Solutions of minimal four-dimensional de Sitter supergravity

J B Gutowski\textsuperscript{1} and W A Sabra\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, King’s College London, Strand, London WC2R 2LS, UK
\textsuperscript{2} Centre for Advanced Mathematical Sciences and Physics Department American University of Beirut, Lebanon

E-mail: jan.gutowski@kcl.ac.uk and ws00@aub.edu.lb

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Abstract
Pseudo-supersymmetric solutions of minimal $N = 2$, $D = 4$ de Sitter supergravity are classified using spinorial geometry techniques. We find three classes of solutions. The first class of solution consists of geometries which are fibrations over a three-dimensional manifold equipped with a Gauduchon–Tod structure. The second class of solution is the cosmological Majumdar–Papapetrou solution of Kastor and Traschen, and the third corresponds to gravitational waves propagating in the Nariai cosmology.

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1. Introduction

The classification of supersymmetric solutions has attracted considerable attention in recent years due to the important role these solutions play in string and M-theory. Many years ago, Tod was able to find all metrics admitting supercovariantly constant spinors in $N = 2$, $D = 4$ ungauged minimal supergravity \cite{Tod}. In recent years and motivated by the work of \cite{Tod}, progress has been made in the classification of supersymmetric supergravity solutions. For example, the classification of supersymmetric solutions of minimal $N = 2$, $D = 4$ gauged supergravity has been performed in \cite{GutowskiSabra1, GutowskiSabra2}. The bosonic part of $N = 2$, $D = 4$ gauged supergravity is basically Einstein–Maxwell theory with a negative cosmological constant. The supersymmetric solutions of this theory are obtained by solving the Killing spinor equation obtained from the vanishing of the gravitini supersymmetry variation. In this paper, we will be interested in finding solutions of Einstein–Maxwell theory with a negative cosmological constant. This theory cannot be embedded in a supergravity theory, as supersymmetry restricts the cosmological constant to be negative. However, one can obtain a fake Killing spinor equation by analytic continuation. Therefore, we will be using fake supersymmetry as a solution generating technique. de Sitter supergravities can also be obtained from type IIB*
theory [4]. Solutions of five-dimensional de Sitter supergravity were recently analysed in [5]. We shall use spinorial geometry techniques to investigate the solutions of the minimal four-dimensional de Sitter supergravity. These have been used to analyse supersymmetric solutions in ten- and 11-dimensional supergravity theories [6–9] as well as in lower dimensions [10].

The plan of the paper is as follows. In section two, a summary of the basic equations of the theory of \( N = 2, D = 4 \) minimal de Sitter supergravity and a brief description of the representatives of the Killing spinors are presented. Sections three, four and five contain a detailed analysis of the Killing spinor equations for the three canonical forms of Dirac spinors, making use of the linear system presented in the appendix. We conclude in section 6.

2. \( N = 2, D = 4 \) minimal de Sitter supergravity

In this section, we present a summary of \( N = 2, D = 4 \) minimal de Sitter supergravity. The bosonic action associated with this theory is [12, 13]

\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{4} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{3}{2\ell^2} \right),
\]

where \( F = dA \) is the Maxwell field strength and \( \ell \) is a non-zero real constant. The signature of the metric is \((- , +, +, +)\). The Einstein and gauge field equations are

\[
R_{\mu\nu} - \frac{3}{2} \ell^{-2} g_{\mu\nu} - 2F_{\mu\rho} F^\rho_\nu + \frac{i}{2} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} = 0
\]

\[
d \star F = 0.
\]

We shall consider solutions which are pseudo-supersymmetric, i.e. those which admit a non-zero Killing spinor \( \epsilon \) satisfying the Killing spinor equation:

\[
D_\mu \epsilon = \left( \partial_\mu + \frac{1}{4} \omega_{\mu, \alpha\beta} \Gamma^\alpha_{\beta\gamma} + \frac{i}{4} F_{\nu\gamma} \Gamma^\alpha_{\mu\nu} - iF_{\alpha\beta} \Gamma^\nu_{\beta\mu} - \frac{i}{2} \ell^{-1} \Gamma^{-1} \Gamma_{\mu} - \ell^{-1} A_\mu \right) \epsilon = 0.
\]

The Killing spinor \( \epsilon \) is a Dirac spinor. We follow the conventions of [14] in dealing with such spinors; for convenience we summarize a number of useful results here.

