THE NORMALIZED CYCLOMATIC QUOTIENT ASSOCIATED WITH PRESENTATIONS OF FINITELY GENERATED GROUPS

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1 Introduction

Given the Cayley graph of a finitely generated group $G$, with respect to a presentation $G^\alpha$ with $n$ generators, the quotient of the rank of the fundamental group of subgraphs of the Cayley graph by the cardinality of the set of vertices of the subgraphs gives rise to the definition of the normalized cyclomatic quotient $\hat{\Xi}(G^\alpha)$. The asymptotic behavior of this quotient is similar to the asymptotic behavior of the quotient of the cardinality of the boundary of the subgraph by the cardinality of the subgraph. Using Følner’s criterion for amenability one gets that $\hat{\Xi}(G^\alpha)$ vanishes for infinite groups if and only if they are amenable. When $G$ is finite then $\hat{\Xi}(G^\alpha) = 1/|G|$, where $|G|$ is the cardinality of $G$, and when $G$ is non-amenable then $1 - n \leq \hat{\Xi}(G^\alpha) < 0$, with $\hat{\Xi}(G^\alpha) = 1 - n$ if and only if $G$ is free of rank $n$. Thus we see that on special cases $\hat{\Xi}(G^\alpha)$ takes the values of the Euler characteristic of $G$. Most of the paper is concerned with formulae for the value of $\hat{\Xi}(G^\alpha)$ with respect to that of subgroups and factor groups, and with respect to the decomposition of the group into direct product and free product. It is not surprising that some of the formulae and bounds we get for $\hat{\Xi}(G^\alpha)$ are similar to those given for the spectral radius of symmetric random walks on the graph of $G^\alpha$ (see [8] and also [8] for the connection of the spectral radius with the “growth-exponent”), but this is not always the case (see e.g. the remark
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after Theorem 3.1). The difference is that we do not explore the graph associated with a given presentation in a “breadth-first” manner but rather in a “depth-first” manner, proceeding in the “best” possible way. In the last section of the paper we define and touch very briefly the balanced cyclomatic quotient, which is defined on concentric balls in the graph. This definition is related to the growth of $G$.

Throughout the paper we assume that when we are given presentations $G_i^\alpha$ of groups $G_i$ and $H$ is a group defined through the $G_i$, then $H$ gets the natural induced presentation $H^\alpha$. Thus, if $H = G_1 \ast G_2$ then the generators and relators of $H^\alpha$ are the union of those of the $G_i^\alpha$, assuming that the generating sets of the $G_i^\alpha$ are disjoint. Similarly for direct products (with the appropriate commutators as extra relators), etc. We do not try, however, to be too precise with regard to the use we make of the notation $G^\alpha$ or $G_i^\alpha$.

The formulae we give for direct and free products give upper and lower bounds for $\hat{\Xi}(G^\alpha)$ in the following sense. Suppose that $G_i^{\alpha_i} = < X_i \mid R_i >$, $i = 1, 2$, and $G^\alpha = < X \mid R >$, with $X = X_1 \cup X_2$, $R \supseteq R_1 \cup R_2$ and both unions disjoint, so that the natural maps $\phi_i : G_i \to G$, $i = 1, 2$, are injective. Then

$$\hat{\Xi}(G^{\alpha_1} \ast G^{\alpha_2}) \leq \hat{\Xi}(G^\alpha) \leq \hat{\Xi}(G^{\alpha_1} \times G^{\alpha_2}).$$

(1)

The left inequality follows from Theorem 3.1 since the Cayley graph associated with $G^\alpha$ is a quotient of that associated with $G^{\alpha_1} \ast G^{\alpha_2}$. The right inequality appears in the proof of Theorem 4.1. These bounds hold for such structures as semi-direct products, amalgamated products, HNN extensions, etc. One can also try and get exact formulae for the structures mentioned above, at least in special cases, but we do not get much into it in the present paper.

We use the following terminology and notation on graphs. The set of vertices of a graph $G$ is denoted by $V(G)$ and the set of vertices by $E(G)$. A path in $G$ is a sequence $v_0, e_1, v_1, e_2, \ldots, v_i \in V(G)$, $e_i \in E(G)$, such that $e_i$ starts at the vertex $v_{i-1}$ and terminates at $v_i$. The length of a path $v_0, e_1, v_1, e_2, \ldots, v_n$ is $n$. A simple path is a path in which the vertices along it are distinct, except possibly for the first and last one, in which case it is a simple closed path or a simple circuit. We assume that each path is reduced, i.e. it is not homotopic to a shorter one when the initial and terminal vertices are kept fixed.

If $H \subseteq G$, i.e. $H$ is a collection of vertices and edges of the graph $G$, then we denote by $< H >$ the subgraph generated by $H$. It is the smallest subgraph of $G$ which contains $H$. That is, we add to $H$ the endpoint vertices of all the edges in $H$. On the other hand, the subgraph of $G$ induced by $H$ is the one whose vertices are those of $H$ and whose edges are all the edges which
join these vertices in \( G \). An induced subgraph is a subgraph which is induced by some \( H \subseteq G \). If \( H_1, H_2 \subseteq G \) then \( H_1 - H_2 \) is the collection of vertices \( V(H_1) - V(H_2) \) and edges \( E(H_1) - E(H_2) \), and it does not necessarily form a subgraph of \( G \), even when \( H_1 \) and \( H_2 \) are subgraphs of \( G \). The boundary of the subgraph \( H \) of \( G \) is \( \partial H = H \cap <G - H> \), and its interior is \( \hat{H} = \hat{H} - \partial H \). The outer boundary of \( H \) (in \( G \)) is the set of vertices of \( G - H \) which are adjacent to \( H \) in \( G \). Assume now that each edge of \( G \) is labeled with some \( x \in X \) in one direction and with \( x^{-1} \in X^{-1} \) in the other direction. Then we define \( E^X_{out}(H) \) to be the set of edges of \( G - H \) whose initial vertices with respect to the directions \( X \) are in \( H \).

Finally, let \( \beta_0(H) = |V(H)| \), let \( \beta_1(H) = |E(H)| \), and let \( \alpha(H) = |\pi_0(H)| \) be the number of the (connected) components of \( H \).

### 2 The Normalized Cyclomatic Quotient

Let \( G^a = \langle X \mid R \rangle \) be a presentation of a group \( G \), with \( X = \{x_1, \ldots, x_n\} \), and let \( G \) be the associated Cayley graph. If \( H \) is the normal closure of \( R \) in \( F = \langle X \rangle \) then it is shown in [8] that the growth function of \( G \) is equivalent to the rank-growth \( rk_H \) of \( H \). The rank-growth is defined by

\[
rk_H(i) = \text{rank}(H_i),
\]

where \( H_i \) is the subgroup of \( F \) generated by the elements of length \( \leq i \) (with respect to \( X \cup X^{-1} \)). We notice that \( H_i \) is the fundamental group of the subgraph of \( G \) of all paths starting at 1 of length \( \leq i \). Thus, there exists an exhausting chain \( (G'_i) \) in \( G \), i.e. a sequence \( G'_1 \subseteq G'_2 \subseteq G'_3 \subseteq \cdots \) of subgraphs of \( G \) whose union is \( G \), with all the \( G'_i \) connected and finite, such that the growth of the function \( \gamma_1(i) = \text{rank}(\pi_1(G'_i)) \) is equivalent to the growth of the function \( \gamma_2(i) = |V(G'_i)| \). We are now interested in the asymptotic behavior of the quotient \( \gamma_1(i)/\gamma_2(i) \), which is related, as we will see, to the quotient \( |V(\partial G'_i)|/|V(G'_i)| \). The latter quotient and its analogs are known and widely studied objects in diverse areas of Mathematics (see e.g. the survey [3]). By Følner’s criterion (see [4]) the group \( G \) is amenable if and only if there exists an exhausting chain \( (G'_i) \) of finite connected subgraphs of \( G \) such that

\[
\limsup_{i \to \infty} \frac{\beta_0(\partial G'_i)}{\beta_0(G'_i)} = 0
\]

(recall that a group \( G \) is amenable if there exists an invariant mean on \( B(G) \), the space of all bounded complex-valued functions on \( G \) with the sup norm \( \|f\|_\infty \), see [2]). We remark that in Følner’s criterion one can consider as well disconnected subgraphs or boundaries of any fixed width \( k \). Notice that [3] implies that if \( G \) is non-amenable then it has exponential growth.
Let us denote by $\beta_2(G')$ the cyclomatic number of $G'$, i.e. the sum of the values of $\text{rank}(\pi_1(H'))$ over all the components $H'$ of the subgraph $G'$ (the notation $\beta_2(G')$ refers to the number of 2-cells of an associated 2-dimensional complex). Let us also use the following notation:

$$\xi(G') = \frac{\beta_2(G')}{\beta_0(G')}.$$  

(4)

$$\mu(G') = \frac{\beta_1(G') + 1}{\beta_0(G')}.$$  

(5)

We denote by $\mathcal{F}(G), \mathcal{F}^*(G), \mathcal{CF}(G), \mathcal{CF}^*(G)$ respectively the sets of finite, non-trivial finite, connected finite, non-trivial connected finite subgraphs of $G$. Here a graph is non-trivial if it contains more than one vertex.

