CONVERGENCE RATES OF THE ALLEN-CAHN EQUATION TO MEAN CURVATURE FLOW: A SHORT PROOF BASED ON RELATIVE ENTROPIES

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Abstract. We give a short and self-contained proof for rates of convergence of the Allen-Cahn equation towards mean curvature flow, assuming that a classical (smooth) solution to the latter exists and starting from well-prepared initial data. Our approach is based on a relative entropy technique. In particular, it does not require a stability analysis for the linearized Allen-Cahn operator. As our analysis also does not rely on the comparison principle, we expect it to be applicable to more complex equations and systems.

1. Introduction

The Allen-Cahn equation

$$\frac{d}{dt} u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon)$$

with a suitable double-well potential $W$ like for instance $W(s) = c (1 - s^2)^2$, $c > 0$ – is the most natural diffuse-interface approximation for (two-phase) mean curvature flow: It is well-known that in the limit of vanishing interface width $\varepsilon \to 0$, the solutions $u_\varepsilon$ to the Allen-Cahn equation (1) converge to a characteristic function $\chi : \mathbb{R}^d \times [0, T] \to \{-1, 1\}$ whose interface evolves by motion by mean curvature. For a proof of this fact in the framework of Brakke solutions to mean curvature flow, we refer to [8], while for the convergence towards the viscosity solution of the level-set formulation under the assumption of non-fattening we refer to [5]. Provided that the total energy converges in the limit $\varepsilon \to 0$, one may prove that the limit is a distributional solution [10]. For a general compactness statement using the gradient-flow structure of (1) and the identification of the limit in the radially symmetric case, we refer the reader to [2]. Under the assumption of the existence of a smooth limiting evolution, rates of convergence may be derived based on a strategy of matched asymptotic expansions and the stability of the linearized Allen-Cahn operator [3, 4].

The Allen-Cahn equation corresponds to the $L^2$ gradient flow of the Ginzburg-Landau energy functional

$$E_\varepsilon[v] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{\varepsilon} W(v) \, dx.$$ 

Solutions to the Allen-Cahn equation (1) satisfy the energy dissipation estimate

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \, dx = - \int_{\mathbb{R}^d} \frac{1}{\varepsilon} |\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon)|^2 \, dx.$$
In the present work, we pursue a strategy of deriving a quantitative convergence result in the sharp-interface limit \( \varepsilon \to 0 \) based purely on the energy dissipation structure. In particular, we give a short proof for the following quantitative convergence of solutions of the Allen-Cahn equation towards a smooth solution of mean curvature flow.

**Theorem 1.** Let \( d \in \mathbb{N} \). Let \( I(t) \subset \mathbb{R}^d \), \( t \in [0, T] \), be a compact interface \( I(t) = \partial \Omega(t) \) evolving smoothly by mean curvature, and let \( \chi : \mathbb{R}^d \times [0, T] \to \{-1, 1\} \) be the corresponding phase indicator function

\[
\chi(x, t) := \begin{cases} 
1 & \text{if } x \in \Omega(t), \\
-1 & \text{if } x \notin \Omega(t).
\end{cases}
\]

Let \( W \) be a standard double-well potential as described below and denote by \( \theta \) the corresponding one-dimensional interface profile. Let \( u_\varepsilon \) be the solution to the Allen-Cahn equation (1) with initial data given by \( u_\varepsilon(x, 0) = \theta(\varepsilon^{-1} \text{dist}^+(x, I(0))) \), where \( \text{dist}^+(x, I(0)) \) is the signed distance function to \( I(0) \) with the convention \( \text{dist}^+(x, I(0)) > 0 \) for \( x \in \Omega(0) \). Define \( \psi_\varepsilon(x, t) := \int_0^{u_\varepsilon(x, t)} \sqrt{2W(s)} \, ds \). Then the error estimate

\[
\sup_{t \in [0, T]} ||\psi_\varepsilon(\cdot, t) - \chi(\cdot, t)||_{L^1(\mathbb{R}^d)} \leq C(d, T, (I(t))_{t \in [0, T]}) \varepsilon
\]

holds.

