Extended phase space of black holes in Lovelock gravity with nonlinear electrodynamics

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In this paper, we consider Lovelock gravity in the presence of two Born–Infeld types of nonlinear electrodynamics and study their thermodynamical behavior. We extend the phase space by considering the cosmological constant as a thermodynamical pressure. We obtain critical values of pressure, volume, and temperature and investigate the effects of both the Lovelock gravity and the nonlinear electrodynamics on these values. We plot $P–v$, $T–v$, and $G–T$ diagrams to study the phase transitions of these thermodynamical systems. We show that the power of the nonlinearity and gravity have opposite effects. We also show how consideration of the cosmological constant, nonlinearity, and Lovelock parameters as thermodynamical variables will modify the Smarr formula and the first law of thermodynamics. In addition, we study the behavior of the universal ratio of $\frac{Pc}{v}$ for different values of the nonlinearity power of electrodynamics as well as the Lovelock coefficients.

Subject Index A70, E03, E04

1. Introduction

In the context of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, it has been proposed that variation of the cosmological constant corresponds to variation of the number of colors in the boundary field theory of Yang–Mills with a chemical potential interpretation [1–4]. On the other hand, according to Teitelboim and Henneaux’s mechanism, coupling 4D gravity with an antisymmetric gauge field without the cosmological constant results in the appearance of the cosmological constant as a constant of motion [5–7], and, therefore, the cosmological constant will be a variable. Recently, the cosmological constant was considered as a state-dependent parameter in 2D dilaton gravity [8]. It was shown that treating the cosmological constant as a $U(1)$ charge with non-minimal coupling leads to confinement of the electrostatic potential. Larranaga showed that consideration of the cosmological constant as a thermodynamical variable can be extended to use of the Smarr formula for the inner and outer horizons of a Bañados–Teitelboim–Zanelli (BTZ) black hole [9].

Recently, there has been increasing interest in the thermodynamical behavior of black holes in asymptotically AdS spacetime. This growing interest comes from the fact that, using the AdS/CFT correspondence, one can find answers regarding a conformal field theory of $d$ dimensions by solving problems that a gravitational field presents in $(d + 1)$-dimensional anti-de Sitter spacetime [10–13]. Hawking and Page, in their pioneering work, showed that the similarity of phase transition between the stable large black hole and thermal gas in AdS space can be interpreted as a
confinement/deconfinement phase transition in the dual strongly coupled gauge theory [14]. Later, in an interesting article, Witten showed that, using the AdS/CFT correspondence, one can study the thermal phase transition and the interpretation of confinement in gauge theories [15]. On the other hand, one may consider the cosmological constant as a thermodynamical pressure (in order to investigate the phase transition of black holes) to extend the phase space (see Ref. [16] for more details) and modify the first law of black hole thermodynamics [17–20]. Another contribution of this consideration is a renewed interpretation of the mass of black holes, which, from internal energy, becomes enthalpy [1]. This interpretation indicates that the mass of black holes plays a more important role in the thermodynamical structure of black holes and contains more information regarding the phase structure of black holes [21].

For a canonical ensemble with a fixed charge, it was found that there exists a phase transition between small and large black holes. This phase transition behaves very like the gas/liquid phase transition in a van der Waals system [22,23]. On the other hand, the phase transitions of small/large black holes in the AdS/CFT correspondence may be interpreted as conductor/superconductor regions of condensed-matter systems [24–26].

Considering the fact that Maxwell theory contains some fundamental problems and that nonlinear electromagnetic fields solve some of these shortcomings [27,28], one is motivated to study different models of nonlinear electrodynamics (NED). One interesting class of these models is the Born–Infeld (BI) type, which is acquired in the low-energy limit of heterotic string theory [29–36]. Therefore, one is motivated to study these theories (of which, in this paper, we have considered logarithmic [37] and exponential forms [38,39]) and the nonlinearity effects of electromagnetic field on critical values representing the phase transition of black holes.

On the other hand, Einstein gravity is not flawless and has some fundamental problems [40–42]. Generalization of the Einstein gravity to higher orders of Lovelock gravity is one way to solve some of these problems [43,44]. Besides, the Lagrangian of Lovelock gravity is obtainable through the use of the low-energy effective action of string theory [29–36]. One can take into account the fact that modification of Einstein gravity may change the conserved quantities of black holes and therefore it is inevitable to see that critical values and phase transitions may depend on the choice of gravity model.

In the literature, there have been various studies regarding higher orders of Lovelock gravity in the presence of different NED [45–53]. Also, the phase transitions and stability conditions of black holes in various gravity models have been studied intensively [54–59]. These investigations lead to interesting consequences and phenomenologies [60,61]. In this paper, we consider higher orders of Lovelock gravity in the presence of two classes of NED and study their phase structure. We investigate the effects of both the nonlinearity of the electrodynamic models and the Lovelock parameters on the phase diagrams and the critical values.

Considering the fact that we are treating black holes as thermodynamical systems and that we interpret the first law of black hole mechanics as the first law of thermodynamics, we expect to see similar thermodynamical behavior for black holes and the usual thermodynamical systems. Therefore, it is crucial to investigate phase transitions and critical values of black holes. Moreover, Lovelock gravity is a generalization of Einstein gravity and it is a theory that solves some of the Einstein gravity problems. Hence, this generalization also gives a correction to the calculated critical values of Einstein gravity. The thermodynamical behavior of this modification should be reasonable and consistent with thermodynamical concepts. Furthermore, as mentioned above, nonlinear electromagnetic fields are introduced to solve the shortcomings of Maxwell theory. So it is reasonable
to investigate the nonlinear effects of NED on the phase transitions of black holes. Also, modifications of electrodynamics like gravitational parts must have consistent thermodynamical behavior.

