BANACH SPACES WHOSE ALGEBRA OF BOUNDED OPERATORS HAS THE INTEGERS AS THEIR $K_0$-GROUP

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Abstract. Let $X$ and $Y$ be Banach spaces such that the ideal of operators which factor through $Y$ has codimension one in the Banach algebra $\mathcal{B}(X)$ of all bounded operators on $X$, and suppose that $Y$ contains a complemented subspace which is isomorphic to $Y \oplus Y$ and that $X$ is isomorphic to $X \oplus Z$ for every complemented subspace $Z$ of $Y$. Then the $K_0$-group of $\mathcal{B}(X)$ is isomorphic to the additive group $\mathbb{Z}$ of integers.

A number of Banach spaces which satisfy the above conditions are identified. Notably, it follows that $K_0(\mathcal{B}(C([0,\omega_1]))) \cong \mathbb{Z}$, where $C([0,\omega_1])$ denotes the Banach space of scalar-valued, continuous functions defined on the compact Hausdorff space of ordinals not exceeding the first uncountable ordinal $\omega_1$, endowed with the order topology.

1. Introduction

The purpose of this note is to prove that, for certain Banach spaces $X$, the $K_0$-group of the Banach algebra $\mathcal{B}(X)$ of (bounded, linear) operators on $X$ is isomorphic to the additive group $\mathbb{Z}$ of integers. More precisely, our main result, which will be proved in Section 3, is as follows.

Theorem 1.1. Let $X$ and $Y$ be Banach spaces such that:

(i) $Y$ contains a complemented subspace which is isomorphic to $Y \oplus Y$;
(ii) $X$ is isomorphic to $X \oplus Z$ for every complemented subspace $Z$ of $Y$; and
(iii) the ideal $\{TS : S \in \mathcal{B}(X,Y), T \in \mathcal{B}(Y,X)\}$ of operators on $X$ that factor through $Y$ has codimension one in $\mathcal{B}(X)$.

Then the mapping

$$n \mapsto n[I_X]_0, \quad \mathbb{Z} \to K_0(\mathcal{B}(X)),$$

is an isomorphism of abelian groups, where $[I_X]_0$ denotes the $K_0$-class of the identity operator on $X$.

As a consequence, we shall deduce in Section 4 that the $K_0$-group of $\mathcal{B}(X)$ is isomorphic to $\mathbb{Z}$ for a number of Banach spaces $X$, including the following:

(i) $X = C([0,\omega_1])$, the Banach space of scalar-valued, continuous functions defined on the compact Hausdorff space of ordinals not exceeding the first uncountable ordinal $\omega_1$, endowed with the order topology;

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(ii) \( X = C(K) \) or \( X = C(K) \oplus Y \), where \( K \) is the compact Hausdorff space constructed by the second-named author in [12], assuming either the Continuum Hypothesis or Martin’s Axiom together with the negation of the Continuum Hypothesis, and where \( Y \) is a Banach space which is isomorphic to the \( \ell_p \)- or \( c_0 \)-direct sum of countably many copies of itself for some \( p \in [1, \infty) \), \( Y \) contains a complemented subspace that is isomorphic to \( c_0 \), and no complemented subspace of \( Y \) is isomorphic to \( C(K) \);

(iii) \( X = X_{\text{AH}} \oplus C_p \), where \( X_{\text{AH}} \) is Argyros and Haydon’s Banach space which solves the scalar-plus-compact problem (see [2]), \( p \in [1, \infty] \), and \( C_p \) is Johnson’s \( p^\text{th} \) universal space through which all approximable operators factor (see [21]);

(iv) \( X = W \oplus Y \), where \( W \) is the non-separable Banach space constructed by Shelah and Steprāns [18] such that every operator on \( W \) is a scalar multiple of the identity plus an operator with separable range, and \( Y \) is the \( \ell_p \)-direct sum of a certain family of separable subspaces of \( W \) for some \( p \in (1, \infty) \); see Example 4.7 for details.

We also obtain a couple of known results as consequences of Theorem 1.1, namely that \( [I_X]_0 \) has infinite order in \( K_0(\mathcal{B}(X)) \). Theorem 1.1 can therefore be viewed as a minimality result for \( [I_X]_0 \); by condition (iii), \( [I_X]_0 \) has infinite order in \( K_0(\mathcal{B}(X)) \), and (I.1) states that this element generates the whole group.

2. Preliminaries

We shall begin by outlining the definition of the \( K_0 \)-group of a unital ring \( \mathcal{A} \); further details can be found in standard texts such as [3] and [17]. For \( m, n \in \mathbb{N} \), we denote by \( M_{m,n}(\mathcal{A}) \) the additive group of \((m \times n)\)-matrices over \( \mathcal{A} \). We write \( M_n(\mathcal{A}) \) instead of \( M_{n,n}(\mathcal{A}) \); this is a unital ring. Define

\[
\text{IP}_n(\mathcal{A}) = \{ P \in M_n(\mathcal{A}) : P^2 = P \},
\]

the set of idempotent \((n \times n)\)-matrices over \( \mathcal{A} \). Given \( P \in \text{IP}_m(\mathcal{A}) \) and \( Q \in \text{IP}_n(\mathcal{A}) \), where \( m, n \in \mathbb{N} \), we say that \( P \) and \( Q \) are algebraically equivalent, written \( P \sim_0 Q \), if \( P = AB \) and \( Q = BA \) for some \( A \in M_{m,n}(\mathcal{A}) \) and \( B \in M_{n,m}(\mathcal{A}) \). This defines an equivalence relation \( \sim_0 \) on the set \( \text{IP}_\infty(\mathcal{A}) = \bigcup_{n \in \mathbb{N}} \text{IP}_n(\mathcal{A}) \), and the quotient \( V(\mathcal{A}) = \text{IP}_\infty(\mathcal{A})/\sim_0 \) is an abelian semigroup with respect to the operation

\[
([P]_V, [Q]_V) \mapsto \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}_V, \quad V(\mathcal{A}) \times V(\mathcal{A}) \to V(\mathcal{A}),
\]

where \([P]_V\) denotes the equivalence class of \( P \in \text{IP}_\infty(\mathcal{A}) \) in \( V(\mathcal{A}) \). The \( K_0 \)-group of \( \mathcal{A} \), denoted by \( K_0(\mathcal{A}) \), is now defined as the Grothendieck group of \( V(\mathcal{A}) \). The fundamental
property of the Grothendieck group implies that we have the following standard picture of $K_0(\mathcal{A})$:

$$K_0(\mathcal{A}) = \{[P]_0 - [Q]_0 : P, Q \in IP_\infty(\mathcal{A})\},$$  \hspace{1cm} (2.2)

where $[P]_0$ is the canonical image of $[P]_V$ in $K_0(\mathcal{A})$.