Dirac spinors can be written as complexified forms on \( \mathbb{R}^2 \); a generic spinor \( \eta \) can therefore be written as [15]

\[
\eta = \lambda e^1 + \mu e^2 + \sigma e_1 e^2,
\]

where \( e_1, e_2 \) are 1-forms on \( \mathbb{R}^2 \) and \( i = 1, 2; e_{12} = e_1 \wedge e_2, \lambda, \mu \) and \( \sigma \) are complex functions. It will be particularly useful to work in a null basis, and set

\[
\Gamma_+ = \sqrt{2} e_2,
\]

\[
\Gamma_- = \sqrt{2} e_2 \wedge
\]

\[
\Gamma_1 = \sqrt{2} e_1 \wedge,
\]

\[
\Gamma_1 = \sqrt{2} e_1 \wedge,
\]

where in the null basis the metric is

\[
d s^2 = 2 e^i e^i + 2 e^i e^j.
\]

We recall from [14] that a spinor \( \epsilon \) can be written, using \( Spin(3, 1) \) gauge transformations, as one of three possible simple canonical forms:

\[
\epsilon = e_2
\]

or

\[
\epsilon = 1 + \mu e_2
\]
or
\[ \epsilon = 1 + \mu^2 e_2. \] (9)
Note that by making use of a \textit{Spin}(3, 1) transformation generated by \( \Gamma_{+-} \), combined with an appropriately chosen \( U(1) \) gauge transformation of \( A \) which together leaves \( 1 \) invariant, one can without loss of generality take \( |\mu|^2 = 1 \) in (9).
To proceed, we evaluate the Killing spinor equation (3) acting on the spinor
\[ \epsilon = \lambda + \mu e_1 . \] (10)
The resulting equations are summarized in the appendix. We then consider the three cases (7), (8) and (9), separately.

3. Solutions with \( \epsilon = e_2 \)
In order to analyse solutions with \( \epsilon = e_2 \), evaluate the equations in the appendix with \( \lambda = \mu^1 = 0 \) and \( \mu^2 = 1 \). One obtains
\[ F_{+-} + F_{1\bar{1}} + \ell^{-1} = 0 \] (11)
and
\[ F_{+-} + F_{1\bar{1}} - \ell^{-1} = 0. \] (12)
It is clear that these equations admit no solution; hence, there are no supersymmetric solutions with \( \epsilon = e_2 \).

4. Solutions with \( \epsilon = 1 + \mu e_1 \)
On evaluating the equations in the appendix with \( \lambda = 1, \mu^1 = \mu, \mu^2 = 0 \) one obtains the conditions:
\[ \partial_1 \mu = \partial_\mu \mu = 0 \]
\[ \partial_1 \mu = \sqrt{2i} \ell^{-1} (1 + \mu \bar{\mu}) \]
\[ \omega_{1,+1} = \omega_{-,+1} = \omega_{1,+1} = \omega_{+,1} = 0 \]
\[ \omega_{-,1} = \frac{(\mu \partial_\mu \bar{\mu} - \bar{\mu} \partial_\mu \mu)}{(1 + \mu \bar{\mu})} \]
\[ \omega_{1,1} = -\mu \sqrt{2i} \ell^{-1} \]
\[ \ell^{-1} A_- = \frac{1}{2} \left( \frac{\mu \partial_\mu \bar{\mu} + \bar{\mu} \partial_\mu \mu}{1 + \mu \bar{\mu}} \right) - \frac{1}{2} \omega_{-,+-} \]
\[ \ell^{-1} A_1 = -\frac{1}{2} \omega_{1,+} - \frac{1}{\sqrt{2}} \mu i \ell^{-1} \]
\[ \ell^{-1} A_+ = -\frac{1}{2} \omega_{+,-} \]
\[ F_{-\bar{1}} = \frac{i}{\sqrt{2}} \frac{\partial_\mu}{(1 + \mu \bar{\mu})} \]
\[ F_{+-} = \ell^{-1} \]
\[ F_{1\bar{1}} = F_{+\bar{1}} = 0. \] (16)
To analyse these conditions, first observe that
\[ d\mathbf{e}^- = -\omega_{A,+} \mathbf{e}^A \wedge \mathbf{e}^- \] (17)
and hence \( \mathbf{e}^- \) is hypersurface orthogonal; one can introduce a co-ordinate \( u \) and function \( H \) such that
\[ \mathbf{e}^- = H du. \] (18)

Next, we examine the constraints on the gauge potential \( A \). Note that
\[ \ell^{-1} A = \frac{1}{2} d \log(1 + |\mu|^2) + \frac{1}{2} d \log H + P du \] (19)
for some function \( P \). Hence, by making a gauge transformation in \( A \), combined with an appropriately chosen \( Spin(3,1) \) transformation generated by \( \Gamma_{+,-} \), which together with the \( A \)-gauge transformation leave \( 1 + \mu \mathbf{e}^1 \) invariant, one can without loss of generality work in a gauge for which
\[ \ell^{-1} A = \left( \frac{1}{2} \frac{\tilde{\mu} \partial_- \mu - \mu \partial_- \tilde{\mu}}{(1 + \mu \tilde{\mu})} - \frac{1}{2} \omega_{-,+} \right) \mathbf{e}^- \] (20)
and moreover
\[ \omega_{+,+} = 0 \] (21)
and
\[ -\frac{1}{2} \omega_{+,+} - \frac{1}{\sqrt{2}} \mu i \ell^{-1} = 0. \] (22)

