Neutrino flavor oscillations in rotating matter

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Abstract

We study the evolution of the neutrinos system in rotating matter. Neutrinos are supposed to be mixed massive particles interacting with background fermions by means of the electroweak forces. First we find the solutions of wave equations for the neutrino mass eigenstates in matter. Then we study the behavior of neutrino flavor eigenstates in background matter. The problems of neutrino bound states and neutrino flavor oscillations are discussed. We also derive the analog of the quantum mechanical evolution equation for the system of two flavor neutrinos in rotating matter and analyze its solution for the particular initial condition for neutrino flavor eigenstates. Keywords: exact solutions of wave equations, neutrino oscillations, rotating matter

1 Introduction

Nowadays it is acknowledged that neutrinos play a significant role in the evolution of massive stars at the ultimate stages of their life. When the mass of a star is about $8 - 9$ or $40 - 60$ solar masses, such a star can explode as a supernova with the emission of great quantity of neutrinos carrying away almost $99\%$ of the initial gravitational energy of a star [1].

Although neutrinos interact rather weakly with background matter, these particles are the key component in the dynamics of a supernova explosion. After the supernova explosion the core a massive star is converted into a compact dense object, a neutron star. Neutrinos are supposed to give the contribution to the subsequent evolution of a neutron star, i.e. causing its cooling [2]. Besides the direct influence to the supernova explosion process neutrinos can also affect various macroscopic characteristics of a neutron star. For example, great peculiar velocities of pulsars can be explained by the asymmetric neutrino emission [3]. It is also supposed that the emission of neutrinos can cause the spin-down of a rotating neutron star [4].

The fact that neutrinos are massive particles has many important consequences. Unlike photons that immediately escape the region where they are created, neutrinos with relatively small
initial energies can form bound states inside or in the vicinity of various astrophysical objects. The most proper candidates for the formation of such non-trivial states are relic neutrinos. The possibility of gravitational clustering of relic neutrinos was studied in Ref. \[5\]. It was discussed in Ref. \[6\] that both Dirac and Majorana neutrinos can also create a superfluid condensate due to the Higgs boson interactions. We considered the situation of gravitational trapping of neutrinos by a massive black hole in Ref. \[7\]. The neutrino trapping in both curved space-time and rotating matter was studied in Ref. \[8\].

There is also a possibility that neutrinos emitted in a neutron star form bound orbits inside the star due to their collective interactions with neutron matter. This issue was studied in Ref. \[9\]. Neutrino trapping inside a rotating neutron star was discussed in Ref. \[10\]. This effect results from the neutrino electroweak interaction with inhomogeneously moving matter.

Recently we developed an approach for the description of neutrino flavor and spin-flavor oscillations in various external fields \[11\]. The Lagrangian for this system is then:

$$\mathcal{L} = \sum_{\lambda=\alpha,\beta} \bar{\nu}_\lambda (i \gamma^\mu \partial_\mu - f^\mu_{\lambda \gamma} P_L) \nu_\lambda - \sum_{\lambda\lambda'=\alpha,\beta} m_{\lambda\lambda'} \bar{\nu}_\lambda \nu_{\lambda'}. \quad (1)$$

In our case of interest, the flavor $\alpha$ will be either $\mu$ or $\tau$, and the flavor $\beta$ will be $e$.

The form of the currents $f^\mu_{\lambda}$ are determined by the neutrino interactions with the medium. In the case of a neutron star the medium consists of electrons, protons, and neutrons, with number densities $n_e$, $n_p$ and $n_n$, respectively, and $n_e = n_p$, corresponding to electrically neutral matter. Therefore, for the standard model neutrino flavors $\alpha$ (either $\mu$ or $\tau$) and $\beta = e$, the corresponding external fields have the following

2 General formulation

Let us first formulate the evolution of two neutrino flavor eigenstates, $\nu_\lambda$, $\lambda = \alpha, \beta$, interacting with moving matter due to the exchange of the electroweak $Z$ and $W^\pm$ bosons. Phenomenologically this interaction with matter can be implemented by means of a set of neutrino wave equations with the external fields $f^\mu_{\lambda}$ \[13\] shown below. In the flavor basis, neutrinos also have a non-diagonal mass matrix ($m_{\lambda\lambda'}$). The Lagrangian for this system is then:

In Sec. 2 we give the general formulation of two neutrino flavors interacting with rotating matter of the type found in neutron stars. In Secs. 3 and 4 we find the solutions for the Dirac equation for a neutrino interacting with moving matter for massless and massive particles respectively. We also compare our solutions with the previously found ones \[10\] \[12\]. In Sec. 5 we discuss the possibility for low energy neutrinos to form bound orbits inside a neutron star. Neutrino flavor oscillations in rotating matter are discussed in Sec. 6. In Appendix A we state the solution of the wave equation for a neutrino in vacuum in cylindrical coordinates. Finally in Sec. 7 we summarize our results.
where the matter interaction term contains the diagonal elements of Eq. (6), so it is an interaction with the medium that mixes the different mass eigenstates.

We solve Eq. (7) treating the last term (the mixing of neutrino types) as a perturbation, so that at zeroth order the neutrino types are decoupled. Also, since the typical energy of a neutrino emitted in a neutron star is \( \sim 10 \text{ MeV} \) whereas the neutrino masses do not exceed a few eV, the neutrinos are ultrarelativistic and we can treat the masses \( m_a \) as also as perturbations.

Now, for a rigid rotating medium, the interaction depends on a velocity \( \mathbf{v}(\mathbf{r}) = \mathbf{\Omega} \times \mathbf{r} \), where \( \mathbf{r} \) is the radius vector from the star center and \( \mathbf{\Omega} \) is the angular velocity of the star. Therefore we define the positive potentials as:

\[
V_a = -g_{aa}^0 = \begin{cases} 
G_{\text{F}}(n_n - 2n_e \sin^2 \theta)/\sqrt{2}, & \text{for } a = 1, \\
G_{\text{F}}(n_n - 2n_e \cos^2 \theta)/\sqrt{2}, & \text{for } a = 2.
\end{cases}
\]  

Note that \( V_a > 0 \) in Eq. (8) since \( n_n \gg n_e \) in a neutron star and \( g_{aa}^0 < 0 \) (\( a = 1, 2 \)).

