The Motion of Massive Test Particles in Dark Matter

with an $a_0/r^2$ Energy Density

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Abstract

The motion of massive test particles in dark matter is studied. It is shown that if the energy density of the dark matter making up a galactic halo has a large $r$ behavior of $1/r^2$, then contrary to intuition the motion of these test particles are not governed by Newtonian gravity, but rather by the equations of geodesic motion from Einstein’s theory of general relativity. Moreover, the rotational velocity curves of orbiting massive test particles in this energy density do not approach a constant value at large $r$ but will instead always increase with the radius of the orbit $r_c$.

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§1. Introduction

One of outstanding problems in astrophysics today is to explain the motion of bodies with orbits in the galactic halo. From Newtonian dynamics one would expect that for a body in a circular orbit

\[ \frac{v_{\phi}^2}{r_c} = \frac{G M_g}{r_c^2} \]

where \( G \) is the gravitational constant, \( M_g \) is the luminous mass of the galaxy, \( r_c \) is the radius of the circular orbit and \( v_{\phi} \) is its rotational velocity. If one were then to plot \( v_{\phi} \) verses \( r \) for various bodies orbiting in a galactic halo one would expect to obtain a rotational velocity curve which decreases as \( 1/\sqrt{r_c} \). This does not, in fact, happen. Experimentally, it is instead found that the rotational velocity curve approaches a constant value at large \( r \). To explain this result, the presence of “dark matter”, matter which has yet to be detected, has traditionally been proposed. (See [1]–[3] and the references contained therein.) Namely, it has been postulated that the galactic halo is filled with a gas of weekly interacting particles with a mass density \( \rho \). With their presence the total mass contained within the radius of the orbiting body changes and the Newtonian equation of motion now becomes

\[ v_{\phi}^2 = G \left( \frac{M_g}{r_c} + \frac{4\pi}{r_c} \int_0^{r_c} \rho r^2 dr \right). \]

If we now take \( \rho \approx a_0 c^2/(G r^2) \) for large \( r \) where \( a_0 \) is a dimensionless constant, then as long as

\[ a_0 \gg \frac{M_g G}{4\pi R_g c^2}, \tag{1} \]

the \( M_g \) term may be neglected and a constant \( v_{\phi} \) with value

\[ \frac{v_{\phi}^2}{c^2} \approx 4\pi a_0, \]
can be obtained. \((R_g\) is the point at which the velocity curves become a constant and is identified as the radius of the luminous galaxy.) Explanation of the rotational velocity curves then reduces to finding and detecting candidates for this dark matter which will have the correct large \(r\) behavior.

The basic premise of this argument is that Newtonian dynamics and gravity will still be valid even after the introduction of a \(1/r^2\) energy density into the system. This need not be true. One should remember that Newtonian gravity is only an approximation of Einstein’s theory of general relativity and that the very act of introducing a mass density \(\rho \sim 1/r^2\) for dark matter introduces an unconfined energy density into the system. Its presence cannot help but have an affect on the geometry of the spacetime in the halo, and, consequently, on the motion of bodies orbiting there.

In this paper we shall show that the standard argument using Newtonian dynamics and gravity for the existence of dark matter with a \(1/r^2\) energy density is inconsistent. The introduction of an energy density which behaves as \(1/r^2\) for large \(r\) changes the geometry of the spacetime so drastically that the motion of bodies in the galactic halo is necessarily non-Newtonian. The basic premise that Newtonian gravity is still valid even after the introduction of this dark matter is incorrect. Furthermore, after using the full geodesic equation from Einstein’s theory of general relativity to analyze circular orbits in the \(1/r^2\) energy density, we find that \(v^2_{\phi} \sim qr_c^q\) for \(0 < q \leq 1\). The rotational velocity always increases with \(r_c\) and will not approach a constant value. In fact, for a static, spherical geometry there are only a very narrow range of essentially unphysical choices for the energy density which will give a constant \(v_\phi\) for \(r > R_g\).

