Variations on known and recent cardinality bounds

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Abstract

Sapirovskii [18] proved that $|X| \leq \pi_X c(X) \psi(X)$, for a regular space $X$. We introduce the $\theta$-pseudocharacter of a Urysohn space $X$, denoted by $\psi_\theta(X)$, and prove that the previous inequality holds for Urysohn spaces replacing the bounds on cellularity $c(X) \leq \kappa$ and on pseudocharacter $\psi(X) \leq \kappa$ with a bound on Urysohn cellularity $Uc(X) \leq \kappa$ (which is a weaker condition because $Uc(X) \leq c(X)$) and on $\theta$-pseudocharacter $\psi_\theta(X) \leq \kappa$ respectively (Note that in general $\psi(\cdot) \leq \psi_\theta(\cdot)$ and in the class of regular spaces $\psi(\cdot) = \psi_\theta(\cdot)$).

Further, in [6] the authors generalized the Dissanayake and Willard’s inequality: $|X| \leq 2^{aLc(X)\chi(X)}$, for Hausdorff spaces $X$ [25], in the class of $n$-Hausdorff spaces and de Groot’s result: $|X| \leq 2^{hL(X)}$, for Hausdorff spaces [11], in the class of $T_1$ spaces (see Theorems 2.22 and 2.23 in [6]). In this paper we restate Theorem 2.22 in [6] in the class of $n$-Urysohn spaces and give a variation of Theorem 2.23 in [6] using new cardinal functions, denoted by $UW(X)$, $\psi_{U\theta}(X)$, $\theta-aL(X)$, $h\theta-aL(X)$, $\theta-aLc(X)$ and $\theta-aL_\theta(X)$. In [5] the authors introduced the Hausdorff point separating weight of a space $X$ denoted by $Hpsw(X)$ and proved a Hausdorff version of Charlesworth’s inequality $|X| \leq psw(X)^{L(X)\psi(X)}$ [7]. In this paper, we introduce the Urysohn point separating weight of a space $X$, denoted by $Upsw(X)$, and prove that $|X| \leq Upsw(X)^{\theta-aLc(X)\psi(X)}$, for a Urysohn space $X$.

Keywords: Urysohn; $\theta$-closure; pseudocharacter; almost Lindelöf degree; Hausdorff point separating weight.

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1 Introduction

We shall follow notations from \[12\] and [14]. Recall that a space \(X\) is Urysohn if for every two distinct points \(x, y \in X\) there are open sets \(U\) and \(V\) such that \(x \in U, y \in V\) and \(\overline{U} \cap \overline{V} = \emptyset\).

For a space \(X\), we denote by \(\chi(X)\) (resp., \(\psi(X)\), \(\pi\chi(X), \pi\psi(X), t(X)\)) the character, (resp., pseudocharacter, \(\pi\)-character, cellularity, tightness) of a space \(X\). \[12\]

The \(\theta\)-closure of a set \(A\) in a space \(X\) is the set \(cl_\theta(A) = \{x \in X : \text{for every neighborhood } U \ni x, \overline{U} \cap A \neq \emptyset\}\); \(A\) is said to be \(\theta\)-closed if \(A = cl_\theta(A)\) \[24\]. Considering the fact that the \(\theta\)-closure operator is not in general idempotent, Bella and Cammaroto defined in \[2\] the \(\theta\)-closed hull of a subset \(A\) of a space \(X\), denoted by \([A]_\theta\), that is the smallest \(\theta\)-closed subset of \(X\) containing \(A\). The \(\theta\)-tightness of \(X\) at \(x \in X\) is \(t_\theta(x, X) = \min\{k : \text{for every } A \subseteq X \text{ with } x \in cl_\theta(A) \text{ there exists } B \subseteq A \text{ such that } |B| \leq k \text{ and } x \in cl_\theta(B)\};\) the \(\theta\)-tightness of \(X\) is \(t_\theta(X) = \sup\{t_\theta(x, X) : x \in X\}\) \[8\]. We have that tightness and \(\theta\)-tightness are independent (see Example 11 and Example 12 in \[9\]), but if \(X\) is a regular space then \(t(X) = t_\theta(X)\). The \(\theta\)-density of \(X\) is \(d_\theta(X) = \min\{k : \text{A dense subset of X and } |A| \leq k\}\). We say that a subset \(A\) of \(X\) is \(\theta\)-dense in \(X\) if \(cl_\theta(A) = X\).

If \(X\) is a Hausdorff space, the closed pseudocharacter of a point \(x\) in \(X\) is \(\psi_c(x, X) = \min\{|U| : U\text{ is a family of open neighborhoods of } x\}\) the intersection of the closure of \(U\); the closed pseudocharacter of \(X\) is \(\psi_c(X) = \sup\{\psi_c(x, X) : x \in X\}\) (see \[19\] where it is called \(S\psi(X)\)). The Urysohn pseudocharacter of \(X\), denoted by \(U\psi(X)\), is the smallest cardinal \(k\) such that for each point \(x \in X\) there is a collection \(\{V(\alpha, x) : \alpha < k\}\) of open neighborhoods of \(x\) such that if \(x \neq y\), then there exist \(\alpha, \beta < k\) such that \(V(\alpha, x) \cap \overline{V(\beta, y)} = \emptyset\) \[20\]; this cardinal function is defined only for Urysohn spaces. The Urysohn-cellularity of a space \(X\) is \(Uc(X) = \sup\{|V| : V \text{ is Urysohn-cellular}\}\) (a collection \(V\) of open subsets of \(X\) is called Urysohn-cellular; if \(O_1, O_2 \in V\) and \(O_1 \neq O_2\) implies \(\overline{O_1} \cap \overline{O_2} = \emptyset\)). Of course, \(Uc(X) \leq c(X)\).

