Contact processes with random vertex weights on oriented lattices

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Abstract:
In this paper we are concerned with contact processes with random vertex weights on oriented lattices. In our model, we assume that each vertex $x$ of $\mathbb{Z}^d$ takes i. i. d. positive random value $\rho(x)$. Vertex $y$ infects vertex $x$ at rate proportional to $\rho(x)\rho(y)$ when and only when there is an oriented edge from $y$ to $x$. We give the definition of the critical value $\lambda_c$ of infection rate under the annealed measure and show that $\lambda_c = [1 + o(1)]/(dE\rho^2)$ as $d$ grows to infinity. Classic contact processes on oriented lattices and contact processes on clusters of oriented site percolation are two special cases of our model.

Keywords: Contact process, random vertex weights, oriented lattice, critical value.

1 Introduction

In this paper we are concerned with contact processes with random vertex weights on oriented lattices. For $d$-dimensional oriented lattice $\mathbb{Z}^d$, there is an oriented edge from $x$ to $x + e_i$ for each $x \in \mathbb{Z}^d$ and $1 \leq i \leq d$, where

$$e_i = (0, \ldots, 0, 1_{i^{th}}, 0, \ldots, 0).$$

For $x, y \in \mathbb{Z}^d$, we denote by $x \to y$ when $y - x \in \{e_i\}_{1 \leq i \leq d}$. We denote by $O$ the origin of $\mathbb{Z}^d$.

Let $\rho$ be a positive random variable such that $P(\rho > 0) > 0$ and $P(\rho \leq M) = 1$ for some $M > 0$. Let $\{\rho(x)\}_{x \in \mathbb{Z}^d}$ be i. i. d. random variables such that $\rho(O)$ and $\rho$ have the same distribution. When $\{\rho(x)\}_{x \in \mathbb{Z}^d}$ is given, the contact process with random vertex weights on oriented lattice $\mathbb{Z}^d$ is a spin system with

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state space \( \{0, 1\}^{Z^d} \) and flip rates function given by

\[
c(x, \eta) = \begin{cases} 
1 & \text{if } \eta(x) = 1, \\
\lambda \sum_{y: y \to x} \rho(x) \rho(y) \eta(y) & \text{if } \eta(x) = 0
\end{cases}
\] (1.1)

for each \((x, \eta) \in Z^d \times \{0, 1\}^{Z^d}\), where \(\lambda > 0\) is a positive parameter called the infection rate. More details of the definition of spin systems can be found in Chapter 3 of [9].

Intuitively, this contact process describes the spread of an infectious disease. Vertices in state 0 are healthy and vertices in state 1 are infected. An infected vertex waits for an exponential time with rate one to become healthy. An healthy vertex \(x\) may be infected by an infected vertex \(y\) when and only when there is an oriented edge from \(y\) to \(x\). The infection between \(y\) and \(x\) occurs at rate in proportional to \(\rho(x)\rho(y)\).

Please note that the assumption \(P(\rho < M) = 1\) ensures the existence of our process according to the basic theory constructed in [8].

The contact processes with random vertex weights is introduced by Peterson in [13] on finite complete graphs. He proves that the infection rate \(\lambda\) has a critical value \(\lambda_c = \frac{1}{E \rho^2}\) such that the disease survives for a long time with high probability when \(\lambda > \lambda_c\) or dies out quickly with high probability when \(\lambda < \lambda_c\).

Recently, contact processes in random environments or random graphs is a popular topic. In [1], Chatterjee and Durrett show that contact processes on random graphs with power law degree distributions have critical value 0. This result disproves the guess in [10, 11] that the critical value is strictly positive according to a non-rigorous mean-field analysis. In [13], Peterson shows that contact processes with random vertex weights on complete graphs have critical value \(\frac{1}{E \rho^2}\), which is consistent with the estimation given by the mean-field calculation. In [2, 16], Yao and Chen shows that complete convergence theorem holds for contact processes in a random environment on \(Z^d \times Z^+\). The random environment they set includes the bond percolation model as a special case.

In our model, if \(\rho\) satisfies that \(P(\rho = 1) = 1 - P(\rho = 0) = p\), then our model can be regarded as contact processes on clusters of oriented site percolation on \(Z^d\). In [7], Kesten shows that site percolation on \(Z^d\) has critical probability \([1 + o(1)]/2d\). We are inspired a lot by this result.

## 2 Main result

Before giving our main results, we introduce some notations. We assume that the random variables \(\{\rho(x)\}_{x \in Z^d}\) are defined on a probability space \((\Omega, \mathcal{F}, P)\). We denote by \(E\) the expectation operator with respect to \(P\).
For any $\omega \in \Omega$, we denote by $P^\omega_\lambda$ the probability measure of our contact process on oriented lattice $\mathbb{Z}^d$ with infection rate $\lambda$ and vertex weights $\{\rho(x, \omega)\}_{x \in \mathbb{Z}^d}$. The probability measure $P^\omega_\lambda$ is called the quenched measure. We denote by $E^\omega_\lambda$ the expectation operator with respect to $P^\omega_\lambda$. We define $P_\lambda,d(\cdot) = E[P^\omega_\lambda(\cdot)]$, which is called the annealed measure. We denote by $E_\lambda,d$ the expectation operator with respect to $P_\lambda,d$.

