TWISTING OF AFFINE ALGEBRAIC GROUPS, II

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Abstract. We use \( \mathcal{G} \) to study the algebra structure of twisted cotriangular Hopf algebras \( j\mathcal{O}(G)_J \), where \( J \) is a Hopf 2-cocycle for a connected nilpotent algebraic group \( G \) over \( \mathbb{C} \). In particular, we show that \( j\mathcal{O}(G)_J \) is an affine Noetherian domain with Gelfand-Kirillov dimension \( \dim(G) \), and that if \( G \) is unipotent and \( J \) is supported on \( G \), then \( j\mathcal{O}(G)_J \cong U(g) \) as algebras, where \( g = \text{Lie}(G) \). We also determine the finite dimensional irreducible representations of \( j\mathcal{O}(G)_J \), by analyzing twisted function algebras on \((H,H)\)-double cosets of the support \( H \subset G \) of \( J \). Finally, we work out several examples to illustrate our results.

Contents

1. Introduction 2
2. Preliminaries 4
2.1. Hopf 2-cocycles 4
2.2. Cotriangular Hopf algebras 5
2.3. Quasi-Frobenius Lie algebras 5
2.4. Ore extensions 6
2.5. Unipotent algebraic groups 6
3. The algebra structure of \( j\mathcal{O}(G)_J \) for unipotent \( G \) 7
3.1. Ring theoretic properties 7
3.2. The minimal case 8
3.3. The general case 9
3.4. One sided twisted algebras 10
3.5. 1-dimensional central extensions 11
3.6. The algebra \( j\mathcal{O}(G)_J \) 12
4. The quotient algebras \( j\mathcal{O}(Z_g)_J \) 12
4.1. \( H \cap C = \{1\} \) 15
4.2. \( C \subset H \) 16
5. Representations of \( j\mathcal{O}(G)_J \) for unipotent \( G \) 19
6. Examples 20

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1. Introduction

Let $G$ be an affine algebraic group over $\mathbb{C}$, and let $O(G)$ be the coordinate algebra of $G$. Then $O(G)$ is a finitely generated commutative Hopf algebra over $\mathbb{C}$. Recall that Drinfeld’s twisting procedure produces (new) cotriangular Hopf algebra structures on the underlying coalgebra of $O(G)$. Namely, if $J \in (O(G)^{\otimes 2})^*$ is a Hopf 2-cocycle for $G$, then there is a cotriangular Hopf algebra $J O(G) J$ which is obtained from $O(G)$ after twisting its ordinary multiplication by means of $J$ and replacing its $R$-form $1 \otimes 1$ with $J^{-1} 21 J$.

In categorical terms, Hopf 2-cocycles for $G$ correspond to tensor structures on the forgetful functor $\text{Rep}(G) \to \text{Vec}$ of the tensor category $\text{Rep}(G)$ of finite dimensional rational representations of $G$ (see, e.g., [EGNO]).

If $J O(G) J = O(G)$ as Hopf algebras then $J$ is called invariant. Invariant Hopf 2-cocycles for $G$ form a group, which was described completely for connected $G$ [EG4, Theorem 7.8]. However, if $J$ is not invariant then the situation becomes much more interesting, since the cotriangular Hopf algebra $J O(G) J$ will be noncommutative. It is thus natural to study the algebra structure and representation theory of $J O(G) J$ in cases where the classification of Hopf 2-cocycles for $G$ is known. For example, for finite groups $G$, this was done in [EG3, Theorem 3.2] and [AEGN, Theorem 3.18].

The classification of Hopf 2-cocycles for connected nilpotent algebraic groups $G$ over $\mathbb{C}$ is also known [G]. For example, Hopf 2-cocycles in the unipotent case are classified by pairs $(H, \omega)$, where $H$ is a closed subgroup of $G$, called the support of $J$, and $\omega \in \wedge^2 \text{Lie}(H)^*$ is a non-degenerate 2-cocycle (equivalently, pairs $(\mathfrak{h}, r)$, where $\mathfrak{h}$ is a quasi-Frobenius Lie subalgebra of $\text{Lie}(G)$ and $r \in \wedge^2 \mathfrak{h}$ is a non-degenerate solution to the CYBE (see 2.3)). This was done in [EG2, Theorem 3.2], using Etingof–Kazhdan quantization theory [EK1, EK2, EK3]. Later, we extended Movshev’s theory on twisting of finite groups [Mov] to the algebraic group case [G, Section 3], and generalized the aforementioned classification to connected nilpotent algebraic groups, without
using Etingof–Kazhdan quantization theory [G, Corollary 5.2, Theorem 5.3].

Thus our goal in this paper is to study the algebra structure and representation theory of the cotriangular Hopf algebras $\mathcal{O}(G)_J$ for connected nilpotent $G$.

The organization of the paper is as follows. In Section 2 we recall some basic notions and results used in the sequel.

In Section 3 we consider the cotriangular Hopf algebras $\mathcal{O}(G)_J$ for unipotent $G$. We first show that $\mathcal{O}(G)_J$ is an iterated Ore extension of $\mathbb{C}$, thus is an affine Noetherian domain with Gelfand-Kirillov dimension $\dim(G)$ (see Corollary 3.2). Secondly, in Theorem 3.4 we prove that if $J$ is minimal (i.e., $J$ is supported on $G$) then $\mathcal{O}(G)_J \cong U(\mathfrak{g})$ as algebras, where $\mathfrak{g} := \text{Lie}(G)$, while in general, $\mathcal{O}(G)_J$ is a crossed product algebra of $\mathcal{O}(H)_J \cong U(\mathfrak{h})$ and the algebra $\mathcal{O}(G/H)$, where $H \subset G$ is the support of $J$ and $\mathfrak{h} := \text{Lie}(H)$ (see Theorem 3.8).

In Section 4 we analyze twisted function algebras on $(H,H)$-double cosets in unipotent $G$, and use [G], to study the quotient algebra $\mathcal{O}(Z)_J$ of $\mathcal{O}(G)_J$ for a double coset $Z$ in $H \backslash G / H$. In Theorems 4.5 and 4.6 we show that $\mathcal{O}(Z)_J$ does not contain a Weyl subalgebra if and only if $\mathcal{O}(Z)_J \cong U(\mathfrak{h})$ as algebras, if and only if, $H$ and $J$ are $g$-invariants for $g \in Z$.

In Section 5 we determine the finite dimensional irreducible representations of $\mathcal{O}(G)_J$ (see Theorem 5.2). Namely, in Theorem 5.1 we show that every irreducible representation of $\mathcal{O}(G)_J$ factors through a unique quotient algebra $\mathcal{O}(Z)_J$, and then deduce from Theorems 4.5 and 4.6 that $\mathcal{O}(Z)_J$ has a finite dimensional irreducible representation if and only if $\mathcal{O}(Z)_J \cong U(\mathfrak{h})$ as algebras, if and only if, $H$ and $J$ are $g$-invariants for $g \in Z$.

In Section 6 we give several examples that illustrate the results from Sections 3–5 (see Examples 6.1–6.5).

Finally, in Section 7 we use the results from Sections 3–5 to describe the algebra structure and representations of the cotriangular Hopf algebras $\mathcal{O}(G)_J$ for connected nilpotent $G$ (see Theorem 7.2).

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\footnote{We stress however, that both the classification of Hopf 2-cocycles and Movshev’s theory for arbitrary affine algebraic groups over $\mathbb{C}$ are still missing (see, e.g., [G] and references therein).}
2. Preliminaries

2.1. Hopf 2-cocycles. Let $A$ be a Hopf algebra over $\mathbb{C}$. A linear form $J : A \otimes A \to \mathbb{C}$ is called a Hopf 2-cocycle for $A$ if it has an inverse $J^{-1}$ under the convolution product $\ast$ in $\text{Hom}_\mathbb{C}(A \otimes A, \mathbb{C})$, and satisfies

$$\sum J(a_1 b_1, c)J(a_2, b_2) = \sum J(a, b_1 c_1)J(b_2, c_2),$$

$$J(a, 1) = \varepsilon(a) = J(1, a)$$

for all $a, b, c \in A$.

Given two Hopf 2-cocycles $K, J$ for $A$, one constructs a new algebra $K_AJ$ as follows. As vector spaces, $K_AJ = A$, and the new multiplication $Km_J$ is given by

$$Km_J(a \otimes b) = \sum K^{-1}(a_1, b_1)a_2 b_2 J(a_3, b_3), \ a, b \in A.$$

In particular, $JA_J$ is a Hopf algebra where $JA_J = A$ as coalgebras and the new multiplication $Jm_J$ is given by

$$(2.1) \quad Jm_J(a \otimes b) = \sum J^{-1}(a_1, b_1)a_2 b_2 J(a_3, b_3), \ a, b \in A.$$

Equivalently, $J$ defines a tensor structure on the forgetful functor $\text{Corep}(A) \to \text{Vec}$.

We also have two new unital associative algebras $A_J := 1A_J$ and $KA := KA_1$, with multiplication rules given respectively by

$$(2.2) \quad m_J(a \otimes b) = \sum a_1 b_1 J(a_2, b_2), \ a, b \in A,$$

and

$$(2.3) \quad Km_J(a \otimes b) = \sum K^{-1}(a_1, b_1)a_2 b_2, \ a, b \in A.$$

(For more details, see, e.g., [EGNO].)

Remark 2.1. The algebras $A_J, KA$ and $KA_J$ are called $(A, JA_J)$-biGalois, $(KA, A)$-biGalois and $(KA, JA_J)$-biGalois algebras, respectively.

Lemma 2.2. The comultiplication map $\Delta$ of $A$ determines an injective algebra homomorphism $\Delta : KA_J \overset{1:1}{\to} KA \otimes A_J$.

Proof. For every $a, b \in A$, we have

$$\Delta(a)\Delta(b) = \sum Km_J(a_1 \otimes b_1) \otimes m_J(a_2 \otimes b_2)$$

$$= \sum K^{-1}(a_1, b_1)a_2 b_2 \otimes a_3 b_3 J(a_4, b_4)$$

$$= \Delta \left( \sum K^{-1}(a_1, b_1)a_2 b_2 J(a_3 b_3) \right) = \Delta(Km_J(a \otimes b)),$$

\footnote{$JA_J$ is denoted also by $A^J$, e.g., in [G].}
as claimed. □

2.2. Cotriangular Hopf algebras. Recall that \((A, R)\) is cotriangular if \(R : A \otimes A \to \mathbb{C}\) is an invertible linear map under \(*\), such that \(R^{-1} = R_{21}\), and for every \(a, b, c \in A\), we have

\[
R(a, bc) = \sum R(a_1, b)R(a_2, c), \quad R(ab, c) = \sum R(b, c_1)R(a, c_2),
\]

and

\[
\sum R(a_1, b_1)b_2a_2 = \sum a_1b_1R(a_2, b_2).
\]

Recall that if \(R\) is non-degenerate, \((A, R)\) is called minimal, and in this case \(R\) defines two injective Hopf algebra maps \(A \overset{1 \rightarrow}{\rightarrow} A^*_{\text{fin}}\) from \(A\) into its finite dual Hopf algebra \(A^*_{\text{fin}}\). Recall also that any cotriangular Hopf algebra \((A, R)\) has a unique minimal cotriangular Hopf algebra quotient \([G, \text{Proposition 2.1}]\).

Given a Hopf 2-cocycle \(J\) for \(A\), \((JA_J, R_J)\) is also cotriangular, where \(R_J := J^{-1} \cdot R \cdot J\). (For more details, see, e.g., \([EGNO]\).)

