Strong-Coupling $\phi^4$-Theory in $4-\epsilon$ Dimensions, and Critical Exponents

Hagen Kleinert*
Institut für Theoretische Physik,
Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

With the help of variational perturbation theory we continue the renormalization constants $\phi^4$-theories in $4-\epsilon$ dimensions to strong bare couplings $g_0$ and find their power behavior in $g_0$, thereby determining all critical exponents without renormalization group techniques.

1. In a recent paper [1] we have shown that there exists a simple way of extracting the strong-coupling properties of a $\phi^4$-theory from perturbation expansions. In particular, we were able to find power behavior of the renormalization constants in the limit of large couplings, and from this all critical exponents of the system. By using the known expansion coefficients of the renormalization constants in three dimensions up to six loops we derived extremely accurate values for the critical exponents. The method is a systematic extension of the Feynman-Kleinert variational approximation to path integrals [2] to arbitrary orders [3]. For an anharmonic oscillator, the derived variational perturbation expansions converge uniformly and exponentially fast, like $e^{-\text{const} \times N^{1/3}}$ in the order $N$ of the approximation [4].

2. Variational perturbation expansions have the important property of possessing a good strong-coupling limit as was first shown for the harmonic oscillator [5,6]. The speed of convergence turned out to be governed by the convergence radius of the strong-coupling expansion [7]. The good strong-coupling properties have enabled us to set up a simple algorithm for deriving uniformly convergent approximations to functions of which one knows a few Taylor coefficients and an important scaling property: they approach a constant value with a given inverse power of the variable. The renormalized coupling constant $\tilde{g}$ of a $\phi^4$-theory has precisely this property as a function of the bare coupling constant $g_0$. In $D = 4-\epsilon$ dimensions, it approaches a constant value $g^*$ for increasing bare coupling constant $g_0$ like

$$g(g_0) = g^* - \frac{\text{const}}{g_0^{\omega/\epsilon}} + \ldots,$$

where $g^*$ is the infrared-stable fixed point and $\omega$ is called the critical exponent of the approach to scaling.

The purpose of this paper is to point out that the theory developed in [1] for a three-dimensional $\phi^4$-theory can easily be applied in $D = 4-\epsilon$ dimensions with beautiful results at the two-loop level.

3. Let us briefly recall the relevant formulas. Consider a function $f(g_0)$ for which we know $N$ expansion terms, $f_N(g_0) = \sum_{n=0}^{N} a_n g_0^n$, and the fact that it approaches a constant value $f^*$ in the form of an inverse power series $f_M(g_0) = \sum_{m=0}^{M} b_m (g_0^{-2/q})^m$ with a finite convergence radius $g_*$ (simple examples were treated in [1]). Then the $N$th approximation to the value $f^*$ is obtained from the formula

$$f_N = \text{opt}_{\tilde{g}_0} \left[ \sum_{j=0}^{N} a_j \tilde{g}_0^j \sum_{k=0}^{N-j} \left( -\frac{qj}{2k} \right)(-1)^k \right].$$

where the expression in brackets has to be optimized in the variational parameter $\tilde{g}_0$. The optimum is the smoothest among all real extrema. If there are no real extrema, the turning points serve the same purpose.

The derivation of this expression is simple: We replace $g_0$ in $f_N(g_0)$ trivially by $\tilde{g}_0 \equiv g_0/\kappa^q$ with $\kappa = 1$. Then we rewrite, again trivially, $\kappa^{-q}$ as $(K^2 + \kappa^2 - K^2)^{-q/2}$ with an arbitrary parameter $K$. Each term is now expanded in powers of $r = (K^2 - K^2)/K^2$ assuming $r$ to be of the order $g_0$. Taking the limit $g_0 \to \infty$ at a fixed ratio $\tilde{g}_0 \equiv g_0/K^q$, so that $K \to \infty$ like $g_0^{1/q}$ and $r \to -1$, we obtain (3).

