Spectra of Complemented Triangulation Graphs

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Abstract: The complemented triangulation graph of a graph G, denoted by CT(G), is defined as the graph obtained from G by adding, for each edge uv of G, a new vertex whose neighbors are the vertices of G other than u and v. In this paper, we first obtain the A-spectra, the L-spectra, and the Q-spectra of the complemented triangulation graphs of regular graphs. By using the results, we construct infinitely many pairs of A-cospectral graphs, L-cospectral graphs, and Q-cospectral graphs. We also obtain the number of spanning trees and the Kirchhoff index of the complemented triangulation graphs of regular graphs.

Keywords: A-spectrum; L-spectrum; Q-spectrum; complemented triangulation graph

MSC: 05C50

1. Introduction

All graphs considered in this paper are undirected and simple. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix of G, denoted by $A(G)$, is the $n \times n$ matrix whose $(i,j)$-entry is 1 if $v_i$ and $v_j$ are adjacent in $G$ and 0 otherwise. We denote by $d_i = d_G(v_i)$ the degree of $v_i$ in $G$ and define $D(G)$ as the diagonal matrix with diagonal entries $d_1, d_2,\ldots, d_n$. The Laplacian matrix of $G$ and the signless Laplacian matrix of $G$ are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. Given an $n \times n$ matrix $M$, we denote by

$$\phi(M; x) = \det(xI_n - M),$$

or, simply, $\phi(M)$, the characteristic polynomial of $M$, where $I_n$ denotes the identity matrix of size $n$. In particular, for a graph $G$, we call $\phi(A(G))$ (respectively, $\phi(L(G))$, $\phi(Q(G))$) the adjacency (respectively, Laplacian, signless Laplacian) characteristic polynomial of $G$, and we call its roots the adjacency (respectively, Laplacian, signless Laplacian) eigenvalues of $G$. The collection of eigenvalues of $A(G)$ together with their multiplicities are called the A-spectrum of $G$. Two graphs are said to be A-cospectral if they have the same A-spectrum. Similar terminology will be used for $L(G)$ and $Q(G)$. So, we obtain the definitions of the L-spectrum, Q-spectrum, L-cospectral graphs, and Q-cospectral graphs. It is known that graph spectra store numerous units of structural information about a graph; see [1–3] and the references therein.

Formulating the characteristic polynomials of graphs, as well as calculating the spectra of graphs, is a fundamental and very meaningful work in spectral graph theory. Up until now, the spectra of several graphs have been investigated. For examples, see [1–10]. It is known that the characteristic polynomials of graphs and the spectra of graphs can enable us to construct infinitely many pairs of A-cospectral (respectively, L-cospectral, Q-cospectral) graphs [7,9,10], as well as to investigate many other properties of graphs, such as the number of spanning trees [1,3], the Kirchhoff index [1,3,11–13], and so on.

In this paper, we investigate the spectra of complemented triangulation graphs of regular graphs, whose definition is given in Definition 1. Our motivation for defining the complemented triangulation graph comes from [14], where the so-called Q-complemented...
graph was introduced and the adjacency characteristic polynomial of the $Q$-complemented graph of a graph was computed.

**Definition 1.** The complemented triangulation graph of a graph $G$, denoted by $CT(G)$, is defined as the graph obtained from $G$ by adding, for each edge $uv$ of $G$, a new vertex whose neighbours are the vertices of $G$ other than $u$ and $v$.

Let $C_n$ denote the cycle on $n$ vertices. Figure 1 depicts the complemented triangulation graph of $C_4$.

![Figure 1. An example of a complemented triangulation graph.](image)

In our work, we first obtain the $A$-spectra, the $L$-spectra, and the $Q$-spectra of complemented triangulation graphs of regular graphs. By using the results, we construct infinitely many pairs of $A$-cospectral graphs, $L$-cospectral graphs, and $Q$-cospectral graphs. We also obtain the number of spanning trees and the Kirchhoff index of the complemented triangulation graphs of regular graphs.

