Abstract: We derive the off-shell nilpotent Becchi-Rouet-Stora-Tyutin (BRST), anti-BRST, (anti-)co-BRST, a bosonic and the ghost-scale symmetry transformations for a couple of equivalent Lagrangian densities of the two (1 + 1)-dimensional (2D) Stueckelberg-modified version of Proca theory which also incorporates a pseudo-scalar field. We also discuss algebraically suitable discrete symmetry transformations of the theory. Finally, we demonstrate the relevance of the above continuous and discrete symmetries in the context of differential geometry and establish that our present massive 2D theory is a tractable field theoretic model for the Hodge theory. One of the key novel observations of our present investigation is the appearance of Curci-Ferrari (CF) type restrictions even in the case of our present massive 2D Abelian 1-form gauge theory. We also point out the mathematical as well as physical implications of the above pseudo-scalar field in our present theory.

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1 Introduction

The Proca theory, in any arbitrary dimension of spacetime, is a generalized version of the Maxwell’s theory which describes a massive bosonic field. In physical four (3 + 1)-dimensions of spacetime, the above bosonic field is endowed with three degrees of freedom due to its mass. The beautiful gauge symmetry of the Maxwell’s theory is, however, not respected by the Proca theory because the latter is endowed with second-class constraints in the language of Dirac’s prescription for classification scheme (see, e.g. [1,2]). It is the Stueckelberg’s formalism (see, e.g. [3]) that converts the above second-class constraints into first-class constraints thereby restoring the original gauge symmetry of the Maxwell’s theory in the case of Proca theory which is modified by the inclusion of a real scalar field.

In our earlier works [4-8] on the $p$-form (with $p = 1, 2, 3$) Abelian gauge theories in $D = 2p$ dimensions of spacetime, we have provided physical realizations of the de Rham cohomological operators* and Hodge duality operation of differential geometry within the framework of BRST formalism where the continuous and discrete symmetry transformations have played a key role. These theories are, however, not massive gauge theories as there is no mass parameter in them. In a very recent set of papers [9-11], we have been able to prove that the $\mathcal{N} = 2$ supersymmetric quantum mechanical models are also the examples of Hodge theory because of their specific discrete and continuous symmetries. Our present 2D modified Proca theory is very special because it is a massive gauge theory.

One of the central themes of our present investigation is to focus on the Stueckelberg-modified version of Proca theory in two $(1 + 1)$-dimensions of spacetime and demonstrate that this theory is a model for the Hodge theory where mass and various kinds of internal symmetries co-exist together in a meaningful manner if we incorporate a pseudo-scalar field in the theory with appropriate mass dimension (in natural units). The inclusion of the latter field enhances the symmetry properties of the theory and it adds aesthetics to the mathematical as well as physical contents of this 2D massive Abelian gauge theory.

Our present kind of studies is physically as well as mathematically useful because, exploiting the key mathematical ideas and theoretical tools of such studies, we have been able to prove that the 2D (non-)Abelian 1-form gauge theories (without any interaction with matter fields) are a new class of topological field theories (TFTs) which capture in their folds a few aspects of Witten type TFT and some salient features of Schwarz type TFT (see, e.g. [4,12,13] for details). In this proof, the idea of Hodge decomposition theorem in the quantum Hilbert space of states has played a key role [14]. Furthermore, we have also been able to show that the 2D Abelian theory with Dirac fields is a model for Hodge theory where a TFT (i.e. the 2D Abelian 1-form gauge field) couples with the matter (Dirac) fields [15,16]. Such studies have also been helpful in showing that the free 4D Abelian 2-form and 6D Abelian 3-form gauge theories are models for the quasi-TFT [5-8].

In our present investigation, we have come across some novel features while proving our modified model of 2D Proca theory to be an example of Hodge theory where the nilpo-

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*On a compact manifold without a boundary, a set of three mathematical operators $(d, \delta, \Delta)$ are called the de Rham cohomological operators of differential geometry where $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) is the exterior derivative, $\delta = \pm * d*$ (with $\delta^2 = 0$) is the co-exterior derivatives and $\Delta = (d + \delta)^2 \equiv \{d, \delta\}$ is the Laplacian operator. Here the $*$ symbol stands for the Hodge duality operation on the compact manifold. The above operators obey a well-known cohomological algebra [cf. equation (37) below].
tent (co-)exterior derivatives of differential geometry have been identified with the off-shell nilpotent (anti-)co-BRST and (anti-)BRST symmetry transformations. For instance, we have found the existence of a set of two equivalent Lagrangian densities for the description of this 2D theory where a new pseudo-scalar field has been introduced on mathematical and physical grounds. Further, we have found that, even for this Abelian 1-form gauge theory, there is existence of CF-type restriction [17] when we discuss the symmetry properties of the equivalent Lagrangian densities. Finally, the pseudo-scalar field appears in the theory with negative kinetic term and, hence, could be a candidate for the dark matter. The above features have, hitherto, not been seen together in the context of BRST approach to the description of $D = 2p$ dimensional Abelian $p$-form ($p = 1, 2, 3$) gauge theories [4-8].

The main motivating factors behind our present investigation are as follows.

First and foremost, we have to generalize our earlier work [18] on modified version of Proca theory where we have discussed the on-shell nilpotent (anti-)BRST, (anti-)co-BRST, a unique bosonic and the ghost-scale symmetries.

Second, it is well-known that the off-shell nilpotent symmetries are more general than the on-shell nilpotent symmetries as the latter are the special case of the former. We have accomplished this goal in our present investigation.

Third, our present endeavor is our modest step towards our main goal of finding the physically interesting 4D massive models for the Hodge theories where mass and various kinds of internal symmetries would co-exist together in a meaningful manner.

Finally, we have to explore the possibility of the existence of CF-type restriction(s) which are the hallmarks (see, e.g. [19,20]) of a given gauge theory described within the framework of BRST formalism. In fact, we do find something like CF-type restrictions when we discuss the symmetry properties of the two equivalent Lagrangian densities. However, in terms of properties, the CF-type restriction of our present theory is somewhat different from the usual CF-condition found in the context of 4D non-Abelian 1-form gauge theory.

The contents of our present investigation are organized as follows. To set up the notations, we recapitulate the bare essentials of the (dual-)gauge symmetries for the gauge-fixed Lagrangian densities of the Proca theory and discuss its generalizations. We also discuss the discrete symmetries of this generalized version of the Lagrangian densities of Proca theory in Sec. 2. Our Sec. 3 is devoted to the derivation of the off-shell nilpotent (anti-)BRST symmetry transformations. In our Sec. 4, we deduce the off-shell nilpotent (anti-)co-BRST symmetry transformations. Our Sec. 5 contains a unique bosonic symmetry of our present theory. In Sec. 6, we discuss the ghost-scale symmetry and full discrete symmetries of our present theory. Our Sec. 7 presents a thorough discussion of the extended BRST algebra and its connections with the cohomological operators. Finally, we make some concluding remarks and point out a few future directions in our Sec. 8.

