A linear-time algorithm for generalized trust region problems*

Rujun Jiang† Duan Li‡

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Abstract

In this paper, we provide the first provable linear-time (in the number of non-zero entries of the input) algorithm for approximately solving the generalized trust region subproblem (GTRS) of minimizing a quadratic function over a quadratic constraint under some regular conditions. Our algorithm is motivated by and extends a recent linear-time algorithm for the trust region subproblem by Hazan and Koren [Math. Program., 2016, 158(1-2): 363-381]. However, due to the non-convexity and non-compactness of the feasible region, such an extension is nontrivial. Our main contribution is to demonstrate that under some regular condition, the optimal solution is in a compact and convex set and lower and upper bounds of the optimal value can be computed in linear time. Using these properties, we develop a linear-time algorithm for the GTRS.

1 Introduction

We consider in this paper the following generalized trust region subproblem (GTRS),

\[(GTRS) \quad \min_{x} x^T Ax + 2a^T x \]
\[\text{s.t.} \quad x^T Bx + 2b^T x + d \leq 0,\]

where \(A\) and \(B\) are \(n \times n\) symmetric matrices which are not necessarily positive semidefinite, \(a, b \in \mathbb{R}^n\) and \(d \in \mathbb{R}\).

When the constraint in (GTRS) is a unit ball, the problem reduces to the classical trust region subproblem (TRS). The TRS first arose in trust region methods for nonlinear optimization [6]. In the meanwhile, the TRS also finds applications in the least square problems [30] and robust optimization [2]. Various approaches have been derived to solve the TRS and its variant with additional linear constraints, see [17, 19, 28, 22, 24, 29, 4, 5, 27]. Recently, Hazan and Koren [9] proposed the first linear-time algorithm (with respect to the nonzero entries in the input) for the TRS, via a linear-time eigenvalue oracle and a linear-time SDP solver based on approximate eigenvector computations [16]. After that, Wang and Xia [25] and Ho-Nguyen and Kilinc-Karzan

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†School of Data Science, Fudan University, Shanghai, China, rjjiang@fudan.edu.cn
‡School of Data Science, City University of Hong Kong, Hong Kong, dli226@cityu.edu.hk
also presented linear-time algorithms to solve the TRS by applying Nesterov’s accelerated gradient descent algorithm to a convex reformation of the TRS, which can also be obtained in linear time.

As a generalization of the TRS, the GTRS has received a lot of attentions in the literature; the GTRS also admits its own applications such as time of arrival problems [10] and subproblems of consensus ADMM in signal processing [12]. Numerous methods have been developed for solving the GTRS under various assumptions; see, for example, [18, 23, 3, 24, 7]. Recently, Ben-Tal and den Hertog [2] showed that if the two matrices in the quadratic forms are simultaneously diagonalizable (SD) (see [13] for more details about SD conditions), the GTRS can be then transformed into an equivalent second order cone programming (SOCP) formulation and thus can be solved efficiently. Jiang et al. [15] derived an SOCP reformulation for the GTRS when the problem has a finite optimal value and further derived a closed form solution when the SD condition fails. Pong and Wolkowicz [21] proposed an efficient algorithm based on minimum generalized eigenvalue of a parameterized matrix pencil for the GTRS, extending the ERW algorithm for the TRS [22]. Recently, Adachi and Nakatsukasa [1] also developed a novel eigenvalue-based algorithm to solve the GTRS. Jiang and Li [14] proposed a novel convex reformulation for the GTRS and derived an efficient first order method to solve the reformulation. However, there is no linear-time algorithm for the GTRS, while Hazan and Koren [9] has already proposed their linear-time algorithm for the TRS. Although it is more general than the TRS, the GTRS still enjoys hidden convexity as the TRS due to the celebrated S-lemma [26, 20]. There is also evidence that the closely related generalized eigenvalue problem for a positive definite matrix pencil can be solved in linear time [8]. Then a natural question is whether or not there exists a linear-time algorithm for the GTRS. We give a positive answer to this question in this paper.

In this paper, we derive a linear-time algorithm to approximately solve the GTRS with high correct probability. The main difficulties in deriving a linear-time algorithm for the GTRS comes from the non-convex constraint of the GTRS; while this challenging point is not present in the TRS as the constraint in the TRS is convex. More specifically, the non-convexity of the constraint implies the unboundedness of the feasible region and makes it hard to derive an initial lower bound for the GTRS in linear-time. These difficulties make the direct generalization of the linear-time algorithm for TRS in [9] inapplicable to the GTRS. By addressing these difficulties, we are able to propose a linear-time algorithm for the GTRS based on the work in [9]. Moreover, our algorithm also inherits good property of the algorithm in [9] that avoids the so called hard case by using approximate eigenvector methods.

The basic idea in our method is to check the feasibility of the following system and then find an $\epsilon$ optimal solution with a binary search over $c$,

\[
\begin{align*}
    x^T A x + 2a^T x & \leq c \\
    x^T B x + 2b^T x + d & \leq 0,
\end{align*}
\]

where $c \in [l, u]$ and $l$ and $u$ are some lower and upper bounds for (GTRS), respectively. In the TRS, $l$ and $u$ can be trivially estimated in linear time [9]. However, a linear-time estimation of the lower and upper bounds are nontrivial in the GTRS. We propose linear-time subroutines that can
find a dual feasible solution that identifies a lower bound for the primal problem by weak duality and an upper bound by constructing a feasible solution to the primal problem. The heart of the binary search is that if system (1) is feasible, then $c$ is an upper bound for (GTRS), otherwise system (1) is infeasible and $c$ is a lower bound. In the case that $c$ is a lower bound, system (1) is infeasible. In addition, to apply the linear time SDP solver in [9], we introduce a shift $\epsilon$ to system (1), i.e.,

$$
\begin{align*}
    x^T A x + 2a^T x &\leq c - \epsilon \\
    x^T B x + 2b^T x + d &\leq -\frac{\epsilon}{K},
\end{align*}
$$

(2)

where $K$ is some parameter that can be estimated in linear-time (to be defined in Lemma 2.6). Introducing the parameter $\epsilon$ in the first inequality shifts the value of the objective function with an error $\epsilon$ and introducing the parameter $\epsilon/K$ in the second inequality shifts the value of the objective function at most $\epsilon$ (Lemma 2.6). We then invoke the linear-time SDP solver in [9] that either returns a vector $x$ satisfying system (1), or correctly declares that the direct SDP relaxation of (2) is infeasible, i.e., a perturbed version of (1) is infeasible. Then via a binary search over $c$, we demonstrate that we can obtain an approximate optimal solution. However, there are still issues to address when borrowing the SDP solver in [9]; the SDP solver in [9] requires the feasible set of $X$ to be $\{X : X \succeq 0, \text{tr}(X) \leq 1\}$ and the direct SDP relaxation of (2) lies in an unbounded feasible region. We will remedy this by showing that the optimal solution of the GTRS must be in a compact set and further that the optimal solution of the corresponding SDP relaxation should also be in a compact set.

