Robust optimality and duality for composite uncertain multiobjective optimization in Asplund spaces with its applications

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Abstract

This article is devoted to investigate a nonsmooth/nonconvex uncertain multiobjective optimization problem with composition fields ((CUP) for brevity) over arbitrary Asplund spaces. Employing some advanced techniques of variational analysis and generalized differentiation, we establish necessary optimality conditions for weakly robust efficient solutions of (CUP) in terms of the limiting subdifferential. Sufficient conditions for the existence of (weakly) robust efficient solutions to such a problem are also driven under the new concept of pseudo-quasi convexity for composite functions. We formulate a Mond-Weir-type robust dual problem to the primal problem (CUP), and explore weak, strong, and converse duality properties. In addition, the obtained results are applied to an approximate uncertain multiobjective problem and a composite uncertain multiobjective problem with linear operators.

Keywords Composite robust multiobjective optimization . Optimality conditions . Duality . Limiting subdifferential . Generalized convexity

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1 Introduction

Composite optimization problems have been widely recognized as a great important class of optimization problems. Their significance stems not only from their ability to cover a wide range of practical optimization problems, including minimax problems and penalty methods for constrained optimization, but also from their provision of a unified framework for studying the convergence behavior of various optimization algorithms and Lagrangian conditions (see, e.g., Fletcher 1987; Burke 1985; Rockafellar 1988). Robust optimization has become a powerful deterministic structure to study optimization problems under data uncertainty (Beck and Ben-Tal 2009; Ben-Tal et al. 2009; Ben-Tal and Nemirovski 2002, 2008; Jeyakumar et al. 2015; Jeyakumar and Li

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An uncertain optimization problem usually associated with its robust counterpart which is known as the problem that the uncertain objective and constraint are satisfied for all possible scenarios within a prescribed uncertainty set. For classic contributions in this field, we refer to Ben-Tal et al. (2009). Robust optimization approach considers the cases in which no probabilistic information about the uncertainties is given. In particular, most practical optimization problems often deal with uncertain data due to measurement errors, unforeseeable future developments, fluctuations, or disturbances, and depend on conflicting goals due to multiobjective decision makers which have different optimization criteria. So, the robust multiobjective optimization is highly interesting in optimization theory and important in applications.

The first idea of robustness in multiobjective optimization was explored by Branke (1998) and provided by Deb and Gupta (2006). Here, the robustness concept is treated as a kind of sensitivity in the objective space against perturbations in the decision space. Kuroiwa and Lee (2012) followed the robust approach (the worst-case approach) for multiobjective convex programming problems under uncertainty in both the objective functions and the constraints, and investigated necessary optimality conditions for weakly and properly robust efficient solutions. Ehrgott et al. (2014) extended the concept as presented by Kuroiwa and Lee, and interpreted a robust solution as a set of feasible solutions to the multiobjective problem of maximizing the objective function over the uncertainty set; see also the paper using the same approach (Chuong 2020). After these works, Ide and Köbis (2014) derived various concepts of efficiency for uncertain multiobjective optimization problems, where only the objective functions were contaminated with different uncertain data, by replacing the set ordering with other set orderings, and presented numerical results on the occurrence of the various concepts.

Recently, Lee and Lee (2018) dealt with robust multiobjective nonlinear semi-infinite programming with uncertain constraints, investigated necessary/sufficient conditions for weakly robust efficient solutions with the worst-case approach, and derived Wolfe-type dual problem and duality results. Chuong (2016) considered uncertain multiobjective optimization problems involving nonsmooth/nonconvex functions, and introduced the concept of (strictly) generalized convexity to establish optimality and duality theories with respect to limiting subdifferential for robust (weakly) Pareto solutions. Chen et al. (2019) studied necessary/sufficient conditions in terms of Clarke subdifferential for weakly and properly robust efficient solutions of nonsmooth multiobjective optimization problems with data uncertainty, formulated Mond-Weir-type dual problem and Wolfe-type dual problem, and explored duality results between the primal one and its dual problems under the generalized convexity assumptions. Fakhar et al. (2018) presented the nonsmooth sufficient optimality conditions for robust (weakly) efficient solutions and Mond-Weir-type duality results by applying the new concept of generalized convexity.

In addition to those stated above, the concept of approximate efficient solutions in multiobjective optimization problems, which can be viewed as feasible points whose objective values display a prescribed error $\varepsilon$ in the optimal values of the vector objective, has been studied widely. Optimality conditions and duality theories of $\varepsilon$-efficient solutions and $\varepsilon$-quasi-efficient solutions for various optimization problems under uncertainty have been presented by Chuong and Kim (2016); Dutta...
To the best of our knowledge, the most important results obtained in these directions did pay attention to finite-dimensional problems not dealing with composite functions. Hence, an infinite-dimensional setting is suitable to induce optimality and duality in composite optimization.

Suppose that \( F : X \to W \) and \( f : W \to Y \) be vector-valued functions between Asplund spaces, and that \( K \subset Y \) be a pointed \((\text{i.e., } K \cap (-K) = \{0\})\) closed convex cone with nonempty topological interior. Consider the following composite multiobjective optimization problem of the form

\[
\text{(CP)} \quad \min_{K} (f \circ F)(x)
\]

\[
\text{s.t. } (g_i \circ G_i)(x) \leq 0, \quad i = 1, 2, \ldots, n,
\]

where \( G = (G_1, G_2, \ldots, G_n) \) and \( g = (g_1, g_2, \ldots, g_n) \) are vector-valued functions on Asplund spaces with components \( G_i : X \to Z \) and \( g_i : Z \to \mathbb{R}, i = 1, 2, \ldots, n \). The problem \((\text{CP})\) with data uncertainty in the constraints can be written by the composite uncertain multiobjective optimization problem

\[
\text{(CUP)} \quad \min_{K} (f \circ F)(x)
\]

\[
\text{s.t. } (g_i \circ G_i)(x, v_i) \leq 0, \quad i = 1, 2, \ldots, n,
\]

where \( x \in X \) is a decision variable, \( v_i \) is an uncertain parameter which belongs to the sequentially compact topological space \( V_i \), and \( G_i : X \times V_i \to Z \times U_i \) and \( g_i : Z \times U_i \to \mathbb{R} \) are given functions for topological space \( U_i \), \( i = 1, 2, \ldots, n \).

For investigating the problem \((\text{CUP})\), we associate with it the so-called robust counterpart

\[
\text{(CRP)} \quad \min_{K} (f \circ F)(x)
\]

\[
\text{s.t. } (g_i \circ G_i)(x, v_i) \leq 0, \quad \forall v_i \in V_i, \quad i = 1, 2, \ldots, n.
\]

Let

\[
C := \{ x \in X \mid (g_i \circ G_i)(x, v_i) \leq 0, \quad \forall v_i \in V_i, \quad i = 1, 2, \ldots, n \}
\]

be the feasible set of the problem \((\text{CRP})\).

Remark 1.1. The problem \((\text{CUP})\) provides a quite general framework for various uncertain multiobjective optimization problems as follows

(i) if \( X = W = Z \), \( V_i = U_i \), \( i = 1, 2, \ldots, n \), and \( F \) and \( G \) are identical maps, the problem \((\text{CUP})\) collapses to the following uncertain multiobjective optimization problem stated by Saadati and Oveisigha (2022b)

\[
\text{(UP)} \quad \min_{K} \left\{ f(x) \mid x \in X, \quad g_i(x, v_i) \leq 0, \quad i = 1, 2, \ldots, n \right\}.
\]

(ii) if \( X = W = Z \), \( V_i = U_i \), \( i = 1, 2, \ldots, n \), \( V_{n+j} = U_{n+j}, \quad j = 1, 2, \ldots, m \), \( F \) and \( G \) are identical maps, and \( g \) is given by

\[
g(x, v) := (g_1(x, v_1), g_2(x, v_2), \ldots, g_n(x, v_n), h_1(x, v_{n+1}), h_2(x, v_{n+2}), \ldots, h_m(x, v_{n+m})),
\]

\[
x \in X,
\]

\[
v_i \in V_i, \quad i = 1, 2, \ldots, n,
\]

\[
v_{n+j} \in V_{n+j}, \quad j = 1, 2, \ldots, m,
\]
then the problem (CUP) reduces to a (standard) uncertain multiobjective optimization problem of the form

\[
(SUP) \quad \min_K \left\{ f(x) \mid x \in X, \ g_i(x, v_i) \leq 0, \ i = 1, 2, \ldots, n, \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad h_j(x, v_{n+j}) = 0, \ j = 1, 2, \ldots, m \right\}.
\]

(iii) if \( X = W = Z, Y := \mathbb{R}^p, K := \mathbb{R}^p_+, V_i = U_i, i = 1, 2, \ldots, m, \) and \( F \) and \( G \) are identical maps, the problem (CUP) collapses to an uncertain multiobjective optimization problem defined by Fakhar et al. (2018).

(iv) if \( X = W = Z := \mathbb{R}^n, Y := \mathbb{R}^m, K := \mathbb{R}^m_+, V_i = U_i, i = 1, 2, \ldots, l, \) and \( F \) and \( G \) are identical maps, then the problem (CUP) reduces to an uncertain multiobjective optimization problem presented by Chuong (2016).

**Definition 1.1.** (i) We say that a vector \( \bar{x} \in X \) is a robust efficient solution of the problem (CUP), and write \( \bar{x} \in S(CRP) \), if \( \bar{x} \) is an efficient solution of the problem (CRP), i.e., \( \bar{x} \in C \) and

\[
(f \circ F)(x) - (f \circ F)(\bar{x}) \notin -K \setminus \{0\}, \quad \forall x \in C.
\]

(ii) A vector \( \bar{x} \in X \) is called a weakly robust efficient solution of the problem (CUP), and write \( \bar{x} \in S^{w}(CRP) \), if \( \bar{x} \) is a weakly efficient solution of the problem (CRP), i.e., \( \bar{x} \in C \) and

\[
(f \circ F)(x) - (f \circ F)(\bar{x}) \notin \text{int} K, \quad \forall x \in C.
\]

The rest of this paper is organized as follows. Section 2 contains some preliminary definitions and several auxiliary results from nonsmooth variational analysis. In Section 3, we establish necessary/sufficient optimality conditions for the existence of weakly robust efficient solutions and also sufficient conditions for robust efficient solutions of the problem (CUP) in terms of the limiting subdifferential. Section 4 is concerned with the duality relations for (weakly) robust efficient solutions between the corresponding problems. The concluding Section 5 provides applications of special composite forms to the robust multiobjective optimization.

## 2 Preliminaries

Throughout this paper, we use the standard notation of variational analysis; see, for example, Mordukhovich 2006. Unless otherwise stated, all the spaces under consideration are Asplund with the norm \( \| \cdot \| \) and the canonical pairing \( \langle \cdot, \cdot \rangle \) between the space \( X \) in question and its dual \( X^* \) equipped with the weak* topology \( w^* \). By \( B_X(x, r) \), we denote the closed ball centered at \( x \in X \) with radius \( r > 0 \), while \( B_X \) and \( B_{X^*} \) stand for the closed unit ball in \( X \) and \( X^* \), respectively. For a given nonempty set \( \Omega \subset X \), the symbols \( \text{co} \Omega, \text{cl} \Omega, \) and \( \text{int} \Omega \) indicate the convex hull, topological closure, and topological interior of \( \Omega \), respectively, while \( \text{cl}^* \Omega \) stands for the weak* topological closure of \( \Omega \subset X^* \). The dual cone of \( \Omega \) is the set

\[
\Omega^+ := \left\{ x^* \in X^* \mid \langle x^*, x \rangle \geq 0, \ \forall x \in \Omega \right\}.
\]
For \( n \in \mathbb{N} := \{1, 2, \ldots \} \), \( \mathbb{R}^n_+ \) denotes the nonnegative orthant of \( \mathbb{R}^n \). Besides, the symbol \( T^\top \) signifies the adjoint operator or conjugate transpose of the linear operator \( T \).

A given set-valued mapping \( H : \Omega \subset X \rightrightarrows X^* \) is called weak* closed at \( \bar{x} \in \Omega \) if for any sequence \( \{x_k\} \subset \Omega \), \( x_k \to \bar{x} \), and any sequence \( \{x_k^*\} \subset X^* \), \( x_k^* \rightharpoonup x^* \), one has \( x^* \in H(\bar{x}) \).

For a set-valued mapping \( H : X \rightrightarrows X^* \), the sequential Painlevé-Kuratowski upper/outer limit of \( H \) as \( x \to \bar{x} \) is defined by

\[
\limsup_{x \to \bar{x}} H(x) := \big\{ x^* \in X^* \mid \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \rightharpoonup x^* \text{ with } x_k^* \in H(x_k) \text{ for all } k \in \mathbb{N} \big\}.
\]

Let \( \Omega \subset X \) be locally closed around \( \bar{x} \in \Omega \), i.e., there is a neighborhood \( U \) of \( \bar{x} \) for which \( \Omega \cap \overline{U} \) is closed. The Fréchet normal cone \( \hat{N}(\bar{x}; \Omega) \) and the Mordukhovich normal cone \( N(\bar{x}; \Omega) \) to \( \Omega \) at \( \bar{x} \in \Omega \) are defined, respectively, by

\[
\hat{N}(\bar{x}; \Omega) := \left\{ x^* \in X^* \mid \limsup_{x \to \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}
\]

and

\[
N(\bar{x}; \Omega) := \limsup_{x \to \bar{x}} \hat{N}(x; \Omega),
\]

where \( x \to \bar{x} \) stands for \( x \to \bar{x} \) with \( x \in \Omega \). If \( \bar{x} \notin \Omega \), we put \( \hat{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega) := \emptyset \).

