Gauge Theory Dynamics and Kähler Potential for Calabi-Yau Complex Moduli

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Abstract

We compute the exact two-sphere partition function and matrix of two-point functions of operators in the chiral ring with their complex conjugates in two-dimensional supersymmetric gauge theories. For gauge theories that flow in the infrared to a Calabi-Yau nonlinear sigma model, these renormalization group invariant observables determine the exact Kähler potential and associated Zamolodchikov metric in the complex structure moduli space of the Calabi-Yau manifold.

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1 Introduction

String theory compactifications on a Calabi-Yau manifold accommodate physically appealing models of particle physics within an ultraviolet complete theory of quantum gravity. In string theory, there is a beautiful and rich interplay between the four dimensional effective field theory capturing the low energy dynamics of string theory on a Calabi-Yau threefold and the two dimensional superconformal field theory (SCFT) on the string worldsheet. Massless scalar fields in four dimensions correspond to exactly marginal operators in the SCFT realizing the Calabi-Yau compactification, while the couplings encoding the effective field theory dynamics are captured by worldsheet correlation functions.

Metric deformations of a Calabi-Yau manifold give rise to massless four dimensional scalar fields. They split into two families: Kähler class deformations and complex structure deformations. The number of corresponding moduli is determined by the two non-trivial Betti numbers of a Calabi-Yau threefold: $h^{1,1}$ and $h^{1,2}$. In the worldsheet theory, the exactly marginal operators associated to Kähler class and complex structure moduli are in different multiplets

$$
\mathcal{O}_i \quad \leftrightarrow \quad \text{Kähler moduli} \quad i = 1, \ldots, h^{1,1}
$$

$$
\mathcal{O}_a \quad \leftrightarrow \quad \text{complex moduli} \quad a = 1, \ldots, h^{1,2}.
$$

The operators $\mathcal{O}_i$ and $\mathcal{O}_a$ are primary operators annihilated by different supercharges ($q_A$ vs. $q_B$) and $U(1)_R \times U(1)_A$ R-symmetry generators in the $\mathcal{N} = (2, 2)$ superconformal algebra

$$
[q_A, \mathcal{O}_i] = [R, \mathcal{O}_i] = 0 \quad (1.1)
$$

$$
[q_B, \mathcal{O}_a] = [A, \mathcal{O}_a] = 0. \quad (1.2)
$$

These operators form a ring in the SCFT: $\mathcal{O}_i$ the twisted chiral ring and $\mathcal{O}_a$ the chiral ring. The conjugate operators $\mathcal{O}_i$ and $\mathcal{O}_a$ are annihilated by conjugate supercharges $q_{\bar{A}}$ and $q_{\bar{B}}$.

Upon compactification on a Calabi-Yau, the metric in field space for the moduli in four dimensions is non-trivial. It is given by the metric in the “quantum” Kähler moduli space $M_K$ and the complex structure moduli space $M_C$ respectively. In the worldsheet theory, the moduli metric is captured by the Zamolodchikov metric of the CFT: the two-point function of the corresponding exactly marginal operators on the two-sphere. The metric in the (complexified) Kähler moduli space is

$$
G^K_{i\bar{j}} = \langle \mathcal{O}_i(N) \mathcal{O}_{\bar{j}}(S) \rangle_{S^2}, \quad (1.3)
$$

while the metric in the complex structure moduli space is

$$
G^C_{ab} = \langle \mathcal{O}_a(N) \mathcal{O}_b(S) \rangle_{S^2}. \quad (1.4)
$$

Operators are inserted at the north pole of the two-sphere and conjugate operators at the south pole. Both metrics are Kähler, and are determined by the Kähler potential $K^K$ and $K^C$ in the Kähler moduli space $M_K$ and complex structure moduli space $M_C$

$$
G^K_{i\bar{j}} = \partial_i \partial_{\bar{j}} K^K \quad G^C_{ab} = \partial_a \partial_b K^C. \quad (1.5)
$$

1These operators also play an important role in massive $\mathcal{N} = (2, 2)$ field theories.

2Superconformal Ward identities determine the two-point functions of the exactly marginal operators in terms of those for the primary operators $\mathcal{O}_i$ and $\mathcal{O}_a$. 
These, in turn, also determine the superpotential (Yukawa) couplings for the four dimensional chiral and antichiral multiplets appearing in heterotic string compactifications.

The computation of these observables in a Calabi-Yau nonlinear sigma model (NLSM) is notoriously difficult. A fruitful approach towards NLSMs is to study instead two dimensional gauge theories [1] – known as gauged linear sigma models (GLSM’s) – which flow in the infrared to a NLSM. In favorable situations, renormalization group invariant observables in the ultraviolet GLSM can be computed exactly by supersymmetric localization [2–4] of functional integrals. In this optimal scenario, computations in the GLSM capture exactly correlators in the elusive infrared NLSM.

Recently, the computation of two dimensional gauge theories in [5] (see also [6]) has resulted in the exact Kähler potential $K_K$ in the “quantum” Kähler moduli space $\mathcal{M}_K$ of a Calabi-Yau manifold in terms of the GLSM two-sphere partition function

$$Z_A = e^{-K_K},$$

as conjectured in [7] and demonstrated in [8]. This provides a physics-based approach to computing, in particular, worldsheet instanton corrections to the Kähler potential that does not rely on mirror symmetry [9–11]. This result has found a variety of applications and generalizations. Most notably the computation of novel Gromov-Witten invariants [7] for which no other method of computation is currently available as well as new insights into mirror symmetry [8] and D-branes [12–14].

In this paper we identify and compute the observable which captures the exact Kähler potential $K_C$ in the complex structure moduli space $\mathcal{M}_C$ of a large class Calabi-Yau manifolds. It is given by a different GLSM two-sphere partition function

$$Z_B = e^{-K_C},$$

which now depends on the complex structure moduli. In this setup, the two-sphere can be enriched by inserting in a supersymmetric way chiral operators $O_a$ at the north pole and their conjugates $O_b$ at the south pole, while in [5,6] the admissible supersymmetric insertions were twisted chiral operators $O_i$ and their conjugates $O_j$ at the poles.

Our results yield a purely gauge theory realization of the Kähler potential $K_C$, which admits a geometrical representation in terms of the nowhere vanishing top holomorphic form $\Omega$ of a Calabi-Yau manifold as

$$e^{-K_C} = i^{\dim M} \int_M \Omega \wedge \overline{\Omega}.$$  

In this paper we focus on gauge theories with an abelian gauge group, which can describe Calabi-Yau manifolds which are complete intersections in toric varieties, leaving the results for non-abelian gauge theories to a separate publication [15].

The plan of the rest of the paper is as follows. In section 2 we explain how gauge theories on a two-sphere can realize two different supersymmetry algebras, and how the choice of supersymmetry determines which operators can be added in the functional integral in a supersymmetric way. This, in turn, determines which supersymmetry algebra has to be realized in a GLSM in order to compute the Zamolodchikov metric for Kähler moduli and for complex structure moduli. In section 3 we write down the supersymmetry transformations and action of two-dimensional gauge theories which allow
the insertion of operators in the chiral ring (and their conjugates). In section 4 we compute the exact two-sphere partition function and matrix of two-point functions of abelian gauge theories by supersymmetric localization. In section 5 we apply our results to the computation of the two-sphere partition of GLSM for various families of Calabi-Yau geometries, and obtain a gauge theory realization of the exact Kähler potential in the complex structure moduli space (1.8). Section 6 contains some discussion. Technical computations and details have been delegated to the appendices.

2 Supersymmetry on \( S^2 \) And Gauged Linear Sigma Models

The nonlinear sigma model describing a Calabi-Yau compactification is a two dimensional \( \mathcal{N} = (2, 2) \) SCFT. Generally, it is only at special loci in the SCFT moduli space – spanned by the twisted chiral and chiral operator marginal couplings – that explicit computations can be performed. This moduli space acquires a beautiful geometric interpretation as the “quantum” Kähler and complex structure moduli space of the Calabi-Yau geometry. A window into the dynamics of such a SCFT throughout moduli space can be provided by a gauged linear sigma model \cite{1}, a super-renormalizable gauge theory which flows to the Calabi-Yau NLSM in the infrared.

The corresponding Zamolodchikov metric on the SCFT moduli spaces (1.4) and (1.3) is the matrix of two-point functions on the two-sphere of chiral and twisted chiral operators with their conjugates. The goal is to realize these as supersymmetric correlators in the ultraviolet GLSM where exact computations can be performed using supersymmetric localization, and infer from them the results for the infrared SCFT. Computing these observables in the GLSM requires constructing \( \mathcal{N} = (2, 2) \) gauge theories on \( S^2 \).

While Calabi-Yau sigma models on the two-sphere have \( \mathcal{N} = (2, 2) \) superconformal symmetry\(^3\), a gauge theory on \( S^2 \) can realize at most an \( SU(2|1) \) subalgebra of the superconformal algebra. An \( SU(2|1) \) subalgebra is generated by supercharges in the \( \mathcal{N} = (2, 2) \) superconformal algebra that close into the \( SU(2) \) isometry generators \( J_m \) of the two-sphere together with \( T \), one of the \( U(1)_R \times U(1)_A \) \( R \)-symmetry generators in the superconformal algebra. It is given by

\[
\begin{align*}
[J_m, J_n] &= i \epsilon_{mnp} J_p \\
[J_m, Q_\alpha] &= -\frac{1}{2} \gamma^\beta_{m \alpha} Q_\beta \\
[J_m, S_\alpha] &= -\frac{1}{2} \gamma^\beta_{m \alpha} S_\beta \\
\{S_\alpha, Q_\beta\} &= \gamma^m_{\alpha \beta} J_m - \frac{1}{2} C_{\alpha \beta} T \\
[T, Q_\alpha] &= -Q_\alpha \\
[T, S_\alpha] &= S_\alpha.
\end{align*}
\]

(2.1)

There are two inequivalent \( SU(2|1) \) subalgebras on the two-sphere: \( SU(2|1)_A \) and \( SU(2|1)_B \). These two algebras are mapped into each other by the \( \mathbb{Z}_2 \) mirror (outer) automorphism \( \sigma \) of the

\(^3\)Classically, a NLSM can be placed on the two-sphere by a Weyl transformation while preserving the full \( \mathcal{N} = (2, 2) \) superconformal algebra. For a NLSM with a Calabi-Yau target space, the superconformal symmetry remains quantum mechanically.
\( \mathcal{N} = (2, 2) \) superconformal algebra\(^4\) [10]

\[
\begin{array}{cc}
SU(2|1)_A & SU(2|1)_B \\
Q_A & Q_B \\
S_A & S_B \\
R & A \\
J_m & J_m
\end{array}
\]

The choice of supersymmetry realized in the gauge theory determines which observables of the theory in the infrared can be captured by localizing correlators in the ultraviolet gauge theory. The \( SU(2|1)_A \) (resp. \( SU(2|1)_B \)) algebra has a supercharge \( Q_A \) (resp. \( Q_B \)) that interpolates between \( q_A \) (resp. \( q_B \)) at the north pole and \( \bar{q}_A \) (resp. \( \bar{q}_B \)) at the south pole of the two-sphere. In \( SU(2|1)_A \) the conserved R-symmetry is \( T = R \) while in \( SU(2|1)_B \) the conserved R-symmetry is \( T = A \). This implies that \( Q_A \) in \( SU(2|1)_A \) annihilates twisted chiral operators \( O_i \) at the north pole and their conjugates \( \bar{O}_i \) at south pole (1.1), while \( Q_B \) in \( SU(2|1)_B \) annihilates chiral operators \( O_a \) at the north pole and their conjugates \( \bar{O}_a \) at the south pole (1.2). As a result, the correlators in the GLSM that flow to the Zamolodchikov metric in the infrared SCFT are supersymmetric.

The \( SU(2|1)_A \) and \( SU(2|1)_B \) invariant gauge theory two-sphere partition function and two-point functions just described are independent of the gauge coupling (see section 4.1 for decoupling theorems). Gauge coupling independence implies that these gauge theory observables are renormalization group invariants. In particular, they coincide with those in the theory in the extreme infrared, where \( g_{\text{YM}}^2 \to \infty \). In particular, for a Calabi-Yau GLSM, this is none other than the sought-after infrared SCFT. Therefore, by virtue of the properties of \( Q_A \) and \( Q_B \), the computation of the Zamolodchikov metric on the Kähler (resp. complex structure) moduli space can be recast in terms of correlators in an \( SU(2|1)_A \) (resp. \( SU(2|1)_B \)) invariant GLSM on \( S^2 \).

| \( Q_A \) | North Pole | \( q_A \) | \( \bar{O}_i \) | Correlation Function | Metric |
| \( O_i \) | South Pole | \( \bar{q}_i \) | \( O_i \) | \( \langle O_i \bar{O}_j \rangle \) | Kähler moduli |
| \( Q_B \) | \( q_B \) | \( O_a \) | \( \bar{O}_b \) | \( \langle O_a \bar{O}_b \rangle \) | Complex moduli |

### 2.1 Gauged and Nonlinear Sigma Models on \( S^2 \)

The multiplets with which to construct \( SU(2|1)_A \) and \( SU(2|1)_B \)-invariant theories are those of two-dimensional \( \mathcal{N} = (2, 2) \) supersymmetry. In flat spacetime, a chiral superfield \( \Phi \) and twisted chiral superfield \( Y \) [16] obey\(^5\)

\[
\bar{D}_+ \Phi = \bar{D}_- \Phi = 0, \quad \bar{D}_+ Y = D_- Y = 0.
\]

\(^4\)See appendix A for the embedding of these two algebras in the \( \mathcal{N} = (2, 2) \) superconformal algebra on \( S^2 \) [5] and the action of the mirror automorphism on the generators.

