PATH REPRESENTATION OF MAXIMAL PARABOLIC KAZHDAN–LUSZTIG POLYNOMIALS

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Abstract. We provide simple rules for the computation of Kazhdan–Lusztig polynomials in the maximal parabolic case. They are obtained by filling regions delimited by paths with “Dyck strips” obeying certain rules. We compare our results with those of Lascoux and Schützenberger.

1. Introduction

Kazhdan–Lusztig polynomials were introduced in [16] as coefficients of the change of basis from the standard basis of the Hecke algebra to a new one, the Kazhdan–Lusztig basis. The latter was motivated by connections to the representation theory of Weyl groups [30] and singularities of Schubert varieties [18] (see e.g. [3] and references therein). However, it reappeared since then in multiple contexts: algebraic combinatorics [17] Lie groups [23], the representation theory of Verma modules [8, 2], and quantum groups [22].

In [11] Deodhar introduced the concept of parabolic Kazhdan–Lusztig polynomials. Roughly, they are associated to certain quotients of the regular representation of the Hecke algebra (q-deformation of the induced representation of one-dimensional representations of parabolic subgroups of the Coxeter group) in the same way as the usual Kazhdan–Lusztig polynomials are associated to the regular representation, and the corresponding bases are projections of certain subsets of the Kazhdan–Lusztig basis. Here we are concerned with type A and a maximal parabolic subgroup, namely with Weyl group $S_N$ and the parabolic subgroup $S_K \times S_{N-K}$.

The maximal parabolically induced representation of the Hecke algebra factors through the Temperley–Lieb algebra [34] and one expects simpler combinatorics than in the general case. Lascoux and Schützenberger [21] gave an algorithm to compute the Kazhdan–Lusztig polynomial for Grassmannian permutations, which is equivalent to the maximal parabolic case (see [35] for a geometric interpretation). Also, there is a natural graphical description of the basis and of the Temperley–Lieb action in terms of tangles and link patterns, as used in models of two-dimensional statistical mechanics [1, 34, 29, 25] and in knot theory [15]. There is an abundant mathematical literature (see e.g. [19, 13, 4, 20, 5, 6]) which provides explicit combinatorial formulae for some of these classes of Kazhdan–Lusztig polynomials.

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Due to the choice of the projection map (see Section 2), we have two types of parabolic Kazhdan–Lusztig polynomials studied in [21, 27, 6]. The goal of the present paper is to provide a unified, self-contained treatment of maximal parabolic Kazhdan–Lusztig polynomials of both types in the language of paths, similar to the one used by Brenti [6]. The main result is their computation according to two graphical rules, denoted by I and II, rule II being equivalent to Brenti’s result. The plan is as follows. In section 2, we introduce Kazhdan–Lusztig polynomials and their maximal parabolic analogues and explain their duality. Section 3 is the heart of the paper, in which we provide diagrammatic rules to compute the maximal parabolic polynomials. In particular, the new rule (I) should be related to the one given by Lascoux and Schützenberger in [21]; and indeed, we provide a bijection between them in section 4. We try to stay close to the conventions of the mathematical physics literature, which is where our motivation comes from. More specifically, one has on the hand the study of the Temperley–Lieb algebra via “link patterns” and its factorization properties [9, 26, 19, 10], which is relevant in calculations that are performed in integrable loop models; on the other hand, there are other explicit formulae [12, 28] which are made in the “standard basis” of spin chains; and we expect our formulae to be useful in connecting these different recent developments in integrable models.

2. Kazhdan–Lusztig polynomials

2.1. Definition. Given a positive integer \( N \), we consider the symmetric group \( S_N \) with generators \( s_i, i = 1, \ldots, N - 1 \). Denote by \( |v| \) the length of \( v \in S_N \), that is \( |v| := \#\{i < j, v(i) > v(j)\} \). \( S_N \) is endowed with the (strong) Bruhat order \( \leq \), that is \( v \leq w \) iff \( w \) can be obtained by a series of multiplications on the left or right by a generator \( s_i \) which each increase length by one.

The Hecke algebra \( H_N \) is the unital associative algebra over the ring \( R := \mathbb{Z}[t, t^{-1}] \) with generators \( T_i, i = 1, \ldots, N - 1 \), and relations

\[
\begin{align*}
(T_i - t)(T_i + t^{-1}) &= 0, & 1 \leq i < N, \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_i+1, & 1 \leq i < N - 1, \\
T_iT_j &= T_jT_i, & |i - j| > 1.
\end{align*}
\]

The standard basis \( (T_v)_{v \in S_N} \) of the Hecke algebra is obtained by writing \( T_v := T_{i_1} \cdots T_{i_k} \) if \( v = s_{i_1} \cdots s_{i_k} \) is a reduced word in the elementary transpositions \( s_i \) (see section 7 of [14]).

Define \( a \to \overline{a} \) to be the involutive ring automorphism of \( H_N \) such that \( \overline{T_i} = T_i^{-1} \) and \( \overline{t} = t^{-1} \). Then

**Theorem 1** (Kazhdan, Lusztig [16]). There exists a unique basis \( (C_w)_{w \in S_n} \) of \( H_N \) such that \( C_w = \overline{C_w} \) and the matrix of change of basis \( (P_{v,w}) \) from the \( T_v \) to the \( C_w \) is “upper triangular” w.r.t. Bruhat order, i.e.

\[
C_w = \sum_{v \in S_N, v \leq w} P_{v,w}(t^{-1})T_v
\]

where the polynomials \( P_{v,w}(t^{-1}) \in t^{-1}\mathbb{Z}[t^{-1}] \) if \( v < w \) and \( P_{v,v} = 1 \).
In fact, $\deg P_{v,w} \leq |w| - |v|$, and the Kazhdan–Lusztig (KL) polynomials are by definition the polynomials $t^{|w| - |v|}P_{v,w}(t^{-1}) \in \mathbb{Z}[t^2]$.

2.2. Maximal parabolic case. Given $0 \leq K \leq N$, we now consider the subgroup $S_K \times S_{N-K} \subset S_N$ with generators $s_i$, $i = 1, \ldots, K - 1, K + 1, \ldots, N - 1$. The set of left cosets $S_N/(S_K \times S_{N-K})$ has a natural induced order: $x \leq y$ iff there exist $v \in x$, $w \in y$ such that $v \leq w$, and a length: $|x| = \min_{v \in x} |v|$.

Let us define $\mathcal{M}_{N,K}$ to be a free $R$-module with basis indexed by $S_N/(S_K \times S_{N-K})$: $\mathcal{M}_{N,K} := \langle m_x, x \in S_N/(S_K \times S_{N-K}) \rangle$. The projection $\varphi$ from $S_N$ to $S_N/(S_K \times S_{N-K})$ induces two natural projection maps $\varphi^\pm$ from $\mathcal{H}_N$ to $\mathcal{M}_{N,K}$, given by $\varphi^\pm(T_i) := (\pm t^{\pm 1})^{|v| - |w|}m_{\varphi(v)}$. Fix $\epsilon \in \{+, -\}$. In order to define a representation of $\mathcal{H}_N$ on the $R$-module $\mathcal{M}_{N,K}$, we require that $\varphi^\epsilon$ commute with the action of the Hecke algebra (cf. lemma 2.2 of [11]), where the latter acts on itself by left multiplication; this leads to the following action of the generators $T_i$ on $\mathcal{M}_{N,K}$:

\begin{equation}
T_im_x = \begin{cases} 
\epsilon t^i m_x & s_i x = x, \\
m_{s_i x} & s_i x \neq x, |s_i x| > |x| \\
(t - t^{-1})m_x + m_{s_i x} & s_i x \neq x, |s_i x| < |x|.
\end{cases}
\end{equation}

This endows $\mathcal{M}_{N,K}$ with the structure of an $\mathcal{H}_N$-module, which is denoted by $\mathcal{M}^\epsilon_{N,K}$.

