SU(8) family unification with boson–fermion balance

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We formulate an SU(8) family unification model motivated by requiring that the theory should incorporate the graviton, gravitinos, and the fermions and gauge fields of the standard model, with boson–fermion balance. Gauge field SU(8) anomalies cancel between the gravitinos and spin $\frac{1}{2}$ fermions. The 56 of scalars breaks SU(8) to SU(3)$_{\text{family}} \times$ SU(5) $\times$ U(1)/Z$_5$, with the fermion representation content needed for “flipped” SU(5) with three families, and with the residual scalars in the representations needed for further gauge symmetry breaking to the standard model. Yukawa couplings of the 56 scalars to the fermions are forbidden by chiral and gauge symmetries. In the limit of vanishing gauge coupling, there are $N = 1$ and $N = 8$ supersymmetries relating the scalars to the fermions, which restrict the form of scalar self-couplings and should improve the convergence of perturbation theory, if not making the theory finite and “calculable”. In an Appendix we give an analysis of symmetry breaking by a Higgs component, such as the $(1,1)(-15)$ of the SU(8) 56 under SU(8) $\supset$ SU(3) $\times$ SU(5) $\times$ U(1), which has nonzero U(1) generator.

I. INTRODUCTION

The presence of bosons and fermions in Nature makes the idea of a fundamental boson–fermion balance appealing, and this has motivated an extensive search for supersymmetric extensions of the standard model. However, since the observed particle mass spectrum is not supersymmetric, supersymmetry breaking must be invoked, and despite much effort a definitive model, and a definitive symmetry breaking mechanism, have yet to emerge. We turn in this paper to another possibility, that boson–fermion balance without full supersymmetry is the relevant property of the unification theory, and construct a model based on this philosophy motivated by SU(8) unification and supergravity.

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II. COUNTING STATES

The model we study is inspired by the state structure of maximal $SO(8)$ supergravity. The usual counting of on-shell states for $N = 8$ supergravity is one graviton with 2 helicity states, 8 Majorana gravitinos with 16 helicity states, 28 vectors with 56 helicity states, 56 Majorana fermions with 112 helicity states, and 70 scalars with 70 helicity states. Thus there are $2+56+70=128$ boson states, and $16+112=128$ fermion states, giving the required boson–fermion balance, and interacting models with this field content exist. Unfortunately, however, these models do not contain the full particle and gauge group content needed for the standard model.

In a fascinating comment in his magisterial work on “Group theory for unified model building”, Richard Slansky wrote [1]: “One may wish to speculate about a future unified theory of all interactions and all elementary particles that would resemble $SO_8$ supergravity but involve sacrificing some principle now held sacred, so that the notion of extended supergravity could be generalized. In such a hypothetical theory, an internal symmetry group $G$ larger than $SO_8$ would be gauged by spin 1 bosons, and both the spin $\frac{3}{2}$ and spin $\frac{1}{2}$ fermions would be assigned to representations of $G$. It is then very natural to suppose that the spin $\frac{3}{2}$ fermions would belong to some basic representation of $G$ and would include only color singlets, triplets and antitriplets. The spin $\frac{1}{2}$ particles would then presumably be assigned to a more complicated representation. These speculations are a major motivation for this review, as they were for ref. [6].” (Slansky’s reference 6 is Gell-Mann, Ramond and Slansky [2].)

The rest of this paper proceeds in the spirit of Slansky’s remarks (which as we shall see, describe the model that we construct.) We begin by noting that if the 70 scalars are eliminated from the counting, and their degrees of freedom are redistributed to the two helicities of 35 vectors, we are left with $28+35=63$ vectors in all, which can be assigned to the adjoint representation of an $SU(8)$ group. The remaining representations in the counting, the 8 and 56, can be interpreted as the fundamental and rank three antisymmetric tensor representations of $SU(8)$, giving an “$SU(8)$ graviton” multiplet consisting of the graviton, the 8 gravitinos, the 63 vectors, and the 56 fermions. There are still 128 boson and 128 fermion helicities in this model, but the state structure is no longer the one corresponding to unitary supersymmetry representations in Hilbert space. Since we are working in 4 dimensions, and the model is not supersymmetric, we switch at this point from Majorana fermions to the usual left chiral (L) Weyl fermions used in grand unification, but the state counting is the same.

There is a long history of $SU(8)$ unification models in the literature; see [3]–[14]. Of particular
interest are the papers of Curtright and Freund [3], C. Kim and Roiesnel [7], and J. Kim and Song [9], which incorporate spin $\frac{1}{2}$ fermions through single left chiral $\bar{8}$, $\overline{28}$, and 56 representations of $SU(8)$. Under breaking to $SU(5)$, the $\overline{28}$ of $SU(8)$ contains three copies of the $\bar{5}$ of $SU(5)$, and the 56 of $SU(8)$ contains three copies of the 10 of $SU(5)$, so this representation content incorporates the three standard model families. Additionally, the paper of Curtright and Freund explicitly ties the representation numbers 8, 28, and 56 to those appearing in $N = 8$ supergravity, with the suggestion that the $SU(8)$ gauge bosons may appear as bound states, as suggested by Cremmer and Julia [15].

Returning to the “$SU(8)$ graviton” multiplet, the $56_L$ of fermions contains three families in the $SU(5)$ 10$L$ representation. In order to incorporate three $SU(5)$ 5$L$ families into a model with boson–fermion balance, we adjoin to the “$SU(8)$ graviton” multiplet a “$SU(8)$ matter” multiplet consisting of a complex scalar field in the 56 representation of $SU(8)$, and two copies of a fermion spin $\frac{1}{2}$ field in the $\overline{28}_L$ representation of $SU(8)$. Use of a complex scalar is necessary since the 56 is a complex representation, and so cannot be assigned to a real scalar multiplet. Boson–fermion balance then requires that we double the number of $\overline{28}_L$ representations, so that the number of spin $\frac{1}{2}$ helicity states is $2 \times 2 \times 28 = 112$, equal to the number of helicity states in a complex 56 scalar. (Although boson–fermion balance could be achieved with a single $28_L$ of fermions and a complex 28 of scalars, $SU(8)$ anomalies would not cancel, and $SU(8)$ could not be broken to $SU(3) \times SU(5)$.) The $SU(8)$ fermion and boson content of the model is summarized in Table I.

