Abstract. In this work, we provide a \(q\)-generalization of flexible algebras and related bialgebraic structures, including center-symmetric (also called antiflexible) algebras, and their bialgebras. Their basic properties are derived and discussed. Their connection with known algebraic structures, previously developed in the literature, is established. A \(q\)-generalization of Myung theorem is given. Main properties related to bimodules, matched pairs and dual bimodules as well as their algebraic consequences are investigated and analyzed. Finally, the equivalence between \(q\)-generalized flexible algebras, their Manin triple and bialgebras is established.

Keywords. Lie algebra, Lie-admissible algebra, flexible algebra, antiflexible algebra, center-symmetric algebra, matched pair, Manin triple, bialgebra.

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1. Introduction

Alternative algebras were introduced by Zorn [29] who established their fundamental identities, studied their nucleus by modifying the characteristic of the field, and investigated their Lie admissibility using the corresponding Jacoby identity. Furthermore, Zorn derived their power associativity conditions. Later, Schafer [26] gave a new formulation of these algebras in terms of left and right multiplication operators, and in terms of division algebras of degree two. He also provided the isotopes of these algebras. Santilli [27] introduced Lie admissible algebras and gave their basic properties. He extended his study to mutation algebras, examined their relation to associative algebras, Lie algebras, Jordan and special Jordan algebras, and established the passage from one type of algebra to another by using a hexahedron with oriented edges. Radicals of flexible Lie admissible algebras were introduced, and some of their properties were established and discussed in [6]. Classes of flexible Lie admissible algebras were also investigated and discussed in [18]. Albert in [1] elaborated fundamental concepts, and studied the isotropy of nonassociative
algebras. Simple and semi-simple algebras, and their characterization from nonassociative algebraic structures were developed and discussed in [2]. For more details, see a self-contained book by Shafer [25] addressing a nice compilation of basic properties of nonassociative algebras.

Contrarily to Lie algebras, and except for some classification based on the characteristics of closed fields (see [7] and references therein), a full classification of nonassociative algebras still remains a tremendous task. Some interesting properties and algebraic identities of antiflexible structures were investigated and discussed in [8]. The properties of simple, semisimple and nearly semisimple antiflexible algebras were also derived and analyzed in [22,24].

Among the nonassociative algebras, the alternative algebras, with an associator preserving certain symmetry by exchanging its elements, play a central role in both mathematics and physics as they possess interesting properties. A nice repertory of their applications in physics, including gauge theory and Yang-Mills gauge theory formulation from nonassociative algebras can be found in [17, (and references therein). A theory of nuclear boson-expansion for odd-fermion system in the context of nonassociative algebras was also examined in [21]. A study on quark structure and octonion algebras was performed in [11]. Further, a generalization of the classical Hamiltonian dynamics to a three-dimensional phase space, generating equations of motion with two Hamiltonians and three canonical variables, was performed with an analog of Poisson bracket realized by means of the associator of nonassociative algebras [19].

Besides, flexible algebras were also investigated in terms of degree of algebras [14]. Other characterizations and applications of nonassociative algebras can be found in [17, 15 and 12 (and references therein).

Similarly to algebraic properties of quantum groups developed by Drinfeld [10], some nonassociative algebras possess interesting identities with applications in physics, and generate the so-called associative or classical Yang-Baxter equations [3,9,13, (and references therein). Furthermore, the bialgebras constructed from Jordan algebras [28] are related to the Lie bialgebras. The left-symmetric algebras, also called pre-Lie algebras [5], are known as Lie admissible algebras, and admit the left multiplication operator as a representation. They can also be used to produce symplectic Lie algebras, while their coboundary bialgebras lead to the identity known as S-equation, and generate para-Kähler Lie algebras. The case of associative algebras also furnished remarkable properties investigated by Aguiar [3] and Bai [4]. The center-symmetric algebras studied in [12] are also Lie admissible algebras.

The present work addresses a q-generalization of flexible algebras and related bialgebraic structures, including center-symmetric (also called antiflexible) algebras, and their bialgebras. Their basic properties are derived and discussed. Their connection with known algebraic structures existing in the literature is established. A q-generalization of Myung theorem is given. Main properties related to bimodules, matched pairs and dual bimodules, and their algebraic consequences are investigated and analyzed. Finally, the Manin triple of q-generalized flexible algebras, and its link to q-generalized flexible bialgebras are built together with the equivalence with the matched pair of q-generalized flexible algebras.

2. Basic properties of a q-generalized flexible algebra

In this section, a q-generalization of algebras encompassing flexible, anti-flexible and associative algebras is provided. New classes of algebras are induced. Their relevant properties and link with known algebras are derived. Jordan identity and Lie admissibility condition are also established.

**Definition 2.1.** Let $\mathcal{A}(q)$, where $q \in \mathbb{K}$, be a finite dimensional vector space. The couple $(\mathcal{A}(q), \cdot)$ is called a q-generalized flexible algebra if, for all $x, y, z \in \mathcal{A}(q)$, the following relation is satisfied:

\[(x, y, z) = q(z, y, x),\]

or, equivalently,

\[\mu \circ (\mu \otimes \text{id}) + q(\mu \circ \tau) \circ ((\mu \circ \tau) \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu) + q(\mu \circ \tau) \circ (\text{id} \otimes (\mu \circ \tau))\]
where \((x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)\) is the associator of the bilinear product \("\cdot\"\) on \(\mathcal{A}(q)\); \(\mu\) is defined by \(\mu(x, y) = x \cdot y\); \(\text{id}\) is the identity map on \(\mathcal{A}(q)\); and \(\tau\) stands for the exchange map on \(\mathcal{A}(q)\) given by \(\tau(x \otimes y) = y \otimes x\).

This can be described by the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{A}(q) \otimes \mathcal{A}(q) \otimes \mathcal{A}(q) & \stackrel{\mu \otimes \text{id} - \text{id} \otimes \mu}{\longrightarrow} & \mathcal{A}(q) \otimes \mathcal{A}(q) \\
\downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\
\mathcal{A}(q) & \stackrel{q \mu \tau}{\longrightarrow} & \mathcal{A}(q)
\end{array}
\]

**Remark 2.2.** We have:

- For \(q = 0\), the algebra \(\mathcal{A}(q)\) is reduced to an associative algebra;
- For \(q = -1\), \(\mathcal{A}(q)\) becomes a flexible algebra;
- For \(q = 1\), \(\mathcal{A}(q)\) turns to be a center-symmetric algebra \([12]\), (also called nonflexible algebra \([15]\)).

In the sequel, \((\mathcal{A}(q), \cdot)\) denotes a \(q\)-generalized flexible algebra over \(\mathbb{K}\). Besides, for notation simplification, we write \(xy\) instead of \(x \cdot y\), for \(x, y \in \mathcal{A}(q)\), i.e. the product \("\cdot\"\) is omitted when there is no confusion.

