THE THETA INVARIANTS AND THE VOLUME FUNCTION ON ARITHMETIC VARIETIES

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ABSTRACT. We introduce a new arithmetic invariant for hermitian line bundles on an arithmetic variety. We use this invariant to measure the variation of the volume function with respect to the metric. The main result of this paper is a generalized Hodge index theorem on arithmetic toric varieties.

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1. INTRODUCTION

Let \( \mathcal{X} \) be an arithmetic variety over \( \text{Spec}(\mathbb{Z}) \), that is a projective, integral and flat scheme over \( \mathbb{Z} \). We assume that \( \mathcal{X}_\mathbb{Q} \) is smooth over \( \mathbb{Q} \). Let \( n + 1 \) be the absolute dimension of \( \mathcal{X} \). Let \( \mathcal{L} = (\mathcal{L}, \| \cdot \|) \) be a continuous hermitian line bundle on \( \mathcal{X} \). For any \( k \in \mathbb{N}_{\geq 1} \), \( k \mathcal{L} \) denotes \( \mathcal{L}^{\otimes k} \).

Moriwaki in [16] introduced the arithmetic volume \( \widehat{\text{vol}}(\mathcal{L}) \) for \( \mathcal{C}^\infty \) hermitian line bundle \( \mathcal{L} \) which is an analogue of the geometric volume function. The arithmetic volume \( \widehat{\text{vol}}(\mathcal{L}) \) is defined by

\[
\widehat{\text{vol}}(\mathcal{L}) = \limsup_{k \to \infty} \frac{\log \# \{ s \in H^0(\mathcal{X}, k \mathcal{L}) \mid \| s \|_{\text{sup}, \phi} \leq 1 \}}{k^{n+1} / (n + 1)!}.
\]

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Chen in [7] proved that the arithmetic volume function is actually a limit.

Gillet and Soulé in [11] defined arithmetic Chow groups \( \hat{CH}^p(X) \) for any \( p \geq 0 \). Let \((E, h)\) be a hermitian vector bundle on \( X \). When \( h \) is smooth, one can attach to \((E, h)\), arithmetic characteristic classes such as the \((n + 1)\)-th power of first Chern character \( \hat{c}_1(E, h)^{n+1} \in \hat{CH}^{n+1}(X)_{\mathbb{Q}} \), see [4, 12].

If \( X \) is regular, \( L \) is ample on \( X \) and the metric of \( \overline{L} \) is smooth with positive first Chern form \( c_1(\overline{L}) \) on \( X \), then

\[
\hat{\text{vol}}(\overline{L}) \geq \hat{\text{deg}}(\hat{c}_1(\overline{L})^{n+1}).
\]

This inequality can be obtained using the arithmetic Riemann-Roch theorem due to Gillet-Soulé [13], or the arithmetic Hilbert-Samuel formula due to Abbes-Bouche [1]. In [17, 18], Moriwaki proved that the arithmetic volume function is continuous with respect to \( \overline{L} \).

The \( \chi \)-arithmetic volume of \( \hat{\text{vol}}_\chi(\overline{L}) \) is defined as follows

\[
\hat{\text{vol}}_\chi(\overline{L}) = \lim_{k \to \infty} \frac{\hat{\chi}(H^0(X, kL), \| \cdot \|_{\sup, k\phi})}{k^{n+1/(n+1)!}},
\]

(for the definition of \( \hat{\chi}(H^0(X, kL), \| \cdot \|_{\sup, k\phi}) \), see for instance [8, 20]).

It is known that the inequality

\[
\hat{\text{vol}}(\overline{L}) \geq \hat{\text{vol}}_\chi(\overline{L}).
\]

holds for any hermitian line bundle \( \overline{L} \) on \( X \).

When \( X = \mathbb{P}^1_\mathbb{Z} \), and \( \overline{L} \) is a toric DSP line bundle on \( \mathbb{P}^1_\mathbb{Z} \), that is a difference of semipositive ones such that metric is invariant under the action of the compact torus of \( \mathbb{P}^1 \) (see [6] for more details), then

\[
\hat{\text{vol}}_\chi(\overline{L}) \geq \hat{\text{deg}}(\hat{c}_1(\overline{L})^2),
\]

(see [14]).

According to [16], there are three kinds of positivity of \( \overline{L} = (L, \| \cdot \|) \).

- **ample**: \( \overline{L} \) is ample if \( L \) is ample on \( X \), the first Chern form \( c_1(\overline{L}) \) is positive on \( X(\mathbb{C}) \) and, for a sufficiently large integer \( k \), \( H^0(X, kL) \) is generated by the set

\[
\{ s \in H^0(X, kL) \mid \| s \|_{\sup} < 1 \},
\]

as a \( \mathbb{Z} \)-module.
• **nef**: \( \mathcal{L} \) is nef if the first Chern form \( c_1(\mathcal{L}) \) is semipositive and \( \hat{\deg}(\mathcal{L}|_{\Gamma}) \geq 0 \) for any 1-dimensional closed subscheme \( \Gamma \) in \( \mathcal{X} \).

• **big**: \( \mathcal{L} \) is big if \( \mathcal{L} \) is big on \( \mathcal{X} \) and \( \hat{\deg}(\mathcal{L}|_{\Gamma}) \geq 0 \) for any 1-dimensional closed subscheme \( \Gamma \) in \( \mathcal{X} \).

Let \( \mathcal{L} = (\mathcal{L}, \| \cdot \|_{\phi}) \) and \( \mathcal{N} = (\mathcal{N}, \| \cdot \|_{\psi}) \) be two \( C^\infty \)-hermitian line bundles on \( \mathcal{X} \). If \( \mathcal{L} \) is ample, we have the following asymptotic expansion

\[
\hat{\chi}(H^0(\mathcal{X}, k\mathcal{L} + \mathcal{N}), \| \cdot \|_{\text{sup}, k\phi+\psi}) = \frac{1}{(n+1)!} \hat{\deg}(\hat{c}_1(\mathcal{L})^{n+1}) k^{n+1} + o(k^{n+1}),
\]

(see [1, 10, 22]). As a consequence,

\[
\hat{h}^0(H^0(\mathcal{X}, k\mathcal{L} + N), \| \cdot \|_{\text{sup}, k\phi+\psi}) = \frac{1}{(n+1)!} \hat{\deg}(\hat{c}_1(\mathcal{L})^{n+1}) k^{n+1} + o(k^{n+1}),
\]

as \( k \to \infty \), see [17] Lemma 3.3. It is worth pointing out that the proof of (3) relies heavily on the asymptotic (2).

Moriwaki generalized (1) to \( C^\infty \)-hermitian line bundles \( \mathcal{L} \) on \( \mathcal{X} \) such that \( \mathcal{L} \) is nef on the generic fiber of \( \mathcal{X} \), \( c_1(\mathcal{L}) \) is semipositive on \( \mathcal{X} \), and \( \mathcal{L} \) has moderate growth of positive even cohomologies, see [17, Theorem A]. If \( \mathcal{L} \) is nef, then

\[ \hat{\text{vol}}(\mathcal{L}) = \hat{\deg}(\hat{c}_1(\mathcal{L})^{n+1}), \]

see [17] Theorem C].

One of the goals of this paper is to recover the results of Abbes-Bouche, Moriwaki, and Zhang using a different approach. Let \( \mathcal{L} = (\mathcal{L}, \| \cdot \|_{\infty}) \) be a hermitian line bundle on an arithmetic variety \( \mathcal{X} \) endowed with a measure \( \mu \). To this data, we attach the invariant \( \Theta(\mu, \phi) \), see Definition 4.4. This new invariant can be seen as an arithmetic analogue of the distortion function \( \rho(\mu, \phi) \). We prove in Corollary 4.8 that

\[
\limsup_{k \to \infty} \frac{\Theta(\mu, k\phi)}{k^n} \leq \mu_{\text{eq}}(\mathcal{X}, \phi),
\]

with equality when \( \mathcal{L} \) is ample, see Theorem 4.9 where \( \mu_{\text{eq}}(\mathcal{X}, \phi) \) is the equilibrium measure of \( \phi \).

