The limit of large mass monopoles

Daniel Fadel and Gonçalo Oliveira

Abstract

In this paper, we consider SU(2) monopoles on an asymptotically conical, oriented, Riemannian 3-manifold with one end. The connected components of the moduli space of monopoles in this setting are labeled by an integer called the charge. We analyze the limiting behavior of sequences of monopoles with fixed charge, and whose sequence of Yang–Mills–Higgs energies is unbounded. We prove that the limiting behavior of such monopoles is characterized by energy concentration along a certain set, which we call the blow-up set. Our work shows that this set is finite, and using a bubbling analysis obtains effective bounds on its cardinality, with such bounds depending solely on the charge of the monopole. Moreover, for such sequences of monopoles there is another naturally associated set, the zero set, which consists of the set at which the zeros of the Higgs fields accumulate. Regarding this, our results show that for such sequences of monopoles, the zero set and the blow-up set coincide. In particular, proving that in this ‘large mass’ limit, the zero set is a finite set of points.

Some of our work extends for sequences of finite mass critical points of the Yang–Mills–Higgs functional for which the Yang–Mills–Higgs energies are $O(m_i)$ as $i \to \infty$, where $m_i$ are the masses of the configurations.

Contents

1. Introduction ........................................ 1531
2. Preliminaries ....................................... 1535
3. Motivating examples ................................ 1538
4. AC and mass-dependent version of Taubes’ small Higgs field estimates ........ 1543
5. $\varepsilon$-regularity estimate ....................... 1544
6. An interior lower bound on the Higgs field .......... 1547
7. The blow-up set and the zero set ................. 1548
8. Bubbling ............................................. 1550
9. Convergence as measures ......................... 1555
Appendix. The proof of assertion (3.10) .............. 1557
References ............................................. 1558

1. Introduction

Let $(X^3, g)$ be an oriented, Riemannian 3-manifold, and $E$ a $G$-bundle over $X$, where $G$ is a compact semi-simple Lie group. Equip the associated adjoint bundle $g_E$ with the inner product arising from the negative of the Killing form on the Lie algebra $g$ of $G$. We denote by $\mathcal{A}(E)$ the space of smooth connections on $E$ and refer to sections of $g_E$ as Higgs fields. A pair $(A, \Phi) \in \mathcal{A}(E) \times \Gamma(g_E)$ is called a configuration on $E$, and any such one is called a monopole if it satisfies the Bogomolnyi equation:

$$* F_A = d_A \Phi,$$

(1.1)
and its Yang–Mills–Higgs (YMH) energy
\[ \delta_X(A, \Phi) := \frac{1}{2} \int_X \left( |F_A|^2 + |d_A \Phi|^2 \right) \text{vol}_X, \]
is finite. The Bogomolnyi equation (1.1) arises from dimensional reduction of the instanton equations in 4 dimensions and monopoles are a special kind of critical point of \( \delta_X \). Indeed, the Euler–Lagrange equations of \( \delta_X \) are
\[ d_A^\dagger F_A = [d_A \Phi, \Phi], \quad \Delta_A \Phi = 0, \tag{1.2} \]
and, using the Bianchi identity \( d_A F_A = 0 \), monopoles are easily seen to satisfy them. In this paper, we shall refer to solutions of (1.2) as YMH configurations. Any such \((A, \Phi)\) satisfies \( \Delta_A \Phi = 0 \) and thus
\[ \Delta \frac{|\Phi|^2}{2} = \langle \Phi, \Delta_A \Phi \rangle - |d_A \Phi|^2 = -|d_A \Phi|^2 \leq 0. \tag{1.3} \]
As a consequence, the function \(|\Phi|^2\) is subharmonic, and so has no local maxima. In particular, if \( X \) was to be a compact manifold without boundary, then \(|\Phi|^2\) would be constant and \( d_A \Phi = 0 = d_A^\dagger F_A \), in which case \( A \) would be a Yang–Mills connection. Thus, if one is to study irreducible YMH configurations, meaning those with \( d_A \Phi \neq 0 \), the manifold \( X \) must be noncompact\(^{\dagger}\).

YMH configurations, more specifically monopoles, have been focus of intense study in conformally flat manifolds such as \( \mathbb{R}^3 \) (some of the earlier references in the mathematics literature are [1, 12, 19]) and \( \mathbb{R}^2 \times S^1 \) (see, for example, [4, 5, 9]), as in these cases the moduli spaces of monopoles are (noncompact) Hyperkähler manifolds. In more general geometries, Braam [3] considered monopoles on asymptotically hyperbolic manifolds, while Floer [8] and Ernst [7] studied monopoles on asymptotically Euclidean (AE) ones, which are natural generalizations of the \( \mathbb{R}^3 \) situation. A further generalization of the \( \mathbb{R}^3 \) situation, which contains the AE case as a subcase, is that of asymptotically conical (AC) manifolds [13, 17]. These are complete Riemannian manifolds which outside of a compact set are asymptotic to a metric cone over a closed 2-dimensional Riemannian manifold, say \( N^2 \). In this paper, we will be considering the case when \( N \) is connected, that is the case when \( X \) has only one end.

In an AC manifold, one can fix a smooth distance function \( \rho : X \to \mathbb{R}^+_0 \), whose level sets exhaust \( X \), and are diffeomorphic to \( N \) for large enough \( \rho \). Then, for a monopole \((A, \Phi)\), the finiteness of \( \delta_X(A, \Phi) \) can be shown to be equivalent (see [12, 15, 20]) to the existence of a constant \( m \in \mathbb{R}^+ \) such that
\[ \lim_{\rho \to \infty} |\Phi| = m. \tag{1.4} \]
This constant, \( m \), is called the mass of the configuration \((A, \Phi)\). As a consequence, for any such \((A, \Phi)\) there is \( r \) sufficiently large so that \( \Phi \) does not vanish in \( \rho^{-1}[r, \infty) \). Thus, in the special case where \( G = \text{SU}(2) \), the bundle \( E \) is trivial and so the various \( \Phi|_{\rho^{-1}(r)} \) yield a well-defined homotopy class of maps \( \rho^{-1}(r) \cong N^2 \to \text{su}(2) \setminus \{0\} \cong S^2 \). The degree of such maps is therefore a well-defined integer \( k \) called the charge of \((A, \Phi)\). Equivalently, the eigenspaces of \( \Phi \) split the bundle in this region as \( E|_{\rho^{-1}(r)} \cong L \oplus L^{-1} \), for some complex line bundle \( L \) over \( N \cong \rho^{-1}(r) \). Moreover, the degree of any such \( L \) does not depend on \( r \) and equals to the charge \( k \) of \((A, \Phi)\).

In particular, in this \( G = \text{SU}(2) \) and AC case, one can rewrite \( \delta_X \) for any finite energy configuration \((A, \Phi)\) satisfying (1.4) as
\[ \delta_X(A, \Phi) = 4\pi mk + \| F_A - d_A \Phi \|^2_{L^2(X)}. \tag{1.5} \]
\(^{\dagger}\)Other options would be to work on manifolds with (nonempty) boundary and/or to consider singular YMH configurations.
Notice that the first term is fixed by the charge and mass while the second is nonnegative and vanishes if and only if \((A, \Phi)\) is a monopole. Thus showing, in particular, that monopoles minimize the YMH energy amongst finite mass configurations. The virtual dimension of the moduli space of monopoles on an AC manifold was computed in [13], and a smooth open set was constructed by a gluing theorem in [17]. Such gluing is an AC version of Taubes’ original gluing of well-separated multi-monopoles in the \(\mathbb{R}^3\) case [12]. In the case of [17], the mass plays the role of a parameter controlling the concentration of the resulting multi-monopole around its centers. Indeed, allowing the mass to vary gives the freedom of bringing these centers as close as one wants. In order to motivate the main results of this paper, we shall now summarize this construction of large mass, charge \(k\) monopoles on \(X\). This goes as follows: Start with \(k\) points in \(X\); insert charge one and mass one monopoles in \(\mathbb{R}^3\) scaled down to fit in small disjoint balls around these points; as a byproduct of having been scaled down the monopoles must have mass larger than \(O(d^{-2})\), where \(d\) is the minimum separation between the \(k\)-points; Then, by making use of a partition of unity, these can be glued with a certain mass \(O(d^{-2})\) monopole in the complement of these balls; the resulting configuration does not solve the monopole equations, but by a version of the contraction mapping principle it can be deformed to a nearby one which does. Moreover, we further remark that this configuration produces monopoles with any mass \(m \geq O(d^{-2})\); for more details and the precise statements, see Theorem 1 in [17] or Theorem 3.2 later in this paper.

The goal of this paper is to take the inverse point of view and consider a sequence of monopoles \(\{(A_i, \Phi_i)\}_{i \in \mathbb{N}}\) with unbounded masses, \(\lim \sup m_i = \infty\), but fixed charge \(k\), over an AC manifold \((X^3, g)\). In this case, the natural expectation would be an inverse construction to that of [17], with the monopoles either ‘escaping’ through the end, or getting concentrated around at most \(k\) points \(x_1, \ldots, x_k\) in \(X\), where a monopole in the Euclidean \(\mathbb{R}^3 \cong T_x X\) bubbles off\(^1\). (See Section 3 in this paper for a plethora of examples motivating this expectation.)

From the analytic point of view, the case when the energies \(\mathcal{E}_X(A_i, \Phi_i)\) of the sequence \(\{(A_i, \Phi_i)\}_{i \in \mathbb{N}}\) are uniformly bounded has a well-known limiting behavior, which is easily understood. In this case, the monopoles are either converging smoothly everywhere on \(X\), or ‘escaping’ to infinity through the end; see, for example [1], for the more general statement in the \(\mathbb{R}^3\) case. In fact, independently whether they are escaping through the end or not, the restriction of such a sequence of monopoles to any compact subset \(K \subset X\) smoothly converges to a monopole. Therefore, the most interesting case is when these energies do not remain bounded. Indeed, the energy formula (1.5) for monopoles \(\mathcal{E}_X(A_i, \Phi_i) = 4\pi km_i\) shows that this is precisely the case under consideration, where the sequence of masses \(m_i\) is unbounded.

We now introduce some preparation needed in order to state our main results. Let \(\{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq A(E) \times \Gamma(g_E)\) be a sequence of finite mass YMH configurations on \((X^3, g)\) whose masses satisfy \(\lim \sup m_i = \infty\). Define the blow-up set \(S\) of \(\{(A_i, \Phi_i)\}\) by

\[
S := \bigcup_{\varepsilon > 0, 0 < r \leq r_0} \bigcap_{i \to \infty} \left\{ x \in X : \liminf_{i \to \infty} m_i^{-1} \mathcal{E}_{B_r(x)}(A_i, \Phi_i) \geq \varepsilon \right\}.
\]

This may be interpreted as the set \(S \subset X\) where the energy of the sequence is concentrating. On the other hand, we have the zero set

\[
Z := \bigcap_{n \geq 1} \bigcup_{i \geq n} \Phi_i^{-1}(0),
\]

\(^1\)We further point out that it should be possible to start this construction by using higher charge monopoles in \(\mathbb{R}^3\) (monopole clusters). A metric version of this gluing have been carried out in [14] for the case of \(\mathbb{R}^3\).

\(^2\)Even though in this introduction, and for motivation purposes, we restrict to the case when the \(\{(A_i, \Phi_i)\}_{i \in \mathbb{N}}\) are monopoles, many of our results hold in the more general case of families of YMH configurations on a fixed \(G\)-bundle.
which consists of the accumulation points of the Higgs fields zeros, that is, the limit set of the zeros. Under suitable assumptions, our main results show that these two sets are equal and the failure of compactness is entirely due to monopole bubbling at its points. In what follows, we shall use $\mathcal{H}^0$ to denote the counting measure on $X$ and $\mathcal{H}^3$ to denote the standard Riemannian measure on $(X^3, g)$.

**Theorem 1.1.** Let $(X^3, g)$ be an AC-oriented Riemannian 3-manifold with one end, $E$ an SU(2)-bundle over $X$, and $\{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{A}(E) \times \Gamma(\text{su}(2)_E)$ a sequence of finite mass monopoles on $(X^3, g)$ with fixed charge $k \neq 0$ and masses $m_i$ satisfying $\limsup m_i = \infty$. Then, after passing to a subsequence, the following hold.

(a) For each $x \in S$, the sequence $(A_i, \Phi_i)$ bubbles off a mass 1 monopole $(A_{x, \Phi_x})$ on $\mathbb{R}^3 \cong T_x X$. Moreover, $\delta_{\mathbb{R}^3}(A_x, \Phi_x) = 4\pi k_x$, where $k_x \in \mathbb{Z}_{>0}$, $k_x \leq k$, is its charge.
(b) The blow-up set $S$ can be written as

$$S = \bigcap_{0 < r \leq \rho_0} \left\{ x \in X : \liminf_{i \to \infty} m_i^{-1} \delta_{B_r(x)}(A_i, \Phi_i) \geq 4\pi \right\}.$$  
Moreover, it coincides with the zero set $Z$:

$$S = Z.$$

(c) $S = Z$ is a finite set of at most $k$ points. In fact, we actually have

$$\mathcal{H}^0(S) \leq \frac{k}{\min_{x \in S} k_x}.$$

(d) The following weak convergence of Radon measures holds:

$$m_i^{-1} e(A_i, \Phi_i) \mathcal{H}^3 \rightharpoonup 4\pi \sum_{x \in S} k_x \delta_x,$$

where $e(A_i, \Phi_i) := |F_{A_i}|^2 + |d_{A_i} \Phi_i|^2$ and $\delta_x$ denotes the Dirac delta measure supported on $\{x\}$.

In the more general case where we have a sequence of YMH configurations on a $G$-bundle, we can still guarantee some of the above results under the assumption that $\delta_X(A_i, \Phi_i) = O(m_i)$ as $i \to \infty$, which amounts to the fixed charge assumption in the case of SU(2)-monopoles.