Since the final result to all orders cannot depend on the arbitrary parameter $K$, we expect the best result to any finite order to be optimal at an extremal value of $K$, i.e., of $\tilde{g}_0$.

The strong-coupling approach to the limiting value $r = -1 + \kappa^2/K^2 = -1 + O(g_0^{-2/q})$ implies the leading correction to $f_N^*$ to be of the order of $g_0^{-2/q}$. Application of the theory to a function with the strong-coupling behavior (1) requires therefore a parameter $q = 2\epsilon/\omega$ in formula (2).

For $N = 2$ and 3 one can give analytic expressions for the strong-coupling limits (3). Setting $\rho \equiv 1 + q/2 = 1 + \epsilon/\omega$, we find for $N = 2$

$$f_2 = \text{opt}_{\tilde{g}_0} \left[ a_0 + a_1 \rho \tilde{g}_0 + a_2 \tilde{g}_0^2 \right] = a_0 - \frac{1}{4} a_2^2 \rho^2.$$

For $N = 3$, we obtain from the extrema

$$f_3 = \text{opt}_{\tilde{g}_0} \left[ a_0 + a_1 \rho (\rho + 1) \tilde{g}_0 + a_2 (2\rho - 1) \tilde{g}_0^2 + a_3 \tilde{g}_0^3 \right]$$
where \( r = \sqrt{1 - 3a_1 a_3 / a_2^2} \) and \( a_3 \equiv -a_2 r (\rho + 1) \) and \( a_2 \equiv a_2(2 \rho - 1) \). The positive square root must be taken to connect \( g_0^4 \) smoothly to \( g_0^2 \) in the limit of a vanishing coefficient of \( g_0^3 \). If the square root is imaginary, the optimum is given by the unique turning point, leading once more to (3) but with \( r = 0 \).

The parameter \( \rho = 1 + \epsilon / \omega \) can be determined from the expansion coefficients of a function \( F(g_0) \) as follows. Assuming \( F(g_0) \) to be constant \( F^* \) in the strong-coupling limit, the logarithmic derivative \( f(g_0) = g_0 F'(g_0)/F(g_0) \) must vanish at \( g_0 = \infty \). If \( F(g_0) \) starts out as \( A_0 + A_1 g_0 + \ldots \) or \( A_1 g_0 + A_2 g_0^2 + \ldots \), the logarithmic derivative is

\[
f(g_0) = A_1 g_0 + (2 A_1' - A_1^2) g_0^2 + (A_1'^3 - 3 A_1' A_2' + 3 A_1 A_3') g_0^3 + \ldots,
\]

where \( A_1' = A_1/A_0 \), or

\[
f(g_0) = 1 + \tilde{A}_2 g_0 + (2 \tilde{A}_3 - \tilde{A}_3^2) g_0^2 + (8 \tilde{A}_2^2 - 3 A_2 \tilde{A}_3 + 3 A_3) g_0^3 + \ldots,
\]

where \( \tilde{A}_1 = A_1/A_0 \). The expansion coefficients on the right-hand sides are then inserted into (3) or (4), and the left-hand sides have to vanish to ensure that \( F(g_0) \rightarrow F^* \).

If the approach \( F(g_0) \rightarrow F^* \) is of the type (10), the function

\[
h(g_0) = g_0 F''(g_0)/F'(g_0) = 2 \tilde{A}_2 g_0 + (4 \tilde{A}_2^2 + 6 \tilde{A}_3) g_0^2 + (8 \tilde{A}_2^3 - 18 \tilde{A}_2 \tilde{A}_3 + 12 \tilde{A}_4) g_0^3 + \ldots
\]

must have the strong-coupling limit

\[
h(g_0) \rightarrow h^* = -\omega / \epsilon - 1.
\]

4. These formulas are now applied to the renormalization constants of the \( \phi^4 \)-theory in \( D = 4 - \epsilon \) dimensions with the bare euclidean action

\[
A = \int d^D x \left\{ \frac{1}{2} [\partial \phi_0(x)]^2 + \frac{1}{2} m_0^2 \phi_0^2(x) + (4\pi)^2 \frac{\lambda_0}{4!} [\phi_0^2(x)]^2 \right\}.
\]

The field \( \phi_0(x) \) is an \( n \)-dimensional vector, and the action is \( O(n) \)-symmetric in this vector space. The Ising model corresponds to \( n = 1 \), the critical behavior of percolation is described by \( n = 0 \), superfluid phase transitions by \( n = 2 \), and classical Heisenberg magnetic systems by \( n = 3 \).

