On a pure hyperbolic alternative to the Navier-Stokes equations

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Abstract

We discuss a pure hyperbolic alternative to the Navier-Stokes equations, which are of parabolic type. As a result of the substitution of the concept of the viscosity coefficient by a microphysics-based temporal characteristic, particle settled life (PSL) time, it becomes possible to formulate a model for viscous fluids in a form of first order hyperbolic partial differential equations. Moreover, the concept of PSL time allows the use the same model for flows of viscous fluids (Newtonian or non-Newtonian) as well as irreversible deformation of solids. In the theory presented, a continuum is interpreted as a system of particles, or in the case of fluids, fluid parcels connected by bonds; the internal resistance to flow is interpreted as elastic stretching of the particle bonds; and a flow is a result of bond destructions and rearrangements of particles. Finally, we examine the model for simple shear flows of Newtonian fluids and demonstrate that Newton’s viscous law can be obtained in the framework of the developed hyperbolic theory as a steady-state limit. A basic relation between the viscosity coefficient, PSL time, and the shear sound velocity is also obtained.

Keywords: Hyperbolic equations, Navier-Stokes equations, solid dynamics, viscous fluids, irreversible deformation, thermodynamics

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1 Introduction

1.1 Solids vs. fluids

Under normal conditions, solids and fluids behave differently; however, in this paper, we demonstrate that from the continuum physics point of view such differences are of a quantitative character and that the dynamics of both these states of matter can be modeled with a single system of partial differential equations (PDEs). In particular, we demonstrate that there is a microphysics-based alternative to the conventional concept of viscosity in its traditional and phenomenological meaning, i.e., as a coefficient of proportionality between the strain rate and stress tensor. With such an alternative, it becomes possible to describe flows of viscous fluids, whether Newtonian or non-Newtonian, from the same basis as irreversible deformation in solids.

Let us first describe an approximate physical model underlying the continuum theory discussed below. We assume that, on a microscopic level, both isotropic solids and fluids can be represented by a disordered system of particles connected by bonds, regardless of their nature: physical, chemical or mechanical. In the case of solids and fluids in a condensed state, known as liquids, this representation may be quite straightforward; however, in the framework of continuum theory we treat a material particle as a volume containing a sufficiently large number of real atoms or molecules. We also should consider an explanation with regard to gases because, in contrast to solids and liquids, there are no real bonds between gas atoms or molecules. However, it is a conventional point of view in continuum mechanics to consider gases as consisting of finite volumes with a constant mass [1]. We shall refer to these finite volumes as parcels. Clearly, this does not preclude the exchange of mass with neighbors. Instead, due to inertia, the diffusion mass flux between two neighboring fluid parcels plays the role of a bond resistant to shearing. In fact, by postulating the existence of fluid parcels, which are a type of imaginary Lagrangian particles, we also impose a solid-like (i.e., particles with

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1In this paper, we consider only dense gases.
2Still large enough to be considered as a continuum.
bonds) structure on gases. Moreover, in some cases, it is reasonable to consider liquids also as a system of parcels, for example, if a liquid has a temperature far above the solidification point. In that state, real bonds between atoms or molecules weak and unstable, and break easily because of intensive particle fluctuations. As a result, the diffusion flux between mesoscale fluid parcels predominates over real particle bonds in their contribution to the friction effect.

With this physical model in mind, it is natural to question how such a system of particles can flow, i.e., deform irreversibly. Clearly, the system can flow only if the bonds between particles can be destroyed, which is the manner in which real materials behave; if the bonds exist forever, the material behaves elastically, which is not the case in this study. If the bonds can be destroyed, then a second question arises: what is a characteristic time \( \tau \) of the existence of particle bonds while the system flows, i.e., the time after which the bonds are destroyed? Clearly, \( \tau \) cannot be zero because that would mean that no bonds exist. Thus, the time \( \tau \) is finite, with \( 0 < \tau < \infty \), which rises the third basic question of what the particle bonds do while the system deforms during time \( t < \tau \)? As we assume that bonds are stretchable, otherwise the system cannot deform, the answer is that bonds are stretched elastically because bond destruction is forbidden at time \( t < \tau \).

Therefore, it becomes clear that the true source of irreversible deformation in a system of connected particles or fluid parcels is the process of particle or parcel rearrangement. Hence, this can be decomposed into three basic subprocesses: elastic bond stretching, bond destruction, and creation of new bonds in a new local particle equilibrium state. Therefore, a physically-based continuum theory for non-stationary irreversible deformation should be based on the micromechanics of these basic subprocesses. In this paper, we assume, however, that the characteristic times for the creation and destruction of bonds are negligibly small if compared with time \( \tau \); therefore, these two subprocesses are eliminated from consideration.

Further to the physical model presented above, in order to characterize the degree of the deformation irreversibility or fluidity, we use the time \( \tau \) of the “settled life” of particles (after Frenkel [2]), i.e., the time during which all bonds of a given particle with its neighbors are conserved; they can stretch but are not destroyed. The introduction of time \( \tau \) allows the clear classification of all types of material responses from the same basis. In fact, a pure elastic response corresponds to \( \tau = \infty \) because all particle bonds are conserved, thus the particles are not rearranged. If particle rearrangements occur, this automatically implies that \( \tau < \infty \), i.e., \( \tau \) becomes finite regardless of whether the nature of a material is a viscous fluid or elastoplastic solid. Finally, the idealized limiting case \( \tau = 0 \) corresponds to inviscid flows, i.e., ideal fluids or ideal plasticity in the case of solids, because bonds do not exist; their lifetime equals to zero. In other words, in order to characterize the degree of irreversibility of deformation, i.e., fluidity, we propose a temporal characteristic \( \tau \) with a clear physical meaning that allows the comprehension of continuum dynamics of solids and fluids under a unified viewpoint.

Unlike the viscosity coefficient, the settled time represents an objective material characteristic that can be observed directly, at least potentially, in experiments without the introduction of any subjective notions invented by the observer, such as stresses. In addition, because of its universality or material-independent character, it covers continuously the full spectrum of material responses, from inviscid flows (\( \tau = 0 \)) to
pure elastic responses \( (\tau = \infty) \).

The second critical material parameter in our theory is connected with the most obvious difference between solid-like and fluid-like behavior, i.e., the ability of solids and inability of fluids to retain static shear loadings. In what follows, we explain that such a difference has a quantitative rather than qualitative character if both solids and fluids are considered as a system of connected particles. In fact, under static loadings, bonds between particles of a solid are able to stretch to some degree and are not destroyed, which means that \( \tau = \infty \), if the loadings are not sufficiently large. In that case, the solid is in an equilibrium static state. If bond stretching exceeds a certain limit, then bonds are destroyed (i.e., \( \tau < \infty \)) and particles rearranged. Thus, we introduce a material parameter \( Y_0 \), which is the bond stretch limit under static loadings. From the previous discussion, it is clear that \( \tau = \tau(Y_0) \) for solids.

In fluids, with a good approximation, \( Y_0 = 0 \), which means simply that \( \tau < \infty \) for any applied external loadings, and the situation when \( \tau = \infty \) is impossible. For example, if a fluid is treated as a system of parcels with bonds, where the bonds are the diffusion mass fluxes between neighboring parcels, then such bonds do not manifest themselves, i.e., \( Y_0 = 0 \), in a static state because there is no change in macroscopic motion state, and thus inertial forces equal to zero.

