Cellularity of endomorphism algebras of Young permutation modules.

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Abstract

Let $E$ be an $n$-dimensional vector space. Then the symmetric group $\text{Sym}(n)$ acts on $E$ by permuting the elements of a basis and hence on the $r$-fold tensor product $E^\otimes r$. Bowman, Doty and Martin ask, in [1], whether the endomorphism algebra $\text{End}_{\text{Sym}(n)}(E^\otimes r)$ is cellular. The module $E^\otimes r$ is the permutation module for a certain Young $\text{Sym}(n)$-set. We shall show that the endomorphism algebra of the permutation module on an arbitrary Young $\text{Sym}(n)$-set is a cellular algebra. We determine, in terms of the point stabilisers which appear, when the endomorphism algebra is quasi-hereditary.

1 Introduction

We fix a positive integer $n$. The symmetric group of degree $n$ is denoted $\text{Sym}(n)$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of $n$ we have the Young subgroup, i.e. the group $\text{Sym}(\lambda) = \text{Sym}(\lambda_1) \times \text{Sym}(\lambda_2) \times \cdots$, regarded as a subgroup of $\text{Sym}(n)$ in the usual way. By a Young $\text{Sym}(n)$-set we mean a finite $\text{Sym}(n)$-set such that each point stabiliser is conjugate to a Young subgroup. Let $R$ be a commutative ring. Our interest is in the endomorphism algebra $\text{End}_{\text{Sym}(n)}(R\Omega)$ of the permutation module $R\Omega$ on a Young $\text{Sym}(n)$-set $\Omega$. We shall show that $\text{End}_{\text{Sym}(n)}(k\Omega)$ has a cellular structure, Theorem 6.4, hence by base change so has $\text{End}_{\text{Sym}(n)}(R\Omega)$, for an arbitrary commutative ring $R$.

Taking the base ring now to be a field $k$ of positive characteristic, we give a criterion for $\text{End}_{\text{Sym}(n)}(k\Omega)$ to be a quasi-hereditary algebra, in terms of the set of partitions $\lambda$ of $n$ for which $\text{Sym}(\lambda)$ occurs as a point stabiliser, and the characteristic $p$ of $k$, see Theorem 6.4. This is applied to the case $\Omega = I(n, r)$, the set of maps from $\{1, \ldots, r\}$ to $\{1, \ldots, n\}$, for a positive integer $r$, with $\text{Sym}(n)$ acting by composition of maps. The permutation module $kI(n, r)$ may be regarded as the $r$th tensor power $E^\otimes r$ of an $n$-dimensional vector space $E$, and we thus determine when $\text{End}_{\text{Sym}(n)}(E^\otimes r)$ is quasi-hereditary, see Proposition 7.3.
Our procedure is to analyse the endomorphism algebra of a Young permutation module in the spirit of the Schur algebra $S(n, r)$ (which is a special case). Of particular importance to us will be the fact that the Schur algebras is quasi-hereditary. There are several approaches to this (see e.g. [5, Section A5] and [18]) but for us the most convenient is that of Green, [9]. This has the advantage of being a purely combinatorial account carried out over an arbitrary commutative base ring. So we regard what follows as a modest generalisation of some aspects of [9]: we follow Green’s approach and notation to a large extent.

2 Preliminaries

We write mod$(S)$ for the category of finitely generated modules over a ring $S$.

Let $G$ be a finite group and $K$ a field of characteristic 0. Let $X$ be a finitely generated $KG$-module. Suppose that all composition factors of $X$ are absolutely irreducible. Let $U_1, \ldots, U_d$ be a complete set of pairwise non-isomorphic composition factors of $X$. We write $X$ as a direct sum of simple modules $X = X_1 \oplus \cdots \oplus X_h$. For $1 \leq i \leq d$ let $m_i$ be the number of elements $r \in \{1, \ldots, h\}$ such that $X_r$ is isomorphic to $U_i$. Let $S = \text{End}_G(X)$. Then $S$ is isomorphic to the product of the matrix algebras $M_{m_1}(K), \ldots, M_{m_d}(K)$. Let the corresponding irreducible modules for $S$ be $L_1, \ldots, L_d$. We have an exact functor from $f : \text{mod}(KG) \to \text{mod}(S)$, given on objects by $f(Z) = \text{Hom}_{\text{Sym}(n)}(X, Z)$. Moreover we have $S = f(X) = \bigoplus_{r=1}^h \text{Hom}_G(X, X_r)$. If follows that the modules $L_i = fU_i = \text{Hom}_G(X, U_i), 1 \leq i \leq d$, form a complete set of pairwise non-isomorphic irreducible $S$-modules.

The situation in positive characteristic is similar, cf. [3] (3.4) Proposition. Suppose now that $F$ is any field which is a splitting field for $G$. Let $Y$ be a finitely generated $KG$-module such that every indecomposable component is absolutely indecomposable. Let $V_1, \ldots, V_e$ be a complete set of pairwise non-isomorphic indecomposable summands of $Y$. We write $Y$ as a direct sum of indecomposable modules $Y = Y_1 \oplus \cdots \oplus Y_k$. For $1 \leq j \leq e$ let $n_j$ be the number of elements $r \in \{1, \ldots, k\}$ such that $X_r$ is isomorphic to $V_j$. Let $T = \text{End}_G(Y)$. Then each $P_j = \text{Hom}_G(Y, V_j)$ is naturally a $T$-module and the modules $P_1, \ldots, P_e$ form a complete set of pairwise non-isomorphic projective $T$-modules. Let $N_j$ be the head of $P_j, 1 \leq j \leq e$. Then the modules $N_1, \ldots, N_e$ form a complete set of pairwise non-isomorphic irreducible $T$-modules. The dimension of $N_j$ over $F$ is $n_j$.

We now fix a positive integer $n$. We write Par$(n)$ for the set of partitions of $n$. By the support $\zeta(\Omega)$ of a Young Sym$(n)$-set $\Omega$ we mean the set of $\lambda \in \text{Par}(n)$ such that the Young subgroup Sym$(\lambda)$ is a point stabiliser. Let
$R$ be a commutative ring. For a Young Sym($n$)-set $\Omega$ we write $S_{\Omega,R}$ for the endomorphism algebra $\text{End}_{\text{Sym}(n)}(R\Omega)$ of the permutation module $R\Omega$. For $\lambda \in \text{Par}(n)$ we write $M(\lambda)_R$ for the permutation module $R\text{Sym}(n)/\text{Sym}(\lambda)$.

We have the usual dominance partial order $\leq$ on $\text{Par}(n)$. Thus, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_a) \in \text{Par}(n)$, we write $\lambda \leq \mu$ if $\lambda_1 + \cdots + \lambda_a \leq \mu_1 + \cdots + \mu_a$ for all $1 \leq a \leq n$.

Recall that the Specht modules $\text{Sp}(\lambda)_Q$, $\lambda \in \text{Par}(n)$, form a complete set of pairwise irreducible $\mathbb{Q}\text{Sym}(n)$-modules. For $\lambda \in \text{Par}(n)$ we have $M(\lambda)_Q = \text{Sp}(\lambda)_Q \oplus C$, where $C$ is a direct sum of modules of the form $\text{Sp}(\mu)$ with $\lambda \not\leq \mu$, and moreover every Specht module $\text{Sp}(\mu)_Q$ with $\lambda \not\leq \mu$ occurs in $C$ (see for example [12, 14.1]).

For a Young Sym($n$)-set $\Omega$ we define

$$\zeta^\supset(\Omega) = \{ \mu \in \text{Par}(n) \mid \mu \trianglerighteq \lambda \text{ for some } \lambda \in \zeta(\Omega) \}.$$ 

Thus the composition factors of $\mathbb{Q}\Omega$ are $\{ \text{Sp}(\mu)_Q \mid \mu \in \zeta^\supset(\Omega) \}$ and, setting $\nabla_{\Omega}(\lambda)_Q = \text{Hom}_{\text{Sym}(n)}(\mathbb{Q}\Omega, \text{Sp}(\mu)_Q)$, we have the following.

**Lemma 2.1.** The modules $\nabla_{\Omega}(\lambda)_Q$, $\lambda \in \zeta^\supset(\Omega)$, form a complete set of pairwise non-isomorphic irreducible $S_{\Omega,Q}$-modules.

**Remark 2.2.** Since $S_{\Omega,Q}$ is a direct sum of matrix algebras over $\mathbb{Q}$ it is semisimple, all irreducible modules are absolutely irreducible and $\dim_\mathbb{Q} S_{\Omega,Q} = \sum_{\lambda \in \zeta^\supset(\Omega)} (\dim_\mathbb{Q} \nabla_{\Omega}(\lambda)_Q)^2$.

