THE VECTOR-VALUED TENT SPACES \(T^1\) AND \(T^\infty\)

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Abstract. Tent spaces of vector-valued functions were recently studied by Hytönen, van Neerven and Portal with an eye on applications to \(H^\infty\)-functional calculi. This paper extends their results to the endpoint cases \(p = 1\) and \(p = \infty\) along the lines of earlier work by Harboure, Torrea and Viviani in the scalar-valued case. The main result of the paper is an atomic decomposition in the case \(p = 1\), which relies on a new geometric argument for cones. A result on the duality of these spaces is also given.

1. Introduction

Coifman, Meyer and Stein introduced in [4] the concept of tent spaces that provides a neat framework for several ideas and techniques in Harmonic Analysis. In particular, they defined the spaces \(T^p\), \(1 \leq p < \infty\), that are relevant for square functions, and consist of functions \(f\) on the upper half-space \(\mathbb{R}^{n+1}_+\) for which the \(L^p\) norm of the conical square function is finite:

\[
\int_{\mathbb{R}^n} \left( \int_{\Gamma(x)} |f(y, t)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{p/2} \, dx < \infty,
\]

where \(\Gamma(x)\) denotes the cone \(\{(y, t) \in \mathbb{R}^{n+1}_+: |x - y| < t\}\) at \(x \in \mathbb{R}^n\). Typical functions in these spaces arise for instance from harmonic extensions \(u\) to \(\mathbb{R}^{n+1}_+\) of \(L^p\) functions on \(\mathbb{R}^n\) according to the formula \(f(y, t) = t \partial_t u(y, t)\).

Tent spaces were approached by Harboure, Torrea and Viviani in [5] as \(L^p\) spaces of \(L^2\)-valued functions, which gave an abstract way to deduce many of their basic properties. Indeed, for \(1 < p < \infty\), the mapping \(Jf(x) = 1_{\Gamma(x)} f\) is readily seen to embed \(T^p\) in \(L^p(\mathbb{R}^n; L^2(\mathbb{R}^{n+1}_+))\), when \(\mathbb{R}^{n+1}_+\) is equipped with the measure \(dy \, dt/t^{n+1}\). Furthermore, they showed that \(T^p\) is embedded as a complemented subspace, which not only implies its completeness, but also gives a way to prove a few other properties, such as equivalence of norms defined by cones of different aperture and the duality \((T^p)^* \simeq T^{p'}\), where \(1/p + 1/p' = 1\).

Treatment of the endpoint cases \(p = 1\) and \(p = \infty\) requires more careful inspection. Firstly, the space \(T^\infty\) was defined in [4] as the space of functions \(g\) on \(\mathbb{R}^{n+1}_+\) for which

\[
\sup_B \frac{1}{|B|} \int_B |g(y, t)|^2 \frac{dy \, dt}{t} < \infty,
\]

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where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and where $\hat{B} \subset \mathbb{R}^{n+1}$ denotes the “tent” over $B$ (see Section 3). The tent space duality is now extended to the endpoint case as $(T^1)^* \simeq T^\infty$. Moreover, functions in $T^1$ admit a decomposition into atoms $a$ each of which is supported in $\hat{B}$ for some ball $B \subset \mathbb{R}^n$ and satisfies

$$\int_{\hat{B}} |a(y,t)|^2 \, dy \, dt \leq \frac{1}{|B|}.$$ 

As for the embeddings, it is proven in [5] that $T^1$ embeds in the $L^2(\mathbb{R}^{n+1})$-valued Hardy space $H^1(\mathbb{R}^n; L^2(\mathbb{R}^{n+1}))$, while $T^\infty$ embeds in BMO($\mathbb{R}^n; L^2(\mathbb{R}^{n+1}))$ – the space of $L^2(\mathbb{R}^{n+1})$-valued functions with bounded mean oscillation.

The study of vector-valued analogues of these spaces was initiated by Hytönen, van Neerven and Portal in [7], where they followed the ideas from [5] and proved the analogous embedding results for $T^p(X)$ with $1 < p < \infty$ under the assumption that $X$ is UMD. It should be noted that, for non-Hilbertian $X$, the $L^2$ integrals had to be replaced by stochastic integrals or some equivalent objects, which in turn required further adjustments in proofs, namely the lattice maximal functions that appeared in [5] were replaced by an appeal to Stein’s inequality for conditional expectation operators. Later on, Hytönen and Weis provided in [8] a scale of vector-valued versions of the quantity appearing above in the definition of $T^\infty$.

This paper continues the work on the endpoint cases and provides definitions for $T^1(X)$ and $T^\infty(X)$. The main result decomposes a $T^1(X)$ function into atoms using a geometric argument for cones. The original decomposition argument in [4] is inherently scalar-valued and not as such suitable for stochastic integrals. Moreover, the spaces $T^1(X)$ and $T^\infty(X)$ are embedded in certain Hardy and BMO spaces, respectively, much in the spirit of [5]. The theory of vector-valued stochastic integration (see van Neerven and Weis [14]) is used throughout the paper.

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**2. Preliminaries**

**Notation.** Random variables are taken to be defined on a fixed probability space whose probability measure and expectation are denoted by $\mathbb{P}$ and $\mathbb{E}$. The integral average (with respect to Lebesgue measure) over a measurable set $A \subset \mathbb{R}^n$ is written as $\bar{f}_A = |A|^{-1} \int_A$, where $|A|$ stands for the Lebesgue measure of $A$. For a ball $B$ in $\mathbb{R}^n$ we write $x_B$ and $r_B$ for its center and radius, respectively. Throughout the paper $X$ is assumed to be a real Banach space and $\langle \xi, \xi^* \rangle$ is used to denote the duality pairing between $\xi \in X$ and $\xi^* \in X^*$. Isomorphism of Banach spaces is expressed using $\simeq$. By $\alpha \lessapprox \beta$ it is meant that there exists a constant $C$ such that $\alpha \leq C \beta$. Quantities $\alpha$ and $\beta$ are comparable, $\alpha \approx \beta$, if $\alpha \lessapprox \beta$ and $\beta \lessapprox \alpha$. 

**Stochastic integration.** We start by discussing the correspondence between Gaussian random measures and stochastic integrals of real-valued functions. Recall that a Gaussian random measure on a $\sigma$-finite measure space $(M, \mu)$ is a mapping $W$ that takes subsets of $M$ with finite measure to (centered) Gaussian random variables in such a manner that

- the variance $\mathbb{E}W(A)^2 = \mu(A)$,
- for all disjoint $A$ and $B$ the random variables $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$ almost surely.

Since for Gaussian random variables the notions of independence and orthogonality are equivalent, it suffices to consider their pairwise independence in the definition above. Given a Gaussian random measure $W$, we obtain a linear isometry from $L^2(M)$ to $L^2(\mathbb{P})$ – our stochastic integral – by first defining $\int_M 1_A \, dW = W(A)$ and then extending by linearity and density to the whole of $L^2(M)$. On the other hand, if we are in possession of such an isometry, we may define a Gaussian random measure $W$ by sending a subset $A$ of $M$ with finite measure to the stochastic integral of $1_A$. For more details, see Janson [9] Chapter 7.

A function $f : M \to X$ is said to be weakly $L^2$ if $\langle f(\cdot), \xi^* \rangle$ is in $L^2(M)$ for all $\xi^* \in X^*$. Such a function is said to be *stochastically integrable* (with respect to a Gaussian random measure $W$) if there exists a (unique) random variable $\int_M f \, dW$ in $X$ so that for all $\xi^* \in X^*$ we have

$$\langle \int_M f \, dW, \xi^* \rangle = \int_M \langle f(t), \xi^* \rangle \, dW(t) \quad \text{almost surely.}$$

We also say that a function $f$ is stochastically integrable over a measurable subset $A$ of $M$ if $1_A f$ is stochastically integrable. Note in particular that each function $f = \sum_k f_k \otimes \xi_k$ in the algebraic tensor product $L^2(M) \otimes X$ is stochastically integrable and that

$$\int_M f \, dW = \sum_k \left( \int_M f_k \, dW \right) \xi_k.$$

A detailed theory of vector-valued stochastic integration can be found in van Neerven and Weis [14], see also Rosiński and Suchanecki [15]. Stochastic integrals have a number of nice properties (see [14]):