Let $n \in \mathbb{N}$, and suppose that $P, Q \in IP_n(\mathcal{A})$ are orthogonal, in the sense that $PQ = 0 = QP$. Then $P + Q$ is idempotent, and the formula for addition in $K_0(\mathcal{A})$ takes the following simple form:

$$[P]_0 + [Q]_0 = [P + Q]_0.$$  \hspace{1cm} (2.3)

We shall require one more basic property of $K_0$: given a ring homomorphism $\varphi: \mathcal{A} \to \mathcal{C}$ (where $\mathcal{C}$, like $\mathcal{A}$, is a unital ring, but $\varphi$ need not be unital) and $n \in \mathbb{N}$, we can define a ring homomorphism $\varphi_n: M_n(\mathcal{A}) \to M_n(\mathcal{C})$ by entrywise application:

$$\varphi_n((A_{j,k})_{j,k=1}^n) = (\varphi(A_{j,k}))_{j,k=1}^n.$$  \hspace{1cm} (2.4)

This induces a group homomorphism $K_0(\varphi): K_0(\mathcal{A}) \to K_0(\mathcal{C})$ which satisfies

$$K_0(\varphi)([P]_0) = [\varphi_n(P)]_0 \quad (n \in \mathbb{N}, P \in IP_n(\mathcal{A})).$$  \hspace{1cm} (2.5)

In Sections 1–4, all Banach spaces and algebras are considered over a fixed scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, whereas in Appendix A, we shall consider complex scalars only.

It is well known and elementary that the standard (unnormalized) trace

$$\text{Tr}_n: (\lambda_{j,k})_{j,k=1}^n \mapsto \sum_{j=1}^n \lambda_{j,j}, \quad M_n(\mathbb{K}) \to \mathbb{K},$$

induces an isomorphism $K_0(\text{Tr}): K_0(\mathbb{K}) \to \mathbb{Z}$ which satisfies

$$K_0(\text{Tr})([P]_0) = \text{Tr}_n(P) \quad (n \in \mathbb{N}, P \in IP_n(\mathbb{K})).$$  \hspace{1cm} (2.6)

(see for instance [17], Example 3.3.2] for a proof for $\mathbb{K} = \mathbb{C}$; the proof for $\mathbb{K} = \mathbb{R}$ is similar).

The symbol $C(K)$ denotes the Banach space of scalar-valued, continuous functions defined on a compact Hausdorff space $K$. By an operator, we understand a bounded, linear mapping between Banach spaces. For Banach spaces $X$ and $Y$, we write $\mathcal{B}(X, Y)$ for the Banach space of all operators from $X$ into $Y$, and we identify $M_{m,n}(\mathcal{B}(X, Y))$ with the Banach space $\mathcal{B}(X^n, Y^m)$ of operators from $X^n$ into $Y^m$, where $X^n$ denotes the direct sum of $n$ copies of $X$, equipped with the norm $\|(x_1, \ldots, x_n)\| = \max\{\|x_1\|, \ldots, \|x_n\|\}$. We write $\mathcal{B}(X)$ instead of $\mathcal{B}(X, X)$; this is a unital Banach algebra. We denote by $I_X$ the identity operator on $X$.

The following easy observation clarifies the meaning of the relation $\sim_0$ in this case.

**Lemma 2.1.** Let $X$ be a Banach space, and let $P, Q \in IP_\infty(\mathcal{B}(X))$. Then $P \sim_0 Q$ if and only if the ranges of $P$ and $Q$ are isomorphic.

We shall also require the following related result.

**Lemma 2.2** ([15], Lemma 3.9(ii)]. Let $X$ and $Y$ be Banach spaces, and let $S \in \mathcal{B}(X, Y)$ and $T \in \mathcal{B}(Y, X)$ be operators such that $ST$ is idempotent. Then $TSTS$ is idempotent, and the ranges of $ST$ and $TSTS$ are isomorphic.
Throughout this section we shall suppose that $X$ is a bounded, unital algebra homomorphism such that the conditions (i)–(iii) of Theorem 1.1. The third of these conditions implies that we can define $Y$ not necessary when $B$ in this case the set $\{TS : S \in \mathcal{B}(X,Y), T \in \mathcal{B}(Y,Z)\}$ is automatically a linear subspace of $\mathcal{B}(X,Z)$. In line with standard practice, we write $\mathcal{B}_Y(X)$ instead of $\mathcal{B}_Y(X,X)$.

3. The proof of Theorem 1.1

Throughout this section we shall suppose that $X$ and $Y$ are Banach spaces which satisfy conditions (ii)–(iii) of Theorem 1.1. The third of these conditions implies that we can define a bounded, unital algebra homomorphism $\varphi : \mathcal{B}(X) \to \mathbb{K}$ by

$$\varphi(\lambda I_X + T) = \lambda \quad (\lambda \in \mathbb{K}, T \in \mathcal{B}_Y(X)). \quad (3.1)$$

Recall that $M_n(\mathcal{B}(X))$ has been identified with $\mathcal{B}(X^n)$ for each $n \in \mathbb{N}$. Under this identification, we have $M_n(\mathcal{B}_Y(X)) = \mathcal{B}_Y(X^n)$ because $\mathcal{B}_Y$ is an operator ideal, and hence

$$\ker \varphi_n = M_n(\ker \varphi) = \mathcal{B}_Y(X^n) = \{TS : S \in \mathcal{B}(X^n,Y), T \in \mathcal{B}(Y,X^n)\}, \quad (3.2)$$

where the final equality follows from the fact that $Y$ satisfies condition (ii).