We now let $k$ be a field of characteristic $p > 0$. For $\lambda \in \text{Par}(n)$ we have the Young module $Y(\lambda)$ for $k\text{Sym}(n)$, labelled by $\lambda$, as described in [5] Section 4.4 for example. Then we have $M(\lambda)_k = Y(\lambda) \oplus C$, where $C$ is a direct sum of Young modules $Y(\mu)$, with $\lambda \not\leq \mu$, see for example [5, Section 4.4 (1) (v)]. A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ will be called $p$-restricted (also called column $p$-regular) if $\lambda_i - \lambda_{i+1} < p$ for all $i \geq 1$. A partition $\lambda$ has a unique expression

$$\lambda = \sum_{i \geq 0} p^i \lambda(i)$$

where each $\lambda(i)$ is a $p$-restricted partition. This is called the base $p$ (or $p$-adic) expansion of $\lambda$.

We write $\Lambda(n)$ for the set of all $n$-tuples of non-negative integers. An expression $\lambda = \sum_{i \geq 0} p^i \gamma(i)$, with all $\gamma(i) \in \Lambda(n)$ (but not necessarily restricted) will be called a weak $p$ expansion.

For an $n$-tuple of non-negative integers $\gamma$ we write $\overline{\gamma}$ for the partition obtained by arranging the entries in descending order.

**Definition 2.3.** For $\lambda, \mu \in \text{Par}(n)$ we shall say that $\mu$ $p$-dominates $\lambda$, and write $\mu \trianglerighteq_p \lambda$ (or $\lambda \triangleleft_p \mu$) if there exists a weak $p$ expansion $\lambda = \sum_{i \geq 0} p^i \gamma(i)$, such that $\mu(i) \trianglerighteq_p \gamma(i)$ for all $i \geq 0$, where $\mu = \sum_{i \geq 0} p^i \mu(i)$ is the base $p$ expansion of $\mu$. 

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Note that $\lambda \preceq_{p} \mu$ implies $\lambda \preceq \mu$.

By [4, Section 3, Remark], for $\lambda, \mu \in \text{Par}(n)$, then module $Y(\mu)$ appears as a component of $M(\lambda)_{k}$ if and only if $\lambda \preceq_{p} \mu$. For a Young $\text{Sym}(n)$-set $\Omega$ we define

$$\zeta^{\preceq_{p}}(\Omega) = \{\mu \in \text{Par}(n) \mid \mu \preceq_{p} \lambda \text{ for some } \lambda \in \zeta(\Omega)\}.$$  

Writing $P(\mu) = \text{Hom}_{\text{Sym}(n)}(k\Omega, Y(\mu))$ and writing $L(\mu)$ for the head of $P(\lambda)$, for $\mu \in \zeta^{\preceq_{p}}(\Omega)$ we have the following.

**Lemma 2.4.** The modules $L(\mu), \mu \in \zeta^{\preceq_{p}}(\Omega)$, form a complete set of pairwise non-isomorphic irreducible $S_{\Omega,k}$-modules.

### 3 Basic Constructions

We fix a positive integer $n$ and a Young $\text{Sym}(n)$-set $\Omega$. Here we assume the base ring $R$ is either the ring integers $\mathbb{Z}$ or the field of rational numbers $\mathbb{Q}$. We write $M_{\Omega,R}$, or just $M_{R}$ for the permutation module $R\Omega$ over $R\text{Sym}(n)$. We also just write $M$ for $M_{\Omega,\mathbb{Z}}$. We shall sometimes write simply $S_{R}$ for $S_{\Omega,R}$ and just $S$ for $S_{\mathbb{Q}}$. We identify $S$ with a subring or $S_{\mathbb{Q}}$ in the natural way.

Let $\{\mathcal{O}_{\alpha} \mid \alpha \in \Lambda_{\Omega}\}$ be a complete set of orbits in $\Omega$. For $\lambda \in \zeta(\Omega)$ we pick $\alpha(\lambda) \in \Lambda_{\Omega}$ such that $\text{Sym}(\lambda)$ is a point stabiliser of some element of $\mathcal{O}_{\alpha}$.

We put $M_{\alpha,R} = R\mathcal{O}_{\alpha}$, and sometimes write just $M_{\alpha}$ for $M_{\alpha,\mathbb{Z}}$, for $\alpha \in \Lambda_{\Omega}$. For $\beta \in \Lambda_{\Omega}$ we define the element $\xi_{\beta}$ of $S_{R}$ to be the projection onto $M_{\beta,R}$ coming from the decomposition $M_{R} = \bigoplus_{\alpha \in \Lambda_{\Omega}} M_{\alpha,R}$. Then each $\xi_{\alpha}$ is idempotent and we have the orthogonal decomposition:

$$1_{S} = \sum_{\alpha \in \Lambda_{\Omega}} \xi_{\alpha}.$$  

For a left $S_{R}$-module $V$ and $\beta \in \Lambda_{\Omega}$ we have the $\beta$ weight space $\beta V = \xi_{\beta}V$ and the weight space decomposition

$$V = \bigoplus_{\alpha \in \Lambda_{\Omega}} \alpha V.$$  

For $\lambda \in \text{Par}(n)$ we define

$$\lambda V = \begin{cases} \xi_{\alpha}(\lambda) V, & \text{if } \lambda \in \zeta(\Omega); \\ 0, & \text{otherwise}. \end{cases}$$

Similar remarks apply to weight spaces of right $S_{R}$-modules.

**Lemma 3.1.** Let $\lambda \in \zeta^{\preceq_{p}}(\Omega)$. Then

(i) $\dim_{\mathbb{Q}} \lambda \nabla_{\Omega}(\lambda)_{\mathbb{Q}} = 1$; and

(ii) if $\mu \in \text{Par}(n)$ and $\mu \nabla_{\Omega}(\lambda)_{\mathbb{Q}} \neq 0$ then $\mu \preceq \lambda$. 

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Proof. Let $\mu \in \text{Par}(n)$ and suppose $\nabla_\Omega(\lambda)_Q \neq 0$. Thus $\xi_\mu \text{Hom}_{\text{Sym}(n)}(M_Q, \text{Sp}(\lambda)_Q) \neq 0$ i.e. $\text{Hom}_{\text{Sym}(n)}(M(\mu)_Q, \text{Sp}(\lambda)_Q) \neq 0$ and so $\mu \geq \lambda$, giving (ii). Moreover

$$
\xi_\lambda \text{Hom}_{\text{Sym}(n)}(M_Q, \text{Sp}(\lambda)_Q) = \text{Hom}_{\text{Sym}(n)}(M(\lambda)_Q, \text{Sp}(\lambda)_Q) = Q
$$
giving (i).

For $\lambda \in \text{Par}(n)$ we set $\xi_\lambda = \left\{ \begin{array}{ll} \xi_{\alpha(\lambda)}, & \text{if } \lambda \in \zeta(\Omega) ; \\
0, & \text{otherwise.} \end{array} \right.$

For $\lambda \in \text{Par}(n)$ we set $S_R(\lambda) = S_R \xi_\lambda S_R$ and for $\sigma \subseteq \text{Par}(n)$ set $S_R(\sigma) = \sum_{\lambda \in \sigma} S_R(\lambda)$.

We also write simply $S(\lambda)$ for $S_\zeta(\lambda)$ and $S(\sigma)$ for $S_\zeta(\sigma)$.

