ON NON-SEPARATED ZERO SEQUENCES OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION

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Abstract  Let \((z_k)\) be a sequence of distinct points in the unit disc \(\mathbb{D}\) without limit points there. We are looking for a function \(a(z)\) analytic in \(\mathbb{D}\) and such that possesses a solution having zeros precisely at the points \(z_k\), and the resulting function \(a(z)\) has ‘minimal’ growth. We focus on the case of non-separated sequences \((z_k)\) in terms of the pseudohyperbolic distance when the coefficient \(a(z)\) is of zero order, but \(\sup_{z \in \mathbb{D}} (1 - |z|)^p |a(z)| = +\infty\) for any \(p > 0\). We established a new estimate for the maximum modulus of \(a(z)\) in terms of the functions \(n_z(t) = \sum_{|z_k - z| \leq t} 1\) and \(N_z(r) = \int_0^r (n_z(t) - 1)^+ / dt\). The estimate is sharp in some sense. The main result relies on a new interpolation theorem.

Keywords: interpolation; unit disc; analytic function; oscillation of solution; differential equation; prescribed zeros

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1. Introduction

The aim of the paper is twofold. On one hand, we are interested in zeros of solutions of

\[ f'' + a(z)f = 0, \tag{1.1} \]

where \(a(z)\) is an analytic function in \(\mathbb{D} = \{ z : |z| < 1 \}\). On the other hand, it leads us to some interpolation problems for corresponding classes of analytic functions in \(\mathbb{D}\).

1.1. Oscillation of solutions

It was proved by Šeda [15] that given a sequence of distinct complex numbers \((z_n)\) with no finite limit points, there exists an entire function \(a(z)\) such that the equation (1.1) has an entire solution \(f\) with the zero sequence \((z_n)\). This result was recently generalized for

\[ f'' + a(z)f = 0, \tag{1.1} \]
an arbitrary domain \( G \subset \mathbb{C} \) when the condition \( f(z_n) = 0 \) is replaced with

\[
f(z_k) = b_k, \quad (1.2)
\]

\((b_n)\) being an arbitrary sequence [17].

We are interested in the description of zero sequences \((z_n)\) of solutions of (1.1) where \(a(z)\) belongs to some growth class. Let \(A^{-p}\) be the Banach space of analytic functions in \(\mathbb{D}\) with the norm

\[
\|a\|_{A^{-p}} := \sup_{|z| < 1} (1 - |z|^2)^p |a(z)|.
\]

In [10], the authors described behaviour of the coefficient when all zero-free solutions belong to some space. We restrict ourselves to the case when \(f\) has infinitely many zeros.

Let \(\varphi(z, w) = z - w / 1 - \bar{z} w\). Let \(\sigma(z, w) = |\varphi_z(w)|\) denote the pseudohyperbolic distance in \(\mathbb{D}\). A sequence \((z_n)\) in the unit disc is called uniformly separated if

\[
\inf_j \prod_{n \neq j} \sigma(z_n, z_j) > 0.
\]

A positive Borel measure \(\mu\) is a Carleson measure if and only if there exists a constant \(K\) such that \(\mu(Q_\delta) \leq K\delta\) for any Carleson box

\[
Q_\delta = \{ \zeta \in \mathbb{D} : |\zeta| \geq 1 - \delta, |\arg \zeta - \varphi| \leq \pi\delta \}.
\]

The following result describes coefficients \(a(z)\) such that zero sequence of a solution (1.1) is uniformly separated.

**Theorem A.** A sequence \(Z\) in the unit disc is the zero-sequence of a non-trivial solution of (1.1) such that \(|a(z)|^2 (1 - |z|^2)^3 dm(z)\) is a Carleson measure if and only if \(Z\) is uniformly separated.

The following problem was formulated in [13].

**Problem.** Let \((z_k)\) be a sequence of distinct points in \(\mathbb{D}\) without limit points there. Find a function \(a(z)\), analytic in \(\mathbb{D}\) such that (1.1) possesses a solution having zeros precisely at the points \(z_k\). Estimate the growth of the resulting function \(a(z)\).

The next result is closely connected to the problem. In order to formulate it, we need more notation. A sequence \(Z = (z_n)\) in the unit disc is called separated if \(\inf_{n \neq j} \sigma(z_n, z_j) > 0\). The uniform density of a sequence \((z_n)\) [16] is defined by

\[
D^+(Z) = \limsup_{r \to 1^-} \sup_{z \in \mathbb{D}} \sum_{\frac{1}{2} < \sigma(z, z_j) < r} \log \frac{1}{\sigma(z, z_j)}.
\]

**Theorem B (Gröhn [8, Theorem 1]).** If \(Z = (z_k)\) is a separated sequence in the unit disc with \(D^+(Z) < 1\) then there exists \(a \in A^{-2}\) such that (1.1) admits a non-trivial solution that vanishes on \(Z\).
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Conversely, if \( a \in A^{-2} \) and \( f \) is a non-trivial solution of (1) whose zero-sequence is \( Z \), then \( Z \) is separated and contains at most one point if \( \|a\|_{A^{-2}} \leq 1 \), while \( D^+(Z) \leq (2\pi + 1)C/(1 - C)^2 \), where \( C = \sqrt{1 - 2\sqrt{\|a\|_{A^{-2}}}/\|a\|_{A^{-2}}} + 1 \).

