Isomorphic Hilbert spaces associated with different Complex Contours of the $\mathcal{PT}$-Symmetric $(-x^4)$ Theory

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Abstract

In this work, we stress the existence of isomorphisms which map complex contours from the upper half to contours in the lower half of the complex plane. The metric operator is found to depend on the chosen contour but the maps connecting different contours are norm-preserving. To elucidate these features, we parametrized the contour $z = -2i\sqrt{1+ix}$ considered in Phys.Rev.D73:085002 (2006) for the study of wrong sign $x^4$ theory. For the parametrized contour of the form $z = a\sqrt{b+ix}$, we found that there exists an equivalent Hermitian Hamiltonian provided that $a^2c$ is taken to be real. The equivalent Hamiltonian is $b$-independent but the metric operator is found to depend on all the parameters $a$, $b$ and $c$. Different values of these parameters generate different metric operators which define different Hilbert spaces. All these Hilbert spaces are isomorphic to each other even for parameters values that define contours with ends in two adjacent wedges. As an example, we showed that the transition amplitudes associated with the contour $z = -2i\sqrt{1+ix}$ are exactly the same as those calculated using the contour $z = \sqrt{1+ix}$, which is not $\mathcal{PT}$-Symmetric and has ends in two adjacent wedges in the complex plane.

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The topic of \( \mathcal{PT} \)-Symmetric theories is believed to solve existing problems in Physics. This topic represents an active research area that addresses different research directions \[1\]–[18]. The main stream of research in this area relies on the fact that there exists a huge number of non-Hermitian theories which have real spectra and thus they might have physical applications. Out of these theories, the wrong sign \((-x^4)\) theory is playing a vital role where its field theoretic \((-\phi^4)\) version represents a prototype example of a one component scalar field theory that possesses the asymptotic freedom property \[1, 6–8\]. This theory has been investigated before and its metric operator is known in a closed form \[19\]. It has been found that the theory has an equivalent Hermitian Hamiltonian with a bounded from below potential. In studying this theory and in all the other studies in the literature of any \( \mathcal{PT} \)-Symmetric theory, a complex contour is to be chosen within what is called the Stokes wedges of the theory \[19, 20\]. For instance, to obtain the Hermitian form of the \( \mathcal{PT} \)-Symmetric \((-x^4)\) theory, Jones and Mateo have employed the complex contour \( z = -2i\sqrt{1+ix} \) (Fig.2). This contour has been chosen so that it does exist in the lower half of the complex plane and starts up in a wedge and ends in a non-adjacent one that is \( \mathcal{PT} \)-Symmetric to the first. It is well known that Schrodinger equation has two independent solutions but both of them can not decay to zero as \( |x| \to \infty \) in two adjacent Stokes wedges. However, in this work, we use simple analysis to show that one can work with contours which are neither \( \mathcal{PT} \)-Symmetric nor lie within non-adjacent Stokes wedges. Nevertheless, we show that these contours keep all the transition amplitudes the same as the ones obtained by using the contour \( z = -2i\sqrt{1+ix} \) used in Ref.\[19\] which connects two non-adjacent wedges that are \( \mathcal{PT} \)-Symmetric to each other. The idea we rely on comes from the theory of orthogonal polynomials where the Hermite functions violate the condition \( H_n (x) \to 0 \) as \( |x| \to \infty \). To illustrate this point, we consider the differential operator in the Hermite differential equation of the form;

\[-\frac{d^2\psi}{dx^2} + (2ixp) \psi = 2\lambda \psi,\]

where \( p = -i\partial/\partial x \). One can introduce the non-Hermitian Hamiltonian \( H \) such that;

\[ H\psi = 2\lambda \psi, \]

\[ H = p^2 + 2ixp. \]

Clearly, the eigen functions \( \psi_n \) are the famous Hermite Polynomials \( H_n (x) \) which do not
vanish at infinity as shown in Fig. 1. We can build up the Hilbert space by introducing the metric operator \( \eta \) such that;

\[
\eta H \eta^{-1} = H^\dagger.
\]

Or

\[
\rho H \rho^{-1} = h,
\]

where \( h \) is the equivalent Hermitian Hamiltonian. Note that;

\[
\eta = \rho^2, \quad \rho = \exp \left( -\frac{1}{2} x^2 \right).
\]

