A characterization of domination weak bicritical graphs
with large diameter

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Abstract

The domination number of a graph $G$, denoted by $\gamma(G)$, is the minimum
cardinality of a dominating set of $G$. A vertex of a graph is called critical if its
deletion decreases the domination number, and a graph is called critical if its all
vertices are critical. A graph $G$ is called weak bicritical if for every non-critical
vertex $x \in V(G)$, $G - x$ is a critical graph with $\gamma(G - x) = \gamma(G)$. In this
paper, we characterize the connected weak bicritical graphs $G$ whose diameter
is exactly $2\gamma(G) - 2$. This is a generalization of some known results concerning
the diameter of graphs with a domination-criticality.

Key words and phrases. weak bicritical graph, critical graph, bicritical graph, diam-
eter

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1 Introduction

All graphs considered in this paper are finite, simple, and undirected.

Let $G$ be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge
set of $G$, respectively. For $x \in V(G)$, we let $N_G(x)$ and $N_G[x]$ denote the open
neighborhood and the closed neighborhood of $x$, respectively; thus $N_G(x) = \{y \in
V(G) : xy \in E(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$. For $x, y \in V(G)$, we let $d_G(x, y)$
denote the distance between $x$ and $y$ in $G$. For $x \in V(G)$ and a non-negative integer
$i$, let $N_G^{(i)}(x) = \{y \in V(G) : d_G(x, y) = i\}$; thus $N_G^{(0)}(x) = \{x\}$ and $N_G^{(1)}(x) = N_G(x)$.
The diameter of $G$, denoted by $\text{diam}(G)$, is defined to be the maximum of $d_G(x, y)$
as \( x, y \) range over \( V(G) \). A vertex \( x \in V(G) \) is \textit{diametrical} if \( \max \{ d_G(x, y) : y \in V(G) \} = \text{diam}(G) \).

We let \( \overline{G} \) denote the \textit{complement} of \( G \). For two graphs \( H_1 \) and \( H_2 \), we let \( H_1 \cup H_2 \) denote the \textit{union} of \( H_1 \) and \( H_2 \). For a graph \( H \) and a non-negative integer \( s \), \( sH \) denote the disjoint union of \( s \) copies of \( H \). We let \( K_n \) and \( P_n \) denote the \textit{complete} graph and the \textit{path} of order \( n \), respectively.

For two subsets \( X, Y \) of \( V(G) \), we say that \( X \) \textit{dominates} \( Y \) if \( Y \subseteq \bigcup_{x \in X} N_G[x] \). A subset of \( V(G) \) which dominates \( V(G) \) is called a \textit{dominating set} of \( G \). The minimum cardinality of a dominating set of \( G \), denoted by \( \gamma(G) \), is called the \textit{domination number} of \( G \). A dominating set of \( G \) with the cardinality \( \gamma(G) \) is called a \( \gamma \)-\textit{set} of \( G \).

For terms and symbols not defined here, we refer the reader to [7].

1.1 Motivations

For a given graph \( G \), we can divide the set \( V(G) \) into the following three subsets;

\[
V^0(G) = \{ x \in V(G) : \gamma(G - x) = \gamma(G) \},
\]
\[
V^+(G) = \{ x \in V(G) : \gamma(G - x) > \gamma(G) \}, \text{ and}
\]
\[
V^-(G) = \{ x \in V(G) : \gamma(G - x) < \gamma(G) \}.
\]

A vertex in \( V^-(G) \) is said to be \textit{critical}. A graph \( G \) is \textit{critical} if every vertex of \( G \) is critical (i.e., \( V(G) = V^-(G) \)), and \( G \) is \textit{k-critical} if \( G \) is critical and \( \gamma(G) = k \). Many researchers have studied critical vertices or critical graphs (for example, see [1, 2, 11, 12, 13]). Among them, we focus on the following theorem which was conjectured by Brigham, Chinn and Dutton [4].

Theorem A (Fulman, Hanson and MacGillivray [8]) Let \( k \geq 2 \) be an integer, and let \( G \) be a connected \( k \)-critical graph. Then \( \text{diam}(G) \leq 2k - 2 \).

After that, Ao [3] characterized the connected \( k \)-critical graphs \( G \) with \( \text{diam}(G) = 2k - 2 \) (see Theorem [2] in Subsection [1,2]).

Now we introduce other criticality for the domination. A graph \( G \) is \textit{bicritical} if \( \gamma(G - \{ x, y \}) < \gamma(G) \) for any pair of distinct vertices \( x, y \in V(G) \), and \( G \) is \textit{k-bicritical} if \( G \) is bicritical and \( \gamma(G) = k \). It is known that for \( k \leq 2 \), the order of a \( k \)-bicritical graph is at most 3 (see [5]), and hence we are interested in \( k \)-bicritical graphs with \( k \geq 3 \). Brigham, Haynes, Henning and Rall [4] gave a conjecture concerning the diameter of bicritical graphs: For \( k \geq 3 \), every connected \( k \)-bicritical graph \( G \) satisfies \( \text{diam}(G) \leq k - 1 \). However, the conjecture was disproved by the following theorem.

