Forecast-Hedging and Calibration

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Abstract

Calibration means that forecasts and average realized frequencies are close. We develop the concept of forecast hedging, which consists of choosing the forecasts so as to guarantee that the expected track record can only improve. This yields all the calibration results by the same simple basic argument, while differentiating between them by the forecast-hedging tools used: deterministic and fixed point based versus stochastic and minimax based. Additional contributions are an improved definition of continuous calibration, ensuing game dynamics that yield Nash equilibria in the long run, and a new calibrated forecasting procedure for binary events that is simpler than all known such procedures.

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1 Introduction

Weather forecasters nowadays no longer say that “it will rain tomorrow” or “it will not rain tomorrow”; rather, they state that “the chance that it will rain tomorrow is x.” As long as x lies strictly between 0 and 1, they cannot be proven wrong tomorrow, whether it rains or not. However, they can be proven wrong over time. This is the case when a forecast, say x = 70%, is repeated many times, and the proportion of rainy days among those days when the forecast was 70% is far from 70%.

A forecaster is said to be (classically) calibrated if, in the long run, the actual proportions of rainy days are close to the forecasts (formally, the average difference between frequencies and forecasts—the calibration score—is small). A surprising result of Foster and Vohra (1998) shows that one may always generate forecasts that are guaranteed to be calibrated, no matter what the weather will actually be. These forecasts must necessarily be stochastic; i.e., in each period the forecast x is chosen by a randomization (e.g., with probability 1/3 the forecaster announces that the chance of rain tomorrow is x = 70%, and with probability 2/3 the forecaster announces that the chance is x = 50%), since deterministic forecasts cannot be calibrated against all possible future rain sequences (cf. Dawid 1982 and Oakes 1985). The analysis is thus from a “worst-case” point of view, which is the same as if one were facing an adversarial “rain-maker.”

Now the calibration score is discontinuous with respect to the forecasts, as it considers days when the forecast was, say, 69.9%, separately from the days when the forecast was 70%. Smoothing out the calibration score by combining, in a continuous manner, the days when the forecast was close to x before comparing the frequency of rain to x yields a continuous calibration score, which we introduce in Section 2.2. The advantage of continuous calibration is that it may be guaranteed by deterministic forecasts (i.e., after every history there is a single x that is forecasted—in contrast to a probabilistic distribution over x in the classic calibration setup of the previous paragraph). Similar concepts that appear in the literature, weak calibration (Kakade and Foster 2004, Foster and Kakade 2006)
and smooth calibration (Foster and Hart 2018), are encompassed by continuous calibration (see Appendix A.2). While the existing proofs of deterministic smooth and weak calibration are complicated, in the present paper we provide a simple proof of deterministic continuous calibration—and so of smooth and weak calibration as well. We thus propose continuous calibration as the more appropriate concept: more natural, and easier to analyze and guarantee.

In the present paper we identify specific conditions, which we refer to as forecast-hedging conditions, that guarantee that the calibration score will essentially not increase, whatever tomorrow’s weather will be. Roughly speaking, they amount to making sure that today’s calibration errors will tend to go in the opposite direction of past calibration errors (thus overshooting, where the forecast is higher than the frequency of rain, is followed by undershooting, and the other way around). This is illustrated in Section 1.2 below by a stylized simple version of forecast-hedging in the basic binary rain/no rain setup. Interestingly, it turns out to yield a new calibrated procedure in this one-dimensional case that is as simple as can be (and is simpler than the one in Foster 1999); see Section 5 for the formal analysis.

We show, first, that the main calibration results in the literature (classic, smooth, weak, almost deterministic, and continuous, introduced here) all follow from the same simple argument based on forecast-hedging. Second, we provide the appropriate forecast-hedging tools. In the classic calibration setup, they correspond to optimal strategies in finite two-person zero-sum games, whose existence follows from von Neumann’s (1928) minimax theorem, and which are mixed (i.e., stochastic) in general. In the continuous calibration setup, they correspond to fixed points of continuous functions, whose existence follows from Brouwer’s (1912) fixed point theorem, and which are deterministic. We refer to the resulting procedures as procedures of type MM and type FP, respectively. This forecast-hedging approach integrates the existing calibration results by deriving them all from the same proof scheme, while clearly differentiating between the MM-procedures and the FP-procedures, both in terms of the tool they use—minimax vs. fixed point—and in terms of being stochastic vs. deterministic. Thus classic calibration is obtained by MM-procedures, whereas continuous calibration, as well as almost deterministic calibration, by FP-procedures. A further benefit of our approach is the simple and straightforward proof that it provides of deterministic continuous calibration, and thus of deterministic smooth calibration (in contrast to the long

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5The use of the term “hedging” here is akin to its use in finance, where one deals with portfolios that are hedged against risks (by using, say, appropriate options and derivatives).
and complicated existing proof).

While calibration is stated in terms of “forecasting,” our forecast-hedging makes it clear that this is a misnomer, as there is no actual prediction of rain or no rain tomorrow (indeed, such a prediction cannot be accomplished without making some assumptions on the behavior of the rain-maker). Rather, calibration obtains by what can be referred to as “backcasting” (instead of forecasting): forecast-hedging guarantees that the past track record can essentially only improve, no matter what the weather will be.

1.1 The Economic Utility of Calibration

Now, why would one consider calibration at all? Though some forecasts are created just for fun (say, predicting a sports winner or a presidential election), other forecasts drive decision making (say, predicting the chance of rain or the chance of selling a million widgets). We will focus on forecasts that have decisions attached to them. If the forecaster is the same person as the decision maker then he can interpret the forecast in any fashion he likes and still be consistent. But, when the forecaster is different from the decision maker, it is desirable for them to be speaking the same language. To make this concrete, consider the rain forecast that a traveler hears on landing in a new city. Should an umbrella be unpacked and made ready? Or is the weather nice enough not to need one? Locals may be perfectly happy with a forecast that implies some set $U$ such that if $x \in U$ then carrying an umbrella makes sense. But, pity our poor traveler who has to figure out the set $U$ without any history. Contrast this with the world where the forecast in each city is known to be calibrated. Then our traveler can figure out a rule, say, $x > 70\%$, and dig his umbrella out if the forecast is higher than 70%. Further, this works for both the timid traveler who has a rule $x > 20\%$ and the outdoors person with a rule of $x > 99\%$. There can be many other wonderful properties of forecasts that we could hope to have (accuracy or martingality to name two), but by merely having calibration the forecasts are connected enough to outcomes to be useful to decision makers.

Calibration thus allows one to separate the problem into two pieces: the first is providing a forecast of the world, and the second is taking an action that is rational given that forecast. This model is a good way of factoring a business since a forecasting team doesn’t need to understand the nuances that go into the decision.

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6That is, the expected benefit of not being wet on a rainy day exceeds the expected cost of carrying the umbrella—and perhaps losing it someplace—on a sunny day.
making, nor does the decision team need to know the details of the most current statistical methods that go into making the forecasts. There are details that the forecasting team will be continuously worrying about, like whether a neural net is more accurate than a decision tree or a simple regression. Likewise there are details that the decision making team will be stressing over, like changing costs and updating constraints. But, as long as they are communicating via calibrated forecasts, these worries don’t need to be exposed to the other team. The forecasting team generates calibrated forecasts, and the optimization team treats these forecasts as if they were probabilities and solves their optimization problem. This factorization localizes information but still generates a globally optimal outcome.

For a concrete example, consider Figure 1 from Foster and Stine (2004). It shows two forecasts of when a customer will go bankrupt. The calibrated forecast (right side) is easy to use: a customer with a forecasted high chance of bankruptcy shouldn’t be extended further credit. The cutoff point can be created using the costs and benefits to the firm. By contrast, constructing a rule based on the un-calibrated forecast (left side) requires actually doing some statistics to figure out what a forecast of, say, “70%” means. The optimization team would have to do

Figure 1: In Foster and Stine (2004) the business problem was to forecast the chance of a person going bankrupt in the next month. Both of the above forecasts are based on a large linear model. The one on the left was obtained by a logistic regression; the one on the right, by a monotone regression. The left-hand forecast is not calibrated, whereas the right-hand forecast is calibrated and so can be used directly for decision making.

7A real-life story from a large online retailer is that an old-fashioned ARMA forecasting model was used for years. It was not calibrated and so the optimization team had learned to buy more than the forecast suggested. When the ARMA model was replaced by a modern neural net that was much more accurate and also calibrated, the retailer lost money—until the optimization team caught up with the change in the forecasting model. If both forecasts had been calibrated, there would have been much less internal stress, and the newer model would have been an easy immediate improvement.
some empirical statistics, and thus we have failed at factoring the problem into two clean pieces.

Figure 1 may incorrectly suggest that all we need to do is map a forecast through an appropriate link function that gives the corresponding average realization and all will be well. This is true for cross-sectional data and for time-series data where the link function is evaluated at a single point in time. But, in general, we would need different such functions at different points in time. Phrased in terms of our intrepid traveler, if he arrives for a second time at the same foreign city, the rule he used on the first visit may no longer apply. But, if the forecasts were calibrated, the same trivial rule would work for both visits. Mathematically, this means that a calibrated forecast must divide an arbitrary sequence into a collection of subsequences (one for each forecast value) all of which have a limit. This is the hard part. The fact that we also require a calibrated forecast to know what this limit is on each of these subsequences is a small restriction compared to guaranteeing that there are no fluctuations over time and all these limits exist.

Let us turn to the decision side of the problem. Sometimes the forecast is so strong for rain that not carrying an umbrella would entail a huge cost. Likewise, it might be that the chance of rain is so low that carrying one would be too costly. Both of these costs are relative to the best possible action one could take. But, sometimes, the forecast is close to the fence and it doesn’t really matter which action is taken. This indifference (equipoise in bio-statistics) allows one to consider randomizing between these two actions. This would cheaply allow estimating the actual costs of each action. It would allow one to compare what would happen if the counterfactual action were taken to what happens if the action that is believed to be the correct action is taken. For these reasons, there are many arguments for randomizing at the boundary. Mathematically it can be thought of as continuously switching from taking an umbrella (at the boundary plus epsilon) to never taking an umbrella (at the boundary minus epsilon). If such a continuous response function is used, then the classic definition of calibration is stronger than it needs to be. Indeed, we only care about what the approximate value of the forecast is since we will behave similarly for all such values. This is where continuous calibration comes in.

Now what is the advantage of using a weaker notion of calibration (continuous calibration is implied by classic calibration), which is also more difficult to obtain

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8We refer to this as “binning”; see Section 2.2.

9While we continue to phrase the discussion in terms of rain for simplicity, think of more meaningful circumstances, such as contextual bandits in machine learning and personalized medicine in clinical trials.
(it requires a fixed point rather than a minimax computation every period; see Section 3.4). The answer is that weakening the calibration requirement allows one to achieve the important property of leakiness of Foster and Hart (2018); namely, the forecasts remain calibrated even if the action in each period depends on the forecast (which is the case when the forecast is revealed, i.e., “leaked,” before the action is chosen). Indeed, for deterministic procedures that yield continuous calibration, the fact that at the start of each period \( t \) the forecast at \( t \) is already known (as it is fully determined by the history before \( t \)) does not matter, as continuous calibration is guaranteed for any action. By contrast, for stochastic procedures that yield classic calibration, at the start of period \( t \) only the distribution of the random forecast at \( t \) is known, and not its actual realization; if the actual realization were known, there would be action choices that would invalidate calibration, as in footnote 3. This distinction is underscored by forecast-hedging, which holds for sure in the deterministic case, and only in expectation in the stochastic case. It is just as in a two-person zero-sum game, where an optimal mixed strategy is no longer optimal if the opponent knows its pure realization, whereas an optimal pure strategy remains so even if known (the same holds for mixed vs. pure Nash equilibria). So to answer our question, we can trade off this weaker requirement of calibration for a guarantee of leakiness. Since the weakening doesn’t decrease the value of the forecast for decision making, we have gained leakiness at minimal cost.

Leakiness turns out to be the crucial property that is needed for game dynamics in general \( n \)-person games to give Nash equilibria rather than correlated equilibria. Specifically, while best replying to calibrated forecasts yields correlated equilibria as the long-run time average of play (see Foster and Vohra 1997), we show in Section 6.1 that best replying to deterministic continuously calibrated forecasts yields Nash equilibria being played in most of the periods (see Kakade and Foster 2004 and Foster and Hart 2018 for earlier, somewhat more complicated, variants of this result).

To return to forecasting, in numerous situations Bayesian methods are optimal. But, if you are using the wrong prior, a lot of the charm of Bayesian methods is lost and estimators that provide robust minimax protection might be preferred. If we could estimate the prior, then a Bayesian approach sounds pretty good. This is one of the motivations for empirical Bayesian methods (see Berger

\(^{10}\) The statements here should be understood with appropriate “approximate” adjectives throughout.

\(^{11}\) Dawid (1982) discusses the connection of calibration to posterior probabilities, whereas here we want to connect it to the priors.
Unfortunately, unless we are observing a sequence of independently and identically distributed problems for which we can truly believe there is a single prior that is common across a string of problems (see Robbins 1956), then figuring out the prior to use for the next problem is not easy. This is where calibration can play a part (see George and Foster 2000). By guaranteeing the connection between the beliefs (our forecasts) and the actual parameters, we can use a calibrated forecast to make stronger claims about priors that are estimated in a sequential empirical Bayes setting.

For a statistician or econometrician, not being calibrated is one of the most embarrassing mistakes to make. Suppose we are trying to predict some variable \( Y \) based on a bunch of \( X_i \)’s. If it turns out that we could get a much better fit by looking at \( X_{17}/X_{12} \) than we currently are getting, that would be considered a great scientific result and no one would fault the previous work that missed it. But, if \( 3 \hat{Y} \), or \( \hat{Y}^3 \), were better forecasts than the \( \hat{Y} \) provided by the statistician, that would be an embarrassing error. Given the numerous ways of correcting uncalibrated forecasts (see Zadrozny and Elkan 2001), people would ask, “Didn’t you look at your forecast at all?” Of course, when dealing with out-of-sample forecasts this can occur since the world might change. Hence, the value of these calibration methods, which sequentially adapt to a changing world, is to ensure we can avoid this embarrassment.

Finally, regarding forecast-hedging: as it is an elementary principle, it might perhaps help dispel some of the mystery behind the prevalence of well-calibrated forecasts, such as the “superforecasters” of the Good Judgement Project (see Tetlock and Gardner 2015 and Mellers et al. 2015), FiveThirtyEight (see Figure 2), ElectionBettingOdds (see Figure 3), and others. Indeed, in most of these cases one forecasts binary yes/no events, where, as we show in Sections 1.2 and 5, forecast-hedging is extremely simple and straightforward to implement.

\[ \text{In such betting / market models, we see that calibration goes part way toward the “weak efficient market hypothesis” (wEMH). For example, take the sequence of times where a stock price is above its seven-day average and we are considering whether to buy it (“momentum”) or sell it (“mean reversion”). If we had a forecast of the “correct price” then these could be expressed as saying “buy” when the forecast is above the price and “sell” when it is below. The property we would then want such a forecast to have is merely calibration. Given how simple it is for forecast-hedging to generate calibration, it is reasonable to expect many traders to all discover something close to the same calibrated forecast and hence push the market in that direction until the price is the same as the forecast (while this would not generate the full wEMH, which requires its holding for all price patterns, it does go in that direction).} \]

\[ \text{Of course, we are not implying that forecast-hedging is what these forecasters consciously do. What we are saying is that since calibration is very easy to achieve, we should not be surprised by its being often obtained. At the same time, it might be of interest to check if there is any balancing of current and past forecasting errors, as in forecast-hedging (see the discussion}} \]
Figure 2: Calibration plots of FiveThirtyEight (projects.fivethirtyeight.com/checking-our-work, updated on June 26, 2019). For example, in the Everything plot the 10% data point (which lies slightly below the diagonal) has the following attached description: “We thought the 107962 observations in this bin had a 10% chance of happening. They happened 9% of the time.”

