Lovelock gravity is equivalent to Einstein

gravity coupled to form fields

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\textbf{Abstract}

Lovelock gravity is a class of higher-derivative gravitational theories whose linearized equations of motion have no more than two time derivatives. Here, it is shown that any Lovelock theory can be effectively described as Einstein gravity coupled to a p-form gauge field. This extends the known example of an $f(R)$ theory of gravity, which can be described as Einstein gravity coupled to a scalar field.
1 Introduction

The Lovelock class of gravity theories are the unique set of higher-derivative extensions to Einstein gravity with two-derivative field equations [1]. When one quantizes small perturbations about a fixed background, this two-derivative limit on the equations of motion is practically equivalent to the unitarity of the theory. We will use the two notions interchangeably.

The interaction terms in a theory of Lovelock gravity are constrained by the dimensionality $D$ of the spacetime. A Lovelock term with $2k$ derivatives is purely topological when $2k = D$ and identically vanishing when $2k > D$. So that, formally, $k \leq k_{\text{max}} \leq \frac{D-1}{2}$. For a more general and recent perspective on this issue, see [2].

Over the years, modified theories of gravity differing from the Lovelock class have been proposed. However, even if unitarity is maintained, all of these include non-trivial modifications to Einstein gravity such as additional gravitons modes (either in the way of massive gravitons or bi-metric theories) and/or spacetimes of dimension less than four. (For a short sample of more recent work, see [3]-[10].) Such proposals are interesting in their own right but, unlike Lovelock gravity, present serious phenomenological issues and have no obvious connection to the realm of string theory.

That unitary gravity is equivalent to Lovelock’s theories may appear to be a powerful assertion to some [11] and a natural one to others [12]. Regardless, we now intend to “raise the ante” and further argue that unitary gravity is essentially equivalent to Einstein’s theory only. The basis for this new claim is that any theory of Lovelock gravity can be effectively described as Einstein gravity coupled to a matter field, much in the same way that any model of
\( f(R) \) gravity can be described as Einstein gravity coupled to a scalar field \([13]\). For \( D \)-dimensional Lovelock gravity, the role of the scalar is played by a \((D-3)\)-form gauge field.

We proceed to elaborate on this picture. After some preliminary considerations, we discuss the cases of \( f(R) \) gravity (as a warmup), Gauss–Bonnet gravity and, then, Lovelock gravity of arbitrary order. The paper concludes with a brief summary.

Recently in \([14]\), the association between Lovelock gravity and form fields was noted and some possible relations to string theory were proposed.

## 2 Lovelock gravity

Lovelock gravity has the following Lagrangian:

\[
L_{LL} = \sum_{k=0}^{k_{\text{max}}} \lambda_k L_k ,
\]

(1)

where \( L_k \) contains terms with \( k \) Riemann tensors contracted together, the \( \lambda \)'s are dimensionful coupling constants and the sum runs up to \( k_{\text{max}} \leq \frac{D-1}{2} \). In this set-up, \( L_0 \) is the cosmological constant \( \Lambda \), \( L_1 = L_E \) is the Ricci scalar and \( L_2 \) is the Gauss–Bonnet extension \([11]\) which will be encountered shortly. The other terms are defined in Eq. \((3)\) below.

Let us define the tensor \( \chi^{abcd} \equiv \frac{\delta L}{\delta R_{abcd}} \), which has the symmetry properties of the Riemann tensor \( R_{abcd} \). Lovelock theories can be defined by the identity \([1]\)

\[
\nabla_a \chi^{abcd} = 0 ,
\]

(2)

valid on- or off-shell and satisfied by each the Lovelock terms separately.
The field equation for a Lovelock theory takes the form \[ L = \frac{1}{2} \mathcal{X}_{abc} \mathcal{R}_{abcd} - \frac{1}{2} \mathcal{G}_{pq} \mathcal{L} = \frac{1}{2} \mathcal{T}_{pq}, \] with \( T_{ab} \) being the stress-tensor for the matter fields. For Einstein gravity, \( L_E = \mathcal{R}_{ab} \mathcal{R}^{ab} \) (we work with \( 16\pi G = 1 \) units throughout) and \( \mathcal{X}_E^{abcd} = \frac{1}{2} \left[ g^{ac} g^{bd} - g^{ad} g^{bc} \right] \), which leads to the standard Einstein equation.

