Abstract

We study the existence of fully nontrivial solutions to the system

\[-\Delta u_i + \lambda_i u_i = \sum_{j=1}^{\ell} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i \quad \text{in } \Omega, \quad i = 1, \ldots, \ell,\]

in a bounded or unbounded domain \(\Omega\) in \(\mathbb{R}^N\), \(N \geq 3\). The \(\lambda_i\)'s are real numbers, and the nonlinear term may have subcritical \((1 < p < \frac{N}{N-2})\), critical \((p = \frac{N}{N-2})\), or supercritical growth \((p > \frac{N}{N-2})\). The matrix \((\beta_{ij})\) is symmetric and admits a block decomposition such that the diagonal entries \(\beta_{ii}\) are positive, the interaction forces within each block are attractive (i.e., all entries \(\beta_{ij}\) in each block are non-negative) and the interaction forces between different blocks are repulsive (i.e., all other entries are non-positive). We obtain new existence and multiplicity results of fully nontrivial solutions, i.e., solutions where every component \(u_i\) is non-trivial. We also find fully synchronized solutions (i.e., \(u_i = c_i u_1\) for all \(i = 2, \ldots, \ell\)) in the purely cooperative case whenever \(p \in (1, 2)\).

**Keywords:** Weakly coupled systems; mixed cooperation and competition; positive and sign-changing solutions; Nehari manifold.

**MSC2020:** 35J47, 35A15.

1 Introduction

We consider the system of nonlinear elliptic equations

\[
\begin{aligned}
-\Delta u_i + \lambda_i u_i &= \sum_{j=1}^{\ell} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i, \\
u_i &\in H, \quad i = 1, \ldots, \ell,
\end{aligned}
\]

where \(H\) is either \(H_0^1(\Omega)\) or \(D_0^{1,2}(\Omega)\), \(\Omega\) is an open subset of \(\mathbb{R}^N\), \(N \geq 3\), \(\lambda_i \in \mathbb{R}\) and \(p > 1\). We assume that

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the operators $-\Delta + \lambda_i$ are well defined and coercive in $H$ for all $i = 1, \ldots, \ell$, 

$(B_1)$ the matrix $(\beta_{ij})$ is symmetric and admits a block decomposition as follows: For some $1 < q < \ell$ there exist $0 = \ell_0 < \ell_1 < \cdots < \ell_{q-1} < \ell_q = \ell$ such that, if we set 

$I_h := \{i \in \{1, \ldots, \ell\} : \ell_{h-1} < i \leq \ell_h\}$, 

$I_h := I_h \times I_h, \quad \mathcal{K}_h := \{(i, j) \in I_h \times I_h : k \in \{1, \ldots, q\} \setminus \{h\}\}$, 

then $\beta_{ii} > 0$, 

$\beta_{ij} \geq 0$ if $(i, j) \in I_h$ and $\beta_{ij} \leq 0$ if $(i, j) \in \mathcal{K}_h, \ h = 1, \ldots, q$. 

This type of systems models some physical phenomena in nonlinear optics and describes the behavior of multi-component Bose-Einstein condensates. The coefficient $\beta_{ij}$ represents the interaction force between the components $u_i$ and $u_j$. The sign of $\beta_{ij}$ determines whether the interaction is attractive or repulsive. If $\beta_{ij} \geq 0$ for all $i \neq j$ (i.e., if $q = 1$) the system (1.1) is called purely cooperative, and it is called purely competitive if $\beta_{ij} \leq 0$ for all $i \neq j$ (i.e., if $q = \ell$).

In the past fifteen years, systems that are either purely cooperative or purely competitive have been extensively studied, particularly those with cubic nonlinearity (i.e., with $p = 2$). It is convenient to consider other powers, specially when dealing with critical systems; see, e.g., [6–9]. We refer to the introduction of the papers [4,10] for an overview on the topic and an ample list of references.

Systems with mixed couplings were considered in the seminal paper [13] by Lin and Wei and more recently in [5,14,15,17–20]. All of these works treat only the cubic nonlinearity $p = 2$. In the present paper, we are mainly concerned with the case $p < 2$.

According to the decomposition given by $(B_1)$, we shall write a solution $u = (u_1, \ldots, u_\ell)$ to (1.1) in block-form as 

$u = (\bar{u}_1, \ldots, \bar{u}_q)$ with $\bar{u}_h = (u_{\ell_{h-1}+1}, \ldots, u_{\ell_h})$.

$u$ is called semitrivial if some but not all of its components $u_i$ are zero and it is said to be fully nontrivial if every component $u_i$ is different from zero. We shall call it block-wise nontrivial if at least one component of each block $\bar{u}_h$ is nontrivial.

We prove the following result.

**Theorem 1.1.** Assume $(A_1)$ and $(B_1)$. Assume further that 

$(A_2)$ the embedding $H \hookrightarrow L^{2p}(\Omega)$ is compact.

Then, the system (1.1) has a least energy block-wise nontrivial solution.

The precise meaning of least energy block-wise nontrivial solution is given in Definition 2.1. The proof of this result is obtained by adapting the variational approach introduced in [9], and is given in Section 2.
If the system is purely competitive (i.e., \( q = \ell \)) any block-wise nontrivial solution is fully nontrivial. On the other hand, for any choice of \( i_h \in I_h \), every solution to the purely competitive system

\[
\begin{cases}
-\Delta v_h + \lambda_{i_h} v_h = \sum_{k=1}^{\ell} \beta_{i_h,k} |v_k|^p |v_h|^{p-2} v_h, \\
v_h \in H, \quad h = 1, \ldots, q,
\end{cases}
\]

gives rise to a block-wise nontrivial solution of (1.1) whose \( i_h \)-th component is \( v_h \), \( h = 1, \ldots, q \), and all other components are 0. If \( q \neq \ell \) this solution is not fully nontrivial. The following result, whose proof is given in Section 3, provides existence of a fully nontrivial solution.

**Theorem 1.2.** Assume (\( A_1 \)) and (\( B_1 \)), and let \( p \leq 2 \). There exists a positive constant \( C_* \) independent of \( (\beta_{ij}) \) - but depending on \( \Omega, \lambda_i, p \) and \( q \) - with the property that, if (\( B_2 \)) either \( p < 2 \) and

\[
\min_{i,j \in I_h \atop i \neq j} \beta_{ij} \left( \min_{k=1, \ldots, q \atop \beta_{ii} \in I_k} \max_{i,j \in I_k} \beta_{ij} \right)^{\frac{p}{p-1}} > C_* (\ell_h - \ell_{h-1} - 1)^{\frac{2p}{p-1}} \sum_{(i,j) \in K_h} |\beta_{ij}|,
\]

for every \( h = 1, \ldots, q \),

or \( p = 2 \), \( \lambda_i =: a_h \) for all \( i \in I_h \), \( \beta_{ij} =: b_h \) for all \( i, j \in I_h \) with \( i \neq j \) and

\[
b_h > \max_{i \in I_h} \beta_{ii} + C_* \left( \frac{\ell_h - \ell_{h-1} - 1}{\min_{k=1, \ldots, q \atop \beta_{ii} \in I_k} \max_{i,j \in I_k} \beta_{ij}} \right)^2 \max_{i,j \in I_h} \sum_{m \in I_h} |\beta_{im} - \beta_{jm}| \]

for every \( h = 1, \ldots, q \),

then every least energy block-wise nontrivial solution to the system (1.1) is fully nontrivial.

If \( \Omega \) is a bounded domain, assumption (\( A_1 \)) holds true if \( \lambda_i > -\lambda_1(\Omega) \), where \( \lambda_1(\Omega) \) is the first Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \), and (\( A_2 \)) is satisfied if the nonlinear term is subcritical. So combining Theorems 1.1 and 1.2 we obtain the following result.

**Theorem 1.3.** If \( \Omega \) is bounded, \( \lambda_i > -\lambda_1(\Omega) \) for all \( i = 1, \ldots, \ell \), \( 1 < p < \frac{N}{N-2} \) and \( (\beta_{ij}) \) satisfies (\( B_1 \)) and (\( B_2 \)), the system

\[
\begin{cases}
-\Delta u_i + \lambda_i u_i = \sum_{j=1}^{\ell} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i, \\
u_i \in H^1_0(\Omega), \quad i = 1, \ldots, \ell,
\end{cases}
\]

has a fully nontrivial solution.
It is well known that compactness is more likely to hold true in a symmetric setting. Symmetries are also helpful to obtain sign-changing solutions. Symmetric versions of Theorems 1.1 and 1.2 yield the following results.

**Theorem 1.4.** If \(1 < p < \frac{N}{N-2}\) and \((\beta_{ij})\) satisfies \((B_1)\) and \((B_2)\), the system
\[
\begin{aligned}
-\Delta u_i + u_i &= \sum_{j=1}^{\ell} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i, \\
u_i &\in H^1(\mathbb{R}^N), \quad i = 1, \ldots, \ell,
\end{aligned}
\]
has a fully nontrivial solution whose components are positive and radial.

If \(N = 4\) or \(N \geq 6\) it has also a fully nontrivial solution whose components are nonradial and change sign.

**Theorem 1.5.** If \(p = \frac{N}{N-2}\) and \((\beta_{ij})\) satisfies \((B_1)\) and \((B_2)\), the critical system
\[
\begin{aligned}
-\Delta u_i &= \sum_{j=1}^{\ell} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i, \\
u_i &\in D^{1,2}(\mathbb{R}^N), \quad i = 1, \ldots, \ell,
\end{aligned}
\]
has a fully nontrivial solution whose components are positive.

If \(N = 3\) or \(N \geq 5\), it has also a fully nontrivial solution whose components change sign.

The proof of the last two theorems and further examples are given in Section 4. They include, for instance, existence and multiplicity results for \((1.1)\) with supercritical nonlinearities \((p > \frac{N}{N-2})\), or in an exterior domain.

