New Type of Regular Black Holes and Particlelike Solutions from Nonlinear Electrodynamics

Alexander Burinskii
Gravity Research Group, NSI Russian Academy of Sciences,
B. Tulskaya 52, 113191 Moscow, Russia

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Sergi R. Hildebrandt †
Institut d’Estudis Espacials de Catalunya (IEEC/CSIC)
Edifici Nexus, Gran Capità 2-4, 08034 Barcelona, Spain

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Abstract
We show that the Bronnikov theorem on the nonexistence of regular electrically charged black holes can be circumvented. In the frame of nonlinear electrodynamics, we present the exact regular black hole solutions of a hybrid type. They are electrically charged, but contain a ‘dual’ core confining a polarization of magnetic charges.

The considered example is based on a modification of the Ayón-Beato & García solution. It represents a very specific realization of the idea on confinement based on dual electrodynamics and dual superconductivity.

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*E-mail: bur@ibrae.ac.ru
†Temporary address: Avda. Marítima, 49, P041E. Candelaria, 38530. S/C. de Tenerife. Spain. E-mail: hildebrandt@ieec.fcr.es
1 Introduction

Last years considerable interest in regular black hole solutions has been renewed [1, 2, 3], and in particular, to those based on nonlinear electrodynamics (NED) [4, 5, 6, 7, 8]. Analyzing the Ayón-Beato & García black hole solutions from nonlinear electrodynamics [4, 5, 6], Bronnikov presented three important results [7]:

(i) - a version of no-go theorem claiming the nonexistence of the regular electrically charged black holes and particlelike solitons with regular center;

(ii) - the existence of cusps and branches in the Lagrangian used for the Ayón-Beato – García solution [4]; and

(iii) - a class of exact regular magnetically charged black holes and solitonic solutions.

He showed that the branch, realized in the Ayón-Beato – García solution near the core, does not tend in the weak field limit to the Maxwell theory and presented two theorems leading to the conclusion that regular black hole and solitonic solutions can be only magnetically charged.

The unusual properties of the Ayón-Beato – García solution obtained by Bronnikov have captured our attention in connection with recent suggestions on a phase transition which has to occur for regular black hole and particlelike solutions in the core region [1, 2, 3]. In this work we show that the conditions of the Bronnikov theorem can be circumvented by dealing with the models having a phase transition near the core of solution, when electric field does not extend to the center of the solution.

After some modification of the Ayón-Beato – García solutions, we obtains a new type of regular, electrically charged black hole solutions. The main peculiarity of these solutions is a ‘magnetic’ core described by dual electrodynamics. This class of solutions deserves interest not only as an explicit example of the regular electrically charged black hole solution, but also as an example of particlelike solution with a very specific realization of the ideas on quark confinement based on dual electrodynamics [4] and dual superconductivity [10, 11, 12, 13, 14].

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1 See also list of references in [1].
2 As well as wormholes or horns.
2 NED and F-P Dual representations

The action for NED in general relativity is

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} [R - \mathcal{L}(F)], \]  

(1)

where \( R \) is the scalar curvature, \( F = F_{\mu\nu}F^{\mu\nu} \), \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( \mathcal{L}(F) \) is a nonlinear function.

The equations, following from Eq. (1) are

\[ \nabla_\mu (L_F F^{\mu\nu}) = 0, \quad \nabla_\mu \star F^{\mu\nu} = 0. \]  

(2)

where \( L_F = d\mathcal{L}/dF \), and the equations \( \nabla_\mu \star F^{\mu\nu} = 0 \) are the Bianchi identities.

