ON A CERTAIN DEGENERATE PARABOLIC EQUATION ASSOCIATED WITH THE INFINITY-LAPLACIAN

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Abstract. The comparison, uniqueness and existence of viscosity solutions to the Cauchy-Dirichlet problem are proved for a degenerate parabolic equation of the form $u_t = \Delta_\infty u$, where $\Delta_\infty$ denotes the so-called infinity-Laplacian given by $\Delta_\infty u = \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j}$. Our proof relies on a coercive regularization of the equation, barrier function arguments and the stability of viscosity solutions.

1. Introduction. Aronsson [2] introduces the so-called infinity-Laplacian $\Delta_\infty$ given by

$$\Delta_\infty \phi(x) = \sum_{i,j=1}^N \frac{\partial \phi}{\partial x_i}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x)$$

(1)

to investigate the existence of absolutely minimizing Lipschitz extensions (AMLE’s for short) of functions $g$ defined only on the boundary $\partial \Omega$ of a domain $\Omega$ in $\mathbb{R}^N$ into $\Omega$. Here the AMLE of $g$ into $\Omega$ means a function $u \in W^{1,\infty}(\Omega)$ satisfying that $u = g$ on $\partial \Omega$ and that for every open subset $U$ of $\Omega$ and $\phi \in W^{1,\infty}(U)$, if $u - \phi \in W^{1,\infty}_0(U)$, then

$$|Du|_{L^\infty(U)} \leq |D\phi|_{L^\infty(U)}.$$

More precisely, the following elliptic problem is proposed in [2] as an Euler equation of the above variational problem for smooth AMLE’s:

$$\Delta_\infty u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial \Omega.$$  

(2)

Aronsson [3] also reveals various properties of classical solutions of (2) in $N = 2$; particularly, it is somewhat important that if $u$ is a non-constant classical solution, then $|\nabla u| > 0$ in $\Omega$, which also implies that in general (2) does not admit classical solutions (this fact is clearly described in [15, p. 55]).

Jensen [15] employs the notion of viscosity solutions as a weak solution of (2) and proves the existence and uniqueness of AMLE’s under somewhat general assumptions, and moreover, it is also shown that $u$ is a viscosity solution of (2) if and only if $u$ is the AMLE of $g$.

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Furthermore, various problems related to elliptic equations associated with the infinity Laplacian, e.g., the regularity of solutions, Harnack’s inequality, limiting problems associated with $p$-Laplacian as $p \to +\infty$, eigenvalue problem, $L^\infty$-inequality of the Poincaré type, have been vigorously studied by many authors (see, e.g., [4], [6], [5], [7], [10], [11], [12], [14], [17], [22]). On the other hand, to the best of the authors’ knowledge, parabolic problems associated with the infinity-Laplacian have not been studied yet except in [9], [21] and [16].

This paper is concerned with the following parabolic problem:

$$u_t = \Delta_\infty u \quad \text{in} \quad Q := \Omega \times (0, T),$$

$$u = \varphi \quad \text{on} \quad \mathcal{P}Q,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with boundary $\partial \Omega$, $\mathcal{P}Q$ denotes the parabolic boundary of $Q = \Omega \times (0, T)$ and $u_t$ denotes the time-derivative of $u = u(x, t)$ (see the notation in the end of this section). The main purpose of this paper is to investigate the comparison, uniqueness and existence of viscosity solutions $u = u(x, t)$ of the Cauchy-Dirichlet problem (3), (4).

Another type of parabolic equation associated with the infinity-Laplacian is also studied by Juutinen and Kawohl in [16], where they treat the following:

$$u_t = \Delta_\infty u |Du|^2 \quad \text{in} \quad Q.$$ (5)

They investigate the existence and uniqueness of solutions of the Cauchy-Dirichlet problem for (5) with initial-boundary data $\varphi$, and moreover, they deal with the Cauchy problem for the case $\Omega = \mathbb{R}^N$ as well. To prove the existence, they introduce approximate problems of the form $(u_{\varepsilon, \delta})_t = \varepsilon \Delta u_{\varepsilon, \delta} + \Delta_\infty u_{\varepsilon, \delta}/(|Du_{\varepsilon, \delta}|^2 + \delta)$ with $\varepsilon, \delta > 0$, and establish boundary Hölder estimates of their solutions by constructing barrier functions.

