On the Coupling of Gravitons to Matter

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Using relationships between open and closed strings, we present a construction of tree-level scattering amplitudes for gravitons minimally coupled to matter in terms of gauge theory partial amplitudes. In particular, we present examples of amplitudes with gravitons coupled to vectors or to a single fermion pair. We also present two examples with massive graviton exchange, as would arise in the presence of large compact dimensions. The gauge charges are represented by flavors of dynamical scalars or fermions. This also leads to an unconventional decomposition of color and kinematics in gauge theories.

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Although gravity and gauge theories both contain local symmetries, their dynamical behavior are rather different. Nonetheless, there are some striking relationships between the theories such as the AdS/CFT correspondence and the perturbative relationships implied by the structure of closed string amplitudes in terms of open string amplitudes. In this letter we demonstrate that at least for the case of a flat background, there is an elegant and powerful way to obtain the interaction of gravitons with matter, based on the structure of heterotic strings. Our construction is valid for the cases of gravity theories, it would be useful to have more general tree amplitudes besides those of pure gravity or supergravity.

Here we discuss amplitudes where gravitons are minimally coupled to gauge theories. Maximally helicity violating (MHV) $n$-point tree amplitudes for pure gravity and for gravitationally dressed Parke-Taylor gluon amplitudes have been presented in refs. Using the KLT relations, we extend these results to the remaining MHV gravitationally dressed gluon amplitudes and to ones with a single fermion pair. Our construction also applies to cases of non-maximal helicity violation, but the MHV cases are the simplest to present. We also present four-point examples with a (massive) graviton exchange.

The KLT equations for field theory gravity tree amplitudes are,

\[ \mathcal{M}_4(1, 2, 3, 4) = -i s_{12} A_4(1, 2, 3, 4) \tilde{A}_4(1, 2, 4, 3), \]
\[ \mathcal{M}_5(1, 2, 3, 4, 5) = i \left[ s_{12} s_{34} A_5(1, 2, 3, 4, 5) \tilde{A}_5(2, 1, 4, 3, 5) \right. \]
\[ \left. + s_{13} s_{24} A_5(1, 3, 2, 4, 5) \tilde{A}_5(3, 1, 4, 2, 5) \right] \]

where $s_{ij} = (k_i + k_j)^2$. The $A_n$ and $\tilde{A}_n$ are tree-level color-ordered gauge theory partial amplitudes. Our notation is to suppress polarizations, $\varepsilon_i$, and momenta, $k_i$, leaving only the label $i$ for each leg. Explicit forms of the KLT equations for an arbitrary number of external
legs may be found in ref. 3. We have dropped the couplings, but for each gauge and gravitational interaction appearing on the left-hand-side, factors of the couplings $g$ and $\kappa/2$ (where $\kappa^2 = 32\pi G_N$) are respectively required.

Although the use of the KLT equations in the field theory limit does not require a fully consistent string theory, some features of string theory are retained. For example, an oriented string will necessarily contain antisymmetric tensor and dilaton states in its massless spectrum. When describing graviton exchanges, the KLT equations will therefore also implicitly contain these states. However, for the cases dealt with in this letter, the antisymmetric tensor and dilaton decouple, so that extra projections are not required to remove these states, if unwanted.

In order to apply the KLT equations to the cases of gravity coupled to matter, we must specify the factorization of each state into states appearing solely in gauge theories. In doing so, we must ensure that the spin and charges for each state are the desired ones. We factorize each graviton into a product of vectors, each gluon into either a product of a vector and a scalar or a product of two fermions and each quark into a fermion and a scalar. As motivated by the structure of the heterotic string, the gauge charges of each physical state are assigned as flavor charges carried by scalars or by fermions. At the linearized level these factorizations are trivial. In a sense, we use the KLT equations to extend them to the fully interacting theories.

As a specific example, consider $SU(N_c)$ gauge theory minimally coupled to Einstein gravity. In fig. 1 we present rules for constructing, via the KLT equations, gluon amplitudes dressed with gravitons. Dashed lines represent scalars and curly lines vectors.