The rest of the paper is organized as follows. In Section 2, we propose the main algorithm and the main result in Theorem 2.7. We then illustrate the subroutines to support our main algorithm in Section 3. Finally, we conclude our paper in Section 4.

**Notations** The notations $A \succ 0$ and $A \succeq 0$ represent that the symmetric matrix $A$ is positive definite and positive semidefinite, respectively. We use the notation $x(i)$ to denote the $i$th entry of a vector $x$. We also denote by $v^*$ the optimal value of problem (GTRS). Notation $\|A\|_2$ denotes the operator norm of matrix $A$. Notations $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest and smallest eigenvalues of matrix $A$, respectively.

## 2 Main results

In this section, we first review some basic properties of the GTRS. Then we propose our main algorithm under some mild regular condition and demonstrate its correctness.

### 2.1 Preliminary

Besides the linear-time algorithm for the TRS, Hazan and Koren [9] also demonstrated that their algorithm can be extended to the GTRS when the quadratic form in the constraint is positive definite. Now let us consider the more general case where $B$ is indefinite. To avoid some degenerate cases, we do not discuss the case that $B$ is positive semidefinite but singular. To simplify notations,
we assume, without loss of generalities, \( \|A\|_2 \leq 1 \) and \( \|B\|_2 \leq 1 \) as in [9]. Otherwise, to achieve this goal, an upper bound \( \rho_A \) (or \( \rho_B \)) of \( \|A\|_2 \) (or \( \|B\|_2 \)) can be estimated in linear time with high probability, by Theorem 5 in [9], and then a scaling of \( A \) (or \( B \)) with \( 1/\rho_A \) (or \( 1/\rho_B \)) achieves this goal, i.e., \( \|A/\rho_A\|_2 \leq 1 \) and \( \|B/\rho_B\|_2 \leq 1 \) (see Section 5.1, [9]).

A central tool in this paper is the following linear-time procedures for approximating eigenvectors of sparse matrices.

**Lemma 2.1** (Lemmas 3 and 5 in [9]). There exists an approximate eigenvector oracle, denoted as \( \text{ApproxmmaxEV} \), that, for given symmetric matrix \( C \in \mathbb{R}^{n \times n} \) with \( \|C\|_2 \leq \rho \) and parameters \( \epsilon, \delta \), returns a unit vector \( x \) with probability at least \( 1 - \delta \) such that \( x^T M x \geq \lambda_{\max}(C) - \epsilon \) and a scalar \( \lambda = x^T M x \) in time

\[
O\left( \frac{N\sqrt{\rho}}{\sqrt{\epsilon}} \log \frac{n}{\delta} \right).
\]

With the same principle, this oracle can be used to compute an approximate eigenvector to the smallest eigenvalue. In fact, the negativity of the approximate largest eigenvalue for \( -C \) gives the smallest approximate eigenvalue of \( C \).

**Lemma 2.2.** There exists an approximate eigenvector oracle, denoted as \( \text{ApproxmminEV} \), that, for given symmetric matrix \( C \in \mathbb{R}^{n \times n} \) with \( \|C\|_2 \leq \rho \) and parameters \( \epsilon, \delta \), returns a unit vector \( x \) with probability at least \( 1 - \delta \) such that \( x^T M x \leq \lambda_{\min}(C) + \epsilon \) and a scalar \( \lambda = x^T M x \) in time

\[
O\left( \frac{N\sqrt{\rho}}{\sqrt{\epsilon}} \log \frac{n}{\delta} \right).
\]

We further make the following assumptions to ensure the boundedness and existence of an optimal solution.

**Assumption 2.3.** Matrix \( B \) has at least one negative eigenvalue with \( \lambda_{\min}(B) \leq -\xi < 0 \) for some \( \xi > 0 \).

**Assumption 2.4.** There exists a \( \mu \) with \( \mu \in (0, 1] \) such that \( \mu A + (1 - \mu) B \succeq \xi I \), where \( \xi \) is the same positive number defined in Assumption 2.3.

Note that the main purpose of the assumption that the two \( \xi \)s in Assumptions 2.3 and 2.4 are the same is only for simplicity consideration. Also note that Assumption 2.4 is closely related to the so called regular condition, i.e., there exists a \( \lambda \geq 0 \) such that \( A + \lambda B > 0 \), under which (GTRS) is bounded below and admits a unique optimal solution [18, 21]. Under Assumption 2.3, Assumption 2.4 also implies that \( \lambda_{\max}(A) > 0 \). We assume in the following that both Assumptions 2.3 and 2.4 hold.

When \( B \) is indefinite, we can always find a positive constant \( K \) in system (2) to ensure that the objective value shifts at most \( \epsilon \) if the constraint shifts \( \epsilon/K \) when system (2) is infeasible. To demonstrate this, let us first recall the celebrated S-lemma [26, 20].

**Lemma 2.5.** Let \( g_i(x) = x^T Q_i x + 2p_i^T x + q_i \), where \( Q_i \) is an \( n \times n \) symmetric matrices, \( p_i \in \mathbb{R}^n \) and \( q_i \in \mathbb{R} \), \( i = 1, 2 \). Assume that there exists an \( \tilde{x} \in \mathbb{R}^n \) such that \( g_2(\tilde{x}) < 0 \). Then the following
two statements are equivalent:

(S1) There is no \( x \in \mathbb{R}^n \) such that \( g_1(x) \leq 0 \) and \( g_2(x) < 0 \).

(S2) There exists a nonnegative multipliers \( \lambda \geq 0 \) such that \( g_1(x) + \lambda g_2(x) \geq 0, \forall x \in \mathbb{R}^n \).

Note that the assumption of the existence of an \( \bar{x} \in \mathbb{R}^n \) such that \( g_2(\bar{x}) < 0 \) automatically holds when \( Q_2 \) has at least one negative eigenvalue. Using the S-lemma, we have the following results.

