GORENSTEIN Q-HOMOLOGY PROJECTIVE PLANES

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Abstract. We present the complete list of all singularity types on Gorenstein
Q-homology projective planes, i.e., normal projective surfaces of second Betti
number one with at worst rational double points. The list consists of 58
possible singularity types, each except two types supported by an example.

1. Introduction

A normal projective surface with quotient singularities whose second Betti num-
ber is 1 is called a Q-homology projective plane. It is easy to see by considering
the Albanese map that such a surface has first Betti number 0, hence has the same
Betti numbers with the complex projective plane. A Q-homology projective plane
has Picard number 1 and either K is ample or K is numerically trivial. The min-
imal resolution of a Q-homology projective plane has p_g = q = 0. A Q-homology
projective plane is said to be Gorenstein if its singularities are all rational double
points. A combination of rational double points will be denoted by its singularity
type, a combination of A_n, D_n and E_n, e.g., the singularity type 2A_3 ⊕ 3A_1
denotes 2 rational double points of type A_3 and 3 of type A_1.

Gorenstein Q-homology projective planes with ample anti-canonical divisor are
called Gorenstein log del Pezzo surfaces of rank 1 and classified by Furushima [5],
Miyanishi and Zhang [14], and by Ye [17]; there are exactly 27 singularity types on
such surfaces and the surfaces with each singularity type were classified. The list
of these 27 types will be denoted by L_{K<0}.

Let L be the list of all singularity types on Gorenstein Q-homology projective
planes, and L_{K=0} (resp. L_{K≠0}) be the sub-list corresponding to the case where
the canonical class K is numerically trivial (resp. not trivial). Then L_{K≠0} contains two
sub-lists L_{K<0} and L_{K>0}, and the two sub-lists may overlap each other. Since a
Gorenstein Q-homology projective plane has rational double points only, it can be
shown (see Section 2) that the rank of the singularity type is at most 9 and is equal
to 9 if and only if K ≡ 0. Thus L is the disjoint union of L_{K=0} and L_{K<0}. Not
much has been known about L except the sub-list L_{K<0}. In this note, we obtain
the list L.

Theorem 1.1. The list L consists of 58 singularity types, each except the two types
2A_3 ⊕ A_2 ⊕ A_1 and A_3 ⊕ 3A_2 supported by an example (Subsection 2.2 and Table
2).

Date: December 27, 2013 and, November 19, 2014 in revised form.
2010 Mathematics Subject Classification. Primary 14J28; Secondary 14J17, 14J25.
Key words and phrases. Q-homology projective plane, Enriques surface, rational double point.
\[ \mathbf{L}_{K \neq 0} = \mathbf{L}_{K < 0} \quad (27 \text{ types}): A_8, A_7 \oplus A_1, A_5 \oplus A_2 \oplus A_1, 2A_4, 2A_3 \oplus 2A_1, 4A_2, E_8, E_7 \oplus A_1, E_6 \oplus A_2, D_9, D_6 \oplus 2A_1, D_5 \oplus A_3, 2D_4, A_7, A_5 \oplus A_2, 2A_4 \oplus A_1, E_7, D_6 \oplus A_1, D_4 \oplus 3A_1, A_5 \oplus A_1, 3A_2, E_6, A_3 \oplus 2A_1, D_5, A_4, A_2 \oplus A_1, A_1. \]

\[ \mathbf{L}_{K = 0} \quad (31 \text{ types}): A_9, A_8 \oplus A_1, A_7 \oplus A_2, A_7 \oplus 2A_1, A_6 \oplus A_2 \oplus A_1, A_5 \oplus A_4, A_5 \oplus A_3 \oplus A_1, A_5 \oplus A_2, A_5 \oplus A_2 \oplus 2A_1, 2A_4 \oplus A_1, A_4 \oplus A_3 \oplus 2A_1, 3A_3, 2A_3 \oplus 3A_1, D_9, D_8 \oplus A_1, D_7 \oplus 2A_1, D_6 \oplus A_3, D_6 \oplus A_2 \oplus A_1, D_6 \oplus 3A_1, D_5 \oplus A_4, D_5 \oplus A_3 \oplus A_1, D_5 \oplus D_4, D_4 \oplus A_3 \oplus 2A_1, 2D_4 \oplus A_1, E_8 \oplus A_1, E_7 \oplus A_2, E_7 \oplus 2A_1, E_6 \oplus A_3, E_6 \oplus A_2 \oplus A_1, 2A_3 \oplus A_2 \oplus A_1, A_3 \oplus 3A_2. \]

We do not know whether any of the last two types realizes or not.

The proof goes as follows.

When \( K \neq 0 \), we only use arithmetic argument and lattice theory, without geometric argument, to obtain a list of 27 types, hence must coincide with the list \( \mathbf{L}_{K < 0} \) of [14] and [17]. Our method is different from theirs which used geometric structure of log del Pezzo surfaces.

When \( K \equiv 0 \), the rank of the singularity type is 9 and we divide into two cases: the number of singular points is at least 5; at most 4. In the first case there is only one type, namely \( 2A_3 \oplus 3A_1 \), by the result of Hwang and Keum ([6], Theorem 1.1), whose proof requires not only arithmetic argument but also certain geometric argument, e.g., the possibility \( D_5 \oplus 4A_1 \) was ruled out by geometric argument. In the second case we only use arithmetic argument to obtain a list of 30 types, which together with the type \( 2A_3 \oplus 3A_1 \) yields the desired list \( \mathbf{L}_{K = 0} \). Since \( K \equiv 0 \) and the surface has rational double points only, the minimal resolution of the surface has \( p_g = q = 0 \) and \( K \equiv 0 \), hence is an Enriques surface. A singularity type gives on the minimal resolution a configuration of the same type of nine smooth rational curves. Each of the 31 types except the two types can be realized by contracting a configuration of nine smooth rational curves on a suitable Enriques surface. For the realization, see Subsection 2.2 where we use the Enriques surfaces constructed by Kondo [10]. We do not know how many Enriques surfaces with given configuration type exist in general. There are only finitely many Enriques surfaces with a configuration of type \( 2A_3 \oplus 3A_1 \) [9]. Section 3 provides the construction of one-dimensional families of Enriques surfaces with a configuration of type \( D_6 \oplus A_1 \) or \( E_7 \oplus 2A_1 \).