To prove the existence for (3), (4), we introduce the following approximate problems with $\varepsilon > 0$:

$$(u_{\varepsilon})_t = \varepsilon \left(|Du_{\varepsilon}|^2 + \varepsilon\right) \Delta u_{\varepsilon} + \Delta_\infty u_{\varepsilon} \quad \text{in} \quad Q$$ (6)

and prove the existence of classical solutions $u_{\varepsilon}$ for the Cauchy-Dirichlet problems for (6) with initial-boundary data $\varphi$. Moreover, as in [16], we employ barrier function arguments to establish a priori estimates for the solutions $u_{\varepsilon}$. Our proof of establishing a priori estimates is inspired by [16].

In the next section, we state our main results on the comparison, uniqueness and existence of viscosity solutions of the Cauchy-Dirichlet problem (3), (4). Section 3 is devoted to our proof of the existence result.

**Notation:** Throughout this paper, we use the following notation: $Q = \Omega \times (0, T)$, $\mathcal{S}Q = \partial \Omega \times (0, T)$, $\mathcal{B}Q = \Omega \times \{0\}$, $\mathcal{C}Q = \partial \Omega \times \{0\}$, $\mathcal{P}Q = \mathcal{S}Q \cup \mathcal{B}Q \cup \mathcal{C}Q$, $\phi_t = \frac{\partial \phi}{\partial t}$, $D_i = \frac{\partial}{\partial x_i}$, $D = (D_1, D_2, \ldots, D_N)$, $D^2 = \frac{\partial^2}{\partial x_i \partial x_j}$, and $D^2$ denotes the $N \times N$ matrix whose $(i, j)$-th element is $D^2_{ij}$. Furthermore, we also use the Einstein summation convention, where we sum over repeated Greek indices. As for the definitions of function spaces such as $C^{2,1}$, $H^\alpha$ and $H^{\ell,1/2}$ and (semi-)norms, we refer the reader to [19, pp. 7-8]. Moreover, we denote by $\text{Lip}(Q)$ the class of Lipschitz continuous functions in $Q$, and we simply denote by $| \cdot |_{\infty}$ the sup-norm in the corresponding space if no confusion arises.
2. Main results. Before stating our main results, we give a couple of notation and definitions to be used. Set
\[ P(s, p, X) := p_i p_j X_{ij} - s, \quad (s, p, X) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N, \]
where \( \mathbb{S}^N \) denotes the set of all symmetric \( N \times N \) matrices. We are then concerned with viscosity solutions of (3) given in the following.

**Definition 1.** Let \( \Omega \) be a domain in \( \mathbb{R}^N \) and let \( Q = \Omega \times (0, T) \). A function \( u \in USC(Q) := \{ \text{upper semicontinuous functions } u : Q \to \mathbb{R} \} \) is said to be a viscosity subsolution in \( Q \) of (3) if
\[
P(\phi_t(\hat{x}, \hat{t}), D\phi(\hat{x}, \hat{t}), D^2\phi(\hat{x}, \hat{t})) \geq 0
\]
for all \( (\hat{x}, \hat{t}) \in Q \) and \( \phi \in C^{2,1}(Q) \) satisfying \( u - \phi \) attains its local maximum at \( (\hat{x}, \hat{t}) \).

A function \( u \in LSC(Q) := \{ \text{lower semicontinuous functions } u : Q \to \mathbb{R} \} \) is said to be a viscosity supersolution in \( Q \) of (3) if
\[
P(\phi_t(\hat{x}, \hat{t}), D\phi(\hat{x}, \hat{t}), D^2\phi(\hat{x}, \hat{t})) \leq 0
\]
for all \( (\hat{x}, \hat{t}) \in Q \) and \( \phi \in C^{2,1}(Q) \) satisfying \( u - \phi \) attains its local minimum at \( (\hat{x}, \hat{t}) \).

Moreover, \( u \in C(Q) \) is said to be a viscosity solution in \( Q \) of (3) if it is both a viscosity subsolution and a viscosity supersolution in \( Q \) of (3).

Furthermore, viscosity solutions of the Cauchy-Dirichlet problem (3), (4) are defined as follows:

**Definition 2.** A function \( u \in USC(Q) \) (resp., \( LSC(Q) \)) is said to be a viscosity subsolution (resp., supersolution) in \( Q \) of (3), (4) if \( u \) is a viscosity subsolution (resp., supersolution) in \( Q \) of (3), \( u \leq \varphi \) (resp., \( u \geq \varphi \)) on \( \mathcal{P}Q \). Furthermore, \( u \in C(Q) \) is a viscosity solution in \( Q \) of (3), (4) if it is both a viscosity subsolution and a viscosity supersolution in \( Q \) of (3), (4).

Applying Theorem 8.2 and related remarks of [8], the comparison principle for (3), (4) is immediately derived, and moreover, it also implies the continuous dependence on initial-boundary data \( \varphi \) and the uniqueness of solutions.

