Inflation of Hamiltonian System: 
The Spinning Top in Projective Space

Holger R. Dullin  
Institut für Theoretische Physik  
Universität Bremen  
Postfach 330440  
28344 Bremen, Germany  
Email: hdullin@physik.uni-bremen.de

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Abstract

We present a method to enlarge the phase space of a canonical Hamiltonian System in order to remove coordinate singularities arising from a nontrivial topology of the configuration space. This “inflation” preserves the canonical structure of the system and generates new constants of motion that realize the constraints. As a first illustrative example the spherical pendulum is inflated by embedding the sphere $S^2$ in the three dimensional Euclidean space. The main application which motivated this work is the derivation of a canonical singularity free Hamiltonian for the general spinning top. The configuration space $SO(3)$ is diffeomorphic to the real projective space $\mathbb{R}P^3$ which is embedded in four dimensions using homogenous coordinates. The procedure can be generalized to $SO(n)$.

1 Introduction

One of the pillars of classical mechanics is the process of deriving equations of motion via the Lagrangian $L$. It starts with some “generalized coordinates” on the configuration space $Q$ as described, e.g., in [4, 5, 6]. The Euler-Lagrange equations of motion are then obtained from the variational principle that $\int L \, dt$ be extremal on solutions of the mechanical system. For most configuration spaces $Q$ there are no global singularity free coordinate systems which could be used as generalized coordinates in the Lagrangian. In the classical examples with compact configuration space (e.g., spherical pendulum, spinning top, geodesic flow on the ellipsoid, see e.g. [4, 5, 6]) global coordinates with singularities are used. This is well suited, e.g., in order to apply the method of separation of variables to solve the equations of motion. For integrable systems the coordinate singularity is typically only encountered by orbits with special values of the constants of motion. In the nonintegrable case the constants of motion are absent, while the coordinate singularity is still present. Moreover, it is unforseeable which orbits will encounter the singularity. Especially for the numerical integration these coordinate systems are therefore not advisable.
There are two possible cures. Since \( Q \) is a manifold there exist local coordinates everywhere, such that we obtain Lagrangians in every chart, supplemented by the transition maps between different charts. This approach is certainly the most general, but it is neither very elegant nor very simple. The second approach starts with an embedding of \( Q \), \( \dim Q = n \), as a submanifold of \( \mathbb{R}^{n+k} \). In order to fulfill the constraints that define the submanifold \( Q \), Lagrange multipliers are introduced, and the equations of motion are obtained by standard procedures, see e.g. [7]. Kozlov [11] introduced the notion of “excessive coordinates”, i.e. the description of a mechanical system with more coordinates than degrees of freedom. The passage from a Lagrangian with multipliers to a Hamiltonian in excessive coordinates is described in [11]. Our approach is different in that we start with the Hamiltonian in (singular) generalized coordinates and employ a transformation to a Hamiltonian in (nonsingular) excessive coordinates, which has all the constraints as additional constants of motion. This “inflation transformation” is described in the first part and illustrated for the case of the spherical pendulum. In the second part we show how to apply this method to the spinning top. After this work, which first appeared in my thesis [2, 3], was completed, I learned that Kozlov also obtained the same Hamiltonian for the spinning top [11]. Nevertheless, the method at hand gives a quite different derivation of these equations, which also generalizes to \( SO(n) \).

## 2 Inflation of Hamiltonian Systems

Consider a Hamiltonian with \( n \) degrees of freedom \( \mathcal{H}(\mathbf{q}, \mathbf{p}) \). We introduce \( n + k \) new coordinates \( \mathbf{Q} \) by

\[
\mathbf{q} = \mathbf{F}(\mathbf{Q}) \\
\mathbf{c_q} = \tilde{\mathbf{F}}(\mathbf{Q}).
\]

We think of the old configuration space, parametrized by \( n \) coordinates \( \mathbf{q} \), as extended by \( k \) additional coordinates \( \mathbf{c_q} \) all of which do not show up in the Hamiltonian, i.e. we introduce \( k \) cyclic variables. The question now is, how to introduce corresponding new canonical momenta \( \mathbf{P} \) in such a way that the desired geometric constraints \( \tilde{\mathbf{F}}(\mathbf{Q}) \) are constants of motion of the dynamics of a new Hamiltonian depending on \( (\mathbf{Q}, \mathbf{P}) \). Note that we must allow the transformation to have singularities at the points where the old coordinates \( \mathbf{q} \) have the coordinate singularities that we are going to remove.