In order to proceed with the description of the evolution of the system (7), we should have the energy levels and wave functions of the mass eigenstates \( \psi_\lambda \) which correspond to the wave equation (7) at the absence of the mixing term \( g_{ab}^0 \). These quantities will be found in Secs. 3 and 4.

expressions [14]:

\[
f_\alpha^\mu = \frac{G_F}{\sqrt{2}} j_\alpha^\mu, \quad f_\beta^\mu = \frac{G_F}{\sqrt{2}} (2j_\alpha^\mu - j_\beta^\mu),
\]

where \( G_F \) is the Fermi constant and

\[
\begin{align*}
\nu_e = (n_e, n_e \mathbf{v}), & \quad j_\nu_e^\mu = (n_e, n_e \mathbf{v}),
\end{align*}
\]

are the hydrodynamical currents of each of the background fermion species. We also assume that all background fermions rotate as a rigid body, i.e., moving with same velocity \( \mathbf{v} \).

To study the evolution of the system (1) we diagonalize the mass matrix \( (m_{\lambda\lambda'}) \) by introducing the set of the neutrino mass eigenstates \( \psi_\lambda \), \( a = 1, 2 \), with help of the matrix transformation:

\[
u_\lambda = \sum_{a=1,2} U_{\lambda a} \psi_a,
\]

\[
(U_{\lambda a}) = \begin{pmatrix} U_{\nu_1} & U_{\nu_2} \\ U_{\nu_1} & U_{\nu_2} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]  

Here \( \theta \) is the vacuum mixing angle, where \( \theta = 0 \) means \( \nu_1 = \nu_\alpha \) and \( \nu_2 = \nu_\beta \). After diagonalization the Lagrangian (1) reads

\[
\mathcal{L} = \sum_{a=1,2} \bar{\psi}_a (i\gamma^\mu \partial_\mu - m_a) \psi_a - \sum_{a,b=1,2} g_{ab}^\mu \bar{\psi}_a \gamma_\mu P_L \psi_b,
\]

where the matter interaction term contains the \( 2 \times 2 \) matrix \( (g_{ab}^\mu) \) in the mass eigenstate basis

\[
(g_{ab}^\mu) = \frac{G_F}{\sqrt{2}} \times \begin{pmatrix} 2j_\nu_1^\mu \sin^2 \theta - j_\nu_2^\mu \\ j_\nu_2^\mu \sin 2\theta \\ j_\nu_2^\mu \sin 2\theta \end{pmatrix}.
\]  

The Dirac equation for the neutrino mass eigenstates, obtained from Eq. (5), has then the form,

\[
(i\gamma^\mu \partial_\mu - m_a - g_{aa}^\mu \gamma_\mu P_L) \psi_a - g_{ab}^\mu \gamma_\mu P_L \psi_b = 0, \quad a \neq b,
\]  

where the last term corresponds to the off-diagonal elements of Eq. (6), so it is an interaction with the medium that mixes the different mass eigenstates.
3 Solution of the wave equation for a neutrino in rotating matter in the limit of zero mass

In this section we find the solution of Eq. (7) at the absence of the mixing between different mass eigenstates due to the interaction with matter, i.e. we put the coefficient $g_{12}^\mu$ to zero.

This case corresponds to a single unmixed neutrino interacting with an external axial-vector field. The wave equation (7) is transformed to the form [14, 15],

$$ (i\gamma^\mu\partial_\mu - m - g^\mu\gamma_\mu P_L)\psi = 0. $$

(9)

Here we are interested in the case of ultrarelativistic neutrinos, i.e when the mass in Eq. (9) is much smaller than the energy.

The equations of motion for the left-handed $\eta$ and right-handed $\xi$ chiral components of the spinor $\psi^T = (\xi, \eta)$ decouple in the $m = 0$ limit.

The mass contribution can be included in perturbation theory (see Sec. 4). Using the chiral basis for the $\gamma^\mu$ matrices in the convention of Ref. [16],

$$ \gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, $$

$$ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, $$

(10)

the Dirac equation for the left-handed component $\eta$ of the neutrino is

$$ i\eta = \{i\sigma \cdot \nabla + \sigma_\mu g^\mu\}\eta, $$

(11)

where the matter interaction term is $\sigma_\mu g^\mu = g_0 + \sigma \cdot g$.

The external field $g^\mu$ with non-zero spatial components corresponds to the external electroweak field due to the moving background matter. In analogy to Eq. (6) we obtain for the three-vector part, $g = g_0 \mathbf{v}$. We want to study the neutrino states inside a neutron star rotating with the angular velocity $\Omega$ directed along the $z$ axis, $\Omega = \Omega e_z$. As we mentioned in Sec. 2, since $g_0 < 0$ in the case of a neutron star, we redefine this potential as $V = -g_0$. Accordingly,

$$ g = V \Omega(e_y - xe_x). $$

(12)

It is natural to use cylindrical coordinates $(r, \phi, z)$ to solve Eq. (11) with

$$ i\sigma \cdot \nabla = i\left( e^{i\phi}[\partial_z + (i/r)\partial_\phi] - e^{-i\phi}[\partial_\phi - (i/r)\partial_z] \right), $$

$$ \sigma \cdot g = V \Omega r \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix}. $$

(13)

Taking into account that Eqs. (11) and (13) do not depend on $z$, we look for a stationary solution of the form $\eta \sim u(r, \phi)e^{-i(k_r - p_z)z}$. Using the analysis of Ref. [17] we can further separate the two coordinates $r$ and $\phi$ in the two-component spinor $u$ using auxiliary functions $F_{1,2}(\rho)$ as

$$ u(r, \phi) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} e^{i(l-1)\phi} F_1(\rho) \\ e^{il\phi} F_2(\rho) \end{pmatrix}, $$

(14)

where $\rho = V \Omega r^2$, and $l$ is an integer so that the function of $\phi$ is single-valued. Since the system is invariant under rotations around the $Z$ axis, $u$ has to definite value of $J_z$ (total angular momentum), which in our notation is equal to $l - 1/2$.