General notations and conventions: Through out the whole body of our text, we shall denote the (anti-)BRST, (anti-)co-BRST, a bosonic and the ghost-scale symmetry transformations by the notations $s_{(a)b}$, $s_{(a)d}$, $s_w$ and $s_g$, respectively. For the two equivalent Lagrangian densities of our present theory, we shall use the superscripts $(1, 2)$ to distinguish them from each-other. We shall confine ourselves to only internal symmetry transformations and we shall not discuss anything connected with the 2D spacetime symmetries of our present theory as the spacetime 2D Minkowski manifold always remains in the background and it does not actively participate in our whole discussion.
2 Local (dual-)gauge transformations

We begin with the modified version of the two \((1 + 1)\)-dimensional (2D) Proca theory which is described by the recently proposed equivalent gauge-fixed Lagrangian densities \(\mathcal{L}_{(1,2)}\) as given below\(^{1}\) (see, e.g. [18] for details)

\[
\mathcal{L}_{(1,2)} = \frac{1}{2} (E \mp m \phi)^2 \pm m E \phi - \frac{1}{2} \partial \mu \phi \partial \nu \phi + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial \mu \phi \partial \nu \phi \pm m A_\mu \partial ^\mu \phi - \frac{1}{2} (\partial \cdot A \pm m \phi)^2,
\]

(1)

where the electric field \(E = -\varepsilon ^{\mu \nu} \partial _\mu A_\nu \equiv F_{01}\) is the only existing component of the 2D curvature tensor \(F_{\mu \nu} = \partial _\mu A_\nu - \partial _\nu A_\mu\), which is derived from the curvature 2-form \(F^{(2)} = dA^{(1)} = [(dx^\mu \wedge dx^\nu)/2!] F_{\mu \nu}\). Here \(d = dx^\mu \partial _\mu\) (with \(d^2 = 0\)) is the exterior derivative and 1-form \(A^{(1)} = dx^\mu A_\mu\) defines the Abelian 1-form connection field \(A_\mu\). In exactly similar fashion, the gauge-fixing term \((\partial \cdot A)\) owes its origin to the co-exterior derivative \(\delta = -*d*\) (with \(\delta^2 = 0\)) because \(\delta A^{(1)} = (\partial \cdot A)\) where \(*\) is the Hodge duality operation on the 2D flat Minkowskian spacetime manifold. There is no magnetic field in the 2D theory.

In the above Lagrangian densities, the basic fields \((A_\mu, \phi, \bar{\phi})\) have mass dimension zero in natural units (i.e. \(\hbar = c = 1\)) and the parameter \(m\) has the dimension of mass. Thus, the latter is the mass parameter in the whole theory. The pair \((\phi, \bar{\phi})\) are the real-scalar and pseudo-scalar fields as are the gauge-fixing term \((\partial \cdot A)\) and the electric field component \(E\), respectively. On mathematical grounds, we are free to add/subtract the pair of fields \((\phi, \bar{\phi})\) to the gauge-fixing and kinetic terms with proper mass dimension. This is what has been precisely achieved in (1). In fact, the signatures of these fields, play very important roles in the discussion of discrete symmetries as would become clear later. It is worth pointing out that the origin of the field \(\phi\) lies in the Stueckelberg formalism which is normally adopted to restore the gauge symmetry in a massive gauge theory (see, e.g. [3]).

The modification of the gauge-fixing term and kinetic term has been done with proper care where the signatures of addition/subtraction of the pair of fields \((\phi, \bar{\phi})\) play very crucial roles. It can be seen that, under the following (dual-)gauge transformations \(\delta_{(d)g}\)

\[
\begin{align*}
\delta_{dg} A_\mu &= -\varepsilon _{\mu \nu} \partial ^\nu \Sigma, & \delta_{dg} E &= \Box \Sigma, & \delta_{dg} \phi &= \pm m \Sigma, \\
\delta_{dg} (\partial \cdot A) &= 0, & \delta_{dg} \phi &= 0, & \delta_{dg} (\partial \cdot A + m \phi) &= 0, \\
\delta_g A_\mu &= \partial _\mu \Lambda, & \delta_g E &= 0, & \delta_g (E - m \phi) &= 0, \\
\delta_g (\partial \cdot A) &= \square \Lambda, & \delta_g \phi &= \pm m \Lambda, & \delta_g \bar{\phi} &= 0,
\end{align*}
\]

(2)

the Lagrangian densities \(\mathcal{L}_{(1,2)}\) transform as follows:

\[
\begin{align*}
\delta_{dg} \mathcal{L}_{(1,2)} &= \partial _\mu \left[ m \varepsilon ^{\mu \nu} (m A_\nu \Sigma \pm \phi \partial _\nu \Sigma) \pm m \bar{\phi} \partial ^\nu \Sigma \right] + (E \mp m \phi) (\Box + m^2) \Sigma, \\
\delta_g \mathcal{L}_{(1,2)} &= -(\partial \cdot A \pm m \phi) (\Box + m^2) \Lambda.
\end{align*}
\]

(4)

\(^{1}\)We adopt here the conventions such that the flat background 2D Minkowskian spacetime manifold is endowed with a metric \(\eta _{\mu \nu}\) with signatures \((+1, -1)\) so that \(A \cdot B = \eta _{\mu \nu} A^\mu B^\nu \equiv A_0 B_0 - A_1 B_1\) is the dot product between two non-null vectors \(A_\mu\) and \(B_\mu\). Here the Greek indices \(\mu, \nu, \lambda, \ldots \) denote the time and space directions of the 2D spacetime manifold. We also use the 2D antisymmetric Levi-Civita tensor \(\varepsilon _{\mu \nu}\) with the convention \(\varepsilon _{01} = +1 = \varepsilon ^{10}\) and \(\varepsilon ^{\mu \nu} \varepsilon _{\mu \nu} = -2!, \varepsilon ^{\mu \nu} \varepsilon _{\mu \lambda} = -1! \delta _\lambda ^\nu, \varepsilon ^{\mu \nu} \varepsilon _{\nu \lambda} = \delta _\lambda ^\mu, \) etc.
Thus, we note that, to attain the perfect (dual-)gauge symmetry, we have to impose exactly similar kind of restrictions on the (dual-)gauge transformation parameters \((\Sigma)\Lambda\). In other words, we have to put the mathematical conditions: \((\Box + m^2)\Sigma = 0, (\Box + m^2)\Lambda = 0\) on the local (dual-)gauge parameters \((\Sigma)\Lambda\) where \(\Box = \partial_0^2 - \partial_i^2\).

There is a perfect discrete symmetry in the theory, even though, there is modification of the kinetic and gauge-fixing terms due to inclusion of the fields \(\tilde{\phi}\) and \(\phi\). For instance, it can be readily checked that under the following discrete symmetry transformations

\[
A_\mu \rightarrow \pm i \varepsilon_{\mu \nu} A^\nu, \quad \phi \rightarrow \pm i \tilde{\phi}, \quad \tilde{\phi} \rightarrow \pm i \phi, \quad E \rightarrow \mp i (\partial \cdot A), \quad (\partial \cdot A) \rightarrow \mp i E,
\]

the Lagrangian densities \(L_{(1,2)}\) remain invariant modulo some total spacetime derivatives. We would like to mention, in passing, that the above discrete symmetries (and their generalizations) would play very important roles in our subsequent discussions.

The kinetic and gauge-fixing terms can be linearized by invoking the Nakanishi-Lautrup type auxiliary fields \((B, \tilde{B}, B, \tilde{B})\). The linearized versions of the above Lagrangian densities \(L_{(1,2)}\) are different in appearance and these are as follows:

\[
L_{(b_1)} = B \left( E - m \tilde{\phi} \right)^2 - \frac{1}{2} B^2 + m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m A_\mu \partial^\mu \phi + B (\partial \cdot A + m \phi) + \frac{1}{2} B^2, \tag{6}
\]

\[
L_{(b_2)} = \tilde{B} \left( E + m \phi \right)^2 - \frac{1}{2} \tilde{B}^2 - m E \phi - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + m A_\mu \partial^\mu \phi + \tilde{B} (\partial \cdot A - m \phi) + \frac{1}{2} \tilde{B}^2. \tag{7}
\]

It is worthwhile to explain that the Lagrangian densities \(L_{(1,2)}\) have been written in independent forms mainly due to the different types of Nakanishi-Lautrup auxiliary fields \((B, \tilde{B}, B, \tilde{B})\) that have been invoked, in the above, for the linearization purposes.

