Thermodynamics of dynamical wormholes

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Abstract. We study thermodynamics of dynamical traversable wormholes. The surface gravity is evaluated on the trapping horizon and the unified first law of thermodynamics is set up. The thermodynamic stability of these wormholes has been investigated. We have also extended these results to charged dynamical wormholes. Our results generalize those that exist for static Morris-Thorne wormholes.

Keywords: Wormholes, GR black holes

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### 1 Introduction

The idea of wormholes is not new and it was discussed in early 20th century by some authors including Flamm [1], Weyl [2] and Einstein and Rosen [3] but the name *wormhole* was first used by Misner and Wheeler [4]. Recently a considerable interest in wormhole physics has been seen in two directions: one with the Euclidean signature metrics and the other with the Lorentzian wormholes [5, 6]. Lorentzian wormholes that are both stable and traversable were first investigated by Morris and Thorne [7] in 1988. Wormholes provide shortcuts to go from one universe to the other or from one part to the other part of the same universe. For a wormhole to be traversable it must not have an event horizon. This requires that the spacetime contains some unusual or exotic matter. This means that the matter has very strong negative pressure and even the energy density is negative according to the static observer. Here, in this paper, the term wormhole would mean a traversable wormhole. Another interesting property of a wormhole is that it can be converted into a time machine if one of its mouth is moved relative to the other [8].

The standard cosmology reveals the fact that the total dominating energy density of the universe is in the form of dark matter and dark energy. The latter is considered to be uniformly distributed all over the universe and associated with a negative pressure and accelerates the expansion of the universe. Dark energy explained entirely on the basis of the cosmological constant is fully consistent with existing observational data. Another candidate is the phantom matter whose energy density increases with the expansion of the universe and which is associated with negative pressure [9–17]. This matter violates the null energy condition and it could be the type of matter which supports wormhole structure [18, 19]. This suggests that wormholes could exist in real universe and that they are not just a mathematical toy spacetime model. Now, exotic matter is considered to be a time-reversed version of
ordinary matter, therefore, one may think of wormhole also to be a time-reversed version of black hole if both show similar thermodynamic behavior. These kinds of analyses will improve the physical status of wormholes greatly \[20, 21\]. The thermodynamic properties of evolving wormholes have been studied in literature at event horizons and apparent horizons \[22–24\].

The main aim of this paper is to investigate dynamical wormholes with particular reference to their thermodynamic properties, which include the generalized surface gravity, unified first law and thermodynamic stability, at trapping horizons which are the hypersurfaces foliated by marginal surfaces. The trapping horizon in the case of static Morris-Thorne wormholes \[20, 21\] is outer due to the flaring out condition which results in positive generalized surface gravity, however, in the present study of dynamical wormholes we note that its nature depends on the scale factor, and thus the generalized surface gravity may be positive, negative or zero depending on which type of scale factor we are using. The need and significance of characterizing black holes by using local considerations has been stressed by Hayward \[25–28\]. Black holes are described by the presence of event horizons, which is the global property and hence cannot be located by observers. Now, trapping horizon is a purely local concept, and in this way the thermodynamic properties of spherically symmetric dynamical black holes were studied using local considerations. For wormholes, we will employ the definition of surface gravity \[29\] where we will use trapping horizon instead of Killing horizon and Kodama vector will play the role of Killing vector. The thermodynamic properties can also be studied for a wormhole by virtue of the presence of trapping horizon, and the results analogous to those of a black hole can be obtained \[30\]. The location of the throat in dynamical wormholes is in the trapped region, unlike the case of static Morris-Thorne wormholes where it coincides with the trapping horizon \[20, 21\]. Thus all the terms of the unified first law, which vanish on the trapping horizon in the case of Morris-Thorne wormholes, behave differently in dynamical wormholes and do not vanish in general. Also in the untrapped region (outside the trapping horizon) only the gradient of Misner-Sharp energy behaves oppositely in the ingoing direction when there is a large amount of exotic matter as compared to the static case. The trapped region (inside the trapping horizon) is formed only in dynamical wormholes where throat also lies. In the trapped and untrapped regions the energy supply term in the outgoing direction, the gradient of Misner-sharp energy for less amount of exotic matter and the work term in the ingoing direction behave opposite to each other.

These dynamical wormholes have been extended by adding charges in these objects, which play the role of extra matter, and we can deal with them in a similar fashion by introducing the definition of effective shape and redshift functions. Here, the energy supply term behaves same as in the uncharged case, however, the work term and the gradient of Misner-Sharp energy show similar behaviour only conditionally (depending on the amount of charge).

In this paper section 2 describes the trapping horizon of a spherical symmetric dynamical wormholes. In section 3 we find the generalized surface gravity for these wormholes on a trapping horizon. The unified first law (UFL) of wormhole thermodynamics is described in section 4. Section 5 deals with the thermodynamic stability of wormholes. In section 6 we derive the expression for the surface gravity following the same approach as in section 3 but now using the areal radius coordinates. In section 7 we discuss charged extension of dynamical wormholes. The generalized surface gravity and UFL for these objects is studied in section 8. Thermodynamic stability is examined in section 9, and finally in section 10 we conclude our work.
2 Trapping horizon

The Hayward formalism uses local quantities to define the properties of real black holes from which one obtains the same results that are yielded by the global considerations in the static case using event horizons and when there is vacuum. It is interesting to note that wormhole thermodynamic properties are similar to those found in black holes when we use local physically relevant quantities. Since event horizon is not present in a traversable wormhole so we use the trapping horizon. Now, the Schwarzschild black hole is the static vacuum solution that has a wormhole extension called the Einstein-Rosen bridge. But it is not traversable as it contains an event horizon. Here we consider a dynamical wormhole in a cosmological background, which is a generalization of the Morris-Thorne wormhole to a time dependent background [31],

\[ ds^2 = -e^{2\Phi(t,r)} dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2 \right], \]

in coordinates \((t, r, \theta, \phi)\) where \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\). The radial coordinate \(r\) ranges in \([r_0, \infty)\). Here the minimum radius \(r = r_0\) corresponds to the throat of the wormhole which connects two regions, each region is \(r_0 < r < \infty\). At \(r \to \infty\) this metric becomes flat, \(a(t)\) is the dimensionless parameter called the scaling factor of the universe. It tells us how our universe is expanding. It is known that the expansion rate of our universe is increasing with time which implies \(\ddot{a}(t) > 0\) or \(\dot{a}(t)\) is an increasing function of time (here over dot represents the time derivative). \(\Phi(t, r)\) is the redshift function as it corresponds to the gravitational redshift. This function should be finite everywhere in order to prevent the existence of an event horizon which is the necessary requirement for a wormhole to be traversable and when \(r \to \infty\) this redshift function should vanish. Here \(b(r)\) is the shape function which describes the shape of a wormhole as can be seen from the embedding space in coordinates \((Z, r, \phi)\), where the 2-surface

