Negativity of the Wigner function and thermal effects of Bell-cat states

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Abstract
We applied the thermofield dynamics formalism to analyze how the non-classical properties of the Bell-cat states are influenced by a gradual change of temperature values, in a thermal equilibrium system. To this purpose we calculate the thermal Wigner functions for these states, whose negative volume is associated with non-classical properties, and we evaluate how these non-classical features vary with temperature. Our results indicate that these properties are almost absent for temperatures of around 2 K.

Keywords: Bell-cat states, thermofield dynamics, Wigner function

(Some figures may appear in colour only in the online journal)

1. Introduction

Years ago, in the celebrated Einstein \textit{et al} paper [1], attention was drawn to the non-local characteristics associated with the formalism of quantum mechanics. Since then, tests of the non-locality of quantum mechanics have been proposed for the analysis and the study of spatially-separated states using theories based on the concept of local realism [2]. In this context one of the most important analysis is the test of the Bell’s inequality, whose violation tipifies non-local properties such as entanglement. The maximally entangled Bell states have been widely used mostly due to their experimental evidences which have been known since the 1980s [3]. Nowadays they play a fundamental role on quantum computation, quantum information and quantum cryptography [4].

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Yurker and Stoler [5, 6] showed that a superposition of coherent states can be obtained by means of the evolution of a coherent state called the Schrödinger’s cat-like state, or only cat state. Spin cat states can be created by a superposition of coherent spins of hundreds of atoms in a system where particle loss can be suppressed, i.e. in Bose–Einstein condensates (BEC’s) [7]. Thus, we can expect BEC’s to be promising for quantum computing in the implementation of quantum logic gates for example, because they are feasible at macroscopically large energy scale [8]. In this context we can also mention the method for implementing dissipative quantum metrology proposed in reference [9] and the demonstration of non linear Mach–Zehnder interferometer (NLMZI) for BEC’s performed on an atom chip [10].

In 1992 Sanders [11] generalized the notion of entangled particle state and introduced the entangled coherent states. Such states can be generate by assembling a macroscopic distinguished superposition of coherent states [12] and the vacuum state in a NLMZI. One feature of these states in contrast with the Bell states is their non-orthogonality between themselves. Thus theoretical efforts were required to understand the importance of non-orthogonal states in the analysis of the entanglement [13, 14]. It turned out that states entangled through non-orthogonal states can reach the maximal entanglement whatever the nature of the states to be entangled [15]. On other hand, we have states introduced by Sanders as a particular case of the quasi Bell states [13], that is formed by coherent states. They can be produced by cat states in a NLMZI and are named quasi Bell entangled states or Bell-cat states.

Teleportation based on coherent states [16–18], quantum computation [19, 20], error free quantum reading [21] and quantum enigma cipher [22] have been investigated by means of Bell-cat states. However, few studies involving the thermal effects of these states have been done. A recent study [23] analyzed the Bell-cat states in the presence of thermal noise, and in particular its degree of entanglement and the error performance in the minimax discrimination problem [22]. However, it did not offer thorough investigation since thermal noise was assumed affecting only one of the two modes of each state.

In this paper we make a theoretical analysis of the non-classicality of the Bell-cat states subjected to the influence of temperature. In this new analysis we will use the procedure of thermofield dynamics [24, 25], in particular the thermal Wigner functions (WF’s) for the Bell-cat states will be considered along with their negativity properties. The thermofield dynamics (TFD) formalism was developed by Umezawa and Takahasi [26, 27] and it is a suitable way to introduce temperature in quantum states [28–32].

We organized the paper as follows: in section 2 we present the Bell-cat states. In section 3 we apply the TFD formalism to construct the thermal Bell-cat states and their corresponding thermal density operators. In section 4 we obtain the thermal WF’s and analyze their behaviour. We then compute negative volumes of the thermal WF’s and establish the relation between temperature and non-classical properties. Finally we present our conclusions in section 5, followed by the acknowledgments and references.

2. The Bell-cat states

A single mode of a stabilized laser can be described by a coherent state

$$|\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

where $\alpha \in \mathbb{C}$ is associated with the average field amplitude and $|n\rangle$ is the number Fock state. Yurker and Stoler [5, 6] showed that a superposition of two coherent states can be obtained by the time evolution of a coherent state, in an amplitude-dispersive medium. These superpositions
were called Schrödinger’s cat-like states or cat states for short. In the form of the superposition of two opposed states, they are written as

\[ |\alpha_{\pm}\rangle = \bar{N}_{\pm} \left[ |\alpha\rangle \pm |-\alpha\rangle \right], \tag{2} \]

where the \(\pm\) sign on the left side of (2) summarizes the two possible states and the normalization constant is given by \(\bar{N}_{\pm} = \left[2 \left(1 \pm e^{-2|\alpha|^2} \right) \right]^{-\frac{1}{2}}\). This notation enables us to treat the even cat state \(|\alpha_{+}\rangle\) and the odd cat state \(|\alpha_{-}\rangle\) in a unified way. By combining the cat states with the vacuum state on a 50/50 beam-splitter, the output are the entangled Bell-cat states \(|\Phi_{\pm}\rangle\) and \(|\Psi_{\pm}\rangle\) which are represented by