Figure 3: Calibration plot of ElectionBettingOdds (electionbettingodds.com/TrackRecord.html, updated on November 13, 2018), which “tracked some 462 different candidate chances across dozens of races and states in 2016 and 2018.”
1.2 Forecast-Hedging: A Simple Illustration

Consider the basic rain/no rain setup—or, for that matter, any sequence of arbitrary, possibly unrelated, yes/no events (as in the above-mentioned projects)—and let the forecasts lie on the equally spaced grid 0, 1/N, 2/N, ..., 1 for some integer \( N \geq 1 \). Take period \( t \). For each forecast \( x \) let \( n(x) \equiv n_{t-1}(x) \) be the number of days that \( x \) has been used in the past \( t-1 \) periods, and let \( r(x) \equiv r_{t-1}(x) \) be the number of rainy days out of those \( n(x) \) days. If the forecast \( x \) is correct there should have been rain on \( x \cdot n(x) \) out of the \( n(x) \) days, and so the excess number of rainy days at \( x \) is \( G(x) \equiv G_{t-1}(x) := r(x) - x \cdot n(x) \). For simplicity consider the sum of squares score \( S \equiv S_{t-1} := \sum_x G(x)^2 \).

Let \( a \equiv a_t \) denote the weather at time \( t \), with \( a = 1 \) standing for rain and \( a = 0 \) for no rain, and let \( c \equiv c_t \) in the interval \([0, 1]\) denote the forecast at time \( t \). The change in the score \( S \) from time \( t-1 \) to time \( t \) is \( S_t - S_{t-1} = (G(c) + a - c)^2 - G(c)^2 \) (the only term that changes in the sum \( S \) is the \( G(c) \) term for the forecasted \( c \)), whose first-order approximation equals \( 2\Delta \) for \( \Delta := G(c) \cdot (a - c) \).

We would like to choose the forecast \( c \) so that

\[
\Delta \equiv G(c) \cdot (a - c) \leq 0 \quad \text{for any } a, \tag{2}
\]

i.e., no matter what the weather will be. This is easy to do when there is a point \( c \) on the grid with \( G(c) = 0 \): just forecast this \( c \). In general, however, we can aim only to make the inequality \( \Delta \leq 0 \) hold on average, by choosing the forecast at above where forecast-hedging is defined, and the illustration in Section 1.2. Finally, we note that forecasters are tested not only by their calibration scores, but by stronger measures of “accuracy” or “skill” (specifically, their Brier scores).

14Think of \( G(x) \) as the total “gap” at \( x \); it may be positive, zero, or negative. The vertical distance from the diagonal in the calibration plot (as in Figures 1 and 2) is the normalized gap \( G(x)/n(x) \).

15We abstract away from technical details, such as the appropriate normalizations, in this illustration; see Sections 4 and 5 for the precise analysis. For the expert reader we note that the calibration score at time \( t \) is \( K_t = \sum_x |G_t(x)|/t \) (see Section 2), which is small when \( S_t/t^2 \) is small (by the Cauchy–Schwarz inequality). Note that a constant forecast of, say, \( c = 1/2 \) yields \( S_t = t^2/4 \) in the worst case (where all days are rainy, or all days are sunny), and thus a calibration score that is bounded away from zero.

16We ignore the term \( (a - c)^2 \), which is bounded by 1, since the total contribution to \( S_t \) of all these terms is at most \( t \), and thus negligible relative to \( t^2 \) (see footnote 15).
This is what we call the forecast-hedging condition (condition (2) is a special case of (3)). Interestingly, this inequality seems to express the idea discussed in the Introduction that the errors \( a - c \) of the current forecast would tend to have the opposite sign of the errors \( G(c) \) of the past forecasts.

How can (3) be obtained? Randomizing between two forecasts, say \( c_1 \) with probability \( p_1 \) and \( c_2 \) with probability \( p_2 = 1 - p_1 \), yields

\[
E[\Delta] = p_1 G(c_1) \cdot (a - c_1) + p_2 G(c_2) \cdot (a - c_2) = [p_1 G(c_1) + p_2 G(c_2)] \cdot (a - c_2) + p_1 G(c_1) \cdot (c_2 - c_1).
\]

We can thus guarantee \( E[\Delta] \) to be small, no matter what \( a \) will be, by choosing the \( c_k \) and \( p_k \) so that, first,

\[
p_1 G(c_1) + p_2 G(c_2) = 0,
\]

and, second, \( c_2 - c_1 \) is small.\footnote{This turns out to suffice because \( c_2 - c_1 \) is multiplied by \( p_1 G(c_1) \), which is of the order of \( t \); again, see Sections 4 and 5 for details. The size of the calibration error is determined by the distance between \( c_1 \) and \( c_2 \).}

Specifically, working on the grid \( 0, 1/N, 2/N, \ldots, 1 \), we obtain these forecasts as follows. If \( G(j/N) = 0 \) for some \( j \), then take \( c = j/N \), which makes \( \Delta = 0 \).
Otherwise \( G(i/N) \neq 0 \) for all \( i \), and so let \( j \geq 1 \) be any index with \( G(j/N) < 0 \) (such a \( j \) exists because \( G(0) > 0 \) and \( G(1) < 0 \)) and take \( c_1 = (j - 1)/N \) and \( c_2 = j/N \) (and thus \( G(c_1) = 0 > G(c_2) \)), with the \( p_k \) inversely proportional to \( |G(c_k)| \) (i.e., as given by (1)). Figure 4 provides two examples of graphs of \( G \) (for \( N = 6 \); the dotted lines provide linear interpolation). On the left we have \( c \) on the grid with \( G(c) = 0 \), which yields the perfect deterministic hedging of (2), and on the right we have adjacent \( c_1, c_2 \) on the grid with \( G(c_1) > 0 > G(c_2) \), which yields the approximate stochastic hedging of (3).

One of course needs to keep track of all the approximation errors, but, surprisingly, the procedure described here does work: it guarantees an average calibration error that goes to 0 as the grid size \( N \) increases; see Section 5. It turns out to be a new addition to the literature, and simpler than any existing calibrated pro-

\[\begin{align*}
E[\Delta] &\equiv E[G(c) \cdot (a - c)] \leq 0 \text{ for any } a.
\end{align*}\]
Figure 4: Examples of forecast-hedging. The forecasts $x$ are marked on the horizontal axis, and the gaps $G(x)$ on the vertical axis. On the left, deterministic forecast-hedging is obtained by forecasting $c$ with $G(c) = 0$; on the right, stochastic forecast-hedging is obtained by randomizing among the forecasts $c_1$ and $c_2$ with probabilities $p_1$ and $p_2$ such that $p_1G(c_1) + p_2G(c_2) = 0$.

Procedure in this one-dimensional binary setup (i.e., rain/no rain). Moreover, while it involves randomizations (it must!), the randomizations are all between two neighboring grid points ($(j - 1)/N$ and $j/N$), and so this procedure is an almost deterministic procedure (Foster 1999, Kakade and Foster 2008).

Forecast-hedging is central to all the calibration results in the present paper, in higher dimensions as well. Specifically:

- For classic calibration, probabilistic weights $p_k$ that ensure that $E[\Delta]$ is small are obtained using a minimax result; this is stochastic forecast-hedging.

- For continuous calibration, where the corresponding function $G$ becomes continuous, a deterministic point $c$ that ensures $\Delta \leq 0$ (a special case of which is $G(c) = 0$) is obtained by a fixed point result; this is deterministic forecast-hedging.

- Again for classic calibration, an almost deterministic forecast is obtained by replacing the fixed point with an appropriate distribution on nearby grid points (as done above in the one-dimensional case); this is almost deterministic forecast-hedging.

1.3 The Organization of the Paper

The paper is organized as follows. Section 2 presents the general calibration setup, and introduces the new concept of “continuous calibration.” Section 3 is devoted to
what we call “outgoing” theorems, which provide the forecast-hedging tools that are used to obtain the calibration results in Section 4. The simple procedure in the one-dimensional case is given in Section 5. In Section 6 we show that the game dynamics of best replying to continuously calibrated forecasts—“continuously calibrated learning”—yield Nash equilibria, and we conclude in Section 7 with the significant distinction made here between the minimax and the fixed point universes. The Appendix provides further details, proofs, and extensions.

2 The Calibration Setup

Let $A$ be a set of possible outcomes, which we call actions (such as $A = \{0, 1\}$, with $a = 1$ standing for rain and $a = 0$ for shine), and let $C$ be the set of forecasts about these actions (such as $C = [0, 1]$, with $c$ in $C$ standing for “the chance of rain is $c$”). We assume that $C \subset \mathbb{R}^m$ is a nonempty compact convex subset of a Euclidean space, and $A \subseteq C$. Some special cases of interest are: (i) $C$ is the set of probability distributions $\Delta(A)$ over a finite set $A$, which is identified with the set of unit vectors in $C$ (and then $C$ is a unit simplex); (ii) $C$ is the convex hull $\text{conv}(A)$ of a finite set of points $A \subset \mathbb{R}^m$ (and then $C$ is a polytope); and (iii) $C = A$ (and then $A$ is already convex). Let $\gamma := \text{diam}(C) \equiv \max_{c, c' \in C} \|c - c'\|$ be the diameter of the set $C$. Let $\delta > 0$; a subset $D$ of $C$ is a $\delta$-grid of $C$ if for every $c \in C$ there is $d \in D$ at distance less than $\delta$ from $c$, i.e., $\|d - c\| < \delta$; a compact set $C$ always has a finite $\delta$-grid (obtained from a finite subcover by open $\delta$-balls).

For each period $t = 1, 2, \ldots$, let $c_t$ in $C$ be the forecast, and let $a_t$ in $A$ be the action. The forecast at time $t$ may well depend on the history $h_{t-1} = (c_1, a_1; c_2, a_2; \ldots; c_{t-1}, a_{t-1}) \in (C \times A)^{t-1}$ of past forecasts and actions. A deterministic (forecasting) procedure $\sigma$ is thus a mapping $\sigma : \bigcup_{t \geq 1} (C \times A)^{t-1} \to C$ that assigns to every history $h_{t-1}$ a forecast $c_t = \sigma(h_{t-1}) \in C$ at time $t$. A stochastic (forecasting) procedure $\sigma$ is a mapping $\sigma : \bigcup_{t \geq 1} (C \times A)^{t-1} \to \Delta(C)$ that assigns to every history $h_{t-1}$ a probability distribution $\sigma(h_{t-1})$ on $C$ according to which the forecast $c_t$ at time $t$ is chosen. Let $\rho > 0$; a stochastic procedure $\sigma$ is $\rho$-almost deterministic if for every history $h_{t-1}$ the support of the distribution $\sigma(h_{t-1})$ of $c_t$ is included in a closed ball of radius $\rho$; that is, the forecast $c_t$ is “deterministic within a precision $\rho$.”

We refer to $a = (a_t)_{t=1}^\infty$, where $a_t \in A$ for every $t$, as an action sequence; the sequence may be anything from a fixed, “oblivious,” sequence, all the way to an
adaptive, “adversarial,” sequence; the latter allows the action \( a_t \) at time \( t \) to be determined by the history \( h_{t-1} \), as well as by the forecasting procedure (i.e., the mapping \( \sigma \)). Let \( a_t = (a_s)_{s=1}^t \) denote the first \( t \) coordinates of \( a \).

### 2.1 Classic Calibration

Fix a time \( t \geq 1 \) and a sequence \((c_s, a_s)_{s=1}^t \in (C \times A)^t\) of forecasts and actions up to time \( t \). For every \( x \in C \) let

\[
 n_t(x) := |\{1 \leq s \leq t : c_s = x\}| = \sum_{s=1}^t 1_x(c_s)
\]

be the number of times that the forecast \( x \) has been used, and, for every \( x \) with \( n_t(x) > 0 \), let

\[
 \bar{a}_t(x) := \frac{1}{n_t(x)} \sum_{s=1}^t 1_x(c_s) a_s
\]

be the average of the actions in all the periods that the forecast \( x \) has been used. The calibration error \( e_t(x) \) of the forecast \( x \) is then

\[
 e_t(x) := \bar{a}_t(x) - x
\]

when the forecast \( x \) has not been used, i.e., \( n_t(x) = 0 \), we put for convenience \( e_t(x) := 0 \).

The classic calibration score is the average calibration error, namely

\[
 K_t := \sum_{x \in C} \left( \frac{n_t(x)}{t} \right) \|e_t(x)\| ; \quad (5)
\]

thus, the error of each \( x \) is weighted in proportion to the number of times \( n_t(x) \) that \( x \) has been used (the weights add up to 1 because \( \sum_x n_t(x) = t \)).

Let \( \varepsilon > 0 \); a (stochastic) procedure \( \sigma \) is \( \varepsilon \)-calibrated (Foster and Vohra 1998)

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\(^{21}\) See the remark following the definition of forecast-hedging in Section 4.1. In the setup of the calibration game (see Foster and Hart 2018), which is a repeated simultaneous game of perfect monitoring and perfect recall between the action player and the calibrating player (the forecaster), the statement “for every action sequence \( a \)” translates to “for every (pure) strategy of the action player.”

\(^{22}\) We write \( 1_x \) for the \( x \)-indicator function, i.e., \( 1_x(c) = 1 \) for \( c = x \) and \( 1_x(c) = 0 \) for \( c \neq x \). The number of elements of a finite set \( Z \) is denoted by \( |Z| \).  

\(^{23}\) The sum is finite as it goes over all \( x \) with \( n_t(x) > 0 \), i.e., over \( x \) in the set \( \{c_1, ..., c_t\} \). In line with standard statistics usage, one may average the squared Euclidean norms \( \|e_t(x)\|^2 \) instead (cf. \( X_t \) in the proof of Theorem S(S)); this will not affect the results.
The expectation $E$ is taken over the random forecasts of $\sigma$. In Appendix A.5 we show that one may make $K_t$ small with probability one (i.e., almost surely), not just in expectation.

### 2.2 Binning and Continuous Calibration

The calibration error $e_t(x)$ can be rewritten as

$$e_t(x) = \sum_{s=1}^{t} \left( \frac{1_x(c_s)}{n_t(x)} \right) (a_s - c_s)$$

(because $\sum_{s=1}^{t} 1_x(c_s)c_s = n_t(x)x$; thus, $e_t(x)$ is the average of the differences $a_s - c_s$ between actions and forecasts, where only the periods $s$ where the forecast was $x$ count.

The calibration score, as defined by (5), can then be interpreted as follows. For each $x$ in $C$ there is a bin, call it the “$x$-bin,” which tracks the errors of the forecast $x$; namely, if at time $s$ the forecast is $c_s$ and the action is $a_s$, then the difference $a_s - c_s$ between the action and the forecast is assigned to the $c_s$-bin. At time $t$ one computes the average error $e_t(x)$ of each $x$-bin, and then the calibration score $K_t$ is the average norm of these errors, where the weight of each $x$-bin is proportional to its size, i.e., to the number of elements $n_t(x)$ that it contains.

