Bi-periodic Fibonacci matrix polynomial and its binomial transforms

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Abstract

In this paper, we consider the matrix polynomial obtained by using bi-periodic Fibonacci matrix polynomial. Then, we give some properties and binomial transforms of the new matrix polynomials.

Keywords: bi-periodic Fibonacci matrix polynomial, bi-periodic Fibonacci matrix sequence, Binet formula, generating function, transform.

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1 Introduction and Preliminaries

The bi-periodic Fibonacci $\{q_n\}_{n\in\mathbb{N}}$ sequence is defined by

\[ q_n = \begin{cases} 
   aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\
   bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} 
\end{cases}, \quad (1.1) \]

where $q_0 = 0$, $q_1 = 1$ and $a, b$ are nonzero real numbers.

Also, the bi-periodic Fibonacci $\{F_n\}_{n\in\mathbb{N}}$ matrix sequence is given as

\[ F_n = \begin{pmatrix} 
   \left( \frac{b}{a} \right)^{e(n)} q_{n+1} & \frac{b}{a}q_n \\
   q_n & \left( \frac{b}{a} \right)^{e(n)} q_{n-1} \end{pmatrix}, \quad (1.2) \]
where $a, b$ are nonzero real numbers and

$$
\varepsilon(n) = \begin{cases} 
1, & n \text{ odd} \\
0, & n \text{ even}
\end{cases}.
$$

(1.3)

In addition to these sequences, the other sequences appear in many branches of science and have attracted the attention of mathematicians (see [1-4, 8-13] and the references cited therein).

Also, the polynomials have attracted the attention of some mathematicians [6, 7, 14]. In [14], the authors gave the bi-periodic Fibonacci polynomial as

$$
q_n(a, b, x) = \begin{cases} 
axq_{n-1}(a, b, x) + q_{n-2}(a, b, x), & \text{if } n \text{ is even} \\
bqx_{n-1}(a, b, x) + q_{n-2}(a, b, x), & \text{if } n \text{ is odd}
\end{cases}
$$

(1.4)

which $q_0(a, b, x) = 0$, $q_1(a, b, x) = 1$ and $a, b$ are nonzero real numbers and they obtained some properties of this polynomial. Hoggatt and Bicknell, in [7], defined the Fibonacci, Tribonacci, Quadranacci, $r$-bonacci polynomials. They generalized Fibonacci polynomials and their relationship to diagonals of Pascal’s triangle. In [6], they give $k$-Fibonacci polynomials and offered the derivatives of these polynomials in the form of convolution of $k$-Fibonacci polynomials.

While on the one hand the sequences and polynomials was defined, on the other hand it was introduced some transorms for the given sequences. Binomial transform, $k$-Binomial transform, rising and falling binomial transforms are a few of these transforms (see [5, 15]).

In this study, firstly, we introduce bi-periodic Fibonacci matrix polynomial and give some properties of this polynomial. In Section 3, we have the new matrix polynomial by using bi-periodic Fibonacci matrix polynomial. And, we get the binomial, $k$-binomial, rising and falling transforms for the matrix polynomial as the first time in the literature. Then, we give the recurrence relations, generating functions and Binet formulas for these generalized Binomial transforms.
In this section, we focus on the bi-periodic matrix polynomial and give some properties of this generalized polynomial. Hence, we first define the bi-periodic Fibonacci matrix polynomials.

**Definition 2.1** For \( n \in \mathbb{N} \) and any two nonzero real numbers \( a, b \), the bi-periodic Fibonacci matrix polynomial \( F_n(a, b, x) \) is defined by

\[
F_n(a, b, x) = \begin{cases} 
axF_{n-1} + F_{n-2}, & \text{if } n \text{ is even} \\
bxF_{n-1} + F_{n-2}, & \text{if } n \text{ is odd}
\end{cases}
\]  

with initial conditions \( F_0(a, b, x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( F_1(a, b, x) = \begin{pmatrix} b & \frac{b}{a} \\ 1 & 0 \end{pmatrix} \).

In Definition 2.1, the matrix \( F_1 \) is analogue to the Fibonacci \( Q \)-matrix which exists for Fibonacci numbers.

**Theorem 2.2** Let \( F_n(a, b, x) \) be as in (2.1). Then the following equalities are valid for all positive integers:

(i) \( F_n(a, b, x) = \left( \begin{array}{cc} \frac{b}{a} & \frac{b}{a}q_n(a, b, x) \\ q_n(a, b, x) & \frac{b}{a}q_{n-1}(a, b, x) \end{array} \right) \),

(ii) \( \det(F_n(a, b, x)) = (-\frac{b}{a})^{\varepsilon(n)} \),

where \( q_n(a, b, x) \) is \( n \)th bi-periodic Fibonacci polynomial.