Let $\leq$ be a partial order on $\text{Par}(n)$ which is a refinement of the dominance partial order. For $\lambda \in \zeta(\Omega)$ we set $S_R(\geq \lambda) = S_R(\sigma)$, where $\sigma = \{ \mu \in \text{Par}(n) \mid \mu \geq \lambda \}$, and $S_R(> \lambda) = S_R(\tau)$, where $\tau = \{ \mu \in \text{Par}(n) \mid \mu > \lambda \}$. Thus

$$
S_R(\geq \lambda) = S_R \xi_\lambda S_R + S(> \lambda).
$$

We set $V_R(\lambda) = S_R(\geq \lambda)/S_R(> \lambda)$. So we have

$$
V_R(\lambda)^\lambda = (S_R \xi_\lambda + S_R(> \lambda))/S_R(> \lambda),
$$

$$
\lambda V_R(\lambda) = (\xi_\lambda S_R + S_R(> \lambda))/S_R(> \lambda)
$$
and the multiplication map $S_R \xi_\lambda \times \xi_\lambda S_R \to S_R$ induces a surjective map

$$
\phi_R(\lambda) : V_R(\lambda)^\lambda \otimes_R \lambda V_R(\lambda) \to V_R(\lambda).
$$

For left $S_R$-modules $P, Q$ and $\lambda \in \text{Par}(n)$ we define $\text{Hom}_\text{Sym}(n)^\lambda(P, Q)$ to be the $R$-submodule of $\text{Hom}_\text{Sym}(n)(P, Q)$ spanned by all composite maps $f \circ g$, with $f \in \text{Hom}_\text{Sym}(n)(M(\lambda)_R, Q)$ and $g \in \text{Hom}_\text{Sym}(n)(P, M(\lambda)_R)$. For a subset $\sigma$ of $\text{Par}(n)$ we set

$$
\text{Hom}_{\text{Sym}(n)}^\sigma(P, Q) = \sum_{\lambda \in \sigma} \text{Hom}_\text{Sym}(n)^\lambda(P, Q).
$$

We note some similarity of our approach here via these groups of homomorphisms with the approach to Schur algebras due to Erdmann. [6] via stratification.
For $\lambda \in \text{Par}(n)$ we define $\text{Hom}_{\text{Sym}(n)}^{\geq \lambda}(P, Q) = \text{Hom}_{\text{Sym}(n)}^{\sigma}(P, Q)$, where
\[
\sigma = \{ \mu \in \text{Par}(n) \mid \mu \geq \lambda \},
\]
and $\text{Hom}_{\text{Sym}(n)}^{\geq \lambda}(P, Q) = \text{Hom}_{\text{Sym}(n)}^{\tau}(P, Q)$, where $\tau = \{ \mu \in \text{Par}(n) \mid \mu > \lambda \}$.

Note that if $\lambda \not\in \zeta(\Omega)$ then $V_{R}(\lambda) = 0$. Suppose $\lambda \in \zeta(\Omega)$. Then we have
\[
S_{R}\xi_{\lambda}S_{R} = \sum_{\alpha, \beta, \gamma, \delta \in \Omega} \text{Hom}_{\text{Sym}(n)}(M_{a, R}, M_{b, R}) \xi_{\lambda} \text{Hom}_{\text{Sym}(n)}(M_{\gamma, R}, M_{\delta, R})
= \sum_{\alpha, \beta \in \Omega} \text{Hom}_{\text{Sym}(n)}(M_{a, R}, M_{\alpha, R}) \xi_{\lambda} \text{Hom}_{\text{Sym}(n)}(M_{\alpha, R}, M_{\beta, R})
= \bigoplus_{\alpha, \beta \in \Omega} \text{Hom}_{\text{Sym}(n)}^{\lambda}(M_{a, R}, M_{\beta, R})
\]
and hence
\[
S_{R}(\sigma) = \bigoplus_{\alpha, \beta \in \Omega} \text{Hom}_{\text{Sym}(n)}^{\sigma}(M_{a, R}, M_{\beta, R})
\tag{1}
\]
for $\sigma \subseteq \text{Par}(n)$. In particular we have
\[
S_{R}(\geq \lambda) = \bigoplus_{\alpha, \beta \in \Omega} \text{Hom}_{\text{Sym}(n)}^{\geq \lambda}(M_{a, R}, M_{\beta, R})
\]
and
\[
S_{R}(> \lambda) = \bigoplus_{\alpha, \beta \in \Omega} \text{Hom}_{\text{Sym}(n)}^{> \lambda}(M_{a, R}, M_{\beta, R})
\]
and hence
\[
V_{R}(\lambda) = \bigoplus_{\alpha, \beta \in \Omega} \text{Hom}_{\text{Sym}(n)}^{\geq \lambda}(M_{a, R}, M_{\beta, R}) / \text{Hom}_{\text{Sym}(n)}^{\geq \lambda}(M_{a, R}, M_{\beta, R})
\tag{2}
\]

**Example 3.2.** Of crucial importance is the motivating example of the usual Schur algebra $S(n, r)$. Let $R$ be a commutative ring and let $E_{R}$ be a free $R$-module of rank $n$. Then $\text{Sym}(r)$ acts on the $r$-fold tensor product $E_{R}^{\otimes r} = E_{R} \otimes \cdots \otimes_{E_{R}} E_{R}$ by place permutation, and the Schur algebra $S_{R}(n, r)$ may be realised as $\text{End}_{\text{Sym}(r)}(E_{R}^{\otimes r})$.

We choose an $R$-basis $e_{1}, \ldots, e_{n}$ of $E_{R}$. We write $I(n, r)$ for the set of maps from $\{1, \ldots, r\}$ to $\{1, \ldots, n\}$. We regard $i \in I(n, r)$ as an $r$-tuple of elements $(i_{1}, \ldots, i_{r})$ with entries in $\{1, \ldots, n\}$ (where $i_{a} = i(a)$, $1 \leq a \leq r$).
The group $\text{Sym}(r)$ acts on $I(n, r)$ composition of maps, i.e. by $w \cdot i = i \circ w^{-1}$, for $w \in \text{Sym}(r)$, $i \in I(n, r)$. Moreover, for $i \in I(n, r)$, $w \in \text{Sym}(r)$, we have $w \cdot e_{i} = e_{i \circ w^{-1}}$.

We may thus regard $E_{R}^{\otimes r}$ as the $R\text{Sym}(r)$ permutation module $R\Omega$ on $\Omega = I(n, r)$. Note that $\zeta(\Omega) = \Lambda^{+}(n, r)$, the set of partitions of $r$ with at most $n$ parts. We write $\Lambda(n, r)$ for the set of weights, i.e. the set of $n$-tuples of non-negative integers $\alpha = (\alpha_{1}, \ldots, \alpha_{n})$ such that $\alpha_{1} + \cdots + \alpha_{n} = r$. An element $i$ of $I(n, r)$ has weight $\text{wt}(i) = (\alpha_{1}, \ldots, \alpha_{n}) \in \Lambda(n, r)$, where $\alpha_{a} = |i^{-1}(a)|$, for $1 \leq a \leq n$. For $\alpha \in \Lambda(n, r)$ we have the orbit $\Omega_{\alpha}$ consisting or all $i \in I(n, r)$ such that $\text{wt}(i) = \alpha$. Then $R\Omega = \bigoplus_{\alpha \in \Lambda(n, r)} R\Omega_{\alpha}$.  

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4 Groups of homomorphisms between Young permutation modules

In the situation of the Example 3.2 it follows from the quasi-hereditary structure of $S_n^Z$ that $V_n^Z(\lambda)$ is a free abelian group - indeed an explicit basis is given by Green in [9, (7.3) Theorem, (ii),(iii)]. Thus, taking $r = n$, from Section 3, (2), we have the following.

**Lemma 4.1.** For all $\lambda, \mu, \tau \in \text{Par}(n)$ the quotient

$\frac{\text{Hom}_{\text{Sym}(n)}^\geq \lambda (M(\mu), M(\tau))}{\text{Hom}_{\text{Sym}(n)}(M(\mu), M(\tau))}$

is torsion free.

We can improve on this somewhat. A subset $\sigma$ of $\text{Par}(n)$ will be called co-saturated (also said to be a co-ideal) if whenever $\lambda, \mu \in \sigma, \lambda \in \sigma$ and $\lambda \triangleright \mu$ then $\mu \in \sigma$.

**Proposition 4.2.** Let $\sigma, \tau$ be co-saturated subsets of $\text{Par}(n)$ with the $\tau \subseteq \sigma$. Then, for all $\mu, \nu \in \text{Par}(n)$, the quotient

$\frac{\text{Hom}_{\text{Sym}(n)}^\sigma (M(\mu), M(\nu))}{\text{Hom}_{\text{Sym}(n)}^\tau (M(\mu), M(\nu))}$

is torsion free.

**Proof.** If there is a co-saturated subset $\theta$ with $\tau \subset \theta \subset \sigma$ (and $\theta \neq \sigma, \tau$) and if

$\frac{\text{Hom}_{\text{Sym}(n)}^\sigma (M(\mu), M(\nu))}{\text{Hom}_{\text{Sym}(n)}^\theta (M(\mu), M(\nu))}$

and

$\frac{\text{Hom}_{\text{Sym}(n)}^\theta (M(\mu), M(\nu))}{\text{Hom}_{\text{Sym}(n)}^\tau (M(\mu), M(\nu))}$

are torsion free then so is

$\frac{\text{Hom}_{\text{Sym}(n)}^\sigma (M(\mu), M(\nu))}{\text{Hom}_{\text{Sym}(n)}^\theta (M(\mu), M(\nu))}$.

