Counting of Teams in First-Order Team Logics

Anselm Haak, Fabian Müller and Heribert Vollmer
Theoretische Informatik
Leibniz Universität Hannover
Appelstraße 4
30167, Germany
haak | fabian.mueller | vollmer@thi.uni-hannover.de

Juha Kontinen and Fan Yang
Department of Mathematics and Statistics
University of Helsinki
Pietari Kalminkatu 5
00014, Finland
juha.kontinen | fan.yang@helsinki.fi

Abstract—We study descriptive complexity of counting complexity classes in the range from \#P to \# \cdot NP. A corollary of Fagin’s characterization of \#P by existential second-order logic is that \#P can be logically described as the class of functions counting satisfying assignments to free relation variables in first-order formulae. In this paper we extend this study to classes beyond \#P and extensions of first-order logic with team semantics. These team-based logics are closely related to existential second-order logic and its fragments, hence our results also shed light on the complexity of counting for extensions of FO in Tarski’s semantics. Our results show that the class \# \cdot NP can be logically characterized by independence logic and existential second-order logic, whereas dependence logic and inclusion logic give rise to subclasses of \# \cdot NP and \#P, respectively. Our main technical result shows that the problem of counting satisfying assignments for monotone Boolean \(\Sigma_1\)-formulae is \# \cdot NP-complete as well as complete for the function class generated by dependence logic.

I. INTRODUCTION

The question of the power of counting arises in propositional and predicate logic in a number of contexts. Counting the number of satisfying assignments for a given propositional formula, \#SAT, is complete for Valiant’s class \#P of functions counting accepting paths of nondeterministic polynomial-time Turing machines \[29\]. Valiant also proved that \#SAT even remains complete when restricted to monotone 2-CNF formulae.

The class \#P can be seen as the counting analogue of NP, which was shown by Fagin \[10\] to correspond to existential second order logic, where the quantified relation encodes accepting computation paths of NP-machines. Hence, if we define \#FO\textsuperscript{rel} to count satisfying assignments to free relational variables in FO-formulae, we obtain \#FO\textsuperscript{rel} = \#P. This result has been refined to prefix classes of FO showing, e.g., that \#\Pi^{rel}_2 = \#P \[27\].

If we define \#FO\textsuperscript{func} in the same fashion as \#FO\textsuperscript{rel} except that we count assignments to function variables instead of relation variables, then obviously \#FO\textsuperscript{func} = \#P. The situation changes for the prefix classes, though. In particular, unlike for \#\Pi^{rel}_2, it holds that \#\Pi^{func}_2 = \#P, and, remarkably, also arithmetic circuit classes like \#AC^0 can be characterized in this context \[17\].

In this paper we consider a different model-theoretic approach to the study of counting processes using the so-called team-based logics. In these logics, formulae with free variables are evaluated not for single assignments to these variables but for sets of such assignments (called teams). Logics with team semantics have been developed for the study of various dependence and independence concepts important in many areas, such as database theory and Bayesian networks (see, e.g., \[15\], \[18\]). Given that model counting is an important inference task in these areas (\[3\], \[25\]), we initiate in this paper the study of counting for team-based logics. In our proofs we utilize the known correspondences between team-based logics and existential second-order logic (\(\Sigma_1\)) and its fragments (see Theorem 2). We want to stress that our results are also novel for existential second-order logic and its fragments.

A team satisfies a first-order formula iff all its members satisfy the formula individually. Interest in teams stems from the introduction of different logical atoms describing properties of teams, called team atoms, such as the value of a variable being functionally dependent on other variables (characterized by the dependence atom \(=\ldots\)), a variable being independent from other variables (characterized by the independence atom \(\perp\)), and the values of a variable occurring as values of some other variable (characterized by the inclusion atom \(\subseteq\)), etc. (\[28\], \[14\], \[11\]). We define \#FO\textsubscript{team} to be the class of functions counting teams that satisfy a given FO-formula, and similarly for extensions of FO by team atoms.

Making use of different team atoms, we give a characterization of \# \cdot NP. While it is relatively easy to see that with every finite set \(A\) of NP-definable team atoms, the class \#FO(A)\textsubscript{team} stays a subclass of \# \cdot NP (Toda’s generalization of \#P, see \[16\] for a survey of counting classes like these), we show that FO extended with the independence atom is actually sufficient to characterize the full class \# \cdot NP:

\[
\#FO(\perp)^{\text{team}} = \#\Sigma_1 = \# \cdot NP.
\]

The situation with inclusion logic and dependence logic is more complex due to their strong closure properties: satisfaction of formulae is closed under union for inclusion logic and is closed downwards for dependence logic. We show that \#FO(\(\subseteq\))\textsubscript{team} is a strict subclass of \#P, unless \(P = NP\). Furthermore, \#FO(\(=\ldots\))\textsubscript{team} is a subclass of \# \cdot NP, which we believe to be strict as well. Interestingly, both classes contain complete problems from their respective superclasses.

In establishing this result for dependence logic, we introduce an interesting class of monotone quantified Boolean formulae and show that the corresponding counting problem where
the all-0-assignment is not counted, \#\Sigma_1\text{-CNF}^+_n\text{-}, is \# \cdot \text{NP}-complete. In order to prove \# \cdot \text{NP}-completeness we also show that the more natural problem of counting all satisfying assignments of the same class of formulae is \# \cdot \text{NP}-complete by introducing a new technique of simultaneous reductions between pairs of counting problems, which we hope will also be useful in other contexts.

For inclusion logic we show that the well-known \#P-complete problem \#2-CNF is in \#\text{FO}(\subseteq)\text{-team} and that the problem of counting assignments for (existentially quantified) dual Horn formulae is hard for \#\text{FO}(\subseteq)\text{-team}.

In related previous work, so-called weighted logics have been used to logically characterize counting complexity classes [11], and the decision-problem analogue PP of \#P and the counting hierarchy have been logically characterized in [19]. [21]. [6]. Counting classes from circuit complexity beyond \#AC^0 have been logically characterized in [6].

II. DEFINITIONS AND PRELIMINARIES

First-order Logic and Team Semantics: Let us start by recalling the syntax of first-order logic (FO). In this work, we only consider relational vocabularies (i.e., vocabularies with no function or constant symbols), and thus the only first-order terms are variables. Formulae of first-order logic are defined by the following grammar:

\[
\varphi ::= \varphi \land \varphi \mid \varphi \lor \varphi \mid \exists x \varphi \mid \forall x \varphi \mid R(\varphi) \mid \neg R(\varphi) \mid x = y \mid x \neq y
\]

where \(x, y\) are variables, \(R\) is a relation symbol, and \(\varphi\) is a tuple of the appropriate number of variables.

The set of free variables of a formula \(\varphi\) is defined as usual, and we sometimes write \(\varphi(x_1, \ldots, x_k)\) to emphasize that the free variables of \(\varphi\) are among \(x_1, \ldots, x_k\). A formula with no free variable is called a sentence. For any \(k\), the fragment \(\Sigma_k\) of FO consists of all formulae of the form \(\exists x_1 \forall x_2 \ldots \forall x_k \varphi\), where \(\varphi\) is quantifier-free and \(Q\) is either \(\exists\) or \(\forall\) depending on whether \(k\) is odd or even; similarly, the fragment \(\Pi_k\) is defined as the class of all formulae \(\forall x_1 \exists x_2 \ldots \forall x_k \varphi\) in prenex normal form with a quantifier prefix with \(k\) alternations starting with universal quantifiers.

We only consider finite structures with a finite relational vocabulary \(\sigma\). Denote the class of all such structures by \text{STRUC}[\sigma], and let \text{dom}(A) denote the universe of a \(\sigma\)-structure \(A\). We will always use structures with universe \(\{0, 1, \ldots, n - 1\}\) for some \(n \in \mathbb{N} \setminus \{0\}\). We assume that our structures contain a built-in binary relation \(\leq\) with the usual interpretation, i.e., \(\leq\) is interpreted in a model of any size as the “less than or equal to” relation on \(\mathbb{N}\). We write enc\(_\sigma\)(A) for the standard binary encoding of a \(\sigma\)-structure \(A\) (see e.g., [18]). Relations are encoded row by row by listing their truth values as 0’s and 1’s. A whole structure is encoded by the concatenation of the encodings of its relations.