**Proof.** By using the iteration, it can be obtained the desired results. □

We obtained the Cassini identity for bi-periodic Fibonacci polynomials [14]. Using the determinant of \( F_n(a, b, x) \) in Theorem 2.2 again we get

\[ a^{1-\varepsilon(n)}b^{\varepsilon(n)}q_{n+1}(a, b, x)q_n(a, b, x) - a^{\varepsilon(n)}b^{1-\varepsilon(n)}q_n^2(a, b, x) = a(-1)^n. \]

**Theorem 2.3** For bi-periodic Fibonacci matrix polynomial, we have the generating function

\[
\sum_{i=0}^{\infty} F_i(a, b, x) t^i = \frac{1}{1 - (abx^2 + 2)t^2 + t^4} \left( 1 + bxt - t^2 \quad \frac{b}{a}t + bxt^2 - \frac{b}{a}t^3 \right). 
\]

3
Proof. Assume that $G(t)$ is the generating function for the polynomial \( \{ F_n(a, b, x) \}_{n \in \mathbb{N}} \). Then, we can write
\[
(1 - bxt - t^2) G(t) = F_0(a, b, x) + t (F_1(a, b, x) - bx F_0(a, b, x))
+ \sum_{i=2}^{\infty} (F_i(a, b, x) - bx F_{i-1}(a, b, x) - F_{i-2}(a, b, x)) t^i.
\]

Since \( F_{2i+1}(a, b, x) = bx F_{2i}(a, b, x) + F_{2i-1}(a, b, x) \), we get
\[
(1 - bxt - t^2) G(t) = F_0(a, b, x) + t (F_1(a, b, x) - bx F_0(a, b, x))
+ \sum_{i=1}^{\infty} (F_{2i}(a, b, x) - bx F_{2i-1}(a, b, x) - F_{2i-2}(a, b, x)) t^{2i}
= F_0(a, b, x) + t (F_1(a, b, x) - bx F_0(a, b, x))
+ (a - b) xt \sum_{i=1}^{\infty} F_{2i-1}(a, b, x) t^{2i-1}.
\]

Now, let
\[
g(t) = \sum_{i=1}^{\infty} F_{2i-1}(a, b, x) t^{2i-1}.
\]

Since
\[
F_{2i+1}(a, b, x) = bx F_{2i}(a, b, x) + F_{2i-1}(a, b, x)
= bx(ax F_{2i-1}(a, b, x) + F_{2i-2}(a, b, x)) + F_{2i-1}(a, b, x)
= (abx^2 + 1) F_{2i-1}(a, b, x) + bx F_{2i-2}(a, b, x)
= (abx^2 + 1) F_{2i-1}(a, b, x) + F_{2i-1}(a, b, x) - F_{2i-3}(a, b, x)
= (abx^2 + 2) F_{2i-1}(a, b, x) - F_{2i-3}(a, b, x),
\]
we have
\[
(1 - (abx^2 + 2) t^2 + t^4) g(t) = F_1(a, b, x) t + F_3(a, b, x) t^3 - (abx^2 + 2) F_1(a, b, x) t^3
+ \sum_{i=3}^{\infty} \left\{ F_{2i-1}(a, b, x) - (abx^2 + 2) F_{2i-3}(a, b, x) \right\} t^{2i-1}.
\]

Therefore,
\[
g(t) = \frac{F_1(a, b, x) t + F_3(a, b, x) t^3 - (abx^2 + 2) F_1(a, b, x) t^3}{1 - (abx^2 + 2) t^2 + t^4}
= \frac{F_1(a, b, x) t + (bx F_0(a, b, x) - F_1(a, b, x)) t^3}{1 - (abx^2 + 2) t^2 + t^4}
\]
and as a result, we get
\[
F_0(a, b, x) + tF_1(a, b, x) + t^2(\alpha xF_1(a, b, x) - F_0(a, b, x) - abx^2F_0(a, b, x)) + t^3(bxF_0(a, b, x) - F_1(a, b, x)) = 1 - (abx^2 + 2)t^2 + t^4.
\]
which is desired equality. □