**Theorem 1** (Comparison and uniqueness). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega \) and let \( u \in USC(Q) \) and \( v \in LSC(Q) \) be a viscosity subsolution and a viscosity supersolution in \( Q = \Omega \times (0, T) \) of (3), respectively, such that \( u \leq v \) on \( \mathcal{P}Q \). Then \( u \leq v \) in \( Q \).

In particular, let \( \varphi_1, \varphi_2 \in C(Q) \) and let \( u_1 \) and \( u_2 \) be viscosity solutions in \( Q \) of (3), (4) with the initial-boundary data \( \varphi_1 \) and \( \varphi_2 \), respectively. Then it follows that
\[
\sup_{(x, t) \in Q} |u_1(x, t) - u_2(x, t)| \leq \sup_{(x, t) \in \mathcal{P}Q} |\varphi_1(x, t) - \varphi_2(x, t)|, \tag{7}
\]
which also implies the uniqueness of solutions.

**Proof of Theorem 1.** Due to Theorem 8.2 of [8], the comparison part follows immediately. Now, let \( u_1 \) and \( u_2 \) be viscosity solutions of (3), (4) with the initial-boundary data \( \varphi_1 \) and \( \varphi_2 \), respectively, and put \( w^\pm(x, t) := u_2(x, t) \pm \sup_{(x, t) \in \mathcal{P}Q} |\varphi_1(x, t) - \varphi_2(x, t)| \). Then the functions \( w^- \) and \( w^+ \) become a viscosity subsolution and a viscosity supersolution of (3), (4) with \( \varphi \) replaced by \( \varphi_1 \), respectively. Thus we have
\[ w^- \leq u_1 \leq w^+ \text{ in } Q, \]
which implies (7). In particular, if \( \varphi_1 = \varphi_2 \) on \( \mathcal{P}Q \), then the uniqueness of solutions follows.

As for the existence of solution, we first introduce the following assumption.

For all \( x_0 \in \partial \Omega \), there exists \( y_0 \in \mathbb{R}^N \) such that \( |x_0 - y_0| = R \) and \( \{ x \in \mathbb{R}^N ; |x - y_0| < R \} \cap \Omega = \emptyset \) for some positive constant \( R \) independent of \( x_0 \).

This assumption is employed only for the construction of approximate solutions in classical sense (see Theorem 4.4 of [19, Chap. VI, p. 560]). Now, our result reads:

**Theorem 2** (Existence). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega \) and let \( Q = \Omega \times (0, T) \). Suppose that (8) is satisfied. Then, for every \( \varphi \in C(\overline{Q}) \), the Cauchy-Dirichlet problem (3), (4) admits a viscosity solution \( u \) in \( \mathcal{C}(\overline{Q}) \) in \( Q \) such that

\[
\sup_{(x,t) \in \Omega} |u(x,t)| \leq \sup_{(x,t) \in \mathcal{P}Q} |\varphi(x,t)|. \tag{9}
\]

3. **Proof of Theorem 2.** In this section, we give a proof of Theorem 2, which is concerned with the existence of viscosity solutions of the Cauchy-Dirichlet problem (3), (4). Firstly we deal with the case \( \varphi \in H^{2+\alpha, 1+\alpha/2}(\overline{Q}) \) for some \( \alpha \in (0, 1) \). We then introduce the following approximation of (3), (4) for each \( \varepsilon \in (0, 1) \).

\[
(u_\varepsilon)_t = \varepsilon (|Du_\varepsilon|^2 + \varepsilon) \Delta u_\varepsilon + \Delta_\infty u_\varepsilon \quad \text{in} \quad Q, \quad u_\varepsilon = \varphi \quad \text{on} \quad \mathcal{P}Q. \tag{10}
\]

Define \( a_{ij}^\varepsilon \in C^\infty(\mathbb{R}^N) \) and \( P_\varepsilon \in C(\mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N) \) by

\[
a_{ij}^\varepsilon(p) := \varepsilon(|p|^2 + \varepsilon) \delta_{ij} + p_ip_j, \quad i, j = 1, 2, \ldots, N, \quad p \in \mathbb{R}^N \]

and

\[
P_\varepsilon(s, p, X) := a_{ij}^\varepsilon(p)X_{ij} - s, \quad (s, p, X) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N.
\]

Then (10) is rewritten into

\[
P_\varepsilon((u_\varepsilon)_t(x, t), Du_\varepsilon(x, t), D^2u_\varepsilon(x, t)) = 0, \quad (x, t) \in Q.
\]

