Quaternion wavelet transform for image denoising

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Abstract. Quaternion wavelet transform (QWT) combines discrete wavelet transform (DWT) and quaternion Fourier transform (QFT). QWT has many applications included image processing. In this research, we discuss about construction, characteristics and implementation of QWT on process of image denoising. We construct denoising algorithm with QWT then we do simulation to know performance of algorithm. We use grayscale test images that have size 512 × 512 pixel with low, medium and high complexity. Experiment removes noise of image successfully. Results of image denoising are used to measure algorithm performance using PSNR (peak signal to noise ratio) value. We compare PSNR values with DWT and QWT for Haar, Biorthogonal, Daubechies and Coiflets wavelet. The method that has the highest PSNR value can be concluded the best performance.

1. Introduction

Quaternion wavelet is a new type of wavelet. Quaternion is proposed by William Rowan Hamilton in 1843. Gai [1] said that quaternion wavelet transform (QWT) is a new multiresolution analysis tool for describing geometric characteristics of the image. Quaternion value is composed of one magnitude and three angle phases. Li [2] said that wavelet transform is based discrete wavelet transform (DWT) and complex wavelet transform (CWT). CWT is a kind of efficient and popular technique for texture representation. Wavelet transform is improvement of Fourier transform. Fourier transform changes a signal domain from spatial domain only to frequency domain. While wavelet transform changes a signal domain from spatial domain to scaling and translation domain.

Yin [3] analyzes quaternion wavelet and applies to image denoising. Firstly, Yin studies standard orthogonal basis of scale space and wavelet space of quaternion wavelet transform (QWT) in spatial $L^2(\mathbb{R}^2)$. Yin proves and presents quaternion wavelet’s scale basis function and wavelet basis function concept in spatial scale $L^2(\mathbb{R}^2; \mathbb{H})$. Next, QWT generalized Gauss distribution is used to model QWT coefficient’s magnitude distribution. Yin uses Bayesian threshold with soft thresholding method to get threshold value. And then, Yin uses “Farrs” filters at level 1 and Q-shift dual-tree filters at higher levels in quaternion wavelet.

Gai [1] proposes a new hidden markov tree (HMT) model utilizing QWT. And then, Gai applies it to image denoising. Next, Kadiri [4] applies QWT to satellite image with noise. Firstly, Kadiri decomposes the noisy image by QWT. Secondly, Kadiri computes one magnitude and three angle phases $(\phi, \theta, \psi)$. Thirdly, Kadiri computes Neigh-Shrink thresholding of magnitude and phases smoothing of angle phases. The last, Kadiri reconstructs denoised image by QWT.
In this paper, we study QWT for image denoising with Haar, Biorthogonal, Daubechies and Coiflets wavelet. We use hard thresholding with VisuShrink method to get threshold value. We also use mean filter for image denoising. This experiment uses grayscale test images that are corrupted by white Gaussian noise with zero mean and variance $\sigma^2$. Test images have size $512 \times 512$ pixel with low, medium and high complexity.

2. Preliminaries

2.1. Discrete wavelet transform

Quaternion wavelet transform (QWT) combines some discrete wavelet transforms (DWT). We discuss construction of DWT with multiresolution analysis.

Definition 1. [5] A sequence of closed subspaces $\{V_j; j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ together with a function $\phi \in V_0$ is called a multiresolution analysis (MRA) if it satisfies the following conditions:

- (Increasing) $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$
- (Density) $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})$.
- (Separation) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$.
- (Scaling) $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$.
- (Orthonormal basis) There exist a scaling function (of the MRA) $\phi \in V_0$ whose integer translates $\{\phi(t-n); n \in \mathbb{Z}\}$ are an orthonormal base for $V_0$.

Wavelet function that constructed by scaling function (father wavelet) is defined:

$$\phi_{j,k}(t) = 2^j \phi(2^j t - k) \in L^2(\mathbb{R})$$  \hspace{1cm} (1)

where $\phi$ is scaling function. Scaling function is defined by

$$\phi(t) = \sum_n h_\phi(n) \sqrt{2} \phi(2t - n), \, n \in \mathbb{Z}$$  \hspace{1cm} (2)

where $h_\phi(n)$ is scaling function coefficient.