We note that this error estimate is of optimal order, as \( \varepsilon \) is the typical width of the diffuse interface in the Allen-Cahn approximation (i.e. the typical width of the region in which the function \( \psi_\varepsilon \) takes values in the range \([-1 + \delta, 1 - \delta]\) for any fixed \( \delta > 0 \)).

The assumptions required for the double-well potential \( W \) are standard: We require \( W \) to satisfy \( W(1) = W(-1) = 0 \) and \( W(s) \geq c \min\{|s-1|^2, |s+1|^2\} \); furthermore, we require \( W \) to be twice continuously differentiable, symmetric around the origin, and subject to the normalization \( \int_{-1}^{1} \sqrt{2W(s)} \, ds = 2 \). The simplest example is the normalized standard double-well potential \( W(s) := \frac{2}{\sqrt{\pi}} (1 - s^2)^2 \). Under these assumptions, we may define the one-dimensional equilibrium profile \( \theta : \mathbb{R} \to \mathbb{R} \) to be the unique odd solution of the ODE \( \theta'(s) = \sqrt{2W(\theta(s))} \) with boundary conditions \( \theta(\pm \infty) = \pm 1 \); the profile \( \theta \) then approaches its boundary values \( \pm 1 \) at \( \pm \infty \) with an exponential rate, see [11].

As our quantitative convergence analysis does not rely on the comparison principle, it may be applicable to more complex models, such as systems of Navier-Stokes-Allen-Cahn type [1]; note that a weak-strong uniqueness theorem for the two-fluid free boundary problem for the Navier-Stokes equation (i.e. the corresponding sharp-interface model) has already been obtained in [6]. We note that a relative entropy concept related to the one in [6] had already been employed by Jerrard and Smets [9] to deduce weak-strong uniqueness of solutions to binormal curvature flow. In the forthcoming work [7], we employ an energy-based strategy to deduce a weak-strong uniqueness theorem for multiphase mean curvature flow.

## 2. Definition of the Relative Energy and Gronwall Estimate

### 2.1. Extending the unit normal vector field of the surface evolving by mean curvature.

Let \( I = I(t) \) be a surface that evolves smoothly by motion by mean curvature. Fix \( r_c > 0 \) small enough depending on \((I(t))_{t \in [0,T]}\). For each
For instance, one can define \( \eta(s) := (1 - cr^2 s^2)\tilde{\eta}(s) \), where \( \tilde{\eta} \) is a standard cut-off which is identically 1 in a neighborhood of 0.

To see that (6a) and (6b) hold, one makes use of the formulas \( n_I(x) = \nabla \text{dist}^\pm(x, I) \) and \( \partial_t \text{dist}^\pm(x, I) = -H_I \cdot n_I(P_Ix) \) valid in a neighborhood of \( I(t) \). Formula (6c) is an immediate consequence of the equality \( H_I = -(\nabla \cdot n_I)n_I \) valid on the interface \( I(t) \) and the Lipschitz continuity of both sides of the equation.

### 2.2. The relative energy inequality.

Our argument is based on a relative energy method. As the Modica-Mortola trick will play an important role in the definition of the relative energy, we introduce the function

\[
\psi_\varepsilon(x, t) := \int_0^{u_\varepsilon(x, t)} \sqrt{2W(s)} \, ds.
\]

Given a smooth solution \( u_\varepsilon \) to the Allen-Cahn equation (1) and a surface \( I(t) \) which evolves smoothly by mean curvature flow, we define the relative energy \( E[u_\varepsilon|I] \) as

\[
E[u_\varepsilon|I] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) - \xi \cdot \nabla \psi_\varepsilon \, dx.
\]

Introducing the short-hand notation

\[
n_\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}
\]