Similarly, requiring that $\varphi^\epsilon$ commute with the bar involution defines uniquely its action on $\mathcal{M}^\epsilon_{N,K}$.

We can now define parabolic analogues of KL basis and polynomials:

**Theorem 2** (Deodhar [11]). There exists a unique basis $(C_x^\pm)_{x \in S_N/(S_K \times S_{N-K})}$ of $\mathcal{M}^\pm_{N,K}$ such that $C_x^\pm = C_x^\mp$ and the matrix of change of basis $(P_{x,y}^\pm)$ from the $m_x$ to the $C_x^\pm$ is “upper triangular” i.e.

$$C_y^\pm = \sum_{x \in S_N/(S_K \times S_{N-K})} P_{x,y}^\pm(t^{-1})m_x$$

where the polynomials $P_{x,y}^\pm(t^{-1}) \in t^{-1}\mathbb{Z}[t^1]$ if $x < y$ and $P_{x,x} = 1$.

In fact, $\deg P_{x,y}^\pm \leq |y| - |x|$, and the parabolic Kazhdan–Lusztig polynomials are by definition the $t^{|y| - |x|}P_{x,y}^\pm(t^{-1}) \in \mathbb{Z}[t^2]$. Here we prefer to use directly the polynomials $P_{x,y}^\pm(t^{-1})$.

2.3. Combinatorial description. There are various ways to describe explicitly the cosets in $S_N/(S_K \times S_{N-K})$. We are of course mostly interested in their *path* representation, but we discuss in this section other useful descriptions.

Let $\epsilon \in \{-, +\}$. We consider the following sets:

\begin{enumerate}
\item[(0)] $S_N/(S_K \times S_{N-K})$
\item[(1)] Binary strings, i.e. elements of $\{1, 2\}^N$, such that there are $K$ 1’s and $N - K$ 2’s.
\item[(2)] Paths from $(0, 0)$ to $(N, \epsilon(2K - N))$ with steps $(1, \pm 1)$.
\item[(3)] Ferrers diagrams inside the rectangle $K \times (N - K)$.
\end{enumerate}
(4) **Link patterns** with at most \( \min(K, N - K) \) pairings, where link patterns are planar pairings of a subset of \( \{1, \ldots, N\} \) in such a way that unpaired vertices belong to the infinite connected component.

(5) **(anti)Grassmannian permutations**, that is permutations \( \sigma \) such that \( 1 \leq i < j \leq K \) or \( K + 1 \leq i < j \leq N \) implies \( \epsilon \sigma(i) > \epsilon \sigma(j) \).

(6) **Standard Young tableaux with at most two rows** (resp. two columns for \( \epsilon = + \)) and with \( N \) boxes, whose second row (resp. column) is of length less or equal to \( \min(K, N - K) \).

as well as the following maps between these sets:

\( (0) \rightarrow (1) \): such binary strings are the orbits under the natural action of \( S_N \) on \( \{1, 2\}^N \), with representative \( (1, \ldots, 1, 2, \ldots, 2) \). The latter has stabilizer \( S_K \times S_{N-K} \).

\( (1) \rightarrow (2) \): a sequence \( v \in \{1, 2\}^N \) is identified with the path with \( i \)th step \( (1, \epsilon(-1)^{i+v_i}) \).

\( (2) \rightarrow (3) \): to a path is associated the (45 degrees rotated) Ferrers diagram located between it and the smallest path for \( \leq \) (corresponding to the binary string \( (1, \ldots, 1, 2, \ldots, 2) \) and to the coset of the identity; it is the lowest path for \( \epsilon = - \), the highest path for \( \epsilon = + \)).

\( (2) \rightarrow (4) \): pair midpoints of steps of equal height such that the horizontal segment between them stays strictly below the path. (see the example below)

\( (0) \rightarrow (5) \): in each coset \( x \), there is exactly one Grassmannian permutation, denoted by \( s(x) \); it is the “shortest representative” (of shortest length). Note that by definition, \( |x| = |s(x)| \), and \( x \preceq y \) iff \( s(x) \preceq s(y) \). Inversely there is exactly one antiGrassmannian permutation in each coset: it is the “longest representative”, and can be written \( s(x) \tilde{w}_0 \), where \( \tilde{w}_0 \) is the longest element of \( S_K \times S_{N-K} \), namely \((K \ldots 1 \quad 1 \ldots K \quad K+1 \ldots N)\).

\( (5) \rightarrow (6) \): applying the Robinson–Schensted algorithm to \( s(x) \tilde{w}_0 \) results in a pair of Young tableaux of same shape with at most 2 rows; keep only the first tableau, the second one being entirely fixed by its shape, say \((N-i,i)\), to be:

\[
\begin{array}{cccccc}
1 & 2 & \cdots & K & K+i+1 & \cdots & N \\
K+1 & \cdots & K+i \\
\end{array}
\]

Similarly, applying the Robinson–Schensted algorithm to \( s(x) \tilde{w}_0 \) results in a pair of Young tableaux of same shape with at most 2 columns; keep only the first tableau, the second one being entirely fixed by its shape. Note that these two tableaux are not transpose of each other.

**Lemma 1.** The maps described above are bijections.

The proofs are standard (see e.g. [32] and online supplements at [31]).

**Example 1.** We choose the \( \epsilon = - \) convention to draw (2), (4).

\( (2) \leftrightarrow (1) \): 

\[
\begin{array}{cccccccccccc}
1 & 2 & \cdots & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

\( \leftrightarrow (2, 1, 1, 2, 2, 1, 2, 1, 1, 1) \)
In what follows, we shall mostly use the path representation, or interchangeably the closely related Ferrers diagram representation. More precisely the bijection to paths with the sign convention $\epsilon \in \{-, +\}$ will be used to index bases of $\mathcal{M}_{N,K}^\epsilon$. The set of paths from $(0, 0)$ to $(N, 2K - N)$ will be denoted by $\mathcal{P}_{N,K}$.

It is perhaps useful to rewrite the action (1) of the Hecke algebra on $\mathcal{M}_{N,K}^\pm$ in terms of local changes of paths: (only the steps $i$ and $i + 1$ are depicted)

$$
\epsilon = - : \quad T_i m \downarrow \ldots = -t^{-1} m \downarrow \ldots \\
T_i m \downarrow \ldots = -t^{-1} m \downarrow \ldots \\
T_i m \uparrow \ldots \downarrow \ldots = m \downarrow \ldots \\
T_i m \uparrow \ldots \downarrow \ldots = (t - t^{-1}) m \downarrow \ldots + m \downarrow \ldots \\
\epsilon = + : \quad T_i m \uparrow \ldots \downarrow \ldots = t m \downarrow \ldots \\
T_i m \uparrow \ldots \downarrow \ldots = t m \downarrow \ldots \\
T_i m \uparrow \ldots \downarrow \ldots = m \downarrow \ldots \\
T_i m \uparrow \ldots \downarrow \ldots = (t - t^{-1}) m \downarrow \ldots + m \downarrow \ldots
$$
In terms of the associated Ferrers diagrams, the third and fourth lines involves adding and removing a box, respectively.