**Theorem 1.2.** Let $(X^3, g)$ be an AC-oriented Riemannian 3-manifold with one end, $E$ a $G$-bundle over $X$ where $G$ is a compact semi-simple Lie group, and $\{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{A}(E) \times \Gamma(\text{g}_E)$ a sequence of finite mass YMH configurations on $(X^3, g)$ whose masses $m_i$ satisfy $\limsup m_i = \infty$. Suppose that

$$m_i^{-1} \delta_X(A_i, \Phi_i) \leq C,$$

for some uniform constant $C > 0$. Then, after passing to a subsequence, the following holds.

(a') For each $x \in S$, the sequence $(A_i, \Phi_i)$ bubbles off a mass 1 YMH configuration $(A_{x, \Phi_x})$ in $\mathbb{R}^3 \cong T_x X$. Moreover, each bubble $(A_{x, \Phi_x})$ has strictly positive energy $\delta_{\mathbb{R}^3}(A_{x, \Phi_x}) > 0$.

(b') The blow-up set $S$ can be written as

$$S = \bigcap_{0 < r \leq \rho_0} \left\{ x \in X : \liminf_{i \to \infty} m_i^{-1} \delta_{B_r(x)}(A_i, \Phi_i) > 0 \right\}.$$
Moreover, it contains the zero set $Z$:

$$Z \subset S.$$  

(c') $S$ (therefore $Z$) is countable.

The proof of these results follows from putting together a number of other results. In order to guide the reader on how these are put together, we shall now explain how this paper is organized. Section 2 is a mix of background definitions such as the notion of AC manifolds or finite mass monopoles and a few technical results which will be useful later. Section 3 gives several examples of families of monopoles whose masses converge to infinity. The results are very illustrative and allow for the realization of all cases in our Theorem 1.1 and give a good intuition for the behavior of large mass monopoles.

The proof of Theorems 1.1 and 1.2 takes up every section from Section 4 to 9, and their content is summarized below.

Having in mind the aim of relating the zero set $Z$ and the blow-up set $S$ of such sequences of large mass monopoles, Section 4 gives an AC version of Taubes’ small Higgs field radius estimate (Theorem 4.1). This provides a way to control how big, in terms of the mass $m \neq 0$ and the charge $k \neq 0$ of the monopole, one needs the radius of a ball in $X$ to be so that the value of the Higgs field outside such a ball is a sufficiently large portion of $m$.

In Section 5, we prove an appropriate $\varepsilon$-regularity theorem (Theorem 5.1) for YMH configurations of finite mass $m \neq 0$, which is an important ingredient to relate the zero set with the blow-up set. Indeed, in Section 6, using a simple argument involving the fundamental theorem of calculus, together with the $\varepsilon$-regularity, we prove that a large mass YMH configuration with (locally) small energy has an interior lower bound on its Higgs field, provided it is bounded from below in some boundary ball. Together with our analogue of Taubes small Higgs field estimate, this is used in Section 7 to prove the inclusion $Z \subseteq S$, that is the last part of (b') in Theorem 1.2; here we also prove (c').

Section 8 uses the scaling properties of the YMH/monopole equations and their ellipticity on a fixed Coulomb gauge to perform the bubbling analysis. In particular, we are able to show that at each point $x \in S$, a mass 1 (YMH/)monopole in $\mathbb{R}^3$ with strictly positive energy bubbles off. This gives part (a) and half of part (b) in Theorem 1.1 and (a') and half of (b') in Theorem 1.2. In the case of monopoles, the energy formula and a degree argument then yield the reverse inclusion $S \subseteq Z$, and thus the equality in part (b) of Theorem 1.1. This part of the proof somewhat resembles Taubes’ proof of the Weinstein conjecture where a degree argument and the energy identity for the vortex equations are used to prove that a certain component of the spinor involved in the Seiberg–Witten equations vanishes (see [21, Section 6.4]).

Using all this and some simple measure theory, Section 9 is dedicated to describe the convergence of the relevant measures as in statement (d) of Theorem 1.1 and an estimate on the maximum number of elements in $S = Z$ follows, depending on the fixed charge $k$ of the sequence and the minimum of the charges $k_x$ of each bubble at $x \in S$; this corresponds to part (c) of Theorem 1.1. All these together gives a full proof of the main Theorems 1.1 and 1.2.

2. Preliminaries

In this short section, we collect a few background facts which will prove useful in the body of the paper. The reader familiar with the notion of AC manifolds is welcome to skip this section and refer back to it as needed.

We start with a basic scaling property of the YMH/monopole equations.
Proposition 2.1. Let \((X^3, g)\) be an oriented Riemannian 3-manifold, \(E\) a \(G\)-bundle over \(X\) and \((A, \Phi)\) a configuration on \(E\). If \((A, \Phi)\) is YMH (respectively a monopole) on \((X^3, g)\), then \((A, \lambda^{-1} \Phi)\) is YMH (respectively a monopole) on \((X^3, g_\lambda := \lambda^2 g)\), for any \(\lambda \in \mathbb{R} \setminus \{0\}\).

Proof. Acting on k-forms, the Hodge-* operators associated to \(g_\lambda\) and \(g\) are related by
\[*_\lambda = \lambda^{3-2k} *_*\]. Therefore, we have \(d_A^* F_\lambda = \lambda^{-2} d_A^* F_A = \lambda^{-2} [d_A \Phi, \Phi]\) in the YMH case and \(*_\lambda F_\lambda = \lambda^{-1} * F_\lambda = \lambda^{-1} d_A \Phi\) in the monopole case. The result follows.

In this article, we shall focus our study on the following class of noncompact Riemannian 3-manifolds.

Definition 1. Let \((X^3, g)\) be a complete, oriented, Riemannian 3-manifold. Then \((X^3, g)\) is called **asymptotically conical** with rate \(\nu < 0\) if there exist a compact set \(K \subset X\), an oriented, closed (compact and without boundary) Riemannian surface \((N^2, g_N)\) and an orientation preserving diffeomorphism
\[ \varphi : C(N) := (1, \infty)_r \times N \to X \setminus K \]
such that the cone metric \(g_C := dr^2 + r^2 g_N\) on \(C(N)\) satisfies
\[ |\nabla^j (\varphi^* g - g_C)|_C = O(r^{\nu-j}), \quad \forall j \in \mathbb{N}_0. \]
Here \(\nabla\) is the Levi-Civita connection of \(g_C\). We say furthermore that \(X\) has **one end** if \(N\) is connected, and we refer to \(X \setminus K\) as the **end** of \(X\). A distance function on \(X\) will be any positive smooth function \(\rho : X \to \mathbb{R}^+\) such that \(\rho|_{X\setminus K} = r \circ \varphi^{-1}\).

On such manifolds, we shall be interested in the following particular class of configurations.

Definition 2. Let \((X^3, g)\) be an AC-oriented Riemannian 3-manifold with one end, as in Definition 1, and let \(E\) be a \(G\)-bundle over \(X\). A configuration \((A, \Phi)\) on \(E\) is said to have **finite mass** if the following holds. There exists a \(G\)-bundle \(E_\infty\) over \(N\) together with an isomorphism of principal bundles \(\varphi^* (E|_{X\setminus K}) \cong \pi^* E_\infty\), where \(\pi : (1, \infty) \times N \to N\) is the projection onto the second factor, and there exists a connection \(A_\infty\) on \(E_\infty\) such that \(A\) is asymptotic to \(A_\infty\) on \(E_\infty\), that is
\[ \varphi^* \nabla_A = \pi^* \nabla_\infty + a, \quad \text{where} \quad |\nabla_\infty^j a| = O(r^{-1-j-\eta}), \quad \forall j \in \mathbb{N}_0 \text{ and for some } \eta > 0, \]
and there is \(m \in \mathbb{R}^+\) with
\[ \lim_{\rho \to \infty} |\Phi| = m. \]
We call the constant \(m\) the **mass** of \((A, \Phi)\).

Remark 2.1. If \((A, \Phi)\) is a YMH configuration, then \(\Delta_A \Phi = 0\) and thus \(|\Phi|\) is subharmonic, as shown in (1.3). So if \((A, \Phi)\) is a YMH configuration with finite mass \(m\), then as \(|\Phi|\) converges to \(m\) along the end of \(X\), the maximum principle yields that either \(|\Phi| < m\) on \(X\) or \(|\Phi|\) is constant equal to \(m\).

The next proposition, on the asymptotic behavior of the Higgs field norm of a finite mass YMH configuration (cf. [15, Section 1.4.1]), will be useful later in the proof of Theorem 4.1.

Proposition 2.2. Let \((X^3, g)\) be an AC-oriented Riemannian 3-manifold with one end, \(E\) a \(G\)-bundle over \(X\) and \((A, \Phi) \in \mathcal{A}(E) \times \Gamma(g_E)\) a finite mass YMH configuration on \((X^3, g)\),
with mass \( m \neq 0 \) and such that \( |d_A \Phi| \in L^2(X) \). Then in a neighborhood \( V(N) \) of the end, we have
\[
|\Phi| = m - \frac{1}{\text{vol}(N)} \frac{||d_A \Phi||^2_{L^2(X)}}{m \rho} + o(\rho^{-1}). \tag{2.1}
\]
In particular, if \( G = SU(2) \) and \( (A, \Phi) \) is a finite mass \( m \) and charge \( k \) monopole on \( (X^3, g) \), then
\[
|\Phi| = m - \frac{4\pi k}{\text{vol}(N)} \rho + o(\rho^{-1}). \tag{2.2}
\]

**Sketch of proof.** By the arguments in [15, proof of Proposition 1.4] (also see [12, Theorem 10.5 in Chapter IV]), one can write
\[
|\Phi| = m - c \rho^{-1} + o(\rho^{-1}) \text{ on } V(N),
\]
for some \( c \in \mathbb{R} \) that we will now compute. Since \( |d_A \Phi| \in L^2(X) \), by dominate convergence we can write
\[
\int_X |d_A \Phi|^2 = \lim_{r \to \infty} \int_{\rho^{-1}(0,r]} |d_A \Phi|^2.
\]
Now, since \( \Delta_A \Phi = 0 \), we know that \( \Delta |\Phi|^2 = -2|d_A \Phi|^2 \). Hence, by Stokes’ theorem,
\[
\int_{\rho^{-1}(0,r]} |d_A \Phi|^2 = \frac{1}{2} \int_{\rho^{-1}(r)} *|d|\Phi|^2.
\]
Therefore, we can compute
\[
\int_X |d_A \Phi|^2 = \lim_{r \to \infty} \int_{\rho^{-1}(r)} *(|\Phi|d|\Phi|)
\]
\[
= \lim_{r \to \infty} \int_{\rho^{-1}(r)} |\Phi|\partial_\rho |\Phi| * d\rho
\]
\[
= \lim_{r \to \infty} \int_{\rho^{-1}(r)} |\Phi|\partial_\rho (m - c \rho^{-1} + o(\rho^{-1})) \rho^2
\]
\[
= \lim_{r \to \infty} \int_{\rho^{-1}(r)} |\Phi|(c + o(1))
\]
\[
= cm\text{vol}(N),
\]
where in the last equality we used that \( (A, \Phi) \) has finite mass equal to \( m \). This proves equation (2.1), which in turn, in the case of monopoles, implies equation (2.2) via the energy formula (1.5). \( \square \)

Finally, we recall an auxiliary result from [17] for later reference. In what follows, for a point \( x \in X \), we let \( \delta_x \in (C^\infty_c(X))^\prime \) denote the Dirac delta distribution supported at \( x \), and we consider the transpose of the Laplace operator, still denoted by \( \Delta \), acting on \( (C^\infty_c(X))^\prime \) in the usual fashion.

**Proposition 2.3** (17, Proposition 2). Let \( (X^3, g) \) be an AC-oriented Riemannian 3-manifold with one end. Then, there are constants \( c_1 > 0 \) and \( c_2 > 0 \) such that for any given point \( x \in X \), there exists a harmonic function \( \phi_x \) on \( X \setminus \{x\} \) such that
In this section, we shall write down the standard mass \( m \) Bogomolnyi-Prasad-Sommerfield (BPS) monopole \((A_m, \Phi_m)\) on \( \mathbb{R}^3 \), constructed by Prasad and Sommerfield in [18]. For any \( m \in \mathbb{R}^+ \), this has a unique zero \( \Phi^{-1}_m(0) = \{0\} \) and is spherically symmetric. Obviously, by considering the sequence letting \( m \to \infty \) we will have \( Z = \{0\} \); however, the interesting thing of considering this specific example is that we shall be able to check the convergence to the delta function on \( Z \) explicitly.

