Learning the Structure and Parameters of Large-Population Graphical Games from Behavioral Data

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Abstract

We consider learning, from strictly behavioral data, the structure and parameters of linear influence games (LIGs), a class of parametric graphical games introduced by [Ir-tan and Ortiz 2011][2013]. LIGs facilitate causal strategic inference: Making inferences from causal interventions on stable behavior in strategic settings. Applications include the identification of the most influential individuals in large (social) networks. Such tasks can also support policy-making analysis. Motivated by the computational work on LIGs, we cast the learning problem as maximum-likelihood estimation (MLE) of a generative model defined by pure-strategy Nash equilibria (PSNE). Our simple formulation uncovers the fundamental interplay between goodness-of-fit and model complexity: good models capture equilibrium behavior within the data while controlling the true number of equilibria, including those unobserved. We provide a generalization bound establishing the sample complexity for MLE in our framework. We propose several algorithms including convex loss minimization (CLM) and sigmoidal approximations. We formally prove LIGs have a small true number of PSNE, with high probability; thus, CLM is sound. We illustrate our approach on synthetic data and real-world U.S. congressional voting records. We briefly discuss our learning framework’s generality and potential applicability to general graphical games.

1 Introduction

Game theory has become a central tool for modeling multi-agent systems in AI. Non-cooperative game theory has been considered as the appropriate mathematical framework
in which to formally study strategic behavior in multi-agent scenarios. The core solution concept of Nash equilibria (NE) [Nash, 1951] serves a descriptive role of the stable outcome of the overall behavior of systems involving self-interested individuals interacting strategically with each other and distributed settings for which no direct global control is possible. NE is also often used in predictive roles as the basis for what one might call causal strategic inference, i.e., inferring the results of causal interventions on stable actions/behavior/outcomes in strategic settings (See, e.g., Ballester et al. 2004, 2006, Heal and Kunreuther 2003, 2006, 2007, Kunreuther and Michel-Kerjan 2007, Ortiz and Kearns 2002, Kearns 2005, Irfan and Ortiz 2011, 2013, and the references therein). Needless to say, the computation and analysis of NE in games is of significant interest to the computational game-theory community within AI.

The introduction of compact representations to game theory over the last decade have extended computational/algorithmic game theory’s potential for large-scale, practical applications often encountered in the real-world. For the most part, such game model representations are analogous to probabilistic graphical models widely used in machine learning and AI. Introduced within the AI community about a decade ago, graphical games [Kearns et al., 2001] constitute an example of one of the first and arguably one of the most influential graphical models for game theory. There has been considerable progress on problems of computing classical equilibrium solution concepts such as NE and correlated equilibria (CE) [Aumann, 1974] in graphical games (see, e.g., Kearns et al. [2001], Vickrey and Koller [2002], Ortiz and Kearns [2002], Blum et al. 2006, Kakade et al. 2003, Papadimitriou and Roughgarden 2008, Jiang and Leyton-Brown 2011 and the references therein). Indeed, graphical games played a prominent role in establishing the computational complexity of computing NE in general normal-form games (see, e.g., Daskalakis et al. 2009 and the references therein).

An example of a recent computational application of non-cooperative game-theoretic graphical modeling and causal strategic inference that motivates the current paper is the work of Irfan and Ortiz [2011, 2013]. They proposed a new approach to the study of influence and the identification of the “most influential” individuals (or nodes) in large (social) networks. Their approach is strictly game-theoretic in the sense that it relies on non-cooperative game theory and the central concept of pure-strategy Nash equilibria (PSNE) approximate predictor of stable behavior in strategic settings, and, unlike other models of

1See, e.g., the survey of Shoham [2008] and the books of Nisan et al. 2007 and Shoham and Leyton-Brown 2009 for more information.

2The fundamental property such compact representation of games exploit is that of conditional independence: each player’s payoff function values are determined by the actions of the player and those of the player’s neighbors only, and thus are conditionally (payoff) independent of the actions of the non-neighboring players, given the action of the neighboring players.

3Other game-theoretic graphical models include game networks [La Mura, 2000], multi-agent influence diagrams (MAIDs) [Koller and Milch 2003], and action-graph games [Jiang and Leyton-Brown 2008].

4In the interest of terseness, we provide a brief description of the work and motivation Irfan and Ortiz 2011, 2013 here, and refer the reader to their article for the details.

5In this paper, because we concern ourselves primarily with PSNE, whenever we use the term “equilibrium” or “equilibria” without qualification, we mean PSNE.
behavior in mathematical sociology, it is not interested and thus avoids explicit modeling of the complex dynamics by which such stable outcomes could have arisen or could be achieved. Instead, it concerns itself with the “bottom-line” end-state stable outcomes (or steady state behavior). Hence, the proposed approach provides an alternative to those based on the diffusion of behavior through a social network (See Kleinberg [2007] for an introduction and discussion targeted to computer scientists, and further references).

The underlying assumption for most work in computational game theory that deals with algorithms for computing equilibrium concepts is that the games under consideration are already available, or have been “hand-designed” by the analyst. While this may be possible for systems involving a handful of players, it is impossible in systems with at least tenths of agent entities, if not more, as we are interested in this paper. For instance, in their paper, Irfan and Ortiz [2011, 2013] propose a class of games, called influence games. In particular, they concentrate on linear influence games (LIGs), and, as briefly mentioned above, study a variety of computational problems resulting from their approach, assuming such games are given as input.

Research in computational game theory has paid relatively little attention to the problem of learning (both the structure and parameters of) graphical games from data. Addressing this problem is essential to the development, potential use and success of game-theoretic models in practical applications. Indeed, we are beginning to see an increase in the availability of data collected from processes that are the result of deliberate actions of agents in complex system. A lot of this data results from the interaction of a large number of individuals, being not only people (i.e., individual human decision-makers), but also companies, governments, groups or engineered autonomous systems (e.g. autonomous trading agents), for which any form of global control is usually weak. The Internet is currently a major source of such data, and the smart grid, with its trumpeted ability to allow individual customers to install autonomous control devices and systems for electricity demand, will likely be another one in the near future.

In this paper, we investigate in considerable technical depth the problem of learning LIGs from strictly behavioral data: We do not assume the availability of utility, payoff or cost information in the data; the problem is precisely to infer that information from

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6Some of these models have recently gained interest and have been studied within computer science, specially those related to diffusion or contagion processes (see, e.g., Granovetter [1978], Morris [2000], Domingos and Richardson [2001], Domingos [2005], Even-Dar and Shapira [2007]).

7It is important that the reader understands and keep in mind that here, and throughout the whole manuscript, we mean “learning structure and parameters” in a sense analogous to learning the structure and parameters of Bayesian networks (BNs). Recall that when we learn a BN, based, for example, on MLE with possibly additional bias/regularization, what we learn is some probabilistic model of the data. Even with an infinite amount of data generated from a ground-truth BN, there is no guarantee that learning will recover/identify the structure and parameters of the ground-truth BN. But instead, that, with enough data, the learned BN will represent exactly the same joint probability distribution as the ground-truth BN. Several times throughout the manuscript, we return to and elaborate on this crucial aspect to understanding our ML-approach to inferring games from strictly behavioral data.

8The machine-learning work presented in this paper complements and is motivated by the computationally and modeling oriented work of Irfan and Ortiz [2011, 2013], but is fundamentally different. Each piece of research work stands on its own and thus appears on their own publication and proper venue. The separation of learning from using models (e.g. for inference) is accepted machine-learning practice, for instance, in Markov random fields (MRFs), Bayesian networks (BNs) and other related probabilistic graphical models Koller and Friedman [2009].
just the joint behavior collected in the data, up to the degree needed to explain the joint behavior itself (see Footnote 7). We expect that, in most cases, the parameters quantifying a utility function or best-response condition are unavailable and hard to determine in real-world settings. The availability of data resulting from the observation of an individual public behavior is arguably a weaker assumption than the availability of individual utility observations, which are often private. In addition, we do not assume prior knowledge of the conditional payoff/utility independence structure as represented by the game graph.

Motivated by the work of Irfan and Ortiz [2011, 2013] on a strictly non-cooperative game-theoretic approach to influence and strategic behavior in networks, we present a formal framework and design algorithms for learning the structure and parameters of LIGs with a large number of players. We concentrate on data about what one might call “the bottom-line”: i.e., data about “end-states”, “steady-states” or final behavior as represented by possibly noisy samples of joint actions/pure-strategies from stable outcomes, which we assume come from a hidden underlying game. Thus, we do not use, consider or assume available any temporal data about the detailed behavioral dynamics. In fact, the data we consider does not contain the dynamics that might have possibly lead to the potentially stable joint-action outcome! Since scalability is one of our main goals, we aim to propose methods that are polynomial-time in the number of players.

Given that LIGs belong to the class of 2-action graphical games [Kearns et al., 2001] with parametric payoff functions, we first needed to deal with the relative dearth of work on the broader problem of learning general graphical games from purely behavioral data. Hence, in addressing this problem, while inspired by the computational approach of Irfan and Ortiz [2011, 2013], the learning problem formulation we propose is in principle applicable to arbitrary games (although, again, the emphasis is on the PSNE of such games). In particular, we introduce a simple statistical generative mixture model, built “on top of” the game-theoretic model, with the only objective being to capture noise in the data. Despite the simplicity of the generative model, we are able to learn games from U.S. congressional voting records, which we use as a source of real-world behavioral data, that, as we will illustrate, seem to capture interesting, non-trivial aspects of the U.S. congress. While such models learned from real-world data are impossible to validate, we argue that there exists a considerable amount of anecdotal evidence for such aspects as captured by the models we learned. Figure 1 provides a brief illustration. (Should there be further need for clarification as to the why we present this figure, please see Footnote 17.)

As a final remark, given that LIGs constitute a non-trivial sub-class of parametric graphical games, we view our work as a step in the direction of addressing the broader problem of learning general graphical games with a large number of players from strictly behavioral data. We also hope our work helps to continue to bring and increase attention from the machine-learning community to the problem of inferring games from behavioral data (in which we attempt to learn a game that would “rationalize” players’ observed behavior).

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9 Of course, because we are motivated by the work of Irfan and Ortiz [2011, 2013], the nature of the games we learn is closely tied to the game’s PSNE, as we will see, which may not align with the basis of other game-theoretic frameworks for causal strategic inference. Hence, our proposed learning framework is intended for learning graphical games of the stated nature and may not be useful or effective for learning games of a different nature or for a different objective.

10 This is a type of problem arising from game theory and economics that is different from the problem of learning in games (in which the focus is the study of how individual players learn to play a game by
Figure 1: 110th US Congress’s LIG (January 3, 2007-09): We provide an illustration of the application of our approach to real congressional voting data. Irfan and Ortiz [2011, 2013] use such games to address a variety of computational problems, including the identification of most influential senators. (We refer the reader to their paper for further details.) We show the graph connectivity of a game learned by independent $\ell_1$-regularized logistic regression (see Section 6.5). The reader should focus on the overall characteristics of the graph and not the details of the connectivity or the actual “influence” weights between senators. We highlight some particularly interesting characteristics consistent with anecdotal evidence. First, senators are more likely to be influenced by members of the same party than by members of the opposite party (the dashed green line denotes the separation between the parties). Republicans were “more strongly united” (tighter connectivity) than Democrats at the time. Second, the current US Vice President Biden (Dem./Delaware) and McCain (Rep./Arizona) are displayed at the “extreme of each party” (Biden at the bottom-right corner, McCain at the bottom-left) eliciting their opposite ideologies. Third, note that Biden, McCain, the current US President Obama (Dem./Illinois) and US Secretary of State Hillary Clinton (Dem./New York) have very few outgoing arcs; e.g., Obama only directly influences Feingold (Dem./Wisconsin), a prominent senior member with strongly liberal stands. One may wonder why do such prominent senators seem to have so little direct influence on others? A possible explanation is that US President Bush was about to complete his second term (the maximum allowed). Both parties had very long presidential primaries. All those senators contended for the presidential candidacy within their parties. Hence, one may posit that those senators were focusing on running their campaigns and that their influence in the day-to-day business of congress was channeled through other prominent senior members of their parties.

1.1 A Framework for Learning Games: Desiderata

The following list summarizes the discussion above and guides our choices in our pursuit of a machine-learning framework for learning game-theoretic graphical models from strictly behavioral data.

- The learning algorithm
  - must output an LIG (which is a special type of graphical game); and
  - should be practical and tractably deal with a large number of players (typically in the 100’s, and certainly at least 4).
The learned model should take a “bottom-line” approach, thus focusing on the prediction of end-state (or steady-state) joint decision-making behavior, and making no explicit attempt or effort to model or predict the temporal behavioral dynamics that might have lead to end-state or the stable steady-state joint behavior, including pre-play “cheap talk,” which are often overly complex processes. (See Appendix A.1 for further discussion.)

The learning framework

- would only have available strictly behavioral data on actual decisions/actions taken. It cannot require or use any kind of payoff-related information.
- should be agnostic as to the type or nature of the decision-maker and does not assume each player is a single human. Players can be institutions or governments, or associated with the decision-making process of a group of individuals representing, e.g., a company (or sub-units, office sites within a company, etc.), a nation state (like in the UN, NATO, etc.), or a voting district. In other words, the recorded behavioral actions of each player may really be a representative of larger entities or groups of individuals, not necessarily a single human.
- must provide computationally efficient learning algorithm with provable guarantees: worst-case polynomial running time in the number of players.
- should be “data efficient” and provide provable guarantees on sample complexity (given in terms of “generalization” bounds).

1.2 Technical Contributions

While our probabilistic model is inspired by the concept of equilibrium from game theory, our technical contributions are not in the field of game theory nor computational game theory. Our technical contributions and the tools that we use are the ones in classical machine learning.

Our technical contributions include a novel generative model of behavioral data in Section 4 for general games. Motivated by the LIGs and the computational game-theoretic framework put forward by Irfan and Ortiz [2011, 2013], we formally define “identifiability” and “triviality” within the context of non-cooperative graphical games based on PSNE as the solution concept for stable outcomes in large strategic systems. We provide conditions that ensure identifiability among non-trivial games. We then present the maximum-likelihood estimation (MLE) problem for general (non-trivial identifiable) games. In Section 5, we show a generalization bound for the MLE problem as well as an upper bound of the functional/strategic complexity (i.e., analogous to the “VC-dimension” in supervised learning) of LIGs. In Section 6, we provide technical evidence justifying the approximation of the original problem by maximizing the number of observed equilibria in the data as suitable for a hypothesis-space of games with small true number of equilibria. We then present our convex loss minimization approach and a baseline sigmoidal approximation for LIGs. For completeness, we also present exhaustive search methods for both general games as well as LIGs. In Section 7, we formally define the concept of absolute-indifference of players and show that our convex loss minimization approach produces games in which all players are
Table 1: Summary of approaches for learning models of behavior. See main text for a discussion. For each method we show its model class (GG: graphical games, NF: normal-form non-graphical games, NGT: non-game-theoretic model); whether it needs observed payoffs, learns utility parameters, learns graphical structure or provides guarantees (e.g., generalization, sample complexity or PAC learnability); its equilibria concept (PSNE: pure strategy or MSNE: mixed strategy Nash equilibria, CE: correlated equilibria, QRE: quantal response equilibria), whether it is dynamic (i.e., behavior predicted from past behavior); and the number of agents in the experimental validation. Note that there are relatively fewer models that do not assume observed payoff; among them, our method is the only one that learns the structure of graphical games, furthermore, we provide guarantees and a polynomial-time algorithm.

| Reference                     | Class | Needs Observed Payoff | Learns Utility Parameters | Learns Graphical Structure | Guarantees | Equilibria Concept | Dyn. | Num. of Agents |
|-------------------------------|-------|------------------------|----------------------------|----------------------------|------------|--------------------|------|----------------|
| Wright and Leyton-Brown 2010  | NF    | N                      | N                          | -                          | N          | QRE                | N    | 2              |
| Wright and Leyton-Brown 2012  | NF    | Y                      | N                          | -                          | N          | QRE                | N    | 2              |
| Gao and Pfieber 2010          | NF    | Y                      | Y                          | -                          | N          | QRE                | N    | 2              |
| Vorobeychik et al. 2005, 2007 | NF    | Y                      | Y                          | -                          | N          | MSNE               | N    | 2-5            |
| Ficici et al. 2008            | NF    | Y                      | Y                          | -                          | N          | MSNE               | N    | 10-200         |
| Duong et al. 2008             | NGT   | N                      | N                          | -                          | N          | N                  | 4,10 |                |
| Duong et al. 2010             | NGT   | Y                      | N                          | -                          | N          | Y                  | 10   |                |
| Duong et al. 2012             | NGT   | Y                      | N                          | -                          | Y          | Y                  | 36   |                |
| Duong et al. 2009             | GG    | Y                      | Y                          | Y                          | N          | PSNE               | N    | 2-13           |
| Kearns and Wortman 2008       | NGT   | N                      | -                          | Y                          | N          | Y                  | 100  |                |
| Ziebart et al. 2010           | NF    | N                      | Y                          | -                          | N          | CE                 | N    | 2-3            |
| Waugh et al. 2011             | NF    | N                      | Y                          | -                          | Y          | CE                 | Y    | 7              |
| **Our approach**              | GG    | N                      | Y                          | Y                          | Y          | PSNE               | N    | 100            |

*non-absolutely-indifferent.* We provide a distribution-free bound which shows that LIGs have small true number of equilibria with high probability.

## 2 Related Work

We provide a brief summary overview of previous work on learning games here, and delay discussion of the work presented below until after we formally present our model; this will provide better context and make “comparing and contrasting” easier for those interested, without affecting expert readers who may want to get to the technical aspects of the paper without much delay.

Table 1 constitutes our best attempt at a simple visualization to fairly present the differences and similarities of previous approaches to modeling behavioral data within the computational game-theory community in AI.

We note that none of the “related” or seemingly “competing” models from the existing literature can learn or output LIGs, which is our foremost objective in this work. The research interest of previous work varies in what they intend to capture in terms of different aspects of behavior (e.g., dynamics, probabilistic vs. strategic) or simply different settings/domains (i.e., modeling “real human behavior,” knowledge of achieved payoff or utility, etc.). Modifications of those models to our setting seem extremely unclear at best.
At the very least, it would certainly require a considerable amount of research work, analysis and proper empirical evaluation that is clearly way beyond the scope of this paper.

With the exception of Ziebart et al. [2010], Waugh et al. [2011], Kearns and Wortman [2008], previous methods assume that the actions as well as corresponding payoffs (or noisy samples from the true payoff function) are observed in the data. Our setting largely differs from Ziebart et al. [2010], Kearns and Wortman [2008] because of their focus on system dynamics, in which future behavior is predicted from a sequence of past behavior. Kearns and Wortman [2008] proposed a learning-theory framework to model collective behavior based on stochastic models.

Our problem is clearly different from methods in quantal response models [McKelvey and Palfrey, 1995, Wright and Leyton-Brown, 2010, 2012] and graphical multiagent models (GMMs) [Duong et al., 2008, 2010] that assume known structure and observed payoffs. Duong et al. [2012] learns the structure of games that are not graphical, i.e. the payoff depends on all other players. Their approach also assumes observed payoff and consider a dynamic consensus scenario, where agents on a network attempt to reach a unanimous vote. In analogy to voting, we do not assume the availability of the dynamics (i.e. the previous actions) that led to the final vote. They also use fixed information on the conditioning sets of neighbors during their search for graph structure. We also note that the work of Vorobeychik et al. [2005, 2007], Gao and Pfeffer [2010], Ziebart et al. [2010] present experimental validation mostly for 2 players only, 7 players in Waugh et al. [2011] and up to 13 players in Duong et al. [2009].

