SOME REMARKS ON SYMMETRIC PERIODIC ORBITS IN THE
RESTRICTED THREE-BODY PROBLEM

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Abstract. The planar circular restricted three body problem (PCRTBP) is symmetric with
respect to the line of masses and there is a corresponding anti-symplectic involution on the
cotangent bundle of the 2-sphere in the regularized PCRTBP. Recently it was shown that
each bounded component of an energy hypersurface with low energy for the regularized
PCRTBP is fiberwise starshaped. This enable us to define a Lagrangian Rabinowitz Floer
homology which is related to periodic orbits symmetric for the anti-symplectic involution
in the regularized PCRTBP and hence to symmetric periodic orbits in the unregularized
problem. In this paper we compute of this homology and discuss about symmetric periodic
orbits.

1. Introduction

The project to apply holomorphic curve techniques to the restricted three body problem just
began, see [AFvKP12, AFFHvK11, AFFvK12, CFvK11]. In particular, Albers-Frauenfelder-
van Koert-Paternain [AFvKP12] proved that each bounded component of the regularized
energy hypersurface is a fiberwise starshaped hypersurface in $T^*S^2$ for energy less than the
first critical value. As they mentioned this opens up the possibility of applying holomorphic
curve techniques. In this paper we compute a related Lagrangian Rabinowitz Floer
homology and using this computation discuss about symmetric periodic orbits which we will introduce
below.

We refer to two massive primaries as the earth and the moon and to the other body with
negligible mass as the satellite. The configuration space is $\mathbb{R}^2 \setminus \{q^E, q^M\}$ and the phase space
is given by $T^*(\mathbb{R}^2 \setminus \{q^E, q^M\}) = (\mathbb{R}^2 \setminus \{q^E, q^M\}) \times \mathbb{R}^2$. Here $q^E$ and $q^M$ lying on the real
line are the positions of the earth and the moon respectively. The Hamiltonian for the planar
circular restricted three body problem (PCRTBP) is given by

$$H(q, p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q - q^M|} - \frac{1 - \mu}{|q - q^E|} + q_2p_1 - q_1p_2$$  (1.1)

where $\mu \in (0, 1)$ is the normalized mass of the moon. The energy hypersurface $H^{-1}(c)$ with
energy $c \in \mathbb{R}$ below the first critical value $H(L_1)$ is composed of three connected components.
Following [AFvKP12], we denote by $\Sigma^E_c$ resp. $\Sigma^M_c$ the bounded component close to the earth
resp. to the moon. Since these components are noncompact due to collisions, we compactify
each of them into $\overline{\Sigma}^E_c$ and $\overline{\Sigma}^M_c$ by Moser regularization. Since the discussions in this paper
go through for both $\overline{\Sigma}^E_c$ and $\overline{\Sigma}^M_c$ with $c < H(L_1)$, we call them $\Sigma$ for convenience. The
regularized phase space is the cotangent bundle of $S^2$ and $\Sigma$ is diffeomorphic to the unit
cotangent bundle of $S^2$. We denote the Hamiltonian function corresponding to $H$ via Moser
regularization by $Q \in C^\infty(T^*S^2)$. More details can be found in Section 2.
An interesting feature of the PCRTBP is that there is an involution such that the problem is symmetric with respect to this involution. More precisely, there exists an anti-symplectic involution $\mathcal{R}$ on $T^*\mathbb{R}^2$ given by

$$\mathcal{R}(q_1, q_2, p_1, p_2) = (q_1, -q_2, -p_1, p_2)$$

such that $H \circ \mathcal{R} = H$. Thus the Hamiltonian vector field of $H$ is invariant under $\mathcal{R}$. Through Moser regularization, $\mathcal{R}$ induces the anti-symplectic involution

$$\mathcal{R} = I \circ T^*\rho : T^*S^2 \to T^*S^2$$

where $I : T^*S^2 \to T^*S^2$ given by $I(\xi, \eta) = (\xi, -\eta)$ and $\rho$ is the reflection on $S^2$ about a great circle. The fixed locus of $\mathcal{R}$ is the conormal bundle of the great circle. In the present paper we are concerned with a periodic orbit of prescribed energy which is carried into itself by $\mathcal{R}$, i.e. $(x, 2T)$ satisfying

$$x : \mathbb{R}/2T\mathbb{Z} \to \Sigma, \quad \dot{x} = X_Q(x), \quad x(T + t) = \mathcal{R}x(T - t) \tag{1.2}$$

which we refer to a symmetric periodic orbit. Here $X_Q$ is the Hamiltonian vector field of $Q$. As mentioned above, $\Sigma$ is shown to be a fiberwise starshaped hypersurface (tight $\mathbb{R}P^3$) in $T^*S^2$ and thus $\text{Fix} \mathcal{R} \cap \Sigma$ is diffeomorphic to the disjoint union of two circles $L_+$ and $L_-$. We note that every symmetric periodic orbit $(x, 2T)$ intersects with $L_+ \cup L_-$ exactly twice at time 0 and $T$ (after an appropriate time shift). It is an interesting question whether a symmetric periodic orbit intersects with both circles or only one of them.

**Definition 1.1.** A symmetric periodic orbit on $\Sigma$ is called of type $I$ if it intersects with both $L_+$ and $L_-$. Otherwise, we call it of type $II$.

For explicit computations, let us consider the case $\Sigma = \Sigma_c^M$ with $c < H(L_1)$. We embed the cotangent bundle of $S^2$ in $\mathbb{R}^6$ as below.

$$T^*S^2 = \{(\xi, \eta) \in \mathbb{R}^6 \mid \xi \in S^2, \xi \cdot \eta = 0\}.$$ 

Then the inverse process of Moser regularization gives a correspondence

$$\mathcal{M} : T^*S^2 \to T^*\mathbb{R}^2,$$

$$(\xi, \eta) \mapsto \left(\eta_1(1 - \xi_0) + \xi_1\eta_0 + q^M_1, \eta_2(1 - \xi_0) + \xi_2\eta_0, \frac{-\xi_1}{1 - \xi_0}, \frac{-\xi_2}{1 - \xi_0}\right)$$

where the moon is located at $q^M = (q^M_1, q^M_2) = (-(-1 - \mu), 0)$ for $\mu \in (0, 1)$. Via this map, the anti-symplectic involution $\mathcal{R}$ on $T^*\mathbb{R}^2$ corresponds to the anti-symplectic involution $\mathcal{R}$ on $T^*S^2$ defined by

$$\mathcal{R}(\xi_0, \xi_1, \xi_2, \eta_0, \eta_1, \eta_2) = (\xi_0, -\xi_1, -\xi_2, -\eta_0, -\eta_1, -\eta_2), \quad (\xi, \eta) \in T^*S^2.$$ 

As mentioned, $\mathcal{R}$ can be regarded as the composition of two involutions

$$I : T^*S^2 \to T^*S^2, \quad I(\xi, \eta) = (\xi, -\eta).$$

and

$$T^*\rho : T^*S^2 \to T^*S^2, \quad T^*\rho(\xi_0, \xi_1, \xi_2, \eta_0, \eta_1, \eta_2) = (\xi_0, -\xi_1, -\xi_2, -\eta_0, -\eta_1, -\eta_2)$$

where $\rho$ is the reflection on $S^2$ about the great circle $L = \{\xi \in S^2 \subset \mathbb{R}^3 \mid \xi_1 = 0\}$. Then the fixed locus of $\mathcal{R}$ is the conormal bundle of $L$.