By introducing the electric intensity \( E \) and magnetic induction \( B \),

\[ E = \{F_{i0}\}, \quad B = \{\star F_{i0}\} = \frac{1}{2} \eta_{0}^{\alpha\beta} F_{\alpha\beta}, \]  

(3)

one can obtain from Bianchi identities \( \nabla_\mu \star F^{\mu\nu} = 0 \) the pair of Maxwell equations \( \nabla B = 0; \quad \nabla \times E = -\partial B/\partial t \). The dynamic equations \( \nabla_\mu (L_F F^{\mu\nu}) = 0 \) can be expressed via the tensor

\[ P^{\mu\nu} = \partial \mathcal{L}/\partial F_{\mu\nu} = L_F F^{\mu\nu}, \]  

(4)

in the form

\[ \nabla_\mu P^{\mu\nu} = 0; \]  

(5)

that, in the terms of the electric induction \( D \) and magnetic intensity \( H \),

\[ D = \{P_{i0}\} = L_F \{F_{i0}\}, \quad H = \{\star P_{0i}\} = L_F \{\star F_{0i}\}, \]  

(6)

yields the second pair of the Maxwell equations \( \nabla D = 0; \quad \nabla \times H = \partial D/\partial t \). These equations show that in NED \( L_F \) plays the role of electric susceptibility \( \epsilon = L_F \) and the magnetic permeability is given by \( \mu = 1/L_F \).

Relation (4) allows one to rewrite the field equations (2) in an equivalent dual P form

\[ \nabla_\mu (P^{\mu\nu}) = 0, \quad \nabla_\mu \frac{1}{L_F} \star P^{\mu\nu} = 0. \]  

(7)

\[ ^3\text{We use signature } -+++. \text{ The Hodge star operator is given by } \star F^{\mu\nu} = \frac{1}{2} \eta_{\mu\nu\alpha\beta} F_{\alpha\beta}, \]  

\[ \eta_{\mu\nu\alpha\beta} \text{ is completely skew-symmetric, } \eta_{0123} = \sqrt{-g}. \]
It should be mentioned that it is a dual description of the same physical system. This description can be obtained from the electromagnetic Lagrangian \( \mathcal{H}(P) \) (expressed via \( P = P_{\mu\nu}P^{\mu\nu} = L_P^2 F \)) determined by Legendre transformation \([3, 7, 15]\)

\[
\mathcal{H}(P) = 2FL_F - \mathcal{L}. \tag{8}
\]

The following relations can be easily obtained

\[
\mathcal{L} = 2PH_P - \mathcal{H}; \quad H_P = d\mathcal{H}/dP = 1/L_F. \tag{9}
\]

The Hodge star duality operation allows one to give a description of the field intensities via dual components \( \tilde{F}_{\mu\nu} = \ast F_{\mu\nu} = \frac{1}{2}\eta_{\mu\alpha\beta}F_{\alpha\beta} \). Since \( \ast\ast = -1 \), by using the Hodge dual description the following relations hold:

\[
F = -\tilde{F}; \quad P = -\tilde{P}; \tag{10}
\]

and also

\[
L_F = -L_{\tilde{F}}; \quad H_P = -H_{\tilde{P}}; \quad \tilde{L}(\tilde{F}) = L(-F); \quad \tilde{\mathcal{H}}(\tilde{P}) = \mathcal{H}(-P); \tag{11}
\]

and the equations [2] take in this description the form

\[
\nabla_\mu(L_F \ast \tilde{F}^{\mu\nu}) = 0, \quad \nabla_\mu \tilde{F}^{\mu\nu} = 0. \tag{12}
\]

Note that the Hodge dual description changes sign of \( L_F \) that can lead to nonphysical values of electric susceptibility \( \epsilon < 0 \).

Vice versa, if the Lagrangian leads to \( L_F < 0 \) it can be considered as a sign to pass to another, physically valid (Hodge) dual description.

### 3 Nonexistence of Electrically Charged Regular Black Holes?

The Bronnikov no-go theorem is based on the assumption that the NED Lagrangian has to lead to Maxwell theory at small \( F \): \( \mathcal{L}(F) \to F \) as \( F \to 0 \). This is equal to \( L_F \to 1 \) at small \( F \). He shows that real behavior of electrically charged solution near the center leads to finite values of \( |FL_F| \)

\[\text{See, for example, [10].}\]
and to divergence of $|F|L_F^2$. Therefore at the center the ratio $\frac{|F|L_F^2}{FL_F} \equiv L_F$ tends to infinity that is non-Maxwellian behavior.

