JACOBIAN SYZYGIES AND TORELLI PROPERTIES FOR
PROJECTIVE HYPERSURFACES WITH ISOLATED
SINGULARITIES

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Abstract. We investigate the relations between the syzygies of the Jacobian ideal of the defining equation for a projective hypersurface \( V \) with isolated singularities and the Torelli properties of \( V \) (in the sense of Dolgachev-Kapranov). We show in particular that hypersurfaces with a small Tjurina numbers are Torelli in this sense. When \( V \) is a plane curve, we briefly discuss the stability of the sheaf of logarithmic vector fields along \( V \) and the freeness of the divisor \( V \).

1. Introduction

Let \( X \) be the complex projective space \( \mathbb{P}^n \) and consider the associated graded \( \mathbb{C} \)-algebra \( S = \bigoplus kS_k \), with \( S_k = H^0(X, \mathcal{O}_X(k)) \). For a nonzero section \( f \in S_N \) with \( N > 1 \), thought of as a homogeneous polynomial of degree \( N \), we consider the hypersurface \( V = V(f) \) in \( X \) given by the zero locus of \( f \) and let \( Y \) denote the singular locus of \( V \), endowed with its natural scheme structure, see [5]. We assume in this paper that \( V \) has isolated singularities.

Let \( \mathcal{I}_Y \subset \mathcal{O}_X \) be the ideal sheaf defining this 0-dimensional subscheme \( Y \subset X \) and consider the graded ideal \( \mathcal{I} = \bigoplus k\mathcal{I}_k \) in \( S \) with \( \mathcal{I}_k = H^0(X, \mathcal{I}_Y(k)) \). Let \( Z = \text{Spec}(S) \) be the corresponding affine space \( \mathbb{C}^{n+1} \) and denote by \( \Omega^k = H^0(Z, \Omega^k_Z) \) the \( S \)-module of global, regular \( k \)-forms on \( Z \). Using a linear coordinate system \( x = (x_0, \ldots, x_n) \) on \( X \), one sees that there is a natural grading on \( \Omega^k \), see [11] for details if necessary.

There is a well defined differential 1-form \( df \in \Omega^1 \) and using it we define two graded \( S \)-submodules in \( \Omega^n \), namely

\[
AR(f) = \ker\{df \wedge : \Omega^n \to \Omega^{n+1}\}
\]

and

\[
KR(f) = \text{im}\{df \wedge : \Omega^{n-1} \to \Omega^n\}.
\]

If one computes in a coordinate system \( x \), then \( AR(f)_m \) is the vector space of all relations of the type

\[
R_m : a_0f_{x_0} + \ldots + a_nf_{x_n} = 0,
\]

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with $f_{x_j}$ being the partial derivative of the polynomial $f$ with respect to $x_j$ and $a_j \in S_m$. Moreover, $KR(f)$ is the module of Koszul relations spanned by obvious relations of the type $f_{x_j} f_{x_i} + (-f_{x_i}) f_{x_j} = 0$ and the quotient
\begin{equation}
ER(f) = AR(f)/KR(f)
\end{equation}
is the graded module of essential relations (which is of course nothing else but the $n$-th cohomology group of the Koszul complex of $f_{x_0}, ..., f_{x_n}$, maybe up to a shift in grading), see [5], [10]. Note also that with this notation, the ideal $I$ is just the saturation of the Jacobian ideal $J_f = (f_{x_0}, ..., f_{x_n}) \subset S = \mathbb{C}[x_0, ..., x_n]$.

Let $\alpha_V$ be the Arnold exponent of the hypersurface $V$, which is by definition the minimum of the Arnold exponents of the singular points of $V$, cite [7], [8]. Using Hodge theory, one can prove that
\begin{equation}
ER(f)_m = 0 \text{ for any } m < \alpha_V N - n,
\end{equation}
under the additional hypothesis that all the singularities of $V$ are weighted homogeneous, see [8] and [15]. It is interesting to note that even though the approaches in [8] and [15] are quite different, the condition that the singularities of $V$ are weighted homogeneous plays a key role in both papers.

While this inequality is the best possible in general, as one can see by considering hypersurfaces with a lot of singularities, see [9], [6], for situations when the hypersurface $V$ has a small number of singularities this result is far from optimal. Our first result gives the following better bound in this case.

