FAT WEDGE FILTRATIONS AND DECOMPOSITION OF POLYHEDRAL PRODUCTS

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Abstract. Bahri, Bendersky, Cohen, and Gitler [BBCG] gave a decomposition of a suspension of the polyhedral product \( Z_K(X,\, X) \), and its desuspension has been studied for special \( K \) in connection with Golodness in algebraic combinatorics. We introduce a filtration of the polyhedral product called the fat wedge filtration, and prove that the above desuspension is equivalent to the triviality of the fat wedge filtration of the real moment-angle complex \( \mathbb{R}Z_K \) for general \( K \). We further investigate a connection between the triviality of the fat wedge filtration of \( \mathbb{R}Z_K \) and the homotopy version of Golodness. We then pass to the special cases, and show that if \( K \) is dual sequentially Cohen-Macaulay over \( \mathbb{Z} \) or \( \lceil \dim K \rceil \)-neighborly, then the fat wedge filtration of \( \mathbb{R}Z_K \) is trivial, so the BBCG decomposition for these complexes desuspends.

1. Introduction

Let \( K \) be an abstract simplicial complex on the vertex set \( [m] := \{1, \ldots, m\} \), and let \((X,\, A)\) be a collection of pairs of spaces indexed by the vertices of \( K \). The space \( Z_K(X,\, A) \) which is now called the polyhedral product, is defined by the union of product spaces constructed from \((X,\, A)\) in accordance with the combinatorial information of \( K \). Polyhedral products were first found in Porter’s work on higher order Whitehead products [P] in 1965, and appear in several fundamental constructions in algebra, geometry, and topology related with combinatorics: the cohomology of \( Z_K(\mathbb{C}P^\infty,\, \ast) \) and \( Z_K(D^2,\, S^1) \) are identified with the Stanley-Reisner ring of \( K \) and its derived algebra, respectively [DJ, BBP, BP]; the fundamental group of \( Z_K(\mathbb{R}P^\infty,\, \ast) \) and \( Z_K(D^1,\, S^0) \) are the right-angled Coxeter group of the 1-skeleton of \( K \) and its commutator subgroup [DO]; the union of the coordinate subspace arrangement in \( \mathbb{R}^m \) associated with \( K \) is \( Z_K(\mathbb{R},\, \ast) \), and its complement has the homotopy type of \( Z_K(D^1,\, S^0) \) [GT1, IK1, BP]. From these examples, one sees that the special polyhedral products \( Z_K(CX,\, X) \) and \( Z_K(X,\, \ast) \) are especially important, where \((CX,\, X)\) and \((X,\, \ast)\) are collections of pairs of cones and their base spaces, and spaces and their basepoints, respectively. There is a homotopy fibration involving these polyhedral products, so they are supplementary to each other in a sense. The object to study in this paper is the polyhedral product \( Z_K(CX,\, X) \), and we are particularly interested in its homotopy type.

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Among other results on the homotopy types of polyhedral products, the work of Bahri, Bendersky, Cohen, and Gitler [BBCG] is remarkable. They proved a decomposition of a suspension of \( \mathcal{Z}_K(\mathbb{X}, A) \) in general, and specializing to the polyhedral product \( \mathcal{Z}_K(C\mathbb{X}, \mathbb{X}) \), they obtained the following decomposition, where the notations will be explained later.

**Theorem 1.1** (Bahri, Bendersky, Gitler, and Cohen [BBCG]). There is a homotopy equivalence

\[
\Sigma \mathcal{Z}_K(C\mathbb{X}, \mathbb{X}) \simeq \Sigma \bigsqcup_{\emptyset \neq I \subseteq [m]} |\Sigma K_I| \wedge \tilde{X}^I.
\]

Let us call the decomposition of this theorem the BBCG decomposition. The proof of the BBCG decomposition is a combination of the decomposition of suspensions of general polyhedral products which they obtained, and a formula of homotopy colimits [ZZ]. Unfortunately, from the original proof, one cannot seize the intrinsic nature of \( \mathcal{Z}_K(C\mathbb{X}, \mathbb{X}) \) which yields the BBCG decomposition, but the BBCG decomposition certainly showed a direction in studying the homotopy type of \( \mathcal{Z}_K(C\mathbb{X}, \mathbb{X}) \), that is, to describe the homotopy type by desuspending the BBCG decomposition. This direction of the study was proposed in [BBCG] when \( K \) is a special simplicial complex called a shifted complex: they conjectured that the previous result of Grbić and Theriault [GT1] on \( \mathcal{Z}_K(D^2, S^1) \) when \( K \) is a shifted complex, can be generalized to a desuspension of the BBCG decomposition. This conjecture was affirmatively resolved independently by Grbić and Theriault [GT2] and the authors [IK1], and these results were generalized to dual vertex-decomposable complexes by Welker and Grujić [GW] by using discrete Morse theory. However, the crucial part of the proofs of these results are over adapted to special properties of the simplicial complex \( K \), so the methods are not applicable to wider classes of simplicial complexes.

The first aim of this paper is to elucidate the intrinsic nature of the polyhedral product \( \mathcal{Z}_K(C\mathbb{X}, \mathbb{X}) \) for general \( K \) which yields the BBCG decomposition and its desuspension. The structure of \( \mathcal{Z}_K(C\mathbb{X}, \mathbb{X}) \) in question is a certain filtration which we call the *fat wedge filtration*. We will see that the BBCG decomposition is actually a consequence of the property of the fat wedge filtration such that it splits after a suspension, so the analysis of the fat wedge filtration naturally shows a way to desuspend the BBCG decomposition. In analyzing the fat wedge filtration, the special polyhedral product \( \mathcal{Z}_K(D^1, S^0) \) which is called the real moment-angle complex for \( K \) and is denoted by \( \mathbb{R}\mathcal{Z}_K \), plays the fundamental role. We will prove that the fat wedge filtration of \( \mathbb{R}\mathcal{Z}_K \) is a cone decomposition of \( \mathbb{R}\mathcal{Z}_K \), and will describe the attaching maps of its cones explicitly in a combinatorial manner. We say that the fat wedge filtration of \( \mathbb{R}\mathcal{Z}_K \) is trivial if all the attaching maps are null homotopic, and now state our first main result.

**Theorem 1.2.** The following conditions are equivalent:

1. The fat wedge filtration of \( \mathbb{R}\mathcal{Z}_K \) is trivial;
(2) For any \( X \), there is a homotopy equivalence

\[
\mathcal{Z}_K(CX, X) \simeq \bigvee_{0 \neq I \subseteq [m]} |\Sigma K_I| \wedge \hat{X}^I.
\]

The second aim of this paper is to examine the triviality of the fat wedge filtration of the real moment-angle complexes for specific simplicial complexes which implies the decomposition of polyhedral products by Theorem 1.2. To this end, we must choose appropriate classes of simplicial complexes. Note that if the BBCG decomposition desuspends, then \( \mathcal{Z}_K(CX, X) \) becomes a suspension, so in particular, all products and higher Massey products in the cohomology of \( \mathcal{Z}_K(D^2, S^1) \) are trivial. As mentioned above, the cohomology of \( \mathcal{Z}_K(D^2, S^1) \) is isomorphic to a certain derived algebra of the Stanley-Reisner ring of \( K \), and the triviality of products and higher Massey products of this derived algebra is called the Goodness of \( K \) which has been extensively studied in combinatorial commutative algebra [HRW, BJ, B]. Then to get the triviality of the fat wedge filtration of \( \mathbb{R}\mathcal{Z}_K \) which implies a desuspend the BBCG decomposition by theorem 1.2, the simplicial complex \( K \) must be Golod, so it is worth clarifying the relation between the triviality of the fat wedge filtration of \( \mathbb{R}\mathcal{Z}_K \) and the Goodness of \( K \) before getting into specific simplicial complexes. By using Hochster’s ring structure of the derived algebra that we are considering together with the result of Berglund and Jöllenbeck [BJ], the Goodness of \( K \) turns out to be equivalent to the triviality of the induced maps between certain complexes constructed from \( K \) in homology, so we can define the stronger notion called the homotopy Golodness by replacing the triviality of maps in homology with the triviality of maps up to homotopy. We will prove:

**Theorem 1.3.** If the fat wedge filtration of \( \mathbb{R}\mathcal{Z}_K \) is trivial, then \( K \) is homotopy Golod.

We now choose subclasses of Golod complexes. For shifted and dual vertex-decomposable complexes, desuspensions of the BBCG decomposition were studied in [GT1, GT2, IK1, GW], where dual shifted complexes are shifted. Originally, shifted and vertex-decomposable complexes were introduced as handy subclasses of shellable complexes in [BW], and shellable complexes form a subclass of sequentially Cohen-Macaulay (SCM, for short) complexes over \( \mathbb{Z} \) [S, BWW] which are a non-pure generalization of Cohen-Macaulay complexes. Then there are implications:

(1.1) shifted \( \Rightarrow \) vertex-decomposable \( \Rightarrow \) shellable \( \Rightarrow \) SCM over \( \mathbb{Z} \)

Then we first choose dual shellable complexes to show the triviality of the fat wedge filtrations of real moment-angle complexes, and then generalize its argument homologically to obtain the following result for dual SCM complexes over \( \mathbb{Z} \), which is a substantial improvement of the previous results [GT1, GT2, IK1, GW]. The theorem will be actually proved for a larger class of simplicial complexes including dual SCM complexes over \( \mathbb{Z} \).
Theorem 1.4. If $K$ is dual SCM over $\mathbb{Z}$, then the fat wedge filtration of $\mathbb{R} \mathcal{Z}_K$ is trivial.

We will prove that $|\Sigma K_I|$ has the homotopy type of a wedge of spheres for any $\emptyset \neq I \subset [m]$ if $K$ is dual SCM over $\mathbb{Z}$, so we obtain the following by Theorem 1.2 and 1.4.

Corollary 1.5. If $K$ is dual SCM over $\mathbb{Z}$, then $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres.

In the proof of the dual shellable version of Theorem 1.4, we can find a good property of dual shellable complexes which is not used for the proof. We will introduce new simplicial complexes, called extractible complexes, having a homological analogue of this property, and will show that the BBCG decomposition desuspends $p$-locally for extractible complexes under some conditions on $X$. This result is somewhat independent from other results in this paper since the fat wedge filtration is implicit.

Let $K$ be the 6-vertex triangulation of $\mathbb{R}P^2$ considered in [GPTW]. Then $K$ is not dual SCM, but $\mathcal{Z}_K(D^2, S^1)$ is shown to decompose into a wedge of sphere. Then in particular, $K$ is Golod, so one may guess that the fat wedge filtration of $\mathbb{R} \mathcal{Z}_K$ should be trivial. Since $K$ has the high neighborliness, we next choose simplicial complexes with high neighborliness to investigate the triviality of the fat wedge filtrations of the real moment-angle complexes, and we obtain the following, where, in fact, the theorem will be proved for a larger class of simplicial complexes including simplicial complexes with high neighborliness as well as dual SCM complexes.

Theorem 1.6. If $K$ is $\lceil \dim K/2 \rceil$-neighborly, then the fat wedge filtration of $\mathbb{R} \mathcal{Z}_K$ is trivial.

This paper is organized as follows. In Section 2 we define polyhedral products, and collect some of their examples and properties which will be used later. In Section 3 and 4, we introduce and study the fat wedge filtration of the polyhedral product $\mathcal{Z}_K(CX, X)$. The key is to show that the fat wedge filtration of $\mathbb{R} \mathcal{Z}_K$ is a cone decomposition such that the attaching maps of its cones are explicitly described in a combinatorial manner, which enables us to pass the topology of polyhedral products to the combinatorics of simplicial complexes. We see that the fat wedge filtration of $\mathcal{Z}_K(CX, X)$ is described by the attaching maps for $\mathbb{R} \mathcal{Z}_K$, and prove the implication (1) $\Rightarrow$ (2) of Theorem 1.2. In Section 5 we study a kind of generalizations of the J.H.C. Whitehead theorem, and apply it to show the implication (2) $\Rightarrow$ (1) of Theorem 1.2. Section 6 deals with a connection between the Golodness of simplicial complexes and polyhedral products, and shows that the triviality of the fat wedge filtration of $\mathbb{R} \mathcal{Z}_K$ is connected with a property of $K$ beyond the Golodness by proving Theorem 1.3. In Section 7 and 8, we give criteria, called the fillability and the homology fillability, for the triviality of the fat wedge filtration of $\mathbb{R} \mathcal{Z}_K$, and apply them to dual shellable complexes and dual sequentially Cohen-Macaulay complexes over $\mathbb{Z}$, proving Theorem 1.4. The criteria are given in terms of the combinatorics of $K$, which is possible because of the combinatorial description of the attaching
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maps for the fat wedge filtration of $\mathbb{R}Z_K$. Section 9 is a spin off of the arguments for dual shellable complexes in Section 8. We introduce a new simplicial complexes called extractible complexes, and prove a $p$-local desuspension of the BBCG decomoposition for them under some conditions on $X$. In Section 10 we give another criterion for the triviality of the fat wedge filtration of $\mathbb{R}Z_K$ in terms of a new notion called the total connectivity of $K$, and apply this criterion to prove Theorem 1.6. Finally in Section 11, we give a list of possible future problems on the fat wedge filtration of polyhedral products.

Throughout the paper, we use the following notations:

- Let $K$ be a simplicial complex on the vertex set $[m]$, where we put $[m] := \{1, \ldots, m\}$;
- Let $X$ be a sequence of spaces with non-degenerate basepoints $\{X_i\}_{i \in [m]}$;
- Put $(CX, X) := \{(CX_i, X_i)\}_{i \in [m]}$, pairs of reduced cones and their base spaces.
- If $(X, A)$ is a pair of spaces, the symbol $(X, A)$ also denotes its $m$-copies ambiguously.

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2. Definition of polyhedral products

In this section, we define polyhedral products, and recall a homotopy fibration involving polyhedral products that we will use.

Definition 2.1. Let $(X, A)$ be a sequence of pairs of spaces $\{(X_i, A_i)\}_{i \in [m]}$. The polyhedral product $Z_K(X, A)$ is defined by

$$Z_K(X, A) := \bigcup_{\sigma \in K} (X, A)^\sigma \subset X_1 \times \cdots \times X_m$$

where $(X, A)^\sigma = Y_1 \times \cdots \times Y_m$ for $Y_i = X_i$ and $A_i$ according as $i \in \sigma$ and $i \notin \sigma$.

The special polyhedral product $Z_K(D^1, S^0)$ is called the real moment-angle complex for $K$ as in the previous section. We here give two easy examples of polyhedral products.

Example 2.2. If $K$ is the simplicial complex with discrete $m$-points, then we have

$$Z_K(X, *) = X_1 \lor \cdots \lor X_m.$$ 

On the other hand, if $K$ is the boundary of the full $(m-1)$-simplex, then $Z_K(X, *)$ is the fat wedge of $X_1, \ldots, X_m$. More generally, if $K$ is the $k$-skeleton of the full $(m-1)$-simplex, then $Z_K(X, *)$ is the $(m-k)^{th}$ generalized fat wedge of $X_1, \ldots, X_m$.

Example 2.3. When $m = 2$ and $K$ is the boundary of the full 1-simplex, we have

$$Z_K(CX, X) = (CX_1 \times X_2) \cup (X_1 \times CX_2) = X_1 \ast X_2$$

where $X \ast Y$ means the join of $X$ and $Y$. For general $m$, if $K$ is the boundary of the full $(m-1)$-simplex, it is proved in [P] that there is a homotopy equivalence

$$Z_K(CX, X) \simeq \Sigma^{m-1}X_1 \land \cdots \land X_m.$$
which can be recovered by the results in [GT2, IK1]. If \( K \) is a skeleton of the full \((m-1)\)-simplex, the homotopy type of \( \mathcal{Z}_K(CX, X) \) can be described also by [GT2, IK1].

**Example 2.4.** We observe the polyhedral product of the joint of two simplicial complexes. We set notation. For simplicial complexes \( K_1, K_2 \) on disjoint vertex sets, their join is defined by

\[
K_1 \ast K_2 := \{ \sigma_1 \sqcup \sigma_2 \mid \sigma_1 \in K_1, \sigma_2 \in K_2 \}.
\]

Let \( I \) be a non-empty subset of \([m]\), and let \( K_I \) denote the full subcomplex of \( K \) on \( I \), that is, \( K_I := \{ \sigma \subset I \mid \sigma \in K \} \). For a sequence of pairs of spaces \( (X, A) = \{(X_i, A_i)\}_{i \in [m]} \), we put \( (X_I, A_I) := \{(X_i, A_i)\}_{i \in I} \). We can deduce the following immediately from the definition of polyhedral products. For \( \emptyset \neq I, J \subset [m] \) with \( I \cap J = \emptyset \) and \( I \cup J = [m] \), we have

\[
\mathcal{Z}_{K_{I \ast J}}(X, A) \cong \mathcal{Z}_{K_I}(X_I, A_I) \times \mathcal{Z}_{K_J}(X_J, A_J).
\]

Then we see that the polyhedral product \( \mathcal{Z}_K(CX, X) \) is not always a suspension: for example, if \( m = 4 \) and \( K \) is a square which is the join of 2-copies of the simplicial complex with discrete 2-points, we have \( \mathbb{R} \mathcal{Z}_K \cong S^1 \times S^1 \) by Example 2.3. This implies that the BBCG decomposition does not always desuspend.

We recall from [DS] a homotopy fibration involving polyhedral products, and we here produce an alternative proof.

**Lemma 2.5** (cf. [Fa, Proposition, pp.180]). Let \( \{ F_i \to E_i \to B \}_{i \in I} \) be a diagram of homotopy fibrations over a fixed base \( B \). Then

\[
\operatorname{hocolim}_{i \in I} F_i \to \operatorname{hocolim}_{i \in I} E_i \to B
\]

is a homotopy fibration.