**Lemma 2.6.** Let \( K > -\lambda_{\text{max}}(A)/\lambda_{\text{min}}(B) \). If the system (2) is infeasible for some \( \epsilon > 0 \), then the following system is also infeasible

\[
\begin{align*}
x^T Ax + 2a^T x & \leq c - 2\epsilon, \\
x^T Bx + 2b^T x + d & \leq 0,
\end{align*}
\]

(3)

**Proof.** Since \( K \) is an upper bound for \( \lambda \) that satisfies \( A + \lambda B \succeq 0 \), we have \( \lambda/K < 1 \). By the S-lemma, the infeasibility of (2) implies that there exists \( \lambda \geq 0 \) such that

\[
0 \leq x^T Ax + 2a^T x - c + \epsilon + \lambda(x^T Bx + 2b^T x + d - \epsilon/K),
\]

\[
= x^T Ax + 2a^T x - c + 2\epsilon + \lambda(x^T Bx + 2b^T x + d) - \epsilon,
\]

for all \( x \in \mathbb{R}^n \), where \( \epsilon = (1 - \lambda/K)\epsilon > 0 \).

When \( \lambda = 0 \), we have

\[
x^T Ax + 2a^T x - c + 2\epsilon + \lambda(x^T Bx + 2b^T x + d - \epsilon/K) > x^T Ax + 2a^T x - c + 2\epsilon + \lambda(x^T Bx + 2b^T x + d) - \epsilon \geq 0.
\]

When \( \lambda > 0 \), due to \( K > \lambda \), we have

\[
x^T Ax + 2a^T x - c + 2\epsilon + \lambda(x^T Bx + 2b^T x + d - \epsilon/K) \geq x^T Ax + 2a^T x - c + 2\epsilon + \lambda(x^T Bx + 2b^T x + d - \epsilon/\lambda) \geq 0.
\]

Hence for all \( \lambda \geq 0 \), we have

\[
x^T Ax + 2a^T x - c + 2\epsilon + \lambda(x^T Bx + 2b^T x + d - \epsilon/K) \geq 0.
\]

From the S-lemma, we conclude that

\[
\begin{align*}
x^T Ax + 2a^T x & \leq c - 2\epsilon, \\
x^T Bx + 2b^T x + d - \epsilon/K & < 0.
\end{align*}
\]

This further implies that the system (3) is infeasible. \( \square \)

The above lemma shows that if (2) is infeasible, then \( c - 2\epsilon \) is a lower bound for (GTRS). Since an upper bound of \( \lambda_{\text{min}}(A) \) is given, i.e., \( \lambda_{\text{max}}(A) \leq 1 \), and \( \lambda_{\text{min}}(B) \) can be approximately estimated in linear time, parameter \( K \) can be estimated in linear time. Also, due to \( -\lambda_{\text{max}}(A)/\lambda_{\text{min}}(B) \leq 1/\xi \), the estimated \( K \) can be restricted to be bounded by some constant (e.g., \( 1/\xi + \epsilon \)). Hence we assume that \( K \) is given in our algorithm.
2.2 Main algorithm

In our main algorithm, Algorithm 1, we first compute an initial estimation for upper and lower bounds for problem (GTRS) and then use bisection techniques, by invoking Algorithm 3 in Subsection 3.3 for at most $O\left(\log\left(\frac{1}{\epsilon}\right)\right)$ iterations, to obtain a feasible solution $\tilde{x}$ with $f(\tilde{x}) \leq v^* + \epsilon$, where $v^*$ is an optimal value of the GTRS under Assumptions 2.3 and 2.4.

Algorithm 1 Find an $\epsilon$-optimal solution for (GTRS)

Input: symmetric $A, B \in \mathbb{R}^n$ with $\|A\|_2 \leq 1$ and $\|B\|_2 \leq 1$, $a, b \in \mathbb{R}^n$, $d \in \mathbb{R}$, $\epsilon, \delta > 0$, and $K \geq -\lambda_{\max}(A)/\lambda_{\min}(B)$

Output: an $\epsilon$-optimal solution; output is correct with probability at least $1 - \delta$

1: function GTRS($A, a, B, b, d, \epsilon, \delta, K$)
2: let $\mu_A = \rho_A + 2\|a\| + |c|$ and $\mu_B = \rho_B + 2\|b\| + |d|$;
3: invoke ($\mu$, $\lambda$, $y$) $\rightarrow$ PsdPencil$(A, B, \xi, \delta/2)$
4: define $\bar{\lambda} = \lambda - \xi/4$
5: set $l = \frac{d\mu_0}{1 - \mu_0} - \frac{\mu_0\|p\|^2}{\bar{\lambda}}$ $\triangleright$ initial lower bound
6: find a solution $k$ of $(ky)^TB(ky) + 2b^T(ky) + d \leq 0$
7: set $u = f(ky)$ $\triangleright$ initial upper bound
8: define $R$ with (7), $c = \frac{l + u}{2}$, $\epsilon' = \frac{\epsilon}{2}$, $T = \log_2\frac{u}{\epsilon'}$ and $\delta' = \frac{\delta}{2T}$
9: for $t = 1 : T$ do
10: invoke Feas($A, B, a, b, c, d, \epsilon', \delta', \mu_A, \mu_B, K, R$)
11: if Feas($A, B, a, b, c, d, \epsilon', \delta', \mu_A, \mu_B, K, R$) returns “infeasible” then
12: set $l = c - 2\epsilon'$; $c = (l + u)/2$
13: else Feas($A, B, a, b, c, d, \epsilon', \delta', \mu_A, \mu_B, K, R$) returns a feasible solution $x$ to (1)
14: set $u = \min\{u, x^TAx + 2a^Tx\}$; $c = (l + u)/2$
15: end if
16: end for
17: return $x$
18: end function

The following theorem shows us the correctness and linear runtime of Algorithm 1, where the subroutines that support this theorem are given in the following sections.

Theorem 2.7. Assume that $\epsilon$ is a sufficiently small positive number. Under Assumptions 2.3 and 2.4, with correct probability at least $1 - \delta$, Algorithm 1 returns an $\epsilon$-optimal solution $\tilde{x}$ to (GTRS), i.e., a feasible solution $\tilde{x}$ with $f(\tilde{x}) \leq f(x^*) + \epsilon$. The total runtime is

$$O\left(\frac{N}{\sqrt{\epsilon}} \log \left(\frac{n}{\delta} \log \frac{1}{\epsilon}\right) \left(\log \frac{1}{\epsilon}\right)^2\right).$$

Proof. Correctness: Section 3.2 shows that the lower and upper bounds $l$ and $u$ can be correctly estimated by Lines 3-7 of Algorithm 1. Note that $l$ and $u$ are constants only depending on...
A, B, a, b, c and ξ.

