MOMENTS OF STUDENT’S T-DISTRIBUTION: A UNIFIED APPROACH

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Abstract. In this paper, we derive new closed form formulae for moments of (generalized) Student’s t-distribution in the one dimensional case as well as in higher dimensions through a unified probability framework. Interestingly, the closed form expressions for the moments of Student’s t-distribution can be written in terms of the familiar Gamma function, Kummer’s confluent hypergeometric function, and the hypergeometric function.

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1. Introduction

In probability and statistics, the location (e.g., mean), spread (e.g., standard deviation), skewness, and kurtosis play an important role in the modeling of random processes. One often uses the mean and standard deviation to construct confidence intervals or conduct hypothesis testing, and significant skewness or kurtosis of a data set indicates deviations from normality. Moreover, moment matching algorithms are among the most widely used fitting procedures in practice. As a result, it is important to be able to find the moments of a given distribution. In his popular note, Winkelbauer (2014) gave the closed form formulae for the moments as well as absolute moments of a normal distribution $N(\mu, \sigma^2)$. The obtained results are beautiful and have been well received. Recently, Ogasawara (2020) provides a unified, non-recursive formulae for moments of normal distribution with strip truncation. Given the close relationship between the normal and Student’s t-distributions, a natural question arises: Can we derive similar formulae for the family of Student’s t-distributions? From the authors’ best knowledge, no such set of formulae exist for (generalized) Student’s t-distributions. The purpose of this note is to provide a complete set of closed form formulae for raw moments, central raw moments, absolute moments, and central absolute moments for (generalized) Student’s t-distributions in the one dimensional case and n-dimensional case. In particular, the formulae given in (2.5) - (2.8) and Proposition 3.1 are new in the literature. In this sense, we unify existing results and provide extensions to higher dimensions, within a common probabilistic framework.

2. Student’s t-distribution: One dimensional case

Recall the probability density function (pdf) of a standard Student’s t-distribution with $\nu > 0$ degrees of freedom, denoted by $St(t|0, 1, \nu)$, is given by

\[
St(t|0, 1, \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu \pi}} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\infty < t < \infty,
\]

where the Gamma function is defined as

\[
\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt.
\]

More generally, we have

\[
St(t|\mu, \sigma, \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\sigma}{\nu \pi}\right)^{\frac{1}{2}} \left(1 + \frac{\sigma}{\nu} (t - \mu)^2\right)^{-\frac{\nu+1}{2}}, \quad -\infty < t < \infty,
\]

where $\mu \in (-\infty, \infty)$ is the location, $\sigma > 0$ determines the scale, and $\nu = 1, 2, 3, \ldots$ represents the degrees of freedom. When $\nu = 1$, the pdf in (2.2) reduces to the pdf of Cauchy($\mu, \sigma$), while the pdf in (2.2) converges to the pdf of the normal $N(\mu, \sigma^{-1})$ as $\nu \to \infty$.

While the tails of the normal distribution decay at an exponential rate, the Student’s t-distribution is heavy-tailed, with a polynomial decay rate. Because of this, the Student’s t-distribution has been widely adopted in robust data analysis including (non) linear regression (Lange et al. 1989), sample selection models (Marchenko & Genton 2012), and linear mixed effect models (Pinheiro et al. 2001). It is also among the most widely applied distributions for financial risk modeling, see McNeil et al. (2015), Shaw (2006), Kwon & Satchell (2020). The reader is invited to refer to Kotz & Nadarajah (2004) for more.
The mean and variance of a Student’s t-distribution are well known and can be found in closed form by using the properties of the Gamma function. However, for higher order raw or central moments, the calculation quickly becomes tedious.

