The positive mass and isoperimetric inequalities for axisymmetric black holes in four and five dimensions

G W Gibbons and G Holzegel

DAMTP, Cambridge University, Wilberforce Road, Cambridge CB3 0WA, UK

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Abstract

In this paper, we revisit Brill’s proof of positive mass for three-dimensional, time-symmetric, axisymmetric initial data and generalize his argument in various directions. In 3 + 1 dimensions, we include an apparent horizon in the initial data and prove the Riemannian Penrose inequality in a wide number of cases by an elementary argument. In the case of 4 + 1 dimensions, we obtain the analogue of Brill’s formula for initial data admitting a generalized form of axisymmetry. Including an apparent horizon in the initial data, the Riemannian Penrose inequality is again proved for a large class of cases. The results may have applications in numerical relativity.

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1. Introduction

Some of the most important results in general relativity are the various positive mass theorems for asymptotically flat and asymptotically anti-de Sitter spacetimes (cf [1–4]). Since the mass in these spacetimes is either constant, or in the case of the Bondi mass, non-increasing, these theorems restrict to some degree the possible evolution of spacetimes. Thus, for example since the mass of an asymptotically flat spacetime is bounded below by zero and this can only be achieved by flat spacetime, these theorems are sometimes taken to show that flat spacetime is stable. However, the mass is a single number, determined only by the asymptotic structure of spacetime. It can give no detailed information about local behaviour of the spacetime geometry. Moreover, one knows from the singularity theorems that under some circumstances in which the mass at infinity is positive the occurrence of spacetime singularities is inevitable.

This situation contrasts sharply with that for classical field theories in flat spacetime. In that case, the mass $M$ may be expressed as an integral

$$M = \int_{\mathbb{R}^3} T_{00} \, d^3x$$

(1)
of a local energy density $T_00$ which is typically positive semi-definite, vanishing only for the trivial field configuration. For example, for a real scalar field $\phi$,

$$T_{00} = \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + V(\phi),$$

where $V(\phi)$ is a positive potential function, with global minimum at the origin of field space $\phi = 0$.

For suitable potential functions $V(\phi)$, the constancy of the energy places powerful restrictions on how the field $\phi$ may evolve. In particular, using Sobolev inequalities it is possible to show that the evolution is smooth for all times.

Now because of the equivalence principle, there is no well defined, generally covariant notion, of energy density in general relativity, and a generally covariant expression like (1) is not available. However, this does not mean that one cannot hope to find an expression analogous to (1) valid in a particular set of coordinate systems. Moreover, if the analogue of $T_{00}$ is positive definite then one may be able to restrict the evolution of the spacetime, and perhaps even prove long-term existence.

Just such a formula was found by Brill [5] for the special case of axisymmetric time-symmetric initial data on $\mathbb{R}^3$ which are also invariant under reversing the azimuthal angle $\theta$. Using his formula, Brill was able to give the first mathematically rigorous proof of the positive mass theorem in this restricted, but nevertheless physically interesting, special case.

It is widely appreciated that Brill’s result may be extended in a straightforward way to maximal data. It is possibly less widely appreciated that the angle reversing assumption can also be dropped. We will review and generalize Brill’s result to this so-called weakly axisymmetric case in section 2. More interesting physically is the possibility of extending his theorem to include an apparent horizon of area $A$, say. In this case one expects, because of the cosmic censorship hypothesis [6], the stronger ‘isoperimetric’ or ‘Riemannian Penrose’ inequality to hold

$$M \geq \sqrt{\frac{A}{16\pi}},$$

with equality only for the Schwarzschild initial data.

In fact, (2) has been proven in the general time-symmetric case by Huisken and Ilmanen [7] using the inverse-mean curvature flow (see also [8]). The purpose of this paper is to show that Brill’s formula may be extended to incorporate an apparent horizon and, at least in a large class of cases to be defined precisely below, to establish the isoperimetric inequality for black holes. These results are presented in section 3.

In recent years, the global behaviour of higher dimensional spacetimes, especially five-dimensional spacetimes, has attracted a great deal of interest because of various applications to string and M-theory, and from particle physics phenomenologists interested in scenarios with small extra dimensions which might permit the production of small black holes in accelerators. It is natural, therefore, to enquire about a possible generalization of Brill’s formula to five dimensions, that is to four-dimensional initial data sets. We will obtain such a generalization in section 4, subject to the assumption that the four-dimensional initial data admit an action by two commuting circle groups, i.e. by the torus group $T^2$.

In five spacetime dimensions, the isoperimetric inequality for black holes becomes

$$M \geq \frac{3\pi}{8} \left( \frac{A}{2\pi^2} \right)^{\frac{3}{2}} = \frac{3}{2} \left( \frac{A}{16\sqrt{\pi}} \right)^{\frac{3}{2}}.$$

1 That is, the metric admits an isometric reversible circle action.
As yet, there is not a lot of mathematical evidence for this inequality. The inverse-mean curvature method is not available because it makes essential use of the two-dimensional Gauss–Bonnet theorem. However, some support is given by the behaviour of collapsing shells [9]. In section 5, we will use our five-dimensional Brill formula to prove the inequality in a wide number of cases.

Finally, we comment on possible applications of our findings to numerical relativity. Three-dimensional axisymmetric initial data sets containing black holes have been studied numerically for quite some time now (see e.g. [10], where the superposition of Brill waves and a black hole is studied). Our results concerning four-dimensional initial data, in particular formula (90), may be useful for controlling the accuracy of numerical codes within the study of higher dimensional gravity.

2. Brill’s proof of positive mass

Consider a maximal axisymmetric initial data set \((\mathbb{R}^3, g, K_{ab})\) for Einstein’s equations. This means in particular that \(g\) and \(K\) satisfy the constraint equations for \(\text{tr} K = 0\)

\[
\begin{align*}
R &= 16\pi \mu + K_{ab} K^{ab}, \\
D^a K_{ab} &= 0.
\end{align*}
\]

Here \(D\) is the covariant derivative with respect to the metric \(g\) and \(\mu\) is the \(T^{00}\) component of the energy momentum tensor. Axisymmetry means that the metric can be written in the form

\[
g = q_{AB} \, dx^A \, dx^B + X^2 (d\phi + A_{\rho} \, d\rho + A_z \, dz)^2,
\]

with \(q_{AB}\) a two-dimensional metric on the orbit space of the Killing field \(\partial/\partial \phi\) and the functions \(X\) and \(A_{\rho}\) not depending on the variable \(\phi\). If the metric is strongly axisymmetric, it enjoys an additional mirror symmetry and \(A_\chi\) has to vanish. Coordinates can be found such that

\[
g = e^{-2U + 2q}(d\rho^2 + dz^2) + \rho^2 e^{-2U} (d\phi + A_{\rho} \, d\rho + A_z \, dz)^2.
\]

This choice of coordinates corresponds to finding a harmonic function on the space of orbits, i.e. a solution to the equation

\[
\Delta q_{AB} \rho = 0.
\]

The solution \(\rho\) is unique once we specify conditions at infinity and the \(z\)-axis, the boundary of the orbit space. Furthermore, to obtain a regular axisymmetric metric, certain conditions have to be imposed on the functions \(U\) and \(q\) (cf [5]). The function \(q\) has to satisfy

\[
q = 0 \quad \text{and} \quad \frac{\partial q}{\partial \rho} = 0 \quad \text{on the } z\text{-axis}.
\]

The first condition is necessary to avoid conical singularities on the axis. Furthermore, we impose that at infinity

\[
q \sim \frac{1}{r^{1+\epsilon}} \quad \text{and} \quad \frac{\partial q}{\partial r} \sim \frac{1}{r^{2+\epsilon}}
\]

for some \(\epsilon > 0\). On the other hand, the \(1/r\)-term of \(U\) contains the ADM mass of the metric:

\[
U \sim -\frac{m}{r} \quad \text{and} \quad \frac{\partial}{\partial r} U \sim \frac{m}{r^2}
\]

at infinity.