We assume that the reader is familiar with the usual Tarskian semantics for first-order formulae, in which formulae are evaluated with respect to single assignments of a structure. In this paper, we also consider the so-called team semantics for first-order formulae, in which formulae are evaluated with respect to teams. A team is a set of assignments of a structure, that is, a set of functions \(s: \{x_1, \ldots, x_k\} \rightarrow \text{dom}(A)\), where we call \(\{x_1, \ldots, x_k\}\) the domain of the team. Note that the empty set \(\emptyset\) is a team, called empty team, and the singleton \(\{\emptyset\}\) containing only the empty assignment is also a team. We denote by team\((A, \{x_1, \ldots, x_k\})\) the set of all teams over a structure \(A\) with the domain \(\{x_1, \ldots, x_k\}\).

We define inductively the notion of a team \(X\) with domain \(\{x_1, \ldots, x_k\}\) of a structure \(A\) with \(A := \text{dom}(A)\) satisfying an FO-formula \(\varphi(x_1, \ldots, x_k)\), denoted by \(A \models X\varphi\), as follows:

\[
\begin{align*}
A & \models X \alpha & \text{if} \alpha \text{ is an atomic formula, iff for all } s \in X, A \models s\alpha \text{ in the usual Tarskian semantics sense.} \\
A & \models X \varphi \lor \psi & \text{iff there are teams } Y, Z \subseteq X \text{ such that } Y \cup Z = X, A \models Y \varphi \text{ and } A \models Z \psi. \\
A & \models X \varphi \land \psi & \text{iff } A \models X \varphi \text{ and } A \models X \psi. \\
A & \models X \exists x \varphi & \text{iff there exists a function } F: X \rightarrow P(A) \setminus \{\emptyset\}, \text{ called supplemet function, such that } A \models X[F / x] \varphi,
\end{align*}
\]

where \(X[F / x] = \{s(a/x) | s \in X \text{ and } a \in F(s)\}\).

A sentence \(\varphi\) is said to be true in \(A\), written \(A \models \varphi\), if \(A \models \{\emptyset\} \varphi\).

First-order formulae \(\varphi\) are flat over team semantics, i.e., \(A \models X \varphi\), iff \(A \models s \varphi\) for all \(s \in X\). In this sense, team semantics is conservative over first-order formulae. We now extend first-order logic by sets of atomic formulae which are not flat. For any sequence \(\pi\) of variables and variable \(y\), the string \((\pi, y)\) is called a dependence atom. For any sequences \(\pi, \gamma, \tau\) of variables, the string \(\gamma \perp \pi \tau\) is called an independence atom. For any two sequences \(\pi, \gamma\) of variables of the same length, the string \(\pi \subseteq \gamma\) is called an inclusion atom. The team semantics of these atoms are defined as follows:

\[
\begin{align*}
A & \models X := (\pi, y) & \text{iff for all } s, s' \in X, \text{ if } s(\pi) = s'(\pi), \text{ then } s(y) = s'(y). \\
A & \models X \gamma \perp \pi \tau & \text{iff for all } s, s' \in X \text{ such that } s(\pi) = s'(\pi), \text{ there exists } s'' \in X \text{ such that } s''(\pi) = s(\pi), s''(\tau) = s(\tau) \text{ and } s''(\gamma) = s'(\gamma). \\
A & \models X \pi \subseteq \gamma & \text{iff for all } s \in X \text{ there is } s' \in X \text{ such that } s(\pi) = s'(\pi).
\end{align*}
\]

For any subset \(A \subseteq \{=, \ldots, \subseteq, \subseteq\}\), we define FO(A) as first-order logic extended by the respective atomic operators, and refer to such a logic as team-based logic. The team-based logic FO(=) is known in the literature as dependence logic [28], FO(\perp) as independence logic [14] and FO(\subseteq) as inclusion logic [11].

We recall some basic properties of these logics from [28], [14]. [11]: Formulae of FO(=) are closed downwards, i.e., \(A \models X \varphi\) and \(Y \subseteq X \implies A \models Y \varphi\), formulae of FO(\subseteq) are closed under unions, i.e., \(A \models X \varphi\) and \(A \models Y \varphi\) imply \(A \models X \cup Y \varphi\), and formulae of any of these logics have the empty team property, i.e., \(A \models \emptyset \varphi\) always holds.

The above atoms expressing team properties can be generalized, as we will do next. Let us first recall below the definition...
of generalized quantifiers, where we follow the notations from [20], [24].

Definition 1. Let $i_1, \ldots, i_n$ ($n > 0$) be a sequence of positive integers, and $\sigma$ a vocabulary consisting of an $i_j$-ary relation symbol for each $1 \leq j \leq n$. A generalized quantifier of type $(i_1, \ldots, i_n)$ is a class $C$ of $\sigma$-structures $(A, B_1, \ldots, B_n)$ such that the following conditions hold:
1) $A \neq \emptyset$ and for each $1 \leq j \leq n$, we have $B_j \subseteq A^{i_j}$.
2) $C$ is closed under isomorphisms, that is, if $(A', B'_1, \ldots, B'_n) \in C$ is isomorphic to $(A, B_1, \ldots, B_n)$, then $(A', B'_1, \ldots, B'_n) \in C$.

Let $Q$ be a generalized quantifier of type $(i_1, \ldots, i_n)$. Let us extend the syntax of first-order logic with an expression $A_Q(x_1, \ldots, x_n)$, where each $x_j$ is a tuple of variables of length $i_j$. We call $A_Q$ a generalized (dependency) atom (of type $(i_1, \ldots, i_n)$), and its team semantics is defined as:

$$A \models x \ A_Q(x_1, \ldots, x_n) \iff \{s(x_1) \mid s \in X\} \cup \{s(x_n) \mid s \in X\} \in Q^A,$$

where $Q^A = \{(B_1, \ldots, B_n) \mid (\text{dom}(A), B_1, \ldots, B_n) \in Q\}$.

We say a generalized dependency atom $A_Q$ is NP-definable if there is an NP-algorithm that decides for a given structure $A$ and a given team $X$ whether $A \models x A_Q(x_1, \ldots, x_n)$ holds or not. A set $A$ of generalized atoms is NP-definable, if every $a \in A$ is NP-definable. For example, the set $A = \{=, (\ldots), \top, \bot\}$ is NP-definable.

Many results in this paper are based on the expressive power of the logics defined above that we shall now recall. We first recall some notions and notations. Existential second-order logic ($\Sigma_2^1$) consists of formulae of the form $\exists R_1 \ldots \exists R_k \varphi$, where $\varphi$ is a first-order formula. Let $\sigma$ be a vocabulary. We write $\sigma(R)$ for the vocabulary that arises by adding a fresh relation symbol $R$ to $\sigma$, and we sometimes write $\varphi(R)$ to emphasize that the relation symbol $R$ occurs in the $\sigma(R)$-formula $\varphi$. If $A$ is a $\sigma$-structure, we write $(A, Q)$ for $A$ expanded into a $\sigma(R)$-structure where the new $k$-relation symbol $R$ is interpreted as $Q \subseteq \text{dom}(A)^k$. A $\sigma(R)$-sentence $\varphi(R)$ of $\Sigma_1^1$ is said to be downward monotone with respect to $R$ if $(A, Q) \models \varphi(R)$ and $Q' \subseteq Q$ imply $(A, Q') \models \varphi(R)$. It is known that $\varphi(R)$ is downward monotone with respect to $R$ if and only if $\varphi(R)$ is equivalent to a sentence where $R$ occurs only negatively (see e.g., [22]). A structure $\mathcal{A}$ and a team $X$ of $\mathcal{A}$ with domain $\{x_1, \ldots, x_k\}$ induce the $k$-ary relation $rel(X)$ on $\text{dom}(\mathcal{A})$ defined as

$$rel(X) := \{(s(x_1), \ldots, s(x_n)) \mid s \in X\}.$$

Theorem 2 (see [12], [28], [22], [11]).

1) For every $\sigma$-formula $\varphi$ of FO($\top$), there is an $\sigma(R)$-sentence $\psi(R)$ of $\Sigma_1^1$ such that for all $\sigma$-structures $\mathcal{A}$ and teams $X$,

$$\mathcal{A} \models x \varphi \iff (\mathcal{A}, rel(X)) \models \psi(R).$$

Conversely, for every $\sigma(R)$-sentence $\psi(R)$ of $\Sigma_1^1$, there is a $\sigma$-formula $\varphi$ of FO($\top$) such that (1) holds for all $\sigma$-structures $\mathcal{A}$ and non-empty teams $X$.

2) The same as the above holds for formulae of FO($\bot$) as well, except that in both directions for FO($\bot$) the relation symbol $R$ is assumed to occur only negatively in the sentence $\psi(R)$.

