Analytic Implicit Functions

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Abstract

In this paper, we introduce a method of converting implicit equations to the usual forms of functions locally without differentiability. For a system of implicit equations which are equipped with continuous functions, if there are unique analytic implicit functions, that satisfies the system in some rectangle, then each analytic function is represented as a power series which is the weak-star limit of partial sums in the space of essentially bounded functions. We also provide numerical examples in order to demonstrate how the theoretical results in this article can be applied in practice and to show the effectiveness of the suggested approaches.

Index terms— implicit function, implicit function theorem, inverse function theorem, analytic function, continuous function, power series, weak-star limit

1 Introduction

Setting up implicit equations and solving them has long been so important that it has virtually come to describe what mathematical analysis and its applications are all about. Many useful mathematical models have expressions of implicit functions, even for problems of minimizing or maximizing functions subject to constraints. From information of a single point solution, the implicit function theorem allows for understanding the relation between variables but, in spite of that, the implicit function theorems do not provide a usual function form defined explicitly. A central issue in this subject is how to derive a power series expansion from an equation involving parameters because the representation as a function form enables us to estimate the mathematical models more easily, in economics, physics, engineering, etc., as well as mathematical development.

In 1669, Issac Newton introduced one of the first instances of analyzing the behavior of an implicitly defined function originated from an equation jointed by mutually correlated variables

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In 1684, Gottfried Leibniz applied implicit differentiation to calculate partial derivatives, which is a way to take the derivative of a term with respect to another variable without having to isolate either variable (11). Joseph Lagrange, in 1770, derived an inversion formula which is one of the fundamental formulas of combinatorics. The formula is closely related to an inverse function theorem, by formulating a formal power series expansion for an implicit function [4, Theorem 2.3.1]. In the 19th century, Augustin-Louis Cauchy was the first to state and solve an implicit equation using a rigorous mathematical form. He gave proof of the implicit function theorem having a form of power series by using Hadamard’s estimate for the Taylor coefficient of a given function, which is induced formally ([4, Theorems 2.4.6 and 6.1.2]). Since then, there have been many improvements on the existence of usual function expressions from implicit equations under suitable assumptions.

Most importantly, in 1877, Ulisse Dini was the first to prove the real variable result of an implicit function theorem ([1]). Conceived over two hundred years ago as a tool for studying mechanics and physics, the implicit function theorem has many formulations and is used in many aspects of mathematics. In addition, there are many particular types of implicit function theorems which are extended to Banach spaces, even under degenerate or non-smooth situations (e.g., [8, 7]). Implicit function theorems, even those that are quite sophisticated, are fundamental and powerful parts of the foundation of modern mathematics. Almost all studies of an implicit function are related to its existence rather than showing how they behave.

This research is devoted to determining a form of function of an implicit function which does not adopt the Taylor series of a function given, as used in Cauchy’s method, but instead of differentiability, we need the integrability of the given function. This article highlights the formulations of a power series representation from an implicit equation such that its partial sum converges weak-star in the space of essentially bounded functions. This method relies essentially on the continuity of a given function. In addition, we present several examples for the computational validity of this study, along with a numerical calculation for each. While there have been too many contributions to the theory of implicit functions to mention them all here, we would like to refer to books [4] and [2] as good overviews.

Throughout the article we use multi-index notations. Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \beta = (\beta_1, \ldots, \beta_n) \) in \( \mathbb{Z}^n \) and \( c \) a scalar. We denote \( \alpha \leq \beta \), \( \alpha \leq c \), and \( \alpha \pm c \) if \( \alpha_k \leq \beta_k \), \( \alpha_k \leq c \), and \( \alpha_1 \pm c, \ldots, \alpha_n \pm c \) for every \( k \), respectively, where the plus–minus sign is replaced by either the plus or minus sign in the same order. Moreover, for a vector-valued function \( f \), \( f_k \) denotes the \( k \)th component of \( f \).

For indexed quantities \( d_\alpha \), \( (d_\alpha)_\alpha \) is called a tensor (or a matrix only when \( \alpha \) has two components) and the determinant of a matrix \( (d_\alpha)_\alpha \) is defined by \( |(d_\alpha)_\alpha| \) or \( \det (d_\alpha)_\alpha \). The subsets \( R \) and \( I \) of \( \mathbb{R}^n \) indicate rectangles which are the Cartesian products of \( n \) compact intervals and \( |R| \) defines the Lebesgue measure of \( R \). Especially, \( I \) represents the codomain of an implicit function. Sometimes the integer \( N \) is considered a multi-index, in which case all components are defined as \( N \). With these the main results are formulated in Theorems 3.2, 4.2, and 5.2.

Before understanding the proposed methods, we must first state the classical implicit function theorem as follows.

**Implicit function theorem.** Let \( f(x, y) : \mathbb{R}^{n+m} \to \mathbb{R}^m \) be continuously differentiable with \( f(a, b) = 0 \). Suppose that \( |J_{f,y}(a, b)| = \det (\partial f/(\partial y)(a, b)) \neq 0 \). Then there is an \( R \times I \) of \( (a, b) \) and
a unique continuously differentiable function \( g : \mathbb{R} \to I \) such that \( f(x, g(x)) = 0 \) in \( R \). Moreover, the partial derivative of \( g \) is given by \( \partial_x g(x) = -J_{f,y}(\partial_y f(x, g(x)))^{-1}_{m \times n}(\partial_x f(x, g(x)))_{m \times 1} \).

This implicit function theorem has been extended to that of analytic functions. If every \( f_k \) is analytic in the implicit function theorem, then every \( g_k \) is also analytic (refer to \cite{4} Theorem 6.1.2).

2 Implicit functions

In this section, we derive the integral representation of an implicit function to isolate a dependent variable even though it is neither transcendental nor algebraic. Let \( \text{sgn} \) be the signum function on \( \mathbb{R} \) and \( f(x, y) \) a function on \( \mathbb{R}^{n+1} \). For each \( x \), define a function \( \text{sgn}_y f(x, y) \) by \( 1 \) if \( f(x, y) \geq 0 \) and \( -1 \) otherwise. We begin with a fundamental condition for the local existence of an implicit function.

**Assumption 1.** Let \( f : \mathbb{R}^{n+1} \to \mathbb{R} \) be continuous. Then there is an \( R \times I \subset \mathbb{R}^{n+1} \) such that for each \( x \in R \), \( \text{sgn}_y f(x, y) \) has only one jump discontinuity on \( I \).

From Assumption 1, by the intermediate value theorem, by the continuity of \( f \), and by the uniqueness of jump discontinuity, there is a continuous function \( g : R \to I \) such that \( f(x, g(x)) = 0 \) in \( R \). Moreover, if \( f \) is continuously differentiable such that \( \partial_y f(a, b) \neq 0 \) with \( f(a, b) = 0 \), then by the implicit function theorem there is an \( R \times I \ni (a, b) \), which has a unique continuously differentiable function \( y = g(x) \) such that \( f(x, g(x)) = 0 \) in \( R \). Also, the Jacobian matrix is invertible in \( R \). This implies that for each \( x \), \( \text{sgn}_y f(x, y) \) must have only one jump discontinuity on \( I \). Thus, the sufficient condition of the implicit function theorem guarantees Assumption 1.

**Example 2.1.** (i) Let \( f(x, y) = x^2 + y^2 - 1 \) be a function with \( f(0, 1) = 0 \). We want to find a function for \( y \) such that \( f = 0 \) in some closed interval of \( x = 0 \). Let us consider \( R \times I = [-1, 1] \times [0, 2] \ni (0, 1) \). Then \( \text{sgn}_y f(x, y) \) has only one jump discontinuity on \( I \). (On the other hand, since \( \partial_y f(0, 1) = 2 \neq 0 \), by application of the implicit function theorem there is an \( R \times I \ni (0, 1) \) such that for each \( x \in R \), \( \text{sgn}_y f(x, y) \) has the only one jump discontinuity on \( I \).)

(ii) Let \( f(x, y) = y^2 - x^4 \) be a function with \( f(0, 0) = 0 \). Although \( \partial_y f(0, 0) = 0 \), by taking \( R \times I = [-1, 1] \times [0, 1] \) or \( [-1, 1] \times [-1, 0] \ni (0, 0) \), we can justify that \( f \) satisfies Assumption 1.

For each \( x \in R \), if we define \( n_y(f(x)) = 1 \) if \( \text{sgn}_y f(x, y) \) is increasing and \( -1 \) if \( \text{sgn}_y f(x, y) \) is decreasing, then, \( n_y(f) \) is constant. Indeed, by the intermediate value theorem we may let \( y(x) \in I \) be a unique solution such that \( f(x, y(x)) = 0 \) for each \( x \in R \). Suppose that there is \( \xi_1 \) and \( \eta_1 \) such that \( n_y(f)(\xi_1)n_y(f)(\eta_1) = -1 \). We may assume that \( n_y(f)(\xi_1) = 1 \) and \( n_y(f)(\eta_1) = -1 \). We choose \( r > 0 \) such that \( y(x) + r \in I \) or \( y(x) - r \in I \) for every \( x \in R \). Let \( x_0 \) be the midpoint of the line segment \( \xi_1\eta_1 \) between \( \xi_1 \) and \( \eta_1 \). If \( f(x_0, y(x_0) + r) > 0 \) or \( f(x_0, y(x_0) - r) < 0 \), then select \( \overline{x_0\eta_1} \), whereas if \( f(x_0, y(x_0) + r) < 0 \) or \( f(x_0, y(x_0) - r) > 0 \),
Lemma 2.2. If equation.

The next result provides a necessary condition for the existence of a function form of an implicit equation. Let \( \Theta \) be the Heaviside step function. For each \( x \), define a function \( \Theta_y f(x, y) \) that assigns 1 if \( f(x, y) \geq 0 \) and 0 otherwise. For a rectangle \( R' \subset R \), we define a quantity as

\[
\iint_{R' \times I} \Theta_y f(x, y) \, dx \, dy = v_{R' \times I}.
\]

(1)

The next result provides a necessary condition for the existence of a function form of an implicit equation.

Lemma 2.2. If \( R \times I \) contains a unique function \( y = g(x) \) such that \( f(x, y) = 0 \), then

\[
v_{R' \times I} = \frac{n_y(f)}{2} |R'| \max I + \frac{n_y(f) - 1}{2} |R'| \min I - n_y(f) \int_{R'} g \, dx
\]

(2)

for every rectangle \( R' \subset R \).