**Theorem 2.4** For every \( n \in \mathbb{N} \), we write the Binet formula for the bi-periodic Fibonacci matrix polynomial as the form
\[
F_n(a, b, x) = A_1(\alpha^n - \beta^n) + B_1\left(\alpha^2\left[\frac{n}{2}\right]^2 - \beta^2\left[\frac{n}{2}\right]^2\right),
\]
where
\[
A_1 = \frac{(ab)\left[\frac{n}{2}\right]}{(ab)\left[\frac{n}{2}\right] + 1 (\alpha - \beta) x^{n+2\varepsilon(n+1)}}\left\{axF_1(a, b, x) - F_0(a, b, x)\right\}^{1-\varepsilon(n)},
\]
\[
B_1 = \frac{(b)\varepsilon(n) F_0(a, b, x)}{(ab)\left[\frac{n}{2}\right] + 1 (\alpha - \beta) x^{n+2\varepsilon(n+1)}}\varepsilon(n) = n - 2 \left\lfloor \frac{n}{2} \right\rfloor
\]
and \( \alpha, \beta \) are roots of \( r^2 - abxr - ab^2 = 0 \) equation.

**Proof.** Using the partial fraction decomposition, we can rewrite \( G(t) \) as
\[
G(t) = \frac{1}{\alpha - \beta} \left\{ \frac{t}{t^2-(\alpha+1)} \left\{ \begin{array}{l}
t \left\{ \beta (F_1(a, b, x) - bxF_0(a, b, x)) - bxF_0(a, b, x) \right\} \\
+ \beta \left( abx^2F_0(a, b, x) + F_0(a, b, x) - axF_1(a, b, x) \right) \\
- abx^2F_0(a, b, x) - axF_1(a, b, x) \\
\end{array} \right\} + \frac{t}{t^2-(\beta+1)} \left\{ \begin{array}{l}
t \left\{ \alpha (xF_0(a, b, x) - F_1(a, b, x)) + bxF_0(a, b, x) \right\} \\
+ \alpha \left( axF_1(a, b, x) - F_0(a, b, x) - abx^2F_0(a, b, x) \right) \\
+ abx^2F_0(a, b, x) - axF_1(a, b, x) \\
\end{array} \right\} \right\}.
\]
Since the Maclaurin series expansion of the function \( \frac{A-Bt}{t^2-C} \) is given by
\[
\frac{A-Bt}{t^2-C} = \sum_{n=0}^{\infty} BC^{-n-1}t^{2n+1} - \sum_{n=0}^{\infty} AC^{-n-1}t^{2n},
\]
Thus, we obtain

the generating function $G(t)$ can also be expressed as

$$
G(t) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left( \frac{1}{abx^2} \right)^n \begin{cases}
-\frac{abx^2}{(\alpha+1)^{n+1}(\beta+1)^{n+1}} \\
+\frac{abx^2}{(\alpha+1)^{n+1}(\beta+1)^{n+1}}
\end{cases} t^{2n+1}
$$

Thus, we obtain

$$
G(t) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left( \frac{1}{abx^2} \right)^n \begin{cases}
-\frac{abx^2}{(\alpha+1)^{n+1}(\beta+1)^{n+1}} \\
+\frac{abx^2}{(\alpha+1)^{n+1}(\beta+1)^{n+1}}
\end{cases} t^{2n+1}
$$

$$
+ \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left( \frac{1}{abx^2} \right)^n \begin{cases}
-\frac{abx^2}{(\alpha+1)^{n+1}(\beta+1)^{n+1}} \\
+\frac{abx^2}{(\alpha+1)^{n+1}(\beta+1)^{n+1}}
\end{cases} t^{2n+1}
$$

Combining the sums, we get

$$
G(t) = \sum_{n=0}^{\infty} \begin{cases}
\{ \mathcal{F}_1(a, b, x) \} \varepsilon(n) \left\{ \frac{-abx^2}{(abx^2)^{n+1}} \right\} t^n
\end{cases}
$$

Therefore, for all $n \geq 0$, from the definition of generating function, we have

$$
\mathcal{F}_n(a, b, x) = A_1 (\alpha^n - \beta^n) + B_1 \left( \alpha^2 \left[ \frac{n}{2} \right] + 2 - \beta^2 \left[ \frac{n}{2} \right] + 2 \right),
$$
which is desired. □

Now, for bi-periodic Fibonacci matrix polynomial, we give the some summations by considering its Binet formula.