In this gauge, we then find
\[ d\mathbf{e}^- = -d \log(1 + |\mu|^2) \wedge \mathbf{e}^- \] (23)
and hence it is most convenient to introduce a local co-ordinate \( u \) such that
\[ \mathbf{e}^- = \frac{1}{1 + |\mu|^2} du. \] (24)

Next, consider the exterior derivative of \( \mathbf{e}^1 \) restricted to hypersurfaces \( u = \text{const} \). One finds that
\[ d\mathbf{e}^1 = -d \log(1 + |\mu|^2) \wedge \mathbf{e}^1 \] (25)
where \( d \) denotes the restriction of the exterior derivative to \( u = \text{const} \). It follows that one can introduce a complex co-ordinate \( z \) such that
\[ \mathbf{e}^1 = \frac{1}{1 + |\mu|^2} (dz + \xi du) \] (26)
for \( \xi \in \mathbb{C} \). The metric can further be simplified by making use of the \( Spin(3,1) \) gauge transformation generated by \( \beta \Gamma_{+1} + \tilde{\beta} \Gamma_{-1} \), for \( \beta \in \mathbb{C} \), which leaves the Killing spinor \( 1 + \mu \mathbf{e}^1 \) invariant. This gauge transformation corresponds to a change of basis of the form
\[ \begin{align*}
(\mathbf{e}^-)' &= \mathbf{e}^- \\
(\mathbf{e}^+)' &= \mathbf{e}^+ + 4|\beta|^2 \mathbf{e}^- - 2\tilde{\beta} \mathbf{e}^1 - 2\beta \mathbf{e}^1 \\
(\mathbf{e}^1)' &= \mathbf{e}^1 - 2\beta \mathbf{e}^-.
\end{align*} \] (27)

By choosing \( \beta \) appropriately, one can, without loss of generality, set \( \xi = 0 \) in (26).
So, on introducing a final local co-ordinate $v$ such that the vector field dual to $e^-$ is $\frac{\partial}{\partial v}$, one finds that one can write the basis in the $u, v, z, \bar{z}$ co-ordinates as

$$e^- = \frac{1}{1 + |\mu|^2} du$$

$$e^1 = \frac{1}{1 + |\mu|^2} dz$$

$$e^+ = dv + \mathcal{H} du + \mathcal{G} dz + \bar{\mathcal{G}} d\bar{z},$$

where $\mathcal{H}$ is a real function, $\mathcal{G}$ is a complex function and $\mu$ does not depend on $v$.

Next consider the constraints implied by (13). In particular, $\mu$ depends only on $\bar{z}$ and $u$, with

$$\frac{\partial \mu}{\partial \bar{z}} = \sqrt{2i\ell^{-1}}.$$  

Hence,

$$\mu = \sqrt{2i\ell^{-1}} \bar{z} + h(u),$$

where $h(u)$ is a function of $u$. By making a change in co-ordinates of the form $\bar{z}' = \bar{z} + \psi(u)$ together with an appropriately chosen $Spin(3, 1)$ transformation generated by $\beta/\Gamma_1 + \bar{\beta}/\Gamma_1$, one can without loss of generality take the basis given in (28) with $\mu = \sqrt{2i\ell^{-1}} \bar{z}$.  

Observe that, in this basis, $\partial_{\bar{z}} \mu = 0$.

To proceed, consider the conditions $\omega_{-,+1} = \omega_{-,-1} = 0$ on the geometry. It is straightforward to show that these imply that

$$\mathcal{G} = -\frac{2\ell^{-2}v\bar{z}}{1 + 2\ell^{-2}z\bar{z}} + \phi$$

where $\phi(u, z, \bar{z})$ is a complex function satisfying

$$\bar{z} \frac{\partial \phi}{\partial z} - z \frac{\partial \phi}{\partial \bar{z}} = 0.$$  

Next, noting that

$$\omega_{-,-} = -(1 + 2\ell^{-2}z\bar{z}) \frac{\partial \mathcal{H}}{\partial v},$$

we impose the Bianchi identity $F = dA$ to obtain the constraints

$$\frac{\partial^2 \mathcal{H}}{\partial v^2} = \frac{2\ell^{-2}}{1 + 2\ell^{-2}z\bar{z}},$$

$$\frac{1}{2} \frac{\partial^2 \mathcal{H}}{\partial z \partial \bar{v}} = \frac{\ell^{-2}}{1 + 2\ell^{-2}z\bar{z}} \left( -\frac{2\ell^{-2}v\bar{z}}{1 + 2\ell^{-2}z\bar{z}} + \phi \right).$$

These can be solved to find

$$\mathcal{H} = \frac{\ell^{-2}v^2}{1 + 2\ell^{-2}z\bar{z}} + \Theta_1 v + \Theta_2,$$

where $\Theta_1, \Theta_2$ do not depend on $v$, and

$$\phi = \frac{1}{2} \ell^2 (1 + 2\ell^{-2}z\bar{z}) \frac{\partial \Theta_1}{\partial z}$$

with $\Theta_1$ constrained by

$$\bar{z} \frac{\partial \Theta_1}{\partial \bar{z}} - z \frac{\partial \Theta_1}{\partial z} = 0.$$  

