A NEW NON-REDUCED MODULI COMPONENT OF
RANK 2 SEMISTABLE SHEAVES ON $\mathbb{P}^3$

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In the present paper we describe new component of the Gieseker-Maruyama moduli space $M(14)$ of coherent semistable rank-2 sheaves with Chern classes $c_1 = 0$, $c_2 = 14$, $c_3 = 0$ on $\mathbb{P}^3$ which is generically non-reduced. The construction of this component is based on the technique of elementary transformations of sheaves and the famous Mumford’s example of a non-reduced component of the Hilbert scheme of smooth space curves of degree 14 and genus 24.

1. Introduction

Let $\mathcal{M}(0, k, 2n)$ be the Gieseker-Maruyama moduli scheme of semistable rank-2 sheaves with Chern classes $c_1 = 0$, $c_2 = k$, $c_3 = 2n$ on the projective space $\mathbb{P}^3$. Denote $\mathcal{M}(k) = \mathcal{M}(0, k, 0)$. By the singular locus of a given $\mathcal{O}_{\mathbb{P}^3}$-sheaf $E$ we understand the set $\text{Sing}(E) = \{x \in \mathbb{P}^3 \mid E \text{ is not locally free at the point } x\}$. $\text{Sing}(E)$ is always a proper closed subset of $\mathbb{P}^3$ and, moreover, if $E$ is a semistable sheaf of nonzero rank, every irreducible component of $\text{Sing}(E)$ has dimension at most 1. For simplicity we will not make a distinction between a stable sheaf $E$ and corresponding isomorphism class $[E]$ as a point of moduli scheme. Also by a general point of an irreducible scheme we understand a closed point belonging to some Zariski open dense subset of this scheme.

Any semistable rank-2 sheaf $[E] \in \mathcal{M}(k)$ is torsion-free, so it fits into the exact triple

$$(1) \quad 0 \rightarrow E \rightarrow E^{\vee \vee} \rightarrow Q \rightarrow 0,$$

where $E^{\vee \vee}$ is a reflexive hull of $E$ and $\dim Q \leq 1$. Conversely, take a reflexive sheaf $F$, a subscheme $X \subset \mathbb{P}^3$, an $\mathcal{O}_X$-sheaf $Q$ and a surjective morphism $\phi : F \rightarrow Q$, then one can show that the kernel sheaf $E := \ker \phi$ is semistable when $F$ and $Q$ satisfy some mild conditions. We call the triple $(F, Q, \phi)$ an elementary transformation data and a sheaf $E$ an elementary transform of $F$ along $X$.

It is interesting that all known irreducible components of the moduli schemes $\mathcal{M}(k)$ general points of which correspond to stable sheaves with singularities were described by using elementary transformations.

More precisely, in [7] there were found two infinite series of irreducible components of the collection $\{\mathcal{M}(k)\}_{k=1}^\infty$ which (generically) parameterize stable sheaves with singularities of dimension 0 and pure dimension 1,
respectively. General points of components of the first series are elementary transforms of stable reflexive sheaves along unions of distinct points in $\mathbb{P}^3$, while those of the second series are elementary transforms of instanton bundles along smooth complete intersection curves.

Next, in [8] there were constructed three components of $\mathcal{M}(3)$ parameterizing sheaves with singularities of mixed dimension. General sheaves of these components are elementary transforms of stable reflexive sheaves with Chern classes $(c_2, c_3) = (2, 2), (2, 4)$ along a disjoint union of a projective line and a collection of points in $\mathbb{P}^3$. This approach was generalized in [9] by doing elementary transformations of stable reflexive sheaves with other Chern classes along a disjoint union of a projective line and a collection of points in order to construct infinite series of components of $\mathcal{M}(-1, c_2, c_3)$.

In [16] the author constructed an infinite series of irreducible moduli components which includes the components parameterizing non-locally free sheaves constructed in [7] and [8] as special cases. General sheaves of these components are obtained by elementary transformations of stable and properly $\mu$-semistable reflexive sheaves along a sheaf $Q = L \oplus O_W$ where $W$ is a collection of points in $\mathbb{P}^3$ and $L$ is a line bundle over a smooth connected curve $C$ which is either rational or a complete intersection curve.

The present paper is devoted to the description of a new component of the moduli scheme $\mathcal{M}(14)$. We prove that this component is generically non-reduced. This yields the first example of a generically non-reduced component of the Gieseker-Maruyama moduli scheme of rank 2 semistable sheaves on $\mathbb{P}^3$. The construction of this component essentially follows the method described in the paper [16], so ideologically the new component can be considered as a particular member of the series of components constructed in [16]. More precisely, a general point of the new component is an elementary transform of the trivial sheaf along a line bundle of degree 51 over a smooth space curve of degree 14 and genus 24. These curves run through a non-reduced component of the corresponding Hilbert scheme which was for the first time described by Mumford in his classical paper [17]. Roughly speaking, new component of $\mathcal{M}(14)$ is fibered over the Mumford component which leads to the non-reducedness of the new component.

The paper is organized as follows. Section 2 is devoted to the construction of a new component of $\mathcal{M}(14)$. This section mainly repeats the basic computations presented in [16]. In Section 3 we remind necessary facts from the deformation theory. In Section 4 we prove that this component is generically non-reduced. More precisely, we consider the first obstruction map for the scheme $\mathcal{M}(14)$ and show that it is non-trivial (see Theorem 2
and its proof). At the end of the paper we conjecture how this result can be generalized.

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2. **Construction of a new component of $\mathcal{M}(14)$**

Let $\text{Hilb}_{14,24}$ denote the open subset of the Hilbert scheme of $\mathbb{P}^3$ parameterising smooth connected curves of degree 14 and genus 24 in $\mathbb{P}^3$. Consider a 56-dimensional irreducible open subscheme $\mathcal{H} \subset (\text{Hilb}_{14,24})_{\text{red}}$ whose general member $C$ is contained in a smooth cubic surface. Mumford [17] showed that the dimension of the tangent space of $\text{Hilb}_{14,24}$ at $[C]$ is equal to 57. Moreover, he proved that $\mathcal{H}$ is maximal as a subvariety of $(\text{Hilb}_{14,24})_{\text{red}}$, and hence $\text{Hilb}_{14,24}$ is non-reduced at all points $\mathcal{H}$. Consider the universal curve $\mathcal{Z} \subset \mathbb{P}^3 \times \mathcal{H}$ of the Hilbert scheme $\text{Hilb}_{14,24}$ restricted onto the subset $\mathcal{H}$.