A note on conventions: “Calligraphic letters” are used to denote four-index tensors. So that, for instance, an \( R \) means the Ricci scalar whereas an \( \mathcal{R} \) is meant as shorthand for \( \mathcal{R}^{abcd} \). A “dot product” will signify a four-fold contraction, \( \mathcal{A} \cdot \mathcal{B} = \mathcal{A}^{abcd} \mathcal{B}_{abcd} \).

The explicit form of the Lovelock term \( L_k \) is given by (with the usual symmetrization factor absorbed into \( \lambda_k \)) \[ L_k = \delta^{a_1 b_1 \ldots a_k b_k} \mathcal{R}^{c_1 d_1 \ldots c_k d_k}_{a_1 b_1 \ldots a_k b_k}, \] where the generalized alternating tensor \( \delta^{\ldots} \) is fully anti-symmetric in both its upper and lower indices. For example, \( \delta^{pqrs} = [\delta^p_q \delta^r_s - \delta^p_s \delta^r_q] \). Hence, we find that \( \mathcal{X}_k^{abcd} = \frac{\delta L_k}{\delta \mathcal{R}_{abcd}} \) goes as
\[
(\mathcal{X}_k)^{pq}_{rs} = k \delta^{pq a_2 b_2 \ldots a_k b_k} \mathcal{R}^{c_2 d_2 \ldots c_k d_k}_{a_2 b_2 \ldots a_k b_k}.
\] (4)

The \( \mathcal{X}_k \)’s satisfy the first and second Bianchi identities. The first Bianchi identity \( (\mathcal{X}_k)^{pq}_{rs} + (\mathcal{X}_k)^{ps}_{qr} + (\mathcal{X}_k)^{pr}_{sq} = 0 \) is trivially satisfied because of the Riemannian symmetry of \( \delta^{pqrs} \).

The second Bianchi identity \( \nabla_m (\mathcal{X}_k)^{pq}_{rs} + \nabla_s (\mathcal{X}_k)^{pq}_{mr} + \nabla_r (\mathcal{X}_k)^{pq}_{sm} = 0 \) follows from the structure of \( \mathcal{X}_k \) and the fact that the Riemann tensor satisfies the second Bianchi identity. Differentiating \( \mathcal{X}_k \), while keeping in mind the product rule and the interchangeability of the Riemann tensors, we
have
\[ \nabla_m (\mathcal{X}_k)^{pq}_{rs} = k(k - 1) \delta^{pqa_2b_2 \ldots a_kb_k} \nabla_m [R_{c_2d_2}^{c_2d_2}] R_{a_3b_3}^{c_3d_3} \cdots R_{a_kb_k}^{c_kb_k}. \]
(5)

Now, permuting the first two pairs of upper indices and using the Riemannian symmetries of the \( \delta \)-symbol, we obtain
\[ \nabla_m (\mathcal{X}_k)^{pq}_{rs} = \cdots [\delta^{a_2b_2}_r \delta^{b_2}_s - \delta^{a_2b_2}_s \delta^{b_2}_r] \nabla_m [R_{c_2d_2}^{c_2d_2}] \cdots = \cdots \nabla_m [R_{c_2d_2}^{c_2d_2}] \cdots. \]
(6)

The second Bianchi identity then follows.

## 3 \( f(R) \) gravity revisited

Let us first consider how \( L(R) = R^n \) models are equivalent to Einstein–scalar theories [13], but from our novel perspective. For these models, \( nL(R) = \mathcal{X} \cdot \mathcal{R} \) such that \( \mathcal{X}^{abcd} = nR^{n-1} \mathcal{X}_E^{abcd} \).

We propose the following effective description:
\[ n \tilde{L}(\mathcal{R}, \mathcal{S}, \mathcal{Y}) = \mathcal{S} \cdot \mathcal{Y} + \mathcal{Y} \cdot (\mathcal{R} - \mathcal{S}) + \mathcal{S} \cdot (\mathcal{X} - \mathcal{Y}) = \mathcal{Y} \cdot (\mathcal{R} - \mathcal{S}) + \mathcal{S} \cdot \mathcal{X} = \mathcal{Y} \cdot \mathcal{R} + \mathcal{S} \cdot (\mathcal{X} - \mathcal{Y}), \]
(7)

with \( \mathcal{S}^{abcd} \) and \( \mathcal{Y}^{abcd} \) serving as auxiliary tensor fields.