In using Baker–Campbell–Hausdorff formula, we get;

\[
\rho H \rho^{-1} = H + [\alpha x^2, H] + \frac{1}{2} [\alpha x^2, [\alpha x^2, H]] + \ldots.
\]

\[
\rho H \rho^{-1} = p^2 + 2ixp + [\alpha x^2, p^2 + 2ixp] + \frac{1}{2} [\alpha x^2, [\alpha x^2, p^2 + 2ixp]]
\]

\[
= p^2 + x^2 - 1 = h.
\]

In fact, the eigen functions \( H_n(x) \) do not vanish at \( x \to \infty \) but the transition amplitudes

\[
T_{ij} = \int \psi_i^* \eta \psi_j dx = \int H_n(x) \exp \left( -x^2 \right) H_m(x) dx,
\]

are finite and calculable although the problem has been treated on the real axis on which the eigen functions \( H_n(x) \) violate the condition \( H_n(x) \to 0 \) as \( |x| \to \infty \). This example shows us that some non-Hermitian theories can be treated on a contour on which the eigen functions do not vanish at its ends. Nevertheless, the structure of the metric operator can turn the transition amplitudes finite. In this work, we will discuss some contours which connect two adjacent Stokes wedges of the \( \mathcal{PT} \)-Symmetric \((-x^4)\) and aim to find the metric that turns the transition amplitudes finite. To achieve our goal, we will follow the method of canonical transformations.

In quantum mechanics, canonical transformations that represent translation and/or scaling of the position variable \( x \) preserve the physical content of a theory \[22\]. So in principle, one might shift and/or scale the real variable \( x \) in the contour \( z = -2i\sqrt{1 + ix} \) in order to obtain another complex contour which might connect two adjacent Stokes wedges of the
theory while the physical content of the theory stays the same. We will show that the Hilbert spaces associated to contours connecting non-adjacent Stokes wedges and those connecting adjacent Stokes wedges are isomorphic to each other.

To start, consider the $\mathcal{P}\mathcal{T}$-symmetric Hamiltonian of the form:

$$H = p^2 - x^4.$$  \hspace{1cm} (1)

The usual recipe of discussing this Hamiltonian is to replace the real variable $x$ by a contour $z(x)$ in the complex plane \cite{19}. Any contour $z(x)$ would represent a canonical transformation that takes the Hamiltonian $H$ to another equivalent non-Hermitian Hamiltonian $H_1$ where,

$$H_1 = \left( \frac{\partial z(x)}{\partial x} \right)^{-1} \left( \frac{\partial z(x)}{\partial x} \right)^{-1} p^2 - i \left( \frac{\partial z(x)}{\partial x} \right)^{-1} \frac{\partial}{\partial x} \left( \frac{\partial z(x)}{\partial x} \right)^{-1} p - (z(x))^4.$$  \hspace{1cm} (2)

Let us parametrize the contour $z(x) = -2i\sqrt{1 + ix}$ chosen by Jones and Mateo in Ref. \cite{19} such that;

$$z(x) = a\sqrt{b + icx}.$$  \hspace{1cm} (3)

Some sets of the parameters $a, b$ and $c$ can define complex contours that connect either adjacent or non-adjacent Stokes wedges. In using the parametrized contour in Eq.(3), we get the Hamiltonian in Eq.(1) transformed to the non-Hermitian form;

$$H_1 = -\frac{4}{a^2c^2} (b + icx)p^2 - \frac{2}{a^2c} x - a^4 (b + icx)^2.$$  \hspace{1cm} (4)

This Hamiltonian might be $\mathcal{P}\mathcal{T}$-symmetric or not depending on the parameters $a, b$ and $c$. Regardless of being $\mathcal{P}\mathcal{T}$-symmetric or not, one may aim to obtain an equivalent Hermitian Hamiltonian by applying a transformation of the form;

$$\rho = \exp \left( fp^3 + gp \right),$$  \hspace{1cm} (5)

where $f$ and $g$ are $C$-number parameters. The transformation $\rho$ transforms $x$ as;

$$\rho x \rho^{-1} = x + \left[ \left( fp^3 + gp \right), x \right] = (x - 3ifp^2 - ig),$$  \hspace{1cm} (6)

and $\rho pp^{-1} = p$. If we set;

$$g = -\frac{b}{c} \text{ and } f = -\frac{2}{3a^6c^3},$$
we get the following Hamiltonian;

\[ h = \frac{4}{a^4c} p^4 + \frac{2}{a^2c} p + a^4 c^2 x^2, \]  