- Khintchine–Kahane inequality: For every stochastically integrable $f$ we have
  $$\left( \mathbb{E} \left\| \int_M f \, dW \right\|_p^p \right)^{1/p} \approx \left( \mathbb{E} \left\| \int_M f \, dW \right\|_q^q \right)^{1/q}$$
  whenever $1 \leq p, q < \infty$.
- Covariance domination: If a function $g \in L^2(M) \otimes X$ is dominated by a function $f \in L^2(M) \otimes X$ in covariance, that is, if
  $$\int_M \langle g(t), \xi^* \rangle^2 \, d\mu(t) \leq \int_M \langle f(t), \xi^* \rangle^2 \, d\mu(t)$$
  for all $\xi^* \in X^*$, then
  $$\mathbb{E} \left\| \int_M g \, dW \right\|^2 \leq \mathbb{E} \left\| \int_M f \, dW \right\|^2.$$
Dominated convergence: If a sequence \((f_k)\) of stochastically integrable functions is dominated in covariance by a single stochastically integrable function and
\[
\int_M \langle f_k(t), \xi^* \rangle^2 \, d\mu(t) \to 0
\]
for all \(\xi^* \in X^*\), then
\[
\mathbb{E} \left\| \int_M f_k \, dW \right\|^2 \to 0.
\]

In particular, if a sequence \((A_k)\) of measurable sets satisfies \(1_{A_k} \to 0\) pointwise almost everywhere, then for every \(f\) in \(L^2(M) \otimes X\) we have
\[
\mathbb{E} \left\| \int_{A_k} f \, dW \right\|^2 \to 0.
\]

The expression
\[
\left( \mathbb{E} \left\| \int_M f \, dW \right\|^2 \right)^{1/2}
\]
defines a norm on the space of (equivalence classes of) strongly measurable stochastically integrable functions \(f : M \to X\). However, the norm is not generally complete, unless \(X\) is a Hilbert space. For convenience, we operate mainly with functions in \(L^2(M) \otimes X\) and denote their completion under the norm above by \(\gamma(M;X)\).

This space can be identified with the space of \(\gamma\)-radonifying operators from \(L^2(M)\) to \(X\) (see [14] and the survey [13]). We note the following facts:

- Given an \(m \in L^\infty(M)\), the multiplication operator \(f \mapsto mf\) on \(L^2(M) \otimes X\) has norm \(\|m\|_{L^\infty(M)}\).
- For \(K\)-convex \(X\) (see [13, Section 10]) the duality \(\gamma(M;X)^* = \gamma(M;X^*)\) holds and realizes for \(f \in L^2(M) \otimes X\) and \(g \in L^2(M) \otimes X^*\) via
\[
\langle f, g \rangle = \int_M \langle f(t), g(t) \rangle \, d\mu(t).
\]

A family \(\mathcal{T}\) of operators in \(\mathcal{L}(X)\) is said to be \(\gamma\)-bounded if for every finite collection of operators \(T_k \in \mathcal{T}\) and vectors \(\xi_k \in X\) we have
\[
\mathbb{E} \left\| \sum_k \gamma_k T_k \xi_k \right\|^2 \lesssim \mathbb{E} \left\| \sum_k \gamma_k \xi_k \right\|^2,
\]
where \((\gamma_k)\) is an independent sequence of standard Gaussians.

Observe, that families of operators obtained by composing operators from (a finite number of) other \(\gamma\)-bounded families are also \(\gamma\)-bounded. It follows from covariance domination and Fubini’s theorem, that the family of operators \(f \mapsto mf\) is \(\gamma\)-bounded on \(L^p(\mathbb{R}^n;X)\) whenever the multipliers \(m\) are chosen from a bounded set in \(L^\infty(\mathbb{R}^n)\).

The following continuous-time result for \(\gamma\)-bounded families is folklore (to be found in Kalton and Weis [10]):

**Lemma 1.** Assume that \(X\) does not contain a closed subspace isomorphic to \(c_0\). If the range of an \(X\)-strongly measurable function \(A : M \to \mathcal{L}(X)\) is \(\gamma\)-bounded, then
for every strongly measurable stochastically integrable function $f : M \to X$ the strongly measurable function $t \mapsto A(t)f(t) : M \to X$ is also stochastically integrable and satisfies

$$\mathbb{E} \left\| \int_M A(t)f(t) \, dW(t) \right\|^2 \lesssim \mathbb{E} \left\| \int_M f(t) \, dW(t) \right\|^2.$$  

Recall that $X$-strong measurability of a function $A : M \to \mathcal{L}(X)$ requires $A(\cdot)\xi : M \to X$ to be strongly measurable for every $\xi \in X$. For simple functions $A : M \to \mathcal{L}(X)$ the lemma above is immediate from the definition of $\gamma$-boundedness and requires no assumption regarding containment of $c_0$, as the function $t \mapsto A(t)f(t) : M \to X$ is also in $L^2(M) \otimes X$. Assuming $A$ to be simple is anyhow too restrictive for applications and to consider nonsimple functions $A$ we need to handle more general stochastically integrable functions than just those in $L^2(M) \otimes X$.

Our choice of $(M, \mu)$ will be the upper half-space $\mathbb{R}_{+}^{n+1} = \mathbb{R}^n \times (0, \infty)$ equipped with the measure $dy \, dt/t^{n+1}$. We will simplify our notation and write $\gamma(X) = \gamma(\mathbb{R}_{+}^{n+1};X)$ — in what follows, stochastic integration is performed on $\mathbb{R}_{+}^{n+1}$.

The UMD-property and averaging operators. It is often necessary to assume that our Banach space $X$ is UMD. This has the crucial implication, known as Stein’s inequality (see Bourgain [2] and Clément et al. [3]), that every increasing family of conditional expectation operators is $\gamma$-bounded on $L^p(X)$ whenever $1 < p < \infty$. While this is proven in the given references only in the case of probability spaces, it can be generalized to the $\sigma$-finite case such as ours with no difficulty. Namely, let us consider filtrations on $\mathbb{R}^n$ generated by systems of dyadic cubes, that is, by collections $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$, where each $\mathcal{D}_k$ is a disjoint cover of $\mathbb{R}^n$ consisting of cubes $Q$ of the form $x_Q + [0, 2^{-k})^n$ and every $Q \in \mathcal{D}_k$ is a union of $2^n$ cubes in $\mathcal{D}_{k+1}$. The conditional expectation operators or averaging operators are then given for each integer $k$ by

$$f \mapsto \sum_{Q \in \mathcal{D}_k} 1_Q \int_Q f, \quad f \in L^1_{\text{loc}}(\mathbb{R}^n;X).$$  

Composing such an operator with multiplication by an indicator $1_Q$ of a dyadic cube $Q$, we arrive through Stein’s inequality to the conclusion that the family $\{A_Q\}_{Q \in \mathcal{D}}$ of localized averaging operators

$$A_Qf = 1_Q \int_Q f,$$

is $\gamma$-bounded on $L^p(\mathbb{R}^n;X)$ whenever $1 < p < \infty$. The following result of Mei [11] allows us to replace dyadic cubes by balls:

**Lemma 2.** There exist $n+1$ systems of dyadic cubes such that every ball $B$ is contained in a dyadic cube $Q_B$ from one of the systems and $|B| \lesssim |Q_B|$.

Stein’s inequality together with the lemma above guarantees that the family $\{A_B : B \text{ ball in } \mathbb{R}^n\}$ is $\gamma$-bounded on $L^p(\mathbb{R}^n;X)$ whenever $1 < p < \infty$. Indeed, for each ball $B$ we can write

$$A_B = 1_B |Q_B| \frac{|Q_B|}{|B|} A_{Q_B} 1_B.$$

This was proven already in [7].
It will be useful to consider smoothed or otherwise different versions of indicators $1_B(x) = 1_{[0,1]}(|x-x_B|/r_B)$. Given a measurable $\psi : [0, \infty) \to \mathbb{R}$ with $1_{[0,1]} \leq |\psi| \leq 1_{[0,\alpha]}$ for some $\alpha > 1$, we define the averaging operators

$$A^\psi_{y,t} f(x) = \psi\left(\frac{|x-y|}{t}\right) \frac{1}{c_\psi t^n} \int_{\mathbb{R}^n} \psi\left(\frac{|z-y|}{t}\right) f(z) \, dz, \quad x \in \mathbb{R}^n,$$

where

$$c_\psi = \int_{\mathbb{R}^n} \psi(|x|)^2 \, dx.$$

Again, under the assumption that $X$ is UMD and $1 < p < \infty$, the $\gamma$-boundedness of the family $\{A^\psi_{y,t} : (y, t) \in \mathbb{R}^{n+1}\}$ of operators on $L^p(\mathbb{R}^n; X)$ follows at once when we write

$$A^\psi_{y,t} = \psi\left(\frac{|\cdot - y|}{t}\right) \frac{|Q_B(y,\alpha t)|}{c_\psi t^n} A_{Q_B(y,\alpha t)}\psi\left(\frac{|\cdot - y|}{t}\right).$$

Observe, that the function $(y, t) \mapsto A^\psi_{y,t}$ from $\mathbb{R}^{n+1}$ to $\mathcal{L}(L^p(\mathbb{R}^n; X))$ is $L^p(\mathbb{R}^n; X)$-strongly measurable. Recall also that every UMD space is K-convex and cannot contain a closed subspace isomorphic to $c_0$.

