ON THE CHARACTER OF CERTAIN IRREDUCIBLE MODULAR REPRESENTATIONS

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1. Let $G$ be an almost simple, simply connected algebraic group over $k$, an algebraically closed field of characteristic $p > 1$. Let $\text{Rep}G$ be the category of finite dimensional $k$-vector spaces with a given rational linear action of $G$ and let $\text{Irr}G$ be a set of representatives for the simple objects of $\text{Rep}G$. We fix a Borel subgroup $B$ of $G$ and a maximal torus $T$ of $B$; let $Y = \text{Hom}(k^*, T)$, $X = \text{Hom}(T, k^*)$ (with group operation written as $+$) and let $\langle , \rangle : Y \times X \rightarrow \mathbb{Z}$ be the obvious pairing. If $V \in \text{Irr}G$ then there is a well defined $\lambda_V \in X$ with the following property: the $T$-action on the unique $B$-stable line in $V$ is through $\lambda_V$; according to Chevalley, $E \mapsto \lambda_E$ is a bijection from $\text{Irr}G$ to a subset of $X$ of the form $X^+ = \{ \lambda \in X; \langle \check{\alpha}_i, \lambda \rangle \in \mathbb{N} \ \forall i \in I \}$ for a well defined basis $\{ \check{\alpha}_i; i \in I \}$ of $Y$. For $\lambda \in X^+$ we shall denote by $V_\lambda$ the object of $\text{Irr}G$ corresponding to $\lambda$. If $V \in \text{Rep}G$, then for any $\mu \in X$ we denote by $n_\mu(V)$ the multiplicity of $\mu$ in $V|_T$; we set $[V] = \sum_{\mu \in X} n_\mu(V)e^\mu \in \mathbb{Z}[X]$ where $\mathbb{Z}[X]$ is the group ring of $X$ (the basis element of $\mathbb{Z}[X]$ corresponding to $\mu \in X$ is denoted by $e^\mu$ so that $e^\mu e^{\mu'} = e^{\mu + \mu'}$ for $\mu, \mu' \in X$). It is of considerable interest to compute explicitly the element $[V_\lambda] \in \mathbb{Z}[X]$ for any $\lambda \in X^+$. Let $h$ be the Coxeter number of $G$. A conjectural formula for $[V_\lambda]$ (assuming that $p \geq c^0_G$ where $c^0_G$ is a constant depending only on the root datum of $G$) was stated in [L1, p.316]. In the early 1990’s it was proved (see [AJS] and the references there) that there exists a (necessarily unique) prime number $c_G \geq c^0_G$ depending only on the root datum of $G$ such that the conjectural formula in [L1, p.316] is true if $p \geq c_G$ and $c_G$ is minimum possible (but $c_G$ was not explicitly determined). In [Fi], Fiebig showed that $c_G \geq c'_G$ where $c'_G$ is an explicitly known but very large constant. In [Wi], Williamson, partly in collaboration with Xuhua He, showed that for infinitely many $G$, $c_G$ is much larger than $c^0_G$. Now the conjecture in [L1, p.316] had an unsatisfactory aspect: it applied only to a finite set of $\lambda \in X^+$ which, after application of Jantzen’s results [Ja] on translation functors, becomes a larger
but still finite set (including all $\lambda$ in $X^+_{\text{red}} = \{ \lambda \in X^+; \langle \check{\alpha}_i, \lambda \rangle \leq p - 1 \ \forall \alpha \in I \}$); then the case of a general $\lambda \in X^+$ had to be obtained by applying the Steinberg tensor product theorem [St]. In this note I want to offer a reformulation of the conjecture in [L1, p.316] (now a known theorem for $p$ large enough) which applies directly to any $\lambda \in X^+$, see 7(b).