This paper is organized as follows. The next section is devoted to an introduction to higher orders of Lovelock gravity and conserved quantities. Then, we extend the phase space by considering the cosmological constant as thermodynamical pressure and study the Smarr formula of these black holes. After that, we calculate critical values and plot related diagrams for different cases. We give a detailed discussion regarding the diagrams, their physical interpretations, and the effects of both NED and gravitational parameters. We finish our paper with some closing remarks.

2. Solutions and thermodynamic quantities of Lovelock gravity

Here, we restrict ourselves to the BI-type NED models that were introduced by Soleng [37] and Hendi [38,39] with the following Lagrangians:

$$
\mathcal{L}(F) = \begin{cases} 
\beta^2 \left( \exp \left( \frac{-F}{\beta^2} \right) - 1 \right), & \text{ENEF} \\
-8 \beta^2 \ln \left( 1 + \frac{F}{8 \beta^2} \right), & \text{LNEF} 
\end{cases},
$$

(1)

where ENEF and LNEF denote \textit{exponential form of nonlinear electromagnetic field} and \textit{logarithmic form of nonlinear electromagnetic field}, respectively, and $\beta$ is the nonlinearity parameter. Considering the fact that we are interested in studying the spherically symmetric spacetime, we employ the following metric:

$$
ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_{d-2}^2,
$$

(2)

where $d\Omega_d$ denotes the standard metric of a $d$-dimensional sphere, $S^d$, with the volume $\omega_d$. It is known that the field equations of Einstein gravity in the presence of NED are in the following forms [62,63]:

$$
G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{L}(F) - 2 \frac{d \mathcal{L}(F)}{dF} F_{\mu\lambda} F^{\lambda\nu},
$$

(3)

$$
\nabla^\mu \left( \frac{d \mathcal{L}(F)}{dF} F^{\mu\nu} \right) = 0,
$$

(4)

where $G_{\mu\nu}$ and $\Lambda$ are the Einstein tensor and cosmological constant, respectively. Einstein gravity in the presence of the aforementioned NED has been studied in Refs. [62,63]. Regardless of gravitational sector, one may consider Eqs. (1) and (4) with the metric (2) to obtain the nonzero components of electromagnetic fields with the following explicit forms [62,63]:

$$
E(r) = F_{tr} = \frac{q}{r^2} \times \begin{cases} 
\frac{L_W}{e^{L_W/2}}, & \text{ENEF} \\
\frac{2}{\Gamma + 1}, & \text{LNEF} 
\end{cases}
$$

(5)

where $q$ is an integration constant that is proportional to the total electric charge of the black hole solutions (regarding $4\pi$ as a proportionality constant) Extended-phase-space thermodynamics and phase diagrams of Einstein gravity in the presence of NED were investigated in Ref. [64]. So, we consider Gauss–Bonnet (GB) and third order of Lovelock (TOL) gravities and investigate the extended-phase-space thermodynamics and critical behavior of these gravities.
2.1. GB gravity

First, we take into account the GB gravity. In order to obtain the field equation of GB gravity, one should add the $G_{\mu \nu}^{GB}$ tensor to the left-hand side of Eq. (3), in which $G_{\mu \nu}^{GB}$ is given by

$$G_{\mu \nu}^{GB} = -\alpha_{GB} \left[ 4 R^{\rho \sigma} R_{\rho \mu \nu \sigma} - 2 R^{\rho \sigma \lambda} R_{\rho \mu \sigma \lambda} - 2 R R_{\mu \nu} + 4 R_{\mu \lambda} R^\lambda_{\nu} + \frac{1}{2} g_{\mu \nu} \mathcal{L}_{GB} \right],$$  
(6)

where $\mathcal{L}_{GB} = R_{\mu \nu \rho \delta} R^{\mu \nu \rho \delta} - 4 R_{\mu \nu} R^{\mu \nu} + R^2$ and $\alpha_{GB}$ is the GB parameter. Considering Eq. (3) with the extra term (6), one can obtain the following solutions [65]:

$$g_{GB}(r) = 1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{\Psi_{GB}(r)} \right),$$  
(7)

where $\alpha = (d - 3) (d - 4) \alpha_{GB}$

and

$$\Psi_{GB}(r) = 1 + \frac{8\Lambda}{(d - 1)(d - 2)} + \frac{4am}{r^{d-1}} + \frac{4\alpha \beta^2 \gamma}{(d - 1)(d - 2)},$$  
(8)

$$\gamma = \left\{ \begin{array}{ll}
1 + \frac{2(d - 1)q}{\beta r^{d-1}} & \text{ENEF}, \\
\frac{8(d - 2)}{(d - 1)} \left[ (2d - 3) (\Gamma - 1) - \frac{(d - 1) \ln \left( \frac{1 + \Gamma}{2} \right)}{d - 2} + \frac{(d - 2)}{d - 3} \mathcal{H} \right] & \text{LNEF}.
\end{array} \right.$$  
(9)

in which $L_W = \text{Lambert} W \left( \frac{4q^2}{\beta r^{d-4}} \right)$ and $m$ is an integration constant that is related to the total mass [65]:

$$M = \frac{\omega_{d-2} (d - 2) m}{16\pi}.$$  
(10)

