THE $t$-ANALOG OF THE BASIC STRING FUNCTION FOR TWISTED AFFINE KAC-MOODY ALGEBRAS

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Abstract. We study Lusztig’s $t$-analog of weight multiplicities associated to level one representations of twisted affine Kac-Moody algebras. An explicit closed form expression is obtained for the corresponding $t$-string function using constant term identities of Macdonald and Cherednik. The closed form involves the generalized exponents of the graded pieces of the twisted affine algebra, considered as modules for the underlying finite dimensional simple Lie algebra. This extends previous work on level 1 $t$-string functions for the untwisted simply-laced affine Kac-Moody algebras.

Let $g$ be an irreducible affine Kac-Moody algebra (over $\mathbb{C}$). If $g$ is untwisted, it can be realized as a central extension of a loop algebra over a finite dimensional simple Lie algebra. Twisted affines are realized as fixed point subalgebras of untwisted affine Lie algebras under finite groups of automorphisms. The category $\mathcal{O}^{\text{int}}(g)$ of integrable $g$-modules in category $\mathcal{O}$ is semisimple, and its simple objects are highest weight representations $L(\lambda)$ indexed by dominant integral weights $\lambda$. The formal character of $L(\lambda)$ is given explicitly by the Weyl-Kac character formula. To understand the structure of the module $L(\lambda)$, one studies the generating function

$$a_{\mu}^\lambda(q) := \sum_{k \geq 0} \dim(L(\lambda)_{\mu - k\delta}) \ q^k$$

of weight multiplicities along $\delta$-strings through dominant weights $\mu$ of $L(\lambda)$. Here $\delta$ is the null root of $g$. One usually also assumes that $\mu$ is maximal, i.e., that $\mu + \delta$ is not a weight of $L(\lambda)$. We will call the $a_{\mu}^\lambda(q)$ string functions of the module $L(\lambda)$ (mildly departing from standard convention [6]).

Among the irreducible modules in $\mathcal{O}^{\text{int}}(g)$, the so called basic representation $L(A_0)$ can be singled out for the unique and important role it plays in the theory. It is the simplest non-trivial representation in $\mathcal{O}^{\text{int}}(g)$, and has many explicit realizations in terms of vertex operators [3,7,9]. If $g$ is an untwisted simply-laced affine Lie algebra, or if $g$ is twisted, then the basic representation is an irreducible $g$-module of level one, and all level one simple modules can be obtained from it by the action of Dynkin diagram automorphisms of $g$, and by tensoring with one-dimensional $g$-modules.

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Further, for such \( \mathfrak{g} \), the basis representation admits a unique string function \( a_{\Lambda_0}^{\Lambda_0}(q) \), which we will refer to as the \textit{basic string function}. This has a well known \cite{BS}, closed form expression as an infinite product (equation (1.2)). For the remaining affine Kac-Moody algebras, i.e., the untwisted ones of types \( B, C, F, G \), there are multiple inequivalent string functions of level 1.

Now, let \( t \) be an indeterminate. The \( t \)-analog \( \mathcal{P}(\beta; t) \) of the Kostant partition function is defined to be the coefficient of \( e^{\beta} \) in the product

\[
\prod_{\alpha \in \Delta_+(\mathfrak{g})} (1 - te^{\alpha})^{-\text{mult} \alpha}
\]

where \( \Delta_+(\mathfrak{g}) \) is the set of positive roots of \( \mathfrak{g} \) and \( \text{mult} \alpha := \dim \mathfrak{g}_\alpha \). In this article, we will be concerned with Lusztig’s \( t \)-analog of weight multiplicity, or (affine) Kostka-Foulkes polynomial \( K_{\lambda \mu}(t) \). Given a dominant integral weight \( \lambda \), and a weight \( \mu \) of \( L(\lambda) \), \( K_{\lambda \mu}(t) \) is defined to be:

\[
K_{\lambda \mu}(t) := \sum_{w \in W} \epsilon(w) \mathcal{P}(w(\lambda + \rho) - (\mu + \rho); t)
\]

where \( \epsilon \) is the sign character of the Weyl group \( W \) of \( \mathfrak{g} \). These polynomials have several important properties: (a) they are the transition coefficients between the bases of affine Hall-Littlewood functions and the formal characters of simple modules \cite{L15} (b) they have non-negative integral coefficients \cite{L14} and (c) \( K_{\lambda \mu}(1) = \dim L(\lambda)\mu \). In view of (c), it is natural to consider the \( t \)-analog of the string function, defined by

\[
a_{\lambda \mu}(t, q) := \sum_{k \geq 0} K_{\lambda,\mu - k\delta}(t) q^k
\]