For later use, we denote the probability density function of a Gamma distribution with parameters \( \alpha > 0, \beta > 0 \) by

\[
\text{Gamma}(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \in (0, \infty).
\]

Similarly, the probability density function of a normal distribution \( X \sim N(\mu, \sigma^2) \) is denoted by

\[
N(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in (-\infty, +\infty).
\]

We will also require two common special functions. The Kummer’s confluent hypergeometric function is defined by

\[
K_1(\alpha, \gamma; z) = {}_1F_1(\alpha, \gamma; z) = \sum_{n=0}^{\infty} \frac{\alpha^n z^n}{\gamma^n n!}.
\]

The hypergeometric function is defined by

\[
{}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{a^n b^n}{c^n n!} z^n,
\]

where

\[
a^n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & n = 0, \\ a(a+1) \ldots (a+n-1) & n > 0. \end{cases}
\]

The following result is surprisingly simple and will be very useful in later derivations. It provides a representation of a conditional Student’s t-distribution in terms of a normal distribution. See, for example, Bishop (2006).

**Lemma 2.1.** Let \( T|\frac{1}{\sigma \lambda} \) be a normal distribution with mean \( \mu \) and variance \( 1/(\sigma \lambda) \). For \( \nu > 0 \), let \( \lambda \sim \text{Gamma}(\nu/2, \nu/2) \). Then the marginal distribution of \( T \) is a \( \text{St}(t|\mu, \sigma, \nu) \) Student’s t-distribution.

**Proof:** As the proof is very concise, we reproduce it here for the reader’s convenience. We have

\[
\begin{align*}
\int_0^\infty N(t|\mu, \frac{1}{\sigma \lambda}) \text{Gamma}(\lambda|\nu/2, \nu/2) \, d\lambda &= \int_0^\infty \sqrt{\frac{\sigma \lambda}{2\pi}} e^{-\frac{\sigma \lambda (t-\mu)^2}{2\nu}} \frac{\lambda^{\nu/2}}{\Gamma(\nu/2)} \lambda^{\nu/2-1} e^{-\frac{\nu}{2} \lambda} \, d\lambda \\
&= \frac{\sqrt{\sigma}}{\sqrt{2\pi}} \frac{\nu^{\nu/2}}{\Gamma(\nu/2)} \frac{\Gamma(\nu/2+1)}{2^{\nu/2} \Gamma(\nu/2)} \int_0^\infty \text{Gamma}(\lambda|\nu/2, \nu/2) \, d\lambda \\
&= \frac{\sqrt{\sigma}}{\sqrt{2\pi}} \frac{\nu^{\nu/2}}{\Gamma(\nu/2)} \frac{\Gamma(\nu/2+1)}{2^{\nu/2} \Gamma(\nu/2)} \left( \frac{\nu}{2} + \frac{\sigma}{2} (t-\mu)^2 \right) \\
&= \text{St}(t|\mu, \sigma, \nu).
\end{align*}
\]

This completes the proof of the Lemma. \( \Box \)

The following results are well known:
Theorem 2.1. We have

1. If \( X \sim N(0, \sigma^2) \) then
   \[
   E(X^m) = \begin{cases} 
   0, & \text{if } m = 2k + 1, \\
   \sigma^m m! & \text{if } m = 2k.
   \end{cases}
   \]

2. If \( X \sim \text{Gamma}(\alpha, \beta) \), then \( E(X^k) = \frac{\beta^{k+1}}{\Gamma(k+\alpha)} \) for \( k > -\alpha \).

With this and Lemma 2.1 above, we are able to find moments of Student’s t-distribution. More specifically, we have the following comprehensive theorem in one dimension.

Theorem 2.2. For \( k \in \mathbb{N}_+, 0 < k < \nu \), the following results hold:

1. For \( T \sim St(t|0, 1, \nu) \), the raw and absolute moments satisfy
   \[
   E(T^k) = \begin{cases} 
   \frac{\Gamma(k+1)}{\sqrt{\pi}} \frac{\nu^{k/2}}{\prod_{i=1}^{k}(\nu/2-i)}, & \text{if } k \text{ even}, \\
   0, & \text{if } k \text{ odd}.
   \end{cases}
   \]

2. \( E(|T|^k) = \frac{\nu^{k/2} \Gamma((k+1)/2) \Gamma((\nu-k)/2)}{\sqrt{\pi} \Gamma(\nu/2)} \).