**Definition 2.1** If $C = (G_1 \subseteq G_2 \subseteq \cdots) \in \mathcal{CF}(G)$, is an exhausting chain, let

$$\eta_C = \limsup_{i \to \infty} \xi(G_i).$$  

(6)

Then we define the cyclomatic quotient of $G^\alpha$ by

$$\Xi(G^\alpha) = \sup_C \eta_C, \quad C \text{ an exhausting chain},$$  

(7)

and the normalized cyclomatic quotient of $G^\alpha$ by

$$\hat{\Xi}(G^\alpha) = 1 - n + \Xi(G^\alpha).$$  

(8)

The following are equivalent definitions of $\hat{\Xi}(G^\alpha)$.

$$\hat{\Xi}(G^\alpha) = 1 - n + \sup_{G' \in \mathcal{F}(G)} \xi(G'),$$  

(9)

$$\hat{\Xi}(G^\alpha) = \sup_{G' \in \mathcal{CF}(G)} \left( 1 - \frac{|E_{\text{out}}(G')|}{\beta_0(G')} \right),$$  

(10)

$$\hat{\Xi}(G^\alpha) = -n + \sup_{G' \in \mathcal{CF}(G)} \mu(G'),$$  

(11)

$$\hat{\Xi}(G^\alpha) = \sup_S \left( 1 + \sum_{j=1}^{n} \left( \frac{|Sx_j \cap S|}{|S|} - 1 \right) \right), \quad S \subseteq G \text{ finite}. $$  

(12)

In case $G$ is infinite then we have

$$\hat{\Xi}(G^\alpha) = -\inf_{G' \in \mathcal{CF}(G)} \frac{|E_{\text{out}}(G')|}{\beta_0(G')},$$  

$$\hat{\Xi}(G^\alpha) = -n + \sup_{G' \in \mathcal{CF}(G)} \frac{\beta_1(G')}{\beta_0(G')},$$  

$$\hat{\Xi}(G^\alpha) = \sup_{S} \sum_{j=1}^{n} \left( \frac{|Sx_j \cap S|}{|S|} - 1 \right), \quad S \subseteq G \text{ finite}. $$  

(13)
The normalized cyclomatic quotient

Clearly definition (9) gives at least the same value as definition (8). To see that these definitions are equivalent we need to show that for every \( G' \in \mathcal{F}(G) \) there exists \( H \in \mathcal{C}\mathcal{F}(G) \) such that \( G'' \subseteq H \) and \( \xi(G') \leq \xi(H) \). But this follows by Lemma 2.2 and by the fact that given any subgraphs \( G_1, G_2 \in \mathcal{F}(G) \), we can cover \( G_2 \) by the induced subgraph of translates of \( G_1 \).

**Lemma 2.2** Let \( G' \in \mathcal{F}(G) \) satisfy \( \xi(G') \geq \xi(H') \) for every \( H' \subseteq G' \). Let \( G'' \in \mathcal{F}(G) \) such that \( \xi(G'') \geq \xi(G') \), and let \( H = G' \cup G'' \). Then \( \xi(H) \geq \xi(G') \).

**Proof.** Clearly

\[
\beta_0(H) = \beta_0(G') + \beta_0(G'') - \beta_0(G' \cap G''),
\]
while

\[
\beta_2(H) \geq \beta_2(G') + \beta_2(G'') - \beta_2(G' \cap G'').
\]

Then by simple calculation we get that

\[
\xi(H) \geq \xi(G')
\]

after clearing denominators in the quotients \( \xi(H) \) and \( \xi(G') \).

Definition (10) follows by

\[
\beta_2(G') = 1 + (n - 1)\beta_0(G') - |E_{\text{out}}^X(G')|, \quad G' \in \mathcal{C}\mathcal{F}(G).
\]

Definition (11) follows by

\[
\beta_2(G') = \alpha(G') - \beta_0(G') + \beta_1(G')
\]

(\( \alpha(G') \) is the number of components of \( G' \)).

Finally, for definition (12) we notice that if \( G' \) is an induced subgraph of \( G \) and that \( S \subseteq G \) is the set of elements of \( G \) which correspond to the vertices of \( G' \) then

\[
\beta_1(G') = \sum_{j=1}^{n} |Sx_j \cap S|.
\]

The slightly simplified expressions (13) when \( G \) is infinite are due to the fact that \( G' \) may be chosen as large as we wish.

**Proposition 2.3** Let \( G^\alpha \) be a presentation of a group \( G \) with \( n \) generators and let \( \mathcal{G} \) be the associated Cayley graph. Then \( 1 - n \leq \hat{\Xi}(G^\alpha) \leq 1 \). Moreover,

(i) if \( G \) is amenable then \( \hat{\Xi}(G^\alpha) = 1/|G| \), where \( 1/|G| \) is defined to be 0 if \( |G| = \infty \);
(ii) if $G$ is non-amenable then $1 - n \leq \hat{\Xi}(G^\alpha) < 0$, with $\hat{\Xi}(G^\alpha) = 1 - n$ if and only if $G$ is free of rank $n \geq 2$.

Proof. If $G$ is finite then any exhausting chain of $G$ stabilizes on $G$. Then by (10) $\hat{\Xi}(G^\alpha) = 1 - \frac{|E_{out}(G)|}{\beta_0(G)} = \frac{1}{|G|}$.

In fact, we see that for every proper subgraph $G'$ of $G$

$$\frac{1 - |E_{out}(G')|}{\beta_0(G')} \leq 0.$$ (21)

Since for every $G' \in F(G)$, $|E_{out}(G')|$ is of the same order as $\beta_0(\partial G')$, then by using Følner’s criterion we obtain from (10) that $\hat{\Xi}(G^\alpha) = 0$ when $G$ is infinite amenable. When $G$ is non-amenable then there exists $c > 0$ such that for every $G' \in F(G)$, $|E_{out}(G')|/\beta_0(G') > c$. Then by (10) $\hat{\Xi}(G^\alpha) \leq -c$, by letting $\beta_0(G') \to \infty$. On the other hand, $\hat{\Xi}(G^\alpha) > 1 - n$ when there exists at least one circuit in $G$. When $G$ contains no circuits then $G$ is free of rank $n$ and $\hat{\Xi}(G^\alpha) = 1 - n$. □

If we use (11) for the definition of $\hat{\Xi}(G^\alpha)$ then by Proposition 2.3 we get the following criterion for amenability: $G$ is amenable if and only if for every $\epsilon > 0$ there exists $G' \in F(G)$ such that

$$\frac{\beta_1(G')}{\beta_0(G')} > n - \epsilon.$$ (22)

In other words, $G$ is amenable if and only if for every $\epsilon > 0$ there is a finite subset $S$ of $G$ such that $|Sx_j \cap S|/|S| > 1 - \epsilon$ for each generator $x_j$. This is easily seen to be equivalent to Følner’s criterion for amenability (11): for every $\epsilon > 0$ and every $w_1, \ldots, w_r \in G$ there is a finite subset $S$ of $G$ such that $|Sw_i \cap S|/|S| > 1 - \epsilon$ for each $i$. (This also shows that a subgroup of an amenable group is amenable.)

The value of $\hat{\Xi}(G^\alpha)$ depends on the presentation $G^\alpha$ of $G$. Let us look at the following example. Let $G^{\alpha_1}$ be the presentation of the free group $G$ of rank 2 with generators $x, y$. Then $\hat{\Xi}(G^{\alpha_1}) = 1 - 2 = -1$. Let $G^{\alpha_2}$ be obtained from $G^{\alpha_1}$ by the Tietze transformation of adding a new generator $x'$ and a relation $x' = w$. If $w = x^k$ for some integer $k$ then $\hat{\Xi}(G^{\alpha_2}) = \hat{\Xi}(G^{\alpha_1})$, as
can be seen from the chain of subgraphs whose vertices consist of increasing powers of $x$. On the other hand, if $w = xy$ then the only simple circuits we get are triangles of the form $x'y^{-1}x^{-1}$ (or cyclic permutations of it) and by forming an increasing chain of subgraphs the best we can do is adding each time 2 new vertices and obtaining a new circuit. Therefore we get that

$$\hat{\Xi}(G^{\alpha_2}) = (1 - 3) + 1/2 = -3/2 < \hat{\Xi}(G^{\alpha_1}).$$

**Proposition 2.4** Let $G^\alpha = < X \mid R >, X = \{x_1, \ldots, x_n\}$.

(i) If $G^{\alpha_1} = < X \cup \{x'\} \mid R, x' = w(X) >$, where $w(X) \in < X >$, then

$$\hat{\Xi}(G^\alpha) - 1 \leq \hat{\Xi}(G^{\alpha_1}) \leq \hat{\Xi}(G^\alpha),$$

with $\hat{\Xi}(G^{\alpha_1}) = \hat{\Xi}(G^\alpha)$ if $w = 1$.

(ii) If $G^{\alpha_2} = < X \cup \{x'_1, \ldots, x'_n\} \mid R, x'_i = x_i, i = 1, \ldots, n >$ then

$$\hat{\Xi}(G^{\alpha_2}) = 2\hat{\Xi}(G^\alpha) - 1/|G|.$$ 

**Proof.** (i) The Cayley graph of $G$ with respect to $G^{\alpha_1}$ is obtained from the Cayley graph of $G$ with respect to $G^\alpha$ by adding the edges in the direction $x'$ from each vertex $v$ to the vertex $vw$. Thus we can increase $\beta_1(G')$ by at most $\beta_0(G')$. The result then follows from (11). When $G$ is (finite or infinite) amenable then $\hat{\Xi}(G^{\alpha_1}) = \hat{\Xi}(G^\alpha)$ by Proposition 2.3. One can also see it directly from (1) by considering the “thickening” of $G'$ to $G''$ by adding to it the outer $d$-boundary, where $d = l(w)$, and noticing that $E_{out}^{X'}(G'')/\beta_0(G'') \to 0$, where $X' = X \cup \{x'\}$. When $w = 1$ we can make sure that the edges going-out in directions $X$ are the same as those going-out in directions $X'$ and thus obtain $\hat{\Xi}(G^{\alpha_1}) = \hat{\Xi}(G^\alpha)$.

