Frame-like gauge invariant formulation for massive high spin particles

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Abstract

In this paper we extend a so called frame-like formulation of massless high spin particles to massive case. We start with two explicit examples of massive spin 2 and spin 3 particles and then construct gauge invariant description for arbitrary integer spin case. Similarly, for the fermionic case we start with first non-trivial example — massive spin 5/2 particle and then construct gauge invariant description for arbitrary half-integer spin case. In all cases we consider massive particles in \((A)dS\) spaces with arbitrary cosmological constant (including flat Minkowski space) and this allows one to investigate all possible massless and partially massless limits for such particles.

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Introduction

As is well known, basically there are two approaches for description of gravity theory — metric one, where the main object is symmetric metric tensor $g_{\mu\nu}$, and tetrad one with tetrad $e_\mu^a$ and Lorentz connection $\omega_{\mu}{}^{ab}$. To a large extent these two approaches are equivalent, but for some concrete task one or another approach can more convenient. In particular, to describe interactions of fermionic fields with gravity (e.g. supergravity) one is forced to use tetrad formulation. Physically, the existence of two such approaches means that for a description of massless spin 2 particles one can use Lagrangians of second or first order in derivatives.

These two approaches admit natural generalization for description of high spin particles (for recent review of high spin theories see e.g. [1, 2, 3, 4]). Generalization of metric approach has been constructed in [5, 6, 7, 8, 9, 10], while generalization of tetrad approach, the so called frame-like formalism, has been constructed in [11, 12, 13] (see also [14, 15, 16, 17, 18, 19, 20, 21, 22]).

As is well known, Lorentz covariant description of massless high spin fields requires a theory to be gauge invariant. This, in particular, lead to so called constructive approach to investigation of consistent interactions of such fields when interaction Lagrangians and appropriate gauge transformations are constructed iteratively by the number of fields. In turn, common description of massive fields requires that some constraints must follow from equation of motion excluding all unphysical degrees of freedom. In this, at least two general problems appear then one tries to switch on interactions. First of all, a number of constraints could change thus leading to a change in a number of degrees of freedom and reappearing of unphysical ones. At second, even if a number of constraints remains to be the same as in free theory, interacting theory very often turns out to be non-casual, i.e. has solution corresponding to non-luminal propagation.

One of the possible solutions is to use gauge invariant description of massive high spin fields. There at least two basic approaches to such description. One of them based on the powerful BRST method [23, 24, 25, 26, 27, 28]. Another one appeared in attempt to generalize to high spins a very well known mechanism of spontaneous gauge symmetry breaking [29, 30, 31, 32] (see also [33, 16, 34, 35, 36]). In such a breaking a set of Goldstone fields with non-homogeneous gauge transformations appear making gauge invariant description of massive gauge fields possible. Such gauge invariant description of massive fields works well not only in flat Minkowski space-time, but in (anti) de Sitter space-times as well. All that one needs to do is to replace ordinary partial derivatives with the covariant ones and take into account commutator of these derivatives which is non-zero now. In particular, this formulation turns out to be very convenient for investigation of so called partially massless theories which appear in de Sitter space [37, 38, 39, 40].

It is evident that in any theory of high spin particles most of them have to be massive (and their gauge symmetries have to be spontaneously broken). It means that in any supersymmetric high spin theory like the superstring these particles must belong to some massive supermultiplet. It may seems strange but though explicit realization of massless supermultiplets with arbitrary spins were known for a long time [40] explicit construction for massive supermultiplets was not available until recently [41] (see also [42, 43]). The main idea is that massive supermultiplet must be easily constructed out of the appropriate set of massless
ones exactly in the same way as massive particle could be constructed using appropriate set of massless ones.

Construction of consistent high spin particles interactions is one of the old, hard and still unsolved problems. For the massless particles it is possible to formulate constructive approach to this problem (for BRST formulation see [44]). In this approach one starts with free Lagrangian for the collection of massless fields with appropriate gauge transformations and tries to construct interacting Lagrangian and modified gauge transformations iteratively by the number of fields so that:

$$\mathcal{L} \sim \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \ldots, \quad \delta \sim \delta_0 + \delta_1 + \delta_2 + \ldots$$

where $\mathcal{L}_1$ — cubic vertex, $\mathcal{L}_2$ — quartic one and so on, while $\delta_1$ — corrections to gauge transformations linear in fields, $\delta_2$ — quadratic in fields and so on. The mere existence of gauge invariant formulation for massive high spin particles allows us to extend such constructive approach for any collection of massive and/or massless particles, see e.g. [45, 46].

Till now, in most of the works on gauge invariant description for massive high spin particles metric-like formulation was used (see, however, [16, 19]). The aim of this paper is to extend frame-like formulation of bosonic and fermionic high spin particles to massive case, in this we will follow a minimalistic approach [30, 31, 32] introducing only minimal number of fields which are absolutely necessary for gauge invariant description of massive particle. We start in section 1 and section 2 with two explicit examples of massive spin 2 and spin 3 particles correspondingly. Then in section 3 we construct gauge invariant description of massive particles with arbitrary integer spin, which in metric-like formalism corresponds to completely symmetric tensors. Similarly, in section 4 we consider explicit construction of first non-trivial fermionic case — massive spin 5/2 particle, and then in section 5 we construct its generalization to arbitrary half-integer spin particles, which in metric-like formalism corresponds to completely symmetric spin-tensors.

## 1 Spin 2

Frame-like formalism is a very natural generalization of well known tetrad formulation of gravity on the case of high spin fields. We have already considered massive spin 2 particles in such formalism [16], however, it is instructive to look how this construction works in this simplest case before turning to generalizations on higher spins.

Two main objects in this case are physical field $h_\mu^a$ and auxiliary field $\omega_\mu^{ab}$, antisymmetric on $ab$. Massless theory has to be invariant under two gauge transformations:

$$\delta h_\mu^a = \partial_\mu \xi^a + \eta_\mu^a, \quad \delta \omega_\mu^{ab} = \partial_\mu \eta^{ab}$$

In this, it is easy to construct a tensor ("torsion") $T_{\mu \nu}^a$ which is invariant under $\xi^a$-transformations:

$$T_{\mu \nu}^a = \partial_\mu h_\nu^a - \partial_\nu h_\mu^a = \partial_{[\mu} h_{\nu]}^a$$

To construct gauge invariant Lagrangian for massless field one can use the following simple trick. Let us consider an expression:

$$\{ \frac{\mu \nu \sigma}{abc} \} \omega_\mu^{ab} T_{\nu \sigma}^c, \quad \{ \frac{\mu \nu \sigma}{abc} \} = \delta_a^e \delta_b^f \delta_c^g \delta_e^\mu \delta_f^\nu \delta_g^\sigma$$
and make a substitution \( T_{\mu \nu}^{a} \rightarrow \omega_{[\mu, \nu]}^{a} \). This gives an expression \( \{ \mu \nu \}_{ab} \omega_{\mu}^{ac} \omega_{\nu}^{bc} \). Now we look for massless Lagrangian in the form

\[
\mathcal{L}_{0} = a_{1} \{ \mu \nu \}_{ab} \omega_{\mu}^{ac} \omega_{\nu}^{bc} + a_{2} \{ \mu \nu \alpha \}_{abc} \omega_{\mu}^{ab} T_{\nu \alpha}^{c}
\]

This Lagrangian is invariant (by construction) under \( \xi^{a} \)-transformations, while invariance under \( \eta^{ab} \)-transformations requires \( a_{1} = -2a_{2} \). In what follows we choose \( a_{1} = \frac{1}{2} \), so finally we get:

\[
\mathcal{L}_{0} = \frac{1}{2} \{ \mu \nu \}_{ab} \omega_{\mu}^{ac} \omega_{\nu}^{bc} - \frac{1}{4} \{ \mu \nu \alpha \}_{abc} \omega_{\mu}^{ab} T_{\nu \alpha}^{c} \quad (3)
\]

Now let us switch to constant curvature \((A)dS_{d}\) space. Here and in what follows we will use the following convention on \((A)dS_{d}\) covariant derivatives:

\[
[D_{\mu}, D_{\nu}]\xi^{a} = -\kappa(e_{\mu}^{a} \xi_{\nu} - e_{\nu}^{a} \xi_{\mu}), \quad \kappa = \frac{2\Lambda}{(d-1)(d-2)} \quad (4)
\]

Now, if we replace all partial derivatives in the Lagrangian and gauge transformations by covariant derivatives:

\[
\delta \mathcal{L}_{0} = \kappa(d-2)[-\omega^{a} \xi_{a} + \eta^{ab} h_{ab}]
\]

where \( \omega^{a} = \omega_{\mu}^{ma} \), but invariance could be restored by adding appropriate corrections to the Lagrangian and gauge transformations:

\[
\Delta \mathcal{L}_{0} = -\frac{\kappa(d-2)}{2} \{ \mu \nu \}_{ab} h_{\mu}^{a} h_{\nu}^{b}, \quad \delta \omega_{\mu}^{ab} = \kappa e_{\mu}^{[a} \xi^{b]} \quad (5)
\]

Now let us consider a massive spin 2 particle in \((A)dS_{d}\) space. As is well known and we have already seen above, even for massless particles in \((A)dS_{d}\) space gauge invariance requires introduction of mass-like terms into Lagrangians as well as appropriate corrections for gauge transformations. So working with massive particles in \((A)dS_{d}\) spaces it turns out to be very convenient to organize a calculations by dimensionality of variations. Let us in general denote physical field as \( \Phi \), auxiliary field as \( \Omega \), parameters of gauge transformations as \( \xi \) and that of local shifts as \( \eta \). Then general structure of massive Lagrangian is \( \mathcal{L} = \mathcal{L}_{0} + \mathcal{L}_{1} + \mathcal{L}_{2} \), where \( \mathcal{L}_{0} \sim \Omega \Phi + \Omega D\Phi, \mathcal{L}_{1} \sim m \Omega \Phi \) and \( \mathcal{L}_{2} \sim m^{2} \Phi \Phi \). At the same time, general form of gauge transformations looks like \( \delta = \delta_{0} + \delta_{1} + \delta_{2} \) where \( \delta_{0} \Phi \sim D\xi + \eta, \delta_{0} \Omega \sim D\eta, \delta_{1} \Phi \sim m \xi, \delta_{1} \Omega \sim m \eta \) and \( \delta_{2} \Omega \sim m^{2} \xi \).