Write \( \mathbb{R}^3 \setminus \{0\} \cong \mathbb{R}_+ \times \mathbb{S}^2 \), and pullback from \( \mathbb{S}^2 \cong \text{SU}(2)/\text{U}(1) \) the homogeneous bundle

\[
P = \text{SU}(2) \times_{\chi} \text{SU}(2),
\]

with \( \chi : \text{U}(1) \to \text{SU}(2) \) the group homomorphism given by \( \chi(e^{i\theta}) = \text{diag}(e^{i\theta}, e^{-i\theta}) \). In this polar form, and actually working on the pullback to the total space of the radially extended Hopf bundle \( \mathbb{R}^+ \times \text{SU}(2) \), the Euclidean metric can be written as

\[
g_E = dr^2 + 4r^2(\omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3),
\]
where $r$ is the radial direction, that is the distance to the origin. Now fix the standard basis $\{S_i\}$ of $\text{su}(2)$ given by the Pauli matrices, and let $\omega_1, \omega_2, \omega_3$ be the dual coframe. The 1-form $S_1 \otimes \omega^1 \in \Omega^1(\text{SU}(2), \text{su}(2))$ equips the Hopf bundle $\text{SU}(2) \to \mathbb{S}^2$ with an $\text{SU}(2)$-invariant connection, which in turn, induces a connection in $P$. Making use of Wang’s theorem [23], one can write any other spherically symmetric connection on $\mathbb{R}^3 \setminus \{0\}$ as

$$A = S_1 \otimes \omega_1 + a(r)(S_2 \otimes \omega^2 + S_3 \otimes \omega^3),$$

for some function $a : \mathbb{R}^+ \to \mathbb{R}$. Similarly, seeing a Higgs field $\Phi(r)$ as a function in the total space with values in $\text{su}(2)$ one can show, see the Appendix in [16], that any spherically symmetric Higgs field must be of the form $\Phi = \phi(r) S_1$, with $\phi : \mathbb{R}^+ \to \mathbb{R}$ some function. A computation yields that

$$F_A = 2(a^2 - 1)S_1 \otimes \omega^{23} + \dot{\phi}(S_2 \otimes dr \wedge \omega^2 + S_3 \otimes dr \wedge \omega^3),$$

$$\nabla_A \Phi = \dot{\phi} S_1 \otimes dr + 2a \phi (S_2 \otimes \omega^3 - S_3 \otimes \omega^2),$$

with the dot denoting differentiation with respect to $r$. The energy density, as a function of $r$, is then

$$e = \frac{(a^2 - 1)^2}{4 r^4} + \frac{\dot{\phi}^2}{2 r^2} + \phi^2 + \frac{2 a^2 \phi^2}{r^2}. \quad (3.1)$$

In this spherically symmetric setting, the monopole equations turn into the following system of ordinary differential equation (ODE):

$$\dot{\phi} = \frac{1}{2 r^2}(a^2 - 1), \quad \dot{a} = 2 a \phi,$$

Some particular solutions are given by the flat connection $(a, \phi) = (\pm 1, 0)$, and the Dirac monopole $(a, \phi) = (0, m - 1/2 r)$, where $m \in \mathbb{R}$. However, the regularity conditions so that the configuration $(A, \Phi)$ smoothly extends over the origin yield that $\phi(0) = 0$ and $a(0) = 1$. One can then show, see the Appendix in [16], that any such solution is given by

$$\phi_m = \frac{1}{2} \left( 1 - \frac{2m}{r \tanh(2mr)} \right), \quad a_m = \frac{2mr}{\sinh(2mr)}, \quad (3.2)$$

for some $m \in \mathbb{R}^+$, which is the mass of the resulting monopole. The resulting formula for the energy density in (3.1) is

$$e_m := \frac{1}{4} \cosh^4(2mr) + (32 m^4 r^4 - 2) \cosh^2(2mr) - 32 \sinh(2mr) \cosh(2mr) m^3 r^3 + 16 m^4 r^4 + 1 \sinh^4(2mr) r^4. \quad (3.3)$$

Recall that in this case we have $Z = \{0\}$. Given the formula above it is easy to see that in $\mathbb{R}^3 \setminus Z$ we have

$$m^{-1} e_m \leq \frac{1}{4m} \coth^4(2mr) + O(m^{-2}) \to 0, \quad \text{as } m \to \infty.$$

On the other hand, using this fact together with the dominated convergence theorem, we have

$$I := \lim_{m \to \infty} \int_{\mathbb{R}^3} m^{-1} e_m \text{vol}_{g_E}$$

$$= \lim_{m \to \infty} \lim_{s \to \infty} \int_{B_s(0)} m^{-1} e_m \text{vol}_{g_E} = 4\pi \lim_{m \to \infty} \lim_{s \to \infty} \int_0^s r^2 m^{-1} e_m(r) dr \quad (3.4)$$

$$= 4\pi \lim_{m \to \infty} \int_0^\infty r^2 m^{-1} e_m(r) dr$$

$$= 4\pi \lim_{m \to \infty} \left( \lim_{r \to \infty} f_m(r) - \lim_{r \to 0^+} f_m(r) \right), \quad (3.5)$$
where 
\[ f_m(r) = -\frac{1}{2r} - \frac{2m}{(e^{4mr} - 1)^2} \left[ (8m^2r^2 - 4mr - 1)e^{8mr} + (8m^2r^2 + 4mr + 2)e^{4mr} - 1 \right]. \]

Thus, inserting this into equation (3.5) shows that \( I = 4\pi \) and thus 
\[ m^{-1}e_m \text{ vol}_{E} \to 4\pi \delta_0, \text{ as } m \to \infty. \]

**Remark 3.1.** Notice that in this case we have explicitly concluded not only that \( Z = \{0\} \) but also that \( e_\infty = 0 \).

### 3.2. Sequences of Taubes’ multi-monopoles on \( \mathbb{R}^3 \) with prescribed \( Z \)

We start by recalling the following Theorem of Taubes (see [12]).

**Theorem 3.1** (Theorems 1.1 and 1.2 in [12]). Let \( k \in \mathbb{N} \). Then, there is \( d_0 > 0 \) and \( c > 0 \) such that for any \( y_1, \ldots, y_k \in \mathbb{R}^3 \) with \( d = \min_{j,l} \text{ dist}(y_i, y_j) > d_0 \), there is a charge \( k \), mass 1, monopole \((A, \Phi)\) in \( \mathbb{R}^3 \). Furthermore, for \( R = cd^{-1/2} \) we have that \( \Phi^{-1}(0) \subset \cup_{i=1}^{k} B_R(y_i) \) and \( \Phi_{|B_R(y_i)} \) has degree 1. In particular, \( \Phi \) does have zeros inside each of the ball’s \( B_R(y_i) \), for \( i = 1, \ldots, k \).

We shall now use this construction to give a number of different examples of sequences of monopoles as those we consider in this paper.

**Proposition 3.1.** Let \( 1 \leq l \leq k \) be integers, \( \{x_1, \ldots, x_l\} \subseteq \mathbb{R}^3 \) a subset of pairwise distinct points, and \( \{m_i\}_{i \in \mathbb{N}} \subseteq \mathbb{R}^+ \) an unbounded increasing sequence, that is, \( m_i \uparrow \infty \). Then, there is a sequence \( \{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \) of charge \( k \), mass \( m_i \) monopoles on \( \mathbb{R}^3 \) with zero set 
\[ Z = \{x_1, \ldots, x_l\}. \]

**Remark 3.2.** In this construction, as will be evident during the proof, we have \( k_{x_j} = 1 \) for all \( j = 1, \ldots, l \). Moreover, it follows from our results that in the case \( l = k \) we must have \( e_\infty = 0 \).

The case \( l < k \) is precisely the case where there are \( k - l \) monopoles ‘escaping through the end’, or ‘run of to infinity’. In the construction below, we shall see that the monopole \((A_i, \Phi_i)\) has a zero 
\[ z_j^i \in B_{cm_i^{-1/2}}(m_ix_j), \]
for \( j = l + 1, \ldots, k \). And the centers of these balls leave any compact set as \( i \to \infty \). Thus, the sequence of zeros \( z_j^i \to \infty \) has no convergent subsequence, and so does not contribute to \( Z \).

In the rest of this subsection, we prove this result by using Theorem 3.1 to construct the monopoles. Let \( \lambda > 0 \) and consider the scaling map \( s_\lambda(x) = \lambda^{-1}x \) for \( x \in \mathbb{R}^3 \). Recall that the Euclidean metric \( g_E \) is invariant by scaling, that is \( g_E = \lambda^2 s_\lambda^* g_E \) for any such positive \( \lambda \). Therefore, by Proposition 2.1, if \((A, \Phi)\) is a charge \( k \) mass 1 monopole, we have that 
\[ (A_\lambda, \Phi_\lambda) = (s_\lambda^* A, \lambda^{-1}s_\lambda^* \Phi) \tag{3.6} \]
is a charge \( k \), mass \( \lambda^{-1} \) monopole.

It is instructive to split the proof in two different cases, the case \( l = k \) and the case \( l < k \). We start with the first.

**Case 1 = k.** We now construct a sequence of charge \( k \), large mass monopoles on \( \mathbb{R}^3 \) with prescribed \( Z = \{x_1, \ldots, x_k\} \) being \( k \) distinct points in \( \mathbb{R}^3 \). After choosing such \( k \) points, we fix a sequence of masses \( m_i \to \infty \) and suppose, with no loss of generality, that \( m_1 \gg 1 \) so
as to \( m_1 \text{dist}(x_j, x_l) > d_0 \), for all \( j, l \in \{1, \ldots, k\} \). Then, we can use Taubes’ Theorem 3.1 to construct a sequence, labeled by \( i \), of charge \( k \), mass 1 monopoles using in the construction the points \( y_j^i = m_i x_j \) for \( j = 1, \ldots, k \). Rescaling these as in equation (3.6) with \( \lambda = m_i^{-1} \), we obtain a sequence of monopoles \((A_i, \Phi_i)\) with charge \( k \), mass \( m_i \) and

\[
\Phi_i^{-1}(0) \subset \bigcup_{j=1}^{k} B_{cm_i^{-1/2}}(x_j), \quad \text{deg}(\Phi_i|_{\partial B_{cm_i^{-1/2}}(x_j)}) = 1. \tag{3.7}
\]

Now, recall from the definition of the zero set that

\[ Z = \bigcap_{n \geq 1} \bigcup_{i \geq n} \Phi_i^{-1}(0). \]

Hence, as the sequence \( \{m_i\}_i \) is increasing, it follows that

\[ \bigcup_{i \geq n} \Phi_i^{-1}(0) \subset \bigcup_{j=1}^{k} B_{cm_n^{-1/2}}(x_j), \]

and thus

\[ Z \subset \bigcap_{n \geq 1} \bigcup_{j=1}^{k} B_{cm_n^{-1/2}}(x_j) = \bigcap_{n \geq 1} \bigcup_{j=1}^{k} B_{cm_n^{-1/2}}(x_j) = \bigcup_{j=1}^{k} \{x_j\}. \]

On the other hand, for every fixed \( j \in \{1, \ldots, k\} \), the degree of the map \( \Phi \) restricted to a normal sphere of radius \( cm_i^{-1/2} \) equals 1 and thus \( \Phi_i \) has a zero

\[ z_i \in B_{cm_i^{-1/2}}(x_j), \]

for each \( i \gg 1 \). Since \( m_i \uparrow \infty \), it follows that \( z_i \to x_j \) as \( i \to \infty \). Thus we get the reverse inclusion \( \{x_1, \ldots, x_k\} \subset Z \), proving that indeed

\[ Z = \{x_1, \ldots, x_k\}. \]

Case \( l < k \). We can modify the above construction in order to make \( l < k \) of the monopoles ‘escape to infinity’. We shall proceed as before and fix \( k \) distinct points \( \{x_1, \ldots, x_k\} \subset \mathbb{R}^3 \). Then, we consider the charge \( k \), mass 1 monopole obtained through Taubes’ Theorem 3.1 using the points \( y_j = m_i x_j \) for \( j = 1, \ldots, l \) and the points \( y_j = m_i^2 x_j \) for \( j = l + 1, \ldots, k \). Then, rescaling this monopole as before, that is using equation (3.6) with \( \lambda = m_i^{-1} \), we obtain a mass \( m_i \), charge \( k \), monopole on \( \mathbb{R}^3 \). This has the property that

\[
\Phi_i^{-1}(0) \subset \left( \bigcup_{j=1}^{l} B_{cm_i^{-1/2}}(x_j) \right) \cup \left( \bigcup_{j=l+1}^{k} B_{cm_i^{-1/2}}(m_i x_j) \right), \quad \text{deg}(\Phi_i|_{\partial B_{cm_i^{-1/2}}(x_j)}) = 1. \tag{3.8}
\]

Similarly to before, we now have

\[ \bigcup_{i \geq n} \Phi_i^{-1}(0) \subset \left( \bigcup_{j=1}^{l} B_{cm_n^{-1/2}}(x_j) \right) \cup \left( \bigcup_{i \geq n} \bigcup_{j=l+1}^{k} B_{cm_n^{-1/2}}(m_i x_j) \right), \]

and for \( n \) sufficiently large the second sets inside parenthesis above are disjoint for \( n \) sufficiently large. It then follows again from the same degree argument as before that

\[ Z = \{x_1, \ldots, x_l\}. \]

\(^1\)By slight modification of this, we can also let the points \( y_j \) for \( j > l \) go off to infinity at different rates.
3.3. An example with $k_x > 1$
In the examples above, we have already seen that it is possible to have $\mathcal{H}^0(Z) < k$ by letting the monopoles ‘escape through the end’. One other possibility would be to have points $x \in Z$ with $k_x > 1$, in this subsection we give the simplest of such examples.

Let $(A, \Phi)$ be a finite mass $SU(2)$-monopole in $(\mathbb{R}^3, g_E)$ with mass $m \neq 0$ and charge $k > 1$. Since $g_E$ is scale-invariant, taking any null-sequence $\lambda_i \downarrow 0$ we get a corresponding sequence

$$(A_{\lambda_i}, \Phi_{\lambda_i}) := (s_{\lambda_i}^\ast A, \lambda_i^{-1} s_{\lambda_i}^\ast \Phi)$$

of monopoles in $(\mathbb{R}^3, g_E)$ with masses $m_i := m\lambda_i^{-1} \to \infty$. Note that for such sequence $Z = \{0\}$ and $k_0 = k > 1$.

3.4. Sequences of monopoles with prescribed $Z$ on any AC 3-manifold with $b^2(X) = 0$
Let $(X, g)$ be an AC-oriented Riemannian 3-manifold with $b^2(X) = 0$, and let $k \in \mathbb{Z}_{>0}$. For a real number $m > 0$, denote by $\mathcal{M}_{k,m}$ the moduli space of mass $m$ and charge $k$ monopoles on $(X, g)$. In this setting, the main result of [17] yields

**Theorem 3.2 (17, Theorem 1).** There is $\mu \in \mathbb{R}$, so that for all $m \geq \mu$ and $X^k(m) \subset X^k$ defined by

$$X^k(m) = \left\{(x_1, \ldots, x_k) \in X^k \mid \text{dist}(x_i, x_j) > 4m^{-1/2}, \text{ for } i \neq j\right\},$$

while $\mathbb{T}^{k-1} = \{(e^{i\theta_1}, \ldots, e^{i\theta_k}) \in \mathbb{T}^k \mid e^{i(\theta_1 + \cdots + \theta_k)} = 1\}$, there is a local diffeomorphism onto its image

$$h_m : X^k(m) \times H^1(X, S^1) \times \mathbb{T}^{k-1} \to \mathcal{M}_{k,m}. \quad (3.9)$$

In order to use this theorem, we shall fix once and for all $\alpha \in H^1(X, S^1)$ and $\theta = (e^{i\theta_1}, \ldots, e^{i\theta_k}) \in \mathbb{T}^k$ satisfying $e^{i(\theta_1 + \cdots + \theta_k)} = 1$. Then, we chose any $k$ disjoint points $(x_1, \ldots, x_k) \in X^k$ and take an increasing sequence of positive real numbers $m_i \uparrow \infty$ with $m_1 > \max\{16 \min_{j,l} \text{dist}(x_j, x_l)^2, \mu\}$ and consider the monopoles

$$(A_i, \Phi_i) = h_{m_i}((x_1, \ldots, x_k), \alpha, \theta).$$

Then, using the results of [17] we have the following.

