Titchmarsh–Weyl formula for the 
spectral density of a class of 
Jacobi matrices in the critical case

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Abstract

We consider a class of Jacobi matrices with unbounded entries in the so called critical 
(double root, Jordan box) case. We prove a formula for the spectral density of the matrix 
which relates its spectral density to the asymptotics of orthogonal polynomials associated 
with the matrix.

Keywords: Jacobi matrix, generalized eigenvector, orthogonal polynomials, Titchmarsh–Weyl 
theory, Levinson theorem, asymptotics, spectral density

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1 Introduction

In the paper [4] A. I. Aptekarev and J. S. Jeronimo for a class of Jacobi matrices with unbounded 
entries found a formula for the spectral density in terms of asymptotics of the orthogonal 
polynomials associated with the matrix. Their class is defined by certain summability conditions 
for the coefficients and is an example of the situation which can be called the non-critical case. 
The critical case for Jacobi matrices with unbounded entries was studied in several papers, 
among them [8, 11, 15, 21]. The distinction between the cases is determined by the limit 
(whenever it exists) of the transfer-matrix of the eigenvector equation: in the non-critical case 
it is diagonalizable, while in the critical case it is similar to a Jordan box. Note that for the 
special case of discrete Schrödinger operator with fast decreasing potential (the main diagonal) 
we have the critical situation only for two values of the spectral parameter \( \lambda = \pm 2 \). However, 
for Jacobi matrices in the critical case we have the similar situation for all values of the spectral 
parameter \( \lambda \). The types of asymptotics of generalized eigenvectors differ significantly in these 
two cases: critical and non-critical. In the papers dealing with the critical case mentioned above 
only asymptotics of the generalized eigenvectors were studied, and one cannot find an analog 
of the formula for the spectral density from [4] for the critical case. The aim of the present 
paper is to fill this gap.

We consider a class of Jacobi matrices in the critical case which is an extension of the class 
studied in [15]. Besides finding the asymptotics of generalized eigenvectors (in a more general

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we obtain a formula for the spectral density of the matrix in terms of the asymptotics of its orthogonal polynomials. Namely, we consider matrices

\[
\mathcal{J} = \begin{pmatrix}
 b_1 & a_1 & 0 & 0 & \cdots \\
 a_1 & b_2 & a_2 & 0 & \cdots \\
 0 & a_2 & b_3 & a_3 & \cdots \\
 0 & 0 & a_2 & b_4 & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]  

(1.1)

with entries

\[ a_n = n^\alpha + p_n, \quad b_n = -2n^\alpha + q_n, \]

(1.2)

where \( a_n > 0, b_n \in \mathbb{R} \), \( \alpha \in (0, 1) \)

and

\[ \left\{ \frac{p_n}{n^{2\alpha}} \right\}_{n=1}^\infty, \left\{ \frac{q_n}{n^\alpha} \right\}_{n=1}^\infty \in l^1. \]

(1.3)

The operators of this family act in the Hilbert space \( l^2(\mathbb{N}) \) and have the following spectral properties: the spectrum on the right half-line is pure point, the spectrum on the left half-line is purely absolutely continuous. This follows from the asymptotics of the generalized eigenvectors by the subordinacy theory \cite{9, 17}. For \( \alpha \in \left( \frac{1}{2}, \frac{2}{3} \right) \) such a family was considered in \cite{15} and for \( \lambda \in \mathbb{R}\setminus\{0\} \) asymptotics of generalized eigenvectors were found. In the present paper to find asymptotics we use a modified version of the remarkable method of R.-J. Kooman \cite{18} which can yield the result for all complex \( \lambda \) in the same manner and works for \( \alpha \in (0, 1) \). In essence, the Kooman’s method is based on a transformation which reduces the discrete linear system to the Levinson (L-diagonal) form and can be considered as an extension of Benzaid–Harris–Lutz methods \cite{5, 10}, see also \cite{6, 12, 24}. All of them are based on the discrete analog of the Levinson asymptotic theorem \cite[Theorem 8.1]{7}.

We call a formula, which relates the spectral density of an ordinary differential or difference operator to the coefficient in asymptotics of the solution of the eigenvector equation which satisfies the boundary condition, the Titchmarsh–Weyl formula by analogy with the classical case of the Schrödinger operator on the half-line with a self-adjoint boundary condition at zero and a summable potential, see the book of E. C. Titchmarsh \cite[Ch. V, 5.6]{34}.

In this study of formulas for the spectral density we have one application in mind, namely, the phenomenon of spectral phase transition. If a family of self-adjoint operators depends on one or several real parameters, it can happen that the space of these parameters is divided into regions where the operators have similar spectral structures. For example, in some regions the spectrum can be purely absolutely continuous, in others it can be discrete. Then on the boundaries of the regions happens a spectral phase transition. We want to see by explicit examples how such transitions happen in terms of the spectral measure. Some examples of spectral phase transitions can be found in the papers \cite{13, 14, 25}. In these works only the “geometry” and type of the spectrum were considered for the lack of suitable methods to analyze the spectral measure. This is where formulas for the spectral density could be useful. Several papers were devoted to establishing such formulas in both discrete and continuous cases, \cite{16, 19, 26}. That analysis has been used to study the behavior of the spectral density of discrete \cite{27} and differential \cite{22, 25} Schrödinger operators with the Wigner–von Neumann potential near the critical points which appear due to that form of the potential. These formulas were derived for special classes of operators, and, moreover, for the non-critical case. The example that we consider in the present paper is taken from a family of Jacobi matrices demonstrating a spectral phase transition, in the case when the parameters belong the boundary of two regions (a line in that case). On that boundary \( (d = \pm 1, \text{ see } (6.2)) \) the Jacobi matrices are in the critical case.
The formula for the spectral density in this case is needed as the first step to understanding the “inner structure” of the spectral phase transition in this family.

We should mention interesting recent works by G. Świderski [29, 30, 31, 32] also devoted, in particular, to the study of the spectral density. However, these considerations did not include the Jordan box case.

The paper is organized as follows. In Section 2 we recall some basic notions and facts related to Jacobi matrices and their generalized eigenvector equations and also explain why in our situation the critical case occurs. In Section 3 we describe the idea of approximating the matrix \( J \) by the “stabilized” matrices \( J_N \), also used in [4], which allows to find the spectral density of \( J \) exploiting the \( * \)-weak convergence of spectral measures. It contains two elementary propositions to be used in the proof of the main theorem. In Section 4 the main result of the paper (Theorem 4.1) is proved: the formula for the spectral density of \( J \) in the critical case.

The paper contains two appendices. In Appendix A we give a proof of the formula for the spectral density of the “stabilized” matrix adjusted to our form of it. And finally, in Appendix B we revisit the result of Aptekarev–Geronimo [4] that gives a formula for the spectral density for a class of unbounded Jacobi matrices in the non-critical case. In the second appendix the proof uses the similar technique which was elaborated for the more complicated critical case.

2 Preliminaries

For the complete definitions of a Jacobi matrix, orthogonal polynomials associated to it and its Weyl function in the limit-point case we refer to the book of N. I. Akhiezer [1].

The operator \( J \), defined by the matrix (1.1)–(1.3), according to the Carleman condition [1] is self-adjoint in \( l^2(N) \). If \( E \) is its projection-valued spectral measure, \( \{ e_n, n \in \mathbb{N} \} \) is the standard basis in \( l^2(N) \), then \( \rho = (Ee_1, e_1) \) is its scalar spectral measure and \( \rho' \), which exists and is finite a.e., is its spectral density. By \( m \) we denote the Weyl function for which the following relations hold:

\[
m(\lambda) = \int_\mathbb{R} \frac{d\rho(x)}{x-\lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

\[
\rho'(\lambda) = \frac{1}{\pi} \text{Im} m(\lambda + i0), \quad \text{a. a.} \quad \lambda \in \mathbb{R}.
\]

Orthogonal polynomials of the first and the second kind, \( P_n(\lambda) \) and \( Q_n(\lambda) \), respectively, are solutions of the eigenvector equation

\[
a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = \lambda u_n, \quad n \geq 2,
\]

and have the initial values \( P_1(\lambda) = 1, P_2(\lambda) = \frac{b_1 - \lambda}{a_1}, Q_1(\lambda) = 0, Q_2(\lambda) = \frac{1}{a_1} \). For \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) their linear combination \( Q_n(\lambda) + m(\lambda)P_n(\lambda) \) is the only (up to multiplication by a constant) solution of (2.3) which belongs to \( l^2(N) \). The weighted Wronskian of two solutions \( u \) and \( v \) of the eigenvector equation (2.3) is defined as

\[
W\{u, v\} := a_n(u_n v_{n+1} - u_{n+1} v_n), \quad n \in \mathbb{N},
\]

being independent of \( n \).

The eigenvector equation (2.3) can be written in the vector form:

\[
\vec{u}_n := \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}, \quad B_n(\lambda) := \begin{pmatrix} 0 & \frac{1}{a_n} \\ -\frac{b_n}{a_n} & \frac{\lambda - b_n}{a_n} \end{pmatrix},
\]

\[
\vec{u}_{n+1} = B_n(\lambda)\vec{u}_n, \quad n \geq 2.
\]

The matrix \( B_n \) is called the transfer-matrix for the equation (2.3).
In our case, (1.1)–(1.3), the transfer-matrix for every \( \lambda \) has the limit
\[
\begin{pmatrix}
0 & 1 \\
-1 & 2
\end{pmatrix}.
\] (2.7)
The eigenvalues of this limit matrix coincide, and, by analogy to the case of constant coefficients, when the roots of the characteristic equation are the same as the eigenvalues of the transfer-matrix, this is called the \textit{double root case}, or \textit{critical case}. The matrix (2.7) is not proportional to the identity, so it is not diagonalizable and is similar to a Jordan box. For this reason our situation can be also called the \textit{Jordan box case}. Asymptotic analysis of solutions can get involved in this case, and several papers were devoted to studying examples of such Jacobi matrices, among them [8, 11, 15, 21].

3 The stabilized matrix

In this section let \( J \) be a Jacobi matrix
\[
J = \begin{pmatrix}
b_1 & a_1 & 0 & 0 & \cdots \\
0 & a_2 & b_3 & a_3 & \cdots \\
0 & 0 & a_3 & b_4 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix},
\] (3.1)
with arbitrary sequences \( \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \) of positive and real numbers, respectively. Consider the bounded matrix \( J_N \) which has the sequence \( a_1, a_2, \ldots, a_{N-1}, b_N, b_N, b_N, \ldots \) of off-diagonal entries and the sequence \( b_1, b_2, \ldots, b_{N-1}, b_N, b_N, b_N, \ldots \) on the main diagonal:
\[
J_N = \begin{pmatrix}
b_1 & a_1 & 0 & 0 & \cdots & 0 & 0 & \vdots \\
0 & a_2 & b_3 & a_3 & \cdots & 0 & 0 & \vdots \\
0 & 0 & a_3 & b_4 & \cdots & 0 & 0 & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}.
\] (3.2)
The matrix \( J_N \) is a scaled and shifted discrete Schrödinger operator perturbed by finitely supported sequences of diagonal and off-diagonal entries (finite rank perturbation). It is known that its spectrum is purely absolutely continuous on the interval \([b_N - 2a_N, b_N + 2a_N]\), which follows, e.g., from the subordinacy theory [17], and discrete and finite on the rest of the real line. One can write an explicit formula for its spectral density in terms of the orthogonal polynomials for the original matrix \( J \).

