Multisoliton solution of 3-waves problem

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Abstract

Multi-soliton solution of the 3-waves problem is represented in explicit determinative form.
1 Introduction

The problem of three waves in two dimensions arises in different forms in many branches of the mathematical physics. For example, this occurs in radio physics and nonlinear optics applications, which can be found in [1].

Three-wave interaction in plasma, stability criteria, and asymptotic behavior for a general system of three interacting waves, the influence of mutually different linear damping coefficients on a system of three interacting waves of equal signs of energy are described in [2], where an accurate studying waves in plasma were performed.

Nonlinear Water Wave Interaction and hydrodynamic turbulence including tree wave problem were investigated in [3, 4, 5].

Furthermore, the three wave problem is also discussed during studying nonlinear lattices [7] and photon crystals [8].

The present paper must be considered as a direct continuation of the previous one [9]. For the convenience of the reader we at first represent necessary results from [9] marking them into the form of an Appendix, putting this on the unequal first place.

2 Appendix

2.1 3-wave problem in dimensionless form and its discrete transformations.

First off, let us consider the system of equations for 6 unknown functions $Q, P, A, D, B, E$

$$
P_1 = -QE, \quad A_2 = -BQ, \quad -(Q_1 + Q_2) \equiv Q_3 = -PA
$$

$$
B_1 = -AD, \quad E_2 = -DP, \quad D_3 = -EB
$$

(1)

Using direct calculation it is not difficult to certain that system (1 is invariant with respect to the following substitution: $T_3$)

$$
\bar{Q} = \frac{1}{D}, \quad \bar{A} = \frac{-B}{D}, \quad \bar{P} = \frac{E}{D},
$$

$$
\bar{B} = D(\frac{B}{D})_2, \quad \bar{E} = -D(\frac{E}{D})_1, \quad \frac{\bar{D}}{D} = DQ - (\ln D)_{1,2}
$$
The transformation is denoted as $T_3$, which is a discrete transformation.

Permutating indexes (1, 3) (altogether with corresponding exchanging unknown functions) make it possible to obtain $T_1$ as a discrete transformation when the system (1) is invariant.

$$\bar{P} = \frac{1}{B}, \quad \bar{Q} = \frac{A}{B}, \quad \bar{E} = -\frac{D}{B},$$

$$\bar{D} = B(D_B^2), \quad \bar{A} = -B(A_B^3), \quad \frac{\bar{B}}{B} = BP - (\ln B)_{2,3}$$

And finally the discrete transformation $T_2$ has the form of:

$$\bar{A} = \frac{1}{E}, \quad \bar{B} = \frac{D}{E}, \quad \bar{Q} = -\frac{P}{E},$$

$$\bar{D} = -E(D_E^1), \quad \bar{P} = E(P_E^3), \quad \bar{E} = EA - (\ln E)_{1,3}$$

In the form presented above, substitutions $T_i$ may be considered as a mapping connected to six initial (unbar) functions with six final (bar) ones. On the other hand, each substitution should be considered as the infinite dimensional chain of equations. For instance, the corresponding chain of equations for the case of $T_1$ substitution has the form of:

$$\frac{B_{n+1}}{B_n} - \frac{B^n}{B^n_{n-1}} = -(\ln B^n)_{2,3}, \quad D^{n+1} = B^n(D^n_B)_{2}, \quad A^{n+1} = -B^n(A^n_B)_{3} \; (2)$$

$$E^{n+1} = -\frac{D^n}{B^n}, \quad Q^{n+1} = \frac{A^n}{B^n}$$

In the first row, we have the lattice as if it were system connected to 3 unknown functions ($B, D, A$) in each point of the lattice. The first chain for $B$ functions is definitely well known as a two dimensional Toda lattice.

### 2.2 Some properties of the discrete transformations.

All of the discrete transformations constructed above are invertable. This means that an unbar unknown function may be presented in terms of the bar ones. For instance, $T_3^{-1}$ looks as:

$$D = \frac{1}{Q}, \quad B = -\frac{A}{Q}, \quad E = \frac{P}{Q},$$
\[ P = \bar{Q}_1 (\bar{Q}_2), \quad A = \bar{Q}_1 (\bar{Q}_2), \quad \bar{Q} = \bar{P} - (\ln \bar{Q})_{1,2} \]

Consequently, it is not difficult to certain by direct computation that discrete transformations \( T_i \) are mutual commutative \( (T_i T_j = T_j T_i) \) in the solutions of the system (1). Moreover, others, such as \( T_1 T_2 = T_2 T_1 = t_3 \), can be also handled in the same way.

