Semi-analytic Continuum Spectra of Type II Supernovae and the Expanding Photosphere Method of Distance Determination
Marcos J. Montes\(^1\) and Robert V. Wagoner\(^2\)
Department of Physics and Center for Space Science and Astrophysics, Stanford University, Stanford, CA 94305-4060

ABSTRACT

We extend the approximate radiative transfer analysis of Hershkowitz, Linder, and Wagoner (1986) to a more general class of supernova model atmospheres, using a simple fit to the effective continuum opacity produced by lines (Wagoner, Perez, and Vasu 1991). At the low densities considered, the populations of the excited states of hydrogen are governed mainly by photoionization and recombination, and scattering dominates absorptive opacity. We match the asymptotic expressions for the spectral energy density \( J_\nu \) at the photosphere, whose location at each frequency is determined by a first-order calculation of the deviation of \( J_\nu \) from the Planck function \( B_\nu \). The emergent spectral luminosity then assumes the form \( L_\nu = 4\pi^2 r^2_\ast \zeta^2 B_\nu(T_p) \), where \( T_p(\nu) \) is the photospheric temperature, \( \zeta \) is the dilution factor, and \( r_\ast \) is a fiducial radius [ultimately taken to be the photospheric radius \( r_p(\nu) \)]. The atmosphere is characterized by an effective temperature \( T_e \propto L^{1/4} r^{-1/2}_\ast \) and hydrogen density \( n_H = n_\ast (r_\ast / r)^\alpha \); and less strongly by the heavy element abundance and velocity gradient. Our major result is the dependence of \( \zeta \) on frequency \( \nu \) and the parameters \( T_p \), \( r_p \), and \( \alpha \).

The resulting understanding of the dependence of the spectral luminosity on observable parameters which characterize the relevant physical conditions will be of particular use in assessing the reliability of the expanding photosphere method of distance determination. This is particularly important at cosmological distances, where no information about the progenitor star will be available. This technique can also be applied to other low-density photospheres.

Subject headings: radiative transfer — stars: atmospheres — stars: supernovae: general — cosmology: distance scale

\(^1\)Internet: marcos@mensch.stanford.edu
\(^2\)Internet: wagoner@leland.stanford.edu

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1. Introduction

The goal of this investigation is to gain a broader understanding of the dependence of the continuum spectral luminosity of Type II supernovae on the physical conditions near their photospheres as reflected by parameters that are in principle observable. We extend the semi-analytic approach of Hershkowitz, Linder, and Wagoner (1986b, hereafter HLW) to a more general class of Type II supernova model atmospheres, using a simple fit to the effective continuum opacity produced by lines (Wagoner, Perez, and Vasu 1991; hereafter WPV). Understanding the continuum spectrum allows us to calculate the dilution factor, a frequency dependent quantity that appears as a correction to the luminosity due to the non-blackbody, reduced flux of the continuum spectrum emitted by supernovae. This dilution factor is arguably the most critical element in the expanding photosphere method (EPM) of distance determination (Kirshner & Kwan 1974; Wagoner 1981; Eastman & Kirshner 1989; Schmidt, Kirshner, & Eastman 1992, hereafter SKE; Wagoner & Montes 1993; Eastman, Schmidt, and Kirshner 1994; Schmidt et al. 1994).

The extensions of this work from that of HLW include: 1) spherically-symmetric power law atmospheres; 2) inclusion the UV opacity of the heavy elements via a simple fit from WPV; 3) allowance for atmospheric regions of two different types: mostly ionized ($n_e \approx n_H$; case I), and mostly neutral ($n_e \approx 0.5n_H$; case N). In addition, at the low temperatures studied ($5,000 \lesssim T_p \lesssim 20,000$K) it is often found that the Lyman continuum is in radiative detailed balance. Then collisional coupling of the $n = 1$ and $n = 2$ levels of hydrogen becomes the most important channel determining the departure coefficient of the ground state. Thus we retain collisional coupling between these levels, although we have found that it is otherwise negligible at the low densities of these photospheres ($n_H \approx 10^{12}$ cm$^{-3}$).

We would like to stress that although our results are approximate, they require no knowledge of the progenitor star and are expressed in terms of observable parameters which fully characterize the relevant physical conditions near the photosphere. The analytical aspect of our results also reflects the pedagogical goal of our investigation.
2. Radiative Transfer Problem

The assumptions which define our radiative transfer problem are listed below.

1. The velocity gradient is small enough to yield a quasi-static photosphere. This is a good approximation for most supernovae after a few days.

2. The supernova is spherically symmetric, with the total hydrogen density given by $n_H = n_*(r_*/r)^\alpha$ near the photosphere. Here $n_*$ and $r_*$ are fiducial values.

3. We consider hydrogen dominated systems but include heavier elements via an effective continuum scattering opacity (from WPV). Hydrogen photoionization and inverse bremsstrahlung account for the absorptive opacity.

4. We consider only low density photospheres, such that collisions may usually be neglected (except for the ground state of hydrogen) and for the optical and UV photons the atmosphere is scattering dominated (by electrons and mostly iron-peak lines).

5. Radiative detailed balance holds for the hydrogen lines, as verified by Hershkowitz and Wagoner (1987).

6. We shall adopt the value of $1/3$ for the Eddington factor near (and below) the photosphere since in scattering dominated photospheres the continuum is formed at an optical depth large enough to make the radiation field essentially isotropic.

7. Energy is solely transported by radiation.

8. In keeping with our approximate analysis, the hydrogen Gaunt factors are set to unity.

Assumptions 2, 3 & 4 are more general than those made in HLW, who also only considered the case of complete ionization and did not examine the dilution factor in any detail. The total opacity and optical depth are

$$\chi_\nu = \chi_{sc} + \kappa_\nu, \quad \tau_\nu = \int_r^\infty \chi_\nu(\nu, r')dr', \quad (1)$$

where the scattering opacity $\chi_{sc} = \chi_{es} + \chi_{lines}$. The absorptive opacity is due to hydrogen photoionization and inverse bremsstrahlung:

$$\kappa_\nu = \sum_i (n_i - n_i^* e^{-h\nu/kT})\alpha_{i\nu}(\nu) + n_e^2 \alpha_{ff}(\nu, T)(1 - e^{-h\nu/kT}). \quad (2)$$

The LTE populations are given by $n_i^* = n_e^2 \Phi_i(T)$, where $\Phi_i(T)$ is the Saha-Boltzmann function for a pure hydrogen gas. We assume (and verify) that most of the free electrons
come from hydrogen, giving
\[
n_H \cong n_e + (1 + d_1)n_1^* ,
\]
where \(d_1\) is the ground-state departure coefficient.

