GORDIAN ADJACENCY FOR TORUS KNOTS

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ABSTRACT. A knot \( K_1 \) is called Gordian adjacent to a knot \( K_2 \) if there exists an unknotting sequence for \( K_2 \) containing \( K_1 \). We provide a sufficient condition for Gordian adjacency of torus knots via the study of knots in the thickened torus \( S^1 \times S^1 \times \mathbb{R} \). We also completely describe Gordian adjacency for torus knots of index 2 and 3 using Levine-Tristram signatures as obstructions to Gordian adjacency. In the last section, Gordian adjacency for torus knots is compared to the notion of adjacency for plane curve singularities.

1. Introduction

Let \( K_1 \) and \( K_2 \) be smooth knots in \( \mathbb{R}^3 \) or \( S^3 \). Their Gordian distance \( d_g(K_1, K_2) \) is the minimal number of crossing changes needed to get from \( K_1 \) to \( K_2 \), see e.g. Murakami \cite{Mur85}. The unknotting number \( u(K) \) of a knot \( K \), which was already studied by Wendt \cite{Wen37}, is the distance \( d_g(K, O) \), where \( O \) denotes the unknot. The Gordian distance induces a metric on the set of isotopy classes of all smooth knots. This discrete metric space is huge, for example, every \( \mathbb{Z}_n \) can be quasi-isometrically embedded into the subspace consisting of all torus knots by a result of Gambaudo and Ghys \cite{GG05}.

In this paper we study the subspace of torus knots and the simple question, ‘When is the triangle inequality \( d_g(K_1, K_2) \geq d_g(K_2, O) - d_g(K_1, O) \) an equality?’

Definition 1. Let \( K_1 \) and \( K_2 \) be knots. We say \( K_1 \) is Gordian adjacent to \( K_2 \), denoted by \( K_1 \leq_g K_2 \), if \( d_g(K_1, K_2) = u(K_2) - u(K_1) \).

Equivalently a knot \( K_1 \) is Gordian adjacent to \( K_2 \) if \( K_2 \) can be unknotted via \( K_1 \), that is, if there exists a unknotting sequence for \( K_2 \) that contains \( K_1 \). An unknotting sequence for a knot \( K \) is a sequence of \( u(K) + 1 \) knots starting with \( K \) and ending with the unknot \( O \) such that any two consecutive knots are related by a crossing change, see Baader \cite{Baa10}. The name ‘Gordian adjacency’ is motivated by the connection to algebraic adjacency, see below. Gordian adjacency is a partial order.

For two coprime natural numbers \( n \geq 2 \) and \( m \geq 2 \) we denote by \( T(n, m) = T(m, n) \) the (positive) torus knot obtained as the standard closure of the \( n \)-strand positive braid \( (\sigma_1 \cdots \sigma_{n-1})^m \) or alternatively as the knot of the singularity \( x^n - y^m \), see Section \( \ref{sec:torus_knots} \). The index of a torus knot \( T(n, m) \) is the minimum of \( n \) and \( m \).

Our main results on Gordian adjacency for torus knots are the following.

Theorem 2. Let \( (n, m) \) and \( (a, b) \) be pairs of coprime natural numbers with \( n \leq a \) and \( m \leq b \). Then the torus knot \( T(n, m) \) is Gordian adjacent to the torus knot \( T(a, b) \).

Theorem 3. Let \( n \) and \( m \) be natural numbers with \( n \) odd and \( m \) not a multiple of 3. Then the torus knot \( T(2, n) \) is Gordian adjacent to \( T(3, m) \) if and only if \( n \leq \frac{4}{3}m + \frac{1}{3} \).

The core of the proof of Theorem 2 is a generalization to knots in \( S^1 \times S^1 \times \mathbb{R} \) of the following elementary fact. If a knot \( K \) in \( \mathbb{R}^3 \) has a knot diagram with \( n \) crossings,
then \( u(K) \leq \frac{2}{7} \). The proof of Theorem 3 relies on explicit constructions of the required adjacencies and on Levine-Tristram signatures as obstructions to Gordian adjacency.

As a consequence of Theorem 2 Gordian adjacency and Gordian distance for torus knots of a fixed index are completely described, i.e. if a natural number \( a \) is fixed, then

\[
T(a, b) \leq_g T(a, c)
\]

for all \( b, c \) coprime to \( a \). Hence

\[
d_g(T(a, b), T(a, c)) = \left| u(T(a, b)) - u(T(a, c)) \right| = \frac{(a - 1)|b - c|}{2},
\]

where the second equation follows from the Milnor conjecture, which determines the unknotting number of torus knots, see equation (1). For torus knots \( T(a, b) \) and \( T(c, d) \) of different indices it is in general not clear how Gordian adjacency is characterized in terms of \( a, b, c, \) and \( d \). Theorem 3(8,16),(989,987) provides such a characterization for the case of indices 2 and 3.

**Remark 4.** To completely determine Gordian adjacency for torus knots of indices 2 and 3, it remains to check which \( T(3, m) \) are Gordian adjacent to torus knots of index 2. Using signatures it is easy to prove that if a torus knot \( T \) is adjacent to a torus knot of index 2, then \( T \) is \( T(3, 4), T(3, 5) \) or \( T(2, n) \) for some \( n \), see Section 4. We conjecture that \( T(3, 4) \) and \( T(3, 5) \) cannot occur and, more generally, that a torus knot cannot be Gordian adjacent to a torus knot of strictly smaller index.

An obvious motivation for finding Gordian adjacencies is that, by definition, every Gordian adjacency determines the Gordian distance of the involved knots. But Gordian adjacencies can also lead to good estimates of Gordian distances between nonadjacent torus knots. For example, the adjacencies \( T(2, 7) \leq_g T(2, 9) \) and \( T(2, 7) \leq_g T(3, 5) \) yield

\[
d_g(T(2, 9), T(3, 5)) \leq u(T(2, 9)) - u(T(2, 7)) + u(T(3, 5)) - u(T(2, 7)) = 4 - 3 + 3 - 2 = 2.
\]

The converse inequality can be proven using signatures, thus \( d_g(T(2, 9), T(3, 5)) = 2 \). Trying to generalize this example for any two torus knots \( T_1 \) and \( T_2 \) we look for the highest unknotting number \( u(K) \) realized by a knot \( K \), adjacent to both \( T_1 \) and \( T_2 \), and ask if \( u(T_1) - u(K) + u(T_2) - u(K) \) is close to the Gordian distance \( d_g(T_1, T_2) \). An ambitious future goal is to use such Gordian adjacencies to determine Gordian distances between all torus knots up to a constant factor, similarly to what was done by Baader for cobordism distance (Baa12).

The **cobordism distance** between two knots \( K_1 \) and \( K_2 \) is defined to be the minimal genus of a connected smooth surface \( F \) in \( S^3 \times [0, 1] \) with \( \partial F = K_1 \times \{0\} \cup K_2 \times \{1\} \). Similar to the unknotting number for the Gordian distance, the *slice genus* or *4-ball genus* of a knot, denoted by \( g_s \), is the cobordism distance to the unknot \( O \). As a crossing change can be realized by a cobordism of genus 1, the Gordian distance is larger than the cobordism distance and a Gordian adjacency between knots \( K_1 \) and \( K_2 \) yields a cobordism of genus \( u(K_2) - u(K_1) \).