**Proposition 2.6** (Denham and Suciu [DS]). There is a homotopy fibration

\[
\mathcal{Z}_K(C\Omega X, \Omega X) \to \mathcal{Z}_K(X, *) \xrightarrow{\text{incl}} X_1 \times \cdots \times X_m.
\]

**Proof.** For any \( \sigma \subset [m] \) there is a homotopy fibration \( (C\Omega X, \Omega X)^\sigma \to (X, *)^\sigma \xrightarrow{\text{incl}} X_1 \times \cdots \times X_m \) which is natural with respect to the inclusions of subsets of \([m]\). Then we have a diagram of homotopy fibrations \( \{(C\Omega X, \Omega X)^\sigma \to (X, *)^\sigma \xrightarrow{\text{incl}} X_1 \times \cdots \times X_m \}_{\sigma \in K} \), so it follows from Lemma 2.5 that there is a homotopy fibration

\[
\operatorname{hocolim}_{\sigma \in K} (C\Omega X, \Omega X)^\sigma \to \operatorname{hocolim}_{\sigma \in K} (X, *)^\sigma \to X_1 \times \cdots \times X_m.
\]

Since the maps \( (C\Omega X, \Omega X)^\sigma \to (C\Omega X, \Omega X)^\tau \) and \( (X, *)^\sigma \to (X, *)^\tau \) are cofibrations for all \( \sigma \subset \tau \subset [m] \), the above homotopy colimits are naturally homotopy equivalent to the colimits which are \( \mathcal{Z}_K(C\Omega X, \Omega X) \) and \( \mathcal{Z}_K(X, *) \), completing the proof. \( \square \)
In this section, we introduce the fat wedge filtration of $Z_K(CX, X)$ and investigate that of the real moment-angle complex $\mathbb{R}Z_K$. We first define the fat wedge filtration of a general subspace of a product of spaces. Recall that the generalized fat wedge of $X$ is defined by

$$T^k := \{(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m \mid \text{at least } m - k \text{ of } x_i \text{ are basepoints}\}$$

for $k = 0, \ldots, m$. Then we get a filtration

$$\ast = T^0 \subset T^1 \subset \cdots \subset T^m = X_1 \times \cdots \times X_m.$$ 

For a subspace $Y \subset X_1 \times \cdots \times X_m$, we put $Y^k := Y \cap T^k$ for $k = 0, \ldots, m$, so we get a filtration

$$\ast = Y^0 \subset Y^1 \subset \cdots \subset Y^m = Y$$

which is called the fat wedge filtration of $Y$.

We give a combinatorial description of the fat wedge filtration $\ast = \mathbb{R}Z^0_K \subset \mathbb{R}Z^1_K \subset \cdots \subset \mathbb{R}Z^m_K = \mathbb{R}Z_K$ of the real moment-angle complex $\mathbb{R}Z_K$, where we choose the point $-1$ to be the basepoint of $S^0 = \{-1, +1\}$. For any $\emptyset \neq I \subset [m]$ we identify $\mathbb{R}Z^i_K$ with the subspace $\{(x_1, \ldots, x_m) \in \mathbb{R}Z_K \mid x_i = -1 \text{ for } i \notin I\}$ of $\mathbb{R}Z_K$. Then by the definition of the fat wedge filtration, we have

$$\mathbb{R}Z^0_K = \{(-1, \ldots, -1)\} \quad \text{and} \quad \mathbb{R}Z^i_K = \bigcup_{I \subset [m], |I| = i} \mathbb{R}Z_K^i$$

for $i = 1, \ldots, m$. In order to describe the fat wedge filtration of $\mathbb{R}Z_K$ combinatorially, we employ the cubical decomposition of a simplicial complex presented in [BP], and we recall it here. To nested subsets $\sigma \subset \tau \subset [m]$, we assign the $(|\tau| - |\sigma|)$-dimensional face $C_{\sigma \subset \tau} := \{(x_1, \ldots, x_m) \in (D^1)^m \mid x_i = -1, +1 \text{ according as } i \in \sigma \text{ and } i \notin \tau\}$ of the cube $(D^1)^m$. Notice that any face of the cube $(D^1)^m$ is expressed by $C_{\sigma \subset \tau}$ for some $\sigma \subset \tau \subset [m]$, and in particular, any vertex of $(D^1)^m$ is given by $C_{\sigma \subset \sigma}$ for some $\sigma \subset [m]$. Let $\text{Sd}L$ denote the barycentric subdivision of a simplicial complex $L$. Then the vertices of $\text{Sd}\Delta^m$ are non-empty subsets of $[m]$, so we can define a piecewise linear map

$$i_c : |\text{Sd}\Delta^m| \to (D^1)^m, \quad \sigma \mapsto C_{\sigma \subset \sigma}$$

which is an embedding onto the union of $(m - 1)$-dimensional faces of $(D^1)^m$ including the vertex $(-1, \ldots, -1)$. This embedding is the cubical decomposition of $\Delta^m$, where one can see the reason for the name “cubical decomposition” from the following figure of $i_c$ for $m = 3$.

We define the cone and the suspension of $K$ by

$$\text{Cone}(K) := \Delta^1 \ast K \quad \text{and} \quad \Sigma K := \partial \Delta^m \ast K$$
as usual. By extending the embedding \( i_c \), we get a piecewise linear homeomorphism

\[
\text{Cone}(i_c) : |\text{Cone}(\text{Sd} \Delta^{[m]})| \to (D^1)^{\times m}
\]

which sends the cone point of \(|\text{Cone}(\text{Sd} \Delta^{[m]})|\) to the vertex \((+1, \ldots, +1) \in (D^1)^{\times m}\). Since the vertex set of \( K \) is \([m] \), \( K \) is a subcomplex of \( \Delta^{[m]} \). Then by restricting \( i_c \) and \( \text{Cone}(i_c) \), we obtain embeddings

\[
i_c : |\text{Sd} K| \to (D^1)^{\times m}, \quad \text{Cone}(i_c) : |\text{Cone}(\text{Sd} K)| \to (D^1)^{\times m}
\]

which are the cubical decompositions of \( K \) and \( \text{Cone}(K) \).

We express the difference \( \text{Cone}(i_c)(|\text{Cone}(\text{Sd} K)|) - i_c(|\text{Sd} K|) \) in terms of the faces \( C_{\sigma \subset \tau} \). For any \( \tau \subset [m] \) we have \( i_c(|\text{Sd}\tau|) = \bigcup_{\phi \neq \sigma \subset \tau} C_{\sigma \subset \tau} \) and \( \text{Cone}(i_c)(|\text{Cone}(\text{Sd}\tau)|) = \bigcup_{\sigma \subset \tau} C_{\sigma \subset \tau} \), so we get

\[
i_c(|\text{Sd} K|) = \bigcup_{\emptyset \neq \sigma \subset \tau \in K} C_{\sigma \subset \tau} \quad \text{and} \quad \text{Cone}(i_c)(|\text{Cone}(\text{Sd} K)|) = \bigcup_{\sigma \subset \tau \in K} C_{\sigma \subset \tau}.
\]

Then it follows that

\[
\text{Cone}(i_c)(|\text{Cone}(\text{Sd} K)|) - i_c(|\text{Sd} K|) = \bigcup_{\tau \in K} C_{\emptyset \subset \tau} - \bigcup_{\emptyset \neq \sigma \subset \tau \in K} C_{\sigma \subset \tau}.
\]

We next express \( \mathbb{R} \mathcal{Z}_K^I \) in terms of the faces \( C_{\sigma \subset \tau} \) as well, and show that the cubical decompositions of full subcomplexes of \( K \) naturally come into the fat wedge filtration of \( \mathbb{R} \mathcal{Z}_K \). We denote by \( (D^1_I, S^0_I) \) the \(|I|\)-copies of the pair \((D^1, S^0)\) for \( I \subset [m] \). Then for \( \mu \subset I \), we have \( (D^1_I, S^0_I)^\mu = \bigcup_{\sigma \subset \tau \subset I, \tau - \sigma = \mu} C_{\sigma \subset \tau} \), where \( C_{\sigma \subset \tau} \) is the face \( C_{\sigma \subset \tau} \) of \((D^1)^{\times I} \). We get

\[
\mathbb{R} \mathcal{Z}_K^I = \bigcup_{\mu \in K, \mu \subset I} (D^1_I, S^0_I)^\mu = \bigcup_{\mu \in K, \mu \subset I} \left( \bigcup_{\sigma \subset \tau \subset I, \tau - \sigma = \mu} C_{\sigma \subset \tau} \right) = \bigcup_{\sigma \subset \tau \subset I} C_{\sigma \subset \tau}
\]

and

\[
\mathbb{R} \mathcal{Z}_K^{|I|^{-1}} = \bigcup_{J \subset I, |J| = |I|^{-1}} \left( \bigcup_{\mu \in K, \mu \subset J} (D^1_I, S^0_I)^\mu \right) = \bigcup_{J \subset I, |J| = |I|^{-1}} \left( \bigcup_{\mu \in K, \mu \subset J} \left( \bigcup_{\sigma \subset \tau \subset J, \tau - \sigma = \mu} C_{\sigma \subset \tau} \right) \right) = \bigcup_{\mu \in K, \mu \subset I} \left( \bigcup_{\emptyset \neq \sigma \subset \tau \subset I, \tau - \sigma = \mu} C_{\sigma \subset \tau} \right) = \bigcup_{\emptyset \neq \sigma \subset \tau \subset I, \tau - \sigma \in K} C_{\sigma \subset \tau}.
\]
Then by (3.2), the embedding Cone($i_\circ$): $|\text{Cone}(|\text{Sd}K_I|)| \rightarrow (D^1)^{x_I}$ descends to a map

\begin{equation}
(3.5) \quad (\text{Cone}(|\text{Sd}K_I|), |\text{Sd}K_I|) \rightarrow (\mathbb{R}Z_{K_I}, \mathbb{R}Z_{K_I}^{\|I\|-1}).
\end{equation}

Moreover since

\[ \mathbb{R}Z_{K_I} - \mathbb{R}Z_{K_I}^{\|I\|-1} = \bigcup_{0 \subseteq \tau \subseteq I, \tau \in K} C^{\|I\|}_\tau - \bigcup_{\emptyset \neq \sigma \subseteq \tau \in K} C^{\|I\|}_\sigma = \text{Cone}(i_\circ)(|\text{Cone}(|\text{Sd}K_I|)|) - i_\circ(|\text{Sd}K_I|) \]

by (3.3), the map (3.5) is actually a relative homeomorphism. Then for

\[ \mathbb{R}Z^i_{K_I} - \mathbb{R}Z^{i-1}_{K_I} = \coprod_{I \subseteq [m], |I| = i} (\mathbb{R}Z_{K_I} - \mathbb{R}Z_{K_I}^{\|I\|-1}), \]

the disjoint union of the maps (3.5)

\[ \coprod_{I \subseteq [m], |I| = i} (\text{Cone}(|\text{Sd}K_I|), |\text{Sd}K_I|) \rightarrow (\mathbb{R}Z^i_{K_I}, \mathbb{R}Z^{i-1}_{K_I}) \]

turns out to be a relative homeomorphism. Let $\varphi_{K_I}$ denote the map $|\text{Sd}K_I| \rightarrow \mathbb{R}Z^{\|I\|-1}_{K_I}$ in (3.5). Then we have established:

**Theorem 3.1.** For $i = 1, \ldots, m$, $\mathbb{R}Z^i_{K_I}$ is obtained from $\mathbb{R}Z^{i-1}_{K_I}$ by attaching a cone to each $j_{K_I} \circ \varphi_{K_I}$ for all $I \subset [m]$ with $|I| = i$, where $j_{K_I}: \mathbb{R}Z^{i-1}_{K_I} \rightarrow \mathbb{R}Z^{i-1}_{K_I}$ is the inclusion.

The above theorem shows that the fat wedge filtration of $\mathbb{R}Z_K$ is a cone decomposition in the usual sense. We say that the fat wedge filtration of $\mathbb{R}Z_K$ is trivial if the maps $\varphi_{K_I}$ are null homotopic for all $\emptyset \neq I \subset [m]$. Since $\mathbb{R}Z^{\|I\|-1}_{K_I}$ is a retract of $\mathbb{R}Z^{\|I\|-1}_{K_I}$, this is equivalent to the composite $j_{K_I} \circ \varphi_{K_I}$ is null homotopic for any $\emptyset \neq I \subset [m]$. We here consider two cases in which the fat wedge filtration of $\mathbb{R}Z_K$ is trivial. We first consider the flag complex of a chordal graph as in \cite{GPTW}. Here graphs mean one dimensional simplicial complexes, and the flag complex of a graph $\Gamma$ is the simplicial complex whose $n$-simplices are complete graphs with $n + 1$ vertices in $\Gamma$. Recall that a graph is called chordal if its minimal cycles are of length at most 3.

**Proposition 3.2.** If $K$ is the flag complex of a chordal graph, then the fat wedge filtration of $\mathbb{R}Z_K$ is trivial.

**Proof.** Suppose $K$ is the flag complex of a graph $\Gamma$. It is known that $\Gamma$ is chordal if and only if each component of $K$ is contractible. Then since $\mathbb{R}Z^{n-1}_K$ is path-connected for any $\emptyset \neq I \subset [m]$, $\varphi_K$ is null homotopic. For any $\emptyset \neq I \subset [m]$, the full subgraph $\Gamma_I$ is chordal, and $K_I$ is the flag complex of $\Gamma_I$. Then we similarly obtain that $\varphi_{K_I}$ is null homotopic for any $\emptyset \neq I \subset [m]$. \qed

We next consider the case $\dim K \geq m - 2$. We start with observing properties of the map $\varphi_K$ for general $K$. In \cite{IK2}, it is proved that the inclusion $\Sigma \mathbb{R}Z^{\|I\|-1}_{K_I} \rightarrow \Sigma \mathbb{R}Z_{K_I}$ admits a left homotopy inverse for $\emptyset \neq I \subset [m]$. Then by Theorem 3.1, we have the following.

**Proposition 3.3.** The maps $\Sigma \varphi_{K_I}$ are null homotopic for all $\emptyset \neq I \subset [m]$. 

For a simplex $\sigma$ of $K$, we denote the deletion and the link of $\sigma$ by $dl_K(\sigma)$ and $lk_K(\sigma)$, that is, $dl_K(\sigma) = \{ \tau \subset \{m\} \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in K \}$. Since
\[
\mathbb{R}Z_K^{m-1} = (\mathbb{R}Z_{dl_K(v)}^{m-2} \times \mathbb{R}Z_{lk_K(v)}^{m-2} \times \{1\}) \cup (\mathbb{R}Z_{dl_K(v)}^{m-2} \times D^1)
\]
for a vertex $v$ of $K$, there is a projection $\mathbb{R}Z_K^{m-1} \to \Sigma\mathbb{R}Z_{lk_K(v)}^{m-2}$. Then it is straightforward to check that through the identification $|\text{Sd}K|/|\text{Sd}(dl_K(v))| = \Sigma|\text{Sd}(lk_K(v))|$, we have a commutative diagram
\[
\begin{array}{ccc}
|\text{Sd}K| & \xrightarrow{\varphi_K} & \mathbb{R}Z_K^{m-1} \\
\text{proj} & & \text{proj} \\
\Sigma|\text{Sd}(lk_K(v))| & \xrightarrow{\Sigma\varphi_{lk_K(v)}} & \Sigma\mathbb{R}Z_{lk_K(v)}^{m-2}
\end{array}
\]
So by Proposition 3.3, we get:

**Corollary 3.4.** The composite $|\text{Sd}K| \xrightarrow{\varphi_K} \mathbb{R}Z_K^{m-1} \xrightarrow{\text{proj}} \Sigma\mathbb{R}Z_{lk_K(v)}^{m-2}$ is null homotopic.

**Proposition 3.5.** If $\dim K \leq m - 2$, then the fat wedge filtration of $\mathbb{R}Z_K$ is trivial.

**Proof.** For $\dim K \leq m - 2$, there is a vertex $v$ of $K$ such that $dl_K(v)$ is the full simplex $\Delta^{[m]-v}$, implying $\mathbb{R}Z_{dl_K(v)}$ is contractible. So the projection $\mathbb{R}Z_K^{m-1} \to \Sigma\mathbb{R}Z_{lk_K(v)}^{m-2}$ is a homotopy equivalence, hence $\varphi_K$ is null homotopic by Corollary 3.4. Since $\dim K \geq |I| - 2$ for all $\emptyset \neq I \subset [m]$, the map $\varphi_K$ is null homotopic for each $\emptyset \neq I \subset [m]$ by the above observation. \(\square\)

### 4. Fat wedge filtration of $Z_K(CX, X)$

In this section, we investigate the fat wedge filtration of $Z_K(CX, X)$ by using the maps $\varphi_{K_i}$ obtained in the previous section, and we prove the implication (1) $\Rightarrow$ (2) of Theorem 1.2.

As well as the real moment-angle complexes, we may regard $Z_{K_I}(CX_I, X_I)$ for $\emptyset \neq I \subset [m]$ as a subspace of $Z_K(CX, X)$ since every $X_i$ has a basepoint, so we have
\[
Z^0_K(CX, X) = \ast \quad \text{and} \quad Z^i_K(CX, X) = \bigcup_{I \subset [m], |I| = i} Z_{K_I}(CX_I, X_I)
\]
for $i = 1, \ldots, m$. We describe $Z_{K_I}(CX_I, X_I)$ by using the map $\varphi_{K_I}$. Let $I = \{j_1 < \cdots < j_i\}$ be a subset of $[m]$ and put $X^{\times I} = X_{j_1} \times \cdots \times X_{j_i}$. Consider the composite of maps
\[
(4.1) \quad |\text{Cone}(\text{Sd}K_I)| \times X^{\times I} \xrightarrow{\text{Cone}(i_I) \times 1} \mathbb{R}Z_{K_I} \times X^{\times I} \to CX_{j_1} \times \cdots \times CX_{j_i}
\]
where the second arrow maps $((t_1, \ldots, t_i); (x_1, \ldots, x_i))$ to $((t_1, x_1); \ldots; (t_i, x_i))$ for $t_k \in D^1$, $x_k \in X_{i_k}$. One easily deduces that the composite descends to a surjection
\[
\Phi_{K_I} : |\text{Cone}(\text{Sd}K_I)| \times X^{\times I} \to Z_{K_I}(CX_I, X_I)
\]
which is homeomorphic on \(|\text{Cone}(\text{Sd}K_I)| \times X^{|I|} - \Phi_{K_I}^{-1}(Z_{K_I}^{i-1}(CX, X)),\) and since we are using reduced cones, we have

\[
\Phi_{K_I}^{-1}(Z_{K_I}^{i-1}(CX, X)) = (|\text{Cone}(\text{Sd}K_I)| \times T^{i-1}(X_I)) \cup (|\text{Sd}K_I| \times X^{|I|}).
\]

So we obtain a relative homeomorphism

\[
\Phi_{K_I}: (\text{Cone}(\text{Sd}K_I), |\text{Sd}K_I|) \times (X^{|I|}, T^{i-1}(X_I)) \to (Z_{K_I}(CX, X), Z_{K_I}^{i-1}(CX, X))
\]

where a product of pairs of spaces are given by \((A, B) \times (C, D) = (A \times B, (A \times D) \cup (B \times C))\) as usual. Then since

\[
Z_K^i(CX, X) = Z_K^{i-1}(CX, X) = \prod_{I \subseteq [m], |I| = i} (Z_K(CX_I, X_I) - Z_K^{i-1}(CX_I, X_I)),
\]

we obtain the following.

