Modelling strains and stresses in continuously stratified rotating neutron stars

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ABSTRACT

We introduce a Newtonian model for the deformations of a compressible and continuously stratified neutron star dropping the widely used Cowling approximation. The model is quite general and can be applied to a number of astrophysical scenarios as it allows to account for a great variety of loading forces. In this first analysis, the model is used to study the impact of a frozen adiabatic index in the calculation of rotation-induced deformations: we assume a polytropic equation of state for the matter at equilibrium but, since chemical reactions may be slow, the perturbations with respect to the unstressed configuration are modeled by using a different adiabatic index. We quantify the impact of a departure of the adiabatic index from its equilibrium value on the stressed stellar configuration and we find that a small perturbation can cause large variations both in displacements and strains. As a first practical application, we estimate the strain developed between two large glitches in the Vela pulsar showing that, starting from an initial unstressed configuration, it is not possible to reach the breaking threshold of the crust, namely to trigger a starquake. In this sense, the hypothesis that starquakes could trigger the unpinning of superfluid vortices is challenged and, for the quake to be a possible trigger, the solid crust must never fully relax after a glitch, making the sequence of starquakes in a neutron star an history-dependent process.

Key words: stars: neutron - pulsars: general

1 INTRODUCTION

The solid crust of a neutron star (NS) may develop stresses under the action of external loads, like the centrifugal force due to rotation (Link et al. 1998; Fattoyev et al. 2018), pinning of superfluid vortices to the crustal lattice (Ruderman 1991a), the presence of mountains (Ushomirsky et al. 2000; Haskell et al. 2006) or intense magnetic fields (Haskell et al. 2008).

The sudden setting of the crust in a strong gravitational field has been invoked in the modeling of many astronomical phenomena related to neutron stars, such as glitches (Ruderman 1976) and flares (Blaes et al. 1989), the possible neutron star precession (Pines 1974; Cutler et al. 2003) and the emission of gravitational waves (Lasky 2015), which may have direct consequences on the observed braking indexes of pulsars (Woan et al. 2018).

The stresses gradually build-up in the crust till a certain threshold, defined by the so-called breaking strain, is reached. At this point the elastic behaviour of the lattice abruptly ceases and a portion of the crust settles to a more relaxed configuration. This process is known as starquake (or crustquake) and its relation to glitches in pulsars (Baym et al. 1969; Ruderman 1991b) and to bursts in magnetars (Thompson & Duncan 1995; Lander et al. 2015; Keer & Jones 2015) is supported by different studies, underlining that the glitch sizes (Melatos et al. 2008; Howitt et al. 2018) and the burst-energy distribution seems to follow a power law (Cheng et al. 1996; Göğüş et al. 2000), as do earthquakes on Earth. The quakes may also drive NSs precession, as explored by Ushomirsky et al. (2000), as well as the evolution of the magnetic field (Link et al. 1998; Lander & Gourgouliatos 2019). On the other hand, the rigid crust can sustain triaxial deformations (referred to as mountains in the literature), which size may be enough to emit gravitational waves detectable in the near future (Abbott et al. 2017).

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Despite such a large use of the crust failure hypothesis, there is still a lack of “realistic” and quantitative models for the study of crust deformations under different types of loading forces. In fact, to date most of the studies rely on different approximations, either in the neutron star structure description, where the star is idealized as a uniform elastic sphere (see e.g. Baym et al. 1969; Franco et al. 2000; Fattoyev et al. 2018), a two-layers sphere (Giliberti et al. 2019) or for using the Cowling approximation (as in Ushomirsky et al. 2000), which is expected to have a considerable impact on the estimates of the deformations and on the related quadrupole moment (Haskell et al. 2006).

As a first step towards a better and more consistent description of elastic deformations in NSs, we adapt a model for deformations of the Earth (Sabadini et al. 2016) to describe neutron stars stresses due to a general loading force. Differently from other works, which are suitable for a specific kind of loading mechanism, the present approach allows to study the effect of a very general force (i.e. it can be easily adapted to model different astrophysical scenarios).

To properly account for stratification, we allow for an arbitrarily large number of layers with different values of the bulk and shear modulus, varying with continuity inside every layer (see Chamel & Haensel 2008, for a comprehensive review of the properties of NSs crust). At the boundaries between two layers the elastic properties of matter may be discontinuous.

The model presented is static, in the sense that we derive and solve the equilibrium equations for the hydro-elastic configuration of a rotating NS. However, the stresses and the corresponding perturbations evolve on a certain typical timescale defined by the dynamics of the specific physical process under study. This is reflected in a non-adiabatic response of matter. For example, several studies on NS oscillations (Meltzer & Thorne 1966; Channugam 1977; Gourgoulhon et al. 1995; Haensel et al. 2002) and thermal fluctuation in accreting neutron stars (Ushomirsky et al. 2000) pointed out that the stellar crust matter can be not in full thermonodynamical equilibrium during the build-up of internal stresses. In particular, beta equilibrium is known to have an important role in the damping of NSs oscillations (Haensel et al. 2002; Yakovlev et al. 2019; Andersson & Pnigouras 2019). In fact, chemical equilibrium is mediated by beta processes on highly neutron rich nuclei and such processes are expected to be extremely slow, so that the crustal layers can be out of beta equilibrium for a very long time, especially for cold (mature) neutron stars (Yakovlev et al. 2001). Hence, the possibility of a non-adiabatic response of matter may be relevant in the calculation of stresses in the crustal layers of a spinning-down (or spinning-up) neutron star.

To model the non-equilibrium response of matter the idea is the following. For a given equation of state (EoS), it is possible to define a related adiabatic index that determines the changes of pressure associated with variations of the local baryon density (Shapiro & Teukolsky 1983). This fundamental quantity appears into the equations governing small-amplitude NSs pulsations (Thorne & Campolattaro 1967) and, in a relativistic context, provides a criteria of stability for cold stars (Meltzer & Thorne 1966; Channugam 1977; Gourgoulhon et al. 1995). When the star is perturbed around the hydrostatic equilibrium configuration, the adiabatic index value must be calculated by taking into account for the possible slowness of the equilibration channels, typically mediated by the weak interaction (Haensel et al. 2007).

If the timescale of the perturbation is comparable or smaller than the ones of reactions, the adiabatic index is not simply the one defined by the equation of state at equilibrium as any change in density disturbs beta equilibrium. Therefore, we will consider a departure of the adiabatic index away from its equilibrium value and discuss how this affects the estimate of the stresses that can be built in a neutron star in which the rotation period is slowly varying. As a first astrophysical application we extend the analysis of stresses in spinning-down pulsars initiated in Giliberti et al. (2019) by means of incompressible models. However, following the recent analysis of Fattoyev et al. (2018), the present model will be used in a future work to discuss the breaking of the crust in spinning-up objects undergoing accretion and its implications on the maximum rotation rate of millisecond pulsars.

The paper is organized as follows. In section 2 we introduce the main equations for compressible deformations of a neutron star. In section 3 we discuss how a departure of the adiabatic index from the value at chemical equilibrium, in the sense provided by Haensel et al. (2002), can be expected during the gradual development of stresses. In section 4 we conclude the description of the model by discussing the boundary conditions and writing the general elastic solution of the problem. Section 5 is devoted to the numerical analysis of the displacements and strains induced by rotation for a polytropic equation of state. Finally, in section 6 we calculate the strain developed by an isolated pulsar between two glitches and we compare the result with the one obtained with the simpler analytical model for incompressible pulsars of Giliberti et al. (2019).

2 MAIN EQUATIONS

Our aim is to construct a framework to calculate the deformations of a compressible and self-gravitating neutron star under the effect of specified forces in a Newtonian context. Building on the analogy with the Earth (Sabadini et al. 2016), we describe the NS as an object with a fluid core topped by a number \( N \) of stratified layers with different elastic properties, as sketched in Fig 1. At the internal boundaries separating two layers of the NS, the material parameters may have step-like discontinuities due to the transition between specific phases of matter.

