Efficient product design using functional equations

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Abstract
We discuss an approach to represent product functions mathematically and derive functional equations as a means to improve product design efficiency. Product design proceeds in a set order: clarification of task and target, concept design, entity design, and detailed design. Concept design belongs to the upstream stage of product design, and consists of procedures to develop the functional structure of the product and create design solutions to realize that functional structure. Many of today’s products are complicated, large-scale systems that combine subsystems from different technology fields. The functional structures of these products are typically developed and design solutions created in concept design. Subsystem engineers then iteratively adjust subsystem design parameters in entity design and detailed design. Finally, efforts are made to realize the function and performance of the product. In the process, the design can change significantly in entity design and detailed design, and there is often a remand in concept design. By formulating product functions mathematically and evaluating the stability and the degree to which the target is achieved, we can predict product optimality in the upstream stage of product design. In this way, the likelihood of redesign in the midstream and downstream stage of product design or of returning to the upstream stage of product design will be reduced. In the design and manufacturing field, the concept of product function has a number of variations. After examining the various alternatives, we focus on generic function of quality engineering as most suitable for formulating product functions and use it to develop our approach, applying it to a combined system composed of different technical fields. We then confirm that the approach is effective in improving product design efficiency.

Keywords: Product design, Concept design, Generic function, Quality engineering, Functional equation, Variational principle, Circuit theory

1. Introduction
Various approaches to product design have been proposed. Among them, the systematic approach, axiomatic design and quality engineering are well known. In the systematic approach, product design proceeds in a specific order: clarification of task and target, concept design, entity design, and detailed design. Concept design belongs to the upstream stage of product design, where the functional structure of the product is determined with an array of partial functions, and design solutions are created through a variety of physical principles. In the midstream and downstream stages, each design solution is then realized with a catalog of partial function structures and shapes. The product is ultimately produced from these design solutions. (Pahl et al., 2007)

In axiomatic design, the design work maps the customer's request to the product. There are three maps, the first of which maps from the customer domain to the functional domain. The second maps from the functional domain to the physical domain; the third maps from the physical domain to the process domain. According to the design equation, which expresses the relationship between the required function in the functional domain and the design parameters in the physical domain, engineers determine the design parameters using their skill and experience. (Suh, 2001)

In quality engineering, product design proceeds in a specific order: system design, parameter design and tolerance design. In system design, the generic functions of the product are defined. In parameter design, product robustness is
achieved. In tolerance design, product specifications are determined. When the product is a complicated, large-scale system, it is divided into a collection of partial generic functions. In parameter design, robustness is then achieved for each partial generic function. Once the robustness of the product having all the partial generic functions is achieved, the product specifications are determined in tolerance design. In other words, quality engineering is to define the generic functions of products in the upstream stage of product design and to optimize the design parameters to satisfy the generic functions in the midstream stage of product design rather than in the upstream stage of product design. (Taguchi et al., 2005)

By the way, there is the 1DCAE concept, method, tool. This will support the design in the upstream stage of product design by expressing the essence of things in a prospective form. There is no general-purpose approach to derive a prospective form. (Ohtomi, 2016)

As shown in Fig. 1, the above three design approaches have several elements in common: First, the structure of the product function is defined and the design solutions of the structure of the product function are created in the upstream stage of product design. Second, the design solution for each partial function is embodied and optimized in the midstream and downstream stages of product design. Finally, these design solutions are combined to realize and optimize the design solution of the product. Importantly, it has been generally accepted that there is no way to quantitatively optimize product functions in the upstream stage of product design.

![Fig. 1. Outline of systematic approach, axiomatic design, quality engineering](image)

Today’s products are often complicated, large-scale systems that combine a variety of technologies. Because engineers tend to focus on incorporating and optimizing design solutions for the individual partial functions that comprise the product, in many cases, the optimum conditions for the various partial functions end up being inconsistent, resulting in the need for repeated adjustments before the product’s intended function is ultimately realized. In such a situation, there are obvious problems. For example, when repeated design changes are required, the realization and optimization of the product can be severely delayed. Looking to the future, it is easy to imagine that the
complexity and scale of products will increase and that the associated problems will grow and multiply. We offer a solution to these problems.

2. Objectives of the study

In order to solve the above problems, we set three objectives for our study:

Objective 1: To represent product functions as mathematical expressions.
Objective 2: To calculate the overall optimum condition of the product function.
Objective 3: To achieve an efficient design process.