(7)

which is Hermitian provided that \( a^2 c \) is real. In this case, the parameter \( f \) in Eq. (5) is also real but \( g \) need not to be real. The Hamiltonian \( h \) in Eq. (7) with \( a^2 c \) real is equivalent to the Hermitian Hamiltonian obtained in Ref. [19] since one can apply the canonical transformation \( x \rightarrow \frac{2p}{a^2 c}, \ p \rightarrow -\frac{a^2 c x}{2} \) to obtain;

\[ h \rightarrow h_1 = p^2 + 4x^4 - 2x, \]  

(8)

which is exactly the Hamiltonian obtained in Ref. [19] ( with the coupling \( g \) there is taken here to equal 1).

In the above discussions, there is no constraint on the parameter \( b \). However, since \( \rho \) should be Hermitian, the ratio \( \frac{b}{c} \) should be real.

If we choose \( b = 1, c = 1 \) and \( a = i \), we obtain \( a^2 c = -1 \) and thus the contour \( z(x) \) ( the contour labeled by (2) in Fig.2) takes the form;

\[ z(x) = a\sqrt{b + icx} = i\sqrt{ix + 1}. \]  

(9)

From Fig.2 this contour lies in the upper half of the complex plane and connects two non-adjacent Stokes wedges that are \( PT \)-symmetric to each other. With this contour, the equivalent Hermitian Hamiltonian in Eq.(7) takes the form;

\[ h = h_1 = 4p^4 - 2p + x^2 \]

\[ \equiv p^2 + 4x^4 - 2x, \]  

(10)

where in the second line we used the canonical transformation \( x \rightarrow -p, \ p \rightarrow x \). This Hamiltonian is the same as the one obtained in Ref. [19] although the contour lies in the upper half. Note that, the WKB approximation, in momentum space, for the wave function \( \tilde{\phi}(p) \) of the Hamiltonian in Eq. (2) gives;

\[ \tilde{\phi}(p) \sim \sqrt[3]{\sqrt{2} (2p^3 - 1) + 2p^3} \exp \left( \frac{-2}{3} p^3 + p - \left| p \right| \sqrt{\frac{2p^2}{3}} (2p^3 - 1) \right), \]  

(11)

which goes to zero as \( |p| \rightarrow \infty \).

Another contour that leads to the same equivalent Hamiltonian is \( z(x) = \sqrt{1 + ix} \) (labeled by (3) in Fig. 2)). This contour is neither \( PT \)-symmetric nor it connects non-adjacent
Stokes wedges and thus the two possible solutions of the Schrödinger equation can not be finite at the two ends of the contour. This contour surprisingly results in the same equivalent Hermitian Hamiltonian. In fact, working with this contour is similar to investigate the Hermite differential equation on the real line where the eigenfunctions blow up at infinity. However, the weight function (metric operator) \( \exp(-x^2) \) turns the probabilities \( H_n(x) \exp\left(-\frac{1}{2}x^2\right) H_n(x) \) finite everywhere on the real line. Similarly, the \( \mathcal{PT} \)-symmetric \( (-x^4) \) theory on the contour \( z(x) = \sqrt{1+ix} \) have eigenfunctions \( \phi(x) \) that violates the condition \( \phi(x) \to 0 \) as \( |x| \to \infty \) but the probability amplitude \( \phi(x) \eta \phi(x) \) is finite as \( |x| \to \infty \). These features can be verified by considering the WKB approximation of momentum space wave function \( \tilde{\phi}(p) \) which takes the form:

\[
\tilde{\phi}(p) = \frac{1}{\sqrt[3]{\sqrt{2p^2} + \sqrt{2p^3} + 1}} \exp\left(\frac{2}{3}p^3 + p - \frac{1}{3} \sqrt{4p^6 + 2p^3}\right). \tag{12}
\]