Assumption \((A_2)\) may be considerably weakened. As we shall see below, the solution given by Theorem 1.1 minimizes a \(C^1\)-functional \(\Psi : \mathcal{U} \to \mathbb{R}\) defined on an open subset \(\mathcal{U}\) of a smooth Hilbert manifold. So compactness is only needed at the level \(c_0 := \inf_{\mathcal{U}} \Psi\); see Theorem 2.5 for the weaker statement.

For \(p = 2\) our condition \((B_2)\) is basically the same as in [18, Theorem 1.5] and it is weaker than the one in [18, Theorem 1.4]. Our approach, however, is different and it has the advantage that it can be used to treat the case \(p < 2\).

If the system \((1.1)\) is purely cooperative (i.e., \(q = 1\)) and \(p < 2\), assumption \((B_2)\) is satisfied and, so, Theorems 1.1 and 1.2 yield the existence of a fully nontrivial solution. This stands in contrast with the situation for \(p = 2\) where purely cooperative systems do not always have a positive solution; see, e.g., [2, Theorem 0.2] or [16, Theorem 1]. For purely cooperative systems with \(p = 2\) our condition \((B_2)\) is basically that in [13, Corollary 2.3].

Finally, we obtain a new result concerning existence of synchronized solutions when all the \(\lambda_i\)'s coincide, i.e., \(\lambda_i := \lambda\) for all \(i = 1, \ldots, \ell\). We say that \(u = (u_1, \ldots, u_\ell)\) is a fully synchronized solution if \(u_i = c_i u\), where \(u\) is a nontrivial solution to the single equation
\[
-\Delta u + \lambda u = |u|^{2p-2} u, \quad u \in H,
\]
and $c = (c_1, \ldots, c_\ell) \in \mathbb{R}^\ell$ solves the algebraic system

\begin{equation}
(1.2) \quad c_i = \sum_{j=1}^{\ell} \beta_{ij} |c_j|^p |c_i|^{p-2} c_i, \quad c_i > 0, \quad \text{for every } i = 1, \ldots, \ell.
\end{equation}

There are some results concerning the solvability of (1.2). The easiest case is when $p = 2$ and $\ell = 2$. Then a solution to (1.2) exists if and only if

$$
\beta_{12} \in (-\sqrt{11 \beta_{12}}, \min\{\beta_{11}, \beta_{22}\}) \cup (\max\{\beta_{11}, \beta_{22}\}, +\infty).
$$

If $p = 2$ and $\ell \geq 2$, a solution to (1.2) exists when $\beta_{ij} := \beta$ for all $i \neq j$ and $\beta \in (\bar{\beta}, \min\{\beta_{ii}\}) \cup (\max\{\beta_{ii}\}, +\infty)$ for some $\bar{\beta} < 0$ (see [1, Proposition 2.1]), while if $p = \frac{N}{N-2} < 2$ and $\ell = 2$ a solution to (1.2) always exists provided $\beta_{12} > 0$ (see [7, Theorem 1.1]). The following theorem complements these results.

**Theorem 1.6.** Let $p < 2$ and assume that the system (1.1) is purely cooperative (i.e., $\beta_{ii} > 0$ and $\beta_{ij} \geq 0$ for all $i, j = 1, \ldots, \ell, i \neq j$) and that $\lambda_i = \lambda$ for all $i = 1, \ldots, \ell$. Then, for each $i$, there exists $c_i > 0$ such that $(c_1 u, \ldots, c_\ell u)$ is a solution to (1.1) for every solution $u$ to the equation

$$
-\Delta u + \lambda u = |u|^{2p-2} u, \quad u \in H.
$$

The proof of this result relies on a simple minimization argument and it is given in Section 5.

## 2 A simple variational approach

We assume throughout that $(A_1)$ and $(B_1)$ hold true.

Recall that $H$ is either $H_0^1(\Omega)$ or $D_0^{1,2}(\Omega)$. Assumption $(A_1)$ asserts that

$$
\|v\| := \left( \int_{\Omega} (|\nabla v|^2 + \lambda_i v^2) \right)^{1/2}
$$

is a norm in $H$, equivalent to the standard one. Let $H_\ell$ denote the space $H$ equipped with this norm and, for the partition in assumption $(B_1)$, define

$$
\mathcal{H}_h := H_{\ell_{h-1}+1} \times \cdots \times H_{\ell_h}, \quad h = 1, \ldots, q,
$$

$$
\mathcal{H} := H_1 \times \cdots \times H_\ell = H_1 \times \cdots \times \mathcal{H}_q.
$$

A point in $\mathcal{H}_h$ will be denoted by $\bar{u}_h$, a point in $\mathcal{H}$ by

$$
u = (\bar{u}_1, \ldots, \bar{u}_q) = (u_1, \ldots, u_\ell) \quad \text{with } \bar{u}_h \in \mathcal{H}_h, \quad u_i \in H_i,
$$

and their norms by

$$
\|\bar{u}_h\| := \left( \sum_{i \in \ell_h} \|u_i\|_i^2 \right)^{1/2} \quad \text{and} \quad \|\nu\| := \left( \sum_{h=1}^q \|\bar{u}_h\|^2 \right)^{1/2}.
$$
Let $J : H \to \mathbb{R}$ be the functional given by

$$J(u_1, \ldots, u_d) := \frac{1}{2} \sum_{i=1}^{d} \|u_i\|^2 - \frac{1}{2p} \sum_{i,j=1}^{d} \beta_{ij} \int_{\Omega} |u_i|^p |u_j|^p.$$

This functional is of class $C^1$ and its critical points are the solutions to the system (1.1). The block-wise nontrivial solutions belong to the set

$$\mathcal{N} := \{u \in H : \|\bar{u}_h\| \neq 0 \text{ and } \partial_{u_h} J(u) \bar{u}_h = 0 \text{ for all } h = 1, \ldots, q\}.$$

Note that

$$\partial_{u_h} J(u) \bar{u}_h = \|\bar{u}_h\|^2 - \sum_{(i,j) \in \mathcal{I}_h} \int_{\Omega} \beta_{ij} |u_i|^p |u_j|^p - \sum_{(i,j) \in \mathcal{K}_h} \int_{\Omega} \beta_{ij} |u_i|^p |u_j|^p,$$

with $\mathcal{I}_h$ and $\mathcal{K}_h$ as defined in assumption $(B_1)$, and that

$$J(u) = (\frac{1}{2} - \frac{1}{2p}) \|u\|^2 \text{ if } u \in \mathcal{N}.$$

**Definition 2.1.** A block-wise nontrivial solution $u$ to the system (1.1) such that $J(u) = \inf_{u \in \mathcal{N}} J$ will be called a least energy block-wise nontrivial solution.

In fact, we will show that any minimizer of $J$ on $\mathcal{N}$ is a critical point of $J$, i.e., a block-wise nontrivial solution to (1.1). We follow the approach introduced in [9].

**Lemma 2.2.** (i) There exists $d_0 > 0$ such that $\min_{h=1, \ldots, q} \|\bar{u}_h\|^2 \geq d_0$ for every $(\bar{u}_1, \ldots, \bar{u}_q) \in \mathcal{N}$. As a consequence, we have that $\mathcal{N}$ is closed in $H$ and $\inf_{u \in \mathcal{N}} J(u) > 0$.

(ii) There exists $d_1 > 0$ independent of $(\beta_{ij})$ such that

$$\inf_{u \in \mathcal{N}} J(u) \leq d_1 \left( \min_{h=1, \ldots, q} \max_{i \in I_h} \beta_{ii} \right)^{-\frac{1}{p}}.$$

**Proof.** (i) : If $(\bar{u}_1, \ldots, \bar{u}_q) \in \mathcal{N}$, the Hölder and the Sobolev inequalities yield

$$\|\bar{u}_h\|^2 \leq \sum_{(i,j) \in \mathcal{I}_h} \int_{\Omega} \beta_{ij} |u_i|^p |u_j|^p \leq C \|\bar{u}_h\|^{2p} \text{ for every } h = 1, \ldots, q.$$

Hence, there exists $d_0 > 0$ such that $\|\bar{u}_h\|^2 \geq d_0$ for every $h = 1, \ldots, q$.

(ii) : Fix $i_h \in I_h$ such that $\beta_{i_hi_h} = \max\{\beta_{ii} : i \in I_h\}$. Let

$$\mathcal{M} := \{(v_1, \ldots, v_q) \in H^q : v_h \neq 0, \|v_h\|_{i_h}^2 = \int_{\Omega} |v_h|^2p, v_kv_k = 0 \text{ if } h \neq k\}$$

and define

$$d_1 := \left(\frac{1}{2} - \frac{1}{2p}\right) \inf_{(v_1, \ldots, v_q) \in \mathcal{M}} \sum_{h=1}^{q} \|v_h\|_{i_h}^2.$$
Given \((v_1, \ldots, v_q) \in \mathcal{M}\), let \(\bar{u}_h \in \mathcal{H}_h\) be the function whose \(i_h\)-th component is \(\beta_i^{1/h} v_h\) and all other components are 0. Then, \(u = (\bar{u}_1, \ldots, \bar{u}_q) \in \mathcal{N}\) and

\[
\inf_{u \in \mathcal{N}} \mathcal{J}(u) \leq \mathcal{J}(u) = \left(\frac{1}{2} - \frac{1}{2p}\right) \sum_{h=1}^{q} \beta_i^{1/h} \|v_h\|_{1,h}^2 \\
\leq \left(\frac{1}{2} - \frac{1}{2p}\right) \left(\min_{h} \beta_i^{1/h}\right) - \frac{1}{p+1} \sum_{h=1}^{q} \|v_h\|_{1,h}^2,
\]

and the inequality in (ii) follows.

