Some Classical Inequalities and their Applications

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Abstract. In this paper, we define analogies of classical Hölder-McCarthy and Young type inequalities in terms of the Berezin symbols of operators on a reproducing kernel Hilbert space \( \mathcal{H} = \mathcal{H}(\Omega) \). These inequalities are applied in proving of some new inequalities for the Berezin number of operators. We also define quasi-paranormal and absolute-\( k \)-quasi paranormal operators and study their properties by using the Berezin symbols.

1. Introduction

Let \( \mathcal{H} = \mathcal{H}(\Omega) \) be a Hilbert space of complex-valued functions on some set \( \Omega \) such that \( f \to f(\lambda) \) is a continuous functional (evaluation functional) for any \( \lambda \) in \( \Omega \). Then, according to the Riesz’s representation theorem there exists uniquely \( k_\lambda \in \mathcal{H} \) such that

\[
 f(\lambda) = \langle f, k_\lambda \rangle
\]

for all \( f \in \mathcal{H} \). The function \( k_\lambda(z), \lambda \in \Omega, \) is called the reproducing kernel of the space \( \mathcal{H} \), and \( \tilde{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|} \) is called the normalized reproducing kernel in \( \mathcal{H} \) (see [2]). The space \( \mathcal{H} \) with the reproducing kernels \( k_\lambda, \lambda \in \Omega \), is called the reproducing kernel Hilbert space (RKHS). For a bounded linear operator \( A \) (i.e., for \( A \in \mathcal{B}(\mathcal{H}) \), the Banach algebra of all bounded linear operators on \( \mathcal{H} \)) its Berezin symbol \( \tilde{A} \) is defined by (Berezin [6, 7])

\[
 \tilde{A}(\lambda) := \langle A\tilde{k}_\lambda, \tilde{k}_\lambda \rangle, \ \lambda \in \Omega.
\]

The Berezin number \( \text{ber}(A) \) of operator \( A \) is the following number:

\[
 \text{ber}(A) := \sup_{\lambda \in \Omega} |\tilde{A}(\lambda)|.
\]
Since $|\hat{A}(\lambda)| \leq \|A\|$ (by the Cauchy-Schwarz inequality) for all $\lambda \in \Omega$, the Berezin number is a finite number and $\text{ber}(A) \leq \|A\|$. Recall that

$$W(A) := \{\langle Ax, x \rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

is the numerical range of operator $A$ and

$$w(A) := \sup \{\|Ax\| : x \in \mathcal{H} \text{ and } \|x\| = 1\} = \sup \{\mu : \mu \in W(A)\}$$

is the numerical radius of $A$ (for more information, see [1, 20–22]). It is well known that

$$\text{Ber}(A) \subset W(A) \text{ and } \text{ber}(A) \leq w(A)$$

for any $A \in \mathcal{B}(\mathcal{H})$. More information about $\text{ber}(A)$ and relations between $\text{ber}(A)$, $w(A)$ and $\|A\|$ can be found in Karaev [16, 18], and also in [3–5, 9–15, 17, 19, 23–25].

In this section, by using the Hölder-McCarthy inequality, we prove some new inequalities for the Berezin number of operators acting on the RKHS $\mathcal{H} = \mathcal{H}(\Omega)$. Some other related questions also will be studied. In general, the present paper is motivated by the paper of Garayev [16], where the McCarthy, Hölder-McCarthy and Kantorovich operator inequalities were extensively used to get some new inequalities for the Berezin number of operators and their powers. Recall that for any positive operator $A$ (i.e., $\langle Ax, x \rangle \geq 0$ for any $x \in \mathcal{H}$, shortly $A \geq 0$), there exists a unique positive operator $R$ such that $R^2 = A$ (denoted by $R = A^{1/2}$).

An operator $T \in \mathcal{B}(\mathcal{H})$ can be decomposed into $T = U P$, where $U$ is a partial isometry and $P = [T] := (T^* T)^{1/2} \text{ (moduli of operator } T) \text{ with } \ker(T) = \ker(P)$ and the last condition uniquely determines $U$ and $P$ of the polar decomposition $T = U P$ (see Furuta [8]). In general, we will refer to the book of Furuta [8] for main definitions and notations.

### 2. Hölder-McCarthy Type Inequalities and Berezin number

In this section, by using the Hölder-McCarthy inequality, we prove some inequalities for the Berezin number of some operators on the RKHS $\mathcal{H}$.