\(^5\)The superspace derivatives are \( D_\pm = \partial_{\theta^\pm} - i \theta^\pm \partial_\pm \) and \( \bar{D}_\pm = -\partial_{\bar{\theta}^\pm} + i \bar{\theta}^\pm \partial_\pm \).
A vector multiplet is a superfield $V = V^\dagger$ subject to gauge redundancy by a chiral multiplet $\Phi$

$$V \simeq V + i (\Phi - \bar{\Phi}), \quad (2.4)$$

while for a twisted vector multiplet the gauge redundancy is by a twisted chiral multiplet $Y$

$$V \simeq V + i (Y - \bar{Y}). \quad (2.5)$$

Conventionally, a NLSM on a Kähler manifold is written in terms of $\mathcal{N} = (2, 2)$ chiral multiplets. Since supersymmetric minimal coupling between chiral multiplets and twisted vector multiplets do not exist, an ultraviolet GLSM flowing to a NLSM is described by an $\mathcal{N} = (2, 2)$ vector multiplet with gauge group $G$ coupled to a chiral multiplet in a (reducible) representation $R$ of $G$.

On chiral multiplets the $U(1)$ R-symmetry $R$ acts vectorially while the $U(1)$ axial R-symmetry $A$ acts axially. Therefore a gauge theory with an $R$-invariant superpotential can be placed on $S^2$ while preserving $SU(2|1)_A$. The current associated to $A$ however, can be anomalous and only gauge theories for which the anomaly cancels can be placed on the two-sphere while preserving $SU(2|1)_B$. If a gauge theory in flat space preserves both $R$ and $A$, it is expected to flow in the infrared to an $\mathcal{N} = (2, 2)$ SCFT. Such gauge theories are the basis for the construction of Calabi-Yau GLSM’s.

In a Calabi-Yau GLSM based on vector and chiral multiplets, the complex structure moduli of the Calabi-Yau are realized by background expectation values for chiral multiplets while the Kähler moduli are realized by background expectation values for twisted chiral multiplets.\(^6\) These moduli appear in the superpotential $W$ and twisted superpotential $\tilde{W}$ of the GLSM, which are $Q_A$-exact and $Q_B$-exact respectively (see section 4.1):

| $W$          | $\tilde{W}$ |
|--------------|--------------|
| $Q_A$-exact  | $Q_B$-exact  |

Therefore, in an $SU(2|1)_A$-invariant GLSM based on vector and chiral multiplets, the partition function depends only on the Kähler moduli, while in an $SU(2|1)_B$-invariant GLSM of vector and chiral multiplets, the partition function depends only on the complex structure moduli.\(^7\)

In [5], the off-shell supersymmetry transformations for a vector multiplet and charged chiral multiplet realizing the full $\mathcal{N} = (2, 2)$ superconformal algebra were written down (see appendix B in [5]). This includes the supersymmetry transformations for both the $SU(2|1)_A$ and the $SU(2|1)_B$ subalgebra of the superconformal algebra. Nevertheless, in this paper we proceed with the computation of the $SU(2|1)_B$-invariant partition function using a different route (the $SU(2|1)_A$ computation was done in [5, 6]).

An alternative way to proceed is to change the coordinates used to write down the GLSM (and ensuing infrared NLSM). In this equivalent approach, the gauge theory realizes the $SU(2|1)_A$ symmetry but is written in terms of twisted vector and twisted chiral multiplets. These multiplets are the

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\(^6\)The FI and topological couplings can be written in terms of the field strength multiplet which is a twisted chiral superfield.

\(^7\)A Calabi-Yau GLSM may not realize all the complex structure and Kähler moduli of the Calabi-Yau.
image of the vector and chiral multiplets under the action of the mirror automorphism \((2, 2)\), which also exchanges \(SU(2|1)_B\) with \(SU(2|1)_A\). The gauge theory constructed by realizing the \(SU(2|1)_A\) on the twisted vector and twisted chiral multiplets is equivalent to the gauge theory constructed by realizing the \(SU(2|1)_B\) on the vector and chiral multiplets.

In this choice of coordinates, the Kähler moduli of the Calabi-Yau are now background expectation values for chiral multiplets\(^8\) while complex structure moduli are background expectation values for twisted chiral multiplets. These appear in superpotential and twisted superpotential terms respectively. Since superpotential terms are \(Q_A\)-exact, the partition function only depends on complex structure moduli.

The two approaches described above are equivalent. If the gauge theory is a Calabi-Yau GLSM, the partition function computes the Kähler potential on the complex structure moduli space.

### 3 Complex Structure Gauged Linear Sigma Model

The goal of this section is to construct \(SU(2|1)_A\)-invariant two-dimensional \(\mathcal{N} = (2, 2)\) gauge theories of twisted vector multiplets and twisted chiral multiplets. In flat space the Lagrangian of a vector coupled to a chiral multiplet is identical to the Lagrangian of a twisted vector coupled to a twisted chiral multiplet. This is no longer the case when the theory is placed on the two-sphere. The background fields \([17]\) and curvature couplings needed to couple the theory to the two-sphere in a supersymmetric way are different, and thus the resulting Lagrangians are different. We now proceed to construct the supersymmetry transformations and invariant couplings for the twisted vector and twisted chiral multiplets.

#### 3.1 Twisted Vector Multiplet

An \(\mathcal{N} = (2, 2)\) twisted vector multiplet consists of a real vector, two complex scalars related by complex conjugation, two complex spinors and a real auxiliary scalar \((A_\mu, \sigma, \bar{\sigma}, \eta, \bar{\eta}, D)\), all of which are valued in the Lie algebra of the gauge group \(G\). While a twisted vector multiplet and a vector multiplet with the same gauge group \(G\) have exactly the same field content, the supersymmetry transformations on the two multiplets are realized differently.

\(^8\)The FI and topological couplings can be written in terms of the twisted field strength multiplet, a chiral superfield.
The $SU(2|1)_A$ supersymmetry transformations on the twisted vector multiplet fields are

$$
\delta \eta = i \slashed{D}(\sigma \epsilon) + \bar{\epsilon}(D + iF) - \frac{i}{2} \gamma^3 \bar{\epsilon}[\sigma, \bar{\sigma}]
$$

$$
\delta \bar{\eta} = i \slashed{D}(\bar{\sigma} \bar{\epsilon}) + \epsilon(D - iF) - \frac{i}{2} \gamma^3 \epsilon[\sigma, \bar{\sigma}]
$$

$$
\delta A_\mu = \frac{i}{2}(\epsilon \gamma^3 \gamma_\mu \eta - \bar{\epsilon} \gamma^3 \gamma_\mu \bar{\eta})
$$

$$
\delta \sigma = \bar{\epsilon} \eta
$$

$$
\delta \bar{\sigma} = \epsilon \bar{\eta}
$$

$$
\delta D = \frac{i}{2} \left\{ D_\mu (\epsilon \gamma^\mu \eta) - [\sigma, \epsilon \gamma^3 \bar{\eta}] \right\} + \frac{i}{2} \left\{ D_\mu (\bar{\epsilon} \gamma^\mu \bar{\eta}) + [\bar{\sigma}, \bar{\epsilon} \gamma^3 \eta] \right\}.
$$

They are parametrized by conformal Killing spinors $\epsilon$ and $\bar{\epsilon}$ obeying

$$
\nabla_\mu \epsilon = \frac{1}{2r} \gamma^\mu \gamma^3 \epsilon, \quad \nabla_\mu \bar{\epsilon} = -\frac{1}{2r} \gamma^\mu \gamma^3 \bar{\epsilon},
$$

where $r$ is the radius of the two-sphere. These transformations realize the $SU(2|1)_A$ algebra off-shell up to gauge transformations. Concretely, the resulting algebra is

$$
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_G(\Lambda)
$$

$$
[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] = \delta_G(\bar{\Lambda})
$$

$$
[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \delta_{SU(2)}(v) + \delta_R(\alpha) + \delta_G(\Omega)
$$

where the $SU(2)$ isometry transformation is constructed from the $S^2$ Killing vector

$$
v = i \bar{\epsilon} \gamma^\mu \epsilon \partial_\mu,
$$

and the $U(1)_R$ transformation is parametrized by the scalar

$$
\alpha = -\frac{1}{2r} \bar{\epsilon} \gamma^3 \epsilon.
$$

The $R$-charges of the various fields are:

| Field | $\sigma$ | $\eta_+$ | $\eta_-$ | $A_\mu$ | D | $\bar{\eta}_+$ | $\bar{\eta}_-$ | $\bar{\sigma}$ |
|-------|-------|-------|-------|-------|---|-------|-------|-------|
|       | -2    | -1    | -1    | 0     | 0 | +1    | +1    | +2    |

Finally, the field dependent gauge transformation parameters generated in the closure of the algebra are

$$
\Lambda = -\epsilon_2 \gamma^3 \epsilon_1 \sigma \quad \bar{\Lambda} = \bar{\epsilon}_2 \gamma^3 \bar{\epsilon}_1 \bar{\sigma} \quad \Omega = -v^\mu A_\mu.
$$

These are the $U(1)_A$ charges of the vector multiplet fields for a vector multiplet of vanishing $U(1)_A$ charge.
3.2 Twisted Chiral Multiplet

The field content of a twisted chiral multiplet is the same as the standard chiral multiplet but also has different supersymmetry transformations. A twisted chiral multiplet can be minimally coupled in a supersymmetric way to a twisted vector multiplet. It transforms in a representation $R$ of the gauge group $G$. The $SU(2|1)_A$ supersymmetry transformations, invariant action and partition function of uncharged twisted chiral multiplets on $S^2$ appeared in [8].

The $SU(2|1)_A$ supersymmetry transformations of charged twisted chiral multiplet fields $(Y, \bar{Y}, \z, \bar{\z}, G, \bar{G})$ are

\[
\begin{align*}
\delta Y &= (\bar{\epsilon}_- - \epsilon_+) \z \\
\delta \bar{Y} &= (\bar{\epsilon}_+ - \epsilon_-) \bar{\z} \\
\delta \z_+ &= -\gamma_+ (i \slashed{D} Y - G) \epsilon + i \gamma_+ \epsilon \sigma Y \\
\delta \z_- &= +\gamma_- (i \slashed{D} Y - G) \epsilon - i \gamma_- \epsilon \sigma Y \\
\delta \bar{\z}_+ &= +\gamma_+ (i \slashed{D} \bar{Y} - \bar{G}) \epsilon - i \gamma_+ \epsilon \bar{Y} \sigma \\
\delta \bar{\z}_- &= -\gamma_- (i \slashed{D} \bar{Y} - \bar{G}) \epsilon - i \gamma_- \epsilon \bar{Y} \sigma \\
\delta G &= +i \epsilon_+ \left( i \slashed{D} \z - \eta Y - \sigma \z - i \epsilon_+ \left( i \slashed{D} \z + \bar{\eta} Y - \bar{\sigma} \z \right) \\
\delta \bar{G} &= +i \epsilon_- \left( i \slashed{D} \bar{\z} - \bar{Y} \eta - \bar{\z} \sigma - i \epsilon_- \left( i \slashed{D} \bar{\z} + \bar{Y} \bar{\eta} - \bar{\z} \bar{\sigma} \right) \right). 
\end{align*}
\]

These supersymmetry transformations realize the off-shell $SU(2|1)$ algebra (3.3) with the same parameters and with the following $R$-charge assignments:

| G | \bar{Y} | \bar{\z}_- | \bar{\z}_+ | \z_- | \z_+ | Y | G |
|---|---|---|---|---|---|---|---|
| 0 | 0 | -1 | +1 | +1 | -1 | 0 | 0 |

The supersymmetry transformations of a twisted chiral multiplet of $U(1)_A$ charge $\Delta$ can be obtained from (3.7) by the field redefinition [8]

\[
G \rightarrow G + \frac{\Delta}{2r} Y. 
\]

Since correlators do not depend on $\Delta$, we take it to vanish.