### III. ANOMALY CANCELATION

To have a consistent $SU(8)$ gauge theory, anomalies must cancel. In the papers of Curtright and Freund [3], Kim and Roiesnel [7], and Kim and Song [9], this is achieved through

\[
\begin{align*}
\text{anomaly}(\bar{8}_L) &= -1, \\
\text{anomaly}(\overline{28}_L) &= -4, \\
\text{anomaly}(56_L) &= 5, \\
\text{total anomaly} &= -1 - 4 + 5 = 0.
\end{align*}
\]

In our model anomaly cancelation involves the same representations, up to conjugation, but different counting. Instead of a spin $\frac{1}{2}$ $\bar{8}_L$, our “$SU(8)$ graviton” multiplet contains a spin $\frac{3}{2}$ $8_L$. Since the chiral anomaly of a spin $\frac{3}{2}$ particle is three times that of the corresponding spin $\frac{1}{2}$ particle [16], [17], the $8_L$ of gravitinos contributes 3 to the anomaly count. The $56_L$ of spin $\frac{1}{2}$ fermions...
TABLE I: Field content of the model, with the top part of the table showing the “SU(8) graviton” multiplet, and the bottom part of the table showing the “SU(8) matter” multiplet. The linearized graviton $h_{\mu \nu}$ is
defined by $g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu}$, with $\eta_{\mu \nu}$ the Minkowski metric and $\kappa$ the gravitational coupling. Branching
rules are from Slansky [1] with $U(1)$ generators (or charges) in parentheses, followed in curly brackets by
 equivalent $U(1)$ generators modulo 5. (The modulo 5 ambiguities in these assignments have been used
to give the assignments needed for flipped SU(5), plus states that can be paired into condensates after
family symmetry breaking, plus states that are neutral with respect to the $SU(3) \times SU(5) \times U(1)/Z_5$ force.)
Square brackets on the field subscripts and superscripts indicate complete antisymmetrization of the enclosed
indices. The indices $\alpha, \beta, \gamma$ range from 1 to 8, the index $A$ runs from 1 to 63, and $\mu, \nu$ are Lorentz indices.

| field | spin | SU(8) rep. | helicities | branching to SU(3) × SU(5) × U(1) |
|-------|------|------------|------------|-----------------------------------|
| $h_{\mu \nu}$ | 2 | 1 | 2 | 1 |
| $\psi^\alpha_\mu$ | Weyl $\frac{3}{2}$ | 8_L | 16 | $(3,1)(-5){0} + (1,5)(3){-2}$ |
| $A^A_\mu$ | 1 | 63 | 126 | $(1,1){0} + (8,1){0} + (3,5){-8}{2} + (3,5){8}{-2} + (1,24){0}$ |
| $\chi^{[\alpha \beta \gamma]}$ | Weyl $\frac{1}{2}$ | 56_L | 112 | $(1,1)(-15){0} + (1,10)(9){-1} + (3,5){-7}{-2} + (3,10)(1){1}$ |
| $\lambda_{[\alpha \beta]}$ | Weyl $\frac{1}{2}$ | $28_L$ | 56 | $(3,1)(10){0} + (1,10)(-6){-1} + (3,5){2}{2}$ |
| $\lambda_{2[\alpha \beta]}$ | Weyl $\frac{1}{2}$ | $28_L$ | 56 | $(3,1)(10){5} + (1,10)(-6){-1} + (3,5)(2){-3}$ |
| $\phi^{[\alpha \beta \gamma]}$ | complex 0 | 56 | 112 | $(1,1)(-15){0} + (1,10)(9){-1} + (3,5){-7}{-2} + (3,10)(1){1}$ |

Contributes 5 as before, while the two $28_L$ of spin $\frac{1}{2}$ fermions contribute $-8$, giving

$$3 \times \text{anomaly}(8_L) = 3$$,
$$2 \times \text{anomaly}(28_L) = -8$$,
$$\text{anomaly}(56_L) = 5$$,

Total anomaly in our model = $3 - 8 + 5 = 0$ .

(2)

So anomalies cancel, but by a different mechanism than in refs. [3], [7], and [9]. Anomaly cancel-
lation with the counting of Eq. (2) (using the conjugate representations $\bar{8}$, 28, and $\bar{56}$) was noted
by Marcus [18] in a study of dynamical gauging of SU(8) in $N = 8$ supergravity.

IV. GAUGE SYMMETRY BREAKING AND STATE CONTENT

We turn to the issue of gauge symmetry breaking. Symmetry breaking in our model is initiated
by a Brout-Englert-Higgs (BEH) mechanism using the complex scalar field in the “SU(8) matter”
multiplet, which is in the 56 of SU(8). (This can be accomplished by either an explicit negative
mass for the scalar in the action, or by an alternative that we favor, the Coleman-Weinberg \[19\] mechanism induced by radiative corrections starting from a massless scalar.) Since the 56 representation of $SU(8)$ branches to the 56$_c$ of $SO(8)$, not to a singlet of $SO(8)$, the symmetry breaking pathway of our model cannot pass through $SO(8) \times U(1)$. Referring to Table I, which gives the branching of the 56 of $SU(8)$ to $SU(3) \times SU(5) \times U(1)$, we see that there is a singlet (1,1) of $SU(3) \times SU(5)$ with a nonzero $U(1)$ generator of $-15$. Hence there are two interesting symmetry breaking pathways. In the first, the BEH mechanism breaks $SU(8)$ to $SU(3) \times SU(5)$, with the $U(1)$ gauge symmetry either completely broken or, as discussed in Appendix A, broken to $U(1)/Z$. It is then natural to identify the $SU(3)$ factor as a family symmetry group, and the $SU(5)$ factor and fermion content as the usual minimal grand unification group \[20\]. In the second, the $U(1)$ gauge symmetry breaks only to $U(1)/Z_5$, that is, after symmetry breaking there is an equivalence between values of $U(1)$ generators that differ by multiples of 5, as a result of a periodicity in the $U(1)$ generator of the broken symmetry ground state, which is discussed in detail in Appendix A. It is again natural to identify the unbroken $SU(3)$ factor as a family symmetry group. An inspection of the $U(1)$ generators modulo 5, given in curly brackets in Table I, shows that the fermion content in this breaking pathway contains all the representations needed for flipped $SU(5)$ grand unification \[21\].

