Underdamped stochastic harmonic oscillator

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We investigate the distribution of potential and kinetic energy in stationary states of the linearly damped stochastic oscillator driven by Lévy noises. In the long time limit distributions of kinetic and potential energies of the oscillator follow the power-law asymptotics and do not fulfill the equipartition theorem. The partition of the mechanical energy is controlled by the damping coefficient. In the limit of vanishing damping a stochastic analogue of the equipartition theorem can be proposed, namely the statistical properties of potential and kinetic energies attain distributions characterized by the same widths. For larger damping coefficient the larger fraction of energy is stored in its potential form. In the limit of very strong damping the contribution of kinetic energy becomes negligible. Finally, we demonstrate that the ratio of instantaneous kinetic and potential energies, which signifies departure from the mechanical energy equipartition, follows universal power-law asymptotics, regardless of the symmetric $\alpha$-stable noise parameters. Altogether our investigations clearly indicate strongly non-equilibrium character of Lévy-stable fluctuations with the stability index $\alpha < 2$.

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I. INTRODUCTION

A damped harmonic oscillator under influence of noise is one of the fundamental conceptual models in non-equilibrium statistical physics [1–5], broadly used to describe relaxation phenomena in the linear regime. Displacement $x(t)$ from the minimum of the potential $V(x)$ is described by the Langevin equation

$$m \frac{d^2x(t)}{dt^2} = -\gamma \frac{dx(t)}{dt} - \lambda x(t) + \sqrt{2C} \xi(t),$$

(1)
in which the interaction with the environment is separated into deterministic dissipative force $\gamma \frac{dx(t)}{dt}$, describing damping, and a noise term $\xi(t)$ describing all the complexity of the interaction between the test particle (or mode) with the rest of the system. In the situation of contact of the system with a single heat bath the (intrinsic) noise corresponding to linear (Stokes) friction has to be assumed Gaussian and white, i.e. it fulfills $\langle \xi(t) \rangle = 0$ and $\langle \xi(t) \xi(s) \rangle = \delta(t-s)$. The coefficient $C$ in Eq. (1) is given by $C = \frac{k_B T}{m}$, where $m$ stands for the particle’s (effective) mass, $T$ is the system’s temperature, and $k_B$ is the Boltzmann constant. The joint probability density $P = P(x, v, t)$ evolves according to the Kramers equation [6]

$$\frac{\partial P}{\partial t} = -v \frac{\partial}{\partial x} \left[ \gamma v + \frac{V'(x)}{m} \right] + \frac{k_B T}{m} \frac{\partial^2}{\partial v^2} P.$$  

(2)
The stationary solution of Eq. (2) has the canonical Boltzmann-Gibbs form [6, 7]

$$P(x, v) = N \exp \left[ -\frac{1}{k_B T} \left( \frac{mv^2}{2} + V(x) \right) \right],$$  

(3)

and factorizes, making position and velocity to be statistically independent random variables. If $V(x) = \lambda x^2$ as in Eq. (1), the stationary solution is a 2D, elliptically contoured normal density. Moreover, the average energy of the oscillator in its stationary (equilibrium) state is then given by the classical equipartition value $\langle \mathcal{E} \rangle = k_B T$ with $\mathcal{E}_k = \frac{mv^2}{2}$ and $\mathcal{E}_p = \frac{\lambda x^2}{2}$.

$$\langle \frac{mv^2}{2} \rangle = \langle \frac{\lambda x^2}{2} \rangle = \frac{k_B T}{2}.$$  

(4)

Within stochastic approach, the equipartition theorem originates due to properties of the noise.

From a physical point of view, the “whiteness” of the noise is a consequence of the large number of statistically independent interactions of a test particle with molecules of heat bath which are bounded in time. Its Gaussian character arises due to the assumption that the interactions are bounded in their strength. In many far-from-equilibrium situations the second assumption fails. The noise still can be considered as white but is now described by heavy-tailed distributions, often of the $\alpha$-stable Lévy type [8]. Heavy-tailed fluctuations of the $\alpha$-stable type have been observed in turbulent fluid flows [9], magnetized plasmas [10], optical lattices [11], heartbeat dynamics [12], neural networks [13], search on a folding polymers [14], animal movement [15], climate dynamics [16], financial time series [17], and even in spreading of diseases and dispersal of banknotes [18]. Such large fluctuations, not appearing in the description of equilibrium bath, can be attributed to the external forcing. The white Lévy noise which naturally appears in description of systems far-from-equilibrium breaks the microscopic reversibility, and changes considerably the properties of the stationary states of the system compared to equilibrium cases [19].

Within the current work we assume that white Gaussian noise in Eq. (1) is replaced by external white Lévy noise.