3. If \( T \sim St(t|\mu, \sigma, \nu) \), the absolute moments satisfy
   \[
   E(|T - \mu|^k) = \frac{(1 + (-1)^k)}{2} (\nu/\sigma)^{k/2} \frac{\Gamma(k+1)}{\sqrt{\pi}} \frac{\Gamma(\nu-k)}{\Gamma(\nu/2)}.
   \]

In general, the moments are undefined when \( k \geq \nu \).

Proof: First assume that \( T \sim St(t|0, 1, \nu) \); we will find \( E(|T|^k) \). The proof for \( E(T^k) \) follows from similar ideas in combination with the result obtained in Theorem 2.1. From the equation (17) in Winkelbauer (2014), we have

\[
E(|T|^k) = \int_{-\infty}^{\infty} |t|^k N(t|0, \frac{1}{\lambda}) \, dt = \frac{1}{\lambda^{k/2}} 2^{k/2} \frac{\Gamma(k+1/2)}{\sqrt{\pi}} K\left(\frac{k}{2}, \frac{1}{2}; 0\right).
\]
Hence we have

\[ \mathbb{E}(|T|^k) = \mathbb{E}(\mathbb{E}(|T|^k|\lambda)) \]

\[ = \int_0^\infty \frac{1}{\sqrt{\pi}} 2^{k/2} \Gamma(k+1) \cdot \frac{\nu^{\nu/2}}{2^{\nu/2} \Gamma(\nu/2)} \lambda^{\nu/2-1} \exp \left( -\frac{\nu}{2} \lambda \right) d\lambda \]

\[ = \int_0^\infty \frac{1}{\sqrt{\pi}} 2^{k/2} \Gamma(k+1) \cdot \frac{\nu^{\nu/2}}{2^{\nu/2} \Gamma(\nu/2)} \lambda^{\nu/2-1} \exp \left( -\frac{\nu}{2} \lambda \right) d\lambda \]

\[ = 2^{k/2} \frac{\Gamma(k+1)}{\Gamma(\nu/2)} \cdot \frac{\nu^{\nu/2}}{2^{\nu/2} \Gamma(\nu/2)} \cdot \frac{\Gamma(\nu-k)}{\Gamma(\nu/2)} \cdot \frac{\Gamma(\nu-k)}{\Gamma(\nu/2)} \cdot \frac{\nu^{\nu/2}}{2^{\nu/2} \Gamma(\nu/2)} \]

\[ = \nu^{k/2} \Gamma((k+1)/2) \Gamma((\nu-k)/2) \frac{\Gamma(\nu-k)}{\sqrt{\pi}\Gamma(\nu/2)}, \]

where we have used the fact that \( K(-\frac{k}{2}, \frac{1}{2}; 0) = 1 \).

Next, assume that \( T \sim St(t|\mu, \sigma, \nu) \). Using the following facts (obtained in Winkelbauer (2014))

\[ \mathbb{E}((T - \mu)^k|\lambda) = \int_{-\infty}^\infty (t - \mu)^k N(t|\mu, \frac{1}{\lambda \sigma}) dt = (1 + (-1)^k) \frac{1}{\lambda^{k/2}} 2^{k/2-1} \sigma^{-k/2} \Gamma(\frac{k+1}{2}) \frac{1}{\sqrt{\pi}}, \]

and

\[ \mathbb{E}(|T - \mu|^k|\lambda) = \int_{-\infty}^\infty |t - \mu|^k N(t|\mu, \frac{1}{\lambda \sigma}) dt = \frac{\sigma^{-k/2}}{\lambda^{k/2}} 2^{k/2} \frac{\Gamma(\frac{k+1}{2})}{\sqrt{\pi}}, \]

the derivations for \( \mathbb{E}((T - \mu)^k) \) and \( \mathbb{E}(|T - \mu|^k) \) follow similarly.