Moreover, we observe that

\[
\varepsilon(|p|^2 + \varepsilon)|\xi|^2 a_{ij}^\varepsilon(p)\xi_i\xi_j \leq \{ \varepsilon(|p|^2 + \varepsilon) + |p|^2 \} |\xi|^2
\]

for all \( \xi \in \mathbb{R}^N \), and furthermore

\[
\left| \frac{\partial a_{ij}^\varepsilon}{\partial p_k} \right| (1 + |p|)^3 \leq C(1 + |p|)^4, \quad i, j, k = 1, 2, \ldots, N.
\]

Thus, since \( \Omega \) satisfies (8), Theorem 4.4 of [19, Chap. VI, p. 560] ensures that the Cauchy-Dirichlet problem (10), (11) admits a classical solution \( u_\varepsilon \in \mathcal{C}(\overline{Q}) \cap H^{2+\alpha, 1+\alpha/2}(Q) \) for each \( \varepsilon \in (0, 1) \).

We now proceed to establish a priori estimates for classical solutions \( u_\varepsilon \) of the Cauchy-Dirichlet problems (10), (11) for each \( \varepsilon \in (0, 1) \). To derive the convergence of \( u_\varepsilon \) as \( \varepsilon \to +0 \), thanks to the stability of viscosity solutions, it suffices to obtain a H"older estimate for \( u_\varepsilon \) on \( \overline{Q} \), which implies the precompactness of \( u_\varepsilon \) in \( \mathcal{C}(\overline{Q}) \). The following lemma provides an \( L^\infty \)-estimate for \( u_\varepsilon \).
Lemma 1 ($L^\infty$-estimate). Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with boundary $\partial \Omega$ and let $u \in C(Q) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0, T)$ of the Cauchy-Dirichlet problem (10), (11) with $\varphi \in C(\overline{Q})$. Then we have

$$|u|_\infty \leq |\varphi|_\infty.$$  

Proof of Lemma 1. The function $w^+(x, t) \equiv |\varphi|_\infty$ (resp., $w^-(x, t) \equiv -|\varphi|_\infty$) becomes a classical supersolution (resp., subsolution) in $Q$ of (10), (11), so the classical comparison principle (see, e.g., Theorem 9.1 of [20, p. 213]) implies that $|u|_\infty \leq |\varphi|_\infty$. \hfill \Box

We have several steps to establish a Hölder estimate for $u_\varepsilon$ in $Q$. The first step is concerned with a Lipschitz estimate for $u_\varepsilon(x, \cdot)$ at $t = 0$ (see Lemma 2), and the second step yields a Lipschitz estimate at any $t \in (0, T)$ (see Lemma 3). In the third step, we estimate a Hölder constant of $u_\varepsilon(\cdot, t)$ on $\partial \Omega$ (see Lemma 4). Hence these three steps imply a boundary Hölder estimate on $PQ$ (see Lemma 5). Finally, we derive a global Hölder estimate for $u_\varepsilon$ in $Q$ from the boundary Hölder estimate (see Lemma 6). Our derivations of these estimates are due to the similar barrier function argument as in [16], and we also employ the translation invariance of the equation (10) to extend Lipschitz and Hölder estimates established only on the boundary, e.g., $t = 0, \partial \Omega, PQ$, as in [18] (a similar argument using the translation invariance of an equation is also found in [13, Corollary 2.11]).

Lemma 2 (Lipschitz estimate for $u_\varepsilon(x, \cdot)$ at $t = 0$). Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with boundary $\partial \Omega$ and let $u \in C(Q) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0, T)$ of the Cauchy-Dirichlet problem (10), (11) with $\varphi \in C^{2,1}(\overline{Q})$. Then it follows that

$$|u(x, t) - \varphi(x, 0)| \leq M_1 t \quad \text{for all } t \in (0, T) \text{ and } x \in \Omega,$$  

where $M_1 := 2(|D^2 \varphi|_\infty^2 + 1)|D^2 \varphi|_\infty + |\varphi_t|_\infty$.

Proof of Lemma 2. Put $w^\pm(x, t) = \varphi(x, 0) \pm M_1 t$ and observe that

\[
P(w^+(x, t), Dw^+(x, t), D^2 w^+(x, t)) = -M_1 + \varepsilon |D^2 \varphi|_\infty + |\varphi_t|_\infty \leq 0
\]

for all $(x, t) \in Q$. Moreover, if $(x, t) \in PQ$, then

\[
w^+(x, t) = \varphi(x, 0) + M_1 t
\]

and

\[
w^-(x, t) = \varphi(x, t) - \varphi(x, 0) + M_1 t
\]

for all $(x, t) \in Q$. Therefore the classical comparison principle ensures that $w^- \leq u \leq w^+$ in $Q$. Hence we obtain (12). \hfill \Box

By using the translation invariance of the equations (10) and the above lemma, we can obtain a Lipschitz estimate for $u_\varepsilon(x, \cdot)$ in $(0, T)$.