**Proposition 3.2.** The zero set

$$Z = \bigcap_{n \geq 1} \bigcup_{i \geq n} \Phi_i^{-1}(0)$$

of the family of monopoles $(A_i, \Phi_i)$ defined above is precisely the set of points $\{x_1, \ldots, x_k\}$.

**Proof.** The result follows from proving that the zeros of the monopole $(A_i, \Phi_i)$ are contained in balls of radius $O(m_i^{-1/2})$ around the points $\{x_1, \ldots, x_k\}$, that is for sufficiently large $i$ we have

$$\Phi_i^{-1}(0) \subset \bigcup_{j=1}^k B_{10m_i^{-1/2}}(x_j) \quad \text{and} \quad \Phi_i^{-1}(0) \cap B_{10m_i^{-1/2}}(x_j) \neq \emptyset, \quad \forall j \in \{1, \ldots, k\}. \quad (3.10)$$
Indeed, if we prove this assertion, then, as the sequence \( \{m_i\} \) is increasing, on the one hand, note that

\[
\bigcup_{i \geq n} \Phi_i^{-1}(0) \subseteq \bigcup_{j=1}^{k} B_{10m_i^{-1/2}}(x_j),
\]

and thus

\[
Z \subseteq \bigcap_{n \geq 1} \bigcup_{j=1}^{k} B_{10m_i^{-1/2}}(x_j) = \bigcup_{j=1}^{k} \bigcup_{n \geq 1} B_{10m_i^{-1/2}}(x_j) = \bigcup_{j=1}^{k} \{x_j\}.
\]

On the other hand, for every fixed \( j_0 \in \{1, \ldots, k\} \), we can find \( z_i \in \Phi_i^{-1}(0) \cap B_{10m_i^{-1/2}}(x_{j_0}) \) for each \( i \gg 1 \). Since \( m_i \uparrow \infty \), it follows that \( z_i \to x_{j_0} \) as \( i \to \infty \). Thus we get the reverse inclusion \( \{x_1, \ldots, x_k\} \subseteq Z \), and equality follows as claimed.

We are thus left with proving the assertion (3.10), which is done in the Appendix. \( \square \)

4. AC and mass-dependent version of Taubes’ small Higgs field estimates

Let \((X^3, g)\) be an AC-oriented Riemannian 3-manifold with one end, let \(E\) be an SU(2)-bundle over \(X\), and let \((A, \Phi)\) be a finite mass SU(2)-monopole on \((X, g)\) with charge \(k \neq 0\) and mass \(m \neq 0\). In [20], Taubes poses and addresses the following question, in the case where \((X^3, g)\) is the Euclidean space \((\mathbb{R}^3, g_E)\):

**Question.** What is the largest radius of a ball in \(X\) that contains only points where \(|\Phi| \ll m|\)?

Below, we shall prove the analogue of Taubes’ result [20, Theorem 1.2] in the case of more general YMH configurations on an asymptotically conical \((X^3, g)\) with one end.

**Theorem 4.1.** Let \((X^3, g)\) be an oriented AC Riemannian 3-manifold with one end, \(E\) be a \(G\)-bundle over \(X\), \(\delta \in (0, 1)\), and \(\Lambda \in (0, \infty)\). Then there is a constant \(m_\ast > 0\), depending only on \(g, \Lambda, \) and \(\delta\), with the following significance. If \((A, \Phi)\) is a finite mass YMH configuration on \(E\) with mass \(m > m_\ast\) and \(m^{-1}\|d_A \Phi\|_{L^2(X)}^2 \leq \Lambda\), then

\[
r_\delta(x) := \sup \left\{ r \in [0, \infty) : \sup_{B_r(x)} |\Phi| < m\delta \right\}
\]

satisfies the upper bound\(^1\)

\[
r_\delta(x) \leq \frac{4\Lambda c_1}{m(1 - \delta)c_2}, \quad (4.1)
\]

where \(c_1, c_2 > 0\) are the constants of Proposition 2.3. In particular, if \(G = \text{SU}(2)\) and \((A, \Phi)\) is a charge \(k\) monopole on \((X^3, g)\), then

\[
r_\delta(x) \leq \frac{16\pi k c_1}{m(1 - \delta)c_2}. \quad (4.2)
\]

\(^1\)It is clear from our proof that such upper bound is not sharp, but for our purposes it suffices to know that \(r_\delta(x) \lesssim \frac{\Lambda}{m(1 - \delta)}\).
Proof. Suppose, by contradiction, that for all \( m_* > 0 \) depending only on the indicated data, there exists a finite mass YMH configuration \((A, \Phi)\) with mass \( m > m_* \) such that

\[
s := \frac{4\Lambda c_1}{m(1-\delta)c_2} < r_\delta(x).
\]

Let \( \phi_x \) be the harmonic function on \( X \setminus \{x\} \) obtained from applying Theorem 2.3, and let \( \phi_0 := c_2^{-1}2\Lambda \phi_x \). Then, for small enough \( r = \text{dist}(x, \cdot) \), equation (2.3) yields

\[
\phi_0|_{V(x)} \geq \frac{m(1-\delta)s}{2r} + c_\Lambda,
\]

(4.3)

for some constant \( c_\Lambda \in \mathbb{R} \), depending only on \( g \) and \( \Lambda \).

Now, as \( s \) is inversely proportional to \( m \), there is \( m_* > 0 \), depending only on \( g, \Lambda, \delta \), so that the expansion (4.3) is valid for \( r = s \). At this point, it is convenient to further define the harmonic function on \( X \setminus \{x\} \) given by \( \phi := \phi_0 + m \). Then, by possibly increasing \( m_* \) so that \( m_* > -2c_\Lambda(1-\delta)^{-1} \), we have

\[
\phi|_{\partial B_r(x)} \geq -\frac{m(1-\delta)}{2} + c_\Lambda + m > m\delta \geq |\Phi|_{\partial B_r(x)},
\]

where in the last inequality we used the assumption that our \( s < r_\delta \). Then, the previous inequality, and the fact that both the harmonic function \( \phi \) and the subharmonic function \( |\Phi| \) converge to \( m \) along the end show that

\[
|\Phi| < \phi \quad \text{in} \quad X \setminus B_r(x).
\]

On the other hand, recall from equations (2.1) and (2.4) that

\[
|\Phi| \geq m - \Lambda \text{vol}(N)^{-1} \rho^{-1} + o(\rho^{-1}), \quad \text{and}
\]

\[
\phi = m - 2\Lambda \text{vol}(N)^{-1} \rho^{-1} + o(\rho^{-1}),
\]

as \( \rho \to \infty \). Putting these together, we conclude that \( \Lambda \text{vol}(N)^{-1} \geq 2\Lambda \text{vol}(N)^{-1} \), hence a contradiction. This completes the proof of (4.1). The case of monopoles (4.2) then follows by using the energy formula (1.5). \( \square \)

Remark 4.1. Under its hypothesis, Theorem 4.1 implies that for any finite mass YMH configuration with mass \( m > m_* \) and \( m - 1 \|dA\Phi\|_{L^2(X)}^2 \leq \Lambda \), whenever \( r > c\Lambda m^{-1}(1 - \delta)^{-1} \) for some suitable constant \( c > 0 \) (depending only on \( g \)) then

\[
\sup_{\partial B_r(x)} |\Phi| \geq \sup_{B_r(x)} |\Phi| \geq m\delta,
\]

where in the first inequality we applied the maximum principle.

5. \( \varepsilon \)-regularity estimate

Notations and scaling. In what follows, it will be convenient to use the following notations. If \((A, \Phi)\) is a configuration on \((X^3, g)\), then we write

\[
e(A, \Phi) := \frac{1}{2}(|F_A|^2 + |d_A\Phi|^2)
\]

for its YMH energy density, so that

\[
\delta_U(A, \Phi) = \int_U e(A, \Phi) \text{vol}.
\]
If we scale \( g \) by \( \lambda^2 \), then for the new metric \( g_\lambda := \lambda^2 g \) we write
- \( B^\lambda_\gamma(x) := \) open \( g_\lambda \)-ball of center \( x \) and radius \( r \);
- \( e_\lambda(A, \lambda^{-1}\Phi) := g_\lambda - \text{YMH energy density of } (A, \lambda^{-1}\Phi) \);
- \( \text{vol}_\lambda := g_\lambda \)-volume form;
- \( \delta_\lambda^U := g_\lambda - \text{YM functional over } U \).

With these notations, note that the following identities hold:
- \( B^\lambda_M(x) = B_r(x) \);
- \( e_\lambda(A, \lambda^{-1}\Phi) = \lambda^{-4} e(A, \Phi) \);
- \( \text{vol}_\lambda = \lambda^3 \text{vol} \);
- \( \delta^U_\lambda(A, \lambda^{-1}\Phi) = \lambda^{-1} \delta^U(A, \Phi) \).

We start with an important rough estimate on the Laplacian of the YMH energy density of a YMH configuration.

**Lemma 5.1** (Bochner-type estimate). Let \((X^n, g)\) be an oriented Riemannian \( n \)-manifold, let \( E \) be a \( G \)-bundle over \( X \), and let \((A, \Phi)\) be a configuration on \( E \) satisfying the second-order equations (1.2) on a ball \( B_r(x) \subseteq (X, g) \), with \( 0 < r \leq \rho_0 \). Then
\[
\Delta e(A, \Phi) \lesssim (|\Phi|^2 + |R|) e(A, \Phi) + e(A, \Phi)^{3/2} \quad \text{on } B_r(x),
\]
where \( R \) is the Riemannian curvature tensor of \( g \).

**Proof.** By standard Bochner–Weitzenböck formulas (cf. [2, Theorems 3.2 and 3.10]), and using the second-order equations (1.2), we can compute on \( B_r(x) \):
\[
\Delta e = \langle \Delta_A \nabla_A F_A, F_A \rangle - |\nabla_A F_A|^2 + \langle \nabla_A \nabla_A (d_A \Phi), d_A \Phi \rangle - |\nabla_A (d_A \Phi)|^2
\leq \langle \Delta_A F_A, F_A \rangle + \langle R, F_A, F_A \rangle + \{F_A, F_A, F_A\}_G
\leq |F_A|^2 |\Phi|^2 + |d_A \Phi|^2 |F_A| + |R||F_A|^2 + |F_A|^3
\leq |F_A| e + |R| e + |\Phi|^2 e + |F_A| e
\leq |\Phi|^2 e + |R| e + e^{3/2}.
\]

**Theorem 5.1** (\( \varepsilon \)-regularity estimate). Let \((X^3, g)\) be an AC-oriented Riemannian 3-manifold with one end, and \( E \) a \( G \)-bundle over \( X \). Then there are scaling invariant constants \( \varepsilon_0 > 0 \) and \( C_0 > 0 \) with the following significance. Let \((A, \Phi) \in \mathcal{A}(E) \times \Gamma(\mathfrak{g}_E)\) be a YMH configuration on \((X^3, g)\) such that \( |\Phi| < m \) on \( X \), for some positive constant \( m > 0 \). For any \( R > 0 \), if \( x \in X \) and \( 0 < r \leq \min\{Rm^{-1}, \rho_0\} \) are such that
\[
\varepsilon := m^{-1} \delta^U_{B_r(x)}(A, \Phi) < \varepsilon_0,
\]
then
\[
\sup_{B^\xi_r(x)} m^{-1} e(A, \Phi) \leq C_R r^{-3} \varepsilon,
\]
where \( C_R := C_0 \max\{1, R^3\} \).

**Proof.** First of all, we note that, by scaling, we may assume \( m = 1 \). Indeed, assume the result is true for \( m = 1 \). Then given a YMH configuration \((A, \Phi)\) with respect to a metric \( g \) and such
that \(|\Phi| < m\), it follows from Proposition 2.1 that \((A, m^{-1}\Phi)\) is a YMH configuration with respect to the scaled metric \(g_m := m^2 g\). Now, by hypothesis, for \(r \in (0, \min \{Rm^{-1}, \rho_0(g)\}]\),
\[
\epsilon_{\mathcal{B}_r}(m^{-1}\Phi) = m^{-1}\epsilon_{\mathcal{B}_r}(A, \Phi) < \varepsilon_0.
\]
Noting that \(\rho_0(g_m) = m\rho_0(g)\), the result for \(m = 1\) implies that
\[
\sup_{\mathcal{B}_{r/m}(x)} e_m(A, m^{-1}\Phi) \leq C_B rm^{-3} \epsilon_{\mathcal{B}_r}(x)(A, m^{-1}\Phi).
\]
Thus, rescaling back to \(g\) we get precisely (5.2). This proves our claim.