**Theorem 4.1.** The map

\[
\prod_{I \subseteq [m], |I| = i} \Phi_{K_I}: \prod_{I \subseteq [m], |I| = i} (\text{Cone}(\text{Sd}K_I), |\text{Sd}K_I|) \times (X^{|I|}, T^{i-1}(X_I)) \to (Z_K^i(CX, X), Z_K^{i-1}(CX, X))
\]

is a relative homeomorphism.

Recall that a categorical sequence of a space \(Y\) in the sense of Fox [Fo] is a filtration \(* = Y_0 \subset Y_1 \subset \cdots \subset Y_m = Y\) such that the inclusion \(Y_i - Y_{i-1} \to Y\) is null homotopic for \(i = 1, \ldots, m\). By the above theorem one can easily deduce that the fat wedge filtration of \(Z_K(CX, X)\) is a categorical sequence whereas the fat wedge filtration of \(\mathbb{R}Z_K\) is a cone decomposition.

One can reprove Theorem 1.1 by using Theorem 4.1, from which one can interpret more directly how full subcomplexes of \(K\) appear in the BBCG decomposition.

**Corollary 4.2** (Bahri, Bendersky, Cohen, and Gitler [BBCG]). There is a homotopy equivalence

\[
\Sigma Z_K(CX, X) \simeq \Sigma \bigvee_{\emptyset \neq I \subseteq [m]} |\Sigma K_I| \wedge \hat{X}^I
\]

which is natural with respect to \(X\) and inclusions of subcomplexes of \(K\), where \(\hat{X}^I = \bigwedge_{i \in I} X_i\).

**Proof.** Note that for \(\emptyset \neq I \subseteq [m], Z_{K_I}(CX_I, X_I)\) is a retract of \(Z_K(CX, X)\) such that for \(\emptyset \neq J \subseteq [m]\) there is a commutative diagram

\[
\begin{array}{ccc}
Z_{K_I}(CX_I, X_I) & \xrightarrow{\text{incl}} & Z_{K_{I \cup J}}(CX_{I \cup J}, X_{I \cup J}) \\
\downarrow \text{proj} & & \downarrow \text{proj} \\
Z_{K_{I \cap J}}(CX_{I \cap J}, X_{I \cap J}) & \xrightarrow{\text{incl}} & Z_{K_J}(CX_J, X_J).
\end{array}
\]

Then \(\{Z_{K_I}(CX_I, X_I)\}_{I \subseteq [m]}\) is a space over the poset \(2^{[m]}\) with natural retractions in the sense of [IK2], where \(2^{[m]}\) is the power set of \([m]\) with the inclusion ordering and we put \(Z_{K_{\emptyset}}(CX_{\emptyset}, X_{\emptyset})\) to be a point. Then the theorem follows from [IK2]. \(\square\)
Remark 4.3. The BBCG decomposition is obtained also by the retractile argument of James [J], but it is hard to get the naturality by it.

From the description of the fat wedge filtration of \( \mathbb{R}Z_K(CX, X) \) in (4.1) and Theorem 4.1, one sees that the attaching maps \( \varphi_{K_i} \) of the cone decomposition of \( \mathbb{R}Z_K \) control the fat wedge filtration of \( \mathbb{R}Z_K(CX, X) \). We further investigate this control in the extreme case that the fat wedge filtration of \( \mathbb{R}Z_K \) is trivial, that is, we prove the implication (1) \( \Rightarrow \) (2) of Theorem 1.2. We prepare a technical lemma.

Lemma 4.4. Let \((X, A), (Y, B)\) be NDR pairs. Suppose that \( Y \) has the quotient topology by a relative homeomorphism \( f: (X, A) \to (Y, B) \), and that the restriction \( f|_A \) is null homotopic in \( B \). Then there is a string of homotopy equivalences

\[
Y \cong D \cong B \vee Y/B
\]

which is natural with respect to the relative homeomorphism between NDR pairs satisfying the same condition and having compatible homotopies.

Proof. Let \( D \) be the double mapping cylinder of \( f|_A \) and the inclusion \( A \to X \). Since \((X, A)\) and \((Y, B)\) are NDR pairs and \( f|_A \) is null homotopic in \( B \), there is a string of homotopy equivalences \( B \vee X/A \cong D \cong B \cup_f X \). Since \( Y \) is given the quotient topology by \( f \), we have \( X/A \cong Y/B \) and \( B \cup_f X \cong Y \). Then we obtain the desired string. The naturality of the string follows from the naturality of double mapping cylinders. \( \square \)

We first consider the special case that the basepoint of \( X_i \) is isolated for each \( i \).

Lemma 4.5. If the fat wedge filtration of \( \mathbb{R}Z_K \) is trivial and the basepoint \( *_i \) of \( X_i \) is isolated for each \( i \), then there is a string of homotopy equivalences

\[
\mathbb{Z}_K(CX, X) \xleftarrow{\delta} D(X) \xrightarrow{\cdot} \bigvee_{\emptyset \neq I \subseteq [m]} |\Sigma K_I| \wedge \hat{X}^I
\]

which is natural with respect to \( X \).

Proof. The lemma follows from Theorem 4.1 and Lemma 4.4 if we show that the restriction

\[
(|\text{Cone}(SdK_I)| \times T_{|I|-1}(X_I)) \cup (|SdK_I| \times X^I) \to Z_{K_I}^{[|I|-1]}(CX_I, X_I)
\]

of \( \Phi_{K_i} \) is null homotopic for all \( \emptyset \neq I \subseteq [m] \) by a homotopy which is natural with respect to \( X \). Since all basepoints are isolated, we have

\[
(|\text{Cone}(SdK_I)| \times T_{|I|-1}(X_I)) \cup (|SdK_I| \times X^I) = (|\text{Cone}(SdK_I)| \times T_{|I|-1}(X_I)) \cup (|SdK_I| \times \hat{X}^I)
\]

where \( \hat{X}^I = (X_{j_1} - *_{j_1}) \times \cdots \times (X_{j_i} - *_{j_i}) \) and \( I = \{j_1 < \cdots < j_i\} \). Then we can consider the restriction of \( \Phi_{K_i} \) to \(|\text{Cone}(SdK_I)| \times T_{|I|-1}(X_I)\) and \(|SdK_I| \times \hat{X}^I\) independently. By deforming \(|\text{Cone}(SdK_I)|\) to its cone point, the restriction of \( \Phi_{K_i} \) to \(|\text{Cone}(SdK_I)| \times T_{|I|-1}(X_I)\)
is naturally homotopic to the inclusion \( T^{[I]-1}(X_I) \to Z^{[I]-1}_{K_I}(CX_I, X_I) \) which is also naturally null homotopic. Note that the restriction of \( \Phi_{K_I} \) to \( |SdK_I| \times \hat{X}_I \) factors so that

\[
|SdK_I| \times \hat{X}_I \xrightarrow{\varphi_{K_I} \times 1} \mathbb{R}Z^{[I]-1}_{K_I} \times \hat{X}_I \to Z^{[I]-1}_{K_I}(CX_I, X_I)
\]

by the construction of \( \Phi_{K_I} \). Then by assumption it is naturally homotopic to the inclusion \( \hat{X}_I \to Z^{[I]-1}_{K_I}(CX_I, X_I) \) which is also naturally null homotopic. Therefore the proof is completed. \( \square \)

We now prove the implication (1) \( \Rightarrow \) (2) of Theorem 1.2.

**Theorem 4.6.** If the fat wedge filtration of \( \mathbb{R}Z_K \) is trivial, then the fat wedge filtration of \( Z_K(CX, X) \) splits for any \( X \) so that there is a homotopy equivalence

\[
Z_K(CX, X) \simeq \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge \hat{X}_I.
\]

**Proof.** For a space \( A \), we always assume that the basepoint of \( A \sqcup * \) is * even when \( A \) itself has a basepoint. We define \( X^k = \{ X^k_i \}_{i \in [m]} \) by

\[
X^k_i := \begin{cases} X_i & \text{if } i \leq k, \\ X_i \sqcup * & \text{if } i > k \end{cases}
\]

for \( k = 0, \ldots, m \), where we may allow \( X_i = \emptyset \) for \( i > k \). We also define \( X^{(k)} = \{ X^{(k)}_i \}_{i \in [m]} \) and \( X^{[k]} = \{ X^{[k]}_i \}_{i \in [m]} \) by

\[
X^{(k)}_i := \begin{cases} X^k_i & \text{if } i \neq k + 1, \\ *_{k+1} \sqcup * & \text{if } i = k + 1 \end{cases}
\]

and \( X^{[k]}_i := \begin{cases} X^k_i & \text{if } i \neq k + 1, \\ *_{k+1} & \text{if } i = k + 1 \end{cases} \)

for \( k = 0, \ldots, m \), where * is the basepoint of \( X_i \) as above. Note that \( X^{(k)} \) and \( X^{[k]} \) are the special cases of \( X^k \) when \( X_{k+1} \) is \( *_{k+1} \) and \( \emptyset \), respectively. Note also that a map \( f: X \to Y \) induces maps \( f^k: X^k \to Y^k \), \( f^{(k)}: X^{(k)} \to Y^{(k)} \) and \( f^{[k]}: X^{[k]} \to Y^{[k]} \). Let \( \iota: X^{(k)} \to X^k \) and \( \pi: X^{(k)} \to X^{[k]} \) denote the inclusion and the projection, respectively. Then \( \iota \) and \( \pi \) are natural with respect to \( X \), that is, \( f^k \circ \iota = \iota \circ f^{(k)} \) and \( f^{[k]} \circ \pi = \pi \circ f^{(k)} \).

Put \( W_K(X) = \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge \hat{X}_I \). We construct a string of homotopy equivalences \( Z_K(CX^k, X^k) \xleftarrow{\delta^k} D(X^k) \xrightarrow{\epsilon^k} W_K(X^k) \) which is natural with respect to \( X \) by induction on \( k \), so we obtain the desired homotopy equivalence when \( k = m \) since \( X^m = X \). We define the string of homotopy equivalences for \( k = 0 \) by Lemma 4.5 which is natural in \( X \). Suppose that we have constructed a string of homotopy equivalences \( Z_K(CX^k, X^k) \xleftarrow{\delta^k} D(X^k) \xrightarrow{\epsilon^k} W_K(X^k) \)
which is natural in \( X \). Then there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{Z}_K(CX^{[k]}, X^{[k]}) & & \mathcal{Z}_K(CX^{(k)}, X^{(k)}) \\
\downarrow \delta^k & & \downarrow \delta^k \\
D(X^{[k]}) & & D(X^{(k)}) \\
\downarrow \epsilon^k & & \downarrow \epsilon^k \\
\mathcal{W}_K(X^{[k]}) & & \mathcal{W}_K(X^{(k)}) \\
\end{array}
\]

which is natural with respect to \( X \). Observe that the pushouts of the top and the bottom rows are \( \mathcal{Z}_K(CX^{k+1}, X^{k+1}) \) and \( \mathcal{W}_K(X^{k+1}) \), respectively. We put \( D(X^{k+1}) \) to be the double mapping cylinder of the middle row and a string of maps \( \mathcal{Z}_K(CX^{k+1}, X^{k+1}) \) by the induced maps. Then since the maps \( \iota \) in the top and the bottom rows are cofibrations, it follows from the standard argument on double mapping cylinders that \( \epsilon^{k+1} \) and \( \delta^{k+1} \) are homotopy equivalences. Moreover, since the diagram is natural with respect to \( X \), so is the new string also. Therefore the induction proceeds. \( \square \)

5. Generalization of the J.H.C. Whitehead theorem

This section studies a generalization of the J.H.C. Whitehead theorem for non-simply connected spaces, which is of independent interest, and prove the implication (2) \( \Rightarrow \) (1) of Theorem 1.2, which together with Theorem 4.6 completes the proof of Theorem 1.2. The J.H.C. Whitehead theorem states that for simply connected CW-complexes \( X, Y \), if a map \( f: X \to Y \) is an isomorphism in homology, then \( f \) is a homotopy equivalence, or equivalently, the induced map in the homotopy sets

\[
f_*: [A, X] \to [A, Y]
\]

is bijective for any space \( A \). There are several generalization of the J.H.C. Whitehead theorem such as Dror’s generalization to nilpotent spaces [D]. Here we generalize the J.H.C. Whitehead theorem to maps inducing isomorphisms in fundamental groups and homology, which we call quasi-equivalences. We motivate ourselves by the following example.

**Example 5.1.** Let \( X := S^1 \vee S^2 \), and consider a map

\[
f: X \to X, \quad \alpha \mapsto \alpha, \quad \beta \mapsto 2\beta - \beta^\alpha
\]

where \( \alpha, \beta \) are generators of \( \pi_1(X) \cong \mathbb{Z} \) and \( \pi_2(X) \cong \mathbb{Z} \), respectively, and \( \beta^\alpha \) means the action of \( \alpha \) on \( \beta \). Then \( f \) is obviously a quasi-equivalence. Let \( \tilde{X} \) be the universal cover of \( X \). Then \( \pi_2(\tilde{X}) \) is a free abelian group with basis \( \{ \tilde{\beta}^n \}_{n \in \mathbb{Z}} \), where \( \tilde{\beta} \) is a lift of \( \beta \). In particular, a lift \( \tilde{f}: \tilde{X} \to \tilde{X} \) of \( f \) satisfies

\[
\tilde{f}_* (\tilde{\beta}^n) = 2\tilde{\beta}^n - \tilde{\beta}^{(n+1)\alpha}
\]
in $\pi_2$ which is injective but is not surjective.

Then as a generalization of the J.H.C. Whitehead theorem, we prove the injectivity of the induced map (5.1) of a quasi-equivalence between wedges of circles and simply connected co-H-spaces. To this end, we recall the decomposition of co-H-spaces due to Wilkerson [W]. We say that a space $X$ is wedge irreducible if $X$ is a retract of $Y \lor Z$, then $X$ is a retract of $Y$ or $Z$ by the composite maps.

**Lemma 5.2** (Wilkerson [W]). *If $X$ is a simply connected co-H-space having the homotopy type of a finite CW-complex, there is a homotopy equivalence

$$X_{(p)} \simeq X_1 \lor \cdots \lor X_n$$

such that each $X_i$ is wedge irreducible, where $-(p)$ denotes the $p$-localization.*

Let $S$ be a wedge of circles and $Y$ be a simply connected co-H-space which is a finite CW-complex. Suppose $f: S \lor Y \to S \lor Y$ is an isomorphism in $\pi_1$ and mod $p$ homology. One can easily modify $f$ to be the identity maps in $\pi_1$ and mod $p$ homology by composing with a self-homotopy equivalence of $S \lor Y$. Then we may assume that $f$ induces the identity maps in $\pi_1$ and mod $p$ homology. We consider the $p$-localization $\tilde{f}: Y \to \tilde{X}$ of the restriction of $f$ at the prime $p$, where $\tilde{X}$ is the universal cover of $S \lor Y$. Note that $\tilde{X}$ is homotopy equivalent to $\bigvee_{\omega \in \pi_1(S)} Y^\omega$ such that the composite of the covering map $\tilde{X} \to S \lor Y$ and the projection $S \lor Y \to Y$ is identified with the folding map $\bigvee_{\omega \in \pi_1(S)} Y^\omega \to Y$ which sends $Y^\omega$ identically onto $Y$, where $Y^\omega$ is the deck transform of $Y$ by $\omega$ and $\pi_1(S)$ is a free group. Since $Y$ is a finite CW-complex, the lift $\tilde{f}$ factors through $V := \bigvee_{i=1}^q Y^{\omega_i} \subset \bigvee_{\omega \in \pi_1(S)} Y^\omega \simeq \tilde{X}$ for some $\omega_1, \ldots, \omega_q \in \pi_1(S)$. Then we regard $\tilde{f}$ as a map into $V$. Let us construct a $p$-local space $Z$ and a map $\pi: V_{(p)} \to Z$ such that the composite $\pi \circ \tilde{f}_{(p)}$ is a homotopy equivalence. It follows from Lemma 5.2 that there is a homotopy equivalence

$$Y_{(p)} \simeq Y_1 \lor \cdots \lor Y_n$$

such that each $Y_i$ is wedge irreducible. Then the composite

$$Y_i \xrightarrow{\tilde{f}_{(p)}} V_{(p)} \xrightarrow{\text{proj}} \bigvee_{j=1}^q Y_i^{\omega_j} \xrightarrow{\bigvee} Y_i$$

is an isomorphism in homology since so is $f$, where $\bigvee$ is the folding map. So this composite is a homotopy equivalence since $Y_i$ is a simply connected CW-complex. Thus for some $1 \leq j_i \leq q$, the composite

$$Y_i \xrightarrow{\tilde{f}_{(p)}} V_{(p)} \xrightarrow{\text{proj}} \bigvee_{j=1}^q Y_i^{\omega_j} \xrightarrow{\text{proj}} Y_i^{\omega_{j_i}} \simeq Y_i$$

is a homotopy equivalence by the wedge irreducibility of $Y_i$. Therefore we obtain:
**Proposition 5.3.** There exist $\mu_1, \ldots, \mu_n \in \pi_1(S)$ such that the composite

$$Y_{(p)} \xrightarrow{\tilde{f}_{(p)}} \tilde{X}_{(p)} \xrightarrow{\text{proj}} Y_1^{\mu_1} \vee \cdots \vee Y_n^{\mu_n} \simeq Y_{(p)}$$

is a homotopy equivalence.

We prepare a technical lemma on localization.

**Lemma 5.4** (Bousfield and Kan [BK, fracture square lemma 6.3]). Let $X$ be a connected finite CW-complex and let $Y$ be a connected nilpotent CW-complex of finite type. If maps $f, g : X \to Y$ satisfy $f_{(p)} \simeq g_{(p)}$ for any prime $p$, then $f \simeq g$.

We now state a generalization of the J.H.C. Whitehead theorem.

**Theorem 5.5.** Let $S_i$ be a wedge of circles and $Y_i$ be a simply connected co-H-space which is a CW-complex of finite type for $i = 1, 2$. If $f : S_1 \vee Y_1 \to S_2 \vee Y_2$ is a quasi-equivalence, then for any connected finite complex $A$ and maps $g_1, g_2 : A \to S_1 \vee Y_1$ which are trivial in $\pi_1$, $f \circ g_1 \simeq f \circ g_2$ implies $g_1 \simeq g_2$.