Varying \( \tilde{L} \) by \( \mathcal{S} \), one obtains \( \mathcal{Y} = \mathcal{X} \). Similarly, \( \mathcal{S} = \mathcal{R} \) is obtained by varying \( \tilde{L}_n \) with respect to \( \mathcal{Y} \). Substituting these relations into \( \tilde{L} \), we find that, when on-shell, \( \tilde{L} = L \) and can therefore be viewed as an equivalent description of the same theory.
Next, we provide the auxiliary fields with physically motivated identities, while respecting the tensorial properties of their on-shell equivalents. To begin, $\mathcal{Y}$ can be expressed in terms of a scalar field $\phi$ times a tensor; say,

$$\mathcal{Y}^{abcd} = \phi X^{abcd}_E.$$  \hspace{1cm} (8)

Then, just like in the standard procedure, $\phi$ becomes a dynamical field that is equivalent to $f'(R)$ on shell ($t$ denotes a derivative with respect to the argument).

As for $\mathcal{S}$, let us make the choice

$$\mathcal{S}^{abcd} = \frac{2\psi}{D(D-1)} X^{abcd}_E,$$  \hspace{1cm} (9)

where $\psi = \psi(\phi)$ is a scalar functional and the “normalization” factor has been chosen to ensure that the on-shell condition $\mathcal{S} = \mathcal{R}$ translates into $\psi = R$.

The effective Lagrangian can now be reformulated as

$$n\tilde{L} = \phi R + \psi(\phi) (nR^{n-1} - \phi)$$

$$= \phi R + \psi(\phi) (n[\psi(\phi)]^{n-1} - \phi)$$

$$= \phi R - V(\phi).$$  \hspace{1cm} (10)

The “potential” $V(\phi)$ is given by $V[\phi, \psi(\phi)] = [\phi - f'(\psi)]\psi$. Its minimum is determined by the equation $V'(\phi) = 0$ and occurs at $\phi_m = nR^{n-1}$, as can be seen from the equation $\mathcal{Y} = \mathcal{X}$. Even though the scalar is constrained on-shell to be at the minimum of the potential, $\phi$ is still fully dynamical and, additionally, $\phi_m$ and $\psi(\phi_m) = R$ are not necessarily spacetime constants. This all follows from the gravitational field equation, which goes as

$$\nabla_p \nabla_q \phi - \phi R_{pq} = \frac{1}{2}g_{pq} [2\nabla^a \nabla_a \phi - \phi R + V(\phi)].$$  

6
And so what we end up with is the well-known, expected result that \( f(R) \) gravity is dynamically equivalent to Einstein gravity coupled to a scalar field.

4 Gauss–Bonnet gravity

The simplest non-trivial Lovelock extension of Einstein gravity is Gauss–Bonnet (GB) gravity. Its Lagrangian is given by \( L_{GB} = L_1 + L_0 + \lambda_2 L_2 \) for which \( \lambda_{GB} = \lambda_E + \lambda_2 \lambda_{abcd} \). Recall that \( L_1 = L_E \), \( L_0 = \Lambda \) and, from Eq. (3),

\[
\lambda_2 L_2 = \lambda_2 [R_{abcd}R_{abcd} - 4R^{ab}R_{cd} + R^2].
\] (11)

The value of the dimensionful coupling constant \( \lambda_2 \) is irrelevant to the current treatment.

Via Eq. (11),

\[
\lambda_2 L_2 = \lambda_2 [R_{abcd}R_{abcd} - 4R^{ab}R_{cd} + R^2].
\] (12)

To be dynamical, the GB theory requires \( D \geq 5 \). Then, \( 2L_2 = R \cdot \lambda_2 \).

We focus on the GB terms \( L_2, \lambda_2 \) and closely follow the discussion of \( f(R) \) gravity until arriving at the step of identifying the auxiliary tensors. For \( S \) (and, likewise, for \( \mathcal{V} \)), what is required is matter that can carry four indices and respect the basic symmetries of the Riemann tensor. We are thus driven to the choice of \( p \)-form gauge fields \( B^{[a_1a_2...ap]} \) and their totally anti-symmetrized \((p + 1)\)-form field-strength tensors \( H^{[a_1a_2...a_{p+1}]} = \nabla^{[a_1} B^{a_2a_3...a_{p+1}]} \), and so

\[
S_{ab}^{cd} = H^{[e_1...e_nab]}H_{[e_1...e_ncd]},
\] (13)
where \( n = p - 1 \). Anti-symmetrization of indices will be implied from now on.