For our extended atmosphere, we use the Mihalas (1978) formulation of the transfer equations:
\[
\frac{\partial (f_\nu q_\nu r^2 J_\nu)}{\partial \tau_{q_\nu}} = \frac{L_\nu}{(4\pi)^2} ,
\]
\[
\frac{\partial L_\nu}{\partial \tau_{\nu}} = \frac{(4\pi r_s^2)}{\chi_\nu}(\kappa_\nu J_\nu - \kappa_\nu^* B_\nu) .
\]
In these equations \(f_\nu\) is the Eddington factor, \(d\tau_{q_\nu} = q_\nu d\tau_{\nu}\), is the ‘spherical optical depth’, and \(B_\nu = (2h\nu^3/c^2)[\exp(h\nu/kT) - 1]^{-1}\) is the Planck function. The sphericity factor \(q_\nu\) is given by \(\ln(r^2 q_\nu) = \int_{r'}^r \left[\frac{3f_\nu - 1}{(r' f_\nu)}\right] dr' + \ln r_s^2\). Since for scattering dominance the total optical depth at the photosphere \(\tau_p(\nu) > 1\), we may use \(f_\nu(\tau_\nu \approx \tau_p) \cong 1/3\), giving \(q_\nu = (r_s/r)^2\). Equation (4) is then simplified to:
\[
\frac{\partial J_\nu}{\partial \tau_{q_\nu}} = \frac{3L_\nu}{(4\pi r_s)^2} .
\]

The constraint of radiative equilibrium imposes the equivalent conditions
\[
\int_0^\infty L_\nu d\nu \equiv 4\pi r_s^2 \sigma_R T_e^4 = \text{constant} ,
\]
\[
\int_0^\infty \left[\kappa_\nu J_\nu - \kappa_\nu^* B_\nu\right] d\nu = 0 .
\]
The effective temperature \(T_e\) and fiducial radius \(r_s\) are parameters characterizing our model photospheres. The second constraint is that the atomic level populations are maintained in statistical equilibrium (not LTE). Because of our assumption of relatively small velocity gradient broadening, we can take the radiative transitions between the bound states to be in detailed balance (Hershkowitz & Wagoner 1987). That is, there is a large optical depth in the lines at the continuum photosphere. Then the non-LTE departure coefficients \(d_l = (n_l - n_l^*)/n_l^*\) are given by
\[
d_l = \left[4\pi \int_{\nu_l}^\infty \left(\alpha_{l\nu_l}/h\nu\right)d\nu + \sum_{j \neq l} \kappa_{ij} C_{ij}\right] - \sum_{j \neq l} d_j C_{ij} = 4\pi \int_{\nu_l}^\infty \left(B_\nu - J_\nu\right)[1 - e^{-(h\nu/kT)}]\left(\alpha_{l\nu}/h\nu\right)d\nu ,
\]
where \(n_l C_{ij}\) is the rate of transitions from level \(l \rightarrow j\) induced by electron collisions (Mihalas 1978).] [Here \(\kappa\) denotes unbound states, \(\nu_l\) are the threshold ionization frequencies, and \(L\) (usually taken to be 10) is the principal quantum number of the highest level included.]
The appropriate second order equation is obtained by substituting equation (6) into equation (5), which yields

\[
\left( \frac{\chi_\nu}{n_e \sigma_{es}} \right) \left( \frac{r^*}{r} \right)^4 \frac{\partial^2 J_\nu}{\partial \tau^2_{q\nu}} = \frac{3}{n_e \sigma_{es}} (\kappa_\nu J_\nu - \kappa^*_\nu B_\nu). \tag{10}
\]

We rewrite the bound-free and free-free absorption coefficients (setting the Gaunt factors to unity) as

\[
\alpha_{ff} = A_{ff}(T)\nu^{-3}, \tag{11}
\]
\[
\Phi_l \alpha_{ll} = A_l(T)\nu^{-3} (\nu > \nu_l). \tag{12}
\]

The constraint (9) of statistical equilibrium can be incorporated directly into the radiative transfer equation (10), yielding for \(\nu_n < \nu < \nu_{n-1}\) our working transfer equation

\[
\left( \frac{r^*}{r} \right)^4 \left( \frac{\chi_\nu}{n_e \sigma_{es}} \right) \frac{\partial^2 J_\nu}{\partial \tau^2_{q\nu}} = \frac{3n_e}{\nu^3 \sigma_{es}} \left\{ A_{ff}(T) + \sum_{l=n}^L A_l(T) \right\} (1 - e^{-h\nu/kT})(J_\nu - B_\nu)

- J_\nu \sum_{l=n}^L A_l(T) \sum_{j=1}^L M_{lj}^{-1} \left[ \int_{\nu_j}^{\infty} (J_\nu - B_\nu)(1 - e^{-h\nu/kT})\nu^{-4}d\nu \right] \right\}. \tag{13}
\]

Here \(M_{lj}^{-1}\) is the inverse of the matrix

\[
M_{ij} = \left( 1 + \sum_{i \neq l}^\kappa C'_{li} \right) \delta_{ij} - C'_{lj}, \tag{14}
\]

which is constructed from the relative transition rates

\[
C'_{lj} = C_{lj} \left[ 4\pi \int_{\nu_j}^{\infty} (\alpha_{ll} J_\nu/H) d\nu \right]^{-1}. \tag{15}
\]
3. Approximate Analysis

We proceed by first matching asymptotic solutions to the transfer equation at the photosphere. This gives an expression for the emergent luminosity which depends upon the location of the photosphere. The location is then determined by solving equation (13) to first order in $|J_\nu - B_\nu|/B_\nu$. Then the relevant physical quantities at the photosphere can be related.