Thus we are reduced to the case $\tau = \sigma \setminus \{\lambda\}$, where $\lambda$ is a maximal element of $\sigma$. We choose a total order $\leq$ on $\text{Par}(n)$ refining $\leq$ such that, writing out the elements of $\text{Par}(n)$ in descending order $\lambda^1 \triangleright \lambda^2 \cdots \triangleright \lambda^h$ we have $\tau = \{\lambda^1, \ldots, \lambda^k\}, \sigma = \{\lambda^1, \ldots, \lambda^{k+1}\}$ (so $\lambda = \lambda^{k+1}$) for some $k$. Then we have

$\frac{\text{Hom}_{\text{Sym}(n)}^\sigma (M(\mu), M(\nu))}{\text{Hom}_{\text{Sym}(n)}^\tau (M(\mu), M(\nu))}$

$= \frac{\text{Hom}_{\text{Sym}(n)}^\geq \lambda (M(\mu), M(\nu))}{\text{Hom}_{\text{Sym}(n)}(M(\mu), M(\nu))}$

which is torsion free by the Lemma.

Returning to the general situation we have, by the Proposition and Section 3, (2), the following results.

**Corollary 4.3.** The $S$-module $V(\lambda)$ is torsion free.

**Corollary 4.4.** Let $\sigma$ be cosaturated set (with respect to $\leq$). Then $S(\sigma)$ is a pure submodule of $S$.
5 Cosaturated Sym\((n)\)-sets

From Corollary 4.4, if \(\sigma\) is any co-saturated subset of \(\text{Par}(n)\) then we may identify \(\mathbb{Q} \otimes_{\mathbb{Z}} S(\sigma)\) with an \(S_{\Omega, \mathbb{Q}}\)-submodule of \(S_{\Omega, \mathbb{Q}}\) via the natural map \(\mathbb{Q} \otimes_{\mathbb{Z}} S(\sigma) \to S_{\Omega, \mathbb{Q}}\).

We now suppose that \(\Omega\) is cosaturated, by which we mean that \(\zeta(\Omega)\) is a cosaturated subset of \(\text{Par}(n)\). We check that much of the structure, described by Green for the Schur algebras in [9], still stands in this more general case.

Let \(\sigma\) be a co-saturated subset of the support \(\zeta(\Omega)\). Let \(\mu \in \zeta(\Omega)\). If \(\nabla_{\Omega}(\mu)\) is a composition factor of \(S(\sigma)\mathbb{Q}\) then it is a composition factor of \(S(\lambda)\mathbb{Q}\) and hence of \(S_{\mathbb{Q}}\xi_{\lambda}\), for some \(\lambda \in \sigma\). Hence we have \(\text{Hom}_{\text{Sym}(n)}(S\xi_{\lambda}, \nabla_{\Omega}(\mu)\mathbb{Q}) \neq 0\) and so \(\mu \geq \lambda\), Lemma 3.1(ii), and therefore \(\mu \in \sigma\).

We fix \(\lambda \in \zeta(\Omega)\). Then \(\text{Hom}_{\text{Sym}(n)}(S\xi_{\lambda}, \nabla_{\Omega}(\lambda)\mathbb{Q}) = \lambda \nabla_{\Omega}(\lambda)\mathbb{Q} = \mathbb{Q}\), by Lemma 3.1(i), so that \(\nabla_{\Omega}(\lambda)\mathbb{Q}\) is a composition factor of \(S(\geq \lambda)\mathbb{Q}\), but not of \(S(> \lambda)\mathbb{Q}\). Now we can write \(S(\geq \lambda)\mathbb{Q} = S(> \lambda) \oplus I\) for some ideal \(I\) which, as a left \(S_{\mathbb{Q}}\)-module, has only the composition factor \(\nabla_{\Omega}(\lambda)\mathbb{Q}\). Hence \(I\) is isomorphic to the matrix algebra \(M_{d}(\mathbb{Q})\), where \(d = \dim \nabla_{\Omega}(\lambda)\mathbb{Q}\), and, as a left \(S_{\mathbb{Q}}\)-module \(S(\geq \lambda)/S(> \lambda)\) is a direct sum of \(d\) copies of \(\nabla_{\Omega}(\lambda)\mathbb{Q}\).

Hence

\[
\dim_{\mathbb{Q}} \lambda V_{\mathbb{Q}}(\lambda) = \dim_{\mathbb{Q}} \text{Hom}_{\text{Sym}(n)}(S\xi_{\lambda}, V_{\mathbb{Q}}(\lambda)) = \dim_{\mathbb{Q}} \text{Hom}_{\text{Sym}(n)}(S\xi_{\lambda}, \nabla_{\Omega}(\lambda)\mathbb{Q}) = d \dim_{\mathbb{Q}} \lambda \nabla_{\Omega}(\lambda)\mathbb{Q} = d.
\]

Thus \(\dim V_{\mathbb{Q}}(\lambda)^{\lambda} \otimes_{\mathbb{Q}} \lambda V_{\mathbb{Q}}(\lambda) = \dim V_{\mathbb{Q}}(\lambda)\) and we have:

the natural map \(V_{\mathbb{Q}}(\lambda)^{\lambda} \otimes_{\mathbb{Q}} \lambda V_{\mathbb{Q}}(\lambda) \to V_{\mathbb{Q}}(\lambda)\) is an isomorphism. (1)

We now consider the integral version. We have the natural surjective map \(V(\lambda)^{\lambda} \otimes_{\mathbb{Z}} \lambda V(\lambda) \to V(\lambda)\). But the rank of \(V(\lambda)^{\lambda}\) is the dimension of \(V(\lambda)^{\lambda}\), the rank of \(\lambda V(\lambda)\) is the dimension of \(\lambda V_{\mathbb{Q}}(\lambda)\), and the rank of \(V(\lambda)\) is the dimension of \(V_{\mathbb{Q}}(\lambda)\) so that, by (1), \(V(\lambda)^{\lambda} \otimes_{\mathbb{Z}} \lambda V(\lambda)\) and \(V(\lambda)\) have the same rank. Thus the surjective map \(V(\lambda)^{\lambda} \otimes_{\mathbb{Z}} \lambda V(\lambda) \to V(\lambda)\) is an isomorphism.

We have shown the following.

**Proposition 5.1.** Assume \(\Omega\) is cosaturated. Then, for each \(\lambda \in \text{Par}(n)\), the map \(V(\lambda)^{\lambda} \otimes_{\mathbb{Z}} \lambda V(\lambda) \to V(\lambda)\) induced by multiplication in \(S\), is an isomorphism.

**Remark 5.2.** If \(k\) is a field then the corresponding algebras \(S_{\Omega, k}\) over \(k\) are Morita equivalent to those considered by Mathas and Soriano in [12]. There they determined blocks of such algebras (for the Schur algebras themselves this was done in [3]), and for the quantised case by Cox in [3]).
6 Cellularity of endomorphism algebras of Young permutation modules

We now establish our main result, namely that the endomorphism algebra of a Young permutation module has the structure of a cellular algebra. We first recall the notion of a cellular algebra due to Graham and Lehrer, [7]. (We have made some minor notational changes to be consistent with the notation above. The most serious of these is the reversal of the partial order from the definition given in [7].)

Definition 6.1. Let $A$ be an algebra over a commutative ring $R$. A cell datum for $(\Lambda^+, N, C, *)$ for $A$ consists of the following.

(C1) A partially ordered set $\Lambda^+$ and for each $\lambda \in \Lambda^+$ a finite set $N(\lambda)$ and an injective map $C : \coprod_{\lambda \in \Lambda^+} N(\lambda) \times N(\lambda) \to A$ with image an $R$-basis of $A$.

(C2) For $\lambda \in \Lambda^+$ and $t, u \in N(\lambda)$ we write $C(t, u) = C^\lambda_{t,u} \in R$. Then $*$ is an $R$-linear anti-involution of $A$ such that $(C^\lambda_{t,u})^* = C^\lambda_{u,t}$.

(C3) If $\lambda \in \Lambda^+$ and $t, u \in N(\lambda)$ then for any element $a \in A$ we have

$$aC^\lambda_{t,u} = \sum_{t' \in N(\lambda)} r_a(t', t)C^\lambda_{t',u} \quad (\text{mod } A(> \lambda))$$

where $r_a(t', t) \in R$ is independent of $u$ and where $A(> \lambda)$ is the $R$-submodule of $A$ generated by $\{C^\mu_{t''',u'''} | \mu \in \Lambda^+, \mu > \lambda$ and $t''', u''' \in N(\mu)\}$.

