(2)}(\bar{x} ; s_\alpha, -\frac{s_\alpha}{2}) dx.
\]

![Proposition 3.3](image)

**Proposition 3.3** Under the relation (1.22) of \(s\) and \(z\), the function

\[
\prod_{1 \leq i < j \leq n} (q^{z_i} - q^{z_j+1}) \times \Phi(s, \xi), \quad (\xi \in \mathcal{S}(K \setminus X))
\]

is holomorphic in \(\mathbb{C}^n\) and \(S_n\)-invariant, hence it is an element of

\[
\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n}.
\]
Proof. By the relation \([1.22]\), \(S_n\) acts on variable \(s\) as follows. Let \(\sigma_\alpha = (\alpha \alpha + 1) \in S_n\), \(1 \leq \alpha \leq n - 1\), then its action on \(\{s_i\}\) is

\[
\begin{align*}
\sigma_\alpha(s_i) &= s_i, \quad \text{unless } i = \alpha, \alpha \pm 1, \\
\sigma_\alpha(s_\alpha) &= -s_\alpha - 4, \\
\sigma(s_j) &= s_j + s_\alpha + 2 \quad \text{if } j = \alpha \pm 1.
\end{align*}
\tag{3.6}
\]

Set \(\mathcal{D}_\alpha = \mathcal{D}_0 \cup \mathcal{D}_{\alpha,1} \cup \mathcal{D}_{\alpha,2}\), where

\[
\mathcal{D}_0 = \{ s \in \mathbb{C}^n \mid \text{Re}(s_i) \geq 0, \ (1 \leq i \leq n - 1) \},
\]

\[
\mathcal{D}_{\alpha,1} = \sigma_\alpha(\mathcal{D}_0) = \left\{ s \in \mathbb{C}^n \left| \begin{array}{l}
\text{Re}(s_i) \geq 0 \text{ for } i \in [1, n - 1], i \neq \alpha, \alpha \pm 1 \\
\text{Re}(s_\alpha) \leq -4 \\
\text{Re}(s_\alpha + s_j + 2) \geq 0 \text{ for } j = \alpha \pm 1 \in [1, n - 1]
\end{array} \right. \right\},
\]

\[
\mathcal{D}_{\alpha,2} = \left\{ s \in \mathbb{C}^n \left| \begin{array}{l}
\text{Re}(s_i) \geq 0, \text{ for } i \in [1, n - 1], i \neq \alpha, \alpha \pm 1, \\
-4 \leq \text{Re}(s_\alpha) \leq 0, \\
\text{Re}(s_\alpha/2 + s_j) \geq 0, \text{ for } j = \alpha \pm 1 \in [1, n - 1]
\end{array} \right. \right\}. \tag{3.7}
\]

By the relation of \(s\) and \(z\) and Proposition \([3.1]\) one has

\[
q^{-\frac{s_\alpha}{2} + \frac{s_\alpha + 1}{2}}(q^{s_{\alpha + 1}} - q^{s_\alpha + 1}) = q^{\frac{s_\alpha + 1}{2} - \frac{s_\alpha}{2}},
\]

\[
(q^{\frac{s_\alpha}{2} + 1} - q^{-\frac{s_\alpha}{2}})\omega^{(\alpha)}(x; s_\alpha, -\frac{s_\alpha}{2}) = (q^{-\frac{s_\alpha}{2} - 1} - q^{-\frac{s_\alpha}{2} + 2})\omega^{(\alpha)}(x; -s_\alpha - 4, \frac{s_\alpha}{2} + 2) \in \mathbb{C}[q^{\pm \frac{s_\alpha}{2}}]^{(\sigma_\alpha)}.
\]

Then by Lemma \([3.2]\) one has for \(s \in \mathcal{D}_0\)

\[
(q^{s_{\alpha + 1}} - q^{s_\alpha + 1})\Phi(s, \xi) = q^{\frac{s_\alpha + s_\alpha + 1}{2}} \int_{X^{\alpha}} \prod_{i \neq \alpha, \alpha + 1} \frac{|d_i(x)|^{s_i}}{i=\alpha} \cdot \prod_{j=\alpha \pm 1} |d_j(x)|^{\frac{s_\alpha}{2} + s_j} \cdot \xi(x)
\]

\[
\times (q^{\frac{s_\alpha}{2} + 1} - q^{-\frac{s_\alpha}{2}})\omega^{(\alpha)}(x; s_\alpha, -\frac{s_\alpha}{2})dx. \tag{3.8}
\]

Since the integrand of RHS of \((3.8)\) is \(\sigma_\alpha\)-invariant, we see the above integral is absolutely convergent for \(s \in \sigma_\alpha(\mathcal{D}) = \mathcal{D}_{\alpha,1}\). The region \(\mathcal{D}_{\alpha,2}\) is \(\sigma_\alpha\)-invariant, and we see that

\[
\prod_{i \neq \alpha, \alpha + 1} |d_i(x)|^{s_i} \cdot \prod_{j=\alpha \pm 1} |d_j(x)|^{\frac{s_\alpha}{2} + s_j} \text{ is bounded for } s \in \mathcal{D}_{\alpha,2},
\]

\[
(q^{\frac{s_\alpha}{2} + 1} - q^{-\frac{s_\alpha}{2}})\omega^{(\alpha)}(x; s_\alpha, -\frac{s_\alpha}{2}) \text{ is a polynomial in } q^{\pm \frac{s_\alpha}{2}}.
\]

Since \(\xi\) is compactly supported, RHS of \((3.8)\) is absolutely convergent also for \(s \in \mathcal{D}_{\alpha,2}\), and so \((q^{s_{\alpha + 1}} - q^{s_\alpha + 1})\Phi(s, \xi)\) is holomorphic in \(\mathcal{D}_\alpha\) and \(\sigma_\alpha\)-invariant. Since

\[
\prod_{1 \leq i < j \leq n} \frac{(q^{s_j} - q^{s_i})}{(q^{s_{\alpha + 1}} - q^{s_\alpha + 1})}
\]

is \(\sigma_\alpha\)-invariant for ant \(\alpha\) and holomorphic in \(\mathbb{C}^n\),

\[
\prod_{1 \leq i < j \leq n} (q^{s_j} - q^{s_i}) \times \Phi(s, \xi) \text{ is holomorphic in } C = \bigcup_{\alpha=1}^{n-1} \mathcal{D}_\alpha \text{ and } S_n\text{-invariant}.
\]
Hence
\[
\prod_{1 \leq i < j \leq n} (q^z_j - q^z_i) \times \Phi(s, \xi)
\] (3.9)
is holomorphic in \( \bigcup_{\sigma \in S_n} \sigma(C) \) and its convex hull \( \mathbb{C}^n \) and \( S_n \)-invariant. Since (3.9) is a rational function of \( q^{z_1}, \ldots, q^{z_n} \), we see that it is a symmetric Laurent polynomial, thus we have
\[
\prod_{1 \leq i < j \leq n} (q^z_j - q^z_i) \times \Phi(s, \xi) \in \mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n}.
\]

Taking the characteristic function of \( K \cdot x \) for \( x \in X_n \) as \( \xi \), we obtain the following theorem.

**Theorem 3.4** Set
\[
\Psi(x; z) = G_n(z) \cdot \omega(x; z), \quad G_n(z) = \prod_{1 \leq i < j \leq n} (q^z_j - q^z_i+1),
\] (3.10)
then \( \Psi(x; z) \) is holomorphic and \( S_n \)-invariant spherical function on \( X \), thus
\[
\Psi(x; z) \in \mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n}.
\]

In consideration of Theorem 3.4, we normalize the spherical Fourier transform \( F_0 \) given in (2.48) as follows:
\[
F : \mathcal{S}(K \backslash X) \to \mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n} (= \mathcal{R}, \text{say})
\]
\[
\varphi \mapsto \hat{\varphi}(z) = \int_X \varphi(x) \cdot \Psi(x^{-1}; z) dx.
\] (3.11)
Then we obtain the following theorem by Theorem 3.4.

