How weak is weak extent?

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Abstract. We show that the extent of a Tychonoff space of countable weak extent can be arbitrary big. The extent of $X$ is $e(X) = \sup\{|F| : F \subset X \text{ is closed and discrete}\}$ while $we(X) = \min\{\tau : \text{ for every open cover } \mathcal{U} \text{ of } X \text{ there is } A \subset X \text{ such that } |A| \leq \tau \text{ and } St(A, \mathcal{U}) = X\}$ is the weak extent of $X$ (also called the star-Lindelöf number of $X$). Also we show that the extent of a normal space with countable weak extent is not greater than $\mathfrak{c}$.

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Recall that the extent of a topological space $X$ is the cardinal $e(X) = \sup\{|F| : F \subset X \text{ is closed and discrete}\}$. The weak extent of $X$ is the cardinal $we(X) = \min\{\tau : \text{ for every open cover } \mathcal{U} \text{ of } X \text{ there is } A \subset X \text{ such that } |A| \leq \tau \text{ and } St(A, \mathcal{U}) = X\}$. The reason for this name is that for any $X \in T_1$, $we(X) \leq e(X)$; indeed, supposing $we(X) > \kappa$, there is an open cover such that for every $A \subset X$ with $|A| \leq \kappa$ one has $St(A, \mathcal{U}) \neq X$; then one can inductively choose points $x_\alpha, \alpha < \kappa$, so that $x_\alpha \notin St(\{x_\beta : \beta < \alpha\}, \mathcal{U})$ for each $\alpha$; once the points have been choosen the set $\{x_\alpha : \alpha < \kappa\}$ is closed, discrete and of cardinality $\kappa$, so $e(X) \geq \kappa$. Note also that $we(X) \leq d(X)$ obviously holds for every $X$. Some cardinal inequalities involving extent can be improved by replacing extent by weak extent. Thus for $X \in T_1$, $|K(X)| \leq we(X)^{psw(X)}$. A natural question was stated in [1], [9]: how big can the difference between the extent and the weak extent of a $T_i$ space be? First, we give the answer for the Tychonoff case.
Theorem 1 For every cardinal $\tau$ there is a Tychonoff space $X$ such that $e(X) \geq \tau$ and $we(X) = \omega$.

Before the paper [5] the cardinal function $we(X)$ was called the star-Lindelöf number [10], [8], [11]. In particular, a space $X$ such that $we(X) = \omega$ is called star-Lindelöf or *Lindelöf, see e.g. [9], [1], [2] [3].

Note that $e(X) \leq 2^{we(X)}$ for every regular space $X$ [1].

To prove Theorem 1, we use a set-theoretic fact in Theorem 2 below. Let $S$ be a set and $\lambda$ a cardinal. A set mapping of order $\lambda$ on $S$ is a mapping that assigns to each $s \in S$ a subset $f(s) \subset S$ so that $|f(s)| < \lambda$ and $s \not\in f(s)$. A subset $T \subset S$ is called $f$-free if $f(t) \cap T = \emptyset$ for every $t \in T$. Answering a question of Erdös, Fodor proved in 1952 ([4], see also [2], Theorem 3.1.5) a general theorem a partial case of which is the following

Theorem 2 (Fodor) Let $S$ be a set of cardinality $\tau$ and let $f$ be a set mapping on $S$ of order $\omega$. Then there is a countable family $\mathcal{H}$ of $f$-free subsets of $S$ such that $\cup \mathcal{H} = S$.

Proof of Theorem 1: Let $\tau$ be an infinite cardinal. For each $\alpha < \tau$, $z_\alpha$ denotes the point in $D^\tau$ with only the $\alpha$-th coordinate equal to 1. Put $Z = \{z_\alpha : \alpha < \tau\}$. Then $Z$ is a discrete subspace of $D^\tau$. Further, let $\kappa$ be a cardinal such that $\text{cf}(\kappa) > \tau$. Put

$$X = (D^\tau \times (k + 1)) \setminus ((D^\tau \setminus Z) \times \{\kappa\}).$$

Also we denote $X_0 = D^\tau \times \kappa$ and $X_1 = Z \times \{\kappa\} = \{(z_\alpha, \kappa) : \alpha < \tau\}$. Then $X = X_0 \cup X_1$.