The following lemma is the key step in the proof of the surjectivity of the mapping (1.1).

**Lemma 3.1.** Let $P \in \text{IP}_n(\mathcal{B}(X))$ for some $n \in \mathbb{N}$, and set $k = \text{Tr}_n \circ \varphi_n(P)$. Then

$$[P]_0 = k \cdot [I_X]_0 \quad \text{in} \quad K_0(\mathcal{B}(X)).$$

**Proof.** We shall first establish the result for $k = 0$. In this case we have $\varphi_n(P) = 0$ because the zero matrix is the only idempotent, scalar-valued matrix with trace zero, so that $P = TS$ for some operators $S : X^n \to Y$ and $T : Y \to X^n$ by (3.2). Lemma 2.2 then implies that the operator $Q = SPT \in \mathcal{B}(Y)$ is idempotent with $Q[Y] \cong P[X^n]$. Combining this with condition (iii), we obtain $X \oplus P[X^n] \cong X \oplus Q[Y] \cong X$; that is, the operators $(I_X^0 \ 0)$ and $I_X$ have isomorphic ranges. Hence $[I_X]_0 + [P]_0 = [I_X]_0$ in $K_0(\mathcal{B}(X))$ by (2.1) and Lemma 2.1 so that $[P]_0 = 0$, as required.

We shall next consider the case where $k = n$. Then $\text{Tr}_n \circ \varphi_n(I_{X^n} - P) = 0$, so that $[I_{X^n} - P]_0 = 0$ by the result established in the first paragraph of the proof. Hence, by (2.3) and (2.1), we conclude that

$$[P]_0 = [I_{X^n} - P]_0 = [I_{X^n}]_0 = n \cdot [I_X]_0 \quad \text{in} \quad K_0(\mathcal{B}(X)).$$

Finally, suppose that $k \in \{1, 2, \ldots, n - 1\}$. Since $\varphi_n(P)$ is an idempotent, scalar-valued matrix, it is diagonalizable, so that there exists $R \in \ker \varphi_n$ such that $\Delta_k + R$ is idempotent and $P \sim_0 \Delta_k + R$, where

$$\Delta_k = \begin{pmatrix} I_X^k & 0 \\ 0 & 0 \end{pmatrix} \in \text{IP}_n(\mathcal{B}(X)).$$

By (3.2), we can find operators $S : X^n \to Y$ and $T : Y \to X^n$ such that $R = TS$. Moreover, $\Delta_k$ has an obvious factorization as $\Delta_k = UV$, where $U : X^n \to X^k$ and $V : X^k \to X^n$ denote the projection onto the first $k$ coordinates and the embedding into the first $k$ coordinates,
respectively. Condition (iii) implies that there exists an isomorphism $W: X^k \to X^k \oplus Y$, and we then have a commutative diagram

\[
\begin{array}{ccc}
X^n & \xrightarrow{\Delta_k + R} & X^n \\
\downarrow{(U \ S)} & & \downarrow{(V \ T)} \\
X^k \oplus Y & \xrightarrow{W^{-1}} & X^k & \xrightarrow{W} & X^k \oplus Y,
\end{array}
\]

where the operators $\begin{pmatrix} U & S \end{pmatrix}$ and $(V \ T)$ are given by $x \mapsto (Ux, Sx)$ and $(x, y) \mapsto Vx + Ty$, respectively. Hence Lemma 2.2 shows that the operator $Q = W^{-1} \begin{pmatrix} U & S \end{pmatrix} (\Delta_k + R) (V \ T) W \in \mathcal{B}(X^k)$ is idempotent and the ranges of $Q$ and $\Delta_k + R$ are isomorphic, so that $Q \sim_0 P$ by Lemma 2.1. The trace property implies that $\text{Tr}_k \circ \varphi_k(Q) = \text{Tr}_n \circ \varphi_n(\Delta_k + R) = k$, and therefore, as shown in the second paragraph of the proof, we have $k \cdot [I_X]_0 = [Q]_0 = [P]_0$, as required.

**Proof of Theorem 1.1.** We shall show below that the group homomorphism

\[ K_0(\text{Tr}) \circ K_0(\varphi) : K_0(\mathcal{B}(X)) \to \mathbb{Z} \]  

is an isomorphism. Since $K_0(\text{Tr}) \circ K_0(\varphi)([I_X]_0) = 1$, which generates the group $\mathbb{Z}$, the mapping given by (1.1) is the inverse of this isomorphism, and hence the conclusion follows.

The surjectivity of the homomorphism (3.3) is immediate because, as observed above, its range contains the generator 1 of the group $\mathbb{Z}$.

To see that the homomorphism (3.3) is injective, suppose that $g \in \ker K_0(\text{Tr}) \circ K_0(\varphi)$. By (2.2), we have $g = [P]_0 - [Q]_0$ for some $P \in \text{IP}_m(\mathcal{B}(X))$ and $Q \in \text{IP}_n(\mathcal{B}(X))$, where $m, n \in \mathbb{N}$. Using (2.5) and (2.6), we obtain

\[ 0 = K_0(\text{Tr}) \circ K_0(\varphi)(g) = \text{Tr}_m \circ \varphi_m(P) - \text{Tr}_n \circ \varphi_n(Q). \]

This implies that $[P]_0 = [Q]_0$ by Lemma 3.1 so that $g = 0$. \[ \square \]

4. Applications

**Example 4.1.** Assuming either the Continuum Hypothesis or Martin’s Axiom together with the negation of the Continuum Hypothesis, the second-named author [12] has constructed a scattered compact Hausdorff space $K$ such that:

1. the ideal $\mathcal{B}(C(K))$ of operators with separable range has codimension one in $\mathcal{B}(C(K))$;
2. every separable subspace of $C(K)$ is contained in a subspace which is isomorphic to $c_0$;
3. whenever $C(K)$ is decomposed into a direct sum of two closed, infinite-dimensional subspaces $A$ and $B$, either $A \cong c_0$ and $B \cong C(K)$, or vice versa.
We claim that $X = C(K)$ and $Y = c_0$ satisfy conditions (i)–(iii) of Theorem 1.1. Indeed, (i) is clear, while (ii) follows from condition (i), above, together with [8, Theorem 5.5], where it is shown that $\mathcal{K}(C(K)) = \mathcal{G}_c(C(K))$. Finally, to verify (iii), we observe that:

- $C(K)$ contains a complemented subspace which is isomorphic to $c_0$ because $K$ is scattered, and consequently $C(K) \cong C(K) \oplus c_0$ by (iii), above; and
- every complemented subspace $Z$ of $c_0$ is either finite-dimensional or isomorphic to $c_0$, and thus $c_0 \oplus Z \cong c_0$.

Hence $C(K) \oplus Z \cong C(K) \oplus c_0 \oplus Z \cong C(K) \oplus c_0 \cong C(K)$, as required. Thus we conclude that $K_0(\mathcal{B}(C(K))) \cong Z$. It is not known whether a compact space with the same properties as $K$ can be constructed within ZFC.

In order to facilitate further applications of Theorem 1.1, we shall show that certain standard properties of Banach spaces ensure that the first two conditions of Theorem 1.1 are satisfied.

**Lemma 4.2.** Let $X$ and $Y$ be Banach spaces such that $X$ is isomorphic to $X \oplus Y$, and suppose that $Y$ satisfies (at least) one of the following two conditions:

1. $Y$ is isomorphic to the $\ell_p$- or the $c_0$-direct sum of countably many copies of itself for some $p \in [1, \infty)$; or
2. $Y$ is primary and contains a complemented subspace which is isomorphic to $Y \oplus Y$.

Then conditions (i)–(iii) of Theorem 1.1 are satisfied.

**Proof.** Condition (i) of Theorem 1.1 is evidently satisfied in both cases.

To verify condition (ii), suppose that $Z$ is a complemented subspace of $Y$.

In case (1), we observe that $Y \cong Y \oplus Y$, so that $Y$ contains a complemented subspace which is isomorphic to $Y \oplus Z$. On the other hand, $Y \oplus Z$ evidently contains a complemented subspace which is isomorphic to $Y$. Hence the Pełczyński decomposition method (as stated in [11, Theorem 2.23(b)], for instance) implies that $Y \cong Y \oplus Z$. Combining this with the assumption that $X \cong X \oplus Y$, we obtain

$$X \oplus Z \cong X \oplus Y \oplus Z \cong X \oplus Y \cong X,$$

as required.

In case (2), either $Y \cong Z$ or $Y \cong Y \oplus Z$ because $Y$ is primary. In the first case, $X \oplus Z \cong X$ is immediate from the fact that $X \cong X \oplus Y$, and in the second case the calculation (4.1), above, applies to give this conclusion. \hfill \Box

**Example 4.3.** For an ordinal $\alpha$, denote by $[0, \alpha]$ the compact Hausdorff space consisting of all ordinals not exceeding $\alpha$, endowed with the order topology, and set $X = C([0, \omega_1])$ and $Y = (\bigoplus_{\alpha < \omega_1} C([0, \alpha]))_{c_0}$, where $\omega_1$ denotes the first uncountable ordinal. Then:

- $X \cong X \oplus Y$ by [9, Lemma 2.14(iv) and Corollary 2.16];
- $Y$ is isomorphic to the $c_0$-direct sum of countably many copies of itself by [9, Lemma 2.12], so that condition (i) of Lemma 1.2 is satisfied (in fact, condition (ii) is also satisfied by [9, Corollary 1.3]);
- the ideal $\mathcal{G}_Y(X)$ has codimension one in $\mathcal{B}(X)$ by [9, Theorem 1.6].
Hence Lemma 4.2 and Theorem 1.1 apply, so that $K_0(\mathcal{B}(X)) \cong \mathbb{Z}$.

This example provided the original motivation behind Theorem 1.1. We have since learnt that a different approach is possible for complex scalars, using a result of Edelstein and Mityagin, and this result also shows that the $K_1$-group of $\mathcal{B}(C([0,\omega_1]))$ vanishes; we refer to Appendix A for details.

**Example 4.4.** Let $p \in (1,\infty)$, and let $X = J_p$ be the $p$th quasi-reflexive James space, which is defined by $J_p = \{x \in c_0 : \|x\|_p < \infty\}$, where

$$\|x\|_p = \sup\left\{\left(\sum_{j=1}^{m} |x_{k_j} - x_{k_{j+1}}|^p\right)^{\frac{1}{p}} : m, k_1, \ldots, k_{m+1} \in \mathbb{N}, k_1 < k_2 < \cdots < k_{m+1}\right\} \in [0,\infty]$$

for each scalar sequence $x = (x_k)_{k \in \mathbb{N}}$. (This space was first considered for $p = 2$ by James [6] and later generalized to arbitrary $p$ by Edelstein and Mityagin [4].) Moreover, let $Y = \left(\bigoplus_{n \in \mathbb{N}} J_p^{(n)}\right)_{\ell_p}$, where $J_p^{(n)}$ denotes the $n$-dimensional subspace of $J_p$ consisting of those elements which vanish from the $(n+1)^{st}$ coordinate onwards. Then $X \cong X \oplus Y$ and $Y$ is isomorphic to the $\ell_p$-direct sum of countably many copies of itself by [4, Lemmas 5–6]. (Note, however, that a key condition appears to be missing in the statement of [4, Lemma 5], namely that the sequence denoted by $\nu$ is unbounded.) Further, the ideal $\mathcal{G}_Y(X)$ has codimension one in $\mathcal{B}(X)$ by [16, Theorem 4.3], so that Lemma 4.2 and Theorem 1.1 show that $K_0(\mathcal{B}(J_p)) \cong \mathbb{Z}$. This reproves [14, Theorem 4.6], whose proof inspired our proof of Theorem 1.1 above.

