UPPER AND LOWER BOUNDS FOR HIGHER MOMENTS OF Theta FUNCTIONS

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Abstract. We obtain optimal lower bounds for moments of theta functions. On the other hand, we also get new upper bounds on individual theta values and moments of theta functions on average over primes. The upper bounds are based on bounds of character sums and in particular on a modification of some recent results of M. Z. Garaev.

1. Introduction

1.1. Background. For a prime p, we denote by $X_p$ the group of multiplicative characters modulo p (we refer to [16] for a background on characters). Denote by $X^+_p$ the subgroup of $X_p$ of order $(p-1)/2$ consisting of even characters $\chi$ (those satisfying $\chi(-1) = 1$) and $X^-_p$ the subset of $X_p$ consisting of odd characters $\chi$ (those satisfying $\chi(-1) = -1$). Furthermore, we use $X^*_p$ to denote the set of nonprincipal characters modulo p.

For real $x > 0$ and $\eta \in \{0, 1\}$ we set

$$\Theta_p(\eta, x, \chi) = \sum_{n=1}^{\infty} \chi(n) n^\eta e^{-\pi n^2 x/p}, \quad \chi \in X_p.$$ 

We note that, if we set $\eta_\chi = 1$ if $\chi$ is odd and $\eta_\chi = 0$ otherwise, then

$$\Theta_p(\eta_\chi, x, \chi) = \vartheta_p(x, \chi)$$

is the classical theta-function of the character $\chi$, see [8] for a background and basic properties.

When computing the root number of $\chi$ appearing in the functional equation of the associated Dirichlet $L$-function, the question of whether $\vartheta_p(1, \chi) \neq 0$ appears naturally (see [17] for details). Numerical computations lead to the conjecture that it never happens if $\chi$ is primitive.

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(see [7] for a counterexample with $\chi$ unprimitive). An algebraic approach based on class field theory, introduced in [3], allows to prove partial results when $p = 2l + 1$ with $p$ and $l$ primes. Nevertheless, it does not give any results in the general case. Thus, a standard way to handle this type of problems analytically is to average over families of characters. As a consequence, we can deduce that the conjecture should be true for a good proportion of characters. The study of moments of theta functions has been initiated in [18], [19], [17] and several conjectures have been stated in [21]. The aim of that paper is to pursue the analytical investigation about theta values.

The paper is divided into two main parts. Firstly, we obtain optimal lower bounds for moments of theta functions. Secondly, we consider the two related problems of giving upper bounds for theta-values individually and on average.

A standard problem in analytic number theory is to study moments of $L$-functions at their central point $s = 1/2$. It is conjectured (see [23, Chapter 5]) that the moments satisfy the following asymptotic formula:

$$M_{2k}(p) = \sum_{\chi \in \chi_p} |L(1/2, \chi)|^{2k} \sim C_k p \log^k p, \quad C_k > 0.$$  

The conjecture (1.1) holds for $k = 1$ (see [24, Remark 3] and [2, Theorem 3], or [13] for a more precise asymptotic expansion), and $k = 2$ (see [14]). Although there are numerical evidence and theoretical reasons sustaining this conjecture, it remains open for $k \geq 3$.

However lower bounds of the expected order of magnitude

$$\sum_{\chi \in \chi_p} |L(1/2, \chi)|^{2k} \gg p \log^k p,$$

have been given by Rudnick and Soundararajan [25].

In a similar way, the moments of theta functions are defined in [18] as follows:

$$S_{2k}^+(p) = \sum_{\chi \in \chi_p^+} |\vartheta(1, \chi)|^{2k} \quad \text{and} \quad S_{2k}^-(p) = \sum_{\chi \in \chi_p^-} |\vartheta(1, \chi)|^{2k}.$$  

It is shown in [18] that

$$S_2^+(p) \sim \frac{p^{3/2}}{4\sqrt{2}}, \quad S_2^-(p) \sim \frac{p^{5/2}}{16\pi \sqrt{2}},$$  

$$S_4^+(p) \sim \frac{3p^2\log p}{16\pi}, \quad S_4^-(p) \sim \frac{3p^4 \log p}{512\pi^3}.$$  

Note that the proof of the asymptotic formulas for the fourth moments in (1.2) is using some ideas of [1].
For higher moments, the trivial character gives the main contribution and it is shown in [21] that

\[ S_{2k}^+(p) \sim c_k p^k, \quad k \geq 3, \]

where \( c_k > 0 \) is a constant. Therefore, it is interesting to pull out the trivial character from the summation and define

\[ T_{2k}^+(p) = \sum_{\chi \in X_p \setminus \chi_0} |\vartheta_p(1, \chi)|^{2k}. \]

We conjecture, based on numerical computation and some theoretical support, that

\[ T_{2k}^+(p) \sim a_k p^{k/2 + 1} (\log p)^{(k-1)^2}, \]

\[ S_{2k}^-(p) \sim b_k p^{3k/2 + 1} (\log p)^{(k-1)^2}, \]

for some positive constants \( a_k \) and \( b_k \), depending only on \( k \).

Indeed, this can be related to recent results of [11] (see also [12]), where the authors obtain the asymptotic behaviour of a Steinhaus random multiplicative function (basically a multiplicative random variable whose values at prime integers are uniformly distributed on the complex unit circle). This can be viewed as a random model for \( \vartheta_p(x, \chi) \).

In fact, the rapidly decaying factor \( e^{-\pi n^2/p} \) is mostly equivalent to restrict the sum over integers \( n \leq n_0(p) \) for some \( n_0(p) \approx \sqrt{p} \) and the averaging behavior of \( \chi(n) \) with \( n \ll p^{1/2} \) is essentially similar to that of a Steinhaus random multiplicative function. Hence, these results are a good support for conjecture (1.3). We obtain results that confirm this heuristic.