We say that $A$ is a cellular $R$-algebra if it admits a cell datum.

Let $G$ be a finite group. Let $\Omega$ be a finite $G$-set and let $R$ be a commutative ring. Now $G$ acts on $\Omega \times \Omega$. If $A \subseteq \Omega \times \Omega$ is $G$-stable then we have an element $a_A \in \text{End}_G(R \Omega)$ satisfying

$$a_A(x) = \sum_y y$$

where the sum is over all $y \in \Omega$ such that $(y, x) \in A$. We write $\text{Orb}_G(\Omega \times \Omega)$ for the set of $G$-orbits in $\Omega \times \Omega$. Then $\text{End}_G(R \Omega)$ free over $R$ on basis $a_A$, $A \in \text{Orb}_G(\Omega \times \Omega)$. We have an involution on $\Omega \times \Omega$ defined by $(x, y)^* = (y, x)$, $x, y \in \Omega$. For a $G$-stable subset $A$ of $\Omega \times \Omega$ we write $A^*$ for the $G$-stable set $\{(x, y)^* | (x, y) \in A\}$.

For $A, B \in \text{Orb}_G(\Omega \times \Omega)$ we have

$$a_{A}a_{B} = \sum_{C \in \text{Orb}_G(\Omega \times \Omega)} n_{A,B}^C a_{C}$$

where, for fixed $x \in A$, $y \in B$, the coefficient $n_{A,B}^C$ is the cardinality of the set $\{z \in C | (x, z) \in A$ and $(z, y) \in B\}$. It follows that $\text{End}_G(R \Omega)$ has an involutory anti-automorphism satisfying $a_{A}^* = a_{A^*}$, for a $G$-stable subset
of $\Omega \times \Omega$. The notion of cellularity has built into it an involutory anti-
automorphism $\ast$ and in the case of endomorphism algebras of permutation
modules, we shall always use the one just defined.

We now restrict to the case $G = \text{Sym}(n)$ with $\Omega$ a Young $\text{Sym}(n)$-set as
usual and label by $\mathcal{O}_\alpha$, $\alpha \in \Lambda_{\Omega}$, the $G$-orbits in $\Omega$. Now, for $\alpha \in \Lambda_{\Omega}$
and $x \in \Omega$ we have

$$\xi_\alpha(x) = \begin{cases} x, & \text{if } x \in \mathcal{O}_\alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\xi_\alpha = a_\mathcal{A}$, where $\mathcal{A} = \{ (x, x) | x \in \mathcal{O}_\alpha \}$ and therefore $\xi_\alpha^* = \xi_\alpha$. In
particular we have $\xi_\lambda^* = \xi_\lambda$ for $\lambda \in \zeta(\Omega)$. Thus we also have $S_{\Omega,R}(\sigma)^* = S_{\Omega,R}(\sigma)$, for $\sigma \subseteq \text{Par}(n)$.

Note that if $\Gamma$ is a $G$-stable subset of $\Omega$ then we have the idempotent $e_\Gamma \in S_{\Omega,R}$ given on elements of $\Omega$ by

$$e_\Gamma(x) = \begin{cases} x, & \text{if } x \in \Gamma; \\ 0, & \text{if } x \not\in \Gamma. \end{cases}$$

Thus $e_\Gamma = a_\mathcal{C}$ where $\mathcal{C} = \{ (y, y) | y \in \Gamma \}$ and $e_\Gamma^* = e_\Gamma$.

So now let $\Gamma$ be a Young $\text{Sym}(n)$-set and let $\Omega$ be a co-saturated Young
Sym($n$)-set containing $\Gamma$. We have the idempotent $e = e_\Gamma \in S_{\Omega,R}$ as above
and $S_{\Gamma,R} = \text{End}_{\text{Sym}(n)}(R\Gamma)$ is naturally identified with $eS_{\Omega,R} e$.

**Lemma 6.2.** For $\lambda \in \zeta(\Omega)$ we have $e\nabla_\Omega(\lambda)_Q \neq 0$ if and only if $\lambda \in \zeta^\ast(\Gamma)$.

**Proof.** We have $e = \sum_{\alpha \in \Lambda_{\Gamma}} \xi_\alpha$. Hence $e\nabla_\Omega(\lambda)_Q \neq 0$ if and only if

$$\xi_\alpha \nabla_\Omega(\lambda)_Q \neq 0 i.e. \sum_{\beta \in \Lambda_{\Gamma}} \xi_\alpha \text{Hom}_{\text{Sym}(n)}(M_{\beta,Q}, \text{Sp}(\lambda)_Q) \neq 0, \text{for some } \alpha \in \Lambda_{\Gamma}.$$ 

Hence $e\nabla_\Omega(\lambda)_Q \neq 0$ if and only if $\text{Hom}_{\text{Sym}(n)}(M_{\beta,Q}, \text{Sp}(\lambda)_Q) \neq 0$ for some $\beta \in \Lambda_{\Gamma}$, i.e. if and only if $\text{Hom}_{\text{Sym}(n)}(M(\mu)_Q, \text{Sp}(\lambda)) \neq 0$ for some $\mu \in \zeta(\Gamma)$, i.e. if and only if there exists $\mu \in \zeta(\Gamma)$ such that $\mu \leq \lambda$. \hfill \Box

We fix a partial order $\leq$ on $\zeta(\Omega)$ refining the partial order $\subseteq$.

Let $\lambda \in \zeta(\Omega)$. We have the section $V(\lambda) = S(\geq \lambda)/S(> \lambda)$ of $S = S_{\Omega}$.

We write $J^\text{op}$ for the opposite ring of a ring $J$. We write $S_{\text{env}}$ for the
enveloping algebra $S \otimes_{\mathbb{Z}} S^\text{op}$. We identify an $(S, S)$-bimodule with a left
$S_{\text{env}}$-module in the usual way.

We have the idempotent $\tilde{e} = e \otimes e \in S_{\text{env}}$ and hence the Schur functor $f : \text{mod}(S_{\text{env}}) \to \text{mod}(\tilde{e}S_{\text{env}} \tilde{e})$ as in [10] Chapter 6]. Moreover,

$$\tilde{e}S_{\text{env}} \tilde{e} = eS \otimes_{\mathbb{Z}} (eSe)^\text{op}. \text{ Now } f \text{ is exact so applying it to the isomorphism}$$

$$V(\lambda) \otimes_{\mathbb{Z}} \lambda V(\lambda) \to V(\lambda) \text{ of Proposition 5.1 we obtain an isomorphism}$$

$$eV(\lambda) \otimes_{\mathbb{Z}} \lambda V(\lambda) e \to eV(\lambda)e \quad (1).$$

Now $\xi_\lambda S + S(> \lambda) = (S\xi_\lambda + S(> \lambda))^*$ so that $eV(\lambda)e \neq 0$ if and only if

$eV(\lambda) \neq 0$. Moreover, $V(\lambda) \neq 0$ is a $\mathbb{Z}$-form of $\nabla(\lambda)_Q$ so that $eV(\lambda)e \neq 0$ if
and only if $e\nabla(\lambda)_Q \neq 0$. Hence by, Lemma 6.2.:
We now assemble our cell data. We have the set $\Lambda^+ = \zeta^\vee(\Gamma)$ with partial order induced from the partial order $\leq$ on $\zeta(\Omega)$ (and also denoted $\leq$). Let $\lambda \in \Lambda^+$. We let $n_\lambda = \dim_{\mathbb{Q}} e \nabla(\lambda)_{\mathbb{Q}}$ and set $N(\lambda) = \{1, \ldots, n_\lambda\}$. The rank of $eV(\lambda)_\Lambda$ is $n_\lambda$. We choose elements $d_{\lambda,1}, \ldots, d_{\lambda,n_\lambda}$ of $eS\xi_\lambda$ such that the elements $d_{\lambda,1} + S(\lambda), \ldots, d_{\lambda,n_\lambda} + S(\lambda)$ form a $\mathbb{Z}$-basis of $eV(\lambda)_\Lambda = (eS\xi_\lambda + S(\lambda))/S(\lambda)$. Then $d_{\lambda,1}^*, \ldots, d_{\lambda,n_\lambda}^*$ are elements of $(eS\xi_\lambda)^* = \xi_\lambda N e$ and the elements $d_{\lambda,1}^* + S(\lambda), \ldots, d_{\lambda,n_\lambda}^* + S(\lambda)$ form a $\mathbb{Z}$-basis of $\lambda V(\lambda)_e = (\xi_\lambda Se + S(\lambda))/S(\lambda)$. Thus $d_{\lambda,tud_{\lambda,tu}^*}$ belongs to $eS\xi_\lambda Se$.