The functions $F_{1,2}$ satisfy the coupled equa-
function Eq. (18), is related to the neutrino energy. The \( \kappa \) is the radial quantum number, and \( \kappa \) is related to the energy in the form, \( \kappa = \frac{(E + V)^2 - p_z^2}{4V\Omega} \). (18)

Let us just solve Eq. (16) for the function \( F_1 \). Eq. (17) can be solved for \( F_2 \) by analogy. Expressing \( F_1(\rho) = e^{-\rho^2/2} \rho^{l-1/2} u(\rho) \), the new function \( u(\rho) \) obeys an associated Laguerre equation,

\[
\rho u'' + (l - \rho) u' + (\kappa - l) u = 0,
\]

whose solutions are the associated Laguerre polynomials \( u(\rho) \sim Q_l^{k-1}(\rho) \). Here \( s = \kappa - l \) is the radial quantum number, and \( \kappa \), defined in Eq. (18), is related to the neutrino energy. The function \( F_1 \) is then a Laguerre function, \( F_1(\rho) = I_{\kappa-1,s}(\rho) \). The Laguerre functions \( I_{n,s}(\rho) \) are defined in terms of the associated Laguerre polynomials \( Q_n^l(\rho) \) (where \( n = s + l \)) as

\[
I_{s+l,s}(\rho) = \frac{1}{\sqrt{(s + l)!s!}} e^{-\rho^2/2} \rho^{l/2} Q_s^l(\rho),
\]

\[
Q_s^l(\rho) = e^\rho \rho^{-l} \frac{d^s}{d\rho^s}(\rho^{s+l} e^{-\rho}).
\]

Another common definition of the associated Laguerre polynomials is \( I_s^l(\rho) = Q_s^l(\rho)/s! \). The Laguerre polynomials satisfy the recursive relation,

\[
Q_{s-1}^l(\rho) = Q_s^l(\rho) - sQ_{s-1}^l(\rho).
\]

We also mention that \( I_{n,s}(\rho) \) and \( Q_s^l(\rho) \) satisfy the integral relations,

\[
\int_0^\infty I_{n,s}(\rho)I_{n-1,s}(\rho)\sqrt{\rho} \ d\rho = \sqrt{n},
\]

\[
\int_0^\infty e^{-\rho} \rho^{l/2} Q_s^l(\rho)Q_s^l(\rho) \ d\rho = \delta_{ss'}s!(l + s)!.
\]

In order to have normalizable functions at the origin, \( l \) must be a non-negative integer. Also, the solution diverges at large radii, unless \( s \) is a non-negative integer. Therefore, in order to have well-behaved solutions, the smaller values of \( s \) and \( \kappa \) must be integers, in which case the lower values of energy, \( E = -V \pm \sqrt{4V\Omega} \kappa + p_z^2 \), are discrete:

\[
\kappa \rightarrow n, \quad E \rightarrow E_n = -V \pm \sqrt{4V\Omega} n + p_z^2.
\]

On the other hand, for large enough energies the divergent behavior of the solution falls outside the star radius, in which case \( \kappa \) has no restrictions and it becomes a continuous variable.

Notice that the energy levels in Eq. (23) are different from the analogous expressions obtained in Ref. [10], \( E = -V \pm \sqrt{2V\Omega n + p_z^2} \). It was claimed in Ref. [10] that there is an analogy between the charged particle dynamics in
an electromagnetic field and the neutrino motion in matter. However this analogy is just superficial. An electromagnetic field is gauge invariant. Therefore when we study the motion of an electron in an external magnetic field $\mathbf{B} = (0,0,B)$, we can choose any gauge. In a gauge that keeps the cylindrical symmetry explicit, the vector potential $\mathbf{A} = (-yB/2,xB/2,0)$ is required. The analog of this gauge is adopted in the present work. If the electron motion is studied in cartesian coordinates, the Landau gauge is more convenient, $\mathbf{A} = (0,xB,0)$. This kind of gauge was used in Ref. [10]. As shown in Ref. [17], in the electromagnetic problem both gauges must give the same energy spectrum for the electron. However for the motion of a neutrino in rotating matter, the situation is different: there is no gauge freedom in this case. If one uses the analog of the Landau gauge, as in Ref. [10], one underestimates the matter contribution to the dispersion relation.

Finally, to derive the lower component of the neutrino spinor $u$, we just need to use the identities (see Ref. [17]),

$$R_1 I_{n-1,s}(\rho) = -\sqrt{4V\Omega n} I_{n,s}(\rho),$$
$$R_2 I_{n,s}(\rho) = \sqrt{4V\Omega n} I_{n-1,s}(\rho),$$

(24)

to find the function $F_2(\rho)$. Thus we get the complete two-component spinor in the form

$$u(r, \phi) = \frac{\sqrt{2V\Omega}}{2\pi} \left( C_1 I_{n-1,s}(\rho) e^{i(l-1)\phi} - i C_2 I_{n,s}(\rho) e^{il\phi} \right).$$

(25)

The coefficients $C_{1,2}$ are related to each other due to Eq. (15) as

$$\sqrt{4V\Omega n} C_1 + (E + V - p_z) C_2 = 0,$$

(26)

and their norm can be chosen to satisfy $C_1^2 + C_2^2 = 1$.

Let us now discuss the limit of small angular velocities to establish the connection with the non-rotating case. For simplicity we study the case $p_z = 0$ which is examined in Sec. 2. The limit $\Omega \to 0$ should be taken together with $n \to \infty$, so that $\Omega \cdot n = \text{constant}$. Using the identity [18],

$$\lim_{n \to \infty} I_{n,n-\ell}(p_\perp^2/4n) = J_\ell(p_\perp r),$$

(27)

we reproduce the neutrino wave functions in vacuum (56). Therefore one can identify $\sqrt{4V\Omega n}$ with the neutrino momentum in the equatorial plane, $p_\perp$, in Eq. (24).

4 Approximate solution of the wave equation for a massive neutrino in rotating matter

In this section we will study the effect of the rotation of matter on the single massive neutrino without mixing, analogously to the treatment of Sec. 3. However, now we will take into account the contribution of the neutrino mass to the energy levels [23] using the perturbation theory.

For a massive particle one should take into account both spinors $\xi$ and $\eta$ in Eq. (9). The coupled wave equations for these spinors have the following form:

$$i\dot{\xi} = (\mathbf{\sigma} \cdot \mathbf{p})\xi - m\eta,$$
$$i\dot{\eta} = - (\mathbf{\sigma} \cdot \mathbf{p})\eta - m\xi + [g_0 + (\mathbf{\sigma} \cdot \mathbf{g})]\eta,$$

(28)

where the vector $\mathbf{g}$ is defined in Eq. (12).