This is achieved by taking (1) as a point transformation from \( \mathbf{Q} \) to the trivially extended coordinates \( (\mathbf{q}, \mathbf{c_q}) \). Using the generating function

\[
S = \mathbf{F}(\mathbf{Q})\mathbf{p} + \tilde{\mathbf{F}}(\mathbf{Q})\mathbf{c_p}
\]

we recover (1) for the coordinates by construction, while the momenta are given by

\[
\mathbf{P} = \left( \frac{\partial(\mathbf{F}, \tilde{\mathbf{F}})}{\partial \mathbf{Q}} \right)^t \left( \begin{array}{c} \mathbf{p} \\ \mathbf{c_p} \end{array} \right).
\]

The inverse yields the desired transformation to the new momenta:

\[
\left( \begin{array}{c} \mathbf{p} \\ \mathbf{c_p} \end{array} \right) = \left( \frac{\partial(\mathbf{F}, \tilde{\mathbf{F}})}{\partial \mathbf{Q}} \right)^{t^{-1}} \mathbf{P} =: \left( \begin{array}{c} \mathbf{G}(\mathbf{Q}, \mathbf{P}) \\ \tilde{\mathbf{G}}(\mathbf{Q}, \mathbf{P}) \end{array} \right).
\]

With this trick (the rest are standard canonical transformations) we can show:
The dynamics of the Hamiltonian System $\mathcal{K}(Q, P) = \mathcal{H}(F(Q), G(Q, P))$ is equivalent to the dynamics of the Hamiltonian System $\mathcal{H}(q, p)$, i.e.:

1. The solutions $(Q(t), P(t))$ of the inflated system are mapped onto solutions of the original system $(q(t), p(t))$ by $(F, G)$.

2. $\tilde{F}(Q)$ and $\tilde{G}(Q, P)$ are constants of motion of the inflated system.

3. The canonical 1-forms are equal on solutions of the systems: $pdq = PdQ$.

These statements are clear, since the transformation is (in an extended sense) a canonical transformation. Once we have the enlarged system, we can transform it back to the old one via a standard canonical transformation, with the unusual result that $k$ coordinates and $k$ momenta become cyclic. This is possible because the new Hamiltonian is singular, i.e. the Hessian of $\mathcal{H}$ with respect to $p$ is degenerate. Because of this we can not pass back to a Lagrangian in these variables. The above three statements are comprised in the transformation of the Poisson bracket. With the notation $X := (Q, P)^t$ and $x := (q, c_q, p, c_p)^t$ we obtain, for any function $f$,

$$\{f, \mathcal{K}\}_X = (\nabla_X f)^t J \nabla X \mathcal{K}$$

$$= \nabla f^t \frac{\partial X}{\partial X} \{f, \mathcal{H}\}_q, p$$.

Since $\mathcal{H}$ does not depend on $c_q, c_p$, in the last row we obtain the bracket on the original space.

The Spherical Pendulum

As a toy example we consider the spherical pendulum described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \left( p_\varphi^2 + \frac{p_\vartheta^2}{\sin^2 \vartheta} \right) - g \cos \vartheta,$$

where the singular coordinates $(\vartheta, \varphi)$ are used to parametrize the configuration space $S^2$. We take the obvious embedding of $S^2$ in $\mathbb{R}^3$ given by $x^2 + y^2 + z^2 = 1$, i.e. we take the coordinates $r = (x, y, z)^t$ of the real $\mathbb{R}^3$ in which the pendulum moves, as excessive coordinates:

$$\begin{pmatrix} \varphi \\ \vartheta \\ c_q \\ c_p \end{pmatrix} = \begin{pmatrix} \arctan y/x \\ \arccos z/r \\ x^2 + y^2 + z^2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.$$  

