Slow dynamo modes in compact Riemannian plasma devices from Brazilian spherical tokamak data

by

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Abstract

Anti-dynamo modes are usually found in spheromaks plasma devices experiments due to the fact that Cowlings anti-dynamo theorem is naturally applied to axisymmetric devices and flows. In this paper full consideration is given to the existence of slow dynamo modes in the case of compact Riemannian plasma devices without boundaries, such as tokamaks, stellarators and torsatrons. It is shown that a perturbed untwisted flow given by a decaying mode magnetic field is able to generate a slow twisted dynamo plasma flow where the unperturbed flow is also a steady flow. When the slow dynamo limit is obtained for high Reynolds magnetic numbers $Rm$, the unperturbed plasma flow achieve equilibrium as in spheromaks. A Riemann metric of a twisted thick magnetic flux tube with a very low aspect ratio is used in the computations. The data of aspect ratio and other data obtained from the Brazilian spherical tokamak at the National institute for space research (INPE) are used to obtain a numerical estimate for the maximum magnetic growth rate of the magnetic field of the slow dynamo action as

$$\gamma_{1\max} \approx \frac{0.26}{a} \times \frac{v_{1\theta}}{B_{1z}}$$

for a toroidal initial magnetic field $B_0 \approx 0.4T$ and an aspect data tokamak of 1.5. For a tokamak internal radius of $a \approx 0.2m$ one obtains a maximum growth rate for the slow dynamo as

$$\gamma_{1\max} \approx 1.3 \times \frac{v_{1\theta}}{B_{1z}}$$

where $v_{1\theta}$ is the poloidal perturbed diffusive flow. The safety factor is given by $q \approx -1$ which is $q < 1$ implying stability of spherical tokamak plasma. Slow dynamos have been recently given in the literature as an example of Vishik’s anti-fast dynamo theorem [Phys Plasmas 15 (2008)], which can also been tested in plasma experimental devices. **Key-words:** anti-dynamo theorems, slow dynamos.
I Introduction

Recently Bellan [1] has been discussed the existence of anti-dynamo modes in plasmas spheromaks. The stability has been given in terms of the safety factor conditions of instability \((q > 1)\) or stability \((q < 1)\). The Cowling’s anti-dynamo theorem [2], which demands the existence of non-axisymmetric magnetic flows for the dynamo existence, can be circumvented in spheromaks as, even in axisymmetric spheromak devices, the existence of non-axisymmetric fluids and fields may exist, which per se, may guarantee dynamo action. Cowling [2], have show us how to recognise dynamo action in most types of magnetized plasma flows. The geometry of spheromaks, may be obtained by taking a very low aspect ratio, which is the ratio between the outer torus radius \(R\) and the cross-section \(a\), or in the language of Riemannian twisted magnetic flux tube, the torus would be thick. Just as an example, in tokamaks in general the aspect ratio can be built in plasma laboratories, as \(A = \frac{R}{a} = 3\), whereas in the Brazilian spherical tokamak [3], the ratio \(A = 1.5\). Actually magnetic plasma stability is favour for aspect ratios of \(A < 2\). Indeed, the magnetic topology of the field lines in the spherical tokamaks are highly torsioned at its center, which allows us these fields to reach values as high as \(4T\). These torsion central lines are used to concentrate a strong magnetic field. This leads us naturally to use the Riemannian geometry of twisted magnetic flux tube [4, 5] to model these spherical tokamaks, provided one considers only thick flux tubes where the aspect ratio is low. Another advantage of its use is that this leads easily to plasma stability. Another advantage is that according to Schuessler [6] astrophysical magnetic dynamos may be modelling by the flux tubes, as happens in the sun and other stars. Also another dynamo action mechanism called stretch-twist and fold (STF) as created by Vainshtein-Zeldovich [7] may naturally appear in the magnetic lines of the spherical tokamaks, since a fundamental ingredient of twist is the Frenet torsion of its lines in the dynamo flow [8]. From the mathematical side, using a dynamo model in Riemannian space has a long and solid tradition in toroidal maps as shown previously by Arnold et al [9] and in more recent papers by the author [5] to model plasma toroidal devices and astrophysical conformal dynamos. Another fundamental ingredient for dynamo existence, the folding, is a quantity which can be associated with Riemann curvature [10] to provide the doubling of the magnetic field intensity by a repetition process that may guarantee dynamo
action. In this paper untwisting flows are initially perturbed to yield slow dynamo action by using the spherical tokamak INPE data. In the presence of diffusion the relation between the growth rate of the perturbed magnetic field and the previous unperturbed one in terms of diffusion coefficient or resistivity. This allows us to show that a slow dynamo is obtained since the growth rate $\gamma$ vanishes when diffusion $\eta \to 0$. Another Riemannian model for dynamo action has been recently obtained by Shukurov et al [11] to model small-scale dynamos such as the Perm toroidal experimental dynamo [12]. Another interesting distinction between the present model and the one addressed in reference (II.12) is that in the case presently considered the perturbation of the magnetic field is not stationary. Recently a modification of Shukurov et al proposal was presented by the author can be considered as a thick Riemannian dynamo as well. Basic difference between our models and Moebius dynamo flow one proposed by Shukurov et al is that theirs is a numerical simulation and ours. Much earlier Mikhailovskii [13] has used a non-diagonal Riemann metric to describe tokamaks and investigate the plasma instabilities.