\section*{§2. Geometry of the \(a_0/r^2\) energy density}

We begin by modeling the galaxy as a sphere of mass \(M_g\) and radius
$R_g$, which is surrounded by a galactic halo made up of dark matter. The most general static, spherically symmetric metric is known to be [4]

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $f$ and $h$ are unknown functions of $r$ only which need to be determined. As usual, we write the energy momentum tensor for the system as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu),$$

where $\rho$ is the energy density of the particles in the halo, $p$ is their pressure, and $u_\mu$ is an unit velocity vector in the direction of the timelike Killing vector for the system. From Einstein’s equations [4] we then obtain

$$8\pi \rho = \frac{1}{r^2} \frac{d}{dr} \left\{ r \left( 1 - \frac{1}{h} \right) \right\},$$

$$8\pi p = \frac{f'}{rfh} - \frac{1}{r^2} \left( 1 - \frac{1}{h} \right),$$

$$8\pi p = \frac{1}{2} \frac{f''}{fh} - \frac{1}{4} \frac{f'}{fh} \left( \frac{f'}{f} + \frac{h'}{h} \right) + \frac{f'}{2rfh} - \frac{h'}{2rh^2},$$

where we are using units in which $G = c = 1$ and the primes denote derivatives with respect to $r$.

Suppose now that the energy density of dark matter is $\rho \approx a_0/r^2$ for $r > R_g$ where $a_0 > 0$ is a (dimensionless) constant. Then the first equation in (2) is trivial to integrate giving

$$h^{-1} = 1 - 8\pi a_0 - \frac{K}{r},$$

where $K$ is an integration constant. Since in the absence of dark matter we would expect to have obtained the Schwarzchild solution, we identify $K$ with $2M_g$. Next, notice that as long as the Newtonian bound (1) on $a_0$ holds

$$a_0 \gg \frac{M_g}{4\pi r},$$
and we can neglect this term in $h$ and can approximate $h^{-1} \approx 1 - 8\pi a_0$ as a constant for $r > R_g$. We shall justify this approximation later.

The difference of the second two equations gives

$$0 = \frac{1}{2} \frac{f''}{f} - \frac{1}{4} \frac{f'}{f} \left( \frac{f'}{f} + \frac{h'}{h} \right) - \frac{1}{2r} \left( \frac{f'}{f} + \frac{h'}{h} \right) + \frac{(h - 1)}{r^2}. \quad (3)$$

Since $h$ is a constant for $r > R_g$, it is straightforward to solve,

$$f(r) \approx \left( k_+ r^{q_+/2} + k_- r^{q_-/2} \right)^2,$$

where

$$q_- = 2 \pm 2 \left( \frac{1 - 16\pi a_0}{1 - 8\pi a_0} \right)^{1/2},$$

and $k_\pm$ are integration constants. Because $f > 0$, $a_0 \leq 1/16\pi$ otherwise $f$ will contain oscillatory solutions. Since we are working in the large $r$ limit, and since $q_+ > q_-$, only one of the two solutions to (2) will survive. Linear combinations of the two will not. We can thus consider each solution independently and for convenience we shall write $f_\pm = k_\pm r^{q_\pm}$. Then

$$8\pi p_\pm \approx \frac{q_\pm + 1 - h}{hr^2},$$

and we see that the pressure also varies as $1/r^2$. Moreover,

$$p = \left( \frac{q_\pm}{4 - q_\pm} \right) \rho,$$

and since $q_+ \geq 2$, $p_+ > \rho$. Because $p \leq \rho/3$, we must therefore exclude the $(+)$ solutions as being unphysical. The only physical solutions are the $(-)$ solutions, and we thus set $q = q_-$, $p = p_-$ and $f = kr^q$. Furthermore, $0 < q \leq 1$ while $M_g/(4\pi R_g) < a_0 \leq 3/(56\pi)$. Consequently, $1 < h \leq 7/4$, and, since $f \sim r^q$, we can see explicitly that spacetime in the presence of the $1/r^2$ energy density is not flat, but is instead quite curved.
We next consider the motion of a massive test particle in a static, spherically symmetric geometry with a velocity $v_\mu$ such that $-1 = v_\mu v^\mu$.