**Proof.** By assumption $S_1 \vee Y_1$ and $S_2 \vee Y_2$ are homotopy equivalent, not necessarily by $f$, so we may assume that $S_1 = S_2$ and $Y_1 = Y_2$. Put $S := S_1$ and $Y := Y_1$. Since $A$ is a finite CW-complex, $g_i$ factors through a finite dimensional skeleton of $S \vee Y$, so we may assume $Y$ is a finite CW-complex since $Y$ is of finite type. Since $g_i$ is trivial in $\pi_1$, it lifts to $\tilde{g}_i : A \to \tilde{X}$, where $\tilde{X}$ is the universal cover of $S \vee Y$. As well as above, $\tilde{X}$ is homotopy equivalent to $\bigvee_{\omega \in \pi_1(S)} Y^\omega$ and the lift $\tilde{f} : A \to \tilde{X}$ factors through $Y^{\omega_1} \vee \cdots \vee Y^{\omega_q}$ for some $\omega_1, \ldots, \omega_q \in \pi_1(S)$.

Localize at the prime $p$. By Lemma 5.2 there is a homotopy equivalence

$$Y_{(p)} \simeq Y_1 \vee \cdots \vee Y_n$$

such that each $Y_i$ is wedge irreducible. It follows from Lemma 5.3 that there are $\mu_1, \ldots, \mu_n$ such that the composite

$$F : Y_{(p)} \xrightarrow{\tilde{f}_{(p)}} \tilde{X}_{(p)} \xrightarrow{\text{proj}} Y_1^{\mu_1} \vee \cdots \vee Y_n^{\mu_n} \simeq Y_{(p)}$$

is a homotopy equivalence. Let $F_j$ be the composite

$$Y^{\omega_1}_{(p)} \vee \cdots \vee Y^{\omega_q}_{(p)} \xrightarrow{\tilde{f}_{(p)}} \tilde{X} \xrightarrow{\text{proj}} \bigvee_{k=1}^j (Y_1^{\mu_1 \omega_k} \vee \cdots \vee Y_n^{\mu_n \omega_k}) \simeq Y^{\omega_1}_{(p)} \vee \cdots \vee Y^{\omega_j}_{(p)}$$

for $j = 1, \ldots, q$. Then there is a homotopy commutative diagram

$$
\begin{array}{ccc}
Y^{\omega_1}_{(p)} \vee \cdots \vee Y^{\omega_{j-1}}_{(p)} & \xrightarrow{\text{incl}} & Y^{\omega_1}_{(p)} \vee \cdots \vee Y^{\omega_j}_{(p)} \\
\downarrow F_{j-1} & & \downarrow F_j \\
Y^{\omega_1}_{(p)} \vee \cdots \vee Y^{\omega_{j-1}}_{(p)} & \xrightarrow{\text{incl}} & Y^{\omega_1}_{(p)} \vee \cdots \vee Y^{\omega_j}_{(p)}
\end{array}
$$
On the other hand, the composite
\[ Y_{(p)}^{\omega_j} \xrightarrow{\text{incl}} Y_{(p)}^{\omega_1} \vee \cdots \vee Y_{(p)}^{\omega_j} \xrightarrow{F_j} Y_{(p)}^{\omega_j} \]
is identified with the map \( F \) in (5.2) through the deck transformation by \( \omega_j \), so the diagram
\[
\begin{array}{ccc}
Y_{(p)}^{\omega_1} \vee \cdots \vee Y_{(p)}^{\omega_j} & \xrightarrow{\text{proj}} & Y_{(p)}^{\omega_j} \\
\downarrow F_j & & \downarrow F \\
Y_{(p)}^{\omega_1} \vee \cdots \vee Y_{(p)}^{\omega_j} & \xrightarrow{\text{proj}} & Y_{(p)}^{\omega_j}
\end{array}
\]
commutes in homology. Thus by juxtaposing (5.3) and (5.4), we obtain a diagram of homotopy cofibrations which commutes in homology. So by induction, we obtain that \( F_q \) is an isomorphism in homology, hence a homotopy equivalence. Now we suppose \( f \circ g_1 \simeq f \circ g_2 \). Then \( \tilde{f}_{(p)} \circ (\tilde{g}_1)_{(p)} \simeq \tilde{f}_{(p)} \circ (\tilde{g}_2)_{(p)} \), where \( \tilde{f}_{(p)} \) has a left homotopy inverse as above. Thus we get \((\tilde{g}_1)_{(p)} \simeq (\tilde{g}_2)_{(p)}\)
implicating \((g_1)_{(p)} \simeq (g_2)_{(p)}\). The proof is completed by Lemma 5.4.

\[\square\]

**Corollary 5.6.** Let \( S_i \) be a wedge of circles and \( Y_i \) be a simply connected co-H-space which is a CW-complex of finite type for \( i = 1, 2 \). If \( f: S_1 \vee Y_1 \to S_2 \vee Y_2 \) is a quasi-equivalence, then for any connected finite complex \( A \) with finite \( \pi_1 \), the induced map
\[ f_*: [A, S_1 \vee Y_1] \to [A, S_2 \vee Y_2] \]
is injective.

**Proof.** Since \( \pi_1(S_1 \vee Y_1) \cong \pi_1(S_1) \) is a free group and \( \pi_1(A) \) is finite, any map \( A \to S_1 \vee Y_1 \) is trivial in \( \pi_1 \), so the proof is completed by Theorem 5.5. \(\square\)

We apply Theorem 5.5 to prove the implication (2) \(\Rightarrow\) (1) of Theorem 1.2.

**Lemma 5.7.** Suppose \( \mathbb{R}Z_{K}^{m-1} \) and \( \mathbb{R}Z_{K} \) are suspensions. If \( K \) is connected, the inclusion
\( \mathbb{R}Z_{K}^{m-1} \to \mathbb{R}Z_{K} \) is an isomorphism in fundamental group.

**Proof.** It follows from Theorem 3.1 that \( \mathbb{R}Z_{K} = \mathbb{R}Z_{K}^{m-1} \cup_{\varphi_K} C|\text{Sd}K| \), so by the van Kampen theorem, \( \pi_1(\mathbb{R}Z_{K}) \cong \pi_1(\mathbb{R}Z_{K}^{m-1})/N \) and the inclusion \( \mathbb{R}Z_{K}^{m-1} \to \mathbb{R}Z_{K} \) in fundamental group is identified with the quotient map
\[ \pi_1(\mathbb{R}Z_{K}^{m-1}) \to \pi_1(\mathbb{R}Z_{K}^{m-1})/N \]
where \( N \) is the smallest normal subgroup including \( \text{Im}\{(\varphi_K)_*: \pi_1(|\text{Sd}K|) \to \pi_1(\mathbb{R}Z_{K}^{m-1})\} \). By Proposition 3.3, \( \Sigma \mathbb{R}Z_{K} \simeq \Sigma \mathbb{R}Z_{K}^{m-1} \vee \Sigma^2|\text{Sd}K| \), implying \( H_1(\mathbb{R}Z_{K}^{m-1}) \cong H_1(\mathbb{R}Z_{K}) \) since \( K \) is connected. Then since \( \mathbb{R}Z_{K}^{m-1} \) and \( \mathbb{R}Z_{K} \) are suspensions, their fundamental groups are free groups of the same rank. Therefore the inclusion \( \mathbb{R}Z_{K}^{m-1} \to \mathbb{R}Z_{K} \) is an isomorphism by the above observation since free groups are Hopfian. \(\square\)
Theorem 5.8. If there is a homotopy equivalence
\[ Z_K(CX, X) \simeq \bigvee_{\emptyset \neq I \subseteq [m]} |\Sigma K_I| \wedge \hat{X}^I \]
for any \( X \), then the fat wedge filtration of \( RZ_K \) is trivial.

Proof. Induct on \( m \). The case \( m = 1 \) is trivial. Suppose the theorem holds for \( m - 1 \). Since
\[ Z_{K_i}(CX, X) = Z_K(CX, X) \simeq \bigvee_{\emptyset \neq I \subseteq [m]} |\Sigma K_I| \wedge \hat{X}^I \]
for \( X_i = * \) with \( i \notin I \), we get \( \varphi_{K_i} \simeq * \) for any \( \emptyset \neq I \subseteq [m] \) by the induction hypothesis. It remains to prove \( \varphi_K \simeq * \). If \( K \) is not connected, we have \( K = K_I \sqcup K_J \) for some disjoint \( \emptyset \neq I, J \subseteq [m] \) with \( I \cup J = [m] \), and by construction, \( \varphi_K \) factors through the map
\[ |\text{Sd}K| = |\text{Sd}K_I| \sqcup |\text{Sd}K_J| \xrightarrow{\varphi_{K_I} \cup \varphi_{K_J}} RZ_{K_I} \sqcup RZ_{K_J} \]
which is trivial since \( |I| < m \) and \( |J| < m \), implying \( \varphi_K \simeq * \). Then we assume \( K \) is connected.

Put \( W := \bigvee_{\emptyset \neq I \subseteq [m]} |\Sigma K_I| \). Since \( W \) is a suspension \( W \simeq S \vee V \), where \( S \) is a wedge of circles and \( V \) is a simply connected suspension. We may assume that \( S \) is a wedge summand of \( RZ_K \).

Define a map \( h \) by the composite
\[ RZ_K \to \bigvee_{\emptyset \neq I \subseteq [m]} RZ_K \xrightarrow{\text{proj}} RZ_K \vee \bigvee_{\emptyset \neq I \subseteq [m]} RZ_{K_I} \xrightarrow{\text{proj}} S \vee W \xrightarrow{\text{proj}} S \vee V \]
where the first arrow is given by the suspension comultiplication. Then \( h \) is a quasi-equivalence. Put \( \hat{W} := \bigvee_{\emptyset \neq I \subseteq [m]} \Sigma K_I \). Then \( \hat{W} \simeq \hat{S} \vee \hat{V} \), where \( \hat{S} \) is a wedge of circles and \( \hat{V} \) is a simply connected suspension as well. By Lemma 5.7, we have \( S = \hat{S} \). We can similarly define a quasi-equivalence
\[ \hat{h} : RZ_K^{m-1} \to S \vee \hat{V} \]
by using the comultiplication of \( RZ_K^{m-1} \), where \( RZ_K^{m-1} \) is a suspension by Theorem 3.1 and the induction hypothesis. Define a map \( r : RZ_K \to RZ_K^{m-1} \) by the composite
\[ RZ_K \xrightarrow{h} S \vee V \xrightarrow{\text{proj}} S \vee \hat{V} \simeq RZ_K^{m-1} \]
where the last homotopy equivalence is given by the induction hypothesis. Then by construction, the diagram
\[ \begin{array}{ccc}
RZ_K^{m-1} & \xrightarrow{\text{incl}} & RZ_K \xrightarrow{r} RZ_K^{m-1} \\
\downarrow h & & \downarrow h \\
\hat{S} \vee \hat{V} & \xrightarrow{\text{incl}} & S \vee V \xrightarrow{\text{proj}} S \vee \hat{V}
\end{array} \]
commutes in fundamental group and homology. Since \( h \) and \( \hat{h} \) are quasi-equivalences, the composite of the top arrows is a quasi-equivalence. Since \( RZ_K \) is the mapping cone of \( \varphi_K \) by
Theorem 3.1, the composite
\[ |\text{Sd}K| \xrightarrow{\varphi_K} \mathbb{R}Z_K^{m-1} \xrightarrow{\text{incl}} \mathbb{R}Z_K \xrightarrow{r} \mathbb{R}Z_K^{m-1} \]
is null homotopic. Since the composite of the last two arrows is a quasi-equivalence as above, \( \varphi_K \) is null homotopic by Theorem 5.5, completing the proof. \( \square \)

Proof of Theorem 1.2. Combine Theorem 4.6 and 5.8. \( \square \)

6. Golodness and fat wedge filtrations

In this section, we study a relation between the Golodness of a simplicial complex \( K \) and the triviality of the fat wedge filtration of \( \mathbb{R}Z_K \). We first recall the definition of the Golodness of simplicial complexes. Let \( k \) be a commutative ring. Recall that the Stanley-Reisner ring of a simplicial complex \( K \) over \( k \) is defined by
\[ k[K] := k[v_1, \ldots, v_m]/I_K, \quad |v_i| = 2 \]
where \( I_K \) is the ideal generated by monomials \( v_{i_1} \cdots v_{i_k} \) for \( \{i_1, \ldots, i_k\} \not\subset K \). As is well-known, Stanley-Reisner rings have been a constant source of interest in algebra and combinatorics, and have been producing a variety of results and applications. See [S] for general structures of Stanley-Reisner rings. A formal definition of the Golodness of \( K \) over \( k \) is given in terms of the Poincaré series of \( k[K] \) and its cohomology [G], but we define the Golodness in a more accessible form. We consider one of the most important derived algebra of the Stanley-Reisner ring \( k[K] \)
\begin{equation}
(6.1) \quad \text{Tor}^*_{k[v_1, \ldots, v_m]}(k[K], k)
\end{equation}
where the product structure is induced from the Koszul resolution of \( k \) over \( k[v_1, \ldots, v_m] \).

**Definition 6.1.** A simplicial complex \( K \) is called Golod over \( k \) if all products and (higher) Massey products in \( \text{Tor}^*_{k[v_1, \ldots, v_m]}(k[K], k) \) vanish.

**Remark 6.2.** Recently it was proved by Berglund and Jöllenbeck [BJ] that the condition on (higher) Massey products is redundant, so we need only to consider products.

There is a combinatorial description of products in \( \text{Tor}^*_{k[v_1, \ldots, v_m]}(k[K], k) \) due to Hochster (cf. [S]), and we recall it here. We start with an isomorphism
\[ \text{Tor}^i_{k[v_1, \ldots, v_m]}(k[K], k) \cong \bigoplus_{I \subset [m]} \tilde{H}^{i-|I|-1}(K_I; k) \]
shown by Hochster, which can be deduced also from the BBCG decomposition. Through this isomorphism, the products in \( \text{Tor}^i_{k[v_1, \ldots, v_m]}(k[K], k) \) splits into maps
\[ \tilde{H}^{i-|I|-1}(K_I; k) \otimes \tilde{H}^{j-|J|-1}(K_J; k) \rightarrow \tilde{H}^{i+j-|I|-|J|-1}(K_{I \cup J}; k) \]
Hochster showed that this map is trivial for \( I \cap J \neq \emptyset \) and is induced from the inclusion \( K_{I \cup J} \to K_I \ast K_J \) for \( I \cap J = \emptyset \). So by the above remark, we obtain:

**Proposition 6.3.** A simplicial complex \( K \) is Golod over \( \mathbb{k} \) if and only if the inclusion \( K_{I \cup J} \to K_I \ast K_J \) is trivial in homology with \( \mathbb{k} \) coefficient for all \( \emptyset \neq I, J \subset [m] \) satisfying \( I \cap J = \emptyset \).

Then one can naively define a notion of simplicial complexes which implies the Golodness.

**Definition 6.4.** A simplicial complex \( K \) is homotopy Golod if the inclusion \( K_{I \cup J} \to K_I \ast K_J \) is null homotopic for all \( \emptyset \neq I, J \subset [m] \) satisfying \( I \cap J = \emptyset \).

By definition, we obviously have:

**Proposition 6.5.** If \( K \) is homotopy Golod, then it is Golod over any ring.

We next consider a connection between Stanley-Reisner rings and polyhedral products. By definition one immediately sees an isomorphism

\[
H^*(\mathbb{Z}_K(CP^n, \ast); \mathbb{k}) \cong \mathbb{k}[K].
\]

This was first found by Davis and Januszkiewicz [DJ], and since then, the combinatorial aspect of polyhedral products have been studied extensively. By the above isomorphism several derived algebras of the Stanley-Reisner ring \( \mathbb{k}[K] \) can be realized by the cohomology of spaces related with polyhedral products. In particular, there is a ring isomorphism

\[
H^*(\mathbb{Z}_K(D^2, S^1); \mathbb{k}) \cong \text{Tor}^*_\mathbb{k}[v_1, \ldots, v_m](\mathbb{k}[K], \mathbb{k})
\]

which was proved by Baskakov, Buchstaber, and Panov [BBP]. (This isomorphism is actually induced from a chain homotopy equivalence between the cochain complex of \( \mathbb{Z}_K(D^2, S^1) \) and the Koszul resolution of \( \mathbb{k} \) over \( \mathbb{k}[v_1, \ldots, v_m] \) tensored with \( \mathbb{k}[K] \).) If the BBCG decomposioon of \( \mathbb{Z}_K(CX, X) \) desuspends, then \( \mathbb{Z}_K(CX, X) \) becomes a suspension, so we obtain:

**Proposition 6.6.** If the BBCG decomposioon of \( \mathbb{Z}_K(D^2, S^1) \) desuspends, then \( K \) is Golod over any ring.

So by Theorem 1.2, the triviality of the fat wedge filtration of \( \mathbb{RZ}_K \) seems too strong to guarantee the Golodness of \( K \). Indeed Cai [C] recently showed that the cohomology of \( \mathbb{RZ}_K \) has a richer structure than \( \mathbb{Z}_K(D^2, S^1) \), so the homotopy decomposition of \( \mathbb{RZ}_K \) seems too much for the Golodness of \( K \). Then we investigate which structure of \( K \) is more directly connected with the triviality of the fat wedge filtration of \( \mathbb{RZ}_K \).

We first relate the attaching maps \( \varphi_{K_I} \) to Whitehead products. For simplicial complexes \( L_1, L_2 \) with disjoint vertex sets, we have

\[
\mathbb{RZ}_{L_1 \ast L_2} = \mathbb{RZ}_{L_1} \times \mathbb{RZ}_{L_2}
\]
by Example 2.4. Then in particular we get
\[ \mathbb{R}Z_{L_1+L_2}^{m_1+m_2-1} = (\mathbb{R}Z_{L_1}^{m_1-1} \times \mathbb{R}Z_{L_2}^{m_2-1}) \cup (\mathbb{R}Z_{L_1} \times \mathbb{R}Z_{L_2}^{m_2-1}) \]
where \( m_1, m_2 \) are the numbers of vertices of \( L_1, L_2 \) respectively. So there is a projection
\[ \mathbb{R}Z_{L_1+L_2}^{m_1+m_2-1} \to \mathbb{R}Z_{L_1} / \mathbb{R}Z_{L_1}^{m_1-1} \vee \mathbb{R}Z_{L_2} / \mathbb{R}Z_{L_2}^{m_2-1} = |\Sigma Sd L_1| \vee |\Sigma Sd L_2| \]
where the last equality holds by Theorem 3.1.

