Equilibrium under TWAP trading with quadratic transaction costs

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Abstract

We study how transaction cost affects to the equilibrium return and optimal stock holdings in equilibrium. To this end, we develop a continuous-time risk-sharing model where heterogeneous agents trade toward terminal target holdings subject to a quadratic transaction cost. The equilibrium stock holdings and trading rate under transaction cost are characterized by a unique solution to a forward-backward stochastic differential equation (FBSDE). The equilibrium return is also characterized as the unique solution of a system of coupled but linear FBSDEs.

Keywords: Equilibrium, Transaction costs, Forward–backward SDE, Targeted-trading, TWAP.

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1 Introduction

We study equilibrium asset pricing models to see the impact of illiquidity on asset returns. That is, the returns are not modeled as exogenous inputs, but determined endogenously by matching supply and demand. This in turn allows us to study how the price characteristics depend on the market’s liquidity.

The analysis of equilibrium models with frictions, however, is notoriously difficult to study and intractable. These difficulties are obviously compounded when equilibrium asset prices are endogenously determined in the presence of transaction costs. As a consequence, most of the existing literature on equilibrium asset pricing model with transaction costs has focused on numerical approaches or on models with very particular simplifying assumptions. For example, Buss and Dumas (2019) [3] and Buss et.al. (2015) [4] propose algorithms for the numerical approximation of equilibrium dynamics in discrete-time, finite-state models. Lo et.al. (2004) [6] study the numerics behind transaction cost equilibria where trading volume fluctuates randomly, but the corresponding equilibrium prices are constant. Vayanos and Vila (1999) [9] obtain explicit formulas in continuous-time models, but focus on settings with deterministic asset prices for tractability. Kim [10] proves the existence of a transaction cost equilibrium in a continuous-time model with deterministic equilibrium annuity prices for tractability. Noh and Kim [7] consider a market with two agents who are incentivized to trade towards a target, and prove the existence of an equilibrium in a model with transaction costs and price impact.

We study a continuous-time multi-agent equilibrium model with quadratic transaction cost. Each agent has their own target holdings at terminal time with one’s own risk aversion coefficients, and this model set-up is most similar to Choi et. al. [5]. In equilibrium, the agents seek to maximize their discounted expected wealth minus both transaction costs and a penalty for deviating from their targets. Then, both optimal trading strategies and stock return in equilibrium that clears the market can be characterized by a unique solution to the system of coupled but linear FBSDEs.

If all agents share the same risk aversion constant, then one can obtain the explicit closed-form expression for the optimal stock holdings as the solution of an ODE. Moreover, if asset supply from noise traders is constant over time, then the frictionless equilibrium returns still clear the market. If we consider a market with no noise traders and two heterogenous strategic agents, then the optimal stock holdings are again explicitly obtained as the solution to an ODE. We can see that trading is more depressed with bigger transaction cost parameter in a simple example.

This paper is organized as follows. Section 2 describes the model. In Section 3, we obtain optimal trading strategy and equilibrium return in frictionless market. Also, optimal trading
strategy in a market with friction is characterized. Section 4 contains results on equilibrium return in a market with transaction cost. In Section 5, special cases are handled with illustration to facilitate understanding. Appendix contains the proof of existence and uniqueness for linear FBSDEs that are used in Section 3 and 5.

2 Model

Throughout, we fix a probability space $(\Omega,\mathcal{F},P)$, and the finite time horizon $T < \infty$. We fix a constant $\delta \geq 0$ and say that an $\mathbb{R}^d$-valued progressively measurable process $(X_t)_{t \in [0,T]}$ belongs to $\mathcal{L}_\delta^p$ for $p \geq 1$ if $\mathbb{E}[\int_0^T e^{-\delta t}||X_t||^p dt] < \infty$, where $|| \cdot ||$ is any norm on $\mathbb{R}^d$.

We consider $N$ agents in a market with a single risky asset (stock). For agent $i$, we denote the trading strategy $\theta_{i,t}$ as the number of shares held at time $t$, and $\dot{\theta}_{i,t}$ as the rate of trading. Let $\theta_{i,-}$ denote the investor’s initial stock holdings. We denote $a_i$ as the terminal target stock holdings for investor $i$, and $a_\Sigma = \sum_{i=1}^N a_i$. So, $a_i - \theta_{i,-}$ is the total amount of stock that investor $i$ eventually wants to trade by the end of the day, and $\theta_{i,t} - \theta_{i,-}$ is the cumulative amount of stock that investor $i$ actually traded by time $t$. We assume $a_i \in \mathcal{L}_\delta^2$ and $\theta_{i,-} \in \mathcal{L}_\delta^2$.

The penalty process for agent $i$ is

$$L_{i,t} = \frac{1}{2} \int_0^t \kappa_i \left( \gamma(s)(a_i - \theta_{i,-}) - (\theta_{i,s} - \theta_{i,-}) \right)^2 ds,$$

where $\kappa_i > 0$ determines the severity of the penalty for each agent, which can also be interpreted as agent $i$’s risk aversion. Here, the target trading trajectory at a generic time $t$ is $\gamma(t)(a_i - \theta_{i,-})$, where $\gamma : [0,T] \to [0,1]$ is assumed to be a càdlàg, nonnegative, non-decreasing deterministic function with $\int_0^T \gamma^2(s)ds < \infty$ for all $t \in [0,T]$. A simple TWAP target has $\gamma_{\text{TWAP}}(t) := \frac{t}{T}$.