The \( H \)'s are identically vanishing unless \( p \leq D - 1 \). To allow for four different indices on an \( S \), the condition \( p \leq D - 3 \) is further required. Even when off-shell, \( S \) satisfies the basic symmetry properties of the Riemann tensor; for instance, \( S_{abcd} = -S_{abdc} \), \( S_{cdab} = +S_{abcd} \).

It can be also be shown that, even when off-shell, \( S \) satisfies the first Bianchi identity. To this end, let us use vierbein formalism to write \( S_{ab}^{cd} = e_i^a e_j^b e_k^c e_l^d H_{ije}^{k \cdots \epsilon_n} H_{kl \epsilon_1 \cdots \epsilon_n} \) or

\[
S_{ab}^{cd} = \left[ A_i^k \right]_c^a \left[ A_j^l \right]_d^b H_{ije}^{k \epsilon_1 \cdots \epsilon_n} H_{k\epsilon_1 \epsilon_2 \cdots \epsilon_n},
\]

where \( \left[ A_i^j \right]_b^a \equiv e_i^a e_j^b \) and the last identity follows from the basic Riemannian structure of \( S \). Given the form \( S_{cd}^{ab} \propto A_a^c A_b^d - A_a^d A_b^c \) for a tensor \( S \), the first Bianchi identity follows.

To obtain the GB version of \( \mathcal{Y} \), we apply the ansatz for \( S \) in Eq. (13) to rewrite Eq. (4) as

\[
\mathcal{Y}_{ab}^{cd} = 2 S_{ab}^{a2b2} S_{c2d2}^{a2b2} ,
\]

The explicit result in terms of \( H \)'s can also be obtained by replacing all the Riemann tensors in Eq. (12) with the corresponding expressions for \( S \),

\[
\mathcal{Y}_{abcd}^{a2b2} = 2 \left[ H_{\epsilon_1 \cdots \epsilon_n}^{\epsilon_1 \cdots \epsilon_n} H_{e_1 \cdots e_n}^{cd} - g^{ac} H_{\epsilon_1 \cdots \epsilon_n}^{\epsilon_1 \cdots \epsilon_n} H_{e_1 \cdots e_p}^{d} - g^{bd} H_{\epsilon_1 \cdots \epsilon_n}^{\epsilon_1 \cdots \epsilon_n} H_{e_1 \cdots e_p}^{c} + g^{ad} H_{\epsilon_1 \cdots \epsilon_n}^{\epsilon_1 \cdots \epsilon_n} H_{e_1 \cdots e_p}^{c} + g^{bc} H_{\epsilon_1 \cdots \epsilon_n}^{\epsilon_1 \cdots \epsilon_n} H_{e_1 \cdots e_p}^{d} + H^2 \mathcal{X}_{abcd}^{E} \right] ,
\]

where \( H^2 = H_{\epsilon_1 \cdots \epsilon_{p+1}}^{\epsilon_1 \cdots \epsilon_{p+1}} H_{e_1 \cdots e_{p+1}} \). Notice that, by its definition, \( \mathcal{Y}_2 \) automatically satisfies the first Bianchi identity.
We will assume that there are no local sources for the field-strength tensor (i.e., no branes). Then, in direct analogy to standard electromagnetism, this tensor must have a vanishing divergence,

$$\nabla_a H^{ae_1 \cdots e_p} = 0,$$

and satisfy a Bianchi-like identity,

$$\nabla_a H_{bc}^{e_1 \cdots e_n} + \nabla_b H_{ca}^{e_1 \cdots e_n} + \nabla_c H_{ab}^{e_1 \cdots e_n} = 0. \quad (18)$$

It turns out that the latter is enough to establish that both $S$ and $Y_2$ satisfy the second Bianchi identity even off-shell. For $S$, this is true because of the Riemannian “double-exchange” symmetry $S^{abcd} = S^{cdab}$, meaning that

$$\nabla_e S_{abcd} = \nabla_e [H^{e_1 \cdots e_n}_{ab} H_{e_1 \cdots e_n cd}] = 2 [\nabla_e H^{e_1 \cdots e_n}_{ab}] H_{e_1 \cdots e_n cd} = 2 H^{e_1 \cdots e_n}_{ab} \nabla_e H_{e_1 \cdots e_n cd}, \quad (19)$$

and the second Bianchi identity follows from Eq. (18) (at least) for the sets $e, a, b$ and $e, c, d$.