**Proposition 3.1.** Consider a Jacobi matrix \( J \) given by (3.1) with some sequences \( \{a_n\}_{n=1}^{\infty} \) of positive numbers and \( \{b_n\}_{n=1}^{\infty} \) of real numbers. Let \( J_N \) be defined by (3.2). Then its spectral density is
\[
\rho_N' (\lambda) = \sqrt{1 - \left( \frac{\lambda - b_N}{2a_N} \right)^2}, \quad \lambda \in (b_N - 2a_N, b_N + 2a_N),
\] (3.3)
where \( \{P_n(\lambda)\}_{n=1}^{\infty} \) are the orthogonal polynomials of the first kind associated with the matrix \( \mathcal{J} \) and
\[
z_N(\lambda) = \frac{\lambda - b_N}{2a_N} - i\sqrt{1 - \left(\frac{\lambda - b_N}{2a_N}\right)^2} \tag{3.4}
\]
is the boundary value of the analytic branch such that \( |z_N(\lambda)| < 1 \) for \( \lambda \in \mathbb{C} \setminus [b_N - 2a_N, b_N + 2a_N] \).

This result is analogous to the classical Titchmarsh–Weyl formula for the differential Schrödinger operator on the half-line with summable potential and is more or less well-known. A version of it is contained in [4], but because of different numbering of entries the stabilized matrix is defined slightly differently there, and hence we cannot literally use that formulation. Proposition 3.3 therefore needs a separate proof which we provide in Appendix A. A much more general version, for the sequences of \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) of bounded variation, can be found in [20], see also [3].

Stabilized matrices approximate the original matrix as \( N \to \infty \), so knowing the spectral density of \( \mathcal{J}_N \) we can pass to the limit and find the spectral density of \( \mathcal{J} \). The next two propositions specify the exact sense of this limit passage, both of them are more or less standard. We use the following notation: for \( -\infty \leq A < B \leq +\infty \)
\[
C_c(A, B) = \{ \varphi \in C(A, B) \mid \text{supp} \varphi \text{ is compact} \},
\]
\[
C_0(A, B) = \{ \varphi \in C(A, B) \mid \forall \varepsilon \exists \text{ compact } K \subset (A, B) : |\varphi(x)| < \varepsilon, \forall x \in (A, B) \setminus K \}. \tag{3.5}
\]
\[
C_0(A, B) \text{ is a Banach space with the norm } \|\varphi\| = \sup_{x \in \mathbb{R}} |\varphi(x)|, \quad C_c(A, B) \text{ is its dense linear subset, the space } C_0^0(A, B) \text{ consists of finite complex-valued Borel measures (automatically regular),} \tag{3.6}
\]
[23, Ch. 3.6]. The following proposition can be found, for example, in [4].

**Proposition 3.2.** Let \( \mathcal{J} \) be a Jacobi matrix \([3.1]\) with some sequences \( \{a_n\}_{n=1}^{\infty} \) of positive numbers, \( \{b_n\}_{n=1}^{\infty} \) of real numbers, such that \( \mathcal{J} \) is in the limit-point case, and \( \mathcal{J}_N \) be defined by \([3.2]\). Then \( \rho_N \to \rho \) in the \( * \)-weak sense as \( N \to \infty \).

The next elementary proposition is also essentially well-known, but it is convenient for us to use it in the following special form.

**Proposition 3.3.** Let \( \mathcal{J} \) be a Jacobi matrix \([3.1]\) with some sequences \( \{a_n\}_{n=1}^{\infty} \) of positive numbers, \( \{b_n\}_{n=1}^{\infty} \) of real numbers, such that \( \mathcal{J} \) is in the limit-point case, and \( \mathcal{J}_N \) be defined by \([3.2]\), \( -\infty \leq A < B \leq +\infty \). If there exists an increasing sequence \( \{N_k\}_{k=1}^{\infty} \) such that \( \rho_{N_k}^{*}(\lambda) \to f(\lambda) \) as \( k \to \infty \) uniformly in \( \lambda \in K \) for every fixed compact set \( K \subset (A, B) \), then the spectral measure \( \rho \) of the operator \( \mathcal{J} \) is absolutely continuous on the interval \((A, B)\) and \( \rho'(\lambda) = f(\lambda) \) for a.a. \( \lambda \in (A, B) \).

**Proof.** By Proposition 3.2 \( \rho_{N_k} \to \rho \) \( * \)-weakly as \( k \to \infty \) in \( C_0^0(\mathbb{R}) \). Hence \( \rho_{N_k}|_{(A, B)} \to \rho|_{(A, B)} \) \( * \)-weakly in \( C_0^0(A, B) \): for any \( \varphi \in C_0(A, B) \) consider its continuation to \( \mathbb{R} \) by zero, \( \tilde{\varphi} \in C_0(\mathbb{R}) \), and convergence follows. Since one sequence in this topology cannot have two limits, it is enough to prove that \( d\rho_{N_k}(\lambda)|_{(A, B)} \to f(\lambda)d\lambda \) \( * \)-weakly in \( C_0^0(A, B) \). For every \( \varphi \in C_c(A, B) \) we have:
\[
\left| \int_A^B \rho_{N_k}'(\lambda)\varphi(\lambda)d\lambda - \int_A^B f(\lambda)\varphi(\lambda)d\lambda \right| = \left| \int_A^B (\rho_{N_k}'(\lambda) - f(\lambda))\varphi(\lambda)d\lambda \right| 
\leq \sup_{x \in \text{supp} \varphi} |\rho_{N_k}'(x) - f(x)| \int_A^B |\varphi(\lambda)|d\lambda \to 0, \quad k \to \infty. \tag{3.7}
\]
By the Banach–Steinhaus theorem, owing to the uniform estimate
\[
\|\rho_{N_k}(\lambda)d\lambda\|_{C_0^0(A, B)} \leq 1, \tag{3.8}
\]
convergence holds for every \( \varphi \in C_0(A, B) \) as well. Thus \( d\rho(\lambda)|_{(A, B)} = f(\lambda)d\lambda \), which completes the proof. \( \square \)
4 Spectral density in the critical case

In this section we formulate and prove the main result of the paper. We investigate the asymptotics of solutions to (2.3) as \( n \to \infty \) locally in \( \lambda \). Let us fix some \( 0 < r < R < \infty \) and consider the open set
\[
\Omega_0 := \{ \lambda \in \mathbb{C} : r < |\lambda| < R \} \setminus \mathbb{R}_-,
\]
see Figure 1. \( \Omega_0 \) denotes the closure containing both sides (considered to be different) of the cut along \( \mathbb{R}_- \) (closure on the Riemannian surface for \( \sqrt{\lambda} \)). Writing \([-R, -r]\) we will always mean the upper side of the cut.

![Figure 1: The domain \( \Omega_0 \)](image)

**Theorem 4.1.** Let the the entries of the Jacobi matrix \( J \) be given by
\[
a_n = n^\alpha + p_n, \quad b_n = -2n^\alpha + q_n
\]
with
\[
\alpha \in (0, 1)
\]
and real sequences \( \{p_n\}_{n=1}^{\infty} \) and \( \{q_n\}_{n=1}^{\infty} \) such that \( a_n > 0 \) for all \( n \) and
\[
\left\{ \frac{p_n}{n^{\frac{\alpha}{2}}} \right\}_{n=1}^{\infty}, \left\{ \frac{q_n}{n^{\frac{\alpha}{2}}} \right\}_{n=1}^{\infty} \in l^1.
\]

Consider the domain \( \Omega_0 \). There exists \( N_0 \in \mathbb{N} \) such that for every \( \lambda \in \overline{\Omega_0} \) the equation
\[
a_{n-1}u_{n-1} + b_nu_n + a_nu_{n+1} = \lambda u_n, \quad n \geq 2,
\]
has a solution \( u^{-}(\lambda) := \{u^{-}_n(\lambda)\}_{n=1}^{\infty} \) with the asymptotics
\[
u^{-}_n(\lambda) = \left( \prod_{l=N_0}^{n} \eta^{-}_l(\lambda) \right) (1 + o(1)), \quad n \to \infty,
\]
uniform in \( \lambda \in \overline{\Omega_0} \), where
\[
\eta^{-}_n(\lambda) = 1 + \frac{\lambda}{2n^\alpha} + \frac{\alpha}{4n} - \frac{\sqrt{\lambda}}{n^{\frac{\alpha}{2}}} \sqrt{1 + \frac{\lambda}{4n^\alpha}} + \frac{\alpha}{\lambda n^{1-\alpha}},
\]
\( \lambda \) the branch of the square roots should be chosen so that they are positive for positive \( \lambda \)
\footnote{the branch of the square roots should be chosen so that they are positive for positive \( \lambda \).}
the value of $N_0$ is chosen so that $\eta_n^{-}(\lambda) \neq 0$ for $\lambda \in \overline{\Omega}_0$, $n \geq N_0$. For every $n$ $u_n^{-}$ is analytic in $\Omega_0$ and continuous in $\overline{\Omega}_0$. For $\lambda \in [-R,-r]$ there exists a non-zero limit
\[
H(\lambda) := \lim_{n \to \infty} n^{\frac{2}{\alpha}} \prod_{i=1}^{n} |\eta_i^{-}(\lambda)|, \quad (4.8)
\]
which is continuous in $\lambda$. For $\lambda \in [-R,-r]$ the sequence $u_n^{+}(\lambda) := \{u_n^{+}(\lambda)\}_{n=1}^{\infty}$ with $u_n^{+}(\lambda) = u_n^{-}(\lambda)$ is another solution of the equation (4.5) and the nonzero Wronskian
\[
W\{u^{+}(\lambda), u^{-}(\lambda)\} = -2i\sqrt{-\lambda}H^2(\lambda), \quad (4.9)
\]
The orthogonal polynomials of the first kind associated with $J$ can be expressed as
\[
P_n(\lambda) = \Psi(\lambda)u_n^{-}(\lambda) + \overline{\Psi(\lambda)}u_n^{+}(\lambda), \quad \lambda \in [-R,-r], \quad n \in \mathbb{N}, \quad (4.10)
\]
where
\[
\Psi(\lambda) = \frac{u_0^{-}(\lambda)}{2i\sqrt{-\lambda}H^2(\lambda)}, \quad (4.11)
\]
$u_0^{-}(\lambda) := (\lambda - b_1)u_1^{-}(\lambda) - a_1u_2^{-}(\lambda)$ (assuming formally in (4.5) that $a_0 := 1$). The spectral density of $J$ is given by the formula
\[
\rho'(\lambda) = \frac{1}{4\pi\sqrt{-\lambda}|\Psi(\lambda)|^2H^2(\lambda)} = \frac{\sqrt{-\lambda}H^2(\lambda)}{\pi|u_0^{-}(\lambda)|^2}, \quad \lambda \in [-R,-r]. \quad (4.12)
\]

Remark 4.2. Another critical case with $b_n = 2n^\alpha + q_n$ can be easily reduced to the situation (4.12) with $\lambda$ replaced by $-\lambda$.

Remark 4.3. The definition of the solution $u_n^{-}(\lambda)$ depends through the value of $N_0$ on the set $\Omega_0$ (i.e., on $r$ and $R$), as well as the coefficients $\Psi$, $H$, $u_0^{-}$; they differ by multiplication by a product of some of the values $\eta_n^{-}(\lambda)$ for “small” $n$, a function of $\lambda$. Unfortunately, this function cannot be taken the same for the whole set $(\overline{\mathbb{C}} \setminus \mathbb{R}_-) \setminus \{0\}$. At the same time, the functions $|\Psi(\lambda)|H(\lambda)/|u_0^{-}(\lambda)|$ and hence $\rho'(\lambda)$ do not depend on $r$ and $R$. Finally, we can take any compact set in $(\overline{\mathbb{C}} \setminus \mathbb{R}_-) \setminus \{0\}$ instead of $\overline{\Omega}_0$.