Remark 1.2. Consider the list \( \mathbf{L}_{K > 0} \) corresponding to the case where \( K \) is ample. Among the 27 possible singularity types in \( \mathbf{L}_{K \neq 0} \) the four types \( A_1, A_2 \oplus A_1, A_3 \oplus 2A_1 \) and \( D_4 \oplus 3A_1 \) can be ruled out by the orbifold version of Bogomolov-Miyaoka-Yau inequality (e.g. [11]). Thus the list \( \mathbf{L}_{K > 0} \) consists of at most 23 types. Since \( K \) is ample, the minimal resolution is a minimal surface of general type with \( p_g = q = 0 \). Examples were found by Keum [8] for the singularity types \( 3A_2 \) and \( 4A_2 \), and by Frapporti [11] for \( 2A_3 \oplus 2A_1 \) and \( 2A_3 \oplus A_1 \). For the remaining 19 types the existence is not known.

Throughout this paper, we work over the field \( \mathbb{C} \) of complex numbers.

2. Proof of Theorem 1.1

2.1. \( \mathbf{L} \) consists of at most 58 types. In this subsection, we prove the following.

Proposition 2.1. Let \( \bar{S} \) be a Gorenstein \( \mathbb{Q} \)-homology projective plane. Then the singularity type of \( \bar{S} \) is one of the 58 types listed in Theorem 1.1.
If $|\text{Sing}(\bar{S})| \geq 5$, then the singularity type of $\bar{S}$ must be $2A_3 \oplus 3A_1$ by the following result.

**Theorem 2.2.** ([6 Theorem 1.1], [3 Theorem 1.2]) Let $\bar{S}$ be a $\mathbb{Q}$-homology projective plane (with quotient singularities). Then $\bar{S}$ contains at most 5 singular points, and if $\bar{S}$ contains 5 singular points, then $\bar{S}$ is a Gorenstein log Enriques surface with singularity type $2A_3 \oplus 3A_1$.

Now assume that

(2.1) $|\text{Sing}(\bar{S})| \leq 4$.

Since $\bar{S}$ has Picard number one, either $\pm K_{\bar{S}}$ is ample or $K_{\bar{S}}$ is numerically trivial, hence $K_{\bar{S}}^2 \geq 0$ with equality iff $K_{\bar{S}} \equiv 0$. Let

\[ f : S \to \bar{S} \]

be the minimal resolution and

\[ R \subset H^2(S,\mathbb{Z})_{\text{free}} := H^2(S,\mathbb{Z})/(\text{torsion}) \]

be the sublattice spanned by the numerical classes of the exceptional curves of $f : S \to \bar{S}$. Then $S$ has second Betti number $1 + \text{rank}(R)$, $p_g = q = 0$, thus $K_S^2 = 9 - \text{rank}(R)$ by Noether’s formula. Since $\bar{S}$ has rational double points only, $K_{\bar{S}} = f^*K_S$ and

\[ K_S^2 = K_{\bar{S}}^2 = 9 - \text{rank}(R) \geq 0, \]

hence

(2.2) $\text{rank}(R) \leq 9$

with equality iff $K_{\bar{S}} \equiv 0$. The lattice $R$ and its singularity type will be denoted by the same notation, a direct sum of root lattices $A_n$, $D_n$ and $E_n$. It is easy to write down all possible singularity types satisfying the conditions (2.1) and (2.2). There are 127 possible types:

- the 57 types in Theorem 1.1 (except the type $2A_3 \oplus 3A_1$),
- the 56 types in Table 1,
- the 14 types in Lemma 2.4.

The 56 types in Table 1 are excluded by the following useful lemma.

**Lemma 2.3.** ([6 Lemma 3.3]) Let $\bar{S}$ be a $\mathbb{Q}$-homology projective plane (with quotient singularities). If $K_{\bar{S}} \neq 0$, then $|\det(R)| \cdot K_{\bar{S}}^2$ is a square number.

It remains to remove the 14 types in Lemma 2.4. The cohomology lattice $H^2(S,\mathbb{Z})_{\text{free}}$ is unimodular by Poincaré duality, and of signature $(1, \text{rank}(R))$. Any indefinite unimodular lattice is up to isometry determined by its signature and parity (i.e., whether it is odd or even) (cf. [2], p. 18).

Suppose that $\text{rank}(R) = 9$. Then $K_{\bar{S}} \equiv 0$ and the minimal resolution $S$ is an Enriques surface. The cohomology lattice $H^2(S,\mathbb{Z})_{\text{free}}$ of an Enriques surface is an even unimodular lattice $II_{1,9}$ of signature $(1, 9)$. Thus

\[ R \subset H^2(S,\mathbb{Z})_{\text{free}} \cong II_{1,9} \cong H \oplus E_8 \]

where $H$ is the even unimodular lattice of signature $(1, 1)$, and $E_8$ is the even unimodular lattice of signature $(0, 8)$.

