REMARKS ON TOEPLITZ PRODUCTS ON SOME DOMAINS

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Abstract. Assuming that one of the symbols satisfies an invariant condition, we prove that the usual Sarason condition is necessary and sufficient for the Toeplitz products to be bounded on Bergman spaces. We also characterize bounded and invertible Toeplitz products on vector weighted Bergman spaces of the unit polydisc. For our purpose, we will need the notion of Békollé-Bonami weights in one and several parameters.

1. Introduction

Let denote by $d\nu$ the Lebesgue measure on the unit disc $\mathbb{D}$ of $\mathbb{C}$. For $\alpha > -1$, we denote by $d\nu_\alpha$ the normalized Lebesgue measure $d\nu_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha d\nu(z)$, $c_\alpha$ being the normalizing constant. The weighted Bergman space $A^p_\alpha(\mathbb{D})$ is the space of holomorphic functions $f$ such that

$$||f||^p_{A^p_\alpha} := \int_{\mathbb{D}} |f(z)|^p d\nu_\alpha(z) < \infty.$$ 

The Bergman space $A^2_\alpha(\mathbb{D})$ is a reproducing kernel Hilbert space with kernel $K^\alpha w(z) = K^\alpha(w, z) = \frac{1}{(1 - wz)^{1/2} + \alpha}$. That is for any $f \in A^2_\alpha(\mathbb{D})$, the following representation holds

$$f(w) = P_\alpha(f) = \langle f, K^\alpha w \rangle = \int_{\mathbb{D}} f(z) K^\alpha(w, z) d\nu_\alpha(z), \ w \in \mathbb{D}.$$ 

For $f \in L^2(\mathbb{D}, d\nu_\alpha)$, we can densely define the Toeplitz $T_f$ with symbol $f$ on $A^2_\alpha(\mathbb{D})$ as follows

$$T_f(g) = P_\alpha(M_f)(g) = P_\alpha(fg)$$

where $M_f$ is the multiplication operator by $f$. The Berezin transform is the operator defined on $L^1(\mathbb{D}, d\nu_\alpha)$ by

$$B_\alpha(f)(w) = \int_{\mathbb{D}} f(z) |k^\alpha_w(z)|^2 d\nu_\alpha(z)$$

where $k^\alpha_w$ is the normalized reproducing kernel of $A^2_\alpha(\mathbb{D})$.

The so-called Sarason conjecture says that given two functions $f, g \in A^2_\alpha(\mathbb{D})$, the product $T_f T_g$ is bounded on $A^2_\alpha(\mathbb{D})$ if and only if the following holds

$$\sup_{w \in \mathbb{D}} (B_\alpha(|f|^2)(w))^{1/2} (B_\alpha(|g|^2)(w))^{1/2} = \sup_{w \in \mathbb{D}} ||fk^\alpha_w||_{2,\alpha} ||gk^\alpha_w||_{2,\alpha} < \infty.$$ 

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We call (3) Sarason condition. For $\alpha = -1$, that is in the case of the Hardy space $H^2(D)$, F. Nazarov has proved that the conjecture fails [14] although (3) is necessary as proved by S. Treil (see [17]). For usual Bergman spaces ($\alpha > -1$), it has been proved in [18] that condition (3) is necessary but the authors of that paper did not manage to prove that the same condition is sufficient or not (see also [8, 12, 15, 19, 20] for related discussions and other domains). In fact A. Aleman, S. Pott and M. C. Reguera recently provided in [1] an example of $f, g \in A^2(D)$ such that (3) holds but the product $T_f T_g$ is not bounded on $A^2(D)$, disproving the conjecture. Nevertheless, in the case of invertible and bounded Toeplitz products, it is known that the conjecture holds. More precisely, the following has been proved in [18, 21].

**Theorem 1.1.** Given $f, g \in A^2_\alpha(D)$, $-1 \leq \alpha < \infty$, the product $T_f T_g$ is bounded and invertible on $A^2_\alpha(D)$ if and only if

$$\inf_{w \in D} |f(w)||g(w)| > 0$$

and (3) holds.

The above result was recently extended to the unit polydisc for $\alpha > -1$ by Z. Sun and Y. Lu in [10]. In [8], Theorem 1.1 was extended to all $A^2_\alpha(D)$ using among others a result of J. Miao [11] and (for $\alpha > -1$) Bekollé-Bonami weights or (for $\alpha = -1$) $A_p$ classes of Muckenhoupt. More precisely, the author of [8] proved the following result.

**Theorem 1.2.** Let $1 < p, q < \infty$, $p = q(p - 1)$ and $-1 \leq \alpha < \infty$. Given $f \in A^p_\alpha(D)$ and $g \in A^q_\alpha(D)$, the product $T_f T_g$ is bounded and invertible on $A^p_\alpha(D)$ if and only if

$$\inf_{w \in D} |f(w)||g(w)| > 0$$

and

$$\sup_{w \in D} B_\alpha(||f(k^\alpha w)^{1 - \frac{2}{p}}||p)(w)B_\alpha(||g(k^\alpha w)^{1 - \frac{2}{p}}||p)(w) < \infty.$$ 

One question that follows after Theorem 1.2 and the result of [10] is how to extend the above results to Bergman spaces with vector weights in the polydisc $D^n$ that is when $\alpha = (\alpha_1, \cdots, \alpha_n)$ where the $\alpha_j$ are not necessarily the same. The other question that one may ask after the result of [1] is to know if there are conditions under which the Sarason condition (3) is necessary and sufficient for the Toeplitz product to be bounded for more general domains. We consider the last question in the setting of the upper-half plane and its generalization and prove that if one of the symbols satisfies an invariant condition, then the Sarason condition is necessary and sufficient for the associated Toeplitz product to be bounded. This idea is suggested by the proof of Theorem 1.2 and the relation between the boundedness of Toeplitz products and the two-weight estimate of the Bergman projection. The first question is also considered using the estimate of the Bergman projection on product domains.

Before going ahead, let us introduce Bergman spaces in the upper-half plane.
The upper-half plane is the set \( H := \{ z = x + iy \in \mathbb{C} : y > 0 \} \). For \( \alpha > -1 \) and \( 1 < p < \infty \), the weighted Bergman space \( A^p_\alpha(H) \) consists of those analytic functions \( f \) on \( H \) such that
\[
\| f \|_{p,\alpha}^p := \int_H |f(x+iy)|^p y^\alpha dxdy < \infty.
\]
We will be also using the notations \( dV_\alpha(z) := y^\alpha dxdy \) for \( \alpha > -1 \).

The Bergman space \( A^2_\alpha(H) \) \((-1 < \alpha < \infty)\) is a reproducing kernel Hilbert space with kernel \( K^\alpha_w(z) = K^\alpha(z, w) = \frac{1}{(z-w)^{2+\alpha}} \). That is for any \( f \in A^2_\alpha(H) \), the following representation holds
\[
f(w) = P_\alpha f(w) = \langle f, K^\alpha_w \rangle = \int_H f(z) K^\alpha(w, z) dV_\alpha(z).
\]
For \( f \in L^2(H, dV_\alpha) \), we can densely define the Toeplitz \( T_f \) with symbol \( f \) on \( A^2_\alpha(H) \) as in (2). We recall that the normalized reproducing kernel \( k^\alpha_w(z) := K^\alpha_w(z) / \| K^\alpha_w \|_{2,\alpha} = \frac{v^{1+\frac{\alpha}{2}}}{(z-w)^{2+\alpha}} \).

It seems natural to ask if the usual characterization of bounded and invertible Toeplitz products in Theorem 1.2 holds on the upper-half plane. We make the easy observation that the answer is no. More precisely, we have the following.

**Proposition 1.3.** Let \( 1 < p, q < \infty \), \( p = q(p-1) \) and \( -1 < \alpha < \infty \). Given \( f \in A^p_\alpha(H) \) and \( g \in A^q_\alpha(H) \), assume that the product \( T_f T_g \) is bounded and invertible at the same time on \( A^p_\alpha(H) \). Then
\[
sup_{w \in \Omega} \| f k^\alpha_w \|_{p,\alpha} \| g k^\alpha_w \|_{q,\alpha} = \infty
\]
or equivalently,
\[
inf_{w \in \Omega} |f(w)| |g(w)| = 0.
\]

The proof of the above proposition follows by making some easy observations: first that a kind of reproducing kernel thesis holds here. More precisely, if we write
\[
u^\alpha_w = \frac{k^\alpha_w}{\| k^\alpha_w \|_{p,\alpha}}
\]
and
\[
u^\alpha_w = \frac{k^\alpha_w}{\| k^\alpha_w \|_{q,\alpha}},
\]
then \( u \) and \( v \) are normalizations of \( k^\alpha_w \) in \( A^p_\alpha(H) \) and \( A^q_\alpha(H) \) respectively and the following necessarily hold
\[
sup_{w \in H} \| T_f T_g u^\alpha_w \|_{p,\alpha} < \infty
\]
and
\[
sup_{w \in H} \| T_f T_g v^\alpha_w \|_{q,\alpha} < \infty.
\]
The second observation is the fact that in the upper-half plane, Bergman spaces do not contain constants except zero since the total measure is not bounded.

We do not know how to characterize bounded and invertible Toeplitz products on this setting. Our guess is that a Toeplitz product can not be bounded and invertible at the same time on the upper-half plane.