The $U(1)_R$ transformation acts chirally on the twisted chiral multiplet fermions $\z$ and $\bar{\z}$. Since the $U(1)$ R-symmetry charge $R$ appears explicitly in the anticommutator of supercharges in $SU(2|1)_A$, anomaly cancellation of $R$ is required to write down an $SU(2|1)_A$ supersymmetric theory of twisted vectors and twisted chirals on the two-sphere. The $R$-current is quantum mechanically conserved whenever the sum of the gauge charges of all charged twisted chiral multiplets vanish for each abelian gauge group factor in $G$. This guarantees that if the flat space gauge theory is also invariant under the R-symmetry $A$, that the gauge theory flows in the infrared to an $\mathcal{N} = (2,2)$ SCFT, and if it has a geometrical phase, to a Calabi-Yau NLSM.

\[\text{These are the same as the } U(1)_A \text{ charges of the components of a chiral superfield with vanishing } U(1)_A \text{ charge.}\]
3.3 Supersymmetric Lagrangian

We now write down the $SU(2|1)_A$-invariant action for a twisted vector multiplet coupled to a charged twisted chiral multiplet. The action has several couplings that are separately supersymmetric

$$S = S_{t.v.m.} + S_{\text{FI}} + S_{\text{top}} + S_{t.c.m.} + S_W + S_{\bar{W}}.$$  (3.9)

The supersymmetrized kinetic terms for the twisted vector multiplets fields are

$$L_{t.v.m.} = \frac{1}{2g_{YM}^2} \text{Tr} \left\{ F^2 + D^\mu \bar{\sigma} D_\mu \sigma + \frac{1}{4} [\sigma, \bar{\sigma}]^2 + D^2 - i \eta \left( \bar{\partial} + \frac{1}{r} \gamma^3 \right) \eta + i \bar{\sigma} (\bar{\eta} \gamma^3 \eta) - i \sigma (\bar{\eta} \gamma^3 \eta) \right\},$$  (3.10)

where $F \equiv \frac{1}{2} \epsilon^\mu\nu F_{\mu\nu}$. The supersymmetric Lagrangian for the charged twisted chiral multiplet fields is

$$L_{t.c.m.} = \bar{Y} \left( - D^2 + iD + \frac{[\sigma, \bar{\sigma}]}{2} \right) Y + \bar{G} G + i \bar{Y} (\bar{\eta}_- - \eta_+) \zeta + i \bar{\zeta} (\bar{\eta}_+ - \eta_-) Y + i \bar{\zeta} (\bar{\partial} - \bar{\sigma}_+ - \sigma_-) \zeta.$$  (3.11)

Twisted chiral multiplets couple via a twisted superpotential\footnote{We use here a convenient normalization.} $W$

$$L_W = \frac{i}{4\pi} \left( W''(Y) \zeta_+ \zeta_- - W'(Y) G + \frac{i}{r} W(Y) \right).$$  (3.12)

Each $U(1)$ factor in the gauge group admits a supersymmetric Fayet-Iliopoulos (FI) and topological term

$$L_{\text{top}} + L_{\text{FI}} = -i \text{Tr} \left( \frac{i}{2\pi} F + \xi D \right).$$  (3.13)

For each abelian factor, the associated field strength multiplet $\Sigma$ is a chiral superfield, and the FI and topological term can be encoded in a linear superpotential $W$

$$W = \frac{i\tau}{2} \Sigma,$$  (3.14)

where

$$L_W = \frac{\partial W}{\partial \Sigma} F_\Sigma - \frac{\partial^2 W}{\partial \Sigma^2} \eta_+ \eta_-.$$  (3.15)

Superpotential couplings are $SU(2|1)_A$ invariant if the superpotential $W$ carries $R$-charge $-2$, which is the charge of $\Sigma$. For twisted vector multiplets on $S^2$, $SU(2|1)_A$-invariance implies that the complexified FI parameter

$$\tau = \frac{i}{2\pi} + i \xi$$  (3.16)

is an exactly marginal coupling.
The action in flat space, obtained by sending $r \to \infty$ in our expressions, has an additional $U(1)_A$ R-symmetry if the charge of the twisted superpotential $W$ is $-2$. On the two-sphere, however, the non-minimal $1/r$ couplings in the action required by supersymmetry break this $U(1)_A$ R-symmetry. This breaking can be understood as arising due to the non-trivial background fields in the supergravity multiplet required to couple the gauge theory to a supersymmetric supergravity background [17].

The parameters of the ultraviolet GLSM are the gauge couplings for each gauge group factor, the complex parameters appearing in the twisted superpotential and the complexified FI parameters appearing in the superpotential. We note that unlike $SU(2|1)_A$-invariant GLSM’s based on vector and chiral multiplets, the twisted chiral multiplets have vanishing twisted masses, since the scalars in the twisted vector multiplet are charged under the $U(1)_R$ symmetry. For a Calabi-Yau GLSM, the complexified FI parameters are the Kähler moduli of the Calabi-Yau while the complex parameters in the twisted superpotential correspond to the complex structure moduli.

4 Localization of the Path Integral

In this section we perform the exact computation of the partition function of the gauge theories constructed in the previous section. This requires choosing a supercharge $Q$ in $SU(2|1)_A$ and a suitable deformation of the Lagrangian

$$\mathcal{L} \to \mathcal{L} + tQV.$$ (4.1)

By the familiar $t$-independence of the path integral (in favorable situations), the path integral reduces to a one-loop integral over the space of saddle points $\mathcal{M}$ of $QV$. The measure of integration is determined by classical action evaluated on the saddle points and by the one-loop determinants $Z_{1\text{-loop}}$ of twisted vector and twisted chiral produced by the deformation term $QV$. The contribution of the gauge fixing multiplet must also be included.

In formulas, for a collection of $Q$-invariant operators collectively denoted by $\mathcal{O}$, we have that

$$\langle \mathcal{O} \rangle = \int_{\mathcal{M}} e^{-S|_{\mathcal{M}}} \mathcal{O}|_{\mathcal{M}} Z_{1\text{-loop}}.$$ (4.2)

In this paper, $\mathcal{O}$ is the two point function of a chiral operator $\mathcal{O}_a$ at the north pole and an anti-chiral operator $\mathcal{O}_{\bar{a}}$ at the south pole of the two-sphere.

4.1 Choice of Supercharge and Decoupling Theorems

We choose the following supercharge\(^{12}\) $Q$ in $SU(2|1)_A$

$$Q = S_1 + Q_2.$$ (4.3)

The $SU(1|1)$ subalgebra that $Q$ generates is

$$Q^2 = J_3 + \frac{R}{2} \quad \left[ J_3 + \frac{R}{2}, Q \right] = 0,$$ (4.4)

\(^{12}\)We drop the index $A$, to avoid cluttering.
where $J_3$ is a $U(1)$ isometry generator of $S^2$, and has two antipodal fixed points which we call the north and south poles of the two-sphere. $R$ is the $U(1)$ R-symmetry generator in $SU(2|1)_A$.

The (Grassmann even) Killing spinors (3.2) parameterizing the transformations generated by $Q$ are

$$
\epsilon = \exp \left( -\frac{i}{2} \theta \gamma^3 + \frac{i}{2} \varphi \right) \epsilon_o, \quad \gamma^3 \epsilon_o = \epsilon_o \tag{4.5}
$$

$$
\bar{\epsilon} = \exp \left( +\frac{i}{2} \theta \gamma^3 - \frac{i}{2} \varphi \right) \bar{\epsilon}_o, \quad \gamma^1 \bar{\epsilon}_o = \epsilon_o,
$$

where $(\theta, \varphi)$ are the canonical coordinates on $S^2$.

At the north pole of the two-sphere, gauge invariant operators $O_a(Y)$ constructed from the lowest component of twisted chiral multiplets are $Q$-invariant. Likewise, at the south pole, operators constructed from the lowest component of twisted anti-chiral multiplets $\bar{O}_a(\bar{Y})$ are also $Q$-invariant. This follows from the supersymmetry transformation (3.7) generated by the spinors (4.5). Therefore the two-point function

$$
\langle O_a(Y) \bar{O}_b(\bar{Y}) \rangle \tag{4.6}
$$

is $Q$-invariant and can be computed by supersymmetric localization.

We now prove that the two-sphere partition function and two-point functions (4.6) are independent of some of the parameters of the Lagrangian. First, we note that the twisted vector multiplet Lagrangian (3.10) as well as the FI and topological terms (3.13) are all $Q$-exact. Explicitly

$$
L_{t.v.m.} = \frac{1}{4g^2_{YM}} Q \bar{Q} \text{Tr} \left( \eta \gamma^3 \bar{\eta} + \frac{i}{r} \sigma \bar{\sigma} \right) - \nabla_{\mu}J_{\mu t.v.m.} \tag{4.7}
$$

where $\bar{Q} = S_1 - Q_2$, a supercharge in $SU(2|1)_A$ parametrized by Killing spinors (4.5) $-\epsilon$ and $\bar{\epsilon}$.\[\text{13}\]

Likewise

$$
L_{FI} = \frac{\xi}{2i} Q \text{Tr} \left( \bar{\epsilon} \gamma^3 \bar{\eta} + \epsilon \gamma^3 \eta \right) - \nabla_{\mu}J_{\mu FI} \tag{4.8}
$$

$$
L_{\text{top}} = \frac{\vartheta}{4\pi} Q \text{Tr} \left( \bar{\epsilon} \gamma^3 \bar{\eta} - \epsilon \gamma^3 \eta \right) - \nabla_{\mu}J_{\mu \text{top}}.
$$

By virtue of equation (3.14), this follows from the more general result that the superpotential $\mathcal{W}$ couplings (3.15) are $Q$-exact $[5, 6]$. The twisted chiral Lagrangian (3.11) is also $Q$-exact

$$
L_{t.c.m.} = \frac{1}{2} Q \bar{Q} \left( GY - \bar{Y} \bar{G} + \frac{i}{r} \bar{Y}Y \right) - \nabla_{\mu}J_{\mu t.c.m.} \tag{4.9}
$$

We note, however, that twisted superpotential couplings (3.12) are not $Q$-exact.

This shows that the gauge theory two-sphere partition function and two-point functions (4.6) are independent of the gauge couplings $g^2_{YM}$ and of the complexified FI parameters $\tau$, but depend on the complex parameters in the twisted superpotential. Gauge coupling independence implies that the two-sphere partition function of a gauge theory is a renormalization group invariant observable. In particular, it coincides with the partition function of a SCFT theory in the extreme infrared, where

\[\text{13}\] The total derivative terms are written down in appendix C.
This is none other than the sought-after Calabi-Yau NLSM when the gauge theory has a geometric phase. Moreover, the Zamolodchikov metric (1.4) of operators in the chiral ring of the \( \mathcal{N} = (2, 2) \) SCFT can be exactly computed in the ultraviolet GLSM, as these correlators have images in the ultraviolet GLSM through (4.6). In conclusion, a gauge theory on the two-sphere computes the Kähler potential and associated Zamolodchikov metric of the infrared SCFT. When the GLSM has a geometric phase, the gauge theory computes these quantities for the complex structure moduli space of the Calabi-Yau.

### 4.2 Q-Exact Deformation Term

We proceed by deforming the gauge theory action by a \( \mathcal{Q} \)-exact term

\[
\mathcal{L} \to \mathcal{L} + t \mathcal{Q} \mathcal{V} .
\]

Following our discussion in the previous subsection, we can take

\[
\mathcal{V} = \frac{1}{4g^2} \bar{Q} \text{Tr} \left( \bar{\eta} \gamma^3 \eta + \frac{i}{r} \sigma \bar{\sigma} \right) - \frac{i}{2} \chi \text{Tr}(\bar{\epsilon} \gamma^3 \eta + \epsilon \gamma^3 \bar{\eta}) + \frac{1}{2} \bar{Q} \left( \bar{G} Y - \bar{Y} G + \frac{i}{r} \bar{Y} Y \right) .
\]

The bosonic part of \( \mathcal{Q} \mathcal{V} \) can be recast into the positive definite form

\[
\frac{1}{2g^2} \text{Tr} \left( |D_\mu \sigma|^2 + \frac{1}{4} [\sigma, \bar{\sigma}]^2 + F^2 + \bar{D}^2 \right) + |D_\mu Y|^2 + |G|^2 + \frac{1}{2} (|\sigma Y|^2 + |\bar{\sigma} Y|^2) + \frac{g^2}{2} (Y \bar{Y} - \chi)^2 ,
\]

where \( \bar{D} = D + ig^2 (Y \bar{Y} - \chi) \). Positive definiteness follows from the reality conditions

\[
\sigma = \bar{\sigma} \quad \bar{D} = \bar{\bar{D}} \quad F = \bar{F} \quad Y = \bar{Y} \quad G = \bar{G} \quad A^\dagger = A_\mu .
\]

By adding this deformation term to the action and taking the limit \( t \to \infty \), we are able to apply the saddle point method, which is exact, and localize the path integral to the extrema of \( \mathcal{Q} \mathcal{V} \). Since the bosonic part of the deformation term is positive definite, all the paths that contribute to the path integral lie at the global minimum surface \( \mathcal{Q} \mathcal{V} = 0 \) in the space of fields. The space of saddle points that we must integrate over in the path integral is therefore

\[
\mathcal{M} = \left\{ Y | Y = Y_0 , \ Y_0 \bar{Y}_0 - \chi = 0 \right\} / G_{\text{global}} ,
\]

with all the other fields vanishing. \( Y = Y_0 \) is constant on the two-sphere. Field configurations related by the residual gauge transformation \( G_{\text{global}} \) (the global part of the gauge group \( G \)) must be identified

\[
Y_0 \simeq e^{i \alpha} Y_0 ,
\]

\[\text{14}\]In our choice of coordinates, where the infrared NLSM is described by twisted chiral multiplets, a chiral ring element in the infrared SCFT is the lowest component of a twisted chiral superfield while an operator in the conjugate ring is the lowest component of a twisted anti-chiral superfield.

\[\text{15}\]For \( \chi = 0 \), \( \sigma \) can be non-zero, but then at least one \( Y \) must vanish. The fermionic superpartner of this field, however, has a fermionic zero mode, and this saddle point does not contribute.

\[\text{16}\]Even though the parameter \( \chi \) enters in the definition of \( \mathcal{M} \), we shall prove that the partition function is independent of \( \chi \), as it should, since it is the coefficient of a \( \mathcal{Q} \)-exact term in (4.11).
where $\alpha$ acts on $Y$ in the corresponding representation $R$ of the gauge group. $\mathcal{M}$ is therefore the Kähler quotient space

$$
\mathcal{M} = \mathbb{C}^{[R]} // G_{\text{global}}.
$$

(4.16)

Localization has to be performed for the gauge fixed functional integral (see appendix B for details). For the background field configurations (4.14), we fix the Lorenz gauge which is compatible with $A_\mu = 0$. For the field fluctuations in the computation of the one-loop determinant however, it is much more convenient to fix an $R_c$-like gauge adapted to the Higgs phase of the theory. This requires introducing gauge fixing terms and a fermionic generator $Q_{\text{BRST}}$. We localize the path integral with respect to the BRST deformed supercharge $\hat{Q} = (Q + Q_{\text{BRST}})$ using as the deformation term $\hat{Q}\mathcal{V}'$, where $\mathcal{V}' = \mathcal{V} + \mathcal{V}_{G.F}$. The space of saddle points of the gauge fixed theory remains unaffected by the inclusion of the gauge fixing terms, however, the gauge fixing terms play an important role in the computation of the measure factor $Z_{\text{1-loop}}$.