We have the following additional easy lemma:

**Lemma 2.** Let \( x, y \in \mathcal{S}_N / (\mathcal{S}_K \times \mathcal{S}_{N-K}) \) and call \( \alpha, \beta \) the associated paths with convention \( \epsilon \). Then \( x \leq y \) iff \( \alpha \) is below \( \beta \) for \( \epsilon = - \), above \( \beta \) for \( \epsilon = + \); and \( |x| \) is the number of boxes of the corresponding Ferrers diagram, also denoted by \( |\alpha| \) in what follows.

**Proof.** Let us prove the case \( \epsilon = - \). If the path \( \beta \) is above \( \alpha \) then \( y \) can be obtained from \( x \) by a series of multiplications on the left by elementary transpositions (as mentioned above, this corresponds to adding one box at a time on top of the path). Therefore \( x \leq y \). Conversely, assume \( x \leq y \) i.e. \( u := s(x) \leq v := s(y) \). Using the simple lemma that \( u \leq v \) and \( v \) Grassmannian implies \( u \) Grassmannian, we can restrict ourselves to the case of one move i.e. \( |v| = |u| + 1 \). There are two possible moves: multiplication on the left by an elementary transposition, which is exactly the case treated above; and multiplication on the right. But because \( v \) can only have descents between \( K \) and \( K + 1 \), multiplication on the right can only be by \( s_K \), and only if this descent is not already there i.e. \( u = 1 \), in which case case multiplication on the right and left are the same.

The first part of the reasoning also shows that increasing the length by one is the same as adding one box under the path, which leads to the second part of the lemma. \( \square \)

### 2.4. Connection between KL and parabolic KL polynomials.

Since the projections \( \varphi^\pm \) commute with the Hecke action and with the bar involution, images of elements \( C_w \) of the Kazhdan–Lusztig basis of \( \mathcal{H}_N \) under \( \varphi^\pm \) are natural candidates for their parabolic counterparts \( C^\pm_w \). And indeed, one can show that \( \varphi^\epsilon(C_w) = C^\epsilon_{\varphi(w)} \) if \( w \) is the shortest (Grassmannian) representative of its coset for \( \epsilon = - \), and the longest representative for \( \epsilon = + \). Note however that the definitions in Theorems 1 and 2 of KL bases break the symmetry in the definition of the Hecke algebra between \( t \) and \( -t^{-1} \) (by requiring the coefficients to be polynomials in \( t^{-1} \)) which is therefore not apparent in the resulting formulae for parabolic KL polynomials:

**Proposition 1** (Deodhar [11]).

\[
P^+_{x,y} = P_{v,w} \quad v = s(x)\tilde{w}_0, \quad w = s(y)\tilde{w}_0 \text{ longest representatives} \\
P^-_{x,y} = \sum_{v \in x} (-t)^{|x|-|v|} P_{v,w} \quad w = s(y) \text{ shortest representative}
\]

### 2.5. Duality.

There is a general duality satisfied by KL polynomials (Theorem 3.1 of [16]). Let \( w^0 \) be the longest element of \( \mathcal{S}_N \), namely \( w^0 = (\frac{N}{1}, \ldots, \frac{1}{N}) \). Reformulated in our language, this result becomes
Theorem 3 (Kazhdan, Lusztig [16]). The following inversion formulae hold:
\[
\sum_{w \in S_N} (-1)^{|w|} P_{u,w} P_{w^{\alpha_\epsilon},w^0} = \delta_{u,v}
\]
\[
\sum_{w \in S_N} (-1)^{|w|} P_{u,v} P_{w^{\alpha_\epsilon}_u,w^0} = \delta_{v,w}
\]

For our purposes it is more convenient to have \(w^0\) act on the right, which amounts to using the opposite product, or to applying the small

Lemma 3. Let \(u^t = w^0 u w^0\). Then \(P_{u^t,v} = P_{u,v}\)

Proof. Firstly, \(\bar{z}\) preserves the Bruhat order. Secondly, extend \(\bar{z}\) into an involution of \(\mathcal{H}_N\) with \(T^\bar{z}_v = T_v\). Noting that \(\bar{z}\) and bar involutions commute, we conclude that \(C_{w^0} = (C_w)^\dagger\), hence the result.

Recall that we also have the longest element in \(S_K \times S_{N-K}\): \(\tilde{w}_0 = (1 \cdots K \cdots N, 1 \cdots K \cdots 1 \cdots N-K \cdots N)\).

Write \(w^0 = \eta \tilde{w}_0\), where \(\eta = (N-K+1 \cdots N \cdots \cdots K \cdots 1 \cdots 1)\).

We now switch as promised to the path indexation. All the paths in this section are in \(\mathcal{P}_{N,K}\), i.e., from \((0,0)\) to \((N,2K-N)\). Let \(\gamma\) be such a path. According to lemma 1, they can be interpreted as either \((\epsilon = -)\) a shortest representative in \(S_N/(S_{N-K} \times S_K)\), say \(w\), or \((\epsilon = +)\) a longest representative in \(S_N/(S_K \times S_{N-K})\), say \(w'\). The key remark is that we have \(w' = w w^0\): indeed multiplying by \(\eta\) on the right flips the path upside down (following the different convention for paths depending on \(\epsilon\)), and multiplying by \(\tilde{w}_0\) turns shortest into longest representative. Therefore, given two paths \(\beta, \gamma \in \mathcal{P}_{N,K}\), one can associate to them \(v\) and \(w\), the shortest representatives as above, and write, applying Proposition 1:

\[
\sum_{\alpha \in \mathcal{P}_{N,K}} (-1)^{|\alpha|+|\beta|} P_{\alpha_\beta}^- P_{\alpha_\gamma}^+ = \sum_{z \in S_N/(S_K \times S_{N-K})} \sum_{u \in z} (-1)^{|u|+|v|} P_{u,v} P_{z|w^0,ww^0} \quad \text{by (2.3.g) of [16]}
\]

where in the application of (2.3.g) of [16] we set \(x = uw^0\), \(y = ww^0\) and use the opposite product.

Writing that \(P_{uw^0,ww^0} = P_{w^{\mu_\nu},w^0}\) (lemma 3) leads to the second identity of theorem 3, so that

\[
\sum_{\alpha \in \mathcal{P}_{N,K}} (-1)^{|\alpha|+|\beta|} P_{\alpha_\beta}^- P_{\alpha_\gamma}^+ = \delta_{\beta,\gamma}
\]

Note that \((-1)^{|\alpha|+|\beta|} P_{\alpha_\beta}^-(t^{-1}) = P_{\alpha_\beta}^-(t^{-1})\). We reach the result

Theorem 4.

\[
\sum_{\alpha \in \mathcal{P}_{N,K}} P_{\alpha_\beta}^- (t^{-1}) P_{\alpha_\gamma}^+ (t^{-1}) = \delta_{\beta,\gamma}
\]
In other words, the change of basis for $\mathcal{M}_{N,K}^+$ is up to $t \to -t$ the inverse transpose of the one for $\mathcal{M}_{N,N-K}^-$, which is just a manifestation of usual (linear algebra) duality.

3. Path representation

3.1. Dyck strips. A Dyck path of length $2l, l \geq 0$, is a path from some $(x, y) \in \mathbb{Z}^2$ to $(x + 2l, y)$ and not crossing below the horizontal line at height $y$. A Dyck strip of length $2l + 1$ is obtained by putting unit boxes (45 degrees rotated) whose centers are at the vertices of a Dyck path of length $2l$ (see some examples on Fig 1).

![Figure 1. Some Dyck strips.](image)

Hereafter, a box $(x, y)$ means a unit box whose center is $(x, y)$. Let $b$ be a box $(x, y)$. Four boxes $(x \pm 1, y \pm 1)$ are neighbors of $b$. The box $(x + 1, y + 1)$ is said to be NE (north-east) of $b$ and similarly the other three boxes are NW, SW and SE of $b$. The two boxes $(x, y \pm 2)$ are said to be just above or just below $b$.

