The muon anomalous magnetic moment

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The magnetic moment

- The magnetic moment $\vec{\mu}$ determines the shift of a particle’s energy in the presence of a magnetic field $\vec{B}$

$$V = -\vec{\mu} \cdot \vec{B}$$

- The intrinsic spin $\vec{S}$ of a particle contributes

$$\vec{\mu} = g \left( \frac{e}{2m} \right) \vec{S}$$

with electric charge $e$, particle mass $m$, and Landé factor $g$. 

Stern & Gerlach, 1922

- Send silver atoms through non-uniform magnetic field, $\vec{F} = -\nabla V$

- Atoms electrically neutral $\Rightarrow$ spin effects can dominate

- Silver has single 5s electron and fully filled shells below $\Rightarrow$ observe $\mu$ of the electron

- $\vec{B} \neq 0$: two distinct lines $\Rightarrow$ quantized spin, distance of lines $\Rightarrow g_e$
The anomalous magnetic moment

► 1924: Stern and Gerlach measured \( g_e = 2.0(2) \)

► 1928: Dirac shows that relativistic quantum mechanics yields \( g_e = 2 \)

► 1947 (Phys. Rev. 72 1256, November 3): Kusch & Foley (Columbia) measure \( g_e = 2.00229(8) \) in the Zeeman spectrum of gallium

► 1947 (Phys. Rev. 73 416, December 30): Schwinger calculates lowest-order radiative photon correction within quantum field theory (QFT): \( g_e = 2 + \frac{\alpha}{\pi} = 2.00232 \ldots \)

Define anomalous magnetic moment \( a_e = (g_e - 2)/2 \) exhibiting effects of QFT
The anomalous magnetic moment

- In QFT \( a \) can be expressed in terms of scattering of particle off a classical photon background

For external photon index \( \mu \) with momentum \( q \) the scattering amplitude can be generally written as

\[
(-ie) \left[ \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q^\nu}{2m} F_2(q^2) \right]
\]

with \( F_2(0) = a \).
Early measurements of $a_\mu$

- Study of $\mu$ decays under varying magnetic field by Garwin, Lederman and Weinrich 1957 (Nevis Cyclotron, Columbia)
  \[ g_\mu = 2.0(2) \]

- Study of stopped muon precession by Garwin, Hutchinson, Penman, Shapiro 1960
  \[ a_\mu = 0.00113 + 0.00016 - 0.00012 \]

- Crucial improvement (magic-momentum method) in CERN-3 experiment 1979
  \[ a_\mu = 0.001165924(9) \]
Magic momentum method

- Send muon in storage ring with uniform magnetic field, observe decays as function of time
- Measure difference of cyclotron frequency $\omega_C$ and spin rotation frequency $\omega_S$ directly with

$$\vec{\omega}_a = \vec{\omega}_S - \vec{\omega}_C = -\frac{Qe}{m} \left[ a_\mu \vec{B} - a_\mu \left( \frac{\gamma}{\gamma + 1} \right) (\vec{\beta} \cdot \vec{B}) \vec{\beta} - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right]$$

(Thomas 1927).

- Minimize uncertainty by tuning $\gamma^2 - 1 \approx 1/a_\mu$ or $p_\mu \approx 3.09$ GeV to suppress effect of electric field; treat $\vec{\beta} \cdot \vec{B}$ term as perturbation

- All experiments discussed in the following use this method
The BNL E821 experiment (2006)

Muons are tiny magnets spinning on axis like tops.

Pions decay to muons.

Muons are fed into a uniform, doughnut-shaped magnetic field and travel in a circle.

After each circle, muon's spin axis changes by 12°, yet it keeps on traveling in the same direction.

One of 24 detectors see an electron, giving the muon spin direction; g-2 is this angle, divided by the magnetic field the muon is traveling through in the ring.

After circling the ring many times, muons spontaneously decay to electron, (plus neutrinos,) in the direction of the muon spin.

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After circling the ring many times, muons spontaneously decay to electron, (plus neutrinos,) in the direction of the muon spin.

\[ \frac{a_{\mu}}{\tilde{\omega}_p} = \frac{\omega_a}{\omega_L} - \frac{a}{\tilde{\omega}_p} = \frac{\omega_a}{\omega_L} / \tilde{\omega}_p - \frac{a}{\tilde{\omega}_p} = R_\lambda - R, \]

where \( \omega_L \) is the Larmor precession frequency of the muon. The ratio \( R = \frac{\omega_a}{\tilde{\omega}_p} \) is measured in our experiment and the muon-to-proton magnetic moment ratio \( \lambda = \frac{\omega_L}{\omega_p} = 3.18334539(10) \) is determined from muonium hyperfine level structure measurements [12, 13].

The BNL experiment was commissioned in 1997 using the same pion injection technique employed by the CERN III experiment. Starting in 1998, muons were injected directly into the ring, resulting in many more stored muons with much less background. Data were collected for 3.6 billion muon decays in the R01 data-taking period. The data is wrapped around modulo 100 μs.

