Alpha cluster condensation in $^{12}\text{C}$ and $^{16}\text{O}$

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A new $\alpha$-cluster wave function is proposed which is of the $\alpha$-particle condensate type. Applications to $^{12}\text{C}$ and $^{16}\text{O}$ show that states of low density close to the 3 resp. 4 $\alpha$-particle threshold in both nuclei are possibly of this kind. It is conjectured that all self-conjugate $4n$ nuclei may show similar features.

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Many properties of finite quantum systems such as nuclei, quantum dots, atomic clusters or ultracold gases in a trap are fairly well described within a mean-field approximation neglecting the correlations between the quasiparticles. However, sometimes correlations become strong, giving rise to the formation of clusters, and they have to be taken into account. An intriguing problem is when bosonic clusters as bound states of fermions are produced, and the Bose character of the composite clusters competes with the fermionic properties of their constituents. As an example, we will discuss the relevance of $\alpha$-like four-nucleon correlations in atomic nuclei. Special attention will be paid to such correlations which correspond to an $\alpha$-type condensate in low-density symmetric nuclear matter, similar to the Bose-Einstein condensation observed for finite numbers of bosonic atoms such as Rb or Na in traps.

It is a well known fact that in light nuclei many states are of the cluster type $\alpha\alpha$. In the case of cluster states of stable nuclei where we have only very few excess nucleons in addition to the clusters, they are all located close to or above the threshold energy of breakup into constituent clusters. This fact which is known as the threshold rule means that the inter-cluster binding is weak in cluster states. The threshold rule can be considered as a necessary condition for the formation of the cluster structure, because if the inter-cluster binding is strong the clusters overlap strongly and the clusters will loose their identities.

For example one of the fundamental questions of the cluster model is what kind of $\alpha$-particle cluster states can be expected to exist around the threshold energy $E_{\alpha\alpha}^{thr} = nE_{\alpha}$ of $n\alpha$ breakup in self-conjugate $4n$ nuclei. One possible answer to this question, which is strongly under debate, is the existence of the cluster state of a linear $n\alpha$ chain structure. The idea of the linear $\alpha$ chain state, originally due to Morinaga, is so fascinating that recently the formation of linear $6\alpha$ chain states in $^{24}\text{Mg}$ was studied extensively by experiments and also theoretically. The possibility of the linear $3\alpha$ chain state in $^{12}\text{C}$, which is the simplest linear $\alpha$ chain state was studied in detail by many authors solving the $3\alpha$ problem microscopically. However, these three-body studies all showed that the $3\alpha$-cluster states around the $3\alpha$ threshold energy $E_{3\alpha}^{thr}$ do not have a linear chain structure. For example it was found that the calculated second $0^+$ state in $^{12}\text{C}$, which corresponds to the observed second $0^+$ state located at 0.39 MeV above the $3\alpha$ threshold energy, has a structure where $\alpha$-clusters interact predominantly in relative S-waves. Thus it was concluded that the cluster state near $E_{3\alpha}^{thr}$ has not a linear chain structure but rather an $\alpha$-particle gas-like structure.

On the other hand there have been recent theoretical investigations on the possibility of $\alpha$-particle condensation in low density nuclear matter. In Röpke et al. made a variational ansatz for the solution of the intermediate 4-body equation. In Beyer et al. solved the Faddeev- Yakubovsky equations for an alpha-like cluster in nuclear matter. The outcome of these studies was that such $\alpha$-condensation can occur only in the low-density region below a fifth of the saturation value. At higher densities rather a state of ordinary $p-n$, $n-n$, or $p-p$ Cooper pairing will prevail. In view of these results it may be a tempting idea that in finite self-conjugate $4n$ nuclei one could expect the existence of excited states of dilute density composed of a weakly interacting gas of $\alpha$-particles. Since the $\alpha$-cluster is a Bose particle, such states could approximately be considered as an $n\alpha$ cluster condensed...
state and eventually excitations thereof.

The purpose of this paper is to report on our study which not only confirms that indeed the second 0+ state in 12C could be considered as such a condensed state but that in addition also in 16O such a state close to the threshold possibly exists. We will therefore then conjecture that the existence of such ω-condensed states might be a general feature in N=Z nuclei.

