We propose a wide universality class of gapless superfluids, and analyze a limit that might be realized in quark matter at intermediate densities. In the breached pairing color superconducting phase heavy s-quarks, with a small Fermi surface, pair with light u or d quarks. The groundstate has a superfluid and a normal Fermi component simultaneously. We expect a second order phase transition, as a function of increasing density, from the breached pairing phase to the conventional color-flavor locked (CFL) phase.

Because the primary one-gluon exchange interaction between high-momentum quarks is attractive for quarks in the color antisymmetric 3 channel, it is a firm prediction of QCD that cold dense quark matter is a color superconductor. At asymptotic dense quark matter is well understood: quarks of all three flavors (ignoring the c, b and t quarks) the ground state is well understood: quarks of all three flavors u, d, and s, pair according to the BCS mechanism, forming the color-flavor locked (CFL) phase. It is much less clear what QCD predicts for the ground state at subasymptotic densities, which could be relevant for describing neutron stars. Differences among the quark masses cause mismatches among the Fermi surfaces of the species which potentially pair. There is no longer an abundance of degenerate low-energy particle-particle or hole-hole states with opposite momenta both near their Fermi surfaces, and so it is less obvious what modes are the best candidates for coherent alignment by attractive interactions.

One much-discussed possibility is the LOFF phase (Larkin-Ovchinnikov-Fulde-Ferrel). Here we suggest quite a different possibility for ordering with mismatched Fermi surfaces, that might be realized at intermediate densities in QCD.

In QCD at intermediate densities, \( \mu \sim 200-300 \text{ MeV} \), there is not only mismatch in quark Fermi surfaces but also a different dispersion relation, since the heavy s-quark, unlike the light u and d quarks, needs not be ultra-relativistic. This difference makes plausible a pairing phase, wherein strange quarks are raised to higher kinetic energies to exploit favorable possibilities for correlation energy through pairing. The opposite limit — pairing of a heavy and light species that heavy species has the larger Fermi surface — is the arena for the interior gap phase discussed in [4].

For illustrative purposes we analyze a toy model with a massive s and a massless u quark. The Fermi momenta are related to chemical potentials as

\[
    p_F^u = \mu - \delta \mu^e, \quad p_F^s = \sqrt{\left(\mu + \delta \mu^e\right)^2 - m_s^2}
\]

where \( \delta \mu^e \) will be tuned to enforce number equality (a stand-in for electric neutrality). We are interested in the case when the Fermi momentum for the s-quark is smaller than that for the u-quark, \( p_F^s < p_F^u \). For simplicity, we shall linearize the s quark dispersion near its Fermi surface. We have checked that this simplification does not alter our results qualitatively. Our simplified model then has dispersion relations

\[
    \epsilon_p^s = V^u(p - p_F^u), \quad \epsilon_p^u = V^s(p - p_F^s)
\]

with \( V^s < V^u \).

In promoting particles of the heavy species to pair around the large Fermi surface of light species, there are two competing energetic factors to consider. These are the single-particle energy cost of such promotion, \( V^s|p_F^u - p_F^s| \) per pair, versus the gain from creating a pair, \( \kappa(V^u + V^s) \), where \( \kappa \) is the momentum gap. There is a net profit when

\[
    |p_F^u - p_F^s| < \frac{V^u + V^s}{V^s}. \quad (3)
\]

For flat dispersion for the s-quark, \( V^s \ll V^u \), promotion of s-quarks to a higher u-quark Fermi surface does not cost much energy, and a paired phase is favored. Translating this into possible effects of non-zero s-quark mass the allowed range for pairing is given by

\[
    |\delta \mu^e - m_s^2| < \frac{\Delta V^u}{V^s}. \quad (4)
\]

It can be more economical to promote heavy particles to higher Fermi momentum than to equalize the two Fermi surfaces by deforming both. In earlier work [4] we suggested the terminology “interior gap” for this phenomenon, motivated by the peaking of the gap parameter interior to the large Fermi surface that carries most of the spectral weight for gapless modes. Following that thought, it would be natural to call the limit we are analyzing here “exterior gap”. But to emphasize the common physical mechanism, we use “breached pairing” to cover both. The pairing is breached, in that it vanishes in a solid annulus in momentum space.