$$\text{Fix} \mathcal{R} = \{(\xi, \eta) \in T^*S^2 \mid \xi_1 = 0, \eta_0 = \eta_2 = 0\} \cong N^*L.$$
Since $\Sigma$ is a fiberwise starshaped hypersurface, $\text{Fix} R \cap \Sigma$ is composed of two circles

$$L^{M}_+ := \{(\xi, \eta) \in \text{Fix} R \mid \eta_1 = f_+(\xi_0, \xi_2)\}, \quad L^{M}_- := \{(\xi, \eta) \in \text{Fix} R \mid \eta_1 = f_-(\xi_0, \xi_2)\}$$

where $f_\pm \colon \{(\xi_0, \xi_2) \mid \xi_0^2 + \xi_2^2 = 1\} \to \mathbb{R}_\pm$ is a positive/negative function. Let $\pi : T^*\mathbb{R}^2 \to \mathbb{R}^2$ be the footpoint projection map. Then the regions of $L^M_+$ and $L^M_-$ in the configuration space of PCRTBP are as below.

$$\mathcal{L}^- := \pi \circ \mathcal{M}(L^-_+) = \{(q_1, 0) \mid q_1 \in \left[ \min_{(\xi, \eta) \in \text{Fix} R} \{f_- (1 - \xi_0) + q_1^M, q_1^M \} \right]\}$$

and

$$\mathcal{L}^+ := \pi \circ \mathcal{M}(L^+_+) = \{(q_1, 0) \mid q_1 \in \left[ \max_{(\xi, \eta) \in \text{Fix} R} \{f_+ (1 - \xi_0) + q_1^M, q_1^M \} \right]\}.$$ 

The region $\mathcal{K}_c^M = \pi(\Sigma_c^M)$ is called the Hill’s region (around the moon) where the satellite with energy $c$ can move. In Figure 1.1 below we depict $\mathcal{K}_c^M$ (the region encircled by the dotted curve) and $\mathcal{L}_\pm$.

![Figure 1.1. Hill’s region near the moon](image)

Suppose that a symmetric periodic orbit $(x, 2T)$ does not pass through the north pole of $S^2$ (i.e. does not collide with the moon). Then there is a periodic solution $((q^x(t), p^x(t)), 2T)$ of the Hamiltonian system of (1.1) corresponding to $(x, 2T)$. Since $(q^x(t), p^x(t))$ passes through $\text{Fix} R$ at time 0 and $T$ and

$$q_1^x T = \frac{\partial H}{\partial p_1} = p_1^x + q_2^x = 0 \quad \text{on} \quad \text{Fix} R = \{(q_1, 0, 0, p_2)\},$$

$q^x(t)$ cuts the $q_1$-axis at right angle at time 0 and $T$. The figures 1.2 and 1.3 describes the geometric motions of symmetric periodic orbits in the configuration space $\mathbb{R}^2$. If $(x, 2T)$ is of type I, $(q_1^x(0) - q_1^M)(q_1^x(T) - q_1^M) < 0$, see Figure 1.2. We note that the Birkhoff retrograde orbit [Bir15] which looks like $X_1$ is of type I. On the other hand, if $(x, 2T)$ is of type II, $(q_1^x(0) - q_1^M)(q_1^x(T) - q_1^M) > 0$, see Figure 1.3. We doubt whether there is a symmetric periodic orbit which does not surround the primary like $X_3$. But we expect Type II symmetric periodic orbits like $X_4$ mostly exist in the PCRTBP for arbitrary $c < H(L_1)$ and $\mu \in (0, 1)$. Indeed, when $\mu = 0$ (the rotating Kepler problem), there always exist such Type II symmetric periodic orbits for every energy below the first critical value: A symmetric periodic orbit which is a $k$-fold covered ellipse in an $l$-fold covered coordinate system (defined in [AFFvK12]) is of type II whenever $k + l$ is odd. Then the perturbation method based on
the implicit function theorem, for instance [Are63, Bar65], ensures survival of them at least for small $\mu \approx 0$.

Since $\Sigma$ is fiberwise starshaped it is a graph of $f \in C^\infty$ over the unit cotangent bundle $S^*S^2$. We note that the Hamiltonian vector field $X_Q$ on $\Sigma$ can be lifted to the Reeb vector field on a starshaped hypersurface in $\mathbb{R}^4$ with respect to the contact form $\alpha := \frac{1}{2}(x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2)$, see e.g. [HP08]. We denote such a double cover of $\Sigma$ by $S \subset \mathbb{R}^4$ and the covering map by $\Pi : S \to \Sigma$. We refer to $S$ dynamically convex if the Conley-Zehnder index (defined in Section 3) of every periodic Reeb orbit is greater than or equal to 3. It was proved that a strictly convex hypersurface is $\mathbb{R}^4$ is dynamically convex, see [HWZ98, Theorem 3.4] or [Lon02, Chapter 15]. This enables us to check the convexity for given mass ratio $\mu \in (0,1)$ and energy $c < H(L_1)$ using computer. Moreover it turns out that $S$ is dynamically convex in some cases [AFFHvK11, AFFvK12] and it is believed that $S$ is dynamically convex for all $\mu \in (0,1)$ and energy $c < H(L_1)$.

**Theorem A.** Suppose that $\Sigma$ is nondegenerate and $S$ is dynamically convex.

(A1) There exist at least two symmetric periodic orbits on $\Sigma$.

(A2) If there are precisely two periodic orbits on $\Sigma$, both are symmetric periodic orbits of type I.

(A3) There exist infinitely many periodic orbits on $\Sigma$ if a type II symmetric periodic exists. 

Aforementioned we meet the assumption on the existence of a type II symmetric periodic orbit in some cases. The nondegeneracy condition will be defined in Section 3 but we expect that this can be removed in the theorem. The assertions (A2) and (A3) are immediate consequences of (A1) and a theorem in [HWZ98], see Remark 1.3. Indeed if there is a type II symmetric periodic orbit $(x, 2T)$, there are two distinct periodic Reeb orbits $(\tilde{x}_1, 2T)$ and $(\tilde{x}_2, 2T)$ on $S$ such that $\pi(\tilde{x}_1) = \pi(\tilde{x}_2) = x$. But there is another periodic Reeb orbit on $S$.
due to (A1) and thus the theorem of [HWZ98] guarantees the existence of infinitely many periodic orbits on $S$ and hence on $\Sigma$ as well.

It is worth mentioning that making use of an idea behind of the proof of (A1), we can find multiple brake orbits (see Remark 1.2) on dynamically convex hypersurfaces in $\mathbb{R}^{2n}$, see [AKM12]. We think that there are a couple of ways to prove (A1). In this paper (A1) will be proved by using the Lagrangian Rabinowitz Floer homology computation

$$RFH_*(\Sigma, \text{Fix } \mathcal{R}, T^*S^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad * \in \mathbb{Z} \setminus \{0,1\}. \quad (1.3)$$

which will be carried out in Theorem 3.6. We hope that this homology computation will provide more information rather than (A1).

**Remark 1.2.** We note that $S$ is a centrally symmetric hypersurface, i.e. $S = -S$ in $\mathbb{R}^4$ and that there is an anti-symplectic involution $\tilde{\mathcal{R}}$ on $\mathbb{R}^4$ given by

$$\tilde{\mathcal{R}}(x_1, x_2, y_1, y_2) := (-x_1, x_2, y_1, -y_2)$$

such that $\Pi \circ \tilde{\mathcal{R}}|_S = \mathcal{R}|_S$. Since we have a map $\Psi(x_1, x_2, y_1, y_2) := (x_1, -y_2, y_1, x_2)$ on $\mathbb{R}^4$ such that $\Psi^* \alpha = \alpha$, the result on $\mathcal{R}$-symmetric periodic orbits on $S$ can be inferred from the result on periodic orbits symmetric with respect to

$$N := \Psi \circ \tilde{\mathcal{R}} \circ \Psi^{-1}, \quad (x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, y_1, y_2)$$
on $\Psi(S)$ which is also centrally symmetric due to $\Psi \circ \text{Id}_{\mathbb{R}^4} = \text{Id}_{\mathbb{R}^4} \circ \Psi$. It is worth remarking that $N$-symmetric periodic orbits are well known as brake orbits in classical mechanics which has a rich history. In particular, we can employ a theorem of [LZZ06] to prove the assertion (A1) when $S$ is strictly convex. We have not checked whether their theorem is applicable to the dynamically convex case. We close the remark by pointing out that centrally symmetric brake orbits resp. centrally asymmetric brake orbits on $\Psi(S)$ correspond to Type I symmetric periodic orbits resp. Type II symmetric periodic orbits in the regularized PCRTBP.