1.2. Our results. We begin with a lower bound of the right order of magnitude, which may be compared to the results obtained for \( L \)-functions by Rudnick and Soundararajan [25, 26].

**Theorem 1.1.** For any fixed integer \( k \geq 1 \), we have

\[ T_{2k}^+(p) \gg p^{1+k/2} \log^{(k-1)^2} p \quad \text{and} \quad S_{2k}^-(p) \gg p^{1+3k/2} (\log p)^{(k-1)^2}. \]

The proof of Theorem 1.6 is given in Section 2.5. Under the assumption of the Generalized Riemann Hypothesis, Munsch [22] obtains the following upper bounds

\[ T_{2k}^+(p) \leq p^{1+k/2} \log^{(k-1)^2+o(1)} p \]

\[ S_{2k}^-(p) \ll p^{1+3k/2} (\log p)^{(k-1)^2+o(1)}. \]

This greatly strengthens our belief in the conjectural asymptotic (1.3).
Even though unconditionally we are far from getting upper bounds of the expected order, we can obtain non trivial upper bounds for almost all primes and also on average over primes if $3 \leq k \leq 6$.

It has been shown in [21] that for any nonprincipal character $\chi$ modulo $p$, the bound
\begin{equation}
|\vartheta_p(1, \chi)| \leq p^{\eta/2+7/16+o(1)}
\end{equation}
holds as $p \to \infty$. The same approach also applies to the more general sums $\Theta_p(\eta, x, \chi)$ for any $\eta \in \{0, 1\}$.

We begin by some improvements for bounds of individual values of theta functions. We use a result of Garaev [9] to improve the bound (1.4) for almost all primes $p$.

**Theorem 1.2.** Let $X \geq 1$ be a sufficient large real number. For any $\eta \in \{0, 1\}$ we have
\[\sum_{p \leq X} \max_{\chi \in X_p^*} |\Theta_p(\eta, 1, \chi)|^8 \leq X^{4\eta+4+o(1)}.
\]

Thus, we immediately derive:

**Corollary 1.3.** Let $X \geq 1$ be a sufficient large real number. For all but $o(X/\log X)$ primes $p \leq X$, for any $\chi \in X_p$ and $\eta \in \{0, 1\}$ we have
\[|\Theta_p(\eta, 1, \chi)| \leq p^{\eta/2+3/8+o(1)}.
\]

Combining Theorem 1.2 with the bounds (1.2), we obtain:

**Theorem 1.4.** Let $X \geq 1$ be a sufficient large real number. For any fixed integer $k$ with $6 \geq k \geq 3$, we have
\[\sum_{p \leq X} T_{2k}^+(p) \leq X^{3k/4+3/2+o(1)} \quad \text{and} \quad \sum_{p \leq X} S_{2k}^- (p) \leq X^{7k/4+3/2+o(1)}.
\]

Finally, for almost all primes $p$, we have nontrivial estimates for arbitrary even moments.

**Theorem 1.5.** Let $X \geq 1$ be a sufficient large real number. For all but $o(X/\log X)$ primes $p \leq X$, and any fixed integer $k \geq 1$, we have
\[T_{2k}^+(p) \leq p^{3k/4+1/2+o(1)} \quad \text{and} \quad S_{2k}^- (p) \leq p^{7k/4+1/2+o(1)}.
\]

Theorems 1.2, 1.4 and 1.5 are proven in Sections 4.1, 4.2 and 4.3, respectively.

As part of our main tools, we use various bounds on the character sums
\begin{equation}
S_p(\chi; t) = \sum_{n \leq t} \chi(n).
\end{equation}
which we define for $\chi \in \mathcal{X}_p$ and a real $t$. As an application of our approach, we also obtain a lower bound on moments of the general Dirichlet polynomials

$$(1.6) \quad \Xi_p(\chi; t) = \sum_{n \leq t} \xi_n \chi(n)$$

with some real coefficient $\xi_n$ that are bounded away from zero.

**Theorem 1.6.** For $1 \leq t < p$ and arbitrary coefficients $\xi_n \gg 1$, $n = 1, 2, \ldots$, we have

$$\sum_{\chi \in \mathcal{X}_p \setminus \chi_0} |\Xi_p(\chi; t)|^{2k} \gg pt^{k/2} \log^{(k-1)^2} p.$$ 

The proof of Theorem 1.6 is given in Section 2.6, immediately after the proof of Theorem 1.1 as it uses very similar ideas.

2. Lower Bounds

2.1. Background on the Riemann zeta-function. First we recall the well known Euler formula

$$(2.1) \quad \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s)$$

for the Riemann zeta-function $\zeta(s)$ where the product is taken over all primes, that holds for any complex $s$ with $\Re s > 1$, see [16, Equation (1.12)].

We also need the following fact about the analytic properties of $\zeta(s)$.

**Lemma 2.1.** For any complex $s = \sigma + it$, with $|\Im s| = |t| \geq 2$ and $1/2 \leq \Re s = \sigma < 1$, we have

$$\zeta(s) \ll |\tau|^{c(1-\sigma)^{3/2}} \log |\tau| \quad \text{and} \quad \zeta(1+it) \ll \log^{2/3} |t|,$$

for some absolute constant $c > 0$.

**Proof.** See [16, Theorem 8.27] and [16, Corollary 8.28], respectively. \(\square\)

The following consequence is important in verifying the assumption of [4, Theorem 1], which is our main tool.