Span of $\{\phi_{j,k}\}$ is defined by

$$V_j = \overline{\text{span}\{\phi_{k}(2^j t)\}} = \overline{\text{span}\{\phi_{j,k}(t)\}}$$  \hspace{1cm} (3)

where $V_j$ is subspace of $L^2(\mathbb{R})$. We can write subspaces of $V_j$ as

$$V_0 \subset V_1 \subset V_2 \subset \ldots$$  \hspace{1cm} (4)

where

$$\lim_{j \to \infty} V_j = \{L_2(\mathbb{R})\}.$$  \hspace{1cm} (5)

Wavelet function that constructed by mother wavelet is defined:

$$\psi_{j,k}(t) = 2^j \psi(2^j t - k) \in L^2(\mathbb{R})$$  \hspace{1cm} (6)

where $\psi$ is mother wavelet. Mother wavelet is defined by

$$\psi(t) = \sum_n h_\psi(n) \sqrt{2} \phi(2t - n), \, n \in \mathbb{Z}$$  \hspace{1cm} (7)

where $h_\psi(n)$ is wavelet function coefficient.

Span of $\{\psi_{j,k}\}$ is defined by

$$W_j = \overline{\text{span}\{\psi_{j,k}(t)\}}.$$  \hspace{1cm} (8)

Relationship of spaces $V_j$ and $W_j$ can be written by $V_{j+1} = V_j \oplus W_j$, where $\oplus$ is direct sums of two spaces. Spaces $V_j$ and $W_j$ also can be written by

$L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \ldots$.  \hspace{1cm} (9)
Wavelet series expansion is defined by

\[ f(t) = \sum_{k} a_{j_0,k} \phi_{j_0,k}(t) + \sum_{j=j_0}^{\infty} \sum_{k} b_{j,k} \psi_{j,k}(t) \]  

(10)

where \( j \geq j_0 \), \( f(t) \in L^2(\mathbb{R}) \). If \( \{\phi_{j_0,k}(t)\} \) and \( \{\psi_{j,k}(t)\} \) are orthonormal in \( L^2(\mathbb{R}) \), then we can write \( a_{j_0,k} \) and \( b_{j,k} \) as

\[ a_{j_0,k} = \langle f(t), \phi_{j_0,k}(t) \rangle = \int_{\mathbb{R}} f(t) \overline{\phi_{j_0,k}(t)} \, dt \]  

(11)

\[ b_{j,k} = \langle f(t), \psi_{j,k}(t) \rangle = \int_{\mathbb{R}} f(t) \overline{\psi_{j,k}(t)} \, dt \]  

(12)

Forward discrete wavelet transform is defined by

\[ W_{\phi_{j_0,k}}(t) = \frac{1}{\sqrt{N}} \sum_{t} f(t) \phi_{j_0,k}(t) \]  

(13)

\[ W_{\psi_{j,k}}(t) = \frac{1}{\sqrt{N}} \sum_{t} f(t) \psi_{j,k}(t) \]  

(14)

where \( f(t) \) is signal with \( t = 0,1, \ldots , N - 1 \) and \( j \geq j_0 \).

Inverse discrete wavelet transform (IDWT) is defined by

\[ f(t) = \frac{1}{\sqrt{N}} \sum_{k} W_{\phi_{j_0,k}}(t) \phi_{j_0,k}(t) + \frac{1}{\sqrt{N}} \sum_{j=j_0}^{\infty} \sum_{k} W_{\psi_{j,k}}(t) \psi_{j,k}(t) \]  

(15)

where \( j_0 = 0, N = 2^l, j = 0,1,2, \ldots , J - 1 \).

2.2. Dual tree complex wavelet transform

Dual tree complex wavelet transform (DTCWT) is based discrete wavelet transforms (DWT). DTCWT decomposes complex signal to real and imaginary. Complex coefficient

\[ d_c(j,n) = d_h(j,n) + id_g(j,n) \]  

(16)

where \( d_g(j,n) \) is Hilbert transform of \( d_h(j,n) \).