By Lemma 2.1, the right-hand side of (2) is well defined.

Proof of Lemma 2.2. Let \( R' \subset R \) be a rectangle. Suppose that \( n_y(f) = 1 \). The integration (1) is calculated as

\[
\iint_{R' \times I} \Theta_y f(x, y) \, dx \, dy = |R'| \max I - \int_{R'} g \, dx.
\]

(3)

If \( n_y(f) = -1 \), then by putting \( F(x, y) = f(x, -y) \), we have \( n_y(F) = 1 \) on \( -I \) with \( y = -g(x) \) which satisfies \( F(x, y) = 0 \). By (3) with \( F \) and \( y = -g(x) \) on \( R \times (-I) \) instead of \( f \) and \( y = g(x) \),

\[
\iint_{R' \times (-I)} \Theta_y F(x, y) \, dx \, dy = |R'| \max(-I) + \int_{R'} g \, dx.
\]

(4)

By the change of variables \((-y \mapsto y)\), (4) equals

\[
\iint_{R' \times I} \Theta_y f(x, y) \, dx \, dy = -|R'| \min(I) + \int_{R'} g \, dx.
\]

(5)

According to (3) and (5), we conclude (2). Therefore, the proof is complete.

Now, we have the following integral representation of an implicit function.
Theorem 2.3. The continuous implicit function \( y = g(x) \) such that \( f = 0 \) in \( R \), is given by

\[
g(x) = \frac{1 + n_y(f)}{2} \max I + \frac{1 - n_y(f)}{2} \min I - n_y(f) \int_I \Theta_y f(x, y) \, dy \quad (x \in R).
\]

Equation (6) does not depend on the choice of a rectangle \( R \times I \) whenever it contains \( y = g(x) \).

Proof of Theorem 2.3. Let \( x \in R \) and take cubes \( Q \subset R \) that shrink to a singleton \( \{x\} \). Divide \( |Q| \) into both sides of \((1)\) and \((2)\) after replacing \( R' \) with \( Q \). By Fubini’s theorem and by the Lebesgue differentiation theorem,

\[
\int_I \Theta_y f(x, y) \, dy = \frac{n_y(f) + 1}{2} \max I + \frac{n_y(f) - 1}{2} \min I - n_y(f)g(x)
\]

for every point of continuity of \( f \). Therefore, the proof is complete.

Now, we set up an algebraic operator that will appear in the main theorems. Let \( C = (c_{ij,...}) \) and \( A = (a_{ij,j_2}) \) be an \( m_1 \times \cdots \times m_N \) tensor and \( n_1 \times n_2 \) matrix, respectively. If \( n_1 = m_k \), then we define a tensor contraction between \( A \) and \( C \) by

\[
A \overset{1 \rightarrow k}{\cdots} C = \left( \sum_{i_k=1}^{m_k} a_{i_k,j_2,...} \right)
\]

which produces an \( m_1 \times \cdots \times m_{k-1} \times n_2 \times m_{k+1} \times \cdots \times m_N \) tensor. If \( n_2 = m_k \), then \( A \overset{2 \rightarrow k}{\cdots} C \) is also defined, and it becomes an \( m_1 \times \cdots \times m_{k-1} \times n_1 \times m_{k+1} \times \cdots \times m_N \) tensor. For another \( n'_1 \times n'_2 \) matrix \( B \), if \( n_1 = m_i \) and \( n'_2 = m_j \) (\( i \neq j \)), then we have the commutative property of

\[
A \overset{1 \rightarrow i}{\cdots} (B \overset{2 \rightarrow j}{\cdots} C) = B \overset{2 \rightarrow j}{\cdots} (A \overset{1 \rightarrow i}{\cdots} C).
\]

Here, the parentheses are omitted providing the commutative property.

3 Polynomial implicit functions

In this section, we derive the implicit function of \( y = g(x) \) such that \( f(x, y) = 0 \), provided \( y = g(x) \) is a unique multivariate polynomial on \( R \times I \). Put \( R = \prod_{k=1}^{n} [\xi_k, \eta_k] \). Let \( N_k \) be a partition number and \( \xi_k < \xi_k + \Delta_k < \xi_k + 2\Delta_k < \cdots < \xi_k + N_k \Delta_k = \eta_k \) a partition on \( [\xi_k, \eta_k] \), where \( \Delta_k = (\eta_k - \xi_k)/N_k \) is a partition size. We define a grid block \( R_\alpha \) by

\[
R_\alpha = \prod_{k=1}^{n} [\xi_k + (\alpha_k - 1)\Delta_k, \xi_k + \alpha_k\Delta_k] \quad (1 \leq \alpha_k \leq N_k),
\]

where \( 1 \leq \alpha_k \leq N_k \) (\( 1 \leq k \leq n \)). The union of all \( R_\alpha \) is \( R \) and \( |R_\alpha| = \prod_{k=1}^{n} (\eta_k - \xi_k)/N_k \). For \( a \in \mathbb{R}^n \), we define a matrix by

\[
V_{N_k} = \left( (\xi_k + \alpha_k \Delta_k - a_k)^j - (\xi_k + (\alpha_k - 1)\Delta_k - a_k)^j \right)_{N_k \times N_k} \quad (1 \leq \alpha_k, j \leq N_k),
\]

where \( \alpha_k \) and \( j \) denote a row and column number, respectively.
Lemma 3.1. For every \( N_k \ (1 \leq k \leq n) \), \(|V_{N_k}|\) is positive and independent of \( a \). Consequently, \( V_{N_k} \) is invertible.

Proof. Fix \( k \ (1 \leq k \leq n) \). Writing \( \gamma_k = \xi_k - a_k \), we get

\[
|V_{N_k}| = \left| \left( \begin{array}{cccc} \alpha_k \Delta_k & \gamma_k - \alpha_k \Delta_k & \cdots & \gamma_N k - \alpha_k \Delta_k \\ \alpha_k + \Delta_k & \gamma_k - \alpha_k \Delta_k & \cdots & \gamma_N k - \alpha_k \Delta_k \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_k + N_k \Delta_k & \gamma_k - \alpha_k \Delta_k & \cdots & \gamma_N k - \alpha_k \Delta_k \end{array} \right) \right| \tag{9}
\]

From the invariant properties of determinants, add the first row to the second, the second row to the third, and so on until the end. Then (9) equals

\[
\begin{array}{cccc}
\Delta_k & (\gamma_k + \Delta_k)^2 - \gamma_k^2 & \cdots & (\gamma_k + \Delta_k)^{N_k} - \gamma_k^{N_k} \\
2\Delta_k & (\gamma_k + 2\Delta_k)^2 - \gamma_k^2 & \cdots & (\gamma_k + 2\Delta_k)^{N_k} - \gamma_k^{N_k} \\
\vdots & \vdots & \ddots & \vdots \\
N_k \Delta_k & (\gamma_k + N_k \Delta_k)^2 - \gamma_k^2 & \cdots & (\gamma_k + N_k \Delta_k)^{N_k} - \gamma_k^{N_k} \\
\end{array}
\tag{10}
\]

which is also equal to

\[
\begin{array}{cccc}
1 & \gamma_k & \gamma_k^2 & \cdots & \gamma_k^{N_k} \\
0 & \Delta_k & (\gamma_k + \Delta_k)^2 - \gamma_k^2 & \cdots & (\gamma_k + \Delta_k)^{N_k} - \gamma_k^{N_k} \\
0 & 2\Delta_k & (\gamma_k + 2\Delta_k)^2 - \gamma_k^2 & \cdots & (\gamma_k + 2\Delta_k)^{N_k} - \gamma_k^{N_k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & N_k \Delta_k & (\gamma_k + N_k \Delta_k)^2 - \gamma_k^2 & \cdots & (\gamma_k + N_k \Delta_k)^{N_k} - \gamma_k^{N_k} \\
\end{array}
\tag{11}
\]

by appending the first row and column of (11) to (10). By adding the first row of (11) to all other rows, we have

\[
\begin{array}{cccc}
1 & \gamma_k & \gamma_k^2 & \cdots & \gamma_k^{N_k} \\
1 & \gamma_k + \Delta_k & (\gamma_k + \Delta_k)^2 & \cdots & (\gamma_k + \Delta_k)^{N_k} \\
1 & \gamma_k + 2\Delta_k & (\gamma_k + 2\Delta_k)^2 & \cdots & (\gamma_k + 2\Delta_k)^{N_k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \gamma_k + N_k \Delta_k & (\gamma_k + N_k \Delta_k)^2 & \cdots & (\gamma_k + N_k \Delta_k)^{N_k} \\
\end{array}
\tag{12}
\]

which is

\[
\Delta_k \prod_{0 \leq i < j \leq N_k} (j - i) > 0,
\]

where the quantity is strictly positive for any \( N_k \) and independent of \( a \). Therefore, the proof is complete. \( \square \)
For notational simplicity, write \( v_{R_a \times I} = v_\alpha \) in (1) and put 
\[
d_\alpha = \frac{1 + n_y(f)}{2} |R_a| \max I + \frac{1 - n_y(f)}{2} |R_a| \min I - n_y(f)v_\alpha
\] (13)
which comes only from \( f \). By Lemma \( \ref{lem:2.2} \) \( d_\alpha \) can also be calculated as 
\[
d_\alpha = \int_{R_a} g \, dx.
\] (14)

Let \( W \) be an \( N_1 \times \cdots \times N_n \) tensor whose \( \alpha \)th component is \( \alpha_1 \cdots \alpha_n \). From now on, set \( N = (N_1, \ldots, N_n) \).

**Theorem 3.2.** If \( y = g(x) \) is a multivariate polynomial from \( R \) to \( I \) such that \( f(x, y) = 0 \), then for \( a \in R \), \( g(x) = \sum_{0 \leq \beta < N} c_\beta(x - a)^\beta \), where \( c_\beta \) is the \((\beta + 1)\)th component of 
\[
W \circ \left[ V_{N_{n_1}}^{-1} 2 \rightarrow n_1 \cdots V_{N_{n_k}}^{-1} 2 \rightarrow n_k \cdot V_{N_{n_1}}^{-1} 2 \rightarrow 1 \cdot (d_\alpha) \right]_\alpha (1 \leq \alpha \leq N),
\] (15)
\( N_k > \) the largest exponent of \( x_k \), and \( \circ \) denotes the Hadamard product between two tensors.