A representative electron decay time histogram is shown in Fig. 2.
To determine $a_\mu$, we divide $\omega_a$ by $\tilde{\omega}_p$, where $\tilde{\omega}_p$ is the measure of the average magnetic field seen by the muons. The magnetic field, measured using NMR, is proportional to the free proton precession frequency, $\omega_p$. The muon anomaly is given by:

$$a_\mu = \frac{\omega_a \omega_L - \omega_a}{\omega_a / \tilde{\omega}_p - \omega_a / \tilde{\omega}_p} = R_\lambda - R,$$

(11)

where $\omega_L$ is the Larmor precession frequency of the muon. The ratio $R = \omega_a / \tilde{\omega}_p$ is measured in our experiment and the muon-to-proton magnetic moment ratio $\lambda = \omega_L / \omega_p = 3.18334539(10)$ (12) is determined from muonium hyperfine level structure measurements [12, 13].

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$$a_\mu^{E821} = 0.00116592089(54)_{\text{stat}}(33)_{\text{sys}}$$
There is a tension of $3.7\sigma$ for the muon

$$a_{\mu}^{E821} - a_{\mu}^{SM} = 27.4 \ (2.7) \ (2.6) \ (0.1) \ (6.3) \times 10^{-10}$$

Hadronic Vacuum Polarization (HVP)

Hadronic Light-by-Light (HLbL)
New experiment: Fermilab E989

\[ a_{\mu}^{E821} - a_{\mu}^{SM} = 27.4 \, (2.7) \, (2.6) \, (0.1) \, (6.3) \times 10^{-10} \]

\text{HVP HLibL other E821}

\[ \delta a_{\mu}^{E989, \, 2019} = 4.5 \times 10^{-10}, \quad \delta a_{\mu}^{E989, \, 2021} = 1.6 \times 10^{-10} \]

Need to improve uncertainties on HVP and HLibL contributions
Experiment
Statistics Run 1 in 2018 and Run 2 in 2019 (talk by N. Tran at FPCP 2019):

- Finished first physics run, Run 1, in July 2018
- Field uniformity 2x better than BNL
- 1.75 \times 10^{10} positrons collected, \sim 2x BNL stats
- 1.4x BNL after data quality cut, \delta_{\omega_a}^{(stat)} \sim 350 ppb
- Analysis in progress
- Half way through the Run 2
- Improvements: muon flux, kicker strength, overall stability, …
Run 1 fit (talk by N. Tran at FPCP 2019):

\[ N(t) = N_0 e^{-t/\tau} \left[ 1 - A \cos (\omega_a t + \phi) \right] \]

Relative unblinding of 6 analyzing groups successful!
HVP contribution
Status of HVP determinations

No new physics

DHMZ 2019 (prelim)
KNT 2018
Jegerlehner 2017
DHMZ 2017
DHMZ 2012
HLMNT 2011
RBC/UKQCD 2018
Mainz 2019
FNAL/HPQCD/MILC 2019
Mainz 2019
RBC/UKQCD 2018
HLMNT 2011
DHMZ 2012
DHMZ 2017
Jegerlehner 2017
KNT 2018
DHMZ 2019 (prelim)
No new physics

\[ a_\mu \times 10^{10} \]
The HVP from dispersion relations

\[ e^+ e^- \rightarrow \text{hadrons}(\gamma) \]

\[ J_\mu = V_{\mu}^{l=1, l_3=0} + V_{\mu}^{l=0, l_3=0} \]

\[ \tau \rightarrow \nu \text{hadrons}(\gamma) \]

\[ J_\mu = V_{\mu}^{l=1, l_3=\pm1} - A_{\mu}^{l=1, l_3=\pm1} \]

Knowledge of isospin-breaking corrections and separation of vector and axial-vector components needed to use \( \tau \) decay data.
Dispersive method - $e^+e^-$ status

Recent results ($\times 10^{10}$) by Keshavarzi et al. 2018, Davier et al. 2017:

| Channel                                      | This work (KNT18) | DHMZ17 [78] | Difference |
|----------------------------------------------|-------------------|-------------|------------|
| $\pi^0\gamma$ (data + ChPT)                 | 4.58 ± 0.10       | 4.29 ± 0.10 | 0.29       |
| $\pi^+\pi^-$ (data + ChPT)                  | 503.74 ± 1.96     | 507.14 ± 2.58 | −3.40     |
| $\pi^+\pi^-\pi^0$ (data + ChPT)            | 47.70 ± 0.89      | 46.20 ± 1.45 | 1.50       |
| $\pi^+\pi^-\pi^+\pi^-$                     | 13.99 ± 0.19      | 13.68 ± 0.31 | 0.31       |

...  

| Total                                        | 693.3 ± 2.5       | 693.1 ± 3.4 | 0.2        |

Good agreement for total, individual channels disagree to some degree. Surprising since they use the same experimental input.
Dispersive method - $e^+e^-$ status

Tension in $2\pi$ experimental input. BaBar and KLOE central values differ by $\delta a_\mu = 9.8(3.5) \times 10^{-10}$, compare to quoted total uncertainties of dispersive results of order $\delta a_\mu = 3 \times 10^{-10}$.

