ADDITIVE UNIT REPRESENTATIONS IN ENDOMORPHISM RINGS AND AN EXTENSION OF A RESULT OF DICKSON AND FULLER

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Dedicated to T. Y. Lam on his 70th Birthday

Abstract. A module is called automorphism-invariant if it is invariant under any automorphism of its injective hull. Dickson and Fuller have shown that if $R$ is a finite-dimensional algebra over a field $F$ with more than two elements then an indecomposable automorphism-invariant right $R$-module must be quasi-injective. In this note, we extend and simplify the proof of this result by showing that any automorphism-invariant module over an algebra over a field with more than two elements is quasi-injective. Our proof is based on the study of the additive unit structure of endomorphism rings.

1. Introduction.

The study of the additive unit structure of rings has a long tradition. The earliest instance may be found in the investigations of Dieudonné on Galois theory of simple and semisimple rings [4]. In [6], Hochschild studied additive unit representations of elements in simple algebras and proved that each element of a simple algebra over any field is a sum of units. Later, Zelinsky [15] proved that every linear transformation of a vector space $V$ over a division ring $D$ is the sum of two invertible linear transformations except when $V$ is one-dimensional over $F_2$. Zelinsky also noted in his paper that this result follows from a previous result of Wolfson [14].

The above mentioned result of Zelinsky has been recently extended by Khurana and Srivastava in [8] where they proved that any element in the endomorphism ring of a continuous module $M$ is a sum of two automorphisms if and only if $\text{End}(M)$ has no factor ring isomorphic to the field of two elements $F_2$. In particular, this means that, in order to check if a module $M$ is invariant under endomorphisms of its injective

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hull $E(M)$, it is enough to check it under automorphisms, provided that $\text{End}(E(M))$ has no factor ring isomorphic to $\mathbb{F}_2$. Recall that a module $M$ is called quasi-injective if every homomorphism from a submodule $L$ of $M$ to $M$ can be extended to an endomorphism of $M$. Johnson and Wong characterized quasi-injective modules as those that are invariant under any endomorphism of their injective hulls [7].

A module $M$ which is invariant under automorphisms of its injective hull is called an automorphism-invariant module. This class of modules was first studied by Dickson and Fuller in [3] for the particular case of finite-dimensional algebras over fields $\mathbb{F}$ with more than two elements. They proved that if $R$ is a finite-dimensional algebra over a field $\mathbb{F}$ with more than two elements then an indecomposable automorphism-invariant right $R$-module must be quasi-injective. And it has been recently shown in [11] that this result fails to hold if $\mathbb{F}$ is a field of two elements. Let us recall that a ring $R$ is said to be of right invariant module type if every indecomposable right $R$-module is quasi-injective. Thus, the result of Dickson and Fuller states that if $R$ is a finite-dimensional algebra over a field $\mathbb{F}$ with more than two elements, then $R$ is of right invariant module type if and only if every indecomposable right $R$-module is automorphism-invariant. Examples of automorphism-invariant modules which are not quasi-injective, can be found in [5] and [13]. And recently, it has been shown in [5] that a module $M$ is automorphism-invariant if and only if every monomorphism from a submodule of $M$ extends to an endomorphism of $M$. For more details on automorphism-invariant modules, see [5], [9], [11], and [12].

The purpose of this note is to exploit the above mentioned result of Khurana and Srivastava in [8] in order to extend, as well as to give a much easier proof, of Dickson and Fuller’s result by showing that if $M$ is any right $R$-module such that there are no ring homomorphisms from $\text{End}_R(M)$ into the field of two elements $\mathbb{F}_2$, then $M_R$ is automorphism-invariant if and only if it is quasi-injective. In particular, we deduce that if $R$ is an algebra over a field $\mathbb{F}$ with more than two elements, then a right $R$-module $M$ is automorphism-invariant if and only if it is quasi-injective.

Throughout this paper, $R$ will always denote an associative ring with identity element and modules will be right unital. We refer to [1] for any undefined notion arising in the text.
EXTENSION OF A RESULT OF DICKSON AND FULLER

Results.

We begin this section by proving a couple of lemmas that we will need in our main result.