A trivial example of bounded Toeplitz products is given by taking one of the symbol to be identically zero. The following second result provides another class of bounded Toeplitz products. It essentially says that if one of the symbols satisfies an invariant condition, then the Sarason condition is necessary and sufficient for the associated Toeplitz product to be bounded.

**Theorem 1.4.** Let $1 < p, q < \infty$, $p = q(p - 1)$, and $-1 < \alpha < \infty$. Suppose that $f$ and $g$ are analytic in $H$ with

$$[f]_{p, \alpha} := \sup_{w \in H} \|fk^\alpha_{w}\|_{p, \alpha}\|f^{-1}k^\alpha_w\|_{q, \alpha} < \infty.$$  

Then $T_f T_g$ is bounded on $A^p_\alpha(H)$ if and only if

$$[f, g]_{p, \alpha} := \sup_{w \in H} \|fk^\alpha_{w}\|_{p, \alpha}\|gk^\alpha_w\|_{q, \alpha} < \infty.$$  

Moreover,

$$\|T_f T_g\| \leq C(p) [f, g]_{p, \alpha} [f]_{p, \alpha}^{\max(p, q)}$$  

and

$$[f, g]_{p, \alpha} \leq C(p) \|T_g T_f\| [f]_{p, \alpha}.$$  

Note that in the above theorem, we do not ask $f$ and $g$ to belong to $A^p_\alpha(H)$ and $A^q_\alpha(H)$ respectively, giving us the flexibility to recover the Bergman projection by taking $f = g = 1$. This assumption is also motivated by the fact that constants (except zero) are not in $A^p_\alpha(H)$. Thus in the case of the unit disc or the unit polydisc, we can suppose that $f \in A^p_\alpha$ and $g \in A^q_\alpha$.

For the proof of the sufficiency part in Theorem 1.4, we will observe with Cruz-Uribe [4] that the boundedness of $T_f T_g$ follows from the boundedness of the operator $P^\alpha_{\alpha}$ from $L^p(H_\alpha, |g(z)|^{-\alpha}dV_\alpha(z))$ to $L^p(H_\alpha, |f(z)|^\alpha dV_\alpha(z))$,

$$P^\alpha_{\alpha} f(z) = \int_{H_\alpha} \frac{f(w)}{|z - w|^{2\alpha}} dV_\alpha(w).$$

We then deduce that $T_f T_g$ is bounded on $A^p_\alpha(H)$ from an upper-half plane version of a result due to D. Békollé and A. Bonami (see [2], [3], [16]) that we will prove in the text.

We recall that $D^n$ is the unit polydisc, where $D$ is the unit disc of $C$. For any two points $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $C^n$, we write

$$\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n,$$

and

$$|z|^2 = |z_1|^2 + \cdots + |z_n|^2.$$  

For any real vector $\alpha = (\alpha_1, \cdots, \alpha_n)$, where every $\alpha_j$ satisfies $\alpha_j > -1$, we consider the measure on $D^n$ given by

$$dv_\alpha(z) = c_\alpha \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j} dv(z_j),$$  

where $c_\alpha$ is a normalization constant.
\[ c_\alpha = \prod_{j=1}^{n} (\alpha_j + 1), \] 
\( dv \) being the Lebesgue area measure on \( \mathbb{D} \), normalized so that the measure of \( \mathbb{D} \) is 1. Let \( H(\mathbb{D}) \) denote the space of analytic functions on \( \mathbb{D} \). For \( 0 < p < \infty \), the (vector) weighted Bergman space \( A^p_\alpha(\mathbb{D}) \) is the intersection \( L^p(\mathbb{D}, dv_\alpha) \cap H(\mathbb{D}) \). The (unweighted) Bergman space \( A^p \) corresponds to \( \alpha = (0, \cdots, 0) \). We will be writing
\[
\|f\|_{p,\alpha} := \left( \int_{\mathbb{D}} |f(z)|^p dv_\alpha(z) \right)^{\frac{1}{p}}
\]
and
\[
\langle f, g \rangle_\alpha := \int_{\mathbb{D}} f(z)\overline{g(z)} dv_\alpha(z).
\]
The Bergman space \( A^2_\alpha \) is a reproducing kernel Hilbert space with its kernel \( K^\alpha \) given by
\[
K^\alpha(z, w) = \prod_{j=1}^{n} \frac{1}{(1 - \bar{z}_j w_j)^{\alpha_j + 2}}.
\]
That is for any \( f \in A^2_\alpha(\mathbb{D}) \), the following representation holds
\[
f(w) = P_\alpha f(w) = \langle f, K^\alpha_\alpha \rangle = \int_{\mathbb{D}} f(z)K^\alpha(z, w) dv_\alpha(z).
\]
For \( f \in L^2(\mathbb{D}, dv_\alpha) \), we can densely define on \( A^2_\alpha \) the Toeplitz \( T_f \) with symbol \( f \) as in (2).

Very recently bounded and invertible Toeplitz products \( T_f T_g \) on \( A^2_\alpha(\mathbb{D}) \) have been characterized in [10]. We aim in this section to extend this characterization to \( A^p_\alpha(\mathbb{D}) \), \( 1 < p < \infty \). More precisely, denoting \( k^\alpha_w \) the normalized \( \alpha \)-characterized \( \mathbb{D} \)-kernel in \( A^2_\alpha(\mathbb{D}) \), we have the following generalization.

**Theorem 1.5.** Let \( \alpha > -1 = (-1, -1, \cdots, -1), \) \( \infty > p > \max\{1, \frac{2}{2+\alpha_1}, \cdots, \frac{2}{2+\alpha_n}\}, \) \( pq = p+q \), and assume that \( f \in A^p_\alpha(\mathbb{D}) \) and \( g \in A^q_\alpha(\mathbb{D}) \). Then the following assertions hold.

(i) If \( T_f T_g \) is bounded and invertible on \( A^p_\alpha(\mathbb{D}) \), then
\[
[f, g]_{p,\alpha} = \sup_{z \in \mathbb{D}} \|f k^\alpha_w\|_{p,\alpha} \|g k^\alpha_w\|_{q,\alpha} < \infty
\]
and this is equivalent to
\[
\eta := \inf_{z \in \mathbb{D}} |f(z)||g(z)| > 0.
\]

(ii) If both (12) and (13) hold, then
\[
[f]_{p,\alpha} := \|f k^\alpha_w\|_{p,\alpha} \|f^{-1} k^\alpha_w\|_{q,\alpha} < \eta^2 [f, g]_{p,\alpha}^2,
\]
and \( T_f T_g \) is bounded and invertible on \( A^p_\alpha(\mathbb{D}) \). Moreover,
\[
\|T_f T_g\| \leq C \eta^{2n \times \max\{p, q\}} [f, g]_{2,\alpha}^{1+2n \times \max\{p, q\}}.
\]

For the proof of the necessity part in Theorem 1.5 we will adapt the proof of the necessity condition (6) for the boundedness of the Toeplitz products in [11]. For the sufficiency part, we define product Békollé-Bonami weights and use the corresponding estimate of the Bergman projection of the polydisc.

In the next section, we provide some useful results needed later. Proposition 1.3 is proved in section 3 while Theorem 1.4 is considered in section 4 where we also provide a proof of the estimate of the positive Bergman
projection that does not use extrapolation. The proof in [16] uses extrapolation. A generalization to the tube domain over the first octant that is the product of upper-half planes is considered in section 5. In section 6, we characterize bounded and invertible Toeplitz products on Bergman spaces of the unit polydisc $\mathbb{D}^n$, generalizing the result of [10]. Some comments are made in the last section.

As usual, given two positive quantities $A$ and $B$, the notation $A \lesssim B$ $(A \gtrsim B)$ means that $A \leq CB$ for some absolute positive constant $C$. The notation $A \eqsim B$ means that $A \lesssim B$ and $B \lesssim A$. We will use $C(p)$ to say that the constant that depends only on $p$.

2. Some useful tools

Given an interval $I \subset \mathbb{R}$, the Carleson box associated to $I$ is the subset $Q_I$ of $\mathbb{H}$ defined by

$Q_I := \{ x + iy \in \mathbb{H} : x \in I \text{ and } 0 < y < |I| \}.$

The center of $Q_I$ is the point $x_I + iy_I$ such that $x_I$ is the center of the interval $I$ and $y_I = \frac{|I|}{2}$. We have the following.

**Lemma 2.1.** Let $I$ be a subinterval of $\mathbb{R}$ and $Q_I$ the associated Carleson box. Then for any $w \in Q_I$,

$$|w_I - \overline{w}| \lesssim y_I$$

where $w_I = x_I + iy_I$ is the center of $Q_I$.

**Proof.** Let $w = x + iy$. On one side, we have

$$|w_I - \overline{w}|^2 = (x_I - x)^2 + (y_I + y)^2 > y_I^2.$$

On the other side, we have

$$|w_I - \overline{w}|^2 = (x_I - x)^2 + (y_I + y)^2 \leq 5y_I^2.$$

□

We observe that if $M$ is the point of $\mathcal{H}$ associated to $z = x + iy$ and $M_0$ the point associated to $x + i0$, then if $I$ is the interval centered at $M_0$ with length $2y$, then $Q_I$ has $M$ as center.

The following is easy to check.

**Lemma 2.2.** Let $1 < p < \infty$, $-1 < \alpha < \infty$. Then

$$\|k_w^\alpha\|_{p,\alpha} \lesssim y^{\left(\frac{2\alpha}{\alpha + 1}\right)(\frac{2}{p} - 1)}, \quad w = x + iy \in \mathcal{D}.$$

We have the following estimate.

