A Survey of HFs Method for Solving Nonlinear Volterra’s IE

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Abstract

Background/Objectives: The numerical results to the nonlinear IE of Volterra’s-type based on projection methods and Hybrid functions are investigated. Methods/Statistical Analysis: In this work, Hybrid functions with a new collocation method is presented. The method of creating HFs are briefly mentioned. Findings: The numerical results confirm that, Hybrid Functions (HFs) method is comparable with other famous techniques. Applications/Improvements: In future, with using another projection method such as Galerk in method and etc., this approach can be modified for Functional IE and IDEs.

Keywords: Block-pulse Functions, Collocation Method, Legendre Polynomials, Nonlinear Integral Equation

1. Introduction

There has been much interest in this problem since Volterra’s integral equations, which came from the electro-magnetic fluid dynamics, yields strong physical background¹–⁷. Moreover, the Fredholm integral equations of second kind are the special case of the Volterra’s integral equations⁸.

We consider the following Volterra’s type:

\[ y(t) = f(t) + \int_{0}^{t} k(t,s)g(s,y(s))ds, \quad t \in [a,b] \] (1)

where, \( f, k \) and \( g \) are given functions and \( y \) is the unknown.

We suppose that:

\[ z(t) = g(t,y(t)), \quad t \in [0,1]. \] (2)

(1) and (2) give the following equation:

\[ y(t) = f(t) + \int_{0}^{t} k(t,s)z(s)ds, \quad t \in [a,b] \] (3)

Then, for \( z(t) \) we have:

\[ z(t) = g(t,f(t) + \int_{0}^{t} k(t,s)z(s)ds), \quad t \in [a,b] \] (4)

With using some collocation points we have an algebraic nonlinear equation. The approximation to \( y(t) \) is \( y_j(t) \), where

\[ y_j(t) = f(t) + \int_{0}^{t} k(t,s)z_j(s)ds \] (5)

By substituting \( z_j(t) \) into the right-hand side of (3) within our method, the solution \( y(t) \) of (1) is obtained⁹.

Following the section 2 is devoted to the creating and computing of HFs that we use in computing approximate solution. We approximate \( y(t) \) by combination both HFS and the collocation method. Our method is carried out for NFIE second kind in the third section. Some theorem and lemmas presented for the error analysis. Finally, we give some numerical examples for showing efficiency of the method.

2. A Survey on Hybrid Functions

Definition 1. 1 An n-set of BPFs \( b_i(t) \), \( i = 1, 2 \ldots n \), on \( [0,1] \) is

\[ b_i(t) = \begin{cases} \frac{i-1}{n}, & \frac{i-1}{n} \leq t < \frac{i}{n}, \\ 0, & \text{otherwise}. \end{cases} \] (6)
With using BPFs, any integrable function \( y(t) \) in the interval \([0, 1)\) can be expressed

\[
y(t) \cong \sum_{i=1}^{n} a_i b_i(t).
\]

**Lemma 2.** Assume that \( y(t) \) is a differentiable function with

\[
|f'(t)| \leq M_1 \quad \forall t \in (0, 1),
\]

then

\[
\left\| y(t) - \sum_{i=1}^{n} a_i b_i(t) \right\|^2 \leq \frac{1}{n^2} M_1^2.
\]

where, \( M_1 \) is a positive integer.

**Definition 3.** The Legendre polynomials \( P_m(t) \) on the interval \([-1, 1]\) are given by the following iteration:

\[
P_0(t) = 1,
\]

\[
P_1(t) = t,
\]

\[
P_m(t) = \frac{2m - 1}{m} t P_{m-1}(t) - \frac{m - 1}{m} P_{m-2}(t), m \geq 2, -1 \leq t \leq 1.
\]

From previous researches, we have:

\[
P_m(\pm 1) = (\pm 1)^m, m \geq 0,
\]

and

\[
P_m(t) = \sum_{j=0}^{m-1} c_{ij} P_j(t)
\]