As discussed in the Introduction, the resulting calibration score is highly discontinuous: forecasts $c$ and $c'$, even when slightly apart, are tracked separately, in distinct bins. To smooth this out and treat them similarly, we have to, first, allow for “fractional” assignments into bins, and, second, make these assignments depend continuously on the forecast $c$.

What then is a general binning system? It is given by the fraction $0 \leq w_i(c) \leq 1$ of each forecast $c$ that goes into each bin $i$, where these fractions add up to 1 over all bins (for each $c$). We assume for convenience that the number of bins is countable, i.e., finite or countably infinite; there is no loss of generality in this assumption, as we show in Appendix A.1. A binning is thus a collection $\Pi = (w_i)^I_{i=1}$, with $I$
finite or \( I = \infty \), of functions \( w_i : C \to [0,1] \) such that

\[
\sum_{i=1}^{I} w_i(c) = 1
\]

for every \( c \in C \); the binning is \textit{continuous} if all the functions \( w_i \) are continuous functions of \( c \).

A continuous binning is obtained, for instance, by taking points \( y_i \) in \( C \) and letting the fraction of forecast \( c \) that goes into the \( y_i \)-bin decrease continuously with the distance between \( c \) and \( y_i \). For a specific example in which only small neighborhoods matter, take \( \{ y_1, ..., y_I \} \) to be a finite \( \delta \)-grid of \( C \) and put

\[
w_i(c) = \Lambda(c, y_i) / \sum_{j=1}^{I} \Lambda(c, y_j)
\]

for each \( 1 \leq i \leq I \), where

\[
\Lambda(c, y) := \delta - \| c - y \|_+.
\]

Next, what is the calibration score \( K_\Pi^t \) with respect to a (continuous) binning \( \Pi = (w_i)_{i=1}^{I} \)? As for the classic calibration score \( K_t \), one first computes the average error in each bin, and then takes the average norm of these errors, in proportion to the total weights accumulated in the bins. The total weight of bin \( i \) is

\[
n_i^t := \sum_{s=1}^{t} w_i(c_s),
\]

the average error of bin \( i \) is

\[
e_i^t := \sum_{s=1}^{t} \left( \frac{w_i(c_s)}{n_i^t} \right) (a_s - c_s)
\]

(again, put \( e_i^t := 0 \) when \( n_i^t = 0 \)), and the \( \Pi \)-calibration score is

\[
K_\Pi^t := \sum_{i=1}^{I} \left( \frac{n_i^t}{t} \right) \| e_i^t \| \quad (6)
\]

(the weights \( n_i^t / t \) add up to 1, because \( \sum_{i=1}^{I} n_i^t = \sum_{s=1}^{t} \sum_{i=1}^{I} w_i(c_s) = t \)).

A deterministic procedure \( \sigma \) is \textit{\( \Pi \)-calibrated} if

\[
\lim_{t \to \infty} \left( \sup_{a_t} K_\Pi^t \right) = 0,
\]

\[\text{For fixed } y, \text{ the graph of the so-called “tent function” } \Lambda(c, y) \text{ looks like the symbol } \Lambda \text{ (with the peak at } c = y).\]

\[\text{For a continuous binning, the bin errors } e_i^t \text{ are continuous averages of the classic calibration errors } e_t(x), \text{ namely,}
\]

\[
e_i^t = \sum_{x \in C} \left( \frac{w_i(x) m_t(x)}{n_i^t} \right) e_t(x);
\]

thus, continuous binnings do indeed capture the idea of smoothing out the calibration errors (as in the above example with the \( \delta \)-grid \( \{ y_i \} \) on \( C \)).
and it is continuously calibrated if it is $\Pi$-calibrated for every continuous binning $\Pi$.

Compared with classic calibration, continuous calibration requires the convergence to be to zero (rather than $\leq \varepsilon$), simultaneously for all continuous $\Pi$.

### 2.3 Gaps and Preliminary Results

Rather than working with the normalized errors, it is convenient to work with unnormalized “gaps.” For every real function on $C$, i.e., $w : C \rightarrow \mathbb{R}$, and $t \geq 1$, let

$$ g_t(w) := \frac{1}{t} \sum_{s=1}^{t} w(c_s)(a_s - c_s) $$

be the (per-period) gap at time $t$ with respect to $w$ (when $w = 1_x$ this is the total gap $G(x)$ of Section 1.2 divided by the number of periods). We extend the definitions of $n_t$ and $e_t$ by

$$ n_t(w) := \sum_{s=1}^{t} w(c_s) \quad \text{and} \quad e_t(w) := \frac{1}{n_t(w)} \sum_{s=1}^{t} w(c_s)(a_s - c_s) $$

for every $w$, and then the relation $g_t(w) = (n_t(w)/t)e_t(w)$ immediately yields

$$ K_t = \sum_{x \in C} \|g_t(1_x)\| \quad \text{and} \quad K^\Pi_t = \sum_{i=1}^{I} \|g_t(w_i)\| $$

(indeed, for $K_t$ we have $n_t(x) \equiv n_t(1_x)$ and $e_t(x) \equiv e_t(1_x)$, and for $K^\Pi_t$ we have $n^i_t \equiv n_t(w_i)$ and $e^i_t \equiv e_t(w_i)$).

For every function $w$ the vectors $e_t(w)$ and $g_t(w)$ are proportional; they differ in that the denominator is $n_t(w)$ in the former, and $t$, which is larger, in the latter. The calibration scores are averages of the norms of $e_t$, and sums of the norms of $g_t$. One advantage of the $g_t$ representation is that we do not need to keep track explicitly of the total weight $n_t$. Another is that, fixing the sequence of actions and forecasts, the mapping $g_t$ is a linear bounded operator: $g_t(\alpha w + \alpha' w') = \alpha g_t(w) + \alpha' g_t(w')$ for scalars $\alpha, \alpha' \in \mathbb{R}$, and, using the supremum norm $\|w\| := \sup_{c \in C} |w(c)|$ for functions $w : C \rightarrow \mathbb{R}$, we have

$$ \|g_t(w)\| \leq \gamma \|w\| $$

$^{27}$One could get uniformity over binnings $\Pi$ by restricting them to a compact space (for instance, by imposing a uniform Lipschitz condition on the $w_i$, as in weak and smooth calibration).

$^{28}$In particular, when $n_t(w)$ vanishes so does $g_t(w)$.
because \( g_t(w) \) is an average of vectors \( w(c_s)(a_s - c_s) \) of norm \( \|w\| \text{diam}(C) = \|w\| \gamma \); therefore

\[
| \|g_t(w)\| - \|g_t(w')\| | \leq \|g_t(w - w')\| = \|g_t(w - w')\| \leq \gamma \|w - w'\|. \tag{7}
\]

Returning to the binning condition, which we can write as \( \sum_{i=1}^{I} w_i = 1 \), it says that \( \Pi = (w_i)_{i=1}^{I} \) is a “partition of unity,” and so the resulting calibration score \( K_t^\Pi \) may be viewed as the “variation” of \( g_t \) with respect to the partition \( \Pi \). In particular, the classic calibration score \( K_t \) is the variation of \( g_t \) with respect to the partition \( \sum_{x \in C} 1_x = 1 \) into indicator functions. Since the indicator partition is the finest partition, \( \sum_{x \in C} 1_x = 1 \), it stands to reason that \( K_t \) would be the maximal possible variation, i.e., the “total variation” of \( g_t \). This is indeed so: for every binning \( \Pi \) we have

\[
K_t^\Pi \leq K_t, \tag{8}
\]

which immediately follows from applying Lemma 1 below to \( \Pi \).

Thus, any notion based on binning—in particular, continuous calibration—is a weakening of classic calibration: if \( K_t \) is small, then so are all the relevant \( K_t^\Pi \).

**Lemma 1** Let \( (w_j)_{j \in J} \) be a countable collection of nonnegative functions on \( C \), i.e., \( w_j : C \to \mathbb{R}_+ \) for every \( j \in J \). Then

\[
\sum_{j \in J} \|g_t(w_j)\| \leq \left\| \sum_{j \in J} w_j \right\| K_t.
\]

**Proof.** Put \( W := \sum_{j \in J} w_j \); using \( w_j = \sum_{x \in C} w_j(x) 1_x \) and the linearity of \( g_t \) we have

\[
\sum_{j \in J} \|g_t(w_j)\| \leq \sum_{j \in J} \sum_{x \in C} w_j(x) \|g_t(1_x)\| = \sum_{x \in C} \sum_{j \in J} w_j(x) \|g_t(1_x)\|
\]

\[
= \sum_{x \in C} W(x) \|g_t(1_x)\| \leq \|W\| \sum_{x \in C} \|g_t(1_x)\| = \|W\| K_t.
\]

\( \Box \)

For another use of this lemma, let \( \Pi = (w_i)_{i=1}^{\infty} \) be an infinite continuous binning. The increasing sequence of continuous functions \( \sum_{i=1}^{k} w_i \) converges pointwise, as \( k \to \infty \), to the continuous function \( 1 \), on the compact set \( C \), and so, by

\( ^{20} \) We write \( 1 \) for the constant 1 function; all indicator and \( w \) functions are defined on \( C \) only.

\( ^{30} \) Any further split into fractions of indicators does not matter since \( g_t(\alpha 1_x) = \alpha g_t(1_x) \).

\( ^{31} \) The sum \( \sum_{x \in C} \) in the proof below is a finite sum (over \( x \in \{c_1, \ldots, c_t\} \)), and so it commutes with \( \sum_{j \in J} \).
Dini’s theorem (see, e.g., Rudin 1976, Theorem 7.13), the convergence is uniform:

$$\lim_{k \to \infty} \left\| \sum_{i=k+1}^{\infty} w_i \right\| = 0.$$  \hspace{1cm} (9)

Using Lemma 1 for every \(a_t\), together with \(K_t \leq \gamma\) (by (5)), yields

$$\lim_{k \to \infty} \left( \sup_{a_t} \sum_{i=k+1}^{\infty} \| g_t(w_i) \| \right) = 0.$$ \hspace{1cm} (10)

Thus, for continuous binning only finitely many \(w_i\) matter, which leads to a simpler characterization of continuous calibration in terms of “pointwise-in-\(w\)” convergence.

**Proposition 2** A deterministic forecasting procedure \(\sigma\) is continuously calibrated if and only if

$$\lim_{t \to \infty} \left( \sup_{a_t} \| g_t(w) \| \right) = 0$$ \hspace{1cm} (11)

for every continuous function \(w : C \to [0, 1]\).

**Proof.** Given a continuous function \(w : C \to [0, 1]\), let \(\Pi\) be the continuous binning \((w, 1 - w)\). Since \(\| g_t(w) \| \leq K^\Pi_t\), continuous calibration implies (11).

Conversely, let \(\Pi = (w_i)_{i=1}^{\infty}\) be a continuous binning. When \(I\) is finite we have

$$\sup_{a_t} K^\Pi_t = \sup_{a_t} \sum_{i=1}^{I} \| g_i(w_i) \| \leq \sum_{i=1}^{I} \sup_{a_t} \| g_i(w_i) \|,$$

which converges to 0 as \(t \to \infty\) by (11). When \(I\) is infinite, for every \(\varepsilon > 0\) there is by (10) a finite \(k\) such that \(\sup_{a_t} K^\Pi_t \leq \sum_{i=1}^{k} \sup_{a_t} \| g_i(w_i) \| + \varepsilon\), which converges to \(\varepsilon\) as \(t \to \infty\) by (11); since \(\varepsilon\) is arbitrary, the limit is 0. \(\square\)

We now construct a continuous binning \(\Pi_0\) such that \(\Pi_0\)-calibration implies \(\Pi\)-calibration for all continuous \(\Pi\) (and so \(\Pi_0\) plays, for continuous calibration, the same role that the indicator binning plays for classic calibration; see (8)).

**Proposition 3** There exists a continuous binning \(\Pi_0\) such that a deterministic forecasting procedure \(\sigma\) is continuously calibrated if and only if it is \(\Pi_0\)-calibrated.

**Proof.** The space of continuous functions from the compact set \(C\) to \([0, 1]\) is separable with respect to the supremum norm; let \((u_i)_{i=1}^{\infty}\) be a dense sequence. Take \(\alpha_i > 0\) such that \(\sum_{i=1}^{\infty} \alpha_i \| u_i \| \leq 1\) (for example, \(\alpha_i = 1/(2^i \| u_i \|)\)), and put \(w_i := \alpha_i u_i\) for all \(i \geq 1\) and \(w_0 := 1 - \sum_{i=1}^{\infty} w_i\) (the function \(w_0\) is continuous because \(\sum_{i=1}^{\infty} w_i(c) \leq \sum_{i=1}^{\infty} \alpha_i \| u_i \| \leq 1\)). Thus \(\Pi_0 = (w_i)_{i=0}^{\infty}\) is a continuous binning, and so continuous calibration implies \(\Pi_0\)-calibration.
Conversely, $\Pi_0$-calibration implies (11) for each $w_i$ in $\Pi_0$ (because $\|g_t(w_i)\| \leq K_{\Pi_t}$), and hence for each $u_i$ by the linearity of $g_t$. This extends from the dense sequence $(u_i)_i$ to any continuous $w : C \to [0, 1]$ by (7), and Proposition 2 completes the proof.

Proposition 2 also implies that continuous calibration is a strengthening of existing Lipschitz-based notions of weak calibration (Kakade and Foster 2004, Foster and Kakade 2006) and smooth calibration (Foster and Hart 2018). Indeed, a continuously calibrated procedure—a simple construction of which we provide in Section 4—is “universally” weakly and smoothly calibrated (by contrast, the known constructions depend on the Lipschitz bound $L$ and the desired calibration error $\varepsilon$). See Proposition 15 in Appendix A.2. Thus, continuous calibration may well be used instead of weak and smooth calibration.

3 Forecast-Hedging Tools

In this section we provide useful variants of Brouwer’s (1912) fixed point theorem and von Neumann’s (1928) minimax theorem; they are used in Section 4 to obtain forecasts that satisfy the forecast-hedging conditions. These conditions, of the form (2) and (3) (see Section 1.2), are referred to as “outgoing” because of their geometric interpretation (see the paragraph following the statement of Theorem 4 below). The reader may skip the proofs in this section at first reading; however, see the important distinction between fixed point and minimax procedures in Section 3.4.

Throughout this section $f : C \to \mathbb{R}^m$ is a function from the nonempty compact and convex subset $C$ of $\mathbb{R}^m$ into $\mathbb{R}^m$ (with the same dimension $m$), which may be interpreted as a vector field “flow” (i.e., think of $x$ as moving to $x + f(x)$, or to $x + \varepsilon f(x)$ for some $\varepsilon > 0$).

3.1 Outgoing Fixed Point

When the function $f$ is continuous we get:

**Theorem 4 (Outgoing Fixed Point)** Let $C \subset \mathbb{R}^m$ be a nonempty compact convex set, and $f : C \to \mathbb{R}^m$ be a continuous function. Then there exists a

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32The traditional way to obtain universal procedures is by restarting them at appropriate times with new values of the parameters (as in Section 4.4 of Kakade and Foster 2004). The procedures that we construct in the present paper are much simpler.
point \( y \) in \( C \) such that
\[
f(y) \cdot (x - y) \leq 0
\] (12)
for all \( x \in C \).