This constraint gives

$$-1 = - f \left(\frac{dt}{d\tau}\right)^2 + h \left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\theta}{d\tau}\right)^2 + r^2 \sin^2 \theta \left(\frac{d\phi}{d\tau}\right)^2,$$

where $\tau$ is the proper time of the particle. Working in the equatorial $\theta = \pi/2$ plane, and using energy and angular momentum conservation,

$$E = f \frac{dt}{d\tau} \quad , \quad L = r^2 \frac{d\phi}{d\tau},$$

(4)

where $E$ and $L$ are the energy and orbital angular momentum per unit mass, respectively, of the particle, we obtain

$$0 = \left(\frac{dr}{dt}\right)^2 + V(r),$$

where

$$V(r) = \frac{f^2}{E^2 h} \left(1 + \frac{L^2}{r^2}\right) - \frac{f}{h},$$

is an effective potential energy. For circular motion, $r = r_c$, the radius of the circular orbit which is a constant in time. Consequently, $V(r_c) = 0$ and $V'(r_c) = 0$, giving

$$E^2 = \frac{2 f^2(r_c)}{2 f(r_c) - r_c f'(r_c)} \quad , \quad \frac{L^2}{r_c^2} = \frac{r_c f'(r_c)}{2 f(r_c) - r_c f'(r_c)}.$$

(5)

Defining the rotational velocity as

$$v_\phi = r \frac{d\phi}{dt}$$

then from (4) and (5) we find that for circular motion,

$$v_\phi^2 = \frac{1}{2} r_c f'(r_c).$$

(6)
This equation holds for any $f$. In particular, if we are dealing with a spherical mass $M$ in free space, then $f = (1 - 2M/r)$ and (6) reduces to what one obtains from Newtonian gravity. If, on the other hand, $\rho \sim 1/r^2$, then from the above $f = kr^q$, so that

$$v_\phi^2 = \frac{1}{2} qkr^q.$$

For $q \neq 0$, $v_\phi$ always increases with the radius of the orbit and never approaches a constant value as one would naively expect from Newtonian gravity.

In the above solution of (3) for $f$ we have neglected the contribution of $h'/h$ in comparison to $f'/f$. We shall now justify this approximation. First, we note that for the galaxy, $2M/r \ll 1$ and, due to the bound (1) on $a_0$, this term is very small in comparison to $1 - 8\pi a_0$. Next, note that while $h'/h \sim 2M/r^2$, the solution we obtained by taking $h$ as a constant gives $f'/f \sim 1/r$. Since $r$ is large, we would expect $h'/h$ to have a very small affect on the $kr^q$ solution of (3). Consequently, we can take $h'/h$ as a small perturbation and solve (3) perturbatively about the $kr^q$ solution. After doing so, we find that to first order in $2M/r$,

$$f = kr^q \left[ 1 - \frac{2Mh_0}{(3-q)r} \left( \frac{q}{2} + 2h_0 + 1 \right) \right],$$

where $h_0^{-1} = 1 - 8\pi a_0$. The inclusion of the $2M/r$ term in $h$ modifies the $kr^q$ solution very slightly since $2M/r \ll 1$, and $1 < h_0 \leq 7/4$. Consequently, we were justified in neglecting this contribution to $f$.

§3. Constant $v_\phi$ energy density

We now ask whether or not it is possible for any physically reasonable $\rho$ to result in a $v_\phi$ which will be constant for $r > R_g$. Using (6), we find that for $v_\phi$ is to be a constant outside of the galaxy, $f$ must then have the
approximate form of \( f_v \approx 2v^2_\phi \log(r/r_0) \) for \( r \) greater than some \( r_0 \). (The subscript \( v \) will denote the fact that we are looking for solutions of Einstein’s equations which will result in a constant \( v_\phi \).) Because \( f_v > 0, r > r_0 \), and since \( v_\phi \) is a constant only outside of the galaxy, we shall identify \( r_0 \) with \( R_g \).