2. Notation. Let $NT$ be the normalizer of $T$ in $G$ and let $W = NT/T$ be the Weyl group. Note that $W$ acts naturally on $T$ hence on $Y, X$ and $\mathbb{Z}[X]$. Let $\check{R} = \{ y \in Y; y = w(\check{\alpha}_i) \text{ for some } w \in W, i \in I \}$ (the set of coroots). Define $\check{\alpha}_0 \in \check{R}$ by the condition that $\check{\alpha}_0 + \check{\alpha}_i \notin \check{R}$ for any $i \in I$. (Thus $\check{\alpha}_0$ is the highest coroot).

For $i \in I \cup \{ 0 \}$ define $\alpha_i \in X$ by the condition that the map $X \to X$, $\lambda \mapsto \lambda - \langle \check{\alpha}_i, \lambda \rangle \alpha_i$, is induced by a (uniquely defined) element $s_i$ of $W$. Let $w \mapsto \epsilon_w$ be the homomorphism $W \to \{ 1, -1 \}$ such that $\epsilon_w = -1$ for any $i \in I$. Define $\rho \in X^+$ by $\langle \check{\alpha}_i, \rho \rangle = 1$ for any $i \in I$. Let $\leq$ be the partial order on $X$ given by $\lambda \leq \lambda'$ whenever $\lambda' - \lambda \leq \sum_{i \in I} \mathbb{N} \alpha_i$.

3. Let

$$\Delta = \{ \lambda \in X; \langle \check{\alpha}_i, \lambda + \rho \rangle \leq 0 \ \forall i \in I, \langle \check{\alpha}_0, \lambda + \rho \rangle \geq -p \}.$$ 

For $i \in I$ we define $s'_i : X \to X$ by $s'_i(\lambda) = \lambda - \langle \check{\alpha}_i, \lambda + \rho \rangle \alpha_i$ (an affine reflection).

We define $s'_0 : X \to X$ by $s'_0(\lambda) = \lambda - (\langle \check{\alpha}_0, \lambda + \rho \rangle + p) \alpha_0$ (an affine reflection). Let $W_a$ be the subgroup of the group of permutations of $X$ generated by $s'_i(i \in I \cup \{ 0 \})$. Then $W_a$ is a Coxeter group on the generators $s'_i(i \in I \cup \{ 0 \})$, with length function $l : W_a \to \mathbb{N}$.

For $\lambda \in X$ we have $w^{-1}(\lambda) \in \Delta$ for some $w \in W_a$ and among all such $w$ there is a unique one, $w_\lambda$, of minimal length.

For $\lambda, \mu \in X^+$ we set $w = w_\lambda$ and

$$P_{\mu, \lambda} = \sum_{w \in W_a; y^{-1}(\mu) = y^{-1}(\lambda)} (-1)^{\langle yw \rangle} P_{y, w}(1) \in \mathbb{Z}$$

where $P_{y, w}$ is the polynomial associated in [KL] to $y, w$ in the Coxeter group $W_a$.

From the definitions we see that $P_{\mu, \lambda} \neq 0 \implies \mu \leq \lambda, P_{\lambda, \lambda} = 1$. Hence for $\lambda, \mu \in X^+$ we can define $q_{\mu, \lambda} \in \mathbb{Z}$ by the requirement

$$\sum_{\nu \in X^+} P_{\mu, \nu} q_{\nu, \lambda} = \delta_{\mu, \lambda}$$

for any $\lambda, \mu$ in $X^+$. We have $q_{\mu, \lambda} \neq 0 \implies \mu \leq \lambda, q_{\lambda, \lambda} = 1$.

4. For any $\lambda \in X^+$ we can write uniquely $\lambda = \sum_{k \geq 0} p^k \lambda^k$ where $\lambda^k \in X^+_{\text{red}}$ for all $k \geq 0$ and $\lambda^k = 0$ for large $k$.