(ii) When $G$ is finite then $\hat{\Xi}(G^{\alpha_2}) = \hat{\Xi}(G^\alpha) = 1/|G|$ by Proposition 2.3. When $G$ is infinite then since the number of out-going edges is doubled we get that

$$\hat{\Xi}(G^{\alpha_2}) = 2\hat{\Xi}(G^\alpha)$$

as $\hat{\Xi}(G^{\alpha_2}) = - \inf_{G'} |E_{out}^{X}(G')|/\beta_0(G'), G' \in C \mathcal{F}(G).$ 

We notice that by Proposition 2.3 and Proposition 2.4 (ii) we get that

$$\hat{\Xi}(G^\alpha)$$

is independent of the presentation if and only if $G$ is amenable.

When $G$ is finite then $\xi$ has, of course, a maximum, and we have seen that the maximum is achieved only at the whole graph $G$ of the presentation. We showed also that in general (for finite and infinite groups), for every proper subgraph $H$ of $G$ there is a subgraph $H'$ which properly contains $H$ and such that $\xi(H') \geq \xi(H)$. We then ask more: does $\xi$ have a maximum in case the group is infinite? It is quite clear that when $G$ is the free product of a finite group and a free group, with a “natural” presentation so that we have generators of the free part which are not involved in any relator, $\xi$ does have a maximum because no circuit involves the generators of the free factor (for
more details on the value of $\xi$ on free products see section 5. Next we will show that in fact this is the only situation where $\xi$ has a maximum.

Given a presentation $G^a = \langle X \mid R \rangle$, with $X = \{x_1, \ldots, x_n\}$ and $R$ not empty, let $c(x_i)$ be the length of a shortest relator in which $x_i$ appears (i.e. the shortest (reduced) circuit in the Cayley graph which contains the edge $x_i$) or 0 if $x_i$ does not appear in any relator. Then let $c(G^a) = \max_i(c(x_i))$.

**Theorem 2.5** Let $G^a = \langle X \mid R \rangle$, $|X| = n$. Then one of the following holds.

**Case 1:** $\hat{\Xi}(G^a) = 1 - n + \xi(G')$ for some $G' \in \mathcal{F}(G)$. Let $X' \subseteq X$ be the set of labels of the edges not going-out of $G'$. Then

(i) if $X' = X$ then $G' = G$ and $G$ is finite;

(ii) if $X' = \emptyset$ then $G$ is the free group on $X$;

(iii) if $|X'| = k$, $1 \leq k \leq n - 1$, then $V(G')$ is the union of left cosets of the finite subgroup $H = G_0(X')$ (which may be trivial) and $\hat{\Xi}(G^a) = \max_0(\beta_0(G'))$.

**Case 2:** $\xi$ does not have a maximum on $\mathcal{F}(G)$. Then for every $G' \in \mathcal{F}(G)$

$$\hat{\Xi}(G^a) \geq 1 - n + \xi(G') + \frac{1}{(c(G^a) - 1)\beta_0(G')}.$$  \hfill (23)

**Proof.** Case 1: it suffices to show that whenever there is a relator involving an edge going-out of $G'$ then $\hat{\Xi}(G^a) > \xi(G')$. For suppose to the contrary that $\hat{\Xi}(G^a) = \xi(G')$ and that $\lambda = v_0, e_1, v_1, e_2, \ldots, e_m, v_0$ is a simple circuit starting at $v_0 \in \partial G'$ and that $e_1 \in E_{out}(G')$. Note that $m \geq 2$ since $\xi(G')$ is maximal. Then by Lemma 2.2 we can add translates of $G''$ along the vertices of $\lambda$ which are not in $G'$ so that the resulting subgraph $G''$ will satisfy $\xi(G'') \geq \xi(G')$. Moreover, if $G''$ does not contain the edge $e_1$ then by adding this edge to $G''$ we increase $\xi$ - in contradiction to assumption. To see that this is indeed possible, let $y_k \in X \cup X^{-1}$ be the label of $e_k$. Then we need to show that for every $k$ for which $v_k \notin V(G')$, there exists $w_k \in V(G')$ such that $u_k w_k^{-1} \notin V(G')$, where $w_k = y_1 \cdots y_k$ (here we look at the vertices as the group elements they represent). But this follows by the assumption that $V(G')$ is not mapped to itself by the map $g \mapsto gw_k$.

Case 2: let $H'_0 \in \mathcal{F}(G)$ and assume that $\xi(H'_0) \geq \xi(H')$ for every $H' \subseteq H'_0$, or otherwise we will take such a subgraph $H' \subseteq H'_0$ and get a better result. Then, as we have shown above, there exists a subgraph $H'_1$, which is a union of translates of $H'_0$, with $\beta_0(H'_1) \leq c(G^a)\beta_0(H'_0)$ and with

$$\xi(H'_1) \geq \xi(H'_0) + \frac{1}{c(G^a)\beta_0(H'_0)}.$$  \hfill (24)
The same process can now be carried out with \( H'_1 \) and so on, obtaining a sequence \( H'_i \) of subgraphs satisfying

\[
\xi(H'_i) \geq \xi(H'_{i-1}) + \frac{1}{c(G^\alpha)^i \beta_0(\mathcal{H}_0)}.
\] (25)

Thus, passing to the limit, we get that

\[
\hat{\Xi}(G^\alpha) \geq 1 - n + \xi(H'_0) + \frac{1}{(c(G^\alpha) - 1) \beta_0(\mathcal{H}_0)}.
\] (26)

We remark that in case 2 of the above theorem we get in particular, by taking \( G' \) to be the trivial subgraph, that

\[
\hat{\Xi}(G^\alpha) \geq 1 - n + \frac{1}{(c(G^\alpha) - 1)}
\] (27)

and that there exists an exhausting chain \( (H'_i), H'_i \in \mathcal{CF}(G) \), such that

\[
\limsup_{i \to \infty} \frac{|E^{\text{out}}(H'_i)|}{\beta_0(\mathcal{H}'_i)} \leq n - 1 - \frac{1}{(c(G^\alpha) - 1)}.
\] (28)

### 3 Subgroups and Factor Groups

When \( G_2 \) is a homomorphic image of \( G \), with the presentation \( G^{\alpha_2}_{2^{\alpha_2}} \) induced by the presentation \( G^{\alpha}_2 \), then the Cayley graph associated with \( G^{\alpha_2}_{2^{\alpha_2}} \) may be regarded as a quotient of the Cayley graph associated with \( G^{\alpha}_2 \), which implies, as expected, that \( \hat{\Xi}(G^{\alpha_2}_{2^{\alpha_2}}) \geq \hat{\Xi}(G^{\alpha}_2) \). In fact we have the following.

**Theorem 3.1** Let \( G_1 \) be a normal subgroup of \( G \) and let \( G_2 = G/G_1 \) with the presentation \( G^{\alpha_2}_{2^{\alpha_2}} \) induced by \( G^\alpha \). Then

\[
\hat{\Xi}(G^\alpha) \leq \hat{\Xi}(G^{\alpha_2}_{2^{\alpha_2}}) - \left( \frac{1}{|G_2|} - \frac{1}{|G|} \right),
\] (29)

with equality holding if \( G_1 \) is amenable.

**Proof.** We may exclude the cases where \( G_1 \) or \( G_2 \) are trivial. When \( G \) is finite the result follows by Proposition 2.3 since \( \hat{\Xi}(G^\alpha) = 1/|G| \). So assume \( G \) is infinite. We will prove first the inequality in (29). Again by Proposition 2.3
the result is clear when $G_2$ is finite. So we further assume that $G_2$ is infinite. Let $\rho: G \to G_2$ be the covering map from the Cayley graph of $G^\alpha$ onto that of $G_2^{\alpha_2}$. If $G' \in \mathcal{CF}(G)$ then one can decompose $G'$, regarded as a topological space, into a finite number of subspaces $\mathcal{E}_i'$, say $i = 1, \ldots, r$, where each $\mathcal{E}_i'$ is in a different sheet, that is the map

$$p|_{\mathcal{E}_i'}: \mathcal{E}_i' \to \mathcal{E}_i'^{\prime}$$

is bijective and continuous but not necessarily a homeomorphism. Each $\mathcal{E}_i'$ is a subgraph of $G_2' = \rho(G')$. Then $\beta_0(G') = \sum_{i=0}^r \beta_0(\mathcal{E}_i')$, but $\beta_2(G') \leq \sum_{i=0}^r \beta_2(\mathcal{E}_i')$, since having a simple circuit of $G'$ which lies in $k$ sheets results in increasing $\sum_{i=0}^r \beta_2(\mathcal{E}_i')$ by $k$. Therefore $\xi(G') \leq \xi(G_2')$. Since this is true for any $G' \in \mathcal{CF}(G)$, we have $\hat{\xi}(G^\alpha) \leq \hat{\xi}(G_2^{\alpha_2})$.

