Stable Recovery of Sparse Signals via \(l_p\)-Minimization

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Abstract

In this paper, we show that, under the assumption that \(\|e\|_2 \leq \epsilon\), every \(k\)-sparse signal \(x \in \mathbb{R}^n\) can be stably (\(\epsilon \neq 0\)) or exactly recovered (\(\epsilon = 0\)) from \(y = Ax + e\) via \(l_p\)-minimization with \(p \in (0, \bar{p}]\), where

\[
\bar{p} = \begin{cases} 
\frac{50}{71}(1 - \delta_{2k}), & \delta_{2k} \in [\sqrt{\frac{2}{71}}, 0.7183] \\
0.4541, & \delta_{2k} \in [0.7183, 0.7729) , \\
(2 - \delta_{2k}), & \delta_{2k} \in [0.7729, 1] 
\end{cases}
\]

even if the restricted isometry constant of \(A\) satisfies \(\delta_{2k} \in [\sqrt{\frac{2}{71}}, 1]\). Furthermore, under the assumption that \(n \leq 4k\), we show that the range of \(p\) can be further improved to \(p \in (0, \frac{3 + 2 \sqrt{2}}{2k} - 1)\). This not only extends some discussions of only the noiseless recovery (Lai et al. and Wu et al.) to the noise recovery, but also greatly improves the best existing results where \(p \in (0, \min\{1, 0.873(1 - \delta_{2k})\})\) (Wu et al.).

Index Terms

Compressed Sensing, restricted isometry constant, \(l_p\)-minimization, sparse signal recovery.

I. INTRODUCTION

In compressed sensing, see, e.g., [1], [2], [3], the following linear model is observed:

\[
y = Ax + e
\]

where \(x \in \mathbb{R}^n\) is an unknown signal, \(y \in \mathbb{R}^m\) is an observation vector, \(A \in \mathbb{R}^{m \times n}\) (with \(m << n\)) is a known sensing matrix and \(e \in \mathbb{R}^m\) is the measurement error vector. For simplicity, in this paper, we only consider \(l_2\) bounded noise, i.e., \(\|e\|_2 \leq \epsilon\) for some \(\epsilon\), see, e.g., [4], [5], [6]. If there is no noise, we take \(\epsilon = 0\).

One of the central goals of compressed sensing is to recover \(x\) based on \(A\) and \(y\). It has been shown that under some suitable conditions, \(x\) can be stably or exactly recovered, see, e.g., [7], [8].

A common method to recover \(x\) from (1) is to solve the following \(l_1\)-minimization problem:

\[
\min_{\gamma \in \mathbb{R}^n} \|\gamma\|_1 : \text{subject to } \|y - A\gamma\|_2 \leq \epsilon.
\]

One of the commonly used frameworks for sparse recovery is the restricted isometry property (RIP) which was introduced in [1]. A vector \(x \in \mathbb{R}^n\) is \(k\)-sparse if \(|\text{supp}(x)| \leq k\), where \(\text{supp}(x) = \{i : x_i \neq 0\}\) is the support of \(x\). For any \(m \times n\) matrix \(A\) and any integer \(k, 1 \leq k \leq n\), the \(k\)-restricted isometry constant (RIC) \(\delta_k\) is defined as the smallest constant such that

\[
(1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2
\]

for all \(k\)-sparse vector \(x\). If \(k + k' \leq n\), then the \(k, k'\)-restricted orthogonality constant (ROC) \(\theta_{k, k'}\) is defined as the smallest constant such that

\[
|\langle Ax, Ax' \rangle| \leq \theta_{k, k'}\|x\|_2\|x'\|_2
\]

for all \(x\) and \(x'\), where \(x\) and \(x'\) are respectively \(k\)-sparse and \(k'\)-sparse and have disjoint supports.

A variety of sufficient conditions based on RIC and ROC for the stable recovery (\(\epsilon \neq 0\)) or exact recovery (\(\epsilon = 0\)) of \(k\)-sparse signal \(x\) have been introduced in the literature. For example, \(\delta_{2k} + \theta_{k, k'} + \theta_{k, 2k} < 1\) in [1] and \(\delta_{2k} + \theta_{k, 2k} < 1\) in [9]. Sufficient conditions based on only RIC have also been given. For example, \(\delta_{2k} + 3\delta_{4k} < 2\) and \(\delta_k < \frac{1}{4}\) were
develop some new techniques to prove the main results. Finally we summarize this paper in section IV.

\[ \min_{\gamma \in \mathbb{R}^n} \|\gamma\|_p : \text{ subject to } \|y - A\gamma\|_2 \leq \epsilon. \]  

(4)

Although the \( l_p \)-minimization problem is more difficult to solve than the \( l_1 \)-minimization problem due to its non-convexity and non-smoothness \[15\], there are some efficient algorithms to solve \[4\], see, e.g., \[11\] and \[15\].

The \( l_p \)-minimization requires weaker condition on \( \delta_{2k} \) than that of the \( l_1 \)-minimization. It was shown in \[19\] that for any \( \delta_{2k+1} \in (0, 1) \), there is some \( p \) such that one can exactly recover the \( k \)-sparse signal \( x \) via solving \( \frac{\|x\|_p}{\|x\|_2} \) with \( \epsilon = 0 \). In \[15\], Sun showed that for any \( \delta_{2k} \in (0, 1) \), one can stably recover (\( \epsilon \neq 0 \)) or exactly recover (\( \epsilon = 0 \)) the \( k \)-sparse signal \( x \) via solving \( \frac{\|x\|_p}{\|x\|_2} \), where \( p \) is about 0.6797(1 - \( \delta_{2k} \)). For the noiseless recovery, the range of \( p \) has been improved to \( p < \min\{1, 1.0873(1 - \delta_{2k})\} \) in \[17\].

As far as we know, \( p < \min\{1, 1.0873(1 - \delta_{2k})\} \) is the best existing results. Therefore, a natural question is to ask whether this condition can be further improved. If so, can the improved condition be extended to the noise recovery?

The answers are affirmative. If \( \delta_{2k} < \frac{3^2}{2} \), then one can choose \( p = 1 \) \[13\]. Therefore, we only need to improve the range of \( p \) for each given \( \delta_{2k} \in \left[ \frac{3^2}{2}, 1 \right) \) for general \( k \), one can stably recover (\( \epsilon \neq 0 \)) or exactly recover (\( \epsilon = 0 \)) the \( k \)-sparse signal \( x \) via solving \( \frac{\|x\|_p}{\|x\|_2} \) with \( p \in (0, \bar{p}] \), where \( \bar{p} \) is defined in \[15\]. Under the assumption that \( k \geq \frac{\sqrt{3^2}}{\sqrt{2}} \), we will show that the range of \( p \) can be further improved to \( p \in (0, \frac{3^2 + 2\sqrt{3^2}}{2}(1 - \delta_{2k})) \). This will not only extend some discussions of only the noiseless recovery \[10\], \[17\] to the noise recovery, but will also greatly improve the best existing results where \( p < \min\{1, 1.0873(1 - \delta_{2k})\} \) in \[17\].