3) In particular, over sentences both FO($\top$) and FO($\bot$) are expressively equivalent to $\Sigma_1^1$, in the sense that every $\sigma$-sentence of FO($\top$) (or FO($\bot$)) is equivalent to a $\sigma$-sentence $\varphi$ of $\Sigma_1^1$, i.e., for any $\sigma$-structure $\mathcal{A}$,

$$\mathcal{A} \models x \varphi \iff \mathcal{A} \models \psi,$$

and vice versa. As a consequence of [10], over finite structures both FO($\top$) and FO($\bot$) capture NP.

4) For any $\sigma$-formula $\varphi(x_1, \ldots, x_k)$ of FO($\subseteq$), there exists a $\sigma(R)$-formula $\psi(R)$ of positive greatest fixed point logic (posGFP) such that for all $\sigma$-structures $\mathcal{A}$ and teams $X$,

$$\mathcal{A} \models x \varphi \iff (\mathcal{A}, rel(X)) \models \psi(R) \text{ for all } s \in X;$$

and vice versa. In particular, over sentences FO($\subseteq$) is expressively equivalent to posGFP. As a consequence of [17], over finite structures, FO($\subseteq$) is expressively equivalent to least fixed point logic (LFP). Thus, by [17], [31], over ordered finite structures, FO($\subseteq$) captures P.

Propositional and Quantified Boolean formulae: In this paper, we will also consider certain classes of propositional and quantified Boolean formulae. As usual, we use CNF to denote the class of propositional formulae in conjunctive normal form and $k$-CNF to denote the class of propositional formulae in conjunctive normal form where each clause contains at most $k$ literals. A formula in CNF is in the class DualHorn, if each of its clauses contains at most one negative literal. For a class $C$ of formulae, we denote by $\Sigma_1^1$-$C$ the class of quantified Boolean formulae in prenex normal form with only existential quantifiers where the quantifier-free part is an element of $C$. With $C^\ast$ (resp. $C^-$) we denote the class of formulae in $C$ whose free variables occur only positively (resp. negatively). For example, $\Sigma_1^1$-$3$CNF$^\ast$ consists of all quantified Boolean formulae in prenex normal form with only existential quantifiers, where the quantifier-free part is in 3-CNF and the free variables occur only negatively.

Counting Problems and Counting Classes: This paper aims to identify model-theoretic characterizations of counting classes in terms of team-based logics. Let us now recall relevant previous results on the descriptive complexity of counting problems. We begin by defining the most important complexity classes for counting problems.

Definition 3. A function $f : \{0,1\}^* \to \mathbb{N}$ is in \#P if there is a non-deterministic polynomial time Turing-machine $M$ such that for all inputs $x \in \{0,1\}^*$,

$$f(x) \text{ is the number of the accepting computation paths of } M \text{ on input } x.$$

This definition can be generalized as follows.
Definition 4. Let $C$ be a complexity class. A function $f : \{0,1\}^* \rightarrow \mathbb{N}$ is said to be in $\# \cdot C$, if there are a language $L \in C$ and a polynomial $p$ such that for all $x \in \{0,1\}^*$: $$f(x) = |\{y : |y| \leq p(|x|)\text{ and } (x,y) \in L\}|.$$ 

Obviously $\#P = \# \cdot P$, and it is well known that $\#P \subseteq \# \cdot NP \subseteq \# \cdot coNP = \#P^{NP}$, where under reasonable complexity-theoretic assumptions, all these inclusions are strict; see [16] for a survey of these issues.

Next, we define the relevant logical counting classes.

Definition 5. A function $f : \{0,1\}^* \rightarrow \mathbb{N}$ is said to be in $\#FO^{rel}$, if there is a vocabulary $\sigma$ with a built-in linear order $\leq$, and an FO-formula $\varphi(R_1, \ldots, R_k, x_1, \ldots, x_\ell)$ over $\sigma$ with free relation variables $R_1, \ldots, R_k$ and free individual variables $x_1, \ldots, x_\ell$ such that for all $\sigma$-structures $A$, $$f(\text{enc}_\sigma(A)) = |\{(S_1, \ldots, S_k, c_1, \ldots, c_\ell) : A \models \varphi(S_1, \ldots, S_k, c_1, \ldots, c_\ell)\}|.$$ 

If the input of $f$ is not of the appropriate form, we assume the output to be 0.

In the same fashion, subclasses of $\#FO^{rel}$, such as $\#\Sigma^rel_k$ and $\#\Pi^rel_k$ for arbitrary $k$, are defined by assuming that the formula $\varphi$ in the above definition is in the corresponding fragments $\Sigma_k$ and $\Pi_k$.

Recall the relationship between the above defined logical counting classes and $\#P$:

Theorem 6 ([27]). $\#\Sigma^rel_0 = \#\Pi^rel_0 \subset \#\Sigma^rel_1 \subset \#\Pi^rel_1 \subset \#\Sigma^rel_2 \subset \#\Pi^rel_2 = \#FO^{rel} = \#P$.

Furthermore, it was shown that $\#\Sigma^rel_0 \not\subseteq FP$.

Complete problems (i.e., hardest problems) for counting classes have also been studied extensively. Let us now recall three reductions that are most relevant in this study. Let $f$ and $h$ be counting problems. We say that $f$ is parsimoniously reducible to $h$ if there is a polynomial-time computable function $g$ such that $f(x) = h(g(x))$ for all inputs $x$. $f$ is Turing reducible to $h$ if $f \in FP^h$, and $f$ is metrically reducible to $h$ if there are polynomial-time computable functions $g_1, g_2$ such that $f(x) = g_2(h(g_1(x)), x)$ for all inputs $x$. Note that metric reductions are thus Turing reductions with one oracle query.

It is often possible to find complete problems in counting classes by counting satisfying assignments for certain (quantified) Boolean formulae. Let $F$ be a class of quantified Boolean formulae. Define the problem $\#F$ as follows:

| Problem: $\#F$ |
|------------------|
| Input: Formula $\varphi \in F$ |
| Output: Number of satisfying assignments of $\varphi$ |

As examples, $\#\text{SAT}$, the function counting the number of satisfying assignments for propositional formulae, as well as its restriction $\#3\text{-CNF}$, are complete for $\#P$ under parsimonious reductions, while $\#2\text{-CNF}^+$ and $\#2\text{-CNF}^-$ are complete for $\#P$ under metric reductions. Aziz et al [2] studied the problem $\#\Sigma^1_1$-SAT under the name projected model counting and noted that it is contained in $\# \cdot NP$.

We end this section by introducing the central class of counting problems for this paper, a counting problem in the context of team-based logics. For any set $A$ of generalized dependency atoms, we define $\#FO(A)^{\text{team}}$ to consist of those functions counting non-empty satisfying teams for FO($A$)-formulae. Note that by the empty team property of dependence, independence, and inclusion logic formulae any function that counts all satisfying teams (including the empty team) could not attain the value 0.

Definition 7. For any set $A$ of generalized atoms, $\#FO(A)^{\text{team}}$ is the class of all functions $f : \{0,1\}^* \rightarrow \mathbb{N}$ for which there is a vocabulary $\sigma$ with a built-in linear order $\leq$ and an FO($A$)-formula $\varphi(\bar{x})$ over $\sigma$ with a tuple $\bar{x}$ of free first-order variables such that for all $\sigma$-structures $A$, $$f(\text{enc}_\sigma(A)) = |\{X \in \text{team}(A, \bar{x}) : X \neq \emptyset \text{ and } A \models \varphi(\bar{x})\}|.$$ We denote by $f_{\varphi}$ the function defined by $\varphi$.

III. A Characterization of the Class $\# \cdot NP$

In this section, we characterize the class $\# \cdot NP$ in terms of team-based logics.

Our first result shows that $\# \cdot NP$ is the largest class attainable by counting teams in team-based logics FO($A$), as long as all generalized atoms in $A$ are NP-definable.

Theorem 8. For any set $A$ of NP-definable generalized atoms, $\#FO(A)^{\text{team}} \subseteq \# \cdot NP$.

Proof. Let $\varphi(x_1, \ldots, x_k) \in FO(A)$. To count for a given input structure $A$ the number of (non-empty) teams $X$ with $A \models x \varphi$ in a $\# \cdot NP$-algorithm, we first non-deterministically guess a team $X$ and then check in NP whether $A \models x \varphi$ holds for this team.