Given the above observation, in order to prove the theorem we are left to prove that, for \(m = 1\), if \(\varepsilon := \epsilon_{\mathcal{B}_r}(A, \Phi)\) is sufficiently small, depending only on \(g\) and \(G\), then
\[
\sup_{\mathcal{B}_{r/m}(x)} e(A, \Phi) \lesssim \max\{1, R^3\} r^{-3}\varepsilon.
\]
(5.3)
The proof we give here is based on the so-called ‘Heinz trick’ and follows [22, Appendix A]. Consider the function \(\theta : \mathcal{B}_{r/2}(x) \to [0, \infty)\) given by
\[
\theta(y) := \left(\frac{r}{2} - d(x, y)\right)^3 e(A, \Phi)(y).
\]
By continuity, \(\theta\) attains a maximum. Since \(\theta\) is non-negative and vanishes on the boundary \(\partial\mathcal{B}_{r/2}(x)\), it achieves its maximum
\[
M := \max_{\mathcal{B}_{r/2}(x)} \theta
\]
in the interior of \(\mathcal{B}_{r/2}(x)\). We will derive a bound for \(M\) of the form \(M \lesssim \max\{1, R^3\} \varepsilon\), from which the assertion of the theorem follows. Let \(y_0 \in \mathcal{B}_{r/2}(x)\) be a point with \(\theta(y_0) = M\), set
\[
e_0 := e(A, \Phi)(y_0)
\]
and
\[
s_0 := \frac{1}{2} \left(\frac{r}{2} - d(x, y_0)\right).
\]
Note that
\[
y \in \mathcal{B}_{s_0}(y_0) \Rightarrow \left(\frac{r}{2} - d(x, y)\right) \geq s_0.
\]
Therefore,
\[
y \in \mathcal{B}_{s_0}(y_0) \Rightarrow e(A, \Phi)(y) \leq s_0^{-3}\theta(y) \leq s_0^{-3}\theta(y_0) \lesssim e_0.
\]
In particular, it follows from Lemma 5.1 and the \(m = 1\) assumption that
\[
\Delta e(A, \Phi) \lesssim e^{3/2}(A, \Phi) + e(A, \Phi) \lesssim e_0^{3/2} + e_0 \quad \text{on} \quad \mathcal{B}_{s_0}(y_0).
\]
(5.4)
Now, since \(e(A, \Phi)\) is a smooth function and \(r \leq \rho_0\), it follows, for example, from [10, Theorem 9.20, p. 244] or [11, Case 2 in the Proof of Theorem B.1] that
\[
e_0 \lesssim s^{-3} \int_{\mathcal{B}_{s_0}(y_0)} e(A, \Phi)\text{vol} + s^2(e_0^{3/2} + e_0), \quad \forall s \leq s_0,
\]
which we rewrite as
\[
s^3 e_0 \lesssim \varepsilon + s^5 (e_0^{3/2} + e_0), \quad \forall s \leq s_0.
\]
(5.5)
We now have two cases.

Case (i). \(e_0 \leq 1\): in this case \(e_0^{3/2} \leq e_0\); hence, for each \(s \leq s_0\), it follows from (5.5) that
\[
s^3 e_0 \leq c\varepsilon + cs^5 e_0,
\]
where \(c\) is some positive constant.
that is
\[
s^3 e_0 \leq \frac{c \varepsilon}{1 - cs^2}.
\] (5.6)

If \(cs_0^3 \leq 1/2\) then we obtain
\[
M = \theta(y_0) \lesssim s_0^3 e_0 \lesssim \varepsilon;
\]
otherwise, setting \(s := (2c)^{-1/2} \leq s_0\) and plugging into (5.6) yields
\[
e_0 \lesssim \varepsilon.
\]
Since \(s_0 \leq r\) and by hypothesis \(r \leq R\), we conclude that \(M \lesssim s_0^3 e_0 \leq R^3 \varepsilon\).

Case (ii). \(e_0 > 1\): In this case \(e_0^{5/3} \geq e_0^{3/2} \geq e_0\), so that from (5.5) we derive
\[
s^3 e_0 \leq \varepsilon c + cs^5 e_0^{5/3} = \varepsilon c + s^3 e_0 c (s^2 e_0^{2/3}), \quad \forall s \leq s_0.
\] (5.7)
Thus, setting \(t = t(s) := s e_0^{1/3}\), the inequality (5.7) can be expressed as
\[
t^3 (1 - ct^2) \leq \varepsilon c.
\]
Now we can choose \(e_0 > 0\) sufficiently small, where the smallness depends only on \(c\), which in turn depends only on \(g\) and \(G\), so that for \(\varepsilon \leq \varepsilon_0\) the corresponding equation \(t^3 (1 - ct^2) = \varepsilon c\) has three small (real) roots \(t_1, t_2, t_3\), which are approximately \(\pm (c \varepsilon)^{1/3}\), and two large (complex) roots. Since \(t(0) = 0\) and \(t\) is continuous, for each \(s \in [0, s_0]\), \(t(s)\) must be less than the smallest positive (real) root. Therefore, \(t(s) \lesssim \varepsilon^{1/3}\) for all \(s \in [0, s_0]\); in particular, \(M \lesssim \varepsilon\). This finishes the proof. \(\square\)

6. An interior lower bound on the Higgs field

The next result is a consequence of the previous \(\varepsilon\)-regularity estimate and will prove to be useful in analyzing large mass YMH configurations.

**Theorem 6.1.** Let \((X^3, g)\) be an AC-oriented Riemannian 3-manifold with one end, let \(E\) be a \(G\)-bundle over \(X\), and let \((A, \Phi)\) be a configuration on \(E\) which is of finite mass \(m \neq 0\) and YMH on \((X^3, g)\). Given \(\delta \in (0, 1)\) and \(R > 0\), set
\[
\varepsilon_{\delta, R} := \min \{C_R^{-1} R \delta^2, \varepsilon_0\}.
\]
Let \(x \in X\). If \(r := Rm^{-1} \leq \rho_0\) and \(\sup_{\partial B_\varepsilon(x)} |\Phi| \geq m \delta\), then
\[
m^{-1} \delta_{B_\varepsilon(x)} (A, \Phi) \leq \varepsilon_{\delta, R} \quad \Rightarrow \quad |\Phi| > \frac{m \delta}{2} \quad \text{on} \quad B_\varepsilon(x).
\]
Here \(C_R\) and \(\varepsilon_0\) are the constants given by Theorem 5.1.

**Proof.** Fix \(q \in \partial B_\varepsilon(x)\) such that the restriction of \(|\Phi|\) to \(\partial B_\varepsilon(x)\) attains its maximum at \(q\). For any \(p \in B_\varepsilon(x)\), we can choose a (smooth by parts) path \(\gamma_p\) in \(B_\varepsilon(x)\) with length \(L(\gamma_p) \leq \frac{r}{2}\) joining \(p\) to \(q\). Thus, using the fundamental theorem of calculus and Kato’s inequality, we get
\[
|\Phi|(q) - |\Phi|(p) \leq \left| \int_{\gamma_p} d|\Phi| \right| \leq \int_{\gamma_p} |dA\Phi| \leq \frac{r}{2} \sup_{B_\varepsilon(x)} |dA\Phi|.
\] (6.1)
On the other hand, by the \( \varepsilon_0 \)-regularity estimate (Theorem 5.1), the hypothesis 
\[
m^{-1} \delta_{B_r(x)}(A, \Phi) < \varepsilon_{\delta,R} \leq \varepsilon_0
\]
implies that
\[
\sup_{B_r(x)} |d_A \Phi| \leq \sup_{B_r(x)} e(A, \Phi)^{\frac{1}{2}} < C \frac{1}{R} r^{-\frac{1}{2}} m^{\frac{1}{2}} e_{\delta,R}^{\frac{1}{2}}.
\]
Putting (6.1) and (6.2) together and using the definitions of \( r \) and \( \varepsilon_{\delta,R} \) along with the lower bound on \( |\Phi|(q) = \sup_{\partial B_r(x)} |\Phi| \) gives the statement. \( \square \)

Combining the above result with Theorem 4.1, we get the following.

**Corollary 6.1.** Let \((X, g)\) be an oriented AC Riemannian 3-manifold with one end, \( E \) be a \( G \)-bundle over \( X \) and \( \Lambda \in (0, \infty) \). Then there are constants \( R_0 > 0 \) and \( \varepsilon_0 > 0 \) such that the following holds. Let \((A, \Phi)\) be a finite mass YMH configuration on \( E \) with mass \( m > m_s \) and \( m^{-1} \|d_A \Phi\|_{L^2IX}^2 \leq \Lambda \). If \( r := R_0 m^{-1} \leq \rho_0 \) then
\[
m^{-1} \delta_{B_r(x)}(A, \Phi) < \varepsilon_0 \implies |\Phi| > \frac{m}{4} \text{ on } B_r(x).
\]

**Proof.** Let
\[
R_\Lambda := c \Lambda \quad \text{and} \quad \varepsilon_\Lambda := \varepsilon_{1/2,R_\Lambda} = \min\{4^{-1} C^{-1} R_\Lambda, \varepsilon_0\}.
\]
Here we choose \( c > 0 \) depending only on \( g \) big enough (for example, \( c := 64 c_1 c_2^{-1} \)) so that by letting \( \delta := 1/2 \) it follows from Theorem 4.1 that
\[
\sup_{\partial B_r(x)} |\Phi| \geq \frac{m}{2}.
\]
Therefore, by hypothesis we can apply Theorem 6.1 with \( R = R_\Lambda \) to get the desired result. \( \square \)

7. **The blow-up set and the zero set**

From now on we will be dealing with a sequence \( \{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{A}(E) \times \Gamma(g_E) \) of finite mass YMH configurations on \((X, g)\) satisfying the uniform bound
\[
m_i^{-1} \mathcal{E}_X(A_i, \Phi_i) \leq C,
\]
for some constant \( C > 0 \), and whose masses \( m_i \) satisfy \( \limsup m_i = \infty \). In fact, for convenience, we may assume (after passing to a subsequence) that \( m_i \uparrow \infty \). We note that in case the \( (A_i, \Phi_i) \) are SU(2)-monopoles of fixed charge \( k \neq 0 \), the energy formula (1.5) guarantees an a priori uniform bound of the form (7.1) with equality for \( C = 4\pi k \).

In order to study the behavior of such sequence of infinitely large mass YMH configurations, it is convenient to consider the corresponding sequence of Radon measures
\[
\mu_i := m_i^{-1} e(A_i, \Phi_i) \mathcal{H}^3.
\]
By (7.1) this sequence is of bounded mass. Thus, after passing to a subsequence which we do not relabel, it converges weakly to a Radon measure \( \mu \). By Fatou’s lemma and Riesz’ representation theorem, we can write
\[
\mu = e_\infty \mathcal{H}^3 + \nu,
\]
where \( e_\infty : X \to [0, \infty] \) is the \( L^1 \)-function
\[
e_\infty := \liminf_{i \to \infty} m_i^{-1} e(A_i, \Phi_i)
\]
and \( \nu \) is some nonnegative Radon measure, singular with respect to \( \mathcal{H}^3 \), called the defect measure.

Let \( \Theta \) be the 0-dimensional density function of \( \mu \), that is
\[
\Theta(x) := \lim_{r \downarrow 0} \mu(B_r(x)), \quad \forall x \in X.
\] (7.4)

Note that \( \Theta \) is well defined and bounded by \( C \). In this section, we will be considering the blow-up set \( S \) of \( \{(A_i, \Phi_i)\} \), which is defined to be
\[
S := \{ x \in X : \Theta(x) > 0 \}.
\]
The fact that \( \{\mu_i\} \) weakly converges to \( \mu \) a priori only implies that \( \mu(B_r(x)) \leq \liminf_{i \to \infty} \mu_i(B_r(x)) \) and \( \mu(B_r(x)) \geq \limsup_{i \to \infty} \mu_i(B_r(x)) \). Thus, for each \( x \in X \), it is convenient to set
\[
\mathcal{R}_x := \{ r \in (0, \rho_0) : \mu(\partial B_r(x)) > 0 \}.
\]
For all \( r \in (0, \rho_0) \setminus \mathcal{R}_x \) one has
\[
\mu(B_r(x)) = \lim_{i \to \infty} \mu_i(B_r(x)).
\]
Since \( \mu \) is locally finite, the set \( \mathcal{R}_x \) is at most countable. In particular, for each point \( x \in X \) we can find a null-sequence \( \{r_i\} \subseteq (0, \rho_0) \setminus \mathcal{R}_x \) so that
\[
\Theta(x) = \lim_{i \to \infty} \mu_i(B_{r_i}(x)).
\]
From these facts, the following is immediate.

**Lemma 7.1.** The blow-up set \( S \) can be written as
\[
S = \bigcup_{j \in \mathbb{N}} S_j,
\]
where
\[
S_j := \bigcap_{0 < r \leq \rho_0} \left\{ x \in X : \liminf_{i \to \infty} m_i^{-1} \phi_{B_r(x)}(A_i, \Phi_i) \geq j^{-1} \right\}.
\]

Our first result relates the blow-up set \( S \) with the accumulation points of the Higgs fields zeros, called the zero set \( Z \), and defined by
\[
Z := \bigcap_{n \geq 1} \bigcup_{i \geq n} \Phi_i^{-1}(0).
\]
In the following statement, we shall use \( \mathcal{H}^0 \) to denote the counting measure.

**Theorem 7.1.** Let \((X^3, g)\) be an AC-oriented Riemannian 3-manifold with one end, \( E \) be an \( G \)-bundle over \( X \), and \( \{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{A}(E) \times \Gamma(g_E) \) be a sequence of finite mass YMH configurations on \((X^3, g)\) satisfying the uniform bound (7.1) and whose masses \( m_i \) satisfy \( \limsup_{i \to \infty} m_i = \infty \). Then, after passing to a subsequence for which \( m_i \uparrow \infty \) and \( \mu_i \to \mu \), where \( \mu_i \) are the Radon measures given by (7.2), the following holds:

(i) \( \mathcal{H}^0(S_j) \leq jC \), for all \( j \in \mathbb{N} \); in particular, each \( S_j \) is finite and \( S \) is countable.