Definition 2. [6] The Hilbert transform \( H(x(t)) \) of a signal \( x(t) \) is defined as

\[ H(x(t)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} \, d\tau \]  

(17)

where \( \tau \) is dummy.

In polar form, complex coefficient is

\[ d_c(j,n) = |d_c(j,n)|e^{i\phi} \]  

(18)

with magnitude

\[ |d_c(j,n)| = \sqrt{|d_h(j,n)|^2 + |d_g(j,n)|^2} \geq 0 \]  

(19)

and phase

\[ \phi = \angle d_c(j,n) = \arctan \left( \frac{d_g(j,n)}{d_h(j,n)} \right) \]  

(20)

A magnitude and angle phase of complex coefficient can be reduced.
3. Results and discussion

3.1. Quaternion wavelet transform

3.1.1. Construction of quaternion wavelet transform

Quaternion wavelet transform (QWT) is based on quaternion algebra, quaternion Fourier transform and Hilbert transform. QWT has characteristics like quaternion Fourier transform (QFT) but QWT about scaling and translation.

QWT has three angle phases so it is suitable for 2-D signal. QWT can be represented in a polar form as

\[ q = |q|e^{i\phi}e^{i\theta}e^{ik\psi} \]

(21)

where \(|q|\) is magnitude of \(q\) and

\[ \phi = \arctan(2(cd + ab)/a^2 - b^2 + c^2 - d^2)/2 \in [-\pi, \pi] \]

(22)

\[ \theta = \arctan(2(bd + ac)/a^2 + b^2 - c^2 - d^2)/2 \in [-\pi/2, \pi/2] \]

(23)

\[ \psi = \arcsin(2(ad - bc))/2 \in [-\pi/4, \pi/4]. \]

(24)

Respectively, \(\phi_h(x)\) and \(\psi_h(x)\) are 1-D scaling function and wavelet function along the \(x\)-axis. Also, \(\phi_h(y)\) and \(\psi_h(y)\) are 1-D scaling function and wavelet function along the \(y\)-axis. Hilbert transforms of \(\psi_h(x)\) and \(\psi_h(y)\) are

\[ \mathcal{H}_x(\psi_h(x)\psi_h(y)) = \psi_g(x)\psi_h(y) \]

(25)

\[ \mathcal{H}_y(\psi_h(x)\psi_h(y)) = \psi_h(x)\psi_g(y) \]

(26)

\[ \mathcal{H}_{xy}(\psi_h(x)\psi_h(y)) = \psi_g(x)\psi_g(y) \]

(27)

where \(\psi_g(x)\) and \(\psi_g(y)\) are Hilbert transforms of \(\psi_h(x)\) and \(\psi_h(y)\), respectively. Also, \(\phi_g(x)\) and \(\phi_g(y)\) are Hilbert transforms of \(\phi_h(x)\) and \(\phi_h(y)\).

After processing of Hilbert transform, 2-D a quaternion scaling function and three quaternion wavelet functions become

\[ \Phi^q(x,y) = \phi_h(x)\phi_h(y) + i\phi_g(x)\phi_h(y) + j\phi_h(x)\phi_g(y) + k\phi_g(x)\phi_g(y) \]

(28)

\[ \Psi_{h}^q(x,y) = \psi_h(x)\phi_h(y) + i\psi_g(x)\phi_h(y) + j\psi_h(x)\phi_g(y) + k\psi_g(x)\phi_g(y) \]

(29)

\[ \Psi_{a}^q(x,y) = \phi_h(x)\psi_h(y) + i\phi_g(x)\psi_h(y) + j\phi_h(x)\psi_g(y) + k\phi_g(x)\psi_g(y) \]

(30)

\[ \Psi_{d}^q(x,y) = \psi_h(x)\psi_g(y) + i\psi_g(x)\psi_g(y) + j\psi_h(x)\psi_g(y) + k\psi_g(x)\psi_g(y). \]

(31)

Definition 3. Function \(f(x,y) \in L^2(\mathbb{R}, \mathbb{H})\), we construct

\[ c^q = \{f(x,y), \Phi^q(x,y)\} \]

(32)

\[ d^q_i = \{f(x,y), \Psi_{i}^q(x,y)\} \]

(33)

where \(i = h, v, d\). Transformation \(c^q\) and \(d^q_i\) \((i = h, v, d)\) are called discrete quaternion wavelet transform of \(f(x,y)\).