### 2. Spectra of Complemented Triangulation Graphs

Before proceeding to the presentation of the main results of this section, we need to state some basic results that will be used frequently later.

Let $G$ be a graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, \ldots, e_m\}$. The (vertex–edge) incidence matrix of $G$, denoted by $R_G = (r_{ij})$, is an $n \times m$ matrix with entry $r_{ij} = 1$ if the vertex $v_i$ is incident the edge $e_j$, and $r_{ij} = 0$ otherwise. It is well known [3] that, if $G$ is an $r$-regular graph, then

$$R_G R_G^\top = A_G + rI_n,$$  \hspace{1cm} (1)

where $R_G^\top$ represents the transpose of $R_G$.

The $M$-coronal $\Gamma_M(x)$ of an $n \times n$ matrix $M$ is defined [4,15] as the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is,

$$\Gamma_M(x) = \mathbf{1}_n^\top (xI_n - M)^{-1} \mathbf{1}_n,$$ where $\mathbf{1}_n$ denotes the column vector of dimension $n$ with all of the components equal to one. It is known [4] (Proposition 2) that, if $M$ is an $n \times n$ matrix with each row sum equal to a constant $t$, then

$$\Gamma_M(x) = n/(x - t).$$  \hspace{1cm} (2)

In particular, since, for any graph $G$ with $n$ vertices, each row sum of $L(G)$ is equal to 0, we have

$$\Gamma_{L(G)}(x) = n/x.$$  \hspace{1cm} (3)
Lemma 1 (see [10]). Let $A$ be an $n \times n$ real matrix, $\alpha$ a real number, and $I_{n \times n}$ the $n \times n$ matrix with all entries equal to one. Then,
\[
\det(xI_n - A - \alpha I_{n \times n}) = (1 - \alpha \Gamma_A(x)) \det(xI_n - A).
\]

Lemma 2 (see [16]). Let $M_1, M_2, M_3,$ and $M_4$ be, respectively, $p \times p$, $p \times q$, $q \times p$, and $q \times q$ matrices, where $M_1$ and $M_4$ are invertible. Then,
\[
\det(M_1 M_2 M_3 M_4) = \det(M_4) \cdot \det(M_1 - M_2 M_4^{-1} M_3),
\]
where $M_1 - M_2 M_4^{-1} M_3$ and $M_4 - M_3 M_1^{-1} M_2$ are called the Schur complements of $M_4$ and $M_1$, respectively.

2.1. $A$-Spectra of Complemented Triangulation Graphs

Theorem 1. Let $G$ be an $r$-regular connected graph with $n$ vertices, $m$ edges, and $r \geq 2$. Suppose that $r = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the adjacency eigenvalues of $G$. Then, the adjacency eigenvalues of $CT(G)$ are stated as follows:

(a) $0$, repeated $m - n$ times;
(b) $r \pm \sqrt{r^2 + 4(2r + mn - 2rn)}$;
(c) $\lambda_i \pm \sqrt{\lambda_i^2 + 4\lambda_i + 4r} / 2$, for $i = 2, 3, \ldots, n$.

Proof. Note that the adjacency matrix of $CT(G)$ is given by
\[
A_{CT(G)} = \begin{pmatrix}
A_G & I_{n \times m} - R_G \\
(I_{n \times m} - R_G)^\top & 0
\end{pmatrix},
\]
where $I_{n \times m}$ denotes the $n \times m$ matrix with all entries equal to one. Then, the adjacency characteristic polynomial of $CT(G)$ is given by
\[
\phi\left(A_{CT(G)}; x\right) = \det\left(
\begin{pmatrix}
xI_n - A_G & -I_{n \times m} + R_G \\
-I_{n \times m} + R_G & xI_m
\end{pmatrix}
\right) = x^m \cdot \det(S),
\]
where
\[
S = xI_n - A_G - \frac{1}{x}(I_{n \times m} - R_G)(I_{n \times m} - R_G)^\top
\]
is the Schur complement of $xI_m$ obtained by Lemma 2. Note (1) that $R_G R_G^\top = rI_n + A_G$. Then, we have
\[
(I_{n \times m} - R_G)(I_{n \times m} - R_G)^\top = (m - 2r)I_{n \times n} + rI_n + A_G.
\]