**Remark 1.1.** The proof of Theorem B uses essentially the interpolation result by K. Seip. In the first part of Theorem B, \( Z \) is a subset of the zero set of \( f \).

To proceed, we need some characteristics that measure growth and zero distribution of analytic functions. Let \( f \) be analytic in \( D \). We write \( M(r, f) = \max\{|f(z)| : |z| = r\} \), \( r \in (0, 1) \). The number of members of the sequence \((z_k)\) satisfying \(|z_k - \zeta| \leq t\) is denoted by \( n_\zeta(t) = \sum_{|z_k - \zeta| \leq t} 1 \). We also define

\[
N_\zeta(r) = \int_0^r \frac{(n_\zeta(t) - 1)^+}{t} \, dt.
\]

For a nondecreasing function \( \psi : [1, +\infty) \to \mathbb{R}_+ \), we write

\[
\tilde{\psi}(x) = \int_1^x \frac{\psi(t)}{t} \, dt.
\]

Note that \( \tilde{\psi}(x) = 1/(p+1) \log^{p+1} x \) if \( \psi(x) = \log^p x \), \( p \geq 0 \), and \( \tilde{\psi}(x) = 1/\rho(x^\rho - 1) \) if \( \psi(x) = x^\rho \), \( \rho > 0 \).

Throughout the paper, we assume that \( \psi(2x) = O(\psi(x)) \), \( x \to +\infty \). (1.3)

**Theorem C ([3]).** Let \((z_n)\) be a sequence of distinct complex numbers in \( D \). Assume that for some nondecreasing unbounded function \( \psi : [1, +\infty) \to \mathbb{R}_+ \) satisfying (1.3) and a constant \( C > 0 \) we have

\[
N_{z_k} \left( \frac{1 - |z_k|}{2} \right) \leq C\psi \left( \frac{1}{1 - |z_k|} \right), \quad k \in \mathbb{N}.
\] (1.4)

Then there exists an analytic function \( a \) in \( D \) and a constant \( C' > 0 \) satisfying

\[
\log M(r, a) \leq C'\tilde{\psi} \left( \frac{1}{1 - r} \right), \quad r \in (0, 1)
\]

such that (1.1) possesses a solution \( f \) having zeros precisely at the points \( z_k, k \in \mathbb{N} \).

**Corollary D.** If for some \( \rho > 0 \) and a constant \( C > 0 \), a sequence \((z_k)\) satisfies the condition

\[
N_{z_k} \left( \frac{1 - |z_k|}{2} \right) \leq C \left( \frac{1}{1 - |z_k|} \right)^\rho, \quad k \in \mathbb{N}
\]

then there exists a function \( a \) analytic in \( D \) such that \( \log M(r, a) = O((1 - r)^{-\rho}) \), \( r \in (0, 1) \) and (1.1) possesses a solution \( f \) having zeros precisely at the points \( z_k, k \in \mathbb{N} \).
The following statement shows that the corollary is sharp (cf. [9]).

**Theorem E.** For arbitrary $\rho > 0$ there exists a sequence of distinct numbers $\{z_n\}$ in $\mathbb{D}$ with the following properties:

(i) $N_{z_k}(1 - |z_k|/2) \leq C(1/1 - |z_k|)^\rho$, $k \in \mathbb{N}$;

(ii) for any $\varepsilon_0 > 0$ the sequence $(z_k)$ cannot be the zero set of a solution of (1.1), where $\log M(r, a) = O((1 - r)^{-\rho + \varepsilon_0})$.

In [9], the case when the coefficient $a \in \mathcal{A}^{-p}$, $p > 2$, is considered. Some other problems on zeros of solutions are considered in a survey [12]. The aforementioned results give a complete solution to the Problem in the cases when $a \in \mathcal{A}^{-2}$ and the order of $a$ is finite and positive. In the intermediate cases, when $a$ is of zero order, but outside of $\mathcal{A}^{-2}$, zero sets of solutions of (1.1) are not described completely. The aim of the paper is to fill this gap. In particular, we improve Theorem C and obtain sharp, in some sense, estimates of $a(z)$ in terms of the zero distribution of a solution of (1.1). Our proof relies on a new interpolation result.

**1.2. Interpolation in the unit disc**

A set $Z = \{z_k\}$ is called an interpolation set for the space $\mathcal{A}^{-n}$, $n > 0$, if for every sequence $(b_k)$ with $(b_k(1 - |z_k|)^n) \in l^\infty$ there is a function $f \in \mathcal{A}^{-n}$ satisfying (1.2). These sets were described by K. Seip in [16]. Namely, he proved that $Z = (z_k)$ is an interpolation set for $\mathcal{A}^{-n}$ if and only if $(z_k)$ is separated, i.e. $\inf_{j \neq k} \sigma(z_k, z_j) > 0$, and $D^+(Z) < n$.

In 2002, A. Hartmann and X. Massaneda [11] established that necessary and sufficient that $Z$ is an interpolation set for a class of growth functions $\eta$ containing all power functions is that

$$\exists \delta \in (0, 1) \exists C > 0 \forall n \in \mathbb{N} : \quad N_{z_n}(\delta(1 - |z_n|)) \leq \eta\left(\frac{C}{1 - |z_n|}\right).$$

Note that the proof of sufficiency in [11] uses a non-constructive method of $L^2$-estimate for the solution to a $\bar{\partial}$-equation. On the other hand, in [3], an interpolating function is constructed explicitly.