The rest of the paper is organized as follows. In section II, we will give our main results. In section III, we will develop some new techniques to prove the main results. Finally we summarize this paper in section IV.

II. MAIN RESULTS

A. Preliminaries

Suppose \( x \) in \[1\] is the real signal which we need to recover and \( x^* \) is the solution of the \( l_p \) minimization problem \[4\]. Like in \[3\], we set \( h = x - x^* \) and denote its \( i \)-th (\( 1 \leq i \leq n \)) component by \( h_i \). Similar to the notation used in \[17\], we respectively assume \( T_b \) be the set \( \{1, 2, \ldots, k\} \), \( T_{0}^c \) be the set \( \{k+1, k+2, \ldots, n\} \) and \( x_{T_0^c} \) be the vector equal to \( x \) on the index set \( T_0^c \) and zero elsewhere. As assumed in \[8\] and \[17\], for simplicity, we assume that \( h_{T_0^c} \) is already sorted in non-increasing order of magnitude, i.e., \( |h_{k+1}| \geq |h_{k+2}| \geq \ldots \geq |h_n| \). We also assume that \( n = (l+1)k \) with \( l \) being a positive integer. Partition the index set \( T_0^c \) as the union of the subsets \( T_i = \{ik+1, ik+2, \ldots, (i+1)k\} \) with \( i \in \{1, 2, \ldots, l\} \).

Let

\[ f(p) = \frac{p}{2} \left( \frac{1}{2} - p \right)^{\frac{1}{2} - \frac{1}{p}}, \quad p \in (0, 1], \]  

(5)

\[ g(p) = \frac{p}{2} \left( 1 - \frac{p}{2} \right)^{\frac{1}{2} - 1}, \quad p \in (0, 1]. \]  

(6)

By some simple calculations, we have

\[ (\ln f(p))' = -\frac{1}{p^2} \ln(2 - p) \leq 0, \quad (\ln g(p))' = -\frac{2}{p^3} \ln(1 - \frac{p}{2}) > 0, \]

\[ f(1) = \sqrt{2}/2, \quad \lim_{p \to -0} f(p) = +\infty, \quad g(1) = \frac{1}{4}, \quad \lim_{p \to -0} g(p) = 0. \]

Therefore,

\[ g(p) \in (0, \frac{1}{4}], \quad \forall p \in (0, 1), \]  

(7)

and \( f(p) = 1 \) has a unique solution. Let \( p^* \) be the unique solution of \( f(p) = 1 \), then

\[ p^* \approx 0.45418, \]  

(8)

and

\[ \begin{cases} f(p) \in [1, +\infty), & p \in (0, p^*] \\ f(p) \in [\sqrt{2}/2, 1), & p \in (p^*, 1], \end{cases} \]  

(9)
By (5) and the aforementioned equation, we have
\[ p^{\frac{1}{2}}(2 - p)^{\frac{1}{2}} > 1, \text{ if } p \in (p^*, 1). \] (10)

In the following, we will give our main results. Like in [8], we divide them into two cases: general case and special case \((n \leq 4k)\).

\textbf{B. General Case}

Let
\[ C(p) = \begin{cases} \frac{\left((2 - \delta_{2k})^{1 - \frac{p}{2} + 2\delta_{2k}}g(p)\right)p/2}{1 - \delta_{2k}}, & p \in (0, p^*) \\ \frac{\left(2 - \delta_{2k}\right)^{\frac{p}{2}}g(p) + 2^{2 - \frac{p}{2}}\delta_{2k}}{1 - \delta_{2k}}, & p \in (p^*, 1) \end{cases} \] (11)

where \(g(p)\) is defined as in (5). Then we have the following result whose proof will be provided in Section III-A.

\textbf{Theorem 1:} Suppose that \(A\) and \(e\) in (1) respectively satisfy the RIP with given \(\delta_{2k} \in \left[\frac{2}{\sqrt{p}}, 1\right]\) and \(\|e\|_2 \leq \epsilon\), then for each \(p \in (0, 1)\) such that
\[ C(p) < 1, \] (12)

the solution \(x^*\) to the \(l_p\)–minimization problem (4) obeys
\[ \|x - x^*\|_p^p \leq C_0\|x_{R^*}\|_p^p + C_1k^{1 - \frac{p}{2}}\epsilon^p, \] (13)

where
\[ C_0 = \frac{2(1 + C(p))}{1 - C(p)}, \quad C_1 = \frac{2^{\frac{4p}{2} + 1}}{(1 - \delta_{2k})^\frac{p}{2}(1 - C(p))} \]

with \(C(p)\) defined as in (11). In particular, if \(\epsilon = 0\) and \(x\) is \(k\)–sparse, then the recovery is exact.

For a given \(\delta_{2k} \in \left[\frac{2}{\sqrt{p}}, 1\right]\), it may be complicate to determine the range of \(p\) such that (12) holds, so in the following, we would like to give a simple rule to determine it. To do this, we need to introduce the following lemma.

For each \(p \in (0, 1)\), let
\[ h(p) = \begin{cases} -0.5p + 1, & p \in (0, p^*) \\ -0.62p + 1, & p \in (p^*, 1) \end{cases} \] (14)

then we have the following result whose proof will be provided in Appendix A.

\textbf{Lemma 1:} For each given \(\delta_{2k} \in \left[\frac{2}{\sqrt{p}}, 1\right]\), if \(\delta_{2k} \leq h(p)\), then (12) holds.

By (8) and (14), we have
\[ \min_{p \in (0, p^*)} h(p) \geq 1 - 0.5p^* > 1 - 0.5 \times 0.4542 = 0.7729; \quad \max_{p \in (p^*, 1)} h(p) \geq 1 - 0.62 \times 0.4542 > 0.7183. \]

Therefore, for each given \(\delta_{2k} \in \left[\frac{2}{\sqrt{p}}, 1\right]\), if \(\delta_{2k} \in \left[\frac{2}{\sqrt{p}}, 0.7183\right]\), then \(\delta_{2k} \leq h(p)\) holds for each \(p \in (0, \frac{50}{48}(1 - \delta_{2k})]\); if \(\delta_{2k} \in [0.7183, 0.7729]\), then \(\delta_{2k} \leq h(p)\) holds for each \(p \in (0, p^*]\); and if \(\delta_{2k} \in [0.7729, 1]\), then \(\delta_{2k} \leq h(p)\) holds for each \(p \in (0, 2(1 - \delta_{2k})]\).