Conflicting input limits the precision and reliability of the dispersive results. First-principles calculation to remove dependence on conflicting input data desirable. (RBC/UKQCD 2018)

Looking for more data and insight: energy-scans update from CMD-3 in Novosibirsk and ISR updates from KLOE2, BaBar, Belle, BESIII and BelleII.
Combined Results Fit [<0.6 GeV] + Data [0.6-1.8 GeV]

| √s range [GeV] | $a_μ^{\text{had}} [10^{-10}]$ All data | $a_μ^{\text{had}} [10^{-10}]$ All but BABAR | $a_μ^{\text{had}} [10^{-10}]$ All but KLOE |
|----------------|----------------------------------------|------------------------------------------|------------------------------------------|
| threshold - 1.8 | $506.9 \pm 1.9_{\text{total}}$         | $505.0 \pm 2.1_{\text{total}}$           | $510.6 \pm 2.2_{\text{total}}$           |

⇒ The difference “All but BABAR” and “All but KLOE” = 5.6 to be compared with 1.9 uncertainty with “All data”
   ➤ The local error inflation is not sufficient to amplify the uncertainty
   ➤ Global tension (normalisation/shape) not previously accounted for
   ➤ Potential underestimated uncertainty in at least one of the measurements?
   ➤ Other measurements not precise enough and are in agreement with BABAR or KLOE

⇒ Given the fact we do not know which dataset is problematic, we decide to
   ➤ Add half of the discrepancy (2.8) as an additional uncertainty (correcting the local PDG inflation to avoid double counting)
   ➤ Take the mean value “All but BABAR” and “All but KLOE” as our central value
Talk by Druzhinin at EPS 2019 (SND experiment preliminary):

\[e^+e^- \rightarrow \pi^+\pi^-\]

\[
\begin{array}{|c|c|c|}
\hline
\text{Experiment} & \alpha_\mu(\pi^+\pi^-) \times 10^{10} \\
\hline
\text{SND & VEPP-2000} & 411.8 \pm 1.0 \pm 3.7 \\
\text{SND & VEPP-2M} & 408.9 \pm 1.3 \pm 5.3 \\
\text{BABAR} & 414.9 \pm 0.3 \pm 2.1 \\
\hline
\end{array}
\]
Dispersive method - $\tau$ status

| Experiment | $2m_\pi \pm 0.36$ GeV | $a^{\text{had,LO}}_{\mu} [\pi\pi, \tau] \times 10^{-10}$ | $0.36 - 1.8$ GeV |
|------------|----------------------|---------------------------------|-----------------|
| ALEPH      | 9.80 ± 0.40 ± 0.05 ± 0.07 | 501.2 ± 4.5 ± 2.7 ± 1.9          |
| CLEO       | 9.65 ± 0.42 ± 0.17 ± 0.07 | 504.5 ± 5.4 ± 8.8 ± 1.9          |
| OPAL       | 11.31 ± 0.76 ± 0.15 ± 0.07 | 515.6 ± 9.9 ± 6.9 ± 1.9          |
| Belle      | 9.74 ± 0.28 ± 0.15 ± 0.07 | 503.9 ± 1.9 ± 7.8 ± 1.9          |
| Combined   | 9.82 ± 0.13 ± 0.04 ± 0.07 | 506.4 ± 1.9 ± 2.2 ± 1.9          |

Davier et al. 2013: $a^{\text{had,LO}}_{\mu} [\pi\pi, \tau] = 516.2(3.5) \times 10^{-10} \ (2m_\pi^\pm - 1.8$ GeV)

Compare to $e^+ e^-$:

$\bullet \ a^{\text{had,LO}}_{\mu} [\pi\pi, e^+ e^-] = 507.1(2.6) \times 10^{-10} \ (\text{DHMZ17}, \ 2m_\pi^\pm - 1.8$ GeV)

$\bullet \ a^{\text{had,LO}}_{\mu} [\pi\pi, e^+ e^-] = 503.7(2.0) \times 10^{-10} \ (\text{KNT18}, \ 2m_\pi^\pm - 1.937$ GeV)

Here treatment of isospin-breaking to relate matrix elements of $V_{\mu}^{l=1, l_3=1}$ to $V_{\mu}^{l=1, l_3=0}$ crucial.

Can calculate from first-principles in lattice QCD+QED (Bruno, Izubuchi, CL, Meyer 2018)
Euclidean Space Representation

Starting from the vector current $J_\mu(x) = i \sum_f Q_f \bar{\Psi}_f(x) \gamma_\mu \Psi_f(x)$ we may write

$$a_{\mu}^{\text{HVP LO}} = \sum_{t=0}^{\infty} w_t C(t)$$

with

$$C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j(\vec{x}, t)J_j(0) \rangle$$

and $w_t$ capturing the photon and muon part of the HVP diagrams.

The correlator $C(t)$ is computed in lattice QCD+QED at physical pion mass with non-degenerate up and down quark masses including up, down, strange, charm, and bottom quark contributions.
Window method (RBC/UKQCD 2018)

We therefore also consider a window method

\[ a_\mu = a_{\mu}^{\text{SD}} + a_{\mu}^{W} + a_{\mu}^{\text{LD}} \]

with

\[ a_{\mu}^{\text{SD}} = \sum_t C(t) w_t [1 - \Theta(t, t_0, \Delta)] , \]

\[ a_{\mu}^{W} = \sum_t C(t) w_t [\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)] , \]

\[ a_{\mu}^{\text{LD}} = \sum_t C(t) w_t \Theta(t, t_1, \Delta) , \]

\[ \Theta(t, t', \Delta) = \left[ 1 + \tanh \left( \frac{(t - t')/\Delta}{2} \right) \right] / 2 . \]

In this version of the calculation, we use

\[ C(t) = \frac{1}{12\pi^2} \int_0^\infty d(\sqrt{s}) R(s) s e^{-\sqrt{s}t} \]

with \( R(s) = \frac{3s}{4\pi\alpha^2} \sigma(s, e^+e^- \rightarrow \text{had}) \)

to compute \( a_{\mu}^{\text{SD}} \) and \( a_{\mu}^{\text{LD}} \) and Lattice QCD+QED for \( a_{\mu}^{W} \).
How does this translate to the time-like region?

