Integrability and exact solution of an electronic model with long range interactions

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Abstract
We present an electronic model with long range interactions. Through the quantum inverse scattering method, integrability of the model is established using a one-parameter family of typical irreducible representations of \( gl(2|1) \). The eigenvalues of the conserved operators are derived in terms of the Bethe ansatz, from which the energy eigenvalues of the Hamiltonian are obtained.

1 Introduction
The Quantum Inverse Scattering Method (QISM) [1] is one of the most powerful tools in the exact study of quantum systems. It can be applied in a number of contexts, including both one-dimensional systems with nearest neighbour interactions such as the Heisenberg [2] and Hubbard [3] models, and also for the analysis of models with long range interactions such as the Gaudin Hamiltonians [4] and extensions [5]. These latter constructions in particular have received renewed attention as it has been realised that the reduced BCS model, which was recently

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proposed to describe superconducting correlations in metallic grains of nanoscale dimensions [5], can be shown to be integrable through the use of Gaudin Hamiltonians in non-uniform external fields [6]. Formulating the reduced BCS model in the framework of the QISM reproduces the exact solution originally obtained by Richardson and Sherman [7], and opens the way for the calculation of form factors and correlation functions [8].

Motivated by this result, one can investigate to what extent the construction can be generalised to yield new classes of models, with some examples already given in [9]. Here, we will consider a case where an underlying superalgebraic structure (i.e., one with both bosonic and fermionic degrees of freedom) is employed to yield an electronic model. The supersymmetric formulation of integrable systems can be traced back to the work of Kulish [10], and recently supersymmetric Gaudin Hamiltonians have been analysed in detail in [11].

In this article we present a Hamiltonian derived through the QISM from a solution of the Yang-Baxter equation (YBE) associated with a typical irreducible representation of the Lie superalgebra $gl(2|1)$, which has the explicit form

$$H = \sum_j^D \epsilon_j n_j - g \sum_{j,k}^{D} \sum_{\sigma = \pm} Q_{j\sigma}^\dagger Q_{k\sigma}. \quad (1)$$

Above, the energy levels $\epsilon_j$ are two-fold degenerate, $g$ is an arbitrary coupling parameter and $D$ is the total number of distinct energy levels. Also, $n_j$ is the fermion number operator for energy level $\epsilon_j$ and for the parameters $\alpha_j$, $j = 1...D$ we define

$$Q_{j\sigma} = c_{j\sigma} \sqrt{\alpha_j + 1} X_{j}^{n_j,\sigma},$$

with $X_j = \sqrt{\alpha_j/(\alpha_j + 1)}$. The $c_{\sigma}^\dagger, c_{\sigma}$, $\sigma = \pm$, are two-fold degenerate Fermi annihilation and creation operators.

The Hamiltonian has a similar form to the reduced BCS model [3]. There, Cooper pairs are scattered into vacant energy levels while the one particle states are blocked from scattering. In the Hamiltonian above, there is correlated scattering depending on the occupation numbers. One of the features of this model is that the scattering couplings can be varied through the choice of the parameters $\alpha_j$. Via the algebraic Bethe ansatz method and using the minimal typical representation of $gl(2|1)$, from which these free parameters arise, we establish the exact solvability of the model. Here we outline the necessary definitions and constructions, while full details will be presented elsewhere.

The Lie superalgebra $gl(2|1)$ has generators $E_j^i$, $i, j = 1, 2, 3$ with supercommutator relations

$$[E_j^i, E_k^l] = \delta_j^k E_j^i - (-1)^{(i)(l)}(j)(k) \delta_j^l E_j^k.$$

Above, the BBF grading $[1] = [2] = 0, [3] = 1$ is chosen and the elements are realised in terms of the Fermi operators through (cf. [12])

$E_2^1 = S^+ = c^\dagger_+ c_-$,
\[ E_1^2 = S^- = c_+^{\dagger} c_+, \quad E_1^1 = -\alpha - n_+, \quad E_2^2 = -\alpha - n_-, \quad E_3^3 = 2\alpha + n, \]
\[ E_1^1 = Q_{+}, \quad E_2^2 = Q_{+}, \quad E_3^3 = Q_{-}, \text{ and we set } S^z = (n_+ - n_-)/2. \]

The Casimir invariant of the algebra, \( C = \sum_{i,j=1}^3 E_i^j \otimes E_j^i (-1)^{[j]}, \) which commutes with all the elements of \( gl(2|1) \), will also be needed, and has the eigenvalue \( \xi_C = -2\alpha (\alpha + 1) \) in the above representation. Below we let \( V(\alpha) \) denote the four-dimensional model on which the representation acts, with the basis \(|+\rangle, |\rangle, |\rangle, |0\rangle\).