**Definition 2.3.** Suppose \(L\) and \(R\) be left and right multiplication operators defined on \(\mathcal{A}(q)\) as:

\[
\begin{align*}
L : & \quad \mathcal{A}(q) \rightarrow \mathfrak{gl}(\mathcal{A}(q)) \\
& x \mapsto L_x : \quad \mathcal{A}(q) \rightarrow \mathcal{A}(q) \quad \mapsto \quad L_x(y) := x \cdot y
\end{align*}
\]  

\[
\begin{align*}
R : & \quad \mathcal{A}(q) \rightarrow \mathfrak{gl}(\mathcal{A}(q)) \\
& x \mapsto R_x : \quad \mathcal{A}(q) \rightarrow \mathcal{A}(q) \quad \mapsto \quad R_x(y) := y \cdot x.
\end{align*}
\]

Then, their associated dual maps are defined as follows:

\[
\begin{align*}
\mathcal{A}(q) & \rightarrow \mathfrak{gl}(\mathcal{A}(q)^*) \\
\mathcal{A}(q)^* & \rightarrow \mathcal{A}(q)^*
\end{align*}
\]

\[
\begin{align*}
L^* : & \quad \mathcal{A}(q)^* \rightarrow \mathcal{A}(q)^* \\
& a \mapsto L^*_x(a) : \quad \mathcal{A}(q) \rightarrow \mathbb{K} \quad \mapsto \quad < L_x(a), y >= \quad \pm < a, L_x(y) >
\end{align*}
\]

\[
\begin{align*}
R^* : & \quad \mathcal{A}(q)^* \rightarrow \mathcal{A}(q)^* \\
& a \mapsto R^*_x(a) : \quad \mathcal{A}(q) \rightarrow \mathbb{K} \quad \mapsto \quad < R_x(a), y >= \quad \pm < a, R_x(y) >
\end{align*}
\]

**Proposition 2.4.** Let \(L\) and \(R\) be the above defined left and right multiplication operators. The following relations are satisfied for all \(x, y, z \in \mathcal{A}(q)\):

\[
L_{xy} - L_x L_y = q(R_x R_y - R_{yx}),
\]

\[
[R_x, L_y] = q[R_y, L_x],
\]

\[
R_x R_y - R_{yx} = q(L_{xy} - L_x L_y).
\]

**Proof:**

Let \(\mathcal{A}(q)\) be a \(q\)-generalized flexible algebra over the field \(\mathbb{K}\). For all \(x, y, z \in \mathcal{A}(q)\), the proof follows from the equivalences:

\[(x, y, z) = q(z, y, x) \iff (xy)z - x(yz) = q(zy)x - qz(yx)\]
\[(x, y, z) = q(z, y, x) \iff (L_{xy} - L_y L_x)(z) = q(R_x R_y - R_{yz})(z)\]
\[(x, y, z) = q(z, y, x) \iff (R_z L_x - L_x R_z)(y) = q(R_x L_z - L_z R_x)(y)\]
\[(x, y, z) = q(z, y, x) \iff [R_z, L_x](y) = q[R_x, L_z](y)\]
\[(x, y, z) = q(z, y, x) \iff (R_z R_y - R_{yz})(x) = q(L_{zy} - L_z L_y)(x)\]

\[\square\]

**Proposition 2.5.** Provided the sub-adjacent algebra \( \mathcal{G}(A(q)) := (A(q), [\cdot, \cdot]) \), where the bilinear product \([\cdot, \cdot]\) is the commutator associated to the product on \( A(q) \), we have, for all \( x, y, z \in A(q) \):
\[
J(x, y, z) := [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = (q-1)\{(x, y, z) + (y, z, x) + (z, x, y)\}. \tag{2.8}
\]

Proof: It stems from a straightforward computation. \( \square \)

**Lemma 2.6.** (1) From the Proposition 2.5, for all \( x, y, z \in A(q) \), the relation
\[
S(x, y, z) := (x, y, z) + (y, z, x) + (z, x, y) = 0 \tag{2.9}
\]
is a sufficient condition for \( A(q) \) to become a Lie admissible algebra, i.e. for \( (A(q), [\cdot, \cdot]) \) to be a Lie algebra. In the particular case where \( q = 1 \), we get a center symmetric algebra which is Lie admissible as developed in [12].

(2) The \( q \)-generalized algebra \( A(q) \) is Lie admissible if and only if, for all \( x, y, z \in A(q) \), we have: \( (q-1)S(x, y, z) = 0 \). In particular, any \( q \)-generalized flexible algebra defined on a field \( \mathbb{K}_{q-1} \) of characteristic \( q-1 \) is Lie admissible.

Proof: Let \( (A(q), \cdot) \) be a \( q \)-generalized flexible algebra.

(1) From the relation (2.9), we have \( J(x, y, z) = (q-1)S(x, y, z) \). Indeed, \( A(q) \) is Lie admissible if and only if \( J(x, y, z) = 0 \) yielding \( S(x, y, z) = 0 \).

(2) The Lie admissible condition \( J(x, y, z) = 0 \) implies that the relation \( (q-1)S(x, y, z) = 0 \) giving \( S(x, y, z) = 0 \), or \( q-1 = 0 \), or, in other words, the field \( \mathbb{K} \) on which \( A(q) \) defines a vector space is of characteristic \( q-1 \). \( \square \)

**Proposition 2.7.** The following relation is satisfied for all \( x, y, z \in A(q) \):
\[
(L_{xy} - L_x L_y + R_z L_y - L_y R_z + R_y R_x - R_{xy}) = q(R_z R_y - R_{yz} + R_y L_x - L_z R_y)(x).
\tag{2.10}
\]
where \( L \) and \( R \) are representations of left and right multiplication operators, respectively.

Proof:

Let us write the associator with the operators \( L \) and \( R \).

For all \( x, y, z \in A(q) \),
\[
(x, y, z) = (xy)z - x(yz) = (L_{xy} - L_x L_y)(z) = (R_z L_x - L_z R_x)(y) = (R_z R_y - R_{yz})(x).
\]

It follows that:
\[
(x, y, z) + (y, z, x) + (z, x, y) = (L_{xy} - L_x L_y + R_z L_y - L_y R_z + R_y R_x - R_{xy})(z) = q(z, y, x) + q(x, z, y) + q(y, x, z) = q(R_z R_y - R_{yz} + R_y L_x - L_z R_y + L_{yz} - L_y L_x)(z).
\]

Therefore, for all \( x, y, z \in A(q) \):
\[
L_{xy} - L_x L_y + R_z L_y - L_y R_z + R_y R_x - R_{xy} = q(R_z R_y - R_{yz} + R_y L_x - L_z R_y + L_{yz} - L_y L_x).
\tag{2.10}
\]

\( \square \)

**Remark 2.8.** The result (2.10) can also be derived from the Proposition 2.4 by summing the relations (2.5), (2.6) and (2.7).
Theorem 2.9. The following relation is satisfied: \( \forall x, y, z \in \mathcal{A}(q), \)
\[
[xy - qyx, z] + [yz - qzy, x] + [zx - qxz, y] = 0, \tag{2.11}
\]
where the bilinear product \([\cdot, \cdot]\) is the commutator associated to the product "." defined on \(\mathcal{A}(q)\).

Proof: By a direct computation. \(\square\)

Remark 2.10. By setting the parameter \(q = 1\), we get the Jacobi identity from the relation \(2.11\) indicating that the underlying algebra is Lie admissible as shown in \([12]\).

We are therefore in right to set the following:

Definition 2.11. Setting \(G(\mathcal{A}(q)) := (\mathcal{A}(q), [\cdot, \cdot])\), where \(\mathcal{A}(q)\) is the underlying vector space associated to a \(q\)-generalized flexible algebra \((\mathcal{A}(q), \cdot)\), the equation \(2.11\) defines a \(q\)-generalized Jacobi identity.

Theorem 2.12. For a \(q\)-generalized flexible algebra \(\mathcal{A}(q)\), the following propositions are equivalent: For all \(x, y, z \in \mathcal{A}(q)\),

1. \[
[z, xy] = \{z, x\} q y - x \{y, z\} q, \tag{2.12}
\]
where \(\{x, y\} := xy + qyx\);

2. \[
[z, x * q y] = [z, x] * q y + x * q [z, y], \tag{2.13}
\]
where \(x * q y = \frac{1}{q}(xy - qyx)\);

3. \(\mathcal{A}(q)\) is a Lie-admissible \(q\)-generalized flexible algebra, i.e.
\[
[[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \tag{2.14}
\]

Remark 2.13. From Theorem 2.12, we observe that:

1. For \(q = -1\), the equation \(2.12\) turns out to be the derivation for the commutator of a flexible algebra as postulated by the well known Myung Theorem \([18], [20]\) (and references therein).