These objectives are explained below.

Objective 1: To represent product functions as mathematical expressions. The intent here is to express product functions as mathematical expressions in the upstream stage of product design—in concept design in the systematic approach, in the functional domain in axiomatic design, and in system design in quality engineering. In the past, products have typically been realized and optimized in the midstream and downstream stages of product design—in entity design and detailed design in the systematic approach, in the physical domain and process domain in axiomatic design, and in parameter design in quality engineering. This has made it difficult to judge the quality of design at the initial product design stage. Expressing the functional structure of the product with a mathematical formula in the upstream stage of product design makes it possible to judge the quality of the design at this initial stage.

Objective 2: To calculate the overall optimum conditions of the product function. The goal here is to calculate the optimum conditions for the product function as a whole. Currently, the optimum conditions for the design solutions of the partial functions of the product are first calculated, after which the optimum conditions for the design solutions of the partial functions of the product are iteratively adjusted, eventually leading to the calculation of the optimum conditions for the product. In the process, multiple design changes are likely and the realization and optimization of the product is prolonged. By optimizing the product function, design changes can be minimized or eliminated and the time needed for the optimization of the design can be shortened.

Objective 3: To achieve an efficient design process. As Objective 1 and Object 2 suggest, the goal here is to produce a more efficient product design process by mathematically expressing and optimizing the product function.

In other words, by formulating the product functions mathematically and evaluating the stability and degree of target achievement, we can predict product optimality in the upstream stage of product design. According to the approach in this research, product functions can be optimized in the upstream stage of product design. As a result, design parameters that realize optimum product functions are only searched in the midstream stage and the downstream stage of product design, and design changes such as review of product functions can be reduced. This will simplify or eliminate the redesign procedure in the midstream or the downstream stage of product design and reduce the need to return to the upstream stage of product design. We can thus expect this approach to improve product design efficiency.

3. Optimization at the upstream stage of product design using functional equations

In developing an approach to achieve our three main objectives, we identified the following three tasks:

Task 1: Formulate the functional equations for the product.
Task 2: Represent product functions that combine different technologies using common symbols.
Task 3: Integrate Task 1 and Task 2 and construct a methodology.

Each task is explained below.

Task 1: Formulate the functional equations for the product. In design and manufacturing, the concept of product function has several variations. In quality engineering (QE), generic function is defined by physical and chemical laws together with energy input and output relationships. It can be represented as formula $Y = \beta M$, where $Y$ is an output characteristic, $\beta$ is a linear coefficient, and $M$ is a signal factor (Taguchi et al., 2005). In value engineering (VE), basic function refers to a function that guarantees product value. In quality function development (QFD), function includes the specification of the product in the house of quality and the numerical value of the performance target. In the systematic approach, product function is defined as the flow from the input to the output of energy, signal and substance. In the axiomatic design, function is defined as the required specification. Except for the case of generic function in quality engineering, these function definitions are qualitative or numerical and are not mathematically
formulated. Given this fact, it is the generic function of quality engineering that appears most suitable for formulating product functions. Consequently, we adopt the generic function of QE. However, since the generic function has no mathematical derivation procedure, and it is well known that most equations in physics and chemistry are derived using the variational principle, we adopted the variational principle as the derivation method of the generic function. Thus, using the general function in QE and the variational principle, we formulate the functional equations of products. However, as the form of the equations differs for each technical field, it is difficult to formulate product functions involving different fields in a unified form.

Task 2: Represent product functions that combine different technologies using common symbols. Since circuit theory is well known as a modeling method with common symbols, it is used in a number of technical fields such as thermodynamics, hydrodynamics, and quantum mechanics. For this reason, we adopt circuit theory while recognizing that there are differences between the variational principle and circuit theory.

Task 3: Integrate Task 1 and Task 2 and construct a methodology. In order to integrate the variational principle with circuit theory, we develop a correspondence between function elements and energy functions; we then establish a four-step approach to establish the functional equations of products that combine different technologies. (Function elements will be described in detail later in the discussion.) The procedure can be outlined as follows:

Step 1: Set the function model for the product [Circuit theory].
Using circuit theory, set up a model that represents the product functions.

Step 2: Set the Lagrange function for the product [Energy representation].
Using the energy function for each function element, set the Lagrange function.