A typical flow model after external loading is applied to a viscous fluid as follows. Because viscous fluids are characterized by \( 0 < \tau < \infty \), there is no flow, i.e., particle rearrangement, at time \( t < \tau \). Therefore, at a given strain rate \( \dot{\varepsilon} \), bonds are stretched elastically during the interval of time \( t \in [0, \tau] \). At the time \( t = \tau \), the bond stretching reaches a value \( Y = Y(\dot{\varepsilon}) > Y_0 = 0 \), called here the dynamic stretch limit, at which bonds are destroyed. In turn, this results in particle rearrangements. Thus, this leads to the key point of our theory, which is the internal resistance to flow (viscosity) of fluids is a result of elastic stretching of microscopic bonds between fluid particles (parcels), and a macroscopic flow is a result of bond destructions and rearrangements of particles.

It is therefore not surprisingly that the mathematical model that we propose to use in Section 2 comes from solid dynamics theory. Thus, for instance, an extra tensorial-type state variable \( A \) that is an elastic measure of bond stretching is used in the theory described as follows together with traditional internal state variables, such as the mass density and entropy.

In summary, with regard to the settled time \( \tau \) and static stretch limit \( Y_0 \), fluids and solids can be ranged as follows:

- \( \tau = \infty \), elastic solids;
- \( \tau(Y_0) = \begin{cases} \tau = \infty, & \text{if bond stretching } < Y_0 \\ 0 < \tau < \infty, & \text{if bond stretching } \geq Y_0 \end{cases} \), elastoplastic solids;
- \( \tau(Y_0) = \begin{cases} \tau = \infty, & \text{if bond stretching } < Y_0 \\ \tau = 0, & \text{if bond stretching } \geq Y_0 \end{cases} \), ideally plastic solids;
- \( 0 < \tau < \infty, Y_0 = 0 \), viscous fluids (Newtonian and non-Newtonian);
- \( \tau = 0, Y_0 = 0 \), ideal fluids;

In general, time \( \tau \) is a function not only of \( Y_0 \), but it also depends on a current flow state, or state variables. In a particular case of Newtonian fluids, we shall demonstrate
in Section 3 that the particle settled life time $\tau$ is a function of only the mass density $\rho$ and entropy $s$ of the fluid. The non-Newtonian behavior of viscous fluids is beyond the scope of this study, but it is clear that such fluids also belong in the class characterized by $0 < \tau < \infty$, $Y_0 = 0$, and the same theory that is described in the following is applicable if the function $\tau$ of the state variables is properly defined.

### 1.2 Navier-Stokes equations and hyperbolicity

It is difficult to overestimate the paramount role of the Navier-Stokes equations (NSEs) in fluid dynamics and it seems that there is a common belief that the NSEs are the single model for viscous flows in the framework of classical continuum theory.

On the other hand, as with any continuum models, the NSEs have limited applicability, and one of the main goals of the fluid dynamics community is to discover these limits beyond which an application of the NSEs are questionable.

There is no doubt that turbulence is one of the challenging and still poorly understood problems in the dynamics of viscous fluids. However, nearly a century of extensive, and mainly unsuccessful, searching for the ultimate Navier-Stokes-based turbulence model [3] may be considered as evidence that turbulent flows might be one of the limits of applicability of the NSEs. Clearly, such a lengthy duration may signify not only the defects in the concept of the viscosity coefficient itself, but the intrinsic extreme difficulties of the turbulence problem.

Regardless of the case, it is clear that the widespread and unconsidered use of the NSEs may be dangerous because of the following drawbacks, which incidentally, can be one of the reasons suppressing further progress in the understanding and modeling of turbulent flows:

- **Stress tensor.** The Navier-Stokes viscous stress tensor was derived from observations of steady and structureless/homogeneous flows. Successive applications of the NSEs to numerous practically important situations also suggests that Newton’s viscous law is a good approximation to non-steady laminar flows. However, it becomes clear that the use of Navier-Stokes-like models for essentially unsteady and non-equilibrium flows, such as strongly sheared and time-dependent turbulent flows, is problematic [4].

  The root of the problem is that the derivation of the NSEs’ stress tensor is based on the strategy where the material responses observed experimentally is straightforwardly mimicked in a model instead of simulating the intrinsic reasons leading to the observable macroscopic response. Clearly, such a mimic strategy is applicable provided the circumstances are simple and we are able to capture the key details of the material response, which unlikely to be the case for turbulence. See the following section for a further discussion.

- **Causality.** Solutions to the NSEs are of a diffusive nature and, hence, do not satisfy causality, *i.e.*, the NSEs admit infinite velocity of disturbance propagation.

\[^3\text{Although linear parabolic diffusion theories based on Fourier, Fick, or Newton’s laws predict that disturbances can propagate at infinite velocity, nonlinear parabolic equations, in some instances, lead to finite velocity of propagation, e.g., see the discussion in [5] and references therein.}\]
tion [5, 6, 7, 8], while by its very nature, a flow of a fluid is a pure hyperbolic process; all disturbances propagate at finite velocities. Thus, to construct a closed, \textit{i.e.}, no more closure relations are required, pure hyperbolic model for viscous flows is of fundamental interest. Clearly, such a model will have several sound velocities: one for longitudinal and two for shear perturbations. Although, shear perturbations decay quickly in viscous fluids, a turbulence model based on such a hyperbolic theory will more precisely/correctly resolve the microstructure of turbulent flows than the NSE-based turbulence models.

Furthermore, it is important to emphasize that in the dynamics of solids, it is a conventional point of view that a continuum model should be an obligatory hyperbolic. Not only because of the causality issue but also because of that the hyperbolicity notion and the local well-posedness of the boundary-initial value problem\footnote{With sufficiently smooth initial data and dissipative boundary conditions.} for a system of PDEs are convertible terms [9, 10, 11]. With regard to this, one questions what is so special in the physics of viscous fluids that makes them intrinsically parabolic. As we shall demonstrate in the following, there are no special characteristics and it is indeed possible to develop a pure hyperbolic theory for viscous flows.

It is worth noting that several attempts were made in order to construct a pure hyperbolic alternative to the NSEs. The most well known are: extended irreversible thermodynamics [7], or Grad’s moment theory, and the hidden variables concept [12]. Both of these theories suffer in that they are too mathematical and their physics backgrounds remain unclear. In addition, extended irreversible thermodynamics suffers from the so-called closure problem, \textit{i.e.}, additional phenomenological closure relations are required in order to start its practical use. Plus, in this theory the stress tensor is used as a state variable, which implies the use of an objective stress rate, which in turn can be defined in many ways resulting in different solutions to the model. With regard to the hidden variable theory, it belongs to the Maxwell-type models, which are discussed in Section 2.7.

Finally, it is important to recall that dissipative processes, such as those described by the Fourier, Darcy, or Fick laws, indeed can be recovered by pure hyperbolic first order PDEs of a relaxation type. In case of heat conduction, it is done with the well-known Cattaneo hyperbolic theory [13] (see also [14, 15]). For the diffusion problem, see, for example, our recent results [15, 16, 17, 18, 19] and other results [20, 21].

1.3 Response modeling vs. microstructure modeling

Response modeling approach. As mentioned previously, the derivation of the NSEs and NSE-like models in general is based on the imitation strategy when a mathematical model mimics the straightforwardly observed macroscopic response of a material in experiments. Such an approach provides quick results in situations closed to the conditions of experiments. However, extrapolation/prediction beyond the framework of these circumstances by means of variation of the model parameters (constants) becomes questionable when flow characteristics are changed non-linearly, \textit{e.g.}, laminar-turbulent transition. In general, such a mimic strategy should imply the analysis of all qualitatively different flows by experiments in order to develop a model that will be
flow-independent, geometry independent, etc. Therefore, there is no guarantee that the mathematical submodels derived for different flow regimes will continuously transform from one into another. In practice, this results in an excessively large number of models for a single material.

