Magnetically actuated and controlled colloidal sphere-pair swimmer

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Abstract
Magnetically actuated swimming of microscopic objects has been attracting attention partly due to its promising applications in the bio-medical field and partly due to interesting physics of swimming in general. While colloidal particles that are free to move in fluid can be an attractive swimming system due its simplicity and ability to assemble in situ, stability of their dynamics and the possibility of stable swimming behavior in periodically varying magnetic fields has not been considered. Dynamic behavior of two magnetically interacting colloidal particles subjected to rotating magnetic field of switching frequency is analyzed here and is shown to result in stable swimming without any stabilizing feedback. A new mechanism of swimming that relies only on rotations of the particles themselves and of the particle pair axis is found to dominate the swimming dynamics of the colloidal particle pair. Simulation results and analytical arguments demonstrate that this swimming strategy compares favorably to dragging the particles with an external magnetic force when colloidal particle sizes are reduced.

1. Introduction

Transport of magnetic colloidal particles finds many applications related to sensing, drug delivery, molecular and cellular separation among others. Ability to actuate and control this transport via remote magnetic field sources seems particularly attractive. Today, for the most part, magnetic transport in fluids, soft media and tissues has been accomplished by dragging magnetic colloidal particles using external magnetic force. Given the fact that magnetized objects have no monopole magnetic moment, the application of such magnetic force requires not just magnetic field, but magnetic field gradients, which by their very nature, decay more rapidly away from sources than the magnetic field itself [1]. This imposes practical limitations on transport of magnetized colloidal particles related to their size and proximity to sources of the magnetic field gradients. It also provides motivation to search for alternative methods of the magnetic particle transport.

Swimming is a possible alternative to dragging. Swimming of microorganisms such as bacteria has received significant attention through the years [2, 3]. In contrast to dragging, it would not require the application of an external net magnetic force to the colloidal particles and could rely instead on transfer of magnetic energy to them, possibly through application of magnetic torque, which could be accomplished using uniform or nearly uniform magnetic field. However, in Newtonian fluids when drag dominates inertial effects (low Reynolds number regime), certain requirements must still be met to make swimming possible. The key requirement is non-reciprocal motion of the swimmer’s parts [4]. This, in turn, requires employing multiple degrees of freedom and some symmetry breaking.

However, in many situations, including those of magnetically controlled swimming, motion of the swimmer’s parts is not directly controlled to enforce non-reciprocal movements. Instead, forces and torques acting on various parts can be varied as functions of time. Under such circumstances it is not always clear that (1) periodic forces and torques will result in stable periodic movements of the swimmer’s parts and (2) that this movement will be non-reciprocal and could lead to stable swimming. In some situations, one may be able to employ feedback to obtain desired non-reciprocal motion. Certain magnetically controlled swimming
structures such as screws [5], waving tails [6, 7], rigidly connected rods [8], and multiple beads connected in various ways [9–14] can execute movement required for swimming in a stable way without the need for feedback on relative positions of different parts. It remains unclear, however, if stable swimming of much simpler structures like disconnected colloidal particles free to move independently is possible without any feedback. Prior experimental work, for example, did employ visual feedback in demonstrating locomotion of colloidal particles pair near surface [15]. Transport of colloidal particles without the need for feedback is important not only as a scientific curiosity, but also in many realistic applications where one may not have access to real time measurement of relative positions of the swimmer parts.

The present work demonstrates that a spherical particle pair stabilized via repulsive colloidal forces can achieve stable swimming when subjected to rotating uniform magnetic field of changing rotation frequency without any feedback on the relative particle positions. Comparison between an analytical model developed here and numerical simulations suggests that the primary mechanism of swimming relies on torques and rotation of each particle with the magnetic field, rather than axial magnetic interaction force between the particles. This new mechanism results in significantly faster swimming rate compared to one that involves radial displacement of the particles relative to each other [16]. It is also demonstrated here that the rate of the colloidal particle pair swimming scales linearly with the particles’ diameter. Therefore, as the particle sizes reduce, swimming becomes inherently advantageous to dragging, the rate of which scales with the square of the particle diameter.