Another motivation for the study of Gordian adjacency comes from the notion of adjacency for singularities of algebraic curves in \( \mathbb{C}^2 \) studied by Arnold (Arn72), which yields a notion of adjacency for algebraic knots, see Section 6. Such an adjacency of algebraic knots \( K_1 \) and \( K_2 \) yields a smooth algebraic curve \( F \) in \( \mathbb{C}^2 \), such that \( K_1 \) and

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1 e.g. a manifold of real dimension 2
$K_2$ are realized as transversal intersection of $F$ with two spheres around the origin of different radii $r_1 < r_2$, i.e.

$$K_i = F \cap \{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 = r_i^2\} \subset \{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 = r_i^2\} \cong S^3.$$  

By a theorem of Kronheimer and Mrowka [KM93, Corollary 1.3], known as the Thom conjecture, the slice genus $g_s(K_2)$ of $K_2$ equals the genus of the intersection of $F$ with the ball centered at the origin of $\mathbb{C}^2$ of radius $r_2$, and the cobordism

$$F \cap \{(x, y) \in \mathbb{C}^2 \mid r_1^2 \leq \|x\|^2 + \|y\|^2 \leq r_2^2\}$$  

in

$$\{(x, y) \in \mathbb{C}^2 \mid r_1^2 \leq \|x\|^2 + \|y\|^2 \leq r_2^2\} \cong S^3 \times [0, 1]$$

has minimal genus $g_s(K_2) - g_s(K_1)$. By the Milnor conjecture (a consequence of the Thom conjecture) the slice genus and the unknotting number of algebraic knots are equal, e.g. for torus knots one has

$$u(T(n, m)) = g_s(T(n, m)) = \frac{(n - 1)(m - 1)}{2}$$

for all coprime natural numbers $n, m$.

In summary we know that $u$ and $g_s$ coincide on algebraic knots and both adjacency notions, which could be thought of as relative versions of $u$ and $g_s$, respectively, have similar properties. Namely for both notions it holds that if $K_1$ is adjacent to $K_2$, then $u(K_1) = g_s(K_1) \leq u(K_2) = g_s(K_2)$ and the cobordism distance equals $u(K_2) - u(K_1) = g_s(K_2) - g_s(K_1)$. Furthermore for both notions $T(n, m)$ is adjacent to $T(a, b)$ if $n \leq a$ and $m \leq b$, see Theorem 2 and Proposition 20. It is then natural to wonder whether the two concepts of adjacency coincide, for example, on torus knots? We answer by the negative in Section 3 but we give a heuristic argument supporting the conjecture that, if two torus knots are Gordian adjacent, then they are algebraically adjacent.

To decide whether a knot is Gordian adjacent to another knot, the unknotting numbers of the involved knots should certainly be known, thus, even ignoring the connection to algebraic adjacency, equality (1) is relevant to the study of Gordian adjacency for torus knots. It is used throughout the text.

Section 2 discusses examples of Gordian adjacent torus knots of index 2 and 3. In Section 3 we study unknotting of knots in $S^1 \times S^1 \times \mathbb{R}$ and use it to prove Theorem 2. Section 4 introduces Levine-Tristram signatures as obstructions to Gordian adjacencies and uses them to prove Theorem 3. Section 5 studies Gordian adjacencies between torus knots of higher indices. The relation between algebraic and Gordian adjacency is discussed in Section 6, in particular Proposition 21 provides a lot of examples of algebraic adjacent torus knots that are not Gordian adjacent.

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2. Examples of Gordian adjacencies.

By definition, the unknot $O$ is adjacent to every knot $K$. Let $k$ be a natural number. The unknotting number of the torus knot $T(2, 2k + 1)$ is $k$. An unknotting sequence of $T(2, 2k + 1)$ is provided by

$$T(2, 2k + 1) \rightarrow T(2, 2k - 1) \rightarrow \cdots \rightarrow T(2, 5) \rightarrow T(2, 3) \rightarrow O.$$  

Consequently $T(2, 2l + 1) \leq g T(2, 2k + 1)$ for all $l \leq k$, an easy instance of Theorem 2. We now construct explicit examples of Gordian adjacencies that are not provided by Theorem 2. Let $\lfloor \cdot \rfloor$ denote the integer part of a real number.
Proposition 5. For every natural number $k$ we have

$$T(2, 2k + 1) \leq T(3, \lceil \frac{3}{2}k + 1 \rceil).$$

Proof. The knot $T(2, 2k+1)$ is the standard closure of the braid

where $k - 3$ denotes the number of the crossings not drawn. We introduce a negative-to-positive crossing change for knots containing a part that looks (in an appropriate diagram) like the above $T(2, 2k+1)$.

$$\sigma_2 \sigma_1 \sigma_2 = \xrightarrow{(2)} x = x = x = x = x,$$

where the first and the two last equalities are obtained by applying the braid relation

$$\sigma_2 \sigma_1 \sigma_2 = x = x = \sigma_1 \sigma_2 \sigma_1.$$

First consider the case when $k$ is odd. We use (2) inductively.

$$T(2, 2k + 1) \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \rightarrow \xrightarrow{\frac{k-5}{2}} \text{crossing changes} \rightarrow T(3, \frac{3k-1}{2} + 2),$$
where every arrow indicates a negative-to-positive crossing change as in (2). Thus
\[
d_g \left( T(2, 2k + 1), T(3, 3\frac{k - 1}{2} + 2) \right) \leq \frac{k - 1}{2} = (3\frac{k - 1}{2} + 1) - k
\]
(1) \quad u(T(3, 3\frac{k - 1}{2} + 2)) - u(T(2, 2k + 1)).

The case when \( k \) is even has essentially the same proof except that the last crossing change does not use (2) but a slight variation of it. □

3. Unknotting on the torus and proof of Theorem 2

Knots in \( \mathbb{R}^3 \) can be studied via knot diagrams on \( \mathbb{R}^2 \) up to Reidemeister equivalence. Similarly, for a surface \( F \), knots in \( F \times \mathbb{R} \) can be studied via knot diagrams on \( F \).

In a knot diagram on \( \mathbb{R}^2 \) with \( n \) crossings one needs to change at most \( \lfloor \frac{n}{2} \rfloor \) of the crossings to get the unknot. This is easily proved geometrically by drawing a knot in \( \mathbb{R}^3 \) that projects to the curve on \( \mathbb{R}^2 \) given by the diagram and that descends (or ascends) monotonically except over one point in the diagram, see Figure 1, and remarking that such a knot is the unknot. To prove Theorem 2 which is a statement entirely about knots in \( \mathbb{R}^3 \), one is surprisingly led to ask whether a similar fact holds for knots in \( S^1 \times S^1 \times \mathbb{R} \). We provide such a result, which we then use to prove Theorem 2.

Let \( F \) be a surface. In what follows a closed smooth curve \( c : [0, 1] \rightarrow F \) is called \textit{presimple} if its lift \( \tilde{c} : \mathbb{R} \rightarrow \tilde{F} \) to the universal cover \( \tilde{F} \) of \( F \) is injective and if \( c \) is homotopic to a simple closed curve. A knot in \( F \times \mathbb{R} \) that is isotopic to a knot that projects to a simple closed curve on \( F \) is called \textit{unknotted}. There is at most one unknot (up to isotopy) in every homotopy class of closed curves in \( F \times \mathbb{R} \).

Lemma 6. For every presimple curve \( c \) in \( S^1 \times S^1 \) there is a knot \( O \) in \( S^1 \times S^1 \times \mathbb{R} \) that projects to \( c \) on \( S^1 \times S^1 \) and that is unknotted.

