DEPINNING IN A RANDOM MEDIUM

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Abstract

We develop a renormalized continuum field theory for a directed polymer interacting with a random medium and a single extended defect. The renormalization group is based on the operator algebra of the pinning potential; it has novel features due to the breakdown of hyperscaling in a random system. There is a second-order transition between a localized and a delocalized phase of the polymer; we obtain analytic results on its critical pinning strength and scaling exponents. Our results are directly related to spatially inhomogeneous Kardar-Parisi-Zhang surface growth.

PACS numbers: 74.40 Ge, 5.40 +j, 64.60 Ak.

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Low-dimensional manifolds in media with quenched disorder are objects encountered in a large variety of different physical realizations. Obvious examples are interfaces in disordered bulk media and random field systems [1, 2] or magnetic flux lines in dirty superconductors [3], but there is also a deep connection to the problem of nonequilibrium surface growth [4, 5] and randomly driven hydrodynamics [6]. Furthermore, the theory serves as a simple paradigm for more complicated, fully frustrated random systems such as spin glasses [7].

A one dimensional manifold in a random medium is a phenomenological continuum model for a (single) magnetic flux line in type-II superconductors with impurities [3, 15], where the flux line interacts with an ensemble of quenched point defects (represented by a random potential). In addition to these point impurities there may also be extended (e.g. columnar or planar) defects in the system. Experiments that systematically probe the effect of this kind of impurities have recently become possible in high temperature superconductors [8]. The statistics of the line configurations is governed by an energetic competition: point defects tend to roughen the flux line; it performs large transversal excursions in order to take advantage of locally favourable regions. An attractive extended defect, on the other hand, suppresses these excursions and, if it is sufficiently strong, localizes the line to within a finite transversal distance $\xi_{\perp}$. The two regimes are separated by a second order phase transition where the localization length $\xi_{\perp}$ diverges. In contrast to temperature-driven transitions, it involves the competition of two different configuration energies rather than energy and entropy and is hence governed by a zero-temperature renormalization group fixed point.

The system is described by an effective Hamiltonian

$$H = \int dt \left\{ \frac{1}{2} \left( \frac{dr}{dt} \right)^2 - V(r, t) + \rho_0 \Phi(t) \right\}. \quad (1)$$

Here $r(t)$ denotes the displacement vector of the flux line (also called directed polymer) in $d'$ transversal dimensions, as a function of the longitudinal “timelike” co-
ordinate \( t \). The random potential \( V(r, t) \), Gauss-distributed with \( \overline{V(r, t)} = 0 \) and 
\( \overline{V(r, t)V(r', t')} = 2\sigma^2 \delta^{dr}(r - r')\delta(t - t') \), models the quenched point disorder. Averages over disorder are denoted by an overbar, thermal averages by brackets \( \langle \ldots \rangle \). The last term in the Hamiltonian describes the interaction with the extended defect. In this letter, we concentrate on columnar defects, \( \Phi(t) = \delta^{dr}(r(t)) \), but most of the results can straightforwardly be generalized to planar defects.

This model gains additional interest since it is related to nonequilibrium critical phenomena [9]. If \( Z(r, t) \) denotes the restricted partition sum over all paths ending at a fixed given point \( (r, t) \), the “height field” \( h(r, t) = \beta^{-1} \log Z(r, t) \) obeys the evolution equation

\[
\frac{\partial h}{\partial t} = \nu_0 \nabla^2 h + \frac{\lambda_0}{2} (\nabla h)^2 + V - \rho_0 \delta^{dr}(r) \tag{2}
\]

with \( \nu_0 = (2\beta)^{-1} \) and \( \lambda_0 = 1 \). This is the Kardar-Parisi-Zhang (KPZ) equation of directed surface growth [10] with an additional term describing a local inhomogeneity in the rate of mass deposition onto the surface.