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[8, 20–22] but the dissipation is still of the Stokes type. The behavior of a damped harmonic oscillator under the influence of noise with $\alpha < 2$ is very different from those for the Gaussian case $\alpha = 2$. Stationary densities are given by bivariate $\alpha$-stable densities [20, 23] for which lines of constant probability are not ellipses. Moreover, in the stationary state, velocity $v$ and position $x$ are not statistically independent [20]. Stationary states for a particle moving in the parabolic potential driven by the white Lévy noise reflect symmetries of the noise, i.e. they are given by the $\alpha$-stable densities both for symmetric [24, 25] and asymmetric noises [26]. The same effect is observed for a 2D parabolic potential perturbed by the bi-variate Lévy noise [27]. In what follows we investigate distributions of kinetic and potential energies of a damped harmonic oscillator and distributions of ratio of their instantaneous values for the case of symmetric Lévy noises.

II. MODEL AND RESULTS

We examine the distributions of the kinetic and potential energy and of their ratio for the case of a damped harmonic oscillator driven by a white $\alpha$-stable noise $\zeta_\alpha(t)$:

$$m\ddot{x}(t) = -\gamma \dot{x}(t) - \lambda x(t) + \zeta_\alpha(t). \quad (5)$$

The white $\alpha$-stable noise $\zeta_\alpha(t)$, which is a formal time derivative of the $\alpha$-stable motion $L_\alpha(t)$ [28], results in stochastic increments which are distributed according to the symmetric $\alpha$-stable density whose characteristic function is given by [8, 20]

$$\phi(k) = \exp[-\sigma^\alpha|k|^\alpha]. \quad (6)$$

The parameter $\alpha$ ($0 < \alpha \leq 2$) is the so called stability index describing asymptotics of $\alpha$-stable densities which for $\alpha < 2$ is of the power-law type $p(x) \propto |x|^{-(\alpha+1)}$. In the $\alpha = 2$ limit the $\alpha$-stable noise is equivalent to the Gaussian white noise. The strength of fluctuations in Eq. (5) is controlled by the scale parameter $\sigma$, see Eq. (6) and its similarity properties are governed by the parameter $\alpha$. Contrary to the $\alpha = 2$ case, due to the divergence of the second moment $\langle x^2 \rangle$, for $0 < \alpha < 2$, there is no fluctuation-dissipation relation of the Smoluchowski-Sutherland-Einstein type [29–33]. Consequently, damping coefficient $\gamma$ and fluctuation intensity $\sigma$ can be viewed as independent parameters. The parameters $m$ and $\lambda$ for free, undamped oscillator define the most convenient units in which the system can be described. We choose $t_0 = \omega_0^{-1} = \sqrt{m/\lambda}$ to define the unit of time. In the dimensionless time $t/t_0$ (for a brief explanation of units, cf. Appendix A) the equation (5) takes the form

$$\ddot{x}(t) = -\gamma \dot{x}(t) - x(t) + \sigma \zeta_\alpha(t), \quad (7)$$

with a damping $\gamma$ replacing the original frequency of dissipation $\gamma = \gamma t_0/m$ (for the clarity, we omit the tilde sign over the original constants). Here the prefactor $\sigma$ of $\zeta_\alpha$, measuring intensity of the noise is $\sigma = \delta t_0^{1+1/\alpha}/m$. Moreover, in Eq. (7), the white Lévy noise $\zeta_\alpha(t)$ with the scale parameter set to unity is used. The instantaneous kinetic and potential energies of the system are denoted by $E_k = v^2/2$ and $E_p = x^2/2$, respectively.

![FIG. 1. The quotient of the corresponding prefactors $D = \frac{v_0}{\omega_0}$ for various values of the stability index $\alpha$. Please note the double logarithmic scale in the main plot and the linear scale in the inset.](image)

The formal solution of Eq. (7) is

$$x(t) = F(t) + \int_{-\infty}^{t} G(t - t')\zeta_\alpha(t')dt', \quad (8)$$

where $G(t)$ is the Green’s (response) function of the corresponding process, and $F(t)$ is a decaying function (a solution of the homogeneous equation under given initial conditions). The solution for $v$ is given by

$$v(t) = F_v(t) + \int_{-\infty}^{t} G_v(t - t')\zeta_\alpha(t')dt', \quad (9)$$

where $G_v(t)$ is the Green’s function of the velocity process

$$G_v(t) = \frac{d}{dt}G(t). \quad (10)$$

In a stationary situation, $t \to \infty$, the $F$-functions in Eqs. (8) and (9) vanish. The Green’s function of Eq. (5) can be easily found e.g. via the Laplace representation, and reads:

$$G(t) = \frac{\exp(-\gamma t/2)}{\sqrt{\omega_0^2 - \gamma^2/4}} \sin \left[ \sqrt{\omega_0^2 - \gamma^2/4} t \right] \quad (11)$$