Given \(u = (\bar{u}_1, \ldots, \bar{u}_q) \in \mathcal{H}\) and \(s = (s_1, \ldots, s_q) \in (0, \infty)^q\), we write

\(su := (s_1 \bar{u}_1, \ldots, s_q \bar{u}_q)\).

Let \(\mathcal{S}_h := \{ \bar{u} \in \mathcal{H}_h : \|\bar{u}\| = 1 \} \) and \(\mathcal{T} := \mathcal{S}_1 \times \cdots \times \mathcal{S}_q\). Define

\(\mathcal{U} := \{ u \in \mathcal{T} : su \in \mathcal{N} \text{ for some } s \in (0, \infty)^q \}\).

Arguing as in [9, Proposition 3.1 and Theorem 3.3] one obtains the following two lemmas. We include their proof for the sake of completeness.

**Lemma 2.3.** (i) Let \(u \in \mathcal{T}\). If there exists \(s_u \in (0, \infty)^q\) such that \(su \in \mathcal{N}\), then \(su\) is unique and satisfies

\[\mathcal{J}(su) = \max_{s \in (0, \infty)^q} \mathcal{J}(su).\]

(ii) \(\mathcal{U}\) is an nonempty open subset of \(\mathcal{T}\), and the map \(\mathcal{U} \to (0, \infty)^q\) given by \(u \mapsto su\) is continuous.

(iii) The map \(\mathcal{U} \to \mathcal{N}\) given by \(u \mapsto su\) is a homeomorphism.

(iv) If \((u_n)\) is a sequence in \(\mathcal{U}\) and \(u_n \to u \in \partial \mathcal{U}\), then \(s_{u_n} \to \infty\).

**Proof.** Given \(u \in \mathcal{H}\) we define \(J_u : (0, \infty)^q \to \mathbb{R}\) by \(J_u(s) := \mathcal{J}(su)\). Then

\[s_h \partial_h J_u(s) = \partial_{\bar{u}_h} \mathcal{J}(su)[s_h \bar{u}_h], \quad h = 1, \ldots, q.
\]

So, if \(\|\bar{u}_h\| \neq 0\) for every \(h = 1, \ldots, q\), then \(su \in \mathcal{N}\) iff \(s\) is a critical point of \(J_u\). The function \(J_u\) can be written as

\[J_u(s) = \sum_{h=1}^{q} a_{u,h} s_h^2 - \sum_{h=1}^{q} b_{u,h} s_h^{2p} + \sum_{k=1}^{q} d_{u,hk} s_k^p s_h^p,
\]

with

\[a_{u,h} := \frac{1}{2} \|\bar{u}_h\|^2, \quad b_{u,h} := \frac{1}{2p} \sum_{(i,j) \in I_h \times I_h} \beta_{ij} \int_{\Omega} |u_i|^p |u_j|^p \]

and

\[d_{u,hk} := -\frac{1}{2p} \sum_{(i,j) \in I_h \times I_k} \beta_{ij} \int_{\Omega} |u_i|^p |u_j|^p.
\]
Lemma 2.4. (i) If \( u \in \mathcal{T} \), then \( a_{u,h} > 0 \). Assumption \((B_1)\) implies that \( b_{u,h} > 0 \) and \( d_{u,h,k} \geq 0 \). By [9, Lemma 2.2], if \( J_u \) has a critical point \( s_u \in (0, \infty)^i \), then it is unique and it is a global maximum of \( J_u \) in \((0, \infty)^i\).

(ii) Let \( v_1, \ldots, v_q \in H \) be such that \( \|v_h\|_{\ell_h} = 1 \) and \( v_h \) and \( v_k \) have disjoint supports if \( h \neq k \), and let \( s_h := (\beta_{\ell_k, \ell_h} \int_{\Omega} |v_h|^{2p})^{-1/(2p-2)} \). Set \( \bar{u}_h := (0, \ldots, 0, v_h) \), \( u := (\bar{u}_1, \ldots, \bar{u}_q) \) and \( s := (s_1, \ldots, s_q) \). Then, \( u \in \mathcal{T} \) and \( s_u \in \mathcal{N} \).

Hence, \( \mathcal{U} \neq \emptyset \).

As \( a_{u,h}, b_{u,h}, d_{u,h,k} \) are continuous functions of \( u \), [9, Lemma 2.3] implies that \( \mathcal{U} \) is open and that the map \( \mathcal{U} \to (0, \infty)^q \) given by \( u \mapsto s_u \) is continuous.

(iii) It follows from (ii) that the map \( \mathcal{U} \to \mathcal{N} \) given by \( u \mapsto s_u u \) is continuous. Its inverse is

\[
(\bar{u}_1, \ldots, \bar{u}_q) \mapsto \left( \frac{\bar{u}_1}{\|\bar{u}_1\|}, \ldots, \frac{\bar{u}_q}{\|\bar{u}_q\|} \right),
\]

which is well defined and continuous.

(iv) Let \( (u_n) \) be a sequence in \( \mathcal{U} \) such that \( u_n \to u \in \partial \mathcal{U} \). If the sequence \( (s_{u_n}) \) were bounded, after passing to a subsequence we would have \( s_{u_n} \to s \). Since \( \mathcal{N} \) is closed, this would imply that \( s_u \in \mathcal{N} \) and, hence, that \( u \in \mathcal{U} \). This is impossible because \( \mathcal{U} \) is open in \( \mathcal{T} \).

Define \( \Psi : \mathcal{U} \to \mathbb{R} \) by

\[
(2.1) \quad \Psi(u) := J(s_u u) = \left( \frac{1}{q} - \frac{1}{2p} \right) \|s_u\|^2.
\]

As \( \mathcal{U} \) is an open subset of the smooth Hilbert submanifold \( \mathcal{T} \) of \( \mathcal{H} \), we may ask whether \( \Psi \) is differentiable. As we shall see below, it is in fact \( C^1 \). We write \( \|\Psi'(u)\|_* \) for the the norm of \( \Psi'(u) \) in the cotangent space \( T_u^*(\mathcal{T}) \) to \( \mathcal{T} \) at \( u \), i.e.,

\[
\|\Psi'(u)\|_* := \sup_{\substack{v \in T_u(\mathcal{U}) \setminus \{0\} \atop \|v\| = 1}} \frac{|\Psi'(u)v|}{\|v\|},
\]

where \( T_u(\mathcal{U}) \) is the tangent space to \( \mathcal{U} \) at \( u \).

Recall that a sequence \( (u_n) \) in \( \mathcal{U} \) is called a \((PS)_c\)-sequence for \( \Psi \) if \( \Psi(u_n) \to c \) and \( \|\Psi'(u_n)\|_* \to 0 \), and \( \Psi \) is said to satisfy the \((PS)_c\)-condition if every such sequence has a convergent subsequence. Similarly, a \((PS)_c\)-sequence for \( \mathcal{J} \) is a sequence \( (u_n) \) in \( \mathcal{H} \) such that \( \mathcal{J}(u_n) \to 0 \) and \( \|\mathcal{J}'(u_n)\|_{H^{-1}} \to 0 \), and \( \mathcal{J} \) satisfies the \((PS)_c\)-condition if any such sequence has a convergent subsequence.

Lemma 2.4. (i) \( \Psi \in C^1(\mathcal{U}, \mathbb{R}) \),

\[
\Psi'(u)v = \mathcal{J}'(s_u u)[s_u v] \quad \text{for all } u \in \mathcal{U} \text{ and } v \in T_u(\mathcal{U}),
\]

and there exists \( d_0 > 0 \) such that

\[
d_0 \|\mathcal{J}'(s_u u)\|_{H^{-1}} \leq \|\Psi'(u)\|_* \leq \|s_u\|_{\infty}\|\mathcal{J}'(s_u u)\|_{H^{-1}} \quad \text{for all } u \in \mathcal{U},
\]

where \( |s|_{\infty} = \max\{|s_1|, \ldots, |s_q|\} \) if \( s = (s_1, \ldots, s_q) \).
(i) If $(u_n)$ is a $(PS)_c$-sequence for $\Psi$, then $(s_n u_n)$ is a $(PS)_c$-sequence for $J$.

(ii) $u$ is a critical point of $\Psi$ if and only if $s_n u$ is a critical point of $\mathcal{J}$.

(iv) If $(u_n)$ is a sequence in $\mathcal{U}$ and $u_n \to u \in \partial \mathcal{U}$, then $\|\Psi(u_n)\| \to \infty$.

Proof. (i): Let $u \in U$, $v \in T_u(U)$, and $\gamma : (-\varepsilon, \varepsilon) \to U$ be smooth and such that $\gamma(0) = u$ and $\gamma'(0) = v$. Fix $t \in (-\varepsilon, \varepsilon)$. Recalling that $\mathcal{J}(s_n u) = \max_{s \in (0, \varepsilon)} \mathcal{J}(s u)$ and applying the mean value theorem to the function $\tau \mapsto \mathcal{J}(s_n u) \gamma'(\tau t)$ we obtain
\[
\Psi(\gamma(t)) - \Psi(u) = \mathcal{J}(s_n u)_\gamma(t) - \mathcal{J}(s_n u) \leq \mathcal{J}(s_n u) \gamma(t) - \mathcal{J}(s_n u) = t \mathcal{J}'(s_n u) \gamma'(\tau t). \]

for some $\tau_1 \in (0, 1)$. Similarly,
\[
\Psi(\gamma(t)) - \Psi(u) \geq t \mathcal{J}'(s_n u) \gamma'(\tau_2 t) \]

for some $\tau_2 \in (0, 1)$. Therefore,
\[
\Psi'(u) v = \lim_{t \to 0} \frac{\Psi(\gamma(t)) - \Psi(u)}{t} = \mathcal{J}'(s_n u)[s_n v].
\]

It follows that $\Psi$ is of class $C^1$ in $\mathcal{U}$.

