INSTABILITY OF REDUCIBLE CRITICAL POINTS OF
THE SEIBERG-WITTEN FUNCTIONAL

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Abstract. The Euler-Lagrange equations for the variational approach
to the Seiberg-Witten equations always admit reducible solutions. In
this context, the existence of unstable reducible solutions is achieved
by assuming the existence of a parallel spinor or the negativenss of a
Perelman-Yamabe type of invariant defined for a spin\textsuperscript{c}-structure.

1. Introduction

Let \((M, g)\) be a closed riemannian four manifold with scalar curvature \(k_g\).
By considering the least eigenvalue \(\lambda_g\) of the operator \(\triangle_g + \frac{k_g}{4}\),
where \(\triangle_g = d^*d\) is the Laplace-Beltrami operator associated to \(g\), Perelman introduced
in \([10, 11]\) the smooth invariant

\[
\bar{\lambda}(M) = \sup_{g \in M} \lambda_g [\text{vol}(M, g)]^{1/2}
\]

where \(M\) is the space of \(C^\infty\)-metrics on \(M\). Let \([g]\) be the conformal class of
\(g \in M\), Kobayashi \([5]\) and Schoen \([13]\) independently introduced the smooth manifold invariant

\[
\mathcal{Y}(M) = \sup_{[g]} \inf_g \int_M k_g dv_g / \text{vol}(M, g)^{1/2}.
\]

Assuming \(\mathcal{Y}(M) < 0\), Akutagawa-Ishida-Le Brun proved \([1]\) the equality
\(\bar{\lambda}(M) = \mathcal{Y}(M)\). A similar quantity turns up by measuring the instability of reducible critical points of the Seiberg-Witten functional, though in this case it depends on a \(\text{spin}^c\)-structure on \(M\). There exist smooth 4-manifolds admitting a \(\text{spin}^c\) structure \(c\) such that the Seiberg-Witten invariant \(SW(c) \neq 0\). These \(\text{spin}^c\) structures are named basic classes and they are in the realm of the 4-dim differential topology. The space \(\text{Spin}^c(M)\) of \(\text{spin}^c\) structures on \(M\) might be identified with

\[
\{c = \alpha_c + \beta_c \in H^2(X, Z) \oplus H^1(X, Z_2) \mid w_2(X) = \alpha \mod 2\}.
\]

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From the analytical point of view, a basic classes carries a \( SW \)-monopole, which is a special solution of a partial differential equation, as it will be defined next. The motivation for this research was to use variational techniques to measure the instability of reducible critical points for the Seiberg-Witten functional; the space of reducible solutions is diffeomorphic to the jacobian torus \( J_M = H^1(M, \mathbb{R})/H^1(M, \mathbb{Z}) \). Based on the fact that the Seiberg-Witten invariants are also expectation values of a \( N = 4 \) supersymmetric twisted gauge theory, \([6]\), one might believe that either there exists a monopole or \( J_M \) achieves the minimum energy.

The isomorphisms \( \text{Spin}_4^c = (SU_2 \times SU_2 \times U_1)/\mathbb{Z}_2 \) and \( \text{Spin}_3^c = U_2 = (SU_2 \times U_1)/\mathbb{Z}_2 \). Let \( P \) be the \( \text{spin}^c \)-principal bundle over \( M \) induced by the class \( c \in \text{Spin}^c(M) \) and \( S^c = P \times_{\rho^c} \mathbb{C}^2 \). In practice, a \( \text{spin}^c \)-structure on \( M \) is means the existence of a pair of rank 2 complex vector bundles \( S^c = P \times_{\rho^c} \mathbb{C}^2 \), which fibers are \( \text{Spin}^c_4 \)-modules, and isomorphisms \( \text{det}(S^c) = \text{det}(S^c) = \mathcal{L}_C \), where \( \text{det}(S^c) \) are the determinant line bundle such that \( c_1(\mathcal{L}_C) = \alpha \in H^2(M, \mathbb{Z}) \). Let \( \Omega^0(\mathcal{L}_C) \) the space of sections on \( S^c \) and \( \mathcal{A}_c \) be the space of \( U_1 \)-connections 1-forms. Each \( A \in \mathcal{A}_c \) induces a covariant derivative \( \nabla_A : \Omega^0(\mathcal{L}_C) \rightarrow \Omega^1(\mathcal{L}_C) \) on \( \mathcal{L}_C \). E.Witten introduced in \([14]\) the coupled system of 1st-order PDE (SW-monopole eqs.),

\[
\begin{align*}
D_A^+ \phi &= 0, & (2.1) \\
F_A^+ &= \sigma(\phi), & (2.2),
\end{align*}
\]

where \( \phi \in \Omega^0(S^c) \), \( D_A^+ \) is the positive component of the Dirac operator, \( F_A^+ \) is the self-dual component of the curvature \( F_A \) and \( \sigma : \Omega^0(S^c) \rightarrow \Omega^2(\mathbb{R}^c) \) is the self-dual 2-form

\[
\sigma(v)(X,Y) = \langle X,Y,v \rangle + \frac{1}{2} \langle X,Y \rangle |v|^2.
\]

performing the coupling between a self-dual 2-form \( F_A \) and a positive spinor field \( v \); \( |\sigma(v)|^2 = \frac{1}{4} |v|^4 \). The configuration space is \( \mathcal{C}_c = \mathcal{A}_c \times \Omega^0(S^c) \).

**Definition 1.0.1.** An element \( (A, \phi) \) is a \( SW_c \)-monopole if it verifies the SW-equations \((4)\). There are two kinds of \( SW_c \)-monopoles (i) irreducible if \( \phi \neq 0 \) and (ii) reducible if \( \phi = 0 \).

The irreducibles exist only for a finite number of classes in \( \text{Spin}^c(M) \). The monopole eqs. \((4)\) fits in a variational formulation whose Euler-Lagrange eqs. are the 2nd-order \( SW \)-equations

\[
\begin{align*}
d^* F_A + 4i \text{Im} \langle \nabla A \phi, \phi \rangle &= 0, \\
\Delta_A \phi + \frac{|\phi|^2 + k_{\phi}}{4} \phi &= 0.
\end{align*}
\]
For that matter, $J_M = \{(A, 0) \in C_c \mid d^* F_A = 0\}$ is the solution set of corresponding to those connections whose curvature is harmonic, and whose existence is guaranteed by Hodge theory. Later, it will be shown that monopoles are the ground state of the theory and are also solutions of eqs (5). In order to measure the instability, for each $c$ the ground state of the theory and are also solutions of eqs (5). In order to measure the instability, for each $c \in Spin^c(M)$, we introduced

$$\bar{\lambda}^c(M) = \sup_{A \in J_M} \left\{ \sup_{g \in \mathcal{M}} \lambda_g^c(A).[\text{vol}(M, g)]^{1/2} \right\}$$

where $M$ is the space of riemannian metrics on $M$. Thus, $J_M$ is defined to be unstable if $\bar{\lambda}^c(M) < 0$.