**Lemma 2.3.** Assume \(A\) is commutative, and let \(J\) be a Hopf 2-cocycle for \(A\). If \(p \in A\) is primitive then for every \(a \in A\), we have

\[
R_J(p, a) = (J - J_{21})(p, a) = (J_{21}^{-1} - J^{-1})(p, a).
\]

**Proof.** Since \(p(1) = 0\), we have

\[
0 = J \cdot J^{-1}(p, a) = \sum J(p_1, a_1)J^{-1}(p_2, a_2)
= \sum J(p, a_1)J^{-1}(1, a_2) + \sum J(1, a_1)J^{-1}(p, a_2)
= (J + J^{-1})(p, a).
\]

Thus, we have

\[
R_J(p, a) = \sum J_{21}^{-1}(p_1, a_1)J(p_2, a_2)
= \sum J_{21}^{-1}(p, a_1)J(1, a_2) + \sum J_{21}^{-1}(1, a_1)J(p, a_2)
= (J + J_{21}^{-1})(p, a) = (J - J_{21})(p, a),
\]

as claimed. □

2.3. Quasi-Frobenius Lie algebras. Recall that a quasi-Frobenius Lie algebra is a Lie algebra \(\mathfrak{h}\) equipped with a non-degenerate skew-symmetric bilinear form \(\omega : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}\) satisfying

\[
\omega([x, y], z) + \omega([z, x], y) + \omega([y, z], x) = 0, \quad x, y, z \in \mathfrak{h}
\]

(i.e., \(\omega\) is a symplectic 2-cocycle on \(\mathfrak{h}\)).
Let $\mathfrak{g}$ be a Lie algebra. Recall that an element $r \in \wedge^2 \mathfrak{g}$ is a solution of the classical Yang-Baxter equation (CYBE) if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$ 

By Drinfeld [D], solutions $r$ of the CYBE in $\wedge^2 \mathfrak{g}$ are classified by pairs $(\mathfrak{h}, \omega)$, via $r = \omega - 1 \in \wedge^2 \mathfrak{h}$, where $\mathfrak{h} \subset \mathfrak{g}$ is a quasi-Frobenius Lie subalgebra with symplectic 2-cocycle $\omega$.

2.4. Ore extensions. Let $A$ be an algebra, and let $\delta : A \to A$ be an algebra derivation of $A$. Recall that the Ore extension $A[y; \delta]$ of $A$ is the algebra generated over $A$ by $y$, subject to the relations $ya - ay = \delta(a)$ for every $a \in A$. (See, e.g., [MR].)

2.5. Unipotent algebraic groups. Let $G$ be a unipotent algebraic group of dimension $m$ over $\mathbb{C}$. Recall that $A := O(G)$ is a finitely generated commutative irreducible pointed Hopf algebra, which is isomorphic to a polynomial algebra in $m$ variables as an algebra.

Assume we have a central extension

$$1 \to C \to G \to G' \to 1,$$

where $C \cong \mathbb{G}_a$ (= additive group). Then we can view $O(G')$ as a Hopf subalgebra of $O(G)$ via $\pi^*$. Let $O(C) = \mathbb{C}[z]$, $z$ is primitive. Choose $W$ in $O(G)$ that maps to $z$ under the surjective Hopf algebra map $\iota^* : O(G) \twoheadrightarrow O(C)$, with minimal possible degree with respect to the coradical filtration. Set

$$q(W) := \Delta(W) - W \otimes 1 - 1 \otimes W.$$ 

Lemma 2.4. $q(W)$ is a coalgebra 2-cocycle in $O(G')^+ \otimes O(G')^+$. 

Proof. Since the components of $q(W)$ have smaller degree than $W$, and are mapped to elements of degree $\leq 1$ in $\mathbb{C}[z]$, it follows that $q(W)$ belongs to $O(G')^+ \otimes O(G')^+$. Finally, $q(W)$ is a coalgebra 2-cocycle since $(\Delta \otimes \text{id})\Delta(W) = (\text{id} \otimes \Delta)\Delta(W)$. □

Lemma 2.5. The polynomial algebra $O(G)[W]$ has a unique Hopf algebra structure such that $O(G')$ is a Hopf subalgebra of $O(G)[W]$, and $\Delta(W) = W \otimes 1 + 1 \otimes W + q(W)$. Moreover, we have $O(G) \cong O(G')[W]$ as Hopf algebras.

Proof. It is clear that $O(G) \cong O(G')[W]$ as algebras, and that the Hopf algebra structure is well defined by Lemma 2.4. Finally, it is clear that $O(G) \cong O(G')[W]$ as Hopf algebras.
Recall that $G$ is obtained from $m$ successive 1-dimensional central extensions with kernel $\mathbb{G}_a$. Thus by Lemma 2.5 $A$ admits a filtration
\[(2.4) \quad \mathbb{C} = A_0 \subset A_1 \subset \cdots \subset A_i \subset \cdots \subset A_m = A\]
by Hopf subalgebras $A_i$, such that for every $1 \leq i \leq m$, $A_i = \mathbb{C}[y_1, \ldots, y_i]$ is a polynomial algebra and $q(y_i)$ is a coalgebra 2-cocycle in $A_{i-1}^* \otimes A_i^*$, with $q(y_1) = q(y_2) = 0$. We will sometime write $q(y_i) = \sum Y_{ij} \otimes Y_{ij}''$ and $(\text{id} \otimes \Delta)(q(y_i)) = \sum Y_{ij} \otimes Y_{ij}'' \otimes Y_{ij}'''.

Finally, let $H \subset G$ be a closed subgroup of codimension 1. It is well known that $H$ is normal in $G$, so $G/H \cong \mathbb{G}_a$ as algebraic groups.

**Lemma 2.6.** The exact sequence $1 \rightarrow H \hookrightarrow G \rightarrow G/H \rightarrow 1$ splits.

**Proof.** It is sufficient to show that the exact sequence of Lie algebras $0 \rightarrow \mathfrak{h} \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h} \rightarrow 0$ splits. But this is clear since $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ of codimension 1, so by choosing a splitting of vector spaces $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{C}x$, $x \in \mathfrak{g}/\mathfrak{h}$, we see that $x$ acts on $\mathfrak{h}$ by derivations. This implies the statement. □

3. The algebra structure of $\mathcal{O}(G)_J$ for unipotent $G$

In sections 3.1 and 3.2 $G$ will denote a unipotent algebraic group over $\mathbb{C}$ of dimension $m$, and $J$ will be a Hopf 2-cocycle for $G$ (i.e., for $\mathcal{O}(G)$).

3.1. Ring theoretic properties. Retain the notation from 2.5 and let $\cdot$ denote the multiplication in $\mathcal{O}(G)_J$. Set $Q := J - J_{21}$.

**Lemma 3.1.** The following hold:

1. \[(2.4)\] determines a Hopf algebra filtration on $\mathcal{O}(G)_J$:
   \[\mathbb{C} = A_0 \subset A_1 \subset \cdots \subset \mathcal{O}(G)_J \subset \cdots \subset \mathcal{O}(G)_J = \mathcal{O}(G)_J\]

2. For every $i$, the Hopf algebra $\mathcal{O}(A_i)_J$ is generated by $y_i$ over $\mathcal{O}(A_{i-1})_J$.

3. For every $i, j$, we have $J(y_i, y_j) + J^{-1}(y_i, y_j) = 0$.

4. For every $j < i$, we have
   \[
y_i \cdot y_j = y_i y_j + \sum Y_{ij} J(Y''_i, y_j) + \sum Y_{ij} J(y_i, Y''_j) + \sum Y_{ij} Y_{ij} J(Y''_i, Y''_j) + \sum J^{-1}(Y_i, Y_j) J(Y''_i, Y''_j) Y''_i Y''_j.
   \]

5. For every $j < i$, we have
   \[
   [y_i, y_j] := y_i \cdot y_j - y_j \cdot y_i
   = \sum Y_{ij} Q(Y''_i, y_j) + \sum Y_{ij} Q(y_i, Y''_j) + \sum Y_{ij} Y_{ij} Q(Y''_i, Y''_j)
   + \sum (J^{-1}(Y_i, Y_j) J(Y''_i, Y''_j) - J^{-1}_2(Y_i, Y_j) J_2(Y''_i, Y''_j)) Y''_i Y''_j.
   \]
Hence, \([y_i, y_j]\) belongs to \(A_{i-1}^+\).

(6) \(y_1, y_2\) are central primitives in \(J A_J\).

(7) The linear map \(\delta_i : J(A_{i-1})_J \rightarrow J(A_i)_{J}, s \mapsto [y_i, s]\), is an algebra derivation of \(J(A_{i-1})_J\) for every \(i\).

(8) For every \(i\), \(J(A_i)_J \cong J(A_{i-1})_J[y_i; \delta_i]\) as Hopf algebras.

Proof. (1)–(2) follow from (2.1) and Lemma 2.5 since each \(A_{i-1}\) is a Hopf subalgebra of \(A_i\). (3)–(4) follow from (2.1) and \(\varepsilon(y_i \cdot y_j) = 0\). (5)–(6) follow from (4), (7) from (5), and (8) from (2) and Lemma 2.5.

Corollary 3.2. The Hopf algebra \(J \mathcal{O}(G)_J\) is an affine Noetherian domain with Gelfand-Kirillov dimension \(\dim(G)\).

Proof. It follows from Lemma 3.1(8), by a simple induction, that \(J \mathcal{O}(G)_J\) is a finitely generated (i.e., affine) Noetherian domain. The claim about Gelfand-Kirillov dimension follows from [MR, Proposition 8.2.11] and Lemma 3.1(8) by simple induction.

Remark 3.3. One shows similarly that for every Hopf 2-cocycle \(K\) for \(G\), the algebra \(K \mathcal{O}(G)_J\) is an affine Noetherian domain with Gelfand-Kirillov dimension \(\dim(G)\).

3.2. The minimal case. Let \(H \subset G\) be the support of \(J\) (see [G, Section 3.1]). Then \(J\) is a minimal Hopf 2-cocycle for \(H\). Let \(\mathfrak{h}\) be the Lie algebra of \(H\).

Theorem 3.4. The \(R\)-form \(R^J\) induces an algebra isomorphism

\[R_+ : J \mathcal{O}(H)_J \cong \mathcal{U}(\mathfrak{h}).\]

Proof. Since \((J \mathcal{O}(H)_J, R^J)\) is a minimal cotriangular Hopf algebra, we have an injective homomorphism of Hopf algebras

\[R_+ : J \mathcal{O}(H)_J \rightarrow (J \mathcal{O}(H)_J)^*_{\text{fin}}, \ R_+(\alpha)(\beta) = R^J(\beta, \alpha)\]

(see 2.2). Since \((J \mathcal{O}(H)_J)^*_{\text{fin}} = J(\mathcal{O}(H)^*_{\text{fin}})_J\) (where the right hand side is a twisted coalgebra), we have \((J \mathcal{O}(H)_J)^*_{\text{fin}} = \mathcal{O}(H)^*_{\text{fin}} = \mathcal{U}(\mathfrak{h}) \rtimes \mathbb{C}[H]\) as algebras.