The paper is organized as follows: In section II the absence of diffusion is shown to lead to non-dynamos or marginal dynamos. In section III diffusion is turn on and the perturbation is shown to lead to a slow dynamo spherical tokamak. Discussions and future prospects are presented in section IV.
II Riemannian plasma non-dynamo devices

This section addresses the mathematical formalism of a general and thick twisted Riemannian magnetic flux tube and show that a marginal dynamo is obtained when a stationary unperturbed model is used when the magnetic field is given along the magnetic lines, which is \( B = B(s)t \), and no slow dynamo is obtained. Here one considers that the growth rate of magnetic perturbed and unperturbed fields given respectively by \( B_1 \) and \( B_0 \) are independently given by \( \gamma_1 \) and \( \gamma_0 \) where the magnetic fields are proportional to

\[
|B_0| \approx e^{\gamma_0 t} \quad \text{II.1}
\]

and

\[
|B_1| \approx e^{\gamma_1 t} \quad \text{II.2}
\]

where the perturbation process is given by

\[
B = B_0 + B_1 \quad \text{II.3}
\]

where \( |B_0| \gg |B_1| \). Here \( t \) is part of the Frenet vector frame along the curve coordinate given by \( s \)-parameter. The complete Frenet frame is given by \((t, n, b)\) where vectors \( n \) and \( b \) are the vectors that lay in the orthogonal plane to the vector \( t \) along the magnetic axis of the toroidal device. In this section one shall consider the diffusionless case, and show that this leads to a non-dynamo or at best to a marginal dynamo where \( \gamma_0 \) vanishes. But before digging into the physics of the problem let us take a moment to consider the Riemannian geometry of flux tubes as given for the first time in the context of solar plasma physics by Ricca [4].

The magnetic flux tube coordinates \((r, \theta_R, s)\), is also used in plasma toroidal devices called tokamaks. Since folding processes in flux tubes can be represented by the Riemann curvature tensor, destructive folding that leads to non-dynamos in diffusive media, can be obtained by the vanishing of folding or vanishing of the Riemann curvature tensor. General flux tube Riemannian metric is

\[
ds_0^2 = dr^2 + r^2 d\theta^2 + K^2(r, s)ds^2 \quad \text{II.4}
\]

The thin Riemann-flat in twisted magnetic flux tube metric is obtained by the constraining the relation \( K^2 := (1 - r\kappa(s)\cos\theta) \) to one. This is obtained as coordinate \( r \) approaches zero. This
happens in the neighbourhood of the torsioned flux tube axis. Coordinate $\theta(s)$ is one of the Riemannian curvilinear coordinates $(r, \theta_R, s)$ and $\theta(s) = \theta_R - \int \tau(s) ds$. The scalar function $\tau$ represents the Frenet torsion. The thin tube metric is

$$ds_0^2 = dr^2 + r^2 d\theta_R^2 + ds^2 \quad (\text{II.5})$$

The torsion term is responsible for the twist of the tube. Solar flux tubes are closed in the inner parts of the Sun, and then the tubes can be considered as compact Riemannian manifold without boundaries. Riemann gradient compact operator is given in general diffusive substrate by

$$\nabla = e_r \partial_r + \frac{e_\theta}{r} \partial_\theta + t \frac{1}{K} \partial_s \quad (\text{II.6})$$

while general self-induction equation is

$$d_t B = (B \cdot \nabla) v + \eta \Delta B \quad (\text{II.7})$$

where in this section the resistivity $\eta = 0$ in the ideal plasma case. Throughout this paper, the magnetic field is strictly confined along and inside the tube, which allows us to simplify the computations by considering that $B_r = 0$ and that $\partial_s B_s = 0$. In the case consider here