**Theorem 1.0.2.** Assume $k_g$ is non-negative. If there exists an irreducible solution $(A, \phi)$ of eqs (5), then $J_M$ is unstable.

**Theorem 1.0.3.** If $c \in Spin^c(M)$ admits a parallel spinor and the Yamabe invariant satisfies $\text{Y}(M) < 0$, then $J_M$ is unstable.

**Theorem 1.0.4.** If $c \in Spin^c(M)$ is a basic class admitting a parallel spinor, and $\alpha^2_c > 0$, then $\bar{\lambda}^c(M) < -\pi \sqrt{\alpha^c_c}$.

A class $c \in Spin^c(M)$ admitting a parallel spinor imposes strong restriction on $M$ ([2], [3]). Assuming $\pi_1(M) = 0$ and $M$ being irreducible as cartesian product, it turn out that either $M$ is Kähler or $M$ is spin Ricci-flat. The former case is characterized by the surjectivity of the Ricci tensor and the existence of an integrable complex structure $J$ on $M$ such that $\alpha_c = c_1(J)$ or $-c_1(J)$. In the last, the Ricci tensor must be null and the manifold spin. The author is not aware of any sort of classification theorem of spin Ricci-flat 4-manifolds, but its importance for physicists. It is a long standing problem to find examples of Ricci-flat manifolds with holonomy $SO_n$.

2. Background

Consider $\pi : E \to M$ a vector bundle with structural group $G$ and denote $F(E)$ the $G$-principal bundle of frames on $E$.

2.1. Gauge Group. Consider $G$ a Lie group with Lie algebra $\mathfrak{g}$. The Gauge group $G_P$ of a principle $G$-bundle $P_G$ is the set of $G$-equivariant automorphism $\Phi : P_G \to P_G$ such that $\pi \circ \Phi = \pi$. A gauge transformation $\Phi$ is better described as a map $s : P_G \to G$ such that $\Phi(p) = p.s(p), s(p.g) = g^{-1}.s(p).g$. Taking the adjoint action $Ad_g : G \to G, Ad_g(x) = g^{-1}.x.g$, and defining the bundle $Ad(G) = P_G \times_{Ad} G$, the gauge group $G$ is the space of sections of $Ad(P)$. The representation $ad : G \to \text{End}(\mathfrak{g}), ad(g) = g^{-1}vg$, induces the associated vector bundle $ad(\mathfrak{g}) = P_G \times_{ad} \mathfrak{g}$. If $G$ is abelian, then $Ad(P) = Map(M, G)$; e.g.: $G = U_1, G = Map(M, U_1)$ and $ad(u_1) = i\mathbb{R}$. The group $G_P$ also acts on an associated vector bundle $E = P \times_\rho V$, $\rho : G \to \text{End}(V)$. The homotopy type of $G_P$ depends on the homotopy type of $P$. 
2.2. Spin and Spin\(^c\) Structures on \(M\). Whenever \(M\) admits a spin structure or a spin\(^c\) structure it carries a Dirac operator useful to study geometric and topological properties on \(M\) by analytical methods. In order to define such structures we consider the Lie groups \(\text{Spin}_4 = SU_2 \times SU_2\), recalling that \(\overline{Ad} : \text{Spin}_4 \to SO_4\) is the universal covering map, and \(\text{Spin}^c_4 = \text{Spin}_4 \times_{\mathbb{Z}_2} U_1\). Let \(\pi : F(M) \to M\) be the frame bundle of \(M\). A spin structure on \(M\) is a principal \(\text{Spin}_4\)-bundle \(P^s\) such that the projection \(\pi' : P^s \to M\) lifts to a map \(\zeta : P^s \to F(M)\) satisfying the following conditions
(i) \(\zeta(p.g) = \zeta(p).\overline{Ad}(g)\), for all \(p \in P^s(E)\) and \(g \in \text{Spin}_4\),
(ii) \(\pi \circ \zeta = \pi' : P^s \to M\).

It turns out that \(M\) admits a spin structure if and only if \(w_2(M) = 0\); in this case the space of spin-structures on \(M\) is \(\text{Spin}(M) = H^1(M, \mathbb{Z}_2)\).

All spin structure on a smooth 4-manifold \(M\) carries a Dirac operator useful to study geometric and topological properties on \(M\) by analytical methods. In order to define an almost complex structure on \(M\) we consider the Lie groups \(\text{Spin}_4\), there exists a class \(\alpha \in H^2(M, \mathbb{Z}_2)\) such that \(w_2(X) \equiv \alpha \mod 2\). Indeed, a spin\(^c\)-structure on \(M\) corresponds to define an almost complex structure on \(M\) by \(\{pt\}\). When \(M\) is a spin manifold we have \(c_1(\mathcal{L}_c) = \alpha_c \in H^2(M, \mathbb{Z}_2)\).

In this case, the bundles \(\mathcal{S}\) and \(\mathcal{L}_c^{1/2}\) are globally defined and \(\mathcal{S}_c = \mathcal{S} \oplus (\mathcal{L}_c)^{1/2}\), where \((\mathcal{L}_c)^{1/2}\) is the square root bundle of \(\mathcal{L}_c\). When \(w_2(M) \neq 0\) the tensor product \(\mathcal{S}_c = \mathcal{S} \otimes (\mathcal{L}_c)^{1/2}\) is globally defined, though the bundles \(\mathcal{S}\) and \((\mathcal{L}_c)^{1/2}\) are not. The bundle \(\mathcal{S}_c\) inherits the decomposition \(\mathcal{S}_c = \mathcal{S}_c^+ \oplus \mathcal{S}_c^-\), where \(\mathcal{S}_c^\pm\) are the \((\pm)\)-complex spinor bundles of rank 2. Moreover,
\[
\begin{align*}
c_1(\mathcal{S}_c^+) &= c_1(\mathcal{L}_c), \quad c_2(\mathcal{S}_c^+) = \frac{1}{4}[c_1^2(\mathcal{L}_c) - 2\chi(M) - 3\sigma(M)], \\
c_1(\mathcal{S}_c^-) &= c_1(\mathcal{L}_c), \quad c_2(\mathcal{S}_c^-) = \frac{1}{4}[c_1^2(\mathcal{L}_c) + 2\chi(M) - 3\sigma(M)].
\end{align*}
\]

3. Geometric Structures

A brief introduction on covariant derivatives and curvature is given in order to fix the concepts and the notations needed along the text.

