Symplectic Parshin-Arakelov inequality

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§1. Introduction

Lefschetz fibration is the smooth analogue of stable holomorphic fibration. In dimension four, its importance stems from its close relations to the mapping class groups and the Deligne-Mumford moduli space of algebraic curves. Recently it has received wide attention because of the discovery, mainly due to Donaldson, that it provides a topological description of symplectic manifolds.

For a relatively minimal genus \( g \) stable holomorphic fibrations over a genus \( h > 0 \) Riemann surface, there is a famous Parshin-Arakelov inequality: \( c_1^2 \geq 8(g - 1)(h - 1) \). In this paper, we will present its symplectic analogue.

**Theorem 1.** Let \( M \) be a relatively minimal genus \( g \) Lefschetz fibration over a genus \( h \) surface. If \( M \) is not rational or ruled, then

\[
c_1^2(M) \geq 2(g - 1)(h - 1).
\]

and it is sharp in the case \( h = 0 \).

When \( h \) is positive, the inequality generalizes Kotschick’s result [K] for surface bundles. We do not know whether it is sharp or not.

In the theorem, the condition that \( M \) not being rational or ruled is necessary. Because when \( h = 0 \), our inequality is \( c_1^2 \geq 2 - 2g \), while there are many Lefschetz fibrations over \( S^2 \) on rational and ruled surfaces with \( c_1^2 = 4 - 4g \). Rational and ruled surfaces are symplectic four-manifolds with exceptional properties and can be characterized among all symplectic four manifolds in several ways (see [L], [Liu], [Mc]). In this paper, we also analyze Lefschetz fibrations on these manifolds. The analysis of Lefschetz fibrations on ruled surfaces, in conjunction with Theorem 1, allows us to obtain a lower bound of the number of irreducible singular fibers for Lefschetz fibrations over \( S^2 \).

**Theorem 2.** The number of irreducible singular fibers of a genus \( g \) Lefschetz fibration over \( S^2 \) is no less than \( g \).

The organization of this paper is as follows. We review Lefschetz fibrations in §2. In §3, we present the proof of the symplectic Parshin-Arakelov inequality. In §4, we first study Lefschetz fibrations on ruled surfaces. We then present the estimate of the minimal number of irreducible singular fibers. Finally we also discuss the applications of our theorems to the mapping class groups and the Deligne-Mumford moduli space of algebraic curves.
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§2. Lefschetz fibrations

Definition 2.1. Let $M$ be a compact, connected, oriented smooth four-manifold. A Lefschetz fibration is a map $\pi : M \rightarrow \Sigma$, where $\Sigma$ is a compact, connected, oriented surface and $\pi^{-1}(\partial \Sigma) = \partial M$, such that

a). the set of critical points $C = \{x_1, \cdots, x_n\}$ of $\pi$ is non-empty and lies in interior of $M$;

b). about each $x_i$ and $\pi(x_i)$, there are orientation-preserving complex local coordinate charts on which

$$\pi(z_1, z_2) = z_1^2 + z_2^2;$$

c). $\pi$ is injective on $C$.

A regular fiber is a closed smooth surface, its genus called the genus of the Lefschetz fibration. Each singular fiber is a transversely immersed surface with a positive double point. A singular fiber is called reducible if the connected component containing the critical point becomes disconnected after the critical point is removed. b) and c) imply that a reducible fiber has exactly two components, each with square $-1$. A Lefschetz fibration is relatively minimal if there is no singular fiber containing a sphere of self-intersection $-1$.

The existence of a Lefschetz fibration $\pi : M \rightarrow \Sigma$ with regular fiber $F$ provides a handlebody description of $M$ (see [K] for more details). A regular neighborhood of a singular fiber is diffeomorphic to $(F \times D^2) \cup H^2$, where $H^2$ is a 2-handle attached along a simple closed curve $\gamma$ in a fiber $F \times \{pt\}$ in the boundary of $F \times D^2$. The attaching circle $\gamma$, well-defined up to isotopy, is called the vanishing cycle. The boundary of $(F \times D^2) \cup H^2$ is diffeomorphic to a $F-$bundle over $S^1$ whose monodromy is given by the right-handed Dehn twist about $\gamma$, $D(\gamma) \in \mathcal{MC}(F)$, where $\mathcal{MC}(F)$ is the mapping class group of $F$. Geometrically, as one approaches the singular fiber, the vanishing cycle is shrunk to the critical point. We see that a separating vanishing cycle corresponds to a reducible fiber.