For an extended real-valued function \( \phi : X \to \mathbb{R} \), the limiting/Mordukhovich subdifferential and the regular/Fréchet subdifferential of \( \phi \) at \( \bar{x} \in X \) are given, respectively, by

\[
\partial \phi(\bar{x}) := \left\{ x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \phi(\bar{x})); \text{epi } \phi) \right\}
\]

and

\[
\hat{\partial} \phi(\bar{x}) := \left\{ x^* \in X^* \mid (x^*, -1) \in \hat{N}((\bar{x}, \phi(\bar{x})); \text{epi } \phi) \right\}.
\]

If \( |\phi(\bar{x})| = \infty \), then one puts \( \partial \phi(\bar{x}) = \hat{\partial} \phi(\bar{x}) = : \emptyset \).

For a vector-valued function \( f : X \to Y \), we apply a scalarization formula with respect to some \( y^* \in Y^* \) defined by

\[
\langle y^*, f \rangle(x) := \langle y^*, f(x) \rangle, \quad x \in X.
\]

We recall another expression of the scalarization scheme in the next lemma.

**Lemma 2.1.** Let \( y^* \in Y^* \), and let \( f : X \to Y \) be Lipschitz continuous around \( \bar{x} \in X \). We have

(i) (See (Mordukhovich et al. 2006, Proposition 3.5)) \( x^* \in \hat{\partial}(y^*, f)(\bar{x}) \iff (x^*, -y^*) \in \hat{N}((\bar{x}, f(\bar{x})); \text{gph } f) \).

(ii) (See (Mordukhovich 2006, Theorem 1.90)) \( x^* \in \partial(y^*, f)(\bar{x}) \iff (x^*, -y^*) \in N((\bar{x}, f(\bar{x})); \text{gph } f) \).

The following lemma gives a **chain rule** for the limiting subdifferential.
Lemma 2.2. (See (Mordukhovich 2006, Corollary 3.43)) Let $f : X \to Y$ be locally Lipschitz at $\bar{x} \in X$, and let $\phi : Y \to \mathbb{R}$ be locally Lipschitz at $f(\bar{x})$. Then one has

$$\partial (\phi \circ f)(\bar{x}) \subset \bigcup_{y^* \in \partial \phi(f(\bar{x}))} \partial (g^*, f)(\bar{x}).$$

The sum rule for the limiting subdifferential will be useful in our analysis.

Lemma 2.3. (See (Mordukhovich 2006, Theorem 3.36)) Let $\phi_i : X \to \mathbb{R}$, $(i \in \{1, 2, \ldots, n\}, n \geq 2)$, be lower semicontinuous around $\bar{x}$, and let all but one of these functions be Lipschitz continuous around $\bar{x} \in X$. Then one has

$$\partial (\phi_1 + \phi_2 + \cdots + \phi_n)(\bar{x}) \subset \partial \phi_1(\bar{x}) + \partial \phi_2(\bar{x}) + \cdots + \partial \phi_n(\bar{x}).$$

It is worth to mention that inspecting the proof of (Chuong 2016, Theorem 3.3) (see also Mordukhovich and Nghia 2013a,b) reveals that this proof contains a formula for the limiting subdifferential of maximum functions in finite-dimensional spaces. The following lemma generalizes the corresponding result in arbitrary Asplund spaces. Its proof is on the straightforward side and similar in some places given in Chuong 2016, and so we omit the details. The notation $\partial_x$ signifies the limiting subdifferential operation with respect to $x$.

Lemma 2.4. Let $V$ be a sequentially compact topological space, and let $g : X \times V \to \mathbb{R}$ be a function such that for each fixed $v \in V$, $g(\cdot, v)$ is Lipschitz continuous around $\bar{x} \in X$ and $g(\bar{x}, \cdot)$ is upper semicontinuous on $V$. Let $\phi(x) := \max_{v \in V} g(x, v)$. If the multifunction $(x, v) \in X \times V \Rightarrow \partial_x g(x, v) \subset X^*$ is weak* closed at $(\bar{x}, \bar{v})$ for each $\bar{v} \in V(\bar{x})$, then the set $\text{cl}^* \text{co} \left( \bigcup \partial_x g(\bar{x}, v) \mid v \in V(\bar{x}) \right)$ is nonempty and

$$\partial \phi(\bar{x}) \subset \text{cl}^* \text{co} \left( \bigcup \partial_x g(\bar{x}, v) \mid v \in V(\bar{x}) \right),$$

where $V(\bar{x}) := \{ v \in V \mid g(\bar{x}, v) = \phi(\bar{x}) \}$.

The next lemma is concerning with the limiting subdifferential for the maximum of finitely many functions in Asplund spaces.

Lemma 2.5. (See (Mordukhovich 2006, Theorem 3.46)) Let $\phi_i : X \to \mathbb{R}$, $(i \in \{1, 2, \ldots, n\}, n \geq 2)$, be Lipschitz continuous around $\bar{x}$. Put $\phi(\bar{x}) := \max_{i \in \{1, 2, \ldots, n\}} \phi_i(\bar{x})$. Then

$$\partial \phi(\bar{x}) \subset \bigcup \left\{ \partial \left( \sum_{i \in I(\bar{x})} \mu_i \phi_i \right)(\bar{x}) \mid (\mu_1, \mu_2, \ldots, \mu_n) \in \Lambda(\bar{x}) \right\},$$

where

$$I(\bar{x}) := \left\{ i \in \{1, 2, \ldots, n\} \mid \phi_i(\bar{x}) = \phi(\bar{x}) \right\}$$

and

$$\Lambda(\bar{x}) := \left\{ (\mu_1, \mu_2, \ldots, \mu_n) \mid \mu_i \geq 0, \sum_{i=1}^n \mu_i = 1, \mu_i (\phi_i(\bar{x}) - \phi(\bar{x})) = 0 \right\}.$$

The following lemma computes the limiting subdifferential of a norm.
Lemma 2.6. (See (Göpfert et al. 2003a, Lemma 4.1.11)) Let $x \in X$ and $\beta > 1$. Then we have

$$
\partial \|x\| = \begin{cases} 
\{ x^* \in X^* \mid \langle x^*, x \rangle = \|x\|, \|x^*\| = 1 \} & \text{if } x \neq 0, \\
\{ x^* \in X^* \mid \|x^*\| \leq 1 \} & \text{if } x = 0,
\end{cases}
$$

and

$$
\partial \left( \frac{1}{\beta} \|x\|^\beta \right) = \{ x^* \in X^* \mid \langle x^*, x \rangle = \|x\|^\beta, \|x^*\| = \|x\|^{\beta-1} \}.
$$

Assumptions. (See (Chuong 2016, p.131))

(A1) For a fixed $\bar{x} \in X$, $F$ is locally Lipschitz at $\bar{x}$ and $f$ is locally Lipschitz at $F(\bar{x})$.

(A2) For each $i = 1, 2, \ldots, n$, $G_i$ is locally Lipschitz at $\bar{x}$ and uniformly on $V_i$, and $g_i$ is Lipschitz continuous on $G_i(\bar{x}, V_i)$.

(A3) For each $i = 1, 2, \ldots, n$, the functions $v_i \in V_i \mapsto G_i(\bar{x}, v_i) \in Z \times U_i$ and $G_i(\bar{x}, v_i) \mapsto g_i(G_i(\bar{x}, v_i)) \in \mathbb{R}$ are locally Lipschitzian.

(A4) For each $i = 1, 2, \ldots, n$, we define the real-valued functions $\phi_i$ and $\phi$ on $X$ via

$$
\phi_i(x) := \max_{v_i \in V_i} (g_i \circ G_i)(x, v_i) \quad \text{and} \quad \phi(x) := \max_{i \in \{1, 2, \ldots, n\}} \phi_i(x),
$$

and we notice that the assumption (A3) implies that $\phi_i$ is well defined on $V_i$. In addition, $\phi_i$ and $\phi$ follow readily that are locally Lipschitz at $\bar{x}$, since each $(g_i \circ G_i)(\bar{x}, v_i)$ is (see, e.g., (Chuong 2016, (H1), p.131) or (Lee and Son 2014, p.290)). Note that the feasible set $C$ can be equivalently characterized by

$$
C = \{ x \in X \mid \phi_i(x) \leq 0, \ i = 1, 2, \ldots, n \} = \{ x \in X \mid \phi(x) \leq 0 \}.
$$

(A5) For each $i = 1, 2, \ldots, n$, the multifunction $(x, v_i) \in X \times V_i \mapsto \partial_x (g_i \circ G_i)(x, v_i) \subset X^*$ is weak$^*$ closed at $(\bar{x}, \bar{v}_i)$ for each $\bar{v}_i \in V_i(\bar{x})$, where $V_i(\bar{x}) = \{ v_i \in V_i \mid (g_i \circ G_i)(\bar{x}, v_i) = \phi_i(\bar{x}) \}$.

3 Robust necessary and sufficient optimality

In this section, we study optimality conditions in composite robust multiobjective optimization problems. More precisely, first by exploiting the nonsmooth version of Fermat’s rule, sum rule, and chain rule for the limiting subdifferential, necessary conditions for weakly robust efficient solutions of the problem (CUP) will be established. We then derive sufficient conditions for the existence of such solutions as well as robust efficient solutions under assumptions of pseudo-quasi convexity for composite vector-valued functions.

The first theorem in this section provides a necessary optimality condition in the sense of the limiting subdifferential for weakly robust efficient solutions of the problem (CUP). To prove, we need to state a fuzzy necessary condition expressed in terms of the Fréchet subdifferential for weakly robust efficient solutions of the problem (UP) as follows.
Theorem 3.1. (See (Chuong 2019, Theorem 3.1)) Let $\bar{x}$ be a weakly robust efficient solution of the problem (UP). Then for each $k \in \mathbb{N}$ there exist $x^{1k} \in B_X(\bar{x}, \frac{1}{k})$, $x^{2k} \in B_X(\bar{x}, \frac{1}{k})$, $y^*_k \in K^+$ with $\|y^*_k\| = 1$, and $\alpha_k \in \mathbb{R}^+$ such that

$$0 \in \partial(y^*_k, f)(x^{1k}) + \alpha_k \partial\phi(x^{2k}) + \frac{1}{k} B_{X^*},$$

$$|\alpha_k \phi(x^{2k})| \leq \frac{1}{k}.$$ 

Theorem 3.2. Suppose that $\bar{x} \in S^w(CRP)$. Then there exist $y^* \in K^+$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^+_n$, with $\|y^*\| + \|\mu\| = 1$, and $\bar{v}_i \in V_i$, $i = 1, 2, \ldots, n$, such that

$$0 \leq \bigcup_{w^* \in \partial(y^*, f)(F(\bar{x}))} \partial(<w^*, F>(\bar{x})) + \sum_{i=1}^{n} \mu_i \partial_x \{ \bigcup_{v^*_i \in \partial_2(G_i(\bar{x}, v_i))} \partial_x \{ v^*_i, G_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \} \},$$

$$\mu_i \max_{v_i \in V_i} g_i(G_i(\bar{x}, v_i)) = \mu_i g_i(G_i(\bar{x}, \bar{v}_i)) = 0, \quad i = 1, 2, \ldots, n.$$  

(3.1)

Proof. Let us put $\tilde{f} := f \circ F$. In this case, the problem (CRP) changes to

$$(\tilde{CRP}) \quad \min_K \left\{ \tilde{f}(x) \mid x \in X, \phi(x) \leq 0 \right\}.$$

Using Theorem 3.1 to the problem (\tilde{CRP}), there exist sequences $x^{1k} \to \bar{x}$, $x^{2k} \to \bar{x}$, $y^*_k \in K^+$ with $\|y^*_k\| = 1$, $\alpha_k \in \mathbb{R}^+$, $x^{1k} \in \partial(y^*_k, f \circ F)(x^{1k})$, and $x^{2k} \in \alpha_k \partial\phi(x^{2k})$ satisfying

$$0 \in x^{1k} + x^{2k} + \frac{1}{k} B_{X^*},$$

$$\alpha_k \phi(x^{2k}) \to 0 \text{ as } k \to \infty.$$  

(3.2)

We now consider the following two possibilities

Case 1: If $\{\alpha_k\}$ is bounded, then without loss of generality we may suppose that $\alpha_k \to \alpha \in \mathbb{R}^+$ as $k \to \infty$. Besides, as the sequence $\{y^*_k\} \subset K^+$ is bounded, by invoking the weak* sequential compactness of bounded sets in duals to Asplund spaces, we don’t restrict the generality by assuming that $y^*_k \to y^*$ in $K^+$ with $\|y^*\| = 1$ as $k \to \infty$. Let $\ell > 0$ be a constant modulus for the locally Lipschitz function $f \circ F$ at $\bar{x}$. It is clear that $\|x^{1k}\| \leq \ell \|y^*_k\| \leq \ell$ for all $k \in \mathbb{N}$ (see (Mordukhovich 2006, Proposition 1.85)). As above, by passing to a subsequence if necessary, that $x^{1k} \to x^*_1 \in X^*$ as $k \to \infty$, and so it follows from (3.2) that $x^{2k} \to x^*_2 := -x^*_1$ as $k \to \infty$. According to Lemma 2.1, we deduce from the inclusion $x^{1k} \in \partial(y^*_k, f \circ F)(x^{1k})$ that

$$(x^{1k}_1, -y^*_k) \in \tilde{N}(x^{1k}, (f \circ F)(x^{1k})); gph (f \circ F), \quad k \in \mathbb{N}.$$ 