2. For \(q = 0\), the equation \(2.12\) becomes an evidence by using the associativity.

3. For \(q = 1\), the equation \(2.12\) leads to the relation \((x, y, z) + (y, z, x) + (z, x, y) = 0\), which characterizes an anti-flexible algebra structure, see \([12]\) (and references therein)

4. For \(q = 1\), the equation \(2.13\) is equivalent to the Jacobi identity in a field of a characteristic 0, what is the case for a center-symmetric (also called anti-flexible) algebra;

5. For \(q = 0\), the equation \(2.13\) describes the derivation property of the commutator (or Lie bracket) of a Lie algebra induced by an associative algebra;

6. For \(q = -1\), the flexibility condition \(2.13\) defines the derivation property of the Jordan product given as \(x \cdot y = \frac{1}{q}(xy + yx)\), see \([20]\).

3. Bimodules and matched pairs of \(q\)-generalized flexible algebras

Definition 3.1. The triple \((l, r, V)\), where \(V\) is a finite dimensional vector space and \(l, r : \mathcal{A}(q) \to \mathfrak{gl}(V)\) are two linear maps satisfying the following relations for all \(x, y \in \mathcal{A}(q)\):

\[
l_{xy} - l_{yx} = q(r_{yx} r_y - r_{yx}), \tag{3.1}
\]

\[
[r_x, l_y] = q[r_y, l_x], \tag{3.2}
\]

\[
r_x r_y - r_y x = q(l_{xy} - l_{yx}), \tag{3.3}
\]
is called a bimodule of \(\mathcal{A}(q)\), also simply denoted by \((l, r)\).

Proposition 3.2. Let \(l, r : \mathcal{A}(q) \to \mathfrak{gl}(V)\) be two linear maps. The couple \((l, r)\) is a bimodule of the \(q\)-generalized flexible algebra \(\mathcal{A}(q)\) if and only if there exists a \(q\)-generalized flexible algebra structure "." on the semi-direct vector space \(\mathcal{A}(q) \oplus V\) given by
\[
(x + u) \ast (y + v) := x \cdot y + l_x v + r_y u, \forall x, y \in \mathcal{A}(q), \text{ and } \forall u, v \in V.
\]
We denote such a $q$-generalized flexible algebra structure "\(*\)" on the semi-direct vector space $A(q) \oplus V$ by $(A(q) \oplus V, * )$ or simply $A(q) \ltimes V$.

**Proof:**

Let $x, y, z \in A(q)$, where $A(q)$ is a generalized flexible algebra, and $u, v, w \in V$, where $V$ is a finite dimensional vector space. Using the bilinear product defined, for all $x, y \in A(q)$, and all $u, v \in V$ by:

$(x + u) * (y + v) := x \cdot y + l_x y + r_y u$, where $l, r : A \rightarrow gl(V)$ are linear maps, the associator of the bilinear product $*$ can be written as:

$$(x + u, y + v, z + w) = ((x + u) * (y + v)) * (z + w) - (x + u) * ((y + v) * (z + w))$$

$$= (x \cdot y + l_x y + r_y u) * (z + w) - (x + u) * (y \cdot z + l_y w + r_z v)$$

$$= (x \cdot y) \cdot z + l_x y w + r_z (l_x v) + r_z (r_y u) - x \cdot (y \cdot z)$$

$$(x + u, y + v, z + w) = (x, y, z) + (l_{x-y} - l_y l_x) w + (z_{x} l_z - l_x r_z) v + (r_z r_y - r_y r_z) u. \quad (3.4)$$

Then, we have, $\forall x, y, z \in A(q)$ and $\forall u, v, w \in V$:

$$(x + u, y + v, z + w) = (x, y, z) + (l_{x-y} - l_y l_x) w + (z_{x} l_z - l_x r_z) v + (r_z r_y - r_y r_z) u. \quad (3.4)$$

Besides,

$$q(z + w, y + v, x + u) = q(z, y, x) + q(l_{x-y} - l_y l_x) u + q(z_{x} l_z - l_x r_z) v + q(r_z r_y - r_y r_z) w. \quad (3.5)$$

By setting $(x + u, y + v, z + w) = q(z + w, y + v, x + u), \forall x, y, z \in A(q)$ and $\forall u, v, w \in V$, i.e. $(A(q) \oplus V, *)$ is a $q$-generalized flexible algebra, we get from the right hand side of the equations $\text{(3.4)}$ and $\text{(3.5)}$:

$$(x, y, z) = q(z, y, x),$$

$$(l_{x-y} - l_y l_x) = q(r_z r_y - r_y r_z),$$

$$(z_{x} l_z - l_x r_z) = q(l_{x-y} - l_y l_x),$$

$$r_z r_y - r_y r_z = q(l_{x-y} - l_y l_x),$$

which are equivalent to the relations $\text{(3.1)}, \text{(3.2)}, \text{(3.3)}$ defining the bimodule of a $q$-generalized flexible algebra.

**Example 3.3.** According to the Proposition 2.3 (L, R), where R and L are the representations of the right and left multiplication operators, respectively, is a bimodule of a $q$-generalized flexible algebra $A(q)$. Indeed, L and R satisfy the equations $\text{(3.1)}, \text{(3.2)}$ and $\text{(3.3)}$.

**Theorem 3.4.** Let $(A(q), \cdot)$ and $(B(q), *)$ be two $q$-generalized flexible algebras. Suppose there exist linear maps $l_A, r_A : A(q) \rightarrow gl(B(q))$ and $l_B, r_B : B(q) \rightarrow gl(A(q))$ satisfying the following relations:

$$(l_B(a)x) \cdot y + l_B(r_B(a)x) y - l_B(a)(x \cdot y) = q(r_B(a)(y \cdot x) - y \cdot (r_B(a)x)) - r_B(l_A(x)a)y, \quad (3.6)$$

$$r_B(a)(x \cdot y) - x \cdot (r_B(a)y) - r_B(l_A(y)a)x = q((l_B(a)y) \cdot x + l_B(r_A(y)a)x - l_A(a)(y \cdot x)), \quad (3.7)$$

$$(r_B(a)x) \cdot y + l_B(l_A(x)a)y - x \cdot (l_B(a)y) - r_B(r_A(y)a)x = q((r_B(a)y) \cdot x + l_B(l_A(a)yx - y \cdot (l_B(a)x)) - r_B(r_A(x)a)y), \quad (3.8)$$

$$(l_A(x)a) * b + l_A(r_A(x)b) - l_A(x)(a * b) = q(r_A(x)(b * a) - b * (r_A(x)a) - r_A(l_B(a)x)b)), \quad (3.9)$$
Then, the \( (3.10) \) and \( (3.11) \). Consider the bilinear product "\( \star \)" defined on the vector space \( \mathcal{A}(q) \oplus \mathcal{B}(q) \) given as: \( (x + a) \star (y + b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \star b + l_A(x)b + r_A(y)a) \).

