Defining integer valued functions in rings of continuous definable functions over a topological field

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Abstract

Let \( K \) be an expansion of either an ordered field \((K, \leq)\), or a valued field \((K, v)\). Given a definable set \( X \subseteq K^m \) let \( C(X) \) be the ring of continuous definable functions from \( X \) to \( K \). Under very mild assumptions on the geometry of \( X \) and on the structure \( K \), in particular when \( K \) is \( o \)-minimal or \( P \)-minimal, or an expansion of a local field, we prove that the ring of integers \( \mathbb{Z} \) is interpretable in \( C(X) \). If \( K \) is \( o \)-minimal and \( X \) is definably connected of pure dimension \( \geq 2 \), then \( C(X) \) defines the subring \( \mathbb{Z} \). If \( K \) is \( P \)-minimal and \( X \) has no isolated points, then there is a discrete ring \( \mathbb{Z} \) contained in \( K \) and naturally isomorphic to \( \mathbb{Z} \), such that the ring of functions \( f \in C(X) \) which take values in \( \mathbb{Z} \) is definable in \( C(X) \).

Contents

1 Introduction 2

2 Model theoretic and topological set up 5

3 Definability of the poset of zero sets 7

4 Basic properties of zero sets 11

5 Constructing integers using limit values and chunks 14

6 Defining integers using local dimension 19

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1 Introduction

We give a first order definition of the ring of integers within rings of continuous functions that are first order definable in expansions of ordered and valued fields. Before describing a more technical outline of the contents, we explain the context of the article and the results in a colloquial way.

Rings of continuous functions on topological spaces are central objects in functional analysis, topology and geometry. To name an example: They are rings of sections for the sheaf of continuous (say, real valued) functions on a topological space and as such play the algebraic role in the study of topological (Hausdorff) spaces.

By a ring of definable continuous functions we mean the following. Let $K$ be an expansion of an ordered field $(K, \leq)$ (e.g. the real field) or a valued field $(K,v)$ (e.g. the $p$-adics). In both cases $K$ carries a topology that turns $K$ into a topological field. Let $X \subseteq K^n$ be definable (with parameters) in $K$ and let $C(X)$ be the set of all functions $X \rightarrow K$ that are continuous and definable in $K$. If $K$ is the real or the $p$-adic field, definable is the same as semi-algebraic. Then $C(X)$ is a ring and similar to the classical case mentioned above, $C(X)$ carries the algebraic information of the definable homeomorphism type of $X$. This is amply illustrated in the case when $K$ is a real closed field: Let $\mathcal{S}$ be the category of semi-algebraic subsets $X \subseteq K^n$, $n \in \mathbb{N}$, with continuous semi-algebraic maps as morphisms. Let $\mathcal{C}$ be the category of all the rings $C(X)$ of continuous $K$-definable functions and $K$-algebra homomorphisms as morphisms. Then the functor $C : \mathcal{S} \rightarrow \mathcal{C}$ that sends $X$ to $C(X)$ and a morphism $f : X \rightarrow Y$ to the $K$-algebra homomorphism $C(Y) \rightarrow C(X)$; $g \mapsto g \circ f$, is an anti-equivalence of categories. This can also be seen in analogy with algebraic geometry, where varieties defined over a field $K$ are anti-equivalent to affine $K$-algebras, leading to the modern language of schemes. For semi-algebraic sets the machinery is developed in this vein in terms of so called semi-algebraic spaces, see [Sch87]. The same connection exists between definable sets $X$ in the $p$-adic context and the rings $C(X)$ for that context. In each case, one may thus think of rings of continuous definable functions as “coordinate rings” when studying topological properties of definable sets.

Model theoretic studies of rings of continuous (definable) functions may be found in [PS02, Tre07] for the case of real closed fields and in [Bel91, Bel95, GT08] in the $p$-adic case. Model theory of rings of continuous functions on topological spaces have been studied in [Che80], which serves as a main source for inspiration for us.

In this article we study model theoretic properties of rings of continuous definable functions under fairly mild assumptions on the base structure. The principal goal is to show that, in most cases, these rings interpret the ring of integers in a uniform way; in particular these rings are not decidable. This is established in Theorem 5.5 Now undecidability was already known in some cases. For example, one can interpret the lattice of closed subsets of $\mathbb{R}^n$ in the ring $C(\mathbb{R}^n)$ of continuous semi-algebraic functions $\mathbb{R}^n \rightarrow \mathbb{R}$. When $n \geq 2$, this lattice itself is undecidable by [Grz51] and indeed interprets the ring of integers, see [Tre17]. However, in this lattice, one cannot interpret $C(\mathbb{R}^n)$ in a uniform way as we will see in Remark 5.8.
In the $p$-adic case, the first author has shown in [Dar06] that the lattice of closed subsets of $\mathbb{Q}_p^n$ is even decidable and it was unknown whether $C(\mathbb{Q}_p^n)$ has a decidable theory at all.

In section 6 we show that in many cases the rings $C(X)$ actually define the subset of constant functions with values in a natural isomorphic copy of $\mathbb{Z}$, see Theorem 6.7 for the precise formulation. When $K$ is an o-minimal expansion of a field and $X$ is definably connected of local dimension $\geq 2$ everywhere, then indeed $C(X)$ defines the ring of constant functions with values in $\mathbb{Z}$. As a consequence, when $K$ expands the real field we obtain that the real field, seen as constant functions $\mathbb{R}^n \to \mathbb{R}$ is definable in $C(X)$. This implies that the projective hierarchy is definable in these rings.

**Detailed description of the main results and set up.**

We consider an expansion $K$ of a topological field $(K, O)$ where $O$ is either the unit interval $[-1, 1]$ of a total order $\leq$ on $K$, or the ring of a non-trivial valuation $v$ on $K$. We endow $K^m$ with the corresponding topology, for every integer $m \geq 0$. We will make almost everywhere the following assumption on $K$.

**(BFin)** Every subset of $O$ that is closed, discrete and parametrically definable in $K$, is finite.

As is well known, every discrete definable subset of $K$ is finite if $K$ is o-minimal [vdD98], P-minimal [HM97] or C-minimal [HM94], and more generally if it is a dp-minimal ordered or valued field [Sim11], [JSW17]. The same holds true if $K$ is any Henselian (non-trivially) valued field of characteristic 0, or any algebraically bounded expansion of such [vdD89]. But property (BFin) is obviously also satisfied if $O$ is compact, hence if $(K, O)$ is any expansion of $\mathbb{R}$, the field of real numbers, or any valued local field (that is a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers, or the field of Laurent series $F((t))$ over a finite field $F$).

Given a definable set $X \subseteq K^m$ and any set $Y \subseteq K^n$ we let $C(X, Y)$ denote the set of continuous functions $X \to Y$ that are definable with parameters in $K$. If $Y$ is a subring of $K$, e.g. when $Y = \mathbb{Z}$, then $C(X, Y)$ is considered as a ring, where addition and multiplication are given point-wise. When $Y = K$, we just write $C(X)$.

Let $\tau^k = \{\tau^k : k \in \mathbb{Z}\}$ for some non-zero $\tau \in O$ with $1/\tau \notin O$. We also furnish $\tau^\mathbb{Z}$ with a ring structure so that the bijection $\mathbb{Z} \to \tau^\mathbb{Z}, k \to \tau^k$ is an isomorphism. Again, pointwise addition and multiplication turns $C(X, \tau^\mathbb{Z})$ into a ring.

Our main result is that, under very general conditions on $K$ and $X$, the ring $C(X, \mathbb{Z})$ (resp. $C(X, \tau^\mathbb{Z})$) is definable (resp. interpretable) in the ring structure of $C(X)$ expanded by the set

$$B = \{s \in C(X) \mid \forall x \in X : s(x) \in O\}.$$  

In many cases, in particular when $K$ is o-minimal or an expansion of a $p$-adically closed field, we will see that $B$ is already definable in the ring $C(X)$, see Remark 4.4. On the other hand $O$ will not be definable in the ring when $K$ is algebraically closed.

A crucial input and starting point of the paper is the definability of the Nullstellensatz in a weak, but surprisingly general form in Theorem 3.4. This says that for almost any
ring \( A \) of functions from some given set \( S \) to a given field \( K \), the \( n+1 \)-ary relation 
\( \{ f_1 = 0 \} \cap \ldots \cap \{ f_n = 0 \} \subseteq \{ g = 0 \} \) of \( A \) is the Jacobson radical relation of \( A \). For example it suffices to ask that \( K \) is not algebraically closed, \( A \) contains the constant functions with value in \( K \) and that all functions in \( A \) without zero are invertible. This is explained in section 3. In the case of \( o \)-minimal or \( P \)-minimal structures it implies that the lattice of closed definable subsets of \( X \) is interpretable in \( C(X) \).

The technical heart of this paper is Section 5 where we prove our first main result (Theorem 5.5 and Corollary 5.6).

**Theorem A** Assume that \( K \) satisfies \([BFin]\). Let \( \tau \) be a non-zero and non-invertible element of \( O \). Let \( X \subseteq K^m \) be a definable set which has arbitrarily many germs\(^4\) at some point \( p_0 \).

- If \( O = [-1,1] \) then the ring of functions \( f \in C(X) \) such that \( f(p_0) \in \mathbb{Z} \) is definable in \( (C(X),B) \).
- In any case the set of functions \( f \in C(X) \) such that \( f(p_0) \in \tau \mathbb{Z} \) is definable in \( (C(X),B) \), and its natural ring structure is interpretable in \( (C(X),B) \).

As a consequence the ring of integers \( \mathbb{Z} \) is interpretable in \( (C(X),B) \).

If \( K' = (K',\ldots) \) is an elementary extension of \( K \), and \( X' \) is the subset of \( K'^m \) defined by the same formula as \( X \), there is a natural embedding of \( (C(X),B) \) into \( (C(X'),B') \). It follows from the above result that, surprisingly enough, this is not an elementary embedding in general (Corollary 5.7).

Note that the above theorem is fairly general: it only assumes that \( K \) satisfies \([BFin]\) and \( X \) has arbitrarily many germs at \( p_0 \). In Section 6 we improve it by assuming, in addition to \([BFin]\), that \( K \) has a good dimension theory for definable sets (see the axioms list \([Dim]\) in Section 2). This holds true if \( K \) is \( o \)-minimal, \( P \)-minimal or \( C \)-minimal, and more generally if it is dp-minimal (see Remark 2.4). We can then prove our second main result.

**Theorem B** Assume that \( K \) satisfies \([Dim]\) and \([BFin]\). Let \( \tau \) be a non-zero and non-invertible element of \( O \). Let \( X \subseteq K^m \) be a definable set of pure dimension \( d \geq 2 \).

1. If \( O = [-1,1] \) then \( C(X,\mathbb{Z}) \) is definable in \( (C(X),B) \).
2. In any case \( C(X,\tau \mathbb{Z}) \) is definable in \( (C(X),B) \).

Theorem 6.7 actually gives a more precise and more general statement. Let us also mention that in the \( P \)-minimal case (among others, see condition \([Z]\) in Section 6) the condition on the pure dimension of \( X \) can be relaxed: the result holds true whenever \( X \) has no isolated point.