Suppose now that $G_1$ is amenable. Then we need to show that $\hat{\xi}(G^\alpha) = \hat{\xi}(G_2^{\alpha_2})$ in case $G_2$ is infinite, or that $G$ is amenable in case $G_2$ is finite. Let $X$ be the generating set of $G^\alpha$ and let $G, G_2$ be the Cayley graphs associated with $G^\alpha, G_2^{\alpha_2}$ respectively. Given $\epsilon > 0$, let $G_2' \in \mathcal{CF}(G_2)$ such that

$$\frac{1 - |E^X_{out}(G_2')|}{\beta_0(G_2')} > \hat{\xi}(G_2^\alpha b) - \frac{\epsilon}{3}. \quad (31)$$

Assume also that $G_2'$ is induced, contains the vertex 1, and that $G_2' = G_2$ in case $G_2$ is finite and otherwise $\beta_0(G_2') > 3/\epsilon$. Let $T_2' \in \mathcal{CF}(G_2)$ be a spanning tree of $G_2'$. Each vertex of $T_2'$ is the $n$ assigned a specific element of $G$ (of which we make use in (32) below). $T_2'$ is embedded as a tree $T' \in \mathcal{CF}(G)$, $T' \subseteq p^{-1}(T_2')$ ( $p$ the covering map) with the same vertex and edge labels. Then we take $G' \in \mathcal{CF}(G)$ to be the subgraph induced by $T'$. Let $H_1$ be the subgroup of $G_1$ generated by the set $Y$ consisting of the non-trivial elements

$$y_{v,x} = vx(p(vx))^{-1} \neq 1, \quad v \in V(T'), \ x \in X, \ p(vx) \in V(T'). \quad (32)$$

If $Y$ is empty we take $H_1$ to be the trivial group. Let $H_1$ be the Cayley graph of $H_1$ with respect to $Y$, and let $H_1' \in \mathcal{CF}(H_1)$ with

$$\frac{1 - |E^Y_{out}(H_1')|}{\beta_0(H_1')} > -\frac{\epsilon \beta_0(G_2')}{3}. \quad (33)$$

Such a subgraph exists by the amenability of $H_1$. Let $G'' \in \mathcal{F}(G)$ be the subgraph induced by (the disjoint union) $\bigcup_{g \in V(H_1')} gG'$ (we look here at $g \in V(H_1')$ as an element of $G$ by writing each $y \in Y$ with the generators $X$ of $G$). Let now $e \in E^X_{out}(G'')$ be an edge labeled with $x \in X$ and starting at
\[ g v \in V(G^n), \ g \in V(H'_1), \ v \in V(T') \]. Then either \( p(e) \in E^X_{\text{out}}(G'_2) \), or else \( p(e) \) joins \( v = p(gv) \) and \( u = p(gu) \), for some \( u \in V(T') \). Then

\[ gv \ x = gy_{v,x}p(vx) = gy_{v,x}u. \quad (34) \]

That is, \( e \) is the unique edge corresponding to an edge \( e' \in E^Y_{\text{out}}(H'_1) \) that starts at \( g \) and is in direction \( y_{v,x} \), and this correspondence is 1-1. Then we have

\[
\frac{1 - |E^X_{\text{out}}(G^n)|}{\beta_0(G^n)} = \frac{1 - (\beta_0(H'_1)|E^X_{\text{out}}(G'_2)| + |E^Y_{\text{out}}(H'_1)|)}{\beta_0(H'_1)\beta_0(G'_2)}
\]

\[
= \frac{1 - |E^X_{\text{out}}(G'_2)|}{\beta_0(G'_2)} + \frac{1 - |E^Y_{\text{out}}(H'_1)|}{\beta_0(H'_1)\beta_0(G'_2)} - \frac{1}{\beta_0(G'_2)}
\]

\[
> \hat{\Xi}(G^{\alpha_2}) - \left( \frac{1}{|G_2|} - \frac{1}{|G|} \right) - \epsilon,
\]

since \( G \) is infinite and \( |E^X_{\text{out}}(G'_2)| = 0 \) if \( G_2 \) is finite. That is, \( \hat{\Xi}(G^\alpha) \geq \hat{\Xi}(G^{\alpha_2}) \) if \( G_2 \) is infinite, and \( \hat{\Xi}(G^\alpha) \geq 0 \) if \( G_2 \) is finite. By the the inequalities in the other directions - these are equalities. \( \square \)

**Corollary 3.2** If \( G^\alpha = G_1^{\alpha_1} \ltimes G_2^{\alpha_2} \) and \( G_1 \) is amenable then

\[
\hat{\Xi}(G^\alpha) = \hat{\Xi}(G^{\alpha_2}) - \left( \frac{1}{|G_2|} - \frac{1}{|G|} \right). \quad (36)
\]

**Proof.** By Proposition 2.4 we may assume that the presentation \( G_2^{\alpha_2} \) is induced by the presentation \( G^\alpha \), by adding the generators \( g_i \) of \( G_1^{\alpha_1} \) and the relations \( g_i = 1 \) for every \( i \). Then the result follows immediately by Theorem 3.1. \( \square \)

Another corollary to Theorem 3.1 is the known fact that if both \( G_1 \) and \( G_2 \) are amenable then \( G \) is also amenable.

We remark that \( G_1 \) may be non-amenable but still \( \hat{\Xi}(G^\alpha) = \hat{\Xi}(G^{\alpha_2}) \). For example, let \( G = H \ast H \ast K \) where \( H \) is a 2-generated finite group and \( K \) is free of rank 2. Let \( K_1 \) be a normal subgroup of \( K \) such that \( K/K_1 \simeq H \), let \( G_1 \) be the normal closure of \( K_1 \) in \( G \), and let \( G_2 = G/G_1 \). Then \( G_2 \simeq H \ast H \ast H \) and by Corollary 5.5 (i), \( \hat{\Xi}(G) = \hat{\Xi}(G_2) \) although \( G_1 \) is non-amenable. This situation does not happen when considering the spectral radii \( \overline{R}, R_2 \) associated with symmetric random walks on \( G, G_2 \) respectively, where \( \overline{R} = R_2 \) if and only if \( G_1 \) is amenable (see [4], Theorem 2).
Definition 3.3 Let $G^\alpha$ be a presentation of $G$ with a generating set $X$. Let $T^\alpha$ be a Schreier transversal for a subgroup $G_1$ of $G$ with respect to $G^\alpha$. Then a Schreier basis $Y$ for $G_1$ with respect to $T^\alpha$ consists of the non-trivial (in $G$) elements of the form

$$y_{v,x} = vx(p(vx))^{-1}, \quad v \in T, \ x \in X,$$

where $p$ is the coset map. Notice that the $y_{v,x}$ are not necessarily distinct elements of $G$.

Proposition 3.4 Let $G^\alpha$ be a presentation of a group $G$ and let $G_1^{\alpha_1}$ be a presentation by a Schreier basis of a subgroup $G_1$ of $G$ of finite index. Then

$$\hat{\Xi}(G_1^{\alpha_1}) \leq |G : G_1|\hat{\Xi}(G^\alpha).$$

Proof. If $G$ is finite then $\hat{\Xi}(G_1^{\alpha_1}) = |G : G_1|\hat{\Xi}(G^\alpha)$. Assume that $G$ is infinite. Let $X$ be the set, of cardinality $n$, of generators of $G^\alpha$, and let $Y$ be the Schreier basis of $G_1^{\alpha_1}$, which is of cardinality $\leq 1 + |G : G_1|(n - 1)$. Let $G, G_1$ be the Cayley graphs associated with $G^\alpha, G_1^{\alpha_1}$ respectively. Let $T' \in CF(G)$ be the Schreier tree by which $Y$ is defined, and let $G'_1 \in CF(G)$ be the subgraph induced by $T'$. Given $\epsilon > 0$, let $G''_1 \in CF(G)$ satisfy

$$1 - \frac{|E_{out}(G''_1)|}{\beta_0(G_1')} > \hat{\Xi}(G_1^{\alpha_1}) - |G : G_1|\epsilon,$$

and let $G'' \in F(G)$ be the subgraph induced by $\bigcup_{g \in V(G_1')} gG'$. The edges $E_{out}(G'')$ are in 1-1 correspondence with the edges $E_{out}(G'_1)$. Then

$$1 - \frac{|E_{out}(G'')|}{\beta_0(G'')} = 1 - \frac{|E_{out}(G'_1)|}{|G : G_1|\beta_0(G'_1)} > \hat{\Xi}(G_1^{\alpha_1}) - \epsilon.$$

Thus we showed that

$$\hat{\Xi}(G_1^{\alpha_1}) \leq |G : G_1|\hat{\Xi}(G^\alpha).$$

For example, when $G$ is free then we get an equality in Proposition 3.4.
4 Direct Products

**Theorem 4.1** Let $G_{i}^{α_i}$ be presentations of non-trivial groups $G_i$, for $i = 1, 2$, with disjoint generating sets of cardinalities $n_i$ respectively, and let $G^{α}$ be the induced presentation of $G = G_1 \times G_2$. Then

$$\hat{\Xi}(G^{α}) = \hat{\Xi}(G_{1}^{α_1}) + \hat{\Xi}(G_{2}^{α_2}) - \left( \frac{1}{|G_1|} + \frac{1}{|G_2|} - \frac{1}{|G|} \right).$$  \hspace{1cm} (42)

*Proof.* The claim is true when both $G_1$ and $G_2$ are finite, because then

$$\hat{\Xi}(G^{α}) = \frac{1}{|G|}. \hspace{1cm} (43)$$

When at least one of the groups is infinite we will show first that (42) is an upper bound for $\hat{\Xi}(G^{α})$. Suppose that $G' \in CF(G)$, where $G$ is the Cayley graph of $G$. Then $G'$ is the union of the subgraphs $H'_1, H'_2$, where $H'_i, i = 1, 2$, is the subgraph generated by the edges with labels in $X_i \cup X_i^{-1}$. We may also assume that both $H'_i$ are not empty, because otherwise we can obtain at most $\max(\hat{\Xi}(G_{i}^{α_1}) - n_2, \hat{\Xi}(G_{2}^{α_2}) - n_1)$ which is less than or equals the right hand side of (42). Then $e$

$$\mu(G') - (n_1 + n_2) \leq (\mu(H'_1) - n_1) + \left( \frac{\beta_1(H'_2)}{\beta_0(H'_2)} - n_2 \right). \hspace{1cm} (44)$$

By (11)

$$\mu(H'_1) - n_1 \leq \hat{\Xi}(G_{1}^{α_1}), \hspace{1cm} (45)$$

and

$$\frac{\beta_1(H'_2)}{\beta_0(H'_2)} - n_2 \leq \min(\hat{\Xi}(G_{2}^{α_2}), 0). \hspace{1cm} (46)$$

Therefore, by symmetry, we get from (44) that

$$\hat{\Xi}(G^{α}) \leq \min(\hat{\Xi}(G_{1}^{α_1}), \hat{\Xi}(G_{2}^{α_2}), \hat{\Xi}(G_{1}^{α_1}) + \hat{\Xi}(G_{2}^{α_2})). \hspace{1cm} (47)$$

But these bounds are exactly the ones that appear in (42) in case at least one of the groups is infinite.

