Pattern Formation in a Substrate–Contactor System with Two Interacting Incompressible Elastic Films

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Abstract

The surface stability of two interacting (for example, by van der Waals forces) incompressible thin films, one bonded to a substrate and the other to a contactor, is studied extending the work of Shenoy and Sharma, Physical Review Letters \textbf{18}, 119–122 (2001). The analysis indicates that the wavelength of the instability depends strongly on the shear moduli and thicknesses of the films but not on the nature and magnitude of the interaction. When the films have equal shear moduli, the wavelength of the instability has an intermediate value between the wavelengths of the instabilities had each of the films been interacting with rigid contactors. On the other hand, if the films have different shear moduli but equal thicknesses, then the wavelength of the instability is identical to that had the films been interacting with rigid contactors. In the more general case when the two films have different shear moduli and thicknesses, the nature of the critical wavelength is more complex. When ratio of the shear moduli of the contacting film to that of the film bonded to the substrate exceeds 5.32, the wavelength of the of the instability jumps from the value close to that determined by the thickness of the film bonded to the substrate to that of the contacting film, as the thickness of the contacting film is increased.

1 Introduction

The mechanics of adhesion and contact between two elastic bodies has attracted attention of researchers over the years. Following the classical work of Hertz, an important step in this area was taken by Johnson, Kendall and Roberts (1971) who identified the importance of the interactions between the contacting surfaces. These interactions can be due to van der Waals forces, electrostatic forces etc. between the contacting surfaces. Indeed, measurements of van der Waals forces between surfaces (Tabor and Winterton 1969) were available when Johnson et al. (1971) formulated their famous JKR theory of contact.

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More recently, two groups (Ghatak, Chaudhury, Shenoy and Sharma 2000, Mönch and Herminghaus 2001) have reported experiments aimed at understanding contact and adhesion mechanics between two elastic bodies with planar topology. The experiments reported by Ghatak et al. (2000) used a configuration where the glass plate was placed in contact with the elastomeric film between two spacer bars creating a small gap between the film and the glass plate. Mönch and Herminghaus (2001) performed experiments that involved the contact of a glass plate with an elastomeric thin film bonded rigidly to a glass substrate. In both these experimental works it was observed that the film lost planarity when the contacting glass plate reached contact proximity (10-50nm), resulting in a pattern with a well defined wavelength. The two key features observed in both sets of experiments are (a) the wavelength of the pattern depends linearly on the thickness of the film (b) the wavelength is not affected by the magnitude and nature of the interactions.

The theoretical interpretation of this results were first reported by Shenoy and Sharma (2001b) who argued that the instability occurs due to a competition between the interaction energy of the film with the contactor and the elastic energy in the film due to inhomogeneous deformation. They showed that instability sets in when the ratio of the “stiffness of interaction” and the elastic stiffness (defined as shear modulus divided by the thickness) exceeds a critical value. In addition, they pointed out that the wavelength of the instability is determined solely by the elastic energy of the film and explained the linear dependence of the wavelength on the thickness of the film. A detailed account of this work can be found in Shenoy and Sharma (2001a). The origins and nature of this instability are different from those known to exist in solid films (Asaro and Tiller 1972, Grinfeld 1993, Srolovitz 1989, Ramirez 1989) and fluid films (Herminghaus, Jacobs, Mecke, Bischof, Fery, Ibn-Elhaj and Schlagowski 1998, Sharma and Khanna 1998, Reiter, Khanna and Sharma 2000, Schäffer, Thurn-Albrecht, Russel and Steiner 2000).

The purpose of this paper is to extend the work of Shenoy and Sharma (2001b) to the case when the contactor also has a film bonded to it as shown in fig. 1. The contacting film may have different properties in that the shear modulus and thickness may be different from the film bonded to the substrate. It is shown that one of key features i. e., the independence of the wavelength of the instability on the nature of interaction is unchanged. The critical wavelength, although does not depend on the nature and strength of the interaction, does depend strongly on the thicknesses and moduli of the film. It is shown that the relative stiffnesses of the films play an important role in the determination of the wavelength of the instability when the films have widely differing properties; the complete dependence of the wavelength on the thicknesses and moduli of the film is obtained. This paper is written with a purpose of motivating further experiments in the kinds of systems indicated in fig. 1 and is organised as follows. The next section contains a stability analysis of two interacting films. Results of the analysis are presented and discussed in section 3. The findings of this paper are summarised in section 4. The appendix at the end of the paper contains a short calculation of the normal traction along the surface of
an incompressible film subjected to a sinusoidal deformation.

