On the fluctuations of Internal DLA on the Sierpinski gasket graph

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Electropolishing
Internal DLA

For an infinite but locally finite connected graph $G$ let

$$\mathcal{I}(1) := \{\circ\},$$

$$\mathcal{I}(i) := \mathcal{I}(i - 1) \cup \{X^i(\sigma^i)\},$$

where $(X^i(t))_{t \geq 0}$ is a sequence of i.i.d. simple random walks on $G$ started in $\circ$ and

$$\sigma^i := \inf\{t > 0 : X^i(t) \notin \mathcal{I}(i - 1)\}.$$
Internal DLA

Theorem [Lawler, Bramson, Griffeath 92]
Let $G = \mathbb{Z}^d$ and $b_n = |B_\circ(n)|$, then for any $\varepsilon > 0$ with probability 1 it holds:

$$B_\circ(n - \varepsilon n) \subseteq \mathcal{I}(b_n) \subseteq B_\circ(n + \varepsilon n)$$
for $n$ large enough.

Theorem [Jerison, Levine, Sheffield 13], [Asselah, Gaudillière 13]
Let $G = \mathbb{Z}^d$ with $d \geq 3$ and $b_n = |B_\circ(n)|$, then there is an absolute constant $c > 0$, such that with probability 1 it holds:

$$B_\circ\left(n - c \sqrt{\log n}\right) \subseteq \mathcal{I}(b_n) \subseteq B_\circ\left(n + c \sqrt{\log n}\right)$$
for $n$ large enough.
Sierpinski graph
Internal DLA on SG

Theorem [Chen, Huss, Sava-Huss, Teplayev 20]
Let $G = SG$ and $b_n = |B_\circ(n)|$, then for any $\varepsilon > 0$ with probability 1 it holds:

$$B_\circ(n - \varepsilon n) \subseteq \mathcal{I}(b_n) \subseteq B_\circ(n + \varepsilon n)$$
for $n$ large enough.

Theorem (improved bounds) [H 21+]
Let $G = SG$, then there is a constant $c > 0$ such that for any $\kappa > 0$ it holds with probability 1

$$B_\circ(n - c n^{\frac{1}{2}} \ln(n)^{\frac{1+\kappa}{2\alpha}}) \subseteq \mathcal{I}(b_n) \subseteq B_\circ(n + c n^{\frac{1}{2}}+\frac{1}{2\alpha} \ln(n)^{(1-\frac{1}{\alpha})\frac{1+\kappa}{2\alpha}})$$
for $n$ large enough and $\alpha = \frac{\ln(3)}{\ln(2)}$.

$$B_\circ(n - cn^{0.5} \ln(n)^{0.63\frac{1+\kappa}{2}}) \subseteq \mathcal{I}(b_n) \subseteq B_\circ(n + cn^{0.82} \ln(n)^{0.23\frac{1+\kappa}{2}}).$$
Divisible Sandpile

Sandpile on $SG$

$\text{Sandpile cluster } S := \{ x^2 | \mu(x) = 1 \}$.

Theorem [Huss, Sava-Huss 17]

For $G = SG$ the resulting sandpile distribution $\mu(x)$ from the starting distribution $\mu_0(x) = b^n(x)$ is given by $\mu(x) = \frac{1}{B^n(x)}$.

Lemma [Levine, Peres 09]

There are functions $\mu, u : V \to \mathbb{R}_{\geq 0}$ with $\mu_k \to \mu$ as well as $u_k \uparrow u$ for $(k \to \infty)$ and it holds:

$$\mu(x) = \mu_0(x) + \Delta u(x) \leq 1.$$ 

Abelian property [Levine, Peres 09]

The functions $\mu, u$ do not depend on the particular choice of the toppling sequence.
Divisible Sandpile

**Lemma [Levine, Peres 09]**
There are functions $\mu, u : V \to \mathbb{R}_{\geq 0}$ with $\mu_k \to \mu$ as well as $u_k \to u$ for $(k \to \infty)$ and it holds:

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**Abelian property [Levine, Peres 09]**
The functions $\mu, u$ do not depend on the particular choice of the toppling sequence.

**Theorem [Huss, Sava-Huss 19]**
For $G = SG$ the resulting sandpile distribution $\mu(x)$ from the starting distribution $\mu_0(x) = b_n \delta_\circ(x)$ is given by

$$\mu(x) = 1_{B_\circ(n)}(x).$$
Proof of inner bound

For fixed $z \in B_\circ(n)$ consider

$$M := M(z) := \#\text{RW hitting } z \text{ before exiting } B_\circ(n),$$
$$L := L(z) := \#\text{RW hitting } z \text{ before exiting } B_\circ(n) \text{ but after settlement } \sigma^i,$$

where $\sigma^i := \inf \{ t > 0 \mid X_t \notin I(i - 1) \}$. Then clearly

$$\mathbb{P}(z \notin I(b_n)) \leq \mathbb{P}(M = L) \leq \mathbb{P}(M \leq a) + \mathbb{P}(L \geq a),$$

for any $a \in \mathbb{R}$.

