ON THE SEMIGROUP $B^{\mathcal{F}}_\omega$ WHICH IS GENERATED BY THE FAMILY $\mathcal{F}$ OF ATOMIC SUBSETS OF $\omega$

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Abstract. We study the semigroup $B^{\mathcal{F}}_\omega$, which is introduced in [O. Gutik and M. Mykhaylenych, On some generalization of the bicyclic monoid, Visnyk Lviv. Univ. Ser. Mech.-Mat. 90 (2020), 5–19], in the case when the family $\mathcal{F}$ of subsets of $\omega$ is subjected only to the condition $p_1 \cdot p_2 = p_1$ for all $p_1, p_2$ in $\mathcal{F}$. We show that $B^{\mathcal{F}}_\omega$ is isomorphic to the semilattice $E_\omega$ of idempotents in $B^{\mathcal{F}}_\omega$. In particular, we prove that $E(B^{\mathcal{F}}_\omega)$ is algebraically complete in the class of Hausdorff semitopological inverse semigroups and every shift-continuous feebly compact $T_1$-topology on $E(B^{\mathcal{F}}_\omega)$ is compact and moreover in this case $E(B^{\mathcal{F}}_\omega)$ is homeomorphic to the one-point Alexandroff compactification of the discrete countable space $\Delta(\omega)$. We prove that the space $(E(B^{\mathcal{F}}_\omega), \tau)$ is homeomorphic to the one-point Alexandroff compactification of the discrete countable space $\Delta(\omega)$. We study the closure of $B^{\mathcal{F}}_\omega$ in a semitopological semigroup. In particular, we prove that $B^{\mathcal{F}}_\omega$ is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion, and a Hausdorff topological inverse semigroup $B^{\mathcal{F}}_\omega$ is closed in any Hausdorff topological semigroup if and only if the band $E(B^{\mathcal{F}}_\omega)$ is compact.

1. Introduction, motivation and main definitions

We shall follow the terminology of [2–5, 19]. By $\omega$ we denote the set of all non-negative integers. Let $\mathcal{P}(\omega)$ be the family of all subsets of $\omega$. For any $F \in \mathcal{P}(\omega)$ and $n, m \in \omega$ we put

$$n - m + F = \{n - m + k: k \in F\}$$

This definition implies that $n - m + F = \emptyset$ if $F = \emptyset$. A subfamily $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called $\omega$-closed if $F_1 \cap (\omega - F_2) \in \mathcal{F}$ for all $F \in \mathcal{F}$ and $F, F_1, F_2 \in \mathcal{F}$.

A semigroup $S$ is called inverse if for any element $x \in S$ there exists a unique $x^{-1} \in S$ (called the inverse of $x$) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. If $S$ is an inverse semigroup, then the function $\text{inv}: S \to S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called the inversion.

If $S$ is a semigroup, then we shall denote the subset of all idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a band (or the band of $S$). The semigroup operation of $S$ determines the following partial order $\leq$ on $E(S)$: $e \leq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents. By $(\omega, \min)$ or $\omega_{\min}$ we denote the set $\omega$ with the semilattice operation $x \cdot y = \min\{x, y\}$.

If $S$ is an inverse semigroup then the semigroup operation on $S$ determines the following partial order $\leq$ on $S$: $s \leq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the natural partial order on $S$ [22].

The bicyclic monoid $\mathcal{C}(p, q)$ is the semigroup with the identity 1 generated by two elements $p$ and $q$ subjected only to the condition $pq = 1$. The semigroup operation on $\mathcal{C}(p, q)$ is determined as follows:

$$q^k \cdot p^l \cdot q^m = q^{k + m - \min\{l, m\}} \cdot p^{l + n - \min\{l, m\}}.$$

It is well known that the bicyclic monoid $\mathcal{C}(p, q)$ is a bisimple (and hence simple) combinatorial $E$-unitary inverse semigroup and every non-trivial congruence on $\mathcal{C}(p, q)$ is a group congruence [3].

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On the set $B_\omega = \omega \times \omega$ we define a semigroup operation “·” in the following way

$$(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} 
(i_1 - j_1 + i_2, j_2), & \text{if } j_1 < i_2; \\
(i_1, j_2), & \text{if } j_1 = i_2; \\
(i_1, j_1 - i_2 + j_2), & \text{if } j_1 > i_2.
\end{cases}$$

It is well known that the semigroup $B_\omega$ is isomorphic to the bicyclic monoid by the mapping $h: \mathcal{C}(p, q) \rightarrow B_\omega$, $q^kp^l \mapsto (k, l)$ (see: [3, Section 1.12] or [18, Exercise IV.1.11(ii)]).

A topological (semitopological) semigroup is a topological space together with a continuous (separately continuous) semigroup operation. If $S$ is a semigroup and $\tau$ is a topology on $S$ such that $(S, \tau)$ is a topological semigroup, then we shall call $\tau$ a semigroup topology on $S$, and if $\tau$ is a topology on $S$ such that $(S, \tau)$ is a semitopological semigroup, then we shall call $\tau$ a shift-continuous topology on $S$. An inverse topological semigroup with the continuous inversion is called a topological inverse semigroup. If $S$ is an inverse semigroup and $\tau$ is a topology on $S$ such that $(S, \tau)$ is a topological inverse semigroup, then we shall call $\tau$ a semigroup inverse topology on $S$.

Next we shall describe the construction which is introduced in [9].

Let $B_\omega$ be the bicyclic monoid and $\mathcal{F}$ be an $\omega$-closed subfamily of $\mathcal{P}(\omega)$. On the set $B_\omega \times \mathcal{F}$ we define the semigroup operation “·” in the following way

$$(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} 
(i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 < i_2; \\
(i_1, j_2, F_1 \cap F_2), & \text{if } j_1 = i_2; \\
(i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 > i_2.
\end{cases}$$

By [9], if the family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is $\omega$-closed, then $(B_\omega \times \mathcal{F}, \cdot)$ is a semigroup. Moreover, if an $\omega$-closed family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ contains the empty set $\emptyset$, then the set

$$I = \{(i, j, \emptyset): i, j \in \omega\}$$

is an ideal of the semigroup $(B_\omega \times \mathcal{F}, \cdot)$. For any $\omega$-closed family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ the following semigroup

$$B_\omega^\mathcal{F} = \begin{cases} 
(B_\omega \times \mathcal{F}, \cdot)/I, & \text{if } \emptyset \in \mathcal{F}; \\
(B_\omega \times \mathcal{F}, \cdot), & \text{if } \emptyset \notin \mathcal{F}
\end{cases}$$

is defined in [9]. The semigroup $B_\omega^\mathcal{F}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [9] that $B_\omega^\mathcal{F}$ is combinatorial inverse semigroup and Green’s relations, the natural partial order on $B_\omega^\mathcal{F}$ and its set of idempotents are described. The criteria of simplicity, 0-simplicity, bisimplicity, 0-bisimplicity of the semigroup $B_\omega^\mathcal{F}$ and when $B_\omega^\mathcal{F}$ has the identity, is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units are given. In particular in [9] it is proved that the semigroup $B_\omega^\mathcal{F}$ is isomorphic to the semigroup of $\omega \times \omega$-matrix units if and only if $\mathcal{F}$ consists of sets of cardinality $\leq 1$ in $\omega$.

Let $\mathcal{F}$ be some family of cardinality $\leq 1$ in $\omega$. In this case we shall say that $\mathcal{F}$ is the family of atomic subsets of $\omega$. It is obvious that if $\mathcal{F} = \{\emptyset\}$ then the semigroup $B_\omega^\mathcal{F}$ is trivial and hence in this paper we assume that the family $\mathcal{F}$ contains at least one singleton subset of $\omega$. It is obvious that in this case $\mathcal{F}$ is an $\omega$-closed subfamily of $\mathcal{P}(\omega)$ and hence $B_\omega^\mathcal{F}$ is an inverse semigroup with zero. Later by $0$ we denote the zero of $B_\omega^\mathcal{F}$ and by $(i, j, \{k\})$ a non-zero element of $B_\omega^\mathcal{F}$ for some $i, j \in \omega$, $\{k\} \in \mathcal{F}$.