We shall refer to such a mimic strategy as the response modeling (RM) approach. The RM approach is inherently an observer-dependent strategy, i.e., subjective. For example, typical state variables in a model developed in the framework of the RM approach are either the pressure $p$, temperature $T$, or stresses $\mathbf{T}$, etc. However, it is clear that these entities do not exist in reality; they were invented by humans according to our inherent way of perception of the macroscopic reality by means of some interpretative procedures of microscopic processes, not to be confused with an averaging.

The classical representatives of the response modeling approach are the Maxwell, Oldroyd-B (e.g., [22]), Bingham, Hershel-Buckley (e.g., see [23]), Navier-Stokes, and Wilkins models [24, 25, 26], etc. See also Section 2.7 for a discussion on the relation between the Maxwell-type models and the model discussed in this paper.

**Microstructure modeling approach.** Intuitively, we understand that nature does not operate in terms of subjective state parameters and that continuum laws of physics should be written in terms of state parameters only, which are in principle, observable, without engaging of any interpretative procedures. Thus, an alternative approach to the RM is not to straightforwardly mimic the macroscopic response of a material, but instead: (i) to comprehend what essential microscopic processes lead to such a macroscopic response; (ii) to determine objective measures, or state parameters, for these processes; and (iii) to discover eventually the macroscopic evolution laws, which are usually hidden from direct observation, for the state parameters defined in the previous step. In order to do this, the experimental observations of macroscopic material responses are no longer sufficient and additional information of a universal character is required, e.g., consistency with thermodynamics, invariance under transformations of the physical space (Galilean or Lorentz invariance), causality, correctness of the initial value problem, etc., in addition to knowledge about the physics of the microstructure. We shall refer to such an approach as the microstructure modeling (MM) approach.

A good example where the RM approach is barely applicable is complex fluids, i.e., fluids for which a complex internal structure, e.g., stretchable long-chain molecules, influences essentially the macroscopic dynamics. In such media, microscopic processes, such as internal friction, rotations, and elastic stretching of molecules, are responsible for a single quantity that is the overall stress tensor. Thus, the relation between the stress tensor, or observable macroscopic material response, and microscopic processes is not a one-to-one relation and therefore, ill-posed. An attempt to derive a constitutive relation for the stress tensor in such a situation for a sufficiently large range of flows might lead to enormous complexity of the model.

Finally, it is interesting to note that, in continuum thermodynamics, these two types of state variables, i.e., objective and subjective, are thermodynamically dual quantities in the following sense. A typical vector of state variables in an RM-based model is $\mathbf{p} = (\mathbf{v}, T, T)$, while in a corresponding MM-based model, a typical vector is $\mathbf{q} = (\mathbf{v}, \mathbf{F}, w_1, w_2, \ldots, B_1, B_2, \ldots, S)$, where $w_i$ and $B_i$ are vectors and tensors representing essential, for macroscopic dynamics, microscopic processes (e.g., see [18]).
In addition, in MM approaches an additional quantity is required, \( i.e. \), a thermodynamic potential \( \mathcal{E} = \mathcal{E}(q) \). Thus, the vectors \( p \) and \( q \) are related as

\[
\mathbf{v} = \mathcal{E}_v, \quad T = \mathcal{E}_F + \mathcal{E}_{w_1} + \mathcal{E}_{w_2} + \ldots + \mathcal{E}_{B_1} + \mathcal{E}_{B_2} + \ldots, \quad T = \mathcal{E}_S.
\]

See also discussions in Sections 3.5 and 3.6 in [18], [27] and concluding remarks in [28].

Well-known examples of continuum models related to the MM approach are the Euler and ideal magnetohydrodynamics equations. Moreover, the model discussed in the following is also representative of the MM approach. For more complex examples, see [29, 30, 31, 18] and references therein.

### 1.4 Motivation

The aim of this paper is to draw the attention of the modeling community to the fact that the continuum description of viscous Newtonian flows is not restricted solely by the NSEs, and it is possible to simulate viscous fluids with a more physically-based model still within the framework of classical continuum mechanics, and which is compatible with the fundamental observations, such as thermodynamics, causality, and mathematical regularity.

The basis of our approach is the elastoplastic model of hyperelastic type with relaxation terms describing plastic deformation in solids, which was derived by Godunov and Romenski in [32, 33, 34] in order to simulate irreversible deformation of metals under extreme loading.

We note that similar theories to [32, 33, 34] were developed independently in [35, 28, 36, 37] (see also [38, 39] and references therein). The common feature of these theories is that irreversible deformations in the Eulerian frame are described using only a measure of the elastic part of the overall, observable deformation.\(^5\) Differences between these theories concern only the choice of an internal energy function and a dissipation mechanism dictated by applications under consideration, \( \text{e.g.} \), elastoplastic solids [32, 33, 37] or polymeric liquids [36]. Eckart [35] appears to have been the first to formulate a theory of irreversible deformation in terms of only an elastic deformation tensor. However, he did not recognize that the internal resistance to flow was already presented in his theory, and the Newtonian viscosity was added. Besseling [28] then resolved this misunderstanding and showed that the viscous Newtonian flows also can be described in the framework of his model.\(^6\)

This paper differs from the previously cited works in the following manner. In general, our goal is a three-dimensional finite-volume numerical implementation of the discussed hyperbolic model for viscous Newtonian flows and its application to the problems where the use of the NSEs might be questionable. Here, only the preliminary work with an emphasis on clear physical interpretation of the model constituents, hyperbolic nature of the model, its fully thermodynamic consistency, and some numerical aspects is reported. In addition, our study is based on the theory of thermodynamically consistent systems of conservation laws [41, 34, 42, 43, 18]. In this theory, the non-dissipative part of the time evolution is generated with the help of only one thermodynamic potential,

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\(^5\)In the Lagrangian frame, the situation is different. The only elastic part of the overall deformation is not enough to compute stresses, see \( \text{e.g.} \), [40, 18].

\(^6\)We discovered this only after our study was complete.
which is assumed to be a convex function of the state variables, in order to satisfy the
first law of thermodynamics. Here, we also show that the dissipative part of the time
evolution obeys the second law of thermodynamics and can be generated by the same
potential. Therefore, in this study only one thermodynamic potential is used, \textit{i.e.}, no
an additional dissipative potential is required.