Note that the morphism $\mathcal{Z} \to \mathcal{H}$ is projective and flat, and its fibers are smooth and connected. So there exists a relative Picard scheme $\text{Pic}_{\mathcal{Z}/\mathcal{H}}$ with a Poincaré sheaf $L$ (see [24, Thm. 4.18.1 and Ex. 4.3]). Consider a component $\widehat{\mathcal{P}} \subset \text{Pic}_{\mathcal{Z}/\mathcal{H}}$ parameterizing invertible sheaves with the Hilbert polynomial $P(k) = 14(k + 2)$. Finally, denote by $\mathcal{P} \subset \widehat{\mathcal{P}}$ an open subset of $\widehat{\mathcal{P}}$ consisting of line bundles $L$ satisfying the following equalities

$$(2) \quad h^1(L) = 0, \quad h^0(\omega_C(4) \otimes L^{-2}) = 0.$$ 

Taking if necessary an open subset of $\mathcal{H}$ we can assume that $\mathcal{P}$ is smooth (see [24, Prop. 5.19]). We will denote also by $L$ the restriction of the Poincaré sheaf onto $\mathcal{P}$. Note that for any $(C, L) \in \mathcal{P}$ we have the equalities

$$(3) \quad h^0(L) = \chi(L) = P(0) = 28, \quad \text{deg}(L) = \chi(L) + g - 1 = 51.$$ 

Also the dimension of the scheme $\mathcal{P}$ is equal to

$$(4) \quad \dim \mathcal{P} = \dim \mathcal{H} + \dim \text{Jac}(C) = 56 + 24 = 80.$$ 

Suppose that $L$ is a line bundle over a smooth curve $C$ representing some point of $\mathcal{P}$. Let $\text{Hom}_e(2\mathcal{O}_{\mathbb{P}^3}, L) \subset \text{Hom}(2\mathcal{O}_{\mathbb{P}^3}, L)$ be the open subset of surjective morphisms $2\mathcal{O}_{\mathbb{P}^3} \twoheadrightarrow L$. Note that the group $\text{Aut}(2\mathcal{O}_{\mathbb{P}^3}) \times \text{Aut}(L) \simeq GL(2) \times k^*$ acts on $\text{Hom}_e(2\mathcal{O}_{\mathbb{P}^3}, L)$ by $(\psi, \zeta) \phi = \zeta \circ \phi \circ \psi^{-1}$ for $(\psi, \zeta) \in \text{Aut}(2\mathcal{O}_{\mathbb{P}^3}) \times \text{Aut}(L)$, $\phi \in \text{Hom}_e(2\mathcal{O}_{\mathbb{P}^3}, L)$.

An epimorphism $\phi \in \text{Hom}_e(2\mathcal{O}_{\mathbb{P}^3}, L)$ defines the kernel sheaf $E := \ker \phi$. Moreover, two epimorphisms $\phi_1, \phi_2 \in \text{Hom}_e(2\mathcal{O}_{\mathbb{P}^3}, L)$ define isomorphic kernel sheaves $\ker \phi_1 \simeq \ker \phi_2$ if and only if there exists some
$$(\psi, \zeta) \in \text{Aut}(2\mathcal{O}_P) \times \text{Aut}(L)$$ such that $(\psi, \zeta)\phi_1 = \phi_2$ (see [4, Corr. 1.5]). Therefore, an element $[\phi]$ of the orbit space

$$\text{Hom}_e(2\mathcal{O}_P, L)/\left( \text{Aut}(2\mathcal{O}_P) \times \text{Aut}(L) \right) \simeq \mathbb{P}\text{Hom}_e(2\mathcal{O}_P, L)/\text{PGL}(2),$$

which we will consider as a set, uniquely defines an isomorphism class $[E := \ker \phi]$. Next, since $c_1(L) = 0$, $c_2(L) = -14$ and $c_3(L) = 0$ we have $c_1(E) = 0$, $c_2(E) = 14$ and $c_3(E) = 0$. Moreover, from Lemma 4.3 in [6] follows that the sheaf $E$ is stable. Overall, these facts imply that the elements of the following set of data of elementary transformations

$$(5) \quad \mathcal{Q} := \left\{ (C, L, [\phi]) \mid (C, L) \in \mathcal{P}, \ [\phi] \in \mathbb{P}\text{Hom}_e(2\mathcal{O}_P, L)/\text{PGL}(2) \right\}$$

are in one-to-one correspondence with some subset of closed points of the moduli scheme $\mathcal{M}(14)$.

**Lemma 1.** There exists an irreducible closed subset $\overline{\mathcal{C}}$ of $\mathcal{M}(14)$ and a dense subset $\mathcal{C} \subset \overline{\mathcal{C}}$ whose closed points are in one-to-one correspondence with the elements of the set $\mathcal{Q}$. The dimension of $\overline{\mathcal{C}}$ is equal to 132.

**Proof:** Let $i : \mathcal{P} \times \mathcal{H} \mathcal{Z} \to \mathcal{P} \times \mathbb{P}^3$ be the natural inclusion and $p : \mathcal{P} \times \mathbb{P}^3 \to \mathcal{P}$ be the projection onto the first term. Consider the $\mathcal{O}_P$-sheaf $\tau := p_*\text{Hom}(2\mathcal{O}_P \times \mathbb{P}^3, i_*L)$. Since the sheaf $L$ is flat over $\mathcal{P}$ the Euler characteristics $\chi(L|_{((C, L) \times \mathcal{C})})$ is constant for all $(C, L) \in \mathcal{P}$. Moreover, by construction of $\mathcal{P}$, we have $h^1(L) = 0$ for any pair $(C, L) \in \mathcal{P}$, so $h^0(L|_{((C, L) \times \mathcal{C})}) = \chi(L|_{((C, L) \times \mathcal{C})})$ is also constant. This means that the fibers $\tau_y \otimes k(y)$, $y \in \mathcal{P}$ are of the same dimension. Now taking into account that the scheme $\mathcal{P}$ is smooth we obtain that the sheaf $\tau$ is actually locally-free, so it can be considered as a vector bundle.

Consider the projective bundle $\mathbf{P}(\tau^\vee) := \text{Proj}(\text{Sym}_{\mathcal{O}_P}(\tau^\vee))$ associated to the vector bundle $\tau$. If the point $u \in \mathcal{P}$ corresponds to the pair $(C, L)$, then the fiber of the projection $\mathbf{P}(\tau^\vee) \longrightarrow \mathcal{P}$ over the point $u$ is the projective space $\mathbb{P}\text{Hom}(2\mathcal{O}_P, L)$. Using this observation and the formulas (4), (3) we can compute dimension of $\mathbf{P}(\tau^\vee)$ as follows

$$(6) \quad \dim \mathbf{P}(\tau^\vee) = \dim \mathcal{P} + \dim \mathbb{P}\text{Hom}(2\mathcal{O}_P, L) = 80 + 2 \cdot 28 - 1 = 135.$$

Let $\mathcal{E} \subset \mathbf{P}(\tau^\vee)$ be the open dense subset of $\mathbf{P}(\tau^\vee)$ consisting of classes of surjective morphisms $[2\mathcal{O}_P \to L]$. Any point $q \in \mathcal{E}$ determines the isomorphism class of the sheaf $[E_q := \ker \psi_q]$, where $[\psi_q] \in \mathbb{P}\text{Hom}_e(2\mathcal{O}_P, L)$. The family $\{E_q, q \in \mathcal{E}\}$ globalizes to the universal sheaf $\mathcal{E}$ over $\mathcal{E} \times \mathbb{P}^3$. In order to show this note that we have the anti-tautological line bundle $\mathcal{O}_{\mathbf{P}(\tau^\vee)}(1)$ over $\mathbf{P}(\tau^\vee)$ such that $\text{pr}_*(\mathcal{O}_{\mathbf{P}(\tau^\vee)}(1)) = \tau^\vee$ where $\text{pr} : \mathbf{P}(\tau^\vee) \longrightarrow \mathcal{P}$.
\( \mathcal{P} \) is the natural projection. The canonical section \( \sigma \in \Gamma(\mathcal{P}, \tau \otimes \tau^\vee) \) comes from a unique section \( \sigma \in \Gamma(\mathcal{P}(\tau^\vee), \text{pr}^*\tau \otimes \mathcal{O}_{\mathcal{P}(\tau^\vee)}(1)) \). The latter defines a morphism of sheaves \( \phi : \text{pr}^*(2\mathcal{O}_{\mathcal{P} \times \mathbb{P}^3}) \rightarrow \text{pr}^*i_*L \otimes \mathcal{O}_{\mathcal{P}(\tau^\vee)}(1) \) and we set \( E := \ker \phi \). So we have the following exact triple:

\[
0 \rightarrow E \rightarrow \text{pr}^*(2\mathcal{O}_{\mathcal{P} \times \mathbb{P}^3}) \rightarrow \text{pr}^*i_*L \otimes \mathcal{O}_{\mathcal{P}(\tau^\vee)}(1) \rightarrow 0.
\]

One can see that the sheaf \( E \) being restricted onto \( E \) is the desired globalization of the family of sheaves \( \{ E_q, q \in \mathcal{E} \} \).