If $\Sigma$ is a two disc, then $M$ is diffeomorphic to $(F \times D^2) \cup H^2 \cup \cdots \cup H^2_n$, where each two-handle $H^2_i$ is attached along a vanishing cycle $\gamma_i$ in a fiber in the boundary of $\Sigma \times D^2$. The boundary of this Lefschetz fibration is a $F-$bundle over $S^1$ whose monodromy is the product $D(\gamma_1) \cdots D(\gamma_n)$. When $\Sigma$ is $S^2$, we get a Lefschetz fibration $M_0$ over $D^2$ by removing a regular neighborhood $U$ from a regular fiber.
Since $U$ is a trivial $F$–bundle over a two disc, the boundary of $M_0$ must be a trivial $F$–bundle over $S^1$. The global monodromy $D(\gamma_1) \cdots D(\gamma_n)$ is therefore trivial. $M$ can thus be described as

$$M = (F \times D^2) \cup H_1^2 \cup \cdots \cup H_n^2 \cup (F \times D^2).$$

(1)

The converse is also true: a relator $D(\gamma_1) \cdots D(\gamma_n) = 1$ gives rise to a Lefschetz fibration over $S^2$.

It is not difficult to prove that Lefschetz fibration of genus zero must be a blow-up of $S^2$–bundle over a closed surface. Genus one Lefschetz fibrations are also well understood thanks to the work of Kas, Moishezon, Mandelbaum, Harper and Matsumoto (see [M1]). The relatively minimal ones are fiber sums of torus bundles and $E(1)$.

Recently, Donaldson [D] obtained a remarkable result concerning the existence of Lefschetz fibrations. Before stating Donaldson’s result, let us first introduce the definition of a symplectic Lefschetz fibration.

**Definition 2.2.** A Lefschetz fibration $M \to \Sigma$ is called a symplectic Lefschetz fibration if there exists a symplectic form $\omega$ on $M$, such that for any $p \in \Sigma$, $\omega$ is nondegenerate at each smooth point on the fiber $F_p$, and that at each double point, $\omega$ is nondegenerate on the two planes contained in the tangent cone.

Donaldson proves that any symplectic four-manifold admits symplectic Lefschetz fibrations over $S^2$ after perhaps blowing up. Gompf proves that (see also [ABKP], [ST]) most Lefschetz fibrations admit a symplectic structure.

**Theorem 2.3 ([GS]).** If a four-manifold admits a Lefschetz fibration $\pi : M \to \Sigma$ and the fiber represents an essential class, then $M$ admits a symplectic Lefschetz fibration structure. In particular, when $g \geq 2$, $M$ admits a symplectic Lefschetz fibration structure.

We now give some elementary lemmas for Lefschetz fibrations over $S^2$, which will be used in §4.

**Lemma 2.4.** Let $M \to S^2$ be a Lefschetz fibration over $S^2$ with regular fiber $F$. Let $l$, $s$ and $n$ be the number of singular fibers, reducible singular fibers and irreducible singular fibers respectively. Then,

1. $n \geq b_1(F) - b_1(M)$, and $n = 0$ iff $b_1(F) = b_1(M)$;
2. $s + 1 \leq b^- \leq l + 1$, $1 \leq b^+ \leq n + 1$;
3. $\sigma = 4k-l$ for some non-negative integer $k$; if all the singular fibers are reducible, then $\sigma = -l$.

**Proof.** Part 1 is well known since non-separating vanishing cycles represent nontrivial classes in $H_1(F)$ and, from the handlebody description, they generate the kernel
of the natural map $H_1(F) \longrightarrow H_1(M)$ induced by inclusion. We first prove part 2. By Gompf’s theorem, $M$ has symplectic structure, and so $b^+ \geq 1$. Since $b_2$ is bounded by $l + 2$, we immediately get the upper bound for $b^-$. To show $b^- \geq s$, let $G_1, \ldots, G_s$ be the connected components of each reducible singular fiber. We know that $G^2_1 = \cdots = G^2_s = -1$ and $G_i \cdot G_j = 0$ for $i \neq j$. Thus the intersection form on the subspace generated by $G_1, \ldots, G_s$ is negative definite. A regular fiber $F$ is orthogonal to this $s-$dimensional subspace and has square zero. Thus $b^- \geq s^- + 1$ because the intersection form is nondegenerate. Since $b_2 \leq l + 2$, the upper bound of $b^+$ follows.

Now we turn to part 3. By the handlebody description of a Lefschetz fibration, there are $l + 2$ two-handles, $2g$ one-handles and $2g$ three-handles. By Poincare Duality, $l + 2 - b_2 = 2(2g - b_1)$. Since $b_2 = 2b^+ - \sigma$ and $b^+ \equiv b_1 - 1 \pmod{2}$, we find that $l \equiv -\sigma \pmod{4}$. If all the singular fibers are reducible, i.e. $s = l$, then $b^-$ must be $l + 1$ and $b^+ = 1$. Therefore $\sigma = -l$.

Lemma 2.4 can also be proved using the signature computation in [O].