Let \(m = \mathcal{O}(H)^+\) be the maximal ideal of 1. We first show that the image of \(R_+\) is contained in the algebra of distributions \(\mathcal{O}(H)^*_{1} = \mathcal{U}(\mathfrak{h})\) supported at 1. Namely, that \(R_+(\alpha)\) vanishes on some power of \(m\) in the algebra \(\mathcal{O}(H)\) for every \(\alpha \in J \mathcal{O}(H)_J\). Indeed, if \(\alpha \in J \mathcal{O}(H)_J\) has degree \(n\) with respect to the coradical filtration of \(\mathcal{O}(H)\), then any summand in \((\alpha\text{-finite expression})\) \(\Delta^{n+1}(\alpha)\) has at least one \(\varepsilon\) as a tensorand. Since \(R^J(\varepsilon, \beta) = R^J(\beta, \varepsilon) = \beta(1) = 0\) for every \(\beta \in m\), it follows that \(R_+(\alpha)\) vanishes on some power of \(m\) in the algebra \(J \mathcal{O}(H)_J\). But it is
clear that every such power contains some power of $m$ in the algebra $\mathcal{O}(H)$, as desired. Thus, we have an injective algebra homomorphism $R_+: j\mathcal{O}(H)_J \xrightarrow{1:1} U(\mathfrak{h})$.

To show that $R_+$ is surjective, it suffices to show that $\mathfrak{h}$ belongs to the image of $R_+$. Indeed, let $V \subset \mathfrak{m}$ be the orthogonal complement of $\mathfrak{m}^2$ (with respect to $\mathfrak{r}^J$). Then $R_+$ maps $V$ injectively into $\mathfrak{h}$ (as $\mathfrak{h} = (\mathfrak{m}/\mathfrak{m}^2)^*$), and since $R^J$ is non-degenerate, $\dim(V) = \dim(\mathfrak{m}/\mathfrak{m}^2)$. Thus $R_+(V) = \mathfrak{h}$, as required.

**Corollary 3.5.** We have an equivalence $\text{Rep}(j\mathcal{O}(H)_J) \cong \text{Rep}(U(\mathfrak{h}))$ of abelian irreducible categories. In particular, $j\mathcal{O}(H)_J$ has a unique finite dimensional irreducible representation (as $\mathfrak{h}$ is nilpotent).

**Remark 3.6.** By [G], Theorems 4.7, 5.1, $\mathcal{O}(H)_J$ and $j\mathcal{O}(H)$ are Weyl algebras with left and right action of $H$ by algebra automorphisms, respectively. The induced action of $\mathfrak{h}$ on $\mathcal{O}(H)_J$ by derivations determines a symplectic 2-cocycle $\omega \in \Lambda^2 \mathfrak{h}^*$. We have $U^\omega(\mathfrak{h}) \cong \mathcal{O}(H)_J$ as $H$-algebras, where $H$ acts on $U^\omega(\mathfrak{h})$ and $\mathcal{O}(H)_J$ by conjugation and left translations, respectively. Similarly, $U^{-\omega}(\mathfrak{h}^{\text{op}}) \cong j\mathcal{O}(H)$ as $H$-algebras. Thus, $j\mathcal{O}(H) \otimes \mathcal{O}(H)_J \cong U^{-\omega}(\mathfrak{h}^{\text{op}} \oplus \mathfrak{h})$ as $H$-algebras.

Now since by Lemma 2.2, we have an algebra isomorphism

$$\Delta : j\mathcal{O}(H)_J \xrightarrow{\cong} (j\mathcal{O}(H) \otimes \mathcal{O}(H)_J)^H,$$

where $h \in H$ acts on $j\mathcal{O}(H) \otimes \mathcal{O}(H)_J$ via $\rho_h \otimes \lambda_h$, it follows that

$$j\mathcal{O}(H)_J \xrightarrow{\cong} U^{-\omega}(\mathfrak{h}^{\text{op}} \oplus \mathfrak{h})^H,$$

as algebras. Thus, by Theorem 3.4, we have an algebra isomorphism

$$U(\mathfrak{h}) \xrightarrow{\cong} U^{-\omega}(\mathfrak{h}^{\text{op}} \oplus \mathfrak{h})^H.$$

3.3. **The general case.** Recall that $\mathcal{O}(G/H)$ and $\mathcal{O}(H\backslash G)$ are left and right coideal subalgebras of $\mathcal{O}(G)$, respectively. It follows that $j\mathcal{O}(G/H)_J = j\mathcal{O}(G/H)$ is a subalgebra of both $j\mathcal{O}(G)_J$ and $j\mathcal{O}(G)$.

**Lemma 3.7.** The subalgebra $\mathcal{O}(H\backslash G/H) \subset j\mathcal{O}(G)_J$ is central.

**Proof.** Follows from $\Delta(\mathcal{O}(H\backslash G/H)) \subset \mathcal{O}(H\backslash G) \otimes \mathcal{O}(G/H)$.

**Theorem 3.8.** The algebra $j\mathcal{O}(G)_J$ is isomorphic to a crossed product algebra

$$j\mathcal{O}(G)_J \cong j\mathcal{O}(G/H) \#^\sigma j\mathcal{O}(H)_J$$

for some invertible 2-cocycle $\sigma : (j\mathcal{O}(H)_J)^{\otimes 2} \to j\mathcal{O}(G/H)$

3I.e., invertible in the convolution algebra Hom((j\mathcal{O}(H)_J)^{\otimes 2}, j\mathcal{O}(G/H)).

4I.e., $\beta \cdot (\alpha \tilde{\alpha}) = (\beta_1 \cdot \alpha)(\beta_2 \cdot \tilde{\alpha})$ and $\beta \cdot (\tilde{\beta} \cdot \alpha) = \sigma(\beta_1, \tilde{\beta}_1)(\beta_2 \tilde{\beta}_2 \cdot \alpha)\sigma^{-1}(\beta_3, \tilde{\beta}_3)$. 

as vector spaces, and the product is given by
\[(\alpha \otimes \beta)(\tilde{\alpha} \otimes \tilde{\beta}) = \alpha(\beta_1 \cdot \tilde{\alpha})\sigma(\beta_2, \tilde{\beta}_2) \otimes \beta_3 \tilde{\beta}_2, \quad \alpha, \tilde{\alpha} \in \mathcal{O}(G/H), \beta, \tilde{\beta} \in \mathcal{O}(H).\]

**Proof.** We have a Hopf quotient \(\iota^* : j\mathcal{O}(G) \to j\mathcal{O}(H),\) with Hopf kernel \(j\mathcal{O}(G/H) = j\mathcal{O}(G/H).\) Thus we have an \(j\mathcal{O}(H)\)-extension \(j\mathcal{O}(G/H) \subset j\mathcal{O}(G)\) of algebras. We claim it is cleft. Indeed, choose a regular section \(j\) to the inclusion morphism \(\iota : H \to G\) (this is possible since \(G\) is unipotent). Then \(\gamma := \varphi : j\mathcal{O}(G) \to j\mathcal{O}(G)\) is an invertible \(j\mathcal{O}(H)\)-comodule map\(^5\) as required. Hence, the statement follows from [Mon, Theorem 7.2.2], with \(\sigma\) and weak action given by
\[\sigma(\beta, \tilde{\beta}) = \gamma(\beta_1)\gamma(\tilde{\beta}_1)\gamma^{-1}(\beta_2 \tilde{\beta}_2), \quad \beta, \tilde{\beta} \in j\mathcal{O}(H),\]
and
\[\beta \cdot \alpha = \gamma(\beta_1) \alpha \gamma^{-1}(\beta_2), \quad \beta \in j\mathcal{O}(H), \alpha \in \mathcal{O}(G/H),\]
where \(\gamma^{-1}\) is the inverse of \(\gamma.\)

**Remark 3.9.** If \(H\) is normal in \(G\) then \(j\mathcal{O}(G/H) = \mathcal{O}(G/H)\) is commutative. However, if \(H\) is not normal in \(G\) then the algebra \(j\mathcal{O}(G/H)\) is typically not commutative (see Example 6.4).

### 3.4. One sided twisted algebras
Let \(L \subset H\) be a closed subgroup. Since \(\mathcal{O}(H/L)\) is a left coideal subalgebra of \(\mathcal{O}(H),\) it follows that \(j\mathcal{O}(H/L)\) is a subalgebra of \(j\mathcal{O}(H).\) Moreover, \(j\mathcal{O}(H/L) = (j\mathcal{O}(H))^L\) is a subalgebra of the Weyl algebra \(j\mathcal{O}(H) \cong U^\omega(h)\) [G, Theorem 4.7].

**Question 3.10.** What is the algebra structure of \(j\mathcal{O}(H/L)\)?

We have the following partial answers to Question 3.10.

**Theorem 3.11.** Let \(N \subset H\) be a closed normal subgroup. Then the following hold:

1. There exists a closed subgroup \(L \subset H\) containing \(N\) such that
   \[j\mathcal{O}(H/N) \cong \mathcal{O}(L/H) \otimes \mathcal{U}_n\]
as algebras, where \(2n = \dim(L) - \dim(N)\).
2. If \(2\dim(N) < \dim(H),\) \(j\mathcal{O}(H/N)\) contains a Weyl subalgebra.

**Proof.** (1) Since \(N\) is normal in \(H,\) \(\mathcal{O}(H/N)\) is a Hopf subalgebra of \(\mathcal{O}(H).\) Thus \(J\) restricts to a Hopf 2-cocycle of \(H/N.\) By [G, Theorem 3.1], there exists a closed subgroup \(L\) of \(H\) containing \(N\) such that \(L/N \subset H/N\) is the support of \(J.\) Then by [G, Theorem 4.7], we have an algebra isomorphism
\[j\mathcal{O}(H/N) \cong \mathcal{O}((L/N)/(H/N)) \otimes \mathcal{U}_n \cong \mathcal{O}(L/H) \otimes \mathcal{U}_n,\]

---

\(^5\)I.e., invertible in the convolution algebra \(\text{Hom}(j\mathcal{O}(H), j\mathcal{O}(G)).\)
as claimed.

(2) By (1), it suffices to show that the restriction of \( J \) to \( O(H/N) \) is not trivial (since then \( n \geq 1 \)). Suppose otherwise. Then \( jO(H/N)_J \) is isotropic with respect to \( R^J \). Thus, \( \dim(H/N) \leq \dim(H)/2 \), which is not the case. \( \square \)

**Corollary 3.12.** Let \( L \subset H \) be a closed subgroup and let \( N \) be its normal closure. If \( 2 \dim(N) < \dim(H) \) then \( jO(H/L) \) contains a Weyl subalgebra, or equivalently, \( jO(H/L) \) is noncommutative. \( \square \)

### 3.5. 1-dimensional central extensions.

Suppose we have a central extension
\[
1 \to C \to G \xrightarrow{\pi} \overline{G} \to 1,
\]
such that \( O(C) = \mathbb{C}[z] \) (\( z \) is primitive), and let \( W \) in \( O(G) \) be as in 2.5. Then by Lemma 3.1, we have an isomorphism of Hopf algebras
\[
jO(G)_J \cong jO(G)_J[W; \delta],
\]
where the derivation \( \delta \) is given by \( \delta(V) = [W, V] \) for every \( V \in jO(G)_J \), and \( q(W) \) is in \( O(G)^+ \otimes O(G)^+ \).

Set \( \overline{H} := \pi(H) \), and let \( L \subset \overline{H} \) be the support of the restriction of \( J \) to \( O(G) \). By [G, Proposition 4.6], \( L \) has codimension \( \leq 1 \) in \( \overline{H} \).