3. OVERVIEW OF TENT SPACES

Tent spaces $T^p(X)$. Let us equip the upper half-space $\mathbb{R}^{n+1}_+$ with the measure $dy \, dt/t^{n+1}$ and a Gaussian random measure $W$. Recall the definition of the cone $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < t\}$ at $x \in \mathbb{R}^n$.

Let $1 \leq p < \infty$. We wish to define a norm on the space of functions $f : \mathbb{R}^{n+1}_+ \to X$ for which $1_{\Gamma(x)} f \in L^2(\mathbb{R}^{n+1}_+; X)$ for almost every $x \in \mathbb{R}^n$ by

$$\|f\|_{T^p(X)} = \left( \int_{\mathbb{R}^n} \left( \mathbb{E}\left\| \int_{\Gamma(x)} f \, dW \right\|^2 \right)^{p/2} \, dx \right)^{1/p},$$

and use the Khintchine–Kahane inequality to write

$$\|f\|_{T^p(X)} \approx \left( \mathbb{E}\left\| \int_{\Gamma(x)} f \, dW \right\|_{L^p(\mathbb{R}^n; X)}^p \right)^{1/p},$$

but issues concerning measurability need closer inspection.

Lemma 3. Suppose that $f : \mathbb{R}^{n+1}_+ \to X$ is such that $1_{\Gamma(x)} f \in L^2(\mathbb{R}^{n+1}_+; X)$ for almost every $x \in \mathbb{R}^n$. Then

1. the function $x \mapsto 1_{\Gamma(x)} f$ is strongly measurable from $\mathbb{R}^n$ to $\gamma(X)$.
2. the function $x \mapsto \int_{\Gamma(x)} f \, dW$ is strongly measurable from $\mathbb{R}^n$ to $L^2(\mathbb{R}^n; X)$ and may be considered, when $\|f\|_{T^p(X)} < \infty$, as a random $L^p(\mathbb{R}^n; X)$ function.
3. the function $x \mapsto (\mathbb{E}\left\| \int_{\Gamma(x)} f \, dW \right\|^2)^{1/2}$ agrees almost everywhere with a lower semicontinuous function so that the set

$$\left\{ x \in \mathbb{R}^n : \left( \mathbb{E}\left\| \int_{\Gamma(x)} f \, dW \right\|^2 \right)^{1/2} > \lambda \right\}$$

is open whenever $\lambda > 0$. 
Proof. Denote by $A_k$ the set $\{(y, t) \in \mathbb{R}_+^{n+1} : t > 1/k\}$ and write $f_k = 1_{A_k} f$. It is clear that for each positive integer $k$, the functions $x \mapsto 1_{\Gamma(x)} f_k$ and $x \mapsto \int_{\Gamma(x)} f_k \, dW$ are strongly measurable and continuous since

$$E \left\| \int_{\Gamma(x) \Delta \Gamma(x')} f_k \, dW \right\|^2 \to 0, \quad x \to x'.$$

Furthermore, $1_{\Gamma(x)} f_k \to 1_{\Gamma(x)} f$ in $\gamma(X)$ for almost every $x \in \mathbb{R}^n$ since

$$E \left\| \int_{\Gamma(x)} (f - f_k) \, dW \right\|^2 = E \left\| \int_{\Gamma(x) \setminus A_k} f \, dW \right\|^2 \to 0.$$

Consequently, $x \mapsto 1_{\Gamma(x)} f$ and $x \mapsto \int_{\Gamma(x)} f \, dW$ are strongly measurable. Moreover, the pointwise limit of an increasing sequence of real-valued continuous functions is lower semicontinuous, which proves the third claim.  

**Definition.** Let $1 \leq p < \infty$. The tent space $T^p(X)$ is defined as the completion under $\| \cdot \|_{T^p(X)}$ of the space of (equivalence classes of) functions $\mathbb{R}_+^{n+1} \to X$ (in what follows, “$T^p(X)$ functions”) such that $1_{\Gamma(x)} f \in L^2(\mathbb{R}_+^{n+1}) \otimes X$ for almost every $x \in \mathbb{R}^n$ and $\|f\|_{T^p(X)} < \infty$.

As was mentioned in the previous section, it is useful to consider the more general situation where the indicator of a ball is replaced by a measurable function $\phi : [0, \infty) \to \mathbb{R}$ with $1_{[0,1)} \leq |\phi| \leq 1_{[0,\alpha)}$ for some $\alpha > 1$. Let us assume in addition, that $\phi$ is continuous at 0. For functions $f : \mathbb{R}_+^{n+1} \to X$ such that

$$(y, t) \mapsto \phi(|x - y|/t) f(y, t) \in L^2(\mathbb{R}_+^{n+1}) \otimes X$$

for almost every $x \in \mathbb{R}^n$, the strong measurability of

$$x \mapsto \left( (y, t) \mapsto \phi\left(\frac{|x - y|}{t}\right) f(y, t) \right) \quad \text{and} \quad x \mapsto \int_{\Gamma(x)} \phi\left(\frac{|x - y|}{t}\right) f(y, t) \, dW(y, t)$$

are treated as in the case of $\phi(|x - y|/t) = 1_{[0,1)}(|x - y|/t) = 1_{\Gamma(x)}(y, t)$.

**Embedding $T^p(X)$ into $L^p(\mathbb{R}^n; \gamma(X))$.** A collection of results from the paper [7] by Hytönen, van Neerven and Portal is presented next. Following the idea of Harboure, Torrea and Viviani [5], the tent spaces are embedded into $L^p$ spaces of $\gamma(X)$-valued functions by

$$Jf(x) = 1_{\Gamma(x)} f, \quad x \in \mathbb{R}^n.$$

Furthermore, for simple $L^2(\mathbb{R}_+^{n+1}) \otimes X$-valued functions $F$ on $\mathbb{R}^n$ we define an operator $N$ by

$$(NF)(x; y, t) = 1_{B(y, t)}(x) \int_{B(y, t)} F(z; y, t) \, dz, \quad x \in \mathbb{R}^n, (y, t) \in \mathbb{R}_+^{n+1}.$$

Assuming that $X$ is UMD, we can now view $T^p(X)$ as a complemented subspace of $L^p(\mathbb{R}^n; \gamma(X))$:

**Theorem 4.** Suppose that $X$ is UMD and let $1 < p < \infty$. Then $N$ extends to a bounded projection on $L^p(\mathbb{R}^n; \gamma(X))$ and $J$ extends to an isometry from $T^p(X)$ onto the image of $L^p(\mathbb{R}^n; \gamma(X))$ under $N$. 

The following result shows the comparability of different tent space norms:

**Theorem 5.** Suppose that X is UMD, let $1 < p < \infty$ and let $1_{(0,1)} \leq |\phi| \leq 1_{(0,\alpha)}$. For every function $f$ in $T^p(X)$ the function $(y,t) \mapsto \phi(|x-y|/t)f(y,t)$ is stochastically integrable for almost every $x \in \mathbb{R}^n$ and

$$
\int_{\mathbb{R}^n} \mathbb{E}\left\| \int_{\mathbb{R}^{n+1}_+} \phi\left(\frac{|x-y|}{t}\right)f(y,t) \, dW(y,t) \right\|^p \, dx \approx \int_{\mathbb{R}^n} \mathbb{E}\left\| \int_{\Gamma(x)} f \, dW \right\|^p \, dx.
$$

The proof relies on the boundedness of modified projection operators

$$(N_\phi F)(x; y, t) = \phi\left(\frac{|x-y|}{t}\right) \int_{B(y,t)} F(z; y, t) \, dz, \quad x \in \mathbb{R}^n, (y, t) \in \mathbb{R}^{n+1}_+.$$ 

and the observation that the embedding

$$J_\phi f(x; y, t) = \phi\left(\frac{|x-y|}{t}\right)f(y,t), \quad x \in \mathbb{R}^n, (y, t) \in \mathbb{R}^{n+1}_+.$$ 

can be written as $J_\phi f = N_\phi J f$. In particular, this shows that the norms given by cones of different apertures are comparable. Indeed, choosing $\phi = 1_{(0,\alpha)}$ gives the norm where $\Gamma(x)$ is replaced by the cone $\Gamma_{\alpha}(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x-y| < \alpha t\}$ with aperture $\alpha > 1$.