In contrast, in this paper, we assume that the joint-actions are the only observable information and that both the game graph structure and payoff functions are unknown, unobserved and unavailable. We present the first techniques for learning the structure and parameters of a non-trivial class of large-population graphical games from joint-actions only. Furthermore, we present experimental validation in games of up to 100 players. Our convex loss minimization approach could potentially be applied to larger problems since it has polynomial time complexity in the number of players.

### 2.1 On Learning Probabilistic Graphical Models

There has been a significant amount of work on learning the structure of probabilistic graphical models from data. We mention only a few references that follow a maximum likelihood approach for Markov random fields [Lee et al., 2006], bounded tree-width distributions [Chow and Liu, 1968, Srebro, 2001], Ising models [Wainwright et al., 2006, Banerjee et al., 2008], Gaussian graphical models [Banerjee et al., 2006], Bayesian networks [Guo and Schuurmans, 2006, Schmidt et al., 2007b], and directed cyclic graphs [Schmidt and Murphy, 2009].
Our approach learns the structure and parameters of games by maximum likelihood estimation on a related probabilistic model. Our probabilistic model does not fit into any of the types described above. Although a (directed) graphical game has a directed cyclic graph, there is a semantic difference with respect to graphical models. Structure in a graphical model implies a factorization of the probabilistic model. In a graphical game, the graph structure implies strategic dependence between players, and has no immediate probabilistic implication. Furthermore, our general model differs from Schmidt and Murphy [2009] since our generative model does not decompose as a multiplication of potential functions.

Finally, it is very important to note that our specific aim is to model behavioral data that is strategic in nature. Hence, our modeling and learning approach deviates from those for probabilistic graphical models which are of course better suited for other types of data, mostly probabilistic in nature (i.e., resulting from a fixed underlying probability distribution). As a consequence, it is also very important to keep in mind that our work is not in competition with the work in probabilistic graphical models, and is not meant to replace it (except in the context of data sets collected from complex strategic behavior just mentioned). Each approach has its own aim, merits and pitfalls in terms of the nature of data sets that each seeks to model.

Thus, we have little interest in determining how much worst game-theoretic models are relative to probabilistic models when applied to data from purely probabilistic processes, without any strategic component, as we think this to be a futile exercise. We believe the same is true for evaluating the quality of a probabilistic graphical model vs. a game-theoretic model when applied to strategic behavioral data, resulting from a process defined by game-theoretic concepts based on the (stable) outcomes of a game. Nevertheless, we summarize some experiments in Figure 2 that should help illustrate the point of the previous paragraph, just in case it was not already evident. In Section 8 (Experimental Results), we also present empirical evidence that establishes the superiority (in terms of data-fit/likelihood) of game-theoretic graphical models over their probabilistic counterparts in the context of what we believe is real-world data on strategic behavior: the congressional voting dataset.

2.2 On Linear Threshold Models and Econometrics

Irfan and Ortiz [2011, 2013] introduced LIGs in the AI community, showed that such games are useful, and addressed a variety of computational problems, including the identification of most influential senators. The class of LIGs is related to the well-known linear threshold model (LTM) in sociology [Granovetter 1978], recently very popular within the social network and theoretical computer science community [Kleinberg 2007]. Irfan and Ortiz [2011, 2013] discusses linear threshold models in depth; we briefly discuss them here for self-containment. LTMs are usually studied as the basis for some kind of diffusion process. A typical problem is the identification of most influential individuals in a social network. An LTM is not in itself a game-theoretic model and, in fact, Granovetter himself argues against this view in the context of the setting and the type of questions in which he was

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11 The plot illustrates our point that “comparing” models can be very misguided and wasteful, as no model is universally better and one should instead try to match the type of model to the nature of the data.

12 L´ opez-Pintado and Watts [2008] also provide an excellent summary of the various models in this area of mathematical social science.
Figure 2: On the Distinction between Game-Theoretic and Probabilistic Models in the Context of “Probabilistic” vs. “Strategic” Behavioral Data. The plot shows the performance of Ising models (in green) vs. LIGs (in red) when we learn models from each respective class from data generated by drawing i.i.d. samples from a mixture model of an Ising model, $p_{\text{Ising}}$, and our PSNE-based generative model, $p_{\text{LIG}}$, with mixing parameter $q_{\text{strat}}$ corresponding to the probability that the sample is drawn from the LIG component. Hence, we can view $q_{\text{strat}}$ as controlling the proportion of the data that is “strategic” in nature.

The graph of the Ising model is an (undirected) chain with 4 variable nodes, while that of the LIG is, as shown in the left, also a chain of 4 players with arcs between every consecutive pair of nodes. The parameters of each mixture component in the “ground-truth” mixture model $p_{\text{mix}}(x) \equiv q_{\text{strat}} p_{\text{LIG}}(x) + (1-q_{\text{strat}}) p_{\text{Ising}}(x)$ are the same: node-potential/bias-threshold parameters are all 0; weights of all the edges is +1. We set the “signal” parameter $q$ of our generative model $p_{\text{LIG}}$ to 0.

The x-axis of the plot in the right-hand side above corresponds to the mixture parameter $q_{\text{strat}}$: so that, as we move from left to right in the plot, more proportion of the data is “strategic” in nature: $q_{\text{strat}} = 0$ means the data is “purely probabilistic” while $q_{\text{strat}} = 1$ means it is “purely strategic.” For each value of $q_{\text{strat}} \in \{0,0.25,0.50,0.75,1\}$, we generated 50 pairs of datasets from $p_{\text{mix}}$, each of size 50, each pair corresponding to a training and a validation dataset, respectively. The learning methods used the validation dataset to estimate their respective $\ell_1$ regularization parameter. We use a simultaneous logistic regression approach, described in Section 6, to learn LIGs. In the y-axis of the plot in the right-hand side is the average, over the 50 repetitions, of the exact KL-divergence between the respective learned model and $p_{\text{mix}}(x)$. We also include (a linear interpolation of the individual) error bars at 95% confidence level. The plot clearly shows that the more “strategic” the data the better the game-theoretic-based generative model. We can see that the learned Ising models (1) do considerably better than the LIG models when the data is purely probabilistic; and (2) are more “robust” across the spectrum, degrading very gracefully as the data becomes more strategic in nature; but (3) seem to need more data to learn when the data comes exclusively from an Ising model than the LIG model does when the data is purely strategic: The LIG achieves KL values much closer to 0 when the data is purely strategic than the Ising model does when the data is purely probabilistic.

most interested [Granovetter, 1978]. Our reading of the relevant literature suggests that subsequent work on LTM has not taken a strictly game-theoretic view either. The problem of learning mathematical models of influence from behavioral data has just started to receive attention. There has been a number of articles in the last couple of years addressing the problem of learning the parameters of a variety of diffusion models of influence [Saito et al., 2008, 2009, 2010, Goyal et al., 2010, Gomez Rodriguez et al., 2010, Cao et al., 2011].

13Often learning consists of estimating the threshold parameter from data given as temporal sequences from “traces” or “action logs.” Sometimes the “influence weights” are estimated assuming a given graph, and almost always the weights are assumed positive and estimated as “probabilities of influence.” For example, Saito et al. [2010] considers a dynamic (continuous time) LTM that has only positive influence weights and
Our model is also related to a particular model of discrete choice with social interactions in econometrics (see, e.g. Brock and Durlauf [2001]). The main difference is that we take a strictly non-cooperative game-theoretic approach within the classical “static”/one-shot game framework and do not use a random utility model. We follow the approach of Irfan and Ortiz [2011, 2013] who takes a strictly non-cooperative game-theoretic approach within the classical “static”/one-shot game framework, and thus we do not use a random utility model. In addition, we do not make the assumption of rational expectations, which in the context of models of discrete choice with social interactions essentially implies the assumption that all players use exactly the same mixed strategy.\[14\]

3 Background: Game Theory and Linear Influence Games

In classical game-theory (see, e.g. Fudenberg and Tirole [1991] for a textbook introduction), a normal-form game is defined by a set of players $V$ (e.g. we can let $V = \{1, \ldots, n\}$ if there are $n$ players), and for each player $i$, a set of actions, or pure-strategies $A_i$, and a payoff function $u_i : x_1 \times x_2 \times \ldots \times x_n \rightarrow \mathbb{R}$ mapping the joint-actions of all the players, given by the Cartesian product $A \equiv \times_{j \in V} A_j$, to a real number. In non-cooperative game theory we assume players are greedy, rational and act independently, by which we mean that each player $i$ always want to maximize their own utility, subject to the actions selected by others, irrespective of how the optimal action chosen help or hurt others.

A core solution concept in non-cooperative game theory is that of an Nash equilibrium. A joint-action $x^* \in A$ is a pure-strategy Nash equilibrium (PSNE) of a non-cooperative game if, for each player $i$, $x^*_i \in \arg\max_{x_i \in A_i} u_i(x_i, x^*_{-i})$; that is, $x^*$ constitutes a mutual best-response, no player $i$ has any incentive to unilaterally deviate from the prescribed action $x^*_i$, given the joint-action of the other players $x^*_{-i} \in \times_{j \in V-\{i\}} A_j$ in the equilibrium. In what follows, we denote a game by $G$, and the set of all pure-strategy Nash equilibria of $G$ by \[15\]

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\mathcal{NE}(G) \equiv \{x^* \mid (\forall i \in V) \ x^*_i \in \arg\max_{x_i \in A_i} u_i(x_i, x^*_{-i})\}.
$$

A (directed) graphical game is a game-theoretic graphical model [Kearns et al., 2001]. It provides a succinct representation of normal-form games. In a graphical game, we have a (directed) graph $G = (V, E)$ in which each node in $V$ corresponds to a player in the game. The interpretation of the edges/arcs $E$ of $G$ is that the payoff function of player $i$ is only a function of the set of parents/neighbors $N_i \equiv \{j \mid (i, j) \in E\}$ in $G$ (i.e. the set of players corresponding to nodes that point to the node corresponding to player $i$ in the graph). In the context of a graphical game, we refer to the $u_i$’s as the local payoff functions/matrices.

Linear influence games (LIGs) [Irfan and Ortiz, 2011, 2013] are a sub-class of 2-action graphical games with parametric payoff functions. For LIGs, we assume that we are given

\[14\] A formal definition of “rational expectations” is beyond the scope of this paper. We refer the reader to the early part of the article by Brock and Durlauf [2001] where they explain why assuming rational expectations leads to the conclusion that all players use exactly the same mixed strategy. That is the relevant part of that work to ours.

\[15\] Because this paper concerns mostly PSNE, we denote the set of PSNE of game $G$ as $\mathcal{NE}(G)$ to simplify notation.
a matrix of influence weights \( W \in \mathbb{R}^{n \times n} \), with \( \text{diag}(W) = 0 \), and a threshold vector \( b \in \mathbb{R}^n \). For each player \( i \), we define the influence function \( f_i(x_{-i}) \equiv \sum_{j \in N_i} w_{ij} x_j - b_i = w_{i,-i}^T x_{-i} - b_i \) and the payoff function \( u_i(x) \equiv x_i f_i(x_{-i}) \). We further assume binary actions: \( A_i \equiv \{-1,+1\} \) for all \( i \). The best response \( x_i^* \) of player \( i \) to the joint-action \( x_{-i} \) of the other players is defined as:

\[
\begin{align*}
    &\{ w_{i,-i}^T x_{-i} > b_i \Rightarrow x_i^* = +1, \\
    &w_{i,-i}^T x_{-i} < b_i \Rightarrow x_i^* = -1 \text{ and } \\
    &w_{i,-i}^T x_{-i} = b_i \Rightarrow x_i^* \in \{-1,+1\} \}
\end{align*}
\]

\( \iff x_i^* (w_{i,-i}^T x_{-i} - b_i) \geq 0 \) (2)

Intuitively, for any other player \( j \), we can think of \( w_{ij} \in \mathbb{R} \) as a weight parameter quantifying the “influence factor” that \( j \) has on \( i \), and we can think of \( b_i \in \mathbb{R} \) as a threshold parameter quantifying the level of “tolerance” that player \( i \) has for playing \(-1\).\(^{16}\)

As discussed in Irfan and Ortiz \[2011, 2013\], LIGs are also a sub-class of polymatrix games \[Janovskaja, 1968\]. Furthermore, in the special case of \( b = 0 \) and symmetric \( W \), a LIG becomes a party-affiliation game \[Fabrikant et al., 2004\].

In this paper, the use of the verb “influence” strictly refers to influences defined by the model.

Figure \[\] provides a preview illustration of the application of our approach to congressional voting.\(^{17}\)

4 Our Proposed Framework for Learning LIGs

Our goal is to learn the structure and parameters of an LIG from observed joint-actions only (i.e., without any payoff data/information).\(^{18}\) Yet, for simplicity, most of the presentation

\(^{16}\)As we formally/mathematically define here, LIGs are 2-action graphical games with linear-quadratic payoff functions. Given our main interest in this paper on the PSNE solution concept, for the most part, we simply view LIGs as compact representations of the PSNE of graphical games that the algorithms of Irfan and Ortiz \[2011, 2013\] use for CIS. (This is in contrast to a perhaps more natural, “intuitive” but still informal description/interpretation one may provide for instructive/pedagogical purposes based on “direct influences,” as we do here.) This view of LIGs is analogous to the modern, predominant view of Bayesian networks as compact representations of joint probability distributions that are also very useful for modeling uncertainty in complex systems and practical for probabilistic inference \[Koller and Friedman, 2009\]. (And also analogous is the “intuitive” descriptions/interpretations of BN structures, used for instructive/pedagogical purposes, based on “causal” interactions between the random variables \[Koller and Friedman, 2009\].)

\(^{17}\)We present this game graph because many people express interest in “seeing” the type of games we learn on this particular dataset. The reader should please understand that by presenting this graph we are definitely not implying or arguing that we can identify the ground-truth graph of “direct influences.” (We say this even in the very unlikely event that the “ground-truth model” be an LIG that faithfully capture the “true direct influences” in this U.S. Congress, something arguably no model could ever do.) As we show later in Section 4.2, LIGs are not identifiable with respect to their local compact parametric representation encoding the game graph through their weights and biases, but only with respect to their PSNE, which are joint-actions capturing a global property of a game that we really care about for CIS. Certainly, we could never validate the model parameters of an LIG at the local, microscopic level of “direct influences” using only the type of observational data we used to learn the model depicted by the graph in the figure. For that, we would need help from domain experts to design controlled experiments that would yield the right type of data for proper/rigorous scientific validation.

\(^{18}\)In principle, the learning framework itself is technically immediately/easily applicable to any class of simultaneous/on-shot games. Generalizing the algorithms and other theoretical results (e.g., on generaliza-
in this section is actually in terms of general 2-action games. While we make sporadic references to LIGs throughout the section, it is not until we reach the end of the section that we present and discuss the particular instantiation of our proposed framework with LIGs.

Our main performance measure will be average log-likelihood (although later we will be considering misclassification-type error measures in the context of simultaneous-classification, as a result of an approximation of the average log-likelihood). Our emphasis on a PSNE-based statistical model for the behavioral data results from the approach to causal strategic inference taken by Irfan and Ortiz [2011, 2013], which is strongly founded on PSNE.

Note that our problem is unsupervised, i.e., we do not know a priori which joint-actions are PSNE and which ones are not. If our only goal were to find a game $G$ in which all the given observed data is an equilibrium, then any “dummy” game, such as the “dummy” LIG $G = (W, b)$, $W = 0, b = 0$, would be an optimal solution because $|\mathcal{NE}(G)| = 2^n$. In this section, we present a probabilistic formulation that allows finding games that maximize the empirical proportion of equilibria in the data while keeping the true proportion of equilibria as low as possible. Furthermore, we show that trivial games such as LIGs with $W = 0, b = 0$, obtain the lowest log-likelihood.

4.1 Our Proposed Generative Model of Behavioral Data

We propose the following simple generative (mixture) model for behavioral data based strictly in the context of “simultaneous”/one-shot play in non-cooperative game theory, again motivated by Irfan and Ortiz [2011, 2013]’s PSNE-based approach to causal strategic inference. Let $G$ be a game. With some probability $0 < q < 1$, a joint-action $x$ is chosen uniformly at random from $\mathcal{NE}(G)$; otherwise, $x$ is chosen uniformly at random from its complement set $\{-1, +1\}^n - \mathcal{NE}(G)$. Hence, the generative model is a mixture model with mixture parameter $q$ corresponding to the probability that a stable outcome (i.e., a PSNE) of the game is observed, uniform over PSNE. Formally, the probability mass function (PMF) over joint-behaviors $\{-1, +1\}^n$ parametrized by $(G, q)$ is

$$p_{(G,q)}(x) = q \frac{1[x \in \mathcal{NE}(G)]}{|\mathcal{NE}(G)|} + (1 - q) \frac{1[x \notin \mathcal{NE}(G)]}{2^n - |\mathcal{NE}(G)|}$$

Note that our problem is unsupervised, i.e., we do not know a priori which joint-actions are PSNE and which ones are not. If our only goal were to find a game $G$ in which all the given observed data is an equilibrium, then any “dummy” game, such as the “dummy” LIG $G = (W, b)$, $W = 0, b = 0$, would be an optimal solution because $|\mathcal{NE}(G)| = 2^n$. Indeed, in our approach, games with no PSNE achieve the lowest possible likelihood within our generative model of the data; said differently, games with PSNE have higher are guaranteed larger likelihoods than those that do not have any PSNE.

Model “simplicity” and “abstractions” are not necessarily a bad thing in practice. More “realism” often leads to more “complexity” in terms of model representation and computation; and to potentially poorer generalization performance as well [Kearns and Vazirani, 1994]. We believe that even if the data could be the result of complex cognitive, behavioral or neuronal processes underlying human decision making and social interactions, the practical guiding principle of model selection in ML, which governs the fundamental tradeoff between model complexity and generalization performance, still applies.
where we can think of $q$ as the “signal” level, and thus $1-q$ as the “noise” level in the data set.

**Remark 1.** Note that in order for eq. [3] to be a valid PMF for any $\mathcal{G}$, we need to enforce the following conditions $|\mathcal{N}(\mathcal{G})| = 0 \Rightarrow q = 0$ and $|\mathcal{N}(\mathcal{G})| = 2^n \Rightarrow q = 1$. Furthermore, note that in both cases ($|\mathcal{N}(\mathcal{G})| \in \{0, 2^n\}$) the PMF becomes a uniform distribution. We also enforce the following condition: \(^22\) if $0 < |\mathcal{N}(\mathcal{G})| < 2^n$ then $q \notin \{0, 1\}$.

### 4.2 On PSNE-Equivalence and PSNE-Identifiability of Games

For any valid value of mixture parameter $q$, the PSNE of a game $\mathcal{G}$ completely determines our generative model $p(\mathcal{G}, q)$. Thus, given any such mixture parameter, two games with the same set of PSNE will induce the same PMF over the space of joint actions. \(^23\)

**Definition 2.** We say that two games $\mathcal{G}_1$ and $\mathcal{G}_2$ are PSNE-equivalent if and only if their PSNE sets are identical, i.e.: $\mathcal{G}_1 \equiv_{\mathcal{N}} \mathcal{G}_2 \iff \mathcal{N}(\mathcal{G}_1) = \mathcal{N}(\mathcal{G}_2)$.