**Theorem 1.1.** Assume that the hypersurface $V : f = 0$ in $\mathbb{P}^n$ has degree $N$ and only isolated singularities. Then
\begin{equation}
ER(f)_m = 0 \text{ for any } m \leq n(N - 2) - \tau(V),
\end{equation}
where $\tau(V)$, the Tjurina number of $V$, is the sum of the Tjurina numbers of all the singularities of $V$.

See also Theorem 2.3 for a stronger result. The proof of these results is elementary (i.e. without Hodge theory) and it does not require the hypothesis of $V$ having weighted homogeneous singularities.

The exact sequence of coherent sheaves on $X$ given by
\[ 0 \to T\langle V \rangle \to \mathcal{O}_{X}(1)^{n+1} \to \mathcal{I}_V(N) \to 0, \]
where the last non-zero morphism is induced by $(a_0, ..., a_n) \mapsto a_0 f_{x_0} + ... + a_n f_{x_n}$ can be used to define the sheaf $T\langle V \rangle$ of logarithmic vector fields along $V$, see [18]. This is a reflexive sheaf, in particular a locally free sheaf $T\langle V \rangle$ (identified to a rank two vector bundle on $X$) in the case $n = 2$. The above exact sequence clearly yields
\begin{equation}
AR(f)_m = H^0(X, T\langle V \rangle(m - 1)),
\end{equation}
for any integer $m$. Recall the following.

**Definition 1.2.** A reduced hypersurface $V \subset X = \mathbb{P}^n$ is called DK-Torelli (where DK stands for Dolgachev-Kapranov) if the hypersurface $V$ can be reconstructed as a subset of $X$ from the sheaf $T\langle V \rangle$. 

For a discussion of this notion and various examples we refer to [13], [20], [12]. In particular, E. Sernesi and the author have shown in [12] that the nodal curves with a small number of nodes are DK-Torelli. In the proof, which follows the line of the proof for smooth hypersurfaces outlined by K. Ueda and M. Yoshinaga in [20], we have used the inequality (1.5) for $n = 2$. Since in the case of small number of singularities the bound obtained in Theorem 1.1 is better, it is natural to see if this new bound gives a slightly stronger result. Theorem 1.1 is hence applied to prove the following result, which slightly improves in the results on the Torelli properties of nodal (or nodal and cuspidal) curves obtained in a recent joint work with E. Sernesi, see [12]. The following result also extends the result by K. Ueda and M. Yoshinaga concerning smooth hypersurfaces in [20] to hypersurfaces having a small Tjurina number.

**Theorem 1.3.** Let $V : f = 0$ be a degree $N \geq 4$ hypersurface in $\mathbb{P}^n$, having only isolated singularities, with local equations $g_i = 0$ for $i = 1, \ldots, s$. If

$$\tau(V) \leq \frac{(n-1)(N-4)}{2} + 1,$$

then one of the following holds.

1. $V$ is DK-Torelli;
2. $V$ is of Sebastiani-Thom type, i.e. in some linear coordinate system $(x_0, \ldots, x_n)$ on $\mathbb{P}^n$, the defining polynomial $f$ for $V$ is written as a sum $f = g + h$, with $g$ (resp. $h$) a polynomial involving only $x_0, \ldots, x_r$ (resp. $x_{r+1}, \ldots, x_n$) for some integer $r$ satisfying $0 \leq r < n$.

In the case $n = 2$, some applications of Theorem 1.1 to the stability of the bundle $T(V)$ and the freeness of the divisor $V$ are also given, in Corollary 3.1 and, respectively, Examples 3.2 and 3.2.

2. The new bound on the minimal degree of a Jacobian syzygy

Let $\mathcal{O}_n$ denote the ring of holomorphic function germs at the origin of $\mathbb{C}^n$ and let $m_n \subset \mathcal{O}_n$ be its unique maximal ideal. For a function germ $g \in \mathcal{O}_n$ defining an isolated hypersurface singularity at the origin of $\mathbb{C}^n$, we introduce an invariant

$$a(g) = \min\{a \in \mathbb{N} : m_n^a \subset J_g + (g)\},$$

where $J_g$ is the Jacobian ideal of $g$ in $\mathcal{O}_n$ and $(g)$ is the principal ideal spanned by $g$ in $\mathcal{O}_n$.