2. System model

The model of the colloidal double sphere swimmer where each can move separately needs to account for hydrodynamic, magnetic and some repulsive interactions needed to maintain colloidal stability. Similar models having been reported in the literature [17]. However, the model employed here differs in the presence of hydrodynamic interactions and fixed magnitude and direction of the magnetic moments in the frame of reference of each particle.

As shown in figure 1, movement of two spherical particles having different radii $a_1$ and $a_2$ and permanent magnetic moments $M_1$ and $M_2$ fixed in the frame of reference of each particle can be described approximately by the equations:

$$V_1 = \frac{1}{6 \pi \eta a_1} F_1 + \frac{1}{8 \pi \eta r^2} \left\{ F_2 + \frac{1}{r^2} (F_2 \cdot \hat{r}) \hat{r} - \frac{\hat{T}_2}{8 \pi \eta r^3} \right\}, \quad (1a)$$

$$V_2 = \frac{1}{6 \pi \eta a_2} F_2 + \frac{1}{8 \pi \eta r^2} \left\{ F_1 + \frac{1}{r^2} (F_1 \cdot \hat{r}) \hat{r} + \frac{\hat{T}_1}{8 \pi \eta r^3} \right\}, \quad (1b)$$

$$\vec{W}_1 = \frac{1}{8 \pi \eta r^3} \times F_2 + \frac{1}{8 \pi \eta} \left\{ 3 (\hat{r} \cdot \hat{T}_2) \hat{r}^2 - \frac{\hat{T}_2}{r^3} \right\} + \frac{\hat{T}_1}{8 \pi \eta (a_1)^3}, \quad (1c)$$

$$\vec{W}_2 = -\frac{1}{8 \pi \eta r^3} \times F_1 + \frac{1}{8 \pi \eta} \left\{ 3 (\hat{r} \cdot \hat{T}_1) \hat{r}^2 - \frac{\hat{T}_1}{r^3} \right\} + \frac{\hat{T}_2}{8 \pi \eta (a_2)^3}, \quad (1d)$$

Figure 1. System structure of the particle pair.

where $\vec{V}_1$, $\vec{V}_2$, $\vec{W}_1$ and $\vec{W}_2$ are the linear and angular velocities of each particle, respectively, $\vec{F}_1$, $\vec{F}_2$, $\vec{T}_1$, and $\vec{T}_2$ are the forces and torques each particle experiences, respectively, $\hat{r} = \vec{r} / \| \vec{r} \|$ is the vector connecting the center of the first to the center of the second particle and $\eta$ is the kinematic viscosity of the fluid.
The system structure and its steady state movement due to rotating magnetic field is illustrated in figure 1. The above description is an approximate one because the hydrodynamic interactions in it are treated to the lowest possible order in the inter-particle distance relative to the particle radii \( r/a_i \) and \( r/a_j \). Similar first-order hydrodynamic interaction description is commonly used [9, 18]. Close approach between the particles will result in stronger hydrodynamic interactions which can be modeled either by including higher order terms in \( r/a_i \) and \( r/a_j \), by employing squeezed film approximations or by other techniques [19]. The first-order approximation has the advantage of relative simplicity in revealing the possibility of swimming. It also provides the lower bound on the swimming rates for a given magnetic field power transfer because true Stoke’s flow around the spheres minimizes dissipation [20].

During swimming, no net force is being applied to the pair of the magnetized particles and the only force acting on each particle is due to magnetic and repulsive interaction (whose nature may be related to the interplay between elastic, steric, double-layer and van der Waals types of forces) with the other particle, i.e.

\[
F_2 = -F_1 = \vec{F}_m + \vec{F}_r,
\]

where \( \vec{F}_r \) is the repulsive interaction force between the particles and \( \vec{F}_m \) is the magnetic interaction which may be repulsive or attractive depending on the relative directions of the particles’ magnetic moments and is given by:

\[
\vec{F}_m = (\bar{M}_1 \cdot \nabla)\bar{B}_1,
\]

where

\[
\bar{B}_1 = \frac{\mu_0}{4\pi \rho} \left( \frac{3(\bar{M}_1 \cdot \bar{r})\bar{r}}{r^5} - \frac{\bar{M}_1}{r^3} \right)
\]

is the dipolar field created by one particle (the external field is uniform and does not contribute to the interaction force for this reason) yielding

\[
\vec{F}_m = \frac{3\mu_0}{4\pi} \left\{ \bar{M}_1 (\bar{M}_2 \cdot \bar{r}) + \bar{M}_2 (\bar{M}_1 \cdot \bar{r}) + (\bar{M}_1 \cdot \bar{M}_2)\bar{r} \right\}.
\]

It is worth noting that the magnetic interaction force is inversely proportional to the to the 4th power of the inter-particle distance.