The almost Lindelöf degree of a subset \(Y\) of a space \(X\) is \(aL(Y, X) = \min\{k : \text{for every cover } V \text{ of } Y \text{ consisting of open subsets of } X\}, \text{there exists } V' \subseteq V \text{ such that } |V'| \leq k \text{ and } \bigcup\{V' : V \in V'\} = Y\}.\) The function \(aL(X, X)\) is called the almost Lindelöf degree of \(X\) and denoted by \(aL(X)\) (see \[25\] and \[15\]). The almost Lindelöf degree of \(X\) with respect to closed subsets of \(X\) is \(aL_c(X) = \sup\{aL(C, X) : C \subseteq X \text{ is closed}\}\).

For a subset \(A\) of a space \(X\) we will denote by \([A]_{\leq \lambda}\) the family of all subsets of \(A\) of cardinality \(\leq \lambda\).

Sapirovskii \[18\] proved that \(|X| \leq \pi\chi(X)c(X)\psi(X)\), for a regular space \(X\).
Later Shu-Hao [19] proved that the previous inequality holds in the class of Hausdorff spaces by replacing the pseudocharacter with the closed pseudocharacter. In Section 2 we introduce the \( \theta \)-pseudocharacter of a Urysohn space \( X \), denoted by \( \psi_\theta(X) \) and prove the following result:

- \( |X| \leq \pi_X(X)^{U_{\psi}(X)} \psi_\theta(X) \) for a Urysohn space \( X \).

A space \( X \) is \( n \)-Urysohn [4] (resp. \( n \)-Hausdorff [3]), \( n \in \omega \), if for every \( x_1, x_2, \ldots, x_n \in X \) there exist open subsets \( U_1, U_2, \ldots, U_n \) of \( X \) such that \( x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n \) and \( \bigcap_{i=1}^{n} U_i = \emptyset \) (resp. \( \bigcap_{i=1}^{n} U_i = \emptyset \)). In [6] the authors generalized the Dissanayake and Willard’s inequality:

\[ |X| \leq 2^{a_{L_{\psi}}(X)} \chi(X), \]

for Hausdorff spaces \( X \) [25], in the class of \( n \)-Hausdorff spaces and de Groot’s result: \( |X| \leq 2^{h_{L}(X)} \), for Hausdorff spaces [11], in the class of \( T_1 \) spaces. In particular, they used two new cardinal functions, denoted by \( HW(X) \), \( \psi w(X) \), to obtain the following results:

- If \( X \) is a \( T_1 \) \( n \)-Hausdorff \( (n \in \omega) \) space, then \( |X| \leq HW(X)2^{a_{L_{\chi}}(X)} \).
- If \( X \) is a \( T_1 \) space, then \( |X| \leq HW(X)\psi w(X)^{h_{aL}(X)} \).

In Section 3 we introduce new cardinal functions, denoted by \( UW(X) \), \( \psi w_{\theta}(X) \), \( \theta-a_{L}(X) \), \( h\theta-a_{L}(X) \), \( \theta-a_{L_{\psi}}(X) \) and \( \theta-a_{L_{\theta}}(X) \) such that \( HW(X) \leq UW(X) \), \( \psi w(X) \leq \psi w_{\theta}(X) \) and \( \theta-a_{L}(X) \leq a_{L}(X) \), restate Theorem 2.22 in [6] in the class of \( n \)-Urysohn spaces and give a variation of Theorem 2.23 in [6]. In particular, we prove the following results:

- If \( X \) is a \( T_1 \) \( n \)-Urysohn \( (n \in \omega) \) space, then \( |X| \leq UW(X)2^{\theta-a_{L_{\chi}}(X)} \).
- If \( X \) is a \( T_1 \) space then \( |X| \leq UW(X)\psi w_{\theta}(X)^{h\theta-a_{L}(X)} \).

In [5] the authors introduced the Hausdorff point separating weight of a space \( X \) denoted by \( H_{psw}(X) \) and proved a Hausdorff version of Charlesworth’s inequality \( |X| \leq p_{sw}(X)^{L(X)\psi(X)} \) [7]. In a similar way, in Section 4 we introduce Urysohn point separating weight of a space \( X \), denoted by \( H_{psw}(X) \), and prove the following result:

- If \( X \) is a Urysohn space, then \( |X| \leq U_{psw}(X)^{\theta-a_{L_{\psi}}(X)} \).

## 2 A generalization of Sapirovskii’s inequality

\[ |X| \leq \pi_X(X)^{c(X)}\psi(X). \]

**Definition 2.1.** If \( X \) is a Urysohn space, we define \( \theta \)-pseudocharacter of a point \( x \in X \) the smallest cardinal \( k \) such that \( \{x\} \) is the intersection of the \( \theta \)-closure of the closure of a family of open neighborhood of \( x \) having cardinality less or equal to \( k \); we denote it with \( \psi_\theta(x, X) \). The \( \theta \)-pseudocharacter of \( X \) is:

\[ \psi_\theta(X) = \sup\{\psi_\theta(x, X) : x \in X\}. \]

The following result is trivial:
**Proposition 2.1.** $X$ is a Urysohn space iff for every $x \in X$, \( \{x\} \) is the intersection of the $\theta$-closure of the closure of a family of open neighborhood of $x$.

