Use of $\delta N$ formalism—difficulties in generating large local-type non-Gaussianity during inflation

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Abstract

We discuss the generation of non-Gaussianity in density perturbation through the super-horizon evolution during inflation by using the so-called $\delta N$ formalism. We first provide a general formula for the nonlinearity parameter generated during inflation. We find that it is proportional to the slow-roll parameters, multiplied by the model-dependent factors that may enhance non-Gaussianity to the observable ranges. Then we discuss three typical examples to illustrate how difficult it is to generate sizable non-Gaussianity through the super-horizon evolution during inflation. The first example is the double inflation model, which shows that temporal violation of slow-roll conditions is not enough for the generation of non-Gaussianity. The second example is the ordinary hybrid inflation model, which illustrates the importance of taking into account perturbations on small scales. Finally, we discuss the Kadota–Stewart model. This model gives an example in which we have to choose rather unnatural initial conditions even if large non-Gaussianity can be generated.

(Some figures in this article are in colour only in the electronic version)

1. Introduction: success of inflation

Inflation solved various cosmological problems such as horizon, homogeneity, the flatness problem, just by assuming that the potential energy of scalar fields dominates the expansion of the universe. Moreover, the inflation scenario naturally explains the origin of the almost scale-invariant density perturbation. Observations in the past few decades verified many of the predictions of inflation. An almost scale-invariant spectrum is now confirmed and we even know that the violation of the exact scale invariance is at a 4% level [1], which is also consistent with the simplest slow-roll inflation scenario.
1.1. Density perturbation

The amplitude of quantum fluctuation of inflaton $\phi$ during slow-roll inflation is determined by a unique relevant mass scale at that time, e.g. the Hubble rate $H$:

$$\delta \phi = \frac{H}{2\pi}.$$  \hspace{1cm} (1)

However, on scales as large as the horizon radius the meaning of the amplitude of field perturbation becomes subtle because it depends on the choice of gauge. For the flat slicing gauge, in which the trace of the spatial curvature is kept unperturbed, the perturbation equation for a scalar field becomes very simple and looks very similar to the one without gravitational perturbation [2]. Therefore, the amplitude mentioned above can be in fact understood as that in the flat slicing gauge. This amplitude of perturbation can be interpreted as the dimensionless curvature perturbation on a uniform energy density surface $\zeta$. The transformation law is given by

$$\zeta = H \delta t = \frac{\delta \phi}{\phi} = \frac{H^2}{2\pi \dot{\phi}}.$$  \hspace{1cm} (2)

where $\delta t$ is the shift in time coordinate for this transformation, which is given by $\delta \phi/\dot{\phi}$ using the time derivative of the background scalar field. Writing the perturbation amplitude in terms of $\zeta$ is useful because the curvature perturbation does not evolve on super-horizon scales when the evolution path of the universe is unique. This constancy of $\zeta$ simplifies the analysis of density perturbation during inflation for a single-inflaton model. An extension of this argument to the case of multi-component inflaton is the $\delta N$ formalism [3–8], as we explain below.

1.2. Further steps from observations

Further detailed comparison between theoretical predictions and future observations is awaited. One important observation will be tensor-type perturbation in a cosmic microwave background. Tensor-type perturbation is the transverse–traceless part of the spatial metric perturbation, which is generated by the same mechanism as the perturbation of the inflaton field. Gravitational action has an overall prefactor $m_{pl}^2$, and hence we define the canonically normalized metric perturbation $\psi_{\mu\nu} = m_{pl} \delta g_{\mu\nu}$ so as to absorb this factor from the quadratic action. Then, the quadratic action for the transverse-traceless part of $\psi_{\mu\nu}$ becomes identical to the one for a massless scalar field. Therefore, the amplitude of perturbation $\psi_{\mu\nu}$ generated through almost de Sitter inflation is also $O(H)$. Using this relation, we have

$$\left| \frac{\delta T}{T} \right|_{\text{tensor}} = O(\delta g_{\mu\nu}) = O\left( \frac{\psi_{\mu\nu}}{m_{pl}} \right) = O\left( \frac{H}{m_{pl}} \right).$$  \hspace{1cm} (3)

Thus, we find that the CMB temperature fluctuation caused by the tensor perturbation directly probes the value of $H$ during inflation, i.e. the energy scale of inflation. This can be discriminated from the scalar-type perturbation by looking at B-mode polarization [9–11].

Another important observation is the non-Gaussianity of temperature fluctuations [12, 13]. The non-Gaussianity is caused by the nonlinear dynamics of cosmological perturbation. Once we have completely understood the evolution of density perturbation at late time, the remaining non-Gaussianity which is not accounted for should have its origin in the earlier universe. The 7 year WMAP results indicate that the local-type non-Gaussianity parameter is given as $f_{NL} = 32 \pm 21$ at 68% confidence level [1]. For the Planck satellite, it is expected that the window of $f_{NL}$ is expected to be reduced to $O(10)$ [9].

Recently, non-Gaussianity of the primordial perturbation has also been studied by many authors [7, 8, 12–48]. The main reason for non-Gaussianity attracting much attention is
the expectation to future observations mentioned above. These observations may bring us valuable information about the dynamics of inflation. Here in this short paper, we would like to clarify the difficulties in generating large non-Gaussianity from the nonlinear dynamics during inflation.