Thus, \( f(y) \cdot y = \max_{x \in C} f(y) \cdot x \). If \( y \) is an interior point of \( C \) then we must have \( f(y) = 0 \) (because \( x - y \) can be proportional to any vector in \( \mathbb{R}^m \)), and if \( y \) is on the boundary of \( C \) then \( f(y) \) is an outgoing normal to the boundary of \( C \) at \( y \). This result is the “variational inequalities” Lemma 8.1 in Border (1985), who attributes it to Hartman and Stampacchia (1966, Lemma 3.1). We provide a short direct proof using Brouwer’s (1912) fixed point theorem.

**Proof.** For every \( z \in \mathbb{R}^m \) let \( \xi(z) \in C \) be the closest point to \( z \) in the set \( C \), i.e., \( \|\xi(z) - z\| = \min_{x \in C} \|x - z\| \). As is well known, because \( C \) is a convex and compact set, \( \xi(z) \) is well defined (i.e., it exists and is unique), the function \( \xi \) is continuous, and
\[
(z - \xi(z)) \cdot (x - \xi(z)) \leq 0
\] (13)
for every \( x \in C \) (when \( z \in C \) it trivially holds because then \( \xi(z) = z \), and when \( z \notin C \) the vector \( z - \xi(z) \) is an outward normal to \( C \) at the boundary point \( \xi(z) \)).

The function \( x \mapsto \xi(x + f(x)) \) is thus a continuous function from \( C \) to \( C \), and so by Brouwer’s fixed point theorem there is \( y \in C \) such that \( y = \xi(y + f(y)) \). Applying (13) to the point \( z = y + f(y) \), for which \( \xi(z) = y \), yields the result. □

### 3.2 Outgoing Minimax

For functions \( f \) that need not be continuous we have:

**Theorem 5 (Outgoing Minimax)** Let \( C \subset \mathbb{R}^m \) be a nonempty compact convex set, let \( D \subset C \) be a finite \( \delta \)-grid of \( C \) for some \( \delta > 0 \), and let \( f : D \to \mathbb{R}^m \). Then there exists a probability distribution \( \eta \) on \( D \) such that
\[
\mathbb{E}_{y \sim \eta} [f(y) \cdot (x - y)] \leq \delta \mathbb{E}_{y \sim \eta} [\|f(y)\|]
\] (14)
for all \( x \in C \). Moreover, the support of \( \eta \) can be taken to consist of at most \( m + 3 \) points of \( D \).

When \( f \) is bounded, by taking \( \delta = \varepsilon / \sup_{x \in C} \|f(x)\| \) we get:

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33Theorem 4 is in fact equivalent to Brower’s fixed point theorem, as the latter is easily proved from the former; see Appendix A.3 which contains various comments on the “outgoing” results.
Corollary 6 Let $C \subset \mathbb{R}^m$ be a nonempty compact convex set, and $f : C \to \mathbb{R}^m$ a bounded function. Then for every $\varepsilon > 0$ there exists a probability distribution $\eta$ on $C$ such that

$$\mathbb{E}_{y \sim \eta} [f(y) \cdot (x - y)] \leq \varepsilon$$

for all $x \in C$. Moreover, the support of $\eta$ can be taken to consist of at most $m + 3$ points of $C$.

Unlike in the Outgoing Fixed Point Theorem 4, in the Outgoing Minimax Theorem 5 $y$ is a random variable and no longer a constant, and the “outgoing” inequality holds in expectation (within an arbitrarily small error). The proof is a finite construct that uses the von Neumann’s (1928) minimax theorem and thus amounts to solving a linear programming problem.

**Proof of Theorem 5.** Let $\delta_0 \equiv \delta_0(D) := \max_{x \in C} \text{dist}(x, D)$ be the farthest away a point in $C$ may be from the $\delta$-grid $D$; the maximum is attained on the compact set $C$ and so $\delta_0 < \delta$. Put $\delta_1 := \delta - \delta_0 > 0$, and take $B \subset C$ to be a finite $\delta_1$-grid of $C$. Consider the finite two-person zero-sum game where the maximizer chooses $b \in B$, the minimizer chooses $y \in D$, and the payoff is $f(y) \cdot (b - y) - \delta_0 \|f(y)\|$. For every mixed strategy $\nu \in \Delta(B)$ of the maximizer, let $\bar{b} := \mathbb{E}_{b \sim \nu}[b] \in C$ be its expectation; the minimizer can make the payoff $\leq 0$ by choosing a point $y$ on the grid $D$ that is within $\delta_0$ of $\bar{b}$:

$$\mathbb{E}_{x \sim \nu} [f(y) \cdot (b - y) - \delta_0 \|f(y)\|] = f(y) \cdot (\bar{b} - y) - \delta_0 \|f(y)\| \leq 0$$

(because $f(y) \cdot (\bar{b} - y) \leq \|f(y)\| \cdot \|\bar{b} - y\| \leq \|f(y)\| \delta_0$). Therefore, by the minimax theorem, the minimizer can guarantee that the payoff is $\leq 0$; i.e., there is a mixed strategy $\eta \in \Delta(D)$ such that

$$\mathbb{E}_{y \sim \eta} [f(y) \cdot (b - y) - \delta_0 \|f(y)\|] \leq 0$$

(15)

for every $b \in B$. Since for every $x \in C$ there is $b \in B$ with $\|x - b\| < \delta_1$, and so $f(y) \cdot (x - b) \leq \delta_1 \|f(y)\|$ for every $y$, adding this inequality to (15) yields, by $\delta_0 + \delta_1 = \delta$, the inequality (14) for every $x \in C$.

For the moreover statement, (14) says that the vector $\mathbb{E}_{y \sim \eta}[F(y)]$ satisfies $\mathbb{E}_{y \sim \eta}[F(y)] \cdot (x, -1, -\delta) \leq 0$ for every $x \in C$, where

$$F(y) := (f(y), f(y) \cdot y, \|f(y)\|) \in \mathbb{R}^{m+2}$$

As we will see in Appendix A.3, Corollary 6 is equivalent to the minimax theorem (as Theorem 4 is equivalent to Brouwer’s fixed point theorem).
for each $y \in D$. By Carathéodory’s theorem, $\mathbb{E}_{y \sim \eta}[F(y)]$ can be expressed as a convex combination of at most $m+3$ points in $\{F(y) : y \in D\}$, and so the support of $\eta$ can be taken to be of size at most $m+3$.

### 3.3 Almost Deterministic Outgoing Fixed Point

We can improve the result of the Outgoing Minimax Theorem and obtain a probability distribution that is “almost deterministic”—i.e., the randomization is between nearby points—by using a fixed point.

A probability distribution $\eta$ is said to be $\rho$-local if its support is included in a closed ball of radius $\rho$; i.e., there exists $x$ such that $\eta(B(x; \rho)) = 1$, where $B(x; \rho) = \{z : \|z - x\| < \rho\}$ and $\overline{B}(x; \rho) = \{z : \|z - x\| \leq \rho\}$ denote, respectively, the open and closed balls of radius $\rho$ around $x$.

**Theorem 7 (Almost Deterministic Outgoing Fixed Point)** Let $C \subset \mathbb{R}^m$ be a nonempty compact convex set, let $D \subset C$ be a finite $\delta$-grid of $C$ for some $\delta > 0$, and let $f : D \to \mathbb{R}^m$. Then there exists a $\delta$-local probability distribution $\eta$ on $D$ such that

$$\mathbb{E}_{y \sim \eta}[f(y) \cdot (x - y)] \leq \delta \mathbb{E}_{y \sim \eta}[\|f(y)\|]$$

for all $x \in C$. Moreover, the support of $\eta$ can be taken to consist of at most $m+1$ points of $D$.

When $f$ is bounded, by taking $\delta = \min\{\varepsilon/\sup_{x \in C} \|f(x)\|, \rho\}$ we get:

**Corollary 8** Let $C \subset \mathbb{R}^m$ be a nonempty compact convex set, and $f : C \to \mathbb{R}^m$ a bounded function. Then for every $\varepsilon > 0$ and $\rho > 0$ there exists a $\rho$-local probability distribution $\eta$ on $C$ such that

$$\mathbb{E}_{y \sim \eta}[f(y) \cdot (x - y)] \leq \varepsilon$$

for all $x \in C$. Moreover, the support of $\eta$ can be taken to consist of at most $m+1$ points of $C$.

**Proof of Theorem 7** From the values of $f$ on $D$ one can generate a continuous function $\tilde{f} : C \to \mathbb{R}^m$ such that $\tilde{f}(x)$ is a weighted average of the values of $f$ on grid points that are within $\delta$ of $x$, i.e.,

$$\tilde{f}(x) \in \text{conv}\{f(d) : d \in D \cap B(x; \delta)\},$$

(16)
for all $x \in C$. For instance, put

$$
\tilde{f}(x) := \frac{\sum_{d \in D} \Lambda(x, d) f(d)}{\sum_{d \in D} \Lambda(x, d)},
$$

where $\Lambda(x, d) := [\delta - \|d - x\|]_+$ (the so-called “tent” function); $\tilde{f}$ is continuous because $D$ is finite, $\Lambda(x, d)$ is continuous in $x$, and the denominator is always positive since $D$ is a $\delta$-grid of $C$; as for (16), it follows since $\|d - x\| \geq \delta$ implies $\Lambda(x, d) = 0$.

Theorem 4 applied to $\tilde{f}(x)$ yields a point $z \in C$ such that $\tilde{f}(z) \cdot (x - z) \leq 0$ for all $x \in C$, and then (16) yields a probability distribution $\eta$ on $D \cap B(z; \delta)$ such that $\tilde{f}(z) = \mathbb{E}_{y \sim \eta}[f(y)]$. The distribution $\eta$ is thus $\delta$-local, and its support can be taken to be of size at most $m + 1$ by Carathéodory’s theorem (because $f(y) \in \mathbb{R}^m$). Now

$$
\mathbb{E}_{y \sim \eta}[f(y) \cdot (x - y)] = \mathbb{E}_{y \sim \eta}[f(y) \cdot (x - z)] + \mathbb{E}_{y \sim \eta}[f(y) \cdot (z - y)];
$$

the first term is $\mathbb{E}_{y \sim \eta}[f(y)] \cdot (x - z) = \tilde{f}(z) \cdot (x - z) \leq 0$ (by the choice of $z$), and the second term is $\leq \delta \mathbb{E}_{y \sim \eta} [\|f(y)\|]$ (because $\|y - z\| \leq \delta$ for every $y$ in the support of $\eta$), which completes the proof. □

### 3.4 FP-Procedures and MM-Procedures

The calibration proofs that we provide below construct procedures where the forecast in each period is given by appealing either to the Outgoing Fixed Point Theorems 4 and 7 or to the Outgoing Minimax Theorem 5, in order to satisfy the corresponding forecast-hedging conditions. We will refer to these two kinds of procedures as procedures of type FP and procedures of type MM, respectively.

This distinction is not just a matter of proof technique. It goes the other way around as well (see Hazan and Kakade 2012 for details and relevant literature): calibration that is obtained by FP-procedures, such as continuous calibration, may be used to get approximate Nash equilibria in non-zero-sum games. Therefore this kind of calibration falls essentially in the PPAD complexity class, which is believed to go beyond the class of polynomially solvable problems, such as minimax problems. The distinction between FP-obtainable calibration and MM-obtainable calibration is a significant distinction, of the non-polynomial vs. polynomial vari-

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35This should come as no surprise since game dynamics where players best reply to continuously calibrated forecasts yield in the long run approximate Nash equilibria for general $n$-person games; see Section 6.
4 Calibrated Procedures

In this section we prove the three main calibration results: deterministic continuous calibration, stochastic classic calibration, and almost deterministic classic calibration. The proofs all run along the same lines: first, we show that appropriate forecast-hedging conditions yield calibration (Theorem 9); and second, we construct, using the outgoing results of Section 3, procedures that satisfy the forecast-hedging conditions (Theorem 10).

We illustrate the idea of the proof (see also Section 1.2) by showing how to construct a deterministic procedure that guarantees that $g_t(w) \rightarrow 0$ as $t \rightarrow \infty$ (see (11)) for a single continuous function $w : C \rightarrow [0, 1]$. By the definition of $g_t$ we have

$$tg_t(w) = (t - 1)g_{t-1}(w) + w(c_t)(a_t - c_t),$$

and so

$$\|tg_t(w)\|^2 = \|(t - 1)g_{t-1}(w)\|^2 + 2(t - 1)g_{t-1}(w) \cdot w(c_t)(a_t - c_t) + w(c_t)^2 \|a_t - c_t\|^2. \quad (17)$$

The last term is $\leq \gamma^2$ (since $w(c_t) \in [0, 1]$ and $a_t, c_t$ belong to $C$, whose diameter is $\gamma$). The middle term is $2(t - 1)\varphi(c_t) \cdot (a_t - c_t)$, where $\varphi(c) := w(c)g_{t-1}(w)$ is a continuous function of $c$ that takes values in $\mathbb{R}^m$ (because $g_{t-1}(w) \in \mathbb{R}^m$). The Outgoing Fixed Point Theorem 4 then yields a point in $C$—which will be our forecast $c_t$—that guarantees that $\varphi(c_t) \cdot (a_t - c_t) \leq 0$, for any action $a_t \in A \subseteq C$. Therefore (17) yields the inequality $\|tg_t(w)\|^2 \leq \|(t - 1)g_{t-1}(w)\|^2 + 2\gamma^2$, which applied recursively gives $\|tg_t(w)\|^2 \leq t\gamma^2$, and thus $\|g_t(w)\| \leq \gamma/\sqrt{t} \rightarrow 0$ as $t \rightarrow \infty$.

The proof is easily extended to handle continuous binnings $(w_i)_i$, such as $\Pi_0$ of Proposition 3, which yields continuous calibration. For classic calibration, where the function $\varphi$ above is in general not continuous, we use the Outgoing Minimax Theorem 8 (for a variant $\psi$ of $\varphi$); finally, using the Outgoing Almost Deterministic Fixed Point Theorem 7 instead yields an almost deterministic procedure for classic calibration.

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36 In this simple case of a single $w$ a fixed point is not really needed: take $c_t$ in $C$ that is maximal in the direction $g_{t-1}(w)$, i.e., $c_t \in \arg \max_{x \in C} x \cdot g_{t-1}(w)$. The fixed point is however needed once we consider multiple $w$'s.
4.1 Forecast-Hedging

Let \( \Pi = (w_i)_{i=1}^I \) be a binning. For every period \( t \geq 2 \) and history \( h_{t-1} \) we define two functions, \( \varphi_{t-1} \) and \( \psi_{t-1} \), from \( C \) to \( \mathbb{R}^m \), by

\[
\varphi_{t-1}(c) := \sum_{i=1}^I w_i(c) g_{t-1}(w_i)
\]

\[
\psi_{t-1}(c) := \sum_{i=1}^I w_i(c) e_{t-1}(w_i)
\]

for every \( c \in C \). Thus \( \varphi_{t-1} \) and \( \psi_{t-1} \) are averages of the vectors \( g_{t-1}(w_i) \) and \( e_{t-1}(w_i) \), respectively, with weights that vary with \( c \) and are given by the binning \( \Pi \). We define:

(D) A deterministic forecasting procedure \( \sigma \) satisfies the \( \Pi \)-deterministic forecast-hedging condition if, for every \( t \geq 2 \) and history \( h_{t-1} \),

\[
\varphi_{t-1}(c_t) \cdot (a - c_t) \leq 0 \quad \text{for every } a \in A,
\]

(D-FH) where \( c_t = \sigma(h_{t-1}) \) is the forecast at time \( t \).