The following theorem gives sufficient conditions for interpolation sequences in classes of analytic functions of moderate growth in the unit disc. It complements Hartmann and Massaneda’s result in the case, when $\psi(t)$ grows slower than any power function.

**Theorem F.** Let $(z_n)$ be a sequence of distinct complex numbers in $\mathbb{D}$. Assume that for some nondecreasing unbounded function $\psi: [1, +\infty) \to \mathbb{R}_+$ satisfying (1.3), the condition (1.4) is valid. Then for any sequence $(b_n)$ satisfying

$$\exists C > 0 : \log |b_n| \leq C\psi\left(\frac{1}{1 - |z_n|}\right), \quad n \in \mathbb{N},$$

(1.5)

there exists an analytic function $f$ in $\mathbb{D}$ with the property (1.2) and

$$\exists C > 0 : \log M(r, f) \leq C\psi\left(\frac{1}{1 - r}\right).$$

(1.6)
The previous theorem becomes a criterion if $\psi$ belongs to the class $\mathcal{R}$ of nondecreasing functions $\psi$ such that $\check{\psi}(r) = O(\hat{\psi}(r))$ as $r \to +\infty$ [3, Theorem 5].

In 2007 A. Borichev, R. Dhuez and K. Kelley [2] described interpolation sets for classes of analytic functions of arbitrary growth in both the complex plane and the unit disc. Let $h : [0, 1) \to [0, +\infty)$ be such that $h(0) = 0$, $h(r) \uparrow \infty$ as $r \to 1-$. The Banach spaces of analytic functions on $\mathbb{D}$ with the norms

$$
\|f\|_h = \sup_{z \in \mathbb{D}} |f(z)| e^{-h(z)} < +\infty, \quad \|f\|_{h,p} = \left( \int_{\mathbb{D}} |f(z)|^p e^{-ph(|z|)} \, dm_2(z) \right)^{1/p},
$$

are denoted by $\mathcal{A}_h$ and $\mathcal{A}_{h,p}^p$, $p > 0$, respectively. Suppose that $h \in C^3([0, 1))$, $\rho(r) := (\Delta h(r))^{-1/2} \searrow 0$, and $\rho'(r) \to 0$ as $r \to 1-$, and for all $K > 0$:

$$
\rho(r + x) \sim \rho(r) \text{ for } |x| \leq K \rho(r), \quad r \to 1^{-}
$$

provided that $K \rho(r) < 1 - r$, and either $\rho(r)(1 - r)^{-c}$ increases for some finite $c$ or $\rho'(r) \log \rho(r) \to 0$ as $r \to 1-$.

Given such an $h$ and a sequence $Z = (z_k)$ in $\mathbb{D}$ we denote by

$$
\mathcal{D}_h^+(Z) = \limsup_{R \to \infty} \limsup_{|z| \to 1-} \frac{\operatorname{card}(Z \cap U(z, R\rho(z)))}{R^2},
$$

where $U(z, t) = \{\zeta \in \mathbb{C} : |\zeta - z| < t\}$.

**Theorem G (Theorem 2.3 [2]).** A sequence $Z$ is an interpolation set for $\mathcal{A}_h(\mathbb{D})$ if and only if $\mathcal{D}_h^+(Z) < \frac{1}{2}$ and

$$
\inf_{k \neq n} \frac{|z_k - z_n|}{\min\{\rho(|z_k|), \rho(|z_n|)\}} > 0.
$$

A similar description holds for interpolation sets for the classes $\mathcal{A}_{h,p}^p(\mathbb{D})$, $p > 0$ [2].

Note that assumptions on $h$ in Theorem G imply $h(r)/\log 1/1 - r \to +\infty$ as $r \to 1-$. 

### 2. Main results

In this paper, we are mostly interested in the case where $\psi(r)$ is a slowly growing function unbounded with respect to $\log 1/1 - r$ as $r \to 1-$, in particular, $\psi \not\in \mathcal{R}$. Theorem F seems no longer to be sharp for such functions $\psi$.

Since our approach relies on properties of canonical products, we need some extra notions and notation. The Pólya order [7] of the function $\psi \in \mathcal{R}$ is defined by

$$
\rho^*[\psi] = \inf\{\rho > 0 : \psi(Cx) \leq C^\rho \psi(x), \quad x, C \to +\infty\}. \tag{2.1}
$$

Let $E(w, 0) = 1 - w$, $E(w, s) = (1 - w) \exp\{w + w^2/2 + \cdots + w^s/s\}$, $s \in \mathbb{N}$, be Weierstrass primary factors. We consider a canonical product of the form

$$
P(z) = P(z, Z, s) = \prod_{n=1}^{\infty} E\left(1 - \frac{|z_n|^2}{1 - \bar{z}_n z}, s\right), \quad s \in \mathbb{N} \cup \{0\}. \tag{2.2}
$$
This product is an analytic function in $D$ with the zero sequence $Z = (z_n)$ provided
\[ \sum_{z_n \in Z} (1 - |z_n|)^{s+1} < \infty. \]

The following result allows to relax the assumption on $N_{z_n}(t)$ in comparison with Theorem F.