The approach developed here allows to consider stresses due to centrifugal forces, but also tidal and non-conservative forces, like vortex pinning to impurities in the crust (Ruderman 1991a). In principle, it is also possible to include loads which account for inhomogeneities inside the star (the so-called bulk loads), as well as surface loads that could be used to model the effect of accretion of matter onto the crust. However, it is known that surface loads need a specific technique to handle the boundary conditions (Sabadini et al. 2016), so that we will not include them here to maintain the description of the model self-contained.

We consider an unstressed and non rotating NS, characterized by the continuous density profile \( \rho_0 \), the pressure \( P_0 \) and the gravitational potential \( \phi_0 \). The momentum and
Poisson equations are
\[ \nabla \rho_0 + \rho_0 \nabla \phi_0 = 0 \] (1)
\[ \nabla^2 \phi_0 = 4 \pi G \rho_0 , \] (2)
which define the initial state of hydrostatic equilibrium. The displacement field \( \mathbf{u} \) describes perturbations with respect to this spherically symmetric configuration and is defined as
\[ \mathbf{r} = \mathbf{x} + \mathbf{u}(\mathbf{x}), \] (3)
where \( \mathbf{x} \) and \( \mathbf{r} \) are the initial and the perturbed positions of the infinitesimal matter elements. Hence, the total Cauchy stress tensor can be expressed as\(^1\) (cf. Zdunik et al. 2008)
\[ \tau(\mathbf{r}) = -P_0(\mathbf{x}) \mathbf{1} + \tau^d(\mathbf{x}) . \] (4)
In the above expression, \( \mathbf{1} \) is the identity tensor and \( \tau^d \) is the material increment of the stress, which can be expanded by means of the usual Hooke’s law (Love 1934; Landau & Lifshitz 1970)
\[ \tau^d = (\kappa - 2/3 \mu) (\nabla \cdot \mathbf{u}) \mathbf{1} + \mu \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) , \] (5)
where \( \kappa \) and \( \mu \) are the bulk and shear moduli respectively. The transpose operation is indicated by the symbol \( T \).
For a stressed configuration, the hydrostatic (1) and Poisson (2) equations read
\[ \nabla \cdot \tau - \rho \nabla \Phi + \mathbf{h} = 0 \] (6)
\[ \nabla^2 \Phi = 4 \pi G (\rho + \rho^t) - 2 \Omega^2 \] (7)
where \( \mathbf{h} \) are the non-conservative body forces, \( \rho \) is the mass density of the deformed NS and \( \rho^t \) the mass density of a possible companion. The potential \( \Phi \) encodes all the conservative body forces and can be split as
\[ \Phi = \phi + \phi^c + \phi^t , \] (8)
where \( \phi \) is the gravitational potential generated by the mass distribution of the NS, \( \phi^c \) is the tidal potential due to the presence of the companion and \( \phi^t \) is the centrifugal potential. In particular, for a NS spinning on its axis with angular velocity \( \Omega \) we have that \( \nabla \times \phi^t = -2 \Omega^2 \).
Let us introduce the local incremental density \( \rho^\Delta \) and the local incremental potential \( \Phi^\Delta \) as
\[ \rho(\mathbf{r}) = \rho_0(\mathbf{r}) + \rho^\Delta(\mathbf{r}) \] (9)
\[ \Phi(\mathbf{r}) = \Phi_0(\mathbf{r}) + \Phi^\Delta(\mathbf{r}) \] (10)
The first quantity is related to the displacement \( \mathbf{u} \) via mass conservation
\[ \rho^\Delta = -\nabla \cdot (\rho_0 \mathbf{u}) , \] (11)
while the latter is given by
\[ \Phi^\Delta = \phi^\Delta + \phi^c + \phi^t , \] (12)
where \( \phi^\Delta \) is the incremental part of the gravitational potential generated only by the deformations of the NS.
We now use the definitions provided in Eqs (5), (9) and (10) and linearize the momentum and Poisson equations. Using Eqs (1) and (2) into the expansion of Eqs (6) and (7), immediately gives that the linearized momentum and Poisson equations read (Sabadini et al. 2016)
\[ \nabla \cdot \tau^d - \nabla \cdot (\rho_0 \mathbf{u} \nabla \phi_0) + \nabla \cdot (\rho_0 \mathbf{u} \nabla \phi_0) - \rho_0 \nabla \Phi^\Delta + \mathbf{h} = 0 \] (13)
\[ \nabla^2 \Phi^\Delta = 4 \pi G ( - \nabla \cdot (\rho_0 \mathbf{u}) + \rho^t ) - 2 \Omega^2 . \] (14)
At this point it is convenient to work with the usual spherical coordinate system \( \{r, \theta, \varphi\} \), that are the radial distance from the center of the NS, the colatitude and the longitude of a point respectively. Clearly, in the initial hydrostatic equilibrium configuration the elastic parameters \( \kappa \) and \( \mu \) are functions of the coordinate \( r \) only. Furthermore, the gradient \( \nabla \phi_0 \) in the above equations can be expressed in terms of the gravity acceleration for a spherically symmetric body,
\[ g(r) = \partial_r \phi_0(r) = \frac{4 \pi G}{r^2} \int_0^r \rho_0(r') r'^2 d r'. \] (15)
To find the stressed configuration we expand the potential \( \Phi^\Delta \) in terms of spherical harmonics as in Eq (A4), see Appendix A for the conventions used. The same is done for the spheroidal and toroidal components of the displacement \( \mathbf{u} \), defined in Eq (A5). With some amount of algebra, Eq (13) can be rearranged as
\[ - \rho_0 \partial_r \Phi_{\ell m} - \rho_0 \partial_r (g U_{\ell m}) + \rho_0 g \chi_{\ell m} + \partial_r \left[ \left( \kappa - \frac{2}{3} \mu \right) \chi_{\ell m} + 2 \mu \partial_r U_{\ell m} \right] + \] \[ \frac{1}{r} \mu \left[ 4 r \partial_r U_{\ell m} - 4 U_{\ell m} + \ell (\ell + 1) (3 V_{\ell m} - U_{\ell m} - r \partial_r V_{\ell m}) \right] + \] \[ h^R_{\ell m} = 0 \] (16)
describe the perturbation of the potential. Finally, in (22) we will refer to the components of the vector $\mathbf{h}$, defined in Eqs (A6) and (A7). Similarly, $h^S_{\ell m}$, $h^R_{\ell m}$, are the expansion coefficients of the non-conservative forces, as given in Eq (A15). Finally, the scalars $\chi_{\ell m}$ are linked to the volume change according to
\[
\nabla \cdot \mathbf{u} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \chi_{\ell m} Y_{\ell m}.
\]

The radial (16) and tangential (17) components of the equilibrium equations are called spheroidal equations while (18) is called toroidal equation. Following a similar procedure, the Poisson equation (14) becomes
\[
\nabla^2 \Phi_{\ell m} = -4\pi G (\rho_0 \chi_{\ell m} + U_{\ell m} \partial_r \rho_0) + 4\pi G \rho^F_{\ell m},
\]
where
\[
\nabla^2 = \partial_r + \frac{2}{r} \partial_r - \frac{\ell (\ell + 1)}{r^2}
\]
and $\rho^F_{\ell m}$ are the spherical harmonics coefficients relative to the mass density of the companion.

Equations (16, 17, 18, 20) hold only for $\ell > 0$; the case $\ell = 0$ needs a specific treatment, as shown in detail in Appendix C. Furthermore, it is useful to notice that the toroidal equation (18) is decoupled from Eqs (16, 17, 20).

We now stick to a more specific physical scenario in which the deformations of the NS are due only to the fact that the star is rotating. This allows us to neglect the toroidal term, so that the vector $\mathbf{h}$ can be set to zero.