To allow quantitative estimates in analytic form we consider a schematic model involving pairing of u and s quarks. Starting with non-interacting degenerate Fermi
gases of \( u \) and \( s \)-quarks, turn on a weak attractive interaction between light and heavy species with a coupling \(-g < 0\). In a basis of light particles and heavy holes the quadratic part of the Hamiltonian is

\[
\mathcal{H}_{\text{quad}} = \left( \psi_{u-p}^\dagger \psi_{s-p} - \frac{\epsilon_{p}^u}{\Delta} \right) \left( \psi_{s-p}^\dagger \psi_{s-p} - \frac{\epsilon_{p}^s}{\Delta} \right),
\]

where the gap parameter is defined as \( \Delta = \frac{1}{(2\pi)^3} \int d^3p \psi_{u-p}^\dagger \psi_{s-p}^\dagger \beta p \), where the groundstate is assumed to be the breached pairing state. \( \mathcal{H}_{\text{quad}} \) can be diagonalized by the Bogoliubov transformation

\[
\left( \begin{array}{c} \psi_{u-p} \\ \psi_{s-p} \end{array} \right) = \left( \begin{array}{cc} \cos \theta_p & -\sin \theta_p \\ \sin \theta_p & \cos \theta_p \end{array} \right) \left( \begin{array}{c} \tilde{\psi}_{u-p} \\ \tilde{\psi}_{s-p} \end{array} \right)
\]

with \( \sin 2\theta_p = \Delta / \sqrt{\epsilon_{p}^u - \epsilon_{p}^s + \Delta^2} \). The new fermion fields \( \tilde{\psi} \) define the form of the interior gap wavefunction up to a relative phase (due to degeneracy) by \( \tilde{\psi}(0)_{\beta p} = 0 \). In the meantime, we obtain two branches of quasi-particle excitations with the spectra \( E_{p}^\pm = \epsilon_{p}^- \pm \sqrt{\epsilon_{p}^+ - \epsilon_{p}^- + \Delta^2} \), where \( \epsilon_{p}^\pm = \frac{1}{2}(\epsilon_{p}^u \pm \epsilon_{p}^s) \). The order parameter is related to \( \theta_p \) by \( \langle \tilde{\psi}_{u-p}^\dagger \tilde{\psi}_{s-p} \rangle_{BP} = \frac{1}{2} \sin 2\theta_p \) outside the breached region.

The energy of the breached pairing state \( |0\rangle_{BP} \) can be computed from the diagonalized Hamiltonian as

\[
\langle H \rangle_{BP} = \frac{1}{g} \Delta^2 - \int \frac{d^3p}{(2\pi)^3} \left( \epsilon_{p}^- + \epsilon_{p}^+ - \Delta^2 \right).
\]

The first term is the mean field potential and the integration is restricted to the area \( D \) defined by \( |p| - \Delta \leq |p_0| - \Delta \leq |p| + \Delta \), where \( p_0^\pm \) are the two roots of \( E_{p}^\pm = 0 \), \( p_0 \) is the momentum defined by \( \epsilon_{p_0}^\pm = 0 \):

\[
p_0 = p_0^+ - \Delta, \quad p_0^- = \Delta, \quad p_0^u = \frac{V^u}{V^u + V^s}, \quad p_0^s = \frac{V^s}{V^u + V^s}, \quad (8)
\]

and \( \lambda \) is the ultraviolet cutoff. (We will always be assuming \( p_0 > \Delta, \lambda, \delta p_F \).) Thus

\[
p_0^\pm = p_0^+ - \Delta, \quad \delta p_F = p_0^+ - p_0^-, \quad (9)
\]

with \( \delta p_F \equiv p_0^+ - p_0^- > 0 \). Varying the groundstate energy with respect to \( \Delta \), \( d\langle H \rangle_{BP}/d\Delta = 0 \), we find the integral equation for the gap parameter

\[
1 = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\epsilon_{p}^u - \epsilon_{p}^s + \Delta^2}}.
\]

Note that the integrand peaks near to \( |p| = p_0 \) where the energy denominator vanishes for \( \Delta \to 0 \).