We often drop the “PSNE-” qualifier when clear from context.

**Definition 3.** We say a set $\Upsilon^*$ of valid parameters $(\mathcal{G}, q)$ for the generative model is PSNE-identifiable with respect to the PMF $p(\mathcal{G}, q)$ defined in eq. [3], if and only if, for every pair $(\mathcal{G}_1, q_1), (\mathcal{G}_2, q_2) \in \Upsilon^*$, if $p(\mathcal{G}_1, q_1)(\mathbf{x}) = p(\mathcal{G}_2, q_2)(\mathbf{x})$ for all $\mathbf{x} \in \{-1, +1\}^n$ then $\mathcal{G}_1 \equiv_{\mathcal{N}} \mathcal{G}_2$ and $q_1 = q_2$. We say a game $\mathcal{G}$ is PSNE-identifiable with respect to $\Upsilon^*$ and the $p(\mathcal{G}, q)$, if and only if, there exists a $q$ such that $(\mathcal{G}, q) \in \Upsilon^*$.

**Definition 4.** We define the true proportion of equilibria in the game $\mathcal{G}$ relative to all possible joint-actions as

$$\pi(\mathcal{G}) \equiv |\mathcal{N}(\mathcal{G})|/2^n.$$ \(^4\)

We also say that a game $\mathcal{G}$ is trivial if and only if $|\mathcal{N}(\mathcal{G})| \in \{0, 2^n\}$ (or equivalently $\pi(\mathcal{G}) \in \{0, 1\}$), and non-trivial if and only if $0 < |\mathcal{N}(\mathcal{G})| < 2^n$ (or equivalently $0 < \pi(\mathcal{G}) < 1$).

The following propositions establish that the condition $q > \pi(\mathcal{G})$ ensures that the probability of an equilibrium is strictly greater than a non-equilibrium. The condition also guarantees that non-trivial games are identifiable.

**Proposition 5.** Given a non-trivial game $\mathcal{G}$, the mixture parameter $q > \pi(\mathcal{G})$ if and only if $p(\mathcal{G}, q)(\mathbf{x}_1) > p(\mathcal{G}, q)(\mathbf{x}_2)$ for any $\mathbf{x}_1 \in \mathcal{N}(\mathcal{G})$ and $\mathbf{x}_2 \notin \mathcal{N}(\mathcal{G})$.

*Proof.* Note that $p(\mathcal{G}, q)(\mathbf{x}_1) = q/|\mathcal{N}(\mathcal{G})| > p(\mathcal{G}, q)(\mathbf{x}_2) = (1-q)/(2^n - |\mathcal{N}(\mathcal{G})|) \Leftrightarrow q > |\mathcal{N}(\mathcal{G})|/2^n$ and given eq. [4], we prove our claim. \(\square\)

\(^22\)We can easily remove this condition at the expense of complicating the theoretical analysis on the generalization bounds because of having to deal with those extreme cases.

\(^23\)It is not hard to come up with examples of multiple games that have the same PSNE set. In fact, later in this section, we show three instances of LIGs with very different weight-matrix parameter that have this property. Note that this is not a roadblock to our objectives of learning LIGs because our main interest is the PSNE of the game, not the individual parameters that define it. We note that this situation is hardly exclusive to game-theoretic models: an analogous issue occurs in probabilistic graphical models (e.g., Bayesian networks).
Proposition 6. Let \((G_1, q_1)\) and \((G_2, q_2)\) be two valid generative-model parameter tuples.

(a) If \(G_1 \equiv_{\mathcal{N}^E} G_2\) and \(q_1 = q_2\) then \((\forall x)\ p_{(G_1, q_1)}(x) = p_{(G_2, q_2)}(x)\).

(b) Let \(G_1\) and \(G_2\) be also two non-trivial games such that \(q_1 > \pi(G_1)\) and \(q_2 > \pi(G_2)\). If \((\forall x)\ p_{(G_1, q_1)}(x) = p_{(G_2, q_2)}(x)\), then \(G_1 \equiv_{\mathcal{N}^E} G_2\) and \(q_1 = q_2\).

Proof. Let \(\mathcal{N}^E_k \equiv \mathcal{N}^E(\mathcal{G}_k)\). First, we prove part (a). By Definition \[2\] \(G_1 \equiv_{\mathcal{N}^E} G_2 \Rightarrow \mathcal{N}^E_1 = \mathcal{N}^E_2\). Note that \(p_{(G_k, q_k)}(x)\) in eq.\(3\) depends only on characteristic functions \(1[x \in \mathcal{N}^E_k]\). Therefore, \((\forall x)\ p_{(G_1, q_1)}(x) = p_{(G_2, q_2)}(x)\).

Second, we prove part (b) by contradiction. Assume, \(\exists x \in \mathcal{N}^E_1 \land x \notin \mathcal{N}^E_2\). Then \(p_{(G_1, q_1)}(x) = p_{(G_2, q_2)}(x)\) implies by eq.\(3\) that \(q_1/|\mathcal{N}^E_1| = (1 - q_2)/(2^n - |\mathcal{N}^E_2|) \Rightarrow q_1/\pi(G_1) = (1 - q_2)/(1 - \pi(G_2))\) by eq.\(4\). By assumption, we have \(q_1 > \pi(G_1)\), which implies that \((1 - q_2)/(1 - \pi(G_2)) > 1 \Rightarrow q_2 < \pi(G_2)\), a contradiction. Hence, we have \(G_1 \equiv_{\mathcal{N}^E} G_2\). Assume, \(q_1 \neq q_2\). Then we have \(p_{(G_1, q_1)}(x) = p_{(G_2, q_2)}(x)\) implies by eq.\(3\) and \(G_1 \equiv_{\mathcal{N}^E} G_2\) (and after some algebraic manipulations) that \((q_1 - q_2)\left(\frac{1|x \in \mathcal{N}^E(G_1)|}{|\mathcal{N}^E(G_1)|} - \frac{1|x \notin \mathcal{N}^E(G_1)|}{2^n - |\mathcal{N}^E(G_1)|}\right) = 0 \Rightarrow \frac{1|x \in \mathcal{N}^E(G_1)|}{|\mathcal{N}^E(G_1)|} = \frac{1|x \notin \mathcal{N}^E(G_1)|}{2^n - |\mathcal{N}^E(G_1)|}\), a contradiction. \(\square\)

The last proposition, along with our definitions of “trivial” (as given in Definition \[4\]) and “identifiable” (Definition \[3\]), allows us to formally define our \textit{hypothesis space}.

Definition 7. Let \(\mathcal{H}\) be a class of games of interest. We call the set \(\Upsilon \equiv \{(G, q) \mid G \in \mathcal{H} \land 0 < \pi(G) < q < 1\}\) the \textit{hypothesis space} of non-trivial identifiable games and mixture parameters. We also refer to a game \(G \in \mathcal{H}\) that is also in some tuple \((G, q) \in \Upsilon\) for some \(q\), as a non-trivial identifiable game.\[24\]

Remark 8. Recall that a trivial game induces a uniform PMF by Remark \[7\]. Therefore, a non-trivial game is not equivalent to a trivial game since by Proposition \[5\] non-trivial games do not induce uniform PMFs.\[25\]

4.2.1 Additional Discussion on Modeling Choices

We now discuss other equilibrium concepts, such as mixed-strategy Nash equilibria (MSNE) and quantal response equilibria (QRE). We also discuss a more sophisticated noise process as well as a generalization of our model to non-uniform distributions; while likely more realistic, the alternative models are mathematically more complex and potentially less tractable computationally.

\[24\]Technically, we should call the set \(\Upsilon\) “the hypothesis space consisting of tuples of non-trivial games from \(\mathcal{H}\) and mixture parameters identifiable up to PSNE, with respect to the probabilistic model defined in eq.\(3\).” Similarly, we should call such game \(G\) “a non-trivial game from \(\mathcal{H}\) identifiable up to PSNE, with respect to the probabilistic model defined in eq.\(3\).” We opted for brevity.

\[25\]In general, Section 4.2 characterizes our hypothesis space (non-trivial identifiable games and mixture parameters) via two specific conditions. The first condition, \textit{non-triviality} (explained in Remark \[1\]), is \(0 < \pi(G) < 1\). The second condition, \textit{identifiability of the PSNE set from its related PMF} (discussed in Propositions \[5\] and \[6\]), is \(\pi(G) < q\). For completeness, in this remark, we clarify that the class of trivial games (uniform PMFs) is different from the class of non-trivial games (non-uniform PMFs). Thus, in the rest of the paper we focus exclusively on non-trivial identifiable games; that is, games that produce non-uniform PMFs and for which the PSNE set is uniquely identified from their PMFs.
On Other Equilibrium Concepts. There is still quite a bit of debate as to the appropriateness of game-theoretic equilibrium concepts to model individual human behavior in a social context. Camerer’s book on behavioral game theory [Camerer, 2003] addresses some of the issues.\textsuperscript{26} Our interpretation of Camerer’s position is not that Nash equilibria is universally a bad predictor but that it is not consistently the best, for reasons that are still not well understood. This point is best illustrated in Chapter 3, Figure 3.1 of Camerer [2003].

(Logit) quantal response equilibria (QRE) [McKelvey and Palfrey, 1995] has been proposed as an alternative to Nash in the context of behavioral game theory.\textsuperscript{27} Models based on QRE have been shown superior during initial play in some experimental settings, but prior work assumes known structure and observed payoffs, and only the “precision/rationality parameter” is estimated, e.g. [Wright and Leyton-Brown, 2010, 2012]. In a logit QRE, the precision parameter, typically denoted by $\lambda$, can be mathematically interpreted as the inverse-temperature parameter of individual Gibbs distributions over the pure-strategy of each player $i$.

It is customary to compute the MLE for $\lambda$ from available data. To the best of our knowledge, all work in QRE assumes exact knowledge of the game payoffs, and thus, no method has been proposed to simultaneously estimate the payoff functions $u_i$ when they are unknown. Indeed, computing MLE for $\lambda$, given the payoff functions, is relatively efficient for normal-form games using standard techniques, but may be hard for graphical games; on the other hand, MLE estimation of the payoff functions themselves within a QRE model of behavior seems like a highly non-trivial optimization problem, and is unclear that it is even computationally tractable, even in normal-form games. At the very least, standard techniques do not apply and more sophisticated optimization algorithms or heuristics would have to be derived. Such extensions are clearly beyond the scope of this paper.\textsuperscript{28}

[Wright and Leyton-Brown, 2012] also considers even more mathematically complex variants of behavioral models that combine QRE with different models that account for constraints in “cognitive levels” of reasoning ability/effort, yet the estimation and usage of such models still assumes knowledge of the payoff functions.

It would be fair to say that most of the human-subject experiments in behavioral game theory involve only a handful of players [Camerer, 2003]. The scalability of those results to games with a large population of players is unclear.

Now, just as an example, we do not necessarily view the Senators final votes as those of a single human individual anyway: after all, such a decision is (or should be) obtained with consultation with their staff and (one would at least hope) the constituency of the state.

\textsuperscript{26}The reader should be mindful of that fact that, as stated in the Introduction, there is a broader view of behavioral data, beyond those generated by individual human behavior (e.g. institutions such as nations and industries, or engineered systems such as autonomous-response devices in residential or commercial properties that are programmed to control electricity usage based on user preferences).

\textsuperscript{27}In this paragraph we discuss only the aspects of QRE relevant to our paper. A thorough discussion of QRE is beyond the scope of this paper.

\textsuperscript{28}Note that despite the apparent similarity in mathematical expression between logit QRE and the PSNE of the LIG we obtain by using individual logistic regression, they are fundamentally different because of the complex correlations that QRE conditions impose on the parameters ($W$, $b$) of the payoff functions. It is unclear how to adapt techniques for logistic regression similar to the ones we used here to efficiently/tractably compute MLE within the logit QRE framework.
they represent. Also, the final voting decision is taken after consultation or meetings between the staff of the different senators. We view this underlying process as one of “cheap talk.” While cheap talk may be an important process to study, in this paper, we just concentrate on the end result: the final vote. The reason is more than just scientific; as the congressional voting setting illustrates, data for such process is sometimes not available, or would seem very hard to infer from the end-states alone. While our experiments concentrate on congressional voting data, because it is publicly available and easy to obtain, the same would hold for other settings such as Supreme court decisions, voluntary vaccination, UN voting records and governmental decisions, to name a few. We speculate that in almost all those cases, only the end-result is likely to be recorded and little information would be available about the “cheap talk” process or “pre-play” period leading to the final decision.

In our work we consider PSNE because of our motivation to provide LIGs for use within the casual strategic inference framework for modeling “influence” in large-population networks of Irfan and Ortiz [2011, 2013]. Note that the universality of MSNE does not diminish the importance of PSNE in game theory. Indeed, a debate still exist within the game theory community as to the justification for randomization, specially in human contexts. While concentrating exclusively on PSNE may not make sense in all settings, it does make sense in some. In addition, were we to introduce mixed-strategies into the inference and learning framework and model, we would be adding a considerable amount of complexity in almost all respects, thus requiring a substantive effort to study on its own.

### On the Noise Process

Here we discuss a more sophisticated noise process as well as a generalization of our model to non-uniform distributions. The problem with these models is that they lead to a significantly more complex expression for the generative model and thus likelihood functions. This is in contrast to the simplicity afforded us by the generative model with a more global noise process defined above. (See Appendix A.2.1 for further discussion.)

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29We are unaware of any controlled behavioral study on the distinctions between final decisions being the result of a single human vs. a group of individuals (i.e., an institution), so it may turn out that the group would behave no different than the human, in which case some of the single-human models of behavior may be better justified than using strict game-theoretic notions of equilibria such as NE.

30Research work on the properties and computation of PSNE include Rosenthal [1973], Gilboa and Zemel [1989], Stanford [1993], Rinott and Scarsini [2000], Fabrikant et al. [2004], Gottlob et al. [2005], Sureka and Wurman [2005], Daskalakis and Papadimitriou [2006], Dunkel [2007], Dunkel and Schulz [2006], Dilkina et al. [2007], Ackermann and Skopalik [2007], Hasan et al. [2008], Hasan and Galiana [2008], Ryan et al. [2010], Chapman et al. [2010], Hasan and Galiana [2010].

31For example, in the context of congressional voting, we believe Senators almost always have full-information about how some, if not all other Senators they care about would vote. Said differently, we believe uncertainty in a Senator’s final vote, by the time the vote is actually taken, is rare, and certainly not the norm. Hence, it is unclear how much there is to gain, in this particular setting, by considering possible randomization in the Senators’ voting strategies.

32For example, note that because in our setting we learn exclusively from observed joint-actions, we could not assume knowledge of the internal mixed-strategies of players. Perhaps we could generalize our model to allow for mixed-strategies by defining a process in which a joint mixed strategy \( p \) from the set of MSNE (or its complement) is drawn according to some distribution, then a (pure-strategy) realization \( x \) is drawn from \( p \) that would correspond to the observed joint-actions. One problem we might face with this approach is that little is known about the structure of MSNE in general multi-player games. For example, it is not even clear that the set of MSNE is always measurable in general!
4.3 Learning Games via MLE

We now formulate the problem of learning games as one of maximum likelihood estimation with respect to our PSNE-based generative model defined in eq. (3) and the hypothesis space of non-trivial identifiable games and mixture parameters (Definition 7). We remind the reader that our problem is unsupervised; that is, we do not know a priori which joint-actions are equilibria and which ones are not. We base our framework on the fact that games are PSNE-identifiable with respect to their induced PMF, under the condition that \( q > \pi(\mathcal{G}) \), by Proposition 6.

First, we introduce a shorthand notation for the Kullback-Leibler (KL) divergence between two Bernoulli distributions parametrized by \( \pi, \hat{\pi} \):

\[
KL(p_1 \| p_2) \equiv KL(Bernoulli(p_1) \| Bernoulli(p_2)) = p_1 \log \frac{p_1}{p_2} + (1 - p_1) \log \frac{1 - p_1}{1 - p_2}
\]

Using this function, we can derive the following expression of the MLE problem.

**Lemma 9.** Given a dataset \( \mathcal{D} = \{x^{(1)}, \ldots, x^{(m)}\} \), define the empirical proportion of equilibria, i.e. the proportion of samples in \( \mathcal{D} \) that are equilibria of \( \mathcal{G} \), as

\[
\hat{\pi}(\mathcal{G}) \equiv \frac{1}{m} \sum_{i} 1[x^{(i)} \in \mathcal{NE}(\mathcal{G})].
\]

The MLE problem for the probabilistic model given in eq. (3) can be expressed as finding

\[
(\hat{\mathcal{G}}, \hat{q}) \in \arg \max_{(\mathcal{G}, q) \in \Upsilon} \hat{L}(\mathcal{G}, q), \text{ where } \hat{L}(\mathcal{G}, q) = KL(\hat{\pi}(\mathcal{G}) \| \pi(\mathcal{G})) - KL(\hat{\pi}(\mathcal{G}) \| q) - n \log 2,
\]

\( \mathcal{H} \) is the class of games of interest, \( \Upsilon = \{(\mathcal{G}, q) \mid \mathcal{G} \in \mathcal{H} \land 0 < \pi(\mathcal{G}) < q < 1\} \) is the hypothesis space of non-trivial identifiable games and mixture parameters, and \( \pi(\mathcal{G}) \) is defined as in eq. (4). Also, the optimal mixture parameter \( \hat{q} = \min(\hat{\pi}(\mathcal{G}), 1 - \frac{1}{2m}) \).

**Proof.** Let \( \mathcal{NE} \equiv \mathcal{NE}(\mathcal{G}) \), \( \pi \equiv \pi(\mathcal{G}) \) and \( \hat{\pi} \equiv \hat{\pi}(\mathcal{G}) \). First, for a non-trivial \( \mathcal{G} \), \( \log p_{(\mathcal{G}, q)}(x^{(i)}) = \log_{\frac{q}{\pi(x)}} \) for \( x^{(i)} \in \mathcal{NE} \), and \( \log p_{(\mathcal{G}, q)}(x^{(i)}) = \log_{\frac{1-q}{1-\pi(x)}} \) for \( x^{(i)} \notin \mathcal{NE} \). The average log-likelihood \( \hat{L}(\mathcal{G}, q) = \frac{1}{m} \sum_{i} \log p_{(\mathcal{G}, q)}(x^{(i)}) = \hat{\pi} \log \frac{\hat{\pi}}{\pi(x)} + (1 - \hat{\pi}) \log \frac{1 - \hat{\pi}}{1 - \pi(x)} = \hat{\pi} \log \frac{\hat{\pi}}{\pi(x)} + (1 - \hat{\pi}) \log \frac{1 - \hat{\pi}}{1 - \pi(x)} - n \log 2 \). By adding \( 0 = -\hat{\pi} \log \hat{\pi} + \pi(\mathcal{G}) \log \hat{\pi}(\mathcal{G}) - (1 - \hat{\pi}) \log (1 - \hat{\pi}) + (1 - \hat{\pi}) \log (1 - \hat{\pi}) \), this can be rewritten as \( \hat{L}(\mathcal{G}, q) = \hat{\pi} \log \frac{\hat{\pi}}{\pi(x)} + (1 - \hat{\pi}) \log \frac{1 - \hat{\pi}}{1 - \pi(x)} = \hat{\pi} \log \frac{\hat{\pi}}{q} - (1 - \hat{\pi}) \log \frac{\hat{\pi}}{q} - (1 - \hat{\pi}) \log \frac{1 - \hat{\pi}}{1 - q} - n \log 2 \), and by using eq. (5) we prove our claim.