Remark 4.4. 1. In the particular case $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$ considered in [13] the formula (4.6) and its conjugate version can be written as
\[
u_n^{-}(\lambda) = \frac{H(\lambda) + o(1)}{n^{\frac{2}{\alpha}}} \exp\left(\pm i \left(\frac{\sqrt{-\lambda}n^{1-\frac{\alpha}{2}}}{1 - \frac{\alpha}{2}} - \frac{n^{\frac{2}{\alpha}}}{\sqrt{-\lambda}} + \frac{(\sqrt{-\lambda})^3n^{1-\frac{3\alpha}{2}}}{24 (1 - \frac{3\alpha}{2})} + \varphi_0(\lambda)\right)\right), \quad (4.13)
\]
where $\varphi_0$ is some real-valued function. The formula (1.13) coincides up to a constant in $n$ multiple with the formula from the work [13]. Note that the restriction $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$ is essential here.

2. It is easy to see that in the case $\alpha \in (0,1) \setminus \left(\frac{1}{2}, \frac{2}{3}\right)$ the asymptotics of $\prod_{i=1}^{n} \eta_i^{-}$ has the form similar to (1.13). The power terms in $n$ in the exponent have orders from the interval $(0,1)$ and correspond to non-summable terms in the asymptotic expansions of $\ln \eta_i^{-}$ as $n \to \infty$. The number of such terms depends on $\alpha$ and grows infinitely as $\alpha$ approaches 0 or 1. At the same time the decay of polynomials is always of the order $\frac{1}{n^{\frac{2}{\alpha}}}$ for $\lambda < 0$, and the asymptotics of the solution $u^{-}$ can be written as
\[
u_n^{+}(\lambda) = \frac{H(\lambda) + o(1)}{n^{\frac{2}{\alpha}}} \exp(\pm i\phi_n(\lambda)), \quad n \to \infty, \quad (4.14)
\]
where \( \phi_n(\lambda) = \sum_{k=0}^{K} \varphi_k(\lambda) n^{\alpha_k} \) with some \( K, 0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_K < 1 \) and some real-valued \( \{ \varphi_k(\lambda) \}_{k=1}^{K} \). Elementary calculations show that the number \( K \sim \left( \frac{2}{\alpha} + \frac{1}{2(1-\alpha)} \right) \) as \( \alpha \to 0 \) or \( \alpha \to 1 \).

**Proof.** Consider \( \lambda \in \overline{\Omega}_0 \). For the system

\[
\vec{u}_{n+1} = B_n(\lambda) \vec{u}_n, \quad n \geq 2,
\]

which is equivalent to the eigenvector equation, we look for a sequence \( \{ S_n(\lambda) \}_{n=N_0}^{\infty} \) of diagonal matrices

\[
S_n(\lambda) = \begin{pmatrix} s_n^+(\lambda) & 0 \\ 0 & s_n^-(\lambda) \end{pmatrix}
\]

such that the transformation

\[
\vec{u}_n = S_n(\lambda) \vec{v}_n, \quad \vec{v}_{n+1} = S_{n+1}^{-1}(\lambda) B_n(\lambda) S_n(\lambda) \vec{v}_n, \quad n \geq N_0,
\]

leads to the system (reminding the system for the discrete Schrödinger operator with the spectral parameter on the boundary of the essential spectrum)

\[
\vec{v}_{n+1} = \begin{pmatrix} 0 & 1 \\ -1 + c_n(\lambda) & 1/2 \end{pmatrix} \vec{v}_n, \quad n \geq N_0,
\]

with some real sequence \( \{ c_n(\lambda) \}_{n=N_0}^{\infty} \). The value \( N_0 \in \mathbb{N} \) will be chosen large enough and uniform in \( \lambda \in \overline{\Omega}_0 \) here and everywhere in the paper. This form corresponds by \( \vec{v}_n = \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} \) to the three-term recurrence relation

\[
x_{n+2} - 2x_{n+1} + (1 - c_n(\lambda)) x_n = 0,
\]

which was studied in the work of R.-J. Kooman, [18, 19]. In order to obtain this we need the following equalities to hold:

\[
\begin{pmatrix} 0 & s_n^- (\lambda) \\ s_n^+ (\lambda) a_{n-1} & s_n^+ (\lambda) a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 + c_n (\lambda) & 1/2 \end{pmatrix}, \quad n \geq N_0.
\]

From the right column

\[
s_n^- (\lambda) = 2, \quad s_n^- (\lambda) = s_{n+1}^+ (\lambda).
\]

Denote

\[
d_n(\lambda) := \frac{\lambda - b_n}{2a_n}, \quad n \geq 1,
\]

then \( \frac{s_{n+1}^+ (\lambda)}{s_n^- (\lambda)} = d_n(\lambda) \). The index \( N_0 \) will be chosen large enough to ensure, in particular, that \( d_n(\lambda) \neq 0 \) for \( \lambda \in \overline{\Omega}_0 \) for \( n \geq N_0 - 1 \). Take

\[
s_n^- (\lambda) := \prod_{l=N_0-1}^{n-1} d_l(\lambda), \quad s_n^+ (\lambda) := \prod_{l=N_0-1}^{n-2} d_l(\lambda), \quad n \geq N_0,
\]

so that

\[
S_n(\lambda) = \begin{pmatrix} \prod_{l=N_0-1}^{n-2} d_l(\lambda) & 1 \\ 0 & d_{n-1}^-(\lambda) \end{pmatrix}, \quad n \geq N_0.
\]

From the equality in the lower-left entries in (4.17) we have

\[
-1 + c_n(\lambda) = -\frac{s_n^+ (\lambda) a_{n-1}}{s_{n+1}^- (\lambda) a_n},
\]

\[
\sum_{k=0}^{K} \varphi_k(\lambda) n^{\alpha_k} \text{ with some } K, 0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_K < 1 \text{ and some real-valued } \{ \varphi_k(\lambda) \}_{k=1}^{K}.
\]
Therefore one has to define
\[ c_n(\lambda) := 1 - \frac{4a_{n-1}^2}{(\lambda - b_{n-1})(\lambda - b_n)}, \quad n \geq N_0. \] (4.24)

Note that \( c_n(\lambda) \to 0 \) as \( n \to \infty \) in the critical case. The system (4.18) by the substitution
\[ \tilde{v}_n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tilde{v}_n \] (4.25)
is further transformed to the system
\[ \tilde{w}_{n+1} = \begin{pmatrix} 1 & 0 \\ c_n(\lambda) & 1 \end{pmatrix} \tilde{w}_n, \quad n \geq N_0, \] (4.26)
which has the same form as in [18]. Following the method of Kooman we look for sequences \( \{g_n(\lambda)\}_{n=N_0}^\infty \) such that the substitution
\[ \tilde{w}_n = \begin{pmatrix} 1 & 1 \\ g_n^+(\lambda) & g_n(\lambda) \end{pmatrix} \bar{y}_n, \] (4.27)
transforms the system (4.26) to
\[ \bar{y}_{n+1} = \left( \begin{array}{cc} 1 + g_n^+ & 0 \\ 0 & 1 + g_n^- \end{array} \right) + \frac{1}{g_{n+1}^+ - g_{n+1}^-} \times \left( \begin{array}{cc} g_{n+1}^- - g_n^+ + g_n^+ g_{n+1}^- & c_n \\ -g_{n+1}^- + g_n^+ + g_n g_{n+1}^+ & -c_n \end{array} \right) \bar{y}_n, \] (4.28)
which has the Levinson (L-diagonal) form [3, 6], if the second term in the coefficient matrix in (4.28) is summable. Combining the substitutions (4.17), (4.25) and (4.27) we see that solutions of the systems (4.15) and (4.28) are related by the equality
\[ \bar{u}_n = S_n(\lambda) \begin{pmatrix} 1 & 1 \\ 1 + g_n^+(\lambda) & 1 + g_n^-(\lambda) \end{pmatrix} \bar{y}_n, \quad n \geq N_0. \] (4.29)

Consider the sequence \( \{c_n(\lambda)\}_{n=N_0}^\infty \). It has the asymptotics
\[ c_n(\lambda) = 1 - \frac{4((n-1)^\alpha + p_{n-1})^2}{(\lambda + 2(n-1)^\alpha - q_{n-1})(\lambda + 2n^\alpha - q_n)} = \frac{\lambda}{n^\alpha} + \frac{\lambda^2}{4n^{2\alpha}} + \frac{\alpha}{n}, \] (4.30)
where \( \{\sup_{\lambda \in \Omega_n} |\nu_n(\lambda)|\}_{n=N_0}^\infty \in l^1 \) for sufficiently large \( N_0 \). Here we use the notation
\[ \chi_n(\lambda) := \frac{\lambda}{n^\alpha} + \frac{\lambda^2}{4n^{2\alpha}} + \frac{\alpha}{n}. \] (4.31)

The index \( N_0 \) will be chosen so that the roots and the poles of \( \chi_n \) and the poles of \( c_n \) lie outside of \( \Omega_0 \) for \( n \geq N_0 \).

An important property which also arises in [18] Theorem 1, case 1) as a condition, and which one can check straightforwardly here, is that
\[ \left\{ \sup_{\lambda \in \Omega_n} \left| \frac{\chi_{n+1}(\lambda) - \chi_n(\lambda)}{\chi_n(\lambda)} + \frac{\alpha}{n} \right| \right\}_{n=N_0}^\infty \in l^1, \] (4.32)
due to the property

\[
\chi_{n+1}(\lambda) = \chi_n(\lambda) \left(1 - \frac{\alpha}{n} + r_n^{(2)}(\lambda)\right) \quad \text{with} \quad \left\{\sup_{\lambda \in \Omega_0} |r_n^{(2)}(\lambda)|\right\}^\infty_{n=N_0} \in l^1
\]  

(4.33)

for sufficiently large \(N_0\). Let us define

\[
g_n^\pm(\lambda) := \pm \sqrt{\chi_n(\lambda)} + \frac{\alpha}{4n}, \quad n \geq N_0.
\]  

(4.34)

To specify the branch of the square root note that the function \(\chi_n\) has two roots \(\lambda_n^\pm := 2n^\alpha (-1 \pm \sqrt{1 - \frac{\alpha}{n}})\) such that \(\lambda_n^- \to -\infty, \lambda_n^+ \to 0\) as \(n \to \infty\), and one pole \(\mu_n := -2n^\alpha\). We choose \(N_0\) so large that \(\lambda_n^- < -3R\) and \(\lambda_n^+ > -r\) for all \(n \geq N_0\). The branch can be chosen in \(\mathbb{C}\setminus[\lambda_n^-, \lambda_n^+]\) by specifying that \(\sqrt{\chi_n(0)} = \sqrt{\frac{2}{n}}\) is a positive number (the principle branch). For every \(n \geq N_0\) the function \(\sqrt{\chi_n(\lambda)}\) is analytic in \(\Omega_0\) and continuous in \(\overline{\Omega}_0\). The same applies to the function

\[
\sqrt{\frac{\lambda_n^\alpha}{n^\alpha} + \frac{\lambda_n^2}{4n^\alpha} + \frac{\alpha}{n}}, \quad n \geq N_0
\]  

(4.35)

which corresponds to the choice of branch in (4.7). Taking into account the arguments of the numerator and the denominator of the second expression in (4.37), one can see (using elementary geometric considerations) that

\[
\text{Re} \sqrt{\chi_n(\lambda)} \geq 0 \quad \text{for} \quad \lambda \in \overline{\Omega}_0, n \geq N_0
\]  