**Example 2.1.** (i) If $g = 0$ is a node, i.e. an $A_1$-singularity, then $a(g) = 1$.
(ii) If $g = 0$ is a cusp, i.e. an $A_2$-singularity, then $a(g) = 2$.
(iii) If $g = 0$ is a $D_4$-singularity, e.g. an ordinary 3-tuple point when $n = 2$, then $a(g) = 3$.
(iv) One always has $a(g) \leq \tau(g)$, where $\tau(g) = \dim \mathcal{O}_n/(J_g + (g))$ is the Tjurina number of $g$. Usually this inequality is strict, for instance when $g = x^d + y^d$ is an ordinary point of multiplicity $d$ and $n = 2$, one has $a(g) = 2d - 3 < (d - 1)^2$ for $d \geq 3$. The case $d = 3$ corresponds to the $D_4$-singularity in dimension 2 mentioned above.
One has a natural morphism $\mathcal{O}_X(k) \to \mathcal{O}_X(k)/\mathcal{I}_Y(k)$ for any integer $k$, inducing an evaluation morphism

$$ev_k : S_k = H^0(X, \mathcal{O}_X(k)) \to H^0(X, \mathcal{O}_X(k)/\mathcal{I}_Y(k)) = H^0(Y, \mathcal{O}_Y),$$

where the last equality comes from the fact that $Y$ is 0-dimensional, i.e. it consists of finitely many points $p_1, \ldots, p_s$. This fact also implies

$$H^0(Y, \mathcal{O}_Y) = \bigoplus_{i=1}^s \mathcal{O}_{Y, p_i}. \quad (2.3)$$

On the other hand, if $g_i = 0$ is a local analytic equation for the hypersurface singularity $(V, p_i)$, one has an isomorphism $\mathcal{O}_{Y, p_i} = \mathcal{O}_n/(J_{g_i} + (g_i))$. The following result is elementary and it has appeared in various forms, see for instance Corollary 2.1 in [1], Proposition (1.3.9) in [4], or section 3 in [16].

**Lemma 2.2.** The evaluation morphism $ev_k : S_k \to \bigoplus_{i=1}^s \mathcal{O}_{Y, p_i}$ is surjective for any $k \geq \sum_{i=1}^s a(g_i) - 1$. In other words, if one defines the $k$-th defect of the singular locus subscheme $Y$ by

$$\text{def}_k Y = \dim \text{coker} ev_k,$$

then $\text{def}_k Y = 0$ for $k \geq \sum_{i=1}^s a(g_i) - 1$.

**Proof.** One considers the following decomosition of the evaluation map $ev_k$

$$S_k \to \bigoplus_{i=1}^s \mathcal{O}_{X, p_i}/m_{n, p_i}^{a(g_i)} \to \bigoplus_{i=1}^s \mathcal{O}_{Y, p_i},$$

and one notices that the first morphism is surjective by Corollary 2.1 in [1], and the second morphism is surjective by the definition of the invariants $a(g_i)$. \hfill \Box

The main result of this section is the following.

**Theorem 2.3.** Assume that the hypersurface $V : f = 0$ in $\mathbb{P}^n$ has degree $N$ and only isolated singularities, with local equations $g_i = 0$ for $i = 1, \ldots, s$. Then

$$ER(f)_m = 0 \quad \text{for any } m \leq n(N - 2) - \sum_{i=1}^s a(g_i).$$

**Proof.** Using Theorem 1 in [5], we see that

$$\dim ER(f)_m = \text{def}_{nN - 2n - 1 - m} Y.$$

The claim follows then from Lemma 2.2. \hfill \Box

Recall that for a homogeneous polynomial $f \in S$ we define its Milnor (or Jacobian) graded algebra to be the quotient $M(f) = S/J_f$. Then the coincidence threshold $ct(V)$ was defined as

$$ct(V) = \max \{ q : \dim_K M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q \},$$

with $f_s$ a homogeneous polynomial in $S$ of degree $N$ such that $V_s : f_s = 0$ is a smooth hypersurface in $\mathbb{P}^n$. Finally, the minimal degree of a nontrivial relation $mdr(V)$ is defined as

$$mdr(V) = \min \{ q : ER(f)_q \neq 0 \}.$$
It is known that one has the equality

\[(2.4) \quad ct(V) = mdr(V) + N - 2,\]

see [10], formula (1.3). Theorem 2.3 clearly implies the following.