As $T_u(U) = \{ (\bar{u}_1, \ldots, \bar{u}_q) \in H : \bar{u}_h = 0 \text{ for each } h = 1, \ldots, q \}$, we have that $H = T_u(U) \oplus \{ (t_1 \bar{u}_1, \ldots, t_q \bar{u}_q) : t_i \in \mathbb{R} \}$ for every $u = (\bar{u}_1, \ldots, \bar{u}_q) \in U$. Since $s_n u \in N$ and $s_h > 0$, we conclude that
\[
\sup_{v \in T_u(U), v \neq 0} \frac{\mathcal{J}'(s_n u)[s_n v]}{\|s_n u\|} = \sup_{w \in H, w \neq 0} \frac{\mathcal{J}'(s_n u)[w]}{\|w\|}.
\]

On the other hand, for every $v \in T_u(U), v \neq 0$, we have that
\[
\min_h s_n u, h \frac{\mathcal{J}'(s_n u)[s_n v]}{\|s_n u\|} \leq \frac{\Psi'(u) v}{\|v\|} \leq \frac{\mathcal{J}'(s_n u)[s_n v]}{\|v\|} \leq \max_h s_n u, h \frac{\mathcal{J}'(s_n u)[s_n v]}{\|s_n u\|},
\]

where $s_n u = (s_n u_1, \ldots, s_n u_d)$. By Lemma 2.2, $\min_{h=1,\ldots,q} s_n u_h \geq d_0$ for every $u \in U$. Taking the supremum over all $v \in T_u(U), v \neq 0$, we obtain the inequalities stated in (i).

Statements (ii) and (iii) follow immediately from (i), and statement (iv) follows from Lemma 2.3(iv) and (2.1).

**Theorem 2.5.** Assume $(A_1)$ and $(B_1)$. If $\mathcal{J}$ satisfies the $(PS)_{c_0}$-condition at $c_0 := \inf_N \mathcal{J}$, then the system (1.1) has a least energy block-wise nontrivial solution.
Proof. Let \((w_n)\) be a minimizing sequence for \(\Psi\) in \(\mathcal{U}\). Lemma 2.4(iv) implies that \(\mathcal{U}\) is positively invariant under the negative pseudogradient flow of \(\Psi\), so the deformation lemma [21, Lemma 5.15] and Ekeland’s variational principle [21, Lemma 8.5] hold true and we may assume that \((w_n)\) is a \((PS)_{c_{0}}\)-sequence for \(\Psi\). As \(\mathcal{J}\) satisfies the \((PS)_{c_{0}}\)-condition, we derive from Lemmas 2.4(ii) and 2.3(iii) that, passing to a subsequence, \(w_n \to w\) and \(w\) is a minimum of \(\Psi\). Then \(u := s_{w}w\) is a minimum of \(\mathcal{J}\) on \(\mathcal{N}\) and Lemma 2.4(iii) asserts that it is a critical point of \(\mathcal{J}\). \(\square\)

Proof of Theorem 1.1. A standard argument shows that, if assumption \((A_2)\) holds true, then \(\mathcal{J}\) satisfies the \((PS)_{c_{0}}\)-condition. So this result follows from Theorem 2.5. \(\square\)

3 Existence of a fully nontrivial solution

We assume throughout \((A_1)\) and \((B_1)\). Our aim is to show that, if \((B_2)\) holds true, no semitrivial function in \(\mathcal{N}\) can be a minimizer of \(\mathcal{J}\) on \(\mathcal{N}\).

Let \(u \in \mathcal{N}\) be semitrivial. To simplify notation we assume that

\[(3.1) \quad u = (0, u_2, \ldots, u_q) = (\bar{u}_1, \ldots, \bar{u}_q) \quad \text{with} \quad u_i \in H_i, \quad \bar{u}_h \in H_q.\]

Given \(\varphi \in H\) and \(\varepsilon > 0\) define

\[u_\varepsilon := (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_q) \quad \text{with} \quad u_\varepsilon, 1 := (\varepsilon \varphi, u_2, \ldots, u_q) \in H_1.\]

Set \(\hat{I}_1 := I_1 \setminus \{1\}\) and \(\hat{I}_h := I_h\) if \(h = 2, \ldots, q\).

Lemma 3.1. There exist \(\varepsilon_0 > 0\) and a \(C^1\)-map \(t : (-\varepsilon_0, \varepsilon_0) \to (0, \infty)^q\) satisfying \(t(\varepsilon)u_\varepsilon \in \mathcal{N}\), \(t(0) = (1, \ldots, 1)\) and \(t'(0) = (0, \ldots, 0)\).

Proof. For \(\varepsilon \in \mathbb{R}\) and \(t = (t_1, \ldots, t_q) \in (0, \infty)^q\) define

\[F_\varepsilon(t) := \partial u_\varepsilon \mathcal{J}(tu_\varepsilon)[t_1 u_\varepsilon, 1]
= t_1^2 \varepsilon^2 \|\varphi\|_1^2 + t_1^2 \|\bar{u}_1\|^2 - t_1^2 \varepsilon \beta_{11} \int_\Omega |\varphi|^{2p}
- 2t_1^p |\varepsilon|^{p_q} \sum_{j \in \hat{I}_1} \beta_{1j} \int_\Omega |\varphi|^p |u_j|^p
- t_1^p \sum_{(i, j) \in \hat{I}_1 \times \hat{I}_1} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p
- t_1^p \sum_{k=2}^q \sum_{j \in \hat{I}_k} t_k^p \beta_{1j} \int_\Omega |\varphi|^p |u_j|^p
- t_1^p \sum_{(i, j) \in \hat{I}_1 \times \hat{I}_k} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p,\]
and for \( h = 2, \ldots, q, \)

\[
F_h(\varepsilon, t) := \partial_{\bar{u}_h} \mathcal{J}(t u_\varepsilon)[t_h \bar{u}_h]
\]

\[
= t_h^2 \|\bar{u}_h\|^2 - t_h^{2p} \sum_{(i,j) \in I_h \times I_h} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p - t_h^p \sum_{i \in I_h} \beta_{ii} \int_\Omega |u_i|^p |\varphi|^p
\]

\[
- t_h^p \sum_{(i,j) \in I_h \times I_1} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p - t_h^p \sum_{k \neq h} \sum_{k = 2}^q t_h^p \sum_{(i,j) \in I_k \times I_k} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p.
\]

We shall apply the implicit function theorem to the \( C^1 \)-map \( F = (F_1, \ldots, F_q) : \mathbb{R} \times (0, \infty)^q \to \mathbb{R}^q. \) Set \( \mathbf{1} = (1, \ldots, 1). \) Note that, as \( \mathbf{u} \in \mathcal{N}, \)

\[
(3.2)
F_h(0, \mathbf{1}) = \|\bar{u}_h\|^2 - \sum_{(i,j) \in I_h \times I_h} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p - \sum_{k \neq h} \sum_{k = 1}^q \sum_{(i,j) \in I_k \times I_k} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p
\]

\[
= \partial_{\bar{u}_h} \mathcal{J}(u_\varepsilon)[\bar{u}_h] = 0 \quad \text{for every} \quad h = 1, \ldots, q.
\]

Using this identity we obtain

\[
(3.3)
a_{hh} := \partial_{\bar{u}_h} F_h(0, \mathbf{1})
\]

\[
= 2 \|\bar{u}_h\|^2 - 2p \sum_{(i,j) \in I_h \times I_h} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p - 2p \sum_{k \neq h} \sum_{(i,j) \in I_k \times I_k} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p
\]

\[
= (2 - 2p) \sum_{(i,j) \in I_h \times I_h} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p + (2 - p) \sum_{k \neq h} \sum_{(i,j) \in I_h \times I_k} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p
\]

\[
< p \sum_{k \neq h} \sum_{(i,j) \in I_h \times I_k} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p \leq 0.
\]

Furthermore, we have

\[
(3.4) \quad a_{hh} := \frac{\partial F_h}{\partial \bar{t}_h}(0, \mathbf{1}) = -p \sum_{(i,j) \in I_h \times I_k} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p > 0 \quad \text{if} \quad h \neq k.
\]

The Jacobian matrix of \( \mathbf{F} \) with respect to \( \mathbf{t} \) at the point \( (0, \mathbf{1}) \) is strictly diagonally dominant, i.e.

\[
|a_{hh}| > \sum_{h \neq k} |a_{hk}| \quad \text{for every} \quad h = 1, \ldots, q.
\]
Indeed, (3.2) yields
\[
|a_{hk}| - \sum_{h \neq k} |a_{hk}| = -a_{hh} - \sum_{h \neq k} a_{hk} \\
= (2p - 2) \sum_{(i,j) \in I_h \times I_k} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p + (p - 2) \sum_{k=1}^q \sum_{\beta_{ij} \neq h} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p \\
+ p \sum_{k=1}^q \sum_{\beta_{ij} \neq h} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p \\
= (2p - 2) \| \bar{u}_h \|^2 > 0.
\]

The Levy-Desplanques theorem asserts that a strictly diagonally dominant matrix is non-singular. So, by the implicit function theorem, there exist \( \varepsilon_0 > 0 \) and a \( C^1 \)-map \( t : (-\varepsilon_0, \varepsilon_0) \to (0, \infty)^q \) satisfying \( t(0) = 1 \),
\[
F_h(\varepsilon, t(\varepsilon)) = 0 \quad \text{for every} \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0), \ h = 1, \ldots, q,
\]
i.e., \( t(\varepsilon) \in N \), and \( t'(\varepsilon) = -\partial_\varepsilon F(\varepsilon, t(\varepsilon))^{-1} \circ \partial_\varepsilon F(\varepsilon, t(\varepsilon)) \). As (3.5)
\[
\partial_\varepsilon F_1(\varepsilon, t) = 2\varepsilon t_1^2 \| \varphi \|^2 - 2p \varepsilon t_1^{2p-1} t_1^p \beta_{11} \int_\Omega |\varphi|^{2p} \\
- (2p) |\varepsilon|^{p-2} t_1^{2p} \sum_{j \in I_1} \beta_{1j} \int_\Omega |\varphi|^{p} |u_j|^p - p |\varepsilon|^{p-2} \varepsilon t_1^p \sum_{k=2}^q \sum_{j \in I_k} \beta_{kj} \int_\Omega |\varphi|^{p} |u_j|^p \\
\partial_\varepsilon F_h(\varepsilon, t) = -p |\varepsilon|^{p-2} t_1^p \sum_{j \in I_h} \beta_{kj} \int_\Omega |u_i|^p |\varphi|^p \quad \text{if} \quad h = 2, \ldots, q,
\]
we conclude that \( t'(0) = 0 \), as claimed. \( \square \)