2. Basic features of generation of non-Gaussianity of local type

2.1. Local non-Gaussianity

Primordial non-Gaussianity gives rise to non-trivial higher order correlation functions in primordial perturbations. In principle, various types of non-Gaussianity are possibly generated [25–27], but here we focus on the so-called local-type non-Gaussianity. The local-type non-Gaussianity is characterized by the existence of one-to-one local map between the physical curvature perturbation $\zeta(x)$ and the variable which follows the Gaussian statistics $\zeta_G(x)$ at respective spatial points. Namely, we can expand the curvature perturbation as

$$\zeta(x) = \zeta_G(x) + \frac{3}{5} f_{NL} \zeta_G^2(x) + \cdots,$$

and the Gaussian variable $\zeta_G$ satisfies ordinary Gaussian statistics:

$$\langle \zeta_G^{k_1} \zeta_G^{k_2} \rangle = \delta^{(3)}(k_1 + k_2) P_\zeta(k_1).$$

In this case the three-point function can be characterized by a single nonlinear parameter $f_{NL}$ [12] as

$$\langle \zeta_k \zeta_{k_2} \zeta_{k_3} \rangle = \delta^{(3)}(k_1 + k_2 + k_3) B_\zeta(k_1, k_2, k_3),$$

with

$$B_\zeta(k_1, k_2, k_3) = \frac{6}{5} \frac{f_{NL} (2\pi)^{3/2}}{k_1 k_2 k_3} \left[ P_\zeta(k_1) P_\zeta(k_2) + P_\zeta(k_2) P_\zeta(k_3) + P_\zeta(k_3) P_\zeta(k_1) \right].$$

In general, other types of non-Gaussianity can be generated. However, when we focus on super-horizon dynamics, only local-type non-Gaussianity can be produced as is explained below.

2.2. Super-horizon dynamics

Here we just present an intuitive explanation about how the density perturbations evolve when the length scale is much larger than the Hubble scale. Super-horizon dynamics is locally described by the Friedmann–Robertson–Walker universe. We consider the evolution of the universe starting with an initial flat hypersurface at $t = t_*$, on which the initial values of the inflaton field have a certain distribution. We evolve the spacetime until reaching the final surface at $t = t_F$ that is characterized by a specified energy density as shown in figure 1. Hence, the final surface is a uniform energy density surface by definition. As each horizon patch is causally disconnected from the others in the inflating universe, its evolution is determined as a local process. If initial conditions are completely given in each horizon patch, we can solve the evolution of its future. Here, for simplicity, we assume that the evolution of the averaged values of fields in each horizon patch is determined by the averaged values of initial data. This assumption is not necessarily true in general, but we have a good reason to assume so in most cases. Initial conditions for the smaller scale perturbations can affect the evolution of the averaged values of variables. However, as there are so many small-scale degrees of freedom, the average of their effects will not largely fluctuate except for rare situations. Of course, even if we can neglect the effect of fluctuations on small scales, this does not mean that the backreaction due to small-scale degrees of freedom does not modify the dynamics of the
averaged values of variables. This backreaction effect arises at the second-order perturbation or higher. Therefore the difference in the backreaction effect between different horizon patches is even higher order in perturbation. Thus, we usually neglect this effect for simplicity. Of course, in the situation like preheating the sub-horizon scale perturbations evolve more rapidly than the averaged values, we definitely need to take the backreaction into account. However, even in this case, it is not necessary to know all the details of small scale perturbations to understand the evolution of the averaged variables.

Anyway, we neglect these effects originating from small-scale perturbations here. Then, the e-folding number between the initial surface and the final one is completely determined by the initial averaged values of the variables. In the scalar-field-dominant universe, such initial conditions are specified by the values of the field components and their time derivatives, $\phi_i(t^*, x)$ and $\dot{\phi}_i(t^*, x)$. We denote $\phi_i(t^*, x)$ together with $\dot{\phi}_i(t^*, x)$ by $\phi^I(t^*, x)$. Then, the e-folding number between the initial surface and the final one,

$$N(t_F, t^*, \phi^I) \equiv \int_{t^*}^{t_F} H \, dt,$$

is given as a function of $\phi^I(t^*, x)$. Then, roughly speaking, the spatial metric on the final uniform energy density surface will be given by

$$ds^2(3) \approx e^{2N(t_F, \phi^I(t^*, x))} \delta_{ij} \, dx^i \, dx^j.$$

The curvature perturbation is in fact the perturbation in the above exponent $N(t_F, t^*, \phi^I(t^*, x))$, and hence we have

$$\zeta(t_F) \approx \delta N(t_F, \phi^I(t^*, x)).$$

This formula is the heart of the $\delta N$ formalism [3–8].

In the slow-roll case, the evolution equation for the scalar field can be approximated by a first-order differential equation in $t$. Therefore, the phase space of the initial conditions $\phi^I(t^*, x)$ is reduced to $\phi^I(t^*, x)$. In this case, if there is only a single component, the trajectory in the field configuration space, which is one dimensional, is necessarily unique. Then, $\delta N(t_F, \phi^I(t^*, x))$ does not depend on the choice of the energy density of the final surface. It can be set to a representative background value at the initial time. Namely,

$$\delta N(t_F, \phi^I(t^*, x)) = \delta N(t^*, \phi^I(t^*, x)).$$

Therefore, nothing non-trivial happens during the super-horizon evolution for slow-roll single-field inflation. Even if the slow-roll conditions are violated, single-field inflation does not produce sizable $\delta N$ for modes far beyond the horizon scale. This is because the trajectories in the configuration space continues to converge as the decaying perturbation mode gets
irrelevant. In order to generate non-Gaussianity through the evolution on super-horizon scales, it is therefore essential to consider a multi-component inflaton field.