\[ |\Phi_{\pm}\rangle = N_{\pm} \left[ |\alpha\rangle, \alpha \pm |\alpha\rangle, -\alpha \right], \tag{3} \]
\[ |\Psi_{\pm}\rangle = N_{\pm} \left[ |\alpha\rangle, -\alpha \pm |\alpha\rangle, -\alpha \right], \tag{4} \]

where \(N_{\pm} = \left[2 \left(1 \pm e^{-4|\alpha|^2} \right) \right]^{-\frac{1}{2}}\) and we have used the notation \(|\beta, \gamma\rangle\) for the tensor product of the coherent states \(|\beta\rangle\) and \(|\gamma\rangle\). Bell-cat states comprise a set of non-orthogonal Bell states and have the property to recover to the regular orthogonal Bell states for larger values of \(\alpha\). For example, for \(|\alpha| = 2\) we have \(|\langle\alpha| - \alpha\rangle| = e^{-4|\alpha|^2} = 1.13 \times 10^{-7}\) such that \(|\alpha\rangle\) and \(|-\alpha\rangle\) can be considered approximately orthogonal states. The Bell states are recovered by encoding logical qubits in the coherent states \(|0\rangle_L = |\alpha\rangle\) and \(|1\rangle_L = |-\alpha\rangle\) for \(|\alpha| \geq 2\). In so doing, the Bell-cat states (3) and (4) can be written as

\[ |\Phi_{\pm}\rangle = N_{\pm} \left[ |1\rangle_L \otimes |1\rangle_L \pm |0\rangle_L \otimes |0\rangle_L \right], \tag{5} \]
\[ |\Psi_{\pm}\rangle = N_{\pm} \left[ |1\rangle_L \otimes |0\rangle_L \pm |0\rangle_L \otimes |1\rangle_L \right], \tag{6} \]

which are in the familiar from of the Bell states.

We can introduce the compact notation

\[ |\psi_{k,\pm}\rangle = N_{\pm} \left[ |\alpha, k\alpha\rangle \pm |-\alpha, -k\alpha\rangle \right] \tag{7} \]

for the Bell-cat states with \(k = \pm 1\), that will be useful for a unified description in the next sections, in the sense that it is possible to express all the four states (3) and (4) in a single notation for the ket \(|\psi_{k,\pm}\rangle\), by combination of the labels \(k\) and \(\pm\).

3. Thermofield dynamics for the Bell-cat states: an approach through Lie algebras

We have structured this section into two subsections. Firstly, we will outline the important steps of the theory of thermofield dynamics in the subsection 3.1. Then we apply it to the Bell-cat states in the subsection 3.2, where we obtain the thermal description of the Bell-cat states and also the analytic expressions for the thermal density operator. Thereafter, we will compute the thermal WF’s relative to the thermal density operator and analyse their thermal features. We will follow the general theoretical developments contained in the references [24–27].

3.1. The thermofield dynamics formalism

The TFD formalism consists in associating the ensemble average of some observable \(A\) to the expectation value over the state \(|0(\beta)\rangle\), namely the thermal vacuum state with \(\beta = 1/K_B T\).
is the temperature of the system in thermal equilibrium and \( K_b \) is the Boltzmann constant, that is
\[
\langle A \rangle = \langle 0(\beta) | A | 0(\beta) \rangle = \frac{1}{Z(\beta)} \text{Tr} \left( e^{-\beta H} A \right), \tag{8}
\]
where \( Z(\beta) = \text{Tr} \left( e^{-\beta H} \right) \) is the partition function of the system described by the Hamiltonian \( H \).

By assuming that the vacuum state can be expanded in the Fock basis, the temperature information can be encoded in the coefficients of the expansion. We then write
\[
| 0(\beta) \rangle = \sum_n g_n(\beta) | n \rangle, \tag{9}
\]
where \( \{ g_n \} \) is a set of temperature dependent complex functions. From (8) and (9) we have
\[
\langle A \rangle = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} \langle n | A | n \rangle = \sum_{n,m} g^*_n(\beta) g_m(\beta) \langle m | A | n \rangle, \tag{10}
\]
with
\[
g^*_n(\beta) g_m(\beta) = \frac{1}{Z(\beta)} e^{-\beta E_n} \delta_{nm}. \tag{11}
\]