It is straightforward to check that there is a perfect discrete symmetry in the theory because the following transformations

\[
A_\mu \rightarrow \pm i \varepsilon_{\mu \nu} A^\nu, \quad \phi \rightarrow \pm i \tilde{\phi}, \quad \tilde{\phi} \rightarrow \pm i \phi, \quad E \rightarrow \mp i (\partial \cdot A), \quad (\partial \cdot A) \rightarrow \mp i E,
\]

\[
B \rightarrow \pm i B, \quad \tilde{B} \rightarrow \pm i \tilde{B}, \quad B \rightarrow \pm i B, \quad \tilde{B} \rightarrow \pm i \tilde{B}, \tag{8}
\]

leave the above Lagrangian densities \(L_{(b_1, b_2)}\) invariant. We point out that the above transformations are nothing but the generalization of our earlier transformations \((5)\). In exactly, similar fashion, it can be seen that, under the following generalized local, continuous and infinitesimal (dual-)gauge transformations \(\delta^{(1)}_{DG}\):

\[
\delta^{(1)}_{DG} A_\mu = -\varepsilon_{\mu \nu} \partial^\nu \Sigma, \quad \delta^{(1)}_{DG} E = \Box \Sigma, \quad \delta^{(1)}_{DG} \tilde{\phi} = -m \Sigma, \quad \delta^{(1)}_{DG} B = \delta^{(1)}_{DG} \tilde{B} = 0,
\]

\[
\delta^{(1)}_{DG} (\partial \cdot A) = 0, \quad \delta^{(1)}_{DG} \phi = 0, \quad \delta^{(1)}_{DG} (\partial \cdot A + m \phi) = 0, \quad \delta^{(1)}_{DG} (E - m \tilde{\phi}) = (\Box + m^2) \Sigma,
\]

\[
\delta^{(1)}_G A_\mu = \partial_\mu \Lambda, \quad \delta^{(1)}_G E = 0, \quad \delta^{(1)}_G (E - m \phi) = 0, \quad \delta^{(1)}_G B = \delta^{(1)}_G \tilde{B} = 0,
\]

\[
\delta^{(1)}_G (\partial \cdot A) = \Box \Lambda, \quad \delta^{(1)}_G \phi = + m \Lambda, \quad \delta^{(1)}_G \tilde{\phi} = 0, \quad \delta^{(1)}_G (\partial \cdot A + m \phi) = (\Box + m^2) \Lambda, \tag{9}
\]
the Lagrangian density $\mathcal{L}_{(b_1)}$ transforms as follows

$$
\delta^{(1)}_{DG} \mathcal{L}_{(b_1)} = \partial_\mu \left[ m \varepsilon^{\mu\nu} (m A_\nu \Sigma + \phi \partial_\nu \Sigma) + m \tilde{\phi} \partial^\mu \Sigma \right] + B (\Box + m^2) \Sigma,
$$

$$
\delta^{(1)}_G \mathcal{L}_{(b_1)} = B (\Box + m^2) \Lambda.
$$

(10)

which shows that, to achieve the (dual-)gauge invariance in the theory, the (dual-)gauge parameters have to restricted in exactly similar fashion [i.e. $(\Box + m^2) \Sigma = 0, \Box + m^2) \Lambda = 0$].

We can discuss the local, continuous and infinitesimal (dual-)gauge transformations for the Lagrangian density $\mathcal{L}_{(b_2)}$, too. These transformations are

$$
\begin{align*}
\delta^{(2)}_{DG} A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu \Sigma, & \delta^{(2)}_{DG} E &= \Box \Sigma, & \delta^{(2)}_G \tilde{\phi} &= +m \Sigma, & \delta^{(2)}_{DG} B &= \delta^{(2)}_{DG} \tilde{B} = 0, \\
\delta^{(2)}_{DG} (\partial \cdot A) &= 0, & \delta^{(2)}_G \phi &= 0, & \delta^{(2)}_{DG} (\partial \cdot A - m \phi) &= 0, & \delta^{(2)}_{DG} (E + m \tilde{\phi}) &= (\Box + m^2) \Sigma, \\
\delta^{(2)}_G A_\mu &= \partial_\mu \Lambda, & \delta^{(2)}_G E &= 0, & \delta^{(2)}_G (E + m \tilde{\phi}) &= 0, & \delta^{(2)}_G \tilde{B} &= \delta^{(2)}_G \tilde{B} = 0, \\
\delta^{(2)}_G (\partial \cdot A) &= \Box \Lambda, & \delta^{(2)}_G \phi &= -m \Lambda, & \delta^{(2)}_G \tilde{\phi} &= 0 & \delta^{(2)}_G (\partial \cdot A - m \phi) &= (\Box + m^2) \Lambda,
\end{align*}
$$

(11)

under which, as expected, the Lagrangian density $\mathcal{L}_{(b_2)}$ transforms as:

$$
\begin{align*}
\delta^{(2)}_{DG} \mathcal{L}_{(b_2)} &= \partial_\mu \left[ m \varepsilon^{\mu\nu} (m A_\nu \Sigma - \phi \partial_\nu \Sigma) - m \tilde{\phi} \partial^\mu \Sigma \right] + \tilde{B} (\Box + m^2) \Sigma, \\
\delta^{(2)}_G \mathcal{L}_{(b_2)} &= \tilde{B} (\Box + m^2) \Lambda.
\end{align*}
$$

(12)

Thus, once again, it is clear that for the sake of maintaining the (dual-)gauge symmetry invariance in the theory, exactly similar kind of restrictions should be imposed on the infinitesimal (dual-)gauge parameters $(\Sigma)\Lambda$ [i.e. $(\Box + m^2) \Sigma = 0, \Box + m^2) \Lambda = 0$].

We wrap up this section with the following comments. First, the nomenclature of the (dual-)gauge transformations is very appropriate because we observe that, under the gauge transformations, it is the total kinetic term, owing its fundamental origin to the exterior derivative, remains invariant. On the other hand, it is the total gauge-fixing term, originating basically from the dual-exterior derivative, that remains unchanged under the dual-gauge transformations. Second, the above restrictions, on the (dual-)gauge transformation parameters, can be taken care of within the framework of BRST formalism where these parameters would be replaced by the (anti-)ghost fields and there would be existence of _perfect_ continuous as well as discrete symmetries in the BRST-invariant theory. Finally, we obtain the following equations of motion from the Lagrangian densities $\mathcal{L}_{(b_1,b_2)}$:

$$
\mathcal{B} = E - m \tilde{\phi}, \quad \tilde{\mathcal{B}} = E + m \tilde{\phi}, \quad \mathcal{B} = -[(\partial \cdot A) - m \phi], \quad \tilde{\mathcal{B}} = -[(\partial \cdot A) + m \phi],
$$

(13)

which show that the Nakanishi-Lautrup type auxiliary fields are connected to one-another, via some relevant fields of our present theory, in the following fashion:

$$
\begin{align*}
\mathcal{B} + \tilde{\mathcal{B}} + 2(\partial \cdot A) &= 0, & \mathcal{B} - \tilde{\mathcal{B}} + 2m \phi &= 0, \\
\mathcal{B} + \tilde{\mathcal{B}} - 2E &= 0, & \mathcal{B} - \tilde{\mathcal{B}} + 2m \tilde{\phi} &= 0.
\end{align*}
$$

(14)

It is interesting to note that, under the _perfect_ discrete symmetry transformations, these relations transform amongst themselves. These relations are like the celebrated Cacci-Ferrari (CF) condition [17] of the non-Abelian 1-form gauge theories which appear in the BRST description of the latter theories. We shall see, in our later discussions, that the above relations do capture some properties of the CF-condition but there are distinct differences, too. We shall dwell on this issue in our forthcoming sections in an explicit manner.
3 Nilpotent (anti-)BRST symmetries

The Lagrangian density $L_{(b)}$ [cf. (6)] can be generalized to the (anti-)BRST invariant Lagrangian density $L_B$ that incorporates the gauge-fixing and Faddeev-Popov ghost terms. The exact expression for this Lagrangian density is:

$$L_{(B)} = B (E - m \bar{\phi}) - \frac{1}{2} B^2 + m E \bar{\phi} - \frac{1}{2} \partial_\mu \bar{\phi} \partial^\mu \bar{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \bar{\phi} \partial^\mu \phi
- m A_\mu \partial^\mu \phi + B (\partial \cdot A + m \phi) + \frac{1}{2} B^2 - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C,$$