Let us compute the scalar curvature of the metric (6). The dreibein is

\[
e^1 = e^{-U + q} \, d\rho, \quad e^2 = e^{-U + q} \, dz, \quad e^3 = \rho \, e^{-U} (d\phi + A_{\rho} \, d\rho + A_z \, dz).
\]
The antisymmetric connection coefficients are
\[
\omega^1_2 = \left( \frac{-U + q}{e^{-U + q}} \right) e^1 - \left( \frac{-U + q}{e^{-U + q}} \right) e^2 + \frac{1}{2} \rho e^{U - 2q} (A_{\rho, \rho} - A_{z, \rho}) e^3, \tag{12}
\]
\[
\omega^3_1 = \left( \frac{1}{\rho - U_{, \rho}} \right) e^{-U + q} e^1 - \frac{1}{2} \rho e^{U - 2q} (A_{\rho, \rho} - A_{z, \rho}) e^2, \tag{13}
\]
\[
\omega^3_2 = -U_{, z} e^{U - q} e^3 + \frac{1}{2} \rho e^{U - 2q} (A_{\rho, \rho} - A_{z, \rho}) e^1. \tag{14}
\]
Note that indices are raised and lowered with the Euclidean metric. The curvature 2-form is obtained from
\[
\mathcal{R}_{ij} = d\omega^k_{ij} + \omega^i_{kj} \wedge \omega^j_{ki} . \tag{15}
\]
Since the scalar curvature is obtained from
\[
R = g^{ab} \mathcal{R}_{ab} = 8 (R^1_{a1b} + R^2_{a2b} + R^3_{a3b}) = 2R_{1212} + 2R_{1313} + 2R_{2323}, \tag{16}
\]
we only need the \( e^1 \wedge e^2 \) part of \( R_{12} \), the \( e^1 \wedge e^3 \) part of \( R_{13} \) and the \( e^2 \wedge e^3 \) part of \( R_{23} \) to compute the scalar curvature. We obtain
\[
R_{1212} = e^{2(U - 2q)} ((U - q),_{\rho \rho} + (U - q),_{zz}) - \frac{3}{4} \rho^2 e^{2(U - 4q)} (A_{\rho, \rho} - A_{z, \rho})^2, \tag{17}
\]
\[
R_{1313} = -\left( \frac{1}{\rho - U_{, \rho}} \right)^2 e^{2U - 2q} \left( \frac{1}{\rho - U_{, \rho}} e^{-q} \right) e^U - q + (-U + q),_z U_{, z} e^{2U - 2q} + \frac{1}{4} \rho^2 e^{2U - 4q} (A_{\rho, \rho} - A_{z, \rho})^2, \tag{18}
\]
\[
R_{2323} = -(U, z)^2 e^{2U - 2q} + (e^{-U - q} U_{, z})_z e^{-U - q} + \frac{1}{4} \rho^2 e^{2U - 4q} (A_{\rho, \rho} - A_{z, \rho})^2 + (U - q),_\rho \left( \frac{1}{\rho - U_{, \rho}} \right) e^{2U - 2q}. \tag{19}
\]
From (16) the scalar curvature then is
\[
(3)^{R} = 2 e^{2U - 2q} ((U - q),_{\rho \rho} + (U - q),_{zz}) - \frac{1}{2} \rho^2 e^{2U - 4q} (A_{\rho, \rho} - A_{z, \rho})^2 + e^{2U - 2q} \frac{3}{4} \rho^2 U_{, \rho}, \tag{20}
\]
which we can write in the form
\[
-\frac{1}{8} (3)^{R} e^{(-2U + 2q)} = -\frac{1}{2} \Delta U + \frac{1}{4} (\nabla U)^2 + \frac{1}{4} \Delta q - \frac{1}{4} \rho \frac{\partial q}{\partial \rho} + \frac{1}{16} \rho^2 e^{-2q} (A_{\rho, \rho} - A_{z, \rho})^2. \tag{21}
\]
The Laplacian and gradient are taken with respect to flat coordinates in \( \mathbb{R}^3 \). We can integrate this expression over \( \mathbb{R}^3 \) to prove positive mass. If we take conditions (8)–(10) into account, the integration yields
\[
M = \frac{1}{16\pi} \int \left[ (3)^{R} + \frac{1}{2} \rho^2 e^{-4q + 2U} (A_{\rho, \rho} - A_{z, \rho})^2 \right] e^{2q} e^{-2U} \, d^3 x + \frac{1}{8\pi} \int (\nabla U)^2 \, d^3 x. \tag{22}
\]
Defining
\[
H_{AB} = A_{A,B} - A_{B,A}, \tag{23}
\]
where $A, B \in \{1, 2\}$ denote indices on the orbit space with metric $q_{AB}$ defined above, and noting that $X = \rho \, e^{-U}$, we can rewrite (22) in the neat form

$$M = \frac{1}{16\pi} \int \left[ \frac{1}{4} R + \frac{1}{4} X^2 H_{AB} H^{AB} \right] e^{2q} \, e^{-2U} \, d^3x + \frac{1}{8\pi} \int (\nabla U)^2 \, d^3x. \quad (24)$$

Since $(3)R = 16\pi \mu + K_{ab}K^{ab} \geq 0$, this proves positive mass for the type of initial data considered above.

3. The generalization to black holes

In this section, we will generalize Brill’s formula to initial data leading to black holes. To determine the event horizon of a black hole in a four-dimensional spacetime, it is necessary to know its entire global structure, i.e. there is no direct way to see an event horizon in the initial data. On the other hand, if we consider initial data whose maximal development admits a complete null-infinity, an apparent horizon present in the initial data will always lead to a regular event horizon in the development (cf [11]). Moreover, in this case there will exist on the initial data slice a component of the event horizon outside the apparent horizon or coinciding with it. We note that for time-symmetric and also for $t$–$\phi$-symmetric initial data the notion of an apparent horizon and a minimal surface coincide. We will restrict to strongly axisymmetric data in the following.

3.1. Static initial data: Weyl’s class

Consider Weyl’s family of axisymmetric static vacuum solutions to Einstein’s equations:

$$d\mathbf{s}^2 = e^{2U(\rho, z)} dt^2 - e^{-2U(\rho, z)} \left[ e^{2q(\rho, z)} \left( d\rho^2 + dz^2 \right) + \rho^2 d\phi^2 \right], \quad (25)$$

with the functions $U$ and $q$ satisfying the equations

$$\begin{align*}
\Delta U &= 0, \\
\partial_\rho q &= \rho \left( (\partial_\rho U)^2 - (\partial_z U)^2 \right), \\
\partial_z q &= 2\rho \partial_\rho U \partial_z U.
\end{align*} \quad (26)$$

Note that the natural spatial slices of (25) are precisely those under consideration (6) as initial data slices. If the topology of the initial data slice is trivial, i.e. $\mathbb{R}^3$, then by Liouville’s theorem, $U = 0$ ($U$ vanishes at infinity due to asymptotic flatness). The only static solution in Weyl’s class with $\mathbb{R}^3$ topology is Minkowski space. To get non-trivial static initial data, we will need to allow non-trivial topology. The standard example is provided by the Schwarzschild data discussed in the following subsection. Prior to this we explicitly check the validity of Brill’s equation (21) for the spatial slices in Weyl’s class. Using equations (26) one easily shows that

$$\frac{1}{4} (\nabla U)^2 + \frac{1}{4} \Delta q - \frac{1}{4\rho} \frac{\partial q}{\partial \rho} = 0. \quad (27)$$

Since also $\Delta U = 0$ and $R = 0$ for the spatial slices in Weyl’s class, equation (21) holds.

