Geometric modeling of offset curves in contour-parallel pocket machining of engineering products

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Abstract. In the present paper analytical solutions to suggested geometric models of equidistant curve formation for flat contours represented by second-order curves and areas bounded by them are considered. Geometric models are spatial and based on cyclographic representation. They differ from the known algebraic models and their solutions to the formation task under consideration in that they present capability to acquire a more complete and vivid spatial representation of interconnection and interrelation of all the geometric objects of the model on the stage of computer spatial visualization. Examples confirming the validity of the suggested geometric models of the considered formation task are provided. The models can be applied in automated cutting tool trajectory design for pocket machining on NC units.

1. Introduction
An equidistant curve constitutes a contour-parallel curve, i.e. a curve of equal distances for all of its points with respect to a given contour curve. Equidistant curve (a family of equidistant curves) finds application in automated instrument trajectory calculation in item machining on NC units and in CAD/CAM systems [1,2].

In practical tasks of equidistant curve family calculation, given a planar area limited by a certain boundary contour, it is required to solve the task of “trimming” of self-intersecting equidistant curves. For this purpose medial axis transform (MAT) of the area is applied. By “medial axis transform” of an area we mean a multitude of all circles of maximum radius inscribed in the area, i.e. tangent to its boundary contour. Center coordinates \((x,y)\) and radius \(R\) of such circles are represented by three numbers, the multitude of which for all the inscribed maximum radius circles is defined by MAT [3,4,5,6]. Centers of all these circles generate medial axis (MA). Therefore, MAT, as a multitude of pairs “point, radius”, allows us to solve the task of equidistant curve trimming in order to perform subsequent transition to automated cutting tool trajectory design applied in contour-parallel pocket machining of specified area with boundary contour on NC units.

There is, however, a different approach to generating a family of equidistant curves, further referred to as a multitude of offset curves (OC). This approach is based on representing MAT as a certain spatial image, bijectively correspondent to a given prototype – a flat area with a boundary contour. In this new approach MAT is represented by a spatial curve, restored by spatial cyclographic representation [7,8] on the basis of geometric information of the area and its boundary contour. The spatial curve is generated upon intersection of various linear \(\alpha\)-surfaces featuring generatrixes inclined to the plane of the contour on angle \(\alpha = 45^\circ\) [9]. These \(\alpha\)-surfaces are formed on the basis of geometric information of the area and its boundary contour. The acquired \(\alpha\)-surfaces are trimmed according to the acquired MAT, which results in formation of a certain \(\alpha\)-shell covering the given area. The \(\alpha\)-shell and the flat area with its boundary contour are bijectively correspondent two-dimensional geometric objects. Further section of the acquired \(\alpha\)-shell by means of a multitude of planes along the \(z\) axis with step \(\Delta z = \delta = \text{const}\) results in generation of a family of level curves, orthogonal projections of which onto the
plane of the given area constitutes a multitude of \( OC \) inside the given area, which is required to perform further automated pocket machining tool trajectory calculation and NC unit programming.

2. Problem definition
Upon comparison of the two approaches to \( OC \) formation – the known one and the one suggested in the present paper – it is possible to notice the difference in formation technology: in the first case \( MA \) is formed on the basis of compacting interpolation - compaction of a multitude of tangent circles filling a flat area - with further construction of \( OC \) multitude and, when necessary, \( MAT \); in the second case, initially, \( MAT \) is formed and a multitude of flat sections of \( \alpha \)-shell are constructed along the \( z \)-axis, then, by means of orthogonal projection onto a flat plane, \( MA \) and \( OC \) multitude are acquired. In the suggested approach the necessity to perform complex analytic operations over a multitude of tangent circles in order to acquire \( MA \), \( MAT \) and construct \( OC \) vanishes. At the same time, it is possible to acquire a more complete and vivid spatial representation of interconnection and interrelation of all the geometric objects taking part in \( OC \) multitude generation. On the basis of the suggested approach to \( OC \) generation the following problem is set: to develop a geometric model featuring an analytic description, that would realize this approach, and to verify it experimentally.

3. Theory of \( \alpha \)-shell formation
An \( \alpha \)-shell constitutes a certain surface limited by \( MAT \) in height, i.e. along the direction of axis \( z \) – and resting over boundary contour of an area on a plane. A bijective correspondence is established between two-dimensional multitudes of points of the \( \alpha \)-shell and the area.