for $\omega_0 = \sqrt{\lambda/m} > \gamma/2$ (underdamped case),

$$G(t) = t \exp(-\gamma t/2) \quad (12)$$

for $\omega_0 = 1 = \gamma/2$ (critical case) and

$$G(t) = \frac{\exp(-\gamma t/2)}{\sqrt{\gamma^2/4 - \omega_0^2}} \sinh \left[ \sqrt{\gamma^2/4 - \omega_0^2} t \right] \quad (13)$$

for $\omega_0 = \sqrt{\lambda/m} = 1 < \gamma/2$ (overdamped case). Note that the functions $G(t)$ vanish both for $t = 0$ and for $t \to \infty$ so that

$$\int_{0}^{\infty} G(t) \left[ \frac{d}{dt}G(t) \right] dt = \frac{1}{2} \int_{0}^{\infty} \left[ \frac{d}{dt}G^2(t) \right] dt \quad (14)$$

$$= \frac{1}{2} G^2(t)|_0^\infty = 0,$$
i.e. \( G(t) \) and \( G_v(t) \) are orthogonal on \([0, \infty)\).

The characteristic function of the stationary distribution of \( x \) and \( v = \dot{x} \) is given by Eq. (17) of Ref. [34]:

\[
f(k, q) = \exp\left[-\sigma^\alpha \int_0^\infty |kG(t) + qG_v(t)|^\alpha \right] \tag{15}
\]

where \( G(t) \) is the Green’s function for the homogeneous part of the equation of motion, see Eqs. (6) – (8) of Ref. [34], \( G_v(t) = \frac{\partial}{\partial t} G(t) \). Moreover, the Eq. (15) is the characteristic function of the 2D \( \alpha \)-stable density [20].

The marginal distributions of \( x \) and \( v \) have the characteristic functions \( f_x(k) = f(k, 0) \) and \( f_v(q) = f(0, q) \) and are the Lévy stable ones with index \( \alpha \) and prefactors

\[
w_x^\alpha = \sigma^\alpha \int_0^\infty |G(t)|^\alpha dt \tag{16}
\]

and

\[
w_v^\alpha = \sigma^\alpha \int_0^\infty |G_v(t)|^\alpha dt. \tag{17}
\]

The corresponding integrals can be easily evaluated numerically for small and moderate values of \( \gamma \) for any \( \alpha > 0 \). Their asymptotic behavior for \( \gamma \to 0 \) and for \( \gamma \to \infty \) will be discussed in the next subsection.

The large \( |x| \) and \( |v| \) asymptotics of the corresponding PDFs are

\[
p_x(x) \propto \frac{w_x^\alpha}{|x|^{1+\alpha}} \tag{18}
\]

and

\[
p_v(v) \propto \frac{w_v^\alpha}{|v|^{1+\alpha}}, \tag{19}
\]

i.e. they are of the of the \( \alpha \)-stable type with the same stability index like the driving noise.

### A. Ratio of distribution widths

The numerically calculated quotient of the corresponding prefactors

\[
D = \frac{w_v^\alpha}{w_x^\alpha} \tag{20}
\]

is depicted in the Fig. 1 as a function of the damping coefficient \( \gamma \). Various curves correspond to different values of the stability index \( \alpha \). The inset shows small \( \gamma \) dependence.

As derived in the Appendix B, the ratio of distributions widths scales as

\[
D = \frac{w_v^\alpha}{w_x^\alpha} = \begin{cases} 
\gamma^{-\alpha} & \text{for } 0 < \alpha < 1 \\
\gamma^{-\alpha-2} & \text{for } 1 \leq \alpha \leq 2
\end{cases} . \tag{21}
\]

The formula (21), accounts also for \( \alpha = 2 \) when \( D = 1 \).

Numerical simulations presented in Fig. 1 perfectly confirm the scaling predicted by Eq. (21). Please note, that results for \( \alpha = 0.5 \) (empty squares) and \( \alpha = 1.5 \) (empty circles) coincide.

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**FIG. 2.** Distributions of the potential \( E_p \) (full symbols) and kinetic \( E_k \) (empty symbols) energies for \( \gamma = 1, 2, 10 \) (from top to bottom). Various curves correspond to various values of the stability index \( \alpha \).