**Remark 1.3.** In order for (A2) and (A3) we observe dynamics on the ellipsoid

$$\mathcal{E}_4(r_1, r_2) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{|z_1|^2}{r_1} + \frac{|z_2|^2}{r_2} = 1, \quad a_2 \geq a_1 > 0 \right\}$$

which is a typical example of a strictly convex hypersurface in $(\mathbb{C}^2, \frac{1}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2))$. The Reeb flow on the ellipsoid $\mathcal{E}_4(r_1, r_2)$ is given by

$$z(t) = (z_1(t), z_2(t)) = (a_1 e^{2\pi i/r_1}, a_2 e^{2\pi i/r_2}), \quad \frac{a_1^2}{r_1} + \frac{a_2^2}{r_2} = 1, \quad a_1, a_2 > 0.$$ 

The minimal periods of $z_1(t)$ and $z_2(t)$ are $T_1 = \pi r_1$ and $T_2 = \pi r_2$ respectively. Thus if $r_1/r_2 \notin \mathbb{Q}$, there are precisely two periodic orbits $(\sqrt{r_1} e^{2\pi i/r_1}, 0)$ and $(0, \sqrt{r_2} e^{2\pi i/r_2})$. In contrast, if $r_1/r_2 \in \mathbb{Q}$, all orbits are periodic with the minimal period $T = \text{lcm}(p, q)T_1/p = \text{lcm}(p, q)T_2/q$ where $p, q \in \mathbb{N}$ satisfy $p/q = r_1/r_2$.

This shows that the ellipsoid possesses either two or infinitely many periodic orbits. In fact this dichotomy remains true for a wider class of 3-dimensional starshaped hypersurfaces: dynamically convex starshaped hypersurfaces [HWZ98]; see also [HWZ03].

**Further discussions.** We close this introductory section with some expectable applications.

1. For $c \in (H(L_1), H(L_2))$ energy between the first critical value and the second critical value, the regularized energy hypersurface $\Sigma_c^{E,M}$ is the connected sum of $\Sigma_c^E$ and $\Sigma_c^M$ which
is embedded in $D^*S^2\#D^*S^2$ the boundary connected sum of two unit disk bundles of $S^2$. It still carries an extended anti-symplectic involution $\mathcal{R}$ and $\text{Fix } \mathcal{R} \cap (\Sigma^E \# \Sigma^M_c)$ is composed of three circles $L^M_+, L^M_+ \# L^E_+$, and $L^E_+$. The corresponding regions $\mathcal{K}_c^{E \# M}$ and $\mathcal{L}^M_-, \mathcal{L}^M_+ \# \mathcal{L}^E_+$, and $\mathcal{L}^E_+$ are described in Figure 1.4.

![Figure 1.4. Hill’s region for $c \in (H(L_1), H(L_2))$](image)

Thus it may be an interesting question to ask what is the Lagrangian Rabinowitz Floer homology

$$\text{RFH}(\Sigma^E \# M_c, \text{Fix } \mathcal{R}_\pi, D^*S^2\#D^*S^2)$$

for energy level $c \in (H(L_1), H(L_2))$ although we can define it only for $c$ slight above $H(L_1)$ at the present, see [AFvKP12]. It is conceivable that the computation of (1.4) can be expressed in terms of the computation (1.3) as in the wrapped Floer homology case [Iri10]; see also the symplectic homology and (periodic) Rabinowitz Floer homology case [Cie02, CFO10, AF12].

2. Since the computation (1.3) only uses the fact that $\Sigma$ is a nondegenerate fiberwise starshaped hypersurface in $T^*S^2$ and the proof of Theorem A continues to hold whenever $\Sigma$ is dynamically convex, the theorem may be applicable to the Hill’s lunar problem [Hil78]. The Hill’s Hamiltonian $H_{\text{Hill}} : T^*(\mathbb{R}^2 \setminus \{(0,0)\}) \to \mathbb{R}$ is given by

$$H_{\text{Hill}}(q, p) := \frac{1}{2}|p|^2 - \frac{1}{2}|q| + q_2p_1 - q_1p_2 - q_1^2 + \frac{1}{2}q_2^2.$$  

Interestingly, the Hill’s Hamiltonian carries an additional involution $\mathcal{R}'$ defined by

$$\mathcal{R}'(q_1, q_2, p_1, p_2) = (-q_1, q_2, p_1, -p_2)$$

as well as $\mathcal{R}$. That is, there are two anti-symplectic involutions $\mathcal{R}$ and $\mathcal{R}'$ on $T^*\mathbb{R}^2$ such that $H_{\text{Hill}} = H_{\text{Hill}} \circ \mathcal{R} = H_{\text{Hill}} \circ \mathcal{R}'$. Via the regularization process as in the PCRTBP case, $\mathcal{R}$ corresponds to an anti-symplectic involution $\mathcal{R} = I \circ T^*\rho$ and $\mathcal{R}'$ corresponds to an anti-symplectic involution $\mathcal{R}' := I \circ T^*\rho'$ where $\rho'$ is a reflection on $S^2$ about the great circle $\{\xi_2 = 0\} \cap S^2$ while $\rho$ is the reflection about another great circle $\{\xi_1 = 0\} \cap S^2$. We denote the regularized energy hypersurface with energy $c < H_{\text{Hill}}(L_1)$ of the Hill’s problem by $\Sigma^H_c$ and its double cover by $S^H_c$. Then $\Sigma^H_c$ is invariant under both $\mathcal{R}$ and $\mathcal{R}'$. A periodic orbit which is symmetric for both involutions is said to be doubly symmetric. It is conceivable that $\Sigma^H_c$ is also fiberwise starshaped. Then as before, $\text{Fix } \mathcal{R} \cap \Sigma^H_c$ is composed of two circles

$$L^H_+ := \{(\xi, \eta) \in \text{Fix } \mathcal{R} \mid \eta_1 = f_+(\xi_0, \xi_2)\}, \quad L^H_- := \{(\xi, \eta) \in \text{Fix } \mathcal{R} \mid \eta_1 = f_-\xi_0, \xi_2)\}$$

and moreover $\text{Fix } \mathcal{R}' \cap \Sigma^H_c$ also consists of two circles

$$L'^H_+ := \{(\xi, \eta) \in \text{Fix } \mathcal{R}' \mid \eta_2 = f'_+\xi_0, \xi_1)\}, \quad L'^H_- := \{(\xi, \eta) \in \text{Fix } \mathcal{R}' \mid \eta_2 = f'_-\xi_0, \xi_1)\}$$
where \( f^\pm \colon \{ (\xi_0, \xi_1) \mid \xi_0^2 + \xi_1^2 = 1 \} \to \mathbb{R}_\pm \) is a positive/negative function. The corresponding regions \( K^H_c, L^H_{\pm}, \text{ and } L^H'_{\pm} \) are illustrated in Figure 1.5.