One might be concerned about cases in which the set of permuted indices starts out on different $H$’s. It is, however, a simple matter to use Riemannian and field-strength (anti-) symmetry properties to manipulate these onto the same $H$.

Since the second Bianchi identity is true for $S$, it is also true for $Y_2$; this, by direct analogy with our previous argument that any $X_k$ satisfies the second Bianchi identity given that $\mathcal{R}$ does (cf, Eqs. (5-6)).
The GB version of the effective Lagrangian (7) can now be put in the form

\[ \tilde{L}_{GB} = L_1 + L_0 - \frac{1}{2(p+1)} H^2 + \lambda_2 \tilde{L}_2 \]

\[ = R + \Lambda - \frac{1}{2(p+1)} H^2 + \lambda_2 \left[ \frac{1}{2} \mathcal{Y}_2(H) \cdot (\mathcal{R} - \mathcal{S}(H)) + \mathcal{S}(H) \cdot \mathcal{X}_2 \right] \]

\[ = R + \Lambda - \frac{1}{2(p+1)} H^2 + \lambda_2 \left[ \frac{1}{2} \mathcal{Y}_2(H) \cdot \mathcal{R} + \frac{1}{2} \mathcal{S}(H) \cdot (\mathcal{X}_2 - \mathcal{Y}_2(H)) \right], \]

(20)

where \( \mathcal{S}(H) \) is given by Eq. (13) and \( \mathcal{Y}_2(H) \), by Eq. (16). The kinetic term \( H^2 \) has been included for completeness.

The equivalence principle is violated by the interaction terms \( \mathcal{Y}_2 \cdot \mathcal{R} \) and \( \mathcal{S} \cdot \mathcal{X}_2 \), as these non-trivially couple the Riemann curvature to the field-strength tensor. But our formulation makes it clear that the violation of the equivalence principle can be attributed to the gauge fields coupling with the Einstein graviton rather than an exotic form of gravitation, in exact analogy with the case of \( f(R) \) gravity.

4.1 Equations of motion

We would now like to understand how unitarity — a maximum of two time derivatives in the linearized equations of motion — is maintained for the effective theory. Of course, the two-derivative constraint on these equations is assured to hold on-shell, as this is when \( \tilde{L}_{GB} \) and \( L_{GB} \) are describing equivalent theories.

Let us begin with the field equation for the gauge field. Varying the
effective action, we obtain the expression
\[
\Delta \tilde{L}_2 = - \frac{1}{(p+1)} \left[ (-1)^p(p-1) \nabla_c \left( \mathcal{X}_{q_1q_2ab} H_{q_3\cdots q_p}^{\ \cdot cab} \right) - 2 \nabla^c \left( \mathcal{X}_{q_1cab} H_{q_2\cdots q_p}^{\ \cdot ab} \right) \right].
\] (21)

One might be concerned by the presence of multi-derivative terms; however, it turns out that this variation is identically vanishing. This outcome follows from the Lovelock identity (2), the vanishing divergence of the field strengths (17), and the realization that both \( \mathcal{X}_2 \) and \( \mathcal{Y}_2 \) are Riemannian tensors.

To understand how all this works, let us start with the first term on the right-hand side of Eq. (21). After imposing Eq. (17), we have (with some indices suppressed for clarity)
\[
\nabla_c \left( H^{\cdot cab} \mathcal{X}_{qrab} \right) = H^{\cdot cab} \nabla_c \mathcal{X}_{qrab}. \tag{22}
\]
Now, since \( H_{\cdot cab} = H_{\cdot abc} = H_{\cdot bca} \), this term can be recast into
\[
H^{\cdot cab} \nabla_c \mathcal{X}_{qrab} = \frac{1}{3} \left[ H^{\cdot cab} \nabla_c \mathcal{X}_{qrab} + H^{\cdot cab} \nabla_a \mathcal{X}_{qrbc} + H^{\cdot cab} \nabla_b \mathcal{X}_{qrca} \right], \tag{23}
\]
which vanishes by virtue of the second Bianchi identity. The same argument can be used to establish that the third term on the right is also vanishing.