We define $C : \prod_{\lambda \in \Lambda^+} N(\lambda) \times N(\lambda) \to eSe$ by $C(t, u) = C_{t, u}^\lambda = d_{\lambda,1}^*, \ldots, d_{\lambda,n_\lambda}^*$, for $t, u \in N(\lambda)$.

Let $M$ be the $\mathbb{Z}$-span of all $C_{t, u}^\lambda$, $\lambda \in \Lambda^+$, $t, u \in N(\lambda)$. We claim that $M = eSe$. We have $S = \sum_{\lambda \in \Lambda_\Omega} S\xi_\lambda$ so that if the claim is false then there exists $\lambda \in \Lambda_\Omega$ such that $eS\xi_\lambda Se \not\subseteq M$. In that case we choose $\lambda$ minimal with this property. First suppose that $\lambda \not\in \zeta^\vee(\Gamma)$. Then we have $eV(\lambda)e = 0$, by (2), i.e., $eS\xi_\lambda Se \subseteq S(\lambda)$ and so $eS\xi_\lambda Se \subseteq eS(\lambda)e$. However, $eS(\lambda)e = \sum_{\mu > \lambda} eS\xi_\mu Se \subseteq M$, by minimality of $\lambda$ and so $eS\xi_\lambda Se \subseteq M$. Thus we have $\lambda \in \Lambda^+ = \zeta^\vee(\Gamma)$.

Now by (1) the map

$$(eS\xi_\lambda + S(\lambda)) \otimes \mathbb{Z} (\xi_\lambda Se + S(\lambda)) \to eS\xi_\lambda Se + S(\lambda)$$

induced by multiplication is surjective. Moreover we have $eS\xi_\lambda + S(\lambda) = \sum_{t=1}^{n_\lambda} \mathbb{Z}d_{\lambda,t} + S(\lambda)$ and $\xi_\lambda Se + S(\lambda) = \sum_{u=1}^{n_\lambda} \mathbb{Z}d_{\lambda,u}^* + S(\lambda)$ so that

$$eS\xi_\lambda Se \subseteq \sum_{t, u=1}^{n_\lambda} \mathbb{Z}d_{\lambda,t}d_{\lambda,u}^* + S(\lambda) = \sum_{t, u=1}^{n_\lambda} \mathbb{Z}C_{t, u}^\lambda + S(\lambda)$$

and hence

$$eS\xi_\lambda Se \subseteq \sum_{t, u=1}^{n_\lambda} \mathbb{Z}C_{t, u}^\lambda + eS(\lambda)e.$$

But now $\sum_{t, u=1}^{n_\lambda} \mathbb{Z}C_{t, u}^\lambda \subseteq M$ by definition and again $eS(\lambda)e \subseteq M$ by the minimality of $\lambda$ so that $eS\xi_\lambda Se \subseteq M$ and the claim is established.

The elements $C_{t, u}^\lambda$, $\lambda \in \Lambda^+$, $1 \leq t, u \leq n_\lambda$ form a spanning set of $eS\xi_\lambda e = S\Gamma$. But the rank of $eSe$ is the $\mathbb{Q}$-dimension of $eS\xi_\lambda e$, i.e., the $\mathbb{Q}$-dimension of $S\Gamma_{\mathbb{Q}}$ and this is $\sum_{\lambda \in \Lambda^+} (\dim e \nabla(\lambda))^2$ by Remark 2.2. Hence the elements $C_{t, u}^\lambda$, with $\lambda \in \Lambda^+$, $1 \leq t, u \leq n_\lambda$, form a $\mathbb{Z}$-basis of $eSe$.

We have now checked the defining properties (C1) and (C2) of cell structure and it remains to check (C3). We fix $\lambda \in \Lambda^+$ and let $1 \leq t, u \leq n_\lambda$. Let $a \in eSe$. Then we have

$$aC_{t, u}^\lambda = ad_{\lambda,t}d_{\lambda,u}^*.$$
Now we have $\sum_{i=1}^{n_\lambda} Zd_{\lambda,i} + S(> \lambda) = eS\xi_\lambda + S(> \lambda)$ so we may write $ad_{\lambda,t} = \sum_{t' = 1}^{n_\lambda} ra(t', t)d_{\lambda,t'} + y$ for some integers $r_a(t', t)$ and an element $y$ of $S(> \lambda)$. Thus we have

$$aC^\lambda_{t,u} = ad_{\lambda,t}d^*_{\lambda,u} = \sum_{t' = 1}^{n_\lambda} ra(t', t)d_{\lambda,t'}d^*_{\lambda,u} + yd^*_{\lambda,u}$$

$$= \sum_{t' = 1}^{n_\lambda} ra(t', t)C^\lambda_{t', u} + yd^*_{\lambda,u}$$

and hence

$$aC^\lambda_{t,u} = \sum_{t' = 1}^{n_\lambda} ra(t', t)C^\lambda_{t', u} \pmod{S(> \lambda)}.$$

We have thus checked defining property (C3) and hence proved the following.

**Theorem 6.3.** Let $\Gamma$ be a Young $\text{Sym}(n)$-set. Then $(\Lambda^+, N, C, *)$ is a cell structure on $\text{St}_{\Gamma, \mathbb{Z}} = eS_{\Omega, \mathbb{Z}}e = \text{End}_{\text{Sym}(n)}(\mathbb{Z}\Gamma)$.

One now obtains a cell structure on $\text{End}_{\text{Sym}(n)}(R\Gamma)$, for any commutative ring $R$ by base change.

There is also the question of when an endomorphism algebra over a field $k$ is quasi-hereditary. If $k$ has characteristic 0 then $\text{End}_{\text{Sym}(n)}(k\Gamma)$ is semisimple and there is nothing to consider. We assume now that the characteristic of $k$ is $p > 0$. By [7, Remark 3.10] (see also [13], [14]) $\text{End}_{\text{Sym}(n)}(k\Gamma)$ is quasi-hereditary if and only if the number of irreducible $\text{End}_{\text{Sym}(n)}(k\Gamma)$-modules (up to isomorphism) is equal to the length of the cell chain, i.e., $|\zeta^p(\Gamma)|$. By Lemma 2.4, the number of irreducible $\text{End}_{\text{Sym}(n)}(k\Gamma)$-modules is $|\zeta^p(\Gamma)|$. Moreover, we have $\zeta^p(\Gamma) \subseteq \zeta(\Gamma)$ and so $\text{End}_{\text{Sym}(n)}(k\Gamma)$ is quasi-hereditary if and only if $\zeta(\Gamma) \subseteq \zeta^p(\Gamma)$. We spell this out in the following result.

**Theorem 6.4.** Let $k$ be a field of characteristic $p > 0$ and let $\Gamma$ be a Young $\text{Sym}(n)$-set. Then the endomorphism algebra $\text{End}_{\text{Sym}(n)}(k\Gamma)$ of the permutation module $k\Gamma$ is quasi-hereditary if and only if for every partition $\lambda$ of $n$ such that the Young subgroup $\text{Sym}(\lambda)$ appears as the stabiliser of a point of $\Gamma$ and every partition $\mu \supseteq \lambda$ there exists a partition $\tau$ such that $\text{Sym}(\tau)$ appears as a point stabiliser and such that $\mu$ $p$-dominates $\tau$, i.e., there exists a weak $p$ expansion $\tau = \sum_{i \geq 0} p^i \gamma(i)$, with $\gamma(i) \in \Lambda(n)$, and $\gamma(i) \leq \mu(i)$ for all $i$ (where $\mu = \sum_{i \geq 0} p^i \mu(i)$ is the base $p$-expansion of $\mu$ and where $\gamma(i)$ is the partition obtained by writing the parts of $\gamma(i)$ in descending order, for $i \geq 0$).

**Remark 6.5.** We emphasise that the above gives a criterion for the endomorphism algebra $\text{End}_{\text{Sym}(n)}(k\Gamma)$ of the Young permutation module $k\Gamma$ to be quasi-hereditary with respect to any labelling of the simple modules by a partially ordered set (which may have nothing to do with those considered
above) thanks to the result of König and Xi, [14, Theorem 3]. Thus if \( \Gamma \) does not satisfy the condition above then \( S_{I,k} \) can not have finite global dimension by [14, Theorem 3] and hence is not quasi-hereditary.