(with \( n_\varepsilon(x, t) \in \mathbb{S}^{d-1} \) arbitrary but fixed in case \( |\nabla u_\varepsilon| = 0 \)) and writing

\[
E[u_\varepsilon|I] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) - |\nabla \psi_\varepsilon| \, dx + \int_{\mathbb{R}^d} (1 - \xi \cdot n_\varepsilon)|\nabla \psi_\varepsilon| \, dx,
\]
we see that the relative energy consists of two contributions: The first term
\[
\int_{\mathbb{R}^d} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) - |\nabla \psi_\varepsilon| \, dx = \int_{\mathbb{R}^d} \frac{1}{2} \sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2 W(u_\varepsilon)} \, dx
\]
controls the local lack of equipartition of energy between the terms \(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2\) and \(\frac{1}{\varepsilon} W(u_\varepsilon)\), while the second term
\[
\int_{\mathbb{R}^d} (1 - \xi \cdot n_\varepsilon) |\nabla \psi_\varepsilon| \, dx \geq \frac{1}{2} \int_{\mathbb{R}^d} |n_\varepsilon - \xi|^2 |\nabla \psi_\varepsilon| \, dx
\]
controls the local deviation of the normals \(n_\varepsilon\) and \(n_I\). Note that the latter term also controls the distance to the interface \(I(t)\) (since \(|\xi| \leq \max\{1 - c \text{dist}^2(x, I), 0\}\).

We furthermore introduce the notation
\[
H_\varepsilon := -\left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon)\right) \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|},
\]
motivated by the fact that \(H_\varepsilon\) will play a role of a curvature vector.

The key step in our analysis is the following Gronwall-type estimate for the relative energy.

**Theorem 2.** Let \(I(t), t \in [0, T]\), be an interface evolving smoothly by mean curvature. Let \(u_\varepsilon\) be a solution to the Allen-Cahn equation (11) with initial data given by \(u_\varepsilon(x, 0) = \theta(\varepsilon^{-1} \text{dist}^2(x, I(0)))\). Then for any \(t \in [0, T]\) the estimate
\[
\frac{d}{dt} E[u_\varepsilon|I] + \int_{\mathbb{R}^d} \frac{1}{4\varepsilon} |H_\varepsilon - H_I| |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon} |n_\varepsilon \cdot H_\varepsilon - (-\nabla \cdot \xi) \sqrt{2 W(u_\varepsilon)}|^2 \, dx \\
\leq C(d, I(t)) E[u_\varepsilon|I]
\]
holds.

### 2.3. Coercivity properties of the relative energy functional.

For the proof of the Gronwall-type inequality of Theorem 2 we shall need the following coercivity properties of the relative energy.

**Lemma 3.** We have the estimates
\[
\begin{align*}
(10a) & \quad \int_{\mathbb{R}^d} \left(\sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2 W(u_\varepsilon)}\right)^2 \, dx \leq 2E[u_\varepsilon|I], \\
(10b) & \quad \int_{\mathbb{R}^d} |n_\varepsilon - \xi|^2 |\nabla \psi_\varepsilon| \, dx \leq 2E[u_\varepsilon|I], \\
(10c) & \quad \int_{\mathbb{R}^d} |n_\varepsilon - \xi|^2 \varepsilon |\nabla u_\varepsilon|^2 \, dx \leq 12E[u_\varepsilon|I], \\
(10d) & \quad \int_{\mathbb{R}^d} \min\{\text{dist}^2(x, I), 1\} \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\right) \, dx \leq CE[u_\varepsilon|I].
\end{align*}
\]

**Proof.** We complete the square to get
\[
E[u_\varepsilon|I] = \int_{\mathbb{R}^d} \frac{1}{2} \left(\sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2 W(u_\varepsilon)}\right)^2 + (1 - \xi \cdot n_\varepsilon) |\nabla \psi_\varepsilon| \, dx.
\]
In particular, we directly obtain (10a) and (10b) by \(|\xi| \leq 1\). By the property (5a) of the cutoff \(\eta\) (and hence \(1 - \xi \cdot n_\varepsilon \geq c \varepsilon \text{dist}^2(x, I)\)), we deduce (10d) with \(|\nabla \psi_\varepsilon|\) instead of the energy density, which we may replace upon using (10a).
Lemma 4. Let \( u_\varepsilon \) be a solution to the Allen-Cahn equation \((1)\) and let \( I = I(t) \) be a smooth solution to mean curvature flow. Let \( \xi \) be as defined in \((3)\). The time evolution of the relative energy functional is then given by