In terms of knot diagrams, Lemma 6 means that if a knot \( K \) in \( S^1 \times S^1 \times \mathbb{R} \) is given by a presimple diagram with \( n \) crossings on \( S^1 \times S^1 \), then one can get the diagram of the unknot by changing at most \( \lfloor \frac{n}{2} \rfloor \) of the \( n \) crossings. To prove this, we use Lemma 6 to get the unknot \( O \) with the same diagram as \( K \), except it differs in the choice of crossings. If this new diagram differs from the original one in less than half of the crossings, we are done. Otherwise we switch all crossings in the diagram of \( O \) yielding a knot diagram of a knot \( \overline{O} \). The knot \( \overline{O} \) is also unknotted, as the following shows. Let \( H_t \) be an isotopy that changes \( O \) to a knot that projects to a simple closed curve on \( S^1 \times S^1 \). Then parametrize \( \overline{O} \) in \( S^1 \times S^1 \times \mathbb{R} \) exactly the same way as \( O \), except changing the sign in the \( \mathbb{R} \) coordinate. The same isotopy \( H_t \) as for \( O \) (with a change of sign in the last coordinate) shows that \( \overline{O} \) is unknotted.
Clearly the assumption that \( c \) is homotopic to a simple closed curve is necessary in Lemma \( 1 \) we conjecture that this is also sufficient and that Lemma \( 1 \) holds for arbitrary orientable surfaces.

**Proof of Lemma 1** Denote \( S^1 \times S^1 \) by \( F \). Our strategy is to construct a presimple homotopy \( h_t \) of \( c \) (meaning \( h_t \) is presimple for every \( t \in [0,1] \)) to a simple closed curve and then to find an isotopy \( H_t \) of knots in \( F \times \mathbb{R} \) that has \( h_t \) as projection.

We first lift the curve \( c \) to a mapping \( \tilde{c} : \mathbb{R} \to \tilde{F} \), where \( \varphi : \tilde{F} \to F \) denotes the universal covering map. As \( c \) is presimple \( \tilde{c} : \mathbb{R} \to \tilde{F} \) is injective and there exists a simple closed curve \( g \) in the homotopy class of \( c \). We take \( g \) such that \( g(0) = c(0) \) and denote by \( \tilde{g} : \mathbb{R} \to \tilde{F} \) its lift to \( \tilde{F} \) with \( \tilde{g}(k) = \tilde{c}(k) \) for all \( k \in \mathbb{Z} \). Let \( \tilde{h}_t : \mathbb{R} \to \tilde{F} \) be an equivariant\footnote{i.e. \( \tilde{h}_t(s+1) = D(\tilde{h}_t(s)) \) for all \( s \) in \( \mathbb{R} \), where \( D \) denotes the unique deck transformation sending \( \tilde{c}(0) \) to \( \tilde{c}(1) \).} isotopy between \( \tilde{c} \) and \( \tilde{g} \) that is constant on \( \mathbb{Z} \), see Figure 2. Of course

\[
h_t = \varphi \circ \tilde{h}_t : [0,1] \to F
\]

is a presimple homotopy.

The idea for building \( H_t \) is to measure how far away from \( g \) points \( p = h_t(s) \) are and then to put this distance \( d(p) \) in the second coordinate of \( H_t \). We need a metric to make this precise and the distance will actually be measured in the universal cover. Put a Riemannian metric on \( F \) with constant curvature 0 such that \( g \) is a simple closed geodesic of length 1. \( \tilde{F} \) is identified with the Euclidean plane \( \mathbb{R}^2 \) such that \( \varphi : \tilde{F} \to F \) is locally an isometry. Let \( d : \tilde{F} \to \mathbb{R} \) denote the oriented distance to the straight line \( \tilde{g} \). We claim that the homotopy

\[
H_t : [0,1] \to F \times \mathbb{R}, s \mapsto (h_t(s), d(\tilde{h}_t(s))),
\]

which projects to the homotopy \( h_t \) on \( F \), is an isotopy and thus finishes the proof.

Assume towards a contradiction that \( H_t \) is not injective for some fixed \( t \). Without loss of generality, we assume \( t = 0 \), i.e. \( \tilde{h}_t = \tilde{c} \). If there exist \( s \neq r \in [0,1] \), such that \( H_0(s) = H_0(r) \), then, by definition of \( H_0 \), the two points \( \tilde{p}_1 = \tilde{c}(s) \) and \( \tilde{p}_2 = \tilde{c}(r) \) in \( \tilde{F} \) satisfy

\[
\varphi(\tilde{p}_1) = \varphi(\tilde{p}_2) \quad \text{and} \quad d(\tilde{p}_1) = d(\tilde{p}_2).
\]

As \( d(\tilde{p}_1) = d(\tilde{p}_2) \), there is a geodesic segment parallel to \( \tilde{g} \) from \( \tilde{p}_1 \) to \( \tilde{p}_2 \). The length of this segment is an integer \( k \), since \( \varphi(\tilde{p}_1) = \varphi(\tilde{p}_2) \). It follows that \( \tilde{p}_2 = \tilde{c}(k + s) \) if

\[
\tilde{c}(1) = \tilde{g}(1)
\]

\[
\tilde{c}(0) = \tilde{g}(0)
\]

**Figure 2.** An equivariant isotopy (green) of \( \tilde{c} \) (black) to \( \tilde{g} \) (red) is indicated.

\footnote{Ordinary Euclidean distance of points in \( \tilde{F} = \mathbb{R}^2 \) to the straight line \( \tilde{g} \) with a sign depending on whether the point is on the left or the right of \( \tilde{g} \).}
the sign of $k$ is chosen correctly. This is seen by lifting $c$ to $\tilde{F}$ such that the lift starts at $\tilde{g}(k) = \tilde{c}(k)$, see Figure 3 for a case with $k = 1$. But $\tilde{c}(r) = \tilde{c}(k+s)$ and $k+s \neq r$ contradict the injectivity of $\tilde{h}_t = \tilde{c}$.

The proof of Theorem 2 is a generalization of the following example.

**Example 7.** Applying Lemma 6 we show that $T(3,4)$ is Gordian adjacent to $T(3,7)$. As $u(T(3,7)) - u(T(3,4)) = 3$ we need to show that we can change $T(3,7)$ to $T(3,4)$ via 3 crossing changes. For this we draw $T(3,7)$ in an $\varepsilon$-neighborhood $N(S^1 \times S^1)$ of the standard torus $S^1 \times S^1 \subset \mathbb{R}^3$ as shown in Figure 4. Projecting $T(3,7)$ to $S^1 \times S^1$ yields a curve with 6 crossings. Thus by Lemma 6 changing 3 of the crossings suffices to produce a knot $K$ in $N(S^1 \times S^1)$ that is unknotted. The knots $K$ and $T(3,4)$ are homotopic unknots in $N(S^1 \times S^1)$ and therefore isotopic in $N(S^1 \times S^1)$. In particular $K$ and $T(3,4)$ are isotopic as knots in $\mathbb{R}^3$.

![Figure 3](image1.png)

*Figure 3.* The curve $\tilde{c}|_{[0,1]}$ (black) intersects $\tilde{c}|_{[k,k+s]}$ (blue) in $\tilde{p}_2$.

