Fracture analysis of cracks terminating at the interface of elastic-piezoelectric bimaterials

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Abstract. In this paper, the fracture behaviors of a piezoelectric-elastic bimaterial with cracks terminating at the interface are investigated by a symplectic approach. In the Hamiltonian system, the Hamiltonian forms of governing equations is derived by the Hamiltonian variational principle and a total unknown vector consisted of generalized displacements and stresses. The interface fracture problem is reduced into a symplectic eigenproblem which can be directly solved by the method of separation of variables. Thus, the total unknown vector is expanded in terms of symplectic eigenfunctions. The unknown coefficients of the symplectic series can be determined from the continuity conditions at the interface and outer boundary conditions. Consequently, exact solutions for the singular electro-elastic fields and explicit expression of electric/elastic intensity factors are obtained simultaneously. Results indicate that the electro-elastic singularities and intensity factors only depend on the first few terms of symplectic eigenfunctions with non-zero eigenvalues. Numerical examples are presented to show the effects of key influencing factors on the singularity orders and intensity factors of such interface cracks. Some new results are given also.

1. Introduction

Due to the intrinsic electro-mechanical coupling feature, the piezoelectric materials have been widely used in the manufacture of smart structures and intelligent devices, such as transducers, sensors and actuators. In the engineering application, the piezoelectric material is usually bonded to an elastic substrate, such as polymers. Crack unavoidably occurs at the interface or in one layer of the laminated structure during long-term service. Therefore, an understanding of the singular electro-elastic fields around the crack tip is of great importance to the evaluation of the piezoelectric-elastic structures. The fracture behaviors of the cracks laying at the interface have been well studied in the open literature. However, the cracks terminating at the interface attracted limited attention, especially for the piezoelectric-elastic laminated structure. Cook and Erdogan considered a problem of two elastic bonded half planes containing a crack perpendicular to the interface [1, 2]. Sung et al. [3, 4] analyzed the order of stress singularities at the tip of a crack terminating normally at an interface between two orthotropic media. Chang and Xu [5] and Chen et al. [6] obtained the singular stress field and stress intensity factors of a crack terminating at a bimaterial interface. Wang et al. [7, 8] studied the piezoelectric layer contains an edge crack that is perpendicular to the surface of medium. Li [9]
investigated the sandwiched piezoelectric composite containing a crack at the center of the piezoelectric strip and normal to the interfaces.

In view of the above-mentioned literature, the fracture analysis of cracks terminating at the interface of piezoelectric-elastic bimaterials is insufficient. The available work was concentrated on the infinite body which may not be suitable for the finite-size engineering structures. To this end, a symplectic approach [10, 11] is introduced to find exact solutions of piezoelectric/elastic bimaterials containing an edge crack terminating at the interface. The analytical solutions of singular electro-­

electric fields are directly obtained and expressed in terms of symplectic eigenfunctions. The explicit expressions of stress/electric intensity factors are obtained simultaneously.

2. Basic equation
Consider two edge-cracked piezoelectric-elastic bimaterials in figure 1. The crack could occur in the elastic layer (Case i) or piezoelectric layers (Case ii). The polar coordinate system is selected and the origin is located at the crack tip. \( \alpha_1, \alpha_2, \alpha_3 \) are the sustained angles. In the absence of body forces, the fundamental equations for the piezoelectric and elastic layers subjected to the anti-plane loads are presented as follows.

![Figure 1. Edge-cracked piezoelectric-elastic biomaterials.](image)

**For the piezoelectric layer:**

the constitutive equations:

\[
\begin{align*}
\begin{bmatrix}
\sigma_{rz}^{(p)} \\
\sigma_{\theta z}^{(p)} \\
D_r \\
D_\theta
\end{bmatrix} &=
\begin{bmatrix}
C_{44} & 0 & -e_{15} & 0 \\
0 & C_{44} & 0 & -e_{15} \\
e_{15} & 0 & \kappa_{11} & 0 \\
0 & e_{15} & 0 & \kappa_{11}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{rz}^{(p)} \\
\varepsilon_{\theta z}^{(p)} \\
E_r \\
E_\theta
\end{bmatrix},
\end{align*}
\]