We call these the \( t \)-string functions. It was shown in \cite{L15} that the \( a_{\lambda}^\mu(t, q) \) are closely related to the constant term identities arising in the theory of Macdonald polynomials. For instance, when \( \mathfrak{g} \) is an untwisted simply-laced affine Kac-Moody algebra, Cherednik’s Macdonald-Mehta constant term identity \cite{Che} allows us to compute the \( t \)-string function \( a_{\Lambda_0}^{\Lambda_0}(t, q) \) of the basic representation, in closed form\footnote{We point out an erratum in \cite{L15}. Theorems 2, 3 and Corollary 3 there are missing the hypothesis that \( \mathfrak{g} \) must be simply laced.}. More precisely \cite[corollary 2]{L15}:

\textbf{Theorem 1.} Let \( \mathfrak{g} \) be one of the simply laced untwisted affine Lie algebras \( A_l^{(1)}, D_l^{(1)}, E_l^{(1)} \). Then \( a_{\Lambda_0}^{\Lambda_0}(t, q) = \prod_{i=1}^l \prod_{n=1}^{\infty} (1 - t^{e_i+1} q^n)^{-1} \) where \( e_i \) (1 \( \leq i \leq l \)) are the exponents of the underlying finite dimensional simple Lie algebra (= \( A_l, D_l \) or \( E_l \) respectively).

The goal of the present work is to prove the twisted version of this theorem. Our main result gives a closed form expression for the basic \( t \)-string function of the twisted affine Kac-Moody algebras.
We recall that every affine Kac-Moody algebra \( g \) admits a natural \( \mathbb{Z} \)-grading into finite dimensional subspaces \( g = \bigoplus_{j \in \mathbb{Z}} g_j \), where \( g_0 \) is a reductive Lie algebra, and the \( g_j \) are irreducible \( g_0 \)-modules. For \( g \) twisted, the basic \( t \)-string function (theorem 2) now involves the generalized exponents of the \( g_j \) (\( j > 0 \)), viewed as representations of the semisimple part of \( g_0 \). When \( g \) is untwisted simply-laced, all the \( g_j \) are isomorphic to the adjoint representation of the underlying finite dimensional simple Lie algebra. Since in this case, the generalized exponents are just the usual exponents, our theorem is in fact a generalization of theorem 1.

The paper is organized as follows. The required preliminaries and the statement of the main theorem appear in §1. The proof for the case \( g \neq A_{2l}^{(2)} \) is in §2. When \( g = A_{2l}^{(2)} \), as an auxiliary step, we first consider the non-reduced affine root system \((C_l^\vee, C_l)\). We show that on passing to an appropriate limit, the Macdonald constant term identity for this root system can be used to derive the Macdonald-Mehta type constant term identity for \( A_{2l}^{(2)} \). This is used to prove the theorem in this case. As an offshoot, we define a two-variable generalization of the Kostka-Foulkes polynomials for \( A_{2l}^{(2)} \), and show that the two variable polynomials associated to the basic representation have non-negative integral coefficients. These details appear in §3. The main results of this article were announced in [13].

1. THE MAIN THEOREM

1.1. Preliminaries. For use in stating our main theorem, we recall the definition of generalized exponents. Let \( g \) be a finite dimensional simple Lie algebra and \( V = V(\lambda) \) be the irreducible finite dimensional \( g \)-module with highest weight \( \lambda \). Fix a triangular decomposition \( g = n^- \oplus h \oplus n^+ \), and let \( E \in n^+ \) be a regular (principal) nilpotent element. Let \( V_0 \) denote the zero weight space of \( V \), and define the Brylinski-Kostant filtration \( [4,10] \) of \( V_0 \) via \( F_p(V_0) := \ker (E^p) \cap V_0 \) for \( p \geq 0 \). Then the multiset \( E(V) \) of generalized exponents of \( V \) is defined via the relation:

\[
\sum_{p \geq 0} \dim(\mathcal{F}^{(p+1)}(V_0)/\mathcal{F}^{(p)}(V_0)) t^p = \sum_{k \in E(V)} t^k.
\]

Next, given a Kac-Moody algebra \( g \) of finite or affine type, we let \( \Delta(g) \), \( \Delta_+(g) \), \( \Delta^{re}(g) \), \( \Delta^{im}(g) \) denote the sets of roots, positive roots, real roots and imaginary roots respectively, and let \( \Delta^{re}_+(g) := \Delta^{re}(g) \cap \Delta_+(g) \), \( \Delta^{im}_+(g) := \Delta^{im}(g) \cap \Delta_+(g) \). The Cherednik kernel \( \hat{\mu} \) of \( g \) is the product

\[
\hat{\mu} := \prod_{\alpha \in \Delta^{re}_+(g)} \frac{1 - e^{-\alpha}}{1 - te^{-\alpha}} \tag{1.1}
\]