Suppose that $\text{rank}(R) < 9$. Then the minimal resolution $S$ is either a rational surface or a minimal surface of general type with $p_g = 0$. The cohomology lattice
Lemma 2.4. (1) Let $R = A_6 \oplus A_3, A_6 \oplus 3A_1, A_4 \oplus A_3 \oplus A_2, A_4 \oplus 2A_2 \oplus A_1, A_7 \oplus A_2, D_5 \oplus 2A_2, D_5 \oplus A_2 \oplus 2A_1, D_4 \oplus A_5, D_4 \oplus 4A \oplus A_1, D_4 \oplus A_3 \oplus A_2, D_4 \oplus 2A_1 \oplus A_1, or E_6 \oplus 3A_1$. Then the lattice $R$ cannot be embedded in the lattice $H \oplus E_8$.

(2) Let $R = D_4 \oplus A_2$ or $D_4 \oplus 2A_2$. Then the lattice $R$ cannot be embedded in the lattice $I_{1,m}$, where $L = \text{rank}(R)$.

Proof. (1) Suppose that $R$ is embedded in $H \oplus E_8$. Let $R^\perp$ be the orthogonal complement of $R$ in $H \oplus E_8$. Then $(R \oplus R^\perp) \otimes \mathbb{Q}_p \cong (H \oplus E_8) \otimes \mathbb{Q}_p$ for any prime number $p$. Thus, by \cite{10} Theorem IV-9, their discriminants must be the same. It
follows that \( d_p(R) d_p(R^\perp) = d_p(R \oplus R^\perp) = d_p(H \oplus E_8) = -1 \) for every prime \( p \). Therefore \((d_p(R), d_p(R^\perp)) = 1\) for every prime \( p \). Again by [16] Theorem IV-9, their epsilon invariants must be the same, i.e., \( \epsilon_p(R \oplus R^\perp) = \epsilon_p(H \oplus E_8) \) for every prime \( p \). Since \( \epsilon_p(R^\perp) = 1 \), \( \epsilon_p(R \oplus R^\perp) = \epsilon_p(R) \epsilon_p(R^\perp) = \epsilon_p(R) \epsilon_p(R) = \epsilon_p(R) \). Since \( \epsilon_p(H \oplus E_8) = 1 \) for any prime \( p \), it is enough to show that \( \epsilon_p(R) = -1 \) for some prime \( p \). For the definition and basic properties of epsilon invariant, see [6] Section 6.

1) The case \( A_6 \oplus A_3 \)
In this case,
\[
\epsilon_7(R) = \epsilon_7(A_6) \epsilon_7(A_3)(d(A_6), d(A_3))_7 = 1 \cdot 1 \cdot (7, -1)_7 = -1.
\]

2) The case \( A_6 \oplus 3A_1 \)
Since \( \epsilon_7(A_6) = \epsilon_7(A_6) \epsilon_7(A_1)(d(A_6), d(A_1))_7 = 1 \cdot 1 \cdot (7, -2)_7 = -1 \) and \( \epsilon_7(2A_1) = (d(A_1), d(A_1))_7 = 1 \), we have
\[
\epsilon_7(R) = \epsilon_7(A_6 \oplus A_1) \epsilon_7(2A_1)(d(A_6 \oplus A_1), d(2A_1))_7 = (1) \cdot 1 \cdot (-14, 1)_7 = -1.
\]

3) The case \( A_4 \oplus A_2 \oplus A_2 \)
Since \( \epsilon_5(A_4 \oplus A_3) = \epsilon_5(A_4) \epsilon_5(A_3)(d(A_4), d(A_3))_5 = 1 \cdot 1 \cdot (5, -3)_5 = 1, \) we have
\[
\epsilon_5(R) = \epsilon_5(A_4 \oplus A_3) \epsilon_5(A_2)(d(A_4 \oplus A_3), d(A_2))_5 = 1 \cdot 1 \cdot (-5, 3)_5 = -1.
\]

4) The case \( A_4 \oplus 2A_2 \oplus A_1 \)
Since \( \epsilon_5(A_4 \oplus A_1) = \epsilon_5(A_4) \epsilon_5(A_1)(d(A_4), d(A_1))_5 = 1 \cdot 1 \cdot (5, -3)_5 = 1 \) and \( \epsilon_5(2A_2) = (d(A_2), d(A_2))_5 = -1 \), we have
\[
\epsilon_3(R) = \epsilon_5(A_4 \oplus A_1) \epsilon_3(2A_2)(d(2A_2), d(A_4 \oplus A_1))_3 = 1 \cdot (-1) \cdot 1 = -1.
\]

5) The case \( D_7 \oplus A_2 \)
In this case,
\[
\epsilon_3(R) = \epsilon_3(D_7) \epsilon_3(A_2)(d(D_7), d(A_2))_3 = 1 \cdot 1 \cdot (-1, 3)_3 = -1.
\]

6) The case \( D_5 \oplus 2A_2 \)
In this case,
\[
\epsilon_3(R) = \epsilon_3(D_5) \epsilon_3(2A_2)(d(D_5), d(2A_2))_3 = 1 \cdot (-1) \cdot (-1, 1)_3 = -1.
\]

7) The case \( D_5 \oplus A_2 \oplus 2A_1 \)
Since \( \epsilon_3(D_5) = \epsilon_3(D_5) \epsilon_3(A_2)(d(D_5), d(A_2))_3 = 1 \cdot 1 \cdot (-1, 3)_3 = -1 \) and \( \epsilon_3(2A_1) = (d(A_1), d(A_1))_3 = (-2, -2)_3 = 1 \), we have
\[
\epsilon_3(R) = \epsilon_3(D_5 \oplus A_2) \epsilon_3(2A_1)(d(D_5 \oplus A_2), d(2A_1))_3 = (1) \cdot 1 \cdot 1 = -1.
\]

8) The case \( D_4 \oplus A_5 \)
In this case,
\[
\epsilon_3(R) = \epsilon_3(D_4) \epsilon_3(A_5)(d(D_4), d(A_5))_3 = 1 \cdot (-1) \cdot (1, -6)_3 = -1.
\]