**Lemma 2.3.** Let $1 < p < \infty$, $-1 < \alpha < \infty$. Then there is a constant $C > 0$ such that for any $f \in A_\alpha^p(\mathcal{H})$,

$$(14) \quad |f(z)|^p \leq C\|k_z^\alpha\|_{p,\alpha}^{-1}\|fk_z^\alpha\|_{p,\alpha}, \quad z \in \mathcal{H}.$$
Proof. Let $\alpha > -1$. If $Q_I$ is centered at $z = x + iy$, then using the mean value property and the two previous lemmas we obtain

$$|f(z)|^p \leq \frac{C}{V_\alpha(Q_I)} \left( \int_{Q_I} |f(w)|^p dV_\alpha(w) \right)^{\alpha} \int_{Q_I} |f(w)|^p dV_\alpha(w)$$

$$\leq \frac{C}{y^{2+\alpha}} \left( \int_{Q_I} |f(w)|^p dV_\alpha(w) \right)^{\alpha} \int_{Q_I} |f(w)|^p dV_\alpha(w)$$

$$= C \left( \frac{y^{2+\alpha}}{y^{1-q}} \right)^p \int_{Q_I} |f(w)|^p dV_\alpha(w)$$

$$\leq C \left( \frac{\|k_\alpha^w\|_{p,\alpha}}{\|f\|_{p,\alpha}} \right)^p \int_{Q_I} |f(w)| k_\alpha^w dV_\alpha(w)$$

$$\leq C \left( \frac{\|k_\alpha^w\|_{p,\alpha}}{\|f\|_{p,\alpha}} \right)^p.$$

$\square$

Let $\omega$ be a positive function defined on $H$ and $\alpha > -1$. We say $\omega$ is a Békollé-Bonami weight (or $\omega$ belongs to the class $B_{p,\alpha}(H)$) if

$$\|\omega\|_{B_{p,\alpha}} := \sup_{I \subset \mathbb{R}} \left( \frac{1}{|I|^{2+\alpha}} \int_{Q_I} \omega(z) dV_\alpha(z) \right)^{p-1} < \infty.$$

Observe that if we put $\sigma = \omega^{1-q}$, and use the notations $|Q_I|_{\omega,\alpha} = \int_{Q_I} \omega dV_\alpha$ and $|Q_I|_\alpha = |Q_I|_{1,\alpha}$, then

$$\|\omega\|_{B_{p,\alpha}} = \sup_{I \subset \mathbb{R}} \frac{|Q_I|_{\omega,\alpha}|Q_I|_{p,\alpha}^{p-1}}{|Q_I|_\alpha^p}.$$

The following can be observed as in [4]. It says that the bounded of a Toeplitz product is equivalent to a two-weight problem for the Bergman projection.

**Proposition 2.4.** Let $1 < p < \infty$, and $-1 < \alpha < \infty$. Then $T_fT_g$ is bounded on $A_p^\alpha(H)$ if and only if $P_\alpha$ is bounded from $L^p(H, |g|^{-p} dV_\alpha)$ to $L^p(H, |f|^{-p} dV_\alpha)$.

3. Proof of Proposition 1.3

Let us start by the following.

**Proposition 3.1.** Let $1 < p, q < \infty$, $p = q(p-1)$ and $-1 \leq \alpha < \infty$. Given $f \in A_p^\alpha(H)$ and $g \in A_p^\alpha(H)$, if the product $T_fT_g$ is bounded and invertible on $A_p^\alpha(H)$, then the two following conditions are equivalent

(i) \( \inf_{w \in H} |f(w)||g(w)| > 0. \)

(ii) \( \sup_{w \in H} \|fk_\alpha^w\|_{p,\alpha} \|gk_\alpha^w\|_{q,\alpha} < \infty. \)
Proof. We assume that \( T_f T_g \) is bounded and invertible on \( A^p_0(\mathcal{H}) \), so that its adjoint \((T_f T_g)^* = T_g T_f \) is bounded and invertible on \( A^q_0(\mathcal{H}) \). Let us first observe that the following conditions obviously hold. They are in fact the reproducing kernel formulations alluded in the first section.

\[
M_1 := \sup_{w \in \mathcal{H}} \| k^\alpha_w \|^{-1}_p \| T_f T_g k^\alpha_w \|_p \alpha < \infty
\]

and

\[
M_2 := \sup_{w \in \mathcal{H}} \| k^\alpha_w \|^{-1}_q \| T_g T_f k^\alpha_w \|_q \alpha < \infty.
\]

Next, observing that given \( w \in \mathcal{H} \), \( T_f T_g k^\alpha_w = g(w) f k^\alpha_w \), we obtain that for any \( w \in \mathcal{H} \),

\[
1 \leq \| k^\alpha_w \|^{-1}_p \| g(w) f k^\alpha_w \|_p \alpha \leq \| k^\alpha_w \|^{-1}_q \| T_f T_g k^\alpha_w \|_q \alpha.
\]

That is

\[
\| k^\alpha_w \|^{-1}_p \| g(w) \| \geq \frac{1}{\| (T_f T_g)^{-1} \|}.
\]

We also have that

\[
1 \leq \| k^\alpha_w \|^{-1}_q \| T_g T_f k^\alpha_w \|_q \alpha \leq \| k^\alpha_w \|^{-1}_q \| f(w) \| \leq \frac{1}{\| (T_f T_g)^{-1} \|}.
\]

Which provides that

\[
\| k^\alpha_w \|^{-1}_q \| f(w) \| \geq \frac{1}{\| (T_f T_g)^{-1} \|}.
\]

Combining (19) and (20), we see that there are two constants \( c_1, c_2 > 0 \) such that for any \( w \in \mathcal{H} \),

\[
\frac{c_1}{\| (T_f T_g)^{-1} \|} \leq \| f(w) \| \| g(w) \| \| f k^\alpha_w \|_p \alpha \leq c_2 M_1 M_2.
\]

Now suppose that

\[
M = \sup_{w \in \mathcal{H}} \| g k^\alpha_w \|_q \alpha \| f k^\alpha_w \|_p \alpha < \infty.
\]

Then from the left inequality in (21) we get that for any \( w \in \mathcal{H} \),

\[
| f(w) | \| g(w) \| \geq \frac{c_1}{M \| (T_f T_g)^{-1} \|} > 0.
\]

Hence (15) holds.

Conversely, if \( \eta = \inf_{w \in \mathcal{H}} | f(w) | \| g(w) | > 0 \), then the right inequality in (21) provides

\[
\| g k^\alpha_w \|_q \alpha \| f k^\alpha_w \|_p \alpha \leq \frac{c_2 M_1 M_2}{\eta} < \infty
\]

for any \( w \in \mathcal{H} \). That is (16) holds. The proof is complete. \( \square \)
REMARK 3.2. The above result holds in more general settings but appears not to have been presented this way anywhere else as far as we know.

Proof of Proposition 1.3. We start by observing that if \( f \equiv 0 \) or \( g \equiv 0 \), \( T_fT_g \equiv 0 \) which is bounded on \( A^p_\alpha(\mathcal{H}) \) but not invertible.

Now let us suppose that \( f \in A^p_\alpha(\mathcal{H}) - \{0\} \) and \( g \in A^q_\alpha(\mathcal{H}) - \{0\} \) so that the product \( T_fT_g \) is bounded and invertible. Then there are two possibilities,

\[
\sup_{w \in \mathcal{H}} \|fk^\alpha_w\|_{p,\alpha} \|gk^\alpha_w\|_{q,\alpha} = \infty
\]
or

\[
M = \sup_{w \in \mathcal{H}} \|fk^\alpha_w\|_{p,\alpha} \|gk^\alpha_w\|_{q,\alpha} < \infty.
\]

Suppose that (23) holds. Then (21) tells us that we will have that there is a constant \( C > 0 \) such that for any \( w \in \mathcal{H} \),

\[
C \leq \|f(w)\| \|g(w)\|.
\]

Integrating both sides of (24) with respect to \( d\nu(\alpha)(w) \) and applying Hölder’s inequality, we obtain that

\[
\infty \leq \int_{\mathcal{H}} |f(w)| \|g(w)|d\nu(\alpha)(w) \leq \|f\|_{p,\alpha} \|g\|_{q,\alpha}
\]

which contradicts the hypotheses \( f \in A^p_\alpha(\mathcal{H}) \) and \( g \in A^q_\alpha(\mathcal{H}) \).

We conclude that if \( T_fT_g \) is bounded and invertible, then (22) holds. The proof is complete. \( \square \)

Conjecture: Let \( 1 < p, q < \infty \), \( p = q(p-1) \) and \( -1 < \alpha < \infty \). Suppose \( f \in A^p_\alpha(\mathcal{H}) \) and \( g \in A^q_\alpha(\mathcal{H}) \). Then the product \( T_fT_g \) can not be bounded and invertible at the same time on \( A^p_\alpha(\mathcal{H}) \).

4. Boundedness of the Toeplitz products and weighted inequalities

We have said that the Sarason conjecture fails for Hardy and Bergman spaces. We will prove here that if one of the symbols satisfies an invariant condition, then the conjecture holds. This means in particular that in the case of the unit disc of \( \mathbb{C} \) or the unit ball of \( \mathbb{C}^n \), if one of the symbols is in the class \( B_{p,\alpha} \), then the Sarason’s conjecture holds.