Next, by construction and by definition of moduli scheme, the sheaf \( E \) defines the modular morphism \( \Phi : E \rightarrow \mathcal{M}(14) \), \( q \mapsto [E_q = \ker \psi_q] \). Now consider the image \( \mathcal{C} := \text{im}(\Phi) \) of the morphism \( \Phi \) as a subset of \( \mathcal{M}(14) \) and take its closure \( \overline{\mathcal{C}} \subset \mathcal{M}(m+d) \). Note that the scheme \( \mathcal{E} \) is irreducible, so the closed subset \( \overline{\mathcal{C}} \) is also irreducible. Moreover, taking if necessary an open dense subset of \( \mathcal{C} \) we can assume that the morphism \( \Phi \) is flat over \( \mathcal{C} \). In particular, this means that for a general point \( [E] = \Phi(x), x \in \mathcal{E} \), we have the following formula for dimension

\[
\dim [E] \overline{\mathcal{C}} = \dim_x \mathcal{E} - \dim_x \Phi^{-1}([E]).
\]

Moreover, we have \( \Phi(x) = \Phi(y) \) if and only if the corresponding equivalence classes of morphisms \( [\phi_x], [\phi_y] \in \text{PHom}_\text{c}(2\mathcal{O}_{\mathbb{P}^3}, L) \) differ by the action of the group \( \text{PGL}(2) \). Since this action is free, the fiber \( \Phi^{-1}([E]) \) is isomorphic to the group \( \text{PGL}(2) \). This implies that the set of closed points of \( \mathcal{C} \) is isomorphic to \( \mathbb{Q} \). Moreover, from formulas (6) and (7) it follows that the dimension of the scheme \( \overline{\mathcal{C}} \subset \mathcal{M}(14) \) is equal to 132.

**Theorem 1.** For any sheaf \( [E] \) from the subset \( \mathcal{C} \subset \mathcal{M}(14) \) we have the equality

\[
\dim T_{[E]}\mathcal{M}(14) = \dim \overline{\mathcal{C}} + 1 = 133.
\]

**Proof:** For the computation of the dimension of the tangent space of the moduli scheme \( \mathcal{M}(14) \) at the point \( [E] \) defined above, we use the standard fact of deformation theory, \( T_{[E]}\mathcal{M}(m+d) \simeq \text{Ext}^1(E, E) \) for a stable sheaf \( E \), and the local-to-global spectral sequence \( H^p(\mathcal{E}xt^q(E, E)) \Rightarrow \text{Ext}^{p+q}(E, E) \), which yields the following exact sequence

\[
0 \rightarrow H^1(\text{Hom}(E, E)) \rightarrow \text{Ext}^1(E, E) \rightarrow H^0(\mathcal{E}xt^1(E, E)) \rightarrow \rightarrow \frac{\phi}{H^2(\text{Hom}(E, E))} \rightarrow \text{Ext}^2(E, E).
\]

According to our construction, any sheaf \( [E] \in \mathcal{C} \) fits into the exact triple of the following form

\[
0 \rightarrow E \rightarrow 2\mathcal{O}_{\mathbb{P}^3} \rightarrow L \rightarrow 0.
\]
Moreover, the first equality of (2) and the triple
\[ H^1(L) \rightarrow H^2(E) \rightarrow H^2(\mathcal{O}_{\mathbb{P}^3}) \]
yields \( H^2(E) = 0 \). Applying the functor \( \mathcal{H}om(-, E) \) to the triple (9) we obtain the following exact sequence
\[ 0 \rightarrow \mathcal{H}om(L, E) \rightarrow \mathcal{H}om(2\mathcal{O}_{\mathbb{P}^3}, E) \rightarrow \mathcal{H}om(E, E) \rightarrow \]
\[ \rightarrow \mathcal{E}xt^1(L, E) \rightarrow \mathcal{E}xt^1(2\mathcal{O}_{\mathbb{P}^3}, E) \rightarrow \mathcal{E}xt^1(E, E) \rightarrow \]
\[ \rightarrow \mathcal{E}xt^2(L, E) \rightarrow \mathcal{E}xt^1(2\mathcal{O}_{\mathbb{P}^3}, E). \]

Since \( E \) is torsion-free sheaf we have \( \mathcal{H}om(L, E) = 0 \). Also we have that the sheaf \( \mathcal{E}xt^{i \geq 1}(2\mathcal{O}_{\mathbb{P}^3}, E) \) is zero, so we obtain the following triple and isomorphism
\[ 0 \rightarrow \mathcal{H}om(2\mathcal{O}_{\mathbb{P}^3}, E) \rightarrow \mathcal{H}om(E, E) \rightarrow \mathcal{E}xt^1(L, E) \rightarrow 0, \]
\[ \mathcal{E}xt^1(E, E) \cong \mathcal{E}xt^2(L, E). \]

Taking into account that the sheaf \( \mathcal{E}xt^i(L, E) \) has dimension at most 1 and \( h^2(E) = 0 \), we have that \( h^2(\mathcal{H}om(E, E)) = 0 \). Therefore, we obtain the following exact triple
\[ 0 \rightarrow H^1(\mathcal{H}om(E, E)) \rightarrow \text{Ext}^1(E, E) \rightarrow H^0(\mathcal{E}xt^1(E, E)) \rightarrow 0. \]

Next, apply the functor \( \mathcal{H}om(L, -) \) to the triple (9)
\[ 0 \rightarrow \mathcal{H}om(L, E) \rightarrow \mathcal{H}om(L, 2\mathcal{O}_{\mathbb{P}^3}) \rightarrow \mathcal{H}om(L, L) \rightarrow \]
\[ \rightarrow \mathcal{E}xt^1(L, E) \rightarrow \mathcal{E}xt^1(L, 2\mathcal{O}_{\mathbb{P}^3}) \rightarrow \mathcal{E}xt^1(L, L) \rightarrow \]
\[ \rightarrow \mathcal{E}xt^2(L, E) \rightarrow \mathcal{E}xt^2(L, 2\mathcal{O}_{\mathbb{P}^3}) \rightarrow \mathcal{E}xt^2(L, L) \rightarrow 0. \]

Since the sheaves \( E \) and \( 2\mathcal{O}_{\mathbb{P}^3} \) are torsion-free we have that \( \mathcal{H}om(L, E) = \mathcal{H}om(L, 2\mathcal{O}_{\mathbb{P}^3}) = 0 \). Note that the smooth curve \( C \) is locally complete intersection, so for any point \( x \in \mathbb{P}^3 \) we have the following
\[ \text{Ext}^1_{\mathbb{P}^3, x}(\mathcal{O}_{C,x}, \mathcal{O}_{\mathbb{P}^3, x}) = 0. \]

