HARMONIC MEASURES OF SLIT SIDES PERPENDICULAR TO THE DOMAIN BOUNDARY

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Abstract. The article is devoted to the geometry of solutions to the chordal Löwner equation which is based on comparison of singular solutions and harmonic measures for the sides of a slit in domains generated by a driving term. It is proved that harmonic measures of two sides of a slit in the upper half-plane which is perpendicular to the real axis are asymptotically equal to each other.

1. Introduction

The Löwner parametric method is one of the powerful tools in geometric function theory. The famous Löwner equation was introduced in 1923 [6]. This article is devoted to the geometry of solutions to the chordal Löwner equation which is based on comparison of singular solutions and harmonic measures for the sides of a slit in domains generated by a driving term.

The chordal version of the Löwner equation deals with the upper half-plane \( \mathbb{H} = \{ z : \text{Im} \, z > 0 \} \), \( \mathbb{R} = \partial \mathbb{H} \), and functions \( f(z, t) \) normalized near infinity by

\[
 f(z, t) = z + \frac{2t}{z} + O \left( \frac{1}{z^2} \right)
\]

which solve the chordal Löwner differential equation

\[
 \frac{df(z, t)}{dt} = \frac{2}{f(z, t) - \lambda(t)}, \quad f(z, 0) \equiv z, \quad t \geq 0,
\]

and map subdomains of \( \mathbb{H} \) onto \( \mathbb{H} \). Here \( \lambda(t) \) is a real-valued continuous driving term.

Let \( \gamma(t) \) be a simple continuous curve in \( \mathbb{H} \cup \{0\} \) with \( \gamma(0) = 0 \) and \( 0 \leq t \leq T \). Then there is a unique map \( f(z, t) : \mathbb{H} \setminus \gamma[0, t] \to \mathbb{H} \) satisfying the chordal Löwner equation. It is known that \( f(z, t) \) can be extended continuously to \( \mathbb{R} \cup \gamma(t) \), and \( f(\gamma(t), t) = \lambda(t) \). Saying that \( \gamma(t) \in C^1 \) we mean that the tangent vector \( \gamma'(t) \) exists and varies continuously on \( [0, T] \).

The extended function \( f(z, t) \) maps \( \gamma[0, t] \) onto a segment \( I = [f_2(0, t), f_1(0, t)] \) while \( \mathbb{R} \) is mapped onto \( \mathbb{R} \setminus I \). The two functions \( f_1(0, t) \) and \( f_2(0, t) \) are singular solutions to the chordal Löwner equation which pass through the singular point \( (f(0, 0), 0) = (0, 0) \) of equation (2). The curve \( \gamma \) has two sides \( \gamma_1 \) and \( \gamma_2 \) which consist of the same points but define different prime ends, except for its tip. The
two parts \([f_2(0, t), \lambda(t)]\) and \([\lambda(t), f_1(0, t)]\) of segment \(I\) are the images of the two sides of the slit \(\gamma[0, t]\) under \(f(z, t)\).

The harmonic measures \(\omega(f^{-1}(i, t); \gamma_k(t), \mathbb{H} \setminus \gamma(t))\) of \(\gamma_k(t)\) at \(f^{-1}(i, t)\) with respect to \(\mathbb{H} \setminus \gamma(t)\) are defined by the functions \(\omega_k\) which are harmonic on \(\mathbb{H} \setminus \gamma(t)\) and continuously extended on its closure except for the endpoints of \(\gamma\), \(\omega_k|_{\mathbb{H} \setminus \gamma(t)} = 1, \omega_k|_{\mathbb{R} \cup \gamma(t)} = 0, k = 1, 2\), see, e.g., [3, §3.6]. Denote

\[
m_k(t) := \omega(f^{-1}(i, t); \gamma_k(t), \mathbb{H} \setminus \gamma(t)), \quad k = 1, 2.
\]

The main result of the article is given in Theorem 1 which we prove in Section 2.

**Theorem 1.** Let \(\gamma(t) \in C^1, \gamma(0) = 0, \Im \gamma(t) > 0 \text{ for } t > 0\), be perpendicular to the real axis \(\mathbb{R}\), and let \(f(z, t)\) map \(\mathbb{H} \setminus \gamma(0, t)\) onto \(\mathbb{H}\) and solve the chordal Löwner differential equation (2) with the diving function \(\lambda(t)\). Then

\[
\lim_{t \to 0} \frac{m_1(t)}{m_2(t)} = 1.
\]

Theorem 1 has an interplay with Theorem 2 and its Corollary 1 proved in Section 4.