In addition, $\mathcal{H}$ and $\Gamma$ are in the following forms:

$$\mathcal{H} = 2F_1 \left( \left[ \frac{1}{2}, \frac{d - 3}{2d - 4} \right], \left[ \frac{3d - 7}{2d - 4}, 1 - \Gamma^2 \right] \right),$$

$$\Gamma = \sqrt{1 + \frac{q^2}{\beta^2 r^{2d-4}}}.$$  

Calculating the Kretschmann scalar, one finds that it diverges at $r = 0$, so the metric function (7) has an essential singularity at $r = 0$ [65]. We should note that these solutions may be interpreted as asymptotically AdS black holes as those of the Einstein case [62]. Now, we take into account the surface gravity interpretation to obtain the Hawking temperature of the aforementioned black hole solutions, yielding [65]

$$T = \frac{-2\Lambda r_+^4 + (d - 2) (d - 3) r_+^2 + (d - 2) (d - 5) \alpha - \sigma r_+}{4\pi r_+ (d - 2) (r_+^2 + 2\alpha)},$$  
(11)

where

$$\sigma = \left\{ \begin{array}{ll}
\beta^2 r_+^3 \left( 1 + \left( \frac{2E}{\beta} \right)^2 \right) e^{-\frac{2E}{\beta}} - 1 & \text{ENEF}, \\
8r_+^3 \beta^2 \ln \left[ 1 - \left( \frac{E}{2\beta} \right)^2 \right] + \frac{4r_+^3 E^2}{1 - \left( \frac{E}{2\beta} \right)^2} & \text{LNEF},
\end{array} \right.$$  
(12)

and $E = E(r) |_{r = r_+}$. Since the obtained solutions are asymptotically AdS, one may obtain the entropy of the black hole solutions by use of the Gibbs–Duhem relation. After some calculations, one can...
obtain [65]
\[
S = \frac{\omega_d-2}{4} \left( r^d-2 + \frac{2 (d-2)}{(d-4)} \alpha r^d + \right),
\]
which confirms that the area law is recovered for \(\alpha = 0\).

### 2.2. TOL gravity

Now, we insert the following TOL term, \(G^\text{TOL}_{\mu\nu}\), to the field equation of GB gravity to obtain the solutions of TOL gravity. The tensor \(G^\text{TOL}_{\mu\nu}\) may be written as

\[
G^\text{TOL}_{\mu\nu} = -\alpha_{\text{TOL}} \left[ 3 \left( 4 R_{\sigma\tau} R_{\sigma\tau} + 8 R_{\sigma\tau} R_{\sigma\tau} R_{\nu\mu} + 2 R_{\nu\mu} R_{\nu\mu} - R_{\sigma\tau} R_{\sigma\tau} \right) - 8 R_{\nu\mu} R_{\nu\mu} - 4 R_{\nu\mu} R_{\nu\mu} - 4 R_{\nu\mu} R_{\nu\mu} - 8 R_{\nu\mu} R_{\nu\mu} - 8 R_{\nu\mu} R_{\nu\mu} \right]
\]

where \(\alpha_{\text{TOL}}\) and \(L_{\text{TOL}}\) are, respectively, the coefficient and Lagrangian of TOL gravity:

\[
L_{\text{TOL}} = \frac{(d-3)(d-4)}{3(d-3)(d-6)} \alpha^2 G_{\text{GB}}
\]

Hereafter, we consider the special case \(\alpha_{\text{TOL}} = \frac{2}{3(d-3)(d-6)} \alpha^2 G_{\text{GB}}\) to simplify the calculations. It has been shown that the metric function of TOL gravity in the presence of NED can be written as [66]

\[
g_{\text{TOL}}(r) = 1 + \frac{r^2}{\alpha} \left( 1 - \Psi_{\text{TOL}}(r)^{\frac{1}{3}} \right),
\]

\[
\Psi_{\text{TOL}}(r) = 1 + \frac{6 \alpha M}{(d-1)(d-2)} + \frac{3 \alpha M}{r^{d-1}} + \frac{3 \alpha M^2}{(d-1)(d-2)}.
\]

The geometric and thermodynamic properties of the asymptotically AdS black holes have been studied before [66]. The finite mass of these solutions is the same as that of Einstein gravity, where \(m\) can be obtained as a function of \(r_+\) from the metric function of TOL gravity. The Hawking temperature and the entropy of the TOL solutions can be calculated as [66]

\[
T = \frac{-6 \Lambda r_+^5 + 3 (d-2) (d-3) r_+^4 + 3 (d-2) (d-5) \alpha r_+^3 + (d-2) (d-7) \alpha^2 - 3 \alpha r_+^3}{12 \pi r_+ (d-2) (r_+^2 + \alpha)^2},
\]

and

\[
S = \frac{\omega_d-2}{4} \left( r_+^{d-2} + \frac{2 (d-2)}{(d-4)} \alpha r_+^{d-4} + \frac{(d-2)}{(d-6)} \alpha^2 r_+^{d-6} \right).
\]

### 3. Extended phase space and phase diagrams

As mentioned in the introduction, there are some motivations to view the cosmological constant as a variable. In addition, there are various theories in which some physical constants (such as gauge coupling constants, the Newton constant, Lovelock coefficients, and the BI parameter) are not fixed but dynamical. In this case, it is logical to consider the variation of these parameters in the first law of black hole thermodynamics [67–70].
In order to investigate the phase structure of these classes of gravities, we employ an approach in which the cosmological constant is a thermodynamical variable (pressure) with the following relation:

\[ P = - \frac{\Lambda}{8\pi}. \]  

(20)

This consideration could be justified due to the fact that, in a quantum context, fundamental fixed parameters could vary. As one can see, the conjugating thermodynamical variable to this assumption (cosmological constant as thermodynamical pressure) will be volume, where, in the literature, the derived volume for different types of black holes is in agreement with the topology of the spacetime [21,54–59,64]. In order to calculate the volume of these thermodynamical systems, we use the following relation:

\[ V = \left( \frac{\partial H}{\partial P} \right)_{S,Q}. \]  

(21)

Also, we should consider the effects of the cosmological constant in the first law of thermodynamics and extend our phase space. In doing so, the total finite mass of the black hole will play the role of enthalpy and hence the corresponding Gibbs free energy will be in the form of

\[ G = H - TS = M - TS. \]  

(22)

Using the aforementioned comments, one can obtain the volume with the following form:

\[ V = \omega \frac{d - 2}{d - 1}, \]  

(23)

which is consistent with the topological structure of spherically symmetric spacetime. Equation (22) was obtained in Einstein gravity [64], which indicates that, although considering Lovelock gravity modifies the metric function and some conserved quantities of the black hole, it does not change the volume of the black hole.