We put $F = \bigcup \mathcal{F}$. Since the semilattice $(\omega, \min)$ is linearly ordered, the set $F$ with the binary operation $xy = \min\{x, y\}$ is a subsemilattice of $(\omega, \min)$ and later by $F_{\min}$ we shall denote the set $F$ with the semilattice operation inherited from $(\omega, \min)$.

We need the following construction from [6].

Let $S$ be a semigroup with zero and $\lambda \geq 1$ be a cardinal. On the set $B_\lambda(S) = (\lambda \times S \times \lambda) \sqcup \{\emptyset\}$ we define a semigroup operation as follows

$$(\alpha, s, \beta) \cdot (\gamma, t, \delta) = \begin{cases} 
(\alpha, st, \delta), & \text{if } \beta = \gamma; \\
\emptyset, & \text{if } \beta \neq \gamma
\end{cases}$$

and

$$(\alpha, s, \beta) \cdot \emptyset = \emptyset \cdot (\alpha, s, \beta) = \emptyset \cdot \emptyset = \emptyset,$$
for all \( \alpha, \beta, \gamma, \delta \in \lambda \) and \( s, t \in S \). The semigroup \( \mathcal{B}_\lambda(S) \) is called the Brandt \( \lambda \)-extension of the semigroup \( S \) [6]. Algebraic properties of \( \mathcal{B}_\lambda(S) \) and its generalization the Brandt \( \lambda^0 \)-extension \( \mathcal{B}_\lambda^0(S) \) are studied in [6, 7, 10, 12].

In this paper we study the semigroup \( B_\omega^\mathcal{F} \) for a family \( \mathcal{F} \) of atomic subsets of \( \omega \). We show that \( B_\omega^\mathcal{F} \) is isomorphic to the subsemigroup \( \mathcal{B}_\omega^\mathcal{F}(F_{\min}) \) of the Brandt \( \omega \)-extension of the semilattice \( F_{\min} \) and describe all shift-continuous feebly compact \( T_1 \)-topologies on the semigroup \( \mathcal{B}_\omega^\mathcal{F}(F_{\min}) \). In particular, we prove that every shift-continuous feebly compact \( T_1 \)-topology \( \tau \) on \( \mathcal{B}_\omega^\mathcal{F}(F_{\min}) \) is compact and moreover in this case the space \( (\mathcal{B}_\omega^\mathcal{F}(F_{\min}), \tau) \) is homeomorphic to the one-point Alexandroff compactification of the discrete countable space \( \mathcal{D}(\omega) \). We study the closure of \( B_\omega^\mathcal{F} \) in a semitopological semigroup. In particularly we show that \( B_\omega^\mathcal{F} \) is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion, and a Hausdorff topological inverse semigroup \( B_\omega^\mathcal{F} \) is closed in any Hausdorff topological semigroup if and only if the band \( E(B_\omega^\mathcal{F}) \) is compact.

Later in this paper we assume that \( \mathcal{F} \) is a non-trivial family of atomic subsets of \( \omega \), i.e., \( \mathcal{F} \) contains at least one nontrivial singleton subset of \( \omega \).

2. Algebraic properties of the semigroup \( B_\omega^\mathcal{F} \)

Proposition 2 of [9] implies the following proposition which describing the natural partial order on \( B_\omega^\mathcal{F} \).

**Proposition 2.1.** Let \( (i_1, j_1, \{k_1\}) \) and \( (i_2, j_2, \{k_2\}) \) be non-zero elements of the semigroup \( B_\omega^\mathcal{F} \). Then \( (i_1, j_1, \{k_1\}) \preceq (i_2, j_2, \{k_2\}) \) if and only if

\[
k_2 - k_1 = i_1 - i_2 = j_1 - j_2 = p
\]

for some \( p \in \omega \).

Since the set \( \omega \) is well ordered by the usual order we enumerate the set \( F = \{k_i : i \in \omega\} \) in the following way \( k_0 < k_1 < \cdots < k_n < k_{n+1} < \cdots \). It is obvious that the set \( F \) is finite if and only if \( F \) contains the maximum.

Proposition 2.1 implies the structure of maximal chains in \( B_\omega^\mathcal{F} \) with the respect to its natural partial order.

**Corollary 2.2.** Let \( i, j \) be arbitrary elements of \( \omega \). Then in the case when the set \( F \) is infinite then the following finite series

\[
0 \preceq (i, j, \{k_0\});
0 \preceq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preceq (i, j, \{k_1\});
0 \preceq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preceq (i + k_2 - k_1, j + k_2 - k_1, \{k_1\}) \preceq (i, j, \{k_2\});
\]

\[
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

\[
0 \preceq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preceq (i + k_2 - k_1, j + k_2 - k_1, \{k_1\}) \preceq \cdots \preceq
\]

\[
\preceq (i + k_{n+1} - k_n, j + k_{n+1} - k_n, \{k_n\}) \preceq (i, j, \{k_{n+1}\});
\]

\[
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

describes maximal chains in the semigroup \( B_\omega^\mathcal{F} \) and in the case when the set \( F \) is finite and contains maximum \( k_n \) then the following finite series

\[
0 \preceq (i, j, \{k_0\});
0 \preceq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preceq (i, j, \{k_1\});
0 \preceq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preceq (i + k_2 - k_1, j + k_2 - k_1, \{k_1\}) \preceq (i, j, \{k_2\});
\]

\[
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

\[
0 \preceq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preceq (i + k_2 - k_1, j + k_2 - k_1, \{k_1\}) \preceq \cdots \preceq
\]

\[
\preceq (i + k_n - k_{n-1}, j + k_n - k_{n-1}, \{k_n\}) \preceq (i, j, \{k_n\})
\]
describes maximal chains in the semigroup \( B_\omega^x \).

We define a map \( f : B_\omega^x \to \mathscr{R}_\omega(F_{\min}) \) by the formulae

\[
(1) \quad f(i, j, \{k\}) = (i + k, k, j + k) \quad \text{and} \quad (0) f = \mathscr{O},
\]

for \( i, j \in \omega \) and \( \{k\} \in \mathcal{F} \setminus \{\emptyset\} \).

**Proposition 2.3.** The map \( f : B_\omega^x \to \mathscr{R}_\omega(F_{\min}) \) is an isomorphic embedding.

**Proof.** It is obvious that the map \( f \) which is defined by formulae (1) is injective.

For arbitrary \( (i_1, j_1, \{k_1\}), (i_2, j_2, \{k_2\}) \in B_\omega^x \) we have that

\[
f((i_1, j_1, \{k_1\}) \cdot (i_2, j_2, \{k_2\})) =
\]

\[
= \begin{cases} 
  f(i_1 - j_1 + i_2, j_2 + \{k_2\}), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  f(i_1, j_2 + \{k_1\}), & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\
  f(i_1, j_1 - i_2, j_2, \{k_1\}), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  f(0), & \text{if } j_1 + k_1 \neq i_2 + k_2
\end{cases}
\]

\[
= \begin{cases} 
  (i_1 - j_1 + i_2 + k_2, j_2 + k_2), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  (i_1 + k_1 + i_2 + j_2 + k_1), & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\
  (i_1 + k_1, j_1 - i_2 + j_2 + k_1), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2
\end{cases}
\]

\[
= \begin{cases} 
  (i_1 + k_1, j_2 + k_2), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  (i_1 + k_1, j_1 + k_2), & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\
  (i_1 + k_1, j_1 + j_2 + k_2), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2
\end{cases}
\]

\[
f((i_1, j_1, \{k_1\}) \cdot (i_2, j_2, \{k_2\})) = (i_1 + k_1, k_1, j_1 + k_1) \cdot (i_2 + k_2, k_2, j_2 + k_2) =
\]

\[
= \begin{cases} 
  (i_1 + k_1, \min\{k_1, k_2\}, j_2 + k_2), & \text{if } j_1 + k_1 = i_2 + k_2; \\
  \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2
\end{cases}
\]

\[
= \begin{cases} 
  (i_1 + k_1, k_2, j_2 + k_2), & \text{if } k_2 < k_1 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  (i_1 + k_1, k_1, j_2 + k_2), & \text{if } k_2 = k_1 \text{ and } k_1 = k_2; \\
  (i_1 + k_1, k_1, j_2 + k_2), & \text{if } k_2 > k_1 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2,
\end{cases}
\]

\[
= \begin{cases} 
  (i_1 + k_1, k_2, j_2 + k_2), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  (i_1 + k_1, k_1, j_2 + k_2), & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\
  (i_1 + k_1, k_1, j_2 + k_2), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2
\end{cases}
\]