**Lemma 2.5.** For any genus $g$ Lefschetz fibrations with $\sigma \geq -l + 4$, $b_1, b_2, b^+$ and $\sigma$ have upper bounds $2g - 2, l - 2, n - 3$ and $n - s - 4$ respectively.

**Proof.** Since we assume that $\sigma \geq -l + 4$, there exists irreducible singular fibers by part 3 of Lemma 2.4, i.e. $n > 0$. So $b_1 = 2g$ is impossible because it would imply that $n = 0$ by part 1 of Lemma 2.4. If $b_1 = 2g - 1$, by part 1 of Lemma 2.4, the non-separating vanishing cycles generate a rank one subgroup of $H_1(F)$. But this is again impossible since the action of a Dehn twist along a non-separating curve on $H_1(F)$ is of infinite order. Thus we have shown that $b_1 \leq 2g - 2$.

This upper bound of $b_1$, plus the handlebody description, implies $b_2 \leq (l + 2) - 2 \cdot 2 = l - 2$. Finally this upper bound of $b_2$ gives the upper bounds of $b^+_2$ and $\sigma$ with part 2 of Lemma 2.4.

To end this section, we describe the connection between Lefschetz fibrations and the Deligne-Mumford moduli space of stable curves $\overline{M}_g$ (see [Sm2]). Recall that for $g \geq 2$, $\overline{M}_g$ is the stable compactification of $M_g$, the moduli space of curves of genus $g$. It is a projective orbifold, and the compactifying divisor $\mathcal{C} = \overline{M}_g - M_g$ consists of stable curves with at least one node.

By choosing a metric compatible with the symplectic form and Kähler in the neighborhood of the singular fibers, we can obtain a smooth map from the two-sphere to $\overline{M}_g$. This map is restricted to intersect with $\mathcal{C}$, and each intersection point is transverse, positive, and lies outside the locus of curves with more than one node. Such a map is well defined up to isotopy preserving the condition on the intersection with $\mathcal{C}$. Stable Kahler fibrations correspond to holomorphic maps. On $\overline{M}_g$, there is a universal bundle $\mathcal{H}_g$, the Hodge line bundle. Smith identifies the sum of the number of singular fibers and the signature to be $<4c_1(\mathcal{H}_g), \phi_*[S^2]>$. 

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In this section, let $\pi : M \to \Sigma$ be a genus $g$ relatively minimal Lefschetz fibration over a Riemann surface $\Sigma$. We will prove Theorem 1.

When $g = 0$, $M$ is a ruled surface which is excluded by our assumption. When $g = 1$, from the classification alluded before, $c_1^2 = 0$ and hence the inequality holds. So let us assume $g \geq 2$. We can then (and will) choose a symplectic Lefschetz fibration structure on $M$ by Theorem 2.3. First, we need to establish the following important fact.

**Lemma 3.1.** There exist compatible almost complex structures on $M$ for which the fibers are pseudo-holomorphic submanifolds.

**Proof.** Near the singular point $x_i \in C$, the symplectic form constructed by Gompf is Kähler with respect to suitable local coordinates on which the projection has the form $\pi(z_1, z_2) = z_1^2 + z_2^2$. Fixing such an integrable complex structure $J_i$ in a closed neighborhood $U_i$ of each singular point, we see that the intersection $F_y \cap U_i$ is clearly holomorphic.

Away from the singular point set $C$, the tangent bundle along the fibers $P$ is a symplectic sub-bundle. Its $\omega$ orthogonal dual $Q$ is also a symplectic subbundle. On the boundary of $(U_i)$, $P$ and $Q$ are both preserved by $J_i$. It is well known that $J_i$ restricted to $P$ can be extended to a compatible complex structure on the complement of $U_i$, and the same is true for $Q$. Thus we obtain a compatible almost complex structure $J$ for which the fibers are pseudo-holomorphic.

Let $F$ denote the class of fibers with complex orientation.

**Proposition 3.2.** Suppose $M$ is not rational or ruled. Let $E$ be a class represented by an embedded sphere with square $-1$, which has positive pairing with $\omega$, then $E \cdot F > 0$.

**Proof.** Take a compatible almost complex structure $J$ constructed in the lemma above, the fibers are $J$–holomorphic curves. For any compatible almost complex structure $J$, $E$ can be represented by a $J$–holomorphic curve $S$. This is true for $b^+ > 1$, as shown in [T]. In the case $b^+ = 1$, this follows from [LL1] with the additional assumption that $K \cdot E = -1$, and we ([L]) have proved that the assumption is always satisfied unless $M$ is rational or ruled.

If $E \cdot F \leq 0$, by the positivity of intersection, $S$ must be contained in some singular fiber $F_s$, with its irreducible components also being irreducible components of $F_s$. This is possible only if $F_s$ is a reducible fiber and one of its irreducible components is a rational curve with square $-1$, since we know $F_s$ has only one node. However, this contradicts with the assumption that $M \to \Sigma$ is relatively minimal, and the proof is finished.