In Theorem \( \ref{thm:3.2} \) every component of (15) vanishes if its index contains a number larger than the largest exponent of \( x_k \) for some \( k \). Now we will call (15) the coefficient tensor for \( g \).

**Proof of Theorem 3.2.** For each \( k \), take a sufficiently large \( N_k \) so that \( N_k > \) the largest exponent of \( x_k \) in \( g \). By (7),
\[
\int_{R_a} g \, dx = \sum_{0 \leq \beta < N} c_\beta \int_{R_a} (x - a)^\beta \, dx
\] (16)
= \[
\sum_{0 \leq \beta < N} c_\beta \prod_{k=1}^n \frac{1}{\beta_k + 1} [ (\xi_k + \alpha_k \Delta_k - a_k)^{\beta_k + 1} - (\xi_k + (\alpha_k - 1) \Delta_k - a_k)^{\beta_k + 1} ].
\] (17)
By (14), the sum of (16) is equal to \( d_\alpha \). Moreover, (16) is the \( \alpha \)th component of 
\[
V_{N_{n_1}}^{-1} 2 \rightarrow 1 \cdot V_{N_{n_2}}^{-1} 2 \rightarrow 2 \cdots V_{N_{n_k}}^{-1} 2 \rightarrow n_k \cdot (c_\beta/(\beta_1 + 1) \cdots (\beta_n + 1)) \beta
\] (18)
where \( 0 \leq \beta \leq N - 1 \). Since \( V_{N_k} \) is invertible by Lemma \( \ref{lem:3.1} \) we have
\[
(c_\beta/(\beta_1 + 1) \cdots (\beta_n + 1)) = V_{N_{n_1}}^{-1} 2 \rightarrow n_1 \cdots V_{N_{n_k}}^{-1} 2 \rightarrow n_k \cdot V_{N_{n_1}}^{-1} 2 \rightarrow 1 \cdot (d_\alpha) \alpha.
\]
So,
\[
(c_\beta) \beta = W \circ \left[ V_{N_{n_1}}^{-1} 2 \rightarrow n_1 \cdots V_{N_{n_k}}^{-1} 2 \rightarrow n_k \cdot V_{N_{n_1}}^{-1} 2 \rightarrow 1 \cdot (d_\alpha) \right]_\alpha.
\] (18)
For the \( N' \)-partition of \( R \) \((N' \geq N)\), according to (7), (8), (13), (14), and (18), let \( (c'_\beta) \beta \) be the coefficient tensor for \( g(x) = \sum_{0 \leq \beta < N'} c'_\beta(x - a)^\beta \) which satisfies \( f = 0 \). From the uniqueness of the implicit function, the nontrivial components of two tensors should be identical, but they are different only in their sizes by adding zero components. This implies that \( c_\beta \) does not depend on the choice of \( N \) whenever \( N_k \) is greater than the largest exponent of \( x_k \) of \( g(x) \). Therefore, (18) results in the desired conclusion (15).
Since the necessary condition of the implicit function theorem with \( m = 1 \) implies Assumption 1, as mentioned before, we have a corollary.

**Corollary 3.2.1.** Suppose that \( f \) is continuously differentiable such that \( \partial_y f(a, b) \neq 0 \) with \( f(a, b) = 0 \). If \( y = g(x) \) is a multivariate polynomial such that \( f(x, y) = 0 \) near \( (a, b) \), then there is an \( R \times I \) so that for \( a \in R \), \( g(x) = \sum_{0 \leq \alpha < \mathcal{N}} c_{\alpha}(x - a)^{\alpha} \) with the coefficient tensor of \([15]\).

The following example is about a polynomial implicit function with two independent variables.

**Example 3.1.** Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be given by

\[
(f(x, y, z) = 0.5x^4 + 0.5x^3y + 0.5x^3 + 2x^2y + 0.5x^2z + 0.5xy^2 - 0.5xyz + 1.5xy - 0.5xz + x + 1.5y^2 - 0.5yz + 2y - z^2 + 3z - 2
\]

with \((2, 0, -2)\) and \((1, 1, 4)\) at which \( f \) vanishes. We want to solve \( f = 0 \) for \( z \) as a function of \( x \) and \( y \).

First, since \( \partial_z f(2, 0, -2) = 8 \neq 0 \), by application of the implicit function theorem there is a rectangle of \((2, 0, -2)\) on which \( \text{sgn}_y f(x, y) \) has the only one jump discontinuity. Choose \( R = [1.5, 2.5] \times [-0.5, 0.5] \ni (2, 0) \) and \( I = [-5, 1] \ni -2 \), so that \( n_x(f) = 1 \) in \( R \times I \). The surface of \( f = 0 \) and \( R \times I \) are depicted in Figure 1. With \( N_1 = N_2 = 3 \), \( V_{N_1}, V_{N_2} \) and \( W \) are calculated as

\[
\begin{pmatrix}
\Delta_x & (-0.5 + \Delta_x)^2 - (-0.5)^2 & (-0.5 + \Delta_x)^3 - (-0.5)^3 \\
\Delta_x & (-0.5 + 2\Delta_x)^2 - (-0.5 + \Delta_x)^2 & (-0.5 + 2\Delta_x)^3 - (-0.5 + \Delta_x)^3 \\
\Delta_x & (-0.5 + 3\Delta_x)^2 - (-0.5 + 2\Delta_x)^2 & (-0.5 + 3\Delta_x)^3 - (-0.5 + 2\Delta_x)^3 \\
\Delta_y & (-0.5 + \Delta_y)^2 - (-0.5)^2 & (-0.5 + \Delta_y)^3 - (-0.5)^3 \\
\Delta_y & (-0.5 + 2\Delta_y)^2 - (-0.5 + \Delta_y)^2 & (-0.5 + 2\Delta_y)^3 - (-0.5 + \Delta_y)^3 \\
\Delta_y & (-0.5 + 3\Delta_y)^2 - (-0.5 + 2\Delta_y)^2 & (-0.5 + 3\Delta_y)^3 - (-0.5 + 2\Delta_y)^3
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 3 \\
2 & 1 & 2 & 2 & 3 \\
3 & 1 & 3 & 2 & 3
\end{pmatrix},
\]

respectively. By \([13]\), the matrix of \( d_{(i,j)} \) is given by

\[
\begin{pmatrix}
-0.0498966767860677 & -0.13631672736493 & -0.22236578394150 & -0.130143979013675 & -0.22236592404210 & -0.31532919630333 \\
-0.22236603139157 & -0.321502042403124 & -0.42067478899573 & -0.0498966767860677 & -0.13631672736493 & -0.22236578394150 \\
-0.130143979013675 & -0.22236592404210 & -0.31532919630333 & -0.13631672736493 & -0.22236578394150 & -0.0498966767860677
\end{pmatrix}.
\]

By Corollary 3.2.1, the desired function \( z = g(x, y) = \sum_{0 \leq \alpha < \mathcal{N}} c_{\alpha}(x - 2)^{\alpha_1}y^{\alpha_2} \) is determined by the coefficient matrix of

\[
W \circ \left[ V_{N_2}^{-1} \begin{pmatrix} 2 & 2 \end{pmatrix} V_{N_1}^{-1} \begin{pmatrix} 2 & 1 \end{pmatrix} (d_{\alpha})_\alpha \right] = \begin{pmatrix} 1 & y & y^2 \\ -2 & -2.5 & 0 \\ -2.50001 & -0.499998 & 2 \times 10^{-6} \\ -0.499999 & -3 \times 10^{-6} & 4 \times 10^{-6} \end{pmatrix}
\]

\[
\begin{pmatrix} 1 \\ x - 2 \\ (x - 2)^2 \end{pmatrix},
\]

\[
1
\]

8
where $-0.499998$, for example, denotes the coefficient of $(x - 2)y$ in the summation of $g$.

On the other hand, since $\partial_z f(1, 1, 4) = -6 \neq 0$, by application of the implicit function theorem there is a rectangle, for example, $R = [0.5, 1.5] \times [0.5, 1.5] \ni (1, 1)$ and $I = [2, 7] \ni 4$, on which $n_z(f) = -1$. For $N_1 = N_2 = 4$ (as shown in Figure 1), we derive the coefficient matrix of

\[
\begin{pmatrix}
1 & y - 1 & (y - 1)^2 & (y - 1)^3 \\
4 & 1.000001 & -1 \cdot 10^{-6} & 1 \cdot 10^{-6} \\
2.000001 & -1 \cdot 10^{-6} & 2 \cdot 10^{-6} & -2 \cdot 10^{-6} \\
0.999999 & 2 \cdot 10^{-6} & -3 \cdot 10^{-6} & 4 \cdot 10^{-6} \\
0 & -1 \cdot 10^{-6} & 3 \cdot 10^{-6} & -4 \cdot 10^{-6}
\end{pmatrix}
\begin{pmatrix}
1 \\
x - 1 \\
(x - 1)^2 \\
(x - 1)^3
\end{pmatrix}
\]

(20)

of $z = g(x, y) = \sum_{0 \leq \alpha < 4} c_\alpha (x - 1)^{\alpha_1} (y - 1)^{\alpha_2}$ that satisfies $f = 0$ in $R$ by the same method shown above.

In fact, $f$ is factorized by

$$f_1 = \frac{5}{2}(x - 2) + \frac{5}{2}y + z + 2 + \frac{1}{2}(x - 2)^2 + \frac{1}{2}(x - 2)y$$

and

$$f_2 = 2(x - 1) + (y - 1) - z + 4 + (x - 1)^2.$$

The coefficient matrices of functions for $z$ such that $f_1 = 0$ and $f_2 = 0$, respectively, are shown as

\[
\begin{pmatrix}
1 & y & y^2 \\
-2 & -2.5 & 0 \\
-2.5 & -0.5 & 0 \\
-0.5 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
x - 2 \\
(x - 2)^2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1 & y - 1 & (y - 1)^2 & (y - 1)^3 \\
4 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
x - 1 \\
(x - 1)^2 \\
(x - 1)^3
\end{pmatrix}
\]

(21)

which are comparable to (19) and (20), respectively.