**Corollary 2.4.** Assume that the hypersurface \( V : f = 0 \) in \( \mathbb{P}^n \) has degree \( N \) and only isolated singularities, with local equations \( g_i = 0 \) for \( i = 1, \ldots, s. \) Then

\[
mdr(V) \geq n(N - 2) - \sum_{i=1,s} a(g_i) + 1
\]

and

\[
ct(V) \geq T - \sum_{i=1,s} a(g_i) + 1,
\]

with \( T = (n + 1)(N - 2). \)

**Example 2.5.** Consider a nodal hypersurface \( V \) in \( \mathbb{P}^n \) having \( \sharp A_1 \) singularities \( A_1 \). In this case \( \alpha_V = n/2, \) see [8], hence the inequality 1.5 yields

\[
ER(f)_m = 0 \text{ for any } m < n(N - 2)/2.
\]

On the other hand, Theorem 2.3 yields

\[
ER(f)_m = 0 \text{ for any } m \leq n(N - 2) - \sharp A_1.
\]

The second vanishing result is stronger than the first one exactly when \( n(N - 2)/2 \leq n(N - 2) - \sharp A_1, \) i.e. if and only if

\[
\sharp A_1 \leq n(N - 2)/2.
\]

For \( \sharp A_1 = 1, \) this implies \( ct(V) \leq n(N - 2) + N - 2 = (n + 1)(N - 2) = T \) and we know that this is in fact an equality by Example 4.3 (i) in [10]. Similarly, Example 4.3 (ii) in [10] shows that \( ct(V) = T - 1 \) when \( \sharp A_1 = 2. \) Hence in these two cases the inequality in Theorem 2.3 is in fact an equality. Example 4.3 (iii) in [10] shows that \( ct(V) = T - 1 \) or \( ct(V) = T - 2 \) when \( \sharp A_1 = 3, \) depending on whether the three nodes are collinear or not. It follows that the bound given by Theorem 2.3 is optimal for \( \sharp A_1 \leq 3. \)

**Example 2.6.** Consider a reduced plane curve \( V : f = 0 \) in \( \mathbb{P}^2 \) having \( n_k \) ordinary singularities of multiplicity \( k \) for \( k = 2, 3, 4 \) and no other singularities. Theorem 2.3 and Example 2.1 yield

\[
ER(f)_m = 0 \text{ for any } m \leq 2(N - 2) - n_2 - 3n_3 - 5n_4.
\]

In the nodal case, i.e. when \( n_3 = n_4 = 0, \) this bound can be better than the one given by the inequality 1.5, but only when \( V \) is irreducible (indeed, otherwise \( ER(f)_{N-2} \neq 0 \) as shown in [10] via Hodge theory and in [14] without Hodge theory and in a more general setting).
3. Stability, free divisors and Torelli type properties

Using Proposition 2.4 in [18] which says that $T(V)$ is stable if and only if $AR(f)_m = 0$ for all $m \leq (N - 1)/2$, we get the following consequence of our Theorem 2.3.

**Corollary 3.1.** Assume that the curve $V : f = 0$ in $X = \mathbb{P}^2$ has degree $N$ and only isolated singularities, with local equations $g_i = 0$ for $i = 1, \ldots, s$. Then the vector bundle $T(V)$ is stable if

$$[(N - 1)/2] \leq 2(N - 2) - \sum_{i=1}^{s} a(g_i),$$

where $[y]$ denotes the largest integer verifying $[y] \leq y$. In particular, since a rank two stable vector bundle is not splittable, it follows that $V$ is not a free divisor when the inequality (3.1) holds.