3.1. Covariant Derivatives and Connections 1-forms on \(\mathcal{S}_c^\pm\). Let’s consider the general case of a smooth vector bundle \(E\) over \(M\). Let \(\mathcal{A}_E\) be the space of connection 1-forms on \(E\). A covariant derivative on a vector bundle \(E\) over \(M\) is a \(\mathbb{R}\)-linear operator \(\nabla : \Omega^0(E) \to \Omega^1(E)\) satisfying the Leibniz rule: for all \(f : M \to \mathbb{R}\) and \(V \in \Omega^0(E)\),
\[
\nabla(fV) = df \wedge V + f \wedge \nabla V.
\]

Using the exterior derivative \(d : \Omega^p(M) \to \Omega^{p+1}\), it can be extended to a linear operator \(d^\nabla : \Omega^p(E) \to \Omega^{p+1}(E)\), by
\[ d\nabla(V \omega) = \nabla V \wedge \omega + V \otimes d\omega. \]

\( \mathcal{A}_E \) is an affine space which turns out to be the vector space \( \Omega^1(ad(g)) \) by fixing an origin at \( \nabla^0 \in \mathcal{A}_E \). Any covariant derivative \( \nabla^A \) can be written as:

\[ \nabla^A = \nabla^0 + A, \quad A \in \Omega^1(ad(g)) = \mathcal{A}_E. \]

Covariant derivatives and connection 1-forms are equivalent. The group \( \mathbf{G} \) acts on \( \mathcal{A}_E \) by \( g.\nabla = g^{-1}\nabla g \), so inducing on \( \Omega^1(ad(g)) \) the \( \mathbf{G} \)-action \( g.A = g^{-1}Ag + g^{-1}dg \). Fix a local chart \( U \subset M \), let \( \nabla : \Omega^0(TM) \to \Omega^1(TM) \) be the riemannian connection on \( M \) and \( \beta_U = \{ e_i \mid 1 \leq i \leq 4 \} \) be a local orthonormal frame of \( TM \) defined on \( U \) having the following properties: for all \( i, j \) (\( \nabla_i = \nabla_{e_i} \))

(i) \( [e_i, e_j] = \nabla_i e_j - \nabla_j e_i = 0 \),

(ii) \( \nabla_i e_k = \sum_l \Gamma_{ik}^l e_l \),

The covariant derivative operator is locally given by

\[ \nabla = \sum_i (\nabla_i)dx^i = d + \Gamma, \]

where \( \nabla_i = \partial_i + \Gamma_i \) and \( \Gamma = \sum_i \Gamma_i dx^i \in \Omega^1(ad(\mathfrak{s}\mathfrak{o}_4)) \) is the connection 1-form. The set of linear maps \( e_k \wedge e_l : \mathbb{R}^4 \to \mathbb{R}^4 \), given by

\[ (e_k \wedge e_l)(v) = v \cdot e_k - v \cdot e_k, \quad (\mathfrak{s}\mathfrak{o}_4 \simeq \Lambda^2(\mathbb{R}^4)), \]

defines a \( \mathfrak{s}\mathfrak{o}_4 \) basis on which \( \Gamma_i = \sum_{k,l} (\Gamma_{ik})_{kl}(e_k \wedge e_l) \). The riemannian connection on \( (M, g) \) induces a connection on \( \mathcal{S} \) as we shown next. Let \( \mathcal{C}(M, g) \) be the Clifford Algebra Bundle and \( c : TM \to \mathcal{C}(M, g) \) be the Clifford map performing the inclusion. A \( \Omega^0(\mathcal{C}(M, g)) \)-module structure is defined on \( \Omega^0(\mathcal{S}) \) by the pointwise product \( (\gamma.\phi)(x) = \gamma(x).\phi(x), \) for all \( \gamma \in \Omega^0(\mathcal{C}(M, g)) \) and \( \phi \in \Omega^0(\mathcal{S}) \). In order to describe a connection (locally defined) on \( \mathcal{S} \), let’s consider \( \gamma_i = c(e_i) \). The whole procedure to induce the connection on \( \mathcal{S} \) relies on the lie algebra isomorphism \( \Theta : \mathfrak{s}\mathfrak{o}_n \to \mathfrak{spin}_n \), \( \Theta(e_k \wedge e_l) = \frac{1}{2} \gamma_k.\gamma_l (\mathbb{I}) \), prop 6.1. In this way, the Christoffel’s symbols of \( M \) induce on \( \mathcal{S} \) the operator \( \Gamma^s : \Omega^0(\mathcal{S}) \to \Omega^1(\mathcal{S}), \Gamma_i^s = \frac{1}{2} \sum_{l,k} \Gamma_{ik}^l (\gamma_k.\gamma_l) \).

The spin connection 1-form on \( \mathcal{S} \) is \( \Gamma^s = \sum_i \Gamma_i^s dx^i \), it induces the covariant derivative \( \nabla^s : \Omega^0(\mathcal{S}) \to \Omega^1(\mathcal{S}) \), \( \nabla^s = d + \Gamma^s \). A covariant derivative operator \( \nabla^A : \Omega^0(\mathcal{S}_c) \to \Omega^1(\mathcal{S}_c) \) is locally defined on \( \mathcal{S}_c \equiv \mathcal{S} \otimes L_{1/2}^c \) by taking the spin connection \( \nabla^s \) on \( \mathcal{S} \) and a \( U_1 \)-connection \( \nabla^A \) on \( L_{1/2}^c \), as follows: let \( \psi \in \Omega^0(\mathcal{S}), \lambda \in \Omega^0(L_{1/2}^c) \) and \( \psi \otimes \lambda \in \Omega^0(\mathcal{S}_c) \),

\[ \nabla^A(\psi \otimes \lambda) = \nabla^s \psi \otimes \lambda + \psi \otimes \nabla^A \lambda. \]

(it can be patched together to define the operator \( \nabla^A \) globally).