Using formula (10), we can express the curvature perturbations on the final surface in terms of the field perturbations on the initial surface as

\[ \zeta(t_c) = \delta N = N_I^* \delta \phi^I_* + \frac{1}{2} N_{IJ}^* \delta \phi^I_* \delta \phi^J_* + \cdots, \]  

(12)

where the subscript * indicates the quantities evaluated at \( \phi^I(t_*) \), and the subscript \( I \) means the differentiation with respect to \( \phi^I(t_*) \):

\[ \begin{align*}
N_I^* & \equiv \frac{\partial N(t, \phi^I)}{\partial \phi^I} \bigg|_{\phi = \bar{\phi}(t_*)}, \\
N_{IJ}^* & \equiv \frac{\partial^2 N(t, \phi^I)}{\partial \phi^I \partial \phi^J} \bigg|_{\phi = \bar{\phi}(t_*)},
\end{align*} \]

(13)

where \( \bar{\phi}^I(t) \) is the background trajectory. With the help of equation (12), we can write the three-point correlation function of \( \zeta \) at \( t = t_F \) as

\[ \langle \zeta(x_1) \zeta(x_2) \zeta(x_3) \rangle = N_I^* N_J^* N_K^* \langle \delta \phi^I_*(x_1) \delta \phi^J_*(x_2) \delta \phi^K_*(x_3) \rangle + \text{(3 perms)} + \cdots. \]

(14)

The first term contains only three \( \delta \phi \)s. If the initial fluctuations at \( t = t_* \) are Gaussian, this term vanishes. Depending on the model of inflation, early generation of non-Gaussianity at around horizon crossing is possible [26]. However, the observed almost scale-invariant and slightly red spectrum of initial curvature perturbation strongly suggests that the inflation is likely to have been in the slow-roll regime at around the horizon crossing. For the slow-roll inflation, it is shown that the early production of non-Gaussianity is strongly suppressed [14–16]. Therefore, it is well motivated to consider non-Gaussianity contained in the second term (or even higher order terms), generated by the nonlinear evolution after the horizon crossing.

Generation of non-Gaussianity can be classified into two classes. One is generation through the super-horizon evolution during inflation. The other is generation at the end of or after inflation [29–31, 34–37, 43, 44]. In either case the non-Gaussianity is of local type. The three-point correlation function is characterized by one parameter \( f_{NL} \) alone, and it is related to the e-folding number \( N(t_F, \phi^I) \) as

\[ 6 f_{NL} \approx \frac{N_I^* N_J^* N_K^*}{(N_I N_J N_K)^2}, \]

(15)

neglecting higher order terms.

### 3. Non-Gaussianity produced at the end of or after inflation

There are various models of inflation that produces large non-Gaussianity of local type at the end of or after inflation. Here we briefly mention a curvaton-type model [51, 52] as a typical example to explain the mechanism to generate large non-Gaussianity. Curvaton is another scalar field that is introduced to explain the origin of primordial curvature perturbation. During inflation, the curvaton field is almost massless but it starts to behave as a massive field later when the Hubble rate decays down to the mass scale of the curvaton field. Then, the curvaton starts to oscillate around the minimum of the potential and eventually decays into radiation. At this point, the fluctuations in the curvaton \( \sigma \) are transferred into curvature perturbations. For simplicity, we consider the case when the density perturbations are dominated by the fluctuations originating from the curvaton. Neglecting a contribution of second order of \( \delta \rho_\sigma \) to \( \zeta \), which turns out to yield only \( f_{NL} = O(1) \), we have [30]

\[ \zeta = \frac{1}{4} \frac{\delta \rho_\sigma}{\rho} = \frac{r}{4} \left( \frac{2 \delta \sigma}{\sigma} + \left( \frac{\delta \sigma}{\sigma} \right)^2 \right), \]

(16)
where \( r = \rho_\sigma/\rho_{\text{tot}} \) is the fraction of the energy density that the decay products of \( \sigma \) takes. Here, we used the estimate \( \rho_\sigma \approx m^2 (\sigma + \delta \sigma)^2 / 2 \) with \( m \) being the mass of the curvaton.

The amplitude of the power spectrum is observationally fixed as

\[
P_\zeta = O(|r\delta \sigma/\sigma|^2) = O(10^{-10}).
\]  

(17)

Therefore, if \( \delta \sigma/\sigma \) is not small, \( r \) can be as small as \( 10^{-5} \). On the other hand, the nonlinear parameter is given by

\[
f_{NL} = O(\langle \zeta^3 \rangle / \langle \zeta^2 \rangle^2) = O(1/r),
\]  

(18)

which can be as large as \( 10^5 \). In this way, the production of large non-Gaussianity at the end of inflation is rather easily achieved.

