A note on relative equilibria of a free multidimensional rigid body

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Abstract

It is well known that a rotation of a free generic three-dimensional rigid body is stationary if and only if it is a rotation around one of three principal axes of inertia. As was noted by many authors, the analogous result is true for a multidimensional body: a rotation is stationary if and only if it is a rotation fixing planes spanned by principal axes of inertia, provided that the eigenvalues of the angular velocity matrix are pairwise distinct. However, if some eigenvalues of the angular velocity matrix of a stationary rotation coincide, then it is possible that this rotation has a different nature. A description of such rotations is given in this paper.

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1. Introduction

Speaking informally, a free multidimensional rigid body is simply a rigid body rotating in a multidimensional space without action of any external forces (i.e. by inertia).

Let us first discuss a three-dimensional free rigid body (the so-called Euler case in the rigid body dynamics). A good model for such a body is a book or a parallelepiped-shaped box. Throw the book in the air spinning it in arbitrary direction. If we neglect the gravity force, then what we get is exactly the Euler case.

Note that a general trajectory of a body is not a rotation in the usual sense. At each moment of time our body is indeed rotating around some axis, but this axis is changing as time goes. What we are interested in, are the relative equilibria of the system, i.e. such trajectories for which the axis of rotation remains fixed. Such rotations are also called stationary.

It is well known that a generic three-dimensional rigid body (i.e. a body with pairwise distinct principal moments of inertia) admits three and only three stationary rotations: these are the rotations around three principal axes of inertia. If we deal with a parallelepiped-shaped body, then these axes coincide with the axes of symmetry. What we want to do is to generalize this result to the case of a multidimensional body.
The equations of a free multidimensional rigid body were first written by Frahm [1]. Arnold [2] wrote these equations in the form of Euler equations on $\mathfrak{so}(n)^*$ and generalized them to the case of an arbitrary Lie algebra. A possibility of generalizing Euler equations to the multidimensional case was also mentioned by Weyl [3]:

‘The above treatment of the problem of rotation may, in contradistinction to the usual method, be transposed, word for word, from three-dimensional space to multidimensional spaces. This is, indeed, irrelevant in practice. On the other hand, the fact that we have freed ourselves from the limitation to a definite dimensional number and that we have formulated physical laws in such a way that the dimensional number appears accidental in them, gives us an assurance that we have succeeded fully in grasping them mathematically.’

The equations of a free multidimensional rigid body are famous for being a completely integrable system. As was shown by Manakov [4], the system admits an $L–A$ pair with a spectral parameter. This allowed him to write down integrals and to show that the system is integrable in Riemann $\theta$-functions. Complete integrability in the Liouville sense was proved by Fomenko and Mishchenko [5, 6] and by Ratiu [7].

2. Rotation of an $n$-dimensional body

First, we shall discuss how an $n$-dimensional body may rotate. At each moment of time, $\mathbb{R}^n$ is decomposed into the sum of $m$ pairwise orthogonal two-dimensional planes $\Pi_1, \ldots, \Pi_m$ and an $(n - 2m)$-dimensional space $\Pi_0$ orthogonal to all these planes:

$$\mathbb{R}^n = \bigoplus_{i=1}^{m} \Pi_i \oplus \Pi_0.$$  

There is an independent rotation in each of the planes $\Pi_1, \ldots, \Pi_m$, while $\Pi_0$ is fixed. This is just a reformulation of the theorem about the canonical form of a skew-symmetric operator. Note that $\Pi_0$ may be zero in the even-dimensional case, which means that there are no fixed axes.

A rotation is stationary if all the planes $\Pi_0, \ldots, \Pi_m$ do not change with time (this condition automatically implies that the velocities of rotations are also constant).

A natural generalization of the three-dimensional result would be the following: a rotation of a generic multidimensional rigid body is stationary if and only if the planes $\Pi_0, \ldots, \Pi_m$ are spanned by the principal axes of inertia. This is true provided that the angular velocities of rotations in planes $\Pi_1, \ldots, \Pi_m$ are pairwise distinct (see [8–10]), however not true in general. The existence of relative equilibria for which $\Pi_0, \ldots, \Pi_m$ are not spanned by the principal axes of inertia was probably first noted in [10]. In this note, we give a complete description of such equilibria. Although this result is very simple, we could not find it in the literature.