### B. Energy distributions

Let us consider the distributions of the kinetic and the potential energies. Through the change of variables \( x = \pm \sqrt{2E_p}, v = \pm \sqrt{2E_k} \) we get

\[
p(E_p) = p_x(\sqrt{2E_p})E_p^{-1/2} \propto \frac{w_x^\alpha}{E_p^{1+\alpha}} . \tag{22}
\]

\[
p(E_k) = p_v(\sqrt{2E_k})E_k^{-1/2} \propto \frac{w_v^\alpha}{E_k^{1+\alpha}} . \tag{23}
\]

The total energy \( E \) in our units is proportional to the square of the amplitude of the phase space vector \( A = \sqrt{x^2 + v^2} \).
This property of the system is closely connected with another phase space property, namely with the phase angle

$$\phi = \arctan \frac{v}{x}$$ \hspace{1cm} (26)

The distribution $p(r)$ has a universal asymptotics, and this universality is closely connected with non-independence of the velocity and coordinate processes, see Eq. (15) and [34].

Imagine the distribution of the phase angle is known and is given by a PDF $p(\phi)$. Then $r = \tan^2 \phi$ and

$$p(r) = p_{\phi}(\pm \phi(r)) \left| \frac{d\phi}{dr} \right|$$

$$= \left[ p_{\phi}(\arctan \sqrt{r}) + p_{\phi}(- \arctan \sqrt{r}) \right] \frac{1}{2(1+r)\sqrt{r}},$$

note that there are two solutions for $\phi$ for a given $r$. For $r \to \infty$, the argument of $p_{\phi}$, $\arctan \sqrt{r}$, tends to $\pi/2$ and therefore the behavior of $p(r)$ depends on whether $p_{\phi}(\phi)$ does or does not have a singularity at $\phi = \pm \pi/2$.

In the case of a driven harmonic oscillator, the association between the velocity and the coordinate processes makes the distribution $p_{\phi}(\phi)$ non-singular at $\pm \pi/2$, which can be seen from the expressions of the corresponding spectral measures as given in [35] which do not show singularities at $\theta = \pm \pi/2$, and can be grasped from the graphical representation of the corresponding level curves. Therefore, for $r \to \infty$

$$p(r) \propto \frac{C}{(1+r)\sqrt{r}} \simeq \frac{1}{r^{3/2}}$$ \hspace{1cm} (28)

with $C = [p_{\phi}(-\pi/2) + p_{\phi}(\pi/2)]/2$, and is independent of $\alpha$. Interestingly enough, the inverse of $r$, the ratio of the potential and the kinetic energy, has exactly the same asymptotic distribution, as it is evident by the explicit change of variables.

The universal $r^{-3/2}$ asymptotics of $p(r)$, see Eq. (28), originates due to lack of independence between position and velocity, see Eq. (15). If $x$ and $v$ were independent the behavior would be very different, due to the fact that the corresponding spectral measure is concentrated (i.e. has singularities) at $\theta = 0, \pm \pi/2$, and $\pi$, i.e. at the intersections of the unit sphere with the axes [20]. The distribution of $r$ can be derived from the distribution of a quotient of two independent symmetric Lévy-stable variables which possesses quite a complicated form [36]. The ratio $r$ of instantaneous kinetic $E_k(t)$ and potential $E_p(t)$ energies has non-universal asymptotics

$$p(r) \simeq \begin{cases} \frac{1}{r^{1+\alpha/2}} & \text{for } 0 < \alpha < 1 \\ \ln r & \text{for } \alpha = 1 \\ \frac{1}{r^{1-2\alpha}} & \text{for } 1 < \alpha \leq 2 \end{cases}$$ \hspace{1cm} (29)

see Appendix C. Eq. (29) should be contrasted with the universal and correct asymptotics given by Eq. (28). For $\alpha = 2$, in the stationary state, position and velocity are independent. Therefore for $\alpha = 2$ asymptotics predicted by Eq. (28) and (29) are the same.
Figure 3 presents the ratio $r$ of instantaneous kinetic $\mathcal{E}_k$ and potential $\mathcal{E}_p$ energy for $\gamma = 1, 2, 10$ (from top to bottom). Various panels correspond to different values of the stability index $\alpha$. Solid lines present theoretical asymptotic given by Eq. (28).

III. SUMMARY AND CONCLUSIONS

In the present work we extend earlier studies [34] of the damped harmonic oscillator driven by Lévy noises in phase space characterized by the position $x$ and velocity $v = \dot{x}$. The stationary state is given by a 2-dim $\alpha$-stable density [20]. First of all, in stationary states, position and velocity are no longer independent [34], leading to a considerable difference from the usual case of the Gaussian white noise. The studied system, i.e. damped harmonic oscillator driven by $\alpha$-stable noise is a highly non-equilibrium system displaying unexpected properties. Presence of non-equilibrium external noise introduces dependence between position and velocity in the stationary state. This dependence is responsible for violation of basic concepts of equilibrium statistical mechanics.