However the relation (11) cannot be satisfied by complex numbers. One possibility to get over this issue is to consider \( g_n(\beta) \) as a vector \( | g_n(\beta) \rangle \) in a replica space \( \tilde{H} \) of the usual Hilbert space \( H \), given by [24–27]
\[
| g_n(\beta) \rangle = f_n(\beta) | \tilde{n} \rangle, \tag{12}
\]
where \( | \tilde{n} \rangle \) is an element of the Fock basis for \( \tilde{H} \). This procedure gives rise to a new definition for the thermal vacuum state as the tensor product between the right-hand sides of (9) and (12), i.e.,
\[
| 0(\beta) \rangle = \sum_n | n \rangle \otimes | g_n(\beta) \rangle = \sum_n f_n(\beta) | n, \tilde{n} \rangle, \tag{13}
\]
with \( | n, \tilde{n} \rangle = | n \rangle \otimes | \tilde{n} \rangle \). Computing the average \( \langle A \rangle \) using the definitions (12) and (13) instead of (9) we get
\[
\langle A \rangle = \langle 0(\beta) | A | 0(\beta) \rangle = \sum_{n,\tilde{n}} f^*_n(\beta) f_\tilde{n}(\beta) \langle m, \tilde{m} | A | n, \tilde{n} \rangle = \sum_n f^*_n(\beta) f_n(\beta) \langle n | A | n \rangle. \tag{14}
\]
Comparing the right-hand side of (14) to the right-hand side of (8) we obtain
\[
f_n(\beta) = \frac{e^{-\beta E_n}}{\sqrt{Z(\beta)}}. \tag{15}
\]
Note that equation (11) is now replaced by \( \langle g_n(\beta) | g_m(\beta) \rangle = \frac{1}{Z(\beta)} e^{-\beta E_n} \delta_{nm} \). Therefore the thermal vacuum state can be written as
\[
| 0(\beta) \rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n} | n, \tilde{n} \rangle. \tag{16}
\]
The thermal vacuum state (16) is a state in the thermal Hilbert space which is the doubled Hilbert space given by \( \mathcal{H}_T = \mathcal{H} \otimes \tilde{\mathcal{H}} \).

We can introduce an unitary transformation that associates the doubled vacuum state \(|0,0\rangle = |0\rangle \otimes |0\rangle\) to the thermal vacuum state \(|0(\beta)\rangle\), namely the Bogoliubov transformation

\[
|0(\beta)\rangle = U(\beta)|0,0\rangle.
\]

This will lead to the definition \( A(\beta) = U(\beta)A U^\dagger(\beta) \) of the thermal operators which preserve the structure of the Hilbert space.

### 3.2. The thermofield dynamics of the Bell-cat state

We are interested in Bell-cat states as described in the TFD formalism applied to the two modes bosonic harmonic oscillator, described by the Hamiltonian

\[
H = \hbar \omega_1 a_1^\dagger a_1 + \hbar \omega_2 a_2^\dagger a_2,
\]

where \( \omega_j \) is the angular frequency, \( a_j \) and \( a_j^\dagger \) the annihilation and creation operators, respectively. They satisfy the usual commutation relations \([a_i, a_j^\dagger] = \delta_{ij}\) and \([a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0\), for the modes \( \ell, j \in \{1, 2\} \). The Hilbert space \( \mathcal{H} \) is now expanded by the two modes number states \(|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle\) and following the TFD description we double the Hilbert space. The replic \( \tilde{\mathcal{H}} \) of the original Hilbert space is expanded by the states \(|\tilde{n}_1, \tilde{n}_2\rangle\).

We have annihilation and creation operators in the tilde Hilbert space satisfying analogous commutation relations \([\tilde{a}_i, \tilde{a}_j^\dagger] = \delta_{ij}\) and \([\tilde{a}_i^\dagger, \tilde{a}_j] = [\tilde{a}_i^\dagger, \tilde{a}_j^\dagger] = 0\). In addition all tilde and non-tilde operators commute among themselves. The thermal Hilbert space for the two modes bosonic oscillator is given by \( \mathcal{H}_T = \mathcal{H} \otimes \tilde{\mathcal{H}} \) and the thermal vacuum state is

\[
|0(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_{n_1,n_2=0}^{\infty} e^{-\frac{\beta \hbar (\omega_1 n_1 + \omega_2 n_2)}{2}} |n_1, n_2, \tilde{n}_1, \tilde{n}_2\rangle = U(\beta)|0,0,\tilde{0},\tilde{0}\rangle,
\]

with \( Z(\beta) = \frac{1}{1-e^{-\beta \hbar \omega_1}} \frac{1}{1-e^{-\beta \hbar \omega_2}} \) and \(|n_1, n_2, \tilde{n}_1, \tilde{n}_2\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |\tilde{n}_1\rangle \otimes |\tilde{n}_2\rangle\). The Bogoliubov transformation \([24, 25]\) is then

\[
U(\beta) = e^{\eta_1(\beta)\tilde{a}_1^\dagger a_1 - \tilde{a}_1 a_1^\dagger} e^{\theta_2(\beta)\tilde{a}_2^\dagger a_2 - \tilde{a}_2 a_2^\dagger},
\]

where \( \theta_j, j \in \{1, 2\} \), are \( \beta \)-dependent parameters defined by the relations

\[
\cosh \theta_j(\beta) = \frac{1}{\sqrt{1-e^{-\beta \hbar \omega_j}}} \equiv u_j(\beta),
\]

\[
\sinh \theta_j(\beta) = \frac{e^{-\beta \hbar \omega_j/2}}{\sqrt{1-e^{-\beta \hbar \omega_j}}} \equiv v_j(\beta).
\]

Using (20) the thermal creation and annihilation operators can be written in terms of the non-thermal ones, that is \([24, 25]\)

\[
a_j(\beta) = u_j(\beta) a_j - v_j(\beta) \tilde{a}_j^\dagger,
\]

\[
a_j^\dagger(\beta) = u_j(\beta) a_j^\dagger - v_j(\beta) \tilde{a}_j,
\]

\[
\tilde{a}_j(\beta) = u_j(\beta) \tilde{a}_j - v_j(\beta) a_j^\dagger.
\]
\[ \tilde{a}_j^\dagger(\beta) = u_j(\beta) \hat{a}_j^\dagger - v_j(\beta) \hat{a}_j, \quad (26) \]