*Proof.* Let $X$ be a Urysohn space and $x \in X$. For every $y \in X \setminus \{x\}$, there exist $U_y$ and $V_y$ open disjoint subsets of $X$ such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. So, $y \notin cl_{\theta}(U_y)$ and \( \{x\} = \bigcap_{y \in X \setminus \{x\}} cl_{\theta}(U_y) \). Viceversa let $x, y$ be distinct points of $X$. By hypothesis there exists an open neighbourhood $V$ of $x$ such that $y \notin cl_{\theta}(V)$. Then there exists an open subset $U$ of $X$ such that $y \in U$ and $\overline{U} \cap V = \emptyset$. So $X$ is Urysohn. 

We have that:

$$
\psi(X) \leq \psi_{c}(X) \leq \psi_{\theta}(X) \leq U\psi(X) \leq \chi(X).
$$

Since for a regular space $X$, $cl_{\theta}(A) = \overline{A}$ for every $A \subseteq X$, we have that for a regular space $X$, $\psi_{c}(X) = \psi_{\theta}(X)$. In general this need not be true for non regular spaces. Indeed if we consider $\mathbb{R}$ with the countable complement topology we have that $\overline{\mathbb{Q}} \neq cl_{\theta}(\mathbb{Q})$.

**Question 2.1.** Is there a Urysohn space such that $\psi_{c}(X) < \psi_{\theta}(X)$?

It was proved in [2] that for Urysohn spaces, \( |cl_{\theta}(A)| \leq |A|^{\chi(X)} \) for every $A \subseteq X$ and further this inequality was used for the estimation of cardinality of Lindelöf spaces. Since $t_{\theta}(X)\psi_{\theta}(X) \leq \chi(X)$, the following proposition improves the result in [2]. (Note that if $X = \omega \cup \{p\}$, with $p \in \omega^{*}$, we have that $\aleph_0 = t_{\theta}(X)\psi_{\theta}(X) < \chi(X)$.)

**Proposition 2.2.** Let $X$ be a Urysohn space such that $t_{\theta}(X)\psi_{\theta}(X) \leq k$. Then for every $A \subseteq X$ we have that $|cl_{\theta}(A)| \leq |A|^{k}$.

*Proof.* Let $x \in cl_{\theta}(A)$, since $\psi_{\theta}(X) \leq k$ there exist a family $\{U_{\alpha}(x)\}_{\alpha<k}$ of neighborhood of $x$ such that $\{x\} = \bigcap_{\alpha<k} \overline{cl_{\theta}(U_{\alpha}(x))}$. We want to prove that $x \in cl_{\theta}(\overline{U_{\alpha}(x)} \cap A)$, \( \forall \alpha < k \). Let $U$ be a neighborhood of $x$ and $\alpha < k$. Then $\emptyset \neq \overline{U} \cap \overline{U_{\alpha}(x)} \cap A \subseteq \overline{U} \cap \overline{U_{\alpha}(x)} \cap A$. This shows that $x \in cl_{\theta}(\overline{U_{\alpha}(x)} \cap A)$. Since $t_{\theta}(X) \leq k$, there exists $A_\alpha \subseteq U_{\alpha}(x) \cap A$ such that $|A_\alpha| \leq k$ and $x \in cl_{\theta}(A_\alpha)$. Then $\{x\} = \bigcap_{\alpha<k} cl_{\theta}(A_\alpha)$ and $\{A_\alpha\}_{\alpha<k} \subseteq [A]^{\leq k}$, so $|cl_{\theta}(A)| \leq \prod_{\alpha<k} |cl_{\theta}(A_\alpha)| \leq |A|^{k}$.

**Corollary 2.1.** [2] If $X$ is a Urysohn space then for every $A \subseteq X$ we have that $|cl_{\theta}(A)| \leq |A|^{\chi(X)}$.

The following result is the analogue of 2.20 in [13] in the case of Urysohn spaces.

**Corollary 2.2.** If $X$ is a Urysohn space and $|X| \leq d_{\theta}(X)t_{\theta}(X)\psi_{\theta}(X)$.
Proof. If $A$ is $\theta$-dense subset of $X$, i.e. $cl_\theta(A) = X$, we have that $|A| \leq d_\theta(X)$ and from the above theorem we have that $|cl_\theta(A)| \leq |A|^{t_\theta(X)\psi_\theta(X)}$, so $|X| \leq d_\theta(X)^{t_\theta(X)\psi_\theta(X)}$. □

The authors know that I. Gotchev obtained independently the results given in Proposition 2.2 and Corollary 2.2.

Now we prove the following result:

**Lemma 2.1.** Let $X$ be a topological space, $B$ a $\pi$-base for $X$ and $W$ a family of open sets. Let $\mathcal{M}$ be a maximal Urysohn cellular subfamily of $\{U \in B : U \subseteq W \text{ for some } W \in W\}$. Then $cl_\theta\left(\bigcup \mathcal{M}\right) \supseteq \bigcup W$.