Identification of tent spaces $T^p(X)$ with complemented subspaces of $L^p(\mathbb{R}^n; \gamma(X))$ gives a powerful way to deduce their duality:

**Theorem 6.** Suppose that X is UMD and let $1 < p < \infty$. Then the dual of $T^p(X)$ is $T^{p'}(X^*)$, where $1/p + 1/p' = 1$, and the duality is realized for functions $f \in T^p(X)$ and $g \in T^{p'}(X^*)$ via

$$\langle f, g \rangle = c_n \int_{\mathbb{R}^{n+1}_+} \langle f(y,t), g(y,t) \rangle \frac{dy \, dt}{t},$$

where $c_n$ is the volume of the unit ball in $\mathbb{R}^n$.

The following theorem combines results from [7, Theorem 4.8] and [8, Corollary 4.3, Theorem 1.3]. The tent space $T^\infty(X)$ is defined in the next section.

**Theorem 7.** Suppose that X is UMD and let $\Psi$ be a Schwartz function with vanishing integral. Then the operator

$$T_\Psi f(y, t) = \Psi_t * f(y)$$

is bounded from $L^p(\mathbb{R}^n; X)$ to $T^p(X)$ whenever $1 < p < \infty$, from $H^1(\mathbb{R}^n; X)$ to $T^1(X)$ and from $\text{BMO}(\mathbb{R}^n; X)$ to $T^\infty(X)$.

4. **Tent spaces $T^1(X)$ and $T^\infty(X)$**

Having completed our overview of tent spaces $T^p(X)$ with $1 < p < \infty$ we turn to the endpoint cases $p = 1$ and $p = \infty$, of which the latter remains to be defined. As for the case $p = 1$, our aim is to show that $T^1(X)$ is isomorphic to a complemented subspace of the Hardy space $H^1(\mathbb{R}^n; \gamma(X))$ of $\gamma(X)$-valued functions on $\mathbb{R}^n$. In the case $p = \infty$, we introduce the space $T^\infty(X)$, which is shown to embed in $\text{BMO}(\mathbb{R}^n; \gamma(X))$, that is, the space of $\gamma(X)$-valued functions whose mean oscillation is bounded. The idea of these embeddings was originally put forward by Harboure et al. in the scalar-valued case (see [5]).
Recall that the tent over an open set \( E \subset \mathbb{R}^n \) is defined by \( \hat{E} = \{(y,t) \in \mathbb{R}^{n+1} : B(y,t) \subset E \} \) or equivalently by
\[
\hat{E} = \mathbb{R}^{n+1}_+ \setminus \bigcup_{x \notin E} \Gamma(x).
\]
Observe that while cones are open, tents are closed. Truncated cones are also needed: For \( x \in \mathbb{R}^n \) and \( r > 0 \) we define \( \Gamma(x;r) = \{(y,t) \in \Gamma(x) : t < r \} \).

In [4] Hytönen and Weis adjusted the quantities that define scalar-valued atoms and \( T^\infty \) functions in terms of tents to more suitable ones that rely on averages of square functions. More precisely for scalar-valued \( g \) on \( \mathbb{R}^{n+1}_+ \) we have
\[
\int_B \int_{\Gamma(x;rB)} |g(y,t)|^2 \frac{dy \, dt}{t^{n+1}} \leq \int_B \int_{\hat{E}} 1_{B(y,t)}(x)|g(y,t)|^2 \frac{dy \, dt}{t^{n+1}} \leq \int_B \int_{\Gamma(x;rB)} |g(y,t)|^2 \frac{dy \, dt}{t^{n+1}},
\]
from which one reads
\[
\int_B |g(y,t)|^2 \frac{dy \, dt}{t} \lesssim \int_B \int_{\Gamma(x;rB)} |g(y,t)|^2 \frac{dy \, dt}{t^{n+1}} \leq \int_{3B} |g(y,t)|^2 \frac{dy \, dt}{t}.
\]

This motivates the definition of a \( T^1(X) \) atom as a function \( a : \mathbb{R}^{n+1}_+ \to X \) such that for some ball \( B \) we have \( \text{supp } a \subset \hat{B} \), \( 1_{\Gamma(x)}a \in L^2(\mathbb{R}^{n+1}_+ \otimes X) \) for almost every \( x \in B \) and
\[
\int_B \mathbb{E}\left[ \left\| \int_{\Gamma(x)} a \, dW \right\|^2 \right] dx \lesssim \frac{1}{|B|}.
\]
Then \( 1_{\Gamma(x)}a \) differs from zero only when \( x \in B \) and so
\[
\|a\|_{T^1(X)} = \int_{\mathbb{R}^n} \left( \mathbb{E}\left[ \int_{\Gamma(x)} a \, dW \right]^2 \right)^{1/2} dx \leq |B|^{1/2} \left( \int_B \mathbb{E}\left[ \left\| \int_{\Gamma(x)} a \, dW \right\|^2 \right] dx \right)^{1/2} \leq 1.
\]

Furthermore, for (equivalence classes of) functions \( g : \mathbb{R}^{n+1}_+ \to X \) such that \( 1_{\Gamma(x;r)}g \in L^2(\mathbb{R}^{n+1}_+ \otimes X) \) for every \( r > 0 \) and almost every \( x \in \mathbb{R}^n \) we define
\[
\|g\|_{T^\infty(X)} = \sup_B \left( \int_B \mathbb{E}\left[ \left\| \int_{\Gamma(x;rB)} g \, dW \right\|^2 \right] dx \right)^{1/2} < \infty,
\]
where the supremum is taken over all balls \( B \subset \mathbb{R}^n \).

**Definition.** The tent space \( T^\infty(X) \) is defined as the completion under \( \| \cdot \|_{T^\infty(X)} \) of the space of (equivalence classes of) functions \( g : \mathbb{R}^{n+1}_+ \to X \) such that \( 1_{\Gamma(x;r)}g \in L^2(\mathbb{R}^{n+1}_+ \otimes X) \) for every \( r > 0 \) and almost every \( x \in \mathbb{R}^n \) and for which \( \|g\|_{T^\infty(X)} < \infty \).

**The atomic decomposition.** In an atomic decomposition, we aim to express a \( T^1(X) \) function as an infinite sum of (multiples of) atoms. The original proof for scalar-valued tent spaces by Coifman, Meyer and Stein [2] Theorem 1 (c)] rests on a lemma that allows one to exchange integration in the upper half-space with “double integration”, which is something unthinkable when “double integration” consists of both standard and stochastic integration. The following argument provides a more geometrical reasoning. We start with a covering lemma:
**Lemma 8.** Suppose that an open set $E \subset \mathbb{R}^n$ has finite measure. Then there exist disjoint balls $B^j \subset E$ such that

$$\widehat{E} \subset \bigcup_{j \geq 1} 5B^j.$$ 

*Proof.* We start by writing $d_1 = \sup_{B \subset E} r_B$ and choosing a ball $B^1 \subset E$ with radius $r_1 > d_1/2$. Then we proceed inductively: Suppose that balls $B^1, \ldots, B^k$ have been chosen and write

$$d_{k+1} = \sup\{r_B : B \subset E, B \cap B^j = \emptyset, j = 1, \ldots, k\}.$$ 

If possible, we choose $B^{k+1} \subset E$ with radius $r_{k+1} > d_{k+1}/2$ so that $B^{k+1}$ is disjoint from all $B^1, \ldots, B^k$. Let then $(y, t) \in \widehat{E}$. In order to show that $B(y, t) \subset 5B^j$ for some $j$ we note that $B(y, t)$ has to intersect some $B^j$: Indeed, if there are only finitely many balls $B^j$, then $y \in \overline{B^j}$ for some $j$. On the other hand, if there are infinitely many balls $B^j$ and they are all disjoint from $B(y, t)$, then $r_j > d_j/2 > t/2$ and $E$ has infinite measure, which is a contradiction. Thus there exists a $j$ for which $B(y, t) \cap B^j \neq \emptyset$ and so $B(y, t) \subset 5B^j$ because $t \leq d_j \leq 2r_j$ by construction. \hfill $\Box$

Given a $0 < \lambda < 1$, we define the extension of a measurable set $E \subset \mathbb{R}^n$ by

$$E^*_\lambda = \{x \in \mathbb{R}^n : M1_E(x) > \lambda\}.$$ 

Here $M$ is the Hardy–Littlewood maximal operator assigning the maximal function

$$Mf(x) = \sup_B \int_B |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$ 

to every locally integrable real-valued $f$. Note that the lower semicontinuity of $Mf$ guarantees that $E^*_\lambda$ is open while the weak $(1, 1)$ inequality for the maximal operator assures us that $|E^*_\lambda| \leq \lambda^{-1}|E|$.