For each \(\delta_{2k} \in \left[\frac{2}{\sqrt{p}}, 1\right]\), let
\[ \bar{p} = \begin{cases} \frac{50}{48}(1 - \delta_{2k}), & \delta_{2k} \in \left[\frac{2}{\sqrt{p}}, 0.7183\right) \\ p^*, & \delta_{2k} \in [0.7183, 0.7729) \\ 2(1 - \delta_{2k}), & \delta_{2k} \in [0.7729, 1) \end{cases} \] (15)

then by the aforementioned analysis, \(\delta_{2k} \leq h(p)\) holds for each \(p \in (0, \bar{p})\). Therefore, by Theorem 1 and Lemma 1, we immediately have the following result.

\textbf{Corollary 1:} Suppose that \(A\) and \(e\) in (1) respectively satisfy the RIP with given \(\delta_{2k} \in \left[\frac{2}{\sqrt{p}}, 1\right]\) and \(\|e\|_2 \leq \epsilon\), then for each \(p \in (0, \bar{p})\), where \(\bar{p}\) is defined as in (15), the solution \(x^*\) to the \(l_p\)–minimization problem (4) obeys (13). In particular, if \(\epsilon = 0\) and \(x\) is \(k\)–sparse, then the recovery is exact.

Theorem 1 and Corollary 1 give the bound on the \(p\)–norm of the error. Like in [6], we also want to bound the \(2\)–norm of the error. Let
\[ D(p) = \begin{cases} \frac{(2 + \delta_{2k})g(p)}{1 - \delta_{2k}}p/2, & p \in (0, p^*) \\ \frac{(2 - \delta_{2k})g(p) + 2^{2 - p/2}\delta_{2k}}{1 - \delta_{2k}}p/2, & p \in (p^*, 1) \end{cases} \] (16)
where \( g(p) \) is defined as in (6). Then we have the following result whose proof will be provided in Section III-A.

Theorem 2: Suppose that \( A \) and \( e \) in (1) respectively satisfy the RIP with given \( \delta_{2k} \in \left[ \sqrt{2} \right], 1 \) and \( \|e\|_2 \leq \epsilon \), then for each \( p \in (0, 1) \) such that (12) holds, the solution \( x^* \) to the \( l_p \)-minimization problem (4) obeys

\[
\|x - x^*\|_p \leq D_0 k^{2p-1} \|x_T^0\|_p + D_1 \epsilon^p,
\]

where

\[
D_0 = \frac{2D(p)}{1 - C(p)}, \quad D_1 = \frac{1}{(1 - \delta_{2k})^2} \left( 2p + \frac{2^p D(p)}{1 - C(p)} \right)
\]

with \( C(p) \) and \( D(p) \) defined as in (11) and (16), respectively. In particular, if \( \epsilon = 0 \) and \( x \) is \( k \)-sparse, then the recovery is exact.

By Theorem 2 and Lemma 1 we have the following result.

Corollary 2: Suppose that \( A \) and \( e \) in (1) respectively satisfy the RIP with given \( \delta_{2k} \in \left[ \sqrt{2} \right], 1 \) and \( \|e\|_2 \leq \epsilon \), then for each \( p \in (0, \bar{p}] \), where \( \bar{p} \) is defined as in (15), the solution \( x^* \) to the \( l_p \)-minimization problem (4) obeys (17). In particular, if \( \epsilon = 0 \) and \( x \) is \( k \)-sparse, then the recovery is exact.

C. Special Case: \( n \leq 4k \)

In the previous subsection, we have obtained some sufficient conditions to guarantee the stable recovery or exactly recovery of the \( k \)-sparse signal \( x \) from (1) via solving (4). In the following, we will show that these conditions can be further improved under the assumption that \( n \leq 4k \). Like in [8], for simplicity, we assume that \( l = 3 \) (i.e., \( n = 4k \)) throughout this case.

Let

\[
\bar{C}(p) = (1 + \delta_{2k}) 2^{p-1} \left( \frac{g(p)}{1 - \delta_{2k}} \right)^{p/2},
\]

where \( g(p) \) is defined as in (6). Then we have the following result whose proof will be provided in Section III-B.

Theorem 3: Suppose that \( A \) and \( e \) in (1) respectively satisfy the RIP with given \( \delta_{2k} \in \left[ \sqrt{2} \right], 1 \) and \( \|e\|_2 \leq \epsilon \), then for each \( p \in (0, 1) \) such that

\[
\bar{C}(p) < 1,
\]

the solution \( x^* \) to the \( l_p \)-minimization problem (4) obeys

\[
\|x - x^*\|_p \leq C_0 \|x_T^0\|_p + C_1 k^{2p-1} \epsilon^p,
\]

where

\[
C_0 = \frac{2(1 + \bar{C}(p))}{1 - C(p)}, \quad C_1 = \frac{2^{p+2}}{(1 - \delta_{2k})^2 (1 - C(p))}
\]

with \( \bar{C}(p) \) defined as in (18). In particular, if \( \epsilon = 0 \) and \( x \) is \( k \)-sparse, then the recovery is exact.

Like in the previous subsection, for a given \( \delta_{2k} \in \left[ \sqrt{2} \right], 1 \), it may be complicate to determine the range of \( p \) such that (19) holds. So in the following, we want to give a simple method to determine it. But we first need to give the following lemma whose proof will be provided in Appendix B.

Lemma 2: For each given \( \delta_{2k} \in \left[ \sqrt{2} \right], 1 \), for each \( p \in (0, 1) \), if

\[
\delta_{2k} \leq -(6 - 4\sqrt{2})p + 1,
\]

then (19) holds.

By Lemma 2 for any given \( \delta_{2k} \in \left[ \sqrt{2} \right], 1 \), (19) hold for all \( p \in \left( \frac{1}{6 - 4\sqrt{2}}, 1 - \delta_{2k} \right) \). Therefore, by Theorem 3 we immediately have the following corollary.