**Proof.** Using Zorn’s Lemma we can say that there exists a maximal Urysohn-cellular subfamily $\mathcal{M}$ of $\{U \in B : U \subseteq W \text{ for some } W \in W\}$. We want to prove that $cl_\theta\left(\bigcup \mathcal{M}\right) \supseteq \bigcup W$. Assume, by the way of contradiction, that $cl_\theta\left(\bigcup \mathcal{M}\right) \nsubseteq \bigcup W$. Let $x \in \bigcup W$ such that $x \notin cl_\theta\left(\bigcup \mathcal{M}\right)$. Then there exists an open set $U$ such that $x \in U$ such that $U \cap \mathcal{M} = \emptyset$, $\forall M \in \mathcal{M}$. So $x \notin M$, $\forall M \in \mathcal{M}$. Let $W \in W$ such that $x \in W$. $\mathcal{M} \cup \{U \cap W\}$ is a Urysohn cellular family. Since $B$ is a $\pi$-base for $X$ and $U \cap W$ is an open set containing $x$, there exists $B \in B$ such that $B \subseteq U \cap W$, so $\mathcal{M}' = \mathcal{M} \cup \{B\}$ is a Urysohn cellular subfamily of $\{U \in B : U \subseteq W \text{ for some } W \in W\}$ containing $\mathcal{M}$; a contradiction. □

**Theorem 2.1.** Let $X$ be a Urysohn space. Then $|X| \leq \pi X(X)^{Uc(X)\psi_\theta(X)}$.

**Proof.** Let $\pi X(X) = \lambda$ and $Uc(X)\psi_\theta(X) = k$; for each $p \in X$, let $U_p$ be a local $\pi$-base at $p$ such that $|U_p| \leq \lambda$.

Construct an increasing chain $\{A_\alpha : \alpha < k^+\}$ of subsets of $X$ and a sequence $\{U_\alpha : 0 < \alpha < k^+\}$ of open collections in $X$ such that:

1. $|A_\alpha| \leq \lambda^k$, $0 \leq \alpha < k^+$;
2. $U_\alpha = \{V \in U_p : p \in \bigcup_{\beta < \alpha} A_\beta\}$, $0 < \alpha < k^+$;
3. for each $\gamma < k$, if $\mathcal{V}_\gamma \subseteq \bigcup_{\alpha \leq k} U_\alpha$ and $W = \bigcup_{\gamma < k} cl_\theta(\bigcup \mathcal{V}_\gamma) \neq X$, then $A_\alpha \setminus W \neq \emptyset$.

The construction is by transfinite induction. Let $0 < \alpha < k^+$ and assume that $\{A_\beta : \beta < \alpha\}$ has already been constructed. Then $U_\alpha$ is defined by 2., i.e., we put $U_\alpha = \{V : \exists p \in \bigcup_{\beta < \alpha} A_\beta, V \in U_p\}$. It follows that $|U_\alpha| \leq \lambda^k$. If $\{\mathcal{V}_\gamma\}_{\gamma < k} \subseteq \bigcup_{\alpha \leq k} U_\alpha$ and $W = \bigcup_{\gamma < k} cl_\theta(\bigcup \mathcal{V}_\gamma) \neq X$, then we can choose one point of $X \setminus W$. Let $S_\alpha$ be the set of points chosen in this way. Note that $|\bigcup_{\alpha \leq k} U_\alpha| \leq \lambda^k$. Define $A_\alpha$ to be the set $S_\alpha \cup (\bigcup_{\beta < \alpha} A_\beta)$. Then $A_\alpha$ satisfies 1., and 3. is also satisfied if $\beta \leq \alpha$. This completes the construction.
Now let \( S = \bigcup_{\alpha < k} A_{\alpha} \); then \(|S| \leq k^+ \lambda^k = \lambda^k\). The proof is complete if \( S = X \). Suppose not and let \( p \in X \setminus S \); since \( \psi_\theta(X) \leq k \), there exist open neighbourhoods \( \{U_\alpha\}_{\alpha < k} \) of \( p \) such that \( \{p\} = \bigcap_{\alpha < k} cl_\theta(U_\alpha) \).

For each \( \alpha < k \), let \( V_\alpha = X \setminus cl_\theta(U_\alpha) \). Then \( S = \bigcup_{\alpha < k} V_\alpha \cap S \). Fix \( \alpha < k \). For each \( q \in V_\alpha \cap S \), there exists \( V_q \in U_\alpha \) such that \( V_q \cap U_\alpha = \emptyset \) (from the definition of \( V_\alpha \)). We have that \( \{V \in U_\alpha \quad V \subseteq V_q\} \) is a local \( \pi \)-base at \( q \). Since \( q \in \bigcup\{V \in U_\alpha \quad V \subseteq V_q\} \), we have that \( S \cap V_\alpha \subseteq \bigcup_{\theta \in S \cap V_\alpha} \bigcup\{V \in U_\alpha \quad V \subseteq V_q\} \subseteq \bigcup\{V \in U_\alpha \quad V \subseteq V_q\}, q \in S \cap V_\alpha \}. \) We put \( W_\alpha = \{V : V \in U_\alpha, V \subseteq V_q, q \in S \cap V_\alpha\} \). Since \( Uc(X) \leq k \), by Lemma \[2\] we have that \( \forall \alpha < k \) there exists a maximal Urysohn cellular family \( \mathcal{W}_\alpha \in [\mathcal{W}_\alpha]^{\leq k} \) such that \( cl_\theta(\bigcup_{\alpha \in W_\alpha}) \supseteq \bigcup_{\alpha \in W_\alpha} \). Since \( cl_\theta(\bigcup_{\alpha \in W_\alpha}) \) is closed, it follows that \( S \cap V_\alpha \subseteq \bigcup_{\alpha \in W_\alpha} \subseteq cl_\theta(\bigcup_{\alpha \in W_\alpha}) = \bigcup_{\alpha \in W_\alpha} \). Then, since \( \bigcup_{\alpha \in W_\alpha} \subseteq \bigcup_{\alpha \in W_\alpha} \cap U_\alpha = \emptyset \) and \( p \notin cl_\theta(\bigcup_{\alpha \in W_\alpha} \cap U_\alpha) \), we have that \( p \notin cl_\theta(\bigcup_{\alpha \in W_\alpha}) \). But \( W \supseteq \bigcup_{\alpha < k} (V_\alpha \cap S) = S \) and \( A_{\alpha_0} \subseteq W \subseteq S \setminus W = \emptyset \); a contradiction.

\[ \square \]

**Corollary 2.3.** \[18\] Let \( X \) be a regular space. Then \(|X| \leq \pi \chi(X)^{c(X)} \psi(X)\).

### 3 Variations of the Dissanayake and Willard’s inequality \(|X| \leq 2^{aLc(X)} \chi(X)\) and of the de Groot’s inequality \(|X| \leq 2^{hL(X)}\) in the class of \( T_1 \) spaces.