For any $t \geq 0$, we denote by $\eta_t$ the configuration of our process at the moment $t$. In this paper, we mainly deal with the case that all the vertices are infected at $t = 0$. In later sections, if we need deal with the case that $A = \{x : \eta_0(x) = 1\} \neq \mathbb{Z}^d$, then we will point out the initial infected set $A$ and write $\eta_t$ as $\eta^A_t$. When $\eta_t$ is with no upper script, we refer to the case that $\{x : \eta_0(x) = 1\} = \mathbb{Z}^d$.

According to basic coupling of spin systems, it is easy to see that

\[ P_\lambda,d(\eta_t(O) = 1) \leq P_{\lambda_1,d}(\eta_s(O) = 1) \]

for $t > s$ and

\[ P_{\lambda_1,d}(\eta_t(O) = 1) \leq P_{\lambda_2,d}(\eta_t(O) = 1) \]

for $\lambda_1 < \lambda_2$. As a result, it is reasonable to define the following critical value of the infection rate.

\[ \lambda_c(d) = \sup \{\lambda : \lim_{t \to +\infty} P_\lambda,d(\eta_t(O) = 1) = 0\}. \] (2.1)

Please note that our process is symmetric under the annealed measure $P_{\lambda,d}$. So, $P_{\lambda,d}(\eta_t(x) = 1)$ does not rely on the choice of $x$. As a result, when $\lambda < \lambda_c(d)$,

\[ \lim_{t \to +\infty} P_{\lambda,d}(\eta_t(x) = 1 \text{ for some } x \in A) = 0 \]

for any finite $A \subseteq \mathbb{Z}^d$ and hence $\eta_t$ converges weakly to the configuration that all the vertices are healthy as $t$ grows to infinity.

Our main result is the following limit theorem of $\lambda_c(d)$.

**Theorem 2.1.** Assume that $P(\rho > 0) > 0$ and $P(\rho < M) = 1$ for some $M > 0$, then

\[ \lim_{d \to +\infty} d\lambda_c(d) = \frac{1}{E\rho^2}. \] (2.2)
Theorem 2.1 shows that the critical value $\lambda_c(d)$ is approximately inversely proportional to the dimension $d$, the ratio of which is the reciprocal of the second moment of $\rho$.

When $\rho \equiv 1$, our process is the classic contact process on oriented lattice. In this case, we write $\lambda_c(d)$ as $\lambda_d$. When $\rho$ satisfies that $P(\rho = 1) = 1 - P(\rho = 0) = p$

for some $p \in (0,1)$, our process is the contact process on clusters of oriented site percolation on $\mathbb{Z}^d$. In this case, we write $\lambda_c(d)$ as $\lambda_c(d, \text{site}, p)$. There are two direct corollaries of Theorem 2.1.

**Corollary 2.2.**

$$\lim_{d \to +\infty} d\lambda_d = 1. \hspace{1cm} (2.3)$$

We say $y$ is $x$’s neighbor when $y \to x$, then Corollary 2.2 shows that $\lambda_d$ is approximately to the reciprocal of the number of neighbors. In [4, 6, 12], Holley, Liggett, Griffeath and Pemantle show that this conclusion holds for contact processes on non-oriented lattices and regular trees. In [14], Xue shows that the same conclusion holds for threshold one contact processes on lattices and regular trees.

**Corollary 2.3.** For $p \in (0,1)$,

$$\lim_{d \to +\infty} dp\lambda_c(d, \text{site}, p) = 1. \hspace{1cm} (2.4)$$

Corollary 2.3 shows that $\lambda_c = [1 + o(1)]/(dp)$ as $d$ grows to infinity for contact processes on clusters of oriented site percolation. In [15], Xue claims that the same conclusion holds for contact process on clusters of oriented bond percolation on $\mathbb{Z}^d$.

Please note that the critical value $\lambda_c(d)$ we define is under the annealed measure $P_{\lambda_d}$. We can also define critical value $\lambda_c(\omega)$ under the quenched measure such that

$$\lambda_c(\omega) = \sup \{ \lambda : \forall x \in \mathbb{Z}^d, \lim_{t \to +\infty} P^\omega_{\lambda}(\eta_t(x) = 1) = 0 \}$$

for any $\omega \in \Omega$. $\lambda_c(\omega)$ is a random variable. However, according to the ergodic theorem for i. i. d. random variables, it is easy to see that

$$P(\omega : \lambda_c(\omega) = \lambda_c(d)) = 1.$$ 

So we only need to deal with the critical value under the annealed measure.
The proof of Theorem 2.1 is divide into two sections. In Section 3, we will prove that
\[
\liminf_{d \to +\infty} d \lambda_c(d) \geq \frac{1}{E \rho^2}.
\]
In Section 4, we will prove that
\[
\limsup_{d \to +\infty} d \lambda_c(d) \leq \frac{1}{E \rho^2}.
\]

3 Lower bound

In this section we give a lower bound of \( \lambda_c(d) \).