To elaborate on this, the basic flipped $SU(5)$ model \[22\] consists of a 10{1} for the quark doublet $Q$, the down quark $d^c$, and the right handed neutrino $N$; a $\overline{5}\{-3\}$ for the lepton doublet $L$ and the up quark $u^c$; and a 1{5} for the charged leptons $e^c$. Referring to Table I, we see that $\chi$ contains a $(3,10){1}$, while $\lambda_2$ contains a $(\overline{3},\overline{5})\{-3\}$ and a $(3,1){5}$. This gives three 3 or $\overline{3}$ families of the states needed for basic flipped $SU(5)$. Note that we have chosen the $U(1)$ charge assignments modulo 5 needed to make this correspondence possible. This guarantees that the correct particle charge assignments are obtained after further breaking to the standard model, and also implies that $SU(5)$ anomalies cancel within the set of spin $\frac{1}{2}$ states assigned to flipped $SU(5)$, without invoking the spin $\frac{3}{2}$ states. The remaining states are the $(\overline{3},5)\{-2\}$ in $\chi$ and the $(\overline{3},\overline{5})\{2\}$ in $\lambda_1$, which after family symmetry breaking can pair to form a condensate; the $(1,1){0}$ in $\chi$ and the $(3,1){0}$ in $\lambda_1$, which do not feel the $SU(5) \times U(1)/Z_5$ forces and could be dark matter candidates; and three $(1,\overline{10})\{-1\}$, one in each of the fermions $\chi$, $\lambda_1$, and $\lambda_2$, which after family symmetry $SU(3)$ breaking can form condensates with the $(3,10){1}$ in $\chi$ to affect the particle mass spectrum.

There is a 2 to 1 asymmetry in these condensates, since the condensates involving $\lambda_{1,2}$ and $\chi$ are bilinear, whereas the one involving $\chi$ twice are quadratic, which could relate to the observed fact that there are two light families and one heavy one. We note finally that an extended version of
flipped $SU(5)$, proposed recently by Barr [23], introduces a vector-like pair $5\{-2\} + \overline{5}\{2\}$ in each family and uses them to argue that proton decay can be rotated away.

There are residual boson states left after the 56 representation boson $\phi$ breaks the group $SU(8)$, with 63 generators, to $SU(3) \times SU(5)$, with $8 + 24 = 32$ generators, plus the single additional generator of the discrete group $U(1)/Z_5$ when $U(1)$ is not completely broken. Since $63 - 33 = 30$ components of $\phi$ are absorbed to form longitudinal components of the broken $SU(8)$ generators, these components can only come from the $(3,10)(1)\{1\}$ representation in the branching expansion of Table I. So the residual boson states necessarily are the representations $(1,\overline{10})(9)\{-1\}$ and $(\overline{3},5)(-7)\{-2\}$, plus the $(1,1)(-15)\{0\}$ when $U(1)$ is only broken to $U(1)/Z_5$. Since breaking minimal $SU(5)$ to the standard model requires a scalar in the 24 representation, the symmetry breaking pathway to $SU(3) \times SU(5)$ with minimal $SU(5)$ requires dynamical generation of this 24, to be further discussed below.

On the other hand, the residual boson states after $SU(8)$ breaking contain the Higgs bosons representations needed to break flipped $SU(5)$ to the standard model [21]. Elaborating on this, the basic flipped $SU(5)$ model uses a $\overline{10}\{-1\}$ of scalars to break flipped $SU(5)$ to the standard model, and a $5\{-2\}$ of scalars to break the electroweak group of the standard model to the electromagnetic $U(1)$ group. The extended flipped $SU(5)$ model of [23] uses a $5\{-2\}$ and a $5\{3\}$ (which is equivalent modulo 5 to a second $5\{-2\}$) to break the electroweak group and generate weak scale masses for quarks and leptons. Table I shows that $\phi$ contains a residual $(1,\overline{10})\{-1\}$ and a family triplet $(\overline{3},5)\{-2\}$. Thus all the “Higgs” representations needed to break $SU(8)$ to the standard model, using the flipped $SU(5)$ state assignments, are contained in the 56 of complex scalars $\phi$.

V. ASYMPTOTIC FREEDOM AND GLOBAL SYMMETRIES

The $SU(8)$ representation content of the model has a small enough spin 0, spin $\frac{1}{2}$, and spin $\frac{3}{2}$ content to keep the theory asymptotically free,

$$\frac{1}{3}[11c(1) - 26c(\text{Weyl } 3/2) - 2c(\text{Weyl } 1/2) - c(\text{complex } 0)]$$

$$= \frac{1}{3}[11 \times 16 - 26 \times 1 - 2 \times (15 + 2 \times 6) - 15] = 27 > 0$$

(3)

with $c(s)$ the index of the $SU(8)$ representation with spin $s$. (For the spin $\frac{3}{2}$ beta function see Curtright [24], Duff [17], and Fradkin and Tseytlin [25]; the index $c$ is tabulated as $\ell$ in the tables of Slansky [1].) Thus the $SU(8)$ coupling increases as the energy decreases, which can trigger
dynamical symmetry breaking in addition to the symmetry breaking provided by the elementary Higgs fields. In addition to a locally gauged $SU(8)$ symmetry, our model admits a number of global chiral symmetries associated with the fermion fields [26]. The first is an overall chiral $U(1)$ symmetry associated with an overall $U(1)$ rephasing of all of the fermion fields, spin $\frac{3}{2}$ as well as spin $\frac{1}{2}$. It will be convenient to regard this phase as associated with the $56_L$ of spin $\frac{1}{2}$ fermions, labeled $\chi$ in Table I. We expect this global symmetry to be broken by the usual instanton and anomaly mechanism that is invoked to solve the “$U(1)$ problem” in QCD [27], [28]. The second global symmetry is an overall $U(1)$ rephasing of the spin $\frac{3}{2}$ gravitino fermion fields relative to the $56_L$ fermions. Finally, since the kinetic Lagrangian contains the doubled $\overline{28}_L$ representation spanned by the fermion basis $\lambda_{1,2}$, there is a global $U(2)$ symmetry associated with mixing of these basis states, relative to the phase of the $56_L$ of fermions. However, the $U(1)$ charge assignments modulo 5 shown in Table I reduce the $U(2)$ global symmetry to $U(1) \times U(1)$.