Traders, however, sometimes intentionally deviate from their target trajectories to achieve trading profits and price improvement. In this case, $\gamma(t)$ would be different from $\gamma_{\text{TWAP}}(t)$. For notational simplicity, let us denote

$$\varphi_i(t) := \gamma(t)(a_i - \theta_{i,-}) + \theta_{i,-}.$$

Then, we can rewrite the penalty process as

$$L_{i,t} = \frac{1}{2} \int_0^t \kappa_i \left( \varphi_i(s) - \theta_{i,-} \right)^2 ds. \quad (2.1)$$

The stock price process $S$ has dynamics

$$dS_t = \mu_t dt + dW_t, \quad S_0 \in \mathbb{R}$$

where the expected return process $(\mu_t)_{t \in [0,T]} \in \mathcal{L}_\delta^2$ is to be determined from equilibrium, and $\{W_t\}_{t \in [0,T]}$ is standard Brownian motion.
Let $\lambda \geq 0$ be the transaction cost parameter. We assume that trades incur costs proportional to the square of the order flow $\dot{\theta}_{i,t} = \frac{d}{dt} \theta_{i,t}$. When the investor $i$ holds $\theta_{i,t}$ shares of stock at hands at time $t$, the agent $i$’s wealth process $\{X_{i,t}^\theta\}_{t \in [0,T]}$ has the following dynamics

$$dX_{i,t}^\theta = \theta_{i,t} dS_t - \lambda \dot{\theta}_{i,t}^2 dt, \quad X_{i,0} \in \mathbb{R},$$

where $X_{i,0}$ is agent $i$’s initial wealth.

The goal of agent $i$ is to choose a trading strategy $\theta_i \in L^2_\delta$ to maximize discounted expected changes of wealth penalized for the deviation from target:

$$J^\theta_i(\theta) = \mathbb{E} \left[ \int_0^T e^{-\delta t} \left( dX_{i,t}^\theta - dL_{i,t} \right) \right] \to \max$$

(2.2)

We assume that there are noise traders whose trading motives are exogenous. Let $w \in L^2_\delta$ denote the noise traders’ supply of the risky assets that the strategic agents must absorb at time $t \in [0,T]$. That is, the stock market clears at time $t$ when agents’ stock holding $\{\theta_{i,t}\}_{i=1}^N$ satisfies

$$\sum_{i=1}^N \theta_{i,t} = w_t, \quad t \in [0,T].$$

### 3 Equilibrium Strategies

In an equilibrium, the stock price is determined so that markets clear when strategic agents invest optimally.

**Definition 3.1.** We say that trading strategies $\theta_i$, $i = 1, 2, \cdots, N$, form an equilibrium if

a. Strategies are optimal: $\theta_i$ solves the maximization problem (2.2) for each $i$.

b. Markets clear: We have $\sum_{i=1}^N \theta_{i,t} = w_t, \quad t \in [0,T]$.

#### 3.1 Frictionless Market

In a frictionless market (i.e. $\lambda = 0$), agent $i$ solves

$$J^i(\theta) = \mathbb{E} \left[ \int_0^T e^{-\delta t} \left( - \frac{1}{2} \kappa_i \theta_{i,t}^2 + \left( \kappa_i \varphi_i(t) + \mu_i \right) \theta_{i,t} - \frac{1}{2} \kappa_i (\varphi_i(t))^2 \right) dt \right] \to \max$$

By pointwise optimization, we have optimal strategies as, for $i = 1, 2, \cdots, N$,

$$\theta_{i,t} = \frac{1}{\kappa_i} \mu_t + \varphi_i(t) = \frac{1}{\kappa_i} \mu_t + \gamma(t)(a_i - \theta_{i,-}) + \theta_{i,-}. \quad (3.1)$$

By the market clearing condition, the frictionless equilibrium expected return is

$$\mu_t = \frac{1}{\sum_{m=1}^N 1/\kappa_m} \left( w_t - w_0 - \gamma(t)(a_\Sigma - w_0) \right), \quad t \in [0,T]. \quad (3.2)$$
3.2 Individual Optimality with Transaction Cost

Now, we consider each agent’s individual optimization problem with transaction cost (i.e. \( \lambda > 0 \)), where agent \( i \) solves (2.2). With frictionless optimizer \( \theta_{i,t} \) in (3.1), we have the following result.

**Lemma 3.2.** The frictional optimization problem (2.2) for agent \( i \) has a unique solution \( \theta^\lambda_{i,t} \), characterized by the FBSDE

\[
\begin{aligned}
&d\theta^\lambda_{i,t} = \dot{\theta}^\lambda_{i,t} dt, & \theta^\lambda_{i,0} = \theta_{i,-}; \\
&d\dot{\theta}^\lambda_{i,t} = dM_{i,t} + \kappa_i (\theta^\lambda_{i,t} - \theta_{i,t}) dt + \delta \dot{\theta}^\lambda_{i,t} dt, & \dot{\theta}^\lambda_{i,T} = 0,
\end{aligned}
\]

(3.3)

where \( M_{i,t} \) is a square-integrable martingale to be determined as part of the solution.

