Remarks on Pickands theorem

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Abstract
In this article we present Pickands theorem and his double sum method. We follow Piterbarg’s proof of this theorem. Since his proof relies on general lemmas we present a complete proof of Pickands theorem using Borell inequality and Slepian lemma. The original Pickands proof is rather complicated and is mixed with upcrossing probabilities for stationary Gaussian processes. We give a lower bound for Pickands constant.

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1 Introduction
James Pickands III (see [4] and [5]) gave an elegant and sophisticated way of finding the asymptotic behavior of the probability

$$\mathbb{P}(\sup_{t \in T} X(t) > u)$$

as $u \to \infty$ where $X$ is a Gaussian process. More precisely for $t \in [0,p]$ let $X(t)$ be a continuous stationary Gaussian process with expected value $\mathbb{E}X(t) = 0$ and covariance

$$r(t) = \mathbb{E}(X(t+s)X(s)) = 1 - |t|^\alpha + o(|t|^\alpha)$$

where $0 < \alpha \leq 2$. Furthermore we assume that $r(t) < 1$ for all $t > 0$. Then

$$\mathbb{P}(\sup_{t \in [0,p]} X(t) > u) = H_{\alpha p} u^{2/\alpha} \Psi(u)(1 + o(1))$$
where \( H_\alpha \) is a positive and finite constant (Pickands constant). We will follow Piterbarg’s proof of this theorem. Since his proof relies on general lemmas and some steps in the proof are not clear we present a complete proof of Pickands theorem using Borel inequality and Slepian lemma. Lemma 5 below is different than Lemma D.2. in Piterbarg [6] that is the constant before exponent depends on \( T \).

The original Pickands proof is rather complicated and is mixed with upcrossing probabilities for Gaussian stationary processes. In his paper this theorem is a lemma (see [5]). The proof of Pickands theorem is based on the elementary Bonferroni inequality which in the literature is in a too strong version. In this paper we present a sharper version of the Bonferroni inequality which has an impact on some lower bounds of Pickands constant (see [2] and [7]). Some upper estimates of Pickands constant can be found in [3].

## 2 Lemmas and auxiliary theorems

In the paper we will consider real-valued stochastic processes and fields. Let us denote

\[
\Psi(u) = 1 - \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{s^2}{2}} ds
\]

and notice

\[
\Psi(u) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{u^2}{2}} (1 + o(1))
\]

as \( u \to \infty \). More precisely for \( u > 0 \)

\[
\left( \frac{1}{u} - \frac{1}{u^3} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} < \Psi(u) < \frac{1}{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}.
\]

**Lemma 1** Let \((X_1, X_2)\) be a Gaussian vector with values in \( \mathbb{R}^2 \) with \( \mathbb{E}X_1 = m_1 \), \( \mathbb{E}X_2 = m_2 \), \( \text{Var } X_1 = \sigma_1^2 \), \( \text{Var } X_2 = \sigma_2^2 \) and \( \rho = \text{Cov}(X_1, X_2) \). Then

\[
X_2 = \alpha X_1 + Z
\]

where

\[
\alpha = \frac{\rho}{\sigma_1^2}
\]

and \( Z \) is independent of \( X_1 \) and is normally distributed with mean \( m_2 - \alpha m_1 \) and variance

\[
\sigma_2^2 = \frac{\rho^2}{\sigma_1^2}.
\]

**Lemma 2** (Bonferroni inequality) Let \((\Omega, \mathcal{S}, \mathbb{P})\) be a probability space and \( A_1, A_2, \ldots, A_n \in \mathcal{S} \) for \( n \geq 2 \). Then

\[
\mathbb{P}(\bigcup_{i=1}^n A_i) \geq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i \cap A_j).
\]
Proof: Our proof will follow by induction. For $n = 2$ we have $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$. Thus let us assume that the inequality is true for $n$. Then

$$
\mathbb{P}\left(\bigcup_{i=1}^{n+1} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) + \mathbb{P}(A_{n+1}) - \mathbb{P}\left(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right)
$$

$$
\geq \sum_{i=1}^{n+1} \mathbb{P}(A_i) - \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i \cap A_j) - \mathbb{P}\left(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right)
$$

$$
\geq \sum_{i=1}^{n+1} \mathbb{P}(A_i) - \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i \cap A_j) - \sum_{i=1}^{n} \mathbb{P}(A_i \cap A_{n+1})
$$

$$
= \sum_{i=1}^{n+1} \mathbb{P}(A_i) - \sum_{1 \leq i < j \leq n+1} \mathbb{P}(A_i \cap A_j)
$$

where in the third line we used induction hypothesis. Thus by induction the inequality is valid for all $n \geq 2$.

\[\square\]

Using above Bonferroni inequality we get a sharper (twice as much bigger) than in [2] a lower bound of Pickands constant whose the proof goes the same way as in [2].

**Theorem 1**

$$
H_\alpha \geq \frac{\alpha}{2^{2+\frac{\alpha}{2}} \Gamma\left(\frac{1}{\alpha}\right)}.
$$

The next theorem is also elementary but very useful.

**Theorem 2** (Slepian inequality) Let Gaussian fields $X(t)$ and $Y(t)$ be separable where $t \in T$ and $T$ is an arbitrary parameter set. Moreover we assume that the covariance functions $r_X(t, s) = \mathbb{E}(X(t) - \mathbb{E}X(t))(X(s) - \mathbb{E}X(s))$ and $r_Y(t, s) = \mathbb{E}(Y(t) - \mathbb{E}Y(t))(Y(s) - \mathbb{E}Y(s))$ satisfy

$$
r_X(t, t) = r_Y(t, t)
$$

$$
r_X(t, s) \leq r_Y(t, s)
$$

for all $t, s \in T$ and their expected values fulfill

$$
\mathbb{E}X(t) = \mathbb{E}Y(t)
$$

for all $t \in T$. Then for any $u$

$$
\mathbb{P}\left(\sup_{t \in T} X_t < u\right) \leq \mathbb{P}\left(\sup_{t \in T} Y_t < u\right).
$$
The next theorem is the most important tool in the theory of Gaussian processes (see [1]).