VI. DYNAMICAL VERSUS ELEMENTARY HIGGS SYMMETRY BREAKING

As already noted, in the $SU(8) \supset SU(3) \times SU(5)$ symmetry breaking pathway, the $SU(5)$ 24 representation needed for breaking to the standard model must be generated dynamically. A quick review of the theory of dynamical symmetry breaking is given in Appendix B. A strategy for getting a 24, following [29], would be to generate a 24 condensate at the unification scale, which violates the chiral symmetries of the theory, and so leads to a 24 Goldstone boson, which could then serve as the 24 Higgs. There are two problems with this scenario. The first is that the only way to generate a 24 representation of $SU(5)$ from the representations in Table I is through either $5 \times 5$ or $\overline{10} \times 10$, both of which contain an $SU(5)$ singlet in addition to a 24. Since the singlet is always the most attractive channel (see Eq. (B.6)), dynamical generation of a 24 seems unlikely [26], [30], [31]. The second problem is that if gauge couplings were strong enough for a 24 condensate to be formed at the unification scale, then one would expect that unification scale condensates involving the wanted fermions in the $(3, 10)$ representation in Table I would also form, removing these states from the low energy spectrum. So getting the standard model from our theory through an $SU(8) \supset SU(3) \times SU(5)$ symmetry breaking pathway is not plausible.

The situation is more favorable for the $SU(8) \supset SU(3) \times SU(5) \times U(1)/Z_5$ symmetry breaking pathway, which as explained in Appendix A involves a ground state that is periodic in the $U(1)$ generator. This pathway does not require dynamical condensates to break $SU(3) \times SU(5) \times U(1)/Z_5$ to the standard model; the residual elementary scalar states in $\phi$ can do this, as well as breaking [32]
the \(SU(3)\) family symmetry. It would then be consistent to suppose that any condensate formation occurs only near or below the electroweak scale, where some non-Abelian running couplings can become large, and where such condensates could play a role in determining the parameter values the standard model.

VII. \textbf{THE GAUGE SECTOR ACTION}

We turn now to writing down the action for the gauge sector of our model. Since all fermion representations are antisymmetrized direct products of fundamental 8 representations, we need only use generators \(t_A\) for the fundamental 8 of \(SU(8)\) to construct covariant derivatives of the fermion fields. We follow here the conventions of [33], and take the \(t_A\) to be anti-self-adjoint, with commutators and trace normalization given by

\[
[t_A, t_B] = f_{ABC} t_C ,
\]

\[
\text{Tr}(t_A t_B) = -\frac{1}{2} \delta_{AB} ,
\]

(4)

with implicit summation on repeated indices.

Defining the gauge variation of the gauge potential by

\[
\delta_G A^A_\mu = \frac{1}{g} \partial_\mu \Theta^A + f_{ABC} A^B_\mu \Theta^C ,
\]

(5)

the gauge covariant field strength \(F^A_{\mu\nu}\) is defined as

\[
F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g f_{ABC} A^B_\mu A^C_\nu ,
\]

(6)

and has the gauge variation

\[
\delta_G F^A_{\mu\nu} = f_{ABC} F^B_{\mu\nu} \Theta^C .
\]

(7)

We can now define covariant derivatives of the fermion fields of the model. Writing

\[
A^\alpha_{\mu \beta} = A^A_\mu (t_A)^\alpha_{\beta} ,
\]

(8)

the covariant derivatives of the fermion fields \(\psi^\alpha_\mu\), \(\chi^{[\alpha\beta\gamma]}\) and \(\lambda_{\alpha\beta} a\), \(a = 1, 2\) of Table I (in the 8,
56, and $28$ representations respectively) are defined by

$$
D_{\nu}^\alpha \psi^\mu_{\beta} = \partial_{\nu} \psi^\alpha_{\beta} + g A^\alpha_{\nu \delta} \psi^\delta_{\mu} ,
D_{\nu} \chi^{[\alpha \beta \gamma]} = \partial_{\nu} \chi^{[\alpha \beta \gamma]} + g (A^\alpha_{\nu \delta} \chi^{[\delta \beta \gamma]} + A^\beta_{\nu \delta} \chi^{[\alpha \delta \gamma]} + A^\gamma_{\nu \delta} \chi^{[\alpha \beta \delta]} ) ,
D_{\nu} \lambda_{a[\alpha \beta]} = \partial_{\nu} \lambda_{a[\alpha \beta]} + g (A^a_{\nu \alpha} \chi^{[\delta \beta]} + A^a_{\nu \beta} \chi^{[\alpha \delta]} ) , \quad a = 1, 2 .
$$

(9)

Similarly, for the scalar field $\phi^{[\alpha \beta \gamma]}$, the covariant derivative is defined by

$$
D_{\nu} \phi^{[\alpha \beta \gamma]} = \partial_{\nu} \phi^{[\alpha \beta \gamma]} + g (A^\alpha_{\nu \delta} \phi^{[\delta \beta \gamma]} + A^\beta_{\nu \delta} \phi^{[\alpha \delta \gamma]} + A^\gamma_{\nu \delta} \phi^{[\alpha \beta \delta]} ) .
$$

(10)

These give

$$
\delta G \psi^\mu_{\alpha} = - \theta A^\alpha_{A \delta} \psi^\delta_{\mu} , \quad \delta G D_{\nu} \psi^\alpha_{\mu} = - \theta A^\alpha_{A \delta} D_{\nu} \psi^\delta_{\mu} ,
$$

(11)

and similarly for the gauge variations of the other fields and their covariant derivatives.

With the $SU(8)$ covariant derivatives of the fields defined, we can now write down the gauge sector action of the model, with gravity treated in the linearized approximation, as follows. The total action is

$$
S(\text{total}) = S(h_{\mu \nu}) + S(\psi_{\mu}) + S(A_{\mu}) + S(\chi) + S(\lambda_{1,2}) + S_{\text{kinetic}}(\phi) + S_{\text{self-coupling}}(\phi) + S_{\text{fermion-coupling}}(\phi, \psi_{\mu}, \chi, \lambda) .
$$

(12)

For $S(h_{\mu \nu})$ we have the usual linearized gravitational action,

$$
S(h_{\mu \nu}) = \frac{1}{8} \int d^4 x h_{\mu \nu} H_{\mu \nu} , \quad H_{\mu \nu} = \partial_{\mu} \partial_{\nu} h_{\lambda} + \Box h_{\mu \nu} - \partial_{\mu} \partial_{\lambda} h_{\nu \lambda} - \partial_{\nu} \partial_{\lambda} h_{\mu \lambda} - \eta_{\mu \nu} \Box h_{\lambda} + \eta_{\mu \nu} \partial_{\rho} \partial^{\rho} h_{\lambda} .
$$

(13)