This derivation is correct. However, there are two points which throw doubts on the initial postulates. The first one is connected with the assumption that the electromagnetic field is extended up to the center of the solution. At least in flat space there is a remarkable example of the electrically charged solution, present only in the region $r \geq a$. It is the classical Dirac electron model, as a charged sphere. The second argument is connected with the demands of Maxwellian behavior. Indeed, the condition $L_F \to 1$ as $F \to 0$ (to recover Maxwell theory) is reasonable at spatial infinity. However, at the core, although the gravitational field is regular everywhere and becomes weaker as one approaches the center (the metric tends to flat space-time), the spatial volume is much smaller (on many orders) than in the asymptotic limit where gravity is again weak and where $L_F \to 1$ makes sense. Thus both types of boundaries are essentially different. Another argument can be a strong vacuum polarization and the presence of other (maybe virtual quantum) fields which have a straight relation to the formation the nonvacuum values of $L_F = \epsilon$. Therefore $\mathcal{L}(F)$ can have a nonlocal behavior depending not only on the value of $F$ but also on full field configuration and proximity to the core.

If one restricts oneself to Lagrangian densities which are local functions of $F$, then, Bronnikov is completely right. But assigning essentially the same physics to the core region as to the asymptotic region (in electromagnetic terms) is not convincing enough. Therefore his restrictions for regular solutions are too especial to be accepted in general. This point of view is suggested in fact by the given Bronnikov analysis showing the existence of cusps and branches in the Lagrangian of the Ayón-Beato – García solution. In fact, his analysis shows that the Ayón-Beato – García Lagrangian has different behavior at the core and far outside the body, although $F \to 0$ in both cases.

The Bronnikov theorem 2 tries to exclude this case showing nonexistence of a horn in space-time. However, there can be other (nonmetrical) mechanisms restricting the penetration of the electromagnetic field into region $r < a$, in particular, superconducting properties of core or a special behavior in NED as it will be shown further.
The electrically charged regular black hole solution has been given by Ayón-Beato – García in [5]. Most investigations in nonlinear electrodynamics are connected with the Born-Infeld type of nonlinearity (see, e.g., [17, 18, 20, 19] and references therein). However, as it was mentioned for regular black hole solutions in [5], the “...Born-Infeld Lagrangian is useless in this topic, since it gives rise to a singular black hole solution, at least in the case of spherical symmetry [21]...”, and Ayón-Beato – García start from the nonlinear function

\[ H(P) = \frac{P}{\cosh^2\left\{s(-2q^2P)^{1/4}\right\}}, \quad (13) \]

where \( s = |q|/2m \) (\( q \) is the electric charge and \( m \) is the mass of the body).

Note, that particular form of nonlinearity \( H(P) \) is not essential, and existence of regular black hole solutions is determined by the asymptotic properties of this function. What is important is the Maxwellian behavior on large distances, \( H(P) \equiv P \) at small \( P \), and a rapid fall-off of this function by \( P \to \infty \). These demands lead to the existence of an extremum of the function \( H \), and moreover to the existence of at least one sector where \( H_P \) is negative. For instance, we use for illustration the simplest function of this

![Figure 1: Dependence \( \mathcal{L}(F) \) for Ayón-Beato – García solution.](image-url)
\[ H(P) = P \exp\{-|P|\}. \tag{14} \]

The corresponding Lagrangian in L description is \[ \mathcal{L}(F) = 2H_P^2 - \mathcal{H}(P), \]
where \( P \) has to be expressed via \( F = P/L_F^2 \), (see Fig. 1 and upper graph in Fig. 2).