Note that by maximizing with respect to the mixture parameter \( q \) and by properties of the KL divergence, we get \( KL(\hat{\pi} \| q) = 0 \iff \hat{q} = \hat{\pi} \). We define our hypothesis space \( \Upsilon \) given
the conditions in Remark 1 and Propositions 5 and 6. For the case \( \hat{\pi} = 1 \), we “shrink” the optimal mixture parameter \( \hat{q} \) to \( 1 - \frac{1}{2m} \) in order to enforce the second condition given in Remark 1.

**Remark 10.** Recall that a trivial game (e.g., LIG \( \mathcal{G} = (W, b), W = 0, b = 0, \pi(\mathcal{G}) = 1 \)) induces a uniform PMF by Remark 1, and therefore its log-likelihood is \( -n \log 2 \). Note that the lowest log-likelihood for non-trivial identifiable games in eq. (7) is \( -n \log 2 \) by setting the optimal mixture parameter \( \hat{q} = \hat{\pi}(\mathcal{G}) \) and given that \( \text{KL}(\hat{\pi}(\mathcal{G}) \parallel \pi(\mathcal{G})) \geq 0 \).

Furthermore, eq. (7) implies that for non-trivial identifiable games \( \mathcal{G} \), we expect the true proportion of equilibria \( \pi(\mathcal{G}) \) to be strictly less than the empirical proportion of equilibria \( \hat{\pi}(\mathcal{G}) \) in the given data. This is by setting the optimal mixture parameter \( \hat{q} = \hat{\pi}(\mathcal{G}) \) and the condition \( q > \pi(\mathcal{G}) \) in our hypothesis space.

### 4.3.1 Learning LIGs via MLE: Model Selection

Our main learning problem consists of inferring the structure and parameters of an LIG from data with the main purpose being modeling the game’s PSNE, as reflected in the generative model. Note that, as we have previously stated, different games (i.e., with different payoff functions) can be PSNE-equivalent. For instance, the three following LIGs, with different weight parameter matrices, induce the same PSNE sets, i.e. \( \mathcal{NE}(W_k, b) = \{(-1, -1, -1), (+1, +1, +1)\} \) for \( k = 1, 2, 3 \):

\[
W_1 = \begin{bmatrix}
0 & 0 & 0 \\
1/2 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}, \quad W_2 = \begin{bmatrix}
0 & 0 & 0 \\
2 & 0 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}, \quad W_3 = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

Thus, not only the MLE may not be unique, but also all such PSNE-equivalent MLE games will achieve the same level of generalization performance. But, as reflected by our generative model, our main interest in the model parameters of the LIGs is only with respect to the PSNE they induce, not the model parameters per se. Hence, all we need is a way to select among PSNE-equivalent LIGs.

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33 Using the formal mathematical definition of “identifiability” in statistics, we would say that the LIG examples prove that the model parameters \((W, b)\) of an LIG \( \mathcal{G} \) are not identifiable with respect to the generative model \( p(\mathcal{G}, q) \) defined in eq. (3). We note that this situation is hardly exclusive to game-theoretic models. As example of an analogous issue in probabilistic graphical models is the fact that two Bayesian networks with different graph structures can represent not only the same conditional independence properties but also *exactly* the same set of joint probability distributions [Chickering, 2002; Koller and Friedman, 2009].

As a side note, we can distinguish this games with respect to their larger set of mixed-strategy Nash equilibria (MSNE), but, as stated previously, we do not consider MSNE in this paper because our primary motivation is the work of Irfan and Ortiz [2011, 2013], which is based on the concept of PSNE.

34 At this point, we remind the reader our main objectives (see Section 1.1), and at the same time, explain why it is still valuable to perform MLE of the LIG model parameters \((W, b)\). We specifically want LIGs because that is what the algorithms for CIS of Irfan and Ortiz [2011, 2013] take as input. LIGs are defined in terms of the influence weight \((W)\) and bias \((b)\) parameters. Thus, LIGs implicitly encode/represent the set of PSNE of games in a compact, parametric form; said differently, LIGs are space-tractable. (This is analogous to probabilistic graphical models.) PSNE forms the basis of CIS. Our generative model thus reflects this explicitly by its definition in terms of the PSNE of games, not model parameters. The PSNE-
In our work, the identifiability or interpretability of exact model parameters of LIGs is not our main interest. That is, in the research presented here, we did not seek or attempt to work on creating alternative generative models with the objective to provide a theoretical guarantee that, given an infinite amount of data, we can recover the model parameters of an unknown ground-truth model generating the data, assuming the ground-truth model is an LIG. We opted for a more practical ML approach in which we just want to learn a single LIG that has good generalization performance (i.e., predictive performance in terms of average log-likelihood) with respect to our generative model. Given the nature of our generative model, this essentially translates to learning an LIG that captures as best as possible the PSNE of the unknown ground-truth game. Unfortunately, as we just illustrated, several LIGs with different model parameter values can have the same set of PSNE. Thus, they all would have the same (generalization) performance ability.

As we all know, model selection is core to ML. One of the reasons we chose an ML-approach to learning games is precisely the elegant way in which ML deals with the problem based generative model defines the likelihood function. We want to learn LIGs, hence the instantiation of the class of games are LIGs. The likelihood function is a function of the LIG model parameters.

Thus, it should be clear by now that the MLE of LIG model parameters allows us to compactly represent the set of PSNE using an LIG, which is what we want to learn. Alternatively, we would have to represent the PSNE as a set in some other form (i.e., a list, perhaps?) which is not only space/representationally inefficient but useless to our main purpose: providing LIGs to algorithms for CIS, such as those of Irfan and Ortiz [2011, 2013]. In addition, in Section 6 we provide algorithms for learning LIGs via approximate MLE of model parameters that are, not only simple in some cases, but also computationally very efficient.

Please remember that our main interest for CIS is the PSNE of the game, not the LIG parameters themselves. This is also analogous to BNs because the basis of probabilistic inference in BNs is the joint probability distribution, implicitly defined and encoded by the structure and parameters of the BN representation, and, like LIGs, BNs are not identifiable with respect to their model parameters (i.e., the DAG and CPDs).

Having said that, we believe the learned parameters could still be valuable, despite the lack of formal guarantees on “interpretability,” as formally defined by statisticians, econometrists, and other social scientists. This is true as long as the domain expert is aware of and understand the characteristics and limitations of LIGs and our ML-approach. For example, we believe it is hard to look at our resulting LIG model parameters that are, not only simple in some cases, but also computationally very efficient.

In short, based on our main interest in this paper, our goal is to provide domain experts with computational tools they can use to perform inference and prediction tasks, such as applying CSI to a learned LIG, as they seek to make new discoveries, to produce new knowledge or perspectives, or to find ways to alter the global behavior of large complex systems resulting from the local interactions of individual entities. But, we also hope that domain experts find the more local/microscopic study of the learned LIG model parameters useful to draw general observations or conclusions, to obtain new intuitions, insights and perspectives, and to generate testable hypotheses, all with high potential for proper validation using data from controlled experiments they would design specifically for that purpose. Unfortunately, when studying LIGs at that “micro” level, the burden for proper/formal validation would be on the domain expert. We believe there is little we can provide as ML researcher in terms of tools with “theoretical guarantees” to study models such as LIGs at very fine-grain levels of granularity, given the state-of-the-art in ML research and the ML philosophy, which typically emphasizes model performance over parameters.

A large number of the econometrics literature concerns the issue of “identifiability” of models from data. For them, for the most part, “identifiability” is defined as the statistical consistency of the estimators of the parameters of structural-equations models (SEM); hence, it only needs to be valid, with high probability, in the limit of infinite data. Establishing the identifiability of a model appears to be taken as a permit to use all the data available for training and evaluation, ignoring the possibility of overfitting because of the finite data sizes.
of how to select among multiple models that achieve the same level of performance: invoke
the principle of Ockham’s razor and select the “simplest” model among those with the
same (generalization) performance. This ML philosophy is not ad hoc. It is instead well
established in practice and well supported by theory. Seminal results from the various
theories of learning, such as computational and statistical learning theory and PAC learning,
support the well-known ML adage that “learning requires bias.” In short, as is by now
standard in an ML-approach, we measure the quality of our data-induced models via their
generalization ability and invoke the principle of Ockham’s razor to bias our search toward
tsimpler models using well-known and -studied regularization techniques.

Now, as we also all know, exactly what “simple” and “bias” means depends on the
problem. In our case, we would prefer games with sparse graphs, if for no reason other than
to simplify analysis, exploration, study, and (visual) “interpretability” of the game model by
human experts, everything else being equal (i.e., models with the same explanatory power
on the data as measured by the likelihoods). For example, among the LIGs presented
above, using structural properties alone, we would generally prefer the former two models
to the latter, all else being equal (e.g., generalization performance).

5 Generalization Bound and VC-Dimension

In this section, we show a generalization bound for the maximum likelihood problem as well
as an upper bound of the VC-dimension of LIGs. Our objective is to establish that with
probability at least $1 - \delta$, for some confidence parameter $0 < \delta < 1$, the maximum likelihood
estimate is within $\epsilon > 0$ of the optimal parameters, in terms of achievable expected log-
likelihood.

Given the ground truth distribution $Q$ of the data, let $\bar{\pi}(G)$ be the expected proportion
of equilibria, i.e.:

$$\bar{\pi}(G) = \mathbb{P}_Q[x \in \mathcal{NE}(G)]$$  \hspace{1cm} (8)

and let $\bar{L}(G, q)$ be the expected log-likelihood of a generative model from game $G$ and mixture
parameter $q$, i.e.:

$$\bar{L}(G, q) = \mathbb{E}_Q[\log p(G, q)(x)]$$  \hspace{1cm} (9)

Note that our hypothesis space $\mathcal{Y}$ in eq.(7) includes a continuous parameter $q$ that could
potentially have infinite VC-dimension. The following lemma will allow us later to prove
that uniform convergence for the extreme values of $q$ implies uniform convergence for all $q$
in the domain.

37Just to be clear, here we mean “interpretability” not in any formal mathematical sense, or as often
used in some areas of the social sciences such as economics. But, instead, as we typically use it in ML/AI
textbooks, such as for example, when referring to shallow/sparse decision trees, generally considered to be
easier to explain and understand. Similarly, the hope here is that the “sparsity” or “simplicity” of the
game graph/model would make it also simpler for human experts to explain or understand what about the
model is leading them to generate novel hypotheses, reach certain conclusions or make certain inferences
about the global strategic behavior of the agents/players, such as those based on the game’s PSNE and
facilitated by CSI. We should also point out that, in preliminary empirical work, we have observed that the
representationally sparser the LIG graph, the computationally easier it is for algorithms and other heuristics
that operate on the LIG, as those of Irfan and Ortiz [2011, 2013] for CIS, for example.
Lemma 11. Consider any game $G$ and, for $0 < q'' < q' < q < 1$, let $\theta = (G, q)$, $\theta' = (G, q')$ and $\theta'' = (G, q'')$. If, for any $\epsilon > 0$ we have $|\hat{L}(\theta) - \bar{L}(\theta)| \leq \epsilon/2$ and $|\hat{L}(\theta'') - \bar{L}(\theta'')| \leq \epsilon/2$, then $|\hat{L}(\theta'') - \bar{L}(\theta'')| \leq \epsilon/2$.

Proof. Let $\mathcal{NE} \equiv \mathcal{NE}(G)$, $\pi \equiv \pi(G)$, $\bar{\pi} \equiv \bar{\pi}(G)$, and $\mathbb{E}[-]$ and $\mathbb{P}[-]$ be the expectation and probability with respect to the ground truth distribution $\mathcal{Q}$ of the data.

First note that for any $\theta = (G, q)$, we have $\hat{L}(\theta) = \mathbb{E}[\log p_G(q)(x)] = \mathbb{E}[1 \mid x \notin \mathcal{NE}] \log \frac{1}{\mathcal{NE}} + \mathbb{P}[x \in \mathcal{NE}] \log \frac{q}{\mathcal{NE}} = \mathbb{P}[x \notin \mathcal{NE}] \log \frac{q}{\mathcal{NE}} + (1 - \bar{\pi}) \log \frac{1 - q}{2^n} = \bar{\pi} \log \left( \frac{q}{1 - q} \cdot \frac{1 - \pi}{1 - \bar{\pi}} \right) + \log \frac{1 - q}{2^n} = \bar{\pi} \log \left( \frac{q}{1 - q} \cdot \frac{1 - \pi}{1 - \bar{\pi}} \right) + n \log 2$.

Similarly, for any $\theta = (G, q)$, we have $\bar{L}(\theta) = \bar{\pi} \log \left( \frac{q}{1 - q} \cdot \frac{1 - \pi}{1 - \bar{\pi}} \right) + n \log 2$. So that $\hat{L}(\theta) - \bar{L}(\theta) = (\bar{\pi} - \bar{\pi}) \log \left( \frac{q}{1 - q} \cdot \frac{1 - \pi}{1 - \bar{\pi}} \right)$.

Furthermore, the function $\frac{q}{1 - q}$ is strictly monotonically increasing for $0 \leq q < 1$. If $\bar{\pi} > \bar{\pi}$ then $-\epsilon/2 \leq \hat{L}(\theta'') - \bar{L}(\theta'') < \hat{L}(\theta'') - \bar{L}(\theta) \leq \bar{L}(\theta) - \bar{L}(\theta) \leq \epsilon/2$. Else, if $\bar{\pi} < \bar{\pi}$, we have $\epsilon/2 \geq \hat{L}(\theta'') - \bar{L}(\theta'') > \hat{L}(\theta) - \bar{L}(\theta) > \hat{L}(\theta) - \bar{L}(\theta) \geq \epsilon/2$. Finally, if $\bar{\pi} = \bar{\pi}$ then $\hat{L}(\theta'') - \bar{L}(\theta'') = \hat{L}(\theta) - \bar{L}(\theta) = 0$.

The following theorem shows that the expected log-likelihood of the maximum likelihood estimate converges in probability to that of the optimal, as the data size $m$ increases.

Theorem 12. Let $\hat{\theta} = (\bar{G}, \bar{q})$ be the maximum likelihood estimate in eq.(7) and $\bar{\theta} = (\bar{G}, \bar{q})$ be the maximum expected likelihood estimate $\bar{\theta} = \arg\max_{\theta \in \Upsilon} \bar{L}(\theta)$, then with probability at least $1 - \delta$:

$$\bar{L}(\theta) \geq \hat{L}(\theta) - \left( \log \max(2m, \frac{1}{1 - q}) + n \log 2 \right) \sqrt{\frac{2}{m} \left( \log d(H) + \log \frac{4}{\delta} \right)}$$

where $\mathcal{H}$ is the class of games of interest, $\Upsilon = \{(G, q) \mid G \in \mathcal{H} \land 0 < \pi(G) < q < 1\}$ is the hypothesis space of non-trivial identifiable games and $d(\mathcal{H}) \equiv |\{G \in \mathcal{H} \mid \mathcal{NE}(G)\}|$ is the number of all possible PSNE sets induced by games in $\mathcal{H}$.

Proof. First our objective is to find a lower bound for $\mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) \geq -\epsilon] \geq \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) \geq \epsilon] \geq \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) \geq -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) \geq -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) \geq -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) \geq -\epsilon] \geq \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) > -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) > -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) > -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) > -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) > -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) > -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) > -\epsilon] = \mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) > -\epsilon]$

Let $\bar{q} \equiv \max(1 - \frac{1}{2m}, q)$. Now, we have $\mathbb{P}[\hat{L}(\theta) - \bar{L}(\theta) > \frac{q}{2} \lor \hat{L}(\theta) - \bar{L}(\theta) < -\frac{q}{2}] \leq \mathbb{P}[(\exists \theta \in \Upsilon, q \leq \bar{q}) |\hat{L}(\theta) - \bar{L}(\theta)| > \frac{q}{2}] = \mathbb{P}[\{\exists \theta \in \Upsilon, q \leq \bar{q} \mid \hat{L}(\theta) - \bar{L}(\theta)| > \frac{q}{2}] = \mathbb{P}[(\exists \theta \in \Upsilon, q \leq \bar{q} \mid \hat{L}(\theta) - \bar{L}(\theta)| > \frac{q}{2}]$. The last equality follows from invoking Lemma [11]

Note that $\mathbb{E}[\hat{L}(\theta)] = \bar{L}(\theta)$ and that since $\pi(G) \leq q \leq \bar{q}$, the log-likelihood is bounded as $(\forall x) - B \leq \log p_G(q)(x) \leq 0$, where $B = \log \frac{1}{1 - q} + n \log 2 = \log \max(2m, \frac{1}{1 - q}) + n \log 2$. Therefore, by Hoeffding’s inequality, we have $\mathbb{P}[|\hat{L}(\theta) - \bar{L}(\theta)| > \frac{q}{2}] \leq 2e^{-\frac{q^2}{2m}}$.

Furthermore, note that there are $2d(\mathcal{H})$ possible parameters $\theta$, since we need to consider only two values of $q \in \{\pi(G), \bar{q}\}$ and because the number of all possible PSNE sets induced by games in $\mathcal{H}$ is $d(\mathcal{H}) \equiv |\{G \in \mathcal{H} \mid \mathcal{NE}(G)\}|$. Therefore, by the union bound we

\[38\] If the ground-truth model belongs to the class of LIGs, then $\hat{\theta}$ is also the ground-truth model, or PSNE-equivalent to it.
get the following uniform convergence $\mathbb{P}[(\exists \theta, G \in \mathcal{H}, q \in \{\pi(G), \bar{q}\}) \mid \hat{L}(\theta) - \bar{L}(\theta) > \frac{\epsilon}{2}] \leq 4d(\mathcal{H})\mathbb{P}[|\hat{L}(\theta) - \bar{L}(\theta)| > \frac{\epsilon}{2}] \leq 4d(\mathcal{H})e^{-\frac{m\epsilon^2}{2n^2}} = \delta$. Finally, by solving for $\delta$ we prove our claim. \hfill \Box

Remark 13. A more elaborate analysis allows to tighten the bound in Theorem 12 from $O(\log \frac{1}{1-q})$ to $O(\log \frac{q}{1-q})$. We chose to provide the former result for clarity of presentation.

The following theorem establishes the complexity of the class of LIGs, which implies that the term $\log d(\mathcal{H})$ of the generalization bound in Theorem 12 is only polynomial in the number of players $n$.

Theorem 14. Let $\mathcal{H}$ be the class of LIGs. Then $d(\mathcal{H}) \equiv |\cup_{G \in \mathcal{H}}\{NE(G)\}| \leq 2^n \frac{n(n+1)}{2} + 1 \leq 2^{n^3}$.

Proof. The logarithm of the number of possible pure-strategy Nash equilibria sets supported by $\mathcal{H}$ (i.e., that can be produced by some game in $\mathcal{H}$) is upper bounded by the VC-dimension of the class of neural networks with a single hidden layer of $n$ units and $n + \binom{n}{2}$ input units, linear threshold activation functions, and constant output weights.