![Figure 4](image2.png)

*Figure 4.* The knot $T(3,7)$ on a torus (green) s.t. the homotopy class is that of the simple closed $T(3,4)$. Four arcs (red) are on the upper half of the torus, the rest of the knot (black) lies on the lower half; the knot has $(7 - 4)(3 - 1) = 6$ crossings on the torus.
Proof of Theorem 2. Let $a < b$ and $n < m$. First the case $a = n$. We need to show that $d_g(T(a, b), T(n, m))$ is less or equal to

$$u(T(a, b)) - u(T(n, m)) = \frac{(b - 1)(a - 1)}{2} - \frac{(m - 1)(n - 1)}{2} = \frac{(b - m)(a - 1)}{2}.$$  

We consider the knot $T(a, b)$ as the standard diagram on $\mathbb{R}^2$ of the closure of the braid $(\sigma_1 \sigma_2 \cdots \sigma_{b-1})^b$. Then we work on the standard torus $S^1 \times S^1$ in $\mathbb{R}^3$. We draw $m$ arcs on the upper half of the torus and the rest of the knot, including arcs do not intersect the rest of the knot on the torus, so we have $(b - m)(a - 1)$ crossings on the torus. By Lemma 6 we need to change at most $\frac{(b - m)(a - 1)}{2}$ crossings in the diagram on the torus to get a knot isotopic to $T(n, m)$ in $S^1 \times S^1 \times \mathbb{R}$. Of course the same changes can be made in the diagram on $\mathbb{R}^2$ and the resulting knot is $T(n, m)$ via the same isotopy as in $S^1 \times S^1 \times \mathbb{R}$ Therefore $d_g(T(a, b), T(n, m)) \leq \frac{(b - m)(a - 1)}{2}$ as we wanted. The same argument works if $m = b$ or $a = m$.

This leaves the case $n < a$. We may assume $m > b - a$, otherwise we replace (inductively) $a, b$ by $a, b - a$, since by the first case $T(a, b - a) \leq_g T(a, b)$. To apply the same idea as before we need to reduce a braid on $a$ or $b$ strands to one on $n$ or $m$ strands. Figure 5 gives the pictorial idea if $m$ equals $b - 1$. More precisely, represent $T(a, b)$ as the closure of the braid

$$(\sigma_1 \cdots \sigma_{b-1})^a = \sigma_a \cdots \sigma_1(\sigma_2 \cdots \sigma_{b-1})^a,$$

which has the same closure as

$$\tau_a^b = \sigma_{a-1} \cdots \sigma_1(\sigma_1 \cdots \sigma_{a-2})^a,$$

see Figure 6. If $m = b - 1$, then we represent $T(a, b)$ as the closure of $\tau_{b-1}$ and draw this similarly as in the first case on the torus with $n$ of the $a$ arcs on the upper half of the torus and the rest of the knot, including $a - 1 + (a - n)(b - 2)$ crossings, on the lower half. Then we use Lemma 6 to get $T(n, m)$ by at most $\frac{a - 1 + (a - n)(b - 2)}{2}$ crossing changes. Thus $d_g(T(n, m), T(a, b))$ is less or equal to

$$\frac{a - 1 + (a - n)(b - 2)}{2} = \frac{(a - 1)(b - 1)}{2} - \frac{(n - 1)(b - 2)}{2} = u(T(a, b)) - u(T(n, m)).$$

\footnote{Here $S^1 \times S^1 \times \mathbb{R}$ is interpreted as an $\varepsilon$-neighborhood of the standard torus $S^3 \times S^1 \subseteq \mathbb{R}^3$.}
Suppose now $m < b - 1$. We change a crossing in $\tau_{b-1}$ to get

\begin{equation}
\sigma_{a-1} \cdots \sigma_2 \sigma_1^{-1} (\sigma_1 \cdots \sigma_{b-2})^a = \sigma_{a-1} \cdots \sigma_2 \sigma_1 (\sigma_1 \cdots \sigma_{b-2})^{a-1}
\end{equation}

and then replace in (3) the part $(\sigma_1 \cdots \sigma_{b-2})^{a-1}$ by $\sigma_{a-1} \cdots \sigma_1 (\sigma_2 \cdots \sigma_{b-2})^{a-1}$ as in (3), which has the same closure as

$$\tau_{b-2} = (\sigma_{a-2} \cdots \sigma_1 \sigma_{b-3})^2 (\sigma_1 \cdots \sigma_{b-3})^{a-2},$$

see Figure 6. If $m = b - 2$ we use Lemma 6 to get $T(n, m)$ from the closure of $\tau_{b-2}$ by changing $\frac{(a-2) + (a-2) + (a-n)(b-3)}{2}$ crossings. Thus $d_g(T(n, m), T(a, b))$ is less or equal to

$$1 + \frac{(a-2) + (a-2) + (a-n)(b-3)}{2} = \frac{2a - 2 + (a-n)(b-3)}{2} = \frac{(a-1)(b-1) - (n-1)(b-3)}{2} = u(T(a, b)) - u(T(n, m)).$$

For general $m > b - a$ it follows similarly that we need to change

$$1 + 2 + \cdots (b - m - 1) = \frac{(b-m)(b-m-1)}{2}$$

crossings of $T(a, b)$ to get the closure of

$$\tau_m = (\sigma_{a-(b-m)} \cdots \sigma_1 \sigma_{b-1})^{b-m} (\sigma_1 \cdots \sigma_{b-1})^{a-(b-m)},$$

see Figure 7. And then we put the closure of $\tau_m$ on the torus, with $n$ arcs on the upper half and $(b-m)(a-(b-m)) + (a-n)(m-1)$ crossings on the lower half. Therefore we get $T(n, m)$ from $\tau_m$ by changing $\frac{(b-m)(a-(b-m)) + (a-n)(m-1)}{2}$ crossings by Lemma 6. Combined we have that $d_g(T(n, m), T(a, b))$ is less or equal to

$$\frac{(b-m)(b-m-1) + (b-m)(a-(b-m)) + (a-n)(m-1)}{2} = u(T(a, b)) - u(T(n, m)).$$

□
Figure 7. The braid $\tau_m$, which can be obtained from $T(a, b)$ by $1 + 2 + \cdots + (b - m - 1)$ crossing changes.

4. Levine-Tristram signatures as obstructions to adjacency

The goal of this section is to prove Theorem 3 using Levine-Tristram signatures [Lev69] [Tri69]. For torus knots they are easy to calculate and yield good obstructions to adjacency, see Lemma 13 and Proposition 11.

Definition 8. [Lev69] [Tri69] Let $A$ be a Seifert matrix of a knot $K$ and $\omega$ in $S^1 \setminus \{1\} \subset \mathbb{C}$. The $\omega$-signature $\sigma_\omega(K) \in \mathbb{Z}$ is defined to be the number of negative eigenvalues minus the number of positive eigenvalues of the Hermitian matrix $(1 - \omega)A + (1 - \overline{\omega})A^t$.

The $\omega$-signature is independent of the choice of Seifert matrix and $\sigma_\omega = \sigma_{\overline{\omega}}$. Setting $\omega = -1$ one recovers the classical signature $\sigma_{-1}$ introduced by Trotter [Tro62]. Note that our nonstandard(!) sign convention of signatures is such that all (positive) torus knots have positive signature, e.g. $\sigma_{-1}(T(2, 3)) = 2$ and not $\sigma_{-1}(T(2, 3)) = -2$.

We say $\omega$ in $S^1 \setminus \{1\}$ is regular for a knot $K$ if $\omega$ is not a root of the Alexander polynomial of $K$. Regular $\omega$'s are all $\omega$ for which $(1 - \omega)A + (1 - \overline{\omega})A^t$ is an invertible matrix, where $A$ is a Seifert matrix for $K$. If $\omega$ is regular for a knot $K$, then the signature $\sigma_\omega(K)$ is even, and if $\omega$ is a root of unity of prime order, then $\omega$ is regular for every knot [Tri69]. For a fixed knot $K$ the signature $\sigma_\omega(K)$ is piecewise-constant in $\omega$, “jumping” at the nonregular $\omega$ [Lev69].