From (23)–(26) we can express a general thermal state \(|\Psi(\beta)\rangle\) in terms of the thermal vacuum state to make explicit the thermal effects, that is,

\[ |\Psi(\beta)\rangle = f(a, a^\dagger; \beta)|0(\beta)\rangle, \quad (27) \]

where \( f \) is a function of the non-tilde operators and of the temperature dependent functions \( u_j(\beta) \) and \( v_j(\beta) \), defined in (21) and (22).

Now we are able to associate the general thermal state (27) with the thermal density operator \( \rho_{\Psi(\beta)} \) using the relation between ensemble average and the expectation value of some operator \( O \), i.e.,

\[ \langle \Psi(\beta) | O | \Psi(\beta) \rangle = \text{Tr} \left( \rho_{\Psi(\beta)} O \right) = \langle 0(\beta) | f^\dagger(a, a^\dagger; \beta) O f(a, a^\dagger; \beta) | 0(\beta) \rangle \]
\[ = \text{Tr} \left( \rho_{\beta} f^\dagger(a, a^\dagger; \beta) O f(a, a^\dagger; \beta) \right) \]
\[ = \text{Tr} \left( f(a, a^\dagger; \beta) \rho_{\beta} f^\dagger(a, a^\dagger; \beta) O \right), \quad (28) \]

where in the last step of (28) we use the cyclic property of the trace. We can now compare the first and the last lines of the above equation and note that

\[ \rho_{\Psi(\beta)} = f(a, a^\dagger; \beta) \rho_{\beta} f^\dagger(a, a^\dagger; \beta), \quad (29) \]

where \( \rho_{\beta} \) is the density operator of associated with the vacuum state and is given by

\[ \rho_{\beta} = \frac{e^{-\beta H}}{Z(\beta)}. \quad (30) \]

An algebraic construction of the thermal Bell-cat states can be done by considering the thermal Lie algebra of the Heisenberg–Weyl group \( H^2_2(\beta) \oplus H^2_1(\beta) \) [33] with the generators \( \{ N_1(\beta), a_1(\beta), a_1(\beta), I_1, N_2(\beta), a_2(\beta), a_2(\beta), I_2 \} \) and the stability subgroup \( U^1_\beta(1) \otimes U^1_\beta(1) \otimes U^1_\beta(1) \otimes U^1_\beta(1) \) whose Lie algebra is spanned by \( \{ N_1(\beta), I_1, N_2(\beta), I_2 \} \). The coset space is

\[ \frac{H^2_2(\beta) \oplus H^2_1(\beta)}{U^1_\beta(1) \otimes U^1_\beta(1) \otimes U^1_\beta(1) \otimes U^1_\beta(1)}, \quad (31) \]

where its representative is the thermal displacement operator

\[ D(\alpha_1, \alpha_2; \beta) = e^{\alpha_1 a_1^\dagger(\beta) - \alpha_1 a_1(\beta) + \alpha_2 a_2^\dagger(\beta) - \alpha_2 a_2(\beta)}D(\alpha_1 a_1 a_2^\dagger a_2(\beta)), \]
\[ = e^{\alpha_1 a_1^\dagger(\beta) - \alpha_1 a_1(\beta) + \alpha_2 a_2^\dagger(\beta) - \alpha_2 a_2(\beta)} = D_1(\alpha; \beta)D_2(\alpha; \beta), \quad (32) \]

where \( \alpha_1 \) and \( \alpha_2 \) are complex parameters and \( D_1(\alpha; \beta) \) and \( D_2(\alpha; \beta) \) are the thermal displacement operators for the two modes. In particular, if we consider \( \alpha_1 = \alpha_2 = \alpha \) we obtain \( D(\alpha; \alpha; \beta) \) and similarly \( D(\alpha, -\alpha; \beta), D(-\alpha, \alpha; \beta) \) and \( D(-\alpha, -\alpha; \beta) \).

Following this algebraic construction the thermal displacement operator associated with the thermal Bell-cat states are \( (k = \pm 1) \)

\[ D_{\Psi_{k,\pm}}(\beta) = D_1(\alpha; \beta)D_2(\alpha; \beta) \pm D_1(-\alpha; \beta)D_2(-k\alpha; \beta), \quad (33) \]