\[ Z(r) = \pm \int \left( \frac{r}{b(r)} - 1 \right)^{-1/2} dr \]

has the same geometry as the 2-surface \(\theta = \pi/2\) and \(t = \text{constant}\) in metric (2.1). The function \(Z(r)\) is called the embedding function. The graph of eq. (2.2), when revolved around the axis of rotation, the \(Z\)-axis, gives the shape of the wormhole [32]. At the wormhole throat a coordinate singularity \(b(r_0) = r_0\) occurs and \(b(r) < r\) for \(r > r_0\). This condition ensures the finiteness of the proper radial distance defined by

\[ l(r) = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \]

where \(\pm\) refers to the two asymptotically flat regions that are connected through the wormhole throat. The flaring out condition for wormholes requires that \(b' < b(r)/r\) at or near the throat which results in violating the null energy condition [7, 33, 34]. These are the conditions on \(\Phi(t, r)\) and \(b(r)\) which provide a traversable wormhole solution. It is clear that when \(\Phi(t, r)\) and \(b(r)/r\) tend to zero then the metric (2.1) becomes the flat Friedmann-Robertson-Walker (FRW) metric, and Morris-Thorne metric is recovered when \(\Phi(t, r) = \Phi(r)\) and \(a(t) \to 1\). Moreover there are conditions that must be satisfied and the forces felt by the observer in the wormhole during his hypothetical travel which has been discussed in detail in ref. [7].
Here in this paper we take $\Phi(t,r) = 0$ so that the wormhole metric (2.1) takes the form

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2 \right].$$

(2.4)

Now for the stress-energy tensor we take the perfect fluid which is completely described by its energy density and pressure [31], with components

$$T_{tt} = -\rho(t,r), \quad T_{rr} = p_r(t,r), \quad T_{\theta\theta} = T_{\phi\phi} = p_t(t,r),$$

(2.5)

where $\rho(t,r)$, $p_r(t,r)$ and $p_t(t,r)$ are, respectively, the energy density, radial pressure and tangential pressure. For isotropic pressure $p_r(t,r) = p_t(t,r)$, otherwise the pressure will be anisotropic.

Using null coordinates $(x^+, x^-)$, the metric (2.4) can be transformed into the form

$$ds^2 = 2g_{+-}dx^+dx^- + R^2 d\Omega^2,$$

(2.6)

where

$$dx^+ = \frac{dt}{a} + \frac{dr}{\sqrt{1 - \frac{b}{r}}},$$

(2.7)

and

$$dx^- = \frac{dt}{a} - \frac{dr}{\sqrt{1 - \frac{b}{r}}}$$

(2.8)

here $x^+$ corresponds to the outgoing radiation and $x^-$ to the ingoing radiation. Here $R$ and $g_{+-} = -a^2/2$ are functions of the null coordinates $(x^+, x^-)$, that correspond to the two preferred null normal directions for the symmetric spheres $\partial_\pm = \partial/\partial x^\pm$, and $R = a(t)r$ is the so-called areal radius [27] and $d\Omega^2$ is the metric for the unit 2-sphere. Now, we define the expansions as

$$\Theta_\pm = \frac{2}{R} \partial_\pm R.$$  

(2.9)

These expansions tell us whether the light rays are expanding ($\Theta > 0$) or contracting ($\Theta < 0$), or equivalently area of the sphere increases or decreases in the null directions. Since the sign of $\Theta_+\Theta_-$ is invariant, a sphere is trapped if $\Theta_+\Theta_- > 0$, which yields

$$H^2 R^2 - 1 + \frac{ab}{R} > 0,$$

(2.10)

untrapped if $\Theta_+\Theta_- < 0$, yielding

$$H^2 R^2 - 1 + \frac{ab}{R} < 0,$$

(2.11)

or marginal if $\Theta_+\Theta_- = 0$, giving

$$H^2 R^2 - 1 + \frac{ab}{R} = 0$$

(2.12)

where $H \equiv \dot{a}/a$ is the Hubble parameter. For fixed $\Theta_+ > 0$ and $\Theta_- < 0$, $\partial_+$ is also fixed outgoing and $\partial_-$ ingoing null normal vector. A surface which is foliated by marginal spheres (a surface where one of the null expansions vanishes) is known as a trapping horizon. This
surface is the boundary of the trapped region (a 4D continuum of trapped surfaces). Thus inside the trapping horizon we have $\Theta_+\Theta_- > 0$, on the trapping horizon $\Theta_+\Theta_- = 0$ and outside the trapping horizon $\Theta_+\Theta_- < 0$. This means that either expansion, outgoing or ingoing, changes sign across the horizon and vanishes on it.

In this paper for the trapping horizon, we choose

$$\Theta_- \cong 0,$$

where the symbol ($\cong$) henceforth shows evaluation on the trapping horizon $R_h = a(t)r_h$ which gives

$$HR - \sqrt{1 - \frac{ab}{R}} \cong 0.$$ (2.14)

Note that the choice $\Theta_- \cong 0$ corresponds to expanding universe ($\dot{a} > 0$), on the other hand if we chose $\Theta_+ \cong 0$, then it will lead us to the contracting universe ($\dot{a} < 0$). Also note that unlike the static Morris-Thorne wormhole, the trapping horizon and the throat of a dynamical wormhole do not coincide. In the case of static Morris-Thorne wormhole the trapping horizon is given by $b(r_0) = r_0$ which is also the value of the shape function at the throat [20]. But in our case, because of the presence of the scaling factor $a(t)$, they do not coincide. On the throat we have $\Theta_+\Theta_- = a^2H^2$, which is positive, and thus the surface is trapped and the throat lies inside the trapping region.

This trapping horizon is future if $\Theta_+ < 0$ (or equivalently $\partial_+R < 0$), giving

$$HR + \sqrt{1 - \frac{ab}{R}} < 0,$$ (2.15)

past if $\Theta_+ > 0$ (or equivalently $\partial_+R > 0$), giving

$$HR + \sqrt{1 - \frac{ab}{R}} > 0,$$ (2.16)

and bifurcating if $\Theta_+ \cong 0$ (or equivalently $\partial_+R \cong 0$), giving

$$HR + \sqrt{1 - \frac{ab}{R}} \cong 0.$$ (2.17)

Note that since we have made choice $\Theta_- \cong 0$ which corresponds to expanding universe ($\dot{a} > 0$), this makes the trapping horizon to be past as can be seen from eq. (2.16), while eqs. (2.15) and (2.17) are not satisfied for $\dot{a} > 0$.