Looking for the stationary solutions of Eq. (28), $\xi \sim e^{-iEt}$ and $\eta \sim e^{-iEt}$, and excluding spinor $\eta$ from Eq. (28) we get the only differential equation for the spinor $\xi$,

$$[E^2 - m^2 - p^2 + VE - E(\mathbf{\sigma} \cdot \mathbf{g}) - V(\mathbf{\sigma} \cdot \mathbf{p}) + (\mathbf{\sigma} \cdot \mathbf{p})(\mathbf{\sigma} \cdot \mathbf{g})]\eta = 0.$$
Note that one should take into account the non-commutativity of the operator $\mathbf{p}$ and vector $\mathbf{g}$ since the latter depends on the spatial coordinates.

As in Sec. 3 we will use cylindrical coordinates $(r, \phi, z)$ to analyze Eq. (29). It is convenient to rewrite this equation for each of the components of the spinor $\eta^T = (\eta_1, \eta_2)$,

\[
\begin{align*}
\left\{ E^2-m^2 + V E + \partial_r^2 + \frac{1}{r} \partial_r + \left( \frac{\partial_\phi}{r} \right)^2 + \partial_z^2 \\
+ iV \Omega \partial_\phi + iV \partial_z + V \Omega (r \partial_r + 2) \right\} \eta_1 \\
= -e^{-i\phi} V \left\{ i \left[ \partial_r - \frac{i}{r} \partial_\phi \right] + \frac{\partial_\phi}{r} \right\} \eta_1 \\
+ \Omega r (i E - \partial_z) \right\} \eta_2,
\end{align*}
\]

\[
\begin{align*}
\left\{ E^2-m^2 + V E + \partial_r^2 + \frac{1}{r} \partial_r + \left( \frac{\partial_\phi}{r} \right)^2 + \partial_z^2 \\
+ iV \Omega \partial_\phi - iV \partial_z - V \Omega (r \partial_r + 2) \right\} \eta_2 \\
= -e^{i\phi} V \left\{ i \left[ \partial_r + \frac{i}{r} \partial_\phi \right] \\
+ \Omega r (i E + \partial_z) \right\} \eta_1.
\end{align*}
\]

We look for the solution of Eq. (30) in the following form:

\[
\eta = e^{ipz} u(r, \phi),
\]

\[
u(r, \phi) = \frac{1}{\sqrt{2\pi}} \left( e^{i(l-1)\phi} F_1(r) \right),
\]

where $F_{1,2}(r)$ are the new unknown functions [see Eq. (14)]. From Eq. (30) we derive the equations for the functions $F_{1,2}(r)$,

\[
\begin{align*}
\left\{ E^2-m^2 - p_z^2 + V E + \partial_r^2 + \frac{1}{r} \partial_r - \frac{(l-1)^2}{r^2} \\
- V \Omega (l-1) - V p_z + V \Omega (r \partial_r + 2) \right\} F_1 \\
= -iV \left\{ \partial_r - \frac{1}{r} \right\} \\
- \Omega r (E - p_z) \right\} F_2,
\end{align*}
\]

\[
\begin{align*}
\left\{ E^2-m^2 - p_z^2 + V E + \partial_r^2 + \frac{1}{r} \partial_r - \frac{l^2}{r^2} \\
- V \Omega l + V p_z - V \Omega (r \partial_r + 2) \right\} F_2 \\
= -iV \left\{ \partial_r - \frac{(l-1)}{r} \right\} \\
+ \Omega r (E + p_z) \right\} F_1.
\end{align*}
\]

Introducing the dimensionless variable $\rho = V/\Omega r^2$, as in Sec. 3 and using the properties of the operators $R_{1,2}$, defined in Eq. (15),

\[
R_1 R_2 = 4V \Omega \left( \rho \partial_\rho + \partial_\rho \\
- \frac{l-1}{2} - \frac{\rho}{4} - \frac{l^2}{4\rho} \right),
\]

\[
R_2 R_1 = 4V \Omega \left( \rho \partial_\rho + \partial_\rho \\
- \frac{l}{2} - \frac{\rho}{4} - \frac{(l-1)^2}{4\rho} \right),
\]

we rewrite Eq. (32) in the following form:

\[
\begin{align*}
\left\{ E^2-m^2 - p_z^2 + V E - V p_z + 3V \Omega \\
+ R_2 R_1 + \sqrt{V \Omega} \rho R_2 \right\} F_1 \\
= -i \left\{ VR_2 - \sqrt{V \Omega} \rho \right\} F_2 \\
\times (V + E - p_z) \right\} F_2,
\end{align*}
\]
\[
\begin{align*}
\{ E^2 - m^2 - p_z^2 + V E - V p_z - 3 V \Omega \\
+ R_1 R_2 - \sqrt{V \Omega \rho R_1} \} F_2 \\
= -i \{ V R_1 + \sqrt{V \Omega \rho} \\
\times (V + E + p_z) \} F_1.
\end{align*}
\] (34)

To study the contribution of the neutrino mass to the neutrino energy spectrum [23] we discuss the situation of the neutrino bound states and take into account neutrino mass with help of the perturbation theory. It means that in Eq. (34) we can use the expressions for the wave functions \( F_{1,2} \) presented in Eq. (25) which correspond to the massless neutrino: \( F_1(\rho) = C_1 I_{n-1,s}(\rho) \) and \( F_2(\rho) = iC_2 I_{n,s}(\rho) \). For this kind of wave functions we get from Eq. (34) the following relations:

\[
\{ C_1[ E^2 - m^2 - p_z^2 \\
+ V(E - p_z + 4 \Omega - 4 \Omega n)] \\
- C_2 V \sqrt{4 \Omega n} I_{n-1,s}(\rho) \\
+ \sqrt{V \Omega \rho} (V + E - p_z) C_2 I_{n,s}(\rho) \\
+ 2 V \Omega \sqrt{\rho(n-1)} C_1 I_{n-2,s}(\rho) = 0,
\]

\[-\{ C_2[ E^2 - m^2 - p_z^2 \\
+ V(E + p_z - 4 \Omega - 4 \Omega n)] \\
- C_1 V \sqrt{4 \Omega n} I_{n,s}(\rho) \\
- \sqrt{V \Omega \rho} (V + E + p_z) C_1 I_{n-1,s}(\rho) \\
- 2 V \Omega \sqrt{\rho(n+1)} C_2 I_{n+1,s}(\rho) = 0. \] (35)

To obtain Eq. (35) we use the known properties of the Laguerre functions,

\[
I_{n+1,s}(\rho) = \sqrt{\frac{\rho}{n+1}} \\
\times \left[ \frac{\rho + n - s}{2\rho} I_{n,s}(\rho) - I'_{n,s}(\rho) \right],
\]

which can be found in Ref. [19].