\(^4\)Roughly speaking, \( X \) has arbitrarily many germs at \( p_0 \) if there exist arbitrarily many disjoint definable subsets \( S \) of \( X \setminus \{ p_0 \} \) such that \( p_0 \in S \). See Section 2 for a precise definition.
2 Model theoretic and topological set up

Let $K$ be a field and let $\mathcal{O}$ be either the unit interval $[-1,1]$ of a total order $\leq$ of $K$, or the ring of a non-trivial valuation $v$ of $K$.

In both cases we let $\mathcal{O}^\times$ denote the set of non-zero $a \in \mathcal{O}$ with $a^{-1} \in \mathcal{O}$. This is a multiplicative subgroup of $K^\times = K \setminus \{0\}$. We let $|\cdot| : K^\times \to K^\times/\mathcal{O}^\times$ be the residue map and extend it by $|0| = 0$. In the ordered case this is just the usual absolute value, in the valued case $|x|$ is a multiplicative notation for the valuation. The set $|K| = \{|x| : x \in K\}$ is totally ordered by the relation $|y| \leq |x|$ if and only if $y \in x \cdot \mathcal{O}$.

Note that in the valued case, $|y| \leq |x|$ if and only if $v(y) \geq v(x)$. The multiplication defined on $|K|$ by $|x| \cdot |y| = |xy|$ extends the multiplication of $|K| \setminus \{0\} = K^\times/\mathcal{O}^\times$. The latter is a totally ordered abelian (multiplicative) group. We denote it by $|K^\times|$, or $v(K^\times)$ when additive notation is more appropriate.

For every $x \in K^m$ we set $\|x\| = \max(|x_1|, \ldots, |x_m|)$, and for all $X \subseteq K^m$ we write $\|X\| = \{|x| : x \in X\}$; if $m = 1$ we simply write $|X|$ (or $v(X)$ in additive notation).

Open and closed balls in $K^m$ with center $c \in K^m$ and radius $r \in K^\times$ are defined as usual; both are clopen in the valued case. We endow $K^m$ with the topology defined by the open balls, and $|K|$ with the image of this topology (which induces the discrete topology on $|K^\times|$ in the valued case). For any set $S$, we write $\mathcal{S}$ for the topological closure of $S$ and $\partial S = \mathcal{S} \setminus S$ for the frontier of $S$.

In this paper, definable in a given structure means “first-order definable with parameters” in that structure. Let $\mathcal{L}_{\text{og}} = \{e, *, \leq\}$ be the language of (additive or multiplicative) ordered groups, let $\mathcal{L}_{\text{ring}} = \{0, 1, +, -, \times\}$ be the language of rings and let $\mathcal{L}$ be an extension of $\mathcal{L}_{\text{ring}}$ containing a unary predicate symbol $\mathcal{O}$. We will be working with expansions $\mathcal{K}$ of an ordered or valued field $K$ to $\mathcal{L}$, where the symbol $\mathcal{O}$ is interpreted as explained above. Definability refers to $\mathcal{L}$ for subsets of $K^m$ and to $\mathcal{L}_{\text{og}}$ for subsets of $|K^\times|$ (hence definable in $|K^\times|$ means definable in the ordered group $|K^\times|$ regardless on whether $\mathcal{K}$ induces even more definable sets on $|K^\times|$).

Now let $\mathcal{C}(X)$ be the ring of continuous definable functions as explained in the introduction. For all $f, g \in \mathcal{C}(X)$ the sets $\{f = g\}, \{f \neq g\}, \{|f| \leq |g|\}$ and so on, are defined as the subsets of $X$ on which the corresponding relation holds true. For example $\{f = 0\} = \{x \in X : f(x) = 0\}$ is the zero-set of $f$. On $\mathcal{C}(X)$ we work with the relation

$$g \sqsubseteq f \iff \forall x \in X \ (g(x) = 0 \Rightarrow f(x) = 0).$$

We prove in Theorem 3.4 that this relation is definable in the ring $\mathcal{C}(X)$.

Definition 2.1 Let $X \subseteq K^m$ be a definable set, $p_0 \in X$ and let $U$ be a definable neighborhood of $p_0$ in $X$. We say that a function $s \in \mathcal{C}(U)$ vanishes on a germ at $p_0$ if $s$ has a non-isolated zero at $p_0$. We say that $s_1, \ldots, s_k \in \mathcal{C}(U)$ vanish on separated germs at $p_0$ if there are $\delta_1, \ldots, \delta_k \in \mathcal{C}(U \setminus \{p_0\})$ with the following properties.

(S1) The sets $S_i = \{s_i = 0\} \setminus \{p_0\}$ are pairwise disjoint.

(S2) Each $s_i$ vanishes on a germ at $p_0$. 

5
For $i \neq j$, the function $\delta_i$ is constantly 1 on $S_i$ and constantly 0 on $S_j$.

Each $\delta_i$ is bounded on $U \setminus \{p_0\}$, i.e., $\delta_i(U \setminus \{p_0\})$ is contained in some $r \cdot O$, $r \in K$.

We call functions $\delta_1, \ldots, \delta_k$ with these properties separating functions for $s_1, \ldots, s_k$.

Finally we say that $X$ has arbitrarily many germs at $p_0$ if for every positive integer $k$ there is a definable neighborhood $U$ of $p_0$ in $X$ and $k$ functions in $C(U)$ that vanish on separated germs at $p_0$.

Examples 2.2 Intuitively this means that $p_0$ can be approached in $X$ through arbitrarily many disjoint ways.

1. For every $r \geq 2$, $X = K^r$ has arbitrarily many germs at the origin (see Proposition 6.4).

2. If $K$ is o-minimal then $X = K$ does not have arbitrarily many germs at 0, because any definable set $S \subseteq K \setminus \{0\}$ whose closure contains 0 will necessarily meet one of the two intervals $(-\infty, 0)$ or $(0, +\infty)$.

3. If $K$ is any expansion of the real field $\mathbb{R}$ including the sin function then $X = K$ has arbitrarily many germs at 0: take $s_i(x) = x \sin((1 - ix)/(kx))$ for $1 \leq i \leq k$.

4. If $K$ is any expansion of a valued field whose value group is a $\mathbb{Z}$-group then $K$ has arbitrarily many germs at 0: take $s_i(x) = x \chi_i(x)$ where $\chi_i$ is the characteristic function of the (clopen) set of elements of $O \setminus \{0\}$ whose valuation is congruent to $i$ modulo $k$.

A coordinate projection $\pi : K^m \to K^r$ is a map of the form $\pi(x) = (x_i)_{i \in I}$ for some $I \subseteq \{1, \ldots, m\}$ of size $r \geq 0$; we write $\pi_I$ when necessary. The dimension of a non-empty set $X$ is the maximal $r \leq m$ such that $\pi(X)$ has non-empty interior for some coordinate projection $\pi : K^m \to K^r$. This is extended to $\dim(\emptyset) = -\infty$. The local dimension of $X$ at a point $x \in K^m$ is

$$\dim(X, x) = \min \{ \dim B \cap X : B \text{ is an open ball centered at } x \}.$$ 

Note that $\dim(X, x) = -\infty$ if and only if $x \notin \overline{X}$, and that $\dim(X, x) = 0$ if and only if $x$ is an isolated point of $X$.

Definition 2.3 For every integer $d \geq 0$ we write

$$\Delta_d(X) = \{ x \in X : \dim(X, x) = d \}.$$ 

Further we write $W_d(X)$ for the set of all $x \in X$ for which there is an open ball $B$ centered at $x$ and a coordinate projection $\pi : K^m \to K^d$ that induces by restriction a homeomorphism between $B \cap X$ and an open subset of $K^d$.

Since the open balls are uniformly definable in $\mathcal{L}_{\text{ring}} \cup \{O\}$ the sets $\Delta_d(X)$ and $W_d(X)$ are definable in $\mathcal{L}$.

We say that $K$ satisfies (Dim) if for every definable set $X \subseteq K^m$ and every definable map $f : X \to K^r$ the following properties hold true.
(Dim1) \( \dim(f(X)) \leq \dim(X) \).

(Dim2) \( \dim(X) = \dim(X) \) and if \( X \neq \emptyset \), then \( \dim(\partial X) < \dim(X) \).

(Dim3) If \( \dim(X) = d \geq 0 \) then \( \dim(X \setminus W_d(X)) < d \).

Remark 2.4 These properties hold true in every dp-minimal expansion of a field which is not strongly minimal ([SW18]). This implies and generalises known results on \( o \)-minimal, \( C \)-minimal and \( P \)-minimal fields (see [vdD98], [HM94], [HM97] and [CKDL17]).

Note that the sets \( \Delta_d(X) \) are pairwise disjoint and that \( \bigcup_{d \geq 0} \Delta_d(X) \) is closed in \( X \) for each \( d \), while \( W_d(X) \) is open in \( X \).

Property 2.5 Property (Dim1) implies that the dimension is preserved by definable bijections and that \( W_d(X) \subseteq \Delta_d(X) \) for every \( d \geq 0 \). In particular the sets \( W_d(X) \) are pairwise disjoint.

Proof: The first assertion is obvious, we prove the second one. If \( x \in W_d(X) \), then there is a definable neighborhood \( U \) of \( x \) in \( X \), a coordinate projection \( \pi : K^m \to K^d \) and an open subset \( V \) of \( K^d \) such that \( \pi|_U \) is a homeomorphism onto \( V \). In particular \( \dim U = d \) by the first assertion, hence \( \dim W_d(X) \geq d \). For every sufficiently small open ball \( B \) centered at \( x \) we have \( B \cap X \subseteq U \), hence \( \pi|_{B \cap X} \) is a homeomorphism onto a non-empty open subset of \( K^d \) and so \( \dim(B \cap X) = d \). This proves \( \dim(X, x) = d \) hence \( W_d(X) \subseteq \Delta_d(X) \). Since the sets \( \Delta_d(X) \) are pairwise disjoint, so are the sets \( W_d(X) \).

3 Definability of the poset of zero sets

Let \( X \) be a set and let \( A \) be a ring of functions \( X \to K \) for some field \( K \). We show in Theorem 3.4 that for a huge class of examples, the \( (n+1) \)-ary relation \( \{ f_1 = 0 \} \cap \ldots \cap \{ f_n = 0 \} \subseteq \{ g = 0 \} \) of \( A \) is equivalent to \( g \) being in the Jacobson radical of the ideal \( (f_1, \ldots, f_n) \). In particular, these relations are \( 0 \)-definable in the ring \( A \). The crucial ingredients are contained in Proposition 3.2 and Proposition 3.3.

Remark 3.1 Recall that for an ideal \( I \) of an arbitrary ring \( A \), the Jacobson radical \( \text{Jac}(I) \) of \( I \) is defined as the intersection of the maximal ideals of \( A \) containing \( I \) (cf. [Mat89, p. 3]). The Jacobson radical of the ring \( A \) is defined as \( \text{Jac}(0) \). One checks easily that \( \text{Jac}(0) = \{ a \in A \mid \forall x \in A : 1 + ax \in A^* \} \), where \( A^* \) denotes the units of \( A \). Translating this description for \( A/I \) back to \( A \) shows that \( \text{Jac}(I) = \{ a \in A \mid \forall x \in A \exists y \in A, z \in I : (1 + ax)y = 1 + z \} \).

\(^2\)In the classical cases of \( o \)-minimal, \( C \)-minimal and \( P \)-minimal structures, the set \( \Delta_d(X) \) is usually considered instead of \( W_d(X) \) in (Dim3). However it is this slightly stronger statement with \( W_d(X) \) which we need in Section 6. It appears in Proposition 4.6 of [SW18].
Proposition 3.2 Let \( K \) be a field.

