Axial-vector exchange contribution to the hadronic light-by-light piece of the muon anomalous magnetic moment

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In this work we study the axial contributions to the hadronic light-by-light piece of the muon anomalous magnetic moment using the framework of resonance chiral theory. As a result, we obtain $a_{HLbL}^{A\mu} = (0.8^{+3.5}_{-0.1}) \cdot 10^{-11}$, that might suggest a smaller value than most recent calculations, underlining the need of future work along this direction. In particular, we find that our results depend critically on the asymptotic behavior of the form factors, and as such, emphasizes the relevance of future experiments for large photon virtualities. In addition, we present general results regarding the involved axial form factors description, comprehensively examining (and relating) the current approaches, that shall be of general interest.

1 Introduction

1.1 Overview of the muon g-2: the importance of hadronic contributions

The anomalous magnetic moment of the muon, $a_\mu = (g_\mu - 2)/2$ is a pure quantum field theory observable that has played a key role since its first measurement showing its non-vanishing value [1, 2], confirmed immediately after with the famous Schwinger computation of $a_\mu = \alpha/(2\pi) + O(\alpha^2)$ [3]. Over the years, it has been (and it still is) one of the most stringent tests of the whole Standard Model, thanks to the increasing accuracy of its determination over time and the improved theoretical computations with reduced uncertainties that became available. This makes it an extremely sensitive probe of new physics that—if heavy—would naively shift $a_\mu$ with the scaling $m_\mu^2/M^2$ ($M$ being the heavy new physics mass). This explains why, despite $a_e$ is measured 2400 times more precisely than $a_\mu$, the latter is still more sensitive to heavy new particles than $a_e$ by a factor $\sim 18$.1 The latest measurements of $a_\mu$ [11–13] yield [14]

\begin{equation}
    a_\mu^{\exp} = (116592091 \pm 63) \cdot 10^{-11},
\end{equation}

1There are exceptions to this counting, see Ref. [4]. Also, there are proposals to measure $a_e$ to a precision high enough as to compete with the future $a_\mu$ experiments [5]. It is nevertheless extremely interesting that, while the measurement of $a_e$ [6] agrees with the former prediction in [7] (that uses as an input the previous values for $\alpha$ from Refs. [8, 9]), it is in tension, at the 2.5 $\sigma$ level, when employing the most recent and precise determination of $\alpha$ [10]. Note at this respect that, as opposed to $a_\mu$, the $a_e$ uncertainty is dominated by that of $\alpha$ [7].
while the weighted average of the most recent evaluations \cite{15–17} of the SM contributions reads\footnote{This arises from $a_\mu^{\text{SM}} = 1.16591783(35)$ \cite{15}, $a_\mu^{\text{SM}} = 1.16591820(35.6)$ \cite{16}, and $a_\mu^{\text{SM}} = 1.16591830(48)$ \cite{15}.}

$$a_\mu^{\text{SM}} = (116591807 \pm 38) \cdot 10^{-11},$$

(2)

showing a tantalizing $3.9\sigma$ discrepancy with respect to the measurement (1). This has motivated two further experiments: one at FNAL, and aiming to achieve an error around $16 \cdot 10^{-11}$ \cite{18}, and a second one at J-PARC, aiming for an error around $50 \cdot 10^{-11}$ \cite{19}.

The amazing precision of the theoretical determination in Eq. (2) is possible thanks to the complete $O(\alpha^5)$ computation of the QED contributions \cite{7, 20, 21} and of the electroweak contributions to two loops (including the leading logarithms from an additional loop) \cite{22–25}, which warrant an associated uncertainty at the level of $\lesssim 1 \times 10^{-11}$. Still, the error of Eq. (1) is a factor $\sim 40$ larger, because of the uncertainties associated to the hadronic contributions \cite{26–28}, as we will discuss next.

There are two main types of hadronic contributions to $a_\mu$: the so-called hadronic vacuum polarization (HVP) and the hadronic light-by-light (HLbL) scattering, which are $O(\alpha^2)$ and $O(\alpha^3)$, respectively. The $3 \cdot 10^{-11}$ uncertainty in the SM prediction of $a_\mu$ above comes from their leading order components (at the next-to-leading order they are known \cite{29, 30} precisely enough). Despite their—dominantly—non-perturbative nature, it has long been known \cite{31, 32} how to obtain a data-driven extraction of the LO HVP contribution via dispersion relations, that provide an immediate connection to the $e^+e^- \rightarrow \text{hadrons}$ cross section. Although the resulting error used to dominate the total uncertainty in Eq. (2)—relegating the HLbL to a second place—the successive improvements on the HVP side (with current errors around $35 \cdot 10^{-11}$ \cite{15–17}) demanded a dedicated theory effort for the HLbL piece (with former errors around $30 \times 10^{-11}$ \cite{26, 33}). More important, to fully benefit from the future measurements of $a_\mu$ at FNAL and J-PARC, a reduction of errors at around $10^{-10}$ is required.

For a long time, the leading order HLbL could not be computed in a data-driven way and it was difficult to evaluate the model-dependence associated to it \cite{34–51}. Recently, there has been a tremendous effort in this direction \cite{52–56} yielding precise numerical results for the two-pion \cite{55, 56}, one-pion \cite{57–59}, and $\eta, \eta'$ \cite{57} contributions—that are the most relevant ones.\footnote{Remarkable progress in the evaluation of the HLbL part of $a_\mu$ on the lattice has been achieved recently \cite{60–62}, as well (see Ref. \cite{63} for a review on this topic).} As a result of this activity, the error on the $\pi^0, \eta, \text{and } \eta'$-exchange contributions is $\lesssim 4 \cdot 10^{-11}$ and $\lesssim 2 \cdot 10^{-11}$ for two-pion contributions. The next ones in size, but with similar errors, are the axial-vector contributions, whose study and evaluation is the aim of this paper.

1.2 Axial-vector contributions to the muon anomalous magnetic moment

Although the Landau-Yang theorem \cite{64, 65} forbids the annihilation of a spin-one particle into a pair of real photons, axial-vector exchange contributions to the HLbL piece of $a_\mu$ are still possible, since at least one photon is off-shell in both axial-$\gamma^*\gamma^*$ vertices in such a contribution. Still, the Landau-Yang theorem imposes non-trivial requirements on the symmetry structure of the involved form factors, as we will see.

Early estimates of the corresponding contributions were carried out both in the extended Nambu-Jona-Lasino model by Bijnens, Pallante and Prades \cite{35–37} and by Hayakawa, Kinoshita and Sanda using Hidden Local Symmetry Lagrangians \cite{38, 39}. The first group obtained $a_\mu^{\text{HLbL;A}} = (2.5 \pm 1.0) \cdot 10^{-11}$, which includes the ballpark value $1.7 \cdot 10^{-11}$, given by the second group.

Melnikov and Vainshtein \cite{45} derived operator product expansion (OPE) constraints on the hadronic light-by-light (HLbL) tensor and built a model where these were saturated by dropping
the momentum dependence of the singly-virtual transition form factors, which increases the contributions to $a_\mu$ from low-photon virtualities in the considered axial exchanges. In addition, it has also been noted that their approach implies too large two-photon $f_1$ and $f'_1$ decay widths [66]. As a result, their evaluation, $a^{H\Lambda L;A}_\mu = (22 \pm 5) \cdot 10^{-11}$, is an order of magnitude larger than the previous estimates.

Recently, there have been a couple of new computations of $a^{H\Lambda L;A}_\mu$ by Jegerlehner [15] and by Pauk and Vanderhaeghen [67]. Despite both claiming Ref. [45] violated the Landau-Yang theorem, this is not the case, as we have verified in Appendix E and will be discussed briefly in our appendix B. Independently of this, both computations reach compatible results: $a^{H\Lambda L;A}_\mu = (6.4 \pm 2.0) \cdot 10^{-11}$ [67] (where only $f_1(1285)$ and $f_1(1420)$ contributions were accounted for) and $a^{H\Lambda L;A}_\mu = (7.6 \pm 2.7) \cdot 10^{-11}$ [15] (where also the $a_1(1260)$ was included as an intermediate state). With the $a_1$ contribution on the order of $1.9 \cdot 10^{-11}$ [15], their agreement is remarkable.

On the experimental side, very little information is available (which again is partly due to the Landau-Yang theorem). Noticeably, the L3 Collaboration at LEP measured the di-photon coupling to the $f_1(1285)$ and $f_1(1420)$ states using their decays to $\pi^+\pi^-\eta$ [68] and $K_S\pi^+\pi^-$ [69] products. In this case, one could study the energy dependence of a linear combination of form factors relevant for $a^{H\Lambda L;A}_\mu$ assuming this information is correlated with the $P^2_T$ of the measured final state, as done in Ref. [67]. We will also use this information in our work.