2 A $t$–$\phi$-symmetric initial data set is defined to be an initial data set leading to a $t$–$\phi$-symmetric spacetime, i.e. a spacetime symmetric under a simultaneous change of sign of the time coordinate $t$ and the azimuthal coordinate $\phi$. 
3.2. Schwarzschild initial data

The function $U$ in (25) leading to Schwarzschild data can be obtained by calculating the Newtonian potential of a rod with unit mass density occupying the $z$-axis from $-M$ to $M$, as depicted in the figure below.

\[ r_1^2 = \rho^2 + (z + M)^2, \quad (28) \]
\[ r_2^2 = \rho^2 + (z - M)^2. \quad (29) \]

The Newtonian potential at the point $(\rho, z)$ is

\[ U_S(\rho, z) = -\frac{1}{2} \int_{-M}^{M} \frac{1}{\sqrt{\rho^2 + (z - \tilde{z})^2}} \, d\tilde{z} \]
\[ = -\frac{1}{2} \log \left( \frac{M + z + r_1}{-M + z + r_2} \right) = -\frac{1}{2} \log \left( \frac{r_1 + r_2 + 2M}{r_1 + r_2 - 2M} \right). \quad (30) \]

The function $q$ is then found from equations (26) to be

\[ q_S(\rho, z) = \frac{1}{2} \log \left( \frac{(r_1 + r_2)^2 - 4M^2}{4r_1r_2} \right). \quad (31) \]

It will be useful to work in spheroidal coordinates $(\lambda, \mu)$. They are defined by the relations

\[ 2\lambda = r_1 + r_2, \quad (32) \]
\[ 2\mu M = r_1 - r_2, \quad (33) \]

with inverse

\[ r_1 = \lambda + \mu M, \quad (34) \]
\[ r_2 = \lambda - \mu M, \quad (35) \]

and hence

\[ \rho^2 = (\lambda^2 - M^2)(1 - \mu^2), \quad (36) \]
\[ z = \lambda \mu. \quad (37) \]

Note that $\mu \in [-1, 1]$. The transformations

\[ \frac{\partial}{\partial \lambda} = \mu \frac{\partial}{\partial z} + \lambda \sqrt{\frac{1 - \mu^2}{\lambda^2 - M^2}} \frac{\partial}{\partial \rho} \quad \text{and} \quad \frac{\partial}{\partial \mu} = \lambda \frac{\partial}{\partial z} - \mu \sqrt{\frac{\lambda^2 - M^2}{1 - \mu^2}} \frac{\partial}{\partial \rho}. \quad (38) \]
and hence
\[
\frac{\partial}{\partial \rho} = \sqrt{(\lambda^2 - M^2)(1 - \mu^2)}(\lambda \partial_\lambda - \mu \partial_\mu),
\]
\[
\frac{\partial}{\partial z} = \frac{1}{\lambda^2 - \mu^2 M^2}(\mu(\lambda^2 - M^2)\partial_\lambda + \lambda(1 - \mu^2)\partial_\mu),
\]
will turn out to be useful later. In spheroidal coordinates, we have
\[
U_S = -\frac{1}{2} \log \left( \frac{\lambda + M}{\lambda - M} \right),
\]
\[
q_S = \frac{1}{2} \log \left( \frac{\lambda^2 - M^2}{\lambda^2 - \mu^2 M^2} \right).
\]
One finds that
\[
d\rho^2 + dz^2 + \rho^2 d\phi^2 = (\lambda^2 - \mu^2 M^2) \left( \frac{dx^2}{\lambda^2 - M^2} + \frac{d\mu^2}{1 - \mu^2} \right) + (\lambda^2 - M^2)(1 - \mu^2) d\phi^2,
\]
and hence the metric (25) becomes
\[
ds^2 = \frac{\lambda - M}{\lambda + M} \left( dt^2 - \frac{\lambda + M}{\lambda - M} dx^2 - \frac{(\lambda + M)^2}{1 - \mu^2} d\mu^2 - ((\lambda + M)^2(1 - \mu^2)) d\phi^2. \]
To see that this is in fact the Schwarzschild metric, we finally transform to coordinates
\[
\lambda = r - M,
\]
\[
\mu = \cos \theta,
\]
to find
\[
ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]

**Summary.** We found that an initial data slice for the Schwarzschild metric can be retrieved from the ansatz (6) for arbitrary axisymmetric initial data by choosing \(U\) and \(q\) as given in (30) and (31).

**Remark.** This is just one possible choice. Another example is provided by choosing \(q = 0\) and \(e^{-2U} = \left(1 + \frac{M}{r} \right)^4\), which leads to a conformally flat, time-symmetric Kruskal slice in the Schwarzschild spacetime. However, these spatial slices are not those used in the Weyl ansatz (25).

We see that that in the Weyl form of the metric the Schwarzschild horizon is represented by a rod in the initial data. On the rod \(\lambda = M\) and \(\mu\) runs from \(-1\) to 1. The rod has area
\[
A = \int_0^{2\pi} \int_{-M}^M e^{-2U_s + qS} \rho \, dz \, d\phi = \int_0^{2\pi} \int_{-1}^1 4M^2 \, d\mu \, d\phi = 16\pi M^2,
\]
and the normal derivative
\[
e^{U_s - qS} \frac{\partial A}{\partial \rho} \bigg|_{\text{rod}} = \frac{1}{2} \sqrt{1 - \mu^2} \int_0^{2\pi} \int_{-M}^M e^{-2U_s + qS} \left(1 + \rho \partial_\rho(-2U_s + qS)\right) \, dz \, d\phi = 0
\]
vanishes on the rod showing that it is indeed a minimal surface. Equation (49) is most easily seen in spheroidal coordinates, where one computes (now using (39) and (40))
\[
\rho \frac{\partial}{\partial \rho} U_S = M\lambda \frac{1 - \mu^2}{\lambda^2 - \mu^2 M^2},
\]
and hence
\[ \frac{\partial}{\partial \rho} U_S \bigg|_{\text{rod}} = 1, \tag{51} \]
as well as
\[ \frac{\partial}{\partial \rho} q_S = M^2 \frac{1 - \mu^2}{\lambda^2 - \mu^2 M^2}, \tag{52} \]
and therefore
\[ \frac{\partial}{\partial \rho} q_S \bigg|_{\text{rod}} = 1, \tag{53} \]
from which (49) follows.

### 3.3. Non-static data: deformations of Schwarzschild

Consider now a deformation of the Schwarzschild data:
\[ U = U_S + \tilde{U} \quad \text{and} \quad q = q_S + \tilde{q}, \tag{54} \]
where we make the following assumptions about the deformation \((\tilde{U}, \tilde{q})\).

1. No new minimal surface should be created by the deformation. (Note, however, that the deformation does not have to be small in general.)
2. The rod should remain a minimal surface. This implies that \(\partial_{\rho}(2\tilde{U} - \tilde{q}) = 0\) will hold on the rod.
3. The area of the rod should be unaltered under the deformation: \[ A = 8\pi M \int_{-M}^{M} \exp(-2\tilde{U} + \tilde{q}) \, dz = 16\pi M^2. \]

We will eventually need the stronger assumption.

3’. The deformation \((\tilde{U}, \tilde{q})\) has support only outside the rod, i.e. in particular \(\tilde{U} = \tilde{q} = 0\) on the rod.

Assumption 3’ implies 2 and 3. Near infinity, we assume the asymptotics
\[ \tilde{U} \sim -\frac{\Delta M}{r} \quad \text{and} \quad \tilde{q} \sim \frac{1}{r^{1+\epsilon}}, \tag{55} \]
and that derivatives lower the power of \(r\) by 1. Integrating (21) now yields the formula
\[
\Delta M = \frac{1}{16\pi} \int_{\text{vol}} R \, e^{-2\tilde{U} + 2\tilde{q}} + \frac{1}{8\pi} \int_{\text{vol}} (\nabla \tilde{U})^2 - \frac{1}{4\pi} \int_{\text{rod}} (-\tilde{n}) \nabla \left( \tilde{U} - \frac{1}{2} \tilde{q} \right) \\
+ \frac{1}{4} \int_{-M}^{M} \, dz \, \tilde{q} + \frac{1}{4\pi} \int_{\text{rod}} \tilde{U} (-\tilde{n}) \nabla U_S - \frac{1}{4\pi} \int_{\text{vol}} \tilde{U} \, \Delta U_S, \tag{56} \]
where the last term vanishes because \(U_S\) is harmonic.