### 4 Analytic implicit functions

In this section, we derive a power series expansion of an implicit function which has one dependent variable, and any number of independent variables as before. The proposed method does not require the size estimate of Taylor coefficients and even differentiability of $f$ as we have seen in Section 3. We start with an assumption that the implicit function $y = g(x)$ for $f(x, y) = 0$, is analytic on a rectangle, which means that $g$ is analytic on some open neighborhood containing the rectangle. By translation and dilation on the independent variables of $f$, we may assume
that the implicit function is analytic on \([-1-\delta, 1+\delta]^n\) for some \(\delta > 0\) and write \(g(x) = \sum \beta c_\beta x^\beta\) which converges in \([-1-\delta, 1+\delta]^n\), where the sum is the power series expansion of \(g\) at the origin. Let \(R = [-1,1]^n\) and put

\[-1 < -1 + \Delta_x < -1 + 2\Delta_x < \cdots < -1 + (N-1)\Delta_x < -1 + N\Delta_x = 1\]

be a partition on \([-1,1]\) \((\Delta_x = 2/N)\). According to (13) and (1), prepare \(d_\alpha\) from \(f\) over \(R_\alpha \times I\), where

\[R_\alpha = \prod_{k=1}^n [-1 + (\alpha_k - 1)\Delta_x, -1 + \alpha_k\Delta_x]\]

for \(\alpha_k = 1, 2, \ldots, N\).

We split \(g\) into two parts: one is a partial sum in which the exponent of a monomial is dominated by a multi-index \(N\), another is the remainder of the series of \(g\), e.g.,

\[g(x) = \sum_{0 \leq \beta < N} c_\beta x^\beta + \sum_{\text{remainder}} c_\beta x^\beta\]

\[(22)\]

\[= g_N(x) + r_N(x), \text{ say.}\]

From analyticity, the series of \(g\) converges uniformly and absolutely. There is a constant \(C\) such that \(|c_\beta x^\beta| \leq C\) uniformly in \(x \in [-1-\delta, 1+\delta]^n\) for every \(\beta\). So, \(|c_\beta (1+\delta)^|\beta| \leq C\) for every \(\beta\) (for the properties of real analytic functions of several variables, refer to [5]).

In \(r_N\) of (22), there is an index \(\beta\) such that its component is greater than or equal to \(N\).
Assume that $\beta_1 \geq N$. Then, for $x \in R$,

$$
\sum_{\text{remainder } \beta} |c_\beta||x^\beta| \leq C \sum_{\text{remainder } \beta} \frac{|x^\beta|}{(1 + \delta)|\beta|}
\leq \sum_{\beta_1=N}^{\infty} \sum_{\beta_k=0}^{\infty} \frac{1}{(1 + \delta)|\beta|}
= \frac{1}{\delta^n(1 + \delta)^N-n}
$$

(23)

which vanishes as $N \to \infty$. Let $\epsilon > 0$. By the triangle inequality and by (23), we have

$$
\|r_N\|_{L^\infty(R)} < \epsilon
$$

and

$$
\int_R |r_N| \, dx \leq |R_\alpha| \epsilon \leq 2^n \epsilon
$$

for every sufficiently large $N$.

For a fixed $N$, we put $\tilde{g}_N(x) = \sum_{0 \leq \alpha < N} \tilde{c}_\alpha x^\alpha$, where $\tilde{c}_\alpha$ is calculated by solving

$$
\int_{R_\alpha} \tilde{g}_N \, dx = d_\alpha
$$

(25)
as the derivation of (15). By (14) and by (25),

$$
\left| \int_{R_\alpha} g_N - \tilde{g}_N \, dx \right| = \left| \int_{R_\alpha} g_N \, dx - d_\alpha \right|
\leq \left| \int_{R_\alpha} g_N - g \, dx \right|
\leq \left| \int_{R_\alpha} r_N \, dx \right|
$$

(26)

where the last equality follows from (22). By the Riesz representation theorem for the Lebesgue spaces (the duality argument), we realize the next lemma.

**Lemma 4.1.** The function $\tilde{g}_N$ converges to $g$ weak-star in $L^\infty(R)$ as $N \to \infty$.

**Proof.** We recall (26). By the triangle inequality,

$$
\int_{R_\alpha} |\tilde{g}_N - g| \, dx \leq \int_{R_\alpha} |\tilde{g}_N - g_N| \, dx + \int_{R_\alpha} |r_N| \, dx \to 0
$$

(27)
uniformly on $\alpha$ as $N \to \infty$. Here, the convergence of (27) follows from (24). Since the collection of finite linear combinations of characteristic functions $\chi_{R_\alpha}$ which are supported on grid blocks $R_\alpha$ is dense in $L^1(R)$, the duality argument (3, 12) from (26) and (27) for Lebesgue spaces yield that $\tilde{g}_N$ goes to $g$ weak-star in $L^\infty(R)$ as $N \to \infty$. Therefore, the proof is complete. \qed

Let $V_N$ and $W$ be a matrix and tensor as in Theorem 3.2 with $V_N = V_{N_k}$ and $N = N_k$. By Lemma 4.1, we conclude the main theorem of this section.
Theorem 4.2. If \( y = g(x) \) is analytic on \( R \) to \( I \) such that \( f(x,y) = 0 \), then \( \tilde{g}_N(x) = \sum_{0 \leq \alpha < N} \tilde{c}_\alpha x^\alpha \) with the coefficient tensor of
\[
W_N \circ \begin{bmatrix} V^{-1}_{N-1} & \cdots & V^{-1}_{N-2} & V_{N-1}^{-1} \end{bmatrix} \left( d_\alpha \right) \alpha,
\]
converges to \( g \) weak-star in \( L^\infty(R) \) as \( N \to \infty \).

As we have seen in Corollary 3.2.1, the next corollary follows.

Corollary 4.2.1. Suppose that \( f \) is continuously differentiable such that \( \partial_y f(0,0) \neq 0 \) with \( f(0,0) = 0 \). If \( y = g(x) \) is analytic on \( R \) to \( I \) such that \( f(x,y) = 0 \), then \( \tilde{g}_N \) converges to \( g \) weak-star in \( L^\infty(R) \) as \( N \to \infty \).

If \( f \) is analytic such that \( \partial_y f(0,0) \neq 0 \) with \( f(0,0) = 0 \), then by the analytic version of the implicit function theorem, Corollary 4.2.1 holds. We now present a numerical approximation of an analytic implicit function.

Example 4.1. Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be given by
\[
f(x,y,z) = x^2 + y^2 + z^2 - 1
\]
with \( f(0,0,1) = 0 \). We want to find a function of \( x \) and \( y \) that satisfies \( f = 0 \) in some rectangle of \( (0,0,1) \). Since \( f \) is analytic on a neighborhood of \( (0,0,1) \) and \( \partial_y f(0,0,1) = 2 > 0 \), by the analytic version of the implicit function theorem there is a rectangle of \( (0,0,1) \) and a unique analytic implicit function for \( z \) such that \( f = 0 \) in the rectangle.

Choose a rectangle \( R \times I \), for example, \( R = [-1/2,1/2] \times [-1/2,1/2] \supset (0,0) \) and \( I = [0,3/2] \ni 1 \), which are depicted in Figure 2. In \( R \times I \), readily \( n_z(f) = 1 \) and with \( N = 6 \), it follows that
\[
V_N = \begin{bmatrix}
\Delta & (-1/2 + \Delta)^2 - (-1/2)^2 & \cdots & (-1/2 + \Delta)^6 - (-1/2)^6 \\
\Delta & (-1/2 + 2\Delta)^2 - (-1/2 + \Delta)^2 & \cdots & (-1/2 + 2\Delta)^6 - (-1/2 + \Delta)^6 \\
\vdots & \vdots & \ddots & \vdots \\
\Delta & (-1/2 + 6\Delta)^2 - (-1/2 + 5\Delta)^2 & \cdots & (-1/2 + 6\Delta)^6 - (-1/2 + 5\Delta)^6
\end{bmatrix},
\]
where \( \Delta = 1/6 \). From [13], the \( 6 \times 6 \) matrix of \( d_{i,j} \) is given by
\[
\begin{pmatrix}
0.022341010165457 & 0.0241922733141742 & 0.025065830482660 & 0.025065830482660 & 0.025065830482660 & 0.022341010165457 \\
0.0241922733141742 & 0.025065830482660 & 0.025065830482660 & 0.025065830482660 & 0.025065830482660 & 0.0241922733141742 \\
0.025065830482660 & 0.026726310029805 & 0.026726310029805 & 0.026726310029805 & 0.026726310029805 & 0.026726310029805 \\
0.026726310029805 & 0.027518589197521 & 0.027518589197521 & 0.027518589197521 & 0.027518589197521 & 0.027518589197521 \\
0.027518589197521 & 0.028350111856260 & 0.028350111856260 & 0.028350111856260 & 0.028350111856260 & 0.028350111856260 \\
0.028350111856260 & 0.029181624571885 & 0.029181624571885 & 0.029181624571885 & 0.029181624571885 & 0.029181624571885
\end{pmatrix}
\]
By Corollary 4.2.1, the coefficient matrix of \( z = \tilde{g}_N(x, y) = \sum_{0 \leq \alpha < 6} c_\alpha x^\alpha y^{\alpha_2} \) which approximates \( z = z(x, y) \) such that \( f(x, y, z(x, y)) = 0 \), is calculated as

\[
W \circ \left[ V_N^{-1} 2 \cdot 2 V_N^{-1} 2 \cdot 1 \left( d_\alpha \right) \right] = \\
\begin{pmatrix}
1 & y & y^2 & y^3 & y^4 & y^5 \\
0.99997 & -0.49880 & 0 & -0.14537 & 0 & x \\
0 & 0 & 0 & 0 & 0 & x^2 \\
0 & -0.49880 & -0.24363 & 0 & -0.23108 & 0 & x^3 \\
0 & 0 & 0 & 0 & 0 & x^4 \\
-0.14537 & 0 & -0.23108 & 0 & -0.50484 & 0 & x^5 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(29)

where \( W = (i_j)_{1 \leq i,j \leq 6} \) and, for example, \(-0.24363\) in (29) is the coefficient of \( x^2 y^2 \) a term of \( g \).