Noteworthy, according to the most recent evaluations of $a^{H\Lambda L;A}_\mu$ in Refs. [15, 67] the axial-vector contributions have a very similar uncertainty ($\sim 3 \cdot 10^{-11}$) as the sum of the tensor and higher-scalar meson contributions [67, 70, 71]. One main motivation of our work is to confirm or disfavor this observation, especially due to the model-dependency of the estimations, which could make further refined studies of $a^{H\Lambda L;A}_\mu$ needed. The other important result of this study is showing how sensitive our result for $a^{H\Lambda L;A}_\mu$ is, depending on the asymptotic behaviour demanded to the relevant form factors. This will be useful towards achieving a reliable model-dependent error estimation of this contribution.

The outline of this paper is as follows: in Section 2 we define our notation and conventions together with the central results, and obtain the relevant form factors in Resonance Chiral Theory (R$\chi$T), which inputs are discussed in Appendix C—accounting for the implications of LEP data. Next, in Section 3, we turn to derive the axial-vector exchange contribution to $a^{H\Lambda L}_\mu$—with details relegated to Appendix D. After that, we evaluate numerically $a^{H\Lambda L;A}_\mu$ for the lowest-lying axial multiplet in Section 4 and state our conclusions in Section 5. Several appendices complete our discussion: Appendix A collects several useful relations derived from Schouten identity; Appendix B includes four other basis (and their relation with ours) for the axial transition form factors; Appendix C summarizes the treatment of $U(3)$ flavor breaking corrections in Resonance Chiral Theory and discusses the determination of the model parameters using short-distance QCD constraints and phenomenological information; finally, Appendix E discusses the implications of the OPE for the axial transition form factors—that in turn is directly related to the discussion concerning the Landau-Yang theorem raised in [15, 67].

We are aware that M. Hoferichter and collaborators have reached a similar conclusion [66].
2 The Axial TFF

2.1 Definitions and main results

Based on parity, charge conjugation and hermiticity, the axial transition matrix element with the electromagnetic currents \( j^\mu = \sum_i Q_i \bar{q}_i \gamma^\mu q_i \) with \( Q_i \) the \( i \)-th quark charge, defined as

\[
i \int \, d^4 x e^{iq x} \langle 0 | T \{ j^\mu (x) j^\nu (0) \} | a^\tau \rangle = \mathcal{M}^{\mu \nu \tau} \varepsilon_{A \tau},
\]

where \( \langle \gamma^\nu (q_1) \gamma^\rho (q_2) | A (p_A) \rangle \equiv (2\pi)^4 \delta (4)(q_1 + q_2 - p_A) i e^2 \mathcal{M}_{A \tau} \varepsilon_{A \tau} (q_1^\ast \varepsilon_1^\ast + q_2^\ast \varepsilon_2^\ast) \), can be generally written as

\[
\mathcal{M}^{\mu \nu \tau} = i \epsilon^{\mu \nu \tau \rho} (q_0^2 A - q_2^2 \bar{A}) + \epsilon^{\mu \nu \rho \tau} q_2 (q_1^\rho B_1 + q_2^\rho B_2) - \epsilon^{\mu \rho \nu \tau} q_1^\rho \bar{B}_1 \right) + i e^{\mu \nu \rho \tau} q_2 (q_1^\rho C + q_2^\rho \bar{C}),
\]

with \( \epsilon^{123} = +1 \) and where, given a form factor \( F = F (q_1^2, q_2^2) \), we define \( \overline{F} = F (q_2^2, q_1^2) \). Note in addition that \( (q_1 + q_2) \varepsilon_A = 0 \) implies that only the antisymmetric part in \( C \) survives on-shell—yet we still keep it for later convenience. Defined in this way, only the tensor structure associated to \( C \) ensures gauge invariance by itself. For the remaining set, gauge invariance implies

\[
A + (q_1 \cdot q_2) B_1 + q_2^2 \bar{B}_1 = \bar{A} + (q_1 \cdot q_2) \bar{B}_1 + q_1^2 \bar{B}_1 = 0.
\]

However, these form factors (FFs) are not independent of \( C \)—they are related via Schouten identities (see Appendix A). This implies that we can dismiss either \( A, B_1 \) and remove it from the equations above; in the following, we relegate \( B_1 \) and obtain \( A = -q_2^2 B_2 \). We arrive at our expression for the TFF parametrization

\[
\mathcal{M}^{\mu \nu \tau} = \epsilon^{\mu \nu \tau \rho} \left( q_2 \cdot q_2 - g_0^2 \right) B_2 + \epsilon^{\mu \nu \rho \tau} q_2 (q_1 \cdot q_1 - g_0^2) \bar{B}_1 + \epsilon^{\mu \rho \nu \tau} q_2 (q_1^\tau C + q_2^\tau \bar{C}),
\]

where \( q_{12} = q_1 + q_2, \ q_{12} = q_1 - q_2, \) and \( C_{A(S)} = (C \mp \bar{C})/2 \) (\( B_{2A,2S} \) can be defined analogously). This definition is that appearing—up to overall factors and the spurious addition of \( C \)—in Ref. [72–74] and roughly in [75], where \( C \rightarrow 0 \) is taken. The relation to alternative existing bases (i.e. different choices for eliminating a form factor via the Schouten identity) are given in Appendix B. Interesting implications from the OPE are discussed in Appendix E. A relevant comment is in order here: since the connection among different bases involves \( C \neq 0 \) terms, different bases will potentially select different off-shell behavior unless \( q_{12}^2 \)-terms vanish in the HLbL amplitude. It will be useful as well in the following to quote the non-vanishing on-shell helicity amplitudes

\[
\mathcal{M}^{\mu \nu \tau} = \varepsilon^{0123} \left[ q_2^2 m_2^2 + q_2^2 - q_2^2 \right] B_2 - q_1^2 m_1^2 + q_1^2 - q_1^2 \bar{B}_2 + 2 q_2^2 m_2 A_C, \quad \mathcal{M}^{0 \pm \pm} = \pm \varepsilon^{0123} \left[ q_1^2 q_2^2 \bar{B}_2 - q_1 \cdot q_2 \right] B_2, \quad \mathcal{M}^{0 \pm \pm} = \pm \varepsilon^{0123} \left[ q_1^2 q_2^2 \bar{B}_2 - q_1 \cdot q_2 \right] B_2, \quad \mathcal{M}^{0 \mp \mp} = \pm \varepsilon^{0123} \left[ q_1^2 q_2^2 \bar{B}_2 - q_1 \cdot q_2 \right] B_2,
\]

\( ^5 \mathcal{M}^{\mu \nu \tau} \) has GeV dimensions; \( B_2 \) and \( C \), GeV\(^{-2} \); \( A \) is dimensionless. We use the notation \( \epsilon_{\mu \nu \rho \sigma} p_0^\rho \equiv \epsilon_{\mu \nu \rho \sigma} \).

\( ^6 \) Lacking massless particles, all the form factors should be regular at \( q_{1,2}^2 = 0 \), implying that \( 2 A (q_1^2, 0) + (q_2^2 - q_1^2)B_1 (q_2^2, 0) = 0 \), while \( B_2 \) is not constrained for vanishing \( q_2^2 \).

\( ^7 \) In order to connect to alternative descriptions, where \( B_1 \) is not relegated, our choice is equivalent to shift \( \Delta C = B_1, \ \Delta A = (q_1 \cdot q_2) B_1 + q_2^2 B_1, \ \Delta B_2 = -B_2 \) and analogously for the barred form factors.