**Proposition 6.7.** For \( \emptyset \neq I, J \subseteq [m] \) satisfying \( I \cap J = \emptyset \), the composite
\[ |\text{Sd}(K_I \ast K_J)| \xrightarrow{\varphi_{K_I \ast K_J}} \mathbb{R}Z_{K_I \ast K_J}^{[I]+[J]-1} \xrightarrow{\text{proj}} |\Sigma \text{Sd} K_I \vee |\Sigma \text{Sd} K_J| \]
is identified with the Whitehead product.

**Proof.** By Theorem 3.1, we have \( \mathbb{R}Z_L = \mathbb{R}Z_L^{[\ell]-1} \cup \varphi_L C(|\text{Sd} L|) \) for any simplicial complex \( L \) with \( \ell \) vertices. Then by
\[ |\text{Sd}(K_I \ast K_J)| = |\text{Sd} K_I| \ast |\text{Sd} K_J| = (|\text{Sd} K_I| \times C|\text{Sd} K_J|) \cup (C|\text{Sd} K_I| \times |\text{Sd} K_J|) \]
and the definition of \( \varphi_{K_I \ast K_J} \), the map \( \varphi_{K_I \ast K_J} \) is identified with the map
\[ (\varphi_{K_I} \times C\varphi_{K_J}) \cup (C\varphi_{K_I} \times \varphi_{K_J}) : (|\text{Sd} K_I| \times C|\text{Sd} K_J|) \cup (C|\text{Sd} K_I| \times |\text{Sd} K_J|) \]
\[ \to (\mathbb{R}Z_{K_I}^{[I]-1} \times (\mathbb{R}Z_{K_J}^{[J]-1} \cup C|\text{Sd} K_J|)) \cup ((\mathbb{R}Z_{K_I}^{[I]-1} \cup C|\text{Sd} K_I|) \times \mathbb{R}Z_{K_J}^{[J]-1}). \]
Thus the proposition follows from an easy inspection. \( \square \)

We now connect the triviality of the fat wedge filtration of \( \mathbb{R}Z_K \) to the homotopy Golodness of \( K \). To this end we prepare a small lemma which immediately follows from the Hilton-Milnor theorem when spaces are path-connected but we are now considering disconnected spaces.

**Lemma 6.8.** Let \( X, Y \) be CW-complexes, not necessarily connected, and let \( w : X \ast Y \to \Sigma X \vee \Sigma Y \) denote the Whitehead product. For any map \( f : A \to X \ast Y \), if \( w \circ f \) is null homotopic, then so is \( f \).

**Proof.** Let \( F \) be the homotopy fiber of the Whitehead product \( w \). It is sufficient to show that the fiber inclusion \( F \to X \ast Y \) is null homotopic. Consider the homotopy fibration sequence
\[ \Omega(X \ast Y) \xrightarrow{\Omega w} \Omega(\Sigma X \vee \Sigma Y) \to F \to X \ast Y. \]
If there is a left homotopy inverse of \( \Omega w \), we get a homotopy equivalence \( \Omega(\Sigma X \vee \Sigma Y) \simeq \Omega(X \ast Y) \times F \) hence a right homotopy inverse of the map \( \Omega(\Sigma X \vee \Sigma Y) \to F \). Then the fiber inclusion \( F \to X \ast Y \) is null homotopic. So we construct a left homotopy inverse of \( \Omega w \). Consider the inclusion \( j : \Sigma X \vee \Sigma Y \to \Sigma X \times \Sigma Y \). Then its homotopy fiber is homotopy equivalent to \( \Omega \Sigma X \ast \Omega \Sigma Y \), and the fiber inclusion \( \Omega \Sigma X \ast \Omega \Sigma Y \to \Sigma X \vee \Sigma Y \) is the Whitehead product of the evaluation maps \( \Sigma \Omega \Sigma X \to \Sigma X \) and \( \Sigma \Omega \Sigma Y \to \Sigma Y \), which we denote by \( \bar{w} \). Since \( \Omega j \) has a right homotopy inverse, \( \Omega \bar{w} \) admits a left homotopy inverse. Then since \( w = \bar{w} \circ (E \ast E) \) for
the suspension map $E: A \to \Omega \Sigma A$ and $E \ast E$ has a left homotopy inverse, $\Omega w$ admits a left homotopy inverse. Therefore the proof is done. \hfill \Box

**Theorem 6.9.** If the fat wedge filtration of $\mathbb{R}Z_K$ is trivial, then $K$ is homotopy Golod.

**Proof.** For $\emptyset \neq I, J \subset [m]$ with $I \cap J = \emptyset$, there is a commutative diagram

$$
\begin{array}{ccc}
|\text{Sd}K_{I \cup J}| & \xrightarrow{\varphi_{K_{I \cup J}}} & \mathbb{R}Z_{K_{I \cup J}}^{[I]+[J]-1} \\
\text{incl} & & \text{incl} \\
|\text{Sd}(K_I \ast K_J)| & \xrightarrow{\varphi_{K_I \ast K_J}} & \mathbb{R}Z_{K_I \ast K_J}^{[I]+[J]-1}.
\end{array}
$$

If the map $\varphi_{K_{I \cup J}}$ is null homotopic, so is the composite

$$
|\text{Sd}K_{I \cup J}| \xrightarrow{\text{incl}} |\text{Sd}(K_I \ast K_J)| \to |\Sigma \text{Sd}K_I| \vee |\Sigma \text{Sd}K_J|
$$

where the last arrow is the Whitehead product by Proposition 6.7. Then the proof is completed by Lemma 6.8. \hfill \Box

7. Homology fillable complexes

In this section, we introduce a new class of simplicial complexes which we call (homology) fillable complexes, and prove that the fat wedge filtration of $\mathbb{R}Z_K$ is trivial if $K$ is (homology) fillable, where homology fillable complexes are a homological generalization of fillable complexes.

7.1. Fillable complexes. We first consider fillable complexes. Recall that a subset $M \subset [m]$ is a minimal non-face of $K$ if $M$ is not a simplex of $K$ but $M - v$ is a simplex of $K$ for each $v \in M$. Notice that if $M$ is a minimal non-face of $K$, then $K \cup M$ is a simplicial complex.

**Definition 7.1.** A simplicial complex $K$ is fillable if there are minimal non-faces $M_1, \ldots, M_r$ of $K$ such that $|K \cup M_1 \cup \cdots \cup M_r|$ is contractible.

**Theorem 7.2.** If $K$ is fillable, then $\varphi_K$ is null homotopic.

**Proof.** We observe how the attaching map $\varphi_K$ behaves with minimal non-faces of $K$. Let $M \subset [m]$ be a minimal non-face of $K$. Then by the definition of the embedding $i_c: |\text{Sd}\Delta^{[m]}| \to (D^1)^m$, we have

$$
i_c(|\text{Sd}\Delta^M|) = \bigcup_{\emptyset \neq \sigma \subset M} C_{\sigma \subset M}.
$$

Since $M$ is a minimal non-face of $K$, $M - \sigma$ is a simplex of $K$ for any $\emptyset \neq \sigma \subset M$. Then by (3.4) we obtain

$$
i_c(|\text{Sd}\Delta^M|) \subset \mathbb{R}Z_{K}^{m-1}.$$
Since $K$ is fillable, there are minimal non-faces $M_1, \ldots, M_r$ such that $|K \cup M_1 \cup \cdots \cup M_r|$ is contractible. By the above observation, the attaching map $\varphi_K: |\text{Sd}K| \to \mathbb{R}\mathbb{Z}_K^{m-1}$ factors as

$$|	ext{Sd}K| \xrightarrow{\text{incl}} |\text{Sd}(K \cup M_1 \cup \cdots \cup M_r)| \xrightarrow{i_c} \mathbb{R}\mathbb{Z}_K^{m-1}.$$ 

Thus $\varphi_K$ is null homotopic.

We immediately obtain the following.

**Corollary 7.3.** If $K_I$ is fillable for any $\emptyset \neq I \subset [m]$, the fat wedge filtration of $\mathbb{R}\mathbb{Z}_K$ is trivial.

By Theorem 1.2 and Corollary 7.3, the BBCG decomposition desuspends for simplicial complexes whose full subcomplexes are fillable, so we get a description of the homotopy types of the corresponding polyhedral products. In order to get a more complete description of the homotopy types, we determine the homotopy type of $|\Sigma K|$ when $K$ is fillable.

**Proposition 7.4.** If $K$ is fillable, then $|\Sigma K|$ is homotopy equivalent to a wedge of spheres.

**Proof.** Since $K$ is fillable, $|K \cup M_1 \cup \cdots \cup M_r|$ becomes contractible for some minimal non-faces $M_1, \ldots, M_r$ of $K$. Then there is a homotopy equivalence $|\Sigma K| \simeq |K \cup M_1 \cup \cdots \cup M_r|/|K|$ because $|K \cup M_1 \cup \cdots \cup M_r|$ is contractible and $(|K \cup M_1 \cup \cdots \cup M_r|, |K|)$ is an NDR pair, where $|K \cup M_1 \cup \cdots \cup M_r|/|K|$ is a wedge of spheres.

**Corollary 7.5.** If $K_I$ is fillable for any $\emptyset \neq I \subset [m]$, then $\mathbb{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres.

**Proof.** Combine Theorem 1.2, Corollary 7.3, and Proposition 7.4.

---

7.2. **Homology fillable complexes.** We next consider a homological generalization of fillability. Recall that $K$ is $i$-acyclic over $\mathbb{k}$ if $\widetilde{H}_*(K; \mathbb{k}) = 0$ for $* \leq i$. If $K$ is $i$-acyclic over $\mathbb{k}$ for any $i$, $K$ is called acyclic over $\mathbb{k}$.

**Definition 7.6.** Let $K_1, \ldots, K_s$ be the connected components of $K$, and let $\tilde{K}_i$ be a simplicial complex obtained from $K_i$ by adding all of its minimal non-faces. A simplicial complex $K$ is homology fillable if

1. for each $i$ and any prime $p$ there are minimal non-faces $M_{i,p}^1, \ldots, M_{i,p}^r$ of $K_i$ such that $K_i \cup M_{i,p}^1 \cup \cdots \cup M_{i,p}^r$ is acyclic over $\mathbb{Z}/p$, and
2. $|\tilde{K}_i|$ is simply connected for each $i$.

Roughly, the first condition of the above definition corresponds to the component-wise fillability at the prime $p$, and the second condition guarantees the integrability of this local contractibility.

**Theorem 7.7.** If $K$ is homology fillable, then $\varphi_K$ is null homotopic.
**Proof.** We have observed in the proof of Theorem 7.2 that the attaching map \( \varphi_K : |SdK| \to \mathbb{R}Z^m_K \) factors through the inclusion

\[
|SdK| = |SdK_1| \sqcup \cdots \sqcup |SdK_s| \to |Sd\tilde{K}_1| \sqcup \cdots \sqcup |Sd\tilde{K}_s|
\]

where \( K_1, \ldots, K_s \) are the connected components of \( K \). Then since \( \mathbb{R}Z^m_K \) is connected, it is sufficient to show that the inclusion \( |K_i| \to |\tilde{K}_i| \) is null homotopic for \( i = 1, \ldots, s \). Since \( K_i \cup M^{i,p}_1 \cup \cdots \cup M^{i,p}_{r(i,p)} \) is of finite type, its acyclicity over \( \mathbb{Z}/p \) implies the acyclicity over \( \mathbb{Z}(p) \), so its \( p \)-localization is contractible. Then the \( p \)-localization of the inclusion \( K_i \to \tilde{K}_i \) is null homotopic since it factors through \( K_i \cup M^{i,p}_1 \cup \cdots \cup M^{i,p}_{r(i,p)} \). Then by Lemma 5.4 and the assumption that \( |\tilde{K}_i| \) is simply connected, we obtain that the inclusion itself is null homotopic, completing the proof.

**Corollary 7.8.** If \( K \) is homology fillable for any \( \emptyset \neq I \subset [m] \), then the fat wedge filtration of \( \mathbb{R}Z_K \) is trivial.

So we obtain the decomposition of polyhedral products and the homotopy Goodness of simplicial complexes whose full subcomplexes are homology fillable by Theorem 1.2 and 1.3. As well as fillable complexes, we can determine the homotopy type of a suspension of a homology fillable complex. We prepare a technical lemma.

**Lemma 7.9.** Let \( X \) be a connected CW-complex of finite type. If \( \Sigma X \) has the \( p \)-local homotopy type of a wedge of spheres for any prime \( p \), then \( \Sigma X \) itself has the homotopy type of a wedge of spheres.

**Proof.** By assumption, \( H_i(\Sigma X; \mathbb{Z}) \) is a free abelian group of finite rank for each \( i \). Choose a basis \( x^i_1, \ldots, x^i_{n_i} \) of \( H_i(\Sigma X; \mathbb{Z}) \) for \( i > 0 \). Using a \( p \)-local homotopy equivalence between \( \Sigma X \) and a wedge of spheres, we can easily construct a map \( p\theta_i^j : S^i \to \Sigma X(p) \) satisfying \( (p\theta_i^j)_*(u_i) = x^j_i \) in homology with coefficient \( \mathbb{Z}(p) \) for any \( i > 0 \) and \( j = 1, \ldots, n_i \), where \( u_i \) is a generator of \( H_i(S^i; \mathbb{Z}) \cong \mathbb{Z} \). Let \( \{p_1, p_2, \ldots\} \) be the set of all primes except for \( p \). It is well-known that the \( p \)-localization \( \Sigma X(p) \) is given by the homotopy colimit of the sequence of maps

\[
\Sigma X \xrightarrow{l_1} \Sigma X \xrightarrow{l_2} \Sigma X \xrightarrow{l_3} \Sigma X \xrightarrow{l_4} \cdots
\]

where \( l_k = p_1 \cdots p_k \) and \( q : \Sigma X \to \Sigma X \) is the degree \( q \) map. By the compactness of \( S^i \), \( p\theta_i^j \) factors through the finite step of the above sequence. Then there is a map \( p\tilde{\theta}_i^j : S^i \to \Sigma X \) satisfying \( (p\tilde{\theta}_i^j)_*(u_i) = p a_i^j x_i^j \) for \( a_i^j \in \mathbb{Z} \) with \( p \nmid p a_i^j \) in integral homology. Now we can choose primes \( q_1, \ldots, q_n \) such that \( q_1 a_i^j, \ldots, q_n a_i^j \) are relatively prime, so there are integers \( d_1, \ldots, d_n \) satisfying \( d_1(q_1 a_i^j) + \cdots + d_n(q_n a_i^j) = 1 \). Then the map

\[
\lambda_i^j := d_1 \circ q_1 \tilde{\theta}_i^j + \cdots + d_n \circ q_n \tilde{\theta}_i^j
\]
satisfies \((\lambda_i^j)_* (u_i) = x_i^j\) in integral homology, where the sum is defined by using the suspension comultiplication of \(\Sigma X\). Thus the map \(\bigvee_{i \geq 1} \bigvee_{j=1}^{n_i} \lambda_i^j : \bigvee_{i \geq 1} \bigvee_{j=1}^{n_i} S^i \to \Sigma X\) is an isomorphism in integral homology, hence a homotopy equivalence by the J.H.C. Whitehead theorem, where \(\Sigma X\) is simply connected since \(X\) is connected. Therefore the proof is completed. \(\Box\)

**Proposition 7.10.** If \(K\) is homology fillable, \(|\Sigma K|\) is homotopy equivalent to a wedge of spheres.

*Proof.* It is sufficient to consider the case that \(K\) is connected. Let \(M_r^1, \ldots, M_r^p\) be minimal non-faces of \(K\) such that \(K \cup M_r^1 \cup \cdots \cup M_r^p\) is acyclic over \(\mathbb{Z}/p\). Since \(|K \cup M_r^1 \cup \cdots \cup M_r^p|\) is a finite complex, it is also acyclic over \(\mathbb{Z}(p)\), so its \(p\)-localization is contractible. Then as in the proof of Proposition 7.4, \(|\Sigma K|_{(p)}\) is homotopy equivalent to \(|K \cup M_r^1 \cup \cdots \cup M_r^p|/\{|K\}|_{(p)}\) which is a wedge of \(p\)-local spheres. (Note that the dimension of each sphere in the wedge is greater than 1 since \(K\) is connected. So we can commute the localization and the wedge.) Thus the proof is completed by Lemma 7.9. \(\Box\)

**Corollary 7.11.** If \(K_I\) is homology fillable for all \(\emptyset \neq I \subset [m]\), then \(Z_K(D^n, S^{n-1})\) is homotopy equivalent to a wedge of spheres.

*Proof.* Combine Theorem 1.2 and 7.8, and Proposition 7.10. \(\Box\)

### 8. Shellable and sequentially Cohen-Macaulay complexes

In this section, we show that the fat wedge filtrations of the real moment-angle complexes for dual shellable and dual sequentially Cohen-Macaulay complexes are trivial by proving their fillability and homology fillability. Our choice of dual shellable and dual sequentially Cohen-Macaulay complexes are motivated by the following, where dual shellable complexes are dual sequentially Cohen-Macaulay complexes as in (1.1), and the easier cases of shifted and dual vertex-decomposable complexes were studied in [GT1, GT2, IK1, GW].

**Proposition 8.1** (Herzog, Reiner, and Welker [HRW]). *The Alexander duals of sequentially Cohen-Macaulay complexes over \(k\) are Golod over \(k\).*

We first consider the case of dual shellable complexes, and next generalize the arguments for dual shellable complexes homologically for dual sequentially Cohen-Macaulay complexes.

#### 8.1. Shellable complex

We first recall the definition of shellable complexes from [BW], where shellability is one of the most active subject studied in combinatorics. Maximal simplices of a simplicial complex are called facets, and if all facets have the same dimension, then the simplicial complex is called pure.

**Definition 8.2.** A simplicial complex \(K\) is shellable if there is an ordering of facets \(F_1, \ldots, F_t\), called a shelling, such that

\[ \langle F_k \rangle \cap \langle F_1, \ldots, F_{k-1} \rangle \]
is pure and $(|F_k| - 2)$-dimensional for $k = 2, \ldots, t$.

Interesting examples of shellable complexes can be found in [BW, H]. We next recall the Alexander dual of a simplicial complex.

**Definition 8.3.** Let $L$ be a simplicial complex whose vertex set is a subset of a finite set $S$. The Alexander dual of $L$ with respect to $S$ is defined by

$$L^\vee := \{ \sigma \subset S \mid S - \sigma \not\in L \}.$$ 

Of course the Alexander dual of $L$ changes if we alter the ambient set $S$, so we must be careful for the ambient set to take the Alexander dual. The Alexander dual of $K$ and $\text{dl}_K(v)$ for $v \in [m]$ will be always taken over $[m]$ and $[m] - v$, respectively. It is easy to verify

$$(L^\vee)^\vee = L$$

where the duals of $L$ and $L^\vee$ are taken over $S$. As well as the topological Alexander dual, the duality of (co)homology holds for the Alexander dual of a simplicial complex.