**Theorem 3.5** The normalized spherical Fourier transform \( F \) is an injective \( \mathcal{H}(G, K) \)-module map, hence one has the commutative diagram
\[
\begin{array}{ccc}
\mathcal{H}(G, K) \times \mathcal{S}(K \backslash X) & \xrightarrow{\ast} & \mathcal{S}(K \backslash X) \\
\lambda_z \downarrow & F \downarrow & \circ \downarrow F \\
\mathcal{R} \times \mathcal{R} & \to & \mathcal{R}
\end{array}
\] (3.12)
where the upper \( \ast \) is the action of \( \mathcal{H}(G, K) \) on \( \mathcal{S}(K \backslash X) \), the lower arrow is the multiplication in \( \mathcal{R} \), and \( \lambda_z \) is the Satake isomorphism defined in (1.23).
§4 Explicit formula for $\omega(x; z)$

4.1. As for the explicit formula of $\omega(x; z)$, it suffices to determine it at each representatives of $K$-orbit in $X$, i.e. at each $\pi^\alpha$, $\alpha \in \Lambda_n$ (cf. [1.10]). We may apply the general expression formula of spherical function on homogeneous spaces (cf. [H2 Prop.1.9], [H4 §2]). In the present case the situation becomes simpler, since $\omega$ and $X$ are $\omega$-spherical. 

As for the explicit formula of $\omega$, assume they are algebraic closure of $k$. For each $\alpha \in \Lambda_n$, we set

$$\lambda_\alpha = (\lambda_i) \in \tilde{\Lambda}_n \quad \text{by} \quad \lambda_i = \left\lfloor \frac{\alpha_i + 1}{2} \right\rfloor,$$

where $[ \ ]$ is the Gauss symbol. If $\alpha$ has an odd entry, odd entries appear in pairs. We assume they are

$$\alpha_{\ell_1}, \alpha_{\ell_1+1}, \ldots, \alpha_{\ell_k}, \alpha_{\ell_k+1}, \quad \ell_1 < \ell_2 < \cdots < \ell_k,$$

and set

$$I_{\text{odd}}(\alpha) = \{\ell_1, \ldots, \ell_k\}, \quad c_{\text{odd}}(\alpha) = (1 - q^{-1})^k \cdot q^{\sum_{\ell \in I_{\text{odd}}(\alpha)}(n-2\ell+1)}.$$

If $\alpha$ has no odd entry we say $\alpha$ is even, and set $I_{\text{odd}}(\alpha) = \emptyset$ and $c_{\text{odd}}(\alpha) = 1$ for convenience. Only if $\alpha$ is even, $\pi^\alpha$ is diagonal and $\lambda_\alpha = \frac{\alpha}{2}$. We define a paring on $\mathbb{Z}^n \times \mathbb{C}^n$ as follows:

$$\langle \lambda, z \rangle = \sum_{i=1}^n \lambda_i z_i, \quad (\lambda \in \mathbb{Z}^n, \ z \in \mathbb{C}^n).$$

Theorem 4.1 (Explicit Formula) For any $\alpha \in \Lambda_n$, one has

$$\Psi(\pi^\alpha; z) = \omega(\pi^\alpha; z) \cdot G_n(z)$$

$$= \frac{(1 - q^{-2})^n \cdot c_{\text{odd}}(\alpha) \cdot q^{\langle \lambda_\alpha, z \rangle}}{w_n(q^{-2})} \sum_{\sigma \in S_n} \sigma \left( \prod_{\ell \in I_{\text{odd}}(\alpha)} q^{\langle \lambda_\alpha, z \rangle} \prod_{i<j} \frac{q^{z_i - q^{z_{i+1}}}(q^{z_i - q^{z_{j+1}}})}{q^{z_i - q^{z_j}} q^{z_i - q^{z_j}}} \right),$$

where $w_n(t) = \prod_{i=1}^n (1 - t^i)$, $G_n(z) = \prod_{1 \leq i < j \leq n} (q^{z_j} - q^{z_i+1})$ (given in Theorem 3.3) and $z_0 = (-n+1, -n+3, \ldots, n-1) \in \mathbb{C}^n$ is the corresponding value in $z$-variable to $s = 0 \in \mathbb{C}^n$. 

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Proof. Applying \[H2\] Prop.1.9] to the present case, we obtain for generic $z$,

$$\omega(x; z) = \frac{1}{Q_n} \sum_{\sigma \in S_n} \gamma(\sigma(z)) \Gamma_\sigma(z) \int_U |d(\nu \cdot x)|^{\sigma(s)} d\nu,$$  \hspace{1cm} (4.6)

where $U$ is the Iwahori subgroup associated with $B$, $d\nu$ is the Haar measure on $U$, $Q_n$ and $\gamma(z)$ are determined by the group $GL_n(D)$ as follows, and $\Gamma_\sigma(z)$ is determined by the functional equation $\omega(x; \sigma(z)) = \Gamma_\sigma(z) \omega(x; z)$. Thus we have

$$Q_n = \sum_{\sigma \in S_n} [U\sigma U : U]^{-1} = \frac{w_n(q^{-2})}{(1 - q^{-2})^n},$$

$$\gamma(z) = \prod_{1 \leq i < j \leq n} \frac{1 - q^{z_i - z_j - 2}}{1 - q^{z_i - z_j}} = \prod_{i < j} \frac{q^{z_j} - q^{z_i - 2}}{q^{z_j} - q^{z_i}},$$

$$\Gamma_\sigma(z) = \frac{G_n(\sigma(z))}{G_n(z)}, \hspace{1cm} \text{(by Theorem 3.4)}$$

and (4.6) becomes

$$\omega(x; z) = \frac{(1 - q^{-2})^n}{w_n(q^{-2}) \cdot G_n(z)} \sum_{\sigma \in S_n} \sigma \left( \prod_{1 \leq i < j \leq n} \frac{(q^{z_j} - q^{z_i + 1})(q^{z_j} - q^{z_i - 2})}{q^{z_j} - q^{z_i}} \delta(x; z) \right), \hspace{1cm} (4.7)$$

where

$$\delta(x; z) = \delta(x; s) = \int_U |d(\nu \cdot x)|^s d\nu.$$  \hspace{1cm} (4.8)

Hence the problem is reduced to the calculation of $\delta(x; z)$. Let $j = j_n \in K$ be the matrix whose all the anti-diagonal entries are 1 and other entries are 0, and set $\tilde{x}^\alpha = j \cdot x^\alpha \in K \cdot x^\alpha$ for each $\alpha \in \Lambda_n$, and $jz = (z_n, z_{n-1}, \ldots, z_1)$ for $z \in \mathbb{C}^n$. We will prove the next proposition in §4.2.

**Proposition 4.2** For any $\alpha \in \Lambda_n$, one has

$$\delta(\tilde{x}^\alpha; z) = \frac{c_{\text{odd}}(\alpha) \cdot q^{(\lambda_n, z_0) + (\lambda_n, jz)}}{\prod_{\ell \in I_{\text{odd}}(\alpha)} (q^{z_{n-\ell + 1}} - q^{z_{n-\ell + 1}})},$$

where $z_0$ is defined in Theorem 4.1.