7 Example: Tensor Powers

Let \( R \) be a commutative ring and let \( E_R \) be a free \( R \)-module on basis \( e_{1,R}, \ldots, e_{n,R} \). Let \( r \) be a positive integer and let \( I(n, r) \) be the set described in Example 3.2. Then the \( r \)-fold tensor product \( E_R^{\otimes r} = E_R \otimes_R \cdots \otimes_R E_R \) has \( R \)-basis \( e_{i,R} = e_{i_1,R} \otimes \cdots \otimes e_{i_r,R}, i \in I(n, r), \) and we thus identify \( E_R^{\otimes r} \) with \( RI(n, r) \), the free \( R \)-module on \( I(n, r) \).

**Remark 7.1.** The symmetric group \( \text{Sym}(r) \) acts on \( E_R^{\otimes r} \) by place permutations, i.e. \( w \cdot e_{i,R} = e_{i \circ w^{-1},R} \), for \( w \in \text{Sym}(r) \), \( i \in I(n, r) \). Thus we may regard \( E_R^{\otimes r} \) as the permutation module \( RI(n, r) \), with \( \text{Sym}(r) \), acting on \( I(n, r) \) by \( w \cdot i = i \circ w^{-1} \). The endomorphism algebra \( \text{End}_{\text{Sym}(r)}(E_R^{\otimes r}) \) is the Schur algebra \( S_R(n, r) \).

The stabiliser of \( i \in I(n, r) \) is the direct product of the symmetric groups on the fibres of \( i \) (regarded as a subgroup of \( \text{Sym}(r) \) in the usual way). Hence \( I(n, r) \) is a Young \( \text{Sym}(r) \)-set. Hence \( E_R^{\otimes r} \) is a Young permutation module and hence \( S_R(n, r) \) is cellular. Moreover, \( \zeta(I(n, r)) \) is the set \( \Lambda^+(n, r) \) of all partitions of \( r \) with at most \( n \) parts. This is a co-saturated set and hence for a prime \( p \) we have \( \zeta(I(n, r)) = \zeta^p(I(n, r)) = \zeta^{\leq r}(I(n, r)) \). Hence, for a field \( k \) of characteristic \( p \) the Schur algebra \( S_k(n, r) \) is quasi-hereditary.

However, this is not a new proof since our treatment relies crucially on a detail from Green’s analysis of \( S_\leq(n, r) \) as in [4], at least in the case \( n = r \). (See Example 3.2 above and the proofs of the results of Section 4.)

We now regard \( E_R \) as an \( R \text{Sym}(n) \)-module with \( \text{Sym}(n) \) permuting the basis \( e_{1,R}, \ldots, e_{n,R} \) in the natural way. This action induces an action on the tensor product \( E_R^{\otimes r} \). Specifically, we have \( w \cdot e_{i,R} = e_{w \circ i,R} \), for \( w \in \text{Sym}(n) \), \( i \in I(n, r) \), and we thus regard \( E_R^{\otimes r} \) as the permutation module \( RI(n, r) \). For \( w \in \text{Sym}(n) \), \( i \in I(n, r) \) we have \( w \circ i = i \) if and only if \( w \) acts as the identity on the image of \( i \), so that the stabiliser of \( i \) is the group of symmetries of the complement of the image of \( i \) in \( \{1, \ldots, n\} \), identified with a subgroup of \( \text{Sym}(n) \) in the usual way. Thus \( I(n, r) \) is a Young \( \text{Sym}(n) \)-set so we have the following consequence of Theorem 6.3, answering a question raised in [1].

**Proposition 7.2.** The endomorphism algebra \( \text{End}_{\text{Sym}(n)}(E_R^{\otimes r}) = \text{End}_{\text{Sym}(n)}(RI(n, r)) \) is a cellular algebra.

The support of \( I(n, r) \) consists of hook partitions, more precisely we have

\[
\zeta(I(n, r)) = \{(a, 1^b) \mid a + b = n, 1 \leq b \leq r\}.
\]
Hence we have
\[ \zeta^E(I(n,r)) = \{ \lambda = (\lambda_1, \lambda_2, \ldots) \in \text{Par}(n) \mid \lambda_1 \geq n - r \}. \]

Let \( k \) be a field of characteristic \( p > 0 \). Then \( \text{End}_{\text{Sym}(n)}(E_k^{\otimes r}) \) is quasi-hereditary if and only if \( \zeta^E(I(n,r)) \subseteq \zeta^E(I(n,r)) \), i.e., if and only for every \( \mu = (\mu_1, \mu_2, \ldots) \in \text{Par}(n) \) with \( \mu_1 \geq n - r \) there exists some \( \lambda = (a, 1^b) \), \( 1 \leq b \leq r \), such that \( \lambda \leq_p \mu \).

We are able to give an explicit list of quasi-hereditary algebras arising in the above manner.

**Proposition 7.3.** Let \( k \) be a field of characteristic \( p > 0 \). Let \( n \) be a positive integer and \( E \) an \( n \)-dimensional \( k \)-vector space with basis \( e_1, \ldots, e_n \).

We regard \( E \) as a \( k\text{Sym}(n) \)-module with \( \text{Sym}(n) \) permuting the basis in the obvious way. For \( r \geq 1 \) we regard the \( r \)th tensor power \( E^{\otimes r} \) as a \( k\text{Sym}(n) \)-module via the usual tensor product action. Then \( \text{End}_{\text{Sym}(n)}(E^{\otimes r}) \) is quasi-hereditary if and only if:

(i) \( p \) does not divide \( n \); and

(ii) either \( n < 2p \) (and \( r \) is arbitrary) or \( n > 2p \) and \( r < p \).

**Proof.** We see this in a number of steps. We regard \( E^{\otimes r} \) as the permutation module \( kI(n,r) \), as above, with \( \text{Sym}(n) \) action by \( w \cdot i = w \circ i \), for \( w \in \text{Sym}(n), i \in I(n,r) \). We shall say that \( I(n,r) \) is quasi-hereditary if \( \text{End}_{\text{Sym}(n)}(E^{\otimes r}) \) is.

**Step 1.** If \( p \) divides \( n \) then \( I(n,r) \) is not quasi-hereditary.

We have \((n-1,1) \in \zeta(I(n,r))\) and \((n,0) \leq (n-1,1)\) so that \((n,0) \in \zeta^E(I(n,r))\).

Now \( n = pm \), for some positive integer \( m \), so that \( \mu = (n,0) = p(m,0) \) has base \( p \) expansion \((n,0) = \sum_{i \geq 0} p^i \mu(i)\), with restricted part \( \mu(0) = 0 \). Thus if \( \tau = (a, 1^b) \) has weak \( p \)-expansion \( \tau = \sum_{i \geq 0} p^i \gamma(i) \) and \( \gamma(0) \leq \mu(i) \), for all \( i \), then \( \gamma(0) = 0 \) and \( \tau \) is divisible by \( p \). However, this is not the case so no such weak \( p \)-expansion exists and \( \mu \in \zeta^E(I(n,r)) \setminus \zeta^E(I(n,r)) \). Thus \( \zeta^E(I(n,r)) \neq \zeta^E(I(n,r)) \) and \( I(n,r) \) is not quasi-hereditary.

**Step 2.** If \( p \) does not divide \( n \) then \( I(n,1) \) is quasi-hereditary.

We have \( \zeta(I(n,1)) = \{(n-1,1)\} \). If \( \mu \in \zeta^E(I(n,1)) \setminus \zeta^E(I(n,r)) \) then \( \mu = (n,0) \).

Now \( n \) has base \( p \) expansion \( n = \sum_{i \geq 0} p^i n_i \), with \( 0 \leq n_i < p \) for all \( i \geq 0 \) and \( n_0 \neq 0 \) and \( \mu \) has base \( p \) expansion \( \mu = \sum_{i \geq 0} p^i \mu(i) \), with \( \mu(i) = (n_i,0) \), for all \( i \geq 0 \).

But now we write
\[
\tau = (n-1,1) = (n_0 - 1,1) + \sum_{i \geq 1} p^i (n_i,0)
\]
and $\tau$ has weak $p$-expansion $\tau = \sum_{i\geq 0} p^i \gamma(i)$, with $\gamma(0) = (n_0 - 1, 1)$, $\gamma(i) = (n_i, 0)$ for $i \geq 1$. Moreover $\gamma(i) \leq \mu(i)$, for all $i$ so that $(n, 0) \in \zeta^{\geq r}(I(n, 1))$. Thus $\zeta^{\geq r}(I(n, 1)) = \zeta^{\geq r}(I(n, 1))$ and $I(n, 1)$ is quasi-hereditary.