3 MODELING THE RESPONSE OF MATTER

Following the standard description for cold catalyzed matter in a NS interior, we consider a barotropic EoS of the kind
\[
P(n_b) = n_b^2 \frac{d}{dn_b} E(n_b) - n_b,
\]
where $E(n_b)$ the ground state energy density at chemical equilibrium composition (Haensel et al. 2007). The equation of state for matter at chemical equilibrium is characterized by the adiabatic index $\gamma_{eq}$, defined as
\[
\gamma_{eq} = \frac{n_b}{P} \frac{\partial P(n_b)}{\partial n_b}.
\]
The use of a barotropic EoS is suitable when pressure-density perturbations are very slow with respect to the typical timescales of the chemical reactions that carry the system towards the full thermodynamic equilibrium. Therefore, $\gamma_{eq}$ regulates the pressure response of matter to perturbations when the typical timescale of the dynamical process considered is larger than that of all the relevant equilibration channels. The opposite limit, in which all the reactions are so slow that the chemical composition is “frozen” has been considered in the context of NS oscillations (Haensel et al. 2002). More generally, we expect that the elastic response of the star can be affected by the slowness of the chemical equilibration reaction when matter does not have enough time to reach the complete thermodynamic equilibrium in the meanwhile stress build-up. In particular, Yakovlev et al. (2001) estimate that the equilibration timescale involving modified URCA processes scales as $\sim (2 \text{ months})/T_9^6$, where $T_9$ is the internal temperature in units of $10^9$ K. Hence, due to the strong dependence on the temperature of the star, the rotation-induced stresses that can develop on the timescale of years in a mature spinning-down (or spinning-up) NS should be calculated by taking into account that the adiabatic index for perturbations of matter may differ from $\gamma_{eq}$. If this is the case, the effective adiabatic index $\gamma_f$ that regulates the pressure-density perturbations, where the subscript $f$ stands for “frozen”, depends explicitly also on the chemical fractions $x_i$ for the $i$-species, namely
\[
\gamma_f(n_b, x_i) = \frac{n_b}{P} \frac{\partial P(n_b, x_i)}{\partial n_b}.
\]
where the derivative is performed at fixed $x_i$ values (Haensel et al. 2002).

It is useful to introduce also a third notion of adiabatic index, that is directed directly from the radial profiles of the thermodynamic quantities: for a given spherically symmetric NS configuration, we define a related effective adiabatic index as

$$\gamma = \rho_0 \frac{\partial \rho_0}{\partial r} = -\frac{\rho_0^2 g}{P_0 \partial_x \rho_0}, \quad (29)$$

where $g$ is the modulus of the gravity acceleration for the spherical NS configuration considered.

For a given barotropic EoS the equilibrium radial profiles $P_0(r)$ and $\rho_0(r)$ can be obtained via Eqs (1) and (2). In this configuration the pressure-density relation supporting the star is characterized by the equilibrium adiabatic-index and $\gamma = \gamma_{eq}$ locally at every radius $r$. This ceases to be true if the EoS employed depends explicitly also on the chemical composition, as in Eq (28): in this case the relation

$$\partial_r \rho_0 = \frac{\partial P}{\partial P} \partial_r P_0 \quad (30)$$

is no more valid\(^2\), so that $\gamma$ differs from both $\gamma_i$ and $\gamma_{eq}$. Hence, the effective adiabatic index can be a useful tool to probe a departure from the barotropic configuration due to a non-equilibrium stratification of matter (Cambiotti & Sabadini 2010).

3.1 Polytropic equation of state

To maintain consistence with the previous analysis of Giliberti et al. (2019), we use a polytropic EoS with polytropic index $n = 1$, namely

$$P(\rho) = K \rho^2 = K n_0^2 n_i^2, \quad (31)$$

Clearly, in this case the adiabatic index is not a function of $n_i$, but takes the constant value $\gamma_{eq} = (n+1)/n = 2$. On the other hand, we have little clues about the actual value of $\gamma_f$. It is expected that $\gamma_f > \gamma_{eq}$, but the actual relation between them strongly depends on the microscopic model underlying the specific EoS (Meltzer & Thorne 1966; Channugam 1977; Haensel et al. 2002; Ushomirsky et al. 2000). However, from the practical point of view, the uncertain value of $\gamma_f$ is not a strong limitation in the present work: we will assume different values and study how the estimated stress and strain change, starting from values that differ by only some percent from $\gamma_{eq}$, up to the incompressible limit $\gamma_f \gg \gamma_{eq}$. Since the main difference between the equilibrium and the frozen adiabatic index is expected to be in the crust (Ushomirsky et al. 2000), we will vary the value of $\gamma_f$ only there, keeping the constant value of two in the core. Finally, for any realistic EoS, both $\gamma_{eq}$ and $\gamma_f$ have a complex dependence on the local properties of matter (see e.g. Douchin & Haensel 2001; Haensel et al. 2002). However, since the equilibrium adiabatic index is just a constant, we will assume a constant $\gamma_f$ in the elastic layer as well.

\(^2\) Instead, we should consider $\frac{\partial \rho_0}{\partial r} = \frac{\partial P}{\partial P} \frac{\partial \rho_0}{\partial r} + \frac{\partial P}{\partial x_i} \frac{\partial x_i}{\partial r}$.

3.2 Strain angle and Tresca criterion

The strain tensor is obtained from the displacement field $u$ via (Love 1934; Landau & Lifshitz 1970)

$$\sigma_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (32)$$

To study the breaking of the elastic crust a failure criterion is needed. We assume the widely used Tresca failure criterion (Christensen 2013). This criterion is based on the strain angle, a local quantity $\alpha(r, \theta)$ that is the difference between the maximum and minimum eigenvalues of the stress tensor at a specific point. The criterion assumes that, locally, the elastic behaviour of a material ceases when the strain angle approaches a particular threshold value $\sigma^{Max}$ known as the breaking strain,

$$\alpha \approx \frac{1}{2} \sigma^{Max}. \quad (33)$$

To date only order-of-magnitude estimates exist for the breaking strain; hence, the particular failure criterion assumed is of secondary importance\(^3\). The molecular dynamics simulations performed by Horowitz & Kadau (2009) suggest that $\sigma^{Max} \sim 10^{-1}$ for a drop of nuclear matter. Using a completely different approach, Baiko & Chugunov (2018) recently found that the maximum strain for a polycrystalline crust is $\sim 0.04$. On the other hand, Ruderman (1991b) reasoned that, if crust has already undergone many cracks events, a macroscopic estimate for $\sigma^{Max}$ should be in the range $10^{-5} \div 10^{-3}$. Hence, to take into account for the large uncertainties on the breaking strain, we will consider constant values of $\sigma^{Max}$ in the whole range $10^{-5} \div 10^{-1}$.

4 BOUNDARY CONDITIONS

To solve Eq (24) we have to impose some boundary conditions. In the unstressed configuration, the material parameters, consisting of the initial mass density $\rho_0$, the bulk modulus $\kappa$ and the shear modulus $\mu$ are continuous functions of the radial distance from the NS center $r$ within each layer. In this first analysis, we assume that matter does not cross the interface between two elastic layers, meaning that material elements do not undergo phase transitions. With these premises, it is known that all the spherical vector solutions in (22) are continuous across the boundaries between different layers (Sabadini et al. 2016), say at $r = r_j$ for $j = 1, ..., N$, so that

$$y_{em} \left( y_j^+ \right) = y_{em} \left( y_j^- \right). \quad (34)$$

This continuity requirement gives us a straightforward way to impose the boundary conditions at the surface of the NS and at the core-crust boundary, as explained in the following subsections.

4.1 Surface-vacuum boundary

We can impose three simple conditions that have to be fulfilled by the spherical solution $y$ at $r = a$, where $a$ is the

\(^3\) An alternative could be the Von Mises criterion, adopted e.g. by Lander et al. (2015) and Ushomirsky et al. (2000).
and sixth components of the spheroidal vector $y$ but, differently with respect to what has been done in the previous subsection, we have to allow for a free slip of the material at the interface (the fluid core can slip under the crust). This request implies that

$$y(r^+_{\text{c}}) = \begin{pmatrix} U_{\text{em}}(r^+_{\text{c}}) \\ R_{\text{em}}(r^+_{\text{c}}) \\ \Phi_{\text{em}}(r^+_{\text{c}}) \\ Q_{\text{em}}(r^+_{\text{c}}) \end{pmatrix} + \begin{pmatrix} 0 \\ C_2 \\ 0 \\ 0 \end{pmatrix}, \quad (47)$$

where $r_\text{c}$ is the core-crust radius and $C_2$ is a constant of integration describing the tangential displacement.