Lemma 3 (Lipschitz estimate for $u_\varepsilon(x, \cdot)$ in $(0, T)$). Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with boundary $\partial \Omega$ and let $u \in C(Q) \cap C^{2,1}(Q)$ be a classical solution in
$Q = \Omega \times (0, T)$ of the Cauchy-Dirichlet problem $(10), (11)$ with $\varphi \in C^{2,1}(\overline{Q})$. Then it follows that

$$|u(x, t) - u(x, s)| \leq M_1|t - s|$$

for all $t, s \in (0, T)$ and $x \in \Omega$, (13)

where $M_1 = 2(|D\varphi|_\infty^2 + 1)|D^2\varphi|_\infty + |\varphi_t|_\infty$.

Proof of Lemma 3. Let $h \in (-T, T)$ be fixed and set $Q_h = \Omega \times (h, T + h)$. Putting $v(x, t) = u(x, t - h)$, we see that $v$ remains to be a classical solution in $Q_h$ of $(10), (11)$ with $\varphi$ replaced by $\varphi(\cdot, - h)$. Hence, by Lemma 2, we infer that

$$|v(x, t) - u(x, t)| \leq M_1|h|$$

for all $(x, t) \in \mathcal{B}(Q \cap Q_h)$.

Here we used the fact that $t = \max\{0, h\}$ if $(x, t) \in \mathcal{B}(Q \cap Q_h)$. Thus we can derive $u \leq v + M_1|h|$ on $\mathcal{B}(Q \cap Q_h)$. Moreover, if $(x, t) \in \mathcal{S}(Q \cap Q_h)$, then we see that $(x, t) \in \mathcal{S}Q$, which implies that

$$v(x, t) + M_1|h| = u(x, t - h) + M_1|h| = \varphi(x, t - h) + M_1|h| \geq \varphi(x, t) = u(x, t).$$

Therefore, since $u(x, t) \leq v(x, t) + M_1|h|$ for all $(x, t) \in \mathcal{P}(Q \cap Q_h)$ and $v + M_1|h|$ also becomes a classical supersolution in $Q \cap Q_h$ of $(10)$, it follows that $u \leq v + M_1|h|$ in $Q \cap Q_h$. Repeating the above argument with $v + M_1|h|$ replaced by $v - M_1|h|$, we can deduce that $v - M_1|h| \leq u \leq v + M_1|h|$ in $Q \cap Q_h$, which also gives $|u(x, t) - u(x, t - h)| \leq M_1|h|$ for all $(x, t) \in Q \cap Q_h$. Furthermore, from the arbitrariness of $h$, we can verify (13). □

We next establish a Hölder estimate for $u_\varepsilon(\cdot, t)$ on $\partial \Omega$.

Lemma 4 (Hölder estimate for $u_\varepsilon(\cdot, t)$ on $\partial \Omega$). Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with boundary $\partial \Omega$ and let $\alpha \in (0, 1)$ and $R > 0$ be fixed. Let $u \in C(\overline{\Omega}) \cap C^{2,1}(\Omega)$ be a classical solution in $Q = \Omega \times (0, T)$ of the Cauchy-Dirichlet problem $(10), (11)$ with $\varphi \in C(\overline{\Omega})$ satisfying

$$|\varphi|_\infty < \infty$$

and

$$(\varphi)_x^\alpha := \sup \left\{ \frac{|\varphi(x, t) - \varphi(y, t)|}{|x - y|^{\alpha}}; x, y \in \Omega, x \neq y, t \in (0, T) \right\} < \infty.$$ 

Then there exist constants $\varepsilon_0 = \varepsilon_0(N, \alpha, R) > 0$ and $M_2 = M_2(|\varphi|_\infty, |\varphi_t|_\infty, (\varphi)_x^\alpha, \Omega, N, \alpha, R) \geq 0$ such that if $\varepsilon < \varepsilon_0$ then

$$|u(x, t) - \varphi(x_0, t_0)| \leq M_2(|x - x_0|^{\alpha} + t_0 - t)$$

for all $(x_0, t_0) \in \mathcal{S}Q$, $x \in \Omega \cap B_R(x_0)$ and $t \in (\max\{0, t_0 - 1\}, t_0)$,

where $B_R(x_0) := \{x \in \mathbb{R}^N; |x - x_0| < R\}$.