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Theorem B (Furuya [9, 10]) Let \( k \geq 3 \) be an integer. Then there exist infinitely many connected \( k \)-bicritical graphs \( G \) with

\[
diam(G) = \begin{cases} 
3 & (k = 3) \\
6 & (k = 5) \\
\frac{3k-1}{2} & (k \text{ is odd and } k \geq 7) \\
\frac{3k-2}{2} & (k \text{ is even})
\end{cases}
\]

Thus one might be interested in an upper bound of the diameter of bicritical graphs. In [10], the author proved the following theorem. (However, it is open to find a sharp upper bound of the diameter of bicritical graphs.)

Theorem C (Furuya [10]) Let \( k \geq 3 \) be an integer, and let \( G \) be a connected \( k \)-bicritical graph. Then \( \text{diam}(G) \leq 2k - 3 \).

For convenience, let \( \mathcal{C} \) and \( \mathcal{C}_B \) denote the family of connected critical graphs and the family of connected bicritical graphs, respectively. Here we compare Theorem A with Theorem C. Although the inequalities in the theorems are similar, the two theorems are essentially different because \( \mathcal{C} \) is different from \( \mathcal{C}_B \):

- We can easily check that the graphs in \( F_k \) defined in Subsection 1.2 are critical and not bicritical.
- It is known that there exist infinitely many connected critical and bicritical graphs (see [5, 9]), and Brigham et al. [5] proved that a graph obtained from a critical and bicritical graph by expanding one vertex is bicritical and not critical. On the other hand, there exist infinitely many connected 4-bicritical graphs which is not critical and not obtained by the above operation (see the graph \( L_s \) in [10]).

In particular, \( \mathcal{C} \) and \( \mathcal{C}_B \) seems to be remotely related.

To treat the criticality and the bicriticality simultaneously, a new critical concept was defined in [10]. A graph \( G \) is weak bicritical if \( V^+(G) = \emptyset \) and \( G - x \) is critical for every \( x \in V^0(G) \), and \( G \) is weak \( k \)-bicritical if \( G \) is weak bicritical and \( \gamma(G) = k \). Since all critical graphs and all bicritical graphs are weak bicritical, the weak bicriticality is a unification of the criticality and the bicriticality. In [10], the author showed the following theorem which is a generalization of Theorem A.

Theorem D (Furuya [10]) Let \( k \geq 2 \) be an integer, and let \( G \) be a connected weak \( k \)-bicritical graph. Then \( \text{diam}(G) \leq 2k - 2 \).

However, Theorem C cannot directly follow from Theorem D. In this paper, our main aim is to give a common generalization of Theorems A and C by characterizing the connected weak \( k \)-bicritical graphs \( G \) with \( \text{diam}(G) = 2k - 2 \).
1.2 Main result

Before we state our main result, we introduce Ao’s characterization.

Let \( k \geq 2 \) be an integer. We define the family \( \mathcal{F}_k \) of graphs as follows: Let \( m_i \geq 2 (1 \leq i \leq k - 1) \) be integers. For each \( 1 \leq i \leq k - 1 \), let \( G_i \) be a graph isomorphic to \( m_iK_2 \) (i.e., \( G_i \) is a graph obtained from the complete graph of order \( 2m_i \) by deleting a perfect matching), and take two vertices \( u_i, v_i \in V(G_i) \) with \( u_iv_i \notin E(G_i) \). Let \( G(m_1, \ldots, m_{k-1}) \) be the graph obtained from \( G_1, \ldots, G_{k-1} \) by identifying \( v_i \) and \( u_{i+1} \) for each \( 1 \leq i \leq k - 2 \), and set

\[
\mathcal{F}_k = \{ G(m_1, \ldots, m_{k-1}) : m_i \geq 2, \ 1 \leq i \leq k - 1 \}.
\]

By the definition of \( \mathcal{F}_k \), we see the following observation.

**Observation 1.1** Let \( k \geq 3, k_1 \geq 2 \) and \( k_2 \geq 2 \) be integers with \( k_1 + k_2 - 1 = k \). Then a graph \( G \) belongs to \( \mathcal{F}_k \) if and only if \( G \) is obtained from two graphs \( H_1 \in \mathcal{F}_{k_1} \) and \( H_2 \in \mathcal{F}_{k_2} \) by identifying diametrical vertices \( u_i \) of \( H_i \) \((i \in \{1, 2\})\).

Ao [3] proved the following theorem. (By using lemmas for our main result, the following theorem can be easily proved. Hence we will give its proof in Section 4).

**Theorem E** (Ao [3]) Let \( k \geq 2 \) be an integer, and let \( G \) be a connected \( k \)-critical graph. Then \( \text{diam}(G) \leq 2k - 2 \), with the equality if and only if \( G \in \mathcal{F}_k \).