Proof:
Let \( (\mathcal{A}(q), \cdot) \) and \( (\mathcal{B}(q), \star) \) be two \( q \)-generalized flexible algebras, \( l_A, r_A : \mathcal{A}(q) \to \mathfrak{gl}(\mathcal{B}(q)) \) and \( l_B, r_B : \mathcal{B}(q) \to \mathfrak{gl}(\mathcal{A}(q)) \) be four linear maps satisfying the relations \( (3.6), (3.7), (3.8), (3.9), (3.10) \) and \( (3.11) \). Consider the bilinear product "\( \star \)" defined on the vector space \( \mathcal{A}(q) \oplus \mathcal{B}(q) \) as:

\[
(x + a) \star (y + b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \star b + l_A(x)b + r_A(y)a), \quad \forall x, y \in \mathcal{A}(q); a, b \in \mathcal{B}(q).
\]

We have:

\[
(x + a, y + b, z + c) = \{(x + a) \star (y + b)\} \star (z + c)
\]

\[
= (x, y, z) + (a, b, c) + \{r_B(c)(x \cdot y) + l_A(x \cdot y)c - x \cdot (r_B(c)y) - l_A(x)(l_A(y)c) - r_B(l_A(y)c)\}
\]

\[
+ \{l_B(c)(r_B(b)x) + l_A(r_B(b)x)c - r_B(b \cdot c)x + l_A(x)b \star c - l_A(x)(b \cdot c)\}
\]

\[
+ \{(r_B(b)x) \star zl_B(l_A(x)b)z + r_A(z)(l_A(x)b) - x \cdot (r_B(b)z) - r_B(r_A(z)b)x - l_A(x)(r_A(z)b)\}
\]

\[
+ \{(l_B(a)y) \cdot z + l_B(r_A(y)a)z + r_A(z)(r_A(y)a)\}
\]

\[
+ \{l_B(a)(y \cdot z) - r_A(y \cdot z)a\} + \{r_B(c)(l_B(a)y)\}
\]

\[
+ \{r_A(y)a \star c + l_A(l_B(a)y)c - r_A(z)(l_B(c)y)c\}
\]

\[
- a \star (l_B(y)c) - r_A(r_B(c)y)a\}
\]

\[
= (x, y, z) + (x, y, c) + (x, b, z) + (x, b, c) + (a, y, z)
\]

\[
+ (a, b, z) + (a, b, c)
\]

Then, the \( q \)-generalized flexibility condition of the bilinear product "\( \star \)" is given as:

\[
(x + a, y + b, z + c) = q(z + c, y + b, x + a) \iff \begin{cases} (x, y, z) = q(z, y, x) \\ (a, b, c) = q(c, b, a) \\ (x, y, c) = q(c, y, x) \\ (x, b, z) = q(z, b, x) \\ (x, b, c) = q(c, b, x) \\ (a, y, z) = q(z, y, a) \\ (a, y, c) = q(c, y, a) \\ (a, b, z) = q(z, b, a) \end{cases}
\]
Indeed, Brem 3.4 is called matched pair of the generalized flexible algebras. Maps are defined as:

\[
(x, y, a) = q(x, y, y)
\]

\[
(x, a, b) = q(b, a, x)
\]

\[
(a, x, y) = q(y, x, a)
\]

\[
(a, x, b) = q(b, x, a)
\]

\[
(a, b, x) = q(x, b, a)
\]

\[
(x + a, y + b, z + c) = q(z + c, y + b, x + a)
\]

Therefore, we obtain a q-generalized flexible algebra structure given, for all \( x, y \in A(q) \) and all \( a, b \in B(q) \), by:

\[
(x + a) \star (y + b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \ast b + l_B(x)b + r_B(y)a)
\]

on the direct sum of the underlying vector spaces \( A(q) \) and \( B(q) \) of the bimodules \((l_A, r_A, B(q))\) and \((l_B, r_B, A(q))\) of the associated q-generalized flexible algebras \( A(q) \) and \( B(q) \).

In this case, the obtained q-generalized flexible algebra \((A(q) \oplus B(q), \ast)\) is denoted by \( A(q) \rtimes_{l_B, r_B} B(q) \), or simply \( A(q) \bowtie B(q) \).

**Definition 3.5.** The sixtuple \((A(q), B(q), l_A, r_A, l_B, r_B)\) satisfying the conditions of Theorem 3.4 is called matched pair of the generalized flexible algebras \( A(q) \) and \( B(q) \).

**Remark 3.6.** Theorem 3.4 is a q-generalization of main theorems, well known in the literature. Indeed,

- For \( q = 0 \), Theorem 3.4 is exactly the fundamental theorem for the matched pair of associative algebras. See [4] and references therein.
- For \( q = 1 \), Theorem 3.4 is reduced to the fundamental theorem for the matched pair of center-symmetric algebras formulated in [12].
- For \( q = -1 \), Theorem 3.4 becomes the fundamental theorem for the matched pair of flexible algebras.

4. Basic properties of the q-generalized flexible algebras

In this section, we construct and discuss the basic definitions and main properties of the q-generalized flexible algebras.

**Definition 4.1.** Let \( l, r : A(q) \rightarrow gl(V) \) be the two above mentioned linear maps. Their dual maps are defined as:

\[
A(q) \rightarrow gl(V^*) \quad V^* \rightarrow V^*
\]

\[
l^* : x \mapsto l_x^* : v^* \mapsto l_x^*(v^*) : u \mapsto \langle l_x^*(v^*), u \rangle, \quad V \rightarrow \mathbb{K}
\]

4.1
Theorem 4.2. For any finite dimensional vector space \(V\), suppose \(l, r : \mathcal{A}(q) \rightarrow \mathfrak{gl}(V)\) be two linear maps such that \(l^\ast\) and \(r^\ast\) are dual maps of \(l\) and \(r\), respectively. The following propositions are equivalent:

(1) \((l, r, V)\) is a bimodule of the \(q\)-generalized flexible algebra \(\mathcal{A}(q)\),

(2) \((r^\ast, l^\ast, V^\ast)\) is a bimodule of the \(q\)-generalized flexible algebra \(\mathcal{A}(q)\).

Remark 4.3. It is worth noticing that the dual bimodule does not depend on the parameter \(q\). Clearly, it is the dual bimodule obtained for the center-symmetric algebra in \([12]\). Therefore, the dual bimodule of center-symmetric algebras is the same as the dual bimodule of flexible algebras.

Proposition 4.4. The quadruple \((R^*, l^*, \mathcal{A}(q)^* )\), where \(\mathcal{A}(q)^*\) is the dual space of \(\mathcal{A}(q)\) and given by \(\mathcal{A}(q)^* = \text{Hom}(\mathcal{A}(q), \mathbb{K})\), is a bimodule of \(\mathcal{A}(q)\).

Proof: By considering Definition \(\ref{def:3.3}\) Proposition \(\ref{prop:4.1}\) and Proposition \(\ref{prop:4.2}\) we deduce that \((R^*, l^*, \mathcal{A}(q)^* )\) is a bimodule of \(\mathcal{A}(q)\). \(\square\)

Theorem 4.5. Let \((\mathcal{A}(q), \cdot)\) be a \(q\)-generalized flexible algebra. Suppose that there is a \(q\)-generalized flexible algebra structure \(\circ^\circ\) on its dual space \(\mathcal{A}(q)^* = \text{Hom}(\mathcal{A}(q), \mathbb{K})\). The sextuple \(\langle \mathcal{A}(q), \mathcal{A}(q)^*, R^*, l^*, R^*_0, l^*_0 \rangle\) is a matched pair of the \(q\)-generalized flexible algebras \(\mathcal{A}(q)\) and \(\mathcal{A}(q)^*\) if and only if the linear maps \(R^*, l^*, R^*_0, l^*_0\) satisfy the following relations for all \(x, y \in \mathcal{A}(q)\) and all \(a, b \in \mathcal{A}(q)^*\):