#### 3.5.1. \( C \cap H = \{1\} \).
In this case, \( L = \overline{H} \). Write
\[
q(W) = \sum W' \otimes W'' \in O(G)^+ \otimes O(G)^+
\]
in the shortest possible way. Set \( S := J^{-1} - J^{-1}_2 \).

**Lemma 3.13.** We can assume that \( W \in O(G/H)^+ \), and then have
\[
[W, V] = \sum S(W', V_1)W''V_2 \text{ for every } V \in jO(G)_J.
\]

**Proof.** The first claim follows from \( C \cap H = \{1\} \). Since \( \mathcal{I}(H) \) is a Hopf ideal and \( O(G/H) \) is a left coideal in \( O(G) \), \( \Delta(W) \) lies in \( \mathcal{I}(H) \otimes O(G) \otimes O(G/H) + O(G)^{\otimes 2} \otimes \mathcal{I}(H) + O(G) \otimes \mathcal{I}(H) \otimes O(G/H) \)
which implies the second claim. \( \square \)

#### 3.5.2. \( C \subset H \).
In this case, \( L \) has codimension 1. Let \( A := \overline{H}/L \). Then \( O(A) = \mathbb{C}[x] \), \( x \) is primitive, and the quotient map \( \sigma : \overline{H} \to A \) induces an injective homomorphism of Hopf algebras \( \sigma^* : \mathbb{C}[x] \xrightarrow{1:1} jO(\overline{H})_J \).

Thus, we can view \( \mathbb{C}[x] \) as a Hopf subalgebra of \( jO(\overline{H})_J \) via \( \sigma^* \). Also, by Lemma 2.6, we can choose a splitting homomorphism of groups \( j : A \xrightarrow{1:1} \overline{H} \) of \( \sigma \), and view \( A \) as a subgroup of \( \overline{H} \subset \overline{G} \) via \( j \). Then \( j^* : O(\overline{H}) \to \mathbb{C}[x] \) is a surjective homomorphism of Hopf algebras.
Lemma 3.14. The Hopf algebra surjective map \( \iota^*_H : \mathcal{O}(G) \to \mathcal{O}(H) \) restricts to an algebra surjective map \( \iota^*_H : \mathcal{O}(L\backslash G/L) \to \mathcal{O}(A) \).

Proof. Clearly, \( \iota^*_H \) maps \( \mathcal{O}(L\backslash G/L) \) onto \( \mathcal{O}(L\backslash H/L) \). Since \( L \) is normal in \( H \), we have \( \mathcal{O}(L\backslash H/L) = \mathcal{O}(A) \), which implies the statement. \( \square \)

Consider \( \mathcal{O}(G)_J \) as a left comodule algebra over \( \mathbb{C}[x] \) via
\[
(j^* \circ (\iota_H^* \otimes \text{id}) \circ \Delta : \mathcal{O}(G)_J \to \mathbb{C}[x] \otimes \mathcal{O}(G)_J).
\]
Let \( B \subset \mathcal{O}(G)_J \) be the coinvariant subalgebra. Pick \( X \in \mathcal{O}(L\backslash G/L) \) such that \( \iota^*_H(X) = x \).

Lemma 3.15. The multiplication map \( B \otimes \mathbb{C}[X] \to \mathcal{O}(G)_J \) is an isomorphism of algebras.

Proof. Follows since by Lemma 3.7 \( X \) is central in \( \mathcal{O}(G)_J \). \( \square \)

3.6. The algebra \( \mathcal{O}(G)_J \). Fix \( g \in G \). Set \( \text{Ad}g := \rho_g \circ \lambda_g \), and \( J^g := J \circ (\text{Ad}g \otimes \text{Ad}g) \).

Lemma 3.16. \( \lambda_{g^{-1}} : \mathcal{O}(G)_J \to \mathcal{O}(G)_J \) is an algebra isomorphism.

Proof. Straightforward. \( \square \)

4. The quotient algebras \( \mathcal{O}(Z_g)_J \)

Retain the notation of Section 3. Every double coset \( Z = HgH \) in \( H\backslash G/H \) is an orbit of the left action of the unipotent algebraic group \( H \times H \) on \( G \), given by \( (a, b) \cdot g := agb^{-1} \). Hence \( Z \) is an irreducible closed subset of \( G \) by the theorem of Kostant and Rosenlicht, and it has dimension \( 2 \dim(H) - \dim(H \cap gHg^{-1}) \) \( (g \in Z) \).

Let \( I(Z) \subset \mathcal{O}(G) \) be the defining ideal of the double coset \( Z \). Since \( Z \) is irreducible, \( I(Z) \) is a prime ideal. Clearly, \( \bigcap_Z I(Z) = 0 \).

Now fix a double coset \( Z_g = HgH \). Set \( H_g := H \cap gHg^{-1} \), and consider the embedding
\[
\theta : H_g \to H \times H, \quad a \mapsto (a, g^{-1}ag),
\]
of \( H_g \) as a closed subgroup of \( H \times H \). The subgroup \( \theta(H_g) \) acts on \( H \times H \) from the right via \( (x, y)\theta(a) = (xa, g^{-1}a^{-1}gy) \), \( x, y \in H \) and \( a \in H_g \). Let \( (x, y) \) denote the orbit of \( (x, y) \) under this action. Then we have an isomorphism of affine varieties
\[
(H \times H)/\theta(H_g) \cong Z_g, \quad (x, y) \mapsto xgy.
\]

The above right action induces a left action of \( \theta(H_g) \) on \( \mathcal{O}(H)^{\otimes 2} \), given by \( (\theta(a)(\alpha \otimes \beta))(x, y) = \alpha(xa)\beta(g^{-1}a^{-1}gy) \), where \( x, y \in H \) and \( a \in H_g \). In other words, the action of \( \theta(a) \) is via \( \rho_a \otimes \lambda_{g^{-1}ag} \), where \( \lambda, \rho \)
are the left, right regular actions of $G$ on $\mathcal{O}(G)$. Let $(\mathcal{O}(H) \otimes \mathcal{O}(H))^{\theta(H_g)}$ denote the subalgebra of invariants under this action. Then we have an algebra isomorphism

$$\mathcal{O}(Z_g) \cong (\mathcal{O}(H) \otimes \mathcal{O}(H))^{\theta(H)}.$$ 

Equivalently, the surjective algebra homomorphism

$$m_g^* := (\iota^* \otimes \iota^*)(\text{id} \otimes \lambda_{g^{-1}})\Delta : \mathcal{O}(G) \to (\mathcal{O}(H) \otimes \mathcal{O}(H))^{\theta(H)}$$

has kernel $\mathcal{I}(Z_g)$.

**Proposition 4.1.** The map $m_g^*$ determines a surjective algebra homomorphism

$$m_g^* : \mathcal{O}(G)_J \to (\mathcal{O}(H) \otimes \mathcal{O}(H))^{\theta(H)}.$$ 

In particular, $\mathcal{I}(Z_g)$ is a two sided ideal in $\mathcal{O}(G)_J$.

**Proof.** Since $\iota^* : \mathcal{O}(G)_J \to \mathcal{O}(H)_J$ is an algebra homomorphism, it suffices to show that $(\text{id} \otimes \lambda_{g^{-1}})\Delta$ is an algebra homomorphism. To this end, notice that we have $\Delta \circ \lambda_{g^{-1}} = (\lambda_{g^{-1}} \otimes \text{id}) \circ \Delta$. Thus, using (2.1)-(2.3), we get that for every $\alpha, \beta \in \mathcal{O}(G)$,

$$(\text{id} \otimes \lambda_{g^{-1}})\Delta(\alpha \beta)$$

$$= (\text{id} \otimes \lambda_{g^{-1}})\Delta \left( \sum J^{-1}(\alpha_1, \beta_1)\alpha_2\beta_2 J(\alpha_3, \beta_3) \right)$$

$$= \sum J^{-1}(\alpha_1, \beta_1)\alpha_2\beta_2 \otimes \lambda_{g^{-1}}(\alpha_3\beta_3) J(\alpha_4, \beta_4)$$

$$= \sum (\alpha_1 \otimes \lambda_{g^{-1}}(\alpha_2)) (\beta_1 \otimes \lambda_{g^{-1}}(\beta_2))$$

$$= (\text{id} \otimes \lambda_{g^{-1}})\Delta(\alpha)(\text{id} \otimes \lambda_{g^{-1}})\Delta(\beta),$$

as required. \qed

For $g \in G$, set $\mathcal{O}(Z_g)_J := \mathcal{O}(G)_J/\mathcal{I}(Z_g)$.

**Corollary 4.2.** For every $g \in G$, the algebra $\mathcal{O}(Z_g)_J$ is an affine Noetherian domain of Gelfand-Kirillov dimension $2\dim(H) - \dim(H_g)$. In particular, $\mathcal{I}(Z_g)$ is a completely prime two sided ideal of $\mathcal{O}(G)_J$.

**Proof.** Since by [G, Theorem 4.7], $\mathcal{O}(H) \otimes \mathcal{O}(H)_J$ is a Weyl algebra, the claim follows from Corollary 3.2 and Proposition 4.1. \qed

**Remark 4.3.** Let $K$ be a Hopf 2-cocycle for $\mathcal{O}(G)$ with support $\tilde{H}$. For every $g \in G$, let $Z_g := \tilde{H}gH$ be the $(\tilde{H}, H)$-double coset of $g$, let $N_g := \tilde{H} \cap gHg^{-1}$, and let

$$d_g := \frac{1}{2} \left( \dim(H) + \dim(\tilde{H}) \right) - \dim(N_g).$$
(By [G] Theorem 5.1, \(\dim(H)\) and \(\dim(\tilde{H})\) are even, so \(d_g\) is an integer.) Then \(\mathcal{I}(Z_g)\) is a completely prime two sided ideal of \(K\mathcal{O}(G)\), and \(K\mathcal{O}(Z_g)_J := (K\mathcal{O}(G))/\mathcal{I}(Z_g)\) is an affine Noetherian domain with Gelfand-Kirillov dimension \(\dim(Z_g) = \dim(H) + \dim(\tilde{H}) - \dim(N_g)\).

**Remark 4.4.** By Proposition 4.1 and [G] Theorem 4.7, if \(H_g\) is trivial then \(\mathcal{O}(Z_g)^J \cong \mathcal{W}^{\omega}(\mathfrak{h})\) is a Weyl algebra.

**Theorem 4.5.** Assume \(H\) is \(g\)-invariant, and let \(\omega_g := \omega^g - \omega\). Then we have an algebra isomorphism

\[
\lambda_{\mathcal{O}(Z_g)} : \mathcal{O}(Z_g)^J \cong \mathcal{O}(H)^J \cong U^{\omega_g}(\mathfrak{h}).
\]

In particular, a maximal Weyl subalgebra of \(\mathcal{O}(Z_g)\) has dimension \(r\), where \(r \in 2\mathbb{Z}^{\geq 0}\) is the rank of \(\omega_g\) restricted to \(\mathfrak{h}\).

**Proof.** Since \(H\) is \(g\)-invariant if and only if \(\text{Ad}g\) defines a Hopf algebra isomorphism \(\mathcal{O}(H) \overset{\sim}{\rightarrow} \mathcal{O}(H)\), \(J^g\) is a minimal Hopf 2-cocycle for \(H\). Clearly, \(J^g\) corresponds to the symplectic 2-cocycle \(\omega^g\) of \(\mathfrak{h}\).

