Abstract

A directed graph $D = (V(D), A(D))$ has a kernel if there exists an independent set $K \subseteq V(D)$ such that every vertex $v \in V(D) - K$ has an ingoing arc $u \rightarrow v$ for some $u \in K$. There are directed graphs that do not have a kernel (e.g. a 3-cycle). A quasi-kernel is an independent set $Q$ such that every vertex can be reached in at most two steps from $Q$. Every directed graph has a quasi-kernel. A conjecture by P.L. Erdős and L.A. Székely (cf. A. Kostochka, R. Luo, and, S. Shan, arxiv:2001.04003v1, 2020) postulates that every source-free directed graph has a quasi-kernel of size at most $|V(D)|/2$, where source-free refers to every vertex having in-degree at least one. In this note it is shown that every source-free directed graph that has a kernel also has a quasi-kernel of size at most $|V(D)|/2$, by means of an induction proof. In addition, all definitions and proofs in this note are formally verified by means of the Coq proof assistant.

In this note, all directed graphs $D = (V(D), A(D))$ are assumed to be finite and without self-loops. The notation $u \rightarrow v$ is used to denote $(u, v) \in A(D)$. A set $I \subseteq V(D)$ is independent if there are no two vertices $u, v \in I$ connected by an arc in any direction. A kernel $K \subseteq V(D)$ is an independent set such that every vertex in $V(D)$ can be reached from a vertex in $K$ in at most one step. A quasi-kernel $Q \subseteq V(D)$ is a weakening of the concept of kernel by requiring that every vertex in $V(D)$ can be reached in at most two steps from a vertex in $Q$. A source in a directed graph is a vertex $u \in V(D)$
having only outgoing arcs. A directed graph is therefore said to be source-free if every vertex has at least one ingoing arc.

Based on these definitions, it is clear that every kernel is also a quasi-kernel. However, there do exist source-free directed graphs that do not have a kernel, for instance an odd directed cycle. Moreover, there are source-free directed graphs that do have a kernel, but none less than or equal to $|V(D)|/2$ in size, as shown by the examples in Figure 1. It is straightforward to prove that every directed graph has a quasi-kernel [2].

The following conjecture is attributed to P.L. Erdős and L.A. Székely (cf. [1, 6, 10]).

**Conjecture 1.** Every source-free digraph has a quasi-kernel of size at most $|V(D)|/2$.

The purpose of this note is to prove Conjecture 1 in the presence of a kernel, by means of an induction proof of moderate technicality. A formal verification of this proof by means of the Coq proof assistant is also supplied with this paper\(^1\).

A kernel in a directed graph is somewhat related to the concept of an independent dominating set in the context of undirected graphs [4], and some

\(^1\)This proof has been checked using version 8.4pl4 of the Coq proof assistant, and is not guaranteed to work in earlier or later versions of Coq.
general terminology in this note is inherited from the theory of domination in undirected graphs [3].

A digraph has a kernel if and only if it does not have a directed odd cycle [9]. Existence of a quasi-kernel of size at most $|V(D)|/2$ is guaranteed in case $D$ is a tournament, semicomplete multipartite, or locally semicomplete [7], or an orientation of a graph with chromatic number at most four [6].

The two definitions below are followed by a brief outline of the solution proposed in this note. It is assumed that $D$ is a directed graph in these two definitions.

**Definition 1.** For $S \subseteq V(D)$ and $u \in S$, an out-neighbor $v \in V(D) - S$ of $u$ is said to be an external private out-neighbor (epon) with regard to $S$, if for all $w \in S$ such that $w \rightarrow v$ it holds that $u = w$.

As an example, the right-most vertex in Figure 1 is an epon with regard to the black vertices. The directed graph on the left does not have an epon.

**Definition 2.** A quasi-kernel $Q \subseteq V(D)$ is inward dominated if for all $w \in Q$ and $v \in V(D) - Q$ such that $v \rightarrow w$, there exists a $u \in Q$ such that $u \rightarrow v$.

The induction proof in Theorem 1 is outlined briefly here. First, Lemma 1 is used to derive existence of an inward dominated quasi-kernel $K$, based on the assumption of a kernel being present. If every vertex $u \in K$ has an epon, then the conclusion $|K| \leq |V(D)|/2$ follows directly from Lemma 3. If there is some vertex $u \in K$ that does not have an epon, then the vertex can be removed, and this process clearly terminates in finitely many steps. The technicality of this solution lies entirely in proving that the premisses for the induction hypothesis are satisfied. In particular, it is not entirely trivial to show that $K - \{u\}$ is again an inward dominated quasi-kernel.

Theorem 1 is supported by the three lemmas below. These are proved here in detailed form, in order to achieve closer resemblance to the way they are formalized in the Coq proof. It is assumed that $D$ is a directed graph in these three lemmas.

**Lemma 1.** If $K \subseteq V(D)$ is a kernel then $K$ is an inward dominated quasi-kernel.

**Proof.** From the definition it is immediate that $K$ is a quasi-kernel. Assume that $w \in K$ and $v \in V(D) - K$ such that $v \rightarrow w$. Then, since $K$ is a kernel, there exists a $u \in K$ such that $u \rightarrow v$. Lemma 1 is formalized as `Lemma kernel_qkernel` and `Lemma kernel_in_dom` in the Coq proof. □
Lemma 2. If \( u \) has an epon with regard to \( T \subseteq V(D) \) and \( S \subseteq T \) such that \( u \in S \) then \( u \) has an epon with regard to \( S \).