\( MAT \) formation
Consider a certain contour \((a,b)\) defined on plane \((xy)\) and consisting of curves \( a \subset C^2, b \subset C^2 \):
\[
\begin{align*}
a : \mathbf{r}_a &= (x_a(t_1), y_a(t_1)); \quad b : \mathbf{r}_b = (x_b(t_2), y_b(t_2)); \quad t_1, t_2 \in R.  
\end{align*}
\]
(1)
Considering that curves \( a \) and \( b \) are involutes, let us put down the equation of their evolutes \( e_a \) and \( e_b \):
\[
\begin{align*}
e_a : \mathbf{r}_{ea}(t_1) &= (x_{ea}(t_1), y_{ea}(t_1)) = \mathbf{r}_a + R_a n_a, \\
e_b : \mathbf{r}_{eb}(t_2) &= (x_{eb}(t_2), y_{eb}(t_2)) = \mathbf{r}_b + R_b n_b.  
\end{align*}
\]
(2)
(3)
Let us construct the following spatial curves \( e_{a1} \) and \( e_{b1} \) for evolutes \( e_a \) and \( e_b \):
\[
\begin{align*}
e_{a1} : \mathbf{r}_{a1}(t_1) &= (x_{a1}, y_{a1}, z_{a1}) = \pm \sqrt{(x_a - x_{ea})^2 + (y_a - y_{ea})^2}, \\
e_{b1} : \mathbf{r}_{b1}(t_2) &= (x_{b1}, y_{b1}, z_{b1}) = \pm \sqrt{(x_b - x_{eb})^2 + (y_b - y_{eb})^2}.  
\end{align*}
\]
(4)
(5)
Curves \( a, e_{a1}, \) and \( b, e_{b1} \) in pairs generate linear \( \alpha \)-surfaces \( Q_1 \) and \( Q_2 \), for which they serve as guiding curves:
\[
\begin{align*}
Q_1 : \mathbf{r}_1(t_1,l_1) &= \mathbf{r}_{a1}(t_1) + l_1 \cdot (\mathbf{r}_a(t_1) - \mathbf{r}_{a1}(t_1)), \\
Q_2 : \mathbf{r}_2(t_2,l_2) &= \mathbf{r}_{b1}(t_2) + l_2 \cdot (\mathbf{r}_b(t_2) - \mathbf{r}_{b1}(t_2)).  
\end{align*}
\]
(6)
(7)
The curve of intersection of the \( \alpha \)-surfaces constitutes \( MAT : Q_1 \cap Q_2 = MAT \). Let us explain this. Let us call the curve of intersection \( s = Q_1 \cap Q_2 \). Then \( \alpha \)-projection of the curve \( s \) of the surface \( Q_1 \) is the curve \( a \). \( \alpha \)-projection of the curve \( s \) of the surface \( Q_2 \) is the curve \( b \). An \( \alpha \)-projection of a curve \( s \) of an \( \alpha \)-surface is understood to be a multitude of points of intersection between the generatrices of the \( \alpha \)-surface and plane \((xy)\). At that, the generatrices are inclined to plane \((xy)\) on angle equal to 45° and pass through the curve \( s \). Therefore, curves \( a \) and \( b \) constitute two branches of the curve of intersection between plane \((xy)\) and envelope of one-parameter set of \( \alpha \)-cones with vertexes locates on curve \( s \) and axes perpendicular to plane \((xy)\). The base of each cone on plane \((xy)\) constitutes a circle of radius \( R \) tangent to curves \( a \) and \( b \). Therefore, a continuous multitude of triples of numbers \((x,y,R=2)\) is generated on plane \((xy)\). This multitude constitutes \( MAT \) according to the above mentioned algebraic definition.
4. Results of experiments
Let us consider MAT and MA construction given curves of second order, assuming we would use them later on as primitives for formation of combined boundary contours. MAT, MA, and OC construction for curves of second order
A contour represented by an ellipse defined by two constituents \( q_1 \) and \( q_2 \) is depicted on figure 1.

The equations of the constituents \( q_1 \) and \( q_2 \) is of the following form:

\[
q_1 : x_1 = a_1 \frac{2t_1}{1 + t_1^2}, \quad y_1 = b_1 \frac{1 - t_1^2}{1 + t_1^2}, \quad -1 \leq t_1 \leq 1, \quad a_1 = 4, \quad b_1 = 2;
\]

\[
q_2 : x_2 = a_2 \frac{2t_2}{1 + t_2^2}, \quad y_2 = b_2 \frac{1 - t_2^2}{1 + t_2^2}, \quad -1 \leq t_2 \leq 1, \quad a_2 = 4, \quad b_2 = 2.
\]

On the basis of these equations and by the use of the above mentioned algorithm of definition of surfaces \( Q_1 \) and \( Q_2 \) (equations (6), (7)), it is possible to acquire these surfaces (figure 2) and MAT curve defined as \( m_{1,2} \). Subsequently, after trimming of the acquired surfaces \( Q_1 \) and \( Q_2 \) along the line \( m_{1,2} \), the \( \alpha \)-shell \( Q_{1,2} \) is formed (figure 3).
Further section of α-shell $Q_{1,2}$ with a bundle of planes $P_i (\Delta z_i = \delta = \text{const})$ results in generation of a multitude of OC (figure 4) on the shell and on plane ($xy$) (figure 5) with $MA(m_{1,2}(xy))$ curve.