By Lemma 1 and (2), we have
\[
\det(S) = \det\left(xI_n - A_G - \frac{1}{x}((m - 2r)J_{n \times n} + rI_n + A_G)\right)
\]
\[
= x^{-n} \det\left((x^2 - r)I_n - (x + 1)A_G - (m - 2r)J_{n \times n}\right)
\]
\[
= x^{-n} \cdot \left(1 - (m - 2r) \cdot \frac{n}{x^2 - r - (x + 1)r}\right) \cdot \det\left((x^2 - r)I_n - (x + 1)A_G\right)
\]
\[
= x^{-n} \cdot \left(x^2 - r - (x + 1)r - (m - 2r)n\right) \cdot \prod_{i=2}^{n} \left(x^2 - r - (x + 1)\lambda_i\right).
\]

Then, the required result follows from \(\phi(A_{CT(G)}; x) = x^n \cdot \det(S)\). This completes the proof. \(\square\)

**Example 1.** Let \(CT(C_4)\) be the graph shown in Figure 1. Note that the adjacency eigenvalues of \(C_4\) are \(\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = -2\). By Theorem 1, the adjacency eigenvalues of \(CT(C_4)\) are \(1 \pm \sqrt{5}, 1 \pm \sqrt{2}, \pm 2\), and 0.

### 2.2. L-Spectra of Complemented Triangulation Graphs

**Theorem 2.** Let \(G\) be an \(r\)-regular connected graph with \(n\) vertices, \(m\) edges, and \(r \geq 2\). Suppose that \(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0\) are the Laplacian eigenvalues of \(G\). Then, the Laplacian eigenvalues of \(CT(G)\) are stated as follows:

(a) \(m - r + n - 2\);

(b) \(n - 2\), repeated \(m - n\) times;

(c) \(\frac{(m - r) + (n - 2) + \mu_i \pm \Delta_i}{2}\), where

\[
\Delta_i = \sqrt{((m - r) + (n - 2) + \mu_i)^2 - 4((m - r)(n - 2) + (n - 1)\mu_i - 2r)},
\]

for \(i = 1, 2, \ldots, n - 1\);

(d) 0.

**Proof.** Note that the Laplacian matrix of \(CT(G)\) is given by

\[
L_{CT(G)} = \begin{pmatrix}
(m - r)I_n + L_G & -J_{n \times m} + R_G \\
-J_{n \times m} + R_G & (n - 2)I_m
\end{pmatrix},
\]

where \(J_{n \times m}\) denotes the \(n \times m\) matrix with all entries equal to one. Thus, the Laplacian characteristic polynomial of \(CT(G)\) is given by

\[
\phi\left(L_{CT(G)}; x\right) = \det\left((x + r - m)I_n - L_G - J_{n \times m} - R_G \right) (x + 2 - n)J_m
\]
\[
= (x + 2 - n)^m \cdot \det(S),
\]

where

\[
S = (x + r - m)I_n - L_G - \frac{1}{x + 2 - m} (J_{n \times m} - R_G)(J_{n \times m} - R_G)^\top
\]
is the Schur complement of \((x + 2 - n)J_m\) obtained by Lemma 2. Note (1) that \(R_G R_G^\top = 2rI_n - L_G\). Then, we have