In particular, the same conclusion also follows with $\Omega \cap B_R(x_0)$ replaced by $\Omega$ by choosing $R > 0$ enough large.

Proof of Lemma 4. Let $(x_0, t_0) \in \mathcal{S}Q$ and $\alpha \in (0, 1)$ be fixed and define

$$w^+(x, t) = \varphi(x_0, t_0) + \kappa|x - x_0|^{\alpha} + \rho(t_0 - t)$$

for all $x \in B_R(x_0) := \{x \in \mathbb{R}^N; |x - x_0| < R\}$ and all $t < t_0$ with positive constants $\kappa$ and $\rho$ which will be determined later. Observing that

$$|w^+(x, t)| = |\rho|, \quad D_i w^+(x, t) = \kappa \alpha |x - x_0|^{\alpha-2}(x - x_0)_i,$$

$$D^2_{ij} w^+(x, t) = \kappa \alpha (\alpha - 2) |x - x_0|^{\alpha-4}(x - x_0)_i(x - x_0)_j + \kappa \alpha |x - x_0|^{\alpha-2} \delta_{ij},$$

and

$$(\varphi)_x^\alpha \leq \frac{1}{|x - x_0|^{\alpha}}.$$ 

we then see that
$$\Delta_{\infty} w^+(x,t) = x_0^4 (\kappa \alpha)^3 (\alpha - 1) |x - x_0|^{3\alpha - 4}.$$ 

Thus it follows that
$$-w_t^+(x,t) + a_{ij}^\epsilon(Dw^+(x,t))D_{ij}^2 w^+(x,t)$$
$$< \rho + (\kappa \alpha)^3 \{ \varepsilon(\alpha - 2 + N) + \alpha - 1 \} |x - x_0|^{3\alpha - 4}$$
$$+ \varepsilon^2 \kappa \alpha (\alpha - 2 + N) |x - x_0|^{\alpha - 2}.$$ 

Here taking \( \varepsilon > 0 \) enough small such that
$$\varepsilon(\alpha - 2 + N) + \alpha - 1 < \frac{1}{2} (\alpha - 1),$$
we have
$$-w_t^+(x,t) + a_{ij}^\epsilon(Dw^+(x,t))D_{ij}^2 w^+(x,t)$$
$$< \rho + \frac{(\kappa \alpha)^3}{2} (\alpha - 1) |x - x_0|^{3\alpha - 4} + \varepsilon^2 \kappa \alpha (\alpha - 2 + N) |x - x_0|^{\alpha - 2}$$
$$= \rho + \kappa \alpha |x - x_0|^{\alpha - 2} \left\{ \frac{(\kappa \alpha)^2}{2} (\alpha - 1) |x - x_0|^{2\alpha - 2} + \varepsilon^2 (\alpha - 2 + N) \right\}$$
$$\leq \rho + \kappa \alpha |x - x_0|^{\alpha - 2} \left\{ \frac{(\kappa \alpha)^2}{2} (\alpha - 1) R^{2\alpha - 2} + \varepsilon^2 (\alpha - 2 + N) \right\},$$
where we used the fact that \( |x - x_0| < R \). Note that
$$\frac{(\kappa \alpha)^2}{2} (\alpha - 1) R^{2\alpha - 2} + \varepsilon^2 (\alpha - 2 + N) \leq \frac{(\kappa \alpha)^2}{4} (\alpha - 1) R^{2\alpha - 2},$$
provided that \( \kappa \geq 1 \) and \( \varepsilon \) is enough small so that
$$\frac{\alpha^2}{4} (\alpha - 1) R^{2\alpha - 2} + \varepsilon^2 (\alpha - 2 + N) \leq 0.$$ 

Thus
$$-w_t^+(x,t) + a_{ij}^\epsilon(Dw^+(x,t))D_{ij}^2 w^+(x,t)$$
$$\leq \rho + \frac{(\kappa \alpha)^3}{4} (\alpha - 1) R^{2\alpha - 2} |x - x_0|^{\alpha - 2}$$
$$\leq \rho + \frac{(\kappa \alpha)^3}{4} (\alpha - 1) R^{3\alpha - 4}.$$ 

Therefore taking \( \kappa \) enough large such that \( 4\rho \leq (\kappa \alpha)^3 (1 - \alpha) R^{3\alpha - 4} \), we conclude that
$$-w_t^+(x,t) + a_{ij}^\epsilon(Dw^+(x,t))D_{ij}^2 w^+(x,t) \leq 0$$
for all \( x \in B_R(x_0) \cap \Omega \) and all \( t < t_0 \).