9) The case \( D_4 \oplus A_4 \oplus A_1 \)
Since \( \epsilon_5(D_4) = \epsilon_5(D_4) \epsilon_5(A_4)(d(D_4), d(A_4))_5 = 1 \cdot 1 \cdot (1, 5)_5 = 1 \), we have
\[
\epsilon_5(R) = \epsilon_5(D_4 \oplus A_4) \epsilon_5(A_1)(d(D_4 \oplus A_4), d(A_1))_5 = 1 \cdot 1 \cdot (5, -2)_5 = -1.
\]

10) The case \( D_4 \oplus A_3 \oplus A_2 \)
Since \( \epsilon_3(D_4 \oplus A_2) = \epsilon_3(D_4) \epsilon_3(A_2)(d(D_4), d(A_2))_3 = 1 \cdot 1 \cdot (1, -3)_3 = 1 \), we have
\[
\epsilon_3(R) = \epsilon_3(D_4 \oplus A_2) \epsilon_3(A_3)(d(D_4 \oplus A_2), d(A_3))_3 = 1 \cdot 1 \cdot (3, -1)_3 = -1.
\]
11) The case $D_4 \oplus 2A_2 \oplus A_1$
Since $\epsilon_3(D_4 \oplus A_1) = \epsilon_3(D_4)\epsilon_3(A_1)(d(D_4), d(A_1))_3 = 1 \cdot 1 = 1$ and $\epsilon_3(2A_2) = -1$,
we have
$$
\epsilon_3(R) = \epsilon_3(D_4 \oplus A_1)\epsilon_3(2A_2)(d(D_4 \oplus A_1), d(2A_2))_3 = 1 \cdot (-1) \cdot 1 = -1.
$$

12) The case $E_6 \oplus 3A_1$
Since $\epsilon_3(E_6 \oplus A_1) = \epsilon_3(E_6)\epsilon_3(A_1)(d(E_6), d(A_1))_3 = (-1) \cdot 1 \cdot (3, -2)_3 = -1$ and
$\epsilon_3(2A_1) = (d(A_1), d(A_1))_3 = 1$,
we have
$$
\epsilon_3(R) = \epsilon_3(E_6 \oplus A_1)\epsilon_3(2A_1)(d(E_6 \oplus A_1), d(2A_1))_3 = (-1) \cdot 1 \cdot (-6, 1)_3 = -1.
$$

(2) Suppose that $R$ is embedded in $I_{1,L}$. Let $R^\perp$ be the orthogonal complement of $R$ in $I_{1,L}$. Then $(R \oplus R^\perp) \otimes Q_p \cong I_{1,L} \otimes Q_p$ for each prime $p$. Again by [10] Theorem IV-9, $\epsilon_p(I_{1,L}) = \epsilon_p(R \oplus R^\perp)$ for any prime $p$. Since $\epsilon_3(I_{1,L}) = 1$ for any positive integer $L$, it is enough to show that $\epsilon_3(R \oplus R^\perp) = -1$.

1) The case $D_4 \oplus A_2$
Since $\epsilon_3(R) = \epsilon_3(D_4 \oplus A_2) = 1$ by the computation in (1), we have
$$
\epsilon_3(R \oplus R^\perp) = \epsilon_3(R)\epsilon_3(R^\perp)(d(R), d(R^\perp))_3 = 1 \cdot 1 \cdot (3, 3)_3 = -1.
$$

2) The case $D_4 \oplus 2A_2$
Since $\epsilon_3(R) = \epsilon_3(D_4 \oplus A_2)\epsilon_3(A_2)(d(D_4 \oplus A_2), d(A_2))_3 = 1 \cdot 1 \cdot (3, 3) = -1$,
we have
$$
\epsilon_3(R \oplus R^\perp) = \epsilon_3(R)\epsilon_3(R^\perp)(d(R), d(R^\perp))_3 = (-1) \cdot 1 \cdot (1, 1)_3 = -1.
$$

2.2. Supporting Examples. We complete the proof of Theorem 1.1 by providing a supporting example for each of the 56 singularity types.

If $R$ is one of the 27 singularity types with $\text{rank}(R) < 9$, then there exists a Gorenstein log del Pezzo surface of rank 1 with the given singularity type $R$ by [17] Theorem 1.2.

If $R$ is one of the types in $L_{R \equiv 0}$ except the two types $2A_3 \oplus A_2 \oplus A_1$ and $A_3 \oplus 3A_2$, then it is supported by an example obtained by contracting suitable nine smooth rational curves on an Enriques surface with finite automorphism group constructed by Kondô ([10]). These examples are listed in Table 2 where we use the same notation for smooth rational curves as in his paper. In Section 3 more impressive examples will be obtained, in a different way, from Enriques surfaces with infinite automorphism group.

Remark 2.5. Neither the type $2A_3 \oplus A_2 \oplus A_1$ nor $A_3 \oplus 3A_2$ can be obtained from Kondô’s examples of Enriques surfaces. In other words, the 29 types in Table 2 are the only types that can be obtained from Kondô’s examples. This can be checked by using computer algebra system, e.g., Maple. The main steps for the algorithm are as follows: for each of the 8 types of Kondô’s examples,