The repulsive interaction \( \vec{F}_r \) should, in principle, describe specific physical/chemical mechanism responsible for the colloidal particle repulsion. However, having tested different forms of interactions they were all found to result in the same qualitative behavior and only the average distance between the particles seems to influence the quantitative results. The results reported here were obtained using purely phenomenological form of repulsive interactions given by:

\[
\vec{F}_r = \begin{cases} 
D n_i^2 \left( 3 - \left( \frac{r}{a_i} \right) \right)^m \frac{r}{r}, & r \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]

where \( D > 0 \) having units of force per meter square is chosen to be sufficiently large so that the magnetic interactions can be balanced by the repulsive ones at some distance of approach between the particles. For a given choice of parameters \( D \) and \( m \) the above repulsive force scales with the surface area of the spheres when the distance between the spheres scales with the sphere radius. The same scaling law applies to the magnetic interaction force.

The torques \( \vec{T}_i \) and \( \vec{T}_j \) in model (1) are assumed to be only due to the magnetism of the particles ignoring the possibility of elastic torques since the particles are taken to be not tethered to each other or to anything else. These are given by:

\[
\vec{T}_i = \bar{M}_i \times (\vec{B}_j + \vec{B}_{\text{ext}}) = \frac{\mu_0}{4\pi} \left\{ 3(\bar{M}_1 \cdot \bar{r})(\bar{M}_i \times \bar{r}) \right\}.
\]

3. Model dynamics under steady rotating field

Dynamics of particles subjected to uniformly rotating magnetic field at various frequencies is considered first. In agreement with experimental observations [21] and previous models [17], there are two different regimes of particles motion under different external field rotating frequencies as illustrated in figure 2. When the frequency is very low, the pair axis rotates synchronously with the rotation of each particle magnetic moment, both of which follow the field rotation, albeit lagging behind slightly. At these lower frequencies, the inter-particle distance increases slowly with the field rotation frequency. The increase in the average distance occurs because
the angle between the magnetic moments and the pair axis increases leading to a reduction of the attractive magnetic force, which permits the repulsive interaction to push the particles apart. As long as the field is relatively strong each particle continues to rotate synchronously with it as the frequency increases. However, at a certain critical frequency particle pair dynamics experiences a bifurcation. This is due to the angle between the magnetic moments and the pair axis becoming sufficiently large to cause a change in the magnetic force to being repulsive for some duration. As shown in figure 2, this causes a bump in the otherwise smooth trajectories of the two particles during which the pair axis rotation effectively speeds up to catch up with the direction of the magnetic moments. The critical frequency \( B \) where the bumps first occur can be called the Mason frequency and is associated with the Mason number \( \lambda \), which is often characterized as the ratio of viscous drag to magnetic forces between the particles. Above the Mason frequency the pair axis no longer rotates synchronously with the magnetic field and, as the field rotation frequency increases, the number of the bumps in the trajectory of the particles relative rotation increases contributing to an increase in the average inter-particle distance. Such transition is shown in figures 2(b) and (c) and is consistent with experimental observations \( \lambda \) validating the simulations reported here.