For the gravitino action we have the $SU(8)$ gauged extension of the usual expression,

$$
S(\psi_{\mu}) = \frac{1}{2} \int d^4 x \overline{\psi_{\mu \alpha}} R^{\mu \alpha} , \quad R^{\mu \alpha} = i \epsilon^{\mu \nu \rho \gamma} \gamma_5 \gamma_\eta D_{\nu} \psi^\alpha_{\rho} = R^{\mu \alpha}_{\text{free}} + R^{\mu \alpha}_{\text{interaction}} , \quad R^{\mu \alpha}_{\text{free}} = i \epsilon^{\mu \nu \rho \gamma} \gamma_5 \gamma_\eta D_{\nu} \psi^\alpha_{\rho} , \quad R^{\mu \alpha}_{\text{interaction}} = i \epsilon^{\mu \nu \rho \gamma} \gamma_5 \gamma_\eta A^\alpha_{\nu \delta} \psi^\delta_{\rho} .
$$

(14)
Since the free gravitino action is invariant under the gravitino gauge transformation $\psi^\alpha_\rho \rightarrow \psi^\alpha_\rho + \partial_\rho \epsilon^\alpha$, a gauge fixing condition is needed to quantize, which can be taken in the covariant form $\gamma^\rho \psi^\alpha_\rho = 0$. The associated ghost fields then play a role in the spin $3/2$ anomaly calculation.

The $SU(8)$ gauge field action has the standard form

$$S(A_\mu) = -\frac{1}{4} \int d^4x F_{\mu\nu}^A F^{A\mu\nu}, \quad (15)$$

and the spin $1/2$ fermion actions are

$$S(\chi) = -\frac{1}{2} \int d^4x \overline{\chi}_{[\alpha\beta\gamma]} \gamma^\nu D_\nu \chi^{[\alpha\beta\gamma]},$$

$$S(\lambda_{1,2}) = -\frac{1}{2} \int d^4x \sum_\alpha \overline{\lambda}_a^{[\alpha\beta]} \gamma^\nu D_\nu \lambda_a_{[\alpha\beta]}, \quad (16)$$

where we have written the second line in a form which exhibits its global $U(2)$ invariance. Finally, for the scalar field kinetic action we have

$$S_{\text{kinetic}}(\phi) = -\frac{1}{2} \int d^4x (D_\nu \phi)^*_{[\alpha\beta\gamma]} D_\nu \phi^{[\alpha\beta\gamma]}, \quad (17)$$

For later use, we note that the equations of motion of the spin $1/2$ fermions and the spin 0 boson, following from these gauged kinetic actions but ignoring for the moment possible additional scalar interaction terms, are

$$\gamma^\nu D_\nu \chi^{[\alpha\beta\gamma]} = 0,$$

$$\gamma^\nu D_\nu \lambda_a_{[\alpha\beta]} = 0, \quad a = 1, 2,$$

$$D^\nu D_\nu \phi^{[\alpha\beta\gamma]} = 0. \quad (18)$$

**VIII. ABSENCE OF SCALAR–FERMION YUKAWA COUPLINGS**

We turn next to possible Yukawa couplings $S_{\text{fermion–coupling}}(\phi, \psi_\mu, \chi, \lambda)$, which we show must all vanish. The chirality requirements for forming nonzero Yukawa couplings are the same as those for forming condensates discussed in Appendix B. Thus, chirality requires that Yukawa couplings of the spin $1/2$ fermions must be of the form $\Psi^T_{L1} \gamma^0 \Psi_{L2} \Phi$, with $\Psi_{1,2}$ any of the spin $1/2$ fermion fields, and $\Phi$ either $\phi$ or $\phi^*$, with $SU(8)$ indices contracted to form a singlet. But this is not possible, since the product of two $\chi$ has 6 upper $SU(8)$ indices, the product of two $\lambda$ has 4 lower $SU(8)$
indices, and the product of $\chi$ with a $\lambda$ has 3 upper $SU(8)$ indices and two lower $SU(8)$ indices, none of which can be contracted with a $\phi$, with three upper indices, or a $\phi^*$, with three lower indices, to form an $SU(8)$ singlet. Hence there are no Yukawa couplings involving the spin $\frac{1}{2}$ fermions by themselves. Additionally, Yukawa couplings of a spin $\frac{3}{2}$ field to a spin $\frac{3}{2}$ field and the scalar are forbidden by a chirality and $SU(8)$ index contraction argument similar to that used in the case of two spin $\frac{1}{2}$ fields. This argument does not forbid couplings of a spin $\frac{3}{2}$ field to a spin $\frac{1}{2}$ field and the scalar of the form $\lambda^{a[\alpha\beta]} \gamma^\nu \psi^{\alpha}_\mu \phi^{*}_{[\alpha\beta]}$ and its conjugate, but these vanish when the gravitino gauge fixing condition $\gamma^\nu \psi^{\alpha}_\mu = 0$ is imposed.

IX. SUPERSYMMETRIES IN THE LIMIT OF ZERO GAUGE COUPLING

Let us now consider the free limit of the theory in which the gauge coupling $g$ vanishes, so that the covariant derivatives $D_\nu$ become ordinary partial derivatives $\partial_\nu$, and the equations of motion of Eq. (18) simplify to

$$\gamma^\nu \partial_\nu \lambda^{[\alpha\beta\gamma]} = 0$$
$$\gamma^\nu \partial_\nu \lambda_a^{[\alpha\beta]} = 0, \quad a = 1, 2$$
$$\partial^\nu \partial_\nu \phi^{[\alpha\beta\gamma]} = 0$$

(19)

One can then form two conserved $SU(8)$ representation 8 supercurrents,

$$J_\alpha^\mu = \gamma^\nu (\partial_\nu \phi^{[\alpha\beta\gamma]} \gamma^\mu \lambda_a^{[\beta\gamma]}$$
$$\partial_\mu J_\alpha^\mu = 0, \quad \alpha = 1, ..., 8 \text{ and } a = 1, 2$$

(20)

and an $SU(8)$ singlet conserved supercurrent,

$$J^\mu = \gamma^\nu (\partial_\nu \phi^{*}_{[\alpha\beta\gamma]} \gamma^\mu \lambda^{[\alpha\beta\gamma]}$$
$$\partial_\mu J^\mu = 0$$

(21)

In deriving supercurrent conservation we have used the equations of motion together with

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \Phi = \eta^\nu \partial_\nu \partial_\mu \Phi$$

(22)
which is a consequence of the commutativity of partial derivatives. The invariance transformation of the free action for which $J^\mu_\alpha$ (with $a = 1$ or 2) is the Noether current is

$$
\delta \phi^{[\alpha\beta\gamma]} = \bar{\epsilon}_{[\alpha} \lambda_{\beta\gamma]} ,
\delta \phi^* \alpha\beta\gamma = \bar{\epsilon} \lambda_{[\alpha\beta\gamma]} ,
\delta \lambda_{\alpha\beta} = \gamma^\nu \partial_\nu \phi^* \alpha\beta\gamma \bar{\epsilon} ,
\delta \bar{\lambda}^\alpha_{\beta} = - \bar{\epsilon} \gamma^\nu \partial_\nu \phi \alpha\beta\gamma ,
$$