For the latter note that for any fixed formula $\varphi$ without disjunctions and existential quantifiers, $A \models x \varphi$ can be checked in nondeterministic polynomial time (where nondeterminism is only needed to handle generalized atoms). Disjunctions can be handled by non-deterministically guessing the subteams for the disjuncts and existential quantifiers can be handled by non-deterministically guessing the supplementing function. Hence, $A \models x \varphi$ can be checked in NP.

Next, we prove the converse inclusion of the above theorem by proving a stronger result: The whole class $\# \cdot NP$ can actually be captured by a single generalized atom, the independence atom.

Theorem 9. $\# \cdot NP \subseteq \#FO(\bot)^{\text{team}}$

Proof. First note that $\#\Sigma^1_1 = \#FO(\bot)^{\text{team}}$, because by Theorem [2,1] any sentence $\varphi(R) \in \Sigma^1_1$ with a $k$-ary relation symbol $R$ can easily be turned into a sentence $\varphi'(R')$ for some $(k+1)$-ary $R'$ such that $\varphi$ and $\varphi'$ define the same
functions and \( \varphi'(R') \) is only satisfied by non-empty relations. It then suffices to show that \( \# \cdot \text{NP} \subseteq \#\Sigma_1 \).

Consider the string vocabulary \( \tau_{\text{string}} = (S^1) \) used to encode binary strings as first-order structures. For any binary string \( w = w_0w_1 \ldots w_{n-1} \in \{0,1\}^* \), we define the structure encoding \( A_w = (\{0,1, \ldots, n-1\}, \text{S}) \), where \( \text{S}(i) = w_i \) for all \( i \). Note that with this definition the standard binary encoding of the structure \( A_w \) is the string \( w \) itself, namely \( \text{enc}_{\tau_{\text{string}}}(A_w) = w \) for all \( w \in \{0,1\}^* \).

Let \( f \in \# \cdot \text{NP} \) via \( L \in \text{NP} \) and the polynomial \( p \), with \( p(n) = n^e + c \) for some \( e, c \in \mathbb{N} \). By definition we have \( f(x) = |\{ y : |y| = p(|x|), (x, y) \in L \}| \). We encode tuples \( (x, y) \) with \( x \in \{0,1\}^* \) and \( y \in \{0,1\}^{|x|} \) by structures over the vocabulary \( \tau_k = \tau_{\text{string}} \cup (R^k) \). Note that such an encoding is possible because we can define the extension of the numerical predicates to tuples in FO (see [13]). For strings \( x \) with \( |x| \geq 2 \) we can choose \( k \) such that \( |x|^k \geq p(|x|) \) (strings of length 1 can be handled separately). Fix such a \( k \). We denote by \( A(x,y) \) the \( \tau_k \)-structure encoding the tuple \( (x, y) \).

Now consider the language

\[
L' := \{ \text{enc}_{\tau_k}(A(x,y)) \mid A(x,y) \in \text{STRUC}[\tau_k], \ y = y_0 \ldots y_{|x|^k-1}, \ y_{p(|x|)} = \ldots = y_{|x|^k-1} = 0 \text{ and } (x, y_0 \ldots y_{p(|x|)-1}) \in L \}.
\]

For any given \( x \), \( \text{enc}_{\tau_k}(A(x,y)) \) is an element in \( L' \) if and only if the first \( p(|x|) \) bits of \( y \) form an input \( z \) such that \( (x, z) \in L \) and the rest of the bits are fixed to be 0. Thus,

\[
f(x) = |\{ y : \text{enc}_{\tau_k}(A(x,y)) \in L' \}|.
\]

Obviously \( L' \in \text{NP} \), which, by Fagin’s Theorem (see [10]), implies that there is a sentence \( \varphi \in \Sigma_1 \) over \( \tau_k \) such that

\[
\text{enc}_{\tau_k}(A(x,y)) \in L' \iff A(x,y) \models \varphi.
\]

Viewing \( \varphi \) as a formula over the vocabulary \( \tau_{\text{string}} \) with free relational variable \( R \) of arity \( k \) we have

\[
A_x \models \varphi(R) \iff \text{enc}_{\tau_k}(A_x, R) \in L'
\]

for all \( x \in \{0,1\}^* \), which yields

\[
f(x) = |\{ R : A_x \models \varphi(R) \}|.
\]

Hence \( f \in \#\Sigma_1 \).

Remark 10. The class \#P can also be characterized by counting teams. In [23] a variant \( L \) of dependence logic was introduced that defines exactly the first-order definable team properties in the sense of Theorem 2. By the result of [27], this logic \( L \) captures \#P. We do not present the details of \( L \) in this paper, but only note that \( L \) has weaker versions of quantifiers and disjunction instead of those standard ones as defined in Section 17.

IV. Counting Teams in Dependence and Inclusion Logic

In this section, we study the smaller classes \#FO(\{=,\ldots\}) and \#FO(\{\subseteq\}) \#NP-complete problem \#\Sigma_1 -CNF -formula \#\Sigma_1 -CNF -formula \#\Sigma_1 -CNF -formula

\[
\text{Problem: } \#\Sigma_1 -CNF -formula
\]

\[
\text{Input: Formula } \varphi(x_1, \ldots, x_k) \in \text{CNF}
\]

\[
\text{Output: Number of satisfying assignments of } \varphi, \text{ disregarding the all-0-assignment}
\]

Note that the all-0-assignment is the assignment assigning the value 0 to each variable.

In first-order logic we encode \( \Sigma_1 -CNF -formulae as structures over the vocabulary \( \tau_{\Sigma_1 -CNF} = (P^1, Q^1, I^1) \),

where the predicates \( P \) and \( Q \) state, respectively, which variables occur free and which occur bound in the encoded formula, and the predicate \( I \) is the incidence relation between clauses and literals. More precisely, an arbitrary \( \Sigma_1 -CNF -formula \varphi(x_1, \ldots, x_k) = \exists y_1 \ldots \exists y_l \psi(x_1, \ldots, x_k, y_1, \ldots, y_l) \)

with \( \psi = \bigwedge_{i=1}^n C_i, C_i = l_{i1} \lor \cdots \lor l_{im}, \) and

\[
l_{ij} \in \{ \neg x_i | i \in \{1, \ldots, k\} \} \cup \{ y_j, \neg y_j | i \in \{1, \ldots, l\}\}
\]

is encoded as the \( \tau_{\Sigma_1 -CNF} \)-structure

\[
A = (\text{dom}(A), P^A, Q^A, I^A)
\]

defined as follows: The elements of \( \text{dom}(A) \) are numerical encodings of the variables and clauses in \( \varphi \). By abuse of notation, we write \( \text{dom}(A) = \{ x_1, \ldots, x_k, y_1, \ldots, y_l, C_1, \ldots, C_n \} \), identifying the variables and clauses with their encoding. The interpretations of the predicate symbols are defined as:

- \( P^A(x) \) iff \( x \) is a free variable in \( \varphi \)
- \( Q^A(x) \) iff \( x \) is a bound variable in \( \varphi \)
- \( I^A(C_i, x, 0) \) iff there is a \( j \) such that \( l_{ij} = \neg x \)
- \( I^A(C_i, x, a) \) for some \( a \neq 0 \) if there is a \( j \) such that \( l_{ij} = x \)

Theorem 11. \( \#\Sigma_1 -CNF \in \#\text{FO}(\{=,\ldots\}) \#NP-complete problem \#\Sigma_1 -CNF -formula \#

Proof. By Theorem 2.2, it suffices to construct a \( \Sigma_1 \) formula \( \psi(T) \) with \( T \) occurring only negatively such that for each \( \tau_{\Sigma_1 -CNF} \)-structure \( A \), the number of relations \( T \) with \( A \models \psi(T) \) is equal to the number of satisfying assignments of the \( \Sigma_1 -CNF -formula encoded by A \). Notice a subtle point in this setting: The all-0-assignment is not counted by \#\Sigma_1 -CNF -formula \#

In the formula \( \psi(T) \), it corresponds to the empty relation, which, in turn, corresponds to the empty team in the translation to \( \text{FO}(\{=,\ldots\}) \) given in Theorem 2.2. Since the empty team is not counted when defining functions in the sense of \#FO(\{=,\ldots\}) \#, this means that the all-0-assignment is not counted in the function \( f_0 \) as desired.

The all-0-assignment should not be counted, since for the \( \Sigma_1 \) formula \( \psi(T) \) it corresponds to the empty relation, which,
in turn, corresponds to the empty team in the translation into FO(=,(...)) given in Theorem 2.2

Let us now define the formula ψ(T). First, let ψ1 be a first-order formula expressing that A is a correct encoding of a Σ₁-CNF⁻⁻ formula (i.e., P and Q correspond to disjoint sets, free variables occur only negatively in the clauses, etc.). We omit here the precise definition of ψ1. Let ψ2 be a formula expressing that T only assigns values to free variables and that each clause must be satisfied by the assignment, that is, formally,

\[ \psi_2(T) = \exists y \forall x \exists y \min x \leq x \land \psi_1(T) \land \psi_2(T) \].