Both \( \tilde{g}_N \) and \( z(x, y) = (1 - x^2 - y^2)^{1/2} \) are plotted in Figure 3.

On the other hand, the coefficient matrix of the partial sum \( g_N \) of the Taylor series of \( z = (1 - x^2 - y^2)^{1/2} \) is as shown as

\[
\begin{pmatrix}
1 & y & y^2 & y^3 & y^4 & y^5 \\
1 & 0 & -0.5 & 0 & -0.125 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 & x^2 \\
-0.5 & 0 & -0.25 & 0 & -0.1875 & 0 & x^3 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-0.125 & 0 & -1.875 & 0 & -0.234375 & 0 & x^4 \\
0 & 0 & 0 & 0 & 0 & 0 & x^5 \\
\end{pmatrix}
\]

As it is indicated on the graph of Figure 4, in the accuracy comparison between \( \tilde{g}_N \) and \( g_N \), the former is more accurate.
In general, the implicit function and inverse function theorems can be thought of as equivalent formulations of a similar basic idea. The next example leads to the polynomial approximation of an inverse function for Kepler’s equation. When an initial value is given, the methods using iterative calculation of a trajectory that satisfies Kepler’s equation are widely used in practice. Although the methods using the formula by a function are very useful, which are formal infinite sums or depend on the expansion method of the Taylor series, there is no proper approach to achieve high-performance computing. The following example provides more accurate numerical values than the other results on Kepler’s equation.

**Example 4.2.** Let $M = E - \epsilon \sin(E)$ be the standard Kepler equation which fixes $(\pi, \pi)$ for every $0 \leq \epsilon \leq 1$, where $M$, $E$, and $\epsilon$ are the mean anomaly, the eccentric anomaly, and the eccentricity, respectively. Kepler’s equation has a unique inverse function for every eccentricity,
since $M$ is increasing monotonically.

Let $K(E,M) = E - \epsilon \sin(E) - M : \mathbb{R}^2 \rightarrow \mathbb{R}$. The Taylor series does not give a good approximation of the inverse function (the implicit function for $E$ of $K = 0$) in the open neighborhood of $M = 0$ and $\pi$, especially when $\epsilon = 1$ because $\partial_E K$ vanishes. (To approximate the inverse function, for example, [4] adopts the Lagrange inverse theorem to obtain a formal series expansion and [6] provides the usability of a solution of Kepler’s equation for $E$ as the most recent result.)

Choose a rectangle of $(\pi, \pi)$, for example, $[0, 2\pi] \times [-\pi, 3\pi]$ on which $n_E(K) = 1$. By Theorem 4.2 for $N = 28$ we have an approximation of $E$ by $\tilde{E}(M) = \sum_{n=0}^{N-1} c_n(M - \pi)^n$ calculated as

$$\tilde{E}(M) = \left( \begin{array}{c} 3.14159265358979 \times 10^{-13} \\ 0.50000000000000 \times 10^{-12} \\ 0.33333333333333 \times 10^{-12} \\ -8.33333333333333 \times 10^{-12} \\ 3.33333333333333 \times 10^{-12} \\ -6.33333333333333 \times 10^{-12} \\ 8.99999999999999 \times 10^{-12} \\ -1.71096401190211 \times 10^{-11} \\ 1.35955555555556 \times 10^{-11} \\ -3.94747474747475 \times 10^{-11} \\ 7.15015555555556 \times 10^{-11} \\ -9.90000000000000 \times 10^{-11} \\ 1.12500000000000 \times 10^{-10} \\ -1.37500000000000 \times 10^{-10} \\ 1.62500000000000 \times 10^{-10} \\ -1.87500000000000 \times 10^{-10} \\ 2.12500000000000 \times 10^{-10} \\ -2.37500000000000 \times 10^{-10} \\ 2.62500000000000 \times 10^{-10} \\ -2.87500000000000 \times 10^{-10} \\ 3.12500000000000 \times 10^{-10} \\ -3.37500000000000 \times 10^{-10} \\ 3.62500000000000 \times 10^{-10} \\ -3.87500000000000 \times 10^{-10} \\ 4.12500000000000 \times 10^{-10} \\ -4.37500000000000 \times 10^{-10} \\ 4.62500000000000 \times 10^{-10} \\ -4.87500000000000 \times 10^{-10} \\ 5.12500000000000 \times 10^{-10} \\ -5.37500000000000 \times 10^{-10} \\ 5.62500000000000 \times 10^{-10} \end{array} \right) \times \left( \begin{array}{c} 1 \\ M - \pi \\ (M - \pi)^2 \\ \vdots \\ (M - \pi)^{26} \\ (M - \pi)^{27} \end{array} \right)_{28 \times 1}.$$

In Figures 5 and 6, $\tilde{E}$ and $K(M, \tilde{E}(M))$ are shown for $\epsilon = 0.8, 0.9$, and 1 (for comparison with related work, e.g., refer to [10, 6]).

![Figure 5: Polynomial approximation $\tilde{E} = \sum_{n=0}^{27} c_n(M - \pi)^n$ for the implicit function for $E$ of $K = 0$ in $[0, 2\pi]$ with $\epsilon = 0.8, 0.9$, and 1.](image)

From Example 4.2 we have seen that the inverse function for $y = f(x)$ is obtained by finding an implicit function for $x$ such that $F(x, y) = 0$ after setting $F(x, y) = y - f(x)$. For a system
of implicit equations, Theorem 5.2 will be applicable for constructing a vector-valued inverse function.

5 System of analytic implicit functions

In this section, we extend the results for a real-valued function in Section 4 to a vector-valued function. We will prove it by a variable-reduction technique which is the process of eliminating the dependent variables one by one. Let $\phi$ be a permutation on $\{1, 2, \ldots, m\}$ that consists of all axis numbers for dependent variables and put $I = \prod_{i=1}^{m} I_i \subset \mathbb{R}^m$, where $I_i$ is an interval. For a vector $b$ and rectangle $I$, the notations $b_{\{j\}}$ and $I_{\{j\}}$ denote the $j$th component and edge deletions of $b$ and $I$, respectively.

Assumption 2. For a continuous vector-valued function $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$, the following conditions are satisfied in descending induction over the number of dependent variables: there is a permutation $\phi$ and an $R \times I \subset \mathbb{R}^{n+m}$ such that

(i) for $(x, y^{\{m\}}) \in R \times I^{\{m\}}$,
\[ \text{sgn}_{y_m} f_{\phi(m)}(x, y^{\{m\}}, y_m) \]
has only one jump discontinuity as a function of $y_m$ on $I_m$,

(ii) for $(x, y^{\{m,m-1\}}) \in R \times I^{\{m,m-1\}}$,
\[ \text{sgn}_{y_{m-1}} f_{\phi(m-1)}(x, y^{\{m,m-1\}}, y_{m-1}, h_{\phi(m)}(x, y^{\{m\}})) \]
has only one jump discontinuity as a function of $y_{m-1}$ on $I_{m-1}$, where $y_m = h_{\phi(m)}$ solves $f_{\phi(m)}(x, y^{\{m\}}, y_m) = 0$ in $R \times I^{\{m,m-1\}}$. 

Figure 6: The non-vanishing values of $K(M, \tilde{E}(M))$ denote errors.
Using (i) and (ii), by the intermediate value theorem, by the continuity of \( f \), and by the uniqueness of jump discontinuity, we know that \( y_m = h_{\phi(m)} \) and \( y_{m-1} = h_{\phi(m-1)} \) are continuous, uniquely determined. Hence, Assumption 2 is well explained by descending induction.

**Example 5.1.** Let \( f = (f_1, f_2) : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be given by \( f_1(x, y, z) = y - 1 \), \( f_2(x, y, z) = x \) with \( f(0, 1, 1) = 0 \). By observation of \( f_1 \) and \( f_2 \), we easily know that there is no rectangle of \((0, 1, 1)\) on which \( \text{sgn}_x f_1(x, y, z), \text{sgn}_z f_1(x, y, z), \text{sgn}_y f_2(x, y, z), \) and \( \text{sgn}_z f_2(x, y, z) \) have only one jump discontinuity. This means that there is no \( \phi \) on a set containing the \( z \)-axis number, which satisfies Assumption 2. On the other hand, since \( \partial_y f_1(0, 1, 1) = 1 \neq 0 \), by the analytic version of the implicit function theorem, there is a rectangle of \((0, 1, 1)\) and a unique analytic function \( y = g(x, z) \) such that \( f_1 = 0 \) in the rectangle on which \( \text{sgn}_y f_1(x, y, z) \) has only one jump discontinuity. Moreover,

\[
\partial_x f_2(x, y(x, z), z) \big|_{(x,z)=(0,1)} = \partial_x f_2(0, 1, 1) + \partial_y f_2(0, 1, 1) \partial_x y(0, 1) = 1 \neq 0.
\]

Thus, \( \text{sgn}_x f_2(x, y(x, z), z) \) has only one jump discontinuity in the rectangle. Now the permutation \( \phi \) is defined by \( \phi(1) = 2 \) and \( \phi(2) = 1 \), where 1 and 2 denote the axis numbers of \( x \) and \( y \), respectively, that satisfies Assumption 2 in some rectangle of \((0, 1, 1)\).

**Example 5.2.** Let \( f = (f_1, f_2) : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be defined by \( f_1(x, y, z) = y - 1 \) and \( f_2(x, y, z) = (z - 1)^2 \) with \( f(0, 1, 1) = 0 \). Since \( f_2 \) is a squared function, \( \text{sgn}_z f_2(x, y, z) \) has no jump discontinuity on any rectangle of \((0, 1, 1)\). However, if we consider \( f_1 \) and \( \partial_z f_2 \) instead of \( f_2 \), then \( \text{sgn}_y f_1 \) and \( \text{sgn}_z \partial_z f_2 \) have only one jump discontinuity, for example, on \([-1, 1] \times [-1, 2] \times [-1, 2] \ni (0, 1, 1)\). We can find \( \phi \) by defining \( \phi(1) = 1 \) and \( \phi(2) = 2 \), which satisfies Assumption 2, where 1 and 2 denote the axis numbers of \( y \) and \( z \), respectively. Note that \((f_1, f_2)\) and \((f_1, \partial_z f_2)\) have the same implicit function for \( y \) and \( z \) in the rectangle.

We need a lemma to prove the main theorem of this section, which will be shown by using a method of variable elimination.