**Theorem 2.** Let

\[
\lambda(t) = c \sqrt{t} + o(\sqrt{t}), \quad t \to 0,
\]

be the driving function in the chordal Löwner differential equation (2). Then the singular solutions \(f_1(0, t), f_2(0, t)\) to (2) satisfy the following conditions

\[
\lim_{t \to 0} \frac{f_1(0, t)}{\sqrt{t}} = \frac{c + \sqrt{c^2 + 16}}{2}, \quad \lim_{t \to 0} \frac{f_2(0, t)}{\sqrt{t}} = \frac{c - \sqrt{c^2 + 16}}{2}.
\]

**Corollary 1.** Let

\[
\lambda(t) = c \sqrt{t} + o(\sqrt{t}), \quad t \to 0, \quad c \neq 0,
\]

be the driving function in the chordal Löwner differential equation (2), and let the solution \(f(z, t)\) to (2) map \(\mathbb{H} \setminus \gamma(t)\) onto \(\mathbb{H}\), where \(\gamma(t)\) is a \(C^1\) simple curve. Then \(\gamma(t)\) is not perpendicular to the real axis \(\mathbb{R}\) at the origin.

The most important argument in the proof of Theorem 1 is the fact [4] that for the arc-length parameter \(s\) of the \(C^1\)-slit which is perpendicular to \(\mathbb{R}\), the function \(s = s(t)\) is expanded as

\[
s = s(t) = A \sqrt{t} + o(\sqrt{t}), \quad A \neq 0, \quad s \to +0.
\]

This result can be compared with the results in [1].

2. Proof of Theorem 1

**Proof of Theorem 1.** Let \(w = f(z, t)\) map \(\mathbb{H} \setminus \gamma(t)\) onto \(\mathbb{H}\), and let the \(C^1\)-slit \(\gamma\) satisfy the conditions of Theorem 1. For the arc-length parameter \(s\), \(\gamma = \gamma(s)\) has the representation

\[
\gamma(s) = is + o(s), \quad s \to +0.
\]
From the other side, the arc-length parameter $s$ is expanded according to (3). Substitute (3) in (4) and obtain

$$\gamma(s(t)) = iA\sqrt{t} + o(\sqrt{t}) = iA\sqrt{t} + \alpha(t)\sqrt{t}, \quad \lim_{t \to +0} \alpha(t) = 0.$$  

The function

$$h(w, t) = \sqrt{w^2 - A^2t}$$

maps $\mathbb{H} \cup \mathbb{R}$ onto $(\mathbb{H} \setminus [0, iA\sqrt{t}]) \cup \mathbb{R}$, $h(0, t) = iA\sqrt{t}$.

Denote

$$z = g_n(w, t) := \sqrt{n}f^{-1}\left(\frac{w}{\sqrt{n}}, \frac{t}{n}\right), \quad n = 1, 2, \ldots.$$  

The function $z = f^{-1}(w, t) = g_1(w, t)$ maps $\mathbb{H}$ onto $\mathbb{H} \setminus \gamma(t)$, $g_1(\lambda(t), t) = \gamma(t)$, $\gamma(t) := \gamma(s(t))$. So the functions $g_n(w, t)$ map $\mathbb{H} \setminus \gamma(t)$ where

$$\gamma_n(t) = \sqrt{n}\gamma\left(\frac{t}{n}\right) = iA\sqrt{t} + \alpha(\frac{t}{n})\sqrt{t}, \quad g_n\left(\sqrt{n}\lambda\left(\frac{t}{n}\right), t\right) = \gamma(t).$$

As soon as

$$|\gamma_n(t) - iA\sqrt{t}| = |\alpha\left(\frac{t}{n}\right)|\sqrt{t}$$

tends to 0 as $n \to \infty$ uniformly with respect to $t \in [0, T]$, the sequence of functions $g_n(w, t)$ converges to $h(w, t)$ as $n \to \infty$ uniformly on $\mathbb{H} \cup \mathbb{R}$ according to the Radó theorem [8] generalized by Markushevich [7], see also [2, p.60].