\( (1a) \)

the geometric equations:

\[
\begin{align*}
\varepsilon_{rz}^{(p)} &= \frac{\partial W_z^{(p)}}{\partial r}, & \varepsilon_{\theta z}^{(p)} &= \frac{1}{r} \frac{\partial W_z^{(p)}}{\partial \theta}, & E_r &= -\frac{\partial \phi}{\partial r}, & E_\theta &= -\frac{1}{r} \frac{\partial \phi}{\partial \theta}
\end{align*}
\]

\( b \)

the governing equations:

\[
\frac{\partial \left( r \sigma_{rz}^{(p)} \right)}{\partial r} + \frac{\partial \left( \sigma_{\theta z}^{(p)} \right)}{\partial \theta} = 0, & \frac{\partial (rD_r)}{\partial r} + \frac{\partial (D_\theta)}{\partial \theta} = 0
\]

\( c \)

**For the elastic layers:**

the constitutive equations:
\[
\begin{align*}
\begin{bmatrix}
\sigma_{rz}^{(e)} \\
\sigma_{r\theta}^{(e)}
\end{bmatrix} &= 
\begin{bmatrix}
G & 0 \\
0 & G
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{rz}^{(e)} \\
\varepsilon_{r\theta}^{(e)}
\end{bmatrix} \\
\text{the geometric equations:}
\varepsilon_{rz}^{(e)} &= \frac{\partial w^{(e)}}{\partial r}, \; \varepsilon_{r\theta}^{(e)} = \frac{1}{r} \frac{\partial w^{(e)}}{\partial \theta}
\end{align*}
\]

the governing equations:
\[
\frac{\partial (r\sigma_{rz}^{(e)})}{\partial r} + \frac{\partial \sigma_{r\theta}^{(e)}}{\partial \theta} = 0
\]

where the superscripts “p” and “e” denote the piezoelectric and elastic layers, respectively; \(\sigma_{rz}, \; \sigma_{r\theta}, \; D_r, \; D_\theta, \; E_r \) and \( E_\theta \) are the components of stress tensor, electric displacement and electric field vectors, respectively; \( C_{44}, \; e_{15} \) and \( \kappa_{11} \) are the elastic constant, piezoelectric constant and dielectric permittivity, respectively.

The boundary conditions at the interface are:
\[
w^{(p)}(r, \theta) = w^{(e)}(r, \theta) \; \; \; \sigma^{(p)}_{rz}(r, \theta) = \sigma^{(e)}_{rz}(r, \theta) \; \; \; \phi(r, \theta) = 0
\]

where \( \theta = \pi - \alpha_1, -\pi + \alpha_3 \).

The boundary conditions are given as follows:
Case i: \( \sigma^{(e)}_{r\theta}(r, \pi) = \sigma^{(e)}_{r\theta}(r, -\pi) = 0 \)  (4)
Case ii: \( \sigma^{(e)}_{r\theta}(r, \pi) = \sigma^{(e)}_{r\theta}(r, -\pi) = 0 \) and \( D_\theta(r, \pi) = D_\theta(r, -\pi) = 0 \) (5)

3. Hamilton system and symplectic eigenproblem

3.1 Hamiltonian dual equation

The basic equations (1) and (2) can be rewritten in the Hamiltonian form by introducing a total unknown vector. For the mode III fracture problem, the total unknown \( \Psi \) for the piezoelectric and elastic layers can be defined as
\[
\Psi^{(p)} = \{w^{(p)}, \phi, r\sigma^{(p)}_{rz}, rD_r\}^T \; \; \; \Psi^{(e)} = \{w^{(e)}, r\sigma^{(e)}_{rz}\}^T
\]

respectively. Therefore, the governing equations in the Hamiltonian form is
\[
\Psi^{(k)} = H^{(k)} \Psi^{(k)} \quad (k = p, \; e)
\]

where the overdot is the represent differentiation with respect to the generalized coordinate \( \eta = \ln r; \)

\[
H^{(p)} = \begin{bmatrix}
0 & \left( M^{(p)} \right)^{-1} \\
\frac{\partial^2 M^{(p)}}{\partial \eta^2} & 0
\end{bmatrix} \quad \text{and} \quad H^{(e)} = \begin{bmatrix}
0 & \frac{1}{G} \\
\frac{\partial^2 G}{\partial \eta^2} & 0
\end{bmatrix}
\]

are the Hamiltonian operator matrices for the piezoelectric and elastic layers, respectively, where \( M^{(p)} = \begin{bmatrix} C_{44} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix} \).