2  The Yang-Baxter equation and integrability

To construct the model, we use the supersymmetric formulation of the QISM [10]. We take the following solution of the YBE which acts on \( W \otimes W \otimes V(\alpha) \), where \( W \) denotes the three-dimensional vector module of \( gl(2|1) \),

\[ R_{12}(u - v)L_{13}(u)L_{23}(v) = L_{23}(v)L_{13}(u)R_{12}(u - v) \]  \( \text{(2)} \)
with

\[ R(u) = I \otimes I + \frac{\eta}{u} \sum_{m,n=1}^{3} (-1)^{[n]} e_m^n \otimes e_n^m, \]  \( \text{(3)} \)

and the L-operator is given by

\[ L(u) = I \otimes I + \frac{\eta}{u} \sum_{m,n=1}^{3} (-1)^{[n]} e_m^n \otimes E_n^m. \]  \( \text{(4)} \)

The representations taken for the operators \( E_n^m \) are as stated above, the variable \( u \) represents the rapidity \( \eta \) is arbitrary and \( I \) is the identity operator.

By the usual procedure of the QISM, we define a transfer matrix acting on the \( D \)-fold tensor product space (for distinct \( \alpha_i \) ) \( V(\alpha_1) \otimes V(\alpha_2) \otimes ... \otimes V(\alpha_D) \) via

\[ t(u) = \text{str}_0 (G_0 L_{0D}(u - \epsilon_D)...L_{01}(u - \epsilon_1)), \]

which gives a mutually commuting family satisfying \([t(u), t(v)] = 0\). Above, \( \text{str}_0 \) denotes the supertrace taken over the auxiliary space labelled by 0 and \( G \) can be any matrix which satisfies \([R(u), G \otimes G] = 0\).

For the BBF grading we choose \( G = \text{diag}(\exp(\beta \eta), \exp(\beta \eta), 1) \) and by employing the algebraic Bethe ansatz method the eigenvalues of the transfer matrix are found to be (cf. [10])

\[ \Lambda(u) = \exp(\beta \eta) \prod_i^{D} \left( 1 - \frac{\eta \alpha_i}{(u - \epsilon_i)} \right) \prod_j^{P} a(v_j - u) \]
\[ + \exp(\beta \eta) \prod_{i}^{D} \left( 1 - \frac{\eta \alpha_i}{u - \epsilon_i} \right) \prod_{j}^{P} a(u - v_j) \prod_{k}^{M} a(\gamma_k - u) \]
\[ - \prod_{i}^{D} \left( 1 - \frac{2\eta \alpha_i}{u - \epsilon_i} \right) \prod_{j}^{M} a(\gamma_j - u), \]  
(5)

where \( a(u) = 1 + \eta/u \). The parameters \( v_i, w_j \) satisfy the Bethe ansatz equations
\[ \prod_{k}^{M} a(\gamma_k - v_j) = -\prod_{i}^{P} \frac{a(v_i - v_j)}{a(v_j - v_i)}, \prod_{i}^{D} \frac{\gamma_l - \epsilon_i - 2\eta \alpha_i}{\gamma_l - \epsilon_i - \eta \alpha_i} = \exp(\beta \eta) \prod_{j}^{P} a(\gamma_l - v_j). \]

We now introduce the operators
\[ T_j = \lim_{u \to \epsilon_j} \frac{(u - \epsilon_j)}{\eta^2} t(u), \quad \text{which satisfy} \quad [T_j, T_k] = 0. \]  
(6)

By taking the quasi-classical expansion \( T_j = \tau_j + o(\eta) \), this leads to
\[ \tau_j = -\beta \psi_j + \sum_{i \neq j} D \frac{\theta_{ji}}{\epsilon_j - \epsilon_i} \]

where \( \theta = \sum_{m,n}^{3} E_{m}^{n} \otimes E_{n}^{m}(-1)^{[m]} \) and \( \psi = E_{3}^{3} \). It is easily deduced that these operators satisfy \([\tau_j, \tau_k] = 0\).