**Corollary 2.5** For $k \geq 0$, the following statements are true:

\(\begin{align*}
\text{(i)} & \quad \sum_{k=0}^{n-1} F_k(a, b, x) = a^{\varepsilon(n)}b^{1-\varepsilon(n)}F_n(a, b, x) + a^{1-\varepsilon(n)}b^{\varepsilon(n)}F_{n-1}(a, b, x) \\
& \quad - aF_1(a, b, x) + abxF_0(a, b, x) - bF_0(a, b, x)\abx,
\end{align*}\)

\(\begin{align*}
\text{(ii)} & \quad \sum_{k=0}^{n} F_k(a, b, x) t^{-k} = \frac{1}{1 - (ab + 2)t^2 + t^4} \begin{pmatrix}
F_{n-1}(a, b, x) - F_{n+1}(a, b, x) \\
F_{n-3}(a, b, x) + F_{n+1}(a, b, x) \\
F_{n-2}(a, b, x) + t^4F_0(a, b, x) + t^3F_1(a, b, x) \\
-t^2[(abx^2 + 1)F_0(a, b, x) - axF_1(a, b, x)] \\
-t(F_1(a, b, x) - bxF_0(a, b, x))
\end{pmatrix},
\end{align*}\)

\(\begin{align*}
\text{(iii)} & \quad \sum_{k=0}^{\infty} F_k(a, b, x) t^{-k} = \frac{t}{1 - (abx^2 + 2)t^2 + t^4} \begin{pmatrix}
t^3 + bxt^2 - t & \frac{b}{a}t^2 + bxt - \frac{b}{a} \\
t^2 + axt - 1 & t^3 - (abx^2 + 1)t + bx
\end{pmatrix},
\end{align*}\)

where $\alpha, \beta$ are roots of $r^2 - abx^2r - abx^2 = 0$ equation and $\varepsilon(n) = n - 2 \left\lfloor \frac{n}{2} \right\rfloor$.

### 3 Binomial transforms for Fibonacci matrix polynomial

In this section, we mainly focus on the new matrix polynomial that obtained by using the bi-periodic Fibonacci matrix polynomial.
Definition 3.1 For \( n \in \mathbb{N} \), the matrix polynomial \( (A_n(a, b, x)) \) obtained by using bi-periodic Fibonacci matrix polynomial is defined by

\[
A_n(a, b, x) = \sqrt{x} a^{\frac{\varepsilon(n)}{2}} b^{\frac{1-\varepsilon(n)}{2}} F_n(a, b, x) \tag{3.1}
\]

where \( a, b \) are nonzero real numbers and \( \varepsilon(n) = n - 2 \lfloor \frac{n}{2} \rfloor \).

In the following, we introduce the binomial transform and \( k \)-binomial transform of the this matrix polynomial.

Definition 3.2 For \( n \in \mathbb{N} \), the binomial and \( k \)-binomial transforms of the matrix polynomial \( (A_n(a, b, x)) \) are defined by

\[
b_n(a, b, x) = \sum_{i=0}^{n} \binom{n}{i} A_i(a, b, x), \tag{3.2}
\]

\[
w_n(a, b, x) = \sum_{i=0}^{n} \binom{n}{i} k^n A_i(a, b, x), \tag{3.3}
\]

respectively, where \( a, b \) are nonzero real numbers.

Throughout this section, we will take \( k = x \sqrt{ab} \).

Now, we give some properties for the binomial transform of the matrix polynomial \( (A_n(a, b, x)) \).

Theorem 3.3 The binomial transform of the matrix polynomial \( (A_n(a, b, x)) \) verifies the following relations:

\[
(i) \quad b_{n+1}(a, b, x) = \sum_{i=0}^{n} \binom{n}{i} \sqrt{x} a^{\frac{\varepsilon(n)}{2}} b^{\frac{1-\varepsilon(n)}{2}} \left( F_i(a, b, x) + a^{\frac{1-2\varepsilon(i)}{2}} b^{\frac{2\varepsilon(i)-1}{2}} F_{i+1}(a, b, x) \right),
\]

\[
(ii) \quad b_{n+1}(a, b, x) = (x \sqrt{ab} + 2) b_n(a, b, x) - x \sqrt{ab} b_{n-1}(a, b, x),
\]

\[
(iii) \quad b_n(a, b, x) = C \left( r_2^n(x) - r_1^n(x) \right) + \frac{\sqrt{x} F_0(a, b, x)}{2} \left( r_2^n(x) + r_1^n(x) \right),
\]
(iv) \( b_n(t) = \frac{\sqrt{b x} F_0(a,b,x) + t(\sqrt{a x} F_1(a,b,x) - \sqrt{b x} (1+x \sqrt{a b}) F_0(a,b,x))}{1 - (x \sqrt{a b + 2}) t + x \sqrt{a b} t^2} \),

(v) \( \sum_{n=0}^{\infty} \frac{b_n(a,b,x)^n}{n!} = C \left( e^{r_2(x)t} - e^{r_1(x)t} \right) + \frac{\sqrt{bx}}{2} F_0(a,b,x) \left( e^{r_2(x)t} + e^{r_1(x)t} \right) \),

where \( r_1(x), r_2(x) \) are roots of the \( r^2 - (x \sqrt{a b} + 2) r + x \sqrt{a b} = 0 \) equation

and \( C = \frac{b x \sqrt{a x} F_0(a,b) - 2 \sqrt{a x} F_1(a,b)}{2 \sqrt{a b} x^2 + 4} \).