In our case on the trapping horizon $\Theta_+ > 0$ and $\Theta_- \cong 0$. Thus, in case of expanding universe, it is the ingoing expansion which changes sign across the trapping horizon and vanishes on it while the outgoing expansion keeps the sign same. Therefore, inside the trapping horizon we have $\Theta_+ > 0$ and outside the trapping horizon we have $\Theta_+ > 0$ but $\Theta_- < 0$. This implies that inside the trapping horizon $HR > \sqrt{1 - ab/R}$, on the trapping horizon $HR \cong \sqrt{1 - ab/R}$ and outside the trapping horizon $HR < \sqrt{1 - ab/R}$.

Further, this trapping horizon is outer if $\partial_+\Theta_- < 0$, giving

$$\frac{\dot{H}}{2} + H^2 - \frac{(ab - Rb')}{4R^3} < 0,$$ (2.18)
inner if $\partial_+ \Theta_- > 0$, giving
\[
\frac{\dot{H}}{2} + H^2 - \frac{(ab - Rb')}{4R^3} > 0,
\] (2.19)
or degenerate if $\partial_+ \Theta_- \approx 0$, giving
\[
\frac{\dot{H}}{2} + H^2 - \frac{(ab - Rb')}{4R^3} \approx 0.
\] (2.20)

3 Generalized surface gravity

According to the equivalence principle the energy produced by the gravitational field of a source is the sum of its mass and energy that is possible to be measured at a distance. This combination of mass of the source and its energy is non-linear, in general, due to non-linearity of the gravitational field, and thus produces the effective (active) gravitational energy. In relativity there is no agreement on the definition of this energy but in spherically symmetric spacetimes the active gravitational energy is the Misner-Sharp energy in spaces. It reduces to Newtonian mass in the Newtonian limit for a perfect fluid. It gives Schwarzschild energy in vacuum. At null and spatial infinity it yields Bondi-Sachs, $E_{BS}$, and Arnowitt-Deser-Misner, $E_{ADM}$, energies, respectively [26]. The Misner-Sharp energy can be expressed as [35]
\[
E = \frac{1}{2} R(1 - \partial^a R\partial_a) = \frac{R}{2} (1 - 2g^+\partial_+ R\partial_- R),
\] (3.1)
which gives
\[
E = \frac{R}{2} \left[ H^2 R^2 + \frac{ab}{R} \right].
\] (3.2)

On a trapping horizon this expression reads $E \approx R/2$.

Now the Einstein’s equations of interest in local coordinates are
\[
\partial_\pm \Theta_\pm = -\frac{1}{2} \Theta_\pm^2 + \Theta_\pm \partial_\pm \log(-g_{+-}) - 2\pi a^2(\rho + p_r),
\] (3.3)
\[
\partial_\pm \Theta_\mp = -\Theta_\mp \Theta_\pm + \frac{1}{R^2} g_{+-} + 2\pi a^2(\rho - p_r),
\] (3.4)
\[
\Theta_+ \Theta_- = -\partial_+ \Theta_- - \partial_- \Theta_+ - 8\pi p_t.
\] (3.5)

In non-stationary spherically symmetric spacetimes we use Kodama vector $K$ instead of Killing vector which was introduced by Kodama [36] and which reduces to a Killing vector in stationary cases. The Kodama vector in null coordinates is given by
\[
K^\pm = -g^{+-}(\partial_+ R\partial_- - \partial_- R\partial_+),
\] (3.6)
which for spacetime (2.4) in covariant form becomes
\[
K^\pm = -\frac{a}{2} \left( \pm HR + \sqrt{1 - \frac{ab}{R}} \right).
\] (3.7)

The norm of $K$ is
\[
norm{K}^2 = \frac{2E}{R} - 1.
\] (3.8)
Note that $\norm{K}^2 \approx 0$ on the trapping horizon $\Theta_- \approx 0$. 

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The trapping horizon is provided by this Kodama vector which is null on a hypersurface \( \partial_{-} R \cong 0 \). In our case of dynamical spacetime, the trapping horizon and the Kodama vector play the same roles as the Killing horizon and the Killing vector play in the static case. In static spacetimes the hypersurface where the Killing vector vanishes is defined as the boundary of the spacetime but here in dynamical spacetimes we use Kodama vector instead. In the above, \( E \) is the Noether charge of Kodama vector. Kodama vector and Killing vector have some similar properties in dynamical and static spacetimes [26], respectively. Now, the generalized surface gravity \( \kappa \) on a trapping horizon can be expressed as [29, 37]

\[
K^{a} \nabla_{[b} K_{a]} \cong \pm \kappa K_{b}.
\] (3.9)

For metric (2.4) the surface gravity on trapping horizon becomes

\[
\kappa \cong -\frac{\dot{H} R}{2} - H^2 R + \frac{1}{4R^2} (ab - b'R),
\] (3.10)

which on using Einstein’s field equations (3.3) and (3.4) can be written as

\[
\kappa \cong -\dot{H} R - H^2 R - 2\pi R (\rho + p_r),
\] (3.11)

and

\[
\kappa \cong \frac{E}{R^2} - 4\pi R \omega = \frac{1}{2R} - 2\pi R (\rho - p_r).
\] (3.12)

This surface gravity, from eq. (3.9), equivalently, can also be expressed as

\[
\kappa \cong \frac{1}{2} g^{ab} \partial_a \partial_b R,
\] (3.13)

on a trapping horizon. It follows that \( \kappa < 0 \), \( \kappa = 0 \) and \( \kappa > 0 \) for inner, degenerate and outer trapping horizons, respectively. As mentioned above, in dynamical spherical spacetimes the Kodama vector is the analogue of a time-like Killing vector. We cannot define surface gravity in dynamical wormholes using Killing vector because it does not vanish everywhere. But still we can use Kodama vector instead and define the generalized surface gravity for static as well as dynamical traversable wormhole on a trapping horizon.

Now the usual surface gravity, defined by the use of Killing vector, means there is a force which acts on a test particle in a gravitational field. In our case, both Killing vector and Kodama vector are present but Kodama vector is more significant as it vanishes on a particular hypersurface unlike Killing vector, and in the vacuum case it reduces to Killing vector as well. Thus, one could suspect that the generalized surface gravity which is defined by using Kodama vector means more than just a force acting on the test particle in a gravitational field, and some extra effects on the test particle could be predicted. However, if these extra effects on a test particle vanish by some kind of symmetry then there is a possibility that such a symmetry would also give rise to a degenerate trapping horizon.