Letting $k \to \infty$ and noticing the definitions (2.1) and (2.2) of the normal cones, we obtain the relation $(x^*_1, -y^*) \in N((\bar{x}, (f \circ F)(\bar{x})); gph (f \circ F))$ which is equivalent to

$$x^*_1 \in \partial(y^*, f \circ F)(\bar{x}),$$

(3.3)

because of Lemma 2.1. Similarly, we get $x^*_2 \in \partial \phi(\bar{x})$. This combined with (3.3) entails that

$$0 \in \partial(y^*, f \circ F)(\bar{x}) + \alpha \partial \phi(\bar{x}),$$

(3.4)
by taking $x_i^* = -x_i^*$. Applying Lemma 2.5 to the limiting subdifferential of the maximum function $\phi$, we arrive at

$$
\partial \phi(\bar{x}) \subset \bigcup \left\{ \partial \left( \sum_{i \in I(\bar{x})} \mu_i \phi_i \right)(\bar{x}) \mid (\mu_1, \mu_2, \ldots, \mu_n) \in \Lambda(\bar{x}) \right\},
$$

(3.5)

where $I(\bar{x}) = \left\{ i \in \{1, 2, \ldots, n\} \mid \phi_i(\bar{x}) = \phi(\bar{x}) \right\}$ and

$$
\Lambda(\bar{x}) = \left\{ (\mu_1, \mu_2, \ldots, \mu_n) \mid \mu_i \geq 0, \sum_{i=1}^n \mu_i = 1, \mu_i (\phi_i(\bar{x}) - \phi(\bar{x})) = 0 \right\}.
$$

Invoking further Lemma 2.4 allows us

$$
\partial \phi_i(\bar{x}) \subset \text{cl}^*\text{co} \left( \bigcup \left\{ \partial_x (g_i \circ G_i)(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right\} \right), \quad i = 1, 2, \ldots, n,
$$

(3.6)

where $\mathcal{V}_i(\bar{x}) = \left\{ v_i \in \mathcal{V}_i \mid (g_i \circ G_i)(\bar{x}, v_i) = \phi_i(\bar{x}) \right\}$ and the set $\text{cl}^*\text{co} \left( \bigcup \left\{ \partial_x (g_i \circ G_i)(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right\} \right)$ is nonempty. By the sum rule of Lemma 2.3, it follows from the relations (3.4)-(3.6) that

$$
0 \in \partial (\bar{y}^*, f \circ F)(\bar{x}) + \alpha \bigcup \left\{ \sum_{i \in I(\bar{x})} \mu_i \text{cl}^*\text{co} \left( \bigcup \left\{ \partial_x (g_i \circ G_i)(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right\} \right) \mid (\mu_1, \mu_2, \ldots, \mu_n) \in \Lambda(\bar{x}) \right\}.
$$

Thus, there exist $\vec{\mu} := (\vec{\mu}_1, \vec{\mu}_2, \ldots, \vec{\mu}_n) \in \Lambda(\bar{x})$, with $\sum_{i=1}^n \vec{\mu}_i = 1$ and $\vec{\mu}_i = 0$ for all $i \in \{1, 2, \ldots, n\} \setminus I(\bar{x})$, satisfying

$$
0 \in \partial (\bar{y}^*, f \circ F)(\bar{x}) + \alpha \sum_{i=1}^n \vec{\mu}_i \text{cl}^*\text{co} \left( \bigcup \left\{ \partial_x (g_i \circ G_i)(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right\} \right).
$$

Dividing the above inclusion by $c := \|\bar{y}^*\| + \alpha \|\vec{\mu}\|$ and then letting $y^* := \frac{\bar{y}^*}{c}$ and $\mu := \frac{\alpha}{c} \vec{\mu}$, we have $y^* \in K^+$ and $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}_n^+$, with $\|y^*\| + \|\mu\| = 1$, such that

$$
0 \in \partial (y^*, f \circ F)(\bar{x}) + \sum_{i=1}^n \mu_i \text{cl}^*\text{co} \left( \bigcup \left\{ \partial_x (g_i \circ G_i)(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right\} \right).
$$

Putting now $\psi := (y^*, f)$, we rewrite the above inclusion as

$$
0 \in \partial (\psi \circ F)(\bar{x}) + \sum_{i=1}^n \mu_i \text{cl}^*\text{co} \left( \bigcup \left\{ \partial_x (g_i \circ G_i)(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right\} \right).
$$

(3.7)

By the assumptions (A1) and (A2), $F$ and $\psi$ are locally Lipschitzian at $\bar{x}$ and $F(\bar{x})$, respectively. And $G_i$ is locally Lipschitzian at $\bar{x}$ and uniformly on $\mathcal{V}_i$, and $g_i$ is Lipschitz continuous on $G_i(\bar{x}, \mathcal{V}_i)$ for each $i = 1, 2, \ldots, n$. So using the limiting subdifferential chain rule of Lemma 2.2 for (3.7), we get the first relation in this theorem.

On the other side, due to the locally Lipschitz continuity of the function $v_i \in \mathcal{V}_i \mapsto (g_i \circ G_i)(\bar{x}, v_i)$ and the sequentially compactness of $\mathcal{V}_i$, there is $\bar{v}_i \in \mathcal{V}_i$ satisfying

$$
(g_i \circ G_i)(\bar{x}, \bar{v}_i) = \max_{v_i \in \mathcal{V}_i} (g_i \circ G_i)(\bar{x}, v_i) = \phi_i(\bar{x}).
$$

(3.8)
In addition, note that $\alpha \phi(\bar{x}) = 0$ due to $\alpha_k \phi(x^{2k}) \to 0$ as $k \to \infty$. Considering $\phi_i(\bar{x}) = \phi(\bar{x})$ for all $i \in I(\bar{x})$, we conclude from (3.8) that

$$
\mu_i (g_i \circ G_i)(\bar{x}, \vbar_i) = \frac{\alpha}{c} \hat{\mu}_i \phi_i(\bar{x}) = \frac{\mu_i}{c} [\alpha \phi(\bar{x})] = 0,
$$
i.e., $\mu_i (g_i \circ G_i)(\bar{x}, \vbar_i) = \mu_i \max_{v_i \in V_i} (g_i \circ G_i)(\bar{x}, v_i) = 0$ for $i = 1, 2, \ldots, n$. This yields the second relation of (3.1).

**Case 2:** Next suppose that $\{\alpha_k\}$ is unbounded. Similar as above, we have from $x_{2k}^* \in \alpha_k \hat{\partial} \phi(x^{2k})$ that $(x_{2k}^*, -\alpha_k) \in \tilde{N}((x^{2k}, \phi(x^{2k})); gph \phi)$ for each $k \in \mathbb{N}$. Hence

$$
\left( \frac{x_{2k}^*}{\alpha_k}, -1 \right) \in \tilde{N}((x^{2k}, \phi(x^{2k})); gph \phi), \quad k \in \mathbb{N}.
$$

Passing $k \to \infty$ and employing again (2.2) give us some $(0, -1) \in \mathcal{N}(\bar{x}, \phi(\bar{x})); gph \phi)$, which is equivalent to 0 $\in \partial \phi(\bar{x})$. To proceed further as in the proof of the Case 1, there exists $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}_+^n \setminus \{0\}$, with $\|\mu\| = 1$, such that

$$
0 \in \sum_{i=1}^n \mu_i cl^*co \left( \bigcup \left\{ \bigcup_{v_i \in \partial_x g_i(\bar{x}, v_i)} \partial_x \{v_i^*, G_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \} \right\} \right).
$$

Moreover, noticing the unboundedness of $\{\alpha_k\}$ and applying $\alpha_k \phi(x^{2k}) \to 0$ as $k \to \infty$, we may take $\vbar_i \in V_i$ such that $\mu_i (g_i \circ G_i)(\bar{x}, \vbar_i) = \mu_i \phi_i(\bar{x}) = \mu_i \phi(\bar{x}) = 0$ for each $i = 1, 2, \ldots, n$. So, (3.1) holds by choosing $y^* := 0 \in K^+$.

**Remark 3.1.** Theorem 3.2 collapses to

(i) (Saadati and Oveisiha 2022b, Theorem 3.2) for the problem (UP) with $X = W = Z$, $V_i = U_i$, $i = 1, 2, \ldots, n$, and identical maps $F$ and $G$,

(ii) (Fakhar et al. 2018, Proposition 3.9) with $X = W = Z$, $Y := \mathbb{R}^p$, $K := \mathbb{R}_+^p$, $V_i = U_i$, $i = 1, 2, \ldots, m$, and identical maps $F$ and $G$, and

(iii) (Chuong 2016, Theorem 3.3) with $X = W = Z := \mathbb{R}^n$, $Y := \mathbb{R}^m$, $K := \mathbb{R}_+^m$, $V_i = U_i$, $i = 1, 2, \ldots, l$, and identical maps $F$ and $G$.

The following corollary provides a Fritz-John optimality condition for weakly robust efficient solutions of the uncertain multiobjective optimization problem (SUP).

**Corollary 3.3.** Let $\bar{x}$ be a weakly robust efficient solution of the problem (SUP). Then there exist $y^* \in K^+$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}_+^n$, $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathbb{R}_+^m$, with $\|y^*\| + \|\mu\| + \|\sigma\| \neq 0$, and $\vbar_i \in V_i$, $i = 1, 2, \ldots, n$, such that

$$
0 \in \partial(y^*, f)(\bar{x}) + \sum_{i=1}^n \mu_i cl^*co \left( \bigcup_{v_i \in \partial_x g_i(\bar{x}, v_i)} \partial_x \{v_i^*, G_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \} \right)
$$

$$
+ \sum_{j=1}^m \sigma_j cl^*co \left( \bigcup_{v_{n+j} \in \partial_x (-h_j)(\bar{x}, v_{n+j})} \partial_x (-h_j)(\bar{x}, v_{n+j}) \mid v_{n+j} \in V_{n+j} \right),
$$

$$
\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = \mu_i g_i(\bar{x}, \vbar_i) = 0, \quad i = 1, 2, \ldots, n.
$$
Proof. Note that the problem (SUP) is a particular case of the composite uncertain multiobjective optimization problem (CUP), where \( g \) is given as in (1.1). First let us define the real-valued functions \( \phi_{n+j}, j = 1, 2, \ldots, m, \) and \( \phi \) on \( X \) via

\[
\phi_{n+j}(x) := \max_{v_{n+j} \in V_{n+j}} |h_j \circ G_{n+j}(x, v_{n+j})| \quad \text{and} \quad \phi(x) := \max_{i=1, 2, \ldots, n} \{ \phi_i(x), \phi_{n+j}(x) \}.
\]

Then proceeding similarly to the proof of Theorem 3.2 and invoking both the subdifferential chain rule

\[
\partial_x h_j \circ G_{n+j}(x, v_{n+j}) \subset \bigcup_{-1 \leq \tau \leq 1} \partial_x (\tau h_j \circ G_{n+j})(x, v_{n+j})
\]

and the inclusion

\[
\partial_x (\tau h_j \circ G_{n+j})(x, v_{n+j}) \subset |\tau|(|\partial_x (h_j \circ G_{n+j})(x, v_{n+j})| \bigcup \partial_x (-h_j \circ G_{n+j})(x, v_{n+j})),
\]

we find \( y^* \in K^+, \gamma := (\mu_1, \mu_2, \ldots, \mu_n, \sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathbb{R}^{n+m}_+, \) with \( \|y^*\| + \|\gamma\| = 1, \) \( \tilde{v}_i \in V_i, \) \( i = 1, 2, \ldots, n, \) and \( \tilde{v}_{n+j} \in V_{n+j}, j = 1, 2, \ldots, m, \) such that

\[
\begin{cases}
0 \in \{ \partial(w^*, F)(\bar{x}) \} + \sum_{i=1}^n \mu_i c_i^* \text{co} \left( \bigcup_{v_i \in V_i} \partial_x g_i(\bar{x}, v_i) \right) \\
| v_i \in V_i(\bar{x}) \} \right) + \sum_{j=1}^m \sigma_j c_j^* \text{co} \left( \bigcup_{v_{n+j} \in V_{n+j} \setminus V_j(\bar{x})} \partial_x (v_{n+j} | G_{n+j}(\bar{x}, v_{n+j}) \bigcup \partial_x (-h_j(\bar{x}, v_{n+j}))) \right),
\end{cases}
\]

\[
\begin{align*}
\sigma_j \max_{v_{n+j} \in V_{n+j}} h_j(G_{n+j}(\bar{x}, v_{n+j})) &= \sigma_j h_j(G_{n+j}(\bar{x}, \tilde{v}_{n+j})) = 0, \quad j = 1, 2, \ldots, m. \\
\end{align*}
\]

In this setting, we see that \( X = W = Z, V_i = U_i, i = 1, 2, \ldots, n, V_{n+j} = U_{n+j}, j = 1, 2, \ldots, m, \) and \( F \) and \( G \) are identical maps, thus the above relations reduces to the following ones

\[
\begin{cases}
0 \in \{ \partial(y^*, f)(\bar{x}) \} + \sum_{i=1}^n \mu_i c_i^* \text{co} \left( \bigcup_{v_i \in V_i(\bar{x})} \partial_x g_i(\bar{x}, v_i) \right) \\
+ \sum_{j=1}^m \sigma_j c_j^* \text{co} \left( \bigcup_{v_{n+j} \in V_{n+j} \setminus V_j(\bar{x})} \partial_x (h_j \circ G_{n+j}(\bar{x}, v_{n+j}) \bigcup \partial_x (-h_j)(\bar{x}, v_{n+j}) \right),
\end{cases}
\]

\[
\begin{align*}
\mu_i \max_{v_i \in V_i(\bar{x})} g_i(\bar{x}, v_i) &= \mu_i g_i(\bar{x}, \tilde{v}_i) = 0, \quad i = 1, 2, \ldots, n, \\
\end{align*}
\]

due to \( h_j(\bar{x}, v_{n+j}) = 0 \) for all \( v_{n+j} \in V_{n+j}, j = 1, 2, \ldots, m. \) Clearly, \( \|y^*\| + \|(\mu_1, \mu_2, \ldots, \mu_n)\| + \|(\sigma_1, \sigma_2, \ldots, \sigma_m)\| \neq 0, \) and so the proof is complete. \( \square \)

We now present an example which illustrates Theorem 3.2 for a composite uncertain multiobjective optimization problem.