**Theorem 2.1.** Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then:

1) $\text{ber}(A^\mu) \geq \text{ber}(A)^\mu$ for any $\mu > 1$.
2) $\text{ber}(A^\mu) \leq \text{ber}(A)^\mu$ for any $\mu \in [0, 1]$.
3) If $A$ is invertible, then $\text{ber}(A^\mu) \geq \text{ber}(A)^\mu$ for any $\mu < 0$.

**Proof.** First we prove 2). Indeed, assume that 2) holds for some $\alpha, \beta \in [0, 1]$. Then we only have to prove 2) holds for $\frac{\alpha + \beta}{2} \in [0, 1]$ by continuity of an operator. In fact, we have for any $\lambda \in \Omega$ that

$$\left(\left|A^{\alpha + \beta} k_1, k_3\right|^2\right)^{\frac{1}{2}} = \left|\langle A^\alpha k_1, A^\beta k_3 \rangle\right|^2 \text{ (by Cauchy-Schwarz inequality)}$$

$$\leq \langle A^\alpha k_1, k_3 \rangle \langle A^\beta k_3, k_3 \rangle \text{ (by assumption)}$$

$$\leq \langle k_1, k_3 \rangle^{\alpha + \beta},$$

so that $A^{\alpha + \beta} (\lambda) \leq A(\lambda)^{\alpha + \beta}$ holds for $\frac{\alpha + \beta}{2} \in [0, 1]$. This implies the desired inequality $\text{ber}(A^\mu) \leq \text{ber}(A)^\mu$ for any $\mu \in [0, 1]$. 

1) Let $\mu > 1$. Then $\frac{1}{\mu} \in [0,1]$. For any $\lambda \in \Omega$

$$\langle \hat{A}k_\lambda, \hat{k}_\lambda \rangle = \langle A^{\frac{1}{\mu}} \hat{A}k_\lambda, \hat{k}_\lambda \rangle$$

$$\leq \langle A^{\frac{1}{\mu}} \hat{A}k_\lambda, \hat{k}_\lambda \rangle^{\frac{1}{\mu}} \text{ by 2),}$$

hence $\langle A^\mu \hat{A}k_\lambda, \hat{k}_\lambda \rangle \geq \langle \hat{A}k_\lambda, \hat{k}_\lambda \rangle^\mu$ for any $\mu > 1$, which shows that $\text{Ber}(A^\mu) \geq \text{Ber}(A)^\mu$ for any $\mu > 1$, as desired.

3) Since $A$ is invertible, we have the following for any $\lambda \in \Omega$

$$1 = \| \hat{k}_\lambda \|^4 = \left\| \langle A^{\frac{1}{\mu}} \hat{A}k_\lambda, A^{-\frac{1}{\mu}} \hat{k}_\lambda \rangle \right\|^2$$

$$\leq \| A^{\frac{1}{\mu}} \hat{A}k_\lambda \|^2 \| A^{-\frac{1}{\mu}} \hat{k}_\lambda \|$$

$$= \langle \hat{A}k_\lambda, \hat{k}_\lambda \rangle \langle A^{-\frac{1}{\mu}} \hat{k}_\lambda, \hat{k}_\lambda \rangle$$

$$= \tilde{A}(\lambda) A^{-1}(\lambda),$$

and hence

$$1 \leq \tilde{A}(\lambda) A^{-1}(\lambda) \text{ for any } \lambda \in \Omega,$$  \hspace{1cm} (1)

which gives us

$$\text{Ber}(A) \text{Ber}(A^{-1}) \geq 1,$$

or equivalently

$$\text{Ber}(A^{-1}) \geq \text{Ber}(A)^{-1}.$$ 

Case: $\mu \in (-\infty,-1)$. Then we have the following for any $\lambda \in \Omega$

$$\langle A^\mu \hat{A}k_\lambda, \hat{k}_\lambda \rangle \geq \langle A^{-\frac{1}{\mu}} \hat{A}k_\lambda, \hat{k}_\lambda \rangle^\mu \text{ by 1) since } |\mu| > 1$$

$$\geq \langle \hat{A}k_\lambda, \hat{k}_\lambda \rangle^{-|\mu|} \text{ by (1)}$$

$$= \langle \hat{A}k_\lambda, \hat{k}_\lambda \rangle^{|\mu|}$$

which implies that $\text{Ber}(A^\mu) \geq \text{Ber}(A)^{\mu}$, as desired.