\( ^8 \) We employ \( q_{1(2)} = (E_{1(2)}, 0, 0, \pm q) \), with \( 2m_A E_{1(2)} = m_A^2 \pm (q_1^2 - q_2^2) \), \( \varepsilon_+^A = \varepsilon_+^\tau = \varepsilon_+^\tau = \pm (0, 1, \pm 1, 0)/\sqrt{2}, \varepsilon_0^A = (0, 0, 0, 1) \) as in Refs. [76–78].
where \( q^2 = \lambda^{1/2}(m_A^2 + q_1^2, q_2^2)(2m_A)^{-1} = [(q_1 \cdot q_2) - q_1^2 q_2^2]^{1/2} m_A^{-1} \) refers to the photon momentum in the axial-vector meson rest frame—find similar results in Ref. [74]. A particularly relevant result is the cross-section for \( \gamma^* \gamma^* \rightarrow A \) that is relevant for \( e^+ e^- \rightarrow e^+ e^- A \) production. Following the definitions in [74, 77, 79, 80], we find that

\[
\sigma_{TT} = \frac{1}{4m_A q^2} \pi \delta(s - m_A^2) \mathcal{M}_{T=\pm}^2, \quad \sigma_{TL} = \frac{1}{2m_A q^2} \pi \delta(s - m_A^2) \mathcal{M}_{T=\pm}^2,
\]

that, in the \( q_1^2 \rightarrow 0 \) limit, produces a cross section \( \sigma_{\gamma\gamma} \approx \sigma_{LT} + \sigma_{TT} \) (find details in [79, 80])

\[
\sigma_{\gamma\gamma} = \delta(s - m_A^2) 16 \pi^2 \frac{3 \Gamma_{\gamma\gamma}^2}{m_A^2} x(1 + x) \left| \tilde{B}_2 \right|^2 x \left( 1 + \frac{x}{2} \right) + \frac{1}{2} |\bar{C}_A|^2 (1 + x)^2 - x(1 + x) \text{Re} \left( \bar{B}_2 \bar{C}_A^* \right), \quad \text{Eq. (10)}
\]

where \( x = Q_2^2/m_A^2 \), \( \tilde{B}_2(\bar{C}_A) = B_2(C_\bar{A})/B_2(0,0) \), and where

\[
\tilde{\Gamma}_{\gamma\gamma} = \lim_{q_{1,2} \rightarrow 0} \frac{m_A^2}{2 Q_2^2} \Gamma_{TL} = \frac{1}{3} \frac{1}{2m_A} \int d\Pi_{\gamma\gamma} \sum_{T=\pm} |e^2 \mathcal{M}_{T=\mp}^2|^2 = \frac{\pi \alpha^2}{12} m_A^5 |B_2(0,0)|^2. \quad \text{Eq. (11)}
\]

Note that in the narrow-width approximation \( \pi \delta(s - m_A^2) \approx m_A \Gamma_A (s - m_A^2)^2 + m_A^2 \Gamma_A^2 \), that allows comparison to Ref. [68], (see Eqs. (1-3) therein). The bracketed expression in Eq. (10) compares to that for the simplified model (i.e., with \( C_A = 0 \), see also Appendix B.2 in Ref. [68], namely \( x(1 + x/2)|F(Q^2)|^2 \). For a dipole form factor \( F(Q^2) \), Ref. [68] finds a reasonable fit to data, suggesting that singly-virtual form factors should not grow faster than \( Q^{-4} \), that has relevant implications as we shall see.

### 2.2 Form factors in \( R\chi T \)

Using the \( R\chi T \) Lagrangian [81, 82] that saturates the \( \mathcal{O}(p^6) \) LECs in the odd-intrinsic parity sector [46], and using the Schouten identities to explicitly show the antisymmetric nature (see Appendix A and comments therein), the leading contribution can be conveniently expressed as

\[
\mathcal{M}^{\mu\nu} \varepsilon_{\lambda T} = \sum_V c_{AV} (q_1^2 - q_2^2) \langle 0 | A^{\mu\nu} | A \rangle (\epsilon_{\mu\nu\rho\sigma} q_{1\lambda} + \epsilon_{\mu\nu\rho\sigma} q_{2\lambda}) - \epsilon_{\rho\sigma q_1 q_2} g_{\mu\lambda} + \epsilon_{\rho\sigma q_1 q_2} g_{\nu\lambda})
\]

\[
\equiv \mathcal{M}^{\mu\nu}_{A^{\rho\sigma}} \langle 0 | A^{\rho\sigma} | A \rangle , \quad \text{Eq. (12)}
\]

The equation above will be the central quantity for determining \( a_{\mu}^{HLbL; A} \). An important advantage of using the Lagrangian formalism of \( R\chi T \) with respect to some previous approaches is that there is no ambiguity in the propagator of the spin-one resonances that enters the computation, which is discussed in the next section. Substituting for the \( \langle 0 | A^{\rho\sigma} | A \rangle \) matrix element in the equation above and applying again the Schouten identities, we obtain for the \( A \rightarrow \gamma^* \gamma^* \) transition, in the isospin \( M_\rho = M_\omega, F_\rho = F_\omega \) limit, and assuming ideal mixing,

\[
\mathcal{M}^{\mu\nu} = \frac{2e^2 c_{AV} M_A^{-1} (q_1^2 - q_2^2)}{(q_1^2 - M_\rho^2)(q_2^2 - M_\rho^2)} (i e^{\mu\lambda\alpha q_1} [q_2^2 q_{2\alpha} - q_2^2 q_{1\alpha}] - i e^{\nu\lambda\alpha q_2} [q_1^2 q_{1\alpha} - q_1^2 q_{2\alpha}] + i e^{\nu\lambda\sigma q_1} q_{1\sigma} q_{2\sigma}), \quad \text{Eq. (13)}
\]

where \( V \rightarrow \rho \omega \) for \( a_1 \) and \( f_1 \) cases, while \( V \rightarrow \phi \) for \( f_1' \). Finally, the form factors read

\[
C_A = B_2 = -\bar{B}_2 = \frac{2c_A}{M_A (q_1^2 - M_\rho^2)(q_2^2 - M_\rho^2)}, \quad c_{(a_1, f_1, f_1')} = \left( \frac{1}{3}, \frac{\sqrt{2}}{3}, \frac{5}{3} \right), \quad \text{Eq. (14)}
\]
Although both, $F_{\rho\omega}$ and $F_\phi$ depart from $F_V$ by $O(m^2_{\pi,K})$ corrections [83], when the appropriate short-distance constraints are required, one recovers $F_{\rho\omega} = F_\phi = F_V$ [83]. We note that the ideal mixing for the spin-one nonets that we have used (that is predicted with $N_C \to \infty$) is supported by the fact that $\text{BR}(f_1 \to \phi\gamma) = (7.4 \pm 2.6) \cdot 10^{-4} \ll \text{BR}(f_1 \to \rho\gamma) = (5.3 \pm 1.2) \cdot 10^{-2}$. For the numerical inputs, we refer to Appendix C.

2.3 Additional vector multiplets in $R\chi T$

As a result of their antisymmetric nature, the singly-virtual form factors will behave as a constant for large space-like virtualities, while the results from L3 Collaboration suggest a $Q^{-4}$ behavior (see Section 2.1). Such behavior requires the inclusion of additional resonances. With an additional multiplet satisfying the condition $F_{V'K_5^{VA}} = -F_{V'K_5^{VA}}(M_{V'}^2/M_{V}^2)$, the asymptotic behaviour is improved but it is not yet satisfactory. However, with a third multiplet fulfilling

$$F_{V'K_5^{VA}} + F_{V''K_5^{VA}} + F_{V''K_5^{VA}} = 0, \quad F_{V'K_5^{VA}} = \frac{F_{V'K_5^{VA}}(M_{V'}^2 - M_{V''}^2)M_{V'}^2}{(M_{V''}^2 - M_{V'}^2)M_{V}^2},$$

the asymptotic behaviour can be considered realistic. The previous equation fixes the relevant combinations $F_{V'K_5^{VA}}$ and $F_{V''K_5^{VA}}$ in terms of $F_{V'K_5^{VA}}$ and the masses of the vector multiplets, which are known phenomenologically, as it is discussed in Appendix C. We note that we have to ensure the normalization of these form factors (with one, two, or three vector resonance multiplets) at zero photon virtualities be the same. This is achieved if the form factor with two vector nonets is multiplied by the factor $M_{V''}^2/(-1 + M_{V'}^2/M_{V}^2)$ and the one with three vector multiplets by $M_{V''}^2/(M_{V''}^2 - M_{V'}^2)/(M_{V''}^2 - M_{V}^2)$. Find further comments in Appendix C.