3. The equations

The motion of a free multidimensional rigid body is described by the Euler–Arnold (or Euler–Frahm) equations on $\mathfrak{so}(n)^*$ (identified with $\mathfrak{so}(n)$). These equations have the form

$$\begin{cases}
M = [M, \Omega] \\
M = \Omega J + J\Omega,
\end{cases}$$  

(1)
where \( M \in \mathfrak{so}(n)^* \) is a skew-symmetric matrix, called the angular momentum matrix; \( J \) is a positive-definite symmetric matrix (see remark 2); and \( \Omega \) is a skew-symmetric matrix, called the angular velocity matrix. It is uniquely defined by the relation

\[
M = \Omega J + J\Omega.
\]

**Remark 1.** Since the map \( J : \mathfrak{so}(n) \to \mathfrak{so}(n) \) given by the formula

\[
J(\Omega) = \Omega J + J\Omega
\]
is invertible, our equations can be rewritten in the \( \Omega \)-coordinates:

\[
\dot{\Omega} = J^{-1}(J(\Omega), \Omega).
\]

However, the explicit formula for \( J^{-1} \) is complicated; therefore, it is convenient to introduce the variable \( M \) and write down the equations in the form (1).

**Remark 2.** In the multidimensional case, \( J \) is sometimes referred to as the ‘inertia tensor’, which seems to be not very precise, because in the three-dimensional case the inertia tensor is not \( J \), but the map \( J : \mathfrak{so}(3) \to \mathfrak{so}(3) \) given by the formula

\[
J(\Omega) = \Omega J + J\Omega.
\]

These two tensors (in the three-dimensional case) have common eigenvectors, but different eigenvalues: the eigenvalues of \( J \) are pairwise sums of the eigenvalues of \( \Omega \).

**Remark 3.** Note that equations (1) describe only the dynamics of the angular velocity matrix. If we want to recover the dynamics in the whole phase space \( T^*\text{SO}(n) \), we should add Poisson equations

\[
\dot{X} = X\Omega, \quad \text{where } X \in \text{SO}(n).
\]

However, we will only be interested in reduced dynamics, given by (1). Note that relative equilibria of a rigid body is nothing else but the equilibrium points of (1).

### 4. Description of relative equilibria

**Theorem 1.** Consider the system of Euler–Arnold equations

\[
\begin{align*}
\dot{M} &= [M, \Omega] \\
M &= \Omega J + J\Omega.
\end{align*}
\]

Suppose that \( J \) has pairwise distinct eigenvalues. Then, \( M \) is an equilibrium point of the system if and only if there exists an orthonormal basis such that \( J \) is diagonal, and \( \Omega \) is block-diagonal of the following form:

\[
\Omega = \begin{pmatrix}
\omega_1 A_1 & & \\
& \ddots & \\
& & \omega_k A_k
\end{pmatrix}, \tag{2}
\]

where \( A_i \in \mathfrak{so}(2m_i) \cap \text{SO}(2m_i) \) for some \( m_i > 0 \), and the \( \omega_i \) are distinct positive real numbers.

Form (2) is unique up to a permutation of blocks.
Proof. We have \([M, \Omega] = [\Omega J + J \Omega, \Omega] = [J, \Omega^2]\); therefore, \(M\) is an equilibrium if and only if \(\Omega^2\) commutes with \(J\).

Assume that we have a basis such that \(J\) is diagonal and \(\Omega\) has the form (2). Then,

\[
A_i^2 = -A_i A_i^t = -E
\]

and

\[
\Omega^2 = \begin{pmatrix}
-\omega_1^2 E & \cdots & 0 \\
\cdots & \ddots & \cdots \\
0 & \cdots & -\omega_k^2 E
\end{pmatrix}.
\]

Therefore, \([\Omega^2, J] = 0\), and our point is an equilibrium point.

Vice versa, let \([\Omega^2, J] = 0\). We shall prove that there exists an orthonormal basis such that \(J\) is diagonal and \(\Omega\) has the form (2).

First, find an orthonormal basis such that \(J\) is diagonal. \(\Omega^2\) is diagonal in this basis as well, since \(J\) has pairwise distinct eigenvalues. Also note that the diagonal entries of \(\Omega^2\) in this basis are non-positive, because \(\Omega\) is skew-symmetric and has only pure imaginary or zero eigenvalues. Now, by a permutation of basis vectors, we can bring \(\Omega^2\) to the form (3), where all \(\omega_i\) are positive and pairwise distinct.

Since \(\Omega^2\) is in the form (3) and \([\Omega^2, \Omega] = 0\), \(\Omega\) has the form

\[
\Omega = \begin{pmatrix}
B_1 \\
\vdots \\
B_k+1
\end{pmatrix},
\]

where \(B_i^2 = -\omega_i^2 E\) for \(i \leq k\), and \(B_{k+1}^2 = 0\).