Thus, from each given initial solution \( W_0 \equiv (A, P, Q, E, B, D) \) of the system (1), it is possible to obtain the chain of solutions labeled by two natural numbers \( (l_1, l_2) \) or \( (l_3) \) and the number of applications of the discrete transformations \( (T_1, T_2, T_3) \) associated to it (as it was noticed above, \( T_1 T_2 = T_2 T_1 = T_3 \)).

Arising chain of equations with respect to \( (D, B, E) \) functions are definitely two-dimensional Toda lattices. Their general solutions in the case of two fixed ends are well-known [13]. As the reader will see, soon this fact allows constructing the many soliton solutions of the 3-wave problem into the most straightforward ways.

### 2.3 Resolving of discrete transformation chains.

#### 2.3.1 Two identities of Yacobi

The first Jacobi identity:

\[
D_n \left( \begin{array}{l}
T_n a^1 \\
b^1 \tau_{11}
\end{array} \right) D_n \left( \begin{array}{l}
T_n a^2 \\
b^2 \tau_{22}
\end{array} \right) - D_n \left( \begin{array}{l}
T_n a^2 \\
b^1 \tau_{12}
\end{array} \right) D_n \left( \begin{array}{l}
T_n a^1 \\
b^2 \tau_{21}
\end{array} \right) =
\]

\[
D_{n-1}(T_{n-1})D_{n+1} \left( \begin{array}{l}
T_n a^1 a^2 \\
b^1 \tau_{11} \tau_{12}
\end{array} \right)
\]

where \( a^i, b^i \) are \((n - 2)\) dimensional columns (rows) vectors, \( \tau_{ij} \) components of 2-th dimensional matrix.

The second Jacobi identity:

\[
D_n \left( \begin{array}{l}
T_n a^1 \\
b^1 \tau
\end{array} \right) D_{n+1} \left( \begin{array}{l}
T_n a^1 a^2 d^1 \nu \mu \\
b^2 \rho \tau
\end{array} \right) - D_n \left( \begin{array}{l}
T_n a^1 \\
b^2 \rho
\end{array} \right) D_{n+1} \left( \begin{array}{l}
T_n a^1 a^2 d^1 \nu \mu \\
b^1 \tau \sigma
\end{array} \right) =
\]

3
$$D_n \left( \frac{T_{n-1}a^1}{d^1 \nu} \right) D_{n+1} \left( \frac{T_{n-1}a^1a^2}{b^2 \rho \tau} \right)$$

These identities can be generalized in the case of arbitrary semi-simple group. The reader can find these results in [11].

2.3.2 Resolving of the discrete lattices

Let us take an initial solution in the form of:

$$Q = A = P = 0, \quad B \equiv B(2), \quad E \equiv E(1), \quad D_3 = -BE$$

Application to this solution to each of the inverse transformations $T_i^{-1}$ is meaningless because of arising zeroes in denominators. The chain of equations under such boundary conditions is what we call the chain with the fixed end from the left (from one side).

The result of applications to such initial solution by $l_3$ times $T_3$ transformation looks as (in order for checking this fact only two Jacobi identities of the previous subsection are necessary).

$$Q^{(l_3)} = (-1)^{l_3-1} \frac{\Delta_{l_3-1}}{\Delta_{l_3}}, \quad D^{(l_3)} = (-1)^{l_3} \frac{\Delta_{l_3+1}}{\Delta_{l_3}}, \quad \Delta_0 = 1$$

$$A^{(l_3)} = (-1)^{l_3} \frac{\Delta_B}{\Delta_{l_3}}, \quad P^{(l_3)} = \frac{\Delta_E}{\Delta_{l_3}}, \quad \Delta_0^B = \Delta_0^E = 0$$

$$B^{(l_3)} = \frac{\Delta_B^{l_3+1}}{\Delta_{l_3}}, \quad E^{(l_3)} = (-1)^{l_3} \frac{\Delta_E^{l_3+1}}{\Delta_{l_3}}, \quad \Delta_{-1} = 0.$$
In the following next notations will be used. $W^{l_3,l_1}$, $(W^{l_3,l_2})$ - this is the result of application of discrete transformation $T^{l_3} T^{l_1} (T^{l_3} T^{l_2})$ to the corresponding component of the 3-wave field. $\Delta_{l_3,l_1}$ $(\Delta_{l_3,l_2})$ - determinant of $l_3 + l_1$ $(l_3 + l_2)$ orders, with the following structure of its determinant matrix. The first $l_3$ rows (columns) of it coinside with matrix of (5) and last $l_1$, $(l_2)$ rows (columns) are constructed from the derivatives of $B$, $(E)$ functions with respect to arguments 2, (1).