3.1.2. Characteristics of quaternion wavelet transform

Quaternion wavelet transform (QWT) is basis like quaternion Fourier transform (QFT) but QWT is specifically scaling and translation basis. Theorem of quaternion Fourier on translation process is

Theorem. [7] Let

\[ F^q(u) = \int_{\mathbb{R}^2} e^{-i2\pi ux}f(x)e^{-j2\pi vy} d^2x \]

(34)

and

\[ F^q_{\tau}(u) = \int_{\mathbb{R}^2} e^{-i2\pi ux}(x - d)e^{-j2\pi vy} d^2x \]

(35)

be the QFT’s of a 2D signal \(f\) and a shifted version of \(f\), respectively. Then, \(F^q(u)\) and \(F^q_{\tau}(u)\) are related by shift through \(d = (d_1, d_2)^{T}\)

\[ F^q_{\tau}(u) = e^{-i2\pi ud_1}F^q(u)e^{-j2\pi ud_2}. \]

(36)
If we denote the phase of $F^q(u)$ by $(\phi(u), \theta(u), \psi(u))^T$. Then, as a result of the shift, the first and the second component of the phase undergo a phase-shift

$$
\begin{pmatrix}
\phi(u) \\
\theta(u) \\
\psi(u)
\end{pmatrix} \rightarrow 
\begin{pmatrix}
\phi(u) - 2\pi u d_1 \\
\theta(u) - 2\pi v d_2 \\
\psi(u)
\end{pmatrix}.
$$

(37)

QWT is one of QFT, therefore translation process of QWT becomes

$$
(\phi(u), \theta(u), \psi(u)) \rightarrow (\phi(u) - 2\pi u d_1, \theta(u) - 2\pi v d_2, \psi(u))
$$

(38)

where $\mathbf{u} = (u, v)$ is QWT domain and $(d_1, d_2)$ is translation.

3.2. Quaternion wavelet transform for image denoising

Quaternion wavelet transform (QWT) decomposes noisy image into sub-bands. In this paper, we use hard thresholding method with VisuShrink method to reduce detail coefficient magnitudes. And also, we use mean filter to approximation coefficient magnitude.

Hard thresholding method is

$$
T_h(y, \lambda) = \begin{cases} 
  y, & \text{if } |y| \geq \lambda \\
  0, & \text{if } |y| < \lambda
\end{cases}
$$

(39)

where $y$ is detail coefficient of image and $\lambda$ is threshold value.

VisuShrink method is

$$
\lambda = \sigma \sqrt{2 \log M}
$$

(40)

where $\sigma$ is deviation standard of noise and $M$ is image size.

Mean filter $3 \times 3$ has mask formula

$$
R = \frac{1}{9} \sum_{i=0}^{9} z_i
$$

(41)

where all number of mask $z_i$ is 1.

Some steps of QWT for image denoising are:

- Decompose noisy image with QWT.
- Calculate imaginary part of quaternion detail coefficient using hard thresholding with VisuShrink.
- Calculate imaginary part of quaternion approximation coefficient using mean filter.
- Calculate quaternion detail coefficient using hard thresholding with VisuShrink.
- Calculate quaternion approximation coefficient using mean filter.
- Reconstruct the denoised by invers quaternion wavelet transform (IQWT).

3.3. Experimental result of quaternion wavelet transform for image denoising

In the experiment, we use discrete wavelet transform (DWT) and quaternion wavelet transform (QWT) with Haar, Biorthogonal 6.8, Daubechies 9 and Coiflets 5 wavelet. Images are corrupted by white Gaussian noise with zero mean and variance $\sigma^2$. White Gaussian noise is one type of noise that is often encountered in digital images. Model of degradation images are

$$
g(x, y) = f(x, y) + e(x, y)
$$

(42)

where $f(x, y)$ is clean image, $g(x, y)$ is noisy image and $e(x, y)$ is noise.