We continue by constructing sectors opening in finite number of directions of our choice. To do this, we fix vectors $v_1, \ldots, v_N$ in the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$ such that

$$\max_{1 \leq m \leq N} v \cdot v_m \geq \frac{\sqrt{3}}{2}$$

for every $v \in \mathbb{S}^{n-1}$. In other words, every $v \in \mathbb{S}^{n-1}$ makes an angle of no more than $30^\circ$ with one of $v_m$’s. We write

$$S_m = \left\{ v \in \mathbb{S}^{n-1} : v \cdot v_m \geq \frac{\sqrt{3}}{2} \right\}$$

and observe that the angle between two $v, v' \in S_m$ is at most $60^\circ$, i.e. $v \cdot v' \geq \frac{1}{2}$. Consequently, $|v - v'| \leq 1$.

For every $x \in \mathbb{R}^n$ and $t > 0$, write

$$R_m(x, t) = \left\{ y \in B(x, t) : \frac{y - x}{|y - x|} \in S_m \text{ or } y = x \right\}$$

for the sector opening from $x$ in the direction of $v_m$. For any two $y, y' \in R_m(x, t)$, the angle between $y - x$ and $y' - x$ is at most $60^\circ$ (when $y$ and $y'$ are different from...
x), implying that $|y - y'| \leq t$. Hence the proportion of $R_m(x, t)$ in $B(y, t)$ for any \( y \in R_m(x, t) \) is a dimensional constant, in symbols,
\[
\frac{|R_m(x, t)|}{|B(y, t)|} = c(n), \quad y \in R_m(x, t).
\]

For every $0 < \lambda < c(n)$ it thus holds that $M1_{R_m(x, t)} > \lambda$ in $B(y, t)$ whenever $y \in R_m(x, t)$. Writing $E^* = E_{(c(n)/2)}^*$ we have now proven the following:

**Lemma 9.** If $E \subset \mathbb{R}^n$ is measurable and $y \in R_m(x, t) \subset E$, then $B(y, t) \subset E^*$.

Note that the next lemma follows easily when $n = 1$ and holds even without the extension. Indeed, if $E$ is an open interval in $\mathbb{R}$ and $x \in E$, then one can choose $x_1$ and $x_2$ to be the endpoints of $E$ and obtain $\Gamma(x) \setminus \hat{E} \subset \Gamma(x_1) \cup \Gamma(x_2)$. On the other hand, for $n \geq 2$ the extension is necessary, which can be seen already by taking $E$ to be an open ball.

**Lemma 10.** Suppose that an open set $E \subset \mathbb{R}^n$ has finite measure. Then for every $x \in E$ there exist $x_1, \ldots, x_N \in \partial E$, with $N$ depending only on the dimension $n$, such that
\[
\Gamma(x) \setminus \hat{E}^* \subset \bigcup_{m=1}^{N} \Gamma(x_m).
\]

**Proof.** For every $1 \leq m \leq N$ we may pick $x_m \in \partial E$ in such a manner that
\[
\frac{x_m - x}{|x_m - x|} \in S_m
\]

and $|x_m - x|$, which we denote by $t_m$, is minimal (while positive, since $E$ is open). In other words, $R_m(x, t_m) \subset E$. We need to show that for every $(y, t) \in \Gamma(x) \setminus \hat{E}^*$ the point $y$ is less than $t$ away from one of the $x_m$’s. Thus, let $(y, t) \in \Gamma(x) \setminus \hat{E}^*$, which translates to $|x - y| < t$ and $B(y, t) \not\subset E^*$.

Consider first the case of $y$ not belonging to any $R_m(x, t_m)$. Then for some $m$,
\[
\frac{y - x}{|y - x|} \in S_m \quad \text{and} \quad |y - x| \geq t_m.
\]

Now the point
\[
z = t_m \frac{y - x}{|y - x|} + x
\]
sits in the line segment connecting $x$ and $y$ and satisfies $|z - x| = t_m$. Hence the calculation
\[
|y - x_m| \leq |y - z| + |z - x_m|
\]
\[
= |y - z| + t_m \left| \frac{z - x}{t_m} - \frac{x_m - x}{t_m} \right|
\]
\[
= |y - z| + |z - x| \left| \frac{z - x}{|z - x|} - \frac{x_m - x}{|x_m - x|} \right|
\]
\[
\leq |y - z| + |z - x| < t,
\]
where we used the fact that $|v - v'| \leq 1$ for any two $v, v' \in S_m$, shows that $(y, t) \in \Gamma(x_m)$.

On the other hand, if $y \in R_m(x, t_m)$ for some $m$, then $|y - x_m| \leq t_m$, since the diameter of $R_m(x, t_m)$ does not exceed $t_m$. Also $B(y, t_m) \subset E^*$ by Lemma 9 so that $t_m < t$ since $B(y, t) \not\subset E^*$, which shows that $(y, t) \in \Gamma(x_m)$. □

We are now ready to state and prove the atomic decomposition for $T^1(X)$ functions.

**Theorem 11.** For every function $f$ in $T^1(X)$ there exist countably many atoms $a_k$ and real numbers $\lambda_k$ such that

$$f = \sum_k \lambda_k a_k \quad \text{and} \quad \sum_k |\lambda_k| \lesssim \|f\|_{T^1(X)}.$$  

**Proof.** Let $f$ be a function in $T^1(X)$ and write

$$E_k = \left\{ x \in \mathbb{R}^n : \left( \mathbb{E} \left\| \int_{\Gamma(x)} f \, dW \right\|^2 \right)^{1/2} > 2^k \right\}$$

for each integer $k$. By Lemma 3 each $E_k$ is open. For each $k$, apply Lemma 8 to the open set $E_k^*$ in order to get disjoint balls $B^j_k \subset E_k^*$ for which

$$\widehat{E_k^*} \subset \bigcup_{j \geq 1} \widehat{5B^j_k}.$$  

Further, for each of these covers, take a (rough) partition of unity, that is, a collection of functions $\chi_k^j$ where

$$0 \leq \chi_k^j \leq 1, \quad \sum_{j=1}^\infty \chi_k^j = 1 \quad \text{on} \quad \widehat{E_k^*} \quad \text{and} \quad \text{supp} \chi_k^j \subset \widehat{5B^j_k}.$$  

For instance, one can define $\chi_k^1$ as the indicator of $\widehat{5B^1_k}$ and $\chi_k^j$ for $j \geq 2$ as the indicator of

$$\widehat{5B^j_k} \setminus \bigcup_{i=1}^{j-1} \widehat{5B^i_k}.$$  

Write $A_k = \widehat{E_k^*} \setminus \widehat{E_{k+1}^*}$. We are now in the position to decompose $f$ as

$$f = \sum_{k \in \mathbb{Z}} 1_{A_k} f = \sum_{k \in \mathbb{Z}} \sum_{j \geq 1} \chi_k^j 1_{A_k} f = \sum_{k \in \mathbb{Z}} \sum_{j \geq 1} \lambda_k^j a_k^j,$$

where

$$\lambda_k^j = |5B_k^j|^{1/2} \left( \int_{5B_k^j} \mathbb{E} \left\| \int_{\Gamma(x) \cap A_k} f \, dW \right\|^2 \, dx \right)^{1/2}.$$  

Observe, that $a_k^j = \chi_k^j 1_{A_k} f / \lambda_k^j$ is an atom supported in $\widehat{5B_k^j}$.

It remains to estimate the sum of $\lambda_k^j$’s. For $x \not\in E_{k+1}$ we have

$$\mathbb{E} \left\| \int_{\Gamma(x) \cap A_k} f \, dW \right\|^2 \, dx \leq 4^{k+1}$$

by the definition of $E_{k+1}$. The cones at points $x \in E_{k+1}$ are the problematic ones and so in order to estimate $\lambda_k^j$’s, we need to exploit the fact that $1_{A_k} f$ vanishes on $\widehat{E_{k+1}^*}$.
Let \( x \in E_{k+1} \) and use Lemma 10 to pick \( x_1, \ldots, x_N \in \partial E_{k+1} \), where \( N \leq c'(n) \), such that

\[
\Gamma(x) \setminus E_{k+1}^* \subset \bigcup_{m=1}^N \Gamma(x_m).
\]

Now \( x_1, \ldots, x_N \not\in E_{k+1} \) which allows us to estimate

\[
\mathbb{E} \| \int_{\Gamma(x) \cap A_k} f \, dW \|^2 \leq \left( \sum_{m=1}^N \left( \mathbb{E} \| \int_{\Gamma(x_m)} f \, dW \|^2 \right)^{1/2} \right)^2 \leq N^2 4^{k+1}.
\]

Hence, integrating over \( 5B_k^j \) we obtain

\[
\int_{5B_k^j} \mathbb{E} \| \int_{\Gamma(x) \cap A_k} f \, dW \|^2 \, dx \leq |5B_k^j| c'(n) 4^{k+1}.
\]

Consequently,

\[
\sum_{k \in \mathbb{Z}} \sum_{j \geq 1} \lambda_k^j \leq c'(n) \sum_{k \in \mathbb{Z}} 2^{k+1} \sum_{j \geq 1} |5B_k^j| \leq c'(n) 5^n \sum_{k \in \mathbb{Z}} 2^{k+1} |E_k^*| \leq c'(n) \lambda(n)^{-1} 5^n \sum_{k \in \mathbb{Z}} 2^{k+1} |E_k| \leq c'(n) \lambda(n)^{-1} 5^n \| f \|_{T^1(X)}.
\]

It is perhaps surprising that the UMD assumption is not needed for the atomic decomposition.