When \( T \sim St(t|\mu, \sigma, \nu) \) for raw absolute moment, from the equation (17) in Winkelbauer (2014), we have

\[ \mathbb{E}(|T|^k|\lambda) = \int_{-\infty}^\infty |t|^k N(t|\mu, \frac{1}{\lambda \sigma}) dt = \frac{1}{\lambda^{k/2}} 2^{k/2} \sigma^{-k/2} \frac{\Gamma(\frac{k+1}{2})}{\sqrt{\pi}} K\left( -\frac{k}{2}, -\frac{1}{2} \cdot \frac{\mu^2}{2} \sigma \lambda \right). \]
Hence, using Part 2) of Theorem 2.1, we have for $k < \nu$

$$
\mathbb{E}(|T|^k) = \mathbb{E}(\mathbb{E}(|T|^k|\lambda))
$$

$$
= \int \frac{1}{\lambda^{k/2}} 2^{k/2} \pi^{-k/2} \frac{\Gamma(k+1)}{\sqrt{\pi}} K\left(-\frac{k}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma}\right) \text{Gamma}(\lambda|\frac{\nu}{2}, \frac{\nu}{2}) d\lambda
$$

$$
= \sigma^{-k/2} 2^{k/2} \pi^{-k/2} \frac{\Gamma(k+1)}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-k/2)^n (-\mu^2/2)^n \sigma^n}{(1/2)^n n!} \int \lambda^{n-k/2} \text{Gamma}(\lambda|\frac{\nu}{2}, \frac{\nu}{2}) d\lambda
$$

$$
= \sigma^{-k/2} 2^{k/2} \pi^{-k/2} \frac{\Gamma(k+1)}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-k/2)^n (-\mu^2/2)^n \sigma^n}{(1/2)^n n!} \frac{\Gamma(n-k/2+\nu/2)}{\Gamma(n/2)}
$$

$$
= \sigma^{-k/2} 2^{k/2} (\nu/2)^k \Gamma(k+1) \frac{\Gamma(\nu/2) - k/2}{\Gamma(\nu/2)} \sum_{n=0}^{\infty} \frac{(-k/2)^n (-\mu^2/2)^n \sigma^n}{(1/2)^n n!} \frac{\Gamma(n-k/2+\nu/2)}{\Gamma(n/2)}
$$

$$
= \frac{(\nu/\sigma)k}{2} \frac{\Gamma(k+1)}{\sqrt{\pi}} \frac{\Gamma(\nu/2) - k/2}{\Gamma(\nu/2)} 2F1\left(-\frac{k}{2}, \frac{\nu}{2} - k/2; -\frac{\mu^2\sigma}{\nu}\right).
$$

Lastly, from the equation (12) in Winkelbauer (2014), we have

$$
\mathbb{E}(T^k|\lambda) = \int_{-\infty}^{\infty} t^k N(t|\mu, \frac{1}{\lambda^2}) dt = \left\{ \begin{array}{ll}
\sigma^{-k/2} 2^{k/2} \pi^{-k/2} \frac{\Gamma(k+1)}{\sqrt{\pi}} \frac{1}{\lambda^{k/2}} K\left(-\frac{k}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma}\right) & \text{k even},\\
\mu \sigma^{-k/2} 2^{(k+1)/2} \pi^{-k/2} \frac{\Gamma(k+1)}{\sqrt{\pi}} \frac{1}{\lambda^{(k-1)/2}} K\left(\frac{1-k}{2}, \frac{3}{2}; -\frac{\mu^2}{2\sigma}\right) & \text{k odd}.
\end{array} \right.
$$