The introduction of \( \Theta(\mu, \phi) \) allows us to recover (1) in the case when \( \mathcal{X} \) is toric, and \( \mathcal{L} \) is an equivariant line bundle on \( \mathcal{X} \). More precisely, we obtain the result.

**Theorem 1.1 (Main Theorem).** (see Theorem 5.5) Let \( \mathcal{X} \) be an arithmetic toric variety over \( \mathbb{Z} \) of relative dimension \( n \). Let \( \mathcal{L} \) be an equivariant ample line bundle on \( \mathcal{X} \). We
assume that the metric of $\mathcal{L}$ is smooth with semi-positive first Chern form $c_1(\mathcal{L})$ on $X$, then
\[ \widehat{\text{vol}}(\mathcal{L}) \geq \widehat{\deg}(\mathcal{c}_1(\mathcal{L})^{n+1}). \]

2. THE DISTORTION FUNCTION AND THE VARIATION OF THE ARITHMETIC DEGREE

Let $X$ be a compact complex manifold of dimension $n$. Let $\mu$ be a probability measure with non-pluripolar support on $X$. Let $L$ be a holomorphic line bundle on $X$. A weight $\phi$ on $L$ is a locally integrable function on the complement of the zero-section in the total space of the dual line bundle $L^*$ satisfying the log-homogeneity property
\[ \phi(\lambda v) = \log |\lambda| + \phi(v) \]
for all non-zero $v \in L^*$, $\lambda \in \mathbb{C}$. Let $\phi$ be a weight function on $L$. $\phi$ defines a hermitian metric on $L$, which we denote by $\| \cdot \|_{\phi}$.

Let $(L, \| \cdot \|_{\phi})$ be a hermitian line bundle on $X$, where $\phi$ is the weight of the metric of $L$. We endow the space of global sections $H^0(X, L)$ with the $L^2$-norm
\[ \|s\|_{(\mu, \phi)}^2 := \int_X \|s(x)\|^2_{\phi} \mu. \]
Also we consider the sup norm defined as follows
\[ \|s\|_{\text{sup}, \phi} := \sup_{x \in X} \|s(x)\|_{\phi}. \]
for any $s \in H^0(X, L)$.

The Bergman distortion function $\rho(\mu, L)$ is by definition the function given at a point $x \in X$ by
\[ \rho(\mu, \phi)(x) = \sup_{s \in H^0(X, L) \setminus \{0\}} \frac{\|s(x)\|^2_{\phi}}{\|s\|^2_{(\mu, \phi)}}. \]
If $\{s_1, \ldots, s_N\}$ is a $(\mu, \phi)$-orthonormal basis of $H^0(X, L)$, then it is well known that
\[ \rho(\mu, \phi)(x) = \sum_{j=1}^N \|s_j(x)\|^2_{\phi} \quad \forall x \in X. \]

We say that $\mu$ has the Bernstein-Markov property with respect to $\| \cdot \|_{\phi}$ if for any $\varepsilon > 0$ we have
\[ \sup_X \rho(\mu, k\phi)^{\frac{1}{2}} = O(e^{k\varepsilon}). \]
If $\mu$ is a smooth positive volume form and $\| \cdot \|_{\phi}$ is a continuous metric on $L$ then $\mu$ has the Bernstein-Markov property with respect to $\| \cdot \|_{\phi}$ (see [3, Lemma 3.2]).
Let $\phi$ be a weight of a continuous hermitian metric $\| \cdot \|_\phi$ on $L$. When $\phi$ is smooth, we define the Monge-Ampère operator as
$$\text{MA}(\phi) := (dd^c \phi)^n.$$

The equilibrium weight of $\phi$ is defined as
$$P_X \phi := \sup \{ \psi \mid \psi \text{ psh weight on } L, \psi \leq \phi \text{ on } X \}.$$ It is known that the equilibrium weight is upper semicontinuous psh weight with minimal singularities. The equilibrium measure of $\phi$ is defined by
$$\mu_{eq}(\phi) := \frac{1}{\text{vol}(L)} \text{MA}(P_X \phi).$$

Let $\mu$ be a smooth positive volume form on $X$ and $\phi$ a $C^2$ weight on $L$. We have
$$\lim_{k \to \infty} \frac{1}{k^{\dim X}} \rho(\mu, k\phi) \mu = \frac{1}{\text{vol}(L)} \text{MA}(P_X \phi),$$
in the weak topology of measures (see for instance [3, Theorem 3.1]).

Let $\| \cdot \|_{L_0}$ and $\| \cdot \|_{L_1}$ be two smooth hermitian metrics on $L$. We define the Monge-Ampère functional $\mathcal{E}$ by the formula

$$\mathcal{E}(\mathcal{T}_1) - \mathcal{E}(\mathcal{T}_0) := \frac{1}{n+1} \sum_{j=0}^{n} \int_X - \log \frac{\| \cdot \|_{\mathcal{T}_1}}{\| \cdot \|_{\mathcal{T}_0}} c_1(\mathcal{T}_0)^j \wedge c_1(\mathcal{T}_1)^{n-j}.$$

An admissible metric $\| \cdot \|$ on a holomorphic line bundle $L$ is, by definition, a uniform limit of a sequence $(\| \cdot \|_n)_{n \in \mathbb{N}}$ of positive and smooth hermitian metrics $L$. An admissible line bundle $(L, \| \cdot \|)$ is a line bundle $L$ endowed with an admissible metric $\| \cdot \|$. We say that $\mathcal{L} = (L, \| \cdot \|)$ is an integrable line bundle if there exist $\mathcal{T}_1$ and $\mathcal{T}_2$, two admissible line bundles on $X$ such that
$$\mathcal{L} = \mathcal{T}_1 \otimes \mathcal{T}_2^{-1}.$$ By the theory of Bedford-Taylor [2], (4) extends to admissible metrics, and hence to integrable ones by polarisation.

Let $\phi_{\mathcal{T}_0}$ and $\phi_{\mathcal{T}_1}$ be the associated weights of $\| \cdot \|_{\mathcal{T}_0}$ and $\| \cdot \|_{\mathcal{T}_1}$ respectively. Following [3], when $L$ is big we set
$$\mathcal{E}_{eq}(\mathcal{T}_1) - \mathcal{E}_{eq}(\mathcal{T}_0) := \frac{1}{\text{Vol}(L)} (\mathcal{E}(\mathcal{T}_1)_X) - \mathcal{E}(\mathcal{T}_0)_X),$$
where $(\mathcal{T}_i)_X$ denotes the line bundle $L$ endowed with the weight $P_X \phi_{\mathcal{T}_i}$, the equilibrium weight of $\phi_{\mathcal{T}_i}$ for $i = 0, 1$. In [3, §1.3], $\mathcal{E}_{eq}(\mathcal{L})$ is called the energy at equilibrium of $(X, \phi_{\mathcal{L}})$.
(ϕ_τ is the weight of L).