**Theorem 2.1.** Let $(z_n)$ be a sequence of distinct complex numbers in $D$. Assume that for some nondecreasing unbounded function $\psi : [1, +\infty) \to \mathbb{R}_+$ satisfying (1.3) we have that
\[ \exists C > 0 \forall z \in D : n_z \left( \frac{1 - |z|}{2} \right) \leq C \psi \left( \frac{1}{1 - |z|} \right), \quad (2.3) \]
and either
\[ \exists C > 0 : \forall n \in \mathbb{N} \quad N_{z_n} \left( \frac{1 - |z_n|}{2} \right) \leq C \tilde{\psi} \left( \frac{1}{1 - |z_n|} \right), \quad (2.4) \]
or
\[ \forall n \in \mathbb{N} : -\log((1 - |z_n|)|P'(z_n)|) \leq C \tilde{\psi} \left( \frac{1}{1 - |z_n|} \right), \quad (2.5) \]
or
\[ \forall n \in \mathbb{N} : -\log|B_n(z_n)| \leq C \tilde{\psi} \left( \frac{1}{1 - |z_n|} \right), \quad (2.6) \]
holds, where $B_n(z) = P(z)/E(1 - |z_n|^2/1 - \bar{z}_n z, s)$, $P$ is the canonical product defined by (2.2), $s \geq [\rho] + 1$, where $\rho$ is Polya order of $\psi$. Then for any sequence $(b_n)$ satisfying (1.5) there exists an analytic function $f$ in $D$ with the properties (1.2) and (1.6).

Hypotheses similar to (2.5) are frequently used in interpolation problems (e.g. [1]). The next theorem addresses the problem formulated in the introduction.

**Theorem 2.2.** Let conditions of Theorem 2.1 be satisfied. Then there exists an analytic function $a$ in $D$ satisfying
\[ \exists C > 0 : \log M(r, a) \leq C \tilde{\psi} \left( \frac{1}{1 - r} \right), \quad r \in (0, 1) \]
such that (1.1) possesses a solution $f$ having zeros precisely at the points $z_k, k \in \mathbb{N}$.

Its proof literally repeats that of Theorem C [3] with the only difference that we apply Theorem 2.1 instead of Theorem F. The same scheme is used in the proof of Theorem 2.5. In particular, after the substitution $f(z) = P(z)e^{g(z)}$ where $g(z)$ is analytic in $D$, $P(z)$ is the canonical product (2.2), the construction of an analytic function $a$ reduces to the interpolation problem of finding an analytic function $h = g'$ satisfying
\[ h(z_k) = b_k := g'(z_k) = -\frac{P''(z_k)}{2P'(z_k)}, \quad k \in \mathbb{N}. \]
At this stage, we apply Theorem 2.1. Finally, the required estimate for the function $a$ follows from the identity (see (3.15))
\[ -a = \frac{P''}{P} + \frac{2P'}{P}g' + (g')^2 + g''. \]
Corollary 2.3. If for some \( \rho > 0 \) and \( \beta > 0 \) a sequence \((z_k)\) satisfies the conditions
\[
\exists C > 0 : n_{z_k} \left( \frac{1 - |z_k|}{2} \right) \leq C \log^\beta \frac{1}{1 - |z_k|};
\]
\[
\exists C > 0 : N_{z_k} \left( \frac{1 - |z_k|}{2} \right) \leq C \log^{\beta+1} \frac{1}{1 - |z_k|},
\]
then there exists a function \( a \) analytic in \( \mathbb{D} \) and satisfying
\[
\log M(r, a) = O \left( \log^{\beta+1} \frac{1}{1 - r} \right), \quad r \in (0, 1)
\]
such that possesses \((1.1)\) a solution \( f \) having zeros precisely at the points \( z_k, k \in \mathbb{N}. \)

This corollary is sharp in the following sense

Theorem 2.4. For arbitrary \( \eta_1, \eta_2 > 0 \) there exists a sequence of distinct numbers \((z_n)\) in \( \mathbb{D} \) with the following properties:

(i) \( \exists C > 0 : n_{z_k} \left( \frac{1 - |z_k|}{2} \right) \leq C \log^{\eta_1} \frac{1}{1 - |z_k|}; \quad k \in \mathbb{N}; \)

(ii) \( \exists C > 0 : N_{z_k} \left( \frac{1 - |z_k|}{2} \right) \leq C \log^{1+\eta_1+\eta_2} \frac{1}{1 - |z_k|}; \quad k \in \mathbb{N}; \)

(iii) \((z_k)\) cannot be the zero sequence of a solution of \((1.1)\) where
\[
\log M(r, a) = O \left( \log^{1+\eta} \frac{1}{1 - r} \right), \quad \eta < \eta_2.
\]

Since Theorem B effectively uses the notion of the uniform density, one may ask whether it is possible to use \( \mathcal{D}^+_{\rho^+} \)-density to solve the Problem. The next theorem gives an estimate of the growth of \( a(z) \) under an assumption in terms of the density introduced by Borichev et al. [2].