Let us denote by $s(K)$ the Rasmussen invariant of a knot $K$ [Ras10]. The next lemma shows how $\omega$-signatures and $s$ behave with respect to crossing changes.

Lemma 9. If $K_-$ is obtained from $K_+$ via one positive-to-negative crossing change and if $\omega$ is regular for $K_1$ and $K_2$, then

$$\sigma_\omega(K_-) \in \{\sigma_\omega(K_+), \sigma_\omega(K_+) - 2\}.$$  

The same holds for the Rasmussen invariant.

Rasmussen used an observation by Livingston [Liv04, Corollary 2 and 3] to prove Lemma 9 for $s$ [Ras10]. At the end of this section we use a variation of this observation to give a proof of Lemma 9.

As a corollary we have the following, see e.g. [GG05].

Corollary 10. Let $K_1$, $K_2$ be knots and let $\omega$ be regular for $K_1$ and $K_2$. Then

$$\left|\frac{\sigma_\omega(K_2) - \sigma_\omega(K_1)}{2}\right| \leq d_g(K_1, K_2).$$

In particular, if $K_1$ is adjacent to $K_2$, then $|\frac{\sigma_\omega(K_2) - \sigma_\omega(K_1)}{2}| \leq u(K_2) - u(K_1)$. 
We apply Corollary 10 to prove that most torus knots are not adjacent to torus knots of index two, as claimed in Remark 1. For index two torus knots the signature equals twice the unknotting number, that is
\[ \sigma_{−1}(T(2,n)) = u(T(2,n)) = \frac{n−1}{2}. \]
This is also true for \( T(3,4) \) and \( T(3,5) \), but for all other torus knots \( T \) there is a signature defect, i.e. \( u(T) > \frac{\sigma_{−1}(T)}{2} \). Thus, by Corollary 10
\[ d_g(T(2,n),T) \geq \frac{\sigma_{−1}(T(2,n))}{2} - \frac{\sigma_{−1}(T)}{2} > u(T(2,n)) − u(T) \]
for all torus knots \( T \) not equal to \( T(3,4) \), \( T(3,5) \) or some \( T(2,m) \).

The following proposition explains how Lemma 9 gives another obstruction to Gordian adjacency of torus knots, which is often better than Corollary 10.

**Proposition 11.** Let \( T_1 \leq_g T_2 \) be a Gordian adjacency of torus knots and let \( \omega \) be regular for \( T_1 \) and \( T_2 \). Then \( \sigma_{\omega}(T_1) \leq \sigma_{\omega}(T_2) \).

**Proof.** For all torus knots \( T \) we have \( s(K) = u(T) \) [Ras10]. Thus Lemma 9 yields that an unknotting sequence of any torus knot involves only positive-to-negative crossing changes, since \( s \) has to drop by 2 with every crossing change. Choose an \( \tilde{\omega} \) that is regular for every knot and such that \( \sigma_{\tilde{\omega}}(T_1) = \sigma_{\tilde{\omega}}(T_1) \) and \( \sigma_{\tilde{\omega}}(T_2) = \sigma_{\tilde{\omega}}(T_2) \). This is for example achieved by a root of unity of prime order that is close to \( \omega \). Let now
\[ T_2 = K_{u(T_2)} \to K_{u(T_2)−1} \to \cdots \to K_1 \to K_0 = O \]
be an unknotting sequence for \( T_2 \) that contains \( T_1 \). As it involves only positive-to-negative crossing changes we have
\[ \sigma_{\tilde{\omega}}(T_2) \geq \sigma_{\tilde{\omega}}(K_{u(T_2)−1}) \geq \cdots \geq \sigma_{\tilde{\omega}}(T_1) \geq \cdots \geq \sigma_{\tilde{\omega}}(O) = 0 \]
by Lemma 9 and thus \( \sigma_{\omega}(T_1) = \sigma_{\tilde{\omega}}(T_1) \leq \sigma_{\tilde{\omega}}(T_2) = \sigma_{\omega}(T_2) \). \( \square \)

**Remark 12.** By the above proof Proposition 11 remains true for any knot \( K \) with \( \frac{s(K)}{2} = u(K) \). For example, all knots that are closures of positive braids, which include algebraic knots.

We prove Theorem 3 using Proposition 11 and the following combinatorial formula for the Levine-Tristram signatures of torus knots, see [GG05], for \( \sigma_{−1} \) it is originally due to Brieskorn [Bri66]. We denote the cardinality of a finite set \( S \) by \( |S| \).

**Lemma 13.** Let \( n \geq 2 \) and \( m \geq 2 \) be coprime natural numbers. Set \( S = \{ \frac{k}{m} \mid \frac{1}{2} \leq k \leq n − 1, 1 \leq l \leq m − 1 \} \subset [0,2] \). Then for \( \theta \in [0,1] \) we have
\[ \sigma_{\theta} = \sigma_{\theta}(T(n,m)) = |S| = |\theta, \theta + 1| - |S\setminus(\theta, \theta + 1)|. \]

**Proof of Theorem 3.** Fix \( n = 2k + 1 \) and note that \( m = \lfloor \frac{3}{2} k + 1 \rfloor \) is minimal with \( n \leq \frac{1}{2} m + \frac{1}{2} \). By Proposition 3 we have \( T(2,2k + 1) \leq_g T(3, \lfloor \frac{3}{2} k + 1 \rfloor) \). Together with \( T(3, \lfloor \frac{3}{2} k + 1 \rfloor) \leq_g T(3,m) \) for all \( m \geq \lfloor \frac{3}{2} k + 1 \rfloor \), an easy instance of Theorem 2 we conclude that \( T(2,2k + 1) \leq_g T(3,m) \) for all \( m \geq \lfloor \frac{3}{2} k + 1 \rfloor \).

For the other direction we let \( n = 2k + 1 \) be any odd number and write \( m = \lfloor \frac{3}{2} k - 1 \rfloor \), which is the largest \( m \) that does not satisfy \( n \leq \frac{1}{2} m + \frac{1}{2} \). Thus we have to show that \( T(2,2k + 1) \leq_g T(3,m) \). For \( k \leq 4 \) calculating unknotting numbers yields \( T(2,5) \leq_g T(3,2) \), \( T(2,7) \leq_g T(3,4) \), and \( T(2,9) \leq_g T(3,5) \). If \( k \geq 5 \) we distinguish two cases. Either \( k \) equals 1 or 2 modulo 4 or \( k \) equals 3 or 4 modulo 4.
For \( k = 1 + 4l, 2 + 4l, \ l \geq 1 \) a calculation using Gordon-Litherland-Murasugi’s reduction formula [GLM81, Theorem 5.2] shows that
\[
\sigma_{-1}(T(3, m)) = 2k - 2,
\]
which is strictly less than
\[
\sigma_{-1}(T(2, 2k + 1)) = 2k.
\]
Thus Proposition [11] yields \( T(2, 2k + 1) \nless T(3, m) \).