1. If \( K \) is not algebraically closed, then there are polynomials \( u(x, y), v(x, y) \) in two variables over \( K \) such that the unique zero of \( xu(x, y) + yv(x, y) \) in \( K^2 \) is \((0,0)\).

2. Assume that \( R \) is a real closed field and \( K = R[i] \) is its algebraic closure. We write \( a^* \) for the complex conjugation of \( a \in K \) with respect to \( R \). Then for the functions \( u(x, y) = x^* \) and \( v(x, y) = y^* \) defined on \( K^2 \), the unique zero of \( xu(x, y) + yv(x, y) \) in \( K^2 \) is \((0,0)\).

3. Assume that \( K \) is a topological field (see [War93]) of characteristic \( \neq 2 \), where a basis of neighborhoods of \( 0 \in K \) is given by the non-zero ideals of a ring \( \mathcal{O} \) with fraction field \( K \) and non-zero Jacobson radical \( \text{Jac}(0) \).

Then there are continuous functions \( u, v : K^2 \to K \) that are definable in the expansion \((K, \mathcal{O})\) of \( K \) by the set \( \mathcal{O} \), such that the unique zero of \( xu(x, y) + yv(x, y) \) in \( K^2 \) is \((0,0)\).

Proof: (1) Since \( K \) is not algebraically closed there is a polynomial \( p(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0 \in K[t] \) without zeroes in \( K \) such that \( d \geq 1 \). Then its homogenization

\[
q(x, y) = x^d + a_{d-1}x^{d-1}y + \ldots + a_0y^d
\]

has a unique zero in \( K^2 \), namely \((0,0)\): If \( b = 0 \), then \( q(a, b) = 0 \) just if \( a = 0 \). If \( b \neq 0 \), then \( q(a, b) = b^d \cdot p(\frac{a}{b}) \neq 0 \).

We choose \( u(x, y) = x^{d-1} \) and \( v(x, y) = a_{d-1}x^{d-1} + \ldots + a_0y^{d-1} \) and see that \( xu(x, y) + yv(x, y) = q(x, y) \) has the required properties.

(2) is clear.

(3) We write \( |K| = \{a \mathcal{O} \mid a \in K\} \) for the set of principal fractional ideals of \( \mathcal{O} \) and \( |a| = a \mathcal{O} \) for \( a \in K \). Further we write \( |a| \leq |b| \) instead of \( |a| \subseteq |b| \).

Claim. For every topological space \( X \), each continuous function \( f : X \to K \) and all \( x_0 \in X \) with \( f(x_0) \neq 0 \) there is an open neighborhood \( U \) of \( x_0 \) in \( X \) such that the restriction of \( |f| : X \to |K| \) to \( U \) is constant.

Proof. Replacing \( f \) by \( \frac{1}{f(x_0)} \cdot f \) if necessary, we may assume that \( f(x_0) = 1 \). By the assumption in (iii) there is a non-zero element \( \varepsilon \) in the Jacobson radical of \( \mathcal{O} \) such that \( 1 + \varepsilon \mathcal{O} \) is an open neighborhood of \( 1 \) in \( K \). By continuity of \( f \) at \( x_0 \) there is a neighborhood \( U \) of \( x_0 \) in \( X \) with \( f(U) \subseteq 1 + \varepsilon \mathcal{O} \). Since \( \varepsilon \mathcal{O} \subseteq \text{Jac}(0) \), no element in \( 1 + \varepsilon \mathcal{O} \) can be in any maximal ideal of \( \mathcal{O} \). Hence \( f(U) \subseteq 1 + \varepsilon \mathcal{O} \subseteq A^x \), which shows that \( |f(x)| = |1| \) for all \( x \in U \).

Returning to the proof of (3) we define

\[
U = \{ |x - y| < |x + y| \}, \quad V = \{ 0 \neq |x - y| \neq |x + y| \}, \quad Z = \{ x - y = x + y = 0 \}.
\]

\footnote{Our principal example here is a proper valuation ring \( \mathcal{O} \) of \( K \).}
This is a partition of $K^2$ in $(K, \mathcal{O})$-definable sets. Note that $x + y \neq 0$ in $U$ and $x - y \neq 0$ in $V$. Using the claim with $X = K^2$ it is easy to see that $U$ and $V$ are open in $K^2$.

Further, the functions $u, v : K^2 \to K$ defined by

$$u(x, y) = \begin{cases} x + y & \text{if } (x, y) \in U, \\
 x - y & \text{if } (x, y) \in K^2 \setminus U \end{cases}$$

and

$$v(x, y) = \begin{cases} x + y & \text{if } (x, y) \in U, \\
 y - x & \text{if } (x, y) \in K^2 \setminus U, \end{cases}$$

are continuous on $U$ and on $V$. Moreover, since $x - y = x + y = 0$ on $Z$, both $u$ and $v$ tend to $0$ at every point of $Z$. Thus $u$ and $v$ are continuous on $X$. By construction they are also definable in $(K, \mathcal{O})$. On $U$, $xu(x, y) + yv(x, y) = (x + y)^2$ has no zero. On $V$, $xu(x, y) + yv(x, y) = (x - y)^2$ has no zeroes.

Since $K$ has characteristic $\neq 2$, the set $Z$ is $\{(0, 0)\}$, which establishes the assertion. $\square$

**Proposition 3.3** Let $K$ be a field and let $u, v : K^2 \to K$ be functions such that the unique zero of $xu(x, y) + yv(x, y)$ in $K^2$ is $(0, 0)$.

Let $X$ be a set and let $A$ be a ring of functions $X \to K$ such that $A$ is closed under composition with $u$ and $v$, i.e., $u \circ (f, g), v \circ (f, g) \in A$ for all $f, g \in A$.

1. For all $f_1, \ldots, f_n \in A$ there are $g_1, \ldots, g_n \in A$ with
   $$\{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} = \{g_1 f_1 + \ldots + g_n f_n = 0\}.$$

2. If every $f \in A$ without zeroes in $X$ is a unit in $A$ then for all $f_1, \ldots, f_n, g \in A$ we have
   $$\{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} \subseteq \{g = 0\} \implies g \in \text{Jac}(f_1, \ldots, f_n).$$

3. If $A$ contains all constant functions $X \to K$, then for all $f_1, \ldots, f_n, g \in A$ we have
   $$g \in \text{Jac}(f_1, \ldots, f_n) \implies \{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} \subseteq \{g = 0\}.$$

**Proof:**

[1] By induction on $n$, where $n = 1$ is trivial. The induction step readily reduces to the claim in the case $n = 2$. By assumption on $A$ we know that $h := f_1 \cdot (u \circ (f_1, f_2)) + f_2 \cdot (v \circ (f_1, f_2)) \in A$. By assumption on $u, v$ we see that the zero set of $h$ in $X$ is $\{f_1 = 0\} \cap \{f_2 = 0\}$. Hence we may take $g_1 = u \circ (f_1, f_2)$ and $g_2 = v \circ (f_1, f_2).

[2] Assume $\bigcap_{i=1}^n \{f_i = 0\} \subseteq \{g = 0\}$ and let $\mathfrak{m}$ be a maximal ideal of $A$ containing $f_1, \ldots, f_n$. We need to show that $g \in \mathfrak{m}$. By [1] there is some $h$ in the ideal $(f_1, \ldots, f_n)$ generated by $f_1, \ldots, f_n$ in $A$ with zero set $\bigcap_{i=1}^n \{f_i = 0\}$. In particular $h \in \mathfrak{m}$ and $\{h = 0\} \subseteq \{g = 0\}$.

Suppose $g \notin \mathfrak{m}$. Then there is some $a \in A$ and some $m \in \mathfrak{m}$ with $ag + m = 1$, in particular $\{g = 0\} \cap \{m = 0\} = \emptyset$. Since $\{h = 0\} \subseteq \{g = 0\}$ we get $\{h = 0\} \cap \{m = 0\} = \emptyset$. By [1] again, there are $b_1, b_2 \in A$ such that $b_1 h + b_2 m$ has no zeroes. But then by assumption on $A$, $b_1 h + b_2 m$ is a unit of $A$, contradicting $h, m \in \mathfrak{m}$.

[3] Let $x \in \bigcap_{i=1}^n \{f_i = 0\}$ and let $e : A \to K$ be the evaluation map at $x$. Since $A$ contains the constant functions we know that $e$ is surjective, hence $\mathfrak{m} = \ker(e)$ is a maximal ideal of $A$. Then $f_1, \ldots, f_n \in \mathfrak{m}$ and so by assumption $g \in \mathfrak{m}$, i.e. $g(x) = 0$. $\square$
**Theorem 3.4** Let $X$ be a set and let $A$ be a ring of functions $X \rightarrow K$ containing the constant functions such that every $f \in A$ without zeroes in $X$ is a unit in $A$. Suppose one of the following conditions hold:

(a) $K$ is not algebraically closed, or,

(b) $K$ is the algebraic closure of a real closed field $R$ and $A$ is closed under conjugation of $K = R[i]$, or,

(c) $K$ is a topological field of characteristic $\neq 2$, where a basis of neighborhoods of $0 \in K$ is given by the non-zero ideals of a ring $O$ with fraction field $K$ and non-zero Jacobson radical. Further assume that $w(f, g) \in A$ for every $(K, O)$-definable continuous function $w : K^2 \rightarrow K$.

The following conditions are equivalent for all $f_1, \ldots, f_n, g \in A$.

1. $\{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} \subseteq \{g = 0\} \iff g \in \text{Jac}(f_1, \ldots, f_n)$.
2. $g \in \text{Jac}(f_1, \ldots, f_n)$.
3. For all $h \in A$, the element $1 + hg$ is a unit modulo the ideal $(f_1, \ldots, f_n)$.

Consequently, the $(n + 1)$-ary relation $\{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} \subseteq \{g = 0\}$ of $A$ is definable in the ring $A$ by the $L_{\text{ring}}$-formula

$$\forall x \exists y_1, \ldots, y_n, z \cdot (1 + x \cdot g) \cdot z = 1 + y_1 f_1 + \ldots + y_n f_n.$$ 

Of particular interest for us is the case $n = 1$. Hence the binary relation $f \sqsubseteq g$ defined as $\{f = 0\} \subseteq \{g = 0\}$, is 0-definable in $A$.

**Proof:** The assumptions in Proposition 3.3 hold by Proposition 3.2. Hence Proposition 3.3(2),(3) imply the equivalence of (1) and (2). The equivalence of (2) and (3) holds by Remark 3.1. $\square$

**Examples 3.5** Let $X$ be a topological space. Then Theorem 3.4 applies to the following rings $A$ of functions $X \rightarrow K$.

(a) $K$ is an ordered field or a $p$-valued field and

- $A$ is the ring of continuous functions $X \rightarrow K$, or,
- $X \subseteq K^n$ is definable in $K$ and $A$ is the ring of definable continuous functions $X \rightarrow K$, or,
- $X \subseteq K^n$ is open (definable) and $A$ is the ring of (definable) $k$-times differentiable functions $X \rightarrow K$, or,
- $X \subseteq K^n$ is a variety and $A$ is the ring of rational functions $X \rightarrow K$ without zeroes on $X$ (sometimes referred to as regular functions in the literature).