This equality implies that the sheaf \( \mathcal{E}xt^1(L, 2\mathcal{O}_{\mathbb{P}^3}) \) is equal to zero, so we have the isomorphism
\[ \mathcal{E}xt^1(L, E) \cong \mathcal{H}om(L, L). \]
and the exact sequence

\[(18) \quad 0 \to \mathcal{E}xt^1(L, L) \to \mathcal{E}xt^2(L, E) \to \mathcal{E}xt^2(L, 2\mathcal{O}_{\mathbb{P}^3}) \to \mathcal{E}xt^2(L, L) \to 0.\]

The following exact triple holds

\[(19) \quad 0 \to L^{-1} \to 2\mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_C \to L \to 0.\]

Note that $\mathcal{E}xt^2(L, 2\mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^2(L, 2\mathcal{O}_C)$. In particular, it means that $\mathcal{E}xt^2(L, 2\mathcal{O}_{\mathbb{P}^3})$ is locally-free $\mathcal{O}_C$-sheaf. Since $\mathcal{E}xt^2(L, L^{-1})$ is also locally-free $\mathcal{O}_C$-sheaf of rank 1, then applying the functor $\mathcal{H}om(L, -)$ to the triple $(19)$ we obtain the following exact triple

\[(20) \quad 0 \to \mathcal{E}xt^2(L, L^{-1}) \to \mathcal{E}xt^2(L, 2\mathcal{O}_{\mathbb{P}^3}) \to \mathcal{E}xt^2(L, L) \to 0.\]

Moreover, we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}xt^2(L, 2\mathcal{O}_{\mathbb{P}^3}) & \longrightarrow & \mathcal{E}xt^2(L, L) \\
\downarrow_{\simeq} & & \downarrow_{=} \\
\mathcal{E}xt^2(L, 2\mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_C) & \longrightarrow & \mathcal{E}xt^2(L, L)
\end{array}
\]

So the morphism $\mathcal{E}xt^2(L, 2\mathcal{O}_{\mathbb{P}^3}) \longrightarrow \mathcal{E}xt^2(L, L)$ in the triple $(20)$ coincides with the last morphism in the exact sequence $(18)$. Therefore, we can simplify $(18)$ as

\[(21) \quad 0 \to \mathcal{E}xt^1(L, L) \to \mathcal{E}xt^2(L, E) \to \mathcal{E}xt^2(L, L^{-1}) \to 0.\]

Note that for any subscheme $C \subset \mathbb{P}^3$ we have that

$$\mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_C) \simeq \mathcal{H}om(I_C, \mathcal{O}_C) \simeq \mathcal{H}om(I_C/I_C^2, \mathcal{O}_C) = N_{C/\mathbb{P}^3}.$$ (the last equality is the definition of the normal sheaf). Besides, if $C$ is a locally complete intersection of the pure dimension 1 then (see [12, Prop. 7.5])

$$\mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_C) \simeq \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^2(\mathcal{O}_C, \omega_{\mathbb{P}^3})(4) \simeq \omega_C(4).$$

Since $L$ is an invertible $\mathcal{O}_C$-sheaf it follows that

$$\mathcal{E}xt^1(L, L) \simeq \mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_C), \quad \mathcal{E}xt^2(L, L^{-1}) \simeq \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_C) \otimes L^{-2}.$$ From these formulas one can deduce the isomorphisms

$$\mathcal{E}xt^1(L, L) \simeq N_{C/\mathbb{P}^3}, \quad \mathcal{E}xt^2(L, L^{-1}) \simeq \omega_C(4) \otimes L^{-2}.$$
Substituting them to (21) we obtain the following exact triple

\[(22) \quad 0 \rightarrow N_{C/P_3} \rightarrow \mathcal{E}xt^2(L, E) \rightarrow \omega_C(4) \otimes L^{-2} \rightarrow 0.\]

Therefore, using the isomorphism (13), the triple (22) and the condition $h^0(\omega_C(4) \otimes L^{-2}) = 0$ from (2) we obtain the isomorphism

\[(23) \quad H^0(\mathcal{E}xt^1(E, E)) \simeq H^0(N_{C/P_3}) \simeq TC\text{Hilb}_{14,24}.\]

After applying the functor $\mathcal{H}om(-,2\mathcal{O}_{P_3})$ to the exact triple (9) we obtain the long exact sequence of sheaves

\[(24) \quad 0 \rightarrow \mathcal{H}om(L,2\mathcal{O}_{P_3}) \rightarrow \mathcal{H}om(E,2\mathcal{O}_{P_3}) \rightarrow \mathcal{H}om(2\mathcal{O}_{P_3},2\mathcal{O}_{P_3}) \rightarrow \]

\[\mathcal{E}xt^1(L,2\mathcal{O}_{P_3}) \rightarrow \mathcal{E}xt^1(E,2\mathcal{O}_{P_3}) \rightarrow 0 \rightarrow \]

\[\mathcal{E}xt^2(L,2\mathcal{O}_{P_3}) \rightarrow \mathcal{E}xt^2(E,2\mathcal{O}_{P_3}) \rightarrow 0.\]

As it was already explained $\mathcal{H}om(L,2\mathcal{O}_{P_3}) = \mathcal{E}xt^1(L,2\mathcal{O}_{P_3}) = 0$ and the sheaf $\mathcal{E}xt^1(2\mathcal{O}_{P_3},2\mathcal{O}_{P_3})$ is zero, so we have the following isomorphisms

\[(25) \quad \mathcal{H}om(E,2\mathcal{O}_{P_3}) \simeq \mathcal{H}om(2\mathcal{O}_{P_3},2\mathcal{O}_{P_3}), \quad \mathcal{E}xt^1(E,2\mathcal{O}_{P_3}) \simeq 0.\]

Consider the part of the commutative diagram with exact rows and columns obtained by applying the bifunctor $\mathcal{H}om(-,-)$ and its derivative $\mathcal{E}xt(-,-)$ to the exact triple (9) which looks as follows

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{H}om(2\mathcal{O}_{P_3},E) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{H}om(2\mathcal{O}_{P_3},2\mathcal{O}_{P_3}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{H}om(E,E) \\
\downarrow & & \downarrow \\
\mathcal{E}xt^1(L,E) & \rightarrow & \mathcal{H}om(E,2\mathcal{O}_{P_3}) \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}
\]

Due to the isomorphism (25) the sheaf coker $\tau$ fits into the exact triple

\[(26) \quad 0 \rightarrow \mathcal{H}om(E,E) \rightarrow \mathcal{H}om(2\mathcal{O}_{P_3},2\mathcal{O}_{P_3}) \rightarrow \text{coker } \tau \rightarrow 0.\]
So we have the following exact sequence
\[(27)\]
\[0 \rightarrow \text{End}(E) \rightarrow \text{End}(2\mathcal{O}_{\mathbb{P}^3}) \rightarrow H^0(\text{coker } \tau) \rightarrow H^1(\mathcal{H}om(E, E)) \rightarrow 0.\]

On the other hand, applying the Snake Lemma to the commutative diagram above and using the isomorphism (17) we have the exact triple
\[(28)\]
\[0 \rightarrow \mathcal{H}om(L, L) \rightarrow \mathcal{H}om(2\mathcal{O}_{\mathbb{P}^3}, L) \rightarrow \text{coker } \tau \rightarrow 0.\]