For every LIG $G = (W, b)$ in $\mathcal{H}$, define the neural network with a single layer of $n$ hidden units, $n$ of the inputs corresponds to the linear terms $x_1, \ldots, x_n$ and $\binom{n}{2}$ corresponds to the quadratic polynomial terms $x_ix_j$ for all pairs of players $(i, j)$, $1 \leq i < j \leq n$. For every hidden unit $i$, the weights corresponding to the linear terms $x_1, \ldots, x_n$ are $-b_1, \ldots, -b_n$, respectively, while the weights corresponding to the quadratic terms $x_ix_j$ are $-w_{ij}$, for all pairs of players $(i, j)$, $1 \leq i < j \leq n$, respectively. The weights of the bias term of all the hidden units are set to 0. All $n$ output weights are set to 1 while the weight of the output bias term is set to 0. The output of the neural network is $1[x \notin NE(G)]$. Note that we define the neural network to classify non-equilibrium as opposed to equilibrium to keep the convention in the neural network literature to define the threshold function to output 0 for input 0. The alternative is to redefine the threshold function to output 1 instead for input 0.

Finally, we use the VC-dimension of neural networks Sontag [1998]. \hfill \Box

From Theorems 12 and 14 we state the generalization bounds for LIGs.

Corollary 15. Let $\hat{\theta} = (\hat{G}, \hat{q})$ be the maximum likelihood estimate in eq. (7) and $\bar{\theta} = (\bar{G}, \bar{q})$ be the maximum expected likelihood estimate, i.e. $\theta = \arg\max_{\theta \in \mathcal{Y}} \bar{L}(\theta)$ and $\bar{\theta} = \arg\max_{\theta \in \mathcal{Y}} \bar{L}(\theta)$, then with probability at least $1 - \delta$:

$$\bar{L}(\hat{\theta}) \geq \bar{L}(\bar{\theta}) - \left(\log \max(2m, \frac{1}{1-q}) + n \log 2\right) \sqrt{\frac{2}{m} \left(n^3 \log 2 + \log \frac{4}{\delta}\right)} \quad (11)$$

where $\mathcal{H}$ is the class of LIGs and $\mathcal{Y} = \{(G, q) \mid G \in \mathcal{H} \land 0 < \pi(G) < q < 1\}$ is the hypothesis space of non-trivial identifiable LIGs and mixture parameters.

6 Algorithms

In this section, we approximate the maximum likelihood problem by maximizing the number of observed equilibria in the data, suitable for a hypothesis space of games with small true
Second, given a fixed number of Nash equilibria the empirical proportion of equilibria and thus leading to a lower log-likelihood in eq.(7). Unobserved joint-action will increase the true proportion of equilibria without increasing Therefore, only joint-actions that are observed in the data are candidates of being Nash equilibria sets, i.e. for \( s = 0, \ldots, 2^n \), we generate all possible sets of size \( s \) with elements from the joint-action space \( \{-1, +1\}^n \). Recall that there exists \( \binom{2^n}{s} \) of such sets of size \( s \) and since \( \sum_{s=0}^{2^n} \binom{2^n}{s} = 2^{2^n} \), the search space is super-exponential in the number of players \( n \).

Based on few observations, we can obtain an \( \mathcal{O}(m \log m) \) algorithm for \( m \) samples. First, note that the above method does not constrain the set of Nash equilibria in any fashion. Therefore, only joint-actions that are observed in the data are candidates of being Nash equilibria in order to maximize the log-likelihood. This is because the introduction of an unobserved joint-action will increase the true proportion of equilibria without increasing the empirical proportion of equilibria and thus leading to a lower log-likelihood in eq.(7). Second, given a fixed number of Nash equilibria \( k \), the best strategy would be to pick the \( k \) joint-actions that appear more frequently in the observed data. This will maximize the empirical proportion of equilibria, which will maximize the log-likelihood. Based on these observations, we propose Algorithm 1.

6.1 An Exact Quasi-Linear Method for General Games: Sample-Picking

As a first approach, consider solving the maximum likelihood estimation problem in eq.(7) by an exact exhaustive search algorithm. This algorithm iterates through all possible Nash equilibria sets, i.e. for \( s = 0, \ldots, 2^n \), we generate all possible sets of size \( s \) with elements from the joint-action space \( \{-1, +1\}^n \). Recall that there exists \( \binom{2^n}{s} \) of such sets of size \( s \) and since \( \sum_{s=0}^{2^n} \binom{2^n}{s} = 2^{2^n} \), the search space is super-exponential in the number of players \( n \).

Based on few observations, we can obtain an \( \mathcal{O}(m \log m) \) algorithm for \( m \) samples. First, note that the above method does not constrain the set of Nash equilibria in any fashion. Therefore, only joint-actions that are observed in the data are candidates of being Nash equilibria in order to maximize the log-likelihood. This is because the introduction of an unobserved joint-action will increase the true proportion of equilibria without increasing the empirical proportion of equilibria and thus leading to a lower log-likelihood in eq.(7). Second, given a fixed number of Nash equilibria \( k \), the best strategy would be to pick the \( k \) joint-actions that appear more frequently in the observed data. This will maximize the empirical proportion of equilibria, which will maximize the log-likelihood. Based on these observations, we propose Algorithm 1.

This is not a disadvantage relative to probabilistic graphical models, since computing the log-likelihood function is also NP-hard for Ising models and Markov random fields in general, while learning is also NP-hard for Bayesian networks.

We show that evaluating the pseudo-likelihood function for our generative model is NP-hard. First, consider a non-trivial LIG \( \mathcal{G} = (W, b) \). Furthermore, assume \( \mathcal{G} \) has a single non-absolutely-indifferent player \( i \) and absolutely-indifferent players \( \forall j \neq i \). Let \( f_i(x_{-i}) \equiv w_{i,-i}^T x_{-i} - b_i \). By Claim i of Lemma 24 we have \( \mathbb{1}[x \in \mathcal{NE}(\mathcal{G})] = \mathbb{1}[x, f_i(x_{-i}) \geq 0] \) and therefore \( \mathbb{P}(\mathcal{G} \mid x) = \frac{1}{\mathcal{Z}^{\mathcal{NE}(\mathcal{G})}} \mathbb{1}[x, f_i(x_{-i}) \geq 0] \mathcal{Z}^{\mathcal{NE}(\mathcal{G})} - \mathbb{1}[x, f_i(x_{-i}) < 0] \mathcal{Z}^{\mathcal{NE}(\mathcal{G})} \).

The results follow because computing \( |\mathcal{NE}(\mathcal{G})| \) is \#P-complete, even for this specific instance of a single non-absolutely-indifferent player. Irfan and Ortiz [2011, 2013].

To prove that counting the number of equilibria is not (Lipschitz) continuous, we show how small changes in the parameters \( \mathcal{G} = (W, b) \) can produce big changes in \( |\mathcal{NE}(\mathcal{G})| \). For instance, consider two games \( \mathcal{G}_1 = (W_1, b_1) \), where \( W_1 = b_1 = 0, |\mathcal{NE}(\mathcal{G}_1)| = 2^n \) and \( W_2 = \varepsilon (11^T - I) \), \( b_2 = 0, |\mathcal{NE}(\mathcal{G}_2)| = 2 \) for \( \varepsilon > 0 \). For \( \varepsilon \to 0 \), any \( \ell_p \)-norm \( \|W_1 - W_2\|_p \to 0 \) but \( |\mathcal{NE}(\mathcal{G}_1)| - |\mathcal{NE}(\mathcal{G}_2)| = 2^n - 2 \) remains constant.

The log-partition function of an Ising model is a trivial bound for counting the number of equilibria. To see this, let \( f_i(x_{-i}) \equiv w_i^T x_{-i} - b_i \), \( |\mathcal{NE}(\mathcal{G})| = \sum_x \mathbb{1}[x, f_i(x_{-i}) \geq 0] \leq \sum_x e^{x^T W x - b^T x} = \mathcal{Z}(\frac{1}{2} W + W^T, b) \), where \( \mathcal{Z} \) denotes the partition function of an Ising model. Given the convexity of \( \mathcal{Z} \) [Koller and Friedman 2009], and that the gradient vanishes at \( W = 0, b = 0 \), we know that \( \mathcal{Z}(\frac{1}{2} (W + W^T), b) \geq 2^n \), which is the maximum \( |\mathcal{NE}(\mathcal{G})| \).
The fact that general games do not constrain the set of Nash equilibria, makes the method more likely to over-fit. On the other hand, LIGs will potentially include unobserved equilibria given the linearity constraints in the search space, and thus they would be less likely to over-fit.\footnote{Here, we are simply stating basic, general ML knowledge about the relationship between the size of hypothesis classes and the potential for overfitting in the context of this algorithm. If the class of games are those explicitly represented/defined by every possible set of PSNE, the hypothesis class would be of size \(2^n\) (doubly exponential in the number of players!). It is well-known that such a huge hypothesis class is very likely to lead to overfitting. This is just a non-controversial ML statement in the general context of this particular algorithm, and has no bearing on the adequacy of other algorithms we propose or our general approach.}

### 6.2 An Exact Super-Exponential Method for LIGs: Exhaustive Search

Note that in the previous subsection, we search in the space of all possible games, not only the LIGs. First note that \textit{sample-picking} for linear games is NP-hard, i.e. at any iteration of \textit{sample-picking}, checking whether the set of Nash equilibria \(\mathcal{NE}\) corresponds to an LIG or not is equivalent to the following constraint satisfaction problem with linear constraints:

\[
\begin{align*}
\min_{W, b} & \quad 1 \\
\text{s.t.} & \quad (\forall x \in \mathcal{NE}) \ x_1 (w_{1,-1}^T x_{-1} - b_1) \geq 0 \land \cdots \land x_n (w_{n,-n}^T x_{-n} - b_n) \geq 0 \\
& \quad (\forall x \notin \mathcal{NE}) \ x_1 (w_{1,-1}^T x_{-1} - b_1) < 0 \lor \cdots \lor x_n (w_{n,-n}^T x_{-n} - b_n) < 0 
\end{align*}
\tag{12}
\]

Note that eq.\ref{eq:12} contains “or” operators in order to account for the non-equilibria. This makes the problem of finding the \((W, b)\) that satisfies such conditions NP-hard for a non-empty complement set \([-1,+1]^n - \mathcal{NE}\). Furthermore, since \textit{sample-picking} only consider observed equilibria, the search is not optimal with respect to the space of LIGs.

Regarding a more refined approach for enumerating LIGs only, note that in an LIG each player separates hypercube vertices with a linear function, i.e. for \(v \equiv (w_{i,-i}, b_i)\) and \(y \equiv (x_i x_{-i}, -x_i) \in \{-1,+1\}^n\) we have \(x_i (w_{i,-i}^T x_{-i} - b_i) = v^T y\). Assume we assign a binary label to each vertex \(y\), then note that not all possible labelings are linearly separable. Labelings which are linearly separable are called \textit{linear threshold functions (LTFs)}. A lower bound of the number of LTFs was first provided in Muroga [1965], which showed that the number of LTFs is at least \(a(n) = 2^{0.33048 n^2}\). Tighter lower bounds were shown later in Yamija and Ibaraki [1965] for \(n \geq 6\) and in Muroga and Toda [1966] for \(n \geq 8\). Regarding

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\begin{algorithm}[h]
\caption{Sample-Picking for General Games}
\begin{algorithmic}
\Input{Dataset \(D = \{x^{(1)}, \ldots, x^{(m)}\}\)}
\State Compute the unique samples \(y^{(1)}, \ldots, y^{(U)}\) and their frequency \(\hat{p}^{(1)} \geq \hat{p}^{(2)} \geq \cdots \geq \hat{p}^{(U)}\) in the dataset \(D\)
\State Sort joint-actions by their frequency such that \(\hat{p}^{(1)} \geq \hat{p}^{(2)} \geq \cdots \geq \hat{p}^{(U)}\)
\For{each unique sample \(k = 1, \ldots, U\)}
\State Define \(G_k\) by the Nash equilibria set \(\mathcal{NE}(G_k) = \{y^{(1)}, \ldots, y^{(k)}\}\)
\State Compute the log-likelihood \(\hat{L}(G_k, \hat{q}_k)\) in eq.\ref{eq:7} (note that \(\hat{q}_k = \hat{\pi}(G) = \frac{1}{\sum} (\hat{p}^{(1)} + \cdots + \hat{p}^{(U)}))\), \(\pi(G) = \frac{k}{2^m}\)
\EndFor
\Output{The game \(G_k\) such that \(\hat{k} = \arg \max_k \hat{L}(G_k, \hat{q}_k)\)}
\end{algorithmic}
\end{algorithm}
an upper bound, Winder [1960] showed that the number of LTFs is at most $\beta(n) \equiv 2^{n^2}$. By using such bounds for all players, we can conclude that there is at least $\alpha(n)^n = 2^{0.33048n^3}$ and at most $\beta(n)^n = 2^{n^3}$ LIGs (which is indeed another upper bound of the VC-dimension of the class of LIGs; the bound in Theorem 14 is tighter and uses bounds of the VC-dimension of neural networks). The bounds discussed above would bound the time-complexity of a search algorithm if we could easily enumerate all LTFs for a single player. Unfortunately, this seems to be far from a trivial problem. By using results in Muroga [1971], a weight vector $v$ with integer entries such that $(\forall i) |v_i| \leq \beta(n) \equiv (n+1)^{(n+1)/2}/2^n$ is sufficient to realize all possible LTFs. Therefore we can conclude that enumerating LIGs takes at most $(2\beta(n)+1)^{n^2} \approx (\sqrt{n+1})^{n^3}$ steps, and we propose the use of this method only for $n \leq 4$.

For $n = 4$ we found that the number of possible PSNE sets induced by LIGs is 23,706. Experimentally, we did not find differences between this method and sample-picking since most of the time, the model with maximum likelihood was an LIG.

### 6.3 From Maximum Likelihood to Maximum Empirical Proportion of Equilibria

We approximately perform maximum likelihood estimation for LIGs, by maximizing the empirical proportion of equilibria, i.e. the equilibria in the observed data. This strategy allows us to avoid computing $\pi(G)$ as in eq. (4) for maximum likelihood estimation (given its dependence on $|NE(G)|$). We propose this approach for games with small true proportion of equilibria with high probability, i.e. with probability at least $1 - \delta$, we have $\pi(G) \leq \frac{\alpha(n)}{\delta}$ for $0 < \kappa < 1$. Particularly, we will show in Section 7 that for LIGs we have $\kappa = 3/4$. Given this, our approximate problem relies on a bound of the log-likelihood that holds with high probability. We also show that under very mild conditions, the parameters $(G,q)$ belong to the hypothesis space of the original problem with high probability.

First, we derive bounds on the log-likelihood function.

**Lemma 16.** Given a non-trivial game $G$ with $0 < \pi(G) < \hat{\pi}(G)$, the KL divergence in the log-likelihood function in eq. (11) is bounded as follows:

$$-\hat{\pi}(G) \log \pi(G) - \log 2 < KL(\hat{\pi}(G)\|\pi(G)) < \hat{\pi}(G) \log \pi(G)$$

(13)

**Proof.** Let $\pi \equiv \pi(G)$ and $\hat{\pi} \equiv \hat{\pi}(G)$. Note that $\alpha(\pi) \equiv \lim_{\pi \to 0} KL(\hat{\pi}||\pi) = 0$ and $\beta(\pi) \equiv \lim_{\pi \to 1} KL(\pi||\pi) = -\log \pi \leq n \log 2$. Since the function is convex we can upper-bound it by $\alpha(\pi) + (\beta(\pi) - \alpha(\pi))\hat{\pi} = -\hat{\pi} \log \pi$.

To find a lower bound, we find the point in which the derivative of the original function is equal to the slope of the upper bound, i.e. $\frac{\partial KL(\hat{\pi}||\pi)}{\partial \hat{\pi}} = \beta(\pi) - \alpha(\pi) = -\log \pi$, which

---

44Here we are making the implicit assumption that $\pi < \hat{\pi}$. This is sensible. For example, in most models learned from the congressional voting data using a variety of learning algorithms we propose, the total number of PSNE would range roughly from 100K—1M; using base 2, this is roughly from $2^{16}$—$2^{20}$. This may look like a huge number until one recognizes that there could potential be $2^{100}$ PSNE. Hence, we have that $\pi$ would be in the range of $2^{-84}$—$2^{-80}$. In fact, we believe this holds more broadly because, as a general objective, we want models that can capture as many PSNE behavior as possible but no more than needed, which tend to reduce the PSNE of the learned models, and thus their $\pi$ values, while simultaneously trying to increase $\hat{\pi}$ as much as possible.
By applying the lower bound in Lemma 16 in eq.(7) to non-trivial games, we have identifiable games and mixture parameters.

Note that the lower and upper bounds are very informative when $\pi(\mathcal{G}) \to 0$ (or in our setting when $n \to +\infty$), since $\log 2$ becomes small when compared to $-\log \pi(\mathcal{G})$, as shown in Figure 3.

Next, we derive the problem of maximizing the empirical proportion of equilibria from the maximum likelihood estimation problem.

**Theorem 17.** Assume that with probability at least $1 - \delta$ we have $\pi(\mathcal{G}) \leq \frac{n^n}{\delta}$ for $0 < \kappa < 1$. Maximizing a lower bound (with high probability) of the log-likelihood in eq.(7) is equivalent to maximizing the empirical proportion of equilibria:

$$\max_{\hat{\mathcal{G}} \in \mathcal{H}} \hat{\pi}(\mathcal{G}) \quad (14)$$

Furthermore, for all games $\mathcal{G}$ such that $\hat{\pi}(\mathcal{G}) \geq \gamma$ for some $0 < \gamma < 1/2$, for sufficiently large $n > \log_\kappa(\delta \gamma)$ and optimal mixture parameter $\hat{q} = \min(\hat{\pi}(\mathcal{G}), 1 - \frac{1}{2m})$, we have $\hat{\pi}(\mathcal{G}) \leq \frac{\kappa^n}{\delta}$, for sufficiently large $n > \log_\kappa(\delta \gamma)$ and optimal mixture parameter $\hat{q} = \min(\hat{\pi}(\mathcal{G}), 1 - \frac{1}{2m})$, we have $\hat{\pi}(\mathcal{G}) \leq \frac{\kappa^n}{\delta}$.

**Proof.** By applying the lower bound in Lemma 16 in eq.(7) to non-trivial games, we have $\hat{\mathcal{L}}(\hat{\pi}(\mathcal{G}), \hat{q}) = KL(\hat{\pi}(\mathcal{G})||\pi(\mathcal{G})) - KL(\hat{\pi}(\mathcal{G})||\hat{q}) - n \log 2 > -\hat{\pi}(\mathcal{G}) \log \pi(\mathcal{G}) - KL(\hat{\pi}(\mathcal{G})||\hat{q}) - (n + 1) \log 2$. Since $\pi(\mathcal{G}) \leq \frac{n^n}{\delta}$, we have $-\log \pi(\mathcal{G}) \geq -\log \frac{n^n}{\delta}$. Therefore $\hat{\mathcal{L}}(\hat{\pi}(\mathcal{G}), \hat{q}) > -\hat{\pi}(\mathcal{G}) \log \frac{n^n}{\delta} - KL(\hat{\pi}(\mathcal{G})||\hat{q}) - (n + 1) \log 2$. Regarding the term $KL(\hat{\pi}(\mathcal{G})||\hat{q})$, if $\hat{\pi}(\mathcal{G}) < 1 \Rightarrow KL(\hat{\pi}(\mathcal{G})||\hat{q}) = KL(\hat{\pi}(\mathcal{G})||\hat{\pi}(\mathcal{G})) = 0$, and if $\hat{\pi}(\mathcal{G}) = 1 \Rightarrow KL(\hat{\pi}(\mathcal{G})||\hat{q}) = KL(1||1 - \frac{1}{2m}) = -\log(1 - \frac{1}{2m}) \leq \log 2$ and approaches 0 when $m \to +\infty$. Maximizing the lower bound of the log-likelihood becomes $\max_{\hat{\mathcal{G}} \in \mathcal{H}} \hat{\pi}(\mathcal{G})$ by removing the constant terms that do not depend on $\mathcal{G}$.