We use test images with low, medium and high complexity. Each test image has size $512 \times 512$ pixel with grayscale characteristic. We use hard thresholding method with VisuShrink method to reduce detail coefficient magnitudes. And also, we use mean filter to approximation coefficient magnitude. The method that has the highest PSNR (peak signal to noise ratio) value can be concluded the best performance.
Denoised image of 100 experiments with $\sigma = 20$ and zero mean are shown in figure 1.

![Original Image](image1.png) ![Noisy Image](image2.png)

**PSNR = 24.75**  **PSNR = 26.09**

![Denoised Image with DWT](image3.png)  ![Denoised Image with QWT](image4.png)

**Figure 1.** Denoised of Lena image with coiflets 5 DWT and QWT.

PSNR values of 100 experiments with zero mean and some deviation standards are shown in figure 2 and table 1.

| Deviation Standard | PSNR Values |
|--------------------|-------------|
| 10                 | DWT: 27.94  |
|                    | QWT: 28.03  |
| 20                 | DWT: 24.75  |
|                    | QWT: 26.09  |
| 30                 | DWT: 22.27  |
|                    | QWT: 24.04  |

**Table 1.** PSNR values of coiflets 5 DWT and QWT.

**Figure 2.** PSNR values of Lena image with coiflets DWT and QWT 5.
Table 1. PSNR values with DWT and QWT for Haar, Biorthogonal 6.8, Daubechies 9 and Coiflets 5.

| Images | σ | Haar DWT | QWT | Biorthogonal 6.8 DWT | QWT | Daubechies 9 DWT | QWT | Coiflets 5 DWT | QWT |
|--------|---|----------|-----|----------------------|-----|------------------|-----|----------------|-----|
| Pantai | 10 | 31.32    | 30.02 | 31.60                | 34.35 | 31.61            | 34.29 | 31.66          | 34.38 |
|        | 20 | 25.33    | 25.94 | 25.35                | 27.48 | 25.36            | 27.46 | 25.40          | 27.51 |
|        | 30 | 21.44    | 22.26 | 21.43                | 23.00 | 21.43            | 22.97 | 21.45          | 23.01 |
| Lena   | 10 | 26.15    | 26.07 | 27.93                | 27.99 | 27.95            | 27.96 | 27.94          | 28.03 |
|        | 20 | 23.71    | 24.60 | 24.73                | 26.10 | 24.74            | 26.09 | 24.75          | 26.09 |
|        | 30 | 21.64    | 23.06 | 22.24                | 24.03 | 22.24            | 24.03 | 22.27          | 24.04 |
| Mandrill | 10 | 22.09    | 21.95 | 22.68                | 22.57 | 22.70            | 22.58 | 22.71          | 22.60 |
|        | 20 | 20.90    | 21.33 | 21.42                | 21.90 | 21.43            | 21.93 | 21.39          | 21.88 |
|        | 30 | 19.80    | 20.72 | 20.17                | 21.16 | 20.17            | 21.20 | 20.14          | 21.14 |

We can see from figure 1, DWT and QWT methods remove noise of image successfully. Initially, noisy image can not be seen properly. After it is done image denoising, image can be seen better. We also can see that result of QWT method better than DWT method. We can see from figure 2 and table 1, we compare PSNR values with DWT and QWT for Haar, Biorthogonal 6.8, Daubechies 9 and Coiflets 5 wavelet. Almost all test images with some deviation standards, QWT method has higher PSNR value than DWT method. Performance results of method are stated quantitatively rather than qualitatively. We can say that QWT method has better performance than DWT method.

4. Concluding remarks
Based on the research, we can obtain some conclusions. Quaternion wavelet transform (QWT) can be constructed by multiresolution analysis. QWT can be represented in a polar form with a magnitude and three angle phases. In a polar form, a magnitude and angle phases of quaternion can be reduced. QWT removes noise of image successfully. We compare PSNR (peak signal to noise ratio) values of DWT and QWT with Haar, Biorthogonal, Daubechies and Coiflets 5 wavelet. QWT method has better performance than discrete wavelet transform (DWT) method for image denoising.

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