Significance of many-body contributions to Casimir energies

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Irreducible many-body contributions to Green’s functions and Casimir energies are defined. We show that the irreducible three-body contribution to Casimir energies is significant and can be more than twenty percent of the total interaction energy. Irreducible three-body contribution for three parallel semitransparent plates in the limit when two plates overlap is obtained in terms of irreducible two-body contributions and shown to be finite and well defined in this limit.

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1. Introduction

Three-body contribution to Casimir energies in the unretarded regime were first considered by Axilrod and Teller[1] and Muto[2]. The three-body contribution to van der Waals-London interaction energy between three identical atoms at the corners of a triangle was found[1] to be negative for configurations forming an acute triangle and positive for configurations forming obtuse triangles. Axilrod[3-5] hoped to explain the cohesion energy of rare-gas crystals by including three-body contributions. In the context of atoms many-body contributions to Casimir-Polder interaction energy in the retarded regime were studied by Aub and Zienau[6]. Three-body contribution to the Casimir-Polder interaction of two spheres above a plate was recently explored in Refs. [7] and [8]. Due to their non-additive nature, irreducible many-body contributions to the total Casimir energy in the strong coupling regime were only considered recently[9,10]. Theorems on finiteness of irreducible many-body contributions to Casimir energies were obtained in Ref.[9] and for scalar fields with potential interactions the sign of irreducible many-body contributions was determined[9]. Explicit expressions for many-body contributions to Casimir energies were derived in Ref.[10]. This used ideas of Faddeev and others[11-14] to solve the many-body Green’s functions[10]. Significance of many-body contributions to Casimir energies becomes apparent by noting that the three-body contribution can be up to 20%
of the total Casimir energy. The importance of such non-additive interactions has been realized by chemists.\cite{chemists15}

We here first review our results on many-body Green’s functions in Ref. \cite{Ref10} and consider some implications of the many-body decomposition of Casimir energies. For (scalar) atom-like potentials above a Dirichlet plate we analytically obtain the three-body contribution to the Casimir force. We consider weakly interacting wedges placed atop Dirichlet plates and show that the irreducible three-body Casimir energy is minimal (and vanishes) when the shorter side of the wedge is perpendicular to the Dirichlet plate.

2. Many-body Green’s functions

The (scalar) Green’s function for \(N\) potentials \(V_i(x), i = 1, 2, \ldots, N\) satisfies the equation

\[
[-\nabla^2 + \zeta^2 + V_1(x) + V_2(x) + \ldots + V_N(x)]G_{1\ldots N}(x, x') = \delta^{(3)}(x - x').
\] (1)

The solution is symbolically written in the form

\[
G_{1\ldots N} = G_0 - G_0 T_{1\ldots N} G_0,
\] (2)

where the free Green’s function \(G_0(x, x')\) satisfies Eq. (1) in the absence of potentials and the \(N\)-body transition matrix \(T_{1\ldots N} \rightarrow T_{1\ldots N}(x, x')\) is of the form

\[
T_{1\ldots N} = (V_1 + V_2 + \cdots + V_N)[1 + G_0(V_1 + V_2 + \cdots + V_N)]^{-1}.
\] (3)

To compactly express the following equations we use the notation

\[
\tilde{G}_i \rightarrow G_i G_0^{-1}, \quad \tilde{V}_i \rightarrow G_0 V_i, \quad \text{and} \quad \tilde{T}_i \rightarrow G_0 T_i,
\] (4)

which is equivalent to setting \(G_0 = 1\). We decompose the \(N\)-body transition matrix in the form,

\[
\tilde{T}_{1\ldots N} = \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{T}^{ij}_{1\ldots N} = \text{Sum}[\tilde{T}_{1\ldots N}],
\] (5)

where the symbol \(\text{Sum}[A]\) stands for the sum of all elements of the matrix \(A\). The matrix form of the \(N\)-body transition operator is,