Writing \( K = \sum_{j,k}^{D} (S_{j}^{+}S_{k}^{-} + S_{k}^{-}S_{j}^{+} + 2S_{j}^{z}S_{k}^{z}) \), which satisfies \([K, \tau_j] = 0, \forall j\), we define the Hamiltonian as follows;
\[ H = \frac{1}{2\beta^2} \sum_{j}^{D} (1 + 2\beta \epsilon_j) \tau_j + \frac{1}{4\beta^3} \sum_{j,k}^{D} \tau_j \tau_k + \frac{1}{2\beta} \sum_{j}^{D} C_j - \frac{K}{2\beta} + 2 \sum_{j}^{D} \epsilon_j (\alpha_j + 1) \]
\[ = \frac{1}{2\beta} \sum_{j}^{D} \sum_{k \neq j} D \theta_{jk} - \frac{1}{2\beta} \sum_{j}^{D} (1 + 2\beta \epsilon_j) \psi_j + \frac{1}{4\beta} \sum_{i,j}^{D} \psi_i \psi_j - \frac{1}{\beta} \sum_{j}^{D} \alpha_j (\alpha_j + 1) \]
\[ - \frac{K}{2\beta} + 2 \sum_{j}^{D} \epsilon_j (\alpha_j + 1). \]

The term involving \( \theta_{jk} \) may be simplified using the Casimir invariant and the commutation relations of the algebra \( gl(2|1) \)
\[ \sum_{j}^{D} \sum_{k \neq j} D \theta_{jk} = \sum_{k,j}^{D} \theta_{jk} - \sum_{j}^{D} C_j, \]
\[ = K - \frac{1}{2} \sum_{j,k}^{D} \psi_j \psi_k + \sum_{j}^{D} \psi_j - 2 \sum_{j,k}^{D} \sum_{\sigma = \pm} Q_{j\sigma}^{+} Q_{k\sigma} + 2 \sum_{j}^{D} \alpha_j (\alpha_j + 1). \]
For \( q = 1/\beta \) we obtain the Hamiltonian (1), which establishes integrability since \([H, \tau_j] = 0, \forall j\).

From (5,6) we obtain the eigenvalues of \( \tau_j \) for the BBF grading,

\[
\lambda_j = -2\beta \alpha_j + \alpha_j \sum_{i}^{M} \frac{1}{\gamma_i - \epsilon_j} - 2 \sum_{i \neq j}^{D} \frac{\alpha_j \alpha_i}{\epsilon_j - \epsilon_i}, \tag{7}
\]

as the quasi-classical limit of the eigenvalues of the transfer matrix. The corresponding Bethe ansatz equations are

\[
\beta + \sum_{j}^{P} \frac{1}{\gamma_l - v_j} = \sum_{i}^{D} \frac{\alpha_i}{\epsilon_i - \gamma_l}, \quad \sum_{l}^{M} \frac{1}{\gamma_l - v_j} = 2 \sum_{i \neq j}^{P} \frac{1}{v_i - v_j}. \tag{8}
\]

For a given solution of the Bethe ansatz equations we find that the number of electrons, \( N = 2D - M, n_+ - n_- = M - 2P \) and the eigenvalue of \( K \) reads

\[
\xi_K = \frac{1}{2} (M - 2P)(M - 2P + 2). \tag{9}
\]

The energy eigenvalues can be computed using (7,8,9) and are given by

\[
E = 2 \sum_{j}^{D} \epsilon_j - \sum_{l}^{M} \gamma_l - 2g \sum_{j}^{D} \alpha_j - gM.
\]

Similar results have been obtained for the FBB and BFB gradings, which will appear elsewhere.

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