**Remark 3.2.** For \( p = 2 \) the map \( t \) is of class \( C^2 \) and its second derivative at 0 solves the system \( \partial_\varepsilon F(0, 1)t''(0) + \partial_\varepsilon F(0, 1) = 0 \), i.e.,
\[
\begin{cases}
\sum_{k=1}^q a_{1k}t''_k(0) = -2\| \varphi \|^2 + 4 \sum_{j \in I_1} \beta_{1j} \int_\Omega |\varphi|^2 |u_j|^2 + 2 \sum_{k=2}^q \sum_{j \in I_k} \beta_{kj} \int_\Omega |\varphi|^2 |u_j|^2 \\
\sum_{k=1}^q a_{hk}t''_k(0) = 2 \sum_{j \in I_h} \beta_{kj} \int_\Omega |\varphi|^2 |u_j|^2 \quad \text{if} \quad h = 2, \ldots, q,
\end{cases}
\]
with \( a_{hk} \) as in the previous lemma, i.e.,
\[
a_{hk} = -2 \sum_{(i,j) \in I_h \times I_k} \beta_{ij} \int_\Omega |u_i|^2 |u_j|^2.
\]
As a consequence,
\[
\sum_{h,k=1}^q a_{hk}t''_k(0) = -2\| \varphi \|^2 + 4 \sum_{h=1}^q \sum_{j \in I_h} \beta_{kj} \int_\Omega |\varphi|^2 |u_j|^2.
\]
Lemma 3.3. If $p < 2$ then, for small enough $\varepsilon > 0$,

$$
\mathcal{J}(t(\varepsilon)u_\varepsilon) - \mathcal{J}(u) = \varepsilon^p \left( - \frac{1}{p} \sum_{k=1}^{q} \sum_{j \in \hat{I}_k} \beta_{ij} \int_{\Omega} |\varphi|^p |u_j|^p + o(1) \right).
$$

**Proof.** Since $u \in \mathcal{N}$ and $t(\varepsilon)u_\varepsilon \in \mathcal{N}$ we have that

$$
\mathcal{J}(u) = \left( \frac{1}{2} - \frac{1}{2p} \right) \left( \sum_{h,k=1}^{q} \sum_{(i,j) \in \hat{I}_h \times \hat{I}_k} \beta_{ij} \int_{\Omega} |u_i|^p |u_j|^p \right)
$$

and, writing $t(\varepsilon) = (t_1(\varepsilon), \ldots, t_q(\varepsilon))$ where $t_h = t_h(\varepsilon)$,

$$
\mathcal{J}(t(\varepsilon)u_\varepsilon) = \left( \frac{1}{2} - \frac{1}{2p} \right) \left( t_{11}^2 \beta_{11} \int_{\Omega} |\varepsilon\varphi|^2 |u_1|^p + 2 \sum_{k=1}^{q} t_k^p t_k^p \sum_{j \in \hat{I}_k} \beta_{1j} \int_{\Omega} |\varepsilon\varphi|^p |u_j|^p \right)
$$

and,

$$
\mathcal{J}(t(\varepsilon)u_\varepsilon) = \left( \frac{1}{2} - \frac{1}{2p} \right) \left( \sum_{h,k=1}^{q} (t_h^p t_k^p - 1) \sum_{(i,j) \in \hat{I}_h \times \hat{I}_k} \beta_{ij} \int_{\Omega} |u_i|^p |u_j|^p \right)
$$

Therefore,

$$
(3.6) \quad \mathcal{J}(t(\varepsilon)u_\varepsilon) - \mathcal{J}(u) = \frac{p - 1}{2p} \left( t_{11}^2 \beta_{11} \int_{\Omega} |\varepsilon\varphi|^2 |u_1|^p + 2 \sum_{k=1}^{q} t_k^p t_k^p \sum_{j \in \hat{I}_k} \beta_{1j} \int_{\Omega} |\varepsilon\varphi|^p |u_j|^p \right)
$$

$$
+ \sum_{h,k=1}^{q} (t_h^p t_k^p - 1) \sum_{(i,j) \in \hat{I}_h \times \hat{I}_k} \beta_{ij} \int_{\Omega} |u_i|^p |u_j|^p,
$$

$$
= \frac{p - 1}{2p} \left( 2\varepsilon^p \sum_{k=1}^{q} \sum_{j \in \hat{I}_k} \beta_{1j} \int_{\Omega} |\varphi|^p |u_j|^p + o(\varepsilon^p) \right)
$$

$$
+ \sum_{h,k=1}^{q} (t_h^p t_k^p - 1) \sum_{(i,j) \in \hat{I}_h \times \hat{I}_k} \beta_{ij} \int_{\Omega} |u_i|^p |u_j|^p.
$$

Set

$$
(3.7) \quad b_{hk} := \sum_{(i,j) \in \hat{I}_h \times \hat{I}_k} \beta_{ij} \int_{\Omega} |u_i|^p |u_j|^p \quad h, k = 1, \ldots, q.
$$

To estimate (3.6) we need to expand the function

$$
\sum_{h,k=1}^{q} (t_h^p(\varepsilon)t_k^p(\varepsilon) - 1) b_{hk}.
$$
As \( p < 2 \) we derive from (3.5) that

(3.8)

\[
T_1 := \lim_{\varepsilon \to 0^+} \varepsilon^{-p} \frac{\partial_{\varepsilon} F_1(\varepsilon, t(\varepsilon))}{\varepsilon - 1} = -2p \sum_{j \in \mathcal{I}_1} \beta_{1j} \int_{\Omega} |\varphi| |u_j|^p - p \sum_{k=2}^{q} \sum_{j \in \mathcal{I}_1} \beta_{1j} \int_{\Omega} |\varphi| |u_j|^p
\]

\[
T_h := \lim_{\varepsilon \to 0^+} \varepsilon^{-p} \frac{\partial_{\varepsilon} F_h(\varepsilon, t(\varepsilon))}{\varepsilon - 1} = -p \sum_{i \in \mathcal{I}_h} \beta_{1i} \int_{\Omega} |u_i|^p |\varphi|^p \quad \text{if} \quad h = 2, \ldots, q.
\]

So, since \( \partial_t F(\varepsilon, t(\varepsilon)) t'(\varepsilon) + \partial_{\varepsilon} F(\varepsilon, t(\varepsilon)) = 0 \), we conclude that

(3.9)

\[
\lim_{\varepsilon \to 0^+} \varepsilon^{-p} t'_h(\varepsilon) = \tau_h \in \mathbb{R} \quad \text{for every} \quad h = 1, \ldots, q,
\]

and

(3.10)

\[
\partial_t F(0, 1) \tau + T = 0,
\]

where \( \tau = (\tau_1, \ldots, \tau_q) \) and \( T = (T_1, \ldots, T_q) \). From (3.9) and L'Hôpital's rule we get

(3.11)

\[
t'_h(\varepsilon) t''_h(\varepsilon) - 1 = (\tau_h + \tau_k) \varepsilon^p + o(\varepsilon^p),
\]

and from (3.10), (3.3), (3.4) and (3.7) we derive

(3.12)

\[
- \sum_{h=1}^{q} T_h = \sum_{h,k=1}^{q} a_{hk} \tau_k
\]

\[
= (2 - 2p) \sum_{h=1}^{q} b_{hh} + (2 - p) \sum_{h,k=1}^{q} b_{hk} \tau_h - \sum_{h,k=1}^{q} b_{hk} \tau_h
\]

\[
= 2(1 - p) \sum_{h,k=1}^{q} b_{hk} \tau_h.
\]

Now, (3.11), (3.12) and (3.8) yield

\[
\sum_{h,k=1}^{q} (t''_h(\varepsilon) t''_k(\varepsilon) - 1) b_{hk} = \varepsilon^p \left( \sum_{h,k=1}^{q} (\tau_h + \tau_k) b_{hk} + o(1) \right)
\]

\[
= \varepsilon^p \left( \frac{1}{p - 1} \sum_{h=1}^{q} T_h + o(1) \right)
\]

\[
= \varepsilon^p \left( - \frac{2p}{p - 1} \sum_{h=1}^{q} \sum_{j \in \mathcal{I}_h} \beta_{1j} \int_{\Omega} |\varphi|^p |u_j|^p + o(1) \right).
\]

Going back to (3.6) we conclude that

\[
\mathcal{J}(t(\varepsilon) u_{\varepsilon}) - \mathcal{J}(u) = \varepsilon^p \left( - \frac{1}{p} \sum_{h=1}^{q} \sum_{j \in \mathcal{I}_h} \beta_{1j} \int_{\Omega} |\varphi|^p |u_j|^p + o(1) \right),
\]

as claimed. □
Lemma 3.4. If $p = 2$ then, for small enough $\varepsilon > 0$,

\[
\mathcal{J}(t(\varepsilon)u_\varepsilon) - \mathcal{J}(u) = \varepsilon^2 \left( \|\varphi\|^2 - \sum_{k=1}^{q} \sum_{j \in I_k} \beta_{1j} \int_\Omega |\varphi|^2 |u_j|^2 + o(1) \right).
\]