1. generate the dual graph $G$ of the intersection matrix of all $(-2)$-curves on the surface (an Enriques surface with finite automorphism group has finitely many $(-2)$-curves),
2. for each set $V$ of 9 vertices of $G$, check if the subgraph $H$ of $G$ generated by $V$ is a disjoint union of graphs of Dynkin type $A_n$, $D_n$, or $E_n$,
3. collect, up to graph isomorphism, all subgraphs $H$ that survived the step (2).
Table 2. Realization of the 29 singularity types in $L_{K^{\equiv 0}}$

| Singularity Type | Kondo’s Example | Nine smooth rational curves |
|------------------|-----------------|----------------------------|
| $A_9$            | Type VII        | $(E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_{12})$ |
| $A_8 \oplus A_1$ | Type I          | $(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_{10}); (F_1)$ |
| $A_7 \oplus A_2$ | Type V          | $(E_1, E_2, E_3, E_4, E_5, E_7, E_8); (E_{14}, E_{15})$ |
| $A_7 \oplus 2A_1$| Type I          | $(F_1, F_2, F_3, F_4, F_5, F_6, F_7); (F_11)$ |
| $A_6 \oplus A_2 \oplus A_1$ | Type V | $(E_1, E_2, E_3, E_4, E_5, E_7); (E_{14}, E_{15})$ |
| $A_5 \oplus A_4$ | Type VI         | $(E_1, E_2, E_3, E_4, E_5); (E_{11})$ |
| $A_5 \oplus A_3 \oplus A_1$ | Type VI | $(E_1, E_2, E_3, E_4, E_5); (E_{11}, E_{14}, E_{15})$ |
| $A_5 \oplus 2A_2$ | Type V          | $(E_1, E_2, E_3, E_4, E_6); (E_{14}, E_{15})$ |
| $A_5 \oplus A_2 \oplus 2A_1$ | Type V | $(E_1, E_2, E_3, E_4, E_6); (E_{14}, E_{15})$ |
| $2A_4 \oplus A_1$ | Type VI         | $(E_1, E_2, E_3, E_4); (E_{13}, E_{14}, E_{15})$ |
| $A_4 \oplus A_3 \oplus 2A_2$ | Type III | $(E_1, E_2, E_3, E_4); (E_{13})$ |
| $3A_3$           | Type II         | $(F_1, F_2, F_3, F_4, F_5, F_6, F_7); (E_{13})$ |
| $2A_3 \oplus 3A_1$ | Type III | $(E_1, E_2, E_3, E_4, E_5, E_6); (E_{12})$ |
| $D_9$            | Type II         | $(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9); (E_{12})$ |
| $D_8 \oplus A_1$ | Type I          | $(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8); (E_{12})$ |
| $D_7 \oplus 2A_1$ | Type III       | $(E_1, E_2, E_3, E_4, E_5, E_6); (E_{11})$ |
| $D_6 \oplus A_3$ | Type II         | $(E_1, E_2, E_3, E_4, E_5, E_6); (E_{11})$ |
| $D_6 \oplus A_2 \oplus A_1$ | Type V | $(E_1, E_2, E_3, E_4, E_5, E_6); (E_{11})$ |
| $D_5 \oplus A_4$ | Type IV         | $(E_1, E_2, E_3, E_4, E_5, E_6); (E_{11})$ |
| $D_5 \oplus A_3 \oplus A_1$ | Type IV | $(E_1, E_2, E_3, E_4, E_5, E_6); (E_{11})$ |
| $D_5 \oplus D_4$ | Type IV         | $(E_1, E_2, E_3, E_4, E_5, E_6); (E_{11})$ |
| $D_4 \oplus A_3 \oplus 2A_1$ | Type III | $(E_1, E_2, E_3, E_4, E_5); (E_{11})$ |
| $2D_4 \oplus A_1$ | Type III       | $(E_1, E_2, E_3, E_4, E_5); (E_{11})$ |
| $E_8 \oplus A_1$ | Type I          | $(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8); (E_{11})$ |
| $E_7 \oplus A_2$ | Type V          | $(E_1, E_2, E_3, E_4, E_5, E_6, E_7); (E_{11})$ |
| $E_7 \oplus 2A_1$ | Type V         | $(E_1, E_2, E_3, E_4, E_5, E_7); (E_{11})$ |
| $E_6 \oplus A_3$ | Type VI         | $(E_1, E_2, E_3, E_4, E_5, E_6); (E_{11})$ |
| $E_6 \oplus A_2 \oplus A_1$ | Type V | $(E_1, E_2, E_3, E_4, E_5, E_6); (E_{11})$ |

If $2A_3 \oplus A_2 \oplus A_1$ or $A_3 \oplus 3A_2$ exists, then it must realize on an Enriques surface with infinite automorphism group.

Remark 2.6. Both lattices $2A_3 \oplus A_2 \oplus A_1$ and $A_3 \oplus 3A_2$ can be embedded into the Enriques lattice $H \oplus E_8$. More precisely,

$$2A_3 \oplus A_1 \oplus A_2 \subset E_7 \oplus A_2 \subset H \oplus E_8,$$

$$3A_2 \oplus 3A_2 \subset A_3 \oplus E_6 \subset H \oplus E_8.$$ 

In fact, these are well-known. To show $2A_3 \oplus A_1 \subset E_7$, let $e_1, e_2, \ldots, e_7$ be $(-2)$-vectors generating the lattice $E_7$, whose Dynkin diagram is given by

$$e_1 - e_2 - e_3 - e_4 - e_5 - e_6 \quad \text{and} \quad e_7$$
Take \( e_0 = -2e_1 - 3e_2 - 4e_3 - 3e_4 - 2e_5 - e_6 - 2e_7 \). Then \( e_0 \in E_7, e_0^2 = -2 \), and we have the following extended Dynkin diagram

\[
e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6
\]

which obviously contains the Dynkin diagram \( 2A_3 \oplus A_1 \). To show \( 3A_2 \subset E_6 \), let \( e_1, e_2, \ldots, e_6 \) be \((-2)\)-vectors generating the lattice \( E_6 \), with Dynkin diagram

\[
e_1 - e_2 - e_3 - e_4 - e_5
\]

Take \( e_0 = -e_1 - 2e_2 - 3e_3 - 2e_4 - e_5 - 2e_6 \). Then \( e_0 \in E_6, e_0^2 = -2 \), and we have the following extended Dynkin diagram

\[
e_1 - e_2 - e_3 - e_4 - e_5
\]

which obviously contains the Dynkin diagram \( 3A_2 \).