Example 3.2. Let \( X := \mathbb{R}^2, W := \mathbb{R}^2, Y := \mathbb{R}^2, Z := \mathbb{R}^2, V_i = U_i := [-1, 1], i = 1, 2, \) and \( K := \mathbb{R}^3_+. \) Consider the following composite uncertain optimization problem

\[
(CUP) \quad \min_{K} \left\{ (f \circ F)(x) \mid x := (x_1, x_2) \in X, (g_i \circ G_i)(x, v_i) \leq 0, i = 1, 2 \right\},
\]

where $F : X \rightarrow W$, $F := (F_1, F_2)$, is defined by $F_1(x_1, x_2) = \frac{1}{2}x_1$ and $F_2(x_1, x_2) = x_2 - 1$, $f : W \rightarrow Y$, $f := (f_1, f_2, f_3)$, is given by

$$
\begin{align*}
&f_1(w_1, w_2) = -2w_1 + |w_2|, \\
f_2(w_1, w_2) = \frac{1}{|w_1| + 1} - 3w_2 + 2, \\
f_3(w_1, w_2) = \frac{1}{\sqrt{|w_1| + 1}} - |w_2 - 1| - 1,
\end{align*}
$$

$G_i : X \times V_i \rightarrow Z \times U_i$, $i = 1, 2$, $G := (G_1, G_2)$, are defined by $G_1(x_1, x_2, v_1) = (x_1 + 1, x_2, v_1)$ and $G_2(x_1, x_2, v_2) = (x_1, 2x_2, v_2)$, and $g_i : Z \times U_i \rightarrow \mathbb{R}$, $i = 1, 2$, $g := (g_1, g_2)$, are given by

$$
\begin{align*}
g_1(z_1, z_2, u_1) &= u_2^2|z_2| + \max \left\{ z_1, 2z_1 \right\} - 3|u_1|, \\
g_2(z_1, z_2, u_2) &= -3|z_1| + u_2z_2 - 2,
\end{align*}
$$

where $v_i \in V_i$ and $u_i \in U_i$, $i = 1, 2$. It is easy to check that

$$
\begin{align*}
\left\{ (x_1, x_2) \in X \mid (g_1 \circ G_1)(x_1, x_2, v_1) \leq 0, \forall v_1 \in V_1 \right\} \\
= \left\{ (x_1, x_2) \in X \mid u_2^2|z_2| + \max \left\{ x_1 + 1, 2x_1 + 2 \right\} - 3|u_1| \leq 0, \forall v_1 \in V_1 \right\} \\
= \left\{ (x_1, x_2) \in X \mid x_1 \leq -1 \text{ and } |x_2| \leq -x_1 + 2 \right\},
\end{align*}
$$

and, since $x_1 \leq -1$, it can be obtained that

$$
\begin{align*}
\left\{ (x_1, x_2) \in X \mid (g_2 \circ G_2)(x_1, x_2, v_2) \leq 0, \forall v_2 \in V_2 \right\} \\
= \left\{ (x_1, x_2) \in X \mid -3|x_1| + 2v_2x_2 - 2 \leq 0, \forall v_2 \in V_2 \right\} \\
= \left\{ (x_1, x_2) \in X \mid x_1 \leq -1 \text{ and } |x_2| \leq -\frac{3}{2}x_1 + 1 \right\}.
\end{align*}
$$

Hence, the robust feasible set is

$$
C = \left\{ (x_1, x_2) \in X \mid -2 \leq x_1 \leq -1 \text{ and } |x_2| \leq -\frac{3}{2}x_1 + 1 \right\} \cup \\
\left\{ (x_1, x_2) \in X \mid x_1 \leq -2 \text{ and } |x_2| \leq -x_1 + 2 \right\}.
$$

Suppose that $\bar{x} := (-1, 1) \in C$ and $x := (x_1, x_2) \in C$. Considering $x_1 \leq -1$, we have $(f_1 \circ F_1)(x) - (f_1 \circ F_1)(\bar{x}) \geq 0$. Therefore

$$
(f \circ F)(x) - (f \circ F)(\bar{x}) \notin \text{int } K, \forall x \in C,
$$

which means that $\bar{x}$ is a weakly robust efficient solution of the problem (CUP). Observe also that

$$
\phi_1(\bar{x}) = \max_{v_1 \in V_1} (g_1 \circ G_1)(\bar{x}, v_1) = \max_{v_1 \in V_1} (u_2^2 - 3|v_1|) = 0,
$$

$$
\phi_2(\bar{x}) = \max_{v_2 \in V_2} (g_2 \circ G_2)(\bar{x}, v_2) = \max_{v_2 \in V_2} (2v_2 - 5) = -3.
$$

Hence $\phi(\bar{x}) = \max \{ \phi_1(\bar{x}), \phi_2(\bar{x}) \} = 0$, $V_1(\bar{x}) = \{0\}$, and $V_2(\bar{x}) = \{1\}$. Performing elementary calculations gives us $\partial F(\bar{x}) = \left(\begin{array}{cc} 1/2 & 0 \\ 0 & 1 \end{array}\right)$ and

$$
\partial (f_1 \circ F_1)(\bar{x}) = \{-1\} \times [-1, 1], \partial (f_2 \circ F_2)(\bar{x}) = \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \{-3\}, \partial (f_3 \circ F_3)(\bar{x}) = \left[ -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right] \times \{-1, 1\}.
$$
Further, we get \( \partial_x G_1(\bar{x}, v_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \partial_x G_2(\bar{x}, v_2) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \) and
\[
\partial_x (g_1 \circ G_1)(\bar{x}, v_1) = [1, 2] \times \{0\}, \quad \partial_x (g_2 \circ G_2)(\bar{x}, v_2) = \{(3, 2), (-3, 2)\}.
\]

So
\[
\begin{aligned}
\{ \text{cl}^* \text{co} \left( \bigcup_{v_1^* \in \partial_x g_1(G_1(\bar{x}, v_1))} \partial_x (v_1^*, G_1)(\bar{x}, v_1) \right) \} &= [1, 2] \times \{0\}, \\
\{ \text{cl}^* \text{co} \left( \bigcup_{v_2^* \in \partial_x g_2(G_2(\bar{x}, v_2))} \partial_x (v_2^*, G_2)(\bar{x}, v_2) \right) \} &= [-3, 3] \times \{4\}.
\end{aligned}
\tag{3.10}
\]

On the other side, due to \( I(\bar{x}) = \{ i \in \{1, 2\} \mid \phi_i(\bar{x}) = \phi(\bar{x}) \} = \{1\} \), it is easy to see from (3.10) that the (CQ), which will be presented later, is satisfied at \( \bar{x} \).

Finally, we can find \( y^* = \left( \frac{\sqrt{2}}{3}, 0, \frac{\sqrt{2}}{3} \right) \in K^+ \setminus \{0\} \) and \( \mu = \left( \frac{1}{3}, 0 \right) \in \mathbb{R}_+^2 \), with \( \|y^*\| + \|\mu\| = 1 \), satisfying
\[
0 = \begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{2}}{3} \end{pmatrix} \begin{pmatrix} -1 & 0 & -1 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 3\frac{\sqrt{2}}{4} & 0 \\ 0 & 4 \end{pmatrix},
\]
and \( \mu_i \max_{v_i \in V_i} g_i(G_i(\bar{x}, v_i)) = 0 \) for \( i = 1, 2 \).

In order to establish sufficient optimality for (weakly) robust efficient solutions of the problem (CUP), we need to define a so-called robust Karush-Kuhn-Tucker (KKT) condition for this problem.

**Definition 3.1.** A point \( \bar{x} \in C \) is termed a robust (KKT) point if there exist \( y^* \in K^+ \setminus \{0\}, \mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}_+^n, \) and \( \bar{v}_i \in V_i, i = 1, 2, \ldots, n, \) such that
\[
\begin{aligned}
0 &\in \bigcup_{w^* \in \partial (y^*, f)(\bar{x})} \partial (w^*, F)(\bar{x}) + \sum_{i=1}^n \mu_i \text{cl}^* \text{co} \left( \bigcup_{v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))} \partial_x (v_i^*, G_i)(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right), \\
\mu_i \max_{v_i \in V_i} (g_i \circ G_i)(\bar{x}, v_i) &= \mu_i (g_i \circ G_i)(\bar{x}, \bar{v}_i) = 0, \quad i = 1, 2, \ldots, n.
\end{aligned}
\]

It is worth to mention here that this condition (KKT) is in the sense of the Wolkowicz constraint qualification in the nonsmooth setting; see, e.g., Wolkowicz 1980; Borwein and Wolkowicz 1986 for more details. Moreover, it can be deduced from Theorem 3.2 that a weakly robust efficient solution of the problem (CUP) becomes a robust (KKT) point under the following constraint qualification (CQ) condition in the sense of robustness.

**Definition 3.2.** (See (Saadati and Oveisah 2022b, Definition 2.3)) Let \( \bar{x} \in C \). We say that the constraint qualification (CQ) is satisfied at \( \bar{x} \) if
\[
0 \notin \text{cl}^* \text{co} \left( \bigcup_{v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))} \partial_x (v_i^*, G_i)(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right), \quad i \in I(\bar{x}),
\]
where \( I(\bar{x}) := \{ i \in \{1, 2, \ldots, n\} \mid \phi_i(\bar{x}) = \phi(\bar{x}) \} \).

In general, a robust feasible point of the problem (CUP) at which the robust (KKT) condition holds may not be a (weakly) robust efficient solution. This motivates us to employ a similar concept of pseudo-quasi convexity in Saadati and Oveisah 2022b for the compositions \( f \circ F \) and \( g \circ G \).
Definition 3.3. (i) We say that \((f \circ F, g \circ G)\) is type I pseudo convex at \(\bar{x} \in X\) if for any \(x \in X\), \(y^* \in K^+\), \(w^* \in \partial(y^*, f)(\bar{F}(\bar{x}))\), \(x^* \in \partial(w^*, F)(\bar{x})\), \(v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))\), and \(\nu \in X\) such that

\[
\langle y^*, f \circ F \rangle(x) < \langle y^*, f \circ F \rangle(\bar{x}) \implies \langle x^*, \nu \rangle < 0,
\]

\[
(g_i \circ G_i)(x, v_i) \leq (g_i \circ G_i)(\bar{x}, v_i) \implies \langle x_i^*, \nu \rangle \leq 0, \quad i = 1, 2, \ldots, n.
\]

(ii) We say that \((f \circ F, g \circ G)\) is type II pseudo convex at \(\bar{x} \in X\) if for any \(x \in X \setminus \{\bar{x}\}\), \(y^* \in K^+ \setminus \{0\}\), \(w^* \in \partial(y^*, f)(\bar{F}(\bar{x}))\), \(x^* \in \partial(w^*, F)(\bar{x})\), \(v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))\), and \(\nu \in X\) such that

\[
\langle y^*, f \circ F \rangle(x) \leq \langle y^*, f \circ F \rangle(\bar{x}) \implies \langle x^*, \nu \rangle < 0,
\]

\[
(g_i \circ G_i)(x, v_i) \leq (g_i \circ G_i)(\bar{x}, v_i) \implies \langle x_i^*, \nu \rangle \leq 0, \quad i = 1, 2, \ldots, n.
\]

Remark 3.3. In Definition 3.3,

(i) if \(X = W = Z, \mathcal{V}_i = \mathcal{U}_i, i = 1, 2, \ldots, n\), and \(F\) and \(G\) are identical maps, then this definition reduces to the corresponding one in (Saadati and Oveisigha 2022b, Definition 2.2). Moreover, as shown in (Saadati and Oveisigha 2022b, Example 2.2), the class of type I pseudo convex functions is properly larger than the class of generalized convex functions.

(ii) if \(X = W = Z, Y := \mathbb{R}^p, K := \mathbb{R}_+^p, \mathcal{V}_i = \mathcal{U}_i, i = 1, 2, \ldots, m\), and \(F\) and \(G\) are identical maps, then this definition collapses to (Fakhar et al. 2018, Definition 3.2). Furthermore, as demonstrated in (Fakhar et al. 2018, Example 3.4), the class of pseudo-quasi generalized convex functions is properly wider than the class of generalized convex functions.

(iii) if \(X = W = Z := \mathbb{R}^n, Y := \mathbb{R}^m, K := \mathbb{R}_+^m, \mathcal{V}_i = \mathcal{U}_i, i = 1, 2, \ldots, l\), and \(F\) and \(G\) are identical maps, then this definition reduces to (Chuong 2016, Definition 3.9). Furthermore, as illustrated in (Chuong 2016, Example 3.10), the class of generalized convex functions contains some nonconvex functions.