**Corollary 2.2.** The function $(s - 1)\zeta(s)$ verifies [4, Equation (1.6)] in the range $\Re(s) > 1/2$.

**Proof.** For $|\Im(s)| \geq 2$, this is a direct consequence of Lemma 2.1 and the fact that for $\Re(s) > 1$, the function $(s - 1)\zeta(s)$ is bounded. In the bounded domain $|\Im(s)| \leq 2$, the function $s\zeta(1-s)$ is holomorphic thus bounded. The conclusion follows easily. \(\square\)
2.2. **Bounds for the restricted divisor function.** The strategy behind the proof of Theorem 1.1 is to ”mollify” theta moments by a short character sum. For that purpose, we need to have good estimates for sums of restricted divisor function. To do this, we employ some results of de la Bretèche [4] on sums of arithmetical functions of many variables. These type of sums appears naturally when we count integer points of bounded height on some varieties. This has been used for example in [5, 6] to prove Manin’s conjecture in some special cases.

**Lemma 2.3.** For any integer \( k \geq 2 \) and any real positive \( \gamma_i \leq 1, i = 1, \ldots, k \), there exists a constant \( \Gamma_k > 0 \) such that

\[
\sum_{a_i, b_i \leq T^{\gamma_i}, i=1, \ldots, k} a_1 \cdots a_k \sim \Gamma_k T^{\gamma} \log^{(k-1)^2} T.
\]

where \( \gamma = \gamma_1 + \ldots + \gamma_k \).

**Proof.** We make use of [4, Theorems 1 and 2] and complete the proof in the following three steps.

**Step 1.** First we prove that Assumption P1 of [4, Theorem 1] is satisfied with

\[
\alpha = \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \in \mathbb{R}^{2k}.
\]

For \( 2k \) positive integers \( (m_1, \ldots, m_{2k}) \) we set \( f(m_1, \ldots, m_{2k}) = 1 \) if

\[
m_1 \cdots m_k = m_{k+1} \cdots m_{2k},
\]

and \( f(m_1, \ldots, m_{2k}) = 0 \), otherwise. We see that \( f \) is multiplicative, that is,

\[
f(m_1 n_1, \ldots, m_{2k} n_{2k}) = f(m_1, \ldots, m_{2k}) f(n_1, \ldots, n_{2k})
\]

whenever \( \gcd(m_1 \cdots m_{2k}, n_1 \cdots n_{2k}) = 1 \).

For a vector \( s = (s_1, \ldots, s_{2k}) \in \mathbb{C}^{2k} \) of \( 2k \) complex numbers, we define the multiple Dirichlet series

\[
F(s) = \sum_{m_1, \ldots, m_{2k} \geq 1} f(m_1, \ldots, m_{2k}) \frac{m_1^{s_1} \cdots m_{2k}^{s_{2k}}}{m_1^{s_1} \cdots m_{2k}^{s_{2k}}},
\]

Let \( d_k(m) \) be the number of ways of writing a positive integer \( m \geq 1 \) as a product of \( k \) positive integers. Since

\[
|m_1^{s_1} \cdots m_{2k}^{s_{2k}}| \leq (m_1 \cdots m_{2k})^{\sigma(s)},
\]

where

\[
\sigma(s) = \min\{\Re s_j : 1 \leq j \leq 2k\},
\]

(2.4)
we have

\[ \sum_{m_1, \ldots, m_k \geq 1} \left| \frac{f(m_1, \ldots, m_{2k})}{m_1^{s_1} \cdots m_{2k}^{s_{2k}}} \right| \leq \sum_{m \geq 1} \frac{(d_k(m))^2}{m^{2\sigma(s)}} = \prod_{p \geq 2} \left( \sum_{a \geq 0} \frac{(a+k-1)^2}{p^{2a\sigma(s)}} \right), \]

which proves the absolute convergence of \( F(s) \) in the range \( \sigma(s) > 1/2 \) and verifies Assumption P1 of [4, Theorem 1] for \( \alpha \) given by (2.3).

**Step 2.** Let us recall some notations used in [4]. We denote by \( \mathcal{L}_{2k}(\mathbb{C}) \) the space of linear forms

\[ \ell(X_1, \ldots, X_{2k}) \in \mathbb{C}[X_1, \ldots, X_{2k}] \]

Following [4], we denote by \( e_j, j = 1 \ldots 2k, \) the canonical basis of \( \mathbb{C}^{2k} \) and \( \{ e_j^* \}_{j=1}^{2k} \) the dual basis in \( \mathcal{L}_{2k}(\mathbb{C}) \). Thus, in our case the linear form \( e_j^* \) are explicitly given by

\[ e_j^*(X_1, \ldots, X_{2k}) = X_j, \quad j = 1 \ldots 2k. \]

We now prove that Assumptions P2 and P3 of [4, Theorem 1] are satisfied with the \( n = k^2 \) linear forms

\[ \ell^{(a,b)} = e_a^* + e_{k+b}^* = X_a + X_{k+b}, \quad 1 \leq a, b \leq k \]

(here there is no linear forms \( h^{(r)} \), in other words \( \mathcal{R} \) is empty in our version of the Assumption P2 of [4, Theorem 1]). Since \( f \) is multiplicative, in this range, we have (where \( p \geq 2 \) runs over the prime numbers):

(2.5) \[ F(s) = \prod_p F_p(s), \]

with

\[ F_p(s) = \sum_{r_1, \ldots, r_{2k} \geq 0} \frac{f(p^{r_1}, \ldots, p^{r_{2k}})}{p^{r_1s_1 + \cdots + r_{2k}s_{2k}}} = \sum_{r_1, \ldots, r_{2k} \geq 0} \frac{1}{p^{r_1s_1 + \cdots + r_{2k}s_{2k}}}. \]