**Proof.** First of all, it is shown in Appendix that the FBSDE (3.3) indeed has a unique solution \( (\theta^\lambda_{i,t}, \dot{\theta}^\lambda_{i,t}, M_{i,t}) \).

Since our functional (2.2) is strictly convex, (2.2) has a unique solution if and only if there exists a (unique) solution to

\[
\langle J^\ddot{\theta} (\dot{\theta}^\lambda), \dot{\phi} \rangle = 0, \quad \text{for all } \phi, \dot{\phi} \in \mathcal{L}_d^2.
\]

(3.4)

The Gateaux derivative of \( J^\ddot{\theta} \) in the direction \( \dot{\phi} \) is given by

\[
\langle J^\ddot{\theta} (\dot{\theta}^\lambda), \dot{\phi} \rangle = \lim_{\rho \to 0} \frac{J^\ddot{\theta} (\dot{\theta}^\lambda + \rho \dot{\phi}) - J^\ddot{\theta} (\dot{\theta}^\lambda)}{\rho}.
\]

(3.5)

By Fubini’s theorem,

\[
\int_0^T e^{-\delta t} A_t \left( \int_0^t \phi_{i,u} du \right) dt = \int_0^T \left( \int_t^T e^{-\delta u} A_u du \right) \dot{\phi}_{i,t} dt.
\]

So, the equation (3.5) becomes

\[
\langle J^\ddot{\theta} (\dot{\theta}^\lambda), \dot{\phi} \rangle = \mathbb{E} \left[ \int_0^T \left( \int_t^T e^{-\delta u} A_u du - 2 e^{-\delta t} \lambda \dot{\theta}^\lambda_{i,t} \right) \dot{\phi}_{i,t} dt \right].
\]

Now, by the tower property, the first-order condition (3.4) can be written as

\[
\mathbb{E} \left[ \int_0^T \left( \mathbb{E} \left[ \int_t^T e^{-\delta u} A_u du \mid \mathcal{F}_t \right] - 2 e^{-\delta t} \lambda \dot{\theta}^\lambda_{i,t} \right) \dot{\phi}_{i,t} dt \right] = 0.
\]
Since it has to hold for any $\dot{\lambda} \in \mathcal{L}_2^\delta$, (2.2) has a (unique) solution $\dot{\theta}_i^\lambda$ if and only if
\begin{equation}
\dot{\theta}_{i,t}^\lambda = \frac{1}{2\lambda} e^{\delta t} \mathbb{E} \left[ \int_t^T e^{-\delta s} \left( \mu_s + \kappa_i (\varphi_i(s) - \theta_{i,s}^\lambda) \right) ds \mid \mathcal{F}_t \right] \tag{3.6}
\end{equation}
has a unique solution. Notice that (3.6) can be written as
\begin{equation}
\dot{\theta}_{i,t}^\lambda = e^{\delta t} \kappa_i \frac{2\lambda}{\mathbb{E} \left[ \int_0^T e^{-\delta s} (\theta_{i,s}^\lambda - \theta_{i,s}) ds \mid \mathcal{F}_t \right]}. \tag{3.7}
\end{equation}

Now, assume that (3.7) has a (unique) solution $\dot{\theta}_i^\lambda$, with terminal condition $\dot{\theta}_{i,T}^\lambda = 0$. Define the square-integrable martingale \( \tilde{M}_{i,t} \) by
\begin{equation}
\tilde{M}_{i,t} = \frac{\kappa_i}{2\lambda} \mathbb{E} \left[ \int_0^T e^{-\delta s} (\theta_{i,s}^\lambda - \theta_{i,s}) ds \mid \mathcal{F}_t \right].
\end{equation}
Then, we have
\begin{equation}
d\dot{\theta}_i^\lambda = e^{\delta t} d\tilde{M}_{i,t} - \frac{\kappa_i}{2\lambda} (\theta_{i,t} - \theta_{i,t}^\lambda) dt + \delta \dot{\theta}_{i,t}^\lambda dt.
\end{equation}
Together with the definition $d\theta_i^\lambda = \dot{\theta}_{i,t}^\lambda dt$, it yields the claimed FBSDE representation (3.3).

Conversely, assume that (3.3) has a (unique) solution $(\theta_i^\lambda, \dot{\theta}_i^\lambda, M_i)$. First, notice that we have
\begin{equation}
e^{-\delta t} \dot{\theta}_i^\lambda = \dot{\theta}_{i,0}^\lambda + \int_0^t e^{-\delta s} \kappa_i \frac{2\lambda}{\mathbb{E} \left[ \int_0^T e^{-\delta s} (\theta_{i,s}^\lambda - \theta_{i,s}) ds \mid \mathcal{F}_t \right]} (\theta_{i,s}^\lambda - \theta_{i,s}) ds + \int_0^t e^{-\delta s} dM_{i,s}, \quad t \in [0, T] \tag{3.8}
\end{equation}
And, one can easily see
\begin{equation}
\dot{\theta}_{i,0}^\lambda = -\int_0^T e^{-\delta s} \kappa_i \frac{2\lambda}{\mathbb{E} \left[ \int_0^T e^{-\delta s} (\theta_{i,s}^\lambda - \theta_{i,s}) ds \mid \mathcal{F}_t \right]} (\theta_{i,s}^\lambda - \theta_{i,s}) ds - \int_0^T e^{-\delta s} dM_{i,s}. \tag{3.9}
\end{equation}
We insert (3.9) into (3.8), and take conditional expectation to get (3.7).