Corollary 3: Suppose that \( A \) and \( e \) in (1) respectively satisfy the RIP with given \( \delta_{2k} \in \left[ \sqrt{2} \right], 1 \) and \( \|e\|_2 \leq \epsilon \), then for each \( p \in \left( \frac{3 + 2\sqrt{2}}{6 - 4\sqrt{2}}, 1 - \delta_{2k} \right) \), the solution \( x^* \) to the \( l_p \)-minimization problem (4) obeys (20). In particular, if \( \epsilon = 0 \) and \( x \) is \( k \)-sparse, then the recovery is exact.

Like in the previous subsection, we also want to bound the \( 2 \)-norm of the error. Let

\[
\bar{D}(p) = \left( \frac{2g(p)}{1 - \delta_{2k}} \right)^{p/2},
\]

(21)
where \( g(p) \) is defined as in (5). Then similarly, we have the following Theorem whose proof will be provided in Section III-B.

**Theorem 4:** Suppose that \( A \) and \( e \) in (1) respectively satisfy the RIP with given \( \delta_{2k} \in [\frac{\sqrt{p}}{2}, 1] \) and \( \|e\|_2 \leq \epsilon \), then for each \( p \in (0, 1) \) such that (19) holds, the solution \( x^* \) to the \( l_p \)-minimization problem (4) obeys

\[
\|x - x^*\|_2^p \leq \frac{2\bar{D}(p)}{1 - C(p)}\|x_{T_0}\|_p^p + D_1 \epsilon^p,
\]

where

\[
\bar{D}_0 = \frac{2\bar{D}(p)}{1 - C(p)}, \quad \bar{D}_1 = \frac{2\bar{D}(p)}{1 - \delta_{2k}}\left(1 + \frac{2\bar{D}(p)}{1 - C(p)}\right)
\]

with \( C(p) \) and \( \bar{D}(p) \) defined as in (18) and (21), respectively. In particular, if \( \epsilon = 0 \) and \( x \) is \( k \)-sparse, then the recovery is exact.

By Theorem 4 and Lemma 2, we immediately have the following corollary.

**Corollary 4:** Suppose that \( A \) and \( e \) in (1) respectively satisfy the RIP with given \( \delta_{2k} \in [\frac{\sqrt{p}}{2}, 1] \) and \( \|e\|_2 \leq \epsilon \), then for each \( p \in (0, \frac{3+2\sqrt{p}}{2}(1 - \delta_{2k})] \), the solution \( x^* \) to the \( l_p \)-minimization problem (4) obeys (22). In particular, if \( \epsilon = 0 \) and \( x \) is \( k \)-sparse, then the recovery is exact.

### III. PROOFS

In this section, we will prove our main results. From now on, we always assume that

\[
\|h_{T_i}\|_p^p = t\|h_{T_0}\|_p^p
\]

for some \( t \in (0, 1] \).

To prove our theorems, we need to use Lemmas 3-8 which were given in [8, 15, 16] respectively.

**Lemma 3:** For \( \forall \delta_{2k} \in (0, 1) \), it holds that \( \|\sum_{i=2}^l Ah_{T_i}\|_2^2 \leq \frac{1}{1 - \delta_{2k}}(2\epsilon + \|\sum_{i=2}^l Ah_{T_i}\|_2^2) \).

**Lemma 4:** For \( \forall p \in (0, 1) \), it holds that \( \|h_{T_0}\|_p^p \leq \|h_{T_0}\|_p^p + 2\|\sum_{i=2}^l Ah_{T_i}\|_p^p \).

**Lemma 5:** For \( \forall p \in (0, 1) \), it holds that \( \sum_{i=2}^l \|Ah_{T_i}\|_2^2 \leq (1 - t)(t^{\frac{1}{p} - 1} + 1 - t)k^{1 - \frac{1}{p}}\|h_{T_0}\|_p^p \).

**Lemma 6:** Let \( u \) be a \( k \) dimensional column vector, then for each \( p \in (0, 1) \), we have \( \|u\|_2 \geq \frac{k^{1 - \frac{1}{p}}}{\sqrt{p}} \|u\|_p \).

**A. Proof of Theorem 1 and Theorem 2**

Before processing to prove Theorem 1 and Theorem 2 we need to introduce the following lemmas, where the proof of Lemma 2 is provided in section Appendix C.

**Lemma 7:** For \( \forall p \in (0, 1) \), it holds that \( \|\sum_{i=2}^l Ah_{T_i}\|_2^2 \leq C_1(t, p)k^{1 - \frac{1}{p}}\|h_{T_0}\|_p^p \), where

\[
C_1(t, p) = \begin{cases} 
(t - 1)t^{\frac{1}{p} - 1} + 2g(p)\epsilon^{2\delta_{2k}}, & p \in (0, p^*) \\
(t - 1)t^{\frac{1}{p} - 1} + 2^{2 - \frac{1}{p}}\epsilon^{2\delta_{2k}}, & p \in (p^*, 1) 
\end{cases}
\]

with \( g(p) \) defined as in (6).

**Remark 1:** The upper bound on \( \|\sum_{i=2}^l Ah_{T_i}\|_2^2 \) given by Lemma 7 is sharper than that of the Lemma 5 in [17], where the bound is \( (1 - t)t^{\frac{1}{p} - 1} + 2\delta_{2k}(1 - \frac{\sqrt{p}}{2})^{\frac{1}{p} - 1}k^{1 - \frac{1}{p}}\|h_{T_0}\|_p^p \). In fact, to show this, it suffices to show the following inequality:

\[
\begin{align*}
\begin{cases}
g(p) \leq p(1 - \frac{\sqrt{p}}{2})^{\frac{1}{p} - 1}, & p \in (0, p^*) \\
2^{2 - \frac{1}{p}} \leq p(1 - \frac{\sqrt{p}}{2})^{\frac{1}{p} - 1}, & p \in (p^*, 1).
\end{cases}
\end{align*}
\]

It is not hard to check that the aforementioned inequality follows from (6) and (10).

**Lemma 8:** For \( \forall \delta_{2k} \in [\frac{\sqrt{p}}{2}, 1) \), for each \( p \in (0, 1) \), it holds that

\[
\|h_{T_0}\|_p^p \leq \frac{2\bar{D}(p)}{1 - \delta_{2k}}k^{1 - \frac{1}{p}}\epsilon^p + C(p)\|h_{T_0}\|_p^p,
\]

where \( C(p) \) is defined as in (11).