Since \( 0 \) and \( \mathscr{O} \) are the zeros of the semigroups \( B_\omega^x \) and \( \mathscr{R}_\omega(F_{\min}) \), respectively, the above equalities imply that the map \( f : B_\omega^x \to \mathscr{R}_\omega(F_{\min}) \) is a homomorphism. This completes the proof of the proposition. \( \square \)

Next we define

\[
\mathscr{R}_\omega(F_{\min}) = \{0\} \cup \{(i + k, j + k) \in \mathscr{R}_\omega(F_{\min}) \setminus \{\emptyset\} : (i, j, \{k\}) \in B_\omega^x\}.
\]

Proposition 2.3 implies

**Theorem 2.4.** Let \( \mathcal{F}^* \) be any family of atomic subsets of \( \omega \). Then the semigroup \( B_\omega^x \) is isomorphic to \( \mathscr{R}_\omega(F_{\min}) \) by the mapping \( f \).

**Proposition 2.5.** Let \( \mathcal{F}^* \) be any family of subsets of \( \omega \) which contains a non-empty set, and \( k_0 = \min \bigcup \mathcal{F}^* \). Then the semigroup \( B_\omega^{\mathcal{F}^*} \) is isomorphic to the semigroup \( B_\omega^{\mathcal{F}^+_0} \) where

\[
\mathcal{F}^+_0 = \{ -k_0 + F : F \in \mathcal{F}^* \}.
\]
Proof. Since the set \( \omega \) with the usual order \( \leq \) is well ordered, the number \( k_0 \) is well defined. This implies that the semigroup \( B_\omega^{\geq 0} \) is well defined, because \( F \subseteq \{ n \in \omega : n \geq k_0 \} \) for any \( F \in \mathcal{F}^* \). Without loss of generality we may assume that \( \emptyset \in \mathcal{F}^* \), which implies that the semigroup \( B_\omega^{\geq 0} \) has zero \( 0 \), and hence the semigroup \( B_\omega^{\geq 0} \) has zero \( 0 \), too.

We define the map \( h : B_\omega^{\geq 0} \to B_\omega^{\geq 0} \) in the following way

\[
(2) \quad h(i, j, \{ k \}) = (i - k_0, j - k_0, \{ k - k_0 \}) \quad \text{and} \quad (0) h = 0
\]

for \( i, j \in \omega \) and \( \{ k \} \in \mathcal{F}^* \setminus \{ \emptyset \} \). It is obvious that such defined map \( h \) is bijective.

For arbitrary \( (i_1, j_1, \{ k_1 \}), (i_2, j_2, \{ k_2 \}) \in B_\omega^{\geq 0} \) we have that

\[
h((i_1, j_1, \{ k_1 \}) \cdot (i_2, j_2, \{ k_2 \})) =
\begin{align*}
&= \begin{cases} 
  h(i_1 - j_1 + i_2, j_2, \{ k_2 - k_0 \}), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  h(i_1, j_1, \{ k_1 \}), & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\
  h(i_1, j_1 - i_2 + j_2, \{ k_1 \}), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  h(0), & \text{if } j_1 + k_1 \neq i_2 + k_2 
\end{cases} \\
&= \begin{cases} 
  (i_1 - j_1 + i_2 - k_0, j_2 - k_0, \{ k_2 - k_0 \}), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  (i_1 - k_0, j_2 - k_0, \{ k_1 - k_0 \}), & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\
  (i_1 - k_0, j_1 - i_2 + j_2 - k_0, \{ k_1 - k_0 \}), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
  0, & \text{if } j_1 + k_1 \neq i_2 + k_2
\end{cases}
\end{align*}
\]

and

\[
h(i_1, j_1, \{ k_1 \}) \cdot h(i_2, j_2, \{ k_2 \}) =
\begin{align*}
&= (i_1 - k_0, j_1 - k_0, \{ k_1 - k_0 \}) \cdot (i_2 - k_0, j_2 - k_0, \{ k_2 - k_0 \}) =
\begin{cases} 
  (i_1 - k_0 - (j_1 - k_0) + i_2 - k_0, j_2 - k_0, \{ k_2 - k_0 \}), & \text{if } j_1 - k_0 < i_2 - k_0 \text{ and } j_1 - k_0 + k_1 = i_2 - k_0 + k_2 - k_0; \\
  (i_1 - k_0, j_2 - k_0, \{ k_1 - k_0 \}), & \text{if } j_1 - k_0 = i_2 - k_0 \text{ and } k_1 - k_0 = k_2 - k_0; \\
  (i_1 - k_0, j_1 - k_0 - (i_2 - k_0) + j_2 - k_0, \{ k_1 - k_0 \}), & \text{if } j_1 - k_0 > i_2 - k_0 \text{ and } j_1 - k_0 + k_1 = i_2 - k_0 + k_2 - k_0; \\
  0, & \text{if } j_1 - k_0 + k_1 = i_2 - k_0 + k_2 - k_0
\end{cases}
\end{align*}
\]

Since \( 0 \) is the zero of both semigroups \( B_\omega^{\geq 0} \) and \( B_\omega^{\geq 0} \), the above equalities imply that such defined map \( h : B_\omega^{\geq 0} \to B_\omega^{\geq 0} \) is a homomorphism. \( \square \)

**Theorem 2.6.** Let \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \) be some families of atomic subsets of \( \omega \). Then the semigroups \( B_\omega^{\geq 1} \) and \( B_\omega^{\geq 2} \) are isomorphic if and only if there exists an integer \( n \) such that

\[
\mathcal{F}^1 = \{ n + F : F \in \mathcal{F}^2 \}.
\]

**Proof.** The implication \( (\Leftarrow) \) follows from Proposition 2.5.

\( (\Rightarrow) \) Put \( \mathcal{F}^1 = \mathcal{F}^1 \) and \( \mathcal{F}^2 = \mathcal{F}^2 \). By Proposition 2.5, without loss of generality we may assume that \( 0 \in \mathcal{F}^1 \cap \mathcal{F}^2 \), i.e., \( \{ 0 \} \in \mathcal{F}^1 \) and \( \{ 0 \} \in \mathcal{F}^2 \).

Suppose to the contrary that the semigroups \( B_\omega^{\geq 1} \) and \( B_\omega^{\geq 2} \) are isomorphic but \( \mathcal{F}^1 \neq \mathcal{F}^2 \). Since \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \) are some families of atomic subsets of \( \omega \), we get that \( \mathcal{F}^1 \neq \mathcal{F}^2 \). Hence without loss of generality we may assume that there exists the minimum positive integer \( m \) of the set \( \mathcal{F}^1 \) such that \( m \notin \mathcal{F}^2 \). Put

\[
\mathcal{F} = \{ k \in \mathcal{F}^2 : k < m \}.
\]
We enumerate the set \( \widetilde{F} = \{k_0, k_1, \ldots, k_n\} \) in the following way
\[
k_0 = 0 < k_1 < \cdots < k_n.
\]
Then we have that \( \widetilde{F} \subset F^1 \).