3.2 Zero eigenfunctions

Assuming \( \Psi^{(k)}(\eta, \theta) = \psi^{(k)}(\theta)e^{i\eta} \), the eigen-equation of equation (7) is \( \mu \Psi^{(e)} = H^{(e)} \Psi^{(e)} \) \( (k = p, \; e) \). Thus, the symplectic eigenfunctions can be divided into two categories: zero eigenfunctions \( (\mu = 0) \) and non-zero eigenfunctions \( (\mu \neq 0) \).

The zero eigenvalue eigenfunctions \( (\mu = 0) \) are
\[ \psi_{0,1}^c = \{1, 0\}^T \quad \text{and} \quad \psi_{0,2}^c = \{1, G_1^c\}^T \]  
\[ \begin{align*} 
\psi_{0,1}^p &= \{1, 0, 0, 0\}^T \\
\psi_{0,2}^p &= \{0, 1, 0, 0\}^T 
\end{align*} \quad \text{and} \quad \begin{align*} 
\psi_{0,3}^c &= \{\eta, X_1, \eta, X_3\}^T \\
\psi_{0,4}^c &= \{X_2, \eta, \eta, 0, X_4\}^T 
\end{align*} \]  
\[ \text{where} \ X_1 = e_1^5 (\kappa_{11}^{-1}), \ X_2 = -e_1^5 (C_{44})^{-1}, \ X_3 = -\det(M^p) (\kappa_{11}^{-1}), \ X_4 = \det(M^p) (C_{44})^{-1}. \]

### 3.3 Non-zero eigenfunctions for Case i
The non-zero eigenvalue eigenfunctions (\(\mu \neq 0\)) for Case i are
\[ \begin{align*} 
\psi_j^c &= \alpha_{1j}^c \cos(\mu, \theta) + \alpha_{2j}^c \sin(\mu, \theta) \quad (i = 1, 3) \\
\psi_j^p &= \alpha_{1j}^p \cos(\mu, \theta) + \alpha_{2j}^p \sin(\mu, \theta) 
\end{align*} \]  
\[ \alpha_1^c = \left\{ \alpha_{11}^c, G\alpha_{11}^c \right\}^T, \quad \alpha_2^c = \left\{ \alpha_{12}^c, G\alpha_{12}^c \right\}^T; \quad \alpha_1^p = \left\{ \alpha_{11}^p, \alpha_{12}^p, \alpha_{13}^p, \alpha_{14}^p \right\}^T, \quad \alpha_2^p = \left\{ \alpha_{12}^p, \alpha_{22}^p, \alpha_{23}^p, \alpha_{24}^p \right\}^T, \quad \mu C^p \left\{ \alpha_{4k}^p, \alpha_{4k}^p \right\}^T = \left\{ \alpha_{3k}^p, \alpha_{4k}^p \right\}^T \quad (k = 1, 2). \]

According to equation (3), we have
\[ \chi_2^{(e)} = \mathbf{R} \chi_1^{(p)} , \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & C_{44} & \frac{e_{15}}{G} \end{bmatrix} \]  
\[ \chi_2^{(e)} = \mathbf{R} \chi_1^{(p)} , \quad \mathbf{T} = \begin{bmatrix} \cos \mu \alpha_2 \mathbf{I}_2 & -\sin \mu \alpha_2 \mathbf{I}_2 \\ \sin \mu \alpha_2 \mathbf{I}_2 & \cos \mu \alpha_2 \mathbf{I}_2 \end{bmatrix} \]  
\[ \text{where} \ \chi_1^{(e)} = \left\{ \alpha_{11}^{(e)}, \alpha_{12}^{(e)} \right\}^T, \quad \chi_1^{(p)} = \left\{ \alpha_{11}^{(p)}, \alpha_{12}^{(p)}, \alpha_{22}^{(p)}, \alpha_{23}^{(p)} \right\}^T, \quad \chi_3^{(e)} = \left\{ \alpha_{11}^{(e)}, \alpha_{12}^{(e)} \right\}^T. \]