As is known \[30\] \[31\], to construct gauge invariant formulation for massive spin 2 field we need two additional physical fields with spin 1 and spin 0. It is natural to use first order formalism for these fields also, so we introduce two pairs \((F_{ab}, A_{\mu})\) and \((\pi^{a}, \varphi)\) and start with the sum of (covariantized) kinetic terms for all three pairs:

\[
\mathcal{L}_{0} = \frac{1}{2} \{ \mu \nu \}_{ab} \omega_{\mu}^{ac} \omega_{\nu}^{bc} + \frac{1}{4} \{ \mu \nu \}_{abc} \omega_{\mu}^{ab} T_{\nu \alpha}^{c} + \frac{1}{4} F_{ab}^{2} - \frac{1}{2} \{ \mu \nu \}_{ab} F^{ab} D_{\mu} A_{\nu} - \frac{1}{2} \{ a \}_{\pi} \pi^{a} D_{\mu} \varphi \quad (6)
\]

and initial gauge transformations:

\[
\delta_{0} h_{\mu}^{a} = D_{\mu} \xi^{a}, \quad \delta_{0} \omega_{\mu}^{ab} = D_{\mu} \eta^{ab}, \quad \delta_{0} A_{\mu} = D_{\mu} \xi \quad (7)
\]
Due to non-commutativity of \((A)dS\) covariant derivatives this Lagrangian is not invariant under initial gauge transformations:

\[
\delta \mathcal{L}_0 = \kappa (d - 2) [-\omega^a \xi_a + \eta^{ab} h_{ab}]
\]

so we have to take this non-invariance into account at appropriate stage of calculations. Now we add additional terms of order \(m\) to the Lagrangian:

\[
\mathcal{L}_1 = a_1 \{\mu \nu\}_{ab} \omega^a_{\mu b} + a_2 \{\mu\}_{a} F^a_{\mu b} + a_3 \{\mu\}_{a} \pi^a A_{\mu}
\]  

(8)

Non-invariance of these terms under initial gauge transformations \(\delta_0 \mathcal{L}_1\) could be compensated by

\[
\delta_1 h^a_{\mu} = \frac{2\alpha_1}{d - 2} e^a_{\mu} \xi,
\delta_1 F^{ab} = -2\alpha_1 \eta^{ab},
\delta_1 A_{\mu} = \alpha_1 \xi_{\mu},
\delta_1 \varphi = \alpha_2 \xi
\]

(9)

provided \(a_1 = a_2 = \alpha_1, a_3 = -\alpha_2\). Thus we obtain \(\delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 = 0\) and this leaves us with variations of order \(m^2\) (taking into account non-invariance due to non-commutativity of covariant derivatives) \(\delta_0 \mathcal{L}_0 + \delta_1 \mathcal{L}_1\). To compensate, we introduce mass-like terms into the Lagrangian as well as appropriate corrections to gauge transformations:

\[
\mathcal{L}_2 = b_1 \{\mu \nu\}_{ab} h^a_{\mu b} + b_2 \{\mu\}_{a} h^a_{\mu \varphi} + b_3 \varphi^2
\]  

(10)

\[
\delta_2 \omega^{ab} = \beta_1 \epsilon^{[a \xi^b]} \quad \delta_2 \pi^a = \beta_2 \xi^a
\]  

(11)

The requirement that total Lagrangian be invariant under the total gauge transformations allows one to express all parameters in terms of \(\alpha_1\) and \(\alpha_2\):

\[
b_1 = \alpha_1^2 - \frac{\kappa (d - 2)}{2}, \quad \beta_1 = -\frac{2b_1}{d - 2}, \quad b_2 = \beta_2 = -\alpha_1 \alpha_2, \quad b_3 = \frac{d}{d - 2} \alpha_1^2
\]

and gives important relation on these two parameters:

\[
4(d - 1)\alpha_1^2 - (d - 2)\alpha_2^2 = 2\kappa (d - 1)(d - 2)
\]

There is no strict definition on what is mass in \((A)dS\) spaces (see for example discussion in [17]), but working with gauge invariant formulation of massive particles it is natural to define massless limit as a limit where all Goldstone fields decouple from the main gauge field. For the case at hand, it means that it is the limit \(\alpha_1 \to 0\) corresponds to massless one. As for the concrete normalization, we will follow the rule ”mass is the parameter that would be mass in flat Minkowski space”. So we put \(\alpha_1^2 = m^2 / 2\). Combining all pieces together, we obtain final Lagrangian:

\[
\mathcal{L} = \frac{1}{2} \{\mu \nu\}_{ab} \omega_{\mu b} \omega_{\nu a} - \frac{1}{4} \{\mu \nu \alpha\}_{ab} \omega_{\mu b} T_{\nu a}^c + \frac{1}{4} F_{a}^{\mu b} - \frac{1}{2} \{\mu \nu\}_{ab} F^{ab} D_{\mu} A_{\nu} - \frac{1}{2} \pi_{a}^2 + \{\mu\}_{a} \pi^a D_{\mu} \varphi
\]

\[
+ \frac{m}{\sqrt{2}} \{\mu \nu\}_{ab} \omega_{\mu b} A_{\nu} + \{\mu\}_{a} F^{ab} h_{\mu b} - \alpha_2 \{\mu\}_{a} \pi^a A_{\mu} + \frac{m^2 - \kappa (d - 2)}{2} \{\mu \nu\}_{ab} h^a_{\mu b} - \frac{m \alpha_2}{\sqrt{2}} \{\mu\}_{a} h^a_{\mu \varphi} + \frac{d}{2(d - 2)} m^2 \varphi^2
\]

(12)
This Lagrangian is invariant under the following gauge transformations:

\[
\begin{align*}
\delta h^a_\mu &= D_\mu \xi^a + \frac{m\sqrt{2}}{d-2} \epsilon^a_\mu \xi, \\
\delta \omega^{ab}_\mu &= D_\mu \eta^{ab} - \left( \frac{m^2}{d-2} - \kappa \right) \epsilon^{[a}_\mu \xi^{b]} \\
\delta A_\mu &= D_\mu \xi + \frac{m}{\sqrt{2}} \xi_\mu, \\
\delta F^{ab} &= -m\sqrt{2}\eta^{ab}, \\
\delta \varphi &= 2\alpha_2 \xi, \\
\delta \pi^a &= -\frac{ma^2}{\sqrt{2}} \xi^a
\end{align*}
\] (13)

where parameter \(\alpha_2\) is defined through the relation:

\[
(d-2)\alpha_2^2 = 2(d-1)[m^2 - \kappa(d-2)]
\] (14)

From the last relation one can see that in \(dS\) space (\(\kappa > 0\)) there is a unitary forbidden region \(m^2 < \kappa(d-2)\). One can assume that it may be some deficiency of gauge invariant description, but it is easy to construct gauge invariant description for this region provided one change the sign of scalar field \(\varphi\) kinetic terms so that this field becomes a ghost. Thus a massless limit is possible in \(AdS\) space only, in this vector and scalar fields decouple and describe massive spin 1 particle. On the other hand, in the \(dS\) space one can put \(\alpha_2 = 0\). In this, scalar field completely decouples, while two other ones gives gauge invariant description of so called partially massless spin 2 particles with the Lagrangian:

\[
\mathcal{L} = \frac{1}{2} \left\{ \begin{array}{l}
\omega^{ac}_\mu \omega^b_\nu - \frac{1}{4} \{ \omega^{\alpha}_\mu \} T^c_{\nu \alpha} + \frac{1}{4} \{ \omega^{\mu}_\nu \} F^{ab}_\mu A_\nu + \\
\frac{m}{\sqrt{2}} \{ \omega^{\mu}_\nu \} F^{ab}_\mu h^b_\mu
\end{array} \right\}
\] (15)

(note the absence of explicit mass terms) which is invariant under the following gauge transformations:

\[
\begin{align*}
\delta h^a_\mu &= D_\mu \xi^a + \frac{m\sqrt{2}}{d-2} \epsilon^a_\mu \xi, \\
\delta \omega^{ab}_\mu &= D_\mu \eta^{ab} - \left( \frac{m^2}{d-2} - \kappa \right) \epsilon^{[a}_\mu \xi^{b]} \\
\delta A_\mu &= D_\mu \xi + \frac{m}{\sqrt{2}} \xi_\mu, \\
\delta F^{ab} &= -m\sqrt{2}\eta^{ab}
\end{align*}
\] (16)