As a mechanism for getting large non-Gaussianity, one can also consider generation of perturbations during the preheating phase [43–47, 49, 50]. In this case, the background evolution is strongly coupled with the evolution of small scale fluctuations because the e-folding number changes depending on how fast the energy of coherent oscillation of fields decays into the small scale perturbations, while in the other models the transmutation of homogeneous isocurvature perturbations on each Hubble patch into adiabatic perturbations is determined just by solving the background evolution of various homogeneous and isotropic universe models. In this sense the mechanism to generate density perturbation during the preheating stage is quite different from the other models.

4. Non-Gaussianity produced during inflation

Here we just present the formula for \( f_{NL} \) in slow-roll inflation with a canonical kinetic term. A systematic derivation for a more general formula, including higher order correlators, is given in [24]. Assuming that the initial perturbations of fields are Gaussian in the following form:

\[
\langle \delta \phi_i \delta \phi_j \rangle = \delta_{ij} \delta_{(3)}(k_1 + k_2) P_{\phi}(k_1),
\]  

(19)

the formula for \( f_{NL} \) is written in terms of the potential of the scalar field \( V \) as

\[
\frac{6}{5} f_{NL} = (N_i^* N_j^*)^{-2} \int_{N_*}^{N_F} dN' N_j(N') Q_{ij}(N') \Theta^i(N').
\]  

(20)

Here, we neglected non-Gaussianity in the relation between \( \delta N \) and \( \delta \phi \) at \( N = N_F \). To define \( N_i \) and \( \Theta^i \), it is convenient to introduce the propagator

\[
\Lambda^i_j(N, N') = \left[ T \exp \left( \int_{N_*}^{N_F} P(N') \ dN'' \right) \right]_j^i,
\]  

(21)

where

\[
P^j_i(N) = \left[ \frac{V^j_i}{V} + \frac{V_i V_j}{V^2} \right]_{\phi=\hat{\phi}(N)}
\]  

(22)

is the potential term that appears in the linear perturbation equation when we use the e-folding number \( N \) as a time coordinate. The indices associated with \( V \) represent differentiation with respect to the scalar field. \( T \) in equation (21) means that the matrices \( P^j_i \) are ordered in time when the exponential is expanded in power of \( P^j_i \). Using this propagator, \( N_i \) and \( \Theta^i \) are defined as

\[
N_i(N) = N_i(N_F) \Lambda^i_j(N_F, N),
\]

\[
\Theta^i(N) = \Lambda^i_j(N, N_*) N^*_j.
\]  

(23)
$N_i(N)$ satisfies the equation of motion adjoint to the linear perturbation equation for $\delta \phi^i$,

$$\frac{d}{dN} N_i(N) = -P_i^j N_j(N),$$

with the boundary condition $N_j(N_F) = (\partial N(N_F, \phi^i)/\partial \phi^j)|_{\phi^i = \bar{\phi}(N_F)}$. $\Theta^i(N)$ satisfies the same equation as $\delta \phi^i$,

$$\frac{d}{dN} \Theta^i(N) = P_i^j \Theta^j(N),$$

and the boundary condition is given by $\Theta^i(N_s) = N^i(N_s)$, where the index is raised by the inverse of the field space metric, which is assumed to be $\delta_{ij}$ here. The three-point interaction $Q_{jk}^i$ is given by

$$Q_{jk}^i(N) \equiv \left[ -\frac{V_{jk}^i V_{ij}}{V^2} + \frac{V_{ik}^j V_{ij}}{V^2} + \frac{V_{ij}^i V_{jk}}{V^2} - \frac{2}{V^3} \frac{V_{ij}^i V_{jk}^j}{V^2} \right]_{\phi^i = \bar{\phi}(N)},$$

In a naive sense, slow-roll conditions require that the potential of the inflaton is a smooth function of $\phi^i$. Hence higher order differentiations with respect to $\phi^i$ are suppressed. More precisely, introducing a slow-roll parameter

$$\epsilon \equiv \frac{1}{2} \frac{V^i V^i}{V^2},$$

where $m_{pl}$ is set to unity, $P_i^j$ and $Q_{jk}^i$ are supposed to be of $O(\epsilon)$ and $O(\epsilon^{3/2})$, respectively. To obtain an almost scale-invariant spectrum, assuming this kind of scaling is natural although it is not strictly required [53]. In general, there might be a very massive isocurvature degree of freedom, which does not satisfy the above simple scaling. However, in this simple example, fluctuations in such a massive degree of freedom decay rapidly, and hence they can be safely neglected.

Duration of the inflation is roughly estimated as

$$N = O\left( \frac{V}{dV/dN} \right) = O\left( \frac{HV}{V^2 \phi} \right) = O(\epsilon^{-1}).$$

Hence, the exponent in the propagator $\Lambda_{ij}^l$ is estimated to be $O(1)$. As it is the exponent, it is crucial whether the amplitude of this factor is slightly bigger or smaller than unity. In principle, therefore $\Lambda_{ij}^l$ itself can be much larger or much smaller than unity. The magnitudes of $N_i$ and $\Theta^i$ depend on the behavior of the propagator, but the value of $N$ at $N = N_F$ is given by the definition so as to satisfy

$$V(\phi(N_F + \delta N)) = V(\bar{\phi}(N_F)).$$

Expanding this equation up to the first order, we obtain

$$N_i(N_F) = O(V V_i / V_j V_j) = O(\epsilon^{-1/2}).$$

If $\Lambda_{ij}^l$ stays of $O(1)$, one can say that $N_i$ and $\Theta^i$ also stay $O(\epsilon^{-1/2})$. Substituting these estimates into equation (20), we find that $f_{NL} = O(\epsilon)$, as the general argument refers in [14–16].