The Hawking temperature [20, 21] is \( T \cong -\kappa/2\pi \) which, in our case from eq. (3.10), becomes

\[
T \cong -\frac{\kappa}{2\pi} = -\frac{1}{2\pi} \left[ -\frac{\dot{H} R}{2} - H^2 R + \frac{1}{4R^2} (ab - b'R) \right],
\] (3.14)

which is negative for the outer trapping horizon since \( \kappa > 0 \). It means the particles coming out of a wormhole have the same properties as that of a phantom energy because this energy is linked with negative temperature as well. Or, we can say that the phantom energy is responsible for this negative temperature [38].
4 Unified first law for dynamical wormholes

We know that we can formulate a UFL of thermodynamics in spherically symmetric spacetimes [27]. This law describes the gradient of the active gravitational energy, using Einstein’s field equations, as a sum of two terms, the energy supply term and the work term. When we project this along the trapping horizon we get the first law of wormhole dynamics. This expression involves the area and surface gravity and has the same form as the wormhole statics if we replace the perturbations by the derivative along the trapping horizon. For the first law of wormhole dynamics we need to define the generalized surface gravity using Kodama vector and trapping horizon in the same manner as the first law of wormhole statics requires the stationary definition of surface gravity using Killing vector and Killing horizon. Also, this expression involves energy at horizon rather than at infinity.

Using the stress-energy tensor of the background fluid we construct a function and a vector in the local coordinates as

\[ \omega = -g_+ T^+ = \frac{\rho - p_r}{2}, \]

and

\[ \psi = T^{++} \partial_+ R \partial_+ + T^{--} \partial_- R \partial_-. \]

In components form it can be written as

\[ \psi_{\pm} = \left( \frac{\rho + p_r}{4} \right) \left( -aHR \pm a\sqrt{1 - \frac{ab}{R}} \right). \]

Now the UFL can be written by taking gradient of the gravitational energy and using Einstein’s field equations as [27]

\[ \partial_{\pm} E = A \psi_{\pm} + \omega \partial_{\pm} V, \]

with

\[ \partial_{\pm} E = 2\pi aR^2 \left( \frac{\pm \rho}{1 - \frac{ab}{R}} - HRp_r \right), \]

where \( A = 4\pi R^2 \) and \( V = 4\pi R^3/3 \) are the area and areal volume of the spheres of symmetry and the corresponding flat space, respectively. We can interpret \( \omega \) and \( \psi \) physically as the energy density and the energy flux (outward flux minus the inward flux). The right hand side of the UFL (4.4) is the sum of two terms, the first term \( A \psi_{\pm} \), called the energy supply term, produces variation in energy of the spacetime and the second term, \( \omega \partial_{\pm} V \), called the work term, supports the spacetime structure.

The variation of gravitational energy is always positive in the outgoing direction because \( \rho > 0 \) and \( p_r < 0 \), however, in the ingoing direction, it is positive inside the trapping horizon while outside the trapping horizon its sign depends on how much exotic matter is present there, for large amount of exotic matter it should be positive. The work term is also positive in the outgoing direction, as energy density \( \omega \) is positive besides the fact that energy conditions are not satisfied. In the ingoing direction this is positive inside the trapping horizon and negative outside the trapping horizon.

The sign of energy supply term depends on the sign of \( \rho + p_r \) (in case of black holes this term corresponds to the fluid which provides energy to the spacetime and respects NEC, hence positive, while it is negative in the case of wormholes where fluid removes energy
from the spacetime and violates NEC). Thus in the outgoing direction it is positive inside the trapping horizon and negative outside the trapping horizon. However, in the ingoing direction this term is always positive in our case ($\rho + p_r < 0$).

On the trapping horizon ($\partial_R R = 0$), in the outgoing direction, energy flux vanishes while both variation of gravitational energy and work term are positive. Thus change in gravitational energy equals the work done in the wormhole on the trapping horizon. In the ingoing direction, on the trapping horizon, work term vanishes while both variation of energy and energy flux are positive. Thus change in gravitational energy equals the energy supply and no work is done on the trapping horizon.

At the throat, all the terms entering in UFL, the variation of gravitational energy, energy supply term and the work term, are always positive both in the outgoing as well as ingoing direction.

Finally, eq. (4.4) when projected along the trapping horizon gives the first law of wormhole dynamics which can be expressed as

$$E' = \frac{\kappa A'}{8\pi} + \omega V', \quad (4.6)$$

where we have used the notation $F' = z \nabla F$. Here $z = z^+ \partial_+ + z^- \partial_-$ is a tangent vector to the trapping horizon. This expression defines a relation between surface area and geometric entropy as

$$S \propto A|_h. \quad (4.7)$$

Using eq. (3.14), eq. (4.6) takes the form

$$E' = -TS' + \omega V', \quad (4.8)$$
on the trapping horizon, where

$$S = \frac{A|_h}{4}. \quad (4.9)$$

The negative sign in front of the first term of the right hand side in eq. (4.8) is due to the energy removal from the wormhole. Thus the first law of wormhole dynamics is stated as: the change in the gravitational energy is equal to the energy that is removed from the wormhole plus the work term which is carried out in the wormhole.

5 Thermodynamic stability

In this section we study the thermodynamic stability of wormholes under consideration using the variables $E, T, S, P$ and $V$. We follow the usual criterion for thermodynamic stability, that is $\frac{\partial \bar{P}}{\partial V} \leq 0$ and $C_P \geq C_V \geq 0$ [39–41], where $\bar{P} = (P_r + 2P_t)/3$ is the average pressure and $C_P$ and $C_V$ are specific heats at constant pressure and volume, respectively.

We subtract eq. (3.11) from (3.12) and rearrange the terms to obtain

$$p_r = -\frac{1}{8\pi R^2} \frac{H + H^2}{4\pi}. \quad (5.1)$$

Eq. (3.5) on the trapping horizon yields

$$2p_t = -\frac{a^2 T}{R}. \quad (5.2)$$
From eqs. (5.1) and (5.2), using the definition of Hawking temperature, we obtain the average pressure $\bar{P}$ as

$$\bar{P} = \frac{p_r + 2p_t}{3} = -\frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi} - \frac{a^2 T}{3R},$$  \hspace{1cm} (5.3)$$

which is the equation of state in three state parameters $T, \bar{P}$ and $V$. From this equation we can analyze the thermodynamic stability of wormhole.