Now,

\[ F_p(s) = 1 + \sum_{a=1}^{k} \sum_{b=k+1}^{2k} \frac{1}{p^{a+b}} + \sum_{r_1, \ldots, r_{2k} \geq 0} \frac{1}{p^{r_1s_1 + \cdots + r_{2k}s_{2k}}}. \]

and, with \( \sigma(s) > 0 \), where \( \sigma(s) \) is given by (2.4), the absolute value of the third term of the right hand side of the above equality is bounded
by
\[
\sum \cdots \sum \frac{1}{p^{(r_1 + \cdots + r_{2k})\sigma(s)}} = \sum_{r \geq 2} \left( \frac{r + k - 1}{k - 1} \right)^2 \frac{1}{p^{2r\sigma(s)}} \ll \sum_{r \geq 2} \frac{r^{2k}}{p^{2r\sigma(s)}}.
\]

Hence
\[
F_p(s) = 1 + \sum_{a=1}^{k} \sum_{b=k+1}^{2k} \frac{1}{p^{s_a+s_b}} + O_A \left( \frac{1}{p^{4\sigma(s)}} \right)
\]
(where the constants in these $O_A$ depend on $A > 0$). Furthermore, for a given $A > 0$ and for $\sigma(s) \geq A$ we have
\[
\prod_{a=1}^{k} \prod_{b=k+1}^{2k} \left( 1 - \frac{1}{p^{s_a+s_b}} \right) = 1 - \sum_{a=1}^{k} \sum_{b=k+1}^{2k} \frac{1}{p^{s_a+s_b}} + O_A \left( \frac{1}{p^{4\sigma(s)}} \right).
\]

Therefore, we see that
\[
(2.6) \quad F_p(s) \prod_{a=1}^{k} \prod_{b=k+1}^{2k} \left( 1 - \frac{1}{p^{s_a+s_b}} \right) = 1 + O_A \left( \frac{1}{p^{4\sigma(s)}} \right).
\]

Taking the product over all primes and using the Euler formula (2.1), we obtain from (2.5) and (2.6) that for $\sigma(s) > 1$ we have
\[
(2.7) \quad F(s) = \psi(s) \prod_{a=1}^{k} \prod_{b=k+1}^{2k} \zeta(s_a+s_b),
\]

where $\psi(s)$ is a holomorphic function for $\sigma(s) \geq A$ for any fixed $A > 1/4$.

We now writing (2.7) as
\[
(2.8) \quad F(s) \prod_{a=1}^{k} \prod_{b=k+1}^{2k} (s_a + s_b - 1) = \psi(s) \prod_{a=1}^{k} \prod_{b=k+1}^{2k} (s_a + s_b - 1) \zeta(s_a+s_b).
\]

Recalling Corollary 2.2 we conclude that the left hand side of (2.8) verifies [4, Equation (1.6)] in the range $\sigma(s) \geq A$, for any $A > 1/4$. Translating each coordinate by $1/2$, we see that
\[
H(s) = F(s + \alpha) \prod_{a=1}^{k} \prod_{b=k+1}^{2k} (s_a + s_b)
\]
verifies [4, Equation (1.6)] in the range \( \sigma(s) \geq B \) for any \( B = A - 1/2 > -1/4 \). Hence, Assumptions P2 and P3 of [4, Theorem 1] are satisfied for \( H(s) \).

**Step 3.** It is easy to verify the last Assumption P4 of [4, Theorem 1]. To conclude, we need a stronger version of [4, Theorem 1] which gives the exact power of \( \log x \) in the asymptotic (2.2). Under the hypothesis of [4, Theorem 1], we show that the extra condition (iv) of [4, Theorem 2] is satisfied with

\[
\beta = (1, \ldots, 1) \in \mathbb{R}^{2k}.
\]

We start with the inequality

\[
H(0, \ldots, 0) = \prod_{p \geq 2} \left( 1 - \frac{1}{p} \right)^{k^2} \left( 1 + \frac{k^2}{p} + \sum_{r \geq 2} \left( \frac{k}{r} + 1 \right)^{2} \frac{1}{p^r} \right) > 0.
\]

Now, we are able to conclude the proof. With the notations of Step 2 above, we clearly have that the linear form \( e_1^* + \cdots + e_{2k}^* \) lies in the positive convex cone of the linear forms \( f(a,b) \). Furthermore we have the equality \( m = \text{rank}(\{ f(a,b), \ 1 \leq a \leq k, \ k + 1 \leq b \leq 2k \}) = 2k - 1 \) and the result follows. \( \square \)

Clearly, the constant \( \Gamma_k \) of Lemma 2.3 can be evaluated explicitly.

### 2.3. Moments of weighted character sums.

Assume we are give some sequences \( \xi_n \) of positive real numbers with \( \xi_n \gg 1 \). We consider the Dirichlet polynomials (1.6). Furthermore, we fix some \( \varepsilon > 0 \), set \( x = p^\varepsilon \), and define

\[
A_\varepsilon(\chi) = S_\rho(x; \chi).
\]

We consider the following two sums

\[
(2.10) \quad \Sigma_1 = \sum_{\chi \in \chi_p^+ \setminus \chi_0} |\Xi(\chi; t)|^2 |A_\varepsilon(\chi)|^{2k-2} \quad \text{and} \quad \Sigma_2 = \sum_{\chi \in \chi_p^+} |A_\varepsilon(\chi)|^{2k}.
\]