It remains to show that (42) is really achieved. Given $\epsilon > 0$ then for $i = 1, 2$ let $H_i \in CF(G_i)$, where $G_i$ is the Cayley graph of $G_{i}^{α_i}$, satisfying

$$\mu(H_i) - n_i > \hat{\Xi}(G_{i}^{α_i}) - \frac{\epsilon}{4}. \hspace{1cm} (48)$$
Suppose also that if $G_i$ is finite then $H_i = G_i$ and otherwise $\beta_0(H_i) > 4/\epsilon$. Let $G'$ be the subgraph of $G$ which is the cartesian product $H_1 \times H_2$. Then

$$\mu(G') - (n_1 + n_2) = \frac{\beta_1(H_1)\beta_0(H_2) + \beta_1(H_2)\beta_0(H_1) + 1}{\beta_0(H_1)\beta_0(H_2)} - (n_1 + n_2)$$

$$= (\mu(H_1) - n_1) + (\mu(H_2) - n_2) - \left( \frac{1}{\beta_0(H_1)} + \frac{1}{\beta_0(H_2)} - \frac{1}{\beta_0(H_1)\beta_0(H_2)} \right)$$

$$> \hat{\Xi}(G_1^{\alpha_1}) + \hat{\Xi}(G_2^{\alpha_2}) - \left( \frac{1}{|G_1|} + \frac{1}{|G_2|} - \frac{1}{|G|} \right) - \epsilon,$$

and the proof is complete.

## 5 Free Products

Let us look at what happens with the computation of $\hat{\Xi}(G^{\alpha})$ for free products. We recall that the decomposition of a group $G$ into non-trivial freely indecomposable factors is unique, up to isomorphism of the factors, as follows by the Kurosh Subgroup Theorem. Such a decomposition contains finitely many factors when $G$ is of finite rank, and in fact, by the corollary to the Grushko-Neumann Theorem, if $G = G_1 \ast G_2$ then $\text{rank}(G) = \text{rank}(G_1) + \text{rank}(G_2)$ (see [7], p. 178). The following expression plays an important role in computing the value of $\hat{\Xi}(G^{\alpha})$.

**Definition 5.1** Let $G^{\alpha}$ be an $n$-generated presentation of a non-trivial group $G$. If $G' \in F^*(G)$ let

$$\psi(G') = \frac{\beta_2(G')}{\beta_0(G') - 1}.$$  

Then we define

$$\Psi(G^{\alpha}) = \sup_{G' \in F^*(G)} \psi(G').$$

and

$$\hat{\Psi}(G^{\alpha}) = 1 - n + \Psi(G^{\alpha}).$$

We call a presentation reduced if none of its generators equals the identity element in the group. Since removing such “redundant generators” does not change the value of $\hat{\Xi}(G^{\alpha})$, no loss of generality is caused when assuming (as we do in Theorem 5.3) that the presentations are reduced. We say that a presentation $G^{\alpha} = \langle X \mid R \rangle$ is minimal if for every proper subset $X'$ of $X$, $Gp(X') \neq G$. (Here $Gp(X')$ is the subgroup of $G$ generated by $X'$.)
Theorem 5.2 Let \( G^\alpha = \langle X \mid R \rangle \) be an \( n \)-generated minimal presentation of a non-trivial group \( G \). Then \( 1 - n \leq \hat{\Psi}(G^\alpha) \leq 1 \). Moreover,

(i) if \( G \) is finite then \( \hat{\Psi}(G^\alpha) = n/(|G| - 1) \), with \( \hat{\Psi}(G^\alpha) = 1 \) if and only if \( |G| = 2 \);

(ii) if \( G \) is infinite then \( \hat{\Psi}(G^\alpha) \leq 0 \), with \( \hat{\Psi}(G^\alpha) = 0 \) if and only if \( G \) is amenable or \( G^\alpha \) is 2-generated and one of the generators is of order 2.

\( \hat{\Psi}(G^\alpha) = 1 - n \) if and only if \( G \) is free of rank \( n \geq 2 \).

If \( H = Gp(Y) \), \( Y \subseteq X \), satisfying

\[ |H| = \min\{|Gp(X')| \mid X' \subseteq X, |X'| = \max\{|X^\prime''| \mid X'' \subseteq X, |Gp(X'')| < \infty\}\} \tag{53} \]

then

\[ \Psi(G^\alpha) = \max(\Xi(G^\alpha), \Psi(H^\alpha)) \], \tag{54} \]

where \( \Psi(H^\alpha) \) is calculated as in (i).

Proof. Let \( G \) be the Cayley graph of \( G^\alpha \).

\[ \hat{\Psi}(G^\alpha) = 1 - n + \sup_{G^\prime \in \mathcal{C}^\alpha(G)} \frac{1 + (n - 1)\beta_0(G^\prime) - |E_{out}^X(G^\prime)|}{\beta_0(G^\prime) - 1} = \sup_{G^\prime} \frac{n - |E_{out}^X(G^\prime)|}{\beta_0(G^\prime) - 1}. \tag{55} \]

Therefore, \( \hat{\Psi}(G^\alpha) \leq 1 \) if and only if

\[ |E_{out}^X(G^\prime)| \geq n + 1 - \beta_0(G^\prime) \tag{56} \]

for every \( G^\prime \in \mathcal{C}^\alpha(G) \). Assume that \( X = \{x_1, \ldots, x_n\} \) and \( X' = \{x_1, \ldots, x_k\} \), \( 0 \leq k \leq n \), is the set of the labels of the edges not going-out of \( G^\prime \). If \( k = 0 \) then \( |E_{out}^X(G^\prime)| \geq n \) and \( \hat{\Psi}(G^\alpha) \leq 0 \). Otherwise, \( G^\prime \) is the union of a finite number of left cosets of \( Gp(X') \). Since the presentation is minimal we have

\[ \beta_0(G^\prime) \geq |Gp(X')| \geq 2^k. \tag{57} \]

Hence

\[ |E_{out}^X(G^\prime)| \geq n - k \geq n - (2^k - 1) \geq n + 1 - \beta_0(G^\prime). \tag{58} \]

Suppose that \( G \) is finite. We will show that \( \psi \) achieves its maximum on \( G \). Let \( G^\prime \in \mathcal{C}^\alpha(G) \) satisfy \( \psi(G^\prime) \geq \psi(H) \) for every \( H \in \mathcal{C}^\alpha(G) \) contained in \( G^\prime \). Let \( G'' \neq G^\prime \) be a left translate of \( G^\prime \) such that \( V(G^\prime \cap G'') \) is not empty, and let \( H' = G' \cup G'' \). Then

\[ \psi(H') \geq \frac{2\beta_2(G^\prime) - \beta_2(G^\prime \cap G'')}{2\beta_0(G^\prime) - \beta_0(G^\prime \cap G'') - 1} \geq \psi(G^\prime), \tag{59} \]
where the right inequality comes from
\[
\frac{a}{b - 1} \geq \frac{c}{d - 1} \iff \frac{2a - c}{2b - d - 1} \geq \frac{a}{b - 1},
\]
(60)
with all denominators positive (the case where \(G'\) meets \(G''\) in a single vertex leads to an equality in (59)).

It is left to examine the case where for every translate \(G''\) of \(G'\), \(V(G' \cap G'')\) is empty. This means that no edge going-out of \(G'\) has the same label as that of an edge of \(G'\). Thus \(G'\) is isomorphic to the Cayley graph of \(H = Gp(X')\), \(|X'| = k\), and by (55)
\[
\psi(G') = k - 1 + \frac{k}{|H| - 1} \leq k.
\]
If \(H\) is then a proper subgroup of \(G\) and \(x_j \notin X'\) then the subgraph \(H'\) which is isomorphic to the Cayley graph of \(H' = Gp(X' \cup X_j)\) satisfies
\[
\psi(H') = k + \frac{k + 1}{|H'| - 1} > k \geq \psi(G').
\]
(62)
We have shown that \(\psi(G') \leq \psi(G)\) for every subgraph \(G' \in CF^*(G)\). Hence
\[
\hat{\Psi}(G^\alpha) = 1 - n + \psi(G) = \frac{n}{|G| - 1}.
\]
(63)
Since by the minimality of the presentation \(|G| \geq 2^n\) then \(\hat{\Psi}(G^\alpha) = 1\) if and only if \(G\) is of order 2.