Let us study the situation when \( E^2 \gg V \Omega \) and \( n \gg 1 \). The former condition is always satisfied for any realistic situations. Indeed, for a neutron star with \( n_n = 10^{38} \) cm\(^{-3} \) we get that \( V \sim 10 \) eV.

If we suppose that the energy has the value of several electron-Volts (we will see in Sec. 5 that a bound state can be formed for such low energy neutrinos) and a neutron star has the angular velocity \( \Omega = 10^3 \) s\(^{-1} \) (\( \sim 10^{-13} \) eV), we get that the condition \( E^2 \gg V \Omega \) is satisfied. The latter condition, \( n \gg 1 \), is also valid (see Sec. 5). It is received there that the critical value of the quantum number \( n \) at which a bound state is still possible is equal to \( 10^{11} \). It means that this condition is fulfilled.

Taking into account the approximations made above we obtain that the energy spectrum has the form,

\[
E = -V \pm \sqrt{4 \Omega n + p_z^2 + m^2}. \] (37)

However, instead of the relation (26), which is valid for the massless case, we have the corrected one,

\[
\sqrt{4 \Omega n} C_1 + (E + V + 4 \Omega - p_z) C_2 = 0. \] (38)

In Sec. 3 we obtained that the quantity \( p_{\text{eff}} = \sqrt{4 \Omega n + p_z^2} \) has the meaning of the effective momentum of a neutrino. Therefore we get in Eq. (37) that in the limit \( n \gg 1 \) the neutrino energy receives the correction \( m^2/2p_{\text{eff}} \) due to the non-zero mass, as it should be. Note that
this energy spectrum coincides with the analogous relation found in Ref. [12].

5 Low energy without flavor mixing: bound states

In Secs. 3 and 4 we found the basis spinors (25) and the energy levels (37) of a single neutrino mass eigenstate interacting with rotating background matter. Now we apply these results for the description of the evolution of the system (7).

Given the axial symmetry of the problem, we use cylindrical coordinates and the eigenfunctions \( \psi_a^{(\pm)}(r, \phi) \) of Eq. (7) must be of the form \( \eta_a(r, \phi) e^{-i(E_a l - p_z z)} \). We find the energy eigenvalues in terms of an index “n” [see Eq. (37)],

\[
E_n^{(a)} = - V_a \pm \sqrt{4 V_a \Omega n + p_z^2 + m_a^2}, \quad n = 0, 1, 2, \ldots
\]

Here \( E_n^{(a)} \) is the energy of a particle (neutrino), containing an attractive potential “\(-V_a\)”, while the negative value \( E_n^{(a)}^- \) must be understood as \(-E_n^{(a)}^+\), the positive energy of an antiparticle (antineutrino), containing a potential term “\(+V_a\)”, which is repulsive. We included the neutrino mass \( m_a \) in the above expression for the energy, although we will neglect it in the spinors.

In a rotating medium, the effect of rotation is largest for neutrinos propagating in the equatorial plane. Choosing \( Z \) as the rotation axis, we then look for solutions which are \( z \)-independent, i.e. \( p_z = 0 \).

The corresponding two-component spinors are given in terms of Laguerre functions \( I_{n,s}(\rho) \) with an energy index “n” and a radial index “s” [see Eqs. (25) and (26)]:

\[
u_{a,n,s}(r, \phi) = \sqrt{\frac{V_a \Omega}{2\pi}} \left( I_{n-1,s}(\rho_a) e^{i(l-1)\phi} + i I_{n,s}(\rho_a) e^{i\phi} \right),
\]

\( l = n - s, \) (40)

and where \( \rho_a = V_a \Omega r^2, a = 1, 2, \) is a dimensionless radial coordinate. For further details of the derivation of Eqs. (39) and (40) the reader is referred to Secs. 3 and 4.

In this case, when “n” is a non-negative integer, the energies in Eq. (39) are discrete values, the energies in Eq. (44), otherwise the wavefunctions would diverge inside the star before reaching the star radius.

Instead, for neutrinos with larger energies, the wavefunction reaches the edge of the star and should continue outside. For those cases the neutrino energy and the index “n” become continuous variables (we change the name \( n \rightarrow \kappa \) in this case). The solution inside the star now has the more general form:

\[
u_a^{(\pm)}(r, \phi) = e^{-\rho_a/2} \sqrt{\frac{2V_a \Omega}{l!}} \left( \frac{\rho_a}{\sqrt{\rho_a}} \right)^{l/2} F(l - \kappa, l, \rho_a) e^{-i\phi} \sum_{\ell=0}^{l}\frac{C^{(\pm)}_\ell}{\sqrt{\pi}} C^{(in)}_{\ell,2}\],

\( \times\) (41)

where \( F(\alpha, \beta, z) \) is a confluent hypergeometric function and the coefficients \( C^{(in)}_{\ell,2} \) satisfy the relation

\[
\sqrt{4V_a \Omega \kappa} C^{(in)}_{\ell,2} + (E_a + V_a - p_z) C^{(in)}_{\ell,2} = 0, \quad \ell = 0, 1, 2, \ldots
\]

considering the energy \( E_a \) with the same expression as in Eq. (39), but with \( n \rightarrow \kappa \), a continuous variable. We derive Eq. (42) from Eq. (26).
changing the norm of the coefficients $C_{1,2}$ there as:

$$C_{1,2}^{(in)} \rightarrow C_{1,2}^{(in)} \sqrt{\frac{n!}{(n-l)!}},$$

(43)

and using the property of the confluent hypergeometric function, $F(l-n, l+1, \rho) = l!Q_{l-n}^{l}(\rho)/n!$.

For the wave function outside the star one must use the outgoing wave solution in vacuum given in Eq. (63), i.e.

$$u_{a}^{(out)}(r, \phi) = \frac{1}{\sqrt{2\pi}} \times \left( \begin{array}{c} C_{1}^{(out)} H_{l-1}^{(1)}(p_{\perp} r) e^{i(l-1)\phi} \\ iC_{2}^{(out)} H_{l}^{(1)}(p_{\perp} r) e^{il\phi} \end{array} \right),$$

(44)

where $H_{l}^{(1)}$ are Hankel functions of the first kind and $p_{\perp} = \sqrt{E^2 - p_z^2 - m_a^2}$ is the momentum perpendicular to the rotation axis.