**Lemma 5.1.** If \( y_i = h_{\phi(i)} \) satisfies

\[
f_{\phi(i)}(x, y^{\{m, m-1, \ldots, i\}}, h_{\phi(i)}(x, y^{\{m, m-1, \ldots, i\}}), h_{\phi(i+1)}(x, y^{\{m, m-1, \ldots, i+1\}}), \ldots, h_{\phi(m)}(x, y^{\{m\}})) = 0
\]

in \( R \times I^{\{m, m-1, \ldots, i\}} \) \((i = m, m-1, \ldots, 1)\), then \( y = g(x) \) such that \( f = 0 \), is continuous and determined uniquely by

\[
g_1(x) = h_{\phi(1)}(x),
\]

\[
g_2(x) = h_{\phi(2)}(x, g_1(x)),
\]

\[
\vdots
\]

\[
g_m(x) = h_{\phi(m)}(x, g_1(x), g_2(x), \ldots, g_{m-1}(x))
\]

in \( R \)

**Proof.** For convenience, we assume that \( \phi = id \). By regarding \((x, y^{\{m\}})\) as the independent variables of \( f \), from (i) there is a unique continuous function \( y_m = h_m(x, y^{\{m\}}) : R \times I^{\{m\}} \rightarrow I_m \) such that \( f_m(x, y^{\{m\}}, y_m) = 0 \) in \( R \times I^{\{m\}} \).
Substitute $y_m = h_m(x, y^{[m]})$ into $f$ and put $f(x, y^{[m]}, h_m(x, y^{[m]})) = f^{[1]}(x, y^{[m]})$. Precisely,
\[
f^{[1]}_1(x, y^{[m]}) = f_1(x, y^{[m]}, h_m(x, y^{[m]})),
\]
\[
\vdots
\]
\[
f^{[1]}_{m-1}(x, y^{[m]}) = f_{m-1}(x, y^{[m]}, h_m(x, y^{[m]})),
\]
\[
f^{[1]}_m(x, y^{[m]}) = 0
\]
on $R \times I^{[m]}$, where the $m - 1$ nontrivial components of $f^{[1]}$ do not depend on the $y_m$-variable.

For every $(x, y^{[m-1]}) \in R \times I^{[m,m-1]}$, the property of (ii) shows that
\[
\text{sgn}_{y_{m-1}} f^{[1]}_{m-1}(x, y^{[m]}) = \text{sgn}_{y_{m-1}} f_{m-1}(x, y^{[m]}, h_m(x, y^{[m]}))
\]
has one jump discontinuity on $I_{m-1}$. This also produces a unique continuous function $y_{m-1} = h_{m-1}(x, y^{[m-1]}): R \times I^{[m,m-1]} \to I_{m-1}$ such that $f^{[1]}_{m-1}(x, y^{[m-1]}, y_{m-1}) = 0$.

Substitute $y_{m-1} = h_{m-1}(x, y^{[m-1]})$ into $f^{[1]}$ and similar to the argument above, put $f^{[1]}(x, y^{[m-1]}, h_{m-1}(x, y^{[m-1]})) = f^{[2]}(x, y^{[m-1]}), i.e.,$
\[
f^{[2]}_1(x, y^{[m-1]}) = f^{[1]}_1(x, y^{[m-1]}, h_{m-1}(x, y^{[m-1]})),
\]
\[
\vdots
\]
\[
f^{[2]}_{m-1}(x, y^{[m-1]}) = 0,
\]
\[
f^{[2]}_m(x, y^{[m-1]}) = 0
\]
on $R \times I^{[m,m-1]}$. Then the $m - 2$ nontrivial components of $f^{[2]}$ clearly do not depend on the variables of $y_m$ and $y_{m-1}$.

Similarly, for $(x, y^{[m-1,m-2]}) \in R \times I^{[m,m-1,m-2]}$, the inductively assumed property of (ii) shows that
\[
\text{sgn}_{m-2} f^{[2]}_{m-2}(x, y^{[m-1]}) = \text{sgn}_{m-2} f^{[1]}_{m-2}(x, y^{[m-1]}, h_{m-1}(x, y^{[m-1]}))
\]
\[
= \text{sgn}_{m-2} f_{m-2}(x, y^{[m-1]}, h_{m-1}(x, y^{[m-1]}), h_m(x, y^{[m]}))
\]
has one jump discontinuity on $I_{m-2}$. Again, we gain a unique continuous function $y_{m-2} = h_{m-2}(x, y^{[m-1,m-2]}): R \times I^{[m,m-1,m-2]} \to I_{m-2}$ which satisfies the implicit equation of $f^{[2]}_{m-2}(x, y^{[m-1,m-2]}, y_{m-2}) = 0$.

Repeating the above variable-reduction process inductively until reaching $y_1$, we get a unique continuous function $y_1 = h_1(x): R_n \to I_1$ such that $f^{[m-1]}_1(x, h_1(x)) = 0$. Eventually, the unique continuous function $y = g(x)$ for $f(x, y) = 0$ is calculated as
\[
y_1 = h_1(x),
\]
\[
y_2 = h_2(x, y_1),
\]
\[
\vdots
\]
\[
y_m = h_m(x, y_1, \ldots, y_{m-1})
\]
in \( R \). Therefore, we obtain (30) and the proof is complete. 

Although \( y_i = h_{\phi(i)} \) in Lemma 5.1 has an integral representation as shown in (30), which does not reveal itself algebraically. From (30) and (31), if every \( h_{\phi(i)} \) is analytic, then every \( g_i \) is also analytic, and vice versa. If these are analytic, then every \( g_i \) is calculated as the limit of a sequence of multivariate polynomials in the weak-star topology on \( L^\infty \). The necessity of the implicit function theorem and the analyticity of implicit functions yield the following main theorem and corollary.

**Theorem 5.2.** If \( f \) is continuously differentiable such that \( |J_{f,y}(a,b)| \neq 0 \) with \( f(a,b) = 0 \) and every component of an implicit function for \( f = 0 \) is analytic on a neighborhood of \((a,b)\), then the implicit function is calculated uniquely in some rectangle of \((a,b)\) as (30).

Since the analytic version of the implicit function theorem guarantees a unique existence of a system of analytic implicit functions in some rectangle, we have a corollary.

**Corollary 5.2.1.** Suppose that every component of \( f \) is analytic such that \( |J_{f,y}(a,b)| \neq 0 \) with \( f(a,b) = 0 \). Then the implicit function for \( f = 0 \) is calculated uniquely in some rectangle of \((a,b)\) as (30).

**Proof of Theorem 5.2.** It suffices to show that there is an \( R \times I \) and a permutation \( \phi \) with which \( y_i = h_{\phi(i)} \) is analytic on \( R \times I^{(m,m-1,...,i)} \) \((i = m, m-1, ..., 1)\), which satisfies Assumption 2. Then by Lemma 5.1, the desired conclusion follows.

Since that \( |J_{f,y}(a,b)| \neq 0 \), the column vectors of \( J_{f,y}(a,b) \) are not all zero. There is \( i \) such that \( \partial y_m f_i(a,b) \neq 0 \) and define \( \phi(m) = i \). Thus there is a unique continuous function \( y_m = h_i(x,y^{(m)}) \) such that \( f_i(x,y^{(m)}, h_i(x,y^{(m)})) = 0 \) for some rectangle of \((a,b)\). In addition, by Corollary 4.2.1 with analyticity, \( y_m = h_i \) is analytic on \( R^{(1)}_1 \times \prod_{k=1}^m I^{(1)}_k \) to \( I^{(1)}_m \) for some rectangle \( R^{(1)}_1 \times \prod_{k=1}^m I^{(1)}_k \ni (a,b) \).

Put \( f^{[i]}(x,y^{(m)}) = f^{[i]}(x,y^{(m)},h_i(x,y^{(m)})) \), where \( f^{[i]} \) is the \( i \)th component removal from \( f \). First, we prove that \( |J_{f^{[i]},y^{(m)}}(a,b^{(m)})| \neq 0 \). Since \( \partial y_{m}, f_i(a,b) \neq 0 \), at \((a,b^{(m)})\) the determinant of

\[
|J_{f^{[i]},y^{(m)}}| = \begin{vmatrix}
\partial y_1 f^{[i]}_1 & \ldots & \partial y_{m-1} f^{[i]}_1 \\
\vdots & \ddots & \vdots \\
\partial y_1 f^{[i]}_{i-1} & \ldots & \partial y_{m-1} f^{[i]}_{i-1} \\
\partial y_1 f^{[i]}_{i+1} & \ldots & \partial y_{m-1} f^{[i]}_{i+1} \\
\vdots & \ddots & \vdots \\
\partial y_1 f^{[i]}_{m} & \ldots & \partial y_{m-1} f^{[i]}_{m}
\end{vmatrix}
\]
is equal to

\[
\frac{(-1)^{i+m}}{\partial_{y_m} f_i(a, b)} \begin{pmatrix}
\partial_{y_1} f_1^{[1]} & \cdots & \partial_{y_{m-1}} f_1^{[1]} & \partial_{y_m} f_1(a, b) \\
\vdots & \ddots & \vdots & \vdots \\
\partial_{y_1} f_i^{[1]} & \cdots & \partial_{y_{m-1}} f_i^{[1]} & \partial_{y_m} f_i(a, b) \\
0 & \cdots & 0 & \partial_{y_m} f_i(a, b) \\
\partial_{y_1} f_{i+1}^{[1]} & \cdots & \partial_{y_{m-1}} f_{i+1}^{[1]} & \partial_{y_m} f_{i+1}(a, b) \\
\vdots & \ddots & \vdots & \vdots \\
\partial_{y_1} f_m^{[1]} & \cdots & \partial_{y_{m-1}} f_m^{[1]} & \partial_{y_m} f_m(a, b)
\end{pmatrix}.
\tag{32}
\]