Now, the spheroidal vector solution in the core can be found by setting $\mu = 0$ and omitting the terms related to the loadings. The first step is to rearrange Eqs (16) and (17) as

$$\frac{\partial R_{\text{em}}}{\rho_0} - \partial_r (gU_{\text{em}}) + g\chi_{\text{em}} - \partial_r \Phi_{\text{em}} = 0 \quad (48)$$

$$\frac{R_{\text{em}}}{\rho_0} - gU_{\text{em}} - \Phi_{\text{em}} = 0. \quad (49)$$

Inserting these two equations into the Poisson one (20), we obtain

$$\nabla^2 \Phi_{\text{em}} = 4\pi G \rho_0 \frac{\Phi_{\text{em}}}{g}. \quad (50)$$

Since $\partial_r \rho_0 = 0$ must hold at the center of the star, the regularity of the potential in $r = 0$ implies

$$\lim_{r \to 0} r^{-\ell} \psi_{\text{em}}(r) = 1, \quad (51)$$

with $\Phi_{\text{em}}(r) = C_1 \psi_{\text{em}}(r)$.

To proceed further, we subtract the radial derivative of Eq (49) from (48) and use the relation

$$R_{\text{em}} = \kappa \chi_{\text{em}}, \quad (52)$$

which is valid in the fluid limit, to obtain the so-called Adams-Williamson relation,

$$\frac{\kappa}{\rho_0} \left( \partial_r \rho_0 + \frac{\rho_0^2 g}{\gamma P} \right) \chi_{\text{em}} = 0. \quad (53)$$

For our purposes, it is convenient to rearrange the above equation as (Cambiotti & Sabadini 2010; Cambiotti et al. 2013)

$$\frac{\kappa}{\rho_0} \frac{\partial_r \rho_0}{\gamma} (\gamma - \gamma_{eq}) \chi_{\text{em}} = 0. \quad (54)$$

Clearly, when $\gamma = \gamma_{eq}$ (i.e. the stratification is compressional, see section 3), the above equation is automatically satisfied. In particular, the Adams-Williamson relation is satisfied in the fluid core, so that the two equations (48) and (49) are not linearly independent; this provides a way to constrain the radial stress at the core-crust interface at $r = r_c$:

$$R_{\text{em}} = \rho_0 g \left[ U_{\text{em}} + \frac{\Phi_{\text{em}}}{g} \right] = \rho_0 g C_3, \quad (55)$$

where the constant $C_3$ is the difference between the radial displacement $U_{\text{em}}$ and the so-called geoid displacement

$$U_{\text{em}}^{\text{geoid}}(r) = -\frac{\Phi_{\text{em}}(r)}{g(r)}, \quad (56)$$

evaluated at $r = r_c$ (Sabadini et al. 2016).
Note that in the case of compositional stratification the volume change within the core is undetermined: below the core-crust interface we cannot specify the displacement and radial stresses with the above assumptions. However, this does not constitute a problem because we are interested only in the deformation of the crust, which is uniquely determined by the boundary conditions.

The constants $C_1$, $C_2$ and $C_3$ define the elastic solution at the core-crust boundary, that can be written as

$$ y_{\ell m}(r_c) = \begin{pmatrix} -C_1 \frac{2n+1}{r_c} + C_3 \\ \rho_0 g C_2 \\ 0 \\ C_1 \psi_{\ell m} \\ C_1 q_{\ell m} + 4\pi G \rho_0 C_3 \end{pmatrix}, \quad (57) $$

where we have defined

$$ q_{\ell m} = \partial_r \psi_{\ell m} + \frac{\ell + 1}{r} \psi_{\ell m} - 4\pi G \rho \psi_{\ell m}. \quad (58) $$

In view of Eq (57), the core-crust boundary conditions (47) are conveniently arranged in the compact form

$$ y_{\ell m}(r_c^+) = I_C \mathbf{C}, \quad (59) $$

where $I_C$ is the $6 \times 3$ matrix

$$ I_C = \begin{pmatrix} -\psi(r_c)/g(r_c) & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & g(r_c) \rho_0 (r_c^+) \\ 0 & 0 & 0 \\ \psi(r_c) & 0 & 0 \\ q_{\ell m}(r_c) & 0 & 4\pi G \rho_0 (r_c^+) \end{pmatrix}. \quad (60) $$

and $\mathbf{C} = (C_1, C_2, C_3)$. \quad (61)

4.3 Elastic solution

The general solution of the differential system (24) reads

$$ y_{\ell m}(r) = \mathbf{I}_\ell(r, r_0) y_{\ell m}(r_0) - \int_{r_0}^{r} \mathbf{I}_\ell(r, r') h_{\ell m}(r') \, dr'. \quad (62) $$

The first terms on the right side of Eq (62) is the homogeneous solution, while the second term is a particular solution which accounts for the external non-conservative forces. Furthermore, the propagator matrix $\mathbf{I}_\ell$ solves the homogeneous equation

$$ \frac{d\mathbf{I}_\ell(r, r')}{dr'} = A_\ell(r) \mathbf{I}_\ell(r, r'), \quad (63) $$

with the condition

$$ \mathbf{I}_\ell(r^+, r') = 1. \quad (64) $$

At each boundary between the layers we have to impose the continuity of the propagator, namely

$$ \mathbf{I}_\ell(r_c^+, r') = \mathbf{I}_\ell(r_c^-, r'). \quad (65) $$

If we choose the core-crust radius as the starting point of the integration in Eq (62), namely $r_0 = r_c^+$, it is easy to use Eq (59) to show that

$$ y_{\ell m}(r) = \mathbf{I}_\ell(r, r_c^+) \mathbf{I}_C \mathbf{C} - \mathbf{w}(r). \quad (66) $$

where

$$ \mathbf{w}(r) = \int_{r_c^+}^{r} \mathbf{I}_\ell(r, r') h_{\ell m}(r') \, dr'. \quad (67) $$

The three constants of integration in the vector $\mathbf{C}$ can be estimated by imposing the conditions at the star surface via Eq (45), so that Eq (66) now reads

$$ y_{\ell m}(r) = \mathbf{I}_\ell(r, r_c) \mathbf{I}_C \left[ \mathbf{P}_\ell \mathbf{I}_\ell (a^-, r_c^+) \right]^{-1} \mathbf{I}_\ell (a^-, b) - \mathbf{w}(r). \quad (68) $$

This equation represents the most general response of the star to the internal (such as vortex pinning), centrifugal and tidal loads and it uniquely determines the spheroidal deformations and the potential within the crust, as well as the radial and tangential spheroidal stresses and the potential stress.

In our explicit case of study, the only external force is conservative (i.e. the centrifugal force) and the solution in Eq (68) assumes the simpler form

$$ y_{\ell m}(r) = \mathbf{I}_\ell(r, r_c) \mathbf{I}_C \left[ \mathbf{P}_\ell \mathbf{I}_\ell (a^-, r_c^+) \right]^{-1} \mathbf{b}. \quad (69) $$

Moreover, when the deformations with respect to the spherical reference configuration are induced only by rotation, the displacement field $\mathbf{u}$ is the sum of only two contributions arising from the zero and second harmonics\(^4\), namely $\mathbf{u} = \mathbf{u}_{00} + \mathbf{u}_{20}$.

This section concludes the general description of the model. In the following we numerically study in details the deformations of rotating neutron stars.

5 NUMERICAL SOLUTION FOR THE POLYTROPE $n = 1$

As anticipated in section 3, we study the behaviour of a neutron star described by a polytrope with $n = 1$. In particular, we are interested in comparing the displacements and strains of neutron stars with different masses and for different values of the frozen adiabatic index.

The peculiar polytropic index $n = 1$ has been chosen because it gives a degenerate mass-radius relation (the radius $a$ and the mass $M$ are independent of each other). Therefore, following the same approach outlined in Giliberti et al. (2019), we can fix the pair $(a, M)$ with the mass and radius values obtained by using a realistic equation of state and solving the Tolman-Oppenheimer-Volkoff equations (TOV). The same can be done for the radius of the core-crust transition.

