Power of Randomization in Automata on Infinite Strings

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Abstract. Probabilistic Büchi Automata (PBA) are randomized, finite state automata that process input strings of infinite length. Based on the threshold chosen for the acceptance probability, different classes of languages can be defined. In this paper, we present a number of results that clarify the power of such machines and properties of the languages they define. The broad themes we focus on are as follows. We precisely characterize the complexity of the emptiness, universality, and language containment problems for such machines, answering canonical questions central to the use of these models in formal verification. Next, we characterize the languages recognized by PBAs topologically, demonstrating that though general PBAs can recognize languages that are not regular, topologically the languages are as simple as $\omega$-regular languages. Finally, we introduce Hierarchical PBAs, which are syntactically restricted forms of PBAs that are tractable and capture exactly the class of $\omega$-regular languages.

1 Introduction

Automata on infinite (length) strings have played a central role in the specification, modeling and verification of non-terminating, reactive and concurrent systems [8, 10, 17, 20, 21]. However, there are classes of systems whose behavior is probabilistic in nature; the probabilistic behavior being either due to the employment of randomization in the algorithms executed by the system or due to other uncertainties in the system, such as failures, that are modeled probabilistically. While Markov Chains and Markov Decision Processes have been used to model such behavior in the formal verification community [15], both these models do not adequately capture open, reactive probabilistic systems that continuously accept inputs from an environment. The most appropriate model for such systems are probabilistic automata on infinite strings, which are the focus of study in this paper.

Probabilistic Büchi Automata (PBA) have been introduced in [3] to capture such computational devices. These automata generalize probabilistic finite automata (PFA) [12, 14, 16] from finite length inputs to infinite length inputs. Informally, PBA’s are like finite-state automata except that they differ in two respects. First, from each state and on each input symbol, the PBA may roll a dice to determine the next state. Second, the notion of acceptance is different because PBAs are probabilistic in nature and have infinite length input strings. The behavior of a PBA on a given infinite input string can be captured by an infinite Markov chain that defines a probability measure on the space of runs/executions of the machine on the given input. Like Büchi automata, a run is considered to be accepting if some accepting state occurs infinitely often, and
therefore, the probability of acceptance of the input is defined to be the measure of all accepting runs on the given input. There are two possible languages that one can associate with a PBA $B$ \[2, 3\] — $L_{>0}(B)$ (called \textit{probable semantics}) consisting of all strings whose probability of acceptance is non-zero, and $L_{=1}(B)$ (called \textit{almost sure semantics}) consisting all strings whose probability of acceptance is 1. Based on these two languages, one can define two classes of languages — $\mathbb{L}(\text{PBA}^{>0})$, and $\mathbb{L}(\text{PBA}^{=1})$ which are the collection of all languages (of infinite length strings) that can be accepted by some PBA with respect to probable, and almost sure semantics, respectively. In this paper we study the expressive power of, and decision problems for these classes of languages.

We present a number of new results that highlight three broad themes. First, we establish the precise complexity of the canonical decision problems in verification, namely, emptiness, universality, and language containment, for the classes $\mathbb{L}(\text{PBA}^{>0})$ and $\mathbb{L}(\text{PBA}^{=1})$. For the decision problems, we focus our attention on RatPBAs which are PBAs in which all transition probabilities are rational. First we show the problem of checking emptiness of the language $L_{=1}(B)$ for a RatPBA $B$ is $\text{PSPACE}$-complete, which substantially improves the result of \[2\] where it was shown to be decidable in $\text{EXPTIME}$ and conjectured to be $\text{EXPTIME}$-hard. This upper bound is established by observing that the complement of the language $L_{=1}(B)$ is recognized by a special PBA $M$ (with probable semantics) called a \textit{finite state probabilistic monitor (FPM)} \[4, 6\] and then exploiting a result in \[6\] that shows that the language of an FPM is non-empty if and only if there is an \textit{ultimately periodic word} in the language. This observation of the existence of ultimately periodic words does not carry over to the class $\mathbb{L}(\text{PBA}^{>0})$. However, we show that $L_{>0}(B)$, for a RatPBA $B$, is non-empty iff it contains a \textit{strongly asymptotic word}, which is a generalization of ultimately periodic word. This allows us to show that the emptiness problem for $\mathbb{L}(\text{PBA}^{>0})$, though undecidable as originally shown in \[2\], is $\Sigma_2^0$-complete, where $\Sigma_2^0$ is a set in the second level of the arithmetic hierarchy. Next we show that the universality problems for $\mathbb{L}(\text{PBA}^{>0})$ and $\mathbb{L}(\text{PBA}^{=1})$ are also $\Sigma_2^0$-complete and $\text{PSPACE}$-complete, respectively. Finally, we show that for both $\mathbb{L}(\text{PBA}^{>0})$ and $\mathbb{L}(\text{PBA}^{=1})$, the language containment problems are $\Sigma_0^0$-complete. This is a surprising observation — given that emptiness and universality are both in $\text{PSPACE}$ for $\mathbb{L}(\text{PBA}^{=1})$, one would expect language containment to be at least decidable.

The second theme brings to sharper focus the correspondence between nondeterminism and probable semantics, and between determinism and almost sure semantics, in the context of automata on infinite words. This correspondence was hinted at in \[2\]. There it was observed that $\mathbb{L}(\text{PBA}^{=1})$ is a strict subset of $\mathbb{L}(\text{PBA}^{>0})$ and that while Büchi, Rabin and Streett acceptance conditions all yield the same class of languages under the probable semantics, they yield different classes of languages under the almost sure semantics. These observations mirror the situation in non-probabilistic automata — languages recognized by deterministic Büchi automata are a strict subset of the class of languages recognized by nondeterministic Büchi automata, and while Büchi, Rabin and Streett acceptances are equivalent for nondeterministic machines, Büchi acceptance is strictly weaker than Rabin and Streett for deterministic machines. In this paper we