2 The model

2.1 State variables

Consider a simple, \textit{i.e.}, no complex internal structure, continuum, regardless of whether
it represents a solid or fluid. Let it be parameterized by the Lagrangian coordinates
\( y = (y_1, y_2, y_3)^T \) at the reference state. The motion \( x = x(y, t) = (x_1, x_2, x_3)^T \) of the
continuum relative to a Cartesian coordinate system, the so-called laboratory coordinate
system, induces two important macroscopic fields, namely the velocity \( v = (v_1, v_2, v_3)^T \)
\[
\frac{dx(y, t)}{dt} = v(x, t), \quad x(y, 0) = y,
\]
and the deformation gradient
\[
F = [F_{ij}] = \frac{\partial x}{\partial y}.
\]

The deformation gradient \( F \) describes the overall, or observable, deformation of the
continuum. In order to describe irreversible deformation, \( F \) is split multiplicatively in
the following way:

\[
F = EP, \quad \det F = \det E > 0,
\]

into elastic part \( E \), which is known as the elastic distortion and associated with the
process of elastic stretching of microscopic particle bonds, and inelastic distortion \( P \),
which stores strains that dissipate due to particle rearrangements. The deformation
gradient \( F \) and distortion \( P \) are not related to the stress state of the continuum but
describe metric properties of the flow; however, they affect the stress field indirectly
via the flow curvature they generate. The microscopic strain field \( E \), or its inverse
\( A = [A_{ij}] = E^{-1} \), is directly related to the process of macroscopic stress generation
by means of elastic bond stretching, and thus the field \( A \) is defined as a state parameter
in our theory.

Equality (1b) will be explained in Section 2.5, where a strain dissipation mecha-
nism, which is a continuum interpretation of the microscopic mechanism of particle
rearrangement, is described.

The mass density \( \rho \) of the continuum is defined as \( \rho = \rho_0 / \det F = \rho_0 \det A \), where
\( \rho_0 \) is the reference state mass density.

We therefore use the following vector of state variables:

\[
(\rho v, A, \rho, \rho s),
\]
where \( \rho v \) is the momentum, \( \rho s \) is the entropy density, and \( s \) is the specific entropy of the system.

### 2.2 Time evolution

Let \( \mathcal{E} \) be the specific total energy of the system, which is a function of state variables (2), i.e., \( \mathcal{E} = \mathcal{E}(\rho v, A, \rho, \rho s) \). In addition, let \( \mathcal{E}_A = [\mathcal{E}_{A_{ij}}] \) denote the matrix for which entries are the partial derivatives, \( \partial \mathcal{E} / \partial A_{ik} \), \( \delta_{ik} \) denotes the Kronecker symbol, and \( \tau = \tau(A, \rho, s, Y_0) \) represents the strain dissipation characteristic time due to microstructure rearrangement, which is a continuum interpretation of the particle settled life time introduced in Section 1, \( Y_0 \) is the static stretch limit (also, see Section 1). Therefore, in the Eulerian framework, the model is

\[
\begin{align*}
\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_k} \left( \rho v_i v_k + \rho^2 \mathcal{E}_\rho \delta_{ik} + \rho A_{mi} \mathcal{E}_{A_{mk}} \right) &= 0, \\
\frac{\partial A_{ik}}{\partial t} + \frac{\partial A_{im} v_m}{\partial x_k} &= -v_j \left( \frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k} \right) - \frac{\Psi_{ik}}{\tau}, \\
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_k}{\partial x_k} &= 0, \\
\frac{\partial \rho s}{\partial t} + \frac{\partial \rho s v_k}{\partial x_k} &= \rho \mathcal{E}_{A_{ij}} \frac{\Psi_{ij}}{\tau}.
\end{align*}
\]

(3a)–(3d)

The first equation represents the momentum conservation. Here, \( T = [T_{ik}] = [-\rho^2 \mathcal{E}_\rho \delta_{ik} - \rho A_{mi} \mathcal{E}_{A_{mk}}] = -\rho^2 \mathcal{E}_\rho I - \rho A^T \mathcal{E}_A = T^T \)

is the symmetric total Cauchy stress tensor, \( \rho^2 \mathcal{E}_\rho = p \) is the pressure, and \( I \) is the identity tensor. Equation (3b) is the time evolution equation for the elastic distortion \( A \), the last term on the right-hand side of (3b) represents the strain dissipation mechanism due to microstructure rearrangements. This equation can be substituted by the time evolution equation for the elastic distortion \( E = A^{-1} \), e.g., see [34, 44]. Equations (3c) and (3d) are the continuity equation and the time evolution for the entropy density, respectively.

On the solution of system (3), an additional conservation law is satisfied, which is the conservation of the total energy density \( \rho \mathcal{E} \) or the first law of thermodynamics:

\[
\frac{\partial \rho \mathcal{E}}{\partial t} + \frac{\partial}{\partial x_k} \left( \rho v_k \mathcal{E} + \rho v_n (\mathcal{E}_\rho \delta_{nk} + A_{mn} \mathcal{E}_{A_{nk}}) \right) = 0,
\]

(5)

It can be derived from system (3) by summing all Eqs. (3) multiplied by the respective multiplicative factors [45, 34, 42, 43, 18]:

\[
(\rho \mathcal{E})_{\rho v_i}, \ (\rho \mathcal{E})_{A_{ik}}, \ \mathcal{E} - v_i \mathcal{E}_{v_i} - V \mathcal{E}_V - s \mathcal{E}_s, \ (\rho \mathcal{E})_{\rho s},
\]

where \( V = 1/\rho \) is the specific volume. This also explains the origins of the source term in the entropy equation, i.e., if this source term is chosen in a different way, then we cannot guarantee the absence of a source term in the energy conservation law,
which clearly would violate the first law of thermodynamics. In addition, we shall show in the following that the source term in the entropy equation is non-negative for an appropriate choice of the strain dissipation function $\Psi$, which makes the model fully thermodynamically consistent.

Remark 2. The mass density $\rho$ has been defined previously by $\rho = \rho_0 \det A$. If we multiply Eq. (3b) by $\rho_A_{ik}$ (note that $\rho_A = [\rho_A_{ik}] = \rho A^{-T} = \rho E^T$) and sum them, we arrive at the mass conservation equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_k}{\partial x_k} = 0. \quad (6)$$

This fact means that the mass conservation law is a consequence of (3b) and plays the role of the involution constraint for the system (3). However, it simplifies the situation, e.g., numerical implementation of the model, if we include the density $\rho$ in the set of state variables (2) for the reasons discussed in [46, 44].

Applications of the model to the modeling of the elastoplastic deformation of solids can be found in [40, 44, 47, 48, 49]. A more sophisticated mechanism of irreversible deformation by means of the stress driven solid-fluid transition is described in [18]. For the remainder of this paper, we discuss applications of the system (3) to viscous Newtonian fluids.

2.3 Equation of state for viscous fluids

To complete the model, we specify the total energy function $\mathcal{E}$, which plays a role of the equation of state or a closure relation in our continuum theory. Let us first note that, as it is seen from the momentum equation, the total stress tensor $T = -\rho^2 \mathcal{E}_\rho I - \rho A^T \mathcal{E}_A$ is generated by the total energy potential. Moreover, we shall show in the following that the strain dissipation function $\Psi$ is also generated with the help of $\mathcal{E}$. Thus, the specification of the total energy is a critical step in the model formulation. It is not surprising, since in continuum physics, the equation of state “absorbs” the information from microscales, which are below the resolution of the continuum scale but are still significant, i.e., they cannot be ignored, for macroscopic dynamics.

Traditionally, we assume the additive decomposition of the specific total energy into kinetic $v^2/2$ and internal energy $E(A, \rho, s)$:

$$\mathcal{E}(\rho v, A, \rho, s) = \frac{v^2}{2} + E(A, \rho, s). \quad (7)$$

In turn, the following decomposition of the internal energy $E$ into the hydrodynamic part $E^h$ and the viscous part $E^v$ is assumed:

$$E = E^h(\rho, s) + E^v(A, \rho, s). \quad (8)$$

For the hydrodynamic part $E^h$, any classical equations of state can be used, e.g., ideal
gas and stiffened gas, and the viscous part $E^v$ is considered in the following form:

$$E^v = c_s^2 \left( I_2 - \frac{T_1^2}{3} \right) = c_s^2 \text{tr}((\text{dev} G)^T \text{dev} G), \quad (9)$$

where

$$I_1 = \text{tr} G, \quad I_2 = \text{tr} G^2, \quad G = A^T A,$$

$$\text{dev} G = G - \frac{\text{tr} G}{3} I,$$

cs = cs(\rho, s)$ is the shear sound velocity at rest, and $I$ is the identity tensor.