It is easy to see that the formula ψ(T) has the desired properties. \( \square \)

Next we show that the problem #Σ₁-CNF⁻⁻ is hard and thus complete for #FO(=,(...)) under parsimonious reductions. Our proof technique is similar to that of [9], where the data complexity of inclusion logic is shown to be polynomial.

**Theorem 12.** #Σ₁-CNF⁻⁻ is complete for #FO(=,(...)) under parsimonious reductions.

**Proof.** By Theorem 11 the problem #Σ₁-CNF⁻⁻ is contained in #FO(=,(...))⁻⁻. It remains to show hardness. Let \( \varphi(x_1, \ldots, x_m) \in FO(=,(...)) \), and let A be a structure with domain \( A = \{0, \ldots, n-1\} \). We reduce computing the value of \( f_\varphi(A) \) to counting the number of satisfying assignments (apart from the all-0-assignment) of a suitable Boolean formula \( \Gamma^\varphi,A \in \Sigma₁-CNF⁻⁻ \). By [23], we may assume without loss of generality that \( \varphi \) is of the form

\[ \forall y_1 \ldots \forall y_k \exists y_{k+1} \ldots \exists y_{k+l} (\bigwedge_t = (\overline{w_t}, 1) \land \theta) \],

where \( \theta \) is a quantifier-free first-order formula, \( w_t \in \{y_{k+1}, \ldots, y_{k+l}\} \), and \( \overline{w_t} \) is a tuple consisting of some of the variables \( y_1, \ldots, y_k \). \( \Gamma^\varphi,A \) is built from the set \( \{X_{s,l} \mid s \in A^m \cup A^{m+1} \cup \cdots \cup A^{m+k+l} \} \) of propositional variables. Observe that each such \( s \) can be identified with a partial first-order assignment over the domain \( \{x_1, \ldots, x_m, y_1, \ldots, y_{k+l}\} \). Clearly, the number of such assignments and consequently also the number of variables \( X_s \) is polynomial. The variables \( X_s \) for \( s \in A^m \cup A^{m+1} \cup \cdots A^{m+k+l} \) will be existentially quantified in \( \Gamma^\varphi,A \), whereas the variables \( X_s \) for \( s \in A^m \) will remain free and occur only negatively.

We now define the set \( C \) of clauses of \( \Gamma^\varphi,A \). For every universally quantified variable \( y_i \) in \( \varphi \), and for every \( s \in A^{m+i-1} \) we introduce to \( C \) the following set of clauses:

\[ \{X_s \rightarrow X_{s'} : s' \in A^{m+i} \land s = s' \mid \{\overline{y_1}, \ldots, y_{i-1}\} \}. \]

For every existentially quantified variable \( y_i \) in \( \varphi \), and for every \( s \in A^{m+i-1} \) we introduce the following clause:

\[ X_s \rightarrow \bigvee_{s' \in A^{m+i} \land \text{ and } s = s'} \].

The quantifier-free part of the formula \( \varphi \) also gives rise to clauses as follows. For each dependence atom \( \sigma(x,m) \) we introduce the set of clauses:

\[ \{X_s \rightarrow \exists y \land \Gamma^\varphi,A \mid s \in A^m \cup A^{m+1} \cup \cdots \cup A^{m+k+l} \land s(w_t) \neq s'(w_t) \}. \]

Finally, for the first-order formula \( \varphi \) the team semantics satisfaction condition stipulates that all assignments \( s \in A^{m+k+l} \) should satisfy \( \theta \). This can be expressed by introducing the following two sets of clauses:

\[ \{X_s \rightarrow \top : s \in A^m \cup A^{m+1} \cup \cdots \cup A^{m+k+l} \land \Gamma^\varphi,A \} \]

and

\[ \{X_s \rightarrow \bot : s \in A^m \cup A^{m+1} \cup \cdots \cup A^{m+k+l} \land \Gamma^\varphi,A \}. \]

Now define \( \Gamma^\varphi,A \in \Sigma₁-CNF⁻⁻ \) with respect to the input \( \varphi \) and \( \Gamma \) as

\[ \Gamma^\varphi,A := \exists \{X_s : s \in \bigcup_{1 \leq l \leq k+l} A^{m+i} \} \land C. \]

Clearly, the formula \( \Gamma^\varphi,A \) can be computed in polynomial time and there is a 1-1-correspondence between teams \( X \) over domain \( \{x_1, \ldots, x_m \} \) and assignments \( S \) of formula \( \Gamma^\varphi,A \). Furthermore it is easy to check that for all teams \( X \) with domain \( \{x_1, \ldots, x_m \} \)

\[ A \models X \varphi(x_1, \ldots, x_m) \Leftrightarrow S \models \Gamma^\varphi,A, \]

where the Boolean assignment \( S \) is defined as \( S(X_s) = \text{true if } s \in X \). \( \square \)

Having proven our results for dependence logic FO(=,(...)), we now turn to inclusion logic FO(\( \subseteq\)). We first prove that #FO(\( \subseteq\)) is a subclass of #P.

**Theorem 13.** #FO(\( \subseteq\)) \( \subseteq \) #P.

**Proof.** To count for a given input structure \( A \) the number of satisfying teams for a formula in FO(\( \subseteq\)), we simply guess a team and verify that it satisfies the formula. The latter step can be done in polynomial time, since model-checking for FO(\( \subseteq\)) is in P by Theorem 2.4. \( \square \)

The above theorem naturally gives rise to the question whether #FO(\( \subseteq\)) actually coincides with #P. However, we identify in the next lemma a particular property of #FO(\( \subseteq\)) functions, making this equivalence unlikely to hold.

**Lemma 14.** Let \( \varphi(\overline{x}) \in FO(\subseteq) \) be a formula over a vocabulary \( \sigma \). Then the language \( L := \{w \mid f_\varphi(w) > 0\} \) is in P.

**Proof.** Let \( w \in \{0, 1\}^* \) and \( A(w) := \text{enc}_\varphi^{-1}(w) \). The condition \( f_\varphi(w) > 0 \) asks whether there is a non-empty team
$X \in \text{team}(A(w),\mathcal{T})$ such that $A(w) \models_{X} \varphi(\mathcal{T})$, which is equivalent to asking whether $A \models_{\emptyset} \exists_{\text{team}} \varphi(\mathcal{T})$ holds. By Theorem 2.3 over ordered structures, the properties definable by FO(≤)-sentences are exactly the properties in P, hence it follows that $L \in P$.

**Corollary 15.** If $P \neq NP$, then $\#FO(\leq)_{\text{team}} \neq P$.

**Proof.** Suppose $\#FO(\leq)_{\text{team}} = P$. This means that $\#3CNF \subseteq \#FO(\leq)_{\text{team}}$. Then, it follows from Lemma 14 that the language $\{w \mid \#3CNF(w) > 0\} = 3CNF$ is in P, which implies P = NP, contradicting the assumption.

Theorem 13 and Corollary 15 indicate that $\#FO(\leq)_{\text{team}}$ is most likely a strict subclass of #P. Nevertheless, we show in the next theorem that $\#FO(\leq)_{\text{team}}$ contains a problem that is complete for #P under Turing reductions, the problem #2-CNF*.

**Theorem 16.** #2-CNF* $\in \#FO(\leq)_{\text{team}}$

**Proof.** Similar to the proof of Theorem 11, we show that #2-CNF* can be defined by counting non-empty teams (corresponding to assignments that evaluate at least one variable to true) satisfying a suitable formula.

Let $\varphi(x_1, \ldots, x_n) = \bigwedge D_i$, where $D_i = \ell_{i,1} \lor \ell_{i,2}$ and $\ell_{i,j} \in \{x_1, \ldots, x_n\}$. Consider the vocabulary $\tau_{2-CNF^*} = \{D^2\}$. Now $\varphi(x_1, \ldots, x_n)$ is encoded by a structure $A = (\{x_1, \ldots, x_n\}, D^A)$ provided $(x,y) \in D^A$ iff the clause $x \lor y$ occurs in $\varphi$. Moreover, an assignment to the variables $x_1, \ldots, x_n$ is encoded by a team $X$ with domain $\{t\}$, where $t$ is a single variable. It is easy to check that the number of non-empty teams $X$ such that

$$A \models_{X} \forall x \forall y (\neg D(x,y) \lor x \subseteq t \lor y \subseteq t)$$

is equal to the number of satisfying assignments of $\varphi$. Hence #2-CNF* $\in \#FO(\leq)_{\text{team}}$.