We next prove that \( w^+ \geq u \) on \( P((B_R(x_0) \cap \Omega) \times (t_0 - 1, t_0)) \) for the case that \( t_0 > 1 \). To do so, we divide our proof to the following three cases:

(i): Let \( x \in (\partial B_R(x_0)) \cap \Omega \) and \( t < t_0 \) be fixed. From the fact that \( |x - x_0| = R \),
we then see that
$$w^+(x,t) = \varphi(x_0,t_0) + \kappa R^\alpha + \rho(t_0 - t) \geq \varphi(x_0,t_0) + \kappa R^\alpha \geq |\varphi|_\infty \geq u(x,t),$$
provided that \( \kappa \geq 2|\varphi|_\infty / R^\alpha \).
Lemma 5. Let \( x \in B_R(x_0) \cap \partial \Omega \) and \( t < t_0 \) be fixed. Since \( \varphi(x,t) = u(x,t) \), it follows that
\[
w^+(x,t) = \varphi(x,t) - \varphi(x,t) + \varphi(x_0,t_0) + \kappa|x-x_0|^\alpha + \rho(t_0-t) \geq u(x,t),
\]
provided that \( \kappa \geq \langle \varphi \rangle^2_{x,Q} \) and \( \rho \geq |\varphi|_\infty \).

(iii): Let \( x \in B_R(x_0) \cap \Omega \) and let \( t = t_0 - 1 \) be fixed. Then
\[
w^+(x,t) = \varphi(x_0,t_0) + \kappa|x-x_0|^\alpha + \rho \geq \varphi(x_0,t_0) + \rho \geq |\varphi|_\infty \geq u(x,t),
\]
provided that \( \rho \geq 2|\varphi|_\infty \).

Now as for the case where \( t_0 < 1 \), we use \( (B_R(x_0) \cap \Omega) \times (0,t_0) \) instead of the cylinder used in the last case. Then it is easily seen that, for \( x \in B_R(x_0) \cap \Omega \) and \( t = 0 \),
\[
w^+(x,0) = \varphi(x_0,t_0) + \kappa|x-x_0|^\alpha + \rho t_0 \geq \varphi(x,0) = u(x,0),
\]
provided that \( \kappa \geq \langle \varphi \rangle^\alpha_{x,Q} \) and \( \rho \geq |\varphi|_\infty \).

Therefore the comparison principle ensures that
\[
u \leq w^+ \text{ on } B_R(x_0) \cap \Omega \times [\max\{0,t_0-1\}, t_0].
\]
Repeating the same argument with the function \( w^-(x,t) := \varphi(x_0,t_0) - \kappa|x-x_0|^\alpha - \rho(t_0-t) \), we can also obtain \( w^- \leq u \) on \( B_R(x_0) \cap \Omega \times [\max\{0,t_0-1\}, t_0] \). Consequently, we can deduce that
\[
|u(x,t) - \varphi(x_0,t_0)| \leq \kappa|x-x_0|^\alpha + \rho(t_0-t)
\]
for all \( (x,t) \in \partial \Omega \) and \( x \in B_R(x_0) \cap \Omega \) and \( t \in [\max\{0,t_0-1\}, t_0] \).

Thus Lemmas 2 and 4 imply the following:

Lemma 5 (Hölder estimate on \( \mathcal{P}Q \)). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega \) and let \( \alpha \in (0,1) \). Suppose that (8) is satisfied. Let \( u \in C(\overline{\Omega}) \cap C^{2,1}(\Omega) \) be a classical solution in \( Q = \Omega \times (0,T) \) of the Cauchy-Dirichlet problem (10), (11) with \( \varin\in (0,\varepsilon_0) \) and \( \varphi \in C^{2,1}(\overline{\Omega}) \). Then it follows that
\[
|u(x,t) - \varphi(x_0,t_0)| \leq M_3 (|x-x_0|^\alpha + |t-t_0|)
\]
for all \( (x,t) \in \mathcal{P}Q \) and \( (x,t) \in Q \),

where \( M_3 = M_1 + M_2 + \langle \varphi \rangle_{x,Q}^{(\alpha)} \).

Proof of Lemma 5. For the case: \( (x_0,t_0) \in \mathcal{S}Q \), by virtue of Lemmas 3 and 4,
\[
|u(x,t) - \varphi(x_0,t_0)| \leq |u(x,t) - u(x,t_0)| + |u(x,t_0) - \varphi(x_0,t_0)| \\
\leq M_1|t_0-t| + M_2|x_0-x|^\alpha.
\]

For the case: \( (x_0,t_0) \in \mathcal{B}Q \), that is, \( t_0 = 0 \), by Lemma 2, we also have
\[
|u(x,t) - \varphi(x_0,t_0)| \leq |u(x,t) - \varphi(x,0)| + |\varphi(x,0) - \varphi(x_0,0)| \\
\leq M_1 t + \langle \varphi \rangle_{x,Q}^{(\alpha)} |x_0-x|^\alpha.
\]

Hence (14) follows.