In order to prove $(\mathcal{G}, \hat{q}) \in \mathcal{Y}$ we need to prove $0 < \pi(\mathcal{G}) < \hat{q} < 1$. For proving the first inequality $0 < \pi(\mathcal{G})$, note that $\hat{\pi}(\mathcal{G}) \geq \gamma > 0$, and therefore $\mathcal{G}$ has at least one equilibria.

For proving the third inequality $\hat{q} < 1$, note that $\hat{q} = \min(\hat{\pi}(\mathcal{G}), 1 - \frac{1}{2m}) < 1$. For proving the second inequality $\pi(\mathcal{G}) < \hat{q}$, we need to prove $\pi(\mathcal{G}) < \hat{\pi}(\mathcal{G})$ and $\pi(\mathcal{G}) < 1 - \frac{1}{2m}$. Since $\pi(\mathcal{G}) \leq \frac{n^n}{\delta}$ and $\gamma \leq \hat{\pi}(\mathcal{G})$, it suffices to prove $\frac{(3/4)^n}{\delta} \leq \gamma \Rightarrow \pi(\mathcal{G}) < \hat{\pi}(\mathcal{G})$. Similarly we need to prove $\frac{(3/4)^n}{\delta} \leq \gamma \Rightarrow \pi(\mathcal{G}) < 1 - \frac{1}{2m}$. Putting both together, we have $\frac{(3/4)^n}{\delta} < \min(\gamma, 1 - \frac{1}{2m}) = \gamma$ since $\gamma < 1/2$ and $1 - \frac{1}{2m} \geq 1/2$. Finally, $\frac{(3/4)^n}{\delta} < \gamma \Leftrightarrow n > \log_\kappa(\delta \gamma)$.
6.4 A Non-Concave Maximization Method: Sigmoidal Approximation

A very simple optimization approach can be devised by using a sigmoid in order to approximate the 0/1 function $1[z \geq 0]$ in the maximum likelihood problem of eq.(7) as well as when maximizing the empirical proportion of equilibria as in eq.(14). We use the following sigmoidal approximation:

$$1[z \geq 0] \approx H_{\alpha,\beta}(z) \equiv \frac{1}{2} (1 + \tanh(\frac{z}{\beta} - \arctanh(1 - 2\alpha^{1/n})))$$  (15)

The additional term $\alpha$ ensures that for $G = (W, b)$, $W = 0, b = 0$ we get $1[x \in \mathcal{N}\mathcal{E}(G)] \approx H_{\alpha,\beta}(0)^n = \alpha$. We perform gradient ascent on these objective functions that have many local maxima. Note that when maximizing the “sigmoidal” likelihood, each step of the gradient ascent is NP-hard due to the “sigmoidal” true proportion of equilibria. Therefore, we propose the use of the sigmoidal maximum likelihood only for $n \leq 15$.

In our implementation, we add an $\ell_1$-norm regularizer $-\rho \|W\|_1$ where $\rho > 0$ to both maximization problems. The $\ell_1$-norm regularizer encourages sparseness and attempts to lower the generalization error by controlling over-fitting.

6.5 Our Proposed Approach: Convex Loss Minimization

From an optimization perspective, it is more convenient to minimize a convex objective instead of a sigmoidal approximation in order to avoid the many local minima.

Note that maximizing the empirical proportion of equilibria in eq.(14) is equivalent to minimizing the empirical proportion of non-equilibria, i.e. $\min_{G \in \mathcal{G}} (1 - \hat{\pi}(G))$. Furthermore, $1 - \hat{\pi}(G) = \frac{1}{m} \sum_l 1[x^{(l)} \notin \mathcal{N}\mathcal{E}(G)]$. Denote by $\ell$ the 0/1 loss, i.e. $\ell(z) = 1[z < 0]$. For LIGs, maximizing the empirical proportion of equilibria in eq.(14) is equivalent to solving the loss minimization problem:

$$\min_{W, b} \frac{1}{m} \sum_l \max_i \ell(x_i^{(l)}(w_{i,-i}^T x_{-i}^{(l)} - b_i))$$  (16)

We can further relax this problem by introducing convex upper bounds of the 0/1 loss. Note that the use of convex losses also avoids the trivial solution of eq.(16), i.e. $W = 0, b = 0$ (which obtains the lowest log-likelihood as discussed in Remark 10). Intuitively speaking, note that minimizing the logistic loss $\ell(z) = \log(1 + e^{-z})$ will make $z \to +\infty$, while minimizing the hinge loss $\ell(z) = \max(0, 1 - z)$ will make $z \to 1$ unlike the 0/1 loss $\ell(z) = 1[z < 0]$ that only requires $z = 0$ in order to be minimized. In what follows, we develop four efficient methods for solving eq.(16) under specific choices of loss functions, i.e. hinge and logistic.

In our implementation, we add an $\ell_1$-norm regularizer $\rho \|W\|_1$ where $\rho > 0$ to all the minimization problems. The $\ell_1$-norm regularizer encourages sparseness and attempts to lower the generalization error by controlling over-fitting.

**Independent Support Vector Machines and Logistic Regression.** We can relax the loss minimization problem in eq.(16) by using the loose bound $\max_i \ell(z_i) \leq \sum_i \ell(z_i)$. This relaxation simplifies the original problem into several independent problems. For each player $i$, we train the weights $(w_{i,-i}, b_i)$ in order to predict independent (disjoint) actions.
This leads to $1$-norm SVMs of Bradley and Mangasarian [1998], Zhu et al. [2003] and $\ell_1$-regularized logistic regression. We solve the latter with the $\ell_1$-projection method of Schmidt et al. [2007a]. While the training is independent, our goal is not the prediction for independent players but the characterization of joint-actions. The use of these well known techniques in our context is novel, since we interpret the output of SVMs and logistic regression as the parameters of an LIGs. Therefore, we use the parameters to measure empirical and true proportion of equilibria, KL divergence and log-likelihood in our probabilistic model.

**Simultaneous Support Vector Machines.** While converting the loss minimization problem in eq. (16) by using loose bounds allow to obtain several independent problems with small number of variables, a second reasonable strategy would be to use tighter bounds at the expense of obtaining a single optimization problem with a higher number of variables.

For the hinge loss $\ell(z) = \max(0, 1 - z)$, we have $\max_i \ell(z_i) = \max(0, 1 - z_1, \ldots, 1 - z_n)$ and the loss minimization problem in eq. (16) becomes the following primal linear program:

$$
\begin{align*}
\min_{W, b, \xi} & \quad \frac{1}{m} \sum_l \xi_l + \rho \|W\|_1 \\
\text{s.t.} & \quad (\forall l, i) x_i^{(l)}(w_{i,-i}^T x_{-i}^{(l)} - b_i) \geq 1 - \xi_l \quad (\forall l) \xi_l \geq 0
\end{align*}
$$

(17)

where $\rho > 0$.

Note that eq. (17) is equivalent to a linear program since we can set $W = W^+ - W^-$, $\|W\|_1 = \sum_{ij} w_{ij}^+ + w_{ij}^-$ and add the constraints $W^+ \geq 0$ and $W^- \geq 0$. We follow the regular SVM derivation by adding slack variables $\xi_l$ for each sample $l$. This problem is a generalization of $1$-norm SVMs of Bradley and Mangasarian [1998], Zhu et al. [2003].

By Lagrangian duality, the dual of the problem in eq. (17) is the following linear program:

$$
\begin{align*}
\max_{\alpha} & \quad \sum_l \sum_i \alpha_{li} \\
\text{s.t.} & \quad \langle i, l \rangle \| \sum_l \alpha_{li} x_i^{(l)} \| \infty \leq \rho \quad (\forall l, i) \alpha_{li} \geq 0 \\
& \quad \langle i, l \rangle \sum_l \alpha_{li} x_i^{(l)} = 0 \quad (\forall l) \sum_i \alpha_{li} \leq \frac{1}{m}
\end{align*}
$$

(18)

Furthermore, strong duality holds in this case. Note that eq. (18) is equivalent to a linear program since we can transform the constraint $\|c\|_\infty \leq \rho$ into $-\rho 1 \leq c \leq \rho 1$.

**Simultaneous Logistic Regression.** For the logistic loss $\ell(z) = \log(1 + e^{-z})$, we could use the non-smooth loss $\max_i \ell(z_i)$ directly. Instead, we chose a smooth upper bound, i.e. $\log(1 + \sum_i e^{-z_i})$. The following discussion and technical lemma provides the reason behind our use of this simultaneous logistic loss.

Given that any loss $\ell(z)$ is a decreasing function, the following identity holds $\max_i \ell(z_i) = \ell(\min_i z_i)$. Hence, we can either upper-bound the max function by the logsumexp function or lower-bound the min function by a negative logsumexp. We chose the latter option for the logistic loss for the following reasons: Claim i of the following technical lemma shows that lower-bounding min generates a loss that is strictly less than upper-bounding max. Claim ii shows that lower-bounding min generates a loss that is strictly less than independently penalizing each player. Claim iii shows that there are some cases in which upper-bounding max generates a loss that is strictly greater than independently penalizing each player.
Lemma 18. For the logistic loss $\ell(z) = \log(1+e^{-z})$ and a set of $n > 1$ numbers $\{z_1, \ldots, z_n\}$:

i. $(\forall z_1, \ldots, z_n)$ $\max_i \ell(z_i) \leq \ell(-\log \sum_i e^{-z_i}) < \log \sum_i e^{\ell(z_i)} \leq \max_i \ell(z_i) + \log n$

ii. $(\forall z_1, \ldots, z_n)$ $\ell(-\log \sum_i e^{-z_i}) < \sum_i \ell(z_i)$

iii. $(\exists z_1, \ldots, z_n)$ $\log \sum_i e^{\ell(z_i)} > \sum_i \ell(z_i)$

Proof. Given a set of numbers $\{a_1, \ldots, a_n\}$, the max function is bounded by the logsumexp function by $\max_i a_i \leq \log \sum_i e^{a_i} \leq \max_i a_i + \log n$. [Boyd and Vandenberghe 2006]. Equivalently, the min function is bounded by $\min_i a_i - \log n \leq -\log \sum_i e^{-a_i} \leq \min_i a_i$.

These identities allow us to prove two inequalities in Claim i, i.e. $\max_i \ell(z_i) = \ell(\min_i z_i) \leq \ell(-\log \sum_i e^{-z_i})$ and $\log \sum_i e^{\ell(z_i)} \leq \max_i \ell(z_i) + \log n$. To prove the remaining inequality $\ell(-\log \sum_i e^{-z_i}) < \log \sum_i e^{\ell(z_i)}$, note that for the logistic loss $\ell(-\log \sum_i e^{-z_i}) = \log(1 + \sum_i e^{-z_i})$ and $\log \sum_i e^{\ell(z_i)} = \log(n + \sum_i e^{-z_i})$. Since $n > 1$, strict inequality holds.

To prove Claim ii, we need to show that $\ell(-\log \sum_i e^{-z_i}) = \log(1 + \sum_i e^{-z_i}) < \sum_i \ell(z_i) = \sum_i \log(1 + e^{-z_i})$. This is equivalent to $1 + \sum_i e^{-z_i} < \prod_i (1 + e^{-z_i}) = \sum_{c \in \{0,1\}^n} e^{-c^T z} = 1 + \sum_i e^{-z_i} + \sum_{c \in \{0,1\}^n, 1^T c > 1} e^{-c^T z}$. Finally, we have $\sum_{c \in \{0,1\}^n, 1^T c > 1} e^{-c^T z} > 0$ because the exponential function is strictly positive.

To prove Claim iii, it suffices to find a set of numbers $\{z_1, \ldots, z_n\}$ for which $\log \sum_i e^{\ell(z_i)} > \sum_i \ell(z_i) = \sum_i \log(1 + e^{-z_i})$. This is equivalent to $n + \sum_i e^{-z_i} > \prod_i (1 + e^{-z_i})$. By setting $(\forall i) z_i = \log n$, we reduce the claim we want to prove to $n + 1 > (1 + \frac{1}{n})^n$. Strict inequality holds for $n > 1$. Furthermore, note that $\lim_{n \to +\infty} (1 + \frac{1}{n})^n = e$.

Returning to our simultaneous logistic regression formulation, the loss minimization problem in eq. (16) becomes

$$\min_{W,b} \frac{1}{m} \sum_l \log(1 + \sum_i e^{-z_i^{(l)}(w_{i-}^T x^{(l)}_{i-} - b_l)}) + \rho \|W\|_1$$

where $\rho > 0$.

In our implementation, we use the $\ell_1$-projection method of Schmidt et al. [2007a] for optimizing eq. (20). This method performs a limited memory Broyden-Fletcher-Goldfarb-Shanno (L-BFGS) step in an expanded model (i.e. $W = W^+ - W^-$, $\|W\|_1 = \sum_{ij} w^+_{ij} + w^-_{ij}$) followed by a projection onto the non-negative orthant to enforce $W^+ \geq 0$ and $W^- \geq 0$.

7 On the True Proportion of Equilibria

In this section, we justify the use of convex loss minimization for learning the structure and parameters of LIGs. We define absolute indifference of players and show that our convex loss minimization approach produces games in which all players are non-absolutely-indifferent. We then provide a bound of the true proportion of equilibria with high probability. Our bound only assumes independence of weight vectors among players. Our bound is distribution-free, i.e. we do not assume a specific distribution for the weight vector of each player. Furthermore, we do not assume any connectivity properties of the underlying graph.
Parallel to our analysis, Daskalakis et al. [2011] analyzed a different setting: random games which structure is drawn from the Erdős-Rényi model (i.e. each edge is present independently with the same probability $p$) and utility functions which are random tables. The analysis in Daskalakis et al. [2011], while more general than ours (which only focus on LIGs), it is at the same time more restricted since it assumes either the Erdős-Rényi model for random structures or connectivity properties for deterministic structures.

7.1 Convex Loss Minimization Produces Non-Absolutely-Indifferent Players

First, we define the notion of absolute indifference of players. Our goal in this subsection is to show that our proposed convex loss algorithms produce LIGs in which all players are non-absolutely-indifferent and therefore every player defines constraints to the true proportion of equilibria.

**Definition 19.** Given an LIG $G = (W, b)$, we say a player $i$ is absolutely indifferent if and only if $(w_{i, -i}, b_i) = 0$, and non-absolutely-indifferent if and only if $(w_{i, -i}, b_i) \neq 0$.

Next, we concentrate on the first ingredient for our bound of the true proportion of equilibria. We show that independent and simultaneous SVM and logistic regression produce games in which all players are non-absolutely-indifferent except for some “degenerate” cases. The following lemma applies to independent SVMs for $c(l) = 0$ and simultaneous SVMs for $c(l) = \max(0, \max_{j \neq i} (1 - x_j(l)(w_{i, -i}^T x_{-i}^T - b_i)))$.

**Lemma 20.** Given $(\forall l) c(l) \geq 0$, the minimization of the hinge training loss $\hat{\ell}(w_{i, -i}, b_i) = \frac{1}{m} \sum_i \max(c(l), 1 - x_i(l)(w_{i, -i}^T x_{-i}^T - b_i))$ guarantees non-absolutely-indifference of player $i$ except for some “degenerate” cases, i.e. the optimal solution $(w_{i, -i}^*, b_i^*) = 0$ if and only if $(\forall j \neq i) \sum_l [x_j(l)^T x_{-j}^T = 1]u(l) = \sum_l 1[x_j(l)^T x_{-j}^T = -1]u(l) and \sum_l 1[x_j(l)^T = 1]u(l) = \sum_l 1[x_j(l)^T = -1]u(l)$ where $u(l)$ is defined as $c(l) > 1 \iff u(l) = 0$, $c(l) < 1 \iff u(l) = 1$ and $c(l) = 1 \iff u(l) \in [0; 1]$.

**Proof.** Let $f_i(x_{-i}) \equiv w_{i, -i}^T x_{-i} - b_i$. By noting that $\max(\alpha, \beta) = \max_{0 \leq u \leq 1} (\alpha + u(\beta - \alpha))$, we can rewrite $\hat{\ell}(w_{i, -i}, b_i) = \frac{1}{m} \sum_i \max_{0 \leq u(l) \leq 1} (c(l) + u(l)) (1 - x_i(l)f_i(x_{-i}) - c(l))$.

Note that $\hat{\ell}$ has the minimizer $(w_{i, -i}^*, b_i^*) = 0$ if and only if 0 belongs to the subdifferential set of the non-smooth function $\hat{\ell}$ at $(w_{i, -i}, b_i) = 0$. In order to maximize $\hat{\ell}$, we have $c(l) > 1 - x_i(l)f_i(x_{-i}) \iff u(l) = 0$, $c(l) < 1 - x_i(l)f_i(x_{-i}) \iff u(l) = 1$ and $c(l) = 1 - x_i(l)f_i(x_{-i}) \iff u(l) \in [0; 1]$. The previous rules simplify at the solution under analysis, since $(w_{i, -i}, b_i) = 0 \Rightarrow f_i(x_{-i}) = 0$.

Let $g_i(w_{i, -i}, b_i) \equiv \frac{\partial}{\partial w_{i, -i}}(w_{i, -i}, b_i)$ and $h(w_{i, -i}, b_i) \equiv \frac{\partial}{\partial b_i}(w_{i, -i}, b_i)$. By making $(\forall j \neq i) 0 \in g_j(0, 0)$ and $0 \in h(0, 0)$, we get $(\forall j \neq i) \sum_l x_j(l)^T x_{-j}^T u(l) = 0$ and $\sum_l x_i(l)^T u(l) = 0$. Finally, by noting that $x_i(l) \in \{-1, 1\}$, we prove our claim.

**Remark 21.** Note that for independent SVMs, the “degenerate” cases in Lemma 20 simplify to $(\forall j \neq i) \sum_l [x_j(l)^T x_{-j}^T = 1] = \frac{m}{2}$ and $\sum_l [x_i(l)^T = 1] = \frac{m}{2}$.
The following lemma applies to independent logistic regression for \( c^0 = 0 \) and simultaneous logistic regression for \( c^0 = \sum_{j \neq i} e^{-x_j^0 w_i^j T x_i - b_i} \).

**Lemma 22.** Given \((\forall l) c^0(l) \geq 0\), the minimization of the logistic training loss \( \hat{\ell}(w_{i,-i}, b_i) = \frac{1}{m} \sum_l \log(c^0(l) + 1 + e^{-x_j^0 w_i^j T x_i - b_i}) \) guarantees non-absolutely-indifference of player \( i \) except for some “degenerate” cases, i.e., the optimal solution \( (w_{i,-i}^*, b_i^*) = 0 \) if and only if \((\forall j \neq i) \sum_l \frac{1[x_j^0 x_i^0=1]}{e^{b_i^0+2}} = \sum_l \frac{1[x_i^0 x_j^0=-1]}{e^{b_i^0+2}} \) and \( \sum_l \frac{1[x_i^0 x_j^0=1]}{e^{b_i^0+2}} = \sum_l \frac{1[x_i^0 x_j^0=-1]}{e^{b_i^0+2}} \).