Expanding the gap equation (10) for small \( \delta p_F \), we find the critical behavior

\[
\Delta \sim (g - g_c)^{\frac{2}{3}},
\]

with \( g_c \) defined in Eq. (12). The quasiparticle excitations of the breached pairing phase differ qualitatively from those of BCS. In the BCS phase, \( E_{p}^\pm \) branches are both gapped with their minimum energies given by \( |\epsilon_{p_0}^- \pm 2\Delta| / (V^u + V^s) \). By contrast, in the breached pairing phase \( E_{p}^\pm \) intersects zero energy only. The \( E_{p}^+ \) branch is fully gapped.

For \( V^s / V^u \ll 1 \) with fixed Fermi momenta, \( g_c \to 0 \) becomes arbitrarily weak.
$N_s(0) \ln [\lambda(V^u + V^s)/\Delta_0]$. We shall use $\Delta_0$ to parameterize the interaction strength below. Performing the integrals

$$\int_{p_0 - \lambda}^{p_0 + \lambda} \frac{dp}{\sqrt{\varepsilon_p^2 + \Delta_0^2}} = \int_{p_0 - \lambda}^{p_0 + \lambda} \frac{dp}{\sqrt{\varepsilon_p^2 + \Delta^2}} - \int_{p_{\Delta}^-}^{p_{\Delta}^+} \frac{dp}{\sqrt{\varepsilon_p^2 + \Delta^2}}$$

where $\varepsilon_p^+ = \frac{1}{2}(V^u + V^s)(p - p_0)$, we obtain

$$\frac{\Delta_0^2}{\Delta^2} = \frac{\varepsilon_{p_0}^s}{\varepsilon_{p_{\Delta}}^s}$$

for $\lambda \to \infty$. After some elementary algebra, the above equation reduces to

$$\frac{\Delta_0^2}{\Delta^2} = \frac{\delta p_F + \sqrt{\delta p_F^2 - 4\Delta^2}}{\delta p_F - \sqrt{\delta p_F^2 - 4\Delta^2}},$$

where $\delta p_{th} = \frac{\Delta_0}{\sqrt{V^u V^s}}$ (threshold value). In this case, the BP state always has a smaller $\Delta$ than (BCS)

$$\Delta = \begin{cases} \Delta_0, & \text{(BCS)} \\ \left[\Delta_0 \sqrt{V^u V^s} \left(\delta p_F - \delta p_{th}\right)\right]^{\frac{1}{2}}, & \text{,(unstable BP)} \end{cases}$$

which is equivalent to $g > g_c$ (see Eq. (12)). In solving Eq. (10), we distinguish three cases:

$I$: Chemical potentials fixed. Then $\delta p_F$ remains unchanged before and after pairing, and the gap equation permits two solutions:

$$\Delta = \begin{cases} \Delta_0, & \text{(BCS)} \\ \left[\Delta_0 \sqrt{V^u V^s} \left(\delta p_F - \delta p_{th}\right)\right]^{\frac{1}{2}}, & \text{,(unstable BP)} \end{cases}$$

II: Overall quark density fixed. In this case one has to adjust the overall chemical potential, $\mu_{oa} \equiv \frac{1}{2}(V^u p_{F}^u + V^s p_{F}^s) = \frac{1}{2}(V^u + V^s)\mu_0$, in order to accommodate a fixed overall density when pairing occurs. We hold the relative chemical potential, $\delta \mu = V^u p_{F}^u - V^s p_{F}^s$ fixed. The mismatched Fermi momentum is then altered, $\delta p_F \to \delta \tilde{p}_F$, with $\tilde{\cdot}$ denoting the new Fermi momenta $\tilde{p}_F^{u,s}$ after pairing.

We find $\delta \tilde{p}_F$ is determined by

$$\delta \tilde{p}_F = \frac{3 - 2\alpha}{1 - 2\alpha} \left[\Delta_0 \sqrt{V^u V^s} \left(\delta p_{th} - \delta p_{F}\right)\right]^{\frac{1}{2}}.$$
used. In the limit $V^*/V^u \to 0$ (so $\alpha \to 1$), the gap equation (10) exhibits the breached pairing phase for all $\delta p_F > 0$; BCS occurs only at the point $\delta p_F = 0$.