For $\tau_\nu \gg \tau_p(\nu)$ we recover LTE populations and a thermalized radiation field, giving $J_\nu = B_\nu$. Integrating equation (6) over frequency, we obtain the familiar result

$$T^4 = \frac{3L_\nu}{16\pi\sigma R^2} (\tau_q^R + C) \equiv \frac{3}{4} T_e^4 (\tau_q^R + C) , \ (C = \text{const}) .$$

Here $\tau_q^R$ is the Rosseland mean spherical optical depth ($d\tau_q^R = -q\chi R dr$). In the other asymptotic limit $\tau_\nu \ll \tau_p(\nu)$, the spectral luminosity is no longer changing ($L_\nu = L_\nu^o$), although scattering will still be important where $\tau_\nu \lesssim 1$. Integrating equation (6) in this regime yields

$$J_\nu = 3L_\nu^o (4\pi r_e)^{-2} (\tau_{qv} + c_\nu) , \ (c_\nu = \text{const.}) .$$

If we now match the asymptotic expressions for $J_\nu$ at $\tau_\nu = \tau_p(\nu)$ we obtain the emergent luminosity

$$L_\nu^o = 4\pi r_e^2 \zeta^2 \pi B_\nu(T_p) ,$$

where the dilution factor $\zeta$ is given by $\zeta^2 = (4/3)(\tau_{qv} + c_\nu)^{-1}_p$. Near the photosphere we take $\tau_{qv} \simeq (\chi_\nu/\chi_R)\tau_q^R \gg c_\nu \sim C \sim 1$, giving

$$\zeta \simeq \left( \frac{\chi_R}{\chi_\nu} \right)^{1/2}_p \left( \frac{T_e}{T_p} \right)^2 .$$

The subscript $p$ indicates that the quantity is evaluated at the (frequency dependent) photosphere.

In order to proceed, we must approximate the dominant opacities in various portions of the spectrum. Employing the results of WPV, we adopt the form

$$\frac{\chi_\nu}{\chi_{es}} = \begin{cases} 
1 + n_0 \Phi_1 \alpha_{1\nu}/\sigma_{es} & \text{if } \nu > \nu_{Lyman} \text{ and } \theta < \theta_{eq} , \\
1 + (n_0 \Phi_1)^{1/2} \alpha_{1\nu}/\sigma_{es} & \text{if } \nu > \nu_{Lyman} \text{ and } \theta > \theta_{eq} , \\
1 + (2500\theta^{16} + 10\theta^4)(\nu/\nu_{Lyman})^{1/2} & \text{if } \nu_{Lyman} > \nu > \nu_{inter} , \\
1 + (2500\theta^{16} + 10\theta^4)(\nu/\nu_{inter})^{10} & \text{if } \nu_{inter} > \nu > \nu_{Paschen} , \\
1 & \text{otherwise} ,
\end{cases}$$

(20)
where $\chi_{es} = n_e \sigma_{es}$, $\theta = 5040/T(\text{K})$ and $\nu_{\text{inter}} = c/3200\text{Å}$. This gives the results shown in Figure 1.

In the temperature range we are considering, photoionization of the ground state of hydrogen usually dominates the total opacity $\chi_{\text{in}}$ in the Lyman continuum. With $\tau_{\nu}(\nu > \nu_{\text{Lyman}}) \gg 1$ at the photosphere corresponding to lower frequencies, we here assume an LTE ground state population. We have used $n_e \simeq n_0$ for case I, and $n_e \simeq (n_0/\Phi)^{1/2}$ for case N, matched at $\theta_{eq}$. We choose $n_0$ as a typical density (in the range $n_0 = n_H = 10^{9} - 10^{12}\,\text{cm}^{-3}$). Since this density only enters in the Lyman continuum it affects our analysis at longer wavelengths only through the Rosseland opacity, and as will be shown later this has only a very small effect on the dilution factor. Since there is little flux in the Lyman continuum in the temperature range we are considering, there are only small changes ($\lesssim 5\%$) in $\chi_R/n_e \sigma_{es}$ as we vary $n_0$ through the range indicated above.

As a first order approximation to the line scattering opacity due to (mostly iron peak) heavy elements, we utilize a fit to the results of WPV [lines 3 and 4 of equation (20)]. Since the strongest dependence of the scattering opacity is on temperature, the fit for the Balmer and Paschen continua was produced at the fiducial values of $n_H = 10^{11}\,\text{cm}^{-3}$, $r/v = 10$ days, and heavy element mass fraction $Z = 0.02$. Varying the coefficients of the fit by a factor of two produced values of $\chi_R/\chi_{es}$ that were at most 15% greater (doubling the coefficients) or at most 10% smaller (halving the coefficients). From WPV we see that changing the coefficients by this factor would also correspond to varying $n_H$ from $10^{10} - 10^{13}\,\text{cm}^{-3}$, $r/v = t - t_0$ from 4 - 30 days, or $Z$ by a factor of 10. The largest effect these variations will have on our later analysis is directly through the quantity $\chi_{\nu}/\chi_{es}$ at the frequencies of interest.

In order to assess the importance of the absorptive part of the continuum opacity due to the lines, we need to estimate the thermalization parameters and the optical depths $\tau_n$ of the lines. An estimate of the thermalization parameter $\epsilon_n$ (appearing in the source function $S_n = (1 - \epsilon_n)\langle J_{\nu}\rangle_n + \epsilon_n B_n$ for line $n$ in a two-level atom without continuum) is provided by equation (5.3) of WPV, giving $\epsilon_n \lesssim 2 \times 10^{-2}$ for $T \gtrsim 5000\,\text{K}$, $n_e \lesssim 10^{11}\,\text{cm}^{-3}$, and $\lambda \lesssim 6000\,\text{Å}$ (where the lines are important). The ratio of effective absorptive to total effective continuum opacity in the lines is then [using equations (3.10) and (3.16) of WPV] $\kappa_{\text{lines}}/\chi_{\text{lines}} \sim \langle \epsilon_n(1 + \tau_n) \rangle$ if $\langle \epsilon_n \tau_n \rangle \lesssim 1$. This is not larger than the ratio of the hydrogen bound-free to total opacity, using the results of WPV which indicate that $\langle \tau_n \rangle \sim 1$. Therefore we shall neglect the contribution of lines (as well as photoionization of heavy elements) to the absorptive opacity.

The assumption of low-density is implicitly introduced when we neglect the free-free and bound-free hydrogen opacities (only in our calculation of $\chi_{\nu}$ and $\chi_R$) for frequencies below
the Lyman limit. This neglect makes $\chi_R$ at most about 12\% too small for $n_H = 10^{11}$ cm$^{-3}$. This implies a corresponding upper limit on $n_H$ in order that our assumption of a scattering dominated system be valid. As one progresses to longer wavelengths, the ratio of absorption to scattering increases; so our analysis becomes less accurate in the infrared.