Most of $\pi\pi$ peak is captured by window from $t_0 = 0.4$ fm to $t_1 = 1.5$ fm, so replacing this region with lattice data reduces the dependence on BaBar versus KLOE data sets.
Calculation of the Hadronic Vacuum Polarization Contribution to the Muon Anomalous Magnetic Moment

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We present a first-principles lattice QCD + QED calculation at physical pion mass of the leading-order hadronic vacuum polarization contribution to the muon anomalous magnetic moment. The total contribution of up, down, strange, and charm quarks including QED and strong isospin breaking effects is $a_{\mu}^{\text{HVP LO}} = 715.4(18.7) \times 10^{-10}$. By supplementing lattice data for very short and long distances with $R$-ratio data, we significantly improve the precision to $a_{\mu}^{\text{HVP LO}} = 692.5(2.7) \times 10^{-10}$. This is the currently most precise determination of $a_{\mu}^{\text{HVP LO}}$.

This method allows us to reduce HVP uncertainty over next years to $\delta a_{\mu}^{\text{LO HVP}} \sim 1 \times 10^{-10}$, below Fermilab E989 uncertainty.
Overview of individual contributions
Diagrams – Isospin limit

FIG. 1. Quark-connected (left) and quark-disconnected (right) diagram for the calculation of $a_{\mu}^{\text{HVP LO}}$. We do not draw gluons but consider each diagram to represent all orders in QCD.
\[ C(t) = \sum_{j=0,1,2} P_j(\phi, t) J_j(0) \]

We can therefore write
\[ a_\mu, ud, \text{conn, isospin} \times 10^{10} \]
with C(t) = 1

With appropriate definition of \( w_t \), we can therefore write

\[ a_{\mu} \cdots \text{small}. \]

FIG. 3. Strong isospin-breaking correction diagrams. The crosses denote the insertion of a scalar operator.

FIG. 2. QED-correction diagrams with external pseudo-scalar vertices.

FIG. 1. Quark-connected (left) and quark-disconnected (right) diagram for the calculation of \( \alpha \). We do not refer to diagrams S and V as the QED-connected and to PNLO as the QED-disconnected contribution. We refer to diagrams S and V as the QED-connected and to \( \alpha \)-disconnected operators, respectively. We keep only the leading corrections to the masses in such a perturbative expansion.

\[ \gamma \frac{\alpha}{\pi} m_\text{light} \gamma \phi \quad \text{and} \quad \gamma \frac{\alpha}{\pi} m_\text{charm} \gamma \phi \]

\[ \text{corrections to the masses} \quad \text{with spatial lattice size smaller than the lattice spacing uncertainty and its experimental value.} \]

The lattice spacing is determined by setting the respective experimental measurements. The lattice spacing of the 4I lattice ensemble described in Ref. [17]. The shift of the meson masses computed in our calculation agree with the experimental value. We perform the calculation as a separate quantum-average of quark loops in diagram F.

For the hadronic vacuum polarization the contribution of negative diagrams are both SU(3) and 1/2. This approximation is estimated to yield an a corresponding 30% uncertainty.

Figure 7: Mass-splitting and HVP 1-photon diagrams. In the former the dots are meson operators, in the latter the dots are external photon vertices. Note that for the HVP some of them (such as F with no gluons between the two quark loops) are counted as HVP NLO instead of HVP LO QED corrections.

Figure 8: Mass-counterterm diagrams for mass-splitting and HVP 1-photon corrections to the meson masses. Diagram O would yield a correction to the HVP as otherwise an internal cut through a single gluon.

HPQCD 2014
Mainz 2017
ETMC 2017
BMW 2017
RBC/UKQCD 2018
SK 2019
Mainz 2019

\[ a_{\mu, s, \text{conn, isospin}} \times 10^{10} \]
with \( C(t) = \frac{1}{3} \sum_{\mathbf{x}} \langle \mathbf{x} | J(t) J(0) | \mathbf{x} \rangle \). With appropriate definition of \( w_t \), we can therefore write
\[ a_\mu \ldots \text{small}. \]

**FIG. 3.** Strong isospin-breaking correction diagrams. The crosses denote the insertion of a scalar operator.
with \( C(t) = \frac{1}{3} \sum_{x} \rho_x \). With appropriate definition of \( w_t \), we can therefore write \( a_{\mu, \text{uds, disc, isospin}} \) (small).

FIG. 3. Strong isospin-breaking correction diagrams. The crosses denote the insertion of a scalar operator.

The correlator is expanded in small contributions to meson masses computed in our calculation agree with the respective experimental measurements. The statistical uncertainties. We find the effect of 1 \( \times 10^{10} \) mass due to the QED correction is significantly smaller than the lattice spacing uncertainty and its entrance in the respective quark lines. The procedure used for the SIB correction is computed by inserting scalar operators with dynamical up, down, and strange quarks and non-diagonal operators.

The correlator \( C(t) = \frac{1}{3} \sum_{x} \rho_x \) is therefore not included separately.
FIG. 3. Strong isospin-breaking correction diagrams. The crosses denote the insertion of a scalar operator.