Lemma 3.1. For each \( d \geq 1 \),
\[
\lambda_c(d) \geq \frac{1}{d E \rho^2}
\]
and hence
\[
\liminf_{d \to +\infty} d \lambda_c(d) \geq \frac{1}{E \rho^2}.
\]

Proof. We use \( f_t \) to denote
\[
E_{\lambda,d} [\rho(O)\eta_t(O)] = E[\rho(O,\omega)P_{\lambda}(\eta_t(O) = 1)].
\]
According to the flip rates function of \( \eta_t \) given by (1.1), \( \rho(O,\omega) \) and \( P_{\lambda}(\eta_t(O) = 1) \) are positive correlated. Therefore,
\[
f_t \geq E \rho(O,\omega)E[P_{\lambda}(\eta_t(O) = 1)] = E \rho P_{\lambda,d}(\eta_t(O) = 1).
\]
Hence,
\[
P_{\lambda,d}(\eta_t(O) = 1) \leq \frac{f_t}{E \rho}.
\] (3.1)

Please note that the assumption \( P(\rho > 0) > 0 \) ensures that \( E \rho > 0 \).

According to Hille-Yosida Theorem and (1.1),
\[
\frac{d}{dt} P_{\lambda}(\eta_t(O) = 1) = -P_{\lambda}(\eta_t(O) = 1)
\]
\[
+ \lambda \sum_{y:y \to O} \rho(O)\rho(y)P_{\lambda}(\eta_t(O) = 0, \eta_t(y) = 1)
\]
\[
\leq -P_{\lambda}(\eta_t(O) = 1) + \lambda \sum_{y:y \to O} \rho(O)\rho(y)P_{\lambda}(\eta_t(y) = 1).
\]
Therefore,
\[
\frac{d}{dt} f_t \leq -f_t + \lambda \sum_{y:y \to O} E[\rho^2(O)\rho(y)P_{\lambda}(\eta_t(y) = 1)].
\] (3.2)
For each $y$ such that $y \to O$, $\eta_t(y)$ is only influenced by the vertices from which there are oriented paths to $y$. Therefore, $\rho(O)$ is independent of $\rho(y) P_\lambda^\omega(\eta_t(y) = 1)$ and hence

$$E[\rho^2(O)\rho(y) P_\lambda^\omega(\eta_t(y) = 1)] = E\rho^2 P_{\lambda,d}(\eta_t(y) = 1) = E\rho^2 f_t. \quad (3.3)$$

By (3.2) and (3.3),

$$\frac{d}{dt} f_t \leq (d\lambda E\rho^2 - 1) f_t. \quad (3.4)$$

According to Greenwood inequality and (3.4),

$$f_t \leq f_0 \exp\{(d\lambda E\rho^2 - 1)t\}$$

and hence

$$\lim_{t \to +\infty} f_t = 0 \quad (3.5)$$

when $\lambda < \frac{1}{d\rho^2}$.

Lemma 3.1 follows (3.1) and (3.5).

4 Upper bound

In this section we will prove that $\liminf_{d \to +\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}$.

First we define the contact process $\hat{\eta}_t$ where disease spreads through the opposite direction of the oriented edges. For any $\omega \in \Omega$, the flip rates of $\hat{\eta}_t$ with random vertex weights $\{\rho(x, \omega)\}_{x \in \mathbb{Z}^d}$ is given by

$$\hat{c}(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \sum_{y : x \to y} \rho(x)\rho(y)\eta(y) & \text{if } \eta(x) = 0. \end{cases}$$

Hence, for $\hat{\eta}_t$, $y$ may infect $x$ when and only when there is an edge from $x$ to $y$.

According to the graphic representation of contact processes introduced by Harris in [5], there is a dual of $\eta_t$ and $\hat{\eta}_t$ such that

$$P_\lambda^\omega(\eta_t(O) = 1) = P_\hat{\lambda}^\omega(\hat{\eta}_t^O \neq \emptyset), \quad (4.1)$$

where $\hat{\eta}_t^O$ is $\hat{\eta}_t$ with that $\{x \in \mathbb{Z}^d : \hat{\eta}_t(x) = 1\} = \{O\}$.

Please note that in (4.1) we utilize the identification of $\hat{\eta}_t^O$ with

$$\{x \in \mathbb{Z}^d : \hat{\eta}_t^O(x) = 1\}.$$ Since $\{\rho(x)\}_{x \in \mathbb{Z}^d}$ are i. i. d., the events $\{\eta_t^O \neq \emptyset\}$ and $\{\hat{\eta}_t^O \neq \emptyset\}$ have the same distribution under the annealed measure $P_{\lambda,d}$. Therefore, according to (4.1),

$$P_{\lambda,d}(\eta_t(O) = 1) = P_{\lambda,d}(\eta_t^O \neq \emptyset). \quad (4.2)$$
To control the size of $\eta^O_t$ from below, we introduce a Markov process $\zeta_t$ with state space $\{-1, 0, 1\}^{Z^2}$. For given $\{\rho(x)\}_{x \in Z^2}$, $\zeta_t$ evolves as follows. For each $x \in Z^d$, if $\zeta(x) = -1$, then $x$ is frozen in the state $-1$ forever. If $\zeta(x) = 1$, then the value of $x$ waits for an exponential time with rate one to become $-1$. If $\zeta(x) = 0$, then the value of $x$ flips to 1 at rate $
abla \sum_{y : y \to x} \rho(x)\rho(y)1_{\{\zeta(y) = 1\}}$.