**Theorem 2.1.** Let \( \mathcal{X} \) be an arithmetic variety over \( \mathbb{Z} \) of dimension \( n + 1 \). Let \( \mathcal{L} \) be a line bundle on \( \mathcal{X} \). Let \( \| \cdot \| \) and \( \| \cdot \|' \) be two integrable metrics on \( \mathcal{L} \). We have

\[
\text{deg}(\hat{c}_1(\mathcal{L}, \| \cdot \|)^{n+1}) - \text{deg}(\hat{c}_1(\mathcal{L}, \| \cdot \|')^{n+1}) = \sum_{i+j=p-1} \int_{\mathcal{X}(\mathbb{C})} \varphi c_1(\mathcal{L}, \| \cdot \|)^i c_1(\mathcal{L}, \| \cdot \|')^j.
\]

where \( \varphi \) is such that \( \| \cdot \|' = e^\varphi \| \cdot \| \)

**Proof.** Let \( \| \cdot \| \) and \( \| \cdot \|' \) be two smooth metrics on \( \mathcal{L} \). where \( \varphi \) is such that \( \| \cdot \|' = e^\varphi \| \cdot \| \), see [4, Proposition 3.2.2]. When the metrics are integrable, then (5) can be generalized to integrable metrics. This is an easy combination of [15, Proposition 5.5.2, (2),(3)], and (5).

\[\square\]

3. **The theta invariants associated with euclidean lattices**

Let \( \nabla \) be a hermitian vector bundle over Spec(\( \mathbb{Z} \)), that is a finitely generated \( \mathbb{Z} \)-module \( V \) which is equipped with a hermitian norm which is invariant under complex conjugation, on the complex vector space

\[ V \otimes_{\mathbb{Z}} \mathbb{C}. \]

Let \( \lambda_{\nabla} \) be the unique translation-invariant Radon measure on \( V_{\mathbb{R}} \) which satisfies the following normalization condition: for every orthonormal basis \( \{ e_1, \ldots, e_N \} \) of \( (V_{\mathbb{R}}, \| \cdot \|_{\nabla}) \),

\[ \lambda_{\nabla} \left( \sum_{i=1}^N [0, 1][e_i] \right) = 1. \]

We set

\[ \text{covol}(\nabla) := \lambda_{\nabla} \sum_{i=1}^N [0, 1][v_i], \]

for every \( \mathbb{Z} \)-basis \( \{ v_1, \ldots, v_N \} \) of \( V \). \( \text{covol}(\nabla) \) is called the covolume of \( \nabla \).

We set

\[ \theta_{\nabla}(t) = \sum_{v \in V} e^{-\pi t \| v \|_{\nabla}^2}. \]

We have the following identity

\[
\sum_{v \in V} e^{-\pi \| v \|_{\nabla}^2} = (\text{covol}(\nabla))^{-1} \sum_{v' \in V'} e^{-\pi \| v' \|_{\nabla'}^2},
\]

which is a consequence of the Poisson formula ([5, (2.1.2)]).
Let $\overline{E} = (E, \| \cdot \|)$ be hermitian vector bundle over $\mathbb{Z}$. $\overline{E}$ is called also an euclidean lattice. We let

$$h_0^0(\overline{E}) := \log \sum_{v \in E} e^{-\pi \|v\|^2_{\overline{E}}},$$

and

$$h_0^1(\overline{E}) := \log \sum_{v^\vee \in E^\vee} e^{-\pi \|v^\vee\|^2_{\overline{E}}},$$

and

$$\widetilde{\deg}(\overline{E}) := -\log \text{covol}(\overline{E}),$$

and

$$\hat{h}_0^0(\overline{E}) := \log |\{v \in E \mid \|v\| \leq 1\}|.$$

The equation (6) may be written in terms of the $\theta$-invariants $\hat{h}_0^0(\overline{E})$ and $\hat{h}_0^1(\overline{E})$, and the Arakelov degree $\widetilde{\deg}(\overline{E})$ as follows

$$h_0^0(\overline{E}) - h_1^0(\overline{E}^\vee) = \widetilde{\deg}(\overline{E}).$$

On the other hand, we have

$$h_0^0(\overline{E}) - \frac{1}{2} \text{rk } E \log \text{rk } E + \log(1 - \frac{1}{2\pi}) \leq \hat{h}_0^0(\overline{E}) \leq h_0^0(\overline{E}) + \pi,$$

(see [5, Theorem 3.1.1]).

**Lemma 3.1.** For $t > 0$,

$$|\log \theta_{\overline{E}}(t) - \log \theta_{\overline{E}}(1)| \leq \frac{1}{2} \text{rk } E \cdot \log t.$$

**Proof.** This lemma follows from the fact that $\log \theta_{\overline{E}}(t)$ is a decreasing function of $t$ in $\mathbb{R}^*_+$, and

$$\log \theta_{\overline{E}}(t) + \frac{1}{2} \text{rk } E \cdot \log t$$

is an increasing function of $t$ in $\mathbb{R}^*_+$, see [5, Lemma 3.1.4].

**Lemma 3.2.** For all $t > 0$, we have

$$\sum_{v \in E} \|v\|^2 e^{-\pi t \|v\|^2} \leq \frac{\text{rank}(E)}{2 \pi t} \sum_{v \in E} e^{-\pi t \|v\|^2}.$$

**Proof.** See [5, (3.1.5)].
4. Theta invariants of hermitian line bundles on arithmetic varieties

Let $\mathcal{X}$ be an arithmetic variety over $\mathbb{Z}$ of dimension $n + 1$. Let $\mathcal{L} = (\mathcal{L}, \| \cdot \|)$ be a hermitian line bundle on $\mathcal{X}$. We assume that the $\mathbb{Z}$-module $H^0(\mathcal{X}, k\mathcal{L})$ is a torsion-free for every $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, $N_k$ denotes the rank of $H^0(\mathcal{X}, k\mathcal{L})$. We set $X := \mathcal{X}(\mathbb{C})$, and $L := \mathcal{L}(\mathbb{C})$. Let $\mu$ be a probability measure with non-pluripolar support on $X$. We denote by $\phi$ the weight of the metric of $\mathcal{L}$, and we write $\| \cdot \|_\phi$ instead of $\| \cdot \|$. 

For any $k \geq 1$, let $\overline{H^0(\mathcal{X}, k\mathcal{L})}_{(\mu,k\phi)}$ (resp. $\overline{H^0(\mathcal{X}, k\mathcal{L})}_{(\sup,k\phi)}$) be the hermitian vector bundle $H^0(\mathcal{X}, k\mathcal{L})$ over $\mathbb{Z}$ equipped with the $L^2$-norm $\| \cdot \|_{(\mu,k\phi)}$ (resp. the sup-norm $\| \cdot \|_{\sup,k\phi}$).

We let

$$h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{(\mu,k\phi)} \right) := \log \sum_{v \in H^0(\mathcal{X}, k\mathcal{L})} e^{-\pi \|v\|^2_{(\mu,k\phi)}},$$

and

$$h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup,k\phi} \right) := \log \sum_{v \in H^0(\mathcal{X}, k\mathcal{L})} e^{-\pi \|v\|^2_{\sup,k\phi}}.$$

**Theorem 4.1.** Assume that $\mu$ has the Bernstein-Markov property with respect to $\| \cdot \|_\phi$. We have

$$\limsup_{k \to \infty} \frac{h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{(\mu,k\phi)} \right)}{k^{n+1}/(n+1)!} = \liminf_{k \to \infty} \frac{h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{(\mu,k\phi)} \right)}{k^{n+1}/(n+1)!} = \hat{\text{vol}}(\mathcal{L}),$$

and

$$\limsup_{k \to \infty} \frac{h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup,k\phi} \right)}{k^{n+1}/(n+1)!} = \limsup_{k \to \infty} \frac{h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup,k\phi} \right)}{k^{n+1}/(n+1)!} = \hat{\text{vol}}(\mathcal{L}).$$

The proof of this theorem is a consequence of the following.