The Mason frequency dependence on particle size and magnetic field strength can be illustrated by taking the particles to be the same size \( a = a_1 = a_2 \). In this case, \( M_1 = M_2 = M = 4/3\pi a^3 M_s \), where \( M_s \) is the magnitude of the particles’ magnetization (saturation magnetization) and represents the property of the magnetic material itself. Using the model equations (1)–(3), one can find the angular rotation rate of the pair axis to be:

\[
(\dot{V}_2 - \dot{V}_1) \times \frac{\vec{r}}{r^2} = \frac{2}{3} \mu_0 \frac{8}{3} \frac{r}{a} \left[ \frac{\dot{M}_s \cdot \hat{r}}{\eta} - 1 \right] \left( \frac{a}{r} \right)^6 \left( \dot{M}_s \cdot \hat{r} \right) \dot{M}_s \times \hat{r} + \left( \frac{a}{r} \right)^{\frac{3}{2}} \frac{M_s \times \vec{B}}{3\eta}.
\]

(4a)

Mason frequency is the largest frequency of the synchronous magnetic field \( \vec{B} \) and magnetization \( \dot{M}_s \) rotation above which the angular rotation rate can no longer remain a constant. The dependence of the Mason frequency
on the field arises only through the second term on the right of the above expression. This dependence is weaker than proportional because, when the field increases in magnitude, the angle between $\vec{M}$ and $\vec{B}$ decreases. In fact, the last term becomes nearly constant at relatively high field. At such high field magnitude, the Mason frequency can be approximated by:

$$W_M = \frac{2 \mu_0 M^2}{3 \eta} \left[ \frac{8 \pi r}{3a} - 1 \right] \left( \frac{a}{r} \right)^6 \approx \frac{16 \mu_0 M^2}{9 \eta} \left( \frac{a}{r} \right)^6.$$  \hspace{1cm} (4b)

The above expression shows that the Mason frequency depends on the magnetic material parameters, fluid viscosity and, in a very sensitive way, on the ratio of the particle sizes to the distance between their centers. However, as the particle sizes increase, so does the distance between their centers making the Mason frequency independent of the particle sizes and, in this sense, scale invariant.

Eventually, as the frequency increases significantly beyond the Mason frequency, the trajectory bumps start occurring one after another and any further increase in the frequency of the field rotation only reduces the magnitude of each trajectory bump as shown in figure 2(d). This observation is also in agreement with the results found in previous work [21]. Thus, there is another frequency at which maximum average steady-state distance between the particles occurs and any further increase in frequency leads to the reduction of the average distance between the particles as can be seen from the simulations shown figure 2(e). The difference in the particle pair dynamics below and above the Mason frequency can also be illustrated using the ratio $\lambda$ of the magnetic field rotation frequency to the rotation frequency of the pair axis. Figure 2(f) shows that, above the Mason frequency that ratio increases rapidly to values significantly above 1, but eventually saturating. The above results apply equally well to particles of the same or different sizes.

Dynamics of the particle pair is also associated with another critical frequency not yet discussed above. In contrast to the Mason frequency, this critical frequency $W_{\text{ind}}$ is associated with the interaction of each particle separately with the external field, rather than with the magnetic interaction of the particles with each other. Indeed, each particle (having the magnetic moment $M$ proportional to the cube of the radius $a$) separately follows the external field rotation only below this individual particle critical frequency and starts oscillating back and forth above it [22].

$$W_{\text{ind}} = \frac{\mu_0 MB}{4\pi\eta a^3} = \frac{\mu_0 MB}{3\eta}.$$  \hspace{1cm} (4c)

When two particles can interact magnetically, they experience increased average magnetic attractive force when they start oscillating, rather than rotating. This, in turn, brings the particles closer together on average. It is interesting to note that, in contrast to the Mason frequency, the individual frequency depends linearly on the magnitude of the rotating field. This means for some small rotating field magnitude, the individual frequency will become smaller than the Mason frequency. When this happens, rotation of the pair axis seizes together with the rotation of each particle. It is also worth noting that the individual frequency is independent of the particle size and the steady state dynamics of the pair of particles whose sizes are not equal is very similar to the pair of equally sized particles.

It is worth emphasizing that the results presented above regarding different critical frequencies and dependence of average inter-particle distance on the field rotation frequency are consistent with both models discussed in the literature [17] and with experimental observations [21] suggesting that the model and its numerical simulation described above are valid. The fact that the Mason frequency $W_M$ and the individual particle critical frequency $W_{\text{ind}}$ are both independent of the particle sizes explains why the frequency range shown in figures 2(e) and (f) is independent of the particle sizes.