It is clear that $X_1$ is closed in $X$ and discrete, so $e(X) \geq \tau$.

It remains to prove that $we(X) = \omega$. First, note that $X_0$ is countably compact, hence star-Lindelöf. So it remains to prove that $X_1$ is relatively star-Lindelöf in $X$, i.e. for every open cover $\mathcal{U}$ of $X$ there is a countable $A \subset X$ such that $St(A, \mathcal{U}) \supset X_1$.

Let $\mathcal{U}$ be an open cover of $X$. For every $\alpha < \tau$ choose an $U_\alpha \in \mathcal{U}$ so that $(z_\alpha, \kappa) \in U_\alpha$. Further, for every $\alpha < \tau$ choose $\xi_\alpha < \kappa$ and $B_\alpha$, an element of the standard base of $D^\tau$, so that $(z_\alpha, \kappa) \in (B_\alpha \times (\xi_\alpha, \kappa]) \cap X \subset U_\alpha$. It remains to check that

$(+)$ there is a countable $C \subset D^\tau$ such that $B_\alpha \cap C \neq \emptyset$ for every $\alpha < \tau$. 

2
Indeed, since \( \text{cf}(\kappa) > \tau \), there is a \( \gamma < \kappa \) such that \( \gamma > \xi_\alpha \) for all \( \alpha < \tau \). Put \( A = C \times \{ \gamma \} \). Then \( U_\alpha \cap A \neq \emptyset \) for all \( \alpha < \tau \), so \( X_1 \subset \text{St}(A, \mathcal{U}) \).

Now we check (+). The set \( B_\alpha \) has the form

\[
B_\alpha = \{ x \in D^\tau : x(\alpha) = 1 \text{ and } x(\alpha') = 0 \text{ for all } \alpha' \in A_\alpha \}
\]

where \( A_\alpha \) is some finite subset of \( \tau \setminus \{ \alpha \} \). Consider the set mapping \( f \) that assigns \( A_\alpha \) to \( \alpha \) for each \( \alpha < \tau \). By Fodor’s theorem, there is a countable, \( f \)-free family \( H = \{ H_n : n \in \omega \} \) of subsets of \( \tau \) such that \( \bigcup H = \tau \). For each \( n \in \omega \), denote by \( c_n \) the indicator function of \( H_n \), i.e. \( c_n(\alpha) = 1 \) iff \( \alpha \in H_n \).

Since \( H_n \) is \( f \)-free, \( B_\alpha \ni c_n \) for all \( \alpha \in H_n \). Put \( C = \{ c_n : n \in \omega \} \). Then \( B_\alpha \cap C \neq \emptyset \) for every \( \alpha < \tau \), i.e. (+) holds.

Pseudocompactness of \( X \) follows from the fact that \( X \) contains a dense countably compact subspace \( X_0 \). \( \blacksquare \)

Now we are going to show that in the normal case the extent of a space of countable weak extent is not greater than \( c \). In fact, we will prove a slightly more general statement. Recall that a family of sets is linked if every two elements have nonempty intersection. The linked-Lindelöf number of \( X \) is the cardinal \( \text{ll}(X) = \min \{ \tau : \text{every open cover of } X \text{ has a subcover representable as the union of at most } \tau \text{ many linked subfamilies} \} \). A space \( X \) with \( \text{ll}(X) = \omega \) is called linked-Lindelöf. It is easy to see that \( \text{ll}(X) \leq \text{we}(X) \) for every \( X \).

**Theorem 3** For every normal space \( X \), \( e(X) \leq 2^{\text{ll}(X)} \).

**Proof:** Let \( \tau \) be an infinite cardinal, \( K \) a closed discrete subspace of a normal space \( X \) and \( |K| = k > 2^\tau \). We have to show that \( \text{ll}(X) > \tau \). It is easy to construct a family \( \mathcal{A} \) of subsets of \( K \) such that \( |\mathcal{A}| = k \) and for every nonempty finite subfamily of \( \mathcal{A} \), say \( A_1, \ldots, A_n, A_{n+1}, \ldots, A_{n+m} \),

\[
(\ast) \quad |A_1 \cap \ldots \cap A_n \cap (K \setminus A_{n+1}) \cap \ldots \cap (K \setminus A_{n+m})| = k.
\]