It is useful to note that if $A$ has a singular value decomposition $A = U \Sigma V^T$ with $U^T U = V^T V = I$ and $\Sigma = \text{diag}(a_1, a_2, a_3)$ the diagonal matrix with the singular values, or principal stretches, $a_i > 0$ on the diagonal, then $E^v$ can be written as

$$E^v = \frac{c_s^2}{3} \left( (a_1^2 - a_2^2)^2 + (a_2^2 - a_3^2)^2 + (a_3^2 - a_1^2)^2 \right). \quad (10)$$

The viscous energy $E^v$ makes a contribution to the overall internal energy $E$ only if shear deformations are present. In turn, shear deformations are absent if and only if

$$G = \frac{\text{tr} G}{3} I \quad \text{or} \quad a_1 = a_2 = a_3. \quad (11)$$

In the incompressible case, $\text{det} A = a_1 a_2 a_3 = 1$, condition (11) is equivalent to $a_1 = a_2 = a_3 = 1$, thus $G = A^T A = I$, which essentially means that $A$ is an orthogonal matrix.

We remark that $c_s$ is the velocity of propagation of pure elastic perturbations, i.e., perturbations that do not destroy bonds between fluid parcels, see Section 4, in the fluid at rest but not the velocity of propagation of observable diffusive shear perturbations. The elastic processes of microscopic bond stretching are weak, and besides are hidden, or masked, by the strong dissipative process of particle rearrangements, which makes the experimental measurements of the sound velocity $c_s$ a problematic procedure. However, it can be estimated using the molecular dynamics simulation technique. In addition, we shall discuss in Section 3 a way that $c_s$ can be calculated for a given value of the particle settled time $\tau$.

Finally, note that if a fluid flows, then there are two local shear sound velocities that become different from $c_s$ due to the factor $I_2 - I_1/3$ in (9). They depend on the intensity of the shear flow, i.e., anisotropy. These local values, at a given state of the fluid, can be computed as the characteristic velocities, or eigenvalues, of a quasilinear form of the system (3); a standard procedure in the theory of hyperbolic PDEs [50].

### 2.4 Stress tensor

The Cauchy stress tensor $T$ for the system (3) is given by formula (4). The details of its derivation can be found in [33, 45, 34, 18]. Here, we only emphasize that the form of the Cauchy stress tensor is dictated by the requirement of compatibility with the first law of thermodynamics, i.e., energy conservation law (5) (e.g., see [18]).
Taking into account (7) and (8), stress tensor (4) can be rewritten as

\[
T = -pI - \rho A^T E_A^v := T^h + T^v,
\]

where \( T^h(\rho, s) = -pI \) and \( T^v(A, \rho, s) = -\rho A^T E_A^v \) are the hydrostatic and viscous parts of the total stress tensor, respectively. For a particular choice of \( E^v \) given by (9), the viscous stress tensor \( T^v \) is

\[
T^v = -4 \rho c_s^2 G \left( G - \frac{T_1}{3} I \right) = -4 \rho c_s^2 G \left( \text{dev} G \right).
\]

As is seen from the definition of \( T^v \), the viscous stresses are absent if \( E_A^v = 0 \) or if (11) holds.

**Remark 3.** In general, the pressure \( p = \rho^2 \delta_\rho = \rho^2 E^h_\rho + \rho^2 E^v_\rho \) does not coincide with the classically defined isotropic pressure \( p^{\text{classic}} = \rho^2 E^h_\rho \). For example, these two pressures will be different for strongly sheared flows due to the term \( \rho^2 E^v_\rho = p - p^{\text{classic}} \). Clearly, the longitudinal sound velocity defined in the framework of the classical theory of viscous fluids \( c_{\text{classic}} = \sqrt{\frac{\partial p^{\text{classic}}}{\partial \rho}} \) and the longitudinal sound velocity \( c_l = \sqrt{\frac{\partial p}{\partial \rho}} \) for model (3), (8) in general are also different.

**Remark 4.** In contrast to the NSEs, the hydrostatic \( T^h \) and the viscous \( T^v \) stresses are generated in a unified manner, i.e. as the partial derivatives of the total energy \( \mathcal{E} \) with respect to the state variables, the so-called thermodynamic forces. This is a standard feature of models developed in the framework of the MM ideology, in particular in the framework of the thermodynamically consistent theory of conservation laws (see [28, 42, 43, 34, 18] and references therein) or in the GENERIC approach [29, 30, 21, 31].

### 2.5 Strain dissipation

Consider the pure dissipative part of the overall time evolution (3), i.e.,

\[
\frac{\partial \mathbf{A}}{\partial t} = -\frac{\Psi}{\tau}, \tag{14a}
\]

\[
\frac{\partial \rho s}{\partial t} = \rho \frac{\mathcal{E}_{Aij}}{\mathcal{E}_s} \frac{\Psi_{ij}}{\tau}. \tag{14b}
\]

We impose the following three conditions on the strain dissipation function \( \Psi = [\Psi_{ij}] \) [34]:

(D1) **Shear stresses \( T^v \) should relax during the strain dissipation process (14a).** This condition originates from the well-known Maxwell conjecture. We note that the conventional material models such as, the Oldroyd-B model in rheology or the Wilkins model in elastoplasticity, are based directly on this property, because they use the stress tensor as a state variable. According to the philosophy of the MM approach (see the Section 1), we avoid the use of such a subjective quantity
as the stress tensor as a state parameter in our theory. Instead, we simulate the cause, i.e., microstructure rearrangement, which then leads to the observable macroscopic process of stress relaxation.

(D2) *The mass density \( \rho \) is not affected by strain dissipation.* This condition is a reflection of the natural assumption that microscopic particle rearrangement cannot generate macroscopic motion. Because of the equality \( \rho = \rho_0 \det A \), this condition can be reformulated as

\[
\frac{\partial \det A}{\partial t} = 0
\]
during the strain dissipation process (14a).

(D3) *Entropy should not decrease during the dissipative time evolution (14), i.e.,* the right-hand side of (14b) should be non-negative:

\[
\frac{\partial \rho s}{\partial t} = \rho \frac{\mathcal{E}_{A_{ij}}}{\mathcal{E}_s} \frac{\Psi_{ij}}{\tau} \geq 0.
\]

(15)

In what follows, we show that if the strain dissipation function \( \Psi \) is chosen to be proportional to \( \mathcal{E}_A = E_A = E_A^r \) with a non-negative proportionality coefficient, then the three conditions are satisfied, which emphasizes the important role of the thermodynamic potential \( \mathcal{E} \). We construct

\[
\frac{\Psi}{\tau} = \frac{3}{4 c_s^2 \tau \Delta} E_A^v = \frac{3}{\tau \Delta} A(\text{dev } G), \quad \Delta = \det A > 0.
\]

(16)

It is clear that condition (D3) is automatically satisfied if \( \Psi \) is given by formula (16) because the right-hand side of (15) becomes a quadratic form with positive coefficients. By comparing (13) and (16), it is obvious that condition (D1) is also satisfied. A proof of condition (D2) can be found in [18].