Admitting Proposition 4.2 for a while, we substitute the value $\delta(\tilde{x}^\alpha; z)$ into (4.7) with $x = x^\alpha$. Then, for any $\alpha \in \Lambda_n$, we have

$$\Psi(\pi^\alpha; z) = \omega(\pi^\alpha; z) \cdot G_n(z) = \omega(\pi^\alpha; z) \cdot G_n(z)$$

$$= \frac{(1 - q^{-2})^n \cdot c_{\text{odd}}(\alpha) \cdot \delta(\tilde{x}^\alpha; z_0)}{w_n(q^{-2})} \sum_{\sigma \in S_n} \sigma \left( \prod_{\ell \in I_{\text{odd}}(\alpha)} (q^{(\lambda_n, z_0)})(q^{(\lambda_n, jz)} \prod_{i < j} \frac{(q^{z_j} - q^{z_i + 1})(q^{z_j} - q^{z_i - 2})}{q^{z_j} - q^{z_i}}) \right),$$

which completes the proof of Theorem 4.1.
Remark 4.3 When \( n = 2 \), the formula in Theorem 4.1 coincides with that in Proposition 3.1. For general \( n \), we take the main term of \( \omega(\pi^\alpha; z) \) for each \( \alpha \in \Lambda_n \), and set
\[
Q(\alpha; z) = \sum_{\sigma \in S_n} \sigma \left( \prod_{\ell \in I_{ad}(\alpha)} q^{(\lambda_\alpha, z)} \prod_{i < j} (q^{z_i} - q^{z_{i+1}+1}) (q^{z_i} - q^{z_{j}}) \right).
\]
(4.9)

Then we see \( Q(\alpha; z) \) is holomorphic for \( z \in \mathbb{C}^n \) and linearly independent with respect to \( \alpha \in \Lambda_n \) (cf. Theorem 3.4, Theorem 3.5). Corresponding main term of the spherical function on \( GL_n(k) \) is a specialization of Hall-Littlewood polynomial
\[
Q(\lambda; z) = \sum_{\sigma \in S_n} \sigma \left( q^{(\lambda, z)} \prod_{i < j} \frac{q^{z_i} - q^{z_{j}+1}}{q^{z_i} - q^{z_{j}}} \right), \quad \lambda \in \tilde{\Lambda_n}.
\]
We note here that specializations of Hall-Littlewood polynomials also appear in the main term of spherical functions on alternating forms \( (X_n \subset GL_{2n}(k)) \) as
\[
Q^{(A)}(\lambda; z) = \sum_{\sigma \in S_n} \sigma \left( q^{(\lambda, z)} \prod_{i < j} \frac{q^{z_i} - q^{z_{j}+2}}{q^{z_i} - q^{z_{j}}} \right), \quad \lambda \in \tilde{\Lambda_n},
\]
and on unramified hermitian forms \( (X_n \subset GL_n(k'), k'/k \) is unramified quadratic \) as
\[
Q^{(H)}(\lambda; z) = \sum_{\sigma \in S_n} \sigma \left( q^{(\lambda, z)} \prod_{i < j} \frac{q^{z_i} + q^{z_{j}+1}}{q^{z_i} - q^{z_{j}}} \right), \quad \lambda \in \tilde{\Lambda_n},
\]
(cf. [H2], [H4], [HS1], [M1]). In the present case, the shape of \( Q(\alpha; z) \) is quite different from them. For \( n = 2 \) and even \( \alpha \), \( Q(\alpha; z) \) has a relation to Askey-Wilson polynomials, for more details, see Remark 3.9.

4.2. In this subsection we prove Proposition 4.2. We decompose \( U = (U \cap B)U_1 \) with
\[
U_1 = \left\{ \nu \in GL_n(\mathcal{O}) \mid \begin{array}{l}
\nu_{ii} = 1 \quad \text{for } 1 \leq i \leq n \\
\nu_{ij} = 0 \quad \text{for } 1 \leq j < i \leq n \\
\nu_{ij} \in \mathcal{P} \quad \text{for } 1 \leq i < j \leq n
\end{array} \right\}.
\]

Since \( |d(\nu \cdot x)|^s = |d(x)|^s \) for \( \nu \in U \cap B \), we see \( \delta(x; s) = \int_{U_1} |d(\nu \cdot x)|^s \, d\nu \). We have only to consider \( \pi^\alpha \) for \( \alpha \in \Lambda_n^+ \), since
\[
\delta(\pi^{\alpha+2e}; z) = q^{-e(s_1+2s_2+\cdots+ns_n)} \delta(\pi^\alpha; s) = q^{e(z_1+\cdots+z_n)} \delta(\pi^\alpha; z), \quad e \in \mathbb{Z}.
\]

Lemma 4.4 Let \( \alpha = (\alpha_i) \in \Lambda_n^+ \) and \( 1 \leq i \leq n \), and assume \( (\alpha_{n-i+2}, \alpha_{n-i+3}, \ldots, \alpha_n) \in \Lambda_{i-1}^+ \) if \( i > 1 \). For \( \nu \in U_1 \), denote by \( c(i, j) \) the \((i, j)\)-entry of \( \nu \cdot \pi^\alpha \).

1. If \( \alpha_{n-i+1} \) is even, say \( 2e \), then \( c(i, i) = \pi^e \) and \( c(i, j) \in \mathcal{P}^{2e+1} \) for \( 1 \leq j < i \).

2. Assume \( \alpha_{n-i+1} = \alpha_{n-i} \) are odd, say \( 2e - 1 \). Then
\[
\begin{pmatrix}
c(i, i) & c(i, i+1) \\
c(i+1, i) & c(i+1, i+1)
\end{pmatrix} = \begin{pmatrix}
\pi^e T_{ld}(u) & -\pi^{2e-1} \\
\pi^{2e-1} & 0
\end{pmatrix},
\]
\[
c(i, j), c(i+1, j) \in \mathcal{P}^{2e} \text{ for } 1 \leq j \leq i-1,
\]
where \( u\Pi \) is the \((i, i+1)\)-entry of \( \nu \).
Proof. We see the results by a direct calculation of $\nu\check{\pi}^\alpha \times \nu^*$, where we notice that diagonal entries belong to $k$.

Lemma 4.5 Let $\alpha = (\alpha_i) \in \Lambda_n^+$ and $m \leq n$. Assume that $\beta = (\alpha_{n-m+1}, \alpha_{n-m+2}, \ldots, \alpha_n) \in \Lambda_m^+$. Then for any $\nu \in U_1$, it holds

$$N_{rd}(\nu \cdot \check{\pi}^\alpha(m)) = N_{rd}(\check{\pi}^\beta), \quad v_\pi(d_m(\nu \cdot \check{\pi}^\alpha)) = v_\pi(d_m(\check{\pi}^\alpha)) = \frac{1}{2} |\beta|$$

where $v_\pi(\cdot)$ is the additive value on $k$.

Proof. By using Lemma 4.4 consecutively for $i \geq 1$, we see for $\nu \in U_1$,

$$d_m(\nu \cdot \check{\pi}^\alpha)^2 = N_{rd}((\check{\pi}^\alpha)(m)) = N_{rd}(\check{\pi}^\beta) \text{ and } v_\pi(d_m(\nu \cdot \check{\pi}^\alpha)) = \frac{1}{2} |\beta|.$$  

Proposition 4.6 If $\alpha$ is even, then $\lambda_\alpha = \frac{1}{2} \alpha$ and one has

$$\delta(\check{\pi}^\alpha; z) = q^{(\lambda_\alpha, jz) + (\lambda_\alpha, z_0)},$$

where $z_0$ is defined in Theorem 4.1.