Let \(1 \leq k \leq m - 1\). Subtract \(\partial_{y_k} h_i\) times the last column of (32) from the other columns. Then the \(k\)th column of (32) is calculated as

\[
\begin{pmatrix}
\partial_{y_k} f_1^{[1]} - \partial_{y_m} f_1(a, b)\partial_{y_k} h_i \\
\vdots \\
\partial_{y_k} f_i^{[1]} - \partial_{y_m} f_i(a, b)\partial_{y_k} h_i \\
-\partial_{y_m} f_1(a, b)\partial_{y_k} h_i \\
\partial_{y_k} f_{i+1}^{[1]} - \partial_{y_m} f_{i+1}(a, b)\partial_{y_k} h_i \\
\vdots \\
\partial_{y_k} f_m^{[1]} - \partial_{y_m} f_m(a, b)\partial_{y_k} h_i
\end{pmatrix} = \begin{pmatrix}
\partial_{y_k} f_1(a, b) \\
\vdots \\
\partial_{y_k} f_i(a, b) \\
-\partial_{y_m} f_i(a, b)\partial_{y_k} h_i \\
\partial_{y_k} f_{i+1}(a, b) \\
\vdots \\
\partial_{y_k} f_m(a, b)
\end{pmatrix}
\tag{33}
\]

by the chain rule, \(\partial_{y_k} f_i^{[1]} = \partial_{y_k} f_i + \partial_{y_m} f_i \partial_{y_k} h_i\). Since \(f_i(x, y^{[m]}, h_i(x, y^{[m]})) = 0\) for \((x, y^{[m]}) \in R^1 \times \prod_{k=1}^{m-1} I_k^1\), we get

\[
0 = \partial_{y_k} f_i(x, y^{[m]}, h_i(x, y^{[m]})) \bigg|_{(x, y^{[m]}) = (a, b^{[m]})} = [\partial_{y_k} f_i^{[1]} - \partial_{y_m} f_i \partial_{y_k} h_i]_{(x, y) = (a, b)}.
\]

This identity and the change of variables of \(\partial_{y_k} f_i^{[1]} = \partial_{y_k} f_i + \partial_{y_m} f_i \partial_{y_k} h_i\) give

\[
-\partial_{y_m} f_i(a, b)\partial_{y_k} h_i(a, b^{[m]}) = \partial_{y_k} f_i(a, b) - \partial_{y_m} f_i(x, y^{[m]}, h_i(x, y^{[m]})) \bigg|_{(x, y^{[m]}) = (a, b^{[m]})} = \partial_{y_k} f_i(a, b).
\]

So, (33) equals

\[
\begin{pmatrix}
\partial_{y_k} f_1(a, b) \\
\vdots \\
\partial_{y_k} f_i(a, b) \\
\partial_{y_k} f_{i+1}(a, b) \\
\vdots \\
\partial_{y_k} f_m(a, b)
\end{pmatrix}
\]

for \(1 \leq k \leq m\). Hence,

\[
|J_{f^{[1]}, y^{[m]}}(a, b^{[m]})| = \frac{(-1)^{i+m}}{\partial_{y_m} f_1(a, b)} |J_{f, y}(a, b)| \neq 0.
\]

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Next, since \( f^{[1]} \) satisfies \(|J_{f^{[1]},y^{(m)}}(a, b^{(m)})| \neq 0\) with \( f^{[1]}(a, b^{(m)}) = 0\), we can take \( i' \) such that \( \partial_{y_{m-1}} f^{[1]}(a, b^{(m)}) \neq 0\). Here, \( i' \neq i \); therefore, put \( \phi(m - 1) = i' \). As before, by Corollary \[4.2.1\] there is a unique analytic function \( y_{m-1} = h_{i'}(x, y^{(m,m-1)}) : R^{(2)} \times \prod_{k=1}^{m-2} I^{(2)}_k \to I^{(2)}_{m-1} \) for some rectangle \( R^{(2)} \times \prod_{k=1}^{m-2} I^{(2)}_k \supseteq (a, b^{(m)}) \) in which \( f^{[1]}_i(x, y^{(m,m-1)}, h_{i'}(x, y^{(m,m-1)})) = 0 \).

Put \( f^{[2]}(x, y^{(m,m-1)}) = f^{[1]}[i'](x, y^{(m,m-1)}, h_{i'}(x, y^{(m,m-1)})) \), where \( f^{[1]}[i'] \) indicates the \( i' \)th component removal from \( f^{[1]} \). By the exact same method above,

\[ |J_{f^{[2]},y^{(m,m-1)}}(a, b^{(m,m-1)})| \neq 0. \]

In descending induction, \( \phi \) is well defined as a permutation on \( \{1, 2, \ldots, m\} \) and take

\[ R = \bigcap_{k=1}^{m} R^{(k)} \quad \text{and} \quad I = \bigcap_{k=1}^{m} I^{(k)}_1 \times \bigcap_{k=1}^{m-1} I^{(k)}_2 \times \cdots \times \bigcap_{k=1}^{2} I^{(k)}_{m-1} \times I^{(m)}_m \]

which contain \( a \) and \( b \), respectively. Finally, we will prove that \( f \) satisfies Assumption 2. Since \( y_i = h_{\phi(i)}(x, y^{(m,m-1,\ldots,i)}) \) is a unique analytic function such that

\[ f_{\phi(i)}(x, y^{(m,m-1,\ldots,i)}, h_{\phi(i)}(x, y^{(m,m-1,\ldots,i)}), \ldots, h_{\phi(m)}(x, y^{(m)}) = f_{\phi(i)}(x, y^{(m,m-1,\ldots,i)}, h_{\phi(i)}(x, y^{(m,m-1,\ldots,i)})) = 0 \]

in \( R \times I^{(m,m-1,\ldots,i)} \), for every \((x, y^{(m,m-1,\ldots,i)}) \in R \times I^{(m,m-1,\ldots,i)} \) there is

\[(y_{i+1}, \ldots, y_m) = (h_{\phi(i+1)}(x, y^{(m,m-1,\ldots,i+1)}), \ldots, h_{\phi(m)}(x, y^{(m)})) \in I^{(i+1,\ldots,i+1)} \]

so that

\[ \text{sgn}_{y_i} f_{\phi(i)}(x, y^{(m,m-1,\ldots,i)}, y_i, y_{i+1}, \ldots, y_m) \]

has only one jump discontinuity on \( I^{(m-i+1)}_i \) for \( i = m, m - 1, \ldots, 1 \). Therefore, the proof is complete. \( \square \)

The following two examples of implicit functions having three variables and two equations will serve to illustrate Theorem \[5.2.2\] and Corollary \[5.2.1\]. In the first example, an implicit function will be approximated when the Jacobian matrix is non-degenerate, while the second is an example which has a degenerate Jacobian matrix.

**Example 5.3.** Let \( f = (f_1, f_2) : \mathbb{R}^3 \to \mathbb{R}^2 \) be defined by \( f_1(x, y, z) = x + y^2 + z^3 - 6 \), \( f_2(x, y, z) = x^3y - z - 1 \) with \( f(1, 2, 1) = 0 \). We focus on calculating \( g(x) = (g_1(x), g_2(x)) \) with \( y = g_1(x) \) and \( z = g_2(x) \) such that \( f = 0 \) in an interval of \( x = 1 \). Since \( f_1 \) and \( f_2 \) are analytic near \((x, y) = (1, 2)\) and \(|J_{f_1(x,y)}(1, 2, 1)| = -7 \neq 0\), by Corollary \[5.2.1\] there is a rectangle, for example, \( R = [0.5, 1.5] \times [1.5, 2.5] \times [-2, 8] \) or \((y, z) = (2, 1)\) on which both \( \text{sgn}_x f_1(x, y, z) \) and \( \text{sgn}_z f_2(x, y, z) \) have only one jump discontinuity. (The surfaces \( f_1 = 0 \) and \( f_2 = 0 \) are illustrated in Figure \[7\] in which \( R \times I \) is also depicted.)

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First, by observation, \( n_z(f_2) = -1 \). For \( N = 4 \), \( z = h_2(x,y) = \sum_{0 \leq \alpha < 4} c_\alpha (x-1)^\alpha (y-2)^{\alpha_2} \) approximates \( z = z(x,y) \) such that \( f_2(x,y, z(x,y)) = 0 \), where \( c_\alpha \) is the \((\alpha + 1)\)th component of \( \text{sgn} \[ \begin{array}{c} 1 \\ y - 2 \\ (y - 2)^2 \\ (y - 2)^3 \\ \hline 0.99999999999997 \\ 6.00000000000016 \\ 6.00000000000000 \\ 1.99999999999848 \end{array} \] \) which is comparable to (34).

\( h(y) \) are the approximations of \( x \) in Example 5.4. Functions is approximated by multivariate polynomials. In that case, Theorem 5.2 still works. We present such an example to show that a system of implicit functions is approximated by multivariate polynomials.

\( (z = h_2(x,y) \) is drawn in Figure [3]. Furthermore, the coefficients of \( z = z(x,y) \) (which is a polynomial in two variables) are given by

\[
\begin{bmatrix}
1 & y - 2 \\
6 & 3 & 0 & 0 \\
2 & 1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
(y - 2)^2 \\
(y - 2)^3 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
(x - 1)^2 \\
(x - 1)^3 \\
\end{bmatrix}
\]

which is comparable to (34).

According to (30), put \( f_1^{[1]}(x,y) = f_1(x,y, h_2(x,y)) = 0 \) (as shown in Figure [9]). Since \( \text{sgn}_y f_1^{[1]}(x,y) \) has only one jump discontinuity and \( n_y(f_1^{[1]}) = 1 \), with \( N = 25 \) we gain \( y = h_1(x) = \sum_{k=0}^{24} c_k (x-1)^k \in [-0.5, 2] \times [0, 5] \), which is an approximation of \( y = y(x) \) such that \( f_1^{[1]}(x,y(x)) = 0 \), where

\[
h_1(x) = \begin{bmatrix}
20400458834 & -14405535077 \\
1987435356 & 25287414678 \\
-402546769168 & -11871539883 \\
-10871539883 & -41976539883 \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
1987435356 & 25287414678 \\
-402546769168 & -11871539883 \\
-10871539883 & -41976539883 \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
\cdots & \cdots \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
\cdots & \cdots \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
\cdots & \cdots \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
\cdots & \cdots \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
\cdots & \cdots \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
\cdots & \cdots \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
25287414678 & -10871539883 \\
-11871539883 & -41976539883 \\
\cdots & \cdots \\
\end{bmatrix}
\]

\( (x - 1)^{24} \)

(shown in Figure [10]). Finally,

\[
y = g_1(x) = h_1(x) \\
z = g_2(x) = h_2(x, h_1(x))
\]

are the approximations of \( y = y(x) \) and \( z = z(x) \) such that \( f(x, y(x), z(x)) = 0 \), respectively, in \( R \). In addition, the quantities \( f_1(x, g_1(x), g_2(x)) \) and \( f_2(x, g_1(x), g_2(x)) \) are illustrated in Figure [11] and the approximated curve \( x \mapsto (g_1(x), g_2(x)) \) for \( f = 0 \) is shown in Figure [12].