Quite a few authors have studied these models. Early numerical work for \( d' = 1 \) indicates a delocalization transition at a finite defect strength \( \rho_0^c \) [11, 12]. In other large scale simulations [13], however, it is found that arbitrarily weak defects localize the polymer for \( d' = 1 \), but a finite defect strength is necessary for \( d' > 1 \). This result is supported by an approximate renormalization treatment for the problem on a hierarchical lattice [13], by scaling arguments [14], and by an approximate functional renormalization [15]. In [16] a Wilson type renormalization is discussed, but its consistency is unclear. All of these approaches rely on nonsystematic approximations; and the status of the transition has remained controversial. The problem has so far defied attempts at an exact solution even for \( d' = 1 \), in contrast to the related problem of disorder-induced depinning from a rigid wall [11, 17].

This letter is devoted to a field-theoretic study of the delocalization transition. The zero-temperature fixed point of directed lines in a random potential (with
\( \rho_0 = 0 \) is a scale-invariant continuum field theory. Its two basic exponents, the roughness exponent \( \zeta \) and the anomalous dimension of the disorder-averaged free energy \(-\omega\), are defined in eqns. (3) and (4) below. This theory is the starting point for a systematic perturbation theory in the pinning potential; it involves an \( \varepsilon \)-expansion with borderline dimension \( d' = 1 \). In contrast to standard cases like \( \phi^4 \)-theory, here even the unperturbed system is a field theory with complicated multipoint correlation functions, due to the non-thermal averaging over the disorder. Nevertheless, two fundamental properties of the local pinning field \( \Phi(t) \) can be obtained in terms of the exponents \( \zeta \) and \( \omega \): (i) its scaling dimension and (ii) the form of its operator product expansion, see eqns. (3) and (5) below. These properties determine the renormalization group equations for the pinning strength to leading order, and hence the phase diagram of the system. We find a transition at \( \rho_0 = 0 \) for \( d' \leq 1 \), and at a finite (nonuniversal) pinning strength \( \rho_0^c \) for \( d' = 2 \). The renormalization may equivalently be carried out in the framework of the KPZ dynamics, eq. (2). This is discussed at the end of this letter, together with some implications for inhomogeneous growth processes.

To derive the renormalization group equations for the depinning problem, it is necessary to study the polymer configurations in the unperturbed random system (i.e. for \( \rho_0 = 0 \)). The leading scaling behavior on large scales of disorder-averaged correlation functions is due to the sample-to-sample fluctuations of the polymer paths of minimal energy [18]. Typical transversal excursions of the paths, given e.g. by the two-point function

\[
\Delta^2(t_1 - t_2) \equiv \left\langle (r(t_1) - r(t_2))^2 \right\rangle \sim |t_1 - t_2|^{2\zeta},
\]

(3)

are characterized by the roughness exponent \( \zeta \). It is larger than for thermal fluctuations, namely \( \zeta = 2/3 \) for \( d' = 1 \) and \( \zeta \approx 5/8 \) for \( d' = 2 \) (see e.g. [3] and references therein). The exponent \(-\omega\) is the anomalous dimension of the disorder-averaged
free energy $F = -\beta^{-1} \log \text{Tr} \exp(-\beta H)$, whose universal part has the scaling form

$$F(T, R) \sim T^\omega F(R/T^\zeta)$$

(4)

in a finite system of transversal size $R$ and longitudinal size $T$. Due to a “tilt symmetry”, these exponents obey the relation $\omega = 2\zeta - 1$ (see e.g. [18, 19, 5]).

This asymptotic scaling is determined by a renormalization group fixed point where the temperature $\beta^{-1}$ is an irrelevant scaling variable, which nevertheless turns out to be crucial to what follows. It governs the crossover from the Gaussian fixed point ($\sigma^2 = \rho_0 = 0$), that describes the free thermal scaling on scales much smaller than the crossover length $\tilde{\xi}_1$, to the disorder fixed point. The finite-size free energy (4) has the crossover scaling form $F(T, \beta^{-1}) = T^\omega \tilde{F}(T/\tilde{\xi}_1)$ at fixed $R/T^\zeta$. In the infrared, $F$ is independent of $\beta^{-1}$, in the ultraviolet, $F \sim \beta^{-1} T^0$.