### 4.3 Partition Function and Zamolodchikov Metric

Calculation of the measure of integration in the space of saddle points $\mathcal{M}$ requires computing the one-loop determinant $Z_{\text{1-loop}}$ of twisted vector, twisted chiral and ghost multiplets around the saddle point configurations $\mathcal{M}$. This is achieved by integrating out to quadratic order in the fluctuations the deformation and gauge fixing terms $\hat{Q}\mathcal{V}'$.

Consider a gauge theory with gauge group $G = U(1)^{N_c}$ coupled to $N_f$ twisted chiral multiplets with charges $Q^a_I$ under $U(1)^{N_c}$, where $a = 1, \ldots, N_c$ and $I = 1, \ldots, N_f$. Supersymmetry on the two-sphere requires anomaly cancelation for the $U(1)_R$ R-symmetry, which yields the constraints

$$
\sum_i Q^a_i = 0 \quad a = 1, \ldots, N_c.
$$

(4.17)

The one-loop determinant around the saddle points (4.14) is given by the determinant of an $N_c \times N_c$ matrix (see appendix D for details)

$$
Z_{\text{1-loop}} = \det(M^\dagger M).
$$

(4.18)

Here $M$ is an the $N_f \times N_c$ mass matrix and $M^\dagger$ is its hermitian conjugate. They are given by

$$
M^a_I = Q^a_i Y_I, \quad M^\dagger_a = Q^a_i \bar{Y}_I.
$$

(4.19)

We note that $N_f \geq N_c$ is a necessary condition for the matrix $M^\dagger M$ to be non-degenerate. For $N_f < N_c$, there is a linear combination of the $U(1)$ generators under which all the twisted chiral fields are neutral, and the associated gaugino has a fermionic zero mode and therefore the path integral vanishes.

---

17 We drop the subindex of $Y_o$ in order to avoid cluttering.

18 Given by $\lambda = \bar{\epsilon}$, where $\bar{\epsilon}$ is the conformal Killing spinor (4.5).
Evaluating the classical action and operator insertions on the saddle points we obtain\(^{19}\)

\[
\langle O_a(N) O_b(S) \rangle = \int \text{vol}_M O_a(Y) O_b(\bar{Y}) Z_{1\text{-loop}} e^{rW(Y) - r\bar{W}(\bar{Y})},
\]

(4.20)

where \(\text{vol}_M\) is the volume form on the space of saddle points \(M\) (4.14). The volume form on \(M\), which is the quotient space (4.16), can be written in terms of the volume form of the ambient flat space \(\mathbb{C}^N_f\) by inserting appropriately normalized Dirac delta distributions and dividing by the volume of the \(U(1)^{N_c}\) gauge orbits:

\[
\text{vol}_M = \frac{\text{d}^N_f Y \wedge \text{d}^N_f \bar{Y}}{\text{vol}(G_{\text{global}})} \det \left[ \frac{\partial F_a}{\partial Y_b} \right] \prod_a \delta(F_a),
\]

(4.21)

where

\[
F_a = \sum_I Q^a_I |Y^I|^2 - \chi_a.
\]

(4.22)

On the ambient space \(\mathbb{C}^N_f\), we can define the Hamiltonian action of the complexification \(U(1)^{N_c}\) of the gauge group. The vector fields that generate the real gauge transformations are

\[
\rho_a = i \sum_I Q^a_I (Y_I \partial_I - \bar{Y}_I \bar{\partial}_I) \quad a = 1, \ldots, N_c,
\]

(4.23)

while

\[
v_a = - \sum_I Q^a_I (Y_I \partial_I + \bar{Y}_I \bar{\partial}_I) \quad a = 1, \ldots, N_c,
\]

(4.24)

generate imaginary gauge transformations. They act, respectively, as

\[
Y_I \rightarrow e^{i \sum \tau_1 Q^a_I \tau_1 Y^I}, \quad Y_I \rightarrow e^{- \sum \tau_2 Q^a_I \tau_2 Y^I} \quad \tau_1, \tau_2 \in \mathbb{R}.
\]

(4.25)

The moment map associated with the imaginary transformation generated by the \(a\)-th \(U(1)\) factor in the gauge group is given by

\[
\mu_a = - \frac{1}{2} \sum_I Q^a_I |Y^I|^2 \quad a = 1, \ldots, N_c,
\]

(4.26)

as it obeys

\[
d\mu_a = v_a \omega,
\]

(4.27)

where \(\omega\) is the Kähler form in \(\mathbb{C}^N_f\). Therefore, the D-term equations entering in the definition of \(M\) in (4.14)

\[
\left\{ \sum_I Q^a_I |Y^I|^2 = \chi \leftrightarrow F_a = 0 ; a = 1, \ldots, N_c \right\},
\]

(4.28)

\(^{19}\)As explained in [5], the partition function is also proportional to \(r^{c/3}\), due to the usual conformal anomaly, where \(c\) is the central charge.
can be interpreted as the moments maps for the imaginary gauge transformations

\[ 2\mu_a + \chi = 0 \quad a = 1, \ldots, N_c. \tag{4.29} \]

We note that these moment maps obey the equations

\[ d\mu_a \cdot d\mu_b = (M^\dagger M)_{ab}, \tag{4.30} \]

where \( d \) denotes the exterior derivative and the inner product \( d\mu_a \cdot d\mu_b \) is the \( \mathbb{C}^{N_f} \) inner product. As a direct consequence of the anomaly cancellation conditions \((4.17)\), the holomorphic and anti-holomorphic factors in the measure

\[ d^{N_f}Y \wedge d^{N_f}\bar{Y} \tag{4.31} \]

are each invariant under the complexified gauge transformations \( U(1)_{C}^{N_c} \). Furthermore, the twisted superpotential \( W(Y) \) and \( \overline{W}(\bar{Y}) \) are also invariant under complex gauge transformations, whereas \( Z_{1\text{-}loop} \) is only invariant under real gauge transformations. This observation suggests a change of coordinates \( \{ Y \} \rightarrow \{ X, \tau \} \), to some gauge invariant coordinates \( X \) and the (complex) gauge orbit coordinates \( \tau \), where the integration over the complex gauge orbits is localized to the real gauge orbits due to the \( \delta \)-distributions arising from the D-term equations.

In computing the volume form \( \text{vol}_M \) we must quotient by the volume of the orbit of \( U(1)^{N_c} \) real gauge transformations. It follows from \((4.23)\) that it is given by

\[ \text{vol}(G_{\text{global}}) = (2\pi)^{N_c} \det (\rho_a \cdot \rho_b)^{1/2}. \tag{4.32} \]

By virtue of \((4.23)\) we have that

\[ \rho_a \cdot \rho_b = 4 (M^\dagger M)_{ab}, \tag{4.33} \]

which combined with \((4.30)\) implies\(^{20}\) that the Jacobian appearing with the delta functions in \((4.21)\) precisely cancels with the volume of the gauge orbit.

Altogether, the correlator \((4.20)\) can be written as

\[ \langle O_a(N)O_b(S) \rangle = \int d^{N_f}Y \wedge d^{N_f}\bar{Y} \frac{(2\pi)^{N_c}}{\det (dF_a \cdot dF_b)} \rho_a(Y) \rho_b(\bar{Y}) \text{det} (M^\dagger M) \prod_a \delta (2\mu_a + \chi_a) \ e^{rW(Y)-r\bar{W}(\bar{Y})}, \tag{4.34} \]

with \( M \) and \( M^\dagger \) defined in \((4.19)\) and \( \mu_a \) in \((4.26)\). The partition function is obtained by placing the identity operator at the north and south poles of the two-sphere, yielding

\[ Z_B = \int d^{N_f}Y \wedge d^{N_f}\bar{Y} \frac{(2\pi)^{N_c}}{\det (dF_a \cdot dF_b)} \text{det} (M^\dagger M) \prod_a \delta (2\mu_a + \chi_a) \ e^{rW(Y)-r\bar{W}(\bar{Y})}. \tag{4.35} \]

\(^{20}\)The Jacobian factor \( J_{(b)} = \text{det} (\partial F_a / \partial Y_b) \) in \((4.21)\) assumes that one carries out the integration over the \( Y_{(b)} \) planes first, treating \( Y_I \) as constant for \( I \neq b \). More covariantly, one may write \( J = \sqrt{\text{det} (dF_a \cdot dF_b)} \) which takes the order of integration into account.
5 Calabi-Yau Geometries

The two-sphere partition function (4.35) of a Calabi-Yau GLSM is expected to compute the Kähler potential $K_C$ for the complex structure moduli of the corresponding Calabi-Yau manifold. Concretely, we expect

$$Z_B = e^{-K_C} = i^{\dim M} \int_M \Omega \wedge \overline{\Omega}, \quad (5.1)$$

where $\Omega$ is the nowhere vanishing holomorphic top form of the corresponding Calabi-Yau. We now turn to explicitly demonstrating this for various families of Calabi-Yau geometries.

5.1 Quintic Hypersurfaces in $\mathbb{C}P^4_{[Q_1, \ldots, Q_5]}$

Consider the partition function (4.35) in the case of a $U(1)$ gauge theory coupled to five twisted chiral multiplets $Y_I$ with charges $Q_I$ and a twisted chiral multiplet $P$ with charge $-q$. The anomaly cancellation condition requires the sum of the charges of all of the twisted chiral multiplets vanish, i.e.

$$q = \sum_I Q_I. \quad (5.2)$$

The twisted superpotential for GLSMs corresponding quintic hypersurfaces in $\mathbb{C}P^4_{[Q_1, \ldots, Q_5]}$ has the general form

$$W = P G_5(Y), \quad (5.3)$$

where $G_5(Y)$ is a transverse polynomial satisfying

$$G_5(\lambda^{Q_I} Y_I) = \lambda^q G_5(Y) \quad \lambda \in \mathbb{C}^*. \quad (5.4)$$

The two-sphere partition function takes the form21

$$Z = \frac{1}{2\pi} \int d^5Y \wedge d^5\overline{Y} \wedge dP \wedge d\overline{P} \ M^\dagger M \ \delta(2\mu + \chi) e^{W - \overline{W}}, \quad (5.5)$$

where the moment map and the mass matrix are given by

$$-2\mu = \sum_I Q_I |Y_I|^2 - q|P|^2,$$

$$M^\dagger M = \sum_I Q_I^2 |Y_I|^2 + q^2 |P|^2. \quad (5.6)$$

We remark the the anomaly cancellation condition (5.2) guarantees that the flat measure and the twisted superpotential factor in (5.5) are invariant under global complex gauge transformation. It is therefore natural to consider the change of variables

$$Y_I = e^{iQ_I \tau} x_I, \quad (5.7)$$

$$P = e^{-iq \tau} p,$$
with $x_5 = \text{constant}$. In these coordinates, complex gauge transformations act only as a shift of the $\tau$ coordinate and therefore $\tau$ is the (complex) gauge orbit coordinate. The invariance of the ambient space volume form and the twisted superpotential under complex gauge transformations generated by $\partial \tau$ becomes manifest in the new coordinates. The volume form of $\mathbb{C}^6$ in the new coordinates is

$$d^5Y \wedge d^5\bar{Y} \wedge dP \wedge d\bar{P} = Q_5^2 |x_5|^2 \, d^4x \wedge d^4\bar{x} \wedge dp \wedge d\bar{p} \wedge d\tau \wedge d\bar{\tau}. \tag{5.8}$$

while the twisted superpotential retains its original form

$$W = PG_5(Y) = pG_5(x). \tag{5.9}$$

The moment map and the mass matrix (5.6), however, depend explicitly on the imaginary $\tau$ direction, denoted by $\tau_2$, as they are only invariant under real gauge transformations, and may be rewritten as

$$-2\mu = \sum_{I=1}^{5} Q_I e^{-2Q_I \tau_2 |x_I|^2} - q e^{2q \tau_2 |p|^2}, \tag{5.10}$$

$$M^\dagger M = \sum_{I=1}^{5} Q_I^2 e^{-2Q_I \tau_2 |x_I|^2} + q^2 e^{2q \tau_2 |p|^2}.$$  