**Proposition 4.2.** Assume that $\mu$ has the Bernstein-Markov property with respect to $\| \cdot \|_\phi$. There exists a positive constant $C \geq 1$ such that for any $\varepsilon > 0$, and any $k \gg 1$,

$$h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{(\mu,k\phi)} \right) - h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup,k\phi} \right) = \varepsilon kO(k^n) + O(k^n).$$

**Proof.** Let $\varepsilon > 0$. We have,

$$\|v\|_{\sup,k\phi} \leq C e^{\varepsilon k} \|v\|_{(\mu,k\phi)}, \quad \forall v \in H^0(\mathcal{X}, k\mathcal{L}) \quad \forall k \gg 1,$$

where $C$ is a positive constant (independent on $k$). Then

$$h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{(\mu,k(\phi + \varepsilon - \log C))} \right) \leq h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup,k\phi} \right) \leq h^0_\theta \left( \overline{H^0(\mathcal{X}, k\mathcal{L})}_{(\mu,k\phi)} \right) \quad \forall k \gg 1.$$
Using Lemma 3.1, we get
\[ 0 \leq h_0^0(\mathcal{X}, kL)_{(\mu, k\phi)} - h_0^0(\mathcal{X}, kL)_{(\mu, k(\phi - \varepsilon - \frac{1}{k}\log C))} \leq N_k(k\varepsilon + C) \quad \forall k \gg 1. \]

Therefore, for \( k \gg 1 \),
\[ h_0^0(\mathcal{X}, kL)_{(\mu, k\phi)} - N_k(k\varepsilon + C) \leq h_0^0(\mathcal{X}, kL)_{\sup, k\phi} \leq h_0^0(\mathcal{X}, kL)_{(\mu, k\phi)}. \]

Since \( N_k = O(k^n) \), we conclude that
\[ h_0^0(\mathcal{X}, kL)_{(\mu, k\phi)} - N_k(k\varepsilon + C) \leq \varepsilon kO(k^n) + O(k^n) \quad \forall k \gg 1. \]

It follows that
\[ \limsup_{k \to \infty} \frac{h_0^0(\mathcal{X}, kL)_{(\mu, k\phi)}}{k^{n+1}/(n+1)!} - \limsup_{k \to \infty} \frac{h_0^0(\mathcal{X}, kL)_{(\sup, k\phi)}}{k^{n+1}/(n+1)!} = O(\varepsilon). \]

Hence
\[ \limsup_{k \to \infty} \frac{h_0^0(\mathcal{X}, kL)_{(\mu, k\phi)}}{k^{n+1}/(n+1)!} = \limsup_{k \to \infty} \frac{h_0^0(\mathcal{X}, kL)_{(\sup, k\phi)}}{k^{n+1}/(n+1)!}. \]

By (7), we see that
\[ \limsup_{k \to \infty} \frac{h_0^0(\mathcal{X}, kL)_{(\sup, k\phi)}}{k^{n+1}/(n+1)!} = \hat{\mathrm{vol}}(\mathcal{L}). \]

This completes the proof of (9). In a similar way, we obtain (10).

\[ \square \]

**Remark 4.3.** For every \( s \in H^0(\mathcal{X}, kL) \), we have
\[ \|s\|_{\sup, k\phi} = \|s\|_{\sup, k(P_X \phi)}, \]
(this is a special case of [3, Proposition 2.8]). It follows from Theorem 4.1 that
\[ \hat{\mathrm{vol}}(\mathcal{L}) = \hat{\mathrm{vol}}(\mathcal{L}_X), \]
where \( \mathcal{L}_X = (\mathcal{L}, \| \cdot \|_{P_X \phi}) \).

**Definition 4.4.** Let \( t > 0 \), and \( x \in \mathcal{X}(\mathbb{C}) \). Set
\[ \Theta(\mu, \phi)(t; x) := 2\pi \sum_{v \in H^0(\mathcal{X}, L)} \|v(x)\|_{\phi}^2 e^{-\pi t\|v\|_{(\mu, \phi)}^2} \sum_{u \in H^0(\mathcal{X}, L)} e^{-\pi t\|u\|_{(\mu, \phi)}^2}. \]

When \( t = 1 \), we write \( \Theta(\mu, k\phi)(x) \) instead of \( \Theta(\mu, k\phi)(1; x) \).
Proposition 4.5. Assume that \( \mu \) has the Bernstein-Markov property with respect to \( \| \cdot \|_{\phi} \). For every \( t > 0 \) and \( k \in \mathbb{N} \)

\[
\Theta(\mu, k\phi)(t; x) < \infty \quad \forall x \in X,
\]

and

\[
\int_X \Theta(\mu, k\phi)(t; x) d\mu \leq \frac{1}{t} N_k.
\]

Moreover,

\[
\int_X \Theta(\mu, k\phi)(t; x) d\mu = U_{H^0(X, k\mathcal{L})_{(\mu, k\phi)}}(t)
\]

see [5, p. 64] for the definition of \( U \).

Proof. Let \( \varepsilon > 0 \). For \( k \gg 1 \)

\[
\|v\|_{\sup, k\phi} \leq C e^{\varepsilon k} \|v\|_{(\mu, k\phi)}, \quad \forall v \in H^0(\mathcal{X}, k\mathcal{L}),
\]

where \( C \) is a positive constant (independent on \( k \)).

Then

\[
\Theta(\mu, k\phi)(t; x) \leq 2\pi C e^{\varepsilon k} \sum_{v \in H^0(\mathcal{X}, k\mathcal{L})} \|v\|_{(\mu, k\phi)}^2 e^{-\pi t \|v\|_{(\mu, k\phi)}^2} \sum_{u \in H^0(\mathcal{X}, k\mathcal{L})} e^{-\pi t \|u\|_{(\mu, k\phi)}^2}.
\]

By Lemma 3.2, the right hand side of the preceding inequality is bounded.

It is easy to see that

\[
\int_X \Theta(\mu, k\phi)(t; x) d\mu = 2\pi \sum_{v \in H^0(\mathcal{X}, k\mathcal{L})} \|v\|_{(\mu, k\phi)}^2 e^{-\pi t \|v\|_{(\mu, k\phi)}^2} \sum_{u \in H^0(\mathcal{X}, k\mathcal{L})} e^{-\pi t \|u\|_{(\mu, k\phi)}^2}.
\]

Using again Lemma 3.2 we conclude that

\[
\int_X \Theta(\mu, k\phi)(t; x) d\mu \leq \frac{1}{t} N_k.
\]

Proposition 4.6. Let \( \psi \) and \( \phi \) be two continuous weights on \( \mathcal{L} \). We have

\[
\hat{h}_0^0(H^0(\mathcal{X}, k\mathcal{L})_{(\mu, k\phi)}) - \hat{h}_0^0(H^0(\mathcal{X}, k\mathcal{L})_{(\mu, k\psi)}) = k \int_X (\phi - \psi) \int_0^1 \Theta(x, k\phi_t) d\mu dt \quad \forall k \in \mathbb{N},
\]

where \( \phi_t = t\phi + (1 - t)\psi \) with \( t \in [0, 1] \).
Proof. Let \( \delta \) be a continuous function on \( X \). For any \( s \in \mathbb{R} \), \( \phi + s\delta \) defines a weight on \( \mathcal{L} \).

It is clear that \( h_0^0(\mathcal{X}, k\mathcal{L})_{(\mu, k(\phi+s\delta))} \) as a function of \( s \in \mathbb{R} \) is differentiable. For any \( s \in \mathbb{R} \),

\[
\frac{d}{ds} h_0^0(\mathcal{X}, k\mathcal{L})_{(\mu, k(\phi+s\delta))} = 2\pi \sum_{v \in H^0(X, k\mathcal{L})} e^{-\pi \|v\|^2_{k(\phi+s\delta)}} \sum_{v \in H^0(X, k\mathcal{L})} \left( \int_X k\delta(x) \|v(x)\|^2_{k(\phi+s\delta)} d\mu \right) e^{-\pi \|v\|^2_{k(\phi+s\delta)}}.
\]

At \( s = 0 \), we obtain

\[
\frac{d}{ds} h_0^0(\mathcal{X}, k\mathcal{L})_{(\mu, k(\phi+s\delta))} |_{s=0} = 2\pi \sum_{v \in H^0(X, k\mathcal{L})} \left( \int_X k\delta(x) \|v(x)\|^2_{k\phi} d\mu \right) e^{-\pi \|v\|^2_{k\phi}}
\]

\[
= k \int_X \delta(x) \Theta(\mu, k\phi) d\mu.
\]

Hence

\[
h_0^0(\mathcal{X}, k\mathcal{L})_{(\mu, k\phi)} - h_0^0(\mathcal{X}, k\mathcal{L})_{(\mu, k\psi)} = k \int_X (\phi - \psi) \int_0^1 \Theta(\mu, k\phi_s) d\mu ds,
\]

where \( \phi_s = \psi + s(\phi - \psi) \) for any \( s \in \mathbb{R} \). \( \square \)