As we are also willing to estimate the norm of Toeplitz products, we will need first to estimate the norm of the positive Bergman operator. This estimation was obtained in [16] and we first give a proof here that avoid extrapolation.

Let us recall some notions and notations. We consider the following system of dyadic grids,

\[
\mathcal{D}^\beta := \{2^j \left[ (0,1) + m + (-1)^j \beta \right] : m \in \mathbb{Z}, \ j \in \mathbb{Z} \}, \ \text{for } \beta \in \{0, 1/3\}.
\]

For more on this system of dyadic grids and its applications, we refer to [1, 9, 7, 9, 10]. We also consider the following positive operators introduced by S. Pott and M. C. Reguera in [16].

\[
Q^\alpha_{\beta} f := \sum_{I \in \mathcal{D}^\beta} \langle f, \frac{1}{|I|^{2+\alpha}} \rangle \alpha 1_Q.
\]
By comparing the positive kernel
\[ K^+_\alpha(z, w) = \frac{1}{|z - w|^{2+\alpha}} \]
and the box-type kernel
\[ K^{\beta}_\alpha(z, w) := \sum_{I \in D^\beta} \frac{1_{Q_I}(z)1_{Q_I}(w)}{|I|^{2+\alpha}}, \]
one proves the following (see [16] for details).

**Proposition 4.1.** There is a constant \( C > 0 \) such that for any \( f \in L^1_{\text{loc}}(H) \), \( f \geq 0 \), and \( z \in H \),
\[ P^+_\alpha f(z) \leq C \sum_{\beta \in \{0, 1/3\}} Q^\beta_\alpha f(z). \]

We have the following characterization of the Békollé-Bonami class of weights.

**Proposition 4.2.** Let \( 1 < p, q < \infty \), \( p = q(p - 1) \) and \( -1 < \alpha < \infty \). Suppose that \( \omega \in B_{p,q}(H) \). Then \( P_\alpha \) is bounded on \( L^p(\omega dV_\alpha) \). Moreover, \( \|P_\alpha\| \leq C(p)[\omega]_{B_{p,q}}^{\max\{1, \frac{q}{p}\}} \).

Proof. Our proof is inspired from the same type of proof for Calderón-Zygmund operators [13]. We start by recalling that given the dyadic grid \( D^\beta \) and a positive weight, the dyadic maximal function \( M_{\omega,\alpha} \) is defined for any \( f \in L^1_{\text{loc}}(H) \) by
\[ M_{\omega,\alpha} f = \sup_{I \in D^\beta} \frac{1_{Q_I}}{|Q_I|^{\omega,\alpha}} \int_{Q_I} |f| \omega dV_\alpha. \]
We observe that using for example techniques in [5], one obtains the following estimate for \( 1 < p < \infty \)
\[ \|M_{\omega,\alpha} f\|_{L^p(\omega dV_\alpha)} \leq C(p)\|f\|_{L^p(\omega dV_\alpha)}. \]

Next recall that given \( Q_I \), its upper-half is
\[ T_I := \{x + iy \in H : x \in I, \quad \text{and} \quad \frac{|I|}{2} < y < |I|\}. \]
It is clear that the family \( \{T_I\}_{I \in D} \) where \( D \) is a dyadic grid in \( \mathbb{R} \) provides a tiling of \( H \).

Now observe that to prove the proposition, it is enough by Proposition 4.1 to prove that the following boundedness holds (with the right estimate of the norm)
\[ Q^\beta_\alpha : L^p(\sigma dV_\alpha) \to L^p(\omega dV_\alpha), \quad \beta \in \{0, 1/3\} \]
and also observe the usual fact that the latter is equivalent to the following
\[ Q^\beta_\alpha(\sigma) : L^p(\sigma dV_\alpha) \to L^p(\omega dV_\alpha), \quad \beta \in \{0, 1/3\}, \quad \sigma = \omega^{1-q}. \]

Let \( f \in L^p(\sigma dV_\alpha) \) and \( g \in L^q(\omega dV_\alpha) \) with \( f, g > 0 \). We aim to estimate
\[ \langle Q^\beta_\alpha(\sigma f), g \omega \rangle_\alpha = \int_H Q^\beta_\alpha(\sigma f) g \omega dV_\alpha. \]
We start by the case $p \geq 2$. Clearly, using the notations

$$B_{\sigma,\alpha}(f, Q_l) = \frac{1}{|Q_l|_{\sigma,\alpha}} \int_{Q_l} f \sigma dV_{\alpha}$$

and

$$B_{\omega,\alpha}(g, Q_l) = \frac{1}{|Q_l|_{\omega,\alpha}} \int_{Q_l} g \omega dV_{\alpha},$$

we obtain

$$\Pi := \langle Q_{\alpha}^3(\sigma f), g\omega \rangle_{\alpha}$$

$$= \sum_{I \in D_{\beta}} \langle \sigma f, 1_{Q_l} \rangle_{\alpha} \langle \omega f, 1_{Q_l} \rangle_{\alpha} |Q_l|^{-1-\frac{2}{p}}$$

$$= \sum_{I \in D_{\beta}} B_{\sigma,\alpha}(f, Q_l)B_{\omega,\alpha}(g, Q_l) \frac{|Q_l|_{\sigma,\alpha}|Q_l|_{\omega,\alpha}}{|Q_l|_{\alpha}}$$

$$\leq [\omega]_{B_{p,\alpha}} \sum_{I \in D_{\beta}} B_{\sigma,\alpha}(f, Q_l)B_{\omega,\alpha}(g, Q_l) \frac{|Q_l|_{\sigma,\alpha}|Q_l|_{\omega,\alpha}}{|Q_l|_{\alpha}} \times 
\frac{|Q_l|^p_{\alpha}}{|Q_l|_{\sigma,\alpha}|Q_l|_{\omega,\alpha}}$$

$$= [\omega]_{B_{p,\alpha}} \sum_{I \in D_{\beta}} |Q_l|^{p-1}_{\alpha}|Q_l|^{2-p}_{\sigma,\alpha} B_{\sigma,\alpha}(f, Q_l)B_{\omega,\alpha}(g, Q_l).$$

We observe that $|Q_l|_{\alpha} \leq |T_I|_{\alpha}$ and as $T_I \subset Q_l$ and $p \geq 2$, $|Q_l|^{2-p}_{\sigma,\alpha} \leq |T_I|^{2-p}_{\sigma,\alpha}$. On the other hand, it is easy to see that

$$|T_I|_{\alpha} \leq |T_I|^{1/q}_{\alpha}|T_I|^{1/p}_{\alpha,\beta}.$$  

Thus

$$|Q_l|^{p-1}_{\alpha}|Q_l|^{2-p}_{\sigma,\alpha} \lesssim |T_I|^{p-1}_{\alpha}|T_I|^{2-p}_{\sigma,\alpha} \lesssim |T_I|^{1/p}_{\alpha,\beta}|T_I|^{1/q}_{\alpha,\beta}.$$  

It follows that

$$\Pi := \langle Q_{\alpha}^3(\sigma f), g\omega \rangle_{\alpha}$$

$$\lesssim [\omega]_{B_{p,\alpha}} \sum_{I \in D_{\beta}} |T_I|^{1/p}_{\sigma,\alpha}|T_I|^{1/q}_{\omega,\alpha} B_{\sigma,\alpha}(f, Q_l)B_{\omega,\alpha}(g, Q_l)$$

$$\leq [\omega]_{B_{p,\alpha}} \left( \sum_{I \in D_{\beta}} |T_I|_{\sigma,\alpha} (B_{\sigma,\alpha}(f, Q_l))^p \right)^{1/p} \left( \sum_{I \in D_{\beta}} |T_I|_{\omega,\alpha} (B_{\omega,\alpha}(g, Q_l))^q \right)^{1/q}$$

$$= [\omega]_{B_{p,\alpha}} \left( \sum_{I \in D_{\beta}} \int_{T_I} (B_{\sigma,\alpha}(f, Q_l))^p \sigma dV_{\alpha} \right)^{1/p} \left( \sum_{I \in D_{\beta}} \int_{T_I} (B_{\omega,\alpha}(g, Q_l))^q \omega dV_{\alpha} \right)^{1/q}$$

$$\leq [\omega]_{B_{p,\alpha}} \| M_{\sigma,\alpha} f \|_{L^p(\sigma dV_{\alpha})} \| M_{\omega,\alpha} g \|_{L^q(\omega dV_{\alpha})}$$

$$\leq C(p)[\omega]_{B_{p,\alpha}} \| f \|_{L^p(\sigma dV_{\alpha})} \| g \|_{L^q(\omega dV_{\alpha})}.$$

For the case $1 < p < 2$, we use the previous and duality. We observe that $Q_{\alpha}^3$ is self-adjoint with respect to the duality pairing $\langle , \rangle$. Hence

$$\| Q_{\alpha}^3 \|_{L^p(\omega dV_{\alpha})} = \| Q_{\alpha}^3 \|_{L^q(\sigma dV_{\alpha})} \leq C(q)[\sigma]_{B_{\sigma,\alpha}} \leq C(p)[\omega]_{B_{p,\alpha}}^{-1}.$$  