It yields the system of electromagnetic equations
\[ \nabla_\mu (P^{\mu\nu}) = 0; \tag{15} \]
and Einstein equations \( G^\nu_\mu = T^\nu_\mu \), where the energy-momentum tensor is given by
\[ T^\nu_\mu = \frac{1}{2}(4L_F F_{\mu\alpha} F^{\nu\alpha} - \delta^\nu_\mu \mathcal{L}). \tag{16} \]

In the case of spherical symmetry the electrically charged solution of Eq. (15) is
\[ P_0 r = -P r_0 = q/r^2 \tag{17} \]
or
\[ F_0 r = -F r_0 = L_F^{-1} q/r^2; \tag{18} \]
the other components being equal to zero. Thus \( P = -2q/r^4 \). The Einstein equations can easily be solved leading to the mass term \( M(r) = m[1 - \tanh(q^2/2mr)] \) and to the metric
\[ ds^2 = -[1 - 2M(r)/r]dt^2 + [1 - 2M(r)/r]^{-1} dr^2 + r^2 d\Omega^2, \tag{19} \]
which is regular by \( r \) going to zero.

As mentioned by Bronnikov [5], the Lagrangian given by the smooth and regular function \( \mathcal{H}(P) \) displays cusps and branches in F description. Appearance of cusps at the points \( P_1 \) and \( P_3 \) is connected with multivaluedness of the inverse map \( P(F) \) (see Fig. 2) and is a general feature of the considered class of nonlinear functions. Very specific behavior of function \( L(F) \) by large \( P \), on the section \( P_2 - P_3 - P_4 \) is determined by the following phenomenon: The magnetic permeability \( \mu = H_F \) is going to zero as \( P \rightarrow P_2 \), the point of \( \mathcal{H} \) extremum. Consequently, electric susceptibility \( \epsilon = 1/H_F \) is divergent at this point. The sphere, determined by the equation \( P = P_2 \) looks like an ideal conducting surface, and electromagnetic field \( F_{\mu\nu} \) (and \( F \)) will tend to zero approaching this surface. The solution, analytically extended inside
Figure 2: Graphs of $\mathcal{H}(P)$ and $F(P)$ for Ayón-Beato – García solution and modified solution $F_{\text{mod}}(P)$.

this sphere departs drastically from a Maxwellian behavior. Nevertheless it still has reasonable properties in many respects. For instance, the electromagnetic field $F_{\mu\nu}$ and the invariant $F$ go both to zero as the very center of the solution is approached (although $L_F \to \infty$ at $P_2$ and $P_4$) and the metric is regular everywhere, leading to regular black hole solutions.

In spite of these properties, the physics inside the core looks rather unusual since the value of electric susceptibility $\epsilon = 1/H_P$ is negative there and does not correspond to any known physical mediums \[16\]. Combining NED and general relativity, one can show from the system of equations (15), (16) that the corresponding contribution of effective charges to the metric will be of opposite sign in the regions with negative $L_F$, as if the charge would be imaginary valued there.

5 Modification of the Ayón-Beato – García Solution

The fact that regularity of black hole solutions can be reached without troubles in the magnetically charged solutions \[7\] suggests the idea to combine
the electrically and magnetically charged solutions by using a phase transition near the core to magnetic phase, regularizing the black hole singularity. In this case, the existence of different branches in the Ayón-Beato – García solution represents a perfect possibility for realization of this idea by means of matching external and internal solutions with necessary properties. As we shall see, the transfer to dual electromagnetic phase near the core of the Ayón-Beato – García solution allows also one to avoid abnormal branch \((P_2 - P_3 - P_4)\) with negative \(L_F\).

Looking on the graph \(L(F)\) (Fig. 1) one sees that it can only be achieved by replacing this branch by the branch obtained by a mirror map with respect to the \(L\) axis, as it is shown in Fig. 3. This replacement takes place in the region of strong field \(|P| > |P_2|\), i.e., inside of the sphere \(r < r_0 = |2q|^{1/4}\) surrounding the singular point of \(P(r)\).

As we shall see this replacement corresponds to “dual” electrodynamics with a polarization of magnetic charges inside the sphere \(r = r_0\), while the solution for \(r > r_0\) survives to be the electrically charged Ayón-Beato – García one. Matching of the modified internal solution and the external Ayón-Beato – García one is smooth with respect to the metric and physical field intensities.
The properties of modified Lagrangian $\tilde{L}(\tilde{F})$ and the modified relations will be the following.