3 Axial contribution to $a_\mu^{HLbL}$ in $R\chi T$

The HLbL contribution to $a_\mu$ in the vanishing external momentum limit can be obtained using the projection techniques outlined in Refs. [42, 71]. Particularly, one finds [42, 71]

$$a_\mu = \frac{1}{48m_\mu} \text{tr}(\not{\rho} + m_\mu)\gamma^\rho\gamma^\sigma(\not{\rho} + m_\mu)\Gamma_{\rho\sigma}(p,p),$$

---

\[^9\]This observation is suggested by the two data points measured in the region $Q^2 \in [0.6, 4]$ GeV$^2$ by the L3 Collaboration [68]. It is hard to draw any conclusion on this issue from Ref. [69], as both $\eta(1475)$ and $f_1(1420)$ states are required to describe the data.

\[^{10}\]We note that here we already perform the change of variables $q_1 \to -q_1$, $q_2 \to q_{12} + k$ at the matrix element level as compared to Refs. [42, 71], where this is performed in a second stage.
where

\[ \Gamma_{\mu \nu}^{\rho}(p, p) = - ie^6 \int \frac{d^4q_1 \, d^4q_2}{(2\pi)^4 (2\pi)^4} \sqrt{\gamma_\mu(q_1 + p_1 + m_\mu) \gamma_\nu(q_2 + m_\mu) \gamma_\lambda} \partial \Pi_{HLbL;A}(q_1, q_2), \]  

(17)

where we have introduced \( \partial \Pi_{HLbL;A}(q_1, q_2) \equiv \lim_{k \to 0} (\partial / \partial k_\mu) \Pi_{HLbL;A}(-q_1, q_1 + k, -q_2, -k) \), with the latter tensor the HLbL tensor. For our case of study, the axial-mesons, their contribution to the latter, after dropping irrelevant k terms, (see Fig. 1) reads [for an alternative—compact—expression for the HLbL tensor, we refer to Eq. (51)]

\[
\Pi_{HLbL;A}^{\mu \nu \lambda}(q_1, q_2, -k) = i M_A^{\mu \nu, \alpha \beta}(q_1, q_2) \Delta^{R}_{F}(q_2) \Delta^{R}_{F}(q_1) \Pi_{HLbL;A}^{\alpha \beta, \alpha \beta}(q_1, q_2, -k) + i M_A^{\mu \nu, \alpha \beta}(q_1, q_2) \Delta^{R}_{F}(q_1) \Delta^{R}_{F}(q_2) \Pi_{HLbL;A}^{\alpha \beta, \alpha \beta}(q_1, q_2, -k),
\]

(18)

for the s-, t-, and u-channels, where \( \Delta^{R}_{F}(q) \) stands for the resonance propagator,\(^{11}\)

\[ \Delta^{R}_{F}(q)^{\mu \nu, \rho \sigma} = - \left[ g^{\mu \rho} q^\nu q^\rho - g^{\nu \sigma} q^\rho q^\rho + g^{\mu \sigma} q^\rho q^\rho (M_R^2 - q^2) \right] - (\mu \leftrightarrow \nu) \]  

(19)

leading to

\[
\partial \Pi_{HLbL;A}^{\mu \nu \lambda \rho} = i \Delta^{R}_{F}(q_2) \Delta^{R}_{F}(q_1) F_{A}(q_1^2, q_2^2) F_{A}(q_2^2, 0) \left[ (\epsilon^{\lambda \sigma \rho \delta} q_2^\delta + \epsilon^{\lambda \delta \sigma \rho} q_2^\delta) + (\epsilon^{\lambda \rho \sigma \delta} g^\lambda q_2 \epsilon^\delta + \epsilon^{\lambda \delta \rho \sigma} q_2^\delta \epsilon^\lambda) \right] \times \left[ (\epsilon^{\nu \rho \alpha \delta} q_1^\delta + \epsilon^{\nu \delta \rho \alpha} q_1^\delta) - (\epsilon^{\nu \rho \alpha \delta} q_2^\delta + \epsilon^{\nu \delta \rho \alpha} q_2^\delta) \right] + (\mu \leftrightarrow \lambda) \left[ (\epsilon^{\nu \rho \alpha \delta} q_1^\delta + \epsilon^{\nu \delta \rho \alpha} q_1^\delta) - (\epsilon^{\nu \rho \alpha \delta} q_2^\delta + \epsilon^{\nu \delta \rho \alpha} q_2^\delta) \right],
\]

(20)

where \( F_{A}(q_1^2, q_2^2) = c_A(q_1^2 - q_2^2)(q_1^2 - M_A^2)^{-1}(q_2^2 - M_A^2)^{-1} \), see Eq. (14). Following the method of Gegenbauer polynomials in Ref. [27] to evaluate the integral, one can show that

\[ a_\mu = \left( \frac{\alpha}{\pi} \right)^3 \frac{2}{3} \int dtdQ_1 dQ_2 \left[ \sqrt{1 - t^2} \frac{Q_1^2 Q_2^3}{Q_1^2 m_\mu^2} \sum_{i=1}^{2} K_i(Q_1^2, Q_2^2, t) \right],
\]

(21)

where the expressions for \( K_i(Q_1^2, Q_2^2, t) \) are given in Appendix D.

4 Numerical evaluation of \( a_\mu^{HLbL;A} \)

We evaluate \( a_\mu^{HLbL;A} \), including the contribution of the lightest axial-vector multiplet with up to three vector multiplets\(^{12}\). The results are obtained upon numerical integration of the formulae

\(^{11}\)As we advanced, one advantage of the Lagrangian formalism is that there is no ambiguity in the resonance propagator. In our case, we choose to represent the spin-one resonances by antisymmetric tensor fields, so the corresponding propagators can be read from Eq. (19). While physical observables are independent of our choice for representing the (axial)-vector meson fields, this does not need to be the case for individual contributions to them if asymptotic constraints are not properly taken into account, and deserves further study in the context of a (see e. g. Refs. [82, 84]).

\(^{12}\)The employed numerical values of the masses and couplings and their relations are discussed at the end of the previous section and in Appendix C.
derived in Section 3. One very important thing to note is that the first contribution, given by the integration of Eq. (21), is not convergent for only one vector multiplet (find comments on this aspect in Appendix D). Because of this, we will only quote our results for either two (2Vs) or three (3Vs) vector multiplets. Although we consider the latter our preferred result, as its asymptotic behaviour seems to agree with the trend shown by L3 data (see Section 2.1), it is nevertheless informative to compare both values and to verify that a more realistic (stronger) asymptotic damping of the relevant form factors yields smaller contributions with three vector multiplets than with only two. Moreover, as we discuss in the following, we will use the resulting difference as an error estimate.

In our evaluation, we are using the restrictions in Eq. (15) (and their analogous for only two vector multiplets) that link the couplings and masses of the different multiplets. As a result of this, we will float $F_V\kappa_S^VA$, $M_V$ and $M_A$ independently and assume $M_V'$ and $M_V''$ to be fully correlated—find details of our inputs in Appendix C. Our results are summarized in Table 1, where the different axial-vector meson contributions, in units of $10^{-11}$, are given. Then, we obtain

$$a_{\mu}^{a_1+f_1+f_1':2Vs} = (4.34_{-0.62}^{+0.62}) \cdot 10^{-11}, \quad a_{\mu}^{a_1+f_1+f_1':3Vs} = (0.81 \pm 0.12) \cdot 10^{-11}. \quad (22)$$

We observe that, while our result with two vector multiplets lies in between the early [35–37, 39, 40] and most recent [15, 67] evaluations, it reveals a much smaller value than all preceding analysis (yet in line with early studies) when three vector multiplets are included. We emphasize that such choice has been adopted in order to satisfy the leading power of the asymptotic behaviour suggested by the last two data points, and as such represents our preferred value. Still, this points out the need of additional data at high energies and a more refined analysis regarding the form factor description there. As a consequence, we prefer to quote as our final value the result for 3Vs, with an additional uncertainty that covers for the results with only 2Vs

$$a_{\mu}^{a_1+f_1+f_1'} = (0.8_{-0.1}^{+3.5}) \cdot 10^{-11}. \quad (23)$$

If confirmed by future (dispersive, lattice, etc.) studies, our finding would imply that axial contributions turn out to be similar in size to the sum of tensor and higher-scalar contributions, with an error that is negligible at the current level of requested accuracy, that underlines the need for further studies regarding the axial contributions to $a_{\mu}^{HL\mu L}$.

## 5 Conclusions

In this article, we have studied the axial-vector contributions to the hadronic light-by-light piece of the muon anomalous magnetic moment, $a_{\mu}^{HL\mu L:A}$. This is a timely enterprise, as we are eagerly awaiting the first publication from the Muon g-2 FNAL Collaboration, which would give $a_{\mu}$ with a comparable uncertainty to the LBNL measurement. In the years to come, both FNAL and the $g-2$ experiment at J-PARC would reach a fourfold improved uncertainty which will challenge
our understanding of the Standard Model and its possible extensions provided a similar reduction can be achieved on the theory prediction, that is dominated by hadronic uncertainties. In fact, the spectacular improvement on the accuracy of the HVP evaluations demands a deeper understanding of the hadronic light-by-light piece, wherein the lightest pole cuts are already known with enough precision. Therefore, subleading contributions which are—however—subject to comparatively large uncertainties, become relevant for this endeavor, and the large relative error of these (otherwise small) contributions coming from heavier intermediate states in the HLBL diagrams need to be reduced. In this context, we have studied the axial-vector contributions to $a_{\mu}^{\text{HLbL};A}$ within R\chi T. Our most important results are discussed in the following.