\[
\tilde{T}_{1\ldots N} = \begin{pmatrix}
\tilde{T}^{11}_{1\ldots N} & \tilde{T}^{12}_{1\ldots N} & \cdots & \tilde{T}^{1N}_{1\ldots N} \\
\tilde{T}^{21}_{1\ldots N} & \tilde{T}^{22}_{1\ldots N} & \cdots & \tilde{T}^{2N}_{1\ldots N} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{T}^{N1}_{1\ldots N} & \tilde{T}^{N2}_{1\ldots N} & \cdots & \tilde{T}^{NN}_{1\ldots N}
\end{pmatrix},
\] (6)

where each component is an integral operator. It was shown\cite{Ref10} that the above \(N\)-body transition matrices satisfy the Faddeev’s equations\cite{Faddeev14,Faddeev11,Faddeev13}

\[
[1 + \tilde{\Theta}_{1\ldots N}] \cdot \tilde{T}_{1\ldots N} = \tilde{T}_{\text{diag}},
\] (7)
where

$$\tilde{\Theta}_{1...N} = \begin{pmatrix} 0 & \tilde{T}_1 & \tilde{T}_1 & \cdots & \tilde{T}_1 \\ \tilde{T}_2 & 0 & \tilde{T}_2 & \cdots & \tilde{T}_2 \\ \tilde{T}_3 & \tilde{T}_3 & 0 & \cdots & \tilde{T}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{T}_N & \tilde{T}_N & \tilde{T}_N & \cdots & 0 \end{pmatrix}, \quad \tilde{T}_{\text{diag}} = \begin{pmatrix} \tilde{T}_1 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{T}_2 & 0 & \cdots & 0 \\ 0 & 0 & \tilde{T}_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{T}_N \end{pmatrix}. \quad (8)$$

Faddeev’s equations of Eq. (7) reduce the problem of solving Eq. (5) for the N-body transition matrix to that of inverting \([1 + \tilde{\Theta}_{1...N}]\) by solving a set of N linear integral equations. Remarkably, \(\tilde{\Theta}_{1...N}\) depends only on single-body transition operators. The norm of \(\tilde{\Theta}_{1...N}\) is less than unity (because the norm of single-body transition matrices is) and Faddeev’s equations can, at least in principle, be solved by (numerical) iteration \[16\].

The two-body transition matrix is obtained by inverting the Faddeev’s equation in Eq. (7) to yield

$$T_{12} = \begin{bmatrix} X_{12} & 0 & 0 \\ 0 & X_{21} \end{bmatrix} \begin{bmatrix} \tilde{T}_1 & -\tilde{T}_1 \tilde{T}_2 \\ -\tilde{T}_2 \tilde{T}_1 & \tilde{T}_2 \end{bmatrix}, \quad (9)$$

where the \(X_{ij}\)’s are solutions to the integral equations,

$$[1 - \tilde{T}_i \tilde{T}_j] X_{ij} = 1. \quad (10)$$

The corresponding three-body transition matrix is

$$T_{123} = \begin{bmatrix} X_{1[23]} & 0 & 0 \\ 0 & X_{2[3]} & 0 \\ 0 & 0 & X_{3[12]} \end{bmatrix} \begin{bmatrix} \tilde{T}_1 & -\tilde{T}_1 \tilde{G}_3 \tilde{T}_2 X_{32} -\tilde{T}_1 \tilde{G}_2 \tilde{T}_3 X_{23} \\ -\tilde{T}_2 \tilde{G}_3 \tilde{T}_1 X_{21} & \tilde{T}_2 & -\tilde{T}_2 \tilde{G}_1 \tilde{T}_3 X_{13} \\ -\tilde{T}_3 \tilde{G}_2 \tilde{T}_1 X_{21} -\tilde{T}_3 \tilde{G}_1 \tilde{T}_2 X_{12} & \tilde{T}_3 \end{bmatrix}, \quad (11)$$

where the three-body effective Green’s functions, \(X_{i[jk]}\), \((i \neq j \neq k)\), satisfy the equation

$$X_{i[jk]} [1 - \tilde{T}_i \tilde{G}_j \tilde{T}_k X_{jk} - \tilde{T}_i \tilde{G}_k \tilde{T}_j X_{kj}] = 1. \quad (12)$$

We refer to Ref. \[10\] for further details and expressions for irreducible transition matrices.