Case: $\mu \in [-1,0)$. For every $\lambda \in \Omega$ we have

$$\tilde{A}(\lambda) = \langle A^\mu \hat{A}k_\lambda, \hat{k}_\lambda \rangle = \langle A^{-\frac{1}{\mu}} \hat{A}k_\lambda, \hat{k}_\lambda \rangle$$

$$\geq \langle A^{-\frac{1}{\mu}} \hat{A}k_\lambda, \hat{k}_\lambda \rangle^{-1} \text{ by (1)}$$

$$\geq \langle \hat{A}k_\lambda, \hat{k}_\lambda \rangle^{-1|\mu|} = \langle \hat{A}k_\lambda, \hat{k}_\lambda \rangle^{|\mu|} = \langle \tilde{A}(\lambda) \rangle^{|\mu|},$$

and the last inequality follows by 2) since $|\mu| \in [0,1]$ and taking inverses of both sides. The theorem is proved.  \hspace{1cm} \Box

Next result proves the equivalence of Hölder-McCarthy type inequality and Young type inequality.
**Theorem 2.2.** For a positive operator $A \in \mathcal{B}(H)$ and $\mu \in [0, 1]$ the following inequalities are equivalent:

1. Hölder-McCarthy type inequality:
   \[
   \widetilde{A}(\lambda)^\mu \geq \widetilde{A}^\mu(\lambda) \quad \text{for all } \lambda \in \Omega.
   \]  
   (2)

2. Young type inequality:
   \[
   [\mu A + I - \mu^{-1}]^{-1} \geq \widetilde{A}^\mu.
   \]  
   (3)

**Proof.** Let us define a scalar function
\[
f(t) := \mu t + 1 - \mu - t^\mu
\]
for positive numbers $t$ and $\mu \in [0, 1]$. Then it is easy to see that $f(t)$ is a nonnegative convex function with the minimum value $f(1) = 0$, so we have
\[
\mu a + 1 - \mu \geq a^\mu
\]
for positive $a$ and $\mu \in [0, 1]$.

(2) $\Rightarrow$ (3). Replacing $a$ by $\widetilde{A}(\lambda) \geq 0$ and $\mu \in [0, 1]$ in (4), we obtain
\[
\mu \widetilde{A}(\lambda) + 1 - \mu \geq A(\lambda)^\mu \geq \widetilde{A}^\mu(\lambda) \quad \text{by (2)},
\]
so we have (3).

(3) $\Rightarrow$ (2). We may assume $\mu \in (0, 1]$. In (3), replace $A$ by $k^{\frac{1}{\mu}}A$ for a positive number $k$, then
\[
\mu k^{\frac{1}{\mu}} \widetilde{A}(\lambda) + 1 - \mu \geq k\widetilde{A}^\mu(\lambda)
\]
(5)
for $\lambda \in \Omega$ by (3). We put $k = \widetilde{A}(\lambda)^{-\mu}$ in (5) if $\widetilde{A}(\lambda) \neq 0$, then we have
\[
\mu \widetilde{A}(\lambda)^{-1} \widetilde{A}(\lambda) + 1 - \mu \geq \widetilde{A}(\lambda)^{-\mu} \widetilde{A}^\mu(\lambda),
\]
that is $A(\lambda)^\mu \geq \widetilde{A}^\mu(\lambda)$ for all $\lambda \in \Omega$ and we get (2). If $\widetilde{A}(\lambda) = 0$, then it means that $A^{\frac{1}{\mu}}k^{\frac{1}{\mu}} = 0$, so $A^k k^{\frac{1}{\mu}} = 0$ for $\mu \in (0, 1]$ by the induction and continuity of $A$, and thus we have (2). The theorem is proved.

**Proposition 2.3.** Let $A \in \mathcal{B}(H)$ be a positive invertible operator and $B \in \mathcal{B}(H)$ be an invertible operator. Then for any real number $\mu$, we have
\[
\text{ber}((BAB^*)^\mu) = \text{ber}\left(BA^{\frac{1}{\mu}} \left(A^{\frac{1}{\mu}} B^* A^{\frac{1}{\mu}}\right)^{\mu-1} A^{\frac{1}{\mu}} B^*\right).
\]  
   (6)

**Proof.** Let $BA^{\frac{1}{\mu}} = UP$ be the polar decomposition of $BA^{\frac{1}{\mu}}$, where $U$ is unitary and $P = \|BA^{\frac{1}{\mu}}\|$. Then it is easy to see that:
\[
(BAB^*)^\mu = (UP^2U^*)^\mu = BA^{\frac{1}{\mu}} P^{-1} P^2 P^{-1} A^{\frac{1}{\mu}} B^* = BA^{\frac{1}{\mu}} \left(A^{\frac{1}{\mu}} B^* A^{\frac{1}{\mu}}\right)^{\mu-1} A^{\frac{1}{\mu}} B^*.
\]
Now (6) is immediate from this equality.
3. Paranormal operators and related problems

Recall that an operator $A$ on a Hilbert space $H$ is called paranormal if $\|Ax\|^2 \geq \|Ax\|^2$ for every unit vector $x \in H$.