\[
(R^*_0(a)x) \cdot y + R^*_0(l^*(x)a)y - R^*_0(a)(x \cdot y) = q(L^*_0(a)(y \cdot x) - y \cdot (L^*_0(a)x) - L^*_0(R^*(y)a)x = q((R^*_0(a)y) \cdot x + R^*_0(l^*(x)a)x - y \cdot (R^*_0(a)x) - L^*_0(R^*(y)a)x - y \cdot (R^*_0(a)x) - L^*_0(R^*(y)a)x).
\]
The following holds:
\[(R_\circ(x)\cdot y + R_\circ(L_\circ(x)\cdot a) - R_\circ(x\cdot y) = q(L_\circ(y\cdot x) - y \cdot (L_\circ(x))
\]
\[-L_\circ(R_\circ(x)\cdot a)y)\]
or, equivalently,
\[L_\circ(R_\circ(x)\cdot b) + b \circ (L_\circ(x)\cdot a) - L_\circ(x)\cdot (b \circ a) = q( (R_\circ(x)\cdot a)\cdot b) - R_\circ(L_\circ(x)\cdot a)y
\]
which is exactly the equation (3.10) by taking the correspondences: \(l_A \rightarrow R_\circ, r_A \rightarrow L_\circ, l_B \rightarrow R_\circ, r_A \rightarrow L_\circ, \). and we have
\[L_\circ(x)\cdot (a \circ b) - a \circ (L_\circ(x)\cdot b) - L_\circ(R_\circ(x)\cdot a)y = q( (R_\circ(x)\cdot b) \circ a
\]
\[+ R_\circ(L_\circ(x)\cdot a) - R_\circ(x)\cdot (b \circ a)). \]
This shows that the relation (4.1.3) is equivalent to (4.1.4). By the same way, we have:
\[< L_\circ(x)\cdot (a \circ b) - x \cdot (L_\circ(x)\cdot a) - L_\circ(R_\circ(x)\cdot a)y - q((R_\circ(x)\cdot a)\cdot x
\]
\[+ R_\circ(L_\circ(x)\cdot a) - R_\circ(x)\cdot (b \circ a)). \]
This exactly gives the equation (3.3) by setting the correspondence \(l_A \rightarrow R_\circ, r_A \rightarrow L_\circ, l_B \rightarrow R_\circ, r_A \rightarrow L_\circ, y \rightarrow x\), and then we get:
\[(R_\circ(x)\cdot a) \circ b + R_\circ(L_\circ(x)\cdot a) - R_\circ(x)\cdot (a \circ b) = q(L_\circ(x)\cdot b \circ a
\]
\[-b \circ (L_\circ(x)\cdot a) - L_\circ(R_\circ(x)\cdot a)y). \]
This shows that the relation (4.4) is equivalent to (4.4.7):
\[< (L_\circ(x)\cdot y + R_\circ(R_\circ(x)\cdot a)y - x \cdot (R_\circ(x)\cdot a)\cdot y
\]
\[-q(L_\circ(x)\cdot a) - x \cdot (R_\circ(x)\cdot a) - q(L_\circ(x)\cdot a)y, b > =< L_\circ(x)\cdot (a \circ b) + R_\circ(L_\circ(x)\cdot a) - R_\circ(x)\cdot (b \circ a)). \]
Therefore, the following relation
\[(L_\circ(x)\cdot y + R_\circ(R_\circ(x)\cdot a)y - x \cdot (R_\circ(x)\cdot a)\cdot y - L_\circ(L_\circ(x)\cdot a)y = q((L_\circ(x)\cdot a)\cdot x
\]
\[+ R_\circ(R_\circ(x)\cdot a) - L_\circ(L_\circ(x)\cdot a)y). \]
is equivalent to
\[L_\circ(L_\circ(x)\cdot a) - R_\circ(R_\circ(x)\cdot a)\circ a - R_\circ(R_\circ(x)\cdot a) - q(a \circ (R_\circ(x)\circ b)
\]
\[L_\circ(L_\circ(x)\cdot a) - R_\circ(R_\circ(x)\cdot a)\circ b - (L_\circ(x)\cdot a) \circ b). \]
which is exactly the equation (3.11) by setting the correspondences \(l_A \rightarrow R_\circ, r_A \rightarrow L_\circ, l_B \rightarrow R_\circ, r_A \rightarrow L_\circ, y \rightarrow x\), and we have
\[(L_\circ(x)\cdot a) \circ b + R_\circ(R_\circ(x)\cdot a)y - a \circ (R_\circ(x)\cdot b) - L_\circ(L_\circ(x)\cdot a)y = q((L_\circ(x)\cdot b) \circ a + R_\circ(R_\circ(x)\cdot b) - a \circ (R_\circ(x)\cdot a) - L_\circ(R_\circ(x)\cdot a)y). \]
Therefore, the sixtuple \((A(q), (A(q))^*, R^*, L^*, R^*_o, L^*_o)\) is a matched pair of the \(q\)-generalized flexible algebras \(A(q)\) and \(A(q)^*\) if and only if the linear maps \(R^*, L^*, R^*_o, L^*_o\) satisfy the equations \(\mathbf{1.35}\), \(\mathbf{1.39}\) and \(\mathbf{1.50}\).

**Remark 4.6.** Theorem 4.5 encompasses particular results known in the literature, namely:

- For \(q = 0\), Theorem 4.5 is exactly reduced to the result obtained by Bai in [4], (see also references therein) giving the construction of the dual matched pair for the associative algebras.
- For \(q = 1\), we recover the theorem relating the dual matched pair of center-symmetric algebras with the dual matched pair of Lie algebras, investigated in [12].
- For \(q = -1\), Theorem 4.5 gives the dual matched pair of flexible algebras. This is a new result, given in this work for the first time, to our best knowledge of the literature.

**Proposition 4.7.** Assume that there is a \(q\)-generalized flexible algebra structure "\(\circ\)" on the dual space \(A(q)^*\). There is a \(q\)-generalized flexible algebra structure "\(\ast\)" on the vector space \(A(q) \oplus A(q)^*\) given, for all \(x, y \in A(q)\) and all \(a, b \in A(q)^*\), by:

\[
(x+a) \ast (y+b) = (x+y + R^*_o(a)y + L^*_o(b)x) + (a \circ b + R^*(x)b + L^*(y)a),
\]

(4.9)

if and only if the sixtuple \((A(q), A(q)^*, R^*, L^*, R^*_o, L^*_o)\) is a matched pair of the \(q\)-generalized flexible algebras \(A(q)\) and \(A(q)^*\).

**Proof:**

It is well known that \(A(q) \oplus A(q)^*\) is a vector space as a direct sum of vector spaces. The product "\(\ast\)" is also a bilinear product by definition. Then, it only remains to show that the \(q\)-generalized flexibility identity for the product "\(\ast\)" is equivalent to the fact that the sixtuple \((A(q), A(q)^*, R^*, L^*, R^*_o, L^*_o)\) is a matched pair of \(A(q)\) and \(A(q)^*\). For all \(x, y, z \in A(q)\), and all \(a, b, c \in A(q)^*\), the left and right hand sides of the associator of the bilinear product "\(\ast\)" are given, respectively, by:

\[
\{(x+a) \ast (y+b)\} \ast (z+c) = \{x \cdot y + R^*_o(a)y + L^*_o(b)x + a \circ b + R^*(x)b + L^*(y)a\} \ast (z+c) = (x \cdot y + R^*_o(a)y + L^*_o(b)x + a \circ b + R^*(x)b + L^*(y)a)
\]