By means of equations (12) and (4), it yields
\[ \mathbf{A}_i^{(p)} \chi_i^{(p)} = 0 \]  
where \( \mathbf{A}_i^{(p)} \) is listed in Appendix.

### 3.4 Non-zero eigenfunctions for Case ii
The non-zero eigenvalue eigenfunctions (\(\mu \neq 0\)) for Case ii are
\[ \begin{align*} 
\psi_j^c &= \alpha_{1j}^c \cos(\mu, \theta) + \alpha_{2j}^c \sin(\mu, \theta) \\
\psi_j^{(p,i)} &= \alpha_{1j}^{(p,i)} \cos(\mu, \theta) + \alpha_{2j}^{(p,i)} \sin(\mu, \theta) \quad (i = 1, 3) 
\end{align*} \]  
\[ \begin{align*} 
\alpha_1^{(e)} &= \left\{ \alpha_{11}^{(e)}, G\alpha_{11}^{(e)} \right\}^T, \quad \alpha_2^{(e)} = \left\{ \alpha_{12}^{(e)}, G\alpha_{12}^{(e)} \right\}^T; \quad \alpha_1^{(p)} = \left\{ \alpha_{11}^{(p)}, \alpha_{12}^{(p)}, \alpha_{22}^{(p)}, \alpha_{23}^{(p)} \right\}^T, \quad \alpha_2^{(p)} = \left\{ \alpha_{12}^{(p)}, \alpha_{22}^{(p)}, \alpha_{23}^{(p)}, \alpha_{24}^{(p)} \right\}^T, \quad \mu C^p \left\{ \alpha_{4k}^{(p)}, \alpha_{4k}^{(p)} \right\}^T = \left\{ \alpha_{3k}^{(p)}, \alpha_{4k}^{(p)} \right\}^T \quad (k = 1, 2). \]

According to equation (3), we have
\[ \chi_2^{(e)} = \mathbf{T}_1^{-1} \mathbf{R} \chi_1^{(p)} , \quad \mathbf{T}_1 = \begin{bmatrix} \cos \mu \alpha_2 & -\sin \mu \alpha_2 \\ \sin \mu \alpha_2 & \cos \mu \alpha_2 \end{bmatrix} \]  
\[ \text{where} \ \chi_1^{(e)} = \left\{ \alpha_{11}^{(e)}, \alpha_{21}^{(e)}, \alpha_{12}^{(e)}, \alpha_{22}^{(e)} \right\}^T, \quad \chi_2^{(e)} = \left\{ \alpha_{11}^{(e)}, \alpha_{12}^{(e)} \right\}^T, \quad \chi_3^{(e)} = \left\{ \alpha_{11}^{(e)}, \alpha_{12}^{(e)} \right\}^T. \]

By means of equations (12) and (5), it yields
\[
A^{(2)} \chi = 0
\]  

where \( \chi = \{ \chi_1^{(e)T}, \chi_2^{(e)T}, \chi_3^{(p)T} \}^T \), \( A^{(2)} \) is listed in Appendix.

Finally, the complete solution of equation (7) is obtained and expressed in terms of zero and non-zero eigenfunctions:

\[
\Psi = \sum_{j=1}^{N} c_{0j} \psi_{0,j} + \sum_{j=1}^{N} c_j \psi_j r^{\tau_j}
\]  

where \( c_{0j} \) and \( c_j \) are unknown coefficients which can be determined by the outer boundary conditions.