which implies that

\[ |\psi_{k,\pm}(\beta)\rangle = N_{\pm} \left[ D_1(\alpha; \beta)D_2(\alpha; \beta) \pm D_1(-\alpha; \beta)D_2(-k\alpha; \beta) \right]|0(\beta)\rangle. \quad (34) \]
Using the notation of equation (7) and that
\begin{equation}
D_1(\alpha; \beta)D_2(\alpha; \beta)|0(\beta)\rangle = U(\beta)e^{-\frac{1}{2}\alpha^2}e^{-\frac{1}{2}\alpha^2} \sum_{n_1,n_2=0}^{\infty} \frac{\alpha^{n_1}}{\sqrt{n_1!}} \frac{\alpha^{n_2}}{\sqrt{n_2!}} |n_1, n_2, \tilde{0}, \tilde{0}\rangle , \tag{35}
\end{equation}
we perform the Bogoliubov transformation onto the two modes doubled state $|n_1, n_2, \tilde{0}, \tilde{0}\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{0}\rangle$, where $|n_1\rangle \otimes |n_2\rangle \in \mathcal{H}$ is a number state associated with the two modes harmonic oscillator Hilbert space $\mathcal{H}$ and $|\tilde{0}\rangle \otimes |\tilde{0}\rangle \in \tilde{\mathcal{H}}$ is the vacuum state in the replic Hilbert space $\tilde{\mathcal{H}}$. Moreover, the thermal Hilbert space associated with the thermal Bell-cat states is $\mathcal{H}_T = \mathcal{H} \otimes \tilde{\mathcal{H}}$. Here we choose to double the Bell-cat states with the vacuum state to recover the original state when $T \to 0$ [28, 29]. The thermalized Bell-cat states are then
\begin{equation}
|\psi_{\pm}(\beta)\rangle = N_{\pm} U(\beta) \left[ |\alpha, k\alpha, \tilde{0}, \tilde{0}\rangle \pm |\alpha, -k\alpha, \tilde{0}, \tilde{0}\rangle \right] \\
= N_{\pm} e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^{n+m} \hbar^m}{\sqrt{n! m!}} \left[ 1 \pm (-1)^{n+m} \right] U(\beta)|n, m, \tilde{0}, \tilde{0}\rangle \\
= N_{\pm} e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^{n+m} \hbar^m}{n! m!} \left[ 1 \pm (-1)^{n+m} \right] \frac{(a_1^\dagger)^n (a_2^\dagger)^m}{(a_1(\beta)^n)(a_2(\beta)^m)} |0(\beta)\rangle. \tag{36}
\end{equation}
It can be noticed in (36) that $|\psi_{\pm}(\beta)\rangle$ are different from zero only in the cases where $n + m$ are even. In another hand $|\psi_{-\pm}(\beta)\rangle$ are different from zero for $n + m$ odd. We can now call $|\psi_{1,+}(\beta)\rangle$ and $|\psi_{1,-}(\beta)\rangle$ even thermal Bell-cat states and $|\psi_{-1,+}(\beta)\rangle$ and $|\psi_{-1,-}(\beta)\rangle$ odd thermal Bell-cat states.

The thermal density operators $\rho_{|\psi_{\pm}(\beta)\rangle}$ that carry the thermal properties of the Bell-cat states can be obtained by using (28) and (36) resulting in
\begin{equation}
\rho_{|\psi_{\pm}(\beta)\rangle} = e^{-2|\alpha|^2} \frac{1 - e^{-\beta \hbar \omega}}{2} \left( 1 + e^{-4|\alpha|^2} \right) \sum_{n,m=0}^{\infty} \frac{\alpha^{n+m} (\alpha^*\alpha)^n \hbar^{m+n}}{n! m! n! m!} \left[ 1 \pm (-1)^{n+m} \right] \left( \sqrt{1 - e^{-\beta \hbar \omega}} \right)^{n+m} \\
\times \left( \sqrt{1 - e^{-\beta \hbar \omega}} \right)^{m+n} \left( e^{-\beta \hbar \omega} \right)^n \sqrt{(n_1 + n)! (n_2 + m)!} (n_1 + m)! (n_2 + m)! \\
\times \sqrt{(n_1 + n)! (n_2 + m)!} |n_1 + n, n_2 + m\rangle \langle n_1 + n, n_2 + m|. \tag{37}
\end{equation}
where we use the symbol $\sum_{[n,m]}$ as short notation for $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty}$. From this expression we can observe that the non-zero components of the thermal density operator associated with the Bell-cat states only remain for both $n + m$ and $n + m$ even or odd.

We finish this section by highlighting that the thermal density operators (37) carry all the information associated with the thermal Bell-cat states. In another words, they describe theoretically the thermal effects of the physical Bell-cat states.

4. Wigner functions of the thermal Bell-cat states and their negative volume

A representation of a quantum state in phase space can be made by evaluating the WF [34], that is a quasi-probability distribution admitting negative values. This negativity of WF is associated
with the non-classical properties of quantum states such as non-locality and entanglement. This feature of the WF is useful in order to characterize theoretically and experimentally states in quantum optics since the negativity of the WF can be measured empirically [35–38].

The WF is defined as a Fourier-like transformation of the one mode density operator $\rho$

$$W(q, p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{+\infty} dv \ e^{ip\frac{v}{\hbar}} \left( q - \frac{v}{2} \right) \rho \left( q + \frac{v}{2} \right).$$  \hspace{1cm} (38)

From this representation, it can be shown that the WF associated with the thermal density operators (37) is given by the product of the WF of the modes of thermal Bell-cat states, that is

$$W_{\alpha, \beta}(q_1, p_1, q_2, p_2) = \frac{1}{(2\pi \hbar)^2} \int_{-\infty}^{+\infty} dv_1 dv_2 \ e^{i\frac{p_1 v_1}{\hbar}} e^{i\frac{p_2 v_2}{\hbar}}$$

$$\times \left( q_1 - \frac{v_1}{2} \right) \left( q_2 - \frac{v_2}{2} \right) \rho_{\psi_{\alpha, \beta}(\gamma)} \left( q_1 + \frac{v_1}{2}, q_2 + \frac{v_2}{2} \right),$$  \hspace{1cm} (39)

where $W_{\alpha, \beta}(q, p)$, $j = 1, 2$, is the WF relative to the $j$th mode of the thermal Bell-cat states.