\[
(J_{n \times m} - R_G)(J_{n \times m} - R_G)^\top = (m - 2r)J_{n \times n} + 2rI_n - L_G.
\]
By Lemma 1 and (3), we have
\[
\det(S) = \det\left( (x + r - m)I_n - L_G - \frac{1}{x + 2 - n}((m - 2r)J_{n \times n} + 2rI_n - L_G) \right) 
\]
\[
= \det(((x + r - m)(x + 2 - n) - 2r)I_n - (x + 1 - n)L_G - (m - 2r)J_{n \times n}) 
\]
\[
= (x + 2 - n)^{-n} \cdot \left( 1 - (m - 2r) \cdot \frac{n}{(x + r - m)(x + 2 - n) - 2r} \right) 
\cdot \det(((x + r - m)(x + 2 - n) - 2r)I_n - (x + 1 - n)L_G) 
\]
\[
= (x + 2 - n)^{-n} \cdot ((x + r - m)(x + 2 - n) - 2r - (m - 2r)n) 
\cdot \prod_{i=1}^{n-1} ((x + r - m)(x + 2 - n) - 2r - (x + 1 - n)\mu_i). 
\]

Then, the required result follows from \( \phi\left( Q_{CT(G)}; x \right) = (x + 2 - n)^m \cdot \det(S) \) and the fact that \( 2m = rn \). This completes the proof. \( \square \)

**Example 2.** Let \( CT(C_4) \) be the graph shown in Figure 1. Note that the Laplacian eigenvalues of \( C_4 \) are \( \mu_1 = 4, \mu_2 = 2, \mu_3 = 2, \) and \( \mu_4 = 0 \). By Theorem 2, the Laplacian eigenvalues of \( CT(C_4) \) are \( 6, 4, 3 \pm \sqrt{3}, 3 \pm \sqrt{3}, \) and \( 0 \).

### 2.3. Q-Spectra of Complemented Triangulation Graphs

**Theorem 3.** Let \( G \) be an \( r \)-regular connected graph with \( n \) vertices, \( m \) edges, and \( r \geq 2 \). Suppose that \( 2r = \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \) are the signless Laplacian eigenvalues of \( G \). Then, the signless Laplacian eigenvalues of \( CT(G) \) are stated as follows:

- (a) \( n - 2 \), repeated \( m - n \) times;
- (b) \( \frac{(r + m + n - 2) \pm \sqrt{(r + m + n - 2)^2 - 8r(n - 3)}}{2} \); 
- (c) \( \frac{(m - r) + (n - 2) + \nu_i \pm \Delta_i}{2} \), where \( \Delta_i = \sqrt{((m - r) + (n - 2) + \nu_i)^2 - 4((m - r)(n - 2) + (n - 3)\nu_i - 2r)} \), for \( i = 2, 3, \ldots, n \).

**Proof.** Note that the signless Laplacian matrix of \( CT(G) \) is given by
\[
Q_{CT(G)} = \left( \begin{array}{cc} (m - r)I_n + Q_G & (J_{n \times m} - R_G)^\top \\ (J_{n \times m} - R_G)^\top I_m \end{array} \right),
\]

where \( I_n \) denotes the identity matrix of size \( n \), and \( J_{n \times m} \) denotes the \( n \times m \) matrix with all entries equal to one. Thus, the signless Laplacian characteristic polynomial of \( CT(G) \) is given by
\[
\phi\left( Q_{CT(G)}; x \right) = \det\left( \begin{array}{cc} (x + r - m)I_n - Q_G & J_{n \times m} - R_G \\ (-J_{n \times m} + R_G)^\top & (x + 2 - n)I_m \end{array} \right) 
\]
\[
= (x + 2 - n)^m \cdot \det(S), 
\]
where

\[
S = (x + r - m)I_n - Q_G - \frac{1}{x + 2 - n}((J_{n \times m} - R_G)(J_{n \times m} - R_G)^T)
\]

is the Schur complement of \((x + 2 - n)I_m\) obtained by Lemma 2. Note (1) that \(R_G^{-1}R_G^T = Q_G\). Then, we have

\[
(J_{n \times m} - R_G)(J_{n \times m} - R_G)^T = (m - 2r)I_n + Q_G.
\]