When \( g \) is affine, we denote the corresponding product over the imaginary positive roots by \( \hat{\mu}^{im} := \prod_{\alpha \in \Delta^{im}_+(g)} \left( \frac{1 - e^{-\alpha}}{1 - te^{-\alpha}} \right)^{\text{mult}_\alpha} = \prod_{n \geq 1} \left( \frac{1 - q^n}{1 - q^n} \right)^{\text{mult}_{n\delta}} \), where we let \( q := e^{-\delta} \) throughout.
Now assume $\mathfrak{g}$ is a twisted affine Kac-Moody algebra, with normalized invariant form $\langle \cdot, \cdot \rangle$ [6, Chap. 8]. Let $\hat{\mathfrak{g}}$ denote the underlying finite dimensional simple Lie algebra, and let $\hat{Q}$ denote its root lattice. We define the lattice $M$ as follows [6]:

$$M := \hat{Q} \text{ if } \mathfrak{g} \neq A_2^{(2)}$$

and

$$M := \mathbb{Z} \text{-span of } 2\alpha/\langle \alpha, \alpha \rangle, \alpha \in \Delta(\hat{\mathfrak{g}}) \text{ (the coroot lattice of } \hat{\mathfrak{g}}) \text{ if } \mathfrak{g} = A_2^{(2)}.$$ 

For a dominant integral weight $\lambda$ of $\mathfrak{g}$, let $\chi_{\lambda} := \sum_{\gamma} \dim(L(\lambda, \gamma)) e^\gamma$ denote the formal character of $L(\lambda)$. Let $\Lambda_0$ denote the fundamental weight corresponding to the zeroth (additional) node of the Dynkin diagram of $\mathfrak{g}$. Then, $L(\Lambda_0)$ is the basic representation of $\mathfrak{g}$. We have the following classical result [6]:

**Proposition 1.** Let $\mathfrak{g}$ be a twisted affine algebra. Then

1. The formal character of the basic representation of $\mathfrak{g}$ is:
   $$e^{-\Lambda_0} \chi_{\Lambda_0} = a_{\Lambda_0}^0(1, q) \Theta$$
   where $\Theta := \sum_{\alpha \in M} e^{\alpha} q^{\langle \alpha, \alpha \rangle / 2}$ is the theta function of the lattice $M$.
2. The basic string function is given by
   $$a_{\Lambda_0}^0(1, q) = \prod_{n \geq 1} (1 - q^n)^{-\text{mult } n\delta}$$

Next, given a formal sum $\xi = \sum_{\alpha \in Q} c_{\alpha} e^\alpha$, define [12] the constant term of $\xi$ to be $\text{ct}(\xi) := \sum_{n \in \mathbb{Z}} c_{n\delta} e^{n\delta}$. The following simple fact [15, (5.8)] can be used to compute the $t$-string functions $a_{\mu}^* (t, q)$:

$$a_{\mu}^*(t, q) = \hat{\mu}^{im} \text{ ct}(e^{-\mu} \chi_{\hat{\mu}})$$

where $\mu$ is a maximal dominant weight of $L(\lambda)$. Putting the above facts together, we obtain the following lemma.

**Lemma 1.** Let $\mathfrak{g}$ be a twisted affine algebra. Then,

1. The $t$-string function of the basic representation of $\mathfrak{g}$ is given by
   $$a_{\Lambda_0}^0(t, q) = a_{\Lambda_0}^0(1, q) \hat{\mu}^{im} \text{ ct}(\hat{\mu} \Theta)$$
2. Further, we have $a_{\Lambda_0}^0(1, q) \hat{\mu}^{im} = \prod_{n \geq 1} (1 - t q^n)^{-\text{mult } (n\delta)}$

Now, suppose that $\mathfrak{g}$ is a twisted affine algebra of type $X_N^{(r)}$; here $X_N$ is a simply laced (A-D-E) Dynkin diagram of finite type with a diagram automorphism $\sigma$ of order $r$ ($r = 2$ or $3$). Let $\mathfrak{m}$ denote the finite dimensional simple Lie algebra with Dynkin diagram $X_N$ and let $\sigma$ also denote the corresponding automorphism of $\mathfrak{m}$. For each $k \in \mathbb{Z}$, let $\mathfrak{m}_k \subset \mathfrak{m}$ be the eigenspace of $\sigma$ for the eigenvalue $\exp(2\pi ki/r)$ (so, $\mathfrak{m}_k = \mathfrak{m}_{k+r}$). Since $\sigma$ acts diagonalizably on $\mathfrak{m}$, we have a $\mathbb{Z}/r\mathbb{Z}$ gradation:

$$\mathfrak{m} = \bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}} \mathfrak{m}_j$$

(1.4)
If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{m}$, let $\mathfrak{h}_j := \mathfrak{h} \cap \mathfrak{m}_j$ for all $j \in \mathbb{Z}$. We collect together the important facts about the decomposition (1.4).