**Proof.** By Lemma 3 and Lemma 7:

\[
(1 - \delta_{2k})\|h_{T_0}\|_2^2 \leq 4\epsilon^2 + 4\epsilon\|\sum_{i=2}^l Ah_{T_i}\|_2 + \|\sum_{i=2}^l Ah_{T_i}\|_2^2 - (1 - \delta_{2k})\|h_{T_0}\|_2^2,
\]

\[
k^{\frac{1}{p} - 1}\|\sum_{i=2}^l Ah_{T_i}\|_2^2 \leq C_1(t, p)\|h_{T_0}\|_p^p.
\]
where $C_1(t, p)$ is defined as in (24). By (23) and Lemma 6

$$k \frac{2}{\delta_1} - 1 \|h_{T_1}\|_2^2 \geq t \frac{2}{\delta_2} \|h_{T_0}\|_p^2.$$  

By (24), (26) and the aforementioned inequality, we have

$$k \frac{2}{\delta_1} - 1 (\sum_{i=2}^l A_h T_i) \leq (1 - \delta_2 k) \|h_{T_1}\|_2^2 \leq C_2(t, p) \|h_{T_0}\|_p^2,$$  

where

$$C_2(t, p) = \begin{cases} (\delta_2 k - 2) t \frac{2}{\delta_1} + t \frac{2}{\delta_2} - 1 + 2 g(p) \delta_2, & p \in (0, p^*], \\ (\delta_2 k - 2) t \frac{2}{\delta_1} + t \frac{2}{\delta_2} - 1 + 2 k \frac{2}{\delta_2} \delta_2, & p \in (p^*, 1). \end{cases}$$

By (6) and (24), one can easily show that

$$C_1(t, p) \leq C_1(1 - \frac{p}{2}, p) = \begin{cases} 1 + 2 \delta_2 k g(p), & p \in (0, p^*], \\ g(p) + 2 k \frac{2}{\delta_2} \delta_2, & p \in (p^*, 1). \end{cases}$$

$$C_2(t, p) \leq C_2(\frac{2 - p}{2(2 - \delta_2)}, p) = \begin{cases} (2 - \delta_2 k) \frac{1}{\delta_1} + 2 g(p) \delta_2, & p \in (0, p^*], \\ (2 - \delta_2 k) \frac{1}{\delta_1} + 2 k \frac{2}{\delta_2} \delta_2, & p \in (p^*, 1). \end{cases}$$

By (7) and (8), for each $\delta_2 k \in [\frac{2^2}{\delta_2}, 1)$ and $p \in (0, 1)$, we have $g(p) \leq 2 \delta_2 k g(p)$ and for each $p \in (p^*, 1)$, we have $g(p) \leq 2 k \frac{2}{\delta_2} \delta_2$. Therefore, by (28) and (29), we have

$$C_1(1 - \frac{p}{2}, p) \leq 2 C_2(\frac{2 - p}{2(2 - \delta_2)}, p).$$

By the aforementioned equation, (26) and (27), we have

$$k \frac{2}{\delta_1} - 1 (\sum_{i=2}^l A_h T_i) \leq \sqrt{2} \sqrt{C_2(\frac{2 - p}{2(2 - \delta_2)}, p) \|h_{T_0}\|_p^2};$$

$$k \frac{2}{\delta_1} - 1 (\sum_{i=2}^l A_h T_i) \leq (1 - \delta_2 k) \|h_{T_1}\|_2^2 \leq C_2(\frac{2 - p}{2(2 - \delta_2)}, p) \|h_{T_0}\|_p^2.$$

By the aforementioned two inequalities and (25),

$$(1 - \delta_2 k) k \frac{2}{\delta_1} - 1 \|h_{T_0}\|_2^2 \leq (2 k \frac{2}{\delta_2} + 2) \|h_{T_0}\|_p^2.$$ 

By (11), (29) and applying Lemma 6 to the aforementioned inequality, we have,

$$\|h_{T_0}\|_p \leq \frac{2^2}{\sqrt{1 - \delta_2}} k \frac{2}{\delta_1} - 1 \|h_{T_0}\|_p + (C(p)) \|h_{T_0}\|_p^2.$$ 

The lemma follows from the aforementioned inequality and the fact that for each fixed $n \in N$, for each $\omega_j \geq 0, 1 \leq j \leq n$ and for each $p \in (0, 1)$, it holds that

$$\sum_{j=1}^n \omega_j \leq \left(\sum_{j=1}^n \omega_j\right) \frac{2^2}{\delta_1}.$$  

In fact, if $\left(\sum_{j=1}^n \omega_j\right)^\frac{2}{\delta_1} = 0$, then (30) obviously holds. Otherwise, we assume $u_j = \frac{\omega_j}{\left(\sum_{j=1}^n \omega_j\right)^\frac{2}{\delta_1}}$. Then $u_j \leq 1$ and

$$\sum_{j=1}^n u_j = 1.$$ 

Since $p \in (0, 1)$, we have $\sum_{j=1}^n u_j \leq \sum_{j=1}^n u_j^p = 1$.

**Proof of Theorem 1**. By Lemma 4, we have,

$$\|h\|_p = \|h_{T_0}\|_p + \|h_{T_0}\|_p^p \leq 2 \|h_{T_0}\|_p + 2 \|h_{T_0}\|_p^p.$$

If (12) holds, then by lemmas 4 and 8, we have

$$\|h_{T_0}\|_p \leq \frac{2 C(p)}{1 - C(p)} \|x_{T_0}\|_p + \frac{2^2}{1 - \delta_2} k \frac{2}{\delta_1} - 1 \|h_{T_0}\|_p^2.$$
The aforementioned two equations imply the theorem.