In addition, it has been shown that the Smarr formula may be extended to Lovelock gravity as well as nonlinear theories of electrodynamics [71–75]. Geometrical techniques (a scaling argument) were used to derive an extension of the first law and its related modified Smarr formula. The result includes variations in the cosmological constant, Lovelock coefficients, and also nonlinearity parameter. In our case, Lovelock gravity in the presence of NED, \( M \) should be a function of entropy, pressure, charge, Lovelock parameters, and BI coupling coefficient [71]. Regarding the previous section, we find that these thermodynamic quantities satisfy the following differential form:

\[ dM = TdS + \Phi dQ + VdP + A'_1 d\alpha_2 + A'_2 d\alpha_3 + Bd\beta, \]  

(24)

where we have achieved \( T \), and one can obtain

\[ \Phi = \left( \frac{\partial M}{\partial Q} \right)_{S,P,\alpha_2,\alpha_3,\beta}, \]

\[ V = \left( \frac{\partial M}{\partial P} \right)_{S,Q,\alpha_2,\alpha_3,\beta}, \]

\[ A'_1 = \left( \frac{\partial M}{\partial \alpha_2} \right)_{S,Q,P,\alpha_3,\beta}, \]
totically AdS solutions in the extended phase space: 

\[ A'_{2} = \left( \frac{\partial M}{\partial \alpha_{2}} \right)_{S,Q,P,\alpha_{2},\beta}, \]

\[ B = \left( \frac{\partial M}{\partial \beta} \right)_{S,Q,P,\alpha_{2},\beta}. \]

Using the redefinition of \( \alpha_{2} \) and \( \alpha_{3} \) with respect to the single parameter \( \alpha \), we can rewrite \( A'_{1}d\alpha_{2} + A'_2d\alpha_3 \) as a single differential form:

\[ d\alpha_{2} = \frac{1}{(d - 3)(d - 4)} d\alpha, \]

\[ d\alpha_{3} = \frac{2\alpha}{3(d - 3)(d - 4)(d - 5)(d - 6)} d\alpha. \]

Moreover, by the scaling argument, we can obtain the generalized Smarr relation for our asymptotically AdS solutions in the extended phase space:

\[ (d - 3)M = (d - 2)TS + (d - 3)Q\Phi - 2PV + 2(A_{1}\alpha + A_{2}\alpha^{2}) - B\beta, \tag{25} \]

where

\[ \Phi = \begin{cases} \frac{\beta r_{+}\sqrt{L_{W_{+}}}}{2(d - 3)(3d - 7)} ((d - 2)D_{+}L_{W_{+}} + 3d - 7), & \text{ENEF} \vspace{1mm} \\ \frac{-2\beta^{2}r_{+}^{d - 1}}{(d - 1)q} (\eta_{+} - 1), & \text{LNEF} \end{cases}, \]

\[ V = \frac{r_{+}^{d - 1}}{(d - 1)}, \]

\[ A_{1} = \frac{(d - 2)k^{2}r_{+}^{d - 5}}{16\pi} - \frac{(d - 2)kTr_{+}^{d - 4}}{2(d - 4)}, \]

\[ A_{2} = \frac{(d - 2)k^{3}r_{+}^{d - 7}}{24\pi} - \frac{(d - 2)k^{2}Tr_{+}^{d - 6}}{2(d - 6)}, \]

\[ B|_{\text{ENEF}} = \frac{q(d - 2)r_{+} + (L_{W_{+}})^{3/2}D_{+}}{8\pi (d - 1)(3d - 7)} - \frac{\beta r_{+}^{d - 1}}{8\pi (d - 1)} + \frac{q\beta r_{+}^{d}\sqrt{L_{W_{+}}}}{8\pi (d - 1) (1 + L_{W_{+}})} \]

\[ + \frac{2qr_{+}}{8\pi (d - 1)\sqrt{L_{W_{+}}}} (1 + L_{W_{+}}), \]

\[ B|_{\text{LNEF}} = \frac{\beta r_{+}^{d - 1}}{2\pi (d - 1)^{2}} \left[ (d - 2) \left( \frac{\Gamma_{+}}{\Gamma_{+} - 1} \right) H_{+} + 2(d - 1) \ln \left( \frac{1 + \Gamma_{+}}{2} \right) + (3d - 5)(1 - \Gamma_{+}) \right], \]

in which

\[ \eta_{+} = 2F_{1} \left[ \left[ -\frac{1}{2}, \frac{1 - d}{2d - 4} \right], \left[ \frac{d - 3}{2d - 4}, 1 - \Gamma_{+}^{2} \right] \right], \]

\[ D_{+} = 1F_{1} \left[ [1], \left[ \frac{5d - 11}{2d - 4}, L_{W_{+}} \right] \right], \]

\[ H_{+} = 2F_{1} \left[ \left[ \frac{1}{2}, \frac{d - 3}{2d - 4} \right], \left[ \frac{3d - 7}{2d - 4}, 1 - \Gamma_{+}^{2} \right] \right]. \]

The next step will be calculating critical values. Due to the relation between the volume and radius of the black hole, we use the horizon radius (specific volume) in order to investigate the critical
behavior of these systems [54–59]. In order to do so, we use the method in which critical values are obtained through the use of \( P-r_+ \) diagrams. First, we use the following relations to obtain the proper equations for the critical radius:

\[
\left( \frac{\partial P}{\partial r_+} \right) = \left( \frac{\partial^2 P}{\partial r_+^2} \right) = 0.
\]

For economical reasons, we will not present the relations obtained to calculate the critical horizon radius. We employ the numerical method for calculating critical values, which results in the following diagrams for different classes of Lovelock gravity. We present various tables in order to plot \( P-r_+ \), \( T-r_+ \), and \( G-T \) and study the effects of the gravitational parameter, which is represented by \( \alpha \), and the NED parameter, which is represented by \( \beta \). In this paper, we have considered two classes of NED. Studying different phase diagrams for these NED shows that they have similar behavior. Therefore, for economical reasons, we will investigate only the LNEF branch and calculate related critical values of this NED model.