**Lemma 2.4.** Let \( t = p^\tau \) with some fixed \( \tau \in (0, 1) \). For any integer \( k \geq 2 \) and a sufficiently small \( \varepsilon > 0 \), there exists a constant \( C(\varepsilon, k) > 0 \) depending only on \( \varepsilon \) and \( k \), such that:

\[
\Sigma_1 \gg p^{1+\varepsilon(k-1)} t \log^{(k-1)^2} p \quad \text{and} \quad \Sigma_2 \sim C(\varepsilon, k) p^{1+\varepsilon k} \log^{(k-1)^2} p.
\]
Proof. We start with proving asymptotic formula for $\Sigma_2$ as the proof is shorter. Using the orthogonality relations for even characters, we obtain:

$$\Sigma_2 = \frac{(p-1)}{2} \sum_{a_1,b_1,\ldots,a_k,b_k \leq x, \ a_1\cdots a_k \equiv \pm b_1\cdots b_k \pmod{p}} 1.$$ 

Choosing $\varepsilon < 1/k$ to ensure that $p^{\varepsilon k} < p$, we see the only possible solutions to $a_1 \cdots a_k \equiv \pm b_1 \cdots b_k \pmod{p}$ are those with $a_1 \cdots a_k = b_1 \cdots b_k$. Hence

$$\Sigma_2 = \frac{(p-1)}{2} \sum_{a_1,b_1,\ldots,a_k,b_k \leq x, \ a_1\cdots a_k = b_1\cdots b_k} 1$$

and we conclude using Lemma 2.3 with $T = p^\varepsilon = x$ and $\gamma_1 = \ldots = \gamma_k = 1$.

We complete the sum $\Sigma_1$ including the trivial character and bound its contribution trivially by

$$|\Xi(\chi_0; t)|^2 |A_\varepsilon(\chi_0)|^{2k-2} = O \left( t^2 p^{2\varepsilon(k-1)} \right).$$

Thus

$$\Sigma_1 = \sum_{\chi \in X^+ / \chi_0} |\Xi(\chi; t)|^2 |A_\varepsilon(\chi)|^{2k-2} + O \left( t^2 p^{2\varepsilon(k-1)} \right).$$

Using the orthogonality of multiplicative characters again, we derive

(2.11) $$\Sigma_1 = \frac{(p-1)}{2} \sum_{a,b \leq t, \ aa_1\cdots a_{k-1} = \pm b b_1\cdots b_{k-1} \pmod{q}} \xi_a \xi_b + O \left( t^2 p^{2\varepsilon(k-1)} \right).$$

Using Lemma 2.3 with $T = p$, $\gamma_1 = \tau$ and $\gamma_i = \varepsilon$ for $2 \leq i \leq k$, and taking a sufficiently small $\varepsilon > 0$, we conclude the proof.

2.4. Moments of the theta function. We define $A_\varepsilon(\chi)$ as in Section 2.3 and consider the following

(2.12) $$\mathcal{G} = \sum_{\chi \in X^h / \chi_0} |\vartheta(1, \chi)|^2 |A_\varepsilon(\chi)|^{2k-2}$$

(with the above choice $x = p^\varepsilon$).

We use the following approximation of $\vartheta(1, \chi)$ by a truncated sum, which easily follows from estimating the tail via the corresponding integral.
Lemma 2.5. Let $\delta > 0$ be a positive number. Then
\[
\vartheta(1, \chi) = \sum_{n \leq p^{1/2+\delta}} \chi(n)e^{-\pi n^2/p} + O(p^{1/2}e^{-p^{\delta}}).
\]

The proof of Theorem 1.1 is derived from the following moment estimate, which in turn follows from Lemmas 2.4 and 2.5.

Lemma 2.6. For any integer $k \geq 2$ and a sufficiently small $\varepsilon > 0$, there exists a constant $C(\varepsilon, k)$ depending only on $\varepsilon$ and $k$, such that:
\[
\mathcal{S} \gg p^{3/2+\varepsilon(k-1)} \log^{(k-1)^2} p.
\]

Proof. Using the trivial bound
\[
\sum_{n \leq p^{2/3}} \chi(n)e^{-\pi n^2/p} \ll p^{2/3}
\]
by Lemma 2.5 (with $\delta = 1/6$) we have
\[
|\vartheta(1, \chi)|^2 = \left| \sum_{n \leq p^{2/3}} \chi(n)e^{-\pi n^2/p} \right|^2 + O\left(p^{5/6}e^{-p^{1/6}}\right).
\]

Therefore,
\[
\mathcal{S} = \sum_{\chi \in \mathcal{X}^+_p \setminus \chi_0} \left| \sum_{n \leq p^{2/3}} \chi(n)e^{-\pi n^2/p} \right|^2 |A_\varepsilon(\chi)|^{2k-2}
\]
\[
+ O\left(p^{5/6}e^{-p^{1/6}} \sum_{\chi \in \mathcal{X}^+_p \setminus \chi_0} |A_\varepsilon(\chi)|^{2k-2}\right).
\]

Then, by the orthogonality of multiplicative characters (or using the analogue of the asymptotic formula of Lemma 2.4 for $\Sigma_2$ with $k-1$ instead of $k$), we obtain
\[
\sum_{\chi \in \mathcal{X}^+_p \setminus \chi_0} |A_\varepsilon(\chi)|^{2k-2} \leq \sum_{\chi \in \mathcal{X}^+_p} |A_\varepsilon(\chi)|^{2k-2}
\]
\[
\ll px^{k-1} \log^{(k-2)^2} p = p^{1+\varepsilon(k-1)} \log^{(k-2)^2} p,
\]
which implies the asymptotic formula
\[
\mathcal{S} = \sum_{\chi \in \mathcal{X}^+_p \setminus \chi_0} \left| \sum_{n \leq p^{2/3}} \chi(n)e^{-\pi n^2/p} \right|^2 |A_\varepsilon(\chi)|^{2k-2} + O(1).
\]
We complete the sum $\mathcal{G}$ including the trivial character and bound its contribution trivially by

$$O \left( |\vartheta(1, \chi_0)|^2 x^{2k-2} \right) = O \left( px^{2k-2} \right) = O \left( p^{1+2\varepsilon(k-1)} \right).$$