If we introduce the quantity
\[ E = \frac{1}{16\pi} \int_{\text{vol}} R \, e^{-2\tilde{U} + 2\tilde{q}} + \frac{1}{8\pi} \int_{\text{vol}} (\nabla \tilde{U})^2 \geq 0, \tag{57} \]
we can write
\[
\Delta M = E - \frac{1}{2} \int_{-M}^{M} \left( \tilde{U} - \frac{1}{2} \tilde{q} \right) \bigg|_{\rho=0} \, dz + \frac{1}{2} \int_{-M}^{M} \rho \frac{\partial}{\partial \rho} \left( \tilde{U} - \frac{1}{2} \tilde{q} \right) \bigg|_{\rho=0} \, dz \tag{58} \]
describing the change of mass under the deformation considered. The last term of (58) vanishes by assumption 2. Unfortunately, assumption 3 will only give an upper bound on the second term of (58), since by assumption 3 and the inequality $e^x \geq 1 + x$ we have

$$A = 16\pi M^2 = 8\pi M \int_{-M}^{M} \exp(-2U + \tilde{q}) \, dz \geq 16\pi M^2 + 8\pi M \int_{-M}^{M} (-2\tilde{U} + \tilde{q}) \, dz,$$

and therefore

$$-\frac{1}{2} \int_{-M}^{M} \left( \tilde{U} - \frac{1}{2} \tilde{q} \right) \, dz \leq 0.$$ 

Consequently, the second term in (58) could in principle spoil the inequality we want to obtain. On the other hand, assuming 3', equation (58) immediately yields a version of the Riemannian Penrose inequality.

Lemma 3.1. For axisymmetric, $t$–$\phi$-symmetric\(^3\) deformations ($\tilde{U}, \tilde{q}$) of Schwarzschild initial data satisfying assumptions 1 and 3', the Riemannian Penrose inequality

$$M \geq \sqrt{\frac{A}{16\pi}}$$

holds.

Proof. If the deformation has support only outside the rod, both integral terms in (58) vanish. Consequently, the mass increases by $E$ while the area remains unaltered. $\square$

One would really like to show more, namely that the inequality (61) does not only hold for arbitrary axisymmetric, $t$–$\phi$-symmetric deformations of Schwarzschild away from the horizon but for general axisymmetric, $t$–$\phi$-symmetric initial data containing a single horizon. One would like to show the following statement, referred to in the future as 'RPI'.

Riemannian Penrose inequality (RPI). Given any asymptotically flat, axisymmetric, $t$–$\phi$-symmetric initial data with apparent horizon of area $A$. Then (61) holds with equality if and only if the initial data is Schwarzschild.

Though we will not be able to establish the above statement by our methods in this paper, we will indicate in the following how one could proceed and what final obstacle one encounters.

To establish RPI it would suffice to show that any axisymmetric, $t$–$\phi$-symmetric initial data can be written in Weyl form as a ‘deformed Schwarzschild data’ with the deformation satisfying $-2\tilde{U} + \tilde{q} \equiv 0$ on the rod. The following lemma shows that it is at least always possible to transform the initial data into Weyl form (62) using a harmonic map. However, we cannot prove that the functions $U$ and $q$ obtained by bringing the data into Weyl form have to agree with their corresponding Schwarzschild values on the rod.

Lemma 3.2. Any axisymmetric, $t$–$\phi$-symmetric initial data set with single connected apparent horizon of area $A$ can—in some coordinates—be written in the form

$$g = e^{-2U + 2q} \left( d\rho^2 + dz^2 \right) + \rho^2 e^{-2U} \, d\phi^2,$$

such that the minimal surface is a rod of area $A$ on the $z$-axis between $-M$ and $M$ for some $M$.

\(^3\) We remind the reader that for $t$–$\phi$-symmetric initial data the notion of an apparent horizon and a minimal surface coincide.
Proof. Given any axisymmetric, $t$-$\phi$-symmetric initial data with apparent horizon of area $A$, we can construct harmonic coordinates $(\hat{\rho}, \hat{z})$ by solving a particular conformal mapping problem. We have to solve

$$\Delta \hat{\rho}(\rho, z) = 0$$

(63)

with boundary condition $\hat{\rho} = 0$ on the $\rho$-axis and on the apparent horizon $T$ and $\hat{\rho}$ should tend to the distance from the axis, i.e. $\rho$, at infinity

This is a classical Dirichlet problem with Lipschitz boundary, admitting a unique solution. The harmonic analogue of $\hat{\rho}, \hat{z}$, follows from the Cauchy–Riemann equations

$$\partial_\rho \hat{z} = -\partial_z \hat{\rho},$$

(64)

$$\partial_z \hat{z} = \partial_\rho \hat{\rho}.$$  

(65)

The coordinate transformation $(\rho, z) \rightarrow (\hat{\rho}, \hat{z})$ maps the minimal surface to a rod on the $z$-axis, lying between $-M$ and $M$, say. In these coordinates, the metric can be written as

$$g = e^{-2U + 2q}(d\hat{\rho}^2 + d\hat{z}^2) + \hat{\rho}^2 e^{-2U} d\phi^2$$

(66)

for some functions $U$ and $q$, unique up to a constant, which is determined by asymptotic flatness. □

Now given any axisymmetric, $t$–$\phi$-symmetric initial data with apparent horizon of area $A$, we can apply lemma 3.2, mapping the minimal surface to a rod between $-M$ and $M$, say, and obtaining unique functions $U$ and $q$, which are obviously singular on the rod. If we split them according to

$$U = U_S + \tilde{U} \quad \text{and} \quad q = q_S + \tilde{q},$$

(67)

where we choose $U_S, q_S$ as in (30) and (31), we find that assumption 2 is satisfied and the second term in (58) vanishes. Unfortunately, we do not know anything about the behaviour of $\tilde{U}, \tilde{q}$ (and hence about the first term in (58)) on the rod. One way to prove RPI would therefore be to show that the functions $U$ and $q$ obtained implicitly by lemma 3.2 satisfy the equation

$$-2U + q = -2U_S + q_S$$

(68)

on the rod. This would mean that the functions $U$ and $q$, however complicated they look like globally, always admit the same blow-up behaviour as in the Schwarzschild case, when restricted to the rod. However, we have been unable to show that this follows from the harmonic map of lemma 3.2 and therefore the general case remains unproved by our methods.

4. Generalization to 4+1 dimensions: regular data

In this section, we consider regular four-dimensional initial data for the five-dimensional theory of Einstein gravity. The initial data should admit two Killing vectors with $U(1)$-orbits.
4.1. Preliminaries

The general metric has the form
\[
g = q_{AB} \, dx^A \, dx^B + Y^2 \, dx^2 + Z^2 \, d\alpha \, d\beta + X^2 \, d\beta^2
\]
\[= e^{2(f + U + V)(\rho, \omega)} (\rho^2 + \rho^2 \cos^2 \omega \, d\alpha \, d\beta) + e^{2U(\rho, \omega)} \rho^2 \sin \omega \cos \omega \, d\alpha \, d\beta + e^{2V(\rho, \omega)} \rho^2 \sin^2 \omega \, d\alpha \, d\beta.
\]  
Equation (69)

Without the cross term we would have rectangular tori. In general, |W| < 1 must hold to ensure that the metric on the torus is positive definite. Note that the choice \(f = U = V = 0\) corresponds to the flat metric

\[
\delta = dx^2 + dy^2 + du^2 + dv^2
\]
written in the coordinates
\[
x + iy = \rho \sin \omega e^{i\beta}, \quad u + iv = \rho \cos \omega e^{i\beta}.
\]
The range of coordinates for (69) is
\[
\alpha \in (0, 2\pi], \quad \beta \in (0, 2\pi], \quad \omega \in \left(0, \frac{\pi}{2}\right).
\]