Proof. Assume that $\alpha$ is even and write $\lambda = \frac{1}{2} \alpha = (\lambda_i)$. Since we may apply Lemma 4.5 for every $m$, we have

$$\delta(\check{\pi}^\alpha; z) = \prod_{i=1}^n |d_i(\check{\pi}^\alpha)|^{s_i} = \prod_{i=1}^n q^{-(\lambda_0 + \lambda_{n-1} + \ldots + \lambda_{n-i+1})s_i}$$

$$= \prod_{i=1}^n q^{-\lambda_i(s_{n-i+1} + \ldots + s_{n-1} + s_n)} = q^{\sum_{i=1}^n (\lambda_i(z_{n-i+1} - n + 2i - 1)}$$

$$= q^{(\lambda, jz) + (\lambda, z_0)}.$$  

Lemma 4.7 Let $\alpha = (\gamma, \beta) \in \Lambda_n^+$ and $\beta \in \Lambda_m^+$ and assume $\beta_1 = \beta_2 = 2e - 1$. For each $\nu \in U_1$, denote by $c_\nu$ the $(m, m)$-entry of the inverse of the $m \times m$-block of $\nu \cdot \check{\pi}^\alpha$. Then one has

$$v_\pi(d_{m-1}(\nu \cdot \check{\pi}^\alpha)) = \frac{1}{2} |\beta| + v_\pi(c_\nu), \quad (4.10)$$

$$\text{vol}(\{ \nu \in U_1 \mid v_\pi(c_\nu) = r \}) = (1 - q^{-1})q^{-r - e + 1}, \quad \text{for } r \geq -e + 1, \quad (4.11)$$

which depends only on the choice of first $m$ rows of $\nu$. Here we normalize the measure on $U_1$ as $\text{vol}(U_1) = 1$.

Proof. By (3.5), Lemma 4.5 and the fact $c_\nu \in k$, we have

$$d_{m-1}(\nu \cdot \check{\pi}^\alpha)^2 = N_{rd}((\nu \cdot \check{\pi}^\alpha)(m-1)) = N_{rd}((\nu \cdot \check{\pi}^\alpha)(m))N_{rd}(c_\nu) = N_{rd}(\check{\pi}^\alpha)c_\nu^2.$$
Then we obtain the identity (4.10) by (1.13). We decompose \( \nu \in U_1 = U_1(n) \) as
\[
\nu = \begin{pmatrix} \nu_1 & \nu_1 w \\ 0 & \nu_2 \end{pmatrix}, \quad \nu_1 \in U_1(m), \; \nu_2 \in U_1(n-m), \; w \in M_{m,n-m}(\mathcal{P}),
\]
then
\[
(\nu \cdot \bar{\pi}^\alpha)^{(m)} = \nu_1(\bar{\pi}^\beta + w \cdot \bar{\pi}^\gamma)\nu_1^*.
\]
(4.12)
For the simplicity of notation, we set \( \beta = (b_m, b_{m-1}, \ldots, b_1) \), where \( 2e-1 = b_m = b_{m-1} (= b, \text{ say}) \) by the assumption. Set \( M = \bar{\pi}^\beta + w \cdot \bar{\pi}^\gamma \in X_m \). Then we may decompose \( M \) as follows:
\[
M = M_1 \times M_2,
\]
\[
M_1 = \text{Diag}(\Pi^{b_1}u_1, \ldots, \Pi^{b_{m-2}}u_{m-2}, -\Pi^{b}u^*, u\Pi^{b}), \quad u_i, u \in \mathcal{O}^v.
\]
and \( M_2 \in K_m \) is congruent modulo \( \mathcal{P}^2 \) to the matrix which has diagonal entry 1 associated to even \( b_i \) and diagonal block \( H_1 \) associated to an odd pair \( b_i, b_{i+1} \) and all the other entries are 0, where \( H_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). For example, if \( \beta = (3,3,1,1,0,0,0) \in \Lambda_7 \) then
\[
M_2 \equiv \begin{pmatrix} 1_3 & 0 & 0 \\ 0 & H_1 & 0 \\ 0 & 0 & H_1 \end{pmatrix} \pmod{\mathcal{P}^2}.
\]
Since any entry of \( w \cdot \bar{\pi}^\gamma \) belongs to \( \mathcal{P}^{b+2} \), we see
\[
\text{(the lowest two rows of } M^{-1}) \equiv \begin{pmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \end{pmatrix} - u^{-1} \Pi^{-b} \begin{pmatrix} \Pi^{-b}u^{-1} \\ \Pi^{-b+1} \end{pmatrix} \pmod{\mathcal{P}^{b+2}}.
\]
(4.14)
Denote by \( \Pi v^* \) the \((m-1,m)\)-entry of \( \nu_1 \in U_1(m) \), where \( v \in \mathcal{O} \). Then the \((m,m-1)\)-entry of \( \nu_1^{-1} \) is \( v\Pi \), and we see by (4.14)
\[
c_\nu = (x_1, x_2, \ldots, x_{m-2}, -u^{-1}\Pi^{-b} + x_{m-1}, v\Pi^{-b+1}u^{-1} + x_m) \begin{pmatrix} y_1 \\ \vdots \\ y_{m-2} \\ -\Pi v^* \\ 1 \end{pmatrix}
\]
\[
\equiv -u^{-1}\Pi^{-b+1}v^* + v\Pi^{-b+1}u^{-1} \equiv \pi^{-e} T_{rd}(vu^{-1}) \pmod{\mathcal{P}^{b+2}},
\]
where \( x_1, \ldots, x_{m-2} \in \mathcal{P}^{-b+1}, \; x_{m-1}, x_m \in \mathcal{P}^{-b+3}, \; y_1, \ldots, y_{m-2} \in \mathcal{P} \), they are determined by \( \nu_1 \) and \( w \), and the above column vector is the \( m \)-th column of \( \nu_1 \). Since \( c_\nu \in k \), we may write
\[
c_\nu = \pi^{-e+1}(T_{rd}(vu^{-1}) + \pi z), \quad z \in \mathfrak{o}.
\]
Here \( z \) is determined by \( \alpha \) and \( w \) and \( \nu_1 \) except the \((m-1,m)\)-entry \( \Pi v^* \), and \( u \in \mathcal{O}^v \) is determined by \( \alpha \) and \( w \) (cf. (4.13)). Hence \( v_{\pi}(c_\nu) \geq -e + 1 \), which is determined by the choice of \( v \in \mathcal{O} \) and independent of the choice of \( \nu_2 \). For \( r \geq -e + 1 \),
\[
\text{vol}(\{ \nu \in U_1 \mid v_{\pi}(c_\nu) = r \}) = \text{vol}(\{ v \in \mathcal{O} \mid T_{rd}(v) \in \pi^{r-e+1} \mathfrak{o}^x \})
\]
\[
= q^{-4\ell \pi} \left\{ \pi \in \mathcal{O}/\mathcal{P}^{2\ell} \mid T_{rd}(u) \in \pi^{r+e-1} \mathfrak{o}^x \right\}
\]
\[
= q^{-r-e+1}(1 - q^{-1}),
\]
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where we take \( \ell \geq r + e \) and we regard \( T_{rd} \) as \((q^{3\ell} : 1)\)-mapping from \( \mathcal{O}/\mathcal{P}^{2\ell} \) onto \( \mathfrak{o}/\mathfrak{p}^\ell \).

Now, we assume \( \alpha \) is has odd entries and set \( I_{odd}(\alpha) = \{ \ell_1, \ldots, \ell_k \} \) (cf. \( 4.3 \)) and \( \alpha_{\ell_j} = 2e_j - 1 \). For \( m = n - \ell_j, 1 \leq j \leq k \), we use Lemma 4.7 and for the other \( m \), we may use Lemma 4.5. Then we obtain

\[
\delta(\pi^{\alpha}; z) = \prod_{m \neq n - \ell_j} q^{-\frac{1}{2}(\alpha_{n-m+1} + \cdots + \alpha_{n})s_{m}} \\
\times \prod_{j=1}^{k} \sum_{r_{j} \geq s_{j} - 1} (1 - q^{-1}) q^{-(r_{j}+s_{j}-1)} q^{-\left(\frac{1}{2}(\alpha_{\ell_j} + \alpha_{\ell_j + 1} + \cdots + \alpha_{n}) + r_{j}\right)s_{n-\ell_j}} \\
= \prod_{m \neq n - \ell_j} q^{-\frac{1}{2}(\alpha_{n-m+1} + \cdots + \alpha_{n})s_{m}} \prod_{j=1}^{k} \frac{1 - q^{-1}}{1 - q^{-1} - s_{n-\ell_j}} q^{-\frac{1}{2}(\alpha_{\ell_j + 1} + \alpha_{\ell_j + 2} + \cdots + \alpha_{n})s_{n-\ell_j}} \quad (\alpha_{\ell_j} = \alpha_{\ell_j + 1} = 2e_j - 1)
\]