Proof of Theorem 2
By Lemma 3, we have
\[
\|h\|_2^2 = (\|h_{T_0}\|_2^2 + \|h_{T_1}\|_2^2) + \sum_{i=2}^{l} \|h_{T_i}\|_2^2 \leq \frac{1}{1 - \delta_{2k}} (2\epsilon + \sum_{i=2}^{l} \|Ah_{T_i}\|_2^2) + \sum_{i=2}^{l} \|h_{T_i}\|_2^2.
\]
Hence,
\[
\|h\|_2 \leq \frac{2\epsilon}{\sqrt{1 - \delta_{2k}}} + \sqrt{\frac{1}{1 - \delta_{2k}} \sum_{i=2}^{l} \|Ah_{T_i}\|_2^2 + \sum_{i=2}^{l} \|h_{T_i}\|_2^2}.
\]
(31)
Then by lemmas 5 and 7, we have
\[
\|h\|_2 \leq \frac{2\epsilon}{\sqrt{1 - \delta_{2k}}} + \sqrt{\frac{1}{1 - \delta_{2k}} \sum_{i=2}^{l} \|Ah_{T_i}\|_2^2 + \sum_{i=2}^{l} \|h_{T_i}\|_2^2}.
\]
where
\[
C_3(t, p) = \begin{cases} 
(2 - \delta_{2k})(1 - t) \frac{2}{\delta} - 1 + 2g(p)\delta_{2k}, & p \in (0, p) \\
(2 - \delta_{2k})(1 - t) \frac{2}{\delta} - 1 + 2^2 \frac{2}{\delta} \delta_{2k}, & p \in (p, 1) 
\end{cases}
\]
Therefore, by (30), we have,
\[
\|h\|_2^2 \leq \frac{2p}{(1 - \delta_{2k})^2} e^p + \left(\frac{C_3(t, p)}{1 - \delta_{2k}}\right)^2 \|h_{T_0}\|_p^p.
\]
By some simple calculations, for each \( t \in [0, 1] \), we have
\[
\left(\frac{C_3(t, p)}{1 - \delta_{2k}}\right)^2 \leq D(p),
\]
where \( D(p) \) is defined as in (16), so
\[
\|h\|_2^2 \leq \frac{2p}{(1 - \delta_{2k})^2} e^p + D(p)k^p \|h_{T_0}\|_p^p.
\]
If (12) holds, then by lemmas 4 and 8, we have
\[
\|h_{T_0}\|_p^p \leq \frac{2}{1 - C(p)} \|x_{T_0}\|_p^p + \frac{2p}{(1 - \delta_{2k})^2} \frac{k^{1 - \frac{p}{2}}}{1 - C(p)} e^p.
\]
The aforementioned two equations imply the theorem.

B. Proof of Theorem 3 and Theorem 4
In this subsection, we will follow the method used in [8] to prove Theorem 3 and Theorem 4. But before proving them, we need to introduce the following lemma.

Lemma 9: For each \( \delta_{2k} \in [\frac{\sqrt{2}}{2}, 1) \), for each \( p \in (0, 1) \),
\[
\|h_{T_0}\|_p^p \leq \frac{2p+1}{(1 - \delta_{2k})^2} \frac{k^{1 - \frac{p}{2}}}{1 - C(p)} e^p + \tilde{C}(p)\|h_{T_0}\|_p^p,
\]
where \( \tilde{C}(p) \) is defined as in (18).

Proof. By Lemma 5
\[
k_p^{1 - \frac{p}{2}} \left( \sum_{i=2}^{3} \|Ah_{T_i}\|_2^2 \right) \leq k_p^{1 - \frac{p}{2}} \left( \sum_{i=2}^{3} \|h_{T_i}\|_2^2 \right) \leq \sqrt{1 + \delta_{2k}} \sqrt{(1 - t)^{\frac{p}{2} - 1}} \|h_{T_0}\|_p \leq \sqrt{1 + \delta_{2k}} \sqrt{g(p)} \|h_{T_0}\|_p,
\]
(32)
By the aforementioned inequality, (25) and Lemma 6, we have
\[
\frac{1}{t} k^\frac{1}{p} - 1 \left( \sum_{i=2}^{3} A h_{T_i} \right)^2 \leq \left[ (1 + \delta_{2k})(1 - t) t^\frac{1}{p} - 1 - (1 - \delta_{2k}) t^\frac{1}{p} \right] \| h_{T_i} \|_p^2
\]
\[
= (1 + \delta_{2k} - 2t) t^\frac{1}{p} - 1 \| h_{T_i} \|_p^2 \leq (1 + \delta_{2k}) \frac{1}{2} \| h_{T_i} \|_p^2.
\]
Obviously, for each \( \delta_{2k} \in [\frac{\sqrt{2}}{2}, 1) \), for each \( p \in (0, 1) \),
\[
2^p \times 2^\frac{1}{p} (1 + \delta_{2k}) \geq \sqrt{1 + \delta_{2k}}.
\]
By the aforementioned inequalities, (25) and (32), we have
\[
(1 - \delta_{2k}^2) k^\frac{1}{p} - 1 \| h_{T_i} \|_p^2 \leq (2^p + 2^\frac{1}{p} \epsilon + (1 + \delta_{2k}) \frac{1}{2} \sqrt{g(p)} \| h_{T_i} \|_p^2).
\]
By (18) and applying Lemma 6 to the aforementioned inequality, we have,
\[
\| h_{T_i} \|_p \leq \frac{21 + \frac{1}{p}}{\sqrt{1 - \delta_{2k}}} k^\frac{1}{p} - 1 \epsilon + (\tilde{C}(p)) \| h_{T_i} \|_p.
\]
By (30) and the aforementioned inequality, the lemma holds. \( \square \)

**Proof of Theorem 3** By Lemma 4, we have
\[
\| h \|_p = \| h_{T_0} \|_p + \| h_{T_i} \|_p^2 \leq 2 \| h_{T_0} \|_p^2 + 2 \| x_{T_0} \|_p^2.
\]
If (19) holds, then by Lemma 4 and 3 we have
\[
\| h_{T_0} \|_p^2 \leq \frac{2\bar{C}(p)}{1 - \bar{C}(p)} \| x_{T_0} \|_p + \frac{2p + 1}{1 - \delta_{2k}} \| h_{T_0} \|_p^2.
\]
The aforementioned two equations imply the theorem.

**Proof of Theorem 4** By (31), (32) and Lemma 5, we have
\[
\| h \|_p \leq \frac{2\epsilon}{\sqrt{1 - \delta_{2k}}} + \frac{k^\frac{1}{p} - 1}{\sqrt{1 - \delta_{2k}}} \sqrt{2(1 - t) t^\frac{1}{p} - 1} \| h_{T_0} \|_p \leq \frac{2\epsilon}{\sqrt{1 - \delta_{2k}}} + \frac{k^\frac{1}{p} - 1}{\sqrt{1 - \delta_{2k}}} \sqrt{2g(p)} \| h_{T_0} \|_p.
\]
Therefore, by (21) and (30), we have
\[
\| h \|_p \leq \frac{2p + 1}{1 - \delta_{2k}} \| h_{T_0} \|_p + D(p) k^\frac{1}{p} - 1 \| h_{T_0} \|_p.
\]
If (19) holds, then by Lemma 4 and 3 we have
\[
\| h_{T_0} \|_p \leq \frac{2}{1 - \bar{C}(p)} \| x_{T_0} \|_p + \frac{2p + 1}{1 - \delta_{2k}} \| h_{T_0} \|_p^2.
\]
where \( \bar{C}(p) \) is defined as in (18). The aforementioned two equations imply the theorem.