Kochanek and the first-named author have observed that the results obtained in [8, Section 3] ensure that the proof of [14, Theorem 4.6] carries over to Edgar’s long James space $J_p(\omega_1)$, originally introduced in [5], so that $K_0(\mathcal{B}(J_p(\omega_1))) \cong \mathbb{Z}$ (see [8, Proposition 3.13]). Our results provide an explicit proof of this conclusion, using [8]. Indeed, let $X = J_p(\omega_1)$, and define $Y = \left(\bigoplus_{\alpha \in L} J_p(\alpha)\right)_{\ell_p}$, where $L$ denotes the set of countably infinite limit ordinals and $J_p(\alpha)$ is the closed subspace of $J_p(\omega_1)$ spanned by the indicator functions of the ordinal intervals $[0,\beta)$ for $\beta < \alpha$. Then [8, Proposition 3.3, Lemma 3.4 and Theorem 3.7] show that $X \cong X \oplus Y$, that $Y$ is isomorphic to the $\ell_p$-direct sum of countably many copies of itself, and that the ideal $\mathcal{G}_Y(X)$ has codimension one in $\mathcal{B}(X)$, so that Lemma 4.2 and Theorem 1.1 apply.

Our final applications of Theorem 1.1 rely on the following general observation.

**Lemma 4.5.** Suppose that $X = W \oplus Y$, where $W$ and $Y$ are Banach spaces such that:

1. $Y$ is isomorphic to the $\ell_p$- or $c_0$-direct sum of countably many copies of itself for some $p \in [1,\infty)$; and
2. the ideal $\mathcal{G}_Y(W)$ has codimension one in $\mathcal{B}(W)$.

Then $X$ and $Y$ satisfy conditions (ii)–(iii) of Theorem 1.1 and hence $K_0(\mathcal{B}(X)) \cong \mathbb{Z}$.

**Proof.** Condition (ii) ensures that $Y \cong Y \oplus Y$, so that $X \cong X \oplus Y$, and Lemma 4.2 therefore shows that conditions (ii)–(iii) of Theorem 1.1 are satisfied.
To verify condition (iii), we use the fact that each operator $T \in \mathcal{B}(X)$ can be represented as a $(2 \times 2)$-matrix

$$T = \begin{pmatrix} T_{1,1}: W \to W & T_{1,2}: Y \to W \\ T_{2,1}: W \to Y & T_{2,2}: Y \to Y \end{pmatrix},$$

and $T$ factors through $Y$ if and only if $T_{j,k}$ does for each pair $j, k \in \{1, 2\}$. Since $T_{j,k}$ trivially factors through $Y$ for $(j, k) \neq (1, 1)$, condition (2) shows that $\mathcal{B}(X)$ has codimension one in $\mathcal{B}(X)$.

**Example 4.6.** Let $W = X_{\text{AH}}$ be Argyros and Haydon’s Banach space which solves the scalar-plus-compact problem (see [2]), and, for some $p \in [1, \infty]$, let $Y = C_p$ be Johnson’s $p^{\text{th}}$ universal space with the property that all approximable operators factor through $C_p$ (see [7]). Then, as noted in [7, p. 341], $Y$ is (isometrically) isomorphic to either the $\ell_p$-direct sum (for $p < \infty$) or the $c_0$-direct sum (for $p = \infty$) of countably many copies of itself.

Moreover, every compact operator $T$ on $W$ is approximable because $W$ has a Schauder basis, and therefore $T$ factors through $Y$ by the fundamental property of $Y$ (see [7, Theorem 1]). Hence we have $\mathcal{K}(W) \subseteq \mathcal{F}_0(W)$. To show that these two ideals are equal, we assume the contrary. Then, as $\mathcal{K}(W)$ has codimension one in $\mathcal{B}(W)$, necessarily $I_W \in \mathcal{F}_0(W)$, so that Lemma 2.2 implies that $Y$ contains a complemented subspace which is isomorphic to $W$. However, as observed in [7, p. 341], every closed, infinite-dimensional subspace of $Y$ contains a subspace which is isometric to $\ell_p$ (for $p < \infty$) or $c_0$ (for $p = \infty$), but no subspace of $W$ is isomorphic to $\ell_p$ or $c_0$ because $W$ is hereditarily indecomposable by [2, Theorem 8.11]. This contradiction proves that $\mathcal{F}_0(W) = \mathcal{K}(W)$. In particular $\mathcal{F}_0(W)$ has codimension one in $\mathcal{B}(W)$, so that Lemma 4.5 implies that $K_0(\mathcal{B}(X_{\text{AH}} \oplus C_p)) \cong \mathbb{Z}$.

**Example 4.7.** Let $W$ be the non-separable Banach space constructed by Shelah and Steprans [13] such that the ideal $\mathcal{K}(W)$ of operators with separable range has codimension one in $\mathcal{B}(W)$, and choose a family $(Y_\gamma)_{\gamma \in \Gamma}$ of closed, separable subspaces of $W$ such that:

1. every closed, separable subspace of $W$ is isomorphic to $Y_\gamma$ for some $\gamma \in \Gamma$; and
2. every subspace $Y_\beta$ is repeated countably many times in the family $(Y_\gamma)_{\gamma \in \Gamma}$, in the sense that the set $\{\gamma \in \Gamma : Y_\gamma = Y_\beta\}$ is countably infinite for each $\beta \in \Gamma$.

Set $Y = (\bigoplus_{\gamma \in \Gamma} Y_\gamma)_{\ell_p}$ for some $p \in (1, \infty)$. Condition (2) ensures that $Y$ is isomorphic to the $\ell_p$-direct sum of countably many copies of itself.