\[
\frac{d}{dt} E[u_\varepsilon | I] = - \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} |H_\varepsilon - H \varepsilon - \xi| \varepsilon |v_\varepsilon| |H_\varepsilon - (\nabla \cdot \xi) \sqrt{2W(u_\varepsilon)}|^2 dx \\
+ \int_{\mathbb{R}^d} |H_\varepsilon| \frac{\varepsilon}{2} |v_\varepsilon|^2 + |\nabla \cdot \xi| \varepsilon |W_\varepsilon + H_\varepsilon \cdot n_\varepsilon (\nabla \cdot \xi)| \varepsilon |v_\varepsilon|^2 dx \\
+ \int_{\mathbb{R}^d} \nabla \cdot H_\varepsilon \left( \frac{\varepsilon}{2} |v_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) - |\nabla \psi_\varepsilon| \right) dx \\
- \int_{\mathbb{R}^d} \nabla H_\varepsilon : n_\varepsilon \otimes n_\varepsilon (\varepsilon |v_\varepsilon|^2 - |\nabla \psi_\varepsilon|) dx \\
- \int_{\mathbb{R}^d} \nabla H_\varepsilon : (n_\varepsilon - \xi) \otimes (n_\varepsilon - \xi)| \nabla \psi_\varepsilon| dx \\
+ \int_{\mathbb{R}^d} \nabla \cdot H_\varepsilon (1 - \xi \cdot n_\varepsilon) | \nabla \psi_\varepsilon| dx \\
- \int_{\mathbb{R}^d} |\nabla \psi_\varepsilon|(n_\varepsilon - \xi) \cdot \left( \frac{d}{dt} \xi + (H_\varepsilon \cdot \nabla) \xi + (\nabla H_\varepsilon) \right) dx \\
- \int_{\mathbb{R}^d} |\nabla \psi_\varepsilon| \xi \cdot \left( \frac{d}{dt} \xi + (H_\varepsilon \cdot \nabla) \xi \right) dx.
\]