(4.9)

\[
\{(x+a) \ast (y+b)\} \ast (z+c) = \{x \cdot y + R^*_o(a)y + L^*_o(b)x + a \circ b + R^*(x)b + L^*(y)a\} \ast (z+c) = (x \cdot y + R^*_o(a)y + L^*_o(b)x + a \circ b + R^*(x)b + L^*(y)a)
\]
and,

\[(x + a) \ast \{(y + b) \ast (z + c)\} = (x + a) \ast \{y \cdot z + R^*_c(b)z + L^*_c(c)y\} + R^*_c(a)(y \cdot z)\]

\[+R^*_c(a)(L^*_c(c)y) + L^*_c(b \circ c)x + L^*_c(R^*_c(y)c)x\]

\[+L^*_c(L^*_c(z)b)x + a \circ (b \circ c) + a \circ (R^*_c(y)c)\]

\[+R^*_c(a)(x \cdot z) + R^*_c(a)(R^*_c(b)z) + R^*_c(a)(L^*_c(c)y) + L^*_c(b \circ c)x\]

\[+\{a \circ (R^*_c(y)c) + L^*_c(L^*_c(c)y)a\}\]

\[+\{a \circ (L^*_c(z)b) + L^*_c(R^*_c(b)z)a\}\]

\[+x \cdot (y \cdot z) + a \circ (b \circ c) + R^*_c(x)(b \circ c)\]

\[+R^*_c(x)(R^*_c(y)c) + R^*_c(x)(L^*_c(z)b) + L^*_c(y \cdot z)a\]

Therefore, the associator can be rewritten as:

\[\{(x + a), (y + b), (z + c)\} = \{(x + a) \ast \{y + b\} \ast (z + c)\}\]

\[= \{(L^*_c(b)x \cdot z + R^*_c(R^*_c(x)b)z - x \cdot (R^*_c(b)z)\}

\[\ast \{L^*_c(c)(x \cdot y) - x \cdot (L^*_c(c)y)\}

\[\ast \{R^*_c(x) \circ c + R^*_c(L^*_c(b)x)c\}

\[\ast \{R^*_c(x) \circ c - R^*_c(x)(R^*_c(y)c)\}

\[\ast \{L^*_c(z)(R^*_c(x)b) - R^*_c(x)(L^*_c(z)b)\}

\[\ast \{L^*_c(z)(L^*_c(y)a) - L^*_c(y \cdot z)a\}\]

\[\ast \{a, b, c\} + (x, y, z) + \{L^*_c(c)(R^*_c(a)y)\}

\[\ast \{L^*_c(a)(L^*_c(c)y)\} + \{L^*_c(c)(L^*_c(b)x) - L^*_c(b \circ c)x\}\]

Similarly, we have:

\[q(z + c, y + b, x + a) = q\{(L^*_c(b)z \cdot x + R^*_c(R^*_c(z)b)x - z \cdot (R^*_c(b)x)\}

\[\ast \{L^*_c(c)(y \cdot x)\} + q\{L^*_c(a)(z \cdot y) - z \cdot (L^*_c(a)y)\}

\[\ast \{R^*_c(x) \circ c - c \circ (L^*_c(x)b)\}

\[\ast \{R^*_c(R^*_c(b)z)x\} + q\{(L^*_c(x)c \circ b) - c \circ (L^*_c(x)b)\}

\[\ast \{-L^*_c(R^*_c(b)z)a + R^*_c(x)(a \circ b)\} + a + q(c, b, a) + q(z, y, x)\]

\[+q\{R^*_c(z \cdot y)a - R^*_c(z)(R^*_c(y)a)\}\]

\[+q\{L^*_c(x)(R^*_c(z)b) - R^*_c(z)(L^*_c(x)b)\}

\[+q\{L^*_c(a)(L^*_c(b)z) - L^*_c(b \circ a)z\}\]
\[\begin{align*}
+q\{L^*_q(a)(R^*_q(c)y) - R^*_q(c)(L^*_q(a)y)\} + q\{R^*_q(c \circ b)x
-R^*_q(c)(R^*_q(b)x)\} + q\{L^*_q(x)(L^*_q(y)c) - L^*_q(y \cdot x)c\}.
\end{align*}\]

Hence, the equality:
\[(x + a, y + b, z + c) = q(z + c, y + b, x + a)\]
is equivalent to the following:
\[
\begin{cases}
(R^*_q, L^*_q, \mathcal{A}(q))^* \text{ and } (R^*_q, L^*_q, \mathcal{A}(q)) \text{ are bimodules of } \mathcal{A}(q) \text{ and } \mathcal{A}(q)^*,
\end{cases}
\]
respectively. \((L^*_q(b)x) \cdot z + R^*_q(R^*_q(x)b)z - x \cdot (R^*_q(b)(z) - L^*_q(L^*_q(x)b)x =
q((L^*_q(b)x) \cdot x + R^*_q(L^*_q(c)b)c - z \cdot (R^*_q(b)(c) - L^*_q(L^*_q(x)b)c),
\]
\[
\begin{split}
L^*_q(c)(x \cdot y) - x \cdot (L^*_q(c)y) - L^*_q(L^*_q(y)c)x =
q((R^*_q(c)y) \cdot x + R^*_q(L^*_q(y)c)x - R^*_q(c)(y \cdot x)),
\end{split}
\]
\[
(R^*_q(a)y) \cdot z + R^*_q(L^*_q(y)a)z - R^*_q(a)(y \cdot z) =
q(L^*_q(a)(z \cdot y) - z \cdot (L^*_q(a)y) - L^*_q(R^*_q(y)a)z),
\]
\[
(R^*_q(x)b) \circ c + R^*_q(L^*_q(c)b)c - R^*_q(c)(b \cdot c) =
q(L^*_q(x)(c \circ b) - c \circ (L^*_q(x)b) - L^*_q(R^*_q(b)x)c),
\]
\[
L^*_q(z)(a \circ b) - a \circ (L^*_q(z)b) - L^*_q(R^*_q(b)z)a =
q((R^*_q(z)b) \circ a + R^*_q(L^*_q(b)z)a - R^*_q(z)(b \circ a)),
\]
\[
(L^*_q(y)a) \circ c + R^*_q(R^*_q(a)y)c - a \circ (R^*_q(y)c) - L^*_q(L^*_q(c)y)a =
q((L^*_q(y)c) \circ a + R^*_q(R^*_q(c)y)a)
-c \circ (R^*_q(y)a) - L^*_q(L^*_q(a)y)c).
\]

Therefore, by Theorem \[\text{[4.3]}\] the bilinear product \(\star\) defines a \(q\)-generalized flexible algebra structure on the vector space \(\mathcal{A}(q) \oplus \mathcal{A}(q)^*\) if and only if \((\mathcal{A}(q), \mathcal{A}(q)^*, R^*_q, L^*_q, R^*_q, L^*_q)\) is a matched pair of the \(q\)-generalized flexible algebras \(\mathcal{A}(q)\) and \(\mathcal{A}(q)^*\).

**Remark 4.8.** From Proposition \[\text{[4.7]}\] we conclude that both the flexible and antiflexible algebras have the same matched pairs given on \(\mathcal{A}(q)\) and \(\mathcal{A}(q)^*\) for \(q = \pm 1\). The same result extends to associative algebras obtained for the parameter \(q = 0\).

**5. Manin triple of \(q\)-generalized flexible algebras and bialgebras**

We start with the following definitions, consistent with analogous formulation for Lie algebras \[\text{[16]}\]:

**Definition 5.1.** Let \((\mathcal{A}(q), \cdot)\) be a \(q\)-generalized flexible algebra. Suppose that there is a \(q\)-generalized flexible algebra structure \(\circ\) on its dual space \(\mathcal{A}(q)^*\). A Manin triple of the \(q\)-generalized flexible algebras \(\mathcal{A}(q)\) and \(\mathcal{A}(q)^*\) associated to a symmetric, non-degenerate, invariant bilinear form \(\mathfrak{B}\) defined on the vector space \(\mathcal{A}(q) \oplus \mathcal{A}(q)^*\) by:
\[
\mathfrak{B}(x + a, y + b) = < x, b > + < y, a >,
\]
for all \(x, y \in \mathcal{A}(q)\) and all \(a, b \in \mathcal{A}(q)^*\), where the bilinear product \(<, >\) is the natural pairing between the vector spaces \(\mathcal{A}(q)\) and \(\mathcal{A}(q)^*\), is a triple \((\mathcal{A}(q) \oplus \mathcal{A}(q)^*, \mathcal{A}(q), \mathcal{A}(q)^*)\) such that the bilinear product \(\star\) defined for all \(x, y \in \mathcal{A}(q)\) and all \(a, b \in \mathcal{A}(q)^*\) by:
\[
(x + a) \star (y + b) = (x \cdot y + R^*_q(a)y + L^*_q(b)x) + (a \circ b + R^*_q(x)b + L^*_q(y)a)
\]
realizes a \(q\)-generalized flexible algebra structure on \(\mathcal{A}(q) \oplus \mathcal{A}(q)^*\).