The coefficients $C_{1,2}^{(in)}$ and $C_{1,2}^{(out)}$ are related due to the continuity of the wave functions (41) and (44) at the neutron star surface, $u_{a,\kappa}^{(in)}(R, \phi) = u_{a}^{(out)}(R, \phi)$. Eqs. (12), (59) and (66) completely define the coefficients for the solution corresponding to an unbound wave function.

As mentioned above, neutrinos could form bound states inside a neutron star, provided the energy is small enough. In those cases the energy (39) assumes discrete values, because otherwise the wave function would diverge before reaching the edge of the star. In Fig. 1 we present an example of wave function [see Eq. (10)] for $l = 10$ and $s = 15$ (solid line). It can be seen that the solution is a rapidly oscillating function in the inner regions and approaches zero for large radii. We also present in Fig. 1 the wave functions for $l = 10$ and $s = 15.1$ (dashed line) as well as for $l = 10$ and $s = 14.9$ (dash-dot line), showing the divergence at large radii.

Figure 1: The solutions of Eq. (19) for $l = 10$ and $s = 15$ (solid line), $l = 10$ and $s = 15.1$ (dashed line) as well as $l = 10$ and $s = 14.9$ (dash-dot line). The function $I_{n,s}(x)$ at non-integer “s” should be understood as the confluent hypergeometric function $F(\kappa - l, l + 1, x)$ in Eq. (41).

From the fact that $E_{n}^{(a)+}$ can be negative for a medium dominated by neutrons, as seen in Eq. (39), neutrinos of low kinetic energy can form bound states inside the star. On the other hand, antineutrinos cannot be bound because in their case the potential is repulsive. Limiting the analysis to $p_z = 0$ only, the energy spectrum for the bound states is discrete. The maximum value of “n” for bound states can be estimated with the equation $E_{n}^{(a)+} = 0$,

$$n \leq \frac{V_{a}}{4\Omega}.$$

(45)

This condition is independent from (but consistent with) the condition that the wave function should not diverge before reaching the star radius. The position of the last maximum in the wave function is approximately at $\rho_{\text{crit}} \sim 2(n + s)$, and this point should be inside the star, i.e. $\rho_{\text{crit}} < V_{a}\Omega R^2$, where $R$ is the star ra-
radius (remember the definition $\rho = V\Omega r^2$). This condition means $n < V_0\Omega R^2/2$.

We can estimate the index “$n$” in these bounds for the case of a neutron star. Taking a reference value for the neutron density $n_n = 10^{38}$ cm$^{-3}$ (close to nuclear density), we get a potential $V_0 \sim 7$ eV. This is much larger than the natural neutrino masses expected from oscillation experiments, so the mass in Eq. (39) and Eq. (45) is indeed negligible. Now, for the angular speed $\Omega$, we can take the Keplerian angular velocity indeed negligible. Now, for the angular speed $\Omega$, we can take the Keplerian angular velocity $\Omega_{\text{max}} = \sqrt{GM/R^3}$ (see Ref. [20]) as a rough upper limit. Taking $M = 1.5M_\odot$ and $R = 10$ km one obtains for $\Omega_{\text{max}} \sim 10^4$ s$^{-1}$ (typical values are two or more orders of magnitude smaller). Eq. (45) then gives $n \sim 10^{11}$, a rather large number of bound states as well as nodes within the star. On the other hand, the condition for the wave function to fade before reaching the edge, $n < V_0\Omega R^2/2$, gives also $n \sim 10^{11}$.

The particles inside the neutron star can be created in the form of neutrino-antineutrino pairs [2]. The fact that low energy neutrinos can be trapped by the star and antineutrinos will be accelerated out of the star makes it possible the existence of the low energy antineutrinos luminosity.

The validity of the wave function method has one main drawback concerning wave coherence: it must be assumed that incoherent scattering can be neglected all over the star. This is particularly unlikely if the number of nodes is as large as $10^{11}$. Another drawback of this treatment concerns neutrino production: most neutrinos are produced with energies much larger than a few electron-Volts, and again, the production is in general an incoherent process.

6 Effect of rotation on flavor mixing of low energy neutrinos

So far we have neglected the off-diagonal elements of Eq. (6) which mix the neutrino mass states. Let us now include them as perturbations to study the effect of rotation in the mixing of neutrino flavors. The general solution of Eq. (7) can be stated in the following form (see also Ref. [11]):

$$
\eta_n(r, \phi, t) = \sum_{n,s=0}^{\infty} \left( a_n^{(a)} u_{a,ns}^+(r, \phi) \exp[-iE_n^{(a)} t] 
+ b_n^{(a)} u_{a,ns}^-(r, \phi) \exp[-iE_n^{(a)} - t] \right),
$$

$$
a = 1, 2.
$$

In Eq. (46) we consider the full state as an expansion in the eigenstates with time-dependent coefficients $a_n^{(a)}(t)$ and $b_n^{(a)}(t)$, requiring the expansion to satisfy the evolution equation (7), and imposing the initial condition that the neutrino wavefunctions at $t = 0$ are of flavor $\beta = e$ only.

Substituting the solution into Eq. (7) and using orthonormality of the spinors in Eq. (40) we obtain the following set of ordinary differential equations for the functions $a_n^{(a)}(t)$:

$$
i \frac{d}{dt} a_n^{(a)}(t) = \sum_{n', s' = 0}^{\infty} \int u_{a,ns}^+(r) \left( \sigma_{\mu} g_{ab} \right) u_{b,a's'}^+(r) d^2r \times \exp[i(E_n^{(a)} - E_{n'}^{(b)} + 1)t] a_{n'}^{(b)}(7)
+ \int u_{a,ns}^+(r) \left( \sigma_{\mu} g_{ab} \right) u_{b,a's'}^+(r) d^2r \times \exp[i(E_n^{(a)} - E_{n'}^{(b)} - 1)t] b_{n'}^{(b)}(t),
$$

$$
a \neq b,
$$

and a similar set of equations for $b_n^{(a)}(t)$. 