**Proof.** We will prove the first two equalities because the proof of the others can be done in similar ways.

(i) By considering the property of binomial numbers, we can write

\[
\begin{align*}
  b_{n+1}(a,b,x) &= \sum_{i=0}^{n+1} \binom{n+1}{i} A_i(a,b,x) = \sum_{i=0}^{n+1} \binom{n+1}{i} \sqrt{x a} \varepsilon(i) b^{1-\varepsilon(i)} F_i(a,b,x) \\
  &= \sum_{i=0}^{n+1} \left[ \binom{n}{i} + \binom{n}{i-1} \right] \sqrt{x a} \varepsilon(i) b^{1-\varepsilon(i)} F_i(a,b,x). 
\end{align*}
\]

If necessary arrangements are made, we have

\[
\begin{align*}
  b_{n+1}(a,b,x) &= \sum_{i=0}^{n} \binom{n}{i} \sqrt{x a} \varepsilon(i) b^{1-\varepsilon(i)} F_i(a,b,x) + \sum_{i=0}^{n+1} \binom{n}{i-1} \sqrt{x a} \varepsilon(i) b^{1-\varepsilon(i)} F_i(a,b,x) \\
  &= \sum_{i=0}^{n} \binom{n}{i} \sqrt{x a} \varepsilon(i) b^{1-\varepsilon(i)} F_i(a,b,x) + \sum_{i=0}^{n} \binom{n}{i} \sqrt{x a} \frac{1-\varepsilon(i)}{2} b^{1-\varepsilon(i)} F_{i+1}(a,b,x) \\
  &= \sum_{i=0}^{n} \binom{n}{i} a^{\frac{1-\varepsilon(n)}{2}} b^{1-\varepsilon(n)} \left( \sqrt{x} F_i(a,b,x) + \sqrt{x a} \frac{1-2\varepsilon(i)}{2} b^{2(i-1)} F_{i+1}(a,b,x) \right). 
\end{align*}
\]

Also, we can write as \( b_{n+1}(a,b,x) = b_n(a,b,x) + \sum_{i=0}^{n} \binom{n}{i} A_{i+1}(a,b,x) \).
(ii) By using the equation (i), we can write
\[
b_{n+1}(a,b,x) = \sum_{i=1}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \sqrt{x a_{i}^{ \frac{\epsilon(i)}{2} b^{\frac{1-\epsilon(i)}{2}}} (F_{i}(a,b,x) + a^{\frac{1-2\epsilon(i)}{2}} b^{\frac{2\epsilon(i)-1}{2}} F_{i+1}(a,b,x)} + \sqrt{bx} F_{0}(a,b,x) + \sqrt{ax} F_{1}(a,b,x) = \left( x \sqrt{ab} + 1 \right) \sum_{i=1}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \sqrt{x a_{i}^{ \frac{\epsilon(i)}{2} b^{\frac{1-\epsilon(i)}{2}}} F_{i}(a,b,x)} + \sum_{i=1}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \sqrt{x a_{i}^{ \frac{\epsilon(i)}{2} b^{\frac{1-\epsilon(i)}{2}}} F_{i-1}(a,b,x) + \sqrt{bx} F_{0}(a,b,x) + \sqrt{ax} F_{1}(a,b,x) = \left( x \sqrt{ab} + 1 \right) b_{n}(a,b,x) + \sum_{i=1}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \sqrt{x a_{i}^{ \frac{\epsilon(i)}{2} b^{\frac{1-\epsilon(i)}{2}}} F_{i-1}(a,b,x)} + \sum_{i=1}^{n-1} \left( \begin{array}{c} n-1 \\ i \end{array} \right) \sqrt{x a_{i}^{ \frac{\epsilon(i)}{2} b^{\frac{1-\epsilon(i)}{2}}} F_{i-1}(a,b,x) - x \sqrt{ab} F_{0}(a,b,x) + \sqrt{ax} F_{1}(a,b,x) = x \sqrt{ab} b_{n-1}(a,b,x) + \sum_{i=1}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \sqrt{x a_{i}^{ \frac{\epsilon(i)}{2} b^{\frac{1-\epsilon(i)}{2}}} F_{i-1}(a,b,x)} + \sum_{i=1}^{n-1} \left( \begin{array}{c} n-1 \\ i \end{array} \right) \sqrt{x a_{i}^{ \frac{\epsilon(i)}{2} b^{\frac{1-\epsilon(i)}{2}}} F_{i-1}(a,b,x)} - x \sqrt{ab} F_{0}(a,b,x) + \sqrt{ax} F_{1}(a,b,x)
\]
Thus, we obtain
\[
b_{n}(a,b,x) = \left( x \sqrt{ab} + 1 \right) b_{n-1}(a,b,x) + \sum_{i=1}^{n-1} \left( \begin{array}{c} n-1 \\ i \end{array} \right) \sqrt{x a_{i}^{ \frac{\epsilon(i)}{2} b^{\frac{1-\epsilon(i)}{2}}} F_{i-1}(a,b,x)} + \sum_{i=1}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \sqrt{x a_{i}^{ \frac{\epsilon(i)}{2} b^{\frac{1-\epsilon(i)}{2}}} F_{i-1}(a,b,x)} - x \sqrt{ab} F_{0}(a,b,x) + \sqrt{ax} F_{1}(a,b,x)
\]
And we get
\[
b_{n+1}(a,b,x) = \left( x \sqrt{ab} + 1 \right) b_{n}(a,b,x) + b_{n}(a,b,x) - x \sqrt{ab} b_{n-1}(a,b,x).
\]
\[\square\]