By Lemma 2 of [9] a non-zero element \((i, j, \{k\})\) of the semigroup \( B^\sharp_\omega \) (or \( B^\sharp_\omega \)) is an idempotent if and only if \( i = j \). This and Corollary 2.2 imply the semigroup \( B^\sharp_\omega \) contains exactly \( m - k_n \) distinct chains (or a chain) of idempotents of the length \( k_n + 2 \), but the semigroup \( B^\sharp_\omega \) contains at least \( m - k_n + 1 \) distinct chains of idempotents of the length \( k_n + 2 \). This contradicts that the semigroups \( B^\sharp_\omega \) are isomorphic. The obtained contradiction implies the implication.

\[\square\]

For any \( i, j \in \omega \) we denote
\[
F^{(i,j)}_{\min} = \{(i, k, j) : (i, k, j) \in \mathcal{B}_\omega(F_{\min})\}
\]
and
\[
\omega^{(i,j)}_{\min} = \{(i, k, j) : (i, k, j) \in \mathcal{B}_\omega(\omega_{\min})\}
\]
where by \( \omega_{\min} \) we denote the semilattice \( (\omega, \min) \).

**Lemma 2.7.** In the semigroup \( B^\varrho_\omega \) both equations \( A \cdot X = B \) and \( X \cdot A = B \) have only finitely many solutions for \( B \neq 0 \).

**Proof.** We show that the equation \( A \cdot X = B \) has finitely many solutions for \( B \neq 0 \) in the semigroup \( \mathcal{B}_\omega(F_{\min}) \). In the case of the equation \( X \cdot A = B \) the proof is similar.

We denote
\[
A = (i_A, k_A, j_A), \quad X = (i_X, k_X, j_X) \quad \text{and} \quad B = (i_B, k_B, j_B),
\]
where \((i_X, k_X, j_X)\) is a variable, \((i_A, k_A, j_A)\) and \((i_B, k_B, j_B)\) are constants of the equation
\[
(i_A, k_A, j_A) \cdot (i_X, k_X, j_X) = (i_B, k_B, j_B).
\]
First we establish the solution of equation (3) in the Brandt \( \omega \)-extension \( \mathcal{B}_\omega(\omega_{\min}) \) of the semilattice \( \omega_{\min} \). The semigroup operation in \( \mathcal{B}_\omega(\omega_{\min}) \) implies that equation (3) has a non-empty set of solutions if and only if \( k_B \lesssim k_A \) in \( \omega_{\min} \) and \( i_A = i_B \). Hence we have that the set of solutions of (3) is a subset of \( \omega^{(i_A,j_A)}_{\min} \). This implies that the set of solutions of equation (3) is a subset of \( F^{(i_A,j_A)}_{\min} \). This and Theorem 2.4 imply the statement of the lemma.

\[\square\]

3. On topologizations of the semigroup \( \mathcal{B}_\omega(F_{\min}) \)

By Proposition 2.5 for any family \( \mathcal{F} \) of atomic subsets of \( \omega \) the semigroup \( B^\varrho_\omega \) is isomorphic to the semigroup \( B^\varrho_\omega \), where \( \mathcal{F}_0 \) is a family of atomic subsets of \( \omega \) such that \( 0 \in \bigcup \mathcal{F}_0 \). Hence later we shall assume that \( 0 \in \mathcal{F} \), i.e., \((i, 0, i) \in \mathcal{B}_\omega(F_{\min}) \) for any \( i, j \in \omega \).

**Proposition 3.1.** Let \( \tau \) be a shift-continuous \( T_1 \)-topology on the semigroup \( \mathcal{B}_\omega(F_{\min}) \). Then every non-zero element of \( \mathcal{B}_\omega(F_{\min}) \) is an isolated point in \( (\mathcal{B}_\omega(F_{\min}), \tau) \).

**Proof.** Fix arbitrary \( i, j \in \omega \). Since
\[
(i, 0, i) \cdot (i, 0, j) \cdot (j, 0, j) = (i, 0, j)
\]
the assumption of the proposition implies that for any open neighbourhood \( W_{(i, 0, j)} \neq \emptyset \) of the point \((i, 0, j)\) there exists its open neighbourhood \( V_{(i, 0, j)} \) in the topological space \( (\mathcal{B}_\omega(F_{\min}), \tau) \) such that
\[
(i, 0, i) \cdot V_{(i, 0, j)} \cdot (j, 0, j) \subseteq W_{(i, 0, j)}.
\]
The definition of the semigroup operation on \( \mathcal{B}_\omega(F_{\min}) \) implies that \( V_{(i, 0, j)} \subseteq F^{(i,j)}_{\min} \). Then \( F^{(i,j)}_{\min} \) is an open subset of the set \((\mathcal{B}_\omega(F_{\min}), \tau)\) because it is the full preimage of \( V_{(i, 0, j)} \) under the mapping
\[
\eta : \mathcal{B}_\omega(F_{\min}) \rightarrow \mathcal{B}_\omega(F_{\min}), \quad x \mapsto (i, 0, i) \cdot x \cdot (j, 0, j).
\]
By Corollary 2.2 the set \( F^{(i,j)}_{\min} \) is finite, which implies the statement of the proposition.

\[\square\]
Next we shall show that the semigroup \( \mathcal{B}_\omega(F_{\text{min}}) \) admits a compact shift-continuous Hausdorff topology.

**Example 3.2.** A topology \( \tau_{\text{Ac}} \) on the semigroup \( \mathcal{B}_\omega(F_{\text{min}}) \) is defined as follows:

a) all nonzero elements of \( \mathcal{B}_\omega(F_{\text{min}}) \) are isolated points in \( (\mathcal{B}_\omega(F_{\text{min}}), \tau_{\text{Ac}}) \);

b) the family

\[
\mathcal{B}_{\text{Ac}}(\emptyset) = \left\{ U(i_1,j_1), \ldots, (i_n,j_n) \in \mathcal{B}_\omega(F_{\text{min}}) \setminus \left( F_{\text{min}}^{(i_1,j_1)} \cup \cdots \cup F_{\text{min}}^{(i_n,j_n)} \right) : n, i_1, j_1, \ldots, i_n, j_n \in \omega \right\}
\]

is the base of the topology \( \tau_{\text{Ac}} \) at the point \( \emptyset \in \mathcal{B}_\omega(F_{\text{min}}) \).

**Corollary 2.2** implies that the set \( F_{\text{min}}^{(i,j)} \) is finite for any \( i, j \in \omega \) which implies that the topological space \( (\mathcal{B}_\omega(F_{\text{min}}), \tau_{\text{Ac}}) \) is homeomorphic to the one-point Alexandroff compactification of the discrete space \( \mathcal{B}_\omega(F_{\text{min}}) \setminus \{ \emptyset \} \).

**Proposition 3.3.** \( (\mathcal{B}_\omega(F_{\text{min}}), \tau_{\text{Ac}}) \) is a Hausdorff compact semitopological semigroup with continuous inversion.