One can simplify the solution considerably by making the co-ordinate transformation

$$v = (1 + 2\ell^{-2}z\bar{z})(v' - \frac{1}{2} \ell^2 \Theta_1).$$
On dropping the prime on \( v \) the solution can then be written as

\[
ds^2 = 2 du [dv + (\ell^{-2} v^2 + \Psi) du] + \frac{2}{(1 + 2 \ell^{-2} z^2)^2} dz d\bar{z}
\]  

(40)

with

\[
F = \ell^{-1} dv \wedge du.
\]  

(41)

The function \( \Psi = \Psi(u, z, \bar{z}) \) appearing in the metric is constrained to be harmonic on \( \mathbb{R}^2 \) by the Einstein equations:

\[
\frac{\partial^2 \Psi}{\partial z \partial \bar{z}} = 0.
\]  

(42)

Observe that the gauge field equations \( d \ast F = 0 \) hold with no further constraints.

5. Solutions with \( \ell = 1 + e^{\imath \alpha} e_2 \)

On evaluating the equations in the appendix with \( \lambda = 1, \mu^1 = 0, \mu^2 = e^{\imath \alpha} \), one obtains the components of the gauge field strength as

\[
F_\pm = \sqrt{2} (\sin \alpha \omega_{+,\pm} - \partial_\pm \alpha \cos \alpha) - \ell^{-1}
\]

\[
F_{11} = i \sqrt{2} (\cos \alpha \omega_{+,+} + \sin \alpha \partial_+ \alpha)
\]

\[
F_{-1} = \frac{i}{\sqrt{2}} e^{\imath \alpha} \omega_{-,1}
\]

\[
F_+ = \frac{i}{\sqrt{2}} e^{-\imath \alpha} \omega_{+,1}.
\]  

(43)

The components of the gauge potential are given by

\[
\ell^{-1} A_- = -\frac{1}{2} \omega_{-,\pm}, \quad \ell^{-1} A_1 = \frac{1}{2} (i \partial_+ \alpha - \omega_{1,1}), \quad \ell^{-1} A_+ = \frac{1}{2} \omega_{+,\pm}.
\]  

(44)

The geometric constraints are given by

\[
\omega_{+,+} + \omega_{-,\pm} = \sqrt{2} \ell^{-1} \sin \alpha
\]

\[
\partial_\pm \alpha - \partial_\pm \alpha = \sqrt{2} \ell^{-1} \cos \alpha
\]

\[
\omega_{1,1} = 2 i \partial_1 \alpha + \omega_{+,1} = \omega_{-,1}
\]

\[
\omega_{1,1} = -\omega_{+,+} - i \partial_+ \alpha - \sqrt{2} e^{\imath \imath \alpha} \ell^{-1}
\]

\[
\omega_{1,-1} = -\omega_{+,+} + i \partial_\pm \alpha
\]

\[
\omega_{1,1} = -i \partial_1 \alpha, \quad \omega_{1,1} = 2 i \partial_1 \alpha
\]

\[
\omega_{-,1} = \omega_{-,1} = \omega_{1,1} = 0.
\]  

(45)

Thus, we can write

\[
de^1 = (-\omega_{+,+} + i \sqrt{2} e^{-i \alpha} \ell^{-1} - i \partial_+ \alpha) e^1 \wedge e^- - (\omega_{+,+} + i \partial_+ \alpha) e^1 \wedge e^+
\]

\[
-2i (\partial_1 \alpha + \omega_{+,1}) e^1 \wedge e^1
\]

\[
de^+ = \omega_{-,\pm} e^+ \wedge e^- + \omega_{-,\pm} e^1 \wedge e^- - i \partial_1 \alpha e^1 \wedge e^+ + \omega_{-,\pm} e^1 \wedge e^- + i \partial_1 \alpha e^1 \wedge e^+
\]

\[
+2i (\partial_1 \alpha + \sqrt{2} \cos \alpha \ell^{-1}) e^1 \wedge e^1
\]

\[
de^- = -\omega_{+,+} e^+ \wedge e^- + i (\partial_\pm \alpha e^1 - \partial_1 \alpha e^1) \wedge e^- + (\omega_{+,+} e^1 + \omega_{+,1} e^1) \wedge e^-
\]

\[
-2i (\partial_1 \alpha + \sqrt{2} \cos \alpha \ell^{-1}) e^1 \wedge e^1.
\]  

(46)
5.1. Solutions with $\cos \alpha \neq 0$

For these solutions, it is convenient to define the 1-form

$$V = \frac{1}{\cos \alpha} (e^+ - e^-)$$  \hspace{1cm} (47)

and introduce a local co-ordinate $t$ such that $V = \frac{\partial}{\partial t}$.