For the purpose of our study we write down a new type of α-cluster wave function describing an α-particle Bose condensed state:

$$\Phi_{na} = (C^\dagger_{na})^n |\text{vac}\rangle$$

where the α-particle creation operator is given by

$$C^\dagger_{\alpha} = \int d^3r_1 e^{-R^2/\beta^2} \int d^3r_2 \cdots d^3r_4 \times \varphi_\alpha(r_1 - R) a^\dagger_{\alpha \uparrow, \tau_1}(r_1) \cdots \varphi_\alpha(r_4 - R) a^\dagger_{\alpha \downarrow, \tau_4}(r_4)$$

with $$\varphi_\alpha(r) = (1/\sqrt{\pi b^2}) e^{-r^2/(2b^2)}$$ and $$a^\dagger_{\sigma \tau}(r)$$ being the creation operator of a nucleon with spin-isospin $$\sigma$$ at the spatial point $$r$$. The total na wave function therefore can be written as

$$\langle r_1 \sigma_1 \tau_1, \cdots r_{4n} \sigma_{4n} \tau_{4n} | \Phi_{na} \rangle \propto A \{ e^{-\frac{b^2}{4}}(X_1^2 + \cdots + X_{4n}^2) \phi(\alpha_1) \cdots \phi(\alpha_n) \},$$

where $$B = (b^2 + 2R_0^2)^{1/2}$$ and $$X_i = (1/4) \sum_{m} r_{in}$$ is the center-of-mass coordinate of the $$i$$-th α-cluster $$\alpha_i$$. The internal wave function of the α-cluster $$\alpha_i$$ is $$\phi(\alpha_i) \propto \exp[-(1/b^2) \sum_{m>n}(r_{im} - r_{in})^2]$$. The wave function of Eq.(3) is totally antisymmerized by the operator $$A$$. It is to be noted that the wave function of Eqs.(1,3) expresses the state where na-clusters occupy the same 0s harmonic oscillator orbit exp$$[-(1/4b^2)X^2]$$ with $$B$$ an independent variational width parameter. For example if $$B$$ is of the size of the whole nucleus whereas $$b$$ remains more or less at the free α-particle value (a situation encountered below), then the wave function (3) describes an na cluster condensed state in the macroscopic limit $$n \rightarrow \infty$$. For finite systems we know from the pairing case that such a wave function still can more or less reflect Bose condensation properties. Of course the total center-of-mass motion can and must be separated out of the wave function of Eq.(1) for finite systems. In the limiting case of $$B = b$$ (i.e. $$R_0 = 0$$), Eq.(3) describes a Slater determinant of harmonic oscillator wave functions. It may also be worth mentioning that for $$B \neq 0$$ the wave function (1,3) is different from the cluster state proposed by Brink [2].

The state $$|\Phi_{na}\rangle$$ has spin-parity 0+. In the limit of $$R_0 = 0$$, the normalized wave function $$|\Phi^N_{na}\rangle = |\Phi_{na}\rangle / \sqrt{|\Phi_{na}\rangle |\Phi_{na}\rangle}$$ is identical to a harmonic oscillator shell model wave function with the oscillator parameter $$b$$. For $$n = 3$$, it is identical to the p-shell wave function $$|(0s)^4(0p)^8|$$, and for $$n = 4$$, it is identical to the double closed shell wave function, $$|(0s)^4(0p)^{12}, 0^+\rangle$$. This is easily proved by noticing that these limit wave functions for $$n = 3$$ and 4 have maximum spatial symmetry [444] and [4444], respectively. Only 8Be has an α-particle structure in its ground state. Heavier 16O nuclei collapse to the dense state in their ground state but the individual α’s may reappear when these nuclei are diluted i.e. excited.