From Theorem 3.5, we know that the subroutine Feas either returns a feasible solution for (1) (yielding a new upper bound $u = \min \{c, x^T Ax + 2a^T x\}$) or declares the infeasibility of (2) (yielding a new lower bound $l = c - 2\epsilon'$ by Lemma 2.6). Now consider the loop in lines 9–16. Let $l_p$ and $u_p$ denote the values of $l$ and $u$ in the end of the $p$th iteration in the “for” loop (particularly, let $l_0$ and $u_0$ be the initial values of $l$ and $u$) and then the length of $u - l$ is at most $\frac{u_0 - l_0}{2^T} + 2\epsilon'$ at the end of the current iteration. At the end of the main loop of Algorithm 1, the length of $u - l$ satisfies

$$u - l \leq \frac{u_0 - l_0}{2^T} + (2 + 1 + \cdots + \frac{1}{2^T-1})\epsilon' \leq 4\epsilon' = 5\epsilon'. $$

From Lemma 2.6 we have $l - 2\epsilon' \leq f(x^*) \leq f(x) = u_T$. Thus $u_T - f(x^*) \leq u_T - (l_T - 2\epsilon') \leq (5 + 2)\epsilon' = 7\epsilon' = \epsilon$. That is, after $O(\log \frac{u-l}{\epsilon})$ iterations of binary search, we obtain an $\epsilon$ optimal solution.

**Runtime:** The main runtime of Algorithm 1 is in subroutines PsdPencil($A, B, \xi/4, \delta/2$) and Feas. From Theorem 3.1 in Subsection 3.1, we know that the main runtime of PsdPencil($A, B, \xi, \delta/2$) is in $O\left(\frac{N}{\sqrt{\xi}} \log \left(\frac{n}{\delta} \log \frac{1}{\xi}\right) \log \frac{1}{\xi}\right) = O\left(\frac{N}{\delta} \log \left(\frac{n}{\delta}\right)\right)$, and the correct probability is $1 - \delta/2$. Section 3.2 also shows that the remaining operators in lines 4–7 run in time $O\left(\frac{N}{\sqrt{\delta}} \log \frac{1}{\delta}\right)$. Note that Algorithm 1 invokes Algorithm 3 $O(\log \frac{u-l}{\epsilon})$ times. Then from Theorem 3.5 in Subsection 3.3, the total time of lines 9-16 is

$$O\left(\frac{NR\sqrt{\kappa}}{\sqrt{\epsilon'}} \log \left(\frac{n}{\delta} \log \frac{\kappa R}{\epsilon'}\right) \log \frac{\kappa R}{\epsilon'} \log \frac{u_l - l}{\epsilon'}\right),$$

and the successful probability is, by noting that $\delta' = \frac{\delta}{2t}$, at least $1 - T \times \frac{\delta}{2T} = 1 - \frac{\delta}{2}$.

Note that $\kappa$ and $R$ are both constants and $\epsilon' = \epsilon/7$. We conclude that the total runtime is

$$O\left(\frac{N}{\sqrt{\epsilon}} \log \left(\frac{n}{\delta} \log \frac{1}{\epsilon}\right) \log \frac{1}{\epsilon} \log \frac{1}{\epsilon}\right),$$

and the correct probability is at least $1 - \delta$. □

### 3 Subroutines

In this section, we present several subroutines to support Algorithm 1.

#### 3.1 Algorithm to compute parameter $\mu$ such that $\mu A + (1 - \mu)B \succeq \frac{\xi}{2} I$

In this subsection, we provide a bisection algorithm to find a $\mu$ with $\mu \in (0, 1]$ such that $\mu A + (1 - \mu)B \succeq \frac{\xi}{2} I$ under Assumption 2.4, which is of linear time. Such a $\mu$ helps us find a lower bound for problem (GTRS) and a compact set in which the optimal solution locates in the following
subsections. We first identify an interval \((\mu_1, \mu_2]\) where the target \(\mu\) locates, if exists. In the algorithm, we initialize \((\mu_1, \mu_2]\) as \((0,1]\). In each step, we invoke the eigenvalue oracle \((\lambda, x) = \text{APPROXMINEV}(\mu A + (1-\mu)B, \xi/4, \delta/T)\) to find an approximate smallest eigenvalue of the midpoint of this interval. We then cut off half of the interval by taking either \((\mu, \mu_2]\) if \(x^T Ax > x^T Bx\), or \((\mu, \mu_2]\) if \(x^T Ax \leq x^T Bx\). The intuition of this step is that for all \(\nu \in (\mu_1, \mu)\), we have \(\lambda_{\text{min}}(\nu A + (1-\nu)B) \leq x^T(\nu A + (1-\nu)B)x \leq x^T(\mu A + (1-\mu)B)x = \lambda \leq 3\xi/4\) when \(x^T Ax > x^T Bx\). This means the target \(\mu\) must be in the other half of the interval, i.e., \((\mu, \mu_2]\), if exists. The other situation of this step follows the same argument. We prove in the following theorem that under Assumption 2.4, such a \(\mu\) can be found correctly in linear time with high probability.

\begin{algorithm}
\textbf{Algorithm 2} Compute parameter \(\mu\) such that \(\mu A + (1-\mu)B \succeq \frac{\xi}{2} I\)
\begin{description}
\item[Input:] symmetric \(A, B \in \mathbb{R}^{n \times n}\) with \(\|A\|_2 \leq 1\) and \(\|B\|_2 \leq 1\), and \(\xi, \delta > 0\)
\item[Output:] \(\mu > 0\) such that \(\mu A + (1-\mu)B \succeq \frac{\xi}{2} I\) and a unit vector \(x\) and \(\lambda = x^T(A + \mu B)x\) such that \(\lambda_{\text{min}}(\mu A + \mu B) \leq \lambda \leq \lambda_{\text{min}}(\mu A + \mu B) + \xi/4\); output is correct with probability at least \(1 - \delta\)
\end{description}
\begin{algorithmic}
\State \textbf{function} PsdPencil\((A, B, \xi, \delta)\)
\State \hspace{0.5em} initialize \(T = \log_2 \frac{16}{\xi}\), \(\mu_1 = 0\), \(\mu_2 = 1\)
\For {\(i = 1 : T\)}
\State \(\mu = (\mu_1 + \mu_2)/2\)
\State \((\lambda, x) = \text{APPROXMINEV}(\mu A + (1-\mu)B, \xi/4, \delta/T)\)
\If {\(\lambda \geq 3\xi/4\)}
\State \textbf{return} \(\lambda\) and \(\mu\)
\ElseIf {\(x^T Ax > x^T Bx\)}
\State update \(\mu_1 \leftarrow \mu\) \Comment{update \(x_1 \leftarrow x\) for analysis in Theorem 3.1}
\Else
\State update \(\mu_2 \leftarrow \mu\) \Comment{update \(x_2 \leftarrow x\) for analysis in Theorem 3.1}
\EndIf
\EndFor
\State \textbf{return} “Assumption 2.4 fails.”
\end{algorithmic}
\end{algorithm}

\textbf{Theorem 3.1.} Assume that \(A\) and \(B\) are symmetric matrices with \(\|A\|_2 \leq 1\) and \(\|B\|_2 \leq 1\). For any \(\epsilon > 0\), Algorithm 2 takes at most \(\log_2 \left( \frac{16}{\xi} \right) \) iterations of APPROXMINEV and returns \(\lambda, \mu, x\) such that \(\mu > 0\), \(\lambda = x^T(\mu A + (1-\mu)B)x \geq 3\xi/4\) and \(\lambda_{\text{min}}(\mu A + (1-\mu)B) \geq \lambda - \xi/4\) with correct probability \(1 - \delta\). And the total runtime is \(O \left( \frac{N}{\sqrt{\xi}} \log \left( \frac{n}{\delta} \log \frac{1}{\xi} \right) \log \frac{1}{\xi} \right) \).