We shall now proceed to show that $\mathcal{K}(W) = \mathcal{F}_0(W)$. Indeed, for each $T \in \mathcal{K}(W)$, we can choose $\gamma \in \Gamma$ such that there is an isomorphism $U$ of $\overline{T(W)}$ onto $Y_\gamma$. Let $\iota_\gamma : Y_\gamma \to Y$ and $\pi_\gamma : Y \to Y_\gamma$ denote the canonical $\gamma^{\text{th}}$ coordinate embedding and projection, respectively. Then we have $T = SR$, where the operators $R$ and $S$ given by $R : w \mapsto \iota_\gamma U Tw$, $W \to Y$, and $S : y \mapsto U^{-1} \pi_\gamma y$, $Y \to W$. This shows that $T \in \mathcal{F}_0(W)$, and therefore the inclusion $\mathcal{K}(W) \subseteq \mathcal{F}_0(W)$ holds.

On the other hand, the Banach space $Y$ is weakly compactly generated, so that the same is true for each of its complemented subspaces. Wark [19, Proposition 2] has shown that $W$ is not weakly compactly generated. Hence no complemented subspace of $Y$ is isomorphic to $W$, so that $I_W \notin \mathcal{F}_0(W)$ by Lemma 2.2. Thus we conclude that $\mathcal{F}_0(W) = \mathcal{K}(W)$, and therefore Lemma 4.5 shows that $K_0(\mathcal{B}(W \oplus Y)) \cong \mathbb{Z}$. 
Example 4.8. Assume either the Continuum Hypothesis or Martin’s Axiom together with the negation of the Continuum Hypothesis, and let \( W = C(K) \), where \( K \) is the scattered compact Hausdorff space described in Example 4.1. Suppose that \( Y \) is a Banach space such that:

1. \( Y \) is isomorphic to the \( \ell_p \)- or \( c_0 \)-direct sum of countably many copies of itself for some \( p \in [1, \infty) \);
2. \( Y \) contains a complemented subspace which is isomorphic to \( c_0 \); and
3. no complemented subspace of \( Y \) is isomorphic to \( W \).

Condition (3) ensures that the ideal \( \mathcal{G}_Y(W) \) is proper, while condition (2) implies that it contains the ideal \( \mathcal{G}_0(W) \), which has codimension one in \( \mathcal{B}(W) \). Hence \( \mathcal{G}_Y(W) = \mathcal{G}_0(W) \), so that \( \mathcal{G}_Y(W) \) has codimension one in \( \mathcal{B}(W) \). The conditions of Lemma 4.5 are therefore satisfied, and thus \( K_0(\mathcal{B}(W + Y)) \cong \mathbb{Z} \).

For instance, conditions (1)-(2), above, are satisfied for \( Y = C(M) \), where \( M \) is any infinite, compact metric space.

Appendix A. An alternative approach for \( \mathcal{B}(C([0, \omega_1])) \)

Edelstein and Mityagin stated in [4] Proposition 4 that the invertible group of the Banach algebra \( \mathcal{B}(C([0, \omega_1])^n) \) is homotopy equivalent to the invertible group of scalar-valued \((n \times n)\)-matrices for each \( n \in \mathbb{N} \). The aim of this appendix is to show how this result can be applied to reprove the conclusion of Example 4.3 that \( K_0(\mathcal{B}(C([0, \omega_1]))) \cong \mathbb{Z} \), and also to deduce from it that the \( K_1 \)-group of \( \mathcal{B}(C([0, \omega_1])) \) vanishes. Note that this approach works for complex scalars only; we shall therefore suppose that the scalar field is \( \mathbb{C} \) throughout this appendix.

The \( K_1 \)-group will be the main object of interest, so we shall begin by defining it formally. In contrast to the purely ring-theoretic definition of \( K_0 \), topology plays a key role here. Suppose that \( \mathcal{A} \) is a complex, unital Banach algebra. For each \( n \in \mathbb{N} \), we turn \( M_n(\mathcal{A}) \) into a Banach algebra by identifying it with its natural image in the Banach algebra \( \mathcal{B}(\mathcal{A}^n) \) of operators acting on the direct sum of \( n \) copies of \( \mathcal{A} \), where \( \mathcal{A}^n \) is equipped with the norm \( \| (A_1, \ldots, A_n) \| = \max \{ \| A_1 \|, \ldots, \| A_n \| \} \), as in Section 2.

Note. For a Banach space \( E \), we have now equipped \( M_n(\mathcal{B}(E)) \) with two potentially different norms, one coming from its identification with \( \mathcal{B}(E^n) \), the other arising from its embedding into \( \mathcal{B}(\mathcal{B}(E)^n) \). Fortunately, these two norms are equal, as is easily checked.

Let \( \text{inv}_n(\mathcal{A}) \) be the group of invertible elements of \( M_n(\mathcal{A}) \), and denote by \( 1_{M_n(\mathcal{A})} \) the \((n \times n)\)-identity matrix over \( \mathcal{A} \). Given \( U \in \text{inv}_m(\mathcal{A}) \) and \( V \in \text{inv}_n(\mathcal{A}) \), where \( m, n \in \mathbb{N} \), we say that \( U \) and \( V \) are \( K_1 \)-equivalent, written \( U \sim_1 V \), if, for some integer \( k \geq \max \{ m, n \} \), there exists a continuous path \( t \mapsto W_t, [0, 1] \mapsto \text{inv}_k(\mathcal{A}) \), such that

\[
W_0 = \begin{pmatrix} U & 0 \\ 0 & 1_{M_{k-m}(\mathcal{A})} \end{pmatrix} \quad \text{and} \quad W_1 = \begin{pmatrix} V & 0 \\ 0 & 1_{M_{k-n}(\mathcal{A})} \end{pmatrix}.
\] (A.1)
This defines an equivalence relation $\sim_1$ on the set $\text{inv}_\infty(\mathcal{A}) = \bigcup_{n \in \mathbb{N}} \text{inv}_n(\mathcal{A})$, and the quotient $K_1(\mathcal{A}) = \text{inv}_\infty(\mathcal{A})/\sim_1$ is an abelian group with respect to the operation

$$([U]_1, [V]_1) \mapsto \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}, \quad K_1(\mathcal{A}) \times K_1(\mathcal{A}) \to K_1(\mathcal{A}),$$

where $[U]_1$ denotes the equivalence class of $U \in \text{inv}_\infty(\mathcal{A})$ in $K_1(\mathcal{A})$.