We now define two relations on the set of Dyck strips as follows. Given an ordered pair of such Dyck strips $(D, D')$, we say that they satisfy rule I/II iff:

**Rule I:** If there exists a box of $D$ just below a box of $D'$, then all boxes just below a box of $D'$ belong to $D$.

**Rule II:** If there exists a box of $D'$ just above, NW or NE of a box of $D$, then all boxes just above, NW and NE of a box of $D$ belong to $D$ or $D'$.

Roughly, Rule I (resp. Rule II) means that we are allowed to pile Dyck strips of smaller or equal (resp. longer) length on top of a Dyck strip.

Let $\alpha, \beta \in \mathcal{P}_{N,K}$ be two paths as defined in section 2.3. We fill the closed domain between these two paths with Dyck strips (such that no Dyck strips overlap, and every unit box is filled). Let us denote by $\text{Conf}(\alpha, \beta)$ the set of all such possible configurations of Dyck strips, and $\text{Conf}^{I/II}(\alpha, \beta)$ the subset of configurations satisfying rule I/II. We denote the number of Dyck strips in a configuration $\mathcal{D}$ by $|\mathcal{D}|$.

![Figure 2. Examples of stacks of Dyck strips satisfying rule I (left) and rule II (right).](image)
Definition 1. The generating function of Dyck strips for the paths $\alpha < \beta$ in $P_{N,K}$ is defined as

$$Q_{\alpha,\beta}^{X_\epsilon}(t^{-1}) = \sum_{D \in \text{Conf}^X(\alpha,\beta)} t^{-|D|}.$$ 

where $X = I, II$ and $\epsilon = \pm$ is the order convention as in Lemma 2. $Q_{\alpha,\alpha}^{X_\epsilon}(t^{-1}) = 1$ and $Q_{\alpha,\beta}^{X_\epsilon}(t^{-1}) = 0$ if $\alpha \not\leq \beta$.

Note that in the case of Rule II, we have at most one configuration for given paths $\alpha$ and $\beta$ due to the condition $l(D) < l(D')$. In other words, the $Q_{\alpha,\beta}^{II}$ are monomials.

Recall that according to Lemma 2, $\alpha \leq \beta$ means $\alpha$ is pictorially above (resp. below) $\beta$ for $\epsilon = +$ (resp. $\epsilon = -$). Therefore, it is obvious that

Lemma 4. $Q_{\alpha,\beta}^{X_+}(t^{-1}) = Q_{\beta,\alpha}^{X_-}(t^{-1})$.

Example 2. When $(\alpha, \beta) = (111212222, 211212221)$,

$$\text{Conf}^I(\alpha, \beta) = \{ \text{other configurations} \}$$

The corresponding generating function is $Q_{\alpha,\beta}^{I_+}(t^{-1}) = t^{-8}(1 + 2t^2 + t^4 + t^6)$.

The relations among the Kazhdan–Lusztig polynomials $P_{\alpha,\beta}^{\pm}$ and the generating functions $Q_{\alpha,\beta}^{X_\epsilon}$ that we shall establish in subsequent sections are summarized as:

Theorem 5 $P_{\alpha,\beta}^- = Q_{\alpha,\beta}^{II_-} \iff \text{transpose} \iff Q_{\alpha,\beta}^{II_+}$

Corollary 1 $Q_{\alpha,\beta}^{I_+} = Q_{\alpha,\beta}^{I_-} \iff \text{inverse}$

Theorem 6 $Q_{\alpha,\beta}^{I_-} \iff \text{transpose} \iff P_{\alpha,\beta}^+= Q_{\alpha,\beta}^{I_+}$

Corollary 2 $P_{\alpha,\beta}^+ = Q_{\alpha,\beta}^{I_+}$

3.2. On the module $M_{N,K}$. For the purposes of this section, we identify a path and binary string of 1 and 2 with convention \(-\) as in section 2.3. We denote by $\alpha = \alpha_1 \ldots \alpha_N$ a binary string of $N$ letters.

Definition 2. For given paths $\alpha, \beta \in P_{N,K}$, we define

$$d(\alpha, \beta) := \#\{ i : \alpha_i \neq \beta_i \}/2.$$ 

Recall that to a path $\beta$ can also be associated a link pattern, that is a set of pairings between indices (possibly leaving some of them unpaired). Each such pairing corresponds to a $\ldots, 2, \ldots, 1, \ldots$ in the corresponding binary string. Define a set of paths by $F(\beta)$ as follows:

$F(\beta) := \{ \alpha \leq \beta : \text{Some pairings of } \beta \text{ are flipped} \}$.

where by flipped we mean replacing $\ldots, 2, \ldots, 1, \ldots$ with $\ldots, 1, \ldots, 2, \ldots$ in the binary string of $\beta$. If the number of pairings of $\beta$ is $r$, then the cardinality of $F(\beta)$ is $2^r$. 


Example 3. \( \beta = 2121 \), that is the link pattern \( \xrightarrow{\text{---}} \). \( \mathcal{F}(\beta) = \{2121, 1221, 2112, 1212\} \).

Remark 1. The set \( \mathcal{F}(\beta) \) can be rephrased in terms of Dyck strips. Let us fix a path \( \beta \). \( \mathcal{F}(\beta) \) is the set of paths \( \{\alpha : \alpha \leq \beta\} \) (with the \(-\) convention) such that the region between them, denoted (following the notation of skew Ferrers diagrams) \( \beta/\alpha \), is filled with Dyck strips according to Rule II.

Note that when \( \alpha \in \mathcal{F}(\beta) \), \( d(\alpha, \beta) \) is equal to the number of flipped pairings in \( \beta \).

On \( \mathcal{M}_{N,K} \), let us define

\[
\tilde{C}_\beta := \sum_{\alpha \in \mathcal{F}(\beta)} t^{-d(\alpha, \beta)} m_\alpha
\]

Let \( s_1s_2 \ldots s_i = s(\beta) \) be a reduced word of the shortest coset representative \( s(\beta) \). We denote this ordered product by \( \prod_{s_i \in s(\beta)} s_i \).

Proposition 2 (see also [19]). The basis \( \tilde{C}_\beta \) for \( \beta \in \mathcal{P}_{N,K} \) is factorized as

\[
\tilde{C}_\beta = \left( \prod_{s_i \in s(\beta)} (T_i + t^{-1}) \right) m_{\beta_0}.
\]

where \( \beta_0 = (1 \ldots 12 \ldots 2) \in \mathcal{P}_{N,K} \).

Proof. We prove the proposition by induction on \( \beta \). We have \( \tilde{C}_{\beta_0} = m_{\beta_0} \) and \( \tilde{C}_{\beta_N, \beta_0} = m_{s_N, \beta_0} + t^{-1}m_{\beta_0} = (T_K + t^{-1})m_{\beta_0} \).

Let \( \beta, \beta' \in \mathcal{P}_{N,K} \) and suppose the statement holds true for all \( \beta' < \beta \). Now let \( s(\beta) = s(s(\beta')) \) with \( |\beta| = |\beta'| + 1 \). This condition is equivalent to \( (\beta_i, \beta_{i+1}) = (\beta'_i, \beta'_i) = (1, 2) \).

Note that the contribution of a pairing to \( m_{\alpha'} \) for \( \alpha' \in \mathcal{F}(\beta') \) is independent of each other. Therefore, it is enough to check the action of \( T_i + t^{-1} \) on a partial path of \( \alpha' \) involving \( \alpha'_i \) and \( \alpha'_{i+1} \). We have three cases.