**Theorem 3** (Borell inequality) Let $X(t)$ be a centered a.s. bounded Gaussian field where $t \in T$ and $T$ is an arbitrary parameter set. Then

$$
\mathbb{E} \sup_{t \in T} X(t) = m < \infty, \quad \sup_{t \in T} \text{Var} X(t) = \sigma^2 < \infty,
$$

and for all $w \geq m$

$$
\mathbb{P}(\sup_{t \in T} X(t) > w) \leq \exp \left(-\frac{(w-m)^2}{2\sigma^2}\right).
$$

We will assume that $0 < \alpha \leq 2$. The next lemma one can find in Piterbarg [6] but it is in more general setting which is not necessary in the proof of Pickands theorem.

**Lemma 3** Let $\chi(t)$ be a continuous Gaussian field where $t = (t_1, t_2) \in \mathbb{R}^2$ with $\mathbb{E}\chi(t) = -|t_1|^\alpha - |t_2|^\alpha$ and $\text{Cov}(\chi(t), \chi(s)) = |t_1|^\alpha + |t_2|^\alpha + |s_1|^\alpha + |s_2|^\alpha - |t_1 - s_1|^\alpha - |t_2 - s_2|^\alpha$ ($s = (s_1, s_2)$) and $X(t)$ be a continuous homogeneous Gaussian field where $t = (t_1, t_2) \in \mathbb{R}^2$ with expected value $\mathbb{E}X(t) = 0$ and covariance

$$
r(t) = \mathbb{E}(X(t+s)X(s)) = 1 - |t_1|^\alpha - |t_2|^\alpha + o(|t_1|^\alpha + |t_2|^\alpha).
$$

Then for any compact set $T \subset \mathbb{R}^2$

$$
\mathbb{P}\left(\sup_{t \in u^{-2/\alpha}T} X(t) > u\right) = \Psi(u)H(T)(1 + o(1))
$$

as $u \to \infty$ where

$$
H(T) = \mathbb{E}\exp(\sup_{t \in T} \chi(t)) < \infty.
$$

**Remark 1** The continuity of the field $\chi(t)$ follows from Sudakov, Dudley and Fernique theorem (see [6]).

**Proof:**

$$
\mathbb{P}\left(\sup_{t \in u^{-2/\alpha}T} X(t) > u\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \mathbb{P}\left(\sup_{t \in u^{-2/\alpha}T} X(t) > u|X(0) = s\right) ds
$$

substituting $s = u - \frac{w}{u}$

$$
= \frac{1}{\sqrt{2\pi u}} e^{-u^2/2} \int_{-\infty}^{\infty} e^{w - \frac{w^2}{u}} \mathbb{P}\left(\sup_{t \in u^{-2/\alpha}T} X(t) > u|X(0) = u - \frac{w}{u}\right) ds.
$$

Let us put

$$
\chi_u(t) = u(X(u^{-2/\alpha}t) - u) + w.
$$
Thus let us rewrite the last integral without the function before the integral (which is $\Psi(u)$ as $u \to \infty$)
\[
\int_{-\infty}^{\infty} e^{w - \frac{u^2}{2w^2}} \mathbb{P}(\sup_{t \in T} \chi_u(t) > w | X(0) = u - \frac{w}{u}) \, ds.
\]
Let us compute the expected value and variance of the distribution $\chi_u(t)$ under condition $X(0) = u - \frac{w}{u}$ (this distribution is Gaussian by Lemma 1). By Lemma 1 we get
\[
\mathbb{E}(\chi_u(t)|X(0)) = u \mathbb{E}(X(u^{-2/\alpha}t)|X(0)) - u^2 + w = u\alpha X(0) - u^2 + w
\]
where $\alpha = r(u^{-2/\alpha}t)$.

And by the assumptions it tends to $-|t_1|^\alpha - |t_2|^\alpha$ as $u \to \infty$. Now let us calculate the variance
\[
\text{Var}(\chi_u(t)|X(0) = u - \frac{w}{u}) = u^2 \text{Var}(X(u^{-2/\alpha}t)|X(0) = u - \frac{w}{u}) = u^2 \text{Var}(Z) = u^2(1 - r^2(u^{-2/\alpha}t)) \tag{3}
\]
where $Z$ in the second line is a suitable random variable from Lemma 1 and by the assumptions it tends to $2(|t_1|^\alpha + |t_2|^\alpha)$ as $u \to \infty$. Similarly we compute
\[
\text{Var}(\chi_u(t) - \chi_u(s)|X(0) = u - \frac{w}{u}) = u^2 \text{Var}(X(u^{-2/\alpha}t) - X(u^{-2/\alpha}s)|X(0) = u - \frac{w}{u})
\]
by Lemma 1
\[
= u^2 [\text{Var}(X(u^{-2/\alpha}t) - X(u^{-2/\alpha}s)) - [r(u^{-2/\alpha}t) - r(u^{-2/\alpha}s)]^2].
\]
Thus we get
\[
\text{Var}(\chi_u(t) - \chi_u(s)|X(0) = u - \frac{w}{u}) = u^2 [2[1 - r(u^{-2/\alpha}(t-s))] - [r(u^{-2/\alpha}t) - r(u^{-2/\alpha}s)]^2]
\]
and one can estimate
\[
\text{Var}(\chi_u(t) - \chi_u(s)|X(0) = u - \frac{w}{u}) \leq 2u^2[1 - r(u^{-2/\alpha}(t-s))]
\]
\[
= 2(|t_1 - s_1|^\alpha + |t_2 - s_2|^\alpha) + u^2 o(u^{-2}||t_1 - s_1|^\alpha + |t_2 - s_2|^\alpha))
\]
\[
= (|t_1 - s_1|^\alpha + |t_2 - s_2|^\alpha)(2 + o(1))
\]
where $o(1) \to 0$ if $u \to \infty$ or $|t_1 - s_1| \to 0$ and $|t_2 - s_2| \to 0$. Hence
\[
\text{Var}(\chi_u(t) - \chi_u(s)|X(0) = u - \frac{w}{u}) \leq 3(|t_1 - s_1|^\alpha + |t_2 - s_2|^\alpha) \tag{4}
\]
for $u$ sufficiently large and $t, s$ belonging to a any bounded set of $\mathbb{R}^2$. One can show also that the covariance of $\chi_u(t)$ and $\chi_u(s)$ under condition $X(0) = u - \frac{w}{u}$ tends to $|t_1|^\alpha + |t_2|^\alpha + |s_1|^\alpha + |s_2|^\alpha - |t_1 - s_1|^\alpha - |t_2 - s_2|^\alpha$. Thus the finite dimensional distributions of the field $\chi_u(t)$ under condition $X(0) = u - \frac{w}{u}$ converge to the finite dimensional distributions of $\chi(t)$ and by (4) the distribution of the field $\chi_u(t)$ under condition $X(0) = u - \frac{w}{u}$ converges weakly to $\chi(t)$ as $u \to \infty$.