The second term on the right of Eq. (21) can similarly be shown to vanish. Here, we start by using Eq. (2) to rewrite this term as
\[
\nabla^c \left( \mathcal{X}_{q_1cab} H_{q_2\cdots q_p}^{\ \cdot ab} \right) = \mathcal{X}_{q_1cab} \nabla^c H_{q_2\cdots q_p}^{\ \cdot ab}. \tag{24}
\]
Now consider that
\[
\nabla^c H_{p...q}^{ab} = \nabla_q p H_{...}^{cab} \tag{25}
\]
because covariant derivatives commute when acting on a $B$ and, since $H_{...}^{cab} = H_{...}^{bac}$,
\[
\nabla^c \left( \mathcal{X}_{q_1}^{cab} H_{...}^{q_1} \right) = \frac{1}{3} \left[ \mathcal{X}_{q_1}^{abc} + \mathcal{X}_{q_1}^{bca} + \mathcal{X}_{q_1}^{cab} \right] \nabla_q p H_{...}^{abc} \tag{26}
\]
which vanishes via the first Bianchi identity.

The fourth term in Eq. (21) vanishes in the same way as the second does except that, in this case, one applies Eq. (17) to move $\mathcal{Y}_2$ outside of the derivative.

We can establish unitarity for the linearized field equation for gravity by showing that its associated $\mathcal{X}$ satisfies the Lovelock identity (2). The variation of $\tilde{L}_{GB}$ with respect to a Riemann tensor yields
\[
\mathcal{X}_{LGB}^{abcd} = \mathcal{X}^{abcd} + \frac{\lambda_2}{2} \mathcal{Y}^{abcd} + \frac{\lambda_2}{2} S_{pqrs} \frac{\delta \mathcal{X}_{2}^{pqrs}}{\delta R_{abcd}} . \tag{27}
\]
Identity (2) is then satisfied, as all three terms on the right-hand side have a vanishing divergence. This is evident for the first term via Eq. (2) and the second term by way of Eq. (17). The divergence of the third term vanishes due to the following argument: As $S$ is functionally independent of the Riemann tensor, we can express the third term as
\[
S_{pqrs} \frac{\delta \mathcal{X}_{2}^{pqrs}}{\delta R_{abcd}} = \frac{\delta S \cdot \mathcal{X}_2}{\delta R_{abcd}} . \tag{28}
\]
Next, applying Eq. (4),
\[
S \cdot \mathcal{X}_2 = 2 \delta_{rsca}^{pqbd} S_{pq}^{rs} R_{abcd}^{r2d} . \tag{29}
\]
which leads to
\[
S_{pqrs} \frac{\delta \lambda^{pqrs}_2}{\delta R_{ab}^{cd}} = 2 \delta_{rscd} S_{pq}^{rs} = Y_{ab}^{cd} \, ,
\]
with the latter equality following from Eq. (15). Hence, this third term is identical to the second, and so unitarity has been established.

5 Lovelock gravity of arbitrary order

For an arbitrary-order term in the Lovelock expansion, things work pretty much the same as for the GB case. The number of $H$’s in the interaction term will increase with increasing $k$, but there are no conceptual differences. Indeed, the generalized version of the effective Lagrangian \([20]\) takes the form
\[
\tilde{L}_{LL} = R + \Lambda - \frac{1}{2(p+1)} H^2 + \sum_{k=2}^{k_{\text{max}}} \lambda_k \tilde{L}_k \, , \text{ with } \kappa \tilde{L}_k = \gamma_k(H) \cdot \mathcal{R} + \mathcal{S}(H) \cdot \mathcal{X}_k - \mathcal{S}(H) \cdot \gamma_k(H) \quad \text{as for the earlier-studied models.}
\]
The function $\gamma_k(H)$ now goes as $H^{2k-2}$ and its specific structure is determined by the structure of $\mathcal{X}_k$.

5.1 Equations of motion

The verification of unitarity follows along similar lines to the GB case, leading to the same basic results. For instance, the $k^{\text{th}}$-order Lovelock term leads to
\[
0 = \frac{\partial \tilde{L}_k}{\partial B^{n \cdots q_p}} = -\frac{2}{k(p+1)} \left[ (-1)^p (p-1) \nabla_c \left( \mathcal{X}_{k, q_1 q_2} H_{q_3 \cdots q_p}^{cab} \right) - 2 \nabla_c \left( \mathcal{X}_{k, q_1 c a b} H_{q_2 \cdots q_p}^{a b} \right) \right]
\]
\[ + (-1)^p (p-1)(k-1) \nabla_c \left( \mathcal{Z}_{k,q_1 q_2 ab} H_{q_3 \cdots q_p}^{\ caffe} \right) - 2(k-1) \nabla_c \left( \mathcal{Z}_{k,q_1 cab} H_{q_2 \cdots q_p}^{\ caffe} \right) \]

\[ - (-1)^p (p-1) k \nabla_c \left( \mathcal{Y}_{k,q_1 q_2 ab} H_{q_3 \cdots q_p}^{\ caffe} \right) + 2k \nabla_c \left( \mathcal{Y}_{k,q_1 cab} H_{q_2 \cdots q_p}^{\ caffe} \right) \right], \]