**Figure 1.5.** Hill’s region for the lunar problem

If \( \Sigma^H_c \) is dynamically convex, we can apply Theorem A. This assumption is not groundless due to [AFFHvK11] and for small energies this is actually true and can be checked by straightforward computations. If this is the case, there exist two \( \mathcal{R} \)-symmetric periodic orbits and two \( \mathcal{R}' \)-symmetric periodic orbits on \( \Sigma^H_c \). But \( \mathcal{R} \)-symmetric periodic orbits may coincide with \( \mathcal{R}' \)-symmetric periodic orbits after time shift (i.e. geometrically the same). Therefore if there are precisely two periodic orbits on \( \Sigma^H_c \), both have to be doubly symmetric. The existence of a doubly symmetric periodic orbit is known thanks to Birkhoff [Bir15] again.

## 2. The regularized restricted three body problem

Though the content of this section can be found in [AFvKP12], we briefly review the regularized PCRTBP to make this paper self-contained. As the name of the PCRTBP indicates, we assume that the moon and the earth rotate in a circular trajectory with center at the center of masses and that the satellite is massless and moves on the plane where the moon and the earth rotate. Let \( m_E \) be the mass of the earth and \( m_M \) be the mass of the moon. We denote the normalized mass of \( m_M \) by \( \mu \), i.e.

\[
\mu = \frac{m_M}{m_E + m_M} \in [0, 1].
\]

In the rotating coordinate system, the earth and the moon are located at

\[
q^E = (\mu, 0), \quad q^M = (-1 + \mu, 0)
\]

respectively and the phase space is given by

\[
T^*(\mathbb{R}^2 \setminus \{(q^E, q^M)\}) = (\mathbb{R}^2 \setminus \{(q^E, q^M)\}) \times \mathbb{R}^2
\]

The positions of the earth and the moon are removed to avoid collisions. The Hamiltonian for the satellite \( H : (\mathbb{R}^2 \setminus \{(q^E, q^M)\}) \times \mathbb{R}^2 \to \mathbb{R} \) is given by

\[
H(q, p) = \frac{1}{2} |p|^2 - \frac{1 - \mu}{|q - q^E|} - \frac{\mu}{|q - q^M|} + q_1 p_2 - q_2 p_1.
\]
The Hamiltonian $H$ carries exactly five critical points $(L_1, L_2, L_3, L_4, L_5)$ called Lagrange points. We may assume that

$$H(L_1) < H(L_2) \leq H(L_3) < H(L_4) = H(L_5).$$

The Hamiltonian $H$ is invariant under the anti-symplectic involution

$$\mathcal{R}: T^*\mathbb{R}^2 \to T^*\mathbb{R}^2, \quad (q_1, q_1, p_1, p_2) \mapsto (q_1, -q_2, -p_1, p_2),$$

which preserves three (colinear) Lagrange points and interchanges two (equilateral) Lagrange points. We denote by the footpoint projection map $\pi : T^*\mathbb{R}^2 \to \mathbb{R}^2$. Then the Hill’s region where the satellite with energy $c$ moves is $\pi(H^{-1}(c))$. The region $\pi(H^{-1}(c))$ for $c < H(L_1)$ is composed of two bounded regions and one unbounded region. We abbreviate the bounded regions by $K^E_c$ and $K^M_c$ such that $q^E \in \operatorname{cl}(K^E_c)$ and $q^M \in \operatorname{cl}(K^M_c)$. Likewise $H^{-1}(c)$ consists of two bounded components and one unbounded component. We denote by $\Sigma^E_c$ resp. $\Sigma^M_c$ the bounded component corresponding to $K^E_c$ resp. $K^M_c$.

In [Mos70], Moser regularized an energy hypersurfaces of the Kepler problem with negative energy into the unit tangent bundle of $S^2$. The PCRTBP (in the rotating coordinate system) can also be regularized in a similar way, see [AFvKP12, Section 6]. In what follows we briefly outline the regularization process for $\Sigma^M_c$ which is the bounded component close to the moon. We first introduce an independent variable

$$s = \int \frac{dt}{|q - q^M|}$$

and define the Hamiltonian $K(q,p)$ by

$$H(q,p) = \frac{K(q,p)}{|q - q^M|} + c.$$

Here $H^{-1}(c)$ is the energy hypersurface to be regularized. One can easily check that the Hamiltonian flow of $K$ at energy level $0$ with time parameter $s$ corresponds to the Hamiltonian flow of $H$ at energy level $c \in \mathbb{R}$ with time parameter $t$. We set $p = -x$, $q - q^M = y$ and perform the inverse of the stereographic projection. Here the stereographic projection

$$S : T^*S^2 = \{ (\xi, \eta) \} \longrightarrow T^*\mathbb{R}^2 = \{ (x, y) \}$$

is given by

$$S(\xi, \eta) = \left( \frac{\xi_1}{1 - \xi_0}, \frac{\xi_2}{1 - \xi_0}, \eta_1(1 - \xi_0) + \xi_1\eta_0, \eta_2(1 - \xi_0) + \xi_2\eta_0 \right)$$

where $\xi = (\xi_0, \xi_1, \xi_2) \in S^2 \subset \mathbb{R}^3$ and $\eta = (\eta_0, \eta_1, \eta_2) \in T^*_\xi S^2$, i.e. $\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 = 0$. Then we obtain the Hamiltonian function $K \circ S$ on $T^*S^2$. Since $K \circ S$ is not smooth at the zero section, we consider instead

$$Q : T^*S^2 \to \mathbb{R}, \quad Q := \frac{1}{2} |\eta|^2(\tilde{K} + \mu)^2.$$

Then one can readily check that Hamiltonian vector fields on $\Sigma^M_c$ are lifted to those of $Q^{-1}(\mu^2 / 2) =: \Sigma^M_c$. In a similar vein we can compactify $\Sigma^E_c$ into $\Sigma^E_c$. A remarkable theorem of Albers-Frauenfelder-van Koert-Paternain [AFvKP12] asserts that $\Sigma^E_c$ and $\Sigma^M_c$ are fiberwise starshaped hypersurfaces in $T^*S^2$. 
3. Rabinowitz Floer homology and Proof of Theorem A

A simple observation shows that the problem (1.2) can be interpreted as the boundary value problem for the Lagrangian submanifold \( \text{Fix } \mathcal{R} \). Indeed if \( (x, T) \) solves
\[
 x : [0, T] \to \Sigma, \quad \dot{x} = X_Q(x), \quad (x(0), x(T)) \in \text{Fix } \mathcal{R} \times \text{Fix } \mathcal{R},
\]
then so does \( x_{\mathcal{R}}(t) := \mathcal{R} x(T - t) \) and we obtain a symmetric periodic orbit
\[
x_{\mathcal{R}}} # x : \mathbb{R}/2T\mathbb{Z} \to \Sigma, \quad x_{\mathcal{R}}} # x(t) := \begin{cases} x(t) & t \in [0, T], \\ x_{\mathcal{R}}(t) & t \in [T, 2T]. \end{cases}
\]

As mentioned in the introduction, \( \Sigma \) can be both \( \Sigma_c^E \) and \( \Sigma_c^M \) for \( c < H(L_1) \). Our approach to this boundary value problem is Lagrangian Rabinowitz Floer homology studied by Merry [Mer10, Mer11], which is the Lagrangian intersection theoretic version of Rabinowitz Floer homology introduced by Cieliebak-Frauenfelder [CF09]. The existence of a symmetric periodic orbit is by now a standard application of Rabinowitz Floer homology theory due to the computation of Lagrangian Rabinowitz Floer homology group
\[
\text{RFH}_* (\Sigma, \text{Fix } \mathcal{R}, T^* S^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad * \in \mathbb{Z} \setminus \{0, 1\},
\]
which will be carried out in Theorem 3.6. Moreover using this computation we shall prove Theorem A.