(4.36)

for \(N_0\) sufficiently large. Then from (4.32) it follows that

\[
\left\{\sup_{\lambda \in \overline{\Omega}_0} \left| \frac{g_n^{+1}(\lambda) - g_n^{\pm}(\lambda) + g_n^{\pm}(\lambda)g_n^{+1}(\lambda) - \chi_n(\lambda)}{g_n^{+1}(\lambda) - g_n^{+1}(\lambda)} \right|\right\}^\infty_{n=N_0} \in l^1.
\]  

(4.37)

Indeed, one can easily check that

\[
g_n^{+1}(\lambda) = \pm \sqrt{\chi_n(\lambda)} \sqrt{1 - \frac{\alpha}{n} + r_n^{(3)}(\lambda) + \frac{\alpha}{4(n+1)}
\]

\[= \pm \sqrt{\chi_n(\lambda)} + \frac{\alpha \sqrt{\chi_n(\lambda)}}{2n} + \frac{\alpha}{4n} + \sqrt{\chi_n(\lambda)} r_n^{(4)}(\lambda),
\]  

(4.38)

\[
g_n^{+1}(\lambda) - g_n^{+}(\lambda) = \sqrt{\chi_n(\lambda)} \left(\pm \frac{\alpha}{2n} + r_n^{(5)}(\lambda)\right),
\]  

(4.39)

\[
g_n^{+1}(\lambda) - g_n^{+1}(\lambda) = -2 \sqrt{\chi_n^{+1}(\lambda)} = \sqrt{\chi_n(\lambda)} \left(-2 + \frac{\alpha}{n} + r_n^{(5)}(\lambda)\right),
\]  

(4.40)

\[
g_n^{+1}(\lambda) g_n^{+1}(\lambda) = \chi_n(\lambda) \pm \frac{\alpha \sqrt{\chi_n(\lambda)}}{2n} + \sqrt{\chi_n(\lambda)} r_n^{(6)}(\lambda)
\]  

(4.41)

with \(\left\{\sup_{\lambda \in \overline{\Omega}_0} \{|r_n^{(3)}(\lambda)|, |r_n^{(4)}(\lambda)|, |r_n^{(5)}(\lambda)|, |r_n^{(6)}(\lambda)|\}\right\}^\infty_{n=N_0} \in l^1\) for sufficiently large \(N_0\), from which (4.37) follows. From (4.30), (4.40) and (4.31) we have

\[
\left\{\sup_{\lambda \in \overline{\Omega}_0} \left| \frac{c_n(\lambda) - \chi_n(\lambda)}{g_n^{+1}(\lambda) - g_n^{+1}(\lambda)} \right|\right\}^\infty_{n=N_0} \in l^1.
\]  

(4.42)
Therefore observing the second term of the system (4.28) we can write it in the form

\[
\vec{y}_{n+1} = \left( \begin{pmatrix} 1 + g_n^+ (\lambda) \\ 0 \\ 1 + g_n^- (\lambda) \end{pmatrix} + R_n (\lambda) \right) \vec{y}_n, \quad n \geq N_0, \tag{4.43}
\]

where \(\sup_{\lambda \in \overline{\Omega}_0} \| R_n (\lambda) \| \) for all \( n \geq N_0 \). Clearly, \( c_n \) and \( g_n^\pm \) are analytic in \( \Omega \) and continuous in \( \overline{\Omega}_0 \) for all \( n \geq N_0 \). Now the tools used to prove the following lemma can be employed to show existence of the solution \( y_n^- (\lambda) \) analytic in \( \Omega_0 \), continuous in \( \overline{\Omega}_0 \) (for all \( n \geq N_0 \)) and with asymptotics uniform in \( \Omega_0 \).

\[
y_n^- (\lambda) = \left( \prod_{l=N_0}^{n-1} (1 + g_l^- (\lambda)) \right) (\vec{e}_- + o(1)), \quad n \to \infty. \tag{4.44}
\]

**Lemma 4.5.** Let the sequence \( \{\lambda_n\}_{n=1}^\infty \) of nonzero complex numbers be such that for some \( C > 0 \) and any \( p, q \in \mathbb{N} \) such that \( p \leq q \),

\[
\prod_{l=p}^{q} |\lambda_l| \geq \frac{1}{C}. \tag{4.45}
\]

Let the sequence \( \{R_n\}_{n=1}^\infty \) of complex \( 2 \times 2 \) matrices be such that

\[
det \left( \begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix} + R_n \right) \neq 0, \quad n \geq 1,
\]

and

\[
\sum_{k=1}^{\infty} |\lambda_k| \|R_k\| < \infty. \tag{4.46}
\]

Then the system

\[
\vec{x}_{n+1} = \left( \begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix} + R_n \right) \vec{x}_n, \quad n \geq 1, \tag{4.47}
\]

has the solution

\[
\vec{x}_n^- = \left( \prod_{l=1}^{n-1} \frac{1}{\lambda_l} \right) (\vec{e}_- + o(1)), \quad n \to \infty, \tag{4.48}
\]

where \( \vec{e}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

**Proof.** One can check that solutions of the system (4.47) are the same as solutions of the integral equations

\[
\vec{x}_n = \begin{pmatrix} \prod_{l=1}^{n-1} \lambda_l & 0 \\ 0 & \prod_{l=1}^{n-1} \frac{1}{\lambda_l} \end{pmatrix} \vec{f} - \sum_{k=n}^{\infty} \begin{pmatrix} \prod_{l=n}^{k} \frac{1}{\lambda_l} & 0 \\ 0 & \prod_{l=n}^{k} \lambda_l \end{pmatrix} R_k \vec{x}_k \tag{4.49}
\]

for arbitrary \( \vec{f} \in \mathbb{C}^2 \) (this is a kind of variation of parameters method). In particular, consider \( \vec{f} = \vec{e}_- \) and

\[
\vec{x}_n^- = \begin{pmatrix} \prod_{l=1}^{n-1} \frac{1}{\lambda_l} \end{pmatrix} \vec{e}_- - \sum_{k=n}^{\infty} \begin{pmatrix} \prod_{l=n}^{k} \frac{1}{\lambda_l} & 0 \\ 0 & \prod_{l=n}^{k} \lambda_l \end{pmatrix} R_k \vec{x}_k^- \tag{4.50}
\]
Then for the new sequence of vectors
\[
\tilde{x}_n := \left( \prod_{l=1}^{n-1} \lambda_l \right) \tilde{x}_n
\]  
the equation \(4.50\) is equivalent to the equation
\[
\tilde{x}_n = e_-- \sum_{k=n}^{\infty} \left( \prod_{l=n}^{k} \lambda_l \right) \lambda_k R_k \tilde{x}_k.
\]  
(4.52)

Denote the family of \(2 \times 2\) matrices
\[
V_{nk} := - \left( \prod_{l=n}^{k} \lambda_l \right) \lambda_k R_k, \quad n, k \geq 1.
\]  
(4.53)

Since
\[
\|V_{nk}\| \leq (C^2 + 1) |\lambda_k||R_k|
\]  
(4.54)
is summable in \(k\) by the conditions \(4.45\) and \(4.46\), the equation \(4.52\) can be written as
\[
\tilde{x}^- = e_- + V \tilde{x}^-\]
(4.55)
with the Volterra operator \(V\) in the Banach space \(l^\infty(N; C^2)\) defined by the matrix-valued kernel \(V_{nk}\). One has
\[
\|V^j\| \leq \left( \sum_{k=1}^{\infty} (C^2 + 1)|\lambda_k||R_k| \right)^j, \quad j \geq 0,
\]  
(4.56)

\[
\|(I - V)^{-1}\| \leq \exp \left( \sum_{k=1}^{\infty} (C^2 + 1)|\lambda_k||R_k| \right),
\]  
(4.57)

\[
\tilde{x}^- = (I - V)^{-1}e_- = \sum_{j=0}^{\infty} V^j e_-.
\]  
(4.58)
From \(4.52\) and \(4.54\) it follows that \(\tilde{x}_n^- \to e_-, n \to \infty\), and from this we have \(4.48\), which completes the proof.

**Remark 4.6.** This is another variation of the discrete asymptotic Levinson theorem, see [5, Lemma 2.1]. However, note that the proof of existence of the “small” solution does not require the dichotomy condition. This is crucial for uniformity of asymptotics and for continuity of solutions in the parameter \(\lambda\) in what follows. Other formulations of smooth and uniform discrete Levinson theorems, as in [24], do not yield the result we need.

Consider the following transformation
\[
\bar{y}_n = \left( \prod_{l=N_0}^{n-1} \sqrt{1 + g_l^+(\lambda)} \sqrt{1 + g_l^-(\lambda)} \right) \bar{x}_n, \quad n \geq N_0.
\]  
(4.59)

For sufficiently large \(N_0\) we have \(|g_n^+(\lambda)| \leq \frac{1}{2}\) for \(\lambda \in \overline{\Omega}, n \geq N_0\), and we can take the principal branch for both square roots. This substitution transforms \(4.43\) to the system
\[
\tilde{x}_{n+1} = \left( \begin{array}{cc} \sqrt{1+g_n} & 0 \\ \sqrt{1+g_n} & \sqrt{1+g_n^2} \end{array} \right) \tilde{x}_n + \frac{R_n}{\sqrt{1+g_n^2} \sqrt{1+g_n}} \tilde{x}_n, \quad n \geq N_0,
\]  
(4.60)
to which Lemma \[4.4\] is applicable in \(\overline{\Omega}_0\), if \(N_0\) is chosen large enough (shifting the index by \(N_0 - 1\) does not change the situation). By the lemma there exists a solution

\[
\bar{x}_n^-(\lambda) = \left( \prod_{l=N_0}^{n-1} \frac{\sqrt{1 + g_l^+(\lambda)}}{\sqrt{1 + g_l(\lambda)}} \right) (\bar{c}_- + o(1)), \quad n \to \infty, 
\]

which together with \[4.59\] gives \[4.44\]. The constant \(C\) from \[4.45\] can be chosen equal to one for every \(\lambda \in \overline{\Omega}_0\); due to \[4.36\] we have \(\left| \frac{1 + \sqrt{\lambda_0(\lambda) + \frac{\alpha}{n}}}{1 - \sqrt{\lambda_0(\lambda) + \frac{\alpha}{n}}} \right| \geq 1\) for all \(\lambda \in \overline{\Omega}_0\) and \(n \geq N_0\). Besides that we have

\[
\left\{ \sup_{\lambda \in \overline{\Omega}_0} \left| \frac{R_n(\lambda)}{1 + g_n^-(\lambda)} \right| \right\}_{n=N_0}^\infty \in l^1, \tag{4.62}
\]

since \(|g_n^-(\lambda)| \leq \frac{1}{2}\) for \(\lambda \in \overline{\Omega}, n \geq N_0\), which ensures that the estimate of the sum in \[4.53\], provided by \[4.54\] is uniform in \(\lambda \in \overline{\Omega}_0\), and hence the asymptotics in \[4.61\] and \[4.44\] are uniform. Furthermore, the sum in \(4.58\) converges absolutely and uniformly (i.e., \(\sum_{j=0}^{\infty} \sup_{\lambda \in \overline{\Omega}_0} \| V^j(\lambda) \bar{c}_- \| < \infty\), the summands are analytic in \(\Omega_0\) and continuous in \(\overline{\Omega}_0\), thus the solutions \(\bar{x}_n^-\) and \(\bar{y}_n^\pm\) are also analytic in \(\Omega_0\) and continuous in \(\overline{\Omega}_0\). Returning to the system \[4.15\] with \[4.27\], \[4.25\], \[4.17\] we obtain the solution of the system \[4.15\] analytic in \(\Omega_0\) and continuous in \(\overline{\Omega}_0\) with asymptotics as \(n \to \infty\) uniform in \(\overline{\Omega}_0\), the new sequence of vectors

\[
\bar{u}_n^-(\lambda) := S_n(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{u}_n^-(\lambda) = S_n(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{y}_n^-(\lambda) = S_n(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \bar{c}_- + o(1), \tag{4.63}
\]

where

\[
h_n^\pm(\lambda) := d_{n-1}(\lambda)(1 + g_n^\pm(\lambda)). \tag{4.64}
\]