### 3.2. Curvature

The curvature of a covariant derivative \( \nabla \) on \( E \) is the \( C^\infty \)-linear operator \( F = d\nabla \circ d\nabla : \Omega^0(E) \to \Omega^2(E) \) defined by

\[ F(V)(X, Y) = (\nabla X \nabla Y - \nabla Y \nabla X - \nabla [Y, X]) V, \]
for all $V \in \Omega^0(E)$ and $X, Y \in \Omega^0(TM)$. In an orthonormal frame $\beta^E = \{f_\alpha | 1 \leq \alpha \leq r\}$ on $E$, such that $\nabla e_i f_\alpha = \sum \beta A^\beta_{i\alpha} f_\beta$, we consider, for each $i, j$, $F_{ij} \in \text{End}(E)$ as the operator

$$F_{ij}(V) = F(e_i, e_j)(V) = \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j] \right)(V),$$

Each $A_i(x)$ being a skew-symmetric operator $\forall x$ implies $F_{ij}(x0 \in \text{End}(E_x)$ is also skew-symmetric. Let $A \in \Omega^1(E)$ and $\nabla = d + A$, the curvature 2-forms $F_A \in \Omega^2(E)$ is $F_A = \sum_{ij} F_{ij} dx^i \wedge dx^j$. The gauge group action on $\Omega^2(E)$ is $g.\omega = g^{-1}.\omega.g$ motivated by the fact that curvature of $gA$ is $g.F_A = g^{-1}.F_{A}g$. When $E = TM$, the curvature 2-form $R : \Omega^0(TM) \rightarrow \Omega^2(TM)$ of the riemannian metric is locally written, using the frame $\beta_U$, as $R = \sum_{ij} R_{ij} dx^i \wedge dx^j$. The components $R_{ij}(e_k) = \sum_l R_{ijk}^l e_l$, $(R_{ij})^l = R_{ijk}^l$ satisfy the identities

$$(i) \ R_{ijk}^l + R_{jki}^l + R_{kij}^l = 0, \quad (iii) \ R_{ijk}^l = -R_{ijl}^k$$
$$(ii) \ R_{ijk}^l = -R_{jik}^l, \quad (iv) \ R_{ijk}^l = R_{kli}^j$$

Using the $\mathfrak{so}_4$ basis $\{e_k \wedge e_l\}$ we have $R_{ij} = \sum_k R_{ijk}^l e_k \wedge e_l$. In this way, the curvature 2-form induced on $S$ by the riemannian connection on $TM$ is

$$R^\xi = \frac{1}{2} \sum_{k,l} \left( \sum_{i,j} R_{ijk}^l dx^i \wedge dx^j \right) \gamma_k \gamma_l \in \Omega^2(S)$$

**Definition 3.2.1.** The Ricci curvature of the riemannian manifold $(M, g)$ is the bilinear form $\text{Ric} : \Omega^0(TM) \times \Omega^0(TM) \rightarrow C^\infty(M)$

$$\text{Ric}(u, v) = \text{trace}_g [w \rightarrow R(u, w)v].$$

Using the frame $\beta_U = \{e_j\}$ on $M$, the Ricci curvature is given by

$$\text{Ric}(u, v) = g(\sum_k R(e_k, v)e_k, u), \ \forall u, v \in TM.$$}

Using the symmetry $R_{ij} = R_{ji}$, we define the linear self-adjoint Ricci operator $\text{Ric} : \Omega^0(TM) \rightarrow \Omega^0(TM)$, $\text{Ric}(u, v) = g(u, \text{Ric}(v))$, locally given by $\text{Ric}(v) = \sum_k R(e_k, v)e_k$. It induces on $S$ the operator

$$\text{Ric}^\xi(v) = \sum_k R(e_k, v)\gamma_k$$

So far, it has been showed how the riemannian connection induces a connection on $S$. The equation (7) induces a connection on $S$, whose curvature 2-form $F_A : \Omega^0(S) \rightarrow \Omega^2(S)$ locally decomposes into
The expression (9) reflects the existence of the decomposition $u_2 = su_2 \oplus u_1$. By projecting the curvature $F_A \in u_2$ into the sub-algebras the component $su_2$ gives part of the Riemann tensor of $M$ and the $u_1$ component gives the curvature of $A$ on $L^1/2$. Thus, the curvature induced on $L_c$ is $2if_A$.

4. Variational Formulation and 2nd Variation

By fixing an origin at $\nabla^0 \in A_\epsilon$ a connection on $L_\epsilon$ is written as $\nabla^A = \nabla^0 + A$, where $A \in \Omega^1(M, i\mathbb{R})$ is a $u_1$-valued 1-forms. A topology on the configuration space $C_\epsilon = A_\epsilon \times \Omega^0(S^+_\epsilon)$ is defined by considering the Sobolev spaces $A_\epsilon = L^{1,2}(\Omega^1(M, i\mathbb{R}))$ and $T(S^+_\epsilon) = L^{1,2}(\Omega^0(X, S^+_\epsilon))$; the gauge group is taken to be $G = L^{2,2}(\text{Map}(X, U_1))$. The $G$ action on $C_\epsilon$ is not free, the isotropy group are $G_{(A,0)} = \{I\}$ and $G_{(A,0)} \simeq U_1$ for all $A \in \mathcal{A}_\epsilon$.

An element $(A, \phi) \in \mathcal{C}_\epsilon$ is named irreducible if $\phi \neq 0$, otherwise is reducible. The subspace of irreducibles $\mathcal{C}_\epsilon^* = \{(A, \phi) \in \mathcal{C}_\epsilon \mid \phi \neq 0\}$ is a universal principal $G$-bundle over the moduli space $\mathcal{B}_\epsilon^* = \mathcal{C}_\epsilon^*/G$. The quotient space $\mathcal{B}_\epsilon^*$ has the same homotopy type of $\mathbb{C}P^{\infty} \times \mathcal{J}_M$. The free action of $U_1 = \{g \in G \mid g \text{ constant}\}$ on $\mathcal{C}_\epsilon^*$ defines a principal $U_1$-bundle over $\mathcal{B}_\epsilon^*$ whose first Chern class $c_1(\mathcal{C}_\epsilon^*) = SW(\mathcal{C}_\epsilon)$ is the generator of subring corresponding to the cohomology of the factor $\mathbb{C}P^{\infty} \times \{p\}$ in $H^\ast(\mathcal{B}_\epsilon^*, \mathbb{Z})$ ($p \in \mathcal{J}_M$).

The riemannian structure on the tangent bundle $T\mathcal{C}_\epsilon = \mathcal{C}_\epsilon \times (\Omega^1(i\mathbb{R}) \oplus \Omega^0(S^+_\epsilon))$ is the product of the following structures on each component;

(i) on $\mathcal{A}_\epsilon$, for all $\eta, \theta \in \Omega^0(M, i\mathbb{R})$,

$$< \eta, \theta > = \int_M (\eta \wedge \ast \theta) dv_g,$$

recalling that the Hodge operator is minus the usual star operator because the forms take values in $i\mathbb{R}$ instead of $\mathbb{R}$.