So for $\zeta_t$, when an infected vertex becomes healthy, then it is removed (in the state $-1$) and will never be infected.

We use $\zeta^O_t$ to denote $\zeta_t$ with $\{x \in Z^d : \zeta_0(x) = 1\} = \{O\}$ and $\{x \in Z^d : \zeta_0(x) = -1\} = \emptyset$. According to the basic coupling of Markov processes, there is a coupling of $\eta_t$ and $\zeta_t$ under quenched measure $P_\lambda$ such that $\eta^O_t \supseteq \{x \in Z^d : \zeta^O_t(x) = 1\}$ (4.3) for any $t > 0$.

We use $C_t$ to denote $\{x \in Z^d : \zeta^O_t(x) = 1\}$. Then, by (4.2) and (4.3),

$$\lim_{t \to +\infty} P_{\lambda, d}(\eta_t(O) = 1) \geq P_{\lambda, d}(\forall t, C_t \neq \emptyset).$$

We give another description of $\{\forall t, C_t \neq \emptyset\}$. Let $\{T_x\}_{x \in Z^2}$ be i.i.d. exponential times with rate 1. For any $x \to y$, let $U_{xy}$ be exponential time with rate $\lambda\rho(x)\rho(y)$. We assume that all these exponential times are independent. For $O = x_0 \to x_1 \to x_2 \to \ldots \to x_n = x$, if $U_{x_i,x_{i+1}} \leq T_{x_i}$ for each $0 \leq j \leq n - 1$, then we say that there is an infected path with length $n$ from $O$ to $x$, which is denoted by $O \Rightarrow_n x$.

In the sense of coupling,

$$\{O \Rightarrow_n x\} = \{\exists t, x \in C_t\}.$$  

Let $I_n = \{x : O \Rightarrow_n x\}$ and $L_n$ be the set of infected paths with length $n$ from $O$. $\{\forall t, C_t \neq \emptyset\}$ is equivalent to that there are infinite many vertices which have ever been infected. Therefore,

$$\{\forall t, C_t \neq \emptyset\} = \{\forall n, I_n \neq \emptyset\},$$

and

$$P_{\lambda, d}(\forall t, C_t \neq \emptyset) = \lim_{n \to +\infty} P_{\lambda, d}(I_n \neq \emptyset) = \lim_{n \to +\infty} P_{\lambda, d}(|L_n| > 0) \geq \limsup_{n \to +\infty} \frac{\left(\mathbb{E}_{\lambda, d}|L_n|\right)^2}{\mathbb{E}_{\lambda, d}|L_n|^2}$$

(4.5)
according to Hölder inequality.

To calculate \( E\lambda,d|L_n| \) and \( E\lambda,d|L_n|^2 \), we utilize the simple random walk \( S_n \) on oriented lattice \( \mathbb{Z}^d \) with \( S_0 = O \) and

\[
P(S_{n+1} - S_n = e_i) = \frac{1}{d}
\]

for \( 1 \leq i \leq d \). Let \( \{\tilde{S}_n\}_{n=0}^{+\infty} \) be an independent copy of \( \{S_n\}_{n=0}^{+\infty} \). We assume that \( \{S_n\}_{n=0}^{+\infty} \) and \( \{\tilde{S}_n\}_{n=0}^{+\infty} \) are defined on probability space \( (\Omega, \mathcal{G}, \tilde{P}) \) and are independent of \( \{\rho(x)\}_{x \in \mathbb{Z}^d}, \{T_x\}_{x \in \mathbb{Z}^d} \) and \( \{U_{xy}\}_{x \rightarrow y} \). We denote by \( \tilde{E} \) the expectation operator with respect to \( \tilde{P} \).

For a given path \( O \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_n \),

\[
P_{\lambda}^{\omega}(U_{x_j,x_{j+1}} < T_{x_j}, \forall 0 \leq j \leq n - 1) = \prod_{j=0}^{n-1} \left[ \frac{\lambda \rho(x_j, \omega) \rho(x_{j+1}, \omega)}{1 + \lambda \rho(x_j, \omega) \rho(x_{j+1}, \omega)} \right].
\]

and hence

\[
P_{\lambda,d}(U_{x_j,x_{j+1}} < T_{x_j}, \forall 0 \leq j \leq n - 1) = E \prod_{j=0}^{n-1} \left[ \frac{\lambda \rho(x_j) \rho(x_{j+1})}{1 + \lambda \rho(x_j) \rho(x_{j+1})} \right]. \quad (4.6)
\]

It is obviously that the right hand side of (4.6) does not rely on the oriented path \( x_1, x_2, \ldots, x_n \) we choose.