There are coordinate singularities at \(\omega = 0\) and \(\omega = \frac{\pi}{2}\). To avoid conical singularities, we have to assume certain behaviour of the functions \(f, U, V, W\). We write the metric in the orthogonal coordinates given above

\[
g = dx^2 \left(\frac{e^{2(f + U + V)} x^2}{x^2 + y^2} + \frac{e^{2U} y^2}{x^2 + y^2}\right) + dy^2 \left(\frac{e^{2(f + U + V)} y^2}{x^2 + y^2} + \frac{e^{2U} x^2}{x^2 + y^2}\right) \\
+ 2dx \, dy \, xy \left(\frac{e^{2(f + U + V)}}{x^2 + y^2} - \frac{e^{2U}}{x^2 + y^2}\right) + 2du \, dv \, \left(\frac{e^{2(f + U + V)} v^2}{u^2 + v^2} - \frac{e^{2V} u^2}{u^2 + v^2}\right) \\
+ du^2 \left(\frac{e^{2(f + U + V)} u^2}{u^2 + v^2} + \frac{e^{2U} v^2}{u^2 + v^2}\right) + dv^2 \left(\frac{e^{2(f + U + V)} v^2}{u^2 + v^2} + \frac{e^{2V} u^2}{u^2 + v^2}\right) \\
- dx \, dv \left(\frac{u \, v \, W \, e^{U + V}}{\sqrt{(x^2 + y^2)(u^2 + v^2)}}\right) - dy \, du \left(\frac{x \, v \, W \, e^{U + V}}{\sqrt{(x^2 + y^2)(u^2 + v^2)}}\right) \\
+ dx \, du \left(\frac{u \, v \, W \, e^{U + V}}{\sqrt{(x^2 + y^2)(u^2 + v^2)}}\right) + dy \, dv \left(\frac{x \, v \, W \, e^{U + V}}{\sqrt{(x^2 + y^2)(u^2 + v^2)}}\right).
\]  
Equation (73)

Regularity (avoidance of conical singularities) requires the cross terms to vanish on the axes of the Clifford tori, i.e.

\[
2f - U + V = 0 \quad \text{for} \quad \omega \to 0,
\]
\[
2f + U - V = 0 \quad \text{for} \quad \omega \to \frac{\pi}{2},
\]  
and
\[
W \to 0 \quad \text{for} \quad \omega \to 0 \quad \text{and} \quad \omega \to \frac{\pi}{2}.
\]  
Equation (74) and (75)

The functions \(f\) and \(W\) should approach zero for \(\rho \to \infty\) like \(f \sim \frac{1}{\rho^2}\), and \(W \sim \frac{1}{\rho^2}\) with \(\epsilon, \delta > 0\) in order not to contribute to the ADM mass. (Again, taking derivatives should lower the order of \(\rho\) by 1.) The latter is encoded in the lowest order term of \(U\) and \(V\):

\[
U \sim \frac{\mu}{\rho^2} = \frac{2M}{3\pi \rho^2} \quad \text{and} \quad V \sim \frac{\mu}{\rho^2} = \frac{2M}{3\pi \rho^2}
\]  
Equation (76)

for \(\rho \to \infty\), where we have defined \(\mu = \frac{3M}{2\pi}\) to keep the notation tidy.
4.2. Proof of positive mass: regular data, $W = 0$

In this subsection, we perform the proof of positive mass for the case $W = 0$, i.e. no cross term in $\alpha$ and $\beta$. Geometrically, this corresponds to an additional $\mathbb{Z}_2$ symmetry. One can think of the Clifford tori as being rectangular in this case. The data are supposed to be regular: they have topology $\mathbb{R}^4$ and do not contain apparent horizons.

We can perform two Kaluza–Klein reductions of the metric (69), first along $\partial_\alpha$, then along $\partial_\beta$. The result is

$$\mathcal{R} = \mathcal{R} - \frac{2}{X} (\nabla^2 X) - \frac{2}{Y} (\nabla^2 Y) - \frac{2}{XY} \nabla_q X \nabla_q Y. \quad (77)$$

Note that

$$\Delta^{(2)} R = -2e^{-2f-U-V} \Delta^{(2)}_{\text{flat}} \left( f + \frac{U}{2} + \frac{V}{2} \right), \quad (78)$$

$$\Delta^{(2)}_{\text{flat}} = e^{-2f-U-V} \Delta^{(2)}_{\text{flat}}, \quad (79)$$

$$\Delta^{(4)}_{\text{flat}} = \frac{\partial^2}{\partial \rho^2} + \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \omega^2} + \frac{1}{\rho^2} \left( \frac{\cos \omega}{\sin \omega} - \frac{\sin \omega}{\cos \omega} \right) \frac{\partial}{\partial \omega}. \quad (80)$$

We compute the terms on the right-hand side of (77). We will omit the index ‘flat’ from now on since all gradients and Laplacians refer to the flat metric.

$$\frac{\Delta^{(2)} X}{X} = \Delta^{(4)} V + (V_\rho)^2 + \frac{V_\rho}{\rho^2} \left( \frac{\cos \omega}{\sin \omega} + \frac{\sin \omega}{\cos \omega} \right), \quad (81)$$

$$\frac{\Delta^{(2)} Y}{Y} = \Delta^{(4)} U + (U_\rho)^2 + \frac{U_\rho}{\rho^2} \left( \frac{\cos \omega}{\sin \omega} + \frac{\sin \omega}{\cos \omega} \right), \quad (82)$$

$$\frac{\nabla X \nabla Y}{XY} = \frac{U_\rho + V_\rho}{\rho} + U_\rho V_\rho + \frac{U_\rho V_\omega}{\rho^2} + \frac{1}{\rho^2} \left( \frac{\cos \omega}{\sin \omega} - \frac{\sin \omega}{\cos \omega} \right) U_\omega \quad (83)$$

$$\mathcal{R} \left( -\frac{1}{2} e^{2f-U+V} \right) = \Delta^{(2)} \left( f + \frac{U}{2} + \frac{V}{2} \right) = \left[ \Delta^{(4)} - \frac{2}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \left( \frac{\cos \omega}{\sin \omega} - \frac{\sin \omega}{\cos \omega} \right) \frac{\partial}{\partial \omega} \right] \left( f + \frac{U}{2} + \frac{V}{2} \right). \quad (84)$$

Hence, equation (77) becomes

$$-\frac{1}{2} \mathcal{R} e^{2f-U+V} = \frac{3}{2} \Delta U + \frac{3}{2} \Delta V + \Delta f + \frac{1}{2} (\nabla U)^2 + \frac{1}{2} (\nabla V)^2 + \frac{1}{2} (\nabla U + \nabla V)^2 + \frac{1}{2 \rho^2 \sin \omega \cos \omega} \partial_{\omega}(U - V) - \frac{2}{\rho} \partial_\rho f + \frac{1}{\rho^2} \left( \frac{\sin \omega}{\cos \omega} - \frac{\cos \omega}{\sin \omega} \right) \partial_{\omega} f. \quad (85)$$

We now integrate over $\mathbb{R}^4$. The term on the left-hand side will be manifestly negative since the Ricci scalar is positive by the constraint equations for maximal data. The first two terms on the right-hand side will give $-8M\pi$ by assumption (76). The term $\Delta f$ vanishes upon integration because $\partial_\rho f$ decays like $1/\rho$. The terms in the second line are manifestly positive. Finally, we will now show that the terms in the last two lines cancel when integrated over $\mathbb{R}^4$.

