Geodesic ball packings generated by regular prism tilings in \textit{Nil} geometry

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Abstract

In this paper we study the regular prism tilings and construct ball packings by geodesic balls related to the above tilings in the projective model of \textit{Nil} geometry. Packings are generated by action of the discrete prism groups $pq2_1$. We prove that these groups are realized by prism tilings in \textit{Nil} space if $(p, q) = (3, 6), (4, 4), (6, 3)$ and determine packing density formulae for geodesic ball packings generated by the above prism groups. Moreover, studying these formulae we determine the conjectured maximal dense packing arrangements and their densities and visualize them in the projective model of \textit{Nil} geometry. We get a dense (conjectured locally densest) geodesic ball arrangement related to the parameters $(p, q) = (6, 3)$ where the kissing number of the packing is 14, similarly to the densest lattice-like \textit{Nil} geodesic ball arrangement investigated by the second author in [22].

1 Introduction and previous results

In mathematics sphere packing problems concern the arrangements of non-overlapping equal spheres which fill a space. Usually the space involved is the three-
dimensional Euclidean space where the famous Kepler conjecture was proved by T. C. Hales and S. P. Ferguson in \cite{5}.

However, ball (sphere) packing problems can be generalized to the other 3-dimensional Thurston geometries.

In an $n$-dimensional space of constant curvature $E^n$, $H^n$, $S^n$ ($n \geq 2$) let $d_n(r)$ be the density of $n + 1$ spheres of radius $r$ mutually touching one another with respect to the simplex spanned by the centres of the spheres. L. Fejes Tóth and H. S. M. Coxeter conjectured that in an $n$-dimensional space of constant curvature the density of packing spheres of radius $r$ can not exceed $d_n(r)$. This conjecture has been proved by C. Roger in the Euclidean space. The 2-dimensional case has been solved by L. Fejes Tóth. In an 3-dimensional space of constant curvature the problem has been investigated by Böröczky and Florian in \cite{2} and it has been studied by K. Böröczky in \cite{3} for $n$-dimensional space of constant curvature ($n \geq 4$).

In \cite{6}, \cite{7}, \cite{24} and \cite{27} we have studied some new aspects of the horoball and hyperball packings in $H^n$ and we have observed that the ball, horoball and hyperball packing problems are not settled yet in the $n$-dimensional ($n \geq 3$) hyperbolic space.

In \cite{28} we generalized the above problem of finding the densest geodesic and translation ball (or sphere) packing to the other 3-dimensional homogeneous geometries (Thurston geometries)

$$\widetilde{SL}_2\mathbb{R}, \text{ Nil, } S^2 \times \mathbb{R}, \ H^2 \times \mathbb{R}, \ Sol,$$

and in the papers \cite{21}, \cite{22}, \cite{23}, \cite{25}, \cite{28} we investigated several interesting ball packing and covering problems in the above geometries. We described in $S^2 \times \mathbb{R}$ geometry (see \cite{28}) a candidate of the densest geodesic and translation ball arrangement whose density is $\approx 0.8750$.

In this paper we consider the Nil geometry that can be derived from W. Heisenberg’s famous real matrix group. This group provides a non-commutative translation group of an affine 3-space. E. Molnár proved in \cite{10}, that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $PS^3$. In this work we will use this projective model of the Nil geometry.

In \cite{22} we investigated the geodesic balls of the Nil space and computed their volume, introduced the notion of the Nil lattice, Nil parallelepiped and the density of the lattice-like ball packing. Moreover, we determined the densest lattice-like geodesic ball packing. The density of this densest packing is $\approx 0.7809$, may be surprising enough in comparison with the Euclidean result $\frac{\pi}{\sqrt{18}} \approx 0.74048$. The kissing number of the balls in this packing is 14.
In [25] we considered the analogue question for translation balls. The notions of translation curve and translation ball were introduced by initiative of E. Molnár (see [14], [21]). We have studied the translation balls of Nil space and computed their volume. Moreover, we have proved that the density of the optimal lattice-like translation ball packing for every natural lattice parameter $1 \leq k \in \mathbb{N}$ is in interval $(0.7808, 0.7889)$ and if $r \in (0, r_d]$ ($r_d \approx 0.7456$) then the optimal density is $\delta_{opt} \approx 0.7808$. Meanwhile we can apply a nice general estimate of L. Fejes Tóth [4] in our proof. The kissing number of the lattice-like ball packings is less than or equal to 14 and the optimal ball packing is realizable in case of equality. We formulated a conjecture for $\delta_{opt}$, where the density of the conjectural densest packing is $\delta_{opt} \approx 0.7808$ for lattice parameter $k = 1$, larger than the Euclidean one ($\frac{\pi}{\sqrt{18}} \approx 0.7405$), but less than the density of the densest lattice-like geodesic ball packing in Nil space known till now [22]. The kissing number of the translation balls in that packing is 14 as well.