2 Stability of Two Interacting Films

2.1 Model Description

A schematic depiction of model considered for the study of stability of interacting thin films is shown fig. 1. The system consists of a substrate-contactor configuration where both the substrate and the contactor have thin films (possibly of different materials) bonded to them. The substrate film (called film $a$) has a thickness $h_a$, while the contactor film (called film $b$) has a thickness $h_b$. A coordinate system described by coordinates $(x_1, x_2)$ is used to describe position vectors. The positive ‘2’ direction is the outward normal of film $a$, while the negative ‘2’ direction point in the normal direction to the surface of the film $b$. We consider only plane strain deformation of the system.

The equilibrium configuration of this system is determined by the potential energy

$$
\Pi = \int_V W(\epsilon) dV - \int_S U((u^a - u^b) \cdot n) dS
$$

(1)

where $W$ is the strain energy density, $\epsilon$ is the strain tensor, $u$ is the displacement vector (the superscript $a$ denotes value in film $a$ etc.) with $V$ being an appropriate measure of the volume (includes volumes of both the films), and $S$ is the interfacial area of the two films over which they interact. The function $U$ represents the interaction potential between the two films; it is this term that gives rise to interesting physics in this system. We make two key physical assumptions in writing the total potential energy: (i) the contribution from the surface energies of the films is negligible (ii) the films are considered to be made of incompressible elastic materials. Both of these assumptions are valid in physical systems where such instabilities are triggered, as shown by (Shenoy and Sharma 2001b).

The physical origins of such an interaction can be any of the several causes – van der Waals forces, electrostatic forces between films etc; the potential $U$ is a generic interaction potential. If the potential is due to the attractive van der Waals interaction, then $U$ is described by

$$
U((u^a - u^b) \cdot n) = \frac{1}{12\pi} \left(\frac{A}{(d_o - (u^a - u^b) \cdot n)^2}\right)
$$

(2)

where $A$ is the Hamaker constant (of the order of $10^{-19} J$), and $d_o$ is the distance between the surfaces of the two films, i.e., the gap thickness. The strength of the interaction potential is determined by the gap thickness $d_o$. The contactor is imagined to be brought in the proximity of the substrate by reducing the distance $d_o$ – the interesting quantity being the distance $d_c$ at which the attractive interactions are strong enough to trigger instability in the system.

Linear stability analysis is performed using linear kinematics and a linearized interaction potential. To this end, the interaction potential is expanded in a power series about the reference
state of the undeformed films and terms of up to quadratic order in \((u^a - u^b) \cdot n\) are retained

\[
U((u^a - u^b) \cdot n) \approx U_0 + F (u^a - u^b) \cdot n + \frac{1}{2} Y ((u^a - u^b) \cdot n)^2
\]

(3)

where

\[
U_0 = U(0), \quad F = U'(0), \quad Y = U''(0).
\]

(4)

The quantity \(Y\), called the *interaction stiffness*, is of importance and governs the stability of the system. The aim of the analysis is to find the condition(s) on \(Y\) under which instability sets in.

The above approximations give an expression for the approximate potential energy as

\[
\Pi_a = \int_V W(\epsilon) dV - \int_S \left( U_0 + F (u^a - u^b) \cdot n + \frac{1}{2} Y ((u^a - u^b) \cdot n)^2 \right) dS.
\]

(5)

The strain energy density \(W\) is

\[
W(\epsilon) = \begin{cases} 
\frac{\mu_a}{2} \epsilon : \epsilon & \text{in film } a \\
\frac{\mu_b}{2} \epsilon : \epsilon & \text{in film } b
\end{cases}
\]

(6)

where \(\mu_a\) and \(\mu_b\) are respectively the shear moduli of the films \(a\) and \(b\). The equilibrium displacement fields in the films minimise the potential energy (5) while satisfying the rigid boundary
conditions at the film-substrate interface in film $a$, and the film contactor interface in film $b$. In addition, the stresses derived from these displacement fields satisfy the condition of vanishing shear stress $\sigma_{12}$ at the surface in both the films. The normal stresses satisfy the condition

$$\sigma_{22}^a = F + Y(u_2^a - u_2^b)$$

$$\sigma_{22}^b = F + Y(u_2^a - u_2^b)$$

at their respective surfaces.