From the definition of binomial and k-binomial transform, we obtain
\[
w_{n}(a,b,x) = k^{n} b_{n}(a,b,x) = \left( x \sqrt{ab} \right)^{n} b_{n}(a,b,x). \text{ Thus, for every } n \in \mathbb{N}, \text{ in the following equalities are true.}
\[ w_{n+1}(a, b, x) = \left( abx^2 + 2x\sqrt{ab} \right) w_n(a, b, x) - abx^3 \sqrt{ab} w_{n-1}(a, b, x), \]

\[ w_n(t) = \sqrt{bx} \mathcal{F}_0(a, b, x) + t \left( ax \sqrt{bx} \mathcal{F}_1(a, b, x) - \left( bx \sqrt{ab} + abx^2 \sqrt{bx} \right) \mathcal{F}_0(a, b, x) \right) \]

\[ 1 - (abx^2 + 2x\sqrt{ab})t + abx^3 \sqrt{ab}t^2, \]

\[ w_n(a, b, x) = C \left( r_3^n(x) - r_3^n(x) \right) + \frac{\sqrt{bx} \mathcal{F}_0(a, b, x)}{2} \left( r_4^n(x) + r_3^n(x) \right), \]

where \( r_3(x) \) and \( r_4(x) \) are roots of \( r^2 - \left( 2x\sqrt{ab} + abx^2 \right) r - x^3 ab \sqrt{ab} = 0 \) equation and \( C = \frac{bx \sqrt{ax} \mathcal{F}_0(a, b, x) - 2 \sqrt{ax} \mathcal{F}_1(a, b, x)}{2 \sqrt{abx^{1/2}}} \).

Now, we introduce the rising \( k \)-binomial transform of the matrix polynomial \( \mathcal{A}_n(a, b, x) \).

**Definition 3.4** For \( n \in \mathbb{N} \), the rising \( k \)-binomial transform of the matrix polynomial \( \mathcal{A}_n(a, b, x) \) is defined by

\[ r_n(a, b, x) = \sum_{i=0}^{n} \binom{n}{i} k^i \mathcal{A}_i(a, b, x) \quad (3.4) \]

where \( a, b \) are nonzero real numbers.

**Theorem 3.5** For every \( n \in \mathbb{N} \), the rising \( k \)-binomial transform of the matrix polynomial \( \mathcal{A}_n(a, b, x) \) is the polynomial \( \mathcal{A}_{2n}(a, b, x) \), that is

\[ r_n(a, b, x) = \mathcal{A}_{2n}(a, b, x) \quad (3.5) \]