**Lemma 4.** If \( f(t) \) be a continuous function defined on \([-1, 1]\) and \( f''(t) \) bounded by \( M_2 \), then

\[
\left\| f(t) - \sum_{j=0}^{m-1} c_{ij} P_j(t) \right\|^2 \leq 12M_2^2 \left( \frac{1}{2j+3} \right)^2.
\]

**Definition 5.** The HFs of block-pulse and Legendre polynomials \( h_{ij}(t) \)

\[
i = 1, 2, ..., n \text{ and } j = 0, 1, ..., m - 1 \text{ on the interval } [0, 1) \text{ is defined as}
\]

\[
h_{ij}(t) = \begin{cases} P_j(2nt - 2i + 1), & \frac{i-1}{n} \leq t < \frac{i}{n}, \\ 0, & \text{otherwise.} \end{cases}
\]

where, \( i \) and \( j \) are the order of BPFs and Legendre polynomials, respectively.

From the definition (7), it is clear that the set of HFs are orthogonal on \([0, 1]\).

3. The expansion of a function

Let \( f(x) \) be a square integrable function defined on \([0, 1]\), then

\[
\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} h_{ij}(x)
\]

is called the HFs expansion of \( f(x) \), where the hybrid coefficients \( c_{ij} \) are determined by

\[
c_{ij} = \frac{(f(x), h_{ij}(x))}{(h_{ij}(x), h_{ij}(x))}.
\]

In Equation (9), \( <.,.> \) denotes the inner product.

If (8) is truncated, then after \( nm \) terms, we have

\[
f(x) \cong \sum_{i=1}^{n} \sum_{j=0}^{m-1} c_{ij} h_{ij}(x) = C^T H(x),
\]

where, \( C \) and \( H(x) \) are \((nm \times 1)\) vectors given by

\[
c = [c_{10}, c_{11}, ..., c_{1m-1}, c_{20}, c_{21}, ..., c_{2m-1}, ..., c_n, c_{n1}, ..., c_{nm-1}],
\]

\[
H(x) = [h_{10}(x), h_{11}(x), ..., h_{1m-1}(x), h_{20}(x), h_{21}(x), ..., h_{2m-1}(x), ..., h_{n0}(x), h_{n1}(x), ..., h_{nm-1}(x)].
\]

Similarly, for the kernel function, \( k(x, t) \), we have:

\[
k(x, t) \cong H^T(x)KH(t),
\]

where, \( K \) is an \((nm \times nm)\) matrix, with

\[
K_{ij} = \frac{(h_{ij}(x), (k(x, t), h_{ij}(t)))}{(h_{ij}(x), h_{ij}(x))(h_{ij}(t), h_{ij}(t))}.
\]

For a positive integer \( p \), \( [y(x)]^p \) may be approximated by HF series as

\[
[y(x)]^p = [Y^T H(x)]^p = Y_p^T H(x),
\]
3. Application of the Method for the Second Kind of NIEs

We consider the following forms

\[ y(x) = f(x) + \int_0^x k(x, t)G(t, y(t))\, dt, \quad x \in [0, 1] \]  \tag{15}

Approximating by HFIs, we obtain

\[ f(x) \approx H^T(x)F, \] \tag{16}

\[ k(x, t) \approx H^T(x)KH(t), \] \tag{17}

where, F and K are HF coefficients vectors of \( f(x) \) and \( k(x, t) \), respectively. The approximation of the unknown function \( y(x) \) and its \( p \)th power will also be obtained in the form of coefficient vectors Y and \( \tilde{Y}_p \), respectively. The vector \( \tilde{Y}_p \) is a column vector which can be expressed as a nonlinear function of the vector Y.