From the weak convergence

$$\mathbb{P}(\sup_{t \in T} \chi_u(t) > w | X(0) = u - \frac{w}{u}) \to \mathbb{P}(\sup_{t \in T} \chi(t) > w)$$

(5)
as $u \to \infty$. Since the process $\chi_u(t)$ under condition $X(0) = u - \frac{w}{u}$ is continuous on $T$ we get by Borell Theorem 3 that

$$\mathbb{E}(\sup_{t \in T} (\chi_u(t) - ex(u, t)) | X(0) = u - \frac{w}{u}) \leq m < \infty,$$

$$\sup_{t \in T} \text{Var} (\chi_u(t) | X(0) = u - \frac{w}{u}) \leq \sigma^2 < \infty$$

where by (2), (3) and (5) $m$ and $\sigma^2$ depend only on $\alpha$ and

$$\mathbb{P}(\sup_{t \in T} (\chi_u(t) - ex(u, t)) > w | X(0) = u - \frac{w}{u}) \leq \exp \left( \frac{-(w - m)^2}{2\sigma^2} \right)$$

(6)

for all $w \geq m$ for sufficiently large $u$. Since

$$\mathbb{P}(\sup_{t \in T} (\chi_u(t) - m) > w | X(0) = u - \frac{w}{u}) \leq \mathbb{P}(\sup_{t \in T} (\chi_u(t) - ex(u, t)) > w | X(0) = u - \frac{w}{u})$$

and by (6) we have

$$\mathbb{P}(\sup_{t \in T} \chi_u(t) > w | X(0) = u - \frac{w}{u}) \leq \exp \left( \frac{-(w - 2m)^2}{2\sigma^2} \right).$$

(7)

Then the dominated convergence theorem yields that

$$\mathbb{E}[\exp(\sup_{t \in T} \chi_u(t)) | X(0) = u - \frac{w}{u}] \to \mathbb{E}[\exp(\sup_{t \in T} \chi(t))]$$
as $u \to \infty$ and $\mathbb{E}[\exp(\sup_{t \in T} \chi(t))] < \infty$ using (7).

Corollary 1 If $T = [a, b] \times [c, d]$ then

$$H(T) \leq [b - a] [d - c] H([0, 1] \times [0, 1])$$

where $[x]$ is the smallest integer larger or equal $x$. 

\[ \square \]
Proof: We increase our rectangle to the rectangle with the sides of the length \([b - a]\) and \([d - c]\). This rectangle can be divided into \([b - a] [d - c]\) unit squares. By the homogeneity of the random field \(X\) we get the assertion.

Reducing one dimension in the previous lemma we get the following lemma.

Lemma 4 Let \(\chi(t)\) be a continuous stochastic Gaussian process where \(t \in \mathbb{R}\) with 

\[
\mathbb{E}\chi(t) = -|t|^\alpha \quad \text{and} \quad \text{Cov}(\chi(t), \chi(s)) = |t|^\alpha + |s|^\alpha - |t - s|^\alpha \quad (s \in \mathbb{R})
\]

and \(X(t)\) be a continuous stationary Gaussian process where \(t \in \mathbb{R}\) with expected value \(\mathbb{E}X(t) = 0\) and covariance

\[
r(t) = \mathbb{E}(X(t + s)X(s)) = 1 - |t|^\alpha + o(|t|^\alpha).
\]

Then for any \(T > 0\)

\[
\mathbb{P}\left( \sup_{t \in [0, u^{-2}/\alpha T]} X(t) > u \right) = \Psi(u)H(T)(1 + o(1))
\]

as \(u \to \infty\) where

\[
H(T) = \mathbb{E}\exp(\sup_{t \in [0, T]} \chi(t)) < \infty.
\]

Remark 2 Let us notice that \(\chi(t) = B_H(t) - |t|^\alpha\) where \(B_H\) is the fractional Brownian motion with Hurst parameter \(H = \alpha/2\) and \(\mathbb{E}B^2_H(1) = 2\).

Proof: The proof goes the same way as the proof of Lemma 3.

\[\square\]

Corollary 2 For \(T > 0\)

\[
H(T) \leq \lceil T \rceil H([0, 1]).
\]

The next lemma is different than Lemma D.2. in Piterbarg [6] that is the constant before exponent depends on \(T\).

Lemma 5 Let \(0 < \epsilon < 1/2\) and \(0 < \epsilon^\alpha < 1/2\) and \(1 - 2|t|^\alpha \leq r(t) \leq 1 - \frac{1}{2}|t|^\alpha\) for all \(t \in [0, \epsilon]\) where \(X(t)\) is defined in Lemma 4. Then for \(T > 0\), \(t_0 > T\) and \(u\) sufficiently large

\[
\mathbb{P}\left( \sup_{t \in [0, u^{-2}/\alpha T]} X(t) > u, \sup_{t \in [u^{-2}/\alpha t_0, u^{-2}/\alpha(t_0 + T)]} X(t) > u \right) \leq C(\alpha, t_0, T) \Psi(u)
\]

where

\[
C(\alpha, t_0, T) = 4[CT] [C(t_0 + T)] \exp\left(-\frac{1}{8}(t_0 - T)^\alpha\right) H([0, 1] \times [0, 1]).
\]

and \(C = \left(\frac{2\sqrt{2}}{\sqrt{\pi}}\right)^{2/\alpha} 16^{1/\alpha}\)

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Remark 3 Let us notice that the assumption \( r(t) = 1 - |t|^{\alpha} + o(|t|^{\alpha}) \) implies that there exists \( \epsilon > 0 \) such that \( 1 - 2|t|^{\alpha} \leq r(t) \leq 1 - \frac{1}{2}|t|^{\alpha} \) for all \( t \in [0, \epsilon] \).