In [28] we studied one type of lattice coverings in the Nil space. We introduced the notion of the density of considered coverings and gave upper and lower estimation to the density of the lowest lattice-like geodesic ball covering. Moreover we formulate a conjecture for the ball arrangement of the least dense lattice-like geodesic ball covering and give its covering density $\Delta \approx 1.4290$.

In papers [25], [14] there are further results related to discrete problems of Nil geometry.

In this paper we study the regular prism tilings and construct ball packings by geodesic balls related to the prism tilings in the projective model of Nil geometry where the packings are generated by action of the discrete prism groups $pq2_1$. We obtain density formulae for calculations for geodesic ball packings. Analyzing these density functions we obtain a conjecture for optimal geodesic ball packing configurations and determine their densities related to the above prismatic tesselations.

The results are summarized in Theorems 3.4, 4.3 and Conjecture 4.4 and the optimal ball configurations are visualized in our model with Figures 2, 3.

2 Basic notions of the Nil geometry

In this Section we summarize the significant notions and denotations of the Nil geometry (see [10], [22]).

The Nil geometry is a homogeneous 3-space derived from the famous real matrix group $L(R)$ discovered by W. Heisenberg. The Lie Theory with the method
of the projective geometry makes possible to investigate and to describe this topic. The left (row-column) multiplication of Heisenberg matrices
\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix} =
\begin{pmatrix}
1 & a + x & c + xb + z \\
0 & 1 & b + y \\
0 & 0 & 1
\end{pmatrix}
\] (2.1)
defines "translations" \(L(R) = \{(x, y, z) : x, y, z \in R\}\) on the points of the space \(\text{Nil} = \{(a, b, c) : a, b, c \in R\}\). These translations are not commutative in general. The matrices \(K(z) \circ L\) of the form
\[
\begin{pmatrix}
1 & 0 & z \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \mapsto (0, 0, z)
\] (2.2)
constitute the one parametric centre, i.e. each of its elements commutes with all elements of \(L\). The elements of \(K\) are called fibre translations. \(\text{Nil}\) geometry of the Heisenberg group can be projectively (affinely) interpreted by the "right translations" on points as the matrix formula
\[
(1; a, b, c) \mapsto (1; a, b, c)
\begin{pmatrix}
1 & x & y & z \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & x \\
0 & 0 & 0 & 1
\end{pmatrix} = (1; x + a, y + b, z + bx + c)
\] (2.3)
shows, according to (1.1). Here we consider \(L\) as projective collineation group with right actions in homogeneous coordinates. We will use the Cartesian homogeneous coordinate simplex \(E_0(e_0), E_1^\infty(e_1), E_2^\infty(e_2), E_3^\infty(e_3), \{e_i\} \subset V^4\) with the unit point \(E(e = e_0 + e_1 + e_2 + e_3)\) which is distinguished by an origin \(E_0\) and by the ideal points of coordinate axes, respectively. Moreover, \(y = cx\) with \(0 < c \in R\) (or \(c \in R \setminus \{0\}\)) defines a point \((x) = (y)\) of the projective 3-sphere \(\mathcal{PS}^3\) (or that of the projective space \(\mathcal{P}^3\) where opposite rays \((x)\) and \((-x)\) are identified). The dual system \(\{(e^i)\}, \{e^i\} \subset V_4\) describes the simplex planes, especially the plane at infinity \((e^0) = E_1^\infty E_2^\infty E_3^\infty\), and generally, \(v = u_c^1\) defines a plane \((u) = (v)\) of \(\mathcal{PS}^3\) (or that of \(\mathcal{P}^3\)). Thus \(0 = xu = yv\) defines the incidence of point \((x) = (y)\) and plane \((u) = (v)\), as \((x)I(u)\) also denotes it. Thus \(\text{Nil}\) can be visualized in the affine 3-space \(A^3\) (so in \(E^3\)) as well.