To approximate Volterra Hammerstein integral equations, for the integral part of Equation (15), we get

\[ \int_0^x k(x, t)z(t)\, dt \approx \int_0^x H^T(x)KH(t)H^T(t)Z\, dt \] \tag{18}

\[ \approx H^T(x)K\left( \int_0^x H^T(t)H^T(t)\, dt \right)Z. \]

Here, we have to simplify \( \int_0^x H(t)H^T(t)\, dt \) for \( 0 \leq x \leq 1 \) for this purpose, we assume

An \( (nm \times m) \) square matrix \( W(x) \) whose elements \( w_{ij} \) are, which can be easily calculated based on a given \( x \):

\[ w_{ij} = \int_0^x h_i(t)h_j(t)\, dt. \] \tag{19}

By substituting Equations (16)–(18) into Equation (15), one has

\[ H^T(x)Y = f(x) + H^T(x)KZ\tilde{Y}_p. \] \tag{20}

\[ H^T(x)Z = G(t, H^T(x)F + H^T(x)KW(x)Z). \] \tag{21}

By evaluating Equation (1) in \( nm \) points \( \{x_i\}_{i=1}^m \) in the interval \([0, 1)\) we have a system of nonlinear equations:

\[ H^T(x_i)Y = f(x_i) + H^T(x_i)KZ\tilde{Y}_p. \] \tag{22}

\[ H^T(x_i)Z = G(t, H^T(x_i)F + H^T(x_i)KW(x_i)Z). \] \tag{23}

The resulting systems can be solved by numerical methods such as Newton iterative method.

4. Convergence Analysis and Accuracy Estimation

An interesting property of the system of HFIs is that the Fourier expansion of any continuous function in this basis converges to this function uniformly on \([0, 1]\). In the following theorems, we indicate the uniform convergence and accuracy estimation of the presented expansion in Section 2.

**Theorem 1.** If a continuous function \( f(x) \) defined on \([0, 1]\) has bounded second derivative, then the HFIs expansion of the function converges uniformly to the function.

**Proof.** From Equation (9),

\[ C_{ij} = \frac{\int_0^1 f(x)h_i(x)\, dx}{\int_0^1 h^2_j(x)\, dx} = \frac{\int_0^1 f(x)P_j(2nx - 2i + 1)\, dx}{\int_0^1 h^2_j(x)\, dx} \]

Now, let \( 2i - 1 = \tilde{t} \) and \( 2nx - t = t \), then

\[ dx = \frac{1}{2n}\, dt. \]

Hence

\[ c_{ij} = \frac{\int_0^{1/2n} f\left( \frac{\tilde{t} + 1}{2n} \right)P_j(\tilde{t})\, d\tilde{t}}{\int_0^{1/2n} P^2_j(\tilde{t})\, d\tilde{t}} = \frac{2j + 1}{2} \int_0^{1/2n} f\left( \frac{\tilde{t} + 1}{2n} \right)P_j(\tilde{t})\, d\tilde{t}. \]

If we let \( u = f\left( \frac{\tilde{t} + 1}{2n} \right) \), then we have

\[ dv = (2j + 1)P_j(\tilde{t})\, d\tilde{t}, \]

\[ u = f\left( \frac{\tilde{t} + 1}{2n} \right) H\, dv = (2j + 1)P_j(\tilde{t})\, d\tilde{t}. \]

Consequently, one has

\[ c_{ij} = \frac{1}{2}\left[ \int_0^{1/2n} f\left( \frac{\tilde{t} + 1}{2n} \right)(P_{j+1}(\tilde{t}) - P_{j-1}(\tilde{t}))\, d\tilde{t} \right] - \frac{1}{2} \int_0^{1/2n} f\left( \frac{\tilde{t} + 1}{2n} \right)(P_{j+1}(\tilde{t}) - P_{j-1}(\tilde{t}))\, d\tilde{t} \]

\[ = \frac{1}{8n^2} \int_{j-1}^{j+1} \left[ \frac{P_{j+1}(t) - P_{j-1}(t)}{2j + 3} - \frac{P_j(t) - P_{j-1}(t)}{2j - 1} \right] dt. \]

\[ = -\frac{1}{4n^2} \int_{j-1}^{j+1} f\left( \frac{t + 1}{2n} \right)(P_{j+1}(t) - P_{j-1}(t))\, dt \]