(31)

where \( \mathcal{Z}_{abcd} \equiv \frac{\partial(Y_k \cdot R)}{\partial S^{abcd}} \). We find that, once again, the equation is automatically satisfied given a vanishing divergence for \( H \) plus the Riemannian symmetries of \( S, Y_2 \) and \( \mathcal{X}_2 \).

The only real subtlety of the generic Lovelock analysis might be in verifying that the last term in the generalized version of Eq. (27),

\[ \frac{1}{\lambda_k} \mathcal{L}_k^{abcd} = \frac{1}{k} Y_k^{abcd} + \frac{1}{k} S_{pqrs} \frac{\delta \mathcal{X}_k^{pqrs}}{\delta R_{abcd}}, \]

(32)

satisfies the Lovelock identity (2). That \( \nabla_a S_{pqrs} \frac{\delta \mathcal{X}_k^{pqrs}}{\delta R_{abcd}} = 0 \) or, equivalently,

\[ \frac{\delta^2 (S \cdot \mathcal{X}_k)}{\delta R_{e f g h} \delta R_{abcd}} \nabla_a R_{e f g h} + \frac{\delta^2 (S \cdot \mathcal{X}_k)}{\delta S_{e f g h} \delta R_{abcd}} \nabla_a S_{e f g h} = 0. \]

(33)

For this purpose, we again call upon the explicit form of a Lovelock term \( \mathcal{X}_k \). Suitably modified, this is

\[ \frac{1}{k} \mathcal{S} \cdot \mathcal{X}_k = \delta^{a_1 b_1 \cdots a_k b_k} \ R^{c_1 d_1} \cdots S^{c_j d_j} \cdots R^{c_k d_k \ a_k b_k}; \]

(34)

where the ellipses indicate Riemann tensors only.

Now twice varying Eq. (34) with the appropriate tensors, we have

\[ \frac{1}{k} \frac{\delta^2 (S \cdot \mathcal{X}_k)}{\delta R_{e f g h} \delta R_{abcd}} = (k-1)(k-2) \delta^{a_1 b_1 \cdots a_k b_k} \ R^{c_1 d_1} \cdots S^{c_j d_j} \cdots R^{c_k d_k \ a_k b_k} \]

and

\[ \frac{1}{k} \frac{\delta^2 (S \cdot \mathcal{X}_k)}{\delta S_{e f g h} \delta R_{abcd}} = (k-1) \delta^{a_1 b_1 \cdots e f \cdots a_k b_k} \ R^{c_1 d_2} \cdots S^{c_j d_j} \cdots R^{c_k d_k \ a_k b_k}; \]

(36)
where, in the last line, the symbol [...] indicates the absence of \( S^{c_j d_j}_{a_j b_j} \) and \( e, f, g, h \) are standing in place of \( a_j, b_j, c_j, d_j \) respectively.

From Eqs. (35) and (36), it follows that both terms on the left side of Eq. (33) vanish by virtue of the second Bianchi identity with respect to permutations of \( a, e \) and \( f \).

In summary, we have shown that any of the Lovelock higher-derivative gravity theories has an effective description as Einstein gravity non-minimally coupled to a \((D - 2)\)-form field-strength tensor. So that, just as an \( f(R) \) theory is Einstein gravity coupled to a scalar, any higher-derivative unitary theory of gravity is Einstein gravity coupled to a \((D - 3)\)-form gauge field. The implication is that, for all practical purposes, Einstein’s is the single unitary theory of gravity. Our constructions would fit in naturally with the myriad of string-theory models that include such higher-form gauge fields.

Acknowledgments

The research of RB was supported by the Israel Science Foundation grant no. 239/10. The research of AJMM was supported by a Rhodes University Discretionary Grant RD11/2012. AJMM thanks Ben Gurion University for their hospitality during his visit.

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