3.1. Construction of Lagrangian Rabinowitz Floer homology.

We briefly introduce Lagrangian Rabinowitz Floer homology and refer to [Mer10, Mer11] for further details. We also refer the reader to [Frau04, Appendix A] for Morse-Bott homology and Floer’s celebrated papers [Flo88a, Flo88b]. Let \( M \) be a closed \( n \)-dimensional manifold and \( Q \) be a closed \( d \)-dimensional submanifold in \( M \). We denote by \( T^* M \) the cotangent bundle of \( M \) and by \( N^* Q \) the conormal bundle of \( Q \). We note that \( N^* Q \) is an exact Lagrangian submanifold in \( (T^* N, d\lambda) \) where \( \lambda \) is the Liouville 1-form. We denote by
\[
P_{N^* Q} T^* M := \{ x \in C^\infty([0, 1], T^* M) \mid x(0), x(1) \in N^* Q \}.
\]

Let \( H \in C^\infty(T^* M) \) such that \( H^{-1}(0) \) is a smooth fiberwise starshaped hypersurface. Then since \( H^{-1}(0) \) splits \( T^* M \) into one bounded component and one unbounded component, we can modify \( H \) to be constant near infinity. For notational convenience we write again \( H \) for the modified Hamiltonian function. Then the Rabinowitz action functional \( \mathcal{A}^H : P_{N^* Q} T^* M \times \mathbb{R} \to \mathbb{R} \) is defined by
\[
\mathcal{A}^H (x, \eta) = - \int_0^1 x^* \lambda - \eta \int_0^1 H(x) dt.
\]

A critical point \( (x, \eta) \in \text{Crit } \mathcal{A}^H \) satisfies
\[
\dot{x} = \eta_X_H (x(t)), \quad x(t) \in H^{-1}(0), \quad t \in [0, 1]
\]
where \( X_H \) is the Hamiltonian vector field associated to \( H \) defined implicitly by \( i_{X_H} \omega = dH \).

Thus if \( (x, \eta) \) is a nontrivial critical point of \( \mathcal{A}^H \), i.e. \( \eta \neq 0 \), \( (x, \eta) \) where \( x_{\eta}(t) := x(t/\eta) \) solves
\[
x_{\eta} : [0, \eta] \to H^{-1}(0), \quad \dot{x}_{\eta} = X_H (x_{\eta}), \quad (x_{\eta}(0), x_{\eta}(\eta)) \in N^* Q \times N^* Q.
\]
We choose an $\omega$-compatible almost complex structure $J$ and define a metric $m_J$ on $P_{N^*Q}T^*M \times \mathbb{R}$ by

$$m_J((\hat{x}_1, \hat{\eta}_1), (\hat{x}_2, \hat{\eta}_2)) := \int_0^1 \omega(\hat{x}_1, J(x)\hat{x}_2)\,dt + \hat{\eta}_1\hat{\eta}_2.$$ 

for $(\hat{x}_1, \hat{\eta}_1), (\hat{x}_2, \hat{\eta}_2) \in T_{(x,\eta)}P_{N^*Q}T^*M \times \mathbb{R}$. Then a map $w \in C^\infty(\mathbb{R}, P_{N^*Q}T^*M \times \mathbb{R})$ which solves

$$\partial_s w + \nabla_{m_J} \mathcal{A}^H(w(s)) = 0$$

is called a gradient flow line of $\mathcal{A}^H$ with respect to the metric $m_J$.

A solution $(x, T)$ for $T \neq 0$ of (3.2) is called nondegenerate if

$$\dim \left( T\phi^T_H(T_{x(0)}N^*Q) \cap T_{x(T)}N^*Q \right) = 0,$$

or equivalently $\phi^T_H(N^*Q)$ transversely intersects with $N^*Q$ at $x(T)$ where $\phi^T_H$ is the flow of $X_H$. A pair $(H^{-1}(0), N^*Q)$ is also called nondegenerate if $H^{-1}(0) \pitchfork N^*Q$ and every nontrivial solution of (3.2) is nondegenerate. From now on we assume that $(H^{-1}(0), N^*Q)$ is nondegenerate. Moreover a (trivial) critical manifold $(H^{-1}(0) \cap N^*Q, 0)$ of $\mathcal{A}^H$ is Morse-Bott, see [Mer10, Lemma 2.5]. In order to define the Morse-Bott homology of $\mathcal{A}^H$ we pick an auxiliary Morse-Smale pair $(f, g)$ where $f \in C^\infty(\text{Crit}(\mathcal{A}^H))$ and $g$ is a Riemannian metric on $\text{Crit}(\mathcal{A}^H)$. The index for critical points of $f$ is defined by

$$\mu_{RFH} : \text{Crit} f \to \mathbb{Z}, \quad \mu_{RFH}(x, \eta) := \begin{cases} \mu_{RS}(x, \eta) + d - \frac{n-1}{2}, & \eta \neq 0, \\ d - n + 1 + i_f(x, 0), & \eta = 0. \end{cases}$$

where $\mu_{RS}$ is the transverse Robbin-Salamon index defined in (3.3) and $i_f$ is the Morse index for $f$, i.e. the number of negative eigenvalues of the Hessian of $f$. We denote by $\text{Crit}_q f$ the set of critical points of $f$ with RFH-index $q \in \mathbb{Z}$. We define a $\mathbb{Z}/2$-vector space

$$\text{CF}_q(\mathcal{A}^H, f) := \left\{ \sum_{c \in \text{Crit}_q f} \xi_c \left| \xi_c \in \mathbb{Z}/2 \right. \right\}$$

with the finiteness condition

$$\# \{ c \in \text{Crit}_q f \left| \xi_c \neq 0, \mathcal{A}^H(c) \geq \kappa \right. \} < \infty.$$ 

Next we recall the Frauenfelder's Morse-Bott boundary operator, namely counting gradient flow lines with cascades. For $c_-, c_+ \in \text{Crit} f$ and $m \in \mathbb{N}$, a flow line from $c_-$ to $c_+$ with $m$ cascades

$$(w, t) = ((w_i)_{1 \leq i \leq m}, (t_i)_{1 \leq i \leq m-1})$$

consists of gradient flow lines of $\mathcal{A}^H$ $w_i \in C^\infty(\mathbb{R}, P_{N^*Q}T^*M \times \mathbb{R})$ and positive real numbers $t_i \in \mathbb{R}_+$ such that

$$\lim_{s \to -\infty} (w_1(-s), w_m(s)) \in W^u(c_-; f) \times W^s(c_+; f), \quad \lim_{s \to -\infty} w_{i+1}(s) = \phi^f_x(\lim_{s \to -\infty} w_i(s))$$

for $i = 1, \ldots, m-1$. Here $W^u(c_-; f)$ resp. $W^s(c_+; f)$ is the unstable manifold resp. the stable manifold and $\phi^f_x$ is the flow of $-\nabla_g f$. It is noteworthy that a flow line with no cascades is nothing but an ordinary negative gradient flow line of $f$. We denote by $\tilde{\mathcal{M}}_m(c_-, c_+)$ the space of flow lines with $m$ cascades from $c_-$ to $c_+$. We divide out the $\mathbb{R}^m$-action on $\tilde{\mathcal{M}}_m(c_-, c_+)$ defined by shifting the $m$ cascades in the $s$-variable. Then we obtain gradient flow lines with
unparametrized cascades and abbreviate $\mathcal{M}_m(c_-, c_+):=\tilde{\mathcal{M}}_m(c_-, c_+)/\mathbb{R}^m$. We define the set of flow lines with cascades from $c_-$ to $c_+$ by

$$\mathcal{M}(c_-, c_+):=\bigcup_{m\in\mathbb{N}\cup\{0\}}\mathcal{M}_m(c_-, c_+).$$

The standard arguments in Floer theory proves that following nontrivial facts. For a generic almost complex structure $J$ and a generic Riemannian metric $g$,

(F1) $\mathcal{M}(c_-, c_+)$ is a smooth manifold of finite dimension $\mu_{RFH}(c_+)-\mu_{RFH}(c_-)-1$. Moreover, if $\mu_{RFH}(c_+)-\mu_{RFH}(c_-)=1$, $\mathcal{M}(c_-, c_+)$ is a finite set.