By Lemma 1 and (2), we have

\[
det(S)
\]

\[
= det\left( (x + r - m)I_n - Q_G - \frac{1}{x + 2 - n}((m - 2r)I_{n \times n} + Q_G) \right)
\]

\[
= det(((x + r - m)(x + 2 - n) - 2r)I_n - (x + 3 - n)Q_G - (m - 2r)I_{n \times n})\)

\[
= (x + 2 - n)^{-n}
\]

\[
\cdot \left( 1 - (m - 2r) \right) \cdot \left( \frac{n}{(x + r - m)(x + 2 - n) - 2r - 2(x + 3 - n)\right)}
\]

\[
\cdot det(((x + r - m)(x + 2 - n) - 2r)I_n - (x + 3 - n)Q_G)
\]

\[
= (x + 2 - n)^{-n}
\]

\[
\cdot ((x + r - m)(x + 2 - n) - 2r - 2(x + 3 - n)r - (m - 2r)n)
\]

\[
\prod_{i=2}^{n} ((x + r - m)(x + 2 - n) - 2r - (x + 3 - n)v_i).
\]

Then, the required result follows from \(\phi(Q_{CT(G)}; x) = (x + 2 - n)^m \cdot det(S)\) and the fact that \(2m = rn\). This completes the proof. \(\square\)

**Example 3.** Let \(CT(C_4)\) be the graph shown in Figure 1. Note that the signless Laplacian eigenvalues of \(C_4\) are \(v_1 = 4, v_2 = 2, v_3 = 2,\) and \(v_4 = 0\). By Theorem 3, the signless Laplacian eigenvalues of \(CT(C_4)\) are \(4 \pm 2\sqrt{3}, 3 \pm \sqrt{7}, 3 \pm \sqrt{7}, 4,\) and 0.

**3. Consequences**

**3.1. Constructing Cospectral Graphs**

It is known that, by using graph operations, we can construct many infinite families of pairs of A-cospectral (respectively, L-cospectral, Q-cospectral) graphs. For example, see [7,9,15]. Here, we use complemented triangulation graphs to construct infinitely many pairs of A-cospectral (respectively, L-cospectral, Q-cospectral) graphs. The following constructions follow from Theorems 1–3. Recall that two graphs are said to be A-cospectral (respectively, L-cospectral, Q-cospectral) if they have the same A-spectrum (respectively, L-spectrum, Q-spectrum).

**Corollary 1.** (a) If \(G_1\) and \(G_2\) are A-cospectral regular graphs, then \(CT(G_1)\) and \(CT(G_2)\) are A-cospectral graphs.

(b) If \(G_1\) and \(G_2\) are L-cospectral regular graphs, then \(CT(G_1)\) and \(CT(G_2)\) are L-cospectral graphs.

(c) If \(G_1\) and \(G_2\) are Q-cospectral regular graphs, then \(CT(G_1)\) and \(CT(G_2)\) are Q-cospectral graphs.

**Example 4.** Let \(G_1\) and \(G_2\) be the two graphs given in [17] (Figure 2). It is known that \(G_1\) and \(G_2\) are A-cospectral (respectively, L-cospectral, Q-cospectral) graphs. By Corollary 1, \(CT(G_1)\) and \(CT(G_2)\) are A-cospectral (respectively, L-cospectral, Q-cospectral) graphs.
3.2. The Number of Spanning Trees

A spanning tree of a graph $G$ is a subgraph that is a tree that includes all of the vertices of $G$ [2]. Let $t(G)$ denote the number of spanning trees of $G$. It is known [2] that, if $G$ is a connected graph on $n$ vertices with the Laplacian spectrum $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$, then

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i(G).$$

Here, we give the number of the spanning trees of $\text{CT}(G)$ for an $r$-regular graph $G$.