Now we recursively define the family \( \mathcal{F}_k^* \) \((k \geq 2)\) of graphs and the identifiable vertices of graphs in \( \mathcal{F}_k^* \). Let

\[
\mathcal{F}_2^* = \{(m+1)K_2, mK_2 \cup K_3, mK_2 \cup P_5 : m \geq 1 \}.
\]

Note that \( \mathcal{F}_2^* \) is equal to the family of connected weak 2-bicritical graphs (see Lemma 1.3 in Subsection 1.3). For each \( G \in \mathcal{F}_2^* \), a vertex \( x \in V(G) \) is identifiable if \( x \in V^-(G) \). Note that if \( G = (m+1)K_2 \), then all vertices of \( G \) are identifiable; if \( G = mK_2 \cup K_3 \), then \( G \) has exactly three non-identifiable vertices; if \( G = mK_2 \cup P_5 \), then \( G \) has exactly two non-identifiable vertices. We assume that \( k \geq 3 \), and for \( 2 \leq k' \leq k - 1 \), the family \( \mathcal{F}_{k'}^* \) and the identifiable vertices of graphs in \( \mathcal{F}_{k'}^* \) have been defined. Let \( \mathcal{F}_k' \) be the family of graphs obtained from two graphs \( H_1 \in \mathcal{F}_{k_1} \) and \( H_2 \in \mathcal{F}_{k_2} \) with \( k_1 \geq 2, k_2 \geq 2 \) and \( k_1 + k_2 - 1 = k \) by identifying a diametrical vertex of \( H_1 \) and an identifiable vertex of \( H_2 \). Let \( m_i \geq 2 \) \((i \in \{1, 2\})\), and let \( u \) be the unique cut vertex of the graph \( G(m_1, m_2) \) \((\in \mathcal{F}_3)\). Let \( G^1(m_1, m_2) \) be the graph obtained from \( G(m_1, m_2) \) by adding a new vertex \( u' \) and joining \( u' \) to all vertices in \( N_{G(m_1, m_2)}(u) \), and let \( G^2(m_1, m_2) = G^1(m_1, m_2) + uu' \). Let

\[
\mathcal{F}_3^* = \{G^1(m_1, m_2), G^2(m_1, m_2) : m_i \geq 2, \ i \in \{1, 2\}\},
\]
and let $F''_k = \emptyset$ for $k \geq 4$. Then by tedious argument, we see that every graph in $F''_3$ is weak 3-bicritical (but we omit detail). Let $F'_k = F'_k \cup F''_k$ for $k \geq 3$. For each $G \in F'_k$, a vertex $x \in V(G)$ is identifiable if $x \in V^-(G)$ and $x$ is a diametrical vertex of $G$. By induction and Lemma 1.6(ii) in Subsection 1.3, we see that every graph $G \in F'_k$ has at least one identifiable vertex, and hence $F'_k$ is well-defined. Furthermore, by the definition of $F_k$ and $F'_k$ and Observation 1.1, we also see that $F_k \subseteq F'_k$ and the diameter of graphs in $F'_k$ is exactly $2k - 2$.

Our main result is the following.

**Theorem 1.2** Let $k \geq 2$ be an integer, and let $G$ be a connected weak $k$-bicritical graph. Then $\text{diam}(G) \leq 2k - 2$, with the equality if and only if $G \in F'_k$.

Theorem 1.2 clearly leads to Theorems A and D. Furthermore, it is not hard to check that no graph in $F'_k$ is bicritical and no graph in $F'_k - F_k$ is critical, and so Theorem 1.2 leads to Theorems C and E. Therefore, Theorem 1.2 is a common generalization of some known results.

### 1.3 Preliminaries

In this subsection, we enumerate some fundamental or preliminary results.

The following has been known property which will be used in our argument.

**Lemma 1.3** Let $G$ be a graph, and let $u, v \in V(G)$. If $N_G[u] \subseteq N_G[v]$, then $v$ is not critical.

In [10], the author showed that the minimum degree of a connected weak bicritical graph of order at least 3 is at least 2. Now we let $G$ be a disconnected weak bicritical graph. Then we can verify that each component of $G$ is weak bicritical. (Indeed, all components of $G$ are critical with at most one exception.) Thus the following lemma holds.

**Lemma 1.4** Let $G$ be a weak bicritical graph, and let $G_1$ be a component of $G$ with $|V(G_1)| \geq 3$. Then the minimum degree of $G_1$ is at least 2.

Since the weak 1-bicritical graphs are only $K_1$ and $K_2$, we are interested in weak $k$-bicritical graphs for $k \geq 2$. The following lemma gives a characterization of weak 2-bicritical graphs (or 2-critical graphs).

**Lemma 1.5** (Furuya [10]) A graph $G$ is weak 2-bicritical if and only if

$$G \in \{mK_2, mK_2 \cup K_3, (m - 1)K_2 \cup P_3 : m \geq 1\}.$$ 

In particular, a graph $G$ is 2-critical if and only if $G \in \{mK_2 : m \geq 1\}$. 

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We next focus on the coalescence of graphs. Let $H_1$ and $H_2$ be two vertex-disjoint graphs, and let $x_1 \in V(H_1)$ and $x_2 \in V(H_2)$. Under this notation, we let $(H_1 \circ H_2)(x_1, x_2; x)$ denote the graph obtained from $H_1$ and $H_2$ by identifying vertices $x_1$ and $x_2$ into a vertex labeled $x$. We call $(H_1 \circ H_2)(x_1, x_2; x)$ the coalescence of $H_1$ and $H_2$ via $x_1$ and $x_2$.