## 4 Equilibrium Return

We now use the above FBSDE characterization (3.3) to determine the equilibrium return $(\mu_i^\lambda)_{t \in [0, T]}$. Since each trader’s trading rate is constrained to be absolutely continuous, we assume the same for the exogenous stock supply:
\begin{align*}
dw_t &= \dot{w}_t dt, \\
d\dot{w}_t &= \mu_i^w dt + dM_i^w,
\end{align*}
where $\mu^w \in \mathcal{L}_2^\delta$ and $M^w$ is a local martingale. We also assume that $w, \dot{w} \in \mathcal{L}_2^\delta$. 
4.1 Equilibrium Return

**Theorem 4.1.** The unique frictional equilibrium return is given by

\[
\mu_t^\lambda = \frac{1}{N} \sum_{i=1}^{N-1} (\kappa_i - \kappa_N) \theta_i^{\lambda,t} - \frac{1}{N} \sum_{i=1}^{N} \kappa_i \varphi_i(t) + \frac{1}{N} \kappa_N \bar{w}_t + \frac{2\lambda}{N} (\delta \hat{w}_t - \mu_t^w). \tag{4.1}
\]

**Proof.** The market clearing condition is \(\sum_{i=1}^{N} \theta_i^{\lambda,t} = w_t\), or equivalently, \(\sum_{i=1}^{N} \dot{\theta}_i^{\lambda,t} = \dot{w}_t\). By the FBSDE (3.3), we have

\[
dM_t + \sum_{i=1}^{N} \frac{\kappa_i}{2\lambda} \left( \theta_i^{\lambda,t} - \left\{ \frac{1}{\kappa_i} \mu_i^\lambda + \varphi_i(t) \right\} \right) dt + \sum_{i=1}^{N} \delta \dot{\theta}_i^{\lambda,t} dt = \mu_t^w dt + dM_t^w
\]

for a local martingale \(M_t\).

Note that \(\sum_{i=1}^{N} a_i = a_\Sigma\) and \(\sum_{i=1}^{N} \theta_i,\_ = w_0\). Since any continuous local martingale of finite variation is constant, we have

\[
\mu_t^\lambda = \frac{1}{N} \sum_{i=1}^{N} \frac{\kappa_i}{N} \left( \theta_i^{\lambda,t} - \varphi_i(t) \right) + \frac{2\lambda}{N} (\delta \hat{w}_t - \mu_t^w).
\]

By using the clearing conditions, we can rewrite \(\mu_t^\lambda\) as

\[
\mu_t^\lambda = \frac{1}{N} \sum_{i=1}^{N-1} (\kappa_i - \kappa_N) \theta_i^{\lambda,t} - \frac{1}{N} \sum_{i=1}^{N} \kappa_i \varphi_i(t) + \frac{1}{N} \kappa_N \bar{w}_t + \frac{2\lambda}{N} (\delta \hat{w}_t - \mu_t^w). \tag{4.2}
\]

Plugging (4.2) back into the individual’s optimality condition (3.3), we deduce that

\[
d\theta_i^{\lambda,t} = \dot{\theta}_i^{\lambda,t} dt, \quad \theta_i^{\lambda,0} = \theta_i,\_;
\]

\[
d\dot{\theta}_i^{\lambda,t} = dM_i,t + \frac{1}{2\lambda} \left( \kappa_i \theta_i^{\lambda,t} + \sum_{j=1}^{N} \frac{\kappa_N - \kappa_j}{N} \theta_j^{\lambda,t} - \kappa_i \varphi_i(t) + \sum_{j=1}^{N} \frac{\kappa_j}{N} \varphi_j(t) \right) dt
\]

\[
+ \frac{1}{N} \left( \mu_t^w - \frac{\kappa_N}{2\lambda} \bar{w}_t - \delta \hat{w}_t \right) dt + \delta \dot{\theta}_i^{\lambda,t} dt, \quad \dot{\theta}_i^{\lambda,T} = 0
\]

for \(i = 1, \cdots, N - 1\). By Lemma 4.2 below, we see that \((\theta_1^{\lambda,t}, \cdots, \theta_{N-1}^{\lambda,t}, \dot{\theta}_1^{\lambda,t}, \cdots, \dot{\theta}_{N-1}^{\lambda,t})\) solves this FBSDE. So, we can say that if an equilibrium exists, it has to be of the form (4.1).

To verify that the proposed return process and trading strategies indeed form an equilibrium, we reverse the above arguments. The market clearing condition holds by the definition of \(\theta_i\). So, it remains to show that \(\theta_i\) and \(\dot{\theta}_i\) are indeed optimal for investor \(i = 1, \cdots, N\). It suffices to show that the individual optimality condition (3.6) is satisfied. If we plug in \(\mu_t^\lambda\) in (4.1) to (3.6), we can see that it is the same as the respective equation in (4.3). This completes the proof.