For every \( A \in \mathcal{A} \) pick a continuous function \( f_A : X \to I \) such that \( f(A) = \{ 1 \} \) and \( f(K \setminus A) = \{ 0 \} \). Denote \( \mathcal{F} = \{ f_A : A \in \mathcal{A} \} \) and \( F = \Delta \mathcal{F} : X \to I^{\mathcal{F}} \). Then \( |\mathcal{F}| = k \). Note that \( F(K) \subset D^\mathcal{F} \). It follows from (\( \ast \)) that \( F(K) \) is dense in \( D^\mathcal{F} \), moreover, every open set in \( D^\mathcal{F} \) contains \( k \) elements of \( F(K) \). There is therefore a bijection \( \varphi : K \to \mathcal{B} \), where \( \mathcal{B} \) is the standard base of
\(D^F\), such that \(\varphi(z) \ni F(z)\) for every \(z \in K\). Every element \(B \in \mathcal{B}\) has the form

\[
B = B_{f_1 \ldots f_n}^{i_1 \ldots i_n} = \{x \in D^F : x(f_1) = i_1, \ldots, x(f_n) = i_n\}
\]

where \(n \in \mathbb{N}, f_1, \ldots, f_n \in \mathcal{F}\) and \(i_1, \ldots, i_n \in D\). Denote

\[
U(B) = \left\{ x \in I^F : \forall j \in \{1, \ldots, n\} \left( x(f_j) > \frac{1}{2} \text{ if } i_j = 1 \right) \text{ or } \left( x(f_j) < \frac{1}{2} \text{ if } i_j = 0 \right) \right\}.
\]

Further, for every \(z \in K\) put \(\tilde{\varphi}(z) = U(\varphi^{-1}(z))\). Then \(\tilde{\varphi}(z)\) is a neighbourhood of \(F(z)\) in \(I^F\). Note that

\[
(**) \quad \tilde{\varphi}(z) \cap \tilde{\varphi}(z') \neq \emptyset \quad \text{iff} \quad \varphi(z) \cap \varphi(z') \neq \emptyset.
\]

Let \(G\) denote the family of all continuous functions form \(X\) to \(I\), \(G = \Delta G : X \rightarrow I^\varnothing, \pi : I^\varnothing \rightarrow I^F\) is the natural projection. For each \(z \in K\) denote \(\tilde{\varphi}(z) = \pi^{-1}(\varphi(z))\). Then \(\tilde{\varphi}(z)\) is a neighbourhood of \(G(z)\) in \(I^\varnothing\) and

\[
(***) \quad \tilde{\varphi}(z) \cap \tilde{\varphi}(z') \neq \emptyset \quad \text{iff} \quad \varphi(z) \cap \varphi(z') \neq \emptyset.
\]

Last, for every \(z \in K\) put \(\tilde{\varphi}(z) = (\tilde{\varphi}(z) \setminus G(K \setminus \{z\})) \cap G(X)\). Then \(\tilde{\varphi}(z)\) is a neighbourhood of \(G(z)\) in \(G(X)\) and

\[
(*)v) \quad \tilde{\varphi}(z) \cap \tilde{\varphi}(z') \neq \emptyset \quad \text{iff} \quad \varphi(z) \cap \varphi(z') \neq \emptyset.
\]

Put \(U_0 = \{\tilde{\varphi}(z) : z \in K\}\). Since \(G\) is a homeomorphic embedding, \(G(K)\) is closed in \(G(X)\), so \(O = G(X) \setminus G(K)\) is open and hence \(U = U_0 \cup \{O\}\) is an open cover of \(G(X)\).

Since \(w(D^F) > 2^n\), \(\mathcal{B}\), a base of \(D^F\), is not representable as the union of at most \(\tau\) many linked subfamilies (see e.g. [4]). By (**), (***) and (*) the same can be said about the family \(U_0\). Note that for every \(z \in K\), \(\tilde{\varphi}(z)\) is the only element of \(U\) that contains \(z\). So \(U\) does not have a subcover representable as the union of at most \(\tau\) many linked subfamilies and thus \(ll(X) = ll(G(X)) > \tau\). \(\square\)

It is not clear whether the inequality in the previous theorem can be made strict, even with star-Lindelöf number instead of linked-Lindelöf.

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