(ii) on $\Omega^0(S^+_\epsilon)$, for any sections $V, W \in \Omega^0(S^+_\epsilon)$, $(z \in \mathbb{C}$, $\Re(z) = \frac{z + \bar{z}}{2}$)

$$< V, W > = \int_X \Re(\langle V, W \rangle) dv_g.$$ 

Thus, the inner product $<, >: T_{(A, \phi)}\mathcal{C}_\epsilon \times T_{(A, \phi)}\mathcal{C}_\epsilon \to \mathbb{R}$ is

$$< \eta + V, \theta + W >= < \eta, \theta > + < V, W > .$$

The Seiberg-Witten equations fit into a variational set up by defining the functional $SW : \mathcal{C}_\epsilon \to \mathbb{R}$,

$$SW_c(A, \phi) = \int_X \left\{ \frac{1}{4} | F_A |^2 + | \nabla^A \phi |^2 + \frac{1}{8}(| \phi |^2 + k_g)^2 - \frac{k_g^2}{8} \right\} dv_g + 2\pi^2 N_c,$$

(10)
where $k_g$ is the scalar curvature of $(X,g)$ and

$$N_c = c_1^2(c) = c_1(c) \wedge c_1(c) = \frac{1}{4\pi^2} \int_X \| F_A^+ \|^2 - | F_A^- |^2 dv_g.$$ 

The functional $SW_c$ is gauge invariant, therefore it is well defined on $\mathcal{B}_c$. Defining $k_g = \max\{0, \sup_{x \in M} (-k_g(x))\}$, it is straightforward from eq. (11) that a necessary condition to the existence of an irreducible monopole is

$$\| \phi \|_{\infty} < \sqrt{k_g}.$$ 

Jost-Peng-Wang proved in [4] the functional $SW_c : \mathcal{B}_c \to \mathbb{R}$ satisfies the Palais-Smale condition, so the critical sets are compact and the minimum is always achieved ($SW_c \geq 0$). Let $\text{grad}(SW_c)(A,\phi)$ be the gradient at $(A,\phi)$, the Euler-Lagrange equations defined by $\text{grad}(SW_c)(A,\phi) = 0$ are ($Im(z) = \frac{\phi}{2i}$)

$$d^* F_A + 4i Im(\langle \nabla^A \phi, \phi \rangle) = 0, \quad \Delta_A \phi + \frac{|\phi|^2 + k_g \phi}{4} = 0.$$ 

These are the $G$-invariant 2nd-order SW-equations because

$$\text{grad}(SW_c)(g, (A,\phi)) = g^{-1} \text{grad}(SW_c)(A,\phi).$$ 

They may admit reducible and reducible solutions. As expected, the SW-monopoles also satisfy eqs (11), as asserted by the identities

$$d^*(F_A) = 2d^* F_A^+ = d^*[\sigma(\phi)] = -4i Im(\langle D^+_A \phi, X, \phi \rangle + \langle \nabla_X^A \phi, \phi \rangle)$$

$$D^+_A \phi = 0 \Rightarrow 0 = D^-_A D^+_A \phi = \Delta_A \phi + \frac{k_g \phi}{4} + \frac{F_A^+}{2} \phi = \Delta_A \phi + \frac{k_g \phi}{4} + \frac{|\phi|^2}{2} \phi.$$ 

Due to the identity $SW_c(A,\phi) = \int_M (\| F_A^+ - \sigma(\phi) \|^2 + | D^+_A \phi |^2) dv_g$, whenever $c \in \text{Spin}^c(X)$ is a basic class the $SW_c$-monopoles are stable critical points. The solution set of eqs. (11) may be singular in the presence of reducible points. Assuming $k_g \geq 0$, the minimum is achieved at reducible points because $SW_c(A,\phi) \geq SW_c(A,0), \forall (A,\phi) \in \mathcal{C}_c$. At a critical point $(A,0)$, the $SW_c$-monopole eqs (11) is $F_A^+ = 0$ and the eqs. (11) reduces to $d^* F_A = 0$. Under the assumption $b_2^+ (M) \geq 2$, independently of the sign of $k_g$, the anti-self-dual solutions can be ruled out. If $d^* F_A = 0$, then $F_A$ is harmonic 2-form. Let $B \in \mathcal{A}_c$ be such that $F_A = F_B$, so $\omega = B - A \in H^1(M,\mathbb{R})$. Moreover, $B$ is gauge equivalent to $A$ if and only if $\omega = g^{-1} dg \in H^1(M,\mathbb{Z})$. Therefore, the space $\mathcal{F}_M = \{(A,0) \in \mathcal{C}_c | d^* F_A = 0\} / \mathcal{G}$ is diffeomorphic to the jacobian torus $T_{\theta_1} = \frac{H^1(M,\mathbb{R})}{H^1(M,\mathbb{Z})}$. A local slice of $\mathcal{B}_c$ at $(A,\phi)$ is given (9) by the kernel $\ker(T^*_\phi) = \ker(d^*) \oplus \phi^\perp$ of the operator

$$T^*_\phi : \Omega^1(X,i\mathbb{R}) \oplus \Omega^0(S^+_c) \to \Omega^0(X,i\mathbb{R}),$$

$$T^*_\phi(\theta, V) = d^* \theta - \langle V, \phi \rangle.$$
Because $(d^*)^2 = 0$, it decomposed into subspaces $\text{ker}(d^*) = d^*(\Omega^2(M, i\mathbb{R})) \oplus H_1$, where the subspace of harmonic 1-forms $H_1 = \{\theta \in \Omega^1(M, i\mathbb{R}) \mid d\theta = d^*\theta = 0\}$ is the tangent space to the Jacobian torus $J_M$ at $(A, 0)$. The instability of $J_M$ is established by performing the analysis of the $2^{nd}$ variation $\frac{\delta^2 SW}{\delta \alpha \delta \beta}$ of the $SW$-functional. The tangent space of $C_\epsilon$ at $(A, \phi)$ is $T_{(A, \phi)}C_\epsilon = \Omega^1(X; i\mathbb{R}) \oplus \Omega^0(S^+_\epsilon)$, so $\frac{\delta^2 SW}{\delta \alpha \delta \beta}$ defines a symmetrical bilinear form $H^S_{(A, \phi)}((\theta_1, V_1), (\theta_2, V_2)) = \langle (\theta_1, V_1), H(\theta_2, V_2) \rangle$, where the operator 

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

has entries given by

$$\frac{\delta^2 SW}{\delta \alpha \delta \theta} \bigg|_{(A, \phi)} (\theta, \Lambda) = \langle \theta, (d^*d\Lambda + 4 <\Lambda(\phi), \phi>) \rangle = \langle \theta, h_{11}(\Lambda) \rangle,$$

$$\frac{\delta^2 SW}{\delta W \delta \theta} \bigg|_{(A, \phi)} (\theta, W) = 2 \langle \nabla^A \phi, \theta(W) \rangle + \langle \nabla^A W, \theta(\phi) \rangle = \langle \theta, h_{12}(W) \rangle, \quad (h_{21} = h_{12})$$

$$\frac{\delta^2 SW}{\delta W \delta V} \bigg|_{(A, \phi)} (V, W) = \langle V, \nabla_A W + \frac{k_4 + |\phi|^2}{4} W + \frac{1}{4} \phi, W \rangle = \langle V, h_{22}(W) \rangle.$$