11
Now, since we are interested in the neutron star matter, we can assume densities close to nuclear matter or above, dominated by neutrons \((n_e \ll n_n)\). Within this approximation, the spinor products in Eq. (47) can be easily calculated,

\[
\int u_{a,ns}^+ (r) (g_{ab}^\mu \tilde{\sigma}_\mu) u_{b,ns'}^+ (r) d^2r = \int u_{a,ns}^- (r) (g_{ab}^\mu \tilde{\sigma}_\mu) u_{b,ns'}^- (r) d^2r
\]

\[
\approx g_{12}^0 \delta_{ll'} \delta_{ss'} - g_{12}^0 \frac{\Omega}{2} \sqrt{\delta_{ll'}} 
\times (2 \delta_{ss'} \sqrt{n} - \sqrt{s + 1} \delta_{s,s' - 1} - \sqrt{s} \delta_{s,s' + 1}),
\]

where \(l = n - s, l' = n' - s'\), \(g_{12}^0\) is the potential that mixes neutrino types, given in Eq. (13), and \(V = G_F n_n / \sqrt{2}\) is the approximation of \(V_1 \approx V_2\) neglecting the electron density \(n_e\). Here we must keep terms linear in \(n_e\), otherwise \(g_{12}^0\) would vanish, and the case of mixing would become trivial. To linear order in \(n_e\), we can take the arguments of the Laguerre functions in Eq. (40) to be equal, \(\rho_1 = \rho_2\), neglecting \(n_e\) in Eq. (41). To derive these results we used the properties (21) and (22) of the Laguerre functions, given in Eq. (22).

The time evolution for \(a_{ns}^{(a)} (t)\) given in Eq. (47), which is due to the flavor off-diagonal perturbation, has contributions from many levels \(\{n,s\}\), making a general analysis rather complicated. One simplification occurs if we disregard neutrino-antineutrino creation or annihilation, which means no transitions between \(u^+\) and \(u^-\) states. In what follows we disregard antineutrinos, so we neglect the coefficients \(b_{ns}^{(a)} (t)\) altogether. Another simplification occurs if we consider wave packets very narrow in energy, so that only the nearest states are involved. A final simplification occurs in the case \(s \gg l\), where level transitions are negligible, thus reducing the problem to solving a two-state quantum system.

The situation \(s \gg l\), occurs when the neutrino angular momentum at emission is small, corresponding to wave functions which are large near the center of the neutron star. In this case the coefficients \(a_{ns}^{(a)}\) are, in practice, functions of just one quantum number \(s\), since \(s = n - l\) while \(l\) is negligible. Assuming \(a_{s}^{(a)}\) depend smoothly on \(s\), i.e. \(a_s^{(a)} \approx a_{s+1}^{(a)}\), and using Eq. (33), we get an evolution equation of the form,

\[
\frac{id}{dt} \begin{pmatrix} a_s^{(1)} \\ a_s^{(2)} \end{pmatrix} = \begin{pmatrix} \omega / 2 & \Delta \\ \Delta & -\omega / 2 \end{pmatrix} \begin{pmatrix} a_s^{(1)} \\ a_s^{(2)} \end{pmatrix},
\]

\[
a_s^{(1)} = a_s^{(1)} e^{-i \omega t / 2}, \quad a_s^{(2)} = a_s^{(2)} e^{i \omega t / 2},
\]

where

\[
\omega = E_s^{(1)+} - E_s^{(2)+} = V_2 - V_1
\]
\[
+ \sqrt{4V_1 \Omega s + m_1^2 - 4V_2 \Omega s + m_2^2},
\]

\[
\Delta = g_{12}^0.
\]

Since, for non-bound states, \(s\) is a very large and continuous variable, we can define a continuous “momentum” variable inside the medium, \(p_{\text{eff}} = \sqrt{4V_1 \Omega s}\), chosen to be the same for both neutrino types, as it is usually done in the treatment of non-rotating media [21]. In terms of
more conventional quantities, $\Delta$ and $\omega$ are then

\[
\omega = \frac{\delta m^2}{2p_{\text{eff}}} - \sqrt{2}G_F n_e \cos 2\theta,
\]

\[
\Delta = \frac{G_F}{\sqrt{2}} n_e \sin 2\theta,
\]

(51)

In Eqs. (50) and (51) we take into account the non-zero masses of the neutrino mass eigenstates $m_{1,2}$ and use the standard notation $\delta m^2 = m_1^2 - m_2^2$. As one can see, Eqs. (49) and (51) are practically independent of the rotation velocity $\Omega$, and so the evolution equation reduces to the known case of neutrino oscillations in non-moving background matter.

For the rotation to have any significant effect on neutrino flavor oscillations the linear velocities of the matter motion should reach very high values. For the realistic angular velocities of pulsars $\sim 10^3 \text{s}^{-1}$ and a neutron star with radius of $\sim 10 - 20 \text{km}$ the matter velocity can be about 0.1, which is not enough to significantly change the transition probability. Therefore the effect potentially interesting from the phenomenological point of view is the trapping of low energy neutrinos discussed in Sec. 5.

7 Conclusion

Summarizing we mention that we studied neutrino flavor oscillations in rotating matter. The analysis was carried out in frames of the relativistic quantum mechanics. We used the exact solutions of the wave equation for a neutrino weakly interacting with inhomogeneously moving matter to solve the initial condition problem for the system of mixed flavor neutrinos. Note that the use of the relativistic quantum mechanics method is essential in describing neutrino flavor oscillations since it takes into account the coordinate dependence of the neutrino wave functions. It was possible to derive the Schrödinger like evolution equation for the two component neutrino wave function for the important case of neutrinos with small angular momentum. This situation corresponds to the particles emitted close to the central region of a neutron star. It was found that rotation does not change the dynamics of neutrino flavor oscillations, i.e. the quantum mechanical description of neutrino oscillations is insensitive to the rotation of background matter.

We have studied the possibility for the existence of neutrino bound states inside a neutron star. The cases of neutrinos and antineutrinos are different. We revealed that low energy neutrinos can be trapped by the rotating neutron star whereas antineutrinos always escape. The applicability of the relativistic quantum mechanics method was analyzed. Despite this approach allows one to account for the interaction with external fields exactly it cannot be used for the description of the evolution of high energy neutrinos in dense rotating matter since these particles experience many incoherent collisions and it is difficult to form the coherent cylindrical wave inside a star. It means that the relativistic quantum mechanics is applicable for later stages of the neutron star evolution when one can neglect the multiple neutrino collisions with background matter.