Theorem 2.5. Let \( h \in C^2[0, 1) \) be an increasing function with \( h(0) = 0 \) and such that
for \( \rho(r) = (\Delta h(r))^{-1/2} \) (1.7) holds and \( (1 - r)h'(r)/h(r) \) is bounded. Let the function
\( \sigma(r) = (1 - r)^2/\rho^2(r) \not\to \infty \) as \( r \not\to 1- \), and satisfy \( \sigma((1 + r)/2) = O(\sigma(r)), r \in [1/2, 1). \)
Suppose that \( \mathcal{D}^+_{\rho^+}(Z) < \infty \) and (1.8) holds. Then there exists an analytic function \( a \) in \( \mathbb{D} \)
satisfying
\[
\exists C > 0 : \log M(r, a) \leq Ch(r), \quad r \in (0, 1)
\]
such that \((1.1)\) possesses a solution \( f \) having zeros precisely at the points \( z_k, k \in \mathbb{N}. \)

Remark 2.6. Note that the assumption \( (1 - r)/\rho(r) \to \infty \) as \( r \to 1- \) implies that
\( h(r)/\log 1/1 - r \to \infty \) as \( r \to 1- \). On the other hand, the boundedness of \((1 - r)h'(r)/h(r) \)
provides that \( h(r)(1 - r)^q \) is bounded for some finite \( q > 0. \)
3. Proofs of the results

Proof of Theorem 2.1. We follow the scheme of the proof of Theorem F from [3]. It follows from the estimate (2.3) and [3, Lemma 9] that

\[
\sum_{n=1}^{\infty} \left| \frac{1 - |z_n|^2}{1 - \bar{z}_n z} \right|^{s+1} \leq C(s) \tilde{\psi} \left( \frac{1}{1 - |z|} \right), \quad z \in \mathbb{D}.
\]

The following two lemmas are important in our arguments.

Lemma H ([3]). For an arbitrary \( \delta \in (0, 1) \), any sequence \( Z \) in \( \mathbb{D} \) satisfying \( \sum_{z_k \in Z} (1 - |z_k|)^{s+1} < \infty \), \( s \in \mathbb{Z}_+ \) there exists a positive constant \( C(\delta, s) \)

\[
|\log |B_k(z_k)| + N_{z_k}(\delta(1 - |z_k|))| \leq C(\delta, s) \sum_{n=1}^{\infty} \left| \frac{1 - |z_n|^2}{1 - \bar{z}_n z} \right|^{s+1}, \quad k \to +\infty.
\]

The next proposition compares some conditions frequently used in interpolation problems (cf. [1]).

Lemma I (Chyzhykov and Sheparovych [3, Proposition 11]). Given a function \( \psi \in \mathcal{R} \) for

\[
\exists C > 0 \quad \forall z \in \mathbb{D} : \quad N_z \left( \frac{1 - |z|}{2} \right) \leq C \psi \left( \frac{1}{1 - |z|} \right)
\]

it is necessary and sufficient that (2.3) and

\[
\forall n \in \mathbb{N} : |\log((1 - |z_n|)|P'(z_n)|)| \leq C \psi \left( \frac{1}{1 - |z_n|} \right), \quad (3.1)
\]

where \( P \) is the canonical product defined by (2.2), \( s = [\rho^*] + 1 \), where \( \rho^* \) is Polya’s order of \( \psi \).

Lemma H directly implies that under the assumption (2.3) the conditions (2.4) and (2.6) are equivalent.

Moreover, one has

\[
P'(z_n)(1 - |z_n|^2) \z_n = -B_n(z_n) \exp \left\{ 1 + \frac{1}{2} + \cdots + \frac{1}{s} \right\}, \quad (3.2)
\]

which yields equivalence of (3.1) and (2.6). Therefore, under assumption (2.3), any of the hypotheses (2.4)–(2.6) imply the other two. Hence, we may assume that the conditions (2.4)–(2.6) hold. Taking into account [3, Lemma 9], which gives an upper estimate of a
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canonical product, and Lemma H we deduce

\[ |\log |B_k(z_k)|| \leq C(\delta, s)\overline{\psi}\left(\frac{1}{1-|z_k|}\right), \quad k \to +\infty. \quad (3.3) \]

The rest of the proof of Theorem 2.1 repeats that of Theorem F [3]. It consists in estimating of the interpolating function

\[ f(z) = \sum_{n=1}^{\infty} \frac{b_n}{z - z_n} \frac{P(z)}{P'(z_n)} \left(\frac{1 - |z_n|^2}{1 - \bar{z}_n z}\right)^{s_n-1}, \]

where \((s_n)\) is an appropriate increasing sequence of natural numbers.

**Proof of Theorem 2.4.** Let \(\eta_1, \eta_2 > 0\) be given. Let \(\varepsilon_n = e^{-(n \log 3)^{1+\eta_2}}\), \(m_n = [(n \log 3)^{\eta_1}]\), \(n \in \mathbb{N}\). Let \((z_{n,k})\) be the sequence defined by

\[ z_{n,k} = 1 - 3^{-n} + k\varepsilon_n/m_n, \quad 0 \leq k \leq m_n - 1. \]

Then

\[ n_{z_{n,k}}(t) = m_n \lesssim \log^{\eta_1} \frac{1}{1 - |z_{n,k}|}, \quad \varepsilon_n \leq t \leq \frac{1 - |z_{n,k}|}{2} \lesssim (n \log 3)^{\eta_1+1+\eta_2}, \quad (3.4) \]

for \(0 \leq k \leq m_n - 1, n \in \mathbb{N}\). On the other hand, for the same range of \(n\) and \(k\), we have

\[ \int_0^{\varepsilon_n} \frac{(n_{z_{n,k}}(t) - 1)^+}{t} \, dt \lesssim m_n \log \frac{1 - |z_{n,k}|}{2\varepsilon_n} \lesssim (n \log 3)^{\eta_1+1+\eta_2}, \quad (3.5) \]

