Schrödinger representations from the viewpoint of monoidal categories

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Abstract

The Drinfel’d double $D(A)$ of a finite-dimensional Hopf algebra $A$ is a Hopf algebraic counterpart of the monoidal center construction. Majid introduced an important representation of $D(A)$, which he called the Schrödinger representation. We study this representation from the viewpoint of the theory of monoidal categories. One of our main results is as follows: If two finite-dimensional Hopf algebras $A$ and $B$ over a field $k$ are monoidally Morita equivalent, i.e., there exists an equivalence $F : A\mathcal{M} \rightarrow B\mathcal{M}$ of $k$-linear monoidal categories, then the equivalence $D(A)\mathcal{M} \cong D(B)\mathcal{M}$ induced by $F$ preserves the Schrödinger representation. Here, $A\mathcal{M}$ for an algebra $A$ means the category of left $A$-modules.

As an application, we construct a family of invariants of finite-dimensional Hopf algebras under the monoidal Morita equivalence. This family is parameterized by braids. The invariant associated to a braid $b$ is, roughly speaking, defined by “coloring” the closure of $b$ by the Schrödinger representation. We investigate what algebraic properties this family have and, in particular, show that the invariant associated to a certain braid closely relates to the number of irreducible representations.

Key words: Hopf algebra, monoidal category, Schrödinger representation, Drinfel’d double
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1 Introduction

Drinfel’d doubles of Hopf algebras [5] are one of the most important objects in not only Hopf algebra theory, but also in other areas including category theory and low-dimensional topology. Let $A$ be a finite-dimensional Hopf algebra over a field $k$, and $(D(A), \mathcal{R})$ be its Drinfel’d double. Due to Majid [25], it is known that there is a canonical representation of $D(A)$ on $A$, which is called the Schrödinger representation (or the Schrödinger module). This representation is an extension of the adjoint representation of $A$, and originates from quantum mechanics as explained in Majid’s book [25, Examples 6.1.4 & 7.1.8] (see Section 2 for the precise definition of the Schrödinger representation). The Schrödinger module is also addressed by Fang [10] as an algebra in the Yetter-Drinfel’d category $A\mathcal{YD}$ via the Miyashita-Ulbrich action.

In this paper, we study the Schrödinger module over the Drinfel’d double from the viewpoint of the theory of monoidal categories. We say that two finite-dimensional Hopf algebras $A$ and $B$ over the same field $k$ are monoidally Morita equivalent if $A\mathcal{M}$ and $B\mathcal{M}$ are equivalent as $k$-linear monoidal categories, where $H\mathcal{M}$ for an algebra $H$ is the category of $H$-modules. One of our main results is that the Schrödinger module is an invariant under the monoidal Morita equivalence.
in the following sense: If $F : A \mathcal{M} \rightarrow B \mathcal{M}$ is an equivalence of $k$-linear monoidal categories, then the equivalence $D(A) \mathcal{M} \approx D(B) \mathcal{M}$ induced by $F$ preserves the Schrödinger modules. To prove this result, we introduce the notion of the Schrödinger object for a monoidal category by using the monoidal center construction. It turns out that the Schrödinger module over $D(A)$ is characterized as the Schrödinger object for $A \mathcal{M}$. Once such a characterization is established, the above result easily follows from general arguments.

As an application of the above category-theoretical understanding of the Schrödinger module, we construct a new family of monoidal Morita invariants, i.e., invariants of finite-dimensional Hopf algebras under the monoidal Morita equivalence. Some monoidal Morita invariants have been discovered and studied; see, e.g., [7, 8, 9, 17, 28, 29, 37, 39]. Our family of invariants is parametrized by braids. Roughly speaking, the invariant associated with a braid $b$ is defined by “coloring” the closure of $b$ by the Schrödinger module. Since the quantum dimension (in the sense of Majid [25]) of the Schrödinger module is a special case of our invariants, we call the invariant associated with $b$ the braided dimension of the Schrödinger module associated with $b$. We investigate what algebraic properties this family of invariants have and, in particular, show that the invariant associated to a certain braid closely relates to the number of irreducible representations.

This paper organizes as follows: In Section 2, we recall the definition of the Schrödinger module over the Drinfel’d double $D(A)$ of a finite-dimensional Hopf algebra $A$. We also describe the definition of another Schrödinger representation of $D(A)$ on $A^{\text{cop}}$, which is introduced by Fang [10]. We refer the corresponding left $D(A)$-module as the dual Schrödinger module. Following [13] we also describe the definition of Radford’s induction functors and its properties. It is shown that the Schrödinger module and the dual Schrödinger module are isomorphic to the images of the trivial left $A$-module and the trivial right $A$-comodule under Radford’s induction functors, respectively. Furthermore, we examine the relationship between the Schrödinger module over $D(A^*)$ and the dual Schrödinger module over $D(A)$.

In Section 3, we study the categorical aspects of the Schrödinger module and the dual Schrödinger module. We introduce a Schrödinger object for a monoidal category $C$ as the object of $\mathcal{Z}(C)$ representing the functor $\text{Hom}_C(\Pi(-), \mathbb{I})$, where $\mathcal{Z}(C)$ is the monoidal center of $C$, $\Pi : \mathcal{Z}(C) \rightarrow C$ is the forgetful functor, and $\mathbb{I}$ is the unit object of $C$. By using the properties of Radford’s induction functor, we show that the Schrödinger module over $D(A)$ is a Schrödinger object for $A \mathcal{M}$ under the identification $\mathcal{Z}(A \mathcal{M}) \approx D(A) \mathcal{M}$. Once this characterization is obtained, the invariance of the Schrödinger module (stated above) is easily proved. A similar result for the dual Schrödinger module is also proved.

In Section 4, based on our category-theoretical understanding of the Schrödinger module, we introduce a family of monoidal Morita invariants parameterized by braids. We give formulas for the invariants associated with a certain series of braids, and give some applications. Note that some monoidal Morita invariants, such as ones introduced in [7] and [37], factor through the
Drinfel’d double construction. Our invariants have an advantage that they do not factor through that. On the other hand, our invariants have a disadvantage in the non-semisimple situation: For any braid $b$, the braided dimension of the Schrödinger module of $D(A)$ associated with $b$ is zero unless $A$ is cosemisimple. From this result, we could say that our invariants are not interesting as monoidal Morita invariants for non-cosemisimple Hopf algebras. However, endomorphisms on the Schrödinger module induced by braids are not generally zero, and thus may have some information about $A$. To demonstrate, we give an example of a morphism induced by a braid, which turns out to be closely related to the unimodularity of $A$.

Throughout this paper, $k$ is an arbitrary field, and all tensor products $\otimes$ are taken over $k$. For $k$-vector spaces $X$ and $Y$, denoted by $T_{X,Y}$ is the $k$-linear isomorphism from $X \otimes Y$ to $Y \otimes X$ defined by $T_{X,Y}(x \otimes y) = y \otimes x$ for all $x \in X$ and $y \in Y$. For a coalgebra $(C, \Delta, \varepsilon)$ and $c \in C$ we use the Sweedler’s notation $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. If $(M, \rho)$ is a right $C$-comodule and $m \in M$, then we also use the notation $\rho(m) = \sum m_{(0)} \otimes m_{(1)}$. For a Hopf algebra $A$, denoted by $\Delta_A$, $\varepsilon_A$ and $S_A$ are the coproduct, the counit, and the antipode of $A$, respectively. If the antipode of $A$ is bijective, then one has two Hopf algebras $A^{\text{cop}}$ and $A^\ast$, which are defined from $A$ by replacing $\Delta_A$ by the opposite coproduct $\Delta_A^{\text{op}} = T_{A,A} \circ \Delta_A$ and replacing the product by the opposite product, respectively. A Hopf algebra map $\alpha : A \longrightarrow B$ induces a Hopf algebra map form $A^{\text{cop}}$ to $B^{\text{cop}}$. The map is denoted by $\alpha^{\text{cop}}$. If $A$ is a finite-dimensional, then the dual vector space $A^\ast$ is also a Hopf algebra so that $\langle pq, a \rangle = \sum p(a_{(1)})q(a_{(2)}), \quad 1_{A^\ast} = \varepsilon_A, \quad \langle \Delta_{A^\ast}(p), a \otimes b \rangle = p(ab), \quad \varepsilon_{A^\ast}(p) = p(1_A), \quad S_{A^\ast}(p) = p \circ S_A$ for $p, q \in A^\ast$ and $a, b \in A$, where $\langle \, , \, \rangle$ stands for the natural pairing between $A$ and $A^\ast$, or $A \otimes A$ and $A^\ast \otimes A^\ast$. Denoted by $\mathcal{M}$ is the $k$-linear monoidal category whose objects are left $A$-modules and morphisms are $A$-module maps, and $\mathcal{M}^A$ is the $k$-linear monoidal category whose objects are right $A$-comodules and morphisms are $A$-comodule maps.

For general facts on Hopf algebras, refer to Abe’s book [1], Montgomery’s book [27] and Sweedler’s book [38], and for general facts on monoidal categories, refer to MacLane’s book [21], Kassel’s book [19] and Joyal and Street’s paper [15].

2 Schrödinger modules of the Drinfel’d double

2.1 Preliminaries: the Drinfel’d double

Let $A$ be a finite-dimensional Hopf algebra $A$ over the field $k$. The Drinfel’d double $D(A)$ of $A$ [5] is the Hopf algebra such that as a coalgebra $D(A) = A^{\text{cop}} \otimes A$, and the multiplication is given by

$$(p \otimes a) \cdot (p' \otimes a') = \sum \langle p'_{(1)}, S_A^{-1}(a_{(3)}) \rangle \langle p'_{(3)}, a_{(1)} \rangle pp'_{(2)} \otimes a_{(2)}a'$$
for all \(p, p' \in A^*\) and \(a, a' \in A\), where \((\Delta_A \otimes \text{id}) \circ \Delta_A)(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}\), \(((\Delta_{A^*} \otimes \text{id}) \circ \Delta_{A^*})(p) = \sum p_{(1)} \otimes p_{(2)} \otimes p_{(3)}\). The unit element is \(\varepsilon_A \otimes 1_A\), and the antipode is given by

\[
S_{D(A)}(p \otimes a) = \sum \langle p_{(1)}, a_{(3)} \rangle \langle S_A^{-1}(p_{(3)}), a_{(1)} \rangle S_A^{-1}(p_{(2)}) \otimes S_A(a_{(2)}).
\]

For all \(p \in A^*\) and \(a \in A\), the element \(p \otimes a\) in \(D(A)\) is frequently written in the form \(p \bowtie a\), since the Drinfel’d double can be viewed as a bicrossed product \(A^{* \bowtie} \bowtie A\) due to Majid [22]. The Drinfel’d double \(D(A)\) has a canonical quasitriangular structure as described below [5]. Let \(\{e_i\}_{i=1}^d\) be a basis for \(A\), and \(\{e_i^*\}_{i=1}^d\) be its dual basis for \(A^*\). Then \(\mathcal{R} = \sum_{i=1}^d (\varepsilon_A \bowtie e_i) \bowtie (e_i^* \bowtie 1_A) \in D(A) \otimes D(A)\) satisfies the following conditions.

1. \(\Delta^{* \bowtie}(x) = \mathcal{R} \cdot \Delta(x) \cdot \mathcal{R}^{-1}\) for all \(x \in D(A)\),
2. \((\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23},\)
3. \((\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}.

Here \(\mathcal{R}_{ij}\) denotes the element in \(D(A) \otimes D(A) \otimes D(A)\) obtained by substituting the first and the second components in \(\mathcal{R}\) to the \(i\)-th and \(j\)-th components, respectively, and substituting 1 to elsewhere. Thus, the pair \((D(A), \mathcal{R})\) is a quasitriangular Hopf algebra [5].

Let \(A\) and \(B\) be two finite-dimensional Hopf algebras over \(k\). If they are isomorphic, then their Drinfel’d doubles are. More precisely, an isomorphism \(f : A \longrightarrow B\) of Hopf algebras yields an isomorphism of quasitriangular Hopf algebras

\[
D(f) := (f^{-1})^* \otimes f : D(A) \longrightarrow D(B).
\]

Let \(A\) be a finite-dimensional Hopf algebra over \(k\). As shown in [33] Theorem 3] and [37] Propositions 2.10 & 3.5], for the Drinfel’d doubles \((D(A), \mathcal{R}), (D(A^{op})^+, \tilde{\mathcal{R}})\) and \((D(A^*), \mathcal{R}')\), there are isomorphisms:

\[
(D(A), \mathcal{R}) \cong (D(A^{op})^+, \tilde{\mathcal{R}}_{21}) \cong (D(A^*)^+, \mathcal{R}_{21}') \cong (D(A^*)^+, \mathcal{R}_{21}).
\]

(2.1)

Here, the first isomorphism \(f_1 : (D(A), \mathcal{R}) \longrightarrow (D(A^{op})^+, \tilde{\mathcal{R}}_{21})\) is given by

\[
f_1(p \bowtie a) = a \bowtie p \quad (p \in A^*, \ a \in A)
\]

under the identification \(A^{**} = A\). The second isomorphism is given by \(f_2 := S \otimes \iota S^{-1} : (D(A^{op})^+, \tilde{\mathcal{R}}_{21}) \longrightarrow (D(A^*)^+, \mathcal{R}_{21}')\) under the identification \(A^{**} = A\), where we regard \(S\) as a Hopf algebra isomorphism from \(A^{op}\) to \(A\). Finally, the third isomorphism is given by the antipode \(S_{D(A^*)}\) [37] Proposition 2.10(1)]. Composing \(f_1, f_2\) and \(S_{D(A^*)}\) we have an isomorphism of quasitriangular Hopf algebras \(\phi_A : (D(A), \mathcal{R}) \longrightarrow (D(A^*)^+, \mathcal{R}_{21}')\) such that

\[
\phi_A(p \bowtie a) = (\iota_A(1_A) \bowtie p) \cdot (\iota_A(a) \bowtie \varepsilon_A) \quad (2.2)
\]

for all \(p \in A^*\) and \(a \in A\), where \(\iota_A : A \longrightarrow A^{**}\) is the usual isomorphism of vector spaces. We note the following property of the isomorphism \(\phi_A\):

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Lemma 2.1. $\phi_{A^*} \circ \phi_A = D(\iota_A)$.