It will be constructive to give a short description regarding the different phase diagrams and the information they contain before presenting the tables and phase diagrams. \( G-T \) diagrams represent the energy levels of different phases between which phase transitions take place and show the changes in energy level before and after phase-transition states. The characteristic swallowtail that is seen in these diagrams shows the process that we know as phase transition. It also gives interesting information regarding the temperature of critical points. \( T-r_+ \) plots contain information regarding critical temperature and horizon radius at which phase transitions takes place. Also, they give some insight into single-state regions, which, in our case, are small/large black holes. Finally, studying \( P-r_+ \) plots gives us information regarding the behavior of pressure as a function of horizon radius, critical pressure, and critical horizon radius (volume) of a phase transition. One of the reasons for studying these diagrams is the similarity between the phase structure of black holes and van der Waals thermodynamical systems.

In what follows, we present various tables to investigate the effects of electrodynamics and gravity models on the critical values of the phase transition. We also plot \( P-r_+ \), \( T-r_+ \), and \( G-T \) diagrams for GB and TOL gravities and interpret them. It is notable to mention that, considering the metric function of GB gravity, one finds that there is an upper limit for the GB parameter to have a real solution.

### 4. Discussion on the results of the diagrams

As one can confirm, higher orders of Lovelock gravity modify the phase diagrams and critical values of volume, pressure, and temperature (see Tables 1–6 and the related Figs. 1–12). It is clear that considering higher orders of Lovelock gravity leads to different kinds of thermodynamical systems; this was also evident through calculated conserved quantities.

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**Table 1.** GB gravity with \( q = 1 \), \( \alpha = 0.1 \), and \( d = 5 \).

| \( \beta \)   | \( v_c \)    | \( T_c \)    | \( P_c \)    | \( \frac{P_{c+}}{T_c} \) |
|-------------|-------------|-------------|-------------|-----------------|
| 0.10000     | 1.03178     | 0.16302     | 0.03904     | 0.24708         |
| 0.50000     | 1.60193     | 0.13789     | 0.02536     | 0.29467         |
| 1.00000     | 1.63292     | 0.13703     | 0.02497     | 0.29765         |
| 1.50000     | 1.63852     | 0.13687     | 0.02490     | 0.29818         |
| 2.00000     | 1.64047     | 0.13682     | 0.02488     | 0.29837         |
In this paper, we consider the thermodynamical effects of Lovelock gravity up to GB and TOL separately. It is evident that critical temperature and pressure are decreasing functions of $\alpha$ and also orders of Lovelock gravity, whereas the critical volume is an increasing function of these two factors (see Figs. 8, 9, 11, and 12). On the other hand, the energy gap between two phases increases drastically by transforming from a lower order of Lovelock gravity to a higher one (see Figs. 8, 9, 11, and 12, right panels) or by increasing $\alpha$ for each order (see Figs. 8 and 9, right panels). In addition, the length of subcritical isobars increases, which means that the single-phase region of small/large black
Fig. 1. $P-v$ (left), $T-v$ (middle), and $G-T$ (right) diagrams in GB gravity for $\alpha = 0.1$, $q = 1$, and $\beta = 1.5$ ($d = 5$). $P-v$ diagram, from top to bottom: $T = 1.2T_c$, $T = 1.1T_c$, $T = T_c$, $T = 0.85T_c$, and $T = 0.75T_c$, respectively. $T-v$ diagram, from top to bottom: $P = 1.2P_c$, $P = 1.1P_c$, $P = P_c$, $P = 0.85P_c$, and $P = 0.75P_c$, respectively. $G-T$ diagram for $P = 0.5P_c$ (solid line), $P = P_c$ (dotted line), and $P = 1.5P_c$ (dashed line).

Fig. 2. $P-v$ (left), $T-v$ (middle), and $G-T$ (right) diagrams in GB gravity for $\beta = 1$, $q = 1$, and $\alpha = 0.5$ ($d = 5$). $P-v$ diagram, from top to bottom: $T = 1.2T_c$, $T = 1.1T_c$, $T = T_c$, $T = 0.85T_c$, and $T = 0.75T_c$, respectively. $T-v$ diagram, from top to bottom: $P = 1.2P_c$, $P = 1.1P_c$, $P = P_c$, $P = 0.85P_c$, and $P = 0.75P_c$, respectively. $G-T$ diagram for $P = 0.5P_c$ (solid line), $P = P_c$ (dotted line), and $P = 1.5P_c$ (dashed line).

Fig. 3. $P-v$ (left), $T-v$ (middle), and $G-T$ (right) diagrams in GB gravity for $\alpha = 0.1$, $q = 1$, and $\beta = 1.5$ ($d = 7$). $P-v$ diagram, from top to bottom: $T = 1.2T_c$, $T = 1.1T_c$, $T = T_c$, $T = 0.85T_c$, and $T = 0.75T_c$, respectively. $T-v$ diagram, from top to bottom: $P = 1.2P_c$, $P = 1.1P_c$, $P = P_c$, $P = 0.85P_c$, and $P = 0.75P_c$, respectively. $G-T$ diagram for $P = 0.5P_c$ (solid line), $P = P_c$ (dotted line), and $P = 1.5P_c$ (dashed line).
Fig. 4. $P-v$ (left), $T-v$ (middle), and $G-T$ (right) diagrams in GB gravity for $\beta = 1$, $q = 1$, and $\alpha = 0.5$ ($d = 7$). $P-v$ diagram, from top to bottom: $T = 1.2 T_c$, $T = 1.1 T_c$, $T = T_c$, $T = 0.85 T_c$, and $T = 0.75 T_c$, respectively. $T-v$ diagram, from top to bottom: $P = 1.2 P_c$, $P = 1.1 P_c$, $P = P_c$, $P = 0.85 P_c$, and $P = 0.75 P_c$, respectively. $G-T$ diagram for $P = 0.5 P_c$ (solid line), $P = P_c$ (dotted line), and $P = 1.5 P_c$ (dashed line).