**Embedding** \( T^1(X) \) **into** \( H^1(\mathbb{R}^n; \gamma(X)) \) **and** \( T^\infty(X) \) **into** \( \text{BMO}(\mathbb{R}^n; \gamma(X)) \). Armed with the atomic decomposition we proceed to the embeddings. Suppose that a smooth function \( \psi : [0, \infty) \to \mathbb{R} \) satisfies \( 1_{[0,1]} \leq |\psi| \leq 1_{[0,\alpha]} \) for some \( \alpha > 2 \) and has \( \int_{\mathbb{R}^n} \psi(|x|) \, dx = 0 \). For functions \( f : \mathbb{R}^{n+1}_+ \to X \) we define

\[
J_{\psi} f(x; y, t) = \psi \left( \frac{|x - y|}{t} \right) f(y, t), \quad x \in \mathbb{R}^n, (y, t) \in \mathbb{R}^{n+1}_+,
\]

and note immediately that \( \int_{\mathbb{R}^n} J_{\psi} f(x) \, dx = 0 \).

Recall also that functions in the Hardy space \( H^1(\mathbb{R}^n; \gamma(X)) \) are composed of atoms \( A : \mathbb{R}^n \to \gamma(X) \) each of which is supported on a ball \( B \subset \mathbb{R}^n \), has zero integral and satisfies

\[
\int_B \mathbb{E} \| \int_{\mathbb{R}^{n+1}_+} A(x; y, t) \, dW(y, t) \|^2 \, dx \leq \frac{1}{|B|}.
\]

For further references, see Blasco [11] and Hytönen [6].

**Theorem 12.** Suppose that \( X \) is UMD. Then \( J_{\psi} \) embeds \( T^1(X) \) into \( H^1(\mathbb{R}^n; \gamma(X)) \) and \( T^\infty(X) \) into \( \text{BMO}(\mathbb{R}^n; \gamma(X)) \).
Proof. We argue that $J_\psi$ takes $T^1(X)$ atoms to (multiples of) $H^1(\mathbb{R}^n; \gamma(X))$ atoms. If a $T^1(X)$ atom $a$ is supported in $\tilde{B}$ for some ball $B \subset \mathbb{R}^n$, then $J_\psi a$ is supported in $\alpha B$ and $\int J_\psi a = 0$. Moreover, since $X$ is UMD, we may use the equivalence of $T^2(X)$ norms (Theorem 5) and write

$$\int_{\alpha B} \mathbb{E} \left( \int_{\mathbb{R}^n+1} \psi \left( \frac{|x-y|}{t} \right) a(y,t) dW(y,t) \right)^2 dx \lesssim \int_B \mathbb{E} \left( \int_{\Gamma(x)} a dW \right)^2 dx \leq \frac{1}{|B|}.$$ 

The boundedness of $J_\psi$ from $T^1(X)$ to $H^1(\mathbb{R}^n; \gamma(X))$ follows. In addition, since $1_{[0,1]} \leq |\psi|$, it follows that $\|f\|_{T^1(X)} \leq \|\psi f\|_{L^1(\mathbb{R}^n; \gamma(X))} \leq \|\psi f\|_{H^1(\mathbb{R}^n; \gamma(X))}$ and so $J_\psi$ is also bounded from below.

To see that $J_\psi$ maps $T^\infty(X)$ boundedly into $\text{BMO}(\mathbb{R}^n; \gamma(X))$, we need to show that

$$\left( \int_B \mathbb{E} \left( \int_{\mathbb{R}^n \times (0,r_B)} (J_\psi g(x,y,t) - \int_B J_\psi g(z,y,t) dz) dW(y,t) \right)^2 dx \right)^{1/2} \lesssim \|g\|_{T^\infty(X)}$$

for all balls $B \subset \mathbb{R}^n$. We partition the upper half-space into $\mathbb{R}^n \times (0,r_B)$ and the sets $A_k = \mathbb{R}^n \times [2^{k-1}r_B, 2^k r_B]$ for positive integers $k$ and study each piece separately.

On $\mathbb{R}^n \times (0,r_B)$ one has

$$\left( \int_B \mathbb{E} \left( \int_{\mathbb{R}^n \times (0,r_B)} \psi \left( \frac{|z-y|}{t} \right) g(y,t) dW(y,t) \right)^2 dz \right)^{1/2} \leq \left( \int_B \mathbb{E} \left( \int_{\Gamma_a(x,r_B)} g dW \right)^2 dx \right)^{1/2} \lesssim \|g\|_{T^\infty}$$

since $|\psi| \leq 1_{[0,1]}$ and the $T^2(X)$ norms are comparable (Theorem 5). Furthermore, as one can justify by approximating $\psi$ with simple functions, we have

$$\left( \mathbb{E} \left( \int_{\mathbb{R}^n \times (0,r_B)} g(y,t) \int_B \psi \left( \frac{|z-y|}{t} \right) dz dW(y,t) \right)^2 \right)^{1/2} \leq \left( \int_B \mathbb{E} \left( \int_{\mathbb{R}^n \times (0,r_B)} \psi \left( \frac{|z-y|}{t} \right) g(y,t) dW(y,t) \right)^2 dz \right)^{1/2},$$

which can be estimated from above by $\|g\|_{T^\infty}$, as above.

For each $k$ and $x \in B$, we claim that

$$\left| \int_B \left( \psi \left( \frac{|x-y|}{t} \right) - \psi \left( \frac{|z-y|}{t} \right) \right) dz \right| \lesssim 2^{-k} 1_{\Gamma_{\alpha+2}(x)}(y,t),$$

whenever $(y,t) \in A_k$. Indeed, if $(y,t) \in A_k \cap \Gamma_{\alpha+2}(x)$, we may use the fact that

$$\left| \psi \left( \frac{|x-y|}{t} \right) - \psi \left( \frac{|z-y|}{t} \right) \right| \lesssim \sup |\psi| \left| \frac{|x-z|}{t} \right| \lesssim \frac{r_B}{2^k r_B} = 2^{-k}$$

for all $z \in B$, while for $(y,t) \in A_k \setminus \Gamma_{\alpha+2}(x)$ we have $|y-x| \geq (\alpha + 2)t \geq \alpha t + 2r_B$ so that $|y-z| \geq |y-x| - |x-z| \geq \alpha t$ for each $z \in B$, which results in

$$\int_B \left( \psi \left( \frac{|x-y|}{t} \right) - \psi \left( \frac{|z-y|}{t} \right) \right) dz = 0.$$
This gives
\[
\left( \int_B \mathbb{E} \left\| \int_{A_k} \frac{g(y, t)}{|B|} \int_B \left( \psi \left( \frac{|x-y|}{t} \right) - \psi \left( \frac{|z-y|}{t} \right) \right) \, dz \, dW(y, t) \right\|^2 \, dx \right)^{1/2} \\
\leq 2^{-k} \left( \int_B \mathbb{E} \left\| \int_{A_k \cap \Gamma_{\alpha+2}(x)} g \, dW \right\|^2 \, dx \right)^{1/2}.
\]

But every $A_k \cap \Gamma_{\alpha+2}(x)$ with $x \in B$ is contained in any $\Gamma_{\alpha+6}(z)$ with $z \in 2^k B$. Indeed, for all $(y, t) \in A_k \cap \Gamma_{\alpha+2}(x)$ we have
\[
|y - z| \leq |y - x| + |x - z| \leq (\alpha + 2)t + (2^k + 1)r_B \leq (\alpha + 6)t.
\]

Hence
\[
\int_B \mathbb{E} \left\| \int_{A_k \cap \Gamma_{\alpha+2}(x)} g \, dW \right\|^2 \, dx \leq \int_{2^k B} \mathbb{E} \left\| \int_{\Gamma_{\alpha+6}(z)} g \, dW \right\|^2 \, dz.
\]