(ii) The blow-up set contains the zero set:
\[
Z \subseteq S.
\]
Proof. (i) Given $0 < r \leq \rho_0$, we can find a countable open covering \( \{B_{5r_i}(x_i)\} \) of $S_j$ with $x_i \in S_j$, $10r_i < r$ and $B_{r_i}(x_i)$ pairwise disjoint. Then

$$
\sum_i (5r_i)^0 \leq j \sum_i \liminf_{i \to \infty} m_i^{-1} \delta_{B_{r_i}(x_i)}(A_i, \Phi_i) \quad (x_i \in S_j)
$$

$$
\leq j \liminf_{i \to \infty} m_i^{-1} \delta_{B_{r_i}(x_i)}(A_i, \Phi_i) \quad \text{(by Fatou’s lemma)}
$$

$$
\leq j \liminf_{i \to \infty} m_i^{-1} \delta_{X}(A_i, \Phi_i) \quad \text{(the $B_{r_i}(x_i)$’s are disjoint)}
$$

$$
= j \liminf_{i \to \infty} m_i^{-1} \delta_{B_{r_0}(x)}(A_i, \Phi_i) < \varepsilon_{\Lambda}.
$$

Since this bound is uniform in $r \in (0, \rho_0]$, it follows that $\mathcal{H}^0(S_j) \leq jC$.

(ii) We shall apply Corollary 6.1 with $\Lambda := 2C$. Let $x_0 \in X \setminus S$. Then $x_0 \in X \setminus S_j$ for all $j \in \mathbb{N}$. In particular, there is $i_0 \in \mathbb{N}$ such that

$$
\liminf_{i \to \infty} m_i^{-1} \delta_{B_{r_0}(x)}(A_i, \Phi_i) < \varepsilon_{\Lambda}.
$$

We may assume $r_0 \notin \mathcal{R}_x$, otherwise we just work in a smaller ball for which we still have the above energy bound. In particular, it follows that there is $i_0 \in \mathbb{N}$ such that

$$
m_i^{-1} \delta_{B_{r_0}(x)}(A_i, \Phi_i) < \varepsilon_{\Lambda}, \quad \forall i \geq i_0.
$$

Since $m_i \uparrow \infty$, by increasing $i_0$ if necessary we may also assume that

$$
m_i > m_* \quad \text{and} \quad r_i := Rm_i^{-1} < \frac{r_0}{2}, \quad \forall i \geq i_0.
$$

Hence, given any $x \in B_{r_{i_0}/2}(x_0)$, it follows that

$$
m_i^{-1} \delta_{B_{r_i}(x)}(A_i, \Phi_i) < \varepsilon_{\Lambda}, \quad \forall i \geq i_0,
$$

so that applying Corollary 6.1 we get that

$$
|\Phi_i|(x) > \frac{m}{4}, \quad \forall i \geq i_0.
$$

Therefore,

$$
\sup_{B_{\frac{r_i}{2}}(x_0)} |\Phi_i| \geq \frac{m}{4} > 0, \quad \forall i \geq i_0.
$$

In particular, it follows that $x_0 \in X \setminus Z$. By the arbitrariness of $x_0 \in X \setminus S$, this shows that $Z \subset S$.

\end{proof}

8. Bubbling

Let $(A, \Phi)$ be a finite mass configuration on $E$ with mass $m \neq 0$, and pick a point $x \in X$. For each $r \in (0, \rho_0]$, consider the geodesic ball $B_r(x) \subset X$. Then, identify $\mathbb{R}^3 \cong T_xX$ and use the exponential map $s_m(\cdot) = \exp(m^{-1}\cdot)$ to define

$$
(A_m, \Phi_m) = (s_m^* A, m^{-1}s_m^* \Phi), \quad g_m = m^2 s_m^* g.
$$

(8.1)

It follows from Proposition 2.1 that if $(A, \Phi)$ is a YMH configuration on $B_r(x) \subset X$, then $(A_m, \Phi_m)$ is a YMH configuration in $B_{rm}(0) \subset \mathbb{R}^3$ with respect to the metric $g_m$\footnote{Here $B_{rm}(0) \subset \mathbb{R}^3$ is a radius $r$ ball with respect to both the metrics $g_m = m^2 s_m^* g$ and $m^2 \exp^* g$, by the Gauss lemma.}. Moreover,
we note that as \( m \to \infty \) the metric \( g_m \) geometrically converges to the Euclidean one, \( g_E \), on compact subsets of \( \mathbb{R}^3 \). The main result of this section is as follows.

**Theorem 8.1.** Let \( (X^3, g) \) be an AC-oriented Riemannian 3-manifold with one end, let \( E \) be a \( G \)-bundle over \( X \), and let \( \{ (A_i, \Phi_i) \}_{i \in \mathbb{N}} \subseteq \mathcal{A}(E) \times \Gamma(g_E) \) be a sequence of finite mass YMH configurations on \( (X^3, g) \) with masses \( m_i \) satisfying \( \limsup m_i = \infty \). Denote by \( S \) and \( Z \) the corresponding blow-up and zero sets. Then, for each \( x \in S \), after passing to a subsequence and changing gauge, the rescaled sequence \( (A_{m_i}, \Phi_{m_i}) \) converges uniformly with derivatives, in compact subsets of \( \mathbb{R}^3 \cong T_x X \), to a YMH configuration \( (A_x, \Phi_x) \) of mass \( m_x \leq 1 \) and strictly positive energy\(^1\) \( \varphi_{E^3}(A_x, \Phi_x) > 0 \). If, moreover, the sequence satisfy \((7.1)\), then \( m_x = 1 \).

Furthermore, in case \( G = SU(2) \) and the \( (A_i, \Phi_i) \) are monopoles with fixed charge \( k \neq 0 \), then the limit \( (A_x, \Phi_x) \) is a monopole of mass \( m_x = 1 \) and charge \( k_x > 0 \), \( k_x \leq k \), and we have \( S = Z \).

The rest of this section is devoted to proving Theorem 8.1. We shall start with two auxiliary results.

**Lemma 8.1.** Let \( r \in (0, \rho_0) \) and \( K \subset \mathbb{R}^3 \) be a compact set. Then, there are constants \( c > 0 \) and \( m_* > 1 \) such that: If \( (A, \Phi) \) is a mass \( m \geq m_* > 1 \) YMH configuration on \( X \), then there is a gauge such that on \( K \subset \mathbb{R}^3 \):

\[
|\Phi_m| < 1 \\
|\nabla_{A_m} \Phi_m|_{g_E} + |\nabla^2_{A_m} \Phi_m|_{g_E} \leq c \\
|A_m|_{g_E} + |\nabla_{A_m} A_m|_{g_E} + |\nabla^2_{A_m} A_m|_{g_E} \leq c.
\]

Furthermore, the following inequalities holds on \( K \):

\[
|F_m|_{g_E} + |\nabla_{A_m} \Phi_m|_{g_E} \leq c \\
|\nabla_{A_m} F_m|_{g_E} + |\nabla^2_{A_m} \Phi_m|_{g_E} \leq c.
\]

**Proof.** We start recalling Remark 2.1, which in terms of \( \Phi_m \) reads \( |\Phi_m| < 1 \).

Now, let \( (x^1, x^2, x^3) \) be geodesic normal coordinates on \( B_1(x) \subset X \) and \( (y^1, y^2, y^3) \) coordinates in \( B_{rm}(0) \subset \mathbb{R}^3 \) so that \( s_m(y^1, y^2, y^3) = (mx^1, mx^2, mx^3) \). In these coordinates we can write the metric \( g \) as

\[
g = \left( \delta_{ij} + \frac{1}{3} R_{iklj} x^k x^l + O(|x|^2) \right) dx^i \otimes dx^j.
\]

Thus, by defining the symmetric 2-tensor \( \gamma = \frac{1}{3} R_{iklj} y^k y^l dy^i \otimes dy^j \), with \( R_{iklj} \) is the Riemann curvature tensor of \( g \), we can write the metric \( g_m \) in \( B_{rm}(0) \subset \mathbb{R}^3 \), as

\[g_m = g_E + m^{-2} \gamma + O(m^{-3}). \quad (8.2)\]

It is at this point that we choose \( m_* \) to be large enough so that \( K \subset B_{rm^{1/2}}(0) \subset B_{rm} \). Hence, given that \( |\gamma| \leq |y|^2 \), in \( K \) we have \( |\nabla^j (g_m - g_E)|_{g_E} \leq O(m^{-1}) \), for all \( j \in \mathbb{N}_0 \). In particular, these metrics are quasi-isometric in \( K \).

Now, the YMH equations \((1.2)\), in Coulomb gauge, give an elliptic system for \( \Phi_m \) and the components of the connection \( A_m \). Furthermore, for \( m_* > 1 \) and \( m \geq m_* \) all the components of such system, written in the coordinates \( y \) on \( K \), are uniformly bounded in \( m \). Thus, elliptic regularity supplied by the \( m \) independent bound \( |\Phi_m| < 1 \) gives \( m \) independent bounds on the

\(^1\)Possibly \( \infty \).
first and second $y$-derivatives of $\Phi_m$ and $A_m$. These bounds can be further iterated to yield bounds on higher $y$-derivatives the connection $A$ and the Higgs field $\Phi$. Moreover, given that the metrics $g_m$ and $g_E$ on $K$ are quasi-isometric it is irrelevant with respect to which of these metric such bounds are written.

Next, we prove the following result which states that as $m \to \infty$ the $(A_m, \Phi_m)$ is not only a YMH configuration with respect to $g_m$ as it approaches one for $g_E$ in compact subsets of $\mathbb{R}^3$.

This is a consequence of the geometric convergence of $g_m$ to $g_E$ but a complete proof is given below.

**Lemma 8.2.** Let $r \in (0, \rho_0]$ and $K \subset \mathbb{R}^3$, then there is $m_\ast \gg 1$ and a constant $c > 0$ with the following significance. If $(A, \Phi)$ is a mass $m$ YMH configuration on $X$, then the inequality

$$|\Delta_m^E \Phi_m|_{g_E} + |d_m^E F_{A_m} - [d_m A_m, \Phi_m]|_{g_E} \leq cm^{-1},$$

holds in $K$. Moreover, in the particular case when $(A, \Phi)$ is actually a monopole, we further have

$$|\ast E F_{A_m} - d_m A_m \Phi_m|_{g_E} \leq cm^{-1}.$$

**Proof.** We shall prove only the case of a YMH configuration, for monopoles the result follows from similar, but somewhat easier computations. We continue to work with the coordinates $y$ introduced during the proof of Lemma 8.1. Start by using equation (8.2) to relate the action of the Hodge-$\ast$ operators of both $g_m$ and $g_E$. Let $\omega$ be a $k$-form and $\text{Ric}_{ij}$ the Ricci curvature of $g$, a computation gives

$$g_m \omega = \left(1 + m^{-2} \gamma(\omega, \omega) - m^{-1} \left(\frac{1}{6} \text{Ric}_{ij} y^i y^j\right) \ast E \omega + O(m^{-3}) \right) (8.3)$$

$$\ast E \omega - m^{-2} \gamma_m(\omega), \quad (8.4)$$

where in the last equality $\gamma_m$ denotes an algebraic operator. This has the property of being uniformly bounded with all derivatives, that is there are $m$-independent constants $c_j > 0$ so that for all $j \in \mathbb{N}_0$ and $m > m_\ast \gg 1$, we have $|\langle \nabla_{E, \gamma_m}(\omega) \rangle|_{g_E} \leq c_j (1 + |y|^2) |\omega|_{g_E}$, where $\nabla_E$ denotes the Levi-Civita connection of the Euclidean metric in $B_{r m}(0) \subset \mathbb{R}^3$. By possibly further increasing $m_\ast$ so that $K \subset B_{r m^{1/2}}(0)$, as in the proof of Lemma 8.1, we have that as a consequence of this we have $|\langle \nabla_{E, \gamma_m}(\omega) \rangle|_{g_E} \leq c_j m |\omega|_{g_E}$ on $K$. Then, we compute

$$\Delta_m^E \Phi_m = \ast E d_m A_m \ast E d_m \Phi_m = \ast E d_m (\ast m d_m A_m \Phi_m + m^{-2} \gamma_m(d_m A_m \Phi_m))$$

$$= \ast E \left(\ast m \Delta_m^m \Phi_m + m^{-2} (\nabla_E \gamma_m)(d_m A_m \Phi_m) + m^{-2} \gamma_m(\nabla_m d_m A_m \Phi_m)\right)$$

$$= \Delta_m^m \Phi_m + \ast m m^{-2} \gamma_m(\nabla_m d_m A_m \Phi_m) + \nabla_{E, \gamma_m}(d_m A_m \Phi_m)$$

$$+ m^{-2} \gamma_m(\ast m \Delta_m^m \Phi_m + m^{-2} (\nabla_E \gamma_m)(d_m A_m \Phi_m) + m^{-2} \gamma_m(\nabla_m d_m A_m \Phi_m)).$$

Recall that $(A_m, \Phi_m)$ is a YMH configuration for $g_m$, $\Delta_m^m \Phi_m = 0$, and so, on $K$ we have

$$|\Delta_m^E \Phi_m|_{g_E} \leq cm^{-1} (|d_m A_m \Phi_m|_{g_E} + |\nabla_{m}^2 \Phi_m|_{g_E}). \quad (8.5)$$

Similarly, we consider

$$d_m^E F_{A_m} = \ast E d_m A_m \ast E F_{A_m} = \ast E d_m (\ast m F_{A_m} + m^{-2} \gamma_m(F_{A_m}))$$

$$= \ast E \left(\ast m d_m A_m F_{A_m} + m^{-2} (\nabla_E \gamma_m)(F_{A_m}) + m^{-2} \gamma_m(\nabla_m F_{A_m})\right)$$

$$= d_m^m A_m + m^{-2} \ast m (\nabla_E \gamma_m)(F_{A_m}) + \gamma_m(\nabla_m F_{A_m})$$

$$+ m^{-2} \gamma_m(\ast m d_m A_m F_{A_m} + m^{-2} (\nabla_E \gamma_m)(F_{A_m}) + m^{-2} \gamma_m(\nabla_m F_{A_m})).$$
Again, the fact that \((A_m, \Phi_m)\) is a YMH configuration for \(g_m\), implies that \(d_{A_m}^* F_{A_m} = [d_{A_m} \Phi_m, \Phi_m]\), which together with the previous computation yields that on \(K\)
\[
|d_{A_m}^* F_{A_m} - [d_{A_m} \Phi_m, \Phi_m]|_{g_E} \leq cm^{-1} (|F_{A_m}|_{g_E} + |\nabla_{A_m} F_{A_m}|_{g_E}). \tag{8.6}
\]

Then, putting together equations (8.5) and (8.6) with the result of Lemma 8.1 we conclude that on \(K\)
\[
|\Delta_{A_m}^E \Phi_m|_{g_E} + |d_{A_m}^* F_{A_m} - [d_{A_m} \Phi_m, \Phi_m]|_{g_E} \leq cm^{-1}, \tag{8.7}
\]
for some \(c > 0\) independent of \(m\).