Large Deviation Results for $M$ and $L$ yield for any $\kappa > 0$

$$\mathbb{P}(M \leq \mathbb{E}(M) - \mathbb{E}(M)^{\frac{1}{2}} \ln(n)^{\frac{1+\kappa}{2}}) + \mathbb{P}(L \geq \mathbb{E}(L) + \mathbb{E}(L)^{\frac{1}{2}} \ln(n)^{\frac{1+\kappa}{2}})$$

$$\leq 4n^{-\frac{1}{4} (\ln n)^\kappa}.$$
Large Deviation Results for $M$ and $L$ yield for any $\kappa > 0$

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\leq 4n^{-1/4} (\ln n)^{\kappa}
$$

So we have to find

$$
a \in I := \left[ \mathbb{E}(L) + \mathbb{E}(L)^{1/2} \ln(n)^{1+\kappa/2}, \mathbb{E}(M) - \mathbb{E}(M)^{1/2} \ln(n)^{1+\kappa/2} \right]
$$

or equivalently show that $I$ is nonempty. And the result follows immediately by Borel-Cantelli.
\[
a \in I := [\mathbb{E}(L) + \mathbb{E}(L)^{\frac{1}{2}} \ln(n)^{\frac{1+\kappa}{2}}, \mathbb{E}(M) - \mathbb{E}(M)^{\frac{1}{2}} \ln(n)^{\frac{1+\kappa}{2}}]
\]

\[
\mathbb{E}(M) - \mathbb{E}(L) \geq \frac{1}{g_n(z, z)} \left( b_n g_n(\circ, z) - \sum_{y \in B_\circ(n)} g_n(y, z) \right),
\]

where \( g_n(x, y) = \mathbb{E}_x \sum_{t=0}^{\tau_{\partial I B_\circ(n)} - 1} \mathbb{1}_{\{X_t = y\}} \).

Now \( f(z) := \left( b_n g_n(\circ, z) - \sum_{y \in B_\circ(n)} g_n(y, z) \right) \) solves the (discrete) Dirichlet problem

\[
\begin{cases}
\Delta f(z) = \left( 1 - b_n \delta_\circ(z) \right), & \text{if } z \in B_\circ(n) \setminus \partial I B_\circ(n) \\
f(z) = 0, & \text{if } z \in \partial I B_\circ(n).
\end{cases}
\]
Recall that for the odometer function $u$ of the divisible Sandpile with starting distribution $\mu_0 = b_n \delta_\circ(z)$ it holds

$$\Delta u(z) = \mu(z) - \mu_0(z) = 1_{B_\circ(n)} - b_n \delta_\circ(z).$$

And since no mass has been distributed outside $B_\circ(n)$ we have

$$u(z) = 0 \text{ for } z \in \partial_1 B_\circ(n).$$

So $u$ and $f$ solve the same Dirichlet problem and by the uniqueness principle

$$u(z) = f(z) \text{ for all } z \in B_\circ(n).$$
Lemma

Let $n, \delta \in \mathbb{N}$ such that $n \gg \delta$ and $u : \text{SG} \to \mathbb{R}$ the odometer function. Then for all $z \in B_{\partial}(n - 3\delta)$ it holds

$$u(z) \geq c \, \delta^\beta$$

for some $c > 0$ and $\beta := \frac{\ln 5}{\ln 2}$.

which then gives

$$\mathbb{E}(M) - \mathbb{E}(L) \geq \frac{1}{g_n(z, z)} u(z) \geq c \frac{1}{g_n(z, z)} d(z, \partial I B_{\partial}(n))^\beta$$
\[ \mathbb{E}(M) - \mathbb{E}(L) \geq \frac{1}{g_n(z, z)} u(z) \geq c \frac{1}{g_n(z, z)} d(z, \partial I B_\circ(n))^\beta \]

Furthermore one can show \( g_n(z, z) \leq c \ d(z, \partial I B_\circ(n))^{\beta - \alpha} \) and we get

\[ \mathbb{E}(M) - \mathbb{E}(L) \geq c \ d(z, \partial I B_\circ(n))^{\alpha} \]

which gives that

\[ I = [\mathbb{E}(L) + \mathbb{E}(L)^{\frac{1}{2}} \ln(n)^{\frac{1+\kappa}{2}}, \mathbb{E}(M) - \mathbb{E}(M)^{\frac{1}{2}} \ln(n)^{\frac{1+\kappa}{2}}] \]

is nonempty (taking \( \mathbb{E}(M) \leq cn^{\alpha} \)) if

\[ d(z, \partial I B_\circ(n))^{\alpha} \geq cn^{\frac{\alpha}{2}} \ln(n)^{\frac{1+\kappa}{2}}. \]
Work in progress

Uniform IDLA: RW’s start uniformly on the existing cluster.
(joint work with Ecaterina Sava-Huss)

Convergence to a continuous model on the fractal blowup.
(joint work with Uta Freiberg, Robin Kaiser, Ecaterina Sava-Huss)

Examine other fractals?