Suppose now that \(G\) is infinite. If \(\psi(G')\) does not have a maximum on \(CF^*(G)\) then \(\hat{\Psi}(G^\alpha) = \hat{\Xi}(G^\alpha)\) since \(|\psi(G') - \xi(G')| \to 0\) as \(\beta_0(G') \to \infty\). The same is true when \(\psi(G')\) does have a maximum but there is no bound to the size of \(G'\) on which \(\psi\) achieves its maximum, e.g. when \(G = H \ast H\) and \(H\) is finite. In fact, by Corollary 5.5 (vii), when \(G = H \ast H\) then \(\Psi(G^\alpha) = \Psi(H^\alpha)\), and if \(H\) is the Cayley graph of \(H\), embedded in \(G\), then when we adjoin \(m\) translates of \(H\) to form \(K \in CF^*(G)\), each translate intersecting the previous one in a single vertex, then
\[
\psi(K) = \frac{m\beta_2(H)}{m\beta_0(H) - (m - 1) - 1} = \psi(H).
\]
(64)
When none of the above occurs then \(\psi\) achieves its maximum on some \(G' \in CF^*(G)\), which is isomorphic to the Cayley graph of \(H = Gp(X')\), \(|X'| = k < n\). Then by (55)
\[
\hat{\Psi}(G^\alpha) = 1 - n + \psi(G') = 1 - n + (k - 1 + \frac{k}{|H| - 1}) = k - n + \frac{k}{|H| - 1} \leq 0.
\]
(65)
In fact, we see that when \( G \) is non-amenable then \( \hat{\Psi}(G^\alpha) = 0 \) if and only if \( G^\alpha \) is 2-generated and one of the generators is of order 2. When \( G \) is infinite amenable then \( \hat{\Psi}(G^\alpha) = 0 \).

We conclude that for both finite and infinite groups \( G \), if \( H = Gp(Y) \), \( Y \subseteq X \), satisfies

\[
|H| = \min\{|Gp(X')| \mid X' \subseteq X, |X'| = \max\{|X''| \mid X'' \subseteq X, |Gp(X'')| < \infty\}\}
\]

then

\[
\Psi(G^\alpha) = \max(\Xi(G^\alpha), \Psi(H^\alpha)), \tag{66}
\]

where \( \Psi(H^\alpha) = m - 1 + m/(|H| - 1) \), with \( m \) being the number of generators of \( H^\alpha \).

Finally, it is clear that \( \hat{\Xi}(G^\alpha) = 1 - n \) if and only if \( G \) is free of rank \( n \geq 2 \). \( \square \)

When the presentation is not minimal the assertions of Theorem 5.2 do not hold. For example, if \( G^\alpha = \langle x_1, \ldots, x_n \mid x_1 = x_2 = \cdots = x_n, x_1 = 1 \rangle \) and \( n \geq 2 \) then \( \Psi(G^\alpha) = 2n - 1 > n \).

**Theorem 5.3** For each \( i \), \( 1 \leq i \leq r \), \( r \geq 2 \), let \( G^\alpha_i = \langle X_i \mid R_i \rangle \) be a reduced \( n_i \)-generated presentation of a non-trivial group \( G_i \) whose Cayley graph is \( G_i \). Let \( G^\alpha = \langle \bigcup_{i=1}^r X_i \mid \bigcup_{i=1}^r R_i \rangle \) be the induced \( n = \sum_{i=1}^r n_i \)-generated presentation of \( G = G_1 * G_2 * \cdots * G_r \). Assume also, without loss of generality, that \( \Psi(G^\alpha_1) \geq \Psi(G^\alpha_2) \geq \cdots \geq \Psi(G^\alpha_r) \), and let \( G^\alpha_{1,2} \) be the induced presentation of \( G_1 * G_2 \).

(i) If \( \Psi(G^\alpha_1) = \Xi(G^\alpha_1) \) then \( \hat{\Xi}(G^\alpha) = 1 - n + \Xi(G^\alpha_1) \).

(ii) If \( \Psi(G^\alpha_1) = \Psi(H^\alpha_1) > \Xi(G^\alpha_1) \), where \( H_1 < G_1 \) is a finite subgroup generated by \( Y_1 \subseteq X_1 \) as in Theorem 5.2, then

\[
\hat{\Xi}(G^\alpha) = 1 - n + \Xi(G^\alpha) = 1 - n + \max\left(\Xi(G^\alpha_1), \Xi(H^\alpha_1) + \frac{\Psi(G^\alpha_2)}{|H_1|}\right). \tag{68}
\]

**Proof.** Let \( \mathcal{G} \) be the Cayley graph corresponding to \( G^\alpha \). Given \( \epsilon > 0 \) let \( \mathcal{G}' \in \mathcal{CF}^*(\mathcal{G}) \) satisfying

\[
\xi(\mathcal{G}') > \Xi(G^\alpha) - \epsilon. \tag{69}
\]

\( \mathcal{G}' \) has the form \( \mathcal{G}' = \bigcup_{i} \mathcal{H}_i \), where each \( \mathcal{H}_i \) is the disjoint union of \( k_i \geq 0 \) subgraphs \( \mathcal{H}_{i,j} \in \mathcal{CF}^*(\mathcal{G}_i) \), \( j = 1, \ldots, k_i \), which we look at as being subgraphs of \( \mathcal{G}_i \), the Cayley graphs of \( G_i^\alpha \). We say that such a subgraph \( \mathcal{H}_{i,j} \) is of type \( i \). Starting from some \( \mathcal{H}_{i_0,j_0} \), \( \mathcal{G}' \) can be constructed inductively, forming a
tree-like structure, by adding at each stage one of the subgraphs \( H_{i,j} \), which meets the subgraph constructed up to that stage at a single vertex, since there are no other simple circuits except the ones in the subgraphs \( H_{i,j} \) (this is where \( \psi \) comes into the picture). This implies that

\[
\xi(G') = \frac{\sum_{i=1}^{r} \beta_2(H_i)}{\sum_{i=1}^{r} \beta_0(H_i) + 1 - \sum_{i=1}^{r} k_i} = \frac{\sum_{i=1}^{r} \sum_{j=1}^{k_i} \beta_2(H_{i,j})}{1 + \sum_{i=1}^{r} \sum_{j=1}^{k_i} (\beta_0(H_{i,j}) - 1)}.
\]  

(70)

By the form of (70) we see that an upper bound for \( \Xi(G') \) is \( \Psi(G') \), and by using only subgraphs of type 1 and 2 we get a lower bound \( \Xi(G') \geq \Psi(G') \). In case \( \Psi(G') = \Psi(G') \) then \( \Xi(G') = \Psi(G') \). We may then assume that \( G_1' \) is involved in \( G' \) when \( \epsilon \) is small enough. We may also assume that except from \( G_1' \), \( G' \) involves edges from other \( G_i' \) (otherwise we have a connected subgraph \( H \) of \( G_1 \) and we can take two copies of it joined by an edge from some \( X_i, i \neq 1 \), so that \( \xi(H) \) is not changed). We look at the decomposition of \( G' \) into the subgraphs \( H_{i,j} \) as above. Then we reconstruct \( \xi(G') \) in the following way. We start from a subgraph \( H_{1,j_0} \) of type 1. Then we add the subgraph of \( G' \) consisting of some \( H_{i,m} \), of type \( i \neq 1 \) and all the new subgraphs (not including the one we started with) of type 1 joined to it (and there may be none of them). We Continue in an inductive way, so that at the \( m \)-th stage we add some new \( H_{i,m} \), of type \( i \neq 1 \), which is joined to the part constructed up to that stage, and all the new subgraphs of type 1 joined to \( H_{i,m} \). We finish after we cover the whole of \( G' \). We show now that there exists a subgraph \( G'' \) of \( G \), decomposed into copies of only two subgraphs \( H \subseteq G_1 \) and \( H' \subseteq G_2 \), such that \( \xi(G'') \geq \xi(G') \). First we notice that if for some \( H_{i,m} \), \( i \neq 1 \), the number of subgraphs of type 1 joined to it is less than \( \beta_0(H_{i,m}) \) then there are subgraphs of type 1 that we can add to it so that the \( \xi \) does not decrease since \( \Psi(G_1') \geq \Psi(G_1') \) for every \( i \). So let us suppose that indeed each such \( H_{i,m} \) is joined to \( \beta_0(H_{i,m}) \) subgraphs of type 1. Let \( H_{1,j}, k = 1, \ldots, p_m = \beta_0(H_{i,m}) - 1 \), be the new subgraphs of type 1 added at the \( m \)-th stage in the reconstruction of \( G' \). That is, at that stage \( \beta_2 \) is increased by

\[
a_m = \beta_2(H_{i,m}) + \sum_{k=1}^{p_m} \beta_2(H_{1,jk})
\]  

(71)

and \( \beta_0 \) is increased by

\[
b_m = \sum_{k=1}^{p_m} \beta_0(H_{1,jk}).
\]  

(72)
Since $\Psi(G^zz_i) \geq \Psi(G^zz_i)$ for every $i > 2$, then there exist finite connected subgraphs $H'_m$ of type 2 and $H''_m$ of type 1, such that

$$\xi(H''_m) + \frac{\psi(H'_m)}{\beta_0(H''_m)} = \frac{(\beta_0(H'_m) - 1)\beta_2(H''_m) + \beta_2(H'_m)}{(\beta_0(H'_m) - 1)\beta_0(H''_m)} \geq \frac{(\beta_0(H_{i,m,j,m}) - 1)\beta_2(H_{i,j,k}) + \beta_2(H_{i,m,j,m})}{(\beta_0(H_{i,m,j,m}) - 1)\beta_0(H_{i,j,k})}$$

for every $1 \leq m \leq t = \sum_{i-2} k_i$ and $1 \leq k \leq p_m$ (first choose $H'_m$ such that $\psi(H'_m) \geq \psi(H_{i,m,j,m})$ for every $m$, and then an appropriate $H''_m$). Hence it follows that

$$\xi(H''_m) + \frac{\psi(H'_m)}{\beta_0(H''_m)} \geq \frac{a_m}{b_m}$$

(as seen after clearing denominators). Let $H \in CF(G_1), H' \in CF(G_2)$ such that

$$\xi(H) + \frac{\psi(H')}{\beta_0(H)} \geq \xi(H''_m) + \frac{\psi(H'_m)}{\beta_0(H''_m)}$$