1. The region of definition of Lagrangian $\tilde{L}$ is extended to positive values of the invariant $\tilde{F} = B^2 - E^2$, and modified electric susceptibility, $\tilde{\epsilon} = \tilde{L}_{\tilde{F}} = -L_F$, is positive.

2. Matching of the new branch with the initial one is smooth and occurs at the point $P_2$ where $F = \tilde{F} = 0$ and $L(F)|_{P_2} = \tilde{L}(\tilde{F})|_{P_2}$.

3. The modified dual intensities are related as
   \[ \tilde{P}^{\mu\nu} = \tilde{L}_F \tilde{F}^{\mu\nu}; \quad \tilde{P} = (\tilde{L}_F)^2 \tilde{F}. \]  
   At the point $P_2$ electric susceptibility is divergent $L_F|_{P_2} = \tilde{L}_F|_{P_2} = +\infty$, while the magnetic permeability $\mu = 1/L_F$ is bounded and $C^1$.

4. As far as in the modified region $\tilde{F} > 0$ we get also $\tilde{P} > 0$. On the other hand $P = H^2 - D^2$ is negative in the region $r > r_0$ (as in the case of an electrically charged solution). Therefore the modified function $P(r)$ will be discontinuous. The change of sign of the invariant $P$ at the point $P_2$ is evidence of a transfer to magnetic phase by $r < r_0$.

Due to the Hodge star operation identity $\star \star = -1$, the equality $\tilde{P} = -P$ can be resolved as $\tilde{P}^{\mu\nu} = \pm \star P^{\mu\nu}$. Whence, the relation (3) shows that the modified Ayón-Beato – García solution (18) by $r < r_0$ describes a field of magnetic induction in radial direction $n$

\[ \tilde{\mathbf{B}} = \mp qn\tilde{L}_F^{-1}/r^2. \]  

(21)

Magnetic induction $\mathbf{B}$ characterizes the true intensity of the field strength in media [16]. In modified solution it tends to zero at $r = 0$ and at $r = r_0$. Contrary to the standard Maxwell equation $\nabla \mathbf{B} = 0$ describing the absence of magnetic charges, we obtain here the magnetic charge distribution

\[ \nabla \tilde{\mathbf{B}} = J_{mag} = \mp (q/r^2)n\nabla \tilde{L}_F = \mp (q/r^2)\tilde{L}_F \frac{d\tilde{F}}{dr}. \]  

(22)

displaying realization of dual electrodynamics in the core region and singular concentration of magnetic charges at $r = 0$ and $r = r_0$.

The initial and modified dependencies $\mathcal{H}(P)$, $F(P)$, and $L(F)$ are given in figs. 1 and 3.
5.1 Stress-energy tensor and metric.

We use the Kerr-Schild form of metric [1, 22]. In the case of spherical symmetry

$$g_{\mu\nu} = \eta_{\mu\nu} + [f(r)/r^2]k_\mu k_\nu,$$

where $\eta_{\mu\nu}$ is the auxiliary Minkowski metric and $k_\mu = \{1, n\}$. Co- and contravariant forms of the metric in the Kerr angular coordinates $\{t, r, \theta, \phi\}$ are given in the Appendix. We have

$$F^{01} = -F_{01}; \quad \tilde{F}_{23} = r^2 \sin \theta F^{01}; \quad \tilde{F}^{23} = \tilde{F}_{23}/(r^4 \sin^2 \theta);$$

which, taking into account Eq. (18), yield

$$\tilde{F} = 2\tilde{F}^{23} \tilde{F}_{23} = 2(F_{01})^2 = 2L_F^2 q^2/r^4.$$  \hspace{1cm} (25)

Similarly,

$$\tilde{P} = 2\tilde{P}^{23} \tilde{P}_{23} = 2(P_{01})^2 = 2q^2/r^4.$$  \hspace{1cm} (26)

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\textsuperscript{6}See also [23] where it was shown that the static solutions of the Einstein equations with spherical symmetry and $T^0_0 = T^1_1$ belong to the Kerr-Schild class.