Proof. As $t$ is of class $C^2$ when $p = 2$, from and Taylor’s expansion for $t_i^2 t_k^2$ we derive

\[
\mathcal{J}(t(\varepsilon)u_\varepsilon) - \mathcal{J}(u) = \varepsilon^2 \left( \sum_{k=1}^{q} \sum_{j \in I_k} \beta_{1j} \int_\Omega |\varphi|^2 |u_j|^2 + o(1) \right)
+ \frac{1}{2} \sum_{h,k=1}^{q} (t_h'k(0) + t_k'h(0)) \sum_{(i,j) \in I_k \times I_k} \beta_{ij} \int_\Omega |u_i|^2 |u_j|^2.
\]

Now we use Remark 3.2 to compute

\[
\sum_{k=1}^{q} \sum_{j \in I_k} \beta_{1j} \int_\Omega |\varphi|^2 |u_j|^2 + \frac{1}{2} \sum_{h,k=1}^{q} (t_h'k(0) + t_k'h(0)) \sum_{(i,j) \in I_k \times I_k} \beta_{ij} \int_\Omega |u_i|^2 |u_j|^2
= \sum_{k=1}^{q} \sum_{j \in I_k} \beta_{1j} \int_\Omega |\varphi|^2 |u_j|^2 - \frac{1}{2} \sum_{h,k=1}^{q} a_{hkt_k'}(0)
= \|\varphi\|^2 - \sum_{h=1}^{q} \sum_{j \in I_h} \beta_{1j} \int_\Omega |\varphi|^2 |u_j|^2.
\]

This completes the proof. \hfill \Box

Proof of Theorem 1.2. Let $u \in \mathcal{N}$ be such that $\mathcal{J}(u) = \inf_{\mathcal{N}} \mathcal{J}$. Set

\[
S := \min_{i=1, \ldots, \ell} \inf_{v \notin H_{\varepsilon\ell}} \frac{\|v\|^2}{|u_i|_{2p}^2}, \quad \text{where } |v|_{2p} := \left( \int_\Omega |v|^{2p} \right)^{\frac{1}{2p}},
\]
and define $C_* := \left( \frac{pd_1}{(p-1)|s|^{\frac{p}{p-1}}} \right)^p$, with $d_1$ as in Lemma 2.2. We distinguish two cases.

1) Let $p < 2$. Arguing by contradiction, assume that that $(B_2)$ holds true and that some component of $u$ is trivial. Without loss of generality, we assume it is the first one, i.e., $u$ is as in (3.1). Fix $i_1^* \in I_1$ such that $\max_{i \in I_1} |u_i|_{2p} = |u_{i_1^*}|_{2p}$. As $u \in \mathcal{N}$ we have that $u_{i_1^*} \neq 0$ and using Hölder’s inequality we obtain

\[
S |u_{i_1^*}|_{2p}^2 \leq \|u_{i_1^*}\|^2 \leq \sum_{i,j \in I_1} \beta_{ij} \int_\Omega |u_i|^p |u_j|^p \leq \beta_{i_1^*} \max_{i,j \in I_1} |u_{i_1^*}|_{2p}^2.
\]
with \( \ell_1 := \ell_1 - 1 \). Therefore,

\[
\sum_{j \in I_1} \beta_{ij} \int_{\Omega} |u_{i1}|^p |u_j|^p \geq \beta_{i1} \int_{\Omega} |u_{i1}|^{2p} \geq \min_{i,j \in I_1, i \neq j} \beta_{ij} \left( \frac{S}{\ell_1^2 \max_{i,j \in I_1} \beta_{ij}} \right)^{\frac{p}{p-1}}.
\]

On the other hand,

\[
S^2 \left( \int_{\Omega} |u_{i1}|^p |u_j|^p \right)^{\frac{p}{p-1}} \leq S^2 |u_{i1}|^{2p} |u_j|^{2p} \leq ||u_{i1}||_{i1}^2 ||u_j||_{j}.\]

As \( J(u) = \inf_{\mathcal{N}} J \) Lemma 2.2 yields

\[
\frac{p-1}{p} ||u_j||_{j}^2 \leq \frac{p-1}{p} ||u||^2 = J(u) = \inf_{\mathcal{N}} J \leq d_1 \left( \min_{h=1,\ldots,n} \max_{i \in I_h} \beta_{ii} \right)^{-\frac{1}{p-1}}.
\]

Therefore,

\[
(3.13) \quad \int_{\Omega} |u_{i1}|^p |u_j|^p \leq \left( \frac{pd_1}{(p-1)S} \right)^{\frac{p}{p-1}} \left( \min_{h=1,\ldots,n} \max_{i \in I_h} \beta_{ii} \right)^{-\frac{1}{p-1}}.
\]

Set \( \varphi := u_{i1} \) and let \( K_1 \) be as in (B1). From (B2) with \( h = 1 \) we get

\[
\sum_{k=1}^q \sum_{j \in I_k} \beta_{ij} \int_{\Omega} |\varphi|^p |u_j|^p = \sum_{j \in I_1} \beta_{ij} \int_{\Omega} |u_{i1}|^p |u_j|^p + \sum_{k=2}^q \sum_{j \in I_k} \beta_{ij} \int_{\Omega} |u_{i1}|^p |u_j|^p \geq \min_{i,j \in I_1} \beta_{ij} \left( \frac{S}{\ell_1^2 \max_{i,j \in I_1} \beta_{ij}} \right)^{\frac{p}{p-1}} - \sum_{(i,j) \in K_1} |\beta_{ij}| \left( \frac{pd_1}{(p-1)S} \right)^{\frac{p}{p-1}} \left( \min_{h=1,\ldots,n} \max_{i \in I_h} \beta_{ii} \right)^{-\frac{1}{p-1}}
\]

\[
= \left( \frac{S}{\ell_1^2} \min_{h=I_1} \max_{i \in I_h} \beta_{ii} \right)^{\frac{p}{p-1}} \left( \min_{i,j \in I_1} \beta_{ij} \left[ \frac{\min_{h=1,\ldots,n} \max_{i \in I_h} \beta_{ii}}{\max_{i,j \in I_1} \beta_{ij}} \right] \right)^{\frac{p}{p-1}} - C_4 \ell_1^{\frac{2p}{p-1}} \sum_{(i,j) \in K_1} |\beta_{ij}| > 0.
\]

Lemma 3.3 asserts that \( t(\varepsilon)u_\varepsilon \in \mathcal{N} \) for small enough \( \varepsilon > 0 \), and from Lemma 3.1 and the previous inequality we derive that

\[
J(t(\varepsilon)u_\varepsilon) < J(u) = \inf_{\mathcal{N}} J.
\]

This is a contradiction.

2) Let \( p = 2 \). Arguing again by contradiction, assume that (B2) holds true and that \( u \) is as in (3.1). Fix \( i_1^* \in I_1 \) such that \( \max_{i \in I_1} |u_i| = |u_{i_1}^*| \). Let us assume, for simplicity, that \( i_1^* = 2 \). Then \( u_2 \neq 0 \). By (B2) we have that \( \beta_{i_2} = b_1 \) for \( i, j \in I_1 \) with \( i \neq j \) and that \( \beta_{22} \leq b_1 \). Therefore,

\[
S|u_2|_i^2 \leq ||u||^2 \leq \sum_{i \in I_1} \beta_{2i} \int_{\Omega} |u_2|^2 |u_i|^2 \leq (\ell_1 - 1)b_1 |u_2|_i^2;
\]
and, so,

\[
\frac{S}{(\ell_1 - 1)b_1} \leq |u_2|^2.
\]

On the other hand, (3.13) reads

\[
\int_{\Omega} |u_2|^2 |u_j|^2 \leq \left( \frac{2d_1}{S} \right)^2 \left( \min_{h \in I_h} \max_{i \in I_h} \beta_{ii} \right)^{-2}.
\]

Now, since \( u \) solves the system (1.1), we know that

\[
\|u_2\|_1^2 = \sum_{h=1}^{q} \sum_{j \in I_h} \beta_{2j} \int_{\Omega} |u_2|^2 |u_j|^2.
\]

Set \( \varphi := u_2 \). As \( \lambda_1 = \lambda_2 \) by (B2), we have that \( \|\varphi\|_1 = \|u_2\|_2 \). Therefore,

\[
\|\varphi\|_2^2 - \sum_{h=1}^{q} \sum_{j \in I_h} \beta_{1j} \int_{\Omega} |u_2|^2 |u_j|^2
\]

\[
= \sum_{h=1}^{q} \sum_{j \in I_h} \beta_{2j} \int_{\Omega} |u_2|^2 |u_j|^2 - \sum_{h=1}^{q} \sum_{j \in I_h} \beta_{1j} \int_{\Omega} |u_2|^2 |u_j|^2
\]

\[
= (\beta_{22} - \beta_{12}) \int_{\Omega} |u_2|^4 + \sum_{j \in I_1 \setminus \{2\}} (\beta_{2j} - \beta_{1j}) \int_{\Omega} |u_2|^2 |u_j|^2
\]

\[
+ \sum_{h=2}^{q} \sum_{j \in I_h} (\beta_{2j} - \beta_{1j}) \int_{\Omega} |u_2|^2 |u_j|^2
\]

\[
\leq \left( \max_{i \in I_1} \beta_{ii} - b_1 \right) \left[ \frac{S}{(\ell_1 - 1)b_1} \right]^2 + \left( \frac{2d_1}{S} \right)^2 \left( \min_{h \in I_h} \max_{i \in I_h} \beta_{ii} \right)^{-2} \sum_{j \in I_h \atop h \geq 2} |\beta_{2j} - \beta_{1j}|
\]

\[
= \left[ \frac{S}{(\ell_1 - 1)b_1} \right]^2 \left( \max_{i \in I_1} \beta_{ii} - b_1 + C_4 \left( \frac{b_1(\ell_1 - 1)}{\min_{h \in I_h} \max_{i \in I_h} \beta_{ii}} \right)^2 \sum_{j \in I_h \atop h \geq 2} |\beta_{2j} - \beta_{1j}| \right)
\]

\[
< 0,
\]

by (B2). Now Lemma 3.4 yields a contradiction.

\[\square\]

4 Systems with symmetries

Let \( G \) be a closed subgroup of the group \( O(N) \) of linear isometries of \( \mathbb{R}^N \) and denote by \( Gx := \{gx : g \in G\} \) the \( G \)-orbit of \( x \in \mathbb{R}^N \). Assume that \( \Omega \) is \( G \)-invariant, i.e., \( Gx \subset \Omega \) for every \( x \in \Omega \). Then, a function \( u : \Omega \to \mathbb{R} \) is called \( G \)-invariant if it is constant on \( Gx \) for every \( x \in \Omega \). Define

\[
H^G := \{ u \in H : u \text{ is } G\text{-invariant} \},
\]

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where, as before, $H$ is either $H^1_0(\Omega)$ or $D^{1,2}_0(\Omega)$.