4. Symplectic adjoint symplectic orthogonality

In the symplectic space, the eigenvalues can be divided into two groups: \((\alpha)\ \mu_j, \ \text{Re} \mu_j < 0 \) or \( \text{Re} \mu_j = 0 \) and \((\beta)\ \mu_{n+j} = -\mu_j \) \( (j = 1, 2, \cdots, n) \). The corresponding eigenfunctions satisfy the following relations:

\[
\langle \psi_m^{(\alpha)}, J, \psi_n^{(\alpha)} \rangle = \langle \psi_m^{(\beta)}, J, \psi_n^{(\beta)} \rangle = 0, \quad \langle \psi_m^{(\alpha)}, J, \psi_n^{(\beta)} \rangle = -\langle \psi_m^{(\beta)}, J, \psi_n^{(\alpha)} \rangle = \delta_{mn}
\]  

where \( \delta_{mn} \) is the Kronecker delta; the inner product is defined as

\[
\langle \psi_m, J, \psi_n \rangle = \int_0^\delta (q_m^{(e)} p_n^{(e)} - q_n^{(e)} p_m^{(e)}) d\theta + \int_0^{\gamma} (q_m^{(e)} p_n^{(e)} - q_n^{(e)} p_m^{(e)}) d\theta, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.
\]

The unknown coefficients in equation (19) can be determined by the above the symplectic adjoint symplectic orthogonality in equation (20).

The boundary condition at the outer boundary can be expressed as follow:

\[
\Psi^{(i)} \bigg|_{r=a} = \begin{cases} 
q^{(i)}(\theta)_{k=1,2,3,4,n} \sum_{k=1}^{N} a_k \phi_k^{(i)}(\theta)_{k=1,2,3,4,n} + \sum_{m=1}^{4} a_{0m} \phi_{0m}^{(i)} \bigg)_{k=1,2,3,4,n} 
& \text{on } \partial \Omega_1 \\
\sum_{k=1}^{N} a_k \phi_k^{(i)}(\theta)_{k=1,2,3,4,n} + \sum_{m=1}^{4} a_{0m} \phi_{0m}^{(i)} \bigg)_{k=1,2,3,4,n} \right \} & \text{on } \partial \Omega_2
\end{cases}
\]

where \( \partial \Omega_1 \) and \( \partial \Omega_2 \) denote the generalized displacements and stresses boundaries, respectively.

By means of equation (20), we have

\[
\begin{cases} 
\sum_{k=1}^{N} a_{0k}^{(i)} A_k + \sum_{k=1}^{N} a_k B_k = E_k \\
\sum_{k=1}^{N} a_{0k}^{(i)} C_k + \sum_{k=1}^{N} a_k D_k = F_i
\end{cases}
\]

where

\[
A_k = \int_{\mathcal{E}_{120}^{(e)}} q_k^{(e)} p_{0j}^{(e)} d\theta - \int_{\mathcal{E}_{120}^{(e)}} p_k^{(e)} q_{0j}^{(e)} d\theta + \int_{\mathcal{E}_{120}^{(e)}} q_k^{(p)} p_{0j}^{(p)} d\theta - \int_{\mathcal{E}_{120}^{(e)}} p_k^{(p)} q_{0j}^{(p)} d\theta, \\
B_k = \left( \int_{\mathcal{E}_{120}^{(e)}} q_k^{(e)} p_{0j}^{(e)} d\theta - \int_{\mathcal{E}_{120}^{(e)}} p_k^{(e)} q_{0j}^{(e)} d\theta \right) a_{0j}^{(e)} + \left( \int_{\mathcal{E}_{120}^{(e)}} q_k^{(p)} p_{0j}^{(p)} d\theta - \int_{\mathcal{E}_{120}^{(e)}} p_k^{(p)} q_{0j}^{(p)} d\theta \right) a_{0j}^{(p)}, \\
E_k = \int_{\mathcal{E}_{120}^{(e)}} q_k^{(e)} (\theta) p_{0j}^{(e)} d\theta - \int_{\mathcal{E}_{120}^{(e)}} p_k^{(e)} (\theta) q_{0j}^{(e)} d\theta + \int_{\mathcal{E}_{120}^{(e)}} q_k^{(p)} (\theta) p_{0j}^{(p)} d\theta - \int_{\mathcal{E}_{120}^{(e)}} p_k^{(p)} (\theta) q_{0j}^{(p)} d\theta, \\
C_k = \int_{\mathcal{E}_{120}^{(e)}} q_k^{(e)} p_{0j}^{(e)} d\theta - \int_{\mathcal{E}_{120}^{(e)}} p_k^{(e)} q_{0j}^{(e)} d\theta + \int_{\mathcal{E}_{120}^{(e)}} q_k^{(p)} p_{0j}^{(p)} d\theta - \int_{\mathcal{E}_{120}^{(e)}} p_k^{(p)} q_{0j}^{(p)} d\theta + \delta_{(4-j)}
\]