(i) Suppose \( \alpha'_i = 1 \) is unpaired and \( (\alpha'_{i+1}, \alpha'_i) = (2, 1) \) is a pairing. In this case, \( \alpha''_i = 1 \) holds true for all \( \alpha'' \in \mathcal{F}(\beta) \).

\[
(T_i + t^{-1})(m_{\ldots 12 \ldots 1} + t^{-1}m_{\ldots 11 \ldots 2}) = m_{\ldots 21 \ldots 1} + t^{-1}m_{\ldots 12 \ldots 1}
\]

where \( T_i \) acts on the underlined places. Now \( \alpha_j = 1 \) becomes an unpaired 1, and \( (\alpha_i, \alpha_{i+1}) \) becomes a pairing in \( \alpha \). Suppose \( \alpha'_{i+1} = 2 \) is unpaired and \( (\alpha'_i, \alpha_i) = (2, 1) \) is a pairing. Similarly as above, we have \( \alpha_j = 2 \) is unpaired and \( (\alpha_i, \alpha_{i+1}) = (2, 1) \) is a pairing.

(ii) \( (\alpha'_k, \alpha'_i) = (\alpha'_{i+1}, \alpha'_i) = (2, 1) \) with \( k < i, i + 1 < l \) and they are pairings.

\[
(T_i + t^{-1})(m_{\ldots 21 \ldots 1} + t^{-1}m_{\ldots 12 \ldots 1} + t^{-1}m_{\ldots 21 \ldots 2} + t^{-2}m_{\ldots 12 \ldots 2})
\]

\[
= m_{\ldots 21 \ldots 1} + t^{-1}m_{\ldots 21 \ldots 2} + t^{-1}m_{\ldots 12 \ldots 2} + t^{-2}m_{\ldots 12 \ldots 2},
\]

This implies that \( (\alpha_k, \alpha_l) = (\alpha_i, \alpha_{i+1}) = (2, 1) \) are pairings in \( \alpha \).
(iii) Suppose both \( \alpha'_i = 2 \) and \( \alpha'_{i+1} = 1 \) are unpaired. We have
\[
(T_i + t^{-1})m_{\ldots 2\ldots} = m_{\ldots 2\ldots} + t^{-1}m_{\ldots 2\ldots},
\]
which means \((\alpha_i, \alpha_{i+1}) = (2, 1)\) is a pairing.

In all cases, obtained expression gives us the set \( \mathcal{F}(\beta) \) and desired coefficients.

**Proposition 3.** The basis \( \tilde{C}_\beta \) is the Kazhdan–Lusztig basis \( C^-_\beta \).

**Proof.** Note that \( \tilde{C}_\beta \) is invariant under the bar involution since \( \bar{T}_i + t^{-1} = T_i + t^{-1} \) and \( \bar{m}_0 = m_0 \).

From Proposition 2, it is clear that the coefficient for \( m_\beta \) is 1 and all other coefficients for \( m_\alpha \) are in \( t^{-1}N[t^{-1}] \) for \( \alpha < \beta \). □

When the region \( \beta/\alpha \) is filled with Dyck strips via Rule II, it is clear that \( d(\alpha, \beta) \) is equal to the number of Dyck strips. From Proposition 3 together with Remark 1, we have the (see also \([6]\))

**Theorem 5.** The Kazhdan–Lusztig polynomial \( P^-_{\alpha,\beta} \) is given by
\[
P^-_{\alpha,\beta}(t^{-1}) = Q^-_{\alpha,\beta}(t^{-1}) = t^{-d(\alpha,\beta)}, \quad \alpha \in \mathcal{F}(\beta).
\]

As mentioned in the introduction, the parabolic KL basis of \( \mathcal{M}^-_{N,K} \) is closely related to the formulation of the Temperley–Lieb algebra in terms of tangles as used in knot theory \([15]\). Indeed, in this basis, the operators \( T_i + t^{-1} \) which appeared in the proof, and which are the usual generators in terms of which the Temperley–Lieb algebra is formulated, have a natural graphical action on link patterns: they correspond to pasting a \( \bigcirc \) to the link pattern, i.e., reconnecting the existing pairings between neighboring sites \( i \) and \( i + 1 \) and creating a new pairing \( (i, i + 1) \).

Descriptions of \( P^- \) that are analogous to Thm. 5 appear under various guises in the literature; see \([7, \text{Eq. (5.12)}]\) for an alternative form of it in terms of oriented cup diagrams, \([33, \text{Lemma 2.2}]\) for an interpretation of this formula in terms of Springer fibres; \([26, \text{Sect. 8}]\) for an appearance in statistical loop models; and \([24, \text{Sect. 8}]\) for a connection to the representation theory of the Brauer algebra.

### 3.3. The inversion formula.

In preparation for the study of the module \( \mathcal{M}^-_{N,K} \), we invert the matrix \( Q^-_{\alpha,\beta} \).

**Theorem 6.**
\[
\sum_{\beta \in \mathcal{P}^-_{N,K}} Q^-_{\alpha,\beta}(t^{-1})Q^-_{\beta,\gamma}(t^{-1})(-1)^{|\beta|+|\gamma|} = \delta_{\alpha,\gamma}
\]

**Proof.** If \( \alpha \nless \gamma \) the l.h.s. is zero, and if \( \alpha = \gamma \) it is one. We now assume \( \alpha < \gamma \). By definition,
\[
\sum_{\beta} Q^-_{\alpha,\beta}(t^{-1})Q^-_{\beta,\gamma}(t^{-1})(-1)^{|\beta|+|\gamma|} = \sum_{\beta,\alpha \leq \beta \leq \gamma} \sum_{D^I \in \text{Conf}^I(\alpha,\beta)} \sum_{D^{II} \in \text{Conf}^{II}(\beta,\gamma)} t^{-|D^I|+|D^{II}|}(-1)^{|D^{II}|}
\]
The sign was obtained by noting that Dyck strips have odd length, so that the number of boxes \(|\gamma| - |\beta|\) and the number of Dyck strips \(|D_I^I|\) of \(D_I^I\) have same parity. Now merge together the two families of Dyck strips \(D_I\) and \(D_I^I\) into a single family \(D\), and switch the summations:

\[
\sum_{\beta} Q_{I,\alpha,\beta}^I Q_{\beta,\gamma}^{I,-}(1)^{|\beta| + |\gamma|} = \sum_{D \in \text{Conf}(\alpha, \beta)} t^{-|D|} \sum_{\beta \in P(D)} (-1)^{|D_I^I(\beta)|}
\]

where \(P(D)\) is the set of paths \(\beta\) between \(\alpha\) and \(\gamma\) such that the \(D \in D\) below \(\beta\) satisfy rule I and those above \(\beta\) satisfy rule II; we denote the corresponding partition \(D = D_I(\beta) \cup D_I^I(\beta)\).

We shall show that for a fixed decomposition \(D\) of \(\gamma/\alpha\) into Dyck strips, the sum over \(\beta\) i.e. over subdivisions of \(D\) into two subsets (one satisfying rule I, the other rule II), is zero. In all that follows, we assume \(P(D) \neq \emptyset\) (otherwise the sum is trivially zero).

In this proof we shall need a relation on Dyck strips in \(D\), which mimics the definition of rule II. We write that \(D < D'\) if all boxes just above, NW or NE of a box of \(D\) belong to \(D\) or \(D'\). This relation has a tree structure in the sense that for given \(D\) there is at most one \(D'\) such that \(D < D'\). If there are no such \(D'\), then \(D\) is called a minimal element (this is just the usual notion of minimality for the associated order relation).