(15)

where the (anti-)ghost fields $(\bar{C})C$ are fermionic (i.e. $C^2 = \bar{C}^2 = 0, C \bar{C} + \bar{C} C = 0$) in nature and they are required in the theory for the validity of unitarity. We observe that, under the following (anti-)BRST transformations $s_{(a)b}^{(1)}$:

$$s_{ab}^{(1)} A_\mu = \partial_\mu \bar{C}, \quad s_{ab}^{(1)} E = s_{ab}^{(1)} \bar{\phi} = s_{ab}^{(1)} \bar{C} = 0, \quad s_{ab}^{(1)} B = s_{ab}^{(1)} B = 0,$$
$$s_{ab}^{(1)} \phi = + m \bar{C}, \quad s_{ab}^{(1)} C = - i B, \quad s_{ab}^{(1)} (\partial \cdot A + m \phi) = (\square + m^2) \bar{C},$$
$$s_b^{(1)} A_\mu = \partial_\mu C, \quad s_b^{(1)} E = s_b^{(1)} \bar{\phi} = s_b^{(1)} C = 0, \quad s_b^{(1)} B = s_b^{(1)} B = 0,$$
$$s_b^{(1)} \phi = + m C, \quad s_b^{(1)} \bar{C} = + i B, \quad s_b^{(1)} (\partial \cdot A + m \phi) = (\square + m^2) C,$$

(16)

the above Lagrangian density transforms to the total spacetime derivatives:

$$s_b^{(1)} L_{(B)} = \partial_\mu \left[ B \partial^\mu C \right], \quad s_{ab}^{(1)} L_{(B)} = \partial_\mu \left[ B \partial^\mu \bar{C} \right].$$

(17)

As a consequence, the action integral $S = \int dx L_{(B)}$ remains invariant.

Exactly in a similar fashion, the Lagrangian density $L_{(b2)}$ can be generalized to an equivalent (i.e. equivalent to $L_{(B)}$) (anti-)BRST invariant Lagrangian density $L_{(B)}$:

$$L_{(B)} = \bar{B} (E + m \bar{\phi}) - \frac{1}{2} \bar{B}^2 + m E \bar{\phi} - \frac{1}{2} \partial_\mu \bar{\phi} \partial^\mu \bar{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \bar{\phi} \partial^\mu \phi
+ m A_\mu \partial^\mu \phi + \bar{B} (\partial \cdot A - m \phi) + \frac{1}{2} \bar{B}^2 - i \partial_\mu \bar{\bar{C}} \partial^\mu \bar{C} + i m^2 \bar{\bar{C}} \bar{C},$$

(18)

which also incorporates the gauge-fixing and Faddeev-Popov ghost terms. Furthermore, it is worth pointing out that the ghost part of the Lagrangian densities $L_{(B, \bar{B})}$ is exactly the same [cf. (15), (18)]. The following (anti-)BRST symmetry transformations:

$$s_{ab}^{(2)} A_\mu = \partial_\mu \bar{\bar{C}}, \quad s_{ab}^{(2)} E = s_{ab}^{(2)} \bar{\bar{\phi}} = s_{ab}^{(2)} \bar{\bar{C}} = 0, \quad s_{ab}^{(2)} \bar{B} = s_{ab}^{(2)} \bar{B} = 0,$$
$$s_{ab}^{(2)} \phi = - m \bar{\bar{C}}, \quad s_{ab}^{(2)} C = - i \bar{B}, \quad s_{ab}^{(2)} (\partial \cdot A - m \phi) = (\square + m^2) \bar{\bar{C}},$$
$$s_b^{(2)} A_\mu = \partial_\mu C, \quad s_b^{(2)} E = s_b^{(2)} \bar{\bar{\phi}} = s_b^{(2)} \bar{\bar{C}} = 0, \quad s_b^{(2)} \bar{B} = s_b^{(2)} \bar{B} = 0,$$
$$s_b^{(2)} \phi = - m C, \quad s_b^{(2)} \bar{\bar{C}} = + i \bar{B}, \quad s_b^{(2)} (\partial \cdot A - m \phi) = (\square + m^2) C,$$

(19)

leave the action integral invariant because the Lagrangian density $L_{(B)}$ transforms to the total spacetime derivatives under the above symmetry transformations:

$$s_b^{(2)} L_{(B)} = \partial_\mu \left[ \bar{B} \partial^\mu C \right], \quad s_{ab}^{(2)} L_{(B)} = \partial_\mu \left[ \bar{B} \partial^\mu \bar{\bar{C}} \right].$$

(20)
The above observation establishes that the (anti-)BRST transformations $s_{(a)b}^{(2)}$ are the symmetry transformations for the Lagrangian density $\mathcal{L}_{(B)}$.

The noteworthy points, at this juncture, are as follows. First, the (anti-)BRST symmetry transformations are off-shell nilpotent [i.e. $(s_{(a)b}^{(1,2)})^2 = 0$] of order two and they are absolutely anticommuting (i.e. $s_{b}^{(1)} s_{ab}^{(1)} + s_{ab}^{(1)} s_{b}^{(1)} = 0$, $s_{b}^{(2)} s_{ab}^{(2)} + s_{ab}^{(2)} s_{b}^{(2)} = 0$) which demonstrates their fermionic nature and linear independence. Second, the total kinetic term, owing its true origin to the exterior derivative of differential geometry, remains invariant under the (anti-)BRST symmetry transformations. Finally, the absolute anticommutativity property of the (anti-)BRST symmetry transformations, however, physically imply that only one of them could be identified with the exterior derivative of differential geometry.

We close this section with the discussion on the transformation properties of the Lagrangian density $\mathcal{L}_{(B)}$ under the (anti-)BRST symmetry transformations $s_{(a)b}^{(1)}$. Furthermore, we also devote time on the discussion of the transformation property of $\mathcal{L}_{(B)}$ under the (anti-)BRST symmetry transformations $s_{(a)b}^{(2)}$ [cf. (19)]. We shall see that, in this exercise, one of the CF-type conditions [cf. (14)] would play an important role. Taking into account the following (anti-)BRST transformations, besides (16) and (19), namely:

$$
\begin{align*}
 s_{b}^{(1)} B &= -2 \Box C, & s_{ab}^{(1)} B &= -2 \Box C, & s_{b}^{(1)} \mathcal{B} &= 0, & s_{ab}^{(1)} \mathcal{B} &= 0, \\
 s_{b}^{(2)} B &= -2 \Box C, & s_{ab}^{(2)} B &= -2 \Box C, & s_{b}^{(2)} \mathcal{B} &= 0, & s_{ab}^{(2)} \mathcal{B} &= 0,
\end{align*}
$$

we obtain the following expressions for the transformation properties of $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(B)}$:

$$
\begin{align*}
 s_{b}^{(1)} \mathcal{L}_{(B)} &= \partial_{\mu} \left[ B \partial^{\mu} C + 2 m^2 A^{\mu} C + 2 m \phi \partial^{\mu} C \right] \\
 &- \left[ B + \tilde{B} + 2 (\partial \cdot A) \right] (\Box + m^2) C, \\
 s_{ab}^{(1)} \mathcal{L}_{(B)} &= \partial_{\mu} \left[ B \partial^{\mu} \tilde{C} + 2 m^2 A^{\mu} \tilde{C} + 2 m \phi \partial^{\mu} \tilde{C} \right] \\
 &- \left[ B + \tilde{B} + 2 (\partial \cdot A) \right] (\Box + m^2) \tilde{C}, \\
 s_{b}^{(2)} \mathcal{L}_{(B)} &= \partial_{\mu} \left[ \tilde{B} \partial^{\mu} C + 2 m^2 A^{\mu} C - 2 m \phi \partial^{\mu} C \right] \\
 &- \left[ B + \tilde{B} + 2 (\partial \cdot A) \right] (\Box + m^2) C, \\
 s_{ab}^{(2)} \mathcal{L}_{(B)} &= \partial_{\mu} \left[ \tilde{B} \partial^{\mu} \tilde{C} + 2 m^2 A^{\mu} \tilde{C} - 2 m \phi \partial^{\mu} \tilde{C} \right] \\
 &- \left[ B + \tilde{B} + 2 (\partial \cdot A) \right] (\Box + m^2) \tilde{C},
\end{align*}
$$

which shows that $s_{(a)b}^{(1,2)}$ are also the symmetry transformations for the Lagrangian densities $\mathcal{L}_{(\tilde{B})}$ and $\mathcal{L}_{(B)}$, respectively, provided we restrict ourselves to a constrained hypersurface in the 2D Minkowskian spacetime manifold where the (anti-)BRST invariant (i.e. $s_{(a)b} \left[ B + \tilde{B} + 2 (\partial \cdot A) \right] = 0$) CF-type restriction $B + \tilde{B} + 2 (\partial \cdot A) = 0$ ([cf. (14)]) is satisfied. This observation is exactly like the role played by the original CF-condition [17] in the BRST description of the 4D non-Abelian 1-form gauge theory (see, e.g. [21,22] for more details). However, the CF-type condition, present in our theory (i.e. $B + \tilde{B} + 2 (\partial \cdot A) = 0$), does not play any role in the proof of anticommutativity of the (anti-)BRST symmetry transformations of our present theory as the latter are, as pointed out earlier, already absolutely anticommuting (i.e. $s_{b}^{(1)} s_{ab}^{(1)} + s_{ab}^{(1)} s_{b}^{(1)} = 0$, $s_{b}^{(2)} s_{ab}^{(2)} + s_{ab}^{(2)} s_{b}^{(2)} = 0$) in nature.
4 Nilpotent (anti-)co-BRST symmetries