Now since \(\lambda_g^{-1}\) maps \(\mathcal{I}(gH)\) isomorphically onto \(\mathcal{I}(H)\), it follows from Lemma 3.16 that \(\lambda_{\mathcal{O}(Z_g)}\) induces an algebra isomorphism

\[
\lambda_{\mathcal{O}(Z_g)} : \mathcal{O}(Z_g)_J \cong \mathcal{O}(G)/\mathcal{I}(gH) \cong \mathcal{O}(G)/\mathcal{I}(H).
\]

Finally, by Theorem 3.4 we have algebra isomorphisms

\[
\mathcal{O}(G)/\mathcal{I}(H) \cong \mathcal{O}(G)/\mathcal{I}(H) \cong \mathcal{O}(G)^J \cong U^{\omega_g}(\mathfrak{h}),
\]

as desired. \(\square\)

Next we consider the case where \(H\) is not \(g\)-invariant, i.e., the case \(d := \dim(H/H_g) > 0\).

**Theorem 4.6.** If \(H\) is not \(g\)-invariant then the algebra \(\mathcal{O}(Z_g)^J\) contains a Weyl subalgebra.

**Proof.** The proof is by induction on the dimension \(m\) of \(G\), \(m \geq 4\).

Assume \(m = 4\). Since \(G\) is not commutative, \(\dim(H) = 2\). Since \(H\) is properly contained in a proper normal subgroup \(N\) of \(G\), it follows that \(N\) is the Heisenberg group of dimension 3. Thus, the induction base is given in Example 6.2.

Assume \(m \geq 5\). Since \(G\) is unipotent, we have a central extension

\[1 \rightarrow C \rightarrow G \rightarrow \tilde{G} \rightarrow 1,
\]

where \(C \cong \mathbb{C}_a\) as in 3.3. Let \(\mathcal{O}(C) = \mathbb{C}[z]\), \(W \in \mathcal{O}(G)\) and \(L \subset \tilde{H} \subset \tilde{G}\) be as in 3.3. Set \(\tilde{g} := \pi(g)\) and \(\tilde{d} := d_g\).

There are two possible cases: Either \(H \cap C\) is trivial or \(C \subset H\).
4.1. $H \cap C = \{1\}$. In this case, we are in the situation of Lemma 4.8. Consider the regular surjective map $\pi : HgH \to L\bar{g}L$. We have $\pi^{-1}(\bar{g}) = gC \cap HgH$, and $\dim(HgH) - \dim(L\bar{g}L) = \dim(\pi^{-1}(\bar{g}))$.

**Lemma 4.7.** Exactly one of the following holds:

1. $\pi^{-1}(\bar{g}) = \{g\}$. Equivalently, $\dim(HgH) = \dim(L\bar{g}L)$.
2. $\pi^{-1}(\bar{g}) = gC$. Equivalently, $\dim(HgH) = \dim(L\bar{g}L) + 1$.

**Proof.** Follows since $gC \cap HgH \subset gC$ and $\dim(gC) = 1$. □

If Lemma 4.7(1) holds, then $\mathcal{O}(Z_g) \cong \mathcal{O}(Z_{\bar{g}})$ and $d = \bar{d}$, so the claim follows by induction.

Otherwise, Lemma 4.7(2) holds. Then $d = \bar{d} + 1$. If $\bar{d} > 0$ then $\mathcal{O}(Z_{\bar{g}})$ contains a Weyl subalgebra by the induction assumption, and since $\mathcal{O}(Z_{\bar{g}})$ is a subalgebra of $\mathcal{O}(Z_g)$, it is a subalgebra of $\mathcal{O}(Z_{\bar{g}})$, so does $\mathcal{O}(Z_{\bar{g}})$.

Otherwise $\bar{d} = 0$. So, $d = 1$. Thus $L$ is $\bar{g}$–invariant, $H_g$ is normal in $H$, $HgH = CgH$, and $\dim(L) = \dim(H_g) + 1$.

**Lemma 4.8.** The following hold:

1. $W$ is not constant on $HgH = CgH$.
2. $\rho_g(W) - W - W(g)$ vanishes on $C$ and $H_g$, but not on $H$.

**Proof.** Since $W = z$ on $C$, each $W'$ vanishes on $C$. Hence by Lemma 3.13 $W(cgh) = W(cg) = W(c) + W(g) + \sum W'(c)W''(g) = c + W(g)$ for every $c \in C$ and $h \in H$, which implies (1) and the first part of (2). Since $\rho_g(W)(ghg^{-1}) = W(gh) = W(g)$ for every $ghg^{-1}$ in $H_g$, and $W$ vanishes on $H$, the second part of (2) follows too. □

For $l \in L$, let $\bar{l} \in L$ be the unique element such that $l = \bar{g}\bar{l}g^{-1}$. Let $h, h \in H$ be the unique elements such that $l = \pi(h)$ and $\bar{l} = \pi(h)$. Let $\tau(l) := ghg^{-1}h^{-1}$. Then $\tau(l) \in C$.

**Lemma 4.9.** $\tau : L \to C$ is a group homomorphism, and we have a splitting exact sequence of algebraic groups $1 \to H_g \xrightarrow{\pi} L \xrightarrow{\tau} C \to 1$.

**Proof.** Follows from Lemma 2.6 since $H_g$ has codimension 1 in $H$. □

By Lemma 4.9 we have an injective homomorphism of Hopf algebras $\tau^* : \mathcal{O}(C) \xrightarrow{1:1} \mathcal{O}(L)$. Let $p := \tau^*(z)$. Then $p$ is a nonzero primitive element in $\mathcal{O}(L)$ that generates the defining ideal of $\pi(H_g)$ in $\mathcal{O}(L)$.

**Lemma 4.10.** We may assume $\tau^*(\rho_g(W) - W - W(g)) = p$.

**Proof.** Consider the surjective Hopf algebra map $f : \mathcal{O}(G) \to \mathcal{O}(CH)$ induced by the inclusion of groups $CH \subset G$. Since $W$ vanishes on $H$, and restricts to $z$ on $C$, it follows that $f(W) = z$, and $f(W')$ is a primitive element in $\mathcal{O}(CH)$ that vanishes on $C$ for every $W'$. □
Thus, \( \psi_L^*(\rho_q(W) - W - W(g)) = \sum \psi_L^*(W')W''(g) \) is a nonzero primitive element in \( \mathcal{O}(L) \) by Lemma 4.8 and since it vanishes on the defining ideal of \( \pi(H_g) \) in \( \mathcal{O}(L) \), it must be proportional to \( p \).

Since \( R^J \) is non-degenerate on \( \mathcal{O}(L)^J \), there exists \( X \in \mathcal{O}(\overline{G}) \) such that \( R^J(p, X) = 1 \). Choose such \( X \) with minimal possible degree with respect to the coradical filtration, and write \( q(X) = \sum_i X_i \otimes Y_i \) in the shortest possible way. Then \( R^J(p, X_i) = 0 \) for every \( i \).

**Proposition 4.11.** We have \( [W, X] \equiv 1 \) on \( HgH \).

**Proof.** Since each \( W'' \) is in \( \mathcal{O}(\overline{G}/L) \), and \( R^J(p, X_i) = 0 \) for every \( i \), we have by Lemma 3.13

\[
[W, X](\bar{g}l) = \sum S(W', X_1)W''(\bar{g}l)X_2(\bar{g}l) = \sum S(W', X_1)W''(g)X_2(\bar{g}l)
= \sum S(W', X)W''(g) + \sum_i S(W', X_i)W''(g)Y_i(\bar{g}l)
= S(\rho_g(W) - W - W(g), X) + \sum_i S(\rho_g(W) - W - W(g), X_i)Y_i(\bar{g}l)
= S(p, X) + \sum_i S(p, X_i)Y_i(\bar{g}l) = R^J(p, X) + \sum_i R^J(p, X_i)Y_i(\bar{g}l)
= R^J(p, X)
\]

for every \( l \in L \), where the equality before last follows from Lemma 2.3. Thus \( [W, X] \equiv 1 \) on \( HgH \), as claimed.

4.2. \( C \subset H \). In this case, we are in the situation of 3.5.2 and \( W \) does not vanish on \( H \).

4.2.1. \( A \) is \( \bar{g} \)-invariant. In this case, we have \( \overline{HgH} = ALgL = L\bar{g}LA \), and \( d = d > 0 \).

**Proposition 4.12.** The algebra \( j\mathcal{O}(Z_g) \) contains a Weyl subalgebra.

**Proof.** Since \( j\mathcal{O}(\overline{HgH}) \) is a subalgebra in \( j\mathcal{O}(Z_g) \) via \( \pi^* \), it suffices to show that \( j\mathcal{O}(\overline{HgH}) \) contains a Weyl subalgebra.

Now since \( \overline{d} > 0 \), the algebra \( j\mathcal{O}(Z_g) \) contains a Weyl subalgebra by the induction assumption. Let \( \alpha, \beta \) in \( j\mathcal{O}(\overline{G}) \), such that \( [\alpha, \beta] \equiv 1 \) on \( L\bar{g}L \). By Lemma 3.15 we can write \( \alpha = \sum_i \alpha_iX^i \) and \( \beta = \sum_i \beta_iX^i \), where \( \alpha_i, \beta_i \) are in \( B \), and \( [\alpha, \beta] = \sum_{i,j} [\alpha_i, \beta_j]X^{i+j} \).

Since \( X \) is \( L \)-bi-invariant, we have \( X \equiv X(g) \) on \( L\bar{g}L \). If \( X(g) = 0 \), then \( X^i(g) = 0 \) for all \( i \geq 1 \), hence \( [\alpha_0, \beta_0] = [\alpha, \beta] \equiv 1 \) on \( L\bar{g}L \). But \( [\alpha_0, \beta_0] \) is in \( B \) (as \( \alpha_0, \beta_0 \) are), so \( [\alpha_0, \beta_0] \) is \( A \)-invariant. Thus, \( [\alpha_0, \beta_0] \equiv 1 \) on \( \overline{HgH} = AL\bar{g}L = L\bar{g}LA \), and we are done.
Otherwise, $X(\bar{g}) \neq 0$. We may assume $X(\bar{g}) = 1$. Then we have
\[ \sum_{i,j} [\alpha_i, \beta_j] = [\alpha, \beta] \equiv 1 \text{ on } L\bar{g}L. \]
Set $\tilde{\alpha} := \sum_i \alpha_i$ and $\tilde{\beta} := \sum_i \beta_i$. Then $\tilde{\alpha}, \tilde{\beta}$ are in $B$, and we have $[\tilde{\alpha}, \tilde{\beta}] \equiv 1$ on $L\bar{g}L$, hence on $\overline{H\bar{g}H}$, as above.

4.2.2. $A$ is not $\bar{g}$-invariant. In this case, $A\bar{g}A$ is 2-dimensional and $d = \bar{d} + 1$. Set $\varphi := j^* \circ \iota_L^* : \mathcal{O}(\overline{G}) \to \mathcal{O}(A)$.

**Lemma 4.13.** There exists $V \in \mathcal{O}(\overline{G})^+$ such that the following hold:

1. $\varphi(\rho_g(V))$ and $\varphi(\lambda_{g^{-1}}(V))$ are algebraically independent, and $V(\bar{g}) = 0$. In particular, $V$ is not primitive.
2. $\varphi(q(V)) = \varphi(q(V))_{21} \neq 0$. In particular, $\varphi(V)$ is not primitive.