□

This lemmata has the following consequence.

**Corollary 8.1.** Let \(\{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{A}(E) \times \Gamma(g_E)\) be a sequence of finite mass YMH configurations on \((X, g)\) with masses \(m_i\) satisfying \(\limsup m_i = \infty\), and let \(x \in X\). Then, after passing to a subsequence and changing gauge, the rescaled sequence \(\{(A_{m_i}, \Phi_{m_i})\}_{i \in \mathbb{N}}\) defined in equation (8.1) converges uniformly with derivatives, in compact subsets of \(\mathbb{R}^3 \cong \mathcal{I}_x X\), to a YMH configuration \((A_x, \Phi_x)\) of mass \(m_x \leq 1\). Moreover, if the \((A_i, \Phi_i)\) satisfy the uniform bound (7.1) then \(m_x = 1\). In particular, if the \((A_i, \Phi_i)\) are monopoles, then so is \((A_x, \Phi_x)\) and \(m_x = 1\).

**Proof.** Lemma 8.1, together with a standard patching argument (see, for example, [6, Section 4.4.2]) and the Arzelà–Ascoli theorem, implies immediately that, after passing to a subsequence and changing gauge, the sequence \((A_{m_i}, \Phi_{m_i})\) converges uniformly with derivatives on compact subsets of \(\mathbb{R}^3\) to a configuration \((A_x, \Phi_x)\) with mass \(m_x \leq 1\). The fact that \((A_x, \Phi_x)\) is a YMH configuration/monopole is then immediate from Lemma 8.2. Finally, to see that we have \(m_x = 1\) in the case (7.1) is satisfied, fix \(r \in (0, \rho_0]\) and note that, since \(\limsup m_i = \infty\), given a sequence \(\{\delta_i\} \subset (0, 1)\) with \(\delta_i \uparrow 1\), then a diagonal argument shows that up to taking a subsequence we can assume that \(m_i > m_\star\) and \(r > 8C_1 c_2^{-1} m_i^{-1} (1 - \delta_i)^{-1}\), so that by Theorem 4.1 we get
\[
1 \geq \sup_{\partial B_{r_{m_i}}(0)} |\Phi_{m_i}| = m_i^{-1} \sup_{\partial B_r(x)} |\Phi_i| > \delta_i.
\]
Taking the limit \(i \to \infty\), we get the desired conclusion.

□

**Remark 8.1.** Since \(g_m\) converges to \(g_E\) in \(C^\infty_{\text{loc}}\) (cf. proof of Lemma 8.1), the first part of Corollary 8.1 could be directly deduced as a consequence of the \(\varepsilon\)-regularity (Theorem 5.1).

**Corollary 8.2.** Let \(\{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{A}(E) \times \Gamma(g_E)\) be a sequence of YMH configurations on \((X^3, g)\) satisfying the uniform bound (7.1) and whose masses satisfy \(\limsup m_i = \infty\). Then, after passing to a subsequence,
\[
S = \bigcap_{0 < r \leq \rho_0} \left\{ x \in X : \liminf_{i \to \infty} m_i^{-1} \mathscr{E}_{B_r}(A_i, \Phi_i) > 0 \right\}. \tag{8.8}
\]
Moreover, in case \(G = \text{SU}(2)\) and the \((A_i, \Phi_i)\) are monopoles of fixed charge \(k \neq 0\), then
\[
S = \bigcap_{0 < r \leq \rho_0} \left\{ x \in X : \liminf_{i \to \infty} m_i^{-1} \mathscr{E}_{B_r}(A_i, \Phi_i) \geq 4\pi \right\}. \tag{8.9}
\]
Proof. We start proving (8.8). One inclusion (⊆) is clear from Lemma 7.1. On the other hand, if \( x \in X \) is such that for all \( r \in (0, \rho_0] \) one has

\[
\varepsilon := \liminf_{i \to \infty} m_i^{-1} \delta_{B_r(x)}(A_i, \Phi_i) > 0,
\]

then by Corollary 8.1 one has that \( \varepsilon > 0 \) is the energy of a YMH configuration \( (A_x, \Phi_x) \) in \( \mathbb{R}^3 \); hence, for \( j = j(x) \in \mathbb{N} \) such that \( \delta_{\mathbb{R}^3}(A_x, \Phi_x) \geq j^{-1} \), we get that \( x \in S_j \subset S \) thereby proving the other inclusion.

In the case of (8.9), the trivial inclusion is (⊇) and it suffices to note that the above \( (A_x, \Phi_x) \) is a positive energy monopole in this case, so that \( \varepsilon = 4\pi k_x \) for some positive integer \( k_x \); hence, \( \varepsilon \geq 4\pi \), as we wanted.

We are now in position to prove the main result of this section.

Proof of Theorem 8.1. Let \( \{ (A_i, \Phi_i) \} \) be a sequence of YMH configurations with masses \( m_i \) satisfying \( \limsup m_i = \infty \). For \( x \in S \), we consider the rescaled sequence \( \{ (A_{m_i}, \Phi_{m_i}) \} \) obtained from the construction in equation (8.1). It follows from the definition of \( S \) that there is \( j = j(x) \in \mathbb{N} \) such that for all \( r \in (0, \rho_0] \) we have

\[
\liminf_{i \to \infty} \int_{B_{rm_i}(0)} e(A_{m_i}, \Phi_{m_i}) \text{vol} \geq j^{-1}.
\]

(8.10)

Moreover, it follows from Corollary 8.1 that, after passing to a subsequence and changing gauge, the \( (A_{m_i}, \Phi_{m_i}) \) converges uniformly with derivatives on compact subsets of \( \mathbb{R}^3 \) to a YMH configuration \( (A_x, \Phi_x) \) with mass \( m_x \leq 1 \) (and \( m_x = 1 \) in case (7.1) holds). Furthermore, equation (8.10) implies that

\[
\int_{\mathbb{R}^3} e(A_x, \Phi_x) \text{vol} \geq j^{-1} > 0.
\]

(8.11)

This condition gives that \( (A_x, \Phi_x) \) has strictly positive energy. Now, in case \( G = SU(2) \) and the \( (A_i, \Phi_i) \) are monopoles with fixed charge \( k \neq 0 \), then \( (A_x, \Phi_x) \) is a monopole of mass \( m_x = 1 \) and the energy formula (1.5) shows that

\[
4\pi k_x \geq \int_{\mathbb{R}^3} e(A_x, \Phi_x) \text{vol} = 4\pi k_x,
\]

(8.12)

where \( k_x \in \mathbb{Z}_{>0} \) is the charge of \( (A_x, \Phi_x) \); in particular, \( k_x \leq k \). Recalling that \( k_x > 0 \) is the degree of \( \Phi_x \) restricted to a large sphere, we conclude that \( \Phi_x \) must have zeros. Thus, by Lemma 8.2, for all sufficiently large \( i \) so does \( (A_{m_i}, \Phi_{m_i}) \) in \( B_{rm_i}(0) \subset \mathbb{R}^3 \) (since as \( i \to \infty \) the \( (A_{m_i}, \Phi_{m_i}) \) becomes as close as one wants of being a positive energy monopole with respect to \( g_E \)). Rescaling back, we have that \( (A_i, \Phi_i) \) must have zeros in \( B_r(x) \subset X \) for \( i \gg 1 \). However, given that the value of \( r \in (0, \rho_0] \) is arbitrary, as \( i \to \infty \) such zeros becomes as close as one wants to \( x \) yielding that \( x \in Z \). This together with Theorem 7.1 shows that \( Z = S \). This finishes the proof of the theorem.

Remark 8.2. Notice that, by the equality (8.8) of Corollary 8.2, if \( x \in X \setminus S \) then rather than equation (8.11) we have

\[
\int_{\mathbb{R}^3} e(A_x, \Phi_x) \text{vol} = 0,
\]

which means that \( (A_x, \Phi_x) \) is gauge equivalent to a flat connection and a constant function \( \mathbb{R}^3 \to \mathfrak{g} \). This means that \( S \) is indeed precisely the set where the bubbling occurs.
9. Convergence as measures

Our aim in this section is to prove the following:

**Theorem 9.1.** Let \((X^3, g)\) be an AC-oriented Riemannian 3-manifold with one end, let \(E\) be a SU(2)-bundle over \(X\), and let \(\{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{A}(E) \times \Gamma(E)\) be a sequence of finite mass monopoles on \((X^3, g)\) with fixed charge \(k \neq 0\) and masses \(m_i\) satisfying \(\limsup m_i = \infty\). Then, up to taking a subsequence, the corresponding blow-up set \(S\) is finite with at most \(k\) points and we have the following weak convergence of Radon measures:

\[
m_i^{-1} e(A_i, \Phi_i) \mathcal{H}^3 \rightharpoonup 4\pi \sum_{x \in S} k_x \delta_x,
\]

where \(\delta_x\) denotes the Dirac delta measure supported on \(\{x\}\).

In what follows we fix a sequence of monopoles \(A_i, \Phi_i\) as in the hypothesis of Theorem 9.1. Recall the sequence \(\{\mu_i\}\) of Radon measures (7.2) which, by the energy formula (1.5), is of bounded mass \(4\pi k\) and hence, after passing to a subsequence which we do not relabel, converges weakly to a Radon measure \(\mu\), where \(\mu\) decomposes as in (7.3).

The first observation we make is that on the one hand \(e_\infty(x) = 0\) for every \(x \in X \setminus S\) by equation (8.8) of Corollary 8.2. On the other hand, it follows from assertion (i) of Theorem 7.1 that \(\mathcal{H}^3(S) = 0\). Therefore, we conclude that \(\mu = \nu\).

Next, we provide some properties of the 0-dimensional density function \(\Theta\) of \(\mu\).

**Proposition 9.1.** The function \(\Theta : X \to [0, 4\pi k]\) satisfies the following properties:

(i) \(\Theta(x) = 4\pi k_x \geq 4\pi\) for all \(x \in S\), where \(k_x\) is the charge of the bubble \((A_x, \Phi_x)\) at \(x\) obtained as in Corollary 8.1.

(ii) \(\Theta\) is upper semicontinuous.

**Proof.** (i) Let \(x \in S\). Then, by Corollary 8.1, after passing to a subsequence and changing gauge, the rescaled sequence \(\{(A_{m_i}, \Phi_{m_i})\}_{i \in \mathbb{N}}\) defined in equation (8.1) converges uniformly with derivatives, in compact subsets of \(\mathbb{R}^3 \cong T_x X\), to a mass 1 monopole \((A_x, \Phi_x)\) of charge \(k_x\). Then, for any \(r \in (0, \rho_0]\),

\[
4\pi k_x = e^{\mathbb{R}^3}(A_x, \Phi_x) = \liminf_{i \to \infty} m_i^{-1} e^{\mathbb{R}^3}_{B_r(x)}(A_i, \Phi_i).
\]

Now recall that for \(r \in (0, \rho_0] \setminus \mathscr{R}_x\) the weak convergence of measures implies

\[
\mu(B_r(x)) = \liminf_{i \to \infty} m_i^{-1} e^{\mathbb{R}^3}_{B_r(x)}(A_i, \Phi_i) = 4\pi k_x.
\]

Moreover, we can find a null-sequence \((r_j) \subset (0, \rho_0] \setminus \mathscr{R}_x\) so that

\[
\Theta(x) = \lim_{j \to \infty} \mu(B_{r_j}(x)) = 4\pi k_x,
\]

as we wanted.

(ii) Suppose \(\{x_n\}\) is a sequence of points in \(X\) with \(x_n \to x \in X\) as \(n \to \infty\). Let \(\delta > 0\) and \(r > 0\). Thus, for \(n \gg 1\) we have

\[
\Theta(x_n) \leq \mu(B_r(x_n)) \leq \mu(B_{r+\delta}(x)),
\]

and so \(\limsup_{n \to \infty} \Theta(x_n) \leq \mu(B_r(x))\). The result then follows from taking the limit as \(r \downarrow 0\). \(\quad \Box\)
Finally, since $B_r$ is closed, we can find a finite open covering $\{B_{2r_j}(x_j)\}_{1 \leq j \leq l}$ of $S \cap K$ with $x_j \in S \cap K$, $2r_j < r$ and $B_{r_j}(x_j)$ pairwise disjoint. Hence,

$$\sum_{j=1}^{l} r_j^0 \leq \frac{1}{4\pi} \sum_{j=1}^{l} \liminf_{i \to \infty} m_i^{-1} e_{B_{r_j}(x_j)}(A_i, \Phi_i) \quad (x_j \in S)$$

$$\leq \frac{1}{4\pi} \liminf_{i \to \infty} m_i^{-1} \sum_{j=1}^{l} e_{B_{r_j}(x_j)}(A_i, \Phi_i)$$

$$\leq \frac{1}{4\pi} \liminf_{i \to \infty} m_i^{-1} e_X(A_i, \Phi_i) \quad \text{(the $B_{r_j}(x_j)$'s are disjoint)}$$

$$= k. \quad \text{(by the energy formula (1.5))}$$

Finally, since $S$ is finite, we can write $S = \{x_1, \ldots, x_l\}$ (for some $l \leq k$) and choose $r \in (0, \rho_0]$ such that the balls $B_{r_j}(x_j), j = 1, \ldots, l,$ are pairwise disjoint. Then

$$\mathcal{H}^0(S) = \sum_{j=1}^{l} r_j^0 = \sum_{j=1}^{l} \frac{1}{4\pi k_{x_j}} e_{B_{x_j}}(A_{x_j}, \Phi_{x_j})$$

$$\leq \frac{1}{4\pi \min_j k_{x_j}} \lim_{i \to \infty} m_i^{-1} \sum_{j=1}^{l} e_{B_{r_j}(x_j)}(A_i, \Phi_i)$$

$$\leq \frac{1}{4\pi \min_j k_{x_j}} \lim_{i \to \infty} m_i^{-1} e_X(A_i, \Phi_i) \quad \text{(the $B_{r_j}(x_j)$ are pairwise disjoint)}$$

$$= \frac{k}{\min_j k_{x_j}}. \quad \text{(by (1.5))} \quad \square$$

Finally, writing $S = \{x_1, \ldots, x_l\}$, for some $l \leq k$ (by Corollary 9.1), we have

**Proposition 9.2.** $\mu = 4\pi \sum_{j=1}^{l} k_{x_j} \delta_{x_j}.$

**Proof.** Firstly, we show that $\text{spt}(\mu) = S$. Indeed, on the one hand, by Proposition 9.1, note that $x \in S$ implies $\Theta(x) \geq 4\pi > 0$ and therefore $x \in \text{spt}(\mu)$. On the other hand, if $x \in X \setminus S$ then $\mu(B_r(x)) \leq \liminf \mu_i(B_r(x)) = 0$ for all $r \in (0, \rho_0]$. Therefore, $x \in X \setminus \text{spt}(\mu)$.