*Homogeneous Solution:* Since both the films are incompressible, the homogeneous solution has displacements vanishing everywhere in both the films, and stress state in both the films is one of constant pressure (equal to $F$). The point in question is the stability of this homogeneous solution, i.e., for what value(s) of $Y$ does instability occur?

### 2.2 Stability Analysis

To study the stability of the system above, the homogeneous solution is perturbed by bifurcation fields (denoted by $u_i$ here and henceforth) such that the surfaces of the film $a$ and $b$, respectively, have displacements

$$u_2^a(x_1) = \alpha \cos(kx_1)$$

$$u_2^b(x_1) = \beta \cos(kx_1).$$

The *additional stresses* produced by these fields satisfy

$$\sigma_{12}^a = 0, \quad \sigma_{12}^b = 0,$$

and

$$\sigma_{22}^a = Y(u_2^a - u_2^b)$$

$$\sigma_{22}^b = Y(u_2^a - u_2^b)$$

along the interacting surfaces of the films. All symbols have obvious meanings.

It is shown in the appendix that the stress $\sigma_{22}^a$ at the surface of a film $a$ whose displacement is $u_2^a(x_1) = \alpha \cos(kx_1)$ with $\sigma_{12}^a = 0$ is given by

$$\sigma_{22}^a(x_1) = 2\mu^a S(h_a k) k\alpha \cos(kx_1),$$

and, the stress $\sigma_{22}^b(x_1)$ along the surface of the film $b$ is

$$\sigma_{22}^b(x_1) = -2\mu^b S(h_b k) k\beta \cos(kx_1),$$
where \( S \) is a nondimensional function defined as
\[
S(\xi) = \frac{1 + \cosh(2\xi) + 2\xi^2}{\sinh(2\xi) - 2\xi}.
\] (16)

For future discussion we note that \( S(\xi) \to 1/\xi^3 \) as \( \xi \to 0 \), and \( S(\xi) \to \coth(\xi) \) when \( \xi \to \infty \).

Substitution of (14) and (15) in (12) and (13) leads to the following homogeneous set of equations for \( \alpha \) and \( \beta \):
\[
\left( \begin{array}{c}
2\mu_a k S(h_a k) - Y \\
- 2\mu_b k S(h_b k)
\end{array} \right) \left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) = \left( \begin{array}{c}
0 \\
0
\end{array} \right).
\] (17)

The condition for the existence of nontrivial bifurcation fields, i.e., the condition for the existence of nontrivial solutions for \( \alpha \) and \( \beta \) is obtained by setting the determinant of the coefficient matrix to zero resulting in a relation between the interaction stiffness \( Y \) and the wavenumber \( k \) of the bifurcation field
\[
Y = \frac{2k \mu_a \mu_b S(h_a k) S(h_b k)}{\mu_a S(h_a k) + \mu_b S(h_b k)}.
\] (18)

If, for a given value of \( Y \), there is at least one real value of \( k \) that solves (18), then the homogeneous solution is unstable and the films deform inhomogeneously. The lowest value of \( Y \) for which bifurcations are possible is called the critical interaction stiffness denoted by \( Y_c \); the wavenumber(s) of the mode(s) that satisfies (18) for \( Y = Y_c \) is(are) called the critical mode(s) and the(thes) wavenumber(s) is(are) denoted by \( k_c \).

At this point, there are four parameters \( \mu_a, \mu_b, h_a, h_b \) that enter into the determination of the stability of the system. The discussion to follow is much simplified on introduction of effective parameters; to this end, we define
\[
\mu = \frac{\mu_a \mu_b}{\mu_a + \mu_b}
\] (19)
and
\[
h = h_a + h_b.
\] (20)

We also introduce nondimensional parameters \( M \) and \( H \) defined as
\[
\frac{\mu}{\mu_a} = M, \quad \frac{\mu}{\mu_b} = (1 - M)
\] (21)
and
\[
h_a = H h, \quad h_b = (1 - H) h.
\] (22)