(F2) Let $\mathcal{M}_c(c_-, c_+)$ be the compactification of $\mathcal{M}(c_-, c_+)$ with respect to the topology of Floer-Gromov convergence. If $\mu_{RFH}(c_+)-\mu_{RFH}(c_-)=2$, $\mathcal{M}_c(c_-, c_+)$ is a compact one-dimensional manifold whose boundary is

$$\partial \mathcal{M}_c(c_-, c_+)=\bigcup_z\mathcal{M}(c_-, z)\times\mathcal{M}(z, c_+)$$

where the union runs over $z\in\text{Crit}\,f$ with $\mu_{RFH}(c_+)-1=\mu_{RFH}(z)$.

Due to (F1), we denote by $n(c_-, c_+)$ the parity of the finite set $\mathcal{M}(c_-, c_+)$ when $\mu_{RFH}(c_+)-\mu_{RFH}(c_-)=1$. Then the boundary operators $\{\partial_q\}_{q\in\mathbb{Z}}$ are defined by

$$\partial_q: CF_q(A^H, f)\rightarrow CF_{q-1}(A^H, f)$$

$$c_-\in\text{Crit}_q f\mapsto \sum_{c_+\in\text{Crit}_{q-1} f} n(c_-, c_+)\cdot c_+.$$  

(F2) yields that $\partial_{q-1}\circ\partial_q=0$ and $(CF_*, (A^H, f), \partial_*)$ is a chain complex indeed. Thus we define Lagrangian Rabinowitz Floer homology by

$$RFH_q(H^{-1}(0), N^*Q, T^*M):=H_q(CF_*(A^H, f), \partial_*).$$

As the above notation indicates, Lagrangian Rabinowitz Floer homology is invariant under the choice of $(H, J, f, g)$ and depends only on $(H^{-1}(0), N^*Q, T^*M)$.

3.2. Computation of Lagrangian Rabinowitz Floer homology.

Making use of the Abbondandolo-Schwartz short exact sequence in [AS09], Merry proved the following theorem in [Mer10, Theorem B] (see also Remark 7.7 and Remark 12.6 in [Mer11]). We should mention that he proved more general statements.

**Theorem 3.1.** Let $Q$ and $M$ be closed manifolds with $d\leq n/2$. Then

$$RFH_*(\Sigma, N^*Q, T^*M)\cong H_*(P_QM;\mathbb{Z}_2)\oplus H^*-\epsilon+2d-n+1(P_QM;\mathbb{Z}_2), \quad \epsilon\in\mathbb{Z}\setminus\{0,1\}$$

where the path space $P_QM$ is defined below.

**Remark 3.2.** Although in proving the above theorem one has to use a Hamiltonian function defining $\Sigma$ which has quadratic growth [AS09, Mer10] or linear growth [CFO10] near infinity, the resulting Floer homology coincides with the Rabinowitz Floer homology defined in the previous subsection, see [AS09, Section 3] and [CFO10, Section 4].

In what follows we compute the singular homology groups in Theorem 3.1 in a special case. Let $Z$ be a connected topological space and $Y$ be a connected subspace. We denote by $\Omega Z$
the based loop space of \( Z \). We further abbreviate relative path spaces by for \( z, z' \in Z \),
\[
\begin{align*}
P_{z,z'} & := \{ \gamma \in C^0([0,1], Z) \mid (\gamma(0), \gamma(1)) = (z, z') \}, \\
P_{z,Y} & := \{ \gamma \in C^0([0,1], Z) \mid (\gamma(0), \gamma(1)) \in \{z\} \times Y \}, \\
P_Y & := \{ \gamma \in C^0([0,1], Z) \mid (\gamma(0), \gamma(1)) \in Y \times Y \}.
\end{align*}
\]
Here we deal with continuous paths but the homotopy types of above path spaces do not change even if we consider \( W^{1,2} \), or \( C^\infty \)-paths instead. Suppose that \( Y \) is contractible to \( z \in Z \) in \( Z \); that is, there exists a continuous map \( F : Y \times I \to Z \) such that
\[
F(\cdot, 0) = \tilde{z}, \quad F(\cdot, 1) = i_Y
\]
where \( \tilde{z} : Y \to \{z\} \) is a constant map and \( i_Y : Y \to Z \) is a canonical inclusion map.

**Proposition 3.3.** Let \( Y \) be contractible to \( z \in Z \) in \( Z \) as above. Then we have the following homotopy equivalences:
\[
P_Y \simeq P_{z,Y} \times Y \simeq \Omega Z \times Y \times Y.
\]

**Proof.** We define for each \( y \in Y \), \( \gamma^y \) a path in \( Z \) by
\[
\gamma^y(t) := F(y, t), \quad t \in [0,1].
\]
In particular, \( \gamma^y(0) = z \) and \( \gamma^y(1) = y \). We set
\[
\tilde{\gamma}^y(t) := \gamma^y(1-t), \quad \gamma_r^y(t) := \gamma^y(rt), \quad r \in [0,1].
\]
We define a map \( \Phi \) which will give the desired homotopy equivalence. Here we abbreviate \( \# \) for the concatenation operation for paths.
\[
\Phi : P_{z,Y} \to \Omega Z \times Y \\
\quad u \mapsto (\tilde{\gamma}^u(1) \# u, u(1))
\]
The map \( \Psi \) below will be a homotopical inverse of \( \Phi \).
\[
\Psi : \Omega Z \times Y \to P_{z,Y} \\
\quad (w, y) \mapsto \gamma^y \# w
\]
Here we consider \( \Omega Z \) as a loop space of \( Z \) with the base point \( z \in Z \). In order to show that \( \Psi \circ \Phi \) is homotopic to the identity, we construct a homotopy
\[
G : P_{z,Y} \times [0,1] \to P_{z,Y} \\
\quad (u, r) \mapsto \gamma^u(1) \# \gamma^u(1) \# u
\]
such that
\[
G(u, 0) = \begin{cases} 
  u(3t) & 0 \leq t \leq 1/3 \\
  u(1) & 1/3 \leq t \leq 1 
\end{cases}, \quad G(\cdot, 1) = \Psi \circ \Phi.
\]
Performing some reparametrizations on \( G \) at time \( r = 0 \), we deduce
\[
\Psi \circ \Phi \simeq \text{Id}_{P_{z,Y}}.
\]
In a similar vein, using the homotopy
\[
R : \Omega Z \times Y \times [0,1] \to \Omega Z \times Y \\
\quad (w, y, r) \mapsto (\tilde{\gamma}^y \# \gamma^y \# w, y)
\]
such that
\[ R(w, y, 0) = \begin{cases} (w(3t), y) & 0 \leq t \leq 1/3 \\ (w(1), y) & 1/3 \leq t \leq 1 \end{cases}, \quad R(\cdot, \cdot, 1) = \Phi \circ \Psi, \]
we obtain after some reparametrizations as before,
\[ \Phi \circ \Psi \simeq \text{Id}_{\Omega Z \times Y}. \]
This proves \( P_{z,Y}Z \simeq \Omega Z \times Y \) and thus the second equivalence is proved. The first equivalence \( P_Y Z \simeq P_{z,Y}Z \times Y \) follows analogously. \( \square \)

**Corollary 3.4.** Let \( S^1 \) be an embedded circle in \( S^2 \). Then we have
\[ P_{S^1} S^2 \simeq \Omega S^2 \times S^1 \times S^1. \]
In particular, we compute
\[ H_n(P_{S^1}(S^2); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & n = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n = 1, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{otherwise}. \end{cases} \]