In each case, condition (a) of Theorem 3.4 applies.
(b) $A$ is the ring of continuous functions $X \to \mathbb{C}$, or, $X \subseteq \mathbb{C}^n$ and $A$ is the ring of continuous semi-algebraic functions $X \to \mathbb{C}$ (here semi-algebraic means that its graph is definable in the real field). In both cases, condition (b) of Theorem 3.4 applies.

(c) $K$ is a valued field of characteristic $\neq 2$, furnished with the valuation topology and $A$ is the ring of continuous functions $X \to K$, or, $X \subseteq K^n$ is definable and $A$ is the ring of definable continuous functions $X \to K$. In both cases, condition (c) of Theorem 3.4 applies, where $\mathcal{O}$ is the valuation ring of $K$.

4 Basic properties of zero sets

We collect a few basic facts, which will be used in the rest of the paper. We continue to work with the set up of the introduction and section 2.

Lemma 4.1  

(1) There is $\nu_m \in \mathcal{C}(K^m)$ such that $|\nu_m(x)| = \|x\|$ for every $x \in K^m$.

Now let $X \subseteq K^m$ be a definable set, $a \in X$ and let $B$ be a closed ball (resp. $B_0$ an open ball) with radius $r \in K^+$ (resp. $r_0 \in K^+$) and center $a$.

(2) Each of the sets $\{a\}$, $B$ and $B_0 = K^m \setminus B_0$ are zero sets of functions from $\mathcal{C}(X)$.

We pick such functions and denote them by $\delta_a$, $\delta_B$ and $\delta_{B_0}$ respectively.

(3) If $B \subseteq B_0$, then there is a function $\delta_{B,B_0} \in \mathcal{C}(X)$ with values in $\mathcal{O}$ such that $\delta_{B,B_0}$ vanishes on $B_0$ and $\{\delta_{B,B_0} = 1\} = B$.

Proof: (1) For every $x = (x_1, \ldots, x_m)$ let $\nu_m(x) = x_1$ if $\|x\| = |x_1|$, $\nu_m(x) = x_2$ if $\|x\| = |x_2| > |x_1|$, and so on. It is obviously definable, and easily seen to be continuous on $K^m$.

(2) and (3) We may take $\delta_a(x) = \nu_m(x - a)$ restricted to $X$, using (1). For the other functions, in the valued case we may take the indicator function of $X \setminus B$ for $\delta_B$, and the indicator function of $B_0$ for $\delta_{B_0}$ as well as for $\delta_{B,B_0}$ (they are continuous because $B$ and $B_0$ are clopen). In the ordered case we have $0 < r_0 < r$ and $|K|$ identifies with the set of non-negative elements of $K$ so we may take $\delta_B(x) = \max\{0, \|x-a\|-|r|\}$, $\delta_{B_0}(x) = \max\{0, r-\|x-a\|\}$ and $\delta_{B,B_0}(x) = \max\{0, \min\{1, u(x)\}\}$ where $u(x) = (r-\|x-a\|)/(r-r_0)$. \hfill \Box

Lemma 4.2 There are formulas $\text{Point}(f)$ and $\text{Inter}(f,g,h)$ in $\mathcal{L}_{\text{ring}}$, and a formula $\text{Isol}(s,p)$ in $\mathcal{L}_{\text{ring}} \cup \{B\}$ such that for every definable set $X \subseteq K^m$ the following equivalences are true.

(1) $\mathcal{C}(X) \models \text{Point}(f) \iff f$ has a single zero in $X$.

(2) $\mathcal{C}(X) \models \text{Inter}(f,g,h) \iff \{f = 0\} \cap \{g = 0\} = \{h = 0\}$.

(3) $\langle \mathcal{C}(X), B \rangle \models \text{Isol}(s,p) \iff \{p = 0\}$ is an isolated point of $\{s = 0\}$. (Recall that $B = \{s \in \mathcal{C}(X) \mid \forall x \in X : s(x) \in \mathcal{O}\}$.)
**Proof:** Let \( \mathcal{V}(X) = \{ \{ s = 0 \} : s \in \mathcal{C}(X) \} \) ordered by inclusion. It is a bounded distributive lattice. Indeed for every \( f, g \in \mathcal{C}(X) \) we have

\[
\{ f = 0 \} \cup \{ g = 0 \} = \{ fg = 0 \}
\]
\[
\{ f = 0 \} \cap \{ g = 0 \} = \{ \nu_2(f, g) = 0 \},
\]

where \( \nu_2 \in \mathcal{C}(K^2) \) is the function given by Lemma 4.1(1). For every \( a \in K^m, \{ a \} = \{ \delta_a = 0 \} \) where \( \delta_a \) is given by Lemma 4.1(2), so the atoms of \( (\mathcal{V}(X), \subseteq) \) are exactly the singletons. The two first points then follow from the fact that \( (\mathcal{V}(X), \subseteq) \) is uniformly interpretable in \( \mathcal{C}(X) \) as the quotient ordered set of the preorder \( \sqsubseteq \), see Theorem 3.4.

For item [3], assume that \( \{ p = 0 \} = \{ p_0 \} \). If \( p_0 \) is an isolated point of \( \{ s = 0 \} \), let \( B_0 \) be an open ball with center \( p_0 \) which is disjoint from \( S = \{ s = 0 \} \setminus \{ p_0 \} \). By Lemma 4.1(2) there is a function \( \delta_{B_0} \in \mathcal{C}(X) \) such that \( \delta_{B_0}(x) = 0 \) if and only if \( x \notin B_0 \). Finally let \( u = \nu_2(s, \delta_{B_0}) \) where \( \nu_2 \) is given by Lemma 4.1(1). Then \( \{ u = 0 \} = \{ s = 0 \} \cap B_0 = S \), hence \( s \notin pu \) and \( s \not\sqsubseteq u \). Conversely if there is \( u \in \mathcal{C}(X) \) such that \( s \subseteq pu \) and \( s \not\sqsubseteq u \) then \( \{ u \neq 0 \} \) is a neighborhood of \( p_0 \) disjoint from \( S \) hence \( p_0 \) is an isolated point \( \{ s = 0 \} \).

So this property is defined by the formula

\[
\text{Isol}(s, p) \equiv \text{Point}(p) \land p \sqsubseteq s \land \exists u (s \sqsubseteq pu \land s \not\sqsubseteq u).
\]

\[\square\]

**Proposition 4.3** For every parameter-free formula \( \varphi(y) \) in \( \mathcal{L}_{\text{ring}} \) (resp. \( \mathcal{L}_{\text{ring}} \cup \{O\} \)) in \( k \) free variables, there is a parameter-free formula \( [\varphi] \) in \( \mathcal{L}_{\text{ring}} \) (resp. \( \mathcal{L}_{\text{ring}} \cup \{B\} \)) in \( k + 1 \) variables such that for every definable set \( X \subseteq K^m \), every \( h \in \mathcal{C}(X)^k \) and every \( s \in \mathcal{C}(X) \) we have

\[
(\mathcal{C}(X), B) \models [\varphi](h, s) \iff \forall x \in \{ s = 0 \}, K \models \varphi(h(x)).
\]

**Proof:** If \( \varphi(y) \) is a polynomial equation \( P(y) = 0 \) (\( y \) a single variable), we may take \( [\varphi](y, s) \) as \( s \sqsubseteq P(y) \). If \( \varphi(y) \) is the formula \( P(y) \in O \), we may take \( [\varphi](y, s) \) as

\[
\forall p \left( \text{Point}(p) \land p \sqsubseteq s \implies \exists f \left( f \in B \land p \sqsubseteq P(y) - f \right) \right),
\]

expressing the fact that a function \( P(y) \) has values in \( O \) on the zero set of \( s \) just if for all \( x \in \{ s = 0 \} \), \( P(y) \) agrees with a function \( f \in B \) at \( x \).

This proves the result for atomic formulas. For arbitrary formulas, \( [\varphi] \) is defined by induction as follows.

- \( [\varphi \land \psi] \) is \([\varphi] \land [\psi] \).
- \( [\neg \varphi](y, s) \) is \( \forall p \left( \text{Point}(p) \land p \sqsubseteq s \rightarrow [\neg \varphi](y, p) \right) \).
- \( [\exists w \varphi](y, s) \) is \( \forall p \left( \text{Point}(p) \land p \sqsubseteq s \rightarrow \exists w [\varphi](w, y, p) \right) \).

We only show the right to left implication of the claimed equivalence in the case of an existential quantifier, all other implications are straightforward. Take \( h \in \mathcal{C}(X)^k \), \( s \in \mathcal{C}(X) \) and assume that \( K \models \exists w \varphi(w, h(x)) \) for every \( x \in \{ s = 0 \} \). We need to
show that \((C(X),\mathcal{B})\models \forall p((\text{Point}(p) \land p \subseteq s) \rightarrow \exists w[\varphi(w,y,p)])\). So take \(p \in C(X)\) whose unique zero \(p_0\) satisfies \(s(p_0) = 0\). By assumption there is some \(c \in K\) with \(K \models \varphi(c,h(p_0))\). Let \(h_0 \in C(X)\) be the constant function with value \(c\). Now, for every \(z \in X\) and every \(q \in C(X)\) that has a unique zero \(z\), if \(p(z) = 0\) then \(K \models \varphi(h_0(z),h(z))\).

By induction, this means \((C(X),\mathcal{B})\models [\varphi](h_0,h,p)\), as required.

Finally, by choice of the formulas \([\varphi]\) we see that \([\varphi]\) is an \(L_{\text{ring}}\)-formula, if \(\varphi\) is an \(L_{\text{ring}}\)-formula. \(\square\)

**Remark 4.4** If \(O\) is definable in \(K\) by a formula \(\varphi(x,a)\) in \(L_{\text{ring}}\), where \(a\) is an \(n\)-tuple of parameters from \(K\), then Proposition 4.3 applied to \(\varphi(x,y)\) implies that \(\mathcal{B}\) is definable in \(C(X)\) by the \(L_{\text{ring}}\)-formula \([\varphi](x,a^*,0)\) with parameters \(a^*\), the constant function of \(C(X)^n\) with value \(a\). There are plenty of such fields with definable orders or valuations (including all the non-algebraically closed local fields):

- real-closed and \(p\)-adically closed fields;
- the field of rational numbers (by Lagrange’s Four Squares Theorem);
- one-variable functions fields over number fields \([\text{MS17}]\);
- dp-minimal valued fields that are not algebraically closed \([\text{Joh15}]\);
- the field of Laurent series \(F((t))\) with the natural valuation \([\text{Ax65}]\);

and numerous others: see \([\text{Fla17}]\) for a summary on Henselian valuation rings definable in their fraction field.

Having arbitrarily many germs at a point is a local property, but it can be made a bit more global as follows.

**Property 4.5** Let \(X \subseteq K^n\) be a definable set and let \(p_0 \in X\). Then \(X\) has arbitrarily many germs at \(p_0\) (see Definition 2.7) if and only if for every positive integer \(k\) there exist \(k\) functions in \(C(X)\) that vanish on separated germs at \(p_0\) with separating functions in \(C(X \setminus \{p_0\})\).