The rectangle \( R \) in Assumption 2 may contain a degenerate critical point. Regardless of the case, Theorem 5.2 still works. We present such an example to show that a system of implicit functions is approximated by multivariate polynomials.

Example 5.4. Consider the pair of equations

\[
\begin{align*}
f_1(x, y, z) &= x^2 + y^2 + (x-1)z^2 \\
f_2(x, y, z) &= x^2 + 2y^2 + (y-1)z^2
\end{align*}
\]
Figure 7: The surfaces of \( f_1(x, y, z) = x + y^2 + z^3 - 6 = 0 \) and \( f_2(x, y, z) = x^3y - z - 1 = 0 \) and \( R \times I \).

Figure 8: The surface \( z = h_2(x, y) = \sum_{0 \leq \alpha < 4} c_\alpha (x - 1)^\alpha (y - 2)^{\alpha_2} \).

with \((0, 0, 0)\) at which \( f_1 \) and \( f_2 \) vanish. Since there is no linear term in (35), at \((0, 0, 0)\) the rank of the Jacobian matrix is zero (as depicted in Figure 13). However, confining a range, for example, to \(0 \leq x \leq 1\), \(0 \leq y \leq 1\), and \(0 \leq z < \infty\) and taking a rectangle \([0, 0.5] \times [0, 0.5] \times [0, 3]\) which has a corner at \((0, 0, 0)\). On the rectangle, by observation, \( \text{sgn}_z f_k(x, y, z) \) has only one jump discontinuity and \( n_z(f_k) = -1, n_x(f_k) = n_y(f_k) = 1 \) \((k = 1, 2)\).

By Theorem 5.2 with \( N = 8 \), \( z = h_1(x, y) \) and \( z = h_2(x, y) \) approximate \( z = z_1(x, y) \) and
Figure 9: A graphical plot of the implicit function of $f_1(x, y, h_2(x, y)) = 0$ in $[-0.5, 2] \times [0, 5]$.

Figure 10: The function $y = h_1(x)$ is denoted in blue in $R \times [1.5, 2.5]$ containing $(1, 2)$, which comes from $f_1(x, y, h_2(x, y)) = 0$. The red indicates a graphical plot of $f_1(x, y, h_2(x, y)) = 0$.

$z = z_2(x, y)$ such that $f_1(x, y, z_1) = 0$ and $f_2(x, y, z_2) = 0$ whose coefficient matrices are

$$
\begin{bmatrix}
1 & y & y^2 & y^3 & y^4 & y^5 & y^6 & y^7 \\
-0.82314068567676 & -0.86313509756101 & -0.86317658279555 & -0.82314068567676 & 2.56792587695851 & -1.48880544364808 & -0.01452447451644 & 21.20846598293292 \\
1.98866944566776 & -1.16909326608108 & 2.19322117467802 & 2.15721284320975 & -3.71952347276957 & 57.48995960279866 & 22.80727484760829 & -292.81766347257 \\
1.63286148938642 & 1.10164116609667 & -1.02243332750225 & 25.84217266471715 & 55.61690089792683 & -52.15355357137594 & 345.242857367544 & 273.680299390402 \\
-0.24424785794297 & 3.86133613090940 & -1.64343390944991 & -161.13774736732696 & 31.81328846992857 & 1290.593805600849 & -5803.0305776689 & 4194.3716512031 \\
2.88626530137803 & -0.2239288374331 & -18.895090946494 & 185.81498673433275 & -565.6289432327 & 21957.4894127041 & 618418.65726907 & -321312.02583376 \\
0.81029650001308 & 4.37919213864819 & -0.15151533940869 & 170.18785658778 & 18624.6873817555 & -12741.92659665 & -153229.07307279 & 1382861.71554844 \\
-3.4326712455283 & 38.643268335212 & 134.544067122128 & -977.910676264299 & 65265.6651965214 & -186091.36165850 & -1100349.02437471 & 445427.49831176 \\
18.767638752790 & -276.72744898198 & 2417.2871285553 & -246.906670877388 & -314854.84257636 & 1459642.19882775 & 2489677.03764666 & -2290429.34325589
\end{bmatrix}
$$
Figure 11: The blue and red curves are drawn by $f_1(x, g_1(x), g_2(x))$ and $f_2(x, g_1(x), g_2(x))$, respectively.

Figure 12: The black curve denotes $x \mapsto (y, z)$, where $y = g_1(x)$ and $z = g_2(x)$, which approximate $y = y(x)$ and $z = z(x)$ such that $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$ in $R \times I$, respectively, and

\[
\begin{array}{cccccccccc}
1 & y & y^2 & y^3 & y^4 & y^5 & y^6 & y^7 \\
1.1562886538154 & 7.3154374102292 & -25.78650985844 & 227.890744545 & -5699.5357921255 & -37753.238770695 & 19891.7756302906 & -229837.32557614 & -3.4229356715044 & 45.9498241577876 & -642.7466258181014 & 1883.35641607184 & 64098.0674666749 & -360910.1404011709 & -774807.9654480028 & 504029.525573144 & 2.84331521739714 & -95.5669077441473 & 2661.2271348121 & -17979.3545605222 & -131490.726839233 & 116633.18379635 & 78264.42794863 & -1177929.46108644 & 1 \n\end{array}
\]
respectively. Put \( h = h_1 - h_2 \) and then \( n_y(h) = -1 \) in \([0, 0.5] \times [0, 0.5] \). With \( N = 8 \), the coefficient matrix of \( y = h(x) \) is calculated as

\[
\begin{pmatrix}
1 & x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 \\
0.116319702320269 & 0.713165167944210 & 0.843601400792723 & -0.552538829658902 & -3.00696939156087 & 19.9372907280740 & 70.0568910222166 & -382.742430825347
\end{pmatrix}.
\]

In Figure 14, the values of \( f_1(x, h(x), h_2(x, h(x))) \) and \( f_2(x, h(x), h_2(x, h(x))) \) are illustrated, where \( y = h(x) \) and \( z = h_2(x, h(x)) \) approximate \( y = y(x) \) and \( z = z(x) \) such that \( f_1(x, y, z) = 0 \) and \( f_2(x, y, z) = 0 \), respectively. Furthermore, the approximated curve \( x \mapsto (h(x), h_2(x, h(x))) \) is shown in Figure 15.

![Figure 13](image)

**Figure 13:** Surfaces of \( f_1(x, y, z) = x^2 + y^2 + (x - 1)z^2 = 0 \), \( f_2(x, y, z) = x^2 + 2y^2 + (y - 1)z^2 = 0 \) with \( f(0,0,0) = (0,0) \) and a rectangle \([0,0.5] \times [0,0.5] \times [0,3] \) which contains a corner \((0,0,0)\). The adjacent netted surfaces illustrate the implicit function \( f_2(x, y, z) = 0 \).

**Remark 5.1.**

(i) Throughout this article the partitions can be chosen differently on each axis.

(ii) We close this article with the application of our formulation to a continuous implicit function. For a continuous function \( f(x, y) : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \), suppose that there is an \( R \times I \) and a unique continuous function \( y = g(x) : R \rightarrow I \) such that \( f = 0 \). We do not know whether the sequence of multivariate polynomials which are calculated in Section 4, converges to the continuous implicit function. Their coefficient matrices depend on the choice of partitions of \( R \). Because of this, by choosing the partitions suitably, the desired convergence can be expected. More precisely, by using dyadic decomposition of partitions, the derived multivariate polynomials can converge to the continuous implicit function in weak-star topology. Indeed, put \( \Gamma_1 = \{R\} \) and let \( \Gamma_2 \) be a collection of grid blocks \( R_{2,\alpha} \) of \( R \), which are caused by dyadic decomposition on \( R \). For a positive integer \( N \), inductively, let \( \Gamma_{N+1} \) consist of all grid blocks \( R_{2^{N+1},\alpha} \) of every element of \( \Gamma_N \) by dyadic decomposition. Now, construct a multivariate polynomial \( g_N \) whose coefficients
Figure 14: For $y = h(x)$ and $z = h_2(x, h(x))$, the values of $f_1(x, h(x), h_2(x, h(x)))$ and $f_2(x, h(x), h_2(x, h(x)))$ are illustrated.

Figure 15: In $(0, 0, 0) \in [0, 0.5] \times [0, 0.5] \times [0, 3]$, two surfaces of $f_1(x, y, z) = x^2 + y^2 + (x-1)z^2 = 0$ and $f_1(x, y, z) = x^2 + 2y^2 + (y-1)z^2 = 0$, and the curve of $(h(x), h_2(x, h(x)))$ are depicted.

are calculate by solving (25) over $\Gamma_N$ according to (15). Let $R_{2N', \alpha} \in \bigcup_N \Gamma_N$. For any $N \geq N'$, there are non-overlapping grid blocks in $\Gamma_N$ whose union equals $R_{2N', \alpha}$, i.e., $R_{2N', \alpha} = \bigcup_{R_{2N_{1}, \beta}} \subseteq R_{2N', \alpha}$. Furthermore, the grid points of $\Gamma_{N'}$ are contained in those
of $\Gamma_N$. By (25) and (14), we have

$$\int_{R_{2N'}^{\alpha}} \tilde{g}_N - g \, dx = \sum_{R_{2N,'}^{\beta} \subset R_{2N'}^{\alpha}} \int_{R_{2N,'}^{\beta}} \tilde{g}_N - g \, dx = 0 \quad (36)$$

for every $N \geq N'$. On the other hand, since the collection of all finite linear combinations of characteristic functions which are supported in dyadic grid blocks in $\bigcup N \Gamma N$ is dense in $L^1(R)$, the multivariate polynomial $\tilde{g}_N$ converges to $g$ weak-star in $L^\infty(R)$ as $N \to \infty$ by the Riesz representation theorem for the Lebesgue spaces.

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