Stable equilibrium of a thermodynamic system requires that $\frac{\partial \bar{P}}{\partial V} |_{T} \leq 0$ where

$$\frac{\partial \bar{P}}{\partial V} |_{T} = \left(\frac{4\pi}{3}\right)^{2/3} \bar{P} + \left(\frac{4\pi}{3}\right)^{1/3} a^2 T.$$  \hspace{1cm} (5.4)$$

Now, to ensure the stable equilibrium we must have

$$T \leq -\frac{1}{4\pi a^2 R},$$  \hspace{1cm} (5.5)$$

thus the temperature assumes negative values everywhere for stable equilibrium which is attributed to the exotic matter. From eq. (5.3) we have

$$\bar{P} \geq \frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi}.$$  \hspace{1cm} (5.6)$$

If the scale factor is a linear function of time then $\ddot{a} = 0$ and then $\bar{P}$ will assume the positive values everywhere, otherwise it could be negative somewhere.

Another condition for stable equilibrium is $C_P \geq C_V \geq 0$. Now since, the constant $V$ means constant $E$ and $S$ so by the definition of $C_V$, $C_V = \frac{\partial E}{\partial T} |_{V} = T \frac{\partial S}{\partial T} |_{V} = 0,$ \hspace{1cm} (5.7)$$

which means we can define heat capacity only at constant pressure as

$$C_P = \frac{\partial E}{\partial T} |_{P} = \frac{(24\pi \bar{P} R^2 + 2\dot{H} R^2 + 2H^2 R^2 + 1)2\pi R^2}{24\pi \bar{P} R^2 + 2\dot{H} R^2 + 2H^2 R^2 - 1},$$  \hspace{1cm} (5.8)$$

where from eq. (5.3),

$$T = \frac{1}{a^2} (3\bar{P} R + \frac{1}{8\pi R} + \frac{\dot{H} R + H^2 R}{4\pi}).$$  \hspace{1cm} (5.9)$$

Now from eq. (5.6), to ensure the stable equilibrium, we can take the value of $\bar{P}$, for any non-negative $\epsilon$, as

$$\bar{P} = \frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi} + \epsilon.$$  \hspace{1cm} (5.10)$$

Thus eq. (5.8) on using eq. (5.10) takes the form

$$C_P = \frac{1}{6\epsilon} + 2\pi R^2.$$  \hspace{1cm} (5.11)$$

which is always positive. Thus the wormholes are thermodynamically stable. This means that for stable equilibrium the average pressure is always positive for linear scale factor, however it may also have negative values for non-linear scale factor while temperature is always negative as is also depicted in ref. [42] in which the possibility of negative temperature emerging from the exotic matter distribution was proposed.
6 Areal radius coordinates

Sometimes it is useful to employ areal radius $R \equiv a(t)r$ as a coordinate instead of $r$. The Schwarzschild-like coordinates are one of this kind of coordinate systems. Also, these systems provide what are called the pseudo-Painleve-Gullstrand coordinates [43]. Using the areal radius, metric (2.4) can be written in the pseudo-Painleve-Gullstrand form as

$$ds^2 = - \left[1 - \frac{ab}{R} - R^2 H^2 \right] dt^2 + \frac{dR^2}{\left(1 - \frac{ab}{R} \right)} - \frac{2HR}{1 - \frac{ab}{R}} dt dR + R^2 d\Omega^2,$$

(6.1)

As required in the Painleve-Gullstrand coordinates the coefficient of $dR^2$ is not unity [44].

To obtain the Schwarzschild-like form we define a new time $T$ by using the transformation

$$dT = \frac{1}{F} \left(dt + \beta dR\right),$$

(6.2)

where $F$ is the integration factor which satisfies

$$\frac{\partial}{\partial R} \left(\frac{1}{F}\right) = \frac{\partial}{\partial t} \left(\beta F\right).$$

(6.3)

Here $\beta(t, R)$ will be chosen later. Using eq. (6.2) in eq. (6.1) implies

$$ds^2 = - \left[1 - \frac{ab}{R} - R^2 H^2 \right] F^2 dt^2 + \left[1 + 2HR\beta - \left(1 - \frac{ab}{R} - R^2 H^2 \right) \beta^2 \right] dR^2$$

$$+ \frac{2F\beta \left(1 - \frac{ab}{R} - R^2 H^2 \right) - 2HRF}{1 - \frac{ab}{R}} dt dR + R^2 d\Omega^2.$$ 

(6.4)

The cross term $dT dR$ is eliminated if we choose

$$\beta = \frac{HR}{1 - \frac{ab}{R} - R^2 H^2}.$$ 

(6.5)

Thus metric (6.4) takes the diagonal form

$$ds^2 = - \left[1 - \frac{ab}{R} - R^2 H^2 \right] F^2 dt^2 + \frac{1}{\left[1 - \frac{ab}{R} - R^2 H^2 \right]} dR^2 + R^2 d\Omega^2,$$

(6.6)

where $F = F(T, R)$, $a$ and $H$ depend on $T$ implicitly.

This metric (6.6) can be put in the form of (2.6) by using null coordinates $x^+ = T + R_*$ and $x^- = T - R_*$ where

$$\frac{dR}{dR_*} = \sqrt{-\frac{g_{TT}}{g_{RR}}} = \frac{1 - \frac{ab}{R} - R^2 H^2}{\sqrt{1 - \frac{ab}{R}}} F.$$ 

(6.7)

The trapping horizon in this case is given by $\Theta_- \approx \frac{2}{R} \partial_- R = 0$ which gives

$$\left(1 - \frac{ab}{R}\right) \approx H^2 R^2.$$ 

(6.8)
Here we have bifurcating trapping horizon as $\Theta_- \equiv 0$ implies $\Theta_+ \equiv 0$.

The Misner-Sharp energy, energy flux and energy density are given, respectively, by

$$E = \frac{R}{2} \left[ 1 - \left( 1 - \frac{ab}{R} - R^2 H^2 \right) F \right], \quad (6.9)$$

$$\psi_\pm = \pm (\rho + p_r) \frac{\left( 1 - \frac{ab}{R} - R^2 H^2 \right) F}{4 \sqrt{1 - \frac{ab}{R}}}, \quad (6.10)$$

$$\omega = \frac{\rho - p_r}{2}. \quad (6.11)$$

It may be noted that $E \sim R/2$ on the trapping horizon only. Now, with the quantity

$$\partial_\pm E = \pm \frac{2 \pi R^2 \rho \left( 1 - \frac{ab}{R} - R^2 H^2 \right) F}{\sqrt{1 - \frac{ab}{R}}}, \quad (6.12)$$

the first law of thermodynamics is satisfied. The Kodama vector in this case takes the form

$$K_\pm = \sqrt{1 - \frac{ab}{R}} F, \quad (6.13)$$

with $\|K\|^2 \equiv 0$ on the trapping horizon. The generalized surface gravity from eq. (3.9) becomes

$$\kappa \equiv - \frac{ab'}{2R} + \frac{ab}{2R^2} - H^2 R, \quad (6.14)$$

which on using Einstein’s field equations takes the form

$$\kappa \equiv -2 \pi R (\rho + p_r) = \frac{E}{R^2} - 4 \pi R \omega. \quad (6.15)$$