Acknowledgments

The work has been supported by Conicyt (Chile), Programa Bicentenario PSD-91-2006 and by FAPESP (Brazil). The author is thankful to C. O. Dib and J. Maalampi for helpful discussions.
A Solution to the wave equation for a neutrino in vacuum in cylindrical coordinates

In this Appendix we find the solution of the wave equation for a neutrino in vacuum in cylindrical coordinates. In the chiral basis for the Dirac equation for a neutrino in vacuum in cylindrical coordinates \((r, \phi, z)\), the spinors \(\xi\) and \(\eta\) satisfy the differential equations

\[
\begin{align*}
    \dot{\xi} &= (\sigma \cdot p)\xi - m\eta, \\
    \dot{\eta} &= - (\sigma \cdot p)\eta - m\xi. \\
\end{align*}
\]

We look for stationary solutions of Eq. (52), which must be of the form,

\[
\begin{align*}
    \xi(r, t) &= e^{-i(Et - p_z z)}w(r, \phi), \\
    \eta(r, t) &= e^{-i(Et - p_z z)}u(r, \phi), \\
\end{align*}
\]

where \(E\) is the energy and \(p_z\) is the \(z\) component of the neutrino momentum. We can separate the variables \((r, \phi)\) introducing radial functions \(G_{1,2}(r)\) and \(F_{1,2}(r)\),

\[
\begin{align*}
    w_1(r, \phi) &= \frac{1}{\sqrt{2\pi}} e^{i(l-1)\phi} G_1(r), \\
    w_2(r, \phi) &= \frac{1}{\sqrt{2\pi}} e^{il\phi} G_2(r), \\
    u_1(r, \phi) &= \frac{1}{\sqrt{2\pi}} e^{i(l-1)\phi} F_1(r), \\
    u_2(r, \phi) &= \frac{1}{\sqrt{2\pi}} e^{il\phi} F_2(r),
\end{align*}
\]

where \(l\) measures the \(z\)-component of the orbital angular momentum. For the radial functions \(G_{1,2}\) and \(F_{1,2}\) we obtain the system of the differential equations,

\[
\begin{align*}
    (E - p_z)G_1 + i \left( \partial_r + \frac{l}{r} \right) G_2 &= -mF_1, \\
    (E + p_z)G_2 + i \left( \partial_r - \frac{l-1}{r} \right) G_1 &= -mF_2, \\
    (E + p_z)F_1 - i \left( \partial_r + \frac{l}{r} \right) F_2 &= -mG_1, \\
    (E - p_z)F_2 - i \left( \partial_r - \frac{l-1}{r} \right) F_1 &= -mG_2. \\
\end{align*}
\]

The solutions of Eq. (55) can be given in terms of Bessel functions. For standing waves that are regular at the origin, the functions \(w(r, \phi)\) and \(u(r, \phi)\) are

\[
\begin{align*}
    w(r, \phi) &= \frac{1}{\sqrt{2\pi}} \left( B_1 J_{l-1}(p_\perp r) e^{i(l-1)\phi} - iB_2 J_l(p_\perp r) e^{il\phi} \right), \\
    u(r, \phi) &= \frac{1}{\sqrt{2\pi}} \left( C_1 J_{l-1}(p_\perp r) e^{i(l-1)\phi} + iC_2 J_l(p_\perp r) e^{il\phi} \right),
\end{align*}
\]

where \(p_\perp^2 = E^2 - p^2 - m^2\), \(B_{1,2}\) and \(C_{1,2}\) are the undefined coefficients.

Using the identity for the Bessel functions,

\[
\begin{align*}
    J'_l(x) + \frac{l}{x} J_l(x) &= J_{l-1}(x), \\
    J'_{l-1}(x) - \frac{l-1}{x} J_{l-1}(x) &= -J_l(x), \\
\end{align*}
\]

and Eq. (55) we find that the coefficients \(B_{1,2}\) and \(C_{1,2}\) obey the system of the following alge-
braic equations:

\[(E - p_z)B_1 - p_\perp B_2 = - mC_1,\]
\[(E + p_z)B_2 - p_\perp B_1 = - mC_2,\]
\[(E + p_z)C_1 + p_\perp C_2 = - mB_1,\]
\[(E - p_z)C_2 + p_\perp C_1 = - mB_2.\]  \(58\)

In the limit of the small neutrino mass one can see that the equations for \(B_1, B_2\) and \(C_1, C_2\) decouple giving one relation only for the coefficients \(C_1, C_2\),

\[\sqrt{E + p_z} C_1 + \sqrt{E - p_z} C_2 = 0.\]  \(59\)

Normalizing the wave function as

\[\int d^2r u_{E, l}(r, \phi) \dagger u_{E', l'}(r, \phi) = E \delta(E - E') \delta_{ll'},\]  \(60\)

and using the integral property of Bessel functions

\[\int_0^\infty dr r J_l(kr) J_l(k'r) = \frac{1}{k} \delta(k - k'),\]  \(61\)

one finds that the coefficients \(C_1, C_2\) also must satisfy the relation

\[C_1^2 + C_2^2 = E^2.\]  \(62\)

Instead of standing waves like Eq. \(56\), we can have outgoing waves. These solutions are similar, but in terms of Hankel functions of the first kind, \(H_l^{(1)}(x)\),

\[u(r, \phi) = \frac{1}{\sqrt{2\pi}} \left( C_1 H_l^{(1)}(p_\perp r) e^{i[l(l-1)]} + i C_2 H_l^{(1)}(p_\perp r) e^{il\phi} \right).\]  \(63\)

The Hankel functions have the asymptotic behaviour of outgoing radial waves in the two dimensional plane

\[H_l^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{-x} \exp \left[ i \left( x - \frac{\pi l}{2} - \frac{\pi}{4} \right) \right].\]  \(64\)

Instead of the normalization \(61\) one has to define the particle flux at large distances, \(j_\infty\). For a radial flow in two dimensions, the flux of particles through a circle of the large radius \(r\) has the form,

\[j_\infty = \lim_{r \to \infty} \int_0^{2\pi} r d\phi u^\dagger(r, \phi) u(r, \phi),\]  \(65\)

where we have assumed relativistic particles. From this expression, one can derive norm for the coefficients \(C_1, C_2\),

\[C_1^2 + C_2^2 = \frac{\pi E j_\infty}{2}.\]  \(66\)

On the other hand, the relation \(59\) is still valid for the wave function \(63\).

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