Proof. As is well-known, $\iota_{A^*} = (\iota_A^{-1})^*$. Hence, for all $a \in A$ and $p \in A^*$,

$$(\phi_{A^*} \circ \phi_A)(p \bowtie a) = \phi_{A^*} \iota_A(1_A) \bowtie p \cdot \phi_A(\iota_A(a) \bowtie \varepsilon_A)$$

$$= (\iota_A(p) \bowtie \iota_A(1_A)) \cdot (\iota_{A^*}(\varepsilon_A) \bowtie \iota_A(a)) = D(\iota_A)(p \bowtie a).$$

\[\blacksquare\]

2.2 Schrödinger modules

The Drinfel’d double $D(A)$ has a canonical representation, which is called the Schrödinger representation as described in Majid’s book [25, Examples 6.1.4 & 7.1.8]. This representation is obtained by unifying the left adjoint action of $A$ and the right $A^*$-action $\leftarrow$ (See below (2.3) and (2.4) for precise definition). It is generalized to quasi-Hopf case by Bulacu and Torrecillas [2, Section 3]. Another generalization is given by Fang [10, Section 2]. He introduced two Schrödinger representations for the Drinfel’d double defined by a generalized Hopf pairing. One corresponds to the original Schrödinger representation, and the other corresponds to a dual version of it. Specializing in our setting, we will describe these representations below.

There are four actions defined as follows.

(a) $a \triangleright c = \sum a_{(1)} c S(a_{(2)}) \quad (a, c \in A)$, \hspace{4cm} (2.3)

(b) $a \leftarrow p = \sum \langle p, a_{(1)} \rangle a_{(2)} \quad (p \in A^*, a \in A)$, \hspace{4cm} (2.4)

(c) $q \downarrow p = \sum S(p_{(1)}) q p_{(2)} \quad (p, q \in A^*)$, \hspace{4cm} (2.5)

(d) $a \rightarrow q = \sum q_{(1)} (q_{(2)}, a) \quad (q \in A^*, a \in A)$. \hspace{4cm} (2.6)

By using these actions two left actions $\bullet$ of $D(A)$ on $A$ and $A^*$ can be defined by

$$(p \bowtie a) \bullet b = (a \triangleright b) \leftarrow S^{-1}(p),$$

$$p \bowtie a \bullet q = (a \rightarrow q) \downarrow S^{-1}(p)$$

for all $a, b \in A^*$ and $p, q \in A^*$. We call the left $D(A)$-modules $(A, \bullet)$ and $(A^*, \bullet)$ the Schrödinger module and the dual Schrödinger module, and denote them by $D(A)A$ and $D(A)A^{cop}$, respectively.

Proposition 2.2. Regard the dual Schrödinger module $D(A^*)(A^{*})^{cop}$ of $D(A^*)$ as a left $D(A)$-module by pull-back along the isomorphism $\phi_A$ defined by (2.2). Then $D(A^*)(A^{*})^{cop}$ is isomorphic to the Schrödinger module $D(A)A$ as a left $D(A)$-module.

Proof. Let $\bullet$ be the left action on $D(A^*)(A^{*})^{cop} = A$ as a left $D(A)$-module by pull-back along
homomorphism. Then, for \( p \in A^* \) and \( a, b \in A \), we have

\[
(p \circlearrowright a) \bullet b = \phi_A(p \circlearrowright a) \bullet b \\
= (1_A \circlearrowright p) \bullet ((a \circlearrowright \varepsilon_A) \bullet b) \\
= (1_A \circlearrowright p) \bullet \left( \sum a_{(2)} b S^{-1}(a_{(1)}) \right) \\
= \sum \langle a_{(4)} b_{(2)} S^{-1}(a_{(1)}), p \rangle a_{(3)} b_{(1)} S^{-1}(a_{(2)}) \\
= \sum \langle a_{(1)} S(b_{(2)}) S(a_{(4)}), S^{-1}(p) \rangle S^{-1}(a_{(2)}) S(b_{(1)}) S(a_{(3)}) \\
= S^{-1}(\langle p \circlearrowright a \rangle \bullet S(b)).
\]

This implies that \( S((p \circlearrowright a) \bullet b) = (p \circlearrowright a) \bullet S(b) \). Thus, the composition \( S \circ \iota^{-1}_A : A^{**} \to A \) gives an isomorphism \( D(A^*)(A^{**})^{\text{cop}} \to D(A)A \) of left \( D(A) \)-modules.

**Corollary 2.3.** Regard the dual Schrödinger module \( D(A)(A)\text{cop} \) as a left \( D(A^*) \)-module by pull-back along the isomorphism \( \phi_A^{-1} : (D(A^*), \mathcal{R}) \to (D(A)\text{cop}, \mathcal{R}_2) \). Then \( D(A)(A)\text{cop} \) is isomorphic to the Schrödinger module \( D(A^*)A^* \) as a left \( D(A^*) \)-module.

**Proof.** Let \( F_1 : D(A)M \to D(A^{**})M \) and \( F_2 : D(A)M \to D(A^*)M \) be the isomorphisms of categories induced by isomorphisms \( D(\iota^{-1}_A) \) and \( \phi_A^{-1} \), respectively. By Lemma \( \ref{lem:iso} \), \( F = F_2 \circ F_1 \) is the isomorphism of categories induced by \( \phi_A^{-1} \). Applying Proposition \( \ref{prop:iso} \) to \( A^* \), we have

\[
F(D(A)(A)\text{cop}) = (F_2 \circ F_1)(D(A)(A)\text{cop}) = F_2(D(A^{**})(A^{**})\text{cop}) = D(A^*)A^*.
\]

### 2.3 Radford’s induction functors and Schrödinger modules

In this subsection we show that the Schrödinger module \( D(A)A \) is isomorphic to the image of the trivial \( A \)-module under Radford’s induction functor. This implies that the Schrödinger modules are remarkable objects from the viewpoint of category theory.

Radford’s induction functors are described by using Yetter-Drinfel’d modules, which were introduced in \( \cite{Yetter-Drinfel’d} \) and called crossed bimodules. Let us recall the definition of Yetter-Drinfel’d modules \( \cite{Radford} \). Let \( A \) be a bialgebra over \( k \). Suppose that \( M \) is a vector space over \( k \) equipped with a left \( A \)-module action - and a right \( A \)-comodule coaction \( \rho \) on it. The triple \( M = (M, \cdot, \rho) \) is called a *Yetter-Drinfel’d \( A \)-module*, if the following compatibility condition, that is called the Yetter-Drinfel’d condition, is satisfied for all \( a \in A \) and \( m \in M \):

\[
\sum (a_{(1)} \cdot m_{(0)}) \otimes (a_{(2)} m_{(1)}) = \sum (a_{(2)} \cdot m)_{(0)} \otimes (a_{(2)} \cdot m)_{(1)} a_{(1)}. \tag{2.9}
\]

A \( k \)-linear map \( f : M \to N \) between two Yetter-Drinfel’d \( A \)-modules is called a *Yetter-Drinfel’d homomorphism* if \( f \) is an \( A \)-module map and an \( A \)-comodule map. For two Yetter-Drinfel’d
A-modules $M$ and $N$, the tensor product $M \otimes N$ becomes a Yetter-Drinfel’d $A$-module with the action and coaction:

$$a \cdot (m \otimes n) = \sum (a^{(1)} \cdot m) \otimes (a^{(2)} \cdot n),$$

$$m \otimes n \mapsto \sum n^{(0)} \otimes n^{(1)} m^{(1)}$$

for all $a \in A$, $m \in M$, $n \in N$. Then, the Yetter-Drinfel’d $A$-modules and the Yetter-Drinfel’d homomorphisms make a monoidal category together with the above tensor products. This category is called the Yetter-Drinfel’d category, and denoted by $A\text{YD}^A$. By Yetter [10] it is proved that the Yetter-Drinfel’d category $A\text{YD}^A$ has a prebraiding $c$, which is a collection of Yetter-Drinfel’d homomorphisms $c_{M,N} : M \otimes N \to N \otimes M$ defined by

$$c_{M,N}(m \otimes n) = \sum n^{(0)} \otimes (n^{(1)} \cdot m)$$

for all $m \in M$, $n \in N$. Furthermore, he also proved that if $A$ is a Hopf algebra, then the prebraiding $c$ is a braiding for $A\text{YD}^A$.

In [34] Radford constructed a functor from $A\text{M}$ to $A\text{YD}^A$, that is a right adjoint of the forgetful functor $R_A : A\text{YD}^A \to A\text{M}$ as shown in [13].

**Lemma 2.4 (Radford[34, Proposition 2], Hu and Zhang[13, Lemma 2.1]).** Let $A$ be a bialgebra over $k$, and suppose that $A^{op}$ has antipode $\overline{S}$. Let $L \in A\text{M}$. Then $L \otimes A \in A\text{YD}^A$, where the left $A$ action · and the right $A$-coaction $\rho$ are given by

$$h \cdot (l \otimes a) = \sum (h^{(2)} \cdot l) \otimes h^{(3)} a \overline{S}(h^{(1)}),$$

$$\rho(l \otimes a) = \sum (l \otimes a^{(1)}) \otimes a^{(2)}$$

for all $h, a \in A$ and $l \in L$. The correspondence $L \mapsto L \otimes A$ is extended to a functor $I_A : A\text{M} \to A\text{YD}^A$ that is a right adjoint of the forgetful functor $R_A$. The functor $I_A$ will be referred to as Radford’s induction functor derived from left $A$-modules.

From the above lemma there is a natural $k$-linear isomorphism $\varphi : \text{Hom}_{A\text{M}}(R_A(M), V) \to \text{Hom}_{A\text{YD}^A}(M, I_A(V))$ for all $M \in A\text{YD}^A$ and $V \in A\text{M}$. This isomorphism is given by

$$f \in \text{Hom}_{A\text{M}}(R_A(M), V) \mapsto \varphi(f) \in \text{Hom}_{A\text{YD}^A}(M, I_A(V)),$$

$$\varphi(f)(m) = \sum f(m^{(0)}) \otimes m^{(1)} \quad (m \in M). \quad (2.10)$$

Now we recall the following easy lemma from the category theory:

**Lemma 2.5.** Let $P : \mathcal{C} \to \mathcal{C}$ and $Q : \mathcal{D} \to \mathcal{D}$ be functors, where $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{D}$ and $\mathcal{D}'$ are arbitrary categories, and suppose that there are equivalences $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{C}' \to \mathcal{D}'$ of categories such that the diagram

$$\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{F'} & \mathcal{D}' \\
\downarrow P & & \downarrow Q \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}$$


commutes up to isomorphism. If \( P \) has a right (left) adjoint \( I \), then \( J = F' \circ I \circ F \) is a right (left) adjoint to \( Q \), where \( F \) is a quasi-inverse functor of \( F' \).

**Proof.** We only prove the case where \( P \) has a right adjoint. Let \( F' \) be a quasi-inverse of \( F' \). Then there are isomorphisms

\[
\text{Hom}_{\mathcal{D}'}(X, J(Y)) \cong \text{Hom}_{\mathcal{C}'}(F'(X), (I \circ F')(Y)) \\
\cong \text{Hom}_{\mathcal{C}}((P \circ F')(X), F(Y)) \\
\cong \text{Hom}_{\mathcal{C}}((F \circ Q)(X), F(Y)) \cong \text{Hom}_{\mathcal{D}}(Q(X), Y),
\]

which are natural in \( X \in \mathcal{D}' \) and \( Y \in \mathcal{D} \). Hence \( J \) is a right adjoint to \( Q \). \( \Box \)

If \( A \) is a finite-dimensional Hopf algebra over \( k \), then a Yetter-Drinfel’d \( A \)-module \( M \) becomes a left \( D(A) \)-module by the action given by \( (p \otimes a) \cdot m = \sum \langle p, (ma)_{(1)}(am)_{(0)} \rangle \) for \( p \in A^*, a \in A \) and \( m \in M \). This construction establishes an isomorphism \( {}_A \mathcal{YD}^A \cong {}_{D(A)} M \) of \( k \)-linear braided monoidal categories (see, e.g., [24]).

For a while, we denote by \( F' : {}_A \mathcal{YD}^A \rightarrow {}_{D(A)} M \) and \( R'_A : {}_{D(A)} M \rightarrow {}_A M \) the above isomorphism and the restriction functor, respectively. Since \( R'_A \circ F' = R_A = \text{id} \circ R_A \), the functor \( I'_A = F' \circ I_A \) is a right adjoint to \( R'_A \) by Lemma [24]. In what follows, based on this observation, we identify \( {}_{D(A)} M \) with \( {}_A \mathcal{YD}^A \) via the isomorphism \( F' \) and, abusing notation, write \( R'_A \) and \( I'_A \) as \( R_A \) and \( I_A \), respectively.