**Definition 5.2.** The triple \((\mathcal{A}(q), A_1(q), A_2(q))\), where:
- \(\mathcal{A}(q)\) is a \(q\)-generalized flexible algebra together with a nondegenerate, invariant and symmetric bilinear form, and
- \(A_1(q)\) and \(A_2(q)\) are two Lagrangian sub-\(q\)-generalized flexible algebras of \(\mathcal{A}(q)\) such that \(\mathcal{A}(q) = A_1(q) \oplus A_2(q)\).

is a \(q\)-generalized flexible bialgebra.
Hence, from the relations (5.3) and (5.4), we have the required result.

Then, we have:

\[B(x, y, z) \in A(q)\]

\[\circ \quad (\circ -\text{generalized flexible algebras } \circ \text{-generalized flexible algebra structure here denoted by } \circ \text{ given, for all } x, y, z \in A(q)\]

\[\text{and all } a, b, c \in A(q)^* \text{, by:}

\[(x + a) \ast (y + b) = (x \cdot y + R_x^*(a)y + L_y^*(b)x) + (a \circ b + R^*(x)b + L^*(y)a).\]

Then, the sixtuple \((A(q), A(q)^*, R^*, L^*, R_x^*, L_y^*)\) is a matched pair of the \(q\)-generalized flexible algebras \((A(q), \cdot, \circ)\) and \((A(q)^*, \circ)\). From Definition 5.1 it remains to show that the bilinear form \(B\) defined by using the natural pairing between \(A(q)\) and its dual in \((5.1)\) satisfies the relation:

\[B((x + a) \ast (y + b), (z + c)) = B((x + a), (y + b) \ast (z + c)).\]  

(5.2)

The left hand side of the equation (5.2) is given, for all \(x, y, z \in A(q)\) and all \(a, b, c \in A(q)^*\), by:

\[B((x + a) \ast (y + b), (z + c)) = B(x \cdot y + R_x^*(a)y + L_y^*(b)x + a \circ b + R^*(x)b + L^*(y)a, z + c)\]

\[= B(x \cdot y + R_x^*(a)y + L_y^*(b)x + a \circ b + R^*(x)b + L^*(y)a, z + c)\]

\[= < x \cdot y + R_x^*(a)y + L_y^*(b)x, c > + < z, a \circ b + R^*(x)b + L^*(y)a, c >\]

\[= < x \cdot y, c > + < R_x^*(a)y, c > + < L_y^*(b)x, c > + < z, a \circ b > + < R^*(x)b, c > + < L^*(y)a, c >\]

\[= < x \cdot y, c > + < y, R_x^*(a)c > + < z, a \circ b > + < R^*(x)b, c > + < L^*(y)a, c >\]

\[= < x \cdot y, c > + < y, c \circ a > + < x, b \circ c > + < z, a \circ b > + < z \cdot x, b > + < y \cdot z, a > .\]

Then, we have:

\[B((x + a) \ast (y + b), (z + c)) = < x \cdot y, c > + < y, c \circ a > + < x, b \circ c > + < z, a \circ b > + < z \cdot x, b > + < y \cdot z, a > .\]  

(5.3)

Besides, the right hand side of the equation (5.2) can be developed as follows:

\[B((x + a), (y + b) \ast (z + c)) = B((x + a), y \cdot z + R_x^*(b)z + L_y^*(c)y + b \circ c + R^*(y)c + L^*(z)b)\]

\[= < x, b \circ c > + < x, R_x^*(y)c > + < x, L_y^*(z)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < R_y^*(x)c > + < L_z^*(y)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < R_y^*(x)c > + < L_z^*(y)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < R_y^*(x)c > + < L_z^*(y)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < R_y^*(x)c > + < L_z^*(y)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < R_y^*(x)c > + < L_z^*(y)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < R_y^*(x)c > + < L_z^*(y)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < R_y^*(x)c > + < L_z^*(y)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

Hence,

\[B((x + a), (y + b) \ast (z + c)) = < x, b \circ c > + < x, R_x^*(y)c > + < x, L_y^*(z)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < x, R_x^*(y)c > + < x, L_y^*(z)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < x, R_x^*(y)c > + < x, L_y^*(z)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

\[= < x, b \circ c > + < x, R_x^*(y)c > + < x, L_y^*(z)b > + < y \cdot z, a > + < R_x^*(b)z, a > + < L_y^*(c)y, a >\]

Therefore, from the relations (5.3) and (5.4), we have the required result. □
THEOREM 5.4. Suppose that there is a \( q \)-generalized flexible algebra structure "\( \circ \)" on the dual space \( A(q)^* \). The following propositions are equivalent:

1. \( (A(q) \oplus A(q)^*, A(q), A(q)^*, \omega) \) is a Manin triple of the \( q \)-generalized flexible algebras \( A(q) \) and \( A(q)^* \) with the nondegenerate symmetric bilinear form \( \omega \) defined on \( A(q) \oplus A(q)^* \), for all \( x, y \in A(q) \) and all \( a, b \in A(q)^* \) by: \( \omega(x + a, y + b) := < x, b > < y, a > \), where \( <, > \) is the natural pairing between \( A(q) \) and \( A(q)^* \).

2. The sextuple \( (A(q), A(q)^*, R^*_1, L^*_1, R^*_c, L^*_c) \) is a matched pair of the \( q \)-generalized flexible algebras \( (A(q), \cdot) \) and \( (A(q)^*, \circ) \).

3. \( (A(q), A(q)^*) \) is a \( q \)-generalized flexible bialgebra.

Proof: By considering the Theorem 5.3, we deduct (1) \( \iff \) (2). By the definition, we also have the equivalence (1) \( \iff \) (3). \( \square \)

6. Application to octonion algebra

DEFINITION 6.1. An octonion algebra \( O \) is an eight dimensional vector space spanned by elements \( \{e_0, e_1, \cdots, e_7\} \) satisfying the following relations: \( \forall i, j, k = 1, \cdots, 7 \),

\[
e_{i}^2 = e_0, e_ie_0 = e_i = e_ie_j = -\delta_{ij} e_0 + c_{ijk} e_k,
\]

where the fully antisymmetric structure constants \( c_{ijk} \) are taken to be 1 for the combination of indexes:

\[
(ijk) \in \{ (124), (137), (156), (235), (267), (346), (457) \},
\]

with the bilinear product given in Table 1.