**Remark 3.5.** In an alternative way, one can directly compute the singular homology of \( P_{S^1} S^2 \) by means of the Leray-Serre spectral sequence. We consider the evaluation map \( ev_1 : P_{z,S^1} S^2 \to S^1 \) defined by \( ev_1(u) = u(1) \). Then we have a fibration
\[ \Omega S^2 \hookrightarrow P_{S^1} S^2 \xrightarrow{ev_1} S^1. \]
We note that the spectral sequence for this fibration degenerates at the second page for dimension reasons, i.e. \( E^\infty_2 = E^2 \). Even though \( S^1 \) is not simply-connected, the \( E^2 \)-page has a simple formula. Since \( S^1 \) is contractible in \( S^2 \), the above fibration has trivial monodromy
\[ \pi_1(S^1) \to \text{Aut}(H_n(\Omega S^2)), \quad \ell \mapsto \text{Id}_{H_n(\Omega S^2)}, \quad \forall \ell \in \pi_1(S^1), \quad n \in \mathbb{N} \cup \{0\}, \]
and thus
\[ E^2_{i,j} \cong H_i(S^1; H_j(\Omega S^2; \mathbb{Z}_2)) \cong H_i(S^1; \mathbb{Z}_2) \otimes H_j(\Omega S^2; \mathbb{Z}_2). \]
Therefore we have
\[ H_n(P_{z,S^1} S^2; \mathbb{Z}_2) \cong \bigoplus_{i+j=n} E^\infty_{i,j} \cong \bigoplus_{i+j=n} H_i(S^1; \mathbb{Z}_2) \otimes H_j(\Omega S^2; \mathbb{Z}_2). \]
The exactly same arguments go through for a fibration
\[ P_{z,S^1} S^2 \hookrightarrow P_{S^1} S^2 \xrightarrow{ev_0} S^1 \]
where \( ev_0 \) is the evaluation map at time zero. Therefore we derive
\[ H_n(P_{S^1} S^2; \mathbb{Z}_2) \cong \bigoplus_{i+j+k=n} H_i(S^1; \mathbb{Z}_2) \otimes H_j(S^1; \mathbb{Z}_2) \otimes H_k(\Omega S^2; \mathbb{Z}_2). \]
Therefore Theorem 3.1 and Proposition 3.3 result in the following.

**Theorem 3.6.** Let \( Q \) and \( M \) be as above and \( Q \) be contractible to a point in \( M \).
\[ \text{RFH}^*_+(\Sigma, N^*Q, T^*M) = \bigoplus_{*_1+*_2+*_3=*} (H_{*_1}(\Omega M; \mathbb{Z}_2) \otimes H_{*_2}(Q; \mathbb{Z}_2) \otimes H_{*_3}(Q; \mathbb{Z}_2)) \]
\[ \bigoplus_{*_1+*_2+*_3=-*+2d-n+1} (H^{*_1}(\Omega M; \mathbb{Z}_2) \otimes H^{*_2}(Q; \mathbb{Z}_2) \otimes H^{*_3}(Q; \mathbb{Z}_2)). \]
In particular if $S^1$ is an embedded circle in $S^2$,
\[ \text{RFH}_*(S^*S^2, N^*S^1, T^*S^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad * \in \mathbb{Z} \setminus \{0, 1\}. \]

3.3. Robbin-Salamon index.

We denote by $\mathcal{L}(\mathbb{R}^{2n})$ the Grassmanian manifold of all Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega_0 = dx \wedge dy)$. Let $V \in \mathcal{L}(\mathbb{R}^{2n})$ and $\Lambda : [0, T] \to \mathcal{L}(\mathbb{R}^{2n})$. We choose $W \in \mathcal{L}(\mathbb{R}^{2n})$ a Lagrangian complement of $\Lambda(t)$ for $t \in [0, T]$. For $v \in \Lambda(t)$ and small $\epsilon$ we can find a unique $w(\epsilon) \in W$ such that $v + w(\epsilon) \in \Lambda(t + \epsilon)$. The crossing form at time $t \in [0, T]$ is defined by
\[ \Gamma(\Lambda, V, t) : \Lambda(t) \cap V \to \mathbb{R} \]
\[ v \mapsto \frac{d}{d\epsilon}_{|\epsilon=0} \omega_0(v, w(t + \epsilon)). \]

It is independent of the choice of $W$. A crossing time $t \in [0, T]$, i.e. $\Lambda(t) \cap V \neq \emptyset$, is said to be regular if $\Gamma(\Lambda, V, t)$ is nondegenerate. Since regular crossings are isolated, the number of crossings for a regular path which has only regular crossings is finite. Thus for a regular path $\Lambda(t) \in \mathcal{L}(\mathbb{R}^{2n})$ and $V \in \mathcal{L}(\mathbb{R}^{2n})$, the Robbin-Salamon index [RS93] can be defined as below.
\[ \mu_{RS}(\Lambda, V) := \frac{1}{2} \text{sign} \Gamma(\Lambda, V, 0) + \sum_{0 < t < T} \text{sign} \Gamma(\Lambda, V, t) + \frac{1}{2} \text{sign} \Gamma(\Lambda, V, T) \]

where the sum is taken over all crossings $t \in [0, T]$ and sign denotes the signature of the crossing form. Since we can always perturb a Lagrangian path to be regular and the Robbin-Salamon index is invariant under homotopies with fixed end points, the Robbin-Salamon index for nonregular paths also can be defined. Let $\Psi : [0, T] \to \text{Sp}(2n)$ be a path of symplectic matrices with $\Psi(0) = \text{Id}_{\mathbb{R}^{2n}}$ and $\text{det}(\text{Id}_{\mathbb{R}^{2n}} - \Psi(T)) \neq 0$. Then the Conley-Zehnder index of $\Psi$ is defined by
\[ \mu_{CZ}(\Psi) := \mu_{RS}((\text{graph } \Psi, \Delta) \]

where $\Delta$ is the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$.

Returning to the regularized PCRTBP, let $(x, t)$ be a solution of (3.1). Then $(x_{\mathcal{R}}, T)$ and $(x_m, mT)$ for $m \in \mathbb{N}$ defined by
\[ x_{\mathcal{R}}(t) := \mathcal{R} x(T-t), \quad x^k := x_{\mathcal{R}} \# \cdots \# x_{\mathcal{R}} \# x, \quad x^{2k+1} := \underbrace{x \# \cdots \# x}_{2k} \# x \]
solve (3.1) as well. Now we associate the Robbin-Salamon index to each solution of (3.1). We first symplectically trivialize the hyperplane field ker $\lambda_{\Sigma}$ (contact structure) by a pair of global sections
\[ X_1 = \frac{\xi \times \eta}{|\xi \times \eta|} - \frac{(\xi \times \eta) \cdot n_\eta}{|\xi \times \eta|} \cdot \partial_\eta, \quad X_2 = -\frac{(\xi \times \eta) \cdot n_\xi}{|\xi \times \eta|} \cdot \partial_\eta + \frac{(\xi \times \eta)}{|\xi \times \eta|} \cdot \partial_\xi \]

where $n_{\Sigma} = (n_\xi \partial_\xi, n_\eta \partial_\eta)$ is the outward pointing normal vector field on $\Sigma$. We denote the induced global symplectic trivialization by
\[ \Phi(\zeta) : \text{ker } \lambda_{\Sigma} \to \mathbb{R}^2, \quad \zeta = (\xi, \eta) \in \Sigma. \]

We note that $\Phi$ is a vertical preserving symplectization trivialization and maps $T\mathcal{R}$ to the reflection about the $X_2$-axis, i.e.
\[ \Phi(\zeta)[T^*\mathcal{R}^* \cap \text{ker } \lambda_{\Sigma}] = (0) \times \mathbb{R} \]
where \( T^vT^*S^2 := \ker T\pi \) be the vertical subbundle of \( TT^*M \) and
\[
\Phi(\mathcal{R}(\zeta)) \circ T\mathcal{R}_\zeta \circ \Phi(\zeta)^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} =: R
\]
We abbreviate the fixed locus of \( R \) by
\[ V := \text{Fix } R = \mathbb{R} \times (0). \]

Then \( T\phi_Q^t \) the linearization of the flow of \( X_Q \) gives a path in \( \text{Sp}(\mathbb{R}^2) \)
\[
\Psi_x(t) := \Phi(x(t))T\phi_Q^t(x(0))\Phi(x(0))^{-1}, \quad t \in [0, T],
\]
and the \textit{Robbin-Salamon index} of \( (x, T) \) a solution of \((3.1)\) is defined by
\[
\mu_{RS}(x, T) := \mu_{RS}(\Psi_x(t)V, V). \tag{3.3}
\]
We remark that the Robbin-Salamon index does not depend on the choice of vertical preserving symplectic trivialization, see [Oh97, Subsection 5.1] or [APS08, Section 3].