**Proof:** One implication is obvious. For the converse, assume that a definable neighborhood \(U\) of \(p_0\) in \(X\) is given together with \(v_1,\ldots,v_k \in C(U)\) that vanish on separated germs at \(p_0\), and with corresponding separating functions \(d_1,\ldots,d_k \in C(U \setminus \{p_0\})\). Restricting \(U\) if necessary, by continuity at \(p_0\), we may assume that each \(v_i\) is bounded on \(U\) and that \(U = B_0 \cap X\) for some open ball \(B_0\) with center \(p_0\). Let \(B\) be a closed ball with center \(p_0\) contained in \(B_0\). By Lemma 4.1(3) we have a bounded function \(h = \delta_{B,B_0} \in C(X)\) with \(h(x) = 0\) on \(X \setminus B_0\) and \(h(x) = 1\) on \(B \cap X\).

For each \(i \leq k\) let \(u_i(x) = 0\) on \(X \setminus B_0\) and \(u_i(x) = h(x)v_i(x)\) on \(B_0 \cap X\). Similarly let \(\delta_i(x) = 0\) on \(X \setminus B_0\) and \(\delta_i(x) = h(x)d_i(x)\) on \(B_0 \cap X \setminus \{p_0\}\). Each \(u_i\) (resp. \(\delta_i\)) is continuous on \(B_0 \cap X\) and tends to 0 at every point in \(\partial(B_0 \cap X)\) (because \(v_i\) and \(d_i\) are bounded) hence \(u_i \in C(X)\) and \(\delta_i \in C(X \setminus \{p_0\})\). Now let \(s_i = v_2(\delta_{B,B_0},u_i)\), where \(v_2,\delta_B \in C(K^2)\) are given by Lemma 4.1(1)(3). By construction \(\delta_i\) is bounded on \(X \setminus \{p_0\}\), \(\delta_i(x) = d_i(x)\) on \(B \cap X \setminus \{p_0\}\) and \(\{s_i = 0\}\) equals \(\{v_i = 0\} \cap B\). So the functions \(s_i,\delta_i\) inherit from \(v_i,\delta_i\) all the properties (S1)–(S4) of functions vanishing on separated germs, cf. Definition 2.1. \(\square\)
5 Constructing integers using limit values and chunks

The present section is devoted to our first main results of interpretability and definability. Our construction is based on the following subset of $K$. Let $X \subseteq K^m$ be a definable set. Let $s, p \in C(X)$ be such that the zero-set of $p$ is a single point $p_0$ and $s$ vanishes on a germ at $p_0$. Then for all $f, g \in C(X)$ for which $g$ has no zeroes in the set $S = \{ s = 0 \} \setminus \{ p_0 \}$, we consider the continuous function $f/g$ on $S$ and write $\Gamma$ for its graph. We define

$$L_{s,p}(f/g) = \{ t \in K : (p_0, t) \in \Gamma \}.$$  

Informally, $L_{s,p}$ is the set of “limit values” at $p_0$ of $f/g$ restricted to $S$. Note that this is always a closed definable subset of $K$. We will also consider the following relations on $C(X)$. Note that they are definable in $\mathcal{L}_{\text{ring}} \cup \{ B \}$ using Proposition 4.3.

$$|f| \leq s \ |g| \iff \{ s = 0 \} \subseteq \{ |f| \leq |g| \}$$

$$|f| < s \ |g| \iff \{ s = 0 \} \subseteq \{ |f| < |g| \}.$$  

**Lemma 5.1** There is a parameter-free formula $\text{Limit}(f, g, h, s, p)$ in $\mathcal{L}_{\text{ring}}$ such that for any definable set $X \subseteq K^m$, we have $C(X) \models \text{Limit}(f, g, h, s, p)$ if and only if

- the zero-set of $p$ is a single point $p_0$,
- $s$ vanishes on a germ at $p_0$,
- $g$ has no zeroes in the set $S = \{ s = 0 \} \setminus \{ p_0 \}$ and
- $h(p_0) \in L_{s,p}(f/g)$.

**Proof:** The first three properties are defined by a conjunction $\chi(g, s, p)$ using the formulas in Lemma 4.2; we may define $\chi(g, s, p)$ as

$$\text{Point}(p) \land \neg \text{Isol}(s, p) \land (\text{Inter}(g, s, p) \lor \text{Inter}(g, s, 1)).$$

Let $f, g, h, s, p \in C(X)$ be such that $C(X) \models \chi(g, s, p)$ and let $p_0$ be the zero of $p$. Pick $\tau \in K^*$ with $|\tau| < 1$. For every $\varepsilon, v \in C(X)$ with $\varepsilon(p_0) \neq 0$ and $v(p_0) \neq 0$, by continuity, there is an open ball $B_0$ centered at $p_0$ such that $B_0 \cap X \subseteq \{ v \neq 0 \}$ and for every $x \in B_0 \cap X$, $|\varepsilon(x)| > |\tau \varepsilon(p_0)|$ and $|h(x) - h(p_0)| < |\tau \varepsilon(p_0)|$. So if $h(p_0) \in L_{s,p}(f/g)$, then there is a point $q_0 \in B_0 \cap X \setminus \{ p_0 \}$ with

$$s(q_0) = 0 \text{ and } \left| \frac{f(q_0)}{g(q_0)} - h(q_0) \right| \leq |\varepsilon(q_0)|.$$  

Note that $q_0 \in B_0 \setminus \{ p_0 \}$ implies that $q \not\subseteq pv$, and that $s(q_0) = 0$ implies $g(q_0) \neq 0$, so the second condition above is equivalent to $|f(q_0) - g(q_0)h(q_0)| \leq |g(q_0)\varepsilon(q_0)|$. Conversely, if there are such points $q_0$ for any $\varepsilon, v \in C(X)$ with $\varepsilon(p_0) \neq 0$ and $v(p_0) \neq 0$, then setting $v = \delta_{B_0}$ for any open ball $B_0$ centered at $p_0$ (with $\delta_{B_0}$ given by Lemma 4.2), we get that $h(p_0) \in L_{s,p}(f/g)$. Thus we can take for $\text{Limit}(f, g, h, s, p)$ the conjunction of $\chi(g, s, p)$ and the formula

$$\forall \varepsilon, v \ (p \not\subseteq \varepsilon v \rightarrow \exists q \ [\text{Point}(q) \land q \not\subseteq pv \land q \subseteq s \land |f - gh| \leq q \ |g\varepsilon|]).$$

$\square$

14
Lemma 5.2 Let \( l_1, \ldots, l_k \in K \) and let \( X \subseteq K^m \) be a definable set. Let \( p \in X \) be such that \( p \) has a single zero \( p_0 \) at which \( X \) has arbitrarily many germs. Let \( s_1, \ldots, s_k \in C(X) \) be vanishing on separated germs at \( p_0 \) and let \( S_i = \{ s_i = 0 \} \setminus \{ p_0 \} \). Then there are \( f, g \in C(X) \) with \( g(x) \neq 0 \) on \( S = S_1 \cup \cdots \cup S_k \) such that the restriction of \( f/g \) to each \( S_i \) has constant value \( l_i \). In particular:

1. \( \{ l_1, \ldots, l_k \} = L_{s,p}(f/g) \), with \( s = s_1 s_2 \cdots s_k \).
2. \( \{ l_1, \ldots, l_k \} = \{ l(p_0) : l \in C(X) \text{ and } p_0 \in S \cap \{ f = g/l \} \} \).

Proof: Let \( \delta_1, \ldots, \delta_s \in C(X \setminus \{ p_0 \}) \) be separating functions for \( s_1, \ldots, s_k \). Clearly \( f = \sum_{i \in k} l_i \delta_i p \) (extended by 0 at \( p_0 \)) and \( g = p \) have the required properties, from which items [1] and [2] follow immediately. For the second item, note that \( p_0 \in S \cap \{ f = g/l \} \) simply means that the function \( l \) takes the same values as \( f/g \) on a subset of \( S \) having arbitrarily close to \( p_0 \), hence \( l(p_0) \in L_{s,p}(f/g) \) by continuity of \( l \). \( \square \)

Definition 5.3 Let \( (G, +, \leq) \) be a totally pre-ordered abelian group, hence \( \leq \) is a total pre-order satisfying \( x \leq y \Rightarrow x + z \leq y + z \). For \( \tau \in G \) with \( \tau > 0 \) we write \( \tau \mathbb{Z} = \{ \tau k : k \in \mathbb{Z} \} \).

We call a subset \( T \) of \( G \) a \( \tau \mathbb{Z} \)-chunk of \( G \) if \( \tau \in T \) and for all \( \alpha, \beta, \gamma \in T \):

1. \( -\alpha \in T \);
2. \( \alpha + \beta \in [-\gamma, \gamma] \Rightarrow \alpha + \beta \in T \);
3. \( \forall u \in [-\gamma, \gamma], \exists ! \xi \in T, \xi \leq u < \xi + \tau \).

This should be seen as a finitary, and hence definable, version of integer parts as studied by [MR93] in the case of real closed fields.

Remark 5.4 For every finite \( \tau \mathbb{Z} \)-chunk \( T \) of \( G \) one checks easily that there is some integer \( n \) with \( T = \{-n \tau, \ldots, -\tau, 0, \tau, 2 \tau, \ldots, n \tau \} \).

In the ordered case (and more generally when there is a total pre-order on \( K \), definable in \( L_{\text{ring}} \cup \{ \mathcal{O} \} \) and compatible with the group structure of \( (K, +) \)), we can consider chunks of \( (K, +, \leq) \). In the valued case however we have to lift chunks from \( |K^x| \) to \( K^x \). So we will consider chunks of the totally pre-ordered multiplicative group \( (K^x, \times, \leq_0) \) where \( \leq_0 \) is the inverse image of \( \leq \) of \( |K^x| \), i.e., \( x \leq_0 y \) is just \( |x| \leq |y| \). In this case we adapt the terminology to multiplicative language. For example, for \( \tau \in K^x \) with \( |\tau| < 1 \) we write \( \tau \mathbb{Z} = \{ \tau k : k \in \mathbb{Z} \} \) instead of \( \tau \mathbb{Z} \).

Theorem 5.5 Assume that \( K \) satisfies \( \text{[BFm]} \). Let \( I = \{ x \in K : |0| < |x| < |1| \} \). There is a parameter-free formula \( \text{Int}^x(p, h, \tau^*) \) in \( L_{\text{ring}} \cup \{ \mathcal{O} \} \) such that for every definable set \( X \subseteq K^m \), if \( p \) has a unique zero \( p_0 \in X \), if \( X \) has arbitrarily many germs at \( p_0 \) and if the value \( \tau = \tau^*(p_0) \) is in \( I \), then

\[
(C(X), \mathcal{B}) \models \text{Int}^x(h, p, \tau^*) \iff h(p_0) \in \tau \mathbb{Z}.
\]
If $K$ has a total order $\preceq$ compatible with $+$ and definably\footnote{This obviously happens in the ordered case, where $K$ is an ordered field and $\mathcal{O}$ is the interval $[-1,1]$. But Theorem \ref{thm:5.5} does not restrict to this case, as it does not assume any relation at all between $\mathcal{O}$ and the order $\preceq$.} in $L_{\text{ring}} \cup \{\mathcal{O}\}$, then there is similarly a formula $\text{Int}_<^+(h,p,\tau^*)$ such that, with the same assumptions on $p$, if $\tau = \tau^*(p_0) \succ 0$ then

$$(C(X),B) \models \text{Int}_<^+(h,p,\tau^*) \iff h(p_0) \in \tau \mathbb{Z}.$$ 

In other words, Theorem \ref{thm:5.5} says essentially that, given any $\tau \in K^X$ with $|\tau| < 1$ and any definable set $X \subseteq K^m$ with arbitrarily many germs at some point $p_0$, then the set of all $h \in C(X)$ taking values in $\tau \mathbb{Z}$ at $p_0$, is definable in $(C(X),B)$; the definition is independent of $X$, $K$ and $m$ and is in the two definable parameters $p_0, \tau$.