3.1. Thermal Structure of the Atmosphere

Following the approach used by HLW, we next need to determine relations among the variables $T$, $r$ and $\tau$ for our two ionization regimes (I,N). A formalism for the general case may be developed in the following way. The derivative of equation (16) with respect to $r$ gives

\[
\frac{d\tau_q^R}{dr} = -\frac{16}{3} \left( \frac{\theta_e}{\theta} \right)^4 \frac{d \ln \theta}{dr} .
\]  

(21)

We also have the definition (also only valid at large optical depths)

\[
\frac{d\tau_q^R}{dr} = -A \left( \frac{r^*_s}{r} \right)^2 n_e \sigma_{es} ,
\]  

(22)

where $A = \chi_R / \chi_{es} \approx A(\theta)$ has a weak dependence on the other properties of the photosphere: density, velocity, and heavy element abundance. In what follows, we sometimes take $A(\theta)$ (seen in Figure 1) to be a slowly varying function, so we may neglect its $\theta$ dependence in integrations involving more rapidly varying functions over restricted ranges of temperature.

If we find expressions for $n_e(\theta, r)$ we may equate equations (21) and (22) and obtain relations for $\theta(r/r_s)$ in both the ionized and neutral cases. Implicit in the following is the assumption that near (and below) the photosphere $1 + d_1 = n_1 / n_1^* \approx 1$, which we find to be verified by complete atmosphere calculations.

For the ionized case $n_H \approx n_e$, giving

\[
n_e = n_s (r_s / r)^\alpha .
\]  

(23)

We then obtain the temperature profile

\[
\left( \frac{\theta_e}{\theta} \right)^4 = G_i \left( \frac{r_s}{r} \right)^{\alpha + 1} + C_i ,
\]  

(24)

with $G_i \equiv \frac{3}{4} A(\theta) n_s \sigma_{es} \Delta r_1$ and $\Delta r_1 \equiv r_s / (\alpha + 1)$. One expects that $G_i \sim \tau_p$, so it is reasonable to neglect $C_i \sim 1$ when scattering dominance produces a photospheric optical depth $\tau_p \gg 1$. 

For the neutral case $n_H \equiv n_+^*$ gives $n_e \equiv (n_H/\Phi_1)^{1/2}$, so we have

$$n_e = 2.94 \times 10^{10} n_+^{1/2} (r_*/r)^{\alpha/2} \theta^{-3/4} \exp(-\gamma \theta),$$

where $2\gamma \equiv (h\nu_1/k)/(5040 \text{ K}) = 31.31$. For this case we are not able to obtain an exact analytical expression. However, since we are working in the regime where $\theta \approx \theta_e \approx 1$, the integral is dominated by the exponential term and we obtain the approximate form

$$
\left(\frac{\theta_e}{\theta}\right)^4 \theta^{-1/4} \exp(\gamma \theta) \approx C_n - G_n \left(\frac{r_*}{r}\right)^{1+\alpha/2},
$$

(26)

where $G_n \equiv 8.63 \times 10^{10} A(\theta)n_+^{1/2} \sigma_{es}\Delta r_2$ and $\Delta r_2 \equiv 2r_*/(\alpha + 2)$.

Since we expect the function $\theta(r)$ for the ionized and neutral regimes to match at $n_e/n_H \approx 0.5$, we determine the location $\theta_m$ of the match by solving for the intersection of equation (24), with $C_i = 0$ and $n_H\Phi_1 = 2$:

$$2\gamma \theta_m + \frac{3 - 5\alpha}{2\alpha + 2} \ln \theta_m = 48.90 - \ln n_+ + \frac{\alpha}{\alpha + 1} (\ln G_i - 4 \ln \theta_e),$$

(27)

with $r_m/r_* = G_i^{1/(\alpha + 1)}(\theta_m/\theta_e)^{4/(\alpha + 1)}$. Since $\ln \theta_m \approx \ln \theta_e \approx 0$, we may approximate equation (27) as

$$\theta_m \approx 3.194 \times 10^{-2} \left[ 48.90 - \ln n_+ + \frac{\alpha}{\alpha + 1} \ln G_i \right].$$

(28)

Equation (28) is used only to indicate the major dependencies of $\theta_m$, and is never used in our numerical work. From equation (27) we find that $\theta_m$ is essentially a universal function of a particular combination of model parameters, indicated in Figure 2.

Also matching equation (26) at this point determines the integration constant to be

$$C_n = \left(\frac{\theta_e}{\theta_m}\right)^4 \theta_m^{-1/4} \exp(\gamma \theta_m) + G_n G_i^{-2(\alpha + 2)/(\alpha + 1)} \left(\frac{\theta_e}{\theta_m}\right)^{2(\alpha + 2)/(\alpha + 1)},$$

(29)

$$= 9.30 \times 10^4 \left(\frac{\theta_e}{\theta_m}\right)^{2\alpha/4} n_{11}^{1/4} \left(4.99 A(\theta)\Delta R_{14}\right)^{\alpha/2\alpha + 2} \left[\sqrt{2\theta_m^{-1} + \gamma \alpha + 1} \right].$$

(30)

We have introduced the dimensionless parameters $n_{11} = n_*/10^{11} \text{ cm}^{-3}$ and $\Delta R_{14} = \Delta r_1/10^{14} \text{ cm} = [r_*/(\alpha + 1)]/10^{14} \text{ cm}$. A more aesthetically pleasing location of the match is provided by requiring that both the temperatures and their derivatives match. This approach yields a value of $f_m \equiv (n_e/n_H)_{\text{match}} = 0.46 - 0.51$ for a wide range of models.

In Figure 3 we plot the (inverse) temperature structure of the atmosphere with the choice $\Delta R_{14} = 1.0$, $n_{11} = 1$, and $\alpha = 8$ (for three different values of the effective
temperature). As shown in this figure, these models also have an asymptotic temperature \( \theta_{\text{max}} \) given by equations (26) and (29) as

\[
\exp(\gamma \theta_{\text{max}}) \theta_{\text{max}}^{-17/4} = \exp(\gamma \theta_{m}) \theta_{m}^{-17/4} \left[ 1 + \frac{\gamma \theta_{m}}{2\sqrt{2}} \left( \frac{\alpha + 1}{\alpha + 2} \right) \right] = C_n \theta_e^4.
\]  

In addition, we obtain a minimum value of the ionization fraction \((n_e/n_H)_{\text{min}} \sim 0.2\) for these atmospheres with a wide range of model parameters. This validates our neglect of the contribution of elements other than hydrogen to the electron density. However, it is important to remind the reader that these temperature structures are only valid for \( \tau_{\nu} > 1 \). At small optical depths the temperature and ionization are governed by the radiation field (formed at large optical depths), which tends to drive the level populations far from their LTE values.