With the notation $\rho^2 = x^2 + y^2$ we obtain for the Jacobian

$$\frac{\partial(F, \tilde{F})}{\partial r} = \begin{pmatrix} \frac{y}{\rho^2} & \frac{x}{\rho^2} & 0 \\ \frac{x}{\rho^2} & \frac{y}{\rho^2} & -\frac{1}{r} \\ 2x & 2y & 2z \end{pmatrix}. \quad (10)$$

3
such that the momenta are given by

\[
\begin{pmatrix}
p_\phi \\
p_\theta \\
p_r
\end{pmatrix} = \begin{pmatrix}
-y & x & 0 \\
xz/\rho & yz/\rho & -\rho \\
x/2r^2 & y/2r^2 & z/2r^2
\end{pmatrix} \begin{pmatrix}
p_x \\
p_y \\
p_z
\end{pmatrix}.
\]

Carrying out the transformation, one obtains a singularity free Hamiltonian for the spherical pendulum in the symmetric form

\[
K = \frac{1}{2} \left( (xp_y - yp_x)^2 + (xp_z - zp_x)^2 + (yp_z - zp_y)^2 \right) - \frac{z}{r}.
\]

A short computation shows that in addition to the old constant of motion \( p_\phi = l_z = xp_y - yp_x \), there are geometric constants of motion \( r^2 \) and \( pr/2r^2 \) representing the constraint. By an appropriate choice of initial conditions we can achieve \( r^2 = 1 \) and \( pr = 0 \), so that the geometric constraints are the sphere and the orientation of its tangent plane in \( \mathbb{R}^3 \), as must be the case.

Neither this coordinate system nor Hamiltonian (12) are new (according to Klein and Sommerfeld [10] it was first used by Hermite). As stated in the introduction, our focus is on the transformation procedure, which allows the application to more complicated systems like the spinning top.

3 The Spinning Top in Projective Space

As usual we take “spinning top” to be a short hand for the motion of a rigid body around a fixed point – with or without a potential. The configuration space of the spinning top is \( SO(3) \). This manifold is parametrized (again with singularities) by Euler angles. To remove these singularities we choose the Cayley parametrization of \( SO(3) \), which immediately generalizes to \( Q = SO(m) \): Consider the antisymmetric \( m \times m \) matrices \( A \) with \( \dim Q = n = m(m-1)/2 \) independent entries \( \xi_1, \xi_2, \ldots, \xi_n \). Now introduce an additional coordinate \( \xi_0 \) and map the \( (n+1) \)-tuple \( \xi := (\xi_0, \xi_1, \ldots, \xi_n) \) onto a matrix \( B \in SO(m) \) via Cayley’s map \( C \)

\[
C : \xi \mapsto B = (\xi_0 1 - A)^{-1}(\xi_0 1 + A).
\]

Obviously every line of \( \xi \) is mapped to the same \( B \), since \( C(\lambda \xi) = C(\xi) \) for any \( \lambda \in \mathbb{R} \setminus \{0\} \). Thus the preimage \( C^{-1}(B) \) is a line through the origin, so we can endow the \( \xi \) with the structure of the projective space \( \mathbb{R}P^n \). The components of \( \xi \) are then interpreted as homogenous coordinates, and \( C \) as a diffeomorphism

\[
C : \mathbb{R}P^n \to SO(m).
\]