Fig. 5. $P-v$ (left), $T-v$ (middle), and $G-T$ (right) diagrams in TOL gravity for $\alpha = 0.1$, $q = 1$, and $\beta = 1.5$ ($d = 7$). $P-v$ diagram, from top to bottom: $T = 1.2 T_c$, $T = 1.1 T_c$, $T = T_c$, $T = 0.85 T_c$, and $T = 0.75 T_c$, respectively. $T-v$ diagram, from top to bottom: $P = 1.2 P_c$, $P = 1.1 P_c$, $P = P_c$, $P = 0.85 P_c$, and $P = 0.75 P_c$, respectively. $G-T$ diagram for $P = 0.5 P_c$ (solid line), $P = P_c$ (dotted line), and $P = 1.5 P_c$ (dashed line).

Fig. 6. $P-v$ (left), $T-v$ (middle), and $G-T$ (right) diagrams in TOL gravity for $\beta = 1$, $q = 1$, and $\alpha = 0.5$ ($d = 7$). $P-v$ diagram, from top to bottom: $T = 1.2 T_c$, $T = 1.1 T_c$, $T = T_c$, $T = 0.85 T_c$, and $T = 0.75 T_c$, respectively. $T-v$ diagram, from top to bottom: $P = 1.2 P_c$, $P = 1.1 P_c$, $P = P_c$, $P = 0.85 P_c$, and $P = 0.75 P_c$, respectively. $G-T$ diagram for $P = 0.5 P_c$ (solid line), $P = P_c$ (dotted line), and $P = 1.5 P_c$ (dashed line).
holes decreases (see Figs. 9 and 12, middle panels). Also, the phase-transition region is an increasing function of \( \alpha \) and order of Lovelock gravity (see Figs. 9 and 12, left panels).

To conclude, the plotted figures show that the highest critical temperature and pressure, and the lowest critical volume and energy gap belong to Einstein gravity. On the other hand, the system needs a higher temperature to undergo phase transition in Einstein gravity, whereas the critical temperature decreases by considering higher orders of Lovelock gravity or increasing Lovelock coefficients. Considering the fact that increasing Lovelock parameters and/or adding higher orders of Lovelock gravity increase the power of gravity, one may say that, in stronger gravitational regimes, phase transitions take place at lower temperatures.

As for the effects of NED, we find the following results. It is evident that the critical temperature at which a swallowtail is formed (see Figs. 7, 11, and 12, right panels) and critical pressure (see Figs. 7, 11, and 12, left panels) are decreasing functions of the nonlinearity parameter, whereas the critical volume is an increasing function. In addition, the length of subcritical isobars is an increasing function of the nonlinearity parameter. This means that the single-phase region of small/large black holes is a decreasing function of \( \beta \) (see Figs. 7, 11, and 12, middle panels). Therefore, for higher values of the nonlinearity parameter (weak nonlinearity strength), black holes need to absorb less mass in order to undergo phase transition. Another issue that must be taken into account is the fact that, as \( \beta \) increases, the gap between isobars decreases, whereas for small \( \beta \) this gap is greater.
Because we take into account BI-type models, for large values of nonlinearity parameter they will lead to Maxwell theory. The obtained results show that the lowest critical temperature and pressure and highest critical volume belong to Maxwell theory. On the other hand, one can conclude that the
Fig. 12. $P-v$ diagram for $T = T_c$ (left), $T-v$ diagram for $P = P_c$ (middle), and $G-T$ diagram for $P = 0.5P_c$ (right) with $q = 1$ and $d = 7$. Solid line: EN gravity with $\beta = 0.5$, solid-bold line: EN gravity with $\beta = 1.5$, dotted line: GB gravity with $\beta = 0.5$, dotted-bold line: GB gravity with $\beta = 1.5$, dashed line: TOL gravity with $\beta = 0.5$, and dashed-bold line: TOL gravity with $\beta = 1.5$.

power of the nonlinearity causes the system to need a higher critical temperature to undergo a phase transition. This effect is the opposite of what was observed for gravity.

In addition, the energy gap between two phases, critical temperature, and Gibbs free energy are increasing functions of dimensions (Fig. 10). In other words, for higher dimensions, the system needs to have more energy to undergo a phase transition. It is worth mentioning that subcritical isobars (and also the critical region) are increasing functions of dimensions (see Fig. 10, middle panel), whereas critical volume is a decreasing function of dimensions. Finally, the ratio $\frac{P_{c,\text{SUB}}}{T_c}$ is a decreasing (increasing) function of $\alpha (\beta)$.

5. Conclusions

In this paper, we have considered both GB and TOL gravities in the presence of two classes of BI-type NED and studied their phase diagrams. We have considered the cosmological constant as pressure and its conjugating quantity as thermodynamical volume. The volume obtained for these cases is consistent with the topological structure of black holes and what was obtained previously [64]. It has been shown that, although both higher orders of Lovelock gravity and NED modify the thermodynamical quantities, the volume of the black holes in these cases is independent of these two modifications and only depends on the topology of the solutions. By employing a numerical method, we have calculated critical thermodynamical values for different cases and studied the effects of gravitational and nonlinear electromagnetic field parameters on these critical values. It has been shown that the black holes under consideration show similar behavior to van der Waals liquid–gas systems.