\[ = \frac{1}{8n^2} \int_{j-1}^{j+1} \left[ \frac{P_{j+1}(t) - P_{j-1}(t)}{2j + 3} - \frac{P_j(t) - P_{j-1}(t)}{2j - 1} \right] dt. \]
Now, let \( P_j(t) + (2j + 3)P_{j-1}(t), \)
where \( c_{ij} = \frac{1}{8n^2(2j + 3)(2j - 1)} \int_0^1 f'' \left( \frac{t + t}{2n} \right) l_j(t) dt \).
Thus, we have
\[
|c_{ij}| \leq \frac{1}{8i^2(2j + 3)(2j - 1)} \int_0^1 \left| f'' \left( \frac{t + t}{2n} \right) \right| |l_j(t)| dt,
\]
\[
\leq \frac{M_2}{8i^2(2j + 3)(2j - 1)} \int_0^1 |l_j(t)| dt.
\]
However,
\[
\left( \int_0^1 |l_j(t)| dt \right)^2 = \left( \int_0^1 \left| (2j - 1)P_{j-1}(t) + (4j + 2)P_j(t) + (2j + 3)P_{j-2}(t) \right| dt \right)^2
\]
\[
= 2 \left[ \frac{(2j - 1)^2}{2j + 5} + \frac{2(4j + 2)^2}{2j + 1} + \frac{2(2j + 3)^2}{2j - 3} \right]
\]
\[
\leq 2 \left[ \frac{2(2j + 3)^2}{2j - 3} + \frac{8(2j + 3)^2}{2j - 3} + \frac{2(2j + 3)^2}{2j - 3} \right]
\]
\[
= \frac{24(2j + 3)^2}{2j - 3}.
\]
Thus, we get
\[
\int_0^1 |l_j(t)| dt \leq \frac{\sqrt{24(2j + 3)}}{\sqrt{2j - 3}}.
\]
Therefore, we obtain
\[
|c_{ij}| \leq \frac{M_2}{8i^2(2j + 3)(2j - 1)} \frac{\sqrt{24(2j + 3)}}{\sqrt{2j - 3}}
\]
\[
= \frac{\sqrt{6}M_2}{4} \frac{1}{i^2(2j - 3)^{3/2}}
\]
and
\[
\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} h_{ij}(x) \to f(x). \]

**Theorem 2.**

Let \( f(x) \) be a continuous function defined on \([0, 1]\), with bounded second derivative \( f(x) \) bounded by \( M_2 \), then we have the following accuracy estimation:
\[
\|f(x) - C^TH(x)\|^2 \leq \frac{3}{8} M_2 \left( \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i^3(2j - 3)^4} \right)^{1/2}.
\]
That results:
\[ \| y^*(t) - y_j(t) \| = \left\| \int_0^1 k(t,s)(z^*(s) - z_j(s)) \, ds \right\|, \quad t \in [0,1] \] \tag{28}
then
\[ \| y^*(t) - y_j(t) \| \leq \| k(t,s) \| \| (z^*(s) - z_j(s)) \|, \quad t \in [0,1] \] \tag{29}
Finally, we deduce the following inequality for the rate of convergence,
\[ \| y^* - y_j \| \leq K \| z^* - z_j \|. \] \tag{30}

5. Numerical Examples

In this section, we give some examples to investigate the effects of choosing HF's. The following examples was solved by our new method with the collocation points chosen to be
\[ \tau_{ji} = \frac{i}{2^j}, \quad i = 0, \ldots, 2^j - 1. \]

We approximated the solution of Volterra's integral equation by combination of HF's and collocation method. Also all programming were carried out using Maple 18.