Also notice that by Remark \ref{rem:4.3}, the set $B$ is definable in $L_{\text{ring}}$ in many cases, hence Theorem \ref{thm:5.5} applies to the pure ring $C(X)$ in all these cases.

**Proof:** In order to ease the notation, in this proof we write $C(X) \models \chi(\ldots)$ instead of $(C(X),B) \models \chi(\ldots)$, for every formula in $L_{\text{ring}} \cup \{B\}$.

We first construct a formula which axiomatizes (uniformly) the property that $\tau = \tau^*(p_0) \in \mathcal{O} \setminus \{0\}$ and $L_{s,p}(f/g)$ is a bounded $\tau^Z$-chunk. We use the formula $\text{Limit}(f,g,h,s,p)$ stated in Lemma \ref{lem:5.1}. It will be convenient to abbreviate it as $\varphi^\sigma(h)$ where $\sigma = (f,g,s,p)$. Thus for every $f,g,v,s,p \in C(X)$ we have

$$v(p_0) \in L_{s,p}(f/g) \iff C(X) \models \varphi^\sigma(v).$$

Let $\psi^\sigma(\alpha,\alpha',\beta,\gamma,p,\tau^*)$ be the conjunction of the following formulas.

1. $p \sqsubseteq \alpha' - 1 \rightarrow \varphi^\sigma(\alpha')$.
2. $|1| \leq_p |\alpha| \leq_p |\gamma^2|$.
3. $\forall u \left(|1| \leq_p |u\gamma| \leq_p |\gamma^2| \rightarrow \exists \varepsilon \left[\varphi^\sigma(\varepsilon) \land |\varepsilon\tau^*| \leq_p |u| \leq_p |\varepsilon|\right]\right)$.\footnote{This obviously happens in the ordered case, where $K$ is an ordered field and $\mathcal{O}$ is the interval $[-1,1]$. But Theorem \ref{thm:5.5} does not restrict to this case, as it does not assume any relation at all between $\mathcal{O}$ and the order $\preceq$.}
4. $\forall u,\xi,\xi' \left([A(u,\xi,p,\tau^*) \land A(u,\xi',p,\tau^*)] \rightarrow p \subseteq \xi - \xi',\right)$, where $A(u,\xi,p,\tau^*)$ stands for $\varphi^\sigma(\xi) \land |\xi\tau^*| \leq_p |u| \leq_p |\xi|$.\footnote{This obviously happens in the ordered case, where $K$ is an ordered field and $\mathcal{O}$ is the interval $[-1,1]$. But Theorem \ref{thm:5.5} does not restrict to this case, as it does not assume any relation at all between $\mathcal{O}$ and the order $\preceq$.}

Clearly $\tau = \tau^*(p_0) \in I$ and $L_{s,p}(f/g)$ is a $\tau^Z$-chunk if and only if $C(X)$ satisfies the conjunction, which we will denote $\text{LimCh}^X(f,g,s,p,\tau^*)$, of $|0| \leq_p |\tau^*| \leq_p |1|$, of $\varphi^\sigma(\tau^*)$ and of

$$\forall \alpha,\alpha',\beta,\gamma \left[\varphi^\sigma(\alpha) \land \varphi^\sigma(\beta) \land \varphi^\sigma(\gamma) \rightarrow \psi^\sigma(\alpha,\alpha',\beta,\gamma,p,\tau^*)\right].$$

The formula $\text{LimBCh}^X(f,g,s,p,\tau^*)$ is then defined as

$$\text{LimCh}^X(f,g,s,p,\tau^*) \land \exists \delta \forall \alpha (\varphi^\sigma(\alpha) \rightarrow |\alpha| \leq_p |\delta|).$$

Clearly it holds true if and only if $\tau = \tau^*(p_0) \in I$ and $L_{s,p}(f/g)$ is a bounded $\tau^Z$-chunk. Finally, let $\text{Int}_<^X(h,p,\tau^*)$ be the formula

$$\exists f,g,s, \text{Limit}(f,g,h,s,p) \land \text{LimBCh}^X(f,g,s,p,\tau^*).$$
If \( \mathcal{C}(X) \models \text{Int}^\times(h, p, \tau^\bullet) \) then \( p \) has a single zero \( p_0 \), \( s \) vanishes on a germ at \( p_0 \) and \( h(p_0) \in L_{s, p}(f/g) \); further, by the above, \( \tau = \tau^\bullet(p_0) \in I \), and \( L_{s, p}(f/g) \) is a bounded \( \tau^\mathbb{Z} \)-chunk (in particular it is a bounded discrete subset of \( K \)). Since \( L_{s, p}(f/g) \) is is a closed definable subset of \( K \) it is then finite by \( \text{BFm} \). Thus by Remark \ref{RemarkBFin} (in multiplicative notation) there is a non-negative integer \( n \) such that \( L_{s, p}(f/g) = \{ \tau^i : -n \leq i \leq n \} \). We obtain \( L_{s, p}(f/g) \subseteq \tau^\mathbb{Z} \) and so \( h(p_0) \in \tau^\mathbb{Z} \).

Conversely, assume that \( p \) has a single zero \( p_0 \) and for this zero we have \( \tau = \tau^\bullet(p_0) \in I \) and \( h(p_0) = r^k \) for some \( k \in \mathbb{Z} \). Let \( T_h = \{ \tau^i : -|k| \leq i \leq |k| \} \). Since \( X \) has arbitrarily many germs at \( p_0 \), we may invoke Property \ref{PropertyDerivative} to obtain functions \( s_i \in \mathcal{C}(X) \) that vanish at separated germs at \( p_0 \), for \( -|k| \leq i \leq |k| \). Let \( s \) be the product of the \( s_i \)'s and \( S = \{ s = 0 \} \). By Lemma \ref{LemmaBFin} there are \( f, g \in \mathcal{C}(X) \) such that \( g(x) \neq 0 \) on \( S \) and

\[
T_h = L_{s, p}(f/g) = \{ l(p_0) : l \in \mathcal{C}(X) \text{ and } p_0 \in S \cap \{ f = gl \} \}.
\]

Thus \( \mathcal{C}(X) \models \text{Limit}(f, g, h, s, p) \land \text{LimBCh}^\times(f, g, s, p, \tau^\bullet) \), hence \( \mathcal{C}(X) \models \text{Int}^\times(h, p, \tau^\bullet) \). This finishes the proof of the first equivalence of the theorem.

If \( K \) has a total order \( \preceq \) that is definable in \( \mathcal{L}_{\text{ring}} \cup \{ \mathcal{O} \} \) and compatible with +, then by Proposition \ref{PropositionDefinableInfty} the following relations are definable in \( \mathcal{L}_{\text{ring}} \cup \{ \mathcal{O} \} \).

\[
g \preceq_s f \iff \{ s = 0 \} \subseteq \{ g \preceq f \}
\]

\[
g \prec_s f \iff \{ s = 0 \} \subseteq \{ g \prec f \}
\]

Let \( \Phi^\sigma(\alpha, \beta, \gamma, p, \tau^\bullet) \) be the conjunction of the following formulas.

1. \( \varphi^\sigma(-\alpha^\cdot) \).
2. \( -\gamma \preceq_p \alpha + \beta \preceq_p \gamma \rightarrow \varphi^\sigma(\alpha + \beta) \).
3. \( \forall u \left[ -\gamma \preceq_p u \preceq_p \gamma \rightarrow \exists \xi (\varphi^\sigma(\xi) \land \xi \preceq_p u \preceq_p \xi + \tau^\bullet) \right] \).
4. \( \forall u, \xi, \xi' \left( B(u, \xi, p, \tau^\bullet) \land B(u, \xi', p, \tau^\bullet) \right) \rightarrow p \subseteq \xi - \xi' \),
   where \( B(u, \xi, p, \tau^\bullet) \) stands for \( \varphi^\sigma(\xi) \land \xi \preceq_p u \preceq_p \xi + \tau^\bullet \).

We then let \( \text{LimCh}^\perp(f, g, s, p, \tau^\bullet) \) be the conjunction of \( 0 \prec_p \tau^\bullet \) with \( \varphi^\sigma(\tau^\bullet) \) and

\[
\forall \alpha, \beta, \gamma \left[ \varphi^\sigma(\alpha) \land \varphi^\sigma(\beta) \land \varphi^\sigma(\gamma) \rightarrow \Phi^\sigma(\alpha, \beta, \gamma, p, \tau^\bullet) \right],
\]

and let \( \text{LimBCh}^\perp(f, g, s, p, \tau^\bullet) \) be

\[
\text{LimCh}^\perp(f, g, s, p, \tau^\bullet) \land \exists \delta \forall \alpha (\varphi^\sigma(\alpha) \rightarrow \alpha \preceq_p \delta).
\]

Finally we take \( \text{Int}^\perp(h, p, \tau^\bullet) \) as \( \exists f, g, s (\text{Limit}(f, g, h, s, p) \land \text{LimBCh}^\times(f, g, s, p, \tau^\bullet)) \). The proof that it satisfies the second equivalence in the Theorem is analogous to the multiplicative case and left to the reader.\footnote{Here \(|k|\) denotes the ordinary absolute value in \( \mathbb{Z} \), not the image of \( k \) in \( |K| \).}
Corollary 5.6 Assume that $\mathcal{K}$ satisfies $[\text{BFin}]$ and take $\tau \in K^\times$ with $|\tau| < |1|$. Let $X \subseteq K^m$ be a definable set having arbitrarily many germs at a given point $p_0$. Let $p, \tau^* \in C(X)$ be such that $p_0$ is the unique zero of $p$ and $\tau^*(p_0) = \tau$. Then the ring of integers $(\mathbb{Z}, +, \times)$ is interpretable in $(C(X), \mathcal{B})$ with parameters $(p, \tau^*)$.