**Proof.** It is obvious that the topology \( \tau_{\text{Ac}} \) is Hausdorff and compact.

Fix any \( U(i_1,j_1), \ldots, (i_n,j_n) \in \mathcal{B}_{\text{Ac}}(\emptyset) \) and \( (i, k, j), (l, m, p) \in \mathcal{B}_\omega(F_{\text{min}}) \setminus \{ \emptyset \} \). Put

\[
K = \{ i, i_1, \ldots, i_n, j_1, \ldots, j_n \} \quad \text{and} \quad U_K = \mathcal{B}_\omega(F_{\text{min}}) \setminus \bigcup_{x,y \in K} F_{\text{min}}^{(x,y)}.
\]

Then we have that \( U_K \in \mathcal{B}_{\text{Ac}}(\emptyset) \) and the following conditions hold

\[
U_K \cdot \{(i,k,j)\} \subseteq U(i_1,j_1), \ldots, (i_n,j_n);
\]

\[
\{(i,k,j)\} \cdot U_K \subseteq U(i_1,j_1), \ldots, (i_n,j_n);
\]

\[
\{\emptyset\} \cdot \{(i,k,j)\} = \{(i,k,j)\} \cdot \{\emptyset\} = \{\emptyset\} \subseteq U(i_1,j_1), \ldots, (i_n,j_n);
\]

\[
\{\emptyset\} \cdot U_{(i_1,j_1), \ldots, (i_n,j_n)} = U(i_1,j_1), \ldots, (i_n,j_n) \cdot \{\emptyset\} = \{\emptyset\} \subseteq U(i_1,j_1), \ldots, (i_n,j_n),
\]

\[
\{(i, k, j)\} \cdot \{(l, m, p)\} = \{\emptyset\} \subseteq U(i_1,j_1), \ldots, (i_n,j_n), \quad \text{if} \quad j \neq l;
\]

\[
\{(i, k, j)\} \cdot \{(l, m, p)\} = \{(i, \min\{k, m\}, p)\}, \quad \text{if} \quad j = l,
\]

\[
(U_{(j_1,i_1), \ldots, (j_n,i_n)})^{-1} \subseteq U(i_1,j_1), \ldots, (i_n,j_n)
\]

Therefore, \( (\mathcal{B}_\omega(F_{\text{min}}), \tau_{\text{Ac}}) \) is a semitopological inverse semigroup with continuous inversion. \( \square \)

We recall that a topological space \( X \) is said to be

- **perfectly normal** if \( X \) is normal and and every closed subset of \( X \) is a \( G_\delta \)-set;
- **scattered** if \( X \) does not contain a non-empty dense-in-itself subspace;
- **hereditarily disconnected** (or **totally disconnected**) if \( X \) does not contain any connected subsets of cardinality larger than one;
- **compact** if each open cover of \( X \) has a finite subcover;
- **countably compact** if each open countable cover of \( X \) has a finite subcover;
- **H-closed** if \( X \) is a closed subspace of every Hausdorff topological space containing \( X \);
- **infra H-closed** provided that any continuous image of \( X \) into any first countable Hausdorff space is closed (see \([15]\));
- **freebly compact** if each locally finite open cover of \( X \) is finite \([1]\);
- **d-freebly compact** (or **DFCC**) if every discrete family of open subsets in \( X \) is finite (see \([17]\));
- **pseudocompact** if \( X \) is Tychonoff and each continuous real-valued function on \( X \) is bounded;
- **\( Y \)-compact** for some topological space \( Y \), if the image \( f(X) \) is compact for any continuous map \( f: X \to Y \).

The relations between above defined compact-like spaces are presented at the diagram in \([14]\).

**Lemma 3.4.** Every shift-continuous \( T_1 \)-topology \( \tau \) on the semigroup \( \mathcal{B}_\omega(F_{\text{min}}) \) is regular.
Proof. By Proposition 3.3 every non-zero element of the semigroup \( \mathcal{B}_\omega(F_{\min}) \) is an isolated point in the space \( (\mathcal{B}_\omega(F_{\min}), \tau) \). Hence every open neighbourhood \( V(\mathcal{0}) \) of the zero \( \mathcal{0} \) is a closed subset in \( (\mathcal{B}_\omega(F_{\min}), \tau) \), which implies that the topological space \( (\mathcal{B}_\omega(F_{\min}), \tau) \) is regular. \( \square \)

Since in any countable \( T_1 \)-space \( X \) every open subset of \( X \) is a \( F_\sigma \)-set, Theorem 1.5.17 from \([5]\) and Lemma 3.4 imply the following corollary:

**Corollary 3.5.** Let \( \tau \) be a shift-continuous \( T_1 \)-topology on the semigroup \( \mathcal{B}_\omega(F_{\min}) \). Then \( (\mathcal{B}_\omega(F_{\min}), \tau) \) is a perfectly normal, scattered, hereditarily disconnected space.

By \( \mathcal{D}(\omega) \) we denote the infinite countable discrete space and by \( \mathbb{R} \) the set of all real numbers with the usual topology.

**Theorem 3.6.** Let \( \tau \) be a shift-continuous \( T_1 \)-topology on the semigroup \( \mathcal{B}_\omega(F_{\min}) \). Then the following statements are equivalent:

1. \( (\mathcal{B}_\omega(F_{\min}), \tau) \) is compact;
2. \( \tau = \tau_{\mathcal{Ac}} \);
3. \( (\mathcal{B}_\omega(F_{\min}), \tau) \) is \( H \)-closed;
4. \( (\mathcal{B}_\omega(F_{\min}), \tau) \) is \( d \)-feebly compact;
5. \( (\mathcal{B}_\omega(F_{\min}), \tau) \) is infra \( H \)-closed;
6. \( (\mathcal{B}_\omega(F_{\min}), \tau) \) is \( \mathcal{D}(\omega) \)-compact.

Proof. Implications \((ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (viii) \Rightarrow (ix)\) and \((i) \Rightarrow (vii) \Rightarrow (iv) \Rightarrow (vi)\) are trivial (see the diagram in \([14]\)). By Lemma 3.4 we get implications \((vi) \Rightarrow (iv)\) and \((iii) \Rightarrow (i)\).

\((ix) \Rightarrow (i)\) Suppose to the contrary that there exists a shift-continuous \( T_1 \)-topology \( \tau \) on the semigroup \( \mathcal{B}_\omega(F_{\min}) \) such that \( (\mathcal{B}_\omega(F_{\min}), \tau) \) is a \( \mathcal{D}(\omega) \)-compact non-compact space. Then there exists an open cover \( \mathcal{U} = \{U_\alpha\} \) of \( (\mathcal{B}_\omega(F_{\min}), \tau) \) which does not contain a finite subcover. Fix \( U_\alpha \in \mathcal{U} \) such that \( \mathcal{0} \in U_\alpha_0 \). Since the space \( (\mathcal{B}_\omega(F_{\min}), \tau) \) is not compact the set \( \mathcal{B}_\omega(F_{\min}) \setminus U_\alpha_0 \) is infinite. We enumerate the set \( \mathcal{B}_\omega(F_{\min}) \setminus U_\alpha_0 \), i.e., put \( \{x_i : i \in \omega\} = \mathcal{B}_\omega(F_{\min}) \setminus U_\alpha_0 \). We identify \( \mathcal{D}(\omega) \) with \( \omega \) and define a map \( f: (\mathcal{B}_\omega(F_{\min}), \tau) \rightarrow \mathcal{D}(\omega) \) by the formula

\[
  f(x) = \begin{cases} 
    0, & \text{if } x \in U_\alpha_0; \\
    i, & \text{if } x = x_i. 
  \end{cases}
\]

Proposition 3.1 implies that such defined map \( f \) is continuous. Also, the image \( f(\mathcal{B}_\omega(F_{\min})) \) is not a compact subset of \( \mathcal{D}(\omega) \), which contradicts the assumption. \( \square \)

**Remark 3.7.**

1. By Proposition 4 of \([9]\) the semigroup \( B_{\omega}^{\mathcal{F}} \) contains an isomorphic copy of the semigroup of \( \omega \times \omega \)-matrix units. Then Theorem 5 from \([11]\) implies that \( B_{\omega}^{\mathcal{F}} \) does not embed into a countably compact Hausdorff topological semigroup.

2. A Hausdorff topological semigroup \( S \) is called \( \Gamma \)-compact if for every \( x \in S \) the closure of the set \( \{x, x^2, x^3, \ldots\} \) is compact in \( S \) (see \([16]\)). The semigroup operation \( B_{\omega}^{\mathcal{F}} \) implies that either \( a \cdot a = a \) or \( a \cdot a = \mathcal{0} \) for any \( a \in B_{\omega}^{\mathcal{F}} \). Hence the semigroup \( B_{\omega}^{\mathcal{F}} \) with any Hausdorff semigroup topology is \( \Gamma \)-compact.