A cornerstone of $K$-theory for complex Banach algebras is that Bott periodicity holds:

$$K_0(\mathcal{A}) \cong K_1(\widetilde{S\mathcal{A}}), \tag{A.2}$$

where $\widetilde{S\mathcal{A}}$ denotes the unitization of the suspension of $\mathcal{A}$, that is,

$$\widetilde{S\mathcal{A}} = \{ f \in C([0, 1], \mathcal{A}) : f(0) = f(1) \in \mathbb{C}1_{\mathcal{A}} \};$$

here $C([0, 1], \mathcal{A})$ denotes the Banach algebra of continuous, $\mathcal{A}$-valued functions defined on the unit interval $[0, 1]$, and $1_{\mathcal{A}}$ is the multiplicative identity of $\mathcal{A}$. We may identify $M_n(C([0, 1], \mathcal{A}))$ with $C([0, 1], M_n(\mathcal{A}))$ for each $n \in \mathbb{N}$; under this identification, we have

$$\text{inv}_n \widetilde{S\mathcal{A}} = \{ f \in C([0, 1], \text{inv}_n \mathcal{A}) : f(0) = f(1) \in M_n(\mathbb{C}1_{\mathcal{A}}) \}. \tag{A.3}$$

Let $\mathcal{A}$ and $\mathcal{C}$ be complex, unital Banach algebras, and let $\varphi : \mathcal{A} \to \mathcal{C}$ be a bounded, unital algebra homomorphism. We shall require the following two homomorphisms associated with $\varphi$. First, in analogy with (2.5), we can define a group homomorphism $K_1(\varphi) : K_1(\mathcal{A}) \to K_1(\mathcal{C})$ by

$$K_1(\varphi)([U]_1) = [\varphi_n(U)]_1 \quad (n \in \mathbb{N}, U \in \text{inv}_n \mathcal{A}), \tag{A.4}$$

where $\varphi_n$ is given by (2.4), and secondly, we obtain a bounded, unital algebra homomorphism $\widetilde{S\varphi} : \widetilde{S\mathcal{A}} \to \widetilde{S\mathcal{C}}$ by the definition $\widetilde{S\varphi}(f) = \varphi \circ f$.

The following result is probably well known to experts, but since we have been unable to locate a precise reference to it, we include a proof.

**Lemma A.1.** Let $\mathcal{A}$ and $\mathcal{C}$ be complex, unital Banach algebras, let $n \in \mathbb{N}$, and let $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{C} \to \mathcal{A}$ be bounded, unital algebra homomorphisms such that the restriction to $\text{inv}_n \mathcal{A}$ of the mapping $\psi_n \circ \varphi_n$ is homotopy equivalent to the identity mapping, in the sense that there exists a continuous mapping $F : [0, 1] \times \text{inv}_n \mathcal{A} \to \text{inv}_n \mathcal{A}$ such that $F(0, U) = U$ and $F(1, U) = \psi_n \circ \varphi_n(U)$ for each $U \in \text{inv}_n \mathcal{A}$. Then

$$K_1(\widetilde{S\psi}) \circ K_1(\widetilde{S\varphi})([f]_1) = [f]_1 \quad (f \in \text{inv}_n \widetilde{S\mathcal{A}}).$$

**Proof.** Given $f \in \text{inv}_n \widetilde{S\mathcal{A}}$, we define $g_t(r) = F(\varphi_n(t), f(0))^{-1}F(\varphi_n(t), f(r)) \in \text{inv}_n \mathcal{A}$ for each pair $r, t \in [0, 1]$, where $F$ is chosen as above. An easy check using (A.3) shows that $g_t \in \text{inv}_n \widetilde{S\mathcal{A}}$ for each $t \in [0, 1]$. Moreover, the mapping $(r, t) \mapsto g_t(r), [0, 1]^2 \to \text{inv}_n \mathcal{A}$, is continuous, and it is therefore uniformly continuous, so that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|g_t(r) - g_t'(r')\|_{M_n(\mathcal{A})} \leq \varepsilon$ whenever $r, r', t, t' \in [0, 1]$ satisfy $\max\{|r - r'|, |t - t'|\} \leq \delta$. This implies that

$$\|g_t - g_t'|_{C([0, 1], M_n(\mathcal{A}))} = \sup_{r \in [0, 1]} \|g_t(r) - g_t'(r)\|_{M_n(\mathcal{A})} \leq \varepsilon \quad (t, t' \in [0, 1], |t - t'| \leq \delta),$$

and
which shows that the mapping \( t \mapsto g_t, [0, 1] \to \text{inv}_n \tilde{S}\mathcal{A} \), is continuous. Hence we have

\[
[f(0)^{-1} \cdot f]_1 = [f(0)^{-1} \cdot (\psi_n \circ \varphi_n \circ f)]_1 \quad \text{in} \quad K_1(\widetilde{S}\mathcal{A}) \tag{A.5}
\]

because \( g_0(r) = f(0)^{-1}f(r) \) and

\[
g_1(r) = (\psi_n \circ \varphi_n)(f(0))^{-1}(\psi_n \circ \varphi_n)(f(r)) = f(0)^{-1}(\psi_n \circ \varphi_n \circ f)(r) \quad (r \in [0, 1]),
\]

where we have used the fact that \( \psi_n \circ \varphi_n(U) = U \) for each \( U \in M_n(\mathbb{C}_{1,\alpha}) \).