We have motivated our conventions for the relevant matrix element and related ours with others previously employed in the literature, clarifying equivalences and providing with a dictionary to translate from one to another. As there are not many studies of this particular topic and a unified treatment has not been adopted yet, we believe our paper can constitute a reference in this respect.

One particular advantage of our approach, by contrast to previous studies, is that—being a Lagrangian formalism—there is no ambiguity in the definition of the spin-one meson propagators. We have quoted in detail the formulae that allow to evaluate $a_{\mu}^{\text{HLbL};A}$ using this formalism. This specific derivation appears here for the first time, to our knowledge.

According to us, the most important result that we have obtained is the large dependence of $a_{\mu}^{\text{HLbL};A}$ on the asymptotic behaviour of the axial transition form factors. The comparison of our two evaluations in Eq. (22) shows neatly that the main systematic uncertainty comes from the lack of data probing the asymptotic region of the axial transition form factors. Therefore, it would be crucial that a number of data points at large $Q^2$ were measured for the $e^+e^- \to e^+e^-A$ cross-section. An interesting and complementary study would be to address the the $e^+e^- \to f_1$ production, that has been recently measured by SND Collaboration [85]. Finally, it might also be interesting to study the sum rules as discussed in [66].

In addition to this, dispersive and lattice evaluations of $a_{\mu}^{\text{HLbL};A}$ would contribute to the understanding of these contributions and to reducing the corresponding uncertainty in the SM prediction of $a_{\mu}$.

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A Schouten identities

It will be useful to employ the Schouten identity

$$\epsilon^{\mu\nu\rho\sigma} g_{\lambda\delta} = \epsilon^{\delta\nu\rho\sigma} g_{\lambda\mu} + \epsilon^{\mu\delta\rho\sigma} g_{\lambda\nu} + \epsilon^{\mu\nu\delta\sigma} g_{\lambda\rho} + \epsilon^{\mu\nu\rho\delta} g_{\lambda\sigma}$$  \hspace{1cm} (24)

in the following; particularly, the latter implies in our case that

$$\epsilon^{\mu\nu q_1 q_2} q_1^\tau = \epsilon^{\tau q_1 q_2} q_1^\mu + \epsilon^{\mu q_1 q_2} q_1^\nu + \epsilon^{\mu\nu q_1 q_2} q_1^2 + \epsilon^{\mu\nu q_1 q_2} (q_1 \cdot q_2),$$  \hspace{1cm} (25)
and analogously for $\mu \leftrightarrow \nu$, 1 $\leftrightarrow$ 2 expression. The former can be conveniently rewritten in terms of gauge invariant terms for later convenience (meaning they are orthogonal to $q_1^\mu$ and $q_2^\nu$) in different ways:

$$
\epsilon^{\mu\nu q_1 q_2} = \epsilon^{\nu\mu q_2 q_1} (q_1^\mu - g^{\mu\nu} q_3^\nu) + q_1^\mu q_2^\nu - g^{\mu\nu} q_2^\mu - \frac{\epsilon^{\mu\nu q_1 q_2}}{q_2^2} [q_1^\mu q_2^\nu - q_2^\mu q_1^\nu],
$$

(26)

$$
\epsilon^{\mu\nu q_1 q_2} = \epsilon^{\nu\mu q_2 q_1} (q_1^\mu - g^{\mu\nu} q_3^\nu) + \epsilon^{\mu\nu q_1 q_2} [q_1^\mu q_2^\nu - q_2^\mu q_1^\nu],
$$

(27)

$$
\epsilon^{\mu\nu q_1 q_2} = \epsilon^{\nu\mu q_2 q_1} (q_1^\mu - g^{\mu\nu} q_3^\nu) - \epsilon^{\mu\nu q_1 q_2} [q_1^\mu q_2^\nu - q_2^\mu q_1^\nu] + \epsilon^{\mu\nu q_1 q_2} [q_2^\mu q_1^\nu - q_1^\mu q_2^\nu],
$$

(28)

and, again, the corresponding $\mu \leftrightarrow \nu$, 1 $\leftrightarrow$ 2 expressions. All of them allow to relate the different possible parametrizations of the axial TFFs. An additional interesting result, that has been used in [46], is the following

$$
\epsilon_{\mu\nu\alpha\beta} g_{\rho\sigma} \langle \{ V^{\mu\nu}, A^{\alpha\beta} \} f_+^{\beta\gamma} \rangle = \epsilon_{\mu\nu\alpha\beta} g_{\rho\sigma} \langle \{ f_+^{\alpha\rho}, V^{\beta\gamma} \} A^{\mu\nu} \rangle.
$$

(29)

Note this implies that, exchanging $V \leftrightarrow f_+$ leads to the same term up to a sign. Further, for $V \propto f_+$, it vanishes, having no contribution to external vector currents nor the presence of a two-resonance term.

B Other bases for the axial transition form factor

B.1 Helicity basis I

A popular choice adopted in Refs. [86–89], with the latter computing $(g-2)^{\text{HLbL-}\Lambda}_{\mu}$, uses

$$
\mathcal{M}^{\mu\nu} = i \epsilon^{\nu q_1 q_2} [q_2^\mu q_1^\nu - q_1^\mu (q_1 \cdot q_2)] F_A + i \epsilon^{\mu q_2 q_1} [q_1^\nu q_2^\mu - q_2^\nu (q_1 \cdot q_2)] F_A' + i \epsilon^{\mu q_1 q_2} (q_1 - q_2)^\tau \frac{1}{2} F_A.
$$

(30)

From the Schouten identities one can show the relations

$$
C = \frac{F_A}{2} + q_2^2 F_A'; \quad \bar{C} = -\frac{F_A}{2} + q_1^2 F_A'; \quad B_2 = -[q_2^2 F_A' + (q_1 \cdot q_2) F_A'] ; \quad \bar{B}_2 = -[q_1^2 F_A' + (q_1 \cdot q_2) F_A'] .
$$

(31)

B.2 Quark-model inspired

Another common choice is to take a single form factor [68, 69, 75, 80, 90]:

$$
\mathcal{M}^{\mu\nu} = i \epsilon^{\nu q_1 q_2} (q_2^\mu q_1^\nu - q_1^\mu q_2^\nu) A(q_1^2, q_2^2) = i [\epsilon^{\nu q_1 q_2} g_4^\nu (-q_1^2) + \epsilon^{\nu q_1 q_2} g_4^\nu (-q_2^2)] A(q_1^2, q_2^2).
$$

(32)

Note however that the formula above is not gauge invariant. The latter can be achieved via

$$
\mathcal{M}^{\mu\nu} = i \epsilon^{\nu q_1 q_2} (q_2^\mu - g_4^\mu q_2^2) A(q_1^2, q_2^2) + i \epsilon^{\nu q_1 q_2} (q_1^\mu - g_4^\mu q_1^2) A(q_1^2, q_2^2),
$$

(33)

allowing to identify $B_2 = \bar{B}_2 = A(q_1^2, q_2^2)$. Particularly, it is the last one that was used in Ref. [67] to compute the contribution to $(g-2)$.  

---

13 Such a choice is equivalent to use the Schouten identities to get rid of $A$: then, the Ward identities imply $(q_1 \cdot q_2) B_2 = q_2^2 B_2 = 0 \rightarrow B_2 = -q_2^2 B_2 = (q_1 \cdot q_2) B_2$, which carries the $q_2^\tau$ suppression we find in Eq. (31). In order not to have it, one would require $B_2 = -q_2^2 (q_1 \cdot q_2)^{-1} B_2$. Note that such additional suppression artificially implies that only the $F_A$ term contributes to $a_{HLbL-\Lambda}^{\mu}$, which is not generally true.

