Log-periodic oscillations in the specific heat behaviour for self-similar Ising type spin systems

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Abstract. The self-similar model of spin-system of the Ising type is formulated. The thermodynamic properties of this model are considered. Analytically and numerically the specific heat of this system is calculated in the nearest neighbor approximation (only the influence of two neighboring spins was taken into account). It is shown that in temperature dependence of the specific heat the log-periodic oscillations are appeared. These oscillations are imposed on the expected power-law dependence.

1. Introduction

Log-periodic oscillations phenomenon can be considered as a general property belonging to a wide class of self-similar systems having discrete scaling invariance [1]. These oscillations have been found in different systems with discrete scale invariance, for example, on lattices (which determine geometrical fractals [2, 3]), or in self-similar distribution of interaction constants [4], or in cases when this invariance takes place in both cases [5, 6]. The first interest to this phenomenon is appeared in the renorm-group theory for description of critical phenomena [7]. But a constant interest to this phenomenon is appeared later because these oscillations have small amplitudes and they cannot be detected easily in evaluation of critical indexes. But the last studying of a wide class of aperiodic models, where fluctuations of interacting constants produce essential changes of the original model, evokes a constant interest to these oscillations again.

The discovery of quasi-crystals in 1984 year [8] evoked again a big interest to quasi-periodic structures that is confirmed by large number of theoretical and experimental papers. The general feature for such structures is the self-similar (fractal) structure of the energy spectrum (see, for example [9]). In general, the behavior of the fractal spectrum exhibits a complex behavior and in order to find and evaluate the thermodynamic peculiarities of these systems some simple models are used. In this sense, in paper [10] as the simplest model of the fractal spectrum the triadic Cantor set was chosen. It was shown the heat capacitance has an interesting behavior: it oscillates log-periodically near the mean value, which is equaled to the fractal dimension of the spectrum considered.

In papers [11] one of the authors of this paper (R.R.N) suggested the theory of dielectric relaxation based on the fractional kinetics. It has been shown that many self-similar dynamic processes
taking place on microscopic scales are averaged on the mesoscopic scales; from the mathematical point of view this averaging procedure leads to power-law dependencies with real or complex-conjugated exponents (log-periodic dependencies). It means that in the range of mesoscopic scales the kinetic equations for the total polarization \( P(i) \) contains differential operators with non-integer or even complex-conjugated values. This general approach helps to understand the reasons of some empirical expressions which were suggested for description of dielectric permittivity in frequency domain.

In this paper the authors suggest new general approach for detection and evaluation of log-periodic oscillations of different thermodynamic values for spin-models using as an example the 1D Ising model. The selection of this model is not random. This model admits the exact solution which, in turn, is expressed in terms of elementary functions. This peculiarity of the selected model gives a possibility to realize almost all analytical calculations and demonstrate all possibilities of the method suggested. The essence of the approach suggested is related to rather general averaging of thermodynamic functions over the self-similar ensemble of copies. The details of this approach are demonstrated on the ensemble of Ising chains with spin \( S = \frac{1}{2} \) and interaction covering the nearest neighbors.

2. Formulation of the model

We consider a system of spins that is located in a medium and in each point the pressure is transmitted. Under an external tension/pressure applied in the medium the strongly nonequilibrium states can be created. Also, in this medium the process of relaxation can have hierarchy subordinated nature. The nature of this subordination is related to the loss of ergodicity. The creation of this hierarchy subordinated states leads finally to a fractal structure of distribution of thermodynamic potential in configuration space. The fractal properties of the system considered essentially change its thermodynamic and kinetic properties. These modifications are related to partition of configuration space on sets of aggregates (valleys) and each valley has own statistical ensemble. So, in this situation the calculation of average values are realized on two stages. At first, it is necessary to average over ensemble belonging to a 'pure' valley and then realize the second averaging over ensemble describing the distribution of all valleys. The kinetic aspect that presents in this system and conditioned on a weak restoration of ergodicity is determined by slow unification of initial valleys to clusters containing large-scale aggregates. As the result of this unification the relaxation process is divided on hierarchy levels. Each level is associated with statistical ensemble of spins and it is located in space region of the certain pressure. Taking into account that the value of the exchange integral depends, in turn, on the value of the hydrostatic pressure applied we obtain finally self-similar ensembles (clusters) of spins that differ from each other by the value of exchange integral and number of spins entering in it. These values we obey to self-similar conditions.