Theorem 4.7. We have

\[
\Theta(\mu, \phi)(x) \leq \rho(\mu, \phi)(x) \quad \forall x \in X.
\]

Proof. Let \( x \in X \). Let \( s \geq 0 \). Let us denote by \( \nabla_s \) the euclidean lattice \( H^0(\mathcal{X}, \mathcal{L}) \) endowed with the norm \( \| \cdot \|_{\nabla_s} \) defined as follows:

\[
\|v\|^2_{\nabla_s} := \|v\|^2_{(\mu, \phi)} + s \|v(x)\|^2_{\phi}, \quad \text{for any } v \in H^0(\mathcal{X}, \mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{R}.
\]

From (10), we get

\[
(11) \quad \sum_{v \in H^0(\mathcal{X}, \mathcal{L})} e^{-\pi \|v\|^2_{\nabla_s}} = \frac{1}{\text{covol}(H^0(\mathcal{X}, \mathcal{L})_s)} \sum_{v^\vee \in H^0(\mathcal{X}, \mathcal{L})^\vee} e^{-\pi \|v^\vee\|^2_{\nabla_s^\vee}}.
\]

Let \( \{s_1, \ldots, s_N\} \) be a \( \mathbb{Z} \)-basis of \( H^0(\mathcal{X}, \mathcal{L}) \). We consider the following matrices.

\[
M := (\langle s_i, s_j \rangle_{(\mu, \phi)})_{1 \leq i, j \leq N} \quad \text{and} \quad C := (\langle s_i(x), s_j(x) \rangle_\phi)_{1 \leq i, j \leq N}.
\]

Then, (11) becomes

\[
\sum_{v \in H^0(\mathcal{X}, \mathcal{L})} e^{-\pi \|v\|^2_{\nabla_s} + s \|v(x)\|^2_{\phi}} = \frac{1}{\det(M + sC)} \sum_{\alpha \in \mathbb{Z}^N} e^{-\pi \alpha^t (M + sC)^{-1} \alpha},
\]
where $a^t$ denotes the transpose of the vector column $a \in \mathbb{Z}^N$.

By taking the derivative with respect to $s$, the preceding equation yields to

\begin{equation}
\sum_{v \in H^0(\mathcal{X}, \mathcal{L})} \|v(x)\|_\phi^2 e^{-\pi \|v\|_\phi^2} = \frac{\operatorname{Tr}(M^{-1}C)}{2\pi} \sum_{v \in H^0(\mathcal{X}, \mathcal{L})} e^{-\pi \|v\|_\phi^2} - \frac{1}{\det(M)^{\frac{1}{2}}} \sum_{a \in \mathbb{Z}^N} (a^t M^{-1} C M^{-1} a) e^{-\pi a^t M^{-1} a}.
\end{equation}

Note that

\begin{equation}
\frac{1}{\det(M)^{\frac{1}{2}}} \sum_{a \in \mathbb{Z}^N} (a^t M^{-1} C M^{-1} a) e^{-\pi a^t M^{-1} a} \sum_{v \in H^0(\mathcal{X}, \mathcal{L})} \phi e^{-\pi \|v\|_\phi^2} = \sum_{a \in \mathbb{Z}^N} e^{-\pi a^t M^{-1} a}.
\end{equation}

Then,

$$\Theta(\mu, \phi)(x) \leq \operatorname{Tr}(M^{-1}C).$$

Let $v_1, \ldots, v_N$ be an orthonormal basis of $H^0(\mathcal{X}, \mathcal{L}) \otimes \mathbb{C}$ with respect to the quadratic form defined by $M$. By basic arguments of linear algebra, there exists a $N \times N$ complex matrix $A = (a_{ij})_{1 \leq i,j \leq N}$ such that $M = A A^t$. More precisely, we have

$$s_i = \sum_{j=1}^{N} a_{ij} v_j, \quad \text{for } i = 1, \ldots, N.$$

It is easy to see that

$$\langle s_i(x), s_j(x) \rangle_\phi \leq \sum_{k=1}^{N} \|v_k(x)\|^2_\phi.$$

Hence

$$\operatorname{Tr}(M^{-1}C) = \sum_{k=1}^{N} \|v_k(x)\|^2_\phi.$$

But, we know that

$$\rho(\mu, \phi)(x) = \sum_{k=1}^{N} \|v_k(x)\|^2_\phi.$$

We conclude that

$$\Theta(\mu, \phi)(x) \leq \rho(\mu, \phi)(x) \quad \text{for all } x \in \mathcal{X}.$$

\[\square\]

**Corollary 4.8.** We assume that $\mathcal{L}_Q$ is big. If $\mu$ is smooth, and $\phi$ is $C^2$, then

(i) $\limsup_{k \to \infty} \frac{\Theta(\mu, k\phi)}{k^n} \mu \leq \mu_{eq}(X, \phi)$.
\[ \sup_{x \in X} \Theta(\mu, k\phi) = O(k^n), \]
as \( k \to \infty. \)

**Proof.** This corollary is a direct consequence of Theorem 4.7 and [3, Theorem 3.1]. \( \square \)

**Theorem 4.9.** Let \( \mathcal{L} = (\mathcal{L}, \| \cdot \|_{\phi}) \) be an ample \( \mathcal{C}^\infty \)-line bundle on \( X \). We assume that \( \mu \) is smooth. We have
\[
\lim_{k \to \infty} \frac{\Theta(\mu, k\phi)}{k^n} \mu = \mu_{eq}(X, \phi).
\]

**Proof.** We keep the same notations as in the proof of Theorem 4.7. There is a constant \( \delta > 0 \), such that
\[
C \leq \delta k^n M, \quad \forall x \in X, \forall k \gg 1.
\]
Indeed, let \( z \in \mathbb{C}^n \setminus \{0\} \), we have
\[
\frac{z^t Cz}{z^t Mz} \leq \rho(k\phi, \mu) \leq \delta k^n,
\]
where the second inequality is because [3, Theorem 3.1].

It follows that
\[
a^t M^{-1} C M^{-1} a \leq \delta k^n a^t M^{-1} a, \quad \forall a \in \mathbb{Z}^N, \forall k \gg 1.
\]

Hence,
\[
\sum_{a \in \mathbb{Z}^N} \frac{(a^t M^{-1} C M^{-1} a) e^{-\pi a^t M^{-1} a}}{\sum_{a \in \mathbb{Z}^N} e^{-\pi a^t M^{-1} a}} \leq \delta k^n \sum_{a \in \mathbb{Z}^N} \frac{(a^t M^{-1} a) e^{-\pi (a^t M^{-1} a)}}{\sum_{a \in \mathbb{Z}^N} e^{-\pi a^t M^{-1} a}}.
\]
By assumption, there is \( k_0 \geq 1 \) and \( 0 < \alpha < 1 \), such that for any \( k \geq k_0 \), the \( \mathbb{Z} \)-module \( H^0(X, k\mathcal{L}) \) is generated by sections \( e_1, \ldots, e_{N_k} \) such that \( \|e_i\|_{\sup, k\phi} \leq \alpha^k \) for \( i = 1, 2, \ldots, N_k \). Let \( \{e_1^\vee, \ldots, e_{N_k}^\vee\} \) be the \( \mathbb{Z} \)-basis of \( H^0(X, k\mathcal{L})^\vee := \text{Hom}_{\mathbb{Z}}(H^0(X, k\mathcal{L}), \mathbb{Z}) \), dual to \( \{e_1, \ldots, e_{N_k}\} \).