The result of additional application to $l_1$ times $T_1$ transformation to the solution (4) looks as:

\begin{align*}
P^{l_3,l_1} &= \frac{\Delta_{l_3,l_1-1}}{\Delta_{l_3,l_1}}, \quad B^{l_3,l_1} = \frac{\Delta_{l_3,l_1+1}}{\Delta_{l_3,l_1}}, \quad \Delta_0 = 1, \quad \Delta_{l_3,-1} = \Delta_{l_1}^E \\
Q^{l_3,l_1} &= (-1)^{l_3+l_1-1} \frac{\Delta_{l_3-1,l_1}}{\Delta_{l_3,l_1}}, \quad D^{l_3,l_1} = (-1)^{l_3+l_1} \frac{\Delta_{l_3+1,l_1}}{\Delta_{l_3,l_1}}, \quad (6) \\
E^{l_3,l_1} &= (-1)^{l_3+l_1} \frac{\Delta_{l_3+1,l_1-1}}{\Delta_{l_3,l_1}}, \quad A^{l_3,l_1} = (-1)^{l_3+l_1+1} \frac{\Delta_{l_3-1,l_1+1}}{\Delta_{l_3,l_1}}.
\end{align*}

We do not present the explicit form for components $W^{l_3,l_2}$, which can be obtained without any difficulties from (6) by corresponding exchanging of the arguments and unknown functions.

3 Multi-soliton solution of the 3-waves problem

3.1 General consideration

The system (1) allows reducing (under additional assumption that all operators of differentiation are the real ones $\partial_\alpha = \partial_\alpha^*$).

\begin{equation*}
P = B^*, \quad A = E^*, \quad Q = D^*
\end{equation*}

In this case the system (1) is reduced to three equations.

\begin{equation*}
B_1 = -DE^*, \quad E_2 = -DB^*, \quad D_3 = -BE
\end{equation*}

for three complex unknown functions $(E, B, D)$. This system of equations is definitely 3-wave problem.
In [9] it was shown that beginning with an initial solution (3) after the $(l_3, l_1)$ steps of discrete transformation leads to a solution of the reduced system (8) if the following conditions are satisfied:

\[
\begin{align*}
\Delta_{2l_3+1,2l_1} &= \Delta_{2l_3+1,2l_1-1} = \Delta_{2l_3,2l_1+1} = 0 \\
\frac{\Delta_{2l_3-1,2l_1+1}}{\Delta_{2l_3,2l_1}} &= (-1)^{l_3+l_1} E^* \\
\frac{\Delta_{2l_3-1,2l_1}}{\Delta_{2l_3,2l_1}} &= (-1)^{l_3-l_1-1} D^* \\
\frac{\Delta_{2l_3,2l_1-1}}{\Delta_{2l_3,2l_1}} &= (-1)^{l_3} B^*
\end{align*}
\]

(9)

Resolving the first line of above equations is as follows:

\[D = \sum_{k=1}^{2l_1} f^k(1)\phi^k(2),\]

To resolve the second line ones the following parametrization is sufficient:

\[
\begin{align*}
f^k &= \sum_{s=1}^{2l_3} c^k_s e^{\lambda^s_1} \\
\phi^k &= \sum_{S=1}^{2l_3+2l_1} d^k_S e^{\mu_S 2} \\
E &= \sum_{s=1}^{2l_3} e_s e^{\lambda^s_1} \\
B &= \sum_{S=1}^{2l_3+2l_1} b_S e^{\mu_S 2}
\end{align*}
\]

All numerical parameters $c^k_s, d^k_S, e_s, b_S$ are connected by equations (9) and one additional equation connects with initial conditions (3).

The initial condition leads to:

\[
(\lambda_s + \mu_S) \sum_{k=1}^{2l_3} c^k_s d^k_S = e_s b_S \\
d^k_S = b_S \sum_{s=1}^{2l_3} e_s (c^{-1})^k_s \lambda^s_s + \mu_S
\]

and finally we obtain for $D$:

\[D = \sum_{s=1, S=1}^{2l_3, 2l_3+1} \frac{e^s e^{\lambda^s_1} b_S e^{\mu_S 2}}{\lambda^s_s + \mu_S}
\]

(10)

Thus, solution to be determined are defined by the pair of $(2l_3)$ parameters $e_s, \lambda_s$ and $2(l_3 + l_1)$ pairs parameters $b_S, \mu_S$. Equations (9) give some additional limitations (of reality) for these parameters. Below, we find these limitations and then $l_3, l_1$ is given by this explicit formula (6), in which conditions of reducing are satisfied.
3.2 Determinant computation. The case of denominator.