Thus

$$\mathcal{G} = \sum_{\chi \in \chi_p^+} \left| \sum_{n \leq p^{2/3}} \chi(n) e^{-\pi n^2/p} \right|^2 |A_\varepsilon(\chi)|^{2k-2} + O \left( p^{1+2\varepsilon(k-1)} \right).$$

Using the orthogonality of multiplicative characters again, we derive

$$\mathcal{G} = \frac{(p-1)}{2} \sum_{a,b \leq p^{1/2}} \sum_{a_1,b_1,\ldots,a_{k-1},b_{k-1} \leq x} e^{-\pi(a^2+b^2)/p} + O \left( p^{1+2\varepsilon(k-1)} \right).$$

Hence, restricting the summation to $a, b \leq p^{1/2}$ and using that in this case $e^{-\pi(a^2+b^2)/p} \gg 1$ we obtain

$$\mathcal{G} \gg p \sum_{a,b \leq p^{1/2}} \sum_{a_1,b_1,\ldots,a_{k-1},b_{k-1} \leq x} 1 + O \left( p^{1+2\varepsilon(k-1)} \right).$$

We now recall the formula (2.11) and use Lemma 2.4 with $\tau = 1/2$. The result now follows. $\square$

2.5. **Proof of Theorem 1.1.** For the sums $\Sigma_2$ and $\mathcal{G}$ given by (2.10) and (2.12), with a sufficiently small $\varepsilon > 0$, by the Hölder inequality, we get

$$\mathcal{G}^k \leq \sum_{\chi \in \chi_p^+ \setminus \chi_0} |\vartheta(1, \chi)|^{2k}.$$

We now recall Lemma 2.4 that gives an upper bound $\Sigma_2$ and Lemma 2.6 that gives a lower bound on $\mathcal{G}$. This yields to the lower bound

$$\sum_{\chi \in \chi_p^+ \setminus \chi_0} |\vartheta(1, \chi)|^{2k} \gg p^{1+k/2} \log^{(k-1)^2} p.$$

The proof in the case of odd characters follows exactly along the same lines.

2.6. **Proof of Theorem 1.6.** We fix $k$ with $(k-1)^2 > A$ and a sufficiently small $\varepsilon > 0$. We then proceed as in the proof of Theorem 1.1 and obtain

$$\Sigma_1^k \leq \Sigma_2^{k-1} \sum_{\chi \in \chi_p^+ \setminus \chi_0} |\Xi_p(\chi; t)|^{2k}.$$
where \( \Sigma_1 \) and \( \Sigma_2 \) are given by (2.10). We now apply Lemma 2.4 and derive

\[
\sum_{\chi \in \mathcal{X}_p^* \setminus \chi_0} |\Xi_\rho(\chi; t)|^{2k} \gg pt^{k/2} \log^{(k-1)^2} p,
\]

which concludes the proof.

3. Bounds of character sums

3.1. Preliminaries. We extend the definitions of \( \mathcal{X}_p \) and \( \mathcal{X}_p^* \) to arbitrary integers \( k \geq 2 \) and use \( \mathcal{X}_k \) and \( \mathcal{X}_k^* \) to denote the sets of all characters and nonprincipal primitive characters modulo \( k \), respectively.

Similarly we defined \( S_k(\chi; t) \) by (1.5) for an arbitrary integer \( k \geq 2 \) and \( \chi \in \mathcal{X}_k \).

We estimate the sums \( S_k(\chi; t) \) given by (1.5) for almost all moduli \( k \) using the ideas of Garaev [9].

We now define the function \( e(z) = \exp(2\pi iz) \). We recall, that for any integer \( z \) and an odd integer \( Q = 2M + 1 \geq 1 \), we have the orthogonality relation

\[
\sum_{b=-M}^{M} e(bz/Q) = \begin{cases} Q, & \text{if } z \equiv 0 \pmod{Q}, \\ 0, & \text{if } z \not\equiv 0 \pmod{Q}, \end{cases}
\]

see [16, Section 3.1].

Furthermore, we also need the bound

\[
\sum_{n=U+1}^{U+V} e(bn/Q) \ll \min \left\{ V, \frac{Q}{|b|} \right\},
\]

which holds for any integers \( b, U \) and \( V \geq 1 \) with \( 0 < |b| \leq Q/2 \), see [16, Bound (8.6)].

First we recall the classical large sieve inequality, see [16, Theorem 7.11]:

**Lemma 3.1.** Let \( a_1, \ldots, a_H \) be an arbitrary sequence of complex numbers and let

\[
A = \sum_{h=1}^{H} |a_h|^2 \quad \text{and} \quad T(u) = \sum_{h=1}^{H} a_h \exp(2\pi iuh).
\]

Then, for an arbitrary \( R \geq 1 \), we have

\[
\sum_{1 \leq r \leq R} \sum_{\gcd(v,r)=1} |T(v/r)|^2 \ll (R^2 + H) A.
\]
The link between multiplicative characters and exponential sums is given by the following well-known identity (see [16, Equation (3.12)]) involving Gauss sums

$$
\tau_k(\chi) = \sum_{v=1}^{k} \chi(v) e(v/k)
$$

defined for a character \(\chi\) modulo an integer \(k \geq 1\):

**Lemma 3.2.** For any primitive multiplicative character \(\chi\) modulo \(k\) and an integer \(b\) with \(\gcd(b, k) = 1\), we have

$$
\chi(b) \tau_k(\chi) = \sum_{\substack{v=1 \\ \gcd(v,k)=1}}^{k} \overline{\chi}(v) e(bv/k),
$$

where \(\overline{\chi}\) is the complex conjugate character to \(\chi\).