As one can see,

\[
d_n(\lambda) = 1 + \frac{\lambda}{2n^2} + \frac{r_n^{(7)}(\lambda)}{n^2} \quad \text{with} \quad \left\{ \sup_{\lambda \in \overline{\Omega}_0} |r_n^{(7)}(\lambda)| \right\}_{n=1}^\infty \in l^1. \tag{4.65}
\]

Using this formula and the definitions \[4.34\], \[4.31\] and \[4.7\] we get

\[
h_n^-(\lambda) = \eta_n^-(\lambda) + r_n^{(8)}(\lambda) \quad \text{with} \quad \left\{ \sup_{\lambda \in \overline{\Omega}_0} |r_n^{(8)}(\lambda)| \right\}_{n=N_0}^\infty \in l^1. \tag{4.66}
\]

By the choice of \(N_0\) we can ensure that for all for \(\lambda \in \overline{\Omega}, n \geq N_0\) we have \(|\eta_n^-(\lambda)| \geq \frac{1}{2}\). Then for every \(n \geq N_0\)

\[
\frac{h_n^-(\lambda)}{\eta_n^-(\lambda)} = 1 + \frac{r_n^{(8)}(\lambda)}{\eta_n^-(\lambda)} = 1 + r_n^{(9)}(\lambda) \quad \text{with} \quad \left\{ \sup_{\lambda \in \overline{\Omega}_0} |r_n^{(9)}(\lambda)| \right\}_{n=N_0}^\infty \in l^1. \tag{4.67}
\]

This quotient is a function analytic in \(\Omega_0\) and continuous in \(\overline{\Omega}_0\) without roots. Therefore the product

\[
\prod_{l=N_0}^{\infty} \frac{h_n^-(\lambda)}{\eta_n^-(\lambda)} =: C_0(\lambda) \tag{4.68}
\]
converges and has the same properties. Define the solution proportional to $\vec{u}_n(\lambda)$

$$
\vec{u}_n^-(\lambda) = \left(\frac{u_{n-1}(\lambda)}{u_n(\lambda)}\right) = \left(\prod_{l=N_0}^{n-1} \frac{\eta_l^-(\lambda)}{\eta_l(\lambda)} \right) \left(\prod_{l=n}^{\infty} \frac{\eta_l^-(\lambda)}{h_l^-(\lambda)} \right) \left(\frac{1}{h_n^+(\lambda)} \frac{1}{h_n^-(\lambda)}\right) (\vec{e}_- + o(1))
$$

$$
= \left(\prod_{l=N_0}^{n-1} \eta_l^-(\lambda) \right) \left(\frac{1}{h_n^+(\lambda)} \frac{1}{h_n^-(\lambda)}\right) (\vec{e}_- + o(1)), \quad (4.69)
$$

from which (4.6) follows. The solution $\vec{u}_n$ is analytic in $\Omega_0$ and continuous in $\overline{\Omega}_0$ for every $n \geq N_0$. Moreover, although this solution is initially defined for $n \geq N_0$, it exists for all $n \geq 2$ (and can be formally defined also for $n = 1$ with $a_0 := 0$) retaining the same properties, because matrices $B_n(\lambda)$ and $B_n^{-1}(\lambda)$ are entire in $\lambda$ for all $n$.

Note that the form of asymptotics $(\prod_{l=N_0}^{n-1} \eta_l^-(\lambda)) \left(1 + o(1)\right)$ implies, but is not equivalent to $(\prod_{l=N_0}^{n-1} \eta_l^-(\lambda)) \left(1 + o(1)\right)$, which would be enough to get (4.6). It contains more information which is lost due to degeneracy of the matrix $\left(\frac{1}{h_n^+(\lambda)} \frac{1}{h_n^-(\lambda)}\right)$. Hence the definition (4.8) is correct.

Throughout this proof we have many times declared that $N_0$ should be chosen sufficiently large so that one one or another property holds true. Evidently, one can choose $N_0$ so that all of them hold at once. This choice is determined only by the values of $r$ and $R$.

For $\lambda \in [-R, -r]$ and large $n$

$$
|\eta_n^-(\lambda)|^2 = \left(1 + \frac{\lambda}{2n^\alpha} + \frac{\alpha}{4n}\right)^2 - \frac{\lambda^2}{4n^{2\alpha}} - \frac{\alpha}{n} = 1 - \frac{\alpha}{2n} + \frac{\alpha \lambda}{4n^{1+\alpha}} + \frac{\alpha^2}{16n^2}. \quad (4.70)
$$

Thus $n^{\frac{\phi}{2}} \prod_{n=N_0}^{\infty} |\eta_n^-(\lambda)|^2$ has a finite limit as $n \to \infty$ which is a continuous function of $\lambda$ without roots in $[-R, -r]$. Hence the definition (4.8) is correct.

Since the eigenvector equation (4.5) has real coefficients, the sequence $u_n^+(\lambda) := u_n^-(\lambda)$ is its solution for $\lambda \in [-R, -r]$. The sequence

$$
\vec{u}_n^+(\lambda) := \left(\frac{u_{n-1}(\lambda)}{u_n^+(\lambda)}\right) = \left(\prod_{l=N_0}^{n-1} \frac{\eta_l^-(\lambda)}{\eta_l(\lambda)} \right) \left(\frac{1}{h_n^+(\lambda)} \frac{1}{h_n^-(\lambda)}\right) (\vec{e}_- + o(1)) \quad (4.71)
$$

is a solution to the system (4.13). For $\lambda \in [-R, -r]$ and large $n$ one has $h_n^+(\lambda) = h_n^-(\lambda)$, hence

$$
\vec{u}_n^+(\lambda) = \left(\prod_{l=N_0}^{n-1} \frac{\eta_l^-(\lambda)}{\eta_l(\lambda)} \right) \left(\frac{1}{h_n^+(\lambda)} 1 \right) (\vec{e}_- + o(1)) = \left(\prod_{l=N_0}^{n-1} \frac{\eta_l^-(\lambda)}{\eta_l(\lambda)} \right) \left(\frac{1}{h_n^+(\lambda)} \frac{1}{h_n^-(\lambda)}\right) \left(\vec{e}_+ + o(1)\right), \quad (4.72)
$$

where $\vec{e}_+ := \left(\begin{array}{c} 1 \\ 0 \end{array}\right)$. The Wronskian of the solutions $u^+(\lambda)$ and $u^-(\lambda)$ is equal for fixed $\lambda \in$
\[-R, -r\] for all \(n \in \mathbb{N}\) to

\[
W\{u^+(\lambda), u^-(\lambda)\} = a_n(u_n^+(\lambda) u_{n+1}^-(\lambda) - v_{n+1}^+(\lambda) u_n^-(\lambda)) = a_n \det \begin{pmatrix}
    u_n^+(\lambda) & v_n^- (\lambda) \\
v_{n+1}^+(\lambda) & u_{n+1}^-(\lambda)
\end{pmatrix}
\]

\[
= \lim_{n \to \infty} a_n \det \begin{pmatrix}
    1 & 1 \\
    h_{n+1}^+ (\lambda) & h_{n+1}^- (\lambda)
\end{pmatrix} (I + o(1)) \prod_{i=N_0}^{n} \begin{pmatrix}
    \eta_i^+(\lambda) & 0 \\
    0 & \eta_i^- (\lambda)
\end{pmatrix}
\]

\[
= \lim_{n \to \infty} a_n (h_{n+1}^- (\lambda) - h_{n+1}^+ (\lambda)) \left( \prod_{i=N_0}^{n} |\eta_i^- (\lambda)|^2 \right) (1 + o(1))
\]

\[
= -2 \lim_{n \to \infty} a_n d_n(\lambda) \sqrt{\chi_{n+1}(\lambda)} \left( \prod_{i=N_0}^{n} |\eta_i^- (\lambda)|^2 \right) (1 + o(1)) = -2i \sqrt{-\lambda H^2(\lambda)}
\] (4.73)

where we used (4.64), (4.69) and (4.72) to get the third equality, (4.64) and (4.34) to get the fifth and (1.2), (4.65), (4.31) and (4.3) for the last equality. Hence \(u^+(\lambda)\) and \(u^-(\lambda)\) are linearly independent for \(\lambda \in [-R, -r]\) and orthogonal polynomials of the first kind can be expressed as

\[
P_n(\lambda) = \Psi(\lambda) u_n^+(\lambda) + \overline{\Psi(\lambda)} u_n^-(\lambda)
\] (4.74)

with

\[
\Psi(\lambda) = \frac{W\{P(\lambda), u^-(\lambda)\}}{W\{u^+(\lambda), u^-(\lambda)\}} = \frac{a_0 (P_0(\lambda) u_1^- (\lambda) - P_1(\lambda) u_0^- (\lambda))}{-2i \sqrt{-\lambda H^2(\lambda)}} = \frac{u_0^- (\lambda)}{2i \sqrt{-\lambda H^2(\lambda)}},
\] (4.75)

where \(P(\lambda) := \{P_n(\lambda)\}_{n=1}^{\infty}\) and \(P_0(\lambda)\) is the formal solution of the eigenvector equation with \(a_0 = 1\), i.e., \(P_0(\lambda) \equiv 0\); one can easily check that constancy of the Wronskian can be extended to \(\mathbb{N} \cup \{0\}\). From this we see that \(\Psi\) and \(u_0^-\) cannot have zeros on \([-R, -r]\).