For any $\lambda \in X^+$ and any $k \in \mathbb{N}$ we define elements $E_\lambda^k \in \mathbb{Z}[X]$ by induction on $k$ as follows:

$$E_\lambda^0 = \sum_{w \in W} \epsilon_we^{w(\lambda + \rho)} / \sum_{w \in W} \epsilon_we^{w(\rho)},$$
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(a) \[ E^k_\lambda = \sum_{\mu \in X^+} p_\mu \sum_{j \geq k} p^{j-k+\lambda_j} E^{k-1}_{\sum_{0 \leq j \leq k-2} p^j \lambda_j + \mu} \] if \( k \geq 1 \).

Note that \( E^k_\lambda \in \mathbb{Z}[X]^W \), the ring of \( W \)-invariants in \( \mathbb{Z}[X] \).

We show that for \( k \geq 0 \) we have

(b) \[ E^k_\lambda = \sum_{\mu \in X^+} q_\mu \sum_{j \geq k} p^{j-k+\lambda_j} E^{k+1}_{\sum_{0 \leq j \leq k-1} p^j \lambda_j + \mu} \sum_{\nu \in X^+} h_{\nu} \sum_{h \geq k} p^{h-k-\mu} \mu h_k - k E^k_{\sum_{0 \leq h \leq k-1} p^h \lambda h + k \nu} \]

For any \( \mu \in X^+ \) and any \( k, h \geq 0 \),

\[ ( \sum_{j:0 \leq j \leq k-1} p^j \lambda_j + p^k \mu )^h \]

is equal to \( \lambda^h \) if \( 0 \leq h \leq k-1 \) and to \( \mu^{h-k} \) if \( h \geq k \); hence, by (a), we have

\[ E^{k+1}_{\sum_{j:0 \leq j \leq k-1} p^j \lambda_j + \mu} = \sum_{\nu \in X^+} p_\nu \sum_{h \geq k} p^{h-k-\mu} h_k - k E^k_{\sum_{0 \leq h \leq k-1} p^h \lambda h + k \nu} \]

Thus the right hand side of (b) is

\[ \sum_{\mu \in X^+} q_\mu \sum_{j \geq k} p^{j-k+\lambda_j} \sum_{\nu \in X^+} p_\nu \sum_{h \geq k} p^{h-k-\mu} h_k - k E^k_{\sum_{0 \leq h \leq k-1} p^h \lambda h + k \nu} \]

\[ = \sum_{\nu \in X^+} \delta_\nu \sum_{j \geq k} p^{j-k+\lambda_j} E^k_{\sum_{0 \leq h \leq k-1} p^h \lambda h + k \nu} \]

\[ = E^k_{\sum_{0 \leq h \leq k-1} p^h \lambda h + k \sum_{j \geq k} p^{j-k+\lambda_j}} \]

This proves (b).

By induction on \( k \) we see, using (a), that for any \( k \geq 1 \) we have

\[ E^k_k = \sum_{\mu_0, \mu_1, \ldots, \mu_{k-1} \in X^+} p_{\mu_0, \lambda^0 + \mu_1 \lambda^1 + \mu_2 \lambda^2 + \cdots + \mu_{k-2} \lambda^{k-2} + \mu_{k-1}} \sum_{j \geq k} p^{j-k+1+\lambda_j} E^0_{\mu_0} \]

By induction on \( k \) we see, using (b), that for any \( k \geq 1 \) we have

\[ E^k_\nu = \sum_{\nu \in X^+} q_\nu \sum_{j \geq k} p^{j-k+1+\lambda_j} E^0_{\nu} \]

\[ \times E^k_{\nu + \nu_0 + \cdots + \nu_{k-1}} \]

(d) \[ \times E^{k_0}_{\nu_0 + \cdots + \nu_{k-1}} \]
5. Let \( \lambda \in X^+ \). We can find \( n \geq 0 \) such that \( \lambda^n = \lambda^{n+1} = \cdots = 0 \). If \( k \geq n \) we have
\[
E^k_\lambda = E^{k+1}_\lambda.
\]
Indeed, if \( \mu \in X^+ \) and \( q, \sum_{j:j \geq k} p^{j-k} \lambda_j \neq 0 \) we have \( q, 0 \neq 0 \) hence \( \mu \leq 0, \mu = 0 \) and \( q, 0 = 1 \); it follows that
\[
E^k_\lambda = E^{k+1}_\lambda = E^{k+1}_\lambda.
\]
Thus we can set \( E^{\infty}_\lambda = E^k_\lambda \) for large \( k \). Clearly, if \( \lambda \in X^+_{\text{red}} \), then \( E^1_\lambda = E^2_\lambda = \cdots = E^{\infty}_\lambda \).