When \( \mu_a \ll \mu_b \), \( M \to 1 \) and \( M \to 0 \) if \( \mu_b \ll \mu_a \); similarly when \( h_a \gg h_b \), \( H \to 1 \), while \( H \to 0 \) implies \( h_b \gg h_a \). Another important quantity of interest is the effective elastic stiffness \( K_{eff} \) of the two-film system defined as
\[
K_{eff} = \frac{\mu_a \mu_b}{h_a h_b} = \frac{\mu_a \mu_b}{\mu_a h_b + \mu_b h_a} = \frac{\mu}{h} \frac{1}{((1 - H)(1 - M) + HM)}.
\] (23)
Based on the above definitions, the expression for the interaction stiffness in (18) can be recast in nondimensional form as

\[
\frac{Y}{K_{eff}} = \frac{2((1 - H)(1 - M) + HM)q S(Hq) S((1 - H)q)}{(1 - M) S(Hq) + M S((1 - H)q)},
\]

where \( q = hk \).

The results for the case when \( \mu_b \gg \mu_a \) obtained by Shenoy and Sharma (2001b) can be immediately recovered. When \( M \to 1 \), \( K_{eff} \to \mu_a h_a \) and

\[
\frac{h_a Y}{\mu_a} = 2 Hq S(Hq) = 2 h_a k S(h_a k)
\]

The critical interaction stiffness can be obtained as \( Y_c = 6.22 \mu_a / h_a \) and \( h_a k_c = 2.12 \), which is precisely the result of Shenoy and Sharma (2001b). Another route to the same result is to let \( h_a \gg h_b \), i.e., as \( H \to 1 \). In this case, \( K_{eff} \to \frac{\mu_a}{h_a} \) again, and

\[
\frac{h_a Y}{\mu_a} = 2 Hq S(Hq) = 2 h_a k S(h_a k)
\]

where the property of the function \( S \) near 0, noted above, is used. It can be easily shown that similar results are obtained when either \( \mu_b \ll \mu_a \) \((M \to 0)\) or \( h_b \gg h_a \) \((H \to 0)\).

In this section, we have derived the condition for the onset of instability in a system with two interacting incompressible elastic films. In addition, it is shown that the results of Shenoy and Sharma (2001b) who treated a film interacting with a rigid contactor, are a special case of the present formulation. The more general case is taken up for discussion in the next section.

3 Results and Discussion

The nondimensional quantities \( M \) and \( H \) introduced in the last section can take any value from 0 to 1. However, it is evident that the results for a given value of \( M \) and \( H \) are physically identical to that of \((1 - M)\) and \((1 - H)\) (as is evident from (24)). Due to this reason, only the regime \( \frac{1}{2} \leq M \leq 1 \) and \( 0 \leq H \leq 1 \) is considered. Results are discussed in three categories: (i) \( \mu_a = \mu_b, h_a \neq h_b \) (ii) \( \mu_a \neq \mu_b, h_a = h_b \) (iii) \( \mu_a \neq \mu_b, h_a \neq h_b \). The results of all these cases are plotted in two sets of graphs shown in fig. 2 and fig. 3.

An important point about the results is that, just as in the case of a film interacting with a rigid contactor, the wavelength of the instability does not depend on the magnitude and nature of the interaction forces, but depends only on the thicknesses of the films and their moduli, i.e., on the parameters \( H \) and \( M \).