\[
= \prod_{i=1}^{n} q^{-\frac{1}{2}\alpha_{i}(s_{n-i+1} + \cdots + s_{n})} \prod_{j=1}^{k} \frac{(1 - q^{-1}) q^{\frac{1}{2}(s_{n-\ell_j} - s_{n-\ell_j + 1})} - q^{s_{n-\ell_j} - s_{n-\ell_j + 1} + 1}}{1 - q^{s_{n-\ell_j} - s_{n-\ell_j + 1} + 1}} \\
= \prod_{i=1}^{n} q^{\frac{1}{2}\alpha_{i}(s_{n-i+1} - n+2t-1)} \prod_{j=1}^{k} \frac{(1 - q^{-1}) q^{\frac{1}{2}(s_{n-\ell_j} + s_{n-\ell_j + 1}) + 1}}{q^{s_{n-\ell_j + 1}} - q^{s_{n-\ell_j} + 1}} \\
= q^{(\lambda_{\alpha} , z_{0} ) - \frac{1}{2} \sum_{j=1}^{k} (-n+2\ell_{j} - 1)+(-n+2(\ell_{j} + 1)-1)} \cdot q^{(\lambda_{\alpha} , jz )} \prod_{j=1}^{k} \frac{(1 - q^{-1}) q}{q^{s_{n-\ell_j + 1}} - q^{s_{n-\ell_j} + 1}} \\
= \frac{(1 - q^{-1}) k^{\sum_{j=1}^{k} (n-2\ell_{j} + 1)}}{\prod_{\ell \in I_{odd}(\alpha)} (q^{s_{n-\ell} + 1} - q^{s_{n-\ell} + 1})} \times q^{(\lambda_{\alpha} , z_{0} ) + (\lambda_{\alpha} , jz )} \\
= \frac{c_{odd}(\alpha)}{\prod_{\ell \in I_{odd}(\alpha)} (q^{s_{n-\ell} + 1} - q^{s_{n-\ell} + 1}) \times q^{(\lambda_{\alpha} , z_{0} ) + (\lambda_{\alpha} , jz )}}.
\]

§5 Schwartz space \( S(K\setminus X) \)

5.1. In this subsection, we study \( \mathcal{H}(G, K) \)-module structure of \( S(K\setminus X) \) through the spherical transform \( F \), so we need to recall Theorem 3.5. For each \( \alpha \in \Lambda_{n} \), we denote by \( \varphi_{\alpha} \in S(K\setminus X) \) the characteristic function of \( K \cdot \pi^{\alpha} \) and \( \varphi(x) = \varphi(x^{-1}) \) for \( \varphi \in S(K\setminus X) \). Then

\[
F(\varphi_{\alpha}) = vol(K \cdot \pi^{\alpha}) \Psi(\pi^{\alpha}; z) \equiv \Psi(\pi^{\alpha}; z),
\]

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where \( \equiv \) means \( \equiv (\mod \mathbb{R}^*) \) in this subsection. The image \( F(S(K\setminus X)) = \langle \Psi(\pi^\alpha; z) : \alpha \in \Lambda_n \rangle \subset \mathbb{C} \) is an ideal of \( R = \mathbb{C}[q^{\pm 1}, \ldots, q^{\pm n}]^{S_n} \), where \( R \) is isomorphic to \( H(G,K) \) by Satake isomorphism (cf. (1.23)).

To make sure of it, we note the results for \( n = 1, 2 \), which is easily seen by definition of spherical function for \( n = 1 \) and Proposition 5.1 for \( n = 2 \).

**Proposition 5.1** Assume \( n = 1 \) or \( 2 \). Then the spherical transform \( F : S(K\setminus X) \to R \) is an \( H(G,K) \)-module isomorphism, and \( S(K\setminus X) \) is generated as an \( H(G,K) \)-module by \( \varphi_0 \) for \( n = 1 \) and \( \varphi(-1,-1) \) for \( n = 2 \).

As a corollary we see the following, where we omit the proof since it can be proved similarly and more easily to Proposition 5.7.

**Proposition 5.2** Assume \( n = 1 \) or \( 2 \). Then any spherical function on \( X \) is a constant multiple of \( \Psi(x; z_0) \) for some \( z_0 \in \mathbb{C}^n/S_n \).

In the rest of this subsection, we consider the case \( n \geq 3 \). For simplicity of notation, we set \( x_i = q^{z_i}, \ 1 \leq i \leq n \) and denote by \( s_i(n) \) the fundamental symmetric polynomial in \( x_1, \ldots, x_n \), for \( 1 \leq i \leq n \). Then

\[
R = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n} = \mathbb{C}[s_1(n), \ldots, s_n(n), s_n(n)^{-1}] = R_0[s_n(n)^{-1}], \tag{5.1}
\]

where \( R_0 = \mathbb{C}[s_1(n), \ldots, s_n(n)] \). We denote by \( J \) the ideal of \( R_0 \) generated by the subset \( \{ \Psi(\pi^\alpha; z) : \alpha \in \Lambda_n, \alpha_n = 0, -1 \} \) of \( R_0 \). Then, since \( \Psi(\pi^\alpha(2e); z) = s_n(n)\Psi(\pi^\alpha; z) \) for \( e \in \mathbb{Z} \), we see that \( F(S(K\setminus X)) = J \otimes_{R_0} R \).

For a (fixed) rational function \( c(x) \) of \( x_1, \ldots, x_n \) and \( \mu \in \mathbb{Z}^n \), we define

\[
P(c(x), \mu; x) = \sum_{\sigma \in S_n} \sigma(x_1^{\mu_1} \cdots x_n^{\mu_n} \cdot c(x)) \in \mathbb{C}(x_1, \ldots, x_n)^{S_n}, \tag{5.2}
\]

where \( S_n \) acts on \( x_1, \ldots, x_n \) by permutation of indices.

**Lemma 5.3** For any \( \lambda \in \tilde{\Lambda}_n \) with \( \lambda_n \geq 0 \), the rational function \( P(c(x), \lambda; x) \) belongs to the \( R_0 \)-module generated by the set \( \{ P(c(x), \mu; x) : \mu \in \Sigma_n \} \), where \( \Sigma_n = \{ \mu = (\mu_i) \in \mathbb{Z}^n : 0 \leq \mu_i \leq n-1 \} \).

**Proof.** Since \( s_i(n + 1) = s_i(n) + s_{i-1}(n)x_{n+1} \), we have the following by induction on \( n \).

\[
x_r^\ell = \sum_{i=1}^{\ell} (-1)^{i-1}s_i(n)x_r^{\ell-i}, \quad (1 \leq \ell \leq n), \quad \text{if} \ r \geq n. \tag{5.3}
\]

For each \( \lambda \in \tilde{\Lambda}_n \), we set \( \lambda^{(\ell,i)} = (\lambda_1, \ldots, \lambda_{\ell-1}, \lambda_\ell - i, \lambda_{\ell+1}, \ldots, \lambda_n) \in \mathbb{Z}^n \). Then, by (5.3), we have

\[
P(c(x), \lambda; x) = \sum_{i=1}^{n} (-1)^i s_i(n) P(c(x), \lambda^{(\ell,i)}; x), \quad \text{if} \ \lambda_\ell \geq n. \tag{5.4}
\]
Taking this procedure for every $\ell$ with $\lambda_\ell \geq n$, we have the result. 