(23)

and the transformation for which $J^\mu$ is the Noether current is

$$
\delta \phi^{[\alpha\beta\gamma]} = \bar{\epsilon} \chi^{[\alpha\beta\gamma]} ,
\delta \phi^* \alpha\beta\gamma = \epsilon \chi_{[\alpha\beta\gamma]} ,
\delta \chi^{[\alpha\beta\gamma]} = \gamma^\nu \partial_\nu \phi^* \alpha\beta\gamma \epsilon ,
\delta \epsilon \chi_\alpha \beta\gamma = - \bar{\epsilon} \gamma^\nu \partial_\nu \phi \alpha\beta\gamma .
$$

(24)

Since covariant derivatives do not commute, when gauge interactions are included there are no longer conserved supercurrents. For example, if we redefine the singlet current as

$$
J^\mu = \gamma^\nu (D_\nu \phi^* \alpha\beta\gamma) \gamma^\mu \chi^{[\alpha\beta\gamma]} ,
$$

(25)

then we find

$$
\partial_\mu J^\mu = (D_\mu \gamma^\nu (D_\nu \phi^* \alpha\beta\gamma)) \gamma^\mu \chi^{[\alpha\beta\gamma]} + \gamma^\nu (D_\nu \phi^* \alpha\beta\gamma) \gamma^\mu D_\mu \chi^{[\alpha\beta\gamma]}
= \frac{1}{2} \left( [D_\mu, D_\nu] \gamma^\mu \phi \alpha\beta\gamma \chi^{[\alpha\beta\gamma]} \right)
= \frac{1}{2} g \gamma^\mu \left( \delta_{\mu\nu}^\alpha \phi^* \alpha\beta\gamma + \delta_{\mu\nu}^\beta \phi^* \alpha\beta\gamma + \delta_{\mu\nu}^\gamma \phi^* \alpha\beta\gamma \right) \chi^{[\alpha\beta\gamma]} ,
$$

(26)

with $\gamma^\mu = \frac{1}{2} \{ \gamma^\nu, \gamma^\mu \}$.

X. SCALAR SECTOR SELF-COUPPLINGS

We consider finally the action terms involving the scalar field without gauging. For the scalar field self-coupling action, taking index permutation possibilities into account, we have

$$
S_{\text{self-coupling}} (\phi) = \phi^* \phi^\ast \alpha\beta\gamma \left( g_1 \phi \phi^{[\alpha\beta\gamma]} \phi^{[\rho\beta\gamma]} + g_2 \phi \phi^{[\alpha\beta\gamma]} \phi^{[\rho\beta\gamma]} \right) ,
$$

(27)
which is a straightforward generalization of the usual real scalar field $\phi^4$ coupling. However, when the gauge coupling $g$ is zero, the kinetic action is invariant under the supersymmetry transformations of Eqs. (23) and (24), which are not invariances of the self-coupling action of Eq. (27). Hence the couplings $g_1$ and $g_2$ must be of order $g^2$ or higher order in the gauge coupling. An important question to be answered is how the invariances of the action affect the renormalization of $g_{1,2}$: In what order of $g^2$ do they contain logarithms of the ultraviolet cutoff, or are they finite and calculable to all orders? Since there are no Yukawa couplings, it is possible that the theory is calculable in the sense suggested by Weinberg [35].

XI. DISCUSSION

Grand unification has been intensively investigated for over forty years, and many different approaches have been tried. The model proposed here involves three ingredients that do not appear in the usual constructions: (1) boson–fermion balance without full supersymmetry, (2) canceling the spin $\frac{1}{2}$ fermion gauge anomalies against the anomaly from a gauged spin $\frac{3}{2}$ gravitino, and (3) using a scalar field representation with non-zero $U(1)$ generator to break the gauge symmetry, through a ground state with periodic $U(1)$ generator structure. The model has a number of promising features: (1) natural incorporation of three families, (2) incorporation of the experimentally viable flipped $SU(5)$ model, (3) a symmetry breaking pathway to the standard model using only the scalar field required by boson-fermion balance, without postulating additional Higgs fields, and (4) vanishing of bare Yukawa couplings and zero gauge coupling supersymmetries, which may improve the predictive power of the theory.

This investigation started from an attempt to base a supersymmetric theory on the state counting of Sec. II. In the free limit of zero $SU(8)$ couplings, we saw that the supercurrents of Eqs. (20) and (21) are conserved, but that the analogous construction does not give a conserved supercurrent when $SU(8)$ gauge interactions are included. Moreover, even in the free limit, there is no corresponding conserved representation 8 supercurrent for the “$SU(8)$ gravity” multiplet, since in $SU(8), 8 \times 63$ does not contain the totally antisymmetric 56 representation. If one instead looks for an $SO(8)$ representation 8 supercurrent, a similar problem arises, since the direct product of 8 with the symmetric 35 of $SO(8)$ again does not contain the totally antisymmetric 56 representation. So for these reasons we abandoned the search for a supersymmetric model, and instead turned to the weaker condition of boson–fermion balance. (Group representation considerations leave open the possibility of constructing $8 \times N = 1$ Lorentz and gauge non-covariant supercurrents in the free limit,
by stacking the two helicity components of the symmetric 35 of the gauge field into an artificial 70 component “scalar" $\tilde{\phi}^{[\alpha\beta\gamma\delta]}$.

Many open issues remain. In Table I, we used the modulo 5 freedom of the $U(1)/Z_5$ charges to assign these charges so that the representations needed for flipped $SU(5)$ have the usual $U(1)$ charge assignments for that model, and so that extra multiplets of fermions are, wherever possible, neutral with respect to the $SU(3) \times SU(5) \times U(1)$ gauging. This recipe is *ad hoc*, and needs further justification from a detailed study of the dynamics of symmetry breaking with a modular ground state, starting from the kinematics given in Appendix A. (For example, it would suffice to show that charge states differing from the wanted ones are separated by a large mass gap, or are absent from the asymptotic spectrum altogether, perhaps from anomaly considerations.) As is clear, our analysis is focussed solely on boson–fermion balance, Lorentz structures, and group theory, and does not address further dynamical issues such as running couplings, proton decay, generating the standard model mass and mixing parameters, $CP$ violation, and flavor changing neutral current constraints on a multiple Higgs structure. Nonetheless, the issues examined are an essential first step in trying to set up a realistic unification model, and the results look promising; the pieces appear to fit together in a jigsaw puzzle-like fashion reminiscent of what one finds in the standard model.