(S) A stochastic procedure \( \sigma \) satisfies the \( (\Pi, \varepsilon) \)-stochastic forecast-hedging condition for \( \varepsilon > 0 \) if, for every \( t \geq 2 \) and history \( h_{t-1} \),

\[
\mathbb{E}_{t-1} \left[ \psi_{t-1}(c_t) \cdot (a - c_t) \right] \leq \varepsilon \mathbb{E}_{t-1} \left[ \| \psi_{t-1}(c_t) \| \right]
\]

for every \( a \in A \), (S-FH)

where \( \mathbb{E}_{t-1} \) denotes expectation with respect to the distribution \( \sigma(h_{t-1}) \) of the forecast \( c_t \) at time \( t \).

Remark. The forecast-hedging conditions (D-FH) and (S-FH) require, for each history \( h_{t-1} \), that the corresponding inequality hold “for every \( a \in A \).” This allows the action \( a_t \) that follows the history \( h_{t-1} \) to depend on \( h_{t-1} \), and thus also on \( \sigma(h_{t-1}) \), which is determined by \( h_{t-1} \). Therefore, when \( \sigma \) is a deterministic procedure, \( a_t \) may depend on \( c_t \) as well; this is the “leaky” setup of Foster and Hart (2018) (when \( \sigma \) is stochastic it may depend on the distribution \( \sigma(c_t) \) of \( c_t \), but not on the actual realization of \( c_t \)). See footnote 21 and Section 6.

Theorem 9

27
(D) If a deterministic procedure $\sigma$ satisfies the $\Pi$-deterministic forecast-hedging condition for a continuous binning $\Pi = (w_i)_i$, then

$$\lim_{t \to \infty} \left( \sup_{a_t} K_t^{\Pi} \right) = 0. \quad (18)$$

(S) If a stochastic procedure $\sigma$ satisfies the $(\Pi, \varepsilon)$-stochastic forecast-hedging condition for a finite binning $\Pi = (w_i)_i$ and $\varepsilon > 0$, then

$$\lim_{t \to \infty} \left( \sup_{a_t} \mathbb{E} \left[ K_t^{\Pi} \right] \right) \leq \varepsilon. \quad (19)$$

Proof. (D) Put $S_t := \sum_{i=1}^I \|tg_t(w_i)\|^2$; we will show that $\lim_{t \to \infty} (1/t)S_t = 0$.

Using (17) for each $w_i$, summing over $i$, and recalling the definition of $\phi_{t-1}$ gives

$$S_t \leq S_{t-1} + 2(t-1)\phi_{t-1}(c_t) \cdot (a_t - c_t) + \gamma^2$$

(the last term is $\sum_i w_i(c_t)^2 \|a_t - c_t\|^2 \leq \gamma^2 \sum_i w_i(c_t) = \gamma^2$ since $w_i(c_t) \in [0, 1]$).

This inequality becomes

$$S_t \leq S_{t-1} + \gamma^2$$

when $\sigma$ satisfies (D-FH); by recursion (starting with $S_0 = 0$) we get $S_t \leq t\gamma^2$. All the inequalities hold for every action sequence $a_t$, because for every history $h_{t-1}$, inequality (D-FH) holds for every $a$.

Thus, dividing by $t^2$, we have

$$\sup_{a_t} \sum_{i=1}^I \|tg_t(w_i)\|^2 \leq \frac{\gamma^2}{t} \xrightarrow{t \to \infty} 0.$$ 

Therefore $\sup_{a_t} \|tg_t(w_i)\| \to 0$ as $t \to \infty$ for every $i \in I$, which yields (18) (by the same argument as in the second part of the proof of Proposition 2, because the binning $\Pi$ is continuous).

(S) Put $X_t := \sum_{i=1}^I n_t(w_i) \|e_t(w_i)\|^2$. We will show that

$$\lim_{t \to \infty} \left( \sup_{a_t} \mathbb{E} \left[ \frac{1}{t}X_t \right] \right) \leq \varepsilon^2;$$

this yields (19) since $(K_t^{\Pi})^2 \leq (1/t)X_t$ by Jensen’s inequality.\(^{38}\)

The proof consists of expressing the one-period increment of $X_t$ as a sum of two terms, a $Y_t$-term, which, by forecast-hedging, is at most $\varepsilon^2$ in expectation, and

\(^{37}\)The score $S_t$ is precisely $S$ of Section 1.2.

\(^{38}\)The score $(1/t)X_t$ is the square-calibration score for $\Pi$, namely, the average of the squared norms of the errors (i.e., replace $\|e_t\|$ with $\|e_t\|^2$ in formula (6) of $K_t^{\Pi}$).
a $Z_t$-term, which converges to zero:

$$X_t - X_{t-1} = Y_t + Z_t,$$

(20)

$$\mathbb{E}_{t-1}[Y_t] \leq \varepsilon^2,$$ and

(21)

$$\sup_{a_t} \sum_{s=1}^{t} Z_s \leq O(\log t)$$

(22)

for every $t \geq 1$ (where $X_0 = 0$). This proves the result, since taking overall expectation of (21) yields $\mathbb{E}[Y_t] \leq \varepsilon^2$, and thus

$$\mathbb{E} \left[ \frac{1}{t} X_t \right] = \mathbb{E} \left[ \frac{1}{t} \sum_{s=1}^{t} (X_s - X_{s-1}) \right] = \frac{1}{t} \sum_{s=1}^{t} \mathbb{E}[Y_s] + \frac{1}{t} \sum_{s=1}^{t} \mathbb{E}[Z_s] \leq \varepsilon^2 + O\left(\frac{\log t}{t}\right) \to \varepsilon^2$$

as $t \to \infty$, uniformly over $a_t$.

- **Proof of (21).** We start with the following easy-to-check identity, for scalars $\alpha, \beta \geq 0$ and vectors $u, v$:

$$\frac{1}{\alpha + \beta} \left( \frac{\alpha u + \beta v}{\alpha + \beta} \right)^2 - \frac{\alpha u - \beta v}{\alpha + \beta} \| u \|^2 = 2\beta u \cdot v - \beta \| u \|^2 - \frac{\beta^2}{\alpha + \beta} \| u - v \|^2.$$

Using this for $\alpha = n_{t-1}(w)$, $\beta = w(c_t)$, $u = e_{t-1}(w)$, and $v = a_t - c_t$ yields

$$n_t(w) \| e_t(w) \|^2 - n_{t-1}(w) \| e_{t-1}(w) \|^2 = y_t(w) + z_t(w),$$

where

$$y_t(w) := 2w(c_t)e_{t-1}(w) \cdot (a_t - c_t) - w(c_t) \| e_{t-1}(w) \|^2$$

and

$$z_t(w) := \frac{w(c_t)^2}{n_t(w)} \| e_{t-1}(w) - (a_t - c_t) \|^2 \leq 4\gamma^2 \frac{w(c_t)^2}{n_t(w)}.$$

(the last inequality because $\| e_{t-1}(w) \| \leq \gamma$ and $\| a_t - c_t \| \leq \gamma$). Applying this to each $w_i$, summing over $i$, and recalling the definition of $X_t$ and $\psi_{t-1}$ gives $X_t - X_{t-1} = Y_t + Z_t$, where

$$Y_t := \sum_{i=1}^{I} y_t(w_i) = 2\psi_{t-1}(c_t) \cdot (a_t - c_t) - \sum_{i=1}^{I} w_i(c_t) \| e_{t-1}(w_i) \|^2,$$ and

$$Z_t := \sum_{i=1}^{I} z_t(w_i) \leq 4\gamma^2 \sum_{i=1}^{I} \frac{w_i(c_t)^2}{n_t(w_i)}.$$

- **Proof of (20).** By the stochastic forecast-hedging condition (S-FH) we have
\[ \mathbb{E}_{t-1} \left[ 2\psi_{t-1}(c_t) \cdot (a_t - c_t) \right] \leq \mathbb{E}_{t-1} \left[ 2\varepsilon \|\psi_{t-1}(c_t)\| \right] ; \text{ now} \]

\[ 2\varepsilon \|\psi_{t-1}(c_t)\| \leq \sum_{i=1}^{I} w_i(c_t) (2\varepsilon \|e_{t-1}(w_i)\|) \leq \sum_{i=1}^{I} w_i(c_t) (\varepsilon^2 + \|e_{t-1}(w_i)\|^2) \]

\[ = \varepsilon^2 + \sum_{i=1}^{I} w_i(c_t) \|e_{t-1}(w_i)\|^2. \]

• **Proof of (22).** We claim that

\[ \sum_{s=1}^{t} \frac{w(c_s)^2}{n_s(w)} \leq \ln n_t(w) + 2 \leq \ln t + 2 \quad (23) \]

for every \( w : C \to [0, 1] \) and \( t \geq 1 \) with \( n_t(w) > 0 \). Indeed, both \( w(c_s) \) and \( w(c_s)/n_s(w) \) are between 0 and 1, and so for every \( 1 \leq r \leq t \) we have

\[ \sum_{s=1}^{r} \frac{w(c_s)^2}{n_s(w)} \leq \sum_{s=1}^{r} w(c_s) = n_r(w) \quad \text{and} \]

\[ \sum_{s=r+1}^{t} \frac{w(c_s)^2}{n_s(w)} \leq \sum_{s=r+1}^{t} \frac{w(c_s)}{n_s(w)} = \sum_{s=r+1}^{t} \left( 1 - \frac{n_{s-1}(w)}{n_s(w)} \right) \]

\[ \leq \sum_{s=r+1}^{t} \ln \left( \frac{n_s(w)}{n_{s-1}(w)} \right) = \ln \left( \frac{n_t(w)}{n_r(w)} \right) \]

(we used \( 1 - 1/x \leq \ln x \) for \( x \geq 1 \)). Taking \( r \) such that \( 1 \leq n_r(w) < 2 \) yields \( < 2 \) in the first inequality and \( \leq \ln n_t(x) \leq \ln t \) in the second, and thus (23); if there is no such \( r \) then \( n_t(w) < 1 \), and the first inequality with \( r = t \) gives \( < 1 \), and thus (23). Applying (23) to each \( w_i \) and summing over \( i \) yields \( \sum_{s=1}^{t} Z_s \leq 4\gamma^2 I(\ln t + 2) \), and thus (22).

This completes the proof of (S). \( \square \)

**Remark.** In (S), using (21) one gets the stronger almost sure convergence; see Appendix A.5.

The reason that the two proofs are slightly different—we use \( S_t \), and thus \( \varphi_{t-1} \), in (D), and \( X_t \), and thus \( \psi_{t-1} \), in (S)—has to do with the limit being 0 in the former, and \( \varepsilon \) in the latter. Roughly speaking, for vectors \( u \) in \( I \)-dimensional space, \( \|u\| = (\sum_i u_i^2)^{1/2} \to 0 \) implies \( \|u\|_1 = \sum_i |u_i| \to 0 \) regardless of the size of \( u \).

---

39One can easily obtain a bound of \( o(t) \) in (23), since \( w(c_t)^2/n_t(w) \leq w(c_t)/n_t \to 0 \) as \( t \to \infty \) (indeed, if \( n_t(w) \to \infty \) then \( w(c_t)/n_t(w) \leq 1/n_t(w) \to 0 \), and if \( n_t(w) \to N < \infty \) then \( w(c_t)/n_t(w) = 1 - n_{t-1}(w)/n_t(w) \to 1 - N/N = 0 \)). Inequality (23) provides a better bound, uniform over all \( w \) and sequences \( c_t \).
whereas \( \|u\| \leq \varepsilon \) yields \( \|u\|_1 \leq \sqrt{I} \varepsilon \), which may not be small when \( I \) increases with \( \varepsilon \); see Appendix [A.4] for further details.

We now show that the outgoing results of Section 3 yield the existence of forecast-hedging procedures.

**Theorem 10**

(D) For every continuous binning \( \Pi \) there exists a deterministic procedure of type FP that satisfies the \( \Pi \)-deterministic forecast-hedging condition.

(S) For every finite binning \( \Pi \), every \( \varepsilon > 0 \), and every finite \( \varepsilon \)-grid \( D \) of \( C \), there exists a stochastic procedure of type MM with forecasts in \( D \) that satisfies the \((\Pi, \varepsilon)\)-stochastic forecast-hedging condition.

(AD) For every finite binning \( \Pi \), every \( \varepsilon > 0 \), and every finite \( \varepsilon \)-grid \( D \) of \( C \), there exists an \( \varepsilon \)-almost deterministic procedure of type FP with forecasts in \( D \) that satisfies the \((\Pi, \varepsilon)\)-stochastic forecast-hedging condition.

**Proof.** (D) When \( \Pi \) is a continuous binning, each function \( \varphi_{t-1} \) is continuous (since each \( w_i \) is continuous and \( \|g_{t-1}(w_i)\| \leq \gamma \); when \( I \) is infinite use the uniform convergence of the corresponding finite sums, as in the second part of the proof of Proposition [2]). Apply the Outgoing Fixed Point Theorem [4] to \( \varphi_{t-1} \) for each history \( h_{t-1} \).

(S) Apply the Outgoing Minimax Theorem [5] to \( \psi_{t-1} \) and \( \delta = \varepsilon \) for each history \( h_{t-1} \).

(AD) Apply the Outgoing Almost Deterministic Fixed Point Theorem [7] to \( \psi_{t-1} \) and \( \delta = \varepsilon \) for each history \( h_{t-1} \).

\( \square \)

### 4.2 Calibration

We now immediately obtain the existence of appropriate calibrated procedures.

**Theorem 11**

(D) There exists a deterministic procedure of type FP that is continuously calibrated.

(S) For every \( \varepsilon > 0 \) there exists a stochastic procedure of type MM that is \( \varepsilon \)-calibrated; moreover, all its forecasts are in \( D \) for any given finite \( \varepsilon \)-grid \( D \) of \( C \).

(AD) For every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-almost deterministic procedure of type FP that is \( \varepsilon \)-calibrated; moreover, all its forecasts are in \( D \) for any given finite \( \varepsilon \)-grid \( D \) of \( C \).
Part (D) implies, by Proposition 15 in Appendix A.2, the results of Foster and Hart (2018) for smooth calibration and of Kakade and Foster (2004) and Foster and Kakade (2006) for weak calibration. Part (S) yields the classic calibration result of Foster and Vohra (1998), and part (AD) the result of Kakade and Foster (2004) for almost deterministic classic calibration.

Proof. (D) Apply Theorem 10(D) and Theorem 9(D) with the continuous binning $\Pi_0$ given by Proposition 3.

(S) Let $D = \{d_1, ..., d_I\}$ be a given finite $\varepsilon$-grid of $C$. Put $w_i := 1_{d_i}$ for $i = 1, ..., I$, and $w_0 := 1_{C \setminus D}$ and let $\Pi$ be the finite binning $(w_i)_{i=0}^I$. When all forecasts are in $D$ we have $K_{\Pi}^t = \sum_{i=1}^I \|g_t(1_{d_i})\| = K_t$ (since $g_t(w_0) = 0$). Apply Theorem 10(S) and Theorem 9(S).