For \( k = 3 + 4l, 4 + 4l, \ l \geq 1 \) we use [GLMS1, Theorem 5.2] similarly to see that
\[
(5) \quad \sigma_{-1}(T(3, m)) = 2k = \sigma_{-1}(T(2, 2k + 1)).
\]
Here \( \sigma_{-1} \) does not suffice as obstruction directly, but we can use (5) to calculate \( \sigma_{\omega}(T(3, m)) \) for \( \omega \) close to \(-1\), which will yield the desired obstruction. More precisely set
\[
(6) \quad \omega = e^{2\pi i \theta} \quad \text{with} \quad \left\{ \begin{array}{ll}
\theta \in \left( \frac{1}{2} - \frac{3}{3m}, \frac{1}{2} - \frac{1}{3m} \right) & \text{for } m \text{ even}, i.e. \ k = 3 + 4l \\
\theta \in \left( \frac{1}{2} - \frac{2}{3m}, \frac{1}{2} - \frac{1}{3m} \right) & \text{for } m \text{ odd}, i.e. \ k = 4 + 4l
\end{array} \right.
\]
By Lemma [14] the value of \( \sigma_{\omega}(T(3, m)) \) is the same for all these \( \omega \).

Claim 14. For all \( k = 3 + 4l, 4 + 4l \) with \( l \geq 1 \) and \( \omega \) as in (6), we have \( \sigma_{\omega}(T(3, m)) = \sigma_{-1}(T(3, m)) - 2 \).

As the above \( \omega \) can be chosen such that \( \sigma_{-1}(T(2, 2k + 1)) = \sigma_{\omega}(T(2, 2k + 1)) \), Claim 14 and (5) yield
\[
\sigma_{\omega}(T(3, m)) = \sigma_{-1}(T(3, m)) - 2 < \sigma_{-1}(T(3, m)) = \sigma_{-1}(T(2, 2k + 1)) = \sigma_{\omega}(T(2, 2k + 1)).
\]
Therefore \( T(2, 2k + 1) \nless T(3, m) \) by Proposition [11]. It remains to prove Claim 14.

For the case when \( m \) is even, Lemma [13] applied to the knot
\[
T = T(3, m) = T(3, \left[ \frac{3}{2} (3 + 4l) - 1 \right]) = T(3, 4 + 6l)
\]
yields
\[
\sigma_{-1}(T) = \sharp S \setminus \left[ \frac{1}{2} - \frac{1}{3m} + \varepsilon, \frac{3}{2} - \frac{1}{3m} + \varepsilon \right] - \sharp S \setminus \left( \frac{1}{2} - \frac{1}{3m} + \varepsilon, \frac{3}{2} - \frac{3}{3m} + \varepsilon \right)
\]
and
\[
\sigma_{\omega}(T) = \sharp S \setminus \left[ \frac{1}{2} - \frac{1}{3m} - \varepsilon, \frac{3}{2} - \frac{1}{3m} - \varepsilon \right] - \sharp S \setminus \left( \frac{1}{2} - \frac{1}{3m} - \varepsilon, \frac{3}{2} - \frac{3}{3m} - \varepsilon \right)
\]
for \( \varepsilon \) small enough. Observe that
\[
3 - \frac{1}{3m} = \frac{2}{3} + \frac{5m - 2}{6m} = \frac{2}{3} + \frac{5(4 + 6l) - 2}{6(4 + 6l)} = \frac{2}{3} + \frac{3 + 5l}{4 + 6l} = \frac{2}{3} + \frac{3 + 5l}{m} \in S
\]
and
\[
\frac{1}{2} - \frac{1}{3m} = \ldots = \frac{1}{3} + \frac{5k}{m} + \frac{11}{3m} \notin S.
\]
This means
\[
\left( S \setminus \left[ \frac{1}{2} - \frac{1}{3m} - \varepsilon, \frac{3}{2} - \frac{1}{3m} - \varepsilon \right] \right) \cup \left\{ \frac{3}{2} - \frac{1}{3m} \right\} = S \setminus \left[ \frac{1}{2} - \frac{1}{3m} + \varepsilon, \frac{3}{2} - \frac{3}{3m} + \varepsilon \right]
\]
and
\[
S \setminus \left( \frac{1}{2} - \frac{1}{3m} - \varepsilon, \frac{3}{2} - \frac{1}{3m} - \varepsilon \right) = \left( \left( S \setminus \left[ \frac{1}{2} - \frac{1}{3m} + \varepsilon, \frac{3}{2} - \frac{3}{3m} + \varepsilon \right] \right) \cup \left\{ \frac{3}{2} - \frac{1}{3m} \right\} \right).
\]
Therefore \( \sigma_{-1}(T) = 2 + \sigma_{\omega}(T) \).

If \( m \) is odd, we have \( m = 5 + 6l \). Similarly to the even case, we get
\[
\frac{3}{2} - \frac{1}{6m} = \frac{2}{3} + \frac{4 + 5l}{m} \in S, \text{ but } \frac{1}{2} - \frac{1}{6m} \notin S.
\]
The rest of the argument is the same. \( \square \)

It remains to prove Lemma [9]. Let \(-K\) denote the mirror image of the knot \( K \) and let \( K_1 \sharp K_2 \) denote the connected sum of two knots \( K_1 \) and \( K_2 \).
Lemma 15. Let \( \tau \) be an integer valued knot invariant satisfying

- \( \tau(K_1 \sharp K_2) = \tau(K_1) + \tau(K_2) \) and \( \tau(-K_1) = -\tau(K_1) \) for all knots \( K_1 \) and \( K_2 \),
- \( \tau(K) \leq g_s(K) \) for all knots \( K \) and
- There exists a knot \( K \) with \( \tau(K) = 1 \) that can be transformed to the unknot \( O \) by a positive-to-negative crossing change.

Then \( \tau \) is a concordance invariant, \( |\tau(K)| \leq g_s(K) \) for all knots \( K \), and

\[ 0 \leq \tau(K^+) - \tau(K^-) \leq 1 \]

whenever \( K^- \) is a knot obtained from \( K^+ \) by a positive-to-negative crossing change.

Lemma 15 is a variation of the statement in [Liv04, Corollary 2 and 3]. The first two assertions are given in [Liv04, Corollary 2]. The proof of the third assertion given in [Liv04] needs to be modified as follows to yield a proof Lemma 15. Replace, in the proof of [Liv04, Corollary 3], the knot \( T(2,3) \) by a knot \( K \) with \( \tau(K) = 1 \) that can be unknotted by changing one positive crossing to a negative one. This is necessary as we do not assume that \( \tau(T(2,3)) = 1 \).

Proof of Lemma 2. Rasmussen proves all conditions of Lemma 15 for \( \tau = \frac{s}{2} \) in [Ras10] (note that \( \frac{4(T(2,3))}{2} = 1 \)). This yields the desired result for \( s \). For every \( \omega \) that is regular for \( K^- \) and \( K^+ \) there is a root of unity \( \tilde{\omega} \) of prime order with \( \sigma_{\tilde{\omega}}(K^-) = \sigma_{\omega}(K^-) \) and \( \sigma_{\tilde{\omega}}(K^+) = \sigma_{\omega}(K^+) \). Thus it is enough to check the three conditions of Lemma 15 for \( \frac{4(T(2,3))}{2} \), where \( \omega \) is a root of unity of prime order.

For signatures \( \sigma_{\omega}(K_1 \sharp K_2) = \sigma_{\omega}(K_1) + \sigma_{\omega}(K_2) \) and \( \sigma_{\omega}(-K_1) = -\sigma_{\omega}(K_1) \) follow from the fact that \( A_1 \oplus A_2 \) is a Seifert matrix for \( K_1 \sharp K_2 \) and \( -A_1 \) is a Seifert matrix for \( -K_1 \) if \( A_1 \) and \( A_2 \) are Seifert matrices for \( K_1 \) and \( K_2 \), respectively. If \( \omega = 1 \), the second condition is proven by Murasugi [Mur65] and for the third condition we can choose \( K \) to be \( T(2,3) \) as \( \sigma_{-1}(T(2,3)) = 2 \). In general fix a root of unity \( \omega \) of prime order.