\[
N_l = N_0 b^l, \quad J_l = J_0 \xi^l, \quad 0 < b, \xi < 1, 0 < l < \infty,
\]

where \( N_0, J_0, b, \xi \) are parameters of the model. Then we realize the averaging procedure of specific free energy of the spin system and find the final expression for the total free energy

\[
\overline{f(T)} = N^{-1} \sum_l N_l f(T, J_l).
\]

The last expression helps to restore other thermodynamic values of the spin-system considered.

3. Log-periodic oscillations the specific heat in the self-similar Ising spin system

Having in mind only pure demonstration purposes we show how to calculate the specific heat of the model spin system defined above. In this model each hierarchy level is associated with ensemble of spins belonging to the Ising chain with \( S = \frac{1}{2} \) and only the interaction between two nearest neighbors is considered. For this model system expression for the specific thermodynamic potential of the \( l \)-th ensemble of spins is determined by well-known expression
\[ f(\beta, J) = -T \ln \left[ 2 \text{ch} \left( \frac{J}{T} \right) \right] = -(J_0 / \beta) \ln \left[ 2 \text{ch} \left( \frac{\beta \xi}{\beta} \right) \right], \]  

where \( \beta = J_0 / T \). Then the averaged thermodynamic potential is determined by the following expression

\[ \overline{f}(\beta) = N^{-1} \sum_i N_i f(\beta, J_i) = -(N_0 J_0 / N) \sum_{i=0}^{\infty} b^i \beta^{-1} \ln \left[ 2 \text{ch} \left( \frac{\beta \xi}{\beta} \right) \right]. \]

where the value of \( N \) determines the total number of spins in the model considered. From the normalization condition \( \sum_{i=0}^{\infty} N_i = N \) we find \( N_0 / N = 1 - b \). We introduce the important characteristic of the self-similar system considered as the "fractal" dimension

\[ d = \ln b / \ln \xi. \]

Then \( b = \xi^d \) and \( d \) is chosen as independent parameter of the system. Taking into account these definitions expression (4) is rewritten in the form

\[ \overline{f}(\beta) = -(1 - \xi^d) J_0 \sum_{i=0}^{\infty} \xi^d i \beta^{-1} \ln \left[ 2 \text{ch} \left( \frac{\beta \xi}{\beta} \right) \right]. \]

The specific heat of the self-similar system of the Ising chains is determined by expression

\[ C(\beta) = -\beta^2 d^2 \left( \frac{\beta f(\beta)}{d \beta^2} \right) / d\beta^2 = (1 - \xi^d) \sum_{i=0}^{\infty} \xi^d i (\beta \xi)^2 \text{ch}^{-2} (\beta \xi). \]  

For calculation of the infinite sum (7) it is convenient to use the Mellin transform. Applying the Mellin transform to both parts of (7) and using the formulae

\[ g(\beta) = \beta^2 \text{ch}^{-2} \beta^{MT} = \hat{Q}(s) = \int_0^\infty \beta^{-s+1} \text{ch}^{-2} \beta d\beta = 2^{-s} (1 - 2^{-s}) \Gamma(s+2) \zeta(s+1), \quad \gamma = \text{Re}\, s > -2, \]

\[ q(\beta \xi) = \xi^{-i\gamma} \hat{Q}(s), \]  

where \( \Gamma(x) \) is gamma-function and \( \zeta(x) \) is zeta Riemann function, then for the Mellin image of the function \( \hat{C}(\beta) \) one can obtain the following expression:

\[ \hat{C}(s) = (1 - \xi^d) \hat{Q}(s) \sum_{i=0}^{\infty} \xi^{d-s} i \frac{(1 - \xi^d) \hat{Q}(s)}{1 - \xi^{d-s}}, \quad -2 < \gamma = \text{Re}\, s < d, \xi < 1. \]