2 Spin 3

In this section we consider less trivial spin 3 case which will show some important features of arbitrary spin case. For the description of massless spin 3 particles one needs main field \(\Phi^{ab}_\mu\), which is symmetric and traceless on "local" indices \(ab\), and auxiliary field \(\Omega^{a,bc}_\mu\) symmetric on \(bc\), traceless on all local indices and satisfying the relation \(\Omega^{a,bc}_\mu = 0\), where round brackets denote symmetrization. To describe right number of physical degrees of freedom, the theory must be invariant under the following gauge transformations:

\[
\begin{align*}
\delta \Phi^{ab}_\mu &= \partial_\mu \xi^{ab} + \eta^{ab}, \\
\delta \Omega^{a,bc}_\mu &= \partial_\mu \eta^{a,bc} + \xi^{a,bc}
\end{align*}
\] (17)

where \(\xi^{ab}\) symmetric and traceless, \(\eta^{a,bc}\) has the same properties as \(\Omega\), while \(\xi^{a,bc,d}\) is symmetric on \(ab\) and \(cd\), traceless on all indices and satisfies a constraint \(\xi^{a,bc,d} = 0\). Note that in general one also needs so called extra filed \(\Omega^{ab,cd}_\mu\) having the same properties on local indices as \(\xi\) and playing the role of gauge field for \(\xi\) transformations. However, such extra
fields do not enter free Lagrangians (though they play important role in construction of interactions) so we will not introduce such fields in this work. As in the previous case, one can easily construct an object out of first derivatives of $\Phi$ which will be invariant under $\xi$-transformations $T_{\mu \nu}^{ab} = \partial_{[\mu} \Phi_{\nu]}^{ab}$. To find a correct structure of massless Lagrangian one can use the same trick:

$$\{ \frac{\mu \nu \alpha}{\mu \nu \alpha} \} \Omega_\mu^{a,bd} T_{\nu \alpha}^{cd} \rightarrow 2 \{ \frac{\mu \nu}{\mu \nu} \} \Omega_\mu^{a,bd} \Omega_{\nu,\alpha}^{cd} = \{ \frac{\mu \nu}{\mu \nu} \} \Omega_\mu^{a,cd} \Omega_{\nu}^{b,cd} + 2 \Omega_\mu^{a,cd} \Omega_{\nu}^{c,cd}$$

So we will look for the appropriate Lagrangian in the form:

$$L_0 = a_1 \{ \frac{\mu \nu}{\mu \nu} \} \Omega_\mu^{a,bd} \Omega_{\nu,\alpha}^{cd} + a_2 \{ \frac{\mu \nu}{\mu \nu} \} \Omega_\mu^{a,bd} T_{\nu \alpha}^{cd}$$

This Lagrangian is invariant under $\xi$-transformations, while invariance under $\eta$-transformations requires $a_2 = -2a_1$ (in this, the Lagrangian is invariant under $\zeta$-transformations as well). We choose $a_1 = -1/6$, $a_2 = 1/3$ and obtain finally:

$$L_0 = -\frac{1}{6} \{ \frac{\mu \nu}{\mu \nu} \} \Omega_\mu^{a,bd} \Omega_{\nu,\alpha}^{cd} + \frac{1}{3} \{ \frac{\mu \nu}{\mu \nu} \} \Omega_\mu^{a,bd} T_{\nu \alpha}^{cd}$$

(18)

Now let us consider deformation to $(A)dS$ space [12]. If one replaces all derivatives in the Lagrangian and gauge transformations by the $(A)dS$-covariant ones, the Lagrangian cease to be invariant:

$$\delta L_0 = \kappa (d-1) \{ \frac{\mu \nu}{\mu \nu} \} \Phi_\mu^{a,bc} \Phi_\nu^{bc}$$

but this non-invariance could easily be compensated by adding appropriate corrections to the Lagrangian and gauge transformations:

$$\Delta L_0 = \kappa (d-1) \{ \frac{\mu \nu}{\mu \nu} \} \Phi_\mu^{a,ac} \Phi_\nu^{bc}$$

(19)

$$\delta \Omega_\mu^{a,bc} = \frac{d-1}{d-2} \kappa [(2e_\mu^{a,bc} - e_\mu^{(bc)c}) + \frac{1}{d-1} (2g^{bc} \sigma_\mu^{a} - g^{a(bc)} \sigma_\mu^{b,c})]$$

(20)

Let us turn to the massive case. Now we need [30, 31] three additional physical fields, corresponding to spin 2, spin 1 and spin 0. Again we will use frame-like formalism for all fields and start with the sum of kinetic terms for all fields where all derivatives are $(A)dS$-covariant ones:

$$L_0 = -\frac{1}{6} \{ \frac{\mu \nu}{\mu \nu} \} \Omega_\mu^{a,bd} \Omega_{\nu,\alpha}^{cd} + \frac{1}{3} \{ \frac{\mu \nu}{\mu \nu} \} \Omega_\mu^{a,bd} T_{\nu \alpha}^{cd} + \frac{1}{2} \{ \frac{\mu \nu}{\mu \nu} \} \omega_\mu^{a,c} \omega_\nu^{c,b} - \frac{1}{4} \{ \frac{\mu \nu}{\mu \nu} \} \omega_\mu^{a,b} T_{\nu \alpha}^{c} + \frac{1}{2} F_{\alpha}^{2} - \frac{1}{4} \{ \frac{\mu \nu}{\mu \nu} \} F_{\alpha}^{ab} D_\mu A_\nu - \frac{1}{2} \pi_\alpha^{2} + \{ \frac{\mu}{\mu} \} \pi_\alpha D_\mu \pi$$

(21)

as well as with full set of initial gauge transformations:

$$\delta_0 \Phi_\mu^{a,b} = D_\mu \eta^{ab}, \quad \delta_0 \Omega_\mu^{a,bc} = D_\mu \eta^{a,bc}, \quad \delta_0 \Omega_\mu^{a,bc} = \frac{1}{2} \pi_\alpha^{2} + \{ \frac{\mu}{\mu} \} \pi_\alpha D_\mu \pi$$

(22)

Again, due to non-commutativity of $(A)dS$ covariant derivatives this Lagrangian is not invariant:

$$\delta_0 L_0 = \kappa (d-1) \{ \frac{\mu \nu}{\mu \nu} \} \Phi_\mu^{a,bc} \Phi_\nu^{bc} - \kappa (d-2) [\omega_\alpha^{a} \xi_\alpha - \eta^{ab} h_{ab}]$$

6
but we will take this non-invariance into account later. Now we add to the Lagrangian all possible terms of order $m$:

$$\mathcal{L}_1 = \{ \mu^\nu, ab \} [a_1 \Omega_{\mu}^{abc} h_{\nu}^c + a_2 \Phi_{\mu}^{ac} \omega_{\nu}^{bc} + a_3 \omega_{\mu}^{ab} A_{\nu}] + \{ \mu_a \} [a_4 h_{\mu}^b F^{ab} + a_5 A_{\mu} \pi^a]$$

(23)

As usual, non-invariance of these terms under the initial gauge transformations $\delta_0 \mathcal{L}_1$ could be compensated by appropriate corrections to gauge transformations:

$$\delta_1 \Omega_{\mu}^{abc} = \frac{3 \alpha_1}{2d} [\eta^{a(b} e_{\mu} c)] + \frac{1}{d-1} (2g^{bc} \eta_{\mu} a - g^{a(b} \eta_{\mu} c)]$$

$$\delta_1 \Phi_{\mu}^{ab} = \frac{3 \alpha_1}{2(d-1)} (e_{\mu}^{(a \xi)} - \frac{2}{d} g^{ab} \xi_{\mu})$$

(24)

$$\delta_1 h_{\mu}^a = \alpha_1 \xi_{\mu}^a + \frac{2 \alpha_2}{d-2} e_{\mu}^a \xi, \quad \delta_1 \omega_{\mu}^{ab} = \alpha_1 \eta^{[a \beta]}_{\mu}$$

$$\delta_1 A_{\mu} = \alpha_2 \xi_{\mu}, \quad \delta F^{ab} = -2 \alpha_2 \eta^{ab}, \quad \delta_1 \varphi = \alpha_3 \xi$$

provided $a_1 = a_2 = -\alpha_1$, $a_3 = a_4 = \alpha_2$, $a_5 = -\alpha_3$. We proceed by adding all possible mass-like terms to the Lagrangian:

$$\mathcal{L}_2 = \{ \mu^\nu, ab \} [b_1 \Phi_{\mu}^{ac} \Phi_{\nu}^{bc} + b_2 h_{\mu}^a h_{\nu}^b] + b_3 \{ \mu_a \} h_{\mu}^a \varphi + b_4 \varphi^2$$

(25)

as well as appropriate corrections to gauge transformations:

$$\delta_2 \Omega_{\mu}^{abc} = \beta_1 [(2e_{\mu}^a \xi_{\nu}^{bc} - e_{\mu}^{(a \xi_{\nu}^c a)}] + \frac{1}{d-1} (2g^{bc} \xi_{\mu} a - g^{a(b \xi_{\mu} c)})]$$

$$\delta_2 \omega_{\mu}^{ab} = \beta_2 e_{\mu}^{[a \xi_{\nu}^b]}, \quad \delta_2 \pi^a = \beta_3 \xi^a$$

(26)

Then cancellation of all remaining variations determines all parameters in the Lagrangian and gauge transformations in terms of three main ones $\alpha_{1,2,3}$:

$$b_1 = -\frac{3 \alpha_1^2}{2} + \kappa (d-1), \quad \beta_1 = \frac{b_1}{d-2}, \quad b_2 = -\frac{3(d^2 - 4)}{4d(d-1)} \alpha_1^2 + \alpha_2^2 - \frac{\kappa (d-2)}{2}, \quad b_3 = \beta_3 = -\alpha_2 \alpha_3, \quad b_4 = \frac{d}{d-2} \alpha_2^2$$

and gives two important relations on these parameters:

$$3(d+1) \alpha_1^2 - d \alpha_2^2 = d(d+1) \kappa, \quad 9d \alpha_1^2 - (d-2) \alpha_3^2 = 6d(d-1) \kappa$$

From this results it follows that it is the limit $\alpha_1 \to 0$ corresponds to the massless one, in this our definition of mass gives $\alpha_1^2 = m^2/3$. Collecting all pieces together we obtain final Lagrangian:

$$\mathcal{L} = \mathcal{L}_0(\Omega_{\mu}^{abc}, \Phi_{\mu}^{ab}) + \mathcal{L}_0(\omega_{\mu}^{ab}, h_{\mu}^a) + \mathcal{L}_0(F^{ab}, A_\mu) + \mathcal{L}_0(\pi^a, \varphi) -$$

$$-\frac{m}{\sqrt{3}} \{ \mu^\nu, ab \} [\Omega_{\mu}^{abc} h_{\nu}^c + \Phi_{\mu}^{ac} \omega_{\nu}^{bc}] + \alpha_2 \{ \mu^\nu, ab \} \omega_{\mu}^{ab} A_{\nu} + \{ \mu_a \} h_{\mu}^b F^{ab} -$$

$$-\frac{m^2 - 2 \kappa (d-1)}{2} \{ \mu^\nu, ab \} [\Phi_{\mu}^{ac} \Phi_{\nu}^{bc} - \frac{3d}{2(d-1)} h_{\mu}^a h_{\nu}^b] -$$

$$-\alpha_2 \alpha_3 \{ \mu_a \} h_{\mu}^a \varphi + \frac{d}{d-2} \alpha_2^2 \varphi^2$$