**Proposition 2.** [6, Chap. 8] With notation as above, we have

1. $\mathfrak{m}_0$ is a simple Lie algebra and $\mathfrak{m}_j$ is an irreducible $\mathfrak{m}_0$-module $\forall j$.
2. $\mathfrak{m}_1 \cong \mathfrak{m}_{-1}$ as $\mathfrak{m}_0$-modules.
3. $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{m}_0$ and its centralizer in $\mathfrak{m}$ is $\mathfrak{h}$.
4. If $\mathfrak{g}$ is not of type $A_{2l}^{(2)}$, then $\mathfrak{m}_0$ and $\mathfrak{g}$ are isomorphic. Further, the highest weight of the $\mathfrak{m}_0$-module $\mathfrak{m}_1$ is the dominant short root $\theta_s$ of $\mathfrak{m}_0$.
5. If $\mathfrak{g}$ is of type $A_{2l}^{(2)}$, then $\mathfrak{m}_0$ is of type $B_l$, while $\mathfrak{g}$ is of type $C_l$. Further, the highest weight of $\mathfrak{m}_1$ is $2\theta_s$, where $\theta_s$ is the dominant short root of $\mathfrak{m}_0$.

We denote $l := \text{rank } \mathfrak{m}_0$, $m := \text{the number of short simple roots of } \mathfrak{m}_0$ and let $\theta_l$ (resp. $\theta_s$) be the dominant long (resp. short) root of $\mathfrak{m}_0$.

**Proposition 3.** (6, Chap. 8) or [1] Let $\mathfrak{g}$ be a twisted affine algebra of type $X_N^{(r)} \neq A_{2l}^{(2)}$. Consider the action of the cyclic group generated by the automorphism $\sigma$, on the nodes of the Dynkin diagram of $X_N$. This has the following properties.

1. Each orbit has cardinality 1 or $r$.
2. The number of orbits equals $l$.
3. The number of orbits of cardinality $r$ is equal to $m$.
4. Thus, $m = \frac{N-l}{r}$.

Next, we recall that the untwisted affine algebra $\hat{\mathcal{L}}(\mathfrak{m}) := \mathbb{C}[z, z^{-1}] \otimes \mathfrak{m} \oplus \mathbb{C}K \oplus \mathbb{C}d$, where $K$ is the central element and $d$ is the degree derivation [6].

We extend $\sigma$ to an automorphism $\tilde{\sigma}$ of $\hat{\mathcal{L}}(\mathfrak{m})$ by $K \mapsto K$, $d \mapsto d$ and $z^j \otimes x \mapsto \exp(-2\pi j/r) z^j \otimes \sigma(x)$ for $j \in \mathbb{Z}, x \in \mathfrak{m}$. The fixed point set of $\tilde{\sigma}$ is the affine Lie algebra $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}(X_N^{(r)})$.

We have a natural $\mathbb{Z}$-grading $\hat{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}} \hat{\mathfrak{g}}_j$ with $\hat{\mathfrak{g}}_0 = \mathfrak{m}_0 + \mathbb{C}K + \mathbb{C}d$ and $\hat{\mathfrak{g}}_j = z^j \otimes \mathfrak{m}_j$ for $j \neq 0$. We observe that for $j \neq 0$, $\hat{\mathfrak{g}}_j \cong \mathfrak{g}_j$ is an irreducible $\mathfrak{m}_0$-module and that $\hat{\mathfrak{g}}_0 \cong \mathfrak{g}_0$ when $j \equiv k \pmod{r}$, $j, k \neq 0$. Let $E_n$ denote the multiset of generalized exponents of the $\mathfrak{m}_0$-module $\mathfrak{m}_n$ for $n \in \mathbb{Z}$. The main result of this paper is the following.

**Theorem 2.** Let $\mathfrak{g}$ be a twisted affine algebra. The $t$-string function of the basic representation of $\mathfrak{g}$ is given by

$$a_{\lambda_0}^\lambda(t, q) = \prod_{n=1}^{\infty} \prod_{e \in E_n} (1 - t^{e+1} q^n)^{-1}$$

The proof will be given in sections 2 and 3. But first, we make some remarks.

**Remark 1.** When $\mathfrak{g}$ is an untwisted simply-laced affine, this result was proved in [15]. In this case, the $\mathfrak{m}_0$-modules $\mathfrak{m}_j$ are all isomorphic to the
Table 1. \( E_n \) for the twisted affines \( g = X^{(r)} \) (\( E_{n+r} = E_n \) for all \( n \)).

| \( g \)         | \( E_0 \)          | \( E_1 = E_{-1} \) |
|-----------------|--------------------|---------------------|
| \( A^2_{2l} \)  | 1, 3, 5, \ldots, 2l − 1 | 2, 4, 6, \ldots, 2l |
| \( A^2_{2l-1} \) | 1, 3, 5, \ldots, 2l − 1 | 2, 4, 6, \ldots, 2l − 2 |
| \( D^{(2)}_{l+1} \) | 1, 3, 5, \ldots, 2l − 1 | \( l \) |
| \( E^{(2)}_6 \) | 1, 5, 7, 11 | 4, 8 |
| \( D^{(3)}_4 \) | 1, 5 | 3 |

adjoint representation of \( m_0 \). Thus \( E_n = E(m_0) \), the set of exponents of \( m_0 \) for all \( n \), and we recover Theorem 1.