Using Proposition 3.3 we come to establishing the limit uniform in \([-R, -r]\) of

\[
\rho_n(\lambda) = \sqrt{1 - d_n^2(\lambda)} \quad \text{for a.a. } \lambda \in (b_n - 2a_n, b_n + 2a_n),
\] (4.76)

where

\[
z_n(\lambda) = d_n(\lambda) - i \sqrt{1 - d_n^2(\lambda)},
\] (4.77)

according to (3.3) and (3.4). Firstly, from (4.65) we have

\[
\sqrt{1 - d_n^2(\lambda)} = \frac{\sqrt{-\lambda}}{\sqrt{n}} \sqrt{1 + \frac{2}{\lambda} n^{1/2} r_n(\lambda) + o(1)},
\] (4.78)

The term \(\frac{2}{\lambda} n^{1/2} r_n(\lambda)\) is not vanishing in general. Moreover, it may behave in an irregular way under our condition (1.3). However, it can be made vanishing on a proper subsequence. Since we are looking for uniform convergence, this subsequence should not depend on \(\lambda\). Let \(\lambda_n := \max \left\{ \frac{\|p_n\|}{\lambda^{1/2}}, \frac{\|q_n\|}{\lambda^{1/2}} \right\}, \left\{\lambda_n\right\}_{n=1}^{\infty} \in l^1\) by the condition (1.3). One can choose an increasing sequence \(\{n_k\}_{k=1}^{\infty}\) such that \(\lambda_n = o\left(\frac{1}{n_k}\right), k \to \infty\). This means that \(p_{n_k}, q_{n_k} = o\left(\frac{1}{n_k}\right)\). Then

\[
d_k(\lambda) = \frac{\lambda + 2n_k^2 - q_{n_k}}{2n_k^2 + p_{n_k}} = 1 + \frac{\lambda}{2n_k^2} + o\left(\frac{1}{n_k^2}\right),
\] (4.79)

\[
\sqrt{1 - d_k^2(\lambda)} = \frac{\sqrt{-\lambda}}{n_k} \left(1 + o\left(\frac{1}{n_k}\right)\right),
\] (4.80)
\[ z_{nk}(\lambda) = 1 - \frac{i\sqrt{-\lambda}}{n_k^\frac{\tau}{2}} + o\left(\frac{1}{n_k^\frac{\tau}{2}}\right), \quad (4.81) \]

and

\[ h_{nk+1}^\pm(\lambda) = 1 \pm \frac{i\sqrt{-\lambda}}{n_k^\frac{\tau}{2}} + o\left(\frac{1}{n_k^\frac{\tau}{2}}\right). \quad (4.82) \]

The error \( o\)-terms are uniform in \( \lambda \in [-R, -r] \). This also gives \( b_{nk} + 2a_{nk} \to 0 \) as \( k \to \infty \) (while for \( b_n - 2a_n \to -\infty \) as \( n \to \infty \) we do not need a subsequence). Thus we can find \( K \in \mathbb{N} \) such that for \( k \geq K \) the inclusion \([ -R, -r ] \subset (b_{nk} - 2a_{nk}, b_{nk} + 2a_{nk}) \) holds and (4.76) is true for a.a. \( \lambda \in [-R, -r] \) and \( n = n_k \). Further, on \([ -R, -r ] \) we have

\[ P_{n+1} - z_n P_n = \Psi(u_{n+1}^+ - z_n u_n^+) + \overline{\Psi}(u_{n+1}^- - z_n u_n^-). \quad (4.83) \]

Using (4.72) we get

\[ u_{n+1}^+ - z_n u_n^+ = (-z_n 1) \left(\prod_{l=N_0}^n \eta_l \right) \left( \frac{1}{h_{n+1}^+} \frac{1}{h_{n+1}^-} \right)(\bar{e}_+ + o(1)) \]

\[ = \left(\prod_{l=N_0}^n \eta_l \right) \left( h_{n+1}^+ - z_n \ h_{n+1}^- - z_n \right) \left( 1 + o(1) \right) \quad (4.84) \]

and analogously

\[ u_{n+1}^- - z_n u_n^- = \left(\prod_{l=N_0}^n \eta_l \right) \left( h_{n+1}^+ - z_n \ h_{n+1}^- - z_n \right) \left( o(1) \right). \quad (4.85) \]

From (4.81) and (4.82) we have

\[ u_{nk+1}^+ - z_{nk} u_{nk}^+ = \left(\prod_{l=N_0}^{n_k} \eta_l \right) \left( h_{nk+1}^+ - z_{nk} \right)(1 + o(1)), \quad (4.86) \]

\[ u_{nk+1}^- - z_{nk} u_{nk}^- = \left(\prod_{l=N_0}^{n_k} \eta_l \right) \left( 1 + o\left(\frac{1}{n_k^\frac{\tau}{2}}\right)\right), \quad (4.87) \]

as \( k \to \infty \), and with (4.8) this all implies that

\[ |u_{nk+1}^+(\lambda) - z_{nk}(\lambda) u_{nk}^+(\lambda)| = \frac{2\sqrt{-\lambda}H(\lambda) + o(1)}{n_k^\frac{\tau}{2}}, \quad (4.88) \]

\[ |u_{nk+1}^-(\lambda) - z_{nk}(\lambda) u_{nk}^-(\lambda)| = o\left(\frac{1}{n_k^\frac{\tau}{2}}\right). \quad (4.89) \]

Therefore

\[ |P_{nk+1}(\lambda) - z_{nk}(\lambda) P_{nk}(\lambda)| = \frac{2\sqrt{-\lambda}\Psi(\lambda)|H(\lambda) + o(1)}{n_k^\frac{\tau}{2}}, \quad k \to \infty, \quad (4.90) \]

uniformly in \( \lambda \in [-R, -r] \). Here we write only the asymptotics of absolute values, because, firstly, this is exactly what we need to proceed, and secondly, because its form should depend on \( \alpha \) and cannot be written explicitly for all \( \alpha \in (0, 1) \), see Remark 4.4. Since \( a_n = n^\alpha(1 + o(1)) \)
due to (4.4), this together with (4.80) is enough to pass to the limit for the subsequence \( \rho'_{n_k} \) using (4.76):

\[
\rho'_{n_k}(\lambda) = \frac{\sqrt{1-d_{n_k}^2(\lambda)}}{\pi a_n |P_{n_k+1}(\lambda) - z_{n_k}(\lambda) P_{n_k}(\lambda)|^2} \rightarrow \frac{1}{4\pi \sqrt{-\lambda} |\Psi(\lambda)|^2 H^2(\lambda)},
\]

and this limit is uniform in \( \lambda \in [-R, -r] \). By Proposition 3.3 the spectral measure \( \rho \) is absolutely continuous on \((-R, -r)\) and

\[
\rho'(\lambda) = \frac{1}{4\pi \sqrt{-\lambda} |\Psi(\lambda)|^2 H^2(\lambda)} = \frac{\sqrt{-\lambda} H^2(\lambda)}{\pi |u_0(\lambda)|^2} \quad \text{for a.a. } \lambda \in (-R, -r).
\]

Finally, note that \( r \) can be chosen arbitrarily small and \( R \) arbitrarily large to cover the whole \( \mathbb{R}_- \).

\[\square\]

5 Appendix A. Proof of Proposition 3.1

Due to different numbering of matrix entries the form of the stabilized matrix \( J_N \) slightly differs from the form used in [4], and for this reason the formula from that work for the spectral density of the stabilized matrix is formally not directly applicable to our situation. This means that we need to provide a proof of Proposition 3.1.

**Proof.** We will restrict ourselves to considering \( \lambda \in \mathbb{C}_+ \cup (b_N - 2a_N, b_N + 2a_N) \), because this is enough for the proof and makes it easy to avoid ambiguous or overcomplicated notations. Consider the eigenvector equation for the matrix \( J_N \). For \( n \leq N \) it coincides with the equation for the matrix \( J \),

\[
a_{n-1}u_{n-1} + b_nu_n + a_nu_{n+1} = \lambda u_n, \quad n \leq N,
\]

for \( n > N \) it has constant coefficients,

\[
a_Nu_{n-1} + b_nu_n + a_Nu_{n+1} = \lambda u_n, \quad n > N.
\]

For every \( \lambda \in \mathbb{C}_+ \cup (b_N - 2a_N, b_N + 2a_N) \) it admits two pairs of solutions: the pair of sequences of polynomials \( P_{N,n}(\lambda) \) and \( Q_{N,n}(\lambda) \) of the first and of the second kind, respectively, and another pair of solutions \( u_{N,n}^{\pm}(\lambda) \) which equal \( z_{N,n}(\lambda) \) for \( n \geq N \) (note that \( z_{N}(\lambda) \) and \( z_{N}^{-1}(\lambda) \) are the characteristic roots of the equation (5.2)) and are defined for \( n < N \) by solving (5.1) backwards. At the same time, the polynomials \( P_{N,n}(\lambda) \) and \( Q_{N,n}(\lambda) \) for the matrix \( J_N \) coincide with the polynomials \( P_n(\lambda) \) and \( Q_n(\lambda) \) for the matrix \( J \) for \( n \leq N + 1 \), which can be seen from (5.1) immediately. These two pairs are related by the following identities:

\[
P_{N,n}(\lambda) = \Phi_N(\lambda)u_{N,n}^+(\lambda) + \Phi_N^{(1)}(\lambda)u_{N,n}^-(\lambda), \quad Q_{N,n}(\lambda) = \Theta_N(\lambda)u_{N,n}^+(\lambda) + \Theta_N^{(1)}(\lambda)u_{N,n}^-(\lambda),
\]

for all \( \lambda \in \mathbb{C}_+ \cup (b_N - 2a_N, b_N + 2a_N) \) and \( n \geq 1 \) with some coefficients \( \Phi_N(\lambda), \Phi_N^{(1)}(\lambda), \Theta_N(\lambda) \) and \( \Theta_N^{(1)}(\lambda) \). By calculation of the Wronskian for large \( n \) we have

\[
W\{u_{N,n}^+(\lambda), u_{N,n}^-(\lambda)\} = a_N \left( z_N(\lambda) - \frac{1}{z_N(\lambda)} \right),
\]

\[
\Phi_N(\lambda) = \frac{W\{P_{N,n}(\lambda), u_{N,n}^-(\lambda)\}}{W\{u_{N,n}^+(\lambda), u_{N,n}^-(\lambda)\}} = \frac{P_N(\lambda)z_{N+1}(\lambda) - P_{N+1}(\lambda)z_N(\lambda)}{z_N(\lambda) - \frac{1}{z_N(\lambda)}},
\]