Proposition 3.2 imply the following result of Stipsicz.
Corollary 3.3 ([S1]). Suppose $\Sigma$ has positive genus and $M \rightarrow \Sigma$ is a relatively minimal Lefschetz fibration, then $M$ is minimal.

Proof. Observe that the intersection number of a surface $S$ with any fiber is simply the degree of the restriction of the projection $\pi: S \rightarrow \Sigma$, and when $S$ is a sphere and the genus of $\Sigma$ is positive, the degree has to be zero. If $M$ is not rational or ruled, it follows from Proposition 3.2 and the observation that $M$ is minimal.

We will finish the proof by showing that $M$ cannot be rational or ruled. Suppose $M$ is rational or ruled. Choose a compatible almost complex structure $J$ as constructed in Lemma 3.1. By [LL1], there exists an irreducible $J$-holomorphic sphere $C$ representing a class $G$ with non-negative square. Since the fibers are $J$-holomorphic, $G \cdot F$ is non-negative. $G \cdot F = 0$ implies that $G$ is an irreducible component of a singular fiber. But this is impossible because any irreducible component of a singular fiber has square $-1$. $G \cdot F$ cannot be positive either by the observation above. Thus the proof of Corollary 3.3 is finished.

We now prove Theorem 1 for the cases $h \geq 1$ and $h = 0$ in Theorems 3.4 and 3.5 separately.

Theorem 3.4. Let $M \rightarrow \Sigma$ be a relatively minimal Lefschetz fibration with fiber $F$. If $g(\Sigma) \geq 1$ and $g(F) \geq 1$, then $c_1^2(M) \geq 2(g(F) - 1)(g(\Sigma) - 1)$.

Proof. Let $K$ denote the canonical class. By Theorem 0.2 (1) in [T], $K$ is represented by a smoothly embedded symplectic submanifold $C$. Furthermore, if $C_1, \ldots, C_k$ are the connected components of $C$, then for each $i$, $C_i^2 \geq -1$. If any $C_i$ is a sphere, since $M$ is minimal by Corollary 3.3, it must have non-negative self-intersection. This would imply that $M$ is rational or ruled according to a Theorem of McDuff ([Mc]), which is excluded by the claim in the second paragraph of Corollary 3.3.

If we project $C_i$ to $\Sigma$, the degree of the projection is $d_i = C_i \cdot F$. Since each $C_i$ has nonzero genus and $\Sigma$ has genus at least one, by a theorem of Kneser (see [Mi]),

$$g(C_i) - 1 \geq d_i(g(\Sigma) - 1).$$

Since the fibers are symplectic, we have the adjunction equality $2(g(F) - 1) = F \cdot F + K \cdot F = K \cdot F$. Similarly, $2(g(C_i) - 1) = C_i \cdot C_i + K \cdot C_i$. Since $\sum_i (g(C_i) - 1) = K^2$ and $\sum_i d_i = K \cdot F$, we have

$$K^2 \geq 2(g(F) - 1)(g(\Sigma) - 1).$$

The theorem is proved because $c_1(M)$ is just $-K$.

Theorem 3.5. Suppose $M$ is not rational or ruled and $M \rightarrow S^2$ is a relatively minimal genus $g$ Lefschetz fibration over $S^2$, then $c_1^2(M) \geq 2 - 2g$, and it is sharp.
Proof. Take a compatible almost complex structure $J$ constructed in Lemma 3.1. Let $E_1, \ldots, E_d$ be the exceptional classes with $J$ holomorphic representatives $S_1, \ldots, S_d$. Since $M$ is not ruled, according to [Mc], $M$ is obtained by blowing up a minimal symplectic manifold $N$ at $d$ points with exceptional curves representing $E_1, \ldots, E_d$. Denote the blow-down map by $p$. Thus

$$K_M = p^*K_N + E_1 + \cdots + E_d,$$

and $K_M^2 = K_N^2 - d$. Since $K_N^2 \geq 0$, by [T] and [Liu], it suffices to show that $d \leq 2g - 2$. To prove $d \leq 2g - 2$, we just need to show that

$$(E_1 + \cdots + E_d) \cdot F \leq 2g - 2,$$

since $E_i \cdot F = S_i \cdot F \geq 1$. By the adjunction formula, $K \cdot F = 2g - 2$. Thus (3.2) is equivalent to the claim that $p^*K_N \cdot F \geq 0$. This clearly is true if $K_N$ is a torsion class.