By the definition of the subscheme \(\mathcal{P} \subset \text{Pic}_{\mathbb{P}^3/H}^5\) we have \(h^1(L) = 0\), so \(h^1(\mathcal{H}om(2\mathcal{O}_{\mathbb{P}^3}, L)) = 0\). Therefore, from the triple (29) we obtain the following exact sequence
\[(29)\]
\[0 \rightarrow \text{End}(L) \rightarrow \text{Hom}(2\mathcal{O}_{\mathbb{P}^3}, L) \rightarrow H^0(\text{coker } \tau) \rightarrow H^1(\mathcal{O}_C) \rightarrow 0.\]

Using these equalities and the fact that the sheaf \(E\) is simple due to its stability, the triple (26) implies the following isomorphism
\[(30)\]
\[H^1(\mathcal{H}om(E, E)) \simeq H^1(\mathcal{O}_C) \oplus \left( \left( \mathcal{H}om(2\mathcal{O}_{\mathbb{P}^3}, L)/\text{End}(L) \right) / \left( \text{End}(2\mathcal{O}_{\mathbb{P}^3})/\text{End}(E) \right) \right).\]

Substituting the isomorphisms (23) and (30) to the exact triple (14) we obtain a non canonical isomorphism
\[(31)\]
\[\text{Ext}^1(E, E) \simeq T_{\mathcal{C}\text{Hilb}_{14,24}} \oplus H^1(\mathcal{O}_C) \oplus \left( \left( \mathcal{H}om(2\mathcal{O}_{\mathbb{P}^3}, L)/\text{End}(L) \right) / \left( \text{End}(2\mathcal{O}_{\mathbb{P}^3})/\text{End}(E) \right) \right).\]

Since the sheaf \(E\) is stable it is also simple, so we have \(\dim \text{End}(E) = 1\). Therefore, the dimension of \(\dim \text{Ext}^1(E, E)\) is equal to
\[(32)\]
\[\dim \text{Ext}^1(E, E) = \dim T_{\mathcal{C}\text{Hilb}_{14,24}} + \dim H^1(\mathcal{O}_C) + \left( 2h^0(L) - 1 \right) - \left( 4 - 1 \right) = 57 + 24 + 55 - 3 = 133.\]

So we have the statement of the Theorem
\[(33)\]
\[\dim T_{[E]\mathcal{M}(14)} = \dim \mathcal{T} + 1.\]
3. Necessary facts from deformation theory

Let us recall some technical details about Ext-groups. Suppose we are given by two objects $A$, $B$ from an abelian category $\mathbf{C}$. Recall that elements of the group $\text{Ext}_C^n(A, B)$ can be represented by equivalence classes of $n$-extensions. More precisely, $\text{Ext}^0(A, B) = \text{Hom}(A, B)$. Next, $\text{Ext}^1(A, B)$ is the set of equivalence classes of extensions of $A$ by $B$, forming an abelian group under the Baer sum. Finally, the elements of higher Ext-groups $\text{Ext}^n(A, B)$ can be defined as equivalence classes of $n$-extensions, which are exact sequences

$$0 \rightarrow B \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow A \rightarrow 0,$$

under the equivalence relation generated by the relation that identifies two extensions

$$\xi : 0 \rightarrow B \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow A \rightarrow 0,$$
$$\xi' : 0 \rightarrow B \rightarrow X'_n \rightarrow \cdots \rightarrow X'_1 \rightarrow A \rightarrow 0,$$

if there are maps $X_m \rightarrow X'_m$ for all $m$ in $\{1, 2, ..., n\}$ so that every resulting square commutes, that is, if there is a chain map $\xi \rightarrow \xi'$ which is the identity on $A$ and $B$.

Next, we have the following pairing between Ext-groups which is called Yoneda product and denoted by $- \cup -$

$$\text{Ext}^n(L, M) \otimes \text{Ext}^m(M, N) \rightarrow \text{Ext}^{n+m}(L, N).$$

This pairing is induced by the map

$$\text{Hom}(L, M) \otimes \text{Hom}(M, N) \rightarrow \text{Hom}(L, N), \ f \otimes g \mapsto g \circ f.$$

In terms of extensions, it can be described as follows. Suppose that an element $\alpha \in \text{Ext}^n(L, M)$ is represented by the extension

$$\alpha : 0 \rightarrow M \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow L \rightarrow 0,$$

and an element $\beta \in \text{Ext}^m(L, M)$ by the extension

$$\beta : 0 \rightarrow N \rightarrow F_0 \rightarrow \cdots \rightarrow F_{m-1} \rightarrow M \rightarrow 0.$$

Then the Yoneda product $\alpha \cup \beta \in \text{Ext}^{m+n}(L, N)$ is represented by the concatenated extension

$$\alpha \cup \beta : 0 \rightarrow N \rightarrow F_0 \rightarrow \cdots \rightarrow F_{m-1} \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow L \rightarrow 0.$$
Now move on to some basic facts on the infinitesimal study of the Hilbert scheme of space curves. Let $C$ be a smooth connected curve in $\mathbb{P}^3$ whose structure sheaf $\mathcal{O}_C$ is defined by the exact triple

\begin{equation}
0 \longrightarrow I_C \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{34}
\end{equation}

Then an (embedded) $n$-th order (infinitesimal) deformation of $C \subset \mathbb{P}^3$ is a closed subscheme $C_n$ of $\mathbb{P}^3 \times \text{Spec}(k[t]/(t^{n+1}))$ which is flat over $k[t]/(t^{n+1})$ and $C_n \otimes_{k[t]/(t^{n+1})} k = C$. The set of all first order deformations of $C \subset \mathbb{P}^3$ is the Zariski tangent space of the Hilbert scheme at the point $[C]$. Let $C_1$ be a first order deformation of $C \subset \mathbb{P}^3$. If there exists no second order deformation $C_2$ of $C \subset \mathbb{P}^3$ such that $C_2 \otimes_{k[t]/(t^3)} k[t]/(t^2) = C_1$, we say $C_1$ is obstructed at the second order. As we know the set of all first order deformations of $C \subset \mathbb{P}^3$ is parametrized by $\text{Hom}(I_C, \mathcal{O}_C)$. Applying the functor $\text{Hom}(I_C, -)$ to the triple (34) we obtain the isomorphism

$$\delta : \text{Hom}(I_C, \mathcal{O}_C) \longrightarrow \text{Ext}^1(I_C, I_C).$$

Let $\phi \in \text{Hom}(I_C, \mathcal{O}_C)$ be a first order deformation of $C \subset \mathbb{P}^3$. Then $\phi$ is obstructed at the second order if and only if the Yoneda product

$$o(\phi) := \delta(\phi) \cup_1 \phi$$

is non-zero. An element $o(\phi) \in \text{Ext}^1(I_C, \mathcal{O}_C)$ is called an obstruction to extend $\phi$ to second order deformations.

Finally, consider the deformation theory of coherent sheaves. Let $X$ be a scheme over $k$, $F$ be a coherent sheaf on $X$ and $S$ be a scheme over $k$ with a marked point 0. Introduce the following definition

**Definition 1.** Deformation of the sheaf $F$ over $S$ is a sheaf $\mathcal{F}$ over $X \times D$ such that $\mathcal{F}$ is flat over $D$, equipped with a homomorphism $p : \mathcal{F} \rightarrow i_* F$ which induces an isomorphism $\mathcal{F} \otimes i_* \mathcal{O}_X \simeq i_* F$ where $i_S : X \simeq X \times \{0\} \hookrightarrow X \times S$.