As a result,

\[
E_{\lambda,d}|L_n| = d^n \prod_{j=0}^{n-1} \left[ \frac{\lambda \rho(S_j) \rho(S_{j+1})}{1 + \lambda \rho(S_j) \rho(S_{j+1})} \right] \quad (4.7)
\]

for any given first \( n \) steps \( (S_0, S_1, \ldots, S_n) \) of the simple random walk.

Please note that we write \( E \) not \( \tilde{E} \times E \) in the right hand side of (4.7). We mean that the right hand side of (4.7) is a random variable with respect to \( \mathcal{G} \) and is a constant with probability one.
To calculate \( E_{\lambda,d} |L_n|^2 \), we introduce the following notations.

\[
\begin{align*}
\tau_1 &= \inf\{ n \geq 0 : S_n = \hat{S}_n, S_{n+1} = \hat{S}_{n+1} \}, \\
\sigma_1 &= \inf\{ n > \tau_1 : S_n = \hat{S}_n, S_{n+1} \neq \hat{S}_{n+1} \}, \\
L_1 &= \sigma_1 - \tau_1 + 1, \\
\tau_2 &= \inf\{ n > \tau_1 : S_n = \hat{S}_n, S_{n+1} = \hat{S}_{n+1} \}, \\
\sigma_2 &= \inf\{ n > \tau_2 : S_n = \hat{S}_n, S_{n+1} \neq \hat{S}_{n+1} \}, \\
L_2 &= \sigma_2 - \tau_2 + 1, \\
&\quad \ldots \ldots \\
\tau_k &= \inf\{ n > \sigma_{k-1} : S_n = \hat{S}_n, S_{n+1} = \hat{S}_{n+1} \}, \\
\sigma_k &= \inf\{ n > \tau_k : S_n = \hat{S}_n, S_{n+1} \neq \hat{S}_{n+1} \}, \\
L_k &= \sigma_k - \tau_k + 1, \\
&\quad \ldots \ldots \\
T &= \sup\{ k : \tau_k < +\infty \}.
\end{align*}
\]

Please note that \( P(T < +\infty) = 1 \) for \( d \geq 4 \) according to the conclusion proven in [3] that \( P(\exists n > 0, S_n = \hat{S}_n) < 1 \) for \( d \geq 4 \). Therefore, \( \tau_k, \sigma_k, L_k \) are finite for \( k \leq T \).

Furthermore, we define

\[
\begin{align*}
A_0 &= \{ 0 \leq n < \tau_1 : S_n = \hat{S}_n, S_{n+1} \neq \hat{S}_{n+1} \}, \\
A_1 &= \{ \sigma_1 < n < \tau_2 : S_n = \hat{S}_n, S_{n+1} \neq \hat{S}_{n+1} \}, \\
&\quad \ldots \ldots \\
A_{T-1} &= \{ \sigma_{T-1} < n < \tau_T : S_n = \hat{S}_n, S_{n+1} \neq \hat{S}_{n+1} \}, \\
A_T &= \{ n > \sigma_T : S_n = \hat{S}_n, S_{n+1} \neq \hat{S}_{n+1} \}.
\end{align*}
\]

For \( 0 \leq i \leq T \), we use \( K_i \) to denote \( |A_i| \).

After all this prepare work, we give a lemma which is crucial for us to give upper bound of \( \lambda_c(d) \).

**Lemma 4.1.** Assume that \( P(\rho > 0) > 0 \) and \( P(\rho < M) = 1 \). If \( \lambda \) makes

\[
E \left[ 2 \frac{T + \sum_{j=0}^{T} K_j}{M} \frac{2 T + 4 \sum_{j=0}^{T} K_j}{M} \frac{2 \sum_{j=1}^{T} L_j + 2 \sum_{j=0}^{T} K_j}{M} \frac{(1 + \lambda M^2) \sum_{j=1}^{T} L_j + 2 \sum_{j=0}^{T} K_j}{M} \right] < +\infty, \tag{4.8}
\]

then

\[
\lambda_c(d) \leq \lambda.
\]
Proof. For each $x \to z_1$ and $y \to z_2$, we define $F(x, y; z_1, z_2)$ as

$$P_n^* (U_{x,z_1} \leq T_x, U_{y,z_2} \leq T_y).$$

By direct calculation,

$$F(x, y; z_1, z_2) = \begin{cases} 
\frac{\lambda^2 \rho(x) \rho(y) \rho(z_1) \rho(z_2)}{(1+\lambda \rho(x) \rho(z_1))(1+\lambda \rho(y) \rho(z_2))} & \text{if } x \neq y \text{ and } z_1 \neq z_2, \\
\frac{\lambda \rho(x) \rho(z_1)}{1+\lambda \rho(x) \rho(z_1)} & \text{if } x \neq y \text{ and } z_1 = z_2, \\
\frac{\rho(x) \rho(z_1)}{1+\lambda \rho(x) \rho(z_1)} & \text{if } x = y \text{ and } z_1 = z_2, \\
\frac{1}{1+\lambda \rho(x) \rho(z_1)} & \text{if } x = y \text{ and } z_1 \neq z_2.
\end{cases}$$