Since \( \text{inv}_n(\mathbb{C}_{1,\alpha}) \) is homeomorphic to \( \text{inv}_n \mathbb{C} \), it is path-connected. We can therefore choose a continuous mapping \( t \mapsto V_t, [0, 1] \to \text{inv}_n(\mathbb{C}_{1,\alpha}) \), such that \( V_0 = 1_{M_n(\mathcal{A})} \) and \( V_1 = f(0)^{-1} \). This implies that the mappings \( t \mapsto V_t \cdot f \) and \( t \mapsto V_t \cdot (\psi_n \circ \varphi_n \circ f) \) of \( [0, 1] \) into \( \text{inv}_n \tilde{S}\mathcal{A} \) are continuous. They connect \( f \) with \( f(0)^{-1} \cdot f \) and \( \psi_n \circ \varphi_n \circ f \) with \( f(0)^{-1} \cdot (\psi_n \circ \varphi_n \circ f) \), respectively. When combined with (A.5), this shows that

\[
[f]_1 = [f(0)^{-1} \cdot f]_1 = [f(0)^{-1} \cdot (\psi_n \circ \varphi_n \circ f)]_1 = [\psi_n \circ \varphi_n \circ f]_1 = K_1(\tilde{S} \psi) \circ K_1(\tilde{S} \varphi)([f]_1),
\]

as required. \( \square \)

**Corollary A.2.** Let \( \mathcal{A} \) and \( \mathcal{C} \) be complex, unital Banach algebras, and suppose that there exist bounded, unital algebra homomorphisms \( \varphi : \mathcal{A} \to \mathcal{C} \) and \( \psi : \mathcal{C} \to \mathcal{A} \) such that the restrictions \( \varphi_n : \text{inv}_n \mathcal{A} \to \text{inv}_n \mathcal{C} \) and \( \psi_n : \text{inv}_n \mathcal{C} \to \text{inv}_n \mathcal{A} \) induce a homotopy equivalence for each \( n \in \mathbb{N} \), in the sense that \( \psi_n \circ \varphi_n \) is homotopy equivalent to the identity mapping on \( \text{inv}_n \mathcal{A} \) and \( \varphi_n \circ \psi_n \) is homotopy equivalent to the identity mapping on \( \text{inv}_n \mathcal{C} \). Then

\[
K_0(\mathcal{A}) \cong K_0(\mathcal{C}) \quad \text{and} \quad K_1(\mathcal{A}) \cong K_1(\mathcal{C}). \tag{A.6}
\]

**Proof.** Using the assumptions in tandem with Lemma [A.1] we see that \( K_1(\tilde{S} \varphi) \) is an isomorphism of \( K_1(\tilde{S}\mathcal{A}) \) onto \( K_1(\tilde{S}\mathcal{C}) \) with inverse \( K_1(\tilde{S} \psi) \). Hence we have

\[
K_0(\mathcal{A}) \cong K_1(\tilde{S}\mathcal{A}) \cong K_1(\tilde{S}\mathcal{C}) \cong K_0(\mathcal{C})
\]

by two applications of Bott periodicity [A.2]. This establishes the first part of (A.6).

The second part is much simpler. Indeed, working straight from the definitions (A.1) and (A.4), we obtain

\[
K_1(\psi) \circ K_1(\varphi)([U]_1) = [\psi_n \circ \varphi_n(U)]_1 = [U]_1 \quad (n \in \mathbb{N}, U \in \text{inv}_n \mathcal{A})
\]

because \( \psi_n \circ \varphi_n \) is homotopy equivalent to the identity mapping on \( \text{inv}_n \mathcal{A} \). A similar argument shows that \( K_1(\varphi) \circ K_1(\psi) \) is equal to the identity on \( K_1(\mathcal{C}) \), and \( K_1(\varphi) \) is therefore an isomorphism of \( K_1(\mathcal{A}) \) onto \( K_1(\mathcal{C}) \) with inverse \( K_1(\psi) \). \( \square \)

Recall from Example [4.3] that, for \( Y = (\bigoplus_{\alpha \leq \omega_1} C([0, \alpha]))_{c_0}, \) the ideal \( \mathcal{B}_Y(C([0, \omega_1])) \) has codimension one in \( \mathcal{B}(C([0, \omega_1])) \), and let \( \varphi : \mathcal{B}(C([0, \omega_1])) \to \mathcal{C} \) be the corresponding algebra homomorphism given by (3.1). Further, define

\[
\psi : \lambda \mapsto \lambda I_{C([0, \omega_1])}, \quad \mathcal{C} \to \mathcal{B}(C([0, \omega_1])).
\]

Then \( \varphi \) and \( \psi \) are bounded, unital algebra homomorphisms such that \( \varphi \circ \psi = I_{\mathcal{C}} \), and hence \( \varphi_n \circ \psi_n = I_{M_n(\mathbb{C})} \) for each \( n \in \mathbb{N} \). The precise statement of [11, Proposition 4] is
that $\psi_n \circ \varphi_n$ is homotopy equivalent to the identity mapping on $\inv_n \mathcal{B}(C([0, \omega_1]))$ for each $n \in \mathbb{N}$, so that we may apply Corollary A.2 to obtain
\[
K_0(\mathcal{B}(C([0, \omega_1]))) \cong K_0(\mathbb{C}) \cong \mathbb{Z} \quad \text{and} \quad K_1(\mathcal{B}(C([0, \omega_1]))) \cong K_1(\mathbb{C}) = \{0\}.
\]
This completes the proof of the statements made at the beginning of this appendix.

A comparison between the above calculation of the $K_0$-group of $\mathcal{B}(C([0, \omega_1]))$ and the one given in Example 4.3 shows some obvious advantages of the former, namely that it is shorter and simultaneously leads to the determination of the $K_1$-group; however, it also has some significant drawbacks:

- it does not apply to real scalars;
- it relies on some very heavy machinery, notably Bott periodicity, but also Edelstein and Mityagin’s highly non-trivial result;
- it is entirely topological, despite the purely ring-theoretic nature of $K_0$.

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