If the model we propose turns out to be the path that Nature follows, there will remain the further question of how the “$SU(8)$ gravity” multiplet and the “$SU(8)$ matter” multiplet of the model are unified in a more fundamental structure, for example, as arising from involutions of a large finite group or from periodic or aperiodic tilings of a large lattice. We note that the “$SU(8)$ gravity” multiplet has 128 boson and fermion helicity states, and the “$SU(8)$ matter” multiplet has 112 boson and fermion helicity states. These numbers respectively match the numbers of half-integer and integer roots of the exceptional group $E(8)$. Is this a numerical coincidence, or a hint of a deep connection with the $E(8)$ root lattice?

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**Appendix A: Higgs mechanism using a representation with nonzero \( U(1) \) charge**

In the usual application of the Higgs mechanism to grand unification, such as in the breaking of minimal \( SU(5) \) to the standard model \( SU(3) \times SU(2) \times U(1)_Y \), a Higgs representation is chosen which contains a component that is a singlet under all three factors of the standard model symmetry group. Thus, the 24 of \( SU(5) \) can be used, since it branches according to \( 24 = (1,1)(0) + (3,1)(0) + (2,3)(-5) + (2,\overline{3})(5) + (1,8)(0) \), which contains the overall singlet \( (1,1)(0) \). This singlet can attain a nonzero expectation in a ground state (the “vacuum”) that has a definite value 0 of the unbroken \( U(1) \) generator.

In the \( SU(8) \) model studied in this paper, only the 56 representation is available as a scalar to break the symmetry to \( SU(3) \times SU(5) \times U(1) \), and the component \( \phi_{(1,1)(-15)} \) that is an \( SU(3) \times SU(5) \) singlet has nonzero \( U(1) \) charge \(-15\). By the generalized Wigner-Eckart theorem, this component cannot acquire a nonzero expectation in a ground state \( |\Omega\rangle \) that is a \( U(1) \) eigenstate with a definite generator value. To get a nonzero expectation, we must take \( |\Omega\rangle \) to be a superposition of at least two \( U(1) \) eigenstates that differ in their \( U(1) \) generators by 15. Anticipating that we want the final result to have a modulo 5 (and not a modulo 15 or modulo 3) structure, we write the ground state as a superposition of \( U(1) \) eigenstates displaced from one another by 5. Let \( G \) be the \( U(1) \) generator, and \( |n\rangle \) a \( SU(3) \times SU(5) \) singlet that is a \( U(1) \) eigenstate with eigenvalue (or \( U(1) \) charge) \( n \), so that \( G|n\rangle = n|n\rangle \). Then we write the ground state \( |\Omega\rangle \) in the form

\[
|\Omega\rangle = \sum_n f(n)|5n\rangle, \tag{A.1}
\]

which for generic \( f(n) \) completely breaks the \( U(1) \) invariance,

\[
\langle \Omega|\phi_{(1,1)(-15)}|\Omega\rangle \neq 0. \tag{A.2}
\]

As in the similar analysis of the ground state structure of quantum chromodynamics, let us now impose the requirement of clustering. In order for the ground state of a tensor product composite system

\[
|\Omega_{A+B}\rangle = \sum_{n_A,n_B} f(n_A+n_B)|A;5n_A\rangle|B;5n_B\rangle \tag{A.3}
\]
to factor when the subsystems $A$, $B$ are widely separated,

$$|\Omega_{A+B}\rangle = |\Omega_A\rangle |\Omega_B\rangle ,$$

$$|\Omega_A\rangle = \sum_{n_A} f(n_A) |A; 5n_A\rangle ,$$

$$|\Omega_B\rangle = \sum_{n_B} f(n_B) |B; 5n_B\rangle ,$$

(A.4)

we must require $f(n)$ to obey

$$f(n_A + n_B) = f(n_A) f(n_B) .$$

(A.5)

This requires that $f(n)$ must have the functional form

$$f(n) = e^{nz}$$

(A.6)

for some complex number $z$. Boundedness as $|n| \to \infty$ requires that $|e^z| = 1$, so $e^z$ is a phase $e^{i\omega}$.

The ground state then has the form

$$|\Omega\rangle = \sum_{n=-\infty}^{\infty} e^{i\omega n} |5n\rangle ,$$

(A.7)

and $U(1)$ charges are only conserved modulo 5. This ground state corresponds to breaking $SU(8)$ to $SU(3) \times SU(5) \times U(1)/Z_5$, which is the “second symmetry breaking pathway” and the one chosen for our analysis. (The “first symmetry breaking pathway” corresponds either to breaking $U(1)$ completely by using a non-exponential $f(n)$ that violates clustering, or to breaking $U(1)$ to $U(1)/Z$ by choosing the ground state $|\Omega\rangle = \sum_{n=-\infty}^{\infty} \exp(i\omega n) |n\rangle$, which equivalences the integer $U(1)$ charges all to zero.)

The full basis of states for the second pathway has the form

$$|k\rangle = \sum_{n=-\infty}^{\infty} e^{i\omega n} |5n + k\rangle ,$$

(A.8)

with $|k = 0\rangle = |\Omega\rangle$. Under a modulo 5 shift we have

$$|k + 5s\rangle = \sum_{n=-\infty}^{\infty} e^{i\omega n} |5n + k + 5s\rangle = e^{-is\omega} \sum_{n=-\infty}^{\infty} e^{i\omega n} |5n + k\rangle = e^{-is\omega} |k\rangle ,$$

(A.9)

and so the state basis has a modulo 5 structure up to overall phases. Denoting by $G_{\pm}$ the raising and lowering operators on the original basis states $|n\rangle$,

$$G_+ |n\rangle = |n + 1\rangle , \quad G_- |n\rangle = |n - 1\rangle ,$$

(A.10)
we can rewrite Eq. (A.9) as

\[ G_+^{5s}|k\rangle = e^{-i\omega}|k\rangle, \quad G_-^{5s}|k\rangle = e^{i\omega}|k\rangle. \tag{A.11} \]

Using the generalized Wigner-Eckart theorem, we can relate the ground state expectation of \( \phi_{(1,1)(-15)} \) to a constant \( K \) times the expectation of \( G_{15}^- \),

\[ \langle \Omega|\phi_{(1,1)(-15)}|\Omega\rangle = K \langle \Omega|G_{15}^-|\Omega\rangle = K e^{3i\omega} \langle \Omega|\Omega\rangle \neq 0. \tag{A.12} \]

So within the modulo 5 state structure, \( \phi_{(1,1)(-15)} \) can attain a nonzero ground state expectation.