TABLE III: Results of the connected light-quark contribution in units of $10^{-10}$ using different fits and cuts. Left: $M$ and $S$ in the SU(3)-breaking variable $\Delta_2 \equiv m_K^2 - m_\pi^2$. The data points for the local-local and the local-conserved discretizations are shown. A linear fit (straight black line), as well as a fit based on ansatz (30) are shown.

Mainz 2019: arXiv:1904.03120; better control of chiral extrapolation could be helpful
For the finite-volume errors, the two-pion states in $d$ are identical to the $I = 1$ contributions of $c$ and can be calculated using the GSL estimate which we use for $c$. For the omega-related finite-volume errors, I will take the fitted $d!$ and $E!$ and use this as the full result at finite-volume and compare it to a GS model with omega mass from the fitted $E!$ and width from the PDG in infinite-volume. I should also compare this to R-ratio results for the $I = 0$ channel.

Do this entire exercise for 24ID and 32ID to estimate discretization errors.

4 QED and SIB diagrams

We will perform a full first-principles calculation of all $O(\alpha)$ and $O(m_u m_d)$ corrections. The corresponding list of diagrams is given in Figs. 1 and 2.

For diagram F we enforce exchange of gluons between the quark loops as otherwise a cut through a single photon line would be possible. This single-photon contribution is counted as part of the HVP NLO and not included for the HVP LO.
For the finite-volume errors, the two-pion states in $d$ are identical to the $I = 1$ contributions of $c$ and can be calculated using the GSL estimate which we use for $c$. For the omega-related finite-volume errors, I will take the fitted $d$ and $E$ and use this as the full result at finite-volume and compare it to a GS model with omega mass from the fitted $E$ and width from the PDG in infinite-volume. I should also compare this to R-ratio results for the $I = 0$ channel.

Do this entire exercise for 24ID and 32ID to estimate discretization errors.

4 QED and SIB diagrams

We will perform a full first-principles calculation of all $O(\alpha)$ and $O(a)$ corrections. The corresponding list of diagrams is given in Figs. 1 and 2.

(a) $V$  (b) $S$  (c) $T$  (d) $T_d$  (e) $D_1$  (f) $D_1_d$

(g) $D_2$  (h) $D_{2_d}$  (i) $F$  (j) $D_3$

Figure 1: QED corrections

(a) $M$  (b) $R$  (c) $R_d$

(d) $O$

Figure 2: SIB corrections

$\alpha \mu$, QED, conn $\times 10^{10}$
For the finite-volume errors, the two-pion states in $d$ are identical to the $I = 1$ contributions of $c$ and can be calculated using the GSL estimate which we use for $c$. For the omega-related finite-volume errors, I will take the fitted $d$ and $E$ and use this as the full result at finite-volume and compare it to a GS model with omega mass from the fitted $E$ and width from the PDG in infinite-volume. I should also compare this to R-ratio results for the $I = 0$ channel. Do this entire exercise for 24ID and 32ID to estimate discretization errors.

4 QED and SIB diagrams

We will perform a full first-principles calculation of all $O(\alpha)$ and $O(\mu_m \mu_d)$ corrections. The corresponding list of diagrams is given in Figs. 1 and 2.

Figure 1: QED corrections

(a) V
(b) S
(c) T
(d) $T_d$
(e) $D_1$
(f) $D_{1d}$

(g) $D_2$
(h) $D_{2d}$
(i) $F$
(j) $D_3$

Figure 2: SIB corrections

(a) $M$
(b) $R$
(c) $R_d$
(d) $O$

$RBC/UKQCD$ 2018

$\mu$, QED, disc $\times 10^{10}$
For the HVP R is negligible since $\Delta m_u \approx -\Delta m_d$ and O is SU(3) and $1/N_c$ suppressed.
and fit d!.

For the finite-volume errors, the two-pion states in d are identical to the \( I = 1 \) contributions of c and can be calculated using the GSL estimate which we use for c. For the omega-related finite-volume errors, I will take the fitted d and \( E \) and use this as the full result at finite-volume and compare it to a GS model with omega mass from the fitted \( E \) and width from the PDG in infinite-volume. I should also compare this to R-ratio results for the \( I =0 \) channel.

Do this entire exercise for 24ID and 32ID to estimate discretization errors.

4 QED and SIB diagrams

We will perform a full first-principles calculation of all \( O(\epsilon) \) and \( O(\mu m) \) corrections. The corresponding list of diagrams is given in Figs. 1 and 2.

(a) V
(b) S
(c) T
d
(e) D1
d
(f) D1
d
(g) D2
d
(h) D2
d
(i) F
(j) D3

Figure 1: QED corrections

(a) M
(b) R
d
(c) Rd

d
(d) O

Figure 2: SIB corrections

ETMC 2019
RBC/UKQCD 2018
FNAL/HPQCD/MILC 2017

\[ a_\mu, \text{SIB} \times 10^{10} \]
Status of RBC/UKQCD HVP effort
The pure lattice calculation of RBC/UKQCD 2018:

\[ 10^{10} \times a_\mu^{\text{HVP LO}} = 715.4(18.7) \]
\[ = 715.4(16.3)_S(7.8)_V(3.0)_C(1.9)_A(3.2)_{\text{other}} \]

(S) statistics, (V) finite-volume errors, (C) the continuum limit extrapolation, (A) scale setting uncertainty;
other \(\supset\) neglected diagrams for QED and SIB, estimate of bottom quark contribution