Exactly like the (anti-)BRST symmetry transformations, there are other fermionic symmetry transformations in the theory. For instance, we observe that, under the following fermionic (anti-)dual-BRST [or (anti-)co-BRST] symmetry transformations $s^{(1)}_{(a)d}$:

\[
\begin{align*}
    s^{(1)}_{(ad)} A_\mu &= -\varepsilon_{\mu \nu} \partial^\nu C, & s^{(1)}_{(ad)} (\partial \cdot A) &= s^{(1)}_{ad} = s^{(1)}_{ad} C = 0, & s^{(1)}_{ad} B &= s^{(1)}_{ad} B = 0, \\
    s^{(1)}_{ad} \phi &= -m C, & s^{(1)}_{ad} C &= +i \bar{B}, & s^{(1)}_{ad} (E - m \tilde{\phi}) &= (\Box + m^2) C, & s^{(1)}_{ad} E &= \Box C, \\
    s^{(1)}_{d} A_\mu &= -\varepsilon_{\mu \nu} \partial^\nu \tilde{C}, & s^{(1)}_{d} (\partial \cdot A) &= s^{(1)}_{d} = s^{(1)}_{d} \tilde{C} = 0, & s^{(1)}_{d} B &= s^{(1)}_{d} B = 0, \\
    s^{(1)}_{d} \phi &= -m \tilde{C}, & s^{(1)}_{d} C &= -i \tilde{B}, & s^{(1)}_{d} (E - m \phi) &= (\Box + m^2) \tilde{C}, & s^{(1)}_{d} E &= \Box \tilde{C},
\end{align*}
\]

the Lagrangian density $L_{(B)}$ transforms to the total spacetime derivatives:

\[
\begin{align*}
    s^{(1)}_{ad} L_{(B)} &= \partial_\mu \left[ B \partial^\mu C + m \varepsilon^{\mu \nu} (m A_\nu C + \phi \partial_\nu C) + m \tilde{\phi} \partial^\mu C \right], \\
    s^{(1)}_{d} L_{(B)} &= \partial_\mu \left[ B \partial^\mu C + m \varepsilon^{\mu \nu} (m A_\nu C + \phi \partial_\nu C) + m \tilde{\phi} \partial^\mu C \right],
\end{align*}
\]

which shows that the above transformations are the symmetry transformations for the action integral $S = \int dx L_{(B)}$ of our present theory.

We can discuss about the (anti-)co-BRST symmetry transformations for the equivalent Lagrangian density $L_{(\bar{B})}$, too. These transformations are listed below:

\[
\begin{align*}
    s^{(2)}_{ad} A_\mu &= -\varepsilon_{\mu \nu} \partial^\nu C, & s^{(2)}_{ad} (\partial \cdot A) &= s^{(2)}_{ad} = s^{(2)}_{ad} C = 0, & s^{(2)}_{ad} \bar{B} &= s^{(2)}_{ad} \bar{B} = 0, \\
    s^{(2)}_{ad} \phi &= +m C, & s^{(2)}_{ad} \tilde{C} &= +i \bar{B}, & s^{(2)}_{ad} (E + m \tilde{\phi}) &= (\Box + m^2) C, & s^{(2)}_{ad} E &= \Box C, \\
    s^{(2)}_{d} A_\mu &= -\varepsilon_{\mu \nu} \partial^\nu \tilde{C}, & s^{(2)}_{d} (\partial \cdot A) &= s^{(2)}_{d} = s^{(2)}_{d} \tilde{C} = 0, & s^{(2)}_{d} \bar{B} &= s^{(2)}_{d} \bar{B} = 0, \\
    s^{(2)}_{d} \phi &= +m \tilde{C}, & s^{(2)}_{d} \tilde{C} &= -i \tilde{B}, & s^{(2)}_{d} (E + m \phi) &= (\Box + m^2) \tilde{C}, & s^{(2)}_{d} E &= \Box \tilde{C}.
\end{align*}
\]

Under the above transformations, the Lagrangian density $L_{(\bar{B})}$ transforms as

\[
\begin{align*}
    s^{(2)}_{ad} L_{(\bar{B})} &= \partial_\mu \left[ \bar{B} \partial^\mu C + m \varepsilon^{\mu \nu} (m A_\nu C - \phi \partial_\nu C) - m \tilde{\phi} \partial^\mu C \right], \\
    s^{(2)}_{d} L_{(\bar{B})} &= \partial_\mu \left[ \bar{B} \partial^\mu C + m \varepsilon^{\mu \nu} (m A_\nu C - \phi \partial_\nu C) - m \tilde{\phi} \partial^\mu C \right],
\end{align*}
\]

which establishes the symmetry invariance of the action integral of our present theory.

At this juncture, a few comments are in order. First and foremost, the nomenclature of the (anti-)co-BRST symmetry transformations is very appropriate as we note that the total gauge-fixing term, owing its fundamental origin to the co-exterior derivative, remains invariant under these transformations. Second, the (anti-)co-BRST symmetry transformations are off-shell nilpotent (i.e. $(s^{(1)}_{(a)d})^2 = 0$) and they are absolutely anticommuting i.e. $s^{(1)}_{ad} s^{(1)}_{ad} + s^{(1)}_{ad} s^{(1)}_{ad} = 0, s^{(2)}_{ad} s^{(2)}_{ad} + s^{(2)}_{ad} s^{(2)}_{ad} = 0$ demonstrating their fermionic and independent nature. Finally, the above absolute anticommutativity property ensures that only one of them could be identified with the co-exterior derivative of differential geometry.

We wrap up this section with the remark that one could also talk about the symmetry properties of the Lagrangian densities $L_{(B)}$ and $L_{(\bar{B})}$ under the transformations $s^{(2)}_{(a)d}$ and
Taking into account the following nilpotent symmetry transformations

\begin{align*}
s_d^{(1)} \hat{B} = 0, & \quad s_d^{(1)} \bar{B} = 0, \quad s_d^{(1)} \hat{B} = 2 \Box \bar{C}, \quad s_d^{(1)} \bar{B} = 2 \Box C, \\
s_d^{(2)} \hat{B} = 0, & \quad s_d^{(2)} \bar{B} = 0, \quad s_d^{(2)} \hat{B} = 2 \Box \bar{C}, \quad s_d^{(2)} \bar{B} = 2 \Box C,
\end{align*}