**Proof.** Note that \( \hat{\ell} \) has the minimizer \((w_{i,-i}^*, b_i^*) = 0 \) if and only if the gradient of the smooth function \( \hat{\ell} \) is \( 0 \) at \((w_{i,-i}, b_i) = 0 \). Let \( g_j(w_{i,-i}, b_i) \equiv \frac{\partial \hat{\ell}}{\partial w_i^j}(w_{i,-i}, b_i) \) and \( h(w_{i,-i}, b_i) \equiv \frac{\partial \hat{\ell}}{\partial b_i}(w_{i,-i}, b_i) \). By making \((\forall j \neq i) g_j(0, 0) = 0 \) and \( h(0, 0) = 0 \), we get \((\forall j \neq i) \sum_l \frac{x_j^0 x_i^0}{e^{b_i^0+2}} = 0 \) and \( \sum_l \frac{x_i^0 x_j^0}{e^{b_i^0+2}} = 0 \). Finally, by noting that \( x_i^0 \in \{-1, 1\} \), we prove our claim. \( \square \)

**Remark 23.** Note that for independent logistic regression, the “degenerate” cases in Lemma 22 simplify to \((\forall j \neq i) \sum_l 1[x_i^0 x_j^0 = 1] = \frac{m}{2} \) and \( \sum_l 1[x_i^0 = 1] = \frac{m}{2} \).

Based on these results, after termination of our proposed algorithms, we fix cases in which the optimal solution \((w_{i,-i}^*, b_i^*) = 0 \) by setting \( b_i^* = 1 \) if the action of player \( i \) was mostly \(-1 \) or \( b_i^* = -1 \) otherwise. We point out to the careful reader that we did not include the \( \ell_1 \)-regularization term in the above proofs since the subdifferential of \( \rho \|w_{i,-i}\|_1 \) vanishes at \( w_{i,-i} = 0 \), and therefore our proofs still hold.

### 7.2 Bounding the True Proportion of Equilibria

In what follows, we concentrate on the second ingredient for our bound of the true proportion of equilibria. We show that for a game with a single non-absolutely-indifferent player, the true proportion of equilibria is bounded by \( 3/4 \).

**Lemma 24.** Given an \( \mathcal{LIG} \mathcal{G} = (\mathbf{W}, \mathbf{b}) \) with non-absolutely-indifferent player \( i \) and absolutely-indifferent players \( \forall j \neq i \), the following statements hold:

1. \( x \in \mathcal{N}(\mathcal{G}) \iff x_i (w_{i,-i}^T x_i - b_i) \geq 0 \)
2. \( |\mathcal{N}(\mathcal{G})| = 2^{n-1} + \sum_{x_{-i}} 1 \{w_{i,-i}^T x_{-i} - b_i = 0\} \)
3. \( \frac{1}{2} \leq \pi(\mathcal{G}) \leq \frac{3}{4} \) \( \quad \) (21)

**Proof.** Let \( f_i(x_{-i}) \equiv w_{i,-i}^T x_{-i} - b_i \). For proving Claim i, note that \( 1[x \in \mathcal{N}(\mathcal{G})] = \min_{j \neq i} 1[x_j f_j(x_{-j}) \geq 0] = 1[x_j f_j(x_{-j}) \geq 0] \min_{j \neq i} 1[x_j f_j(x_{-j}) \geq 0] \). Since all players except \( i \) are absolutely-indifferent, we have \((\forall j \neq i) (w_j, b_j) = 0 \) \( \Rightarrow f_j(x_{-j}) = 0 \) which implies that \( \min_{j \neq i} 1[x_j f_j(x_{-j}) \geq 0] = 1 \). Therefore, \( 1[x \in \mathcal{N}(\mathcal{G})] = 1[x_i f_i(x_{-i}) \geq 0] \).

For proving Claim ii, we have \( |\mathcal{N}(\mathcal{G})| = \sum x_i 1[x_i f_i(x_{-i}) \geq 0] \). We can rewrite \( |\mathcal{N}(\mathcal{G})| = \sum x_i 1[x_i = +1] 1[f_i(x_{-i}) \geq 0] + \sum x_i 1[x_i = 0] 1[f_i(x_{-i}) \geq 0] \) or equivalently \( |\mathcal{N}(\mathcal{G})| = \sum x_{-i} 1[f_i(x_{-i}) \geq 0] + \sum x_{-i} 1[f_i(x_{-i}) \leq 0] = 2^{n-1} + \sum x_{-i} 1[f_i(x_{-i}) \leq 0] \).

For proving Claim iii, by eq.(4) and Claim ii we have \( \pi(\mathcal{G}) = \frac{|\mathcal{N}(\mathcal{G})|}{2^n} = \frac{1}{2} + \frac{1}{2^n} \alpha(w_{i,-i}, b_i) \), where \( \alpha(w_{i,-i}, b_i) \equiv \sum x_{-i} 1[w_{i,-i}^T x_{-i} - b_i = 0] \). This proves the lower bound \( \pi(\mathcal{G}) \geq \frac{1}{2} \).
Geometrically speaking, \( \alpha(w_i, b_i) \) is the number of vertices of the \((n - 1)\)-dimensional hypercube that are covered by the hyperplane with normal \( w_i \) and bias \( b_i \). Recall that \( (w_i, b_i) \neq 0 \). If \( w_i = 0 \) and \( b_i \neq 0 \) then \( \alpha(w_i, b_i) = \sum_{x} 1[|b_i| = 0] = 0 \Rightarrow \pi(G) = \frac{1}{2} \).

If \( w_i \neq 0 \) then as noted in [Aichholzer and Aurenhammer 1996] a hyperplane with \( n - 2 \) zeros on \( w_i \) (i.e. a \((n - 2)\)-parallel hyperplane) covers exactly half of the \( 2^{n-1} \) vertices, the maximum possible. Therefore, \( \pi(G) = \frac{1}{2} + \frac{1}{2^n} \alpha(w_i, b_i) \leq \frac{1}{2} + \frac{2^{n-2}}{2^n} = \frac{3}{4}. \)

Next, we present our bound for the true proportion of equilibria of games in which all players are non-absolutely-indifferent.

**Theorem 25.** If all players are non-absolutely-indifferent and if the rows of an LIG \( G = (W, b) \) are independent (but not necessarily identically distributed) random vectors, i.e. for every player \( i \), \( (w_i, b_i) \) is independently drawn from an arbitrary distribution \( \mathcal{P}_i \), then the expected true proportion of equilibria is bounded as follows:

\[
(1/2)^n \leq \mathbb{E}_{\mathcal{P}_1, \ldots, \mathcal{P}_n}[\pi(G)] \leq (3/4)^n
\]  

furthermore, the following high probability statement holds:

\[
\mathbb{P}_{\mathcal{P}_1, \ldots, \mathcal{P}_n}[\pi(G) \leq \frac{(3/4)^n}{\delta}] \geq 1 - \delta
\]

**Proof.** Let \( y_i \equiv 1[x_i(w_{i,i}^T x_{-i} - b_i) \geq 0], \mathcal{P} \equiv \{\mathcal{P}_1, \ldots, \mathcal{P}_n\} \) and \( \mathcal{U} \) the uniform distribution for \( x \in \{-1, +1\}^n \). By eq. (1), \( \mathbb{E}_{\mathcal{P}}[\pi(G)] = \mathbb{E}_{\mathcal{P}}[\mathbb{E}_{\mathcal{U}}[\prod_i y_i]] = \mathbb{E}_{\mathcal{U}}[\mathbb{E}_{\mathcal{P}}[\prod_i y_i]] \). Note that each \( y_i \) is independent since each \( (w_i, b_i) \) is independently distributed. Therefore, \( \mathbb{E}_{\mathcal{P}}[\pi(G)] = \mathbb{E}_{\mathcal{U}}[\prod_i \mathbb{E}_{\mathcal{P}_i}[y_i]]. \) Similarly each \( z_i \equiv \mathbb{E}_{\mathcal{P}_i}[y_i] \) is independent since each \( (w_i, b_i) \) is independently distributed. Therefore, \( \mathbb{E}_{\mathcal{P}}[\pi(G)] = \mathbb{E}_{\mathcal{U}}[\prod_i z_i] = \mathbb{E}_{\mathcal{U}}[\prod_i \mathbb{E}_{\mathcal{P}_i}[y_i]] = \prod_i \mathbb{E}_{\mathcal{P}_i}[\mathbb{E}_{\mathcal{U}}[y_i]]. \) Note that \( \mathbb{E}_{\mathcal{U}}[y_i] \) is the true proportion of equilibria of an LIG with non-absolutely-indifferent player \( i \) and absolutely-indifferent players \( \forall j \neq i \), and therefore \( 1/2 \leq \mathbb{E}_{\mathcal{U}}[y_i] \leq 3/4 \) by Claim iii of Lemma 24. Finally, we have \( \mathbb{E}_{\mathcal{P}}[\pi(G)] \geq \prod_i \mathbb{E}_{\mathcal{P}_i}[1/2] = (1/2)^n \) and similarly \( \mathbb{E}_{\mathcal{P}}[\pi(G)] \leq \prod_i \mathbb{E}_{\mathcal{P}_i}[3/4] = (3/4)^n. \)

By Markov’s inequality, given that \( \pi(G) \geq 0 \), we have \( \mathbb{P}_{\mathcal{P}}[\pi(G) \geq c] \leq \frac{\mathbb{E}_{\mathcal{P}}[\pi(G)]}{c} \leq \frac{(3/4)^n}{c}. \)

For \( c = \frac{(3/4)^n}{\delta} \Rightarrow \mathbb{P}_{\mathcal{P}}[\pi(G) \geq \frac{(3/4)^n}{\delta}] \leq \delta \Rightarrow \mathbb{P}_{\mathcal{P}}[\pi(G) \leq \frac{(3/4)^n}{\delta}] \geq 1 - \delta. \)

**Remark 26.** Under the same assumptions of Theorem 25, it is possible to prove that with probability at least \( 1 - \delta \) we have \( \pi(G) \leq (3/4)^n + 3/8 \sqrt{2 \log \frac{1}{\delta}} \) by using Hoeffding’s lemma.

We point out that such a bound is not better than the Markov’s bound derived above.

8 Experimental Results

Before presenting our experimental evaluations, let us address the issue of the perceived need for comparisons to other models within the context and scope of this paper.

8.1 A Side Note on Experimental Comparisons

We fully motivate and discuss the significance of addressing the specific task of learning the structure and parameters of LIGs in the Introduction. Our goal is to output a game,
not a Markov random field or any other model (Section 1.1). While simple, our learning framework and generative model is well founded. The learning problems that arise from our framework and model are far from computationally trivial, thus warranting scientific analysis in their own right.

We point out and illustrated the distinct “abilities” of Ising models and LIGs with respect to the corresponding nature of the data they seek to model (See Section 2.1 and Figure 2). Our work is not in competition with the work in probabilistic graphical models, e.g. Ising models. We compared LIGs to Ising models because Ising models have been previously applied to the same real-world data we use here on congressional voting records. The “comparison” results from scientific curiosity. While we illustrate the benefit of our model in the U.S. congressional voting records, we believe that each model has its own benefits. If the practitioner “believes” that the data at hand is generated by a particular class of models, then the corresponding interpretation of the learned model allows us to obtain insights into the problem or data at hand.

While generalization on unseen data is a very important measurement tool, and we clearly believe it is a useful guiding principle for model evaluation and selection in ML practice, such methods are not full-proof: There is always the (albeit small) chance that even a model with better generalization might not really correspond to the “ground truth model” of the real-world data at hand.

There is no other method that learns the structure of a graphical game in our setting of strictly behavioral data, as discussed in Section 2. But even if there were methods for learning the structure of other different classes of games, such comparisons between different models while proposing a new method for structure learning of a specifically different model is distracting and not standard practice in ML research or its scientific literature. Such experimental comparisons would still be outside the scope of the current paper.

8.2 Experimental Evaluation

For learning LIGs we used our convex loss methods: independent and simultaneous SVM and logistic regression (See Section 6.5). Additionally, we used the (super-exponential) exhaustive search method (See Section 6.2) only for $n \leq 4$. As a baseline, we used the (NP-hard) sigmoidal maximum likelihood only for $n \leq 15$ as well as the sigmoidal maximum empirical proportion of equilibria (See Section 6.4). Regarding the parameters $\alpha$ and $\beta$ our sigmoidal function in eq.(15), we found experimentally that $\alpha = 0.1$ and $\beta = 0.001$ achieved the best results.

This discussion is really “academic” at this point because, at least until now, such “hypothetical” comparisons are actually impossible given that such methods do not yet exist for our setting.

Most work in probabilistic graphical models follow the reasonable approach of focusing in the structure learning problem by assuming a specific model, and not comparing different models in the same publication. This should at least be clear from textbooks on probabilistic graphical models such as that of Koller and Friedman [2009]. For instance, when proposing a new method for learning the structure of Gaussian graphical models, researchers do not compare Gaussian versus other continuous distributions (see, e.g., the work on Markov random fields [Lee et al., 2006]; bounded tree-width distributions [Chow and Liu, 1968; Srebro, 2001]; Ising models [Wainwright et al., 2006; Banerjee et al., 2008]; Höfling and Tibshirani, 2009]; Gaussian graphical models [Banerjee et al., 2006]; Bayesian networks [Guo and Schuurmans, 2006; Schmidt et al., 2007]; and directed cyclic graphs [Schmidt and Murphy, 2009].
For scientific curiosity, we compare LIGs to learning Ising models. Our goal is not to show the superiority of either games or Ising models, as discussed in Section 8.1. For \( n \leq 15 \) players, we perform exact \( \ell_1 \)-regularized maximum likelihood estimation by using the FOBOS algorithm \cite{Duchi2009a,Duchi2009b} and exact gradients of the log-likelihood of the Ising model. Since the computation of the exact gradient at each step is NP-hard, we used this method only for \( n \leq 15 \). For \( n > 15 \) players, we use the Höfling-Tibshirani method \cite{Hoefting2009}, which uses a sequence of first-order approximations of the exact log-likelihood. We also used a two-step algorithm, by first learning the structure by \( \ell_1 \)-regularized logistic regression \cite{Wainwright2006} and then using the FOBOS algorithm \cite{Duchi2009a,Duchi2009b} with belief propagation for gradient approximation. We did not find a statistically significant difference between the test log-likelihood of both algorithms and therefore we only report the latter.

Our experimental setup is as follows: after learning a model for different values of the regularization parameter \( \rho \) in a training set, we select the value of \( \rho \) that maximizes the log-likelihood in a validation set, and report statistics in a test set. For synthetic experiments, we report the Kullback-Leibler (KL) divergence, average precision (one minus the fraction of falsely included equilibria), average recall (one minus the fraction of falsely excluded equilibria) in order to measure the closeness of the recovered models to the ground truth. For real-world experiments, we report the log-likelihood. In both synthetic and real-world experiments, we report the number of equilibria and the empirical proportion of equilibria. Our results are statistically significant, we avoided showing error bars for clarity of presentation since error bars and markers overlapped.

### 8.3 Experiments on Synthetic Data

We first test the ability of the proposed methods to recover the PSNE induced by ground-truth games from data when those games are LIGs.\footnote{We do not perform any synthetic experiments testing whether our algorithms can recover LIGs with the exact parameter values as those of the synthetic ground-truth models. We already established in Section 4.2 that LIGs are not identifiable with respect to the parameters \( W \) and \( b \), and we show example LIGs that illustrate this. We consistently bring up the analogy with other models and approaches in ML, such as those regarding Bayesian networks. We remind the reader that our main interest is the PSNE of games. We motivate this interest from the start of the introduction, where we talk about our intended application to the CSI framework proposed by Irfan and Ortiz \cite{Irfan2011,Irfan2013}. All of our learning objectives and theoretical results we provide throughout the manuscript reflect that, including those regarding learning generalization performance.} We use a small first synthetic model in order to compare with the (super-exponential) exhaustive search method. The ground-truth model \( G_g = (W_g, b_g) \) has \( n = 4 \) players and 4 Nash equilibria (i.e. \( \pi(G_g)=0.25 \)), \( W_g \) was set according to Figure 4 (the weight of each edge was set to +1) and \( b_g = 0 \). The mixture parameter of the ground-truth model \( q_g \) was set to 0.5,0.7,0.9. For each of 50 repetitions, we generated a training, a validation and a test set of 50 samples each. Figure 4 shows that our convex loss methods and sigmoidal maximum likelihood outperform (lower KL) exhaustive search, sigmoidal maximum empirical proportion of equilibria and Ising models. Note that the exhaustive search method which performs exact maximum likelihood suffers...
Figure 4: Closeness of the Recovered Models to the Ground-Truth Synthetic Model for Different Mixture Parameters $q_g$. Our convex loss methods (IS,SS: independent and simultaneous SVM, IL,SL: independent and simultaneous logistic regression) and sigmoidal maximum likelihood (S1) have lower KL than exhaustive search (EX), sigmoidal maximum empirical proportion of equilibria (S2) and Ising models (IM). For all methods, the recovery of equilibria is perfect for $q_g = 0.9$ (number of equilibria equal to the ground truth, equilibrium precision and recall equal to 1) and the empirical proportion of equilibria resembles the mixture parameter of the ground truth $q_g$.

from over-fitting and consequently does not produce the lowest KL. From all convex loss methods, simultaneous logistic regression achieves the lowest KL. For all methods, the recovery of equilibria is perfect for $q_g = 0.9$ (number of equilibria equal to the ground truth, equilibrium precision and recall equal to 1). Additionally, the empirical proportion of equilibria resembles the mixture parameter of the ground-truth model $q_g$.

Next, we use a slightly larger second synthetic model with more complex interactions. We still keep the model small enough in order to compare with the (NP-hard) sigmoidal maximum likelihood method. The ground truth model $G_g = (W_g, b_g)$ has $n = 9$ players and 16 Nash equilibria (i.e. $\pi(G_g) = 0.03125$), $W_g$ was set according to Figure 5 (the weight of each blue and red edge was set to $+1$ and $-1$ respectively) and $b_g = 0$. The mixture parameter of the ground truth $q_g$ was set to 0.5,0.7,0.9. For each of 50 repetitions, we generated a training, a validation and a test set of 50 samples each. Figure 5 shows that our convex loss methods outperform (lower KL) sigmoidal methods and Ising models. From all convex loss methods, simultaneous logistic regression achieves the lowest KL. For convex loss methods, the equilibrium recovery is better than the remaining methods (number of equilibria equal to the ground truth, higher equilibrium precision and recall). Additionally,

\footnote{This is of course completely consistent with basic, well-known ML findings and results, and thus expected: the MLE tend to overfit when the number of samples is not “large enough.” Hence the common practice to add regularization to the likelihood function in an attempt to improve generalization performance.}
Figure 5: Closeness of the recovered models to the ground truth synthetic model for different mixture parameters $q_g$. Our convex loss methods (IS, SS: independent and simultaneous SVM, IL, SL: independent and simultaneous logistic regression) have lower KL than sigmoidal maximum likelihood (S1), sigmoidal maximum empirical proportion of equilibria (S2) and Ising models (IM). For convex loss methods, the equilibrium recovery is better than the remaining methods (number of equilibria equal to the ground truth, higher equilibrium precision and recall) and the empirical proportion of equilibria resembles the mixture parameter of the ground truth $q_g$.

the empirical proportion of equilibria resembles the mixture parameter of the ground truth $q_g$.