Realizing the inverse Mellin transform of (9) we get

\[ \overline{C}(\beta) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{C}(s)\beta^{-s} ds = \varepsilon(1 - \xi^d) \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{Q}(s)\beta^{-s} ds, \]

where the parameter \( \gamma \) is chosen in accordance with condition (9). The value of \( \varepsilon = \pm 1 \) determines the chosen contour of integration. For calculation of the Mellin-Barns integral in the right-hand side of (10) the integration line we complete up to the closed counter forming correspondingly the left (\( \varepsilon = +1 \)) or the right (\( \varepsilon = -1 \)) semicircle and after that we use the Cauchy theorem

\[ \overline{C}(\beta) = \varepsilon(1 - \xi^d) \sum_{\text{Res}} \left[ \hat{Q}(s)\beta^{-s} \left( 1 - \xi^{d-s} \right)^{-1} \right]. \]

Here \( s_\chi \) are the poles belonging to the chosen contour and form two groups: \( s_\chi \) are the poles of the function \( \hat{Q}(s) \) (that is determined by expression (8)) and \( s_\chi^* \) are zeros of the function \( 1 - \xi^{d-s} \) which in turn are determined by expression

\[ s_\chi^* = d + 2\pi ik / \ln \xi, \quad k \in \mathbb{Z}. \]
The poles of the function $\hat{Q}(s)$ are determined by the poles of gamma-function figuring in (8) and these poles are well-known and equaled to $s_k^* = -k - 2 \ (k = 0, 1, 2, \ldots)$. For the case $\xi < 1$ the poles $s_k^* = d + 2\pi ik / \ln \xi$ are located on the right-hand side from $\gamma$, so the temperature interval is divided onto regions: $T < T_m$ and $T > T_m$, where $T_m$ corresponds to the temperature of the heat capacitance global maximum. For region $T < T_m$ the contribution to $\tilde{C}(\beta)$ gives only poles $s_k^*$ and so the integration line in (10) we close by the right semicircle. In the result we have

$$\tilde{C}(T) = -(1 - \xi^d) \sum_k \text{Res}_{s_k^*} \left[ \hat{Q}(s) \beta^{s_k^*} \left(1 - \xi^{(d-s)}\right)^{-s_k^*} \right] = \ln^{-1}(1/\xi) \left(1 - \xi^d\right)(T/J_0) \xi^{\xi}(\ln T / \ln(1/\xi)), \quad (13)$$

gde $w_d(x)$ - periodic function having the unit period.

$$w_d(x) = \sum_{k=-\infty}^{\infty} \hat{Q}(d + 2\pi ik / \ln(1/\xi)) e^{\pi ik x}. \quad (14)$$

In the region of temperatures $T > T_m$ the integration line is closed by the left semicircle and contribution to heat capacitance comes from the poles $s_k'$. Taking into account the known relationship $\text{Res}\Gamma(s + 2) = (-1)^k / k!$ we obtain

$$\tilde{C}(\beta) = (1 - \xi^d) \sum_k \text{Res}_{s_k'} \left[ \hat{Q}(s) \beta^{s_k'} \right] = (1 - \xi^d) \sum_{k=0}^{\infty} \frac{2^{k+2}(2^{k+2} - 1)(-1)^{k+1}\xi(k - 1)\beta^{k+2}}{k!(1 - \xi^{(2d+k)})}. \quad (15)$$

Taking into account the values $\xi(-2m) = 0$, $\xi(-(2m-1)) = -B_{2m}/2m$, $m = 1, 2, \ldots$, where $B_{2m}$ are Bernoulli numbers, expression (15) can be rewritten in the form

$$\tilde{C}(\beta) = (1 - \xi^d) \sum_{k=1}^{\infty} \frac{4^k(4^k - 1)B_{2k}\beta^{2k}}{2k(2k - 2)(1 - \xi^{(d+2k)})}. \quad (16)$$