It is straightforward to see that the supersymmetry constraints imply that

$$\frac{\partial \alpha}{\partial t} = \sqrt{2} \ell^{-1}$$  \hspace{1cm} (48)

and furthermore

$$\mathcal{L}_V e^1 = \frac{\sqrt{2} i \ell^{-1} e^{-i\omega}}{\cos \alpha} e^1$$
$$\mathcal{L}_V (e^+ + e^-) = \sqrt{2} \ell^{-1} \tan \alpha (e^+ + e^-).$$  \hspace{1cm} (49)

These constraints imply that one can write

$$e^1 = (1 + i \tan \alpha) \hat{e}^1$$
$$e^+ + e^- = \frac{\sqrt{2}}{\cos \alpha} e^2,$$  \hspace{1cm} (50)

where

$$\mathcal{L}_V \hat{e}^1 = 0, \quad \mathcal{L}_V e^2 = 0.$$  \hspace{1cm} (51)

Note, furthermore, that

$$d\hat{e}^1 = \left( \sqrt{2} \sec \alpha \omega_{+,+} - \frac{1}{\sqrt{2} \cos^2 \alpha} (\partial_{+,+} + \partial_{-,+}) - \ell^{-1} \tan \alpha - 2i \ell^{-1} \right) e^2 \wedge \hat{e}^1$$
$$+ \sec^2 \alpha (-i \partial_{1} \alpha - \cos \alpha e^{-i\omega_{+,1}}) \hat{e}^1 \wedge \hat{e}^1,$$  \hspace{1cm} (52)

and

$$d e^2 = \frac{1}{2} \sec^2 \alpha (e^{2i\omega_{+,1} + \omega_{-,1}}) \hat{e}^1 + (e^{-2i\omega_{+,1} + \omega_{-,1}}) \hat{e}^1 \wedge \hat{e}^1 \wedge e^2 - 2i \ell^{-1} e^1 \wedge \hat{e}^1.$$  \hspace{1cm} (53)

Note then that the metric can be written as

$$d s^2 = -\frac{1}{2} (e^+ - e^-)^2 + \sec^2 \alpha d s^2_{GT},$$  \hspace{1cm} (54)

where

$$d s^2_{GT} = (e^2)^2 + 2 \hat{e}^1 \wedge \hat{e}^1,$$  \hspace{1cm} (55)

The metric on the manifold $GT$ does not depend on $t$, and (52) and (53) imply that $GT$ admits a $t$-independent basis $E^i$ for $i = 1, 2, 3$ satisfying

$$d E^i = B \wedge E^i - \ell^{-1} e^{-ij} E^j \wedge E^i,$$  \hspace{1cm} (56)

where

$$B = \frac{1}{2} \sec^2 \alpha ((\omega_{-,1} + e^{2i\omega_{+,1}}) \hat{e}^1 + (\omega_{-,1} + e^{-2i\omega_{+,1}}) \hat{e}^1)$$
$$+ \left( \sqrt{2} \sec \alpha \omega_{+,+} - \frac{1}{\sqrt{2} \cos^2 \alpha} (\partial_{+,+} + \partial_{-,+}) - \ell^{-1} \tan \alpha \right) e^2.$$  \hspace{1cm} (57)

Note in particular that (56) implies that $B$ must be independent of $t$, and furthermore, must satisfy

$$d B = 2 \ell^{-1} \wedge B,$$  \hspace{1cm} (58)
where $\star_3$ denotes the Hodge dual on $GT$ (in our conventions, the volume form on $GT$ is $i\hat{e}^1 \wedge \hat{e}^1 \wedge e^2$). In turn, (58) implies that
\[
\star_3 B = 0.
\]
(59)
Condition (56) implies that $GT$ admits a Gauduchon–Tod structure. Such structures arise in the context of four-dimensional hyper-Kähler with torson manifolds which admit a triholomorphic isometry, and have been analysed in [16] and [17].