Since \( f_v \) is now given and \( h_v \) unknown, (3) becomes a differential equation for \( h_v \) which may be written as

\[
0 = \frac{d}{dy} \left( \frac{1}{h_v} \right) - \frac{1 + 2y}{y} \left( \frac{1}{h_v} \right) + \frac{4y}{1 + 2y},
\]

where \( y = \log(r/R_g) > 0 \). Its solution is straightforward,

\[
\frac{1}{h_v} = ye^{2y} \left( c_h + 2 \int_{2y}^{\infty} \frac{e^{-t}}{1 + t} dt \right),
\]

where \( c_h \) is an integration constant. Then from (2) we find that:

\[
8\pi r^2 \rho_v = 1 + \frac{4y}{1 + 2y} - (1 + 3y)e^{2y} \left( c_h + 2 \int_{2y}^{\infty} \frac{e^{-t}}{1 + t} dt \right),
\]

\[
8\pi r^2 p_v = (1 + y)e^{2y} \left( c_h + 2 \int_{2y}^{\infty} \frac{e^{-t}}{1 + t} dt \right) - 1. \tag{7}
\]

Physically, \( \rho_v \geq 3p_v \). This gives an upper bound of 0.0013 for \( c_h \). As \( h > 0 \) for all \( y > 0, c_h \geq 0 \). Consequently, \( 0 \leq c_h \leq 0.0013 \) and there are only a very narrow range of values for \( c_h \) which will result in a physically reasonable \( \rho_v \) and \( p_v \). Moreover, if \( c_h > 0 \), then it is only when \( y \) is between some \( y_{min} \) and \( y_{max} \) that \( \rho_v \geq 3p_v \). Outside of these two bounds the energy density must have a different form and \( v_\phi \) cannot be a constant. If, on the other hand, \( c_h = 0 \) then \( \rho_v \geq 3p_v \) for all \( y > 0.627 \). As long as \( r > e^{0.627}R_g \) the energy density given in (7) for \( c_h = 0 \) will result in a constant rotational velocity curve outside of the galaxy. For \( r < e^{0.627}R_g \) the energy density will have a different form and \( v_\phi \) will not be a constant, as expected.

§4. Concluding Remarks
We have thus shown that if the energy density \( \rho \) of dark matter behaves as \( 1/r^2 \) for large \( r \), then \( f \approx kr^q \) for some constant \( k \) while \( h \approx 1 - 8\pi a_0 \). Consequently, the premise that \( \rho \sim 1/r^2 \) for large \( r \) contradicts the premise that Newtonian gravity is valid in the halo. The introduction of this energy density, which is not confined but spread over a large area, alters the geometry of spacetime so drastically that Newtonian dynamics is no longer valid in the halo. The assumption that Newtonian gravity is valid even after the introduction of dark matter with a \( 1/r^2 \) energy density is incorrect. In fact, contrary to what is expected from Newtonian gravity, circular orbits for this energy density have a \( v^2_\phi = qkr^q/2 \) which always increases with \( r_c \). It never approaches a constant, although for very small \( q \) it increases very slowly.

From a physicist point of view, however, the major problem with using a \( 1/r^2 \) energy density for dark matter is not that there was an inconsistency in the argument for its introduction since this is straightforwardly resolved by using general relativity instead of Newtonian gravity to analyze the system; nor is it that \( v_\phi \) always increases with \( r_c \) since \( q \) may be taken to be quite small so that any increase in \( v_\phi \) occurs very gradually (with one caveat; see [5]). It is rather that the spacetime in a \( 1/r^2 \) energy density is so curved that Newtonian dynamics is no longer valid. If the energy density of the dark matter in the galactic halo truly does behave as \( 1/r^2 \) for large \( r \), this will present great difficulties in interpreting extragalactic astronomical observations. As most galaxies have a halo, including presumably our own, light from a distant galaxy would then first have to pass through its own halo, a region of curved spacetime, and then our through own galactic halo, another region of curved spacetime, before we can observe it. Since the usual assumption is that light from other galaxies passes through a spacetime which is essentially flat before it reaches us, if the energy density
of dark matter has a $1/r^2$ large $r$ behavior, then all of the extragalactic observational data would have to be re-evaluated and interpreted. This is just one of the many problems that would arise from using a $1/r^2$ energy density for dark matter which have yet to be addressed.