Note that \( \tilde{\phi}(p) \to \infty \) as \( p \to \infty \) and \( \tilde{\phi}(p) \to 0 \) as \( p \to -\infty \). In fact, this is expected as the contour \( z(x) = \sqrt{1+ix} \) connects two adjacent Stokes wedges that are not \( \mathcal{PT} \)-symmetric to one another. However, the probability takes the form,

\[
\tilde{\phi}(p) \eta \tilde{\phi}(p) \sim \left(\frac{1}{\sqrt[3]{\sqrt{2p^2} + \sqrt{2p^3} + 1}} \exp\left(-\frac{1}{3} \sqrt{4p^6 + 2p^3}\right)\right)^2, \tag{13}
\]

where \( \eta = e^{-\frac{1}{3}p^3-2p} \) is the metric operator. Accordingly, although the wave function \( \tilde{\phi}(p) \) blows up at one end of the contour, the metric turns the theory finite the same way the weight function does with the Hermite polynomials.

The last case we study here is the contour \( \sqrt{ix} \) which also have the same Hermitian Hamiltonian as the contour \( -2i\sqrt{1+ix} \). In fact, this contour can be considered in the lower or the upper half of the complex plane. This is because, for each value of \( x \), there exist two roots one of positive imaginary part and the other of negative imaginary one. Either taking the upper or the lower root will result in a wave function \( \tilde{\phi}(p) \) that goes to zero as \( |p| \to \infty \). This can be easily checked by the WKB approximation which results in:

\[
\tilde{\phi}(p) = \frac{1}{\sqrt[3]{\sqrt{2p^2} + \sqrt{2p^3} + 1}} \exp\left(\frac{2}{3}p^3 - \frac{1}{3} \sqrt{4p^6 + 2p^3}\right). \tag{14}
\]

Now, we have shown that contours from upper and lower halves in the complex \( x \) plane can lead to the same equivalent Hermitian Hamiltonian. To prove the equivalence, one has
to show that all the transition amplitudes are also the same for all contours. To show this, one consider the metric operator \( \eta \) from Eq. (3) which can be written as:

\[
\eta = \rho^2 = \exp \left( -\frac{4p^3}{3a^6c^4} - \frac{2b}{c}p \right). \tag{15}
\]

Thus for different parameters (i.e., different contours), different metric operators will define different Hilbert spaces and one may wonder if these Hilbert spaces are equivalent (isomorphic). To discuss this point; let \( \psi_i \) are the eigen functions of the differential equation associated with the Hamiltonian \( H_1 \),

\[
\left( \left( \frac{\partial z_1 (x)}{\partial x} \right)^{-1} \left( \frac{\partial z_1 (x)}{\partial x} \right)^{-1} p^2 - i \left( \frac{\partial z_1 (x)}{\partial x} \right)^{-1} \frac{\partial}{\partial x} \left( \frac{\partial z_1 (x)}{\partial x} \right)^{-1} p - (z_1 (x))^4 \right) \psi_i = E_i \psi_i, \tag{16}
\]

for the complex contour \( z_1 (x) = a_1 \sqrt{b_1 + ic_1 x} \) and \( \phi_i \) are the eigen functions of the differential equation;

\[
\left( \left( \frac{\partial z_2 (x)}{\partial x} \right)^{-1} \left( \frac{\partial z_2 (x)}{\partial x} \right)^{-1} p^2 - i \left( \frac{\partial z_2 (x)}{\partial x} \right)^{-1} \frac{\partial}{\partial x} \left( \frac{\partial z_2 (x)}{\partial x} \right)^{-1} p - (z_2 (x))^4 \right) \phi_i = E_i \phi_i,
\]

associated with the contour \( z_2 (x) = a_2 \sqrt{b_2 + ic_2 x} \). To obtain the map that transforms \( z_1 (x) \) to \( z_2 (x) \), one rewrites \( z_2 (x) \) in the form;

\[
z_2 (x) = \sqrt{a_2^2b_2 + ia_2^2c_2x} = \sqrt{a_2^2b_1 + \frac{a_2^2c_2}{a_1^2c_1}a_1^2c_1 \left( x - i \left( \frac{b_2}{c_2} - \frac{a_1^2b_1}{a_2^2c_2} \right) \right)}. \tag{17}
\]