Denote by $D := \text{Spec}(k[t]/t^2)$ the spectrum of the ring of dual numbers. Note that there is the one-to-one correspondence between the elements of the group $\text{Ext}^1(F, F)$ and the pairs $(\mathcal{F}, p)$ where $\mathcal{F}$ is a deformation of the sheaf $F$ over $D$ and $p : \mathcal{F} \rightarrow i_* F$ is a morphism which induces an isomorphism $\mathcal{F} \otimes i_* \mathcal{O}_X \simeq i_* F$. In particular, if we denote by $\text{pr} : X \times D \rightarrow X$ the projection onto the first term then we obtain the exact triple

\begin{equation}
0 \longrightarrow F \longrightarrow \text{pr}_* F \longrightarrow F \longrightarrow 0, \tag{36}
\end{equation}

which can be considered as an element of the group $\text{Ext}^1(F, F)$. 
Next, let \((S, 0)\) be a germ of scheme \(S\) at a point \(0 \in S\). Deformation of a sheaf over a germ of scheme can be defined in the similar way as in Definition 1. Recall that a morphism between deformations \(F_1\) and \(F_2\) of a sheaf \(F\) over \((S_1, 0)\) and \((S_2, 0)\), respectively, can be defined as a morphism of germs \(\phi : (S_1, 0) \to (S_2, 0)\) such that there exists an isomorphism \(\phi^* F_2 \cong F_1\) satisfying the following commutative diagram

\[ \begin{array}{ccc}
\phi^* F_2 & \cong & F_1 \\
\downarrow & & \downarrow \\
\phi^* i_{S_2*} F & = & i_{S_1*} F
\end{array} \]

**Definition 2.** Deformation \(F\) of the sheaf \(F\) over the germ of scheme \((S, 0)\) is called versal if it satisfies the following property. Let \(F'\) and \(F''\) be some deformations over \((S', 0)\) and \((S'', 0)\), respectively. Suppose that we have a morphism \(\phi : (S', 0) \to (S'', 0)\) from \(F'\) to \(F''\) and a \(\psi : (S', 0) \to (S, 0)\) from \(F'\) to \(F\). Then there exists a morphism \(\theta : (S'', 0) \to (S, 0)\) from \(F''\) to \(F\) such that \(\psi = \theta \circ \phi\).

To any deformation \(F\) of a sheaf \(F\) over \((S, 0)\) we can associate the Kodaira-Spencer map \(\tau_F : T_0 S \to \text{Ext}^1(F, F)\) from the tangent space \(T_0 S\) to the set of deformations of \(F\) over \(D\) which can be identified with the group \(\text{Ext}^1(F, F)\) as it was mentioned above.

**Definition 3.** Deformation \(F\) of the sheaf \(F\) over the germ of scheme \((S, 0)\) is called semi-universal if it is versal and the corresponding Kodaira-Spencer map \(\tau_F\) is an isomorphism.

**Lemma 2.** Let \(X\) be a smooth projective variety, \(F\) a coherent sheaf on \(X\). Then there exists a germ of a nonsingular algebraic variety \((M, 0)\) together with a morphism \(\mathcal{Y}_F : (M, 0) \to (\text{Ext}^2(F, F), 0)\), called the obstruction map, such that the following properties are verified:

(i) \((\mathcal{Y}_F^{-1}(0), 0)\) is the base of a semi-universal deformation \(F\) of the sheaf \(F\). The Kodaira-Spencer map of this deformation provides a natural isomorphism \(\text{KS} : T_0 M \cong \text{Ext}^1(F, F)\).

(ii) Let

\[ \mathcal{Y}_F = \sum_{i=0}^{\infty} \mathcal{Y}_{F,i}, \quad \mathcal{Y}_{F,i} \in \text{Hom}
\left(S^i(T_0 M), \text{Ext}^2(F, F)\right) \]

be a Taylor expansion of \(\mathcal{Y}_F\). Then \(\mathcal{Y}_{F,1} = 0\) and \(\mathcal{Y}_{F,2}\) is the composition

\[ T_0 M \xrightarrow{(KS, KS)} \text{Ext}^1(F, F) \times \text{Ext}^1(F, F) \xrightarrow{\langle \xi, \eta \rangle \mapsto \xi \cup \eta} \text{Ext}^2(F, F) \]

where \(\xi \cup \eta\) denotes the Yoneda product of two elements of \(\text{Ext}^1(F, F)\).
Proof: The Appendix of Bingener to [18] provides the following scheme of the proof of this statement. The existence of a formal versal deformation was proven in [19]. By [20], the formal versal deformation is the formal completion of a genuine versal deformation. For the construction of $Y_{F,i}$ for all $i$, see Proposition A.1 of [23]. The identification of the obstruction $Y_{F,2}$ on the formal level with the Yoneda product was done in [21, 22]. □

4. The proof of non-reducibility

Below we will prove that the obstruction map $Y_E$ for any sheaf $[E] \in \mathcal{C}$ is not trivial. This will immediately imply the main result of the paper. The idea of the proof is to connect the obstruction map $Y_E$ for the sheaf $E$ with the obstruction map $Y_L$ for the sheaf $L$ from the triple (9). First of all, let us make sure that the obstruction map $Y_L$ is not trivial.

Lemma 3. For any sheaf $L \in \mathcal{P}$ the following Yoneda product

\[\text{Ext}^1(L, L) \times \text{Ext}^1(L, L) \xrightarrow{\cup_2} \text{Ext}^2(L, L)\]

is non-trivial. Therefore, the first obstruction map $Y_{L,2}$ is non-zero.

Proof: According to [15, Prop. 3. 1, remark on the page 133], any curve $C \in \mathcal{H}$ has an embedded first order infinitesimal deformation $\phi_C \in \text{Hom}(I_C, \mathcal{O}_C)$ which is obstructed at the second order. In terms of the previous section this means that the Yoneda product (35) is not trivial.

On the other hand, we have the isomorphism $\xi_{1,2} : \text{Ext}^{1,2}(\mathcal{O}_C, \mathcal{O}_C) \simeq \text{Ext}^{1,2}(L, L)$. Note that from the local-to-global spectral sequence it follows that

\[\dim \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = h^1(\text{Hom}(\mathcal{O}_C, \mathcal{O}_C)) + h^0(\text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C)).\]

Since $\text{Hom}(\mathcal{O}_C, \mathcal{O}_C) \simeq \mathcal{O}_C$ and $\text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \simeq \text{Hom}(I_C, \mathcal{O}_C)$ we have

\[\dim \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = h^1(\mathcal{O}_C) + \dim \text{Hom}(I_C, \mathcal{O}_C).\]

Now applying the functor $\text{Hom}(-, \mathcal{O}_C)$ to the triple (34) and using the equality (38) we obtain the following exact triple

\[0 \rightarrow \text{Hom}(I_C, \mathcal{O}_C) \xrightarrow{\sigma} \text{Ext}^1(I_C, \mathcal{O}_C) \rightarrow H^1(\mathcal{O}_C) \rightarrow 0,\]

and the isomorphism $\zeta : \text{Ext}^1(I_C, \mathcal{O}_C) \simeq \text{Ext}^2(\mathcal{O}_C, \mathcal{O}_C)$. Using the maps $\delta, \xi_{1,2}, \sigma, \zeta$ we obtain the following commutative diagram

\[\begin{array}{ccc}
\text{Ext}^1(I_C, I_C) \times \text{Hom}(I_C, \mathcal{O}_C) & \xrightarrow{\cup_1} & \text{Ext}^1(I_C, \mathcal{O}_C) \\
\downarrow_{(\xi_1 \circ \sigma \circ \delta^{-1}) \times (\xi_1 \circ \sigma)} & & \downarrow_{\xi_2 \circ \zeta} \\
\text{Ext}^1(L, L) \times \text{Ext}^1(L, L) & \xrightarrow{\cup_2} & \text{Ext}^2(L, L)
\end{array}\]
Since the left vertical map is injective, the right vertical map is an isomorphism and the upper horizontal map is non-trivial we obtain that the bottom horizontal map is also non-trivial.