The second condition is proven by Tristram [Tri69] \(^5\). It remains to find a suitable \( K \) for the third condition. For a natural number \( k \) let \( T(2k - 1) \) be the positive twist knot with \( 2k - 1 \) half-twists. These knots can be unknotted by a positive-to-negative crossing change. One checks that \( A = \begin{bmatrix} -k & 1 \\ 0 & -1 \end{bmatrix} \) is a Seifert matrix for \( T(2k - 1) \). For sufficiently large \( k \) both eigenvalues of the Hermitian matrix \( (1 - \omega)A + (1 - \bar{\omega})A^t \) are negative. Thus \( K \) can be chosen to be \( T(2k - 1) \) for a sufficiently large \( k \).

5 Note that Tristram formulates this result only for \( \omega = -1 \) and \( \omega = e^{-\frac{i\pi(p-1)}{p}} \), where \( p \) is any odd prime; however, his proof works for every root of unity of prime order.

5 A bound on Gordian adjacency for torus knots of higher indices

This section is concerned with the question, when is \( T(a,n) \leq g(T(b,m)) \) for fixed \( a < b \) and \( n, m \) large? Concretely we study the numbers

\[
\varepsilon(a,b) = \lim_{m \to \infty} \inf \frac{n(m)}{m} \quad \text{and} \quad \varepsilon'(a,b) = \lim_{m \to \infty} \sup \frac{n(m)}{m},
\]

where \( n(m) \) denotes the largest integer such that \( T(a,n(m)) \leq g(T(b,m)) \). We suspect, but cannot prove, that \( \varepsilon(a,b) = \varepsilon'(a,b) \) for all \( a < b \in \mathbb{N} \). Certainly \( \varepsilon(2,3) = \varepsilon(2,3) = \frac{1}{3} \) by Theorem 3. Also note that \( 1 \leq \varepsilon(a,b) \) by Theorem 2 and \( \varepsilon(a,b) \leq \frac{b-1}{m-1} \) as

\[
\frac{(a - 1)(n(m) - 1)}{2} = u(T(a,n(m))) \leq u(T(b,m)) = \frac{(b-1)(m-1)}{2}.
\]

Using Levine-Tristram signatures we get an upper bound for \( \varepsilon'(a,b) \) that is strictly better than \( \frac{b-1}{m-1} \).
Proposition 16. If $a \leq b \in \mathbb{N}$, then
\[
\overline{c}(a, b) \leq \frac{a \left\lfloor \frac{b}{a} \right\rfloor^2 - (a + 2b) \left\lfloor \frac{b}{a} \right\rfloor + b(b + 1)}{(a - 1)b} \leq \frac{b}{a}.
\]

A calculation shows that $\frac{a \left\lfloor \frac{b}{a} \right\rfloor^2 - (a + 2b) \left\lfloor \frac{b}{a} \right\rfloor + b(b + 1)}{(a - 1)b} = \frac{b}{a}$ if and only if $a$ divides $b$. If for example $b - a$ equals 1, Proposition 16 yields
\[
\overline{c}(a, a + 1) \leq \frac{a + 2}{a + 1}.
\]

This is better than $\frac{b}{a} = \frac{a + 1}{a}$ or even $\frac{b - 1}{a - 1}$, but we only know it to be optimal for $a = 2$, namely $\underline{c}(2, 3) = \overline{c}(2, 3) = \frac{4}{3}$.

Proof. We will use an approximation given by Gambaudo and Ghys in [GG05, Proposition 5.2]. Let $l$ be a natural number, then
\[
|\sigma_{2+\omega}(T(b, m)) - m \left(2(b - (2l - 1))\theta + \frac{2l(l - 1)}{b}\right)| \leq 2b \text{ for } l \leq \frac{l - 1}{b} < \theta \leq \frac{l}{b}.
\]

Proposition 11 yields $\sigma_{\omega}(T(b, m)) - \sigma_{\omega}(T(a, n(m))) \geq 0$ if $\omega$ is regular for $T(b, m(n))$ and $T(a, n)$. By the approximation we get
\[
\begin{align*}
&\frac{m}{2} \left(2(b - (2l - 1))\theta + \frac{2l(l - 1)}{b}\right) - n(m) \left(2(a - (2l' - 1))\theta + \frac{2l'(l' - 1)}{a}\right) \\
&\quad \geq -2(a + b),
\end{align*}
\]

where $l$ and $l'$ are natural numbers with $\frac{l - 1}{b} < \theta \leq \frac{l}{b}$ and $\frac{l' - 1}{a} < \theta \leq \frac{l'}{a}$, respectively. Choosing $\theta = \frac{1}{a}$ the inequality (7) becomes
\[
\begin{align*}
&\frac{m}{2} \left(2(b - (2\left\lfloor \frac{b}{a} \right\rfloor - 1)) + 2 \left\lfloor \frac{b}{a} \right\rfloor (\left\lfloor \frac{b}{a} \right\rfloor - 1)\right) - n(m)2\frac{a - 1}{a} \\
&\quad \geq -2(a + b)
\end{align*}
\]
or equivalently
\[
\frac{n(m)}{m} \leq \frac{a \left\lfloor \frac{b}{a} \right\rfloor^2 - (a + 2b) \left\lfloor \frac{b}{a} \right\rfloor + b(b + 1)}{(a - 1)b} + \frac{a(a + b)}{m(a - 1)}.
\]

This proves the first inequality. The second inequality can be checked by a calculation.

\[\text{□}\]

Remark 17. Our choice $\theta = \frac{1}{a}$ is the best possible and yields the optimal bound for $\overline{c}(a, b)$ that can be achieved using the properties of signatures from Lemma 9. This can be checked using the above approximation from [GG05].

In order to determine $\underline{c}(a, b)$ and $\overline{c}(a, b)$ for $(a, b) \neq (2, 3)$, we now wish to find geometric constructions in the spirit of Section 2 which at least for some $a$ and $b$ yield a lower bound for $\underline{c}(a, b)$ that is equal to the upper bound given Proposition 10. So far we have only found constructions giving lower bounds that do not coincide with the upper bounds, e.g. $\frac{5}{3} \leq \underline{c}(2, 4) \leq \overline{c}(2, 4) \leq 2$ and $\frac{5}{3} \leq \underline{c}(3, 4) \leq \overline{c}(3, 4) \leq \frac{5}{2}$.

Note the following technical point. If $\omega = e^{2\pi i \theta}$ is nonregular for $T(b, m(n))$ or $T(a, n)$, then Lemma 2 cannot be applied as above. Instead one chooses a sequence of $\theta_k$ tending to $\frac{\pi}{a}$ such that every $e^{2\pi i \theta_k}$ is regular for $T(b, m(n))$ and $T(a, n)$. Inequality (8) is then obtained by choosing $\theta = \theta_k$ in inequality (7) and setting $k$ tend to infinity.
6. Algebraic adjacency

In this section we compare \( \leq_g \) with an adjacency notion for plane curve singularities. We first recall the notion of an algebraic knot following Milnor [Mil68]. Let \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be a polynomial function or a holomorphic function germ that is irreducible in the ring of holomorphic function germs \( \mathbb{C}\{x, y\} \) and has an isolated singularity at the origin. The transversal intersection of its zero set \( V(f) \subseteq \mathbb{C}^2 \) with a sufficiently small sphere around the origin \( S^3_\varepsilon = \{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 = \varepsilon^2\} \) is a knot in \( S^3_\varepsilon \cong S^3 \) called the knot of the singularity of \( f \). For example, the torus knot \( T(n, m) \) is the knot of the singularity of \( x^n - y^m \). In this case the small sphere can be taken to be the standard unit sphere \( S^3 \) and thus \( T(n, m) = S^3 \cap \{(x, y) \in \mathbb{C}^2 \mid x^n - y^m = 0\} \subseteq S^3 \). Knots that can occur as knots of singularities are called algebraic.