**Proposition 2.6.** Let \( A \) be a finite-dimensional Hopf algebra over \( k \). Then, the \( k \)-linear map \( \Phi : {}_{D(A)} A \rightarrow I_A(k) \) defined by \( \Phi(a) = 1 \otimes S^{-1}(a) \) for all \( a \in A \) is an isomorphism of left \( D(A) \)-modules. Here, \( k \) in \( I_A(k) \) means the trivial left \( A \)-module.

**Proof.** Let \( V \) be a left \( A \)-module. The left \( A^* \) action corresponding to the right coaction of \( I_A(V) \in {}_A \mathcal{YD}^A \) is given by

\[
p \cdot (v \otimes a) = \sum \langle p, (v \otimes a)_{(1)} \rangle (v \otimes a)_{(0)} = \sum \langle p, a_{(2)} \rangle v \otimes a_{(1)}
\]

for all \( p \in A^*, v \in V, a \in A \). Thus, the left \( D(A) \)-action on \( I_A(V) \) is given as follows.

\[
(p \otimes h) \cdot (v \otimes a) = p \cdot (h \cdot (v \otimes a)) \\
= \sum p \cdot ((h_{(2)} \cdot v) \otimes h_{(3)} a S^{-1}(h_{(1)})) \\
= \sum \langle p, (h_{(3)} a S^{-1}(h_{(1)}))_{(2)} \rangle (h_{(2)} \cdot v) \otimes (h_{(3)} a S^{-1}(h_{(1)}))_{(1)} \\
= \sum \langle p, h_{(5)} a_{(2)} S^{-1}(h_{(1)}) \rangle (h_{(3)} \cdot v) \otimes (h_{(4)} a_{(1)} S^{-1}(h_{(2)}))
\]

for all \( p \in A^*, h, a \in A, v \in V \). In particular, when \( V \) is the trivial \( A \)-module \( k \),

\[
(p \otimes h) \cdot (1 \otimes a) = \sum \langle p, h_{(4)} a_{(2)} S^{-1}(h_{(1)}) \rangle 1 \otimes (h_{(3)} a_{(1)} S^{-1}(h_{(2)})) \\
= \sum \langle S^{-1}(p), h_{(1)} S(a_{(2)}) S(h_{(4)}) \rangle 1 \otimes S^{-1}(h_{(2)} S(a_{(1)}) S(h_{(3)})).
\]

8
So, identifying $k \otimes H = H$, we have
\[(p \bowtie h) \cdot a = \sum (S^{-1}(p), h(1)S(a(2))S(h(4))) S^{-1}(h(2))S(a(1))S(h(3))).\]
This is equivalent to
\[S((p \bowtie h) \cdot S^{-1}(a)) = \sum (S^{-1}(p), h(1)a(1)S(h(4))) h(2)a(2)S(h(3)).\]
(2.11)
The right-hand side of the above equation coincides with the left action of the Schrödinger module $D(A)A$.
\[\square\]

As in a similar to Lemma 2.4, the following holds.

**Lemma 2.7 (Radford [34, Proposition 1], Hu and Zhang [13, Remark 2.2]).** Let $A$ be a bialgebra over $k$, and suppose that $A^{op}$ has antipode $S$. Let $N \in M^A$. Then $A \otimes N$ is an $A^{op}$-comodule.

For all $h, a \in A$ and $n \in N$. The correspondence $N \mapsto A \otimes N$ is extended to a functor $I^A : M^A \to A^{cop}D^A$ that is a left adjoint of the forgetful functor $R^A$. The functor $I^A$ will be referred to as Radford’s induction functor derived from right $A$-comodules.

Let $A$ be a finite-dimensional Hopf algebra over $k$. The Hopf algebra embedding $j : A^{cop} \hookrightarrow D(A)$ defined by $j(p) = p \bowtie 1_A$ induces a monoidal functor $D(A)M \to A^{cop}M \cong (M^A)^{rev}$. Here, $(M^A)^{rev}$ means the reversed monoidal category of $M^A$. Under the identification $A^{op}D^A = D(A)M$, the above functor $D(A)M \to (M^A)^{rev}$ coincides with the forgetful functor $R^A$ given in Lemma 2.7. So, we denote this functor by $R^A$, again. By Lemmas 2.5 and 2.7, we see that the Radford’s induction functor $I^A : (M^A)^{rev} \to A^{op}D^A = D(A)M$ is a left adjoint of $R^A : D(A)M \to (M^A)^{rev}$.

Let $A$ be a finite-dimensional Hopf algebra over $k$. Then the bialgebra $A^{op}$ has an antipode since the antipode $S$ of $A$ is bijective [33, Theorem 1]. Let $M = (M, \cdot, \rho_M)$ be a finite-dimensional Yetter-Drinfel’d $A$-module. Then the dual vector space $M^*$ becomes a Yetter-Drinfel’d $A$-module with respect to the following action · and coaction $\rho_{M^*}$:
\[(a \cdot \alpha)(m) = \alpha(S(a) \cdot m) \quad (a \in A, \alpha \in M^*, m \in M),\]
\[\rho_{M^*}(e_j^*) = \sum_{i=1}^d e_i^* \otimes S^{-1}(a_{ji}) \quad (j = 1, \ldots, d),\]
where $\{e_i\}_{i=1}^d$ is a basis for $M$ and $\{e_i^*\}_{i=1}^d$ is its dual basis, and $\rho_M(e_j) = \sum_{i=1}^d e_i \otimes a_{ij}$ for $j = 1, \ldots, d$. 

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**Proposition 2.8.** Let $A$ be a finite-dimensional Hopf algebra over $k$. Then, the $k$-linear map $\Phi : D_\mathcal{A} A^{\text{cop}} \to (I^A(k))^*$ defined by $\Phi(q) = S^{-1}(q) \otimes 1$ for all $q \in A^*$ is an isomorphism of left $D_\mathcal{A}$-modules. Here, $k$ in $I^A(k)$ means the trivial right $A$-comodule.

**Proof.** Let $W$ be a finite-dimensional right $A$-comodule. Applying Radford’s induction functor to the dual $A$-comodule $W^*$, we have a Yetter-Drinfel’d $A$-module $I^A(W^*) = A \otimes W^*$. As mentioned before Proposition 2.8, the dual $(I^A(W^*))^*$ is also a Yetter-Drinfel’d $A$-module whose left action is given by

$$(h \cdot f)(a \otimes \alpha) = f(S(h) \cdot (a \otimes \alpha)) = f(S(h) a \otimes \alpha) \quad (2.12)$$

for all $f \in (A \otimes W^*)^*$, $h, a \in H$, $\alpha \in W^*$. The coaction $\psi$ of $(I^A(W^*))^*$ is given as follows. Let $\{x_s\}_{s=1}^r$ and $\{e_i\}_{i=1}^d$ be bases for $A$ and $W$, respectively. Let $\{f_{si}\}_{1 \leq s \leq r \atop 1 \leq i \leq d}$ be the dual basis for $\{x_s \otimes e_i^*\}_{1 \leq s \leq r, 1 \leq i \leq d}$, that is, $f_{si} \in (A \otimes W^*)^*$ and $f_{si}(x_t \otimes e_j^*) = \delta_{si} \delta_{tj}$ for all $i, j = 1, \ldots, d$ and $s, t = 1, \ldots, r$. Then,

$$\psi(f_{si}) = \sum_{t=1}^r \sum_{j=1}^d f_{sj} \otimes S^{-1}(a_{(s,i),(t,j)}),$$

where $a_{(t,j),(s,i)} \in A$ is defined by $\rho(x_t \otimes e_i^*) = \sum_{j=1}^d \sum_{i=1}^r (x_t \otimes e_j^*) \otimes a_{(t,j),(s,i)}$. Let $\rho_W$ be the coaction of $W$, and write $\rho_W(e_j) = \sum_{i=1}^d e_i \otimes a_{ij} \ (a_{ij} \in A)$. Since

$$\rho(x_s \otimes e_i^*) = \sum_s (x_s(2) \otimes e_i^*) \otimes (x_s(3))^{-1}(a_{ij}s^{-1}(x_s(1))) = \sum_s (x_s(2), x_t^*) \otimes (x_s(3))^{-1}(x_s(1)a_{ij})$$

by Lemma 2.7(1), we see that $a_{(s,i),(s,i)} = \sum_s (x_s(2), x_t^*) (x_s(3))^{-1}(x_s(1)a_{ij})$, and whence

$$\psi(f_{si}) = \sum_{t=1}^r \sum_{j=1}^d (x_t(2) \otimes x_t^*) f_{tj} \otimes S^{-1}((x_t(3))^{-1}(x_t(1)a_{ij})).$$

Under the canonical identification $(A \otimes W^*)^* \cong W^{**} \otimes A^* \cong W \otimes A^*$ as vector spaces, we regard $W \otimes A^*$ as a Yetter-Drinfel’d $A$-module. Then, from the above observation it follows that the action and coaction on $W \otimes A^*$ are given by

$$a \cdot (w \otimes p) = w \otimes (a \cdot p),$$

$$\psi(w \otimes p) = \sum_{t=1}^r (p, x_t(2)) (w_0 \otimes x_t^*) \otimes S^{-1}((x_t(3))^{-1}(x_t(1)w_1))$$

for all $a \in A, w \in W, p \in A^*$, where $a \cdot p$ is the action of $A$ on $A^*$ as the dual $A$-module of the left regular $A$-module.
Considering the case where \( W \) is the trivial right \( A \)-comodule \( k \), we have a Yetter-Drinfel’d \( A \)-module structure on \( A^* \) whose action and coaction are given by

\[
a \cdot q = \sum_{t=1}^{r} \langle a \cdot q, x_t \rangle x_t^* = \sum_{t=1}^{r} \langle q, S(a)x_t \rangle x_t^* = \sum_{t=1}^{r} \langle q(1), S(a)q(2) \rangle,
\]

\[
\psi(p) = \sum_{t=1}^{r} \langle p, (x_t)(2) \rangle x_t^* \otimes S^{-1}((x_t)(3))S^{-1}((x_t)(1))
\]

So, the corresponding left \( D(A) \)-module structure on \( A^* \) is given by

\[
(p \triangleright h) \cdot q = p \cdot (h \cdot q)
\]

\[
= \sum_{t=1}^{r} \sum_{q} \langle h \cdot q, (x_t)(2) \rangle \langle p, S^{-1}((x_t)(3))S^{-1}((x_t)(1)) \rangle x_t^*
\]

\[
= \sum_{t=1}^{r} \sum_{q} \langle h \cdot q, (x_t)(2) \rangle \langle S^{-1}(p(2)), (x_t)(3) \rangle \langle S^{-2}(p(1)), (x_t)(1) \rangle x_t^*
\]

\[
= \sum_{t=1}^{r} S^{-2}(p(1)) \langle h \cdot q \rangle S^{-1}(p(2)), x_t x_t^*
\]

\[
= \sum_{t=1}^{r} \langle q(1), S(h)S^{-2}(p(1))q(2)S^{-1}(p(2)) \rangle
\]

for all \( p, q \in A^* \) and \( a \in A \). Therefore, the left \( D(A) \) action on \( A^* \cong (I^A(k^*))^* \) satisfies the equation

\[
S((p \triangleright h) \cdot S^{-1}(q)) = \sum q(2), h)p(2)q(1)S^{-1}(p(1)).
\]

(2.13)

The right-hand side coincides with the left \( D(A) \) action on the dual Schrödinger module \( D(A)^{A^{\text{cop}}} \). Since \( k^* \cong k \) as right \( A \)-comodules, the equation (2.13) gives rise to an isomorphism \( (I^A(k^*))^* \cong D(A)^{A^{\text{cop}}} \). \( \square \)

### 2.4 Tensor products of Schrödinger modules

In this subsection, we compute the tensor product of Schrödinger modules by using Propositions 2.2 and 2.8. First, we provide the following lemma, which is proved straightforwardly.

**Lemma 2.9.** Let \( A \) be a Hopf algebra over \( k \) with bijective antipode. Then:

1. There is a natural isomorphism of Yetter-Drinfel’d modules

\[
\Phi : I_A(V) \otimes M \longrightarrow I_A(V \otimes R_A(M)) \quad (V \in \text{\textit{AM}}, M \in \text{\textit{VD}})
\]

\[
given \ by \ \Phi(v \otimes a \otimes m) = \sum v \otimes m(0) \otimes m(1)a \ \text{for} \ v \in V, a \in A, \text{and} \ m \in M.
\]

2. There is a natural isomorphism of Yetter-Drinfel’d modules

\[
\Psi : I^A(V \otimes R^A(M)) \longrightarrow I^A(V) \otimes M \quad (V \in \text{\textit{MA}}, M \in \text{\textit{VD}})
\]

\[
given \ by \ \Psi(a \otimes v \otimes m) = \sum a(1) \otimes v \otimes a(2)m \ \text{for} \ v \in V, a \in A, \text{and} \ m \in M.
\]
For a Hopf algebra $A$, we denote by $A_{ad}$ the adjoint representation of $A$, i.e., the vector space $A$ endowed with the left $A$-module structure given by (2.3).