The partition function (5.5) in the new coordinates is

$$Z = -2iQ_5^2 |x_5|^2 \int d^4x \wedge d^4\bar{x} \wedge dp \wedge d\bar{p} e^{pG_5(x) - \bar{p}\bar{G}_5(x)} \int d\tau_2 \, M^\dagger M \delta (2\mu + \chi) \right) , \tag{5.11}$$

where we have carried out the integration over $\tau_1$ which only contributes a factor of $2\pi$. It is clear from (5.10) that $M^\dagger M$ is the Jacobian $\partial_{\tau_2} \mu$, that is

$$d\tau_2 M^\dagger M = d\tau_2 \frac{\partial \mu}{\partial \tau_2} \approx d\mu \tag{5.12}$$

keeping $x_I$ and $p$ constant. This implies that the integration over $\tau_2$ can be readily carried out yielding

$$\int d\tau_2 \, M^\dagger M \delta (2\mu + \chi) \approx \int d\mu \delta (2\mu + \chi) = 1/2. \tag{5.13}$$

The partition function (5.11) can be put into the proposed form in terms of an integral over the holomorphic three-form by performing the integration over the complex variables $p$ and $\bar{p}$ as well as the integration over one of the $x$ planes, say $x_4$. Integrating over $p$ imposes the embedding equation $G_5 = 0$ in $\mathbb{CP}^4_{[Q_1, \ldots, Q_5]}$ via $\delta$ distributions

$$\int dp \wedge d\bar{p} \, e^{pG_5 - \bar{p}\bar{G}_5} = -\frac{1}{4\pi^2} \delta(G_5)\delta(\bar{G}_5) \tag{5.14}$$

\footnote{In general, the equations $2\mu + \chi = 0$ have multiple solutions for $\tau_2$; this only introduces a multiplicative factor which we ignore.}
and finally, integrating over $x_4$ and $\bar{x}_4$ yields

$$Z = i \frac{Q_5^2 |x_5|^2}{4\pi^2} \sum_{\{x_4|G_5=0\}} \sum_{\{\bar{x}_4|\bar{G}_5=0\}} \int \frac{dx_1 \wedge dx_2 \wedge dx_3}{\partial_4 G_5(x)} \wedge \frac{d\bar{x}_1 \wedge d\bar{x}_2 \wedge d\bar{x}_3}{\partial_4 \bar{G}_5(\bar{x})}.$$  \hfill (5.15)

From (5.15) we can read off the holomorphic three-form to be

$$\Omega = \frac{Q_5}{2\pi} \frac{x_5}{\partial_4 G_5(x)} dx_1 \wedge dx_2 \wedge dx_3,$$  \hfill (5.16)

which matches the well known formulae for the holomorphic three-form presented in [18,19] of quintic hypersurfaces in $\mathbb{CP}^4_{[Q_1,...,Q_5]}$. We remark that although (5.16) appears to have singularities whenever $\partial_4 G_5 = 0$, via a simple change of coordinates, corresponding to integrating (5.14) with respect to $x_1$ instead of $x_4$, it may be written as

$$\Omega = -\frac{Q_5}{2\pi} \frac{x_5}{\partial_1 G_5(x)} dx_2 \wedge dx_3 \wedge dx_4.$$  \hfill (5.17)

Since the polynomial $G_5(x)$ is transversal and $x_5 \neq 0$, it follows that the holomorphic three-form $\Omega$ is non-singular and nowhere vanishing.

**Mirror Quintic Complex Structure Kähler Potential**

In [8] the $SU(2|1)_A$-invariant partition function for the familiar quintic three-fold in $\mathbb{CP}^4$ was shown to coincide with the $SU(2|1)_A$-invariant partition function of the Hori and Vafa mirror theory [20]. This is a $U(1)$ vector multiplet coupled to twisted chiral multiplets $(Y_1, \ldots, Y_5, Y_P)$ with a twisted superpotential

$$W = \left[ i \Sigma \left( \sum_{a=1}^{5} Y^a - 5Y_P + 2\pi i \tau \right) - \left( \sum_{a=1}^{5} e^{-Y^a} + e^{-Y^P} \right) \right],$$  \hfill (5.18)

where $\Sigma$ is the field strength multiplet. As shown in [8], the relation to the Mellin-Barnes like formula for $SU(2|1)_A$ invariant gauge theories derived in [5, 6] follows by integrating out the twisted chiral multiplet fields. Explicitly, decomposing the integral into contours

$$\int_{Y^*=-\bar{Y}} dY d\bar{Y} e^{-Y+iQ\Sigma} e^{\bar{Y}+iQ\bar{\Sigma}} = \int_{C} dt e^{-t} t^{-iQ\Sigma-1} \int_{C} dt e^{\bar{t}+iQ\bar{\Sigma}-1},$$  \hfill (5.19)

and with the identities

$$\int_{0}^{\infty} dt e^{-t} t^{-iQ\Sigma-1} = \frac{2\pi i}{\Gamma(1 + iQ\Sigma)}, \quad \int_{C} dt e^{\bar{t}+iQ\bar{\Sigma}-1} = \frac{2\pi i}{\Gamma(1 + iQ\Sigma)},$$  \hfill (5.20)

one arrives at the gauge theory result [8].

$^{23}$C is the Hankel contour, which starts at $-\infty - i\epsilon$, then goes around the branch cut along the negative real $t$ axis, and ends up at $\infty + i\epsilon$.

$^{24}$This is a streamlined version of the identity derived in [8].
The two-sphere partition function of the mirror theory can be reduced to an orbifold Landau-Ginzburg model by integrating out $\Sigma$. This yields [8]

$$Z_{L.G.} = \int \prod_{a=1}^{5} d\tilde{X}_a d\overline{\tilde{X}}_a e^{-W_{\text{eff}}+\overline{W}_{\text{eff}}},$$

(5.21)

where the effective twisted superpotential is

$$W_{\text{eff}} = \sum_{a} \tilde{X}_a^5 + e^{-2\pi i r/5} \prod_{a} \tilde{X}_a.$$  

(5.22)

The canonical variables $\tilde{X}_a$ are given by

$$\tilde{X}_a = e^{-\frac{1}{5} Y_a},$$

(5.23)

and therefore we must orbifold by

$$\tilde{X}_a \simeq e^{2\pi i/5} \tilde{X}_a.$$  

(5.24)

This orbifold Landau-Ginzburg model realizes the mirror Calabi-Yau geometry: the mirror quintic $W$. Indeed, it is easy to show that the orbifold Landau-Ginzburg model partition function also computes the Kähler potential on the complex structure moduli space of the mirror quintic $W$

$$Z_{L.G.} = i \int_{W} \Omega \wedge \overline{\Omega}.$$  

(5.25)

### 5.2 Complete Intersection Surfaces in $\mathbb{CP}^{n}_{[Q_1,\ldots,Q_{n+1}]}$

The analysis of the last section can be easily generalized to intersection of multiple hypersurfaces in $\mathbb{CP}^{n}_{[Q_1,\ldots,Q_{n+1}]}$. As the analysis is quite parallel to that of the last section, some details are omitted here.

Consider the partition function (4.35) in the case of a $U(1)$ gauge theory, this time coupled to $n+1$ twisted chiral multiplets $Y_I$ with charges $Q_I$ and $m$ twisted chiral multiplet $P_\alpha$ with charges $-q_\alpha$. Imposing the anomaly cancellation condition restricts the charges to satisfy

$$\sum_{\alpha} q_\alpha = \sum_{I} Q_I.$$  

(5.26)

The partition function takes the form

$$Z = \frac{i^{n+m+1}}{2\pi} \int d^{n+1}Y \wedge d^{n+1}\overline{Y} \wedge d^m P \wedge d^m \overline{P} M^\dagger M \delta (2\mu + \chi) e^{W-W}$$

(5.27)

where the twisted superpotential is linear in $P_\alpha$ and is a polynomial in $Y_I$,

$$W = \sum_{\alpha} P_\alpha G_\alpha(Y),$$

(5.28)
with the polynomials $G_\alpha$ satisfying

$$G_\alpha(\lambda^Q y_I) = \lambda^{q_\alpha} G_\alpha(y) .$$

(5.29)

We emphasize again that both the twisted superpotential term and the volume form for the ambient space $\mathbb{C}^{n+1+m}$ are invariant under complex gauge transformations. The change of variables

$$Y_I = e^{iQ_I x_I} ,
\quad P_\alpha = e^{-iq_\alpha} p_\alpha ,$$

(5.30)

with $x_{n+1} = \text{constant}$, makes this invariance manifest as the gauge transformations in the new coordinates act simply as a shift in $\tau$. The twisted superpotential in the new coordinates assumes the $\tau$-independent form

$$W = \sum_\alpha p_\alpha G_\alpha(x)$$

(5.31)

and the volume form is

$$2i^{n+m} Q^2_{n+1} |x_{n+1}|^2 d^n x \wedge d^n \bar{x} \wedge d^m p \wedge d^m \bar{p} \wedge d\tau_1 \wedge d\tau_2 .$$

(5.32)

Here $\tau_1$ and $\tau_2$ are the real and imaginary parts of the $\tau$ coordinate parameterizing the compact and non-compact directions of the gauge orbit surface. The moment map, which has an explicit $\tau_2$ dependence takes the form

$$-2\mu = \sum I e^{-2Q_I \tau_2} Q_I |x_I|^2 - \sum_\alpha q_\alpha e^{2q_\alpha \tau_2} p_\alpha ,$$

(5.33)

while $M^\dagger M$ can be related to the moment map, as in the case of quintic hypersurfaces, by $\tau_2$ differentiation of the latter,

$$M^\dagger M = \frac{\partial \mu}{\partial \tau_2} .$$

(5.34)

The integration over $\tau_1$ and $\tau_2$ may then be carried out as was done for quintic hypersurfaces (5.11), yielding

$$Z = \frac{i^{n+m} Q^2_{n+1} |x_{n+1}|^2}{(2\pi)^{2m}} \int d^n x \wedge d^n \bar{x} \wedge d^m p \wedge d^m \bar{p} \left( \sum_\alpha (p_\alpha G_\alpha - \bar{p}_\alpha \bar{G}_\alpha) \delta(2\mu + \chi) \right)$$

(5.35)

This is a simple generalization of the case of a hypersurface defined by a single embedding equation studied in the last section, with multiple $p$ fields, one for each constraint. Integration over the $p$ planes then imposes all the constraints leading to

$$Z = \frac{i^{n-m} Q^2_{n+1} |x_{n+1}|^2}{(2\pi)^{2m}} \int d^n x \wedge d^n \bar{x} \prod_\alpha \delta(G_\alpha) \delta(\bar{G}_\alpha) .$$

(5.36)
Carrying out the integration over the \( m \) dimensional space \( \{ x_I | I = n - m + 1, \ldots, n \} \) we arrive at the desired expression

\[
Z = i^{n-m} \frac{Q_{n+1}^2 |x_{n+1}|^2}{(2\pi)^{2m}} \sum_{\{ x_{n-m+\beta} \}} \frac{\int d x_1 \wedge \cdots \wedge d x_{n-m} \wedge \frac{d x_1 \wedge \cdots \wedge d x_{n-m}}{\det (\partial_{n-m+\beta} G_\alpha(x))}}{\det (\partial_{n-m+\beta} G_\alpha(\bar{x}))}, \tag{5.37}
\]

where each determinant in the denominator is computed over the \( \alpha \) and \( \beta \) indices. This yields the holomorphic \( n - m \) form

\[
\Omega = \frac{Q_{n+1}}{(2\pi)^m} \frac{x_{n+1} d x_1 \wedge \cdots \wedge d x_{n-m}}{\det (\partial_{n-m+\beta} G_\alpha(x))} \tag{5.38}
\]

for the intersection of \( m \) hypersurfaces in \( \mathbb{C}P^{Q_{n+1}}_{[Q_1, \ldots, Q_{n+1}]} \) [18, 19]. That \( \Omega \) appears to be singular whenever \( \det (\partial_{n-m+\beta} G_\alpha(x)) = 0 \) is an artifact of the choice of coordinates. For these points on the manifold, there is a different choice \( \{ x_{a(\alpha)}, \alpha = 1, \ldots, m \} \) of coordinates to integrate the \( \delta \)-distributions in (5.36), such that (5.38) is non-singular.