The following lemma shows the existence and uniqueness of solution to an FBSDE.
Lemma 4.2. There exists a unique solution

\[(\theta_{i,t}^\lambda, \dot{\theta}_{i,t}^\lambda) = (\theta_{1,t}^\lambda, \cdots, \theta_{N-1,t}^\lambda, \dot{\theta}_{1,t}^\lambda, \cdots, \dot{\theta}_{N-1,t}^\lambda)\]

of the FBSDE

\[
d\theta_{i,t}^\lambda = \dot{\theta}_{i,t}^\lambda dt, \quad \theta_{i,0}^\lambda = \theta_{i,-};

d\dot{\theta}_{i,t}^\lambda = M_t + \left(\frac{A}{2\lambda} \theta_{i,t}^\lambda + \delta \dot{\theta}_{i,t}^\lambda + \frac{B}{2\lambda} \varphi_i(t) + \eta_t\right) dt, \quad \dot{\theta}_{i,T}^\lambda = 0,
\]

where \((M_t)_{t \in [0,T]}\) is an \(\mathbb{R}^{N-1}\)-valued martingale with finite second moments, and \((N-1) \times (N-1)\) matrix \(A\) and \((N-1) \times N\) matrix \(B\) are given as

\[
A = \begin{bmatrix}
\kappa_1 + \frac{\kappa_{N-1}}{N} & \cdots & \frac{\kappa_{N-1}}{N} \\
\vdots & \ddots & \vdots \\
\frac{\kappa_{N-1}}{N} & \cdots & \kappa_N + \frac{\kappa_{N-1}}{N}
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
\frac{\kappa_1}{N} - \kappa_1 & \cdots & \frac{\kappa_{N-1}}{N} & \frac{\kappa_N}{N} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\kappa_1}{N} & \cdots & \frac{\kappa_{N-1}}{N} - \kappa_N & \frac{\kappa_N}{N}
\end{bmatrix};
\]

and the \(\mathbb{R}^{N-1}\)-valued process \((\eta_t)_{t \in [0,T]}\) is

\[
\eta_t = \frac{1}{N} \left(\mu_t^w - \frac{\kappa_N}{2\lambda} w_t - \delta \dot{w}_t\right) \left[1, 1, \ldots, 1\right]^T.
\]

Proof. The eigenvalues of the matrix \(A\) are all real and positive, and \(A\) is invertible, by Lemma A.5 in [2]. Also, \(\eta \in L^2_\mathcal{Q}\). Therefore, this Lemma holds by Theorem A.4 in [2].

4.2 Liquidity Premium

We define ‘liquidity premium’ as the difference between the frictional equilibrium return in (4.1) and their frictionless counterparts (3.2). To this end, let us denote \(\bar{\theta}_i\) as the frictionless optimal strategy from (3.1) with the frictionless equilibrium return (3.2):

\[
\bar{\theta}_{i,t} = \frac{1}{\sum_{m=1}^{N} 1/\kappa_m} \frac{1}{\kappa_i} \left(w_t - w_0 - \gamma(t)(a_{\Sigma} - w_0)\right) + \varphi_i(t).
\]

Then, the frictionless equilibrium return in (3.2) can be written as

\[
\mu_t = \sum_{i=1}^{N} \frac{\kappa_i}{N} \left(\bar{\theta}_{i,t} - \varphi_i(t)\right).
\]
We can also notice that clearing condition $\sum_{i=1}^{N} \bar{\theta}_{i,t} = w_t$ holds. So, we have the following expression for the liquidity premium $(LP)_t := \mu_t^\lambda - \mu_t$

$$
(\lambda_i)_{t} := \mu_t^\lambda - \mu_t \\
= \sum_{i=1}^{N} \frac{\kappa_i}{N} \left( \theta_{i,t}^\lambda - \varphi_i(t) \right) + \frac{2\lambda}{N} (\delta \dot{w}_t - \mu_t^w) - \sum_{i=1}^{N} \frac{\kappa_i}{N} \left( \bar{\theta}_{i,t} - \varphi_i(t) \right) \\
= \sum_{i=1}^{N} \frac{\kappa_i}{N} \left( \theta_{i,t}^\lambda - \bar{\theta}_{i,t} \right) + \frac{2\lambda}{N} (\delta \dot{w}_t - \mu_t^w) \\
= \sum_{i=1}^{N} \frac{1}{N} (\kappa_i - \bar{\kappa}) (\theta_{i,t}^\lambda - \bar{\theta}_{i,t}) + \frac{2\lambda}{N} (\delta \dot{w}_t - \mu_t^w),
$$

(4.5)

where $\bar{\kappa} = \frac{1}{N} \sum_{i=1}^{N} \kappa_i$.

To have an interpretation of liquidity premium, let us assume there are no noise traders ($w \equiv 0$). Then, we can see that the liquidity premium is positive if the more risk-averse agents hold larger risky positions than in the frictionless equilibrium, or the less risk-averse agents hold smaller risky positions than in the frictionless equilibrium. We can see it more clearly in Section 5 under simplified settings.

Now consider a market with homogenous agents in the sense that $\kappa_i = \kappa$ for all $i = 1, 2, \cdots, N$ and for some $\kappa > 0$. Then we have the following result for the equilibrium liquidity premium.

**Corollary 4.3.** If all strategic agents share the same constant $\kappa$, then

$$(LP)_t = \frac{2\lambda}{N} (\delta \dot{w}_t - \mu_t^w), \quad t \in [0, T].$$

Moreover, if there are no noise traders ($w \equiv 0$), then there is no liquidity premium. That is to say, the frictionless equilibrium returns also clear the market with transaction costs.