(27)
in addition to the other transformations listed in (23) and (25), we observe the following interesting transformations for the Lagrangian densities:

\begin{align*}
s_d^{(1)} \mathcal{L}_{(B)} &= \partial_\mu \left[ \mathcal{B} \partial^\mu \bar{C} + m \varepsilon^{\mu\nu} (m A_\nu \bar{C} - \phi \partial_\nu \bar{C}) + m \bar{\phi} \partial^\mu \bar{C} \right] \\
&\quad - \left[ \mathcal{B} + \bar{\mathcal{B}} - 2E \right] (\Box + m^2)\bar{C}, \\
s_d^{(1)} \mathcal{L}_{(\bar{B})} &= \partial_\mu \left[ \mathcal{B} \partial^\mu C + m \varepsilon^{\mu\nu} (m A_\nu C - \phi \partial_\nu C) + m \bar{\phi} \partial^\mu C \right] \\
&\quad - \left[ \mathcal{B} + \bar{\mathcal{B}} - 2E \right] (\Box + m^2)C, \\
s_d^{(2)} \mathcal{L}_{(B)} &= \partial_\mu \left[ \mathcal{B} \partial^\mu \bar{C} + m \varepsilon^{\mu\nu} (m A_\nu \bar{C} + \phi \partial_\nu \bar{C}) - m \bar{\phi} \partial^\mu \bar{C} \right] \\
&\quad - \left[ \mathcal{B} + \bar{\mathcal{B}} - 2E \right] (\Box + m^2)\bar{C}, \\
s_d^{(2)} \mathcal{L}_{(\bar{B})} &= \partial_\mu \left[ \mathcal{B} \partial^\mu C + m \varepsilon^{\mu\nu} (m A_\nu C + \phi \partial_\nu C) - m \bar{\phi} \partial^\mu C \right] \\
&\quad - \left[ \mathcal{B} + \bar{\mathcal{B}} - 2E \right] (\Box + m^2)C,
\end{align*}

(28)

which demonstrate that we have symmetry invariance of the Lagrangian densities \( \mathcal{L}_{(B)} \) and \( \mathcal{L}_{(\bar{B})} \) under the (anti-)co-BRST symmetry transformations \( s_{(a)d}^{(2)} \) and \( s_{(a)d}^{(1)} \), too, provided we confine ourselves to a constrained hypersurface that is described by the CF-type of field equations \( \mathcal{B} + \bar{\mathcal{B}} - 2E = 0 \) in the 2D Minkowskian spacetime manifold. This observation is exactly like the role played by the original CF-condition [17] in the context of 4D non-Abelian gauge theory discussed under the purview of BRST formalism (see, e.g. [21,22] for details). However, we lay stress on the fact that the CF-type condition of our present theory does not play any role in the proof of anti-commutativity property of the (anti-)co-BRST symmetry transformations as is the case with the original CF-condition [17] which does play a crucial role in the above proof for the (anti-)BRST symmetry transformations in the description of the 4D 1-form \( (A^{(1)} = dx^\mu A_\mu) \) non-Abelian gauge theory.

5 \hspace{1em} Bosonic symmetries

We have seen that there are four fermionic \([ (s_{(a)b})^2 = 0, (s_{(a)d})^2 = 0 ]\) type symmetries for each of the Lagrangian densities \( \mathcal{L}_{(B)} \) and \( \mathcal{L}_{(\bar{B})} \). It can be readily checked that, in their operator form, these transformations obey the following relationships:

\begin{align*}
\{ s_b^{(1)}, s_{ab}^{(1)} \} = \{ s_b^{(1)}, s_{ad}^{(1)} \} = \{ s_d^{(1)}, s_{ad}^{(1)} \} = \{ s_d^{(1)}, s_{ab}^{(1)} \} = 0, \\
\{ s_b^{(2)}, s_{ab}^{(2)} \} = \{ s_b^{(2)}, s_{ad}^{(2)} \} = \{ s_d^{(2)}, s_{ad}^{(2)} \} = \{ s_d^{(2)}, s_{ab}^{(2)} \} = 0.
\end{align*}

(29)

The following two independent anticommutators define the two bosonic symmetry transformations for the two equivalent Lagrangian densities. These are:

\begin{align*}
\{ s_b^{(1)}, s_d^{(1)} \} = s_w^{(1)} = - \{ s_{ab}^{(1)}, s_{ad}^{(1)} \}, \\
\{ s_b^{(2)}, s_d^{(2)} \} = s_w^{(2)} = - \{ s_{ab}^{(2)}, s_{ad}^{(2)} \}.
\end{align*}

(30)
Thus, we note that there are unique bosonic symmetry transformations for each of the Lagrangian densities $L_B$ and $L_{\bar{B}}$ of our present theory.

The relevant fields of the above Lagrangian densities transform, under the infinitesimal and continuous version of the bosonic transformations $s_w^{(1)}$, as:

\[
\begin{align*}
    s_w^{(1)} A_\mu &= \partial_\mu B + \varepsilon_{\mu\nu} \partial^{\nu} B, \\
    s_w^{(1)} B &= m B, \\
    s_w^{(1)} \phi &= m \phi, \\
    s_w^{(1)} \Phi &= m \Phi,
\end{align*}
\]

\[
\begin{align*}
    s_w^{(1)} (\theta \cdot A) &= \Box B, \\
    s_w^{(1)} E &= -\Box B, \\
    s_w^{(1)} [B, \bar{B}, C, \bar{C}] &= 0,
\end{align*}
\]

\[
\begin{align*}
    s_w^{(2)} A_\mu &= \partial_\mu \bar{B} + \varepsilon_{\mu\nu} \partial^{\nu} \bar{B}, \\
    s_w^{(2)} B &= -m \bar{B}, \\
    s_w^{(2)} \phi &= -m \phi, \\
    s_w^{(2)} \Phi &= -m \Phi,
\end{align*}
\]

\[
\begin{align*}
    s_w^{(2)} (\theta \cdot A) &= \Box \bar{B}, \\
    s_w^{(2)} E &= -\Box \bar{B}, \\
    s_w^{(2)} [B, \bar{B}, C, \bar{C}] &= 0.
\end{align*}
\]  

One of the decisive features of these bosonic symmetry transformations is the observation that the ghost part of Lagrangian densities $L_B$ and $L_{\bar{B}}$ remains invariant. It can be readily checked that the above Lagrangian densities transform to the total spacetime derivatives under the bosonic symmetry transformations. These can be expressed mathematically as:

\[
\begin{align*}
    s_w^{(1)} L_B &= \partial_\mu \left[ B \partial^\mu B - \bar{B} \partial^\mu B - m \varepsilon^{\mu\nu} (\phi \partial_\nu B + m A_\nu B) - m \phi \partial^{\mu} B \right], \\
    s_w^{(2)} L_B &= \partial_\mu \left[ \bar{B} \partial^\mu \bar{B} - \bar{B} \partial^\mu \bar{B} + m \varepsilon^{\mu\nu} (\phi \partial_\nu \bar{B} - m A_\nu \bar{B}) + m \bar{\phi} \partial^{\mu} \bar{B} \right],
\end{align*}
\]

ensuring the invariance of the action integrals for our current theory under consideration.

A few key points, at this stage, are as follows. First, the bosonic symmetry transformations are unique because their two representations are equal modulo a sign factor. In other words, we have: $s_w^{(1)} = \{ s_w^{(1)}, s_w^{(2)} \} \equiv -\{ s_w^{(1)}, s_w^{(2)} \}$. Second, the bosonic symmetry transformations are derived from the fundamental off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations. Third, it is clear that the bosonic symmetry transformations provide the physical realization of the Laplacian operator of differential geometry if one of the (anti-)BRST symmetry transformations is chosen to be the exterior derivative and a single transformation from the (anti-)co-BRST symmetry transformations is selected to represent the co-exterior derivative. Finally, we would like to state that the bosonic transformations (31) are written modulo an overall $(-i)$ factor so that the conserved Noether charge, corresponding to these transformations, could turn out to be real. This is due to the fact that (identified with the Laplacian operator), this Noether charge should be Hermitian and it should produce the positive eigen values.