**Lemma 1.6** ([4, 5, 6, 9]) Let $H_1$ and $H_2$ be graphs, and for each $i \in \{1, 2\}$, let $x_i$ be a non-isolated vertex of $H_i$. Let $G = (H_1 \circ H_2)(x_1, x_2; x)$. Then the following hold.

(i) We have $\gamma(H_1) + \gamma(H_2) - 1 \leq \gamma(G) \leq \gamma(H_1) + \gamma(H_2)$. If $x_i$ is a critical vertex of $H_i$ for some $i \in \{1, 2\}$, then $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.

(ii) If $x_i$ is a critical vertex of $H_i$ for each $i \in \{1, 2\}$, then

\[
V^-(G) = (V^-(H_1) - \{x_1\}) \cup (V^-(H_2) - \{x_2\}) \cup \{x\}.
\]

In particular, the graph $G$ is critical if and only if both $H_1$ and $H_2$ are critical.

## 2 Coalescences

In this section, we prove the following theorem.

**Theorem 2.1** Let $H_1$ and $H_2$ be graphs, and for each $i \in \{1, 2\}$, let $x_i$ be a non-isolated vertex of $H_i$. Let $G = (H_1 \circ H_2)(x_1, x_2; x)$. Then $G$ is weak bicritical if and only if for some $i \in \{1, 2\}$,

(1) $H_i$ is critical,

(2) $H_{3-i}$ is weak bicritical, and

(3) $x_{3-i}$ is a critical vertex of $H_{3-i}$.

Furthermore, if $G$ is weak bicritical, then $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.

**Proof.** We first assume that $G$ is weak bicritical, and show that $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$ and (1)–(3) hold.

**Claim 2.1** The vertex $x$ belongs to $V^-(G)$.

**Proof.** Suppose that $x \notin V^-(G)$. Then $x \in V^0(G)$ and $G - x$ is critical. Since $G - x$ is the union of $H_1 - x_1$ and $H_2 - x_2$, $\gamma(G) = \gamma(H_1 - x_1) + \gamma(H_2 - x_2)$ and $H_i - x_i$ is critical for each $i \in \{1, 2\}$. For $i \in \{1, 2\}$, let $y_i \in N_{H_i}(x_i)$, and let $S_i$ be a $\gamma$-set of $H_i - \{x_i, y_i\}$. Then $\gamma(H_i - \{x_i, y_i\}) \leq \gamma(H_i - x_i) - 1$. Since $S_1 \cup S_2 \cup \{x\}$ is a dominating
set of $G$, we have $\gamma(H_1 - \{x_1, y_1\}) + \gamma(H_2 - \{x_2, y_2\}) + 1 = |S_1| + |S_2| + |\{x\}| \geq \gamma(G)$. Consequently,

\[
\gamma(G) = \gamma(G - x) \\
= \gamma(H_1 - x_1) + \gamma(H_2 - x_2) \\
\geq \gamma(H_1 - \{x_1, y_1\}) + \gamma(H_2 - \{x_2, y_2\}) + 2 \\
\geq \gamma(G) + 1,
\]

which is a contradiction. \qed

**Claim 2.2** For $i \in \{1, 2\}$, $x_i$ is a critical vertex of $H_i$.

**Proof.** Let $S$ be a $\gamma$-set of $G - x$. Then by Claim 2.1 and Lemma 1.6(i), $|S| \leq \gamma(G) - 1 \leq \gamma(H_1) + \gamma(H_2) - 1$. Since $\{S \cap V(H_1), S \cap V(H_2)\}$ is a partition of $S$, we have $|S \cap V(H_i)| \leq \gamma(H_i) - 1$ for some $i \in \{1, 2\}$. Without loss of generality, we may assume that $|S \cap V(H_1)| \leq \gamma(H_1) - 1$. Since removing a vertex can decrease the domination number at most by one and $S \cap V(H_1)$ is a dominating set of $H_1 - x_1$, this implies that $\gamma(H_1 - x_1) = |S \cap V(H_1)| = \gamma(H_1) - 1$ and $x_1$ is a critical vertex of $H_1$. Again by Lemma 1.6(i), $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$, and hence $|S| \leq \gamma(G) - 1 = \gamma(H_1) + \gamma(H_2) - 2$. Consequently

\[
|S \cap V(H_2)| = |S| - |S \cap V(H_1)| \\
\leq (\gamma(H_1) + \gamma(H_2) - 2) - (\gamma(H_1) - 1) \\
= \gamma(H_2) - 1.
\]

Since $S \cap V(H_2)$ is a dominating set of $H_2 - x_2$, $\gamma(H_2 - x_2) \leq |S \cap V(H_2)| \leq \gamma(H_2) - 1$ and $x_2$ is a critical vertex of $H_2$. \qed

By Lemma 1.6 and Claim 2.2

\[
\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1 \quad (2.1)
\]

and

\[
V^-(G) = (V^-(H_1) - \{x_1\}) \cup (V^-(H_2) - \{x_2\}) \cup \{x\}. \quad (2.2)
\]

If $H_1$ and $H_2$ are critical, then (1)–(3) hold. Thus, without loss of generality, we may assume that $H_1$ is not critical (i.e., $V(H_1) - V^-(H_1) \neq \emptyset$). Let $y \in V(H_1) - V^-(H_1)$. By (2.2), $y \notin V^-(G)$, and hence $G - y$ is critical.