We have

$$4\pi^2 \cdot \frac{1}{2} \int \rho \partial_\omega(U - V) \, d\rho \, d\omega = 4\pi^2 \cdot \frac{1}{2} \int \rho \left[ (U - V) \left( \rho, \frac{\pi}{2} \right) - (U - V)(\rho, 0) \right] d\rho \quad (86)$$
and

\[-4\pi^2 \cdot 2 \int \rho^2 f,\rho \sin \omega \cos \omega \, d\rho \, d\omega = -4\pi^2 \cdot 2 \int \left[ \rho^2 f(\rho,\omega) \right]_{\rho=0}^{\rho=\infty} \sin \omega \cos \omega \, d\omega \]

\[+ 4\pi^2 \cdot 4 \int \rho f \sin \omega \cos \omega \, d\rho \, d\omega, \quad (87)\]

and

\[4\pi^2 \int \rho (\sin^2 \omega - \cos^2 \omega) \partial_\omega f \, d\rho \, d\omega = 4\pi^2 \int \rho \left[ f \left( \rho, \frac{\pi}{2} \right) + f(\rho,0) \right] \, d\rho \]

\[- 4\pi^2 \cdot 4 \int \rho f \sin \omega \cos \omega \, d\rho \, d\omega. \quad (88)\]

The last term of (88) cancels with the last term of (87). The sum of expression (86) and the first term of (88) vanishes due to the regularity conditions (74) imposed on the axes. Finally, the first term of (87) vanishes because \( f \) falls off faster than \( \frac{1}{\rho^2} \) at infinity as assumed above.

We can summarize the result in the formula

\[M = -\frac{1}{16\pi} \int_{\mathbb{R}^4} \left( e^{c_1 + c_2} \, d^4x + \frac{1}{8\pi} \right) \int_{\mathbb{R}^4} \left[ \frac{1}{2} (\nabla U)^2 + \frac{1}{2} (\nabla V)^2 + \frac{1}{2} (\nabla W + \nabla V)^2 \right] \, d^4x, \quad (89)\]

which clearly shows \( M \geq 0 \).

4.3. The case \( W \neq 0 \)

In the \( W \neq 0 \) case, the calculations are much more involved but straightforward. The analogue of (85) reads

\[-\frac{1}{2} \int_R e^{c_1 + c_2} \, d^4x + \frac{1}{8\pi} \int \left[ \frac{1}{2} (\nabla U)^2 + \frac{1}{2} (\nabla V)^2 + \frac{1}{2} (\nabla W + \nabla V)^2 \right] \, d^4x, \quad (90)\]

We already know how to deal with the terms of the first four lines from the previous section. We will now use partial integration to handle the terms in the fifth line including the function \( W \) only. We observe that

\[\int \left( -\frac{W \triangle W}{1-W^2} - \frac{3}{4} \frac{(\nabla W)^2}{(1-W^2)^2} - \frac{W^2(\nabla W)^2}{(1-W^2)^2} - \frac{W}{1-W^2} \frac{1}{\rho} \right) \rho^3 \sin \omega \cos \omega \, d\rho \, d\omega \, d\alpha \, d\beta \]

\[= \int \left( \frac{1}{4} \frac{(\nabla W)^2}{(1-W^2)^2} + \log(1-W^2) \frac{1}{\rho^2} \right) \rho^3 \sin \omega \cos \omega \, d\rho \, d\omega \, d\alpha \, d\beta. \quad (91)\]

All terms on the right-hand side are manifestly non-negative. We will need the first and the second term of (91) in the following to make other expressions manifestly non-negative. Next,
we collect the terms of (90) involving a derivative of $U$ or $V$ in $\rho$, borrowing also one of the manifestly non-negative terms we just obtained in (91). We will show non-negativity of the following expression:

$$(U,\rho)^2 + (V,\rho)^2 + U,\rho V,\rho + \frac{W^2(W,\rho)^2}{(1 - W^2)^2}$$

$$- \frac{3}{2} \frac{W}{1 - W^2} (U,\rho W,\rho + V,\rho W,\rho) + \frac{1}{4} \frac{W^2}{1 - W^2} (U,\rho - V,\rho)^2 \geq 0.$$  

(92)

(The last term in the first line of (92) comes from (91).) Defining $a = U,\rho$, $b = V,\rho$, $c = \frac{W W,\rho}{W^2}$, we need to show

$$a^2 + b^2 + c^2 + ab - \frac{3}{2} bc - \frac{3}{2} ac \geq 0.$$  

For this, we simply write the left-hand side as

$$\frac{1}{4} ((a - b)^2 + c^2) + \frac{3}{4} (a + b - c)^2 \geq 0.$$  

(94)

Next, we turn to the terms of (90) involving an $\omega$ derivative and the term just involving trigonometric functions. Collecting these terms (again borrowing two manifestly negative terms of (91)) we have to show non-negativity of the expression

$$\frac{1}{4 \rho^2} \frac{W^2}{1 - W^2} \left( \begin{array}{c}
U,\omega - V,\omega - \frac{W,\omega}{W} \cos(2\omega) + \frac{1}{\sin \omega \cos \omega} \\
+ \left[ - \frac{1}{4 \rho^2} (W,\omega)^2 \cos^2(2\omega) + \frac{1}{4 \rho^2} \frac{(W,\omega)^2}{1 - W^2} \right] \\
+ \frac{1}{2} \frac{W U,\omega W,\omega}{\rho^2 (1 - W^2)} \cos(2\omega) - \frac{1}{2} \frac{W V,\omega W,\omega}{\rho^2 (1 - W^2)} \cos(2\omega) + \frac{W^2 (W,\omega)^2}{\rho^2 (1 - W^2)^2} \\
+ \frac{1}{\rho^2} \left( (U,\omega)^2 + (V,\omega)^2 + U,\omega V,\omega \right) - \frac{3}{2} \frac{W}{\rho^2 (1 - W^2)} (U,\omega W,\omega + V,\omega W,\omega) \geq 0.
\end{array} \right.$$  

(95)

(Note that the second term in the second line and the last term in the third line of (95) are taken from (91).) The terms in the first and second lines of the expression (95) are already manifestly non-negative. For the remaining terms, we define $A = \frac{U,\omega}{\rho}$, $B = \frac{V,\omega}{\rho}$ and $C = \frac{W W,\omega}{\rho (1 - W^2)}$ and show that

$$A^2 + B^2 + C^2 + AB - AC \left( \frac{1}{2} - \frac{1}{4} \cos(2\omega) \right) + BC \left( \frac{1}{2} + \frac{1}{4} \cos(2\omega) \right) \geq 0.$$  

(96)

To see this, we just write the left-hand side as

$$\frac{1}{2} (A + B - C)^2 + \frac{3}{4} (A + B - C)^2 + \frac{3}{4} (A - B - C)^2 + \frac{3}{4} (A + B - C)^2$$

$$+ \frac{1}{4} (A - B + C \cos(2\omega))^2 + \frac{1}{4} C^2 (1 - \cos^2(2\omega)) \geq 0,$$  

(97)

making it manifestly non-negative. We finally obtain the formula

$$M = \frac{1}{16 \pi} \int_{\mathbb{R}^4} \frac{1}{4 e^{2/4} + V} d^4 x$$

$$+ \frac{1}{8 \pi} \int_{\mathbb{R}^4} \left[ - \log(1 - W^2) \frac{1}{\rho^2} + \frac{1}{4} \frac{(W,\rho)^2}{1 - W^2} + \frac{1}{4 \rho^2} \frac{(W,\omega)^2}{1 - W^2} \right]$$

$$+ \frac{1}{4} ((a - b)^2 + c^2) + \frac{3}{4} (a + b - c)^2 + \frac{3}{4} (A - B - C)^2$$

$$+ \frac{1}{4} (A - B + C \cos(2\omega))^2 + \frac{1}{4} C^2 (1 - \cos^2(2\omega))$$

$$+ \frac{1}{4} \frac{W^2}{1 - W^2} \left( A - B - \frac{W,\omega}{W \rho} \cos(2\omega) + \frac{1}{\rho \sin \omega \cos \omega} \right)^2$$

$$+ \frac{1}{4} \frac{W^2}{1 - W^2} (a - b)^2.$$  

(98)
5. Generalization to 4 + 1 dimensions: black holes

In this section, we generalize the Brill formula to include black holes and derive a Riemannian Penrose inequality for the initial data (cf lemma 5.1). We restrict to the case of rectangular tori.