Proof. By direct computation, we obtain

\[
\frac{d}{dt} E[u_\varepsilon | I] = \frac{d}{dt} \int_{\mathbb{R}^d} \frac{\varepsilon}{2} |v_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) - \xi \cdot \nabla \psi_\varepsilon dx \\
= \int_{\mathbb{R}^d} \frac{1}{\varepsilon} |\Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon)|^2 dx \\
- \int_{\mathbb{R}^d} \nabla \psi_\varepsilon \cdot \frac{d}{dt} \xi dx + \int_{\mathbb{R}^d} \sqrt{2W(u_\varepsilon)} \left( \Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon) \right) \nabla \cdot \xi dx.
\]
With the definitions (9a) and (9b), we obtain
\[
\frac{d}{dt} E[u_\varepsilon|I] = \int_{\mathbb{R}^d} -\frac{1}{\varepsilon} |H_\varepsilon|^2 + n_\varepsilon \cdot H_\varepsilon (-\nabla \cdot \xi) \frac{1}{\varepsilon} \sqrt{2W(u_\varepsilon)} dx
\]
\[
+ \int_{\mathbb{R}^d} \nabla H_I : \xi \otimes n_\varepsilon |\nabla \psi_\varepsilon| dx
\]
\[
+ \int_{\mathbb{R}^d} (H_I \cdot \nabla)\xi \cdot \nabla \psi_\varepsilon dx
\]
\[
- \int_{\mathbb{R}^d} \nabla \psi_\varepsilon \cdot \left( \frac{d}{dt} \xi + (H_I \cdot \nabla)\xi + (\nabla H_I)^T \xi \right) dx.
\]
We exploit the symmetry of the Hessian $\nabla^2 \psi_\varepsilon$
\[
\int_{\mathbb{R}^d} (H_I \cdot \nabla)\xi \cdot \nabla \psi_\varepsilon dx
\]
\[
= - \int_{\mathbb{R}^d} \nabla \cdot H_I \xi \cdot \nabla \psi_\varepsilon dx - \int_{\mathbb{R}^d} H_I \otimes \xi : \nabla^2 \psi_\varepsilon dx
\]
\[
= \int_{\mathbb{R}^d} (\nabla \cdot H_I - \nabla \cdot H_I \xi) \cdot \nabla \psi_\varepsilon dx + \int_{\mathbb{R}^d} (\xi \cdot \nabla) H_I \cdot \nabla \psi_\varepsilon dx,
\]
which yields
\[
\frac{d}{dt} E[u_\varepsilon|I] = \int_{\mathbb{R}^d} -\frac{1}{\varepsilon} |H_\varepsilon|^2 + n_\varepsilon \cdot H_\varepsilon (-\nabla \cdot \xi) \frac{1}{\varepsilon} \sqrt{2W(u_\varepsilon)} dx
\]
\[
+ \int_{\mathbb{R}^d} \nabla H_I : \xi \otimes n_\varepsilon |\nabla \psi_\varepsilon| dx
\]
\[
+ \int_{\mathbb{R}^d} (\nabla \cdot H_I - \nabla \cdot H_I \xi) \cdot \nabla \psi_\varepsilon dx
\]
\[
+ \int_{\mathbb{R}^d} (\xi \cdot \nabla) H_I \cdot n_\varepsilon |\nabla \psi_\varepsilon| dx
\]
\[
- \int_{\mathbb{R}^d} \nabla \psi_\varepsilon \cdot \left( \frac{d}{dt} \xi + (H_I \cdot \nabla)\xi + (\nabla H_I)^T \xi \right) dx.
\]
The computation (13) below then implies by adding zero
\[
\frac{d}{dt} E[u_\varepsilon|I] = \int_{\mathbb{R}^d} -\frac{1}{\varepsilon} |H_\varepsilon|^2 + H_\varepsilon \cdot H_I |\nabla u_\varepsilon| + n_\varepsilon \cdot H_\varepsilon (-\nabla \cdot \xi) \frac{1}{\varepsilon} \sqrt{2W(u_\varepsilon)} dx
\]
\[
+ \int_{\mathbb{R}^d} \nabla \cdot H_I |\nabla \psi_\varepsilon| dx
\]
\[
+ \int_{\mathbb{R}^d} \nabla \cdot H_I \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) - |\nabla \psi_\varepsilon| \right) dx
\]
\[
- \int_{\mathbb{R}^d} \nabla H_I : n_\varepsilon \otimes n_\varepsilon (\varepsilon |\nabla u_\varepsilon|^2 - |\nabla \psi_\varepsilon|) dx
\]
\[
- \int_{\mathbb{R}^d} \nabla H_I : (n_\varepsilon - \xi) \otimes (n_\varepsilon - \xi) |\nabla \psi_\varepsilon| dx
\]
\[
+ \int_{\mathbb{R}^d} (\nabla \cdot H_I - \nabla \cdot H_I \xi) \cdot \nabla \psi_\varepsilon dx
\]
\[
+ \int_{\mathbb{R}^d} (\xi \cdot \nabla) H_I \cdot \xi |\nabla \psi_\varepsilon| dx
\]
Indeed, due to definition (9b) we have
\[ 2.5. \]
Auxiliary computation. Completing the squares and adding zero, we obtain (12).
\[ \square \]
the formula
\[ \hat{\|\nabla \psi_x \|} \]
on the right-hand side of the identity (12). Using (6a), (6b), and the bound
\[ 2.6. \]
Proof of Theorem 2. Derivation of the Gronwall inequality.
\[ I \]
\[ \|\nabla \cdot \mathbf{H}_t \| \leq C(I(t)), \]
the last four lines of (12) may be estimated by
\[ C(I(t)) \int_{R^d} \min\{\text{dist}^2(x, I), 1\} |\nabla \psi_x| + |n_x - \xi| |\nabla \psi_x| + (1 - n_x \cdot \xi) |\nabla \psi_x| \, dx, \]
which by (10) and (10) is bounded by \( C(I(t))E[u_x | I] \).
\[ \text{The third line on the right-hand side of (12) can be estimated as} \]
\[ 2.6. \]
Derivation of the Gronwall inequality.
\[ \text{Proof of Theorem 2. Using the estimates of Lemma 3 we can control the terms on the right-hand side of the identity (12). Using (6a), (6b), and the bound } \|\nabla \mathbf{H}_t\|_{L^\infty} \leq C(I(t)), \text{ the last four lines of (12) may be estimated by} \]
\[ \int_{R^d} \nabla \cdot \mathbf{H}_t \left( \frac{\xi}{2} |\nabla u_x|^2 + \frac{1}{\varepsilon} W(u_x) - |\nabla \psi_x| \right) \, dx \leq \|\nabla \cdot \mathbf{H}_t\|_{L^\infty} E[u_x | I]. \]
Thus, it only remains to estimate the second and the fourth term on the right-hand side of (12).