The $1/r^2$ energy density is an unconfined energy density which, presumably, extends for large distances into the spacetime. As the universe is known to be expanding and thus changing with time, one may question the validity of using a static solution of Einstein’s equations to analyze the motion of test particles in the dark matter as we have done. As, however, the galactic rotation curves only extend out to a few galactic radii, we are only interested in the behavior of the motion of bodies relatively close to the galaxy and in this region a static approximation is certainly valid. Of course, at some $r \gg R_0$ the $1/r^2$ energy density must be cut off-ed and our analysis will no longer be valid, but this will only happen outside the region we are interested in. We should also note that precisely the same static approximation is made in the standard analysis of the motion test particles in the galactic halo using Newtonian gravity. The important point here is not whether the use of a static, $1/r^2$ energy density to model the galactic halo is a valid approximation or not, but rather that our analysis of the motion of test particles in a $1/r^2$ energy density using general relativity holds in precisely the same regime in which the standard analysis using Newtonian gravity was presumed to be valid.

As (3) is a non-linear second order differential equation in $f$, it may be that we truly cannot neglect the $h'/h$ term in (3) no matter how small $2M/r$ is. In §3, however, we have shown that in a static, spherical geometry a rotational velocity curve which is truly a constant for $r > R_0$ is obtainable only for the very special choices of the $\rho_v$ given in (7). The only approx-
imation made in obtaining (7) was once again that the system is static, and spherically symmetric. Notice, however, that if (7) is truly the energy density of dark matter, this would mean that no matter what the internal properties of the galaxy are, once one leaves it the energy density of the particles making up its’ halo has to be determined within one part in 1000. This is extraordinarily and prohibitively restrictive. The form that \( \rho_v \) takes is very particular and it is difficult to imagine a physical process which will not only reproduce it, but also determine \( \rho_v \) to such a high degree of accuracy. On this basis alone we would tend to rule out \( \rho_v \) as a physically viable energy density for dark matter. We also note, however, that because \( f \sim \log(\frac{r}{R_g}) \), once again the spacetime with this energy density is curved and in using \( \rho_v \) as the energy density of dark matter we would once again be faced with the problem of interpreting the observational data.

The analysis done in this paper was done for a very special system under some very restrictive conditions. For example, although we have used a spherically symmetric geometry to model the galaxy, most observed galaxies are axisymmetric and have a definite angular velocity. It is, moreover, not even clear whether the experimental data gives rotational velocity curves which are truly a constant or whether they are instead slightly increasing or decreasing with \( r_c \). All that we are comfortable concluding from this analysis, therefore, is that one must be much more careful about introducing any unconfined energy density for dark matter into the system. Not only must it be able to explain the experimental rotational velocity curves within the framework of general relativity, but one must also consider the subsequent affects of this energy density on the geometry of the spacetime. As we have seen, for the \( 1/r^2 \) energy density these affects are considerable.
Acknowledgements

ADS would like to thank K.-W. Ng for many helpful discussions while this paper was being written. This work is supported by the National Science Council of the Republic of China under contract number NSC 82-0208-M-001-086.

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[5] Since we have taken $a_0 \gg M_g G/(4\pi R_g c^2)$, there is a lower bound below which we would not be able to decrease $q$. For a typical galaxy with $M_g \approx 10^{12} M_\odot$ and $R_g \approx 5$ kpc, this bound turns out to be $\sim 2 \times 10^{-5}$. Note also that because $p = q/(4 - q) \rho$, very small $q$ would correspond to very “cold” dark matter which would have a temperature which is very much smaller than its mass.