(27)
where parameters $\alpha_2$ and $\alpha_3$ are defined through the relations:

$$d\alpha_2^2 = (d + 1)[m^2 - dk], \quad (d - 2)\alpha_3^2 = 3d[m^2 - 2(d - 1)k]$$ (28)

From the last relations we see that in $dS$ space we again obtain unitary forbidden region $m^2 < 2(d - 1)k$, so that massless limit is possible in $AdS$ space only. The boundary of forbidden region corresponds to $\alpha_3 = 0$, in this scalar field decouples, while the remaining fields provides gauge invariant description of the first partially massless theory (in $d = 4$ it is a particle with helicities $\pm 3, \pm 2, \pm 1$) with the Lagrangian (note the absence of explicit mass-like terms):

$$\mathcal{L} = \mathcal{L}_0(\Omega_{\mu}^{a,bc}, \Phi_{\mu}^{ab}) + \mathcal{L}_0(\omega_{\mu}^{ab}, h_{\mu}^{a}) + \mathcal{L}_0(F_{\mu}^{ab}, A_{\mu}) -$$

$$-\frac{m}{\sqrt{3}} \left\{ \frac{\mu}{ab} \right\} [\Omega_{\mu}^{a,bc}h_{\nu}^{c} + \Phi_{\mu}^{ac}\omega_{\nu}^{bc}] + \alpha_2 \left\{ \frac{\mu}{ab} \right\} \omega_{\mu}^{ab}A_{\nu} + \{ \mu \} h_{\mu}^{b}F_{\nu}^{ab}$$ (29)

where $d\alpha_2^2 = (d + 1)(d - 2)k$.

Another example of partially massless theory which "lives" in a forbidden region we obtain by setting $m^2 = dk$, i.e. $\alpha_2 = 0$. In this case both vector field $A_{\mu}$ and scalar one $\varphi$ decouple, while remaining $\Phi_{\mu}^{ab}$ and $h_{\mu}^{a}$ provides gauge invariant description of partially massless particle (in $d = 4$ it has helicities $\pm 3, \pm 2$) with the Lagrangian:

$$\mathcal{L} = \mathcal{L}_0(\Omega_{\mu}^{a,bc}, \Phi_{\mu}^{ab}) + \mathcal{L}_0(\omega_{\mu}^{ab}, h_{\mu}^{a}) -\frac{m}{\sqrt{3}} \left\{ \frac{\mu}{ab} \right\} [\Omega_{\mu}^{a,bc}h_{\nu}^{c} + \Phi_{\mu}^{ac}\omega_{\nu}^{bc}] +$$

$$+\frac{d - 2}{2d}m^2 \{ \frac{\mu}{ab} \} [\Phi_{\mu}^{ac}\Phi_{\nu}^{bc} - \frac{3d}{2(d - 1)}h_{\mu}^{a}h_{\nu}^{b}]$$ (30)

which is invariant under the following gauge transformations:

$$\delta\Phi_{\mu}^{ab} = D_{\mu}\xi^{ab} + \eta_{\mu}^{ab} + \frac{m\sqrt{3}}{2(d - 1)}(e_{\mu}^{(a}\xi^{b)} - \frac{2}{d}g^{ab}\xi_{\mu})$$

$$\delta\Omega_{\mu}^{a,bc} = D_{\mu}\eta^{a,bc} + \frac{m\sqrt{3}}{2d}(e_{\mu}^{a}\xi_{bc}^{c}) + \frac{1}{d - 1}(2g^{bc}\eta_{\mu}^{a} - g^{a,b}\eta_{\mu}^{c}) +$$

$$+\frac{m^2}{2d}(2e_{\mu}^{a}\xi_{bc}^{c} - e_{\mu}^{(bc}\xi^{a)}) + \frac{1}{d - 1}(2g^{bc}\xi_{\mu}^{a} - g^{a,b}\xi_{\mu}^{c})$$ (31)

$$\delta h_{\mu}^{a} = D_{\mu}\xi^{a} + \eta_{\mu}^{a} + \frac{m}{\sqrt{3}}\xi_{\mu}^{a}, \quad \delta\omega_{\mu}^{ab} = D_{\mu}\eta^{ab} + \frac{m}{\sqrt{3}}\eta_{\mu}^{[a,b]} + \frac{3m^2}{2(d - 1)e_{\mu}^{[a}\xi^{b]}}$$

3 Arbitrary integer spin

For description of massless spin $s$ particles one needs [11] [12] main physical field $\Phi_{\mu}^{a_1...a_{s-1}}$ symmetric and traceless on local indices and auxiliary field $\Omega_{\mu}^{a,a_{1}...a_{s-1}}$ which must be symmetric on last $s - 1$ local indices, traceless on all local indices and satisfy condition $\Omega_{\mu}^{(a_{1}...a_{s-1})} = 0$ (as in the spin 3 case we will not introduce any extra fields here). To have a correct number of physical degrees of freedom theory must be invariant under the following gauge transformations:

$$\delta\Phi_{\mu}^{a_1...a_{s-1}} = \partial_{\mu}\xi^{a_1...a_{s-1}} + \eta_{\mu}^{a_1...a_{s-1}}, \quad \delta\Omega_{\mu}^{a,a_{1}...a_{s-1}} = \partial_{\mu}\eta^{a,a_{1}...a_{s-1}} + \xi_{\mu}^{a,a_{1}...a_{s-1}}$$ (32)
where parameter $\xi$ — symmetric and traceless, parameter $\eta$ has the same properties on local indices as $\Omega$, while $\zeta^{bc,a_1\ldots a_{s-1}}$ is symmetric on both groups of indices, completely traceless and satisfies a constraint $\zeta^{b(c,a_1\ldots a_{s-1})} = 0$. As in the previous cases, we introduce an object

$$T_{\mu\nu}^{a_1\ldots a_{s-1}} = \partial_{[a}\Phi_{\nu]}^{a_1\ldots a_{s-1}}$$

invariant under $\xi$-transformations and consider an expression:

$$\{\mu^{\alpha\nu} \} \Omega_{\mu}^{ab}{a_2\ldots a_{s-1}} T_{\nu\alpha}^{ca_2\ldots a_{s-1}} \Rightarrow \{\mu^{\alpha\nu} \} \Omega_{\mu}^{ab}{a_2\ldots a_{s-1}}\Omega_{\nu,\alpha}^{ca_2\ldots a_{s-1}} =$$

$$= \{\mu^{\alpha\nu} \} [\Omega_{\mu}^{c,a_2\ldots a_{s-1}}\Omega_{\nu}^{c,b_2\ldots a_{s-1}} + \frac{1}{s-1}\Omega_{\mu}^{a_1\ldots a_{s-1}}\Omega_{\nu}^{b_1\ldots a_{s-1}}]$$

Let us introduce condensed notations for tensor objects like $\Phi_{\mu}^{a_1\ldots a_{s-1}} = \Phi_{\mu}^{(s-1)}$ and $\Omega_{\mu}^{a_1\ldots a_{s-1}} = \Omega_{\mu}^{a,(s-1)}$. In this notations our candidate for massless Lagrangian will looks as follows:

$$\mathcal{L}_0 = a_1 \{\mu^{\alpha\nu} \} [\Omega_{\mu}^{c,a(s-2)}\Omega_{\nu}^{c,b(s-2)} + \frac{1}{s-1}\Omega_{\mu}^{a(s-1)}\Omega_{\nu}^{b(s-1)}] + a_2 \{\mu^{\alpha\nu} \} \Omega_{\mu}^{a,b(s-2)} T_{\nu\alpha}^{c(s-2)}$$

It is (by construction) invariant under $\xi$-transformations, while invariance under $\eta$-transformations requires $a_2 = -a_1$. For simplicity we will use non-canonical normalization of fields and choose the following final form for our massless Lagrangian:

$$(-1)^s \mathcal{L}_0 = \{\mu^{\alpha\nu} \} [\Omega_{\mu}^{c,a(s-2)}\Omega_{\nu}^{c,b(s-2)} + \frac{1}{s-1}\Omega_{\mu}^{a(s-1)}\Omega_{\nu}^{b(s-1)}] - \{\mu^{\alpha\nu} \} \Omega_{\mu}^{a,b(s-2)} T_{\nu\alpha}^{c(s-2)}$$

Now we consider deformation to $(A)dS$ space [12]. If one replaces all derivatives in the Lagrangian and gauge transformations by $(A)dS$ covariant ones, the Lagrangian cease to be invariant:

$$(-1)^s \delta \mathcal{L}_0 = \frac{2s(d+s-4)}{s-1}\kappa[\Phi_{\mu}^{(s-1)}\eta^{\mu,(s-1)} - \Omega_{\mu}^{\mu,(s-1)}\xi^{(s-1)}]$$

but this non-invariance could be compensated by adding appropriate corrections to Lagrangian and gauge transformations:

$$(-1)^s \Delta \mathcal{L}_0 = -s(d+s-4)\kappa \{\mu^{\nu} \} \Phi_{\mu}^{a(s-2)}\Phi_{\nu}^{b(s-2)}$$

$$\delta \Omega_{\mu}^{a,(s-1)} = \frac{d+s-4}{d-2}\kappa[\kappa'(s-1)e_{\mu}^a\xi^{(s-1)} - e_{\mu}^{(1)}\xi^{(s-2)}a - \frac{1}{d+s-4}((s-2)g^{a(1)}\xi^{(s-2)} - 2g^{(12)}\xi^{(s-3)}a)]$$