More explicitly, we consider a realistic equation of state, say the SLy4 (Douchin & Haensel 2001), solve the TOV equations, and use the obtained values of $a$ and $r_c$ to fix the mass and the radius of the polytropic configuration. In this way, despite the fact that the radial density profile is the one obtained in a Newtonian framework by using the polytrope $n = 1$ (that contains no information about the core-crust transition), in the following numerical analysis we still have

\(^4\) The fact that $\mathbf{u}$ is a sum of only the $\ell = 0, 2$ and $m = 0$ harmonic contributions depends on the fact that we have assumed a constant rotation axis. Taking into account also for a possible nutation would require the contributions of other $m \neq 0$ harmonics.
The crust-core transition is set at the fiducial density $1$. For a given radius $a$ and mass $M$ in the range $1M_\odot \div 2M_\odot$ we can calculate the corresponding value of $K$ in Eq (75) and of the central density $\rho_{ce} = \rho(r = 0)$ as

$$K = \frac{2}{a^2} G \rho_{ce} = \frac{M}{4\pi a^2} \left( \frac{K}{2\pi G} \right)^{-3/2}.$$  

(70)

The crust-core transition is set at the fiducial density $1.5 \times 10^{34}$ g/cm$^3$, giving a core-crust transition at $r_c \approx 0.90$ a for a standard neutron star with $M = 1.4M_\odot$, as can be seen in Fig 2.

Now that the equilibrium structure of the NS has been fixed, we have to find the corresponding spheroidal solution $\mathbf{y}$ for a given angular velocity $\Omega$. In the following, if not otherwise stated, we show the results for a $1.4M_\odot$ NS, which characteristics are reported in Tab 1.

In the numerical calculations the physical quantities are normalized by using the stellar radius $a$, the central density $\rho_{ce}$, the shear modulus $\mu_c$ at the core-crust interface and the angular velocity $\Omega$ of the particular NS under consideration. The first two parameters vary with $M$, while the other two are fixed. In particular, the vector solution $\mathbf{y}$ is rescaled as

$$\mathbf{y} = d(\Omega) \times \begin{pmatrix} a \\ a \\ \mu_c \\ \mu_c \\ \mu_c \\ \mu_c \\ a^2/a \end{pmatrix} \tilde{\mathbf{y}},$$  

(71)

where the tilde superscript indicates the dimensionless quantities. An overall scaling is given by the dimensionless factor $d(\Omega) = \frac{1}{3} \frac{\Omega^2 a^2}{v^2},$  

(72)

where the velocity $v$ is defined as $v = \sqrt{\mu_c/\rho_{ce}}$. The above ratio sets the relative importance of the centrifugal force with respect to the elastic reaction of matter and, for practical purposes, can be expressed as

$$d(\Omega) = 6.3 \times 10^{-4} \left( \frac{\Omega}{1 \text{ rad/s}} \right)^2 \left( \frac{\rho_{ce}}{1.38 \times 10^{15} \text{ g/cm}^3} \right) \times$$

$$\left( \frac{\mu_c}{10^{30} \text{ dyn/cm}^2} \right)^{-1} \left( \frac{a}{1.17 \times 10^9 \text{ cm}} \right)^2,$$  

(73)

calculated by using the parameters in Table 1 as a reference.

The elastic response of the star is fixed by the EoS and by the slowly known parameters $\kappa$ and $\mu$ in the crust. For the shear modulus we follow the same prescription guessed by Cutler et al. (2003), namely

$$\mu(r) = 10^{-2} \times P(r).$$  

(74)

On the other hand, the elastic modulus $\kappa$ is linked to the adiabatic index by the relation

$$\kappa = \gamma_{eq,f} P,,$$  

(75)

where the choice of $\gamma_{eq}$ or $\gamma_f$ depends on the slowness of the loading mechanism.

To summarize, the stellar mass, together with the EoS, the adiabatic index governing perturbations, the shear modulus and the core-crust transition density completely fix the elastic behaviour of the star. We can now study the effects of the centrifugal force, starting from a non-rotating and unstrained reference configuration.

### 5.1 Slow dynamics

We set at first the equilibrium bulk modulus in (75), namely

$$\kappa(r) = \gamma_{eq} P(r) = 2P(r).$$  

(76)

With this choice we are implicitly assuming that the rotational dynamics of the NS has a much longer timescale compared to the one of the chemical reactions near equilibrium.

Using the parameters given in Table 1, the displacement with respect to the non-rotating configuration turns out to be $u_\ell(r = a, \theta = \pi/2) \approx 4.2 \times 10^{-3}$ cm at the equator.

The centrifugal potential is particular as its expansion consists of only two harmonic parts having $m = 0$ and $l = 0, 2$. The one with degree $\ell = 0$ gives a smaller contribution to the total displacement with respect to the $\ell = 2$ contribution as we find that $2.5 \leq |U_{20}(r)/U_{00}(r)| \leq 2.9$. Furthermore, the $\ell = 0$ contribution corresponds to a global increase of volume of the star since it is positive everywhere.

We now explore the possibility of crust failure, by calculating the strain angle $\alpha$ and using the Tresca criterion in Eq (33). In Fig 3, the normalized strain angle $\tilde{\alpha} = \alpha/d(\Omega)$ is shown as a function of the colatitude $\theta$ and of the normalized radius $r/a$. Our analysis shows that:

(i) Contrary to the uniform and incompressible model studied by Franco et al. (2000) and Fattoyev et al. (2018),

Figure 2. Normalized core-crust radius $r_c/a$ as a function of the stellar mass $M$, obtained for the SLy EoS.

|        | $\rho_{ce}$ | $\mu_c$ | $\Omega$ | $d(\Omega)$ |
|--------|-------------|---------|---------|-------------|
| $a$ cm | $1.17 \times 10^6$ | $1.38 \times 10^{15}$ | $10^{30}$ | $1$ | $6.3 \times 10^{-4}$ |

Table 1. Parameters for the reference configuration of a neutron star used in the numerical analysis. Since the rotation induced deformations and stresses are proportional to $\Omega$, the angular velocity is set to the value $1$ rad/s so that the numerical results can be easily rescaled for different angular velocities.
the strain angle is an increasing function of the radius. This happens because in the present model the shear modulus is not a constant but a decreasing function of the stellar radius, implying that the strain is expected to be larger near the surface, as was also noticed by Cutler et al. (2003).

(ii) Differently from the incompressible and uniform model, where $\alpha(r, \theta)$ is always peaked at the equator (as shown by Giliberti et al. (2019)), we find that the maximum value of the strain angle $\alpha^{\text{Max}}$ occurs at the poles.

(iii) Concerning the dependence on the stellar mass, $\alpha^{\text{Max}}$ is a decreasing function of $M$, as can be seen in Fig 4. In this case it is not convenient to plot the normalized strain angle $\tilde{\alpha}$, since also the normalizing factor $d$ depends on the stellar mass through $a$ and $v$. This behaviour of $\alpha^{\text{Max}}(M)$ can be understood by considering a very simplified description of a NS with only one uniform elastic layer extending from the center to the crust. In this case the displacement $u$ turns out to be proportional to the dimensionless factor (Baym & Pines 1971; Link et al. 1998)

$$W = \frac{\Omega^2 a^2}{v_K^2},$$

(77)

which is the ratio between the squared equatorial velocity $\Omega^2 a^2$ and the Keplerian one $v_K = \sqrt{GM/a}$. The dependence of $\alpha^{\text{Max}}$ on $M$ is almost entirely due to $W$, which is proportional to

$$W \propto \frac{a^3}{GM} \propto \frac{1}{\rho},$$

(78)

where $\rho$ is the density of the uniform star (Giliberti et al. 2019). Therefore, for more massive (i.e. denser) stars, smaller displacements are expected. This leads us to conclude that this reasonable behaviour, which could appear as a by-product of the simplified uniform model of Baym & Pines (1971) is also a feature of more refined models.

5.2 Fast dynamics

We now investigate what happens when the dynamical timescale of the perturbations is fast with respect to the one of chemical reactions, so that

$$\kappa(r) = \gamma_f P(r).$$

(79)

The polytropic EoS that we employ is purely phenomenological and does not carry any information about the value for the non-equilibrium adiabatic index. Therefore, we study the displacements, stresses and strains in three different cases, characterized by the adiabatic indexes $\gamma_f = 2.1, 200$. For comparison purposes, we consider also the limiting case discussed in the previous section in which $\gamma_{eq}$ is used. The value $\gamma_f = 2.1$ is 5% larger than the equilibrium adiabatic index, while the choice $\gamma_f = 200$ is just a numerical counterpart of the analytical incompressible limit given by $\gamma \to \infty$.