The proof is complete. \qed
\textbf{Theorem 4.3.} Let $1 < p, q < \infty$, $p = q(p-1)$, and $-1 < \alpha < \infty$. Suppose that $f$ and $g$ are holomorphic in $\mathcal{H}$ with
\begin{equation}
[f]_{p,\alpha} := \sup_{w \in \mathcal{H}} \|fk^\alpha_w\|_{p,\alpha}\|f^{-1}k^\alpha_w\|_{q,\alpha} < \infty.
\end{equation}
Then $T_f T_g$ is bounded on $A^p_{\alpha}(\mathcal{H})$ if and only if
\begin{equation}
[f, g]_{p,\alpha} := \sup_{w \in \mathcal{H}} \|fk^\alpha_w\|_{p,\alpha}\|gk^\alpha_w\|_{q,\alpha} < \infty.
\end{equation}
Moreover,
\[\|T_f T_g\| \leq C(p)[f, g]_{p,\alpha}[f]^{\max\{p, q\}}\]
and
\[\|f, g\|_{p,\alpha} \leq C(p)\|T_g T_f\|[f]_{p,\alpha}.
\]

\textit{Proof.} Let us start with the sufficiency part. We first observe that the condition on $f$ provides in particular that the weight $\omega = |f|^p$ is in $B_{p,\alpha}(\mathcal{H})$. Clearly, for the interval $I \subset \mathbb{R}$, let $Q_I$ be its associated Carleson box and $w$ its center. Then
\[
\infty > [f]_{p,\alpha} > \left( \int_{Q_I} |f|^p|k^\alpha_w|^p dV_\alpha \right)^{1/p} \left( \int_{Q_I} |f|^{-q}|k^\alpha_w|^q dV_\alpha \right)^{1/q} \leq \left( \int_{Q_I} |f|^p|k^\alpha_w|^p dV_\alpha \right)^{1/p} \left( \int_{Q_I} \frac{1}{|Q_1|^\alpha} \int_{Q_1} |f|^{-q} dV_\alpha \right)^{1/q} = \left( \int_{Q_I} \omega dV_\alpha \left( \frac{1}{|Q_1|^\alpha} \int_{Q_1} \omega^{1-q} \right)^{p-1} \right)^{1/p}.
\]
Hence if $\omega = |f|^p$, then
\[|\omega|_{B_{p,\alpha}} \lesssim [f]_{p,\alpha}^p.
\]
Next using Lemma 2.3, we obtain
\[
|T_f T_g h(z)| = |f(z)||P_{\alpha}(\overline{h})(z)| \leq |f(z)| \int_{\mathcal{H}} |g(w)||h(w)| |z - w|^{-2\alpha} dV_\alpha(w) = |f(z)| \int_{\mathcal{H}} |g(w)||f(w)||f(w)|^{-1}|h(w)| |z - w|^{-2\alpha} dV_\alpha(w) \leq [f, g]_{p,\alpha} [f(z)] \int_{\mathcal{H}} \frac{|f(w)|^{-1}|h(w)|}{|z - w|^{-2\alpha}} dV_\alpha(w).
\]
Hence to prove that $T_f T_g$ is bounded on $A^p_{\alpha}(\mathcal{H})$, it is enough to prove that the positive operator
\[h \mapsto |f(z)| \int_{\mathcal{H}} \frac{|f(w)|^{-1}|h(w)|}{|z - w|^{-2\alpha}} dV_\alpha(w), \quad z \in \mathcal{H}
\]
is bounded on $L^p(\mathcal{H}, dV_\alpha)$. The boundedness of the latter is equivalent to the boundedness of $P^\alpha_{\alpha}$ on $L^p(\mathcal{H}, |f|^p dV_\alpha)$ which holds by Proposition 4.2.
since the weight \( \omega = |f|^p \) is in the class \( B_{p,\alpha}(\mathcal{H}) \) and with the right estimate. Thus

\[
\|T_f T_{\mathcal{F}}\| \leq C(p)\|\omega\|_{p,\alpha}^{\max\{1, \frac{1}{p}\}}|f, g|_{p,\alpha} \leq C(p)|f, g|_{p,\alpha}^{\max\{p, q\}}.
\]

Let us now suppose that \( T_f T_{\mathcal{F}} \) is bounded on \( A^p_{q,\alpha}(\mathcal{H}) \). Then in particular we have

\[
\|k_w^\alpha\|_{q,\alpha}^{-1} \|f(w)\|g_k^\alpha_{q,\alpha} = \|k_w^\alpha\|_{q,\alpha}^{-1} \|f(w)\|g_k^\alpha_{p,\alpha}\| g_k^\alpha_{q,\alpha}
\leq \|T_g T_{\mathcal{F}}\|.
\]

It follows using Lemma 2.3 again that

\[
\|f k_w^\alpha_{p,\alpha}\|g_k^\alpha_{q,\alpha} \leq \|k_w^\alpha_{p,\alpha}\|_{q,\alpha}^{-1} \|f(w)\|\|f(w)\|^{-1} \|f k_w^\alpha_{p,\alpha}\|g_k^\alpha_{q,\alpha}
\leq C\|T_g T_{\mathcal{F}}\|\|k_w^\alpha_{p,\alpha}\|_{q,\alpha}^{-1} \|f(w)\|^{-1} \|f k_w^\alpha_{p,\alpha}\|_{p,\alpha}
\leq C\|T_g T_{\mathcal{F}}\|\|k_w^\alpha_{p,\alpha}\|_{p,\alpha} \|k_w^\alpha_{q,\alpha}\|_{q,\alpha} \|f k_w^\alpha_{p,\alpha}\|_{p,\alpha} \|f^{-1} k_w^\alpha_{w}\|_{q,\alpha}
\leq C\|T_g T_{\mathcal{F}}\|\|f\|_{p,\alpha} < \infty.
\]

The proof is complete. \( \square \)

5. THE CASE OF THE TUBE OVER THE FIRST OCTANT

Let \( n \geq 2 \), the first octant is the set

\[
(0, \infty)^n := \{(y_1, \cdots, y_n) \in \mathbb{R}^n : y_j > 0, j = 1, \cdots, n\}
\]

and the tube domain over it is

\[
\mathcal{H}^n := \mathbb{R}^n + i(0, \infty)^n = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n : \exists z_j > 0, j = 1, \cdots, n\}.
\]

We consider the problems of the previous sections on the tube domain over the first octant.

Given \( \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n \) and \( \beta = (\beta_1, \cdots, \beta_n) \in \mathbb{R}^2 \), the notation \( \alpha < \beta \) (resp. \( \alpha = \beta \)) means that \( \alpha_j < \beta_j \) (resp. \( \alpha_j = \beta_j \)), \( j = 1, \cdots, n \). The notation \( \alpha > \beta \) is equivalent to \( \beta < \alpha \) and \( \alpha \leq \beta \) is equivalent to \( \alpha < \beta \) or \( \alpha = \beta \). We also use the notations \( 0 = (0, \cdots, 0), -1 = (-1, \cdots, -1), \) and \( \infty = (\infty, \cdots, \infty) \).

For \( \alpha > -1 \) and \( 1 < p < \infty \), the weighted Bergman space \( A^p_{q,\alpha}(\mathcal{H}^n) \) consists of those analytic functions \( f \) on \( \mathcal{H}^n \) such that

\[
\|f\|_{p,\alpha} := \int_{\mathcal{H}^n} |f(z_1, \cdots, z_n)|^p dV_{\alpha_1}(z_1) \cdots dV_{\alpha_n}(z_n) < \infty.
\]

We will be also using the notation

\[
dV_{\alpha}(z) = dV_{\alpha_1}(z_1) \cdots dV_{\alpha_n}(z_n) := y_1^{\alpha_1} \cdots y_n^{\alpha_n} dx_1 \cdots dx_n dy_1 \cdots dy_n
\]

for \( \alpha > -1 \).

The Bergman space \( A^2_{q,\alpha}(\mathcal{H}^n) \) \((-1 < \alpha < \infty)\) is a reproducing kernel Hilbert space with kernel

\[
K_w^\alpha(z) = K^{\alpha_1}(z_1, w_1) \cdots K^{\alpha_n}(z_n, w_n) = \frac{1}{(z_1 - w_1)^{2+\alpha_1}} \cdots \frac{1}{(z_n - w_n)^{2+\alpha_n}}.
\]
That is for any $f \in A^2_\alpha(\mathcal{H}^n)$, the following representation holds
\begin{equation}
(33) \quad f(w) = P_\alpha f(w) = \langle f, K^\alpha_w \rangle_\alpha = \int_{\mathcal{H}^n} f(z) K^\alpha(w, z) dV_{\alpha}(z).
\end{equation}

For $f \in L^2(\mathcal{H}^n, dV_{\alpha})$, we can densely define the Toeplitz $T_f$ with symbol $f$ on $A^2_\alpha(\mathcal{H}^n)$ as in (2). We recall that the normalized reproducing kernel $k^\alpha_w$ of $A^2_\alpha(\mathcal{H}^n)$ is given by
\begin{equation}
k^\alpha_w(z) := \frac{K^\alpha_{w_1}(z_1) \ldots K^\alpha_{w_n}(z_n)}{\|K^\alpha_{w_1}\|_{2,\alpha} \ldots \|K^\alpha_{w_n}\|_{2,\alpha}}.
\end{equation}

Let us introduce the class $\mathcal{B}_{p,\alpha}(\mathcal{H}^n), 1 < p < \infty, \mathbb{R}^n \ni \alpha = (\alpha_1, \ldots, \alpha_n) > -1$. We say a positively locally integrable function $\omega$ on $\mathcal{H}^n$ belongs to $\mathcal{B}_{p,\alpha}(\mathcal{H}^n)$ if there is a constant $C > 0$ such that for any $k \in \{1, \ldots, n\}$,
\begin{equation}
(34) \quad \sup_{w=(w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n) \in H^{n-1}} [\omega(w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n)]_{\mathcal{B}_{p,\alpha}(\mathcal{H})} \leq C.
\end{equation}
We denote by $[\omega]_{\mathcal{B}_{p,\alpha}(\mathcal{H}^n)}$ the infimum of the constants $C$ in (34). Note that the class $\mathcal{B}_{p,\alpha}(\mathcal{H}^n)$ is not empty; one easily checks that the weight $\omega = \prod_{j=1}^n \omega_j$ where $\omega_j \in B_{p,\alpha_j}(\mathcal{H})$ belongs to $\mathcal{B}_{p,\alpha}(\mathcal{H}^n)$.