Proof: Let us consider a Gaussian field \( Y(t, s) = X(t) + X(s) \). Then

\[
\mathbb{P}(\sup_{t \in A} X(t) > u, \sup_{t \in B} X(t) > u) \leq \mathbb{P}(\sup_{(t,s) \in A \times B} Y(t, s) > 2u) \tag{8}
\]

where \( A = [0, u^{-2/\alpha}T] \) and \( B = [u^{-2/\alpha}t_0, u^{-2/\alpha}(t_0 + T)] \). Let us notice

\[
\sigma^2(t, s) = \text{Var} Y(t, s) = 2 + 2r(t - s) = 4 - 2(1 - r(t - s)). \tag{9}
\]

From the assumptions of the lemma for \( |t - s| \leq \epsilon \) we have

\[
\frac{1}{2}|t - s|^{\alpha} \leq 1 - r(t - s) \leq 2|t - s|^{\alpha}
\]

which gives

\[4 - 4|t - s|^{\alpha} \leq \sigma^2(t, s) \leq 4 - |t - s|^{\alpha}.
\]

Thus for sufficiently large \( u \) we get

\[
\inf_{(t,s) \in (A \times B)} \sigma^2(t, s) \geq 4 - 4 \sup_{(t,s) \in (A \times B)} |t - s|^{\alpha} \geq 4 - 4\epsilon^{\alpha} \geq 2 \tag{10}
\]

where in the last inequality we used the assumption of the lemma. Similarly for sufficiently large \( u \) we obtain

\[
\sup_{(t,s) \in (A \times B)} \sigma^2(t, s) \leq 4 - \inf_{(t,s) \in (A \times B)} |t - s|^{\alpha} \leq 4 - |u^{-2/\alpha}(t_0 - T)|^{\alpha} = 4 - u^{-2}(t_0 - T)^{\alpha}. \tag{11}
\]

Let us put

\[Y^*(t, s) = \frac{Y(t, s)}{\sigma(t, s)}\]

where \( \sigma(t, s) \) is defined in (9). Let us estimate the right hand side of (8). Thus for sufficiently large \( u \) we have

\[
\mathbb{P}(\sup_{(t,s) \in A \times B} Y(t, s) > 2u) \leq \mathbb{P}(\exists(t,s) \in A \times B : \frac{Y(t, s)}{\sigma(t, s)} > \frac{2u}{\sigma(t, s)})
\]

\[
\leq \mathbb{P}(\sup_{(t,s) \in A \times B} Y^*(t, s) > \frac{2u}{\sqrt{4 - u^{-2}(t_0 - T)^{\alpha}}}) \tag{12}
\]
where in the last line we used (11). Let us compute the following expectation for \((t, s) \in A \times B\) and \((t_1, s_1) \in A \times B\)

\[
\mathbb{E}[Y^*(t, s) - Y^*(t_1, s_1)]^2 = \mathbb{E}\left[\frac{Y(t, s) - Y(t_1, s_1)}{\sigma(t, s)} + \frac{Y(t_1, s_1)}{\sigma(t, s)} - \frac{Y(t_1, s_1)}{\sigma(t_1, s_1)}\right]^2
\]

\[
\leq 2\mathbb{E}\left[\frac{Y(t, s) - Y(t_1, s_1)}{\sigma(t, s)}\right]^2 + 2\left[\frac{1}{\sigma(t, s)} - \frac{1}{\sigma(t_1, s_1)}\right]^2 \mathbb{E}Y^2(t_1, s_1)
\]

where in the last inequality we used that \((a + b)^2 \leq 2a^2 + 2b^2\) and continuing

\[
\leq \frac{2}{\text{inf}_{(t, s) \in A \times B} \sigma^2(t, s)} \mathbb{E}[Y(t, s) - Y(t_1, s_1)]^2 + 2\left[\frac{1}{\sigma(t, s)} - \frac{1}{\sigma(t_1, s_1)}\right]^2 \sigma^2(t_1, s_1)
\]

\[
= \frac{2}{\text{inf}_{(t, s) \in A \times B} \sigma^2(t, s)} \mathbb{E}[Y(t, s) - Y(t_1, s_1)]^2 + 2\left[\frac{\sigma(t_1, s_1) - \sigma(t, s)}{\sigma(t, s)}\right]^2
\]

using (10) for sufficiently large \(u\) we get

\[
\leq \mathbb{E}[Y(t, s) - Y(t_1, s_1)]^2 + [\sigma(t_1, s_1) - \sigma(t, s)]^2
\]

\[
= \mathbb{E}[X(t) - X(t_1) + X(s) - X(s_1)]^2 + [\sigma(t_1, s_1) - \sigma(t, s)]^2
\]

\[
\leq 2\mathbb{E}[X(t) - X(t_1)]^2 + 2\mathbb{E}[X(s) - X(s_1)]^2 + [\sigma(t_1, s_1) - \sigma(t, s)]^2
\]

where in the last inequality we used that \((a + b)^2 \leq 2a^2 + 2b^2\) and continuing

\[
= 2\mathbb{E}[X(t) - X(t_1)]^2 + 2\mathbb{E}[X(s) - X(s_1)]^2 + [\sigma(t_1, s_1) - \sigma(t, s)]^2
\]

\[
\sigma^2(t_1, s_1) - 2\sigma(t_1, s_1)\sigma(t, s) + \sigma^2(t, s)
\]