### 7 Dynamical charged wormholes

In this section we extend the formalism described above to dynamical wormholes containing electric charge. The charged extension of metric (2.1) can be written as

$$ds^2 = -(2e^{\Phi(t,r)} + q^2/r^2) dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{b(r)}{r}} + \frac{q^2}{r^2} + r^2 d\Omega^2 \right]. \quad (7.1)$$

If we take $a(t) = 1$ and $\Phi(t, r) = \Phi(r)$ then we obtain the (static) charged wormhole [45]. The charge act as extra matter. Now, the effective redshift function is described as $\Phi_{\text{eff}}(t, r) = \frac{1}{2} \ln(e^{2\Phi(t, r)} + q^2/r^2)$, which should be finite everywhere for the absence of event horizon and should vanish at infinity. Here we will consider $\Phi_{\text{eff}}(t, r) = 0$. The traversability conditions now depend on effective shape function that can be written as $b_{\text{eff}}(r) = b(r) - q^2/r$. The radial coordinate $r$ varies in $\tilde{r}_0 \leq r < \infty$, where the minimum radius $\tilde{r}_0$ is the effective throat on which there occurs coordinate singularity $b_{\text{eff}}(\tilde{r}_0) = \tilde{r}_0$, which simplifies to [45]

$$\tilde{r}_0 = \frac{1}{2}(b \pm \sqrt{b^2 - 4q^2}). \quad (7.2)$$
Other than $r = \tilde{r}_0$, we have $b_{\text{eff}}(r) < r$ giving $b < r + q^2/r$. The flaring out condition $b'_{\text{eff}} \leq b_{\text{eff}}/r$ and positiveness condition, $b_{\text{eff}}(r) > 0$, give

$$b' \leq \frac{b - 2q^2}{r^2},$$

(7.3)

and

$$b > \frac{q^2}{r}.$$  

(7.4)

Now, Lagrangian due to the charge can be written [45] as $L^{(e)} = \frac{1}{16\pi} F_{\alpha\beta} F^\alpha_{\gamma\sigma} g^{\alpha\gamma} g^{\beta\sigma}$, with

$$F_{\mu\nu} = \varepsilon(r) \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

(7.5)

The electromagnetic stress-energy tensor

$$T^{(e)}_{\mu\nu} = \frac{1}{4\pi} (F_{\mu\lambda} F^\lambda_{\nu} - \frac{1}{4} g_{\mu\nu} F^\lambda_{\sigma} F^{\lambda\sigma}),$$

(7.6)

with the use of eq. (7.5) takes the form

$$T^{(e)}_{\mu\nu} = \frac{\varepsilon^2}{8\pi} \text{diag}(3,3,1,1) \gamma \delta,$$

(7.7)

where $\varepsilon = \varepsilon(r) = (q/r^2) \sqrt{|g_{00}g_{11}|}$ is the radial component of the electric field, while $\gamma = r^{-2}$ and $\delta = r^2 - rb(r) + q^2$.

Now metric (7.1) can be transformed into the form (2.6) by introducing

$$dx^+ = \frac{dt}{a} + \frac{dr}{\sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}}},$$

(7.8)

$$dx^- = \frac{dt}{a} - \frac{dr}{\sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}}},$$

(7.9)

and

$$g_{+-} = -\frac{a^2}{2}.$$  

(7.10)

The expansions in this case take the form

$$\Theta_{\pm} = \frac{2}{R} \partial_{\pm} R = \frac{a}{R} \left\{ HR \pm \sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}} \right\}.$$  

(7.11)

For trapping horizon we choose $\Theta_- \cong 0$, which gives

$$HR - \sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}} \cong 0.$$  

(7.12)

On the trapping horizon we have $\Theta_+ \cong 2aH$ which is positive thus the trapping horizon is past. Thus, in our case on the trapping horizon $\Theta_+ > 0$ and $\Theta_- \cong 0$. Inside the trapping...
horizon we have $\Theta_+ > 0$ and outside the trapping horizon we have $\Theta_- < 0$. Therefore, inside the trapping horizon $HR > \sqrt{1 - ab/R + a^2q^2/R^2}$, on the trapping horizon $HR \approx \sqrt{1 - ab/R + a^2q^2/R^2}$ and outside the trapping horizon $HR < \sqrt{1 - ab/R + a^2q^2/R^2}$. However, it may be outer, inner or degenerate depending on the sign of $\partial_+ \Theta_-$ as positive, negative or zero, respectively, on the trapping horizon, given by

$$\partial_+ \Theta_- \approx a^2 \left\{ H^2 + \frac{\dot{H}}{2} - \frac{ab - b'R}{4R^3} + \frac{2a^2q^2}{4R^4} \right\}. \quad (7.13)$$

### 8 Generalized surface gravity and UFL for dynamical charged wormholes

The Misner-Sharp energy for dynamical charged wormholes takes the form

$$E = \frac{R}{2} \left\{ H^2 R^2 + \frac{b}{r} - \frac{a^2q^2}{R^2} \right\}, \quad (8.1)$$

which is positive. This expression, on trapping horizon, becomes $E \approx R/2$. The Einstein-Maxwell field equations of interest are

$$\partial_\pm \Theta_\pm = -\frac{1}{2} \Theta_\pm^2 + \Theta_\pm \partial_\pm \log(-g_{+-}) - 2\pi a^2(\rho + p_r), \quad (8.2)$$

$$\partial_\pm \Theta_\mp = -\Theta_+ \Theta_- + \frac{1}{R^2} g_{+-} + 2\pi a^2 \left\{ \rho - p_r - \frac{3a^4q^2}{4\pi R^4} \right\}, \quad (8.3)$$

$$\Theta_+ \Theta_- = -\partial_+ \Theta_- - \partial_- \Theta_+ - 8\pi \left\{ \rho - \frac{3a^4q^2}{8\pi R^4} \right\}. \quad (8.4)$$

Solving eq. (3.9) on the trapping horizon, the surface gravity $\kappa$, in this case, is found to be

$$\kappa \approx -\frac{\dot{H}R}{2} - H^2 R + \frac{ab - b'R}{4R^3} - \frac{a^2q^2}{2R^3}. \quad (8.5)$$