The translation group \(L\) defined by formula (2.3) can be extended to a larger group \(G\) of collineations, preserving the fibering, that will be equivalent to the (orientation preserving) isometry group of \(\text{Nil}\). In [11] E. Molnár has shown that
a rotation through angle $\omega$ about the $z$-axis at the origin, as isometry of \( \text{Nil} \), keeping invariant the Riemann metric everywhere, will be a quadratic mapping in $x, y$ to $z$-image $\overline{z}$ as follows:

$$
\mathbf{r}(O, \omega) : (1; x, y, z) \rightarrow (1; \overline{x}, \overline{y}, \overline{z});
\overline{x} = x \cos \omega - y \sin \omega, \quad \overline{y} = x \sin \omega + y \cos \omega,
\overline{z} = z - \frac{1}{2} xy + \frac{1}{4} (x^2 - y^2) \sin 2\omega + \frac{1}{2} xy \cos 2\omega.
$$

(2.4)

This rotation formula, however, is conjugate by the quadratic mapping $\mathcal{M}$

$$
x \rightarrow x' = x, \quad y \rightarrow y' = y, \quad z \rightarrow z' = z - \frac{1}{2} xy \quad \text{to}
$$

$$
(1; x', y', z') \rightarrow (1; x', y', z') \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \omega & \sin \omega & 0 \\
0 & -\sin \omega & \cos \omega & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = (1; x'', y'', z''),
$$

(2.5)

with $x'' \rightarrow \overline{x} = x''$, $y'' \rightarrow \overline{y} = y''$, $z'' \rightarrow \overline{z} = z'' + \frac{1}{2} x'' y''$.

i.e. to the linear rotation formula. This quadratic conjugacy modifies the \( \text{Nil} \) translations in (2.3), as well. We shall use the following important classification theorem.

**Theorem 2.1 (E. Molnár [11])**

1. Any group of \( \text{Nil} \) isometries, containing a 3-dimensional translation lattice, is conjugate by the quadratic mapping in (2.5) to an affine group of the affine (or Euclidean) space $A^3 = E^3$ whose projection onto the (x,y) plane is an isometry group of $E^2$. Such an affine group preserves a plane $\rightarrow$ point polarity of signature $(0, 0, \pm 0, +)$.

2. Of course, the involutive line reflection about the $y$ axis

$$
(1; x, y, z) \rightarrow (1; -x, y, -z),
$$

preserving the Riemann metric, and its conjugates by the above isometries in 1 (those of the identity component) are also \( \text{Nil} \)-isometries. There does not exist orientation reversing \( \text{Nil} \)-isometry.

**Remark 2.2** We obtain from the above described projective model a new model of \( \text{Nil} \) geometry derived by the quadratic mapping $\mathcal{M}$. This is the linearized model of \( \text{Nil} \) space (see [11]).
2.1 Geodesic curves and spheres

The geodesic curves of the Nil geometry are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves \( g(x(t), y(t), z(t)) \) in our model can be determined by the general theory of Riemann geometry. We can assume, that the starting point of a geodesic curve is the origin because we can transform a curve into an arbitrary starting point by translation (2.1);

\[
x(0) = y(0) = z(0) = 0; \quad \dot{x}(0) = c \cos \alpha, \quad \dot{y}(0) = c \sin \alpha, \quad \dot{z}(0) = w; \quad -\pi \leq \alpha \leq \pi.
\]

The arc length parameter \( s \) is introduced by

\[
s = \sqrt{c^2 + w^2 \cdot t}, \quad \text{where} \quad w = \sin \theta, \quad c = \cos \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},
\]

i.e. unit velocity can be assumed.

**Remark 2.3** Thus we have harmonized the scales along the coordinate axes.

The equation systems of a helix-like geodesic curves \( g(x(t), y(t), z(t)) \) if \( 0 < |w| < 1 \):

\[
x(t) = \frac{2c}{w} \sin \frac{wt}{2} \cos \left( \frac{wt}{2} + \alpha \right), \quad y(t) = \frac{2c}{w} \sin \frac{wt}{2} \sin \left( \frac{wt}{2} + \alpha \right),
\]

\[
z(t) = wt \cdot \left\{ 1 + \frac{c^2}{2w^2} \left[ \left( 1 - \sin(2wt + 2\alpha) \right) - \frac{\sin(2wt)}{2wt} \right] \right\} = \left\{ 1 + \frac{c^2}{2w^2} \left[ \left( 1 - \frac{\sin(wt)}{wt} \right) + \frac{1 - \cos(2wt)}{2wt} \sin(wt + 2\alpha) \right] \right\}.
\]

(2.6)

In the cases \( w = 0 \) the geodesic curve is the following:

\[
x(t) = c \cdot t \cos \alpha, \quad y(t) = c \cdot t \sin \alpha, \quad z(t) = \frac{1}{2} c^2 \cdot t^2 \cos \alpha \sin \alpha.
\]

(2.7)

The cases \( |w| = 1 \) are trivial: \((x, y) = (0, 0), z = w \cdot t\).