Arnold studied adjacency of singular function germs [Arn72, Definition 2.1], see also [Sie74]. As we are interested in knots, we study singular function germs only up to topological type, i.e. up to the isotopy class of their knots of singularity, see e.g. [BK86]. Thus we use the following version of adjacency.

A deformation of \( f \in \mathbb{C}\{x, y\} \) is a smooth family \( h_t \in \mathbb{C}\{x, y\} \), defined for small enough real \( t \geq 0 \), with \( h_0 = f \).

**Definition 18.** Let \( K_1 \) and \( K_2 \) be algebraic knots. We say \( K_1 \) is algebraically adjacent to \( K_2 \), denoted by \( K_1 \preceq_a K_2 \), if there exists a germ \( f \in \mathbb{C}\{x, y\} \) with \( K_2 \) as knot of the singularity and a deformation \( h_t \) of \( f \), such that for small nonzero \( t \) the germ \( h_t \) has \( K_1 \) as knot of the singularity.

**Remark 19.** It is not important whether one studies polynomials or holomorphic germs, since every holomorphic germ yields the same knot as its Taylor polynomials of large enough degrees.

Isotopy classes of algebraic knots can be identified canonically with \( \mu \)-constant-homotopy classes of irreducible germs \( (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \), where \( \mu \) is the Milnor number. With this identification the above notion of adjacency for algebraic knots corresponds to the concept of \( \mu \)-adjacency studied by Siersma in [Sie74]. We sketch this identification. If two plane curves can be connected by a \( \mu \)-constant path, then the associated algebraic knots are isotopic [TR76]. For the converse assume that two irreducible germs \( f_0 \) and \( f_1 \) have the same knot of singularity. After coordinate changes they are both of the form \( y^m + c_{m-1}(x)y^{m-1} + \cdots + c_0(x) \), where \( m \) is the multiplicity of \( f_0 \) and \( f_1 \), and where the \( c_k \in \mathbb{C}\{x\} \) are holomorphic germs with \( c_k(0) = 0 \). If \( f_0 \) and \( f_1 \) have the same knot of singularity, then they have the same essential terms in their corresponding Puiseux expansions \( y_0(x^{\frac{1}{m}}) \) and \( y_1(x^{\frac{1}{m}}) \), see e.g. [BK86]. Thus the two Puiseux expansions can be connected by a family of Puiseux expansions \( y_t(x^{\frac{1}{m}}) \) with the same essential terms. This yields a \( \mu \)-constant family of germs

\[
 f_t = \prod_{\xi^m=1} \left( y - y_t(\xi x^{\frac{1}{m}}) \right) \in \mathbb{C}\{x, y\}
\]

that connects \( f_0 \) to \( f_1 \).

As described in the introduction both notions of adjacency have a lot of properties in common. For example, the statement of Theorem 2 holds for \( \leq_a \) and is easy to show.

**Proposition 20.** If \( n \leq a \) and \( m \leq b \), then \( T(n, m) \leq_a T(a, b) \).

\( \text{With the weaker assumption ‘squarefree’ most of what is done in this section still works, but we get links instead of knots.} \)
Proof. Regard $T(a, b)$ as the knot of the singularity of $y^a - x^b$ and choose as deformation $h_t(x, y) = y^a - x^b + t(y^n - x^m)$. For $t$ small (but fix) we make, in a small chart around the origin, a biholomorphic coordinate change, which does not change the topological type of the singularity, such that $h_t = y^{n}(t + y^{a-n}) - x^{m}(t + x^{b-m})$ becomes $y^n - x^m$. □

The obstruction to Gordian adjacency given in Corollary 10 also holds for algebraic adjacency. Actually Corollary 10 and its counterpart for $\leq a$ are a consequence of the fact that $|\sigma_{c}(K_1) - \sigma_{c}(K_2)|$ is less or equal to the cobordism distance of $K_1$ and $K_2$, and the following. For algebraic knots both $K_1 \leq_g K_2$ and $K_1 \leq_a K_2$ yield a cobordism in $S^3 \times [0, 1]$ between $K_1$ and $K_2$ of minimal genus $u(K_2) - u(K_1) = g_s(K_2) - g_s(K_1)$. For an algebraic adjacency given by a deformation $h_t$ this cobordism is given as follows. Let $S_2$ be a sufficiently small sphere with $K_2 = S_2 \cap V(h_0)$. Then, by transversality, $t$ can be chosen small enough such that $S_2 \cap V(h_t)$ is still $K_2$ and $K_1 = S_1 \cap V(h_t)$ for a small enough sphere $S_1$. By a small perturbation of $h_t$ the zero set $V(h_t)$ becomes a smooth algebraic curve $F$ with $K_2 = S_2 \cap F$ and $K_1 = S_1 \cap F$. The cobordism between $K_1$ and $K_2$, which is given by $F$, has minimal genus $u(K_2) - u(K_1) = g_s(K_2) - g_s(K_1)$ by the Thom conjecture.

Despite these similarities the two notions of adjacency do not agree for algebraic knots or even torus knots. The obstruction given in Proposition 11 that the signature decreases on Gordian adjacent torus knots, does not hold for algebraic adjacency. Concretely we have $T(2, 15) \leq_a T(3, 10)$, but $T(2, 15) \not\leq_g T(3, 10)$ by Theorem 3. We generalize an example of an algebraic adjacency calculated by Arnold [Arn72, A5 ← E6]. This gives a large class of examples of algebraic adjacencies of torus knots, including $T(2, 15) \leq T(3, 10)$, which are not covered by Proposition 20.

**Proposition 21.** Let $a, b, c$ be natural numbers with $a \leq b$, then $T(a, bc) \leq_a T(b, ac)$. In particular $T(2, 3c) \leq_a T(3, 2c)$.

**Proof.** Suppose $a < b$, regard $T(b, ac)$ as the knot of the singularity of $y^b - x^{ac}$ and choose as a deformation $h_t = y^b - (x^c + ty)^a$. Now fix some small $t > 0$. We change coordinates by $x \mapsto x, y \mapsto \frac{y - x^c}{x}$ such that $h_t = (\frac{y - x^c}{x})^b - y^a$. Remarking that $h_t$ has the same Newton polyhedron as $x^{bc} - y^a$ we conclude that they have the same knot of singularity by a result of Kouchnirenko [Kou76, Corollaire 1.22]. □

**Remark 22.** Proposition 21 gives an algebraic proof of an observation by Baader, which states that the cobordism distance of $T(a, bc)$ and $T(b, ac)$ is equal to $\frac{bc + a - ac - b}{2}$ and which is a key proposition in [Baa12].

Proposition 21 shows that, if we define an algebraic counterpart of $c(a, b)$ in Section 5 it is larger or equal to $\frac{b}{a}$ whereas in the Gordian setting $c(a, b)$ is smaller or equal to $\frac{b}{a}$ by Proposition 10. Thus, asymptotically, whenever $T(a, n) \leq_g T(b, m)$ for $a \leq b$ we get roughly $n \leq \frac{b}{a} m$ and therefore $T(a, n) \leq_a T(b, m)$. We take this as evidence to conjecture that for torus knots Gordian adjacency implies algebraic adjacency.

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