Letting \( k \to \infty \) in 4(c),(d), we deduce that
\[
E^{\infty}_\lambda = \sum_{\mu_0, \mu_1, \mu_2, \ldots \text{ in } X^+; \mu_h = 0 \text{ for large } h} (P_{\mu_0, \lambda^0} + P_{\mu_1, \lambda^1} + P_{\mu_2, \lambda^2} + p \mu_3 + \cdots) E^0_{\mu_0},
\]
(note that for large \( h \) we have \( P_{\mu_h, \lambda^h} + p \mu_{h+1} = 0 \) so that the infinite product makes sense) and
\[
E^0_\lambda = \sum_{\nu_0, \nu_1, \nu_2, \ldots \text{ in } X^+} (q_{\nu_0, \lambda} q_{\nu_1, (\nu_0 - \nu_0^0)/p} q_{\nu_2, (\nu_1 - \nu_1^0)/p} \cdots) E^{\infty}_{\nu_0^0 + \nu_1^0 + \cdots}
\]
(note that for large \( h \) we have \( \nu_h = 0 \) hence \( q_{\nu_{h+1}, (\nu_h - \nu_h^0)/p} = q_0, 0 = 1 \) so that the infinite product makes sense).

6. It is known since the early 1990’s that, if \( p \) is not very small, then the conjecture 8.2 in [L2] on quantum groups at a \( p \)-th root of 1 holds. In particular for \( \lambda \in X^+ \), the element \( E^1_\lambda \) describes the character of an irreducible finite dimensional representation of such a quantum group and the tensor product theorem [L2, 7.4] holds for it.

Thus, if for any \( \xi = \sum_{\lambda \in X} c_\lambda e^\lambda \in Z[X] \) (with \( c_\lambda \in Z \)) and any \( h \geq 0 \) we set \( \xi^{(h)} = \sum_{\lambda \in X} c_\lambda e^{h \lambda} \in Z[X] \), then for any \( \lambda \in X^+ \) we have the equality
\[
(a) \quad E^1_\lambda = E^0_{\lambda^0} (E^1_{\sum_{j \geq 1} p^{j-1} \lambda_j})^{(1)}.
\]
We show by induction on \( k \geq 1 \) that for any \( \lambda \in X^+ \) we have
\[
(b) \quad E^k_\lambda = E^1_{\lambda^0} (E^1_{\lambda^1})^{(1)} \cdots (E^1_{\lambda^{k-1}})^{(k-1)} (E^0_{\sum_{j \geq k} p^{j-k} \lambda_j})^{(k)}.
\]
By (a), we can assume that \( k \geq 2 \). Using 4(a), it is enough to show that
\[
\sum_{\mu \in X^+} P_{\mu, \sum_{j:j \geq k-1} p^{j-k+1} \lambda_j} E^k_{\sum_{j:0 \leq j \leq k-2} p^j \lambda_j + p^{k-1} \mu} = E^1_{\lambda^0} (E^1_{\lambda^1})^{(1)} \cdots (E^1_{\lambda^{k-1}})^{(k-1)} (E^0_{\sum_{j \geq k} p^{j-k} \lambda_j})^{(k)}.
\]
Replacing here

\[ E^{k-1}_{\sum_{j, 0 \leq j \leq k-2} p^j \lambda^j + p^{k-1} \mu} = E^1_{\lambda^0}(E^1_{\lambda^1})(1) \ldots (E^1_{\lambda^{k-2}})(k-2)(E^{0}_{\sum_{j \geq k-1} p^j \lambda^j})^{(k-1)} \]

which is known from the induction hypothesis, we see that it is enough to show that

\[ \sum_{\mu \in X^+} P_{\mu} \sum_{j, \lambda_j \geq k-1} p^j \lambda_j \lambda_j^1 \left( E^0_{\sum_{j \geq k-1} p^j \lambda^j} \right)^{k-1} = E^1_{\lambda^{k-1}}(E^0_{\sum_{j \geq k} p^j \lambda^j})^{k} \]