We have found that critical temperature and pressure are decreasing functions of orders and/or coefficients of Lovelock gravity, and critical volume and energy gap are increasing functions of them. In other words, black holes with higher orders of Lovelock gravity undergo phase transitions and reach stable states at lower temperatures compared to the Einstein case. On the other hand, the length of the subcritical isobars and region of the phase transition are increasing functions of the orders and/or coefficients of Lovelock gravity. The orders of Lovelock gravity denote different powers of the curvature scalar. From what we have obtained, one can argue that the critical temperature, pressure, and the region of the small/large black holes are decreasing functions of the power of the curvature scalar, while the critical volume, subcritical isobars, and region of the phase transition are increasing.
functions of it. Therefore, the power of the curvature scalar indeed has a crucial role in the variation of critical values. It is also noted that consideration of the higher orders of Lovelock theory increases the power of gravity.

It has been shown that critical temperature and pressure are decreasing functions of $\beta$ whereas the critical volume is an increasing function of it. In a comparison between BI-type and Maxwell electrodynamics, it was found that the lowest critical pressure and temperature and the largest critical volume belong to linear (Maxwell) theory.

The gravitational and electromagnetic fields have opposite effects on critical pressure and also phase transition. According to these results, one can say that phase transition is fundamentally related to both gravity and electrodynamics and their powers. As the power of the gravitational field (electromagnetic field) increases (decreases), the critical temperature decreases (increases). Interaction between the gravitational and electrodynamic sectors of a charged black hole may be found through the metric function. In addition, investigations of the phase diagrams confirm that they weaken each other’s effects.

As for dimensionality, we found that, as it increases, the energy gap, critical temperature, and critical pressure increase. The universal ratio of $\frac{P_c}{V_c}$ was a decreasing function of $\beta$ and an increasing function of dimensions and Lovelock parameter.

For higher orders of Lovelock gravity, we have higher values of entropy ($\alpha > 0$), which indicates that the thermodynamical system that the gravity describes contains a higher value of disorder. In addition, the consideration of higher orders of Lovelock gravity will cause the black holes to have a higher degree of complexity in their geometrical structure. If one considers the complexity of the geometrical structure as a disorder measurement of the system, it is logical to expect to see higher values of entropy for higher orders of Lovelock gravity. On the other hand, higher values of entropy mean that our systems will undergo phase transition at lower critical temperatures. This is the result that we have derived through our numerical calculations.

It has been shown [75] that GB gravity in the presence of a Maxwell field has no phase transition for arbitrary electric charge in higher than 5 dimensions, while we found that, on adjusting the nonlinearity parameter, $\beta$, there is a phase transition for various values of $q$ and $d \geq 5$. In other words, the nonlinearity parameter of electrodynamics has modified both the electrodynamics and the thermodynamical behavior of a black hole system.

Due to the opposite effects of gravitational power in Lovelock gravity and the power of nonlinearity in electrodynamics, it is constructive to find the dominant effect for various domains of thermodynamical systems. Also, one can study whether these two different fields cancel each other’s effects or not for certain values of $\alpha$ and $\beta$.

Considering the effects of Hawking radiation, we expect to see different behavior for higher orders of Lovelock gravity. This indicates that, in order to investigate black hole evaporation and phase transition of black holes, one could take both effects into account simultaneously. Another interesting issue is studying the connection between the complexity of the spacetime (topological structure of the black hole) and the entropy of the system, and the interpretation of entropy as a geometrical property. We leave these problems for future work.