(4.9)

We denote by $P_n$ the set of all the oriented paths from $O$ with length $n$, then

$$E_{\lambda,d}|L_n|^2 = \sum_{x \in P_n} \sum_{y \in P_n} \lambda \rho(x) \rho(y) \rho(z_1) \rho(z_2) (\forall 0 \leq i \leq n-1, U_{x,x_i+1} \leq T_{x_i}, U_{y,y_i+1} \leq T_{y_i})$$

$$= \sum_{x \in P_n} \sum_{y \in P_n} E \sum_{i=0}^{n-1} F(x_i, y_i; x_{i+1}, y_{i+1})$$

$$= \sum_{x \in P_n} \sum_{y \in P_n} E \sum_{i=0}^{n-1} \lambda \rho(x) \rho(y) \rho(z_1) \rho(z_2)$$

$$= d^{2n} \sum_{x \in P_n} \sum_{y \in P_n} \lambda \rho(x) \rho(y) \rho(z_1) \rho(z_2)$$

$$= d^{2n} \sum_{x \in P_n} \sum_{y \in P_n} \lambda \rho(x) \rho(y) \rho(z_1) \rho(z_2)$$

$$= d^{2n} \sum_{x \in P_n} \sum_{y \in P_n} \lambda \rho(x) \rho(y) \rho(z_1) \rho(z_2)$$

$$= d^{2n} \sum_{x \in P_n} \sum_{y \in P_n} \lambda \rho(x) \rho(y) \rho(z_1) \rho(z_2)$$

Therefore, by (4.9),

$$\frac{(E_{\lambda,d}|L_n|^2)^2}{E_{\lambda,d}|L_n|^2} = \left\{ \frac{\lambda \rho(x) \rho(y) \rho(z_1) \rho(z_2)}{(1+\lambda \rho(x) \rho(z_1))(1+\lambda \rho(y) \rho(z_2))} \right\}^{-1}. $$

(4.10)

Then by (4.3) and (4.4), $\lambda \geq \lambda_c(d)$ when

$$\limsup_{n \to +\infty} \tilde{E} \left[ \frac{\sum_{i=0}^{n-1} \lambda \rho(x) \rho(y) \rho(z_1) \rho(z_2)}{(1+\lambda \rho(x) \rho(z_1))(1+\lambda \rho(y) \rho(z_2))} \right] < +\infty.$$

Now we control

$$\tilde{E} \left[ \frac{\sum_{i=0}^{n-1} \lambda \rho(x) \rho(y) \rho(z_1) \rho(z_2)}{(1+\lambda \rho(x) \rho(z_1))(1+\lambda \rho(y) \rho(z_2))} \right]$$
we can abstract to

After all these operations, we can cancel many common factors in the numerator and denominator. For example, if \( i \) or \( i+1 \), then we narrow the factor \( \frac{1}{1+\lambda \rho_{i} \rho(S_{i+1})} \) to 1. If \( i \in A_{k} \) for some \( k \), then by (1.9), we enlarge the factors

\[ 2\lambda^2 \rho^2(S_{i}) \rho(S_{i+1}) \rho(\hat{S}_{i+1}) \] and \( \lambda^2 \rho(S_{i-1}) \rho(\hat{S}_{i-1}) \rho^2(S_{i}) \)

to

\[ 2\lambda^2 M^2 \rho(S_{i+1}) \rho(\hat{S}_{i+1}) \] and \( \lambda^2 \rho(S_{i-1}) \rho(\hat{S}_{i-1}) M^2 \).

If \( i = \tau_{k} \) for some \( k \), then we enlarge the factors

\[ \lambda^2 \rho(S_{i-1}) \rho(\hat{S}_{i-1}) \rho^2(S_{i}) \] and \( \lambda \rho(S_{i}) \rho(S_{i+1}) \)

to

\[ \lambda^2 \rho(S_{i-1}) \rho(\hat{S}_{i-1}) M^2 \] and \( \lambda M \rho(S_{i+1}) \).

If \( i = \sigma_{k} \) for some \( k \), then we enlarge the factors

\[ \lambda \rho(S_{i-1}) \rho(S_{i}) \] and \( 2\lambda^2 \rho^2(S_{i}) \rho(S_{i+1}) \rho(\hat{S}_{i+1}) \)

to

\[ \lambda \rho(S_{i-1}) M \] and \( 2\lambda^2 M^2 \rho(S_{i+1}) \rho(\hat{S}_{i+1}) \).