**Proof.** (1) Since $A$ is not $\bar{g}$-invariant, the first claim follows, and replacing $V$ by $V - V(\bar{g})$ if necessary, we may assume $V(\bar{g}) = 0$. Since either $\varphi(\rho_g(V))$ or $\varphi(\lambda_{g^{-1}}(V))$ is not primitive, $V$ is not primitive.

(2) The first claim follows from $\Delta(\varphi(V)) = \Delta^{op}(\varphi(V))$ (as $A$ is commutative). If $\varphi(q(V)) = \varphi(q(V))_{21} = 0$, then $\varphi(\rho_g(V)) = \varphi(V)$ and $\varphi(\lambda_{g^{-1}}(V)) = \varphi(V)$, which is a contradiction. Thus, $\varphi(V)$ is not primitive, as claimed.

Pick $V \in \mathcal{O}(\overline{G})^+$ as in Lemma 4.13 with minimal possible degree $\ell \geq 2$ with respect to the coradical filtration. By Lemma 3.14, we may assume $V \in \mathcal{O}(L \backslash \overline{G}/L)^+$. Write $q(V) = \sum_i X_i \otimes Y_i$. Then for every $i$, we have $X_i \in \mathcal{O}(L \backslash \overline{G})^+$ and $Y_i \in \mathcal{O}(\overline{G}/L)^+$.

**Lemma 4.14.** We have $[W, V] = \sum_i S(W, X_i)Y_i - \sum_i S(W, Y_i)X_i$.

**Proof.** Since $Y_i$ and $V$ lie in $\mathcal{O}(\overline{G}/L)$, it follows from (2.1) that
\[ [W, V] = \sum_i S(W, X_i)Y_i - \sum_i S(W, Y_i)X_i + \sum S(W', X_i)W''Y_i. \]
Moreover, since $X_i \in \mathcal{O}(L \backslash \overline{G})^+$ for every $i$, and $W' \in \mathcal{O}(\overline{G})$, we have $S(W', X_i) = 0$ for every $i$ and $W'$, so the claim follows.

**Lemma 4.15.** $q(V) = X \otimes Y$, where $X$ and $Y$ are primitive elements in the defining ideal of $L$, and $\iota_H^*(X) = \iota_H^*(Y)$.

**Proof.** Suppose $\iota_H^*(X_i) \neq 0$. Then since $X_i$ vanishes on $L$, it cannot vanish on $A$. So, $\varphi(X_i) \neq 0$. Moreover, since the degree of $X_i$ is $< \ell$, $X_i$ must be primitive by minimality of $\ell$. Similarly, if $\iota_H^*(Y_i) \neq 0$ then $Y_i$ must be primitive. Thus $\ell = 2$, which implies the statement.

Set $p := \iota^*_H(X) = \iota^*_H(Y)$. Then $p$ is primitive in $\mathcal{O}(H)$.

**Lemma 4.16.** We have $[W, V] = R^j(W, p)(Y - X)$. 
Proof. By Lemmas 4.14 and 4.15(2), we have

\[
[W, V] = S(W, \iota_H^*(X))Y - S(W, \iota_H^*(Y))X
= S(W, \iota_H^*(X))Y - S(W, \iota_H^*(Y))X = S(W, p)(Y - X)
= R^j(W, p)(Y - X),
\]
as claimed, where the last equation holds by Lemma 2.3. □

Set \(c := R^j(W, p)(Y - X)(g) \in \mathbb{C}\).

Proposition 4.17. We have \([W, V] \equiv c \neq 0\) on \(HgH\). Thus, we may assume \([W, V] \equiv 1\) on \(HgH\).

Proof. We first show that \(c \neq 0\). To this end, we have to show that \(X(g) \neq Y(g)\) and \(R^j(W, p) \neq 0\). Since \(\varphi(\rho_g(V)) = \varphi(V) + \varphi(X)Y(g)\), \(\varphi(\lambda_{g^{-1}}(V)) = \varphi(V) + \varphi(Y)X(g)\), \(\varphi(\rho_g(V)) \neq \varphi(\lambda_{g^{-1}}(V))\) by Lemma 4.13 and \(\varphi(X) = \varphi(Y)\) by Lemma 4.15, we have \(X(g) \neq Y(g)\). Furthermore, since \(p\) vanishes on \(L\) by Lemma 4.15, \(p\) is orthogonal to \(JQ_{\mathfrak{h}}\) inside \(JQ_{\mathfrak{h}}\). Thus, \(R^j(W, p) \neq 0\) by the non-degeneracy of \(R^j\) on \(JQ_{\mathfrak{h}}\).

Next we show that \([W, V] \equiv c\) on \(\overline{HgH}\). Since by Lemma 4.15, \(X\) is primitive in \(\mathcal{I}(L)\), we have \(X(a_1l_1g^a_2l_2) = X(a_1) + X(g) + X(a_2)\) for every \(a_1, a_2 \in A\) and \(l_1, l_2 \in L\), and similarly for \(Y\). Thus, since by Lemma 4.15, \(X = Y\) on \(A\), we have \((Y - X)(a_1l_1g^a_2l_2) = (Y - X)(g)\) for every \(a_1, a_2 \in A\) and \(l_1, l_2 \in L\), which implies that \([W, V] \equiv c\) on \(\overline{HgH}\), as claimed. □

The proof of Theorem 4.6 is complete. □

Question 4.18. Fix \(g \in G\), and set \(A := JQ(Z_g)\).

1. Is it true that \(A \cong U^\omega_{\mathfrak{h}}(\mathfrak{h}_g) \otimes \mathcal{W}\) as algebras, where \(\mathcal{W}\) is a Weyl subalgebra with \(\text{GKdim}(\mathcal{W}) = 2d_g\)?

2. Is it true that \(A\) contains a subalgebra \(\mathcal{W} \cong U^\omega_{\mathfrak{h}}(\mathfrak{h}_g)\), and a Weyl subalgebra \(\mathcal{W}\) with \(\text{GKdim}(\mathcal{W}) = 2d_g\), such that \(\mathcal{W} \cap \mathcal{W}\) is trivial?

3. Is it true that a maximal Weyl subalgebra of \(A\) has Gelfand-Kirillov dimension \(2d_g + r\), where \(r \in 2\mathbb{Z}_{\geq 0}\) is the rank of \(\omega_g\) restricted to \(\mathfrak{h}_g\)?

(See, e.g., the end of Example 6.5)

Remark 4.19. By Proposition 4.1, Question 4.18 is a special case of Question 3.10.
5. REPRESENTATIONS OF $J\mathcal{O}(G)_J$ FOR UNIPOTENT $G$

Retain the notation of Sections 3 and 4.

**Theorem 5.1.** Every irreducible representation $V$ of $J\mathcal{O}(G)_J$ factors through a unique quotient $J\mathcal{O}(Z)_J$.

**Proof.** By Lemma 3.7, the central subalgebra $\mathcal{O}(H/G/H) \subset J\mathcal{O}(G)_J$ acts on $V$ by a certain central character $\chi_0: \mathcal{O}(H/G/H) \to \mathbb{C}$. Let $K_0 := \text{Ker}(\chi_0)$, let $I_0 \subset \mathcal{O}(G)$ be the ideal generated by $K_0$, and let $Z_0 \subset G$ be the closed subscheme defined by $I_0$. Then $Z_0$ is an affine scheme of finite type (i.e., $\mathcal{O}(Z_0)$ can be non-reduced and have nilpotents) with an $H \times H$-action, and all orbits of $H \times H$ on the underlying variety of $Z_0$ are closed by the theorem of Kostant and Rosenlicht since $H$ is unipotent. Pick an orbit $Y$ in $Z_0$. If $Y = Z_0$ then $Z_0 = HgH$ is a single $H \times H$-orbit, so $V$ factors through $J\mathcal{O}(Z_g)_J$, and we are done.

Otherwise, let $\tilde{I}_0$ be the ideal of functions on $Z_0$ vanishing on $Y$. Then $\tilde{I}_0$ is invariant under $H \times H$, and is a union of finite dimensional $H \times H$-modules, so it has a fixed vector $f \neq 0$ (as $H$ is unipotent), and this $f$ cannot be constant since it vanishes on $Y$. Thus, $\mathcal{O}(H/Z_0/H)$ is nontrivial.

Now consider the nontrivial central subalgebra $\mathcal{O}(H/Z_0/H) \subset \mathcal{O}(Z_0)$. It has a maximal ideal $n$, so $\mathcal{O}(H/Z_0/H)/n$ is a field extension of $\mathbb{C}$. But it is countably dimensional, so has to be $\mathbb{C}$. Thus, we have a central character $\chi_1: \mathcal{O}(H/Z_0/H) \to \mathbb{C}$ by which $\mathcal{O}(H/Z_0/H)$ acts on $V$, as above. Let $K_1 := \text{Ker}(\chi_1)$, let $I_1 \subset \mathcal{O}(G)/I_0$ be the ideal generated by $K_1$, and let $Z_1 \subset Z_0$ be the closed subscheme defined by $I_1$. Then $Z_1$ is $H \times H$-stable. Thus we can proceed as above by looking at the orbits of $H \times H$ in $Z_1$. However, by the Hilbert basis theorem, the sequence $Z_0 \supset Z_1 \supset \cdots$ must stabilize. Hence the process will end, and we will reach a single $H \times H$-orbit $HgH$, as desired.

Finally, since $\mathcal{I}(Z) + \mathcal{I}(Z') = J\mathcal{O}(G)_J$ for every two distinct double cosets $Z$ and $Z'$, $\mathcal{I}(Z)$ and $\mathcal{I}(Z')$ cannot both annihilate $V$. □

Let $N_G(H, J)$ be the subgroup of the normalizer $N_G(H)$ of $H$ in $G$, consisting of all elements $g \in N_G(H)$ such that $J$ is $g$-invariant.

**Theorem 5.2.** There is one to one correspondence between isomorphism classes of finite dimensional irreducible representations of $J\mathcal{O}(G)_J$ and elements of the group $N_G(H, J)$.

**Proof.** Follows from Theorems 4.3, 4.6 and 5.1, and the facts that Weyl algebras of degree $\geq 1$ have no finite dimensional representations,
and nilpotent Lie algebras have only the trivial finite dimensional irreducible representation.

\[ \square \]

**Remark 5.3.** Retain the notation from Remark 4.3. Then every irreducible representation of the algebra \( K \mathcal{O}(G)_J \) factors through a unique quotient algebra \( K \mathcal{O}(Z_g)_J \). Furthermore, if \( d_g = 0 \) then \( K \) and \( J \) are gauge equivalent, and \( K \mathcal{O}(G)_J \) has finite dimensional irreducible representations if and only if \( J \) is \( g \)-invariant. Indeed, note that \( d_g = 0 \) if and only if \( N_g = \tilde{H} = gHg^{-1} \). But the later implies that \( K, J \) are gauge equivalent. Thus we are reduced to Theorem 5.2.

### 6. Examples

Let \( A = G^2 \) with \( \mathcal{O}(A) = \mathbb{C}[X, V] \), and let \( \mathfrak{a} \) be the Lie algebra of \( A \), with basis \( \{ \frac{\partial}{\partial X}, \frac{\partial}{\partial V} \} \). Let \( r := \frac{\partial}{\partial X} \wedge \frac{\partial}{\partial V} \). Then the composition

\[ J : \mathcal{O}(A) \otimes \mathcal{O}(A) \xrightarrow{e^{r/2}} \mathcal{O}(A) \otimes \mathcal{O}(A) \xrightarrow{\varepsilon \otimes \varepsilon} \mathbb{C} \]

is a minimal Hopf 2-cocycle for \( A \) \cite[Section 4]{Etingof}, and it is straightforward to verify that

\[ J(X, V) = J^{-1}(V, X) = 1/2, \quad J(V, X) = J^{-1}(X, V) = -1/2. \]

Clearly, we have \( J \mathcal{O}(A)_J = \mathcal{O}(A) \cong U(\mathfrak{a}) \) as Hopf algebras.