Let \( k \geq k_0 \). Let \( v^\vee \in H^0(X, k\mathcal{L})^\vee \). We write \( v^\vee = \sum_{i=1}^{N_k} a_i e_i^\vee \) where \( a_1, \ldots, a_{N_k} \in \mathbb{Z} \). Set \( a = (a_1, \ldots, a_{N_k})^t \). We have
(a^t M^{-1} a)^{\frac{1}{2}} = \|v^\vee\|_{H^0(\mathcal{X}, k\mathcal{L})^{\vee}_{(\mu, k\phi)}}^{\frac{1}{2}}

= \sup_{u \in H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R}} \frac{|v^\vee(u)|}{\|u\|_{(\mu, k\phi)}}

\geq \max_{i=1, \ldots, N_k} \frac{|a_i|}{\|e_i\|_{(\mu, k\phi)}}

\geq \frac{1}{N_k} \left( \sum_{i=1}^{N_k} \frac{|a_i|^2}{\|e_i\|^2_{(\mu, k\phi)}} \right)^{\frac{1}{2}}

\geq \frac{1}{N_k \alpha_k} \left( \sum_{i=1}^{N_k} |a_i|^2 \right)^{\frac{1}{2}},

(\text{where } H^0(\mathcal{X}, k\mathcal{L})^{\vee}_{(\mu, k\phi)} \text{ denotes the hermitian vector bundle dual to } H^0(\mathcal{X}, k\mathcal{L})_{(\mu, k\phi)}).)

Hence,

(14) \quad a^t M^{-1} a \geq \frac{1}{N_k^2 \alpha_k^2} a^t a.

From this inequality, we get

(1 - \alpha^2_k) a^t M^{-1} a \geq \frac{a^t a}{N_k^2} \left( \frac{1}{\alpha^2_k} - 1 \right).

That is

a^t M^{-1} a \geq \left( \frac{1}{\alpha^2_k} - 1 \right) \frac{a^t a}{N_k^2} + \alpha^2_k a^t M^{-1} a.

On the other hand, we have

\|v^\vee\|_{H^0(\mathcal{X}, k\mathcal{L})^{\vee}_{(\mu, k\phi)}} \leq N_k \max_i |a_i| \|e_i^\vee\|_{(\mu, k\phi)} \leq N_k \left( \sum_i |a_i|^2 \|e_i^\vee\|^2_{(\mu, k\phi)} \right)^{\frac{1}{2}}.
We have

\[
\sum_{a \in \mathbb{Z}^{N_k}} (a^t M^{-1} a) e^{-\pi a^t M^{-1} a} \leq \sum_{a \in \mathbb{Z}^{N_k}} (a^t M^{-1} a) e^{-\pi \left(\frac{(a_2 - 2k - 1) a_2}{N_k} + \alpha 2k a^t M^{-1} a\right)} \quad \text{by (14)}
\]

\[
\leq e^{-\pi \left(\frac{(a_2 - 2k - 1)}{N_k} - \frac{\alpha}{N_k} a^t a\right)} \sum_{a \in \mathbb{Z}^{N_k} \setminus \{0\}} (a^t M^{-1} a) e^{-\pi a^t a} \quad \text{by (8)}
\]

\[
\leq \frac{N_k}{2\pi \alpha 2k} e^{-\pi \left(\frac{a_2 - 2k - 1}{N_k} - \frac{\alpha}{N_k} a^t a\right)} \sum_{a \in \mathbb{Z}^{N_k}} e^{-\pi \frac{a^t a}{N_k}} \quad \text{by (14)}
\]

\[
= \frac{N_k}{2\pi \alpha 2k} e^{-\pi \left(\frac{a_2 - 2k - 1}{N_k} - \frac{\alpha}{N_k} a^t a\right)} \left(\sum_{n \in \mathbb{Z}} e^{-\pi \frac{n^2}{N_k}}\right)^{N_k}
\]

\[
= \frac{N_k}{2\pi \alpha 2k} e^{-\pi \left(\frac{a_2 - 2k - 1}{N_k} - \frac{\alpha}{N_k} a^t a\right)} N_k \left(\sum_{n \in \mathbb{Z}} e^{-\pi N_k^2 n^2}\right)^{N_k}
\]

\[
\leq \frac{N_k^{N_k + 1}}{2\pi \alpha 2k} e^{-\pi \left(\frac{a_2 - 2k - 1}{N_k} - \frac{\alpha}{N_k} a^t a\right)} \left(1 + 2e^{-\pi N_k^2} \sum_{n = 1}^{\infty} e^{-\pi n^2}\right)^{N_k}
\]

where we have used \( \theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right) \) where \( \theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} \quad (t > 0) \).

Note that

\[
\lim_{k \to \infty} \left(1 + 2e^{-\pi N_k^2} \sum_{n = 1}^{\infty} e^{-\pi n^2}\right)^{N_k} = 1.
\]

Let \( l \in \mathbb{R} \). We have

\[
\lim_{k \to \infty} \frac{\alpha^{2(k + 1)} N_k^{l+1} \log N_k^{k+1}}{\alpha^{2k} N_k^{l} \log N_k} = \alpha^2,
\]

where we have used that \( N_k = c k^n + O\left(k^{n-1}\right) \).

By letting, \( x_k = 1 - \frac{1}{\alpha^{2k} N_k^2} = 1 - \frac{\pi}{N_k^2 \alpha^{2k}} \) for any \( k \in \mathbb{N} \), we deduce that

\[
\lim_{k \to \infty} x_k = 1.
\]

Note that

\[
N_k \log N_k - \frac{\pi}{N_k^2 \alpha^{2k}} = -\frac{\pi}{N_k^2 \alpha^{2k}} x_k.
\]
Then, for any \( l \in \mathbb{R} \),
\[
\frac{N_k^{l+N_k+1}}{\alpha^{2k+1}} e^{-\left(\pi \frac{\alpha^{2(k+1)}-1}{N_k^{2}}\right)} \quad \text{and} \quad \frac{N_k^{l+N_k}}{\alpha^{2k}} e^{-\left(\pi \frac{\alpha^{2k}-1}{N_k^{2}}\right)}
\]
\[
= \alpha^{-2} \frac{N_k^{l+1}}{N_k^{2}} \exp\left( N_k^{l+1} \log N_k^{l+1} - N_k \log N_k - \pi \frac{1}{\alpha^{2(k+1)}N_{k+1}^{2}} + \pi \frac{1}{\alpha^{2k}N_k^{2}} + \pi \frac{1}{N_{k+1}^{2}} - \pi \frac{1}{N_k^{2}}\right)
\]
\[
= \alpha^{-2} \frac{N_k^{l+1}}{N_k^{2}} \exp\left( N_k^{l+1} \log N_k^{l+1} - N_k \log N_k - \pi \frac{1}{\alpha^{2(k+1)}N_{k+1}^{2}} + \pi \frac{1}{\alpha^{2k}N_k^{2}}\right)
\]
\[
= \alpha^{-2} \frac{N_k^{l+1}}{N_k^{2}} \exp\left( N_k^{l+1} \log N_k^{l+1} - N_k \log N_k - \pi \frac{1}{\alpha^{2(k+1)}N_{k+1}^{2}} + \pi \frac{1}{\alpha^{2k}N_k^{2}}\right)
\]
\[
= \alpha^{-2} \frac{N_k^{l+1}}{N_k^{2}} \exp\left( -\pi \frac{x_{k+1}}{N_{k+1}^{2}N_k^{2}}\right)\left(1 - \alpha^2\frac{x_{k+1}^2}{x_{k+1}N_k^{2}}\right).\]

That is
\[
\frac{N_k^{l+N_k+1}}{\alpha^{2(k+1)}} e^{-\left(\pi \frac{\alpha^{2(k+1)}-1}{N_k^{2}}\right)} = \alpha^{-2} \frac{N_k^{l+1}}{N_k^{2}} \exp\left( -\pi \frac{x_{k+1}}{N_{k+1}^{2}N_k^{2}}\right)\left(1 - \alpha^2\frac{x_{k+1}^2}{x_{k+1}N_k^{2}}\right)
\]

It follows that
\[
\frac{N_k^{l+N_k+1}}{\alpha^{2(k+1)}} e^{-\left(\pi \frac{\alpha^{2(k+1)}-1}{N_k^{2}}\right)} = \alpha^{-2} \exp\left( -\pi \frac{x_{k+1}}{N_{k+1}^{2}N_k^{2}}\right)\left(1 - \alpha^2\right),\]
as \( k \to \infty \).