(AD) Same as (S), applying Theorem 10(AD). □

5 A Simple Calibrated Procedure for Binary Events

This section shows how to obtain classic calibration in the one-dimensional case, where the actions are binary yes/no outcomes (such as win/lose in politics and sport events, or rain/shine, and so on), by a procedure that is as simple as can be; it is simpler than any existing procedure, including the one in Foster (1999). The procedure is moreover almost deterministic, with all randomizations being between two neighboring points on a fixed grid. It is essentially the procedure described in Section 1.2 in the Introduction, except that we work with the normalized errors $e$ instead of the gaps $G$.

We are thus in the one-dimensional case ($m = 1$), with $A = \{0, 1\}$ (with, say, 1 for “rain” and 0 for “no rain”) and $C = [0, 1]$. Fix an integer $N \geq 1$, and let $D := \{0, 1/N, 2/N, ..., 1\}$ be the grid on which the forecasts lie. Consider a history $h_{t-1}$. For every $i = 0, 1, ..., N$, the error of the forecast $i/N$ is $e^i := e_{t-1}(i/N) = r^i/n^i - i/N$, where $n^i$ is the number of times that the forecast $i/N$ has been used in the first $t - 1$ periods, and $r^i$ is the number of rainy periods among these $n^i$ periods (with $e^i = 0$ when $n^i = 0$). The procedure $\sigma$ chooses the forecast $c_t$ as follows (as in Figure 4, with $e$ instead of $G$):

- **Case 1:** There is $j$ such that $e^j = 0$. Put $y := j/N$ and let the (deterministic) forecast be $c_t = y$.

\footnote{Since $e^j = 0$ for unused forecasts $j/N$, in the first periods we try each point on the grid once; alternatively, assume that there is some initial data for each possible forecast (all this does not matter, of course, in the long run).}
• Case 2: \( e^i \neq 0 \) for all \( i \). In this case \( e^0 > 0 \) (because \( r^0 \geq 0 \)) and \( e^N < 0 \) (because \( r^N \leq u^N \)), and so let \( j \geq 1 \) be, for concreteness, the smallest index with \( e^j < 0 \); thus \( e^{j-1} > 0 > e^j \). Put \( y_1 := (j-1)/N \) and \( y_2 := j/N \), and let the forecast be \( c_t = y_1 \) with probability \( p_1 := |e^j|/(|e^{j-1}| + |e^j|) \) and \( c_t = y_2 \) with the remaining probability \( p_2 := |e^{j-1}|/(|e^{j-1}| + |e^j|) \); thus, \( p_1 e_{t-1}(y_1) + p_2 e_{t-1}(y_2) = 0 \) (cf. (I)), and \( y_2 - y_1 = 1/N \).

The above construction amounts to linearly interpolating the function \( e_{t-1} \) from the finite grid \( D \) to the whole interval \([0, 1]\), and then taking a point where this function vanishes (\( y \) in Case 1, and \( p_1 y_1 + p_2 y_2 \) in Case 2) and using it for the forecast (\( y \) itself in Case 1, and the \( p_1, p_2 \) probabilistic mixture of \( y_1 \) and \( y_2 \) in Case 2). We thus have \( E_{t-1}[e_{t-1}(c_t)] = 0 \) in both cases, where \( E_{t-1} \) stands for \( E [\cdot | h_{t-1}] \).

**Theorem 12** The above procedure \( \sigma \) is \( 1/(2N) \)-almost deterministic and \( 1/(2N) \)-calibrated.

**Proof.** Put \( \bar{y} := y \) in Case 1 and \( \bar{y} := (y_1 + y_2)/2 \) in Case 2. Then \( |\bar{y} - c_t| \leq 1/(2N) \) in both cases, which implies that \( E_{t-1}[e_{t-1}(c_t) \cdot (\bar{y} - c_t)] \leq (1/2N) E_{t-1}[|e_{t-1}(c_t)|] \).

Now \( E_{t-1}[e_{t-1}(c_t) \cdot (a - \bar{y})] = 0 \) for every \( a \) (because \( a - \bar{y} \) is constant given \( h_{t-1} \), and \( E_{t-1}[e_{t-1}(c_t)] = 0 \) by the construction of \( \sigma \); adding to the previous inequality gives the \((\Pi, \varepsilon)\)-stochastic forecast-hedging condition (S-FH), where \( \Pi \) is the same as in the proof of Theorem (S), and \( \varepsilon = 1/(2N) \). Therefore \( \sigma \) is \( 1/(2N) \)-calibrated by Theorem (S); in addition, \( \sigma \) is \( 1/(2N) \)-almost deterministic because we always have \( |c_t - \bar{y}| \leq 1/(2N) \). \( \square \)

The calibration bound of \( 1/(2N) \) is the best that one can achieve with forecasts on the grid \( D \): consider for instance the action sequence where \( a_t \) equals 1 with probability \( 1/(2N) \), independently over \( t \).

### 6 Calibration and Game Dynamics

Forecasts are a useful tool for dynamic multi-player interactions. Consider a game that is played repeatedly. A natural type of game dynamic is one where in each period the players make forecasts on what will happen next and then choose their actions in response to these forecasts. Interesting long-run behavior obtains when

\(^{41}\)Any \( j \) for which \( e^{j-1} \) and \( e^j \) have opposite signs will work here. In fact, a \( j \) for which the signs are reversed, i.e., \( e^{j-1} < 0 < e^j \) (however, such a \( j \) need not exist in general), will work even better, as it yields 0 on the right-hand side of the forecast-hedging condition (S-FH).
the forecasts are “good”—i.e., calibrated—and the responses to the forecasts are “good”—i.e., best responses.

The “calibrated learning” of Foster and Vohra (1997), on the one hand, and the “publicly calibrated learning” of Kakade and Foster (2004) and the “smooth calibrated learning” of Foster and Hart (2018), on the other hand, are two such types of game dynamics. The main difference between the two types is that in the former each player uses a stochastic classically calibrated forecasting procedure, whereas in the latter all players use the same deterministic weakly, or smoothly, calibrated forecasting procedure. In the long run, the former yields correlated equilibria as the time average of play, whereas the latter yields Nash equilibria as the period-by-period behavior (of course, everything should be understood with appropriate “approximate” adjectives); see Foster and Hart (2018) for a more extensive discussion. If we replace the deterministic weakly and smoothly calibrated procedures with the stronger, but easier to obtain, deterministic continuously calibrated procedures (see Proposition 15 in Appendix A.1), we obtain the same long-run result: period-by-period behavior that is close to Nash equilibria. The simplicity of continuous calibration allows for a simple result and proof; see Theorem 13 below.

The game dynamics results underscore the importance of deterministic procedures, which are “leaky” (see Foster and Hart 2018) and thus remain calibrated even if in each period the forecast is revealed before the action is chosen. By contrast, stochastic procedures are no longer calibrated if the actual realization of the random forecast is revealed before the action is chosen.

### 6.1 Continuously Calibrated Learning

A finite game is given by a finite set of players $N$, and, for each player $i \in N$, a finite set of pure strategies $A^i$ and a payoff function $u^i : A \to \mathbb{R}$, where $A := \prod_{i \in N} A^i$ denotes the set of strategy combinations of all players. Let $n := |N|$ be the number of players, $m^i := |A^i|$ the number of pure strategies of player $i$, and $m := \sum_{i \in N} m^i$. The set of mixed strategies of player $i$ is $X^i := \Delta(A^i)$, the unit simplex (i.e., the set of probability distributions) on $A^i$; we identify the pure strategies in $A^i$ with the unit vectors of $X^i$, and so $A^i \subseteq X^i$. Put $C \equiv X := \prod_{i \in N} X^i$ for the set of mixed-strategy combinations (i.e., $N$-tuples of mixed strategies). The payoff functions $u^i$ are multilinearly extended to $X$, and thus $u^i : X \to \mathbb{R}$.

For each player $i$ and combination of mixed strategies of the other players $x^{-i} = (x^j)_{j \neq i} \in \prod_{j \neq i} X^j =: X^{-i}$, let $\bar{u}^i(x^{-i}) := \max_{y^i \in X^i} u^i(y^i, x^{-i}) =$
max_{a' \in A_i} u^i(a', x^{-i}) be the maximal payoff that $i$ can obtain against $x^{-i}$; for every $\varepsilon \geq 0$, let $\text{BR}_t^i(x^{-i}) := \{x^i \in X^i : u^i(x^i, x^{-i}) \geq \bar{u}^i(x^{-i}) - \varepsilon \}$ denote the set of $\varepsilon$-best replies of $i$ to $x^{-i}$. A (mixed) strategy combination $x \in X$ is a Nash $\varepsilon$-equilibrium if $x^i \in \text{BR}_t^i(x^{-i})$ for every $i \in N$; let $\text{NE}(\varepsilon) \subseteq X$ denote the set of Nash $\varepsilon$-equilibria of the game.

A (discrete-time) dynamic consists of each player $i \in N$ playing a pure strategy $a^i_t \in A^i$ at each time period $t = 1, 2, \ldots$; put $a_t = (a^i_t)_{i \in N} \in A$. There is perfect monitoring: at the end of period $t$ all players observe $a_t$. The dynamic is uncoupled (Hart and Mas-Colell 2003, 2006, 2013) if the play of every player $i$ may depend only on player $i$'s payoff function $u^i$ (and not on the other players’ payoff functions). Formally, such a dynamic is given by a mapping for each player $i$ from the history $h_{t-1} = (a_1, \ldots, a_{t-1})$ and his own payoff function $u^i$ into $X^i = \Delta(A^i)$ (player $i$’s choice may be random); we will call such mappings uncoupled. Let $x^i_t \in X^i$ denote the mixed action that player $i$ plays at time $t$, and put $x_t = (x^i_t)_{i \in N} \in X$.

The dynamics we consider are continuous variants of the “calibrated learning” introduced by Foster and Vohra (1997). Calibrated learning consists of each player best replying to calibrated forecasts on the other players’ strategies; it results in the joint distribution of play (i.e., the time average of the $N$-tuples of strategies $a_t$) converging in the long run to the set of correlated equilibria of the game. We consider continuously calibrated learning, where stochastic classic calibration is replaced with deterministic continuous calibration, and best replying is replaced with continuous approximate best replying. Moreover, the forecasts are now $N$-tuples of mixed strategies (in $\prod_i \Delta(A^i)$), rather than correlated mixtures (in $\Delta(\prod_i A^i)$).

Formally, given $\varepsilon > 0$ a continuously calibrated $\varepsilon$-learning dynamic is given by:

(I) A deterministic continuously calibrated procedure on $X$, which yields at each time $t$ a forecast $c_t = (c^i_t)_{i \in N} \in X$ on the distribution of strategies of each player.

(II) For each player $i \in N$ a continuous $\varepsilon$-best-reply function $\beta^i : X \to X^i$; i.e., $\beta^i(x) \in \text{BR}_t^i(x^{-i})$ for every $x^{-i} \in X^{-i}$.

The dynamic consists of each player running the procedure in (I), generating at time $t$ a forecast $c_t \in X$; then each player $i$ plays at period $t$ the mixed strategy $x^i_t := \beta^i(c_t) \in X^i$, where $\beta^i$ is given by (II). All players observe the strategy $x_t = (x^i_t)_{i \in N} \in X$.

\[ \text{Thus } P[a_t = a \mid h_{t-1}] = \prod_{i \in N} x^i_t(a^i) \text{ for every } a = (a^i)_{i \in N} \in A, \text{ where } h_{t-1} \text{ is the history and } x^i_t(a^i) \text{ is the probability that } x^i_t \in \Delta(A^i) \text{ assigns to the pure strategy } a^i \in A^i. \]
combination \( a_t = (a^i_t)_{i \in N} \in A \) that has actually been played, and remember it. Let \( \beta(x) = (\beta^i(x))_{i \in N} \); thus, \( \beta : X \to X \) is a continuous function. We refer to \( c_t \in X \) as the forecast, \( x_t = \beta(c_t) \in X \) the behavior (i.e., the mixed strategies played), and \( a_t \in A \) the actions (i.e., the realized pure strategies played (\( c_t, x_t \), and \( a_t \) depend on the history).

Since for each player \( i \) the approximate best reply condition in (II) makes use only of player \( i \)'s payoff function \( u^i \), we can without loss of generality choose \( \beta^i \) so as to depend only on \( u^i \), which makes the dynamic uncoupled (see above).

The existence of a deterministic continuously calibrated procedure in 1 is given by Theorem \( \Pi(D) \); the existence of \( \varepsilon \)-approximate continuous best-reply mappings in (II) is well known.

Our result is:

\begin{unsrttheo}
\textbf{Theorem 13} Let \( \Gamma = (N, (A^i)_{i \in N}, (u^i)_{i \in N}) \) be a finite game. For every \( \varepsilon > 0 \), a continuously calibrated \( \varepsilon \)-learning dynamic is an uncoupled dynamic and satisfies almost surely

\[
\lim_{t \to \infty} \frac{1}{t} |\{s \leq t : x_s \in \text{NE}(\varepsilon')\}| = 1 \quad (24)
\]

for every \( \varepsilon' > \varepsilon \).
\end{unsrttheo}

The proof goes by the following three claims. (i) If the forecasts \( c_t \) are continuously calibrated for the sequence of pure strategies \( a_t \), they are continuously calibrated also for the sequence of mixed strategies \( x_t \) (because, by the law of large numbers, the long-run averages of the \( a_t \)'s and of the \( x_t \)'s are close, as \( x_t \) is the expectation of \( a_t \) conditional on the history). (ii) For every \( c \), in every period where the forecast is \( c \) the mixed play is the same, namely, \( x = \beta(c) \), and so if the sequence \( c_t \) is continuously calibrated for the sequence \( x_t \) then \( c_t \approx x_t = \beta(c_t) \).

(iii) From \( c_t \approx x_t \) we immediately get \( x_t = \beta(c_t) \approx \beta(x_t) \) (apply the continuous map \( \beta \) to both sides), which says that the approximate best reply to \( x_t \) is \( x_t \) itself, and thus \( x_t \) is an approximate Nash equilibrium.

The crucial feature of our dynamic is that continuous calibration is preserved despite the fact that the actions depend on the forecasts (this leakiness property does not hold for classic, probabilistic, calibration); in addition, in each period all players have the same (deterministic) forecast.

In Appendix \( A.6 \) we provide a number of comments and extensions.

\( ^{43} \)It does not follow that we can take \( \varepsilon' = \varepsilon \); for instance, consider the case where at time \( t \) we have an \( (\varepsilon + 1/t) \)-equilibrium. “Almost surely” applies to all \( \varepsilon' > \varepsilon \) simultaneously (take a sequence \( \varepsilon'_n \) decreasing to \( \varepsilon \)).
Proof. For every \( w : X \to [0, 1] \) let \( \tilde{g}_t(w) \) be the per-period gap for the mixed \( x_t \) instead of the pure \( a_t \), i.e.,

\[
\tilde{g}_t(w) := \frac{1}{t} \sum_{s=1}^{t} w(c_s)(x_s - c_s) = g_t(w) + \frac{1}{t} \sum_{s=1}^{t} w(c_s)(a_s - x_s).
\]

(25)

- Claim (i). Let \( W_0 \) be a countable collection of continuous functions \( w : X \to [0, 1] \). Then for almost all infinite histories \( h_\infty = (c_t, a_t)_{t=1}^{\infty} \) we have

\[
\lim_{t \to \infty} \tilde{g}_t(w) = 0 \text{ for all } w \in W_0.
\]

Proof. First, for every \( h_\infty \) we have \( \lim_{t \to \infty} g_t(w) = 0 \) for all \( w \in W_0 \) by continuous calibration (see Proposition 2).