**Definition 3.1.** We will say that $A$ is a quasi-paranormal operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$, if $\|A^2_k\lambda\|^2 \geq \|A_{\lambda}k\|^2$ for any $\lambda \in \Omega$.

**Definition 3.2.** An operator $T$ belongs to class $\tilde{A}$ if $\left\|\tilde{A}^2\right\|^2 \geq \|A\|^2$.

**Definition 3.3.** For each $k > 0$, an operator $T$ is absolute-$k$-quasi-paranormal if

$$\left\|\left|T\right|^k \tilde{T}_{k\lambda}\right\|^2 \geq \left\|\tilde{T}_{k\lambda}\right\|^{k+1}$$

for every $\lambda \in \Omega$.

It follows from these definitions that:

(a) If $A$ is quasi-paranormal, then

$$\text{ber}\left(\left\|A^2\right\|^2\right) \geq \text{ber}\left(\|A\|^2\right)^2$$

(b) If $A$ belongs to class $\tilde{A}$, then

$$\text{ber}\left(\left\|A^2\right\|^2\right) \geq \text{ber}\left(\|A\|^2\right)$$

(c) If $A$ is absolute-$k$-quasi-paranormal, then

$$\text{ber}\left(\left\|A^k A\right|^2\right) \geq \text{ber}\left(\|A\|^{k+1}\right)$$

In this section, to prove some inequalities for the Berezin number of such operators, we need to other properties of these operators.

**Proposition 3.4.** Every operator in $\tilde{A}$ is a quasi-paranormal operator on a RKHS.

**Proof.** Suppose $A \in \tilde{A}$, i.e.,

$$\left\|A^2\right\| \geq \|A\|^2.$$  \hspace{1cm} (8)

Then for every $\lambda \in \Omega$, we have $\left\|A^2\right\|^2(\lambda) \geq \|A\|^2(\lambda)$, and therefore it follows from the proof of Theorem 2.1 that

$$\left\|A^2_{k\lambda}\right\|^2 = \left\langle A^2_{k\lambda}, A^2_{k\lambda} \right\rangle = \left\langle \left(\left\|A^2\right\|^2\right) A^2_{k\lambda}, A^2_{k\lambda} \right\rangle$$

$$\geq \left\langle \left\|A^2_{k\lambda}, A^2_{k\lambda} \right\rangle \right\rangle \geq \left\langle \left\|A^2_{k\lambda}, A^2_{k\lambda} \right\rangle \right\rangle \geq \left\langle \left\|A^2_{k\lambda}, A^2_{k\lambda} \right\rangle \right\rangle \geq \left\langle \left\|A^2_{k\lambda}, A^2_{k\lambda} \right\rangle \right\rangle = \left\|A^2_{k\lambda}\right\|^2$$

Hence

$$\left\|A^2_{k\lambda}\right\|^2 \geq \left\|A^2_{k\lambda}\right\|^2$$

for every $\lambda \in \Omega$, so that $A$ is quasi-paranormal, which proves the proposition. \hfill \blacksquare
Definition 3.5. For each $k > 0$, we say that an operator $A$ belongs to class $\widetilde{\mathcal{A}}(k)$ if

$$
\left( (A^* |A|^{2k} A)^{\frac{1}{2k}} \right)^{\frac{k}{k+1}} \geq |A|^2.
$$

The proof of Theorem 2.1 allows us also prove the following.

Proposition 3.6. (a) Every quasi-paranormal operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ is an absolute-$k$-quasi-paranormal operator for $k \geq 1$.

(b) For each $k > 0$, every class $\widetilde{\mathcal{A}}(k)$ operator is an absolute-$k$-quasi-paranormal operator.