However, if the amplitude of $\Lambda_{ij}^l$ largely deviates from unity, $N_i$ and hence $\Theta^i$ are also largely enhanced or suppressed, resulting in large enhancement or suppression of the integrand in equation (20). Then, the order of magnitude of $f_{NL}$ is not necessarily suppressed like $O(\epsilon)$. Furthermore, if slow-roll conditions are violated, we have more chances to generate large non-Gaussianity. However, it is not so easy to construct a viable model which generates large non-Gaussianity from the super-horizon dynamics during inflation as we shall see below.
5. Is there any model which produces large non-Gaussianity during inflation?

5.1. Temporal slow-roll violation is not sufficient

First we consider the double-inflation model [54], which is a simple two-field model. The potential is given by

\[ V(\phi, \chi) = \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2} m_\chi^2 \chi^2. \]  

There is no direct coupling between the two fields, \( \phi \) and \( \chi \), except for the gravitational one. Here we follow the analysis of [21]. We show the background trajectory for \( m_\chi / m_\phi = 20 \) in figure 2 taken from the same reference. Since the mass of the \( \chi \)-field is much larger, the field moves rapidly in the \( \chi \)-direction first. Arriving at around the minimum in the \( \chi \)-direction, the slow-roll conditions are violated and the field oscillates several times. This violation of slow-roll conditions at the intermediate step is possible when the mass ratio is large enough. The trajectory becomes just a smooth curve and slow-roll conditions are not violated in the middle. One may expect that this temporal violation of slow roll produces large non-Gaussianity. However, this is not the case.

During the period when the \( \chi \)-field moves rapidly, \( \phi \)-field stays almost constant because the mass of the \( \phi \)-field is small compared to the Hubble rate which is dominantly determined by the term \( m_\chi^2 \chi^2 \) in the potential. Hence, the total e-folding number is simply given by the sum of e-foldings for two stages of inflation:

\[ N = N^{(\phi)} + N^{(\chi)} \approx \phi^2 + \chi^2. \]  

Applying formula (15), we find

\[ 6 \frac{f_{NL}}{5} = \frac{N_i N_j N^{ij}}{(N_i N^i)^2} = \frac{2}{\phi_i^2 + \chi_i^2} = O \left( \frac{1}{N} \right). \]  

Hence, as in the usual slow-roll case, the non-Gaussianity is suppressed by the slow-roll parameter \( \epsilon = O(1/N) \).
5.2. Simple hybrid inflation does not give large non-Gaussianity

As a second example, let us consider a hybrid inflation model [55] whose potential is given by

\[ V(\phi, \chi) = \frac{\lambda}{4}(\chi^2 - v^2)^2 + \frac{m^2}{2}\phi^2 + \frac{g^2}{2}\phi^2\chi^2. \]  

(34)

Usually \( \phi \) is the inflaton field and \( \chi \) stays at \( \chi = 0 \) during inflation. The effective mass of the \( \chi \)-field is a function of \( \phi \) given by

\[ m^2_{\chi} = -\lambda v^2 + g^2\phi^2, \]  

(35)

assuming \( \chi = 0 \). The mass of the \( \chi \)-field changes its sign at some point, which we denote by \( N = N_c \). Then, tachyonic instability in the \( \chi \)-direction occurs, leading to the end of inflation.

To analyze the dynamics of the \( \chi \)-field analytically at this critical point, we approximate the mass of the \( \chi \)-field as

\[ m^2_{\chi} = -\mu^2(N - N_c), \]  

(36)

where \( \mu \) is a parameter that depends on the choice of the model potential.

If \( \mu^2 \gtrsim H^2 \), the \( \chi \)-field is massive during \( \phi \)-field inflation. Thus, the fluctuation of the \( \chi \)-field decays on large scales. If \( \mu^2 \lesssim H^2 \), the \( \chi \)-field can stay nearly massless during \( \phi \)-field inflation. In this case, the fluctuation of the \( \chi \)-field may play an important role. As the mass scale of \( \chi \) is small, we can apply the slow-roll approximation to the \( \chi \)-field, as well. Then, we have

\[ 3H^2\partial_N \chi = \mu^2(N - N_c)\chi. \]  

(37)

This equation can be solved to obtain

\[ \ln \frac{\chi}{\chi_*} = \frac{\mu^2}{6H^2}[(N - N_c)^2 - (N_* - N_c)^2]. \]  

(38)

Applying formula \((6/5)f_{NL} \approx N_{\chi}/N^2\)\(\chi\), which is valid when the fluctuation of the \( \chi \)-field dominates, we have

\[ \frac{6}{5}f_{NL} \approx \frac{\mu^2}{3H^2}(N_F - N_c). \]  

(39)

Since we are restricted to the case when \( \mu^2 \lesssim H^2 \) and \( N_F - N_c \) cannot be very large, \( f_{NL} \) cannot be larger than 20 or so. If the fluctuations of the \( \phi \)-field also contribute, which is usually required in order to produce an almost scale-invariant spectrum, the value of \( f_{NL} \) becomes even smaller.

Furthermore, a more stringent constraint can be set by considering the following condition.