Comparing these two limits gives $\beta^{-1} \sim \tilde{\xi}_1^\omega$, i.e. $-\omega$ is the anomalous dimension of the temperature at the disorder fixed point. In an analogous way, the two-point function (3) has the scaling form $\Delta^2(t, \beta^{-1}) = t^{2\xi} J(\beta^{-1} t^{-\omega})$ with a temperature-independent infrared limit and the ultraviolet asymptotic behavior $\Delta^2(t, \beta^{-1}) = \beta^{-1} t$ given by the Gaussian theory. Hence the corrections to scaling of (3) due to a small temperature (which are illustrated in fig. 1) are given by

$$\Delta^2(t, \beta^{-1}) = t^{2\xi} + \tilde{c} \beta^{-1} t^{2\xi-\omega} + \ldots$$

(5)

with $2\xi - \omega = 1$, assuming analyticity of the scaling function $J$.

Now we discuss the correlation functions of the local pinning field $\Phi(t)$ at the disorder fixed point. The one-point function $\langle \Phi(t) \rangle$ gives the probability that at time $t$, the polymer is at the origin $r = 0$, averaged over thermal and disorder fluctuations. In a system of infinite longitudinal size, but finite transversal size $R$ (and periodic boundary conditions), one has by translational invariance

$$\langle \Phi(t) \rangle = L^{-x} \quad \text{with} \quad x = d' \zeta,$$
where \( L \) is a longitudinal scale defined by \( L \equiv R^{1/\zeta} \). In the renormalization discussed below, \( L \) has the rôle of an infrared cutoff, and it generates the renormalization group flow. The exponent \( x \) is the scaling dimension of the field \( \Phi \) at the disorder fixed point.

The full multipoint correlation functions \( \langle \Phi(t_1) \cdots \Phi(t_m) \rangle \) give the probability that a (single) path crosses the line \( r = 0 \) at given times \( t_1 \ldots t_m \). To discuss the short distance properties of these objects, specifically consider the two-point function \( \langle \Phi(t_1)\Phi(t_2) \rangle \) for \( |t_1 - t_2| \ll L \). In this limit, it depends on the infrared cutoff as \( L^{-x} \), i.e. in the same way as the one-point-function (4). Hence asymptotically, it factorizes into \( \langle \Phi(t_1) \rangle \) and the \( L \)-independent “return probability” to the origin (which is simply the inverse spread of the paths \( \sim \Delta(t)\sim \) given by (3)),

\[
\langle \Phi(t_1)\Phi(t_2) \rangle \sim |t_1 - t_2|^{-d'\zeta}(1 - \tilde{c}\beta^{-1}|t_1 - t_2|^{-\omega} + \ldots)\langle \Phi(t_1) \rangle .
\]  

(7)

Again the leading singularity is due to the sample-to-sample fluctuations of the minimal energy paths, while the correction term is due to the thermal fluctuations around these paths. The leading, temperature-independent singularity equally occurs in the thermally disconnected two-point function \( \langle \Phi(t_1) \rangle \langle \Phi(t_2) \rangle \) [20]. Hence in the connected two-point function \( \langle \Phi(t_1)\Phi(t_2) \rangle^c \), only the subleading singularity survives. An analogous argument applies to the singularities in any correlation function \( \langle \ldots \Phi(t)\Phi(t')\ldots \rangle \) as \( |t - t'| \to 0 \). Therefore the relation

\[
\Phi(t)\Phi(t') \sim c \beta^{-1}|t - t'|^{-x-\omega} \Phi(t)
\]  

(8)

(with a constant \( c > 0 \)) is valid as an operator identity, i.e. inserted into an arbitrary connected correlation function \( \langle \ldots \Phi(t)\Phi(t')\ldots \rangle^c \). The notion of an operator algebra that encodes the universal short-distance properties of correlation functions is familiar in field theory [21]. The new feature of (8) is that the leading singularity is governed by a correction-to-scaling exponent. This is a consequence of the breakdown of hyperscaling at the zero-temperature disorder fixed point and has direct implications for the renormalization of the pinning problem to which we now turn.
The renormalized perturbation theory for the Hamiltonian (1) is constructed along the lines of ref. [22]. The universal part of the disorder-averaged free energy density $\overline{F} \equiv \lim_{T \to \infty} F / TR^{d'}$ can be expanded in powers of the dimensionful bare coupling constant $\rho_0$,

$$\overline{f}(\rho_0, L) - \overline{f}(0, L) = \beta^{-1} R^{-d'} \sum_{m=1}^{\infty} \frac{(-\beta \rho_0)^m}{m!} J_m , \quad (9)$$

where

$$J_m = \int dt_2 \cdots dt_m \langle \Phi(0) \Phi(t_2) \cdots \Phi(t_m) \rangle^c \quad (10)$$