**Proposition 2.10.** Let $A$ be a finite-dimensional Hopf algebra over $k$. Then

$$(D(A)A)^{\otimes n} \cong I_A(A_{ad}^{\otimes (n-1)}).$$

**Proof.** By Proposition 2.6 and Lemma 2.9

$$(D(A)A)^{\otimes n} \cong I_A(k) \otimes (D(A)A)^{\otimes (n-1)} \cong (I_A \circ R_A)((D(A)A)^{\otimes (n-1)}) \cong I_A(A_{ad}^{\otimes (n-1)}).$$

The following result is a non-semisimple generalization of a part of [3, Proposition 4].

**Proposition 2.11.** $(D(A)A) \otimes (D(A)A^{*\text{cop}}) \cong D(A)$ as left $D(A)$-modules.

**Proof.** Let, in general, $H$ be a finite-dimensional Hopf algebra. As is well-known, there are natural isomorphisms of vector spaces

$$\text{Hom}_H(X \otimes H, Y) \cong \text{Hom}_k(X, Y) \cong \text{Hom}_H(X, Y \otimes H) \quad (X, Y \in H^M).$$

(2.14)

Since $(R_A \circ I^A)(k) \cong A$ as left $A$-modules, there are natural isomorphisms of vector spaces

$$\text{Hom}_{D(A)}(X, (D(A)A) \otimes (D(A)A^{*\text{cop}})) \cong \text{Hom}_{D(A)}(X, I_A(k) \otimes I^A(k)^*)$$

$$\cong \text{Hom}_{D(A)}(X \otimes I^A(k), I_A(k))$$

$$\cong \text{Hom}_A(R_A(X) \otimes A, k)$$

$$\cong R_A(X)^* = X^* \quad (\text{by } (2.14))$$

for $X \in (D(A)A)^M$. On the other hand, $\text{Hom}_{D(A)}(X, D(A)) \cong X^*$ by (2.14). Hence the result follows from Yoneda’s lemma.

$$\square$$

### 3 Categorical aspects of Schrödinger modules

#### 3.1 The center of a monoidal category

We first recall the center construction for monoidal categories, which was introduced by Drinfel’d (see Joyal and Street [16] and Majid [24]). Let $C = (\mathcal{C}, \otimes, \mathbb{I}, a, r, l)$ be a monoidal category, and let $V \in \mathcal{C}$ be an object. A *half-braiding* for $V$ is a natural isomorphism $c_{-, V} : (-) \otimes V \to V \otimes (-)$ such that the diagram

$$
\begin{array}{c}
(X \otimes Y) \otimes V \xrightarrow{c_{X \otimes Y,V}} V \otimes (X \otimes Y) \xleftarrow{a_{V,X,Y}^{-1}} (V \otimes X) \otimes Y \\
\downarrow a_{X,Y,V} \qquad \qquad \downarrow c_{X,V \otimes \text{id}Y} \\
X \otimes (Y \otimes V) \xrightarrow{\text{id}_X \otimes c_{Y,V}} X \otimes (V \otimes Y) \xrightarrow{a_{X,V,Y}^{-1}} (X \otimes V) \otimes Y
\end{array}
$$
commutes for all $X, Y \in \mathcal{C}$. The center of $\mathcal{C}$, denoted by $\mathcal{Z}(\mathcal{C})$, is the category defined as follows: An object of $\mathcal{Z}(\mathcal{C})$ is a pair $(V, c_{-,V})$ consisting of an object $V \in \mathcal{C}$ and a half braiding $c_{-,V}$ for $V$. If $(V, c_{-,V})$ and $(W, c_{-,W})$ are objects of $\mathcal{Z}(\mathcal{C})$, then a morphism from $(V, c_{-,V})$ to $(W, c_{-,W})$ is a morphism $f : V \to W$ in $\mathcal{C}$ such that

$$(f \otimes \text{id}_X) \circ c_{X,V} = c_{X,W} \circ (\text{id}_Y \otimes f)$$

for all $X \in \mathcal{C}$. The composition of morphisms in $\mathcal{Z}(\mathcal{C})$ is given in an obvious way. Forgetting the half-braiding defines a faithful functor $\Pi_{\mathcal{C}} : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, which will be referred to as the forgetful functor from $\mathcal{Z}(\mathcal{C})$ to $\mathcal{C}$.

The category $\mathcal{Z}(\mathcal{C})$ is in fact a braided monoidal category: First, the tensor product of objects of $\mathcal{Z}(\mathcal{C})$ is defined by $(V, c_{-,V}) \otimes (W, c_{-,W}) = (V \otimes W, c_{-,V \otimes W})$, where the half-braiding $c_{-,V \otimes W}$ for $V \otimes W$ is a unique natural isomorphism making the diagram

$$
\begin{array}{ccc}
X \otimes (V \otimes W) & \xrightarrow{c_{X,V \otimes W}} & (V \otimes W) \otimes X \\
\downarrow a_{X,Y,V} & & \downarrow \text{id}_W \otimes c_{X,W} \\
(X \otimes V) \otimes W & \xrightarrow{c_{X,V} \otimes \text{id}_X} & (V \otimes X) \otimes W \\
& & \xrightarrow{a_{V,X,W}} V \otimes (X \otimes W)
\end{array}
$$

commute for all $X \in \mathcal{C}$. The unit object is $I_{\mathcal{Z}(\mathcal{C})} := (I, l^{-1} \circ r)$. The associativity and the unit constraints for $\mathcal{Z}(\mathcal{C})$ are defined so that the forgetful functor $\Pi_{\mathcal{C}}$ is strict monoidal. Finally, the braiding of $\mathcal{Z}(\mathcal{C})$ is given by

$$c = \left\{c_{V,W} : (V, c_{-,V}) \otimes (W, c_{-,W}) \to (W, c_{-,W}) \otimes (V, c_{-,V})\right\}_{(V, c_{-,V}), (W, c_{-,W}) \in \mathcal{Z}(\mathcal{C})}.$$

Since the center construction is described purely in terms of monoidal categories, it is natural to expect that equivalent monoidal categories have equivalent centers. We omit to give a proof of the following well-known fact, since the proof is easy but quite long.

**Lemma 3.1.** For each monoidal equivalence $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories $\mathcal{C}$ and $\mathcal{D}$, there exists a unique braided monoidal equivalence

$$\mathcal{Z}(F) : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$$

such that $\Pi_{\mathcal{D}} \circ \mathcal{Z}(F) = \Pi_{\mathcal{C}} \circ F$ as monoidal functors. The assignment $F \mapsto \mathcal{Z}(F)$ enjoys the following properties:

1. If $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ is a sequence of monoidal equivalences between monoidal categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, then $\mathcal{Z}(G \circ F) = \mathcal{Z}(G) \circ \mathcal{Z}(F)$.

2. Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ be monoidal equivalences. For each monoidal natural transformation $\alpha : F_1 \to F_2$, a monoidal natural transformation

$$\mathcal{Z}(\alpha) : \mathcal{Z}(F_1) \to \mathcal{Z}(F_2)$$

enjoys the following properties:

(a) $\Pi_{\mathcal{D}} \circ \mathcal{Z}(\alpha) = \Pi_{\mathcal{C}} \circ \alpha$.

(b) For each $V \in \mathcal{Z}(\mathcal{C})$, the induced natural transformation $\alpha_{V} : F_1(V) \to F_2(V)$ is a monoidal natural transformation in $\mathcal{D}$.
is defined by $Z(\alpha)_X = \alpha_X$ for $X = (X, \sigma_X) \in Z(C)$. Moreover, $\alpha \mapsto Z(\alpha)$ preserves the horizontal and the vertical compositions of natural transformations.

By a $k$-linear monoidal category, we mean a monoidal category $C$ such that $\text{Hom}_C(X, Y)$ is a vector space over $k$ for all objects $X, Y \in C$, and the composition and the tensor product of morphisms in $C$ are linear in each variables. We note that if $C$ is a $k$-linear monoidal category, then so is $Z(C)$ in such a way that the functor $\Pi_C$ is $k$-linear. If $F : C \rightarrow D$ is a $k$-linear monoidal equivalence between $k$-linear monoidal categories, then $Z(F)$ is $k$-linear.

### 3.2 Schrödinger modules in monoidal categories

Let $C$ be a monoidal category. We say that an object $A \in Z(C)$ is a 
Schrödinger object if there exists a bijection $\text{Hom}_C(\Pi_C(X), I_{Z(C)}) \cong \text{Hom}_{Z(C)}(X, A)$, which is natural in the variable $X \in Z(C)$. Note that such an object is unique up to isomorphism by Yoneda’s lemma (if it exists).

**Lemma 3.2.** Let $C$ and $D$ be monoidal categories.

1. Suppose that $\Pi_C : Z(C) \rightarrow C$ has a right adjoint functor $I_C : C \rightarrow Z(C)$. Then the object $I_C(I)$ is a Schrödinger object for $C$.

2. Suppose that $A_C$ is a Schrödinger object for $C$, and there exists an equivalence $F : C \rightarrow D$ of monoidal categories. Then $Z(F)(A_C)$ is a Schrödinger object for $D$.

**Proof.** Part (1) follows immediately from the definition of the adjoint functor. Part (2) is shown as follows: Let $\overline{F}$ be a quasi-inverse of $F$. Then $Z(\overline{F})$ is a quasi-inverse of $Z(F)$. Hence,

$$\text{Hom}_{Z(D)}(X, Z(F)(A_C)) \cong \text{Hom}_{Z(C)}(Z(\overline{F})(X), A_C) \cong \text{Hom}_C((\Pi_C \circ Z(F))(X), I)$$

$$\cong \text{Hom}_D((F \circ \Pi_D)(X), I) \cong \text{Hom}_D(\Pi_D(X), \overline{F}(I)) \cong \text{Hom}_D(\Pi_D(X), I)$$

for all $X \in Z(C)$. Therefore, $Z(F)(A_C)$ is the Schrödinger object for $D$.

Now, let $A$ be a Hopf algebra over $k$ with bijective antipode. Then

$$\Phi_A : Z(A) \rightarrow A \mathcal{YD}^A, \quad (V, c_-, V) \mapsto (V, \rho), \quad \quad (3.1)$$

where $\rho : V \rightarrow V \otimes A$ is the linear map given by $\rho(v) = c_{A,V}(1_A \otimes v)$ ($v \in V$), is an isomorphism of $k$-linear braided monoidal categories [10, 24].

**Theorem 3.3.** Let $A$ and $B$ be Hopf algebras over $k$ with bijective antipode. Suppose that there is an equivalence $F : A \rightarrow B$ of $k$-linear monoidal categories.

1. $S_A := (\Phi_A^{-1} \circ I_A)(k) \in Z(A)$ is a Schrödinger object for $A$.
(2) There uniquely exists an equivalence \( \tilde{F} : \text{A}\mathcal{YD}^A \to \text{B}\mathcal{YD}^B \) of \( k \)-linear braided monoidal categories such that \( R_B \circ \tilde{F} = F \circ R_A \) as monoidal functors.

(3) The functor \( \tilde{F} \) in Part (2) satisfies \( I_B \circ F \cong \tilde{F} \circ I_A \).

**Proof.** To prove Part (1), note that the diagram

\[
\begin{array}{ccc}
\mathcal{Z}(A\mathcal{M}) & \xrightarrow{\Phi_A} & A\mathcal{YD}^A \\
\Pi_{A\mathcal{M}} & & \downarrow R_A \\
A\mathcal{M} & \xrightarrow{id} & A\mathcal{M}
\end{array}
\]  

(3.2)

commutes. Applying Lemma 2.5 to this diagram, we see that \( \Phi_A^{-1} \circ I_A \) is a right adjoint functor to \( \Pi_{A\mathcal{M}} \). Hence \( S_A \) is a Schrödinger object for \( A\mathcal{M} \) by Part (1) of Lemma 3.2.

Now we show Part (2). It is easy to see that \( \tilde{F} = \Phi_B^{-1} \circ Z(F) \circ \Phi_A \) satisfies the required conditions. To show the uniqueness, note that (3.2) is in fact a commutative diagram of monoidal functors. Thus, if \( \tilde{F} \) satisfies the required conditions, then, by (3.2),

\[
\Pi_{B\mathcal{M}} \circ (\Phi_B \circ \tilde{F} \circ \Phi_A^{-1}) = R_B \circ \tilde{F} \circ \Phi_A^{-1} = F \circ R_A \circ \Phi_A^{-1} = F \circ \Pi_{A\mathcal{M}}
\]

as monoidal functors. Hence \( \tilde{F} = \Phi_B^{-1} \circ Z(F) \circ \Phi_A \) by the uniqueness part of Lemma 3.1.

To prove Part (3), we recall that \( I_A \) is a right adjoint to \( R_A \). Let \( \overline{F} \) be a quasi-inverse of \( F \). By Part (2) and Lemma 2.5 \( I' := \tilde{F} \circ I_A \circ \overline{F} \) is a right adjoint to \( R_A \). By the uniqueness of the right adjoint functor, we have \( I' \cong I_B \). Hence, \( I_B \circ F \cong \tilde{F} \circ I_A \).