\textbf{Proof.} \textbf{Runtime:} First note that the algorithm invokes APPROXMINEV for at most \(T = \log_2 \frac{16}{\xi}\) iterations. In each iteration, we invoke APPROXMINEV once for matrix \(\mu A + (1-\mu)B\), whose norm is not larger than 1, and other main operators (i.e., the matrix vector products \(x^T Ax\) and \(x^T Bx\)) are dominated by APPROXMINEV. Since \(\|\mu A + (1-\mu)B\|_2 \leq \mu \|A\|_2 + (1-\mu) \|B\|_2 \leq 1\), the time
for each call of APPROXMINEV is $O(N\sqrt[\gamma]{\xi} \log \frac{nT}{\delta})$. Hence, from Lemma 2.1, each iteration runs in time $O\left(\frac{N}{\sqrt[\gamma]{\xi}} \log \left(\frac{n}{\delta} \log \frac{1}{\xi}\right)\right)$. Thus, the total runtime is

$$O\left(\frac{N}{\sqrt[\gamma]{\xi}} \log \left(\frac{n}{\delta} \log \frac{1}{\xi}\right)\right).$$

**Correctness:** If the algorithm returns $\lambda$ and $\mu$ for some iteration $i \leq T$, then the returned $\mu$ is the one as required, i.e., $\lambda_{\min}(\mu A + (1 - \mu)B) \geq \lambda - \xi/4 \geq \xi/2$. Now it suffices to prove that under Assumption 2.4, the algorithm must terminate in some iteration $i \leq T$.

Recall that Assumption 2.4 states that $\exists \mu_0 > 0$ such that $\mu_0 A + (1 - \mu_0)B \succeq \xi I$. Since $\|A\|_2 \leq 1$ and $\|B\|_2 \leq 1$, we have $A + B \preceq 2I$. Hence for any $\varrho \in [-\frac{\xi}{8}, \frac{\xi}{8}]$, we have $(\mu_0 + \varrho)A + (1 - \mu_0 - \varrho)B \succeq \xi I - 2|\varrho|I \succeq \frac{\xi}{8}I$. So for all $\mu' \in [\mu_0 - \frac{\xi}{8}, \mu_0 + \frac{\xi}{8}] \cap (0, 1)$, we have $\mu'A + (1 - \mu')B \succeq \frac{\xi}{8}I$. The length of the interval $[\mu_0 - \frac{\xi}{8}, \mu_0 + \frac{\xi}{8}] \cap (0, 1]$ is between $\frac{\xi}{8}$ and $\frac{\xi}{4}$. From the above analysis we know that under Assumption 2.4, the interval length of $\{\mu : \mu A + (1 - \mu)B \succeq \frac{\xi}{8}I\}$ is at least $\frac{\xi}{8}$.

If the algorithm does not terminate in the “for loop”, at the end of the loop, we have $x_1^T C(\mu_1)x_1 < 3\xi/4$ and $x_2^T C(\mu_2)x_2 < 3\xi/4$ (note that $x_1$ and $x_2$ are defined in the comments in lines 9 and 11 of Algorithm 2, respectively), where $C(\mu_i) = \mu_i A + (1 - \mu_i)B$ ($i = 1, 2$). Then $x_1^T C(\mu_1)x_1 = \mu_1 x_1^T Ax_1 + (1 - \mu_1)x_1^T Bx_1 \leq \mu_1 x_1^T Ax_1 + (1 - \mu_1)x_1^T Bx_1 < 3\xi/4$ for $\mu \leq \mu_1$ because $x_1^T Ax_1 > x_1^T Bx_1$. Furthermore, $\lambda_{\min}(C(\mu)) \leq x_1^T C(\mu)x_1 < 3\xi/4$ for $\mu \leq \mu_1$. Similarly, we have $\lambda_{\min}(C(\mu)) < 3\xi/4$ for $\mu \geq \mu_2$. So if the algorithm terminates in Line 14, we have $\mu_2 - \mu_1 \leq \frac{\xi}{16}$ as the binary search runs for $\log_2\frac{16}{\xi}$ iterations and the initial length of $\mu_2 - \mu_1 = 1$. Since for any $\mu$ outside the interval $[\mu_1, \mu_2]$, we have $\lambda_{\min}(\mu A + (1 - \mu)B) < 3\xi/4$, the length of interval $\{\mu : \lambda_{\min}(\mu A + (1 - \mu)B) \geq 3\xi/4\}$ is then at most $\mu_2 - \mu_1$, which is less than $\xi/16$. This contradicts Assumption 2.4.

The correct probability $1 - \delta$ can be obtained, by the union bound, from that correct probability of each APPROXMINEV is $1 - \delta/T$ and that the total iteration number is at most $T$.

### 3.2 Lower and upper bounds

In this subsection, we will show that Lines 4-7 in Algorithm 1 supply an estimation of initial lower and upper bounds in linear time.

More specifically, we demonstrate that initial lower and upper bounds for (GTRS) can be computed in $O\left(\frac{N}{\sqrt[\gamma]{\xi}} \log \frac{n}{\delta}\right)$ time. The main principle here is that an upper bound can be found by a feasible solution of (GTRS) and a lower bound can be found by a feasible solution of the Lagrangian dual problem of (GTRS).