4. On the closure of \( B_{\omega}^{\mathcal{F}} \) in a (semi)topological semigroup

**Lemma 4.1.** Let \( S \) be a dense subsemigroup of a \( T_1 \)-semitopological semigroup \( T \) and \( 0 \) be the zero of \( S \). Then the element \( 0 \) is the zero of \( T \).

Proof. Suppose to the contrary that there exists \( a \in T \setminus S \) such that \( 0 \cdot a = b \neq 0 \). Then for every open neighbourhood \( U(b) \neq 0 \) in \( T \) there exists an open neighbourhood \( V(a) \neq 0 \) of the point \( a \) in \( T \) such
that $0 \cdot V(a) \subseteq U(b)$. But $|V(a) \cap S| \geq \omega$, and hence $0 \in 0 \cdot V(a) \subseteq U(b)$. This contradicts the choice of the neighbourhood $U(b)$. Therefore $0 \cdot a = 0$ for all $a \in T \setminus S$.

The proof of the equality $a \cdot 0 = 0$ is similar. 

\textbf{Theorem 4.2.} Let $T$ be a $T_1$-semitopological semigroup which contains the semigroup $B^F_\omega$ as a dense proper subsemigroup. Then $I = \left( T \setminus B^F_\omega \right) \cup \{0\}$ is an ideal of $T$.

\textbf{Proof.} Lemma 4.1 implies that 0 is the zero of the semigroup $T$. Since $T$ is a $T_1$-topological space, the set $B^F_\omega \setminus \{0\}$ is dense in $T$. By Lemma 3 [13], $B^F_\omega \setminus \{0\}$ is an open subspace of $T$.

Fix an arbitrary non-zero element $y \in I$. If $x \cdot y = z \notin I$ for some $x \in B^F_\omega \setminus \{0\}$ then there exists an open neighbourhood $U(y)$ of the point $y$ in the space $T$ such that

$$\{x \cdot U(y) = \{z \} \subset B^F_\omega \setminus \{0\}. $$

By Lemma 2.7 the open neighbourhood $U(y)$ should contain finitely many elements of the set $B^F_\omega \setminus \{0\}$ which contradicts our assumption. Hence $x \cdot y \in I$ for all $x \in B^F_\omega \setminus \{0\}$ and $y \in I$. The proof of the statement that $y \cdot x \in I$ for all $x \in B^F_\omega \setminus \{0\}$ and $y \in I$ is similar.

Suppose to the contrary that $x \cdot y = w \notin I$ for some non-zero elements $x, y \in I$. Then $w \in B^F_\omega \setminus \{0\}$ and the separate continuity of the semigroup operation in $T$ yields open neighbourhoods $U(x)$ and $U(y)$ of the points $x$ and $y$ in the space $T$, respectively, such that $\{x \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$. Since both neighbourhoods $U(x)$ and $U(y)$ contain infinitely many elements of the set $B^F_\omega \setminus \{0\}$, equalities $\{x \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$ do not hold, because $\{x \cdot (U(y) \cap B^F_\omega \setminus \{0\}) \subseteq I$. The obtained contradiction implies that $x \cdot y \in I$. 

A subset $D$ of a semigroup $S$ is said to be $\omega$-unstable if $D$ is infinite and $aB \cup Ba \subseteq D$ for any $a \in D$ and any infinite subset $B \subseteq D$.

\textbf{Definition 4.3} ([8]). An ideal series (see, for example, [3,4]) for a semigroup $S$ is a chain of ideals

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = S.$$ 

We call the ideal series tight if $I_0$ is a finite set and $D_k = I_k \setminus I_{k-1}$ is an $\omega$-unstable subset for each $k = 1, \ldots, n$.

\textbf{Lemma 4.4.} The ideal series $I_0 = \{ \emptyset \} \subset I_1 = \mathcal{F}_\omega(F_{\text{min}})$ is tight for the semigroup $\mathcal{F}_\omega(F_{\text{min}})$.

\textbf{Proof.} Fix any infinite subset $D \subseteq \mathcal{F}_\omega(F_{\text{min}}) \setminus \{ \emptyset \}$ and any element $a \in \mathcal{F}_\omega(F_{\text{min}}) \setminus \{ \emptyset \}$. Since the set $D$ is infinite and the set $F_{\text{min}}^{(i,j)}$ is finite for any $i, j \in \omega$, at least one of the following conditions holds:

(i) there exist infinitely many $i_n \in \omega$ such that $(i_n, k_n, j_n) \in D$ for some $j_n \in \omega$ and $k_n \in F_{\text{min}}$;

(ii) there exist infinitely many $j_n \in \omega$ such that $(i_n, k_n, j_n) \in D$ for some $i_n \in \omega$ and $k_n \in F_{\text{min}}$.

Both above conditions and the semigroup operation of $\mathcal{F}_\omega(F_{\text{min}})$ imply that $\emptyset \in (i, k, j) \cdot D \cup D \cdot (i, k, j)$, which completes the proof of the lemma. 

Let $\mathcal{S}$ be a class of semitopological semigroups. A semigroup $S \in \mathcal{S}$ is called $\mathcal{S}$-closed, if $S$ is a closed subsemigroup of any semitopological semigroup $T \in \mathcal{S}$ which contains $S$ both as a subsemigroup and as a topological space. $\mathcal{H \mathcal{T} \mathcal{F}}$-closed topological semigroups, where $\mathcal{H \mathcal{T} \mathcal{F}}$ is the class of Hausdorff topological semigroups, are introduced by Stepp in [20], and there they were called maximal semigroups. An algebraic semigroup $S$ is called algebraically complete in $\mathcal{S}$, if $S$ with any Hausdorff topology $\tau$ such that $(S, \tau) \in \mathcal{S}$ is $\mathcal{S}$-closed.

By Proposition 10 from [8], every inverse semigroup $S$ with a tight ideal series is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion. Hence Theorem 2.4 and Lemma 4.4 imply the following theorem.

\textbf{Theorem 4.5.} Let $\mathcal{F}$ be a family of atomic subsets of $\omega$. Then the semigroup $B^\mathcal{F}_\omega$ is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion.

The following lemma describes the closure of the semigroup $\mathcal{F}_\omega(F_{\text{min}})$ in a $T_1$-topological semigroup.
Lemma 4.6. Let $S$ be a $T_1$-topological semigroup which contains the semigroup $B'_ω(F_{\text{min}})$ as a dense subsemigroup. Then the following conditions hold:

(i) if $S \setminus B'_ω(F_{\text{min}}) \neq \emptyset$ then $x^2 = 0$ for all $x \in S \setminus B'_ω(F_{\text{min}})$;

(ii) $E(S) = E(B'_ω(F_{\text{min}}))$.

Proof. (i) By Lemma 4.1 the element $0$ is the zero of the semigroup $S$. Suppose to the contrary that there exists $x \in S \setminus B'_ω(F_{\text{min}})$ such that $x^2 = y = 0$. Since $S$ is a $T_1$-space there exists an open neighbourhood $U(y)$ of the point $y$ in $S$ such that $0 \notin U(y)$. The continuity of the semigroup operation in $S$ implies that there exists an open neighbourhood $V(x)$ of the point $x$ in the space $S$ such that $V(x) \cdot V(x) \subseteq U(y)$. By Corollary 2.2 the set $F_{\text{min}}^{(i,j)}$ is finite for any $i, j \in ω$. Since the set $V(x) \cap B'_ω(F_{\text{min}})$ is infinite, the above arguments and the definition of the semigroup operation in $B'_ω(F_{\text{min}})$ imply that $0 \in V(x) \cdot V(x) \subseteq U(y)$, a contradiction.

Statement (ii) follows from (i). □

Lemma 4.7. Let $B'_ω(F_{\text{min}})$ be a Hausdorff topological semigroup with the compact band $E(B'_ω(F_{\text{min}}))$. If a Hausdorff topological semigroup $S$ contains $B'_ω(F_{\text{min}})$ as a subsemigroup then $B'_ω(F_{\text{min}})$ is a closed subset of $S$.