**Remark 2.** The cardinality of \( E_n \) is the dimension of the zero weight space of \( g_n \). From proposition 2, it follows that \( |E_n| = \dim(z^n \otimes h_n) \). Since \( z^n \otimes h_n \) is the root space of \( g \) corresponding to the imaginary root \( n\delta \), we deduce that \( |E_n| = \text{mult}(n\delta) \). Thus, this expression is a \( t \)-deformation of the expression for the basic string function (equation (1.2)).

**Remark 3.** From the explicit description of the Chevalley generators of \( m_0 \) in terms of those of \( m \) [6, Chap. 8], it is clear that a principal nilpotent element of \( m_0 \) is also a principal nilpotent of \( m \). This observation, together with proposition 2 implies the following equality of multisets:

\[
E(m) = \bigsqcup_{j=1}^{r} E_j
\]

where the left hand side is the multiset of exponents of the Lie algebra \( m \), i.e., the generalized exponents of its adjoint representation. Further, since \( m_r = m_0 \), we have \( E_r = E(m_0) \). Thus, the sets \( E(m) \) and \( E(m_0) \) determine the \( E_n \) for all \( n \); this is clear for \( r = 2 \), while for \( r = 3 \) it follows from the further fact that \( E_1 = E_2 \). Table 1 lists the \( E_n \) for all twisted affine algebras.

1.2. We derive an interesting corollary of theorem 2. If \( g \) is an affine Kac-Moody algebra of rank \( l + 1 \), and \( e_i, f_i \) (\( i = 0, \ldots, l \)) are the Chevalley generators, the principal Heisenberg subalgebra \( s \) of \( g \) is defined to be

\[
s := \{ x \in g : \sum_{i=0}^{l} e_i x e_i \in \mathbb{C}K \}
\]

where \( K \) is the central element of \( g \) [6]. The principal gradation of \( g \) induces a gradation \( s = \bigoplus_{j \in \mathbb{Z}} s_j \). If \( g \) is an untwisted simply-laced or twisted affine algebra, the basic representation \( L(\Lambda_0) \), as a \( s \)-module, is irreducible. The *exponents* of the affine algebra \( g \) are the elements of the (infinite) multiset
The multisets $E^+(g)$ and $E_n$. The following lemma relates

**Lemma 2.** Let $g$ be a twisted affine algebra or an untwisted simply-laced affine algebra of type $X_N^{(r)}$, with Coxeter number $h$. Then

$$E^+(g) = \{ e + hn : n \geq 0, e \in E_n \}$$

**Proof:** Follows easily from [6, Chap. 14] and table 4. □

We deduce the following nice formula for the specialization of the $t$-string function $a_{\Lambda_0}^g(t,q)$ at $t \mapsto q, q \mapsto q^h$.

**Corollary 1.** Let $g$ be a twisted affine algebra or an untwisted simply-laced affine algebra, with Coxeter number $h$. Let $\hat{g}$ be its underlying finite dimensional simple Lie algebra. Then

$$a_{\Lambda_0}^g(q,q^h) = \prod_{e \in E(\hat{g})} (1 - q^e + 1) \prod_{e \in E^+(g)} (1 - q^{e+1})$$

(1.5)

where $E(\hat{g})$ is the (finite) multiset of exponents of $\hat{g}$.

**Proof:** Applying the specialization $t \mapsto q, q \mapsto q^h$ to theorem 2 and using lemma 2 we obtain equation (1.5), but with $m_0$ in place of $\hat{g}$. Proposition 2 implies that $m_0$ and $\hat{g}$ are either isomorphic or dual. Since dual algebras have the same exponents, the result follows in all cases. □

2. **Proof of Theorem 2 for $g \neq A_{2l}^{(2)}$**

Throughout this section, we take $g$ to be a twisted affine algebra, $g \neq A_{2l}^{(2)}$. Let $\langle \cdot, \cdot \rangle$ denote the normalized invariant form of $g$ [6]. We then have $\langle \alpha, \alpha \rangle = 2$ for all short real roots of $g$. We recall that the height of a root $\alpha$ (written $ht\alpha$) is the sum of the coefficients obtained when $\alpha$ is written as a linear combination of simple roots. The following result is a special case of Cherednik’s difference Macdonald-Mehta constant term identity [2].