We next have the following result.

**Proposition 5.1.** Let $1 < p, q < \infty, p = q(p-1)$ and $1 < \alpha = (\alpha_1, \ldots, \alpha_n) < \infty$. Suppose that $\omega \in \mathcal{B}_{p,\alpha}(\mathcal{H}^n)$. Then both $P_\alpha$ and $P_\alpha^+$ are bounded on $L^p(\mathcal{H}^n, dV_{\alpha}(z))$. Moreover,
\begin{equation}
\|P_\alpha\|, \|P_\alpha^+\| \leq C[\omega]_{\mathcal{B}_{p,\alpha}(\mathcal{H}^n)}^{n \times \max\{1, \frac{\alpha}{p}\}}.
\end{equation}

**Proof.** Observe that $P_\alpha = P_{\alpha_1} \cdots P_{\alpha_n}$ where $P_{\alpha_j}$ is the one parameter Bergman projection in the $j$-th variable. It follows using the one parameter estimate of these projections in Proposition 4.2 that for any $f \in L^p(\mathcal{H}^n, dV_{\alpha_1}(z_1) \cdots dV_{\alpha_n}(z_n))$,
\begin{align*}
\|P_\alpha f\|_{p,\alpha} &= \|P_{\alpha_1} \cdots P_{\alpha_n} f\|_{p,\alpha} \\
&\leq C[\omega]_{\mathcal{B}_{p,\alpha}(\mathcal{H}^n)}^{\max\{1, \frac{\alpha}{p}\}} \|P_{\alpha_2} \cdots P_{\alpha_n} f\|_{p,\alpha} \\
&\leq C[\omega]_{\mathcal{B}_{p,\alpha}(\mathcal{H}^n)}^{\max\{1, \frac{\alpha}{p}\}} \|f\|_{p,\alpha}.
\end{align*}
The same observations hold for the positive operator $P_\alpha^+$. The proof is complete. \hfill \square

Let us observe also the following.

**Lemma 5.2.** Let $1 < \alpha < \infty$. Assume that $p > \max\{1, \frac{2}{2+\alpha_1}, \ldots, \frac{2}{2+\alpha_n}\}$ and put $p = q(p-1)$. If $f$ is analytic in $\mathcal{H}^n$ with
\begin{equation}
[f]_{p,\alpha} := \sup_{w \in \mathcal{H}^n} \|f k^\alpha_w\|_{p,\alpha} \|f^{-1} k^\alpha_w\|_{q,\alpha} < \infty,
\end{equation}
then there is a constant $C > 0$ such that for any $1 \leq k \leq n$, and any $w = (w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_{n-1}) \in \mathcal{H}^{n-1}$,
\begin{equation}
\sup_{z \in \mathcal{H}} \|f_{w_k} k^\alpha_z\|_{p,\alpha} \|f_{w_k}^{-1} k^\alpha_z\|_{q,\alpha} \leq C \sup_{\zeta \in \mathcal{H}^n} \|f \zeta\|_{p,\alpha} \|f^{-1} \zeta\|_{q,\alpha},
\end{equation}
where $f_w(z) = f(\zeta_1, \ldots, \zeta_n), \zeta_j = w_j$ for $j \neq k$ and $\zeta_k = z$. \hfill \square
Proof. We may suppose that \( k = 1 \). Let \( w = (w_1, \cdots, w_n) \in \mathcal{H}^n \) be given. For any \( z = x + iy \in \mathcal{H} \) given, we put \( \zeta = (z, w_1, \cdots, w_n) \) and \( \tilde{\alpha} = (\alpha_2, \cdots, \alpha_n) \). Then using Lemma 2.2 and Lemma 2.3, we obtain
\[
\| f_w k_z^\alpha \|_{p, \alpha}^p = \int_\mathcal{H} |f(u + iv, w)|^p |k_z^\alpha (u + iv)|^p v^\alpha_1 dV(u + iv)
\]
\[
\lesssim |k_w^\alpha|^{-p}_{p, \tilde{\alpha}} \| f \|_{p, \alpha}^p \int_\mathcal{H} |k_z^\alpha (u + iv)|^p v \left( \frac{2 + \alpha_1}{2} \right) (p - 2 + \alpha_1) dV(u + iv)
\]
\[
= |k_w^\alpha|^{-p}_{p, \tilde{\alpha}} \| f \|_{p, \alpha}^p \int_\mathcal{H} \frac{v \left( \frac{2 + \alpha_1}{2} \right) (p - 2 + \alpha_1)}{v^\alpha_1} dV(u + iv)
\]
\[
\lesssim |k_w^\alpha|^{-p}_{p, \tilde{\alpha}} \| f \|_{p, \alpha}^p.
\]

In the same way, we obtain
\[
\| f_w^{-1} k_z^\alpha \|_{q, \alpha} \lesssim |k_w^\alpha|^{-q}_{q, \tilde{\alpha}} \| f^{-1} k_\xi^\alpha \|_{q, \alpha}.
\]

Thus as \( |k_w^\alpha|^{-p}_{p, \tilde{\alpha}} |k_w^\alpha|^{-q}_{q, \tilde{\alpha}} \lesssim 1 \), we obtain
\[
\| f_w k_z^\alpha \|_{p, \alpha} \| f_w^{-1} k_z^\alpha \|_{q, \alpha} \lesssim \sup_{\xi \in \mathcal{H}^n} \| f_k^\alpha \|_{p, \alpha} \| f^{-1} k_\xi^\alpha \|_{q, \alpha}.
\]

The proof is complete. \( \square \)

We have the following extension of Theorem 4.3 to the multi-parameter case.

**Theorem 5.3.** Let \(-1 < \alpha < \infty\). Assume that \( p > \max \{1, \frac{2}{2 + \alpha_1}, \cdots, \frac{2}{2 + \alpha_n} \} \) and put \( p = q(p - 1) \). Suppose that \( f \) and \( g \) are analytic in \( \mathcal{H}^n \) with
\[
[f]_{p, \alpha} := \sup_{w \in \mathcal{H}^n} \| f_k^\alpha \|_{p, \alpha} \| f^{-1} k_\xi^\alpha \|_{q, \alpha} < \infty.
\]

Then \( T_f T_g \) is bounded on \( \mathcal{A}^\alpha_p (\mathcal{H}^n) \) if and only if
\[
[f, g]_{p, \alpha} := \sup_{w \in \mathcal{H}^n} \| f_k^\alpha \|_{p, \alpha} \| g k_\alpha^\alpha \|_{q, \alpha} < \infty.
\]

Moreover,
\[
\| T_f T_g \| \leq C(p) [f, g]_{p, \alpha} [f]_{p, \alpha}^{\max \{p, q\}}
\]
and
\[
[f, g]_{p, \alpha} \leq C(p) \| T_f T_g \| [f]_{p, \alpha}.
\]

**Proof.** From Lemma 5.2 and the beginning of the proof of Theorem 4.3 we obtain that the condition \( \| \omega \|_{B_{p, \alpha}(\mathcal{H}^n)} \lesssim [f]_{p, \alpha}^p \)

\습니다. Following the steps of the proof of Theorem 4.3 and using Proposition 5.1, we obtain the result. \( \square \)
6. Bounded and invertible Toeplitz products on the unit Polydisc of \( \mathbb{C}^n \)

Before proving Theorem 1.5, let us prove some useful results. We start by the following lemma.