Combining (3.4) and (3.5), we deduce

\[ N_{z_{n,k}} \left(\frac{1 - |z_{n,k}|}{2}\right) = \left(\int_{0}^{\varepsilon_n} + \int_{\varepsilon_n}^{1 - |z_{n,k}|/2} \right) \frac{(n_{z_{n,k}}(t) - 1)^+}{t} \, dt \lesssim (n \log 3)^{\eta_1+1+\eta_2} \]

\[ \sim \log^{\eta_1+1+\eta_2} \frac{1}{1 - |z_{n,k}|}, \quad 0 \leq k \leq m_n - 1, n \to \infty. \]

Thus, assertion (ii) is proved.
To prove assertion (iii), we assume on the contrary that there exists a solution \( f = Be^g \) of (1.1) having the zero sequence \((z_n)\), where \( B \) is the Blaschke product, and such that

\[
\log M(r, a) \leq C_0 \log^{1+\eta} \frac{1}{1-r}, \quad r \in [0,1),
\]

(3.6)

where \( \eta_2 > \eta > 0 \) and \( C_0 \) is a positive constant. The function \( \varphi(r) = \exp \log^{1+\eta} 1/1-r \) has infinite logarithmic order, that is

\[
\limsup_{r \to 1^-} \frac{\log \varphi(r)}{\log \log \frac{1}{1-r}} = \infty.
\]

Using the notation from [5]

\[
\sigma_\varphi(M_{1/2}(r, a)^{1/2} (1-r)) = \limsup_{r \to 1^-} \frac{\log(M_{1/2}(r, a)^{1/2} (1-r))}{\log \varphi(r)}
\]

\[
= \limsup_{r \to 1^-} \frac{\log \left( \frac{1}{2\pi} \int_0^{2\pi} |a(re^{i\theta})|^{1/2} d\theta (1-r) \right)}{\log^{1+\eta} \frac{1}{1-r}}
\]

\[
\leq \limsup_{r \to 1^-} \frac{1}{2} \log M(r, a) + \log(1-r) \leq \frac{1}{2} \log \frac{1}{1-r}
\]

Then by [5, Theorem 1] \( \limsup_{r \to 1^-} \log T(r, f)/\log \varphi(r) \leq C_0/2 \). Hence

\[
\log \log M(r, f) \leq \log \left( T \left( \frac{1+r}{2}, f \right) \frac{2}{1-r} \right) \leq \left( \frac{C_0}{2} + o(1) \right) \log \varphi(r) \sim \frac{C_0}{2} \log^{1+\eta} \frac{1}{1-r}, \quad r \to 1^-.
\]

(3.7)

We write \( R_n = 1 - 2 \cdot 3^{-n-1} \) and \( \delta = \frac{1}{4} \). Taking into account that \( n_{R_n e^{i\theta}} (1 - R_n/4) = N_{R_n e^{i\theta}} (1 - R_n/4) = 0 \), and applying Lemma H, we deduce that

\[
\Re g(R_n e^{i\theta}) \leq \log M(R_n, f) + |\log |B(R_n e^{i\theta})||
\]

\[
\leq \log M(R_n, f) + C \left( \frac{1}{4}, 1 \right) \sum_{j,k} \frac{1 - |z_{j,k}|^2}{|1 - z_{j,k} R_n e^{i\theta}|}
\]

\[
\leq e^{(C_0/2+o(1))} \log^{1+\eta} \sum_{|z_{j,k}| \leq R_n} \frac{1 - |z_{j,k}|}{1 - R_n} + C \left( \frac{1}{4}, 1 \right) \sum_{|z_{j,k}| > R_n} \frac{1 - |z_{j,k}|}{1 - R_n}
\]
\[ \leq \exp \left\{ \left( \frac{C_0}{2} + o(1) \right) \log^{\eta} \frac{1}{1 - R_n} \right\} + O\left( \log^{1+\eta} \frac{1}{1 - R_n} \right) \]

\[ \sim \exp \left\{ \left( \frac{C_0}{2} + o(1) \right) \log^{\eta} \frac{1}{1 - R_n} \right\}, \quad n \to +\infty. \quad (3.8) \]

Since \( \Re g \) is harmonic, \( B(r, \Re g) = \max\{\Re g(re^{i\theta}) : \theta \in [0, 2\pi]\} \) is an increasing function. It follows from (3.8) and the relation \( 1 - R_n \approx 1 - R_{n+1} \) that

\[ B(r, \Re g) \leq \exp \left\{ \left( \frac{C_0}{2} + o(1) \right) \log^{1+\eta} \frac{1}{1 - r} \right\}, \quad r \to 1^-. \]

The last estimate and Caratheodory’s inequality [14, Chapter 1, §6] imply

\[ M(r, g) \leq \exp \left\{ \left( \frac{C_0}{2} + o(1) \right) \log^{1+\eta} \frac{1}{1 - r} \right\}, \quad r \to 1^- . \]

Then, applying Cauchy’s integral formula, we obtain

\[ M(r, g') \leq \frac{C}{1 - r} \exp \left\{ \left( \frac{C_0}{2} + o(1) \right) \log^{1+\eta} \frac{1}{1 - r} \right\}, \quad r \to 1^- , \]

which yields

\[ \log M(r, g') \leq \left( \frac{C_0}{2} + o(1) \right) \log^{1+\eta} \frac{1}{1 - r}, \quad r \to 1^- . \quad (3.9) \]