A weak defect potential distorts the minimal energy paths of the unperturbed system, the dominant paths reorganize exploiting the low-lying excitations. As discussed above, the statistics of these excitations is encoded in the connected correlation functions that appear in $J_m$. The leading ultraviolet singularities of these integrals are dictated by the operator algebra (8). In an analytic continuation to arbitrary $d'$, these show up as poles in $\varepsilon(d') \equiv 1 - (d' \zeta + \omega)$, which serves as expansion parameter. Inserting (8) and (8) into (9), we find to second order

$$\overline{f}(\rho_0, L) - \overline{f}(0, L) = -L^{\omega-1} \langle \Phi \rangle \left( w_0 - \frac{c}{\varepsilon(d')} w_0^2 \right) + O \left( w_0^3, \varepsilon_0 w_0^2 \right) , \quad (11)$$

where $w_0 \equiv \rho_0 L^{\varepsilon}$ is the dimensionless bare coupling constant. The pole in $\varepsilon(d')$ can be absorbed into a renormalized coupling constant $w = Z(w) w_0$ with $Z(w) = 1 - (c/\varepsilon) w + O \left( w^2 \right)$. Its renormalization group flow \[23\]

$$L \partial_L w = \varepsilon(d') w - c w^2 + O \left( w^3 \right) \quad (12)$$

determines the large scale behavior of the perturbed system. For $\varepsilon(d') > 0$, i.e. for $d' < 1$, the perturbation is relevant: for any attractive bare defect potential, the renormalized coupling is driven towards large attractive values. Hence the flux line is localized by an arbitrary weak attractive columnar defect. The localization length diverges as $\xi_\perp \sim |\rho_0|^{-\nu_\perp}$ with $\nu_\perp = \zeta(d') / \varepsilon(d')$ when the defect strength approaches zero from below. In the borderline dimension $d' = 1$ an attractive defect potential is
marginally relevant: the line is still localized by an arbitrary weak columnar defect, but with an essential singularity in the localization length \( \xi_\perp \sim \exp(2/3c|w|) \).

For \( \varepsilon(d') < 0 \), i.e. for \( d' > 1 \), a weak defect is an irrelevant perturbation. The transition to a pinned state now takes place at a finite critical strength \( \rho_0 \) (which however depends on the microscopic scales of the system and is hence nonuniversal). It is governed by the nontrivial fixed point \( w^* = \varepsilon(d')/c < 0 \) of (12). Close to the transition, the localization length diverges as \( \xi_\perp \sim |\rho_0 - \rho_0^0|^{-\nu_\perp} \), where \( \nu_\perp = \zeta(d')/y^*(d') \) and \( y^*(d') \) is given by the \( \varepsilon \)-expansion \( y^* = -\varepsilon(d') + O(\varepsilon^2) \).

Additional insight into this problem may be gained by the mapping onto the KPZ equation (1). From this stochastic equation, one constructs in a standard way the generating functional \( \text{Tr} \exp(-S[h, \tilde{h}]) \) of the dynamical correlation functions (denoted by \( \langle \langle \ldots \rangle \rangle \)) in terms of the height field \( h \) and the "conjugate" field \( \tilde{h} \). Insertions of this field generate response functions, e.g. \( \langle \langle h(r, t) \Pi_i \tilde{h}(r_i, t_i) \rangle \rangle = \langle \langle \delta h(r, t)/\Pi_i \delta V(r_i, t_i) \rangle \rangle \). In the dynamical action, there is a term \( S_i = \rho_0 \int dt \tilde{h}(0, t) \), the analogue of the pinning term in (1). The perturbation series (9) for the restricted free energy \( \beta^{-1} \log Z(r, t) = \langle \langle h(r, t) \rangle \rangle \) is in one-to-one correspondence with the dynamical perturbation series; we have

\[
\beta^{m-1} \langle \langle \Phi(t_1) \cdots \Phi(t_m) \rangle \rangle = \langle \langle h(r, t) \tilde{h}(0, t_1) \cdots \tilde{h}(0, t_m) \rangle \rangle.
\]  