This, on using eqs. (8.2) and (8.3), can also be written as

$$\kappa \approx -\dot{H}R - H^2 R - 2\pi R(\rho + p_r), \quad (8.6)$$

and

$$\kappa \approx \frac{E}{R^2} - 4\pi R\omega = \frac{1}{2R} - 2\pi R \left\{ \rho - p_r - \frac{3a^4q^2}{4\pi R^4} \right\}. \quad (8.7)$$

The derivative of Misner-sharp energy, on using eqs. (8.2) and (8.3), can be written as

$$\partial_\pm E = 2\pi a R^2 \left\{ \pm \left( \rho - \frac{3a^4q^2}{8\pi R^4} \right) \sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2} - HR \left( \rho_r + \frac{3a^4q^2}{8\pi R^4} \right)} \right\}. \quad (8.8)$$

From the stress-energy tensor, $T_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(e)}_{\mu\nu}$, we can construct a function

$$\omega = -g_{+-} T^{+-} = \frac{\rho - p_r}{2} - \frac{3a^4q^2}{8\pi R^4}, \quad (8.9)$$


and a vector
\[ \psi_\pm = \frac{a(p + p_r)}{4} \left\{ \pm \sqrt{1 - \frac{ab}{R} + \frac{a^2 q^2}{R^2} - HR} \right\}. \]  
(8.10)

Thus the UFL (4.4) can be formulated using eqs. (8.8), (8.9) and (8.10).

We must be careful about the conditions that govern the signs of the terms involved in UFL. We note that the energy supply term is always positive in the ingoing direction as \( \rho + p_r < 0 \). In the outgoing direction, it is positive inside the trapping horizon and negative outside the trapping horizon. The work term, in the outgoing direction, is positive if \( (\rho - p_r) > 3a^4q^2/4\pi R^4 \) and negative otherwise. In the ingoing direction, inside the trapping horizon it keeps the same behaviour while outside the trapping horizon its behaviour reverses. The gradient of Misner-Sharp energy, in the outgoing direction, is positive if \( \rho > 3a^4q^2/8\pi R^4 \) and \(-p_r > 3a^4q^2/8\pi R^4 \) while it is negative if \( \rho < 3a^4q^2/8\pi R^4 \) and \(-p_r < 3a^4q^2/8\pi R^4 \). In the ingoing direction, inside the trapping horizon it is positive when \( \rho > 3a^4q^2/8\pi R^4 \) and \(-p_r > 3a^4q^2/8\pi R^4 \) and in the range \( \rho < 3a^4q^2/8\pi R^4 < -p_r \). Outside the trapping horizon it is positive when \( \rho < 3a^4q^2/8\pi R^4 \) and \(-p_r < 3a^4q^2/8\pi R^4 \), and in the range \( \rho < 3a^4q^2/8\pi R^4 < -p_r \).

On the trapping horizon, in the outgoing direction, the gradient of Misner-Sharp energy and the work term are equal and they can be positive or negative, depending on the amount of charge in the wormhole. In the absence of charge these terms are positive, also if \( \rho - p_r > 3a^4q^2/4\pi R^4 \) then these terms are positive. However, they can get negative if large amount of charge is present in the wormhole such that \( \rho - p_r < 3a^4q^2/4\pi R^4 \). The energy supply term vanishes on the trapping horizon in the outgoing direction always. In the ingoing direction, the gradient of Misner-Sharp energy and the energy supply terms is same and always positive while the work term vanishes on the trapping horizon.

At the throat, in both the directions, ingoing and outgoing, the energy supply term is always positive, however, the other two terms are positive for small quantity of charge while negative for large quantity. Thus, when there is less amount of charge in the wormhole such that \( (\rho - p_r) > 3a^4q^2/4\pi R^4 \) then all the terms appearing in the UFL are positive. If more charge is added such that \(-p_r = 3a^4q^2/8\pi R^4 \) then the gradient of Misner-Sharp energy vanishes while the energy supply term and the work term become equal in magnitude but their signs are not same. If \( (\rho - p_r) = 3a^4q^2/4\pi R^4 \) then work density vanishes while gradient of Misner-Sharp energy and energy supply term becomes same. Here we also note that if both the conditions \( -p_r = 3a^4q^2/8\pi R^4 \) and \( (\rho - p_r) = 3a^4q^2/4\pi R^4 \) are met at the same time then this will ensure that \( \rho = -p_r \), and thus it will respect the NEC. Thus exotic nature of the material supporting the wormhole may be lost. Thus both the conditions cannot be met at the same time in the case of a wormhole supported by exotic matter. If more charge is added to the wormhole such that \( -p_r < 3a^4q^2/8\pi R^4 \), then the gradient of Misner-Sharp energy and the work term become negative, however, the energy supply term is still positive. This means that increasing charge induces an increase in work density in the negative direction.

9 Thermodynamic stability of dynamical charged wormholes

We will examine thermodynamic stability of dynamical charged wormholes, in this section, using the same criterion discussed earlier for uncharged dynamical wormholes. We subtract eq. (8.7) from (8.6), obtaining
\[ p_r = \frac{1}{8\pi R^2} \left( \frac{\dot{H} + H^2}{4\pi} - \frac{3a^4q^2}{8\pi R^4} \right). \]  
(9.1)
Solving eq. (8.4) on the trapping horizon, using the definition of surface gravity, we get

\[ 2p_t = -\frac{a^2 T}{R} - \frac{a^4 q^2}{4\pi R^4}. \]  

(9.2)

From eqs. (9.1) and (9.2), the average pressure \( \bar{P} \) can be found as

\[ \bar{P} = \frac{p_r + 2p_t}{3} = -\frac{1}{24\pi R^2} \frac{\dot{H} + H^2}{12\pi} - \frac{a^2 T}{3R} - \frac{5a^4 q^2}{24\pi R^4}. \]  

(9.3)

The thermodynamic stability can be analyzed from equation of state (9.3). Taking derivative of this with respect to \( V \) at constant temperature, we get

\[ \frac{\partial \bar{P}}{\partial V} |_T = \frac{(4\pi/3)^{2/3}}{36\pi V^{5/3}} + \frac{(4\pi/3)^{1/3} a^2 T}{9V^{4/3}} + 5a^4 q^2 \left( \frac{32\pi}{59049 V^7} \right)^{1/3}. \]  

(9.4)

Now, for stable equilibrium of a thermodynamic system, we must have \( \frac{\partial \bar{P}}{\partial V} |_T \leq 0 \) which is ensured for \( T \leq -\frac{R^2 + 10a^4 q^2}{4\pi a^2 R^3} \).

(9.5)

This negative temperature can be attributed to exotic matter. Solving Eqs. (9.3) and (9.5), we find

\[ \bar{P} \geq \frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi} + \frac{15a^4 q^2}{24\pi R^4}. \]  

(9.6)

If the scale factor is a linear function of time then \( \ddot{a} = 0 \) and then \( \bar{P} \) will assume positive values everywhere, otherwise it could be negative somewhere.