In Proposition \[2.1\] it was shown that Urysohn axiom is equivalent to \( \{x\} = \bigcap\{cl_\theta(U) : U \text{ open}, x \in U\} \), for every point \( x \) of the space. The following example shows that in spaces which are not Urysohn the previous intersection can be large.

**Example 3.1.** Any infinite space \( X \) with the cofinite topology is a \( T_1 \), not Hausdorff space for which there is a point \( x \) such that \( \bigcap\{cl_\theta(U) : x \in U\} \) has large cardinality.

The example above gives a motivation to introduce the following definition:

**Definition 3.1.** Let \( X \) be a \( T_1 \) topological space and for all \( x \in X \), let

\[ Uw(x) = \bigcap\{cl_\theta(U) : x \in U, U \text{ open}\}. \]
The Urysohn width is:

\[ UW(X) = \sup\{|Uw(x)| : x \in X\}. \]

It is clear that if \( X \) is a Urysohn space then \( UW(X) = 1 \).

Recall that \( HW(X) = \sup\{|Hw(x)| : x \in X\} \) is the Hausdorff width, where \( Hw(x) = \bigcap\{U : x \in U, U \text{ open}\} \). Since the \( \theta \)-closure of a set contains its closure we have that \( HW(X) \leq UW(X) \).

**Question 3.1.** Is \( HW(X) = UW(X) \) in some class of non regular spaces?

**Definition 3.2.** \[6\] Let \( X \) be a space and \( x \in X \).

\[ \psi w(x) = \min\{|U_x| : \bigcap\{U : U \in U_x\} = Hw(x), U_x \text{ is a family of open neighborhood of } x\}; \]

and

\[ \psi w(X) = \sup\{\psi w(x) : x \in X\}. \]

Similarly, we introduce the following definition.

**Definition 3.3.** Let \( X \) be a space and \( x \in X \).

\[ \psi w_\theta(x) = \min\{|U_x| : \bigcap\{cl_\theta(U) : U \in U_x\} = Uw(x), U_x \text{ is a family of open neighborhood of } x\}; \]

and

\[ \psi w_\theta(X) = \sup\{\psi w_\theta(x) : x \in X\}. \]

Of course, if \( X \) is a \( T_1 \) space then \( \psi w(X) \leq \psi w_\theta(X) \leq \chi(X) \); further if \( X \) is a Urysohn space then we have that \( \psi w_\theta(X) = \psi_\theta(X) \).

We introduce the following definition:

**Definition 3.4.** Let \( Y \) be a subset of a space \( X \).

The \( \theta \)-almost Lindelöf degree of a subset \( Y \) of a space \( X \) is

\[ \theta-aL(Y, X) = \min\{k : \text{for every cover } \mathcal{V} \text{ of } Y \text{ consisting of open subsets of } X, \text{there exists } \mathcal{V}' \subseteq \mathcal{V} \text{ such that } |\mathcal{V}'| \leq k \text{ and } \bigcup\{cl_\theta(V) : V \in \mathcal{V}'\} = Y\}. \]

The function \( \theta-aL(X, X) \) is called \( \theta \)-almost Lindelöf degree of the space \( X \) and denoted by \( \theta-aL(X) \).

The \( \theta \)-almost Lindelöf degree with respect to closed subsets of \( X \), denoted by \( \theta-aL_c(X) \), is the cardinal \( \sup\{\theta-aL(C, X) : C \subseteq X \text{ is closed}\} \).

The \( \theta \)-almost Lindelöf degree with respect to \( \theta \)-closed subsets of \( X \), denoted by \( \theta-aL_\theta(X) \), is the cardinal \( \sup\{\theta-aL(B, X) : B \subseteq X \text{ is } \theta \text{-closed}\} \).
Of course $\theta-aL(X) \leq aL(X)$, for every space $X$. Using a slight modification of Example 2.3 in [1] we prove that the previous inequality can be strict.

**Example 3.2.** A space $X$ such that $\theta-aL(X) < aL(X)$.

Let $k$ be any uncountable cardinal, let $\mathbb{Q}$ be the set of all the rationals and let $\mathbb{P}$ be the set of the irrationals. Put $X = (\mathbb{Q} \times k) \cup \mathbb{P}$. We topologized $X$ as follows. If $q \in \mathbb{Q}$ and $\alpha < k$ then a neighborhood base at $(q, \alpha)$ is $U(q, \alpha) = \{U_n(q, \alpha) : n \in \omega\}$ where

$$U_n(q, \alpha) = \{(r, \alpha) : r \in \mathbb{Q} \text{ and } |r-q| < \frac{1}{n}\}.$$ 

If $p \in \mathbb{P}$ a neighborhood base at $p$ takes the form:

$$\{(b \in \mathbb{P} : |b-p| < \frac{1}{n}) \cup \{(q, \alpha) : \alpha < k \text{ and } |q-p| < \frac{1}{n}\} : n \in \omega\}.$$ 

For every $q \in \mathbb{Q}$, $\alpha < k$ and $n \in \omega$ we have that:

$$\overline{U_n(q, \alpha)} = U_n(q, \alpha) \bigcup \{(r, \alpha) : r \in \mathbb{Q}, |r-q| < \frac{1}{n}\} \bigcup \{p \in \mathbb{P} : |q-p| < \frac{1}{n}\};$$ 

and:

$$\text{cl}_\theta \overline{U_n(q, \alpha)} = \overline{U_n(q, \alpha)} \bigcup \{(r, \beta) : |r-q| < \frac{1}{n}, \beta < k \text{ and } \beta \neq \alpha\}.$$ 

Let $\alpha < k$, we have that $X = \bigcup_{q \in \mathbb{Q}} \text{cl}_\theta \overline{(U(q, \alpha))}$ and so $\theta-aL(X) = \aleph_0$ but we have that $aL(X) = 2^{\aleph_0}$.