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References

[1] D. Kastor, S. Ray and J. Traschen, Classical Quantum Gravity 26, 195011 (2009).
[2] C. V. Johnson, Classical Quantum Gravity 31, 205002 (2014).
[3] B. P. Dolan, [arXiv:1408.4023 [gr-qc]] [Search in tnsPIRE].
[4] B. P. Dolan, J. High Energy Phys. 10, 179 (2014).
[5] C. Teitelboim, Phys. Lett. B 158, 293 (1985).
[6] M. Henneaux and C. Teitelboim, Commun. Math. Phys. 98, 391 (1985).
[7] J. D. Brown and C. Teitelboim, Nucl. Phys. B 297, 787 (1988).
[8] D. Grumiller, R. McNees and J. Salzer, Phys. Rev. D 90, 044032 (2014).
[9] A. Larrañaga, [arXiv:0711.0012 [gr-qc]] [Search in tnsPIRE].
[10] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).
[11] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998).
[12] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998).
[13] B. P. Dolan, J. High Energy Phys. 10, 179 (2014).
[14] C. Teitelboim, Phys. Lett. B 158, 293 (1985).
[15] M. Henneaux and C. Teitelboim, Commun. Math. Phys. 98, 391 (1985).
[16] J. D. Brown and C. Teitelboim, Nucl. Phys. B 297, 787 (1988).
[17] D. Grumiller, R. McNees and J. Salzer, Phys. Rev. D 90, 044032 (2014).
[18] A. Larrañaga, [arXiv:0711.0012 [gr-qc]] [Search in tnsPIRE].
[19] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).
[20] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998).
[21] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998).
[22] B. P. Dolan, J. High Energy Phys. 10, 179 (2014).
[23] B. P. Dolan, J. High Energy Phys. 10, 179 (2014).
[24] J. Creighton and R. B. Mann, Phys. Rev. D 52, 4569 (1995).
[25] G. W. Gibbons, R. Kallosh and B. Kol, Phys. Rev. Lett. 77, 4992 (1996).
[26] M. Cvetic, G. Gibbons, D. Kubiznak and C. Pope, Phys. Rev. D 84, 024037 (2011).
[27] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, Phys. Rev. D 60, 064018 (1999).
[28] J. Y. Shen, R. G. Cai, B. Wang and R. K. Su, Int. J. Mod. Phys. A 22, 11 (2007).
[29] Q. Pan and B. Wang, Phys. Lett. B 693, 159 (2010).
[30] R. G. Cai, Z. Y. Nie and H. Q. Zhang, Phys. Rev. D 82, 066007 (2010).
[31] J. Jing, L. Wang, Q. Pan and S. Chen, Phys. Rev. D 83, 066010 (2011).
[32] M. Born and L. Infeld, Proc. R. Soc. Lond. A 143, 410 (1934).
[33] M. Born and L. Infeld, Proc. R. Soc. Lond. A 144, 425 (1934).
[34] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B 163, 123 (1985).
[35] R. R. Matsaev, M. A. Rahmanov and A. A. Tseytlin, Phys. Lett. B 193, 207 (1987).
[36] E. Bergshoff, E. Sezgin, C. Pope and P. Townsend, Phys. Lett. B 188, 70 (1987).
[37] N. Seiberg and E. Witten, J. High Energy Phys. 09, 032 (1999).
[38] J. Ambjorn, Y. M. Makeenko, J. Nishimura and R. J. Szabo, Phys. Lett. B 480, 399 (2000).
[39] Y. Kats, L. Motl and M. Padi, J. High Energy Phys. 12, 068 (2007).
[40] R. G. Cai, Z. Y. Nie and W. Y. Sun, Phys. Rev. D 78, 126007 (2008).
[41] D. Anninos and G. Pastras, J. High Energy Phys. 07, 030 (2009).
[42] H. H. Soleng, Phys. Rev. D 52, 6178 (1995).
[43] S. H. Hendi, J. High Energy Phys. 03, 065 (2012).
[44] S. H. Hendi and A. Sheykhi, Phys. Rev. D 88, 044044 (2013).
[45] K. S. Stelle, Gen. Rel. Gravitation 9, 353 (1978).
[46] W. Maluf, Gen. Rel. Gravitation 19, 57 (1987).
[47] M. Farhoudi, Gen. Rel. Gravitation 38, 1261 (2006).
[48] D. Lovelock, J. Math. Phys. 12, 498 (1971).
[49] D. Lovelock, J. Math. Phys. 13, 874 (1972).
[50] M. Aiello, R. Ferraro and G. Giribet, Phys. Rev. D 70, 104014 (2004).
[51] M. H. Dehghani and S. H. Hendi, Int. J. Mod. Phys. D 16, 1829 (2007).
[52] M. H. Dehghani, N. Alirejaved and S. H. Hendi, Phys. Rev. D 77, 104025 (2008).
[53] H. Maeda, M. Hassaine and C. Martinez, Phys. Rev. D 79, 044012 (2009).
[54] S. H. Hendi and B. Eslam Panah, Phys. Lett. B 684, 77 (2010).
[55] O. Miskovic and R. Olea, Phys. Rev. D 83, 024011 (2011).
[56] O. Miskovic and R. Olea, Phys. Rev. D 83, 064017 (2011).
[57] P. Li, R. H. Yue and D. C. Zou, Commun. Theor. Phys. 56, 845 (2011).
[58] S. H. Hendi, S. Panahiyan and H. Mohammdpour, Eur. Phys. J. C 72, 2184 (2012).
[59] S. Gunasekaran, D. Kubiznak and R. B. Mann, J. High Energy Phys. 11, 110 (2012).
[55] S. H. Hendi and M. H. Vahidinia, Phys. Rev. D 88, 084045 (2013).
[56] S. Chen, X. Liu, C. Liu and J. Jing, Chin. Phys. Lett. 30, 060401 (2013).
[57] D. C. Zou, S. J. Zhang and B. Wang, Phys. Rev. D 89, 044002 (2014).
[58] J. X. Mo and W. B. Liu, Eur. Phys. J. C 74, 2836 (2014).
[59] D. C. Zou, Y. Liu and B. Wang, Phys. Rev. D 90, 044063 (2014).
[60] R. Zhao, H. H. Zhao, M. S. Ma and L. C. Zhang, Eur. Phys. J. C 73, 2645 (2013).
[61] N. Altamirano, D. Kubiznak and R. B. Mann, Phys. Rev. D 88, 101502 (2013).
[62] S. H. Hendi, Ann. Phys. (N.Y.) 333, 282 (2013).
[63] S. H. Hendi, Ann. Phys. (N.Y.) 346, 42 (2014).
[64] S. H. Hendi, S. Panahiyan and B. Eslam Panah, [arXiv:1410.0352 [gr-qc]] [Search inSPIRE].
[65] S. H. Hendi, S. Panahiyan and E. Mahmoudi, Eur. Phys. J. C 74, 3079 (2014).
[66] S. H. Hendi and A. Dehghani, Phys. Rev. D 91, 064045 (2015).
[67] G. W. Gibbons, R. Kallosh and B. Kol, Phys. Rev. Lett. 77, 4992 (1996).
[68] J. D. E. Creighton and R. B. Mann, Phys. Rev. D 52, 4569 (1995).
[69] D. A. Rasheed, [arXiv:hep-th/9702087] [Search inSPIRE].
[70] N. Breton, Gen. Rel. Gravitation 37, 643 (2005).
[71] D. Kastor, S. Ray and J. Traschen, Classical Quantum Gravity 27, 235014 (2010).
[72] R. G. Cai, L. M. Cao, L. Li and R. Q. Yang, J. High Energy Phys. 09, 005 (2013).
[73] D. C. Zou, S. J. Zhang and B. Wang, Phys. Rev. D 89, 044002 (2014).
[74] Z. Sherkatghanad, B. Mirza, Z. Mirzaeyan and S. A. H. Mansoori, [arXiv:1412.5028 [gr-qc]] [Search inSPIRE].
[75] R. G. Cai, L. M. Cao, L. Li and R. Q. Yang, J. High Energy Phys. 09, 005 (2013).