**Proposition 4.** Let $g$ be a twisted affine algebra, $g \neq A_{2l}^{(2)}$. Let $\hat{g}$ be the underlying finite dimensional simple Lie algebra and let $\langle \cdot, \cdot \rangle$ denote the normalized invariant form of $g$. Then we have

$$\text{ct}(\hat{g} \Theta) = \prod_{\alpha \in \Delta^+(\hat{g})} \prod_{j=1}^{\infty} \left( \frac{1 - q^{ht\alpha} \langle \alpha, \alpha \rangle^j}{1 - q^{ht\alpha+1} \langle \alpha, \alpha \rangle^j} \right)$$

(2.1)
Let us now separate the contributions of long and short roots in equation (2.1). Applying [2, theorem 5.3] with $R$ chosen to be the coroot system of $\hat{g}$ yields equation (2.1).

To simplify notation, we let $(a_1, a_2, \ldots, a_p; x)_\infty := \prod_{j=1}^p \prod_{n=0}^\infty (1 - a_j x^n)$. Let us now separate the contributions of long and short roots in equation (2.1).

Define

\[
K_s(q) \quad \text{(resp. } K_l(q)) := \prod_{\alpha \in \Delta^+ (\hat{g})} \frac{(\text{ht } \alpha; q)_\infty}{(\text{ht } \alpha + 1; q)_\infty}
\]

Since $\langle \alpha, \alpha \rangle / 2$ is 1 (resp. $r$) if $\alpha$ is short (resp. long), proposition 4 implies

\[
\text{ct}(\hat{\mu} \Theta) = K_s(q) K_l(q^r)
\]

Now, for each $k \geq 1$, let $n_k$ (resp. $n_k(s)$) denote the number of positive roots (resp. short positive roots) of $\hat{g}$ of height $k$. This gives

\[
K(q) := K_s(q) K_l(q) = \prod_{p \geq 1} \frac{(tq; q)_\infty^{l_p}}{(t^{p+1}q; q)_\infty^{n_p - n_{p+1}}}
\]

where $l$ is the number of simple roots of $\hat{g}$. Similarly,

\[
K_s(q) = \prod_{p \geq 1} \frac{(tq; q)_\infty^{m_p}}{\prod_{p \geq 1} (t^{p+1}q; q)_{n_p(s) - n_{p+1}(s)}}
\]

where $m$ is the number of short simple roots of $\hat{g}$. We recall the following classical result (see, for example, [5]) relating the $n_k$ and $n_k(s)$ to generalized exponents of certain representations of $\hat{g}$.

**Proposition 5.** With notation as above, $n_p - n_{p+1}$ is the number of times $p$ occurs as an exponent of $\hat{g}$ (i.e., as a generalized exponent of the adjoint representation $V(\theta_i)$). Similarly, $n_p(s) - n_{p+1}(s)$ is the number of times $p$ occurs as a generalized exponent of the representation $V(\theta_s)$ of $\hat{g}$.

Now, rewriting $K_s(q) K_l(q^r) = K(q^r) \frac{K_s(q)}{K_s(q^r)}$ and using propositions 2 and 3 we get

\[
\text{ct}(\hat{\mu} \Theta) = (tq^r; q^r)_\infty \frac{(tq; q)_\infty}{(tq^r; q^r)_\infty} \prod_{p \in \mathbb{Z}_0} \frac{1}{(t^{p+1}q; q^r)_\infty} \prod_{p \in \mathbb{Z}_1} \frac{(t^{p+1}q^r; q^r)_\infty}{(t^{p+1}q; q)_\infty}
\]
We now observe that \( l = \text{mult} j \delta \) for \( j \equiv 0 \) (mod \( r \)) and \( m = \frac{N-l}{r-1} = \text{mult} j \delta \) for \( j \not\equiv 0 \) (mod \( r \)). Thus, the above equation can be rewritten as:

\[
ct(\tilde{\mu} \Theta) = \prod_{n \geq 1} (1 - t q^n)^{\text{mult}(n \delta)} \prod_{n \geq 1} e_{\in \mathbb{Z}_n} \frac{1}{1 - t e^{1} q^n}
\]

Lemma \( \ref{lem:main} \) now completes the proof of theorem \( \ref{thm:main} \) for all twisted affine algebras \( \mathfrak{g} \not\equiv A^{(2)}_{2\ell} \).