Remark 3.4. It follows from Definition 3.3 that if \((f \circ F, g \circ G)\) is type II pseudo convex at \(\bar{x} \in X\), then \((f \circ F, g \circ G)\) is type I pseudo convex at \(\bar{x} \in X\), but converse is not true (see (Saadati and Oveisigha 2022b, Example 2.2) considering special case of Remark 3.3(i)).

The forthcoming proposition shows that the class of (resp., type II) type I pseudo convex composite functions includes (resp., strictly) convex composite functions.

Proposition 3.4. Let \(\bar{x} \in X\), and let \(f, F, g, G\) be such that \(\langle y^*, f \rangle\) is convex for every \(y^* \in K^+\), \(\langle w^*, F \rangle\) is convex for every \(w^* \in \partial(y^*, f)(\bar{F}(\bar{x}))\), \(g_i\) is a convex function, and \(\langle v_i^*, G_i \rangle\) is convex for every \(v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))\), \(v_i \in \mathcal{V}_i(\bar{x})\), \(i = 1, 2, \ldots, n\). Then

(i) \((f \circ F, g \circ G)\) is type I pseudo convex at \(\bar{x}\).

(ii) If \(\langle y^*, f \rangle\) is strictly convex for every \(y^* \in K^+ \setminus \{0\}\) and \(F\) is injective, then \((f \circ F, g \circ G)\) is type II pseudo convex at \(\bar{x}\).
Proof. (i) Suppose that \( x \in X, y^* \in K^+, w^* \in \partial(y^*, f)(F(\bar{x})), x^* \in \partial(w^*, F)(\bar{x}), v^*_i \in \partial_x g_i(G_i(\bar{x}, v_i)), \) and \( x_i^* \in \partial_x (v^*_i, G_i)(\bar{x}, v_i), v_i \in V_i(\bar{x}), i = 1, 2, \ldots, n, \) and that
\[
\langle y^*, f \circ F \rangle(x) < \langle y^*, f \circ F \rangle(\bar{x}), \tag{3.11}
\]
\[
(g_i \circ G_i)(x, v_i) \leq (g_i \circ G_i)(\bar{x}, v_i), \quad i = 1, 2, \ldots, n. \tag{3.12}
\]
Take \( \nu := x - \bar{x}. \) Since \( \langle w^*, F \rangle \) is a convex function, we have
\[
\langle x^*, \nu \rangle = \langle x^*, x - \bar{x} \rangle \leq \langle w^*, F \rangle(x) - \langle y^*, f \rangle(F(\bar{x})) = \langle y^*, f \circ F \rangle(x) - \langle y^*, f \circ F \rangle(\bar{x}).
\]
On the other hand, since \( \langle y^*, f \rangle \) is convex, we obtain
\[
\langle w^*, F(x) - F(\bar{x}) \rangle \leq \langle y^*, f \rangle(F(x)) - \langle y^*, f \rangle(F(\bar{x})) = \langle y^*, f \circ F \rangle(x) - \langle y^*, f \circ F \rangle(\bar{x}).
\]
Combining now the both latter inequalities with (3.11), one gets
\[
\langle x^*, \nu \rangle \leq \langle y^*, f \circ F \rangle(x) - \langle y^*, f \circ F \rangle(\bar{x}) < 0.
\]
Similarly, using the convexity of functions \( g_i \) and \( \langle v^*_i, G_i \rangle \) and applying (3.12) give us
\[
\langle x_i^*, \nu \rangle \leq (g_i \circ G_i)(x, v_i) - (g_i \circ G_i)(\bar{x}, v_i) \leq 0, \quad i = 1, 2, \ldots, n.
\]
Consequently, \( f \circ F, g \circ G \) is type I pseudo convex at \( \bar{x}. \)

(ii) Suppose that \( x \in X \setminus \{\bar{x}\}, y^* \in K^+ \setminus \{0\}, w^* \in \partial(y^*, f)(F(\bar{x})), x^* \in \partial(w^*, F)(\bar{x}), v^*_i \in \partial_x g_i(G_i(\bar{x}, v_i)), \) and \( x_i^* \in \partial_x (v^*_i, G_i)(\bar{x}, v_i), v_i \in V_i(\bar{x}), i = 1, 2, \ldots, n, \) and that
\[
\langle y^*, f \circ F \rangle(x) \leq \langle y^*, f \circ F \rangle(\bar{x}), \tag{3.13}
\]
\[
(g_i \circ G_i)(x, v_i) \leq (g_i \circ G_i)(\bar{x}, v_i), \quad i = 1, 2, \ldots, n.
\]
Choosing \( \nu := x - \bar{x} \) and proceeding as in the part (i), it holds that
\[
\langle x^*, \nu \rangle \leq \langle w^*, F(x) - F(\bar{x}) \rangle \tag{3.14}
\]
and
\[
\langle x_i^*, \nu \rangle \leq (g_i \circ G_i)(x, v_i) - (g_i \circ G_i)(\bar{x}, v_i) \leq 0, \quad i = 1, 2, \ldots, n.
\]
Note here that \( F(x) \neq F(\bar{x}) \) due to the injectivity of \( F. \) Since \( \langle y^*, f \rangle \) is strictly convex, it follows from (Hiriart-Urruty and Lemaréchal 1993, Proposition 6.1.3) that
\[
\langle w^*, F(x) - F(\bar{x}) \rangle < \langle y^*, f \rangle(F(x)) - \langle y^*, f \rangle(F(\bar{x})) = \langle y^*, f \circ F \rangle(x) - \langle y^*, f \circ F \rangle(\bar{x}).
\]
This combined with (3.13) and (3.14) yields that
\[
\langle x^*, \nu \rangle < \langle y^*, f \circ F \rangle(x) - \langle y^*, f \circ F \rangle(\bar{x}) \leq 0.
\]
So, \( f \circ F, g \circ G \) is type II pseudo convex at \( \bar{x}. \)

Now we are ready to derive a robust (KKT) sufficient condition for (weakly) robust efficient solutions of the problem (CUP).
Theorem 3.5. Assume that $\bar{x} \in C$ satisfies the robust (KKT) condition.

(i) If $(f \circ F, g \circ G)$ is type I pseudo convex at $\bar{x}$, then $\bar{x} \in S^w(CR) \cup P$.

(ii) If $(f \circ F, g \circ G)$ is type II pseudo convex at $\bar{x}$, then $\bar{x} \in S(CR)$.

Proof. As $\bar{x} \in C$ is a robust (KKT) point of the problem (CUP), we can find $y^* \in K_+ \setminus \{0\}$, $w^* \in \partial y^*, f(F(\bar{x})), x^* \in \partial w^*, F(\bar{x})$, $\mu_i \geq 0$, and $u^*_i \in \text{cl}^* \text{co} \left\{ \bigcup \left\{ \text{max} \sum_{v_i \in G_i(\bar{x}, v_i)} g_i(\bar{x}, v_i) \bigg| v_i \in V(\bar{x}) \right\} \right\}$, $i = 1, 2, \ldots, n, n$, such that

\begin{align}
0 &= x^* + \sum_{i=1}^n \mu_i u^*_i, \\
\mu_i \text{ max } g_i &\in G_i(\bar{x}, v_i) = 0, \quad i = 1, 2, \ldots, n.
\end{align}

First prove (i). On the contrary, suppose that $\bar{x} \notin S^w(CR) \cup P$. This means by definition that there exists $\tilde{x} \in C$ such that $(f \circ F)(\tilde{x}) - (f \circ F)(\bar{x}) \in -\text{int } K$. It follows from (Jahn 2004, Lemma 3.21) that $\langle y^*, (f \circ F)(\tilde{x}) - (f \circ F)(\bar{x}) \rangle < 0$. Noting that $(f \circ F, g \circ G)$ is the type I pseudo convex at $\bar{x}$, we conclude from the latter inequality that there is $\nu \in X$ such that

\begin{align}
\langle x^*, \nu \rangle < 0.
\end{align}

In addition, taking (3.15) into account gives us

\begin{align}
0 = \langle x^*, \nu \rangle + \sum_{i=1}^n \mu_i \langle u^*_i, \nu \rangle
\end{align}

for $\nu$ above. So, the relationships in (3.17) and (3.18) lead to

\begin{align}
\sum_{i=1}^n \mu_i \langle u^*_i, \nu \rangle > 0.
\end{align}

To proceed, first pick $i_0 \in \left\{1, 2, \ldots, n\right\}$ satisfying $\mu_{i_0} \langle u^*_{i_0}, \nu \rangle > 0$. On the one side, due to

\begin{align}
u^*_{i_0} \in \text{cl}^* \text{co} \left( \bigcup \left\{ \bigcup \left\{ \text{max} \sum_{v_i \in G_i(\bar{x}, v_i)} g_i(\bar{x}, v_i) \bigg| v_i \in V_i(\bar{x}) \right\} \right\} \right),
\end{align}

we have a sequence $\left\{ u^*_{i_0_k} \right\} \subset \text{co} \left( \bigcup \left\{ \bigcup \left\{ \text{max} \sum_{v_i \in G_i(\bar{x}, v_i)} g_i(\bar{x}, v_i) \bigg| v_i \in V_i(\bar{x}) \right\} \right\} \right)$ such that $u^*_{i_0_k} \xrightarrow{w^*} u^*_{i_0}$. Since $\mu_{i_0} > 0$, there exists $k_0 \in \mathbb{N}$ such that

\begin{align}
\langle u^*_{i_0_{k_0}}, \nu \rangle > 0.
\end{align}

On the other side, due to $u^*_{i_0_{k_0}} \in \text{co} \left( \bigcup \left\{ \bigcup \left\{ \text{max} \sum_{v_i \in G_i(\bar{x}, v_i)} g_i(\bar{x}, v_i) \bigg| v_i \in V_i(\bar{x}) \right\} \right\} \right)$, we get $u^*_p \in \bigcup \left\{ \bigcup \left\{ \text{max} \sum_{v_i \in G_i(\bar{x}, v_i)} g_i(\bar{x}, v_i) \bigg| v_i \in V_i(\bar{x}) \right\} \right\}$ and $\mu_p \geq 0$ with $\sum_{p=1}^n \mu_p =$
1, \( p = 1, 2, \ldots , s \), \( s \in \mathbb{N} \), satisfying \( u^*_{i_0 k_0} = \sum_{p=1}^{s} \mu_p u^*_p \). This combined with (3.19) entails that
\[
\sum_{p=1}^{s} \mu_p \langle u^*_p, \nu \rangle > 0.
\]
Then we may choose \( p_0 \in \{1, 2, \ldots , s\} \) so that
\[
\langle u^*_{p_0}, \nu \rangle > 0,
\]
and take \( \bar{v}_{i_0} \in \mathcal{V}_{i_0}(\bar{x}) \) and \( \bar{v}^*_{i_0} \in \partial_x g_{i_0}(G_{i_0}(\bar{x}, \bar{v}_{i_0})) \) such that \( u^*_{p_0} \in \partial_x \langle \bar{v}^*_{i_0}, G_{i_0}\rangle(\bar{x}, \bar{v}_{i_0}) \) due to \( u^*_{p_0} \in \bigcup \left\{ \bigcup \partial_x \langle \bar{v}^*_{i_0}, G_{i_0}\rangle(\bar{x}, \bar{v}_{i_0}) \big | \bar{v}_{i_0} \in \mathcal{V}_{i_0}(\bar{x}) \right\} \). Employing now the type I pseudo-convexity of \( (f \circ F, g \circ G) \) at \( \bar{x} \) and applying (3.20), one has
\[
(g_{i_0} \circ G_{i_0})(\bar{x}, \bar{v}_{i_0}) > (g_{i_0} \circ G_{i_0})(\bar{x}, \bar{v}_{i_0}).
\]
Since \( \bar{v}_{i_0} \in \mathcal{V}_{i_0}(\bar{x}) \), it implies that \( g_{i_0}(G_{i_0}(\bar{x}, \bar{v}_{i_0})) = \max_{\bar{v}_{i_0} \in \mathcal{V}_{i_0}} g_{i_0}(G_{i_0}(\bar{x}, \bar{v}_{i_0})) \) which gives by (3.16) that
\[
\mu_{i_0} g_{i_0}(G_{i_0}(\bar{x}, \bar{v}_{i_0})) = 0.
\]
Using the last equality together with (3.21) allows us \( \mu_{i_0} (g_{i_0} \circ G_{i_0})(\bar{x}, \bar{v}_{i_0}) > 0 \), and so \( (g_{i_0} \circ G_{i_0})(\bar{x}, \bar{v}_{i_0}) > 0 \), which contradicts with the fact that \( \bar{x} \in C \) and ends the proof of (i).

The verification of assertion (ii) is similar to the part (i). Let \( \bar{x} \notin S(CRP) \). Then there exists \( \hat{x} \in C \) satisfying \( (f \circ F)(\hat{x}) - (f \circ F)(\bar{x}) \in -K \setminus \{0\} \). This holds that \( \hat{x} \neq \bar{x} \) and \( \langle y^*, (f \circ F)(\hat{x}) - (f \circ F)(\bar{x}) \rangle \leq 0 \). Finally, involving the type II pseudo convexity of \( (f \circ F, g \circ G) \) at \( \bar{x} \), we arrive at the result.