Statistical noise mostly from isospin symmetric light quark connected (14.2) and disconnected (3.3), QED (5.7), SIB (4.3)

RBC/UKQCD 2019 update (in preparation):

- Improved methodology
- A lot of new data
The RBC & UKQCD collaborations

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- Jonathan Flynn
- Ryan Hill
- Andreas Jüttner
- James Richings
- Chris Sachrajda

**Stony Brook University**
- Jun-Sik Yoo
- Sergey Syritsyn (RBRC)

**MIT**
- David Murphy

**UC Boulder**
- Oliver Witzel

**Columbia University**
- Ryan Abbot
- Norman Christ
- Duo Guo
- Christopher Kelly
- Bob Mawhinney
- Masaaki Tomii
- Jiqun Tu

**KEK**
- Julien Frison

**University of Liverpool**
- Nicolas Garron

**Peking University**
- Xu Feng

Aaron & Mattia joined since 2018 paper
The correlator in finite volume

\[ C(t) = \sum_n |\langle 0 | V | n \rangle|^2 e^{-E_n t}. \]

We can bound this correlator at each \( t \) from above and below by the correlators

\[ \tilde{C}(t; T, \tilde{E}) = \begin{cases} C(t) & t < T, \\ C(T) e^{-(t-T)\tilde{E}} & t \geq T \end{cases} \]

for proper choice of \( \tilde{E} \). We can chose \( \tilde{E} = E_0 \) (assuming \( E_0 < E_1 < \ldots \)) to create a strict upper bound and any \( \tilde{E} \) larger than the local effective mass to define a strict lower bound.
Improved Bounding Method  

Therefore if we had precise knowledge of the lowest \( n = 0, \ldots, N \) values of \( |\langle 0 | V | n \rangle| \) and \( E_n \), we could define a new correlator

\[
C^N(t) = C(t) - \sum_{n=0}^{N} |\langle 0 | V | n \rangle|^2 e^{-E_n t}
\]

which we could bound much more strongly through the larger lowest energy \( E_{N+1} \gg E_0 \).

New method: do a GEVP study of FV spectrum to perform this subtraction

- 10 operator basis including two \( 4\pi \) operators
- Automatic group theory by A. Meyer
- Automatic contractions/evaluations using distillation:  
  https://github.com/lehner/Wick

Reduces statistical error of light quark contribution by more than a factor of 3.
Other improvements:

- FV corrections both directly calculated at physical pion mass \(a_\mu(L = 6.22 \text{ fm}) - a_\mu(L = 4.66 \text{ fm})\), GSL\(^2\) method, update of Hansen and Patella.

- HVP QED from re-analysis of HLbL point-source data (see also RBC/UKQCD \(\tau\) project, Bruno et al. 1811.00508) reduces statistical noise by \(\approx 10\times\) for V and S

- Infinite-volume and continuum limit also for diagram V, S, and F

- First results for T, D1, and R; other sub-leading in preparation

- Global fit combined with calculation of mass derivatives gives much reduced uncertainty for diagrams M and O (connected and disconnected SIB)
Ensembles at physical pion mass:

48I (1.73 GeV, 5.5fm), 64I (2.359 GeV, 5.4fm), 24ID (1 GeV, 4.7fm), 32ID (1 GeV, 6.2fm), 48ID (1 GeV, 9.3fm), 32IDf (1.37 GeV, 4.6fm)

RBC/UKQCD 2019 (data for light quarks, changes from 2018):

- A2A data for connected isospin symmetric: 48I (127 conf → 400 conf), 64I (160 conf → 250 conf), 24ID (new 130 conf, multi mass), 32ID (new 88 conf, multi mass)

- A2A data (tadpole fields) for disconnected: 48I (33 conf), 24ID (new 260 conf, multi mass), 32IDf (new 103 conf)

- QED and SIB corrections to meson and Ω masses, $Z_V$: 48I (30 conf) and 64I (new 30 conf)

- QED and SIB from HLbL point sources on 48I, 24ID, 32ID, 32IDf (on order of 20 conf each, 2000 points per config)

- Distillation data on 48I (33 conf), 64I (in progr.), 24ID (33 conf), 32ID (11 conf, multi-mass)

- New Ω mass operators (excited states control): 48I (130 conf)
Add $a^{-1} = 2.77$ GeV lattice spacing

- Third lattice spacing for strange data ($a^{-1} = 2.77$ GeV with $m_\pi = 234$ MeV with sea light-quark mass corrected from global fit):

> For light quark need new ensemble at physical pion mass. Started run on Summit Machine at Oak Ridge this year ($a^{-1} = 2.77$ GeV with $m_\pi = 139$ MeV).
HLbL contribution
Current HLbL value is model estimate

Contributions to $a_{\mu}^{\text{HLbL}} \times 10^{10}$

| Contribution          | PdRV09  | JN09    | FJ17    |
|-----------------------|---------|---------|---------|
| $\pi^0, \eta, \eta'$  | 11.4(1.3)| 9.9(1.6)| 9.5(1.2)|
| $\pi, K$ loops        | -1.9(1.9)| -1.9(1.3)| -2.0(5)|
| axial-vector          | 1.5(1.0) | 2.2(5)  | 0.8(3)  |
| scalar                | -0.7(7)  | -0.7(2) | -0.6(1) |
| quark loops           | 0.2 (charm)| 2.1(3) | 2.2(4)  |
| tensor                | 0.1(0)   |         | 0.3(2)  |
| NLO                   |         |         |         |
| Total                 | 10.5(4.9)| 11.6(3.9)| 10.3(2.9)|

$10.5(2.6)$ (quadrature)

Potential double-counting and ad-hoc uncertainties
Two new avenues for a model-independent value for the HLbL

Dispersive analysis + Experimental/lattice input

Direct lattice calculation

... How to estimate uncertainty of truncation of cuts/states?