Let us consider massive case now. For gauge invariant description of massive spin $s$ particle one needs a set of fields with spins $k$, $0 \leq k \leq s$. General formulas given above work for $k \geq 2$ only, so spin 1 and spin 0 must be treated separately. As before, we start with the sum of (covariantized) kinetic terms for all fields:

$$\mathcal{L}_0 = \sum_{k=2}^{s} \mathcal{L}_0(\Phi_k) + \frac{1}{4} F_{ab}^2 - \frac{1}{2} \{\mu^{\alpha\nu} \} F^{ab} D_{\mu} A_{\nu} - \frac{1}{2} \pi_{a}^2 + \{ \mu^{\alpha} \} \pi^{a} D_{\mu} \varphi$$
where
\[(1)^k L_0(\Phi_k) = \{ \mu^\nu \} \{ a^c_{a,k-2}\Omega^c_{\mu} a^b_{b,k-2} + \frac{1}{k-1} \Omega^a_{\mu} a^b_{b,k-1}\Omega^b_{\nu} a^c_{c,k-1} \} - \{ \mu^\nu \} \Omega^a_{\mu} a^b_{b,k-2} T^c_{\nu a} c^{k-2}\]
and corresponding set of initial gauge transformations:
\[\delta_0 \Phi_{\mu}^{(k)} = D_\mu \xi_{\mu}^{(k)} + \eta_{\mu}^{(k)} \quad \delta_0 \Omega_{\mu}^{a,(k)} = D_\mu \eta_{\mu}^{a,(k)} \quad 1 \leq k \leq s - 1 \quad \delta_0 A_\mu = D_\mu \xi \quad (37)\]
Due to non-commutativity of covariant derivatives our Lagrangian is not invariant under the initial gauge transformations:
\[\delta_0 L_0 = \sum_{k=1}^{s-1} (-1)^k 2(k+1)(d+k-3) \frac{k}{k} \xi_{\mu}^{(k)} \Omega_{\mu}^{a,(k)} \Omega_{\nu}^{b,(k-1)} - b_1(A\pi)\]
so we have to take this non-invariance into account at appropriate stage of calculations.

Now we proceed by adding to the Lagrangian all possible terms of order \(s\):
\[L_1 = \sum_{k=2}^{s} (-1)^k \{ \mu^\nu \} \left[ a_k \Omega_{\mu}^{a,(k-2)} \Phi_{\nu}^{(k-2)} + b_k \Phi_{\mu}^{a,(k-2)} \Omega_{\nu}^{b,(k-2)} \right] - b_1(A\pi) \quad (38)\]
Consider gauge transformations for field \(\Phi_k\). In general case \((k \neq s \text{ and } k \neq 1)\) this field enters \(L_1\) as follows:
\[(-1)^k \{ \mu^\nu \} \left[ -a_{k+1} \Omega_{\mu}^{a,(k-1)} \Phi_{\nu}^{(k-1)} - b_{k+1} \Phi_{\mu}^{a,(k-1)} \Omega_{\nu}^{b,(k-1)} + a_k \Omega_{\mu}^{a,(k-2)} \Phi_{\nu}^{(k-2)} + b_k \Phi_{\mu}^{a,(k-2)} \Omega_{\nu}^{b,(k-2)} \right]\]
As usual, non-invariance of these terms under the initial gauge transformations of \(\Phi_k\) field could be compensated by the following corrections to gauge transformations:
\[\delta_1 \Phi_{\mu}^{(k)} = \frac{k\alpha_k}{(k-1)(d+k-3)} \left[ e_\mu^{(1)\xi_{(k-1)}} - \frac{2}{d+2k-4} g_{(2)\Phi_{\mu}^{(k-2)}} \right] \]
\[\delta_1 \Omega_{\mu}^{a,(k)} = \frac{k\alpha_k}{(k-1)(d+k-2)} \left[ e_\mu^{(1)\eta_{a,(k-1)}} - \frac{1}{d+k-3} g_{a(1)\eta_{\mu}^{a,(k-1)}} - \frac{2}{d+2k-4} g_{(12)\eta_{\mu}^{a,(k-2)}} + \frac{2}{d+k-3}(d+2k-4) g_{(12)\eta_{\mu}^{a,(k-2)}} \right] \quad (39)\]
\[\delta_1 \Phi_{\mu}^{(k-2)} = \frac{\alpha_k}{k-2} \xi_{\mu}^{(k-2)} \quad \delta_1 \Omega_{\mu}^{a,(k-2)} = \frac{k-1}{k-2} \alpha_{k-1} \left[ \eta_{a,(k-2)} - \frac{1}{k-1} \eta_{\mu}^{a,(k-2)} \right] \]
provided:
\[a_{k+1} = b_{k+1} = \frac{2k}{k-1} \alpha_k\]
Now, collecting together all corrections for the field \(\Phi_k\), we obtain:
\[\delta_1 \Phi_{\mu}^{(k-1)} = \frac{\alpha_k}{k-1} \xi_{\mu}^{(k-1)} + \frac{(k-1)\alpha_{k-1}}{(k-2)(d+k-4)} \left[ e_\mu^{(1)\xi_{(k-2)}} - \frac{2}{d+2k-6} g_{(12)\xi_{\mu}^{(k-3)}} \right] \]
\[\delta_1 \Omega_{\mu}^{a,(k-1)} = \alpha_{k-1} \left[ e_\mu^{(1)\eta_{a,(k-1)}} - \frac{1}{d+k-4} g_{a(1)\eta_{\mu}^{a,(k-2)}} - \frac{2}{d+2k-6} g_{(12)\eta_{\mu}^{a,(k-3)}} \right] \quad (40)\]
This formulas work for the \( k = s \) if we assume \( \alpha_s = 0 \), while \( k = 1, 2 \) cases have to be considered separately. Let us collect all terms in \( \mathcal{L}_1 \) containing \( \Phi_2 \):

\[
\{_{\mu
u}^{ab}\} \left[-a_3\Omega_{\mu u}^{ab} h_{\nu r} + b_3\Phi_{\mu u}^{ac} \omega_{\nu r}^{bc} + a_2\omega_{\mu u}^{ab} A_{\nu r} + b_2 h_{\mu u} a F_{\nu r}^b \right]
\]

Their non-invariance could be compensated by the following corrections to gauge transformations:

\[
\begin{align*}
\delta_1\Phi_{\mu u}^{ab} &= \frac{2\alpha_2}{d-1} \left[ e_{\mu u}^{(a}\xi^{b)} - \frac{2}{d} g_{\mu u}^{a b} \xi \right] \\
\delta_1\Omega_{\mu u}^{a, b c} &= \frac{2\alpha_2}{d} \left[ \eta^{a(b} e_{\mu u}^{c) d} + \frac{1}{d-1} (2g_{b c}^d \eta^a_{\mu u} - g^a_{(b} \eta^c_{\mu u}) \right] \\
\delta_1 A_{\mu u} &= \alpha_1 \xi_{\mu u}, \quad \delta F_{u v}^{a b} = -2\alpha_1 \eta_{u v}^{a b}
\end{align*}
\]

provided:

\[
a_3 = b_3 = 4\alpha_2, \quad a_2 = b_2 = \alpha_1
\]

At last, non-invariance of the terms containing \( \Phi_1 \):

\[
\{_{\mu
u}^{ab}\} \left[a_2\omega_{\mu u}^{ab} A_{\nu r} + b_2 h_{\mu u} a F_{\nu r}^b \right] - b_1 (A \pi)
\]

could be compensated with:

\[
\begin{align*}
\delta h_{\mu u}^a &= \frac{\alpha_1}{2(d-2)} e_{\mu u}^a \xi, \quad \delta \varphi = \alpha_0 \xi, \quad b_1 = \alpha_0
\end{align*}
\]

We proceed by adding mass-like terms to the Lagrangian:

\[
\mathcal{L}_2 = \sum_{k=2}^s (-1)^k \{_{\mu
u}^{ab}\} c_k \Phi_{\mu u}^{a(k-2)} \Phi_{\nu r}^{b(k-2)} + c_1 h \varphi + c_0 \varphi^2
\]

as well as appropriate corrections to gauge transformations:

\[
\begin{align*}
\delta_2\Omega_{\mu u}^{a, b c} &= \beta_k \left[ k e_{\mu u}^{a} \xi^{(b)} - e_{\mu u}^{(b} \xi^{a)} - \frac{1}{d + k - 3} \left[ (k - 1) g^a_{(b} \xi^{(c)} - 2g_{b}^{12} \xi^{(c)} (k-2)a) \right] \right], \quad k > 1 \\
\delta_2\Phi_{\mu u}^{a b} &= \beta_1 \left[ e_{\mu u}^{a} \xi_{b} - e_{\mu u}^{b} \xi_{a} \right], \quad \delta_2 \pi^a = \beta_0 \xi^a
\end{align*}
\]

Cancellation of variations at order \( m^2 \) (taking into account non-invariance of kinetic terms due to non-commutativity of covariant derivatives) requires:

\[
c_k = -\frac{k^2(d + 2k - 2)(d + k - 4)}{(k - 1)(d + k - 3)(d + 2k - 4)} \alpha_k^2 + k(k-1) \frac{1}{k-2} \alpha_{k-1}^2 - k(d + k - 4) \kappa, \quad k > 2
\]

\[
c_2 = -\frac{4(d^2 - 4)}{d(d-1)} \alpha_2^2 + \alpha_1^2 - 2(d - 2) \kappa \quad \quad c_1 = -\alpha_1 \alpha_0
\]

\[
\beta_k = -\frac{c_{k+1}}{(k+1)(d-2)}, \quad k > 0, \quad \beta_0 = c_1
\]