Suppose that \( A \) is finite-dimensional. Then we obtain an isomorphism

\[
\Psi_A : \mathcal{Z}(A\mathcal{M}) \xrightarrow{\cong} _{D(A)}\mathcal{M}
\]

(3.3)
of \( k \)-linear braided monoidal categories by composing (3.1) and the isomorphism \( \text{A}\mathcal{YD}^A \cong _{D(A)}\mathcal{M} \) (which we have recalled in Section 2). Note that

\[
\Pi_{A\mathcal{M}} = R_A \circ \Psi_A
\]
as monoidal functors, where \( R_A \) is the restriction functor \( _{D(A)}\mathcal{M} \to A\mathcal{M} \) (cf. (3.1) and (3.2)). Let \( I_A \) be the right adjoint functor of \( R_A \) given in Section 2. Recall from Proposition 2.8 that the Schrödinger module \( _{D(A)}A \) is isomorphic to \( I_A(k) \). By the same way as the previous theorem, we prove:

**Theorem 3.4.** Let \( A \) and \( B \) be finite-dimensional Hopf algebras over the same field \( k \). Suppose that there is an equivalence \( F : A\mathcal{M} \to B\mathcal{M} \) of \( k \)-linear monoidal categories.

(1) \( S_A := \Psi_A^{-1}(D(A)) \in \mathcal{Z}(A\mathcal{M}) \) is a Schrödinger object for \( A\mathcal{M} \).
(2) There uniquely exists an equivalence $\tilde{F} : D(A)M \rightarrow D(B)M$ of $k$-linear braided monoidal categories such that $R_B \circ \tilde{F} = F \circ R_A$ as monoidal functors.

(3) The functor $\tilde{F}$ in Part (2) satisfies $I_B \circ F \cong \tilde{F} \circ I_A$.

An important corollary is the following invariance of the Schrödinger module:

**Corollary 3.5.** The equivalence $\tilde{F} : D(A)M \rightarrow D(B)M$ of Theorem 3.4 preserves the Schrödinger module, i.e., $\tilde{F}(D(A)A) \cong D(B)B$.

**Remark 3.6.** An equivalence $G : D(A)M \rightarrow D(B)M$ of $k$-linear braided monoidal categories does not preserve the Schrödinger module in general.

**Remark 3.7.** As Masuoka pointed out to us, Corollaries 3.5 and 3.9 below can be also derived from the point of view of cocycle deformations by using the action given in [26, Proposition 5.1].

### 3.3 The dual Schrödinger module as a Schrödinger object

Let $A$ be a finite-dimensional Hopf algebra over $k$. Recall from Section 2 that there exists an isomorphism $D(A) \cong D(A^*)^\text{cop}$ of quasitriangular Hopf algebras. Since $M^A$ is isomorphic to $A^*M$ as $k$-linear monoidal categories, we have isomorphisms

$$Z(M^A)^\text{rev} \cong Z(A^*M)^\text{rev} \cong (D(A^*)^\text{rev}) \cong D(A^*)^\text{cop}M \cong D(A)M$$

(3.4)

of $k$-linear braided monoidal categories. Now, let $\Xi_A : Z(M^A)^\text{rev} \rightarrow D(A)M$ be the composition of the above isomorphisms. One can check that the diagram

$$\begin{array}{ccc}
Z(M^A)^\text{rev} & \xrightarrow{\Xi_A} & D(A)M \\
\downarrow_{R^A} & & \downarrow_{R^A} \\
(M^A)^\text{rev} & \xrightarrow{\text{id}} & (M^A)^\text{rev}
\end{array}$$

(3.5)

commutes, where $R^A$ is the functor used in Subsection 2.1. By Lemma 2.7 the functor $I^A : M^A \rightarrow D(A)M$ is a left adjoint to $R^A$. The following theorem is proved by the same way as Theorem 3.4.

**Theorem 3.8.** Let $A$ and $B$ be finite-dimensional Hopf algebras over the same field $k$. Suppose that there is an equivalence $F : M^A \rightarrow M^B$ of $k$-linear monoidal categories.

(1) There uniquely exists an equivalence $\tilde{F} : D(A)M \rightarrow D(B)M$ of $k$-linear braided monoidal categories such that $R^B \circ \tilde{F} = F \circ R^A$ as monoidal functors.

(2) The functor $\tilde{F}$ in Part (1) satisfies $I^B \circ F \cong \tilde{F} \circ I^A$.  

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In Proposition 2.8 it is shown that the dual Schrödinger module $D(A)A^\text{cop}$ is isomorphic to the left dual of $I^A(k)$. The following corollary is obtained by Part (2) of the above theorem, and the fact that a monoidal equivalence preserves left duals.

**Corollary 3.9.** The equivalence $\tilde{F} : D(A)M \to D(B)M$ of Theorem 3.8 preserves the dual Schrödinger module, i.e., $\tilde{F}(D(A)A^\text{cop}) \cong D(B)B^\text{cop}$.

We have introduced the notion of a Schrödinger object to explain categorical nature of the Schrödinger module. The following theorem claims that also the dual Schrödinger object can be interpreted in terms of a Schrödinger object:

**Theorem 3.10.** $S_A := \Xi^{-1}_A (D(A)A^\text{cop}) \in Z(M^A)$ is a Schrödinger object for $M^A$.

**Proof.** For simplicity, write $\text{Hom}_{M^A}(X,Y)$ as $\text{Hom}^A(X,Y)$. By Proposition 2.8 and Lemma 2.9 we obtain natural isomorphisms

$$\text{Hom}_{D(A)}(X, D(A)A^\text{cop}) \cong \text{Hom}_{D(A)}(I^A(k) \otimes X, k)$$

$$\cong \text{Hom}_{D(A)}(I^A(k \otimes R^A(X)), k) \cong \text{Hom}^A(R^A(X), k)$$

for $X \in D(A)M$. Hence there are natural isomorphisms

$$\text{Hom}_{Z(M^A)}(X, S_A) \cong \text{Hom}_{D(A)}(\Xi_A(X), D(A)A^\text{cop})$$

$$\cong \text{Hom}^A((R^A \circ \Xi_A)(X), k) \cong \text{Hom}^A(\Pi_{M^A}(X), k)$$

for $X \in Z(M^A)$. This shows that $S_A \in Z(M^A)$ is a Schrödinger object for $M^A$. \hfill \Box

The following theorem is a slight weak version of Proposition 2.2 and its corollary. Emphasizing the role of the Schrödinger object, we now reexamine the proof of Proposition 2.2 and its corollary.

**Theorem 3.11.** For a finite-dimensional Hopf algebra $A$ over $k$, we denote by

$$F_A : (D(A^\ast)M)^\text{rev} = D(A^\ast)^\text{cop}M \to D(A)M$$

(3.6)

the isomorphism of $k$-linear braided monoidal categories induced by the isomorphism $\phi_A$ given by (2.2). Then there are isomorphisms

$$F_A : (D(A^\ast)A^\ast)^\text{cop} \cong D(A)A^\text{cop} \quad \text{and} \quad F_A((D(A^\ast))A^\ast)^\text{cop} \cong D(A)A.$$

**Proof.** To establish the first isomorphism, we first note that the category isomorphism $F_A$ is expressed as follows (cf. (3.4)):

$$F_A : D(A^\ast)^\text{cop}M \cong (D(A^\ast)M)^\text{rev} \xrightarrow{\Psi_{A^\ast}} Z_{(A^\ast)M}^\text{rev} \cong Z(M^A)^\text{rev} \xrightarrow{\Xi_{A^\ast}} D(A)M,$$
where the arrow $\mathcal{Z}(A \circ M)^{\text{rev}} \rightarrow \mathcal{Z}(A^\circ M)^{\text{rev}}$ is the isomorphism induced by $A \circ M \cong M^A$. Theorem [3.4] tells us that $D(A^\circ A)^* \rightarrow M(A^\circ A)^*$ is an object corresponding to a Schrödinger object for $A \circ M$. On the other hand, Theorem [3.10] tells us that $D(A^\circ A)^* \rightarrow M(A^\circ A)^*$ is an object corresponding to a Schrödinger object for $M^A$. Hence, by Lemma [3.2] we have $F_A(D(A^\circ A)^*) \cong D(A^\circ A)^{\text{cop}}$.

Once the first isomorphism is obtained, the second isomorphism can be obtained by replacing $A$ with $A^*$ (cf. the proof of Corollary 2.3).

4 Applications

Motivated by the construction of quantum representations of the $n$-strand braid group $B_n$ due to Reshetikhin and Turaev [36], a family of monoidal Morita invariants of a finite-dimensional Hopf algebra, which is indexed by braids, can be obtained from the Schrödinger module.

Let $A$ be a finite-dimensional Hopf algebra. It turns out that the invariant associated with the identity element $1 \in B_1$ is equal to the quantum dimension the Schrödinger module $D(A^\circ A)$ in the sense of Majid [25], and thus equal to $\text{Tr}(S^2)$ by [25, Example 9.3.8], where $S$ is the antipode of $A$ (see Bulacu-Torrecillas [2] for the case of quasi-Hopf algebras). As is well-known, $\text{Tr}(S^2)$ has the following representation-theoretic meaning: $\text{Tr}(S^2) \neq 0$ if and only if $A$ is semisimple and cosemisimple [30]. In this section, we show that the invariants derived from other braids, like \( \lambda \), involve further interesting results connecting with representation theory.

The invariant associated with a braid $b$ is, roughly speaking, defined by “coloring” the closure of $b$ by the Schrödinger module as if we were computing the quantum invariant of a (framed) link. Such an operation is not allowed in general since $D(A^\circ A)$ may not be a ribbon category. So we will use the (partial) braided trace, introduced below, to define invariants.

4.1 Partial traces in braided monoidal categories

We briefly recall the concept of duals in a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{I}, a, r, l)$. For an object $X \in \mathcal{C}$ the triple $(X^*, e_X, n_X)$ consisting of an object $X^* \in \mathcal{C}$ and morphisms $e_X : X^* \otimes X \rightarrow \mathbb{I}$, $n_X : \mathbb{I} \rightarrow X \otimes X^*$ in $\mathcal{C}$ is said to be a left dual if two compositions

\[
X \xrightarrow{l^{-1}} \mathbb{I} \otimes X \xrightarrow{n_X \otimes \text{id}} (X \otimes X^*) \otimes X \xrightarrow{\alpha} X \otimes (X^* \otimes X) \xrightarrow{\text{id} \otimes e_X} X \otimes \mathbb{I} \xrightarrow{r} X,
\]

\[
X^* \xrightarrow{r^{-1}} X^* \otimes \mathbb{I} \xrightarrow{\text{id} \otimes n_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}} (X^* \otimes X) \otimes X^* \xrightarrow{e_X \otimes \text{id}} \mathbb{I} \otimes X \xrightarrow{l} X
\]

are equal to $\text{id}_X$ and $\text{id}_{X^*}$, respectively. If all objects in $\mathcal{C}$ have left duals, then the monoidal category is called left rigid.

From now on, all monoidal categories are assumed to be strict although almost all definitions and results are not needed this assumption.
Let \((\mathcal{C}, c)\) is a left rigid braided monoidal category. We choose a left dual \((X^*, e_X, n_X)\) for each object \(X \in \mathcal{C}\). Let \(f : X \otimes Y \rightarrow X \otimes Z\) be an morphism in \(\mathcal{C}\). Then the following composition \(\text{Tr}^l_{c,X}(f) : Y \rightarrow Z\) can be defined:

\[
Y \xrightarrow{r_Y^{-1}} Y \otimes I \xrightarrow{n_X \otimes \text{id}_Y} X \otimes X^* \otimes Y \xrightarrow{\text{id}_X \otimes f} X^* \otimes X \otimes Y \xrightarrow{e_{X^*, X} \otimes \text{id}_Y} X \otimes X^* \otimes I \xrightarrow{\text{id}_X \otimes f} X \otimes X^* \otimes Y \xrightarrow{n_X \otimes \text{id}_Y} X \otimes X^* \otimes Z \xrightarrow{e_X \otimes \text{id}_Z} I \otimes Z \xrightarrow{\text{id}_Z \otimes l_Z} Z.
\]

The morphism \(\text{Tr}^l_{c,X}(f) : Y \rightarrow Z\) is said to be the left partial braided trace of \(f\) on \(X\).