Similar to the calculation above, we have

$$
\mathbb{E}(T^k) = \left\{ \begin{array}{ll}
\frac{(\nu/\sigma)k}{2} \frac{\Gamma(k+1)}{\sqrt{\pi}} \frac{\Gamma(\nu/2) - k/2}{\Gamma(\nu/2)} 2F1\left(-\frac{k}{2}, \frac{\nu}{2} - k/2; -\frac{\mu^2\sigma}{\nu}\right) & \text{k even},\\
2\mu (\nu/\sigma)^{(k-1)/2} \frac{\Gamma(k+1)}{\sqrt{\pi}} \frac{\Gamma(\nu/2) - k}{\Gamma(\nu/2)} 2F1\left(\frac{1-k}{2}, \frac{\nu}{2} - k/2; -\frac{\mu^2\sigma}{\nu}\right) & \text{k odd}.
\end{array} \right.
$$

This completes the proof of the theorem. \(\square\)

**Remark 2.1.**

1. The formulae given in (2.5) - (2.8) are new in the literature. Also when $T \sim \text{St}(t|0,1,\nu)$, $\mathbb{E}(T^k)$ is well known. Moreover, one can directly use the definition to find $\mathbb{E}(|T|^k)$ through the class of $\beta$-functions defined in Section 6.2 of Abramowitz & Stegun (1948) and arrive at the same formula. However, this direct approach no longer works for expectations of the form $\mathbb{E}(|T|^k)$, $\mathbb{E}(T^k)$ when $T \sim \text{St}(t|\mu, \sigma, \nu)$, or for higher dimensional moments considered in Section 3. Also, clearly (2.5) is reduced to (2.3) and (2.7) is reduced to (2.4) when $\mu = 0$ and $\sigma = 1$, which shows the consistency of the method.

2. If $T \sim \text{St}(t|\mu, \sigma, \nu)$, and once $\mathbb{E}((T - \mu)^i), 0 \leq i \leq k$ have been computed, we can use them to compute $\mathbb{E}(T^k)$ for $k < \nu$ using the expansion

$$
\mathbb{E}(T^k) = \mathbb{E}((T - \mu + \mu)^k) = \sum_{i=0}^{k} \mu^{k-i} \binom{k}{i} \mathbb{E}((T - \mu)^i).
$$
3. Higher dimensional case

Now we consider the case when $n \geq 2$. Denote $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$. Denote the pdf of $n$-dimensional Normal random variable as

$$N(x|\mu, \Sigma) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)},$$

where $|\Sigma|$ is the determinant of $\Sigma$. Similar to Lemma 2.1, we have the probability density of $n$-dimensional Student’s $t$-distribution is given by

$$St(t|\mu, \Sigma, \nu) = \int_0^{\infty} N(t|\mu, (\eta \Sigma)^{-1}) \text{Gamma}(\eta \frac{\nu}{2}, \frac{\nu}{2}) d\eta,$$

where $\mu$ is called the location, $\Sigma$ is the scale matrix, and $\nu$ is the degrees of freedom parameter. It can be shown that the pdf of the $n$-dimensional multivariate Student’s $t$-distribution is given by

$$St(t|\mu, \Sigma, \nu) = \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})} \frac{|\Sigma|^{1/2}}{(\nu \pi)^{n/2}} \left( 1 + \frac{1}{\nu} (t - \mu)^T \Sigma (t - \mu) \right)^{-\frac{\nu+n}{2}}.$$

Note that in the standardized case of $\mu = 0$ and $\Sigma = I$, the representation in (3.1) is reduced to

$$St(t|0, I, \nu) = \int_0^{\infty} N(t|0, \frac{1}{\eta} I) \text{Gamma}(\eta \frac{\nu}{2}, \frac{\nu}{2}) d\eta.$$

Let $T = (T_1, T_2, \ldots, T_n)$, and $k = (k_1, k_2, \ldots, k_n)$ with $0 \leq k_i \in \mathbb{N}$. The $k$ moment of $T$ is defined as

$$E(T^k) = \int t_1^{k_1} t_2^{k_2} \ldots t_n^{k_n} \cdot St(t|\mu, \Sigma, \nu) dt_1 \ldots dt_n.$$