An upper bound for problem (GTRS) is given by $u = f(ky)$, where $k$ is the positive solution of the following quadratic inequality and $y$ is the unit eigenvector returned by APPROXMINEV$(B, \xi/2, \delta),$

$$(ky)^TB(ky) + 2b^T(ky) + d \leq 0. \tag{4}$$

Note that $y^TB y \leq \lambda_{\min}(B) + \xi/2$. Hence under Assumption 2.3, we have $y^TB y \leq \lambda_{\min}(B) + \xi/2 \leq -\xi/2$, which further implies the feasibility of the quadratic inequality (4). Any feasible $k$ yields an
upper bound $f(kx)$, which is bounded by a constant depending only on $A, B, a, b$ and $d$. The vector vector product $b^T y$ can be done in $O(n)$ and finding a feasible solution of the quadratic system can be done in $O(1)$. The runtime for computing the approximate smallest eigenvalue is $O\left(\frac{N}{\sqrt{\xi}} \log \frac{n}{\delta} \right)$ (note that $\|B\|_2 \leq 1$ as assumed). Hence the cost in finding the upper bound is bounded by $O\left(\frac{N}{\sqrt{\xi}} \log \frac{n}{\delta} \right)$.

Next we illustrate that a lower bound can be found by a feasible solution of the Lagrangian dual problem. Let $(\mu_0, x, \lambda) = \text{PsdPencil}(A, B, \xi, \delta/2)$. From Theorem 3.1 in the next section, $\lambda$ is a $\xi/4$ approximate smallest eigenvalue and satisfies $\lambda \geq 3\xi/4$ under Assumption 2.4. Hence, the true smallest eigenvalue satisfies $\lambda_{\min}(\mu_0 A + (1 - \mu_0)B) \geq \lambda - \xi/4 \geq \xi/2$. The Lagrangian dual problem of (GTRS) is

$$\begin{align*}
\text{(L)} & \quad \max_{\nu \geq 0} \min_x f(x) + \nu h(x) \\
& = \max_{\nu \geq 0} (x(\nu))^T (A + \nu B)x(\nu) + 2(a + \nu b)^T x(\nu) + d\nu,
\end{align*}$$

where $x(\nu)$ is the optimal solution of the inner minimization problem. Letting $\nu_0 = (1 - \mu_0)/\mu_0$, we have $P = A + \nu_0 B \succeq \frac{\xi}{2\mu_0} I$. So $\nu_0$ is a feasible solution for problem (L). Thus from weak duality, we have the following inequality

$$v(\text{L}(\nu_0)) \leq \max_{\nu} v(\text{L}(\nu)) \leq v(\text{GTRS}).$$

Letting $p = a + \nu_0 b$, a lower bound of $L(\nu_0)$ can be found by the following formulation,

$$v(\text{L}(\nu_0)) = \min_x (x + P^{-1} p)^T P (x + P^{-1} p) + d\nu_0 - p^T P^{-1} p$$

$$\geq d\nu_0 - \frac{\|p\|^2}{\lambda_{\min}(P)}$$

$$\geq d\nu_0 - \frac{\mu_0 \|p\|^2}{\lambda},$$

where $\bar{\lambda} = \lambda - \xi/4$ satisfies $\lambda_{\min}(P) \geq \bar{\lambda} \geq \xi/2$ (by Theorem 3.1). And as $\mu_0$ and $p$ are bounded by constants depending only on $A, B, a, b$ and $d$, we conclude that the lower bound is bounded by constant. Note also that the main time cost for computing the lower bound is in calling PsdPencil, which runs in time, by Theorem 3.1

$$O\left(\frac{N}{\sqrt{\xi}} \log \left(\frac{n}{\delta} \log \frac{1}{\xi}\right) \log \frac{1}{\xi}\right).$$

### 3.3 Identify feasibility of quadratic systems

Our main subroutine in Algorithm 1 is to utilize a linear-time SDP solver, RELAXSOLVE developed in [9], to approximately solve the following feasibility problem:

$$D_i \bullet X \succeq \epsilon, \quad i = 1, 2, \quad X \in \mathcal{K},$$

where $D_i, i = 1, 2$, are symmetric matrices with $\|D_i\|_2 \leq 1$ and $\mathcal{K} = \{X: X \succeq 0, \text{tr}(X) \leq 1\}$.  

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Lemma 3.2 (Theorem 2 in [9]). Given symmetric matrices \( D_1, D_2 \in \mathbb{R}^{n \times n} \) with \( \| D_i \|_2 \leq 1 \) and \( \epsilon, \delta > 0 \), with probability at least \( 1 - \delta \), RELAXSOLVE outputs a matrix \( X \in \mathcal{K} \) of rank 2 that satisfies \( D_i \cdot X \geq \epsilon/2, i = 1, 2 \) or correctly declares that \( (GTRS) \) is infeasible. The algorithm calls the oracle APPROXMAXEV at most \( O(\log \frac{1}{\epsilon}) \) times and can be implemented to run in total time

\[
O \left( \frac{N}{\sqrt{\epsilon}} \log \frac{1}{\epsilon} \log \left( \frac{n}{\delta} \log \frac{1}{\epsilon} \right) \right).
\]

The direct SDP relaxation of \( (2) \) is

\[
P_i \cdot X \geq \epsilon, \quad i = 1, 2, \quad X \succeq 0,
\]

where \( P_1 = \left( \begin{array}{cc} -A & -a \\ -a^T & c \end{array} \right), P_2 = K \left( \begin{array}{cc} -B & -b \\ -b^T & -d \end{array} \right) \). However, the feasible region for problem (GTRS) may be not contained in \( \mathcal{K} \) and thus we cannot directly utilize the SDP solver in [9]. To address this issue, we prove in the next theorem that the optimal solution of (GTRS) must locate in a compact set.

**Theorem 3.3.** Assume that i) \( (\mu, \lambda, x) \rightarrow \text{PsDPencil}(A, B, \xi, \delta) \), ii) \( u \) is an upper bound of problem (GTRS) and iii) \( x^* \) is the optimal solution of problem (GTRS). Then, under Assumptions 2.3 and 2.4, it holds that \( x^* \in Q := \{ x : \| x \| \leq \bar{R} \} \) with probability \( 1 - \delta \), where

\[
\bar{R} = -\frac{\mu \| a + vb \|_\lambda}{\lambda} + \sqrt{\left( \frac{\| \mu (a + vb) \|^2_\lambda}{\lambda} + \frac{\mu (u - \nu d)}{\lambda} \right)},
\]

where \( \bar{\lambda} = \lambda - \xi/4 \) and \( \nu = (1 - \mu)/\mu \). Moreover, \( R \) is also upper bounded by some constant.