At last, all variations at order \( m^3 \) cancel provided:

\[
k(d + k - 2) c_{k+1} = (k-1)(d + k - 3) c_k, \quad k > 1
\]
\[ 4(d-1)c_2 = (d-2)c_0^2, \quad c_0 = \frac{d}{d-2}a_1^2 \]

These last relations give us recurrent relations on the main parameters \( \alpha \):

\[ \frac{(k+1)^2(d+2k)}{d+2k-2}a_{k+1}^2 - \frac{2k^3(d+2k-3)}{(k-1)(d+2k-4)}a_k^2 + \frac{k(k-1)^2}{k-2}a_{k-1}^2 + 2k(d+2k-3)\kappa = 0 \]

To solve these relations, recall that \( \alpha_{s} = 0 \), while our definition of mass gives in this case

\[ \alpha_{s-1}^2 = \frac{s^2}{(s-1)^2}m^2. \]

We obtain:

\[ \alpha_k^2 = \frac{(k-1)(s-k)(d+s+k-3)}{k^2(d+2k-2)}[m^2 - (s-k-1)(d+s+k-4)\kappa], \quad k > 1 \]
\[ \alpha_1^2 = \frac{2(s-1)(d+s-2)}{d}[m^2 - (s-2)(d+s-3)\kappa] \]
\[ \alpha_0^2 = \frac{4s(d+s-3)}{(d-2)}[m^2 - (s-1)(d+s-4)\kappa] \]

Using this solution we get final expressions for parameters \( c \):

\[ c_k = \frac{s(d+s-3)}{(k-1)(d+k-3)}[m^2 - (s-1)(d+s-4)\kappa], \quad k > 1 \]
\[ c_1 = -\alpha_1\alpha_0, \quad c_0 = \frac{d}{d-2}a_1^2 \]

Collecting all results we obtain final Lagrangian for gauge invariant description of massive spin \( s \) particles in \((A)dS_d\) space:

\[ \mathcal{L} = \sum_{k=2}^{s} \mathcal{L}_0(\Phi_k) + \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\{\mu^{\nu}\} F_{\mu\nu}D_{\alpha}A_{\nu} - \frac{1}{2}\pi_{\mu}^2 + \{\pi_{\mu}\} \pi_{\mu}D_{\mu}\varphi + \]
\[ + \sum_{k=2}^{s} (-1)^k \{\mu^{\nu}\} a_k[\Omega_{\mu}a^{k-2}\Phi_{\nu}^{(k-2)} + \Phi_{\mu}^{(a)}b^{(k-2)}] - \alpha_0(\Lambda_{\pi}) + \]
\[ + \sum_{k=2}^{s} (-1)^k \{\mu^{\nu}\} c_k\Phi_{\mu}^{(a)(k-2)} - \alpha_1\alpha_0h\varphi + \frac{d}{d-2}a_1^2\varphi^2 \quad (45) \]
\[ a_k = \frac{2(k-1)}{k-2}\alpha_{k-1}, \quad k > 2, \quad a_2 = \alpha_1, \quad c_k = \frac{s(d+s-3)}{(k-1)(d+k-3)}M^2 \]

where \( M^2 = m^2 - (s-1)(d+s-4)\kappa \). As for the gauge transformations leaving this Lagrangian invariant, we have seen that the most complicated part is related with gauge transformations of auxiliary fields \( \Omega \). So we reproduce here gauge transformations for physical fields only:

\[ \delta \Phi_{\mu}^{(k)} = D_{\mu}\xi^{(k)} + \eta_{\mu}^{(k)} + \alpha_{k+1}\xi_{\mu}^{(k)} + \frac{k\alpha_k}{(k-1)(d+k-3)}[e_{\mu}^{(1)}\xi^{(k-1)}\kappa] - \frac{2}{d+2k-4}\eta_{\mu}^{(2)}\xi^{(k-2)} \]
\[ \delta h_{\mu}^{\alpha} = D_{\mu}\xi^{\alpha} + \eta_{\mu}^{\alpha} + \alpha_2\xi_{\mu}^{\alpha} + \frac{\alpha_1}{2(d-2)}e_{\mu}^{\alpha}\xi, \quad \delta A_{\mu} = D_{\mu}\xi + \alpha_1\xi_{\mu}, \quad \delta \varphi = \alpha_0\xi \quad (46) \]
As in the spin 2 and spin 3 cases, we see that massless limit is possible in AdS (and Minkowski) space only. In this, massive spin $s$ particle decompose into massless spin $s$ and massive spin $s - 1$ ones. In $dS$ space we again find unitary forbidden region $m^2 < (s - 1)(d + s - 4)\kappa$. At the boundary of this region $m^2 = (s - 1)(d + s - 4)\kappa$ scalar field decouples and we obtain first partially massless theory (note that in this case all explicit mass-like terms vanish). Inside this forbidden region we obtain a number of partially massless theories. Namely, if one of the $\alpha_k = 0$, then all fields $\Phi_l$ with $0 \leq l \leq k$ decouple, while the remaining fields with $k + 1 \leq l \leq s$ give gauge invariant description of corresponding partially massless particle. Let us give here only one concrete example — the most simple one where only two fields $\Phi_s$ and $\Phi_{s-1}$ remain. It happens when $m^2 = (d + 2s - 6)\kappa$.

Corresponding Lagrangian looks as:

$$
\mathcal{L} = \mathcal{L}_0(\Phi_s) + \mathcal{L}_0(\Phi_{s-1}) + (-1)^s \frac{2m}{\sqrt{s-2}} \left\{ \eta_{ab} \right\} \left[ \Omega^{a,(s-2)} \Phi^{(s-2)} + \Phi^{a(s-2)} \Omega^{b,(s-2)} \right] -
\frac{(-1)^s s(s + 5)\kappa}{s-1} \left\{ \eta_{ab} \right\} \left[ \frac{s-2}{s} \Phi^{a(s-2)} \Phi^{b(s-2)} - \frac{d + s - 3}{d + s - 4} \Phi^{a(s-3)} \Phi^{b(s-3)} \right]
$$

(47)

while gauge transformations leaving it invariant have the form:

$$
\begin{align*}
\delta \Phi^{(s-1)}_{\mu} &= D_{\mu} \xi^{(s-1)} + \eta^{(s-1)}_{\mu} + \frac{m}{\sqrt{s-2}(d+s-4)} \left[ e_{\mu}(1 \xi^{s-2}) - \frac{2}{d+2s-6} g^{(2 \xi^{s-3})} \right] \\
\delta \Phi^{(s-2)}_{\mu} &= D_{\mu} \xi^{(s-2)} + \eta^{(s-2)}_{\mu} + \frac{m\sqrt{s-2}}{s-1} \xi^{(s-2)}
\end{align*}
$$

(48)

4 Spin 5/2

In this and in the next section we will work in four-dimensional space-time assuming that all fermionic objects are Majorana ones, however all results could be easily generalized to arbitrary dimension with appropriate spinors. We start here with the first non-trivial example — spin 5/2. For the description of free massless spin 5/2 particles [13] there is no need to introduce any auxiliary fields, though they play important role for construction of interactions. So the only object we need — spin-tensor $\Psi^{a}_{\mu}$ which is $\gamma$-transverse $\gamma^a \Psi^{a}_{\mu} = 0$.

To describe correct number of physical degrees of freedom, theory must be invariant under the following gauge transformations:

$$
\begin{align*}
\delta \Psi^{a}_{\mu} &= \partial_{\mu} \xi^{a} + \eta^{a}_{\mu}, \quad \gamma^a \xi^{a} = 0, \quad \eta^{ab} = -\eta^{ba}, \quad \gamma^a \eta^{ab} = 0
\end{align*}
$$

(49)

It is not hard to construct gauge invariant Lagrangian describing massless particle:

$$
\mathcal{L}_0 = \frac{i}{2} \left\{ \eta_{abc} \right\} \left[ \bar{\Psi}^{a}_{\mu} \gamma^{c}_{\gamma} \gamma^{b}_{\mu} \partial_{\nu} \Psi^{a}_{\mu} d - 6 \bar{\Psi}^{a}_{\mu} \gamma^{b}_{\mu} \partial_{\nu} \Psi^{a}_{\nu} \right]
$$

(50)

where relative coefficient is determined by the $\eta$-invariance. Let us first of all consider deformation to AdS space. Working with covariant derivatives one has to take into account (implicit) spinor indices on fermionic objects, e.g.:

$$
[D_{\mu}, D_{\nu}] \xi^{a} = -\kappa (e^{a}_{\mu} \xi^{\nu} - e^{a}_{\nu} \xi^{\mu} + \frac{1}{2} \gamma^{a}_{\mu\nu} \xi^{a})
$$

(51)
As is well known, in AdS space gauge transformations has to be modified:

$$\delta \Psi_\mu^a = D_\mu \xi^a + i \alpha_0 \gamma_\mu \xi^a + \eta_\mu^a,$$

and, for this transformations to be compatible with the constraint $$\gamma^a \Psi_\mu^a = 0$$, the constraint on $$\eta$$ parameter also has to be changed:

$$\gamma^a \eta^{ab} = 0 \Rightarrow \gamma^a \eta^{ab} = 2i \alpha_0 \xi^b$$

Non-invariance of (covariantized) massless Lagrangian under new gauge transformations (taking into account contribution from $$\eta$$-transformations due to constraint) has the form:

$$\delta_\xi L_0 = -12 \alpha_0 \left\{ \mu^\nu \right\} \left[ \bar{\Psi}_\mu \gamma^a \gamma^b D_\nu \xi^c + 2 \bar{\Psi}_\mu^a D_\nu \xi^b \right] + 30i \kappa (\bar{\Psi} \gamma)^a \xi^a$$

where $$(\bar{\Psi} \gamma)^a = \bar{\Psi}^a \gamma^\mu$$ and could be compensated by adding to the Lagrangian mass-like terms:

$$L_1 = \left\{ \mu^\nu \right\} \left[ a_1 \bar{\Psi} \gamma^a \gamma^b \Psi^c + a_2 \bar{\Psi}^a \Psi^b \right]$$

provided $$a_1 = 6 \alpha_0$$, $$a_2 = 12 \alpha_0$$, $$\alpha_0^2 = -\frac{5}{4}$$. Thus, the Lagrangian for massless spin 5/2 particle and gauge transformations leaving it invariant have the form:

$$\mathcal{L} = \frac{i}{2} \left\{ \mu^\nu \right\} \left[ \bar{\Psi}_\mu \gamma^a \gamma^b \gamma^c D_\nu \Psi^d - 6 \bar{\Psi}_\mu \gamma^a \gamma^b D_\nu \Psi^c \right] + 3 \sqrt{-\kappa} \left\{ \mu^\nu \right\} \left[ \bar{\Psi}_\mu \gamma^a \gamma^b \Psi^c + 2 \bar{\Psi}^a \Psi^b \right]$$

$$\delta \Psi_\mu^a = D_\mu \xi^a + i \sqrt{-\kappa} \frac{1}{2} \gamma_\mu \xi^a + \eta_\mu^a, \quad \gamma^a \eta^{ab} = i \sqrt{-\kappa} \xi^b$$

Note that working with massless fermions in AdS space very often one introduces a convenient generalized covariant derivative, e.g.

$$\nabla_\mu \xi^a = D_\mu \xi^a + i \sqrt{-\kappa} \frac{1}{2} \gamma_\mu \xi^a$$

As we will see later on, for massive case parameter $$\alpha_0$$ will depends both on mass and cosmological constant, so we will not introduce such covariant derivative here.