$$d_t B = \gamma B - \tau_0 B_0 n \quad (\text{II.8})$$

whose extra term is a non-inertial term similar to one that is introduced into a inertial frame by the use of curvilinear coordinates or Coriolis force in the frame. Therefore the diffusionless self-induction equation is given by

$$[\gamma_0 t - \tau_0^2 n] B_0 = B_0 v_0 \partial_s n \quad (\text{II.9})$$

where due to the highly torsioned character of the internal spherical tokamak, one has used the helical hypothesis of circular helices where the torsion equals the Frenet curvature $\kappa_0$ and are constants. By comparison of the both sides of equation (II.9) one obtains

$$\gamma_0 = 0 \quad (\text{II.10})$$

and

$$v_0 = -\tau_0 \quad (\text{II.11})$$
Here the Frenet equations

\[ dt = \kappa n \]  
(II.12)

\[ dn = -\kappa t + \tau b \]  
(II.13)

and

\[ db = -\tau b \]  
(II.14)

have been used to obtain the above results \( v_0 \) is the constant flow and the RHS of the equation (II.9) represents the stretching of the flow. Actually \( v = v_0 t \) and the incompressibility condition of the flow

\[ \nabla \cdot v = 0 \]  
(II.15)

This implies actually that \( v_0 \) be constant. Thus from expression (II.10) one must conclude that no dynamo action is possible in for a constant modulus magnetic initial field in diffusionless media with a constant modulus stretching flow. Thus in the net section, we observe that by perturbing these magnetic fields with a non-stationary unsteady magnetic field a dynamo action is present but the dynamo is still slow.

### III Riemannian spherical tokamaks. slow dynamos and safety factor

The above perturbation scheme described in the last section, is used in the above magnetic self-induced equation, where \( B_0 \) and its perturbation \( B_1 \) obey the following zero and first-order equations

\[ dt B_0 = (B_0, \nabla)v_0 + \eta \Delta B_0 \]  
(III.16)

and

\[ dt B_1 = (B_1, \nabla)v_0 + (B_0, \nabla)v_1 + \eta \Delta B_1 \]  
(III.17)

where \( \Delta = \nabla^2 \) is the Laplacian Riemannian operator given by

\[ \Delta = [\partial^2_r + \frac{1}{r} \partial_r + \frac{\tau_0^{-2} (1 + \cos \theta)^2}{\cos^2 \theta} \partial_s^2] \]  
(III.18)
Applying the following perturbed magnetic field $B_1$ into the above Riemannian Laplacian operator in the form

$$B_1 = B_{1\theta}e_{\theta} + B_{1s}t \quad (\text{III.19})$$

where the dynamo mode $m$ of the spatial toroidal coordinate-s appears as

$$B_{1s} = b_s e^{\gamma_1 t + ims} \quad (\text{III.20})$$

where $m \in \mathbb{Z}$, where $\mathbb{Z}$ is the field of integers numbers. Here $b_s$ is a constant with magnetic field dimensions. Note that the poloidal magnetic field perturbation $B_{1\theta}$ is given by the same expression with the only difference that the constant $b_s$ of the toroidal field is now replaced by $b_{1\theta}$. Note that the above choice seemingly hide the coordinate-$\theta$, but actually this is present in the $ms$ exponent since the poloidal angular coordinate inside the tokamak, does depend on coordinate-s as $\theta = \theta(s)$ as contained above. The reason one is not using a radial mode in the magnetic field exponent is that one shall be considering here only compact spherical tokamak surfaces, where the radial dependence of the magnetic fields does not exist and the internal cross-section radius of the tokamak is $a$. Bellow one shall show that the slow dynamo mode $m$ vanishes. Let us now display the expressions for the Laplacian of the initial field $B_0 = B_0 t$ which by the solenoidality property

$$\nabla \cdot B_0 = 0 \quad (\text{III.21})$$

implies that $B_0$ does not depend on coordinate-s, since however the tangent vector $t = t(s)$, the vector field $B_0$ is a non-uniform unsteady magnetic field. Due to the absence of radial dependence in the magnetic field over the tokamak Riemannian compact surface without boundaries the Laplacian of $B_0$ simplifies to

$$\Delta B_0 = \left[ \frac{(1 + \cos \theta)^2}{a^2 \cos^2 \theta} B_0 (-t + n) \right] \quad (\text{III.22})$$

From this expression, after some algebra, the complete diffusive self-induction equation results in the following growth rate

$$\gamma_0 = -\eta \frac{(1 + \cos \theta)^2}{(a \cos \theta)^2} \quad (\text{III.23})$$