\[
= 2\mathbb{E}[X(t) - X(t_1)]^2 + 2\mathbb{E}[X(s) - X(s_1)]^2 + \mathbb{E}Y^2(t_1, s_1) - 2\sqrt{\mathbb{E}Y^2(t_1, s_1)}\mathbb{E}Y^2(t, s) + \mathbb{E}Y^2(t, s)
\]

by Schwarz inequality we obtain

\[
\leq 2\mathbb{E}[X(t) - X(t_1)]^2 + 2\mathbb{E}[X(s) - X(s_1)]^2 + \mathbb{E}Y^2(t_1, s_1) - 2\mathbb{E}[Y(t_1, s_1)Y(t, s)] + \mathbb{E}Y^2(t, s)
\]

\[
= 2\mathbb{E}[X(t) - X(t_1)]^2 + 2\mathbb{E}[X(s) - X(s_1)]^2 + \mathbb{E}[Y(t, s) - Y(t_1, s_1)]^2
\]

\[
= 2\mathbb{E}[X(t) - X(t_1)]^2 + 2\mathbb{E}[X(s) - X(s_1)]^2 + \mathbb{E}[X(t) - X(t_1) + X(s) - X(s_1)]^2
\]
using the inequality \((a+b)^2 \leq 2a^2 + 2b^2\) we get
\[
\leq 4\mathbb{E}[X(t) - X(t_1)]^2 + 4\mathbb{E}[X(s) - X(s_1)]^2.
\] (13)

Since for \(|t - t_1| \leq \epsilon\)
\[
\mathbb{E}[X(t) - X(t_1)]^2 = 2 - 2r(|t - t_1|)
\leq 4|t - t_1|^\alpha
\] (14)

where in the last inequality we used the assumption of the lemma. Thus by (13) and (14) we have for \((t, s) \in A \times B\) and \((t_1, s_1) \in A \times B\) and \(u\) sufficiently large
\[
\mathbb{E}[Y^*(t, s) - Y^*(t_1, s_1)]^2 \leq 16[|t - t_1|^\alpha + |s - s_1|^\alpha].
\] (15)

Since \(\mathbb{E}[Y^*(t, s)]^2 = 1\) and by (15)
\[
\mathbb{E}[Y^*(t, s)Y^*(t_1, s_1)] \geq 1 - 8|t - t_1|^\alpha - 8|s - s_1|^\alpha.
\] (16)

Let us define the following random field
\[
Z(t, s) = \frac{1}{\sqrt{2}}(\eta_1(t) + \eta_2(s))
\] (17)
where \(\eta_1\) and \(\eta_2\) are independent Gaussian stationary processes with \(\mathbb{E}\eta_1(t) = \mathbb{E}\eta_2(t) = 0\) and \(\mathbb{E}[\eta_i(t)\eta_i(s)] = \exp(-32|t - s|^\alpha)\) for \(i = 1, 2\). Hence
\[
\mathbb{E}[Z(t, s)Z(t_1, s_1)] = \frac{1}{2}(\mathbb{E}[\eta_1(t)\eta_1(s) + \mathbb{E}[\eta_2(t)\eta_2(s)])
\leq \frac{1}{2}[\exp(-32|t - t_1|^\alpha) + \exp(-32|s - s_1|^\alpha)]
\leq 1 - 16|t - t_1|^\alpha - 16|s - s_1|^\alpha + o(\sqrt{|t - t_1|^{2\alpha} + |s - s_1|^{2\alpha}})
\] (18)

where in the last line we used Taylor formula \(\frac{1}{2}(e^{-x} + e^{-y}) = 1 - \frac{1}{2}x - \frac{1}{2}y + o(\sqrt{x^2 + y^2})\). Let us notice that for \(a \geq 0\) and \(b \geq 0\) we get \(a + b \geq \sqrt{a^2 + b^2}\) and additionally for \(a\) and \(b\) sufficiently small we have \(\sqrt{a^2 + b^2} \geq o(\sqrt{a^2 + b^2})\). Thus for \(a \geq 0\) and \(b \geq 0\) and sufficiently small we obtain \(a + b \geq o(\sqrt{a^2 + b^2})\). Hence for sufficiently small \(|t - t_1|\) and \(|s - s_1|\) we get
\[
1 - 8|t - t_1|^\alpha - 8|s - s_1|^\alpha \geq 1 - 16|t - t_1|^\alpha - 16|s - s_1|^\alpha + o(\sqrt{|t - t_1|^{2\alpha} + |s - s_1|^{2\alpha}}).
\]