### 3.2. Location of the Photosphere

Following the approach of HLW, we now operate on our master equation (13) with \( \int_{\nu_h}^{\infty} \nu^{-1} d\nu \), obtaining

\[
\int_{\nu_j}^{\infty} \frac{X_{\nu}}{n_e \sigma_{es}} \left( \frac{r_*}{r} \right) \nu^4 \frac{\partial^2 J_{\nu}}{\partial \tau_{2\nu}^2} \nu^{-1} d\nu = \frac{3n_e}{\sigma_{es} A_{ff}} D_j + \sum_{l=1}^{j-1} A'_{l} \left( D_l - E'_{l} \sum_{k=1}^{L} M_{lk}^{-1} D_k / E'_{k} \right)
\]

\[
+ \sum_{l=j}^{L} A'_{l} \left( D_j - E'_{j} \sum_{k=1}^{L} M_{jk}^{-1} D_k / E'_{k} \right) = \frac{3n_e}{\sigma_{es} A_{ff}} \sum_{k} Q_{jk} D_k,
\]

where

\[
D_k = \int_{\nu_k}^{\infty} (J_{\nu} - B_{\nu})(1 - e^{-h\nu/kT})^{-1} \nu^{-4} d\nu,
\]

\[
E'_{k} = (c^2/2h) \int_{\nu_k}^{\infty} J_{\nu} \nu^{-4} d\nu,
\]

and \( A'_{l} = A_{l} / A_{ff} \). Next, we invert the above matrix equation to obtain the first-order quantities \( D_k \) and substitute the expression into the equivalent quantities in equation (13).

We next approximate the resulting transfer equation as the photosphere is approached from below by replacing all occurrences of \( J_{\nu} \) with \( B_{\nu} \), but keeping terms in \( J_{\nu} - B_{\nu} \) as the
first order correction. This yields (for \( \nu_n < \nu < \nu_{n-1} \))

\[
(1 + \sum_{l=n}^{L} A_l^f) (1 - e^{-h\nu/kT})(J_\nu - B_\nu) = 
\frac{n_e \sigma_{es}}{3 A_{ff}} \left[ \nu^3 N(\nu, \theta) + B_\nu \sum_{l=n}^{L} A_l^f \sum_{j=1}^{L} M_{ij}^{-1} \frac{\sum_{k=1}^{L} Q_{jk}^{-1} \int_{\nu_k}^{\infty} N(\nu, \theta) d\nu}{E_j^\nu} \right],
\]

(33)

with

\[
N(\nu, \theta) = \left( \frac{\chi_\nu}{n_e \sigma_{es}} \right) q^2 \frac{\partial^2 B_\nu}{\partial \tau_{q\nu}^2},
\]

(34)

and with the integral in (dimensionless) \( E_j^\nu \) now over \( B_\nu \) instead of \( J_\nu \). Since all the Gaunt
factors are unity,

\[
A_{ff} = 5.20 \times 10^6 \theta^{1/2} cm^5 s^{-3},
\]

\[
A_l^f = 62.5 l^{-3} \theta \exp(31.31 \theta/l^2).
\]

(35)

The structure of the matrices \( Q_{ij} \) and \( M_{ij} \) depends upon the approximations employed. Naively, in the limit of very low densities collisions may be neglected, giving \( M_{ij} = \delta_{ij} \), as in HLW. However, since the Lyman continuum is usually in radiative detailed balance due to its large absorptive opacity, we keep the collisional coupling between the \( n = 1 \) and \( 2 \) levels of hydrogen. This modifies \( M_{11}^{-1} \) by the inclusion of the term \( u = C'_{12}/(1 + C'_{12}) \) in two elements, \( M_{11}^{-1} = 1 - u \) and \( M_{21}^{-1} = u \), with the rest of the matrix being unchanged. Because of the relative lack of photons in the Lyman continuum at the temperatures of interest, \( u \) is not necessarily small, even at our low densities. We have explored the range \( 0 \leq u \leq 1 \), and the results we report for the optical and IR do not change as we vary \( u \). The largest change occurs at the highest temperatures (\( T \gtrsim 12,000 \) K), but even then the changes are relatively small and are restricted to the Lyman continuum.

We now investigate the properties of the photosphere in terms of the major model parameters: \( \theta_e, \Delta R_{14}, n_{11}^*, \) and \( \alpha \). In keeping with our first-order analysis, we may use the zeroth order expression (16) for the relation between optical depth and temperature in simplifying the expression for \( \partial^2 B_\nu/\partial \tau_{q\nu}^2 \). We obtain

\[
\partial^2 B_\nu/\partial \tau_{q\nu}^2 \cong (9/16)(\chi_R/\chi_\nu)^2 (\theta/\theta_e)^8 B_\nu f(h\nu/kT),
\]

(36)

where

\[
f(x) = \frac{[x - 5 + (x + 5) e^{-x}] x}{16(1 - e^{-x})^2}.
\]

(37)

The first order deviation of the average intensity from blackbody is then given by

\[
\frac{|J_\nu - B_\nu|}{B_\nu} = \left[ \frac{2.778 \times 10^{10}}{n_e(cm^3)} \right] \left( \frac{r_s}{r} \right)^4 \frac{\theta_e^{9/2} S_2(\theta)}{\theta_e^8} S_3(\theta, \nu)
\]

(38)
in the interval \( \nu_n < \nu < \nu_{n-1} \), with
\[
S_1(\theta, \nu) = \frac{\chi R}{\chi e} ,
\]
\[
S_2(\theta) = \frac{\chi R}{\chi es} ,
\]
\[
S_3(\theta, \nu) = \frac{|y^3 f(y) S_1(\nu, \theta) + Q_n|}{(1 - e^{-y}) \left(1 + \sum_{l=n}^{L} A_l^l\right)} ,
\]
\[
Q_n \equiv \sum_{l=n}^{L} A_l^l \sum_{j=1}^{L} (M^{-1}_{ij}/E_j^v) \sum_{j=1}^{L} Q^{-1}_{jk} \int_{y_k}^{\infty} S_1(\nu, \theta) f(x) (e^x - 1)^{-1} x^2 dx ,
\]
and \( y \equiv h\nu/kT \). In order to further utilize these equations, we must determine the function \( n_{e}^{-1}(r^*/r)^4 \), which is straightforward in the two limits of ionization.