14 While added terms are irrelevant when connecting to on-shell currents, such as in $e^+ e^-$ production, this is not the case in $(g-2)$ where, in a general $R^\xi$ gauge, the photon propagator demands to keep those terms in order to obtain a $\xi$-independent result.
B.3 Helicity basis II

Finally, we find a different choice in Ref. [76, 80] based on helicities. Defining \( X = (q_1 \cdot q_2)^2 - q_1^2 q_2^2 \) and \( R^{\mu \nu} = - g^{\mu \nu} + \frac{1}{X} (q_1 \cdot q_2)(q_1^\mu q_2^\nu + q_1^\nu q_2^\mu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu) \), the form factor is defined as

\[
\mathcal{M}^{\mu \nu \tau} = i \epsilon_{\rho \sigma \alpha} \left[ R^{\mu \rho} R^{\nu \sigma} (q_1 - q_2) \alpha q_1 q_2 \frac{1}{m_A^2} F_A^{(0)} (q_1, q_2^2) + R^{\nu \rho} \left( q_1^\mu q_2^\nu \frac{1}{m_A^2} F_A^{(1)} (q_1, q_2^2) \right. \\
+ \left. R^{\rho \mu} \left( q_2^\nu - q_2^\sigma q_1^\nu \right) q_1 q_2^\sigma \frac{1}{m_A^2} F_A^{(1)} (q_1, q_2^2) \right) \right]. \tag{34}
\]

The outcome can be conveniently recast via the Schouten identities as

\[
\mathcal{M}^{\mu \nu \tau} = i \epsilon^{\nu q_1 q_2} \left[ q_1^\mu - q_2^\mu \frac{q_1^2}{q_1 \cdot q_2} \frac{1}{m_A^2} F_A^{(1)} + i \epsilon^{\mu q_2 q_1} \left[ q_2^\nu - q_1^\nu \frac{q_2^2}{q_1 \cdot q_2} \frac{1}{m_A^2} F_A^{(1)} \right] \right] + i \epsilon^{\mu \nu q_0} \frac{q_1 \cdot q_2}{2 m_A^2} \left[ \bar{q}_1 \bar{q}_2 - q_1^2 \right] F_A^{(0)}. \tag{35}
\]

The last piece, containing \( \bar{q}_{12} \), vanishes on-shell and the analogy to Eq. (30) is clear. For \( F_A^{(1)} \), the result is analogous to \( F'_A \) up to the \((q_1 \cdot q_2)\) overall term—that we argued was more natural. Finally, \( F_A^{(0)} \) is, up to the additional \( ad-hoc \) \( q_{12}^2 \)-dependency induced, analogous to \( F_A \) in Eq. (30). Using Eq. (31) we obtain

\[
C = \frac{1}{m_A^2} \left[ \frac{(q_1 \cdot q_2)^2 - q_2^2 (q_1 \cdot q_2)}{X} F_A^{(0)} - \frac{q_2^2}{q_1 \cdot q_2} \bar{F}_A^{(1)} \right], \quad B_2 = \frac{1}{m_A^2} \left[ \bar{F}_A^{(1)} + \frac{q_1^2}{q_1 \cdot q_2} F_A^{(1)} \right], \tag{36}
\]

\[
\tilde{C} = \frac{1}{m_A^2} \left[ \frac{(q_1 \cdot q_2)^2 - q_1^2 (q_1 \cdot q_2)}{X} F_A^{(0)} - \frac{q_1^2}{q_1 \cdot q_2} \bar{F}_A^{(1)} \right], \quad \bar{B}_2 = \frac{1}{m_A^2} \left[ F_A^{(1)} + \frac{q_2^2}{q_1 \cdot q_2} \bar{F}_A^{(1)} \right]. \tag{37}
\]

Note however that such form factors have not been used so far to compute the contribution to \((g - 2)\); instead, the ones in the previous subsection were employed [67].

B.4 \( \langle VVA \rangle \) basis

It can be shown that, up to overall factors, the axial meson contributions to the \( \langle VVA \rangle \) Green’s function corresponds to that of the axial meson transition form factors times an additional \((1/i)\sqrt{2} F_A M_A (q_{12}^2 - M_A^2)^{-1} \delta^{abc}/2\) factor, that makes interesting to study the connection to the standard tensor basis for \( \langle VVA \rangle \) that is employed in Refs. [33, 91, 92] \[16\]

\[
\langle V_\mu (q_1) V_\nu (q_2) A_\tau \rangle = \frac{e^{123}}{8 \pi^2} \left\{ - w_L \epsilon_{\mu \nu q_1 q_2} q_{12 \tau} + w_T^{(+)} t^{(+)}_{\mu \nu \tau} + w_T^{(-)} t^{(-)}_{\mu \nu \tau} + \bar{w}_T^{(-)} \bar{t}^{(-)}_{\mu \nu \tau} \right\}, \tag{38}
\]

\[
t^{(+)}_{\mu \nu \tau} = \epsilon_{q_1 q_2 \mu \nu} q_{1 \tau} - \epsilon_{q_1 q_2 \nu \tau} q_{2 \mu} - (q_1 \cdot q_2) \epsilon_{\mu \nu \tau q_{12}} - \frac{2 (q_1 \cdot q_2)}{q_{12}^2} \epsilon_{\nu q_1 q_2 q_{12 \tau}}, \tag{39}
\]

\[
t^{(-)}_{\mu \nu \tau} = \epsilon_{\mu \nu q_{12}} \left[ q_{12} - \frac{q_{12} \cdot \bar{q}_{12}}{q_{12}^2} q_{12 \tau} \right], \tag{40}
\]

\[
\bar{t}^{(-)}_{\mu \nu \tau} = \epsilon_{q_1 q_2 \mu \nu} q_{1 \tau} + \epsilon_{q_1 q_2 \nu \tau} q_{2 \mu} - (q_1 \cdot q_2) \epsilon_{\mu \nu \tau q_1} + \epsilon_{\mu \nu \tau q_2}. \tag{41}
\]

\[15\]Note however that off-shell effects will be relevant for \((g - 2)\) unless the propagator in Ref. [89] is taken.

\[16\]Ref. [33] uses \( e^{123} = -1 \) instead, that we adapt. Further, we omitted \( i \) overall terms as they cancel in the transition from the axial form factors to the \( \langle VVA \rangle \) function, and the overall \((8 \pi^2)^{-1}\) in Eq. (38).
Comparing to Eq. (4), one can identify the form factors and recast them via Schouten identities in terms of those in Eq. (6), showing that

$$C_A = w_T^{(-)} + \tilde{w}_T^{(-)} \quad B_{2A} = \tilde{w}_T^{(-)} \quad B_{2S} = -w_T^{(+)} \quad C_S = -w_L + \frac{q_1^2 + q_2^2}{q_{12}} w_T^{(+)} \quad - \frac{q_{12}}{q_{12}^2} w_T^{(-)}$$ (42)

$$w_T^{(-)} = C_A - B_{2A} \quad \tilde{w}_T^{(-)} = B_{2A} \quad w_T^{(+)} = -B_{2S} \quad w_L = -[C_S + \frac{q_1^2 + q_2^2}{q_{12}} B_{2S} + \frac{q_{12}}{q_{12}^2} (C_A - B_{2A})].$$

C Phenomenological information on the relevant parameters of the RχT Lagrangian

For the spin-one meson nonets, in application of the large-$N_C$ limit [93], we have considered ideal mixing between the isoscalar component of the octet and the additional isosinglet state completing the nonet. This way, we will have the following diagonal elements of the nonets in flavor space:

$$(V_{11}, V_{22}, V_{33})^{\mu \nu} = \left( \frac{\rho^0 + \omega}{\sqrt{2}}, \frac{-\rho^0 + \omega}{\sqrt{2}}, \phi \right)^{\mu \nu}, \quad (A_{11}, A_{22}, A_{33})^{\mu \nu} = \left( \frac{a_0^1 + f_1}{\sqrt{2}}, \frac{-a_0^1 + f_1}{\sqrt{2}}, f_1' \right)^{\mu \nu},$$ (43)

where $f_1 \sim f_1(1285)$ and $f_1' \sim f_1(1420)$. The leading breaking of the $U(3)$ symmetry splits the heaviest components of each nonet ($\phi$ and $f_1'$) from its partners. In the large-$N_C$ and isospin symmetry limits, the Lagrangian bi-linear in the spin-one fields of the same type (either $VV$ or $AA$) in the even-intrinsic parity sector [94] produces the mass splittings [95, 96] ($M_V$ and $M_A$ are the large-$N_C$ masses of the whole nonet before the symmetry breaking, which is induced by non-vanishing $e_m^V$)

$$M_\rho^2 = M_V^2 - 4e_m^V m_\pi^2 = M_\omega^2, \quad M_\phi^2 = M_V^2 - 4e_m^V (2m_K^2 - m_\pi^2),$$
$$M_{a_1}^2 = M_A^2 - 4e_m^A m_\pi^2 = M_{f_1}^2, \quad M_{f_1}^2 = M_A^2 - 4e_m^A (2m_K^2 - m_\pi^2).$$ (44)