Suppose $K_N$ is not a torsion class. When $b^+(M) > 1$, by Theorem 0.2 in [T] and by the blow-up formula of Gromov-Taubes invariants in [LL2], the class $p^*K_N$ can be represented by a $J$ holomorphic curve with components $T_1, \ldots, T_l$. Since $F$ is represented by an irreducible $J$ holomorphic curve with square zero, the positivity of intersection gives $T_i \cdot F \geq 0$ and hence the claim. When $b^+(M) = 1$, $GT_N(2K_N)$ is shown to be nontrivial (see [LL3]). Again by the blow-up formula of Gromov-Taubes invariants, $GT_M(p^*2K_N) = GT_N(2K_N)$. Thus $p^*2K_N$ can also be represented by $J$ holomorphic curve (possibly disconnected) and we have the claim by similar argument.

This inequality is sharp for many stable holomorphic Lefschetz fibrations on blowups of K3 surfaces. Consider a generic pencil in a very ample system with square $2h$. The base locus consists of $2h$ points. The generic members are embedded curves with genus $h + 1$ and the only singularity of each singular member is a nodal point. Blow up the base locus, we obtain a genus $h + 1$ holomorphic stable Lefschetz fibration over $S^2$ on $K3#2h\overline{CP}^2$. Clearly, $c_1^2$ and $2 - 2(1 + h)$ are both equal to $-2h$. Theorem 3.5 is proved.

It is proved in [S1] by a self fiber sum argument that $c_1^2 \geq 4 - 4g$ for any Lefschetz fibrations. Examples of $M$ supporting Lefschetz fibrations over $S^2$ with $c_1^2 = 4 - 4g$, as constructed in [GS], are necessarily rational or ruled by Theorem 3.5.

Theorem 3.4 and 3.5 complete Theorem 1.

Theorem 1 provides the symplectic analogue of Iitaka’s conjecture $C_{2,1}$ concerning the Kodaira dimensions of the total space, the fiber and the base of a stable
holomorphic Lefschetz fibration. We first introduce the definition of the symplectic analogue of the Kodaira dimension.

**Definition 3.6.** The Kodaira dimension \( k(M) \) of a minimal symplectic 2−manifold or a 4−manifold with symplectic form \( \omega \) and symplectic canonical class \( K \) is defined in the following way,

\[
\begin{align*}
    k(M) &= -\infty & \text{if } K \cdot \omega < 0; \\
    k(M) &= 0 & \text{if } K \cdot \omega = 0; \\
    k(M) &= 1 & \text{if } K \cdot \omega > 0 \text{ and } K^2 = 0; \\
    k(M) &= 2 & \text{if } K \cdot \omega > 0 \text{ and } K^2 > 0.
\end{align*}
\]

The Kodaira dimension of a non-minimal symplectic 4−manifold is the Kodaira dimension of one of its minimal models.

The definition for non-minimal symplectic 4−manifolds does not depend on the choice of the minimal model. This is because that only rational and ruled symplectic four manifolds have more than one minimal models (see [Mc] and [L]), which all have Kodaira dimension \(-\infty\).

The symplectic manifolds with Kodaira dimension \(-\infty\) have been classified. They are just the rational or ruled surfaces by results of Taubes and Liu ([T], [Liu]). We speculate that four-manifolds with Kodaira dimension zero either have Kähler structure or are torus bundles over torus.

With the above definition, the following is immediate from Theorem 1.

**Corollary 3.7.** The Kodaira dimension of a Lefschetz fibration is subadditive, i.e. if \( M \to \Sigma \) is a Lefschetz fibration with fiber \( F \), then \( k(M) \geq k(F) + k(\Sigma) \).

**§4. The number of singular fibers**

In this section, let \( M \to S^2 \) be a Lefschetz fibration over \( S^2 \). We assume \( g \geq 2 \), since the cases for \( g = 0 \) and \( g = 1 \) are well understood. We have described in §2 the correspondence between Lefschetz fibrations over \( S^2 \) and relators consisting of positive Dehn twists in the mapping class groups. Given a Lefschetz fibration, the number of singular fibers \( l \) is just the length of the corresponding relator, and the number of irreducible singular fibers \( n \) and the number of reducible singular fibers \( s \) are the numbers of positive Dehn twists along nonseparating curves and separating curves in the relator respectively. Here we study the lower bounds of \( n, l \) and \( s \).

The story for \( s \) is very simple – there are Lefschetz fibrations with no reducible singular fibers for each \( g \). In this section, we will focus on the lower bound of \( n \). We will also provide an estimate of the lower bound of \( l \).

For the lower bound of \( n \), we need to establish the lower bound of the signature.

**Proposition 4.1.** There are no Lefschetz fibrations over \( S^2 \) with \( \sigma = -l \).
Lemma 4.2. Let $M \to S^2$ be a genus $g$ Lefschetz fibration on a ruled surface over a genus $h$ Riemann surface $W$. Then $g \geq 2h - 1$.