The uniform convergence of $g_n(w, t)$ to $h(w, t)$ implies convergence of corresponding coefficient sequences in boundary hydrodynamic normalization (3). Definition of $g_n(w, t)$ gives the expansion

$$g_n(w, t) = w - \frac{2t}{w} + O\left(\frac{1}{w^2}\right), \quad w \to \infty.$$  

The function $h(w, t)$ is expanded as

$$h(w, t) = w - \frac{A^2t}{2w} + O\left(\frac{1}{w^2}\right), \quad w \to \infty.$$  

Hence $A = 2$.

Denote by $\Gamma_1(t)$ the "right" side of the segment $[0, i2\sqrt{t}]$ and by $\Gamma_2(t)$ the "left" side of this segment.

Let $\gamma_1(t), \gamma_2(t)$ be the two sides of $\gamma(t)$ which are mapped onto the segments $I_1 \subset \mathbb{R}$, $I_2 \subset \mathbb{R}$ under $g_n^{-1}(z, t)$, respectively. The uniform convergence of $g_n$ to $h$ implies that $I_1(t)$ tends to $[0, A\sqrt{t}]$ and $I_2(t)$ tends to $[-A\sqrt{t}, 0]$ as $n \to \infty$.

Let $\gamma_1'(t)$, $\gamma_2'(t)$ be the two sides of $\gamma(t)$ which are mapped onto the segments $I_1' \subset \mathbb{R}$, $I_2' \subset \mathbb{R}$ under $f(z, \frac{t}{n})$, respectively. Comparing $I_{kn}$ and $I_{kn}'$, we see that $\operatorname{meas}I_{kn} = \sqrt{n}\operatorname{meas}I_{kn}'$, $k = 1, 2$, $n \geq 1$.

The harmonic measures $\omega(i; I_{kn}(t), \mathbb{H})$ of $I_{kn}(t)$ at $i$ with respect to $\mathbb{H}$ equal the angle divided over $\pi$ under which the segment $I_{kn}'(t)$ is seen from the point $i$. Similarly, the harmonic measures $\omega(i; I_{kn}(t), \mathbb{H})$ of $I_{kn}(t)$ at $i$ with respect to $\mathbb{H}$ equal the angle divided over $\pi$ under which the segment $I_{kn}(t)$ is seen from the point $i$, $k = 1, 2$, $n \geq 1$, see, e.g., [2, p.334].
Now the following equalities

\[
\lim_{t \to +0} \frac{m_1(t)}{m_2(t)} = \lim_{n \to \infty} \frac{m_1\left(\frac{t}{n}\right)}{m_2\left(\frac{t}{n}\right)} = \lim_{n \to \infty} \frac{\omega(f^{-1}(i, \frac{t}{n}), \gamma_1\left(\frac{t}{n}\right), \mathbb{H} \setminus \gamma\left(\frac{t}{n}\right))}{\omega(f^{-1}(i, \frac{t}{n}), \gamma_2\left(\frac{t}{n}\right), \mathbb{H} \setminus \gamma\left(\frac{t}{n}\right))}
\]

\[
\lim_{n \to \infty} \frac{\omega(i, I_{1n}(t), \mathbb{H})}{\omega(i, I_{2n}(t), \mathbb{H})} = \lim_{n \to \infty} \frac{\tan(\pi \omega(i, I_{1n}(t), \mathbb{H}))}{\tan(\pi \omega(i, I_{2n}(t), \mathbb{H}))} = \lim_{n \to \infty} \frac{\text{meas}_{I_{1n}}}{\text{meas}_{I_{2n}}}
\]

\[
\lim_{n \to \infty} \frac{\text{meas}_{I_{1n}}}{\text{meas}_{I_{2n}}} = \frac{\text{meas}[0, i2\sqrt{7}]}{\text{meas}[-i2\sqrt{7}, 0]} = 1
\]

lead to the conclusion desired in Theorem 1.

This chain contains 8 equality signs. We have to comment almost each of them.

The first equality sign needs a more strict explanation. In order to reduce the limit as \( t \to +0 \) to the limit as \( n \to \infty \) it is necessary to choose an arbitrary sequence \( \{j_n\} \) of positive numbers \( j_n \) such that \( j_n \to \infty \) as \( n \to \infty \). All the arguments for \( g_n \) should be repeated for \( g_{jn} \). We omitted the details and chose the sequence of natural numbers because of simplicity and evidence of repeating the arguments.