5
$$D_{ij} = \left( \int_{\partial G} q^{(e)}_{j} p^{(e)}_{i} d\theta \right) - \left( \int_{\partial G} q^{(p)}_{j} p^{(p)}_{i} d\theta \right) + \left( \int_{\partial G} q^{(p)}_{j} p^{(e)}_{i} d\theta \right),$$

$$F_{i} = \int_{\partial G} q^{(p)}_{i} d\theta - \int_{\partial G} q^{(e)}_{i} d\theta + \int_{\partial G} q^{(p)}_{i} d\theta - \int_{\partial G} q^{(e)}_{i} d\theta.$$

Here, $N$ is the number of eigenfunction taken in the computation. Through solving equation (22), the remain unknown coefficients in complete solution in equation (19) is achieved.

5. Fracture parameters

According to equation (19), the singular fields around the crack tip can be represented by

$$a_{i} \psi_{j}^{(f)} r^{\kappa} (i = p, e)$$

Therefore, the intensity factors for the singular fields can be defined as [12]:

$$K_{s} = \lim_{r \to 0} \sqrt{2 \pi r} \sigma_{\alpha\alpha} \bigg|_{\theta = \theta_{0}}$$

$$K_{D} = \lim_{r \to 0} \sqrt{2 \pi r} D_{\alpha\alpha} \bigg|_{\theta = \theta_{0}}$$

where $\theta_{0} = \pi - \alpha_{1}$, $-\pi + \alpha_{3}$; $K_{S}$ and $K_{D}$ are the stress intensity factors (SIFs) and electric displacement intensity factors (EDIFs), respectively.

6. Numerical examples

In this section, several examples are presented to show the accuracy and performance of the proposed method. The material properties are tabulated in table 1. The non-dimensional forms are used in the following examples. Let $\bar{r} = r/\alpha$, $\bar{w} = w/\alpha$, $\bar{\phi} = \phi/(\alpha \times 10^{10} \text{N/C})$, $\bar{\kappa}_{11} = \kappa_{11}/(aC_{44})$ and $\bar{\kappa}_{11} = \kappa_{11}/(aC_{44})$. The corresponding non-dimensional material properties are

$\bar{\kappa}_{11} = \kappa_{11}/(aC_{44})$, $\bar{\kappa}_{11} = \kappa_{11}/(aC_{44})$, $\bar{C}_{44} = 1$ and $\bar{G} = G/C_{44}$.

**Table 1.** Material properties.

|        | $C_{44}$ (N/m²) | $e_{15}$ (C/m²) | $\kappa_{11}$ (C/V·m) |
|--------|-----------------|-----------------|-----------------------|
| Al     | 2.65×10¹⁰      | -               | -                     |
| PZT-4  | -               | 2.56×10¹⁰      | 12.7                  | -64.6×10⁻¹⁰                  |

6.1 Verification

As the first example, the accuracy of the symplectic method is validated by comparing with the available reference [13]. Since there is no result of the piezoelectric-elastic bimaterials in the open literature, a pure circular piezoelectric shaft subjected to a pair of concentrated forces and in-plane charge $F_{i} = Q_{i} = 1$ is considered here. $\alpha$ is the load angle. The eigenfunctions for case $i$ with $\alpha_{i} = \alpha_{3} = 0$ is used here. The variation of stress intensity factors $K_{S}$ versus various load angles $\omega$ is plotted in figure 3. The present solutions are in excellent agreement with the reference data [13]. In addition, a convergence study of the number of eigenfunctions is performed in figure 4. It is seen that $N$ does not affect the SIFs when $N > 4$. Therefore, we taken $N = 10$ eigenfunctions in the following computation to ensure the accuracy of fracture parameters.
Figure 2. An edge-cracked circular piezoelectric shaft.