It is easy to show that the almost Lindelöf degree is hereditary with respect to $\theta$-closed subsets. It is natural to ask:

**Question 3.2.** Is the $\theta$-almost Lindelöf degree hereditary with respect to $\theta$-closed subsets?

We find out (Proposition 3.1) that the $\theta$-almost Lindelöf degree is hereditary with respect to a new class of spaces that we call $\gamma$-closed.

**Definition 3.5.** Let $X$ be a topological space and $A \subseteq X$. The $\gamma$-closure of the set $A$ is

$$cl_\gamma(A) = \{x : \text{for every open neighborhood of } X, \text{ cl}_\theta \overline{(U)} \cap A \neq \emptyset\}.$$ 

$A$ is said to be $\gamma$-closed if $A = cl_\gamma(A)$.

The following example shows that the $\gamma$-closure and the $\theta$-closure of a subset of a topological space can be different.
Example 3.3. A Urysohn space $X$ having a subset $Y$ such that $\text{cl}_\gamma(Y) \neq \text{cl}_\delta(Y)$.

Proof. Let $\mathbb{R} = A \cup B \cup C \cup D$ where $A, B, C, D$ are pairwise disjoint and each is dense in $\mathbb{R}$. Let $A'$ be a topological copy of $A$; points in $A'$ are denoted as $a'$ where $a \in A$.

Let $a, b \in \mathbb{R}$. A base for $X$ is generated by these families of open sets:

1. $\{(a, b) \cap A : a, b \in \mathbb{R}, a < b\}$
2. $\{(a, b) \cap C : a, b \in \mathbb{R}, a < b\}$
3. $\{(a, b) \cap A' : a, b \in \mathbb{R}, a < b\}$
4. $\{(a, b) \cap (A \cup B \cup C) : a, b \in \mathbb{R}, a < b\}$, and
5. $\{(a, b) \cap (C \cup D \cup A') : a, b \in \mathbb{R}, a < b\}$.

Note that for every $a, b \in \mathbb{R}$, $(a, b) \cap A = [a, b] \cap (A \cup B)$, $(a, b) \cap A' = [a, b] \cap (A' \cup D)$, $(a, b) \cap C = [a, b] \cap (B \cup C \cup D)$, $\text{cl}_\theta((a, b) \cap A) = [a, b] \cap (A \cup B \cup C)$ and $\text{cl}_\theta((a, b) \cap A') = [a, b] \cap (A' \cup D \cup C)$. For these reasons we can say that if $a, b \in \mathbb{R}$ and if we put $Y = (a, b) \cap C$, we have that $\text{cl}_\theta(Y) = [a, b] \cap (B \cup C \cup D)$ and $\text{cl}_\gamma(Y) = [a, b] \cap (A \cup B \cup C \cup D \cup A')$. \qed

We have the following:

Proposition 3.1. The $\theta$-almost Lindelöf degree is hereditary with respect to $\gamma$-closed subsets.

Proof. Let $X$ be a topological space such that $\theta-aL(X) \leq k$ and let $C \subseteq X$ be $\gamma$-closed set. $\forall x \in X \setminus C$ we have that there exists an open neighborhood $U_x$ of $x$ such that $\text{cl}_\theta(U) \subseteq X \setminus C$. Let $\mathcal{U}$ be a cover of $C$ consisting of open subsets of $X$. Then $\mathcal{V} = \mathcal{U} \cup \{U_x : x \in X \setminus C\}$ is an open cover of $X$ and since $\theta-aL(X) \leq k$, there exists $\mathcal{V}' \in [\mathcal{V}]^{\leq k}$ such that $X = \bigcup \{\text{cl}_\theta(V) : V \in \mathcal{V}'\}$. Then there exists $\mathcal{V}'' \in [\mathcal{U}]^{\leq k}$ such that $C \subseteq \bigcup \{\text{cl}_\theta(V) : V \in \mathcal{V}''\}$; this proves that $\theta-aL(C) \leq k$. \qed

Now we use $UW(X)$ and $\theta-aL_\theta(X)$ to restate Theorem 2.22 in \cite{6} in the class of $n$-Urysohn spaces. The proof follows step by step the proof of Theorem 2.22 in \cite{5}.

Theorem 3.1. If $X$ is a $T_1$ $n$-Urysohn ($n \in \omega$) space, then $|X| \leq UW(X)2^{\theta-aL_\theta(X)}\chi(X)$.