Proof: For every $k, l \in \mathbb{Z}$ let $l \mid k$ denote the divisibility relation. It is well known (see for example [Rich99]) that multiplication is 0-definable in $(\mathbb{Z}, +, |, \leq)$, hence it suffices to interpret the latter structure in $(C(X), \mathcal{B})$. Let $\mathcal{Z} = \{f \in C(X) : f(p_0) \in \tau^\mathbb{Z}\}$, and for every $f \in \mathcal{Z}$ let $\sigma(f)$ be the unique $k \in \mathbb{Z}$ such that $f(p_0) = \tau^k$. Then $f(p_0) \in \tau^\mathbb{Z} \iff (C(X), \mathcal{B}) \models \text{Int}^{\times}(f, p, \tau^*)$

by Theorem 5.5 and therefore $\mathcal{Z}$ is definable in $(C(X), \mathcal{B})$. The equivalence relation $\sigma(f) = \sigma(g)$ on $\mathcal{Z}$ is also definable, because

$$\sigma(f) = \sigma(g) \iff f(p_0) = g(p_0) \iff (C(X), \mathcal{B}) \models p \sqsubseteq f - g.$$ 

For every $f, g \in \mathcal{Z}$ we have obviously $\sigma(f) + \sigma(g) = \sigma(fg)$ and $\sigma(g) \geq \sigma(f) \iff |g(p_0)| \leq |f(p_0)| \iff (C(X), \mathcal{B}) \models |g| \leq_p |f|$. This gives an interpretation of $(\mathbb{Z}, +, |, \leq)$ in $(C(X), \mathcal{B})$. Moreover $\sigma(g)$ divides $\sigma(f)$ if and only if $f(p_0) \in (\tau^{\sigma(g)})^\mathbb{Z} = (g(p_0))^\mathbb{Z}$. We obtain $\sigma(f) | \sigma(g) \iff (C(X), \mathcal{B}) \models \text{Int}^{\times}(f, p, g)$ in the terminology of Theorem 5.5. Consequently $(\mathbb{Z}, +, |, \leq)$ is interpretable in $(C(X), \mathcal{B})$, from which the result follows. □

We say that $|K^\times|$ is $\tau$-archimedean for some $\tau \in K$ if $\tau^\mathbb{Z}$ is coinitial in $|K^\times|$, i.e., for every $x \in K^\times$ there is some $k \in \mathbb{Z}$ with $|\tau^k| \leq |x|$. If this is the case, then $|K^\times|$ is also $\tau^{-1}$-archimedean and obviously $\tau \neq 0, 1$; so we may always assume that $|0| < |\tau| < |1|$. In the ordered case, $\mathcal{K}$ is 1/2-archimedean if and only if it is archimedean in the sense that $\mathcal{Z}$ is cofinal in $K$.

Corollary 5.7 Let $\mathcal{K}' = (K', \ldots)$ be any elementary extension of $\mathcal{K}$ and assume that $\mathcal{K}$ and $\mathcal{K}'$ satisfy $[\text{BFin}]$. Let $X \subseteq K^m$ be a definable set having arbitrarily many germs at a given point $p_0$. Let $X' \subseteq K'^m$ be defined by a formula defining $X$. If $|K^\times|$ is $\tau$-archimedean for some $\tau \in K$ and $|K^\times'|$ is not $\tau$-archimedean, then the natural $L_{\text{ring}} \cup \{\mathcal{B}\}$-embedding of $C(X)$ into $C(X')$ is not an elementary embedding.

Proof: We may assume that $|0| < |\tau| < |1|$. Let $p \in C(X)$ be such that $\{p = 0\} = \{p_0\}$, and let $\tau^*$ be the constant function on $X$ with value $\tau$. For every $f \in C(X)$, there exists $k \in \mathbb{Z}$ such that $|f(p_0)| \leq |\tau^k|$, so there exists $g \in C(X)$ such that $|f(p_0)| \leq |g(p_0)|$ and $g(p_0) \in \tau^\mathbb{Z}$ (it suffices to take $g = (\tau^*)^k$). Thus by Theorem 5.5

$$\forall f \exists g, |f| \leq_p |g| \land \text{Int}^{\times}(g, p, \tau^*).$$

(*)

On the other hand, since $K'$ is not $\tau$-archimedean, there exists an element $a' \in K'$ such that $|a'| \not\leq |\tau^k|$ for every $k \in \mathbb{Z}$. Let $f'$ be the constant function on $X$ with value $a'$. For every $g' \in C(X')$ such that $g'(p_0) \in \tau^\mathbb{Z}$, $|f'(p_0)| \not\leq |g'(p_0)|$, hence the formula in (*) is not satisfied in $(C(X'), \sqsubseteq, \mathcal{B})$. □
Remark 5.8 Let $R \subseteq S$ be real closed fields. Let $L(R^n)$ be the lattice of closed and semi-algebraic subsets of $R^n$. In $\text{Ast}13$ it is shown that the natural embedding $L(R^n) \to L(S^n)$ is an elementary map. However, if $n \geq 2$ and $R = \mathbb{R} \subset S$, then we know from 5.7 that the natural embedding $C(R^n) \to C(S^n)$ is not elementary. Consequently for real closed fields $R$, there is no interpretation of the ring $C(R^n)$ in the lattice $L(R^n)$ that is independent of $R$.

6 Defining integers using local dimension

In this section, assuming (Dim) we show that the set of functions in $\mathcal{C}(X)$ that take values in $\tau^\mathbb{Z}$ or $\tau^\mathbb{Z}$ for some $\tau \in K^\times$ at some point $p_0$ as in Theorem 5.5 is definable in $(\mathcal{C}(X), \mathcal{B})$, provided $X$ is not of small local dimension at $p_0$. In order to carry this out, we first prove that the points at which $X$ has arbitrarily many germs are dense among those at which $X$ has local dimension $\geq 2$.

Recall from Definition 2.3 that $\Delta_k(X)$ denotes the set of $x \in X$ where $X$ is of local dimension $k$ and $W_k(X)$ denotes the set of $x \in X$ such that there is an open ball $B$ centered at $x$ and a coordinate projection $\pi : K^m \to K^k$ which induces by restriction a homeomorphism between $B \cap X$ and an open subset of $K^k$.

Lemma 6.1 Assume that $K$ satisfies (Dim) For every definable set $X \subseteq K^m$ and every integer $k \geq 0$, $W_k(X)$ is a dense subset of $\Delta_k(X)$. If non-empty, both of them have dimension $k$.

Proof: We already know that $W_k(X) \subseteq \Delta_k(X)$ by Property 2.5. Pick $x \in \Delta_k(X)$ and an open ball $B \subseteq K^m$ centered at $x$. By shrinking $B$ if necessary we may assume that $\dim(B \cap X) = k$. From (Dim2) we know $W_k(B \cap X) \neq \emptyset$. On the other hand, $W_k(B \cap X) \subseteq W_k(X)$ because $B \cap X$ is open in $X$. Consequently $W_k(B \cap X) \subseteq B \cap W_k(X)$ and so $B \cap W_k(X) \neq \emptyset$. This proves density.

By (Dim2) it only remains to prove that $W_k(X)$ has dimension $k$, provided it is not empty. Clearly $\dim W_k(X) \geq k$. If $\dim W_k(X) = l > k$ then by (Dim3) $W_l(W_k(X))$ is non-empty. But $W_l(X)$ is open in $X$, hence $W_l(W_k(X))$ is contained in $W_l(X)$. So $W_l(W_k(X))$ is contained both in $W_l(X)$ and in $W_k(X)$, a contradiction since $W_l(X)$ and $W_k(X)$ are disjoint by Property 2.5.

For any two subsets $A, B$ of a topological space $X$, the strong order $\sqsubseteq$ is defined by

$$ B \sqsubseteq A \iff B \subseteq A \setminus B. $$

If $X \subseteq K^m$ is definable and $f, g \in \mathcal{C}(X)$ we define

$$ g \sqsubseteq f \iff \{g = 0\} \subseteq \{f = 0\}. $$

Lemma 6.2 For every definable set $X \subseteq K^m$ and every $f, g \in \mathcal{C}(X)$

$$ g \sqsubseteq f \iff \mathcal{C}(X) \models \forall h (f \sqsubseteq gh \to g \sqsubseteq h). $$

In particular $\sqsubseteq$ is definable in $L_{\text{ring}}$.
Proof: For every definable set $U \subseteq X$ that is open in $X$ and each $p_0 \in U$ there is a function $h \in C(X)$ with $h(p_0) \neq 0$ and $h(x) = 0$ on $X \setminus U$ (using Lemma 4.1 it suffices to take $h = \delta_{B_0^c}$ for any open ball $B_0$ with center $p_0$ such that $B \cap X$ is contained in $U$), hence $p_0 \in \{h \neq 0\} \subseteq U$. Hence the sets $\{h = 0\}$ with $h \in C(X)$ form a basis of closed sets of the topology of $X$. Since $g \in f$ just if every closed definable set containing $\{f = 0\} \setminus \{g = 0\}$ also contains $\{g = 0\}$, we see that $g \in f$ if and only if for all $h \in C(X)$

$$\{f = 0\} \setminus \{g = 0\} \subseteq \{h = 0\} \Rightarrow \{g = 0\} \subseteq \{h = 0\}.$$ 

This is equivalent to $\{f = 0\} \subseteq \{g = 0\} \cup \{h = 0\} \Rightarrow \{g = 0\} \subseteq \{h = 0\}$, in other words, $f \subseteq gh \Rightarrow g \in h$. \hfill \Box

Proposition 6.3 Assume that $\mathcal{K}$ satisfies [Dim]. For every integer $k \geq 0$ and every $p_0 \in X$ the following are equivalent.

1. $\dim(X, p_0) \geq k$.
2. $\forall v \in C(X), v(p_0) \neq 0 \Rightarrow \exists f_0 \in f_1 \in \cdots \in f_k \subseteq C(X)$ such that $f_0 \notin C(X)^x$ and $\{f_k = 0\}$ is disjoint from $\{v = 0\}$.

Furthermore, there is a parameter-free formula $\chi_k(p)$ in $\mathcal{L}_{\text{ring}}$ such that $C(X) \models \chi_k(p)$ if and only if $p$ has a single zero $p_0 \in X$ and $\dim(X, p_0) = k$.

Proof: \([1] \Rightarrow [2]\) Assume that $\dim(X, p_0) = l \geq k$ and take $v \in C(X)$ with $v(p_0) \neq 0$. Then $\{v \neq 0\}$ is a neighborhood of $p_0$ in $X$ hence by Lemma 6.1 there is a point $q_0 \in \{v \neq 0\} \cap W_l(X)$. Let $V$ be a definable neighborhood of $q_0$ in $X$, contained in $\{v \neq 0\} \cap W_l(X)$ and homeomorphic to an open subset $W$ of $K^l$ via the restriction of a coordinate projection $\pi : K^m \to K^l$.

Let $\varphi = (\varphi_1, \ldots, \varphi_l) : K^l \to K^l$ be defined by $\varphi(y) = y - q_0$. For $1 \leq i \leq l$ let $g_i = \varphi_i - \varphi_i$, where $\varphi_i \subseteq C(K^i)$ is the map defined in Lemma 4.1\([1]\). Each $g_i$ is in $C(K^i)$ and $\{g_i = 0\}$ is the affine subspace of $K^m$ defined by $\varphi_1 = \cdots = \varphi_i = 0$, hence $g_0 = \cdots \in g_i$. Let $B$ be a closed ball in $K^m$ centered at $q_0$ and contained in $V$. Take $\delta_B \subseteq C(K^m)$ with $\{\delta_B = 0\} = B$ as given by Lemma 4.1\([2]\). Finally let $f_i$ be the restriction of $\nu_2(\psi_i \circ \pi, \delta_B)$ to $X$, where $\nu_2$ is defined in Lemma 4.1\([1]\). Clearly $f_i \subseteq C(X)$, $f_0(q_0) = 0$ hence $f_0 \notin C(X)^x$, $\{f_k = 0\} \subseteq B$ is disjoint from $\{v = 0\}$, and $f_0 \in \cdots \in f_k$ because $\pi$ maps the zero-set of each $f_i$ homeomorphically to $\{g_i = 0\} \cap \pi(B)$.