**Theorem 8.4** (cf. [BT]). Let $L$ be a simplicial complex whose vertex set is a subset of a finite set $S$. Then for any $i$,

$$\tilde{H}_i(L; k) \cong \tilde{H}^{[S]_i - 3}(L^\vee; k)$$

where the Alexander dual of $L$ is taken over $S$.

The following properties of the Alexander duals will play a fundamental role in showing the fillability of dual shellable complexes.

**Lemma 8.5.** The following holds:

1. $F$ is a facet of $K^\vee$ if and only if $F^\vee := [m] - F$ is a minimal non-face of $K$;
2. $\text{dl}_K(v)^\vee = \text{lk}_{K^\vee}(v)$ for any $v \in [m]$.

**Proof.** (1) $F$ is a facet of $K^\vee$ if and only if $F^\vee \not\in K$ and $(F \cup v)^\vee \in K$ for any $v \not\in F$. Since $F^\vee - v = [m] - (F \cup v) = (F \cup v)^\vee$ for any $v \in F^\vee$, the proof is done.

(2) For any $v \in [m]$, we have

$$\text{dl}_K(v)^\vee = \{ \sigma \subset [m] - v \mid (\sigma \cup v) \not\in k \} = \{ \sigma \subset [m] - v \mid [m] - (\sigma \cup v) \not\in K \} = \{ \tau \in K^\vee \mid v \not\in \tau \text{ and } \tau \cup v \in K^\vee \} = \text{lk}_{K^\vee}(v).$$

$\square$

We show that the dual shellability is preserved by a vertex deletion.

**Lemma 8.6** (Björner and Wachs [BW, Proposition 10.14]). For a shellable complex $L$ and its vertex $v$, the link $\text{lk}_L(v)$ is shellable.
Proof. Let $F_1, \ldots, F_t$ be a shelling of $L$ such that $F_{i_1}, \ldots, F_{i_r}$ are all facets including the vertex $v$ with $i_1 < \cdots < i_r$. Put $G_k = F_{i_k} - v$. Then $G_1, \ldots, G_r$ are all facets of $lk_L(v)$. Since $F_1, \ldots, F_t$ is a shelling of $L$, there exists $j < k$ for $k = 2, \ldots, t$ and $w \in F_k$, such that $F_k - w \subset F_j$, implying that $G_1, \ldots, G_r$ is a shelling of $lk_L(v)$. □

**Proposition 8.7.** If $K^\vee$ is shellable, then so is $dl_K(v)^\vee$ for any $v \in [m]$.

*Proof.* If $[m] - v$ is a simplex of $K$, $dl_K(v)^\vee$ is trivially shellable. Then we may assume that $[m] - v$ is not a simplex of $K$, or equivalently $v$ is a vertex of $K^\vee$. Thus the proof is done by combining Lemma 8.5 and 8.6. □

We next show the fillability of dual shellable complexes.

**Lemma 8.8.** If the Alexander dual of $K$ is collapsible, then $|K|$ is contractible.

*Proof.* Suppose that for $\sigma \subset \tau \subset [m]$, $\tau^\vee$ is a free face of $K^\vee$ such that $\sigma^\vee$ is the only simplex of $K^\vee$ satisfying $\tau^\vee \subset \sigma^\vee$ and $\dim \sigma^\vee = \dim \tau^\vee + 1$. Then by inspection, one deduces that $K \cup \{\sigma, \tau\}$ is a simplicial complex and $\sigma$ is a free face of $K \cup \{\sigma, \tau\}$ such that $\tau$ is the only simplex of $K \cup \{\sigma, \tau\}$ satisfying $\sigma \subset \tau$ with $\dim \tau = \dim \sigma + 1$. In particular $|K|$ and $|K \cup \{\sigma, \tau\}|$ have the same homotopy type. On the other hand, we have

$$(K \cup \{\sigma, \tau\})^\vee = K^\vee - \{\sigma^\vee, \tau^\vee\},$$

where the right hand side is the elementary collapse of $K^\vee$ with respect to the free face $\tau^\vee$. Then since $K^\vee$ is collapsible, by iterating the above procedure, we see that $|K|$ is homotopy equivalent to the Alexander dual of the 0-simplex $\Delta^{\{v\}}$ for some $v \in [m]$, where the dual of $\Delta^{\{v\}}$ is taken over $[m]$. This Alexander dual of $\Delta^{\{v\}}$ is obviously the star of the vertex $v$ in $\Delta^{[m]}$ which is contractible. Therefore the proof is completed. □

**Proposition 8.9.** If the Alexander dual of $K$ is shellable, then $K$ is fillable.

*Proof.* Let $F_1, \ldots, F_t$ be a shelling of $K^\vee$ such that $F_{i_1}, \ldots, F_{i_r}$ are all facets of $K^\vee$ satisfying $<F_{i_k}> \cap \langle F_1, \ldots, F_{i_{k-1}} \rangle = \partial F_{i_k}$, which are called the spanning facets. Then one immediately sees that $K^\vee - \{F_{i_1}, \ldots, F_{i_r}\}$ is collapsible. By Lemma 8.5, each $F_{i_r}^\vee$ is a minimal non-face of $K$, implying that $K \cup \{F_{i_1}^\vee, \ldots, F_{i_r}^\vee\}$ is a simplicial complex. Since

$$(K \cup \{F_{i_1}^\vee, \ldots, F_{i_r}^\vee\})^\vee = K^\vee - \{F_{i_1}, \ldots, F_{i_r}\},$$

it follows from Lemma 8.8 that $|K \cup \{F_{i_1}^\vee, \ldots, F_{i_r}^\vee\}|$ is contractible, completing the proof. □

We now obtain:

**Theorem 8.10.** If the Alexander dual of $K$ is shellable, then $K_1$ is fillable for any $\emptyset \neq I \subset [m]$.

*Proof.* Combine Proposition 8.7 and 8.10, where every full subcomplex is obtained by consecutive vertex deletions. □
Corollary 8.11. If the Alexander dual of $K$ is shellable, the fat wedge filtration of $\mathbb{R}Z_K$ is trivial.

Proof. Combine Corollary 7.3 and Theorem 8.10.

Corollary 8.12. If the Alexander dual of $K$ is shellable, then $Z_K(D^n, S^{n-1})$ has the homotopy type of a wedge of spheres.

Proof. Combine Theorem 1.2, Proposition 7.4, and Corollary 8.11.

8.2. Sequentially Cohen-Macaulay complex. Recall that a simplicial complex $K$ is Cohen-Macaulay (CM, for short) over a ring $k$ if its Stanley-Reisner ring $k[K]$ is a Cohen-Macaulay ring, that is, the Krull dimension and the depth of $k[K]$ are the same. By definition CM complexes are pure, and sequentially Cohen-Macaulay (SCM, for short) complexes were introduced as a non-pure generalization of CM complexes [S].

Definition 8.13. A simplicial complex $K$ is sequentially Cohen-Macaulay over $k$ if the subcomplex of $K$ generated by $i$-dimensional faces is Cohen-Macaulay over $k$ for $i \geq 0$.

By definition, we have

$pure \text{ and SCM over } k \iff CM \text{ over } k$.

As well as CM complexes, there is a useful homological characterization of SCM complexes. For a simplicial complex $L$ and $i \geq 0$, let $L^{(i)}$ denote the subcomplex of $L$ generated by faces of dimension $\geq i$.

Proposition 8.14 (Björner and Wachs [BW]). A simplicial complex $K$ is SCM over $k$ if and only if for any $\sigma \in K$ and $i \geq 0$, $lk_K(\sigma)^{(i)}$ is $(i-1)$-acyclic over $k$.

Pure shellability of simplicial complexes were introduced as a combinatorial criterion for CMness, and we also have an implication (1.1) in the non-pure case. We now start to show all full subcomplexes of a dual SCM complex over $\mathbb{Z}$ are homology fillable by generalizing the above arguments for dual shellable complexes. The key is the following homological generalization of spanning facets which play the important role in the proof of Proposition 8.10. Facets $F_1, \ldots, F_r$ of a simplicial complex $L$ is called homology spanning facets over $k$ if $L - \{F_1, \ldots, F_r\}$ is acyclic over $k$, where $L - F$ is a subcomplex of $L$ whenever $F$ is a facet of $L$. Let us search for homology spanning facets of SCM complexes.

Lemma 8.15. Let $k$ be a field and $L$ be a simplicial complex satisfying $\tilde{H}_i(L^{(i+1)}; k) = 0$. Then any non-boundary $i$-cycle of $L$ over $k$ involves a facet of dimension $i$.

Proof. Let $x$ be an $i$-cycle of $L$ over $k$. If $x$ involves no facet of dimension $i$, it is a cycle of $L^{(i+1)}$ over $k$. Then since $\tilde{H}_i(L^{(i+1)}; k) = 0$, $x$ is a boundary, completing the proof.
Proposition 8.16. If $L$ is an SCM complex over a field $\mathbb{k}$, then it has homology spanning facets over $\mathbb{k}$.

Proof. Choose a basis $x_1^i, \ldots, x_{n_i}^i$ of $\tilde{H}_i(L; \mathbb{k})$ for $i \geq 0$. By Lemma 8.15, $x_1^i$ involves a facet $F_1^i$, and by subtracting a multiple of $x_1^i$ from $x_2^i, \ldots, x_{n_i}^i$ if necessary, we may assume that $x_2^i, \ldots, x_{n_i}^i$ do not involve $F_1^i$. Then by induction, we see that for $j = 1, \ldots, n_i$ and $k \neq j$, $x_j^i$ involves a facet $F_j^i$ and $x_k^i$ does not involve a facet $F_j^i$. We shall show that facets $F_0^i, \ldots, F_0^{n_1}, \ldots, F_d^i, \ldots, F_d^{n_d}$ are homology spanning facets of $L$ over $\mathbb{k}$, where $d = \dim L$. Put $\Delta = L - \{F_0^1, \ldots, F_0^{n_1}, \ldots, F_d^1, \ldots, F_d^{n_d}\}$. Then we have $$|L|/|\Delta| = \bigvee_{i=1}^d \bigwedge_{j=1}^{n_i} |F_j^i|/|\partial F_j^i| = \bigvee_{i=1}^d S_i^j$$ where $S_j^i$ is a copy of $S_i$. Note that the projection $|L| \to |L|/|\Delta|$ sends $x_j^i$ to a generator of $H_i(S_j^i; \mathbb{k})$. Thus this projection is an isomorphism in homology with coefficient $\mathbb{k}$, hence the proof is completed by the Puppe exact sequence of the homotopy cofibration $|\Delta| \to |L| \to |L|/|\Delta|$. \hfill $\square$

Regarding the second condition of homology fillability, we prove the following. Recall from Section 7 that for a simplicial complex $L$, the simplicial complex $\hat{L}$ is defined by the disjoint union of $\hat{L}_1, \ldots, \hat{L}_s$, where $L_1, \ldots, L_s$ are the connected component of $L$ and $\hat{L}_i$ is a simplicial complex constructed from $L$ by adding all of its minimal non-faces.

Proposition 8.17. If a connected simplicial complex $L$ is Golod over some ring $\mathbb{k}$, then $|\hat{L}|$ is simply connected.

Proof. If there is a minimal cycle in $L$ of length $\geq 4$, say $C$, then $C$ is a full subcomplex of $L$, hence $Z_C(D^2, S^1)$ is a retract of $Z_L(D^2, S^1)$. It follows from [BP, Proposition 7.23] that there is a non-trivial product in $\tilde{H}^*(Z_C(D^2, S^1); \mathbb{k})$, and then so is in $\tilde{H}^*(Z_L(D^2, S^1); \mathbb{k})$. This contradicts to the assumption by the ring isomorphism (6.1). Hence we get that the 1-skeleton of $L$ is chordal, that is, every minimal cycle in $L$ is of length $\leq 3$. In particular we get that the 2-skeleton of $\hat{L}$ is isomorphic to the 2-skeleton of the flag complex of a chordal graph. Thus since the flag complex of a connected chordal graph is contractible, $|\hat{L}|$ is simply connected. \hfill $\square$

By definition, any connected component of a dual SCM complex over $\mathbb{k}$ is SCM over $\mathbb{k}$. Then by Proposition 8.1 and 8.17, we get:

Corollary 8.18. If the Alexander dual of $K$ is SCM over some ring, then every connected component of $|\hat{K}|$ is simply connected.

We now prove homology fillability of dual SCM complexes over $\mathbb{Z}$.

Proposition 8.19. If the Alexander dual of $K$ is SCM over $\mathbb{Z}$, then $K$ is homology fillable.
Proof. Let $L_1, \ldots, L_s$ be the connected components of $K^\vee$. As mentioned above, each $L_i$ is SCM over $\mathbb{Z}$, so by Proposition 8.16, $L_i$ has homology spanning facets $F_1^{i,p}, \ldots, F_{r(i,p)}^{i,p}$ over $\mathbb{Z}/p$ for any prime $p$. Put $\Delta^{i,p} = L_i - \{F_1^{i,p}, \ldots, F_{r(i,p)}^{i,p}\}$. Since $\partial F_j^{i,p} \subset \Delta^{i,p}$, $F_j^{i,p}$ is a minimal non-face of $\Delta^{i,p}$ for all $j$. Then as in the proof of Proposition 8.10, we have
\[
(\Delta^{i,p})^\vee = L_i^\vee \cup (F_1^{i,p})^\vee \cup \cdots \cup (F_{r(i,p)}^{i,p})^\vee
\]
where the dual is taken over the vertex set of $L_i$. Since $(F_j^{i,p})^\vee$ is a facet of $(\Delta^{i,p})^\vee$ for all $j$, $(F_j^{i,p})^\vee$ is a minimal non-face of $L_i^\vee$ for all $j$ by Lemma 8.5. By Theorem 8.4 we also have
\[
\tilde{H}_i((\Delta^{i,p})^\vee; \mathbb{Z}/p) \cong \tilde{H}^{m-i-3}(\Delta^{i,p}; \mathbb{Z}/p)
\]
implying $\tilde{H}_i((\Delta^{i,p})^\vee; \mathbb{Z}/p) = 0$. Then the first condition of homology fillability is satisfied. The second condition is also satisfied by Corollary 8.18, completing the proof. □

The dual SCMness is preserved by vertex deletions as well as dual shellability.

Proposition 8.20. If the Alexander dual of $K$ is SCM over $\mathbb{k}$, then so is $\text{dl}_K(v)^\vee$ for any $v \in [m]$.

Proof. By Lemma 8.5, $\text{dl}_K(v)^\vee = \text{lk}_{K^\vee}(v)$, and by Proposition 8.14, $\text{lk}_{K^\vee}(v)$ is SCM over $\mathbb{k}$. □

Summarizing, we have established:

Theorem 8.21. If the Alexander dual of $K$ is SCM over $\mathbb{Z}$, then $K_I$ is homology fillable for any $\emptyset \neq I \subset [m]$.

Proof. Combine Proposition 8.21 and 8.20. □

Corollary 8.22 (Theorem 1.4). If the Alexander dual of $K$ is SCM over $\mathbb{Z}$, then the fat wedge filtration of $\mathbb{R}\mathbb{Z}_K$ is trivial.

Proof. Combine Corollary 7.8 and Theorem 8.21. □

Corollary 8.23 (Corollary 1.5). If the Alexander dual of $K$ is SCM over $\mathbb{Z}$, then $\mathbb{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres.

Proof. Combine Corollary 7.10 and Theorem 8.21. □

9. Extractible complexes

We have seen that a dual shellable complex is fillable by describing the homotopy type of its suspension as a wedge of spheres. This description actually has a further property regarding vertex deletions as follows. Suppose $K$ is dual shellable such that $F_1, \ldots, F_t$ is a shelling of $K^\vee$. If $F_{i_1}, \ldots, F_{i_s}$ are all facets including the vertex $v$ of $K^\vee$ with $i_1 < \ldots < i_s$, then as in the proof of Lemma 8.6, $F_{i_1} - v, \ldots, F_{i_s} - v$ is a shelling of $\text{lk}_{K^\vee}(v)$. Suppose $F_{j_1}, \ldots, F_{j_r}$ are spanning
facets of $K^\vee$ such that $F_{j_1}, \ldots, F_{j_q}$ include the vertex $v$ for $q \leq r$, so $F_{j_1} - v, \ldots, F_{j_q} - v$ are, not necessarily all, spanning facets of $\text{lk}_{K^\vee}(v)$. Then the proof of Proposition 8.10 shows that there are homotopy equivalences

$$|\Sigma K| \simeq |K \cup F_{j_1}^\vee \cup \cdots \cup F_{j_r}^\vee|/|K| = \bigvee_{i=1}^r S^{m-|F_i|}-1$$

and

$$|\Sigma dl_K(v)| \simeq |dl_K(v) \cup (F_{j_1} - v)^\vee \cup \cdots \cup (F_{j_q} - v)^\vee \cup G_1 \cup \cdots \cup G_u|/|dl_K(v)| = \bigvee_{i=1}^q S^{m-|F_i|}-1 \bigvee_{i=1}^u S^{G_i}-1$$

for some minimal non-faces $G_1, \ldots, G_u$ of $dl_K(v)$, where $F_i^\vee = [m] - F_i$ and $(F_i - v)^\vee = ([m] - v) - (F_i - v) = [m] - F_i$ as above. Then the inclusion $|\Sigma dl_K(v)| \to |\Sigma K|$ restricts to the inclusion $\bigvee_{i=1}^q S^{m-|F_i|}-1 \to \bigvee_{i=1}^r S^{m-|F_i|}-1$. Thus we can easily deduce the following.

**Proposition 9.1.** If the Alexander dual of $K$ is shellable, then the wedge of inclusions

$$\bigvee_{v \in [m]} |\Sigma dl_K(v)| \to |\Sigma K|$$

admits a right homotopy inverse.

This section proves a homological generalization of the property of Proposition 9.1 guarantees a $p$-local desuspension of the BBCG decomposition under some conditions on $X$. Before generalizing the property of Proposition 9.1, it is helpful to recall some properties of the Bousfield-Kan (almost) localization. For a space $X$, there is a canonical homotopy equivalence

$$\Sigma X \simeq S \vee Y$$

where $S$ is a bouquet of circles and $Y$ is simply connected. Then we can define the almost $p$-localization of $\Sigma X$ by

$$(\Sigma X)_{(p)} := S \vee Y_{(p)}$$

which is natural with respect to $X$. Although we ambiguously use the same notation for the usual localization and the almost localization of suspensions, there will be no confusion since we will deal only with simply connected spaces except for suspensions. We freely use the following properties of the (almost) $p$-localization, where the property of the usual localization of wedges of simply connected spaces was already used in Section 5 implicitly.