The restriction of the $2^{nd}$-variation to the slice of $B_\epsilon$ at $(A, \phi)$ is an elliptic operator $H : \text{ker}(T^*_A) \rightarrow \text{ker}(T^*_A)$ whose leading terms $d^*d = \nabla$ and $\nabla_A = -(\nabla A)^* \nabla A$ are laplacians and whose tail is a compact operator. Thus, $H$ is a self-adjoint Fredholm operator. The spectrum $\sigma(H)$ is a discrete set such that each eigenvalue has finite multiplicity and no accumulation points, besides, there are but a finite number of eigenvalues below any given number. At $(A, 0)$, the hessian operator becomes $H = \begin{pmatrix} d^*d & 0 \\ 0 & L_A \end{pmatrix}$, where $L_A : \Omega^0(S^+_\epsilon) \rightarrow \Omega^0(S^+_\epsilon)$ is the elliptic self-adjoint operator

$$L_A(V) = \nabla_A V + \frac{k_4}{4} V.$$

For each $\lambda \in \sigma(L_A)$, the corresponding eingenspace $\mathcal{V}_\lambda \subset T_{(A,0)}B_\epsilon$ has finite dimension. In this way, $\text{ker}(H) = T_{(A,0)}J_X \oplus \mathcal{V}_0$. The lower eigenvalue of $L_A$ given by Rayleigh’s quotient

$$\lambda_g(A) = \inf_{V \in S^+_\epsilon} \frac{\int_M \{ |\nabla^A V|^2 + \frac{k_4}{4} |V|^2 \} dv_g}{\int_M |V|^2 dv_g}$$

is bounded below, so is the spectrum $\sigma(L_A)$.

4.1. Parallel Spinor. A spinor $\psi \in \Omega^0(S^+_\epsilon)$ is parallel with respect to a connection $\nabla$ if $\nabla \psi = 0$. In general, it is difficult to pull off information about $\sigma(L_A)$, but using Kato’s inequality it can be compare with the
spectrum of $L = \triangle_g + \frac{k}{2}$ defined on functions $f : M \to \mathbb{R}$ ($\triangle_g$=Laplace-Beltrami). Consider on $M$ a smooth atlas $A(M) = \{(U_\lambda, \xi_\lambda) \mid \lambda \in \Lambda\}$ such that, for each $\lambda \in \Lambda$, (i) $U_\lambda$ is convex, (ii) the local coordinates are $\{(x_1, x_2, x_3, x_4) \in U_\lambda \mid x_i \in \mathbb{R}\}$ and (iii) attached to $U_\lambda$ there exists a local orthonormal frame $\beta_\lambda = \{e_i \mid e_i = \partial_i, \, 1 \leq i \leq 4\}$.

**Proposition 4.1.1.** (Kato’s ineq.) Let $A \in \mathcal{A}_\nu$ and $V \in \Omega^0(S^+_c)$. Then,

$$| \nabla | V |^2 | \leq | \nabla^A V |^2$$

(15)

The equality holds if, and only if, there exists a 1-form $\omega \in \Omega^1(M)$ such that $\nabla^A V = \omega V$.

**Proof.** Taking the orthonormal frame $\beta = \{e_i \mid 1 \leq i \leq 4\}$, locally we get $| \nabla | V |^2 = \sum_i | \nabla_i | V |^2$ and $| \nabla^A V |^2 = \sum_i | \nabla^A_i V |^2$. From the identities $\nabla_i | V |^2 = 2 | V | \cdot \nabla_i | V |$ and $\nabla_i | V |^2 = \nabla_i < V, V > = 2 < \nabla^A_i V, V >$, we have $| V | \cdot \nabla_i | V | = < \nabla^A_i V, V >$. Assuming $V \neq 0$ and applying Cauchy-Schwartz inequality it follows the inequality $| \nabla_i | V | \leq | \nabla^A_i V |$. Hence, ineq. (15) is verified. The equality is attained whenever there exists functions $\alpha_i : M \to \mathbb{C}$ such that $\nabla^A_i V = \alpha_i V$, that is,

$$\nabla^A V = \sum_i \nabla^A_i V dx^i = \left[ \sum_i \alpha_i dx^i \right] V = \omega V$$

□

If $V$ is a harmonic spinor ($D_A V = 0$) and $\nabla^A V = \omega V$, then $\nabla^A V = 0$. It is rather restrictive to assume $V$ as a harmonic spinor, but under an extra assumption on the functions $\alpha_i : M \to \mathbb{C}$ the existence of a parallel spinor can be achieved. The reverse claim is also true;

**Proposition 4.1.2.** There exists a parallel spinor $V \in \Omega^0(S^+_c)$ if, and only if, there exists a spinor $V_0 \in \Omega^0(S^+_c)$ and a class $\omega \in H^1_{dR}(M)$ such that $\nabla^A V_0 = \omega V_0$.

**Proof.** Suppose $V \in \Omega^0(S^+_c)$ is parallel, $\nabla^A V = 0$. So, $V$ has constant length. Let $V = f V_0$, where $f : M \to \mathbb{C}$, so $f(x) \neq 0$ and $V_0(x) \neq 0$, $\forall x \in M$. Furthermore,

$$\nabla^A V = df \wedge V_0 + f \wedge \nabla^A V_0 = 0 \Rightarrow \nabla^A V_0 = -\frac{df}{f} V_0.$$  

(16)

The 1-form $\omega = -\frac{df}{f} = -d(ln(f))$ is exact. Now, let’s prove the reverse assuming that $\nabla V_0 = \omega V_0$ and $d\omega = 0$. The equation $\nabla^A (f V_0) = 0$ is equivalent to $df \wedge f\omega = 0$; in this case $\omega = -d(ln(f))$. Taking a local chart $(U_\lambda, \phi_\lambda)$ from the atlas defined at the beginning of this section, say a chart $(U_\lambda, \phi_\lambda)$ with frame $\beta_\lambda = \{e_i \mid 1 \leq i \leq 4\}$, and defining $\alpha_i = w(e_i)$, we get $\omega = \sum_i \alpha_i dx^i$. In this way, the equation $df \wedge f\omega = 0$ becomes locally
Definition 4.1.3. Let $\Omega^2$ satisfy the conditions $\partial_j \alpha_i = \partial_i \alpha_j$, for all $i, j$. Thus, the necessary condition $\partial_j \partial_i f = \partial_i \partial_j f$ to the existence of $f$ is easily verified, since

$$\partial_j \partial_i f = -(\partial_j \alpha_i) f - \alpha_i \alpha_j f = \partial_i \partial_j f.$$ 