\([2] \Rightarrow [1]\) Let $B_0$ be an open ball in $K^m$ centered at $p_0$. We have to prove that $\dim(B_0 \cap X) \geq k$. Let $v \in C(X)$ be such that $v(x) = 0$ on $X \setminus B_0$ and $v(p_0) \neq 0$ (e.g. one can choose $v = \delta_B$ as in Lemma 4.1\([2]\)). By assumption there are $f_0 \in \cdots \in f_k \subseteq C(X)$ such that $f_0 \notin C(X)^x$ and $\{f_k = 0\}$ is disjoint from $\{v = 0\}$, hence contained in $B_0 \cap X$. It then suffices to prove that each set $Z_i = \{f_i = 0\}$ has dimension $\geq i$. This holds for $Z_0$ because $f_0 \notin C(X)^x$ implies $Z_0 \neq \emptyset$. When $i > 0$, the set $Z_i$ is non-empty since it contains $Z_0$; further $Z_{i-1} \subseteq \partial(Z_i \setminus Z_{i-1})$ because $Z_{i-1} \subseteq Z_i$; consequently $\dim(Z_{i-1}) < \dim Z_i$ by (Dim2). Item \([1]\) now follows by induction on $i$.

Using Lemma 6.3 and Lemma 4.2 the equivalence of \([1]\) and \([2]\) implies the existence of a parameter-free formula $\chi_{\geq k}(p)$ in $\mathcal{L}_{\text{ring}}$, such that for every $p \in C(X)$ we have
\( \mathcal{C}(X) \models \chi_{\geq k}(p) \) just if \( p \) has a single zero \( p_0 \in X \) and \( \dim(X, p_0) \geq k \). We then take \( \chi_k \) as \( \chi_{\geq k} \land \neg \chi_{\geq k+1} \).

**Proposition 6.4** Every definable open subset \( U \) of \( K^r \) with \( r \geq 2 \) has arbitrarily many germs at every point \( p_0 \in U \).

**Proof:** For each \( k \geq 1 \) we need to define \( s_i \) and \( \delta_i \) verifying conditions \((\text{S}1)-(\text{S}4)\) of Definition 2.1. It suffices to do the case of \( K^r \) (and then take restrictions to \( U \) of the functions \( s_i \) and \( \delta_i \) built for \( K^r \)). Let \( V_L = K^r \), take a hyperplane \( H \subset K^r \) containing \( p_0 \) and \( k \) distinct lines \( L_1, \ldots, L_k \) passing through \( p_0 \) and not contained in \( H \). Let \( \sigma_i : K^r \to L_i \) be the projection onto \( L_i \) along \( H \). Now let \( s_i(x) = \nu_r(x - \sigma_i(x)) \) where \( \nu_r \) is given in Lemma 4.1(1). Conditions \((\text{S}1)\) and \((\text{S}2)\) are fulfilled since \( \{s_i = 0\} = L_i \) for each \( i \). Finally let \( \delta_i = \prod_{j \neq i} \delta_{i,j} \) where \( \delta_{i,j}(x) = s_j(x)/s_j(\sigma_i(x)) \). Clearly \( \delta_{i,j}(x) \in C(K^r \setminus \{p_0\}) \) and we have \( \delta_{i,j}(x) = 1 \) on \( L_i \setminus \{p_0\} \) and \( \delta_{i,j}(x) = 0 \) on \( L_j \setminus \{p_0\} \), hence \( \delta_i \) has properties \((\text{S}3)\) and \((\text{S}4)\).

The intuition coming from real geometry suggests that having local dimension \( \geq 2 \) should be necessary (and sufficient) for a definable set to have arbitrarily many germs at a point. In contrast, the next result shows that local dimension \( \geq 1 \) (which is obviously necessary) is sufficient at least if \( K \) has the following property, which holds true for example in \( P \)-minimal structures like the \( p \)-adics.

\((Z)\) \( v(K^\infty) \) is elementarily equivalent to \((\mathbb{Z}, +, \leq)\) (i.e., it is a \( Z \)-group) and for every definable set \( X \subseteq K^\infty \), \( v(X) \) is definable in \( v(K^\infty) \).

**Proposition 6.5** If \( K \) satisfies \((Z)\), then every definable open subset \( U \) of \( K \) has arbitrarily many germs at every point of \( U \).

**Proof:** For any \( p_0 \in U \), the set \( \{v(u - p_0) : u \in U \setminus \{p_0\}\} \) is a Presburger set by \((Z)\) and it is not bounded above in \( v(K^\infty) \). Hence it contains some set

\[ A = \{\xi \in v(K^\infty) : \alpha \leq \xi \text{ and } \xi \equiv a [N]\} \]

for some \( \alpha \in v(K^\infty) \) and some integers \( a \) and \( N \geq 1 \) (where \( \xi \equiv a [N] \) denotes the usual congruence relation \( \xi - a \in Nv(K^\infty) \)). Given an integer \( k \geq 1 \), for \( 1 \leq i \leq k \) let

\[ A_i = \{\xi \in A : \alpha \leq \xi \text{ and } \xi \equiv a + iN [kN]\} \]

and let \( S_i = v^{-1}(A_i) \cap U \). The sets \( S_i \) are pairwise disjoint, clopen in \( U \setminus \{p_0\} \), and \( p_0 \) belongs to the closure of each of them. So the functions defined by \( s_i(x) = 0 \) if \( x \in S_i \) and \( s_i(x) = \delta_{p_0}(x) \) otherwise (where \( \{\delta_{p_0} = 0\} = \{p_0\} \), see Lemma 4.1) are continuous on \( U \). Clearly each of them vanishes on a germ at \( p_0 \) since \( \{s_i = 0\} = S_i \cup \{p_0\} \).

**Remark 6.6** For all definable sets \( X \subseteq K^m \), \( Y \subseteq K^n \) and every point \( p_0 \in X \), if there is a definable homeomorphism \( \varphi : U \to V \) such that \( U \) (resp. \( V \)) is a definable neighborhood of \( p_0 \) in \( X \) (resp. of \( \varphi(p_0) \) in \( Y \)) then \( X \) has arbitrarily many germs at \( p_0 \) if and only if \( Y \) has arbitrarily many germs at \( \varphi(p_0) \), because this property is local. It then follows from Proposition 6.4 that for every definable set \( X \subseteq K^m \) and every integer \( k \geq 2 \), \( X \) has arbitrarily many germs at every point of \( W_k(X) \) for every \( k \geq 2 \) (and even for \( k = 1 \) if \( K \) satisfies \((Z)\) by Proposition 6.5).
For every integer \( k \geq 0 \), every definable set \( X \subseteq K^m \) and every \( Y \subseteq K \) let \( C_k(X, Y) \) (resp. \( C_{\geq k}(X, Y) \)) be the set of functions \( f \in \mathcal{C}(X) \) such that \( f(x) \in Y \) for every \( x \in \Delta_k(X) \) (resp. \( x \in \bigcup_{i \geq k} \Delta_i(X) \)), and \( f(x) = 0 \) otherwise.

**Theorem 6.7** Assume that \( \mathcal{K} \) satisfies \( [\text{Dim}] \) and \( [\text{BFin}] \). Let \( X \subseteq K^m \) be a definable set and let \( \tau \) be a non-zero element of \( K \).

1. If \( |\tau| \neq |1| \) then the set \( C_k(X, \tau Z) \) is definable in \( \mathcal{L}_{\text{ring}} \cup \{ B \} \) for every integer \( k \geq 2 \), hence so is \( C_{\geq 2}(X, \tau Z) \). If moreover \( \mathcal{K} \) satisfies \( [Z] \) the same holds true for \( k \geq 1 \).

2. If \( K \) has a total order \( \prec \) compatible with \( + \) and definable in \( \mathcal{L}_{\text{ring}} \cup \{ O \} \) then the set \( C_k(X, \tau Z) \) is definable in \( \mathcal{L}_{\text{ring}} \cup \{ B \} \) for every integer \( k \geq 2 \), hence so is \( C_{\geq 2}(X, \tau Z) \).

By Remark 4.4, Theorem 6.7 leads to definability in \( \mathcal{L}_{\text{ring}} \) in many cases (in particular if \( \mathcal{K} \) is an expansion of a local field or if the theory of \( \mathcal{K} \) is \( dp \)-minimal) provided \( K \) is not algebraically closed.

**Remark 6.8** The formulas given by the proof of Theorem 6.7 involve only one parameter: the function with constant value \( \tau \). Thus if \( \tau \in \mathbb{Z} \) these definitions of \( C_k(X, \tau Z) \) and \( C_{\geq 2}(X, \tau Z) \) are parameter-free.

**Proof:** Replacing \( \tau \) by \( 1/\tau \) if necessary we may assume that \( |\tau| < |1| \). Let \( \chi_k(p) \) and \( \text{Int}^\times(h, p, \tau^*) \) be the formulas given by Proposition 6.3 and Theorem 5.5 respectively. Consider the formula \( Z_k(h, \tau^*) \) defined as

\[
\forall p \left[ \chi_k(p) \rightarrow \forall v (p \not\subseteq v \rightarrow \exists q \not\subseteq v \land \text{Int}^\times(h, q, \tau^*)) \right].
\]

Given \( \tau \in K^\times \) with \( |0| < |\tau| < |1| \) let \( \tau^* \in \mathcal{C}(X) \) be the function with constant value \( \tau \). Let \( h \) be any function in \( \mathcal{C}(X) \) and \( D_h = \{ x \in X : h(x) \in \tau Z \} \).

Assume that \( \mathcal{C}(X) \models Z_k(h, \tau^*) \). For any point \( p_0 \in \Delta_k(X) \) and any open ball \( B_0 \subseteq K^m \) centered at \( p_0 \) let \( p = \delta_{p_0} \) and \( v = \delta_{B_0} \) be the functions from Lemma 4.1. Then \( \mathcal{C}(X) \) satisfies \( \chi_k(p) \) and \( p \not\subseteq v \); hence \( Z_k(h, \tau^*) \) gives \( q \in \mathcal{C}(X) \) having a single zero \( q_0 \) with \( q_0 \in B \) and \( h(q_0) \in \tau Z \). In other words \( p_0 \) belongs to the closure of \( D_h \). By continuity we obtain \( h(p_0) \in \tau Z \), hence \( h \in C_k(X, \tau Z) \).

Conversely assume that \( h \in C_k(X, \tau Z) \). Take \( p \in \mathcal{C}(X) \) with \( \mathcal{C}(X) \models \chi_k(p) \). Then \( h \) has a single zero \( p_0 \) and \( p_0 \in \Delta_k(X) \). For any \( v \in \mathcal{C}(X) \) with \( \mathcal{C}(X) \models \chi_k(p) \), by continuity, the set \( V = \{ v \neq 0 \} \) is a neighborhood of \( p_0 \) in \( X \). Hence by Lemma 6.1 there is a point \( q_0 \in V \cap W_k(X) \). In particular \( q_0 \in \Delta_k(X) \) by Property 2.5 and therefore \( h(q_0) \in \tau Z \). By Remark 6.6 the assumptions on \( \mathcal{K} \) ensure that \( X \) has arbitrarily many germs at \( p_0 \). Thus \( \mathcal{C}(X) \models \text{Int}^\times(h, q, \tau^*) \) with \( q = \delta_{q_0} \) defined in Lemma 4.1(2), which shows that \( \mathcal{C}(X) \models Z_k(h, \tau^*) \).

The proof of the second statement is similar. Replacing \( \tau \) by \( -\tau \) if necessary we may assume that \( \tau > 0 \). Then let \( \tau^* \in \mathcal{C}(X) \) be the function with constant value \( \tau \) and replace \( \text{Int}^\times(h, q, \tau^*) \) by \( \text{Int}^{\tau^*}_\prec(h, q, \tau^*) \) in \( Z_k(h, \tau^*) \). \( \Box \)
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