Thus we have \( k = 1 \) for the global homogeneous coordinates, where the constraint is obtained from choosing a representative for every projective line on \( S^n \), by restricting \( \xi \) to unit length. Antipodal points of \( S^n \) have the same image under \( C \). If we identify these points we obtain a model of \( SO(m) \) as

\[
S^n/\{\pm 1\} \simeq \mathbb{R}P^n \simeq SO(m).
\]
For \( m = n = 3 \) our equations live on \( S^3 \cong SU(2) \), the universal covering space of \( \mathbb{R}P^3 \cong SO(3) \). In this case the coordinates \( \xi \) are usually called Euler parameters. For the rotation matrix we obtain

\[
B^i = \frac{1}{\xi^2} \begin{pmatrix}
2\xi_0^2 + 2\xi_1^2 - \xi_2^2 & 2\xi_1\xi_2 - 2\xi_0\xi_3 & 2\xi_0\xi_2 + 2\xi_1\xi_3 \\
2\xi_1\xi_2 + 2\xi_0\xi_3 & 2\xi_0^2 + 2\xi_2^2 - \xi_1^2 & -2\xi_0\xi_1 + 2\xi_2\xi_3 \\
-2\xi_0\xi_2 + 2\xi_1\xi_3 & 2\xi_0\xi_1 + 2\xi_2\xi_3 & 2\xi_0^2 + 2\xi_3^2 - \xi_1^2
\end{pmatrix}.
\]

(16)

The vector \((\xi_1, \xi_2, \xi_3)^i\) is an eigenvector of \( B \) with eigenvalue 1. Thus it is the fixed axis of rotation described by \( B \). The angle of rotation \( \alpha \) is given by \( \cos(\alpha/2) = \xi_0/2 \). The description of rotations by the 4-vectors \( \xi \) is directly related to the Pauli spin matrices respectively the algebra of quaternions, see e.g. [13, 14, 15]. We only remark that both choices of “spin”, \(+\xi\) and \(-\xi\), by construction give the same \( B \).

Comparing the matrix \( B^i \) with an \( SO(3) \) matrix given in Euler angles in the convention used in [7], we can read off the inflation transformation to obtain a Hamiltonian with \( \xi \) as coordinates for the configuration space and constraint \( \xi^2 = 1 \):

\[
\varphi = \arctan \frac{\xi_0\xi_2 - \xi_1\xi_3}{\xi_0\xi_1 + \xi_2\xi_3}
\]

(17)

\[
\vartheta = \arccos \frac{\xi_0^2 - \xi_1^2 - \xi_2^2 + \xi_3^2}{\xi^2}
\]

(18)

\[
\psi = \arctan \frac{\xi_0\xi_2 + \xi_1\xi_3}{-\xi_0\xi_1 + \xi_2\xi_3}
\]

(19)

\[
c_r = \sqrt{\xi^2}
\]

(20)

The old momenta given by the new ones \( \pi \) are

\[
p_\varphi = \frac{1}{2}(\xi_3\pi_0 - \xi_2\pi_1 + \xi_1\pi_2 - \xi_0\pi_3)
\]

(21)

\[
p_\vartheta = \frac{1}{2}\sqrt{\xi_0^2 + \xi_3^2}(\xi_1\pi_1 - \xi_2\pi_2) - \frac{1}{2}\sqrt{\xi_1^2 + \xi_3^2}(\xi_0\pi_0 - \xi_3\pi_3)
\]

(22)

\[
p_\psi = \frac{1}{2}(\xi_3\pi_0 + \xi_2\pi_1 + \xi_1\pi_2 + \xi_0\pi_3)
\]

(23)

\[
c_p = \xi^1\pi/\xi.
\]

(24)

Inserting this into the original Hamiltonian in Euler angles (see, e.g., [13, 14]) the coordinate singularities are removed and we obtain the very symmetric Hamiltonian

\[
H = \frac{(\pi_0\xi_1 - \pi_1\xi_0 + \pi_2\xi_3 - \pi_3\xi_2)^2}{8\Theta_1} + \frac{(\pi_0\xi_2 - \pi_1\xi_3 - \pi_2\xi_0 + \pi_3\xi_1)^2}{8\Theta_2} + \frac{(\pi_0\xi_3 + \pi_1\xi_2 - \pi_2\xi_1 - \pi_3\xi_0)^2}{8\Theta_3} + \frac{1}{\xi^2}\{2\xi_0\xi_2 + 2\xi_1\xi_3\} + (-2\xi_0\xi_1 + 2\xi_2\xi_3)S_2 + (2\xi_0^2 + 2\xi_3^2 - \xi^2)S_3,
\]