Corollary 2. Let $G$ be an $r$-regular graph on $n$ vertices and $m$ edges. Suppose that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the Laplacian eigenvalues of $G$. Then,

$$t(\text{CT}(G)) = \frac{(n-2)^m \cdot (m-r + n - 2) \cdot \prod_{i=1}^{n-1} ((m-r)(n-2) + (n-1)\mu_i - 2r)}{m + n}. $$

Proof. By Theorem 2, we define

$$\theta_i^\pm = \frac{(m-r) + (n-2) + \mu_i \pm \Delta_i}{2},$$

where

$$\Delta_i = \sqrt{((m-r) + (n-2) + \mu_i)^2 - 4((m-r)(n-2) + (n-1)\mu_i - 2r)},$$

for $i = 1, 2, \ldots, n-1$. One can easily verify that

$$\theta_i^+ \theta_i^- = (m-r)(n-2) + (n-1)\mu_i - 2r.$$

Therefore, $t(\text{CT}(G))$ is obtained by the definition of the number of spanning trees of a graph. \□

Example 5. Let $\text{CT}(C_4)$ be the graph shown in Figure 1. Note that the Laplacian eigenvalues of $C_4$ are $\mu_1 = 4, \mu_2 = 2, \mu_3 = 2$, and $\mu_4 = 0$. By Corollary 2,

$$t(\text{CT}(C_4)) = \frac{4 \cdot \prod_{j=1}^{3} (3\mu_j)}{8} = 216.
$$

3.3. The Kirchhoff Index

The Kirchhoff index of a graph $G$, denoted by $Kf(G)$, is defined as the sum of the resistance distances between all pairs of vertices [18,19]. At almost exactly the same time, Gutman et al. [20] and Zhu et al. [21] proved that the Kirchhoff index of a connected graph $G$ with $n \ (n \geq 2)$ vertices can be expressed as

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i(G)},$$

where $\mu_1(G), \ldots, \mu_{n-1}(G)$ are the non-zero Laplacian eigenvalues of $G$. By Theorem 2, we obtain the Kirchhoff index of $\text{CT}(G)$ for an $r$-regular graph $G$. 


Corollary 3. Let $G$ be an $r$-regular graph on $n$ vertices and $m$ edges. Suppose that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the Laplacian eigenvalues of $G$. Then,

$$Kf(CT(G)) = (m + n) \left( \frac{m - n}{n - 2} + \frac{1}{m - r + n - 2} + \sum_{i=1}^{n-1} \frac{(m - r) + (n - 2) + \mu_i}{(m - r)(n - 2) + (n - 1)\mu_i - 2r} \right).$$

Proof. By Theorem 2, we define

$$\theta_i^\pm = \frac{(m - r) + (n - 2) + \mu_i \pm \Delta_i}{2},$$

where

$$\Delta_i = \sqrt{((m - r) + (n - 2) + \mu_i)^2 - 4((m - r)(n - 2) + (n - 1)\mu_i - 2r)},$$

for $i = 1, 2, \ldots, n - 1$. One can easily verify that

$$\frac{1}{\theta_i^+} + \frac{1}{\theta_i^-} = \frac{(m - r) + (n - 2) + \mu_i}{(m - r)(n - 2) + (n - 1)\mu_i - 2r}.$$

Thus, $Kf(CT(G))$ is obtained by the definition of the Kirchhoff index of a graph. \qed

Example 6. Let $CT(C_4)$ be the graph shown in Figure 1. Note that the Laplacian eigenvalues of $C_4$ are $\mu_1 = 4$, $\mu_2 = 2$, $\mu_3 = 2$, and $\mu_4 = 0$. By Corollary 3,

$$Kf(CT(C_4)) = 8 \left( \frac{1}{4} + \sum_{j=1}^{3} \frac{4 + \mu_j}{3\mu_j} \right) = \frac{70}{3}.$$

4. Conclusions

Computing the spectra of graphs is an important and interesting work in spectral graph theory. In this paper, we determined the spectra of complemented triangulation graphs of regular graphs. As applications, we constructed infinitely many pairs of cospectral graphs. We also calculated the number of spanning trees and the Kirchhoff index of complemented triangulation graphs. We are sure that the methods used in this paper can also be applied to other graphs, and any attempts on other graphs should be welcome.

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