Proof. Let $UW(X) \leq k$, $\theta-aL_\theta(X)\chi(X) \leq \tau$. For all $x \in X$, let $\mathcal{U}_x$ be a local base and $|\mathcal{U}_x| \leq \tau$. Note that for all $x \in X$, $Uw(x) = \bigcap \{\text{cl}_\delta(U) : U \in \mathcal{U}_x\}$. Construct $\{H_\alpha : \alpha \in \tau^+\}$ and $\{B_\alpha : \alpha \in \tau^+\}$ such that:

1. $H_\alpha \subseteq H_\beta \subseteq X$, for all $\alpha, \beta \in \tau^+$;
2. $H_\alpha$ is $\theta$-closed for all $\alpha \in \tau^+$;
3. $|H_\alpha| \leq 2^\tau$ for all $\alpha \in \tau^+$;

4. if $\{H_\beta : \beta \in \alpha\}$ are defined for some $\alpha \in \tau^+$, then $B_\alpha = \bigcup \{U_x : x \in \bigcup \{H_\beta : \beta \in \alpha\}\}$;

5. if $\alpha \in \tau^+$ and $W \in [B_\alpha]^{\leq \tau}$ is such that $X \setminus (\bigcup \{cl_\theta(U) : U \in W\}) \neq \emptyset$ then $H_\alpha \setminus (\bigcup \{cl_\theta(U) : U \in W\}) \neq \emptyset$.

Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ be already defined. For all $W$ as in 5., choose a point $x(W) \in X \setminus (\bigcup \{cl_\theta(U) : U \in W\})$ and let $C_\alpha$ be the set of these points. Let $H_\alpha = \bigcup \{H_\beta : \beta \in \alpha\} \cup C_\alpha$. Considering the fact that if $X$ is a $n$-Urysohn space we have that for every $A \subseteq X$, $|A|_\theta \leq |A|^{|X|}$ we have that $|H_\alpha| \leq 2^\tau$. Let $H = \bigcup \{H_\beta : \beta \in \tau^+\}$. Since $t_\theta(X) \leq \chi(X) \leq \tau$, $\tau^+$ is regular and $\{H_\alpha : \alpha \in \tau^+\}$ is an increasing family of my$\theta$-closed sets of length $\tau^+$, we have that $H$ is $\theta$-closed. Also $|H| \leq 2^\tau$. Let $H^* = \bigcup \{Uw(x) : x \in H\} \supseteq H$. Then $|H^*| \leq k2^\tau$.

We want to prove that $X = H^*$. Suppose that there exists a point $q \in X \setminus H^* \subseteq X \setminus H$. Then for all $x \in H$ there is $U(x) \in U_\alpha$ such that $q \notin cl_\theta(U(x))$. From $\theta$-$aL_\theta(X) \leq \tau$ choose $H' \in [H]^{\leq \tau}$ such that $H \subseteq \bigcup \{cl_\theta(U(x)) : x \in H'\}$. Then $H' \subseteq H_\alpha$ for some $\alpha \in \tau^+$ and hence $W' = \{cl_\theta(U(x)) : x \in H'\} \subseteq [B_\alpha]^{\leq \tau}$ and $q \in X \setminus (\bigcup \{cl_\theta(U) : U \in W\}) \neq \emptyset$. Hence we have already chosen $x(W) \in H_\alpha+1 \cap (H \setminus \bigcup \{cl_\theta(U(x)) : x \in H'\}) \subseteq H \cap (X \setminus H)$ a contradiction. Hence $X = H^*$ and $|X| \leq k2^\tau$. 

Now we use $UW(X), \psi w_\theta(X)$ and $h\theta$-$aL(X)$ to present a variation of the Theorem 2.23 in [6]. The proof of Theorem 3.2 follows step by step the proof of Theorem 2.23 in [6].

**Theorem 3.2.** If $X$ is a $T_1$ space then $|X| \leq UW(X)\psi w_\theta(X)^{h\theta-aL(X)}$.

**Proof.** Let $UW(X) \leq k$, $h\theta$-$aL(X) \leq \tau$ and $\psi w_\theta(X) \leq \lambda$. For all $x \in X$, let $U_\alpha$ be a family of open neighborhood of $x$ such that $|U_\alpha| \leq \lambda$ and $Uw(x) = \bigcap \{cl_\theta(U) : U \in U_\alpha\}$. By trasfinite induction we construct two families $\{H_\alpha : \alpha \in \tau^+\}$ and $\{B_\alpha : \alpha \in \tau^+\}$ such that:

1. $\{H_\alpha : \alpha \in \tau^+\}$ is an increasing sequence of subsets of $X$;

2. $|H_\alpha| \leq k\lambda^\tau$ for all $\alpha \in \tau^+$;

3. if $\{H_\beta : \beta \in \alpha\}$ are defined for some $\alpha \in \tau^+$, then $B_\alpha = \bigcup \{U_x : x \in \bigcup \{Uw(y) : y \in \bigcup \{H_\beta : \beta \in \alpha\}\}\}$;

4. if $\alpha \in \tau^+$ and $W \in [B_\alpha]^{\leq \tau}$ is such that $X \setminus (\bigcup \{cl_\theta(U) : U \in W\}) \neq \emptyset$ then $H_\alpha - (\bigcup \{cl_\theta(U) : U \in W\}) \neq \emptyset$. 

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Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ be already defined. For all $W$ as in 4., choose a point $x(W) \in X \setminus (\bigcup\{cl_\theta(U) : U \in W\})$ and let $C_\alpha$ be the set of these points.

Let $H_\alpha = \bigcup\{H_\beta : \beta \in \alpha\} \cup C_\alpha$. Then $|H_\alpha| \leq k\lambda^\tau$.

Let $H = \bigcup\{H_\alpha : \alpha \in \tau^+\}$ and $H^* = \bigcup\{U_w(x) : x \in H\} \supseteq H$. Then $|H^*| \leq k\lambda^\tau$.