To proceed further, we next consider the constraints which (46) impose on $e^+-e^-$. It will be convenient to write
\[
e^+-e^- = -2\sec\alpha (dt + \Omega)
\]
and to set
\[
\alpha = \sqrt{2}\ell^{-1} t + \Phi,
\]
(61)
where $\Omega$ is a $t$-dependent 1-form on $GT$ and $\Phi$ is a $t$-independent function. Then (46) implies that
\[
\mathcal{L}_V \Omega + 2\sqrt{2}\ell^{-1} \tan\alpha \Omega - B - 2\tan\alpha d\Phi = 0.
\]
(62)
This condition can be integrated up, and on changing co-ordinates from $t$ to $\alpha$, one obtains
\[
e^+-e^- = -\sqrt{2}\ell \sec\alpha d\alpha - \sqrt{2}\ell \sin\alpha B - 2\cos\alpha \psi
\]
(63)
where $\psi$ is an $\alpha$-independent 1-form on $GT$. Note that $\psi$ is defined in terms of the basis $e^1, \hat{e}^1, e^2$ as
\[
\psi = \frac{\ell}{2\sqrt{2}\cos^2\alpha} (i(-\omega_{-1} + e^{2i\alpha} \omega_{+1})\hat{e}^1 - i(-\omega_{-1} + e^{-2i\alpha} \omega_{-1})\hat{e}^1)
\]
\[
+\frac{1}{2} \left( -\ell \sec\alpha (\partial_\alpha \alpha + \partial_\alpha \alpha) - \sqrt{2}\ell \frac{\sin\alpha}{\cos^2\alpha} (\sqrt{2}\omega_{+1} - \ell^{-1} \sin\alpha) \right) e^2.
\]
(64)
The remaining content of (46) imposes an additional condition on $\psi$:
\[
d\psi + B \wedge \psi - 2\ell^{-1} \star_3 \psi = 0.
\]
(65)
It remains to consider the constraints on the fluxes. Note first that (44) implies that
\[
A = -\frac{\ell}{2} \tan\alpha \, d\alpha + \frac{\ell}{2} \cos 2\alpha B - \frac{1}{\sqrt{2}} \sin 2\alpha \psi.
\]
(66)
It is straightforward to show that on applying the exterior derivative to (66), one obtains the components of the field strength given in (43), with no further constraint. In order to evaluate the gauge field equations, observe that the above conditions imply that the Hodge dual of $F$ is given by
\[
\star F = -\sqrt{2}\ell d\alpha \wedge \left( \frac{1}{\sqrt{2}} \cos 2\alpha B - \ell^{-1} \sin 2\alpha \psi \right) + \sqrt{2}\cos^2\alpha B \wedge \psi - \sin 2\alpha \star_3 B
\]
\[
-\sqrt{2}\ell^{-1} \cos 2\alpha \star_3 \psi.
\]
(67)
The conditions obtained previously imply that the RHS of this expression is closed with no additional constraint; hence, the gauge equations are satisfied.

In order to examine the Einstein equations we follow the reasoning presented (in the context of solutions of the Anti de Sitter minimal gauged supergravity) in appendix E of [2]. In particular, the integrability conditions of the Killing spinor equation associated with a pseudo-supersymmetric solution for which the Maxwell field strength $F$ satisfies the Bianchi identity and gauge field equations imply that
\[
E_{\mu\nu} \Gamma^\nu \epsilon = 0,
\]
(68)
where
\[ E_{\mu\nu} = R_{\mu\nu} - 3\ell^{-2}g_{\mu\nu} - 2F_{\mu\rho}F_{\nu}^{\rho} + \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}. \]  
(69)

Evaluating (68) acting on the Killing spinor \( \epsilon = 1 + e^{\alpha}e_2 \), one finds that all components of \( E_{\mu\nu} \) are constrained to vanish, i.e. the Einstein equations hold automatically.

To summarize, the solutions with Killing spinor \( 1 + e^{\alpha}e_2 \) and \( \cos \alpha \neq 0 \) have metric
\[ ds^2 = -\left( \ell \sec \alpha \, d\alpha + \ell \sin \alpha \, B + \sqrt{2} \cos \alpha \, \psi \right)^2 + \sec^2 \alpha \, ds^2_{GT}. \]  
(70)

where \( ds^2_{GT} \) is an \( \alpha \)-independent metric on a three-dimensional manifold which has a Gauduchon–Tod structure. The 3-manifold GT admits a \( \alpha \)-independent basis \( E' \) and an \( \alpha \)-independent 1-form \( B \) satisfying (56) (with associated integrability conditions (58) and (59)). GT also admits an \( \alpha \)-independent 1-form \( \psi \) satisfying (65). The flux is then given by
\[ F = d\left( \frac{\ell}{2} \cos 2\alpha \, B - \frac{1}{\sqrt{2}} \sin 2\alpha \, \psi \right). \]  
(71)

Finally, we remark that on making the coordinate transformation \( t' = \ell \tan \alpha \), one finds that the metric can be written in the form originally obtained in [11].

5.2. Solutions with \( \cos \alpha = 0 \)

Suppose that \( \sin \alpha = \pm 1 \); then
\[ F = (\pm \sqrt{2} \omega_{\alpha,+} - \ell^{-2}) e^* \wedge e^- \pm \frac{1}{\sqrt{2}} (e^* - e^-) \wedge (\omega_{\alpha,+} e^1 + \omega_{\alpha,-} e^1) \]  
(72)

and
\[ \ell^{-1} A = \frac{1}{2} \omega_{\alpha,+} e^* - \frac{1}{2} \omega_{\alpha,-} e^- + \frac{1}{2} \omega_{\alpha,+} e^1 + \frac{1}{2} \omega_{\alpha,-} e^1 \]  
(73)

and
\[ \omega_{\alpha,+} + \omega_{\alpha,-} = \pm \sqrt{2} \ell^{-1} \]
\[ \omega_{1,1} = \omega_{\alpha,+} = \omega_{\alpha,-} \]
\[ \omega_{1,1} + \omega_{\alpha,+} = \pm \sqrt{2} \ell^{-1} \]
\[ \omega_{1,-} = -\omega_{\alpha,+} \]
\[ \omega_{1,+} = \omega_{1,1} = \omega_{-1} = \omega_{-1} = \omega_{1,1} = \omega_{1,-} = \omega_{1,-} = 0. \]  
(74)