Let us turn to the massive case. Gauge invariant description of massive spin 5/2 particle requires introduction of two additional fields with spin 3/2 and spin 1/2. We consider a Lagrangian which is a sum of covariantized kinetic terms for all three fields plus all possible mass-like terms:

$$\mathcal{L} = \frac{i}{2} \left\{ \mu^\nu \right\} \left[ \bar{\Psi}_\mu \gamma^a \gamma^b \gamma^c D_\nu \Psi^d - 6 \bar{\Psi}_\mu \gamma^a \gamma^b D_\nu \Psi^c \right] + \frac{i}{2} \bar{\chi} \hat{D} \chi + \left\{ \mu^\nu \right\} \left[ a_1 \bar{\Psi} \gamma^a \gamma^b \Psi^c + 2 a_1 \bar{\Psi}^a \Psi^b + i a_2 \bar{\Psi}^a \gamma^b \Psi^c + a_3 \bar{\Psi}^a \gamma^b \Psi^c \right]$$

as well as the most general form of corresponding gauge transformations:

$$\delta \Psi_\mu^a = D_\mu \xi^a + i \alpha_1 \gamma_\mu \xi^a + \alpha_2 e_\mu^a \xi + \eta_\mu^a, \quad (\gamma^a \eta)^a = 2i \alpha_1 \xi^a + \alpha_2 \gamma^a \xi$$

$$\delta \psi_\mu = D_\mu \xi + i \alpha_3 \gamma_\mu \xi + \alpha_4 \xi_\mu, \quad \delta \chi = \alpha_5 \xi$$
Note that the constraint on $\eta$ parameter was changed. First of all, the requirement that Lagrangian has to be invariant under such gauge transformations allows one to express all parameters in the Lagrangian and gauge transformations in terms of three main parameters $\alpha_1, \alpha_4$ and $\alpha_5$:

$$a_1 = 6\alpha_1, \  a_2 = 6\alpha_4, \  a_3 = -9\alpha_1, \  a_4 = \alpha_5, \  a_5 = -6\alpha_1, \  \alpha_2 = \frac{\alpha_4}{4}, \  \alpha_3 = 3\alpha_1$$

Already from these formulas we see that it is the limit $\alpha_4 \to 0$ that corresponds to the massless limit. Our usual convention on the mass normalization gives here $\alpha_4^2 = 5m^2/4$. In this, two other main parameters are determined by the relations:

$$16\alpha_1^2 = m^2 - 4\kappa, \  \alpha_5^2 = 24(m^2 - 3\kappa)$$ (56)

So the resulting Lagrangian and gauge transformations could be written as follows:

$$\mathcal{L} = \frac{i}{2} \{ \mu^\alpha_{\ ab} \} \left[ \bar{\Psi}_\mu \gamma^a \gamma^b c D_\mu \Psi_\alpha d - 6 \bar{\Psi}_\mu \gamma^a b D_\mu \Psi_\alpha c - \bar{\Psi}_\mu \gamma^a \gamma^b c D_\nu \psi_\alpha \right] + \frac{i}{2} \hat{\chi} \hat{D} \chi +$$

$$+ \{ \mu^\alpha_{\ ab} \} \left[ 6 \alpha_1 \bar{\Psi}_\mu \gamma^a \gamma^b \Psi_\nu c + 12 \alpha_1 \bar{\Psi}_\mu \gamma^a b \psi_\nu + i \frac{3\sqrt{5}}{2} m \bar{\Psi}_\mu \gamma^a b \psi_\nu - 9 \alpha_1 \bar{\psi}_\mu \gamma^a b \gamma^b \psi_\nu \right] +$$

$$+ i \alpha_5 (\bar{\psi} \gamma) \chi - 6 \alpha_1 \bar{\chi} \chi$$ (57)

$$\delta \Psi_\mu^a = D_\mu \xi^a + i \alpha_1 \gamma_\mu \xi^a + \frac{\sqrt{5}}{8} m \xi^a \bar{\xi} + \eta^a$$

$$\delta \psi_\mu = D_\mu \xi + 3i \alpha_1 \gamma_\mu \xi + \frac{\sqrt{5}}{2} m \xi_\mu, \  \delta \chi = \alpha_5 \xi$$ (58)

From these results we see that massless limit is once again possible in $AdS$ space ($\kappa < 0$) only, in this the whole system decomposes into massless spin $5/2$ particle and massive spin $3/2$ one. In $dS$ space we also obtain unitary forbidden region $m^2 < 4\kappa$. At the boundary of this region, i.e. $\alpha_1 = 0$, the theory greatly simplifies:

$$\mathcal{L} = \frac{i}{2} \{ \mu^\alpha_{\ ab} \} \left[ \bar{\Psi}_\mu \gamma^a \gamma^b c D_\mu \Psi_\alpha d - 6 \bar{\Psi}_\mu \gamma^a b D_\mu \Psi_\alpha c - \bar{\Psi}_\mu \gamma^a \gamma^b c D_\nu \psi_\alpha \right] + \frac{i}{2} \hat{\chi} \hat{D} \chi +$$

$$+ i \frac{3\sqrt{5}}{2} m \{ \mu^\alpha_{\ ab} \} \bar{\Psi}_\mu \gamma^a b \psi_\nu + i \sqrt{6} m (\bar{\psi} \gamma) \chi$$ (59)

$$\delta \Psi_\mu^a = D_\mu \xi^a + \frac{\sqrt{5}}{8} m \xi^a \bar{\xi} + \eta^a, \  \delta \psi_\mu = D_\mu \xi + \frac{\sqrt{5}}{2} m \xi_\mu, \  \delta \chi = \sqrt{6} m \xi$$ (60)

Inside the forbidden region there is a special value $m^2 = 3\kappa$ when $\alpha_5 = 0$. In this case spinor field decouples, while two other give gauge invariant description of partially massless particle (with helicities $\pm 5/2, \pm 3/2$) with the Lagrangian:

$$\mathcal{L} = \frac{i}{2} \{ \mu^\alpha_{\ ab} \} \left[ \bar{\Psi}_\mu \gamma^a \gamma^b c D_\mu \Psi_\alpha d - 6 \bar{\Psi}_\mu \gamma^a b D_\mu \Psi_\alpha c - \bar{\Psi}_\mu \gamma^a \gamma^b c D_\nu \psi_\alpha \right] +$$

$$+ \{ \mu^\alpha_{\ ab} \} \left[ 6 \alpha_1 \bar{\Psi}_\mu \gamma^a \gamma^b \Psi_\nu c + 12 \alpha_1 \bar{\Psi}_\mu \gamma^a b \psi_\nu + i \frac{3\sqrt{5}}{2} m \bar{\Psi}_\mu \gamma^a b \psi_\nu - 9 \alpha_1 \bar{\psi}_\mu \gamma^a b \gamma^b \psi_\nu \right]$$ (61)
which is invariant under the following gauge transformations:

$$\delta \Psi^a = D_\mu \xi^a + i \alpha_1 \gamma_\mu \xi^a + \frac{\sqrt{5}}{8} m e^a_\mu \xi + \eta^a$$ \hspace{1cm} \delta \bar{\psi}_\mu = D_\mu \xi + 3i \alpha_1 \gamma_\mu \xi + \frac{\sqrt{5}}{2} m \xi$$

(62)

However, in this case $\alpha_1^2 = -\kappa/16 < 0$.