In this section we use slightly different units, adapting to the works [22, 3, 11] and setting $G = 1$. The reason for this is the possibility of using the Kerr-Schild formalism for further extension to rotating solutions.
The modified energy-momentum tensor inside the core acquires the form

\[ T^{(\text{in})}_{\mu} = -\frac{1}{2}(\delta^\nu_{\mu}\tilde{L} - 4\tilde{P}_{\mu\alpha}\tilde{P}^{\nu\alpha}/\tilde{L}) = -\frac{1}{2}\text{diag}\{\tilde{L}, \tilde{L}, \tilde{L} - 2\tilde{P}/\tilde{L}, \tilde{L} - 2\tilde{P}/\tilde{L}\}. \]  

(27)

Outside the core, by \( r > r_0 \) it is

\[ T^{(\text{ext})}_{\mu} = -\frac{1}{2}(\delta^\nu_{\mu}L - 4P_{\mu\alpha}P^{\nu\alpha}/L_F) = -\frac{1}{2}\text{diag}\{L - 2P/L_F, L - 2P/L_F, L, L\}, \]  

(28)

where \( P = -\tilde{P} = -2q^2/r^4 \). As far as contributions of terms \( 2P/L_F \) and \( 2\tilde{P}/\tilde{L}_F \), together with their first derivative, tend to zero approaching \( r_0 \), one sees that these expressions match smoothly at \( r_0 \).

\[ T^\nu_{\mu} = (8\pi)^{-1}\text{diag}\{-2G, -2G, D + 2G, D + 2G\}, \]  

(29)

where (see the Appendix)

\[ D = -f''/r^2, \quad G = (f/r)'/r^2 \]  

(30)

Figure 5: Radial dependencies of \(-L\) and \(\mathcal{H}\).

Comparing corresponding expressions for the generalized Kerr-Schild class of metrics

\[ T^\nu_{\mu} = (8\pi)^{-1}\text{diag}\{-2G, -2G, D + 2G, D + 2G\}, \]  

(29)

where (see the Appendix)

\[ D = -f''/r^2, \quad G = (f/r)'/r^2 \]  

(30)
and \( \therefore \equiv d/dr \) gives the relations:

For external field

\[
- \mathcal{H}(r) = \frac{(f/r)'}{2\pi r^2} = 2\rho(r),
\]

(31)

where \( \rho(r) \) is the energy density in this region.

For the external part of the function \( f(r) \)

\[
f(r) = r M_{\text{ext}}(r), \quad M_{\text{ext}}(y) = -2\pi \int_{r_0}^{y} \mathcal{H}(r) r^2 dr + C,
\]

(32)

where \( C \) is the integration constant.

For function \( f(r) \) inside the core, \( r < r_0 \), we obtain

\[
\tilde{L}(r) = \frac{(f/r)'}{2\pi r^2} = 2\rho(r),
\]

(33)

that gives rise for the internal region to

\[
f(r) = r M_{\text{int}}(r), \quad M_{\text{int}}(y) = 2\pi \int_{0}^{y} \tilde{L}(r) r^2 dr.
\]

(34)

Each of the functions \( M_{\text{ext/int}}(y) \) plays the role of the mass confined inside the sphere \( y < r \). Constant \( C \) can be determined by matching both sectors of \( f(r) \) at \( r = r_0 \) that yields \( C = M_{\text{int}}(r_0) \). This matching is smooth as far as \( \mathcal{H}(r_0) = -\tilde{L}(r_0) \), thus the metric (23) is smooth everywhere.

By analyzing dependencies \( L(r) \) and \( \mathcal{H}(r) \), Figs. 4 and 5., one sees that there is a core region \( (r < 0.6r_0) \) which does not give contribution to mass at all. The space is flat in this core since \( f(r) \approx 0 \) there. The region \( 0.6 \leq r/r_0 \leq 1 \) is a transitional one with a dual magnetic phase, whereas the region \( 1 \leq r/r_0 \leq 1.6 \) is one with an electric phase. In the region \( r > 1.6r_0 \) we have \( L \sim 1 \), it is practically a Coulomb phase.