We calculated the energy surfaces in the two parameter space, $$R_0$$ and $$b$$, $$E_{\text{na}}(R_0, b) = \langle \Phi^N_{na}(R_0, b) | H | \Phi^N_{na}(R_0, b) \rangle$$, for $$n = 3$$ and 4. The Hamiltonian $$H$$ consists of the kinetic energy, the Coulomb energy, and the effective nuclear force named F1 which was proposed by one of the authors and contains a finite range three-nucleon force in addition to the finite range two-nucleon force [10]. This force reproduces reasonably well the binding energy and radius of the α-particle, the α-α phase shifts of various partial waves, and the binding energy and density of nuclear matter. As we will see below this force also gives good results for binding energies and radii of 13C and 16O.

In Fig.1 we give the contour maps of the energy surfaces $$E_{\text{na}}(R_0, b)$$ for 12C and 16O. The qualitative features of both surfaces are similar. They show a valley running from the outer region with large $$R_0 > 11$$ fm and $$b \approx b_0 = 1.44$$ fm to the inner region with small $$R_0$$ and $$b > b_0$$, where $$b_0$$ is the oscillator parameter of the free α-particle. The valleys have a saddle point at $$R_0 \approx 10$$ fm for $$n = 3$$ and at $$R_0 \approx 10.6$$ fm for $$n = 4$$. Beyond the saddle point, $$E_{\text{na}}(R_0, b_0) \approx E_{\text{thr}} = nE_\alpha$$, where $$E_\alpha = -27.5$$ MeV is the theoretical binding energy of the free α particle by the present F1 force in Hartree-Fock approximation. Therefore we have $$3E_\alpha = -82.5$$ MeV and $$4E_\alpha = -110$$ MeV. In the asymptotic region the average inter-α distance is large and the kinetic energy of the center-of-mass motion of an α-cluster (3h^2/4mB^2) is very small which leads to $$E_{\text{na}}(R_0, b_0)$$ being more or less equal to $$E_{\text{thr}}$$, the height of the saddle point measured from the theoretical threshold energy is about 1.4 MeV for $$n = 3$$ and 2.2 MeV for $$n = 4$$. The appearance of the saddle point is due to the increase of the Coulomb energy and kinetic energy towards the inward direction which is not yet compensated by the gain in potential energy around the saddle point region. This saddle point will help to stabilize the possible α-condensed state around $$E_{\text{thr}}$$. The minimum of the energy surface is located at $$R_0 \approx 2$$ fm for $$n = 3$$ and at $$R_0 \approx 1$$ fm for $$n = 4$$. Since $$R_0 = 0$$ means the shell model limit, we thus see that the wave function even at the energy minimum point deviates from the shell model limit and shows rather strong α-particle correlations. The gain in energy from the shell model limit is 10.3 MeV for 12C and 4.7 MeV for 16O. Before comparing numbers with experiments we have to make a quantum mechanical calculation. This will be achieved via a standard Hill-Wheeler ansatz taking $$R_0$$ and $$b$$ as the Hill-Wheeler coordinates. However, in order to re-
duce the complexity of the calculation and because the valleys run essentially parallel to the $R_0$ axis at $b = b_\alpha$ we take $b = b_\alpha$ = constant and only discretise the $R_0$ variable. We therefore have

$$|\Psi_{\alpha,k}(R_0,b)\rangle = \sum_j f_k((R_0)_j,b_\alpha)|\Phi_{n\alpha}^N((R_0)_j,b_\alpha)\rangle. \quad (4)$$

The normalization of $f_k((R_0)_j,b_\alpha)$ is so that the $k$-th eigen-function $|\Psi_{\alpha,k}(R_0,b)\rangle$ is normalized. The adopted mesh size of $R_0$ values is typically 0.5 fm. In order to see the character of the obtained wave function $|\Psi_{\alpha,k}(R_0,b)\rangle$, we introduce the overlap amplitude

$$A_{\alpha,k}(R_0,b) = \langle \Phi_{n\alpha}^N(R_0,b) | \Psi_{\alpha,k}(R_0,b) \rangle. \quad (5)$$

From this overlap amplitude we can estimate the relevant values of the variational parameter $R_0$ in the different states $k$ as will be discussed below.