5.1. The generalized Weyl class

The analogue of the four-dimensional Weyl class for orthogonal tori was obtained in [12]. Here we will use a slightly different representation and state the relevant formulae in both toroidal and standard \( r-z \) orbit space coordinates. The metric is

\[
g = -e^{-2U(\rho, \omega) - 2V(\rho, \omega)} \, dt^2 + e^{2f(\rho, \omega)+U(\rho, \omega)+V(\rho, \omega)} (d\rho^2 + \rho^2 \, d\omega^2) + e^{2U(\rho, \omega)} \rho^2 \sin^2 \omega \, d\alpha^2 + e^{2V(\rho, \omega)} \rho^2 \cos^2 \omega \, d\beta^2,
\]

with Einstein equations

\[
\Delta_{\text{flat}}^{(4)} U = \left[ \frac{\partial^2}{\partial \rho^2} + \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \omega^2} + \frac{1}{\rho^2} \left( \frac{\cos \omega}{\sin \omega} - \frac{\sin \omega}{\cos \omega} \right) \frac{\partial}{\partial \omega} \right] U(\rho, \omega) = 0,
\]

\[
\Delta_{\text{flat}}^{(4)} V = \left[ \frac{\partial^2}{\partial \rho^2} + \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \omega^2} + \frac{1}{\rho^2} \left( \frac{\cos \omega}{\sin \omega} - \frac{\sin \omega}{\cos \omega} \right) \frac{\partial}{\partial \omega} \right] U(\rho, \omega) = 0,
\]

and

\[
-\frac{4}{\rho} \partial_\rho f + \frac{4}{\rho^2} \frac{\cos(2\omega)}{\sin(2\omega)} \partial_\omega f = -2[(\partial_\rho U)^2 + \partial_\rho U \partial_\rho V + (\partial_\rho V)^2]
+ \frac{2}{\rho^2} [(\partial_\alpha U)^2 + \partial_\alpha U \partial_\alpha V + (\partial_\alpha V)^2] + \frac{1}{\rho^2 \cos \omega \sin \omega} (\partial_\omega U - \partial_\alpha V),
\]

\[
-\frac{2}{\rho} \frac{\cos(2\omega)}{\sin(2\omega)} \partial_\rho f - \frac{2}{\rho^2} \partial_\omega f = -\frac{1}{\rho} (2\partial_\rho U \partial_\rho U + 2\partial_\rho V \partial_\rho V + \partial_\rho V \partial_\alpha U + \partial_\rho U \partial_\alpha V)
+ \frac{1}{\rho \sin(2\omega)} (\partial_\alpha V - \partial_\alpha U).
\]

The relation of the toroidal variables \( \rho \) and \( \omega \) to the standard \( r-z \) orbit space variables used in [12] is given by

\[
r = \frac{1}{4} \rho^2 \sin(2\omega),
\]

\[
z = \frac{1}{4} \rho^2 \cos(2\omega),
\]

such that in \( z-r \) coordinates the metric (99) reads

\[
g = -e^{-2U(r,z) - 2V(r,z)} \, dt^2 + e^{2f(r,z)+U(r,z)+V(r,z)} (dz^2 + dr^2) + e^{2U(r,z)} (-z + \sqrt{r^2 + z^2}) \, da^2 + e^{2V(r,z)} (z + \sqrt{r^2 + z^2}) \, db^2.
\]

The equations for the functions \( U \) and \( V \) translate to

\[
2\sqrt{r^2 + z^2} \left[ \partial_r^2 + \frac{1}{r} \partial_r + \partial_t^2 \right] U(r, z) = 0,
\]

\[
2\sqrt{r^2 + z^2} \left[ \partial_r^2 + \frac{1}{r} \partial_r + \partial_t^2 \right] V(r, z) = 0,
\]
involving the three-dimensional flat Laplacian just like in the three-dimensional case. It is clear how to translate equations (102) and (103) into \( r-z \) coordinates.

Using equations (100)–(103) and the fact that the Ricci scalar of the initial data slice vanishes because of the time symmetry of the slice and vacuum, one easily checks that the differential Brill formula (85) holds for the spatial slices in the five-dimensional Weyl class (99).

5.2. Schwarzschild–Tangherlini initial data

A conformally flat time-symmetric initial data slice for the Schwarzschild–Tangherlini [13] metric can be obtained by choosing \( f = W = 0 \) and

\[
U = V = \log \left( 1 + \frac{\mu}{\rho r} \right) = \log \left( 1 + \frac{2M}{3\pi \rho^2} \right)
\]

(109)
in the ansatz (69). This would lead to a spatial slice of the form

\[
g = \left( 1 + \frac{2M}{3\pi \rho^2} \right)^2 (d\rho^2 + \rho^2 d\omega^2)
\]

(110)
for which we find a minimal surface at \( \rho = \sqrt{\frac{2M}{3\pi}} \) with area

\[
A = 16\sqrt{\pi} \left( \frac{2M}{3} \right)^{\frac{3}{2}}.
\]

(111)
In particular, this slice covers both asymptotically flat ends.

However, this slice is not part of the slicing implicit in the Weyl ansatz (99). The Schwarzschild metric in Weyl coordinates is given by a certain rod representation, just as in the four-dimensional case. We will now turn to this representation, giving all formulae in \( r-z \) coordinates, i.e. the metric functions for the ansatz (106). They can easily be translated to toroidal coordinates using (104). The harmonic functions \( U \) and \( V \) are

\[
U_S = \frac{1}{2} \log \frac{\mu - z + \sqrt{(\mu - z)^2 + r^2}}{-z + \sqrt{r^2 + z^2}},
\]

(112)

\[
V_S = \frac{1}{2} \log \frac{\mu + z + \sqrt{(\mu + z)^2 + r^2}}{z + \sqrt{r^2 + z^2}}.
\]

(113)
The function \( U_S \) corresponds to the Newtonian potential induced by a rod lying on the \( z \)-axis in the interval \([0, \mu]\) whereas \( V_S \) corresponds to a rod in \([-\mu, 0]\). Asymptotically,

\[
U_S \sim \frac{\mu}{2\sqrt{r^2 + z^2}}, \quad V_S \sim \frac{\mu}{2\sqrt{r^2 + z^2}},
\]

(114)
as it should be. The function \( f_S \) is found to be given by the expression

\[
f_S = -\frac{U_S}{2} - \frac{V_S}{2} + \frac{1}{2} \log \left( \sqrt{r^2 + z^2} \right) + \frac{1}{2} \log \left( \frac{r_1 r_2}{r_1 + \mu} \right),
\]

(115)
where we have defined the distances from the rod

\[
r_1 = \sqrt{(z + \mu)^2 + r^2}, \quad r_2 = \sqrt{(z - \mu)^2 + r^2}
\]

(116)
along with the four-dimensional case. The function \( f_S \) is unique up to a constant, which is determined by asymptotic flatness. Asymptotically,

\[
f_S \sim \frac{\mu^2}{2(r^2 + z^2)} = \frac{2\mu^2}{\rho^4},
\]

(117)
as required. Inserting \( U_S, V_S \) and \( f_S \) into (106) will yield the Schwarzschild metric. Transforming to toroidal coordinates (104) and then applying the coordinate transformation (cf [12])

\[
\omega = \frac{1}{2} \arctan \left[ \frac{\sqrt{1 - \frac{4\mu}{R^2}} \tan(2\theta)}{1 - \frac{2\mu}{R^3}} \right],
\]

\[
\rho = \left( R^4 - 4\mu R^2 + 4\mu^2 \cos(2\theta) \right)^\frac{1}{4}
\]

will finally lead to the Schwarzschild metric in the standard \((t, R, \theta, \alpha, \beta)\) coordinates:

\[
g_S = -\left( 1 - \frac{4\mu}{R^2} \right) dt^2 + \left( 1 - \frac{4\mu}{R^2} \right)^{-1} dR^2 + R^2(d\theta^2 + \sin^2 \theta \, d\alpha^2 + \cos^2 \theta \, d\beta^2),
\]

where \( \mu = \frac{2M}{\pi} \) as defined above.