Similarly, for a morphism \(f : \otimes X \rightarrow \otimes X\), the right partial braided trace of \(f\) on \(X\) is defined by the composition

\[
Y \xrightarrow{r_Y^{-1}} Y \otimes I \xrightarrow{\text{id}_X \otimes n_X} \otimes X \otimes X^* \xrightarrow{\text{id} \otimes f} \otimes X \otimes X^* \xrightarrow{e_{X^*, X} \otimes \text{id}_I} \otimes X \otimes X^* \otimes \otimes X \xrightarrow{\text{id} \otimes e_X} \otimes \otimes X \xrightarrow{\text{id}_X \otimes f} \otimes X \otimes X^* \otimes I \xrightarrow{\text{id}_X \otimes f} \otimes X \otimes X^* \otimes Z \xrightarrow{\text{id}_X \otimes e_Z} I \otimes Z \xrightarrow{\text{id}_Z \otimes r_Z} Z.
\]

For an endomorphism \(f : \rightarrow \) in \((\mathcal{C}, c)\), the left braided trace \(\text{Tr}^l_{c}(f)\) and the right braided trace \(\text{Tr}^r_{c}(f)\) are defined by \(\text{Tr}^l_{c}(f) := \text{Tr}^l_{c,X}(f \otimes \text{id}_I)\) and \(\text{Tr}^r_{c}(f) := \text{Tr}^r_{c,X}(\text{id}_I \otimes f)\). They coincide with the following compositions, respectively.

\[
\text{Tr}^l_{c}(f) : I \xrightarrow{n_X} X \otimes X^* \xrightarrow{\text{id}_X \otimes f} X^* \otimes X \xrightarrow{e_X} I,
\]

\[
\text{Tr}^r_{c}(f) : I \xrightarrow{n_X} X \otimes X^* \xrightarrow{f \otimes \text{id}_X} \otimes X \otimes X^* \xrightarrow{e_X \otimes \text{id}_I} I.
\]
Lemma 4.1. Let \((C, c)\) and \((D, c')\) be left rigid braided monoidal categories, and \((F, \phi, \omega) : C \to D\) be a braided monoidal functor. Then

(1) For any morphism \(f : X \otimes Y \to X \otimes Z\) in \(C\)

\[
F(\mathcal{T}_{c,F}^{X}(f)) = \mathcal{T}_{c',F(X)}^{X}(\phi_{X,Z}^{-1} \circ F(f) \circ \phi_{X,Y}).
\]  
(4.3)

(2) For any morphism \(f : Y \otimes X \to Z \otimes X\) in \(C\)

\[
F(\mathcal{T}_{c,F}^{Y}(f)) = \mathcal{T}_{c,F(X)}^{X}(\phi_{X,Y}^{-1} \circ F(f) \circ \phi_{X,Y}).
\]  
(4.4)

Proof. For each \(X \in C\) we choose a left dual \((X^*, e_X, n_X)\). Then \((F(X^*), e'_{F(X)}, n'_{F(X)})\) is a left dual of \(F(X)\), where

\[
e_{F(X)}' := \omega^{-1} \circ F(e_X) \circ \phi_{X^*,X} : F(X^*) \otimes F(X) \to I',
\]

\[
n'_{F(X)} := \phi_{X^*,X}^{-1} \circ F(n_X) \circ \omega : I' \to F(X) \otimes F(X^*).
\]

By using this left dual of \(F(X)\) and computing the partial braided trace \(\mathcal{T}_{c}^{X}(\phi_{X,Z}^{-1} \circ F(f) \circ \phi_{X,Y})\), we have the desired equation (4.3). Part (2) is also proved as in the same manner with Part (1).

\[\square\]

Let us describe a relationship between left and right partial traces. Let \(c\) be a braiding for a monoidal category \(C\). Then, the collection \(\overline{c}\) consisting of isomorphisms \(\overline{c}_{X,Y} := c_{Y,X}^{-1} : X \otimes Y \to Y \otimes X\) over all pairs \((X, Y)\) of objects in \(C\) is also a braiding for \(C\).

Let \((C, c)\) be a left rigid braided monoidal category chosen left duals \((X^*, e_X, n_X)\) for all objects \(X\) in \(C\). Then, for two objects \(X, Y\) in \(C\) there is a natural isomorphism \(j_{X,Y} : Y^* \otimes X^* \to (X \otimes Y)^*\) such that \(e_{X \otimes Y} \circ (j_{X,Y} \circ \text{id}_{X \otimes Y}) = e_Y \circ (\text{id}_{Y^*} \otimes e_X \otimes \text{id}_Y)\) [12]. For any morphism \(f : X \to Y\) in \((C, c)\), there is a unique morphism \(t'f : Y^* \to X^*\) in \(C\), which is characterized by \(e_X \circ (t'f \otimes \text{id}_X) = e_Y \circ (\text{id}_{Y^*} \otimes f)\). The morphism \(t'f\) is called the left transpose of \(f\). The left and right partial traces are related as follows.

Lemma 4.2. For any morphism \(f : X \otimes Y \to X \otimes Z\) in a left rigid braided monoidal category \((C, c)\),

\[
\mathcal{T}_{c,X^*}^{X}(-) = t'(\mathcal{T}_{c}^{X}(f)).
\]

Proof. The equation of the lemma is obtained from a graphical calculus depicted as in Figure [2]

\[\square\]

Let \(M\) be an object in a strict left rigid braided monoidal category \((C, c)\). For each endomorphism \(f \in \text{End}(M^\otimes n)\) and each positive integer \(k\) \((1 \leq k \leq n)\), we set \(\mathcal{T}_{c}^{M^\otimes k}(f) := \mathcal{T}_{c}^{M^\otimes k}(f)\), \(\mathcal{T}_{c}^{r,M^\otimes k}(f) := \mathcal{T}_{c}^{r,M^\otimes k}(f)\), and

\[
\tilde{T}_{c}(f) := (\mathcal{T}_{c}^{1}(f) \circ \cdots \circ \mathcal{T}_{c}^{1}(f)) \circ (\mathcal{T}_{c}^{1}(f) \circ \cdots \circ \mathcal{T}_{c}^{1}(f)).
\]

\[\text{(4.5)}\]
The modified traces (4.5) are preserved by a braided monoidal functor. More precisely:

**Proposition 4.3.** Let $\mathcal{C} = (\mathcal{C}, c)$ and $\mathcal{D} = (\mathcal{D}, c')$ be strict left rigid braided monoidal categories, and $(F, \phi, \omega): \mathcal{C} \to \mathcal{D}$ be a braided monoidal functor. Let $M$ be an object in $\mathcal{C}$, and $k$ be a positive integer, and define the isomorphism $\phi(k): F(M)^{\otimes k} \to F(M^{\otimes k})$ in $\mathcal{D}$ by

$$\phi(1) := \text{id}_{F(M)}, \quad \phi(k) := \phi_{M, M^{\otimes (k-1)}} \circ (\text{id}_{F(M)} \otimes \phi(k-1)) \quad (k \geq 2).$$

Then for an endomorphism $f$ on $M^{\otimes n}$ in $\mathcal{C}$, the following equations hold.

$$\tilde{\text{Tr}}^I_{c'}((\phi(n))^{-1} \circ F(f) \circ \phi(n)) = \omega^{-1} \circ (F(\tilde{\text{Tr}}^I_{c'}(f))) \circ \omega, \quad (4.6)$$

$$\tilde{\text{Tr}}^r_{c'}((\phi(n))^{-1} \circ F(f) \circ \phi(n)) = \omega^{-1} \circ (F(\tilde{\text{Tr}}^r_{c'}(f))) \circ \omega. \quad (4.7)$$

Therefore, if $\mathcal{C}$, $\mathcal{D}$ and $(F, \phi, \omega)$ are $k$-linear, then

$$\tilde{\text{Tr}}^I_{c'}((\phi(n))^{-1} \circ F(f) \circ \phi(n)) = \tilde{\text{Tr}}^I_{c'}(f), \quad \tilde{\text{Tr}}^r_{c'}((\phi(n))^{-1} \circ F(f) \circ \phi(n)) = \tilde{\text{Tr}}^r_{c'}(f)$$

as elements in $k$.

**Proof.** We set $g := (\phi(n))^{-1} \circ F(f) \circ \phi(n)$. By (4.4) and Lemma 4.1 we have

$$\tilde{\text{Tr}}^{I}_{c'}(g) = (\phi(n-1))^{-1} \circ \left(\tilde{\text{Tr}}^{I}_{c'}(\phi^{-1}_{M, M^{\otimes (n-1)}} \circ F(f) \circ \phi_{M, M^{\otimes (n-1)}})\right) \circ \phi(n-1)$$

$$= (\phi(n-1))^{-1} \circ F\left(\tilde{\text{Tr}}^{I}_{c'}(f)\right) \circ \phi(n-1).$$

The same arguments for $f_1 := \tilde{\text{Tr}}^{I}_{c'}(f)$ and $g_1 := \tilde{\text{Tr}}^{I}_{c'}(g)$ provide the equation

$$\tilde{\text{Tr}}^{I}_{c'}(\tilde{\text{Tr}}^{I}_{c'}(g)) = (\phi(n-2))^{-1} \circ F\left(\tilde{\text{Tr}}^{I}_{c'}(\tilde{\text{Tr}}^{I}_{c'}(f))\right) \circ \phi(n-2).$$

By repeating the same arguments, the equation

$$\tilde{\text{Tr}}^{I}_{c'}(\cdots \tilde{\text{Tr}}^{I}_{c'}(g)) = F\left(\tilde{\text{Tr}}^{I}_{c'}(\cdots \tilde{\text{Tr}}^{I}_{c'}(f))\right) \quad (4.8)$$

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is obtained. Setting \( f_{n-1} := (\prod_{i=1}^{n-1} f_i) (f) \) and applying \( \tilde{\text{Tr}}^l \) to the equation \( (4.8) \), we have the desired equation

\[
\tilde{\text{Tr}}^l_r (g) = \tilde{\text{Tr}}^l_r \left( F(f_{n-1}) \right) = \omega^{-1} \circ F(\tilde{\text{Tr}}^l_r (f_{n-1})) \circ \omega = \omega^{-1} \circ F(\tilde{\text{Tr}}^l_r (f)) \circ \omega.
\]

As in a similar way, the equation \( (4.7) \) can be shown by using \( \phi^{(k)} = \phi_{\mathcal{M} \otimes (k-1), \mathcal{M}} \circ (\phi^{(k-1)} \otimes \text{id}_{F(\mathcal{M})}) \). The last two equations are immediately derived from \( (4.6) \) and \( (4.7) \) by setting \( \tilde{\text{Tr}}^n_r = \lambda \text{id}_r \) and \( \tilde{\text{Tr}}^r_r = \lambda' \text{id}_r \) for some \( \lambda, \lambda' \in \mathbb{k} \).

As the same manner of the proof of the above proposition with help from Lemma \( 4.2 \) we have:

**Proposition 4.4.** Let \( \mathcal{C} = (\mathcal{C}, c) \) be a \( k \)-linear strict left rigid braided monoidal category. Let \( M \) be an object in \( \mathcal{C} \), and \( k \) be a positive integer, and define the isomorphism \( j^{(k)} : (M^*)^\otimes k \rightarrow (M^\otimes k)^* \) in \( \mathcal{C} \) by

\[
j^{(1)} := \text{id}_{M^*}, \quad j^{(k)} := j_{M^\otimes (k-1), M} \circ (\text{id}_{M^*} \otimes j^{(k-1)}) \quad (k \geq 2).
\]

Then for an endomorphism \( f \) on \( M^\otimes n \) in \( \mathcal{C} \), the following equation holds as elements in \( \mathbb{k} \):

\[
\tilde{\text{Tr}}^r_r ((j^{(n)})^{-1} \circ tf \circ j^{(n)}) = \tilde{\text{Tr}}^r_r (f).
\]

Given a quasitriangular Hopf algebra \( (A, R) \), a braiding \( c^R = \{ c^R_{X,Y} : X \otimes Y \rightarrow Y \otimes X \} \}_{X,Y \in \mathcal{A}} \) is defined by

\[
(c^R)_{X,Y} (x \otimes y) = T_{X,Y} (R \cdot (x \otimes y))
\]

for all \( x \in X \) and \( y \in Y \), where \( R \cdot (x \otimes y) \) is the diagonal action of \( R \) to \( X \otimes Y \). Denoted by \( (A, R) \mathcal{M} \) is the braided monoidal category \( (\mathcal{A} \mathcal{M}, c^R) \). We use the notation \( \text{Tr}_R \) instead of \( \text{Tr}_L \).

**Example 4.5.** Let \( (A, R) \) a quasitriangular Hopf algebra over \( \mathbb{k} \), and \( u \) be the Drinfeld’s element of it. It is well-known that the Drinfeld’s element \( u \) is invertible, and when \( R \) is written as \( R = \sum_j \alpha_j \otimes \beta_j \), the inverse is given by \( R^{-1} = (S \otimes \text{id})(R) = \sum_j S(\alpha_j) \otimes \beta_j \), and \( u = \sum_j S(\beta_j) \alpha_j \) and \( u^{-1} = \sum_j \beta_j S^2(\alpha_j) \).

Let \( M \) be a finite-dimensional left \( A \)-module. For any \( a \in A \) the action of \( a \) on \( M \) is denoted by \( a_M \). Then for any \( A \)-module endomorphism \( f \) on \( M^\otimes n \) the following formulas hold:

\[
\tilde{\text{Tr}}^R_r (f) = \text{Tr}((u_M^{-1} \otimes \cdots \otimes u_M^{-1}) \circ f), \quad (4.10)
\]

\[
\tilde{\text{Tr}}^r_r (f) = \text{Tr}((u_M \otimes \cdots \otimes u_M) \circ f), \quad (4.11)
\]

where \( \text{Tr} \) in the right-hand side stands for the usual trace on linear transformations.
Proof. Here, we only prove the first equation since the second equation can be proved by the same argument. The equation (4.10) can be shown by induction on $n$ as follows.

Let $\{e_i\}_{i=1}^d$ be a basis for $M$. For any $a \in A$, $a \cdot e_i$ is expressed as $a \cdot e_i = \sum_{i'=1}^d M_{i,i'}(a) e_{i'}$ for some $M_{i,i'}(a) \in k$. Then $\tilde{\text{Tr}}_R^l(f) = \text{Tr}_R^l(\rho(f)) = \sum M_{i,j}(\beta_j) M_{k,i}(\alpha_j) = \sum M_{i,j}(\tilde{u}_M^{-1} \circ f) = \text{Tr}(\tilde{u}_M^{-1} \circ f)$.