Proof. (a) Suppose that $A$ is a quasi-paranormal operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$. Then, for any $\lambda \in \Omega$ and $k \geq 1$, we have

$$
\left\| |A|^k \hat{A}_k \right\| = \left\langle |A|^{2k} \hat{A}_k, \hat{A}_k \right\rangle
\geq \left( |A|^2 \hat{A}_k, \hat{A}_k \right)^k \left\| \hat{A}_k \right\|^{2(1-k)} \quad \text{(see the proof of Theorem 2.1, 1)}
= \left\| A^* \hat{A}_k \right\|^2 \left\| \hat{A}_k \right\|^{2(1-k)}
\geq \left\| \hat{A}_k \right\|^k \left\| \hat{A}_k \right\|^{2(1-k)} \quad \text{(by quasi-paranalormality of $A$)}
\geq \left\| \hat{A}_k \right\|^{2(k+1)},
$$

and hence

$$
\left\| |A|^k \hat{A}_k \right\| \geq \left\| \hat{A}_k \right\|^{k+1}
$$

for all $\lambda \in \Omega$ and $k \geq 1$, so that $A$ is absolute-$k$-quasi-paranormal operator for $k \geq 1$.

(b) Let $A \in \widetilde{\mathcal{A}}(k)$ for $k > 0$, that is

$$
\left( (A^* |A|^{2k} A)^{\frac{1}{2k}} \right)^{\frac{k}{k+1}} \geq |A|^2 \quad \text{for } k > 0.
$$

Then for any $\lambda \in \Omega$,

$$
\left\| |A|^k \hat{A}_k \right\| = \left\langle A^* |A|^{2k} \hat{A}_k, \hat{A}_k \right\rangle
\geq \left\langle \left( A^* |A|^{2k} A \right)^{\frac{1}{2k}} \hat{A}_k, \hat{A}_k \right\|^{k+1}
\geq \left\langle |A|^2 \hat{A}_k, \hat{A}_k \right\|^{k+1} \quad \text{(by (9))}
\geq \left\| \hat{A}_k \right\|^{2(k+1)},
$$

from which

$$
\left\| |A|^k \hat{A}_k \right\| \geq \left\| \hat{A}_k \right\|^{k+1}
$$

for all $\lambda \in \Omega$,

so that $A$ is absolute-$k$-quasi-paranormal operator for $k > 0$. This completes the proof. □

As further extension of previous results, we prove the following result.
Theorem 3.7. Let $A \in B(H(\Omega))$ be an absolute-$k$-quasi-paranormal operator for $k > 0$. Then for every $\lambda \in \Omega$,

$$F(\ell) = \left\| A^{\ell} \widehat{A} \right\|^{\frac{1}{\ell}}$$

is increasing for $\ell > k > 0$, and the following inequality holds:

$$F(\ell) \geq \left\| A \widehat{A} \right\|,$$

i.e., $A$ is absolute-$\ell$-quasi-paranormal operator for $\ell \geq k > 0$.

Proof. Assume that $A$ is an absolute-$k$-quasi-paranormal operator on $H = H(\Omega)$ for $k > 0$, i.e.,

$$\left\| A^{\ell} \widehat{A} \right\| \geq \left\| A \widehat{A} \right\|^{k+1}$$

for every $\lambda \in \Omega$. Clearly, (10) holds if and only if

$$F(k) = \left\| A^{k} \widehat{A} \right\|^{\frac{1}{k}} \geq \left\| A \widehat{A} \right\|$$

for any $\lambda \in \Omega$. Then for every $\lambda \in \Omega$ and any $\ell$ such that $\ell \geq k > 0$, we have

$$F(\ell) = \left\| A^{\ell} \widehat{A} \right\|^{\frac{1}{\ell}} = \left\| A^{2k} \widehat{A} \right\|^{\frac{1}{2k}} \left\| A^{k} \widehat{A} \right\|^{\frac{1}{k}} \geq \left\| A \widehat{A} \right\|^{1+1}$$

and hence

$$F(\ell) = \left\| A^{\ell} \widehat{A} \right\|^{\frac{1}{\ell}} \geq \left\| A \widehat{A} \right\|$$

for every $\lambda \in \Omega$ and $\ell \geq k$, so that $A$ is absolute-$\ell$-quasi-paranormal for $\ell \geq k > 0$.

Now we prove that, $F(\ell)$ is increasing for $\ell \geq k > 0$. Indeed, for any $\lambda \in \Omega$, $m$ and $\ell$ such that $m \geq \ell \geq k > 0$, we have:

$$F(m) = \left\| A^{m} \widehat{A} \right\|^{\frac{1}{m}} = \left\| A^{2m} \widehat{A} \right\|^{\frac{1}{2m}} \left\| A^{m} \widehat{A} \right\|^{\frac{1}{m}} \geq \left\| A \widehat{A} \right\|^{1+1}$$

and hence $F(m) \geq F(\ell)$, that is $F(\ell)$ is increasing for $\ell \geq k > 0$. This proves the theorem. □
Corollary 3.8. \( F(\ell) \geq \sqrt{\det(\|A\|^2)} \) for \( \ell \geq k > 0 \).