Note that at this time, though the unperturbed magnetic field growth rate does not vanish the situation is even worse here, cause $\gamma_0$ is negative and this means that the initial toroidal
magnetic field decays as happens with some primordial magnetic fields in the universe [14]. The other equation yields the value of torsion $\tau$ as

$$\tau(s) \approx \frac{\eta s}{a^2}$$  \hspace{1cm} (III.24)

This result seems to be rather interesting since it shows that folding and twisting of the magnetic axis depends on a straightforward manner from the Let us now show that this does not happens with the perturbed field $B_1$ growth rate $\gamma_1$. The solenoidal property of the perturbed field is

$$\nabla \cdot B_1 = 0$$  \hspace{1cm} (III.25)

Application of the appropriate magnetic field into this equation yields

$$B_{1\theta} \over B_{1s} = \cos \theta$$  \hspace{1cm} (III.26)

and

$$im(\cos^2 \theta - 1)B_{1s} = 0$$  \hspace{1cm} (III.27)

this last equation leads to the constraint $m = 0$ for the dynamo action mode if it exists at all. Substitution of this mode into the Riemannian Laplacian expression for the perturbed field reduces it to

$$\eta \Delta B_1 = -\frac{\eta^2 (1 + \cos \theta)^2}{a^4 \cos^2 \theta} \left[ \frac{\sin 2\theta}{2} - s \right] B_\theta$$  \hspace{1cm} (III.28)

This yields the following growth rate as

$$\gamma_1 = \frac{\eta B_0}{a^2 B_{1s}} \sin \theta - \frac{2\eta^2 (1 + \cos \theta)^2}{a^4 \cos^2 \theta} \left[ \frac{\sin 2\theta}{2} - s \right]$$  \hspace{1cm} (III.29)

Note that in the limit $\eta \to 0$ the growth rate $\gamma_1 \to 0$ which characterizes the slow dynamo model. Thus one may say that the mode $m = 0$ represents a slow dynamo mode. More general modes may be found which may represent a fast dynamo action in the spherical tokamak. Let us now consider the data involved in the INPE brazilian spherical tokamak experiment, in the case of maximum value of the growth rate $\gamma_{1\text{max}}$, by taking into account the case when the magnetic Reynolds number $Rm$ is high or when the dynamo flow is highly conductive. In this case the diffusion $\eta$ is small but finite and in this case the term of second order in $\eta$ may be dropped and finally the last expression reduces to

$$\gamma_{1\text{max}} \approx \frac{2 B_{1\theta} B_0}{3 a B_{1s}}$$  \hspace{1cm} (III.30)
Note that by using the INPE spherical tokamak data $a = 0.2m$ and initial toroidal field $B_0 = 0.4 T$, this growth rate of slow dynamo is

$$\gamma_{1\text{max}} \approx \frac{0.26}{a} \frac{v_1}{B_{1s}} \approx 1.3 \frac{v_1}{B_{1s}}$$

(III.31)

here the torsion has been computed as

$$\tau_0 \approx \frac{1}{a} \approx 3.3 m^{-1}$$

(III.32)

Finally let us compute the safety factor $q$ of the tokamak as

$$q = \frac{d\Phi}{d\theta} = -\frac{1}{R \tau ds}$$

(III.33)

where we have used the expression above relating $\theta$ and torsion integral $\int \tau(s) ds$. Since Frenet curvature is given by

$$\kappa_0 = \frac{1}{R}$$

(III.34)

These two last expressions yields

$$q = -\frac{\kappa_0}{\tau_0}$$

(III.35)

which in the circular helix case is given by $q = -1 < 1$ and thus guarantees the tokamak plasma stability.
IV Conclusions

Anti-dynamo modes in several plasma devices have been known in the literature. In this paper it is shown that slow dynamo modes can be obtained in non-turbulent flows diffusive plasma media. It is also shown that the absence of diffusion forbids the presence of a fast dynamo action in the case of initially toroidal flows where initial flows are aligned with the magnetic field. Data from the brazilian spherical tokamak operating in INPE is given to estimate the value of the growth rate of perturbed flows magnetic field. It is shown that this depend upon directly of the inverse of the magnetic Reynolds number which displays an explicitly slow dynamo behaviour. The Riemannian flux tube model in the thick case lead us to transform a toroidal tokamak into a spherical one, and the growth rate of the perturbed field is also proportional to the perturbed flow. When the perturbed flow vanishes the growth rate vanishes as well and no dynamo action is found whatsoever. Several other modes may be investigated in spherical tokamaks with the hope fast dynamo action may be found in future experiments. For example a more complicated model to INPE tokamak would be given by a tube in the center of the spherical tokamak where the torsion of the magnetic field could be confined around a Riemannian torus.

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