The second step uses the definition of \( m_k(t) \). The third step uses the invariance of harmonic measures with respect to conformal map \( f(z, \frac{t}{n}) \). The following step uses the limit property of the ratio of infinitesimal functions. The next step is based on elementary trigonometric formulas and limit procedures.

The 6-th step takes into account that the segments \( I_{kn} \) and \( I'_{kn} \) are proportional.

The 7-th resulting step appeared because of uniform convergence of \( g_n(w, t) \) to \( h(w, t) \), which implies that the pre-images \( I_{kn}(t) \) of the sides \( \gamma_{kn}(t) \) of \( \gamma_n(t) \) under \( g_n(w, t) \) tend to the corresponding pre-images of the sides \( \Gamma_k(t) \) of the segment \([0, i2\sqrt{7}]\) under \( h(w, t), k = 1, 2 \). The final step is clear.

This completes the proof.

Theorem 1 can be generalized for \( C^1 \)-slits \( \gamma(t) \subset \mathbb{H}, \gamma(0) = 0 \), which are tangential at the origin to the straight line under the angle \( \frac{\pi}{2}(1-c) \) to \( \mathbb{R}, -1 < c < 1 \), provided the asymptotic relation (3) is valid.

**Proposition 1.** Let \( \gamma(t) \in C^1, \gamma(0) = 0, \text{Im} \gamma(t) > 0 \) for \( t > 0 \), be tangential at the origin to the straight line under the angle \( \frac{\pi}{2}(1-c) \), \( -1 < c < 1 \), to the real axis \( \mathbb{R} \), and let \( f(z, t) \) map \( \mathbb{H} \setminus \gamma[0, t] \) onto \( \mathbb{H} \) and solve the chordal Löwner differential equation (2). Then

\[
\lim_{t \to +0} \frac{M_1(t)}{M_2(t)} = \frac{1-c}{1+c},
\]

where

\[
M_k(t) := \omega(f^{-1}(i, t); \gamma_k(t), \mathbb{H} \setminus \gamma(t)), \quad k = 1, 2,
\]

\( \gamma_1(t) \) and \( \gamma_2(t) \) are the two sides of \( \gamma(t) \), provided the asymptotic relation (3) is valid.

We omit the proof which can follow the steps of Theorem 1 where the slit perpendicular to \( \mathbb{R} \) was generated by the trivial explicit map \( h(w, t) \). Instead, the straight line under the angle \( \frac{\pi}{2}(1-c) \) is generated by the implicitly given map in (3) which solves the Löwner differential equation (2) with the driving function \( \lambda(t) = c^t \sqrt{t} \). Therefore the proof is more complicated technically but does not contain any new ideas.
3. Driving terms with higher Lipschitz orders

**Theorem 3.** Let \( f(z, t) \) be a solution to the chordal Löwner equation (2) with the driving function 
\[
\lambda(t) = At^\alpha + o(t^\alpha), \quad t \to 0, \quad \alpha > \frac{1}{2}, \quad A \neq 0.
\]
Then the singular solutions \( f_1(0, t) \) and \( f_2(0, t) \) to (2) satisfy the following relations
\[
f_1(0, t) = 2\sqrt{t} + o(\sqrt{t}), \quad f_2(0, t) = -2\sqrt{t} + o(\sqrt{t}), \quad t \to 0.
\]