**IV. SUMMARY AND FUTURE WORK**

In this paper, we showed that, under the assumption that \( \| e \|_2 \leq \epsilon \), every \( k \)-sparse signal \( x \in \mathbb{R}^n \) can be stably recovered \( (\epsilon = 0) \) or exactly recovered \( (\epsilon = 0) \) from (1) via \( p \)-minimization with \( p \in (0, \bar{p}) \), where \( \bar{p} \) is defined as in (15), even if \( \delta_{2k} \in [\frac{\sqrt{2}}{2}, 1) \). Furthermore, under the assumption that \( n \leq 4k \), we showed that the range of \( p \) can be further improved to \( p \in (0, \frac{1 + \sqrt{2}}{2(1 - \delta_{2k})} \] \( ) \). This not only extended some discussions of only the noiseless recovery [16], [17] to the noise recovery, but also greatly improves the best existing results where \( p < \min\{1, 1.0873(1 - \delta_{2k})\} \) [17].

In the future, we will discuss the largest possible ranges of \( p \) for a given \( \delta_{2k} \in [\frac{\sqrt{2}}{2}, 1) \) and how to chose \( p \) if \( \delta_{2k} \) is not given.

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APPENDIX A
PROOF OF LEMMA 1

Before proving Lemma 1, we need to give the following lemma.

Lemma 10: For each \( t \in (0, 1) \), let

\[
\phi_1(t) = \left(\frac{1 - t}{1 + t}\right)^{\frac{1}{t} - 1} \quad \text{and} \quad \phi_2(t) = (1 - t)^{\frac{1}{t}}.
\] (34)

Then \( \phi_1 \) and \( \phi_2 \) are respectively monotonically increasing and decreasing functions.

Proof. Since

\[
(\ln \phi_1(t))' = -\frac{2}{t(1 + t)} - \frac{1}{t^2} \ln\left(\frac{1 - t}{1 + t}\right) \quad \text{and} \quad (\ln \phi_2(t))' = -\frac{1}{t(1 - t)} - \frac{1}{t^2} \ln(1 - t),
\]

it is equivalent to show

\[
\ln\left(\frac{1 + t}{1 - t}\right) > \frac{2t}{1 + t} \quad \text{and} \quad -\ln(1 - t) < \frac{t}{1 - t}.
\]

One can easily show that the above inequalities hold for each \( t \in (0, 1) \). □

In the following, we will prove Lemma 1.

Proof of Lemma 1. Firstly, we prove the lemma holds for \( p \in (0, p^\star] \). Obviously, it suffices to show

\[
((2 - \delta_{2k})^{1 - \frac{1}{p}} + 2\delta_{2k})g(p) + \delta_{2k} < 1.
\]

By direct calculation and (7), for each fixed \( p \), the left hand side of the above inequality is a monotonically increasing function of \( \delta_{2k} \), so it suffices to show

\[
\left[(1 + \frac{p}{2}^{1 - \frac{1}{p}} + 2(1 - \frac{p}{2})\right]g(p) < \frac{p}{2}.
\]

By (8) and Lemma 10, it is equivalent to show

\[
\phi_1\left(\frac{p}{2}\right) + 2\phi_2\left(\frac{p}{2}\right) < 1.
\]

By (8) and Lemma 10 for each \( p \in (0, p^\star] \), we have

\[
\phi_1\left(\frac{p}{2}\right) + 2\phi_2\left(\frac{p}{2}\right) \leq \phi_1(0.25) + 2\lim_{p \to 0^+} \phi_2(p) < 1,
\]

so the lemma holds in this case.

Secondly, we prove the lemma holds for \( p \in (p^\star, 1) \). Similarly, it suffices to show

\[
(1 + 0.62p)^{1 - \frac{1}{p}}g(p) + 2^{2 - \frac{1}{p}}(1 - 0.62p) < 0.62p.
\]

By (8), we only need to show

\[
(1 + 0.62p)^{1 - \frac{1}{p}}\left(1 - \frac{p}{2}\right)^{\frac{1}{p} - 1} + 2^{3 - \frac{1}{p}}\left(\frac{1}{p} - 0.62\right) < 1.24.
\] (35)

By (34) and Lemma 10 it is easy to check that for each \( p \in (p^\star, 1) \), we have

\[
(1 + 0.62p)^{1 - \frac{1}{p}}\left(1 - \frac{p}{2}\right)^{\frac{1}{p} - 1} < \phi_1\left(\frac{1}{2}\right) = \frac{1}{3}.
\]

It is easy to verify that \( 2^{3 - \frac{1}{p}}\left(\frac{1}{p} - 0.62\right) \) achieves the maximal value at \( \frac{1}{p} = 0.62 + \frac{1}{2\ln(2)} \), therefore for each \( p \in (p^\star, 1) \), we have

\[
2^{3 - \frac{1}{p}}\left(\frac{1}{p} - 0.62\right) < 2^{1.76 - \frac{1}{2\ln(2)}} \cdot \frac{1}{2\ln(2)} < 0.8988.
\]

By the aforementioned two inequations, (35) holds and this finishes the proof. □
APPENDIX B

PROOF OF LEMMA 2

Before proving Lemma 2 we need to give the following lemma.

Lemma 11: For each $p \in (0,1]$, let

$$
\varphi(p) = (1 - (3 - 2\sqrt{2})p)\frac{2}{2} \cdot (1 - \frac{p}{2})^{\frac{1}{2}} - 1,
$$

then $\varphi(p)$ is a monotonically increasing function on $(0,1]$.

Proof: By some simple calculations, we have

$$
(\ln(\varphi(p)))' = -\frac{1}{p}[\frac{2(3 - 2\sqrt{2})}{1 - (3 - 2\sqrt{2})p}] + 1 - \frac{2}{p^2} \ln(1 - (3 - 2\sqrt{2})p) + \ln(1 - \frac{p}{2}).
$$

So it suffices to show, for each $p \in (0,1]$,

$$
\varphi(p) = \frac{(3 - 2\sqrt{2})p}{1 - (3 - 2\sqrt{2})p} + \frac{p}{2} + \ln(1 - (3 - 2\sqrt{2})p) + \ln(1 - \frac{p}{2}) \leq 0.
$$

One can easily show that for each $p \in (0,1]$,

$$
\varphi'(p) = \frac{(3 - 2\sqrt{2})}{[1 - (3 - 2\sqrt{2})p]^2} - \frac{1}{2} = \frac{(3 - 2\sqrt{2})^2 p}{[1 - (3 - 2\sqrt{2})p]^2} + \frac{1}{2} - \frac{1}{2 - p} \leq 0.
$$