3. Casimir energies for parallel semitransparent \(\delta\)-plates

For scalar fields semitransparent plates are described by \(\delta\)-function potentials

$$V_i(x) = \lambda_i \delta (z - a_i), \quad (13)$$

where \(a_i\) specifies the position of the \(i\)-th plate on the \(z\)-axis, and \(\lambda_i > 0\) is the coupling parameter. In the strong coupling limit, \(\lambda_i \to \infty\), the potential of Eq. (13) simulates a plate with Dirichlet boundary conditions. The total energy \(E_i\) for a single semitransparent plate is

$$E_i(\lambda_i) = E_0 + \Delta E_i(\lambda_i) \quad (14)$$
and the total energy $E_{12}$ of two parallel semitransparent plates may be decomposed as,

$$E_{12}(\lambda_1, \lambda_2, a_{12}) = E_0 + \Delta E_1(\lambda_1) + \Delta E_2(\lambda_2) + \Delta E_{12}(\lambda_1, \lambda_2, a_{12}),$$

(15)

where $a_{12}$ is the distance between the two plates and

$$E_0 = \frac{1}{12\pi^2} \int_0^\infty \kappa^3 d\kappa,$$

(16)

$$\Delta E_i = \frac{1}{12\pi^2} \int_0^\infty \kappa^2 d\kappa \frac{\lambda_i}{\lambda_i + 2\kappa} \to \frac{\lambda_i}{\lambda_i + 2\kappa},$$

(17)

$$\Delta E_{ij} = \frac{1}{12\pi^2} \int_0^\infty \kappa^2 d\kappa \left[ \frac{2\kappa a_{ij} + (1 - \bar{t}_i) + (1 - \bar{t}_j)}{\lambda_i - 1} \right] \to \frac{\pi^2}{1440 a_{ij}^3}. \tag{18}$$

The single-body dimensionally reduced transition matrix and the two-body determinants are

$$\bar{t}_i = \frac{\lambda_i}{\lambda_i + 2\kappa} \to 1, \quad \Delta_{ij} = 1 - \bar{t}_i \bar{t}_j e^{-2\kappa a_{ij}} \to (1 - e^{-2\kappa a_{ij}}). \tag{19}$$

The Casimir energy for free space $E_0$ is divergent and not well defined, but the trace-log formula formally gives a negative expression. For a single plate the irreducible single-body Casimir energy also is divergent and the corresponding expression is positive. The irreducible two-body Casimir energy is unambiguously finite and negative. The above expressions also gives the same behavior in the Dirichlet limit.

It is instructive to analyze the two-body contribution to the the energy in the limit $a_{12} \to 0$. In this limit the two plates overlap and we are interested in the distinction between a single plate versus two overlapping plates. Two overlapping plates treated as a single body have the vacuum energy

$$E_{(1+2)}(\lambda_1 + \lambda_2) = E_0 + \Delta E_{(1+2)}(\lambda_1 + \lambda_2).$$

(20)

Comparing with the same vacuum energy, given by Eq. (15) in this limit, expressed in terms of irreducible one- and two-body contributions, we identify

$$\Delta E_{12}(\lambda_1, \lambda_2; a_{12} \to 0) = \Delta E_{(1+2)}(\lambda_1 + \lambda_2) - \Delta E_1(\lambda_1) - \Delta E_2(\lambda_2),$$

(21)

where the $(1 + 2)$ in the subscript denotes the two overlapping plates. In the limit $a_{12} \to 0$ the irreducible two-body Casimir energy thus is formally a (divergent) difference of single-body Casimir energies. The above analysis and conclusions survive in the Dirichlet limit on either plate or on both plates, if the limit $a_{12} \to 0$ is taken before the strong coupling limit. Although Eq. (21) compares divergent expressions we will see in the following that the limit of two coinciding plates is well defined for the irreducible three-body Casimir energy.
The total Casimir energy for three parallel semitransparent plates is decomposed in terms of the irreducible many-body Casimir energies as

\[ E_{123} = E_0 + \Delta E_1 + \Delta E_2 + \Delta E_3 + \Delta E_{12} + \Delta E_{23} + \Delta E_{31} + \Delta E_{123}, \]

