A study of two qubits system with Quantum operator formalism

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Abstract. An open system consists of the decay of the excited states of two two-level atoms due to the stimulated emission of photons is studied in a quantum operator formalism, the approach of Kraus operators. We have constructed in the work the Kraus operators which will be applied to derive the time evolution of the density matrix for analyzing the stability of the entanglement of two qubits system.

1. Introduction
In quantum mechanics, two interesting and very useful phenomena are quantum superposition and quantum entanglement, which have been recognized as key resources of quantum computing and quantum information processing [1, 2, 3, 4]. The simplest entanglement is the Bell state [5] in which a pair of qubits (or quantum bits) is formed. The Bell system has been studied intensively. For instance, the Bell singlet state is explored in Ref. [6] in the suppression of disentanglement. The system consists of two atoms inside the same cavity, where the interaction of these atoms is allowed by exchanging photons inside a lossless cavity (environment).

Our research follows the work in Ref. [6] but in the approach of Kraus operators in the interaction picture rather than the Schrodinger picture applied in Ref. [6] in which the master equation of the density matrix is solved directly. In general, it is difficult (or say, impossible) to directly solve the master equation of a system with interactions. It is believed that the Kraus operator representation may provide a perturbative approach for solving the master equation of interacting systems. In general, a composite system [7, 8] (open system + environment) in terms of the interaction Hamiltonian does not allow the Kraus operator representation form of the dynamical open system [9]. The Kraus operators [10] for the one-qubit system are derived [11], but it is still unclear to apply the Kraus operator representation to two-qubit interacting systems.

The paper is arranged as follows: In Section 2, we introduce two-qubit systems and give the Kraus operators of the system in the interaction picture. Discuss and conclusions are given in Section 3. The detailed calculations of the Kraus operators are derived in Appendix.

2. Kraus operators representation of a qubit system
The model of our system consists of two two-level atoms inside the same cavity. We assume that these atoms are identical and allowed to interact via photon exchange with the cavity. The
The total Hamiltonian is given by, \( (\hbar = 1) \):

\[
H_{\text{total}} = H_{\text{atoms}} + H_\gamma + H_{\text{int}}
\]

\[
= \frac{1}{2} \hbar \omega_0 \Sigma_z + \sum_i \hbar \omega_i (a_i^\dagger a_i) + \sum_i (\lambda a_i \Sigma_+ + \lambda^* a_i^\dagger \Sigma_-) \tag{1}
\]

where \( \Sigma_k = (\sigma_k + \tau_k) \), \( k \) can be either \( \{x, y, z\} \) or \( \{+, -\} \) for raising and lowering operations, \( a_i (a_i^\dagger) \) are the annihilation (creation) operators of photons, and \( \lambda \) and \( \lambda^* \) are the photon-atom coupling constants with the same strength. The spin operators of atoms \( A \) and \( B \) are represented by \( \sigma, \tau \). The dynamical equation (master equation) may be solved directly, as in Ref. [6], or in the Klaus operator representation.

The time evolution of a closed quantum system can be described by a unitary operator \( U(t) \). For open quantum systems, however, the time evolution is not unitary. The evolution of an open system is usually described by the Kraus representation, which is constructed by considering a larger (closed) system. Normally, the general solutions for the master equation can be derived from the method of the Kraus representation, which allows a perturbative analysis of the disentanglement for arbitrary states.

The solution for the master equation of the density matrix \( \rho_s(t) \) can be written in terms of the Kraus operators in the form,

\[
\rho_s(t) = \sum_{\eta=0}^{\infty} K_\eta(t) \rho_s(0) K_\eta^\dagger(t) \tag{2}
\]

where \( \rho(0) \) is the density of the initial state of the two atoms, and the summation is over all possible photon states. The Kraus operators \( (K_\eta) \) are given by

\[
K_\eta = \langle \eta | V(t) | 0 \rangle , \tag{3}
\]

where \( |\eta\rangle \) are the states of photons with \( \eta \) being the number of photons, and the interaction Hamiltonian \( (V(t)) \) can be written as

\[
V(t) = \exp \frac{-i H_{\text{int}} t}{\hbar} = \exp \frac{-i \lambda (\Sigma_+ a + \Sigma_- a^\dagger)}{\hbar} . \tag{4}
\]

Note that the relation \( \sum_\eta K_\eta^\dagger(t) K_\eta(t) = I \) must be satisfied.