After outlining the results for the new kind of wave function for $^{12}$C and $^{16}$O, we will discuss whether the obtained condensed states correspond to the states found in these nuclei. We first consider $^{12}$C, i.e. $n = 3$, see Table I. The calculated lowest two eigenenergies are situated at -85.9 and -82.0 MeV. The lowest energy state corresponds to the ground state of $^{12}$C and is only slightly lower than the minimum point of the energy surface located at -85.5 MeV. However, the calculated ground state is still above the observed binding energy of $^{12}$C which is at -92.16 MeV. An increase of mesh points will certainly lower the energy but, as has been discussed by many people, in order to reproduce the observed $^{12}$C binding energy satisfactorily we have to extend our functional space as to include the spatial symmetry broken wave functions which allow to incorporate the effect of the spin-orbit force adequately. The second eigenvalue lies 0.36 MeV above our theoretical 3o threshold energy, $E_{3o}^{thr} = -82.5$ MeV, and we believe that it corresponds to the observed second $0^+$ state of $^{12}$C which lies 0.5 MeV above $E_{3o}^{thr}$. As seen in Table I the rms radius of the obtained wave function $|\Psi_{3o,2}(R_0,b)\rangle$ is 4.29 fm which is much larger than the one of the ground state which is 2.97 fm, slightly greater than the experimental value 2.45 fm but in agreement with the missing binding of 6.75 MeV. We thus see that the second $0^+$ state corresponds to a very dilute system of average density which is only about a fifth of the experimental ground state density.

To characterise the wave function by a typical value of the width parameter $R_0$, we consider the overlap amplitude $A_{3o,k}(R_0,b_\alpha)$ given in Eq.(6) as a function of $R_0$ at fixed $b_\alpha$. Whereas the ground state ($k = 1$) wave function is almost exhausted by one $\Phi_{3o}^N(R_0,b_\alpha)$ with $R_0 \approx 2$ fm which is quite close to the wave function of the minimum energy point of the energy surface, the second $0^+_k$ state has the largest overlap amplitude (about 0.87) with $R_0 \approx 4.5$ fm. This rather large value implies that the distribution of the center-of-mass momenta is rather narrow, in a certain approximation to an $\alpha$ condensate in infinite nuclear matter where all $\alpha$-clusters populate the same state $P = 0$ of the center-of-mass momentum. The fact that the calculated $0^+_k$ state is of dilute density is in agreement with nuclear matter calculations [30] where it was shown that a condensate of $\alpha$-like particles (quartetting) is possible only in matter with density $\rho \leq 0.03$ fm$^{-3}$. The average distance of the $\alpha$-clusters in the dilute $0^+_k$ state is in agreement with this value for low density nuclear matter, where the overlap of the $\alpha$-clusters is small so that the Pauli blocking effects are weak.

Let us now discuss the case of $^{16}$O, i.e. $n = 4$. The energies of the lowest observed $0^+$ states are shown in Table II, together with the corresponding widths. The first excited $0^+_2$ state at 6.06 MeV is very well known to have $\alpha$-clustering character [11] and is well described by the $^{12}$C + $\alpha$ microscopic cluster model as having the structure where the $\alpha$-cluster moves in a S state around the $^{12}$C-cluster in its ground state [12] though also other cluster states have been proposed [33]. Similarly, the third excited $0^+_3$ state at 12.05 MeV can be described by the same model where the $\alpha$-cluster moves in a D state around the $^{12}$C-cluster in its first $2^+$ excited state [12]. We will exclude these well understood states from our further discussion. The excited states $0^+_4$ at 11.26 MeV and $0^+_5$ state at 14.0 MeV observed in $^{12}$C + $\alpha$ elastic scattering [13] cannot be described by such a model. Furthermore, they have very large decay widths, not typical for the other states. These states may be described by our new wave function as condensed states.