### 6 Ghost-scale and discrete symmetries

We note that the ghost part of the Lagrangian densities has the discrete symmetry transformations: $C \rightarrow \mp i \bar{C}, \bar{C} \rightarrow \mp i C$. Furthermore, there are continuous scale transformations in the theory, under which, the relevant fields of the theory transform as:

\[
C \rightarrow e^{(1+i)\Omega} C, \quad \bar{C} \rightarrow e^{(1-i)\Omega} C, \quad \Psi \rightarrow e^{(0)\Omega} \Psi, \quad \Psi \equiv A_\mu, B, \bar{B}, \bar{B}, \phi, \Phi,
\]  

which are called as the ghost-scale symmetry transformations because only the (anti-)ghost fields $(C)\bar{C}$, with ghost numbers $(\mp 1)$, transform and rest of the fields of the theory, with zero ghost number, remain unchanged. The ghost numbers are reflected in the above scale transformations as the coefficients of the parameter $\Omega$ in the exponentials. In the above, the
transformation scale parameter $\Omega$ is global (i.e. spacetime independent). The infinitesimal version of these transformations, denoted by $s_g$, are as follows:

$$s_g C = + C, \quad s_g \bar{C} = - \bar{C}, \quad s_g \Psi = 0,$$

(34)

where we have chosen $\Omega = 1$ for the sake of brevity (for our further discussions). We have not used any superscript for $s_g$ because the ghost part of the Lagrangian densities (15) and (18) is the same and infinitesimal transformations (34) is true for both of them.

There are a couple of discrete symmetry transformations in our theory which, as we shall see later, play very important roles in our whole discussion. In this context, let us recall the discrete transformations (8), under which, the Lagrangian densities (without the ghost part) were found to be invariant. It can be checked that its generalized form

$$A_\mu \rightarrow \pm i\varepsilon_{\mu\nu}A^\nu, \quad \phi \rightarrow \pm i\tilde{\phi}, \quad \bar{\phi} \rightarrow \pm i\phi, \quad E \rightarrow \mp i(\partial \cdot A), \quad (\partial \cdot A) \rightarrow \mp iE,$$

$$B \rightarrow \pm iB, \quad \bar{B} \rightarrow \pm i\bar{B}, \quad B \rightarrow \pm iB, \quad \bar{B} \rightarrow \pm i\bar{B}, \quad C \rightarrow \mp i\bar{C}, \quad \bar{C} \rightarrow \mp iC,$$

(35)

leaves the (anti-)BRST invariant Lagrangian densities (15) and (18) absolutely invariant. We shall see that the above discrete symmetry transformations would provide the physical realization of the Hodge duality operation of differential geometry (see, Sec. 7 below).

7 Symmetries and cohomological operators

The six (i.e. four fermionic and two bosonic) continuous symmetry transformations of our present theory obey the following algebra in their operator form$^4$:

$$s_b^2 = s_d^2 = s_{ab}^2 = s_{ad}^2 = 0, \quad \{s_b, s_{ad}\} = 0, \quad \{s_d, s_{ab}\} = 0, \quad \{s_b, s_{ad}\} = 0, \quad \{s_d, s_{ab}\} = 0,$$

$$\{s_b, s_{ad}\} = 0, \quad \{s_b, s_d\} = s_w = -\{s_{ab}, s_{ad}\}, \quad [s_w, s_r] = 0, \quad r = (b, ab, d, ad, g),$$

$$[s_g, s_b] = +s_b, \quad [s_g, s_d] = -s_d, \quad [s_g, s_{ad}] = +s_{ad}, \quad [s_g, s_{ab}] = -s_{ab}.$$

(36)

The noteworthy points, at this stage, are as follows. First, we note that there are four fermionic and two bosonic type of continuous symmetries in our theory. Second, the bosonic symmetry operator $s_w$ is the Casimir operator for the whole algebra. Finally, it is very important to note that the ghost symmetry transformation $s_g$ has exactly similar kind of algebra with the pair of symmetry operators $(s_b, s_{ad})$ and $(s_d, s_{ab})$.

A close look at the above algebra, satisfied by the continuous symmetry operators, demonstrates that the structure of this algebra is very similar to the algebra of de Rham cohomological operators $(d, \delta, \Delta)$ of differential geometry where, as we have seen earlier, the operators $(\delta)d$ are the (co-)exterior derivatives and $\Delta = (d + \delta)^2$ is the Laplacian operator [23-25]. They obey the following well-known algebra

$$d^2 = 0, \quad \delta^2 = 0, \quad \Delta = \{d, \delta\}, \quad [\Delta, d] = 0, \quad [\Delta, \delta] = 0.$$

(37)

$^4$We ignore here the superscripts (1, 2) on the continuous and infinitesimal (anti-)BRST, (anti-)co-BRST and bosonic symmetry transformations because our statements, in this section, are general in nature and true for both of them. In fact, our observations are valid for any arbitrary model of Hodge theory.
Thus, we note that the Laplacian operator $\Delta$ is the Casimir operator for the whole algebra and it is very similar to the bosonic symmetry transformation $s_w$ of the algebra in (36). However, as we have seen earlier, there are two realizations of $s_w$ in terms of the fermionic symmetry transformations: $s_w = \{s_b, s_d\} = -\{s_{ab}, s_{ad}\}$. In contrast, we have only a single realization of $\Delta$ because $\Delta = \{d, \delta\}$ in differential geometry where $(\delta)d$ are nilpotent of order two (i.e. $d^2 = \delta^2 = 0$) just like the above fermionic symmetry operators. This shows that there is one-to-two mapping: $\Delta \Rightarrow \{s_b, s_d\} \equiv -\{s_{ab}, s_{ad}\}$.

The above analysis leads us to make a guess that there should be two realizations of $d$ and $\delta$ as well in the language of the symmetry transformations. It turns out that, this precisely is the case, in our 2D theory. Thus, for our 2D model for the Hodge theory, the mapping is one-to-two from the cohomological operators to the symmetry operators, as

$$d \Rightarrow (s_b, s_{ad}), \quad \delta \Rightarrow (s_d, s_{ab}), \quad \Delta \Rightarrow s_w = \{s_b, s_d\} = -\{s_{ab}, s_{ad}\}. \quad (38)$$

We wish to lay emphasis on the fact that it is the nature of the commutation relations with $s_g$ that decides the grouping of the transformations $(s_b, s_{ad})$ and $(s_d, s_{ab})$ in providing the realizations of the exterior and co-exterior derivatives of differential geometry.

We know that (co-)exterior derivatives $(\delta)d$ are connected by the relationship: $\delta = - \ast d \ast$ in the even dimensional spacetime manifold [23-25]. This relationship could be also realized in the language of symmetry transformations. For instance, it can be checked that the interplay between the continuous and discrete symmetry transformations provide the realizations of the above relation (of differential geometry) in the physical language of symmetries (of our present theory), as:

$$s_{(a)d} \Psi = - \ast s_{(a)b} \ast \Psi, \quad \Psi = A_\mu, \phi, \tilde{\phi}, C, \bar{C}, B, \bar{B}, \mathcal{B}, \mathcal{B}, E, (\partial \cdot A), \quad (39)$$

where $\Psi$ is the generic field of our present theory and $\ast$ operation is nothing but the operation of the discrete symmetry transformations (35). The minus sign, present on the r.h.s. of equation (39), is actually dictated by the two successive operations of the discrete symmetry transformations on any individual field. For instance, in our 2D theory, it can be checked that the generic field $\Psi$ transforms as follows under the two successive operations of the discrete symmetry transformations (35), namely;

$$\ast (\ast \Psi) = - \Psi, \quad \Psi = A_\mu, \phi, \tilde{\phi}, C, \bar{C}, B, \bar{B}, \mathcal{B}, \mathcal{B}, E, (\partial \cdot A). \quad (40)$$

Thus, we conclude that the relationship, quoted in equation (39), is correct as per the rules and prescriptions laid down by the perfect duality invariant theory (see, e.g. [26]).

To sum up, we have captured all the relevant abstract mathematical relations, obeyed by the de Rham cohomological operators of differential geometry, in the language of the discrete and continuous symmetry transformations of our present theory which is unique, in some sense, because here the mass and analogue of the gauge symmetries (and other symmetries) co-exist together in a meaningful and complementary manner.

8 Conclusions

We have been able to establish, within the framework of BRST formalism, the 1D, 2D, 4D and 6D physically interesting models to be the tractable physical examples of Hodge
theory. In this context, mention can be made of the 1D model of a rigid rotor [27], 2D free (non-)Abelian gauge theories (without any interaction with matter fields) and the modified version of 2D anomalous gauge theory [28], free 4D Abelian 2-form gauge theory [5,6] and free 6D model of Abelian 3-form gauge theory [7,8]. Recently, we have shown the supersymmetric quantum mechanical models to be an interesting set of examples for the Hodge theory [9-11]. However, we wish to re-emphasize that our present model of the modified 2D Proca theory is a very special example because, in this model, mass and various kinds of continuous and discrete symmetries co-exist together in a meaningful manner.