\[
\Theta_N(\lambda) = \frac{W\{Q_{N,n}(\lambda), u_{N,n}^-(\lambda)\}}{W\{u_{N,n}^+(\lambda), u_{N,n}^-(\lambda)\}} = \frac{Q_N(\lambda)z_{N+1}(\lambda) - Q_{N+1}(\lambda)z_N(\lambda)}{z_N(\lambda) - \frac{1}{z_N(\lambda)}},
\]
which shows that the functions \( \Phi_N \) and \( \Theta_N \) are analytic in \( \mathbb{C}_+ \) and continuous in \( \mathbb{C}_+ \cup (b_N - 2a_N, b_N + 2a_N) \). For \( \lambda \in (b_N - 2a_N, b_N + 2a_N) \) and \( n \geq N \) \((5.3)\) becomes
\[
P_{N,n}(\lambda) = \frac{\Phi_N(\lambda)}{z_N^n(\lambda)} + \overline{\Phi_N(\lambda)}z_N^n(\lambda), \quad Q_{N,n}(\lambda) = \frac{\Theta_N(\lambda)}{z_N^n(\lambda)} + \overline{\Theta_N(\lambda)}z_N^n(\lambda).
\]
For \( \lambda \in \mathbb{C}_+ \) the solution
\[
Q_{N,n}(\lambda) + m_N(\lambda)P_{N,n}(\lambda) = \frac{\Theta_N(\lambda) + m_N(\lambda)\Phi_N(\lambda)}{z_N^n(\lambda)} + (\Theta_N^{(1)}(\lambda) + m_N(\lambda)\Phi_N^{(1)}(\lambda))z_N^n(\lambda)
\]
has to belong to \( l^2 \), hence \( \Theta_N(\lambda) + m_N(\lambda)\Phi_N(\lambda) = 0 \) (recall that \( |z_N| < 1 \) for \( \lambda \in \mathbb{C}_+ \), or
\[
m_N(\lambda) = -\frac{\Theta_N(\lambda)}{\Phi_N(\lambda)}, \quad \lambda \in \mathbb{C}_+.
\]
This equality can be continued to \( \lambda \in (b_N - 2a_N, b_N + 2a_N) \) where it implies \([2]\) that
\[
\rho_N'(\lambda) = \frac{\text{Im} m_N(\lambda)}{\pi} = \frac{\Phi_N(\lambda)\Theta_N(\lambda) - \overline{\Phi_N(\lambda)}\overline{\Theta_N(\lambda)}}{2\pi i|\Phi_N(\lambda)|^2}.
\]
Calculations of the Wronskian for \( n = 1 \) and \( n \geq N \) using \((5.7)\) yield for \( \lambda \in (b_N - 2a_N, b_N + 2a_N) \):
\[
1 = W\{P_{N,n}, Q_{N,n}\} = a_N \left( \frac{\Phi_N}{z_N^n} + \overline{\Phi_N}z_N^n \right) \left( \frac{\Theta_N}{z_N^{n+1}} + \overline{\Theta_N}z_N^{n+1} \right) - a_N \left( \frac{\Phi_N}{z_N^{n+1}} + \overline{\Phi_N}z_N^{n+1} \right) \left( \frac{\Theta_N}{z_N^n} + \overline{\Theta_N}z_N^n \right) = a_N(\Phi_N\Theta_N - \overline{\Phi_N}\overline{\Theta_N})(z_N - \frac{1}{z_N})
\]
and therefore
\[
\Phi_N(\lambda)\Theta_N(\lambda) - \Phi_N(\lambda)\overline{\Theta_N(\lambda)} = \frac{1}{a_N(z_N(\lambda) - \frac{1}{z_N(\lambda)})}.
\]
So we have
\[
\rho_N'(\lambda) = \frac{1}{2\pi i a_N(z_N(\lambda) - \frac{1}{z_N(\lambda)})|\Phi_N(\lambda)|^2}, \quad \lambda \in (b_N - 2a_N, b_N + 2a_N).
\]
Taking the absolute value of \((5.5)\), we arrive at the equality
\[
|\Phi_N(\lambda)| = \frac{|P_{N+1}(\lambda) - z_N(\lambda)P_N(\lambda)|}{z_N(\lambda) - \frac{1}{z_N(\lambda)}}, \quad \lambda \in (b_N - 2a_N, b_N + 2a_N).
\]
Together with the formula \((5.4)\) this gives
\[
\rho_N'(\lambda) = \frac{z_N(\lambda) - \frac{1}{z_N(\lambda)}}{2\pi a_N|P_{N+1}(\lambda) - z_N(\lambda)P_N(\lambda)|^2} = \frac{1 - \left(\frac{\lambda - b_N}{2a_N}\right)^2}{\pi a_N|P_{N+1}(\lambda) - z_N(\lambda)P_N(\lambda)|^2},
\]
which completes the proof. \(\square\)
6 Appendix B. Spectral density in the non-critical case
(Aptekarev–Geronimo theorem revisited)

In this appendix we consider the class of Jacobi matrices from the paper by Aptekarev and Geronimo\cite{AptekarevGeronimo} and show how to prove their formula for the spectral density using the technique of the proof in the critical case above. This gives a somewhat new proof of the known fact. Note that the application of our technique in the non-critical case is much simpler than in the critical one.

We use the same notation for the entries of a Jacobi matrix \(\{a_n\}_{n=1}^{\infty}\) and \(\{b_n\}_{n=1}^{\infty}\) and impose different assumptions on them. We hope that this will not lead to misunderstanding.

**Theorem 6.1.** Let the entries \(a_n > 0\) and \(b_n \in \mathbb{R}\), \(n \in \mathbb{N}\), of the Jacobi matrix \(J\) be such that

\[
\left\{ \frac{b_n}{a_n} \right\}_{n=1}^{\infty}, \left\{ \frac{1}{a_n} \right\}_{n=1}^{\infty}, \left\{ \frac{a_{n-1}}{a_n} \right\}_{n=1}^{\infty} \tag{6.1}
\]

are sequences of bounded variation,

\[
\frac{b_n}{a_n} \to 2d, \quad \frac{1}{a_n} \to 0, \quad \frac{a_{n-1}}{a_n} \to 1 \quad \text{as} \quad n \to \infty \tag{6.2}
\]

with

\[
d \in (-1, 1) \tag{6.3}
\]

and

\[
\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty. \tag{6.4}
\]

Then for every \(\lambda \in \mathbb{C}_+\) the eigenvector equation for \(J\),

\[
a_{n-1}u_{n-1} + b_nu_n + a_nu_{n+1} = \lambda u_n, \quad n \geq 2, \tag{6.5}
\]

has a solution \(u_n^-(\lambda)\) with the asymptotics

\[
u_n^-(\lambda) = \left( \prod_{l=2}^{n} \mu_l^-\left(\lambda\right) \right) (1 + o(1)), \quad n \to \infty, \tag{6.6}
\]

where

\[
\mu_l^-\left(\lambda\right) := \frac{\lambda - b_n}{2a_n} - i\sqrt{\frac{a_{n-1}}{a_n} - \left(\frac{\lambda - b_n}{2a_n}\right)^2} \to -d - i\sqrt{1-d^2}, \quad n \to \infty. \tag{6.7}
\]

The asymptotics (6.6) is uniform in every compact set \(K \subset \mathbb{C}_+\). For any \(n\) the vector component \(u_n^-\) is analytic in \(\mathbb{C}_+\) and continuous in \(\overline{\mathbb{C}_+}\). The limit

\[
M(\lambda) := \lim_{n \to \infty} \sqrt{a_n} \prod_{l=2}^{n} |\mu_l^-\left(\lambda\right)|, \quad \lambda \in \mathbb{R}, \tag{6.8}
\]

exists and is finite, nonzero and continuous in \(\lambda \in \mathbb{R}\). The sequence \(u_n^+(\lambda) := u_n^-(\lambda)\) for \(\lambda \in \mathbb{R}\) is also a solution to the equation (6.5) and the Wronskian of solutions \(u^\pm(\lambda) = \{u_n^\pm(\lambda)\}_{n=1}^{\infty}\)

\[
W\{u^+(\lambda), u^-\} = -2i\sqrt{1-d^2}M^2(\lambda), \tag{6.9}
\]

therefore \(u^+(\lambda)\) and \(u^-\) are linearly independent. The orthogonal polynomials can be expressed for \(\lambda \in \mathbb{R}\) as

\[
P_n(\lambda) = \Psi(\lambda)u_n^+(\lambda) + \overline{\Psi(\lambda)}u_n^-(\lambda), \tag{6.10}
\]
where
\[ \Psi(\lambda) = \frac{u_0(\lambda)}{2i\sqrt{1 - d^2}M^2(\lambda)}. \quad (6.11) \]
\[ u_0(\lambda) := (\lambda - b_1)u_1^- (\lambda) - a_1u_2^- (\lambda) \] (assuming formally in (6.5) that \( a_0 := 1 \)). Finally,
\[ \rho'(\lambda) = \frac{1}{4\pi\sqrt{1 - d^2} |\Psi(\lambda)|^2 M^2(\lambda)} = \frac{\sqrt{1 - d^2}M^2(\lambda)}{\pi |u_0(\lambda)|^2}, \quad \lambda \in \mathbb{R}. \quad (6.12) \]

**Remark 6.2.** The critical case corresponds to \( d = \pm 1 \), and the formula (6.12) then fails. For \(|d| > 1\) the spectrum of \( J \) is discrete, since we are in the situation of dominating main diagonal. This can be shown by estimating quadratic forms of truncated matrices \([13, 33]\).

**Remark 6.3.** Note that the conditions (6.1)–(6.2) are much weaker compared to the condition (4.4) in the critical case. Surely, it is not surprising and, moreover, the similar situation happens for discrete Schrödinger operator with decreasing potential at the edges of the essential spectrum, the points \( \lambda = \pm 2 \).

**Proof.** From the assumption (6.2) it immediately follows that in the formula for the spectral density (3.3) of the stabilized matrix \( J_n \),
\[ \rho'_n(\lambda) = \frac{\sqrt{1 - (\frac{\lambda - b_n}{2a_n})^2}}{\pi a_n |P_{n+1}(\lambda) - z_n(\lambda) P_n(\lambda)|^2}, \quad (6.13) \]
the numerator converges to \( \sqrt{1 - d^2} \) as \( n \to \infty \). Note that the complex numbers
\[ \mu_n^\pm(\lambda) := \frac{\lambda - b_n}{2a_n} \pm i \sqrt{\frac{a_{n-1}}{a_n} - \left( \frac{\lambda - b_n}{2a_n} \right)^2}, \quad (6.14) \]
are exactly the eigenvalues of the transfer-matrix
\[ B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -\frac{a_{n-1}}{a_n} & \frac{\lambda - b_n}{a_n} \end{pmatrix}, \quad (6.15) \]
Here the branches for \( \lambda \in \mathbb{C}_+ \) are chosen so that \( \sqrt{a_{n-1}/a_n} \mu_n^+(\lambda) \in \mathbb{C}_+ \setminus \mathbb{D} \) and \( \sqrt{a_n/a_{n-1}} \mu_n^-(\lambda) \in (\mathbb{C}_- \cap \mathbb{D}) \setminus \{0\} \), \( \mathbb{D} \) denotes the open unit disc. The determinant of the transfer-matrix equals
\[ \mu_n^+(\lambda) \mu_n^-(\lambda) = \frac{a_{n-1}}{a_n}, \quad n \geq 2. \quad (6.16) \]

From (6.2) we also see that
\[ B_n(\lambda) \to \begin{pmatrix} 0 & 1 \\ -1 & -2d \end{pmatrix}, \quad \mu_n^+(\lambda) \to -d \pm i\sqrt{1 - d^2}, \quad (6.17) \]
\[ z_n(\lambda) \to -d - i\sqrt{1 - d^2}, \quad \frac{1}{z_n(\lambda)} \to -d + i\sqrt{1 - d^2}, \quad (6.18) \]
as \( n \to \infty \), and for \( \lambda \in \mathbb{R} \) the eigenvalues of \( B_n(\lambda) \) are in the elliptic case for large \( n \):
\[ \mu_n^+(\lambda) = \mu_n^-(\lambda), \quad |\mu_n^-(\lambda)| = |\mu_n^+(\lambda)| = \sqrt{\frac{a_{n-1}}{a_n}}. \quad (6.19) \]
Therefore the limit of the sequence \( \lim_{n \to \infty} \sqrt{a_n} \prod_{i=2}^{n} |\mu_i(\lambda)| \) as \( n \to \infty \) exists for every \( \lambda \in \mathbb{R} \) and is a continuous function of \( \lambda \) without zeros. Denote

\[
z := -d - i\sqrt{1 - d^2}.
\]

At the real points

\[
\lambda_n^\pm := b_n \pm 2\sqrt{a_{n-1}a_n}
\]

we have \( \mu_n^+ \equiv \mu_n^- \) and the transfer-matrix \( B_n(\lambda) \) is not diagonalizable, while it is diagonalizable for all other \( \lambda \in \mathbb{C} \). Denote

\[
X_N := \{ \lambda_n^+, n \geq N \}.
\]

Fix an arbitrary compact set \( K \subset \mathbb{R}_+ \). Since \( \lambda_n^+ \sim 2a_n(d \pm 1) \) as \( n \to \infty \), for every \( d \in (-1, 1) \) there exists \( N(K) \in \mathbb{N} \) such that \( K \cap X_{N(K)} = \emptyset \). This means that for all \( n \geq N(K) \) and \( \lambda \in K \) one can diagonalize the matrix \( B_n(\lambda) \),

\[
B_n(\lambda) = \begin{pmatrix}
1 & 1 \\
\mu_n^+(\lambda) & \mu_n^-(\lambda)
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & \mu_n^-(\lambda)
\end{pmatrix} \begin{pmatrix}
1 & 1 \\
\mu_n^+(\lambda) & \mu_n^-(\lambda)
\end{pmatrix}^{-1}, \quad n \geq N(K).
\]