Figure 3. Variation of stress intensity factors versus various load angles.

Figure 4. Convergence study on the number of symplectic eigenfunctions $N$: (a) SIFs; (b) displacement at the point $\left(0, \frac{\pi}{2}\right)$. 
6.2 Singularity orders

After verifying the proposed method, the effects of geometrical parameters and material properties on the singularity orders of the piezoelectric-elastic bimaterial is investigated in figures 5 and 6. Figure 5 depicts the variations of singularity orders versus the sustained angle $\alpha_1$. It is observed that the bimaterial of Case i only has one singularity order while that of Case ii has two. In other words, the crack in piezoelectric material will produce one more singularity order, which may be contributed by the singular electric fields of the piezoelectric material. It is also noted that, when $\alpha_1$ increases, the singular orders for Case i show a decreasing trend while those for Case ii show an increasing trend. It implies the crack in the piezoelectric material will lead to higher singularity orders.

To reveal the effect of material properties, define $G = \beta G$, $C_{44} = \beta C_{44}$, $e_{15} = \beta e_{15}$, $\kappa_{11} = \beta \kappa_{11}$ where $\beta$ is the adjustment coefficient. The variations of singularity orders versus adjustment coefficients $\beta$ for different material constants are plotted in figure 6. It is seen from the curves that $e_{15}$ and $\kappa_{11}$ almost have no influence on the singularity orders for both Cases i and ii. It is also interesting to find that the effects of $G$ and $C_{44}$ are completely different for the two cases. The increasing $C_{44}$ increase the singularity orders for Case i (figure 6a) while decrease the singularity orders for Case ii (figure 6b). The similar effects of $G$ are also found. These observations indicate that the elastic constants of the elastic and piezoelectric materials are the key influencing factors on the singularity orders of the bimaterial.

![Figure 5](image1.png)

**Figure 5.** Variations of singularity orders versus sustained angle $\alpha_1$: (a) Case i; (b) Case ii.

![Figure 6](image2.png)

**Figure 6.** Variations of singularity orders versus adjustment coefficients $\beta$: (a) Case i; (b) Case ii.
6.3 An edge-cracked piezoelectric-elastic bimaterial subjected to coupled loads
Consider an edge cracked bimaterial made of PZT-4 and Aluminum in a uniform electric field $E_y = 1/\kappa_{11}$, as shown in figure 7 ($\alpha_1 = \alpha_3 = \pi/2$ and $\alpha_2 = \pi$). The material combinations are selected as: $M_1$ is Aluminum and $M_2$ is PZT-4 for Case i; $M_1$ is PZT-4 and $M_2$ is Aluminum for Case ii. A pair of concentrated forces is applied at the outer boundary with a load angle $\omega$. Due to the symmetry of the bimaterial, $K_S |_{\theta = \pi/2} = K_S |_{\theta = -\pi/2}$ and $K_D |_{\theta = \pi/2} = K_D |_{\theta = -\pi/2}$. Therefore, only the intensity factors along the interface $\theta = \pi/2$ is considered in the following computation. The stress and electric displacement intensity factors for various load angles are tabulated in tables 2 and 3, respectively. It is clear that, regardless of the cases, all intensity factors decrease with the increase of load angles. For a fixed $\omega$, the values of intensity factors for the cases are almost the same. It demonstrates that, for the crack terminating at the interface, the crack position (Case i or ii) does not have a significant influence on the fracture parameters. Furthermore, the contour plots of stress components are presented in figure 8. The discontinuous are clearly observed.