### 4. Mechanism of spherical particle pair swimming

As discussed above, when the field rotates at some frequency below Mason frequency, each particle rotates synchronously with the field and the particle pair axis rotates as well, although not necessarily at the same rate as the field. The situation considered next is what happens when the rate of the field rotation changes during the cycle of the pair axis rotation. Specifically, the case when the field rotation frequency switches two times during the pair axis rotation cycle is discussed. Simulation of this behavior is shown in figure 3(b). After each field rotation frequency change, the particle dynamics stabilizes into a periodic trajectory and can be described using the analysis carried out above for a fixed field rotation frequency. Although exact periodicity of steady-state trajectories is hard to test numerically, the observed periodicity in simulations are consistent with the fact that, for high magnetic field magnitude, the system practically has only two degrees of freedom, pair axis orientation and inter-particle distance, which would not permit chaotic non-periodic trajectories that may be possible for weaker fields. Regular switching between two well-defined particle trajectories results in a well-defined pair displacement after each complete pair axis rotation cycle. The direction of displacement is controlled in these
simulations by the timing of rotating field frequency changes with periodic frequency switching leading to stable swimming dynamics. The mechanism of swimming is analyzed below using some analytical approximations.

There are two salient modes of magnetic energy utilization needed to maintain the pair movement. One mode relies on the component of the magnetic interaction force directed along the particles’ axis that performs work when the inter-particle distance changes. This force applies no torque to the individual particles or to the pair. Consequently, this mode of magnetic energy utilization can be called torque-free. Alternatively, if one ignores changes in the inter-particle distance or if the net axial inter-particle force is zero, the pair motion can be powered by the magnetic torques applied to each particle and to the pair as a whole. Such mode of magnetic energy utilization can be called axial force-free mode.

These two modes can be conveniently described by using the following equations obtained from 1(a) and (b):

\[
(V_1 + V_2) = \frac{1}{6\pi \eta} \left( \frac{1}{a_2} - \frac{1}{a_1} \right) \bar{F} - \frac{(\bar{T}_1 - \bar{T}_2) \times \bar{r}}{8\pi \mu \mu r^3},
\]

\[
(V_2 - V_1) = \left[ \frac{1}{6\pi \eta} \left( \frac{1}{a_2} + \frac{1}{a_1} \right) - \frac{1}{4\pi \mu \mu r} \left( I + \frac{\bar{r}}{r^2} (\bar{r} \cdot \bar{r}) \right) \right] \bar{F} + \frac{(\bar{T}_1 + \bar{T}_2) \times \bar{r}}{8\pi \mu \mu r^3},
\]

where \( \bar{F} \) is the total force of interaction between the particles (taken here to be the force acting on particle 2).

Dropping the torques \( \bar{T}_1 \) and \( \bar{T}_2 \) in (5) to analyze the torque-free mode results in:

\[
(V_1 + V_2) = \frac{1}{6\pi \eta} \left( \frac{1}{a_2} - \frac{1}{a_1} \right) \bar{F},
\]

\[
(V_2 - V_1) = \left[ \frac{1}{6\pi \eta} \left( \frac{1}{a_2} + \frac{1}{a_1} \right) - \frac{1}{4\pi \mu \mu r} \left( I + \frac{\bar{r}}{r^2} (\bar{r} \cdot \bar{r}) \right) \right] \bar{F}.
\]

Let us now suppose for a moment that vector \( \bar{r} \) rotates without changing its magnitude. In this case, given that over the cycle of the pair axis rotation, velocity difference \( (V_2 - V_1) \) is periodic with zero average (there is no net relative displacement of the particles in a cycle), the interaction force \( \bar{F} \) would also have to be periodic with zero average value. Using this periodic zero average force in (a) would yield periodic and zero-time average \( (V_1 + V_2) \).