3.1 Films with Equal Shear Moduli

When the shear moduli of the two films are equal \((\mu_a = \mu_b)\), \( M = \frac{1}{2} \), \( \mu = \mu_a / 2 \) and \( K_{eff} = H \mu_a / h_a \). The results for \( Y_c \) and \( k_c \) correspond to \( M = 0.5 \) in fig. 2 and fig. 3. The magnitude
Figure 2: The critical interaction stiffness as a function of $H$ for various values of $M$. 
of non dimensional critical interaction stiffness \( Y_c/K_{eff} \) increases with decreasing \( H \) attains a peak and falls as \( H \) approaches \( \frac{1}{2} \). If \( h_a \) is kept fixed and \( h_b \) is changed to change \( H \), then increasing \( Y_c/K_{eff} \) does not imply an increase in \( Y_c \), since \( K_{eff} \) falls linearly with \( H \). Thus for a fixed \( h_a \), the effective stiffness of the system reduces with decreasing \( H \). For example, when \( H = 3/4 \), i.e., \( h_b/h_a = 1/3 \), the value of \( K_e = 3\mu_a/4h_a \) and the value of \( Y_c/K_e = 6.92 \) or \( Y_c = 5.19\mu_a/h_a \) which is less than \( Y_c = 6.22\mu_a/h_a \) when the film \( a \) interacts with a rigid contactor. Looking at the same example from the point of view of film \( b \), we note that \( K_e = (1-H)\mu_b/h_b \) and \( Y_c = 1.73\mu_b/h_b \) which is again less than the value \( 6.22\mu_b/h_b \). In fact, when films have equal thicknesses and equal shear moduli, the value of \( Y_c = 3.11\mu_a/h_a \). The conclusion of these observations is that the magnitude of the critical interaction stiffness is less than that when either of the film is interacting with a rigid contactor. Physically, this can be understood by the fact that introduction of a second film on the contactor makes the system more compliant, i.e., the effective elastic stiffness of the system comes down. Thus, the elastic energy penalty required to cause an inhomogeneous deformation in the system also comes down. As a consequence, the absolute value of the interaction stiffness required to trigger the instability is reduced. Moreover, the actual value of \( Y_c \) is governed by the more compliant film.

Turning now to a discussion of the wavenumber of the critical mode, we note that as \( H \) decreases from 1, \( h_ak_c \) increases and then falls subsequently. For values of \( H \) close to 1, the instability triggers a shorter wavelength mode in film \( a \) as compared to when the film \( a \) interacts with a rigid contactor. For example, when \( H = 3/4 \), the value of \( h_ak_c = 2.45 \); in terms the film \( b \) the value of \( k_c \) is \( h_bk_c = 0.82 \), i.e., the wavelength of the critical mode is much larger than that if the film \( b \) had been interacting with a rigid contactor. The key point is that the wavelength of the instability in this regime is intermediate to critical wavelengths determined by the thicknesses of the participating films. As the thicknesses are made equal, i.e., when \( H = \frac{1}{2} \), the critical wavelength becomes \( h_ak_c = 2.12 \) which is exactly the wavelength had either of the films been interacting with a rigid contactor. When \( M = 1/2 \), the case of \( H < 1/2 \) is physically identical to the case \( H > 1/2 \). These results can be interpreted physically as follows. The dependence elastic energy in the film on the wavenumber \( k \) is determined solely be the thickness of the film, more precisely by the function \( S \) (defined in (16)) which depends on the thickness of the film, the shear modulus is a mere multiplicative parameter. The elastic energy (per unit length of the film) is a minimum when the wavenumber is equal to \( 2.12/(\text{thickness}) \). However, when there are two films with very different thicknesses, the total elastic energy in the system (sum of elastic energies in the two films) can be minimised by choosing an intermediate wavelength.

### 3.2 Films with Equal Thicknesses

When the films are made of equal thickness \( h_a = h_b \), the parameter \( H = \frac{1}{2} \). For this case, the effective stiffness \( K_e = \mu/h_a \), and the nondimensional critical interaction stiffness is independent of the value of \( M \), with \( Y_c = 6.22\mu/h_a \). Since \( \mu < \mu_a \), and \( \mu < \mu_b \), the actual magnitude of
Figure 3: The wavelength of the critical mode as a function of $H$ for various values of $M$. 
the critical interaction stiffness is less than the critical interaction stiffness of the films if they had been interacting with rigid contactors. This can, again, be understood by the argument that the effective stiffness of the system comes down on introduction of the second film on the contactor. The critical wavenumber, however, does not depend on $M$ as is evident from fig. 3 and is equal to the wavelength had the films been interacting with a rigid contactor. Again, this can be understood from the argument stated above that the elastic energy of a mode is determined by the thickness of the film via the function $S$ and the moduli are multiplicative constants in the energy expression. This idea is evident from (24) on substitution of $H = \frac{1}{2}$.