Further, let $\phi: G \to \mathbb{Z}_2 := \{-1, 1\}$ be a continuous homomorphism of groups and let $K := \ker \phi$ be its kernel. Assume

$(\phi)$ there exists $x_0 \in \Omega$ such that $Kx_0 \neq Gx_0$.

A function $u: \Omega \to \mathbb{R}$ is called $\phi$-equivariant if $u(gx) = \phi(g)u(x)$ for every $g \in G$ and $x \in \Omega$. Define

$$H^\phi := \{u \in H : u \text{ is } \phi\text{-equivariant}\}.$$ 

Assumption $(\phi)$ guarantees that this space has infinite dimension. Moreover, it implies that $K \neq G$, i.e., that $\phi$ is surjective. Therefore, every nontrivial $\phi$-equivariant function is nonradial and changes sign.

By the principle of symmetric criticality [21, Theorem 1.28] the critical points of the restriction of $J$ to the either $(H^G)^J$ or to $(H^\phi)^J$ are critical points of $J$, i.e., they solve system (1.1). Clearly, all results in the previous sections go through if we take $H$ to be one of these spaces. So Theorems 2.5, 1.1 and 1.2 hold true for $H^G$ and $H^\phi$ as well.

Next, we give some applications.

### 4.1 Systems in bounded domains

Let $G$ be a closed subgroup of $O(N)$ and $\Omega$ a $G$-invariant domain. Let $\lambda^G_i(\Omega)$ be the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)^G$ and $d := \min\{\dim(Gx) : x \in \Omega\}$. Then we have the following result.

**Theorem 4.1.** If $\Omega$ is bounded, $\lambda_i > -\lambda^G_i(\Omega)$, $1 < p < \frac{N-d}{N-d-2}$ when $N > d+2$, and $(\beta_{ij})$ satisfies $(B_1)$ and $(B_2)$, the system

$$(4.1)$$

$$
\begin{aligned}
-\Delta u_i + \lambda_i u_i &= \sum_{j=1}^\ell \beta_{ij} |u_j|^p |u_i|^{p-2} u_i, \\
&H^1_0(\Omega), \\
i &= 1, \ldots, \ell,
\end{aligned}
$$

has a fully nontrivial solution whose components are positive and $G$-invariant.

Moreover, if there exists a continuous homomorphism of groups $\phi: G \to \mathbb{Z}_2$ satisfying $(\phi)$, then (4.1) has has a fully nontrivial solution whose components are $\phi$-equivariant and, thus, change sign.

**Proof.** It is shown in [11, Corollary 2] that, if $\Omega$ is bounded and $1 < p < \frac{N-d}{N-d-2}$, the embedding $H^1_0(\Omega)^G \hookrightarrow L^p(\Omega)$ is compact. So Theorems 1.1 and 1.2 yield a least energy fully nontrivial solution $(u_1, \ldots, u_\ell)$ with $u_i \in H^1_0(\Omega)^G$. As $|u| \in H^1_0(\Omega)^G$ for every $u \in H^1_0(\Omega)^G$, $(|u_1|, \ldots, |u_\ell|)$ is also a least energy fully nontrivial solution.

On the other hand, since $\dim(Kx) = \dim(Gx)$ for all $x \in \Omega$, $H^1_0(\Omega)^K \hookrightarrow L^p(\Omega)$ is also compact and, as $H^1_0(\Omega)^\phi \subset H^1_0(\Omega)^K$, the embedding $H^1_0(\Omega)^\phi \hookrightarrow L^p(\Omega)$ is compact. So Theorems 1.1 and 1.2 yield a least energy fully nontrivial solution $(u_1, \ldots, u_\ell)$ with $u_i \in H^1_0(\Omega)^\phi$. Since $\phi$ is surjective, $u_i$ changes sign. \qed
Note that this result includes the case when $\Omega$ has no symmetries. Then $d = 0$ and the system is subcritical. Observe also that the system is supercritical if $d \geq 1$.

For highly symmetric domains Theorem 4.1 yields infinitely many solutions.

**Theorem 4.2.** If $\Omega$ is a ball or an annulus, $\lambda_i > -\lambda_1(\Omega)$, $1 < p < \frac{N}{N-2}$ and $(\beta_{ij})$ satisfies $(B_1)$ and $(B_2)$, the system (4.1) has infinitely many fully nontrivial solutions. All components of one of them are radial and positive, and all components of the rest are nonradial and change sign.

**Proof.** For each $m \in \mathbb{N}$ let $G_m$ be the group generated by $\vartheta_m := \pi/2m$ on $R^2$ acting on the first factor of $R^2 \times R^{N-2} \equiv R^N$, and let $\phi_m : G_m \rightarrow Z_2$ be the homomorphism given by $\phi_m(\vartheta_m) := -1$. Theorem 4.1 applied to $O(N)$ yields a fully nontrivial solution whose components are positive and radial, and applied to $\phi_m$ yields a fully nontrivial solution whose components $u_{m,i}$ satisfy

$$u_{m,i}(\vartheta_m x, y) = -u_{m,i}(x, y) \quad \text{for every } (x, y) \in R^2 \times R^{N-2}.$$ 

Hence, $u_{m,i}$ is nonradial and it changes sign. It is easy to see that $u_{m,i} \neq u_{n,i}$ if $m \neq n$.

### 4.2 Subcritical systems in exterior domains

Let $G$ be a closed subgroup of $O(N)$ such that the $G$-orbit $Gx$ of every point $x \in \mathbb{R}^N \setminus \{0\}$ is an infinite set, and let $\Omega$ be a $G$-invariant exterior domain (i.e., $\mathbb{R}^N \setminus \Omega$ is bounded, possibly empty). Under these assumptions we have the following result.

**Theorem 4.3.** If $\lambda_i > 0$, $1 < p < \frac{N}{N-2}$ and $(\beta_{ij})$ satisfies $(B_1)$ and $(B_2)$, the system (4.1) has a fully nontrivial solution whose components are positive and $G$-invariant.

If, in addition, there exists a continuous homomorphism of groups $\phi : G \rightarrow \mathbb{Z}_2$ satisfying $(\phi)$, then (4.1) has a fully nontrivial solution whose components are $\phi$-equivariant and, thus, change sign.

**Proof.** Since the $G$-orbit of every point $x \in \mathbb{R}^N \setminus \{0\}$ is an infinite set, the embedding $H^1_0(\Omega)^G \hookrightarrow L^p(\Omega)$ is compact [9, Lemma 4.3]. If $K = \ker \phi$, the $K$-orbit of every $x \in \mathbb{R}^N \setminus \{0\}$ is also infinite. Hence, $H^1_0(\Omega)^K \hookrightarrow L^p(\Omega)$ is also compact and so is $H^1_0(\Omega)^\phi \hookrightarrow H^1_0(\Omega)^K \hookrightarrow L^p(\Omega)$. The result now follows from Theorems 1.1 and 1.2.

Theorem 1.4 is a special case of this result.

**Proof of Theorem 1.4.** The first statement follows from Theorem 4.3 with $\Omega = \mathbb{R}^N$ and $G = O(N)$.

For the second one we take $G$ to be the group generated by $K \cup \{\vartheta\}$, where $K := O(2) \times O(2) \times O(N-2)$ and $\vartheta$ is the reflection given by $\vartheta(x, y, z) := (y, x, z)$ for $(x, y, z) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{N-4}$, $N \geq 4$. Note that the $K$-orbit of $x = (x, y, z)$
is $S^1_{|x|} \times S^1_{|y|} \times S^{N-5}_r$, where $S^{N-1}_r := \{ x \in \mathbb{R}^n : |x| = r \}$. So $K \vec{x}$ is infinite for every $\vec{x} \neq 0$ if $N \neq 5$. Let $\phi$ be the homomorphism given by $\phi(g) = 1$ if $g \in K$ and $\phi(g) = -1$. The result now follows from Theorem 4.3.

The latter symmetries were introduced in [3] to prove existence of nonradial solutions to a Schrödinger equation, see also [21, Theorem 1.37].

### 4.3 Entire solutions of critical systems

In the critical case linear group actions do not provide compactness. One needs to consider conformal actions.