Proof. Suppose to the contrary that there exists a Hausdorff topological semigroup $S$ which contains $B'_ω(F_{\text{min}})$ as a non-closed subsemigroup. Since the closure of a subsemigroup of $S$ is again a subsemigroup in $S$ (see [2, page 9]), without loss of generality we may assume that $B'_ω(F_{\text{min}})$ is a dense subsemigroup of $S$ and $S \setminus B'_ω(F_{\text{min}}) \neq \emptyset$. By Lemma 4.1 the element $0$ is the zero of $S$.

Fix an arbitrary $x \in S \setminus B'_ω(F_{\text{min}})$. By Hausdorffness of $S$ there exist open neighbourhoods $U(x)$ and $U(0)$ of the points $x$ and $0$ in $S$, respectively, such that $U(x) \cap U(0) = \emptyset$. Since $x \cdot 0 = 0 \cdot x = 0$, there exist open neighbourhoods $V(x)$ and $V(0)$ of the points $x$ and $0$ in the space $S$, respectively, such that

$$V(x) \cdot V(0) \subseteq U(0), \quad V(0) \cdot V(x) \subseteq U(0), \quad V(x) \subseteq U(x) \quad \text{and} \quad V(0) \subseteq U(0).$$

The compactness of $E(B'_ω(F_{\text{min}}))$ and Proposition 3.1 imply that the set $E(B'_ω(F_{\text{min}})) \setminus V(0)$ is finite. Also, by Corollary 2.2 the set $F_{\text{min}}^{(i,j)}$ is finite for any $i, j \in ω$. Since the set $V(x) \cap B'_ω(F_{\text{min}})$ is infinite, the above arguments and the definition of the semigroup operation in $B'_ω(F_{\text{min}})$ imply that there exists $(i, k, j) \in V(x)$ such that $(i, k, i) \in V(0)$ or $(j, k, j) \in V(0)$. Therefore, we have that at least one of the following conditions holds:

$$(V(x) \cdot V(0)) \cap V(x) \neq \emptyset, \quad (V(0) \cdot V(x)) \cap V(x) \neq \emptyset.$$  

Since $V(x) \subseteq U(x)$, this contradicts the assumption $U(x) \cap U(0) = \emptyset$. The obtained contradiction implies the statement of the lemma. □

Later by $\mathcal{HT}$ we denote the class of all Hausdorff topological semigroups.

The following lemma shows that the converse statement to Lemma 4.7 is true in the case when $B'_ω(F_{\text{min}})$ is a topological inverse semigroup.

Lemma 4.8. Let $(B'_ω(F_{\text{min}}), τ)$ be a Hausdorff topological inverse semigroup. If $(B'_ω(F_{\text{min}}), τ)$ is an $\mathcal{HT}$-closed topological semigroup then the band $E(B'_ω(F_{\text{min}}))$ is compact.

Proof. Suppose to the contrary that there exists a Hausdorff semigroup inverse topology $τ$ on the semigroup $B'_ω(F_{\text{min}})$ such that $(B'_ω(F_{\text{min}}), τ)$ is an $\mathcal{HT}$-closed topological semigroup and the band $E(B'_ω(F_{\text{min}}))$ is not compact. By Proposition 3.1 every non-zero element of $B'_ω(F_{\text{min}})$ is an isolated point in $(B'_ω(F_{\text{min}}), τ)$ and hence there exists an open neighbourhood $V(0)$ of the zero $0$ in the space $(B'_ω(F_{\text{min}}), τ)$ such that $M = E(B'_ω(F_{\text{min}})) \setminus V(0)$ is an infinite subset of the band $E(B'_ω(F_{\text{min}}))$. Since the semigroup $B'_ω(F_{\text{min}})$ is countable, so is the set $M$. Next we enumerate elements of the set $M$ by positive integers:

$$M = \{(i_n, k_n, i_n) : n = 1, 2, 3, \ldots\}.$$
By Corollary 2.2 the set $F^{(i,j)}_{\min}$ is finite for any $i, j \in \omega$, and hence without loss of generality we may assume that $i_m < i_n$ for any positive integers $m < n$. Since $(\mathcal{B}_\omega(F_{\min}), \tau)$ is a topological inverse semigroup the maps $\varphi: \mathcal{B}_\omega(F_{\min}) \to E(\mathcal{B}_\omega(F_{\min}))$ and $\psi: \mathcal{B}_\omega(F_{\min}) \to E(\mathcal{B}_\omega(F_{\min}))$ defined by the formulae $\varphi(x) = x \cdot x^{-1}$ and $\psi(x) = x^{-1} \cdot x$, respectively, are continuous, and hence $I_M = \varphi^{-1}(M) \cup \psi^{-1}(M)$ is a closed subset in the topological space $(\mathcal{B}_\omega(F_{\min}), \tau)$.

Let $y \notin \mathcal{B}_\omega(F_{\min})$. Put $S = \mathcal{B}_\omega(F_{\min}) \cup \{y\}$. We extend the semigroup operation from $\mathcal{B}_\omega(F_{\min})$ onto $S$ as follows:

$$y \cdot y = y \cdot x = x \cdot y = \emptyset,$$

for all $x \in \mathcal{B}_\omega(F_{\min})$.

Simple verifications show that so extended binary operation is associative.

We put

$$M_n = \{(i_{2j-1}, k_{2j-1}, i_{2j}) : j = n, n + 1, n + 2, \ldots\}$$

for any positive integer $n$. We define a topology $\tau_S$ on $S$ in the following way:

(i) for every $x \in \mathcal{B}_\omega(F_{\min})$ the bases of topologies $\tau$ and $\tau_S$ at the point $x$ coincide; and

(ii) the family $\mathcal{B} = \{U_n(y) = \{y\} \cup M_n : n = 1, 2, 3, \ldots\}$ is the base of the topology $\tau_S$ at the point $y$.

Since $M_n \subset I_M$ for any positive integer $n$, $\tau_S$ is a Hausdorff topology on $S$.

For any open neighbourhood $V(\emptyset)$ of the zero $\emptyset$ such that $V(\emptyset) \subseteq U(\emptyset)$ and any positive integer $n$ we have that

$$V(\emptyset) \cdot U_n(y) = U_n(y) \cdot V(\emptyset) = U_n(y) \cdot U_n(y) = \{\emptyset\} \subseteq V(\emptyset).$$

We remark that the definition of the set $M_n$ implies that for any non-zero element $(i, k, j)$ of the semigroup $\mathcal{B}_\omega(F_{\min})$ there exists the smallest positive integer $n_{(i,k,j)}$ such that

$$(i, k, j) \cdot M_{n_{(i,k,j)}} = M_{n_{(i,k,j)}} \cdot (i, k, j) = \{\emptyset\}.$$

This implies that

$$(i, k, j) \cdot U_{n_{(i,k,j)}}(y) = U_{n_{(i,k,j)}}(y) \cdot (i, k, j) = \{\emptyset\} \subseteq V(\emptyset).$$

Therefore $(S, \tau_S)$ is a Hausdorff topological semigroup which contains $(\mathcal{B}_\omega(F_{\min}), \tau)$ as a proper dense subsemigroup, which contradicts the assumption of the lemma. The obtained contradiction implies that the band $E(\mathcal{B}_\omega(F_{\min}))$ is compact.

The proof of Lemma 4.8 implies Proposition 4.9, which gives the sufficient conditions on the topological semigroup $(\mathcal{B}_\omega(F_{\min}), \tau)$ to be non-$\mathcal{H}F\mathcal{I}$-closed.