In the next experiment, we show that the performance of convex loss minimization improves as the number of samples increases. We used random graphs with slightly more variables and varying number of samples (10, 30, 100, 300). The ground truth model $G_g = (W_g, b_g)$ contains $n = 20$ players. For each of 20 repetitions, we generate edges in the ground truth model $W_g$ with a required density (either 0.2, 0.5, 0.8). For simplicity, the weight of each edge is set to +1 with probability $P(+1)$ and to −1 with probability $1 - P(+1)$. 49Hence, the Nash equilibria of the generated games does not depend on the

49Part of the reason for using such “simple”/”limited” binary set of weight values in this synthetic experiment regards the ability to generate “interesting” LIGs; that is, games with interesting sets of PSNE. As a word of caution, this is not as simple as it appears at first glance. LIGs with weights and biases generated uniformly at random from some set of real values are almost always not interesting, often having only 1 or 2 PSNE [Irfan and Ortiz 2011, 2013]. It is not until we move to more “special”/restricted classes of games, such as that used in this experiments, that more interesting PSNE structure arises from randomly generated LIGs. That is in large part why we concentrated our experiments in games with those simple properties. (Simplicity itself also had a role in our decision, of course.) Please understand that we are not saying that LIGs that use a larger set of integers, or non-integer real-valued weights $w_{ij}$’s or $b_i$’s are not interesting, as the LIGs we learn from the real-world data demonstrate. What we are saying is that we do not yet have a good understanding on how to randomly generate “interesting” synthetic games from the standpoint of their induced PSNE. We leave a comprehensive evaluation of our MLE-based algorithms’ ability to recover the PSNE of randomly generated synthetic LIGs, which would

37
Figure 6: KL divergence between the recovered models and the ground truth for datasets of different number of samples. Each chart shows the density of the ground truth, probability $P(+1)$ that an edge has weight +1, and average number of equilibria (NE). Our convex loss methods (IS,SS: independent and simultaneous SVM, IL,SL: independent and simultaneous logistic regression) have lower KL than sigmoidal maximum empirical proportion of equilibria (S2) and Ising models (IM). The results are remarkably better when the number of equilibria in the ground truth model is small (e.g. for NE < 20).

Magnitude of the weights, just on their sign. We set the bias $b_g = 0$ and the mixture parameter of the ground truth $q_g = 0.7$. We then generated a training and a validation set with the same number of samples. Figure 6 shows that our convex loss methods outperform (lower KL) sigmoidal maximum empirical proportion of equilibria and Ising models (except for the synthetic model with high true proportion of equilibria: density 0.8, $P(+1) = 0$, NE> 1000). The results are remarkably better when the number of equilibria in the ground truth model is small (e.g. for NE < 20). From all convex loss methods, simultaneous logistic regression achieves the lowest KL.

involve a diversity of synthetic game graph structures, influence weights and biases that induce “interesting” sets of PSNE, for future work.
In the next experiment, we evaluate two effects in our approximation methods. First, we evaluate the impact of removing the true proportion of equilibria from our objective function, i.e. the use of maximum empirical proportion of equilibria instead of maximum likelihood. Second, we evaluate the impact of using convex losses instead of a sigmoidal approximation of the 0/1 loss. We used random graphs with varying number of players and 50 samples. The ground truth model \( G_g = (W_g, b_g) \) contains \( n = 4, 6, 8, 10, 12 \) players. For each of 20 repetitions, we generate edges in the ground truth model \( W_g \) with a required density (either 0.2, 0.5, 0.8). As in the previous experiment, the weight of each edge is set to +1 with probability \( P(+1) \) and to -1 with probability \( 1 - P(+1) \). We set the bias \( b_g = 0 \) and the mixture parameter of the ground truth \( q_g = 0.7 \). We then generated a training and a validation set with the same number of samples. Figure 7 shows that in general, convex loss methods outperform (lower KL) sigmoidal maximum empirical proportion of equilibria, and the latter one outperforms sigmoidal maximum likelihood. A different effect is observed for mild (0.5) to high (0.8) density and \( P(+1) = 1 \) in which the sigmoidal maximum likelihood obtains the lowest KL. In a closer inspection, we found that the ground truth games usually have only 2 equilibria: \((+1, \ldots, +1)\) and \((-1, \ldots, -1)\), which seems to present a challenge for convex loss methods. It seems that for these specific cases, removing the true proportion of equilibria from the objective function negatively impacts the estimation process, but note that sigmoidal maximum likelihood is not computationally feasible for \( n > 15 \).

8.4 Experiments on Real-World Data: U.S. Congressional Voting Records

We used the U.S. congressional voting records in order to measure the generalization performance of convex loss minimization in a real-world dataset. The dataset is publicly available at [http://www.senate.gov/](http://www.senate.gov/). We used the first session of the 104th congress (Jan 1995 to Jan 1996, 613 votes), the first session of the 107th congress (Jan 2001 to Dec 2001, 380 votes) and the second session of the 110th congress (Jan 2008 to Jan 2009, 215 votes). Following on other researchers who have experimented with this data set (e.g., [Banerjee et al. 2008]), abstentions were replaced with negative votes. Since reporting the log-likelihood requires computing the number of equilibria (which is NP-hard), we selected only 20 senators by stratified random sampling. We randomly split the data into three parts. We performed six repetitions by making each third of the data take turns as training, validation and testing sets. Figure 8 shows that our convex loss methods outperform (higher log-likelihood) sigmoidal maximum empirical proportion of equilibria and Ising models. From all convex loss methods, simultaneous logistic regression achieves the lowest KL. For all methods, the number of equilibria (and so the true proportion of equilibria) is low.

We apply convex loss minimization to larger problems, by learning structures of games from all 100 senators. Figure 9 shows that simultaneous logistic regression produce

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The objective of last part of this section is to illustrate general characteristics of the games we learn. We remind the reader that when we use the words "hypothesis" and "validate" in the reminder of this section is only with respect to LIGs, not to imply that our learning algorithms truly and formally validate a real-world hypothesis, regardless of how much anecdotal evidence there is supporting that hypothesis (e.g., increasing negative "direct influences" between parties). As we point out in the Introduction as we introduce Figure 1 and Footnote 17 among other places sporadically throughout this article, there is no way to validate the models we learn on this data in terms of specific game-model parameters, except informally via anecdotal evidence. This is not only because the games are not identifiable, but also because this is data collected.
Figure 7: KL divergence between the recovered models and the ground truth for datasets of different number of players. Each chart shows the density of the ground truth, probability $P(+1)$ that an edge has weight +1, and average number of equilibria (NE) for $n = 2,n = 14$. In general, simultaneous logistic regression (SL) has lower KL than sigmoidal maximum empirical proportion of equilibria (S2), and the latter one has lower KL than sigmoidal maximum likelihood (S1). Other convex losses behave the same as simultaneous logistic regression (omitted for clarity of presentation).

Figure 8: Statistics for games learnt from 20 senators from the first session of the 104th congress, first session of the 107th congress and second session of the 110th congress. The log-likelihood of our convex loss methods (IS,SS: independent and simultaneous SVM, IL,SL: independent and simultaneous logistic regression) is higher than sigmoidal maximum empirical proportion of equilibria (S2) and Ising models (IM). For all methods, the number of equilibria (and so the true proportion of equilibria) is low.
structures that are sparser than its independent counterpart. The simultaneous method better elicits the bipartisan structure of the congress. We define the \textit{(aggregate) direct influence} of player \(j\) to all other players as \(\sum_i |w_{ij}|\) after normalizing all weights, i.e., for each player \(i\) we divide \((w_{i, -i}, b_i)\) by \(\|w_{i, -i}\|_1 + |b_i|\). Note that Jeffords and Clinton are one of the 5 most directly-influential as well as 5 least directly-influenceable (high bias) senators, in the 107th and 110th congress respectively. McCain and Feingold are both in the list of 5 most directly-influential senators in the 104th and 107th congress. McCain appears again in the list of 5 least \textit{directly} influenceable senators in the 110th congress (as defined above in the context of the LIG model).

We test the hypothesis that the aggregate direct influence, \textit{as defined by our model}, between senators of the same party are stronger than senators of different party. We learn structures of games from all 100 senators from the 101th congress to the 111th congress (Jan 1989 to Dec 2010). The number of votes casted for each session were average: 337, minimum: 215, maximum: 613. Figure 10 validates our hypothesis and more interestingly, it shows that influence between different parties is decreasing over time. Note that the influence from Obama to Republicans increased in the last sessions, while McCain’s influence to Republicans decreased.

9 Concluding Remarks

In Section 6, we present a variety of algorithms to learn LIGs from strictly behavioral data, including what we call \textit{independent logistic regression (ILR)}. There is a very popular technique for learning Ising models that uses independent regularized logistic regression to compute the individual conditional probabilities \textit{as a step toward computing a globally coherent joint probability distribution}. However, this approach is inherently problematic, as some authors have previously pointed out (see, e.g., [Guo et al., 2010]). Without getting too technical, the main roadblock is that there is no guarantee that estimates of the weights produced by the individual regressions be symmetric: \(\hat{w}_{ij} = \hat{w}_{ji}\) for all \(i, j\). Learning an Ising model requires the enforcement of this condition, and a variety of heuristics have been proposed. (Please see Section 2.1 for relevant work and references in this area.)

We also \textit{apply ILR in exactly} the same manner but for a \textit{different objective}: learning LIGs. Some seem to think that this diminishes the significance of our contributions. We strongly believe the opposite is true: That we can learn games by using such simple, practical, efficient and well-studied techniques is a significant plus in our view. Again, without getting too technical, the estimates of ILR need not be symmetric for LIG models, and are always perfectly consistent with the LIG definition. In fact, asymmetric estimates are common in practice (the LIG for the 110th Congress depicted in Figure 1 is an example).
And we believe this makes the model more interesting. In the ILR-learned LIG, a player may have a positive, negative or no direct effect on another players utility, and vice versa.

Thus, despite the process of estimation of model parameters being similar, the view of the output of that estimation process is radically different in each case. Our experiments show that our generative model with LIGs built from ILR estimates achieves higher generalization likelihoods than standard probabilistic models such as Ising models that may also use ILR. This fact, that the generative model defined in terms of game-theoretic equilibrium concepts can explain the data better than traditional probabilistic models, provides further evidence supporting such a game-theoretic “view” of the ILR estimated parameters and yields additional confidence in their use in game-theoretic models.

In short, ILR is a thoroughly studied method with a long tradition and an extensive literature from which we can only benefit. We find it to be a good, unexpected outcome of our research in this work, and thus a reasonably significant contribution, that we can successfully and effectively use ILR, a very simple and practical estimation technique for learning probabilistic graphical models, to learn game-theoretic graphical models too.

9.1 Extensions and Future Work

There are several ways of extending this research. We can extend our approach to \( \epsilon \)-approximate PSNE. In this case, for each player instead of one condition, we will have two best-response conditions which are still linear in \( W \) and \( b \). Additionally, we can extend our approach to a broader class of graphical games and non-Boolean actions. Note that our analysis does not rely on binary actions, but on binary features of one player \( 1[x_i = 1] \) or two players \( 1[x_i = x_j] \). We can use features of three players \( 1[x_i = x_j = x_k] \) or of non-Boolean actions \( 1[x_i = 3, x_j = 7] \). This kernelized version is still linear \( W \) and \( b \). These extensions are possible because our algorithms and analysis rely on linearity and binary features; additionally, we can obtain a new upper-bound on the “VC-dimension” by changing the inputs of the neural-network architecture. We can easily extend our approach to parameter learning for fixed structures by using a \( \ell_2^2 \) regularizer instead.

Future work should also consider and study more sophisticated noise processes, MSNE, and the analysis of different upper bounds for the 0/1 loss (e.g. exponential, smooth hinge). Finally, we should consider other slightly more complex versions of our model based on Bayesian or stochastic games to account for possible variations of the influence-weights and bias-threshold parameters (e.g., versions of our model for congressional voting that would explicitly capture game differences in terms of influences and biases that depend on the nature or topic of each specific bill being voted on, as well as Senators’ time-changing preferences and trends).

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51 It is easy to come up with examples of such opposing/antagonistic interactions between individuals in real-world settings. (See, e.g., “parents with teenagers,” perhaps a more timely example is the U.S. Congress in recent times.)

52 By definition, given \( \epsilon \geq 0 \), a joint pure-strategy \( x^* \) is an \( \epsilon \)-approximate PSNE if for each player \( i \), we have \( u_i(x^*) \geq \max_{x_i} u_i(x_i, x^*_{-i}) - \epsilon \); in other words, no players can gain more than \( \epsilon \) in payoff/utility value from unilaterally deviating from \( x_i^\ast \), assuming the other players play \( x^*_{-i} \). Using this definition, we can see that a PSNE is simply a 0-approximate PSNE.
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## A Additional Discussion

### A.1 On End-State Predictions without Explicitly Modeling Dynamics

We believe this additional complexity of temporal dynamics, while possibly more “realistic,” might easily weaken the power of the “bottom-line” prediction of the possible stable final outcomes because the resulting models can get side-tracked by the details of the interaction. We believe the difficulty of modeling such details, specially with the relatively limited amount of the data available, leads to poor prediction performance on what we really care about from an *engineering* standpoint: We would like to know or predict what will end up happening, and have little or no interest on the *how or why* this happens.

We recognize the *scientific* significance and importance of research in social and behavioral sciences such as sociology, psychology and, in some cases economics, on explaining, at a higher level of abstraction, often going as low as the “cognitive” or “neuroscience” level, the process by which final decisions are reached.
We believe the sudden growth of interest from industry (both online and physical companies), government and other national and international institutions on predicting “behavior” for the purpose of revenue, improving efficiency, instituting effective policies with minimal regulations, etc., should shift the focus of the study of “behavior” closer to an engineering endeavor. We believe such entities are after the “bottom-line” and will care more about the end-goal than how or why a specific outcome is achieved, modulo, of course, having simple enough and tractable computational models that provide reasonably accurate predictions of final end-state behavior, or at least accurate enough for their purposes.

A.2 On Alternative Noise Models

A.2.1 On Generalizations of the PSNE-based Mixture Noise Models

A simple extension to our model in eq. (3) is to allow for more general distributions for the PSNE and the non-PSNE sets. That is, with some probability $0 < q < 1$, a joint-action $x$ is chosen from $\text{NE}(G)$ by following a distribution $P_\alpha$ parametrized by $\alpha$; otherwise, $x$ is chosen from its complement set $\{-1, +1\}^n - \text{NE}(G)$ by following a distribution $P_\beta$ parametrized by $\beta$. The corresponding probability mass function (PMF) over joint-behaviors $\{-1, +1\}^n$ parametrized by $(G, q, \alpha, \beta)$ is:

$$p(G, q, \alpha, \beta)(x) = q \frac{p_\alpha(x)}{\sum_{y \in \text{NE}(G)} p_\alpha(y)} + (1 - q) \frac{p_\beta(x)}{\sum_{z \in \text{NE}(G)} p_\beta(z)}$$

where $G$ is a game, $p_\alpha(x)$ and $p_\beta(x)$ are PMFs over $\{-1, +1\}^n$.

One reasonable technique to learn such a model from data is to perform an alternate method. Compared to our simpler model, this model requires a step that maximizes the likelihood by changing $\alpha$ and $\beta$ while keeping $G$ and $q$ constant. The complexity of this step will depend on how we parameterize the PMFs $P_\alpha$ and $P_\beta$ but will very likely be an NP-hard problem because of the partition function. Furthermore, the problem of maximizing the likelihood by changing $\text{NE}(G)$ (while keeping $\alpha$, $\beta$ and $q$ constant) is combinatorial in nature and in this paper, we provided a tractable approximation method with provable guarantees (for the case of uniform distributions). Other approaches for maximizing the likelihood by changing $G$ are very likely to be exponential as we discuss briefly at the start of Section 6.

A.2.2 On Individual-Player Noise Models

As an example, consider a the generative model in which we first randomly select a PSNE $x$ of the game from a distribution $P_\alpha$ parametrized by $\alpha$, and then each player $i$, independently, acts according to $x_i$ with probability $q_i$ and switches its action with probability $1 - q_i$. The corresponding probability mass function (PMF) over joint-behaviors $\{-1, +1\}^n$ parametrized by $(G, q_1, \ldots, q_n, \alpha)$ is:

$$p_G, q, \alpha(x) = \sum_{y \in \text{NE}(G)} \frac{p_\alpha(y)}{\sum_{z \in \text{NE}(G)} p_\alpha(z)} \prod_i q_i^{1[x_i = y_i]}(1 - q_i)^{1[x_i \neq y_i]}$$

where $G$ is a game, $q = (q_1, \ldots, q_n)$ and $p_\alpha(y)$ is a PMF over $\{-1, +1\}^n$.

One reasonable technique to learn such a model from data is to perform an alternate method. In one step, we maximize the likelihood by changing $\text{NE}(G)$ while keeping $q$
and $\alpha$ constant, which could be tractably performed by applying Jensen’s inequality since the problem is combinatorial in nature. In another step, we maximize the likelihood by changing $q$ while keeping $G$ and $\alpha$ constant, which could be performed by gradient ascent (the complexity of this step depends on the size of $\mathcal{N}\mathcal{E}(G)$ which could be exponential!). In yet another step, we maximize the likelihood by changing $\alpha$ while keeping $G$ and $q$ constant (the complexity of this step will depend on how we parameterize the PMF $P_\alpha$ but will very likely be an NP-hard problem because of the partition function). The main problem with this technique is that it can be formally proved that the first step (Jensen’s inequality) will almost surely pick a single equilibria for the model (i.e. $|\mathcal{N}\mathcal{E}(G)| = 1$).
Figure 9: (Top) Matrices of (direct) influence weights W for games learned from all 100 senators, from the first session of the 104th congress (left), first session of the 107th congress (center) and second session of the 110th congress (right), by using our independent (a) and simultaneous (b) logistic regression methods. A row represents how much every other senator directly-influences the senator in such row, in terms of the influence weights of the learned LIG. Positive influence-weight parameter values are shown in blue; negative values are in red. Democrats are shown in the top/left corner, while Republicans are shown in the bottom/right corner. Note that simultaneous method produce structures that are sparser than its independent counterpart. (c) Partial view of the graph for simultaneous logistic regression. (d) Most directly-influential and (e) least directly-influenceable senators, respectively. [Please note that, in the interest of keeping the labels and titles of all the plots simple, we leave out the adjective “direct” and the adverb “directly” in front of the words “influence” and “influenceable,” respectively.] Regularization parameter $\rho = 0.0006$. 

52
Figure 10: Direct influence between parties and direct influences from Obama and McCain. Games were learned from all 100 senators from the 101th congress (Jan 1989) to the 111th congress (Dec 2010) by using our simultaneous logistic regression method. Direct influence between senators of the same party are stronger than senators of different party, which is also decreasing over time. In the last sessions, direct influence from Obama to Republicans increased, and influence from McCain to both parties decreased. Regularization parameter $\rho = 0.0006$. 