By calculating the Feynman integrals regularized via an expansion in \( \epsilon = 4 - D \) with the help of an arbitrary mass scale \( \mu \), one obtains renormalized values of mass, coupling constant, and field related to the bare input quantities by renormalization constants \( Z_m, Z_m, Z_\phi \):

\[
m_0^2 = m^2 Z_m Z_\phi^{-1}, \quad \lambda_0 = \lambda Z_\phi^{-2}, \quad \phi_0 = \phi Z_\phi^{-2/3}.
\]

Up to two loops, perturbation theory yields the following expansions in powers of the dimensionless reduced coupling constant \( \lambda / \mu^2 \):

\[
g = g_0 - \frac{n + 8}{3 \epsilon} g_0^2 + \frac{[n + 8]^2}{9 \epsilon^2} + \frac{3 n + 14}{6 \epsilon} g_0^3,
\]

\[
m_0^2 = 1 - \frac{n + 2}{3 \epsilon} g_0^2 + \frac{n + 2}{9} \left[ \frac{n + 5}{\epsilon^2} + \frac{5}{4 \epsilon} \right] g_0^2,
\]

\[
\phi_0^2 = 1 + \frac{n + 2}{36} g_0^2 \epsilon.
\]

We now set the scale parameter \( \mu \) equal to \( m \) and consider all quantities as functions of \( g_0 = \lambda / m^2 \). In order to describe second-order phase transitions, we let \( m_0^2 \) go to zero like \( \tau = \text{const}(T - T_c) \) as the temperature \( T \) approaches the critical temperature \( T_c \). Then also \( m_0^2 \) will go to zero, and thus \( g_0 \) to infinity. Assuming the theory to scale as suggested by experiments, we now determine the value of the renormalized coupling constant \( g \) in the strong-coupling limit \( g_0 \rightarrow \infty \), and the approach to it, assuming the behavior (10). For this we apply formula (2) to \( g(g_0) \) to find \( g^* \), and use the vanishing of (1) or (3) with (9) at strong couplings to determine \( \omega \). Under the scaling assumption, the ratios \( m_0^2 / m_0^2 \) and \( \phi_0^2 / \phi_0^2 \) have the limiting power behavior for small \( m_0^2 \):

\[
\frac{m_0^2}{m_0^2} \sim g_0^{-\eta_m / \epsilon} \sim m_0^{-\eta_m / \epsilon}, \quad \frac{\phi_0^2}{\phi_0^2} \sim g_0^{\eta / \epsilon} \sim m_0^{-\eta / \epsilon}.
\]

The powers can therefore be calculated from the strong-coupling limits of the logarithmic derivatives

\[
\frac{d}{d \log g_0} \log m_0^2 = \eta_m(g_0) = -\epsilon \frac{d}{d \log g_0} \log \phi_0^2.
\]

From (12) and (13) find the expansions

\[
\eta_m(g_0) = \frac{n + 2}{3} g_0 - \frac{n + 2}{18} \left[ \frac{n + 8}{\epsilon} \right] g_0^2,
\]

\[
\eta(g_0) = \frac{n + 2}{18} g_0^2.
\]