As seen from Table I, the experimental value of the ground state ($0^+_1$) of $^{16}$O at -127.62 MeV is well reproduced by the calculated energy value for the ground state ($0^+_1$) at -124.8 MeV. The calculated energy is above the minimum energy of the energy surface. It is because the $b$ value of the minimum energy point is fairly larger than $b_\alpha$ and the minimum energy point is not covered by the adopted mesh points. In order to have a better wave function for the ground state we, of course, need to include mesh points around the minimum energy in our generator coordinate calculation. When we adopt $b = 1.57$ fm which is the $b$ value of the energy minimum point, the generator coordinate calculation gives -128.0 MeV as the lowest eigen energy. The rms radius of the calculated ($0^+_1$) state is 2.59 fm and is slightly smaller than the observed ($0^+_1$) rms radius, 2.73 fm, of $^{16}$O. The second ($0^+_2$) state of our calculation is bound by 6 MeV below the theoretical 4o threshold energy. The rms radius of this state is 3.12 fm and this state has the largest overlap amplitude (about 0.86) with $\Phi_{4o}^N(R_0,b_\alpha)$ with $R_0 \approx 2.5$ fm. We conjecture that this state corresponds to the observed $0^+_1$ state situated at 3.18 MeV below the observed 4o threshold energy. Indeed one may argue that there will be some mixing between the second ($0^+_2$) state and the $^{12}$C+$\alpha$ state, bringing theoretical and experimental energies closer together. The third ($0^+_3$) state of our
calculation is bound by 0.7 MeV below the theoretical $4\alpha$ threshold energy. We think that it may correspond to the measured $0^+_5$ state situated at 0.44 MeV below the observed $4\alpha$ threshold energy. This state has a very large rms radius of 3.94 fm and has the largest overlap amplitude (about 0.80) with $\Phi^N_4(R_0, b_\alpha)$ with $R_0 \approx 4.1$ fm. In analogy to the case of $^{12}\text{C}$, these values indicate that this state of dilute density should be considered as just the $4\alpha$-cluster condensed state that we expected. One should point out that the ease with which we get the $0^+_\text{states}$ around $E_{\text{thr}}^{\text{n}\alpha}$ is a strong indication that our wave function (1,3) grasps the essential physics because otherwise the threshold states are very difficult to obtain. We also would like to mention that the present formalism yields very good results for the groundstate of $^8\text{Be}$ as well.

In conclusion, our present study thus predicts in $^{12}\text{C}$ and $^{16}\text{O}$ the existence of near-$n\alpha$-threshold states which are the finite system analogues to $\alpha$-cluster condensation in infinite matter. They are characterized by low density states so that the $\alpha$-clusters are not strongly overlapping and by an $n$-fold occupation of their identical S-wave center-of-mass wave function. Therefore these states are quite similar in structure to the Bose-Einstein condensed states of bosonic atoms in magnetic traps where all atoms populate the same lowest S-wave quantum orbital. Because of the short life-time of the $\alpha$-condensed states, the predicted large values for the rms radii may be verified by indirect methods. The measurements of the spectra of the emitted $\alpha$ particles should allow to determine the Coulomb barrier which is expected to be small for the low density states. Of particular interest would be $\alpha-\alpha$ coincidence measurement of decaying condensed states.

We conjecture that such condensed $\alpha$-cluster states near the $n\alpha$ threshold may also occur in other heavier $4n$ self-conjugate nuclei. For example condensed $6\alpha$ states of $^{24}\text{Mg}$ could be deformed and the measurement of a reduced moment of inertia over the rigid body value would be a strong indication for $\alpha$-particle superfluidity. The wave function we have proposed in this work is very flexible and can straightforwardly be adopted for the description of other condensation phenomena such as for example ordinary Cooper pairing or a mixture of Cooper pair and $\alpha$-particle condensation.