As may be expected, this implies that no swimming is possible in the torque-free mode when \( \bar{r} \) experiences only rotation without changes of its magnitude. Swimming is possible in this mode if \( \bar{r} \) changes its magnitude as it rotates. In the simulations described above, the magnitude \( r \) experiences a small average change when the field rotation rate changes abruptly two times in the pair axis rotation cycle. Swimming of particle pairs that move along such trajectories was analyzed previously [16] and it was found that the net displacement per the pair axis cycle can be described by:

\[\text{Figure 3. Trajectory and displacement magnitude of the pair swimmer (using same parameter as given for figure 2) with field rotation frequencies switched between 1 and 5 Hz.}\]
\[ D_{\text{ff}} (d_{\text{max}}, d_{\text{min}}) = \frac{3(a_1a_2)(a_2 - a_1)}{(a_1 + a_2)} \times \left( \frac{3a_1a_2(d_{\text{max}} - d_{\text{min}})}{(2d_{\text{max}}(a_1 + a_2) - 3a_1a_2)(2d_{\text{min}}(a_1 + a_2) - 3a_1a_2)} + \frac{1}{a_1 + a_2} \ln \frac{d_{\text{max}}(a_1 + a_2) - 3a_1a_2}{d_{\text{min}}(a_1 + a_2) - 3a_1a_2} \right) \]

where \( D_{\text{ff}} \) is the net displacement in the torque-free cycle and \( d_{\text{max}} \) and \( d_{\text{min}} \) are the maximum and minimum inter-particle distances within the cycle. It can be shown that the above net displacement per cycle is a fraction of \((d_{\text{max}} - d_{\text{min}})\). Thus, when the inter-particle distance changes by a relatively small amount relative to the overall pair size, the net displacement per cycle is a small fraction of the pair size. This is natural since change in magnetic interaction energy will be relatively small portion of the overall magnetic interaction energy and only a fraction of the change of magnetic interaction energy be used to obtain net pair displacement. Furthermore, the net displacement per cycle will depend on the operating frequencies of the field rotation only because the time averaged inter-particle distance will be dependent of the operating frequencies of the field rotation frequency. The simulation results indicate that the displacement obtained from torque-free mode only contribute around 1% of the total displacement.

In the axial force-free mode, when the inter-particle distance remains constant, the net force in the direction \( \hat{r} \) is zero and equations (5) reduce to:

\[
(V_1 + V_2) = \frac{1}{6\pi\eta} \left( \frac{1}{a_2} - \frac{1}{a_1} \right) F^\perp - \frac{(T_1 - T_2) \times \hat{r}}{8\pi\eta r^3}, \quad \text{(8a)}
\]

\[
(V_2 - V_1) = \frac{1}{6\pi\eta} \left[ \left( \frac{1}{a_2} + \frac{1}{a_1} \right) - \frac{3}{2r} \right] F^\perp + \frac{(T_1 + T_2) \times \hat{r}}{8\pi\eta r^3}, \quad \text{(8b)}
\]

where \( F^\perp \) is the interaction force perpendicular to \( \hat{r} \) which, upon elimination of \( F^\perp \) from the above equations, yields:

\[
(V_1 + V_2) = \left( \frac{1}{a_2} - \frac{1}{a_1} \right) \left( \frac{1}{a_2} + \frac{1}{a_1} - \frac{3}{2r} \right) \frac{(T_1 + T_2) \times \hat{r}}{r^3} + \frac{(T_1 - T_2) \times \hat{r}}{r^3}.
\]

Since \( r \) remains constant, integrating the above over one cycle of the pair axis rotation and taking into account the fact that \((V_2 - V_1)\) integrates to zero,

\[
D_{\text{ff}} = \left[ \int_{t_1}^{0} \left( V_1 + V_2 \right) dt \right] = \left[ 0 \right] + \left[ \int_{\tau_1}^{\tau_2} \left( \frac{1}{a_2} - \frac{1}{a_1} \right) \left( \frac{1}{a_2} + \frac{1}{a_1} - \frac{3}{2r} \right) \frac{(T_1 + T_2) \times \hat{r}}{r^3} + \frac{(T_1 - T_2) \times \hat{r}}{r^3} \right] dt,
\]

where \( D_{\text{ff}} \) is the net displacement in the axial force-free cycle and \( \tau \) is the pair rotation cycle time. If torques \( T_1 \) and \( T_2 \) remain constant as the magnetic field rotates at constant rate throughout the entire pair axis rotation cycle, their integrals will be zero and no net displacement of the pair will occur within the cycle. If, however, torques change throughout the pair rotation cycle, time integrals of \((T_1 \times \hat{r})\) and \((T_2 \times \hat{r})\) will be non-zero and the net displacement of the pair will also be non-zero. The following approximation could be used to describe torque when the magnetic field rotates perpendicularly to the z-axis at one rate during the first half of the pair axis rotation cycle ending at \( t = \tau_1 \) and then rotates at another rate causing the second half of the pair axis rotation cycle to complete within time \( \tau_2 \):