For the ionized case (I) we may write equation (38) in the following for \( m \) after using equations (23) and (24) to obtain \( n_{e}(\theta) \) and \( r(\theta) \):
\[
|J_{\nu} - B_{\nu}|/B_{\nu} = F_1(\Delta R_{14}, n_{11}, \alpha) F_2(\theta_e, \alpha) F_3(\theta, \nu, \alpha) ,
\]
where
\[
F_1 = (4.99n_{11})^{-5/(\alpha+1)} \Delta R_{14}^{(\alpha-4)/(\alpha+1)} ,
\]
\[
F_2 = \theta_e^{(8-12\alpha)/(\alpha+1)} ,
\]
\[
F_3 = 1.3865S_2^{(2\alpha-3)/(\alpha+1)} \theta^{(17\alpha-23)/(2\alpha+2)} S_3(\nu, \theta) .
\]
In the case where \( M_{ij}^{-1} = \delta_{ij} \) (no collisions), \( \alpha \to \infty \) (sharp atmosphere), and \( \chi_{e} = n_{e}\sigma_{es} \) (no line scattering), this case corresponds to the treatment of HLW.

We estimate the location of the photosphere at each frequency as the depth at which the mean intensity differs from the Planck function by of order the Planck function,
\[
|J_{\nu} - B_{\nu}| \approx B_{\nu} .
\]
As in HLW, we fit a smooth function through the temperature region where \( J_{\nu} - B_{\nu} \) changes sign. This procedure should reflect the smooth dependence of the location of the photosphere on our model parameters. Numerical calculation of \( S_1, S_2, \) and \( S_3 \) allows us to determine the location of the photosphere, represented by \( \theta_p \) (for fixed \( \nu \) and \( \alpha \)), as the particular function of model parameters \( \theta_e, \Delta R_{14}, \) and \( n_{11} \) indicated in the above functions \( F_1 \) and \( F_2 \) and shown in Figure 4a.

The neutral case (N) is much more difficult. Inspection of equations (29) and (30) shows us that we cannot separate the dependencies as completely as in equation (41) since model parameters occur in both \( C_n \) and \( G_n \). Nevertheless, using equations (23) and (24), equation (38) now assumes the form
\[
\frac{J_{\nu} - B_{\nu}}{B_{\nu}} = 10.85 \left( \Delta R_{14} \frac{\alpha + 1}{\alpha + 2} \right)^{\frac{\alpha - 8}{\alpha + 2}} \left( 3.63 \times 10^{6} n_{11}^{\frac{1}{2}} \right)^{\frac{10}{\alpha + 2}} \theta_e^{-8} .
\]
\[ S_2^{2\alpha-6} \theta_\gamma^{21} e^{\gamma \theta} \left( C_n - \theta_c^{4\theta-17} e^{\gamma \theta} \right)^{\frac{n-6}{n+2}} S_3. \] (43)

Figure 4b is a plot of the numerically calculated \( \theta_p \) [from setting equation (43) equal to unity] for a range of model parameters, with the constraint that \( \theta_m \leq \theta_p \leq \theta_{\text{max}} \). If \( T_p \) is only slightly less than \( T_m \), then \( \theta_p \) should predominantly be a function of the combination

\[ X = \theta_c^{(10\alpha-8)/(\alpha+1)} \Delta R_{14}^{(8-\alpha)/(2\alpha+2)} n_{p11}^{9/(2\alpha+2)}. \]

Notice that except for the special case of \( \alpha = 8 \), there is some spread in the relation. However, for \( \alpha = 6 \) we obtain a narrow spread in the relation \( \theta_p \) vs. \( X \), corresponding to \( \theta_p/\theta_m < 1.1 \). The spread is much greater for \( \alpha = 10 \).

### 3.3. Calculation of the Dilution Factor

Now we have all the ingredients necessary to calculate the dilution factor \( \zeta \) from equation (13) in terms of model parameters. In the previous section we indicated how the function \( \theta_p(\theta_e, \Delta R_{14}, n_{p11}, \alpha; \nu) \) is obtained. Thus for any desired set of frequencies we may replace \( \theta_e \) through this function in equation (19), which yields

\[ \zeta = \zeta(\theta_p, n_{p11}, \Delta R_{14}, \alpha, \nu). \]

Thus far, the model parameters have referred to a fiducial radius \( r_* \), and thus are not observed quantities. We now calculate the dilution factor in terms of parameters evaluated at the photosphere, which are (potentially) observable. In addition to \( \theta_p(\nu) \) we employ the photospheric radius \( r_p(\nu) = v_p(t - t_0) \), where the velocity \( v_p(\nu) \) at the photosphere is obtained from analysis of an appropriate line profile (most reliably from the sharp minimum of a weak line). The photospheric density is then \( n_H(r_p) = n_p(\nu) \).

We now choose our fiducial radius to be the photospheric radius, so that \( r_* = r_p(\nu) = r_{p14} \times 10^{14} \) cm and \( n_* = n_p(\nu) = n_{p11} \times 10^{11} \) cm\(^{-3} \). We also define the photospheric scale height

\[ \Delta R_{14} = r_{p14}/(\alpha + 1). \]

Setting equation (38) equal to unity, we then obtain relations among the photospheric quantities:

\[ 0.2779 n_{p11}^{-1} \theta_e^{-8} \theta_p^{0/2} S_2(\theta_p) S_3(\theta_p) = 1 \] (44)

for the ionized case and

\[ 2.988 \times 10^{-6} n_{p11}^{-1/2} \theta_e^{-8} \theta_p^{21/4} e^{\gamma \theta_p} S_2(\theta_p) S_3(\theta_p, \nu) = 1 \] (45)

for the neutral case. We use these two equations to eliminate \( n_{p11} \) from equations (24) and (26) at \( r = r_* = r_p \).