3. Examples from Enriques surfaces with infinite automorphism group

In this section, we give further examples of Gorenstein \( \mathbb{Q} \)-homology projective planes with numerically trivial canonical divisor. In this case, the minimal resolution is an Enriques surface. In the previous section we have already seen plenty of such examples arising from Enriques surfaces with finite automorphism group. Here we construct new ones from Enriques surfaces with infinite automorphism group. We summarize the result as follows.

**Theorem 3.1.** Let \( R \) be one of the two types \( D_8 \oplus A_1, E_7 \oplus 2A_1 \) in the list \( L_{K \equiv 0} \). Then there exists a one-dimensional family of Gorenstein \( \mathbb{Q} \)-homology projective planes with singularity type \( R \) such that the minimal resolution of a very general member is an Enriques surface with infinite automorphism group.

**Remark 3.2.** Any Gorenstein \( \mathbb{Q} \)-homology projective plane is an example of a Mori dream space, namely its Cox ring is finitely generated. Is the minimal resolution of a \( \mathbb{Q} \)-homology projective plane also a Mori dream space? The finite generation of the Cox ring of \( S \) implies the finiteness of the automorphism group \( \text{Aut}(S) \), so Theorem 3.1 gives a negative answer to the question. The question is still open for rational surfaces (see [7, Question 6.7]).

Our Enriques surfaces arise from Kummer surfaces of special type. It is known that a very general Kummer surface of product type \( \text{Km}(E \times F) \) cover essentially two kinds of Enriques surfaces, namely the quotients by Lieberman involutions and Kondo-Mukai involutions [15]. The idea is to specialize them to the case of self-product type \( \text{Km}(E \times E) \).
Now let the elliptic curve $E_\lambda$ be

$$E_\lambda : y^2 = x(x-1)(x-\lambda), \quad (\lambda \neq 0, 1)$$

defined in the Legendre form. To the abelian surface $A = E_\lambda \times E_\lambda$, we can associate the Kummer surface of self-product type $X = Km(A)$ as the minimal resolution of the quotient surface $\overline{X} = A/(−1)_A$. Since the double covering $E_\lambda \to E_\lambda/(−1)_E_\lambda \simeq \mathbb{P}^1 \times \mathbb{P}^1$ branches at the four points $x = 0, 1, \lambda, \infty$, the natural map $\overline{X} \to (E_\lambda/(−1)_E_\lambda) \times (E_\lambda/(−1)_E_\lambda) = \mathbb{P}^1 \times \mathbb{P}^1 = R$ is the double covering branched along the eight lines

$$l_1 : x = 0, \quad l_2 : x = 1, \quad l_3 : x = \lambda, \quad l_4 : x = \infty,$$

$$l'_1 : x' = 0, \quad l'_2 : x' = 1, \quad l'_3 : x' = \lambda, \quad l'_4 : x' = \infty$$

where $x$ and $x'$ are the inhomogeneous coordinates of two $\mathbb{P}^1$s. This gives the another visible construction of $X$ as the desingularization of the double covering of $R$. We write $l_i \cap l'_j = \{p_{ij}\}$. See Figure 1 and also [12, Section 3]. We denote by $L_i$ and $L'_i$ the smooth rational curves on $X$ dominating $l_i$ and $l'_i$ respectively. We denote by $E_{ij}$ the exceptional curve on $X$ which resolves the node on $\overline{X}$ corresponding to the point $p_{ij}$. Thus we get 24 smooth rational curves on $X$, which are often called the double Kummer configuration of $X$.

By our choice of the abelian surface $A$, we can obtain further rational curves on $X$. Let us look at the four curves of bidegree $(1, 1)$ on $R$ defined by the following equations.

$$c_1 : x' = x \quad \text{(through } p_{11}, p_{22}, p_{33}, p_{44})$$

$$c_2 : xx' = \lambda \quad \text{(through } p_{14}, p_{23}, p_{32}, p_{41})$$

$$c_3 : (x-1)x' = x - \lambda \quad \text{(through } p_{13}, p_{24}, p_{31}, p_{42})$$

$$c_4 : (x-\lambda)x' = \lambda(x-1) \quad \text{(through } p_{12}, p_{21}, p_{34}, p_{43})$$

The special property owned by them is that each $c_k$ passes through four of the intersections $p_{ij}$ as indicated. Therefore, after discarding exceptional curves, the inverse image of $c_k$ on $X$ decomposes into two smooth rational curves $C_k$ and $C'_k$. We obtain eight additional smooth rational curves on $X$ in this way. Their configuration is partially described as follows.

**Lemma 3.3.** The curves $C_k$ and $C'_k$ are disjoint for $k = 1, \ldots, 4$. Let $k_1 \neq k_2$. Then, the four curves $C_{k_1}, C'_{k_1}, C_{k_2}, C'_{k_2}$ form a disjoint union of two Kodaira fibers of type $I_2$. 

![Figure 1. Branches of $\overline{X} \to \mathbb{P}^1 \times \mathbb{P}^1$]
Proof. The first assertion is clear. We prove the second statement for $(k_1, k_2) = (1, 2)$. The proof for other pairs is similar.