By the energy formula, we know that $\Theta(x) \leq 4\pi k$ for all $x \in X$. In particular, given $A \subseteq \text{spt}(\mu) = S$, it follows that

$$\mu(A) = \sum_{j=1}^{l} \lim_{r \uparrow 0} \mu(A \cap B_r(x_j)) \leq 4\pi k \mathcal{H}^0(S).$$
Hence, $\mu$ is absolutely continuous with respect to $\mathcal{H}^0$. Putting these facts together, the Radon–Nikodym theorem implies that we can write $\mu = \theta \mathcal{H}^0|_S$, for some $L^1$-function $\theta : S \to \mathbb{R}^+$. Since $S$ is finite, by the definition of the density function $\Theta$ it immediately follows that $\theta = \Theta|_S$, thereby proving the desired statement. \hfill \square

This finishes the proof of Theorem 9.1.

Appendix. The proof of assertion (3.10)

In this section, we shall prove assertion (3.10), which says that the zeros of the monopoles constructed via Theorem 3.2 are contained in balls of radius $10m^{-1/2}$ around the $k$-points in $X$ used in the construction. This requires a number of technical ingredients from [17], and so we decided to include this section as an appendix.

It follows from [17, Proposition 6] that the monopole $(A_i, \Phi_i)$ can be written as $(A_i, \Phi_i) = (A_i^0, \Phi_i^0) + (a_i, \phi_i)$, where $(A_i^0, \Phi_i^0)$ is an approximate monopole constructed in [17, Proposition 4]. Moreover, by its own construction, we have that the restriction

$$\Phi_i^0 : X \setminus \cup_{j=1}^k B_{10m_i^{-1/2}}(x_j) \to su(2),$$

satisfies $|\Phi_i^0| \geq m_i/2$ has no zeros and yields a splitting of the trivial rank-2 complex vector bundle $\mathbb{C}^2 \cong L \oplus L^{-1}$, where the complex line bundle $L$ is such that

$$\deg(L|_{\partial B_{10m_i^{-1/2}}(x_j)}) = 1,$$

for all $j = 1, \ldots, k$. In other words, the restricted map

$$\Phi_i^0 : \partial B_{10m_i^{-1/2}}(x_j) \to su(2) \setminus \{0\},$$

has degree 1.

A.a $(a_i, \phi_i) \in \Gamma((\Lambda^1 \oplus \Lambda^0) \otimes su(2))$ satisfies an elliptic equation, when in a certain Coulomb gauge (see [17, Lemma 13]). Moreover, from [17, Proposition 6], it satisfies

$$\| (a_i, \phi_i) \|_{H^{1,-1/2}} \lesssim m_i^{-7/4}, \quad (A.1)$$

where $H^{1,-1/2}$ is a certain Sobolev space.

The Sobolev space $H^{1,\nu+1}$, with $\nu = -3/2$ here, is one of several $H_{n,\nu+n}$ constructed using the approximate monopole $(A_i^0, \Phi_i^0)$. These are well adapted to solving the monopole equation and have the property that, in a certain gauge (see [17, Section 5]), one can iterate estimate (A.1) to obtain that

$$\| (a_i, \phi_i) \|_{H_{n,\nu+n}} \lesssim m_i^{-7/4},$$

for all $n \in \mathbb{N}$. Moreover, once restricted to certain subsets of $X$, these spaces satisfy a number of interesting properties. Some of these can be easily read from the definition in [17, Section 4.1], and we summarize them below.

B.a Restricted to compact set $K \subset X$, the norm $H^{1,1+\nu}(K)$ is equivalent to the usual $L^{2,1}(K)$. However, not in an $m_i$-independent way. In fact, there is a constant $c_n$, only depending on $g$ and $K$, not $m_i$, so that

$$\| (a_i, \phi_i) \|_{L^{2,1}(K)} \leq c_n(K) m_i^2 \| (a_i, \phi_i) \|_{H^{1,1+\nu}(K)} \lesssim m_i^{1/4}. \quad (A.2)$$

B.b For $\epsilon > 0$ we consider

$$C_\epsilon = X \setminus \cup_{j=1}^k B_\epsilon(x_j),$$

that is the complement of the balls of radius $\epsilon$ centered at the points $x_i$. Let $4d = \min_{j \neq i} \text{dist}(x_j, x_i)$, then the balls of radius $d$ around the points $x_i$ are disjoint. Using $d$, we shall consider $C_d$. Then, certain weight functions $W_n$, on which the spaces $H_{n,v+n}$ depend, can be arranged so that

$$\|(a_i, \phi_i)\|_{L^{2,n}(K)} \leq c_n(K) \|(a_i, \phi_i)\|_{H_{n,v+n}(K)} \lesssim m_i^{-7/4}.$$  \hfill (A.3)

for any $K \subset C_d$.

B.c On $C$ we can use the fact that $\Phi^0_i > 0$, as mentioned in A.a, to write any $\mathfrak{su}(2)$-valued tensor $f$ as $f = f^{\parallel} + f^{\perp}$, with the $f^{\parallel}$ denoting the component parallel to $\Phi^0_i$ and $f^{\perp}$ the orthogonal one. On $C$, and for large $m_i$, we can write

$$\|(a_i^\parallel, \phi_i^\parallel)\|_{L^{2,n}(C)} + \|(a_i^\perp, \phi_i^\perp)\|_{L^{2,n}(C)} = \|(a_i, \phi_i)\|_{H_{n,v+n}(C)} \lesssim m_i^{-7/4},$$  \hfill (A.4)

where the spaces $L^{2,n}_{v+n}$ are the more standard Lockhart–McOwen conically weighted spaces.

Combining item B.a above and the Sobolev embedding $L^{2,n}(K) \hookrightarrow C^{n-2}(K)$, one obtains that

$$\|(a_i, \phi_i)\|_{C^{n-2}(K)} \lesssim m_i^{1/4} \Rightarrow \|\Phi_i - \Phi^0_i\|_{C^0(K)} \lesssim m_i^{1/4},$$  \hfill (A.5)

for any compact set $K \subset X$. In particular $(a_i, \phi)$ is smooth. Moreover, as mentioned in A.a, $|\Phi^0_i| \geq m_i/2$ in $C_{10m_i^{-1/2}}$ and thus

$$|\Phi_i| \geq |\Phi^0_i| - \|\phi_i\|_{C^0} \geq \frac{m_i}{2} - cm_i^{1/4}, \text{ in any compact } K \subset C_{10m_i^{-1/2}}$$

and so, for $m_i \gg 1$, does not vanish in $\partial B_{10m_i^{-1/2}}(x_j)$ for any $j \in \{1, \ldots, k\}$. In particular, putting this together with the estimate (A.1) in A.b, which shows that $\phi_i$ is decaying, we conclude that $\Phi_i$ does not vanish in $C$, and so any of its zeros must be inside one of the balls of radius $10m_i^{-1/2}$ around the points $x_i$. Furthermore, this estimate shows that 1-parameter family of maps

$$\Phi_t^i = \Phi^0_i + t\phi_i : C_{10m_i^{-1/2}} \to \mathfrak{su}(2) \setminus \{0\},$$

gives an homotopy between $\Phi^0_i$ and $\Phi_i$. Combining this with the discussion in A.a, we conclude that $\Phi_i$

$$\deg(\Phi^0_i|_{\partial B_{10m_i^{-1/2}}(x_j)}) = \deg(\Phi^0_i|_{\partial B_{10m_i^{-1/2}}(x_j)}) = 1.$$

Thus, $\Phi_i$ does have zeros inside $B_{10m_i^{-1/2}}(x_j)$.

**Acknowledgements.** The authors would like to thank Henrique Bursztyn and Vinicius Ramos for the warm hospitality and for making possible the research visits of the first author to IMPA, where the major part of this work was carried out. We also would like to thank Ákos Nagy, Thomas Walpuski, and an anonymous referee for their mathematical comments on an earlier version. These allowed us to correct an error in Theorem 5.1 and to correct a misstatement in Proposition 2.3.

**References**

1. M. Atiyah and N. Hitchin, *The geometry and dynamics of magnetic monopoles*, M. B. Porter Lectures (Princeton University Press, Princeton, NJ, 1988).
2. J. P. Bourguignon and H. B. Lawson Jr, ‘Stability and isolation phenomena for Yang–Mills fields’, Comm. Math. Phys. 79 (1981) 189–230.
3. P. J. Braam, ‘Magnetic monopoles on three-manifolds’, J. Differential Geom. 30 (1989) 425–464.
4. S. Cherkis and A. Kapustin, ‘Nahm transform for periodic monopoles and $N = 2$ Super Yang–Mills theory’, Comm. Math. Phys. 218 (2001) 333–371.
5. S. Cherkis and A. Kapustin, ‘Hyperkähler metrics from periodic monopoles’, Phys. Rev. D 65 (2002) 084015.
6. S. K. Donaldson and P. B. Kronheimer, The geometry of four–manifolds (Clarendon Press, Oxford, 1990).
7. K. D. Ernst, The ends of the monopole moduli space over $\mathbb{R}^3 \# \text{homology sphere}$: Part I, The Floer memorial volume (eds H. Hofer, C. H. Taubes, A. Weinstein and E. Zehnder; Birkhäuser, Basel, 1995) 355–408.
8. A. Floer, Monopoles on asymptotically flat manifolds, The Floer memorial volume (eds H. Hofer, C. H. Taubes, A. Weinstein and E. Zehnder; Birkhäuser, Basel, 1995) 3–41.
9. L. Foscolo, ‘A gluing construction for periodic monopoles’, Int. Math. Res. Not. 2017 (2016) 7504–7550.
10. D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics Series 224 (Springer, Berlin, 2001).
11. S. Hohloch, G. Noetzel and D. A. Salamon, ‘Hypercontact structures and Floer homology’, Geom. Topol. 13 (2009) 2543–2617.
12. A. Jaffe and C. H. Taubes, Vortices and monopoles: structure of static gauge theories, Progress in Mathematics 2 (Boston-Basel-Stuttgart, Birkhäuser, Boston, 1980).
13. C. Kottke, ‘Dimension of monopoles on asymptotically conic 3-manifolds’, Bull. Lond. Math. Soc. 47 (2015) 818–834.
14. C. Kottke and M. Singer, ‘Partial compactification of monopoles and metric asymptotics’, Preprint, 2015, arXiv:1512.02979.
15. G. Oliveira, ‘Monopoles in higher dimensions’, PhD Thesis, Imperial College London, 2014.
16. G. Oliveira, ‘Monopoles on the Bryant–Salamon $G_2$–manifolds’, J. Geom. Phys. 86 (2014) 599–632.
17. G. Oliveira, ‘Monopoles on AC 3-manifolds’, J. Lond. Math. Soc. 93 (2016) 785–810.
18. M. K. Prasad and C. M. Sommerfield, ‘Exact classical solution for the ’t Hooft monopole and the Julia-Zee dyon’, Phys. Rev. Lett. 35 (1975) 760–762.
19. C. H. Taubes, ‘The existence of a non-minimal solution to the SU(2) Yang–Mills–Higgs equations on $\mathbb{R}^3$. Part I’, Comm. Math. Phys. 86 (1982) 257–298.
20. C. H. Taubes, ‘Magnetic bag like solutions to the SU(2) monopole equations on $\mathbb{R}^3$, Comm. Math. Phys. 330 (2014) 539–580.
21. C. H. Taubes, ‘The Seiberg–Witten equations and the Weinstein conjecture’, Geom. Topol. 11 (2007) 2117–2202.
22. T. Walpuski, ‘A compactness theorem for Fueter sections’, Comment. Math. Helv. 92 (2017) 751–776.
23. H. C. Wang, ‘On invariant connections over a principal fibre bundle’, Nagoya Math. J. 13 (1958) 1–19.

Daniel Fadel
IMECC-UNICAMP
Rua Sérgio Buarque de Holanda
651 | CEP 13083-859
Campinas SP
Brazil
fadel.daniel@ime.unicamp.br

Goncalo Oliveira
Instituto de Matemática e Estatística
Departamento de Matemática Aplicada (GMA)
Universidade Federal Fluminense
Campus Gragoatá
Rua Prof. Marcos Waldemar de Freitas
Reis, s/n, São Domingos
Niterói, RJ, 24210-201
Brazil
galato97@gmail.com
https://sites.google.com/view/goncalo-oliveira-math-webpage/home

The Proceedings of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.