If \( \hat{\chi}(N_c) \gtrsim H \) is not satisfied, fluctuations at the horizon scale modes at that time determine on which side the field rolls down. Then, the tachyonic instability does not keep the coherence of the \( \chi \)-field on the super-horizon scales, leading to the so-called spinodal decomposition. As a result, the long wavelength fluctuation is, roughly speaking, completely erased. Hence, there is no chance to generate large non-Gaussianity originating from the fluctuation of the \( \chi \)-field. The condition \( \hat{\chi}(N_c) \gtrsim H \) becomes

\[ \frac{\mu^2}{6H^2}(N_F - N_c)^2 \approx \ln \frac{\chi_F}{\chi_c} \lesssim \ln \frac{v}{H}, \]  

(40)

where we used the estimate \( \chi_F \approx v \). Using this inequality, the above estimate for \( f_{NL} \) (39) is bounded by

\[ \frac{6}{5}f_{NL} \lesssim \frac{\mu}{H} \sqrt{\frac{2}{3}} \ln \frac{m_{pl}}{H}. \]  

(41)
where we used $H^2 = O(\lambda v^4/m_{pl}^2)$ and $|m_\chi|^2|_{\phi=0} = O(\lambda v^2) \gg O(H)$. The latter condition is required to avoid topological inflation which occurs at $\phi = \chi = 0$. The factor $\sqrt{\frac{2}{3}} \ln \frac{m_{\phi}^2}{H} \approx 7$ even if we reduce the energy scale of inflation $\sqrt{m_{pl}H}$ to the TeV scale. Here, as $\mu^2 \ll H^2$ is also required, we conclude that $f_{NL}$ cannot be much larger than unity.

5.3. Modular inflation

In the preceding subsection we have observed that the large mass scale for the isocurvature modes is necessary to generate large non-Gaussianity. However, the mass squared should not be large positive at the early stage of inflation when interesting scales exit the horizon. This requirement is not easily compromised unless we invent a very artificial form of potential just to produce large non-Gaussianity.

Here we give one rather natural model which satisfies our requirement. This model was proposed by Kadota and Stewart [56]. The basic model potential takes the form

$$V = V_0 - m_\chi^2 |\Phi|^2 + \frac{1}{4} m_\phi^2 (\Phi^4 + \text{h.c.}) + m_\phi^2 |\Phi|^4,$$

(42)

where again we set $m_{pl} = 1$. $\Phi = (\phi + i\chi)/\sqrt{2}$ is a complex scalar field. This potential is modified near $\Phi = 0$, the point with enhanced symmetry, due to loop correction of the Coleman–Weinberg type: $V_{\text{loop}} = -\beta m_\phi^2 |\Phi|^2 \ln|\Phi/\Phi_\ast|$. This correction produces ring-shaped maximum around $\Phi = 0$, where topological inflation occurs. From the edges of the region of topological inflation, the field slowly rolls down the hill.

The contour plot of the potential is given in figure 3. We focus on the trajectories which pass near the saddle point as shown in the same figure with a red curve. In this model mass squared in the $\chi$-direction is negative from the beginning. However, since it is the direction of the approximate $U(1)$ symmetry, the trajectories in any direction are almost equally preferred at this stage. As $\phi$ increases, the fluctuation in the $\chi$-direction is enhanced by the ratio of $\phi$ compared with its initial value. After the direction of trajectory changes, this fluctuation in the $\chi$-direction is transferred to the curvature perturbation. In this way the fluctuations in the $\chi$-direction can rather easily dominate the final density perturbations. Then, the amplitude of
fluctuations in the $\chi$-direction stays almost constant near the ring-shaped top of the potential, and therefore the scale-invariant spectrum is realized.

Near the saddle point, the equation in the $\chi$-direction can be approximated by

$$3H^2 \partial_N \chi = -m^2 \chi,$$

with $m^2$ being the mass squared of the $\chi$-field evaluated at the saddle point. Using the approximation $H^2 = \text{constant}$, this equation can be solved to give

$$N = \frac{3H^2}{-m^2} \ln \frac{\chi}{\chi_*},$$

where $\chi_*$ is the initial value of $\chi$. Applying the formula $(6/5)f_{NL} \approx \frac{N\chi}{N^2}$, we obtain

$$\frac{6}{5} f_{NL} \approx \frac{m^2}{3H^2}.$$  \hspace{1cm} (45)

This can be large if $m^2 \gg H^2$. However, to achieve this by natural form of potential, the potential minima as well as the saddles points should be located close to the origin in the Planck unit. Then, inflation does not occur except for the top of the potential or near the saddle points. To sustain a sufficiently large e-folding number, inflation in these specific regions should last long. However, this is not allowed. If inflation lasts long near the top of the potential, which is ring-shaped, the motion in the radial direction becomes very slow. As a result, the amplitude of curvature perturbation originating from the fluctuation in $\phi$ direction dominates. For inflation to last long near the saddle point, the trajectory should pass very close to the saddle point. However, in such cases significant fraction of the universe will fall into the other side of the saddle point. As a result domain walls are formed, which leads to the problematic domain-wall-dominated universe.