3. \( \mathfrak{g} = A^{(2)}_{2\ell} \)

3.1. In this section, we consider the case \( \mathfrak{g} = A^{(2)}_{2\ell} \). The underlying finite dimensional simple Lie algebra is \( \mathfrak{g} = C_\ell \). Let \( \hat{\Delta} \) be the set of roots of \( \hat{\mathfrak{g}} \). Letting \( \langle \cdot , \cdot \rangle \) denote the standard inner product in \( \mathbb{R}^l \) and \( \epsilon_i (1 \leq i \leq l) \) be the standard orthonormal basis, we can take \( \hat{\Delta} = \{ \pm \epsilon_i \pm \epsilon_j : 1 \leq i < j \leq l \} \cup \{ \pm 2 \epsilon_i : 1 \leq i \leq l \} \). We observe that the coroot lattice \( M \) is just \( M = \bigoplus_{i=1}^l \mathbb{Z} \epsilon_i \). The set of real roots of \( \mathfrak{g} \) is given by \( \Delta^\text{re} = S_1 \cup S_2 \cup S_4 \) where \( S_1 = \{ \frac{1}{2}(\alpha + (2n-1)\delta) : \alpha \in \hat{\Delta}_l \} \), \( S_2 = \{ \alpha + n\delta : \alpha \in \hat{\Delta}_s \} \) and \( S_4 = \{ \alpha + 2n\delta : \alpha \in \hat{\Delta}_l \} \), where \( \delta \) is the null root of \( \mathfrak{g} \) and \( \hat{\Delta}_l \) (resp. \( \hat{\Delta}_s \)) denotes the set of long (resp. short) roots in \( \hat{\Delta} \). The elements of \( S_n \) have norm \( n \) (\( n = 1, 2, 4 \)), and each \( S_n \) is invariant under the Weyl group \( W \) of \( \mathfrak{g} \).

Let \( \tilde{\mu} \) denote the Cherednik kernel of \( A^{(2)}_{2\ell} \), given by equation \( \ref{eq:cherednik_kernel} \).

Now enlarge \( \Delta^\text{re} \) by defining:

\[
\Phi := \bigcup_{i=1}^5 \Phi_i \text{ where } \Phi_1 := (1/2)S_4, \Phi_2 := S_4, \Phi_3 := S_1, \Phi_4 := 2S_1, \Phi_5 := S_2
\]

The set \( \Phi \) is the non-reduced irreducible affine root system of type \( (C^{\gamma}_\ell, C_\ell) \) in the classification of Macdonald \( \ref{ref:macdonald} \). Observe that \( \Phi \) is \( W \)-invariant, with each \( \Phi_i \) being a \( W \)-orbit. Following the notation of Macdonald \( \ref{ref:macdonald} \), define \( R_1^i := \{ \epsilon_1, \ldots, \epsilon_l \} \) and \( R_2^i := \{ \epsilon_i \pm \epsilon_j : 1 \leq i < j \leq l \} \). We now let \( k_i (1 \leq i \leq 5) \) be arbitrary parameters, and let \( (u_1, u_2, u_3, u_4) = (q^{k_1}, -q^{k_2}, q^{k_3+\frac{1}{2}}, -q^{k_4+\frac{1}{2}}) \) and \( (u'_1, u'_2, u'_3, u'_4) = (qu_1, qu_2, u_3, u_4) \). The Cherednik kernel \( \Delta \) (with parameters \( k_i \)) for the non-reduced affine root system \( \Phi \) then becomes \( \Delta := \Delta^{(1)} \Delta^{(2)} \) where

\[
\Delta^{(1)} = \prod_{\alpha \in R_1^i} \frac{(e^{-2\alpha}; q^{2\alpha}; q)_\infty}{\prod_{i=1}^4 (u_i e^{-\alpha}; q)_\infty (u'_i e^{\alpha}; q)_\infty}
\]

\[
\Delta^{(2)} = \prod_{\alpha \in R_2^i} \frac{(e^{-\alpha}; q^{\alpha}; q)_\infty}{(q^{-k_5 e^{-\alpha}}; q^{k_5+1} e^{\alpha}; q)_\infty}
\]

\( \ref{ref:macdonald} \) (5.1.14)]. The following lemma relates the kernels \( \Delta \) and \( \tilde{\mu} \).
Lemma 3. Define \( t := q^{k_5} \), and let the parameters \( k_i \) satisfy the relations \( k_3 = k_5 = 2k_1 = 2k_2 \). We then have:

1. If \( k_4 = 0 \), then \( \Delta = \hat{\mu} \).
2. If \( k_4 \to \infty \) (i.e \( q^{k_4} \to 0 \)), then \( \Delta \to \frac{\hat{\mu} \Theta_M}{(q; q)_\infty} \) where \( M := \bigoplus_{i=1}^{l} \mathbb{Z} e_i \) and \( \Theta_M := \sum_{\alpha \in M} e^{\alpha} q^{(\alpha, \alpha)/2} \) is its theta function.

Proof: The first statement is easy; in fact one can recover the Cherednik formula for \( \text{ct} \left( \hat{\mu} \Theta_M \right) \) to specialize the well-known formula for \( \text{ct} \left( \hat{\mu} \Theta_M \right) \) to recover the Cherednik constant term identity for \( A_2^{(2)} \) (i.e an expression for \( \text{ct} \left( \hat{\mu} \Theta_M \right) \)).