**Lemma 1.** Let $M$ be a right $R$-module such that $\text{End}(M)$ has no factor isomorphic to $\mathbb{F}_2$. Then $\text{End}(E(M))$ has no factor isomorphic to $\mathbb{F}_2$ either.

**Proof.** Let $M$ be any right $R$-module such that $\text{End}(M)$ has no factor isomorphic to $\mathbb{F}_2$ and let $S = \text{End}(E(M))$. We want to show that $S$ has no factor isomorphic to $\mathbb{F}_2$. Assume to the contrary that $\psi : S \to \mathbb{F}_2$ is a ring homomorphism. As $\mathbb{F}_2 \cong \text{End}_Z(\mathbb{F}_2)$, the above ring homomorphism yields a right $S$-module structure to $\mathbb{F}_2$. Under this right $S$-module structure, $\psi : S \to \mathbb{F}_2$ becomes a homomorphism of $S$-modules. Moreover, as $\mathbb{F}_2$ is simple as $\mathbb{Z}$-module, so is as right $S$-module. Therefore, $\ker(\psi)$ contains the Jacobson radical $J(S)$ of $S$ and thus, it factors through a ring homomorphism $\psi' : S/J(S) \to \mathbb{F}_2$.

On the other hand, given any endomorphism $f : M \to M$, it extends by injectivity to a (non-unique) endomorphism $\varphi_f : E(M) \to E(M)$

\[
\begin{array}{ccc}
M & \xrightarrow{f} & M \\
\downarrow & & \downarrow \\
E(M) & \xrightarrow{\varphi_f} & E(M).
\end{array}
\]

Now define $\eta : \text{End}(M) \to \frac{S}{J(S)}$ by $\eta(f) = \varphi_f + J(S)$. It may be easily checked that $\eta$ is a ring homomorphism. Clearly, then $\eta \circ \psi' : \text{End}(M) \to \mathbb{F}_2$ is a ring homomorphism. This shows that $\text{End}(M)$ has a factor isomorphic to $\mathbb{F}_2$, a contradiction to our hypothesis. Hence, $\text{End}(E(M))$ has no factor isomorphic to $\mathbb{F}_2$. \qed

**Lemma 2.** ([8]) Let $M$ be a continuous right module over any ring $S$. Then each element of the endomorphism ring $R = \text{End}(M_S)$ is the sum of two units if and only if $R$ has no factor isomorphic to $\mathbb{F}_2$.

We can now prove our main result.

**Theorem 3.** Let $M$ be any right $R$-module such that $\text{End}(M)$ has no factor isomorphic to $\mathbb{F}_2$, then $M$ is quasi-injective if and only if $M$ is automorphism-invariant.

**Proof.** Let $M$ be an automorphism-invariant right $R$-module such that $\text{End}(M)$ has no factor isomorphic to $\mathbb{F}_2$. Then by Lemma 1, $\text{End}(E(M))$
has no factor isomorphic to \( \mathbb{F}_2 \). Now by Lemma 2, each element of \( \text{End}(E(M)) \) is a sum of two units. Therefore, for every endomorphism \( \lambda \in \text{End}(E(M)) \), we have \( \lambda = u_1 + u_2 \) where \( u_1, u_2 \) are automorphisms in \( \text{End}(E(M)) \). As \( M \) is an automorphism-invariant module, it is invariant under both \( u_1 \) and \( u_2 \), and we get that \( M \) is invariant under \( \lambda \). This shows that \( M \) is quasi-injective. The converse is obvious. \( \square \)

**Lemma 4.** Let \( R \) be any ring and \( S \), a subring of its center \( Z(R) \). If \( \mathbb{F}_2 \) does not admit a structure of right \( S \)-module, then for any right \( R \)-module \( M \), the endomorphism ring \( \text{End}(M) \) has no factor isomorphic to \( \mathbb{F}_2 \).