Kinetic and potential energies of a harmonic oscillator driven by a symmetric $\alpha$-stable noise have the same power-law asymptotics of $\mathcal{E}^{-\left(1+\alpha/2\right)}$ type determined by the noise type. Contrary to the classical Gaussian case, showing the equipartition between the kinetic and the potential energy, $\langle \mathcal{E}_k \rangle = \langle \mathcal{E}_p \rangle$, we demonstrate that no such equipartition in a whatever statistical sense is observed for the Lévy noise, $\langle E \rangle$. With the increasing damping larger fraction of energy is stored in the form of the potential energy. Consequently, with increasing $\gamma$ the ratio of kinetic and potential energy distributions’ widths decreases, and is given by a power-law in $\gamma$ with the exponent depending on the stability index $\alpha$. In the limit of $\gamma \to \infty$ the system is fully overdamped. In such a case stochastic oscillator is fully characterized by its position only [25], and the kinetic energy vanishes.

Finally, we have studied the distribution of the ratio $r = \mathcal{E}_k(t)/\mathcal{E}_p(t)$ of instantaneous kinetic and potential energies in the stationary state. We show that this ratio has a universal $r^{-\alpha/2}$ asymptotics independent on the stability index $\alpha$, which differs strikingly from the situation when the position and the velocity of the oscillator were independent.

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Appendix A: Units

The white Lévy noise $\zeta_\alpha(t)$ is by definition a formal derivative of the strictly $\alpha$-stable Lévy process $\zeta_\alpha(t) \equiv \frac{dL_\alpha(t)}{dt}$ whose increments are independent stationary variables distributed according to the symmetric $\alpha$-stable density, see Eq. (6). The self-similarity of the Lévy motion (Lévy process) signifies that its realizations fulfill the condition $L_\alpha(t) = t^{1/\alpha} L_\alpha(1)$. In view of the above, scaling properties of the Lévy white noise assume the change of variables according to $\zeta_\alpha(t_0 0) \to t_0^{1/\alpha-1} \zeta_\alpha(t)$.

Appendix B: Ratio of distribution widths $D$

Small ($\gamma \to 0$) and large ($\gamma \to \infty$) asymptotics of the quotient $D = \frac{w_\alpha^2}{w_\alpha^2}$, see Eq. (20), can be calculated analytically. For $\gamma \to 0$ the system is strongly underdamped, for which

$$G(t) = \frac{\exp(-\gamma t/2)}{\sqrt{1 - \gamma^2/4}} \sin \left[\sqrt{1 - \gamma^2/4} t\right]$$

and

$$G_v(t) \simeq \exp(-\gamma t/2) \left[ -\frac{\gamma}{2} \sin t + \cos t \right]$$

in the lowest order in $\gamma$. The last expression can be rewritten as

$$G_v(t) \simeq \exp(-\gamma t/2) \sqrt{1 + \frac{\gamma^2}{4}} \cos(1 + \phi)$$

with $\phi = \arccos(1/\sqrt{1 + \gamma^2/4})$.

For $\gamma \to 0$ we have

$$w_\alpha^2 \simeq \sigma^\alpha \int_0^\infty \exp(-\gamma t/2) |\sin t|^\alpha dt$$

and a similar expression (with a cosine) for $w_\alpha^2$. For $\gamma \to 0$ the exponential hardly changes on the period of oscillations of the trigonometric function, so we can average over these oscillations. Essentially what we do is to split the domain of integration into the $\pi$-intervals, which are the domains of periodicity of the absolute value of the trigonometric function and rewrite the total integral as the sum

$$w_\alpha^2 = \sum_{n=0}^\infty \exp(n\pi) \int_0^{(n+1)\pi} \exp(-\gamma \alpha t/2) |\sin t|^\alpha dt$$

and

$$= \sum_{n=0}^\infty \exp(n\pi \gamma/2) \int_0^\pi \exp(-\gamma \alpha t/2) |\sin t|^\alpha dt$$