A straightforward calculation of (39) can be performed by using the representation in terms of Hermite polynomials whose integrals lead to the associated Laguerre polynomials $L_n^{\alpha}(x)$ [39, 40]. We then obtain the following expression for the WF associated with the thermal Bell-cat states:

$$W_{\alpha, \beta}(q_1, p_1, q_2, p_2) = \frac{(1 - e^{-\beta h\omega_1}) (1 - e^{-\beta h\omega_2})}{2\pi^2 h^2 (2^q_i) (\pm e^{-2\omega_i^2})}$$

$$\times \sum_{\{n,m\}} \frac{\alpha_{n+m} (\alpha_{n+m}^*) e^{i\omega_i (n+m) + \omega_i (m+n)}}{n! m! n! m! n! n!} \left[ 1 \pm (-1)^{n+m} \right] \left[ 1 \pm (-1)^{m+n} \right]$$

$$\times \left( \sqrt{1 - e^{-\beta h\omega_1}} \right)^{n+m} \left( \sqrt{1 - e^{-\beta h\omega_2}} \right)^{m+n} (e^{-\beta h\omega_1})^n$$

$$\times (e^{-\beta h\omega_2})^m (n_1 + \min(n, m)) (n_2 + \min(n, m)) (-1)^{n_1 + m}$$

$$\times (-1)^{n_2 + m - \min(0, n - m) - \min(0, m - n)} (\sqrt{2})^{n_1 - m}$$

$$\times (\sqrt{2})^{m-n} \chi_1^{m-n} \chi_2^{m-n} L_{n_1 + \min(n, m)}^{m-n} \left( \frac{2q_1^2}{b_1^2} + \frac{2p_1^2}{b_1^2} \right)$$

$$\times L_{n_2 + \min(n, m)}^{m-n} \left( \frac{2q_2^2}{b_2^2} + \frac{2p_2^2}{b_2^2} \right),$$  \hspace{1cm} (40)

where we defined the constants $b_j^2 = \frac{\hbar}{m_j}$ and $\chi_j = \begin{cases} \frac{q_j + i p_j b_j}{b_j} & \text{if } n \geq \bar{n} \\ \frac{q_j - i p_j b_j}{b_j} & \text{if } n < \bar{n} \end{cases}$ with $j = \{1, 2\}$.

The interest in obtaining the WF of the quantum states used in quantum optics is related to its negativity properties. Our WF’s are functions in four dimensions and cannot be completely
Figure 1. WF of thermal Bell-cat states with $\alpha = 1$.

visualized in a three dimensional plot unless we specify values for $q_j$ and $p_j$. This can be done in several ways. In figures 1–3 we choose to represent $x_1 = \frac{q_1}{b_1} + i\frac{p_1}{b_1}$ and $x_2 = \frac{q_2}{b_2} + i\frac{p_2}{b_2}$ so that we can visualize the $W_{\alpha,\beta}(x_1, x_2)$ portion of the complete WF. Based in the references [41–43] we choose $\frac{2\pi}{\omega_1} = \frac{2\pi}{\omega_2} = 5.5$ GHz and performed the calculation of the function (40) numerically using the R Language [44]. The results are presented in the figures 1–3.

These figures show the plots of the WF associated with the states $|\psi_{-1,+}(\beta)\rangle$ and $|\psi_{1,-}(\beta)\rangle$ with the parameter $\alpha$ possessing the values: $\alpha = 1$, $\alpha = 1 + i$ and $\alpha = 2$, and temperature values: $T = 0.01$ K, $T = 1$ K and $T = 10$ K. We choose only the states $|\psi_{-1,+}(\beta)\rangle$ and $|\psi_{1,-}(\beta)\rangle$ in our analysis, because these states correspond to a rotation in the phase space of the states $|\Phi_{\pm}\rangle$ and $|\Psi_{\pm}\rangle$ respectively. Note we have used the notation $|\psi_{1,+}(\beta)\rangle$ and $|\psi_{-1,-}(\beta)\rangle$ which correspond to the TFD formalism applied to the Bell-cat states $|\Phi_{\pm}\rangle$ and $|\Psi_{\pm}\rangle$. 
respectively, defined in (3) and (4). Our graphics coincide with those obtained in reference [45] for lower temperature, reinforcing the results.

Varying the temperature we can see in more details the importance of thermal effects in the analysis of Bell-cat states. In figures 1–3 we observe a decrease in the range of both positive and negative values of the thermal WF as the temperature goes higher. For temperatures smaller than 0.01 K we have a similar behavior, which indicates that thermal effects have less influence in these temperatures. As the temperature increases, such for example at 10 K, the Bell-cat states lose their properties. This is suggested by the deformed shape of the thermal WF graphics showed in the subfigures (e) and (f) of the figures 1–3.