When approaching the second-order phase transitions, where \( m_0^2 \) vanishes like \( \tau \equiv (T - T_c) \), \( m_0^2 \) vanishes with a different power of \( \tau \). This power is obtained from the first equation in (14) which shows that \( m \sim \tau^{1/(2 - n_\eta)} \). Experiments observe that the coherence length of fluctuations \( \xi = 1 / m \) increases near \( T_c \) like \( \tau^{-\nu} \), so that we derive for the critical exponent \( \nu \) a value \( 1/(2 - n_\eta) \). Similarly we see from the first equation in (14) that \( m_0^2 / 2 \) of the free field \( \phi_0 \) for \( T \rightarrow T_c \) is changed, in the strong-coupling limit \( g_0 \rightarrow \infty \), to \( m_0^2 / 2 \rightarrow 1 + \eta / 2 \), the number \( \eta \) being the so-called anomalous dimension of the field. This implies a change in the large-distance behavior of the correlation functions \( \langle \phi(x) \phi(0) \rangle \) at \( T_c \) from the free-field behavior \( r^{-D+2} \) to \( r^{-D+2-\eta} \). The magnetic
susceptibility is determined by the integrated correlation function \( \langle \phi_0(x)\phi_0(0) \rangle \). At zero coupling constant \( g_0 \), this is proportional to \( 1/m^2 \propto \tau^{-1} \), which is changed by fluctuations to \( m^{-2} \phi_0^2/\phi^2 \). This has a temperature behavior \( m^{-(2-\eta)} = \tau^{-\nu(2-2\eta)} = \tau^{-\gamma} \), which defines the critical exponent \( \gamma = \nu(2-\eta) \) observable in magnetic experiments. Using \( \nu = 1/(2-\eta_m) \) and the expansions (19), (17), we obtain for \( \gamma(g_0) \) the perturbation expansion up to second order in \( g_0 \):

\[
\gamma(g_0) = 1 + \frac{n+2}{6} g_0 + \frac{n+2}{36} \left( n - 4 - 2 \frac{n+8}{\epsilon} \right) g_0^2.
\]  

All calculations in this note will be restricted to the two loop level, which will be sufficient to demonstrate the power and beauty of the new strong-coupling theory with analytical results.

5. We begin by calculating the critical exponent \( \omega \) from the requirement that \( g(g_0) \) has a constant strong-coupling limit, implying the vanishing of \( \langle \rangle \) for \( g_0 \to \infty \). From the expansion (11) we obtain a logarithmic derivative (1) up to the term \( g_0^2 \), so that Eq. (3) can be used to find the scaling condition

\[
0 = 1 - \frac{A_2^2}{4(2A_3 - A_2^2)} \rho^2.
\]  

This gives

\[
\rho = \sqrt{8A_3/A_2^2 - 4}.
\]  

Since \( \omega \) must be greater than zero, only the positive square root is physical. With the explicit coefficients \( A_1, A_2, A_3 \) of expansion (11), this becomes

\[
\rho = 2 \sqrt{1 + 3 \frac{3n+14}{(n+8)^2} \epsilon}.
\]  

The associated critical exponent \( \omega = \epsilon/(\rho - 1) \) is plotted in Fig. 1. It has the \( \epsilon \)-expansion

\[
\omega = \epsilon - 3 \frac{3n+14}{(n+8)^2} \epsilon^2 + \ldots,
\]  

which is also shown in Fig. 1, and agrees with the first two terms obtained from renormalization group calculations (1).

From Eqs. (8), (9), and (1) we obtain for the critical exponent \( \omega \) a further equation

\[
- \frac{\omega}{\epsilon} - 1 = - \frac{\rho}{\rho - 1} = - \frac{1}{2} \frac{A_2^2 \rho^2}{3A_3 - 2A_2}.
\]  

which is solved by

\[
\rho = \frac{1}{2} + \sqrt{\frac{6A_3}{A_2^2} - \frac{15}{4}}.
\]  

with the positive sign of the square root ensuring a positive \( \omega \). Inserting the coefficients of (11), this becomes

\[
\rho = \frac{1}{2} + \frac{3}{2} \sqrt{1 + 4 \frac{3n+14}{(n+8)^2} \epsilon}.
\]  

The associated critical exponent \( \omega = \epsilon/(\rho - 1) \) has the same \( \epsilon \)-expansion (23) as the previous approximation (24). The full approximations based on (23) is indistinguishable from the earlier one in the plot of Fig. 1.