Second, for each \( w \) we have

\[
\mathbb{E}[w(c_s)a_s | h_{s-1}] = w(c_s)\mathbb{E}[a_s | h_{s-1}] = w(c_s)x_s
\]

(given \( h_{s-1} \) the forecast \( c_s \), and thus \( w(c_s) \), is determined, and so only \( a_s \) is random; its conditional expectation is \( \mathbb{E}[a_s | h_{s-1}] = \beta(c_s) = x_s \)). The Strong Law of Large Numbers for Dependent Random Variables (Theorem 32.1.E in Loève, 1978) says that

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} (Y_s - \mathbb{E}[Y_s|h_{s-1}]) = 0 \text{ (a.s.)}
\]

(26)

for bounded random variables \( Y_t \); since the \( w(c_s)a_s \) are all bounded by \( \gamma \), and there are countably many \( w \) in \( W_0 \), we obtain

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} (w(c_s)a_s - w(c_s)x_s) = 0 \text{ for all } w \in W_0 \text{ (a.s.).}
\]

Using (25) yields the claim. \( \square \)

- Claim (ii). For every \( \delta > 0 \) we have

\[
\lim_{t \to \infty} \frac{1}{t} \left| \{s \leq t : \|\beta(c_s) - c_s\| \geq \delta \} \right| = 0
\]

for almost every \( h_\infty \).

Proof. For every \( d \in X \) and \( \ell > 0 \) let \( w_{d,\ell}(x) := [1 - \ell \|x - d\|]_+ \) (a “tent” function on \( X \)); thus, \( w_{d,\ell}(x) > 0 \) if and only if \( x \in B(d; 1/\ell) \). Let \( D \) be the set of points \( x \) with rational coordinates; put \( W_0 := \{w_{d,\ell} : d \in D, \ell \geq 1\} \); then \( W_0 \) is a countable collection of continuous functions from \( X \) to \([0, 1]\), and so Claim (i) applies to it.

---

\[ 44 \] One can show, as in Section 2.2, that \( \lim_{t \to \infty} \tilde{g}_t(w) = 0 \) for all continuous \( w : X \to [0, 1] \) holds for almost all infinite histories (however, there is no uniformity over the action sequences).

\[ 45 \] Recall that we identify the pure actions \( a^i \in A^i \) with the unit vectors in the simplex \( X^i \).
Take $\delta > 0$; the function $\alpha(x) := \beta(x) - x$ is uniformly continuous on the compact set $X$, and so there is an integer $\ell > 0$ such that $\|x - y\| \leq 1/\ell$ implies $\|\alpha(x) - \alpha(y)\| \leq \delta$. If $d \in D$ satisfies $\|\alpha(d)\| \geq 2\delta$, then for every $x$ with $w_{d,\ell}(x) > 0$, i.e., $x \in B(d; 1/\ell)$, we have $\|\alpha(x) - \alpha(d)\| \leq \delta$, which yields

\[
\left\| \sum_{s=1}^{t} w_{d,\ell}(c_s)\alpha(c_s) - \sum_{s=1}^{t} w_{d,\ell}(c_s)\alpha(d) \right\| \leq \delta \sum_{s=1}^{t} w_{d,\ell}(c_s),
\]

that is, $\|t\tilde{g}_t(w_{d,\ell}) - \alpha(d)n_t(w_{d,\ell})\| \leq \delta n_t(w_{d,\ell})$. Therefore

\[
\|t\tilde{g}_t(w_{d,\ell})\| \geq (\|\alpha(d)\| - \delta) n_t(w_{d,\ell}) \geq \delta n_t(w_{d,\ell}).
\]

By Claim (i), this implies that

\[
\frac{1}{t}\tilde{g}_t(w_{d,\ell}) \to 0 \quad \text{(27)}
\]

almost surely as $t \to \infty$.

Take a finite set $D_0 \subset D$ such that $\cup_{d \in D_0} B(d; 1/\ell) \supset X$, and put $D_1 := \{d \in D_0 : \|\alpha(d)\| \geq 2\delta\}$. The compact set $Y := \{x \in X : \|\alpha(x)\| \geq 3\delta\}$ is covered by $\cup_{d \in D_1} B(d; 1/\ell)$ (because $\|\alpha(x)\| \geq 3\delta$ implies that there is $d \in D_0$ such that $y \in B(d; 1/\ell)$, and then $\|\alpha(d)\| \geq \|\alpha(x)\| - \delta \geq 2\delta$), and the continuous function $\sum_{d \in D_1} w_{d,\ell}(x)$ is positive on $Y$, and thus it is $\geq \eta$ for some $\eta > 0$, yielding

\[
\sum_{d \in D_1} n_t(w_{d,\ell}) = \sum_{s=1}^{t} \sum_{d \in D_1} w_{d,\ell}(c_s) \geq \eta \cdot |\{s \leq t : \|\alpha(c_s)\| \geq 3\delta\}|.
\]

Using (27) and replacing $\delta$ with $\delta/3$ completes the proof. □

- Claim (iii). For every $\varepsilon' > \varepsilon$ there is $\delta > 0$ such that $\|\beta(c) - c\| \leq \delta$ implies that $\beta(c)$ is a Nash $\varepsilon'$-equilibrium.

Proof. By the uniform continuity of the functions $\beta^i$ and $u^i$, let $\delta > 0$ be such that $\|x - y\| \leq \delta$ implies $|u^i(\beta^i(x), x^{-i}) - u^i(\beta^i(y), x^{-i})| \leq \varepsilon' - \varepsilon$ for every $i$. Taking $x = \beta(c)$ and $y = c$ yields $|u^i(\beta^i(x), x^{-i}) - u^i(x)| \leq \varepsilon' - \varepsilon$, which together with $u^i(\beta^i(x), x^{-i}) \geq \max_y u^i(y^i, x^{-i}) - \varepsilon$ by the choice of $\beta^i$ as an $\varepsilon$-best reply proves the claim. □

The theorem follows from Claims (ii) and (iii). □
7 The Minimax Universe vs. the Fixed Point Universe

The forecast-hedging integration of the various calibration approaches that we have carried out has pointed to a clear distinction between two separate, parallel, universes: the MINIMAX universe and the FIXED POINT universe.\footnote{This applies to dimension $m \geq 2$ (there is no distinction for dimension $m = 1$, where both minimax and fixed point reduce to the intermediate value theorem).} Table 1 summarizes the differences exhibited in the present paper.

|                          | MINIMAX | FIXED POINT |
|--------------------------|---------|-------------|
| forecast-hedging         | stochastic | deterministic |
| procedure type           | MM      | FP          |
| calibration              | classic | continuous  |
| equilibrium              | correlated | Nash        |
| dynamic result           | time average | period-by-period |

Table 1: The minimax and the fixed point universes

A Appendix

A.1 General Binnings

In this appendix we show that the limitation to countable binnings is without loss of generality.

Sums over arbitrary sets are defined, as usual, as the supremum over all finite sums, i.e., $\sum_{i \in I} z_i := \sup \{ \sum_{i \in J} z_i : J \subseteq I, |J| < \infty \}$ (for real $z_i$).

Define a general binning as $\Pi = (w_i)_{i \in I}$, where $I$ is an arbitrary set of bins and $w_i : C \to [0,1]$ for every $i \in I$, such that $\sum_{i \in I} w_i(c) = 1$ for every $c \in C$. The general binning $\Pi$ is continuous if all $w_i$ are continuous functions. The $\Pi$-calibration score is $K^\Pi_t := \sum_{i \in I} \| g_t(w_i) \|$.

For classic calibration, $K_t$ is the maximal score, i.e.,

$$K_t = \max_\Pi K^\Pi_t,$$

where $\Pi$ ranges over all general binnings. Indeed, Lemma $\Pi$ holds for arbitrary
collections \((w_j)_{j \in J}\) (apply it to finite sets and then take the supremum), and so \(K^\Pi_t \leq K_t\) for every general binning \(\Pi\).

For continuous calibration, which is defined as \(\Pi\)-calibration for every *countable* continuous binning \(\Pi\), we show that it implies \(\Pi\)-calibration for every continuous *general* binning \(\Pi\) as well.

**Proposition 14** If the deterministic procedure \(\sigma\) is continuously calibrated then it is \(\Pi\)-calibrated for every continuous general binning \(\Pi\).

**Proof.** Let \(\Pi = (w_i)_{i \in I}\) be a continuous general binning.

We claim that for every \(\varepsilon > 0\) there is a finite set \(J^* \subseteq I\) such that

\[
\left\| \sum_{i \in I \setminus J^*} w_i \right\| \leq \varepsilon. \tag{28}
\]

This follows from Dini’s theorem for nets (instead of sequences); the proof is the same, and as it is short we provide it here for completeness. Let \(J^*\) denote the collection of finite subsets of \(I\). For every \(J \in J\) let \(D_J := \{c \in C : \sum_{i \in J} w_i(c) > 1 - \varepsilon\}\); then \(D_J\) is an open set (because \(J\) is finite and so \(\sum_{i \in J} w_i\) is continuous), and \(\cup_{J \in J} D_J = C\) (because for every \(c\) we have \(\sup_{J \in J} \sum_{i \in J} w_i(c) = 1\), and so there is \(J \in J\) for which the sum is \(1 - \varepsilon\)). The set \(C\) is compact, and so there is a finite subcover \(\cup_{k=1}^r D_{J_k} = C\). Put \(J^* := \cup_{k=1}^r J_k\); then \(J^*\) is a finite set, and \(\sup_{J \in J^*} \sum_{i \in J} w_i(c) = 1\), and so \(\sum_{i \in I \setminus J^*} w_i < \varepsilon\), which yields (28).

Therefore, by Lemma 1

\[
\sum_{i \in I \setminus J^*} \left\| g_t(w_i) \right\| \leq \varepsilon K_t \leq \gamma \varepsilon.
\]

For any \(J \in J\) we then have

\[
\sup_{a_t} \sum_{i \in J} \left\| g_t(w_i) \right\| \leq \sum_{i \in J \cap J^*} \sup_{a_t} \left\| g_t(w_i) \right\| + \sup_{a_t} \sum_{i \in J \setminus J^*} \left\| g_t(w_i) \right\|
\]

\[
\leq \sum_{i \in J^*} \sup_{a_t} \left\| g_t(w_i) \right\| + \gamma \varepsilon.
\]

Taking the supremum over \(J \in J\) yields

\[
\sup_{a_t} \sum_{i \in I} \left\| g_t(w_i) \right\| \leq \sum_{i \in J^*} \sup_{a_t} \left\| g_t(w_i) \right\| + \gamma \varepsilon;
\]

the right-hand side converges to \(\gamma \varepsilon\) as \(t \to \infty\) by (11) of Proposition 2 (as \(J^*\) is
finite). Since $\varepsilon > 0$ is arbitrary, the limit of the left-hand side is 0.

□

A.2 Continuous Calibration Implies Smooth and Weak Calibration

This appendix recalls the definitions of the existing concepts of smooth and weak calibration, and proves that they are both implied by the stronger concept of continuous calibration (see Section 2).

Let $\varepsilon \geq 0$ and $L < \infty$. For a collection $\Lambda = (\Lambda_x)_{x \in C}$ of $L$-Lipschitz functions $\Lambda_x : C \to [0, 1]$, let

$$\tilde{K}_t^\Lambda := \frac{1}{t} \sum_{x \in C} n_t(x) \| e_t(\Lambda_x) \|.$$  

(29)

A deterministic procedure is $(\varepsilon, L)$-smoothly calibrated (Foster and Hart 2018) if

$$\lim_{t \to \infty} \left( \sup_{a_t, \Lambda} \tilde{K}_t^\Lambda \right) \leq \varepsilon,$$

where the supremum is over all action sequences $a$ and all collections of $L$-Lipschitz functions $\Lambda = (\Lambda_x)_{x \in C}$ as above; it is $(\varepsilon, L)$-weakly calibrated (Kakade and Foster 2004, Foster and Kakade 2006) if

$$\lim_{t \to \infty} \left( \sup_{a_t, w} \| g_t(w) \| \right) \leq \varepsilon,$$

where the supremum is over all action sequences $a$ and all $L$-Lipschitz functions $w : C \to [0, 1]$.

While formula (29) for $\tilde{K}_t^\Lambda$ resembles formula (6) for $K_t^\Pi$, there are two differences. The first is that the weight of $\| e_t(\Lambda_x) \|$ in $\tilde{K}_t^\Lambda$ is not the total weight $n_t(\Lambda_x)$ of $\Lambda_x$ (which is the denominator of $e_t(\Lambda_x)$), but rather the number of times $n_t(x)$ that $x$ has been used as a forecast up to time $t$ (the sum in $\tilde{K}_t^\Lambda$ is thus the finite sum over $x \in \{c_1, \ldots, c_t\}$). The second is that the functions $\Lambda_x$ do not form a binning; i.e., they do not add up to 1. The second difference does not really matter (it can be addressed, for instance, by rescaling the $\Lambda_x$ functions, which does not affect the $e_t(\Lambda_x)$, because $e_t(w)$ is homogeneous of degree 0 in $w$). The first difference is more significant; it necessitates the use of certain approximations, such as the small cubes in Lemma 11 in Foster and Hart (2018) and the resulting Proposition 13 there.

A function $f$ is $L$-Lipschitz if $|f(z) - f(z')| \leq L \| z - z' \|$ for all $z, z'$ in the domain of $f$.

48 The bound on $\tilde{K}_t^\Lambda$ that is obtained in the proof of Proposition 13 in Foster and Hart (2018) plays the same role as Proposition 2 here.
By contrast, continuous calibration uses the more appropriate weights $n_t(\Lambda_x)$; this streamlines the analysis and simplifies the proofs. Moreover, continuous calibration yields a “universal” smoothly and weakly calibrated procedure for all parameter values $(\varepsilon, L)$ at once (recall footnote 32).

**Proposition 15** A deterministic procedure $\sigma$ that is continuously calibrated is $(0, L)$-smoothly calibrated and $(0, L)$-weakly calibrated for every $0 < L < \infty$.

**Proof.** The convergence to zero in (11) is uniform over any finite set of continuous $w$’s, and thus, by (7), over any compact set of $w$’s—in particular, the set of $L$-Lipschitz functions $w : C \to [0, 1]$, which is compact by the Arzelà–Ascoli theorem. This is precisely $(0, L)$-weak calibration; by Proposition 13 in Foster and Hart (2018), it implies $(0, L)$-smooth calibration. □

### A.3 Outgoing Results

We provide here a number of comments and extensions to the results of Section 3.

**Remarks on Theorem 4.**

(a) Theorem 4 was proved using Brouwer’s fixed point theorem; conversely, Brouwer’s theorem can be proved using Theorem 4. Indeed, let $g : C \to C$ be a continuous function. Theorem 4 applied to $f(x) = g(x) - x$ yields $y \in C$ such that, in particular, $f(y) \cdot (g(y) - y) \leq 0$ (because $g(y) \in C$); this is $f(y) \cdot f(y) \leq 0$, and so $f(y) = 0$, i.e., $g(y) = y$.