### 5 Illustration: Market with Two Agents

In this section, we present simple deterministic examples to facilitate understanding optimal stock holdings and stock return in equilibrium, assuming that there are only two strategic agents and no noise traders.

Let us assume that two agents have $\kappa_1 < \kappa_2$. In case of frictionless market, we have the following from (3.1), (3.2), and market clearing condition:

$$
\theta_{1,t} = -\frac{\kappa_2}{\kappa_1} \gamma(t) a_{\Sigma} + \gamma(t) (a_1 - \theta_{1,-}) + \theta_{1,-}, \quad \theta_{2,t} = -\theta_{1,t};
$$

$$
\mu_t = -\frac{\kappa_1 \kappa_2}{\kappa_1 \Sigma} \gamma(t) a_{\Sigma}.
$$
When there is positive transaction cost, the equilibrium return is
\[ \mu_t^\lambda = \frac{1}{2} (\kappa_1 - \kappa_2) \theta_{1,t}^\lambda - \frac{1}{2} (\kappa_1 \varphi_1(t) + \kappa_2 \varphi_2(t)), \]
and the optimal stock holding for agent 1 satisfies the following FBSDE:
\[
\begin{cases}
    d \theta_{1,t}^\lambda = \dot{\theta}_{1,t}^\lambda dt, & \theta_{1,0} = \theta_{1,-} \\
    d \dot{\theta}_{1,t}^\lambda = dM_{1,t} + \frac{\kappa_1}{2\lambda} \left( \theta_{1,t}^\lambda - \left( \varphi_1(t) - \frac{\kappa_2}{\kappa_1 + \kappa_2} \gamma(t)a_\Sigma \right) \right) dt + \delta \dot{\theta}_{1,t}^\lambda dt, & \dot{\theta}_{1,T}^\lambda = 0.
\end{cases}
\]
This is FBSDE (A.1) with
\[ \alpha = \frac{\kappa_1}{2\lambda}, \quad \beta = \alpha + \frac{1}{4} \delta^2, \quad \xi_t = \varphi_1(t) - \frac{\kappa_2}{\kappa_1 + \kappa_2} \gamma(t)a_\Sigma. \]
So, by Theorem A.1, the optimal stock holding for agent 1 is explicitly given by (A.2).

To illustrate the result, first notice that if \( a_1, a_2 \) are deterministic, then so are \( \xi \) and \( \psi \). So, in Figure 1, we use the parametric values:
\[ a_1 = 0.25, \quad a_2 = -0.2, \quad \kappa_1 = 0.2, \quad \kappa_2 = 0.5, \quad \delta = 0, \quad T = 1, \quad \theta_{1,-} = 0. \]

![Figure 1: Optimal stock holdings for agent 1 in a frictionless market and with transaction cost parameter \( \lambda = 0.1 \) and \( \lambda = 0.5 \) (left), and liquidity premium in this case (right).](image)

That is, we assume that agent 1 is a net buyer and agent 2 a net seller, and agent 2 is more risk averse. Also, we assume both agents’ initial stock holdings are zero. Since \( a_1 + a_2 > 0 \) and \( \kappa_1 < \kappa_2 \), one can expect positive liquidity premium to clear the market (right figure). That is, it
corresponds to the case that the less risk-averse agent (i.e. agent 1) holds less amount of stock compared to the frictionless case, which makes liquidity premium positive. Also, from the left figure, one can see the effect of transaction cost to the optimal number of stock holdings. Here we use the simple TWAP target ratio function $\gamma^{TWAP}(t) = t$.

As a special case, we can consider a market with homogenous agents; that is, $\kappa_1 = \kappa_2 = \kappa$. Then, simple algebra in (A.5) shows that the optimal strategy $\hat{\theta}^{\lambda}_{i,t}$ satisfies the linear ODE

$$d\hat{\theta}^{\lambda}_{i,t} = \sqrt{\beta} \tanh(\tau^\beta(t))(\hat{\theta}_{i,t} - \hat{\theta}^{\lambda}_{i,t}) dt,$$

where, for $t \in [0, T)$, we let

$$\tau^\beta(t) \triangleq \sqrt{\beta}(T - t), \quad \hat{\theta}_{i,t} \triangleq \mathbb{E}\left[ \int_t^T \theta_{i,u} K(t,u) du \mid \mathcal{F}_t \right],$$

with the kernel

$$K(t,u) \triangleq \frac{\sqrt{\beta} \cosh(\tau^\beta(u))}{\sinh(\tau^\beta(t))}, \quad 0 \leq t \leq u < T.$$

Note that rather than towards the target position $a_i$, the optimal frictional rules prescribe to trade towards weighted average $\hat{\theta}_{i,t}$ of expected future frictionless position $\theta_{i,.}$. Indeed, for each $t \in [0, T]$, the kernel $K(t, \cdot)$ specifies non-negative kernel which integrates to 1 over $[t, T]$, so $\hat{\theta}$ averages out the expected future position of $\theta$.

To demonstrate the result, we use the same parametric values as those used in Figure 1, except $\kappa$ values; here we use $\kappa_1 = \kappa_2 = 0.4$.