\[
T_1 \times \hat{r} = \begin{cases} 
T_{1f} \left[ \sin \left( \frac{\pi t}{\tau_1} \right) \hat{x} + \cos \left( \frac{\pi t}{\tau_1} \right) \hat{y} \right], & 0 < t < \tau_1 \\
- T_{1s} \left[ \sin \left( \frac{\pi (t - \tau_1)}{\tau_2} \right) \hat{x} + \cos \left( \frac{\pi (t - \tau_1)}{\tau_2} \right) \hat{y} \right], & \tau_1 < t < \tau_1 + \tau_2 = \tau 
\end{cases}
\]

\[ (11a) \]
\[
T_1^2 \times \vec{r} = \begin{cases}
T_{1f} \tau_1 \left[ \sin \left( \frac{\pi t}{\tau_1} \right) \hat{x} + \cos \left( \frac{\pi t}{\tau_1} \right) \hat{y} \right], & 0 < t < \tau_1 \\
- T_{1s} \tau_1 \left[ \sin \left( \frac{\pi (t - \tau_1)}{\tau_s} \right) \hat{x} + \cos \left( \frac{\pi (t - \tau_1)}{\tau_s} \right) \hat{y} \right], & \tau_1 < t < \tau_1 + \tau_s = \tau
\end{cases}
\]

(11b)

where \( T_{1f}, T_{1s}, T_{2f} \) and \( T_{2s} \) are the torque magnitudes for each of the particles in the first and second half of the pair rotation cycle, respectively. The above approximation ignores an important effect related to transient dynamics of the particle pair following any switching of the external field rotation rate. Transients will essentially reduce the times \( \tau_f \) and \( \tau_s \) resulting in \( \tau_1 + \tau_s < \tau \). This, in turn, implies that actual displacement will be smaller than one calculated using the approximation (11). Nevertheless, this approximation is useful at least as an upper bound on the swimming displacement in the axial force-free mode. At higher magnetic field magnitudes, the magnitudes of particle torques can be roughly related to the magnetic field angular frequencies \( W_f \) and \( W_s \) in the first and second halves of the pair axis cycle from (c) and (d) by ignoring hydrodynamic interactions of the particles:

\[
8\pi \eta (a_1)^3 W_f \approx T_{1f}, \quad 8\pi \eta (a_2)^3 W_f \approx T_{1s}, \quad 8\pi \eta (a_1)^3 W_s \approx T_{2f}, \quad 8\pi \eta (a_2)^3 W_s \approx T_{2s}
\]

(12)

Substituting (11) and (12) into the integrals in (10) and simplifying yields:

\[
D_{\text{eff}} = 2a_1 \left[ \lambda_f - \lambda_s \right] \left( \frac{a_1}{r} \right)^2 \left( \frac{d_1}{a_2} - 1 \right) \left( 1 + \left( \frac{d_2}{a_1} \right)^3 \right) + \left( 1 - \left( \frac{d_2}{a_1} \right)^3 \right), \quad \quad (13)
\]

where \( \lambda_f \) and \( \lambda_s \) are the ratio between external rotating frequency over pair axis rotation frequencies in the first and second halves of the cycle, which can be obtained from figure 2(f). Figure 4 shows the comparison between simulation and approximate result for the axial force-free displacement. As discussed above, the analytical approximation can be viewed as an upper bound on the swimming displacement. This bound exceeds the numerically simulated swimming by about a factor of 4. Nevertheless, this analytically obtained bound is quite useful in revealing the mechanism of axial force-free swimming. For example, the bound in (13) shows that the swimming rate scales with the square of the ratio of the particle sizes to the distance between them. When one particle becomes several times larger than the others, this ratio reduces dramatically and the rate of swimming becomes very slow.