**Definition 2.4** The distance \( d(P_1, P_2) \) between the points \( P_1 \) and \( P_2 \) is defined by the arc length of geodesic curve from \( P_1 \) to \( P_2 \).
In our work [22] we introduced the following definitions:

**Definition 2.5** The geodesic sphere of radius $R$ with centre at the point $P_1$ is defined as the set of all points $P_2$ in the space with the condition $d(P_1, P_2) = R$. Moreover, we require that the geodesic sphere is a simply connected surface without self-intersection in the Nil space.

**Remark 2.6** We shall see that this last condition depends on radius $R$.

**Definition 2.7** The body of the geodesic sphere of centre $P_1$ and of radius $R$ in the Nil space is called geodesic ball, denoted by $B_{P_1}(R)$, i.e. $Q \in B_{P_1}(R)$ iff $0 \leq d(P_1, Q) \leq R$.

**Remark 2.8** Henceforth, typically we choose the origin as centre of the sphere and its ball, by the homogeneity of Nil.

We have denoted by $B(S)$ the body of the Nil sphere $S$, furthermore we have denoted their volumes by $\text{Vol}(B(S))$.

In [22] we have proved the following theorem:

**Theorem 2.9** The geodesic sphere and ball of radius $R$ exists in the Nil space if and only if $R \in [0, 2\pi]$.

We obtain the volume of the geodesic ball of radius $R$ by the following integral (see 2.8):

$$
\text{Vol}(B(S)) = 2\pi \int_0^{\pi} \sin^2 \theta \, d\theta = 
$$

$$
= 2\pi \int_0^{\pi} \left( \frac{2 \cos \theta \sin \frac{(R \sin \theta)}{2}}{\sin \theta} \right)^2 \cdot \left( -\frac{1}{2} \frac{R \cos^3 \theta}{\sin^2 \theta} + \frac{\cos \theta \sin (R \sin \theta)}{\sin \theta} + \frac{\cos^3 \theta \sin (R \sin \theta)}{\sin^3 \theta} - \frac{1}{2} \frac{R \cos^3 \theta \cos (R \sin \theta)}{\sin^2 \theta} \right) d\theta.
$$

(2.8)

The parametric equation system of the geodesic sphere $S(R)$ in our model (see [22]):

$$
x(R, \theta, \phi) = \frac{2c}{w} \sin \frac{wR}{2} \cdot \cos \phi = \frac{2 \cos \theta}{\sin \theta} \cdot \frac{R \sin \theta}{2} \cdot \cos \phi,
$$

$$
y(R, \theta, \phi) = \frac{2c}{w} \sin \frac{wR}{2} \cdot \sin \phi = \frac{2 \cos \theta}{\sin \theta} \cdot \frac{R \sin \theta}{2} \cdot \sin \phi,
$$

$$
z(R, \theta, \phi) = wR + \frac{c^2 R}{2w} - \frac{c^2}{2w^2} \sin wR + \frac{1}{4} \left( \frac{2c}{w} \sin \frac{wR}{2} \right)^2 \sin 2\phi =
$$

$$
= R \sin \theta + \frac{R \cos^2 \theta}{2 \sin \theta} - \frac{\cos^2 \theta}{2 \sin^2 \theta} \sin (R \sin \theta) + \frac{1}{4} \left( \frac{2 \cos \theta}{\sin \theta} \sin \frac{R \sin \theta}{2} \right)^2 \sin 2\phi
$$
\[ -\pi < \phi \leq \pi, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad \text{and} \quad \theta \neq 0. \]

If \( \theta = 0 \) then
\[
\begin{align*}
x(R, 0, \phi) &= R \cos \phi, \\
y(R, 0, \phi) &= R \sin \phi, \\
z(R, 0, \phi) &= \frac{1}{2} R^2 \cos \phi \sin \phi.
\end{align*}
\]

We have obtained by the derivatives of these parametrically represented functions (by intensive and careful computations with Maple through the second fundamental form) the following theorem (see [22]):

**Theorem 2.10** The geodesic Nil ball \( B(S(R)) \) is convex in affine-Euclidean sense in our model if and only if \( R \in [0, \frac{\pi}{2}] \).