Another initial contour – parabola $(q_1, q_2)$ – is depicted on figure 6. Equations of its constituents are of form

\[ q_1 : x_1 = -t_1, y_1 = \frac{t_1^2}{2p_1}, p_1 = \frac{1}{4}, -1 \leq t_1 \leq 0; \]

\[ q_2 : x_2 = -t_2, y_2 = \frac{t_2^2}{2p_2}, p_2 = \frac{1}{4}, 0 \leq t_2 \leq 1. \]
On the basis of the initial contour – parabola – $MAT (m_{1,2})$ and $\alpha$-shell $Q_{1,2}$ are constructed (figure 7). Subsequently, the procedure of section of $\alpha$-shell $Q_{1,2}$ with a bundle of planes $P_1 (\Delta z = \delta = \text{const})$ is performed and a multitude of $OC$ is acquired on plane $(xy)$ (figure 8).

The next initial contour – hyperbola $(q_1, q_2, q_3, q_4)$ (figure 9) – is defined by equations of its constituents:

$$q_1 : x_1 = \frac{e^{t_1} + e^{-t_1}}{2}, y_1 = \frac{e^{t_1} - e^{-t_1}}{2}, -1 \leq t_1 \leq 0; \quad q_2 : x_2 = \frac{e^{t_2} + e^{-t_2}}{2}, y_2 = \frac{e^{t_2} - e^{-t_2}}{2}, 0 \leq t_2 \leq 1;$$

$$q_3 : x_3 = \frac{e^{t_3} + e^{-t_3}}{2}, y_3 = \frac{e^{t_3} - e^{-t_3}}{2}, -1 \leq t_3 \leq 0; \quad q_4 : x_4 = \frac{e^{t_4} + e^{-t_4}}{2}, y_4 = \frac{e^{t_4} - e^{-t_4}}{2}, 0 \leq t_4 \leq 1.$$

Figure 6. Initial contour: parabola $(q_1, q_2)$.

Figure 7. Section of $\alpha$-shell over parabolic contour $(q_1, q_2)$ with a bundle of planes.

Figure 8. Aggregate result: a multitude of $OC$ of parabola.
As in the previous examples, let us begin with formation of \( \alpha \)-surfaces \( Q_1, Q_2, Q_3 \) and \( Q_4 \) and corresponding \( \alpha \)-shells \( Q_{1,2}, Q_{3,4} \). Subsequent section of the acquired shells with a bundle of planes \( P_i (\Delta z = \delta = \text{const}) \) allows us to acquire a multitude of OC on plane \((xy)\) (figure 10).

![Figure 9. Initial contour: hyperbola \((q_1, q_2, q_3, q_4)\).](image)

![Figure 10. Aggregate result: a multitude of OC of hyperbola.](image)

The results of solution of a more complex task of OC formation are presented on the example of symmetrical compound closed contour \( \partial \Omega \) bounding area \( \Omega \) on plane \((xy)\) (figure 11). The contour \( \partial \Omega \) consists of arcs of ellipse \((q_1, q_1')\), arcs of two hyperbolas \((q_{21}, q_{22})\) and \((q_{21}', q_{22}')\), arcs of two hyperbolas \((q_{31}, q_{32})\) and \((q_{31}', q_{32}')\), arcs of two hyperbolas \((q_{41}, q_{42})\) and \((q_{41}', q_{42}')\). In junction points \( T_1, \ldots, T_5 \) and \( T'_1, T'_2, T'_3, T'_4, T'_5 \) the second-order arcs are connected with the second order of smoothness. The formation of \( \alpha \)-surfaces is performed given the compound contour \( \partial \Omega \) in accordance with the mentioned algorithm (figure 12). Subsequently, trimming is performed in order to acquire a compound \( \alpha \)-shell (figure 13), which is then dissected with a bundle of planes \( P_i (\Delta z = \delta = \text{const}) \) in order to acquire a multitude of level curves with respect to plane \((xy)\), symmetrical with respect to plane \((xz)\) (figure 14). Further orthogonal projection of the level curves onto plane \((xy)\) allows us to acquire the multitude of OC of area \( \Omega \) with contour \( \partial \Omega \) (figure 15).

5. Consideration of results
Geometric models and their analytical description for acquiring the multitude of OC of second-order curves on the basis of cyclographic representation have, as follows from the considered examples, an analytic solution. This allows us to consider such curves and multitudes of OC as certain primitives that can be utilized to construct multitudes of OC for compound contours consisting of sections – arcs of curves of second order – connected with order of smoothness \( C^2 \). The example of solution of such task provided in the present paper proves the above.

![Figure 11. The initial compound contour \( \partial \Omega \).](image)

![Figure 12. Formation of \( \alpha \)-surface over compound contour \( \partial \Omega \).](image)
6. Conclusion

It is possible to acquire an analytical solution to the task of formation of multitude of offset curves given a compound contour constituting arcs of second order and area bounded by it. The contour in question represents pocket surface profile of an engineering product. This significantly simplifies the computer-aided tool trajectory calculation for machining of such surfaces on NC units and in CAD/CAM systems. In the line of research touched upon in the present paper, the application of the considered approach to offset curves formation given a compound contour constituting arcs of certain splines arouses theoretical and practical interest.

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