Proposition 4.9. Let $\tau$ be a semigroup topology on the semigroup $\mathcal{B}_\omega(F_{\min})$. Let $\varphi: \mathcal{B}_\omega(F_{\min}) \to E(\mathcal{B}_\omega(F_{\min}))$ and $\psi: \mathcal{B}_\omega(F_{\min}) \to E(\mathcal{B}_\omega(F_{\min}))$ be the maps which are defined by the formulae $\varphi(x) = x \cdot x^{-1}$ and $\psi(x) = x^{-1} \cdot x$. If there exists an open neighbourhood $U(\emptyset)$ of zero in $(\mathcal{B}_\omega(F_{\min}), \tau)$ such that

$$(\varphi^{-1}(M) \cup \psi^{-1}(M)) \cap U(\emptyset) = \emptyset$$

for some infinite subset $M$ of the band $E(\mathcal{B}_\omega(F_{\min}))$, then $(\mathcal{B}_\omega(F_{\min}), \tau)$ is not an $\mathcal{H}F\mathcal{I}$-closed topological semigroup.

Theorem 2.4 and Lemmas 4.7, 4.8 imply

Theorem 4.10. Let $\mathcal{F}$ be a some family of atomic subsets of $\omega$. Then a Hausdorff topological semigroup $\mathcal{B}_\omega^{\mathcal{F}}$ with the compact band is an $\mathcal{H}F\mathcal{I}$-closed topological semigroup. Moreover, a Hausdorff topological inverse semigroup $\mathcal{B}_\omega^{\mathcal{F}}$ is an $\mathcal{H}F\mathcal{I}$-closed topological semigroup if and only the band $E(\mathcal{B}_\omega^{\mathcal{F}})$ is compact.

Example 4.11 and Proposition 4.12 imply that the converse statement to Lemma 4.7 (and hence to the first statement of Theorem 2.4) is not true.

Example 4.11. For any positive integer $n$ we denote

$$U_n(\emptyset) = \{\emptyset\} \cup \bigcup \left\{F^{(i,j)}_{\min} : n \leq i < j\right\}.$$

We define a topology $\tau_1$ on the semigroup $\mathcal{B}_\omega(F_{\min})$ in the following way:
(i) any non-zero element of the semigroup $\mathcal{B}_n'(F_{\min})$ is an isolated point in $(\mathcal{B}_n'(F_{\min}), \tau_1)$;
(ii) the family $\mathcal{B}_n'(\mathcal{O}) = \{U_n(\mathcal{O}) : n \in \omega\}$ is the base of the topology $\tau_1$ at the zero $\mathcal{O}$.

It is obvious that $(\mathcal{B}_n'(F_{\min}), \tau_1)$ is a Hausdorff topological space.

**Proposition 4.12.** $(\mathcal{B}_n'(F_{\min}), \tau_1)$ is an $\mathcal{HTT}$-closed topological semigroup.

**Proof.** First we show that the semigroup operation is continuous in $(\mathcal{B}_n'(F_{\min}), \tau_1)$. Since every non-zero element of the semigroup $(\mathcal{B}_n'(F_{\min}), \tau_1)$ is an isolated point, it is complete to show that the semigroup operation in $(\mathcal{B}_n'(F_{\min}), \tau_1)$ is continuous at zero. Fix an arbitrary $(i, k, j) \in \mathcal{B}_n'(F_{\min}) \setminus \{\mathcal{O}\}$. Then for $n = \max\{i, j\} + 1$ we have that

$$(i, k, j) \cdot U_n(\mathcal{O}) = U_n(\mathcal{O}) \cdot (i, k, j) = \{\mathcal{O}\} \subset U_n(\mathcal{O}).$$

Also for any $n \in \omega$ we have that

$$U_n(\mathcal{O}) \cdot U_n(\mathcal{O}) \subseteq U_n(\mathcal{O}).$$

Therefore $(\mathcal{B}_n'(F_{\min}), \tau_1)$ is a topological semigroup.

Suppose to the contrary that there exists a Hausdorff topological semigroup $S$ which contains $(\mathcal{B}_n'(F_{\min}), \tau_1)$ as a non-closed subsemigroup. Since the closure of a subsemigroup in a topological semigroup is a subsemigroup (see [2, page 9]), without loss of generality we can assume that $(\mathcal{B}_n'(F_{\min}), \tau_1)$ is a dense proper subsemigroup of $S$.

Fix an arbitrary $x \in S \setminus \mathcal{B}_n'(F_{\min})$. By Lemmas 4.1 and 4.6 we have that

$$x \cdot x = x \cdot \mathcal{O} = \mathcal{O} \cdot x = \mathcal{O}.$$ 

Fix any positive integer $n$. Let $W(\mathcal{O})$ be an open neighbourhood of zero $\mathcal{O}$ in $S$ such that $W(\mathcal{O}) \cap \mathcal{B}_n'(F_{\min}) = U_n(\mathcal{O})$. The continuity of the semigroup operation in $S$ implies that there exist open neighbourhoods $V(x)$, $V(\mathcal{O})$ and $U(\mathcal{O})$ of the points $x$ and $\mathcal{O}$ in the space $S$, respectively, such that

$$V(x) \cdot V(\mathcal{O}) \subseteq U(\mathcal{O}), \quad V(\mathcal{O}) \cdot V(x) \subseteq U(\mathcal{O}), \quad V(x) \cdot V(x) \subseteq U(\mathcal{O}),$$

$$V(x) \cap U(\mathcal{O}) = \emptyset \quad \text{and} \quad V(\mathcal{O}) \subseteq U(\mathcal{O}) \subseteq W(\mathcal{O}).$$

Theorem 9 of [21] implies that $E(\mathcal{B}_n'(F_{\min}))$ is a closed subset of $S$. Hence, we may assume that $V(x) \cap E(\mathcal{B}_n'(F_{\min})) = \emptyset$, and moreover $U(\mathcal{O}) \cap \mathcal{B}_n'(F_{\min}) = U_m(\mathcal{O})$ and $V(\mathcal{O}) \cap \mathcal{B}_n'(F_{\min}) = U_l(\mathcal{O})$ for some positive integers $l$ and $m$ such that $l > m > n$.

Then conditions

$$V(x) \cdot V(\mathcal{O}) \subseteq U(\mathcal{O}) \quad \text{and} \quad V(x) \cap U(\mathcal{O}) = \emptyset$$

imply that there exists an open neighbourhood $V_1(x) \subseteq V(x)$ of the point $x$ in the space $S$ such that

$$V_1(x) \cap \left( \bigcup \left\{ F_{\min}^{(i,s)} : s \in \omega \right\} \right) = \emptyset$$

for any non-negative integer $i < m$. This and Theorem 9 of [21] imply that there exists an open neighbourhood $V_2(x) \subseteq V(x)$ of the point $x$ in $S$ such that

$$V_2(x) \cap \mathcal{B}_n'(F_{\min}) \subseteq \bigcup \left\{ F_{\min}^{(i,j)} : i > j, \ i, j \in \omega \right\}.$$ 

Hence there exists an infinite sequence $\{(i_p, k_p, j_p)\}_{p \in \omega}$ in $V_2(x)$ such that the sequence $\{i_p\}_{p \in \omega}$ is increasing and $j_p < i_p - 1$ for any $p \in \omega$. The definition of the topology $\tau_1$ implies that there exists an element $(i_p, k_p, j_p)$ of the sequence $\{(i, k, j)\}_{p \in \omega}$ such that

$$F_{\min}^{(i_p-1,j_p)} \subseteq U_l(\mathcal{O}) \subseteq V(\mathcal{O}).$$

Then we have that

$$F_{\min}^{(i_p-1,j_p)} : (i_p, k_p, j_p) \subseteq F_{\min}^{(i_p-1,j_p)} \notin U_m(\mathcal{O}),$$

which contradicts the inclusion $V(\mathcal{O}) \cdot V(x) \subseteq U(\mathcal{O})$. The obtained contradiction implies that $x$ is not an accumulation point of $\mathcal{B}_n'(F_{\min})$ in the topological space $S$, and hence the statement of the proposition holds. \qed
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