for each $m = 1, \ldots, t$. By the last two inequalities the subgraph $G''$ constructed by starting with $H_{1,j_0}$ and adjoining $t$ times the subgraph consisting of a copy of $H'$ and $(\beta_0(H') - 1)$ copies of $H$ satisfies

$$\xi(G'') \geq \xi(G').$$

Since $\epsilon$ was chosen arbitrarily, we get by the form of $G''$ that

$$\hat{\Xi}(G^\alpha) = 1 - n + \sup_{H'_1 \in CF(G_1), H'_2 \in CF(G_2)} \left( \xi(H'_1) + \frac{\psi(H'_2)}{\beta_0(H'_1)} \right)$$

$$= 1 - n + \sup_{H'_1 \in CF(G_1)} \left( \xi(H'_1) + \frac{\Psi(G^zz_{1,2})}{\beta_0(H'_1)} \right)$$

$$= 1 - n + \Xi(G^\alpha_{1,2})$$

Let us define $\zeta(H'_1)$ on $CF(G_1)$ by

$$\zeta(H'_1) = \xi(H'_1) + \frac{\Psi(G^zz_{1,2})}{\beta_0(H'_1)}.$$

Thus we need to find

$$\Xi(G^\alpha) = \sup_{H'_1 \in CF(G_1)} \zeta(H'_1).$$
If \( \Psi(G_1^{α_1}) = \Psi(G_2^{α_2}) \) then as we have seen \( Ξ(G_1) = \Psi(G_1^{α_1}) = ζ(1) \), where 1 is the trivial subgraph. Also, when \( Ξ(G_1^{α_1}) = \Psi(G_1^{α_1}) \) then \( Ξ(G_1) = Ξ(G_1^{α_1}) \) because \( \Psi(G_1^{α_1}) = \Psi(G_1^{α_1}) \) by (70) and \( Ξ(G_1) ≥ Ξ(G_1^{α_1}) \) by the embedding of \( G_1 \) in \( G \). When \( Ξ(G_1^{α_1}) < \Psi(G_1^{α_1}) \) but an arbitrarily large subgraph \( H' \) can be chosen without decreasing \( ζ(H') \) then we get that \( Ξ(G_1^{α_1}) = Ξ(G_1^{α_1}) \) as the second summand in the expression defining \( ζ(H') \) tends to zero when \( β_0(H') \to ∞ \). It remains to check the case where \( ζ \) achieves its maximum on a finite number of members of \( CF(G_1) \). Let \( H'_1 \) be maximal (with respect to the number of vertices) among these subgraphs. We will show that \( H'_1 \) is isomorphic to the Cayley graph of a finite subgroup \( H_1 \) of \( G_1 \) of the form described in Theorem 5.2 (ii) on which \( \psi \) achieves its maximum. For we are given that \( H'_1 \) satisfies \( ζ(H'_1) ≥ ζ(H') \) for every \( H' \in CF(G) \) which is contained in \( H'_1 \). Suppose there exists \( H''_1 ≠ H'_1 \) a left translate of \( H'_1 \) such that \( H'_1 \cap H''_1 ≠ \emptyset \). Let \( H_1 = H'_1 \cup H''_1 \). Then

\[
ζ(H_1) ≥ \frac{2β_2(H'_1) - β_2(H'_1 \cap H''_1) + \Psi(G_2^{α_2})}{2β_0(H'_1) - β_0(H'_1 \cap H''_1)} ≥ ζ(H'_1),
\]

where the right inequality comes from

\[
\frac{a + \Psi}{b} ≥ \frac{c + \Psi}{d} ⇔ \frac{2a - c + \Psi}{2b - d} ≥ \frac{a + \Psi}{b},
\]

with all denominators positive. But this contradicts the maximality of \( H'_1 \). Thus the set of labels of the edges of \( H'_1 \) is disjoint from the set of labels of its outer edges. This means that \( H'_1 \) is isomorphic to the Cayley graph of a subgroup of \( G_1 \). We need to show that \( \psi \) too achieves its maximum on \( H'_1 \). Recall that we are in the case where \( Ψ(G_1^{α_1}) > Ξ(G_1^{α_1}) \), and so \( \psi \) achieves its maximum on some subgraph \( H_1 \) which is isomorphic to the Cayley graph of a finite subgroup \( H_1 < G_1 \). Let \( K_1 \in CF^{s}(G_1) \) be isomorphic to the Cayley graph of a finite subgroup \( K_1 < G_1 \). Thus \( Ψ(H_1^{α_1}) ≥ Ψ(K_1^{α_1}) \). If \( |H_1| ≤ |K_1| \) then since for finite groups

\[
Ψ(H_1^{α_1}) ≥ Ψ(K_1^{α_1}) ⇔ Ξ(H_1^{α_1}) ≥ Ξ(K_1^{α_1}),
\]

we get that

\[
ζ(H_1) = Ξ(H_1^{α_1}) + \frac{Ψ(G_2^{α_2})}{|H_1|} ≥ Ξ(K_1^{α_1}) + \frac{Ψ(G_2^{α_2})}{|K_1|} = ζ(K_1).
\]

So assume \( |H_1| > |K_1| \). Then \( ψ(H_1) ≥ ψ(K_1) \) is equivalent to

\[
ζ(H_1) + \frac{ψ(H_1)}{β_0(H_1)} ≥ ζ(K_1) + \frac{ψ(K_1)}{β_0(K_1)}.
\]
Since $\psi(\mathcal{H}_1) \geq \Psi(G_2^{\alpha_2})$ we have

$$\xi(\mathcal{H}_1) - \xi(\mathcal{K}_1) \geq \psi(\mathcal{H}_1) \left( \frac{1}{\beta_0(\mathcal{K}_1)} - \frac{1}{\beta_0(\mathcal{H}_1)} \right) \geq \Psi(G_2^{\alpha_2}) \left( \frac{1}{\beta_0(\mathcal{K}_1)} - \frac{1}{\beta_0(\mathcal{H}_1)} \right).$$

That is

$$\xi(\mathcal{H}_1) - \xi(\mathcal{K}_1) \geq \Psi(G_2^{\alpha_2}) \left( \frac{1}{\beta_0(\mathcal{K}_1)} - \frac{1}{\beta_0(\mathcal{H}_1)} \right).$$

We conclude that if $H_1$ is a finite subgroup of $G_1$ ($G_1$ may be finite or infinite) generated by some set $Y \subseteq X_1$ satisfying

$$|H_1| = \min\{|Gp(X_1')| \mid X_1' \subseteq X_1, |X_1'| = \max\{|X_1''| \mid X_1'' \subseteq X_1, |Gp(X_1'')| < \infty\}\}$$

then

$$\hat{\Xi}(G^\alpha) = 1 - n + \max \left( \Xi(G_1^{\alpha_1}), \Xi(H_1^{\alpha_1}) + \frac{\Psi(G_2^{\alpha_2})}{|H_1|} \right).$$

If, on the other hand, $Gp(X_1')$ is infinite for every $X_1' \subseteq X_1$ then $\hat{\Xi}(G^\alpha) = 1 - n + \max \Xi(G_1^{\alpha_1})$. □

**Example 5.4** Let $G = G_1 \ast G_2$, where $G_1 = C_2 \ast C_3$ ($C_i$ the cyclic group of order $i$) and $G_2 = C_4$, with all cyclic factors single-generated. Then $\Psi(G_1) = \Psi(C_2) = 1, \Psi(G_2) = 1/3, \Xi(G_1) = \Xi(C_2) + \Psi(C_3)/|C_2| = 1/2 + 1/(2 \cdot 2) = 3/4$, and $\Xi(G) = \Xi(G_1) = 3/4 > 2/3 = 1/2 + 1/(3 \cdot 2) = \Xi(C_2) + \Psi(G_2)/|C_2|$. On the other hand, suppose that $G_1 = C_2 \ast C_4, G_2 = C_3$ and $G = G_1 \ast G_2$. Then $\Psi(G_1) = 1, \Psi(G_2) = 1/2, \Xi(G_1) = 1/2 + 1/(3 \cdot 2) = 2/3$ and $\Xi(G) = 3/4 = 1/2 + 1/(2 \cdot 2) = 1/\Xi(C_2) + \Psi(G_2)/|C_2| > \Xi(G_1)$. We see that when $\Psi(G_1) > \Xi(G_1)$ then either of the possibilities for $\Xi(G)$ that are stated in Theorem 5.3 can occur.

In the following corollary the results either follow immediately from Theorem 5.3 or are already stated within the proof of Theorem 5.3 or follow from the arguments there. Therefore only a partial proof is given. We further assume that the presentations are reduced.

**Corollary 5.5** Let $G = G_1 \ast G_2 \ast \cdots \ast G_r$ with the assumptions of Theorem 5.3. Then the following claims hold.

(i) $\Psi(G_2^{\alpha_2}) \leq \Xi(G^\alpha) \leq \Psi(G_1^{\alpha_1})$.

(ii) $\Xi(G^\alpha) \geq \max \{\Xi(G_i^{\alpha_i})\}$.

$\Xi(G^\alpha) = \Xi(G_1^{\alpha_1})$ if
(a) \( \Xi(G_{1}^{\alpha}) = \Psi(G_{1}^{\alpha}) \); or

(b) \( G_{p}(X_{1}') \) is infinite for every \( X_{1}' \subseteq X_{1} \), in particular if \( G_{1} \) is torsion-free; or

(c) \( \xi(H) \) does not have a maximum on \( \mathcal{F}(G_{1}) \) and \( (c(G_{1}^{\alpha})-1)\Psi(G_{2}^{\alpha_2}) \leq 1 \); or

(d) \( \Psi(K_{1}^{\alpha_1}) \geq \Psi(G_{2}^{\alpha_2}) \), where \( K_{1} = G_{p}(Z_{1}) \), \( Z_{1} = X_{1} - Y_{1} \) is non-empty and \( Y_{1} \subseteq X_{1} \) generates a finite subgroup \( H_{1} \) for which \( \Psi(G_{1}^{\alpha_1}) = \Psi(H_{1}^{\alpha_1}) > \Xi(G_{1}^{\alpha_1}) \).