It still remains to specify the strain dissipation time \( \tau \). For example, interpolation formulas for \( \tau = \tau(A, \rho, s, Y_0) \) in metals can be found in [51, 52, 53, 37, 44, 48, 49]. As we have already discussed in Section 1, the static stretching limit \( Y_0 = 0 \) in case of fluids, which means that the strain dissipation mechanism (14) is turned on immediately as a flow starts. Moreover, in Section 3, we demonstrate that for the Newtonian fluids \( \tau = \tau_0(\rho, s) \), i.e., \( \tau \) is constant for given values of \( \rho \) and \( s \) and does not depend on \( A \).

Finally, note that the equation (16) provides that the distortion \( A \) and dissipation function \( \Psi \) are coaxial. This means that the nine ordinary differential equations (14a) equivalent to the following three differential equations:

\[
\frac{\partial a_i}{\partial t} = -\frac{(2 a_i^2 - a_m^2 - a_n^2)}{\tau a_m a_n}, \quad i \neq m \neq n \neq i,
\]

written in the terms of singular values \( a_i \) of the elastic distortion \( A \).

2.6 Hyperbolicity and stability of the equilibrium state

Because of the Galilean invariance of the model [54], the condition of hyperbolicity for system (3) in the three-dimensional case is equivalent to hyperbolicity in a one-
dimensional case. In one-dimension, say $x_1$, system (3) can be transformed into an equivalent quasilinear symmetric hyperbolic form if the total energy $E$ is a convex function with respect to the state variables $\rho v$, $\rho$, $\rho s$, and only the first column $A_{i1}$ of the elastic distortion $A$ [45, 40]; or the second or third if $x_2$ or $x_3$-direction are chosen. In particular, viscous energy (9) is a convex function of $A_{i1}$ if the following inequalities are satisfied [40]; here, we use Eq. (10)

$$ E_{a_1 a_1} = \frac{4}{3} c_s^2 \left( 6a_1^2 - a_2^2 - a_3^2 \right) > 0, $$

$$ E_{a_1 a_2} - E_{a_2 a_2} \frac{a_1 - a_2}{a_1 + a_2} + E_{a_1 a_1} \frac{a_1 + a_2}{a_1 + a_2} = \frac{8}{3} c_s^2 \left( 2a_1^2 + 2a_2^2 - a_3^2 \right) > 0, $$

$$ E_{a_3 a_3} - E_{a_1 a_1} \frac{a_3 - a_1}{a_3 + a_1} + E_{a_3 a_3} \frac{a_3 + a_1}{a_3 + a_1} = \frac{8}{3} c_s^2 \left( 2a_1^2 + 2a_3^2 - a_2^2 \right) > 0, $$

or if

$$ a_1^2 > \frac{(a_2^2 + a_3^2)}{6} \text{ and } a_3^2 < 2(a_1^2 + a_2^2) \text{ and } a_2^2 < 2(a_1^2 + a_3^2), $$

(19)

which is sufficient for fluid dynamics applications because the singular values are close to one (see Section 3 for a typical order of $a_i - 1$). Characteristic analysis of the model can be found in [44, 47].

It is important to emphasize that convexity conditions (19) also guarantee that the shear stress free equilibrium state characterized by the equality $E_{AA} = 0$ or $a_1 = a_2 = a_3$, which is the steady point of differential equation (14a), (16) or (17), is stable, i.e., the steady point is attractor.

### 2.7 Relation to Maxwell fluid

Despite the fact that the development of the discussed hyperbolic theory in [32, 33, 34, 51, 52, 53] was inspired by the Maxwell material model for viscoelastic fluids, these two theories have little in common.

For the discussion that follows, it is useful to recall that in the Maxwell-type models the stress tensor plays the role of a state variable. This fact is automatically attributed such models to the RM approach (see Section 1.3). A typical time evolution for the shear stress tensor is

$$ \lambda \frac{D \Sigma}{Dt} + \Sigma = 2\eta D, $$

(20)

where $\lambda$ is the so-called stress relaxation time, $D/Dt$ is a frame invariant time derivative, $\Sigma$ is the shear stress tensor, $\eta$ is the Newtonian viscosity, and $D$ is the symmetric part of the velocity gradient.

**Stress relaxation vs. strain dissipation.** As it is seen in (20), Maxwell type-models are primarily based on the concept of the viscosity coefficient, i.e., the NSEs, or viscous Newtonian flows, are the relaxation limit of a Maxwell-like model as $\lambda \rightarrow 0$. In contrast, as we have already discussed in Section 1 and shall demonstrate numerically in the following section, the strain dissipation time, or PSL time, $\tau \neq 0$ always for viscous flows, and $\tau = 0$ only in the case of inviscid fluids. Therefore, there cannot be a one-to-one relation between the times $\lambda$ and $\tau$. This is the most essential difference.
The second discrepancy is the consequence of the first. The fact that the Maxwell-like models are based on the NSEs results in the unlimited growth of the velocity of shear perturbations as the stress relaxation time $\lambda$ tends to zero, which in turn makes such models inconsistent with experimental observations on wave propagation. Thus, for example, the shear sound velocity $c_s$ for the upper-convected Maxwell fluid is \([55]\): $c_s = \sqrt{\eta / (\rho \lambda)}$. In particular, according to this formula, for some small value of $\lambda$, the shear sound velocity may become greater than the velocity $c_l$ of propagation of longitudinal sound waves, which is also physically meaningless.

In contrast, the shear sound velocity $c_s$ for model (3) is always finite. In fact, it is dictated by the requirement of the consistency with linear elasticity (see also the remark on the elastic response of Newtonian fluids in Section 4) that the difference $c_l^2 - \frac{4}{3}c_s^2$ must be positive, hence $c_s < c_l$. See also formulas (26) and (27) in the following section for the relation between shear sound velocity $c_s$, Newtonian viscosity $\eta$, and the strain dissipation time $\tau$.

\textbf{Relation to thermodynamics.} Although equivalence between the second law of thermodynamics and criteria for well-posedness of the Cauchy problem has been demonstrated in \([56]\) for the upper-convected Maxwell fluids, it is proved in \([57]\) that this equivalence does not hold, in general, for other Maxwell-type models. The situation is different in the discussed hyperbolic theory. As mentioned in Section 2.2, 2.5, and 2.6, although, without rigorous proof, a proper choice of the thermodynamic potential $\mathcal{E}$ provides both properties simultaneously: complete compatibility with the thermodynamics and mathematical regularity of the model.

\textbf{Physical interpretation.} All elements of our continuum hyperbolic theory, such as elastic distortion and strain dissipation time, have microphysics counterparts, such as bond stretching and particle settled life time, \textit{i.e.}, they are observable in principle. In contrast, the Maxwell-type models are representatives of the RM approach (see Section 1) and hence, phenomenological.

\section{Numerical example}

The main goal of this section is to demonstrate that Newton’s viscous law can be recovered with our hyperbolic theory. The second goal is to discover the functional dependence $\tau = \tau(A, \rho, s)$ for which this can be done.