Step 3: Derive the Lagrange equation for the product [Variational principle].
Using the variational principle, derive the Lagrange equation for the product.

Step 4: Define the functional equation for the product.
Treating the state variables of the Lagrange equation as the input signal $M$ and the output characteristic $Y$ of the generic function $Y = \beta M$, define the functional equation.

The new approach in this research is formulating product functions mathematically with the above procedure and evaluating the stability and the degree to which the target is achieved, we can predict product optimality in the upstream stage of product design.

Following these four essential steps, the derivation of a product’s functional equation is described below.

**Step 1: Set the function model for the product [Circuit theory]**

In circuit theory, the correspondence between the three mechanical elements of springs, dampers, and masses and the three passive electrical elements of capacitors, resistors, and coils is well known. In our proposed approach, we define three product function elements corresponding to these mechanical and electrical elements, as shown in Table 1 (Meusel, 1966). In our proposed approach, it is assumed that these correspondences can be established not only between the electrical system and the mechanical system but between various technical systems. Other correspondence relationships will be confirmed as necessary in the future.

| Function elements | Function symbols | Mechanical system | Electrical system |
|-------------------|-----------------|-------------------|-------------------|
| Elasticity $K$    | $K$             | Spring constant $k$ | Capacitance $C$   |
| Attenuation $A$   | $A$             | Damper $R_m$      | Resistance $R_e$  |
| Inertia $M$       | $M$             | Mass $m$          | Coil/Inductance $L$ |

Elasticity $K$, Attenuation $A$ and Inertia $M$ are function elements. Elasticity $K$ is defined as a generalized functional element that includes Spring constant $k$ of the mechanical system and Capacitance $C$ of the electrical system. Attenuation $A$ is defined as a generalized functional element that includes Damper $R_m$ of the mechanical system and Resistance $R_e$ of the electrical system. Inertia $M$ is defined as a generalized functional element that includes Mass $m$ of the mechanical system and Coil/Inductance $L$ of the electrical system.
To illustrate our proposed approach, we consider a product system in which at least two of the subsystems involve different technical fields. $F$ is the external force applied to each subsystem, and $f$ is the interaction between each subsystem. The system, then, is a combined system. Fig. 2 shows a combined system example, and then the subscripts indicate subsystems. Although a serial connection is shown here between the combined systems, a parallel connection is also possible.

Fig. 2. Function model of combined system

Step 2: Set the Lagrange function for the product [Energy representation]

In circuit theory, the correspondence between the energy of the three mechanical and electrical elements is well known. By expanding the correspondence relationship between the energy of the mechanical system and the energy of the electrical system and defining a generalized energy function, we define the energy of our three function elements as shown in Table 2.

Table 2. Correspondence among energy of functional elements.

| Energy items       | Function elements | Characteristic coefficients | Mechanical system | Electrical system |
|--------------------|-------------------|-----------------------------|--------------------|-------------------|
| Potential energy   | Elasticity        | Spring constant             | $\frac{1}{2}Kx^2$  | Capacitance       |
|                    |                   |                             |                    | $\frac{1}{2C}Q^2$ |
| Dissipation energy | Attenuation       | Damper                      | $\frac{1}{2}Rm^2$  | Resistance        |
|                    |                   |                             |                    | $\frac{1}{2}R_eI^2$ |
| Kinetic energy     | Inertia           | Mass                        | $\frac{1}{2}m\dot{x}^2$ | Coil/Inductance   |
|                    |                   |                             |                    | $\frac{1}{2}U^2$   |

The variables $q$ and $\dot{q}$ express the general state, where $q$ is a function of the coordinate $x$ and the charge $Q$, and $\dot{q}$ is a function of the velocity $\dot{x}$ and the current $I$. The energy of the elasticity function is associated with potential energy, and the energy of the inertia function is associated with kinetic energy, and the energy of the attenuation function is associated with dissipated energy.

We express the Lagrange function in terms of potential energy, kinetic energy, dissipated energy, and external force. For the combined system shown in Fig. 2, the Lagrange function $\mathcal{L}_j$ for subsystem $j$ is

$$\mathcal{L}_j = (E_{jk} - E_{ju}) + E_{dis} - (F_j - f_{ij})q_j = \left(\frac{1}{2}M\dot{q}_j^2 - \frac{1}{2}K_jq_j^2\right) + \frac{1}{2}A_j\dot{q}_j^2 - (F_j - f_{ij})q_j, j < l$$

where $E_{jk}$ is the kinetic energy, $E_{ju}$ is the potential energy, $E_{dis}$ is the dissipated energy, $F_j$ is the external force, $f_{ij}$ is the interaction, and $q_j$ is the displacement. Note that the first term expresses the exchange energy between the kinetic energy and the potential energy in the system. The second term indicates the dissipation energy from the system, while the third term expresses external forces from the external system and internal interactions in the system.