7 quark-level topologies
Dispersive analysis - recent results

- PRD94(2016)074507 (Mainz): Pion-pole contribution
  \[ a_\mu^{\pi-\text{pole}} = 6.50(83) \times 10^{-10} \]
  using a model parametrization of the \( \pi \rightarrow \gamma^*\gamma^* \)
  form factor constrained by lattice data
  \[
  F_{\pi^0\gamma^*\gamma^*}^{\text{LMD+V}}(q_1^2, q_2^2) = \frac{\tilde{h}_0 q_1^2 q_2^2 (q_1^2 + q_2^2) + \tilde{h}_1 (q_1^2 + q_2^2)^2 + \tilde{h}_2 q_1^2 q_2^2 + \tilde{h}_5 M_{V_1}^2 M_{V_2}^2 (q_1^2 + q_2^2) + \alpha M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_2^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}
  \]

- JHEP1704(2017)161 (Colangelo et al.): Pion-box plus S-wave rescattering
  \[ a_\mu^{\pi-\text{box}} + a_\mu^{\pi\pi,\pi-\text{pole}}^{\text{LHC},J=0} = -2.4(1) \times 10^{-10} \]

- PRL121(2018)112002 (Hoferichter et al.); 1808.04823: Pion-pole contribution
  \[ a_\mu^{\pi-\text{pole}} = 6.26(30) \times 10^{-10} \]
  reconstructing \( \pi \rightarrow \gamma^*\gamma^* \) form factor from
  \( e^+ e^- \rightarrow 3\pi, e^+ e^- \pi^0 \) and \( \pi^0 \rightarrow \gamma\gamma \) width

Combining these results one finds: \[ a_\mu^{\pi-\text{pole}} + a_\mu^{\pi-\text{box}} + a_\mu^{\pi\pi} = 3.9(3) \times 10^{-10} \]

Further estimates: \[ a_\mu^{\eta,\eta'} \approx 3 \times 10^{-10}, a_\mu^{\text{axial vector}} \approx 1 \times 10^{-10}, a_\mu^{\text{short distance}} \approx 1 \times 10^{-10} \]

Control of truncation error very important.
7 quark-level topologies of direct lattice calculation

Hierarchy imposed by QED charges of dominant up- and down-quark contribution

\[ Q_u^4 + Q_d^4 = \frac{17}{81} \]

\[ (Q_u^2 + Q_d^2)^2 = \frac{25}{81} \]

\[ (Q_u^3 + Q_d^3)(Q_u + Q_d) = \frac{9}{81} \]

\[ (Q_u^2 + Q_d^2)(Q_u + Q_d)^2 = \frac{5}{81} \]

\[ (Q_u + Q_d)^4 = \frac{1}{81} \]

Further insight for magnitude of individual topologies can be gained by studying long-distance behavior of QCD correlation functions (Bijnens, RBC, . . .)
7 quark-level topologies of direct lattice calculation

Hierarchy imposed by QED charges of dominant up- and down-quark contribution

\[ Q_u^4 + Q_d^4 = 17/81 \]

\[ (Q_u^2 + Q_d^2)^2 = 25/81 \]

Dominant diagrams in top row: connected and leading disconnected diagram

\[ (Q_u^3 + Q_d^3)(Q_u + Q_d) = 9/81 \]

\[ (Q_u^2 + Q_d^2)(Q_u + Q_d)^2 = 5/81 \]

\[ (Q_u + Q_d)^4 = 1/81 \]

Further insight for magnitude of individual topologies can be gained by studying long-distance behavior of QCD correlation functions (Bijnens, RBC, . . .)
New sampling strategy with 10x reduced noise for same cost (red versus black):

Stochastically evaluate the sum over vertices $x$ and $y$:

- Pick random point $x$ on lattice
- Sample all points $y$ up to a specific distance $r = |x - y|$
- Pick $y$ following a distribution $P(|x - y|)$ that is peaked at short distances

Figure 9. A comparison of the results for $F_2(q^2)$.
Calculation at physical pion mass with finite-volume QED prescription (QED$_L$) at single lattice cutoff of $a^{-1} = 1.73$ GeV and lattice size $L = 5.5$ fm.

Connected diagram:

$\mu\mu = 11.6(0.96) \times 10^{-10}$

Leading disconnected diagram:

$\mu\mu = -6.25(0.80) \times 10^{-10}$

Large cancellation expected from pion-pole-dominance considerations is realized:

$\mu\mu = \mu + \mu = 5.35(1.35) \times 10^{-10}$

Potentially large systematics due to finite-volume QED!
Preliminary results for infinite-volume extrapolation

Graphs showing the behavior of $a_{\mu} dHbl x 10^{10}$ and $1/(m_{\mu} L)^2$ for different QED configurations.