(22)

where the parameter dependences have been suppressed. (See Fig. 1) \( \Delta E_{123} \) of three parallel semitransparent plates was obtained in Ref. 10 which is finite and positive. In the Dirichlet limit for all three plates the irreducible three-body Casimir energy cancels the well-known two-body interaction between the outer Dirichlet plates,

\[ \frac{\Delta E_{123}^D}{L_x L_y} = \frac{\pi^2}{1440} \frac{1}{a_{13}^3} = -\frac{\Delta E_{13}^D}{L_x L_y}, \]

(23)

where \( a_{13} \) is the distance between the outer two plates. The previous analysis of overlapping plates can be extended to three plates when two of the plates overlap. In this case we find

\[ \Delta E_{123}(\lambda_1, \lambda_2, \lambda_3; a_{12} \to 0, a_{13}) = \Delta E_{(1+2)3}(\lambda_1 + \lambda_2, \lambda_3; a_{13}) - \Delta E_{13}(\lambda_1, \lambda_3; a_{13}) - \Delta E_{23}(\lambda_2, \lambda_3; a_{23}). \]

(24)

Thus, remarkably, in the limit when two plates overlap, the irreducible three-body contribution can be written as a difference of finite irreducible two-body contributions. The above conclusion survives the strong coupling limit if the limit of overlapping plates is taken before the Dirichlet limit.

4. (Scalar) atom-like localized potentials above a Dirichlet plate

In Refs. 7 and 8 the configuration of two spheres above a surface was considered. We here investigate the scalar analog which is further simplified by considering atom-like potentials. The analogous interaction of atoms on a dielectric plate was explored in Ref. 17. The following scalar analysis might explain why certain bonds between molecules are weakened in the presence of a metal sheet.
Scalar atom-like potentials will be described by
\[ V_\iota(x) = \lambda_\iota \delta(x - x_\iota), \quad i = 1, 2, \]  \tag{25}
where \( \lambda_\iota \) now has dimensions of length and \( x_\iota \) gives the position of the individual atom. We chose the two atoms to be at the same height \( h \) above the Dirichlet plate and separated by distance \( a \). (See Fig. 2.) The expressions for two-body and three-body Casimir energies in Ref. 10 in this case lead to
\[ \Delta E_{12} = -\frac{\lambda_1 \lambda_2}{64\pi^3 a^3}, \quad \Delta E_{i3} = -\frac{\lambda_i}{32\pi^2 h^2}, \quad \Delta E_{123} = +\frac{\lambda_1 \lambda_2}{64\pi^3 a^3} g(\beta), \]  \tag{26}
where
\[ g(\beta) = \frac{2}{\beta (1 + \beta)} - \frac{1}{\beta^3}, \quad \text{with} \quad \beta = \sqrt{1 + \left(\frac{2h}{a}\right)^2}. \]  \tag{27}

The correction to the force-component in the direction of \( \hat{a} \) is plotted in Fig. 2 and around \( a \sim 2h \) reduces the attraction between the scalar atoms by more than 20%.
5. Weak potentials interacting with a Dirichlet plate

Let us next consider the more general case of two independent potentials, \( V_i(x), i = 1, 2 \), describing two objects interacting weakly with each other and with a Dirichlet plate placed at \( z = 0 \). \( V_3 = \lambda_3 \delta(z - a_3) \), with \( \lambda_3 \rightarrow \infty \). We shall consider only two-dimensional problems and exclusively deal with potentials that are translationally symmetric in the \( x \)-direction. It is convenient to define Casimir energy per unit length (in \( x \)-direction), \( \mathcal{E} \), and subtract the single body energies and \( E_0 \) ab initio to write

\[
\mathcal{E} = \Delta \mathcal{E}_{12} + \Delta \mathcal{E}_{23} + \Delta \mathcal{E}_{31} + \Delta \mathcal{E}_{123},
\]

(31)

where \( \mathcal{E} \) on the left hand side is the total interaction energy per unit length of the three objects. The two-body Casimir interactions with the Dirichlet plate are given by the extremely simple expressions\[10\]

\[
\Delta \mathcal{E}_{i3}^W = -\frac{1}{32 \pi^3} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{V_i(y, z)}{|z|^2}, \quad (i = 1, 2),
\]