Even if we can avoid the domain wall formation (by considering models with higher symmetry), it is not easy to construct models which explain the naturalness of such a special trajectory. The probability distribution of the universes with different values of $\chi_*$ will be affected by the volume expansion factor during inflation, which is proportional to $e^{3N}$. If $3H^2/|m^2|$ is small as is requested for large non-Gaussianity, $\chi_*$ must be extremely small to gain a large e-folding there. Then, the probability of having such a small value of $\chi_*$ is extremely small. As long as we consider models which realize sufficiently large e-foldings for the natural choice of the fiducial value of $\chi_*$, $f_{NL}$ will not largely exceed unity. Further variation of this model is possible by introducing loop correction at around the saddle points, but the final conclusion does not change much. We will report on the detailed analysis about this model in our future publication.

6. Conclusion

Multi-field inflation generates entropy perturbation as well as the adiabatic one. This entropy perturbation in general affects the evolution of the curvature perturbation even after the horizon crossing. Possible mechanisms of production of non-Gaussianity in the super-horizon regime can be classified into two cases. One is the production of non-Gaussianity during inflation and the other is that at the end of or after inflation. Although we could not give a general proof, it seems very difficult to produce large non-Gaussianity during inflation as far as we consider potential without fine-tuning, though there are several claims that suggest that it is possible. In the examples given in [40, 41], it is not clear if we should say that the non-Gaussianity is generated during super-horizon evolution during inflation because distribution in field space stays Gaussian. In fact, the origin of non-Gaussianity is concentrated in the term neglected in equation (20).
Here are several subtle issues. In this paper we claimed that it is difficult to construct a natural model in which large non-Gaussianity originating from the nonlinear evolution is produced before inflation terminates. By contrast, generation of large non-Gaussianity at the end of or after inflation is rather easy. However, it may not be so trivial in general to identify when non-Gaussianity is generated. We gave an example of double inflation in section 5.1. Although we did not mention it in the text, in this model the nonlinear parameter $f_{NL}$ as a function of $t_F$ becomes very large temporally when the background trajectory changes its direction, but this large $f_{NL}$ does not persist long. This means that the time dependence of $f_{NL}$ is not always appropriate to read when non-Gaussianity is mainly generated.

Another important point to note is the limitation of neglecting small-scale perturbations. The small-scale perturbations often become important for the field components that become massive during inflation. A typical example is the hybrid inflation model discussed in section 5.2. To take into account the effect of small scale perturbations, $\delta N$ formalism based on the background evolution for spatially homogeneous spacetime is not sufficient. We need to extend $\delta N$ formalism incorporating collective variables that characterize statistical state such as the averaged magnitude of small-scale fluctuation.

Finally we should mention the naturalness of the initial conditions. It might be possible to obtain large non-Gaussianity by tuning the initial conditions, but it will not be realized in reality if they are extremely fine-tuned. As an example, we discussed the Kadota–Stewart model in section 5.3. However, once we begin to discuss the likeliness of the chosen initial conditions, the long-standing measure problem arises. At the moment we are very far from a conclusive answer on this issue.