![Figure 2: Optimal stock holdings for agent 1 is plotted. Each plot corresponds to a different level of transaction cost $\lambda$ when $\kappa_1 = \kappa_2 = 0.4.$](image)

Figure 2 shows the optimal stock holdings for agent 1 in frictionless and frictional market. In a frictionless market, optimal strategy is linear in time because so is target ratio function $\gamma$. If
\( \lambda > 0 \), then transaction cost suppresses motive to trade, so agent holds smaller amount than in frictionless market.

\section*{A Appendix}

In this Appendix, the existence and uniqueness of solution to the following FBSDE is proved:

\[
\begin{aligned}
    d\phi_t &= \dot{\phi} dt, \quad \phi_0 = \phi_-
    \\
    d\dot{\phi} &= dM_t + \alpha(\phi_t - \xi_t) dt + \delta \dot{\phi} dt, \quad \dot{\phi}_T = 0,
\end{aligned}
\]  

(A.1)

where \( \alpha > 0 \) a constant, \( \delta \geq 0 \), and \( \xi \in L^2_\delta \).

To this end, denote

\[
\beta = \alpha + \frac{1}{4}\delta^2, \quad F(t) = \cosh(\sqrt{\beta}(T-t)), \quad F'(t) = -\sqrt{\beta} \sinh(\sqrt{\beta}(T-t)).
\]

\[G(t) = -\frac{\alpha F'(t)}{\beta F(t) - \delta^2 F'(t)}, \quad \psi_t = \frac{1}{\beta F(t) - \delta^2 F'(t)} \mathbb{E}\left[ \int_t^T \left( \beta F(s) - \frac{\delta}{2} F'(s) \right) \alpha e^{-\frac{\delta}{2}(s-t)} \xi_s ds \mid \mathcal{F}_t \right].\]

\[\phi_t = e^{-\int_0^t G(s) ds} \phi_- + \int_0^t e^{-\int_s^t G(u) du} \psi_s ds,
\]

(A.2)

\textbf{Theorem A.1.} The FBSDE (A.1) has the unique solution

Proof. Firstly, let us prove that if a solution of the FBSDE (A.1) exists, then it is of the form (A.2). Let \((\phi_t, \dot{\phi}_t)\) be a solution to FBSDE (A.1), and set \(\bar{\phi}_t := e^{-\frac{\delta}{2}t} \phi_t\). Using

\[
\dot{\bar{\phi}}_t = -\frac{\delta}{2} \bar{\phi}_t + e^{-\frac{\delta}{2}t} \dot{\phi}_t, \quad d\dot{\bar{\phi}}_t = \frac{\delta}{2} \phi dt + e^{-\frac{\delta}{2}t} d\phi_t - \frac{\delta}{2} e^{-\frac{\delta}{2}t} \phi_t dt,
\]

(A.3)

one can see that \((\bar{\phi}, \dot{\bar{\phi}})\) solves the FBSDE

\[
\begin{aligned}
    d\bar{\phi}_t &= \dot{\bar{\phi}} dt, \quad \bar{\phi}_0 = \phi_-;
    \\
    d\dot{\bar{\phi}}_t &= dM_t + \alpha(\bar{\phi}_t - \bar{\xi}_t) dt + \delta \dot{\bar{\phi}}_t dt, \quad \dot{\bar{\phi}}_T = -\frac{\delta}{2} \bar{\phi}_T.
\end{aligned}
\]

Here, \(d\bar{M}_t = e^{-\frac{\delta}{2}t} dM_t\) is a square-integrable martingale and \(\bar{\xi}_t = e^{-\frac{\delta}{2}t} \xi_t\). With matrix notation, this FBSDE can be written as

\[
d \begin{bmatrix} \bar{\phi}_t \\ \dot{\bar{\phi}}_t \end{bmatrix} = C_1 dM_t + C_2 \begin{bmatrix} \dot{\bar{\phi}}_t \\ \phi_t \end{bmatrix} dt - C_3 \bar{\xi}_t dt,
\]
where

\[
C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \\ \beta & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}.
\]

By integration by parts, we have

\[
d\left( e^{C_2(T-t)} \begin{bmatrix} \tilde{\phi}_t \\ \dot{\tilde{\phi}}_t \end{bmatrix} \right) = e^{C_2(T-t)}C_1d\tilde{M}_t - e^{C_2(T-t)}C_3\tilde{\xi}_t dt,
\]

and hence

\[
\begin{bmatrix} \tilde{\phi}_T \\ \dot{\tilde{\phi}}_T \end{bmatrix} = e^{C_2(T-t)} \begin{bmatrix} \tilde{\phi}_t \\ \dot{\tilde{\phi}}_t \end{bmatrix} + \int_t^T e^{C_2(T-s)}C_1d\tilde{M}_s - \int_t^T e^{C_2(T-s)}C_3\tilde{\xi}_s ds. \tag{A.4}
\]

Note that by matrix exponential, we have

\[
e^{C_1(T-t)} = \begin{bmatrix} F(t) & -\frac{F'(t)}{\beta} \\ -F'(t) & F(t) \end{bmatrix}.
\]