### 5.3 Complete Intersection of Hypersurfaces in Product of Weighted Projective Spaces

As a much more general class of complete intersections with abelian GLSM realization, we now consider the partition function (4.35) in the case of \( U(1)^{N_c} \) gauge theory with \( N_f = n + m + N_c \) twisted chiral multiplets \( Y_I \) with charge matrix \( \{ Q_I^a | a = 1, \ldots, N_c; I = 1, \ldots, N_f \} \). The anomaly cancellation conditions restrict the charge matrix to obey

\[
\sum_I Q_I^a = 0 \quad \text{for all } a. \tag{5.39}
\]

The partition function has the general form (4.35) where the superpotential is a polynomial in \( \{ X_I = Y_I | I = 1, \ldots, N_f - m \} \) and is linear in \( \{ P_\alpha = Y_\alpha | \alpha = N_f - m + 1, \ldots, N_f \} \),

\[
W = \sum_\alpha P_\alpha G_\alpha(X). \tag{5.40}
\]

The polynomials \( G_\alpha \) satisfy

\[
G_\alpha (\lambda^{Q_I^a} X_I) = \lambda^{-Q_I^a} G_\alpha(X), \tag{5.41}
\]

which guarantees the invariance of the twisted superpotential under \( U(1)^{N_c} \) gauge transformations. As before, we introduce the complex \( \tau^a \) coordinates, one for each \( U(1) \) factor in the gauge group, via

\[
X_I = e^{i \sum_a Q_I^a \tau^a} x_I, \quad P_\alpha = e^{i \sum_a Q_\alpha^a \tau^a} p_\alpha, \tag{5.42}
\]

and with \( x_{n+1} = \cdots = x_{n+N_c} = 1 \). This isolates the action of each \( U(1)_a \) factor in the gauge group to a shift in \( \tau_a \) and highlights the gauge invariance of the twisted superpotential

\[
W = \sum_\alpha p_\alpha G_\alpha(x). \tag{5.43}
\]

\[25\]This amounts to choosing inhomogeneous coordinates on the Calabi-Yau.
To write the volume form of $\mathbb{C}^{N_f}$ in the new coordinates, first consider the volume form of the subspace $\mathbb{C}^{N_c}$ of constant $x_I$. The holomorphic part of this volume form may be written as

$$dX_{n+1} \wedge \cdots \wedge dX_{n+N_c} = i^{N_c} \sum_{\alpha_1, \ldots, \alpha_{N_c}} Q^\alpha_{n+1} \cdots Q^{\alpha_{N_c}}_{n+N_c} \tau^{\alpha_1} \wedge \cdots \wedge \tau^{\alpha_{N_c}} = \det (iQ^a_{n+b}) d^{N_c} \tau,$$

(5.44)

where the determinant is over the $a$ and $b$ indices. The partition function may then be written as

$$Z = i^{N_f} \frac{\det(Q^a_{n+b})^2}{(2\pi)^{N_c}} \int d^n x \wedge d^n \bar{x} \wedge d^m p \wedge d^m \bar{p} \ e^{W - \bar{W}} \int d^{N_c} \tau \wedge d^{N_c} \bar{\tau} \det(M\dagger M) \prod_a \delta(2\mu_a + \chi_a),$$

(5.45)

where the moment map depends on the imaginary part of $\tau^a$ according to

$$-2\mu_a = \sum_{I=1}^n e^{-2\sum_b Q_I^a Q_I^b |x_I|^2} + \sum_{I=n+1}^{n+N_c} Q_I^a e^{-2\sum_b Q_I^a Q_I^b} + \sum_{\alpha=N_f-m+1} N_f Q^a e^{-2\sum_b Q^a Q^b} |p_\alpha|^2,$$

(5.46)

and the mass matrix $M\dagger M$ can be expressed in terms of the moment maps via

$$(M\dagger M)_{ab} = \frac{\partial \mu_a}{\partial \tau^b_2}.$$  

(5.47)

This last relation implies that $\det(M\dagger M)$ is precisely the inverse of the Jacobian factor produced by the coordinate transformation $\{\tau_2^a\} \rightarrow \{\mu_a\}$. Consequently, the integration over the space of complex gauge orbits can be carried out leading to the numerical factor

$$\int d^{N_c} \tau \wedge d^{N_c} \bar{\tau} \det(M\dagger M) \prod_a \delta(2\mu_a + \chi_a) = (-2i\pi)^{N_c}.\ 

(5.48)$$

With the space of gauge orbits integrated out, the partition function (5.45) assumes the simple form

$$Z = i^{n+m} \det(Q^a_{n+b})^2 \int d^n x \wedge d^n \bar{x} \wedge d^m p \wedge d^m \bar{p} \ e^{\sum_a (p_a G_a - \bar{p}_a \bar{G}_a)}.$$  

(5.49)

As in the last two examples, the $p$ integrals impose the embedding equation constraints $G_a = 0$ which can be used to solve for $m$ of the coordinates $x_I$. This leads to the partition function

$$Z = i^{n-m} \frac{\det(Q^a_{n+b})^2}{(2\pi)^{2m}} \sum_{\{x_{n+m+\beta} | G_a = 0\}} \sum_{\{x_{n+m+\beta} | G_a = 0\}} \int dx_1 \wedge \cdots \wedge dx_{n-m} \det (\partial_{n-m+\beta} G_a (x)) \wedge \frac{dx_1 \wedge \cdots \wedge dx_{n-m}}{\det (\partial_{n-m+\beta} G_a (\bar{x}))},$$

(5.50)

The resulting nowhere vanishing holomorphic $n-m$ form $\Omega$ is given by

$$\Omega = \frac{\det(Q_{n+b})}{(2\pi)^m} \frac{dx_1 \wedge \cdots \wedge dx_{n-m}}{\det (\partial_{n-m+\beta} G_a (x))},$$

(5.51)

where the determinant in the denominator is over the $\alpha$ and $\beta$ indices, thus realizing from gauge theory the formulae for the holomorphic form on a Calabi-Yau in [18, 19, 21].
6 Discussion

In this paper we have computed the exact partition function of abelian two dimensional $\mathcal{N} = (2, 2)$ gauge theories capturing the Kähler potential in the complex structure moduli space of Calabi-Yau manifolds (1.7)(1.8). These path integrals can be enriched by the insertion of operators in the chiral ring and their conjugates, which compute the metric in the moduli space. It would be interesting to extend the computations in this paper by considering the hemisphere function and to compute the associated D-brane charges, as well as to probe the geometric interpretation of conformal defects in the Calabi-Yau NLSM.

The results of this paper, combined with those in [5,6], give the exact answer to the inequivalent supersymmetric partition functions on the two-sphere. While the metric in the moduli space is unambiguously defined, the two-sphere partition functions suffer from some ambiguities, and can be thought of as generating functions for the metrics on Kähler and complex structure moduli space. In particular, the partition function has an ambiguity associated to turning on an arbitrary twisted superpotential for the complex structure moduli, and this modifies the result of the partition function.

The computation of the Kähler potentials [7,8] in both the Kähler moduli space and the complex structure moduli space are achieved by a direct gauge theory computation. This GLSM approach does not rely on mirror symmetry, and in fact has been used in [7] to compute the Kähler potential for Calabi-Yau manifolds for which a geometrical mirror is not currently known.

It is nevertheless of interest to try to use all these results to find new mirror manifolds. A geometric mirror manifold is expected to exist whenever a Calabi-Yau manifold has a point in its moduli space of maximal unipotent monodromy. Mirror symmetry maps the Kähler potential in Kähler moduli space of a Calabi-Yau $M$ to the Kähler potential in complex structure moduli space of the mirror Calabi-Yau $W$, and vice versa. In terms of the various two-sphere partition functions, mirror symmetry implies that

$$Z_A(M) = Z_B(W) \quad Z_B(M) = Z_A(W).$$

(6.1)

The existence of explicit formulae for these partition functions provides a window in which this problem can be addressed.

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\footnote{The more invariant quantity is the product of the two partition functions $Z_A \cdot Z_B$.}
A Supersymmetry Algebra on \( S^2 \)

The superconformal algebra in the canonical basis on \( S^2 \) was written down in [5]. Here we present explicitly the map between the \( S^2 \) basis and the standard Virasoro basis and extract the two distinct \( SU(2|1) \) subalgebras. The map between the two basis was also presented in [14].

A.1 The Superconformal Algebra in the Standard Basis

The globally defined \( \mathcal{N} = (2, 2) \) superconformal group in two dimensions is generated by the bosonic symmetries \( \{ J_0, L_0, L_{\pm}, \bar{J}_0, \bar{L}_0, \bar{L}_{\pm} \} \) and the fermionic generators \( \{ G_{\pm}^s, \bar{G}_{\pm}^s \} \) satisfying the (anti-) commutation relations [22]

\[
\begin{align*}
[L_0, G_\pm^s] &= \mp \frac{1}{2} G_\pm^s \\
[L_\pm, G_\mp^s] &= \pm G_\pm^s \\
[J_0, G_\mp^s] &= \pm G_\pm^s \\
[L_m, L_n] &= (m - n)L_{m+n} \\
\{ G_\pm^+, G_\pm^- \} &= 2L_\pm \\
\{ G_\pm^+, \bar{G}_\mp^- \} &= 2\bar{L}_\pm
\end{align*}
\]

with all the other (anti-) commutations vanishing. This algebra admits an automorphism \( \sigma \) whose action on the generators is given by

\[
\sigma (G_\pm^s) = G_\mp^s, \quad \sigma(J_0) = -J_0, \quad \sigma = 1 \text{ otherwise.}
\]

We shall see below that this is precisely the map between the A-type and the B-type subalgebras.

A.2 The Superconformal Algebra in the \( S^2 \) Basis

The \( \mathcal{N} = (2, 2) \) superconformal algebra in the \( S^2 \) basis is spanned by the bosonic generators \( J_m, K_m, R, A \),

\[
J_m, K_m, R, A,
\]

and the supercharges \( Q_\alpha, S_\alpha, \bar{Q}_\alpha, \bar{S}_\alpha \).

\[
J_m \text{ generate the } SU(2) \text{ isometries of } S^2 \text{ while } K_m \text{ generate the conformal symmetries of } S^2. \text{ } R \text{ and } A \text{ are each a } U(1) \text{ } R\text{-symmetry generator, the first being non-chiral and the latter being chiral. These}
\]
generators are related to the standard generators introduced above via

\[ J_1 = \frac{i}{2} (L_- + L_+ + \bar{L}_- + \bar{L}_+) \quad K_1 = -\frac{1}{2} (L_- + L_+ - L_- - L_+) \]
\[ J_2 = \frac{i}{2} (L_- - L_+ + \bar{L}_- + \bar{L}_+) \quad K_2 = \frac{i}{2} (L_- - L_+ + \bar{L}_- - \bar{L}_+) \]
\[ J_3 = L_0 - \bar{L}_0 \quad K_3 = i \left( L_0 + \bar{L}_0 \right) \]
\[ R = J_0 + \bar{J}_0 \quad A = -J_0 + \bar{J}_0 \quad \text{(A.5)} \]
\[ S = \frac{1}{\sqrt{2}} \left( G_+^+ + i\bar{G}_+^+ \right) \quad \bar{S} = \frac{1}{\sqrt{2}} \left( G_+^- + i\bar{G}_+^- \right) \]
\[ Q = \frac{1}{\sqrt{2}} \left( -iG_+^- + \bar{G}_+^- \right) \quad \bar{Q} = \frac{1}{\sqrt{2}} \left( iG_+^+ + \bar{G}_+^+ \right) \]

and satisfy the algebra\(^{27}\)

\[ \{S_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R \quad [J_m, S^\alpha] = -\frac{1}{2} \gamma^m_{\alpha\beta} S_\beta \quad [R, S_\alpha] = +S_\alpha \]
\[ \{\bar{S}_\alpha, \bar{Q}_\beta\} = -\gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R \quad [J_m, Q^\alpha] = -\frac{1}{2} \gamma^m_{\alpha\beta} Q_\beta \quad [R, Q_\alpha] = -Q_\alpha \]
\[ \{Q_\alpha, \bar{Q}_\beta\} = \gamma^m_{\alpha\beta} K_m - \frac{i}{2} C_{\alpha\beta} A \quad [J_m, Q^\alpha] = -\frac{1}{2} \gamma^m_{\alpha\beta} \bar{Q}_\beta \quad [R, \bar{Q}_\alpha] = +\bar{Q}_\alpha \]
\[ \{S_\alpha, \bar{S}_\beta\} = \gamma^m_{\alpha\beta} K_m + \frac{i}{2} C_{\alpha\beta} A \quad [J_m, \bar{S}^\alpha] = -\frac{1}{2} \gamma^m_{\alpha\beta} \bar{S}_\beta \quad [R, \bar{S}_\alpha] = -\bar{S}_\alpha \quad \text{(A.6)} \]
\[ [J_m, J_n] = i\epsilon_{mnp} J^p \quad [K_m, S^\alpha] = -\frac{1}{2} \gamma^m_{\alpha\beta} \bar{Q}_\beta \quad [A, S_\alpha] = i\bar{Q}_\alpha \]
\[ [K_m, K_n] = -i\epsilon_{mnp} J^p \quad [K_m, Q^\alpha] = -\frac{1}{2} \gamma^m_{\alpha\beta} \bar{S}_\beta \quad [A, Q_\alpha] = -i\bar{S}_\alpha \]
\[ [J_m, K_n] = i\epsilon_{mnp} K^p \quad [K_m, Q^\alpha] = -\frac{1}{2} \gamma^m_{\alpha\beta} S_\beta \quad [A, Q_\alpha] = -iS_\alpha \]
\[ [K_m, \bar{S}^\alpha] = -\frac{1}{2} \gamma^m_{\alpha\beta} Q_\beta \quad [A, \bar{S}_\alpha] = iQ_\alpha . \]