The Lagrange function $\mathcal{L}_T$ for the combined system can then be written as the sum of the Lagrange functions $\mathcal{L}_j$ for the subsystems.

$$\mathcal{L}_T \equiv \sum_{j=1}^{n} \mathcal{L}_j = \sum_{j=1}^{n} \left\{ \left(\frac{1}{2}M\dot{q}_j^2 - \frac{1}{2}K_jq_j^2\right) + \frac{1}{2}A_j\dot{q}_j^2 - (F_j - f_{ij})q_j \right\}, j < l \leq n$$

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Step 3: Derive the Lagrange equation for the product [Variational principle]

We derive the Lagrange equation using the Lagrange function from Step 2. The variational principle, with dissipated energy $\mathcal{L}_{j\text{dis}}$, external force $F_j$, and interaction $f_{ij}$ between subsystems, is

$$\int_{t_1}^{t_2} \delta \mathcal{L}_d dt = \int_{t_1}^{t_2} \delta \left( \sum_{j=1}^{n} \{ \mathcal{L}_{j\text{sys}} + \mathcal{L}_{j\text{dis}} - (F_j - f_{ij})q_j \} \right) dt = 0, j < l \leq n$$

(3)

where $\delta \mathcal{L}_{j\text{dis}}$ is

$$\delta \mathcal{L}_{j\text{dis}} = \frac{\partial \mathcal{L}_{j\text{dis}}}{\partial q_j} \delta q_j$$

(4)

The Lagrange equation is then written as

$$\sum_{j=1}^{n} \left( \frac{d}{dt} \frac{\partial \mathcal{L}_{j\text{sys}}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}_{j\text{sys}}}{\partial q_j} + \frac{\partial \mathcal{L}_{j\text{dis}}}{\partial q_j} - (F_j - f_{ij}) \right) = 0, j < l \leq n$$

(5)

Finally, substituting the energy function, the Lagrange equation is written as

$$\sum_{j=1}^{n} \left( \frac{d}{dt} \frac{\partial \mathcal{L}_{j\text{sys}}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}_{j\text{sys}}}{\partial q_j} + \frac{\partial \mathcal{L}_{j\text{dis}}}{\partial q_j} - (F_j - f_{ij}) \right) = 0, j < l \leq n$$

(6)

Step 4: Define the functional equation for the product

We define the functional equation by associating the Lagrange equation obtained in Step 2 and 3 with the generic function $Y = \beta M$. Here, the state variables are considered to be $\dot{q}(t) = \dot{q}e^{j\omega t}, F(t) = F e^{j\omega t}$. From the Lagrange equation (6), we have

$$\sum_{j=1}^{n} \left( iw M_j + \frac{1}{iw} K_j + A_j \right) \dot{q}_j(t) = \sum_{j=1}^{n} (F_j(t) - f_{ij}(t))$$

(7)

We can then define the functional equation as

$$\dot{q}_n(t) \equiv \sum_{j=1}^{n} (F_j(t) - f_{ij}(t))$$

(8)

Where $M \equiv \sum_{j=1}^{n} \left( F_j(t) - f_{ij}(t) \right)$ is the input signal, $Y \equiv \dot{q}_n(t)$ is the output characteristic, and $\beta \equiv 1/ \sum_{j=1}^{n} \left( iw M_j + \frac{1}{iw} K_j + A_j \right)$ is the coefficient. This follows a similar format to the generic function $Y = \beta M$ that is the basis of our approach.