One of the very important observations of our present investigation is the existence of a couple of equivalent Lagrangian densities for the massive 2D Abelian 1-form gauge theory where a set of (anti-)BRST invariant CF-type restrictions exist. Even though the latter do not play any important role in the proof of anticommutativity of the off-shell nilpotent (i.e. $s^2_{(a)b} = s^2_{(a)d} = 0$) (anti-)BRST and (anti-)co-BRST transformations of our present theory, they appear when we discuss the symmetry properties of the Lagrangian densities together under the (anti-)BRST and (anti-)co-BRST transformations. These are completely novel observations in the context of the application of BRST formalism to the description of a massive Abelian gauge theory. In our present investigation, we have laid emphasis on these issues (in the main body of our text) in the language of the off-shell nilpotent (anti-)BRST, (anti-)co-BRST, a unique bosonic and the ghost-scale symmetries of the theory.

It is very interesting (and physically important) to point out that the kinetic term for the pseudo-scalar field $\tilde{\phi}$, in our theory, carries a negative sign. In the literature, such kinds of fields and particles have been discussed within the realm of quantum field theory and quantum mechanics (see, e.g. [29,30] for details). If the symmetries are the guiding principles for a beautiful theory, it is essential to invoke a pseudo-scalar field, with a negative kinetic term, in our theory so that we could have a set of perfect discrete symmetry transformations [e.g. equation (35)] in our theory. We note that the equation of motion for this field [i.e. $(\Box + m^2) \tilde{\phi} = 0$] shows that this intriguing field carries a physical mass $m$. As a consequence, this field will provide a candidate for the dark matter. Such kinds of massless fields have also appeared in the context of 4D free Abelian 2-form gauge theory when we have proved it to provide a tractable model for the Hodge theory [7].

It is worthwhile to point out that the Stueckelberg real-scalar field, with positive kinetic term, couples basically with the gauge-fixing $(\partial \cdot A)$ term as is clear from a close look at the (anti-)BRST invariant Lagrangian densities (15) and (18). On the contrary, the pseudo-scalar field, with negative kinetic term, couples with the electric field which happens to be a pseudo-scalar field in our present 2D theory. We note that the coupling constant of these interaction terms is nothing but the mass $m$ itself. This demonstrates that the fields $(\phi, \tilde{\phi})$ can interact only via gravitational interaction. It would be very interesting to explore more details about these fields in the physical four dimensions of spacetime which might give some clues about the nature of the dark matter and its interaction with gauge fields. This aspect of our observation would be a key topic of research in our future endeavors.

During the past, we have exploited the theoretical power and potential of the augmented version [31-33] of the usual superfield formalism [34-37] to derive the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations for the gauge, corresponding (anti-)ghost and matter fields of a given gauge/reparametrization invariant theory. We are currently deeply involved with this aspect of investigation in the context of the modified version of
2D Proca theory and we have already obtained some novel results that are connected with the geometrical interpretation of (and inter-relations between) the nilpotency and absolute anticommutativity properties associated with the (anti-)BRST and (anti-)co-BRST charges. We hope to report about the key results of our work in the near future.

It would be very nice to extend our present analysis and ideas to 3D and 4D massive gauge theories like Jackiew-Pi [38] and Freedman-Townsend [39] models of non-Abelian 1-form and 2-form theories. The 4D Abelian model of the topologically massive gauge theory (with celebrated $B \wedge F$ term) is another model where our ideas could be applied. The above models of physical interest could be also proven, perhaps, to be the physical examples of Hodge theory. These are the issues that are presently under investigation and our results would be reported in our future publications [40].

References

[1] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University Press, New York (1964)

[2] K. Sundermeyer, *Constrained Dynamics: Lecture Notes in Physics*, vol. 169, Springer-Verlag, Berlin (1982)

[3] See, e.g., Henri Ruegg, Marti Ruiz-Altaba, Int. J. Mod. Phys. A 19, 3265 (2004)

[4] See, e.g., for a review, R. P. Malik, Int. J. Mod. Phys. A 22, 3521 (2007)

[5] S. Gupta, R. P. Malik, Eur. Phys. J.C 58, 517 (2008)

[6] E. Harikumar, R. P. Malik, M. Sivakumar, J. Phys. A: Math. Gen. 33, 7149 (2000)

[7] R. Kumar, S. Krishna, A. Shukla, R. P. Malik, Eur. Phys. J. C 72 1980 (2012)

[8] R. Kumar, S. Krishna, A. Shukla, R. P. Malik, arXiv: arXiv: 1203.5519 [hep-th]

[9] R. Kumar, R. P. Malik, Euro. Phys. Lett. 98, 11002 (2012)

[10] R. P. Malik, Avinash Khare, Annals of Physics 334, 142 (2013)

[11] R. Kumar, R. P. Malik, arXiv: 1303.5253 [hep-th]

[12] R. P. Malik, J. Phys. A: Math. Gen. 34, 4167 (2001)

[13] R. P. Malik, J. Phys. A: Math. Gen. 35, 6919 (2002)

[14] R. P. Malik, Int. J. Mod. Phys. A 15, 1685 (2000)

[15] R. P. Malik, Mod. Phys. Lett. A 15, 2079 (2000)

[16] R. P. Malik, Mod. Phys. Lett. A 16, 477 (2001)

[17] G. Curci, R. Ferrari, Phys. Lett. B 63, 91 (1976)
[18] T. Bhanja, D. Shukla, R. P. Malik, arXiv: 1305.1013 [hep-th]
[19] L. Bonora, R. P. Malik, Phys. Lett. B 655, 75 (2007)
[20] L. Bonora, R. P. Malik, J. Phys. A: Math. Theor. 43, 375403 (2010)
[21] See, e.g., N. Nakanishi, I. Ojima, Covariant Operator Formalism of Gauge Theory and Quantum Gravity, World Scientific, Singapore (1990)
[22] K. Nishijima, Czechoslov. J. Phys. 46, 1 (1996)
[23] T. Eguchi, P. B. Gilkey, A. Hanson, Phys. Rep. 66, 213 (1980)
[24] S. Mukhi, N. Mukunda, Introduction to Topology, Differential Geometry and Group Theory for Physicists, Wiley Eastern Private Limited, New Delhi (1990)
[25] J. W. van Holten, Phys. Rev. Lett. 64, 2863 (1990)
[26] S. Deser, A. Gomberoff, M. Henneaux, C. Teitelboim, Phys. Lett. B 400, 80 (1997)
[27] Saurabh Gupta, R. P. Malik, Eur. Phys. J C 68, 325 (2010)
[28] S. Gupta, R. Kumar, R. P. Malik, Eur. Phys. J. C 65, 311 (2010)
[29] See, e.g., V. M. Zhuravlev, D. A. Kornilov, E. P. Savelova, Gen. Rel. Grav. 36, 1719 (2004)
[30] See, e.g., Y. Aharonov, S. Popescu, D. Rohrlich, L. Vaidman, Phys. Rev. D 48, 4084 (1993)
[31] R. P. Malik, Eur. Phys. J. C 45, 513 (2006)
[32] R. P. Malik, J. Phys. A: Math. Gen. 40, 4877 (2007)
[33] R. P. Malik, Eur. Phys. J. C 47, 227 (2006)
[34] L. Bonora, M. Tonin, Phys. Lett. B 98, 48 (1981)
[35] L. Bonora, P. Pasti, M. Tonin, Nuovo Cimento A 63, 353 (1981)
[36] R. Delbourgo, P. D. Jarvis, J. Phys. A: Math. Gen 15, 611 (1981)
[37] R. Delbourgo, P. D. Jarvis, G. Thompson, Phys. Lett. B 109, 25 (1982)
[38] R. Jackiw, S-Y. Pi, Phys. Lett. B 403, 297 (1997)
[39] D. Z. Freedman, P. K. Townsend, Nucl. Phys. B 177, 282 (1981)
[40] R. P. Malik, etal., in preparation