As discussed in section 3, we vary only the crust adiabatic index, while the core maintains its equilibrium compressibility also in the limiting case in which the crust is treated as incompressible. Hence, also in the limiting case $\gamma_f = \infty$ the radial displacement is different from zero because the core modifies its shape during the spin-down, loading the crust.

The numerical solution of the model shows that even a small departure from the equilibrium value of the adiabatic index carries the system to a configuration similar to the incompressible one, as can be seen in Figs 5 and 6. In particular, Fig 5 shows the normalized values of $\tilde{U}_{20}$ and $\tilde{V}_{20}$ for a $M = 1.4 M_\odot$ rotating neutron star, according to the values listed in Tab 1. We notice that the response of the star to the same change of the centrifugal force is quite different in the equilibrium scenario with respect to the frozen ones. The same consideration is valid for the radial stress $R_{20}$ and the tangential stress $S_{20}$, as shown in Fig 6.

For the same stellar configuration, Fig 7 represents the radial displacement $\tilde{U}_{90}$. In all cases we find that $\tilde{U}_{90} > 0$, so that the star undergoes a global increase of volume due to the effect of rotation, as expected. However, there is a marked difference between the equilibrium scenario and the incompressible limit, with the change of the slope of the plotted curve.

This change of behaviour of $\tilde{U}_{90}$ when going from the equilibrium limit to the incompressible one can be explained by recalling that in a neutron star (Chamel & Haensel 2008)

$$\frac{\mu}{\kappa} \ll 1.$$  

(80)

Therefore, the key physical aspects of the problem are already present by studying the $\mu \to 0$ limit: if the effective
adiabatic index is such that \( \gamma \neq \gamma_{eq} \) the Adama-Williamson equation 54 requires \( \chi_{tm} \) to be zero, implying that there can be no volume changes. From Eq (52) it follows that

\[
R_{tm} = \kappa \chi_{tm} = 0 \quad (81)
\]

and via Eq (49) it is possible to show that the radial displacement must coincide with the geoid displacement defined in Eq (56). Therefore, we expect that

\[
U_{tm} = -\frac{\Phi_{tm}}{g} = r_{geoid}^\ell m \quad (82)
\]

when \( \gamma \neq \gamma_{eq} \). To check numerically this assertion, we first focus on the \( \ell = 2 \) harmonic contribution \( U_{20} \) to the radial displacement and on the geoid radial displacement \( U_{20}^{\text{geoid}} \). For the shear modulus in Eq (74), the difference between the radial and the geoid displacements decreases by increasing the adiabatic index, as can be seen in Fig 8: in the \( \gamma_f = \gamma_{eq} \) case (panel-a) we have a clear departure from the geoid, but if \( \gamma_f = 200 \) (panel-d) the two radial displacements almost coincide. As expected, the cases \( \gamma_f = 200 \) and \( \gamma_f = \infty \) give in practice the same result.

Secondly, we numerically check the harmonic contribution of degree \( \ell = 0 \) for increasing values of \( \gamma_f \). From the analytical point of view, the incompressibility assumption requires \( \chi_{tm} = 0 \), implying that

\[
\chi_{tm} = \partial_r U_{tm} + \frac{2}{r} U_{tm} - \frac{\ell(\ell + 1)}{r} V_{tm} = 0. \quad (83)
\]

Since the \( \ell = 0 \) displacement is purely radial (i.e. \( V_{tm} = 0 \)), the above equation becomes

\[
\partial_r U_{00} = -2 \frac{U_{00}}{r}. \quad (84)
\]

In Fig (9) both \( \partial_r U_{00} \) and \(-2U_{00}/r\) are shown for the four different adiabatic indexes considered: again, we approach the incompressible limit as \( \gamma_f \) increases.

We can conclude that our numerical solution behaves as expected when varying the adiabatic index: by solving numerically the model we find that all the harmonic contributions to the radial displacement tend towards the geoid ones in the incompressible limit, as happens exactly in a purely fluid medium.

We now discuss the strain angle in the crust. In Fig 10 we plot the normalized strain angle \( \bar{\alpha} \) as a function of \( r \) and \( \theta \). The color maps representing the various values of \( \bar{\alpha} \) differ significantly when going from the equilibrium to the non-equilibrium configurations.

The value of \( \gamma_f \) influences the slope of the strain angle curve: if for \( \gamma_f = 2, \alpha \) is an increasing function of \( r \) (see Fig 3), for \( \gamma_f = 2, 200 \) and \( \infty \) the opposite is true. This is an interesting result, since one could expect the strain to be maximum at the star’s surface, where \( \mu \) is smaller (Cutler et al. 2003). However, the shear modulus is so small with respect to the bulk modulus that the global crustal deformation is ruled essentially by the value of the adiabatic index, so that even for small deviations of \( \gamma_f \) from the equilibrium
value the strain angle behaves qualitatively as in the incompressible model studied in \cite{Giliberti et al. 2019}.

Finally, we estimate the importance of the stellar mass parameter $M$ by studying the function $\alpha_{\text{Max}}(M)$. As in section 5.1 we have to evaluate the coefficient $d$ for stellar configurations of different mass and we fix $\Omega = 1 \text{ rad/s}$, so that the results can easily rescaled for different rotation rates. We find that the strain angle dependence on mass is almost the same for every value of the frozen adiabatic index. We checked this behaviour by employing a larger set of values for $\gamma_f$, with values ranging from $\gamma_f = \gamma_{\text{eq}}$ to the incompressible limit, as reported in Fig 11.

The comparison between different adiabatic indexes allows to calculate the ratio $\alpha_{\text{Max}}(1M_\odot)/\alpha_{\text{Max}}(2M_\odot)$ for different scenarios, going from the equilibrium to the incompressible one. This is shown in Fig 12. As we can see that ratio has large values when a $\gamma_f$ near $\gamma_{\text{eq}}$ is employed, but it rapidly decreases towards the asymptote $\alpha_{\text{Max}}(1M_\odot)/\alpha_{\text{Max}}(2M_\odot) \simeq 2.6$ as the incompressible limit is approached. This latter value resembles the one obtained with the homogeneous two-density incompressible model where $\alpha_{\text{Max}}(1M_\odot)/\alpha_{\text{Max}}(2M_\odot) \simeq 2/3$ (Giliberti et al. 2019).

5.3 Effective adiabatic index: the effect of General Relativity

We end this section with a technical note, expanding the motivation behind our basic working assumption of a non-relativistic framework.

The Adams-Williamson equation (53) tells us that the adiabatic index (both the equilibrium one as well as the more uncertain frozen one) depends strongly on the stratification: this can be envisaged by comparing the effective adiabatic index of Eq (29), calculated by using the equilibrium configuration of a non-rotating star with the equilibrium one that is intrinsic to the barotropic EoS used.

We consider the usual $M = 1.4M_\odot$ star described by a polytrope $n = 1$, which stratification has been calculated both assuming Newtonian gravity and using the TOV equa-
Figure 9. The functions $\partial_r U_{00}$ (orange) and $-2V_{00}/r$ (purple) calculated for $\gamma_f = 2$ (a), $\gamma_f = 2.1$ (b), $\gamma_f = 200$ (c) and $\gamma_f = \infty$ (d). As the adiabatic index increases the difference between $\partial_r U_{00}$ and $-2V_{00}/r$ goes to zero, as expected.

Figure 10. Color map of the normalized strain angle $\tilde{\alpha}$ as a function of the colatitude $\theta$ and of the normalized radius $r/a$. The region shown here refers to the crustal layer, from $r = r_c$ to $r = a$. Our reference star of $M = 1.4 M_\odot$ has been used, with different adiabatic indexes governing the perturbations: $\gamma = 2.1$ (top) and $\gamma = 200$ (bottom). Here $\alpha$ for $\gamma = \infty$ is not reported because it has the same shape and values of the case $\gamma = 200$.