5 Arbitrary half-integer spin

For the description of massless spin $s + \frac{1}{2}$ particle ($s = 1, 2, \ldots$) one needs [13] spin-tensor $\Psi^{a_1 \ldots a_{s-1}}$ symmetric on local indices and satisfying a constraint $\gamma^{a_1} \Psi^{a_1 \ldots a_{s-1}} = 0$. In this section we will use the same condensed notations as before, e.g. our main field will be denoted as $\Psi^{(s-1)}$. Free massless Lagrangian has to be invariant under the following gauge transformations:

$$\delta \Psi^{(s-1)} = \partial_\mu \xi^{(s-1)} + \eta_\mu^{(s-1)}, \hspace{0.5cm} \gamma^a_\mu \xi^{(s-2)} = 0, \hspace{0.5cm} \gamma^a_\mu \eta^{(s-1)} = \gamma^b \eta^{a_1 a_2 (s-2)} = 0, \hspace{0.5cm} \eta^{(s-1)} = 0$$

(63)

Such a Lagrangian could be written in the following form:

$$(-1)^s \mathcal{L}_0 = \frac{i}{2} \{ \frac{\mu \nu \alpha}{\lambda \beta \epsilon} \} [\bar{\Psi}^{(s-1)}(s-1) \gamma^a_\mu \gamma^b_\nu \gamma^c_\beta \partial_\epsilon \Psi^{(s-1)} - 6(s-1) \bar{\Psi}^{(s-2)}(s-2) \gamma^b \partial_\nu \Psi^{(s-2)}]$$

(64)

In order to describe massless particle in AdS space one has first of all change gauge transformations and constraint on $\eta$ parameter:

$$\delta \Psi^{(s-1)} = D_\mu \xi^{(s-1)} + i \alpha_0 \gamma_\mu \xi^{(s-1)} + \eta_\mu^{(s-1)}$$

(65)

and supplement the Lagrangian with additional mass-like terms of the form:

$$(-1)^s \mathcal{L}_1 = 3s \alpha_0 \{ \frac{\mu \nu}{ab} \} [\bar{\Psi}^{(s-1)}(s-1) \gamma^a_\mu \gamma^b_\nu \Psi^{(s-1)} + 2(s-1) \bar{\Psi}^{(s-2)} a_\mu b_\nu (s-2)]$$

(66)

Resulting Lagrangian will be gauge invariant provided $\alpha_0^2 = -\frac{s}{4}$.

Now let us turn to the massive case. To construct gauge invariant description of massive spin $s + \frac{1}{2}$ particle one needs [32] a set of fields with spins $s + \frac{1}{2}, s - \frac{1}{2}, \ldots \frac{1}{2}$. Thus we introduce fields $\Psi^{(k)}_\mu, 0 \leq k \leq s - 1$ and spinor $\chi$. Consider a Lagrangian which is a sum of covariantized kinetic terms for all fields plus the most general mass-like terms:

$$\mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} \frac{i}{2} \{ \frac{\mu \nu \alpha}{ab} \} [\bar{\Psi}^{(k)} \gamma^a_\mu \gamma^b_\nu \Psi^{(k)} - 6k \bar{\Psi}^{(k-1)}(s-1) \gamma^b \partial_\nu \Psi^{(k-1)}] + \frac{i}{2} \bar{\chi} D\chi +$$

$$\sum_{k=1}^{s-1} (-1)^{k+1} \{ \frac{\mu \nu}{ab} \} [a_k \bar{\Psi}^{(k)}(s-1) \gamma^a_\mu \gamma^b_\nu \Psi^{(k)} + 2k a_k \bar{\Psi}^{(k-1)} \gamma^b \partial_\nu \Psi^{(k-1)} + ib_k \bar{\Psi}^{(k-1)} \gamma^b \Psi^{(k-1)}] -$$

$$-a_0 \{ \frac{\mu \nu}{ab} \} \bar{\Psi}^{(k)} \gamma^a_\mu \gamma^b_\nu \Psi^{(k)} + \bar{b}_0 (\bar{\Psi} \gamma) \chi + c_0 \bar{\chi} \chi$$

(67)

as well as the following ansatz for the gauge transformations (note the modification of constraints on $\eta$ parameters):

$$\delta \Psi^{(k)}_\mu = D_\mu \xi^{(k)} + \alpha_{k+1} \xi^{(k)} + i \beta_k \gamma_\mu \xi^{(k)} + \rho_k [\epsilon_\mu (\xi^{(k-1)} - \frac{1}{k} g_{\mu \nu} (\xi^{(k-2)}) \nu)] + \eta^{(k)}_\mu$$
\[ \delta \chi = \alpha_0 \xi, \quad (\gamma \eta)^{(k)} = 2ik\beta_k \xi^{(k)} + \frac{1}{k}\rho_k \gamma^{(1)} \xi^{k-1} \]  

(68)

Variations of order \( m \) give:

\[ a_k = 3(k+1)\beta_k, \quad b_k = 6k\alpha_k, \quad \rho_k = \frac{k\alpha_k}{(k+1)^2}, \quad b_0 = \alpha_0 \]

while variations of order \( m^2 \) (including contributions form the commutators of covariant derivatives) cancel provided:

\[ (k+1)\beta_k = (k+3)\beta_{k+1} \]

\[ 4(k+1)(2k+3)\beta_k^2 - 2k\alpha_k^2 + \frac{2(k+1)^2}{k+2}\alpha_{k+1}^2 + (2k+3)(k+1)\kappa = 0 \]

\[ \alpha_0^2 = 3\alpha_1^2 + 36\beta_0^2 + 9\kappa, \quad c_0 = -2\beta_0 \]

To solve these relations we proceed as follows. First of all, noting that \( \alpha_s = 0 \) while our definition of mass gives this time \( \alpha_{s-1}^2 = \frac{(2s+1)m^2}{2s(s-1)} \), from the second relation with \( k = s-1 \) we obtain:

\[ \beta_{s-1}^2 = \frac{m^2 - s^2 \kappa}{4s^2} \]

Then solving the first relation recurrently we get all other \( \beta \)-s:

\[ \beta_k^2 = \frac{(s+1)^2(m^2 - s^2 \kappa)}{4(k+1)^2(k+2)^2} \]

Now the second relation becomes recurrent relation on \( \alpha \)-s and can easily be solved. Result:

\[ \alpha_k^2 = \frac{(s-k)(s+k+2)}{2k(k+1)}[m^2 - (s^2 - (k+1)^2)\kappa] \]

\[ \alpha_0^2 = 3s(s+2)[m^2 - (s^2 - 1)\kappa] \]

The final Lagrangian and gauge transformations in terms of \( \alpha \) and \( \beta \) look as follows:

\[
\mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} \frac{i}{2} \left\{ \mu_{\alpha} \right\} \left[ \bar{\Psi}_{\mu}^{(k)} \gamma^\alpha \gamma^b \gamma^c D_{\nu} \Psi_{\alpha}^{(k)} - 6k\bar{\Psi}_{\mu} a^{(k-1)\gamma^c D_{\nu} \Psi_{\alpha}^{c(k-1)}} \right] + \frac{i}{2} \hat{\bar{\chi}} \hat{D} \chi + \\
\sum_{k=1}^{s-1} (-1)^{k+1} \left\{ \mu_{\mu} \right\} \left[ 3(k+1)\beta_k \bar{\Psi}_{\mu}^{(k)} \gamma^\alpha \gamma^b \Psi_{\nu}^{(k)} + 6k(k+1)\beta_k \bar{\Psi}_{\mu} a^{(k-1)\gamma^c D_{\nu} \Psi_{\alpha}^{c(k-1)}} + \\
+ 6ik\alpha_k \bar{\Psi}_{\mu} a^{(k-1)\gamma^c D_{\nu} \Psi_{\alpha}^{c(k-1)}} - 3\beta_0 \left\{ \mu_{\mu} \right\} \bar{\Psi}_{\mu} \gamma^\alpha \gamma^b \Psi_{\nu} + i\alpha_0 \left( \bar{\Psi}_{\gamma} \gamma \right) - 2\beta_0 \bar{\chi} \chi \right] \]

(69)

\[ \delta \Psi_{\mu}^{(k)} = D_{\mu} \xi^{(k)} + \alpha_{k+1} \xi_{\mu}^{(k)} + i\beta_k \gamma_{\mu} \xi^{(k)} + \frac{k\alpha_k}{(k+1)^2} \left( \xi_{\mu}^{(k-1)} - \frac{1}{k} \gamma \right)^{(1)} \xi^{k-1} \]

\[ \delta \chi = \alpha_0 \xi, \quad (\gamma \eta)^{(k)} = 2ik\beta_k \xi^{(k)} + \frac{2\alpha_k}{k(k+1)^2} \gamma^{(1)} \xi^{k-1} \]

(70)

Now we are ready to analyze main properties of the theory obtained. It is hardly come as a surprise that massless limit turns out to be possible in AdS space only, in this massive spin \( s + \frac{1}{2} \) particle decompose into massless spin \( s + \frac{1}{2} \) and massive \( s - \frac{1}{2} \) ones. In dS space
we once again find unitary forbidden region $m^2 < s^2 \kappa$. At the boundary of this region all parameters $\beta$ become zero and theory greatly simplifies:

$$
\mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} \frac{i}{2} \left\{ \mu^{\alpha} \left[ \bar{\Psi}_\mu \left( k \right) \gamma^\alpha \gamma^\beta D_\nu \Psi_\alpha \left( k \right) - 6k \bar{\Psi}_\mu \left( k-1 \right) \gamma^h D_\nu \Psi_\alpha \left( k-1 \right) \right] \right\} + \frac{i}{2} \chi \bar{D} \chi + i \sum_{k=1}^{s-1} (-1)^{k+1} 6k \alpha_k \left\{ \mu^{ab} \bar{\Psi}_\mu \left( k-1 \right) \gamma^b \Psi_\nu \left( k-1 \right) \right\} + i \alpha_0 (\bar{\Psi} \gamma) \chi
$$

(71)

$$
\delta \Psi_\mu \left( k \right) = D_\mu \xi \left( k \right) + \alpha_{k+1} \xi \left( k \right) + \frac{k \alpha_k}{(k+1)^2} \left[ e_\mu (1 \xi_{k-1}) - \frac{1}{k} g(12 \xi_{k-2})_{\mu} \right] + \eta_\mu \left( k \right)
$$

$$
\delta \chi = \alpha_0 \xi, \quad (\gamma \eta) \left( k \right) = \frac{2 \alpha_k}{k(k+1)^2} \gamma (1 \xi_{k-1})
$$

(72)

Note the essential difference between integer and half-integer cases [47]: for the integer spin at the boundary of unitary forbidden region spin 0 field decouples and we obtain first partially massless theory, while for the half-integer spin all the partially massless theories "live" inside the forbidden region. Indeed, for any value of mass where one of the parameters $\alpha_k$ becomes zero, all fields with spins $l + \frac{1}{2}$ for $0 \leq l \leq k$ decouple, while remaining fields describe partially massless theory.

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