Since \( \lim_{k \to \infty} N_k^{2} \alpha^{2k} = 0 \), and \( 0 < \alpha < 1 \), we deduce that
\[
\lim_{k \to \infty} \frac{N_k^{l+N_k+1}}{\alpha^{2(k+1)}} e^{-\left(\pi \frac{\alpha^{2(k+1)}-1}{N_k^{2}}\right)} N_{k+1}^{l} = 0.
\]

We conclude, using (13), that
\[
\lim_{k \to \infty} \frac{1}{k^n} \sum_{a \in \mathbb{Z}^N} (a^t M^{-1} CM^{-1} a) e^{-\pi a^t M^{-1} a} = 0.
\]

Then, the theorem follows from (12). \(\square\)
Theorem 4.10. Let $\mu$ be a probability measure which has the Bernstein-Markov property with respect to the metric of $\mathcal{L}$. We have
\[
\hat{\text{vol}}(\mathcal{L}) = \hat{\text{vol}}(\mu, k \phi)(\mathcal{L}).
\]

Proof. The proof is essentially the same as in [19, Lemma 2.1]. □

Theorem 4.11. Let $$(\mathcal{L}, \| \cdot \|_\phi)$$ and $$(\mathcal{L}, \| \cdot \|_\psi)$$ be two $C^\infty$ hermitian line bundles on $X$. We assume $\psi \leq \phi$ and $$(\mathcal{L}, \| \cdot \|_\psi)$$ is generated by small sections. We have
\[
\hat{\text{vol}}(\mathcal{L}, \| \cdot \|_\phi) - \hat{\deg}(\hat{c}_1(\mathcal{L}, \| \cdot \|_\phi))^{n+1} = \hat{\text{vol}}(\mathcal{L}, \| \cdot \|_\psi) - \hat{\deg}(\hat{c}_1(\mathcal{L}, \| \cdot \|_\psi))^{n+1}.
\]

Proof. Let $\epsilon > 0$. Let $t \in [0, 1]$, the hermitian line $$(\mathcal{L}, \| \cdot \|_{t \phi + (1-t) \psi})$$ is generated by strictly small sections.
Combining Proposition 4.6, Corollary 4.8, Theorems 4.9 and 4.10, we get
\[
\hat{\text{vol}}(\mathcal{L}, \| \cdot \|_\phi) - \hat{\text{vol}}(\mathcal{L}, \| \cdot \|_\psi) = \int_X (\phi - \psi) \int_0^1 \mu_{eq}(X, \phi_t) dt.
\]

It is known that
\[
\int_X (\phi - \psi) \int_0^1 \mu_{eq}(X, \phi_t) dt = \int_X (\phi - \psi) \sum_{i=0}^n (dd^c P_X \phi)_i (dd^c P_X \psi)^{n-1},
\]
(see [3, Proposition 4.1]).

Using (5), we conclude the proof of the theorem. □

5. A generalized Hodge index theorem on arithmetic toric varieties over $\mathbb{Z}$

Let $X$ be an arithmetic toric variety over $\mathbb{Z}$. Let $\mathcal{L}$ be an equivariant line bundle on $X$. It is known that $\mathcal{L}$ possesses a canonical and continuous metric which is described uniquely in terms of the combinatorial structure of $X$ (see [15, 23]). Let $\phi_\infty$ be the weight of the canonical metric of $\mathcal{L}$. We denote by $\| \cdot \|_{\phi_\infty}$ the canonical metric of $\mathcal{L}$, and we set $\mathcal{L}_\infty := (\mathcal{L}, \| \cdot \|_{\phi_\infty})$.

Example 5.1. Let $n \in \mathbb{N}^\ast$. We endow the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ with the metric
\[
\|s(x)\|_{\phi_\infty} = \frac{|s(x)|}{\max(|x_0|, \ldots, |x_n|)},
\]
where $s$ is a local holomorphic section of $\mathcal{O}(1)$. Then $\| \cdot \|_{\phi_\infty}$ is admissible.
We assume moreover that $\mathcal{L}$ is big. If $\mathcal{L}$ is generated by its global sections, then the current $c_1(\mathcal{T}_\infty)$ is semi-positive, and it defines a measure $\mu_\mathcal{L}$ on $X$:

$$
\mu_\mathcal{L} := \frac{1}{\text{vol}(L)} c_1(\mathcal{T}_\infty)^n.
$$

One can attach to $\mathcal{L}$ a convex polytope denoted by $\Delta_\mathcal{L}$ which describes the global sections of $k\mathcal{L}$, more precisely

$$
H^0(\mathcal{X}, k\mathcal{L}) = \bigoplus_{m \in (k\Delta) \cap \mathbb{Z}^n} \mathbb{Z}\chi^m
$$

(see [9, 21] for more details).

**Proposition 5.2.** Let $\mathcal{X}$ be an arithmetic toric variety over $\mathbb{Z}$. Let $\mathcal{L}$ be an equivariant big line bundle generated by its global sections on $\mathcal{X}$. We consider the continuous hermitian line bundle $\mathcal{L}_\infty = (\mathcal{L}, \| \cdot \|_{\phi_\infty})$. Let $k \geq 1$. There exists an orthogonal basis $\{s_1, \ldots, s_{N_k}\}$ of elements of $H^0(\mathcal{X}, k\mathcal{L})$ with respect to the scalar product $<\cdot, \cdot>_{(\mu_\mathcal{L}, k\phi_\infty)}$ such that

$$
\int_X \frac{s_i(x)s_j(x)}{\max(|s_1(x)|, \ldots, |s_N(x)|)^2} \mu_\mathcal{L} = 0 \quad \forall 1 \leq i \neq j \leq N_1.
$$

**Proof.** We know that

$$
\|s\|_{\phi_\infty}(x) = \frac{|s(x)|}{\max_{m \in \Delta_\mathcal{L} \cap \mathbb{Z}^n}(|\chi^m(x)|)}, \quad \forall s \in H^0(X, L), \quad \forall x \in X,
$$

(see [15, §3.3.3]).

Let $m, m' \in \Delta_\mathcal{L} \cap \mathbb{Z}^n$. From [15] Corollaire 6.3.5 and the fact that $|\chi^m(x)| = 1$ on $S_X$ the compact torus of $X$, we get

$$
\int_X \frac{\chi^m(x)\chi^{m'}(x)}{\max(|\chi^m(x)|, \ldots, |\chi^N(x)|)^2} \mu_\mathcal{L} = \int_{S_X} \chi^m(x)\chi^{m'}(x)\mu_\mathcal{L} = \delta_{m,m'},
$$

(see [15, §3.3.3]).

**Theorem 5.3.** Let $\mathcal{X}$ be an arithmetic toric variety over $\mathbb{Z}$. Let $\mathcal{T}_\infty = (\mathcal{L}, \| \cdot \|_{\phi_\infty})$ be an equivariant ample line bundle on $\mathcal{X}$. We have

(i) \[
\sup_{x \in X} \rho(\mu_\mathcal{L}, k\phi_\infty)(x) = \#((k\Delta_L) \cap \mathbb{Z}^n) = O(k^n) \quad \forall k \in \mathbb{N}.
\]

(ii) \[
\lim_{k \to \infty} \frac{h^0(H^0(\mathcal{X}, k\mathcal{L})_{(\mu_\mathcal{L}, k\phi_\infty)})}{k^n/n!} = \text{vol}(\mathcal{L}) \log \sum_{n \in \mathbb{Z}} e^{-\pi n^2},
\]
(iii) \[ \widehat{\mathrm{vol}}(\mathcal{L}_\infty) = 0. \]

**Proof.**

(i) Recall that
\[ \rho(\mu_L, k\phi_\infty)(x) = \sup_{s \in H^0(X, kL)} \frac{\|s(x)\|^2_{\phi_\infty}}{\|s\|^2_{(\mu_L, k\phi_\infty)}}, \]