From the best fit in Ref. [83] one has $M_V = (791 \pm 6)$ MeV and $e_m^V = -0.36 \pm 0.10$, which deviate clearly from the fit to mass spectrum that is obtained if one identifies the states in the large-$N_C$ limit with the physical states, yielding $M_V \sim 764.3$ MeV and $e_m^V \sim -0.28$ [96]. It is well understood, however, that such departures occur [97]. In absence of data on the axial-vector transition form factors that could help us to verify in which way $M_A$ and $e_m^A$ differ from the naive values that are obtained fitting the axial-vector meson nonet with the above formulae, and as $M_V^2$ and $M_A^2$ are connected by short-distance constraints [98], we will assume that the shift induced is analogous to the one for the vector mesons. In this way, we obtain $M_A = (1310 \pm 44)$ MeV and $e_m^A = -0.35 \pm 0.13$, where the conservative error is estimated so as to include the naive values of $M_A$ and $e_m^A$ at one standard deviation. According to the preceding discussion, we will use in the following

$$M_V = (791 \pm 6) \text{ MeV}, \quad e_m^V = -0.36 \pm 0.10, \quad M_A = (1310 \pm 44) \text{ MeV}, \quad e_m^A = -0.35 \pm 0.13,$$ (45)

so that, in this limit, the common mass for the isotriplet and isoscalar states of the spin-one octets is

$$M_{\rho \omega} = (808 \pm 8) \text{ MeV}, \quad M_{a_1 f_1} = (1320 \pm 44) \text{ MeV},$$ (46)

\footnote{Note that, provided $q_{12}, M^{\mu \nu} = 0$ as conjectured, $w_L \equiv 0$ to all orders for the axial resonance contribution.}
and the common mass for the extra isoscalar state is

\[ M_0 = (1144 \pm 80) \text{ MeV}, \quad M_{f_1} = (1543 \pm 96) \text{ MeV}. \]  

(47)

We observe that \( M_{\rho\omega} \) is \( \sim 4\% \) larger than its experimental (isospin-averaged) value, while \( M_0 \) is \( \sim 12\% \) larger than its measurement. We will assume a similar deviation for the corresponding states in excited multiplets.

The considered flavor-symmetry breaking also affects the coupling of the vector meson resonances to the photon (encoded in the \( F_V \) couplings). However, as shown in ref. [83], the corresponding leading shifts are given in terms of a single coupling (\( \lambda_A^{V_A} \) in ref. [94]), that vanishes according to short-distance QCD constraints [83]. Thus, \( F_\rho \sim F_\omega \sim F_0 \sim F_V \), within our setting (and similarly for the excited vector resonances). Since \( F_{a_1} \neq F_{f_1} \neq F_{f_1'} \) is induced in complete analogy, we will take the coupling of the axial-vector resonances to the axial current \( (F_A) \) in the \( U(3) \) symmetry limit, as their breaking given by \( \lambda_A^{V_A} \) vanishes by asymptotic conditions [83].

As noted before, the axial-vector contribution to the hadronic light-by-light piece of the muon anomalous magnetic moment, within \( R_{\chi T} \), only depends on the product of couplings \( F_{V_{V_A}}^{V_A} \) for the different \( i = 1, 2, 3 \ldots \) vector multiplets and on the (axial-)vector-resonance masses. Moreover, the high-energy behavior of our form factor links additional \( F_{V_{V_A}}^{V_A} \) factors to \( F_{\pi_{V_A}}^{V_A} \), while the masses of the corresponding multiplets are needed inputs in this case, see Eq. (15). In order to determine \( F_{\pi_{V_A}}^{V_A} \), we follow Refs. [92, 99], where the OPE condition for the \( VVA \) Green’s function up to \( O(1/p^4) \) demands—when matching the \( R_{\chi T} \) result to it—that

\[ \kappa_{V_A}^{V_A} = \kappa_{3_{VV}}^{V_V} \frac{F_V}{F_A} = -\frac{N_C M_{V}^2}{64 \pi^2 F_V F_A} \rightarrow F_V \kappa_{V_A}^{V_A} = -\frac{N_C M_{V}^2}{64 \pi^2 F_A}, \]  

(48)

where the second equality follows from the constraint for \( \kappa_{3_{VV}}^{V_V} \) in ref. [100] and \( F_A \in [130, 150] \text{ MeV} \) [101, 102]. We note that, with only one vector and one axial-vector multiplet, the first Weinberg rule is \( F_{V A}^2 - F_{A}^2 = F^2 \) that -with \( F_V = \sqrt{3} F \sim 160 \text{ MeV} \) [100], which is quite well satisfied phenomenologically [101–103]- yields \( F_A = \sqrt{2} F \sim 130 \text{ MeV} \). Employing the previous values in Eq. (48), one finds \( F_V \kappa_{V_A}^{V_A} \sim (-21.3 \pm 1.5) \text{ MeV} \), with reasonably little uncertainty and that we shall employ in our calculations.

Instead, one could use \( A \rightarrow V \gamma \) decays, whose amplitude reads

\[ \Gamma(A \rightarrow V \gamma) = \frac{2}{3} \alpha |\kappa_{V A}^{V_A}|^2 m_A \left(1 - \frac{m_2^2}{m_A^2}\right)^3 \left(1 + \frac{m_2^2}{m_1^2}\right) \left[\text{tr}(\{V, A\}Q)\right]^2. \]  

(49)

Employing the \( f_1(1285) \rightarrow \rho \gamma \) branching fraction [14] we bind \( |\kappa_{V A}^{V_A}| = 0.45 \pm 0.06 \). However, a recent measurement by CLAS Collaboration [104] implies a much smaller width, that would imply \( |\kappa_{V A}^{V_A}| = 0.27 \pm 0.06 \), much closer to the value \( |\kappa_{V A}^{V_A}| = -0.12 \pm 0.02 \) obtained using short-distance constraints [81, 100–102] or phenomenological determinations of \( F_V \) and \( F_A \) in Eq. (48). Further, given the \( \rho \)-meson width, additional operators involving pion fields might be relevant as well. For these reasons, we advocate to adopt the value implied by the short-distance constraints and emphasize the need for future measurements.

The last input to be fixed are the masses of the vector meson excitations. This will be done using the corresponding generalization of Appendix C and assuming that \( M_{\rho'\omega'} \) and \( M_{\rho'\omega'} \) exceed their (isospin-averaged) PDG values by \( \sim 4\% \) and \( \sim 12\% \), respectively (as it happens with the lightest vector multiplet), and analogously with \( M_{\rho'\omega'} \) and \( M_{\rho'\omega'} \). In this way, we estimate

\[ M_{\rho'\omega'} = (1.51 \pm 0.03) \text{ GeV}, \quad M_{\rho'} = (1.88 \pm 0.03) \text{ GeV}, \]  

\[ M_{\rho'\omega'} = (1.78 \pm 0.03) \text{ GeV}, \quad M_{\rho'} = (2.45 \pm 0.03) \text{ GeV}. \]  

(50)
D Functions involved in the HlbL computation

The axial-meson contribution to the HlbL tensor can be succinctly expressed—in an obvious gauge invariant way—as follows:

\[
\Pi^\text{HlbL,A}_{\mu\nu\rho\sigma} \epsilon_1^{\mu} \epsilon_2^{\nu} \epsilon_3^{\rho} \epsilon_4^{\sigma} = -4i F_A(q_1^2, q_2^2) F_A(q_3^2, q_4^2) \Delta_{H}^{R}(q_{12})^{\alpha\beta,\alpha\beta} (\tilde{F}_2 F_1)_{\alpha\beta} (\tilde{F}_4 F_3)_{\alpha\beta} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4),
\]

where \( F^{\alpha\beta} = (q_1^2 \epsilon_1^{\beta} - q_i^2 \epsilon_i^{\alpha} \) and the additional terms corresponds to the t- and u-channels (see Fig. 1). Above, we have used the notation \( (\tilde{F}_i F_j)^{\alpha\beta} = \tilde{F}_i^{\alpha\mu} F_j^{\nu\beta} g_{\mu\nu} \). In this respect, it is useful to note that \( (\tilde{F}_i F_j)_{\alpha\beta} = -(\tilde{F}_j F_i)_{\alpha\beta} - g_{\alpha\beta} \tilde{F}_i F_j \), where \( F^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}/2 \), and \( \tilde{F}_i F_j = \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F^{\rho\sigma}/2 \).