![Figure 7. An edge-cracked piezoelectric-elastic bimaterial subjected to coupled loads.](image)

| $\omega$ | 0  | 30 | 60 | 90 | 120 | 150 | 180 |
|----------|----|----|----|----|-----|-----|-----|
| $K_S$    | 1.1186 | 1.0825 | 0.9743 | 0.8013 | 0.5676 | 0.2961 | 0   |
| $K_D$    | 5.5491 | 5.3703 | 4.8335 | 3.9750 | 2.8159 | 1.4691 | 0   |

Table 2. Intensity factors for various load angles for Case i.

| $\omega$ | 0  | 30 | 60 | 90 | 120 | 150 | 180 |
|----------|----|----|----|----|-----|-----|-----|
| $K_S$    | 1.1380 | 1.1016 | 0.9886 | 0.8067 | 0.5755 | 0.3045 | 0   |
| $K_D$    | 5.6455 | 5.4652 | 4.9043 | 4.0017 | 2.8549 | 1.5106 | 0   |

Table 3. Intensity factors for various load angles for Case ii.
Figure 8. Distributions of stress components: (a) $\sigma_{\theta z}$ (Case i); (b) $\sigma_{\theta z}$ (Case i); (c) $\sigma_{\theta z}$ (Case ii); (d) $\sigma_{\theta z}$ (Case ii).

7. Conclusion

Exact solutions for a piezoelectric-elastic bimaterial containing an edge crack terminating at the interface is obtained under the framework of Hamiltonian mechanics. The explicit expressions of stress and electric displacement intensity factors are derived. By introducing a total unknown vector, the high-order governing differential equations are reduced into a set of low-order ordinary equations. Therefore, the fracture problem is regarded as an eigenproblem and the exact solutions are expressed in terms of symplectic eigenfunctions. The stress and electric displacement intensity factors only depend on the coefficients of the singular items. Comparisons are presented to show the accuracy and stability of the proposed method. The singularity orders and intensity factors are computed and discussed also.

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Appendices

$$A^{(1)} = \begin{bmatrix} -\sin \mu \alpha_1 & 0 & \frac{C_{44}}{G} \cos \mu \alpha_1 & \frac{e_{15}}{G} \cos \mu \alpha_1 \\ \sin \mu \alpha_3 & 0 & \frac{C_{44}}{G} \cos \mu \alpha_3 & \frac{e_{15}}{G} \cos \mu \alpha_3 \\ 0 & 1 & 0 & 0 \\ 0 & \cos \mu \alpha_2 & 0 & -\sin \mu \alpha_2 \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{11} = \begin{bmatrix} -C_{44} \sin \mu \alpha_1 & -e_{15} \sin \mu \alpha_1 & C_{44} \cos \mu \alpha_1 & e_{15} \cos \mu \alpha_1 & 0 \\ -e_{15} \sin \mu \alpha_1 & \kappa_{11} \sin \mu \alpha_i & e_{15} \cos \mu \alpha_1 & -\kappa_{11} \cos \mu \alpha_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix},$$
\[
A_{21} = \begin{bmatrix}
0 & 0 & C_{44} / G & e_{15} / G & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\quad
A_{12} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} / e_{15} & 0 \\
0 & 0 & 0 & e_{15} / G & -\kappa_{11} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
A_{22} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & \delta_{11} - \delta_{22} C_{44} / G & -\delta_{22} e_{15} / G & \delta_{12} + \delta_{21} C_{44} / G & \delta_{21} e_{15} / G \\
-1 & -\delta_{21} - \delta_{12} C_{44} / G & -\delta_{12} e_{15} / G & -\delta_{22} + \delta_{11} C_{44} / G & \delta_{11} e_{15} / G \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \mu \alpha_3 & 0 & \sin \mu \alpha_3
\end{bmatrix},
\]

\[\delta_{11} = \cos \mu \alpha_2 \cos \mu \alpha_3, \quad \delta_{12} = \cos \mu \alpha_2 \sin \mu \alpha_3, \quad \delta_{21} = \sin \mu \alpha_2 \cos \mu \alpha_3, \quad \delta_{22} = \sin \mu \alpha_2 \sin \mu \alpha_3.\]

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