Next, assume that the equation $\tilde{\text{Tr}}_R^l(g) = \text{Tr}(\tilde{u}_M^{-1} \circ g)$ holds for any $A$-module endomorphism $g$ on $M^{\otimes(n-1)}$. Let $f$ be an $A$-module endomorphism on $M^{\otimes n}$. Then $g := \text{Tr}_R^l(f)$ is an $A$-module endomorphism on $M^{\otimes(n-1)}$. Applying the induction hypothesis, we have $\tilde{\text{Tr}}_R^l(f) = \tilde{\text{Tr}}_R^l(g) = \text{Tr}(\tilde{u}_M^{-1} \circ g) = \text{Tr}(\tilde{u}_M^{-1} \circ f)$. \hfill \qed

### 4.2 Construction of monoidal Morita invariants

In this subsection we introduce a family of monoidal Morita invariants of a finite-dimensional Hopf algebra by using partial braided traces.

Let $\mathcal{C} = (\mathcal{C}, c)$ be a strict left rigid braided monoidal category, and $M$ be an object in $\mathcal{C}$. Then there is a representation $\rho_M : B_n \rightarrow \text{GL}(M^{\otimes n})$ of the $n$-strand braid group $B_n$ such that each positive crossing and negative crossing correspond to $c_{M,M}$ and $c_{M,M}^{-1}$, respectively [36]. For each $b \in B_n$ we set

$$\mathbf{b} \cdot \text{dim}_c^l(M) := \text{Tr}^l(\rho_M(b)), \quad \mathbf{b} \cdot \text{dim}_c^r(M) := \text{Tr}^r(\rho_M(b)).$$

By Example [4.5],

$$\mathbf{1} \cdot \text{dim}_c^r(M) = (\text{the quantum dimension of } M) \quad (4.12)$$

in the sense of [25], where 1 is the identity element of $B_1$.

**Lemma 4.6.** Let $M$ and $N$ be two objects in $\mathcal{C}$.

1. If $M$ and $N$ are isomorphic, then $\mathbf{b} \cdot \text{dim}_c^l(M) = \mathbf{b} \cdot \text{dim}_c^l(N)$, $\mathbf{b} \cdot \text{dim}_c^r(M) = \mathbf{b} \cdot \text{dim}_c^r(N)$.

2. $\mathbf{b} \cdot \text{dim}_c^r(M^*) = \mathbf{b} \cdot \text{dim}_c^l(M)$.

**Proof.** (1) Let $\varphi : M \rightarrow N$ be an isomorphism. The map $\varphi^{\otimes n} : M^{\otimes n} \rightarrow N^{\otimes n}$ is also an isomorphism. Let $\rho_M : B_n \rightarrow \text{GL}(M^{\otimes n})$ and $\rho_N : B_n \rightarrow \text{GL}(N^{\otimes n})$ be the representations induced from the braiding $c$. Since $c_{N,N} \circ (\varphi \otimes \varphi) = (\varphi \otimes \varphi) \circ c_{M,M}$ from naturality of $c$, the endomorphisms $f := \rho_M(b)$ and $g := \rho_N(b)$ satisfy $g \circ \varphi^{\otimes n} = \varphi^{\otimes n} \circ f$. Thus, $g$ is expressed as $g = (\varphi^{\otimes n}) \circ f \circ (\varphi^{\otimes n})^{-1}$, and it follows from Proposition [4.3] that $\mathbf{b} \cdot \text{dim}_c^l(N) = \text{Tr}^l(g) = \text{Tr}^l(f) = \mathbf{b} \cdot \text{dim}_c^l(M)$. The equation $\mathbf{b} \cdot \text{dim}_c^r(M) = \mathbf{b} \cdot \text{dim}_c^r(N)$ is also shown by the same argument.

(2) By the definition of the natural isomorphism $j_{M,N} : N^* \otimes M^* \rightarrow (M \otimes N)^*$, it is easy to see that $j_{M,N} \circ c_{M^*,N^*} = \imath(c_{M,N}) \circ j_{N,M}$. It follows that the representation $\rho_{M^*} : B_n \rightarrow$
GL((M*)⊗n) induced from the braiding c satisfies j(n) ◦ ρ_{M\cdot}(b) = \ell(ρ_M(b)) ◦ j(n), where j(n) is the isomorphism defined in Proposition 4.4. Thus we have b-dim^l(M^*) = \tilde{\text{Tr}}_r((j(n))^{-1} ◦ \ell(ρ_M(b)) = b-dim^l(M).

Let (A, R) be a quasitriangular Hopf algebra over k, and M be a finite-dimensional left A-module. For each b ∈ B_n we set

$$b\text{-dim}^l(M) := b\text{-dim}^l_R(M), \quad b\text{-dim}^r(M) := b\text{-dim}^r_R(M).$$

In particular, in the case where (A, R) = (D(H), R) for some finite-dimensional Hopf algebra H over k, we denote

$$b\text{-dim}^l(M) := b\text{-dim}^l_R(M), \quad b\text{-dim}^r(M) := b\text{-dim}^r_R(M).$$

Now we will show that b-dim^l(D(H)A) and b-dim^r(D(H)A) are preserved under k-linear monoidal equivalences of the module categories.

**Theorem 4.7.** Let A and B be finite-dimensional Hopf algebras over k. If \( A M \) and \( B M \) are equivalent as k-linear monoidal categories, then \( b\text{-dim}^l(D(A)A) = b\text{-dim}^r(D(B)B) \) for all \( b \in B_n \). The same statement holds for \( b\text{-dim}^l \).

**Proof.** Let \( (F, φ, ω) : (AM, c) → (BM, c') \) be a k-linear braided monoidal functor. Then b-dim^r_F(M) = b-dim^r_F for all finite-dimensional left A-module M and b ∈ B_n. This equation can be shown as follows. Since \( ρ_{M,M} ◦ c'_{F(M),F(M)} = F(c_{M,M}) ◦ φ_{M,M} \), and \( f := ρ_M(b) : M⊗n → M⊗n \) is expressed by compositions and tensor products of \( c_{M,M}^\pm \) and id_M, the isomorphism \( φ^{(n)} : F(M)⊗n → F(M⊗n) \) defined in Proposition 4.2 satisfies \( φ^{(n)} ◦ ρ_{F(M)}(b) = F(f) ◦ φ^{(n)} \). Therefore, by Proposition 4.2 we have

$$b\text{-dim}^r_F(M) = \tilde{\text{Tr}}_r((φ^{(n)})^{-1} ◦ F(f) ◦ φ^{(n)}) = \tilde{\text{Tr}}_r(f) = b\text{-dim}^r(M).$$

Suppose that \( (F, φ, ω) : AM → BM \) is an equivalence of k-linear monoidal categories. By Corollary 3.5 \( \hat{F}(D(A)A) \cong D(B)B \) as left D(B)-modules. It follows from Lemma 4.6 that \( b\text{-dim}^r_{\hat{F}}(D(A)A) = b\text{-dim}^r_{\hat{F}} \hat{F}(D(A)A) = b\text{-dim}^r_{\hat{F}}(D(B)B), \) where \( R \) and \( R' \) are the canonical universal R-matrices of D(A) and D(B), respectively.

By using Theorem 3.11 the monoidal Morita invariants \( b\text{-dim}^l \) and \( b\text{-dim}^r \) of the dual Schrödinger modules \( D(A)^* \) and \( D(A^*)^\text{cop} \) are computable from the monoidal Morita invariants of the Schrödinger modules \( D(A^*) \) and \( D(A) \), respectively.

**Proposition 4.8.** Let A be a finite-dimensional Hopf algebra over k, and R, R' be the canonical universal R-matrices of D(A), D(A^*), respectively. For any b ∈ B_n, the following equations hold.

1. \( b\text{-dim}^l(D(A^*)^\text{cop}) = b\text{-dim}^l(D(A)A), \quad b\text{-dim}^r(D(A^*)^\text{cop}) = b\text{-dim}^r(D(A)A). \)
2. \( b\text{-dim}^l(D(A)A^*) = b\text{-dim}^l(D(A)A^\text{cop}), \quad b\text{-dim}^r(D(A^*)A^*) = b\text{-dim}^r(D(A)A^\text{cop}). \)
Proof. (1) Let $F_A: (D(A^*)^\text{rev}) \rightarrow (D(A), c^\mathcal{R})$ be the equivalence of braided monoidal categories defined in Theorem 3.11. Setting $M := (A^*)^{\text{cop}}$, we have $b\text{-dim}_{\mathcal{R}}^{\text{rev}}(F_A(M)) = b\text{-dim}_{\mathcal{R}}^{\text{re}}(M)$ from the proof of Theorem 3.7. Since $F_A(M)$ and $D(A)A$ are isomorphic as left $D(A)$-modules by Proposition 2.2, it follows from Lemma 4.6(1) that $b\text{-dim}_{\mathcal{R}}^{\text{re}}(F_A(M)) = b\text{-dim}_{\mathcal{R}}^{\text{re}}(D(A)A)$. Thus, the second equation is obtained. Similarly, the equation $b\text{-dim}_{\mathcal{R}}^{\text{re}}((D(A^*)^\text{cop}) = b\text{-dim}_{\mathcal{R}}^{\text{re}}(D(A))$ can be proved.

Part (2) can be proved as in the proof of (1) by using Corollary 2.3 instead of Proposition 2.2.

Let $A$ be a finite-dimensional Hopf algebra. In view of Example 4.5, it is important to know the action of the Drinfel’d element $u \in D(A)$ on a given $D(A)$-module $M$ to compute the braided dimension of $M$. Below we give formulas for the actions of $u$ and $S(u)$ on the Schrödinger module $D(A)A$.

Recall that a left integral in $A$ is an element $\Lambda \in A$ such that $a\Lambda = \varepsilon(a)\Lambda$ for all $a \in A$. A right integral in $A$ is a left integral in $A^{\text{cop}}$. It is known that a non-zero left integral $\Lambda \in A$ always exists (under our assumption that $A$ is finite-dimensional), and is unique up to a scalar multiple. Hence one can define $\alpha \in A^*$ by $\Lambda a = \langle \alpha, a \rangle \Lambda$ for $a \in A$. The map $\alpha$ is in fact an algebra map, and does not depend on the choice of $\Lambda$. We call $\alpha$ the distinguished grouplike element of $A^*$.

The Hopf algebra $A$ is said to be unimodular if the distinguished grouplike element $\alpha \in A^*$ is the counit of $A$, or, equivalently, $\Lambda \in A$ is central.

Lemma 4.9. With the above notations, we have

$$u \rightarrow a = \sum S^2(a_{(1)}) \langle \alpha^{-1}, a_{(2)} \rangle$$

and $S(u) \rightarrow a = S^{-2}(a)$

for all $a \in D(A)A$, where $\alpha^{-1} = \alpha \circ S$.

Proof. Recall that $u = \sum_i S^{-1}_A(e_i^*) \bowtie e_i$, where $\{e_i\}$ is a fixed basis of $A$, and $\{e_i^*\}$ is the dual basis to $\{e_i\}$. We first compute the action of $S(u)$. Since $S^2(u) = u$,

$$S(u) = S^{-1}(u) = \sum_i (\varepsilon \bowtie S_A^{-1}(e_i)) \cdot (e_i^* \bowtie 1).$$

Hence, for all $a \in D(A)A$, we have

$$S(u) \rightarrow a = \sum_i \langle e_i^*, S^{-1}(a_{(1)}) \rangle \cdot \left( S^{-1}(e_i) \rightarrow a_{(2)} \right)$$

$$= \sum_i \langle e_i^*, S^{-1}(a_{(1)}) \rangle \cdot S^{-1}(e_{i(2)})a_{(2)}e_{i(1)}$$

$$= \sum S^{-1}\left( S^{-1}(a_{(1)})_{(2)} \right) a_{(2)}S^{-1}(a_{(1)})_{(1)} = S^{-2}(a).$$

Next, we compute the action of $u$. Fix a non-zero right integral $\lambda \in A^*$, and define $g \in A$ to be the unique element such that $p\lambda = \langle p, g \rangle \lambda$ for all $p \in A^*$ (i.e., the distinguished grouplike
element of \((A^\text{cop})^\star = (A^\star)^\text{cop}\) regarded as an element of \(A\). Radford showed in [30] that \(D(A)\) is unimodular, and \(g \bowtie \alpha\) is the distinguished grouplike element of \(D(A)^\text{cop}\). Hence, by [32] Theorem 2, we have \(u \rightarrow a = S(u) \rightarrow (g \bowtie \alpha) \rightarrow a\) for all \(a \in D(A)A\). Using the formula of the fourth power of the antipode [31], we obtain

\[
 u \rightarrow a = \sum \langle \alpha, S^{-1}(a_{(1)}) \rangle S^{-2}(ga_{(2)}g^{-1}) = \sum S^2(a_{(1)}) \langle \alpha^{-1}, a_{(2)} \rangle.
\]

Combining Example 4.5 and Lemma 4.9, we obtain the following proposition:

**Proposition 4.10.** Let \(A\) be a finite-dimensional Hopf algebra over \(k\). If \(A\) is involutory (i.e. the square of the antipode is the identity) and unimodular, then we have

\[
 b \cdot \dim^l(D(A)A) = b \cdot \dim^r(D(A)A) = \text{Tr}(\rho(b))
\]

for all \(b \in B_n\), where \(\rho : B_n \rightarrow \text{GL}((D(A)A)^{\otimes n})\) is the braid group action.

### 4.3 Examples

We denote by \(\sigma_i \in B_n\) \((i = 1, \ldots, n - 1)\) the braid of \(n\) strands with only one positive crossing between the \(i\)-th and the \((i + 1)\)-st strands. For integers \(p\) and \(q\) with \(p \geq 2\), the braid

\[
 t_{p,q} := (\sigma_1 \sigma_2 \cdots \sigma_{p-1})^q \in B_p
\]

is called the \((p,q)\)-torus braid, as its closure is the \((p,q)\)-torus link. The below is an example of the computation of the braided dimension associated with \(b = t_{2,q}\).