It starts with a gauge theory amplitude only to wind up at the same amplitude. It will, however, easily generalize to cases including gravitons.

With the standard color decomposition 4, the four-gluon tree amplitude is expressed as,

\[
\mathcal{M}_4(1^a,2^a,3^a,4^a) = g^2 \sum_\sigma \text{Tr}[T^{a\sigma(1)}T^{a\sigma(2)}T^{a\sigma(3)}T^{a\sigma(4)}] \times A_4(\sigma(1),\sigma(2),\sigma(3),\sigma(4)),
\]

where $\sigma$ runs over the set of non-cyclic permutations and the $A_4$ are color-ordered partial amplitudes. We denote gluon, graviton, scalar and quark legs by the subscripts $g$, $h$, $s$ and $q$. (Our fundamental representation color matrices are normalized by $\text{Tr}[T^aT^b] = \delta^{ab}$.)

By applying the KLT formula 3 we express the four-gluon amplitude with only gluon exchanges as

\[
\mathcal{M}_4(1^a,2^a,3^a,4^a) = -ig^2 \frac{s_12}{s_1} \times A_4(1^a,2^a,3^a,4^a) \times A_4(1^s,2^s,3^s,4^s),
\]

where the diagrams for obtaining $A_4(1^a,2^a,3^a,4^a)$ and $A_4(1^s,2^s,3^s,4^s)$ are given in figs. 3 and 3. Note that in each of the two cases the labels on the diagrams follow the cyclic ordering of the arguments of the amplitudes. The value of the first of these partial amplitudes is well known and may be found in, for example, ref. 4. Evaluating the diagrams in fig. 3, the four-scalar partial amplitude is,

\[
\tilde{A}_4(1^s,2^s,3^s,4^s) = \frac{2if^{abc}}{s_1} + \frac{2if^{abc}}{s_13},
\]

where $f^{abc} = -i\text{Tr}([T^a,T^b]T^c)/\sqrt{2}$ are the group structure constants. This type of decomposition of group theory and kinematics is rather unconventional. Nevertheless, it is straightforward to verify that the amplitude in eq. 3 is equal to that in eq. 2. Similar decompositions hold for any number of external legs, by applying

\[\begin{align*}
\frac{1}{2} \sum^3_{\mu}\frac{3}{\sqrt{2}}(\kappa_1 - \kappa_2)^\mu + \text{cyclic}
\end{align*}\]
$n$-point KLT equations. In this way both graviton and full gauge theory amplitudes satisfy KLT equations, providing a natural way to obtain mixed amplitudes of states appearing in gravity and in gauge theories.

\[ A(1_g, 2_g, 3_g, 4_g) \]

**FIG. 2.** The color-ordered Feynman diagrams contributing to $A(1_g, 2_g, 3_g, 4_g)$.

\[ \begin{align*}
A_{d}(1_{d}^{+}, 2_{d}^{+}, 3_{d}^{+}, 4_{d}) & = \text{if gauge theory partial amplitudes } \\
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A_{d}(1_{d}^{+}, 2_{d}^{+}, 3_{d}^{+}, 4_{d}) & = \text{if gauge theory partial amplitudes }
\end{align*} \]

**FIG. 3.** The two diagrams contributing to the scalar partial amplitude $A_{d}(1_{d}^{+}, 2_{d}^{+}, 3_{d}^{+}, 4_{d}^{+})$. Each diagram is dressed with flavor group theory factors.

For example, we can express $\mathcal{M}_4(1_g, 2_g, 3_g, 4_h)$, in terms of the gauge theory partial amplitudes $A_4(1_g, 2_g, 3_g, 4_h)$, whose diagrams are given in fig. 2, and the three-scalar one-vector partial amplitude $A_4(1_{d}^{+}, 2_{d}^{+}, 3_{d}^{+}, 4_{d}^{+})$ whose diagrams are the same as those in fig. 2, except that leg 4 is replaced with a vector.