Having determined \( \omega \), we can now calculate \( g^* \). Inserting the first two coefficients of the expansion (11) into (3) we obtain

\[
g_2^* = a_0 - \frac{1}{4} \frac{a_1^2}{\rho^2}.
\]  

Inserting (21), this yields

\[
g_2^* = \frac{3}{n + 8} \epsilon + 9 \frac{3n+14}{(n+8)^3} \epsilon^2,
\]  

which is precisely the well-known \( \epsilon \)-expansion of \( g^* \) in renormalization group calculations up to the second order. Including the next coefficient, we can use formula (4) to calculate the next approximation \( g_3^* \). At \( \epsilon = 1 \), the square root turns out to be imaginary, so that it has to be omitted (corresponding to the turning point as optimum). The resulting curve lies slightly (\( \approx 8\% \)) above the curve (24), i.e., represents a worse approximation than (27). Indeed, the \( \epsilon^3 \)-term in \( g_3^* \) is \( 81(3n+14)^2/(n+8)^5 \) and disagrees in sign with the exact term \( \epsilon^3[3(-33n^3+110n^2+1760n+4544)/(8 \cdot 36\zeta(3)(n+8)(5n+22))/(n+8)^5) \), which we would find by calculating \( \rho \) from an expansion (11) with one more power in \( g_0 \).

We now turn to the critical exponent \( \nu \). Taking the expansion (11) to \( g_0 \to \infty \), we obtain from formula (3) the limiting value

\[
\eta_m = \epsilon \frac{n+2}{4n+8+5\epsilon/2} \rho^2.
\]  

The corresponding \( \nu = 1/(2 - \eta_m) \) is plotted in Fig. 1. With the approximation (21) for \( \rho \) we find for \( \nu \) the \( \epsilon \)-expansion

\[
\nu = \frac{1}{2} + \frac{1}{4} \frac{n+2}{n+8} \epsilon + \frac{(n+2)(n+3)(n+20)}{8(n+8)^3} \epsilon^2 + \ldots,
\]  

which is also shown in Fig. 1, and agrees with renormalization group results to this order.

As a third independent critical exponent we calculate \( \gamma = (2 - \eta)/(2 - \eta_m) \) by inserting the coefficients of the expansion (18) into formula (3), which yields

\[
\gamma = 1 + \frac{\epsilon}{8(n+8-2(n-4)\epsilon/2} \rho^2,
\]  

which is also shown in Fig. 1.
plotted in Fig. 1. This has an \( \epsilon \)-expansion
\[
\gamma = 1 + \frac{1 + n + 2}{2n + 8} + \frac{1}{4} \frac{(n + 2)(n^2 + 22n + 52)}{(n + 8)^3} \epsilon^2 + \ldots ,
\]
shown again in Fig. 1, and agreeing with renormalization group results to this order. The full approximation is shown again in Fig. 1, and agreeing with renormalization discussions.