Thus, it is enough to show that

\[ \sum_{\mu \in X^+} P_{\mu} \sum_{j, \lambda_j \geq k-1} p^j \lambda_j \lambda_j^1 \left( E^0_{\sum_{j \geq k-1} p^j \lambda^j} \right)^{(k-1)} = (E^1_{\lambda^{k-1}})(k-1)(E^0_{\sum_{j \geq k} p^j \lambda^j})^{(k)} \]

or that

\[ \sum_{\mu \in X^+} P_{\mu} \sum_{j, \lambda_j \geq k-1} p^j \lambda_j \lambda_j^1 \left( E^0_{\sum_{j \geq k-1} p^j \lambda^j} \right)^{(k-1)} = E^1_{\lambda^{k-1}}(E^0_{\sum_{j \geq k} p^j \lambda^j})^{(1)} \]

Using (a), the right hand side is \( E^1_{\lambda^{k-1}} \cdot E^0_{\sum_{j \geq k} p^j \lambda^j} \). This is equal to the left hand side, by 4(a). This proves (b).

Letting \( k \to \infty \) in (b) we obtain for any \( \lambda \in X^+ \):

\[ (c) \quad E^\infty_{\lambda} = E^1_{\lambda^0}(E^1_{\lambda^1})(1)(E^1_{\lambda^2})(2) \ldots \]

Note that for large \( h \) we have \( \lambda^h = 0 \) hence \( E^{1}_{\lambda^h} = 1 \) so that the infinite product makes sense. (We have also used that \( E^0_{\sum_{j \geq k} p^j \lambda^j} = E^0_0 = 1 \) for large \( h \).)

7. We now assume that \( p \geq c_G \). Then, by the first paragraph of no.6, we have

\[ (a) \quad [V_{\lambda}] = E^1_{\lambda} \text{ for any } \lambda \in X^+_{\text{red}}. \]

Using the Steinberg tensor product theorem [St] and (a), we see that for any \( \lambda \in X^+ \) we have

\[ [V_{\lambda}] = [V_{\lambda^0}][V_{\lambda^1}](1)[V_{\lambda^2}](2) \ldots = E^1_{\lambda^0}(E^1_{\lambda^1})(1)(E^1_{\lambda^2})(2) \ldots \]

Using this and 6(c) we deduce

\[ (b) \quad [V_{\lambda}] = E^\infty_{\lambda}. \]
8. We preserve the setup of no.7. In the identity

\[ E_\lambda^0 = \sum_{\mu \in X^+} q_{\mu, \lambda} E_\mu^1 \]

(see 4(b)) the coefficient \( q_{\mu, \lambda} \) can be interpreted as the multiplicity of an irreducible representation of a quantum group at a \( p \)-th root of 1 in a not necessarily irreducible representation of that quantum group. In particular we have

(a) \( q_{\mu, \lambda} \in \mathbb{N} \)

for any \( \lambda, \mu \) in \( X^+ \). We show by descending induction on \( k \) that for any \( \lambda \in X^+ \) and any \( k \geq 0 \) we have

(b) \( E_\lambda^k = [V_\lambda(k)] \)

for some \( V_\lambda(k) \in \text{Rep}G \).

If \( k \) is large, we have \( E_\lambda^k = E_\lambda^\infty \) and (b) follows from 7(b). Assume now that \( k \geq 0 \) and that (b) is known when \( k \) is replaced by \( k + 1 \). Then (b) holds for \( k \) by (a), 4(b) and the induction hypothesis.

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