After all these operations, we can cancel many common factors in the numerator and denominator. For example, if \( i, j \in A_{k} \) and \( l \notin A_{k} \) for each \( i < l < j \), then we can abstract

\[ \left[ \frac{\prod_{l=i}^{j-1} \rho_l^2}{\prod_{l=i}^{j-i-2} (1 + \lambda \rho_{l} \rho_{l+1})} \right]^2 \]

from both numerator and denominator and cancel this common factor, where \( \{\rho_l\}^{j-1}_{l=i} \) are i. i. d. and have the same distribution as that of \( \rho \).

Therefore, after all the above operations, it is easy to see that

\[
\limsup_{n \to +\infty} \frac{E \prod_{i=0}^{n-1} \lambda \rho(S_{i}) \rho(S_{i+1})}{E \prod_{i=0}^{n-1} \lambda \rho(S_{i}) \rho(S_{i+1})} \leq 2 \frac{T^{\sum_{j=0}^{T} K_{j}}}{M^{T+4 \sum_{j=0}^{T} K_{j}}} \frac{6T+4 \sum_{j=0}^{T} K_{j}}{L_{j}+2T+2 \sum_{j=0}^{T} K_{j}} \left( E \rho^3 \right)^{\sum_{j=0}^{L_{j}-T} L_{j}+2T+2 \sum_{j=0}^{T} K_{j}}}
\]
and

\[
\limsup_{n \to +\infty} E \left[ \frac{E \prod_{i=0}^{n-1} F(S_i, \hat{S}_i; S_{i+1}, \hat{S}_{i+1})}{\left( E \prod_{i=0}^{n-1} \lambda \rho(S_i) \rho(S_{i+1}) \right)^2} \right] \\
\leq E \left[ \frac{T + T \sum_{j=0}^{T} K_j}{\lambda} \right] \frac{M}{\rho} \left( 1 + \lambda M^2 \right)^2 \sum_{j=1}^{T} L_j + 2 \sum_{j=0}^{T} K_j \sum_{j=1}^{T} L_j + 2 \sum_{j=0}^{T} K_j \right] \right) \quad (4.11)
\]

Lemma 4.1 follows (4.4), (4.5), (4.10) and (4.11).

Finally, we give the proof of \( \limsup_{n \to +\infty} d\lambda_c(d) \leq \frac{1}{E\rho^2} \).

**Proof of \( \limsup_{n \to +\infty} d\lambda_c(d) \leq \frac{1}{E\rho^2} \).** Let

\[ \tau = \inf \{ n > 0 : S_n = \hat{S}_n \} \]

Then according to (2.9) of [3],

\[ P(2 \leq \tau < +\infty) \leq \frac{C_1}{d^2} \]

where \( C_1 \) does not depend on \( d \). Therefore, according to strong Markov property,

\[ P(T = m, K_i = k_i \text{ for } 0 \leq i \leq m, L_i = l_i \text{ for } 1 \leq i \leq m) \]

\[ \leq \left( \frac{C_1}{d^2} \right)^m (k_0 + \cdots + k_{m-1}) \left( \frac{1}{d} \right)^{m-1} \sum_{i=1}^{m} l_{i-1} \quad (4.12) \]

for all possible \( m, k_i, l_i \). Please note that \( k_0 \) may take 0 but \( l_i \geq 1 \) and \( k_i \geq 1 \) for \( 1 \leq i \leq m \).

Let

\[ \lambda = \frac{\gamma}{dE\rho^2} \]
for fixed $\gamma > 1$, then by (4.12),

$$
\tilde{E} \left[ \frac{T+\sum_{j=0}^{T} K_j}{M} \frac{6T+4}{\lambda^T} \frac{\sum_{j=0}^{T} K_j}{(E\rho^2)^{T+1}} \left( 1 + \lambda M^2 \right) \frac{2}{\lambda^2} \frac{T+2}{\lambda^2} \frac{\sum_{j=0}^{T} K_j}{(E\rho^2)^{T+2}} \left( 1 + \lambda M^2 \right) \frac{2}{\lambda^2} \frac{T+2}{\lambda^2} \frac{\sum_{j=0}^{T} K_j}{(E\rho^2)^{T+2}} \left( 1 + \lambda M^2 \right) \frac{2}{\lambda^2} \frac{T+2}{\lambda^2} \frac{\sum_{j=0}^{T} K_j}{(E\rho^2)^{T+2}} \right] 
$$

$$
\leq \sum_{m=0}^{+\infty} \sum_{k_0=0}^{+\infty} \cdots \sum_{k_m=1}^{+\infty} \sum_{l_1=1}^{+\infty} \sum_{l_m=1}^{+\infty} \sum_{l_m=1}^{+\infty} \frac{(C_1^2)}{d^2} \left( \sum_{i=0}^{m} \frac{k_i + m - 1}{d} \right) \sum_{i=1}^{l_i - m} 
$$

$$
\times \sum_{m=0}^{2m+\sum_{j=0}^{m} k_j} \sum_{M=0}^{6m+4} \sum_{k_j=0}^{+\infty} \sum_{l_j=1}^{+\infty} \sum_{l_j=1}^{+\infty} \frac{(1 + \lambda M^2)}{(E\rho^2)^{l_j}} \left[ \sum_{k_0=0}^{+\infty} \sum_{l_0=0}^{+\infty} \sum_{l_0=0}^{+\infty} \frac{(1 + \lambda M^2)^{2}}{(d\lambda E\rho^2)^{2}} \right] m \frac{d^2}{C_1}. 
$$