As a check, we compute the area of the rod using (99). The right part of the rod, located on the z-axis between 0 and \( \mu \), has area

\[
A = \int_0^{\sqrt{\pi}} (e^{f_S + \frac{1}{2}U_S + \frac{1}{2}V_S} \rho^3 \cos \omega \sin \omega) \, d\rho \, d\alpha \, d\beta.
\]

Note that this will only give half of the area of the full rod, since we also need to take the left part (located at \( \omega = \frac{\pi}{2} \)) into account. Inserting the functions \( U_S, V_S \) and \( f_S \) defined above, we find that

\[
e^{f_S + \frac{1}{2}U_S + \frac{1}{2}V_S} \rho^3 \cos \omega \sin \omega |_{\text{rod}} = 2\sqrt{\mu} \rho
\]

and hence

\[
A = 8\sqrt{\pi} \left( \frac{2M}{3} \right)^\frac{1}{2}
\]

which—if we take both halves of the rod into account—is consistent with (111), the area obtained for the conformally flat slice.

Finally, using (121) one checks that the rod between \(-\mu \) and \( \mu \) is indeed a minimal surface, namely that

\[
\frac{1}{\rho} \frac{\partial}{\partial \omega} A = \pm \rho \frac{\partial}{\partial \rho} A = 0
\]

holds on the rod.

5.3. Brill formula

Just as in the four-dimensional case, we can consider deformations of the Schwarzschild initial data:

\[
U = U_S + \tilde{U}, \quad V = V_S + \tilde{V}, \quad f = f_S + \tilde{f},
\]

where we assume the following asymptotics near infinity:

\[
\tilde{U} \sim \frac{2\Delta M}{3\pi \rho^2}, \quad \tilde{V} \sim \frac{2\Delta M}{3\pi \rho^2}, \quad \tilde{f} \sim \frac{1}{\rho^{2\mu}}.
\]
and that taking derivatives lowers the order of $\rho$ by 1. Integrating the differential Brill formula (85) will now lead to

$$\Delta M = \frac{1}{16\pi} \int \frac{4}{8\pi} \int \left[ \frac{1}{2} (\nabla U)^2 + \frac{1}{2} (\nabla V)^2 \right]$$

$$+ \frac{1}{2} \int_{\rho}^{\pi} \left[ \rho \sin \omega \cos \omega \partial_\omega \left( \frac{3}{2} \tilde{U} + \frac{3}{2} \tilde{V} \right) \right] (\rho, \frac{\pi}{2}) d\rho$$

$$- \frac{1}{2} \int_{\rho}^{\pi} \left[ \rho \sin \omega \cos \omega \partial_\omega \left( \frac{3}{2} \tilde{U} + \frac{3}{2} \tilde{V} \right) \right] (\rho, 0) d\rho$$

$$+ \frac{1}{2} \int_{\rho}^{\pi} \rho \left( \frac{3}{2} \tilde{U} + \frac{3}{2} \tilde{V} + \tilde{f} \right) \left( \rho, \frac{\pi}{2} \right) d\rho$$

$$+ \frac{1}{2} \int_{\rho}^{\pi} \rho \left( \frac{3}{2} \tilde{U} + \frac{3}{2} \tilde{V} + \tilde{f} \right) \left( \rho, 0 \right) d\rho,$$

which can be translated to $r$-$z$ coordinates, where we find the expression familiar from the three-dimensional case (cf equation (58))

$$\Delta M = \frac{1}{16\pi} \int_{\text{vol}} \left[ \frac{4}{8\pi} \int \left[ \frac{1}{2} (\nabla U)^2 + \frac{1}{2} (\nabla V)^2 \right] \right]$$

$$- \pi \int_{-\mu}^{\mu} r \frac{\partial}{\partial r} \left( \frac{3}{2} \tilde{U} + \frac{3}{2} \tilde{V} + \tilde{f} \right) \bigg|_{r=0} dz + \pi \int_{-\mu}^{\mu} \left( \frac{3}{2} \tilde{U} + \frac{3}{2} \tilde{V} + \tilde{f} \right) \bigg|_{r=0} dz. \quad (128)$$

To obtain these formulae, one uses

$$-\vec{n} \cdot \vec{n}_{\text{rod}} = \mp \frac{1}{\rho} \frac{\partial}{\partial \omega} = -\rho \frac{\partial}{\partial r}, \quad (129)$$

where the upper (lower) sign stands for the right (left) part of the rod, $\omega = 0 \ (\omega = \frac{\pi}{2})$. The vector $-\vec{n}$ is the unit-normal vector on the rod. Furthermore,

$$r \frac{\partial}{\partial r} U_{\text{axis}} \bigg|_{r=0} = \begin{cases} -1, & \text{on the right part of the rod,} \\ 0, & \text{else} \end{cases},$$

$$r \frac{\partial}{\partial r} V_{\text{axis}} \bigg|_{r=0} = \begin{cases} -1, & \text{on the left part of the rod,} \\ 0, & \text{else}. \end{cases}$$

From (127) one infers the following analogue of the lemma 3.1.

**Lemma 5.1.** For $T^2$-symmetric deformations of Schwarzschild–Tangherlini initial data $(\tilde{f}, \tilde{U}, \tilde{V})$ admitting support only outside the minimal surface, i.e. $\tilde{U} = \tilde{V} = \tilde{f} = 0$ on
the rod, the mass of the new initial data will increase, whilst the area will stay the same. Hence,

\[ M \geq \frac{3}{2} \left( \frac{A}{16\sqrt{\pi}} \right)^{\frac{3}{2}} \] (130)

holds for deformations away from the minimal surface.

To prove the four-dimensional Riemannian Penrose inequality in full generality in the \( T^2 \)-symmetric setting, it remains to show that any biaxisymmetric initial data can be written in deformed Schwarzschild form such that \( \frac{3}{2} \tilde{U} + \frac{3}{2} \tilde{V} + \tilde{f} = 0 \) always holds on the rod, leading us to the same obstacles encountered in section 3.

6. Conclusion

We have presented various generalizations of an argument used by Brill to prove positive mass of three-dimensional, axisymmetric, time-symmetric, regular initial data for Einstein’s equations. We first noted that the symmetry assumptions can be slightly weakened. Furthermore, an apparent horizon was introduced in the data, which was seen to be represented by a rod, when the data were put into Weyl’s form. A simple calculation then established a version of the Riemannian Penrose inequality, namely that all axisymmetric, \( t-\phi \)-symmetric deformations (not necessarily small) of the Schwarzschild metric away from the horizon will increase the mass while leaving the area invariant. We indicated how we hope to proceed with an elementary proof of the general case in this axisymmetric, \( t-\phi \)-symmetric class. While the Penrose inequality has been proven for the time-symmetric case in great generality, there is so far not much mathematical evidence that it will hold in higher dimensions. Here we were able to show the version of the Riemannian Penrose inequality mentioned above for four-dimensional initial data admitting an action by the torus group \( T^2 \). For the proof we derived the generalization of Brill’s formula for \( 4+1 \) dimensions and also obtained the positive mass theorem for data with trivial topology.

Although the primary objective will be the removal of the gap preventing us to prove the Riemannian Penrose inequality in full generality using our method of rods, the techniques used in this paper should also allow generalization to further interesting cases, which we hope to address in the future. On the one hand, one could include more than one apparent horizon and try to extract quantitative information about the interaction energies and the gravitational radiation produced in the development of the data (see [14] for an attempt in this direction). On the other hand, one could enquire about the inclusion of a cosmological constant or a study of the electrovacuum case. In the latter case, one may hope to prove the stronger inequality

\[ M \geq \sqrt{\frac{A}{16\pi} + q^2} \sqrt{\frac{\pi}{A}} \] (131)

as suggested in [15]. All generalizations mentioned can be attempted for both \( 3+1 \) and \( 4+1 \) dimensions.

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