The restriction of the plot of the WF, corresponding to Bell-cat states, to three dimensions is not the best method to analyze the non-classicality, because these plots are particular situations

Figure 2. WF of thermal Bell-cat states with $\alpha = 1 + i$. 
and consequently do not show all information. Another way to analyze the behaviour of the WF is to use the volume of the negative part. We highlight two measures of negativity of WF and apply to our thermal expression (40). The first one was proposed as the phase space integral of the difference between the absolute value and the actual value of the WF [46], defined as

$$\delta_{k,\pm}^{\alpha,\beta} = \iiint \left[ |W_{k,\pm}^{\alpha,\beta}(q_1, p_1, q_2, p_2)| - W_{k,\pm}^{\alpha,\beta}(q_1, p_1, q_2, p_2) \right] dq_1 dp_1 dq_2 dp_2. \quad (41)$$

The second one is based on the rate of the sum and difference between the positives and negatives volumes of the WF [48, 49], which when applied to our expression yields

$$\nu_{k,\pm}^{\alpha,\beta} = 1 - \frac{I_{+}(W_{k,\pm}^{\alpha,\beta}) - I_{-}(W_{k,\pm}^{\alpha,\beta})}{I_{+}(W_{\alpha,\beta}) + I_{-}(W_{\alpha,\beta})}. \quad (42)$$

Figure 3. WF of thermal Bell-cat states with $\alpha = 2$. 
Figure 4. Negativity of thermal WF of thermal Bell-cat states with $\alpha = \{1, 1+i, 2\}$.

where $I_+ (W^{\pm}_{\alpha,\beta})$ and $I_- (W^{\pm}_{\alpha,\beta})$ are the volume modules of positive and negative parts of the thermal WF’s (40), respectively. If we consider the normalization $I_+ (W^{\pm}_{\alpha,\beta}) - I_- (W^{\pm}_{\alpha,\beta}) = 1$, we can relate these parameters through the following expression:

$$\nu^{\pm}_{\alpha,\beta} = \frac{2 I_- (W^{\pm}_{\alpha,\beta})}{1 + 2 I_- (W^{\pm}_{\alpha,\beta})} = \frac{\delta^{\pm}_{\alpha,\beta}}{1 + \delta^{\pm}_{\alpha,\beta}} \quad (43)$$

We plot in figure 4 the curve of the $\nu^{\pm}_{\alpha,\beta}$ parameter for several temperature values and visualize how non-classicality evolves towards gradual changes of thermal equilibrium of the system. Numerical integration of the thermal WF (40) was performed by R Language [44], for temperature values varying from 0 K to 2 K and was used to calculate the parameter $\nu^{\pm}_{\alpha,\beta}$ corresponding to the states $|\psi_{-1,\pm}(\beta)\rangle$ and $|\psi_{1,\pm}(\beta)\rangle$, with $\alpha = 1$, $\alpha = 1+i$ and $\alpha = 2$. We can observe that the parameter $\nu^{\pm}_{\alpha,\beta}$ measuring non-classicality remains stable until approximately $T \approx 0.3$ K, which coincides with the values found in the references [46, 47]. From this temperature onwards $\nu^{\pm}_{\alpha,\beta}$ presents a gradual decrease getting closer to zero as the temperature grows. For the state $|\psi_{-1,\pm}(\beta)\rangle$ the highest deviation to non-classical properties is reached for $\alpha = 1+i$ whereas for $|\psi_{1,\pm}(\beta)\rangle$ the highest deviation is reached for $\alpha = 1$ exhibiting the largest of the non-classical properties among all we have analyzed. These properties remain over a larger temperature range.
Despite the limitations of three dimensional plot analysis of a four dimensional function, the behavior discussed above could be read from the figures 1–3 where the changing in the peaks of the WF’s is observed as the value of temperature increases from $T = 0.01$ K to $T = 10$ K. We further note the decrease of the negative peaks, which carry the non-classicality of the states, for the higher temperature.

5. Conclusions

In this paper we have used the framework of thermofield dynamics in order to include temperature in the study of Bell-cat states. We have found the analytical expressions for the thermal density operators as well as the thermal WF’s associated with these states. The analyses of the three dimensional graphics of the thermal WF’s associated with the states $|\psi^{-1,\alpha}_{1,\beta}\rangle$ and $|\psi^{\alpha,1}_{1,-\beta}\rangle$, with $\alpha = 1$, $\alpha = 1 + i$ and $\alpha = 2$, and for the temperature of $T = \{0.01\ \text{K}, 1\ \text{K}, 10\ \text{K}\}$, suggest that, for an increase in temperature, there is a decrease in the values of the thermal WF’s. Such behavior can be interpreted as a gradual loss in its properties as temperature increases. To complement our analysis we proposed to use the negativity parameter idealized by Benedict et al. [48, 49] to investigate the non-classicality of the thermal WF’s associated with the two modes Bell-cat states. This concept applied to thermal WF’s results in a temperature dependent parameter $\nu_{\alpha,\beta}^{k,\pm}$ for the temperature range made explicit above. The $\nu_{\alpha,\beta}^{k,\pm}$ parameter shows that for temperatures higher than $0.3$ K the Bell-cat states gradually loose their non-classical properties until around $2$ K, where the read of the $\nu_{\alpha,\beta}^{k,\pm}$ suggests that these properties subside. Due to the exponential behavior of the TFD formalism, the thermal WF shows sigmoidal (S-shaped) behaviour exhibiting an inflection point, which determines two thermal ranges that are sensitive to temperature variation in different ways. Such behaviour describes the smooth blending between those thermal ranges. Combining Benedict’s parameter with the thermal WF seems to be promising strategy since the non-classical properties can be associated with the negativity of the volume associated with the WF. This proved helpful in the analysis of the non-classicality behaviour with respect to the temperature. This analysis can be further applied to states such as those of interest in quantum optics and quantum information. In addition this method can be extended by looking for analytical expression for the limits where non-classical properties of the systems remain preserved.