or

$$= \exp(-\gamma \alpha t/2) \frac{1}{1 - \exp(\pi \gamma \alpha/2)} \int_0^\pi |\sin t|^\alpha dt$$
with \(0 < t^s < \pi\). The expression for \(w_v\), in the lowest order in \(\gamma\) is
\[
w_v^\alpha = \frac{\exp(-\gamma\alpha t^s/2)}{1 - \exp(\pi\gamma\alpha/2)} \int_0^\pi |\cos(t + \phi)|^\alpha \, dt,
\]
and only differs with respect to the position of the intermediate point \(0 < t^s < \pi\). The integrals over the trigonometric functions are the same, and are given by
\[
\int_0^\pi |\sin t|^\alpha \, dt = \int_0^\pi |\cos(t + \phi)|^\alpha \, dt = \sqrt{\pi} \frac{|\Gamma((\alpha + 1)/2)|}{\Gamma((\alpha + 2)/2 + 1)}.
\]
Therefore
\[
D = \frac{w_v^\alpha}{w_x^\alpha} = \exp[-\gamma\alpha(t^s - t^s)/2] \rightarrow 1
\]
for \(\gamma \rightarrow 0\). For small friction we have a “kind of” equipartition, i.e. both densities \(p(x)\) and \(p(v)\) are characterized by the same width. In the lowest order in \(\gamma\) this can be obtained by expanding the denominator:
\[
w_v^\alpha = w_x^\alpha = \frac{2}{\sqrt{\pi\gamma\alpha}} \frac{\Gamma((\alpha + 1)/2)}{\Gamma((\alpha + 2)/2 + 1)}.
\]
For \(\gamma \rightarrow \infty\) we start from the explicit expression
\[
G(t) = \frac{\exp(-\gamma t/2)}{\sqrt{\gamma^2/4 - 1}} \sinh \left[ \sqrt{\gamma^2/4 - 1} \, t \right]
\]
\[
= \frac{1}{2\sqrt{\gamma^2/4 - 1}} \left[ e^{(\sqrt{\gamma^2/4 - 1} - \gamma/2)t} - e^{-(\sqrt{\gamma^2/4 - 1} + \gamma/2)t} \right]
\]
\[
\approx \frac{1}{\gamma} \left( e^{-\gamma/2} - e^{-\gamma t} \right)
\]
where in the last line only the leading contributions in \(\gamma\) in the exponential are retained. For \(\gamma \rightarrow \infty\) the first exponential is decaying very slowly, and gives the major contribution to the time integral
\[
G(t) \approx \frac{1}{\gamma} e^{-\gamma/2}. \quad (B11)
\]
We now can estimate the integral for \(w_x^\alpha\), and get
\[
w_x^\alpha \approx \frac{\gamma^{1-\alpha}}{\alpha}. \quad (B12)
\]
Similarly for \(G_v(t) = \frac{d}{dt} G(t)\) we get
\[
G_v(t) \approx \frac{1}{\gamma^2} e^{-\gamma/2} + e^{-\gamma t}. \quad (B13)
\]
In order to calculate \(w_v^\alpha\) special care is required. The second term in Eq. (B13) cannot be neglected because for \(\gamma \gg 1\) it is larger than the first one
\[
w_v^\alpha = \int_0^\infty |G_v(t)|^\alpha \, dt = \int_0^\infty \left[ \frac{1}{\gamma^2} e^{-\gamma/2} + e^{-\gamma t} \right]^\alpha \, dt.
\]
\[
\int_0^\infty |G_v(t)|^\alpha \, dt = \int_0^\infty \frac{1}{\gamma^2} e^{-\gamma/2} + e^{-\gamma t} \left[ \frac{1}{\gamma^2} e^{-\gamma/2} + e^{-\gamma t} \right]^\alpha \, dt.
\]
The integrand (B14) shows a crossover between two types of asymptotics: for \(\gamma \gg 1\) the short time behavior is dominated by the fast decay \(e^{-\alpha t}\), while at long times it is dominated by the slow decay \(\gamma^{-2} e^{-\alpha t/\gamma}\). The crossover between two regimes takes place at time \(t_c = 2\gamma^{-1} \ln \gamma\). Consequently, the total integral can be estimated as
\[
w_v^\alpha \approx \int_0^{t_c} e^{-\alpha t} \, dt + \gamma^{-2} \int_{t_c}^\infty e^{-\alpha t/\gamma} \, dt
\]
resulting in
\[
w_v^\alpha \approx \frac{1}{\alpha\gamma} \left[ 1 - \frac{1}{\gamma^{2\alpha}} \right] + \frac{1}{\alpha\gamma^{2\alpha/\gamma}}. \quad (B16)
\]
For \(\gamma \gg 1\) the above expression can be further simplified to
\[
w_v^\alpha \approx \frac{1}{\alpha\gamma} + \frac{1-\alpha}{\alpha\gamma^{2\alpha/\gamma}} \quad (B17)
\]
leading to the dominating terms
\[
w_v^\alpha \approx \begin{cases} 
\frac{\gamma^{1-\alpha}}{\alpha} & \text{for } 0 < \alpha < 1 \\
\frac{\gamma^{1-\alpha}}{\alpha} & \text{for } 1 \leq \alpha \leq 2
\end{cases} \quad (B18)
\]
Finally, the ratio of distributions widths scales as
\[
D = \frac{w_v^\alpha}{w_x^\alpha} = \begin{cases} 
\gamma^{-\alpha} & \text{for } 0 < \alpha < 1 \\
\gamma^{-\alpha-2} & \text{for } 1 \leq \alpha \leq 2
\end{cases} \quad (B19)
\]
For \(\alpha = 2\), from Eq. (B10) and the definition
\[
D = 1, \quad (B20)
\]
as predicted by Eq. (B19).