Summing up, we obtain
\[
\sum_{k=1}^{\infty} \left( \int_B \mathbb{E} \left\| \int_{A_k} g(y, t) \int_B \left( \psi \left( \frac{|x-y|}{t} \right) - \psi \left( \frac{|z-y|}{t} \right) \right) \, dz \, dW(y, t) \right\|^2 \, dx \right)^{1/2} \\
\leq \sum_{k=1}^{\infty} 2^{-k} \left( \int_{2^k B} \mathbb{E} \left\| \int_{\Gamma_{\alpha+6}(z)} g \, dW \right\|^2 \, dz \right)^{1/2} \lesssim \|g\|_{T^\infty(X)}.
\]

To see that $\|g\|_{T^\infty(X)} \lesssim \|J_\psi g\|_{\text{BMO}(\mathbb{R}^n; \gamma(X))}$ it suffices to fix a ball $B \subset \mathbb{R}^n$ and show, that for every $x \in B$ we have
\[
1_{\Gamma(x; r_B)}(y, t) \leq \left| \psi \left( \frac{|x-y|}{t} \right) - \int_{(\alpha+2)B} \psi \left( \frac{|z-y|}{t} \right) \, dz \right|,
\]

since this gives us
\[
\int_B \mathbb{E} \left\| \int_{\Gamma(x; r_B)} g \, dW \right\|^2 \, dx \\
\leq \int_B \mathbb{E} \left\| \int_{\mathbb{R}^n} g(y, t) \left( \psi \left( \frac{|x-y|}{t} \right) - \int_{(\alpha+2)B} \psi \left( \frac{|z-y|}{t} \right) \, dz \right) \, dx \right\|^2 \, dx \\
\leq (\alpha + 2)^n \|J_\psi g\|_{\text{BMO}(\mathbb{R}^n; \gamma(X))}.
\]

Now that $1_{[0, 1]} \leq |\psi|$ and $\int_{\mathbb{R}^n} \psi(|x|) \, dx = 0$, it is enough to prove for a fixed $x \in B$, that
\[
\text{supp } \psi \left( \frac{|\cdot - y|}{t} \right) \subset (\alpha + 2)B
\]
for every $(y, t) \in \Gamma(x; r_B)$, i.e. that $B(y, \alpha t) \subset (\alpha + 2)B$ whenever $|x - y| < t < r_B$. This is indeed true, as every $z \in B(y, \alpha t)$ satisfies
\[
|z - x| \leq |z - y| + |y - x| < (\alpha + 1)r_B.
\]
We have established that, also in this case, $J_\psi$ is bounded from below. \qed
It follows that different $T^1(X)$ norms are equivalent in the sense that whenever $1_{[0,1)} \leq |\phi| \leq 1_{[0,\alpha)}$ for some $\alpha > 1$, we can take smooth $\psi : [0, \infty) \to \mathbb{R}$ with $|\phi| \leq |\psi| \leq 1_{[0,2\alpha)}$ to obtain

$$
\|f\|_{T^1(X)} \leq \|J_\psi f\|_{L^1(\mathbb{R}^n;\gamma(X))} \leq \|J_\psi f\|_{L^1(\mathbb{R}^n;\gamma(X))} \leq \|J_\psi f\|_{H^1(\mathbb{R}^n;\gamma(X))} \lesssim \|f\|_{T^1(X)}.
$$

To identify $T^1(X)$ as a complemented subspace of $H^1(\mathbb{R}^n;\gamma(X))$ we define a projection first on the level of test functions. Let us write

$$
T(X) = \{ f : \mathbb{R}^n_{+1} \to X : 1_{\Gamma(x)}f \in L^2(\mathbb{R}^n_{+1}) \otimes X \text{ for almost every } x \in \mathbb{R}^n \}
$$

and

$$
S(\gamma(X)) = \text{span} \{ F : \mathbb{R}^n \times \mathbb{R}^n_{+1} \to X : F(x; y, t) = \Psi(x; y, t)f(y, t) \}
$$

for some $\Psi \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n_{+1})$ and $f \in T(X)$. Observe, that $J_\psi$ maps $T(X)$ into $S(\gamma(X))$ and that $S(\gamma(X))$ intersects $L^p(\mathbb{R}^n;\gamma(X))$ densely for all $1 < p < \infty$ and likewise for $H^1(\mathbb{R}^n;\gamma(X))$.

For $F$ in $S(\gamma(X))$ we define

$$
(N_\psi F)(x; y, t) = \psi\left(\frac{|x - y|}{t}\right) \frac{1}{c_\psi} \int_{\mathbb{R}^n} \psi\left(\frac{|z - y|}{t}\right) F(z; y, t) \, dz,
$$

where $c_\psi = \int_{\mathbb{R}^n} \psi(|x|)^2 \, dx$. Now $N_\psi$ is a projection and satisfies $N_\psi J_\psi = J_\psi$. Also, for every $F \in S(\gamma(X))$ we find an $f \in T(X)$ so that $N_\psi F = J_\psi f$, namely

$$
f(y, t) = \frac{1}{c_\psi} \int_{\mathbb{R}^n} \psi\left(\frac{|z - y|}{t}\right) F(z; y, t) \, dz, \quad (y, t) \in \mathbb{R}^n_{+1}.
$$

**Theorem 13.** Suppose that $X$ is UMD. Then $N_\psi$ extends to a bounded projection on $H^1(\mathbb{R}^n;\gamma(X))$ and $J_\psi$ extends to an isomorphism from $T^1(X)$ onto the image of $H^1(\mathbb{R}^n;\gamma(X))$ under $N_\psi$.

**Proof.** Let $1 < p < \infty$. For simple $L^2(\mathbb{R}^n_{+1}) \otimes X$ -valued functions $F$ defined on $\mathbb{R}^n$ the mapping $(y, t) \mapsto F(\cdot; y, t) : \mathbb{R}^n_{+1} \to L^p(\mathbb{R}^n; X)$ is in $L^2(\mathbb{R}^n_{+1}) \otimes L^p(\mathbb{R}^n; X)$ and we may express $N_\psi$ using the averaging operators as

$$
(N_\psi F)(\cdot; y, t) = A^\psi_{y,t}(F(\cdot; y, t)).
$$

Since $X$ is UMD, Stein’s inequality guarantees $\gamma$-boundedness for the range of the strongly $L^p(\mathbb{R}^n; X)$-measurable function $(y, t) \mapsto A^\psi_{y,t}$, and so by Lemma 1

$$
\mathbb{E} \left\| \int_{\mathbb{R}^n_{+1}} A^\psi_{y,t}(F(\cdot; y, t)) \, dW(y, t) \right\|_{L^p(\mathbb{R}^n; X)}^p \lesssim \mathbb{E} \left\| \int_{\mathbb{R}^n_{+1}} F(\cdot; y, t) \, dW(y, t) \right\|_{L^p(\mathbb{R}^n; X)}^p.
$$

In other words, $\|N_\psi F\|_{L^p(\mathbb{R}^n;\gamma(X))} \lesssim \|F\|_{L^p(\mathbb{R}^n;\gamma(X))}$.

We wish to define a suitable $\mathcal{L}(\gamma(X))$-valued kernel $K$ that allows us to express $N_\psi$ as a Calderón–Zygmund operator

$$
N_\psi F(x) = \int_{\mathbb{R}^n} K(x, z) F(z) \, dz, \quad F \in L^p(\mathbb{R}^n;\gamma(X)).
$$
For distinct \(x, z \in \mathbb{R}^n\) and we define \(K(x, z)\) as multiplication by

\[
(y, t) \mapsto \psi\left(\frac{|x - y|}{t}\right) \frac{1}{c_\psi t^n} \psi\left(\frac{|z - y|}{t}\right),
\]

and so

\[
\|K(x, z)\|_{L(\gamma(X))} = \sup_{(y, t) \in \mathbb{R}^{n+1}} \left|\psi\left(\frac{|x - y|}{t}\right) \frac{1}{c_\psi t^n} \psi\left(\frac{|z - y|}{t}\right)\right|.
\]

For \(|x - z| > \alpha t\) we have

\[
\psi\left(\frac{|x - y|}{t}\right) \frac{1}{c_\psi t^n} \psi\left(\frac{|z - y|}{t}\right) = 0
\]

while \(|x - z| \leq \alpha t\) guarantees that

\[
\left|\psi\left(\frac{|x - y|}{t}\right) \frac{1}{c_\psi t^n} \psi\left(\frac{|z - y|}{t}\right)\right| \leq \frac{1}{c_\psi} \leq \frac{\alpha^n}{c_\psi |x - z|^n}.
\]

Hence

\[
\|K(x, z)\|_{L(\gamma(X))} \lesssim \frac{1}{|x - z|^n}.
\]

Similarly,

\[
\|\nabla_x K(x, z)\|_{L(\gamma(X))} = \sup_{(y, t) \in \mathbb{R}^{n+1}} \left|\psi\left(\frac{|x - y|}{t}\right) \frac{1}{c_\psi t^{n+1}} \psi\left(\frac{|z - y|}{t}\right)\right| \lesssim \frac{1}{|x - z|^{n+1}}.
\]

Thus \(K\) is indeed a Calderón–Zygmund kernel.