Hence \( \tilde{h}(0, t) = \beta \Phi(t) \) is a field of scaling dimension \( \bar{x} = d' \zeta + \omega \); by virtue of (8), it obeys the short-distance algebra

\[
\tilde{h}(0, t) \tilde{h}(0, t') \sim c |t - t'|^{-\bar{x}} \tilde{h}(0, t') + \ldots .
\]

Its leading singularity is no longer a correction-to-scaling exponent; the peculiarity of the correlation functions written in terms of the \( \tilde{h} \) fields is rather that they have to be computed in the nontrivial "vacuum" state \( h(r, t) \). This makes the nonunitarity of this theory manifest, which is generated by the averaging over disorder.

For \( \rho_0 < 0 \), the term \( S_i \) describes an excess mass deposition onto the growing surface at \( r = 0 \). This term breaks the translational invariance. For stationary
growth $\langle h(r, t) \rangle = vt + H(r)$ in a one-dimensional system of size $R$, it results in an excess growth velocity and an approximately triangular surface profile $H(r)$ \cite{9}. From the mapping to the polymer system, one concludes that for $\xi_\perp(w) \ll R$, the excess velocity scales as $(v(w, R) - v(0, R)) \sim \xi_\perp^{(-1+\omega)}/\zeta \sim \exp(2/3cw)$. The same essential singularity shows up in the slope $|\partial H/\partial r| \sim (v(w, R) - v(0, R))^{1/2}$. The response function has the form $\langle \langle h(r, t) \tilde{h}(r_1, t_1) \rangle \rangle = \xi_\perp^{-1} G(r_1/\xi_\perp)$ for $t_0 \to \infty$ and $r, \xi_\perp \ll R$. All of these quantities are accessible in numerical simulations which could provide a useful test of the results discussed in this letter.

In summary, we have shown that a class of field theories with quenched randomness shares with conventional field theories the notion of a short-distance algebra of its scaling operators, whence they are amenable to systematic renormalization.

\textit{Note added:} In the final stages of this work we received a preprint by Hwa and Nattermann \cite{26} on the depinning problem. They exploit the mapping onto the KPZ equation \cite{2}. In $d' = 1$, they obtain the algebra \cite{14} by using a mode-coupling approximation for the dynamical correlation functions.

We thank J. Krug, R. Lipowsky, and L.-H. Tang for useful discussions.
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In general there are also contributions of paths crossing $r = 0$ at one of the two times, but the leading singularity for $|t_1 - t_2| \to 0$ again comes from paths that cross the line at both times.

For example, the algebra of the pinning field at the Gaussian fixed point, which is relevant to temperature-driven unbinding transitions, reads $\Phi(t)\Phi(t') \sim |t - t'|^{-x_0}\Phi(t) + \ldots$ with $x_0 = d'/2$. See ref. [22].

Further primitive singularities are expected at higher orders in the perturbation expansion; hence in contrast to thermal depinning [24], the flow equation does not terminate at this order.

For the general formalism, see e.g. P.C. Martin, E.D. Siggia, H.A. Rose, Phys. Rev. A8 (1973) 423; R. Bausch, H.K. Janssen, H. Wagner, Z. Phys. B24 (1976), 113.

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Figure 1

Ensemble of paths with a common starting point in a typical disordered sample. The thick line denotes the path of minimal energy; its transversal fluctuations are given by $\Delta^2(t) \sim t^{2\zeta}$. The thin lines represent thermal fluctuations of width $\delta^2(t) \equiv \langle r^2(t) \rangle^c \sim \beta^{-1}t$ around the minimal path.
Figure 1

H. Kinzelbach, M. Lassig: Depinning in a random potential