We also use equation (19) to eliminate \( \theta_e \) in favor of \( \zeta \), yielding expressions for the dilution factor for both cases in terms of photospheric quantities. For the ionized case, we obtain

\[ \zeta = \zeta_i = 0.947 \Delta R_{p14}^{-1/6} \theta_p^{7/12} S_1^{1/2} S_2^{-1/3} S_3^{-1/6}. \] (46)
and for the neutral case we obtain, for $\zeta = \zeta_n$,

$$
\zeta^{-6} \Delta R_{p14}^{-1} \theta_p^{5/2} S_{1p}^3 S_{2p}^{-1} S_{3p}^{-1} + 10.85 \left( \frac{\alpha + 1}{\alpha + 2} \right) S_{2p} = S_{2m} \left[ 10.85 \left( \frac{\alpha + 1}{\alpha + 2} \right) + \frac{1.96}{\theta_m} \right]
\times \left[ 1.498 \times 10^5 \theta_m^{-2} S_{2m}^{-1/2} \zeta^{-5} \Delta R_{p14}^{-1/2} \theta_p^{10/4} e^{-\gamma_p} S_{1p}^{5/2} S_{2p}^{-1} S_{3p}^{-1} \left( \frac{\alpha + 2}{\alpha + 1} \right) \right].
$$

(47)

where the subscripts $p$ and $m$ indicate that the function is evaluated at $\theta_p$ and $\theta_m$, respectively. These equations represent the most important results of this paper. The dilution factors for the mostly ionized and mostly neutral cases are presented in Figures 5a and 5b.

For the ionized case we have a universal dependence on photospheric scale height, with a unique dependence on photospheric temperature for each wavelength, as shown in Figure 5a. The decrease in the dilution factor (at fixed scale height) with decreasing temperature and wavelength is due to the increase in UV opacity as the temperature drops, although the (Rosseland) mean opacity is barely changing.

For the neutral case there is no such simple scaling. In order to obtain the dilution factor from equation (17) we employ the same steps as above to eliminate $n_e$, $\Delta r_1$, and $\theta_e$ in equation (27) for $\theta_m$ in favor of $\zeta$, $\Delta R_{p14}$, and $\theta_p$. We then iteratively solve equation (47), which is easily done because $\theta_m$ is a weak function of the parameters, as indicated in Figure 2. In addition, we only accept solutions for which $\theta_m \leq \theta_p \leq \theta_{max}$. For $\alpha > 4$ it is found that there are either two or no solutions for $\zeta_n$. The smaller root always occurs where $\theta_p/\theta_{max} \approx 0.99$, which corresponds to the outer crossing of the $n_e/n_H = 0.5$ and temperature curves, as seen in Figure 3. We see that the physical location of this root is in the outer region of the atmosphere that is not accurately modelled. In addition, this smaller root does not match $\zeta_i$ as the photosphere becomes ionized. For these reasons we always choose the larger of the two values of $\zeta_n$.

Figure 5b shows how $\zeta_n$ varies with photospheric temperature, scale height, and wavelength; and its match to $\zeta_i$. We also find that for $\theta_p$ only slightly greater than $\theta_m$, we obtain a dependence $\zeta_n \propto \Delta R_{p14}^{-1/10}$, as expected when the first term in equation (17) is negligible. For fixed scale height $\Delta R_{p14} = r_{p14}/(\alpha + 1)$, the behavior of the dilution factor shown in Figure 5b is relatively insensitive to the value of $\alpha$ (at least for the range $6 \leq \alpha \leq 10$ investigated).

A particular frequency dependence that we have investigated is that across the Balmer ionization threshold at $\lambda = 3646$ Å, motivated by the dependence on $\Delta R$ found by Hershkowitz, Linder, & Wagoner (1986a). We find that the absolute value of the fractional jump in the photospheric temperature (corresponding to the results in Figures 4a and 4b) is less than 0.03 for our ranges of the photospheric parameters. The fractional jump in the
dilution factor (corresponding to the results in Figures 5a and 5b) is likewise found to be less than 0.04. Therefore it appears that the line scattering (and atmospheric extension) has reduced the jump from the values found with only electron scattering in a sharp photosphere by Hershkowitz, Linder, & Wagoner (1986a).
4. Discussion

The determination of the distance of a supernova via EPM would proceed as follows within the above formulation, which assumes nothing (except spherical symmetry) about the nature of its progenitor. (The other assumptions we have listed in section 2 can be checked after one obtains the photospheric conditions from our analysis.) From each observed spectrum, the photospheric temperature \( T_p(\nu) \) is first estimated by fitting Planck functions to the continuum in the neighborhood of various frequencies. As has been indicated above, the radius \( r_p(\nu) \) of the photosphere is obtained from (the weaker) line profiles. The remaining parameter to be determined is \( \alpha \), the total hydrogen density power-law index.

A comparison of detailed model atmospheres with spectra of SN1987A obtained during days 2-10 after explosion led Eastman & Kirshner (1989) to conclude that \( 7 \leq \alpha \leq 11 \). It was found that the UV continuum as well as the line shapes were sensitive to this parameter, although not greatly so in the range indicated. However, Branch (1980) has shown that the effects of optical depth (i.e., heavy element abundance) and density profile on line shapes may be difficult to separate. In addition, nonLTE effects (which he did not include) will be important for some lines.

Since it may be difficult to determine \( \alpha \) accurately from lines, let us consider whether it might be obtained in another way from observations of the continuum. If one follows a fiducial volume element which always contains the same nuclei, its radius and density soon obey \( r_s \propto t - t_0 \) and \( n_s \propto (t - t_0)^{-3} \). It then follows that the corresponding photospheric quantities are related by \( n_p(\nu, t) \propto (t - t_0)^{(\alpha - 3)} r_p^{-\alpha}(\nu, t) \). From this relation, one then sees that

\[
\alpha(t) = \left[ 3 + \frac{\partial \ln n_p}{\partial \ln(t - t_0)} \right] \left[ 1 - \frac{\partial \ln r_p}{\partial \ln(t - t_0)} \right]^{-1}. \tag{48}
\]

In principle, one could obtain the function \( n_p(\theta_p, r_p, \alpha) \) by using equation (19) to write equations (44) and (45) in the form

\[
n_p = 2.78 \times 10^{10} \zeta^{1/2} \nu^{-7/2} S_{1p} S_{2p} S_{3p} \ \text{cm}^{-3} \tag{49}
\]

for the ionized case and

\[
n_p = 0.893 \zeta n_e^{1/2} \theta_p^{-11/2} \nu^{-7/6} S_{1p}^{-5/3} S_{2p} S_{3p} \ \text{cm}^{-3} \tag{50}
\]

for the neutral case. Our results for the dilution factor \( \zeta(\theta_p, r_p, \alpha; \nu) \) would then be inserted into these relations. For instance, one would then obtain \( n_{11} = 0.224 \Delta R_{14}^{2/3} \theta_p^{-7/6} S_{2p}^{-1/3} S_{3p}^{1/3} \). 
for the ionized case. We also note that SKE quoted the same relation between $\zeta$ and $n_p$ as seen in equation (49), for fixed $\theta_p$. Of course, since $n_p$ depends upon $\alpha$, the solution of equation (48) would require iteration, using values of $\theta_p$ and $r_p$ obtained at various epochs. One could assume that $\alpha(t)$ was a slowly varying function. However, a test of the practical viability of this method is beyond the scope of this paper.