Define

\[
\mathcal{I}(D) = \left( \bigcup_{\beta \in P(D)} D_I^I(\beta) \right) \cap \left( \bigcup_{\beta \in P(D)} D_I^I(\beta) \right)
\]

i.e. the set of Dyck strips which can be on either side of the boundary between zones I and II. We have the first observation

**Lemma 5.** If \(D, D' \in \mathcal{I}(D), D \neq D'\), then the \(x\) coordinates of boxes of \(D\) and \(D'\) are distant by at least 2.

**Proof.** Assume the \(x\) coordinates are distant by less than 2. Then there is a box \((x, y)\) of one of the two Dyck strips, say \(D\), which is above a box \((x', y')\) of \(D'\) in the sense that \(y > y'\) and \(x = x' \pm 1\). But note that this excludes the possibility of finding a path between \(\alpha\) and \(\gamma\) such that \(D\) is below it and \(D'\) is above it. Therefore, choosing \(\beta, \beta', \beta''\) such that \(D \in D_I^I(\beta) \cap D_I^I(\beta')\) and \(D' \in D_I^I(\beta'') \cap D_I^I(\beta''')\), we conclude that \(D' \in D_I^I(\beta)\) and \(D \in D_I^I(\beta'')\). But this implies that there is a region containing both \(D\) and \(D'\), namely the region below \(\beta\) and above \(\beta''\), in which both rule I and rule II apply. The rule II and the relative position of \(D\) and \(D'\) imply that there is a chain \(D < D_1 < \cdots < D_k < D\); but this in turn implies that two successive Dyck strips in the chain also satisfy the conditions of applicability of rule I. These two facts are contradictory because they imply opposite inequalities on the lengths, i.e. \(l(D') < l(D_1) < \cdots < l(D_k) < l(D)\) and \(l(D) > l(D_1) > \cdots > l(D_k) > l(D)\). \(\square\)

We conclude immediately that distinct elements of \(\mathcal{I}\) “do not interact” with each other in the sense that they can be added/removed independently from \(D_I^I, D_I^I\). More explicitly, note that since \(P(D) \neq \emptyset\), \(\bigcup_{\beta \in P(D)} D_I^I(\beta) \neq \emptyset\); and its lower boundary is again a path, say \(\alpha_0\). Similarly one can define \(\gamma_0\) which is the upper boundary of \(\bigcup_{\beta \in P(D)} D_I^I(\beta)\). Then for any subset \(\mathcal{J} \subset \mathcal{I}(D)\), there is a path \(\beta \in P(D)\) such that \(D_I^I(\beta) = D_I^I(\gamma_0) \sqcup \mathcal{J}\) and \(D_I^I(\beta) = D_I^I(\alpha_0) \sqcup (\mathcal{I}(D) \setminus \mathcal{J})\). Indeed, rules I and II cannot apply to two elements of \(\mathcal{I}(D)\)
because they are too far apart, and in all other cases one easily checks that these rules are already satisfied by definition.

To summarize, we have found that the summation over $\beta$ is structured as follows: $P(D)$ is of cardinality $2^{|I(D)|}$, corresponding to whether $D \in I(D)$, is above or below the path separating regions I and II. Furthermore, we have the following key fact:

**Proposition 4.**

$$I(D) \neq \emptyset$$

**Proof.** We shall in fact provide an explicit description of $I(D)$ using the relation $\preceq$. Recall from the structure of $P(D)$ described above that there is a path $\alpha_0 \in P(D)$ such that $\bigcup_{\beta \in P(D)} D_{II}(\beta) = D_{II}(\alpha_0)$.

We claim that $I(D)$ is exactly the set min $D_{II}(\alpha_0)$ of minimal elements (in the sense of $\preceq$) of $D_{II}(\alpha_0)$. $I(D) \subset \text{min } D_{II}(\alpha_0)$ is evident by definition of $I(D)$. Let us now prove the reverse inclusion, i.e. prove that any minimal element of $D_{II}(\alpha_0)$ can also be moved to the region I.

**Figure 3.** Illustration of the proof of Prop. 4. The thick lines represent the boundaries $\alpha_0$ and $\gamma_0$ of the maximal/minimal regions I/II, so that the Dyck strips in between the two form $I(D)$.

Pick such a minimal element $D_{\text{min}} \in D_{II}(\alpha_0)$. Due to the way we defined $\preceq$, it is easy to see that $D_{II}(\alpha_0) \setminus D_{\text{min}} = D_{II}(\beta)$ for some $\beta$ above $\alpha_0$. We now claim that the Dyck strips below $\beta$ satisfy rule I. These Dyck strips consist of the Dyck strips below $\alpha_0$, which by construction satisfy rule I, plus the additional $D_{\text{min}}$. To a box of $D_{\text{min}}$ with coordinates $(x, y)$ associate $D_x$, which is the Dyck strip to which belongs the box right below i.e. $(x, y - 2)$, or $\emptyset$ if this box is below the bottom line $\alpha$. Rule I means that this function should be constant. The proof is by contradiction. Suppose there is an $x$ such that $D_x \neq D_{x+1}$. We can assume up to reflection w.r.t. the $y$ axis that the higher of the two boxes is the first, i.e. $(x, y - 2) \in D_x \neq \emptyset$ and $(x + 1, y - 3) \in D_{x+1}$ if $D_{x+1} \neq \emptyset$. Now note that $y - 3 \geq h - 2$, where $h$ is the minimum $y$ coordinate of boxes of $D_{\text{min}}$; so that $y - 2 \geq h - 1$. Therefore the Dyck strip $D_x$ cannot pass below the endpoints of $D_{\text{min}}$ (whose $y$ coordinates are $h$); in other words, its $x$ span is strictly included in that of $D_{\text{min}}$ and it touches $D_{\text{min}}$ at its two boundaries.

Now introduce another relation $\rightarrow$ as follows: $D \rightarrow D'$ if there exists a box of $D'$ NW, NE or above a box of $D$. There is a naturally associated order relation, which we simply denote $D \rightarrow \cdots \rightarrow D'$, obtained by forming chains. We can consider $\mathcal{X} = \{D \in D_I(\alpha_0) : D_x \rightarrow \cdots \rightarrow D\}$. $\mathcal{X}$ is non empty because $D_x \in \mathcal{X}$; a maximal element $D$ of it (for the order
relation $\rightarrow \cdots \rightarrow$) is such that all boxes NW, NE and above it are outside $D^I(\alpha_0)$; but since its $x$ span is strictly included in that of $D_{\min}$, these boxes must belong to $D_{\min}$. Therefore $D \prec D_{\min}$, contradicting the minimality of $D_{\min}$.

Thus, since $D^I(\alpha_0) \neq \emptyset$, $I(D) = \min D^I(\alpha_0)$.

Note finally that the possible paths $\beta \in P(D)$ correspond to $D^I(\beta) = D^I(\gamma_0) \cup J$ where $J$ is any subset of $I(D)$, so that we can compute the sum over $\beta \in P(D)$ by rewriting it $\sum_{J \subset I(D)} (-1)^{|J|} = 0$.