Let \( s \in H^0(X, kL) \setminus \{0\} \). We have \( s = \sum_{m \in (k\Delta) \cap \mathbb{Z}^n} a_m \chi^m \) where \( a_m \in \mathbb{C} \) for any \( m \in (k\Delta) \cap \mathbb{Z}^n \). We have
\[
\frac{\|s(x)\|^2_{\phi_\infty}}{\|s\|^2_{(\mu_L, k\phi_\infty)}} = \frac{\sum_{m, m' \in (k\Delta) \cap \mathbb{Z}^n} a_m a_m' \chi^m(x) \chi^{m'}(x) e^{-2k\phi_\infty}}{\left(\sum_{m \in (k\Delta) \cap \mathbb{Z}^n} |a_m|^2\right)} \leq \sum_{m \in (k\Delta) \cap \mathbb{Z}^n} \|\chi^m(x)\|^2_{k\phi_\infty}.
\]

We conclude that
\[ \rho(\mu_L, k\phi_\infty)(x) = \sum_{m \in (k\Delta) \cap \mathbb{Z}^n} \|\chi^m(x)\|^2_{k\phi_\infty}. \]

It follows that
\[ \sup_{x \in \mathcal{X}(\mathbb{C})} \rho(\mu_L, k\phi_\infty)(x) = \#((k\Delta) \cap \mathbb{Z}^n) = O(k^n). \]

(ii) From Proposition 5.2, we see that
\[
\sum_{s \in H^0(X, kL)} e^{-\pi \|s\|^2_{(\mu_L, k\phi_\infty)}} = \sum_{(a_m)_{m \in (k\Delta) \cap \mathbb{Z}^n} \in \mathbb{N}_k^N} e^{-\pi \sum_{m \in (k\Delta) \cap \mathbb{Z}^n} a_m^2} = \left(\sum_{n \in \mathbb{Z}} e^{-\pi n^2}\right)^{N_k}.
\]

Hence
\[
\lim_{k \to \infty} \frac{h^0(\mathcal{X}, k\mathcal{L})_{(\mu, k\phi_\infty)}}{k^n/n!} = \mathrm{vol}(\mathcal{L}) \log \sum_{n \in \mathbb{Z}} e^{-\pi n^2}.
\]

(iii) By Theorem 4.1 and (ii), we conclude that
\[ \widehat{\mathrm{vol}}(\mathcal{L}_\infty) = \lim_{k \to \infty} \frac{N_k}{k^{n+1}} \log \left(\sum_{n \in \mathbb{Z}} e^{-\pi n^2}\right) = 0. \]

□
**Theorem 5.4.** Let $\mathcal{X}$ be an arithmetic toric variety over $\mathbb{Z}$. Let $(\mathcal{L}, \| \cdot \|_\phi)$ be an equivariant ample hermitian line bundle on $\mathcal{X}$. We have

$$\hat{\text{vol}}(\mathcal{L}, \| \cdot \|_\phi) = \hat{\text{deg}}(\hat{c}_1(\mathcal{L}, \| \cdot \|_{P_X\phi})^{n+1}).$$

If moreover, the metric of $(\mathcal{L}, \| \cdot \|_\phi)$ is semipositive, then

$$\hat{\text{vol}}(\mathcal{L}, \| \cdot \|_\phi) = \hat{\text{deg}}(\hat{c}_1(\mathcal{L}, \| \cdot \|_\phi)^{n+1}).$$

**Proof.** There exists $0 < \alpha \leq 1$ such that

$$\alpha \| \cdot \|_\phi \leq \| \cdot \|_{\phi_\infty}$$

where $\| \cdot \|_{\phi_\infty}$ is the canonical metric of $\mathcal{L}$.

By Theorem 4.11 and a continuity argument, we can show that

$$\hat{\text{vol}}(\mathcal{L}, \alpha \| \cdot \|_\phi) - \hat{\text{deg}}(\hat{c}_1(\mathcal{L}, \alpha \| \cdot \|_{P_X\phi})^{n+1}) = \hat{\text{vol}}(\mathcal{L}, \| \cdot \|_{\phi_\infty}) - \hat{\text{deg}}(\hat{c}_1(\mathcal{L}, \| \cdot \|_{P_X\phi_\infty})^{n+1}),$$

and

$$\hat{\text{vol}}(\mathcal{L}, \alpha \| \cdot \|_\phi) - \hat{\text{deg}}(\hat{c}_1(\mathcal{L}, \alpha \| \cdot \|_{P_X\phi})^{n+1}) = \hat{\text{vol}}(\mathcal{L}, \| \cdot \|) - \hat{\text{deg}}(\hat{c}_1(\mathcal{L}, \| \cdot \|_{P_X\phi})^{n+1}).$$

By Theorem 4.11 (ii) of Theorem 5.3 and [15, Proposition 7.1.1], we deduce

$$\hat{\text{vol}}(\mathcal{L}, \| \cdot \|) = \hat{\text{deg}}(\hat{c}_1(\mathcal{L}, \| \cdot \|_{P_X\phi})^{n+1}).$$

\[\square\]

**Theorem 5.5.** [A generalized Hodge index theorem] Let $\mathcal{X}$ be an arithmetic toric variety over $\mathbb{Z}$ of relative dimension $n$. Let $\mathcal{L}$ be an equivariant ample line bundle on $\mathcal{X}$. We assume that the metric of $\overline{\mathcal{L}}$ is smooth with semi-positive first Chern form $c_1(\overline{\mathcal{L}})$ on $\mathcal{X}$, then

$$\hat{\text{vol}}(\overline{\mathcal{L}}) \geq \hat{\text{deg}}(\hat{c}_1(\overline{\mathcal{L}})^{n+1}).$$

**Proof.** There exist $k_0 \in \mathbb{N}$, and $s_1, \ldots, s_{N_{k_0}}$ elements of $H^0(\mathcal{X}, k_0 \mathcal{L})$ such that the $\mathbb{Z}$-algebra $\bigoplus_{k \geq 0} H^0(\mathcal{X}, kk_0 \mathcal{L})$ is generated by $s_1, \ldots, s_{N_{k_0}}$.

Since, for any $k = 1, 2, \ldots$,

$$\hat{\text{vol}}(k \overline{\mathcal{L}}) = k^{n+1} \hat{\text{vol}}(\overline{\mathcal{L}}) \quad \text{and} \quad \hat{\text{deg}}(\hat{c}_1(k \overline{\mathcal{L}})^n) = k^{n+1} \hat{\text{deg}}(\hat{c}_1(\overline{\mathcal{L}})^n),$$

we may assume that $k_0 = 1$. Let $\alpha$ be a positive real number such that

$$0 < \alpha < 1 \quad \text{and} \quad \alpha \| s_i \|_{\sup, \phi} < 1 \quad \text{for} \quad i = 1, 2, \ldots, N_1.$$

It follows that the hermitian line bundle $(\mathcal{L}, \alpha \| \cdot \|_\phi)$ is ample. From Theorem 5.4 we get

$$\hat{\text{vol}}(\mathcal{L}, \alpha \| \cdot \|_\phi) = \hat{\text{deg}}(\hat{c}_1(\mathcal{L}, \alpha \| \cdot \|_\phi)^{n+1}).$$
The combination of Proposition 4.6 and Theorem 4.7 gives the following inequality
\[ \widehat{\operatorname{vol}}(L, \| \cdot \|_\phi) - \widehat{\deg}(\hat{c}_1(L, \| \cdot \|_\phi)^{n+1}) \geq \widehat{\operatorname{vol}}(L, \alpha \| \cdot \|_\phi) - \widehat{\deg}(\hat{c}_1(L, \alpha \| \cdot \|_\phi)^{n+1}). \]
Therefore
\[ \widehat{\operatorname{vol}}(L, \| \cdot \|_\phi) \geq \widehat{\deg}(\hat{c}_1(L, \| \cdot \|_\phi)^{n+1}). \]

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