The functions \( K_i(Q_1^2, Q_2^2, t) \) introduced in Eq. (21), arising in the \( a_i^\text{HlbL:A} \) evaluation, are given by

\[
K_1(Q_1^2, Q_2^2, t) = \frac{8F_A(Q_1^2, Q_2^2) F_A(Q_3^2, 0)}{(Q_1^2 + m_A^2)} \left[ \frac{2m_\mu^2}{Q_1^2} \left( \frac{Q_1 (t^2 - 1)}{Q_1^2} (2Q_1 + Q_2 t) - \frac{2Q_1^2 + 3Q_1 Q_2 t + Q_2^2}{Q_2^2} \right) \right. \\
+ \left. (1 - R_{m_1}) \left( \frac{2Q_1^2 (t^2 - 1) + Q_1 Q_2 t (t^2 - 5) + 2Q_2^2 (t^2 - 3)}{m_A^2} - \frac{2Q_1^2 + 7Q_1 Q_2 t + 6Q_2^2}{Q_2^2} \right) \right. \\
- \left. \frac{(1 - R_{m_2})}{m_A^2 Q_1^2} \left( 6Q_1^2 + 8Q_1 Q_2 t + Q_2^2 \right) + 4Q_1 Q_2^2 (Q_1 + Q_2 t) \right. \\
+ \left. 4X \left( -\frac{Q_2^2 (2Q_1^2 + Q_2^2 (1 - t^2)) - 3Q_3^2 + m_\mu^2 \left( \frac{2Q_2 t}{Q_1} + \frac{4Q_1 t}{Q_2} + 4t^2 + 2 \right) }{m_A^2} \right) \right],
\]

\[
K_2(Q_1^2, Q_2^2, t) = \frac{4F_A(Q_1^2, Q_2^2) F_A(Q_3^2, 0)}{(Q_1^2 + m_A^2)} \left[ \frac{4 (Q_1^2 - Q_2^2) X (Q_1 Q_2^2 + 2m_\mu^2 m_\rho^2 t)}{m_A^2 Q_1 Q_2} \right. \\
+ \left. (1 - R_{m_1}) \left( m_A^2 Q_1 (Q_2 t - Q_1) + Q_2 \left( 2Q_1^2 t + Q_1 Q_2 (3t^2 - 1) + Q_1 Q_2^2 t (t^2 - 3) - 2Q_3^2 \right) \right) \right. \\
- \left. \frac{(1 - R_{m_2})}{m_A^2 Q_1^2} \left( 2m_\mu^2 (Q_1^2 - Q_2^2) \left( m_A^2 + Q_1 Q_2 t (t^2 - 1) \right) \right) \right. \\
- \left. \frac{2m_\mu^2 (Q_1^2 - Q_2^2) \left( m_A^2 + Q_1 Q_2 t (t^2 - 1) \right)}{m_A^2 Q_1^2 Q_2^2} \right],
\]

where the following functions, together with \( Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1 Q_2 t \), have been employed

\[
R_{m_i} = \sqrt{1 + \frac{4m_\mu^2}{Q_i^2}}, \quad z = \frac{Q_1 Q_2}{4m_\mu^2} (1 - R_{m_1})(1 - R_{m_2}), \quad X = \frac{(1 - t^2)^{-1/2}}{Q_1 Q_2} \arctan \left( \frac{\sqrt{1 - t^2}}{1 - z t} \right).
\]

It is also interesting to discuss the asymptotic behavior of the integrands. From the definition
in Eq. (21), and for constant $F_A(Q_1^2, Q_2^2) \to 1$ form factors we obtain for $w_1$

$$\lim_{Q \to \infty} \int_{-1}^{1} dt \, w_1(Q, Q, t) = \frac{70\pi^2 m_{\mu}^2}{3 m_A^2},$$

$$\lim_{Q_1 \to \infty} \int_{-1}^{1} dt \, w_1(Q_1, Q_2, t) = \frac{8\pi^2 Q_2^3}{3 m_A^2 Q_1(m^2 + Q_2^2)} \left[3(Q_2^2 - m_{\mu}^2) + R_{m_2}(6m_{\mu}^2 + 4Q_2^2)\right],$$

$$\lim_{Q_2 \to \infty} \int_{-1}^{1} dt \, w_1(Q_1, Q_2, t) = \frac{2\pi^2 Q_1^3}{9 m_A^2 Q_2} \left[68R_{m_1} - 42 + \frac{13Q_2^2}{m^2}(1 - R_{m_1})\right];$$

while for the second case, $w_2$, we find

$$\lim_{Q \to \infty} \int_{-1}^{1} dt \, w_2(Q, Q, t) = 0,$$

$$\lim_{Q_1(2) \to \infty} \int_{-1}^{1} dt \, w_2(Q_1, Q_2, t) = \pm \frac{\pi^2 Q_2^3}{3 m^2 A Q_1(2)} \left[14 - 8R_{m_2(1)} + \frac{3Q_2^2}{m^2}(1 - R_{m_2(1)})\right],$$

the first result and the relations among the large-$Q_{1,2}$ limits due to the antisymmetric properties of the integrand. Clearly, the asymptotic results set constraints on the form factor asymptotic behavior. In particular, we find that the large $Q_{1(2)}$ limits require the form factors to fall, at least, as $Q_{1(2)}^{-1}$ that, due to the antisymmetric nature of the form factor, demands at least a dipole form.

### E  Operator product expansion

For two highly virtual photons, $q_{1,2} \simeq \lambda q$, with $\lambda \to \infty$, so that $q_1 + q_2 = O(1)$, while $q_1 - q_2 = O(\lambda)$, one can use the operator product expansion, which is valid for large space-like momenta. As a result, one finds (see also [45])

$$(2\pi)^4 \delta^{(4)}(q_1 + q_2 - q_A) M^{\mu\nu \tau \varepsilon}_{\epsilon A} = i \int d^4x d^4y \, e^{iq_1 \cdot x} e^{iq_2 \cdot y} \langle 0 | T\{j^\mu(x) j^\nu(y)\} | A\rangle$$

$$= -\frac{2i}{q^2} e^{i\mu \rho \lambda} \int d^4z \, e^{i(q_1 + q_2) \cdot z} \langle 0 | j_5 \rho | A\rangle,$$

where $\hat{q} = (q_1 - q_2)/2$ and $j_5 \rho = q_\rho \gamma^5 \hat{Q}^2 q$, with $\hat{Q}$ the charge operator. This implies, adopting $\langle 0 | j_5 | A\rangle \equiv \sqrt{2} F_A m_A \epsilon_{A \rho} \text{tr}(\hat{Q}^2 A)$, that

$$M^{\mu\nu \tau \varepsilon}_{\epsilon A} \to -\frac{4i}{(q_1 - q_2)^2} \sqrt{2} F_A m_A \text{tr}(\hat{Q}^2 A) \epsilon_{\mu\nu \epsilon \Lambda \rho \sigma} = -\frac{i}{q^2} \sqrt{2} F_A m_A \text{tr}(\hat{Q}^2 A) \epsilon_{\mu\nu \epsilon \Lambda \rho \sigma}.$$  

Comparing to Eq. (6), the former puts the following constraint

$$\lim_{Q^2 \to \infty} = B_{2S}(-Q^2, -Q^2) = \frac{\sqrt{2} F_A m_A}{Q^4} + O(Q^{-6}).$$

while the antisymmetric form factors and the single-virtual of $B_{2S}$ remain unconstrained. Experimental data however, seems to favor, for all single-virtual form factors, a $Q^{-4}$ high-energy scaling as well.
Incidentally, this shows that the form factor in Ref. [45] $\phi_T^{(a)}(q_1^2, q_2^2)$, that appears when saturating their OPE constraint via axial-vector resonances is, up to overall factors, nothing but $B_{2S}$. This is indeed symmetric as opposed to what was claimed in Refs. [15, 67]. As a final comment, the second form factor $w_T^{(a)}(q_1^2, q_2^2)$, that appears at the external photon vertex, corresponds to the sum of the $w_T^{(+)}$, $w_T^{(-)}$ form factors in Appendix B.4, which tensor structures are the same for $a_{\mu}^{HLbL}$ kinematics (the $w_T^{(-)}$ form factor does not contributes to $a_{\mu}^{HLbL}$). This means that $w_T^{(a)}(q_1^2, q_2^2)$ has mixed symmetry and both, symmetric and antisymmetric, form factors do contribute. We find then, that there was no contradiction in Ref. [45] with the Landau-Yang theorem. Particularly, this means that the OPE cannot be used to put constraints for the antisymmetric form factor, as it is done in [15] and as it is obvious from Eq. (63). Indeed, as explained in Appendix B.1, the fact that only the antisymmetric form factors appear in $a_{\mu}^{HLbL,A}$ is artificial.

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