**Example 3.2.** For $N \geq 5$, consider the family of plane curves $V_N : f_N = 0$ in $\mathbb{P}^2$ given by the equation

$$f_N = x^2 y^2 z^{N-4} + x^5 z^{N-5} + y^5 z^{N-5} + x^N + y^N = 0.$$ 

Then $V_N$ has a unique singularity at $p_1 = (0, 0, 1)$ which is isomorphic to the singularity $g(u, v) = u^2 v^2 + u^5 + v^5$. It follows that $\tau(V_N) = \tau(g) = 10$ and $a(g) = 5$, see for instance Example (6.56) in [3]. Moreover, this singularity, usually denoted by $T_{2,5,5}$ in Arnold’s classification, is not weighted homogeneous, since $11 = \mu(g) > \tau(g) = 10$.

Theorem 2.3 yields $mdr(V_N) \geq 2N - 8$, while Theorem 1.1 yields the weaker bound $mdr(V_N) \geq 2N - 13$. A direct computation of the Jacobian syzygies in the case $5 \leq N \leq 10$ using Singular shows that $mdr(V_N) = 2N - 7$. Therefore Theorem 2.3 is almost sharp in these cases.

Using Corollary 3.1, this computation also implies that the curves $V_N$ have the property that the associated bundle $T(V_N)$ is stable for any $N \geq 5$.

**Example 3.3.** If $V$ is an irreducible free divisor in $X = \mathbb{P}^2$ with degree $N$ and only isolated singularities, with local equations $g_i = 0$ for $i = 1, \ldots, s$, it follows from Corollary 3.1 that one has

$$\sum_{i=1}^{s} a(g_i) > 2(N - 2) - [(N - 1)/2].$$

In other words, such a curve should have a lot of singularities (or singularities with large invariants $a(g_i)$) and this explains the difficulty and the interest in constructing such examples, see for instance [2], [17], [19]. The example $V_5$ above shows that the inequality (3.2) is not sufficient to imply the freeness of the divisor.

Now we turn to the proof of Theorem 1.3 stated in the Introduction. This proof follows closely the proof of the corresponding result in [12]. We repeat below the main steps, for the reader’s convenience and also to point out the new facts necessary to treat the $n$-dimensional case.

**Lemma 3.4.** With the above notation and hypothesis, the sheaf $T(V)$ determines the vector subspace $J_{f,N-1} \subset S_{N-1}$. 

To prove this Lemma, let $E : g = 0$ be a (possibly nonreduced) hypersurface in $X = \mathbb{P}^n$ of degree $N - 1$. For any $k \in \mathbb{Z}$, consider the exact sequence

$$0 \to O_X(k - N + 1) \to O_X(k) \to O_E(k) \to 0,$$

where the first morphism is induced by the multiplication by $g$. Tensoring this sequence of locally free sheaves by $T\langle V \rangle$, we get a new short exact sequence

$$0 \to T\langle V \rangle(k - N + 1) \to T\langle V \rangle(k) \to T\langle V \rangle(k) \otimes O_E \to 0.$$

The associated long exact sequence of cohomology groups looks like

$$0 \to H^0(T\langle V \rangle(k - N + 1)) \to H^0(T\langle V \rangle(k)) \to H^0(T\langle V \rangle(k) \otimes O_E) \to$$

$$\to H^1(T\langle V \rangle(k - N + 1)) \to H^1(T\langle V \rangle(k)) \to \cdots$$

Then, using the formula (1.6), we see that

$$\delta_k = \dim H^0(T\langle V \rangle(k)) - \dim H^0(T\langle V \rangle(k-N+1)) = \dim AR(f)_{k+1} - \dim AR(f)_{k-N+2}$$

depends only on $f$ but not on $g$. Next note that the morphism

$$H^1(T\langle C \rangle(k - N + 1)) \to H^1(T\langle C \rangle(k))$$

in the above exact sequence can be identified, using the formulas (5) and (9) in [18] with the morphism

$$g^*_k : (I/J_f)_{k+1} \to (I/J_f)_{k+N}$$

induced by the multiplication by $g$ (we recall that $I$ is the saturation of the Jacobian ideal $J_f$). The above proves the following equality.

(3.3) \hspace{1cm} \dim H^0(T\langle C \rangle(k) \otimes O_E) = \delta_k + \dim \ker g^*_k + 1.

Let $m$ be the largest integer such that $2m \leq N - 2$. Since clearly $m < N - 1$, it follows that $J_{f,m} = 0$ and hence $g^*_m$ is defined on $I_m$. If $g \in J_f$, then clearly $g^*_m = 0$, and hence its kernel has maximal possible dimension.