Remark 3.5. Theorem 3.5 reduces to

(i) (Saadati and Oveisgha 2022b, Theorem 3.4) under pseudo-quasi convex assumptions,

(ii) (Fakhar et al. 2018, Theorem 3.10) where the involved functions are pseudo-quasi generalized convexity, and

(iii) (Chuong 2016, Theorem 3.11) with generalized convex functions,

when the last two conditions with the finite-dimensional frameworks.

The following corollary of Theorem 3.5 concerns a convex problem of uncertain multiobjective optimization to reobtain the robust (KKT) sufficient optimality for (weakly) robust efficient solutions.

Corollary 3.6. Let \( f \) and \( g_i \), \( i = 1, 2, \ldots , n \), be convex functions and \( h_j \), \( j = 1, 2, \ldots , m \), be affine functions. Suppose that \( \bar{x} \) is a robust (KKT) point of the problem (SUP), i.e., (3.9) holds with \( y^* \neq 0 \). Then \( \bar{x} \) is a weakly robust efficient solution of the problem (SUP). If \( f \) is a strictly convex function, then \( \bar{x} \) is a robust efficient solution of such problem.

Proof. Observe, as in the proof of Corollary 3.3, that the problem (SUP) is a special case of the composite uncertain multiobjective optimization problem (CUP), where \( g \) is given as in (1.1). Invoking the convexity of \( f \) and \( g_i \) and the affineness of \( h_j \), it follows from Proposition 3.4(i) that \( (f \circ F, g \circ G) \) is type II pseudo convex at \( \bar{x} \). If suppose that \( f \) is a strict convex function, then \( (f \circ F, g \circ G) \) is type II pseudo convex at \( \bar{x} \) as shown by Proposition 3.4(ii). So, we directly arrive at the desired conclusions by applying Theorem 3.5. \( \square \)
4 Robust duality

In this section, we address a Mond-Weir-type dual problem for the composite robust multiobjective optimization problem (CRP), and study the weak, strong, and converse duality relations between the corresponding problems under the assumptions of pseudo-quasi convexity.

Let \( z \in X, \ y^* \in K^+ \setminus \{0\} \), and \( \mu \in \mathbb{R}^n_+ \). In connection with the problem (CRP), we introduce a dual robust problem of the form

\[
\text{(CRD)} \quad \max_K \{ \bar{f}(z, y^*, \mu) := (f \circ F)(z) \mid (z, y^*, \mu) \in C_D \},
\]

where \( C_D \) is the feasible set defined by

\[
C_D := \left\{ (z, y^*, \mu) \in X \times (K^+ \setminus \{0\}) \times \mathbb{R}^n_+ \mid 0 \in \bigcup_{w^* \in \partial(y^*, f)(\bar{x})} \partial(w^*, F)(\bar{x}) + \sum_{i=1}^{n} \mu_i \text{cl}^* \co \left( \bigcup_{v_i^* \in \partial g_i(G_i(\bar{x}, v_i))} \partial_x (v_i^*, G_i)(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right) \right\},
\]

\[
\mu_i g_i(G_i(z, v_i)) \geq 0, \quad i = 1, 2, \ldots, n.
\]

It should be noticed here that the notions of robust efficient solutions (resp., weakly robust efficient solutions) of the dual problem (CRD) are understood as in Definition 1.1 by replacing \(-K\) (resp., \(-\text{int } K\)) by \(K\) (resp., \(\text{int } K\)). Besides, denote by \(\mathcal{S}(\text{CRD})\) (resp., \(\mathcal{S}^w(\text{CRD})\)) the set of robust efficient solutions (resp., weakly robust efficient solutions) of the problem (CRD). And make for convenience the standard notations

\[
u \prec v \Leftrightarrow u - v \in -\text{int } K, \quad u \preceq v \Leftrightarrow u - v \in -K \setminus \{0\}.
\]

Weak duality theorem which holds between the primal problem (CRP) and the dual problem (CRD) is represented as follows.

**Theorem 4.1. (Weak Duality)** Let \( x \in C \), and let \( (z, y^*, \mu) \in C_D \).

(i) If \((f \circ F, g \circ G)\) is type I pseudo convex at \( z \), then \((f \circ F)(x) \not\preceq \bar{f}(z, y^*, \mu)\).

(ii) If \((f \circ F, g \circ G)\) is type II pseudo convex at \( z \), then \((f \circ F)(x) \not\leq \bar{f}(z, y^*, \mu)\).

*Proof.* Since \((z, y^*, \mu) \in C_D\), there exist \(y^* \in K^+ \setminus \{0\}, w^* \in \partial(y^*, f)(\bar{x}), x^* \in \partial(w^*, F)(\bar{x}), \mu_i \geq 0\), and \(u_i^* \in \text{cl}^* \co \left( \bigcup_{v_i^* \in \partial g_i(G_i(\bar{x}, v_i))} \partial_x (v_i^*, G_i)(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right) \), \(i = 1, 2, \ldots, n\), satisfying

\[
0 = x^* + \sum_{i=1}^{n} \mu_i u_i^*, \tag{4.1}
\]

\[
\mu_i \max_{v_i \in \mathcal{V}_i} g_i(G_i(\bar{x}, v_i)) = 0, \quad i = 1, 2, \ldots, n.
\]

To prove (i) by contradiction, suppose that \((f \circ F)(x) \prec \bar{f}(z, y^*, \mu)\). Then \((y^*, (f \circ F)(x) - \bar{f}(z, y^*, \mu)) \prec 0\) due to \(y^* \neq 0\). This is nothing else but \((y^*, (f \circ F)(x) - (f \circ F)(z)) \prec 0\). Using the
type I pseudo convexity of \((f \circ F, g \circ G)\) at \(z\), we infer from the latter inequality that there is \(\nu \in X\) such that
\[
\langle x^*, \nu \rangle < 0.
\]
On the other side, applying \((4.1)\) allows us
\[
0 = \langle x^*, \nu \rangle + \sum_{i=1}^{n} \mu_i \langle u_i^*, \nu \rangle
\]
for \(\nu\) above. Combining these two relations, we arrive at
\[
\sum_{i=1}^{n} \mu_i \langle x_i^*, \nu \rangle > 0.
\]
Now assume that there exists \(i_0 \in \{1, 2, \ldots, n\}\) such that \(\mu_{i_0} \langle x_{i_0}^*, \nu \rangle > 0\). Proceeding as in the proof of Theorem 3.5(i) and changing from \(\tilde{x} - \bar{x}\) to \(x - z\) yield \((g_{i_0} \circ G_{i_0})(x, \bar{u}_{i_0}) > 0\), which contradicts with \(x \in C\).

Next to justify (ii), we follow the proof of the part (i) by definition of the type II pseudo convexity of \((f \circ F, g \circ G)\) at \(z\) and observe that the condition \((f \circ F)(x) \leq \bar{f}(z, y^*, \mu)\) leads us to a contradiction. \(\square\)

The two forthcoming theorems declare strong duality relationships between the primal problem (CRP) and the dual problem (CRD).

**Theorem 4.2. (Strong Duality)** Let \(\bar{x} \in S^{\nu}(\text{CRP})\) be such that the (CQ) is satisfied at this point. Then there exists \((\tilde{y}^*, \tilde{\mu}) \in K^+ \setminus \{0\} \times \mathbb{R}^n_+\) such that \((\bar{x}, \tilde{y}^*, \tilde{\mu}) \in C_D\). Furthermore,

(i) If \((f \circ F, g \circ G)\) is type I pseudo convex at \(z\) for all \(z \in X\), then \((\bar{x}, \tilde{y}^*, \tilde{\mu}) \in S^{\nu}(\text{CRD})\).

(ii) If \((f \circ F, g \circ G)\) is type II pseudo convex at \(z\) for all \(z \in X\), then \((\bar{x}, \tilde{y}^*, \tilde{\mu}) \in S(\text{CRD})\).

**Proof.** According to Theorem 3.2, there exist \(y^* \in K^+ \setminus \{0\}, \mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+\), with \(\|y^*\| + \|\mu\| = 1\), and \(\bar{v}_i \in \mathcal{V}_i, i = 1, 2, \ldots, n\), satisfying
\[
0 \leq \bigcup_{w^* \in \partial(y^*, f)(\bar{x})} \partial\langle w^*, F \rangle(\bar{x}) + \sum_{i=1}^{n} \mu_i \text{cl}^* \text{co} \left( \bigcup_{v_i^* \in \partial G_i(\bar{x}, \bar{v}_i)} \partial_x(v_i^*, G_i)(\bar{x}, v_i) \right),
\]
\[
\mu_i \text{ max}_{v_i \in \mathcal{V}_i} g_i(G_i(\bar{x}, v_i)) = 0, \quad i = 1, 2, \ldots, n. \tag{4.2}
\]
Letting \(\tilde{y}^* := y^*\) and \(\tilde{\mu} := (\mu_1, \mu_2, \ldots, \mu_n)\), we get \((\tilde{y}^*, \tilde{\mu}) \in K^+ \setminus \{0\} \times \mathbb{R}^n_+\). In addition, the inclusion \(v_i \in \mathcal{V}_i(\bar{x})\) means that, for all \(i \in \{1, 2, \ldots, n\}\), one has \(g_i(G_i(\bar{x}, v_i)) = \max_{u_i \in \mathcal{V}_i} g_i(G_i(\bar{x}, u_i))\). Hence, it follows from (4.2) that \(\mu_i g_i(G_i(\bar{x}, v_i)) = 0, \quad i = 1, 2, \ldots, n\). So \((\bar{x}, \tilde{y}^*, \tilde{\mu}) \in C_D\).

(i) Let \((f \circ F, g \circ G)\) be type I pseudo convex at \(z\) for all \(z \in X\). For each \((z, y^*, \mu) \in C_D\), employing Theorem 4.1(i), we arrive at
\[
\bar{f}(\bar{x}, \tilde{y}^*, \tilde{\mu}) = (f \circ F)(\bar{x}) \neq f(z, y^*, \mu).
\]
Therefore \((\bar{x}, \tilde{y}^*, \tilde{\mu}) \in S^{\nu}(\text{CRD})\).
(ii) Let \((f \circ F, g \circ G)\) be type II pseudo convex at \(z\) for all \(z \in X\). Similarly, employing Theorem 4.1(ii), we obtain
\[
\bar{f}(\bar{x}, \bar{y}^*, \bar{\mu}) \not\leq \bar{f}(z, y^*, \mu)
\]
for each \((z, y^*, \mu) \in C_D\). Thus \((\bar{x}, \bar{y}^*, \bar{\mu}) \in \mathcal{S}(\text{CRD}).\)

Remark 4.1. Theorem 4.1 and Theorem 4.2 improve both (Saadati and Oveisiha 2022b, Theorem 4.1) and (Saadati and Oveisiha 2022b, Theorem 4.2), both (Fakhar et al. 2018, Theorem 5.2) and (Fakhar et al. 2018, Corollary 5.4), and both (Chuong 2016, Theorem 4.1) and (Chuong 2016, Theorem 4.3).

**Theorem 4.3. (Strong Duality)** Let \(\bar{x} \in C\) be such that the robust (KKT) condition is satisfied at this point. Then there exists \((\bar{y}^*, \bar{\mu}) \in K_+ \setminus \{0\} \times \mathbb{R}_+^n\) such that \((\bar{x}, \bar{y}^*, \bar{\mu}) \in C_D\). Moreover,

(i) If \((f \circ F, g \circ G)\) is type I pseudo convex at \(z\) for all \(z \in X\), then \((\bar{x}, \bar{y}^*, \bar{\mu}) \in \mathcal{S}^w(\text{CRD})\) and \(\bar{x} \in \mathcal{S}^w(\text{CRP}).\)

(ii) If \((f \circ F, g \circ G)\) is type II pseudo convex at \(z\) for all \(z \in X\), then \((\bar{x}, \bar{y}^*, \bar{\mu}) \in \mathcal{S}(\text{CRD})\) and \(\bar{x} \in \mathcal{S}(\text{CRP}).\)

**Proof.** Using Definition 3.1, one can proceed similarly to the proof of Theorem 4.2.

Remark 4.2. Theorem 4.1 and Theorem 4.3 develop both (Saadati and Oveisiha 2022b, Theorem 4.1) and (Saadati and Oveisiha 2022b, Theorem 4.3), and both (Fakhar et al. 2018, Theorem 5.2) and (Fakhar et al. 2018, Theorem 5.3).

Finally, we establish converse duality relations between the problems (CRP) and (CRD).

**Theorem 4.4. (Converse Duality)** Let \((\bar{x}, \bar{y}^*, \bar{\mu}) \in C_D\) be such that \(\bar{x} \in C\).

(i) If \((f \circ F, g \circ G)\) is type I pseudo convex at \(\bar{x}\), then \(\bar{x} \in \mathcal{S}^w(\text{CRP}).\)

(ii) If \((f \circ F, g \circ G)\) is type II pseudo convex at \(\bar{x}\), then \(\bar{x} \in \mathcal{S}(\text{CRP}).\)

**Proof.** Since \((\bar{x}, \bar{y}^*, \bar{\mu}) \in C_D\), we find \(\bar{y}^* \in K_+ \setminus \{0\}\) and \(\bar{\mu}_i \in \mathbb{R}_+^n, i = 1, 2, \ldots, n\), such that
\[
0 \in \bigcup_{w^* \in \partial (g^*, f)(\bar{x})} \partial (w^*, F)(\bar{x}) + \sum_{i=1}^n \bar{\mu}_i \mathcal{C}^* \mathcal{C} \left\{ \bigcup_{v_i \in \partial _i (G_i, G_i^*)} \partial _i (v_i^*, G_i^*)(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right\},
\]
\[
\bar{\mu}_i g_i(G_i(\bar{x}, v_i)) \geq 0, \quad i = 1, 2, \ldots, n.
\]
(4.3)

On the other hand, since \(\bar{x} \in C\), we have \((g_i \circ G_i)(\bar{x}, v_i) \leq 0\) for all \(v_i \in V_i, i = 1, 2, \ldots, n\). Noting that \(\bar{\mu}_i g_i(G_i(\bar{x}, v_i)) \geq 0\) due to \(\bar{\mu}_i \geq 0\). Hence, it follows from (4.3) that \(\bar{\mu}_i g_i(G_i(\bar{x}, v_i)) = 0\), and yields by taking into account Definition 3.1 that \(\bar{x}\) is a (KKT) point of the problem (CRP). To finish the proof, it remains to apply Theorem 3.5.