(25)

where we have used a linear potential acting on the center of mass \( S \), and the moments of
inertia $\Theta$. Introducing the notation

$$\xi = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \quad \Leftrightarrow \quad \dot{\xi}, \ddot{\xi} = \begin{pmatrix} \xi_1 - \xi_0 \pm \xi_3 \mp \xi_2 \\ \pm \xi_3 - \xi_0 \pm \xi_1 \\ \pm \xi_2 \mp \xi_1 \mp \xi_0 \end{pmatrix},$$

(26)

(the upper signs belong to $\dot{\xi}$ and the lower to $\ddot{\xi}$) and $\ddot{\xi}_z$ for the third row of $\ddot{\xi}$ we can write the Hamiltonian most compactly as

$$\mathcal{H} = \frac{1}{8} \pi^t \xi^t \Theta^{-1} \dot{\xi} \pi + \frac{1}{\xi_z} \hat{\xi} \ddot{\xi}_z \cdot S.$$ 

(27)

The constants of motion induced by the constraint are $\xi^2$ and $\xi^t \pi$, which can be fixed to 1 and 0 respectively, by an appropriate choice of initial conditions. These two constants of motion are related to the identities $\xi \xi = \dot{\xi} = 0$ and $\xi \pi + \dot{\pi} = 0$. The angular momentum in body coordinates is $L = \dot{\xi} \pi / 2$, and in space fixed coordinates $\ddot{\xi} \pi / 2$. The reason for these symmetric expressions is the relation $\ddot{\xi} = B \hat{\xi}$. Using $\ddot{\xi} = \xi_2 \hat{\xi}$ this gives a factorization of $B = \ddot{\xi} / \xi_2$. The total angular momentum is given by $4t^2 = \xi_2 \pi^2 - (\xi^t \pi)^2$, and with the above choice of values for the geometric constraints it just gives $l^2 = \pi^2 / 4$.

Note that the Hamiltonian (and the constants of motion) are invariant under a change of “spin”, i.e. under the transformation $(\xi, \pi) \rightarrow (-\xi, -\pi)$, which is induced by the transformation to the antipodal point on $S^3$. Our coordinates are from $T^*S^3$, but the dynamics takes place in $T^*SO(3)$. E.g. for a periodic orbit which is non contractable in $SO(3)$ we will find twice its period in $S^3$. Most notably this applies to all the relative equilibria.

If we introduce $\gamma = \ddot{\xi} \xi_z / \xi^2$ for the coordinates of the space fixed unit vector in $z$-direction we can write the equations of motion as

$$\dot{\xi} = \frac{1}{2} \xi^t \Theta^{-1} L \quad \dot{\pi} = \frac{1}{2} \pi^t \Theta^{-1} L - \left( \frac{\partial \gamma}{\partial \xi} \right)^t S.$$ 

(28)

Finally we remark that since $l_z$ is also a constant of motion, we can introduce new coordinates $(\gamma, L)$ as given above, and perform a reduction to the standard Euler-Poisson-equations on $T^*S^2$.

4 Discussion

We have shown how to describe a Hamiltonian system in a configuration space with higher dimension than the number of degrees of freedom. The resulting description with excessive coordinates preserves the Hamiltonian structure. This description is always useful when the configuration space has a nontrivial topology such that there do not exist global coordinates. Our main motivation was to obtain singularity free and Hamiltonian equations of motion, which are needed, e.g. for the numerical calculation of actions in integrable cases as described in [4, 3]. The use of Euler parameters, which are used as a global coordinate system, has a long history. Weierstraß [13] and Klein & Sommerfeld [10] used them to obtain explicit solutions for integrable cases. Also Whittaker [14] and Goldstein [7] give an introduction to the description of rotations with Euler parameters. However they don’t
give a Hamiltonian in these variables. Related descriptions are also given by Kirchgraber and Stiefel for the use in perturbation theory, and in for numerical integration. Let us again remark that Kozlov gives the same equations, but with a different derivation. Our approach using Cayley’s map directly generalizes to $SO(n)$.

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