**Example 6.1.** Let

\[ G = \{ 1 + xE_{12} + vE_{13} + yE_{23} | x, v, y \in \mathbb{C} \} \subset U_3 \]

be the Heisenberg group of dimension 3. Its coordinate Hopf algebra is a polynomial algebra \( \mathcal{O}(G) = \mathbb{C}[X, Y, V] \), with \( X, Y \) being primitive, and \( \Delta(V) = V \otimes 1 + 1 \otimes V + X \otimes Y \).

Set

\[ a := \frac{\partial}{\partial X}, \quad b := \frac{\partial}{\partial Y}, \quad c := \frac{\partial}{\partial V}. \]

Then \( \mathfrak{g} := \text{Lie}(G) \) has basis \( a, b, c \), with bracket \([a, b] = c\). The element \( r := a \wedge c \) is a non-degenerate \( \mathfrak{g} \)-invariant solution to the CYBE in \( \wedge^2 \mathfrak{h} \), where \( \mathfrak{h} \subset \mathfrak{g} \) is the (abelian) Lie subalgebra spanned by \( a, c \). Thus, \( J := e^{r/2} \) is a minimal Hopf 2-cocycle for \( H \), where

\[ H = \{ 1 + xE_{12} + vE_{13} | x, v \in \mathbb{C} \} \subset G \]

is the (normal abelian) subgroup with \( \text{Lie}(H) = \mathfrak{h} \), and we have that \( J \mathcal{O}(H)_J = \mathcal{O}(H) \cong U(\mathfrak{h}) \) as Hopf algebras.

We now view \( J \) as a (non-minimal) Hopf 2-cocycle for \( G \) by pulling it back along the obvious Hopf algebra surjective map \( \mathcal{O}(G) \rightarrow \mathcal{O}(H) \) determined by \( Y \mapsto 0, X \mapsto X \), and \( V \mapsto V \). Since \( J \) is an invariant
Hopf 2-cocycle for $G$, $\mathcal{O}(G)_J = \mathcal{O}(G) \cong \mathcal{O}(G/H) \otimes \mathcal{O}(H)_J$ as algebras (so, $\mathcal{O}(G)_J$ is isomorphic to $U(\mathbb{C}^3)$ as an algebra, but not to $U(g)$), and $\mathcal{O}(Z_g)_J \cong \mathcal{O}(H)_J = \mathcal{O}(H) \cong U(h)$ for every $g \in G$ since $H$ is normal and $r$ is $g$-invariant.

**Example 6.2.** (The induction base in the proof of Theorem 4.6) Let

$$G = \left\{ \begin{pmatrix} 1 & x & v & w \\ 0 & 1 & y & \frac{w^2}{2} \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x, v, y, w \in \mathbb{C} \right\} \subset U_4.$$  

Then $G$ is a 4-dimensional unipotent algebraic group over $\mathbb{C}$. Its coordinate Hopf algebra $\mathcal{O}(G) = \mathbb{C}[X, Y, V, W]$ is a polynomial algebra, with $X, Y$ being primitive, and

$$\Delta(V) = V \otimes 1 + 1 \otimes V + X \otimes Y,$$

$$\Delta(W) = W \otimes 1 + 1 \otimes W + V \otimes Y + X \otimes Y^2/2.$$  

Let $C := \{1 + wE_{14} \mid w \in \mathbb{C}\}$. Then $C \cong \mathbb{G}_a$ is a closed central subgroup of $G$. Set

$$a := \frac{\partial}{\partial X}, \quad b := \frac{\partial}{\partial Y}, \quad c := \frac{\partial}{\partial V}, \quad d := \frac{\partial}{\partial W}.$$  

Then $\mathfrak{g} := \text{Lie}(G)$ has basis $a, b, c, d$, with brackets $[a, b] = c$, $[c, b] = d$.

Let

$$H := \{1 + xE_{12} + vE_{13} \mid x, v \in \mathbb{C} \right\} \subset G.$$  

Then $H \cong \mathbb{G}_a^2$ and $\mathcal{O}(H) = \mathbb{C}[X, V]$ is a polynomial Hopf algebra. Since $C \cap H = \{1\}$, we have $H \cong L$ (see 4.1). Let $\mathfrak{h} := \text{Lie}(H)$ with basis $a, c$, let $r := a \wedge c$, and let $J := e^{r/2}$ as above. We have $\mathcal{O}(H)_J = \mathcal{O}(H)$ as algebras.

Pull $J$ back to $\mathcal{O}(G)$ along the obvious Hopf algebra surjective map $\mathcal{O}(G) \twoheadrightarrow \mathcal{O}(H)$ determined by $Y, W \mapsto 0$, $X \mapsto X$, and $V \mapsto V$. By (2.1) and (6.1), it is straightforward to find out that $\mathcal{O}(G)_J$ is generated as an algebra by $X, Y, V, W$, subject to the relations

$$[X, Y] = [X, V] = [Y, V] = [Y, W] = 0, \quad [W, X] = Y, \quad [W, V] = Y^2/2.$$  

In particular $X, V$ span a Lie algebra isomorphic to $\mathfrak{h}$, $W, Y$ span an abelian Lie algebra $\mathfrak{a}$, and we have algebra isomorphisms

$$\mathcal{O}(G)_J \cong U(\mathfrak{a}) \rtimes U(\mathfrak{h}) \cong \mathcal{O}(G/H) \# \mathcal{O}(H)_J,$$

where $[X, W] = -Y$, $[X, Y] = 0$, $[V, Y] = 0$ and $[V, W] = -Y^2/2$. (See Theorem 3.8)

Take $g := g(x_0, v_0, w_0, 0) \in G$. Then $H_g = H$, $\mathcal{T}(Z_g) = (Y, W - w_0)$, and $\mathcal{O}(Z_g)_J \cong \mathbb{C}[X, Y] \cong U(\mathfrak{h}_g) = U(\mathfrak{h})$ as algebras.
Take \( g := g(0, 0, 0, y_0) \), \( y_0 \neq 0 \). Then \( g^{-1} = g(0, 0, 0, -y_0) \). We have
\[
H_g = \{1 + x E_{12} - \frac{y_0}{2} x E_{13} \mid x \in \mathbb{C} \} \cong \mathbb{G}_a,
\]
and
\[
H g H = \left\{ \begin{pmatrix} 1 & x & v & w \\ 0 & 1 & y_0 & \frac{y_0^2}{2} \\ 0 & 0 & 1 & y_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left| x, v, w \in \mathbb{C} \right. \right\}.
\]
It follows that \( \mathcal{I}(Z_g) = (Y - y_0) \), and \([W, V] \equiv \frac{y_0^2}{2} \neq 0 \) on \( H g H \). Thus, we have
\[
j \mathcal{O}(Z_g) \cong \mathbb{C}[X] \otimes \mathbb{C}[V][W; d/dV] \cong U(h_g) \otimes \mathcal{H}_1
\]
as algebras.

Finally, note that we have \( p = \frac{y_0^2}{2} X + V, \iota^*_L(\rho_g(W) - W - W(g)) = y_0 p \), \( p \) vanishes on \( H_g \), and \( R^J(p, \iota^*(X)) = 1 \). Also, for \( l = l(x, v) \in L \), we have \( \tau(l) = 1 - y_0(y_0 x/2 + v)E_{14} \in C \) (see 4.1).

**Example 6.3.** Let \( G \) and \( g \) be as in Example 6.2. Set \( r := a \wedge c + d \wedge b \).

Then \( r \) is a non-degenerate solution of the CYBE in \( \wedge^2 g \), corresponding to the symplectic structure \( \omega \) on \( g \) determined by \( \omega(a, c) = \omega(d, b) = 1 \).

Let \( J := 1 + r/2 + \cdots \) be the corresponding minimal Hopf 2-cocycle for \( G \).

It is straightforward to verify that
\[
J(X, V) = J^{-1}(V, X) = J(W, Y) = J^{-1}(Y, W) = 1/2,
J(V, X) = J^{-1}(X, V) = J(Y, W) = J^{-1}(W, Y) = -1/2,
\]
and \( J, J^{-1} \) vanish on other pairs of generators.

By (2.1), it is straightforward to find out that the minimal cotriangular Hopf algebra \( j \mathcal{O}(G)_J \) is generated as an algebra by \( X, Y, V, W \), such that
\[
[W, X] = Y, \quad [W, V] = Y^2/2 + X,
\]
and other pairs of generators commute. Set \( X' := Y^2/2 + X \). Then \( X', Y, V, W \) span a Lie algebra of dimension 4 such that \([W, V] = X'\) and \([W, X'] = Y\), hence isomorphic to \( g \). Thus, \( j \mathcal{O}(G)_J \cong U(g) \) as algebras (see Theorem 3.4).

**Example 6.4.** Let \( G := U_4 \) be the 6-dimensional unipotent algebraic group of 4 by 4 upper triangular matrices over \( \mathbb{C} \). Its coordinate Hopf

\[\text{It is known that } J \text{ has this form (see, e.g., [EG1]).}\]
algebra $O(G) = \mathbb{C}[F_{12}, F_{23}, F_{34}, F_{13}, F_{24}, F_{14}]$ is a polynomial algebra, with $F_{12}, F_{23}, F_{34}$ being primitive,

$$\Delta(F_{13}) = F_{13} \otimes 1 + 1 \otimes F_{13} + F_{12} \otimes F_{23}, \quad \Delta(F_{24}) = F_{24} \otimes 1 + 1 \otimes F_{24} + F_{23} \otimes F_{34}$$

and

$$\Delta(F_{14}) = F_{14} \otimes 1 + 1 \otimes F_{14} + F_{13} \otimes F_{34} + F_{12} \otimes F_{24}.$$  

Let $H := \{1 + x E_{12} + u E_{34} | x, u \in \mathbb{C}\} \subset G$. Then $H \cong \mathbb{G}_a^2$ is a closed subgroup of $G$, and $O(H) = \mathbb{C}[X, U]$ is a polynomial Hopf algebra. Let $a := \frac{\partial}{\partial x}, c := \frac{\partial}{\partial u}, r := a \wedge c$, and $J := e^{r/2}$.

Pull $J$ back to $O(G)$ along the Hopf algebra surjective homomorphism $O(G) \to O(H)$ determined by $F_{23}, F_{13}, F_{24}, F_{14} \mapsto 0, F_{12} \mapsto X, F_{34} \mapsto U$. Then it is straightforward to verify that $jO(G/H)$ is generated as an algebra by $F_{23}, F_{13}, Y, V$, where $Y := F_{24} - F_{23}F_{34}$ and $V := F_{14} - F_{13}F_{34}$, such that

$$[Y, F_{13}] = F_{23}^2, \quad [F_{13}, V] = F_{23}F_{13}, \quad [Y, V] = F_{23}Y,$$

and other pairs of generators commute. In particular, $jO(G/H)$ is not commutative (see Remark 3.9).

**Example 6.5.** Let $G := U_4$ be as in Example 6.4. Let

$$H = \left\{ \begin{pmatrix} 1 & x & v & w \\ 0 & 1 & y & \frac{y^2}{2} \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \bigg| x, v, w, y \in \mathbb{C} \right\},$$

and $J$ be as in Example 6.3 (where $H$ is denoted there by $G$).