**Lemma 4.11.** Let \((A,R)\) be a quasitriangular Hopf algebra over \(k\), and \(u\) be the Drinfel’d element of it. For each non-negative integer \(m\) and finite-dimensional left \(A\)-module \(X\),

\[
 t_{2,q} \cdot \dim^l_{\otimes X} = \begin{cases} 
 \sum \text{Tr}(u^{m-1}(u^{-m})_{(1)}X)\text{Tr}(u^{m-1}(u^{-m})_{(2)}X) & \text{if } q = 2m, \\
 \sum \text{Tr}((u^{m-1} \otimes u^{-m})\Delta(u^{-m})R_{21}X \otimes X \circ T_{X,X}) & \text{if } q = 2m + 1,
\end{cases}
\]

\[
 t_{2,q} \cdot \dim^r_{\otimes X} = \begin{cases} 
 \sum \text{Tr}(u^{m+1}(u^{-m})_{(1)}X)\text{Tr}(u^{m+1}(u^{-m})_{(2)}X) & \text{if } q = 2m, \\
 \sum \text{Tr}((u^{m+1} \otimes u^{-m+1})\Delta(u^{-m})R_{21}X \otimes X \circ T_{X,X}) & \text{if } q = 2m + 1.
\end{cases}
\]

Here, for elements \(a, b \in A\) the notation \(a \otimes bX \otimes X\) stands for the left action on \(X \otimes X\) defined by \(x \otimes y \mapsto (a \cdot x) \otimes (b \cdot y)\) for all \(x, y \in X\).

**Proof.** The formula for \(t_{2,q} \cdot \dim^l_{\otimes X}\) can be obtained as follows.

Let \(\{e_s\}_{s=1}^d\) be a basis for \(X\), and \(\{e^*_s\}_{s=1}^d\) be its dual basis. Let \(R^{(q)}\) be the element in \(A \otimes A\) defined by

\[
 R^{(q)} = \begin{cases} 
 (R_{21}R)^m & \text{if } q = 2m, \\
 (R_{21}R)^mR_{21} & \text{if } q = 2m + 1.
\end{cases}
\]

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By Example 4.5, we see that
\[ t_{2,q} \cdot \dim R X = \left\{ \begin{array}{ll}
\text{Tr}((u^{-1} \otimes u^{-1}) R^q \otimes X) & \text{if } q \text{ is even}, \\
\text{Tr}((u^{-1} \otimes u^{-1}) R^q \otimes X \circ T_{X,X}) & \text{if } q \text{ is odd}.
\end{array} \right. \tag{4.13} \]

Since \( R_{21} = \Delta(u^{-1})(u \otimes u) = (u \otimes u) \Delta(u^{-1}) \) \[6\], it follows that \( (R_{21} R)^m = (u^m \otimes u^m) \Delta(u^{-m}) \). Substituting this equation to (4.13) we obtain the formula for \( t_{2,q} \cdot \dim R X \) in the lemma. By a similar consideration, the formula for \( t_{2,q} \cdot \dim R X \) can be obtained. \( \square \)

In the case where \( A \) is semisimple, the braided dimension of the Schrödinger module associated with \( t_{2,2} \) has the following representation-theoretic meaning:

**Theorem 4.12.** Suppose that \( k \) is an algebraically closed field of characteristic zero. If \( A \) is a finite-dimensional semisimple Hopf algebra over \( k \), then
\[ t_{2,2} \cdot \dim (D(A)A) = t_{2,2} \cdot \dim' (D(A)A) = \dim(A) \# \text{Irr}(A), \]
where \( \# \text{Irr}(A) \) is the number of isomorphism classes of irreducible \( A \)-modules.

**Proof.** It is sufficient to show \( t_{2,2} \cdot \dim (D(A)A) = \dim(A) \# \text{Irr}(A) \) in view of Proposition 4.10. By the assumption, Radford induction functor \( I_A : A \mathcal{M} \to D(A) \mathcal{M} \) is isomorphic to the functor \( D(A) \otimes_A (-) \) by [14, Lemma 2.3]. Combining this fact with Proposition 4.9, we have
\[ (D(A)A) \otimes (D(A)A) \cong I_A^1(A_{ad}) \cong D(A) \otimes_A A_{ad}. \]
Hence, by Lemma 4.11
\[ t_{2,2} \cdot \dim (D(A)A) = \sum \text{Tr}((u^{-1})_{(1)}_{D(A)A}) \text{Tr}((u^{-1})_{(2)}_{D(A)A}) = \text{Tr}(u^{-1}_{D(A)A} \otimes (D(A)A)) = \text{Tr}(u^{-1}_{D(A)A} \otimes (D(A)A_{ad})). \tag{4.14} \]

We use some results on the Frobenius-Schur indicator [20]. Let \( V \) be a finite-dimensional left \( A \)-module. The “third formula” [18, §6.4] of the \( n \)-th Frobenius-Schur indicator \( \nu_n(V) \) \((n = 1, 2, \ldots)\) expresses \( \nu_n(V) \) by using the Drinfel’d element, as
\[ \nu_n(V) = \frac{1}{\dim(A)} \text{Tr}(u^n_{D(A) \otimes A V}). \]
Since \( u \) is of finite order [4], \( \dim(A) \nu_n(V) \in \mathbb{Z}[\xi] \subset k \), where \( \xi \in k \) is a root of unity of the same order as \( u \). Hence, if we denote by \( z \to \xi \) the ring automorphism of \( \mathbb{Z}[\xi] \) defined by \( \xi \to \xi^{-1} \), then we have
\[ \text{Tr}(u^{-n}_{D(A) \otimes A V}) = \dim(A) \nu_n(V). \]
On the other hand, the “first formula” [18, §2.3] yields \( \nu_1(V) = \dim(\text{Hom}_A(k, V)) \). Considering the case where \( V \) is the adjoint representation \( A_{ad} \), we obtain
\[ \text{Tr}(u^{-1}_{D(A) \otimes A A_{ad}}) = \dim(A) \nu_1(A_{ad}) = \dim(\text{Hom}(k, A_{ad})) = \dim(A) \# \text{Irr}(A). \]
Now the result follows from [4.11]. \( \square \)
As this theorem suggests, the Schrödinger module $D(A)A$ has much information about the category of $A$-modules, at least, in the semisimple case. However, the computation of the braided dimension is not easy in general. Fortunately, if $A$ is a group algebra, then the braided dimension of $D(A)A$ closely relates to the link group of the closure of the braid, and can be computed in the following way:

**Theorem 4.13.** Let $b \in B_n$. If $A = k[G]$ is the group algebra of a finite group $G$, then

$$b \cdot \dim(D(A)A) = b \cdot \dim^r(D(A)A) = \sharp \Hom(\pi_1(\mathbb{R}^3 \setminus \tilde{b}), G)$$

in $k$, where $\tilde{b}$ is the link obtained by closing the braid $b$, and $\pi_1$ means the fundamental group.

**Proof.** Set $X = D(A)A$ for simplicity. Then the braiding $c_{X,X}$ is given by

$$c_{X,X}(g \otimes h) = h \otimes (h^{-1} \triangleright g) = h \otimes h^{-1}gh \quad (g, h \in G).$$

Let $B_n$ act on $G^n$ by

$$g(\sigma_1)(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \ldots, g_n) \quad (g_1, \ldots, g_n \in G).$$

By Proposition 4.10, $b \cdot \dim(D(A)A)$ and $b \cdot \dim^r(D(A)A)$ are equal to the number of fixed points of $g(b)$ regarded as an element of $k$. On the other hand, the number of fixed points of $g(b)$ has been studied by Freyd and Yetter [11] in relation with link invariants arising from crossed $G$-sets. The claim of this theorem follows from [11, Proposition 4.2.5].

**Example 4.14.** We consider the case where $A = k[G]$ is the group algebra of a finite group $G$. If $k$ is an algebraically closed field of characteristic zero, then we obtain

$$t_{2,2} \cdot \dim(D(A)A) = t_{2,2} \cdot \dim^r(D(A)A) = |G| \cdot \#\text{Conj}(G),$$

$$t_{2,2} \cdot \dim(D(A^*)A^*) = t_{2,2} \cdot \dim^r(D(A^*)A^*) = |G|^2$$

by Theorem 4.12, where $\text{Conj}(G)$ is the set of conjugacy classes of $G$. In particular,

$$t_{2,2} \cdot \dim(D(A)A) \neq t_{2,2} \cdot \dim(D(A^*)A^*)$$

whenever $G$ is non-abelian. This result is interesting from the viewpoint that some other monoidal Morita invariants, such as ones introduced in [7] and [37], are in fact invariants of the braided monoidal category of the representations of the Drinfel’d double.

In topology, the link $H = \hat{t}_{2,2}$ is known as the Hopf link. Since $\pi_1(\mathbb{R} \setminus H)$ is the free abelian group of rank two, we have

$$t_{2,2} \cdot \dim(D(A)A) = t_{2,2} \cdot \dim^r(D(A)A) = \#\text{Comm}(G)$$

by Theorem 4.13, where $\text{Comm}(G) = \{(x, y) \in G \times G \mid xy = yx\}$. Comparing (4.15) with (4.16), we get $|G| \cdot \#\text{Conj}(G) = \#\text{Comm}(G)$. Although this formula itself is well-known in finite
group theory, we expect that some non-trivial formulas for finite groups (or, more generally, for finite-dimensional semisimple Hopf algebras) would be obtained via the investigation of the braided dimension.

By (4.12) and [25, Example 9.3.8], we have 

\[ \dim_r(D(A)) = \text{Tr}(S^2) \] 

(see [2] for the quasi-Hopf case). In particular, \( \dim_r(D(A)) = 0 \) whenever \( A \) is not cosemisimple. More strongly, we have the following theorem:

**Theorem 4.15.** Let \( A \) be a finite-dimensional Hopf algebra. If \( A \) is not cosemisimple, then we have \( \dim_l(D(A)) = \dim_r(D(A)) = 0 \) for all braids \( b \).

**Proof.** Let, in general, \( X \) be a finite-dimensional Hopf algebra, let \( \Lambda \in X \) be a left integral, and let \( \lambda \in X^* \) be the right integral such that \( \langle \lambda, \Lambda \rangle = 1 \). By [30, Proposition 2 (a)],

\[ \text{Tr}(X \rightarrow X; x \mapsto S^2(x_{(2)})\langle p, x_{(1)} \rangle) = \langle \lambda, 1 \rangle \langle p, \Lambda \rangle \]

for all \( p \in X^* \). Applying this formula to \( X = A^{\text{op cop}} \), we have

\[ \text{Tr}(A \rightarrow A; a \mapsto S^2(a_{(1)})\langle p, a_{(2)} \rangle) = 0 \]  

(4.17)

for all \( p \in A^* \), since the Hopf algebra \( A \) is not cosemisimple and thus \( \langle \lambda, 1 \rangle = 0 \).

Now, let \( b \in B_n \) be a braid. By (4.8),

\[ \dim_l(D(A)) = \text{Tr}(f), \]

where

\[ f = (\text{Tr}_1 \circ \cdots \circ \text{Tr}_1)(\rho(b)). \]

Let \( f : A_{\text{ad}} \rightarrow k \) be the \( A \)-linear map corresponding to \( \tilde{f} \) under the isomorphism

\[ \text{End}_{D(A)}(D(A)A) \rightarrow \text{Hom}_{D(A)}(D(A)A, I_A(k)) \rightarrow \text{Hom}_A(A_{\text{ad}}, k) \]

given by (2.10) and Proposition 2.6. Then we have \( \tilde{f}(a) = \sum f(a_{(2)})a_{(1)} \) for all \( a \in D(A)A \), and therefore \( \dim_l(D(A)) \) is equal to the trace of the linear map

\[ A \rightarrow A; \ a \mapsto \tilde{f}(a) = \sum S^2(a_{(1)})\langle a^{-1}, a_{(2)} \rangle f(a_{(3)}) \]  

(\( a \in A \)).

Hence, \( \dim_l(D(A)) = 0 \) by (4.17). The equation \( \dim_r(D(A)) = 0 \) is proved in a similar way. \( \square \)

By this theorem, we could say that the braided dimension of the Schrödinger module is not interesting as a monoidal Morita invariant for non-cosemisimple Hopf algebras. However, the endomorphism of \( D(A)A \) induced by a braid, such as \( \tilde{f} \) in the above proof, is not generally zero, and thus may have some information about \( A \). For example, let us consider the map

\[ z_M := \text{Tr}_{e, 1}(\rho_M(\sigma_1)) : M \rightarrow M \]  

(4.18)
for finite-dimensional $M \in D(A)\mathbf{M}$, where $\rho_M : \mathcal{B}_2 \rightarrow \text{GL}(M^{\otimes 2})$ is the action of $\mathcal{B}_2$. One can check that $z_M$ is given by the action of $z := uS(u)$ on $M$. Hence, if $M = D(A)A$, then

$$z_M(a) = z \cdot a = a_{(1)}(\alpha^{-1}, a_{(2)}) \quad (a \in D(A)A) \quad (4.19)$$

by Lemma 4.9, where $\alpha \in A^*$ is the distinguished grouplike element. Therefore this map has the following information: $z_M$ for $M = D(A)A$ is the identity if and only if $A$ is unimodular.

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