3.4. On the module $\mathcal{M}_{N,K}^+$. The two families of Kazhdan–Lusztig polynomials $P_{x,y}(t^{-1})$ on the modules $\mathcal{M}_{N,K}^\pm$ are related by the duality theorem 4. Together with Lemma 4, we have

$$(P_\alpha^+)^{-1}(t^{-1}) = (-1)^{|\alpha|+|\beta|} P_{\beta,\alpha}(t^{-1}) = Q_{\alpha,\beta}^I(-t^{-1}),$$

where we have once again used that $(-1)^{|\alpha|+|\beta|} = (-1)^{|\alpha|-|\beta|}$ and that the length of a Dyck strip is always odd. Hence,

**Corollary 1.** On $\mathcal{M}_{N,K}^+$, the monomial basis $m_\alpha$ is expressed in terms of the Kazhdan–Lusztig basis as

$$m_\beta = \sum_{\alpha \leq \beta} Q_{\alpha,\beta}^I(-t^{-1}) C_\alpha^+.$$  

A slightly more explicit version of this formula is provided in appendix C.

More importantly, Theorem 6 allows us to invert this relation to obtain the Kazhdan–Lusztig basis $C_\beta^+$ in terms of the monomial basis $m_\alpha$:

**Corollary 2.** The Kazhdan–Lusztig polynomial $P_{\alpha,\beta}^+$ is the generating function of Dyck strips according to Rule I, that is,

$$P_{\alpha,\beta}^+(t^{-1}) = Q_{\alpha,\beta}^I(t^{-1}).$$

Examples can be found in appendix A.

Formulæ for such polynomials were of course already known: see in particular [13] for a similar approach in a more general setting; and [21], the combinatorial description of which is described in the next section and shown to be in bijection with ours.

4. Relation to Lascoux–Schützenberger rule

4.1. Lascoux–Schützenberger binary trees. We briefly review the construction of the binary trees of Lascoux–Schützenberger to compute the Kazhdan–Lusztig polynomials for Grassmannian permutations [21]. In our setup, they correspond to the polynomials $P_{\alpha,\beta}^+(t^{-1})$ from Corollary 2.

Let $\mathcal{Z}$ be a set such that $\emptyset \in \mathcal{Z}$, $z \in \mathcal{Z} \Rightarrow 1z2 \in \mathcal{Z}$ and $z_1, z_2 \in \mathcal{Z} \Rightarrow z_1z_2 \in \mathcal{Z}$. We define inductively a rooted tree $A(w)$ for $w$ an arbitrary binary string by:

- $A(\emptyset)$ is the empty tree,
• \( A(2w) = A(w1) = A(w) \),
• \( A(zw) \) is obtained by attaching the trees for \( A(z) \) and \( A(w) \) at their roots, \( z \in \mathbb{Z} \),
• \( A(1z2) \) is obtained by attaching an edge just below the tree \( A(z) \), \( z \in \mathbb{Z} \).

We denote by \( \|\alpha\| \) the length of a binary string \( \alpha \) and by \( \|\alpha\|_\sigma \) the number of \( \sigma \) in a string \( \alpha \). Let \( v, w \in \{1, 2\}^N \) with \( v \leq w \) and \( v = v'\alpha\beta v'' \), \( w = w'12w'' \) with \( \|\alpha'\| = \|w'\| \) and \( \alpha, \beta \in \{1, 2\} \). A capacity of the edge corresponding to the underlined 1 and 2 is defined by
\[
\text{cap}(12) := \|v'\alpha\|_1 - \|w'1\|_1
\]
The condition \( v \leq w \) implies a capacity is always non-negative.

The capacity of \( v \) with respect to \( w \) is the collection of capacities of pairs of adjacent 1 and 2 in \( w \) and called as the relative capacities.

We denote by \( A(w/v) \) the rooted tree with relative capacities. \( A(w/v) \) is obtained from the tree \( A(w) \) by putting corresponding capacities at leaves (end points) of the tree, see Fig. 4(a).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{tree.png}
\caption{(a) A tree with capacities. (b) A labelled tree.}
\end{figure}

A labelling of the tree \( A(w/v) \) is a set of non-negative integers on edges of \( A(w) \) satisfying
• An integer on a leaf is less than or equal to its capacity,
• Integers on edges are non-increasing from leaves to the root.

See Fig. 4(b).

The analysis of the recursive relations for both the Kazhdan–Lusztig polynomials and the generating function of the tree \( A(w/v) \) led Lascoux and Schützenberger to the following theorem, formulated here in our notations (in particular we identify as before binary strings and paths with convention +):

**Theorem 7** (Lascoux, Schützenberger).
\[
P_{\alpha,\beta}^+(t^{-1}) = t^{\|\alpha\| - \|\beta\|} \sum_\nu t^{2\Sigma(\nu)}.
\]
where \( \nu \) runs all possible labellings of \( A(\beta/\alpha) \) and \( \Sigma(\nu) \) is the sum of labels of \( \nu \).

Below, we produce a bijection between a labelling of \( A(\beta/\alpha) \) and a configuration of Dyck strips between paths \( \alpha \) and \( \beta \) (i.e. in the skew-diagram \( \beta/\alpha \)).
4.2. **From trees to link patterns.** In the previous section we have introduced, following Lascoux and Schützenberger, binary trees starting from a binary string. Using the bijections of section 2.3, we can equivalently start from a path, or from a link pattern. The latter correspondence is particularly natural, since the binary tree is the *dual graph* of the link pattern, cf Fig. 5(a) (with the same example as in Fig. 4). Note that there is a bijective map \( p \) which to an edge \( e \) associates a pairing \( p(e) \) of the link pattern. However, unless the link pattern has maximum number of pairings, the map from link patterns to trees is not one-to-one: when we take the dual graph, we ignore the unpaired vertices. In what follows we denote by \( \pi(\beta) \) the link pattern associated to the path (or binary string) \( \beta \).

![Figure 5](image.png)

**Figure 5.** (a) Link pattern and binary tree. (b) Labelling of the link pattern.

It is also convenient to attach labellings to the link pattern as follows. Given a labelling of \( A(\beta/\alpha) \) and an edge \( e \) with label \( n(e) \), we put the label \( n'(p(e)) = n(e) - n(e') \) on the corresponding pairing \( p(e) \), where the edge \( e' \) is the parent edge of \( e \), unless there is no parent edge (edge connected to the root) in which case we put \( n(e) \). See Fig. 5(b) (with the same labelling as in Fig. 4(b)).

Labellings of the link pattern \( \pi(\beta) \) thus obtained from a labelling of \( A(\beta/\alpha) \) are defined by the following conditions:

- All labels are non-negative integers.
- Given a smallest planar pairing \( p(e) \) (a pairing of neighbors), the sum of all labels on planar pairings which surround \( p(e) \) is less than or equal to the capacity of \( e \).

4.3. **From labelled link patterns to Dyck strips.** We now consider a pair of paths \( \alpha \) and \( \beta \), with \( \alpha \) above \( \beta \), and the associated link pattern \( \pi(\beta) \) along with a labelling as above. We associate to it a collection of Dyck strips between paths \( \alpha \) and \( \beta \) as follows. Recall that a Dyck strip is characterized by a Dyck path. To each pairing \( p \) of \( \pi(\beta) \) we associate Dyck paths which start a half-step to the left of the left point of the pairing and a half-step to the right of its right point. More precisely, if \( p \) has label \( n'(p) \) we then stack \( n'(p) \) such Dyck paths on top of each other, forming parallel layers above \( \beta \). We then repeat the process for every pairing, respecting the order which is to start with the largest arches and end with the smallest arches (this way we respect rule I). See Fig. 6(a) for the same example as in previous figures. Note that some Dyck paths may have coinciding starting or end points, in which case they are merged into a larger Dyck path.