Appendix C: Ratio of instantaneous kinetic and potential energies \(r\)

For pedagogical reason it is interesting to assume that \(v\) and \(x\) are independent. In such a case, the qualitative discussion of the asymptotic behavior of \(r = \mathcal{E}_k(t)/\mathcal{E}_p(t)\) is however very simple. The large values of
\[
z = \frac{x}{y} \quad (C1)
\]
will typically appear either due to the very large values of the numerator or to very small values of denominator (the possibility that both occurs simultaneously is very small, and plays the role only in the case of the Cauchy distribution, vide infra). In the symmetric Lévy case the probability density of large values of numerator decays as
\[
p(x) \approx \frac{1}{|x|^{1+\alpha}}, \quad (C2)
\]
while the distribution of \(q = 1/y\) is given by the variable transformation
\[
p(q) = p\left(\frac{1}{q}\right) \frac{1}{q^2}. \quad (C3)
\]
Since $p(x)$ is non-singular and does not vanish at zero, the tail of the PDF $p(q)$ is universal and of the same type as for the Cauchy distribution:

$$p(q) \propto \frac{1}{q^2}. \quad (\text{C4})$$

For $\alpha > 1$ the tail of $p(z)$ is dominated by the tail of $p(q)$ and therefore $p(z) \propto z^{-2}$. The variable transformation to $r = z^2/2$ transforms this tail into

$$p(r) = \frac{1}{r^{3/2}}, \quad (\text{C5})$$

exactly as in the case of the correlated variables above. An explicitly solvable example is given by the Gaussian case $\alpha = 2$ for which the distribution of $z$ is known explicitly: it is a Cauchy distribution

$$p(z) = \frac{1}{\pi (1 + z^2)}. \quad (\text{C6})$$

The variable transformation to $r$ gives

$$p(r) = \frac{1}{\pi (r + 1)^{3/2}}. \quad (\text{C7})$$

For $\alpha < 1$ the tail of the ratio $z$ is dominated by the tail of the enumerator, so that

$$p(z) \simeq \frac{1}{|z|^{1+\alpha}}, \quad (\text{C8})$$

and the variable transformation gives

$$p(r) \simeq \frac{1}{r^{1+\alpha}}. \quad (\text{C9})$$

The transition between the two regimes happens at $\alpha = 1$, i.e. for the Cauchy distribution, for which the distribution of the ratio of the two variables is again explicitly known [37]

$$p(z) = \frac{1}{\pi^2 z^2 - 1} \ln z^2, \quad (\text{C10})$$

so that

$$p(r) = \frac{1}{2\pi^2 (r - 1) \sqrt{r}} \ln r \quad (\text{C11})$$

and involves a logarithmic correction: its asymptotic behavior is

$$p(r) \simeq \frac{\ln r}{r^{\frac{\alpha}{2}}}. \quad (\text{C12})$$

Consequently, if the velocity $v$ and position $x$ would be independent, the ratio of instantaneous kinetic $E_k$ and potential $E_p$ energy, has the following non-universal asymptotics

$$p(r) \simeq \begin{cases} \frac{1}{r^{1+\alpha/2}} & \text{for } 0 < \alpha < 1 \\ \frac{\ln r}{r^{\frac{\alpha}{2}}} & \text{for } \alpha = 1 \\ \frac{1}{r^{\frac{\alpha}{2}}} & \text{for } 1 < \alpha < 2 \end{cases}. \quad (\text{C13})$$

Contrary to this $\alpha$-dependent behavior, the correct asymptotic behavior of $p(r)$ for the harmonic Lévy oscillator, where $v$ and $x$ are not independent, is universal:

$$p(r) \simeq r^{-3/2}. \quad (\text{C14})$$

[1] J. L. Doob, Ann. of Math. 43, 351 (1942).
[2] B. J. West and V. Seshadri, Physica A 113, 203 (1982).
[3] A. V. Chechkin and V. Y. Gonchar, J. Exp. Theor. Phys. 91, 635 (2000).
[4] M. Gitterman, The noisy oscillator: the first hundred years, from Einstein until now (World Scientific Publishing, Singapore, 2005).
[5] N. Lin and S. Lototsky, Commun. Stoch. Anal. 5, 233 (2011).
[6] H. Risken, The Fokker-Planck equation. Methods of solution and application (Springer Verlag, Berlin, 1984).
[7] S. Chandrasekhar, Rev. Mod. Phys. 15, 1 (1943).
[8] A. Janicki and A. Weron, Simulation and chaotic behavior of $\alpha$-stable stochastic processes (Marcel Dekker, New York, 1994); A. V. Chechkin, V. Y. Gonchar, J. Klafter, and R. Metzler, in Fractals, Diffusion, and Relaxation in Disordered Complex Systems: Advances in Chemical Physics, Part B, Vol. 133, edited by W. T. Coffey and Y. P. Kalmykov (John Wiley & Sons, New York, 2006) pp. 439–496; A. A. Dubkov, B. Spagnolo, and V. V. Uchaikin, Int. J. Bifurcation Chaos. Appl. Sci. Eng. 18, 2649 (2008).
[9] M. F. Shlesinger, G. M. Zaslavski, and J. Klafter, Nature (London) 363, 31 (1993); J. Klafter, M. F. Shlesinger, and G. Zumofen, Phys. Today 49, 33 (1996); T. H. Solomon, E. R. Weeks, and H. L. Swinney, Phys. Rev. Lett. 71, 3975 (1993); D. del Castillo-Negrepe, Phys. Fluids 10, 576 (1998).
[10] A. V. Chechkin, V. Y. Gonchar, and M. Szydłowski, Phys. Plasmas 9, 78 (2002); D. del Castillo-Negrepe, B. A. Carreras, and V. E. Lynch, Phys. Rev. Lett. 94, 065003 (2005).
[11] H. Katori, S. Schlipf, and H. Walther, Phys. Rev. Lett. 79, 2221 (1997).
[12] C.-K. Peng, J. Mietus, J. M. Hausdorff, S. Havlin, H. E. Stanley, and A. L. Goldberger, Phys. Rev. Lett. 70, 1343 (1993).
[13] R. Segev, M. Benveniste, E. Hulata, N. Cohen, A. Palevski, E. Kapon, Y. Shapira, and E. Ben-Jacob, Phys. Rev. Lett. 88, 118102 (2002).
[14] M. A. Lombhorst, T. Ambjörnsson, and R. Metzler, Phys. Rev. Lett. 95, 260603 (2005).
[15] G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Mur-
phy, P. A. Prince, and H. E. Stanley, Nature (London) **381**, 413 (1996).

[16] P. D. Ditlevsen, Geophys. Res. Lett. **26**, 1441 (1999).

[17] R. N. Mantegna and H. E. Stanley, *An introduction to econophysics. Correlations and complexity in finance* (Cambridge University Press, Cambridge, 2000).

[18] D. Brockmann, L. Hufnagel, and T. Geisel, Nature (London) **439**, 462 (2006).

[19] Ł. Kusmierz, A. Chechkin, E. Gudowska-Nowak, and M. Bier, EPL (Europhysics Letters) **114**, 60009 (2016).

[20] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian random processes: Stochastic models with infinite variance* (Chapman and Hall, New York, 1994).

[21] A. Janicki, *Numerical and statistical approximation of stochastic differential equations with non-Gaussian measures* (Hugo Steinhaus Centre for Stochastic Methods, Wroclaw, 1996).

[22] M. Vahabi, J. H. P. Schulz, B. Shokri, and R. Metzler, Phys. Rev. E **87**, 042136 (2013).

[23] S. J. Press, J. Multivariate Anal **2**, 444 (1972).

[24] S. Jespersen, R. Metzler, and H. C. Fogedby, Phys. Rev. E **59**, 2736 (1999).

[25] A. V. Chechkin, J. Klafter, V. Y. Gonchar, R. Metzler, and L. V. Tanatarov, Chem. Phys. **284**, 233 (2002).

[26] B. Dybiec, E. Gudowska-Nowak, and I. M. Sokolov, Phys. Rev. E **76**, 041122 (2007).

[27] K. Szczepaniec and B. Dybiec, Phys. Rev. E **90**, 032128 (2014).

[28] A. Janicki and A. Weron, Stat. Sci. **9**, 109 (1994).

[29] M. Von Smoluchowski, Ann. Phys. **21**, 756 (1906).

[30] A. Einstein, Ann. Phys. **17**, 549 (1905).

[31] H. Touchette and E. G. D. Cohen, Phys. Rev. E **76**, 020101 (2007).

[32] A. V. Chechkin and R. Klages, J. Stat. Mech. , L03002 (2009).

[33] L. Kusmierz, J. Rubi, and E. Gudowska-Nowak, J. Stat. Mech. **38**, 1263 (2009).

[34] I. M. Sokolov, B. Dybiec, and W. Ebeling, Phys. Rev. E **83**, 041118 (2011).

[35] S. Zozor and C. Vignat, Phys. Rev. E **84**, 031115 (2011).

[36] P. N. Rathie, L. C. de S. M. Ozelim, and C. E. G. Otiniano, Comm. Nonlinear. Sci. Numer. Simulat. **16**, 204 (2011).

[37] P. R. Ride, Amer. Math. Monthly **72**, 303 (1965).