4. Case study for a speaker system

To illustrate, we derive the functional equation for a speaker system using our proposed approach. A speaker system is a combined system composed of an electrical subsystem, a mechanical subsystem and an acoustic subsystem. The electrical subsystem supplies an excitation force to the coil, the mechanical subsystem connects the speaker cone to the coil, and the acoustic subsystem corresponds to air vibrations near the cone. A schematic of the speaker system is shown in Fig. 3. The equivalent coefficients are added. In the mechanical subsystem and the acoustic subsystem, the equivalent model are drawn using a dotted line and the equivalent coefficients are added. The subscripts $e$, $m$, and $a$, represent the electrical, mechanical, and acoustic subsystems, respectively.
Step 1: Set the function model for the speaker system

In circuit theory, the characteristic coefficients in the electrical, mechanical and acoustic subsystems are related to the elasticity, attenuation, and inertia functions. In addition, the state variables for the electrical, mechanical and acoustic subsystems are related to the generalized coordinates, velocity, and force.

| Function elements | Characteristic coefficients | Generalized variables | State variables |
|-------------------|-----------------------------|-----------------------|----------------|
| Elasticity        | Electrical: 1/C, Mechanical: k, Acoustic: 1/\kappa | Coordinates: Q, x, V | Electrical: E, Velocity: I, v, Force: F, Acoustic: P |
| Attenuation       | R_E, R_m, R_a               | Velocity: I, v, U     |
| Inertia           | L, m, 1/\rho                | Force: E, F, P        |

Table 3 shows the characteristic coefficients and state variables, where $C$ is the equivalent capacitance, $k$ is the equivalent spring constant, $\kappa$ is the equivalent compressibility, $R_E$ is the equivalent electric resistance, $R_m$ is the equivalent mechanical resistance, $R_a$ is the equivalent viscosity coefficient, $L$ is the equivalent inductance, $m$ is the equivalent mass, $\rho$ is the equivalent density for the characteristic coefficients, and $Q$ is the charge, $x$ is the displacement, $V$ is the volume, $I$ is the current, $v$ is the velocity, $U$ is the volumetric flow, $E$ is the voltage, $F$ is the force, and $P$ is the pressure for the state variables of the generalized coordinates.

From the above, we can define the characteristic coefficients in the three subsystems. We can then let the characteristic coefficients be function elements and connect the three subsystems with their interactions. Finally, the function model for the speaker system can be represented as shown in Fig. 4. In the model, an electrical subsystem, a mechanical subsystem, and an acoustic subsystem are connected in series.
Step 2: Set the Lagrange function for the speaker system

We can match the function elements with the energy functions in the three subsystems, i.e., the electrical, mechanical, and acoustic systems, as shown in Table 4.

| Energy items   | Function elements | Energy representations | Characteristic coefficients | Electrical system |
|----------------|-------------------|------------------------|-----------------------------|-------------------|
| Potential Energy | Elasticity $\frac{1}{2} K q^2$ | Compressibility $\frac{1}{2} V^2$ | Spring constant $\frac{1}{2} k x^2$ | Capacitance $\frac{1}{2} C Q^2$ |
| Dissipation energy | Attenuation $\frac{1}{2} A q^2$ | Viscosity coefficient $\frac{1}{2} R_a U^2$ | Damper $\frac{1}{2} R_m v^2$ | Resistance $\frac{1}{2} R_a l^2$ |
| Kinetic energy | Inertia $\frac{1}{2} M \dot{q}^2$ | Density $\frac{1}{2} \rho V^2$ | Mass $\frac{1}{2} m v^2$ | Coil Inductance $\frac{1}{2} L l^2$ |

Then, we express the Lagrange function in terms of the potential energy, kinetic energy, the dissipated energy, and the external forces and interactions. According to (1), (2) and Table 4, the Lagrange function $\mathcal{L}_T$ for the speaker system is regarded as the sum of the Lagrange functions for the electrical, mechanical and acoustic subsystems.

$$\mathcal{L}_T = \mathcal{L}_e + \mathcal{L}_m + \mathcal{L}_a$$

$$\mathcal{L}_e = \frac{1}{2} LI^2 - \frac{1}{2} C Q^2 + \frac{1}{2} R_e l^2 - (E - e) Q$$

$$\mathcal{L}_m = \frac{1}{2} m v^2 - \frac{1}{2} k x^2 + \frac{1}{2} R_m v^2 - (F - f) x$$

$$\mathcal{L}_a = \left( \frac{1}{2} \rho U^2 - \frac{1}{2} k V^2 \right) + \frac{1}{2} R_a U^2 - PV$$