So, we can obtain \( z_2 \) from scaling \( x \) in \( z_1 \) by \( \frac{a_2^2c_2}{a_1^2c_1} \) and then shifting \( x \) by \(-i \left( \frac{b_2}{c_2} - \frac{a_1^2b_1}{a_2^2c_2} \right)\). These operations can be represented by the consequent transformations; \( \zeta_1 = \exp \left( \frac{i}{2} \ln \beta \{ x, p \} \right) \) and \( \zeta_2 = \exp (\gamma p) \) where;

\[
\beta = \frac{a_2^2c_2}{a_1^2c_1}, \quad \gamma = \frac{b_2}{c_2} - \frac{a_1^2b_1}{a_2^2c_2}. \tag{18}
\]

In other words, the map \( \zeta = \zeta_2 \zeta_1 \) is the transformation mapping the complex contour \( z_1 (x) \) to \( z_2 (x) \) and thus we have \( \phi_i = \zeta \psi_i \). To show that \( \zeta \) is an isomorphism, we need to show that \( \zeta \) preserve the inner product or in other words to show that the transition amplitudes equality of the form; \( \langle \psi_i | \eta_1 | \psi_j \rangle = \langle \phi_i | \eta_2 | \phi_j \rangle \) holds. For that, consider the transition amplitudes associated with \( z_1 (x) = a_1 \sqrt{b_1 + ic_1 x} \) to be;

\[
\langle \psi_i | \eta_1 | \psi_j \rangle = \langle \psi_i | \exp \left( -\frac{4}{3a_1^6c_1^4}p^3 - \frac{2b_1}{c_1}p \right) | \psi_j \rangle.
\]
Using the map $\zeta$ one can show that;

$$\langle \psi_i | \eta_1 | \psi_j \rangle = \langle \zeta^{-1} \phi_i | \exp \left( -\frac{4}{3} \frac{a_2^2 c_2 p^3}{a_1^2 c_1} \right) | \zeta^{-1} \phi_j \rangle$$

$$= \langle \zeta_i^{-1} \zeta_2^{-1} \phi_i | \exp \left( -\frac{4}{3} \frac{a_2^6 c_2^3 p^3}{a_1^6 c_1^3} \right) | \zeta_i^{-1} \zeta_2^{-1} \phi_j \rangle$$

$$= \langle \phi_i | \exp \left( -\frac{4}{3} \frac{p^3}{a_1^6 c_1^3} \right) - \frac{2 b_1}{c_1} (p \beta) - 2 \gamma p | \phi_j \rangle$$

$$= \langle \phi_i | \eta_2 | \phi_j \rangle$$

where $\eta_2 = \exp \left( -\frac{4}{3} \frac{p^3}{a_2^6 c_2^3} - \frac{2 b_2}{c_2^3} p \right)$. This shows that the transformation $\zeta$ is an isomorphism that maps the Hilbert space associated with $z_1(x)$ to the Hilbert space associated with $z_2(x)$.

Let us give some examples for the contours $z_1$ and keep always $z_2$ to be the one chosen in Ref. [19], $z_2 = -2i\sqrt{1 + ix}$. Let $z_1 = i\sqrt{1 + ix}$, the metric operator is then given by;

$$\eta_1 = \exp \left( \frac{4}{3} p^3 - 2p \right).$$

The transition amplitudes are then

$$\langle \psi_i | \eta_1 | \psi_j \rangle = \langle \psi_i | \exp \left( \frac{4}{3} p^3 - 2p \right) | \psi_j \rangle.$$

Based on the above discussion, the map

$$\zeta = \exp (\gamma p) \exp \left( \frac{i}{2} \ln \beta \{x, p\} \right),$$

transforms the contour $z_1 = i\sqrt{1 + ix}$ to the contour $z_2 = -2i\sqrt{1 + ix}$ where

$$\beta = \frac{a_2^2 c_2}{a_1^2 c_1}, \quad \gamma = \frac{b_2}{c_2} - \frac{a_1^2 b_1}{a_2^2 c_2},$$

$$a_2 = -2i, \quad b_2 = 1, \quad c_2 = 1.$$

So,

$$\langle \psi_i | \eta_1 | \psi_j \rangle = \langle \psi_i | \exp \left( \frac{4}{3} p^3 - 2p \right) | \psi_j \rangle$$

$$= \langle \phi_i | \left( \zeta^{-1} \right)^\dagger \exp \left( \frac{4}{3} p^3 - 2p \right) \zeta^{-1} | \phi_j \rangle$$

$$= \langle \phi_i | \exp \left( \frac{1}{48} p^3 - 2p \right) | \phi_j \rangle.$$