(32)

and the two-body interaction between the two objects is given by\[18\]

\[
\Delta \mathcal{E}_{12}^W = -\frac{1}{32 \pi^3} \int d^2 r_1 \int d^2 r_2 \frac{V_1(r_1) V_2(r_2)}{r_{12}}.
\]

(33)

The corresponding three-body contribution to the Casimir energy is\[10\]

\[
\Delta \mathcal{E}_{123}^W = \frac{1}{32 \pi^3} \int d^2 r_1 \int d^2 r_2 \frac{V_1(r_1) V_2(r_2)}{r_{12}} \frac{Q_3(\bar{r}_{12})}{r_{12}},
\]

(34)

where the distances are defined as \( r_{12}^2 = (y_1 - y_2)^2 + (z_1 - z_2)^2 \) and \( (\bar{r}_{12})^2 = (y_1 - y_2)^2 + (|z_1| + |z_2|)^2 \), and the kernel \( Q_3 \) is

\[
Q_3(x) = \frac{2 \ln x}{(1 - x)} - 1.
\]

(35)

In Ref.\[10\] we considered a triangular-wedge with two sides described by weak potentials atop a Dirichlet plate at \( z = 0 \), forming a waveguide of triangular cross-section:

\[
\begin{align*}
V_1(y, z) &= \lambda_1 \delta(-z + m_\alpha(y - a)) \theta_1, \quad (36a) \\
V_2(y, z) &= \lambda_2 \delta(-z + m_\beta(y - b)) \theta_2, \quad (36b) \\
V_3(z) &= \lambda_3 \delta(z), \quad \text{with } \lambda_3 \rightarrow \infty, \quad (36c)
\end{align*}
\]

where \( \theta_1 \equiv \theta(y - \min[0, a]) \theta(\max[0, a] - y) \) and \( \theta_2 \equiv \theta(y - \min[0, b]) \theta(\max[0, b] - y) \). The sides of the wedge have lengths \( \sqrt{h^2 + a^2} \) and \( \sqrt{h^2 + b^2} \). The constraint \( m_\alpha a = m_\beta b = -h \) forces the sides to intersect at \( (y = 0, z = h) \), where \( h \) is the height of the triangle. The base of the triangle formed then measures \( |b - a| \). Note that the Dirichlet plate at \( z = 0 \) is of infinite extent. This triangular-wedge on a Dirichlet plate is depicted in FIG.\[8\] Suitable parameters for describing the triangular waveguide are \( (h, \bar{a} = a/h, \bar{b} = b/h) \). Without loss of generality we
measure all lengths in multiples of the height $h$. The triangle then has height $h = 1$ and the parameter space for the triangle is $-\infty < a, b < \infty$.

In FIG. 3 we plot the three-body energy for the above configuration as a function of $\hat{a}$ for fixed area: $A = h^2$, or $|\hat{b} - \hat{a}| = 2$. The three-body Casimir energy for a waveguide of given cross-sectional area is minimal when the shorter side of the wedge is perpendicular to the Dirichlet plate ($\hat{a} = 0$ or $\hat{b} = 0$). In the intermediate region the energy is extremal for an isosceles triangle ($-\hat{a} = \hat{b} = 1$) with $\Delta \xi_{123}(-1, 1) = 0.893112\ldots$. The dashed curve in FIG. 3 represents the approximation $\Delta \xi_{123}(\hat{a}, \hat{b}) \sim |\hat{a}\hat{b}|$. Remarkably, this extremely simple expression for the irreducible three-body energy is accurate to better than ten percent everywhere.

A similar configuration involving parabolic surfaces was also analyzed in Ref. 10. No qualitative change in the three-body Casimir energy was observed for parabolic surfaces.

6. Future extensions

For scalar case we have shown that irreducible three-body parts of Casimir energies can contribute significantly. The formalism and expressions for the many-body Green’s functions and Casimir energies readily generalize to the electromagnetic case and we intend to study two real atoms above a metal surface using the methods developed in Refs. 10 and 17. We intend to check if three-body effects contribute significantly in the electromagnetic case.
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