Consider a simple shear flow of a layer of a Newtonian fluid in the Lagrangian direction $y_2$. In this case, the velocity vector is $v = (0, v_2, 0)^T$ and the overall deformation gradient $F = [F_{ij}]$ has the form

\[ F = \begin{pmatrix} 1 & 0 & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{21} \]

where $\varepsilon = v_2t/\varepsilon_0$, $t$ is the time of observation, $\varepsilon_0$ equals to 1 and denotes the reference
length in the \( y_2 \) direction. The Lagrangian time evolution of \( F \)

\[
\frac{dF_{ij}}{dt} = \frac{\partial v_i}{\partial y_j}
\]

reduces to one equation:

\[
\frac{d\varepsilon}{dt} = \frac{\partial v_2}{\partial y_1},
\]

where \( d/dt \) denotes the material time derivative. The Eulerian form of this equation is

\[
\dot{\varepsilon} = \frac{\partial v_2}{\partial x_1},
\]

where \( \dot{\varepsilon} = \partial\varepsilon/\partial t \) is the rate of strain.

Thus, we rewrite Eq. (3b) for elastic distortion \( A \) in the form

\[
\frac{\partial A_{ij}}{\partial t} + v_k \frac{\partial A_{ij}}{\partial x_k} + A_{ik} \frac{\partial v_k}{\partial x_j} = -\frac{\Psi_{ij}}{\tau}.
\] (22)

Given that \( v_1 = v_3 \equiv 0 \) and \( \partial/\partial x_2 = \partial/\partial x_2 \equiv 0 \), the equations for the first column of \( A \) have the form

\[
\frac{\partial A_{i1}}{\partial t} + \dot{\varepsilon} A_{i2} = -\frac{\Psi_{i1}}{\tau},
\] (23)

and equations for the second and third columns are

\[
\frac{\partial A_{ij}}{\partial t} = -\frac{\Psi_{ij}}{\tau}, \quad i = 1, 2, 3, \quad j = 2, 3.
\] (24)

Finally, we solve the system (23) and (24) of nine ordinary differential equations with the rate of strain \( \dot{\varepsilon} \) as a parameter. In what follows, formulas (16) and (13) for the strain dissipation function \( \Psi \) and the viscous stress tensor \( T^v \), respectively, are used. For our study, heat effects can be neglected, and because the flow (21) is incompressible \((\rho = \rho_0 / \det F \) and \( \det F = 1 \)), the shear sound velocity \( c_s \) is assumed to be constant.

Since strain dissipation occurs in the direction of minimization of \( \|\text{dev} \, G\|^7 \), we can define the norm of \( \text{dev} \, G \) as the measure of the dynamical stretch limit \( Y \) for a given fluid parcel, \textit{i.e.},

\[
Y = \|\text{dev} \, G\| = \sqrt{\text{tr}((\text{dev} \, G)^T \text{dev} \, G)}.
\]

For the numerical resolution of (23), (24), the open source ODE solver LSODE [58] and the solver ODE15S of the commercial software MATLAB [59] for stiff ODEs were used. Both solvers provided identical results.

### 3.1 Results

As a representative of Newtonian fluids, we consider air with the viscosity \( \eta = 18.21 \cdot 10^{-6} \) Pa·s at the temperature 20 °C. Thus, we solve system (23), (24) with the initial

\footnote{In general, the directions of minimization of \( \|\text{dev} \, A\| \) and \( \|\text{dev} \, G\| \) do not coincide.}
Figure 1: Results of the numerical solution of system (23), (24) for the rate of strain \( \dot{\varepsilon} = 0, 2, 4, 6, 8, 10 \text{ s}^{-1} \) for air with the viscosity coefficient \( \eta = 18.21 \cdot 10^{-6} \text{ Pa} \cdot \text{s} \): (a) time evolutions of the first and third singular values of matrix \( \mathbf{A} \) (the second singular value equals identically to one in this test), (b) profiles of the shear stress \( T_{21} \), (c) comparison of Newton’s viscous law for air (green solid line) with results obtained with the hyperbolic model (blue circles, the steady-state values for \( T_{21} \) extracted from plot (b)).

Our goal is to fit the shear stress \( T_{21} = T_{21}(\mathbf{A}) = T_{12} \) obtained as a solution to the hyperbolic model, \textit{i.e.} as a solution to system (23), (24), to Newton’s viscous law

\[
T_{21} = \eta \dot{\varepsilon}.
\]

The result of the fitting is presented in Fig. 1(c). It appears that, in order to fit the law (25), the strain dissipation time can be assumed to be constant for a given density and temperature. In particular, for air \( \tau = 2.7315 \cdot 10^{-9} \text{ s} \), provided the shear sound velocity\(^8\) is \( c_s = 100 \text{ m/s} \). Results for other typical Newtonian fluids, such as water and a liquid with the viscosity coefficient close to a value typical for honey, are given in Table 1.

In the following, we present the first observation in our numerical experiments.

---

\(^8\)Recall that the longitudinal sound velocity of air is about 343 m/s at 20 °C.
Observation 1. Different values of the shear sound velocity provide different values of \( \tau \), but our calculations show that the quantity \( \tau c_s^2 \) remains constant,

\[
\tau c_s^2 = \nu_0,
\]  

for a given Newtonian fluid. Clearly, this is because we are fitting one parametric law (25) with the two parameter model. The physical dimension of \( \nu_0 \) is \( m^2/s \).

It may seem that any values of the shear sound velocity are appropriate provided that equality (26) is fulfilled. However, it is important to recall that \( \tau \) cannot be arbitrary, since it is an objective quantity with a definite physical meaning, and thus can be estimated theoretically or in experiments. For non-Newtonian fluids, \( \nu_0 \) is not a constant, since law (25) is nonlinear.

The key feature of the model is that the microscopic elastic deformation, characterized by \( A \), of a fluid parcel reaches rapidly (during the stage of elastic stretching \( t < \tau \), see the vertical solid line on Fig. 1(b)) toward a certain value corresponding to the dynamical stretch limit \( Y \) and then remains constant while the observable macroscopic deformation \( F \) of the fluid layer continues to increase proportionally with respect to the time of observation (see (21)). Fig. 1(a), (b) shows the time evolution of the first and third singular values \( a_i \) (\( a_2 \equiv 1 \) for this test case) of elastic distortion \( A \) (for the sake of convenient scaling, we plot \( a_i - 1 \) instead of \( a_i \)) and the shear stress \( T_{21} \), respectively for different values of \( \dot{\varepsilon} \).

This steady-state behavior of the singular values \( a_i \) is in agreement with our interpretation of elastic distortion \( A \) as a parameter describing elastic stretching of microscopic bonds between fluid parcels. Therefore, parts of bonds break because they have reached the dynamic stretch limit \( Y(\dot{\varepsilon}) \), while newly created bonds, for which the stretch is still below \( Y \), continue to stretch. This process of bond stretching and destruction occurs permanently, but on average, it provides the current macroscopic steady-state value of the stress \( T_{21} \). In these particular computations, \( Y(\dot{\varepsilon}) = \tau \dot{\varepsilon}/(3\sqrt{2}) \). We also note that the steady-state values for the singular values \( a_i \) remain quite close to 1, i.e., \( a_i - 1 \) is of order \( 10^{-9} \) (see Fig. 1(a)).

Fig. 1 (b) depicts the time evolutions of the shear stress \( T_{21} \) for different values of \( \dot{\varepsilon} \). The steady-state values of \( T_{21} \) are then plotted on Fig. 1(c) with blue circles.

Our second observation is:

Observation 2. The relation between the settled time \( \tau \) and the viscosity coefficient \( \eta \) is

\[
\eta = \varrho_0 \tau c_s^2 = \varrho_0 \nu_0,
\]

where the coefficient of proportionality \( \varrho_0 \), for any values of \( \eta \), or for any Newtonian fluids, is \( \varrho_0 = 2/3 \) and has the physical dimension of density \( g/m^3 \).