To each Dyck path (where Dyck paths which touch have been merged) we now associate the corresponding Dyck strip. Note that such strips necessarily have length greater or equal to 3. We claim that these strips remain under the path \( \alpha \). Indeed, let \( p \) be a smallest planar pairing, that is, connecting \( i \) and \( i + 1 \). Then the difference of heights of \( \alpha \) and \( \beta \) at the
center of the pairing (i.e. the depth of the \&-corner in the skew Ferrers diagram) is by direct computation exactly the capacity of the corresponding edge $e$ in the tree $A(\beta/\alpha)$. Therefore the number of Dyck strips above that point, that is the sum of labels of pairings surround $p$, which is nothing but the label of $e$ in the tree $A(\beta/\alpha)$, is less or equal to the capacity i.e. the difference of heights. Therefore the Dyck strips remain below $\alpha$ at every local maximum of $\beta$, therefore everywhere.

The last stage is to declare that the boxes of $\beta/\alpha$ that do not belong to any of the Dyck strips above are by definition Dyck strips of length one (consisting of a single box). See Fig. 6(b) for the final result.

![Figure 6. Dyck paths and Dyck strips.](image)

It is easy to show that the correspondence above is bijective. Therefore we have proved the

**Theorem 8.** There exists a bijection between labellings of the tree $A(\beta/\alpha)$ and configurations of Dyck strips between paths $\alpha$ and $\beta$ satisfying rule I.

In order to show that Corollary 2 and Theorem 7 are equivalent, we still need to compare powers of $t^{-1}$, which naively look quite different. Let us start from a configuration of Dyck strips between paths $\alpha$ and $\beta$. Consider a Dyck strip of length $\geq 3$. It is obtained from one or possibly several Dyck paths each associated to a certain pairing, say $p_1, \ldots, p_k$. The number of boxes of the Dyck strip is equal to $1 + \sum_{i=1}^{k} d(p_i)$, where $d(p)$ is the distance between the two endpoints of $p$. This formula still holds for Dyck strips of size one provided one associates to it zero Dyck paths. We now write the number of boxes between $\alpha$ and $\beta$ as:

$$|\beta| - |\alpha| = \sum_{\text{Dyck strip formed from paths } p_1, \ldots, p_k} \left(1 + \sum_{i=1}^{k} d(p_i)\right)$$

$$= \text{number of Dyck strips} + \sum_{p \text{ pairing}} n'(p)d(p)$$

$$= \text{number of Dyck strips} + \sum_{e \text{ edge}} (n(e) - n(e'))(2 + 2 \times \text{number of descendents of } e)$$

$$= \text{number of Dyck strips} + 2 \sum_{e \text{ edge}} n(e)$$
where we have used the fact that \(d(p) = 2 + 2\) times the number of pairings surrounded by \(p\) and translated it into the language of trees. We then write \(n'(p(e)) = n(e) - n(e')\) where \(e'\) is the parent of \(e\). The final equality provides the required identification of powers of \(t^{-1}\).

Appendix B provides the full computation of a KL polynomial in the various formulations (path, tree).

**APPENDIX A. TABLE OF POLYNOMIALS AT \(N = 4, K = 2\)**

Here are a few examples in small size. Blank entries correspond to zeroes due to violation of the order.

**Table of \(P^-\) for \(N = 4, K = 2\)**

|       |       |       |       |       |
|-------|-------|-------|-------|-------|
| \(\downarrow\) | \(\uparrow\) | \(\downarrow\) | \(\uparrow\) | \(\downarrow\) |
|       | 1     | \(t^{-1}\) | 0     | 0     | \(t^{-2}\) |
|       | 1     | \(t^{-1}\) | \(t^{-1}\) | \(t^{-2}\) | \(t^{-1}\) |
|       | 1     | \(t^{-1}\) | \(t^{-1}\) | 0     |       |
|       | 1     | \(t^{-1}\) | 0     |       |       |
|       | 1     | \(t^{-1}\) |       |       | 1     |

**Table of \(P^+\) for \(N = 4, K = 2\)**

|       |       |       |       |       |
|-------|-------|-------|-------|-------|
| \(\downarrow\) | \(\uparrow\) | \(\downarrow\) | \(\uparrow\) | \(\downarrow\) |
|       | 1     | \(t^{-1}\) | \(t^{-2}\) | \(t^{-2}\) | \(t^{-3}(1 + t^2)\) | \(t^{-4}\) |
|       | 1     | \(t^{-1}\) | \(t^{-1}\) | \(t^{-2}\) | \(t^{-3}\) |       |
|       | 1     | \(t^{-1}\) | \(t^{-2}\) |       |       |       |
|       | 1     | \(t^{-1}\) | \(t^{-2}\) |       |       |       |
|       | 1     | \(t^{-1}\) |       |       |       |       |
|       |       |       |       |       |       | 1     |
The only non-monomial polynomial in $P^+$ corresponds to the two Dyck strip decompositions and .

**Appendix B. An example of rule I**

**Appendix C. A more explicit formula for $(P^+)^{-1}$**

Throughout this section, we again identify paths and binary strings. We describe the set $\mathcal{L}(\beta)$, which is the “transposed” set of $\mathcal{F}(\beta)$. 
A linkage $w$ of a path $\beta$ is a set of pairs of integers from $[N] := \{1, \ldots, N\}$ satisfying:

1. Each integer in $N$ appears exactly once in $w$.
2. If a pair $(i, j) \in w$, $i, j \in [N]$, then $\beta_i = 1$ and $\beta_j = 2$.
3. Suppose $i$ and $j$, ($i < j$) are paired. Then, there is no pair of $k$ and $l$ ($k < l$) such that $i < k < j < l$ or $k < i < l < j$.

Note that there are several linkages for a given path $v$, however, we recover a path from a given linkage.

**Definition 3.** $\mathfrak{m}(\beta)$ is a set of all possible linkages of the path $\beta$.

We need some terminology for pairs to define a map from an element $w \in \mathfrak{m}(\beta)$ to the set of paths $L(\beta)$.

1. A pair $(i, j)$ is said to an ordered (resp. reversed) pair if $i < j$ (resp. $i > j$).
2. A pair of $k$ and $l$, $k < l$, i.e., a pair $(k, l)$ or $(l, k)$, is said to be inside of a pair of $i$ and $j$ if $i < k < l < j$, where $i, j, k, l \in [N]$.

We define an operation $r$-flip acting on a reversed pair $P$ in a linkage $w$ as follows. We flip $i$ and $j$ in $P$, all reversed pairs inside of $P$ and keep all ordered pairs unchanged.

**Definition 4.** $L'(\beta; w)$ be the all possible paths recovered from linkages obtained by $r$-flipping (or without $r$-flipping) the linkage $w \in \mathfrak{m}(\beta)$.

**Definition 5.** The set of paths by taking the union of $L'(\beta; w)$ with respect to $w$:

$$L(\beta) := \bigcup_{w \in \mathfrak{m}(\beta)} L'(\beta; w).$$

In general, $L'(\beta; w) \cap L'(\beta; w') \neq \emptyset$ if $w, w' \in \mathfrak{m}(\beta)$. Let $\alpha, \beta$ be two paths and $\alpha \in L(\beta)$. The function $d(\alpha, \beta)$ depends only on the two paths, and this function counts the number of flipped r-pairs in $w \in \mathfrak{m}(\beta)$ to obtain the path $\alpha$. Therefore, the number of flipped r-pairs to obtain $\alpha$ from $\beta$ are independent of the choice of a linkage.

It is not hard to see that the set $L(\beta)$ describes exactly the set of $\alpha$ for which the summand in the formula of Corollary 1 is non-zero. Therefore we have the slightly more explicit formula:

$$m_\beta = \sum_{\alpha \in L(\beta)} (-t)^{-d(\alpha, \beta)} C^+_{\alpha}$$

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