### 3 Nil prisms and prism tilings

The prisms and prism-like tilings have been thoroughly investigated in \( S^2 \times \mathbb{R}, H^2 \times \mathbb{R} \) and \( \widetilde{\text{SL}_2\mathbb{R}} \) spaces in papers [16], [19], [30]. Here we consider the analogous problem in Nil geometry. In the following the plane of \( x, y \) axis are called base plane of the model and if we say plane then it is a plane in Euclidean sense.

**Definition 3.1** Let \( \mathcal{P}^i \) be an infinite solid bounded by planes, that are determined by fibre-lines passing through the points of a \( p \)-gon (\( p \geq 3 \), integer parameter) \( \mathcal{P}^b \) lying in the base-plane. The images of \( \mathcal{P}^i \) by Nil isometries are called infinite \( p \)-sided prisms.

The common part of \( \mathcal{P}^i \) with the base plane is defined as the base figure \( \mathcal{P}^b \) of the prism.

Let \( \mathcal{F} \) be the \( \mathcal{M}^{-1} \) image of the base plane in the Nil-space (see Remark 2.2) and let \( \tau \) be a fibre translation (2.2).

**Definition 3.2** Let \( \mathcal{P}^i \) be an infinite \( p \)-sided prism, that is trimmed by the surface \( \mathcal{F} \) and its translated copy \( \mathcal{F}^\tau \). The parts of \( \mathcal{F} \) and \( \mathcal{F}^\tau \) inside the infinite prism are called cover faces and are denoted by \( C_\mathcal{F} \) and \( C_{\mathcal{F}^\tau} \).

The \( p \)-sided bounded prism is the part of \( \mathcal{P}^i \) between the cover faces \( C_\mathcal{F} \) and \( C_{\mathcal{F}^\tau} \).

**Definition 3.3** A bounded or infinite \( p \)-sided prism is said to be regular if its side surfaces are congruent to each other under Nil rotations with angle \( \frac{2\pi}{p} \) (see (2.4) and (2.5)) about the central fibre line of the prism.
3.1 Regular bounded prism tilings

In this section we will investigate the existence of regular bounded prism tilings \( T_p(q) \) of Nil space. In this case the prism tiles are regular bounded prisms having \( p \)-gonal base figures \((p \geq 3)\). The prism itself is a topological polyhedron with \( 2p \) vertices, and having at every vertex one \( p \)-gonal cover face and two quadrangle side faces (traced by fibre lines). We are looking such prism tilings of Nil space where at each side edge of the prism (which are fibre lines going through vertices of the base figure) meet \( q \) prisms regularly, by Nil rotations with angle \( \frac{2\pi}{q} \) \((q \geq 3, \text{integer parameter})\).

We shall see in Theorem 3.4 that the regular prism tiling \( T_p(q) \) exists for some parameters \((p, q)\). Let \( \mathcal{P}_p(q) \) one of its tiles with with vertices \( A_1A_2 \ldots A_p B_1B_2 \ldots B_p \). We may assume that \( A_1 \) lies on the \( x \)-axis. It is clear that the side curves \( c_{A_iA_{i+1}} \) \((i = 1 \ldots p, A_{p+1} \equiv A_1)\) are derived from each other by \( \frac{2\pi}{p} \) rotation about the \( x \) axis. The corresponding vertices \( B_1B_2 \ldots B_p \) are generated by a fibre translation \( \tau \) with a positive real parameter. The cover faces \( A_1, \ldots, A_p, B_1, \ldots, B_p \) and the side surfaces form a \( p \)-sided regular prism \( \mathcal{P}_p(q) \) in Nil. \( T_p(q) \) will be generated by its rotational isometry group \( \Gamma_p(q) = pq2_1 \subset Isom(\text{Nil}) \) (if these tiling there exist see Theorem 3.4) which is given by its fundamental domain \( \mathcal{F}_p(q) = A_1A_2OA_1sA_2sO^s, A_1s = B_p \). Here, \( A_2 = B_1, O^s = O^\tau, \text{and } \mathcal{F}_p(q) \) is a piece-wise linear topological polyhedron. The group presentation can be determined by a standard procedure \[8], called Poincaré algorithm. The generators will pair the bent (piecewise linear) faces of \( \mathcal{F} \):