| \( \wedge \) | \( e_0 \) | \( e_1 \) | \( e_2 \) | \( e_3 \) | \( e_4 \) | \( e_5 \) | \( e_6 \) | \( e_7 \) |
|---|---|---|---|---|---|---|---|---|
| \( e_0 \) | \( e_0 \) | \( e_1 \) | \( e_2 \) | \( e_3 \) | \( e_4 \) | \( e_5 \) | \( e_6 \) | \( e_7 \) |
| \( e_1 \) | \( -e_0 \) | \( e_4 \) | \( e_7 \) | \( -e_2 \) | \( e_6 \) | \( -e_5 \) | \( -e_3 \) | \( e_7 \) |
| \( e_2 \) | \( -e_4 \) | \( -e_0 \) | \( e_5 \) | \( e_1 \) | \( -e_3 \) | \( e_7 \) | \( -e_6 \) | \( e_7 \) |
| \( e_3 \) | \( -e_7 \) | \( -e_5 \) | \( -e_0 \) | \( e_6 \) | \( -e_4 \) | \( e_1 \) | \( e_7 \) | \( e_7 \) |
| \( e_4 \) | \( e_4 \) | \( -e_1 \) | \( -e_6 \) | \( -e_0 \) | \( e_7 \) | \( e_3 \) | \( -e_5 \) | \( e_1 \) |
| \( e_5 \) | \( -e_6 \) | \( e_3 \) | \( e_2 \) | \( -e_7 \) | \( -e_0 \) | \( e_1 \) | \( e_2 \) | \( e_2 \) |
| \( e_6 \) | \( e_6 \) | \( e_5 \) | \( -e_7 \) | \( e_4 \) | \( -e_3 \) | \( -e_1 \) | \( -e_0 \) | \( e_2 \) |
| \( e_7 \) | \( e_7 \) | \( e_7 \) | \( e_6 \) | \( -e_1 \) | \( e_5 \) | \( -e_4 \) | \( -e_2 \) | \( -e_6 \) |

TABLE 6.1. Multiplication table of octonion algebra.

The associator of the octonion algebra \( O = Span\{e_0, e_1, \cdots, e_7\} \) defined as:

\[
e_{ijk} := (e_i, e_j, e_k) = (e_i e_j) e_k - (e_i e_k) e_j, \forall i, j, k \in \{0, 1, 2, \cdots, 7\}
\]

obeys the following relations: \( \forall i, j, k \in \{1, 2, \cdots, 7\} \),

\[
(e_0, e_i, e_j) = (e_i, e_0, e_j) = (e_i, e_j, e_0) = (e_i, e_i, e_j) = (e_i, e_j, e_i) = 0
\]

\[
(e_i, e_j, e_k) = \sum_{m=1}^{7} (c_{im} \delta_{mk} - c_{jm} \delta_{im}) e_0 + \sum_{n=1, m=1}^{7} (c_{imn} c_{mkn} - c_{jkm} c_{imn}) e_n,
\]

where the associator is written as \( (e_i, e_j, e_k) := (e_i e_j) e_k - e_i (e_j e_k) := c_{ijk} e_k \).

PROPOSITION 6.2. Let \( O \) be an octonion algebra with basis \( \{e_0, e_1, \cdots, e_7\} \). We have:

1. The 4 dimensional sub-algebras, spanned by the elements \( \{e_0, e_i, e_j, e_k\} \) where the index \( (ijk) \in \{ (124), (137), (156), (235), (267), (346), (457) \} \), are associative, i.e., their associator vanishes. So far, the associator \( (e_i, e_j, e_k) \) such that indexes are repeated, or contain zero, also vanishes, and the vector space \( \{e_0, e_i, e_j, e_k\} \) does not have a sub-algebra property.
Figure 1. Realization of octonion algebra

| $e_j$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $e_{00}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_{01}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_{11}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_{12}$ | $0$ | $0$ | $0$ | $-2e_6$ | $0$ | $2e_7$ | $2e_4$ | $-2e_5$ |
| $e_{22}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_{23}$ | $0$ | $-2e_6$ | $0$ | $0$ | $-2e_7$ | $0$ | $2e_1$ | $2e_4$ |
| $e_{33}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_{34}$ | $0$ | $2e_5$ | $-2e_7$ | $0$ | $0$ | $-2e_1$ | $0$ | $2e_2$ |
| $e_{44}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_{45}$ | $0$ | $2e_3$ | $2e_6$ | $-2e_1$ | $0$ | $0$ | $-2e_2$ | $0$ |
| $e_{55}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_{56}$ | $0$ | $0$ | $2e_4$ | $2e_7$ | $-2e_2$ | $0$ | $0$ | $-2e_3$ |
| $e_{66}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_{67}$ | $0$ | $-2e_4$ | $0$ | $2e_5$ | $2e_1$ | $-2e_3$ | $0$ | $0$ |
| $e_{77}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |

Table 6.2. Table of composition of associator of octonion.

(2) Other associators $(e_i, e_j, e_k)$, where the indexes $(ijk)$ are fully skew-symmetric for the combinations:

$\{(123), (125), (126), (127), (234), (236), (237), (341), (342), (345), (347), (451), (452), (453), (456), (562), (563), (564), (567), (671), (673), (674), (675)\}$

do not vanish, and are anti-left symmetric, anti-right symmetric and anti-center symmetric.

**Definition 6.3.** Let $\mathcal{O}$ be an octonion algebra. Consider the triple $(l, r, V)$, where $V$ is a finite dimensional vector space, and $l, r : \mathcal{O} \rightarrow gl(V)$ are two linear maps. Then, the following relations are satisfied: for all $e_i \in \mathcal{O}, i = 0, 1, \ldots, 7$,

$$l_{e_0} = id = r_{e_0}, l_{e_i} = -r_{e_i}, \quad (6.5)$$

$$[r_{e_i}, l_{e_j}] = [r_{e_j}, l_{e_i}], \quad (6.6)$$

$$\delta_{ij} + l_{e_i}l_{e_j} = c_{ijk}l_{e_k}, \quad (6.7)$$
where the structure constants \( c_{ijk} \), given in the equation (6.1), are well defined.

**Proposition 6.4.** Let \( \mathcal{O} \) be an octonion algebra, and \( l, r : \mathcal{O} \to \mathfrak{gl}(V) \) be two linear maps. The couple \((l, r)\) is a bimodule of the octonion algebra \( \mathcal{O} \) if and only if there exists an octonion algebra structure "*" on the semi-direct vector space \( \mathcal{O} \oplus V \) given by
\[
(e_i + u) \ast (e_j + v) := e_i e_j + l_{e_i} v + r_{e_j} u, \forall e_i, e_j \in \mathcal{O}, \forall u, v \in V, i, j = 0, 1, \cdots 7.
\]

**Theorem 6.5.** For an octonion algebra \( \mathcal{O} \) spanned by \( \{e_0, e_1, \cdots, e_7\} \), the following relations are equivalent:

1. \[
[e_k, e_i e_j] = [e_k, e_i] e_j + e_i [e_k, e_j],
\]
2. \[
c_{ijm} c_{kml} = c_{kim} c_{mjl} + c_{jkm} c_{iml},
\]
where the \( C_{ijk} \) are defined in the equation (6.1), for all \( i, j, k \in \{0, 1, \cdots 7\} \).

**Theorem 6.5** is known as Myung Theorem. For more details, see [18][20].

**Proposition 6.6.** Let \( \mathcal{O} \) be an octonion algebra with basis \( \{e_0, e_1, \cdots, e_7\} \). The following relation is satisfied:
\[
2\delta_{ij} + c_{ijk} r_{ek} + 2r_{ej} r_{ei} = 0,
\]
or, equivalently,
\[
2\delta_{ij} - c_{ijk} l_{ek} + 2l_{ej} l_{ei} = 0,
\]
\( \forall i, j, k = 0, 1, \cdots 7 \), where \( c_{ijk} \) are defined in the equation (6.1), and \( l_{e_i}, r_{e_i} \) are linear operators satisfying the relations (6.1), (6.6) and (6.7).

### 7. Concluding remarks

In this work, we have provided a \( q \)-generalization of flexible algebras, including center-symmetric, (also called antiflexible), algebras, and their bialgebras. Basic properties have been derived and discussed, as well as their connection to known algebraic structures investigated in the literature. A \( q \)-generalization of Myung theorem has been given. Main properties related to bimodules, matched pairs, dual bimodules, and their algebraic consequences have been derived and discussed. Finally, the equivalence between \( q \)-generalized flexible algebras, their Manin triple and bialgebras has been elucidated.

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