The following corollary is well known (see, for instance, [8]).

Lemma 3.9. Let \( a \) and \( b \) be positive real numbers. Then,
\[
a^\alpha b^\beta \leq \lambda a + \mu b
\]
holds for \( \lambda > 0 \) and \( \mu > 0 \) such that \( \lambda + \mu = 1 \).

Our next result characterizes absolute-\( k \)-quasi-paranormal operators \( A \) on the RKHS \( \mathcal{H} = \mathcal{H}(\Omega) \).

Theorem 3.10. For each \( k > 0 \), an operator \( A \) on \( \mathcal{H} \) is absolute-\( k \)-quasi-paranormal if and only if
\[
\left(A^*|A|^2k - (k+1)a^\alpha|A|^2 + k\alpha^{k+1}\right) \geq 0
\]
holds for all \( \alpha > 0 \).

Proof. \( \Rightarrow \). Suppose that \( A \) is absolute-\( k \)-quasi-paranormal for \( k > 0 \), i.e.,
\[
\left\|A^kA_{k,\lambda}\right\| \geq \left\|A_{k,\lambda}\right\|^{k+1}
\]
(12)
for every \( \lambda \in \Omega \). Inequality (12) holds if and only if
\[
\left\|A^kA_{k,\lambda}\right\|^{\frac{k}{k+1}}\left\|\mathcal{K}_{\lambda}\right\|^{\frac{1}{k+1}} \geq \left\|A_{k,\lambda}\right\|
\]
for all \( \lambda \in \Omega \), or equivalently
\[
\left\langle A^*|A|^2k, k_{\lambda}\right\rangle^{\frac{1}{k+1}}\left\langle k_{\lambda}, k_{\lambda}\right\rangle^{\frac{k}{k+1}} \geq \left\langle |A|^2k, k_{\lambda}\right\rangle
\]
for all \( \lambda \in \Omega \). By Lemma 3.9, we have:
\[
\left\langle A^*|A|^2k, k_{\lambda}\right\rangle^{\frac{1}{k+1}}\left\langle k_{\lambda}, k_{\lambda}\right\rangle^{\frac{k}{k+1}} = \left\langle \left(\frac{1}{\alpha}\right)^k A^*|A|^2k, k_{\lambda}\right\rangle^{\frac{1}{k+1}}\left\langle k_{\lambda}, k_{\lambda}\right\rangle^{\frac{k}{k+1}} \leq \frac{1}{k+1} \left(\frac{1}{\alpha}\right)^{k+1} A^*|A|^2k, k_{\lambda} + \frac{k}{k+1} \alpha\left\langle k_{\lambda}, k_{\lambda}\right\rangle
\]
(13)
for all \( \lambda \in \Omega \) and \( \alpha > 0 \), so that (12) ensures the following inequality by (13):
\[
\frac{1}{k+1} \left(\frac{1}{\alpha}\right)^{k+1} A^*|A|^2k, k_{\lambda} + \frac{k}{k+1} \alpha\left\langle k_{\lambda}, k_{\lambda}\right\rangle \geq \left\langle |A|^2k, k_{\lambda}\right\rangle
\]
(14)
for all \( \lambda \in \Omega \) and \( \alpha > 0 \).

Conversely, (14) implies (12) by putting \( \alpha = \left(\frac{A^*|A|^2k, k_{\lambda}}{\left\langle k_{\lambda}, k_{\lambda}\right\rangle}\right)^{\frac{1}{k+1}} \); in case \( A^*|A|^2k, k_{\lambda} = 0 \), let \( \alpha \to 0 \). Hence (14) holds if and only if
\[
\left(A^*|A|^2k - (k+1)a^\alpha|A|^2 + k\alpha^{k+1}\right) \geq 0
\]
holds for all \( \alpha > 0 \), which completes the proof of the theorem. \( \square \)

Since absolute-1-quasi-paranormal is quasi-paranormal, the following is immediate from Theorem 3.10.
Corollary 3.11. An operator $A$ is quasi-paranormal if and only if
\[
(A^2 A^* - 2a A^* A + a^2) \sim \geq 0
\]
holds for all $\alpha > 0$.

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