This algebra admits a \(Z_2\) automorphism, under which

\[ J_m, R, Q_\alpha, S_\alpha \rightarrow J_m, R, Q_\alpha, S_\alpha \quad \text{and} \quad K_m, A, \bar{Q}_\alpha, \bar{S}_\alpha \rightarrow -K_m, -A, -\bar{Q}_\alpha, -\bar{S}_\alpha . \quad \text{(A.7)} \]

The generators \(\{J_m, R, S, Q\}\) form a subalgebra which is the A-type \(SU(2|1)\) algebra used in [5]. In addition to this automorphism, the algebra \(\text{(A.6)}\) inherits the automorphism \(\sigma\) defined in \(\text{(A.2)}\). This implies that \(\{\sigma(J_m), \sigma(R), \sigma(S), \sigma(Q)\}\) given by

\[ \sigma(J_m) = J_m \quad \sigma(S) = \frac{S + \bar{S}}{2} + \frac{iQ + \bar{Q}}{2} \]
\[ \sigma(R) = A \quad \sigma(Q) = -i\frac{S - \bar{S}}{2} + \frac{Q - \bar{Q}}{2} \quad \text{(A.8)} \]

\(^{27}\)The generator of the \(U(1)\) axial symmetry \(A\) used here defers from the one used in [5] by a factor of \(i\).
also form a $SU(2|1)$ subalgebra with the (anti-) commutation relations

\[
\begin{align*}
[J_m, J_n] &= i \epsilon_{mnp} J_p, \\
[J_m, \sigma(Q)] &= -\frac{1}{2} \gamma_m \sigma(Q), \\
[J_m, \sigma(S)] &= -\frac{1}{2} \gamma_m \sigma(S), \\
\{\sigma(S)_{\alpha}, \sigma(Q)_{\beta}\} &= \gamma_{\alpha \beta}^m J_m - \frac{1}{2} C_{\alpha \beta} A \\
[A, \sigma(Q)] &= -\sigma(Q), \\
[A, \sigma(S)] &= \sigma(S),
\end{align*}
\]

which is precisely the B-type algebra we sought. One can check that the realization of this algebra on the vector and chiral multiplets matches the realization of the A-type $SU(2|1)$ algebra on the twisted vector and twisted chiral multiplets up to field redefinitions.

### A.3 Choice of Localizing Supercharge

Our choice of localizing supercharge for the theory of twisted vector and twisted chiral multiplets is $Q = S_1 + Q_2$. The corresponding choice of killing spinors is given by (4.5) and the parameters of the $U(1) \times U(1)_{R} \times G$ transformations generated by $Q^2$ are completely determined through (3.4)(3.5)(3.6). The Killing vector (3.4)

\[
v^\mu \partial_\mu = i \bar{\epsilon} \gamma^\mu \epsilon \partial_\mu = \frac{1}{r} \partial_\varphi
\]

has indeed fixed points at the poles of the two-sphere. The $U(1)_{R}$ transformation parameter (3.5) is given in terms of the radius of the two-sphere

\[
\alpha = -\frac{1}{2r} \bar{\epsilon} \gamma^3 \epsilon = -\frac{1}{2r}.
\]

The induced gauge transformations (3.6) are

\[
\begin{align*}
\Lambda &= -\frac{1}{2} \sin \theta e^{+i \varphi} \sigma \\
\bar{\Lambda} &= \frac{1}{2} \sin \theta e^{-i \varphi} \bar{\sigma} \\
\Omega &= -\frac{1}{r} A_\varphi.
\end{align*}
\]

The gauge transformation parameters are invariant under $Q^2 = J_3 + \frac{R}{2}$.

### B BRST Supercharge and Gauge Fixing

As in any gauge theory, the formalism we have used has built in it a large redundancy which we need to remove in order to proceed with our computation of the partition function. This is achieved by introducing the supercharge $Q_{\text{BRST}}$ and the ghost and anti-ghost multiplets \{\(c, a_o\)\} and \{\(\bar{c}, b\)\}, where \(c\) and \(\bar{c}\) are Grassmann odd and \(a_o\) and \(b\) are Grassmann even scalars and they all have vanishing R-charge.

In terms of the ghost multiplet fields, the BRST operator is defined as

\[
Q_{\text{BRST}} = \delta_G(c), \quad Q^2_{\text{BRST}} = \delta_G(a_o),
\]

where \(a_o\) is assumed to be supersymmetric \(i.e.\) $Q a_o = 0$. By construction, adding the BRST supercharge to the supersymmetry algebra (4.4) leaves the algebra invariant up to gauge transformations.
We therefore define the supercharge $\hat{Q} = Q + Q_{\text{BRST}}$ and require that it realizes the $su(1|1)$ algebra (4.4) as

$$\hat{Q}^2 = \mathcal{L}_v - \frac{i}{2r} R + \delta_G(a_o)$$  \hspace{1cm} (B.2)

where $\mathcal{L}_v$ denotes the Lie(-Lorenz) derivative along $v = 1/r \partial_\varphi$ and $R$ is the generator of the $U(1)_R$ symmetry. This fixes the supersymmetry transformation rule for the ghost and anti-ghost multiplet fields completely. The action of $\hat{Q}$ on the ghost multiplet fields is found to be

$$\hat{Q}c = a + i\epsilon c + v^\mu A_\mu + \frac{1}{2} \sin \theta \left( \sigma e^{i\varphi} - \bar{\sigma} e^{-i\varphi} \right), \hspace{1cm} \hat{Q}a_o = i[c, a_o],$$  \hspace{1cm} (B.3)

while the anti-ghost multiplet fields transform as

$$\hat{Q}\bar{c} = ib, \hspace{1cm} \hat{Q}b = -i(\mathcal{L}_v + i[a, \cdot])\bar{c}.$$  \hspace{1cm} (B.4)

We remark that by construction the action of $\hat{Q}$ and $Q$ coincide on all gauge invariant objects. In particular the deformation term (4.11) satisfies $\hat{Q}V = QV$.

For a choice of gauge fixing functional $G[A, \Phi]$, the gauge fixing condition, $G = 0$, can be imposed on the path integral in a supersymmetric way by adding the deformation term $\hat{Q}V_{\text{G.F.}}$ to the action

$$V_{\text{G.F.}} = \frac{1}{2} \int d^2 x \sqrt{h} \text{Tr} \left\{ \bar{c} \left( G - i b \right) \right\}.$$  \hspace{1cm} (B.5)

Being exact in $\hat{Q}$, this choice of deformation term guarantees the independence of the path integral from the choice of gauge fixing functional $G$, provided that the ghost kinetic term, $\bar{c}Q_{\text{BRST}}G$, is non-degenerate.

In the presence of a Higgs branch, such as the saddle points (4.14), a particularly convenient choice for the gauge fixing functional $G$ is the so called $R_\xi$ gauge (with $\xi = 1$)

$$G = \nabla_\mu A^\mu + i \left( Y\bar{Y} - Y_o\bar{Y} \right)$$  \hspace{1cm} (B.6)

We remark that the gauge fixing condition on the saddle points reduces to the usual Lorenz gauge $\nabla_\mu A^\mu = 0$ which is compatible with the choice $A_\mu = 0$ in (4.14).

C. $\hat{Q}$-Exact Deformation Term

Here we spell out the precise deformation term $\hat{Q}V'$, including all the total derivative terms, which we use for the localization computation. We break $V'$ into four pieces corresponding to the twisted vector, twisted chiral, Fayet-Iliopoulos and gauge fixing terms

$$V' = V_{\text{t.v.m.}} + V_{\text{G.F.}} + V_{\text{F.I.}} + \sum_I V_{\text{t.c.m.}}^I.$$  \hspace{1cm} (C.1)

For concreteness, let $\{ T_a, a = 1, \ldots, \text{dim} \mathfrak{g} \}$ be the set of normalized generators of the gauge algebra $\mathfrak{g}$. The twisted vector multiplet as well as the ghost fields are valued in the adjoint representation of
while the twisted chiral multiplet fields live in a representation \( r \) of \( g \). Suppressing the integration over the sphere, the various terms in \( V' \) are given by

\[
V_{\text{t.v.m.}} = \frac{1}{4} \text{Tr} \left\{ \bar{\epsilon} \gamma^3 \bar{\eta} (D + iF) + e \gamma^3 \eta (D - iF) + \frac{i}{2} (\bar{\epsilon} \eta + \epsilon \bar{\eta}) [\sigma, \bar{\sigma}] - i \bar{\epsilon} \gamma^3 \bar{\eta} \bar{\sigma} \eta - ie \gamma^3 \bar{\eta} \sigma \bar{\eta} \right\},
\]

\[
V_{\text{G.F.}} = \frac{1}{2} \text{Tr} \left\{ \bar{c} \left( \nabla_{\mu} A^\mu + ig^2 (Y \bar{Y}_o - Y_o \bar{Y}) - \frac{i}{4} b \right) \right\},
\]

\[
V_{\text{F.I.}} = \frac{\chi}{2i} \text{Tr} \left( \bar{\epsilon} \gamma^3 \bar{\eta} + e \gamma^3 \eta \right),
\]

\[
V_{\text{t.c.m.}} = \frac{1}{2} \left[ \bar{G} (\bar{\epsilon} \zeta_+ + \epsilon \zeta_+) - (\bar{\epsilon} \zeta_+ + \epsilon \zeta_-) G + i \bar{Y} D_{\mu} (\bar{\epsilon} \gamma^\mu \zeta_+ + e \gamma^\mu \zeta_-) - i \bar{Y} (\bar{\sigma} \bar{\epsilon} \zeta_+ + \sigma \epsilon \zeta_-)
\]

\[
- i D_{\mu} (\bar{\epsilon} \gamma^\mu \bar{\zeta}_+ + e \gamma^\mu \bar{\zeta}_-) Y + i (\bar{\epsilon} \zeta_+ \bar{\sigma} + \epsilon \zeta_- \sigma) Y + i \bar{Y} \left( \bar{\epsilon} \gamma^3 \bar{\eta} + e \gamma^3 \eta \right) Y \right],
\]

where there is an independent Fayet-Iliopoulos parameter \( \chi_a \) for each \( U(1) \) factor in the gauge group.\(^{28}\) As we alluded to in section 4, the twisted vector and twisted chiral terms may be written in a more compact form as

\[
V_{\text{t.v.m.}} = \frac{1}{4} \bar{Q} \text{Tr} \left( \eta \gamma^3 \bar{\eta} + \frac{i}{r} \sigma \bar{\sigma} \right),
\]

\[
V_{\text{t.c.m.}} = \frac{1}{2} \bar{Q} \left( \bar{G} - \bar{\bar{Y}} G + \frac{i}{r} \bar{Y} Y \right),
\]

where \( \bar{Q} = S_1 - Q_2 \). Using (C.2), the deformation term may be split into bosonic and fermionic pieces, up to a total derivative term, \( \text{i.e.} \)

\[
\hat{Q} V = \hat{Q} V|_{\text{bos.}} + \hat{Q} V|_{\text{fer.}} + \nabla_{\mu} J^\mu
\]

where the bosonic part is given by

\[
t \hat{Q} V|_{\text{bos.}} = \frac{t}{2} \sum_a \left\{ F^a_{\mu \nu} + (D^a \sigma)_{\alpha} (D^a \bar{\sigma})_{\dot{\alpha}} + \frac{1}{4} [\sigma, \bar{\sigma}]^2 + \tilde{D}^a + \tilde{b} + \bar{G} a + 2 (\bar{Y} T_a Y - \chi)^2 \right\}
\]

\[
+ t \left( \bar{G} G + D^a \bar{Y} D^a Y + \frac{1}{2} \bar{Y} \{\sigma, \bar{\sigma}\} Y \right)
\]

with \( \tilde{D} = D_a + i(\bar{Y} T_a Y - \chi) \) and \( \tilde{b} = b/2 + i G \). The fermionic part of \( \hat{Q} V \) given by

\[
t \hat{Q} V|_{\text{fer.}} = - \frac{it}{2} \text{Tr} \left\{ \bar{\eta} \left( \bar{\psi} + \frac{1}{r} \gamma^3 \right) \eta + \sigma \bar{\eta} \gamma^3 \eta - \bar{\sigma} \eta \gamma^3 \bar{\eta} + i \bar{\epsilon} \bar{Q} G + \frac{i}{4} \bar{\epsilon} (v^\mu \partial_\mu + i [a_\nu, \cdot]) \bar{c} \right\}
\]

\[
+ it (\bar{Y} (\bar{\eta} + \eta) \zeta + \tilde{\zeta} (\bar{\eta} + \eta) Y) + \zeta (\bar{\psi} - \sigma \gamma_- - \bar{\sigma} \gamma_+) \zeta
\]

\(^{28}\)Without loss of generality, we have not chosen an independent parameter for all the different \( \hat{Q} \)-exact pieces in the deformation term since \( \hat{Q} \)-exactness guarantees that the final result will be independent of such parameters.
and the total derivative term may be written as

\[ J^\mu = \frac{i}{2} \text{Tr} \left\{ (\bar{e}e)\varepsilon^\mu \bar{\sigma} D_\nu \sigma + \frac{1}{2r} v^\mu \bar{\sigma} \sigma + \frac{1}{2} (\bar{e} \gamma^\mu \bar{\eta}) \left( \varepsilon \gamma^3 \eta \right) - \frac{1}{2} (\bar{e} \eta) \left( \varepsilon \gamma^3 \gamma^\mu \eta \right) \right\} \]

\[ + \sum_I \left( (\bar{e} \gamma^-_e) D^\mu \bar{Y}_I Y_I - (\bar{e} \gamma^+ e) \bar{Y}_I D^\mu Y_I - \frac{i}{2} (\bar{e} \gamma^3 \gamma^\mu \bar{e}) \left( \bar{G}_I Y_I + \bar{Y}_I G_I \right) + i (\bar{e} \gamma_- e) \left( \bar{\zeta}_I \gamma^\mu \zeta_I \right) \right) \]

\[ + \frac{\lambda}{2} \text{Tr} \left\{ (\bar{e} \gamma^3 \gamma^\mu \bar{e}) \bar{\sigma} + (\varepsilon \gamma^3 \gamma^\mu \varepsilon) \sigma \right\}. \]

(C.7)

D One-Loop Determinant

Consider the Abelian gauge theory with gauge group \( U(1)^{N_c} \), minimally coupled to \( N_f \) twisted chiral multiplets with generic charges \( \{Q^a_I | a = 1, \ldots, N_c; I = 1, \ldots, N_f \} \). We assume \( N_f \geq N_c \) since, as will become clear, the one-loop determinant vanishes for \( N_c > N_f \) due to fermionic zero modes.