Remark 4.3. Theorem 4.4 reduces to (Saadati and Oveisiha 2022b, Theorem 4.4) for the case of problem (UP).
5 Applications

In this section, applications to necessary optimality conditions for an approximate uncertain multiobjective problem involving equality and inequality constraints and for a composite uncertain multiobjective problem with linear operators are given.

The class of multiobjective approximation problems plays a crucial role in optimization theory, as it provides a convenient framework that can effectively address a wide range of practical problems (see, e.g., Göpfert et al. 2003b; Jahn 2004). Various problem domains, such as location problems, surrogate problems, inverse Stefan type problems, and optimal control problems, can be treated as particular models of multiobjective approximation problems.

Let \( X \) and \( Y_k, k = 1, 2, \ldots, l \), be Asplund spaces, let \( r = (r_1, r_2, \ldots, r_l) : X \to \mathbb{R}^l \) be a vector-valued function, and let \( T_k : X \to Y_k, k = 1, 2, \ldots, l \), be linear operators. We select \( y^k_0 \in Y_k \), \( \alpha_k \geq 0 \), and \( \beta_k \geq 1, k = 1, 2, \ldots, l \), and formulate the following approximate uncertain multiobjective optimization problem

\[
\text{(AUP)} \quad \min_{x \in X_+} \left\{ r(x) + (\alpha_1 \|T_1 x - y^1_0\|^{\beta_1}, \alpha_2 \|T_2 x - y^2_0\|^{\beta_2}, \ldots, \alpha_l \|T_l x - y^l_0\|^{\beta_l}) \mid x \in C \right\}.
\]

Here the feasible set \( C \) is given by

\[
C := \left\{ x \in X \mid g_i(x, v_i) \leq 0, \ i = 1, 2, \ldots, n, \right. \\
\left. h_j(x, v_{n+j}) = 0, \ j = 1, 2, \ldots, m \right\},
\]

where functions \( g_i : X \times V_i \to \mathbb{R} \) and \( h_j : X \times V_{n+j} \to \mathbb{R} \) define the constraints, \( v_i \) and \( v_{n+j} \) are uncertain parameters, and \( v_i \in V_i \) and \( v_{n+j} \in V_{n+j} \) for sequentially compact topological spaces. As mentioned before, we suppose that \( r_k, k = 1, 2, \ldots, l \), \( g_i, i = 1, 2, \ldots, n \), and \( h_j, j = 1, 2, \ldots, m \), are locally Lipschitzian at the point under consideration.

Now we are in a position to provide necessary conditions of the Fritz-John type for weakly robust efficient solutions of the problem (AUP), where the notions of such solutions are introduced similarly to the corresponding ones from Definition 1.1.

**Theorem 5.1.** Let \( \bar{x} \) be a weakly robust efficient solution of the problem (AUP). Then there exist \( \lambda_k \geq 0, k = 1, 2, \ldots, l \), \( \mu_i \geq 0, i = 1, 2, \ldots, n \), \( \sigma_j \geq 0, j = 1, 2, \ldots, m, \) not all zero, \( y^*_k \in Y^*_k, k = 1, 2, \ldots, l, \) with

\[
\langle y^*_k, T_k \bar{x} - y^*_0 \rangle = \|T_k \bar{x} - y^*_0\|^{\beta_k}, \begin{cases} \|y^*_k\| \leq 1 & \text{if } \beta_k = 1 \text{ and } T_k \bar{x} = y^*_0, \\ \|y^*_k\| = \|T_k \bar{x} - y^*_0\|^{\beta_k - 1} & \text{otherwise}, \end{cases}
\]

and \( \bar{v}_i \in V_i, i = 1, 2, \ldots, n, \) such that

\[
0 \in \sum_{k=1}^l \lambda_k \partial r_k(\bar{x}) + \sum_{k=1}^l \lambda_k \alpha_k \beta_k T_k^\top y^*_k + \sum_{i=1}^n \mu_i \partial^* g_i(\bar{x}, v_i) \left( \bigcup_{v_i \in V_i} \partial g_i(\bar{x}, v_i) \right) + \sum_{j=1}^m \sigma_j \partial^* h_j(\bar{x}, v_{n+j}) \bigcup \partial h_j(-h_j(\bar{x}, v_{n+j})) \left( v_{n+j} \in V_{n+j} \right), \tag{5.1}
\]

\[
\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = \mu_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, 2, \ldots, n.
\]
Proof. For each $k = 1, 2, \ldots, l$, put $\tilde{r}_k(x) := \alpha_k \|T_k x - y_0^k\| \beta_k$, $x \in X$, and define vector-valued functions $F : X \to \mathbb{R}^l \times \mathbb{R}^l$, $f : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}^l$, $G_i : X \times V_i \to X \times V_i$, $i = 1, 2, \ldots, n$, $G_{n+j} : X \times V_{n+j} \to X \times V_{n+j}$, $j = 1, 2, \ldots, m$, $g_i : X \times V_i \to \mathbb{R}$, $i = 1, 2, \ldots, n$, and $g_{n+j} : X \times V_{n+j} \to \mathbb{R}$, $j = 1, 2, \ldots, m$, by

$$F(x) := (r_1(x), r_2(x), \ldots, r_l(x), \tilde{r}_1(x), \tilde{r}_2(x), \ldots, \tilde{r}_l(x)), \quad x \in X,$$

$$f(w) := (w_1 + \bar{w}_1, w_2 + \bar{w}_2, \ldots, w_l + \bar{w}_l), \quad w := (w_1, w_2, \ldots, w_l, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_l) \in \mathbb{R}^l \times \mathbb{R}^l,$$

$$G_i(x, v_i) := (x, v_i), \quad x \in X, \quad v_i \in V_i, i = 1, 2, \ldots, n,$$

$$G_{n+j}(x, v_{n+j}) := (x, v_{n+j}), \quad x \in X, \quad v_{n+j} \in V_{n+j}, j = 1, 2, \ldots, m,$$

$$g(z, u) := (g_1(z, u_1), g_2(z, u_2), \ldots, g_l(z, u_l), h_1(z, u_{l+1}), h_2(z, u_{l+2}), \ldots, h_m(z, u_{l+m})), \quad z \in X,$$

$$u_i \in V_i, \quad i = 1, 2, \ldots, n,$$

$$u_{n+j} \in V_{n+j}, \quad j = 1, 2, \ldots, m.$$

One can check that the problem (AUP) is a particular case of the problem (CUP) with $K := \mathbb{R}_+^l$. Employing Theorem 3.2 and proceeding as in the proof of Corollary 3.3, we can find $y^* := (\lambda_1, \lambda_2, \ldots, \lambda_l) \in K^+ = \mathbb{R}_+^l$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}_+^n$, $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathbb{R}_+^m$, with $\|y^*\| + \|\mu\| + \|\sigma\| \neq 0$, $\bar{v}_i \in V_i$, $i = 1, 2, \ldots, n$, and $\bar{v}_{n+j} \in V_{n+j}$, $j = 1, 2, \ldots, m$, satisfying

$$0 \in \bigcup_{w^* \in \partial(y^*, f)(F(x))} \partial(w^*, F)(\bar{x}) + \sum_{i=1}^n \mu_i \operatorname{cl}^* \sum_{i=1}^n \left\{ \partial x \langle v_i^*, G_i \rangle(\bar{x}, v_i) \right\} + \sum_{j=1}^m \sigma_j \operatorname{cl}^* \sum_{j=1}^m \left\{ \partial x \langle v_{n+j}^*, G_{n+j} \rangle(\bar{x}, v_{n+j}) \right\}$$

$$\partial x \langle v_{n+j}^*, G_{n+j} \rangle(\bar{x}, v_{n+j}) \mid v_{n+j} \in V_{n+j}(\bar{x}) \right\}$$

(5.2)

and

$$\begin{cases} \mu_i \max_{v_i \in V_i} g_i(G_i(\bar{x}, v_i)) = \mu_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, 2, \ldots, n, \\
\sigma_j \max_{v_{n+j} \in V_{n+j}} h_j(G_{n+j}(\bar{x}, v_{n+j})) = \sigma_j h_j(\bar{x}, \bar{v}_{n+j}) = 0, \quad j = 1, 2, \ldots, m. \end{cases}$$

(5.3)

It follows from the above definition that for each $w \in \mathbb{R}^l \times \mathbb{R}^l$ one has

$$\langle y^*, f \rangle(w) = \sum_{k=1}^l \lambda_k (w_k + \bar{w}_k),$$

and hence

$$\partial \langle y^*, f \rangle(w) = \{ (\lambda_1, \lambda_2, \ldots, \lambda_l, \lambda_1, \lambda_2, \ldots, \lambda_l) \}.$$
Thus applying the sum rule of Lemma 2.3, we get
\[ \partial(w^*, F)(\bar{x}) = \partial \left( \sum_{k=1}^{l} \lambda_k r_k + \sum_{k=1}^{l} \lambda_k \tilde{r}_k \right)(\bar{x}) \subset \sum_{k=1}^{l} \lambda_k \partial r_k(\bar{x}) + \sum_{k=1}^{l} \lambda_k \partial \tilde{r}_k(\bar{x}). \]

In addition putting
\[ p_k(x) := T_k x - y^k, \quad x \in X, \quad q_k(y) := \alpha_k \|y\|^{2k}, \quad y \in Y_k, \]
we have from Lemmas 2.2 and 2.6 that
\[ \partial \tilde{r}_k(\bar{x}) = \partial (p_k \circ q_k)(\bar{x}) \subset \{ \alpha_k \beta_k T_k y_k^* \mid y_k^* \in Y_k^* \} \]
with
\[
\begin{cases}
\|y_k^*\| \leq 1 & \text{if } \beta_k = 1 \text{ and } T_k \bar{x} = y_k^0, \\
\langle y_k^*, T_k \bar{x} - y_k^0 \rangle = \|T_k \bar{x} - y_k^0\|^{2k} \text{ and } \|y_k^*\| = \|T_k \bar{x} - y_k^0\|^{\beta_k - 1} & \text{otherwise.}
\end{cases}
\]

So, we arrive at the following inclusion
\[
\bigcup_{w^* \in \partial(w^*, F)(\bar{x})} \partial(w^*, F)(\bar{x}) \subset \sum_{k=1}^{l} \lambda_k \partial r_k(\bar{x}) + \left\{ \sum_{k=1}^{l} \lambda_k \alpha_k \beta_k T_k y_k^* \mid y_k^* \in Y_k^* \right\},
\]
where \( y_k^* \) for \( k = 1, 2, \ldots, l \) satisfy (5.4). Similarly to the above, we obtain
\[
\bigcup_{v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))} \partial_x \langle v_i^*, G_i \rangle(\bar{x}, v_i) = \partial_x g_i(\bar{x}, v_i), \quad i = 1, 2, \ldots, n,
\]
and
\[
\bigcup_{v_{n+j}^* \in \partial_x h_j(G_{n+j}(\bar{x}, v_{n+j}))} \partial_x \langle v_{n+j}^*, G_{n+j} \rangle(\bar{x}, v_{n+j}) 
\subset \partial_x h_j(\bar{x}, v_{n+j}) \cup \partial_x (-h_j)(\bar{x}, v_{n+j}), \quad j = 1, 2, \ldots, m.
\]

Combining (5.2) with (5.5)-(5.7), we finally get (5.1). Note that in our setting (5.3) collapses to the following
\[ \mu_i \max_{\bar{v}_i \in V_i} g_i(\bar{x}, \bar{v}_i) = \mu_i g_i(\bar{x}, \bar{v}_i) = 0 \]
for each \( i = 1, 2, \ldots, n \). This justifies the last statement of the theorem and completes the proof.