To complete the proof of Lemma 3.4 it is enough to show that the converse also holds. To do this, we show first that there are two elements $h_1, h_2 \in I_m$ having no irreducible factor in common. Otherwise, all the elements in $I_m$ are divisible by a homogeneous polynomial, and hence in particular one has $\dim I_m \leq \dim S_{m-1}$ which implies

(3.4) \hspace{1cm} \tau(V) \geq \dim S_m/I_m \geq \binom{m+n}{n} - \binom{m+n-1}{n} = \binom{m+n-1}{n-1}.

One also has the inequality (perhaps well known)

(3.5) \hspace{1cm} \binom{m+n-1}{n-1} \geq (n-1)m + 1,

which can be proved by looking at the subsets $E'$ of cardinal $n - 1$ of a set $E$ which is a disjoint union $E = E_1 \cup E_2$, with $\sharp E_1 = m$, $\sharp E_2 = n - 1$ and count how many subsets $E'$ satisfy $\sharp (E' \cap E_1) \leq 1$. It follows that

$$2m \leq 2 \cdot \frac{\tau(V) - 1}{n - 1} \leq N - 4,$$
a contradiction with the choice of $m$. This shows that there are two elements $h_1, h_2 \in I_m$ having no irreducible factor in common.

Then $g_m^* = 0$ implies $gh_1 = \sum_{j=0,n} a_j f_{x_j}$ and $gh_2 = \sum_{j=0,n} b_j f_{x_j}$ for some polynomials $a_j, b_j \in S_m$. It follows that

$$\sum_{j=0,n} (a_j h_2 - b_j h_1) f_{x_j} = 0.$$ 

Since

$$\sum_{i=1,s} a(g_i) \leq \tau(V) = \sum_{i=1,s} \tau(g_i) \leq \frac{(n-1)(N-4)}{2} + 1 \leq (n-1)(N-2)$$

it follows that

$$2m \leq N - 2 \leq n(N - 2) - \sum_{i=1,s} a(g_i).$$

Theorem 2.3 implies that the only syzygy of degree $2m$ is the trivial one, i.e. $a_j h_2 = b_j h_1$ for any $j$. These relations imply that the polynomials $a_j$'s are all divisible by $h_1$ in $S$, and hence $g \in J_f$.

It follows that $g \in J_{f,N-1}$ if and only if

$$\dim H^0(T\langle V \rangle (m - 1) \otimes \mathcal{O}_E) = \delta_{m-1} + \dim I_m,$$

i.e. the sheaf $T\langle V \rangle$ determines the homogeneous component $J_{f,N-1}$ of the Jacobian ideal $J_f$, and this completes the proof of Lemma 3.4.

To finish the proof of Theorem 1.3 it is enough to use Theorem 1.1 in Zhenjian Wang paper [21], which generalizes a Lemma in [12] covering the case $n = 2$. Indeed, this Theorem says that we can have the following situations.

(A) The Jacobian ideal $J_f$ (or its homogeneous component $J_{f,N-1}$ determines $f$ up to a multiplicative nonzero constant. In this case $V$ is DK-Torelli.

(B) $V$ is of Sebastiani-Thom type.

(C) $V$ has at least one singular point $p_i$ with multiplicity $N - 1$, i.e. there is a local equation $g_i$ such that $g_i \in m_{n,p_i}^{N-1}$. However this is impossible in our conditions as we show now. If $g_i \in m_{n,p_i}^{N-1}$, then the monomials in the corresponding local coordinates $u_1, \ldots, u_n$ of degree $\leq N - 3$ are linearly independent in $\mathcal{O}_n/(J_i + (g_i))$. This implies that

$$\tau(V) \geq \tau(g_i) \geq \dim S_{N-3} = \binom{N - 3 + n}{n} \geq n(N - 3) + 1,$$

as in (3.5). But this is in contradiction with the hypothesis

$$\tau(V) \leq \frac{(n-1)(N-4)}{2} + 1,$$

so the proof of Theorem 1.3 is complete.

**Corollary 3.5.** Assume that the curve $V : f = 0$ in $\mathbb{P}^2$ has degree $N \geq 4$, $\nu$ nodes, $\kappa$ cusps and no other singularities. If

$$\nu + 2\kappa \leq \frac{N - 2}{2}$$
then the curve $V$ is DK-Torelli.