Step 3: Derive the Lagrange equation for the speaker system

According to (3), the variational principle for the speaker system composed of the electrical, mechanical and acoustic subsystems is

$$\int_{t_1}^{t_2} \delta \mathcal{L}_T dt = \int_{t_1}^{t_2} \delta (\mathcal{L}_e + \mathcal{L}_m + \mathcal{L}_a) dt = 0$$

(10)

According to (4) and (5), the Lagrange equation is

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}_{sys}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}_{sys}}{\partial q} = - (E - e)$$

$$+ \frac{d}{dt} \left( \frac{\partial \mathcal{L}_{sys}}{\partial \dot{v}} \right) - \frac{\partial \mathcal{L}_{sys}}{\partial v} = - (F - f)$$

$$+ \frac{d}{dt} \left( \frac{\partial \mathcal{L}_{sys}}{\partial \dot{U}} \right) - \frac{\partial \mathcal{L}_{sys}}{\partial U} = - P = 0$$

(11)

Substituting (9) into (11), we derive the Lagrange equation for the speaker system, where the driving voltage $E$ is applied externally and the interactions $e, F, f$ are between the electrical, mechanical, and acoustic subsystems in the speaker system are in effect.

$$\left( LI + \frac{1}{C} Q + R_e I \right) + \left( m \ddot{v} + k x + R_m v \right) + \left( \frac{1}{\rho} \dddot{U} + \frac{1}{k} V + R_a U \right) = (E - e) + (F - f) + P$$

(12)

Step 4: Define the functional equation for the speaker system

We define the driving voltage $E$ as the input signal and the volumetric flow $U$ as the output characteristic. We also denote the driving voltage as $E(t) = E e^{i\omega t}$, the current as $I(t) = I e^{i\omega t}$, the velocity as $v(t) = v e^{i\omega t}$, the pressure as $P(t) = P e^{i\omega t}$, and the volumetric flow as $U(t) = U e^{i\omega t}$. 

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According to (12), the Lagrange equation is
\[
\dot{\mathbf{L}} + \frac{1}{\dot{\mathbf{C}}} \mathbf{I} + \mathbf{R} \dot{\mathbf{v}} + \dot{\mathbf{r}} \mathbf{v} + \frac{k}{\dot{\mathbf{w}}} \mathbf{v} + \mathbf{R} \mathbf{e} + \mathbf{R} \dot{\mathbf{m}} \mathbf{v} + \mathbf{k} \mathbf{e} + \mathbf{R} = \left( E - e \right) + \left( F - f \right) + P
\] (13)
The variables \( l, v, e, F, f, \) and \( P \) are eliminated using the mutual coupling relations \( e = B l v, F = B l I, \) and \( f = P S, U = v s \) between the electrical, mechanical and acoustic subsystems. The relational expression connecting the volumetric flow \( U \) and the driving voltage \( E \) can then be written as
\[
U(t) = \frac{B l S}{\left( \dot{\mathbf{w}} + \frac{1}{\dot{\mathbf{C}}} + \mathbf{R} \right) \left( \dot{\mathbf{w}} m + \frac{k}{\mathbf{w}} + \mathbf{R} \mathbf{m} \right) + \left( B l \right)^2 + \left( \dot{\mathbf{w}} L + \frac{1}{\dot{\mathbf{C}}} + \mathbf{R} \right) \left( \frac{\dot{\mathbf{w}}}{\mathbf{P}} + \frac{1}{\dot{\mathbf{C}}} + \mathbf{R} \right) S^2}
\] (14)
Thus, the functional equation for the speaker system is derived.

5. The effectiveness of our proposed approach

The effectiveness of our proposed approach can be demonstrated by applying it to the process of designing the speaker system. Having selected the parameter design of quality engineering, one of the more effective and practical optimization design methods, we can calculate the product’s optimum condition, which here is the maximum condition of the SN ratio. We illustrate the proposed approach and conventional approach as follows:

In the proposed approach, we use expression (14) to show the square of each output characteristic of the speaker system as
\[
|U(t)|^2 = |\beta|^2 |E(t)|^2
\] (15)
where
\[
\beta = \frac{B l S}{\left( \dot{\mathbf{w}} L + \frac{1}{\dot{\mathbf{C}}} + \mathbf{R} \right) \left( \dot{\mathbf{w}} m + \frac{k}{\mathbf{w}} + \mathbf{R} \mathbf{m} \right) + \left( B l \right)^2 + \left( \dot{\mathbf{w}} L + \frac{1}{\dot{\mathbf{C}}} + \mathbf{R} \right) \left( \frac{\dot{\mathbf{w}}}{\mathbf{P}} + \frac{1}{\dot{\mathbf{C}}} + \mathbf{R} \right) S^2}
\] (16)
We can then calculate the optimum condition using the functional equations (15) and (16) for the entire speaker system. The speaker system has twelve design parameters. Its output characteristic is volumetric flow. We simulate the calculation of the optimum condition— that is, the maximum condition of the SN ratio—using the L27 orthogonal table. By utilizing the parameter design, we can obtain the optimal stability condition for the square of the output characteristics of the speaker system with the proposed approach. These optimal stability conditions are the maximum signal-to-noise (SN) ratios. The results are shown in Figs. 5. With the factor effect diagram, it is possible to easily confirm the combination of the design parameters that maximizes the SN ratio of volumetric flow that is the output characteristic of the speaker system. Here, it can be seen that the design parameters \( \rho, P_a, S, B, l, L, C \) and \( P_e \) are significant for the SN ratio.

In Figs. 5, the circle symbols represent the optimal conditions for maximizing the SN ratio. The level numbers 1, 2, and 3 of the control factor are marked as small, middle, and large, respectively.

We can contrast this proposed approach with the conventional approach. In the conventional approach, each engineer first embodies and optimizes the design solution for the individual partial functions. In most cases, the optimum conditions of these various design solutions are inconsistent. Consequently, repeated adjustments are needed. Then, in order to simulate the optimization of the design solution of the partial functions in the midstream or the downstream stage of product design, the optimum condition of each subsystem is calculated by using the functional equations of the electrical system, the mechanical system, and the acoustic system. These functional equations are as follows:
\[ I(t) = \frac{1}{i\omega L + \frac{1}{i\omega C} + R_e} (E(t) - e(t)) \]  
(17)

\[ v(t) = \frac{1}{i\omega m + \frac{k}{i\omega} + R_m} (F(t) - f(t)) \]  
(18)

\[ U(t) = \frac{1}{i\omega \rho + \frac{1}{i\omega \kappa} + R_a} P(t) \]  
(19)

Using (17), (18) and (19), the square of the output characteristic of each of the three subsystems is given as

\[ |I(t)|^2 = \left| \frac{1}{i\omega L + \frac{1}{i\omega C} + R_e} \right|^2 (E(t) - e(t))^2 \]  
(20)

\[ |v(t)|^2 = \left| \frac{1}{i\omega m + \frac{k}{i\omega} + R_m} \right|^2 (F(t) - f(t))^2 \]  
(21)

\[ |U(t)|^2 = \left| \frac{1}{i\omega \rho + \frac{1}{i\omega \kappa} + R_a} \right|^2 P(t)^2 \]  
(22)

We calculate each optimum condition using functional equations (20), (21), (22) for the electrical, mechanical, and acoustic subsystems. Since each subsystem has three design parameters, the three subsystems have nine design parameters. The output characteristics of the each subsystem are the current, the velocity, and the volumetric flow. We then simulate the calculation of each of the optimum conditions that are the maximum conditions of the SN ratio using three L9 orthogonal tables. By utilizing the parameter design, we can conveniently obtain each partial optimal stability condition for the square of the electrical, mechanical, and acoustic subsystems. Each of these partial optimal stability conditions is the maximum signal-to-noise (SN) ratio. The results are shown in Fig. 6. According to the respective factor effect diagrams, it is possible to identify the combinations of design parameters that maximize the SN ratios of the current, the velocity, and the volumetric flow, which are the respective output characteristics of the electrical system, the mechanical system, and the acoustic system. Here, it is found that the design parameters \( L \) and \( P_e \) are significant with respect to the SN ratio of the electrical system and \( m \) and \( P_m \) are significant with respect to the SN ratio of the mechanical system.

![Fig. 6. Factor effect diagrams for the electrical, mechanical, acoustic subsystems.](image)

In Fig. 6, the circle symbols represent the optimal conditions for maximizing the SN ratio. The level numbers 1, 2, and 3 of the control factors are marked as small, middle, and large, respectively.