Thus by (16) and (18) it follows
\[
\mathbb{E}[Y^*(t, s)Y^*(t_1, s_1)] \geq \mathbb{E}[Z(t, s)Z(t_1, s_1)]
\] (19)
for sufficiently small \(|t - t_1|\) and \(|s - s_1|\). Hence by Slepian inequality we have for large \(u\)
\[
\mathbb{P}\left(\sup_{(t,s) \in A \times B} Y^*(t, s) > u^*\right) \leq \mathbb{P}\left(\sup_{(t,s) \in A \times B} Z(t, s) > u^*\right)
\] (20)
where
\[ u^* = \frac{2u}{\sqrt{4 - u^{-2}(t_0 - T)^\alpha}} \]
(see (12)). Let us put
\[ \eta(t, s) = Z \left( \frac{t}{16^{1/\alpha}}, \frac{s}{16^{1/\alpha}} \right) \]
then
\[ \mathbb{P}( \sup_{(t,s)\in A\times B} Z(t, s) > u^*) = \mathbb{P}( \sup_{(t,s)\in A'\times B'} \eta(t, s) > u^*) \] (21)
where \( A' = [0, u^{-2/\alpha}T16^{1/\alpha}] \) and \( B' = [u^{-2/\alpha}t_0 16^{1/\alpha}, u^{-2/\alpha}(t_0 + T)16^{1/\alpha}] \). Let us notice that \( \eta(t, s) \) satisfies the assumptions of the Lemma 3. For
\[ u \geq u_0 = \left[ \frac{(t_0 - T)^{\alpha/2}}{\epsilon} \right] \]
we get
\[ \frac{u^*}{u} = \frac{2}{\sqrt{4 - u^{-2}(t_0 - T)^\alpha}} \leq \frac{2}{\sqrt{4 - u_0^{-2}(t_0 - T)^\alpha}} = \frac{2}{\sqrt{4 - \epsilon^2}} < \frac{2\sqrt{2}}{\sqrt{7}} \]
where in the last inequality we used the assumption of the lemma that \( \epsilon^\alpha < \frac{1}{2} \). Thus it follows that \( A' \subset [0, (u^* \sqrt{\frac{7}{2\sqrt{2}}})^{-2/\alpha} T 16^{1/\alpha}] \) and \( B' \subset [0, (u^* \sqrt{\frac{7}{2\sqrt{2}}})^{-2/\alpha}(t_0 + T)16^{1/\alpha}] \).
Let us define \( T = [0, (\sqrt{\frac{7}{2\sqrt{2}}})^{-2/\alpha} T 16^{1/\alpha}] \times [0, (\sqrt{\frac{7}{2\sqrt{2}}})^{-2/\alpha}(t_0 + T)16^{1/\alpha}] \). Hence
\[ \mathbb{P}( \sup_{(t,s)\in A'\times B'} \eta(t, s) > u^*) \leq \mathbb{P}( \sup_{(t,s)\in (u^*)^{-2/\alpha}T} \eta(t, s) > u^*) \]
\[ = \Psi(u^*) H(T)(1 + o(1)) \] (22)
as \( u \to \infty \) where in the last line we used Lemma 3. By the fact that \( \frac{1}{1-x} \geq 1 + x \) for \( x < 1 \) we get for sufficiently large \( u \)
\[ (u^*)^2 = \frac{4u^2}{4 - u^{-2}(t_0 - T)^\alpha} \geq u^2[1 + \frac{1}{4}u^{-2}(t_0 - T)^\alpha] = u^2 + \frac{1}{4}(t_0 - T)^\alpha \geq u^2. \]
Thus using (1) we deduce that for sufficiently large \( u \)
\[ \Psi(u^*) \leq 2\Psi(u) \exp(-\frac{1}{8}(t_0 - T)^\alpha). \]
Hence by (22) it follows for sufficiently large \( u \)
\[ \mathbb{P}( \sup_{(t,s)\in A'\times B'} \eta(t, s) > u^*) \leq 2\Psi(u) \exp(-\frac{1}{8}(t_0 - T)^\alpha) H(T)(1 + o(1)) \]
\[ \leq 4\Psi(u) \exp(-\frac{1}{8}(t_0 - T)^\alpha) H(T). \] (23)
From Corollary 1 we have that
\[ H(T) \leq H([0, 1] \times [0, 1])[(\sqrt{\frac{7}{2\sqrt{2}}})^{-2/\alpha} T 16^{1/\alpha}][(\sqrt{\frac{7}{2\sqrt{2}}})^{-2/\alpha}(t_0 + T)16^{1/\alpha}] \] (24)
Thus collecting (8), (12), (20), (21), (23) and (24) we get the assertion of the lemma. \( \square \)
3 Pickands theorem

**Theorem 4** (Pickands) Let $X(t)$ where $t \in [0, p]$ be a continuous stationary Gaussian process with expected value $\mathbb{E} X(t) = 0$ and covariance

$$r(t) = \mathbb{E}(X(t + s)X(s)) = 1 - |t|^\alpha + o(|t|^\alpha).$$

Furthermore we assume that $r(t) < 1$ for all $t > 0$. Then

$$\mathbb{P}(\sup_{t \in [0, p]} X(t) > u) = H_\alpha p u^{2/\alpha} \Psi(u)(1 + o(1))$$

as $u \to \infty$ where

$$H_\alpha = \lim_{T \to \infty} \frac{H(T)}{T}$$

is positive and finite (Pickands constant).

**Complete proof:** Put

$$\Delta_k = [ku^{-2/\alpha}T, (k + 1)u^{-2/\alpha}T]$$

where $k \in \mathbb{N}$ and $T \geq p$ and $N_p = \left\lfloor \frac{p}{u^{-2/\alpha}T} \right\rfloor$. Thus

$$\mathbb{P}(\sup_{t \in [0, p]} X(t) > u) \leq \sum_{k=0}^{N_p} \mathbb{P}(\sup_{t \in \Delta_k} X(t) > u) = (N_p + 1)\mathbb{P}(\sup_{t \in \Delta_0} X(t) > u)$$

where in the last equality we use stationarity of the process $X$. Thus using Lemma 4 we get

$$\limsup_{u \to \infty} \frac{\mathbb{P}(\sup_{t \in [0, p]} X(t) > u)}{u^{2/\alpha} \Psi(u)} \leq \frac{p}{T} H(T). \quad (25)$$

Let us estimate our probability from below

$$\mathbb{P}(\sup_{t \in [0, p]} X(t) > u) \geq \mathbb{P}(\bigcup_{k=0}^{N_p-1} \{\sup_{t \in \Delta_k} X(t) > u\}) \geq N_p \mathbb{P}(\sup_{t \in \Delta_0} X(t) > u)$$

$$- \sum_{0 \leq i < j \leq N_p-1} \mathbb{P}(\sup_{t \in \Delta_i} X(t) > u, \sup_{t \in \Delta_j} X(t) > u) \quad (26)$$

where in the last inequality we applied Lemma 2. Let us consider the last double sum (that is why the method is called double sum method)

$$\Sigma_2 = \sum_{0 \leq i < j \leq N_p-1} \mathbb{P}(\sup_{t \in \Delta_i} X(t) > u, \sup_{t \in \Delta_j} X(t) > u)$$

$$= \sum_{k=1}^{N_p-1} (N_p - k)\mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in \Delta_k} X(t) > u)$$