By using the Taylor expansion, one may rewrite the Kraus operators in the form,

\[
K_\eta(t) = \langle \eta | \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{-i \lambda t}{\hbar} \right)^m (\Sigma_+ a + \Sigma_- a^\dagger)^m | 0 \rangle . \tag{5}
\]

After some tedious calculations, as shown in Appendix, we derive,

\[
K_\eta(t) = \delta_{\eta 0} [I - (\Sigma_+ \Sigma_+ + \Sigma_- \Sigma_-) + \cos(2\alpha)(\Sigma_+ \Sigma_- + \Sigma_- \Sigma_+)] \\
+ \delta_{\eta 1} [-(\cos(\alpha) \Sigma_-) + \delta_{\eta 2} [(\cos(2\alpha) - 1)(\Sigma_- \Sigma_-)] \] \tag{6}
\]
where $\alpha \equiv \frac{\lambda t}{\hbar}$. More explicitly, the Kraus operators are as follows,

$$
K_0(t) = \begin{pmatrix}
(2 \cos(2\alpha) - 1) & 0 & 0 & 0 \\
0 & (2 \cos(2\alpha) - 1) & (2 \cos(2\alpha) - 2) & 0 \\
0 & (2 \cos(2\alpha) - 2) & (2 \cos(2\alpha) - 1) & 0 \\
0 & 0 & 0 & (2 \cos(2\alpha) - 1)
\end{pmatrix},
$$

$$
K_1(t) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-\cos(\alpha) \sin(\alpha) & 0 & 0 & 0 \\
-\cos(\alpha) \sin(\alpha) & 0 & 0 & 0 \\
0 & -\cos(\alpha) \sin(\alpha) & -\cos(\alpha) \sin(\alpha) & 0
\end{pmatrix},
$$

$$
K_2(t) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(2 \cos(2\alpha) - 2) & 0 & 0 & 0
\end{pmatrix}.
$$

The relation $\sum_{\eta=0}^{2} K^\dagger_{\eta}(t)K_{\eta}(t) = I$ is time independent and satisfied by Kraus operators. Note that we have followed the condition of the two-qubit system in our calculation, that is, the parameter $\lambda \neq 0$ (any two qubits are not perpendicular to each other, in other words, they are inseparable).

3. Conclusion
In this work, we have derived the Kraus operators of the entangled system consist of two two-level atoms interacting with the same cavity. The application of the Klaus operators to analyze the time evolution of the density matrix is underway.

Acknowledgments
This work was supported by Suranaree University of Technology (SUT) and by the Office of the Higher Education Commission under NRU project of Thailand. Siriratchanee Thammasuwan acknowledges support by SUT-OROG scholarship (contract no. 20/2557).

Appendix: Kraus operator-sum representation
We calculate the Kraus operators here. The evaluation of the Kraus operators is to perform a partial trace over the environment to obtain the reduced state of the system, $\rho_s(t)$:

$$
\rho_s(t) = Tr_E(V(t)\rho_s(0) \otimes \rho_E(0)V^\dagger(t))
= \sum_{\eta} \langle \eta | V(t) | 0 \rangle \rho_s(0) \langle 0 | V^\dagger(t) | \eta \rangle = \sum_{\eta} K^\dagger_{\eta}(t)\rho_s(0)K_{\eta}(t)
$$

(8)

where $\rho_s$ is the density of the two atoms and $\rho_E$ is the density of the vacuum state of the cavity, and

$$
V(t) = \exp^{-iH_{\text{total}}t} = \exp^{-i\hbar(\Sigma_+ a + \Sigma_- a^\dagger)}.
$$

(9)

The density matrix is defined as

$$
\rho_{ij} = |i \rangle \langle j |,
$$

(10)

with $|i\rangle = |0\rangle |0\rangle, |0\rangle |1\rangle, |1\rangle |0\rangle, |1\rangle |1\rangle$, where $|0\rangle$ and $|1\rangle$ represent the ground state and the excited state of the atoms.
In the interaction picture, the Kraus operators take the form:

\[
K_\eta = \langle \eta | \exp \left(-\frac{i\lambda t}{\hbar} (\Sigma_+ a + \Sigma_- a^\dagger)\right) |0\rangle \\
= \langle \eta | \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{i\lambda t}{\hbar}\right)^m (\Sigma_+ a + \Sigma_- a^\dagger)^m |0\rangle .
\]

(11)

By considering the properties of the operators $\Sigma_{\pm}$, $a$ and $a^\dagger$, our detailed calculations lead to

\[
K_\eta = \{(I - (\Sigma_+ \Sigma_- + \Sigma_- \Sigma_+)) + \left[ \sum_{\eta=0}^{\infty} \frac{(2\eta)!}{\eta!} \right] (\Sigma_+ \Sigma_- + \Sigma_- \Sigma_+) \} \delta_{\eta 0} \\
+ \left\{ \left[ \sum_{\eta=0}^{\infty} \frac{(2\eta)!}{\eta! (2\eta+1)!} \right] (-i\Sigma_-) \right\} \delta_{\eta 1} + \left\{ \left[ \sum_{\eta=0}^{\infty} \frac{(2\eta)!}{\eta! (2\eta+1)!} \right] - 1 \right\} (\Sigma_- \Sigma_-) \right\} \delta_{\eta 2},
\]

(12)

where $\alpha \equiv \frac{\lambda t}{\hbar}$.

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