**Claim 2.3** We have $y \in V^0(H_1)$.
Proof. Note that $\gamma(G - \{x, y\}) < \gamma(G)$, and $\gamma(H_2 - x_2) = \gamma(H_2) - 1$ because $x_2$ is a critical vertex of $H_2$ and removing a vertex can decrease the domination number at most by one. Since $G - \{x, y\}$ is the union of $H_1 - \{x, y\}$ and $H_2 - x_2$, this together with (2.1) leads to

\[
\gamma(H_1) + \gamma(H_2) - 2 = \gamma(G) - 1 \\
\geq \gamma(G - \{x, y\}) \\
= \gamma(H_1 - \{x, y\}) + \gamma(H_2 - x_2) \\
= \gamma(H_1 - \{x, y\}) + \gamma(H_2) - 1,
\]

and so $\gamma(H_1 - \{x, y\}) \leq \gamma(H_1) - 1$. Since $S_1 \cup \{x\}$ is a dominating set of $H_1 - y$ for a $\gamma$-set $S_1$ of $H_1 - \{x, y\}$, we have

\[
\gamma(H_1 - y) \leq \gamma(H_1 - \{x, y\}) + 1 \leq \gamma(H_1).
\]

Since $y \notin V^-(H_1)$, the desired conclusion holds. \(\square\)

Since $y$ is an arbitrary vertex in $V(H_1) - V^-(H_1)$, it suffices to show that both $H_1 - y$ and $H_2$ are critical. Note that $y \neq x_1$. Now we show that

\[x_1\text{ is a non-isolated vertex of } H_1 - y.\]  

(2.3)

By way of contradiction, we suppose that $x_1$ is an isolated vertex of $H_1 - y$. Since $x_1$ is a non-isolated vertex of $H_1$, $N_{H_1}(x_1) = \{y\}$. Since $G$ is weak bicritical and $x_2$ is a non-isolated vertex of $H_2$, the component of $G$ containing $y$ has at least three vertices. This together with Lemma 1.4 implies $N_{H_1}(y) - \{x_1\} \neq \emptyset$. Let $y' \in N_{H_1}(y) - \{x_1\}$.

Since $G - y$ is critical, $\gamma(G - \{y, y'\}) \leq \gamma(G) - 1 = \gamma(H_1) + \gamma(H_2) - 2$. Let $S$ be a $\gamma$-set of $G - \{y, y'\}$. If $x \in S$, let $S' = ((S - \{x\}) \cap V(H_2)) \cup \{x_2\}$; if $x \notin S$, let $S' = S \cap V(H_2)$. In either case, $S'$ is a dominating set of $H_2$, and hence $|(S - \{x\}) \cap V(H_1)| = |S| - |S'| \leq (\gamma(H_1) + \gamma(H_2) - 2) - \gamma(H_2) = \gamma(H_1) - 2$. Since $(S - \{x\}) \cap V(H_1)$ is a dominating set of $H_1 - \{x, y, y'\}$, $S'' = ((S - \{x\}) \cap V(H_1)) \cup \{y\}$ is a dominating set of $H_1$ with $|S''| \leq \gamma(H_1) - 1$, which is a contradiction. Thus (2.3) holds.

Recall that $G - y$ is critical. Since $G - y = ((H_1 - y) \bullet H_2)(x_1, x_2; x)$, it follows from Lemma 1.4(ii) and (2.3) that $H_1 - y$ and $H_2$ are critical.

We next assume that (1)–(3) hold, and show that $G$ is weak bicritical. We may assume that $i = 1$ (i.e., $H_1$ is critical, $H_2$ is weak bicritical, and $x_2$ is a critical vertex of $H_2$). By Lemma 1.6(i), $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$. If $G$ is critical, then the desired conclusion holds. Thus $V(G) - V^-(G) \neq \emptyset$. Let $y \in V(G) - V^-(G)$. By Lemma 1.6(ii), $y \in V^0(H_2)$, and hence $H_2 - y$ is critical.
Claim 2.4 We have \( y \in V^0(G) \).