The identity $\partial_j \alpha_i = \partial_i \alpha_j$ allow us to integrate and write

$$\alpha_i(x_1, x_2, x_3, x_4) = \int_0^{x_1} \partial_i \alpha_1(t, x_2, x_3, x_4) dt, \quad 2 \leq i \leq 4.$$ 

Therefore, the function

$$f(x_1, x_2, x_3, x_4) = e^{\int_0^{x_1} \alpha_1(t, x_2, x_3, x_4) dt},$$

satisfies $\partial_i f - \alpha_i f = 0$, for $1 \leq i \leq 4$, and is $C^\infty$. The function $f$ is globally defined because it depends only on the 1-form $\omega$. □

As before, consider $\beta = \{e_\alpha; 1 \leq \alpha \leq 4\}$ an orthonormal frame on $M$ and $\gamma_\alpha = c(e_\alpha)$. The Ricci operator induces the operator $c(Ric(\cdot)) : TX \to Cl(X)$, $c(Ric(X)) = \sum_{\alpha} R^\alpha(e_\alpha, X) \gamma_\alpha$, such that

$$[c(Ric(X))]^2 = -\sum_{\alpha} |R^\alpha(e_\alpha, X)|^2 = -|Ric(\cdot)|^2.$$ 

**Definition 4.1.3.** Let $A \in A_c$ be a connection 1-form with curvature if $A \in \Omega^2(M, i\mathbb{R})$;

1. Let $H_A : TX \times \Omega^0(S_c) \to \Omega^0(S_c)$ be the linear operator defined by

$$H_A(X, \psi) = -\frac{1}{2} \sum_{\alpha} \gamma_\alpha F^\alpha_A(e_\alpha, X)(\psi),$$ 

2. Let $I_A : \Omega^0(TM) \to \Omega^0(TM)$ be the skew-symmetric operator

$$(17) \quad I_A(X) = \sum_{\alpha} [(X \flat, f_A)(e_\alpha)] e_\alpha$$

Using the local frame $\beta = \{e_\alpha\}$ we have $(I_A)_{\alpha \beta} = i(e_\beta \flat, f_A)(e_\alpha) = 2i f_{\beta \alpha}$. Let $A \in A_c$ and assume $\psi \in \Omega^0(S_c)$ is a parallel spinor. Global information about $M$ can be draw, as we shall see next, upon the existence of a parallel spinor $\psi$. In this case, it follows that $H_A(X, \psi) = 0$, for all $X \in TM$. The first consequence is the identity $||Ric(\cdot)||^2 = ||f_A||^2$ whose proof in [3], chap 3) relies strongly on the identity

$$(18) \quad [c(Ric(X)) - ic(I_A(X))], \psi = 0, \forall X \in TM.$$ 

So, $f_A = 0$ if and only if $Ric = 0$. Moreover, (i) $|c(Ric(X))| = |c(I_A(X))|$ and (ii) $<c(Ric(X)), ic(I_A(X))> = 0$. The identity $c(Ric(X))(\psi) = ic(I_A(X))(\psi)$ is a key point. Defining $R = \{Ric(X) \mid X \in \Omega^0(TM)\),$
from the self-adjointness of Ricci operator there exists the decomposition
\[ TX = R \oplus ker(I_A) \quad (ker(I_A) = R^\perp) \]. Let’s consider the operators
\[ \Psi : TX \to \Omega^0(S) \quad \Psi^* : TX \to \Omega^0(S) \]
\[ X \to \text{c}(X,\psi) \quad X \to i\text{c}(X,\psi) \]
and the vector space \( E = \Psi^{-1}(\text{Im}(\Psi) \cap \text{Im}(\Psi^*)) \). The existence of a parallel spinor \( \psi \) means that \( R \subset E \) and so \( E^\perp \subset ker(I_A) \). These spaces define the distributions \( E = \{ E_x \mid x \in M \} \) and \( E^\perp = \{ E_x^\perp \mid x \in M \} \).

**Proposition 4.1.4.** If \( \psi \) is a parallel spinor, then the distributions \( E \) and \( E^\perp \) are integrable.

**Proof.** For all \( X \in E \) there exist an unique \( Y \in TX \) such that \( X.\psi = iY.\psi \). The space \( E \) is closed under the action of the covariant derivative because \( X.\psi = iY.\psi \) implies \( (\nabla X).\psi = i(\nabla Y).\psi \).

(i) \( E^\perp \) is integrable.

Note that for all \( X \in E^\perp \) we get \( X_c f_A = 0 \), in particular \( f_A \) annihilates \( X \). Since \( df_A = 0 \), the distribution \( E^\perp \) is integrable.

(ii) \( E \) is integrable.

Taking \( X, Y \in E \), it follows from the \( \nabla \)-invariance of \( E \) that the commutator \( [X, Y] = \nabla_X Y - \nabla_Y X \in E \).

\[ \square \]

**Corollary 4.1.5.** There exist submanifolds \( M_1, M_2 \subset M \) such that \( M_1 \) is Kähler and \( M_2 \) is spin.

**Proof.** Let \( M_1 \) be the submanifold whose tangent space \( T_x M_1 = E_x \). Since for each \( X \in E \) there exists only one \( Y \) such that \( X.\psi = iY.\psi \), define the automorphism \( J : TX \to TX \), \( X.\psi = iJ(X).\psi \). Thus, for each \( x \in M \), \( J : T_x M_1 \to T_x M_1 \) defines a complex structure since

\[ iJ^2(X).\psi = iJ(J(X)).\psi = J(X).\psi = -iX.\psi \Rightarrow J^2 = -I. \]

Moreover, \( \nabla J = 0 \) because \( J(\nabla X) = \nabla Y \) and, for all \( X, Y \in \Omega^0(TM) \),

\[ (\nabla X).\psi = [(\nabla J)X + J(\nabla X)].\psi = iJ(\nabla X).\psi \Rightarrow \nabla J = 0. \]

From the identity

\[ [J(X), J(Y) + J(Y)J(X)] \quad \psi = (XY + YX).\psi + 2i[g(J(X), Y) + g(J(Y), X)] \quad \psi \]

we get \( g(J(X), J(Y)).\psi = \{ g(X, Y) + i[g(J(X), Y) + g(J(Y), X)] \} \quad \psi \), and so, \( g(J(X), J(Y)) = g(X, Y) \) and \( g(J(X), Y) = -g(J(Y), X) \). Therefore, \( M_1 \) is Kähler, and \( M_2 \) is spin Ricci-flat because it follows from \( f_A \mid M_2 = 0 \) that \( \kappa |_{M_2} = 0 \), hence \( w_2(M_2) = 0 \), and \( \| Ric \|^2 = \| f_A \|^2 = 0 \) on \( M_2 \). \( \square \)