Proof. Recall that \( \text{PsDPencil}(A, B, \xi, \delta) \) returns \( \lambda \) such that \( \lambda = x^T (\mu A + (1 - \mu) B) x \geq 3\xi/4 \) and \( \lambda_{\text{min}} (\mu A + (1 - \mu) B) \geq \lambda - \xi/4 \) with correct probability \( 1 - \delta \). Hence \( \bar{\lambda} = \lambda - \xi/4 \geq \xi/2 > 0 \) is a lower bound of the smallest eigenvalue of \( \mu A + (1 - \mu) B \). So the smallest eigenvalue of \( A + \nu B \) is larger than \( \bar{\lambda}/\mu \), where \( \nu = 1/\mu - 1 \). From the optimality of \( x^* \), we have \( f_1(x^*) - u + \nu f_2(x^*) \leq 0 \) and hence

\[
(x^*)^T (A + \nu B)x^* + 2(a + vb)^T x^* - u + \nu d \leq 0.
\]

Let \( \mathcal{X} = \{ x : x^T (A + \nu B)x + 2(a+vb)^T x - u + \nu d \leq 0 \} \). Then \( \mathcal{X} \subset \{ x : \frac{1}{\mu} x^T x + 2(a+vb)^T x - u + \nu d \leq 0 \} \). This further implies

\[
\mathcal{X} \subset \left\{ x : \| x \| \leq -\frac{\mu \| a + vb \|_\lambda}{\lambda} + \sqrt{\left( \frac{\| \mu (a + vb) \|^2_\lambda}{\lambda} + \frac{\mu (u - \nu d)}{\lambda} \right)} \right\}.
\]

From Section 3.2, \( \mu \) is bounded by some constant. Hence \( R \) is also upper bounded by some constant.

\[\Box\]

By defining \( Y^* = \begin{pmatrix} x^* \\ 1 \end{pmatrix} \begin{pmatrix} x^* \\ 1 \end{pmatrix}^T / S \), we have \( \text{trace}(Y^*) = \text{trace}(\begin{pmatrix} x^* \\ 1 \end{pmatrix} \begin{pmatrix} x^* \\ 1 \end{pmatrix}^T / S) = \frac{\| x^* \|^2 + 1}{S} \leq 1 \), i.e., \( Y^* \in \mathcal{K} \), where \( S = R^2 + 1 \). This motivates us to solve the following SDP problem instead of (6),

\[
\frac{1}{\kappa} P_i \cdot Y \geq \frac{\epsilon}{\kappa S}, \quad i = 1, 2, \quad Y \in \mathcal{K},
\]

\[\text{(8)}\]
where $\kappa = \max\{\mu_A, K\mu_B\}$, $\mu_A = 1 + 2\|a\| + |c|$, $\mu_B = 1 + 2\|b\| + |d|$. These parameters make $\|P_i/\kappa\|_2 \leq 1$, $i = 1, 2$ and the optimal solution, if exists, $Y^* \in K$. Therefore, the SDP feasibility problem (8) can be solved by the linear-time SDP solver, RELAXSOLVE, in [9]. Then due to Lemma 3.2 with probability $1 - \delta$, RELAXSOLVE($P_i/\kappa$, $P_i/\kappa$, $\epsilon/2\kappa S$, $\delta$) either declares that (8) is infeasible, or returns $Y \in K$ such that $\frac{1}{\kappa}P_i \cdot Y \geq \frac{\epsilon}{2\kappa S}$, $i = 1, 2$, which is further equivalent to that $X = SY$ satisfies $P_i \cdot X \geq \epsilon/2$, $i = 1, 2$, $X \succeq 0$. When RELAXSOLVE($P_i/\kappa$, $P_i/\kappa$, $\epsilon/2\kappa S$, $\delta$) returns $Y \in K$ such that $\frac{1}{\kappa}P_i \cdot Y \geq \frac{\epsilon}{2\kappa S}$, we further invoke the SZROTATION algorithm in [9], which is a variant of the matrix decomposition procedure in Sturm and Zhang [24], to find a vector $z$ such that $z^TP_iz \geq \epsilon/2r$, $i = 1, 2$ with $r = \text{rank}(Y)$.

**Lemma 3.4** ([9]). Given a decomposition $X = \sum_{i=1}^r x_i x_i^T$ of a positive semidefinite matrix $X$ of rank $r$ and an arbitrary matrix $M$ with $M \cdot X \succeq a$, SZROTATION outputs a decomposition $X = \sum_{i=1}^r y_i y_i^T$ such that $y_i^TMY_i \succeq a/r$ for all $i = 1, \ldots, r$. The procedure runs in time $O(Nr)$, where $N \geq n$ is the number of non-zero entries in $M$.

**Theorem 3.5.** Given the linear-time SDP solver RELAXSOLVE, parameters $\epsilon, \delta > 0$, and $Q_1$ and $Q_2$ defined in [9], Algorithm 3, with probability at least $1 - \delta$, returns a vector $x \in \mathbb{R}^n$ satisfying system (7), or correctly declares that (3) is infeasible. The total runtime of Algorithm 3 is

$$O\left(\frac{NR\sqrt{\kappa}}{\sqrt{\epsilon}} \log \left(\frac{n}{\delta} \log \frac{\kappa R}{\epsilon} \right) \log \frac{\kappa R}{\epsilon}\right),$$

where $R$ is defined in [7].

Proof. **Runtime:** The main runtime is costed by the two sub-algorithms RELAXSOLVE and SZROTATION with $O\left(\frac{N}{\sqrt{\epsilon'}} \log \left(\frac{n}{\delta} \log \frac{1}{\epsilon'} \right) \log \frac{1}{\epsilon'}\right)$ and $O(Nr)$, respectively, where $\epsilon' = \epsilon/(\kappa S)$ and $r \leq 2$ is the rank of $X$ returned by RELAXSOLVE. Since $O(Nr)$ is dominated by $O\left(\frac{N}{\sqrt{\epsilon'}} \log \left(\frac{n}{\delta} \log \frac{1}{\epsilon'} \right) \log \frac{1}{\epsilon'}\right)$, the total runtime is

$$O\left(\frac{NR\sqrt{\kappa}}{\sqrt{\epsilon}} \log \left(\frac{n}{\delta} \log \frac{\kappa R}{\epsilon} \right) \log \frac{\kappa R}{\epsilon}\right).$$

**Correctness:** If RELAXSOLVE returns “infeasible”, the SDP relaxation (8) is infeasible and thus (6) is infeasible. This implies the infeasibility of (2) since (1) is a relaxation of (2). And the infeasibility of (2) further implies the infeasibility of (3) by Lemma 2.6.