Proof. Let \( S_1 \) be a \( \gamma \)-set of \( H_1 \), and let \( S_2 \) be a \( \gamma \)-set of \( H_2 - \{ x_2, y \} \). If \( x_1 \in S_1 \), let \( S = (S_1 - \{ x_1 \}) \cup S_2 \cup \{ x \} \); if \( x_1 \notin S_1 \), let \( S = S_1 \cup S_2 \). In either case, \( S \) is a dominating set of \( G - y \). Since \( |S| = \gamma(H_1) + \gamma(H_2 - \{ x_2, y \}) \leq \gamma(H_1) + (\gamma(H_2) - 1) = \gamma(G) \), we have \( \gamma(G - y) \leq \gamma(G) \). Since \( y /\in V^-(G) \), the desired conclusion holds. \( \square \)

Since \( y \) is an arbitrary vertex in \( V(G) - V^-(G) \), it suffices to show that \( G - y \) is critical. Note that \( y \neq x \). Now we show that

\[ x_2 \text{ is a non-isolated vertex of } H_2 - y. \tag{2.4} \]

Recall that \( x_2 \) is a non-isolated vertex of \( H_2 \). Furthermore, since \( x_2 \) is a critical vertex of \( H_2 \), the component of \( H_2 \) containing \( x_2 \) is not isomorphic to \( K_2 \), and hence the component of \( H_2 \) containing \( x_2 \) has at least three vertices. This together with Lemma [1.4] implies that the degree of \( x_2 \) in \( H_2 \) is at least 2, and so the degree of \( x_2 \) in \( H_2 - y \) is at least 2. Thus (2.4) holds.

Recall that both \( H_1 \) and \( H_2 - y \) are critical. Since \( G - y = (H_1 \bullet (H_2 - y))(x_1, x_2; x) \), it follows from Lemma [1.6 ii) and (2.4) that \( G - y \) is critical.

This completes the proof of Theorem 2.1. \( \square \)

3 Sufficient pairs

Let \( l \geq 3 \) be an integer, and let \( G \) be a connected graph. A pair \((x, j)\) of a vertex \( x \in V(G) \) and an integer \( j \geq 2 \) is \( l \)-sufficient if \( x \) is a diametrical vertex of \( G \) and there exists a \( \gamma \)-set \( S \) of \( G \) with \( |S \cap (\bigcup_{0 \leq i \leq j} N_G^{(i)}(x))| \geq (j + l)/2 \).

Lemma 3.1 (Furuya [10]) Let \( k \geq 3 \) and \( l \geq 3 \) be integers, and let \( G \) be a connected weak \( k \)-bicritical graph having an \( l \)-sufficient pair. Then \( \text{diam}(G) \leq 2k - l + 1 \).

Theorem 3.2 Let \( k \geq 3 \) be an integer, and let \( G \) be a connected weak \( k \)-bicritical graph. If \( G \) has a diametrical vertex \( x \) such that \( \bigcup_{1 \leq i \leq 3} N_G^{(i)}(x) \subseteq V^-(G) \) and \( |N_G^{(2)}(x)| \geq 2 \), then \( \text{diam}(G) \leq 2k - 3 \).

Proof. We show that \( \text{diam}(G) \leq 3 \) or \( G \) has a 4-sufficient pair. By way of contradiction, we suppose that \( \text{diam}(G) \geq 4 \) and \( G \) has no 4-sufficient pair. For each \( i \geq 0 \), let \( X_i = N_G^{(i)}(x) \) and \( U_i = X_0 \cup X_1 \cup \cdots \cup X_i \).

Claim 3.1 If a set \( S \subseteq V(G) \) dominates \( N_G[x] \) and \( |S \cap U_2| \leq 1 \), then \( x \) is the unique vertex of \( S \cap U_2 \).
Proof. By the assumption of the claim, there exists a vertex \( z \in N_G[x] \) dominating \( N_G[x] \) in \( G \). Since \( N_G[x] \subseteq N_G[z] \), if \( z \neq x \), then \( z \in N_G^{(1)}(x) \) and \( z \) is not a critical vertex of \( G \) by Lemma 1.3, which contradicts the assumption of the theorem. \( \square \)

Let \( w_2, w'_2 \in X_2 \) be distinct vertices, and let \( S_1 \) be a \( \gamma \)-set of \( G - w_2 \). Note that \( S_1 \cup \{w_2\} \) is a \( \gamma \)-set of \( G \) because \( w_3w'_2 \in E(G) \). Since \( G \) has no 4-sufficient pair, \( |\{S_1 \cup \{w_2\}\} \cap U_2| < (2 + 4)/2 = 3 \), and so \( |S_1 \cap U_2| \leq 1 \). Since \( S_1 \) dominates \( N_G[x] \) in \( G \), it follows from Claim 3.1 that \( x \) is the unique vertex in \( S_1 \cap U_2 \). Since \( G \) has no 4-sufficient pair, \( |\{S_1 \cup \{w_2\}\} \cap U_4| < (4 + 4)/2 = 4 \), and so \( |S_1 \cap U_4| \leq 2 \). Since \( |X_2| \geq 2 \) and \( S_2 \) dominates \( (X_2 \cup X_3) - \{w_2\} \), there exists a vertex \( w_3 \in X_3 \) dominating \( (X_2 \cup X_3) - \{w_2\} \) in \( G - w_2 \).