(b) Brouwer’s fixed point theorem is widely used to prove results in many areas. Most such proofs use ingenious constructions, which are needed to make the values of the continuous function lie in its domain, i.e., have the function map $C$ into $C$. By contrast, Theorem 4 puts no restriction on the range of the function (beyond it being in the Euclidean space of the same dimension); one only needs to ensure that a point $y$ that satisfies (12) has the desired properties.

To demonstrate how Theorem 4 may yield simpler proofs, consider the famous result on the existence of Nash equilibria in finite games (Nash 1950). Let

\[(N, (S_i)_{i \in N}, (u^i)_{i \in N})\]

be a finite game in strategic form. Let $C := \prod_{i \in N} \Delta(S^i) \subset \mathbb{R}^m$ where $m := \sum_{i \in N} |S^i|$, and, for every $x = (x^i)_{i \in N} \in C$, put $f^i(x) := (u^i(s^i, x^i))_{s^i \in S^i}$ (this is the vector of $i$’s payoffs for all his pure strategies against $x^{-i}$), and $f(x) := (f^i(x))_{i \in N}$. The function $f : C \to \mathbb{R}^m$ is a polynomial and thus continuous, and so Theorem 4 gives $y \in C$ such that $f(y) \cdot (c - y) \leq 0$ for every
for any $i \in \mathbb{N}$ and $x^i \in \Delta(S')$, we get $0 \geq f(y) \cdot (c-y) = f^i(y) \cdot (x^i-y^i) = u^i(x^i, y^{-i})$, which shows that $y$ is a Nash equilibrium. Moreover, when the game is symmetric, putting $C := \Delta(S^1)$ and $f(x) := (u^i(s, x, ..., x))_{s \in S}$ for every $x \in C$ yields the existence of a symmetric Nash equilibrium. Compare this short proof to the usual proofs that are based directly on Brouwer’s fixed point theorem, which are much more intricate.

**Remarks on Theorem 5.**

(a) The factor $\delta$ on the right-hand side of (14) can be lowered to $\delta_0 \equiv \delta_0(D) < \delta$ (see the proof of Theorem 5) by a limit argument, which is however no longer a finite minimax construct. Indeed, take a sequence $B_n$ of finite $\delta_n$-grids of $C$ with $\delta_n$ decreasing to 0; we then get a sequence of probability distributions $\eta_n \in \Delta(D)$ such that

$$\mathbb{E}_{y \sim \eta_n} [f(y) \cdot (x - y)] \leq (\delta_0 + \delta_n) \mathbb{E}_{y \sim \eta_n} [||f(y)||]$$

(30)

for every $n \geq 1$ and every $x \in C$. Since $D$ is a finite set the sequence $\eta_n$ has a limit point $\eta \in \Delta(D)$, say $\eta_n' \to \eta$ for a subsequence $n' \to \infty$; for each $x \in C$ taking the limit of (30) as $n' \to \infty$ then yields

$$\mathbb{E}_{y \sim \eta} [f(y) \cdot (x - y)] \leq \delta_0 \mathbb{E}_{y \sim \eta} [||f(y)||].$$

(31)

(b) The bound in (31) is tight: $\delta_0$ cannot be lowered. Indeed, take a point $x_0 \in C$ for which $\text{dist}(x_0, D) = \delta_0$, and consider the function $f : D \to \mathbb{R}^m$ defined by $f(y) = (x_0 - y)/||x_0 - y||$ for every $y \in D$; we have $||f(y)|| = 1$ and $f(y) \cdot (x_0 - y) = ||x_0 - y|| \geq \delta_0$ for every $y \in D$.

**Remarks on Corollary 6.**

(a) In Corollary 6 one can get $\eta \in \Delta(C)$ with support of size at most $m + 2$ (rather than $m + 3$), because when using Carathéodory’s theorem the last coordinate of $F(y)$, namely, $||f(y)||$, is no longer needed as it is replaced by the constant $\sup_{x \in C}||f(x)||$.

(b) If $f$ is a continuous function then the result of Corollary 6 holds also for $\epsilon = 0$. Indeed, take a sequence $\epsilon_n \to 0^+$. For each $n$, Corollary 6 yields a distribution $\eta_n$ on $C$ such that $\mathbb{E}_{y \sim \eta_n} [f(y) \cdot (c - y)] \leq \epsilon_n$ for every $c \in C$. All the

49 The subsequence $n'$ is such that $\eta_{n'}(y)$ is a convergent subsequence, with limit $\eta(y)$, for each one of the finitely many elements $y$ of $D$; then $\mathbb{E}_{y \sim \eta} [g(y)] = \sum y \in D \eta_{n'}(y)g(y) \to \sum y \in D \eta(y)g(y) = \mathbb{E}_{y \sim \eta} [g(y)]$ as $n' \to \infty$ for every real function $g$ on $D$.

50 Of course, Theorem 4 yields in this case a stronger result, i.e., a point $y$ rather than a distribution $\eta$. However, the result for $\epsilon = 0$ is obtained here by a minimax, rather than a fixed point, theorem.
distributions \( \eta_n \) can be taken to have support of size at most \( m+2 \) (see Remark (a) above), and so the sequence \( \eta_n \) has a limit point \( \eta \), which is also a distribution on \( C \) with support of size at most \( m+2 \); then \( \mathbb{E}_{\eta_n} [f(y) \cdot (c-y)] \leq 0 \) for every \( c \in C \) (because \( \eta_n \to \eta \) implies \( \mathbb{E}_{\eta_n} [f(y) \cdot (c-y)] \to \mathbb{E}_\eta [f(y) \cdot (c-y)] \), since \( f(y) \cdot (c-y) \) is a continuous function of \( y \)).

(c) If \( f \) is not continuous the result of Corollary 6 need not hold for \( \varepsilon = 0 \); take for example \( C = [0,2] \), and \( f(x) = 1 \) if \( x < 1 \) and \( f(x) = -1 \) if \( x \geq 1 \). Assume that \( \eta \in \Delta(C) \) satisfies \( \mathbb{E}_{\eta} [f(y) \cdot (c-y)] \leq 0 \) for all \( c \in C \). Taking \( c = 1 \) gives \( \mathbb{E}_{\eta} [f(y) \cdot (1-y)] \leq 0 \); but \( f(y) \cdot (1-y) \geq 0 \) for all \( y \in [0,2] \), with equality only for \( y = 1 \), and so \( \eta \) must put unit mass on \( y = 1 \); but then \( \mathbb{E}_{\eta} [f(y) \cdot (c-y)] = f(1) \cdot (c-1) = 1-c \), which is positive for \( c < 1 \).

(d) The minimax theorem follows from Corollary 6. First, consider a symmetric finite two-person zero-sum game, given by an \( m \times m \) payoff matrix \( B \) that is skew-symmetric (i.e., \( B^\top = -B \)). Take \( C \) to be the unit simplex in \( \mathbb{R}^m \) (i.e., the set of mixed strategies), and let \( f : C \to \mathbb{R}^m \) be given by \( f(x) := Bx \). Corollary 6 together with Remark (b) above implies that there exists a distribution \( \eta \) on \( C \) (with finite support) such that \( \mathbb{E}_{\eta} [y^\top B^\top (c-y)] = \mathbb{E}_{\eta} [B y \cdot (c-y)] \leq 0 \) for every \( c \in C \). Now \( y^\top B^\top y = 0 \) for every \( y \in C \) by symmetry (i.e., \( B^\top = -B \)), and so \( \mathbb{E}_{\eta} [y^\top B^\top c] \leq 0 \) for every \( c \in C \). Thus \( z := \mathbb{E}_{\eta} [y] \in C \) satisfies \( z^\top Bc = -z B^\top c \geq 0 \) for every \( c \in C \), and so \( z \) is a minimax strategy that guarantees the value 0; by symmetry, \( z \) is also a maximin strategy that guarantees the value 0, and we are done. Finally, for a general two-person zero-sum game, use a standard symmetrization argument (e.g., Luce and Raiffa 1957, A6.8).

**Remark on Theorem 7**

If \( C \) is a convex polytope and the set \( D \) consists of the vertices of a simplicial subdivision of \( C \), then we can define \( \tilde{f} \) by linearly interpolating inside each simplex; this implies that we moreover have \( \mathbb{E}_{\tilde{f}} [y] = z \) (however, to keep satisfying this additional property may require \( \eta \) to have support of size \( 2m+1 \) instead of \( m+1 \)).

### A.4 Deterministic and Stochastic Forecast-Hedging

We explain here why the proofs of (D) and (S) of Theorem 9 are somewhat different: we use \( S_t \) and the derived \( \varphi_{t-1} \) in (D), and \( X_t \) and the derived \( \psi_{t-1} \) in (S).

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51 Take a subsequence \( n' \) where all the \( m+2 \) values and all the \( m+2 \) probabilities converge (thus we do not need to appeal to Prokhorov’s theorem); denote by \( \eta \) the limit distribution. Then \( \mathbb{E}_{\eta_n'} [g(y)] \to \mathbb{E}_{\eta} [g(y)] \) for any continuous function \( g \) (because then \( p_{n'} \to p \) and \( y_{n'} \to y \) implies \( p_{n'} g(y_{n'}) \to p g(y) \)).
One can check that the $S_t$ approach in the (S) setup gives $\lim_t (1/t^2) S_t = \lim_t \sum_{i=1}^t \|g_t(w_i)\|^2 \leq \varepsilon^2$. What this yields is $\lim_t \mathbb{E}[K^\Pi_t] = \lim_t \mathbb{E}\left[ \sum_{i=1}^t \|g_t(w_i)\| \right] \leq \varepsilon \sqrt{T}$ (consider for instance the case where the $\|g_t(w_i)\|^2$ are all equal to $\varepsilon^2 / I$), which however does not suffice. Indeed, for classic calibration the binning comes from an $\varepsilon$-grid of $C$ (see the proof of Theorem 11(S)), and so its size $I$ is of the order of $1/\varepsilon^m$, which makes the bound $\varepsilon \sqrt{T}$ not useful beyond dimension $m = 1$. The more delicate approach with $X_t$ gets rid of this annoying $\sqrt{T}$ factor. The issue does not arise in (D), since there we have $\varepsilon = 0$, and so $\varepsilon \sqrt{T} = 0$ for every finite binning, which extends to countable continuous binnings by (10).

Going in the other direction, while we could use the $X_t$ approach for (D) as well (it will not affect the result), the $S_t$ approach is preferable as it is shorter and simpler.

### A.5 Calibration with Probability One

In this appendix we show how to strengthen the results on classic calibration (Theorem 11(S) and (AD) in Section 4) from convergence in expectation to convergence almost surely (“a.s.”).

The definition of classic calibration in Section 2.1 requires that the calibration score $K_t$ be small in expectation (i.e., that $\mathbb{E}[K_t]$ be less than $\varepsilon$ in the limit). One may require in addition that $K_t$ be small almost surely (i.e., with probability one); that is, for every action sequence $a$,

$$\lim_{t \to \infty} K_t \leq \varepsilon \quad \text{(a.s.).}$$

We now show that the procedures constructed in Section 4 do indeed satisfy this additional requirement.

In the proof of Theorem 9(S), the sequence $Y_t$ is uniformly bounded (by $2\gamma \cdot \gamma + \gamma^2 = 3\gamma^2$), and so we can apply the Strong Law of Large Numbers for Dependent Random Variables (see (26)):

$$\frac{1}{t} \sum_{s=1}^t (Y_s - \mathbb{E}[Y_s|h_{s-1}]) \to_{t \to \infty} 0 \quad \text{(a.s.).}$$

Since $\mathbb{E}[Y_s|h_{s-1}] = \mathbb{E}_{s-1}[Y_s] \leq \varepsilon^2$ by (21), it follows that $\lim_{t \to \infty} (1/t) \sum_{s=1}^t Y_s \leq \varepsilon^2$ (a.s.). Together with $\lim_{t \to \infty} (1/t) \sum_{s=1}^t Z_s = 0$ by (22), we get $\lim_{t \to \infty} (1/t)X_t \leq \varepsilon^2$ (a.s.), and thus $\lim_{t \to \infty} K^\Pi_t \leq \varepsilon$ (a.s.) (because $(K^\Pi)^2 \leq (1/t)X_t$). Applying this to the binning $\Pi$ of Theorem 10(S) yields (32), for stochastic classic calibration (Theorem 11(S)) as well as for almost deterministic classic calibration (Theorem 45).
A.6 Continuously Calibrated Learning

In this appendix we provide a number of comments and extensions on the result on game dynamics of Section 6.

Remarks on Theorem 13.

(a) The forecasts are also approximate Nash equilibria:

\[ \lim_{t \to \infty} \frac{1}{t} \left| \{ s \leq t : c_s \in \text{NE}(\varepsilon') \} \right| = 1 \quad (\text{a.s.}) \]

for every \( \varepsilon' > \varepsilon \). This follows by replacing Claim (iii) with:

Claim (iii'). For every \( \varepsilon' > \varepsilon \) there is \( \delta > 0 \) such that \( \| \beta(c) - c \| \leq \delta \) implies that \( c \in \text{NE}(\varepsilon') \) (for the proof, take \( \delta > 0 \) such that \( \| x - y \| \leq \delta \) implies \( |u^i(x^i, y^{-i}) - u^i(y)| \leq \varepsilon' - \varepsilon \) for every \( i \)).

(b) A statement that is equivalent to (24) is

\[ \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} \text{dist}(x_s, \text{NE}(\varepsilon)) = 0, \quad (33) \]

which is the way it appears in Kakade and Foster (2004) (and the same applies to the statement in (a) above). Indeed, for every \( \varepsilon' > \varepsilon \) let \( \delta(\varepsilon') := \inf_{x \notin \text{NE}(\varepsilon')} \text{dist}(x, \text{NE}(\varepsilon)) \) and \( \rho(\varepsilon') := \sup_{x \in \text{NE}(\varepsilon')} \text{dist}(x, \text{NE}(\varepsilon)) \); then it is straightforward to see that \( \delta(\varepsilon') > 0 \) and \( \lim_{\varepsilon' \searrow \varepsilon} \rho(\varepsilon') = 0 \) (use the compactness of \( X \) and the continuity of the functions \( u^i \)). Therefore \( \delta(\varepsilon') \mathbf{1}_{x \notin \text{NE}(\varepsilon')} \leq \text{dist}(x, \text{NE}(\varepsilon)) \leq \rho(\varepsilon') + \sqrt{m} \mathbf{1}_{x \notin \text{NE}(\varepsilon')} \) (because \( \sup_{x, y \in X} \| x - y \| \leq \sqrt{m} \)). Using the first inequality for each \( x_s \) shows that (33) implies (24), and using the second inequality for each \( x_s \) shows that (24) implies (33) (the limit is \( \leq \rho(\varepsilon') \) for every \( \varepsilon' > \varepsilon \), and thus 0, because \( \lim_{\varepsilon' \searrow \varepsilon} \rho(\varepsilon') = 0 \)).

(c) The forecasting procedure in (I) depends only on the sizes of the strategy spaces \( (m^i)_{i \in N} \).

(d) The play in each period \( t \) need not be independent across the players, so long as the marginals are \( (\beta^i(c_t))_{i \in N} \).

52 Which is not surprising, as \( c_t \) and \( x_t \) are close (see Claim (ii)). Of course, what we care about are not the forecasts, but the behaviors; this is why the result in Theorem 13 is stated for \( x_t \).
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