Since $\lambda = \frac{2}{dE\rho^2}$ for $\gamma > 1$,

$$
\frac{2C_1 M^6 \lambda}{d(E\rho^2)^2} \leq \frac{C_2}{d^2} 
$$

and

$$
\frac{2C_1 M^4 (1 + \lambda M^2)^2}{d^2 (E\rho^2)^2} \leq \frac{C_3}{d^2} 
$$

for sufficiently large $d$, where $C_2$ and $C_3$ does not depend on $d$ (but may depend on $\gamma$ and $\rho$).

We choose $\hat{\gamma}$ such that $1 < \hat{\gamma} < \gamma$, then for sufficiently large $d$,

$$
\frac{(1 + \lambda M^2)^2}{d\lambda E\rho^2} = \frac{(1 + \frac{\lambda M^2}{dE\rho^2})^2}{\gamma} < \frac{1}{\hat{\gamma}}. 
$$

Then, by (4.13),

$$
\tilde{E} \left[ \frac{T+\sum_{j=0}^{T} K_j}{M} \frac{6T+4}{\lambda^T} \frac{\sum_{j=0}^{T} K_j}{(E\rho^2)^{T+1}} \left( 1 + \lambda M^2 \right) \frac{2}{\lambda^2} \frac{T+2}{\lambda^2} \frac{\sum_{j=0}^{T} K_j}{(E\rho^2)^{T+2}} \left( 1 + \lambda M^2 \right) \frac{2}{\lambda^2} \frac{T+2}{\lambda^2} \frac{\sum_{j=0}^{T} K_j}{(E\rho^2)^{T+2}} \right] 
$$

$$
\leq \frac{d^2}{C_1} \sum_{m=0}^{+\infty} \sum_{k_0=0}^{+\infty} \frac{C_2}{d^2} \left[ \sum_{l=1}^{+\infty} \frac{C_3}{d^2} \right] \left[ \sum_{l=1}^{+\infty} \frac{1}{\hat{\gamma}} \right]^m. 
$$
For sufficiently large $d$,
\[ \sum_{l=1}^{+\infty} \frac{C_3 d^l}{d^2} = \frac{C_3}{d^2 - C_3} \leq \frac{C_4}{d^2} \]
and
\[ \sum_{k_0=0}^{+\infty} \left( \frac{C_3 d^2}{d^2 - C_3} \right)^{k_0} = \frac{d^2}{d^2 - C_3} \leq 2, \]
where $C_4$ does not depend on $d$.

Therefore,
\[ \tilde{E} \left[ \frac{2^{T+\sum_{j=0}^{T} K_j} M^{6T+4 \sum_{j=0}^{T} K_j}}{\lambda^{\sum_{j=0}^{T} L_j - T} \left( E\rho^2 \right)^{\sum_{j=0}^{T} L_j + 2T + 2 \sum_{j=0}^{T} K_j}} \left( 1 + \lambda M^2 \right)^{2 \sum_{j=1}^{T} L_j + 2 \sum_{j=0}^{T} K_j} \right] \]
\[ \leq \frac{2d^2}{C_1} \sum_{m=0}^{+\infty} \left( \frac{C_2 d^2}{d^2} \right)^m \left( \frac{C_3 d^2}{d^2} \right)^m \left[ \frac{1}{\gamma - 1} \right]^m \]
\[ = \frac{2d^2}{C_1} \sum_{m=0}^{+\infty} \left[ \frac{C_2 C_4}{d^2 (\gamma - 1)} \right]^m. \]
(4.14)

For sufficiently large $d$, $\frac{C_2 C_4}{d^2 (\gamma - 1)} < 1$ and therefore
\[ \tilde{E} \left[ \frac{2^{T+\sum_{j=0}^{T} K_j} M^{6T+4 \sum_{j=0}^{T} K_j}}{\lambda^{\sum_{j=0}^{T} L_j - T} \left( E\rho^2 \right)^{\sum_{j=0}^{T} L_j + 2T + 2 \sum_{j=0}^{T} K_j}} \left( 1 + \lambda M^2 \right)^{2 \sum_{j=1}^{T} L_j + 2 \sum_{j=0}^{T} K_j} \right] < +\infty \]
(4.15)
when $\lambda = \frac{\gamma}{E\rho^2}$.

Then according to Lemma 4.1
\[ \lambda_c(d) \leq \frac{\gamma}{E\rho^2} \]
for sufficiently large $d$ and hence
\[ \limsup_{d \to +\infty} d\lambda_c(d) \leq \frac{\gamma}{E\rho^2} \]
for any $\gamma > 1$.

Let $\gamma$ decrease to 1, then we accomplish the proof.

Since we have shown that $\liminf_{d \to +\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}$ in Section 3, the whole proof of Theorem 2.1 is completed.

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