Once the dilution factor $\zeta(\theta_p, r_p, \alpha; \nu)$ has been determined, its relation to the luminosity [equation (18), with $r_* = r_p$] could be applied in two steps. First, because of the frequency dependence of the dilution factor, the previous estimate of the photospheric temperature should be improved by iterating the fit of equation (18) to the observed continuum. Second, once the fundamental parameters $T_p(\nu)$, $r_p(\nu)$, and $\alpha$ have been determined, equation (18) can be employed to obtain the luminosity distance. Of course, if the supernova is in the Hubble flow, the redshift $Z$ (of the supernova or parent galaxy) will then produce a value of the Hubble constant [and for redshifts $Z \geq 0.4$, the deceleration parameter (Wagoner, 1977)].

Another important question is the sensitivity of the dilution factor to the coefficients of the line-scattering opacity. We have both increased and decreased the coefficients by a factor of two. The dilution factor is most affected in the Balmer continuum (where the effects of the line opacity are strongest). At the highest temperatures, the dilution factors in the Paschen continuum only vary by a few percent. However, as the scattering increase at lower temperatures, we see much larger affects. Equations (46) and (39) show that $\zeta_i \propto S_{1p}^{1/3}$, so doubling the coefficients leads to variations of about 25% when the line scattering is greater than the electron scattering. However, doubling the coefficients corresponds to increasing the heavy element abundance by about an order of magnitude, as shown by WPV. In principle, this abundance can be determined by the behavior of the UV portion of the spectrum.

Like the heavy element abundance, the extinction of the spectrum by our galaxy and the supernova parent galaxy must be determined before a reliable luminosity and distance is obtained. If the dependence of the extinction on wavelength is universal, then its magnitude can be obtained by including this dependence in the fit of the spectrum. However, this procedure only becomes reliable and sensitive if the observations extend from the IR to at least the near UV.

A major challenge that faces us is to reconcile our dilution factor with that obtained by SKE. There is qualitative agreement in the ionized regime, although the temperature dependence is somewhat different. However, in the recombination era our dilution factor decreases as the temperature decreases, whereas the SKE dilution factor [as well as that recently obtained by Baron, et al. (1994)] increases. We can understand our result
based on the fact that as the ionization fraction decreases, the opacity due to absorption (proportional to $n_e^2$) decreases faster than the electron and line scattering opacity (roughly proportional to $n_e$), producing a more dilute radiation field. Another point to make in comparing our work to that of SKE is that the color temperature they determined may not correspond to our photospheric temperature. These temperatures should be similar as long as there is no net flux in the lines, the frequency dependence of the dilution factor is negligible, and a continuum can be uniquely determined. The increase in the density of lines toward the UV make the determination of a photospheric temperature potentially more difficult in the B and U bands.

Some other crucial questions remain, which can only be answered by detailed comparisons between the spectral luminosity predicted by this method and the observed flux of a variety of Type II supernovae at various epochs. This is the next step in our program. Some of these questions are:

1) How valid are the approximations (such as scattering dominance) that we have made in this analysis? (If the dilution factor approaches unity, as probably occurs in the infrared wavelengths, the photosphere is no longer scattering dominated.)

2) How closely does our definition of the photosphere correspond to reality?

3) How tightly can the luminosity be determined from the ranges of the parameters constrained by the spectral fit?

In spite of these uncertainties, we believe we have developed a potentially useful tool for determining the luminosity of Type II supernovae directly from observables in a model-independent manner. While the results for the ionized regime seem fairly robust, more work is needed to understand more fully the applicability of our results in the neutral regime.

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Fig. 1.— The temperature dependence of the ratio of Rosseland mean to total opacity, choosing fiducial parameters $n_H = 10^{11}$ cm$^{-3}$, $r/v = 10$ days, and $Z = 0.02$. The curves correspond to the following wavelengths, from top to bottom: 9000 Å, 7000 Å, 5500 Å, 4400 Å, 3650 Å, and 3000 Å. The top curve also gives $\chi_R/\chi_{es}$, since for $\lambda > \lambda_{\text{Paschen}}$, $\chi_\nu \approx \chi_{es}$.

Fig. 2.— The (tight) dependence of the matching temperature $\theta_m$ on a particular combination of model parameters. This plot has points for $\alpha = 6$, 8, and 10. The range of the other parameters is $0.01 \leq \Delta R_{14} \leq 100$, $0.01 \leq n_1 \leq 10$, $0.6615 \leq \theta_e \leq 1.26$.

Fig. 3.— The (inverse) temperature profile as a function of the scaled radius for model atmospheres with $\alpha = 8$, $n_{11} = 1$ and $\Delta R_{14} = 1$; for several values of effective temperature. Note that the curves match smoothly at the fitting point $n_e/n_H = 0.5$. The ionization fraction remains greater than 10% for a wide range of models. temperatures.

Fig. 4a.— The dependence of the photospheric temperature, in the ionized case, on a combination of model parameters for three choices of wavelength. The dashed and solid curves are for $\alpha = 10$ and $\alpha = 6$, respectively. The results for $\lambda = 3650$ Å and 5500 Å are essentially identical to that for 4400 Å.

Fig. 4b.— The dependence of the photospheric temperature, in the neutral case, on a combination of model parameters for $\lambda = 5500$ Å. The ranges of model parameters are the same as in Figure 2. There is no photospheric solution when the abscissa is less than about −1.5 (representing low densities and high temperatures, the ionized case).

Fig. 5a.— The (scaled) dilution factor for the ionized case as a function of photospheric temperature. The behavior at short wavelengths reflects that of $\chi_R/\chi_\nu$.

Fig. 5b.— The dilution factor as a function of photospheric temperature for $\alpha = 8$, at three wavelengths. Both the ionized regime (dashed) and the neutral regime (solid) are shown. For each wavelength, the upper curve is for $\Delta R_{p14} = 1$ while the lower curve is for $\Delta R_{p14} = 10$. 