Similarly,

$$E(|T|^k) = \int |t_1|^{k_1} |t_2|^{k_2} \ldots |t_n|^{k_n} \cdot St(t|\mu, \Sigma, \nu) dt_1 \ldots dt_n.$$

From the authors’ best knowledge, the following is new:

**Theorem 3.1.** For $\sum k_i < \nu$, we have

1. If $T \sim St(t|0, I, \nu)$ then
   - The raw moments satisfy
     $$E(T^k) = \begin{cases} 
     0, & \text{if at least one } k_i \text{ is odd}, \\
     \nu \sum_{k_i} \frac{\Gamma(\frac{\nu-\sum k_i}{2})}{\Gamma(\frac{\nu}{2})} \frac{\prod (k_i)!}{2^{(\sum k_i)} \prod (k_i)!}, & \text{if } \sum k_i \text{ is even}.
     \end{cases}$$
   - The absolute moments satisfy
     $$E(|T|^k) = \nu \sum_{k_i} \frac{\Gamma(\frac{\nu-\sum k_i}{2})}{\Gamma(\frac{\nu}{2})} \frac{\prod (k_i) + 1}{\sqrt{\pi}}.$$

2. If $T \sim St(t|\mu, \Sigma, \nu)$, let $\Sigma^{-1} = (\sigma_{ij})$ and $e_i = (0, \ldots, 1, \ldots, 0) - \text{the } i\text{th unit vector of } \mathbb{R}^n$. Then we have the following recursive formula to compute the moments of $T$:

$$E(T^{k+e_i}) = \mu_i E(T^k) + \frac{\nu}{2} \Gamma(\frac{\nu}{2}) \sum_{j=1}^{n} \sigma_{ij} k_j E(T^{k-e_j}).$$
Similar to Theorem 1 in Kan & Robotti (2017), we have

\[ E(T^k) = \int_0^\infty E(X^k|0, \frac{1}{t} I) \Gamma(t \frac{\nu}{2}, \frac{\nu}{2}) dt, \]

where \( E(X^k|0, \frac{1}{t} I) \) is the \( k \) moment of a \( N(0, \frac{1}{t} I) \). Using Theorem 2.1, we have

\[ E(X^k|0, \frac{1}{t} I) = \prod_{i=1}^n E(X_i^{k_i}|0, \frac{1}{t}) = \begin{cases} 0, & \text{if at least one } k_i \text{ is odd,} \\ \frac{t^{-\sum k_i/2} \prod (k_i)!}{2^{(\sum k_i)/2} \prod (k_i)!}, & \text{if all } k_i \text{ are even.} \end{cases} \]

As a result,

\[ E(T^k) = \begin{cases} 0, & \text{if at least one } k_i \text{ is odd,} \\ \frac{t^{-\sum k_i/2} \prod (k_i)!}{2^{(\sum k_i)/2} \prod (k_i)!} \int_0^\infty t^{-\sum k_i/2} \Gamma(t \frac{\nu}{2}, \frac{\nu}{2}) dt, & \text{if all } k_i \text{ are even.} \end{cases} \]

Similarly, we have

\[ E(|T|^k) = \int_0^\infty E(|X_1|^{k_1} |X_2|^{k_2} \ldots |X_n|^{k_n}|0, \frac{1}{t} I) \Gamma(t \frac{\nu}{2}, \frac{\nu}{2}) dt \]

where

\[ E(|X_1|^{k_1} |X_2|^{k_2} \ldots |X_n|^{k_n}|0, \frac{1}{t} I) = \prod_{i=1}^n E(|X_i|^{k_i}|0, \frac{1}{t}) = \prod_{i=1}^n \frac{1}{\sqrt{\pi}} \frac{\Gamma(k_i/2)}{\Gamma(k_i/2)}. \]