**Proof.** For all \( t > 0, \)
\[
f_2(0, t) < \lambda(t) < f_1(0, t) \quad \text{and} \quad f_2(0, t) < 0 < f_1(0, t).
\]
For given \( \epsilon > 0, \) \( |\lambda(t)| < \epsilon \sqrt{t} \) for \( t > 0 \) small enough. Therefore, for such \( t > 0, \)
\[
\frac{df_1(0, t)}{dt} = \frac{2}{f_1(0, t) - \lambda(t)} > \frac{2}{f_1(0, t) + \epsilon \sqrt{t}}.
\]
Denote
\[
g(t) = \frac{f_1(0, t)}{\sqrt{t}}
\]
and obtain from (5) that
\[
\frac{dg}{dt} > \frac{1}{t} \frac{4 - g^2(t) - \epsilon g(t)}{2(g(t) + \epsilon)}, \quad 0 < t < T(\epsilon).
\]
The polynomial \( 4 - g^2 - \epsilon g \) has two roots \( g_1(\epsilon) < 0 \) and
\[
g_1(\epsilon) = \frac{\sqrt{\epsilon^2 + 16} - \epsilon}{2}, \quad 0 < g_1(\epsilon) < 2.
\]
Suppose that there exists \( t_0 \in (0, T(\epsilon)) \) such that \( g(t_0) < g_1(\epsilon) \). It follows from (6) that \( g'(t_0) > 0, \) and \( g(t) \) decreases together with \( t \) varying from \( t_0 \) to 0. This implies that
\[
\frac{4 - g^2(t) - \epsilon g(t)}{2(g(t) + \epsilon)} > c > 0, \quad 0 < t \leq t_0,
\]
and inequality (6) reduces to
\[
\frac{dg}{dt} > \frac{c}{t}, \quad 0 < t \leq t_0.
\]
Integrating (7) from \( \delta > 0 \) to \( t_0 \), we obtain
\[
g(t_0) > g(\delta) + c \log \frac{t_0}{\delta} > c \log \frac{t_0}{\delta},
\]
which contradicts the condition \( g(t_0) < g_1(\epsilon) \) for \( \delta \) small enough.
So, for all \( t \in (0, T(\epsilon)), \) \( g(t) \geq g_1(\epsilon) \). Going back to \( f_1(0, t), \) we see that
\[
f_1(0, t) \geq g_1(\epsilon) \sqrt{t}, \quad t \in (0, T(\epsilon)).
\]
Substitute (8) in (2) and obtain that
\[
\frac{df_1(0, t)}{dt} = \frac{2}{f_1(0, t) - \lambda(t)} \leq \frac{2}{g_1(\epsilon) \sqrt{t} - \epsilon \sqrt{t}}, \quad 0 < t < T(\epsilon).
\]
Integrate this inequality from 0 to $t$ and obtain that

\[ f_1(0, t) \leq \frac{4}{g_1(\epsilon) - \epsilon} \sqrt{t}, \quad 0 < t < T(\epsilon). \]  

Inequalities (8) and (9) mean together that

\[ \lambda(t) = o(f_1(0, t)), \quad t \to 0, \]

and

\[ \frac{df_1(0, t)}{dt} = \frac{2}{f_1(0, t) + o(f_1(0, t))}. \]

This leads to the first statement of Theorem 3 for $f_1(0, t)$.

The second statement of Theorem 3 for $f_2(0, t)$ is proved similarly.

\[ \square \]

4. PROOF OF THEOREM 2 AND COROLLARY 1

Proof of Theorem 2. For given $\epsilon' > 0$, $|\lambda(t) - c\sqrt{t}| < \epsilon'\sqrt{t}$ for $t > 0$ small enough. Therefore, for such $t > 0$,

\[ \frac{df_1(0, t)}{dt} = \frac{2}{f_1(0, t) - \lambda(t)} > \frac{2}{f_1(0, t) - (c - \epsilon')\sqrt{t}}. \]

Denote

\[ g(t) = \frac{f_1(0, t)}{\sqrt{t}} \]

and obtain from (10) that

\[ \frac{dg(t)}{dt} > \frac{1}{t} \frac{4 - g^2(t) + (c - \epsilon')g(t)}{2(g(t) - c + \epsilon')}, \quad 0 < t < T(\epsilon'). \]

The polynomial $4 - g^2 + (c - \epsilon')g$ has two roots $g_2(\epsilon') < 0$ and $g_1(\epsilon') = \frac{\sqrt{(c - \epsilon')^2 + 16 + c - \epsilon'}}{2}, \quad 0 < g_1(\epsilon') < 2$.