It follows that
\[ de^* = -\omega_{\alpha,-} e^* \wedge e^- + (\omega_{\alpha,+} e^1 + \omega_{\alpha,-} e^1) \wedge e^- \]
\[ de^- = -\omega_{\alpha,+} e^* \wedge e^- + (\omega_{\alpha,+} e^1 + \omega_{\alpha,-} e^1) \wedge e^* \]
\[ de^1 = [\omega_{\alpha,+} + \sqrt{2} \ell^{-1}) e^* + \omega_{\alpha,-} e^- + \omega_{\alpha,+} e^1] \wedge e^1. \]  
(75)

To proceed, note that (75) implies that
\[ (e^* + e^-) \wedge d(e^* + e^-) = 0, \quad (e^* - e^-) \wedge d(e^* - e^-) = 0. \]  
(76)

Hence, there exist real functions \( H, B, z, t \) such that
\[ e^* = \frac{1}{\sqrt{2}} (Hdz - Bdt), \quad e^- = \frac{1}{\sqrt{2}} (Hdz + Bdt). \]  
(77)

Next, note that (72) and (75) imply that
\[ F = \pm \frac{1}{\sqrt{2}} d(e^* - e^-). \]  
(78)
On comparing this expression with (73), one finds that there exists a function \( C \) such that
\[
\frac{1}{\sqrt{2}} (e^+ - e^-) = \frac{\ell}{2} (\omega_{x,+} e^x - \omega_{x,-} e^x + \omega_{x,+1} e^{1} + \omega_{x,+1} e^{1}) - \frac{\ell}{2} d \log C. \tag{79}
\]
Substituting this expression back into (75) one finds
\[
d e^1 = d \log C \wedge e^1 \tag{80}
\]
and so there exist real functions \( C, x, y \) such that
\[
e^1 = \frac{1}{\sqrt{2}} C (d x + i d y). \tag{81}
\]
It is then straightforward to show that (75) implies that
\[
H = C f_1(z), \quad B = C^{-1} f_2(t), \tag{82}
\]
where \( f_1 \) and \( f_2 \) are arbitrary functions of \( z, t \). By making appropriate \( z, t \) co-ordinate transformations, one can without loss of generality take \( f_1 = f_2 = 1 \). Furthermore, (75) implies that
\[
\frac{\partial C}{\partial t} = \pm \ell^{-1} \tag{83}
\]
so that
\[
C = \pm \ell^{-1} t + V \tag{84}
\]
for \( V = V(x, y, z) \). The metric and gauge field strength are then given by
\[
ds^2 = -\frac{1}{(V \pm \ell^{-1} t)^2} d t^2 + (V \pm \ell^{-1} t)^2 (d x^2 + d y^2 + d z^2) \tag{85}
\]
and
\[
F = \mp d \left( \frac{1}{V \pm \ell^{-1} t} d t \right). \tag{86}
\]
Finally, we impose the gauge field equations \( d \star F = 0 \), which imply that \( V \) is harmonic on \( \mathbb{R}^3 \):
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V = 0, \tag{87}
\]
and we remark that, from the reasoning used in the previous sub-section, this condition is sufficient to ensure that the Einstein equations hold automatically.

This solution is the cosmological Majumdar–Papapetrou black hole solution found in [18]. Observe that on taking the limit \( \ell \to \infty \) one recovers the standard Majumdar–Papapetrou solution. The cosmological solution (85) is obtained by shifting the harmonic function by a term linear in \( t \); this method of obtaining solutions in de Sitter supergravity has also been investigated in [5, 19].

6. Conclusions

Using spinorial geometry techniques, all pseudo-supersymmetric solutions of minimal de Sitter \( N = 2, D = 4 \) supergravity have been classified. There are three classes of solutions.
(i) The first class of solution has metric and field strength
\[ ds^2 = 2du[dv + (\ell^{-2}v^2 + \Psi)du] + \frac{2}{(1 + 2\ell^{-2}v^2)^2}dzd\xi \]  
(88)

with
\[ F = \ell^{-1}dv \wedge du , \]  
(89)

where \( \Psi = \Psi(u, z, \xi) \) satisfies
\[ \frac{\partial^2 \Psi}{\partial z \partial \xi} = 0. \]  
(90)

(ii) The second class of solution has metric
\[ ds^2 = -(\ell \sec \alpha \, da + \ell \sin \alpha \, B + \sqrt{2} \cos \alpha \, \psi)^2 + \sec^2 \alpha \, ds_{GT}^2, \]  
(91)

where \( ds_{GT}^2 \) is an \( \alpha \)-independent metric on a three-dimensional manifold which has a Gauduchon–Tod structure. The 3-manifold GT admits a \( \alpha \)-independent basis \( E^i \) and an \( \alpha \)-independent 1-form \( B \) satisfying
\[ dE^i = B \wedge E^i - \ell^{-1} \epsilon^{ijk} E^j \wedge E^k \]  
(92)

together with an \( \alpha \)-independent 1-form \( \psi \) satisfying
\[ d\psi + B \wedge \psi - 2\ell^{-1} \star_3 \psi = 0. \]  
(93)