\[
\begin{align*}
\mathbf{a} : & \quad OA_1B_pO^s(O) \rightarrow OA_2B_1O^s(O), \\
\mathbf{b} : & \quad A_1A_2B_1(A_1) \rightarrow A_1B_pB_1(A_1), \quad \mathbf{s} : \quad OA_1A_2(O) \rightarrow O^sB_pB_1(O^s)
\end{align*}
\]

mapping \( \mathcal{F}_p(q) \) onto its neighbours \( \mathcal{F}_p(q)^a, \mathcal{F}_p(q)^b, \mathcal{F}_p(q)^s \), respectively. E.g. for the face \( a^{-1} \) a point \( A \) (relative freely, e.g. in the segment \( OB_p \)) is taken. Then the union of triangles \( AO^s, AO^sB_p, AB_pA_1, OA_1O \) will be the face \( a^{-1} \).

Then the \( a \)-image \( A^a \) is taken in \( OB_1 \) for the face \( a = A^aOO^s \cup A^aO^sB_1 \cup A^aB_1A_2 \cup A^aA_2O, \) as usual. The relations are induced by the edge equivalence classes \{\( OO'\); \{\( A_1B_1\); \{\( OA_1, OA_2, O'B_1, O'B_p\); \{\( A_1A_2, A_1B_p, A_2B_1, B_pB_1\)\}. So we get the group

\[
pq2_1 = \{a, b : a^p = b^q = ababa^{-1}b^{-1}a^{-1}b^{-1} = 1\}, \tag{3.1}
\]

where \( a \) is a \( p \) rotation about the fibre line through the origin (the \( z \)-axis), \( b \) is a \( q \) rotation about a side fibre line of \( \mathcal{P}_p(q) \) (through a vertex of its base figure).
Notice, that \( \text{bab} \) is a screw motion, and thus \( \tau := \text{abab} = \text{baba} \) is the fibre translation connecting the cover faces.

Our first question is the following: For which \( 3 \leq p, q \in \mathbb{N} \) is \( \Gamma_p(q) = pq2_1 \subset \text{Isom}((\text{Nil})) \)?

The following Theorem answers it:

**Theorem 3.4** In \( \text{Nil} \) there exist 3 regular \( p \)-gonal non-face-to-face prism tilings \( T_p(q) \) with \( \text{Nil} \) isometry group \( \Gamma_p(q) = pq2_1 \) for integer parameters \( p, q \geq 3 \):

- the regular triangular prism tiling with \( (p, q) = (3, 6) \),
- the regular square prism tiling with \( (p, q) = (4, 4) \),
- the regular hexagonal prism tiling with \( (p, q) = (6, 3) \),

and each group \( \Gamma_p(q) \) has a free parameter \( x_p(q) \in \mathbb{R}^+ \).

Figure 1: The regular triangular and rectangular prisms in \( \text{Nil} \)-space, with parameters \( x_3(6) = \frac{2}{\sqrt{3}} \) and \( x_4(4) = \sqrt{2} \) respectively

**Proof:**

Let \( A_1 = (1; x_p(q), 0, 0) \) be a "bottom" vertex of the regular bounded prism. Then the other vertices of the bottom cover face can be generated by the \( \text{Nil} \)
rotation formula (see (2.4), (2.5)): 

\[ A_2 = A_1^a = \left(1; x_p(q) \cos \left(\frac{2\pi}{p}\right), x_p(q) \sin \left(\frac{2\pi}{p}\right), \frac{1}{4}x_p(q)^2 \sin \left(\frac{4\pi}{p}\right)\right), \]

\[ A_3 = A_2^a = A_1^{a^2} = \left(1; x_p(q) \cos \left(\frac{4\pi}{p}\right), x_p(q) \sin \left(\frac{4\pi}{p}\right), \frac{1}{4}x_p(q)^2 \sin \left(\frac{8\pi}{p}\right)\right), \]

\[ \ldots \]

\[ A_p = A_{p-1}^a = A_1^{a^{p-1}}. \]

Then the condition for the existence of the tiling is the following:

\[ A_3 = A_2^a \equiv A_1^{b^{-1}}, \quad (3.2) \]

where a is a p rotation about the fibre line through the origin and b is a q rotation about the side fibre line of \( P_p(q) \) through the vertex \( A_2 \). \( \equiv \) means that the corresponding points lie on the same fibre lines.

\[ 1 = \cos^2 \left(\frac{\pi}{p}\right) + \cos^2 \left(\frac{\pi}{q}\right), \]

where p and q are positive integers. This equation only has the following integer solutions:

\( (p, q) = (4, 4), (3, 6) \) or \( (6, 3) \).