By computer calculation, it is possible to express symmetric polynomials in terms of $s_i(n)$, $1 \leq i \leq n$, if the variable $n$ and degrees of polynomials are small. The author owes Satoshi Murai for a program using Macaulay2(\cite{Mac2}), which worked well for size $n \leq 4$. Further it is possible to check for a polynomial whether it is contained in a fixed ideal of $\mathcal{R}_0$ or not, by Macaulay2. Thus we have the following proposition.

**Proposition 5.4** Assume $n = 3$.

(1) $F(\mathcal{S}(K\setminus X))$ is an ideal of $\mathcal{R}$ generated by $\Psi(1_3; z) \equiv s_1s_2 - q^2(q^{-2} + q^{-1} + 1)^2s_3$ and $\Psi(\pi^{(0,1,-1)}; z) \equiv s_1^2 - q^2(q^{-2} + q^{-1} + 1)^2s_2$, where $s_i = s_i(3)$, $1 \leq i \leq 3$, and it is non-principal.

(2) The $\mathcal{H}(G, K)$-module $\mathcal{S}(K\setminus X)$ is generated by $\phi_0$ and $\phi_{(1,1,0)}$, and it is not monomial.

**Proof.** (1) In consideration of the explicit formula of $\Psi(\pi^\alpha; z)$ (Theorem 4.1), we set

$$c_3(x) = \prod_{1 \leq i < j \leq 3} \frac{(x_i - qx_j)(x_i - q^{-2}x_j)}{x_i - x_j},$$

$$P1(a, b; x) = P(c_3(x), (a, b, 0); x) \equiv \Psi(\pi^{(2a,2b,0)}; z),$$

$$P2(a; x) = P(c_3(x), (a, 0, 0); x) \equiv \Psi(\pi^{(2a,-1,1)}; z),$$

$$P3(a; x) = P(c_3(x), (a, a, 0); x) \equiv \Psi(\pi^{(2a-1,2a-1,0)}; z).$$ (5.5)

Then $J$ is generated by the set $\{P1(a, b; x) \mid a \geq b \geq 0\} \cup \{P2(a; x) \mid a \geq 0\} \cup \{P3(a; x) \mid a \geq 1\}$. Let $J_0$ be the ideal of $\mathcal{R}_0$ generated $P1(0, 0; x) \equiv \Psi(\pi^{(0,0,0)}; z) = \Psi(1_3; z)$ and $P2(0; x) \equiv \Psi(\pi^{(0,-1,1)}; z)$, where their values are calculated as above. It is clear that $J_0$ is not principal. We may check by computer that the set $\{P1(a, b; x) \mid a, b = 0, 1, 2\} \cup \{P2(a; x) \mid a = 1, 2\} \cup \{P3(a; x) \mid a = 1, 2\}$ is contained in $J_0$, which brings $J = J_0$ by Lemma 5.3. As for $P3$, the above set is enough, since we may use $x_1^3x_2^2 = s_2x_1^2x_2^2 - s_1s_3x_1x_2 + s_2^3$ instead of (5.3).

(2) Since $F$ is an injective $\mathcal{H}(G, K)$-module map and $F(\phi_0) \equiv \Psi(\pi^\alpha; z)$, we see that $\mathcal{S}(K\setminus X)$ is generated by $\phi_0 = \phi_0$ and $\phi_{(0,1,1)} = \phi_{(1,1,0)}$ and is not monomial. 

**Remark 5.5** In the above, any polynomial in $\{P1(a, b; x) \mid 0 \leq a < b \leq 2\}$ is not combined with $\Psi(\pi^\alpha; z), \alpha \in \Lambda_3$, it is not assured to be contained even in $J$. Fortunately, it is contained in $J_0$.

In the similar way to size 3, we obtain the following result for size 4. We have to consider 5 types of polynomials associated with $\Psi(\pi^\alpha; z)$ according to the location of odd entries of $\alpha \in \Lambda_4$ (cf. Theorem 4.1). Then, by Lemma 5.3, it is enough to verify a certain finite set of polynomials. In this case, as with the case of size 3, some polynomials are not combined with $\Psi(\pi^\alpha; z)$, but they are contained in the ideal fortunately. The second claim follows from the first one and the fact the spherical transform $F$ is an injective $\mathcal{H}(G, K)$-module map.
Proposition 5.6 Assume $n = 4$.

1. $F(S(K\setminus X))$ is an ideal of $R$ generated by two elements

$$\Psi(1_4; z) \equiv s_1 s_2 s_3 - q^2(q^{-2} + q^{-1} + 1)^2 s_1 s_4$$

$$-q^2(q^{-2} + q^{-1} + 1)^2 s_2^2 + q^3(q^{-2} + 1)(q^{-1} + 1)^4 s_2 s_4,$$

$$\Psi(\pi^{-1,-1,-1,1}; z) \equiv s_2^2 - q(q^{-2} + q^{-1} + 1) s_1 s_3 + q^3(q^{-2} + 1)^2(q^{-2} + q^{-1} + 1)s_4,$$

where $s_i = s_i(4)$, $1 \leq i \leq 4$, and it is non-principal.

2. The $H(G, K)$-module $S(K\setminus X)$ is generated by $\varphi_0$ and $\varphi_{(1,1,1,1)}$, and not monomial.

As a corollary of Propositions 5.4 and 5.6 we have the following.

Proposition 5.7 Assume $n = 3, 4$, and set $\beta = (1, 1, 0)$ for $n = 3$ and $\beta = (1, 1, 1, 1)$ for $n = 4$. If $\Psi(1_n; z_0) \neq 0$ and $\Psi(\pi^\beta; z_0) \neq 0$, then any spherical function on $X$ corresponding to $z_0$ is a constant multiple of $\Psi(x; z_0)$.

Proof. Any spherical function is associated with some $z_0 \in \mathbb{C}^n$ by its eigenvalue (cf. the comment at the end of §1). We introduce the pairing on $S(K\setminus X) \times C^\infty(K\setminus X)$ by

$$\langle \varphi, \Phi \rangle = \int_X \varphi(x) \Phi(x) dx, \quad \varphi \in S(K\setminus X), \ \Phi \in C^\infty(K\setminus X)$$

where $dx$ is the $G$-invariant measure. Then it satisfies for any $f \in H(G, K)$,

$$\langle f * \varphi, \Phi \rangle = \langle \varphi, \hat{f} * \Phi \rangle, \quad \hat{f}(g) = f(g^{-1}).$$

Assume $\Phi$ is a spherical function on $X$ corresponding to $z_0$ which satisfies the assumption above. Denote $\varphi_0 = \varphi_0$ and $\varphi_1 = \varphi_\beta$. Then for any $f \in H(G, K)$ and $i = 0, 1$, one has by (5.6) and (5.7)

$$\langle f * \varphi_i, \Phi \rangle = \lambda_{z_0}(\hat{f}) \langle \varphi_i, \Phi \rangle,$$

$$\langle f * \varphi_i, \Psi(\ ; z_0) \rangle = \lambda_{z_0}(\hat{f}) \langle \varphi_i, \Psi(\ ; z_0) \rangle,$$

where $\langle \varphi_i, \Psi(\ ; z_0) \rangle \neq 0$ by the choice of $z_0$. Thus

$$\langle f * \varphi_i, \Phi \rangle = \frac{\langle \varphi_i, \Phi \rangle}{\langle \varphi_i, \Psi(\ ; z_0) \rangle} \langle f * \varphi_i, \Psi(\ ; z_0) \rangle, \quad f \in H(G, K), \ i = 0, 1. \quad (5.8)$$

On the other hand, by the commutative diagram (3.12), there is some $g_i \in H(G, K)$, $i = 0, 1$ such that

$$\lambda_z(g_1) F(\varphi_0) = \lambda_z(g_0) F(\varphi_1) \neq 0$$

thus it holds

$$g_1 * \varphi_0 = g_0 * \varphi_1 (\neq 0), \quad (5.9)$$
By \((5.8), (5.9)\) and \((5.10)\), one sees

\[
\langle g_1 * \varphi_0, \Psi(\cdot ; z_0) \rangle = \langle g_0 * \varphi_1, \Psi(\cdot ; z_0) \rangle (\neq 0), \tag{5.10}
\]