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References

[1] Einstein A, Podolsky B and Rosen N 1935 Phys. Rev. 47 777
[2] Bell J S 1964 Physics 1 195
[3] Aspect A, Grangier P and Roger G 1982 Phys. Rev. Lett. 49 91
    Aspect A, Dalibard J and Roger G 1982 Phys. Rev. Lett. 49 1804
[4] Artur B D E and Anton Z 2000 The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation (Berlin: Springer)
[5] Yurke B and Stoler D 1986 Phys. Rev. Lett. 57 13
[6] Yurke B and Stoler D 1987 Phys. Rev. A 35 4846
[7] Lau H W, Dutton Z, Wang T and Simon C 2014 Phys. Rev. Lett. 113 090401
[8] Byrnes T, Wen K and Yamamoto Y 2012 Phys. Rev. A 85 040306
[9] Huang J, Qin X, Zhoung T, Ke Y and Lee C 2016 Sci. Rep. 5 17894
[10] Berrada T, Frank S, Bücker R, Schumm T, Schaff J F and Schmiedmayer J 2013 Nat. Commun. 4 2077
[11] Sanders B C 1992 Phys. Rev. A 45 6811
[12] Leibfried D E et al 2005 Nature 438 639
[13] Knill O and Sasaki M 2002 Quantum Communication, Computing, and Measurement 3 (Boston, MA: Springer) pp 359–66
[14] Gilchris A, Nemoto K, Munro W J, Ralph T C, Glancy S, Braunstein S L and Milburn G J 2004 J. Opt. B: Quantum Semiclass. Opt. 6 S828
[15] Hirota O, Enk S J, Nakamura K, Sohma M and Kato K 2001 Entangled nonorthogonal states and their decoherence properties (arXiv:quant-ph/0101096)
[16] Enk S J and Hirota O 2001 Phys. Rev. A 64 022313
[17] Jeong H, Kim M S and Lee J 2001 Phys. Rev. A 64 052308
[18] Wang X 2001 Phys. Rev. A 64 022302
[19] Jeong H and Kim M S 2002 Phys. Rev. A 65 042305
[20] Ralph T C, Munro W J and Milburn G J 2002 Quantum Opt. Comput. Commun. 4917 1
[21] Hirota O 2011 Error free quantum reading by quasi Bell state of entangled coherent states (arXiv:1108.4163)
[22] Hirota O 2016 J. Laser Opt. Photonics 3 129
[23] Kato K 2015 Quantum Commun. Quantum Imaging XIII 9615 96150N
[24] Umezawa H 1995 Advanced Field Theory: Micro, Macro, and Thermal Physics (New York: AIP Press)
[25] Khanna F C, Malboisson A P C, Malboisson J M C and Santana A E 2009 Thermal Quantum Field Theory: Algebraic Aspects and Application (Singapore: World Scientific)
[26] Takahashi Y and Umezawa H 1996 Int. J. Mod. Phys. B 10 1755
[27] Takahashi Y and Umezawa H 1975 Collect. Phenom. 2 55
[28] Trindade M A S, Silva Filho L M, Santos L C, Martins M G R and Vianna J D M 2013 Int. J. Mod. Phys. B 27 1350133
[29] Floquet S, Trindade M A S and Vianna J D M 2017 Int. J. Mod. Phys. A 32 1750015
[30] Kitajima S, Arimitsu T, Obinata M and Yoshida K 2014 Physica A 404 242
[31] Prudêncio T, Filho T M R and Santana A E 2019 Phys. Scr. 94 095102
[32] Prudêncio T 2012 Int. J. Quantum Inf. 10 1230001
[33] Zhang W-M, Feng D H and Gilmore R 1990 Rev. Mod. Phys. 62 867
[34] Wigner E 1932 Phys. Rev. 40 749
[35] Bell J S 1987 Speakable and Unspeakable in Quantum Mechanics (Cambridge: Cambridge University Press) pp 196–200
[36] McConnell R, Zhang H, Hu J, Cuk S and Vuletić V 2015 Nature 519 439
[37] Benedict M G and Czirjak A 1999 Phys. Rev. A 60 4034
[38] Fo¨ldi P, Czirjak A, Molnár B and Benedict M G 2002 Opt. Express 10 376