In the context of the arguments above, if \( \text{Ricc} : TM \to TM \) is onto, then \( M \) is Kähler. Thus, taking the restriction \( L_1 = L_\|_{M_1} \), the canonical class of \((M_1, J)\) is \( \kappa_J = L_1 \). Assuming \( \pi_1(M) = 0 \), Morianu \([2]\) proved \( \kappa_J \) and \( -\kappa_J \) to
be the only spin$^c$ classes on $M$ carrying parallel spinors, which are known to be the only basic class in $M$ ([9]). Using de Rham’s decomposition theorem, Moroianu concluded that a simply connected manifold carries a parallel spinor if, and only if, it is isometric to the Riemannian product $M_1 \times M_2$ where $M_1$ is Kähler and $M_2$ is spin Ricci-flat. Of course, if we assume $M$ is irreducible as cartesian product, then either $M$ is Kähler or $M$ is spin. To the best of author’s knowledge there is no example of a Ricci-flat spin manifold with holonomy $SO_4$ and it is not known the classification of spin Ricci-flat 4-manifolds beyond the one quoted in [8].

5. Conclusion

In this section, Kato’s inequality ([15]) is used to compare the lower eigenvalue $\lambda_g^c(A)$ of operator $L_A$ in (13) with the lower eigenvalue $\lambda_g$ of operator $L = \Delta_g + \frac{k_g}{4}$ acting on functions $f : M \to \mathbb{R}$. By the Rayleigh’s formula, each lower eigenvalue is given by

\begin{equation}
\lambda_g = \inf_{f \in \Omega^0(M)} \frac{\int_M |\nabla f|^2 + \frac{k_g}{4} |f|^2} {\int_M |f|^2} dv_g
\end{equation}

\begin{equation}
\lambda_g^c(A) = \inf_{V \in \mathcal{S}_c} \frac{\int_M |\nabla A V|^2 + \frac{k_g}{4} |V|^2} {\int_M |V|^2} dv_g
\end{equation}

**Definition 5.0.6.** The Perelman-Yamabe smooth invariant of $M$ is

\begin{equation}
\bar{\lambda}(M) = \sup_g \lambda_g [vol(M,g)]^{1/2}
\end{equation}

Let $\mathcal{M}_M$ be the space of Riemannian metrics on $M$ and $[g] = \{\zeta.g \mid \zeta : M \to (0,\infty)\}$ the conformal class of $g$. The Yamabe constant of $[g]$ is defined by

\begin{equation}
Y_{[g]} = \inf_{\hat{g} \in [g]} \frac{\int_M k_{\hat{g}}} {vol(M,\hat{g})^{1/2}}.
\end{equation}

The condition $Y_{[g]} \leq 0$ implies the existence of unique metric realizing the Yamabe constant ([12]). The smooth Yamabe invariant is defined as $\mathcal{Y}(M) = \sup_{[g] \subset \mathcal{M}} Y_{[g]}$. Under the hypothesis $\mathcal{Y}(M) \leq 0$, Akutagawa-Ishida-LeBrun proved in [11] the identity $\mathcal{Y}(M) = \bar{\lambda}(M)$. By analogy, associated to the operator $L_A$ we consider

$$
\bar{\lambda}^c(A) = \sup_{g \in \mathcal{M}} \lambda_g^c(A), [vol(M,g)]^{1/2}.
$$

and $\bar{\lambda}^c(M) = \sup_{A \in \mathcal{J}_M} \bar{\lambda}^c(A)$. Assuming that $\mathcal{Y}(M) < 0$ and $e \in Spin^c(X)$ is a class carrying a parallel spinor $\psi \in \Omega^0(S^+_e)$, so $\bar{\lambda}^c(M) < 0$ and $\mathcal{J}_M$ is
unstable, concluding Theorem 1.0.3. If an irreducible solution \((A, \phi)\) exists, it follows from the \(SW\)-equations that

\[
\int_X \left[ |\nabla A|^2 + \frac{k_g}{4} |\phi|^2 \right] dv_g = -\frac{1}{4} \int_X |\phi|^4 dv_g.
\]

Consider \(k_g\) isn’t non-negative and let \((A, \phi)\) be an irreducible solution of the eqs. (11), so from eq. (20) we have

\[
\lambda_g^\epsilon(A). \int_M |\phi|^2 dv_g \leq -\frac{1}{4} \int_M |\phi|^4 dv_g = \lambda_g^\epsilon(A) < 0.
\]

Applying Cauchy-Schwartz inequality we get

\[
\int_M |\phi|^2 dv_g \leq \left[ \text{vol}(M, g) \right]^{1/2} \left[ \int_M |\phi|^4 dv_g \right]^{1/2},
\]

and so

\[
\tilde{\lambda}_g^\epsilon(M). \left[ \int_M |\phi|^4 dv_g \right]^{1/2} \leq \lambda_g^\epsilon(A). \int_M |\phi|^2 dv_g \leq -\frac{1}{4} \int_X |\phi|^4 dv_g.
\]

Hence, \(\tilde{\lambda}^\epsilon(M) \leq 0\), proving theorem 1.0.2 Whenever there exists a \(SW\)-monopole \((A, \phi)\), then

\[
\frac{1}{4} \int_M |\phi|^4 dv_g = \int_M |F_A^+|^2 dv_g = 4\pi^2 c_1^2(J)[M] + \int_M |F_A^-|^2 dv_g
\]

\[
-\frac{1}{4} \left[ \int_X |\phi|^4 dv_g \right]^{1/2} = -\frac{1}{2} \left[ \int_M |F_A^+|^2 dv_g \right]^{1/2} \leq -\pi \sqrt{\alpha^2[J^\epsilon]}[M].
\]

Let’s consider the case \(c_1^2(L_c)[M] > 0\), otherwise \(\phi = 0\). So, defining \(\tilde{\lambda}^\epsilon(M) = \sup_{A \in \mathcal{J}_M} \tilde{\lambda}^\epsilon(A)\), we get the upper bound in theorem 1.0.4

\[
\tilde{\lambda}^\epsilon(M) \leq -\pi \sqrt{c_1^2(L_c)[M]}.
\]

Therefore, if \(M\) admits a \(SW\)-monopole, then \(\lambda^\epsilon(M) < 0\).

It ought to be checked if \(\tilde{\lambda}^\epsilon(M)\) is a smooth invariant, for each \(\epsilon \in Spin^\epsilon(X)\).

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