Let \( S_2 \) be a \( \gamma \)-set of \( G - w_3 \). Note that \( S_2 \cup \{w'_2\} \) is a \( \gamma \)-set of \( G \) because \( w_3w'_2 \in E(G) \). Since \( G \) has no 4-sufficient pair, \( |\{S_2 \cup \{w'_2\}\} \cap U_2| < (2 + 4)/2 = 3 \), and so \( |S_2 \cap U_2| \leq 1 \). Since \( S_2 \) dominates \( N_G[x] \) in \( G \), it follows from Claim 3.1 that \( x \) is the unique vertex in \( S_2 \cap U_2 \). Since \( G \) has no 4-sufficient pair, \( |\{S_2 \cup \{w'_2\}\} \cap U_4| < (4 + 4)/2 = 4 \), and so \( |S_2 \cap U_4| \leq 2 \). Since \( S_2 \) dominates \( (X_2 \cup X_3) - \{w_3\} \), there exists a vertex \( w'_3 \in X_3 \) dominating \( (X_2 \cup X_3) - \{w_3\} \) in \( G - w_3 \). Recall that \( w_3 \) dominates \( X_3 \) in \( G - w_2 \). Thus \( w_3w'_3 \in E(G) \), and hence \( S_2 \) is a dominating set of \( G \), which is a contradiction.

Consequently \( \text{diam}(G) \leq 3 \) or \( G \) has a 4-sufficient pair. In either case, it follows from Lemma 3.1 that the desired conclusion holds. \( \square \)

4 Proof of Theorems E and 1.2

In this section, we prove Theorems E and 1.2. As we mentioned in Subsection 1.2, \( \mathcal{F}_k \subseteq \mathcal{F}'_k \) and the diameter of graphs in \( \mathcal{F}'_k \) is exactly \( 2k - 2 \). By Lemma 1.5, \( \mathcal{F}_2 \) is equal to the family of connected 2-critical graphs. Thus by induction and Lemma 1.6(ii), we see that all graphs in \( \mathcal{F}_k \) are \( k \)-critical, and so

if a graph \( G \) belongs to \( \mathcal{F}_k \), then \( G \) is \( k \)-critical and \( \text{diam}(G) = 2k - 2 \). \( \text{(4.1)} \)

Recall that every graph in \( \mathcal{F}'_2 \) is weak 2-bicritical and every graph in \( \mathcal{F}'_3 \) is weak 3-bicritical. This together with induction and Theorem 2.1 implies that all graphs in \( \mathcal{F}'_k \) are weak \( k \)-bicritical, and so

if a graph \( G \) belongs to \( \mathcal{F}'_k \), then \( G \) is weak \( k \)-bicritical and \( \text{diam}(G) = 2k - 2 \). \( \text{(4.2)} \)

Proof of Theorem E Let \( k \) and \( G \) be as in Theorem E. By (4.1), it suffices to show

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that

\[
\text{if }\text{diam}(G) \geq 2k - 2, \text{ then } G \in \mathcal{F}_k. \quad (4.3)
\]

We proceed by induction on \(k\).

If \(k = 2\), then Lemma 1.20 leads to (4.3). Thus we may assume that \(k \geq 3\). Suppose that \(\text{diam}(G) \geq 2k - 2\). Let \(w\) be a diametrical vertex of \(G\). If \(|N_G^{(2)}(w)| \geq 2\), then \(\text{diam}(G) \leq 2k - 3\) by Theorem 3.2, which is a contradiction. Thus \(|N_G^{(2)}(w)| = 1\).

In particular, \(G\) has a cut vertex \(x\). Hence we can write \(G = (H_1 \cdot H_2)(x_1, x_2; x)\) for two graphs \(H_1\) and \(H_2\) and vertices \(x_i \in V(H_i)\ (i \in \{1, 2\})\). For each \(i \in \{1, 2\}\), set \(k_i = \gamma(H_i)\). By Lemma 1.6 \(H_1\) and \(H_2\) are critical and \(k_1 + k_2 - 1 = \gamma(H_1) + \gamma(H_2) - 1 = \gamma(G) = k\). Furthermore, we have \(\text{diam}(G) \leq \text{diam}(H_1) + \text{diam}(H_2)\).

By induction hypothesis, \(\text{diam}(H_i) \leq 2k_i - 2\), with the equality if and only if \(H_i \in \mathcal{F}_{k_i}\). Consequently, we have \(2k - 2 \leq \text{diam}(G) \leq (2k_1 - 2) + (2k_2 - 2) = 2k - 2\). This implies that \(H_i \in \mathcal{F}_{k_i}\) and \(x_i\) is a diametrical vertex of \(H_i\). Then by Observation 1.1 we have \(G \in \mathcal{F}_k\).

This completes the proof of Theorem 1.2. \(\Box\)

**Proof of Theorem 1.2.** Let \(k\) and \(G\) be as in Theorem 1.2. By (4.2), it suffices to show that

\[
\text{if }\text{diam}(G) \geq 2k - 2, \text{ then } G \in \mathcal{F}^*_k. \quad (4.4)
\]

We proceed by induction on \(k\).