The divisor $f = c_1 + c_2$ has bidegree $(2, 2)$ on $R$ and passes through eight points $p_{11}, p_{14}, p_{22}, \ldots, p_{44}$. On the other hand, the divisors $f_1 = l_1 + l_4 + l_2' + l_3'$ and $f_2 = l_2 + l_3 + l_1' + l_4'$ also satisfy the same conditions. Therefore these three divisors are in the elliptic pencil

$$L = |\mathcal{O}_R(2, 2) − p_{11} − p_{14} − p_{22} − p_{32} − p_{33} − p_{41} − p_{44}|.$$ 

It defines the rational elliptic surface $\hat{R} \to \mathbb{P}^1$. The surface $X$ is nothing but (the resolution of) the double cover of $\hat{R}$ branched along the two fibers $f_1$ and $f_2$. Thus, the fiber $f$ decomposes into two $I_2$ fibers on $X$ by this double cover, as stated. □

From now on we assume that $\lambda$ is very general.

**Proposition 3.4.** If $\lambda$ is very general, then the transcendental lattice of the Kummer surface $X$ has the Gram matrix

$$
\begin{pmatrix}
0 & 2 & 0 \\
2 & 0 & 0 \\
0 & 0 & 4
\end{pmatrix}.
$$

Therefore any Enriques surface covered by $X$ has an infinite automorphism group.

Proof. If $\lambda$ is very general, the abelian surface $A = E_\lambda \times E_\lambda$ has Picard number 3. It is generated by the axes $E_\lambda \times \{0\}$ and $\{0\} \times E_\lambda$ and the diagonal $\Delta = \{(a, a) \mid a \in E_\lambda\}$. Their intersection numbers are given by the following Gram matrix

$$
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
$$

and the Neron-Severi lattice of $A$ is isomorphic to $U \oplus A_1$ (where $U$ is the hyperbolic plane and $A_1 = \langle -2 \rangle$). Therefore the transcendental lattice of $A$ is isomorphic to $U \oplus \langle 2 \rangle$. By passing to the Kummer surface, the transcendental lattice is multiplied by two. Thus we get the result. The last sentence follows from the classification of Kondo [10] of Enriques surfaces with finite automorphism groups. □

In what follows we give two kinds of Enriques quotients of $X$.

**The Lieberman involution.** Let us give the first free involution $\varepsilon_1$ on $X$, which gives the Enriques quotient $S_1$. Recall that the points $p_{ij}$ are in one-to-one correspondence with 2-torsion points of $A$. The zero element is at $p_{44}$. We consider the translation automorphism $\tau \in \text{Aut}(A)$ defined by the 2-torsion point corresponding to $p_{33}$. Then we can see that the composite $\tau \circ (1_{E_\lambda}, -1_{E_\lambda}) \in \text{Aut}(A)$ descends to $\overline{X}$ and to $X$ to give a fixed-point-free involution $\varepsilon_1$. This is one of what is known as the Lieberman involution [13, 15]. As the quotient, we obtain the Enriques surface $S_1 = X/\varepsilon_1$. In terms of the double covering $X \to R$, $\varepsilon_1$ is one of the two lifts of the small involution

$$(x, x') \mapsto \left( \frac{\lambda(x - 1)}{x - \lambda}, \frac{\lambda(x' - 1)}{x' - \lambda} \right).$$

To see how $\varepsilon_1$ permutes smooth rational curves on $X$, we set $\sigma = (12)(34) \in S_4$. The following facts can be verified.
The involution \( \varepsilon_1 \) permutes the curves \( L_i, L'_i, E_{ij} \) by the action of \( \sigma \) on indices; \( \varepsilon_1(L_1) = L_2, \varepsilon_1(E_{13}) = E_{24} \) for example.

The involution \( \varepsilon_1 \) exchanges \( C_k \) and \( C'_k \) since \( C_k \cup C'_k \) is preserved and \( \varepsilon_1 \) exchanges the two sheets of \( \overline{X}/R \).

From these facts, we can draw the dual graph of smooth rational curves on the Enriques quotient \( S_1 = X/\varepsilon_1 \). We denote by \( L_{12}, L_{34}, L'_{12}, L'_{34} \) the rational curves on \( S_1 \) corresponding to \( L_1 + L_2, L_3 + L_4, L'_1 + L'_2, L'_3 + L'_4 \) on \( X \) respectively. We denote by \( F_{ij} \) the rational curve on \( S_1 \) corresponding to \( E_{ij} + E_{\sigma(i)\sigma(j)} \) (our choice is such that \( i < \sigma(i) \)). Also we denote by \( D_k \) the curve on \( S_1 \) corresponding to \( C_k + C'_k \). The dual graph of the the set of curves \( \{L_{ij}, L'_{ij}, F_{ij}\} \) is depicted in Figure 2.

![Figure 2. Smooth rational curves on \( S_1 \)](image)

On the other hand, by using Lemma 3.3, we see that the dual graph \( K'_4 \) of the curves \( D_k (k = 1, \ldots, 4) \) is the complete graph in four vertices with edges doubled. There are edges connecting Figure 2 and \( K'_4 \) as follows: each \( D_k \) has two double edges connecting to two vertices in Figure 2 which can be detected from (3.2). To be explicit, we have

\[
(D_1, F_{11}) = (D_1, F_{33}) = 2, \quad (D_2, F_{14}) = (D_2, F_{32}) = 2
\]

\[
(D_3, F_{13}) = (D_3, F_{31}) = 2, \quad (D_4, F_{12}) = (D_4, F_{34}) = 2
\]

and there are no other edges between \( K'_4 \) and Figure 2. We summarize as follows.

**Proposition 3.5.** The Enriques surface \( S_1 \) has \( 12 + 4 = 16 \) smooth rational curves on it. Their configuration is described by Figure 2, the graph \( K'_4 \) and the relation (3.3).

We can choose the rational curves on \( S_1 \) defined for example by

\[
F_{14} + F_{13} + L'_{34} + F_{34} + L_{34} + F_{31} + L'_{12} + F_{12} \cup D_1.
\]

This gives the resolution graph of rational double points of type \( D_8 \) and \( A_1 \). By Artin’s contractability theorem ([1]), we obtain the following.