Figure 13. The difference between the two effective adiabatic indexes is shown in Fig 13.

In the Newtonian case the effective index is clearly given by $\gamma = \gamma_{eq}$ as it should be. However, when the radial profiles are integrated by using the TOV equations, there is a departure of the effective adiabatic index $\gamma$ from the equilibrium case, exceeding the 5% in the crust.

As discussed in the previous section, the behavior of the stresses and displacements calculated for $\gamma_f = 2.1$ is different from the equilibrium case. This warns against using an equilibrium stellar configuration calculated via the TOV equations in a Newtonian model for stresses and strains: in this way an artificial effective adiabatic index that does not correspond to the physical response of the star is introduced.

6 APPLICATION TO PULSAR GLITCHES

Glitches are sudden jumps in the rotational frequency of a pulsar followed by a period of slow recovery that can last for several weeks or months. The first attempts to explain the glitches activity indicated starquakes as a possible mechanism, but soon it became clear that large glitches, as the one of the Vela pulsar, cannot be only due to a change of the NS moment of inertia. (Baym et al. 1969). Nowadays it is thought that glitches are due to an angular momentum exchange between the superfluid core and the elastic crust, following a sudden unpinning of many vortices contained in the superfluid permeating the inner crust and the outer core of a NS (Haskell & Melatos 2015). Within this latter see-
Following the same approach of Giliberti et al. (2019), we use the present and more refined model to estimate
the accumulated strain due to the spin-down between two glitches in the Vela, which parameters \( \Omega \) and spin-down rate \( \dot{\Omega} \) are given in Table 2.

The absolute difference in angular velocity between two Vela glitches can be estimated as:
\[
\Delta \Omega = |\dot{\Omega}(t_{gl}) - \dot{\Omega}(t_{gl})|
\]
where \( t_{gl} \) is the typical observed inter-glitch time. Assuming that the only loading mechanism is due to the slowly-changing angular velocity, the strain developed during the inter-glitch time is roughly given by:
\[
\alpha = (d(\Omega) - d(\Omega - \omega)) \tilde{\alpha} \approx \frac{2\Omega_\omega \omega^2}{3v^2 \tilde{\alpha}},
\]
where \( \tilde{\alpha} \) is the normalized strain angle and \( \alpha \) is given by the polytrope
\[
\alpha = \frac{\dot{\Omega}(\Omega)}{\dot{\Omega}(\Omega - \omega)} - 2\Omega \approx \frac{\dot{\Omega}(\Omega)}{\dot{\Omega}(\Omega - \omega)} - 2\Omega.
\]
This is an extremely small value if compared to the assumed breaking strain in the range \( 10^{-5} \div 10^{-1} \); therefore, the crust’s failure may be a viable trigger for glitches only in the eventuality that the crust is always stressed and very near to the breaking threshold. In other words, in order to trigger a sequence of glitches, the crust-quakes must release only an extremely small portion of the crustal stresses that have been accumulated up to that point.

Alternatively, very intense internal loads other than the pure spin-down process should take part in the process, like the pinning of many vortex lines in the crust (Ruderman 1991a), a possibility that will be investigated in a future work.

7 CONCLUSION
The first aim of the present work is to introduce a Newtonian model to study the deformation of a self-gravitating and compressible neutron star under a variety of loads, like tidal forces and non-conservative forces. In this sense, we considerably extended the model of Giliberti et al. (2019),
in which the NS has been modeled as an incompressible body with two uniform layers. On the other hand, in the present treatment we considered a continuous stratification of matter that is consistent with the assumed EoS and self-gravitation.

In the numerical analysis of the model, the equilibrium structure of the NS has been fixed by assuming a polytropic EoS with $n = 1$ (i.e. $\gamma_{eq} = 2$), which mass-radius relation has been set to be the one of the SLy EoS Douchin & Haensel (2001). In this way our Newtonian model is consistent, in the sense that no spurious effects due to an artificial effective adiabatic index are introduced (as explained in section 5.3), but the mass and radius of the star have realistic values obtained via the integration of the TOV equations. In particular, we studied the dependence of the maximum strain angle $\alpha_{\text{Max}}$ on the stellar mass for compressible NS, finding that the strain caused by the change in the rotation rate of a pulsar is a decreasing function of the stellar mass. Hence, the crust of a very compact and massive NS is more difficult to deform and, ultimately, to break with respect to a $\sim 1M_{\odot}$ NS.

As a first application of our model, we studied the effect of a possible departure of the adiabatic index from its equilibrium value (Haensel et al. 2002) during the build-up of stresses induced by the slowly changing rotation rate of the NS. The elastic response of a neutron star in the case of a non-equilibrium adiabatic index $\gamma_f$ shows interesting features. We find that, despite the small difference in the value of adiabatic indexes in the cases $\gamma_{eq} = 2$ and $\gamma_f = 2.1$, the dynamical responses of the model are clearly distinguishable, either for displacement, stresses and strain angles. This fact is closely related to the smallness of the shear modulus with respect to the bulk modulus in a NS, as discussed in section 5.2.

Secondly, we checked that the sensitivity of strains and stresses to the actual value of the adiabatic index governing perturbations puts a severe warning against the naive use of density profiles obtained via integration of the TOV equations in Newtonian models for stellar deformations. In fact, given the same EoS, the effective adiabatic index calculated by employing the relativistic stratification turns out to be quite different with respect to the value $\gamma_{eq}$, that is intrinsic to the EoS. Therefore, in order to clearly separate the effect of inconsistent stratification from to the one arising from the use of an frozen adiabatic index, a completely relativistic model is needed.

Finally, we explored to what extent the spin-down of a pulsar between two consecutive glitches can load the solid crust. With the present, much more refined model for stellar deformations, we confirm the results of Giliberti et al. (2019): the loading of the solid crust caused by the spin-down between glitches is not enough to reach the breaking threshold and trigger a crustquake. In fact, the difference in angular velocity between glitches in the Vela gives rise to extremely small maximum strain angles, of the order of angular velocity between glitches in the Vela gives rise to extremely small maximum strain angles, of the order of $10^{-9}$ for a typical $M = 1.4M_{\odot}$ NS. These values are orders of magnitude smaller even with respect to the smallest breaking strain that is theoretically expected, of the order of $10^{-13}$. This result challenges the idea that crustquakes are the cause of the sudden unpinning of the superfluid vortices in the crust and, ultimately, that crustquakes are the trigger of vortex-mediated glitches in pulsars: this could be a possibility only if the crust never relieves all the accumulated stresses (i.e. the strain angle in some regions of the crust is always close to the breaking threshold) or some internal load that is orders of magnitude stronger than the variation in the centrifugal force (maybe due to vortex pinning or to the magnetic field) is operating in the crust.

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APPENDIX A: spherical harmonics

In this work spherical harmonics are defined as
\[ Y_{\ell m} (\theta, \phi) = P_{\ell m} \cos \theta \, e^{i m \phi}, \tag{A1} \]
where \( P_{\ell m} \) are the Legendre polynomials, given by
\[ P_{\ell m}(x) \begin{cases} \frac{1}{2^\ell \ell!} \left( \frac{d}{dx} \right)^m \left( x^2 - 1 \right)^\ell & \text{if } m \geq 0, \\ (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_{\ell m}(x) & \text{if } m < 0. \end{cases} \]

The spherical harmonics are the eigenvalues of the angular part of the Laplacian, i.e.
\[ \nabla^2 Y_{\ell m} = -\frac{\ell (\ell + 1)}{\sin \theta} Y_{\ell m}, \tag{A2} \]
and are normalized as
\[ \int_0^\Omega Y_{\ell m} Y_{\ell m}^* d\Omega = \frac{4\pi}{2\ell + 1} (\ell + m)! (\ell - m)! \delta_{\ell \ell'} \delta_{m m'}. \tag{A3} \]

where \( d\Omega = \sin \theta d\theta d\phi. \) Consistently with the previous definitions, we expand the total incremental potential \( \Phi^\delta \) (as well as all the scalar functions) as
\[ \Phi^\delta (r, \theta, \phi) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \Phi_{\ell m} (r) Y_{\ell m} (\theta, \phi). \tag{A4} \]