More generally, by applying the Feynman rules in fig. 2 we have obtained four-, five-, and six-point mixed amplitudes for gluons and gravitons. A sampling of these is,

\[ M_5(1_g^+, 2_g^+, 3_g^+, 4_g^+, 5_h^-) = -ig^2\left(\frac{\kappa}{2}\right)\frac{45}{41} \]

\[ M_5(1_g^-, 2_g^+, 3_g^+, 4_g^+, 5_h^-) = ig\left(\frac{\kappa}{2}\right)^2\frac{45}{31} \]

\[ M_5(1_g^-, 2_g^+, 3_g^+, 4_g^-, 5_h^0) = 0 \]

\[ M_5(1_g^-, 2_g^+, 3_g^+, 4_g^+, 5_h^-) = i\left(\frac{\kappa}{2}\right)^3\frac{45}{2} \]

for the coefficient of the color traces $\text{Tr}[T^{a_1} \cdots T^{a_n}]$ which follow the ordering of gluon legs. The complete amplitudes are given by summing over non-cyclic permutations of the gluon legs after multiplying by the color traces.

The function $h(a, \{2P\}, b)$ appearing in the last amplitude is the half-soft function defined in ref. 3. Our convention for the ‘±’ helicity labeling is to take all legs to be outgoing. In these amplitudes graviton exchanges between gluons are not incorporated; such contributions are suppressed by additional powers of the gravitational coupling. We have expressed the amplitudes in terms of $D = 4$ spinor inner products. (See e.g. ref. 3.) These are denoted by $(ij) = (i^-|j^+)$ and $[ij] = (i^+|j^-)$, where $|i^\pm\rangle$ are massless Weyl spinors of momentum $k_i$, labeled with the sign of the helicity. They are antisymmetric, with norm $|(ij)| = |[ij]| = \sqrt{\kappa}.$

Although $D = 4$ spinor helicity is quite useful, the KLT equations do not require its use, and are valid in arbitrary dimensions. Likewise, the conditions of maximal helicity violation are not essential, although they considerably simplify amplitudes. We have, for example, obtained all the independent amplitudes with five gluons and one graviton using the KLT equations and the six-gluon partial amplitudes given in ref. 14.

The cases corresponding to gravitationally dressed Parke-Taylor amplitudes match those previously computed by Selivanov 13 using a generating function method. Selivanov’s $n$-point expression for the coefficient of $\text{Tr}[T^{a_1}T^{a_2} \cdots T^{a_n}]$ is

\[ M_n(1_g^-, 2_g^+, \ldots, i_g^-, m_g^+(m+1)_h^+, \ldots, n_h^0) = ig^{n-2}\left(\frac{\kappa}{2}\right)^{n-m} \frac{(1)^4}{(12)(23) \cdots (m1)}S(1,i,\{h^+\},\{g^+\}) \]

where all legs have positive helicity except 1 and i and

\[ S(i,j,\{h^+\},\{g^+\}) = \left( \prod_{m \in \{h^+\}} \frac{d}{da_m} \right) \times \prod_{i \in \{g^+\}} \exp \left[ \sum_{n_1 \in \{h^+\}} \frac{a_{n_1}(\langle i|l\rangle \langle j|l\rangle)}{(n_1 i)(n_1 j)} \right] \times \prod_{n_2 \in \{h^+\}} \exp \left[ \sum_{n_2 \in \{h^+\}} a_{n_2}(\langle n_1 i|n_1 j\rangle) \frac{(n_1 i)(n_1 j)}{(n_2 i)(n_2 j)} \exp \left( \ldots \right) \right] \right|_{a_i=0} \]

In this factor legs $i$ and $j$ are the negative helicity ones while the sets $\{h^+\}$ and $\{g^+\}$ represent the positive helicity graviton and gluon legs.