By [16, Lemma 3.1] we also have:

**Lemma 3.3.** For any primitive multiplicative character \(\chi\) modulo an integer \(k \geq 1\) we have

$$
|\tau_k(\chi)| = k^{1/2}.
$$

3.2. **Bounds for almost moduli.** We use some ideas of Garaev [9, Theorem 10], which we adapt to our purposes and specific relations between the parameters.

**Lemma 3.4.** For \(Q = X^{1/2+o(1)}\), we have

$$
\sum_{k \in [X, 2X]} \max_{\chi \in \chi_k^*} \max_{t \leq Q} |S_k(\chi; t)|^8 \ll X^{4+o(1)}.
$$

**Proof.** We follow the ideas of Garaev [9, Theorem 3].

For each \(k \in [X, 2X]\) we choose a primitive multiplicative character \(\chi_k\) modulo \(k\) and \(t_k \leq Q\) such that with the largest values of

$$
|S_k(\chi_k; t_k)| = \max_{\chi \in \chi_k^*} \max_{t \leq Q} |S_k(\chi; t)|
$$

Without loss of generality we can assume that \(Q = 2M + 1\) is an odd integer. Then using (3.1), for \(t_k \leq Q\) we write

$$
S_k(\chi_k; t_k) = \sum_{m=1}^{Q} \chi_k(m) \frac{1}{Q} \sum_{n=1}^{t_k} \sum_{b=-M}^{M} e(b(m - n)/Q)
$$

$$
= \frac{1}{Q} \sum_{b=-M}^{M} \sum_{n=1}^{t_k} e(-bn/Q) \sum_{m=1}^{Q} \chi_k(m) e(bm/Q).
$$
Recalling (3.2), we derive
\[ S_k(\chi_k; t_k) \ll \sum_{b=-M}^{M} \frac{1}{|b| + 1} \left| \sum_{m=1}^{Q} \chi_k(m)e(bm/Q) \right|. \]

Therefore, writing
\[ |b| + 1 = (|b| + 1)^{7/8} (|b| + 1)^{1/8}, \]
the Hölder inequality yields the bound
\[ (3.3) \sum_{k \in [X,2X]} |S_k(\chi_k; t_k)|^8 \ll (\log Q)^7 \sum_{b=-M}^{M} \frac{1}{|b| + 1} U_b, \]
where
\[ U_b = \sum_{k \in [X,2X]} \left| \sum_{m=1}^{Q} \chi_k(m)e(bm/Q) \right|^8. \]

We now note that
\[ \left( \sum_{m=1}^{Q} \chi_k(m)e(bm/Q) \right)^4 = \sum_{h=1}^{H} \rho_b(h) \chi_k(h), \]
where \( H = Q^4 \) and
\[ \rho_b(h) = \sum_{m_1,m_2,m_3,m_4=1}^{Q} e(b(m_1 + m_2 + m_3 + m_4)/Q). \]

Using Lemma 3.2, we write
\[ \left( \sum_{m=1}^{Q} \chi_k(m)e(bm/Q) \right)^4 = \sum_{h=1}^{H} \rho_b(h) \frac{1}{\tau_k(\chi_k)} \sum_{\tau=1}^{k} \tau_{k}(v)e(hv/k). \]

Changing the order of summation, by Lemma 3.3 and the Cauchy inequality, we obtain,
\[ \left| \sum_{m=1}^{M} \chi_k(m)e(bm/Q) \right|^8 \leq \sum_{\tau=1}^{k} \left| \sum_{v=1}^{H} \rho_b(h)e(hv/k) \right|^2. \]

Therefore
\[ U_b \leq \sum_{k \in [X,2X]} \left( \sum_{\tau=1}^{k} \left| \sum_{v=1}^{H} \rho_b(h)e(hv/k) \right|^2 \right). \]
Using the standard upper bound on the divisor function, see, for example, [16, Bound (1.81)], we conclude that

\[ |\rho_b(h)| \leq \sum_{m_1m_2m_3m_4=h} 1 = h^{\omega(1)} \]

as \( h \to \infty \). Hence, we now derive from Lemma 3.1

\[ U_b \leq (X^2 + H) H X^{\omega(1)} \leq (X^2 + Q^4) Q^4 X^{\omega(1)} \leq X^{4+\omega(1)}, \]

which after substitution in (3.3) implies

\[ \sum_{k \in [X,2X]} |S_k(\chi_k; t_k)|^8 \ll X^{4+\omega(1)} \]

and concludes the proof. \( \square \)

4. Upper Bounds

4.1. Proof of Theorem 1.2. We first obtain a bound on for almost all primes in dyadic intervals.

**Lemma 4.1.** Let \( X \geq 1 \) be a sufficient large real number. For any \( \eta \in \{0,1\} \) we have

\[ \sum_{p \in [X,2X]} \max_{\chi \in \chi_p^*} |\Theta_p(\eta, 1, \chi)|^8 \leq X^{4\eta+4+\omega(1)}. \]

**Proof.** Using partial summation, we obtain

\[ \Theta_p(\eta, 1, \chi) = \frac{2\pi}{p} \int_1^{+\infty} \left( t^{1+\eta} + \frac{p}{2\pi} \right) S_p(\chi; t)e^{-\pi t^2/p} \, dt. \]

where \( S_p(\chi; t) \) is given by (1.5).