In mean field theory there is a single second-order phase transition between the BCS and BP phases (Fig 1). At fixed mismatch in Fermi momenta $\delta p_F$ and as $\Delta_0$ increases, one first encounters BP and then the BCS phase. The transition between the BP and BCS phases occurs at $\delta p_F = \delta p^*_F$. Since (21) is symmetric with respect to interchange of $V^u$ and $V^s$, the expression for critical points holds equally for both interior and “exterior” gap limits.

**III: Relative quark density fixed.** In this case, we also hold the overall chemical potential fixed while allowing the overall density to vary. In the limit $p_0 \gg \lambda \gg \Delta, \delta p_F$, we find

$$\delta p_F = \sqrt{\delta p^2_F + \frac{4\Delta^2}{V^u V^s}}.$$  

The gap equation (10) (with $\delta p_F \to \delta p^*_F$) has no BCS but BP solution for any $\delta p_F > 0$,

$$\Delta = \left[\Delta_0 \sqrt{\frac{V^u V^s}{V^u V^s}} (\delta p_{th} - \delta p_F)\right]^{\frac{1}{2}}.  \quad \text{(BP)}$$

The BCS solution only occurs at a single point, $\delta p_F = 0$

**Condensation energy.** To leading order in $\Delta/\delta p_F$, the energy difference between the breached pairing and normal ($\Delta = 0$) phases is

$$E_{BP} - E_N = \begin{cases} + \frac{N_s(0)\Delta^4}{2\delta p^2_F \sqrt{V^u V^s}}, & \text{(I)} \\ - \frac{N_s(0)\Delta^4}{2\delta p^2_F \sqrt{V^u V^s}}, & \text{(II)} \\ - \frac{N_s(0)\Delta^4}{2\delta p^2_F \sqrt{V^u V^s}}, & \text{(III)} \end{cases}.$$  

The BP state is energetically favorable in case II if $V^*/V^u < 3 - 2\sqrt{2}$ (with overall density and $\delta \mu$ fixed), and it is always favorable in case III (with the relative density and overall $\mu$ fixed).

In the breached pairing scenario pairing occurs at zero total momenta and peaks near the light species Fermi surface. It resembles standard BCS in that the pairing occurs at all angles, there is no necessity for translation or rotation symmetry breaking, and there is a branch of gapped excitations. There are, nevertheless, also gapless modes at two new “effective” Fermi surfaces, between which pairing is suppressed. For small $\Delta$ the energy gain of the breached pairing phase with respect to unpaired normal matter is $\sim \Delta^4$, parametrically less than $\sim \Delta^2$ in the BCS case.

**Discussion.** In the context of QCD, enforcing charge neutrality will make case III relevant for the $u$ and $d$ quarks. For pairing between strange and light quarks, case II is relevant. Thus, breached pairing allows all flavors to form condensates $(ud)_{BP}, (us)_{BP}, (ds)_{BP}$ of breached pairing type. We suspect that there is a density range where it is energetically favorable compared to unpaired matter or the 2SC phase with only $(ud)_{BCS} \neq 0$ condensation. Alford and Rajagopal [2] showed the absence of 2SC phase in compact stars, and suggested that there should be some non-BCS pattern of pairing including possible unpaired matter in the range of densities between hadronic and CFL matter. The breached pairing version of CFL appears to be a serious candidate for these intermediate densities. Since there is a simple second order phase transition from BP to BCS superfluidity, as a function of increasing density, it is natural to expect a single phase transition from breached pairing CFL to standard CFL quark matter. Detailed calculations, including shifts of Fermi surfaces due to electric and color neutrality, will be required to determine the range, if any, over which breached pairing CFL is favored (in preparation). At strong coupling one could even speculate about breached pairing CFL states at zero strangeness, where the strange chemical potential falls between $\pm m_s$.

The possibility that one might rearrange the free-particle Fermi surfaces to take advantage of opportunities for pairing was considered long ago in condensed matter literature by Sarma and others [3]. Superfluidity with quasiparticle dispersion relation crossing zero, thus leading to gapless modes, was encountered in a model resembling ours in Ref. [3]. In neither case, however, was the possibility of a stable breached pairing state recognized. (See also a recent study in cold atoms [3].) As this work was being completed reference Ref. [10] appeared, in which a related problem, imposing a charge neutrality condition directly, was analyzed.

We have phrased our discussion in terms of QCD and quarks, but one can easily imagine closely related universality classes for condensed matter or cold atom systems.

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