Another condition for stable equilibrium is \( C_P \geq C_V \geq 0 \). Now, since constant \( V \) means constant \( E \) and \( S \) so by the definition of \( C_V \),

\[ C_V = \frac{\partial E}{\partial T} |_{V} = T \frac{\partial S}{\partial T} |_{V} = 0, \]  

(9.7)

which means we can define heat capacity only at constant pressure as

\[ C_P = T \frac{\partial S}{\partial T} |_{P} = \frac{(24\pi \bar{P} R^4 + 2 \dot{H} R^4 + 2H^2 R^4 + R^2 + 5a^4 q^2)2\pi R^2}{24\pi PR^4 + 2HR^3 + 2H^2 R^3 - R^2 - 15a^4 q^2}, \]  

(9.8)

where from eq. (9.3),

\[ T = -\frac{1}{a^2} \left( 3R \bar{P} + \frac{1}{8\pi R} \frac{\dot{H} R + H^2 R}{4\pi} + \frac{5a^4 q^2}{8\pi R^3} \right). \]  

(9.9)

Now, from eq. (9.6), to ensure the stable equilibrium, we can take the value of \( \bar{P} \), for any non-negative \( \epsilon \), as

\[ \bar{P} = \frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi} + \frac{15a^4 q^2}{24\pi R^4} + \epsilon. \]  

(9.10)

Thus eq. (9.8) on using eq. (9.10) takes the form

\[ C_P = \frac{1}{6\epsilon} + 2\pi R^2 + \frac{5a^4 q^2}{3R^2 \epsilon}. \]  

(9.11)

All the terms appearing in the above equation are positive which ensures \( C_P \geq C_V \geq 0 \). Thus dynamical charged wormholes are thermodynamically stable.
10 Conclusion

In this paper we have investigated dynamical traversable wormholes, which are the time generalization of Morris-Thorne wormholes, and studied their thermodynamics and the laws of mechanics. In dynamical spacetimes the Kodama vector and the trapping horizon replace the role of the Killing vector and Killing horizon, respectively. The Kodama vector reduces to the Killing vector for static vacuum case. However, this is not possible for non-vacuum cases. There is no Killing horizon (even though we do have the Killing vector) present to find the surface gravity in wormholes. So, we find the generalized surface gravity with the help of the trapping horizon. Our results generalize the results available in the literature for the Morris-Thorne wormholes.

We have derived the generalized surface gravity for a dynamical traversable wormhole on the trapping horizon. This surface gravity is positive, negative or zero for outer, inner or degenerate trapping horizons, respectively. When we compare the results for black holes and wormholes we get useful information about these dynamical wormholes and hence about the exotic matter which supports the construction of these spacetimes. Now from UFL it is depicted that there is variation of sign in different regions of spacetime unlike the case of a static wormhole. In the static case all the terms, entering in UFL, vanish on the trapping horizon (throat) of a wormhole, thus resulting in no evolution of the throat, and in general, the variation of gravitational energy and the work term have same sign (positive in outgoing direction and negative in ingoing direction) opposite to energy supply term [20]. But here in dynamical wormholes the situation is different. First of all the trapping horizon and throat do not coincide here and different behaviour of terms appearing in UFL is observed. We observe that, on the throat and inside the trapping horizon, all the terms entering in UFL are positive both in the ingoing and outgoing directions. Thus, the direction does not matter on the throat and inside the trapping horizon in our case of dynamical wormholes. However, on the trapping horizon, \( \partial_{-} R = 0 \), and we observe that the energy supply term and the work term vanish in the outgoing and ingoing directions, respectively. This means that the variation of the gravitational energy and the work terms are equal and positive in the outgoing direction, and in the ingoing direction variation of gravitational energy and energy supply terms are equal and positive. Outside the trapping horizon, in the outgoing direction, energy supply term becomes negative while rest of the terms do not change their behaviour, however, in the ingoing direction, work term becomes negative while rest of the terms do not change their behaviour but energy supply term may become negative as well if exotic matter reduces in the wormhole.

The projection of UFL along the trapping horizon gives the first law of wormhole thermodynamics which is stated as ‘the change in the gravitational energy equals the energy removed from the wormhole plus the work term’. The gravitational energy and the work term appearing in the first law have same signs while the energy supply term is negative. This means that the matter content takes energy from the spacetime and from this energy it does work for maintaining the structure of the wormhole unlike the situation in a black hole where the sign of the energy supply term is positive such that it gives energy to the black hole spacetime.

We have discussed thermodynamic stability of wormholes and have shown that, for linear scale factor, average pressure assumes positive values everywhere (\( \bar{P} \geq 1/24\pi R^2 \)) which is a natural requirement in the usual thermodynamic systems. In the case of non-linear scale factor, the pressure could also have negative values depending on the value of
the second derivative of the scale factor which is also possible in gravitational systems such as in the case of dark energy. The temperature is always negative \( T \leq -1/4\pi R \) for stable thermodynamic equilibrium which could be attributed to the exotic matter.

We also have extended our results to dynamical charged wormholes. We observe that our analysis is valid in this case also by replacing the shape function with an effective shape function and stress-energy tensor \( T_{\mu\nu} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(e)} \). By carefully observing the terms in the UFL we note that the behaviour of the energy supply term is the same as in the case of uncharged dynamical wormholes in each region and in both the directions. However, gradient of Misner-Sharp energy and work terms are affected by different amounts of charge. When \( \rho - p_r > 3a^4q^2/4\pi R^4 \) then both the terms show similar behaviour as in the case of uncharged dynamical wormholes. If \( \rho - p_r < 3a^4q^2/4\pi R^4 \) then the sign of the work term becomes opposite to that in the case of uncharged dynamical wormholes. However, the gradient of Misner-Sharp energy reverses its sign always in the outgoing direction. In the ingoing direction, its sign is same on the trapping horizon and also other than trapping horizon (provided \( \rho < 3a^4q^2/8\pi R^4 < -p_r \) holds) as in uncharged dynamical wormholes, but it reverses its sign on the throat and other than trapping horizon (provided \( \rho < 3a^4q^2/8\pi R^4 < -p_r \) does not hold).

Thermodynamic equilibrium is maintained for negative values of temperature, \( T \leq -(R^2 + 10a^4q^2)/4\pi a^2 R^3 \), however average pressure is always positive \( \bar{P} \geq 1/24\pi R^2 + 15a^4q^2/24\pi R^4 \), in the case when the scale factor is linear.

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