**Example 5.1.** Let \( X := \mathbb{R}^2, Y_k := \mathbb{R}^2, k = 1, 2, 3, Y_i := [-1, -
1/4], i = 1, 2, \) and \( Y_{2+j} := [-1, -1/4], j = 1, 2 \). Take the following approximate uncertain multiobjective optimization problem
\[
\text{(AUP)} \quad \min_{\mathbb{R}^3} \left\{ r(x) + (\alpha_1 \|T_1 x - y_0^1\|^{\beta_1}, \alpha_2 \|T_2 x - y_0^2\|^{\beta_2}, \alpha_3 \|T_3 x - y_0^3\|^{\beta_3}) \mid x := (x_1, x_2) \in X, \right.
\]
\[
g_i(x, v_i) \leq 0, \quad i = 1, 2 \quad \text{and} \quad h_j(x, v_{2+j}) = 0, \quad j = 1, 2 \left\},
\]
where \( r : X \to \mathbb{R}^3, r := (r_1, r_2, r_3), \) is defined by
\[
\begin{align*}
r_1(x_1, x_2) &:= 3|x_1| + \frac{2}{5}x_2 + \frac{4}{5}, \\
r_2(x_1, x_2) &:= \frac{1}{4}x_1^2 + 2, \\
r_3(x_1, x_2) &:= 2|x_1| - \frac{1}{8}x_2^2 + 1,
\end{align*}
\]
$g_i : X \times \mathcal{V}_i \to \mathbb{R}$, $i = 1, 2$, and $h_j : X \times \mathcal{V}_{2+j} \to \mathbb{R}$, $j = 1, 2$, are given respectively by

\[
\begin{align*}
    g_1(x_1, x_2, v_1) &:= \frac{1}{4} v_1^2 |x_1| + \frac{1}{2} v_1^2 x_2 + \frac{1}{4} |v_1|, \\
    g_2(x_1, x_2, v_2) &:= \frac{1}{8} x_1^2 + |v_2| x_2 + |v_2| + \frac{1}{4}, \\
    h_1(x_1, x_2, v_3) &:= v_3(-3x_1 + x_2 + 2), \\
    h_2(x_1, x_2, v_4) &:= v_4(-3x_1 - x_2 - 2),
\end{align*}
\]

$T_1 := \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix}$, $T_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$, $T_3 := \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}$, $y_1 := (-1, 0) \in Y_1$, $y_2 := (0, -1) \in Y_2$, $y_3 := (-1, -1) \in Y_3$, $\alpha_1 := 2$, $\alpha_2 := \alpha_3 := 1$, and $\beta_1 := \beta_2 := \beta_3 := 1$. Note that from the constraints of inequality and equality one get

$$C = \{(x_1, x_2) \in X \mid -20 \leq x_2 \leq -2, \quad \text{and} \quad x_2 = 3x_1 - 2, \quad x_2 = -3x_1 - 2\}.$$ 

Now, suppose that $\bar{x} := (0, -2) \in C$ and $x := (x_1, x_2) \in C$. It is easy to check that $\bar{x}$ is a weakly robust efficient solution of the problem (AUP). One can also see that

$$\phi_1(\bar{x}) = \max_{v_1 \in \mathcal{V}_1} g_1(\bar{x}, v_1) = \max_{v_1 \in \mathcal{V}_1} (-v_1^2 + \frac{1}{4} |v_1|) = 0,$$
$$\phi_2(\bar{x}) = \max_{v_2 \in \mathcal{V}_2} g_2(\bar{x}, v_2) = \max_{v_2 \in \mathcal{V}_2} (-|v_2| + \frac{1}{4}) = 0.$$

Hence $\phi(\bar{x}) = \max\{\phi_1(\bar{x}), \phi_2(\bar{x})\} = 0$, $\mathcal{V}_1(\bar{x}) = \{-\frac{1}{4}\}$, and $\mathcal{V}_2(\bar{x}) = \{-\frac{1}{4}\}$. Performing elementary calculations gives us from the definition that $\partial r_1(\bar{x}) = [-3, 3] \times \{\frac{2}{5}\}$, $\partial r_2(\bar{x}) = (0, 0)$, $\partial r_3(\bar{x}) = [-2, 2] \times \{\frac{1}{2}\}$,

$$\text{cl}^* \text{co} \left( \partial_x g_1(\bar{x}, v_1) = -\frac{1}{4} \right) = \left[-\frac{1}{64}, \frac{1}{64}\right] \times \left[\frac{1}{32}\right], \quad \text{cl}^* \text{co} \left( \partial_x g_2(\bar{x}, v_2) = -\frac{1}{4} \right) = \left(0, \frac{1}{4}\right),$$

and

$$\text{cl}^* \text{co} \left( \bigcup \left\{ \partial_x h_1(\bar{x}, v_3) \cup \partial_x h_2(\bar{x}, v_4) \mid v_3 \in \mathcal{V}_3 \right\} \right) = \left\{(v_3, -\frac{1}{3}v_3) \mid v_3 \in [-3, 3]\right\},$$

$$\text{cl}^* \text{co} \left( \bigcup \left\{ \partial_x h_1(\bar{x}, v_4) \cup \partial_x h_2(\bar{x}, v_4) \mid v_4 \in \mathcal{V}_4 \right\} \right) = \left\{(v_4, \frac{1}{3}v_4) \mid v_4 \in [-3, 3]\right\}.$$ 

Finally, observe that there exist $\lambda = (1, 0, 1) \in \mathbb{R}_+^3$, $\mu = (0, 1) \in \mathbb{R}_+^2$, $\sigma = (1, 0) \in \mathbb{R}_+^2$, $y_1^* = (-\frac{2}{5}, 0) \in \mathcal{Y}_1^*$, $y_2^* = (0, 0) \in \mathcal{Y}_2^*$, and $y_3^* = (-\frac{1}{2}, -\frac{1}{2}) \in \mathcal{Y}_3^*$ satisfying

$$0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{32} & 1 & 1 \\ 1 & \frac{1}{4} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix},$$

$$+ 2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix},$$

and $\mu_i \max_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) = 0$ for $i = 1, 2$. 

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Next let $T_k : X \to Y_k$, $k = 1, 2, \ldots, l$ be linear operators and $f_k : Y_k \to \mathbb{R}$, $k = 1, 2, \ldots, l$ be local Lipschitz functions between Asplund spaces. We consider the following composite uncertain multiobjective optimization problem with linear operators

$$(\text{CUL}) \quad \min_{\bar{x} \in X} \left\{ (f_1(T_1x), f_2(T_2x), \ldots, f_l(T_lx)) \mid x \in C \right\}. $$

Here the feasible set $C$ is given by

$$C := \left\{ x \in X \mid g_i(x, v_i) \leq 0, \ i = 1, 2, \ldots, n \right\}, $$

where function $g_i : X \times V_i \to \mathbb{R}$ is locally Lipschitz, $v_i$ is uncertain parameter, and $v_i \in V_i$ for sequentially compact topological spaces.

We now derive necessary conditions of the Fritz-John form for weakly robust efficient solutions of the problem (CUL).

**Theorem 5.2.** Let $\bar{x}$ be a weakly robust efficient solution of the problem (CUL). Then there exist $\lambda_k \geq 0$, $k = 1, 2, \ldots, l$, $\mu_i \geq 0$, $i = 1, 2, \ldots, n$, with $\|(\lambda_1, \lambda_2, \ldots, \lambda_l)\| + \|(\mu_1, \mu_2, \ldots, \mu_n)\| = 1$, and $\bar{v}_i \in V_i$, $i = 1, 2, \ldots, n$, such that

$$0 \in \sum_{k=1}^l \lambda_k T_k^T \partial f_k(\bar{y}_k) + \sum_{i=1}^n \mu_i c^i \co \left( \bigcup_{v_i \in V_i} \partial_x g_i(\bar{x}, v_i) \right), $$

$$\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = \mu_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, 2, \ldots, n, $$

where $\bar{y}_k = T_k \bar{x}$ for $k = 1, 2, \ldots, l$.

**Proof.** We first observe that the problem (CUL) is a special case of the composite uncertain multiobjective optimization problem (CUP), where $Z = X$, $G = (G_1, G_2, \ldots, G_n)$ is an identical map, the functions $F : X \to W := Y_1 \times Y_2 \times \cdots \times Y_l$ and $f : W \to Y := \mathbb{R}^l$ are defined, respectively, by

$$F(x) := (T_1x, T_2x, \ldots, T_lx), \quad x \in X,$n

$$f(w) := (f_1(w_1), f_2(w_2), \ldots, f_l(w_l)), \quad w := (w_1, w_2, \ldots, w_l) \in Y_1 \times Y_2 \times \cdots \times Y_l,$n

and $K := \mathbb{R}^l_+$. Applying Theorem 3.2, we find $y^* := (\lambda_1, \lambda_2, \ldots, \lambda_l) \in K^+ = \mathbb{R}^l_+$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, with $\|y^*\| + \|\mu\| = 1$, and $\bar{v}_i \in V_i$, $i = 1, 2, \ldots, n$, such that

$$0 \in \bigcup_{w^* \in \partial (y^*, f)(\bar{x})} \partial (w^*, F)(\bar{x}) + \sum_{i=1}^n \mu_i c^i \co \left( \bigcup_{v_i^* \in \partial_x g_i(\bar{x}, v_i)} \partial_x (v_i^*, G_i)(\bar{x}, v_i) \right), $$

$$\mu_i \max_{v_i \in V_i} g_i(G_i(\bar{x}, v_i)) = \mu_i g_i(G_i(\bar{x}, \bar{v}_i)) = 0, \quad i = 1, 2, \ldots, n. $$

For each $k = 1, 2, \ldots, l$, consider a mapping $\Psi_k : Y_1 \times Y_2 \times \cdots \times Y_l \to Y_k$ given by $\Psi_k(w) := w_k$, $w := (w_1, w_2, \ldots, w_l) \in Y_1 \times Y_2 \times \cdots \times Y_l$. Using the definition, we have

$$\langle y^*, f \rangle (w) = \sum_{k=1}^l \lambda_k (f_k \circ \Psi_k)(w), \quad w \in Y_1 \times Y_2 \times \cdots \times Y_l, $$
and hence invoking Lemmas 2.2 and 2.3, 
\[
\partial(y^*, f)(w) \subset \sum_{k=1}^l \lambda_k (f_k \circ \Psi_k)(w) 
\subset \lambda_1 \partial f_1(w_1) \times \lambda_2 \partial f_2(w_2) \times \cdots \times \lambda_l \partial f_l(w_l).
\]
This gives 
\[
\partial(y^*, f)(F(\bar{x})) \subset \lambda_1 \partial f_1(y_1) \times \lambda_2 \partial f_2(y_2) \times \cdots \times \lambda_l \partial f_l(y_l),
\]
where \(y_k = T_k x\) for \(k = 1, 2, \ldots, l\).

Now picking any \(w^* \in \partial(y^*, f)(F(\bar{x}))\) and taking (5.12) into account, there exist \(w_k^* \in \partial f_k(\bar{y}_k)\), \(k = 1, 2, \ldots, l\), with \(\bar{y}_k = T_k \bar{x}\) satisfying \(w^* = (\lambda_1 w_1^T, \lambda_2 w_2^T, \ldots, \lambda_l w_l^T)\). Then it follows from the latter that 
\[
\langle w^*, F \rangle(x) = \sum_{k=1}^l \lambda_k w_k^T T_k x, \quad x \in X,
\]
and thus \(\partial \langle w^*, F \rangle(\bar{x}) = \sum_{k=1}^l \lambda_k T_k^T w_k^*\). So, we get the inclusion 
\[
\bigcup_{w^* \in \partial(y^*, f)(F(\bar{x}))} \partial \langle w^*, F \rangle(\bar{x}) \subset \sum_{k=1}^l \lambda_k T_k^T \partial f_k(\bar{y}_k).
\]
Similarly, we arrive at 
\[
\bigcup_{v_i^* \in \partial_x G_i(G(\bar{x}, v_i))} \partial_x v_i^* G_i(\bar{x}, v_i) = \partial_x g_i(\bar{x}, v_i), \quad i = 1, 2, \ldots, n,
\]
since \(G\) is an identical map. Finally, combining (5.10) with the relations in (5.13) and (5.14), we obtain (5.8). Furthermore, in our setting (5.11) reduces to (5.9), and so the proof of the theorem is complete.

**Example 5.2.** Let \(X, Y_k, k = 1, 2, 3, V_i, i = 1, 2, T_k, k = 1, 2, 3, r_k, k = 1, 2, 3, \) and \(g_i, i = 1, 2, \) be the same as Example 5.1, and let \(f_k := r_k, k = 1, 2, 3.\) Take the following composite uncertain multiobjective optimization problem with linear operators

\[
(CUL) \quad \min_{\mathbb{R}_+^3} \left\{ (f_1(T_1 x), f_2(T_2 x), f_3(T_3 x)) \mid x := (x_1, x_2) \in X, \quad g_i(x, v_i) \leq 0, \quad i = 1, 2 \right\}.
\]

Note that the robust feasible set is given by 
\[
C = \left\{ (x_1, x_2) \in X \mid |x_1| \leq 1 \text{ and } x_2 \leq -\frac{1}{2} |x_1| - 2 \right\} \bigcup \left\{ (x_1, x_2) \in X \mid |x_1| > 1 \text{ and } x_2 \leq -\frac{1}{2} x_1^2 - 2 \right\}.
\]
Suppose that \(\bar{x} := (0, -2) \in C.\) It is easy to check that \(f_2(T_2 x) - f_2(T_2 \bar{x}) \geq 0\) for all \(x \in C,\) i.e., \(\bar{x}\) is a weakly robust efficient solution of the problem (CUL). Taking into account that \(\bar{y}_1 = T_1 \bar{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\) 
\(\bar{y}_2 = T_2 \bar{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\) and \(\bar{y}_3 = T_3 \bar{x} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},\) we have \(\partial r_1(\bar{y}) = [-3, 3] \times \left\{ \frac{2}{5} \right\}, \partial r_2(\bar{y}) = (0, 0),\) and 
\(\partial r_3(\bar{y}) = [-2, 2] \times \left\{ \frac{1}{4} \right\}.\) Finally, we can find \(\lambda = (0, 0, \frac{1}{2}) \in \mathbb{R}_+^3\) and \(\mu = (0, \frac{1}{2}) \in \mathbb{R}_+^2,\) with \(|\lambda| + |\mu| = 1,\) such that
\[
0 = 0 \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{2}{5} \end{pmatrix} + 0 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{3}{4} \\ \frac{1}{4} \end{pmatrix} + 0 \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{32} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
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