**Proof.** From the Theorem 2.4, we can write

\[
\begin{align*}
\mathcal{A}_n(a, b, x) &= \sum_{i=0}^{n} \binom{n}{i} \left( x \sqrt{ab} \right)^i \sqrt{a} \mathcal{F}_0(a, b) + \sqrt{b} \mathcal{F}_1(a, b) \\
&= \sum_{i=0}^{n} \binom{n}{i} \left( ab \right)^{\frac{i}{2}} x^{i+\frac{1}{2}} a^{\frac{\epsilon(i)}{2}} b^{\frac{1-\epsilon(i)}{2}} \left[ A_1 \left( \alpha^i - \beta^i \right) + B_1 \left( \alpha^{2\left(\frac{i}{2}\right)+2} - \beta^{2\left(\frac{i}{2}\right)+2} \right) \right].
\end{align*}
\]

Consequently, making the necessary arrangements, we have

\[
\mathcal{A}_n(a, b, x) = \sqrt{bx} \left[ A_1 \left( \alpha^{2n} - \beta^{2n} \right) + B_1 \left( \alpha^{2n+2} - \beta^{2n+2} \right) \right] = \sqrt{bx} \mathcal{F}_{2n}(a, b, x) = \mathcal{A}_{2n}(a, b, x).
\]

\[ \Box \]
Theorem 3.6  For every $n \in \mathbb{N}$, the recurrence relation for rising $k$-binomial transform of the matrix polynomial $(A_n(a,b,x))$,

$$r_{n+1}(a,b,x) = (abx^2 + 2) r_n(a,b,x) - r_{n-1}(a,b,x), \quad (3.6)$$

where $r_0 = \sqrt{bx} \mathcal{F}_0(a,b,x)$ and $r_1 = \sqrt{bx} \mathcal{F}_0(a,b,x) + ax \sqrt{bx} \mathcal{F}_1(a,b,x)$.

Proof. For the matrix polynomial $(A_{2n}(a,b,x))$, the following relation can be written

$$A_{2n+2}(a,b,x) = \sqrt{bx} \mathcal{F}_{2n+2}(a,b,x) = \sqrt{bx} (ax \mathcal{F}_{2n+1}(a,b,x) + \mathcal{F}_{2n}(a,b,x))$$

$$= \sqrt{bx} (ax (bx \mathcal{F}_{2n}(a,b,x) + \mathcal{F}_{2n-1}(a,b,x)) + \mathcal{F}_{2n}(a,b,x))$$

$$= \sqrt{bx} ((abx^2 + 1) \mathcal{F}_{2n}(a,b,x) + \mathcal{F}_{2n}(a,b,x) - \mathcal{F}_{2n-2}(a,b,x))$$

$$= (abx^2 + 2) \left( \sqrt{bx} \mathcal{F}_{2n}(a,b,x) \right) - \sqrt{bx} \mathcal{F}_{2n-2}(a,b,x)$$

$$= (abx^2 + 2) A_{2n}(a,b,x) - A_{2n-2}(a,b,x).$$

Therefore, from the Theorem 3.5 we find the desired result. □

In the following, we introduce the falling $k$-binomial transform of the matrix polynomial $(A_n(a,b,x))$.

Definition 3.7  For $n \in \mathbb{N}$, the falling $k$-binomial transform of the matrix polynomial $(A_n(a,b,x))$ is defined by

$$f_n(a,b,x) = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} A_i(a,b,x) \quad (3.7)$$

where $a,b$ are nonzero real numbers.

Theorem 3.8  For every $n \in \mathbb{N}$, the recurrence relation for falling $k$-binomial transform of the matrix polynomial $(A_n(a,b,x))$, 

$$f_{n+1}(a,b,x) = 3x \sqrt{ab} f_n(a,b,x) - (2abx^2 - 1) f_{n-1}(a,b,x),$$

where $f_0(a,b,x) = \sqrt{bx} \mathcal{F}_0(a,b,x)$ and $f_1(a,b,x) = bx \sqrt{ax} \mathcal{F}_0(a,b,x) + \sqrt{ax} \mathcal{F}_1(a,b,x)$.
**Proof.** Firstly, we prove that

\[ f_{n+1}(a, b, x) = \sum_{i=0}^{n} \binom{n}{i} (ab)^{\frac{n+1-i}{2}} x^{n+1-i} (A_{i+1}(a, b, x) + A_i(a, b, x)) \]

Thus, similar to the (ii) of Theorem 3.3, the proof can be done. □

**Theorem 3.9** For every \( n \in \mathbb{N} \), the Binet formula for falling and rising \( k \)-binomial transform of the matrix polynomial \( (A_n(a, b, x)) \),

\[ f_n(a, b, x) = C(r_5^n(x) - r_6^n(x)) + \frac{\sqrt{bx}F_0(a, b, x)}{2} (r_8^n(x) + r_7^n(x)), \]

\[ r_n(a, b, x) = C(r_0^n(x) - r_5^n(x)) + \frac{\sqrt{bx}F_0(a, b, x)}{2} (r_6^n(x) + r_8^n(x)), \]

where \( r_5(x) \) and \( r_6(x) \) are roots of \( r^2 - (abx^2 + 2)r + 1 = 0 \) and \( C = \frac{bx\sqrt{ax}F_0(a,b,x)-2\sqrt{bx}F_1(a,b,x)}{2\sqrt{abx^2+4}} \) also \( r_7(x) \) and \( r_8(x) \) are roots of \( r^2 - 3x\sqrt{abr} + 2abx^2 - 1 = 0 \).