We end this section by exhibiting a hard problem for the class $\#FO(\leq)_{\text{team}}$. It is an open question whether the problem is definable by an inclusion logic formula.

**Theorem 17.** #Σ₁-DualHorn is hard for $\#FO(\leq)_{\text{team}}$ with respect to parsimonious reductions.

**Proof.** The proof is analogous to that of Theorem 12 (see also 9). The only additional ingredient needed is the fact that inclusion atoms $\mathcal{T} \subseteq \mathcal{Y}$ can be expressed via adding the following type of DualHorn clauses:

$$X_{s} \rightarrow \bigvee_{s' \in M^{m+k} \text{ and } s(\mathcal{T}) = s'(\mathcal{Y})} X_{s'}.$$ 

V. COMPLETE PROBLEMS FOR #NP

In this section we show that #Σ₁-CNF* is #NP-complete. To this end, we first observe that #Σ₁-CNF is #NP-complete. Afterwards we show that the smaller class #Σ₁-CNF remains #NP-complete by adapting the proof for #NP-completeness of #2-CNF* given by Valiant [30]. We conclude this section with a reduction from #Σ₁-CNF* to #Σ₁-CNF, showing the #NP-completeness of the latter.

**Lemma 18.** #Σ₁-SAT and #Σ₁-CNF are #NP-complete under parsimonious reductions.

**Proof.** Membership of #Σ₁-SAT in #NP is due to Aziz et al [2]. Since #Σ₁-CNF is a restriction of #Σ₁-SAT, membership for #Σ₁-CNF also follows immediately. A simple adaptation of Cook’s proof of #NP-completeness of SAT [4] shows that both problems are hard for #NP with respect to parsimonious reductions.

In order to show that #Σ₁-CNF is #NP-complete as well, we define the following two counting problems:

**Problem:** #(3-CNFS, Σ₁-3CNF)

**Input:** Formula $\varphi(x_1, \ldots, x_k) \in 3-CNFS$ and formula $\psi(x_1, \ldots, x_k) \in \#\Sigma_{1}-3CNF^-$

**Output:** Number of satisfying assignments of $\varphi \land \psi$

We will define additional auxiliary counting problems in the proof.

**Theorem 19.** #Σ₁-CNF is #NP complete under Turing reductions.

**Proof.** Membership follows from [18] since #Σ₁-CNF is a special case of #Σ₁-CNF. For the hardness proof, we first show a chain of reductions adapted from the one used by Valiant [30] to show the #P-completeness of #2-CNF*. Recall that the main steps of Valiant’s chain of reductions are:

$$#3CNF \leq \text{PERMANENT} \leq \text{PERFECT-MATCHING} \leq \text{IMPERFECT-MATCHING} \leq #2-CNF^*.$$

Our idea is to add a Σ₁-3CNF-formula to the input of each problem in the above chain of reductions, and to express certain properties of the respective inputs in the added formula. We then count only the solutions to the input that also satisfy the added formula.

We first reduce #Σ₁-3CNF to #(3-CNFS, Σ₁-3CNF^-), and then apply the above chain of reductions with the added formulae to reduce #(3-CNFS, Σ₁-3CNF^-) to #(2-CNFS-, Σ₁-3CNF^-), which will be further reduced to #Σ₁-CNF^-*. All of these reductions will be parsimonious, except for the one from perfect matchings to imperfect matchings.

#Σ₁-3CNF ≤ #(3-CNFS, Σ₁-3CNF^-);

Let $\varphi(x_1, \ldots, x_k) \in \#\Sigma_{1}-3CNF$ with $k \in \mathbb{N}$ and

$$\varphi(x_1, \ldots, x_k) = \exists y_1 \ldots \exists y_k \bigwedge C_i \land \bigwedge D_i \land \bigwedge E_i,$$
where $\text{Var}(C_i) \subseteq \{x_1, \ldots, x_k\}$, $\text{Var}(D_i) \subseteq \{y_1, \ldots, y_r\}$, $\text{Var}(E_i) \cap \{x_1, \ldots, x_k\} \neq \emptyset$ and $\text{Var}(E_i) \cap \{y_1, \ldots, y_r\} \neq \emptyset$. We now construct two formulae $\varphi' \in 3\text{-CNF}$ and $\psi \in \Sigma_1\text{-3CNF}^-$ such that $\#\Sigma_1\text{-3CNF}(\varphi) = \#(3\text{-CNF}, \Sigma_1\text{-3CNF}^-)(\varphi', \psi)$. Define

$$ \varphi'(x_1, \ldots, x_k, e_1, \ldots, e_m) = \bigwedge \left(C_i \wedge \bigwedge (\neg e_i \leftrightarrow E_i \mid \{x_1, \ldots, x_k\}) \right) $$

and

$$ \psi(e_1, \ldots, e_m) = \exists y_1 \ldots \exists y_r \bigwedge D_i \wedge \bigwedge (e_i \rightarrow E_i \mid \{y_1, \ldots, y_r\}) $$

where $m$ is the number of the clauses $E_i$, and $C \mid_V$ is the clause $C = \ell_1 \lor \ell_2 \lor \ell_3$ restricted to variables in $V$, that is,

$$ C \mid_V := \bigvee_{i \in \{1, 2, 3\}} \ell_i, \exists x \in V, \ell_i = x \text{ or } \ell_i = \neg x $$

Note that in these two formulae the new implications and bimpressions can be trivially transformed to 3-CNF-formulae, and in $\psi$ the free variables only occur negatively. Intuitively, through the new variables $e_i$, the formula $\varphi'$ expresses that the assignment to the variables $x_1, \ldots, x_k$ does not satisfy any literal in $E_i$, and thus, as expressed in $\psi$, the clause $E_i$ has to be satisfied by an appropriate assignment to the variables $y_1, \ldots, y_r$. Since the assignments to the new variables $e_i$ are uniquely determined by the assignments to the variables $x_1, \ldots, x_k$, the formula $\varphi' \land \psi$ has the same satisfying assignments as the original formula $\varphi$.

Problem: $\#(\text{CYCLE-COVER}, \Sigma_1\text{-3CNF}^-)$

Input: Directed Graph $G = (V, E)$ with $E = \{e_1, \ldots, e_n\}$ and formula $\varphi(e_1, \ldots, e_n) \in \Sigma_1\text{-3CNF}^-$

Output: Number of cycle covers $E' \subseteq E$ of $G$ with $c_{E'} \models \varphi$, where $c_{E'}$ is the characteristic function of $E'$ with respect to $E$

$\#(3\text{-CNF}, \Sigma_1\text{-3CNF}^-) \leq \#(\text{CYCLE-COVER}, \Sigma_1\text{-3CNF}^-)$:

Let $\varphi(x_1, \ldots, x_k) \in 3\text{-CNF}$, $\psi(x_1, \ldots, x_k) \in \Sigma_1\text{-3CNF}^-$. We map $\varphi$ to an instance $G$ of $\#\text{CYCLE-COVER}$ by a variant of the reduction from $[30]$. In Valiant’s reduction in $[30]$, certain pairs of nodes are connected by so-called junctions, which are essentially two edges connecting the nodes in both directions. The goal then is to count only “good” cycle covers, namely those cycle covers containing for each junction at most one of these edges. In the original proof this is achieved by replacing junctions by a certain gadget. In our case, we can instead use a formula to express the crucial condition: A junction consisting of two edges $e_1, e_2$ is used appropriately if and only if one of the edges $e_1$ and $e_2$ is not contained in the cycle cover.

In Valiant’s construction, each satisfying assignment of $\varphi$ corresponds to exactly one good cycle cover of $G$, and vice versa. In particular, for each variable $x$ of $\varphi$, there is a certain edge $e$ in $G$ such that $e$ is contained in each good cycle cover of $G$ if and only if the variable $x$ is assigned to 1 by the corresponding assignment.