The expansion of vectorial quantities is more subtle. In particular, the total displacement \( \mathbf{u} \) decomposed in terms of the spheroidal \( \mathbf{u}_S \) and the toroidal \( \mathbf{u}_T \) displacements as
\[ \mathbf{u} = \mathbf{u}_S + \mathbf{u}_T, \tag{A5} \]
where
\[ \mathbf{u}_S (r) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \left[ U_{\ell m} (r) \mathbf{R}_{\ell m} (\theta, \phi) + V_{\ell m} (r) \mathbf{S}_{\ell m} (\theta, \phi) \right] \tag{A6} \]
\[ \mathbf{u}_T (r) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \left[ W_{\ell m} (r) \mathbf{T}_{\ell m} (\theta, \phi) \right]. \tag{A7} \]

In the above expansions, the symbols \( \mathbf{R}_{\ell m}, \mathbf{S}_{\ell m}, \mathbf{T}_{\ell m} \) are vectorial quantities defined by
\[ \mathbf{R}_{\ell m} = Y_{\ell m} \mathbf{e}_r, \tag{A8} \]
\[ \mathbf{S}_{\ell m} = r \nabla Y_{\ell m} = \partial_r Y_{\ell m} e_\theta + \frac{1}{\sin \theta} \partial_\theta Y_{\ell m} e_\phi, \tag{A9} \]
\[ \mathbf{T}_{\ell m} = \nabla \times (r Y_{\ell m}) = \frac{1}{\sin \theta} \partial_\phi Y_{\ell m} e_\theta - \partial_r Y_{\ell m} e_\phi, \tag{A10} \]
where \( \mathbf{e}_r, \mathbf{e}_\theta \) and \( \mathbf{e}_\phi \) are the usual unit vectors of the spherical coordinate system.

The incremental stress acting on a spherical surface element with outward normal \( \mathbf{e}_r \) can be computed as
\[ \sigma^\delta \cdot \mathbf{e}_r = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \left( R_{\ell m} R_{\ell m}^* + S_{\ell m} S_{\ell m}^* + T_{\ell m} T_{\ell m}^* \right) \tag{A11} \]
where
\[ R_{\ell m} = \lambda \chi_{\ell m} + 2 \mu \partial_r U_{\ell m}, \tag{A12} \]
\[ S_{\ell m} = \mu \left( \partial_r W_{\ell m} + \frac{U_{\ell m} - V_{\ell m}}{r} \right), \tag{A13} \]
\[ T_{\ell m} = \mu \left( \partial_r W_{\ell m} - \frac{W_{\ell m}}{r} \right). \tag{A14} \]

In the main text, \( R_{\ell m} \) and \( S_{\ell m} \) are referred to as the radial and tangential spherical stresses respectively. On the other hand, \( T_{\ell m} \) is the toroidal stress.

Finally, a generic non-conservative force \( \mathbf{h} \) can be expanded in terms of three real and independent sets of coefficients \( h_{\ell m}^R, h_{\ell m}^S, h_{\ell m}^T \) according to the formula
\[ \mathbf{h} = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \left( h_{\ell m}^R \mathbf{R}_{\ell m} + h_{\ell m}^S \mathbf{S}_{\ell m} + h_{\ell m}^T \mathbf{T}_{\ell m} \right). \tag{A15} \]

APPENDIX B: general form of the matrix \( A \)

In Eq (24), the spherical equations are written in a compact notation thanks to the use of the \( 6 \times 6 \) matrix \( A \), which general form is
\[ A_r (r) = \begin{pmatrix} -\frac{2\lambda}{r^2} & \frac{\ell (\ell + 1)\lambda}{r^3} & \frac{\ell (\ell + 1)\lambda}{r^3} & 0 & 1 & 0 \\ \frac{2}{r} & -\frac{\lambda_0}{r^2} & \frac{\ell (\ell + 1)\lambda}{r^3} & \frac{\ell (\ell + 1)\lambda}{r^3} & 0 & 1 \\ -\frac{\lambda_0}{r^2} & \frac{\ell (\ell + 1)\lambda}{r^3} & -\frac{\lambda_0}{r^2} & \frac{\ell (\ell + 1)\lambda}{r^3} & 0 & 1 \\ 0 & 0 & 0 & -\frac{\lambda_0}{r^2} & \frac{\ell (\ell + 1)\lambda}{r^3} & 0 \\ 0 & 0 & 0 & -\frac{\lambda_0}{r^2} & \frac{\ell (\ell + 1)\lambda}{r^3} & 0 \\ 0 & 0 & 0 & 0 & -\frac{\lambda_0}{r^2} & \frac{\ell (\ell + 1)\lambda}{r^3} \end{pmatrix} \tag{B1} \]
where \( \beta = \kappa + \frac{4}{3} \mu \) and \( \lambda = \kappa - \frac{2}{3} \mu \).
APPENDIX C: the $\ell = 0$ harmonic

The matrix $A_\ell (r)$, for the harmonic $\ell = 0$ is

$$A_0 (r) = \begin{pmatrix} -\frac{2(\kappa - 2\rho)}{\sqrt{3}} & \frac{\beta}{r} & 0 & 0 \\ \frac{4\pi G\rho_0}{r} & -\frac{4}{7} & -\frac{\rho_0}{r} & \rho_0 \\ 0 & -\frac{1}{r} & 1 & 0 \\ -\frac{4\pi G\rho_0}{r} & 0 & 0 & -\frac{1}{r} \end{pmatrix}. \quad (C1)$$

This expression can be obtained from the general form for the matrix $A$ (see appendix B) by putting $\ell = 0$ and neglecting the tangential displacement and stress. The equation that we have to solve is

$$\frac{dw_{00}}{dr} = A_0 w_{00} + f_{cen}, \quad (C2)$$

where $f_{cen}$ is the centrifugal force vector

$$f_{cen} = (0, f_{cen}, 0, 0) \quad (C3)$$

and the spherical solution $w_{00}$ is the four-vector

$$w_{00} (r) = (U_{00} (r), R_{00} (r), \Phi_{00} (r), Q_{00} (r)). \quad (C4)$$

We underline that in this case the approach is different with respect to the one presented in Eq (12) for $\ell \geq 2$, since here the potential $\Phi^\Delta$ coincides with the perturbed gravitational potential $\phi^\Delta$. It can be shown that the system (C2) can be simplified in a system for $U$ and $R$. In this respect the general vector solution of Eq C2 can be written as

$$w_{00} (r) = \begin{pmatrix} U^{reg} (r) \\ R^{reg} (r) \\ \Phi^{reg} (r) \\ \phi^{reg} (r) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ 1 \\ \frac{1}{r} \end{pmatrix} \begin{pmatrix} U^f \\ R^f \\ \Phi^f \\ Q^f \end{pmatrix}. \quad (C5)$$

The superscript $reg$ indicates the regular solution of the associated homogeneous system; while $f$ indicates a particular solution of (C2). Note that the constant $C_2^f$ does not affect the radial displacement and the radial stress, that can be found by fixing the other constant $C_1^f$. This can be done by imposing the boundary conditions for the radial stress, namely

$$R_{00} (a) = 0. \quad (C6)$$

To find the solutions of Eq (C5), we need some initial conditions. Here, for simplicity, we are looking only for the explicit solution for $U_{00}$ and $R_{00}$. In the innermost part of the star, we can state that all the quantities $\rho$, $\kappa$ and $\mu$ are constant, therefore

$$\rho = \rho_0, \quad \kappa = \kappa_0, \quad \mu = \mu_0 = 0. \quad (C7)$$

With this assumption we find

$$U^{reg} (x) = \frac{-x \cos x + \sin x}{x^2},$$

$$R^{reg} (x) = \frac{4\sqrt{\pi/3} \sqrt{G\kappa_0 \rho_0} \sin x}{x},$$

$$U^f (x) = -\frac{3x \sqrt{3\kappa_0 M}^2}{32G^{3/2} \pi^{3/2} \rho_0^2},$$

$$R^f (x) = -\frac{9\kappa_0 M^2}{8\pi G \rho_0},$$

where $x = 4 \sqrt{G\pi/(3\kappa_0)} r \rho_0$.

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