Therefore $\varphi(p) \leq \varphi(0) = 0$. Thus the lemma is proved. □

Proof of Lemma 2: By (18) and (19), obviously, it suffices to show

$$(1 + \delta_{2k})\frac{1}{2} - \frac{1}{2} g(p) + \delta_{2k} < 1.$$

By (7), the left hand side of the above inequality is a monotonically increasing function of $\delta_{2k}$, so it suffices to show

$$(2 - (6 - 4\sqrt{2})p)\frac{1}{2} - \frac{1}{2} g(p) < (6 - 4\sqrt{2})p. $$

By (6), we only need to show

$$(1 - (3 - 2\sqrt{2})p)\frac{1}{2} - \frac{1}{2} (1 - \frac{p}{2})^{\frac{1}{2}} < (6 - 4\sqrt{2}).$$

By Lemma 11 for each $p \in (0,1)$, we have

$$(1 - (3 - 2\sqrt{2})p)\frac{1}{2} - \frac{1}{2} (1 - \frac{p}{2})^{\frac{1}{2}} < (6 - 4\sqrt{2}),$$

so the lemma holds. □

APPENDIX C

PROOF OF LEMMA 7

Before proving Lemma 7 we need to introduce the following lemma whose proof will be provided in the latter part of this subsection.

Lemma 12: For $\forall p \in (0,1)$, it holds that,

$$
\sum_{i=2}^{l} ||h_{T_i}||_2 \leq \sqrt{2} C_1(p) k^{\frac{1}{2} - \frac{1}{p}} ||h_{T_{1}}||_p,
$$

where

$$
C_1(p) = \begin{cases} \frac{(p^{\frac{1}{2}})^{\frac{1}{2}}}{\frac{1}{2} - \frac{1}{p}}, & p \in (0,p^*) \\ \frac{1}{\frac{1}{2} - \frac{1}{p}}, & p \in (p^*,1). \end{cases}
$$

Remark 2: The bound given by Lemma 12 is sharper than the corresponding bound given in [16] (lemma 4), in [17] (lemma 4) and in [20] (lemma 2.4). To save the space, we do not give the details.

Proof of Lemma 2: For each $i, j \geq 1$ and $i \neq j$, $T_i \cap T_j = \phi$, therefore, by Lemma 2.1 in [4], we have

$$
|\langle Ah_{T_i}, Ah_{T_j} \rangle| \leq \delta_{2k}||h_{T_i}||_2 ||h_{T_j}||_2.
$$
By the aforementioned equation and (3), we have,
\[
\|\sum_{i=2}^{l} Ah_{T_i}\|_2^2 = \sum_{i,j \geq 2} \langle Ah_{T_i}, Ah_{T_j} \rangle \leq \sum_{i=2}^{l} \|\langle Ah_{T_i}, Ah_{T_i} \rangle \| + 2 \sum_{i>j \geq 2} \|\langle Ah_{T_i}, Ah_{T_j} \rangle \|
\]
\[
\leq \sum_{i=2}^{l} (1 + \delta_{2k})\|h_{T_i}\|_2^2 + \delta_{2k} \sum_{i>j \geq 2} \|h_{T_i}\|_2 \|h_{T_j}\|_2 = \sum_{i=2}^{l} \|h_{T_i}\|_2^2 + \delta_{2k} (\sum_{i=2}^{l} \|h_{T_i}\|_2^2).
\]
So the lemma follows from Lemmas 12 and 5. □

Before following the methods used in [20] and [17] to prove Lemma 12, we introduce the following lemma which was provided in [21].

**Lemma 13:** If \( p \in (0, 2) \) and \( u_1 \geq \ldots \geq u_i \geq u_{i+1} \geq \ldots \geq u_r \geq u_{r+1} \geq \ldots \geq u_{r+l} \geq 0 \), then
\[
(\sum_{i=l+1}^{r} u_i^2)^{1/p} \leq C (\sum_{i=1}^{r} u_i^p)^{1/p},
\]
where \( C = \max \left\{ \frac{1}{2}, \left( \frac{2}{r-p} \right) \frac{1}{r-p} \right\} \).

By (5), (9) and Lemma 13 we immediately obtain the following corollary.

**Corollary 5:** If \( p \in (0, 1) \) and \( u_1 \geq \ldots \geq u_k \geq u_{k+1} \geq \ldots \geq u_{2k} \geq u_{2k+1} \geq \ldots \geq u_{3k} \geq 0 \), then
\[
(\sum_{i=k+1}^{3k} u_i^2)^{1/2} \leq C_1(p) k^{1/2} \left( \sum_{i=1}^{2k} u_i^p \right)^{1/p},
\]
where \( C_1(p) \) is defined as in (37).

**Remark 3:** In corollary 1 in [17], \( C_1(p) = p^{\frac{1}{2}} \left( \frac{2}{2-p} \right)^{\frac{1}{2-p}} \). By (10), \( p^{\frac{1}{2}} \left( \frac{2}{2-p} \right)^{\frac{1}{2-p}} \geq 2^{\frac{1}{2-p}} \) for each \( p \in (p^*, 1) \), so our bound on \( (\sum_{i=k+1}^{3k} u_i^2)^{1/2} \) is sharper.

**Proof of Lemma 12** For every even \( j \in \{2, 4, \ldots, \} \), obviously, \( T_j \cap T_{j+1} = \emptyset \). Therefore, one can easily show that
\[
\|h_{T_j}\|_2 + \|h_{T_{j+1}}\|_2 \leq \sqrt{2} \|h_{T_j \cup T_{j+1}}\|_2.
\]
Summing up all the aforementioned inequalities for \( j = 2, 4, \ldots \), yields
\[
\sum_{j=2}^{l} \|h_{T_j}\|_2 \leq \sqrt{2} \sum_{j=1}^{l} \|h_{T_{2j-1} \cup T_{2j}}\|_2.
\]
Since \( |T_j| = k \) for each \( j \geq 1 \). By Corollary 5 we have,
\[
\|h_{T_{2j-1} \cup T_{2j}}\|_2 \leq C_1(p) k^{1/2} \|h_{T_{2j-1} \cup T_{2j}}\|_p.
\]
By the aforementioned inequalities, we have
\[
\sum_{j=2}^{l} \|h_{T_j}\|_2 \leq \sqrt{2} C_1(p) k^{1/2} \sum_{j=1}^{l} \|h_{T_{2j-1} \cup T_{2j}}\|_p.
\]
The lemma follows from the aforementioned equation and (30). □

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