We obtain from the above computations, that the existence of the above regular prism tilings is independent from the parameter \( x_p(q) \in \mathbb{R}^+ \), so we have proven the Theorem. \( \square \)

Remembering that \( \tau = abab \) is the "vertical" translation of the group, we can also compute the height of the regular bounded prism corresponding to the group tiling, since: \( O^{abab} = O^{\tau} \), where O is the origin. Using this, we can also give a metric representation of the group, allowing the visualization of the corresponding prism and prism tiling (see Fig. 1. and Fig. 2.).

4 The optimal geodesic ball packings under group \( pq^2_1 \)

The sphere packing problem deals with the arrangements of non-overlapping equal spheres, or balls, which fill the space. While the usual problem is in the \( n \)-dimensional Euclidean-space \( (n \geq 2) \), it can be generalized to the other 3-dimensional Thurston spaces (see [29]). In this paper we investigate the optimal ball packings of Nil generated by the above described \( pq^2_1 \) group.
Let $\mathcal{T}_p(x_p(q), q)$ (where $(p, q) = (3, 6), (4, 4)$ or $(6, 3)$ as stated above and $x_p(q) \in \mathbb{R}^+)$ be a regular prism tiling, and let $\mathcal{P}_p(x_p(q), q)$ be one of its tiles that is centered at the origin, with a base face given by the vertices $A_1, A_2, \ldots, A_p$. The corresponding vertices $B_1, B_2, \ldots, B_p$ of the prism are generated by fibre translations $\tau = a b a b$.

We can assume by symmetry, that the optimal geodesic ball is centered at the origin. The volume of a geodesic ball with radius $R$ can be determined by the formula (2.8).

We study only one case of the multiply transitive geodesic ball packings where the fundamental domains of the Nil space groups $pq2_1$ are not prisms. Let the fundamental domains be derived by the Dirichlet — Voronoi cells (D-V cells) where their centers are images of the origin. The volume of the $p$-times fundamental domain and of the D-V cell is the same, respectively, as in the prism case (for any above $(p, q, x_p(q))$ fixed). It is easy to see by the formulas (2.5), using the quadratic mapping $\mathcal{M}$, that the volume of the Dirichlet — Voronoi cell (or the corresponding prism) is

$$\text{Vol}(\mathcal{P}_p(x_p(q), q)) = \frac{p}{2} x_p^2(q) \sin \left(\frac{2\pi}{p}\right) \text{d}(OO^r).$$

(4.1)

These locally densest geodesic ball packings can be determined for all possible fixed integer parameters $p, q, x_p(q)$. The optimal radius $R_{\text{opt}}(x_p(q))$ is

$$R_{\text{opt}}(x_p(q), p, q) = \min \left\{ \text{d}(O A_1), \frac{\text{d}(OO^r)}{2}, \frac{\text{d}(O, O^{ab})}{2} \right\},$$

(4.2)

where $\text{d}$ is the geodesic distance function of Nil geometry (see Definition 2.4).

Since the congruent images of $\mathcal{P}_p(x_p(q), q)$ under the discrete group $pq2_1$ cover the Nil space, therefore for the density of the ball packing it is sufficient to relate the volume of the ball to the volume of the prism:

**Definition 4.1** The maximal density $\delta_p(x_p(q), q)$ of the above multiply transitive ball packing for given parameters $(p, q, x_p(q)) ((p, q) = (3, 6), (4, 4), (6, 3)$ and $x_p(q) \in \mathbb{R}^+)$:

$$\delta_p(x_p(q), q) = \frac{\text{Vol}(B(R_{\text{opt}}))}{\text{Vol}(\mathcal{P}_p(x_p(q), q))} = \frac{\text{Vol}(B(R_{\text{opt}}))}{\frac{p}{2} x_p^2(q) \sin \left(\frac{2\pi}{p}\right) \text{d}(OO^r)}.$$  

(4.3)