Now in order to establish a connection between the obstruction map $\mathcal{Y}_E$ for the sheaf $E$ with the obstruction map $\mathcal{Y}_L$ for the sheaf $L$ we need the following technical lemma.

**Lemma 4.** Suppose that we have the following commutative diagram

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & & \downarrow & \\
0 & \to E & \to F & \to L & \to 0 \\
& \downarrow \phi_1 & \downarrow \phi_2 & \downarrow \phi_3 & \\
0 & \to E_0 & \to F_0 & \to L_0 & \to 0 \\
& \downarrow \psi_1 & \downarrow \psi_2 & \downarrow \psi_3 & \\
0 & \to E & \to F & \to L & \to 0 \\
& \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
$$

(40)

Each vertical extension in this diagram represents the element of the group $\text{Ext}^1(E, E)$, $\text{Ext}^1(F, F)$, and $\text{Ext}^1(L, L)$, respectively. Consider these extensions as 3-complexes and denote them by $\mathcal{E}$, $\mathcal{F}$, $\mathcal{L}$. Then the Yoneda squares of these complexes also fits into the exact triple

$$
\begin{array}{c}
0 \to \mathcal{E} \cup \mathcal{E} \to \mathcal{F} \cup \mathcal{F} \to \mathcal{L} \cup \mathcal{L} \to 0.
\end{array}
$$

(41)
**Proof:** The triple \( (41) \) of 4-complexes can be read as the following diagram of sheaves

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & & & & & & \\
\downarrow & & & & & & & & \\
0 & E & \xi & F & \zeta & L & \rightarrow & 0 \\
\downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 \\
0 & E_0 & \eta & F_0 & \gamma & L_0 & \rightarrow & 0 \\
\downarrow \phi_1 \circ \psi_1 & & \downarrow \phi_2 \circ \psi_2 & & \downarrow \phi_3 \circ \psi_3 \\
0 & E_0 & \eta & F_0 & \gamma & L_0 & \rightarrow & 0 \\
\downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_3 \\
0 & E & \xi & F & \zeta & L & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & 0 & 0 & & & & & & \\
\end{array}
\]  

(42)

In order to prove its commutativity it is sufficient to show that the following squares are commutative

\[
\begin{array}{ccc}
E_0 & \eta & F_0 \\
\downarrow \phi_1 \circ \psi_1 & & \downarrow \phi_2 \circ \psi_2 \\
E_0 & \eta & F_0 \\
\end{array}
\]  

(43)

\[
\begin{array}{ccc}
F_0 & \gamma & L_0 \\
\downarrow \phi_2 \circ \psi_2 & & \downarrow \phi_3 \circ \psi_3 \\
F_0 & \gamma & L_0 \\
\end{array}
\]  

The commutativity of these squares can be derived from the commutative diagram \( (40) \) as follows

\[
\begin{align*}
\eta \circ \phi_1 \circ \psi_1 &= \phi_2 \circ \xi \circ \psi_1 = \phi_2 \circ \psi_2 \circ \eta, \\
\gamma \circ \phi_2 \circ \psi_2 &= \phi_3 \circ \zeta \circ \psi_1 = \phi_3 \circ \psi_3 \circ \gamma.
\end{align*}
\]  

(44)

(45)

Finally, using the previous lemma we are able to prove the main result of the paper.

**Theorem 2.** For any sheaf \([E] \in C\) the following Yoneda product

\[
\cup : \text{Ext}^1(E, E) \times \text{Ext}^1(E, E) \rightarrow \text{Ext}^2(E, E)
\]

is non-trivial. Therefore, the first obstruction map \( Y_{E,2} \) is non-zero.
**Proof:** By the definition of the subset \( \mathcal{C} \), a sheaf \([E] \in \mathcal{C}\) fits into the exact sequence

\[
0 \longrightarrow E \longrightarrow 2\mathcal{O}_{\mathbb{P}^3} \overset{\phi}{\longrightarrow} L \longrightarrow 0,
\]

where \([L] \in \mathcal{P}\) is a line bundle over a smooth curve \( C \) from \( \mathcal{H} \). Using Lemma 3 we can fix the deformation \( \text{L} \rightarrow L \) of the sheaf \( L \) over dual numbers such that the Yoneda square of the corresponding element of the group \( \text{Ext}^1(L, L) \) is non-zero. Consider also the trivial deformation \( 2\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_D \rightarrow 2\mathcal{O}_{\mathbb{P}^3} \) and arbitrary surjective morphism \( f : 2\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_D \rightarrow L \) such that \( f \otimes k = \phi \). Then the kernel \( E := \ker f \) satisfies the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & E & \longrightarrow & 2\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_D & \overset{f}{\longrightarrow} L & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & 2\mathcal{O}_{\mathbb{P}^3} & \overset{\phi}{\longrightarrow} L & \longrightarrow 0 
\end{array}
\]

(46)

Since \( 2\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_D \) and \( L \) are flat over \( D \), then \( E \) is also flat over \( D \). Moreover, the equality \( f \otimes k = \phi \) implies that the morphism \( E \rightarrow E \) induces an isomorphism \( E \otimes k \simeq E \). Therefore, \( E \rightarrow E \) is the deformation of the sheaf \( E \) and it defines an element of the group \( \text{Ext}^1(E, E) \).

Similar to (36) consider the corresponding exact triples for the sheaves \( \text{pr}_*E, \text{pr}_*(2\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_D), \text{pr}_*L \) as 3-term complexes and denote them by \( \mathcal{E}, \mathcal{F}, \mathcal{L} \), respectively:

\[
\begin{align*}
\mathcal{E} : & \quad 0 \longrightarrow E \longrightarrow \text{pr}_*E \longrightarrow E \longrightarrow 0, \\
\mathcal{F} : & \quad 0 \longrightarrow 2\mathcal{O}_{\mathbb{P}^3} \longrightarrow \text{pr}_*(2\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_D) \longrightarrow 2\mathcal{O}_{\mathbb{P}^3} \longrightarrow 0, \\
\mathcal{L} : & \quad 0 \longrightarrow L \longrightarrow \text{pr}_*L \longrightarrow L \longrightarrow 0.
\end{align*}
\]

Note that due to our assumption we have that

\[
(47) \quad [\mathcal{L} \cup \mathcal{L}] \neq 0 \in \text{Ext}^2(L, L).
\]

Taking pushforward of the upper exact triple of (46) we obtain the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & E & \longrightarrow & 2\mathcal{O}_{\mathbb{P}^3} & \overset{\phi}{\longrightarrow} L & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{pr}_*E & \longrightarrow & \text{pr}_*(2\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_D) & \overset{f}{\longrightarrow} \text{pr}_*L & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & 2\mathcal{O}_{\mathbb{P}^3} & \overset{\phi}{\longrightarrow} L & \longrightarrow 0
\end{array}
\]

(48)
which can be read as the exact triple of 3-complexes
\[ 0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{L} \to 0. \]