**Appendix B: Review of condensate formation**

We first review the Lorentz kinematics of forming condensates from Dirac spinors, and then turn to the dynamics of condensate formation. For any two Dirac spinors \( \Psi_1 \) and \( \Psi_2 \), both \( \overline{\Psi}_1 \Psi_2 \) and \( \overline{\Psi}_1 \Psi_2 \) are Lorentz scalars, with \( c \) denoting charge conjugation and with \( \overline{\Psi} = \Psi^\dagger i\gamma^0 \). In analyzing condensate formation, it is convenient to use real Majorana representation \( \gamma \mu \) matrices, with \( \gamma_5 \) self-adjoint and skew symmetric, and \( \gamma^0 \) skew symmetric. The chiral projectors \( P_L, P_R \) defined by

\[ P_L = \frac{1}{2}(1 + \gamma_5), \quad P_R = \frac{1}{2}(1 - \gamma_5), \tag{B.1} \]

then obey

\[ P_L^\dagger = P_L, \quad P_R^\dagger = P_R, \]
\[ P_L^T = P_R, \quad P_R^T = P_L, \tag{B.2} \]

with \( \dagger \) the adjoint and \( T \) the Dirac transpose. Charge conjugation now reduces to complex conjugation, and so we have

\[ \Psi^c = \Psi^*, \]
\[ \overline{\Psi}^c = (\Psi^c)^\dagger i\gamma^0 = \Psi^T i\gamma^0. \tag{B.3} \]

For left chiral spinors, \( \overline{\Psi}_L \Psi_L = 0 \), while Eqs. (B.2) and (B.3) imply that

\[ \overline{\Psi}_L^c \Psi_L = \Psi_L^T i\gamma^0 \Psi_L \neq 0. \tag{B.4} \]
Thus Eq. (B.4) gives the general Lorentz structure of scalar condensates constructed from left chiral spinors. Since $\gamma_0$ is skew symmetric, and since spinors anticommute, Eq. (B.4) has the same form when state labels 1, 2 are interchanged,
\[
\Psi^T_{L1} i\gamma^0 \Psi_{L2} = \Psi^T_{L2} i\gamma^0 \Psi_{L1} \quad .
\] (B.5)
Because this equation involves no complex conjugation, the group representation content of the condensate is simply the direct product of the representation content of $\Psi_1$ and $\Psi_2$.

The only way to rigorously determine if condensates form in a theory is to calculate the effective action governing condensate formation, and this is generally not feasible. So to study the dynamics of condensate formation, one falls back on simple rules of thumb, such as determining whether the leading order perturbation theory force between the constituents is attractive. The single gluon exchange potential produced when a vector gluon mediates the reaction $A + B \rightarrow A + B$ is
\[
V = \frac{g^2 K(A + B; A, B)}{2r} ,
\]
\[
K(A + B; A, B) = C_2(A + B) - C_2(A) - C_2(B) ,
\] (B.6)
with $g$ the gauge coupling and the $C_2$ the relevant Casimirs. (The Casimir for a representation $R$ is calculated from the index $\ell(R)$, the dimension $N(R)$, and the dimension of the adjoint representation $N(\text{adjoint})$ by $C_2(R) = \ell(R)N(\text{adjoint})/N(R)$; see Slansky [1].) When more than one non-Abelian group acts on the fermions forming the condensate, the one gluon exchange potentials associated with each are added.

[1] R. Slansky, Phys. Reports 79, 1 (1981).
[2] M. Gell-Mann, P. Ramond, and R. Slansky, Rev. Mod. Phys. 50, 721 (1978).
[3] T. L. Curtright and P. G. O. Freund, “SU(8) Unification and Supergravity”, in Supergravity, P. van Nieuwenhuizen and D. Z. Freedman eds., North-Holland (1979).
[4] P. Ramond, “The Family Group in Grand Unified Theories”, invited talk at the Sannibel Symposia, Feb. 1979, also arXiv:hep-ph/9809459.
[5] P. H. Frampton, Phys. Lett. B 89, 352 (1980).
[6] J. Chakrabarti, M Popović, and R. N. Mohapatra, Phys. Rev. D 21, 3212 (1980).
[7] C. W. Kim and C. Roiesnel, Phys. Lett. B 93, 343 (1980).
[8] Dzh. L. Chkareuli, Pis’ma Zh. Eksp. Teor. Fiz. 32, 684 (1980).
[9] J. E. Kim and H. S. Song, Phys. Rev. D 25, 2996 (1982).
[10] S. K. Yun, Phys. Rev. D 29, 1494 (1984).
[11] S. K. Yun, Phys. Rev. D 30, 1598 (1984).
[12] J. L. Chkareuli, Phys. Lett. B 300, 361 (1993).
[13] S. M. Barr, Phys. Rev. D 78, 075001 (2008).
[14] R. Martinez, F. Ochoa, and P. Fonseca, “A 3-3-1 model with $SU(8)$ unification”, arXiv:1105.4623.
[15] E. Cremmer and B. Julia, Phys. Lett. B 80, 48 (1978).
[16] N. K. Nielsen and H. Römer, Phys. Lett. B 154, 141 (1985).
[17] J. L. Chkareuli, Phys. Lett. B 300, 361 (1993).
[18] S. M. Barr, Phys. Rev. D 78, 075001 (2008).
[19] E. S. Fradkin and A. A. Tseytlin, “One-Loop Divergences and $\beta$-Functions in Supergravity Theories”, in Nara 1982 Proceedings, Gauge Theory and Gravitation, pp. 293-300. The beta function coefficient value $13$ given by this reference is $\frac{1}{2}$ the coefficient value $26$ given by refs. 24 and 17; we use the larger of the two.
[20] S. M. Barr, Phys. Rev. D 78, 075001 (2008).
[21] T. L. Curtright, Phys. Lett. B 102, 17 (1981).
[22] S. Coleman, Aspects of Symmetry, Chapter 7 “The uses of instantons”, Cambridge (1985).
[33] D. Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge (2012).

[34] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234, 269 (1983)

[35] S. Weinberg, Phys. Rev. Lett. 29, 388 (1972).

[36] J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).