By \((5.8), (5.9)\) and \((5.10)\), one sees

\[
c_\Phi := \frac{\langle \varphi_0, \Phi \rangle}{\langle \varphi_0, \Psi(\cdot ; z_0) \rangle} = \frac{\langle \varphi_1, \Phi \rangle}{\langle \varphi_1, \Psi(\cdot ; z_0) \rangle} (\neq 0).
\]

Then, since \(S(K \backslash X) = H(G, K) * \varphi_0 + H(G, K) * \varphi_1\), we have

\[
\langle \varphi, \Phi \rangle = c_\Phi \langle \varphi, \Psi(\cdot ; z_0) \rangle = \langle \varphi, c_\Phi \Psi(\cdot ; z_0) \rangle, \quad \varphi \in S(K \backslash X),
\]

which yields \(\Phi(x) = c_\Phi \Psi(x; z_0)\) in \(C^\infty(K \backslash X)\) as required.

\[\text{Remark 5.8}\]

It is expected that, for general size \(n \geq 5\), the \(H(G, K)\)-module \(S(K \backslash X)\) is generated by \(\varphi_0\) and \(\varphi_\beta\), where \(\beta = (1, \ldots, 1)\) or \((1, \ldots, 1, 0)\) according to the parity of \(n\), and is not monomial. In other words, it is expected that the ideal \(F(S(K \backslash X))\) of \(R\) is generated by \(\Psi(\pi^0; z) = \Psi(1_n; z)\) and \(\Psi(\pi^\beta; z)\) where \(\beta' = (-1, \ldots, -1)\) or \((0, -1, \ldots, -1)\), and is not principal. If this is true for \(n\), then the parallel result for \(n\) to Proposition \(5.7\) holds.

\[\text{5.2.} \]

We introduce the Plancherel formula for size 2 proved by Y. Komori. Throughout this subsection we only consider the case of size 2, hence \(X \subset G = GL_2(D)\).

\[\text{5.2.1}\]

Since \(G_n\)-invariant measure \(dx\) on \(X_n\) of size \(n\) is determined up to constant by the differential form

\[
\frac{\wedge^n_{i=1} dx_{ii} \wedge \wedge_{1 \leq i < j \leq n} dx_{ij}}{|N_{D}(x)|^{\frac{n-1}{2}}} = (x = (x_{ij}) \in X, \ x_{ii} \in k, \ x_{ij} \in D), \tag{5.11}
\]

volume \(v(K \cdot \pi^\alpha)\) in the case of size 2 is a constant multiple of \(q^{\frac{3}{2}|\alpha|}/\mu(\pi^\alpha, \pi^\alpha)\), and by Theorem \(2.3\) we have

\[
\mu(\pi^\alpha, \pi^\alpha) = \begin{cases} 
q^{6\lambda_1}(1 + q^{-1})(1 - q^{-2}) & \text{if } \alpha = (2\lambda_1, 2\lambda_1) \\
q^{\lambda_1 + 5\lambda_2}(1 + q^{-1})^2 & \text{if } \alpha = (2\lambda_1, 2\lambda_2), \ \lambda_1 > \lambda_2 \\
q^{6\ell-2}(1 - q^{-4}) & \text{if } \alpha = (2e - 1, 2e - 1) \tag{5.12}
\end{cases}
\]

We normalize \(dx\) as \(v(K \cdot 1_n) = 1\). Then, for the characteristic function \(\varphi_\alpha\) of \(K \cdot \pi^\alpha, \ \alpha \in \Lambda_2\) and \(\varphi_\alpha(x) = \varphi_\alpha(x^{-1})\), we see

\[
\int_X \varphi_\alpha(x) \overline{\varphi_\beta(x)} dx = \delta_{\alpha, \beta} \times v(K \cdot \pi^{(-\alpha_2, -\alpha_1)}) = \delta_{\alpha, \beta} \times v(K \cdot \pi^\alpha)
\]

\[
= \delta_{\alpha, \beta} \times \begin{cases} 
1 & \text{if } \alpha_1 = \alpha_2 \in 2\mathbb{Z} \\
q^{2(\lambda_1 - \lambda_2)}(1 - q^{-1}) & \text{if } \alpha = (2\lambda_1, 2\lambda_2), \ \lambda_1 > \lambda_2 \\
q^{\lambda_1 - \lambda_2}(1 + q^{-1}) - q^{-1} & \text{if } \alpha_1 = \alpha_2 \notin 2\mathbb{Z} \tag{5.13}
\end{cases}
\]

\[30\]
On the other hand, by the definition of $F$ (cf. (3.10), (3.11)) and Proposition 3.1 we have

$$F(\tilde{\varphi}_\alpha) = v(K \cdot \pi^\alpha) \Psi(\pi^\alpha; z) = v(K \cdot \pi^\alpha)\omega(\pi^\alpha, z)(q^{z_2} - q^{z_1 + 1})$$

$$= \begin{cases} 
\frac{1}{1 + q^{-z}} Q_\lambda(z) & \text{if } \alpha = 2\lambda, \lambda_1 = \lambda_2, \\
\frac{q^{\lambda_1 - \lambda_2}(1 - q^{-1})}{1 + q^{-z}} Q_\lambda(z) & \text{if } \alpha = 2\lambda, \lambda_1 > \lambda_2, \\
\frac{1 - q^{-2}}{1 + q^{-z}} q^{\ell(z_1 + z_2)} & \text{if } \alpha = (2\varepsilon - 1, 2\varepsilon - 1).
\end{cases} \quad (5.14)$$

where

$$Q_\lambda(z) = (q^{z_1} + q^{z_2}) \sum_{\sigma \in S_2} \sigma \left( q^{\lambda_1 z_1 + \lambda_2 z_2} \frac{(1 - q^{z_2 - z_1 + 1})(1 - q^{z_2 - z_1 - 2})}{1 - q^{2(z_2 - z_1)}} \right). \quad (5.15)$$

5.2.2 Fix $u_i$ as $0 < u_i < 1$, $i = 1, 2$ and set

$$H_\ell(y) = \sum_{\sigma \in S_2} \sigma \left( q^{-\ell y} \frac{(1 - u_1 q^{2y})(1 - u_2 q^{2y})}{1 - q^{2y}} \right). \quad (\ell \in \mathbb{N}), \quad (5.16)$$

$$w(y) = \frac{1}{1 - u_1 q^{2y}(1 - u_2 q^{2y})} \frac{1 - q^{-2y}}{(1 - u_1 q^{-2y})(1 - u_2 q^{-2y})}. \quad (5.17)$$

Take $U = \{ y = \sqrt{-1}t \mid 0 \leq t \leq 2\pi \log q \}$ and a (suitably normalized) measure $dy$ on $U$, one has

$$\int_U H_\ell(y) H_m(y) w(y) dy = \begin{cases} 
1 - u_1 u_2 & (\ell = m = 1), \\
1 & (\ell = m > 1), \\
0 & (\ell \neq m),
\end{cases} \quad (\ell \geq 1), \quad (5.18)$$

$$\int_U H_\ell(y) w(y) dy = 0 \quad (\ell \geq 1), \quad (5.19)$$

$$\int_U w(y) dy = \frac{1}{(1 + u_1)(1 + u_2)(1 - u_1 u_2)}. \quad (5.20)$$

where $\overline{H_m(y)}$ is the complex conjugate of $H_m(y)$. As for (5.20), we calculate the integral

$$\frac{1}{2\pi \sqrt{-1}} \int_{|Y| = 1} \frac{1 - Y^{-2}}{(1 - u_1 Y^{-2})(1 - u_2 Y^{-2})} \cdot \frac{1 - Y^2}{(1 - u_1 Y^2)(1 - u_2 Y^2)} dY = \frac{2}{(1 + u_1)(1 + u_2)(1 - u_1 u_2)}.$$