Proof. When $h$ is 0, the statement is obvious. So we assume $h > 0$. Let $F$ be a fiber. The composition of blowing down $p$ and projection $q$ to the base of the $S^2$-bundle $N$ gives rise to a smooth map $q \circ p : F \to W$. It is shown in [LL1] that the fiber of any $S^2$-bundle has pseudo-holomorphic representative for any compatible almost complex structure. Thus the map $q \circ p$ must have positive degree because of the positivity of intersection. This implies $g \geq h$. If we assume that $h \leq g < 2h - 1$, then any orientation-preserving map from $\Sigma_g$ to $\Sigma_h$ must be of degree 1.

Let us first assume that $N$ is the trivial $S^2$-bundle. Let $U$ be a fiber and $V$ the section class of the $S^2$-bundle $N$ such that $U^2 = V^2 = 0$ and $U \cdot V = 1$. Let $E_1, \ldots, E_k$ be the exceptional classes. We denote the class of fibers of the Lefschetz fibration also by $F$. Since $U$, $V$, $E_1, \ldots, E_k$ form a basis of $H_2(M; \mathbb{Z})$,

$$F = aU + bV + c_1E_1 + \cdots + c_kE_k$$

for some integers $a, b, c_1, \ldots, c_k$, where $c_i \leq 0$. Since the degree of $q \circ p : F \to W$ is $b$, $b$ must be 1. From $F \cdot F = 0$, we get $2a - c_1^2 - \cdots - c_k^2 = 0$. Recall that the canonical bundle is given by $K = (2h-2)U - 2V + E_1 + \cdots + E_k$. From the adjunction formula, we find $2h - 2 - 2a - c_1 - \cdots - c_k = 2g - 2$. Thus $(c_1^2 + c_1) + \cdots + (c_k^2 + c_k) = 2h - 2g$. Under the assumption that $h \leq g$, this is possible only if $h = g$. However, in the case $h = g$, if $F$ is a reducible fiber, each of its component has genus less than $h$; if $F$ is an irreducible fiber, its normalization has genus $h - 1$. In either case, $b$ is forced to be 0 rather than 1, which leads to contradiction.

Similar argument applies to the case when $N$ is the nontrivial $S^2$-bundle. The lemma is proved.

We now prove Proposition 4.1.

Proof. Let $M \to S^2$ be a genus $g$ Lefschetz fibration such that $\sigma = -l$. It is easy to see that $b^+ = 1$ and $b_1 = 2g$. Then $M$ is the blow-up of a $S^2$-bundle over a genus $g$ surface by Theorem A in [Liu]. But under the assumption that $g \geq 2$, this contradicts Lemma 4.2. The proposition is proved.

Together with part 3 of Lemma 2.4, we immediately have the following corollary.

Corollary 4.3. Any Lefschetz fibration over $S^2$ has at least one irreducible singular fiber.

In [ABKP], the authors conjectured that the monodromy group is not contained in the Torelli group. This conjecture was proved in [Sm2]. Since the Torelli group is generated by Dehn twists along separating curves, their conjecture is also a consequence of Corollary 4.3.
Let $\mu(M)$ be the lowest genus of Lefschetz fibrations over $S^2$ on blowups of $M$. The $\mu$ invariant is zero for $\mathbb{C}P^2$ and $S^2 \times S^2$, one for elliptic surfaces, and three for the four-torus (see [Sm1]). In fact, with a little more effort we can determine $\mu$ for ruled surfaces.

**Proposition 4.4.** Let $M$ be a ruled surface over a genus $h$ Riemann surface. Then $\mu = 2h$.

**Proof.** When $h = 0$, $\mu = 2h$ is obvious. So we assume $h \geq 1$. Suppose there is a genus $g = 2h - 1$ Lefschetz fibration. By Kneser’s theorem, the degree of $q \circ p$ is at most two. On the other hand, by the argument in Lemma 4.2, we can rule out the case when the degree of $q \circ p$ is one. Therefore it must be exactly two. Consider an irreducible singular fiber, whose existence is due to Corollary 4.3. Its normalization is a surface of genus $2h - 1$, and therefore does not admit a degree two map to a genus $h$ surface. Thus we have shown $g$ has to be greater than $2h - 1$.

To show that $\mu = 2h$, we need to construct $g = 2h$ Lefschetz fibrations. There are many constructions of such fibrations generalizing the example of Matsumoto ([M2]). We sketch one here. Take the trivial fibration $S^2 \times \Sigma_h$, and let $U$ and $V$ be the fiber class and the section class as above. Consider the divisor class $U + 2V$, which has square four and its smooth members have genus $g = 2h$. Blowing up four times, we obtain a genus $2h$ Lefschetz fibrations.