This is a direct consequence of Theorem 1.3, since a curve with only nodes and cusps and of degree at least 4 cannot satisfy the property (2), see also [12].

For $V$ irreducible and $\kappa = 0$, this result coincides with the result given in [12]. For the remaining cases, Corollary 3.3 is a slight improvement over the corresponding results given in [12]. In particular, Corollary 3.3 shows that a curve with $\nu = 0$ and $\kappa = 1$ is DK-Torelli as soon as $N \geq 6$, while the bound given in [12] for the same result was $N \geq 8$.

References

[1] M. Beltrametti, A.J. Sommese, On $k$-jet ampleness. In: Ancona, Silva (eds.) Complex analysis and geometry, pp. 355-376, Plenum Press, NY (1993).
[2] R.O. Buchweitz, A. Conca: New free divisors from old. arXiv:1211.4327v1
[3] A. Dimca, Topics on Real and Complex Singularities, Vieweg Advanced Lecture in Mathematics, Friedr. Vieweg und Sohn, Braunschweig, 1987.242+xvii pp.
[4] A. Dimca, Singularities and Topology of Hypersurfaces, Universitext, Springer-Verlag, 1992.
[5] A. Dimca, Syzygies of Jacobian ideals and defects of linear systems, Bull. Math. Soc. Sci. Math. Roumaine Tome 56(104) No. 2, 2013, 191–203.
[6] A. Dimca, On the syzygies and Hodge theory of nodal hypersurfaces, arXiv:1310.5344.
[7] A. Dimca, M. Saito, Graded Koszul cohomology and spectrum of certain homogeneous polynomials, arXiv:1212.1081v3.
[8] A. Dimca, M. Saito, Generalization of theorems of Griffiths and Steenbrink to hypersurfaces with ordinary double points, arXiv:1403.4563v4.
[9] A. Dimca, G. Sticlaru, On the syzygies and Alexander polynomials of nodal hypersurfaces, Math. Nachr. 285 (2012), 2120–2128.
[10] A. Dimca, G. Sticlaru, Koszul complexes and pole order filtrations, arXiv:1108.3976, to appear in Proc. Edinburgh Math. Soc.
[11] A. Dimca, G. Sticlaru, Syzygies of Jacobian ideals and weighted homogeneous singularities, arXiv:1407.0168.
[12] A. Dimca, E. Sernesi: Syzygies and logarithmic vector fields along plane curves, arXiv:1401.6838 (to appear in Journal de l’École polytechnique-Mathématiques).
[13] I. Dolgachev, M. Kapranov: Arrangements of hyperplanes and vector bundles on $\mathbb{P}^n$. Duke Math. J. 71 (1993), no. 3, 633664.
[14] D. Eisenbud and B. Ulrich, Regularity of the conductor, in: A Celebration of Algebraic Geometry: A Conference in Honor of Joe Harris 60th Birthday, Harvard University Cambridge, MA August 2528, 2011, Editors: Brendan Hassett, James McKernan, Jason Starr, Ravi Vakil, Clay Mathematics Proceedings, Volume 18, 2013, pages 267-280.
[15] R. Kloosterman, On the relation between Alexander polynomials and Mordell-Weil ranks, equianalytic deformations and a variant of Nagata’s conjecture (preprint in preparation).
[16] D. Mégé, Sections hyperplanes à singularités simples et exemples de variations de structure de Hodge, Math. Ann. 353 (2012), 633–661.
[17] R. Nanduri, A family of irreducible free divisors in $\mathbb{P}^2$, arxiv:1305.7464.
[18] E. Sernesi: The local cohomology of the jacobian ring, Documenta Mathematica, 19 (2014), 541-565.
[19] A. Simis, S.O. Tohaneanu: Homology of homogeneous divisors. arXiv:1207.5862.
[20] K. Ueda, M. Yoshinaga: Logarithmic vector fields along smooth divisors in projective spaces, Hokkaido Math. J. 38 (2009), 409-415.
[21] Zhenjian Wang, On homogeneous polynomials determined by their Jacobian ideal, arXiv:1402.3810.
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