Suppose that there exists $t_0 \in (0, T(\epsilon'))$ such that $g(t_0) < g_1(\epsilon')$. It follows from (11) that $g'(t_0) > 0$, and $g(t)$ decreases together with $t$ varying from $t_0$ to 0. This implies that

\[ \frac{4 - g^2(t) + (c - \epsilon')g(t)}{2(g(t) + \epsilon')} > p > 0, \quad 0 < t \leq t_0, \]

and inequality (11) reduces to

\[ \frac{dg}{dt} > \frac{p}{t}, \quad 0 < t \leq t_0. \]

Integrating (12) from $\delta > 0$ to $t_0$, we obtain

\[ g(t_0) > g(\delta) + p \log \frac{t_0}{\delta} > p \log \frac{t_0}{\delta}, \]

which contradicts the condition $g(t_0) < g_1(\epsilon')$ for $\delta$ small enough.
So, for all \( t \in (0, T(\epsilon')) \), \( g(t) \geq g_1(\epsilon') \). Going back to \( f_1(0, t) \), we see that
\[
 f_1(0, t) \geq g_1(\epsilon') \sqrt{t}, \quad t \in (0, T(\epsilon')).
\]
Substitute (13) in (2) and obtain that
\[
 \frac{df_1(0, t)}{dt} = \frac{2}{f_1(0, t) - \lambda(t)} \leq \frac{2}{g_1(\epsilon') \sqrt{t} - (c + \epsilon') \sqrt{t}}, \quad 0 < t < T(\epsilon').
\]
Integrate this inequality from 0 to \( t \) and obtain that
\[
 f_1(0, t) \leq \frac{4}{g_1(\epsilon') - (c + \epsilon')} \sqrt{t}, \quad t \in (0, T(\epsilon')).
\]
If \( c = 0 \), then inequalities (13) and (14) prove the first statement of Theorem 2 for \( f_1(0, t) \).
Let \( c \neq 0 \). Write two inequalities (13) and (14) in the form
\[
 k_1' \sqrt{t} < f_1(0, t) < k_1'' \sqrt{t}, \quad t \in (0, T(\epsilon')).
\]
Show that inequalities (15) admit a recurrent improvement converging to the desired point. Indeed, substitute the left inequality (15) in (2) and obtain
\[
 \frac{df_1(0, t)}{dt} = \frac{2}{f_1(0, t) - c \sqrt{t} + o(\sqrt{t})} < \frac{2}{(k_1' - c) \sqrt{t} + o(\sqrt{t})}
\]
which gives after integration the improved right inequality (15)
\[
 f_1(0, t) < k_2'' \sqrt{t} + o(\sqrt{t}), \quad t \to 0, \quad k_2'' = \frac{4}{k_1' - c}.
\]
Now substitute this inequality in (2) and obtain
\[
 \frac{df_1(0, t)}{dt} > \frac{2}{(k_2'' - c) \sqrt{t} + o(\sqrt{t})}
\]
which gives after integration the improved left inequality (15)
\[
 f_1(0, t) > k_2' \sqrt{t} + o(\sqrt{t}), \quad t \to 0, \quad k_2' = \frac{4}{k_2'' - c}.
\]
Repeat the procedure and obtain at the \( n \)-th step
\[
 k_n' \sqrt{t} + o(\sqrt{t}) < f_1(0, t) < k_n'' \sqrt{t} + o(\sqrt{t}),
\]
where
\[
 k_n' = \frac{4}{k_{n-1}' - c}, \quad k_n'' = \frac{4}{k_{n-1}'' - c}.
\]
Inequalities \( k_n' > c, k_n'' > c, 4 > c(k_n' - c), 4 > c(k_n'' - c) \) are verified by elementary sources. Both sequences \( \{k_n'\} \) and \( \{k_n''\} \) converge monotonically to \( (c + \sqrt{c^2 + 16})/2 \). To every \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) such that
\[
 0 < \frac{c + \sqrt{c^2 + 16}}{2} - k_n' < \frac{\epsilon}{2}, \quad 0 < k_n'' - \frac{c + \sqrt{c^2 + 16}}{2} < \frac{\epsilon}{2}.
\]
and
\[
\frac{o(\sqrt{t})}{\sqrt{t}} < \frac{\epsilon}{2}
\]
for both terms \(o(\sqrt{t})\) in (16). This proves the first statement of Theorem 2 for \(f_1(0, t)\).

The second statement of Theorem 2 for \(f_2(0, t)\) is proved similarly.

**Proof of Corollary 1.** According to Theorem 1, for \(C^1\) curves \(\gamma(t)\) which are perpendicular to \(\mathbb{R}\), \(f_1(0, t)\) and \(f_2(0, t)\) have the same main term in the asymptotic expansion. From the other side, according to Theorem 2, \(f_1(0, t)\) and \(f_2(0, t)\) differ by their main asymptotic terms. This proves Corollary 1.

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