Therefore,

\[ E(|T|^k) = 2^{\sum k_i/2} \prod \frac{\Gamma(k_i/2)}{\sqrt{\pi}} \int_0^\infty t^{-\sum k_i/2} \Gamma(t \frac{\nu}{2}, \frac{\nu}{2}) dt \]

\[ = \frac{\nu^{\sum k_i/2}}{\sqrt{\pi}} \prod \frac{\Gamma(k_i/2)}{\Gamma(k_i/2)} \text{ if } \sum k_i < \nu. \]

For 2), from (3.3)

\[ E(T^k) = \int_0^\infty E(X^k|\mu, \frac{1}{t} \Sigma^{-1}) \Gamma(t \frac{\nu}{2}, \frac{\nu}{2}) dt, \]

where \( E(X^k) \equiv E(X^k|\mu, \frac{1}{t} \Sigma^{-1}) \) is the \( k \) moment of \( N(\mu, \frac{1}{t} \Sigma^{-1}) \). Recall the pdf of \( N(\mu, \frac{1}{t} \Sigma^{-1}) \) is given by

\[ N(x|\mu, \frac{1}{t} \Sigma^{-1}) = \frac{1}{(2\pi)^{n/2} |\frac{1}{t} \Sigma^{-1}|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}. \]

Similar to Theorem 1 in Kan & Robotti (2017), we have

\[ -\frac{\partial N(x|\mu, \frac{1}{t} \Sigma^{-1})}{\partial x} = t \Sigma (x - \mu) N(x|\mu, \frac{1}{t} \Sigma^{-1}). \]
Hence
\[- \int x^k \frac{\partial N(x|\mu, \frac{1}{t} \Sigma^{-1})}{\partial x} dx = \int x^k t \Sigma(x - \mu) N(x|\mu, \frac{1}{t} \Sigma^{-1}) dx.\]

By integration by parts, we arrive at
\[\int k_j x^{k-e_j} N(x|\mu, \frac{1}{t} \Sigma^{-1}) dx = \int x^k t \Sigma(x - \mu) N(x|\mu, \frac{1}{t} \Sigma^{-1}) dx.\]

Or equivalently
\[\int x^k (x - \mu) N(x|\mu, \frac{1}{t} \Sigma^{-1}) dx = \frac{1}{t} \Sigma^{-1} \int k_j x^{k-e_j} N(x|\mu, \frac{1}{t} \Sigma^{-1}) dx.\]

This in turn implies that
\[\mathbb{E}(X^{k+e_i}) = \mu_i \mathbb{E}(X^k) + \frac{1}{t} \sum_{j=1}^n \sigma_{ij} k_j \mathbb{E}(X^{k-e_j}).\]

Plugging this into the equation (3.4), we have the following recursive equation
\[\mathbb{E}(T^{k+e_i}) = \mu_i \mathbb{E}(T^k) + \sum_{j=1}^n \sigma_{ij} k_j \mathbb{E}(T^{k-e_j}) \int_0^\infty \frac{1}{t} \Gamma(t; \frac{\nu}{2}, \frac{\nu}{2}) dt \]
\[\quad = \mu_i \mathbb{E}(T^k) + \frac{\nu}{2} \left(1 - \Gamma\left(\frac{\nu}{2}, \frac{\nu}{2}\right)\right) \sum_{j=1}^n \sigma_{ij} k_j \mathbb{E}(T^{k-e_j}).\]

This completes the proof of the theorem. \( \square \)

Lastly, for \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \), let \( a_{(j)} \) is the vector obtained from \( a \) by deleting the \( j \)th element of \( a \). For \( \Sigma = (\sigma_{ij}) \), let \( \sigma_i^2 = \sigma_{ii} \) and \( \Sigma_{(i,j)} \) stand for the \( i \)th row and \( j \)th column removed. Analogously, let \( \Sigma_{(i),(j)} \) stand for the matrix \( \Sigma \) with \( i \)th row and \( j \)th column removed.