*Proof.* Assume to the contrary that there is a ring homomorphism \( \psi : \text{End}_R(M) \to \mathbb{F}_2 \). Now, define a map \( \varphi : S \to \text{End}_R(M) \) by the rule \( \varphi(r) = \varphi_r \), for each \( r \in S \), where \( \varphi_r : M \to M \) is given as \( \varphi_r(m) = mr \). Clearly \( \varphi \) is a ring homomorphism since \( S \subseteq Z(R) \) and so, the composition \( \varphi \circ f \) gives a nonzero ring homomorphism from \( S \) to \( \mathbb{F}_2 \), yielding a contradiction to the assumption that \( \mathbb{F}_2 \) does not admit a structure of right \( S \)-module. \( \square \)

We can now extend the above mentioned result of Dickson and Fuller.

**Theorem 5.** Let \( A \) be an algebra over a field \( \mathbb{F} \) with more than two elements. Then any right \( A \)-module \( M \) is automorphism-invariant if and only if \( M \) is quasi-injective.

*Proof.* Let \( M \) be an automorphism-invariant right \( A \)-module. Since \( A \) is an algebra over a field \( \mathbb{F} \) with more than two elements, by Lemma 4 it follows that \( \mathbb{F}_2 \) does not admit a structure of right \( Z(A) \)-module and therefore \( \text{End}(M) \) has no factor isomorphic to \( \mathbb{F}_2 \). Now, by Theorem 3 \( M \) must be quasi-injective. The converse is obvious. \( \square \)

As a consequence we have the following

**Corollary 6.** Let \( R \) be any algebra over a field \( \mathbb{F} \) with more than two elements. Then \( R \) is of right invariant module type if and only if every indecomposable right \( R \)-module is automorphism-invariant.

**Corollary 7.** If \( A \) is an algebra over a field \( \mathbb{F} \) with more than two elements such that \( A \) is automorphism-invariant as a right \( A \)-module, then \( A \) is right self-injective.

It is well-known that a group ring \( R[G] \) is right self-injective if and only if \( R \) is right self-injective and \( G \) is finite (see [2], [10]). Thus, in particular, we have the following

**Corollary 8.** Let \( K[G] \) be automorphism-invariant, where \( K \) is a field with more than two elements. Then \( G \) must be finite.
References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, New York, 1992.

[2] I. G. Connell, *On the Group Ring*, Can. J. Math. 15 (1963), 650-685.

[3] S. E. Dickson, K. R. Fuller, *Algebras for which every indecomposable right module is invariant in its injective envelope*, Pacific J. Math., vol. 31, no. 3 (1969), 655-658.

[4] J. Dieudonné, *La théorie de Galois des anneaux simples et semi-simples*, Comment. Math. Helv., 21 (1948), 154-184.

[5] N. Er, S. Singh, A. K. Srivastava, *Rings and modules which are stable under automorphisms of their injective hulls*, J. Algebra, 379 (2013), 223-229.

[6] G. Hochschild, *Automorphisms of simple algebras*, Trans. Amer. Math. Soc. 69 (1950), 292-301.

[7] R. E. Johnson, E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc. 36 (1961), 260-268.

[8] D. Khurana and A. K. Srivastava, *Right self-injective rings in which each element is sum of two units*, J. Alg. Appl., vol. 6, no. 2 (2007), 281-286.

[9] T. K. Lee, Y. Zhou, *Modules which are invariant under automorphisms of their injective hulls*, J. Alg. Appl., to appear.

[10] G. Renault, *Sur les anneaux de groupes*, C. R. Acad. Sci. Paris Sér. A-B 273 (1971), 84-87.

[11] S. Singh, A. K. Srivastava, *Dual automorphism-invariant modules*, J. Algebra, 371 (2012), 262-275.

[12] S. Singh and A. K. Srivastava, *Rings of invariant module type and automorphism-invariant modules* Contemp. Math., Amer. Math. Soc., to appear (available on [http://arxiv.org/pdf/1207.5370.pdf](http://arxiv.org/pdf/1207.5370.pdf))

[13] M. L. Teply, *Pseudo-injective modules which are not quasi-injective*, Proc. Amer. Math. Soc., vol. 49, no. 2 (1975), 305-310.

[14] K. G. Wolfson, *An ideal theoretic characterization of the ring of all linear transformations*, Amer. J. Math. 75 (1953), 358-386.

[15] D. Zelinsky, *Every Linear Transformation is Sum of Nonsingular Ones*, Proc. Amer. Math. Soc. 5 (1954), 627-630.

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