Next, from Lemma 4 it follows that the Yoneda squares also fit into the exact triple (41), so we have
\[ 0 \to \mathcal{E} \cup \mathcal{E} \to \mathcal{F} \cup \mathcal{F} \to \mathcal{L} \cup \mathcal{L} \to 0. \]

Now suppose that \([\mathcal{E} \cup \mathcal{E}] \in \text{Ext}^2(E, E)\) is trivial. By definition it means that there is a commutative diagram (roof) of the following form
\[
\begin{array}{ccccccccc}
0 & \to & E & \to & E' & \to & E' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E & \to & E_0 & \to & E_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E & \to & \text{pr}_*E & \to & \text{pr}_*E & \to & 0
\end{array}
\]

Denote the middle cochain complex of this diagram by \(\tilde{E}\). Taking the composition of morphisms \(\mathcal{E} \to \mathcal{F}\) and \(\tilde{E} \to \mathcal{E}\) we obtain the morphism \(\tilde{E} \to \mathcal{F}\). This morphism and its cokernel complex \(\tilde{\mathcal{L}} := \text{coker}(\tilde{E} \to \mathcal{F})\) can be included using the Snake lemma into the following commutative diagram
\[
\begin{array}{ccccccccc}
\tilde{E} & \to & \mathcal{F} \cup \mathcal{F} & \to & \tilde{\mathcal{L}} & \to & 0 \\
\downarrow & & \| & & \downarrow & & \\
0 & \to & \mathcal{E} \cup \mathcal{E} & \to & \mathcal{F} \cup \mathcal{F} & \to & \mathcal{L} \cup \mathcal{L} & \to & 0
\end{array}
\]

Assume that the 4-complex \(\tilde{\mathcal{L}}\) has the following form
\[
\tilde{\mathcal{L}} : \quad 0 \to L \to L_0 \to L_1 \underset{\eta}{\to} L \to 0.
\]

Note that from the commutative square of the commutative diagram (49)
\[
\begin{array}{cccccc}
E & \to & E \\
\| & & \| \\
E & \to & E_0
\end{array}
\]

follows existence of a splitting morphism \(\phi : E_0 \to E\), so \(E_0 \simeq E \oplus \tilde{E}\) where \(\tilde{E}\) is some sheaf. Consider the upper exact triple of the diagram (50) as
the following diagram of sheaves

\[
\begin{array}{c}
0 \longrightarrow E \longrightarrow 2\mathcal{O}_{\mathbb{P}^3} \longrightarrow L \longrightarrow 0 \\
\downarrow \phi \downarrow \psi \downarrow \psi \downarrow \psi \\
E_0 \longrightarrow 2\mathcal{O}_{\mathbb{P}^3} \oplus 2\mathcal{O}_{\mathbb{P}^3} \longrightarrow L_0 \longrightarrow 0 \\
\downarrow \downarrow \downarrow \downarrow \\
E_1 \longrightarrow 2\mathcal{O}_{\mathbb{P}^3} \oplus 2\mathcal{O}_{\mathbb{P}^3} \longrightarrow L_1 \longrightarrow 0 \\
\downarrow \downarrow \downarrow \downarrow \\
0 \longrightarrow E \longrightarrow 2\mathcal{O}_{\mathbb{P}^3} \longrightarrow L \longrightarrow 0 \\
\end{array}
\] (53)

Again using Snake lemma the splitting morphisms $\phi$ and $\psi$ imply the existence of a splitting morphism $\zeta : L_0 \to L$, so we have an isomorphism $L_0 \simeq L \oplus \tilde{L}$ for some sheaf $\tilde{L}$. The morphism $\zeta$ can be included into the following commutative diagram

\[
\begin{array}{c}
0 \longrightarrow L \longrightarrow L_0 \longrightarrow L_1 \xrightarrow{\eta} L \longrightarrow 0 \\
0 \longrightarrow L \xrightarrow{\zeta} L \\
0 \longrightarrow L \longrightarrow 0 \\
\end{array}
\] (54)

where the morphism $\eta$ comes from the definition (51) of the complex $\tilde{L}$. This means that the equivalence class $[\tilde{L}]$ in $\text{Ext}^2(L, L)$ is zero. Since the extensions $L$ and $\tilde{L}$ are equivalent, the class $[L \cup L] \in \text{Ext}^2(L, L)$ is also zero. However, it contradicts to the assumption (47), so the class $[E \cup E] \in \text{Ext}^2(E, E)$ cannot be trivial. □

Corollary 1. Let $\mathfrak{V} \subset \mathcal{M}(14)$ be a maximal closed subscheme of $\mathcal{M}(14)$ containing the closed subset $\overline{C} \subset \mathcal{M}(14)$. Then the underlying topological subspace of $\mathfrak{V}$ coincides with $\overline{C}$. Moreover, $\mathfrak{V}$ is generically non-reduced.

Proof: Consider the obstruction map $\mathcal{Y} = \sum_{i=1}^{\infty} \mathcal{Y}_i$ defined in Lemma 2

Theorem 1 states that $\mathcal{Y}_2 \neq 0$, so $\mathcal{Y}_2 \neq 0$ and $\dim \mathcal{Y}_2(0) < 133$. On the other hand, from Lemma 2 it follows that the germ of scheme $(\mathcal{Y}_2^{-1}(0), 0)$ can be immersed in the germ $(\mathcal{M}(14), [E])$. However, the germ $(\mathcal{C}, [E])$ is subgerm of $(\mathcal{M}(14), [E])$, so we have the following inequality

$$132 = \dim \mathcal{C} \leq \dim \mathcal{Y}_2^{-1}(0) < 133.$$ 

Therefore, we obtain that $\dim \mathcal{Y}_2^{-1}(0) = 132$. This means that the subset $\mathcal{C}$ is not contained in some component of the scheme $\mathcal{M}(14)$ whose dimension
is more than 132. So the underlying topological space of $\mathcal{M}$ coincides with $\overline{\mathcal{C}}$. Also as it was shown in Theorem 1, we have $\dim T_{[E]}\mathcal{M}(14) = 133$ for any sheaf $[E] \in \mathcal{C}$. So the subscheme $\mathfrak{M} \subset \mathcal{M}(14)$ is an irreducible component which is generically non-reduced.

In conclusion we conjecture that all computations above can be generalized such that the constructed component will be included into a series of generically non-reduced moduli components. More precisely, consider some infinite series of components $\{\mathcal{R}_i\}$ of moduli scheme of reflexive sheaves and series (finite or infinite) of generically non-reduced components $\{\mathcal{H}_j\}$ of the Hilbert scheme which parameterize smooth curves in $\mathbb{P}^3$. Now we can construct a family of stable sheaves by taking elementary transforms of the form

$$0 \to E \to F \to L \oplus O_W \to 0,$$

where $[F] \in \mathcal{R}_i$, $L$ is a line bundle over curve $C \in \mathcal{H}_j$ and $W$ is a 0-dimensional subscheme of $\mathbb{P}^3$. If we now provide the second condition from (2) and some other mild conditions (see [10] Lemma 4) then the isomorphism (31) will hold. This means that the difference between dimension of the tangent space $T_{[E]}\mathcal{M}$ and dimension of the family itself is equal to $\dim T_C\mathcal{H}_j - \dim (\mathcal{H}_j)_{\text{red}}$. So there should exist obstructed deformations of the stable sheaf $E$. Therefore, the corresponding component of the Gieseker-Maruyama moduli scheme is also generically non-reduced.

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