The above simple model of a torque based axial force-free swimming mode of the unequal size spheres illustrates that this mode of swimming can be implemented in a variety of ways. For example, one may change direction of the magnetic field rotation as well as frequency after half the pair axis cycle completion. One may also connect the spheres by a rigid axis, while allowing them to rotate. In this case, the shape of the swimmer surfaces will not change at all during swimming illustrating the fact that change of shape is not required for swimming. Instead, one can think of this swimming device as being completely described by two degrees of freedom: angle of the pair’s rigid axis and angle of the individual spheres.
To explain swimming of this device more intuitively, consider back and forth rotation of the pair axis at the same rate by 180°, as illustrated in figure 5(a). Forward axis rotation by a half turn creates fluid vorticity far away from the pair and also net fluid displacement due to unequal sphere sizes if the pair is held pinned at the rotation center. Similar fluid flow occurs in the opposite direction during the backward rotation of the pair if its rotation center is pinned. However, in the forward rotation, when the individual spheres rotate faster than the axis, they create stronger fluid vorticity and stronger net flow due to unequal sphere sizes far away from the pair. Thus, the added sphere rotation in the forward rotation creates an imbalance in the net fluid displacement far away from the pair by the end of the cycle. This also implies that the pair will experience a net displacement at the end of the cycle if its center of rotation is free to move in a stationary fluid. The trajectory in the space of the degrees of freedom shown in figure 5(b) illustrates the cycle. The non-zero area enclosed by the trajectory in the degree of freedom space is what makes swimming possible. The area enclosed is a measure of the net displacement per cycle.

It is worth emphasizing that much of the above reasoning applies to particle pairs subjected to a magnetic field magnitude high enough to force both the larger and the smaller sphere to have nearly the same orientation of their magnetic moments. At such field, magnetic interaction torques between the particles are much smaller than the torques due to the interaction with the external field. As long as these conditions hold, the displacement per cycle of the pair axis is virtually independent of the field magnitude. At lower values, field magnitude as well as its rotational frequency can be used to control the pair swimming. This regime is not specifically addressed in the present work and will be investigated in the future.

Equation (13) also demonstrates an important feature of many swimming systems in Newtonian fluids: displacement per cycle scales linearly with the characteristic linear dimension. In this case, this characteristic linear dimension can be related to the larger spherical particle radius $a_1$. For a given frequency of the magnetic field rotation, linear scaling of the displacement per cycle leads to linear scaling of the average swimming velocity with the particle radii. It is interesting to contrast this linear scaling with the quadratic scaling law for the pair dragging velocity using external field gradients. Dragging velocity $V_d$ can be roughly estimated by ignoring hydrodynamic interactions between the particles:

$$V_d = \left( \frac{F_1}{6\pi \eta a_1} + \frac{F_2}{6\pi \eta a_2} \right) \approx \frac{1}{6\pi \eta} \left( \frac{M_1}{a_1} + \frac{M_2}{a_2} \right) \frac{dB_{ext}}{dz} = \frac{2M_1}{9\eta} (a_1^2 + a_2^2) \frac{dB_{ext}}{dz}$$

Favorable (linear versus quadratic) scaling of the pair velocity in swimming versus dragging is one very important advantage that makes swimming strategy potentially very attractive as particle sizes reduce.

5. Conclusion

Swimming of colloidal spherical particles requires kinematic non-reversibility of the particle motions. However, motions are often not directly controlled. Instead, interaction forces and interaction/externally applied torques
are controlled in many situations of practical interest, including those that involve magnetic fields and magnetic particles. This work demonstrates that, rotating uniform magnetic field of sufficiently large magnitude results in stable dynamics of permanently magnetized colloidal spheres when repulsive forces limit the inter-particle spacing. It also demonstrates that stable swimming dynamics is obtained by periodically switching magnetic field rotation rate. A new mechanism of swimming is proposed which relies only on rotations of unequal size particles in pairs. Comparison with simulations indicates that this is the dominant mechanism of swimming when the particle pairs are subjected to rotated relatively strong magnetic fields of periodically switched rotation frequency.

It is worth emphasizing again that simulations and an analytical model developed in this work focuses on the regime where the magnetic field is strong enough to ignore differences in the magnetic moment orientations of the two particles. Weaker magnetic fields may give rise to additional phenomena and will be discussed in future work where extensions to larger number of the colloidal particles will be considered as well.

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