Consider the following truncated \( k \) moment
\[F^n_k(a, b; \mu, \Sigma, \nu) = \int_a^b t^k S(x|\mu, \Lambda, \nu) dt.\]

We have
\[F^n_k(a, b; \mu, \Sigma, \nu) = \int_0^\infty \mathbb{E} \left[ 1_{\{a \leq X \leq b\}} X^k | \mu, \frac{1}{t} \Sigma^{-1} \right] \Gamma(t; \frac{\nu}{2}, \frac{\nu}{2}) dt,\]

Using Theorem 1 in Kan & Robotti (2017), we have for \( n > 1 \)
\[\mathbb{E}(X^n_{k+e_i}; a, b, \mu, \frac{1}{t} \Sigma^{-1}) := \mathbb{E} \left[ 1_{\{a \leq X \leq b\}} X^n_k | \mu, \frac{1}{t} \Sigma^{-1} \right] \]
\[\quad = \mu_i \mathbb{E} \left[ 1_{\{a \leq X \leq b\}} X^n_k | \mu, \frac{1}{t} \Sigma^{-1} \right] + \frac{1}{t} e_i^\top \Sigma^{-1} c_k,\]

and \( c_k \) satisfies
\[c_{k,j} = k_j \mathbb{E}(X^n_{k-e_i}; a, b, \mu, \frac{1}{t} \Sigma^{-1}) + a_{j} k_j N(a_j | \mu_j, \sigma_j^2) \mathbb{E}(X^n_{k-j}; a_{(j)}, b_{(j)}, \mu_j^a, \frac{1}{t} \Sigma^{-1}) \]
\[- b_{j} k_j N(b_j | \mu_j, \sigma_j^2) \mathbb{E}(X^{n-1}_{k-j}; a_{(j)}, b_{(j)}, \mu_j^b, \frac{1}{t} \Sigma^{-1})\]
with

\[
\begin{align*}
\hat{\mu}_j^a &= \mu_j + \Sigma_{(j),j}^{-1} a_j - \mu_j,
\hat{\mu}_j^b &= \mu_j + \Sigma_{(j),j}^{-1} b_j - \mu_j,
\hat{\Sigma}^{-1} &= \Sigma_{(j),j}^{-1} - \frac{1}{\sigma_j^2} \Sigma_{(j),j}^{-1} \Sigma_{(j),j}^{-1}.
\end{align*}
\]

(3.7)

Thus, we have the following recursive formula

\[
F_{n+k} \left( a, b; \mu, \Sigma, \nu \right) = \mu_i F_{n+k} \left( a, b; \mu, \Sigma, \nu \right) + \frac{\nu}{\nu - 2} e_i^\top \Sigma^{-1} d_k,
\]

where

\[
d_{k,j} = k_j F_{k+e_i} \left( a, b; \mu, \Sigma, \nu \right) + a_j k_j N(a_j | \mu_j, \sigma_j^2) F_{k} \left( a_j, \Sigma_{(j),j}^{-1} \right) \left( a_j, \Sigma_{(j),j}^{-1} \mu_j, \Sigma_{(j),j}^{-1} \right)
\]

\[
- b_j k_j N(b_j | \mu_j, \sigma_j^2) F_{k} \left( a_j, \Sigma_{(j),j}^{-1} \right) \left( b_j, \Sigma_{(j),j}^{-1} \mu_j, \Sigma_{(j),j}^{-1} \right).
\]

Note by convention that the first term, second term, and third term in the expression of \(d_{k,j}\) equal 0 when \(k_j = 0, a_j = \infty, b_j = -\infty\) respectively.

4. Conclusion

We derive the closed form formulae for the raw moments, absolute moments, and central moments of Student’s t-distribution with arbitrary degrees of freedom. We provide results in one and \(n\)-dimensions, which unify and extend the existing literature for the Student’s t-distribution. It would be interesting to investigate tail quantile approximation or asymptotic tail properties of higher (generalized) Student’s T-distribution as done in Schlüter & Fischer (2012) and Finner et al. (2008). We leave it as an interesting project for future studies.
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