It should be noted that the results presented obtained for a particular, but quite general, choice (9) of the viscous part \( E^v \) of the internal energy, e.g., we suspect that the value 2/3 for \( \varrho_0 \) might be different for other choices of energy. However, it can be expected that the form of relations (26) and (27) may have a universal character for Newtonian fluids, or at least may help to determine more universal relations.

As it is required for our theory (see Section 1) that the strain dissipation time satisfies the inequality \( \tau > 0 \) for viscous flows, it then follows from relations (26) and
that the shear sound velocity $c_s$ is always finite.

We finally note that the off-diagonal entries $A_{ij}$, $i \neq j$ of the elastic distortion $A$ do not exhibit steady-state behavior as depicted in Figs. 1(a) and (b) for singular values $a_i$ and the shear stress $T_{21}$. This is because distortion $A$ contains not only the information about the deformation of the fluid parcel, but $A$ also comprises the information about rotations of the parcel after it rearranges with its neighbors (see also (11) and a comment immediately after that). In contrast, the deformation tensor $G = [G_{ij}] = A^T A$ does not contain such rotations; it describes solely the deformation of the parcel, and hence, all entries $G_{ij}$ of the matrix $G$ exhibit a steady-state behavior similarly to the singular values $a_i$ and the stress $T_{12}$. An example is given in Fig. 2, where the typical time evolution profiles of $A_{ij}$, $A_{ii}$, $G_{ij}$ and $T_{ij}^v$ for air ($\tau = 2.731 \cdot 10^{-9}$ s) are given. The rate of strain tensor

$$\dot{\epsilon} = \left[ \frac{\partial v_i}{\partial x_j} \right] = \begin{bmatrix} -0.47 & 1.53 & -4.50 \\ 6.34 & -1.37 & 0.13 \\ -3.18 & 4.62 & 1.84 \end{bmatrix} \text{s}^{-1}, \quad \text{tr} \dot{\epsilon} = 0, \quad (28)$$

was determined randomly in this example. Instead of system (23) and (24), we solve Eq. (22):

$$\frac{dA}{dt} + A \dot{\epsilon} = -\frac{\Psi}{\tau},$$

where $d/dt$ is the material time derivative. Fig. 2 shows that the deformation of fluid parcels preserved (Figs. 2(a) and (d)) and the shear stresses remain constant (Fig. 2(c)) after time $t = \tau$, but parcels continue to rotate at a constant angular velocity; slopes of $A_{ij}$, $i \neq j$ are constant, Fig. 2(b).

**Remark 7.** As long as a fluid under consideration is simple and a flow is laminar, the information about rotations of fluid parcels stored in the distortion $A$ (see Fig. 2, (b)) can be ignored and, for example, the deformation tensor $G = A^T A$ can be used as the state variable instead of $A$. The situation becomes quite different if we consider complex fluids, e.g., liquid crystals [60], where the orientation of parcels plays an important role, and therefore parcel rotations cannot be ignored. One may expect that the same is true for turbulent flows of simple Newtonian fluids [61, 62].

## 4 Discussion

**Relation to the NSEs.** As was shown previously, Newton’s viscous law, and subsequently the NSEs *per se*, is obtained in our hyperbolic theory as a steady-state limiting case if the time of observation and a characteristic time of mechanical fluctuations are sufficiently larger than the settled time $\tau$. However, it is also clear that, in general, solutions to the hyperbolic model (non-equilibrium model) and NSEs (equilibrium model) may not coincide. For example, one may expect that solutions to these models may diverge for strongly sheared and essentially time-dependent flows.

In addition, the hyperbolic model and the NSEs, which are of parabolic type, are two quite different mathematical objects. For example, it is well known that the type of PDEs influences strongly the choice of a numerical method, the way of solution of
Table 1: Values of the settled time $\tau$ for a given shear sound velocity $c_s$ for three typical Newtonian fluids.

|       | $\eta$, [Pa·s] | $c_s$, [m/s] | $\tau$, [s] |
|-------|---------------|--------------|-------------|
| air   | $18.21 \cdot 10^{-6}$ | 100 | $2.731 \cdot 10^{-9}$ |
| water | $1.002 \cdot 10^{-3}$ | 1000 | $1.503 \cdot 10^{-9}$ |
| “honey” | 5 | 1000 | $7.5 \cdot 10^{-6}$ |

Elastic response of Newtonian fluids. According to the definition of the PSL time $\tau$, Newtonian fluids can exhibit a pure elastic response if a characteristic time $t'$ of shear fluctuations is less than $\tau$. In fact, if the loading reverses its direction of action for time $t' < \tau$, then no bond destructions and subsequent particle rearrangements occur, thus, the fluid cannot flow but exhibits a pure elastic response. This time is, however, extremely small for such fluids as water and air; of order $10^{-9}$ s, which corresponds to a perturbation frequency of order 1 GHz, and this is far below the time scale of the typical day life.

Numerical implementation. Time step. It may seem that such a small time scale for the strain dissipation characteristic time ($\tau \approx 10^{-9}$ s) prevents practical use of the model for such fluids as air or water because the integration time step $\Delta t$ for models of relaxation type should be of the order of $\tau$ or less. However, the situation is not so unfavorable. Because the solutions to (22) exhibit a steady-state behavior in the time interval $\tau < t < t'$, where $t'$ is the moment of time before which the assumption $\dot{\varepsilon} = const$ gives a satisfactory approximation, $\Delta t$ can be taken of the order of $t'$. Therefore, the task is to invent a procedure that determines the values of $A$ corresponding to this steady-state regime. After that, it remains to compute the viscous stress tensor using (12), (13). Such a strategy is used, for example, in multiphase flow problems [63].

Lack of divergence form. One of the drawbacks of the model is that Eq. (3b) for elastic distortion is not in a divergence form. A conservative extension of the model based on a continuum notion of flow defects, i.e., microscopic slips of one cluster of particles relative to another, is discussed in [18].
Figure 2: Results of numerical solutions of Eq. (22) for air for an arbitrary chosen rate of strain tensor given by (28). Solution profiles on (a), (c), and (d) demonstrate a steady-state behavior at times \( t > \tau \) while the off-diagonal entries \( A_{ij}, i \neq j \) of the elastic distortion \( A \) continue to grow, (b), which essentially means that fluid parcels rotate during the parcel rearrangement process.

5 Summary

Using the conventional continuum interpretation of fluids and solids as a system of microscopic particles connected by bonds, we discussed an alternative to the concept of the viscosity coefficient: the settled particle life time \( \tau \) [2], or strain dissipation characteristic time, in order to characterize the internal resistance to flow for such systems. The proposed characteristic allows us to develop a pure hyperbolic theory for flows of viscous fluids. The base of our theory was the non-stationary equations of non-linear elastoplasticity derived in [32, 33]. The internal resistance to flow was interpreted as elastic stretching of particle bonds; a flow is a result of particle rearrangements, or strain dissipation, after bonds are destroyed.

We examined the theory for the simple shear flow of Newtonian fluids, such as air, water, and honey, and demonstrated that Newton’s viscous law can be recovered in
the framework of the hyperbolic theory as the steady-state limit. Basic relations (26) and (27) for the viscosity coefficient, strain dissipation time \( \tau \), and shear sound velocity were obtained.

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