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\[ \leq \, N_p \mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in \Delta_1} X(t) > u) \\
+ N_p \sum_{k=2}^{N/4-1} \mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in \Delta_k} X(t) > u) \\
+ N_p \sum_{k=N/4}^{N-1} \mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in \Delta_k} X(t) > u). \]

Let us denote the last three terms \( A_1, A_2 \) and \( A_3 \), respectively. First let us consider \( A_3 \) and take \( u \) such that \( \frac{u}{\alpha T} \leq \epsilon/16 \). Then it is easy to notice that the distance of the intervals \( \Delta_0 \) and \( \Delta_k \) is at least \( \epsilon/4 \) in \( A_3 \). Hence in \( A_3 \) (for \( k \) from \( A_3 \)) for \((t, s) \in \Delta_0 \times \Delta_k\) we have
\[
\text{Var} (X(t) + X(s)) = 2 + 2r(t - s) \\
= 4 - 2(1-r(t-s)) \\
\leq 4 - 2 \inf_{s \geq \epsilon/4} (1-r(s)) \\
= 4 - \delta < 4 \tag{27}
\]
where \( \delta = 2 \inf_{s \geq \epsilon/4} (1-r(s)) \). Let us notice that \( X(t) + X(s) \) is a continuous Gaussian field on \([0, T] \times [0, T]\) which implies by Borell Theorem 3 that
\[
\mathbb{E} \sup_{(t,s) \in \Delta_0 \times \Delta_k} (X(t) + X(s)) \leq m \tag{28}
\]
and by (27) and (28) we get
\[
\mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in \Delta_k} X(t) > u) \leq \mathbb{P}(\sup_{(t,s) \in \Delta_0 \times \Delta_k} X(t) + X(s) > 2u) \\
\leq \exp \left( -\frac{(2u-m)^2}{2(4-\delta)} \right) \\
= \exp \left( -\frac{(u-m/2)^2}{2(1-\delta/4)} \right) \\
\leq \exp \left( -\frac{1}{2} \left( \frac{u-m/2}{1-\delta/8} \right)^2 \right)
\]
where in the last inequality we used the fact that \( 1 - \delta/4 \leq (1 - \delta/8)^2 \). Hence
\[
\limsup_{u \to \infty} \frac{A_3}{N_p \Psi(u)} \leq \limsup_{u \to \infty} \frac{N_p^2 \exp \left( -\frac{1}{2} \left( \frac{u-m/2}{1-\delta/8} \right)^2 \right)}{N_p \Psi(u)} \\
= \lim_{u \to \infty} \left| \frac{p}{u^{-2/\alpha T}} \right| \sqrt{2\pi} u \exp\left( -\frac{1}{2} \left( \frac{u-a/2}{1-\delta/8} \right)^2 + \frac{1}{2} u^2 \right) \\
= 0 \tag{29}
\]
where the second line follows from (1) and the fact that \( 1 - \delta/8 < 1 \) (by the assumption \( r(t) < 1 \) for \( t > 0 \)).
Now let us consider $A_2$. For $k \geq 2$ we have from Lemma 5 ($C_1$ and $C_2$ constants depending on $\alpha$)

\[ \mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in \Delta_k} X(t) > u) \leq C_1 \left[ C_2 T \right] \left[ C_2 (k + 1) T \right] \exp(-\frac{1}{8}(k-1)^{\alpha}T^\alpha) \Psi(u). \]

Thus

\[ A_2 \leq C_1 \left[ C_2 T \right] \Psi(u) N_p \sum_{k=2}^{N_{/4}} \left[ C_2 (k + 1) T \right] \exp(-\frac{1}{8}(k-1)^{\alpha}T^\alpha) \]

and let us estimate

\[ \sum_{k=2}^{N_{/4}} \left[ C_2 (k + 1) T \right] \exp(-\frac{1}{8}(k-1)^{\alpha}T^\alpha). \]

We have

\[ \sum_{k=2}^{N_{/4}} \left[ C_2 (k + 1) T \right] \exp(-\frac{1}{8}(k-1)^{\alpha}T^\alpha) \leq \sum_{k=2}^{\infty} \left[ C_2 (k + 1) T \right] \exp(-\frac{1}{8}(k-1)^{\alpha}T^\alpha) \]

\[ \leq \left[ C_2 T \right] \sum_{k=2}^{\infty} (k + 1) \exp(-\frac{1}{8}(k-1)^{\alpha}T^\alpha) \]

\[ = \left[ C_2 T \right] \sum_{k=1}^{\infty} (k + 2) \exp(-\frac{1}{8}k^aT^\alpha) \]

\[ \leq 3 \left[ C_2 T \right] \sum_{k=1}^{\infty} k \exp(-\frac{1}{8}k^aT^\alpha) \]

\[ \leq 6 \left[ C_2 T \right] \int_{1/2}^{\infty} (s + 1) \exp(-\frac{1}{8}s^aT^\alpha) ds \]

substituting $t = \frac{1}{6}s^aT^\alpha$ we continue (from now on $C$, $C'$, $C_1$ and $C_2$ will be any positive constant depending on $\alpha$ and their values can change from line to line)

\[ \leq \frac{C [T]}{T} \int_{\alpha/(8,2\alpha)}^{\infty} \left( \frac{8^{1/\alpha}}{T} t^{2/a-1} + t^{1/a-1} \right) \exp(-t) dt \]

using the following property of the incomplete gamma function

\[ \int_{u}^{\infty} s^p e^{-s} ds = u^p e^{-u}(1 + O(1/u)) \]

for $u \to \infty$ where $p \in \mathbb{R}$ we keep on estimating

\[ \leq \frac{C_1 [T]}{T^\alpha} \exp(-CT^\alpha)(1 + O(T^{-\alpha})) + \frac{C_2 [T]}{T^\alpha} \exp(-CT^\alpha)(1 + O(T^{-\alpha})) \]

\[ = \frac{C' [T]}{T^\alpha} \exp(-CT^\alpha)(1 + O(T^{-\alpha})). \]