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[1] H. Kleinert, APS E-Print aps1997jun25_001.
[2] R.P. Feynman and H. Kleinert, Phys. Rev. A34, 5080 (1986). A similar approach has been pursued independently by R. Giachetti and V. Tognetti, Phys. Rev. Lett. 55, 912 (1985); Int. J. Magn. Mater. 54-57, 861 (1986); R. Giachetti, V. Tognetti, and R. Vaia, Phys. Rev. B33, 7647 (1986).
[3] H. Kleinert, Phys. Lett. A173, 332 (1993).
[4] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics and Polymer Physics, World Scientific, Singapore 1995.
[5] For earlier similar expansions see by R. Seznec and J. Zinn-Justin, J. Math. Phys. 20, 1398 (1979); T. Barnes and G.I. Ghandour, Phys. Rev. D22, 924 (1980); B.S. Shaverdyan and A.G. Ushveridze, Phys. Lett. B123, 316 (1983); P.M. Stevenson, Phys. Rev. D30, 1712 (1985); D32, 1389 (1985); P.M. Stevenson and R. Tarrach, Phys. Lett. B176, 436 (1986); A. Okopinska, Phys. Rev. D35, 1835 (1987); D36, 2415 (1987); W. Namgung, P.M. Stevenson, and J.F. Reed, Z. Phys. C45, 47 (1989); U. Ritschel, Phys. Lett. B227, 44 (1989); Z. Phys. C51, 469 (1991); M.H. Thoma, Z. Phys. C44, 343 (1991); I. Stancu and P.M. Stevenson, Phys. Rev. D42, 2710 (1991); R. Tarrach, Phys. Lett. B262, 294 (1991); H. Haugerud and F. Raundal, Phys. Rev. D43, 2736 (1991); A.N. Sissakian, I.L. Solovtsov, and O.Y. Sheychenko, Phys. Lett. B313, 367 (1993); A. Duncan and H.F. Jones, Phys. Rev. D47, 2560 (1993).
[6] W. Janke and H. Kleinert, Phys. Lett. A199, 287 (1995).
[7] W. Janke and H. Kleinert, Phys. Rev. Lett. 75, 2787 (1995). (quant-ph/9502019).
That paper contains references to earlier, less accurate calculations of strong-coupling expansion coefficients from weak-coupling perturbation theory, in particular F.M. Fernández and R. Guardiola, J. Phys. A26, 7169 (1993); F.M. Fernández, Phys. Lett. A166, 173 (1992); R. Guardiola, M.A. Solís, and J. Ros, Nuovo Cimento B107, 713 (1992). A.V. Turbiner and A.G. Ushveridze, J. Math. Phys. 29, 2053 (1988); B. Bonnier, M. Hontebery, and E.H. Ticembal, J. Math. Phys. 26, 3048 (1985); Those works did not extract the exponential law of convergence from their data.
[8] H. Kleinert and W. Janke, Phys. Lett. A206, 283 (1995) (quant-ph/9509005).
[9] A convergence proof which is completely equivalent our results in was given by R. Guida, K. Konishi, and H. Suzuki, Annals Phys. 249, 109 (1996) (hep-th/9505050).
Predecessors of these works which did not explain the exponentially fast convergence in the strong-couplings limit observed in Ref. are I.R.C. Buckley, A. Duncan, and H.F. Jones, Phys. Rev. D47, 2554 (1993); C.M. Bender, A. Duncan, and H.F. Jones, Phys. Rev. D49, 4219 (1994); A. Duncan and H.F. Jones, Phys. Rev. D47, 2560 (1993); C. Arvanitis, H.F. Jones, and C.S. Parker, Phys. Rev. D52, 3704 (1995) (hep-th/9502386); R. Guida, K. Konishi, and H. Suzuki, Annals Phys. 241, 152 (1995) (hep-th/9407027).
[10] H. Kleinert, Phys. Lett. A207, 133 (1995) (quant-ph/9507005).
[11] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, and S.A. Larin, Phys. Lett. B272, 39 (1991) (hep-th/9503230); H. Kleinert and V. Schulte-Frohlinde, Phys. Lett. B342, 284 (1995) (cond-mat/9503038).
FIG. 1. For the Ising universality class \( n = 1 \), the first figure shows the renormalized coupling at infinite bare coupling as a function of \( \epsilon = 4 - D \) calculated via variational perturbation theory from the first two perturbative expansion terms. The curve coincides with the \( \epsilon \)-expansion up to order \( \epsilon^2 \). The dashed curve indicates the linear term. The other figures show the critical exponents \( \omega, \nu, \) and \( \gamma \). Dashed curves indicate linear and quadratic \( \epsilon \)-expansions. The dots mark presently accepted values of \( g^* \approx 0.48 \pm 0.003 \), \( \omega \approx 0.802 \pm 0.003 \), \( \nu = 0.630 \pm 0.002 \), and \( \gamma = 1.241 \pm 0.004 \) obtained from six-loop calculations.\[5\]