Now, we extend the above Hölder estimate on the parabolic boundary \( \mathcal{P}Q \) into the parabolic domain \( Q \) in the following lemma, which is derived from Theorem 6 of \cite{18}, but for the completeness we give a proof.
Lemma 6 (Global Hölder estimate). Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with boundary $\partial \Omega$ and let $\alpha \in (0, 1)$. Suppose that (8) is satisfied. Let $u \in C(\overline{\Omega}) \cap C^{2,1}(\Omega)$ be a classical solution in $Q = \Omega \times (0, T)$ of the Cauchy-Dirichlet problem (10), (11) with $\varepsilon \in (0, \varepsilon_0)$ and $\varphi \in C^{2,1}(\overline{\Omega})$. Then it follows that
\[ |u(x, t) - u(y, s)| \leq M_3 (|x - y|^\alpha + |t - s|) \text{ for all } (x, t), (y, s) \in Q, \tag{15} \]
where $M_3 = M_1 + M_2 + (\varphi(x, t))^{(\alpha)}$.

Proof of Lemma 6. Let $h := (h_x, h_t) \in \mathbb{R}^N \times \mathbb{R}$ be fixed and let $Q + h := \{(x, t) \in \mathbb{R}^{N+1}, (x - h_x, t - h_t) \in \Omega \}$. Moreover, put $v(x, t) = u(x - h_x, t - h_t)$. We then find that $v$ still remains to be a classical solution in $Q + h$ of (10), (11) with $\varphi$ replaced by $\varphi(-h_x, \cdots, -h_t)$. Then, by Lemma 5, we can assure that, for $(x, t) \in P(Q \cap (Q + h))$, $|v(x, t) - u(x, t)| \leq M_3 |h|_{\alpha, 1}$, where $|h|_{\alpha, 1} := |h_x|^\alpha + |h_t|$. Hence, $v \leq v + M_3 |h|_{\alpha, 1}$ also become classical solutions in $Q \cap (Q + h)$ of (10), the classical comparison theorem ensures that $v \leq v + M_3 |h|_{\alpha, 1}$ also become classical solutions in $Q \cap (Q + h)$ of (10), the classical comparison theorem ensures that $v \leq v + M_3 |h|_{\alpha, 1}$ in $Q \cap (Q + h)$. From the arbitrariness of $h$, we can verify (15). \hfill \Box

By virtue of the global Hölder estimate for $u_\varepsilon$ in Lemma 6 and Ascoli-Arzela's compactness theorem, taking a sequence $\varepsilon_n \to +0$, we can deduce that
\[ u_{\varepsilon_n} \to u \text{ uniformly on } \overline{\Omega} \tag{16} \]
as $\varepsilon_n \to +0$. We also note that $P_\varepsilon(s, p, X) \to P(s, p, X)$ as $\varepsilon \to +0$, for all $(s, p, X) \in \mathbb{R} \times \mathbb{R}^N \times S^N$.

Therefore the stability of viscosity solutions (see, e.g., Section 6 of [8]) ensures that the limit $u$ becomes a viscosity solution of (3), (4).

Secondly we proceed to the case $\varphi \in C(\overline{\Omega})$. By virtue of Weierstrass's approximation theorem (see, e.g., 1.29 Corollary of [1, p. 10]), we can take an approximate sequence $\varphi_n \in H^{2+\alpha, 1+\alpha/2}(\overline{\Omega})$ such that $\varphi_n \to \varphi$ uniformly on $\overline{\Omega}$. Hence, due to the last case, there exists a viscosity solution $u_n$ of (3), (4) with $\varphi$ replaced by $\varphi_n$. Moreover, by Theorem 1,
\[ \sup_{(x, t) \in Q} |u_n(x, t) - u_m(x, t)| \leq \sup_{(x, t) \in P_Q} |\varphi_n(x, t) - \varphi_m(x, t)| \to 0 \]
as $n, m \to +\infty$. Thus $(u_n)$ forms a Cauchy sequence in $C'(\overline{\Omega})$, so $u_n \to u$ uniformly on $\overline{\Omega}$. Therefore, from the stability of viscosity solution, $u$ also becomes a viscosity solution of (3), (4) with the initial data $\varphi \in C(\overline{\Omega})$. Furthermore, as in Lemma 1, (9) follows immediately. This completes our proof of Theorem 2.

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