The substitution

\[
\vec{u}_n = \begin{pmatrix} 1 \\ \mu_n^+(\lambda) \\ \mu_n^-(\lambda) \end{pmatrix} \vec{v}_n
\]

transforms for \( n \geq N(K) \) the system

\[
\vec{u}_{n+1} = B_n(\lambda)\vec{u}_n,
\]

which is equivalent to the eigenfunction equation \((6.5)\), to the system

\[
\vec{v}_{n+1} = \begin{pmatrix} 1 \\ \mu_{n+1}^+(\lambda) \\ \mu_{n+1}^-(\lambda) \end{pmatrix}^{-1} B_n(\lambda) \begin{pmatrix} 1 \\ \mu_n^+(\lambda) \\ \mu_n^-(\lambda) \end{pmatrix} \vec{v}_n
\]

\[
= \begin{pmatrix} 1 \\ \mu_{n+1}^+(\lambda) \\ \mu_{n+1}^-(\lambda) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mu_n^+(\lambda) \\ \mu_n^-(\lambda) \end{pmatrix} \begin{pmatrix} \mu_n^+(\lambda) \\ 0 \\ \mu_n^-(\lambda) \end{pmatrix} \vec{v}_n, \quad n \geq N(K).
\]

Using the condition \((6.1)\) one can show by an elementary calculation that

\[
\left\{ \sup_{\lambda \in K} \left\| \begin{pmatrix} 1 \\ \mu_{n+1}^+(\lambda) \\ \mu_{n+1}^-(\lambda) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mu_n^+(\lambda) \\ \mu_n^-(\lambda) \end{pmatrix} - I \right\| \right\}_n^{N(K)} \in l^1
\]

\((6.26)\)

\[
\]

\[
\text{(here we essentially use that } |d| < 1 \text{). Therefore the system } (6.26) \text{ can be written as}
\]

\[
\vec{v}_{n+1} = \left( \begin{pmatrix} \mu_n^+(\lambda) \\ 0 \\ \mu_n^-(\lambda) \end{pmatrix} + R_n(\lambda) \right) \vec{v}_n, \quad n \geq N(K),
\]

\[(6.27)\]

\[
\text{with } \left\{ \sup_{\lambda \in K} \| R_n(\lambda) \| \right\}_n^{N(K)} \in l^1. \text{ Consider the second substitution } \vec{v}_n \to \vec{x}_n:
\]

\[
\vec{v}_n = \prod_{l=N(K)}^{n-1} \sqrt{\mu_l^+(\lambda)\mu_l^-} \vec{x}_n, \quad n \geq N(K),
\]

\[
(6.28)\]

\[
\text{this transforms the system } (6.28) \text{ to the system}
\]

\[
\vec{x}_{n+1} = \left( \begin{pmatrix} \mu_n^+(\lambda) \\ 0 \\ \mu_n^-(\lambda) \end{pmatrix} + R_n(\lambda) \sqrt{\mu_n^+(\lambda)\mu_n^-} \right) \vec{x}_n, \quad n \geq N(K).
\]

\[
(6.29)\]
Lemma 4.5 is applicable to this system for \( n \geq N(K) \): there exists \( C(K) > 0 \) such that for every \( p, q \in \mathbb{N} \) such that \( N(K) \leq p \leq q \)

\[
\prod_{l=p}^{q} \sqrt{\frac{\mu_l^+(\lambda)}{\mu_l^-(\lambda)}} \geq \frac{1}{C(K)} \quad (6.31)
\]

(provided by the fact that we choose \( |\mu_n^+(\lambda)| \geq |\mu_n^-(\lambda)| \) for large \( n \) and)

\[
\left\{ \sup_{\lambda \in K} \left| R_n(\lambda) \right| \right\} \bigg|_{n=N(K)} \in l^1, \quad (6.32)
\]

which gives the condition (4.46) of Lemma 4.5. By the lemma there exists a solution \( \tilde{x}^{K,-}(\lambda) \) of the system (6.30)

\[
\tilde{x}^{K,-}_n(\lambda) = \left( \prod_{l=N(K)}^{n-1} \sqrt{\frac{\mu_l^+(\lambda)}{\mu_l^-(\lambda)}} \right) \left( \bar{e}_- + o(1) \right), \quad n \to \infty, \quad (6.33)
\]

which is analytic in \( \text{int} \, K \), continuous in \( K \) and has uniform in \( \lambda \in K \) asymptotics. Correspondingly the system (6.25) has a solution

\[
\tilde{u}^{K,-}_n(\lambda) := \begin{pmatrix} 1 \\ \mu_n^-(\lambda) \end{pmatrix} \left( \prod_{l=N(K)}^{n-1} \sqrt{\frac{\mu_l^+(\lambda)}{\mu_l^-(\lambda)}} \right) \tilde{x}^{K,-}_n(\lambda) \quad (6.34)
\]

with the same properties,

\[
\tilde{u}^{K,-}_n(\lambda) = \begin{pmatrix} u_{n-1}^{K,-}(\lambda) \\ u_n^{K,-}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ \mu_n^-(\lambda) \end{pmatrix} \left( \prod_{l=N(K)}^{n-1} \mu_l^- (\lambda) \right) \left( \bar{e}_- + o(1) \right) = \begin{pmatrix} u_{n-1}^{K,-}(\lambda) \\ u_n^{K,-}(\lambda) \end{pmatrix} \left( \left( \frac{1}{z} \right) + o(1) \right), \quad n \to \infty. \quad (6.35)
\]

This solution is formally defined only for \( n \geq N(K) \), but obviously exists for all \( n \geq 2 \) retaining its properties, because the matrices \( B_n(\lambda) \) and \( B_n^{-1}(\lambda) \) are entire functions of \( \lambda \) for all \( n \). Let us observe that the solution proportional to \( \tilde{u}^{K,-}_n(\lambda) \)

\[
\tilde{u}^-_n(\lambda) := \begin{pmatrix} (N(K)-1) \\ \mu_l^- (\lambda) \end{pmatrix} \tilde{u}^{K,-}_n(\lambda), \quad (6.36)
\]

\[
\tilde{u}^-_n(\lambda) = \begin{pmatrix} u_{n-1}^-(\lambda) \\ u_n^-(\lambda) \end{pmatrix} = \begin{pmatrix} \mu_l^- (\lambda) \end{pmatrix} \left( \left( \frac{1}{z} \right) + o(1) \right), \quad n \to \infty, \quad (6.37)
\]

does not depend on \( K \) and hence is analytic in \( \mathbb{C}_+ \) and continuous in \( \overline{\mathbb{C}}_+ \). It is enough to show that for any compact sets \( K_1, K_2 \) such that \( K_1 \subset K_2 \subset \overline{\mathbb{C}}_+ \) for every \( \lambda \in K_1 \) solutions \( \tilde{u}^{K_1,-}(\lambda) \) and \( \tilde{u}^{K_2,-}(\lambda) \) are proportional (and not only have proportional asymptotics). To this end consider the Wronskian: for any \( n \geq 1 \) and \( \lambda \in K_1 \)

\[
W\{u^{K_1,-}(\lambda), u^{K_2,-}(\lambda)\} = \lim_{n \to \infty} \det \begin{pmatrix} u_{n+1}^{K_1,-}(\lambda) & u_{n+1}^{K_2,-}(\lambda) \\ u_n^{K_1,-}(\lambda) & u_n^{K_2,-}(\lambda) \end{pmatrix} = \lim_{n \to \infty} \det \begin{pmatrix} u_{n+1}^{K_1,-}(\lambda) & u_{n+1}^{K_2,-}(\lambda) \\ u_n^{K_1,-}(\lambda) & u_n^{K_2,-}(\lambda) \end{pmatrix} = 0 \quad (6.38)
\]
using (6.38). For \( \lambda \in \mathbb{R} \) the sequence \( u_+^+(\lambda) := \overline{u_n^-(\lambda)} \) is a solution of (6.25) and, since \( |z| = 1 \),
\[ z = \frac{1}{z}, \]

\[
W\{u^+(\lambda), u^-(\lambda)\} = a_n \det \begin{pmatrix} u_+^+(\lambda) & u_n^-(\lambda) \\ u_{n+1}^+(\lambda) & u_{n+1}^-(\lambda) \end{pmatrix} \\
= \lim_{n \to \infty} a_n \det \left[ \left( \begin{pmatrix} 1 \\ \frac{1}{z} \end{pmatrix} + o(1) \right) \prod_{l=2}^{n} \begin{pmatrix} \mu_l^-(\lambda) & 0 \\ 0 & \mu_l^-(\lambda) \end{pmatrix} \right] \\
= \lim_{n \to \infty} a_n \left( z - \frac{1}{z} + o(1) \right) \left( \prod_{l=2}^{n} |\mu_l^-(\lambda)|^2 \right) = -2i \sqrt{1 - d^2} M^2(\lambda). \tag{6.39} \]

Therefore \( u^+(\lambda) \) and \( u^-(\lambda) \) form a basis of solutions of (6.35). We have for \( \lambda \in \mathbb{R} \)
\[
P_n(\lambda) = \Psi(\lambda)u_+^+(\lambda) + \overline{\Psi(\lambda)}u_n^-(\lambda) \tag{6.40} \]
with
\[
\Psi(\lambda) = \frac{W\{P(\lambda), u^-(\lambda)\}}{W\{u^+(\lambda), u^-(\lambda)\}} = \frac{u_0^-(\lambda)}{2i \sqrt{1 - d^2} M^2(\lambda)}. \tag{6.41} \]

Recall that \( u_0^-(\lambda) = (\lambda - b_1)u_1^-(\lambda) - a_1 u_2^-(\lambda) \), then
\[
P_{n+1}(\lambda) - z_n(\lambda)P_n(\lambda) = \Psi(\lambda)(u_{n+1}^+(\lambda) - z_n(\lambda)u_n^+(\lambda)) + \overline{\Psi(\lambda)}(u_{n+1}^-(\lambda) - z_n(\lambda)u_n^-(\lambda)) \\
= \Psi(\lambda) \left( \prod_{l=2}^{n} \mu_l^-\lambda(-) \right) \left( \mu_{n+1}^-\lambda(-) - z_n(\lambda) + o(1) \right) + \overline{\Psi(\lambda)} \left( \prod_{l=2}^{n} \mu_l^-\lambda(+) \right) \left( \mu_{n+1}^-\lambda(+) - z_n(\lambda) + o(1) \right) \\
= \Psi(\lambda) \left( \prod_{l=2}^{n} \mu_l^-\lambda(+) \right) \left( \frac{1}{z} - z + o(1) \right), \tag{6.42} \]

since \( \mu_{n+1}^-\lambda(-) - z_n(\lambda) \to 0 \) and \( \mu_{n+1}^-\lambda(+) - z_n(\lambda) \to \frac{1}{z} - z \) as \( n \to \infty \). Therefore
\[
\sqrt{a_n}|P_{n+1}(\lambda) - z_n(\lambda)P_n(\lambda)| \to |\Psi(\lambda)| \lim_{n \to \infty} \left( \sqrt{a_n} \prod_{l=2}^{n} |\mu_l^-\lambda(+)\right) \left| \frac{1}{z} - z \right| = 2|\Psi(\lambda)|M(\lambda) \sqrt{1 - d^2}, \tag{6.43} \]

and the denominator of (6.13) converges to \( 4\pi |\Psi(\lambda)|^2 M^2(\lambda)(1 - d^2) \) as \( n \to \infty \). By Proposition 3.3 we arrive at the formula (6.12). This completes the proof. \( \square \)

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