If \(k = 2\), then Lemma 1.20 leads to (4.4). Thus we may assume that \(k \geq 3\). Suppose that \(\text{diam}(G) \geq 2k - 2\). If \(G\) is critical, then it follows from Theorem 1.2 that \(G \in \mathcal{F}_k \subseteq \mathcal{F}^*_k\), as desired. Thus we may assume that \(G\) is not critical (i.e., \(V^0(G) \neq \emptyset\)). Let \(w, w' \in V(G)\) be vertices with \(d_G(w, w') = \text{diam}(G)\).

**Claim 4.1** If \(G\) has no cut vertex, then \(G \in \mathcal{F}^*_k\).

**Proof.** Note that \(|N^{(2)}(w)| \geq 2\). If \(V^0(G) \subseteq \{w, w'\}\) (i.e., \(V(G) - \{w, w'\} \subseteq V^{-}(G)\)), then by Theorem 3.2 we have \(\text{diam}(G) \leq 2k - 3\), which is a contradiction. Thus \(V^0(G) - \{w, w'\} \neq \emptyset\). Let \(z \in V^0(G) - \{w, w'\}\). Then \(G - z\) is a connected critical graph and

\[
\text{diam}(G - z) \geq d_{G-z}(w, w') \geq d_G(w, w') = \text{diam}(G) \geq 2k - 2.
\]

This together with Theorem 1.2 forces \(G - z \in \mathcal{F}_k\) and \(\text{diam}(G - z) = d_{G-z}(w, w') = \text{diam}(G) = 2k - 2\). By the definition of \(\mathcal{F}_k\), we have \(|N_{G-z}^{(2)}(w)| = |N_{G-z}^{(4)}(w)| = 1\). Write \(N_{G-z}^{(2)}(w) = \{z'\}\). Since \(G\) has no cut vertex, the following hold:
\[\begin{align*}
  \bullet & \quad k = 3, \\
  \bullet & \quad z \text{ is adjacent to a vertex in } N_{G-z}^{(1)}(w) \text{ and a vertex in } N_{G}^{(3)}(w), \text{ and} \\
  \bullet & \quad N_{G}(z) \subseteq \bigcup_{1 \leq i \leq 3} N_{G-z}^{(i)}(w).
\end{align*}\]

Suppose that \(z'\) is a critical vertex of \(G\), and let \(S\) be a \(\gamma\)-set of \(G - z'\). Since \(N_{G}(z) \subseteq N_{G}[z']\) and \(S\) is not a dominating set of \(G\), this forces \(zz' \notin E(G)\) and \(z \in S\). Since \(S\) dominates \(w\), \(S \cap N_{G}[w] \neq \emptyset\). In particular, \(|(S \cup \{z'\}) \cap (\bigcup_{0 \leq i \leq 2} N_{G}^{(i)}(w))| \geq 3\). Since \(S \cup \{z'\}\) is a \(\gamma\)-set, \((w, 2)\) is a 4-sufficient pair. This together with Lemma 3.1 implies that \(\text{diam}(G) \leq 2k - 3\), which is a contradiction. Thus \(z'\) is not a critical vertex of \(G\) (i.e., \(z' \in V^0(G)\)).

Replacing the role of \(z\) and \(z'\), we have \(G - z' \in \mathcal{F}_k\) and \(N_{G-z'}(z) = N_{G-z'}^{(1)}(w) \cup N_{G-z'}^{(3)}(w)\). Hence \(G\) is isomorphic to a graph in \(\mathcal{F}_3^k\) (\(\subseteq \mathcal{F}_3^k\)). □

By Claim 4.1 we may assume that \(G\) has a cut vertex \(x\). Then we can write \(G = (H_1 \bullet H_2)(x_1, x_2; x)\) for two graphs \(H_1\) and \(H_2\) and vertices \(x_i \in V(H_i)\) \((i \in \{1, 2\})\). For each \(i \in \{1, 2\}\), set \(k_i = \gamma(H_i)\). Having Theorem 2.1 in mind, we may assume that \(H_1\) is critical, \(H_2\) is weak bicritical and \(x_2\) is a critical vertex of \(H_2\). Furthermore, \(k_1 + k_2 - 1 = \gamma(H_1) + \gamma(H_2) - 1 = \gamma(G) = k\). By induction hypothesis, \(\text{diam}(H_1) \leq 2k_1 - 2\), with the equality if and only if \(H_1 \in \mathcal{F}_{k_1}\). By Theorem 1.4 \(\text{diam}(H_2) \leq 2k_2 - 2\), with the equality if and only if \(H_2 \in \mathcal{F}_{k_2}\). Since \(\text{diam}(G) \leq \text{diam}(H_1) + \text{diam}(H_2)\), we have \(2k - 2 \leq \text{diam}(G) \leq (2k_1 - 2) + (2k_2 - 2) = 2k - 2\). This implies that \(H_1 \in \mathcal{F}_{k_1}\), \(H_2 \in \mathcal{F}_{k_2}\) and \(x_i\) is a diametrical vertex of \(H_i\). Since \(x_2\) is a critical vertex of \(H_2\), it follows from the definition of \(\mathcal{F}_{k}^*\), we have \(G \in \mathcal{F}_{k}^*\).

This completes the proof of Theorem 1.2 □

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