Pull $J$ back to $O(G)$ along the surjective Hopf algebra homomorphism $O(G) \to O(H)$ determined by

$$F_{12} \mapsto X, \quad F_{13} \mapsto V, \quad F_{14} \mapsto W, \quad F_{23}, F_{34} \mapsto Y, \quad F_{24} \mapsto Y^2/2.$$  

Then using (6.1), it is straightforward to verify that $jO(G)_{J}$ is generated as an algebra by $\{F_{ij}\}$, such that

$$[F_{14}, F_{12}] = F_{34}, \quad [F_{13}, F_{14}] = F_{24} - F_{12} - F_{23}F_{34}, \quad [F_{14}, F_{24}] = F_{23} - F_{34},$$

and other pairs of generators commute.

Let $C := \{1 + wE_{14} | w \in \mathbb{C}\}$. Then $C \subset H$ is central in $G$ (see 4.2). We have

$$L = \left\{ \begin{pmatrix} 1 & x & v & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \bigg| x, v \in \mathbb{C} \right\} \subset \mathcal{H} = \left\{ \begin{pmatrix} 1 & x & v & 0 \\ 0 & 1 & y & \frac{y^2}{2} \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \bigg| x, v, y \in \mathbb{C} \right\}$$
Set $A$ denote the Weyl subalgebra generated by $W$ and $J$. In particular, $A = gA$ is a Weyl subalgebra in $J$, $I$, $A, B, C, T, Q$, such that $A, [-A, T, Q] = -C$, $[B, Q] = T - A - C(C - 1)$, $[T, Q] = 1$, and other pairs of generators commute. Thus, replacing $A$ with $-A$ and setting $P := T - C(C - 1)$, we see that we have an algebra isomorphism $j\mathcal{O}(Z_g) \cong \mathbb{C}[A, B, C, P][Q, \delta]$, $\delta := C \frac{\partial}{\partial A} + (P + A) \frac{\partial}{\partial B} + \frac{\partial}{\partial P}$. Set $A' := A - CP$ and $B' := B - (P + A)P + (C + 1)P^2/2$, and let $\mathcal{W}$ denote the Weyl subalgebra generated by $P, Q$. Then we have an
algebra isomorphism

\[ j\mathcal{O}(Z_g) \cong C[A', B', C] \otimes \mathcal{W} \cong U^{\omega g}(\mathfrak{h}_g) \otimes \mathcal{W}. \]

(See Question 4.18.)

7. The Hopf Algebra \( j\mathcal{O}(G) \) for Connected Nilpotent \( G \)

In this section, we let \( G = T \times U \) be a connected nilpotent algebraic group over \( C \), where \( T \) is the maximal torus of \( G \) and \( U \) is the unipotent radical of \( G \). Let \( A := \mathcal{O}(G) \). Let \( \mathcal{O}(U) = C[y_1, \ldots, y_m] \) be as in Section 3 and let \( A_0 := \mathcal{O}(T) = C[X(T)] = C[x_1^{\pm 1}, \ldots, x_k^{\pm 1}] \). Recall that \( A = \mathcal{O}(T) \otimes \mathcal{O}(U) \) as Hopf algebras. Finally, set \( A_i := A_0[y_1, \ldots, y_i], 1 \leq i \leq m \) (so, \( A_m = A \)).

**Lemma 7.1.** Let \( J \) be a Hopf 2-cocycle for \( A \), and let \( \cdot \) denote the multiplication in \( jA_J \). The following hold:

1. We have a Hopf filtration on \( jA_J \):

\[
A_0 \subset j(A_1)_J \subset \cdots \subset j(A_i)_J \subset \cdots \subset j(A_m)_J = jA_J.
\]

2. For every \( i \), the Hopf algebra \( j(A_i)_J \) is generated by \( y_i \) over \( j(A_{i-1})_J \).

3. For every \( j < i \), we have

\[
[y_i, y_j] := y_i \cdot y_j - y_j \cdot y_i = \sum Q(Y''_i, y_j)Y'_{j_i} + \sum [J^{-1}(Y_i', x_j)J(Y''_{i2}, x_j) - J^{-1}(Y_i', x_j)J_1(Y''_{i2}, x_j)]Y''_{i1}Y''_{j1}.
\]

Hence, \( [y_i, y_j] \) belongs to \( A^+_{i-1} \).

4. \( y_1, y_2 \) are central primitives in \( jA_J \).

5. For every \( i, j \), we have

\[
[y_i, x_j] = x_j \sum Q(Y''_i, x_j)Y'_{i_j} + x_j \sum [J^{-1}(Y_i', x_j)J(Y''_{i2}, x_j) - J^{-1}(Y_i', x_j)J_1(Y''_{i2}, x_j)]Y''_{i1}.
\]

6. The linear map \( \delta_i : j(A_{i-1})_J \to j(A_{i-1})_J, \ s \mapsto [y_i, s] \), is an algebra derivation of \( j(A_{i-1})_J \) for every \( i \).

7. For every \( i \), \( j(A_i)_J \cong j(A_{i-1})_J[y_i; \delta_i] \) as Hopf algebras.

**Proof.** (1)–(4) are similar to Lemma 3.1. As for (5), we have

\[
[y_i, x_j] = x_j \left( J^{-1}(y_j, x_j) - J^{-1}(y_j, x_j) + Q(y_j, x_j) \right) + x_j \sum Q(Y''_i, x_j)Y'_{i_j} + \sum J^{-1}(Y_i', x_j)Y''_{i1}J(Y''_{i2}, x_j) - x_j \sum J^{-1}(Y_i', x_j)Y''_{i1}J_1(Y''_{i2}, x_j),
\]
and since \( \epsilon([y_i, x_j]) = 0 \), \( J^{-1}(y_j, x_j) - J^{-1}_{21}(y_j, x_j) + Q(y_j, x_j) = 0 \). Finally, (6) follows from (3) and (5), and (7) from (2) and (6). \( \square \)

For every \( i, j \), define \( p_{ij} \in \mathcal{O}(U)^+ \) as follows:

\[
p_{ij} := \sum Q(Y''_i, x_j) Y'_i + \sum \left( J^{-1}(Y'_i, x_j) J(Y''_i, x_j) - J^{-1}_{21}(Y'_i, x_j) J_{21}(Y''_i, x_j) \right) Y''_i.
\]

**Theorem 7.2.** The following hold:

1. \( \mathcal{O}(T) \) and \( \mathcal{O}(U)_J \) are Hopf subalgebras of \( \mathcal{O}(G)_J \).
2. The group \( X(T) \) acts on \( \mathcal{O}(U)_J \) by automorphisms via \( x^{-1}_j y_i x_j = y_i + p_{ij} \).
3. \( \mathcal{O}(G)_J \cong \mathcal{O}(U)_J \rtimes \mathbb{C}[X(T)] \) is a smash product algebra.
4. We have \( \text{Rep}(\mathcal{O}(G)_J) = \text{Rep}(\mathcal{O}(U)_J)^{X(T)} \).
5. The Hopf algebra \( \mathcal{O}(G)_J \) is an affine Noetherian domain with Gelfand-Kirillov dimension \( \text{dim}(G) \).

**Proof.** (1) and (2) follow from Lemma [7.1], (3) follows from (2), and (4)–(5) follow from (3) and Corollary [3.2]. \( \square \)

**Remark 7.3.** Theorems [5.2, 7.2(4)] imply a classification of finite-dimensional irreducible representations of \( \mathcal{O}(G)_J \).

**Example 7.4.** Let \( U \) be the Heisenberg group as in Example [6.1] (except, there it is denoted by \( G \)). Let \( G := \mathbb{G}_m \times U \). Then \( G \) is a connected (non-unipotent) nilpotent algebraic group over \( \mathbb{C} \), and we have \( \mathcal{O}(G) = \mathbb{C}[F^\pm, X, Y, V] \), where \( F \) is a grouplike element and \( X, Y, V \) are as in Example [6.1]. The Lie algebra \( \mathfrak{g} \) of \( G \) has basis \( f, a, b, c \), where \( f := F \frac{\partial}{\partial F} \), and \( a, b, c \) are as in Example [6.1]. By [G] Theorem 5.3 & Proposition 5.4, the classical \( r \)-matrix \( r := f \wedge (a + b) \) for \( \mathfrak{g} \) corresponds to a Hopf 2-cocycle \( J \) for \( G \). It is straightforward to verify that \( \mathcal{O}(G)_J \) is generated as an algebra by \( F, X, Y, V \), such that \( [V, F] = F(Y - X) \), or equivalently, \( F^{-1}VF = V + Y - X \), and other pairs of generators commute. Thus, we have \( \mathcal{O}(G)_J \cong \mathbb{C}[X, Y, V] \rtimes \mathbb{C}[F^\pm] \) as algebras, with nontrivial action.

**References**

[AEGN] E. Aljadeff, P. Etingof, S. Gelaki and D. Nikshych. On twisting of finite-dimensional Hopf algebras. *Journal of Algebra* **256** (2002), 484–501.

[D] V. Drinfeld. Constant quasiclassical solutions of the Yang-Baxter quantum equation. (Russian) *Dokl. Akad. Nauk SSSR* **273** (1983), no. 3, 531–535.

[EG1] P. Etingof and S. Gelaki. On cotriangular Hopf algebras. *American Journal of Mathematics* **123** (2001), 699–713.
[EG2] P. Etingof and S. Gelaki. Quasisymmetric and unipotent tensor categories. *Math. Res. Lett.*, 15 (2008), no. 5, 857–866.

[EG3] P. Etingof and S. Gelaki. The representation theory of cotriangular semisimple Hopf algebras. *International Mathematics Research Notices* 7 (1999), 387–394.

[EG4] P. Etingof and S. Gelaki. Invariant Hopf 2-cocycles for affine algebraic groups. *International Mathematics Research Notices*, Vol. 2020, No. 2, 344–366.

[EK1] P. Etingof and D. Kazhdan. Quantization of Lie Bialgebras, I. *Selecta Mathematica* 2 (1996), Vol. 1, 1–41.

[EK2] P. Etingof and D. Kazhdan. Quantization of Lie Bialgebras, II, III. *Selecta Mathematica* 4 (1998), 213–231, 233-260.

[EK3] P. Etingof and D. Kazhdan. Quantization of Poisson algebraic groups and Poisson homogeneous spaces. Symmetries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 935–946.

[EGNO] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik. Tensor Categories. *AMS Mathematical Surveys and Monographs book series* 205 (2015), 362 pp.

[G] S. Gelaki. Twisting of affine algebraic groups, I. *International Mathematics Research Notices*, Vol. 2015, No. 16, 7552-7574.

[Mou] S. Montgomery. Hopf algebras and their actions on rings. *CBMS Regional Conference Series in Mathematics* 82 (1993), 238 pp.

[Mov] M. V. Movshev. Twisting in group algebras of finite groups. (Russian) *Funktsional. Anal. i Prilozhen.* 27 (1993), no. 4, 17–23, 95; translation in *Funct. Anal. Appl.* 27 (1993), no. 4, 240-244 (1994).

[MR] J. C. McConnell and J. C. Robson. Noncommutative Noetherian rings. With the cooperation of L. W. Small. Revised edition. *Graduate Studies in Mathematics*, 30. American Mathematical Society, Providence, RI, (2001).

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