**Proposition 9.2** (Bousfield and Kan [BK, Proposition 4.6, Chapter V]).

1. If $X$ and $Y$ are simply connected,

$$(X \vee Y)_{(p)} \simeq X_{(p)} \vee Y_{(p)} \quad \text{and} \quad (X \wedge Y)_{(p)} \simeq X_{(p)} \wedge Y_{(p)}.$$  

2. For any spaces $X, Y$ and a simply connected space $Z$, it holds that

$$(\Sigma X)_{(p)} \vee (\Sigma Y)_{(p)} \simeq \Sigma (X \vee Y)_{(p)} \quad \text{and} \quad (\Sigma X)_{(p)} \wedge Z_{(p)} \simeq (\Sigma X \wedge Z)_{(p)}.$$
We now generalize the property of Proposition 9.1, and define extractible complexes.

**Definition 9.3.** A simplicial complex $K$ is extractible over $\mathbb{Z}/p$ if

1. the vertex deletion $dl_K(v)$ is a simplex for some vertex $v$, or
2. there is a map $\theta: |\Sigma K|_{(p)} \to \bigvee_{i=1}^{m} |\Sigma dl_K(i)|_{(p)}$ such that the composite

$$|\Sigma K|_{(p)} \xrightarrow{\theta} \bigvee_{i=1}^{m} |\Sigma dl_K(i)|_{(p)} \to |\Sigma K|_{(p)}$$

is the identity map in mod $p$ homology, where the second map is the wedge of inclusions.

By Proposition 9.1, dual shellable complexes are extractible over $\mathbb{Z}/p$ for any prime $p$. So it is natural to ask whether dual SCM complexes over $\mathbb{Z}/p$ are extractible over $\mathbb{Z}/p$ or not. We shall prove the answer is yes. For a chain $x = \sum_{\sigma \in K} a_\sigma \sigma$ ($a_\sigma \in k$) of a simplicial complex $K$, we put $x_v = \sum_{\sigma \in K, v \notin \sigma} a_\sigma \sigma$. Note that if a cycle $x$ of $K$ includes a facet of $K$, then $x$ is not a boundary. We consider a relation between cycles of $K$ and of $\text{lk}_K(v)$.

**Lemma 9.4.** Let $x$ be a cycle of $K$ over a ring $k$ which involves a facet $F$. For $v \in F$, $\partial(x_v)$ is a cycle of $\text{lk}_K(v)$ over $k$ involving $F-v$ which is a facet of $\text{lk}_K(v)$.

**Proof.** In the Mayer-Vietoris exact sequence

$$\cdots \to H_{*+1}(K; k) \xrightarrow{\delta} H_* (\text{lk}_K(v); k) \to H_* (dl_K(v); k) \oplus H_* (\text{st}_K(v); k) \to H_* (K; k) \to \cdots$$

we have $\delta x = \partial(x_v)$ for the boundary map $\partial$, so the proof is completed by an easy inspection.  \qed

**Proposition 9.5.** If the Alexander dual of $K$ is SCM over $\mathbb{Z}/p$, $K$ is extractible over $\mathbb{Z}/p$.

**Proof.** The argument is quite similar to the dual shellable case. By Proposition 8.20 and the definition of extractibility, we only need to prove the proposition for each connected component of $K$, so we may assume that $K$ is connected. It follows from Proposition 8.16 that there are homology spanning facets $F_1, \ldots, F_r$ of $K^\vee$ over $\mathbb{Z}/p$. Suppose that $F_{i_1}, \ldots, F_{i_s}$ involves a vertex $v$. Then it follows from Lemma 9.4 that there are homology spanning facets of $\text{lk}_K(v)$ over $\mathbb{Z}/p$ including $F_{i_1} - v, \ldots, F_{i_s} - v$. Then $K \cup F_{i_1}^\vee \cup \cdots \cup F_r^\vee$ and $dl_K(v) \cup (F_{i_1} - v)^\vee \cup \cdots \cup (F_{i_s} - v)^\vee \cup G_1 \cup \cdots \cup G_t$ are acyclic over $\mathbb{Z}/p$ for some minimal non-faces $G_1, \ldots, G_t$ of $dl_K(v)$, where $(F_i - v)^\vee = ([m] - v) - (F_i - v) = [m] - F_i$. Then since these simplicial complexes are of finite type, they are also acyclic over $\mathbb{Z}(p)$, so they are contractible after $p$-localization. Hence as well as the dual shellable case, there are homotopy equivalences

$$|\Sigma K|_{(p)} \simeq \bigvee_{i=1}^{r} S^{m-|F_i|-1}$$

and

$$|\Sigma dl_K(v)|_{(p)} \simeq \bigvee_{k=1}^{s} S^{m-|F_k|-1} \bigvee_{i=1}^{t} S^{G_i-1}$$

such that the inclusion $|\Sigma dl_K(v)| \to |\Sigma K|$ restricts to the inclusion $(\bigvee_{k=1}^{s} S^{m-|F_k|-1})_{(p)} \to (\bigvee_{i=1}^{r} S^{m-|F_i|-1})_{(p)}$. Now the construction of the map $\theta$ is straightforward.  \qed
We now prove a $p$-local desuspension of the BBCG decomposition for extractible complexes over $\mathbb{Z}/p$ by assuming some conditions on $X_i$. Since we will deduce a homotopy equivalence from homology, the polyhedral product $\mathcal{Z}_K(CX, X)$ should be simply connected. The following proposition gives a sufficient condition for this.

**Proposition 9.6.** If each $X_i$ is path-connected, then $\mathcal{Z}_K(CX, X)$ is simply connected.

**Proof.** As in [GT1, GT2, IK1, GW], for a vertex $v$ of $K$, the pushout of simplicial complexes

\[
\begin{array}{ccc}
\text{lk}_K(v) & \longrightarrow & \text{st}_K(v) \\
\downarrow & & \downarrow \\
\text{dl}_K(v) & \longrightarrow & K
\end{array}
\]

induces a pushout of spaces

\[
(9.1) \quad \mathcal{Z}_{\text{lk}_K(v)}(CX_{[m]-v}, X_{[m]-v}) \times X_v \longrightarrow \mathcal{Z}_{\text{lk}_K(v)}(CX_{[m]-v}, X_{[m]-v}) \times CX_v
\]

\[
\downarrow \downarrow
\]

\[
\mathcal{Z}_{\text{dl}_K(v)}(CX_{[m]-v}, X_{[m]-v}) \times X_v \longrightarrow \mathcal{Z}_K(CX, X)
\]

where $\text{st}_K(v)$ denotes the star of $v$ in $K$, i.e. $\text{st}_K(v) := \text{lk}_K(v) * \{v\}$, and all arrows in (9.1) are cofibrations. The proof is completed by an inductive application of the van Kampen theorem to the pushout (9.1) which is also a homotopy pushout. $\square$

**Theorem 9.7.** Suppose that each $X_i$ is a connected CW-complex. If $K$ is extractible over $\mathbb{Z}/p$, then there is a homotopy equivalence

\[
\mathcal{Z}_K(CX, X)(p) \simeq \bigvee_{\emptyset \neq I \subset [m]} (|\Sigma K| \wedge \hat{X}_I)(p).
\]

**Proof.** First of all, recall from [IK2] that the homotopy equivalence of Theorem 4.2 is given by the composite of maps

\[
\Sigma \mathcal{Z}_K(CX, X) \to \Sigma \bigvee_{\emptyset \neq I \subset [m]} \mathcal{Z}_K(CX, X) \overset{\text{proj}}{\longrightarrow} \Sigma \bigvee_{\emptyset \neq I \subset [m]} \mathcal{Z}_{K_I}(CX_I, X_I) \overset{\text{proj}}{\longrightarrow} \Sigma \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge \hat{X}_I
\]

which we denote by $\rho_K$, where the first map is defined by the suspension comultiplication. Since each $X_i$ is a connected CW-complex, $\mathcal{Z}_K(CX, X)$ and $\bigvee_{\emptyset \neq I \subset [m]} |\Sigma K| \wedge \hat{X}_I$ are simply connected CW-complexes by Proposition 9.6. Then in order to prove the theorem, it is sufficient to construct a map

\[
\epsilon_K: \bigvee_{\emptyset \neq I \subset [m]} (|\Sigma K| \wedge \hat{X}_I)(p) \to \mathcal{Z}_K(CX, X)(p)
\]

which coincides with $\rho_K^{-1}$ in homology.
We consider the first case in the definition of extractible complexes. By Theorem 1.2 and Proposition 3.5, $\mathcal{Z}_K(CX, X)$ is a suspension, so we can define the composite

$$\mathcal{Z}_K(CX, X) \to \Sigma \bigvee_{\emptyset \neq I \subseteq [m]} \mathcal{Z}_K(CX, X) \xrightarrow{\text{proj}} \bigvee_{\emptyset \neq I \subseteq [m]} \mathcal{Z}_K_i(CX_I, X_I) \xrightarrow{\text{proj}} \bigvee_{\emptyset \neq I \subseteq [m]} |\Sigma K_I| \land \breve{X}^I$$

which we denote by $\bar{\rho}_K$, where the first arrow is defined by the comultiplication of $\mathcal{Z}_K(CX, X)$. Obviously $\Sigma \bar{\rho}_K$ is homotopic to $\rho_K$. Then $\bar{\rho}_K$ is an isomorphism in homology, hence a homotopy equivalence by the J.H.C. Whitehead theorem. Thus $(\bar{\rho}_K)^{-1}$ is the desired map $\epsilon_K$.

We next consider the second case in the definition of extractible complexes. Induct on $m$. For $m = 1$, both $\bigvee_{\emptyset \neq I \subseteq [m]} |\Sigma K| \land \breve{X}^I$ and $\mathcal{Z}_K(CX, X)$ are contractible, so we put $\epsilon_K$ to be the constant map. Suppose that we have constructed the desired map for extractible complexes over $\mathbb{Z}/p$ with vertices less than $m$. Let

$$\hat{\epsilon}_K : \bigvee_{I \subseteq [m], I \neq \emptyset} (|\Sigma K| \land \breve{X}^I)(p) \to \mathcal{Z}_K(CX, X)(p)$$

be a wedge of the composite of maps

$$(|\Sigma K| \land \breve{X}^m)(p) \xrightarrow{\epsilon_K} \mathcal{Z}_K_i(CX_I, X_I)(p) \xrightarrow{\text{incl}} \mathcal{Z}_K(CX, X)(p).$$

for $\emptyset \neq I \subseteq [m]$, where we have the map $\epsilon_K$, by the induction hypothesis. Then by the naturality of $\rho_K$ in Corollary 4.2, $\hat{\epsilon}_K$ is the restriction of $\rho_K^{-1}$ in homology. Then by the construction of $\rho_K$, we need only to construct a map $\Theta : (|\Sigma K| \land \breve{X}^m)(p) \to \mathcal{Z}_K(CX, X)(p)$ such that the composite

$$(|\Sigma K| \land \breve{X}^m)(p) \xrightarrow{\Theta} \mathcal{Z}_K(CX, X)(p) \xrightarrow{\text{proj}} (\mathcal{Z}_K(CX, X)/\mathcal{Z}_K^{m-1}(CX, X))(p) = (|\Sigma K| \land \breve{X}^m)(p)$$

is the identity map in homology with coefficient $\mathbb{Z}(p)$. For $v \in [m]$, define a map $\Theta_v : (|\Sigma d_{\mathcal{K}}(v)| \land \breve{X}^m)(p) \to \mathcal{Z}_K(CX, X)(p)$ by the composite

$$(|\Sigma d_{\mathcal{K}}(v)| \land \breve{X}^m)(p) \to (|\Sigma d_{\mathcal{K}}(v)| \land \breve{X}^{m-v})(p) \times X_v \xrightarrow{\text{incl}} \bigvee_{\emptyset \neq I \subseteq [m]-v} (|\Sigma d_{\mathcal{K}}(v)| \land \breve{X}^I)(p) \times X_v$$

$$(\theta_{d_{\mathcal{K}}(v)} \times 1)_{p} \xrightarrow{\epsilon_{d_{\mathcal{K}}(v)}} \mathcal{Z}_{d_{\mathcal{K}}(v)}(CX_{m-v}, X_{m-v})(p) \times X_v \xrightarrow{\text{incl}} (\mathcal{Z}_K(CX, X)/CX_v)(p) \simeq \mathcal{Z}_K(CX, X)(p)$$

where we use a homotopy equivalence $\Sigma A \times B \simeq \Sigma A \lor (\Sigma A \land B)$ for the first arrow. Then the naturality of $\rho_K$ in Corollary 4.2 shows that $\Sigma \Theta_v$ is homotopic to the composite

$$\Sigma(|\Sigma d_{\mathcal{K}}(v)| \land \breve{X}^m)(p) \xrightarrow{\text{incl}} \Sigma(|\Sigma K| \land \breve{X}^m)(p) \xrightarrow{\text{incl}} \bigvee_{\emptyset \neq I \subseteq [m]} (|\Sigma K_i| \land \breve{X}^I)(p) \xrightarrow{\rho_K^{-1}} \Sigma \mathcal{Z}_K(CX, X)(p).$$

Thus the composite

$$f : |\Sigma K| \land \breve{X}^m(p) \xrightarrow{\theta \land 1} \bigvee_{i \in [m]} |\Sigma d_{\mathcal{K}}(i)| \land \breve{X}^m(p) \xrightarrow{\bigvee_{i \in [m]} \Theta_i} \mathcal{Z}_K(CX, X)(p) \xrightarrow{\text{proj}} (|\Sigma K| \land \breve{X}^m)(p)$$
is the identity map in mod $p$ homology, so it is an isomorphism in homology with coefficient $\mathbb{Z}_p$ since spaces are of finite type. Hence the above composite is a homotopy equivalence, and therefore $(\theta \land 1) \circ f^{-1}$ is the desired map, completing the proof.

We obtain a $p$-local homotopy decomposition of $\mathbb{Z}_K(CX, X)$ when $K$ is dual SCM over $\mathbb{Z}/p$.

**Corollary 9.8.** If the Alexander dual of $K$ is SCM over $\mathbb{Z}/p$ and each $X_i$ is a connected CW-complex, then there is a homotopy equivalence

$$\mathbb{Z}_K(CX, X)_{(p)} \simeq \bigvee_{\emptyset \neq I \subset [m]} (|K| \land \hat{X}^I)_{(p)}.$$  

**Proof.** Combine Proposition 9.5 and Theorem 9.7. □

10. Total Connectivity

In this section, we introduce the total connectivity of simplicial complexes, and consider the triviality of the fat wedge filtrations of the real moment-angle complexes in connection with the total connectivity. We first observe a property of the attaching map $\varphi_{K_I}$ when $\varphi_{K_J}$ is null homotopic for all $\emptyset \neq J \subset [m]$ with $|J| < |I|$. Suppose that $\varphi_{K_J}$ is null homotopic for all $\emptyset \neq J \subset [m]$ with $|J| < i$. Then it follows from Theorem 3.1 that there is a homotopy equivalence

$$\mathbb{R}Z_{K_I}^{i-1} \simeq \bigvee_{\emptyset \neq J \subset [m], |J| < i} |\Sigma K_J|$$

such that the composite $\mathbb{R}Z_{K_I}^{i-1} \simeq \bigvee_{\emptyset \neq J \subset [m], |J| < i} |\Sigma K_J| \xrightarrow{\text{proj}} |\Sigma K_J|$ is homotopic to the composite $\mathbb{R}Z_{K_I}^{i-1} \xrightarrow{\text{proj}} \mathbb{R}Z_{K_J} \xrightarrow{\text{proj}} |\Sigma K_J|.

**Lemma 10.1.** Suppose that $\varphi_{K_J}$ is null homotopic for any $\emptyset \neq J \subset [m]$ with $|J| < i$. Then for $I \subset [m]$ with $|I| = i$, the composite

$$|\text{Sd}K_I| \xrightarrow{\varphi_{K_I}} \mathbb{R}Z_{K_I}^{i-1} \simeq \bigvee_{\emptyset \neq J \subset [m], |J| < i} |\Sigma K_J| \xrightarrow{\text{proj}} |\Sigma K_H|$$

is null homotopic for any $\emptyset \neq H \subset [m]$ with $|H| < i$.

**Proof.** Since the projection $\mathbb{R}Z_{K_I}^{i-1} \to \mathbb{R}Z_{K_H}$ factors as $\mathbb{R}Z_{K_I}^{i-1} \to \mathbb{R}Z_{dK_I(v)} \to \mathbb{R}Z_{K_H}$ for $v \in I - H$, where $\emptyset \neq H \subset [m]$ with $|H| < i$. It is sufficient to show that the composite $|\text{Sd}K_I| \xrightarrow{\varphi_{K_I}} \mathbb{R}Z_{K_I}^{i-1} \xrightarrow{\text{proj}} \mathbb{R}Z_{dK_I(v)}$ is null homotopic for all $v \in I$. Consider the join $\{v\} \ast dK_I(v)$ for $v \in I$. Then by the definition of $\varphi_{K_I}$, there is a commutative diagram

$$\begin{array}{ccc}
|\text{Sd}K_I| & \xrightarrow{\varphi_{K_I}} & \mathbb{R}Z_{K_I}^{i-1} \\
\downarrow \text{incl} & & \downarrow \text{incl} \\
|\text{Sd}(\{v\} \ast dK_I(v))| & \xrightarrow{\varphi_{(v)} \ast dK_I(v)} & \mathbb{R}Z_{(v) \ast dK_I(v)}^{i-1}\end{array}$$
Then since the projection $\mathbb{R}Z_{K_i}^{i-1} \rightarrow \mathbb{R}Z_{dlK_i}(v)$ factors as $\mathbb{R}Z_{K_i}^{i-1} \xrightarrow{\text{incl}} \mathbb{R}Z_{\{v\}+dlK_i(v)}^{i-1} \xrightarrow{\text{proj}} \mathbb{R}Z_{dlK_i(v)}$, the composite $|\text{Sd}K_i| \xrightarrow{\varphi_{K_i}} \mathbb{R}Z_{K_i}^{i-1} \xrightarrow{\text{proj}} \mathbb{R}Z_{dlK_i(v)}$ factors through a contractible space $|\text{Sd}(\{v\} * dlK_i(v))|$, completing the proof.