3.3 Films with Unequal Thicknesses and Moduli

The physics of the instability in the general case can be understood based on the results of the previous two cases. When $M > \frac{1}{2}$, the film $a$ is softer than the film $b$, i.e., $\mu_a < \mu_b$. When $M$ is close to $1/2$, the qualitative features of the instability are unchanged. The nondimensional interaction stiffness $Y_c/K_{eff}$ increases with decreasing $H$ from 1, and falls as $H$ approaches $\frac{1}{2}$. On further decrease of $H$, $Y_c/K_{eff}$ increases and decreases again to 6.22 as $H$ approaches 0. The behaviour of the critical wavelength is qualitatively identical to that of the case when $M = \frac{1}{2}$.

As $M$ is made close to 1 ($M < 1$), a qualitative change appears in the solution. In fact, when $M = 0.8418$ (corresponds to $\mu_b/\mu_a = 5.32$), the critical wavenumber changes suddenly as a function of $H$ at about $H = 0.1756$ (corresponds to $h_b/h_a = 4.69$). In fact, for larger values of $M$, as $H$ is reduced, the critical wavenumber remains a constant (and close to the wavenumber $h_a k_c = 2.12$) but jumps suddenly to a much lower value – the value of $H$ at which the jump occurs is called $H_J$. This is evident when $M = 0.9$ (shown in fig. 3) where the wavenumber of the critical mode $h_a k_c \approx 1.79$ when $H < 0.25$; however, at $H = 0.095 = H_J$ (corresponding to $h_b/h_a = 9.52$), the critical wavelength becomes $h_a k_c = 0.22 \rightarrow h_b k_c = 2.1$, i.e., the wavenumber of the instability changes to the critical wavenumber of the film $b$ had it been interacting with a rigid contactor. The value of $H_J$, i.e., the nondimensional value of $H$ at which the critical wavenumber jumps from a higher value $k_h$ to a lower value $k_l$ depends on the ratio of the shear moduli; this dependence is plotted in fig. 5. It is evident that $H_J$ decreases with $M$, as is expected from the physical arguments presented below. The two wavenumbers at $H_J$ are plotted in fig. 6. Again, as $M$ approaches 1, $h_a k_h$ approaches 2.12, the value determined by the thickness of the film $a$, and $h_a k_l$ approaches 0 (it can, in fact, be shown that $h_b k_l$ approaches 2.12).

From a physical perspective, the total elastic energy is the sum of elastic energies in the films which are products of their respective shear moduli with the $S$ function defined in (16). When the elastic moduli are very different, the total energy of the system can have multiple minima, which implies that $Y$ as a function of $q$ defined in (24) has multiple minima. Indeed, a plot of $Y$ as a function of $h_a k$ shown in fig. 4 clearly shows the presence of multiple minima; additionally, the value of $h_a k$ where the minimum is attained changes to a smaller value of $h_a k$ as $H$ is reduced. The key physical idea is that instability is governed by the film that has a smaller
Figure 4: The interaction stiffness $Y/K_{eff}$ as a function $h_ak$ showing multiple minima for different values of $H$. Note how the value of $h_ak$ at which $Y/K_{eff}$ attains a minimum changes as $H$ is reduced.
Figure 5: The nondimensional parameter $H_J$ (that characterises the difference in the thicknesses of the films) at which the critical wavenumber jumps, as a function of $M$. 
stiffness (the compliant film). Although $\mu_b$ is much larger than $\mu_a$, $h_b$ can be made larger than $h_a$ to the extent that the stiffness of the film $b$ is smaller than that of $a$. Thus when $H$ is lowered, the critical wavenumber of the instability of the system jumps from a wavenumber close to the critical wavenumber determined by film $a$ (as if interacting with a rigid contactor) to that of the film $b$ (as if interacting with a rigid substrate).

4 Summary and Conclusion

This paper extends the work of Shenoy and Sharma (2001b), who studied the interaction of a soft thin film with a rigid contactor, to the case where the contactor also has a soft film bonded to it. This new analysis uncovers some interesting new results that are summarised below.
1. Instability sets in when the interaction stiffness exceeds a critical value which is determined by the shear moduli and thicknesses of the interacting films. The magnitude of the critical interaction stiffness is smaller than that of the critical interaction stiffnesses of the individual films had they been interacting with rigid contactors. Thus it is “easier” for the instability to occur in the two-film system.

2. In all the cases, the wavelength of the instability does not depend of the nature and magnitude of the interaction. The critical wavelength is determined only by the thicknesses and shear moduli of the films.