**Proposition 3.6.** There exists a birational morphism \( S_1 \rightarrow \overline{S}_1 \) which contracts the nine curves above (3.4) to rational double points of type \( D_8 \oplus A_1 \). The surface \( \overline{S}_1 \) is a Gorenstein \( \mathbb{Q} \)-homology projective plane.

This gives the first surface in Theorem 3.1.

**The Kondo-Mukai involution.** The second free involution \( \varepsilon_2 \) is constructed as follows. As is introduced in [15], we project the quadric surface \( R \subset \mathbb{P}^3 \) onto \( \mathbb{P}^2 \) from the point \( p_{44} \). Then the double cover \( \overline{X}/R \) is birationally transformed into
the second double cover $\overline{X} \to \mathbb{P}^2 = R'$. If we use the homogeneous coordinates $(x_0 : x_1 : x_2)$ of $R'$ with the birational isomorphism $x_1 / x_0 = x, x_2 / x_0 = x'$ to $R$, then the branch of $\overline{X} / R'$ is given by the union of lines

$$l_1 : x_1 = 0; \quad l_2 : x_1 = x_0; \quad l_3 : x_1 = \lambda x_0;$$

$$l'_1 : x_2 = 0; \quad l'_2 : x_2 = x_0; \quad l'_3 : x_2 = \lambda x_0.$$ 

Here and in what follows, we denote the curves and points of $R'$ corresponding to those of $R$ by the same letter. We note that the point $p_{44} \in R$ is blown up to the line at infinity of $R'$.

The Kondo-Mukai involution $\varepsilon_2$ of $X$ is induced from the Cremona involution of $R'$ whose center is $\{p_{12}, p_{21}, p_{33}\}$ and which interchanges the two points $(0 : 1 : 0)$ and $(0 : 0 : 1)$. The latter two points are the blow-downs of lines $l_4$ and $l'_4$ respectively. By the change of coordinates

$$y_0 = x_1 + x_2 - x_0,$$

$$y_1 = x_1 - (1 - \lambda^{-1})x_2 - x_0,$$

$$y_2 = x_2 - (1 - \lambda^{-1})x_1 - x_0,$$

the Cremona involution is given by the formula

$$(y_0 : y_1 : y_2) \mapsto \left( \frac{1}{y_0} : \frac{-1 + \lambda^{-1}}{y_1} : \frac{-1 + \lambda^{-1}}{y_2} \right).$$

Since the four isolated fixed points are not on the branch lines, we see in fact that a suitable lift $\varepsilon_2$ is a fixed point free involution of $X$. This is one of what is known as the Kondo-Mukai involution.

**Proposition 3.7.** The involution $\varepsilon_2$ acts on the curves on $X$ as follows. We also define the notation of the corresponding curves on $S_2 = X / \varepsilon_2$ by the table.

| On $X$  | Notation  |
|---------|-----------|
| $E_{11}$ ↔ $E_{22}$ | $F_{11}$ |
| $E_{13}$ ↔ $E_{32}$ | $F_{13}$ |
| $E_{14}$ ↔ $E_{42}$ | $F_{14}$ |
| $E_{23}$ ↔ $E_{31}$ | $F_{23}$ |
| $E_{24}$ ↔ $E_{41}$ | $F_{24}$ |
| $E_{34}$ ↔ $E_{43}$ | $F_{34}$ |
| $L_1$ ↔ $L'_2$ | $L_{12}$ |
| $L_2$ ↔ $L'_1$ | $L_{21}$ |
| $L_3$ ↔ $L'_3$ | $L_{33}$ |
| $L_4$ ↔ $L'_4$ | $L_{44}$ |

**Figure 3.** The action of $\varepsilon_2$ on the 20 rational curves on $X$

**Proof.** All cases can be verified by computations and easy combinatorial observations. □

We also see that $\varepsilon_2$ exchanges $C_1$ and $C'_1$, which produce the smooth rational curve $D_1$ on $S_2$. The image $\varepsilon_2(E_{12})$ of the curve $E_{12}$ is the inverse image of the
Figure 4. Smooth rational curves on $S_2$

line connecting $p_{21}$ and $p_{33}$ and they produce the smooth rational curve $B$ on $S_2$. The dual graph of these twelve curves introduced so far is depicted in Figure 4.

We can choose the rational curves on $S_2$ defined by

$$F_{14} + L_{44} + F_{34} + L_{33} + F_{13} + F_{23} + L_{21} \cup B \cup D_1.$$ 

This gives the resolution graph of rational double points of type $E_7 \oplus 2A_1$. As in the first example, we obtain the following second surface of theorem 3.1.

**Proposition 3.8.** There exists a birational morphism $S_2 \to \mathfrak{X}_2$ which contracts the nine curves above (3.5) to rational double points of type $E_7 \oplus 2A_1$. The surface $\mathfrak{X}_2$ is a Gorenstein $Q$-homology projective plane.

This completes the proof of Theorem 3.1.

**Acknowledgments.** This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF-2011-0022904) and by the National Research Foundation of Korea (NRF-2007-C00002). The authors are grateful to the referees for careful reading and crucial suggestions. DongSeon Hwang would like to thank Kenji Hashimoto for useful discussion.

**Added in Proof.** The authors have been informed that the type $A_3 \oplus 3A_2$ realizes on the Enriques surface constructed by Slawomir Rams and Matthias Schütt [On Enriques surfaces with four cusps, arXiv:1404.3924, Example 3.9]. The Enriques surface admits an elliptic fibration with four $I_3$-fibres, one of them a double fibre, such that the configuration of the 12 nodal curves and a suitable double section contains the configuration $A_3 \oplus 3A_2$.

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