[1] K. Wildermuth and Y. C. Tang, *A Unified Theory of the Nucleus* (Vieweg Braunschweig, 1977).
[2] D. M. Brink, Proc. Int. School Phys. Enrico Fermi 36 (Academic Press, New York, 1966);
[3] G. F. Bertsch and W. Bertozzi, Nucl. Phys. A165, 199 (1971);
[4] Y. Fujiwara, H. Horiuchi, K. Ikeda, M. Kamimura, K. Kato, Y. Suzuki, and E. Uegaki, Prog. Theor. Phys. Supplement No.68, 29 (1980).
[5] K. Ikeda, N. Takigawa, and H. Horiuchi, Prog. Theor. Phys. Supplement Extra Number, 464 (1968).
[6] H. Morinaga, Phys. Rev. 101, 254 (1956); Phys. Letters 21, 78 (1966).
[7] B. R. Fulton, Proc. 7th Int. Conf. on Clustering Aspects in Nuclear Structure and Dynamics, Rab (1999), edited by M. Koroliya, Z. Basrak, and R. Čaplar, (World Scientific, Singapore), p.122.
[8] G. Röpke, A. Schnell, P. Schuck, and P. Nozières, Phys. Rev. Letters 80, 3177 (1998).
[9] M. Beyer, S. A. Sofianos, C. Kuhrts, G. Röpke, and P. Schuck, Phys. Letters B488, 247 (2000).
[10] A. Tohsaki, Phys. Rev. C49, 1814 (1994).
[11] F. Ajzenberg-Selove, Nucl. Phys. A460, 1 (1986).
[12] Y. Suzuki, Prog. Theor. Phys. 55, 1751 (1976); 56, 111 (1976).
\[ E_{\text{exc}} = E_k - E_{\text{thr}}^k = \langle r^2 \rangle \text{exp} \left( \frac{E}{\text{MeV}} \right) (\text{fm}) \]

|        | \( E_k \) (MeV) | \( E_{\text{exp}} \) (MeV) | \( E_k - E_{\text{thr}}^k \) (MeV) | \( (E - E_{\text{thr}}^k)_{\text{exp}} \) (MeV) | \( \sqrt{\langle r^2 \rangle} \) (fm) | \( \sqrt{\langle r^2 \rangle}_{\text{exp}} \) (fm) |
|--------|-----------------|-----------------|-----------------|-----------------|----------------|----------------|
| \(^{14}\text{C}\) | \( k = 1 \) | -85.9 | -92.16 \( (0^+_1) \) | -3.4 | -7.27 | 2.97 | 2.65 |
|        | \( k = 2 \) | -82.0 | -84.51 \( (0^+_2) \) | +0.5 | 0.38 | 4.29 |
|        | \( E_{\text{thr}}^k \) | -82.5 | -84.89 |        |       |       |
| \(^{16}\text{O}\) | \( k = 1 \) | -124.8 | -127.62 \( (0^+_1) \) | -14.8 | -14.44 | 2.59 | 2.73 |
|        | (\(-128.0\))* | (\(-18.0\))* |        |       |       |
|        | \( k = 2 \) | -116.0 | -116.36 \( (0^+_2) \) | -6.0 | -3.18 | 3.16 |
|        | \( k = 3 \) | -110.7 | -113.62 \( (0^+_3) \) | -0.7 | -0.44 | 3.97 |
|        | \( E_{\text{thr}}^k \) | -110.0 | -113.18 |        |       |       |

**TABLE I.** Comparison of the generator coordinate method calculations with experimental values. \( E_{\text{thr}}^k = nE_\alpha \) denotes the threshold energy for the decay into \( \alpha \)-clusters, the values marked by * correspond to a refined mesh, see main text.

|        | \( E_{\text{exc}} \) (MeV) | \( \Gamma \) (MeV) |
|--------|-----------------|----------------|
| \( 0^+_2 \) | 6.06 |       |
| \( 0^+_3 \) | 11.26 | 2.6 |
| \( 0^+_4 \) | 12.05 | 1.6\times10^{-3} |
| \( 0^+_5 \) | 14.0 | 4.8 |
| \( 0^+_6 \) | 14.03 | 2.0\times10^{-1} |

**TABLE II.** Observed excitation energies \( E_{\text{exc}} \) and widths \( \Gamma \) of the lowest five \( 0^+ \) excited states in \(^{16}\text{O}\).
FIG. 1. Contour map of the energy surface $E_{3\alpha}(R_0,b)$ for $^{12}$C. Numbers attached to the contour lines are the binding energies.

FIG. 2. Contour map of the energy surface $E_{4\alpha}(R_0,b)$ for $^{16}$O. Numbers attached to the contour lines are the binding energies.