Step 3. If $\mu \in \zeta^{\geq r}(I(n, r))$ is $p$-restricted then $\mu \in \zeta^{\geq r}(I(n, r))$

We have $\mu \geq (a, b)$ for some $n = a + b$, $1 \leq b \leq r$. The partition $\mu$ has base $p$ expansion $\mu = \sum_{i\geq 0} p^i \mu(i)$, with $\mu(i) = 0$ for all $i \geq 1$.

But now $\tau = (a, 1^b)$ has week $p$-expansion $\tau = \sum_{i\geq 0} p^i \gamma(i)$, with $\gamma(0) = (a, 1^b)$ and $\gamma(i) = 0$ for all $i \geq 1$. Furthermore we have $\gamma(i) \leq \mu(i)$ for all $i \geq 0$ so $\mu \in \zeta^{\geq r}(I(n, r))$.

Step 4. If $n < p$ then $I(n, r)$ is quasi-hereditary.

This follows from Step 3 all since elements of $\text{Par}(n)$ are restricted.

Step 5. If $p < n < 2p$ then $I(n, r)$ is quasi-hereditary.

For a contradiction suppose not and let $\mu = (\mu_1, \mu_2, \ldots) \in \zeta^{\geq r}(I(n, n)) \cap \zeta^{\geq r}(I(n, r))$. We have $\mu \geq (a, 1^b)$ for some $a, b$ with $n = a + b$, $1 \leq b \leq r$. Choose $a, b$ with this property with $b \geq 1$ minimal. If $b = 1$ then $\mu \in \zeta^{\geq r}(I(n, 1))$, which by Step 2 is $\zeta^{\geq r}(I(n, 1))$. Thus we have $b \geq 2$.

We claim that $\mu_1 = a$. Since $\mu \geq (a, 1^b)$ the length $l$, say, of $\mu$ is at most $l = (a, 1^b)$, i.e. $b + 1$. Put $\xi = (\xi_1, \xi_2, \ldots) = (a + 1, 1^{b-1})$. If $\mu_1 > a$ then $\mu_1 \geq \xi_1$ and, for $1 < i \leq l$, we have

$$\mu_1 + \cdots + \mu_i \geq a + 1 + (i - 1) = a + i = \xi_1 + \cdots + \xi_i.$$ 

So $\mu \geq \xi = (a + 1, 1^{b-1})$, which is a contradiction, and the claim is established.

Note that $\mu$ is non-restricted, by Step 3, and, since $\mu$ is a partition of $n < 2p$ in the base $p$ expansion $\mu = \sum_{i\geq 0} p^i \mu(i)$ of $\mu$, we must have $\mu(1) = (1, 0)$ and $\mu(i) = 0$ for $i \geq 2$. Let $\tau = (a, 1^b)$. Then $\tau \leq \mu$ implies that $\tau - (p, 0) \leq \mu - (p, 0) = \mu(0)$. But now

$$\tau = (a, 1^b) = (a - p, 1^b) + p(1, 0)$$

so we have the weak $p$ expansion $\tau = \sum_{i\geq 0} p^i \gamma(i)$ with $\gamma(0) = (a - p, 1^b)$, $\gamma(1) = (1, 0)$ and $\gamma(i) = 0$ for $i > 1$. Since $\gamma(i) \leq \mu(i)$ for all $i \geq 0$ we have $(a, 1^b) \leq_p \mu$ and so $\mu \in \zeta^{\geq r}(I(n, r))$, a contradiction.

Step 6. If $n > 2p$ and $r \geq p$ then $I(n, r)$ is not quasi-hereditary.

Note that $\zeta(I(n, r))$ contains $(n - p, 1^p)$ and hence $\zeta^{\geq r}(I(n, r))$ contains $\mu = (n - p, p)$. Now we have $\mu = (n - 2p, 0) + p(1, 1)$ and so $\mu = \mu(0) + p \xi$, where $\mu(0)$ has at most one part and $\xi$ has two parts. Hence in the base $p$ expansion $\mu = \sum_{i\geq 0} p^i \mu(i)$, there is for some $j \geq 1$, such that $\mu(j)$ has two parts.
Now if $\mu \in \zeta^\varphi(I(n,r))$ there there exists some $\tau = (a,1^b)$ with weak $p$ expansion $\tau = \sum_{i \geq 0} p^i \gamma(i)$ such that $\gamma(i) \leq \mu(i)$ for all $i \geq 0$. But then $\gamma(j)$ must have at least two parts. Since $j \geq 1$, the partition $\tau = (a,1^b)$ has two parts of size at least $p$. This is not the case so there is no such weak $p$ expansion and $\mu \not\in \zeta^\varphi(I(n,r))$. Thus $\zeta^\varphi(I(n,r)) \neq \zeta^\varphi(I(n,r))$ and $I(n,r)$ is not quasi-hereditary.

Step 7. If $n > 2p$, if $p$ does not divide $n$ and if $r < p$, then $I(n,r)$ is quasi-hereditary.

If not there exists $\mu = (\mu_1, \mu_2, \ldots) \in \zeta^\varphi(I(n,r)) \setminus \zeta^\varphi(I(n,r))$. Thus $\mu \geq (a,1^b)$, for some $n = a + b$, $b \geq 1$ and, as in Step 5, we choose such $(a,1^b)$ with $b$ minimal. Again, by Step 2, we have $b \geq 2$.

We claim that $\mu_1 = a$. If not, we get $\mu \geq (a + 1,1^{b-1})$ as in Step 5, contradicting the minimality of $b$.

Thus we have $\mu_2 + \cdots + \mu_n = n - \mu_1 = b < p$, in particular we have $\mu_i < p$ for all $i \geq 1$. Hence in the base $p$ expansion $\mu = \sum_{i \geq 0} p^i \mu(i)$, for all $i \geq 1$ we have $\mu(i) = (c_i,0,\ldots,0)$, for some $0 \leq c_i < p$. Also, $\mu(0) = (k,\mu_2,\ldots,\mu_n)$, for some $k > 0$.

Now we have

$$\tau = (a,1^b) = (k + \sum_{i \geq 1} p^i c_i,1^b) = (k,1^b) + \sum_{i \geq 1} p^i (c_i,0,\ldots,0).$$

Thus we have the weak $p$-expansion $\tau = \sum_{i \geq 0} p^i \gamma(i)$, with $\gamma(0) = (k,1^b)$ and $\gamma(i) = (c_i,0,\ldots,0)$, for $i \geq 1$. Furthermore, $\gamma(i) \leq \mu(i)$, for all $i \geq 1$ so that $\mu \in \zeta^\varphi(I(n,r))$ and therefore $\zeta^\varphi(I(n,r)) = \zeta^\varphi(I(n,r))$ and $I(n,r)$ is quasi-hereditary.

Let $k$ be a field. Recall that, for $\delta \in k$, and $r$ a positive integer we have the partition algebra $P_r(\delta)$ over $k$. One may find a detailed account of the construction and properties of $P_r(\delta)$ in for example the papers by Paul P. Martin, [16], [17], and [11], [1]. Suppose now that $k$ has characteristic $p > 0$ and $\delta = n1_k$, for some positive integer $n$. Let $E_n$ be an $n$-dimensional vector space with basis $e_1,\ldots,e_n$. Then $P_r(n) = P_r(n1_k)$ acts on $E_n^{\otimes r}$. By a result of Halverson-Ram, [11] Theorem 3.6 the image of the representation $P_r(n) \to \text{End}_k(E_n^{\otimes r})$ is $\text{End}_{\text{Sym}(n)}(E_n^{\otimes r})$. Moreover, for $n \gg 0$ the action of $P_r(n)$ is faithful. Let $N = n + ps$, for $s$ suitably large, so that $P_r(n) = P_r(N)$ acts faithfully on $E_N^{\otimes r}$. Thus $P_r(n)$ is quasi-hereditary if and only if $\text{End}_{\text{Sym}(N)}(E_N^{\otimes r})$ is faithful. Hence from Proposition 7.3 we have the following, which is a special case of a result of König and Xi, [14] Theorem 1.4.

**Corollary 7.4.** The partition algebra $P_r(n)$ is quasi-hereditary if and only if $n$ is prime to $p$ and $r < p$. 

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