By plugging it into (A.4), we obtain

\[
\begin{align*}
\tilde{\phi}_T &= F(t)\tilde{\phi}_t - \frac{F'(t)}{\beta}\dot{\tilde{\phi}}_t - \int_t^T \frac{F'(s)}{\beta}d\tilde{M}_s + \int_t^T \frac{F'(s)}{\beta}\alpha\tilde{\xi}_s ds, \\
\dot{\tilde{\phi}}_T &= -F'(t)\tilde{\phi}_t + F(t)\dot{\tilde{\phi}}_t + \int_t^T F(s)d\tilde{M}_s - \int_t^T F(s)\alpha\tilde{\xi}_s ds.
\end{align*}
\]

Due to the terminal condition \( \dot{\tilde{\phi}}_T = -\frac{\delta}{2}\tilde{\phi}_T \), it gives

\[
0 = \left( \frac{\delta}{2}F(t) - \frac{\delta}{2}F'(t) \right)\tilde{\phi}_t + \left( -\frac{\delta}{2\beta}F'(t) + F(t) \right)\dot{\tilde{\phi}}_t \\
+ \int_t^T \left( -\frac{\delta}{2\beta}F'(s) + F(s) \right)d\tilde{M}_s + \int_t^T \left( \frac{\delta}{2\beta}F'(s) - F(s) \right)\alpha\tilde{\xi}_s ds.
\]

Multiplying \( \beta \) and taking conditional expectation yields

\[
\begin{align*}
\left( \beta F(t) - \frac{\delta}{2}F'(t) \right)\dot{\tilde{\phi}}_t &= \mathbb{E}\left[ \int_t^T \left( \beta F(s) - \frac{\delta}{2}F'(s) \right)\alpha\tilde{\xi}_s ds \mid \mathcal{F}_t \right] \\
+ \left( \beta F'(t) - \frac{\delta}{2}\beta F(t) \right)\dot{\tilde{\phi}}_t.
\end{align*}
\]

Now, applying (A.3) and rearranging, it follows that

\[
\begin{align*}
\left( \beta F(t) - \frac{\delta}{2}F'(t) \right)e^{-\frac{\delta}{2}t}\tilde{\phi}_t &= \mathbb{E}\left[ \int_t^T \left( \beta F(s) - \frac{\delta}{2}F'(s) \right)\alpha e^{-\frac{\delta}{2}s}\tilde{\xi}_s ds \mid \mathcal{F}_t \right] \\
+ \left( \beta - \frac{\delta^2}{4} \right) F'(t)e^{-\frac{\delta}{2}t}\tilde{\phi}_t.
\end{align*}
\]
Here one can easily see $\beta F(t) - \frac{\delta}{2} F'(t) > 0$. So, by dividing both sides of above equation by $\beta F(t) - \frac{\delta}{2} F'(t)$, we have

$$\dot{\phi}_t = \psi_t - G(t)\phi_t. \quad \text{(A.5)}$$

This linear ODE has a unique solution (A.2) by the variations of constant formula. So, if FBSDE (A.1) has a solution, then it must be of the form (A.2).

Now, it remains to show that (A.2) indeed solves FBSDE (A.1). First note that $\dot{\phi}_t \in L^2_\delta$ since $\xi$ and hence $\psi$ is in $L^2_\delta$. Next, one can see $\dot{\phi}_t$ satisfies (A.5) using integration by parts, and it shows $\dot{\phi}_t \in L^2_\delta$. Now define square-integrable martingale $(\bar{M}_t)_{t \in [0,T]}$ by

$$\bar{M}_t := \mathbb{E} \left[ \int_0^T (\beta F(s) - \frac{\delta}{2} F'(s)) \alpha \xi_s ds \mid \mathcal{F}_t \right].$$

Multiply $\left( \beta F(t) - \frac{\delta}{2} F'(t) \right) e^{-\frac{\delta}{2} t}$ on both sides of (A.5), and rearrange the terms to have

$$\left( \beta F(t) - \frac{\delta}{2} F'(t) \right) \dot{\phi}_t = \bar{M}_t - \int_0^t \left( \beta F(s) - \frac{\delta}{2} F'(s) \right) \alpha \xi_s ds + \left( \beta F'(t) - \frac{\delta}{2} \beta F(t) \right) \bar{\phi}_t.$$

Taking differentials and using the relationship $F''(t) = \beta F(t)$, we obtain

$$\left( \beta F(t) - \frac{\delta}{2} F'(t) \right) d\dot{\phi}_t = d\bar{M}_t - \left( \beta F(t) - \frac{\delta}{2} F'(t) \right) \alpha \xi_t dt + \left( \beta F'(t) - \frac{\delta}{2} \beta F(t) \right) \beta \bar{\phi}_t dt.$$

Multiplying $e^{\frac{\delta}{2} t} \left( \beta F(t) - \frac{\delta}{2} F'(t) \right)^{-1}$ and using the relationship $\beta = \alpha + \frac{\delta^2}{4}$, we have

$$d\dot{\phi}_t = d\bar{M}_t + \alpha (\phi_t - \xi_t) dt + \delta \dot{\phi}_t dt.$$

which is FBSDE (A.1), where the square-integrable martingale $M_t$ is defined by

$$dM_t = e^{\frac{\delta}{2} t} \left( \beta F(t) - \frac{\delta}{2} F'(t) \right)^{-1} d\bar{M}_t, \quad M_0 = \bar{M}_0.$$

So, one can conclude that FBSDE (A.1) has unique solution as (A.2).

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