(20)
One can realize that $\eta_2 = \exp \left( \frac{1}{48} p^3 - 2p \right)$ is exactly the metric operator obtained in Ref. [19]. For the contour $z_1 = \sqrt{1 + ix}$ with the contour $z_2 = -2i \sqrt{1 + ix}$, we have also the equivalence relation;

$$\langle \psi_i | \eta_1 | \psi_j \rangle = \langle \psi_i \left| \exp \left( -\frac{4}{3} p^3 - 2p \right) \right| \psi_j \rangle = \langle \phi_i \left| \exp \left( \frac{1}{48} p^3 - 2p \right) \right| \phi_j \rangle.$$  

(21)

For the contour $z_1 = \sqrt{ix}$, we get the equivalence relation;

$$\langle \psi_i | \eta_1 | \psi_j \rangle = \langle \psi_i \left| \exp \left( -\frac{4}{3} p^3 \right) \right| \psi_j \rangle = \langle \phi_i \left| \exp \left( \frac{1}{48} p^3 - 2p \right) \right| \phi_j \rangle.$$  

(22)

In conclusion, we showed that one can employ complex contours in the upper half of the complex plane for the $\mathcal{PT}$-symmetric $(-x^4)$ theory. For these contours, the amplitudes stay the same as those associated with $\mathcal{PT}$-symmetric contours in the lower half. Besides, we showed that a complex contour that is symmetric about the real line can preserve the physical content of the theory. For this contour, the solutions of the Schrodinger equation cannot be finite at both ends of the contour but the metric operator has been shown to turn the transition amplitudes finite.

To elucidate our idea, we parametrized the contour $z_2 (x) = -2i \sqrt{1 + ix}$ studied in Ref. [19]. The parametrized contour takes the form $z(x) = a \sqrt{b + icx}$, where $a$, $b$ and $c$ are $C$-number parameters. The parameters $a$, $b$ and $c$ can be varied in such a way that results in contours connecting adjacent Stokes wedges.

For the parametrized contour $a \sqrt{b + icx}$, we found that there exists equivalent Hermitian Hamiltonian if $a^2 c$ is kept real. This Hermitian Hamiltonian is found to be $b$-independent and is exactly the same as the one associated with the contour $z_2 (x) = -2i \sqrt{1 + ix}$ from Ref. [19].

We showed that all the Hilbert spaces associated with different parameter choices are isomorphic to each other. The most interesting realization is that when considering the contour $z(x) = \sqrt{1 + ix}$ which connects two adjacent Stokes wedges. For this contour, the WKB approximation for the wave function shows that it blows up at one end of the contour.
In calculating the transition amplitudes using this contour, we found that they are exactly the same as those obtained by considering the contour $z(x) = -2i\sqrt{1+ix}$ which is $\mathcal{PT}$-symmetric as well as connecting two non-adjacent Stokes wedges. The point is that while the wave function associated with the contour $\sqrt{1+ix}$ violates the condition $\phi(x) \rightarrow 0$ as $|x| \rightarrow \infty$, the metric turns the probability amplitude of the form $\phi^*(x) \eta \phi(x)$ finite everywhere and thus the theory is square integrable.
FIG. 1: The first four Hermite polynomials $H_n(x)$, $n = 0, 1, 2, 3$. One can realize that none of these eigen functions verify the condition $H_n(x) \to 0$ as $|x| \to \infty$.

FIG. 2: In the lower half in this figure we have the contour $-2i\sqrt{1+ix}$ labeled by 1. Also, we have one of the possible roots that defines the contour $\sqrt{ix}$ labeled by 4 while the stokes lines are represented by solid straight lines. Also, we have half of the contour $\sqrt{1+ix}$ labeled by 3 and its other half lies in the upper half of the complex plane. In the upper half, we have the contour $i\sqrt{1+ix}$ labeled by 2, one of the possible roots that defines the contour $\sqrt{ix}$ (dashed-straight lines) and the other half of the contour $\sqrt{1+ix}$. Except for the contour $\sqrt{1+ix}$, all the contours are $\mathcal{PT}$-symmetric and connects two non-adjacent wedges. The contour $\sqrt{1+ix}$ is not $\mathcal{PT}$-symmetric and connects two adjacent wedges which mean that none of the solutions is finite at both ends of the contour.

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