We write (cf. [13])

\[ -g'(z_{n,0}) = \frac{B''(z_{n,0})}{2B'(z_{n,0})} = \sum_{k=1}^{m_n-1} \frac{1}{z_{n,0} - z_{n,k}} \frac{1}{1 - |z_{n,k}|^2} \]

\[ + \sum_{j \neq n} \sum_{k=0}^{m_j-1} \frac{1}{z_{j,k} - z_{n,0}} \frac{1}{1 - |z_{j,k}|^2} \]

\[ =: I_1 + I_2. \quad (3.10) \]

It is easy to see that

\[ |I_1| \geq \frac{1}{2} m_n \sum_{k=1}^{m_n-1} \frac{1}{k} \]

\[ \geq C \exp \left\{ \log^{n+1} \frac{1}{1 - |z_{n,0}|} \right\} \log^{n} \frac{1}{1 - |z_{n,0}|} \log \log \frac{1}{1 - |z_{n,0}|}, \quad n \to +\infty. \quad (3.11) \]
Then

\[ |I_2| \leq \sum_{j \neq n} \sum_{k=0}^{m_j-1} \frac{1}{|z_{j,k}-z_{n,0}|} \left( \frac{1-|z_{j,k}|}{1-\bar{z}_{j,k}} \right) \]

\[ \leq \sum_{j=1}^{n-1} \sum_{k=0}^{m_j-1} \frac{4}{1-|z_{j,0}|} \sum_{j=n+1}^{\infty} \sum_{k=0}^{m_j-1} \frac{4(1-|z_{j,k}|)}{(1-|z_{n,0}|)^2} \]

\[ \leq 4 \sum_{j=1}^{n-1} 3^{j+1} (j \log 3)^n + \sum_{j=n+1}^{\infty} 3^{-j} (j \log 3)^n \]

\[ \leq C 3^n (n \log 3)^n + \frac{C}{(1-|z_{n,0}|)^2} \frac{(n \log 3)^n}{3^n} \leq \frac{C \log n}{1-|z_{n,0}|}. \]

Therefore

\[ |g'(z_{n,0})| = \left| \frac{B''(z_{2n})}{2B'(z_{2n})} \right| \]

\[ \geq C \exp \left\{ \log^{n_2+1} \frac{1}{1-|z_{n,0}|} \right\} \log^n \frac{1}{1-|z_{n,0}|} \log \log \frac{1}{1-|z_{n,0}|}, \quad n \to +\infty. \]

Hence,

\[ \log M(|z_{n,0}|, g') \geq (1 + o(1)) \log^{n_2+1} \frac{1}{1-|z_{n,0}|}, \quad n \to +\infty. \quad (3.12) \]

This contradicts with (3.9). The theorem is proved. \( \square \)

**Proof of Theorem 2.5.** Let \( D_\rho^+(Z) < D < \infty \). It follows from the definition of \( D_\rho^+(Z) \) that under the assumptions of Theorem G there exist \( R_0 > 0 \) and \( r_0 \in (0, 1) \) such that \( n_z(R \rho(z)) < DR^2 \), \( R > R_0 \) and \( |z| \in (r_0, 1) \). Separation condition (1.8) and the property (1.7) imply that each disk \( U(z, \rho(z)/3) \) contains at most one point \( z_k \), and there exists a constant \( D_1 \geq D \) such that \( n_z(R \rho(z)) \leq D_1 R^2 \) for \( 0 < R \leq R_0 \) and \( |z| \in (r_1, 1) \) for some \( r_1 \in (r_0, 1) \). Therefore

\[ n_z(R \rho(z))0, \quad |z| \in (r_1, 1). \quad (3.13) \]

The last estimate directly implies

\[ n_z \left( \frac{1-|z|}{2} \right) \leq \frac{D_1}{4} \frac{(1-|z|)^2}{\rho^2(z)} = \frac{D_1}{4} (1-|z|)^2 \Delta h(r), \quad |z| = r \in (r_1, 1). \]

We write \( \psi(t) = t^{-2} \Delta h(r) \bigg|_{r=1-t^{-1}}^{t}. \) It follows from the definition of \( \sigma(r) \) that \( \psi \) is a nondecreasing function on \([1, \infty)\) and \( \psi(2t) = O(\psi(t)), \ t \geq 2. \) Let us estimate \( \psi(T) = \)
\[\int_{1}^{T} \psi(t)/t \, dt.\]

We have
\[
\tilde{\psi}(T) = \int_{1}^{T} \frac{\Delta h(r)}{r^{3}} \, dr = \int_{0}^{1-T^{-1}} (1 - r) \Delta h(r) \, dr \\
= \int_{0}^{1-T^{-1}} (1 - r) \frac{(r h'(r))'}{r} \, dr \\
= (1 - r) h'(r) \bigg|_{0}^{1-T^{-1}} + \int_{0}^{1-T^{-1}} \frac{h'(r)}{r} \, dr \\
\leq \frac{h'(1-T^{-1})}{T} + 2h(1-T^{-1}), \quad T \geq 2.
\]

Since \(h'(r)/h(r) = O((1-r)^{-1})\), \(r \to 1-\), we deduce that
\[
\tilde{\psi}\left(\frac{1}{1-r}\right) = O(h(r)), \quad r \to 1-.
\]