Now, by the Jacobi triple product identity, we have

\[
(q^{1/2} e^{\epsilon_i}, -q^{1/2} e^{-\epsilon_i}; q)_\infty = (q; q)_\infty^{-1} \sum_{n \in \mathbb{Z}} q^{n^2/2} e^{n \epsilon_i} = (q; q)_\infty^{-1} \Theta_Z e_i
\]

Since the theta function \( \Theta_M \) of the rectangular lattice \( M = \bigoplus_{i=1}^{l} \mathbb{Z} e_i \) is just the product \( \prod_{i=1}^{l} \Theta_Z e_i \), the result follows.

To obtain the explicit form of the Cherednik-Macdonald-Mehata constant term identity for \( A_2^{(2)} \) (i.e an expression for \( \text{ct} \left( \hat{\mu} \Theta_M \right) \)) it thus only remains to specialize the well-known formula for \( \text{ct} \left( \Delta \right) \) [11, (5.8.20)] at \( k_3 = k_5 = 2k_1 = 2k_2 \) and \( k_4 \to \infty \) (and letting \( q^{k_5} = t \)). We give the result of this (long, but straightforward) calculation below.

Proposition 6. For \( g = A_2^{(2)} \), we have:

\[
\text{ct}(\hat{\mu} \Theta_M) = \frac{(tq; q)_\infty}{(t^2 q^2, t^4 q^4, \ldots, t^{2l} q^{2l}; q^2)_\infty} (t^3 q, t^5 q, \ldots, t^{2l+1} q; q^2)_\infty
\]

Now, for \( A_2^{(2)} \), recall that \( \text{mult} (j\delta) = l \forall j \geq 1 \). This fact, together with proposition [5] lemma [1] and Table [1] proves theorem [2] for \( g = A_2^{(2)} \).

3.2. Two variable generalization. One can prove a slightly more general, two-variable version of theorem [2] for \( g = A_2^{(2)} \). To state this, let \( s, t \) be indeterminates, and define the two-variable Kostant partition function \( \mathcal{P}(\beta; s, t) \) to be the coefficient of \( e^\beta \) in the product \( \prod_{\alpha \in \Delta_+(g)} (1 - u_\alpha e^\alpha)^{-\text{mult} \alpha} \)

where \( u_\alpha := s \) if \( \alpha \) is a real root of norm 1 (=shortest root length) and \( u_\alpha := t \) for all other roots (i.e imaginary roots, and real roots of norms 2 and 4). For a dominant integral weight \( \lambda \) of \( A_2^{(2)} \), and a maximal dominant weight \( \mu \) of \( L(\lambda) \), define the two variable Kostka-Foulkes polynomial

\[
K_{\lambda \mu}(s, t) := \sum_{w \in W} e(w) \mathcal{P}(w(\lambda + \rho) - (\mu + \rho); s, t)
\]
and let the corresponding \((s, t)\)-string function for the basic representation be 
\[ a_{\lambda_0}^0(s, t, q) := \sum_{p \geq 0} K_{\lambda_0, \lambda_0, -p}(s, t) q^p. \]
The following is the two variable version of theorem 2.

**Proposition 7.** For \( g = A_{2l}^{(2)} \),
\[ a_{\lambda_0}^0(s, t, q) = \prod_{j=1}^{2l} (t^j q^2; q^2)_j \in \mathbb{Z}_{\geq 0}[s, t] \text{ for all } p \geq 0. \]

**Proof:** The proof is along the exact same lines as that of proposition 8 but now with parameters chosen differently. We choose \( k_5 = 2k_1 = 2k_2, k_4 \to \infty \), but leave \( k_3 \) as a free parameter. We then take \( t := q^{k_5} \) and \( s := q^{k_3} \). The remaining details are easily checked. \( \square \)

**Corollary 2.** \( K_{\lambda_0, \lambda_0, -p}(s, t) \in \mathbb{Z}_{\geq 0}[s, t] \) for all \( p \geq 0. \)

Finally, we remark that it would be of interest to find a more natural explanation for the positivity result of the above corollary (or more generally, for \( K_{\lambda_0}(s, t) \)) in terms of a Brylinski-Kostant type filtration, as is known for the usual (one variable) affine Kostka-Foulkes polynomials \( [14] \). We also note that the two variable Kostka-Foulkes polynomials can be defined for all twisted affines (in fact, for any affine root system with more than one root length) and in more than one way (corresponding to different choices of the \( u_\alpha \) in the definition). But it appears, from preliminary calculations, that only \( A_{2l}^{(2)} \) (with the given choice of \( u_\alpha \)) exhibits the positivity property of corollary 2.

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