Thus we get

\[ A_2 \leq \frac{C' [T]^2}{T^\alpha} \Psi(u) N_p \exp(-CT^\alpha)(1 + O(T^{-\alpha})) \]

which yields

\[ \limsup_{u \to \infty} \frac{A_2}{\Psi(u) N_p} \leq \frac{C' [T]^2}{T^\alpha} \exp(-CT^\alpha)(1 + O(T^{-\alpha})). \]
Now let us consider the $A_1$ term. Thus

\[
\mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in \Delta_1} X(t) > u) \leq \mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in u^{-2/\alpha}[T, T+\sqrt{T}]} X(t) > u) \\
+ \mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in u^{-2/\alpha}[T+\sqrt{T}, 2T+\sqrt{T}]} X(t) > u) \\
\leq \mathbb{P}(\sup_{t \in u^{-2/\alpha}[T, T+\sqrt{T}]} X(t) > u) + \mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in u^{-2/\alpha}[T+\sqrt{T}, 2T+\sqrt{T}]} X(t) > u) \\
= \mathbb{P}(\sup_{t \in [0, u^{-2/\alpha}\sqrt{T}]} X(t) > u) + \mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in u^{-2/\alpha}[T+\sqrt{T}, 2T+\sqrt{T}]} X(t) > u).
\]

(31)

First let us consider the second term from (31). By Lemma 5 we have

\[
\mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in u^{-2/\alpha}[T+\sqrt{T}, 2T+\sqrt{T}]} X(t) > u) \\
\leq 4[C T][C (2T + \sqrt{T})]\exp(-\frac{1}{8} T^{\alpha/2})H([0, 1] \times [0, 1])\Psi(u).
\]

The first term from (31) can be estimated by Lemma (4)

\[
\mathbb{P}(\sup_{t \in [0, u^{-2/\alpha}\sqrt{T}]} X(t) > u) = \Psi(u)H(\sqrt{T})(1 + o(1)).
\]

Hence we obtain

\[
\mathbb{P}(\sup_{t \in \Delta_0} X(t) > u, \sup_{t \in \Delta_1} X(t) > u) \leq \Psi(u)H_\alpha(\sqrt{T})(1 + o(1)) \\
+ C[T][2T + \sqrt{T}]\exp(-\frac{1}{8} T^{\alpha/2})\Psi(u) \\
\leq \Psi(u)[\sqrt{T}]H_\alpha(1) + o(1) \\
+ C[T][2T + \sqrt{T}]\exp(-\frac{1}{8} T^{\alpha/2})\Psi(u)
\]

(32)

where in the last inequality we used Corollary 2. Thus we get

\[
\limsup_{u \to \infty} \frac{A_1}{N_p\Psi(u)} \leq [\sqrt{T}]H_\alpha(1) + C[T][2T + \sqrt{T}]\exp(-\frac{1}{8} T^{\alpha/2}).
\]

(33)

Thus let us consider the lower bound

\[
\liminf_{u \to \infty} \frac{\mathbb{P}(\sup_{t \in [0,p]} X(t) > u)}{p u^{2/\alpha}\Psi(u)} = \liminf_{u \to \infty} \frac{\mathbb{P}(\sup_{t \in [0,p]} X(t) > u)}{N_pT\Psi(u)}
\]

which by Lemma 4, (25), (29), (30) and (33) is bigger or equal than

\[
f(T) = \frac{H(T)}{T} - \frac{C'}{T^{\alpha+1}}\exp(-CT^{\alpha})(1 + O(T^{-\alpha}))
\]

(34)
\[-\frac{\sqrt{T}}{T} H(1) - C\frac{T}{T} \left[2T + \sqrt{T}\right] \exp\left(-\frac{1}{8} T^{\alpha/2}\right).\]

Let us assume that \(\limsup_{T \to \infty} \frac{H(T)}{T} > 0\) then by (25) and (34) we get

\[
\frac{H(T)}{T} \geq \limsup_{u \to \infty} \frac{\mathbb{P}(\sup_{t \in [0,1]} X(t) > u)}{u^{2/\alpha} \Psi(u)} \geq \liminf_{u \to \infty} \frac{\mathbb{P}(\sup_{t \in [0,1]} X(t) > u)}{u^{2/\alpha} \Psi(u)} \geq \limsup_{S \to \infty} f(S) = \limsup_{S \to \infty} \frac{H(S)}{S}
\]

which implies

\[
\infty > \liminf_{T \to \infty} \frac{H(T)}{T} = \limsup_{T \to \infty} \frac{H(T)}{T} > 0
\]

and

\[
\lim_{T \to \infty} \frac{H(T)}{T}
\]

exists and is finite and positive. It remains to prove that \(\limsup_{T \to \infty} \frac{H(T)}{T} > 0\).

Thus let us put \(D = \bigcup_{j=0}^{\infty} \Delta_j \cap [0,1]\) then

\[
\mathbb{P}(\sup_{t \in [0,1]} X(t) > u) \geq \mathbb{P}(\sup_{t \in D} X(t) > u).
\]

Applying Bonferroni inequality for the set \(D\) (Lemma 2 and see (26) and using Lemma 4 and bounds for \(A_2\) and \(A_3\) (note that \(A_1\) disappears by the definition of the set \(D\)) we get

\[
\frac{H(T)}{T} \geq \limsup_{u \to \infty} \frac{\mathbb{P}(\sup_{t \in [0,1]} X(t) > u)}{u^{2/\alpha} \Psi(u)} \geq \frac{H(S)}{2S} - \frac{C'' [S]^2}{S^{\alpha+1}} \exp(-CS^\alpha)(1 + O(S^{-\alpha}))
\]

\[
= S^{-1} \left(\frac{H(S)}{2} - \frac{C'' [S]^2}{S^\alpha} \exp(-CS^\alpha)(1 + O(S^{-\alpha}))\right)
\]

which is positive for sufficiently large \(S\) because \(H(S)\) is increasing function of \(S\) and \(\frac{C'' [S]^2}{S^\alpha} \exp(-CS^\alpha)(1 + O(S^{-\alpha}))\) tends to 0 when \(S \to \infty\).

\[\square\]

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