Since \(B_{k+1} = 0\) and \(B_{k+1}\) is skew-symmetric, we have \(B_{k+1} = 0\). For \(i \leq k\), set

\[
A_i = \frac{1}{\omega_i} B_i.
\]

On the one hand, \(A_i \in so(l_i)\) for some \(l_i\). On the other hand,

\[
A_i A_i^t = -A_i^2 = -\frac{1}{\omega_i^2} B_i^2 = E,
\]

which means that \(A_i \in SO(l_i)\). But \(so(l_i) \cap SO(l_i)\) is empty for odd \(l_i\); therefore, \(l_i = 2m\), and \(\Omega\) has the form (2). \(\square\)

Remark 4. Note that \(so(2m) \cap SO(2m)\) is the homogeneous space \(O(2m)/U(m)\), which is identified with the space of complex structures compatible with the standard Euclidian metrics.

Corollary 1. A relative equilibrium of Euler–Arnold equations is defined by the following ways.

(i) Choosing a decomposition

\[
\mathbb{R}^n = \left( \bigoplus_{i=1}^{\ell} \Pi_i \right) \oplus \Pi_0,
\]

where all \(\Pi_i\) are spanned by the main axes of inertia and all \(\Pi_i\) for \(i > 0\) are even dimensional.
(ii) Assigning an angular velocity $\omega_i > 0$ and a complex structure compatible with the Euclidian metrics to each $\Pi_i$ for $i > 0$.

Corollary 2 (Well known, see [8–10]). Suppose that $M$ is a relative equilibrium, and all eigenvalues of $J$ are pairwise distinct. Moreover, let all non-zero eigenvalues of $\Omega$ be pairwise distinct. Then, there exists an orthonormal basis such that $J$ is diagonal, while $\Omega$ and $M$ are block diagonal with $2 \times 2$ blocks on the diagonal.

In other words, a stationary rotation with pairwise distinct eigenfrequencies is a rotation fixing planes spanned by principal axes of inertia.

Remark 5. Sometimes (see e.g. [9]) the result of corollary 2 is formulated in the following way: if a relative equilibrium belongs to a regular (co)adjoint orbit, then it is a rotation fixing planes spanned by principal axes of inertia. This is not very precise, because ‘belongs to a regular (co)adjoint orbit’ means that $M$ is regular. However, regularity of $M$ does not imply regularity of $\Omega$, and vice versa.

We suggest the following.

Definition 1. We will say that an equilibrium $M$ is regular if there exists an orthonormal basis such that $J$ is diagonal and $\Omega$ is block-diagonal with $2 \times 2$ blocks on the diagonal (i.e. this equilibrium is a rotation in the principal axes of inertia). Otherwise, we will say that $M$ is exotic.

Corollary 2 says that all stationary rotations with pairwise distinct eigenfrequencies are regular.

Example 1. Consider an exotic equilibrium of a four-dimensional body. Then, $\Omega = \omega A$, where $A \in \mathfrak{s}\mathfrak{o}(4) \cap \text{SO}(4)$. This condition is satisfied if and only if $\Omega$ is a singular element in $\mathfrak{s}\mathfrak{o}(4)$. Since $\mathfrak{s}\mathfrak{o}(4) = \mathfrak{s}\mathfrak{o}(3) \oplus \mathfrak{s}\mathfrak{o}(3)$, such elements form two three-dimensional planes in $\mathfrak{s}\mathfrak{o}(4)$. These are exactly two families $s_\pm$ described in [10].

In [11], Bolsinov and Oshemkov study those equilibria of (1) that are equilibrium points simultaneously for all the integrals of the system. It is proven that the set of such equilibria coincides with the set of regular equilibria in our terminology. Consequently, for each exotic equilibrium $M$, we can find an integral $f$ such that the Hamiltonian vector field $v$ generated by $f$ does not vanish at $M$. All the points belonging to the trajectory of $v$ passing through $M$ will be equilibrium points of (1). Consequently, exotic equilibria are not isolated on the co-adjoint orbits of $\mathfrak{s}\mathfrak{o}(n)$, but form smooth submanifolds of equilibrium points (while regular equilibria are, on the contrary, always isolated on a given orbit). This can be used to prove that exotic equilibria are always Lyapunov unstable.

Note that while each exotic equilibrium is unstable, exotic equilibria can be stable as a family, i.e. stable modulo drift along the set of equilibria (see [10]).

Remark 6. The problem of stability of relative equilibria of a multidimensional rigid body was studied by many authors. See [8–10, 12, 13].

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