Let us estimate characteristic scale \( r_0 \) for the Ayón-Beato – García type of nonlinearity (13). Radius \( r_0 \) is determined by the equation \( H_P = 0 \), or by the root of equation \( \zeta \tanh(\zeta) = 2 \). It yields \( \zeta = 2.066 \), and

\[
r_0 = \frac{2^{1/4}q^2}{2m\zeta} = 0.48r_e,
\]

(35)

where \( r_e = q^2/2m \) is a classical electromagnetic radius.
For comparison, in the Born-Infeld theory (in our notations), Lagrangian is

\[ \mathcal{L}_{BI} = \frac{1}{b^2} (\sqrt{1 + 2b^2 F} - 1), \]

and has a dual Lagrangian

\[ \mathcal{H}_{BI} = \frac{1}{b^2} (1 - \sqrt{1 - 2b^2 P}). \]

The typical NED branches are absent here, and the sphere \( H_P = 0 \) shrinks to the point \( r = 0 \). Therefore the considered above type of solutions is absent in Born-Infeld theory.

Black holes in Born-Infeld theory were actively studied (see [18] which includes the original references). The Born-Infeld black hole solution has finite total energy \( E = m \approx 1.236e^2/r_0 \) and a smooth distribution of effective charge \( \rho_{eff} = \frac{1}{4\pi} \text{div} \mathbf{E} \) in the region \( r_0 = \sqrt{eb} \). Parameter \( b^{-1} = e/r_0^2 \) characterizes a critical value of the electromagnetic field.

Considering particlelike solutions, one sees that, similarly to NED, the effective radius of the elementary particle in Born-Infeld theory is close to classical electromagnetic one

\[ r_0 = 2.472r_e \sim 10^{-13} \text{ cm}. \]

However, in contrast to NED, the Born-Infeld solution has a \( \delta \)-function source at the origin, leading to a conical singularity in curvature. Therefore the Born-Infeld theory, as well as NED, has classical radius \( r_e \) as one of the characteristic scales. However, regularizing (or smearing) the black hole singularities, both the theories display that their minimal (space) scale is extended up to Planck sizes.

On the other hand, the Born-Infeld theory can be derived from open string theory [19, 18], and parameter \( b \) takes in this case the value \( b = 2\pi\alpha' = T^{-1} \) defined by Regge slope \( \alpha' \). On the level of particlelike solutions, it does not contradict with \( b^{-1} \sim e/r_0^2 \), leading to approximate value of the fine structure constant \( \alpha \approx \frac{1}{4\pi} \) (see, e.g, [18]). However, in the modern superstring theory tension \( T_{(ss)} \) defines the critical scale for all field strength, including curvature, and characteristic radius \( r_{crit} = \sqrt{\alpha'_{(ss)}} \) corresponds to the Planck scale. It shows that the scale range of nonlinearity in NED action is much larger than the Planck scale that governs higher order corrections in curvature.
6 Concluding remarks

The Ayón-Beato – García solution contains distribution of electric charges \( J^\nu_e = \nabla_\mu F^{\mu\nu} \) near the sphere \( r = r_0 \) and in the center. In the presented modified solution the charges inside the sphere \( r = r_0 \) are magnetic \( J^\nu_m = \nabla_\mu \tilde{F}^{\mu\nu} \) fulfilling the Dirac idea on dual electrodynamics [7]. These charges are confined inside the sphere \( r = r_0 \), forming a magnetic vacuum polarization. For the external observer this solution exhibits only electric charge. Since vacuum of the core expels electric charges one can speculate that it possesses superconducting properties, whereas the vacuum of the external observer demonstrates the dual superconductivity, expelling the magnetic charges. The considered matching of the electric and magnetic phases corresponds to a sharp phase transition between these vacua (London limit of superconductivity).