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References

[1] Komatsu E et al 2010 arXiv:1001.4538
[2] e.g., Mukhanov V F, Feldman H A and Brandenberger R H 1992 Phys. Rept. 2 215 203
[3] Starobinsky A A 1985 JETP Lett. 42 152
Starobinsky A A 1985 Pis’ma Zh. Eksp. Teor. Fiz. 42 124
[4] Salopek D S and Bond J R 1990 Phys. Rev. D 42 3936
[5] Sasaki M and Stewart E D 1996 Prog. Theor. Phys. 95 71 (arXiv:astro-ph/9507001)
[6] Sasaki M and Tanaka T 1998 Prog. Theor. Phys. 99 763 (arXiv:gr-qc/9801017)
[7] Lyth D H, Malik K A and Sasaki M 2005 J. Cosmol. Astropart. Phys. JCAP05(2005)004 (arXiv:astro-ph/0411220)
[8] Lyth D H and Rodriguez Y 2005 Phys. Rev. Lett. 95 121302 (arXiv:astro-ph/0504045)
[9] Mendes N et al 2010 arXiv:1001.2657.
[10] Baumann D et al (CMBPol Study Team Collaboration) 2009 AIP Conf. Proc. 1141 10 (arXiv:0811.3919)
[11] http://quiet.uchicago.edu/
[12] Komatsu E and Spergel D N 2001 Phys. Rev. D 63 063002 (arXiv:astro-ph/0005036)
[13] Bartolo N, Komatsu E, Matarrese S and Riotto A 2004 Phys. Rep. 402 103 (arXiv:astro-ph/0406398)
[14] Maldacena J M 2003 J. High Energy Phys. JHEP05(2003)013 (arXiv:astro-ph/0210603)
[15] Seery D and Lidsey J E 2005 J. Cosmol. Astropart. Phys. JCAP06(2005)003 (arXiv:astro-ph/0503692)
[16] Seery D, Lidsey J E and Sloth M S 2007 J. Cosmol. Astropart. Phys. JCAP01(2007)027 (arXiv:astro-ph/0610210)
[17] Seery D and Lidsey J E 2005 J. Cosmol. Astropart. Phys. JCAP09(2005)011 (arXiv:astro-ph/0506056)
[18] Malik K A and Wands D 2004 Class. Quantum. Grav. 21 L65 (arXiv:astro-ph/0307055)
[19] Malik K A 2005 J. Cosmol. Astropart. Phys. JCAP11(2005)005 (arXiv:astro-ph/0506532)
[20] Yokoyama S, Suyama T and Tanaka T 2007 J. Cosmol. Astropart. Phys. JCAP07(2007)013 (arXiv:0705.3178)
[21] Yokoyama S, Suyama T and Tanaka T 2008 Phys. Rev. D 77 083511 (arXiv:0711.2920)
[22] Byrnes C T, Sasaki M and Wands D 2006 Phys. Rev. D 74 123519 (arXiv:astro-ph/0611075)
[23] Yokoyama S, Suyama T and Tanaka T 2009 J. Cosmol. Astropart. Phys. JCAP02(2009)004 (arXiv:0810.2471)
[24] Yokoyama S, Suyama T and Tanaka T 2008 Phys. Rev. D 77 123511 (arXiv:0711.2920)
[25] Creminelli P 2003 J. Cosmol. Astropart. Phys. JCAP10(2003)003 (arXiv:astro-ph/0306122)
[26] Alishahiha M, Silverstein E and Tong D 2004 Phys. Rev. D 70 123505 (arXiv:hep-th/0404084)
[27] Koyama K 2010 arXiv:1002.0600.
[28] Alabidi L 2006 J. Cosmol. Astropart. Phys. JCAP10(2006)015 (arXiv:astro-ph/0604611)
[29] Malik K A and Lyth D H 2006 J. Cosmol. Astropart. Phys. JCAP09(2006)008 (arXiv:astro-ph/0604387)
[30] Sasaki M, Valiviita J and Wands D 2006 Phys. Rev. D 74 103503 (arXiv:astro-ph/0607627)
[31] Dvali G, Gruzinov A and Zaldarriaga M 2004 Phys. Rev. D 69 023505 (arXiv:astro-ph/0303591)
[32] Bernardeau F and Uzan J P 2002 Phys. Rev. D 66 103506 (arXiv:hep-th/0202329)
[33] Bernardeau F and Uzan J P 2003 Phys. Rev. D 67 121301 (arXiv:astro-ph/0209330)
[34] Lyth D H 2005 J. Cosmol. Astropart. Phys. JCAP01(2005)006 (arXiv:astro-ph/0510443)
[35] Salem M P 2005 Phys. Rev. D 72 123516 (arXiv:astro-ph/0511146)
[36] Seery D and Lidsey J E 2007 J. Cosmol. Astropart. Phys. JCAP07(2007)008 (arXiv:astro-ph/0611034)
[37] Alabidi L and Lyth D 2006 J. Cosmol. Astropart. Phys. JCAP08(2006)006 (arXiv:astro-ph/0604569)
[38] Byrnes C T and Tasinato G 2009 Non-Gaussianity beyond slow roll in multi-field inflation J. Cosmol. Astropart. Phys. JCAP(2009)011 (arXiv:0906.0767)
[39] Byrnes C T, Choi K Y and Hall L M H 2008 J. Cosmol. Astropart. Phys. JCAP10(2008)008 (arXiv:0807.1101)
[40] Byrnes C T and Choi K Y 2010 arXiv:1002.3110
[41] Cogollo H R S, Rodriguez Y and Valenzuela Toledo C A 2008 J. Cosmol. Astropart. Phys. JCAP08(2008)029 (arXiv:0806.1546)
[42] Rodriguez Y and Valenzuela Toledo C A 2010 Phys. Rev. D 81 023531 (arXiv:0811.4092)
[43] Enqvist K, Jokinen A, Mazumdar A, Multamaki T and Vaihkonen A 2005 Phys. Rev. Lett. 94 161301 (arXiv:astro-ph/0411394)
[44] Jokinen A and Mazumdar A 2006 J. Cosmol. Astropart. Phys. JCAP04(2006)003 (arXiv:astro-ph/0512368)
[45] Chambers A and Rajantie A 2008 Phys. Rev. Lett. 100 041302 (arXiv:0710.4133)
[46] Chambers A and Rajantie A 2008 Phys. Rev. Lett. 101 149903 (erratum)
[47] Chambers A and Rajantie A 2008 J. Cosmol. Astropart. Phys. JCAP08(2008)002 (arXiv:0805.4795)
[48] Barnes N and Cline J M 2006 Phys. Rev. D 73 106012 (arXiv:astro-ph/0601481)
[49] Melrose D, Seery D and Wesley D 2009 arXiv:0911.3550.
[50] Tanaka T and Bassett B 2003 Proc. 12th Workshop on General Relativity and Gravitation, (arXiv:astro-ph/0302544)
[51] Suyama T and Yokoyama S 2007 Class. Quantum. Grav. 24 1615 (arXiv:astro-ph/0606228)
[52] Moroi T and Takahashi T 2001 Phys. Lett. B 522 215
[53] Moroi T and Takahashi T 2002 Phys. Lett. B 539 303 (arXiv:hep-ph/0110006) (erratum)
[54] Lyth D H and Wands D 2002 Phys. Rev. D 65 103508 (arXiv:astro-ph/0110322)
[55] Silk J and Turner M S 1987 Phys. Rev. D 35 419
[56] Linde A D 1994 Phys. Rev. D 49 748 (arXiv:astro-ph/9307002)
[57] Kadoya K and Stewart E D 2003 J. High Energy Phys. JHEP07(2003)013 (arXiv:hep-ph/0304127)