\textbf{Lemma 3.7.} If \((x, T)\) is a solution of \((3.1)\)
\[
\mu_{RS}(x, T) = \mu_{RS}(x_{\mathcal{R}}, T).
\]

\textbf{Proof.} Since \( x_{\mathcal{R}}(t) = \mathcal{R}x(T-t) \), we have
\[
\Psi_{x_{\mathcal{R}}}(t) = \Phi(x_{\mathcal{R}}(t))T\phi_X^{t}(x_{\mathcal{R}}(0))\Phi(x_{\mathcal{R}}(0))^{-1}
\]
\[
= R \circ \Phi(x(T-t))T\mathcal{R}_{x_{\mathcal{R}}}(t)T\phi_X^t(x_{\mathcal{R}}(0))\circ T\mathcal{R}_{x_{\mathcal{R}}}(0)\circ \Phi(x(T))^{-1} \circ R
\]
\[
= R \circ \Phi(x(T-t))T\phi_X^{T-t}(x(T))\circ \Phi(x(T))^{-1} \circ R.
\]
Then we deduce
\[
\mu_{RS}(x_{\mathcal{R}}, T) = \mu_{RS}(\Psi_{x_{\mathcal{R}}}(t)V, V) = \mu_{RS}(R \circ \Psi_{x}(T-t)RV, V)
\]
\[
= -\mu_{RS}(\Psi_{x}(T-t)RV, RV) = \mu_{RS}(\Psi_{x}(t)V, V) = \mu_{RS}(x, T).
\]

In [LZZ06, Section 6], Long-Zhang-Zhu proved the limit
\[
\hat{\mu}_{RS}(x, T) := \lim_{m \to \infty} \frac{\mu_{RS}(x^m, mT)}{m},
\]
exists\(^1\) and furthermore they proved the following identity
\[
\hat{\mu}_{RS}(x, T) = \frac{1}{2} \hat{\mu}_{CZ}(x^2, 2T)
\]
where
\[
\hat{\mu}_{CZ}(x^2, 2T) := \lim_{m \to \infty} \frac{\mu_{CZ}(x^{2m}, 2mT)}{m}
\]
for a periodic solution \((x^2, 2T)\), see [Lon02]. In consequence, we obtain the following proposition which plays crucial roles in proving Theorem A.

\textbf{Proposition 3.8.} If \( S \), the double cover of \( \Sigma \), is dynamically convex,
\[
\hat{\mu}_{RS}(x, T) > \frac{1}{2}.
\]
\(^1\)Actually they used \( \mu_1 \)-index which is different from \( \mu_{RS} \)-index by a constant \( n/2 \), see [LZ00, Theorem 3.1].
Proof. We note that \((x^4, 4T)\) is a periodic Reeb orbit on \((S, \alpha)\). It can be readily verified that \(\hat{\mu}_{CZ}(x^4, 4T) > 2\) by the assumption \(\mu(x^4, 4T) \geq 3\) and the index iteration formulas, see [Lon02, Section 8].\(^2\) Thus we have
\[
\lim_{m \to \infty} \frac{\mu_{RS}(x^{2m}, 2mT)}{m} = \hat{\mu}_{RS}(x^2, 2T) = \frac{1}{2} \hat{\mu}_{CZ}(x^4, 4T) > 1
\]
and that \(\hat{\mu}_{RS}(x, T) > 1/2\) is proved since the limit \(\hat{\mu}_{RS}(x, T)\) also exists. \(\square\)

3.4. Proof of Theorem A.

Since the Lagrangian Rabinowitz Floer homology (1.3) is different from the singular homology \(H_*(\Sigma \cap N*S^1; \mathbb{Z}_2)\), there exists a nontrivial solution \((x, \eta)\) of
\[
\dot{x} = \eta X_Q(x(t)), \quad x(t) \in \Sigma = Q^{-1}(\mu^2/2), \quad (x(0), x(1)) \in \text{Fix} \mathcal{R} \times \text{Fix} \mathcal{R}, \quad \forall t \in [0, 1].
\]
Assume on the contrary that \((x_\eta^2, 2\eta)\) is the only (geometrically distinct) symmetric periodic orbit. Since \(\mu_{RFH}(x, \eta) = \mu_{RS}(x, \eta) + 1/2\),
\[
\lim_{m \to \infty} \frac{\mu_{RFH}(x^m, m\eta)}{m} > 1/2
\]
due to Proposition 3.8 and thus there exist \(r, N \in \mathbb{N}, N \geq 4\) such that
\[
\mu_{RFH}(x^m, m\eta) > \frac{m}{2} + \frac{m}{r}, \quad \text{for all } m \geq N.
\]
We set
\[
P_1 := \min \left\{ n \in \mathbb{N} \Big| \frac{N}{2} + \frac{N}{r} \leq n \right\}
\]
and
\[
P_2 := \max\{\mu_{RFH}(x^\ell, \ell\eta) \big| \ell < N\}
\]
By definition,
\[
\mu_{RFH}(x^m, m\eta) > P_1, \quad m \geq N.
\]
We abbreviate
\[
J = \#\{(x^\ell, \ell\eta) \big| \ell < N, \mu_{RFH}(x^\ell, \ell\eta) \in [P_1, P_2]\} \in \mathbb{N} \cup \{0\}
\]
Since \(\mu_{RFH}(x^m, m\eta) = \mu_{RFH}((x_\eta^m)\eta, m\eta)\) for all \(m \in \mathbb{Z}\) due to Proposition 3.7 and
\[
\mu_{RFH}(x^{N+2rJ}, (N + 2rJ)\eta) > P_1 + rJ + 2J,
\]
we obtain
\[
\dim_{\mathbb{Z}/2} \bigoplus_{q \in [P_1, P_1+rJ+2J]} \text{CF}_q(A^H, f) \leq 2J + 2\#\{(x^N, N\eta), \ldots, (x^{N+2rJ-1}, (N + 2rJ - 1)\eta)\}
\]
\[
= 2J + 4rJ.
\]
But on account of (1.3), we have a contradictory inequality
\[
\dim_{\mathbb{Z}/2} \bigoplus_{q \in [P_1, P_1+rJ+2J]} \text{CF}_q(A^H, f) \geq 4(rJ + 2J + 1)
\]
and this proves the theorem. \(\square\)

\(^2\) For instance if \((x^4, T^4)\) is an elliptic periodic orbit on \(S, \hat{\mu}_{CZ}(x^4, 4T) = \mu_{CZ}(x^4, 4T) - 1 + \theta\) for some \(\theta \in (0, 1)\) and thus \(\hat{\mu}_{CZ}(x^4, 4T) > 2\).
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