- $QED_\infty$, 48I
- $QED_L$, 24D
- $QED_L$, 48I
- $QED_L$, 32D
- $QED_L$, 48D

Diagrams illustrating the contributions of various Feynman diagrams to the calculation of $a_{\mu}$. 
Preliminary results for infinite-volume extrapolation

Data used for finite-volume result in PRL118(2016)022005
Hadronic light-by-light contribution to the muon anomalous magnetic moment from lattice QCD

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We report preliminary results for the hadronic light-by-light scattering contribution to the muon anomalous magnetic moment. Several ensembles using 2+1 flavors of Möbius domain-wall fermions, generated by the RBC/UKQCD collaborations, are employed to take the continuum and infinite volume limits of finite volume lattice QED+QCD. We find $a_{\mu}^{\text{HLbL}} = (7.41 \pm 6.33) \times 10^{-10}$.
Next steps in first-principles calculation of HLbL

- Further reduce statistical and finite-volume errors

- Take infinite-volume limit also with finite-volume QCD+infinite-volume QED mixed approach

PRD96(2017)034515 (Blum, Christ, Hayakawa, Izubuchi, Jin, Jung, and CL)

Continued effort using these methods to reduce HLbL uncertainty over next years to $\delta a_{\mu}^{HLbL} \sim 1 \times 10^{-10}$, below Fermilab E989 uncertainty
g-2 theory initiative
Muon g-2 Theory Initiative – Goals

Theory Support for the Fermilab E989 experiment to maximize its impact:

▶ Work towards reduction and scrutiny of uncertainties of hadronic contributions

▶ Provide summary of theory calculations of the hadronic contributions

⇒ Write report (whitepaper) before Fermilab experiment has first results (target December 2019)

▶ Steering Committee: Colangelo, Davier, Eidelman, El-Khadra, Lehner, Mibe, Nyffeler, Roberts, Teubner
Plenary and working-group workshops:

- 3-6 June 2017, near Fermilab, first plenary workshop
- 12-14 February 2018, KEK, HVP working group workshop
- 12-14 March 2018, University of Connecticut, HLbL WG workshop
- 18-22 June 2018, Mainz, 2018 plenary meeting
- 9-13 September 2019, Seattle, 2019 workshop with focus on whitepapers

As whitepapers are being finalized, there are still opportunities to participate in the effort!
A tale of two anomalies
Assuming further improvements solidify the tensions

\[ a_e^{\text{EXP}} - a_e^{\text{SM}} = -88 \begin{pmatrix} \alpha \\ \text{SM} \\ \text{EXP} \end{pmatrix} \times 10^{-14} \]

and

\[ a_\mu^{\text{EXP}} - a_\mu^{\text{SM}} = 27.4 \begin{pmatrix} \text{HVP} \\ \text{HLbL} \\ \text{other} \\ \text{EXP} \end{pmatrix} \times 10^{-10}, \]

is there a plausible BSM scenario?
Davoudiasl & Marciano 2018: Light new physics

\[ \mathcal{L}_\phi = -\frac{1}{2} m_\phi^2 \phi^2 - \sum_f \lambda_f \phi \bar{f} f - \frac{\kappa_\gamma}{4} \phi F_{\mu\nu} F^{\mu\nu} \]

- 1-loop $\Delta a_\mu$, 2-loop (Barr-Zee) for $\Delta a_e$ gives opposite signs!
- Real scalar $\phi$; $\phi \gamma \gamma$ coupling from integrating out heavy fermion
- For $m_\phi = 250$ MeV, $\lambda_\mu = 10^{-3}$, $\lambda_e = 4 \times 10^{-6}$, $\lambda_\tau = 0.06$, can obtain both anomalies. This parameter space is not yet ruled out by other experiments.
- This model can be tested in $e^+ e^- \rightarrow \tau^+ \tau^- \phi \rightarrow \tau^+ \tau^- \ell^+ \ell^-$ decays at Belle II (Batell et al. 2016)
Stöckinger et al. 2015: MSSM ($\tan\beta \to \infty$) with radiative muon mass

$\mu_R \tilde{\nu}_\mu \mu_L$

$m_{\mu}^{\text{Pole}} \sim y_\mu v_d + y_\mu v_u \times \text{loop}$ and $a_{\mu}^{\text{SUSY}} \sim y_\mu v_u \times \text{loop}$

Idea: $v_d = 0$ then mass and $a_\mu$ diagrams scale identically

$M_{\text{SUSY}} = \ldots = m_{\tilde{e}_R} = 500$ GeV and $m_{\tilde{\mu}_R} \approx 10 \times M_{\text{SUSY}}$, then

$$\Delta a_e = -7 \times 10^{-13}, \quad \Delta a_\mu = 30 \times 10^{-10}.$$
Conclusions and Outlook
Expect experimental results from Fermilab E989 before end of year

Concerted effort of theory community both lattice and non-lattice methods (g-2 theory initiative whitepaper to appear before experimental result)

Interplay of lattice and non-lattice methods for both HVP and HLbL useful to address leading systematics in dispersive approaches

Pure lattice QCD calculations for HVP have made significant progress and may soon rival precision of dispersive approach

RBC/UKQCD:

- **HVP**: New methods to reduce statistical and systematic errors and a lot of additional data, by end of year first-principles lattice result could have uncertainty of $O(5 \times 10^{-10})$

- **HLbL**: First ab-initio calculation with complete error budget in preparation with uncertainty of $O(5 \times 10^{-10})$, publish before end of year