The series (16) converges at $T > T_m = 0,63694J_0$. (The convergence radius is found for power law series as the limit of the ratio $a_{k+1}/a_k$ at $k \to \infty$). From expression (13) it follows that in the interval $T < T_m$ we have power-law dependence $T^d$ in behaviour of heat capacitance that is modified by log-periodic oscillations. Above this temperature $T_m$ the capacitance is decreasing monotonically with increasing of temperature. On Fig. 1 (a) we give temperature dependencies of heat capacitance calculated in accordance with expression (7) at $\xi = 0,25$ for different values of fractal dimension $d$ and Fig. 1 (b) demonstrate its behavior at $d=0,25$ for different values of parameter $\xi$. These two plots clearly shows the oscillating character of heat capacitance at low temperatures and the character of this oscillations is changed with increasing of dimension $d$ and they decay clearly at increasing of the parameter $\xi$.  

4
Figure 1. The dependence of the specific heat with respect to the dimensionless temperature \( T/J_0 \) at various values of the parameters \( \xi \) and \( d \).

Figs. 2 show the dependence of the function \( C(T) \cdot T^{-d} \) from temperature for some values of fractal dimension \( d \) (Fig. 2 (a)) and parameter \( \xi \) (Fig. 2 (b)). Fig. 2 (b) clearly shows the decay of these oscillations with increasing of the value of the scaling parameter \( \xi \). The region of parameters \( \xi \) and \( d \) where the influence of log-periodic oscillations is essential one can evaluate as the ratio of the first decomposition coefficient to the zero one in behaviour of the log-periodic function \( w_0(x) \) (14).

Figure 2. The dependence \( C(T) \cdot T^{-d} \) against the temperature for \( \xi = 0.25 \) at various values of the fractal dimension \( d \) (Fig. 2 (a)) and for \( d=1.5 \) at various values of \( \xi \) (Fig. 2 (b)).

4. Conclusions

In this paper we considered the thermodynamics of the spin system of the Ising-type having self-similar structure. The self-similar structure is presented by rather simple model: self- similar conditions are fulfilled for the constants of the exchange interaction and number of spins in a cluster having the same value of the exchange integral. It is clear that the suggested model of such type is not
a specific one. It can be applied for a wide class of spin and electron many-body systems. The suggested model of the self-similar medium is characterized by only two parameters: scaling parameter $\xi$ characterizing the hierarchy of the constants of interaction and the generalized fractal dimension $d$.

The equilibrium properties of the model suggested were based on the analysis of the specific heat depending on temperature, scaling parameter $\xi$ and fractal dimension $d$. The averaging of the specific heat over self-similar ensemble allows to receive the log-periodic oscillations which form a mixed combination with power-law temperature dependence. Temperature region of the existence of the log-periodic oscillations is determined by the parameter $\xi$. In the case $\xi < 1$ which is considered in the paper the desired oscillations take place at low temperatures but for $\xi > 1$ these oscillations are appeared at high-temperatures. In the last case these oscillations have small amplitude because the log-periodic component is appeared together with monotone dependence. The character of the monotonic component is determined presumably by the value of the fractal dimension $d$, and contains alongside with power-law terms the logarithmic corrections also. So, one can state that log-periodic oscillations associate closely with the fractal (self-similar) structure of the system considered, which, in turn, accompanies a wide class of different processes. So, any accurate and general approach that enables to consider and model of hierarchy structure of disordered systems opens new possibilities in analysis of a wide class of physical phenomena that take place in the systems of such kind. The suggested approach combining the simplicity of the calculated scheme allows to receive the important characteristics of the self-similar system as the log-periodic oscillations of a wide class of thermodynamic values. It creates a general base for calculation and analytical evaluations of equilibrium values of various disordered structures based, in turn, on different self-similar (fractal) models.

5. Acknowledgements

The work was done in the frame of the research plan "Dielectric spectroscopy and kinetics of complex systems" (Number: 12-18 02 0210 021000018) that has been accepted by the KFU for 2012 year.

6. References

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