The KLT equations immediately show that mixed gluon and graviton amplitudes vanish if all legs have identical helicity or if there is a single leg of opposite helicity, since the corresponding pure gluon amplitudes vanish. We also extrapolate our results to $n$-legs to obtain the remaining non-vanishing MHV gluon amplitudes dressed with gravitons. Doing so yields,

\[ M_n(1_g^-, 2_g^+, \ldots, m_g^+(m+1)_h^+, \ldots, n_h^0) = ig^{n-2}\left(\frac{\kappa}{2}\right)^{n-m} \frac{(1)^4}{(12)(23) \cdots (m1)} \times S(1,m+1,\{h^+\},\{g^+\}) \]

for the coefficient of the color factor $\text{Tr}[T^{a_1}T^{a_2} \cdots T^{a_n}]$. Furthermore, the amplitudes where the gluons are all of identical helicity vanish, independent of the graviton helicity configuration. These amplitudes were constructed by demanding that they satisfy the required factorization properties in all channels: The amplitudes have no multi-particle poles, obey the collinear and soft factorization for gluons given in ref. 14 and the ones for gravitons...
given in ref. [1] Moreover, they have the correct phase singularity for a gluon becoming collinear with a graviton. Although this does not constitute a complete proof that these results are valid for all $n$, in the past similar analytic requirements have invariably led to the correct results. (See e.g. refs. [2][3].)

The KLT equation (1) can also be directly applied to the case of a single fermion pair, by using the vertex $i\gamma^\mu/\sqrt{2}$ for the coupling of a fermion pair to a vector. For example, in eq. (6) the result of replacing legs 1 and 2 by gluinos is given by multiplying by the overall factor $(2m+1)/(1m+1)$. This is consistent with the result obtained by applying $N=1$ supersymmetry Ward identities [4]. For fundamental representation quarks one must also modify the associated color factor to $(T^{a}\cdots T^{n})_{ij}$. Cases with a single gravitino pair are similar. On the other hand, cases with multiple fermion pairs are more intricate. In particular, for the KLT factorization to work, auxiliary rules for assigning global charges in the color-ordered amplitudes appear to be necessary.

Although, we do not have a general formalism, it is interesting that a KLT factorization can also be found for examples with a massive graviton. For example, the four-gluon amplitude with a massive graviton exchange can be factorized as

$$M_{4}^{\text{ex}}(1^{-a_{1}}, 2^{a_{2}}, 3^{-a_{3}}, 4^{+a_{4}})$$
$$= -i\left(\frac{k}{2}\right)^{2} (s_{12} - m^{2}) A_{4}(1^{-a_{1}}, 2^{a_{2}}, 3^{-a_{3}}, 4^{+a_{4}})$$
$$\times \tilde{A}_{4}(1^{a_{1}}, 2^{a_{2}}, 3^{+a_{3}}, 4^{a_{4}})$$
$$= -i\left(\frac{k}{2}\right)^{2} [4][2][1]^{3/2} \delta_{s_{12} - m^{2}}^{a_{1}a_{2}a_{3}a_{4}a_{5}}$$

On the right-hand-side the color-ordered amplitudes are those for quark scattering via massive vector boson exchanges. (This factorization works because of our particular assignment of $a_{i}$ charges.) Similarly, the scattering of fundamental representation quarks by gluons via massive graviton exchange is

$$M_{4}^{\text{ex}}(1^{-a_{1}}, 2^{a_{2}}, 3^{-a_{3}}, 4^{+a_{4}})$$
$$= -i\left(\frac{k}{2}\right)^{2} (s_{12} - m^{2}) A_{4}(1^{i_{1}}, 2^{i_{2}}, 3^{a_{3}}, 4^{+a_{4}})$$
$$\times \tilde{A}_{4}(1^{a_{1}}, 2^{a_{2}}, 3^{+a_{3}}, 4^{a_{4}})$$
$$= i\left(\frac{k}{2}\right)^{2} (13^{2}) [4][2][1]^{3/2} \delta_{s_{12} - m^{2}}^{a_{1}a_{2}a_{3}a_{4}a_{5}}$$

where $q$ and $Q$ are distinct massless fermion species. For both amplitudes (6) and (7) we have verified that both sides of the equations are equal by explicitly computing the amplitudes on the left-hand-side using conventional Feynman rules. The examples (6) and (7) suggest that it should be possible to obtain a general formalism for (massive) graviton exchanges. With multiple fermion pairs the antisymmetric tensor and dilaton will decouple only with extra projections.

It would be interesting to see if the relationships between gravity and gauge theories discussed here can be extended to more general field configurations.

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\[\text{References}\]

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