3. When the films have equal shear moduli, the critical wavenumber of the instability depends on the relative thicknesses of the films. When the thicknesses of the films are very different, the wavelength of the instability is intermediate between the wavelength of the instability in each of the films had they been interacting with rigid contactors. However, when the thicknesses of the film are equal, the wavelength of the instability is exactly same as that when the films interact with rigid contactors.

4. When the films have equal thicknesses, but different shear moduli, the magnitude of the critical interaction stiffness is less than that of each of the film had they been interacting with rigid contactors. The wavelength of the critical mode, however, is equal to that when the films interact with rigid contactors and is independent of the shear moduli of the films.

5. When the films are of both unequal thicknesses and moduli, the behaviour is dominated by the less stiff films (stiffness being the ratio of the shear modulus to the thickness of the film). As long as the modulus of the contacting film is less than about five times that of the substrate film, the behaviour of the solution is qualitatively same as that discussed in the above cases. However, when the modulus of the contacting film is larger than 5.32 times the thickness of the film on the substrate, the wavelength of the instability depends strongly on the thicknesses of the films. As the thickness of the contacting film is increased, the wavelength of the instability jumps from a value determined by the thickness of the substrate film to that determined by the contacting film.

It is hoped that the paper will stimulate further experiments along the lines of Ghatak et al. (2000) and Mönch and Herminghaus (2001) to verify the results presented here. The analysis presented here neglects the effect of surface energies and the compressibility of the films. The inclusion of these effects along with the viscosity of the films will be presented elsewhere (Sarkar, Shenoy and Sharma 2001).

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A Stresses Along the Surface of an Incompressible Film Bonded to a Substrate with Sinusoidal Surface Deformation

The aim of this appendix is to outline the determination of the normal traction along the surface of an incompressible film bonded to a rigid substrate, the surface of which is deformed sinusoidally. The main result of this appendix is the formula used in (14) and (15).

Consider an incompressible elastic film described by coordinates \((x_1, x_2)\) of thickness \(h\) bonded to a rigid substrate such that the free surface of the film has coordinate \(x_2 = 0\) and the interface between the film and the substrate has coordinate \(x_2 = -h\). The film has a shear modulus \(\mu\). The boundary value problem has the following boundary conditions:

\[
\begin{align*}
  u_1(x_1, -h) &= u_2(x_1, -h) = 0, \\
  u_2(x_1, 0) &= \alpha \cos(kx_1), \quad \sigma_{12}(x_1, 0) = 0.
\end{align*}
\]

The equilibrium equation in terms of the displacements is

\[
\mu u_{i, jj} + p_{,i} = 0,
\]

where \(p\) is the pressure field. The incompressibility condition is expressed as

\[
u_{i,i} = 0.
\]

A general solution of the set of differential equations that anticipates the boundary conditions (27) and (28) is

\[
\begin{align*}
  u_1(x_1, x_2) &= -\frac{\alpha}{k} \left( (B + k(A + Bx_2))e^{kx_2} + (D - k(C + Dx_2))e^{-kx_2} \right) \sin(kx_1) \\
  u_2(x_1, x_2) &= \alpha \left( (A + Bx_2)e^{kx_2} + (C + Dx_2)e^{-kx_2} \right) \cos(kx_1) \\
  p(x_1, x_2) &= -2\mu \alpha \left( Be^{kx_2} + De^{-kx_2} \right) \cos(kx_1)
\end{align*}
\]

The constants \(A, B, C\) and \(D\) can be determined from the boundary conditions (27) and (28). The solution is

\[
\begin{align*}
  A &= \frac{1 + e^{2kh} - 2kh(1 - kh)}{2 \sinh(2kh) - 4kh}, \quad C = 1 - A, \\
  B &= -\frac{k(1 + e^{2kh} - 2kh)}{2 \sinh(2kh) - 4kh}, \quad D = k + B.
\end{align*}
\]
From these relations, the expression for the normal component of traction along the surface of the film can be derived as

\[ \sigma_{22}(x_1, 0) = 2\mu u_{2,2} + p = 2\mu (2A - 1)ka \cos(kx_1). \] (36)

On substitution of the solution for \( A \) from (34), the expression simplifies to

\[ \sigma_{22}(x_1, 0) = 2\mu S(hk) ka \cos(kx_1) \] (37)

where the function \( S \) is defined in (16).

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