Now, let $\varphi'$ be the formula obtained from $\psi$ by replacing all occurrences of the free variables by the corresponding edges. Let $J$ be the set of junctions in $G$, each of which can be given as the set of its edges. Define

$$ \psi'' = \psi' \land \bigwedge (\neg j_1 \lor \neg j_2) $$

Note that the free variables in $\psi''$ only occur negatively. Then, we have

$$ \#(\text{CYCLE-COVER}, \Sigma_1\text{-3CNF}^-)(G, \psi'') = \#(3\text{-CNF}, \Sigma_1\text{-3CNF}^-)(\varphi, \psi). $$

Problem: $\#(\text{PERFECT-MATCHING}, \Sigma_1\text{-3CNF}^-)$

Input: Bipartite Graph $G = (V_1, V_2, E)$ with $E = \{e_1, \ldots, e_n\}$ and formula $\varphi(e_1, \ldots, e_n) \in \Sigma_1\text{-3CNF}^-$

Output: Number of perfect matchings $E' \subseteq E$ of $G$ with $c_{E'} \models \varphi$, where $c_{E'}$ is the characteristic function of $E'$ with respect to $E$

$\#(\text{CYCLE-COVER}, \Sigma_1\text{-3CNF}^-) \leq \#(\text{PERFECT-MATCHING}, \Sigma_1\text{-3CNF}^-)$:

Following the 1-to-1 correspondence between cycle covers of directed graphs and perfect matchings of bipartite graphs, the reduction can be given as follows:

$$ ((V, E), \varphi) \mapsto ((V, \{v' \mid v \in V\}, \{\{v_1, v'_2\} \mid (v_1, v_2) \in E\}), \varphi'), $$

where $\varphi'$ is obtained from $\varphi$ by replacing all occurrences of variables $\{v_1, v_2\}$ by the corresponding new edges $\{v_1, v'_2\}$, which are variables in $\varphi$.

Problem: $\#(\text{IMPERFECT-MATCHING}, \Sigma_1\text{-3CNF}^-)$

Input: Bipartite Graph $G = (V_1, V_2, E)$ with $E = \{e_1, \ldots, e_n\}$ and formula $\varphi(e_1, \ldots, e_n) \in \Sigma_1\text{-3CNF}^-$

Output: Number of matchings $E' \subseteq E$ of $G$ with $c_{E'} \models \varphi$, where $c_{E'}$ is the characteristic function of $E'$ with respect to $E$

$\#(\text{PERFECT-MATCHING}, \Sigma_1\text{-3CNF}^-) \leq \#(\text{IMPERFECT-MATCHING}, \Sigma_1\text{-3CNF}^-)$:

Let $G = (V_1, V_2, E)$ be a bipartite graph with $E = \{e_1, \ldots, e_n\}$ and $\varphi(e_1, \ldots, e_n) \in \Sigma_1\text{-3CNF}^-$. For the reduction $\text{PERFECT-MATCHING} \leq \text{IMPERFECT-MATCHING}$, Valiant constructs bipartite graphs $G_k$ for $1 \leq k \leq |V_1| + 1$ from $G$ by adding copies of all nodes in $V_1$ as follows:

$$ G_k = (V_1, V_2, E_k) $$

where

$$ V_{1,k} = V_1 \cup \{u_{ij} \mid 1 \leq i \leq |V_1|, 1 \leq j \leq k\} $$

and

$$ E_k = E \cup \{u_{ij}, v_i \mid 1 \leq i \leq |V_1|, 1 \leq j \leq k\} $$
Let $A_r$ be the number of matchings of $G$ of size $|V_1| - r$. Then $G_k$ has exactly $\sum_{r=1}^{k} A_r \cdot (k + 1)^r$ matchings. Using the number of matchings of all graphs $G_k$ we get a system of linear equations that allows us to compute $A_0$, the number of perfect matchings of $G$. Note that each matching of $G$ corresponds to a number of matchings in each $G_k$ (those consisting only of copies of the edges from the original matching).

To compute the number of perfect matchings $E'$ of $G$ with $c_{E'} \models \psi$, we now associate each graph $G_k$ with a formula $\psi_k$ such that $c_{E'} \models \psi_k$ holds for those matchings $E''$ of $G_k$ corresponding to a matching $E'$ of $G$ with $c_{E'} \models \psi$. Let $e_i = \{v_1, v_2\}$ be an edge of $G$. A matching $E''$ of $G_k$ corresponds to a matching $E'$ of $G$ that does not use edge $e_i$ if and only if it does not even use the edge $\{v_1, v_2\}$ nor any of the edges $e_{ij}$, where $e_{ij} = \{u_{1j}, v_2\}$. Formally this can be written as

$$c_{E'} \models \neg e_i \iff c_{E''} \models \neg \{v_1, v_2\} \land \bigwedge_{1 \leq j \leq k} \neg e_{ij}.$$ 

Now in any clause $(-e_i \lor f_1 \lor f_2)$ where $f_1$ and $f_2$ are literals of bound variables of $\psi$ we can replace $-e_i$ by $\bigwedge_{1 \leq j \leq k} \neg e_{ij}$.

The resulting formula is equivalent to

$$\bigwedge_{1 \leq j \leq k} (\neg e_{ij} \lor f_1 \lor f_2),$$

which is of the desired form. Similarly we can replace any clause of the form $(-e_{i1} \lor -e_{i2} \lor f_1)$ by $\bigwedge_{1 \leq j \leq k} \neg e_{i1,j} \lor \bigwedge_{1 \leq j \leq k} \neg e_{i2,j} \lor f_1$, resulting in the formula

$$\bigwedge_{(j_1,j_2) \in \{1,\ldots,k\}^2} (\neg e_{i1,j_1} \lor \neg e_{i2,j_2} \lor f_1).$$

Analogously we can handle clauses of the form $(-e_{i1} \lor -e_{i2} \lor -e_{i3})$.

Let $\psi'$ be $\psi$ after applying the above changes. We have that any matching $E''$ of $G_k$ corresponds to a matching of $E'$ of $G$ with $c_{E'} \models \psi$ if and only if $c_{E''} \models \psi'$. Now, we can proceed as in [30]: Let $A'_r$ be the number of matchings $E'$ of $G$ of size $|V_1| - r$ with $c_{E'} \models \psi$. Then $G_k$ has exactly $\sum_{r=0}^{k} A'_r \cdot (k + 1)^r$ matchings $E''$ with $c_{E''} \models \psi'$. Using the number of such matchings for all graphs $G_k$ we get a system of linear equations allowing us to compute $A'_r$, the number of perfect matchings $E'$ of $G$ with $c_{E'} \models \psi$.

$$\text{#}(\text{IMPERFECT-MATCHING}, \Sigma_1 \cdot 3\text{-CNF}) \leq \text{#}(2\text{-CNF}^-, \Sigma_1 \cdot 3\text{-CNF}^-):$$

Let $G = (V_1, V_2, E)$ be a bipartite graph with $E = \{e_1, \ldots, e_n\}$ and $\psi(e_1, \ldots, e_n) \in \Sigma_1 \cdot 3\text{-CNF}^-$. The reduction works completely analogously to the proof by Valiant: We define a 2-CNF formula $\varphi(e_1, \ldots, e_n)$ expressing that each node of the graph is only matched once as:

$$\varphi(e_1, \ldots, e_n) = \bigvee_{(e_1,e_2) \in E \times E} \neg e_1 \lor -e_2,$$

$$e_1 \neq e_2 \text{ and } e_1 \lor e_2 \neq \emptyset.$$

Then

$$\text{#}(\text{IMPERFECT-MATCHING}, \Sigma_1 \cdot 3\text{-CNF}^-)(G, \psi) = \text{#}(2\text{-CNF}^-, \Sigma_1 \cdot 3\text{-CNF}^-)(\varphi, \psi).$$

### VI. CONCLUSION

We have studied in this paper the following hierarchy of classes defined by counting problems for team-based logics:

$$\text{#}^{\text{FO}(\bot)^\text{team}} \subseteq \text{#}^{\text{FO}(\bot)^\text{team}} \subseteq \text{#}^{\text{FO}(\bot)^\text{team}} \subseteq \text{SO}.$$ 

In this hierarchy, the inclusion of $\text{#}^{\text{FO}(\bot)^\text{team}}$ in $\text{#}^{\text{FO}(\bot)^\text{team}}$ is strict unless $P = NP$. We have also shown that $\text{#}^{\text{FO}(\bot)^\text{team}}$ and $\text{#}^{\text{FO}(\bot)^\text{team}}$ contain a complete problem from their respective superclass. The latter problem is complete for $\text{#}^{\text{FO}(\bot)^\text{team}}$ even under parsimonious reductions.

We end the paper by stating some open problems and pointing out further directions of our work.

The connection between $\text{#}^{\text{FO}(\bot)^\text{team}}$ and the classes $P$ and $\#NP$ is not yet clear. While we know that a complete problem from $\#NP$ is contained, it is open whether the class...
This question is in a similar vein to results from [27] where with team semantics.