The gauge field strength is
\[ F = \pm d\left( \ell \frac{2}{\cos \alpha \, B - \frac{1}{\sqrt{2}} \sin \alpha \, \psi} \right). \]  
(94)

(iii) The third class of solution consists of the cosmological Majumdar–Papapetrou black hole solution found in [18] with
\[ ds^2 = -\frac{1}{(V \pm \ell^{-1}t)^2} dt^2 + (V \pm \ell^{-1}t)^2(dx^2 + dy^2 + dz^2) \]  
(95)

and
\[ F = \pm d\left( \frac{1}{V \pm \ell^{-1}t} dt \right) \]  
(96)

where \( V = V(x, y, z) \) satisfies
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V = 0. \]  
(97)

We remark that the solution given in (i) for the special case \( \Psi = 0 \) was found in [11]; where it is noted that the spacetime is the Nariai solution [20]; solution (88) corresponds to gravitational waves propagating in this background [21].

7. Appendix. The linear system

In this appendix we present the decomposition of the Killing spinor equation acting on the spinor \( \epsilon = \lambda 1 + \mu e_i \); we obtain the following constraints:
\[ \partial_\alpha \lambda + \lambda \left( \frac{1}{2} \omega_{+,+} - \frac{1}{2} \omega_{+,1} - \ell^{-1} A_+ \right) - \frac{i}{\sqrt{2}} \mu \left( F_+ + F_{11} + \ell^{-1} \right) = 0 \]
\[ \partial_\alpha \mu + \mu \left( -\frac{1}{2} \omega_{+,+} + \frac{1}{2} \omega_{+,1} - \ell^{-1} A_+ \right) - \omega_{-,1} \mu^2 = 0 \]
\[
\partial_\mu \mu^2 + \omega_\nu \partial_\mu \mu^1 + \mu^2 \left( \frac{1}{2} \omega_{\nu} + \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_\nu \right) = 0
\]
\[
\omega_{\lambda,11} + \sqrt{2i} F_{\nu} = 0
\]
\[
\partial_\nu \lambda + \left( \frac{1}{2} \omega_{\nu} - \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_\nu \right) - \sqrt{2i} F_{\nu} = 0
\]
\[
\partial_\nu \mu^1 = -\sqrt{2i} F_{\nu} \lambda + \mu^1 \left( \frac{1}{2} \omega_{\nu} + \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_\nu \right) - \omega_{\lambda,11} \mu^2 = 0
\]
\[
\partial_\nu \mu^2 + \frac{1}{\sqrt{2}} \lambda (F_{\nu} - F_{11} - \ell^{-1}) + \omega_{\nu,1} \mu^1 + \mu^2 \left( \frac{1}{2} \omega_{\nu} + \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_\nu \right) = 0
\]
\[-\omega_{\nu,1} \lambda - \frac{1}{\sqrt{2}} \mu^1 (F_{\nu} + F_{11} - \ell^{-1}) = 0
\]
\[
\partial_\nu \lambda + \left( \frac{1}{2} \omega_{\nu} - \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_\nu \right) - \frac{1}{\sqrt{2}} \mu^1 (F_{\nu} + F_{11} + \ell^{-1}) = 0
\]
\[
\partial_\nu \mu^1 = \mu^1 \left( \frac{1}{2} \omega_{\nu} + \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_1 \right) + \omega_{\nu,1} \mu^2 = 0
\]
\[
\partial_\nu \mu^2 + \omega_{\nu,1} \mu^1 + \mu^2 \left( \frac{1}{2} \omega_{\nu} - \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_1 \right) = 0
\]
\[
\omega_{\nu,1} \lambda + \sqrt{2i} F_{\nu} \mu^1 = 0
\]
\[
\partial_\nu \lambda + \left( \frac{1}{2} \omega_{\nu} - \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_1 \right) + \sqrt{2i} F_{-1} \mu^2 = 0
\]
\[
\partial_\nu \mu^1 + \lambda \left( -F_{\nu} + F_{11} - \ell^{-1} \right) + \mu^1 \left( \frac{1}{2} \omega_{\nu} + \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_1 \right) = 0
\]
\[
\partial_\nu \mu^2 + \sqrt{2i} F_{\nu} \lambda + \omega_{\nu,1} \mu^1 + \mu^2 \left( \frac{1}{2} \omega_{\nu} - \frac{1}{2} \omega_{\lambda,11} - \ell^{-1} A_1 \right) = 0
\]
\[-\omega_{\nu,1} \lambda + \frac{1}{\sqrt{2}} \mu^2 (F_{\nu} + F_{11} - \ell^{-1}) = 0.
\]

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