**Corollary 4.5.** On any genus $g$ Lefschetz fibration over $S^2$ of the blowup of an $S^2$–bundle, there are at least $2g$ singular fibers and $g$ irreducible singular fibers.

**Proof.** Let $M$ be a blowup of an $S^2$–bundle over a surface of genus $h$. Suppose $M$ admits a Lefschetz fibration of genus $g$ with $l$ singular fibers. Since the Euler number of an $S^2$–bundle over a surface of genus $h$ is $-2(2h - 2)$, we find that $l + -2(2g - 2) \geq -2(2h - 2)$. Since $g \geq 2h$, we have $l \geq 2g$. And by Lemma 2.4, we have $n \geq 2g - 2h \geq g$.

**Corollary 4.6.** Let $M \to S^2$ be a genus $g$ Lefschetz fibration. If $M$ is not rational or ruled, $n \geq (6g + 6)/5 + s/5$.

**Proof.** By Theorem 3.5, we have $c_1^2 \geq 2 - 2g$. Since $c_1^2 = 2e + 3\sigma$ and $e = 4(g - 1)(-1) + l$, $\sigma = (-2l - 8(g - 1)(-1) + c_1^2)/3$. By Lemma 2.5 and Proposition 4.1, $\sigma \leq n - s - 4$, thus we find $n \geq (6g + 6)/5 + s/5$.

Now Theorem 2 follows from Corollaries 4.5 and 4.6.

We want to remark that when $g$ is odd, we can in fact show $n \geq g + 1$ with a more detailed analysis on ruled surfaces. A stronger bound for Lefschetz fibrations on ruled surfaces is recently obtained in [S2].

When $g$ is low, it should be possible to determine the exact lower bound of $n$. We believe that the exact bound is six when $g = 2$, and it is twelve when $g = 3$. Examples with those numbers of irreducible singular fibers include genus
two Lefschetz fibration on \( S^2 \times T^2 \# 4\overline{CP}^2 \) in \([M2]\) and genus three fibrations on some torus bundles in \([Sm1]\). We are not yet able to prove the exact bound, but we will present the best estimate in the following proposition.

**Proposition 4.7.** The number of irreducible singular fibers of a genus \( g \) Lefschetz fibration over \( S^2 \) is no less than four. And it is no less than six if \( g \geq 3 \).

**Proof.** By Lemma 2.4 and Lemma 2.5, \(-n-s+4 \leq \sigma \leq n-s-4\). So there are at least four irreducible singular fibers. If \( g \geq 3 \), the statement follows similarly from Lemma 2.5 and the following lemma.

**Lemma 4.8.** If \( \sigma(M) = -l + 4 \), then \( g \leq 2 \). And

1. if \( g = 1 \), \( M \) is the rational elliptic surface \( E(1) \);
2. if \( g = 2 \), \( M \) has \( b^+ = 1 \) and \( b_1 = 2 \).

**Proof.** Let us assume that \( \sigma = -l + 4 \). By Lemma 2.5, \( b^+ = 1 \) and \( b_1 = 2g - 2 \). The last statement of the lemma follows. If \( g \geq 3 \), \( M \) is the blow up of an \( S^2 \)-bundle over a genus \( g-1 \) surface according to \([Liu]\). But when \( g \geq 3 \), \( g \leq 2(g-1) - 1 = 2g - 3 \), which is ruled out by Proposition 4.4.

If \( g = 1 \), we know \( M \) is diffeomorphic to \( E(k) \) hence \( \sigma = -8k \) and \( l = 12k \). Therefore \( \sigma = 4 - l = 4 - 12k \) implies that \( k = 1 \), so \( M \) is \( E(1) \).

Now we state a lower bound of the number of singular fibers, which follows from Cor. 4.5 and 4.6.

**Proposition 4.9.** The number of singular fibers in a genus \( g \) Lefschetz fibration over \( S^2 \) is at least \((6g + 6)/5\).

The best estimate, due to Stipsicz ([S2]), is \( l \geq 8/5g \). We believe that the optimal bound of \( l \) is of the order \( 2g \). In fact, Gompf conjectures that the Euler number of a symplectic manifold \( M \) is non-negative if \( M \) is not a blow-up of an \( S^2 \)-bundle over a surface of genus at least two. If this conjecture holds, then it is easy to see that there are at least \( 4g - 4 \) singular fibers for any Lefschetz fibration on any manifold which is not an \( S^2 \) bundle. With Corollary 4.5, we will be able to conclude that there are at least \( 2g + 2 \) singular fibers if \( g \) is odd and \( 2g \) singular fibers if \( g \) is even.