Now let us assume that RELAXSOLVE returns $Y \in K$ such that $Q_i \cdot Y \geq \epsilon/2\kappa$, $i = 1, 2$. Then $X = SY$ satisfies $P_i \cdot X \succeq \epsilon/2\kappa$, $i = 1, 2$, $X \succeq 0$. As shown in RELAXSOLVE in [9], we have $Y = qy_1y_1^T + (1 - q)y_2y_2^T$ for $q \in [0, 1]$, which means that $\text{rank}(Y)$ is at most 2. Particularly, we have $Y_{11} = qy_1(1)^2 + (1 - q)y_2(1)^2 \leq 1$, because $y_1$ and $y_2$ are both unit vectors as shown by the last line of RELAXSOLVE in [9]. Then we have $X_{11} = SY_{11} \leq S$. Invoking SZROTATION($Q_1, X$) yields a solution $X = \sum_{i=1}^r z_i z_i^T$ such that $z_i^TQ_1z_i \geq \epsilon/2r\kappa$, $i = 1, 2$ [9]. Together with $Q_2 \cdot X \succeq \epsilon/2\kappa$, we conclude that at least one of
Algorithm 3 Find a feasible solution for (2) or declare the infeasibility of (3)

**Input:** symmetric \( A, B \in \mathbb{R}^{n \times n} \) with \( \|A\|_2 \leq 1 \) and \( \|B\|_2 \leq 1 \), \( a, b, c, d, \epsilon, \delta, \mu_A, \mu_B, K, R > 0 \)

**Output:** find a feasible solution for (2) or declare the infeasibility of (3); output is correct with probability at least \( 1 - \delta \)

1: function FEAS\((A, B, a, b, c, d, \epsilon, \delta, \mu_A, \mu_B, K, R)\)
2: define \( S = (R + 1)^2 \), \( \mu_A = 1 + 2 \|a\| + |c| \) and \( \mu_B = 1 + 2 \|b\| + |d| \)
3: let \( \kappa = \max\{\mu_A, K \mu_B\} \)
4: define \((n + 1) \times (n + 1)\) symmetric matrices
\[
Q_1 = \frac{1}{\kappa} \begin{pmatrix} -A & -a \\ -a^T & c \end{pmatrix} \quad \text{and} \quad Q_2 = \frac{K}{\kappa} \begin{pmatrix} -B & -b \\ -b^T & d \end{pmatrix}
\] (9)
5: invoke RELAXSOLVE\((Q_1, Q_2, \epsilon/(\kappa S), \delta)\)
6: if RELAXSOLVE returns “infeasible” then
7: return “infeasible”
8: else \{RELAXSOLVE returns \( Y \) such that \( Q_i \bullet Y \geq \epsilon/S, i = 1, 2, \}\}
9: invoke SZROTATION\((Q_1, SY)\) that return \( X = \sum_{i=1}^r z_i z_i^T \) as output
10: if \( r = 1 \) then
11: \( z = z_1 \)
12: else \( r = 2 \)
13: find a vector \( z \in \{z_1, z_2, \ldots, z_r\} \) for which \( z^T Q_2 z \geq \epsilon/2r \) and let \( \hat{z} = z(2 : n + 1) \)
14: end if
15: if \( z(1) \neq 0 \) then
16: \( x = \hat{z}/z(1) \)
17: else
18: set \( \alpha = \min\{\frac{\kappa}{2r(\|2b^T \hat{z}\| + |d|)}, \frac{\kappa}{2Kr(\|2a^T \hat{z}\| + |c|)}\} \}; \) \( x = \hat{z}/\alpha \)
19: end if
20: return \( x \)
21: end if
22: end function
$z_i$ satisfies $z_i^T Q_2 z_i \geq \epsilon/2r\kappa$. So $\text{SZRotation}(Q_1, X)$ indeed finds a vector $z \in \{z_1, z_2\}$ such that $z^T Q_j z \geq \epsilon/2r\kappa, \; (j = 1, 2)$, which further implies that,

$$
\begin{align*}
\tilde{z}^T A \tilde{z} + 2 z(1) a^T \tilde{z} & \leq cz(1)^2 - \epsilon/2r\kappa, \\
\tilde{z}^T B \tilde{z} + 2 z(1) b^T \tilde{z} + z(1)^2 d & \leq -\epsilon/2rK\kappa,
\end{align*}
$$

where $\tilde{z} = z(2:n+1)$. If $z(1) \neq 0$, by dividing $z(1)^2$ on both sides of the two inequalities in the above system and letting $x = \tilde{z}/z(1)$, we have

$$
\begin{align*}
x^T A x + 2 a^T x - c & \leq -\epsilon/2r\kappa z(1)^2, \\
x^T B x + 2 b^T x + d & \leq -\epsilon/2rK\kappa z(1)^2.
\end{align*}
$$

Then we have

$$
\begin{align*}
x^T A x + 2 a^T x & \leq c, \\
x^T B x + 2 b^T x + d & \leq 0
\end{align*}
$$
as required.

Else if $y(1) = 0$, noting that

$$
\begin{align*}
\tilde{z}^T A \tilde{z} & \leq -\epsilon/2r\kappa, \\
\tilde{z}^T B \tilde{z} & \leq -\epsilon/2rK\kappa.
\end{align*}
$$

By setting $\alpha = \min\{\frac{\epsilon}{2r\kappa(2|a^T \tilde{z}| + |c|)}, \frac{\epsilon}{2rK\kappa(2|b^T \tilde{z}| + |d|)}\} \leq 1$ and $x = \tilde{z}/\alpha$, we have

$$
\begin{align*}
x^T A x & = \frac{\tilde{z}^T A \tilde{z}}{\alpha^2} \\
& \leq -\frac{\epsilon}{2r\kappa\alpha^2} \\
& \leq -\frac{|2a^T \tilde{z}| + |c|}{\alpha} \quad (\text{due to } -1/\alpha \leq -\frac{2r\kappa(|2a^T \tilde{z}| + |c|)}{\epsilon}) \\
& \leq -|2a^T \tilde{z}/\alpha| - |c|. \quad (\text{due to } \alpha \leq 1)
\end{align*}
$$

Thus, $x^T A x + 2 a^T x - c \leq x^T A x + 2|a^T \tilde{y}|/\alpha + |c| \leq 0$. Similarly, we have

$$
\begin{align*}
x^T B x + 2 b^T x + d & \leq 0.
\end{align*}
$$

Hence $x$ is indeed a solution to system (1). \[ \square \]

4 Conclusion

In this paper, we presented the first linear-time algorithm to approximately solve the generalize trust region subproblem, which extends the recent result in [9] for the trust region subproblem. Our algorithm avoids diagonalization or factorization of matrices as that in [9]. Our algorithm also has the same time complexity as in the linear-time algorithm for TRS in [9] as well as in generalized eigenvector computation [8]. Similar to [9], our algorithm avoids the “hard case” by using an approximate linear-time SDP solver. Our future research will focus on extending the current algorithm to some variants of the GTRS with additional linear constraints or an additional unit ball constraint.

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