Remark 5.9 The set $\{H_m\}$ essentially coincides with a special case of the Hall-Littlewood limit of the Askey-Wilson polynomials [KS], that is, the limit $q \to 0$ in the context of $q$-orthogonal polynomials. This follows from the fact that the Hall-Littlewood limit of the Askey-Wilson polynomials satisfies the orthogonal conditions (5.23) and (5.24) and that such polynomials with the leading terms $Y^m$ are unique.
Set \( x = \frac{z_1 + z_2}{2}, \ y = \frac{z_1 - z_2}{2} \) and \( u_1 = q, \ u_2 = q^{-2} \). Then for \( \lambda = (\lambda_1, \lambda_2) \in \hat{\Lambda}_2 \), we have

\[
Q_\lambda(z) = q^{(\lambda_1 + \lambda_2 + 1)x} (q^y + q^{-y}) \sum_{\sigma \in S_2} \sigma \left( q^{-(\lambda_1 - \lambda_2)y} \frac{(1 - u_1 q^{2y})(1 - u_2 q^{2y})}{1 - q^{4y}} \right)
\]

\[
= q^{(|\lambda| + 1)x} \sum_{\sigma \in S_2} \sigma \left( q^{-(\lambda_1 - \lambda_2 + 1)y} \frac{(1 - u_1 q^{2y})(1 - u_2 q^{2y})}{1 - q^{2y}} \right)
\]

\[
= q^{(|\lambda| + 1)x} H_{\lambda_1 - \lambda_2 + 1}(y), \quad (5.21)
\]

where we set \(|\lambda| = \lambda_1 + \lambda_2\). For \( e \in \mathbb{Z} \), we define

\[
R_e(x, y) = q^{(e+1)(z_1 + z_2)} = q^{2(e+1)x}. \quad (5.22)
\]

Now, for a while, we consider \( u_1, u_2 \) are independent of \( q \) and still keep the condition \( 0 < u_1, u_2 < 1 \). We set \( T = \{ x = \sqrt{-1t} \mid 0 \leq t \leq 2\pi \log q \} \) and \( U = \{ y = \sqrt{-1t} \mid 0 \leq t \leq 2\pi \log q \} \), and define the inner product on \( \mathbb{C}[q^x, q^{-x}, q^y, q^{-y}] \) by

\[
\langle f, g \rangle = \int_T dx \int_U f \overline{g} w(y) dy. \quad (5.23)
\]

Then we see, (cf. (5.18) also)

\[
\langle R_e, R_{e'} \rangle = 0, \quad \text{for } e, e' \in \mathbb{Z}, \ e \neq e', \quad (5.24)
\]

\[
\langle Q_\lambda, Q_\mu \rangle = 0, \quad \text{for } \lambda, \mu \in \hat{\Lambda}_2, \ \lambda \neq \mu. \quad (5.25)
\]

If \(|\lambda| = \lambda_1 + \lambda_2\) is even, by the integral with respect to \( x \), we see \( \langle Q_\lambda, H_e \rangle = 0 \). Assume \(|\lambda|\) is odd. Then \( \langle Q_\lambda, H_e \rangle = 0 \) unless \(|\lambda| = 2e + 1\). When \(|\lambda| = 2e + 1, \ \lambda_1 - \lambda_2 = 2n - 1 > 0, \) and \( Q_\lambda(x, y) = q^{(|\lambda| + 1)x} H_{2n}(y) \). Then \( \langle Q_\lambda, R_e \rangle = 0 \) by (5.19).

We have to consider the case \( u_1 = q \) and \( u_2 = q^{-2} \). We fix \( u_2 = q^{-2} \) and change \( u_1 \) continuously from \( 0 < u_1 < 1 \) to \( q \). When \( 0 < u_1 < 1 \), the poles of \( w(y) \) are, written by variable \( Y = q^y \),

\[
Y = \pm \sqrt{u_1}, \ \pm \sqrt{u_2} = \pm q^{-1} \quad \text{within } |Y| < 1, \quad (5.26)
\]

\[
Y = \pm \sqrt{u_1^{-1}}, \ \pm \sqrt{u_2^{-1}} = \pm q \quad \text{outside of } |Y| = 1. \quad (5.27)
\]

According to the change of \( u_1 \), we change the integration path \( |Y| = 1 \) as the path does not change the way around these poles (cf. Figures 1,2).
5.2.3 We calculate the norm \( \langle F(\varphi_\alpha), F(\varphi_\alpha) \rangle \) by using (5.14), (5.18), (5.20), and (5.21). When \( \alpha = (2e, 2e) \),
\[
\langle F(\varphi_\alpha), F(\varphi_\alpha) \rangle = \frac{(1 - q^{-1})}{(1 + q^{-2})^2}, \quad (v(K \cdot \pi^\alpha) = 1); \quad (5.28)
\]
when \( \alpha = 2\lambda, \lambda_1 > \lambda_2 \), the value is
\[
\frac{q^{2(\lambda_1 - \lambda_2)}(1 - q^{-1})^2}{(1 + q^{-2})^2} = v(K \cdot \pi^\alpha) \times \frac{1 - q^{-1}}{(1 + q^{-2})^2}; \quad (5.29)
\]
when \( \alpha = (2e - 1, 2e - 1) \), the value is
\[
\frac{(1 - q^{-2})^2}{(1 + q^{-2})^2(1 + q)(1 + q^{-2})(1 - q^{-1})} = \frac{q^{-1}(1 - q^{-2})}{(1 + q^{-2})^3} = v(K \cdot \pi^\alpha) \times \frac{1 - q^{-1}}{(1 + q^{-2})^2}. \quad (5.30)
\]
By comparison with (5.13), we normalize the inner product (5.23) by multiplying \((1 + q^{-2})^2/(1 - q^{-1})\) and keep the notation, then we obtain
\[
\int_X \overline{\varphi_\alpha(x) \varphi_\beta(x)} dx = \langle F(\varphi_\alpha), F(\varphi_\beta) \rangle, \quad (\alpha, \beta \in \Lambda_2). \quad (5.31)
\]
Thus we have the Plancherel formula as following.

**Theorem 5.10 (Plancherel Formula)** Define the inner product on \( C[q^x, q^{-x}, q^y + q^{-y}] \) by
\[
\langle f, g \rangle = \int_T dx \int_U f \overline{w(y)} dy, \quad (5.32)
\]
where
\[
x = \frac{z_2 + z_1}{2}, \quad y = \frac{z_2 - z_1}{2}, \quad T = \{ x = \sqrt{-1}t \mid 0 \leq t \leq 2\pi \log q \},
\]
\( U \) is the set indicated in Figure 2, \( \int_{T \times U} dx dy = \frac{(1 + q^{-2})^2}{2(1 - q^{-1})}; \)
\[
w(y) = \frac{1 - q^{2y}}{(1 - q^{2y+1})(1 - q^{2y-2})} \cdot \frac{1 - q^{-2y}}{(1 - q^{-2y+1})(1 - q^{-2y-2})}.
\]
Then, for any $\varphi, \psi \in \mathcal{S}(K\backslash X)$, the following identity holds:

$$
\int_X \varphi(x)\overline{\psi(x)}dx = \langle F(\varphi), F(\psi) \rangle.
$$

(5.33)

As a corollary of Plancherel formula, we have

**Corollary 5.11 (Inversion Formula)** For any $\varphi \in \mathcal{S}(K\backslash X)$, the following identity holds:

$$
\varphi(x) = \frac{1}{v(K \cdot x)} \langle F(\varphi), F(ch_x) \rangle, \quad (x \in X),
$$

where $ch_x$ is the characteristic function of $K \cdot x$ in $\mathcal{S}(K\backslash X)$.

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