For every $p, q, x_p(q)$ parameters the locally densest geodesic ball packing can be determined.
If we fixed the parameters \( p \) and \( q \) then the distance function \( d(x_p(q)) \) is a continuous function. Therefore it is easy to prove the following Lemma:

**Lemma 4.2** In Nil-space for the rotation group \( \Gamma_p(q) = pq2_1 \) there always exist \( x_p(q) \in \mathbb{R}^+ \) for given parameters \((p, q) = (4, 4), (3, 6), (6, 3)\) where

\[
d(O, O^{ab}) = d(O, O^{abab}) = d(O, O^r).
\] (4.4)

Figure 2: Some balls of the optimal ball arrangements for the square and hexagonal tilings with parameters \((p, q) = (4, 4)\) and \((p, q) = (3, 6)\).

The system of equations (4.4) in Lemma 4.2 can be solved by numerical methods and the corresponding ball arrangements are denoted by \( B_p(q) \). We obtain - using the formulas (4.1-3) - that in Nil-space for the rotation group \( \Gamma_p(q) = pq2_1 \) the metric data of the godesic ball arrangements \( B_p(q) \) are the following:

**Theorem 4.3** If the system of equation (4.4) holds then the maximal radii and densities of the optimal ball packings are the following:

- If \((p, q) = (3, 6)\), then \( \delta_p(q) \approx 0.2593 \), with \( R_{opt}(p, q) \approx 0.7389 \),
- If \((p, q) = (4, 4)\), then \( \delta_p(q) \approx 0.6512 \), with \( R_{opt}(p, q) \approx 1.2154 \),
- If \((p, q) = (6, 3)\), then \( \delta_p(q) \approx 0.7272 \), with \( R_{opt}(p, q) \approx 1.9601 \).
If we vary the parameter $x_p(q)$ in the above cases then the corresponding radius $R_{\text{opt}}(p, q)$ and the density $\delta_p(q)$ also change. The following table shows that probably the $B_p(q)$ ball packings with maximal kissing numbers provide the optimal ball packing densities.

| $(p, q)$ | Radius | Prism volume | Density | Kissing number |
|----------|--------|--------------|---------|----------------|
| (3,6)    | 0.5876 | 4.1446       | 0.2063  | 2              |
|          | 0.6392 | 4.9032       | 0.2246  | 2              |
|          | 0.6929 | 5.7616       | 0.2438  | 2              |
|          | 0.7389 | 6.5517       | 0.2593  | 8              |
|          | 0.7787 | 7.8111       | 0.2558  | 6              |
|          | 0.8132 | 9.0201       | 0.2525  | 6              |
|          | 0.8481 | 10.3641      | 0.2495  | 6              |
| (4,4)    | 0.9927 | 7.8849       | 0.5283  | 2              |
|          | 1.0644 | 9.0650       | 0.5678  | 2              |
|          | 1.1386 | 10.3729      | 0.6090  | 2              |
|          | 1.2154 | 11.8175      | 0.6512  | 10             |
|          | 1.2594 | 13.4079      | 0.6404  | 8              |
|          | 1.3036 | 15.1538      | 0.6295  | 8              |
|          | 1.3480 | 17.0647      | 0.6194  | 8              |
| (6,3)    | 1.6934 | 34.4141      | 0.6190  | 2              |
|          | 1.7801 | 38.0287      | 0.6537  | 2              |
|          | 1.8690 | 41.9209      | 0.6897  | 2              |
|          | 1.9601 | 46.1044      | 0.7272  | 14             |
|          | 2.0087 | 50.5935      | 0.7153  | 12             |
|          | 2.0573 | 55.4028      | 0.7038  | 12             |
|          | 2.1059 | 60.5470      | 0.6929  | 12             |

Therefore, we can formulate by the above results the following conjecture:

**Conjecture 4.4** The ball arrangements $B_p(q)$ provide the densest ball packing arrangements related to $\Gamma_p(q) = pq2_1$ Nil isometry group with parameters $(p, q) = (4, 4), (3, 6), (6, 3)$.

**Remark 4.5** The optimal ball packing in the case of $(p, q) = (6, 3)$ has a kissing number of 14, which is greater than the maximal kissing number 12 in the Euclidean 3-dimensional space. In fact, this is the second ball packing arrangement in Nil that has this high of a kissing number (see [22]).
Figure 3: Some balls of the optimal ball arrangements for the square and hexagonal tilings shown from the direction of the $z$-axis.

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