Example 1. For the first example, solve (1) with
\[ k(s,t) = 4ts + \pi t \sin \frac{\pi}{2} s, \]
\[ f(t) = \sin \frac{\pi t}{2} - 2t \ln \frac{t}{3}, \]
\[ z(t) = \frac{1}{y^2(t) + t^2 + 1} \]
where, exact solution is
\[ y(t) = \sin \frac{\pi t}{2}. \]

The maximum errors listed in Table 1.

| j  | \[ \| z^* - z_j \| \_\infty \] | \[ \| y^* - y_j \| \_\infty \] |
|----|-------------------------------|-------------------------------|
| 6  | 4.980–2                       | 6.880–2                       |
| 8  | 7.010–3                       | 2.040–2                       |
| 12 | 8.250–4                       | 9.840–2                       |
| 15 | 1.980–6                       | 3.750–3                       |
| 20 | 6.940–10                      | 1.720–4                       |
| 25 | 3.360–12                      | 1.710–6                       |
| 30 | 6.220–15                      | 9.260–7                       |

Example 2. For the second example consider
\[ g(t, y(t)) = e^{-y^2(t)}; f(t) = \frac{t}{e}, k(s, t) = 2ts \]
where, exact solution is \[ y(t) = t. \] The computed errors for the solutions are shown in Table 2.

| j  | \[ \| z^* - z_j \| \_\infty \] | \[ \| y^* - y_j \| \_\infty \] |
|----|-------------------------------|-------------------------------|
| 6  | 2.082–2                       | 4.739–2                       |
| 8  | 2.590–3                       | 1.660–2                       |
| 12 | 1.244–4                       | 7.948–3                       |
| 15 | 3.390–5                       | 6.300–3                       |
| 20 | 1.313–9                       | 9.924–6                       |
| 25 | 1.770–12                      | 2.700–6                       |
| 30 | 2.890–15                      | 1.940–8                       |

Example 3. We solve (1) with
\[ g(t, y(t)) = \frac{1}{1 + y^2(t)}; f(t) = t^2 \left( \frac{1}{2} - \ln 2 \right) + \sqrt{t}, k(s, t) = (ts)^2 \]
where, exact solution is \[ y(t) = \sqrt{t}. \] The computed errors for the solutions are shown in Table 3.

| j  | \[ \| z^* - z_j \| \_\infty \] | \[ \| y^* - y_j \| \_\infty \] |
|----|-------------------------------|-------------------------------|
| 6  | 3.470–3                       | 1.060–2                       |
| 8  | 2.110–7                       | 5.110–3                       |
| 12 | 1.800–11                      | 7.948–3                       |
| 15 | 1.190–13                      | 6.300–3                       |
| 20 | 2.210–14                      | 9.924–6                       |
| 25 | 2.990–15                      | 2.330–15                      |

6. Conclusion

In this paper, we have presented and tested a numerically combined type method for handling (1) based on HF's and the collocation method. From Table 1–3 we can conclude convergence be fastest when \[ j \to +\infty. \]

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8. References

1. Anselone P. Nonlinear integral equations, University of Wisconsin: Madison; 1964.
2. Atkinson K. A survey of numerical methods for solving nonlinear integral equations. Journal of Integral Equations and Applications. 1992; 4(1):15–46.
3. Argyros IK. Quadratic equations and applications to Chandrasekhar and related equations. Bulletin of the Australian Mathematical Society. 1985; 32(2):275–92.
4. Abdou MA, Badr AA. On a method for solving an integral equation in the displacement contact problem. Applied Mathematics and Computation. 2002; 127(1):65–78.
5. Brunner H, Crisci MR, Russo E, Recchio A. A family of methods for Abel integral equations of the second kind. Journal of Computational Applied Mathematics. 1991; 34(2):211–19.
6. Kilbas AA, Saigo M. On solution of nonlinear Abel–Volterra integral equation. Journal of Mathematical Analysis and Applications. 1999; 229(1):41–60.
7. Jiang ZH, Schaufelberger W. Block pulse functions and their applications in control systems. Springer-Verlag Berlin Heidelberg; 1992.
8. Maleknejad K, Derili H, Sohrabi S. Numerical solution of Urysohn integral equations using the iterated collocation method. International Journal of Computer Mathematics. 2008; 85(1):143–54.
9. Maleknejad K, Derili H. The collocation method for Volterra’s integral equation by Daubecies Wavelets. Applied Mathematics and Computation. 2006; 172(2):846–64.