Now \(\int_{\mathbb{R}^n} \psi(|x|) \, dx = 0\) implies that \(\int_{\mathbb{R}^n} N_\psi F(x) \, dx = 0\) for \(F \in H^1(\mathbb{R}^n; \gamma(X))\), which guarantees that \(N_\psi\) maps \(H^1(\mathbb{R}^n; \gamma(X))\) into itself (see Meyer and Coifman [12, Chapter 7, Section 4]).

We proceed to the question of duality of \(T^1(X)\) and \(T^\infty(X^*)\). Assuming that \(X\) is UMD, it is both reflexive and \(K\)-convex so that the duality

\[
H^1(\mathbb{R}^n; \gamma(X))^* \simeq \text{BMO}(\mathbb{R}^n; \gamma(X)^*) \simeq \text{BMO}(\mathbb{R}^n; \gamma(X^*))
\]

holds (recall the discussion in Section 2) and we may define the adjoint of \(N_\psi\) by

\[
\langle F, N_\psi^* G \rangle = \langle N_\psi F, G \rangle, \quad \text{where } F \in H^1(\mathbb{R}^n; \gamma(X)) \text{ and } G \in \text{BMO}(\mathbb{R}^n; \gamma(X^*)).
\]

Moreover, as \(T^1(X)\) is isomorphic to the image of \(H^1(\mathbb{R}^n; \gamma(X))\) under \(N_\psi\), its dual \(T^1(X)^*\) is isomorphic to the image of \(\text{BMO}(\mathbb{R}^n; \gamma(X^*))\) under the adjoint \(N_\psi^*\) and the question arises whether the latter is isomorphic to \(T^\infty(X^*)\). For \(J_\psi\) to give this isomorphism (and to be onto) one could try and follow the proof strategy of the case \(1 < p < \infty\) and give an explicit definition of \(N_\psi^*\) on a dense subspace of \(\text{BMO}(\mathbb{R}^n; \gamma(X^*))\). Even though the properties of the kernel \(K\) of \(N_\psi\) guarantee that \(N_\psi^*\) formally agrees with \(N_\psi\) on \(L^p(\mathbb{R}^n; \gamma(X^*))\), it is problematic to find suitable dense subspaces of \(\text{BMO}(\mathbb{R}^n; \gamma(X^*))\).

In order to address these issues in more detail, we specify another pair of test function classes, namely

\[
\widetilde{T}(X) = \{g : \mathbb{R}^{n+1}_+ \to X : 1_{\Gamma(x,r)}g \in L^2(\mathbb{R}^{n+1}_+) \otimes X \text{ for every } r > 0 \text{ and for almost every } x \in \mathbb{R}^n\}
\]
and

\[ \tilde{S}(\gamma(X)) = \text{span}\{ G : \mathbb{R}^n \times \mathbb{R}_{+}^{n+1} \to X : G(x; y, t) = \Psi(x; y, t)g(y, t) \text{ for some} \]
\[ \Psi \in L^\infty(\mathbb{R}^n \times \mathbb{R}_{+}^{n+1}) \text{ and } g \in \tilde{T}(X) \}/ \{ \text{constant functions} \}. \]

Since \( \int_{\mathbb{R}^n} \psi(|x|) \, dx = 0 \), the projection \( N_{\psi} \) is well-defined on \( \tilde{S}(\gamma(X)) \). Moreover, given any \( G \in \tilde{S}(\gamma(X)) \) we can write

\[ g(y, t) = \frac{1}{c_{\psi} t^n} \int_{\mathbb{R}^n} \psi \left( \frac{|z - y|}{t} \right) G(z; y, t) \, dz \]

to define a function \( g \in \tilde{T}(X) \) for which \( N_{\psi} G = J_{\psi} g \). But \( \tilde{S}(\gamma(X)) \) has only weak*-dense intersection with \( \text{BMO}(\mathbb{R}^n; \gamma(X)) \) (recall that \( X \simeq X^{**} \)). Nevertheless, \( J_{\psi} \) is an isomorphism from \( T^\infty(X) \) onto the closure of the image of \( \tilde{S}(\gamma(X)) \cap \text{BMO}(\mathbb{R}^n; \gamma(X)) \) under \( N_{\psi} \). It is not clear whether test functions are dense in the closure of their image under the projection.

The following relaxed duality result is still valid:

**Theorem 14.** Suppose that \( X \) is UMD. Then \( T^\infty(X^*) \) is isomorphic to a norming subspace of \( T^1(X)^* \) and its action is realized for functions \( f \in T^1(X) \) and \( g \in T^\infty(X^*) \) via

\[ \langle f, g \rangle = c \int_{\mathbb{R}^{n+1}_+} \langle f(y, t), g(y, t) \rangle \frac{dy \, dt}{t}, \]

where \( c \) depends on the dimension \( n \).

**Proof.** Fix a smooth \( \psi : [0, \infty) \to \mathbb{R} \) such that \( 1_{[0,1]} \leq |\psi| \leq 1_{[0,\alpha]} \) for some \( \alpha > 2 \) and \( \int_{\mathbb{R}^n} \psi(|x|) \, dx = 0 \). By Theorem 13 \( T^1(X) \) is isomorphic to the image of \( H^1(\mathbb{R}^n; \gamma(X)) \) under \( N_{\psi} \), from which it follows that the dual \( T^1(X)^* \) is isomorphic to the image of \( \text{BMO}(\mathbb{R}^n; \gamma(X^*)) \) under the adjoint projection \( N_{\psi}^* \), which formally agrees with \( N_{\psi} \).

The space \( T^\infty(X^*) \), on the other hand, is isomorphic to the closure of the image of \( \tilde{S}(\gamma(X^*)) \cap \text{BMO}(\mathbb{R}^n; \gamma(X^*)) \) under \( N_{\psi} \) in \( \text{BMO}(\mathbb{R}^n; \gamma(X^*)) \) and hence is a closed subspace of \( T^1(X)^* \). We can pair a function \( f \in T^1(X) \) with a function \( g \in T^\infty(X^*) \) using the pairing of \( J_{\psi} f \) and \( J_{\psi} g \) and the atomic decomposition of \( f \) to get:

\[ \langle f, g \rangle = \sum_k \langle J_{\psi} a_k, J_{\psi} g \rangle = \sum_k \lambda_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+1}} \psi \left( \frac{|x - y|}{t} \right)^2 \langle a_k(y, t), g(y, t) \rangle \frac{dy \, dt}{t^{n+1}} \]
\[ = c_n c_{\psi} \sum_k \lambda_k \int_{\mathbb{R}^{n+1}} \langle a_k(y, t), g(y, t) \rangle \frac{dy \, dt}{t} \]
\[ = c_n c_{\psi} \int_{\mathbb{R}^{n+1}} \langle f(y, t), g(y, t) \rangle \frac{dy \, dt}{t}, \]

where \( c_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \). The subspace \( L^\infty(\mathbb{R}^n) \otimes L^2(\mathbb{R}_{+}^{n+1}) \otimes X^* \) is weak*-dense in \( \text{BMO}(\mathbb{R}^n; \gamma(X^*)) \) and hence a norming subspace for \( H^1(\mathbb{R}^n; \gamma(X)) \).
As it is contained in $\widetilde{S}(\gamma(X^*)) \cap \text{BMO}(\mathbb{R}^n; \gamma(X^*))$, we obtain
\
\|f\|_{T^1(X)} \approx \|J_\psi f\|_{H^1(\mathbb{R}^n; \gamma(X))} = \sup_G |\langle J_\psi f, G \rangle| = \sup_G |\langle N_\psi J_\psi f, G \rangle|
\
= \sup_G |\langle J_\psi f, N_\psi^* G \rangle| \approx \sup_g |\langle J_\psi f, J_\psi g \rangle| = \sup_g |\langle f, g \rangle|,
\
where the suprema are taken over $G \in \widetilde{S}(\gamma(X^*)) \cap \text{BMO}(\mathbb{R}^n; \gamma(X^*))$ with $\|G\|_{\text{BMO}(\mathbb{R}^n; \gamma(X^*))} \leq 1$ and $g \in T^\infty(X^*)$ with $\|g\|_{T^\infty(X^*)} \leq 1$. □

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