At first, let us calculate the common determinant (see (9)) $-\Delta_{2l_3,2l_1}$ using explicit expressions for initial functions $D$, $E$, $B$. The determinant matrix of dimension $2(l_3 + l_1) \times 2(l_3 + l_1)$ may be presented as product of two matrices. The first one has the block form $2l_3 \times 2l_3$ matrix $L(e, \lambda; 1)$ in upper left angle and zeroes on other places except of the unity on main diagonal. The elements of the matrix $L$ are the following $L_{ik} = \lambda_i^{-1} e_k e^{\lambda_k}$. The result is obvious:

$$\Delta^{(1)}_{2l_3,2l_1} = Det(L) = \prod_{s=1}^{2l_3} e_s W_{2l_3}(\lambda_1, ... \lambda_{2l_3}) e^{(\sum_{k=1}^{2l_3} \lambda_k)}$$  \hspace{1cm} (11)

where $W$ is a Vandermond determinant. The second matrix in its turn can be represented as product of another two ones. The first $2l_3$ lines of the first matrix have the matrix elements $\frac{1}{\lambda_s + \mu_S}$ (s number of line, $S$ number of column). The remaining $2l_1$ lines of this matrix are the usual Vandermond matrices $W_{i,S} = \mu_i^{S-1}$. Computation of this determinant is not complicate problem with the following result:

$$\Delta^{(2)}_{2l_3,2l_1} = W_{2l_3}(\lambda_1, ... \lambda_{2l_3}) W_{2(l_3+l_1)}(\mu_1, ... \mu_{2(l_3+l_1)}) \prod_{s=1,S=1}^{s=2l_3,S=2(l_3+l_1)} \frac{1}{\lambda_s + \mu_S}$$

Finally, the matrix elements of last $2(l_3 + l_1) \times 2(l_3 + l_1)$ matrix have the following analitical structure $b_l e^{\mu_S} \mu_i^{-1}$. And in consequence:

$$\Delta^{(3)}_{2l_3,2l_1} = \prod_{S=1}^{2(l_3+l_1)} b_s W_{2l_3}(\mu_1, ... \mu_{2l_3+l_1}) e^{(\sum_{s=1}^{2l_3+l_1} \mu_S)}$$

Thus, for $\Delta_{2l_3,2l_1}$ summatimg all results above we obtain:

$$W_{2l_3}^{2}(\lambda_1, ... \lambda_{2l_3}) W_{2(l_3+l_1)}^{2}(\mu_1, ... \mu_{2l_3+l_1}) \prod_{s=1,S=1}^{s=2l_3,S=2l_3+l_1} \frac{1}{\lambda_s + \mu_S} \prod_{s=1}^{2l_3} e_s e^{\lambda_1} \prod_{S=1}^{2l_3} b_S e^{\mu_S^2}$$ \hspace{1cm} (12)

3.3 Determinant computation. The case of $B$ function

In the process of computation of $\Delta_{2l_3,2l_1}-1$, the determinant $(2(l_3 + l_1) - 1) \times (2(l_3 + l_1) - 1)$ matrix may be represented as product of two matrices. The
The first one coincides with matrix $L$ of the previous subsection. Determinant of this matrix has been already calculated in (11). Second matrix in its turn may be represented in the form of the product of two rectangular matrices of dimensions $(2(l_3 + l_1) - 1) \times 2(l_3 + l_1)$ and $2(l_3 + l_1) \times (2(l_3 + l_1) - 1)$, correspondingly. The structure of these matrices are described in the previous subsection. The determinant of the degree $2(l_3 + l_1) - 1$ is equal to sum of products of all minors of $2(l_3 + l_1)$ orders of both matrices. These determinants were computed also in the previous subsection. The finally result is as follows:

$$
\sum_{i=1}^{2(l_3 + l_1)} W_{2l_3}^{2} (\lambda_1, \ldots, \lambda_{2l_3}) W_{2(l_3 + l_1) - 1}^{2} \prod_{S=1, S \neq i}^{2(l_3 + l_1)} \frac{1}{\lambda_s + \mu_s} \prod_{s=1, S=1, S \neq i}^{2(l_3 + l_1)} b_S e^{\mu_S} 2^{2(l_3 + l_1)} \prod_{s=1}^{2l_3} \prod_{s=1}^{2(l_3 + l_1)} e^{-\mu_S^2} \frac{1}{b_S}
$$