Let $\Gamma$ be a closed subgroup of $O(N + 1)$. Then $\Gamma$ acts isometrically on the unit sphere $\mathbb{S}^N := \{ x \in \mathbb{R}^{N+1} : |x| = 1 \}$. The stereographic projection $\sigma : \mathbb{S}^N \to \mathbb{R}^N \cup \{ \infty \}$ induces a conformal action of $\Gamma$ on $\mathbb{R}^N$, given by

$$(\gamma, x) \mapsto \tilde{\gamma}x,$$

where $\tilde{\gamma} := \sigma \circ \gamma^{-1} \circ \sigma^{-1} : \mathbb{R}^N \to \mathbb{R}^N$,

which is well defined except at a single point. The group $\Gamma$ acts on the Sobolev space $D^{1,2}(\mathbb{R}^N)$ by linear isometries as follows:

$$\gamma u := |\det \tilde{\gamma}|^{1/2} u \circ \tilde{\gamma},$$

for any $\gamma \in \Gamma$ and $u \in D^{1,2}(\mathbb{R}^N)$; see [8, Section 3]. Set

$$D^{1,2}(\mathbb{R}^N)_\Gamma := \{ u \in D^{1,2}(\mathbb{R}^N) : \gamma u = u \text{ for all } \gamma \in \Gamma \}. $$

The argument in [8, Lemma 3.2] shows that this space is infinite dimensional if $\Gamma \xi \neq \mathbb{S}^N$ for every $\xi \in \mathbb{S}^N$. If $\phi : \Gamma \to \mathbb{Z}_2$ is a continuous homomorphism of groups satisfying $(\phi)$, we define

$$D^{1,2}(\mathbb{R}^N)_\phi := \{ u \in D^{1,2}(\mathbb{R}^N) : \gamma u = \phi(\gamma) u \text{ for all } \gamma \in \Gamma \}. $$

Clearly, Theorems 1.1 and 1.2 hold true for $D^{1,2}(\mathbb{R}^N)_\Gamma$ and $D^{1,2}(\mathbb{R}^N)_\phi$ as well.

**Proof of Theorem 1.5.** Let $\Gamma = O(m) \times O(n)$ with $m + n = N + 1$ and $m, n \geq 2$ act on $\mathbb{R}^{N+1} \equiv \mathbb{R}^m \times \mathbb{R}^n$ in the obvious way. Then $D^{1,2}(\mathbb{R}^N)_\Gamma \hookrightarrow L^{\frac{N+1}{N-1}}(\mathbb{R}^N)$ is a compact embedding [8, Proposition 3.3 and Example 3.4]. So Theorems 1.1 and 1.2 yield a fully nontrivial least energy solution whose components belong to $D^{1,2}(\mathbb{R}^N)_\Gamma$. Replacing each component by its absolute value gives a solution whose components are positive.

To prove the second statement we write $\mathbb{R}^{N+1} \equiv \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{N-3}$ and consider the group $\Gamma$ generated by $K := O(2) \times O(2) \times O(N - 3)$ and the reflection given by $\phi(x, y, z) := (y, x, z)$ for $(x, y, z) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{N-3}$, and we take $\phi$ to be the homomorphism defined by $\phi(g) = 1$ if $g \in K$ and $\phi(g) = -1$. Then $D^{1,2}(\mathbb{R}^N)_K \hookrightarrow L^{\frac{2N}{N+1}}(\mathbb{R}^N)$ is a compact embedding if $N = 3$ or $N \geq 5$, and Theorems 1.1 and 1.2 yield a fully nontrivial solution whose components belong to $D^{1,2}(\mathbb{R}^N)_\phi$. 

$\square$
5 Synchronized solutions of cooperative systems

Throughout this section we assume that the system (1.1) is purely cooperative. We also assume that \( \lambda_i = \lambda \) for all \( i \).

Let \( u \) be a nontrivial solution to the equation

\[-\Delta u + \lambda u = |u|^{2p-2}u, \quad u \in H,\]

Then \( u = (c_1 u, \ldots, c_\ell u) \) is a solution to the system (1.1) iff \( c = (c_1, \ldots, c_\ell) \in \mathbb{R}^\ell \) solves the algebraic system (1.2). The solutions to (1.2) are the critical points of the \( C^1 \)-function \( J : \mathbb{R}^\ell \to \mathbb{R} \) defined by

\[J(c) := \frac{1}{2} |c|^2 - \frac{1}{2p} \sum_{i,j=1}^{\ell} \beta_{ij} |c_j|^p |c_i|^p, \quad \text{with} \quad |c|^2 := \sum_{i=1}^{\ell} c_i^2.\]

The nontrivial ones belong to the set

\[M := \{ c \in \mathbb{R}^\ell : c \neq 0, \langle \nabla J(c), c \rangle = 0 \},\]

which is a closed \( C^1 \)-submanifold of \( \mathbb{R}^\ell \). Hence, there exists \( c \in M \) such that

\[\min_M J = J(c)\]

and \( c \) is a solution to the algebraic system (1.2). Observe that

\[\langle \nabla J(c), c \rangle = |c|^2 - \sum_{i,j=1}^{\ell} \beta_{ij} |c_j|^p |c_i|^p\]

and so

\[J(c) = \frac{p-1}{p} |c|^2 \quad \text{for any} \quad c \in M.\]

Following the idea we used to prove Theorem 1.2, we analyze whether all components of a minimizing solution to (1.2) are nontrivial.

**Lemma 5.1.** Let \( c = (c_1, \ldots, c_\ell) \in M \) be such that \( J(c) = \min_M J \).

(i) If \( p < 2 \), then \( c_i \neq 0 \) for every \( i = 1, \ldots, \ell \).

(ii) If \( p = 2 \), then \( \sum_{j \neq i} \beta_{ij} c_j^2 \leq 1 \) for every \( i = 1, \ldots, \ell \).

**Proof.** (i) Let \( p < 2 \) and assume by contradiction that one component of \( c \) is zero, say \( c_1 = 0 \). Set \( c_\varepsilon := (\varepsilon, c_2, \ldots, c_\ell) \). As \( c \in M \), we have

\[|c|^2 = \sum_{i=2}^{\ell} c_i^2 = \sum_{i,j=2}^{\ell} \beta_{ij} |c_j|^p |c_i|^p.\]

Hence, there exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \) there is a unique \( t = t(\varepsilon) > 0 \) solving

\[t^2 (\varepsilon^2 + \sum_{i \geq 2} c_i^2) = t^{2p} \left( \beta_{11} \varepsilon^{2p} + 2 \varepsilon^p \sum_{j \geq 2} \beta_{1j} |c_j|^p + \sum_{i,j \geq 2} \beta_{ij} |c_j|^p |c_i|^p \right) \]

\[= t^{2p} \left( \beta_{11} \varepsilon^{2p} + 2 \varepsilon^p \sum_{j \geq 2} \beta_{1j} |c_j|^p + \sum_{i \geq 2} c_i^2 \right),\]
namely,

\[ t(\varepsilon) = \left( \frac{\varepsilon^2 + |c|^2}{\beta_{11}\varepsilon^2 p + 2\varepsilon p \sum_{j \geq 2} \beta_{1j}|c_j|^p + |c|^2} \right)^{\frac{1}{p-2}}. \]

Moreover, \( \varepsilon \mapsto t(\varepsilon) \) is a \( C^1 \)-function, \( t(0) = 1 \) and \( t'(0) = 0 \).

So, if \( p < 2 \), then

\[ t(\varepsilon) = 1 - \frac{1}{p-1} \varepsilon^p \frac{1}{|c|^2} \sum_{j \geq 2} \beta_{1j}|c_j|^p + o(\varepsilon^p). \]

Therefore,

\[ J(c) - J(c_\varepsilon) = \frac{p-1}{p} \left[ \sum_{i \geq 2} c_i^2 - t^2 \left( \varepsilon^2 + \sum_{i \geq 2} c_i^2 \right) \right] \]

\[ = \frac{p-1}{p} \left[ (1 - t^2)|c|^2 - t^2 \varepsilon^2 \right] \]

\[ = \frac{p-1}{p} \left( \frac{4}{2p-2} \varepsilon^p \sum_{j \geq 2} \beta_{1j}|c_j|^p + o(\varepsilon^p) \right) \]

\[ = \frac{2}{p} \varepsilon^p \left( \sum_{j \geq 2} \beta_{1j}|c_j|^p + o(1) \right) > 0, \]

which is a contradiction.

(ii) : Let \( p = 2 \) and assume by contradiction that \( \sum_{j \neq i} \beta_{ij}c_j^2 > 1 \) for some \( i \). Then, as \( c \) satisfies (1.2), we have that \( c_i = 0 \). Let us assume, for simplicity, that \( i = 1 \). Then, defining \( c_\varepsilon \) and \( t(\varepsilon) \) as before we have that

\[ t(\varepsilon) = 1 + \frac{1}{2} \varepsilon^2 \frac{1}{|c|^2} \left( 1 - 2 \sum_{j \geq 2} \beta_{1j}|c_j|^2 \right) + o(\varepsilon^2) \]

and, therefore,

\[ J(c) - J(c_\varepsilon) = \frac{1}{2} \left[ (1 - t^2) \sum_{i \geq 2} c_i^2 - t^2 \varepsilon^2 \right] \]

\[ = \frac{1}{2} \left( \varepsilon^2 (2 \sum_{j \geq 2} \beta_{1j}c_j^2 - 2) + o(\varepsilon^2) \right) \]

\[ = \varepsilon^2 \left( \sum_{j \geq 2} \beta_{1j}c_j^2 - 1 + o(1) \right) > 0. \]

This is a contradiction. \( \square \)

These computations highlight the different behavior of the system for \( p < 2 \) and \( p = 2 \). To complete the picture observe that, if \( p > 2 \), then

\[ t(\varepsilon) = 1 + \frac{1}{2p-2} \varepsilon^2 \frac{1}{|c|^2} + o(\varepsilon^2). \]
Therefore
\[ J(c) - J(c_\varepsilon) = \frac{p-1}{p} \left[ (1 - t^2)|c|^2 - t^2 \varepsilon^2 \right] \]
\[ = \frac{p-1}{p} \left[ -\varepsilon^2 + o(\varepsilon^2) \right]. \]

We conjecture that for \( p > 2 \) there is no fully synchronized solution.

**Proof of Theorem 1.6.** This result follows immediately from Lemma 5.1(i).

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