Comparing the SN ratio calculation for the speaker system and those for the three subsystems enables us to evaluate the effectiveness of the proposed approach. Tables 5 and 6 summarize the optimal conditions for the speaker system and the three subsystems, respectively. The overall optimal condition for the speaker system determined by our proposed approach is different from the partial optimal conditions for the three subsystems determined by the conventional approach. Additionally, the significant design parameters in the speaker system from the proposed approach are different from the partial optimal conditions for the three subsystems determined by the conventional approach. In the conventional approach, further adjustment of the design parameters is necessary to obtain the overall optimum condition.
Table 5. Optimal conditions for the speaker system

| Parameter | Speaker system |
|-----------|----------------|
| $\rho$    | 2 (3)          |
| $\kappa$  | 1 (3)          |
| $R_a$     | 1 (3)          |
| $S_m$     | 3 (2)          |
| $k_m$     | 3 (2)          |
| $B$       | 3 (2)          |
| $L$       | 3 (2)          |
| $C$       | 3 (2)          |
| $R_e$     | 3 (2)          |

Table 6. Optimal conditions for the three subsystems

| Parameter | Acoustic subsystem | Mechanical subsystem | Electrical subsystem |
|-----------|--------------------|----------------------|---------------------|
| $\rho$    |                    |                      |                     |
| $\kappa$  |                    |                      |                     |
| $R_a$     |                    |                      |                     |
| $m$       | 1 (3)              |                      |                     |
| $k_m$     |                    |                      |                     |
| $R_m$     |                    |                      |                     |
| $L$       |                    |                      | 1 (2)               |
| $C$       |                    |                      |                     |
| $R_e$     |                    |                      |                     |

In Table 5 and Table 6, "." indicates that the design parameters are not significant. Numbers enclosed by ( ) indicate the level of the design parameters at which the SN ratio is the maximum.

With proposed approach, the overall optimum condition for the speaker system can be easily calculated. We calculate the maximum SN ratio condition for the speaker system using an L27 orthogonal table once, and then we calculate the maximum SN ratio condition for each subsystem using L9 orthogonal table three times. Thus, both procedures require 27 calculations. Because the speaker system calculation includes subsystem interactions, the maximum SN ratio condition is the overall optimum condition for the system. On the other hand, because the three subsystem calculations do not include subsystem interactions, the maximum SN ratio condition for each of the subsystems is only a partial optimum. We would need to calculate the influence of the interactions between the various subsystems in order to determine the overall optimum condition. Using the proposed approach, we can derive the functional equation for the total product and easily determine the overall optimum condition.

6. Conclusions

We develop a new approach to derive the functional equation for a product. The approach uses the generic function in QE, the variational principle, circuit theory and the correspondence between the variational principle and circuit theory. The functional equation for the combined system is derived. To demonstrate the approach, the functional equation for a speaker system is developed. According to the conventional approach, the partial optimization for the three subsystems requires three orthogonal experiments L9 (9 conditions) × 3 = 27 conditions and furthermore parameters adjustment experiments between each subsystem are necessary. According to the approach in this research, only one orthogonal experiment L27 (27 conditions) is required for the overall optimization of the speaker system. Therefore, we confirm that the number of experiments of the approach in this research is smaller than the number of experiments of the conventional approach. By applying the proposed approach to the design process, the optimum condition for the speaker system can be easily calculated. We confirm that the proposed approach is effective for improving product design efficiency.

At the concept design stage, the specific shape, structure and physical properties of a product have not yet been precisely determined. By using the proposed approach, we can obtain the most suitable functions for achieving the desired target. This approach is effective for developing new products or replacing one function with another in existing products.

We confirm the effectiveness of our proposed approach for a combined system involving various technical fields. Recognizing that most products are likely to involve systems that are much more complicated and on a larger scale than our illustrative speaker system, we aim to show that our approach has general applicability. In future studies, we will confirm the effectiveness of the approach for a hierarchical system composed of various subsystems.

We develop the new approach in this research using the variational principle, circuit theory and the generic function in QE. Therefore, we expect that it can be applied to various technical fields besides mechanics and electricity, such as fluid, heat and chemical. In order to apply in various technical fields, we assume that it is necessary to build a framework of general correspondence relationships on "quantities representing conditions" that are handled in each technical field. (Takahashi, 1974)

In modern product design, software is more important than mechanics and electricity. In order to apply the new approach in this research to software, we assume that we need to develop a method to formulate the input / output...
relationship of signals to products by utilizing the theory and principle of information. We would like to develop specific solution for application to software in a future, by investigating and studying the theory and principle of information.

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