We then estimate \(N_{z_k}(t)\) for \(t \leq (1-|z_k|)/2\). Using the separation condition (1.8) and estimate (3.13), we deduce
\[
N_{z_k}((1-|z_k|)/2) = \int_{0}^{(1-|z_k|)/2} \frac{(n_\zeta(t) - 1)^+}{t} \, dt \leq \int_{\rho(z_k)/2}^{(1-|z_k|)/2} \frac{n_\zeta(t)}{t} \, dt \\
= \int_{1/2}^{r(1-|z_k|)/2} \frac{n_\zeta(\rho(z_k)\tau)}{\tau} \, d\tau \leq \int_{1/2}^{r(1-|z_k|)/2(2\rho(z_k))} D_1 \tau \, d\tau \\
\leq \frac{D_1}{8} \frac{(1-|z_k|)^2}{\rho^2(z_k)} = \frac{1}{2} \tilde{\psi}\left(\frac{1}{1-r}\right).
\]

Since \(n_\zeta((1-|z|)/2) = O(\psi(1/1-|z|))\), \(|z| \in (r_1, 1)\), applying Lemma 9 from [3], we get the following estimate of the canonical product
\[
\log |P(z)| \leq C\tilde{\psi}\left(\frac{1}{1-|z|}\right) \leq Ch(|z|). \quad (3.14)
\]

Any analytic function \(f\) in \(\mathbb{D}\) with the zero sequence \(Z = (z_k)\) can be written in the form \(f(z) = P(z)e^{g(z)}\), where \(g\) is analytic in \(\mathbb{D}\). If \(f\) is a solution of (1.1), then
\[
P'' + 2P'g' + (g'^2 + g' + a)P = 0, \quad (3.15)
\]
and, consequently
\[
g'(z_k) = -\frac{P''(z_k)}{2P'(z_k)} =: b_k, \quad k \in \mathbb{N}. \quad (3.16)
\]

Therefore, in order to find a solution of (1.1) with the zero sequence \(Z\), we have to find an analytic function \(h = g'\) solving the interpolation problem \(h(z_k) = b_k, \quad k \in \mathbb{N}\). Using
Cauchy’s integral theorem and (3.14), we deduce
\[
|P''(z_k)| \leq \frac{8}{(1 - |z_k|)^2} \max_{|z|=1+|z_k|/2}|P(z)| \leq \frac{8}{(1 - |z_k|)^2}e^{C\tilde{\psi}(2/1-|z_k|)}.
\]
On the other hand, (3.2) and (3.3) imply (cf. (3.1)) that
\[
\frac{1}{|P''(z_k)|} \leq (1 - |z_k|)e^{C\tilde{\psi}(1/1-|z_k|)}.
\]
Hence
\[
|b_k| = \frac{P''(z_k)}{2P'(z_k)} \leq \frac{4}{1 - |z_k|}e^{C\tilde{\psi}(1/1-|z_k|)}
= e^{C\tilde{\psi}(2/1-|z_k|)+\log\frac{4}{1 - |z_k|}} \leq e^{C\tilde{\psi}(1/1-|z_k|)}, \quad k \in \mathbb{N},
\]
because \(\tilde{\psi}(t)/\log t \rightarrow +\infty \ (t \rightarrow +\infty)\). Since the assumptions of Theorem F are satisfied, there exists a function \(h\) analytic in \(\mathbb{D}\) such that \(h(z_k) = b_k\) and \(\log M(r, h) \leq C\tilde{\psi}(1/1-r), r \rightarrow 1-\), i.e. \(\log M(r, g') \leq C\tilde{\psi}(1/1-r), r \rightarrow 1-\).

Then, applying Cauchy’s integral theorem once more, we get that
\[
M(r, g'') \leq \frac{2}{1-r}M\left(\frac{1 + r}{2}, g'\right) \leq e^{C\tilde{\psi}(1/1-r)}, \quad r \rightarrow 1-.
\]
From (3.15), we obtain
\[
|a(z)| \leq \frac{|P''(z)|}{|P'(z)|} + 2|g'(z)| \frac{|P'(z)|}{|P'(z)|} + |g'(z)|^2 + |g''(z)|.
\]
It follows from results of [4] (or [6]) that for any \(\delta > 0\) there exists a set \(E_\delta \subset [0, 1)\) such that
\[
\max \left\{ \frac{|P''(z)|}{|P'(z)|}, \frac{|P'(z)|}{|P'(z)|} \right\} \leq \frac{1}{(1 - |z|)^q}, \quad |z| \in [0, 1) \setminus E_\delta,
\]
where \(q \in (0, +\infty), \) and \(m_1(E_\delta \cap [r, 1)) \leq \delta (1-r)\) as \(r \uparrow 1\). Thus,
\[
|a(z)| \leq e^{C\tilde{\psi}(1/1-|z|)}, \quad |z| \in [0, 1) \setminus E. \tag{3.17}
\]
Since \(M(r, a)\) increases, condition (1.3) and Lemma 4.1 from [6] imply that inequality (3.17) holds for all \(z \in \mathbb{D}\) for an appropriate choice of \(\tilde{C}\). \hfill \Box

**Remark 3.1.** Though the hypotheses of Theorem 2.5 are similar to those of Theorem G, we do not use the interpolating function constructed in [2]. We need an interpolating function \(f\) to have the property \(f(z) \asymp \text{dist}(z, Z_f)p(z)\), where \(Z_f\) is the zero set of \(F\), \(p(z)\) is some nonvanishing continuous function in \(\mathbb{D}\). But it seems that it is not the case.

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