(iii) \( \hat{\Xi}(G^{\alpha}) \leq 1 - r + \sum_{i=1}^{r} \hat{\Xi}(G_{i}^{\alpha_i}) \), with equality holding if and only if either \( G_{2} \) is free or \( G_{1} \simeq G_{2} \simeq C_{2} \) and \( G_{3} \) (if exists) is free.

(iv) \( \hat{\Xi}(G^{\alpha}) = n_{1} - n \) if \( G_{1} \) is infinite amenable.

(v) \( \Xi(G^{\alpha}) = \Xi(G_{1}^{\alpha_1}) + \frac{\Psi(G_{2}^{\alpha_2})}{|G_{1}|} = n_{1} - 1 + \frac{\Psi(G_{2}^{\alpha_2})+1}{|G_{1}|} \) if \( G_{1} \) is finite.

\( \Xi(G^{\alpha}) = n_{1} - 1 + \frac{n_{2}|G_{2}|}{|G_{1}|(|G_{2}|-1)} \) if \( G_{1} \) and \( G_{2} \) are finite.

(vi) \( \hat{\Xi}(G^{\alpha}) = 1 - r + \frac{1}{|G_{1}|(1 - \frac{1}{|G_{2}|})} \) if all the factors are (finite or infinite) cyclic and single-generated.

(vii) \( \Psi(G^{\alpha}) = \Psi(G_{1}^{\alpha_1}) \).

**Proof.** (ii) (c) For every \( \epsilon > 0 \) there exists \( H \in \mathcal{C}(G_{1}) \) such that

\[ \Xi(G^{\alpha}) \leq \xi(H) + \epsilon = \xi(H) + \frac{\Psi(G_{2}^{\alpha_2})}{\beta_{0}(H)} + \epsilon. \]  

When \( \xi(H) \) does not have a maximum on \( \mathcal{F}(G_{1}) \) (which is the case “in general”) then by Theorem 3.3

\[ \Xi(G_{1}^{\alpha_1}) \geq \xi(H) + \frac{1}{(c(G_{1}^{\alpha_1}) - 1)\beta_{0}(H)}. \]

If, in addition, \( (c(G_{1}^{\alpha_1})-1)\Psi(G_{2}^{\alpha_2}) \leq 1 \) then we get from the two inequalities that

\[ \Xi(G^{\alpha}) \leq \Xi(G_{1}^{\alpha_1}), \]

and by the inequality in the other direction this is an equality.

(ii) (d) \( G_{p}(Y_{1} \cup Z_{1}) < G_{1} \) is a quotient of \( L_{1} = H_{1} \ast K_{1} \) and therefore, by Theorem 3.1 \( \Xi(G_{1}^{\alpha_1}) \geq \Xi(L_{1}^{\alpha_1}) \geq \Xi(H_{1}^{\alpha_1}) + \frac{\Psi(L_{1}^{\alpha_1})}{|H_{1}|} \geq \Xi(H_{1}^{\alpha_1}) + \frac{\Psi(G_{2}^{\alpha_2})}{|H_{1}|} \). Thus, by Theorem 3.3, \( \Xi(G^{\alpha}) = \Xi(G_{1}^{\alpha_1}). \)
(iii) $\hat{\Xi}(G^\alpha) \leq 1 - r + \sum_{i=1}^r \hat{\Xi}(G^\alpha_i)$ since by (70)

$$\xi(G') = \sum_{i=1}^r \frac{\beta_2(H_i)}{\beta_0(G')} \leq \sum_{i=1}^r \xi(H_i).$$

(92)

A necessary condition for equality in (iii) is that for every $\epsilon > 0$ there exists $G'$ such that for every non-free factor $G_i$, $\beta_0(H_i)/\beta_0(G') > 1 - \epsilon$. But this is possible if and only if there is either only one non-free factor, or there are two such factors and each is isomorphic to $C_2$, the group of order 2.

(iv) If $G_1$ is infinite amenable then $\Xi(G_1^\alpha) = \Psi(G_1^\alpha)$ and therefore $\hat{\Xi}(G^\alpha) = 1 - n + \Xi(G_1^\alpha) = n_1 - n$.

(v) When $G_1$ is finite then $\zeta(H'_1)$, $H'_1$ in $\mathcal{CF}(G_1)$ achieves its maximum on $G_1$.

(vi) When $G_1$ is cyclic then $\Xi(G_1^\alpha) = (|G_1|)^{-1} - 1$ and $\Psi(G_1^\alpha) = (|G_1| - 1)^{-1}$. Then when $G_1^\alpha, G_2^\alpha$ is the induced presentation of $G_1 * G_2$ we get from Theorem 5.3 that

$$\hat{\Xi}(G^\alpha) = 1 - r + \Xi(G_1^\alpha, G_2^\alpha) = 1 - r + \frac{1}{|G_1|} + \frac{1}{(|G_2| - 1)|G_1|} = -1 + \frac{1}{|G_1|(1 - \frac{1}{|G_2|})}.$$  

(93)

(vii) This is also clear by (70).

6 The Normalized Balanced Cyclomatic Quotient

The definition we used for $\hat{\Xi}(G^\alpha)$ says that its value has to be looked for in the “best chain” we can find in the graph. If we look on the other hand only on the chain which consists of the concentric balls around 1, we get some “averaging” and in this case the value we calculate depends on the growth of the group.

**Definition 6.1** Let $G^\alpha$ be $n$-generated and let $G$ be the corresponding Cayley graph. Let $B_i$ be the (induced) concentric balls around 1 of radius $i$ in $G$. Then we define the normalized balanced cyclomatic quotient of $G^\alpha$ by

$$\hat{\Theta}(G^\alpha) = 1 - n + \limsup_{i \to \infty} \xi(B_i).$$  

(94)

Clearly $\hat{\Theta}(G^\alpha) \leq \hat{\Xi}(G^\alpha)$, and equality holds when $G$ has subexponential growth, as seen from the proposition below. When $G$ is infinite then $\hat{\Theta}(G^\alpha)$ takes values between $1 - n$ and 0.
Proposition 6.2  The growth of $G$ is exponential if and only if $\hat{\Theta}(G^a) < 0$.

Proof. This follows by having $1 - n + \xi(B_i) = (1 - |E_{out}^X(B_i)|)/\beta_0(B_i)$, and $\beta_0(B_i), i = 0, 1, 2, \ldots$ is (a representative of the equivalent class of) the growth function of $G$. \square

We say that a non-empty subgraph $H \subseteq G$ has thickness $\geq r$ if $H$ contains a non-empty subgraph $H'$ such that $d(v, H') = r$ for every $v \in \partial H$. The supremum on all such $r$ is the thickness of $H$. When $\partial H$ is empty then the thickness of $H$ is $\infty$. Thus every subgraph has thickness $\geq 0$, and it has thickness $\geq 1$ if and only if every vertex on its boundary is adjacent to an interior vertex.

We call a subgraph $H < F$ a supnormal subgroup if the maximal normal subgroup $N$ of $F$ which is contained in $H$ is non-trivial. In the next lemma we refer to the length of a shortest word in $N$, which is the girth of the Cayley graph of $F/H$ in case $H$ is normal.

Lemma 6.3 Let $H$ be a supnormal subgroup of the free group $F$ of rank $n$ and let $m > 0$ be the length of a shortest word in the maximal normal subgroup of $F$ contained in $H$. Then for every finite subgraph $G'$ of the cosets graph of $H$ with thickness $> m/2$

$$\xi(G') \geq \frac{2(n - 1)}{m((2n - 1)^{1+m/2} - 1)}.$$  \hspace{1cm} (95)

Proof. Clearly it is enough to show the claim holds for connected subgraphs. Let $\lambda$ be a fixed circuit of length $m$ corresponding to an element as in the lemma. Let $G'$ be a connected finite subgraph of thickness $\geq m/2$ in the cosets graph of $H$, and let $G''$ be the subgraph of $G'$ induced in $G'$ by the set of vertices of distance $\geq m/2$ from $\partial G'$. We cover $G''$ with copies of $\lambda$, so that each new circuit begins in a vertex not covered yet. Part of the circuits may extend beyond $G''$, but not beyond $G'$. Then

$$\beta_2(G') \geq \frac{1}{m}\beta_0(G'')$$ \hspace{1cm} (96)

since each circuit was counted at most $m$ times. As for the cardinality of $G'$, we have

$$\beta_0(G') \leq \beta_0(G'') + a\beta_0(\partial G'') \leq (1 + a)\beta_0(G''),$$ \hspace{1cm} (97)

where

$$a = \sum_{j=1}^{m/2} (2n - 1)^j.$$ \hspace{1cm} (98)
Therefore
\[
\xi(G') \geq \frac{\beta_0(G''')}{m(1 + a)\beta_0(G''')} = \frac{2(n - 1)}{m((2n - 1)^{1+m/2} - 1)}.
\]
(99)

As an immediate corollary we have

**Proposition 6.4** Let \( G^\alpha = \langle X \mid R \rangle \) be a presentation of a non-free group \( G \) with \(|X| = n\). Let \( m \) be the length of a shortest relator. Then
\[
\hat{\Theta}(G^\alpha) \geq 1 - n + \frac{2(n - 1)}{m((2n - 1)^{1+m/2} - 1)}.
\]
(100)

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