**Proposition 10.2.** Suppose that $\varphi_{K_J}$ is null homotopic for all $\emptyset \neq J \subset [m]$ with $|J| < i$. Then for $I \subset [m]$ with $|I| = i$, the composite

$$|\text{Sd}K_I| \xrightarrow{\varphi_{K_i}} \mathbb{R}Z_{K_i}^{i-1} \cong \bigvee_{\emptyset \neq J \subset [m], |J| < i} |\Sigma K_J| \rightarrow \prod_{\emptyset \neq J \subset [m], |J| < i} |\Sigma K_J|$$

is null homotopic, where the last arrow is the inclusion.

**Proof.** The composite of maps in the statement is the product of the composite of maps in Lemma 10.1, so we obtain the desired result. \Box

Let $F_i$ be the homotopy fiber of the inclusion $\bigvee_{\emptyset \neq J \subset [m], |J| < i} |\Sigma K_J| \rightarrow \prod_{\emptyset \neq J \subset [m], |J| < i} |\Sigma K_J|$. Then as an immediate corollary of Proposition 10.2, we get:

**Corollary 10.3.** Suppose that $\varphi_{K_J}$ is null homotopic for all $\emptyset \neq J \subset [m]$ with $|J| < i$. Then for $I \subset [m]$ with $|I| = i$, the attaching map $\varphi_{K_I}$ lifts to $F_i$.

Then we check the triviality of the attaching map $\varphi_{K_I}$ for $I \subset [m]$ with $|I| = i$ by looking at its lift to $F_i$ together with Corollary 10.3 and Proposition 10.5. The easiest case that this lift is trivial, is when the connectivity of $F_i$ exceeds the dimension of $K_I$, which we record here.

**Proposition 10.4.** Suppose that $\varphi_{K_J}$ is null homotopic for all $J \subset [m]$ with $|J| < i$. For $I \subset [m]$ with $|I| = i$, if $\dim K_I \leq \text{conn} F_i$, then the attaching map $\varphi_{K_I}$ is null homotopic.

In order to apply Proposition 10.4, we describe the homotopy type of the homotopy fiber $F_i$.

**Proposition 10.5.** The homotopy fiber $F_i$ is homotopy equivalent to

$$\bigvee_{r \geq 2} \left( \bigvee_{I_1, \ldots, I_r \subset [m], |I_1| < i, \ldots, |I_r| < i} \Sigma(\Omega|\Sigma K_{I_1}| \wedge \cdots \wedge \Omega|\Sigma K_{I_r}|) \right).$$

**Proof.** Let $L$ be the discrete simplicial complex on the vertex set $\{J \subset [m] | |J| < i\}$, and let $K$ be a collection of spaces $\{|K_J|\}_{J \in L}$, where we put $|K_{\emptyset}|$ to be a point. Then as in Example 2.2, we have

$$Z_L(\Sigma K, \ast) = \bigvee_{J \subset [m], |J| < i} |\Sigma K_J|$$

and hence $F_i$ is the homotopy fiber of the inclusion $Z_L(\Sigma K, \ast) \rightarrow \prod_{J \in L} |\Sigma K_J|$. So by Lemma 2.6, $F_i$ is homotopy equivalent to the polyhedral product $Z_L(C\Omega K, \Omega K)$. Now $L^\circ$ is a skeleton of a simplex, hence shelly. Thus the proof is done by Theorem 1.2 and Corollary 8.10. \Box
Corollary 10.6. The homotopy fiber $F_i$ is $2(\min\{\text{conn } K_J \mid J \subset [m], |J| < i\} + 1)$-connected.

We now define the notion of total connectivity.

Definition 10.7. A simplicial complex $K$ is totally $n$-connected if all proper full subcomplexes of $K$ are $n$-connected.

Theorem 10.8. If $K$ is totally $\left\lceil \frac{\dim K}{2} \right\rceil - 1$-connected, the fat wedge filtration of $\mathbb{R}Z_K$ is trivial.

Proof. Let $I \subset [m]$ with $|I| = i$. We prove the triviality of the attaching map $\varphi_{K_I}$ by induction on $i$. For $i = 1$, $\varphi_{K_I}$ is obviously trivial. Suppose that $\varphi_{K_J}$ is null homotopic for all $\emptyset \neq J \subset [m]$ with $|J| < i$. By assumption and Corollary 10.6, the connectivity of the homotopy fiber $F_i$ is greater than the dimension of $K_I$ since $\dim K \geq \dim K_I$, so by Proposition 10.4, $\varphi_{K_I}$ is null homotopic, completing the proof.

We apply Theorem 10.8 to more specific simplicial complexes.

Definition 10.9. A simplicial complex $K$ is $k$-neighborly if any subset of $I \subset [m]$ with $|I| = k + 1$ is a simplex of $K$, that is, $K$ includes the $k$-skeleton of $\Delta[m]$.

Lemma 10.10. If $K$ is $k$-neighborly, then $K$ is totally $(k - 1)$-connected.

Proof. Since $K$ is $k$-neighborly, its $k$-skeleton is the $k$-skeleton of the full simplex $\Delta[m]$ which is $(k - 1)$-connected. Any map $S^n \to |K|$ factors through the $n$-skeleton of $|K|$ by the cellular approximation theorem, so if $K$ is $k$-neighborly, then $K$ is $(k - 1)$-connected. Since any full subcomplex of $K$ is also $k$-neighborly, the proof is completed.

Theorem 10.11 (Theorem 1.6). If $K$ is $\left\lceil \frac{\dim K}{2} \right\rceil$-neighborly, then the fat wedge filtration of $\mathbb{R}Z_K$ is trivial.

Proof. Combine Theorem 10.8 and Lemma 10.10.

Example 10.12. Let $K$ be the 6 vertex triangulation of $\mathbb{R}P^2$ illustrated below.
It is shown in [GPTW] that the BBCG decomposition of the special polyhedral product \( Z_K(D^2, S^1) \) desuspends. However their argument is quite ad-hoc and depends heavily on the pair \((D^2, S^1)\), so it is not applicable to \( Z_K(C\bigvee_i X_i) \) in general. Now \( K \) is 1-neighborly and \( \dim K = 2 \), so we can apply Theorem 1.2 and 1.6 to obtain a desuspension of the BBCG decomposition such that

\[
Z_K(C\bigvee_i X_i) \simeq \left( \bigsqcup_{I \in S} \Sigma^2 \tilde{X}_I \right) \lor \left( \bigsqcup_{I \subseteq [6], |I| = 4, 5} \Sigma^2 \tilde{X}_I \right) \lor (\bigvee (\Sigma \mathbb{R} P^2 \wedge \tilde{X}^{[6]})
\]

where \( S = \{\{4, 5, 6\}, \{3, 4, 5\}, \{2, 4, 6\}, \{2, 3, 6\}, \{2, 3, 5\}, \{1, 5, 6\}, \{1, 3, 6\}, \{1, 3, 4\}, \{1, 2, 5\}, \{1, 2, 4\}\} \).

We give a generalization of Theorem 10.8 by replacing the dimension of \( K \) with the homology dimension of \( K \).

**Definition 10.13.** The homology dimension of a space \( X \), denoted by \( \text{hodim} X \), is less than \( n \) if and only if \( \tilde{H}^* (X; A) = 0 \) for \( * < n \) and any finitely generated abelian group \( A \).

We prepare two technical lemmas.

**Lemma 10.14.** If \( G \) is a perfect group and \( F \) is a free group, then any homomorphism \( G \to F \) is trivial.

**Proof.** For a homomorphism \( f : G \to F \) the image \( f(G) \) is a perfect subgroup of \( F \). By the Nielsen-Schreier theorem, \( f(G) \) is also a free group, then \( f(G) \) must be trivial. \( \square \)

**Lemma 10.15.** Let \( X \) be a finite CW-complex and \( Y \) be an \( n \)-connected space of finite type. If \( \text{hodim} X < n \) and additionally \( \pi_1(Y) \) is free for \( n = 0 \), then any map \( X \to Y \) is null homotopic.

**Proof.** Consider the Postnikov tower of \( Y \):

\[
\cdots \to Y_k \hookrightarrow Y_{k-1} \to \cdots \to Y_2 \to Y_1 = K(\pi_1(Y), 1)
\]

Since \( X \) is a finite CW-complex, the triviality of the homotopy set \([X, Y_k]\) is implied by the triviality of \([X, Y_k]\) for all \( k \). It follows from Lemma 10.14 that \([X, K(\pi_1(Y), 1)] = \ast \) for \( n = 0 \), and \([X, K(\pi_1(Y), 1)]\) is obviously trivial for \( n > 0 \). So the homotopy exact sequence associated with the homotopy fibration \( K(\pi_k(Y), k) \to Y_k \to Y_{k-1} \) shows that \([X, Y_k] = \ast \) for all \( k \). \( \square \)

Put \( d(K) = \max\{\text{hodim} K_I | I \subseteq [m], I \neq \emptyset, [m]\} \). Obviously we have \( d(K) \leq \dim K \).

**Theorem 10.16.** If \( K \) is totally \( \lceil \frac{d(K)}{2} \rceil \)-connected, the fat wedge filtration of \( \mathbb{R}Z_K \) is trivial.

Quite similarly to Theorem 10.11, we can prove the following.

**Theorem 10.17.** If \( K \) is \( \lceil \frac{d(K)}{2} \rceil \)-neighborly, then the fat wedge filtration of \( \mathbb{R}Z_K \) is trivial.
Proof. This follows from Lemma 10.10 and Theorem 10.16.

Example 10.18. In [B] Berglund considered a simplicial complex $K$ on the vertex set $[10]$ whose minimal non-faces are

$$\{1, 2, 6, 7\}, \{2, 3, 7, 8\}, \{3, 4, 8, 9\}, \{4, 5, 9, 10\}, \{1, 5, 6, 10\}, \{6, 7, 8, 9, 10\}. $$

It was proved that $K^\vee$ is not SCM over $\mathbb{Z}$ but $K$ is Golod, so we cannot apply Corollary 8.21 to decompose the polyhedral product $Z_K(CX, X)$. Note that $K$ is 2-neighborly but is not 3-neighborly, and $\dim K = 6$. Then we cannot apply Theorem 10.8 to this case either. We shall show $d(K) \leq 4$, and apply Theorem 10.17, which implies that our generalization from $\dim K$ to $d(K)$ is meaningful.

Let $I$ be a non-empty subset of $[m]$.

1. For $|I| \leq 6$, $|K_I|$ is contractible, so $\text{hodim } K_I = 0$.
2. For $|I| = 7$, $|K_I|$ is homotopy equivalent to a CW-complex of dimension $\leq 4$ since $K_I$ is not the boundary of the 6-simplex. Then $\text{hodim } K_I \leq 4$.
3. For $|I| = 8$, it is a routine work to check that $(K_I)^\vee$ is contractible or homotopy equivalent to $S^1$ since $(K_I)^\vee$ has at most three facets. Then $K_I$ is contractible or homotopy equivalent to $S^4$ by Theorem 8.4 and the fact that $K_I$ is simply connected.
4. For $|I| \geq 9$, $K_I$ is contractible. The proof is divided into two cases. If $I = [10] - \{i\}$ for $i = 6, \cdots, 10$, then $K_I$ is a cone which is contractible. For example, $K_{[9]}$ is a cone with apex 5. For the other case, we only consider the whole complex $K$ since other cases are similar.

Consider the cofibration

$$|\text{lk}_K(10)| \to |K_{[9]}| \to |K|.$$ 

Since $|K_{[9]}|$ is contractible, $|K| \simeq \Sigma |\text{lk}_K(10)|$. Similarly we consider the cofibration

$$|\text{lk}_K(\{9, 10\})| \to |(\text{lk}_K(10))_{[8]}| \to |\text{lk}_K(10)|,$$

where $(\text{lk}_K(10))_{[8]}$ is a simplicial complex on the vertex set $[8]$ with the minimal non-faces

$$\{1, 2, 6, 7\}, \{2, 3, 7, 8\}, \{1, 5, 6\}. $$

Then $(\text{lk}_K(10))_{[8]}$ is a cone with apex 4 which is contractible. Then we get $|K| \simeq \Sigma |\text{lk}_K(10)| \simeq \Sigma^2 |\text{lk}_K(\{9, 10\})|$ as above. Furthermore, we can see that $|K| \simeq \Sigma^4 |\text{lk}_K(\{7, 8, 9, 10\})|$ in the same way, where $\text{lk}_K(\{7, 8, 9, 10\})$ is a simplicial complex on the vertex set $[6]$, with the minimal non-faces

$$\{2, 3\}, \{3, 4\}, \{4, 5\}, \{6\}. $$

Since it is a cone with apex 1, $|\text{lk}_K(\{7, 8, 9, 10\})|$ is contractible, and therefore $|K|$ is also contractible.

Summarizing, we conclude $d(K) = 4$. 


11. Further problems

In this section, we list possible future problems on the homotopy type of the polyhedral product $\mathbb{Z}_K(C\!\setminus\! X, X)$ mainly related with the fat wedge filtrations.

11.1. Homotopy Golodness. In Section 6, we have proved that there is an implication:

\[(11.1) \quad \text{triviality of the fat wedge filtration of } \mathbb{R}\mathbb{Z}_K \implies \text{homotopy Golodness of } K\]

The proof of this implication, i.e. the proof of Theorem 6.9, only deals with the top filter of the associated fat wedge filtration of the real moment-angle complexes, so the implication seems to be strict. Then it is worth studying the gap of this implication to get a further interpretation of the fat wedge filtrations, so we ask:

**Problem 11.1.** Find a simplicial complex for which the implication (11.1) is strict or equality.

The Golodness is defined by the triviality of certain maps in homology, and the homotopy Golodness is defined by replacing the triviality in homology with the triviality up to homotopy. Then we can define the stable homotopy Golodness by the triviality of maps in the definition of the Golodness with the triviality of the maps up to homotopy after stabilization. Then we have implications:

\[(11.2) \quad \text{homotopy Golodness} \implies \text{stable homotopy Golodness} \implies \text{Golodness}\]

We next ask the following question which seems quite combinatorial.

**Problem 11.2.** Find simplicial complexes for which the implications (11.2) are strict or equality.

Here we give an example of a class of simplicial complexes for which all implications in (11.1) and (11.2) are equalities.

**Theorem 11.3.** If $K$ is a flag complex, then the fat wedge filtration of $\mathbb{R}\mathbb{Z}_K$ is trivial if and only if $K$ is Golod over $\mathbb{Z}$.

**Proof.** If the fat wedge filtration of $\mathbb{R}\mathbb{Z}_K$ is trivial, then $K$ is Golod over any ring by Theorem 1.2 and Proposition 6.6. Then we show the converse holds. As mentioned in the proof of Proposition 8.17, if $K$ is the underlying graph of $K$ is not chordal, $K$ is not Golod. Thus the proof is completed by Proposition 3.2.

\[\square\]

11.2. Strong gcd-condition. We first pose a general problem.

**Problem 11.4.** Find a class of simplicial complexes for which the fat wedge filtrations of the real moment-angle complexes are trivial.
One of our choice for the above problem in this paper is dual SCM complexes, where the choice is motivated by the Golodness. As mentioned above, SCM complexes can be thought of as a generalization of shellable complexes, and there is another generalization of dual shellability keeping the Golodness, called the strong gcd-condition. We here recall the definition of the strong gcd-condition.

**Definition 11.5.** A simplicial complex $K$ satisfies the strong gcd-condition if minimal non-faces of $K$ admit an ordering $M_1, \ldots, M_r$, called a strong gcd-order, such that whenever $1 \leq i < j \leq r$ and $M_i \cap M_j = \emptyset$, $M_k \subset M_i \cup M_j$ for some $k$ with $i < k \neq j$.

**Proposition 11.6** (Berglund [B]).

1. If the Alexander dual of $K$ is shellable, then $K$ satisfies the strong gcd-condition.
2. If $K$ satisfies the strong gcd-condition, then it is Golod over any ring.

Then we choose simplicial complexes satisfying the strong gcd-condition to attack Problem 11.4, so we ask:

**Problem 11.7.** Show that the fat wedge filtration of $\mathbb{R}Z_K$ trivial if $K$ satisfies the strong gcd-condition.

The simplicial complex in Example 10.18 satisfies the strong gcd-condition, and we have seen that the fat wedge filtration of its real moment-angle complex is trivial. Besides this example, we show supporting evidences for Problem 11.7 by applying Theorem 10.11. It is useful to recall from [B] the weak shellability which is the Alexander dual of the strong gcd-condition.

**Definition 11.8.** A simplicial complex $K$ is called weakly shellable if there is an ordering $F_1, \ldots, F_r$ of the facets of $K$, called a weak shelling, such that if $F_i \cup F_j = [m]$ for $i < j$, then there is $F_i \cap F_j \subset F_k$ for some $k$ with $i \neq k < j$.

**Proposition 11.9** (Berglund [B]). An ordering $M_1, \ldots, M_r$ of subsets of $[m]$ is a strong gcd-order of $K$ if and only if the ordering $M'_1, \ldots, M'_r$ is a weak shelling of $K'$.

If $2 \dim K + 2 < m$, then $K'$ is weakly shellable by any ordering of facets, hence $K$ satisfies the strong gcd-condition.

**Proposition 11.10.** If $2 \dim K + 2 < m$, then the fat wedge filtration of $\mathbb{R}Z_K$ is trivial.

**Proof.** If $\dim K \geq m - 2$, the fat wedge filtration of $\mathbb{R}Z_K$ is trivial by Proposition 3.5, so we assume $\dim K \leq m - 3$. Since all simplices of $K'$ are of dimension at most $d = \dim K'$, all $(m-d-3)$-dimensional simplices of $\Delta^{[m]}$ belong to $K$, hence $K$ is $(m-d-3)$-neighborly. Since $2d + 2 < m$, we have $\lceil \frac{m-3}{2} \rceil \leq m - d - 3$. Thus by Theorem 10.11 the fat wedge filtration of $\mathbb{R}Z_K$ is trivial. 

□
Remark 11.11. One can easily see that if $K^\vee$ is connected, the condition $2 \dim K^\vee + 2 < m$ in Proposition 11.10 can be improved by one such that $2 \dim K^\vee + 1 < m$.

Corollary 11.12. If $K^\vee = SdL$ for a simplicial complex $L$ with $\dim L \geq 2$, then the fat wedge filtration of $\mathbb{R}Z_K$ is trivial.

Proof. If $\dim L = d$, then $SdL$ has at least $2^{d+1} - 1$ vertices, and for $d \geq 2$, we have $2d + 2 < 2^{d+1} - 1 \leq m$. Thus since $\dim SdL = \dim L = d$, the proof is completed by Proposition 11.10. \qed

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