The similar baglike structures with a phase transition to dual superconductivity (approximate solutions) were recently discussed for regular rotating black holes to supergravity [3]. It was assumed that superconductivity is caused by the Higgs (chiral) fields forming two dual (supersymmetric) vacuum states and a smooth domain wall interpolating between them. It is remarkable that nonlinear electrodynamics allows one to get \textit{exact and self-consistent} solutions of this sort belonging to the Kerr-Schild class of metrics. However, it could seem wonderful since the Higgs fields providing the vacuum states and phase transition are absent here. On the other hand, the NED Lagrangian \( L(F) \) can be represented in the form \( \frac{L(F)}{F} F \) showing that the factor \( \frac{L(F)}{F} \) can be identified as a dilaton (corresponding to a metric on the moduli space in supergravity). It allows one to suppose that NED can be assumed as a limiting case of supergravity when the potential is very sharp and chiral fields are dropped out. The presented exact solution from NED shows that dilaton has to be very essential in the baglike models.

A parallelism between the above baglike models and the field models of dual superconductivity should also be mentioned. Indeed, there are known the dual superconductivity models of two type: the models with Higgs fields, leading to solutions with a soft phase transition (finite thickness of transitional layer), see, e.g., [12, 13]; and the similar to NED models without Higgs fields, see, e.g., [11, 14], demonstrating the sharp phase transition connected with the London limit of superconductivity. The field models of the
first type represent in fact a truncation of the four-dimensional supergravity, while the models of the second type represent, apparently, its singular limit. However, these questions demand more detailed consideration elsewhere.

It would also be very interesting to look on NED from a high dimensional point of view, getting it as a result of compactification or as a four-dimensional slice of a multidimensional space-time in the spirit of early brane world work [24].

Finally, taking into account nontrivial topology of the Kerr geometry, its exceptional character with respect to dual rotations [15] and relationships to the spinning particlelike solutions (see, e.g., [25] and references therein), the treatment of corresponding regular rotating solutions is very important and could bring some surprising results. In particular, the appearance of a closed vortex string on the place of the Kerr singular ring is expected.

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**Appendix. The Kerr-Schild form of metric in the case of spherical symmetry**

The Kerr-Schild form of metric [22] in the case of spherical symmetry is

\[ g_{\mu \nu} = \eta_{\mu \nu} + 2h_k k_{\nu}, \]  

(39)

where \( \eta_{\mu \nu} \) is the auxiliary Minkowski metric, \( h = f(r)/r^2 \), and \( k_\mu = \{1, n\} \). In the Kerr angular coordinates \( \{t, r, \theta, \phi\} \) we have

\[
    g_{\mu \nu} = \begin{pmatrix}
    2h - 1 & 2h & 0 & 0 \\
    2h & 1 + 2h & 0 & 0 \\
    0 & 0 & r^2 & 0 \\
    0 & 0 & 0 & r^2 \sin^2 \theta
    \end{pmatrix},
\]  

(40)

\footnotetext{Note that effective Higgs fields can also be introduced in the models of the second type [14].}
where \( h(r) = f(r)/r^2 \), \( \sqrt{-g} = r^2 \sin \theta \), and the contravariant form of metric is

\[
g^{\mu\nu} = \begin{pmatrix}
-(1 + 2h) & 2h & 0 & 0 \\
2h & 1 - 2h & 0 & 0 \\
0 & 0 & 1/r^2 & 0 \\
0 & 0 & 0 & (r^2 \sin^2 \theta)^{-1}
\end{pmatrix}, \tag{41}
\]

The energy-momentum tensor for the generalized Kerr-Schild class of metrics is[1, 3]

\[
T^\nu_\mu = (8\pi)^{-1} \text{diag}\{-2G, -2G, D + 2G, D + 2G\}, \tag{42}
\]

where

\[
D = -f''/r^2, \quad G = (f/r)'/r^2, \tag{43}
\]

the prima means \( d/dr \), and energy density is \( \rho = 2G/8\pi \).

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