Proof. Suppose that \( u \in S \) has an epon \( v \in V(D) - T \) with regard to \( T \) and assume towards a contradiction that there exists a \( w \in S \) such that \( w \neq u \) and \( w \rightarrow v \). Then clearly \( w \in T \) and therefore \( v \) cannot be an epon with regard to \( T \). Lemma 2 is encoded as Lemma has\_epon\_incl in the Coq proof.

Lemma 3. If all vertices in \( S \subseteq V(D) \) have an epon with regard to \( S \), then \( |S| \leq |V(D)|/2 \).

Proof. In general, say that \( R \subseteq X \times Y \) is a binary total injective relation if

(1) for all \( x \in X \) there exists a \( y \in Y \) such that \( R(x, y) \), and

(2) for all \( x, x' \in X \) and \( y \in Y \) such that \( R(x, y) \) and \( R(x', y) \) it holds that \( x = x' \).

Clearly, \( |X| \leq |Y| \) holds here due to injectivity.

Say that \( T \subseteq V(D) \) contains the vertices that are an epon with regard to the set \( S \), as given in the statement of this lemma. Define \( R \subseteq S \times T \) as

\[
R = \{(u, v) \in S \times T \mid v \text{ is an epon of } u \text{ with regard to } S\}
\]

and observe that \( R \) is a binary total injective relation. It then follows that \( |S| \leq |T| \) and hence, as \( S \) and \( T \) are disjoint, \( |S| + |T| \leq |V(D)| \) and thus \( 2|S| \leq |V(D)| \). Lemma 3 is encoded as Lemma all\_epon\_half\_size (using Lemma inj\_leq) in the Coq proof.

Conjecture 1, in case a kernel is present, is proved in Theorem 1. This part of the proof is formalized as Theorem main at the end of the Coq code.

Theorem 1. If a source-free digraph \( D \) has a kernel, then \( D \) has a quasi-kernel of size at most \( |V(D)|/2 \).

Proof. Assume \( D \) is a source-free directed graph that has a kernel \( K \). From Lemma 1 it is clear that \( K \) is also an inward dominated quasi-kernel. By induction, the following will be shown: if \( K \) is an inward dominated quasi-kernel, then there exists a quasi-kernel of size at most \( |V(D)|/2 \). For this purpose, define \( S \subseteq V(D) \) as follows:

\[
S = \{u \in K \mid u \text{ does not have an epon with regard to } K\},
\]
and apply induction towards $|S|$, thereby generalizing over all other variables.

If $|S| = 0$ then every vertex in $K$ has an EPON and by Lemma 3 it then follows that $|K| \leq |V(D)|/2$.

Assume that $|S| > 0$ and assume that there exists a vertex $u \in S$ such that $u$ does not have an EPON with regard to $S$ in $D$. Now define $R \subseteq K - \{u\}$ as follows:

$$R = \{v \in K - \{u\} \mid v \text{ does not have an EPON with regard to } K - \{u\}\}.$$ 

Now, three premisses are required to be able to apply the induction hypothesis and thereby complete the proof: (1) $|R| < |S|$, (2) $K - \{u\}$ is a quasi-kernel, and, (3) $K - \{u\}$ is inward dominated. These will be proved here one by one.

(1) Clearly, it holds that $u \in S$ and $u \notin R$, and therefore it suffices to show that $R \subseteq S$. Assume that $v \in R$ such that $v$ does not have an EPON with regard to $K - \{u\}$. Then, by contraposition of Lemma 2, $v$ cannot have an EPON with regard to $K$, hence $v \in S$.

(2) It is immediate that $K - \{u\}$ is also independent. Assume that $v \in V(D)$ and distinguish between the following cases.

- If $u = v$ then, as $D$ is source-free, there must exist a vertex $u' \in V(D)$ such that $u' \rightarrow u$ and $u' \notin K$. As $K$ is an inward dominated quasi-kernel, there must exist a vertex $w \in K$ such that $w \rightarrow u'$. Clearly, only the case $w = u$ is relevant. Assume there does not exist some alternative vertex $w' \in K$ such that $w' \rightarrow u'$. Then, $u'$ is an EPON of $u$ with regard to $K$, thereby contradicting the assumption $u \in S$.

- Now consider the case $u \neq v$, and distinguish between the cases following from the assumption that $K$ is a quasi-kernel.
  - If $v \in K$ then $v \in K - \{u\}$.
  - Now suppose that $v \notin K$ and there exists a vertex $w \in K$ such that $w \rightarrow v$. If $w = u$ then either there exists an alternative vertex $w' \in K$ such that $w' \rightarrow v$, or $v$ is an EPON of $u$ with regard to $K$.
  - For the final case corresponding to $v \notin K$, assume there exist vertices $x \in K$ and $w \notin K$ such that $x \rightarrow w$ and $w \rightarrow v$. If $x = u$ then either there exists an alternative vertex $x' \in K$ such that $x' \rightarrow w$, or $w$ is an EPON of $u$ with regard to $K$. 

(3) Assume that $w \in V(D) - (K - \{u\})$ and $v \in K - \{u\}$ such that $w \rightarrow v$. Since $K$ is inward-dominated, there must exist some vertex $x \in K$ such that $x \rightarrow w$. Now assume $x = u$. If there does not exist some alternative vertex $y \in K$ such that $y \rightarrow w$, then $w$ is an epon of $u$ with regard to $K$, contradicting $u \in S$.

Conjecture 1 remains open if there is no kernel present. Results concerning the presence of multiple distinct quasi-kernels in this situation are known [8, 5]. However, these distinct quasi-kernels are not necessarily disjoint. The reasonable complexity of the proof in this note leads to the suggestion that Conjecture 1, in its unconditional form, is perhaps also accessible via elementary methods.

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