Taking into account the last equality from (9) and substituting in it all the results for determinant calculations above, we obtain:

$$
\sum_{S=1}^{2(l_3 + l_1)} \prod_{s=1}^{2l_3} (\lambda_s + \mu_S) \prod_{K=1, K \neq S}^{2(l_3 + l_1)} (\mu_K - \mu_S)^{-2} e^{-\mu_S^2} \frac{1}{b_S} = (-1)^{l_3} \sum_{S=1}^{2(l_3 + l_1)} b_S^* e^{\mu_S^2}
$$

The last equality can be satisfied only under condition:

$$
\mu_S^* = -\mu_{PS}, \quad S \rightarrow (PS)
$$

where $P$ operator of permutation of $2(l_3 + l_1)$ numbers with obvious property $P^2 = 1$. Comparing terms under the same exponents, we obtain:

$$
(-1)^{l_3} b_{PS}^* = \prod_{s=1}^{2l_3} (\lambda_s + \mu_S) \prod_{K=1, K \neq S}^{2(l_3 + l_1)} (\mu_K - \mu_S)^{-2} e^{-\mu_S^2} \frac{1}{b_S}
$$

This is typical for soliton theory connection between the "energy" and amplitudes.

### 3.4 Determinant computation. The case of $E$ function.

Numerical parameters $c_s$ are connected to the first equation (9). To find these relations it is necessary to calculate $\Delta_{2l_3 - 1, 2l_1 + 1}$. The technique of
these calculations are the same as in previous subsection and finally equality (9) for $E$ functions looks as:

$$\sum_{s=1}^{2l_3} e_s^* e^{\lambda_s^*} = \sum_{s=1}^{2l_3} e_s e^{-\lambda_s^*} \prod_{S=1}^{2(l_3+1)} (\lambda_s + \mu_S) \prod_{k=1, k \neq s}^{2l_3} (\lambda_k - \lambda_s)^{-2}$$

As in previous subsection, the equality above restricted values of parameters $\lambda$ as:

$$\lambda_s^* = -\lambda_{P_s}$$

where $P$ is some permutation of the group of $2l_3$ numbers satisfying the condition $P^2 = 1$. Comparison the coefficients at the same exponents in both sides of the last equality leads to the following conditions on parameters $e_s$:

$$e_s^* = \prod_{S=1}^{2(l_3+1)} (\lambda_{P_s} + \mu_S) \prod_{k=1, k \neq s}^{2l_3} (\lambda_k - \lambda_s)^{-2} \frac{1}{c(P_s)}$$

### 3.5 Determinant computation. The case of $D$ function.

Calculation of the determinant defining $D$ function and corresponding condition of reality (9) does not add new limitations to the numerical parameters obtained above in the last two subsections.

### 4 Outlook

No one of the authors is a specialist in the field of applications of 3-wave problem to certain physical phenomenon. Due to this reason, we can not discuss the usefulness of such applications.

From a computational point of view, it seems that here we have the first paper having a systematical investigation of a multicomponent integrable system. It is interesting to compare the results in the case of multicomponent (the simplest one) integrable system connected with the algebra $A_2$ and numerous integrable systems connected with $A_1$ algebra. The strategy of computations in both cases is basically the same. Beginning from some simple solution connected with the upper triangular nilpotent subalgebra ($P = A = Q = 0$) in the case of the present paper after corresponding $2n$ steps of discrete transformation, we come to solution connected with the
lower triangular nilpotent subalgebra. Connection with hermitianity of these solutions leads to limitation of (arbitrary up to now) numerical parameters of the problem. In the case of two component integrable system, for instance in nonlinear Scrodinger equation, this dependence looks as \[ ? \].

\[
c^*_s = \prod_{k=1,k\neq s}^{2l_3} (\lambda_k - \lambda_s)^{-2} \frac{1}{c(P_s)}, \quad \lambda^*_s = -\lambda_P
\]

In the case under consideration, this dependence modified by the formulas of subsections B, E. This comparison offers possibility assuming that each simple root of semisimple algebra borns its own systems of pairs of parameters amplitude-phase and numbers of these pairs are arbitrary and only by them multi-soliton solutions are defined in the case of multicomponent integrable systems.

In our consideration, it was unable to connect constructed solutions with L-A pair formalism and authors have no idea how this can be done (if possible).

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