**Proof.** Using the initial conditions, the Theorem can be proved. □

**Conclusion**

In this paper, we define the matrix polynomial and give new equalities for it. Then, defining the transforms for this matrix polynomial, we get some properties of this transforms. Thus, it is obtained a new generalization for the polynomials, matrix sequences and number sequences that have the similar recurrence relation in the literature. That is,

- If we take \( a = b = 1 \) in Section 2, we get the some properties of the Fibonacci polynomial.
- If we take \( a = b = 2 \) in Section 2, we get the some properties of the Pell polynomial.
• If we take \( a = b = k \) in Section 2, we get the some properties of the \( k \)-Fibonacci polynomial.

If we choose \( x = 1 \) in Section 3, then we obtain some properties for binomial transforms of bi-periodic Fibonacci matrix sequence and bi-periodic Fibonacci numbers.

Also, for different values of \( a \) and \( b \), we obtain the some properties of binomial transforms of the well-known matrix sequence and number sequence in the literature:

• If we choose \( a = b = 1 \), we obtain the some properties for binomial transforms of Fibonacci matrix sequence and Fibonacci numbers.

• If we choose \( a = b = 2 \), we obtain the some properties for binomial transforms of Pell matrix sequence and Pell numbers.

• If we choose \( a = b = k \), we obtain the some properties for binomial transforms of \( k \)-Fibonacci matrix sequence and \( k \)-Fibonacci numbers.

References

[1] Bilgici G., Two generalizations of Lucas sequence, Applied Mathematics and Computation, 245 (2014), 526-538.

[2] Civciv H., Türkmen R., On the \((s, t)\)-Fibonacci and Fibonacci matrix sequences, Ars Combinatoria, 87 (2008), 161-173.

[3] Coskun A., Taskara N., The matrix sequence of bi-periodic Fibonacci numbers, arXiv, 1603.07487, (2016).

[4] Edson M., Yayanie O., A new Generalization of Fibonacci sequence and Extended Binet’s Formula, Integers, 9 (2009), 639-654.

[5] Falcon S., Plaza A., Binomial Transforms of the \( k \)-Fibonacci Sequence, Int. J. of Nonlinear Sciences & Numerical Simulation, 10 (2009), 1527-1538.
[6] Falcon S., Plaza A., On k-Fibonacci sequences and polynomials and their derivatives, *Chaos, Solitons & Fractal*, 39 (2009), 1005-1019.

[7] Hoggatt Jr V. E., Bicknell M., Generalized Fibonacci polynomials., *Fibonacci Quart.*, 11(5) (1973), 457-465.

[8] Horadam A.F., A generalized Fibonacci sequence, *Math. Mag.*, 68 (1961), 455–459.

[9] Koshy T., *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons Inc, NY, 2001.

[10] Ocal A.A., Tuglu N., Altinisik E., On the representation of k-generalized Fibonacci and Lucas numbers, *Applied Mathematics and Computations*, 170 (1) (2005), 584–596.

[11] Tasci D., Firengiz M.C., Incomplete Fibonacci and Lucas p-numbers, *Mathematical and Computer Modelling*, 52(9) (2010), 1763-1770.

[12] Uslu K., Uygun S., The (s,t) Jacobsthal and (s,t) Jacobsthal-Lucas Matrix Sequences, *Ars Combinatoria*, 108 (2013), 13-22.

[13] Yazlik Y., Taskara N., A note on generalized k-Horadam sequence, *Computers and Mathematics with Applications*, 63 (2012), 36-41.

[14] Yilmaz N., Coskun A., Taskara N., On properties of bi-periodic Fibonacci and Lucas Polynomials, *ICNAAM 2016 (14th International Conference of Numerical Analysis and Applied Mathematics 2016)*, 19-25 September, Rhodes, Greece, 2016.

[15] Yilmaz N., Taskara N., Binomial Transforms of the Padovan and Perrin Matrix Sequences, *Abstract and Applied Analysis*, 2013, (2013).