**Lemma 6.1.** Let \( \alpha > -1 \), \( 1 < p < \infty \), \( qp = p + q \), and let \( f \in A^p_\alpha(\mathbb{D}^n) \), \( g \in A^q_\alpha(\mathbb{D}^n) \). Then

\[
\|f\|_{p,\alpha} \|g\|_{q,\alpha} \lesssim \|T_f T_g\|.
\]

**Proof.** Let \( f \in A^p_\alpha(\mathbb{D}^n) \) and \( g \in A^q_\alpha(\mathbb{D}^n) \). Consider on \( A^p_\alpha(\mathbb{D}^n) \) the operator \( f \otimes g \) defined by

\[
(f \otimes g) h = \langle h, g \rangle f,
\]

for \( h \in A^p_\alpha(\mathbb{D}^n) \). It is easily proved that \( f \otimes g \) is bounded on \( A^p_\alpha(\mathbb{D}^n) \) with norm equal to

\[
\|f \otimes g\| = \|f\|_{p,\alpha} \|g\|_{q,\alpha}.
\]

Next recall that the Berezin transform of an operator \( S \) defined on \( A^2_\alpha(\mathbb{D}^n) \) is given by

\[
B_\alpha(S)(w) = \langle S k_w^\alpha, k_w^\alpha \rangle.
\]

Observing that \( T_f K_w^\alpha = \overline{f}(w) K_w^\alpha \), we easily obtain

\[
B_t(T_f T_g)(w) = f(w) \overline{g}(w).
\]

One also proves as in the one parameter case that

\[
B_\alpha(f \otimes g)(w) = \langle (f \otimes g) k_w^\alpha, k_w^\alpha \rangle = \prod_{j=1}^n (1 - |w_j|^2)^{\alpha_j + 2} f(w) g(w).
\]

Note also that if \( (\lambda)_{\alpha,k} \) is a sequence defined so that the Taylor expansion of \( (1-x)^{2+\alpha} \) in a neighborhood of the origin is

\[
(1-x)^{2+\alpha} = \sum_{k=0}^\infty \lambda_{\alpha,k} x^k,
\]

then for \( \alpha > -1 \) the series in the right hand side of (41) is absolutely convergent in the closed unit disc of the complex plane. Hence, if \( x = (x_1, \ldots, x_n) \) and \( \lambda_{\alpha,l} = \lambda_{\alpha,1l_1} \cdots \lambda_{\alpha,nl_n} \) we have

\[
\prod_{j=1}^n (1-x_j)^{2+\alpha_j} = \sum_{l \in \mathbb{N}^n} \lambda_{\alpha,l} x^l,
\]

where \( x^l = x_1^{l_1} \cdots x_n^{l_n} \) and the series in (42) is absolutely convergent in the closed unit polydisc.

It follows from (39) that for \( t = (t_1, \ldots, t_n) \), the Berezin transform of the product \( T_{z^2} T_f T_{\overline{g}} T_{z^2} \) is given by

\[
B_\alpha(T_{z^2} T_f T_{\overline{g}} T_{z^2})(w) = B_t(T_{z^2} f T_{\overline{g}})(w) = w^t f(w) g(w) w^t.
\]
Thus using the expansion (42), equation (40) and the injectivity of the 
Berezin transform, we obtain

\[ f \otimes g = \sum_{\lambda \in \mathbb{N}^n} \lambda \cdot T_{\lambda} f \otimes T_{\lambda} g = \sum_{\lambda \in \mathbb{N}^n} \lambda \cdot T_{\lambda_{1} \cdots \lambda_{n}} f \otimes T_{\lambda_{1} \cdots \lambda_{n}} g. \]

It follows that if \( s_{\alpha} = \sum_{\lambda \in \mathbb{N}^n} |\lambda_{\alpha}| \), then as \( ||T_{\lambda}|| = 1 \), we have

\[ ||f||_{p,\alpha} ||g||_{q,\alpha} = ||f \otimes g|| \leq s_{\alpha} ||T_{\lambda} f|| \]

for \( f \in A_{p}^{\alpha}(\mathbb{D}^n) \) and \( g \in A_{q}^{\alpha}(\mathbb{D}^n) \). The proof is complete. \( \square \)

We also need the following necessary condition for the boundedness of the Toeplitz product. It extends the one-parameter result in [11].

**Proposition 6.2.** Let \( \alpha > -1 \), \( 1 < p < \infty \), \( qp = p + q \), and let \( f \in A_{p}^{\alpha}(\mathbb{D}^n) \), \( g \in A_{q}^{\alpha}(\mathbb{D}^n) \). If \( T_{f} T_{g} \) is bounded on \( A_{p}^{\alpha}(\mathbb{D}^n) \), then (12) holds. Moreover,

\[ [f, g]_{p,\alpha} \lesssim ||T_{f} T_{g}||. \]

**Proof.** The proof is essentially the same as in the one-parameter in [11]. Let us give it here for completeness. We start by recalling that for \( a \in \mathbb{D} \), the transform \( z \mapsto \varphi_{a}(z) = \frac{a - z}{1 - \bar{a}z} \) is an automorphism of the unit disc \( \mathbb{D} \) with inverse \( \varphi_{a}^{-1} = \varphi_{a} \), and such that \( \varphi_{a}(0) = a \) and \( \varphi_{a}(a) = 0 \). Now for \( a = (a_{1}, \cdots, a_{n}) \in \mathbb{D}^{n} \), the corresponding mapping in the unit polydisc \( \mathbb{D}^{n} \) is the transform \( z = (z_{1}, \cdots, z_{n}) \in \mathbb{D}^{n} \mapsto \varphi_{a}(z) = (\varphi_{a_{1}}(z_{1}), \cdots, \varphi_{a_{n}}(z_{n})) \). This is an automorphism of \( \mathbb{D}^{n} \) and its real Jacobian is \( \prod_{j=1}^{n} |\varphi_{a_{j}}(z_{j})| \) (see [19]). It follows that as in the one-parameter case, the change of variables \( \varphi_{a}(z) \) works in the polydisc and the following holds:

\[ \int_{\mathbb{D}^{n}} f(\varphi_{a}(z)) dv_{a}(z) = \int_{\mathbb{D}^{n}} f(z)\left| k_{a}^{\alpha}(z) \right|^{2} dv_{a}(z). \]

Let us consider the mapping \( U_{a}(h) = (h \circ \varphi_{a}) k_{a}^{\alpha} \). It is easy to check that \( U_{a} \) is an isometry of \( A_{p}^{\alpha}(\mathbb{D}^n) \). Also, \( U_{a} \) is unitary and commutes with the Toeplitz operators in the following sense:

\[ U_{a} T_{f} = T_{f \circ \varphi_{a}} U_{a}, \quad f \in A_{p}^{\alpha}(\mathbb{D}^n) \]

and consequently,

\[ T_{f} U_{a} = U_{a} T_{f \circ \varphi_{a}} \]

(see [19]). As a consequence, one obtains that for any \( f_{1} \in A_{p}^{\alpha}(\mathbb{D}^n) \) and \( g_{1} \in A_{q}^{\alpha}(\mathbb{D}^n) \),

\[ T_{f_{1} \circ \varphi_{a}} T_{g_{1} \circ \varphi_{a}} = U_{a} (T_{f_{1}} T_{g_{1}}) U_{a}. \]

As said above, \( U_{a} \) is isometry of \( A_{p}^{\alpha}(\mathbb{D}^n) \). Using (15), one checks that the following operator provides also an isometry of \( A_{p}^{\alpha}(\mathbb{D}^n) \),

\[ U_{a}^{p}(h) = (h \circ \varphi_{a}) k_{a}^{\alpha} (\left| k_{a}^{\alpha} \right|^{2})^{2/p-1}. \]

The following operator was introduced in one-parameter case by J. Miao in [11]:

\[ V_{a}^{p}(h) = P_{\alpha} (U_{a}^{p}(h)) = P_{\alpha} \left( (h \circ \varphi_{a}) k_{a}^{\alpha} (\left| k_{a}^{\alpha} \right|^{2})^{2/p-1} \right). \]
Using the following identity which holds for any \( f_2 \in A^{p}_a(\mathbb{D}^n) \) and any \( g_2 \in A^{q}_a(\mathbb{D}^n) \),
\[
T_{f_2}P_\alpha (g_2) = P_\alpha (T_{g_2} f),
\]
we obtain for any \( h \in A^{p}_a(\mathbb{D}^n) \),
\[
T_{\left( \frac{1}{k_a^q} \right)}^{1-2/p} V^p_a(h) = P_\alpha \left( \left( \frac{1}{k_a^q} \right)^{1-2/p} (h \circ \varphi_a) k_a^\alpha \left( \frac{1}{k_a^a} \right)^{2/p-1} \right)
= (h \circ \varphi_a) k_a^\alpha.
\]
That is
\[
T_{\left( \frac{1}{k_a^q} \right)}^{1-2/p} V^p_a(h) = U_a(h).
\]
From (47), we have for any \( u \in A^{p}_a(\mathbb{D}^n) \) and any \( v \in A^{q}_a(\mathbb{D}^n) \),
\[
\langle T_{f_1} \circ \varphi_a , T_{g_1} \circ \varphi_a, u, v \rangle_{\alpha} = \langle T_{f_1} (U_a(u)), T_{g_1} (U_a(v)) \rangle_{\alpha}.
\]
It follows from (50) and (51) that for any \( u \in A^{p}_a(\mathbb{D}^n) \) and any \( v \in A^{q}_a(\mathbb{D}^n) \),
\[
\langle T_{f_1} \circ \varphi_a , T_{g_1} \circ \varphi_a, u, v \rangle_{\alpha} = \langle T_{T_{f_1}^\alpha (k_a^q)}^{1-2/p} V^p_a(u), T_{T_{g_1}^\alpha (k_a^q)}^{1-2/p} V^q_a(v) \rangle_{\alpha}.
\]
Let us take
\[
f_1 = \frac{f}{(k_a^q)^{1-2/p}}, \quad g_1 = \frac{g}{(k_a^q)^{1-2/p}}.
\]
Clearly, \( f_1 \) and \( g_1 \) are holomorphic and consequently,
\[
T_{f_1} T_{\left( \frac{1}{k_a^q} \right)}^{1-2/p} = T_{\frac{1}{f_1(k_a^q)}}^{1-2/q} = T_f
\]
and
\[
T_{g_1} T_{\left( \frac{1}{k_a^q} \right)}^{1-2/p} = T_{\frac{1}{g_1(k_a^q)}}^{1-2/p} = T_g.
\]
It follows from the Hölder’s inequality that
\[
|\langle T_{f_1} \circ \varphi_a , T_{g_1} \circ \varphi_a, u, v \rangle_{\alpha}| = |\langle T_{T_{f_1}^\alpha (k_a^q)}^{1-2/p} V^p_a(u), T_{T_{g_1}^\alpha (k_a^q)}^{1-2/p} V^q_a(v) \rangle_{\alpha}|
\leq ||T_f T_{\frac{1}{f_1(k_a^q)}}^{1-2/q}||_p ||V^p_a(u)||_{p,\alpha} ||V^q_a(v)||_{q,\alpha}
\leq ||T_f T_{\frac{1}{g_1(k_a^q)}}^{1-2/p}||_p ||U^p_a(u)||_{p,\alpha} ||U^q_a(v)||_{q,\alpha}
= ||T_f T_{\frac{1}{g_1(k_a^q)}}^{1-2/p}||_p ||u||_{p,\alpha} ||v||_{q,\alpha}
\]
Thus using Lemma 6.1,
\[
||f_1 \circ \varphi_a||_{p,\alpha} ||g_1 \circ \varphi_a||_{q,\alpha} = ||\langle T_{f_1} \circ \varphi_a , T_{g_1} \circ \varphi_a, u, v \rangle_{\alpha}|| \leq ||T_f T_{\frac{1}{g_1(k_a^q)}}^{1-2/p}||_p.
\]
Next we observe that
\[
f_1 \circ \varphi_a = (f \circ \varphi_a)(k_a^\alpha \circ \varphi_a)^{-1+\frac{2}{p}} = (f \circ \varphi_a)(k_a^\alpha \circ \varphi_a)^{1-\frac{2}{p}}
\]
and consequently using (15) that
\[
||f_1 \circ \varphi_a||_{p,\alpha} = ||f \circ \varphi_a||_{p,\alpha}.
\]
In the same way, we obtain
\[
||g_1 \circ \varphi_a||_{p,\alpha} = ||g \circ \varphi_a||_{p,\alpha}.
\]
We conclude that
\[
||f \circ \varphi_a||_{p,\alpha} ||g \circ \varphi_a||_{q,\alpha} \lesssim ||T_f T_{\frac{1}{g_1(k_a^q)}}^{1-2/p}||_p.
\]
The proof is complete.
Proof of Theorem 4.3. Note that if $T_f T_{\overline{g}}$ is bounded on $A^p_w(\mathbb{D}^n)$, then (12) holds by Proposition 6.2. Hence if $T_f T_{\overline{g}}$ is bounded and invertible on $A^p_w(\mathbb{D}^n)$, (12) holds and by Proposition 4.1 and the remark after its proof, this is equivalent to (13).

Before proving assertion (ii), we need to recall some notations and make some remarks about Békollé-Bonami weights on the polydisc. We start by recalling if $T$ is the unit circle, the Carleson box associated to an interval $I \subset \mathbb{T}$ is defined by

$$Q_I : \{re^{i\theta} : 1 - |I| < r < 1, \text{ and } e^{i\theta} \in I\}.$$ 

Békollé-Bonami weights in the unit disc are defined as in section 2 using the Carleson boxes defined by (52), moreover, Proposition 4.2 holds the same in this setting (see also [1]). The class $B_{p,\alpha}(\mathbb{D}^n)$ is also defined as in the previous section and the result in Lemma 5.2 and Proposition 5.1 hold the same in the unit polydisc.

Now let us suppose that both (12) and (13) are satisfied. We first prove that the weight $\omega = |f|^p$ belongs to the class $B_{p,\alpha}(\mathbb{D}^n)$. Let us observe for this that as $\eta = \inf_{z \in \mathbb{D}^n} |f(z)| |g(z)| > 0$, we have that for any $z \in \mathbb{D}^n$, $|f(z)||g(z)| > \eta$ and consequently, that $|g(z)| > \eta |f(z)|^{-1}$. It follows using the analogue of the estimate (14) for the polydisc that for any $w \in \mathbb{D}^n$,

$$[f]_{p,\alpha} := \|f k^\alpha_w\|_{p,\alpha} \|f^{-1} k^\alpha_w\|_{q,\alpha} = \|f k^\alpha_w\|_{p,\alpha} |f(w)||f(w)|^{-1} \|f^{-1} k^\alpha_w\|_{q,\alpha} \leq \eta^2 \|f k^\alpha_w\|_{p,\alpha} |f(w)||g(w)||g k^\alpha_w\|_{q,\alpha} |\leq \eta^2 \|f k^\alpha_w\|_{p,\alpha} \|g k^\alpha_w\|_{q,\alpha} \rangle^2 \leq \eta^2 \langle f, g \rangle_{p,\alpha}^2.$$

That is

$$\sup_{w \in \mathbb{D}^n} \|f k^\alpha_w\|_{p,\alpha} \|f^{-1} k^\alpha_w\|_{q,\alpha} \leq \eta^2 \langle f, g \rangle_{p,\alpha}^2,$$

and as in the previous sections, this implies that $\omega = |f|^p$ belongs to the class $B_{p,\alpha}(\mathbb{D}^n)$. Thus as in the last section, we obtain that $T_f T_{\overline{g}}$ is bounded and

$$\|T_f T_{\overline{g}}\| \leq C\langle f, g \rangle_{p,\alpha} \|f\|_{p,\alpha} \|g\|_{p,\alpha} \leq \eta^{2n \times \max\{p, q\}} \|f, g\|_{2,\alpha}^{1 + 2n \times \max\{p, q\}}.$$

To prove that $T_f T_{\overline{g}}$ is invertible, we first observe with the right hand inequality in (21) that there is a constant $c_1 > 0$ such that for any $w \in \mathbb{D}^n$,

$$c_1 \leq |f(w)||g(w)|\|f k^\alpha_w\|_{p,\alpha} \|g k^\alpha_w\|_{q,\alpha} \leq \|f(w)||g(w)||f, g\|_{p,\alpha}.$$

Hence that the function $(f\overline{g})^{-1}$ is bounded on $\mathbb{D}^n$. Thus the Toeplitz operator $T_{(f\overline{g})^{-1}}$ is bounded on $A^p_w(\mathbb{D}^n)$. Hence, denoting $I$ the identity operator, it follows that

$$T_f T_{\overline{g}} T_{(f\overline{g})^{-1}} = I = T_{(f\overline{g})^{-1}} T_f T_{\overline{g}}.$$

The proof is complete. □

7. Some comments

Now that the Sarason conjecture has been disproved, it seems natural to look for conditions on the symbols that are necessary and sufficient for the associated Toeplitz products to be bounded. The result in Theorem 4.3 is not surprising for specialists in two-weight problems because the Sarason
condition (6) is the right substitute of what we can call joint Békollé-Bonami condition, that is defined given two weights \( \sigma, \omega \) by

\[
[\sigma, \omega]_{B_{p,\alpha}} := \sup_I \frac{|Q_I|_{\omega,\alpha} |Q_I|_{\sigma,\alpha}^{p-1}}{|Q_I|_{\sigma,\alpha}^p} < \infty.
\]

Also, assumption that one of the symbol satisfies an invariant condition comes from the need of having the weight \( \sigma \) in the class \( B_{p,\alpha} \) so that the condition (53) becomes sufficient for the boundedness of the Bergman projection from \( L^p(\sigma^{-1/p-1}) \) to \( L^p(\omega) \). This fact is well known in real harmonic analysis and will be discussed elsewhere.

To finish, let us observe that when \( g = 1/f \), the invariant condition above is actually necessary and sufficient and we can even deduce more from what has been proved in the previous sections.

**Proposition 7.1.** Let \( 1 < p < \infty \) and \( f \in A^p_\alpha(\mathbb{D}) \) so that \( \frac{1}{p} \in A^q_\alpha(\mathbb{D}) \), \( pq = p + q \). Then the following are equivalent.

(i) \( T_f T_{1/f} \) is bounded on \( A^p_\alpha(\mathbb{D}) \).

(ii) \[
\sup_{w \in \mathbb{D}} \|f k_{\alpha,w}^0\|_{p,\alpha} \|f^{-1} k_{\omega,w}^0\|_{q,\alpha} < \infty.
\]

(iii) If \( \omega = |f|^p \) and \( \sigma = \omega^{1-q} = |f|^{-q} \), then

\[
\sup_I \frac{|Q_I|_{\omega,\alpha} |Q_I|_{\sigma,\alpha}^{p-1}}{|Q_I|_{\sigma,\alpha}^p} < \infty.
\]

(iv) \( P^+_\alpha \) is bounded from \( L^p(|f|^{-p}d\nu_\alpha) \) to \( L^p(|f|^p d\nu_\alpha) \).

We note that the above proposition also holds in the case of the unit polydisc. That (i) \( \Rightarrow \) (ii) is Proposition 6.2. (ii) \( \Rightarrow \) (iii) is the beginning of the proof of Theorem 4.3 while (iii) \( \Rightarrow \) (iv) is Proposition 4.2. The implication (iv) \( \Rightarrow \) (i) is Proposition 2.4. There is a more direct proof of the equivalence (ii) \( \Leftrightarrow \) (iii) that can be found in [8].

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