Deforming the path integral by adding the deformation term \( t\mathcal{QV} \) to the action and taking the large \( t \) limit, the path integral localizes to the saddlepoints which are constant maps subject to the D-term constraints

\[ \left\{ Y = \text{constant} \mid \sum_I Q^a_I |Y_I|^2 = \chi^a \right\}. \]

(D.1)

The measure of integration is defined by the one-loop – with respect to \( t \) – fluctuations of the fields around these saddle points. To extract this measure, we expand \( \mathcal{QV} \) to quadratic order around the saddle points (D.1), we therefore redefine the fields as

\[ \Phi \rightarrow \frac{1}{\sqrt{t}} \Phi \]

(D.2)

for twisted vector fields and

\[ Y_I \rightarrow Y_I + \frac{1}{\sqrt{t}} y_I, \quad G_I \rightarrow \frac{1}{\sqrt{t}} G_I, \quad \zeta_I \rightarrow \frac{1}{\sqrt{t}} \zeta \]

(D.3)

for twisted chiral fields. Imposing the \( R_\xi \) gauge on the gauge field fluctuations

\[ \mathcal{G}_a = \frac{1}{\sqrt{t}} \left( \nabla_\mu A^\mu_a - i \sum_I Q^a_I (\bar{y}_I Y_I - \bar{Y}_I y_I) \right) \]

(D.4)

the quadratic part of the deformation term can be cast into the following form

\[ t\mathcal{QV}|_{\text{quad}} = \frac{1}{2r^2} \sum_{a,b} [A^a_b (2M^2_{ab} + \delta_{ab} - r^2 \delta_{ab} \nabla^2) A^b_a + (\bar{\sigma}_a, \bar{c}_a) \left( 2M^2_{ab} - r^2 \delta_{ab} \nabla^2 \right) (\sigma_b, c_b)^T] \]

\[ \quad + \frac{1}{r^2} \sum_{I,J} (\bar{y}_I (2M^2_{IJ} - r^2 \delta_{IJ} \nabla^2) y_J + i \sum_{a,I} Q^a_I [\bar{\eta}_a (\bar{Y}_I \zeta^I_+ + \bar{\zeta}_I Y_I) - \eta_a (\bar{Y}_I \zeta^I_- + \bar{\zeta}_I Y_I)] \]

\[ \quad - \frac{i}{2r} \sum_a [r(\bar{\nabla} + \gamma^3) \eta_a + i \sum_I \bar{\zeta}_I \nabla \zeta_I + \sum_I \bar{G}_I G_I + \frac{1}{2} \sum_a (\bar{D}^2 + \bar{b}^2) + \sum_a \bar{c}_a K_a \]

(D.5)
where $\bar{c}K$ summerizes the fermionic terms that do not contribute to the one-loop determinant. Explicitely, $K_a$ is given by

$$K_a = \frac{1}{8} \nu^\mu \partial_\mu \bar{c}_a - \frac{i}{4} \nabla_\mu \left( \epsilon^3 \gamma^\mu \eta_a - \bar{\epsilon}^3 \gamma^\mu \bar{\eta}_a \right) + \frac{i}{2} \sum_I Q^a_I \left( (\bar{\epsilon}^I - \epsilon^I) Y_o - \bar{Y}_o (\bar{\epsilon}^I - \epsilon^I) \right).$$

We define the $N_f \times N_c$ matrix $M$ and it’s hermitian conjugate $M^\dagger$ as

$$ M^a_I = r Q^a_I Y_I, \quad M^a_I = r Q^a_I \bar{Y}_I. \quad (D.6) $$

The mass matrices $M^2_{ab}$ and $M^2_{IJ}$ appearing in (D.5) are then given by

$$ M^2_{ab} = (M^\dagger M)_{ab}, \quad M^2_{IJ} = (MM^\dagger)_{IJ}. \quad (D.7) $$

For generic charges $Q^a_I$ and with $N_f \geq N_c$, both of these matrices are of rank $N_c$. Furthermore, one can easily check that they have the same eigenvalues since for any eigenvector $u$ of $MM^\dagger$, the vector $M^\dagger u$ is an eigenvector of $M^\dagger M$ with the same eigenvalue.

From (D.5), it is evident that the path integral over the auxiliary fields $\tilde{D}_a, \tilde{b}_a$ and $G_I$ is Gaussian and yields a trivial factor. It is also clear that the path integration over the twisted vector scalars $\{\sigma, \bar{\sigma}\}$ and the ghost and anti-ghost fields $\{c, \bar{c}\}$ yield canceling contributions.

As for the rest of the field, we begin our analysis by diagonalizing the Laplacian on the gauge field. Using the spectrum of the Laplacian operator on $T^*S^2$,

$$\text{spectum}(-r^2 \nabla^2 |_{T^*S^2}) = \{(J^2 + J - 1)^{4J+2}; J = 1, 2, \ldots \}, \quad (D.8)$$

we may compute the contribution of the gauge fields to the one-loop determinant to be

$$\Delta^{-1}_A = \prod_{J=1}^{\infty} \det \left( \frac{J(J+1)}{2} + 2M^\dagger M \right)^{2J+1}. \quad (D.9)$$

In order to compute the contribution to the one-loop determinant arising from the fluctuations of the twisted chiral scalar fields, we first need to isolate the zero modes satisfying

$$\nabla_\mu y_I = 0 \quad \text{and} \quad \sum_J M^2_{IJ} y_J = 0. \quad (D.10)$$

These are the longitudinal fluctuations that lie in the space of saddle points (D.1) and need to be excluded from the one-loop analysis. This amounts to removing the vanishing eigen values of $MM^\dagger$ from the $J = 0$ mode contribution. The contribution from the twisted chiral scalars is then

$$\Delta^{-1}_y = \det'(2MM^\dagger) \prod_{J=1}^{\infty} \det \left( J(J+1) + 2MM^\dagger \right)^{2J+1}$$

$$= \det(2M^\dagger M) \prod_{J=1}^{\infty} \left[ J^{4J(N_f-N_c)} \det \left( J(J+1) + 2M^\dagger M \right)^{2J+1} \right]. \quad (D.11)$$

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Putting (D.9) and (D.11) together, the boson and ghost contributions to the one-loop determinant takes the form

$$\Delta_b^{-1} = \det(2M^\dagger M)^{\sum J \in N_f} \prod_{J=1}^{\infty} \left( \frac{1}{2} \right)^{(2J+1)N_c} \prod_{J=1}^{\infty} \left[ J^{4J(N_f-N_c)} \det \left( J(J+1) + 2M^\dagger M \right)^{4J+2} \right].$$  \hspace{1cm} (D.12)

To compute the contribution to the one-loop determinant due to fermionic fields, consider the field redefinition

$$\psi = \tilde{\zeta} + \zeta, \quad \bar{\psi} = \bar{\zeta} + \bar{\zeta}.$$ \hspace{1cm} (D.13)

In terms of $\psi$ and $\bar{\psi}$, we may rewrite the quadratic fermion part of $tQV$ as

$$\bar{f}Df = \left( \oplus_a \bar{\eta}_a \right)^T \left( \begin{array}{cc} 1 \otimes \left( -\frac{i}{2} \nabla - \frac{1}{2} \gamma^3 \right) & i(M^\dagger \otimes \gamma_+ + M^T \otimes \gamma_-) \\ -i(M \otimes \gamma_- + M^* \otimes \gamma_+) & 1 \otimes i\nabla \end{array} \right) \left( \oplus_a \eta_a \right),$$ \hspace{1cm} (D.14)

where the operator $D_f$ is block diagonal, i.e. it does not mix the eigenmodes of the Dirac operator. Exploiting this fact we may consider each block separately. In the $J$th mode, the Dirac operator is diagonal while the chirality operator $\gamma^3$ has only non-zero off-diagonal elements. Explicitly, we have

$$i\nabla|_J = \left( J + \frac{1}{2} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^3|_J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$ \hspace{1cm} (D.15)

in the basis of eigenspinors of the Dirac operator. In this basis, the $J$th block of the operator $D_f$ in (D.14) takes the form

$$D_f[J] = \left( \begin{array}{cc} 1 \otimes \left( -\frac{J+1/2}{2} \nabla - \frac{1}{2} \gamma^3 \right) & i \left( M^\dagger + M^T \right) \left( M^\dagger - M^T \right) \\ -i \left( M^* + M \right) \left( M^* - M \right) & 1 \otimes \left( J + \frac{1}{2} \right) \end{array} \right),$$ \hspace{1cm} (D.16)

and the fermion contribution to the one-loop determinant takes the form

$$\Delta_f = \prod_{J=1/2}^{\infty} \left| D_f[J] \right|^{2J+1} = \prod_{J=1}^{\infty} \left| D_f[J - 1/2] \right|^{2J}. \hspace{1cm} (D.17)$$

The finite dimensional determinant $\left| D_f[J - 1/2] \right|$ can easily be computed since the bottom right $N_f \times N_f$ block of (D.16) is diagonal which allows us to put the matrix $D_f[J]$ in a lower triangular form. This is achieved via the non-degenerate matrix

$$U[J - 1/2] = \left( \begin{array}{cc} 1 & -\frac{i}{2} \left( M^\dagger + M^T \right) \left( M^\dagger - M^T \right) \\ 0 & 1 \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \otimes \left( J^{-1} \right) \end{array} \right)$$ \hspace{1cm} (D.18)

whose determinant is given by

$$\left| U[J - 1/2] \right| = (-1)^{N_f} J^{-2N_f}. \hspace{1cm} (D.19)$$

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Using this matrix, \(|D_f[J]|\) in (D.17) decomposes as
\[
|D_f[J - 1/2]| = \frac{1}{|U|} |UD_f| = \frac{1}{|U|} \begin{vmatrix} D_f' & 0 \\ U' & 1 \end{vmatrix} = (-1)^{N_f} J^{2N_f} |D_f'[J - 1/2]| \tag{D.20}
\]
where \(D_f'[J - 1/2]\) is given by
\[
D_f'[J - 1/2] = 1 \otimes \left( \begin{array}{cc} -\frac{i}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{i}{2} \end{array} \right) - \frac{1}{4J} \left( \begin{array}{cc} M^\dagger + M^T & M_1 - M^T \\ M^\dagger - M^T & M_1 + M^T \end{array} \right) \left( \begin{array}{cc} M^* + M & M^* - M \\ M - M^* & -M^* - M \end{array} \right) \tag{D.21}
\]
and its determinant is given by
\[
|D_f'[J - 1/2]| = \left( \frac{-1}{(2J)^2} \right)^{N_c} \det \left[ (J(J + 1) + 2M^\dagger M) (J(J - 1) + 2M^\dagger M) \right]. \tag{D.22}
\]
Using this result, substituting (D.20) in (D.17) yields
\[
\Delta_f = \prod_{J=1}^{\infty} 2^{-4J N_c} \prod_{J=1}^{\infty} J^{4J(N_f - N_c)} \prod_{J=0}^{\infty} \det (J(J + 1) + 2M^\dagger M)^{4J^2}. \tag{D.23}
\]
Combining (D.12) and (D.23), the one-loop determinant is given by
\[
\Delta = \det(M^\dagger M) \tag{D.24}
\]
up to an irrelevant divergent factor which may be regularized via zeta function regularization to \(2^{2N_c/3}\).
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