We want to prove that $X = H^*$. Suppose that there exists a point $q \in X \setminus H^*$. Then $q \notin U_w(x), \forall x \in H$. Hence for all $x \in H$ there is $U(x) \in U_x$ such that $q \notin cl_\theta(U(x))$. From $h^\theta-aL(X) \leq \tau$ choose $H' \subseteq [H]^{\leq \tau}$ such that $H \subseteq \bigcup\{cl_\theta(U(x)) : x \in H'\}$. Let $W = \{U(x) : x \in H'\}$. We have that $H' \subseteq H_\alpha$ for some $\alpha \in \tau^+$ and $W \in [B_{\alpha+1}]^{\leq \tau}$ and $X \setminus (\bigcup\{cl_\theta(U) : U \in W\}) \neq \emptyset$. Hence we have already chosen $x(W) \in X \setminus (\bigcup\{cl_\theta(U) : U \in W\}) \subseteq X \setminus H$ and $x(W) \in H$ a contradiction. Hence $X = H^*$ and $|X| \leq k\lambda^\tau$.

**Corollary 3.1.** If $X$ is a Urysohn space then $|X| \leq \psi(X)^{h^\theta-aL(X)}$.

## 4 The Urysohn point separating weight

**Definition 4.1.** [5] A Hausdorff point separating open cover $S$ for a space $X$ is an open cover of $X$ having the property that for each distinct points $x, y \in X$ there exists $S \in S$ such that $x \in S$ and $y \notin S$.

The Hausdorff point separating weight of a space $X$ is

$$Hpsw(X) = \min\{\tau : X \text{ has a Hausdorff point separating open cover } S \text{ such that each point of } X \text{ is contained in at most } \tau \text{ elements of } S\}.$$  

Following the same idea as in [5] we introduce the following definition:

**Definition 4.2.** A Urysohn point separating open cover $S$ for a space $X$ is an open cover of $X$ having the property that for each distinct points $x, y \in X$ there exists $S \in S$ such that $x \in S$ and $y \notin cl_\theta(S)$.

**Definition 4.3.** The Urysohn point separating weight of a Urysohn space $X$ is the cardinal:

$$Upsw(X) = \min\{\tau : X \text{ has a Urysohn point separating open cover } S \text{ such that each point of } X \text{ is contained in at most } \tau \text{ elements of } S\} + \aleph_0.$$  

Note that $Hpsw(X) \leq Upsw(X)$, for every Urysohn space $X$.

The proof of the following theorem follows step by step the proof of Theorem 20 in [5].

**Theorem 4.1.** If $X$ is a Urysohn space then $nw(X) \leq Upsw(X)^{h^\theta-aLc(X)}$.
Proof. Let $\theta-aL_c(X) = k$ and $S$ a Urysohn point separating open cover for $X$ such that for each $x \in X$, $|S_x| \leq \lambda$, where $S_x$ is the collection of members of $S$ containing $x$.

We first show that $d(X) \leq \lambda^k$. $\forall \alpha < k$ construct a subset $D_\alpha$ of $X$ such that:

1. $|D_\alpha| \leq \lambda^k$;
2. if $\mathcal{U}$ is a subcollection of $\bigcup\{S_x : x \in \bigcup_{\beta < \alpha} D_\beta\}$ such that $|\mathcal{U}| \leq k$ and $X \setminus \bigcup cl_\theta(\mathcal{U}) \neq \emptyset$ we have that $D_\alpha \setminus \bigcup cl_\theta(\mathcal{U}) \neq \emptyset$.

Such a $D_\alpha$ can be constructed since the member of possible $\mathcal{U}$'s at the $\alpha$th stage of construction is $\leq (\lambda^k\lambda)^k = \lambda^k$.

Let $D = \bigcup_{\alpha < k} D_\alpha$. We have that $|D| \leq \lambda^k$. We want to prove that $\overline{D} = X$. Suppose that there exists $p \in X \setminus \overline{D}$, since $Upsw(X) \leq \lambda$, $\forall x \in \overline{D}$, there exists $V_x \in S_x$: $p \notin cl_\theta(V_x)$. Since $x \in \overline{D}$, $V_x \cap D \neq \emptyset$. Let $y \in V_x \cap D$, so $V_x \in \bigcup\{S_y : y \in D\}$. Put $\mathcal{W} = \{V_x : x \in \overline{D}\} \subseteq \bigcup\{S_y : y \in D\}$. $\mathcal{W}$ is an open cover of $\overline{D}$ and since $\theta-aL_c(X) \leq k$, there exists $\mathcal{W}' \subseteq \mathcal{W}$ with $|\mathcal{W}'| \leq k$ such that $\overline{D} \subseteq \bigcup\{cl_\theta(V) : V \in \mathcal{W}'\}$ and $p \notin \bigcup\{cl_\theta(V) : V \in \mathcal{W}'\}$ and this contradicts 2.

Since $d(X) \leq \lambda^k$ we have that $|S| \leq \lambda^k$.

Let $\mathcal{N} = \{X \setminus S : S is the union of at most k members of S\}$. $|\mathcal{N}| \leq \lambda^k$ and $\mathcal{N}$ is a network for $X$.

\[ \square \]

**Theorem 4.2.** If $X$ is a Urysohn space then $|X| \leq Upsw(X)^{\theta-aL_c(X)}\psi(X)$.

*Proof.* If $X$ is a $T_1$ space then $|X| \leq nw(X)^\psi(X)$ and using the theorem above we have that $|X| \leq nw(X)^\psi(X) \leq Upsw(X)^{\theta-aL_c(X)}\psi(X)$.

\[ \square \]

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