Recall that at the end of §2, we introduced the geometric approach viewing genus \( g \) Lefschetz fibrations as isotopy classes of smooth maps from the two sphere to the Deligne-Mumford moduli space of curves \( \overline{M}_g \) which have transverse positive intersections with \( C \). On \( \overline{M}_g \), there is a universal bundle \( \mathcal{H}_g \), the Hodge line bundle. Following from Smith’s signature formula and Corollary 4.6, we have a positive lower bound of \( c_1(\mathcal{H}_g) \) linearly in the genus.

**Corollary 4.10.** Suppose a genus \( g \) Lefschetz fibration over \( S^2 \) corresponds to a smooth map \( \phi : S^2 \longrightarrow \overline{M}_g \).

\[
< c_1(\mathcal{H}_g), \phi_*[S^2] > \geq \frac{1}{12} l + \frac{g-1}{3} \geq \frac{3g-2}{6}.
\]
Notice that for holomorphic Lefschetz fibration, the positivity is obvious since $\mathcal{H}_g$ is an ample line bundle and $\phi$ is a holomorphic map. As remarked in [Sm1], this is not a purely homological statement. Since by Wolpert’s ([W]) computation of the homology of $\mathcal{M}_g$, there are two dimensional homology classes which have positive intersections with all the components of $C$ but not with $c_1(\mathcal{H}_g)$.

In [ABKP], the authors ask whether the pairing is still non-negative when the genus $h$ of the base surface is positive. From Theorem 1 we can similarly derive

$$< c_1(\mathcal{H}_g), \phi^* [\Sigma_h] > \geq -\frac{1}{2} (h - 1) (g - 1) + \frac{1}{12} l.$$

This provides an affirmative answer to their question when $h = 1$.

References.

[ABKP] J. Amoros, F. Bogomolov, L. Katzarkov, T. Pantev, Symplectic Lefschetz fibrations with arbitrary fundamental groups.

[D] S. Donaldson, Lefschetz fibrations in symplectic geometry. Doc. Math. J. DMV., Extra Volume ICMII (1998) 309-314.

[E] H. Endo, Meyer’s signature and hyperelliptic fibrations, preprint.

[GS] R. Gompf and A. Stipsicz, A introduction to 4-manifolds and Kirby calculus, book in preparation.

[K] A. Kas, On the handlebody decomposition associated to a Lefschetz fibration, Pacific. J. Math. 89 (1980) 89-104.

[Ko] D. Kotschick, Signatures, monopoles and mapping class groups, Math. Research Letter 5 (1998) 227-235.

[L] T. J. Li, Smoothly embedded spheres in symplectic four manifolds, Proc. AMS. 127 (1999) 609-613.

[Liu] A-K. Liu, Some new applications of the general wall crossing formula, Math. Research Letters, 3 (1996) 569-585.

[LL1], T. J. Li and A. K. Liu, Symplectic structures on ruled surfaces and a generalized adjunction inequality, Math. Research Letters 2(1995) 453-471.

[LL2], T. J. Li and A. K. Liu, On the equivalence between SW and Gr in the case $b^+ = 1$, IMRN, (1999) 335-345.

[LL3], T. J. Li and A. K. Liu, Symplectic submanifolds in symplectic four manifolds, in preparation.

[M1] Y. Matsumoto, Diffeomorphism types of elliptic surfaces, Topology 25 (1986) 549-563.
[M2] Y. Matsumoto, Lefschetz fibrations of genus two—a topological approach, Proceedings of the 37th Taniguchi Symposium on Topology and Teichmuller Spaces, ed. S. Kojima et al., World Scientific (1996) 123-148.

[Mc]. D. McDuff, The structure of rational and ruled symplectic 4—manifold, Jour. AMS. v.1. no.3. (1990), 679-710.

[Mi] J. W. Milnor, On the existence of a connection with curvature zero, Comm. Math. Helv. 32 (1959) 215-223.

[O] B. Ozbagci, Signatures of Lefschetz fibrations, preprint.

[Sm1] I. Smith, Symplectic geometry of Lefschetz fibrations, Oxford thesis, 1998.

[Sm2] I. Smith, Lefschetz fibrations and the Hodge bundle, Geometry and Topology 3 (1999) 211-233.

[S1] A. Stipsicz, Chern numbers of certain Lefschetz fibrations, Proc. AMS. to appear.

[S2] A. Stipsicz, Singular fibers in Lefschetz fibrations on manifolds with $b^+ = 1$, preprint.

[ST] B. Siebert and G. Tian, On hyperelliptic $C^\infty$—Lefschetz fibrations of four manifolds, Commun. Contemp. Math. 1 (1999) no.2 255-280.

[T] C. Taubes, $SW \Rightarrow Gr$: From Seiberg-Witten equations to pseudoholomorphic curves, JAMS 9 (1996) 845-918.

[W] S. Wolpert, On the homology of the moduli space of stable curves, Ann. of Math., 118 (1983) 491-523.

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