We conjecture that the answer to both questions is negative, since the defining logic has closure properties that make it unlikely to contain counting versions of non-monotone problems from #P. Since we do not know whether $\text{FO}(\{\ldots\})^{\text{team}}$ is closed under parsimonious reductions the completeness of $\Sigma_1\text{-CNF}^-$ with respect to suitable logical reductions would be a step forward. Besides [13] for inclusion logic, there has not been a systematic study of logical reductions for logics with team semantics.

Regarding $\text{FO}(\subseteq)^{\text{team}}$, the search for a complete problem could be interesting, since we only showed that the problem $\#2\text{-CNF}^+$ is contained in this class and the problem $\#\Sigma_1\text{-DualHorn}$ is hard for this class, but neither of the problems is known to be complete. Another interesting class from counting complexity is TOTP [26]. While it can easily be seen that this class contains $\text{FO}(\subseteq)^{\text{team}}$, another possible direction for further study is to investigate whether these classes actually coincide.

The lower end of our hierarchy deserves further study. The class $\#\text{FO}^{\text{team}}$, i.e., the class not containing any dependency atom at all, can be shown to be a subclass of TC$^0$, the class of functions computable by families of polynomial size constant depth majority circuits (see [32]). The circuit-based counting class $\#AC^0$, counting proof trees in polynomial size constant depth unbounded fan-in circuits [32], was characterized in a model-theoretic manner by counting assignments to free function symbols in certain quantifier-restricted FO formulae [7]. A similar quantifier restriction for $\text{FO}(A)^{\text{team}}$, where $A$ consists of the dependency atom plus a totality atom (not further studied in the present paper), also leads to a characterization of $\#AC^0$. This suggests that low level counting classes and circuit classes in the context of counting problems for team-based logics might be worth studying. Also, the question arises which generalized dependency atoms lead to interesting relations to complexity classes; besides the aforementioned totality atom, the constancy or the exclusion [11] atom should be examined. In particular, it is an open question to find an atom $A$ such that $\text{FO}(A)^{\text{team}} = \#P$ as the logic $L$ of [23] is not of this form.

From an efficiency point of view, the question which of the presumably hard counting classes allows an approximation scheme or randomized approximation scheme is interesting. This question is in a similar vein to results from [27] where subclasses of #P with efficient approximation schemes were identified.

Our proof of the completeness of $\Sigma_1\text{-CNF}^-$ for #NP introduces problems that arise from “pairing decision problems” and giving simultaneous reductions between such pairs. This idea might be helpful in other contexts as well; in particular it should lead to more interesting complete problems for #NP or higher levels $\#\Sigma_k$ of the counting polynomial-time hierarchy.

**References**

[1] Marcelo Arenas, Martin Muñoz, and Cristian Riveros. Descriptive complexity for counting complexity classes. In LICS, pages 1–12. IEEE Computer Society, 2017.

[2] Rehan Abdul Aziz, Geoffrey Chu, Christian J. Muisse, and Peter J. Stuckey. #3SAT: Projected model counting. In SAT, volume 9340 of Lecture Notes in Computer Science, pages 121–137. Springer, 2015.

[3] Fahiem Bacchus, Shannon Dalmao, and Toniann Pitassi. Solving #SAT and Bayesian inference with backtracking search. CoRR, abs/1401.3458, 2014.

[4] Stephen A. Cook. The complexity of theorem-proving procedures. In STOC, pages 151–158. ACM, 1971.

[5] Jukka Corander, Antti Hyytinne, Juha Kontinen, Johan Pensar, and Jouko Väänänen. A logical approach to context-specific independence. In WoLLIC, volume 9803 of Lecture Notes in Computer Science, pages 165–182. Springer, 2016.

[6] Arnaud Durand, Johannes Ebbing, Juha Kontinen, and Heribert Vollmer. Dependency logic with a majority quantifier. Journal of Logic, Language and Information, 24(3):289–305, 2015. URL: https://doi.org/10.1007/s10849-015-9218-3
doi:10.1007/s10849-015-9218-3

[7] Arnaud Durand, Anselm Haak, Juha Kontinen, and Heribert Vollmer. Descriptive complexity of #AC$^0$ functions. In CSL, volume 62 of LIPIcs, pages 20:1–20:16. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

[8] Arnaud Durand, Anselm Haak, and Heribert Vollmer. Model-theoretic characterization of Boolean and arithmetic circuit classes of small depth. In LICS, pages 354–363. ACM, 2018.

[9] Arnaud Durand, Juha Kontinen, Nicolas de Rugy-Altherre, and Jouko Väänänen. Tractability frontier of data complexity in team semantics. In Gandalf, volume 193 of EPTCS, pages 73–85, 2015.

[10] Ronald Fagin. Generalized first-order spectra, and polynomial time recognizable sets. SIAM-AMS Proceedings, 7:43–73, 1974.

[11] Pietro Galliani. Inclusion and exclusion dependencies in team semantics - on some logics of imperfect information. Ann. Pure Appl. Logic, 163(1):68–84, 2012.

[12] Pietro Galliani and Lauri Hella. Inclusion logic and fixed-point logic. In CSL, volume 23 of LIPIcs, pages 281–295. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013.

[13] Erich Grädel. Games for inclusion logic and fixed-point logic. In Samson Abramsky, Juha Kontinen, Jouko Väänänen, and Heribert Vollmer, editors, Dependence Logic: Theory and Applications, pages 73–98. Springer International Publishing, Cham, 2016. URL: http://dx.doi.org/10.1007/978-3-319-31803-5_5
doi:10.1007/978-3-319-31803-5_5

[14] Erich Grädel and Jouko A. Väänänen. Dependence and independence. Studia Logica, 101(2):399–410, 2013.

[15] Mikula Hannula and Juha Kontinen. A finite axiomatization of conditional independence and inclusion dependencies. Inf. Comput., 249:121–137, 2016.

[16] Lane A. Hemaspaandra and Heribert Vollmer. The satanic notations: counting classes beyond #P and other definitional adventures. SIGACT News, 26(1):2–13, 1995.

[17] Neil Immerman. Relational queries computable in polynomial time. Information and Control, 68(1-3):86–104, 1986.

[18] Neil Immerman. Descriptive Complexity. Graduate texts in computer science. Springer, 1999.

[19] Juha Kontinen. A logical characterization of the counting hierarchy. ACM Trans. Comput. Log., 10(1):1:1–1:21, 2009.

[20] Juha Kontinen, Antti Kuusisto, and Jonni Virtema. Decidability of predicate logics with team semantics. In MFCS, volume 58 of LIPIcs, pages 60:1–60:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

[21] Juha Kontinen and Hannu Niemistö. Extensions of MSO and the monadic counting hierarchy. Inf. Comput., 209(1):1–19, 2011.

[22] Juha Kontinen and Jouko A. Väänänen. On definability in dependence logic. Journal of Logic, Language and Information, 18(3):317–332, 2009.

[23] Juha Kontinen and Fan Yang. First-order definable team properties. In preparation, 2018.

[24] Antti Kuusisto. A double team semantics for generalized quantifiers. Journal of Logic, Language and Information, 24(2):149–191, 2015.
[25] Antti Kuusisto and Carsten Lutz. Weighted model counting beyond two-variable logic. In LICS, pages 619–628. ACM, 2018.

[26] Aris Pagourtzis and Stathis Zachos. The complexity of counting functions with easy decision version. In MFCS, volume 4162 of Lecture Notes in Computer Science, pages 741–752. Springer, 2006.

[27] Sanjeev Saluja, K. V. Subrahmanyam, and Madhukar N. Thakur. Descriptive complexity of #P functions. J. Comput. Syst. Sci., 50(3):493–505, 1995.

[28] Jouko A. Väänänen. Dependence Logic - A New Approach to Independence Friendly Logic, volume 70 of London Mathematical Society student texts. Cambridge University Press, 2007.

[29] Leslie G. Valiant. The complexity of computing the permanent. Theor. Comput. Sci., 8:189–201, 1979.

[30] Leslie G. Valiant. The complexity of enumeration and reliability problems. SIAM J. Comput., 8(3):410–421, 1979.

[31] Moshe Y. Vardi. The complexity of relational query languages (extended abstract). In STOC, pages 137–146. ACM, 1982.

[32] Heribert Vollmer. Introduction to Circuit Complexity - A Uniform Approach. Texts in Theoretical Computer Science. An EATCS Series. Springer, 1999.