First, we remark that it suffices to bound the above integral for \( t \leq 2(X \log X)^{1/2} \). Indeed, we bound the tail for \( t \geq 2(X \log X)^{1/2} \) trivially, using that we always have \( |S_p(\chi; t)| \leq p \),

\[ \frac{2\pi}{p} \int_{2(X \log X)^{1/2}}^{+\infty} \left( t^{1+\eta} + \frac{p}{2\pi} \right) S_p(\chi; t)e^{-\pi t^2/p} \, dt \]

\[ \ll \int_{2(X \log X)^{1/2}}^{+\infty} t^3 e^{-\pi t^2/p} \, dt = I_1 + I_2 \]

where

\[ I_1 = \int_{2(X \log X)^{1/2}}^X t^3 e^{-\pi t^2/p} \, dt \quad \text{and} \quad I_2 = \int_X^{+\infty} t^3 e^{-\pi t^2/p} \, dt. \]
We now have the following elementary estimates:

\[
I_1 = X^3 \int_{2(X \log X)^{1/2}}^{X} e^{-\pi t^2 / p} dt \ll X^3 \exp \left( -\pi \frac{4X \log X}{p} \right) \leq X^3 \exp (-\pi \log X) \ll 1,
\]

\[
I_2 \leq \int_{X}^{+\infty} e^{-0.5\pi t^2 / p} dt \ll 1.
\]

Substituting the above estimates of \(I_1\) and \(I_2\) in (4.1), we now conclude
\[
\Theta_p(\eta, 1, \chi) \ll X^{-1} \left( |S_p(\chi; t)| e^{-\pi t^2 / p} \right) dt + 1
\]

Hence,
\[
\Theta_p(\eta, 1, \chi) \ll X^{-1} (X^{1/2} \log X) (\eta X^{(1+\eta)/2} + \eta X)
\]

(4.2)

We now remark that for \(\eta \in \{0, 1\}\) we have
\[
\eta X \leq (X \log X)^{(1+\eta)/2}.
\]

Hence, we derive from (4.2) that
\[
\Theta_p(\eta, 1, \chi) \ll X^{-1} (X^{1/2} \log X)^{(3+\eta)/2} \max_{t \leq 2(X \log X)^{1/2}} |S_p(\chi; t)| + 1
\]

(4.3)

(clearly the term 1 can be dropped as, for example, \(S_p(\chi; 1) = 1\)). We now see from (4.3) that
\[
\sum_{p \in [X, 2X]} \max_{\chi \in \chi_p^*} |\Theta_p(\eta, 1, \chi)|^8 \leq X^{4\eta + o(1)} \sum_{p \in [X, 2X]} \max_{\chi \in \chi_p^*} \max_{t \leq 2(X \log X)^{1/2}} |S_p(\chi; t)|^8,
\]

and using Lemma 3.4 we conclude the proof. \(\square\)

We deduce Theorem 1.2 from Lemma 4.1 by splitting \([1, X]\) in dyadic intervals \([X/2^{k+1}, X/2^k]\).
4.2. **Proof of Theorem 1.4.** For even characters we write,

\[
\sum_{p \leq X} \sum_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)|^{2k} \leq \sum_{p \leq X} \max_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)|^{2k-4} \sum_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)|^4.
\]

Using (1.2), we obtain,

\[
\sum_{p \leq X} \sum_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)|^{2k} \leq X^{2+o(1)} \sum_{p \leq X} \max_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)|^{2k-4}.
\]

Finally, since \(3 \leq k \leq 6\) then by the Hölder inequality

\[
\sum_{p \leq X} \max_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)|^{2k-4} \leq \left( \sum_{p \leq X} 1 \right)^{(6-k)/4} \left( \sum_{p \leq X} \max_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)|^8 \right)^{(k-2)/4},
\]

and using Theorem 1.2, after simple calculations we obtain the result for the even characters. A similar argument also implies the desired estimate for the odd characters.

4.3. **Proof of Theorem 1.5.** For \(k = 1\) and \(k = 2\) the result is contained in (1.2). Let \(X \geq 1\) be a sufficient large real number. For all but \(o(X/\log X)\) primes \(p \leq X\), we see from Corollary 1.3 that for any \(\chi \in \mathcal{X}_p\) we have

\[
\max_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)| \leq p^{3/8+o(1)}.
\]

So for these primes, for \(k \geq 3\), we have

\[
\sum_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)|^{2k} \leq p^{3k/4+o(1)} \sum_{\chi \in \mathcal{X}_p^+ \setminus \chi_0} |\psi_p(1, \chi)|^4.
\]

Using (1.2), we obtain the result for the even characters. A similar argument also implies the desired estimate for the odd characters.

5. **Concluding remarks**

We remark that under the Generalised Riemann Hypothesis, instead of Lemma 3.4 we can use the well-known bound

\[
(5.1) \quad \max_{\chi \in \mathcal{X}_p} |S_p(\chi; t)| \leq t^{1/2} p^{o(1)},
\]

see [20, Section 1]; it can also be derived from [10, Theorem 2].
Hence, substituting (5.1) in (4.3) (with $X = p$) we obtain a more realistic individual bound
\[
\Theta_p(\eta, 1, \chi) \leq p^{1/4 + \eta/2 + o(1)}.
\]
It is worthwhile to notice that this is consistent with the asymptotic conjectural formula (1.3).

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