Concerning the second term, we use the fact that $(\xi \cdot \nabla)H_I \equiv 0$ holds in a neighborhood of $I(t)$, Young’s inequality, and (7) to deduce

$$
\int_{\mathbb{R}^d} |\nabla H_I : n_\varepsilon \otimes n_\varepsilon (\varepsilon |\nabla u_\varepsilon|^2 - |\nabla \psi_\varepsilon|)| \, dx
$$

$$
\leq \int_{\mathbb{R}^d} |\nabla H_I : n_\varepsilon \otimes (n_\varepsilon - \xi) (\varepsilon |\nabla u_\varepsilon|^2 - |\nabla \psi_\varepsilon|)| \, dx
$$

$$
+ C \int_{\mathbb{R}^d} \min\{\text{dist}^2(x, I), 1\} (\varepsilon |\nabla u_\varepsilon|^2 + |\nabla \psi_\varepsilon|) \, dx
$$

$$
\leq \|\nabla H_I\|_\infty \left( \int_{\mathbb{R}^d} |n_\varepsilon - \xi|^2 \varepsilon |\nabla u_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} \left( \varepsilon |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right)^2 \, dx \right)^{\frac{1}{2}}
$$

$$
+ C \int_{\mathbb{R}^d} \min\{\text{dist}^2(x, I), 1\} (\varepsilon |\nabla u_\varepsilon|^2 + |\nabla \psi_\varepsilon|) \, dx.
$$

Consequently, Lemma 3 implies that the fourth line on the right-hand side of (12) is bounded by $CE[u_\varepsilon |I|$.

It only remains to bound the term in the second line of the right-hand side of (12). To this aim, we complete the square and estimate

$$
\int_{\mathbb{R}^d} |H_I|^2 \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + |\nabla \cdot \xi|^2 \frac{1}{\varepsilon} W(u_\varepsilon) + H_I \cdot n_\varepsilon \nabla \cdot \xi |\nabla \psi_\varepsilon| \, dx
$$

$$
= \int_{\mathbb{R}^d} \frac{1}{2} \left| \sqrt{\varepsilon} |\nabla u_\varepsilon| H_I + \frac{1}{\sqrt{\varepsilon}} \nabla \cdot \xi \sqrt{2W(u_\varepsilon)} n_\varepsilon \right|^2 \, dx
$$

$$
\leq \frac{3}{2} \int_{\mathbb{R}^d} \left| \nabla \cdot \xi n_\varepsilon \left( \sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right) \right|^2 \, dx
$$

$$
+ \frac{3}{2} \int_{\mathbb{R}^d} \left| \nabla \cdot \xi (n_\varepsilon - \xi) \sqrt{\varepsilon} |\nabla u_\varepsilon| \right|^2 \, dx
$$

$$
+ \frac{3}{2} \int_{\mathbb{R}^d} \left| H_I + (\nabla \cdot \xi) \xi \sqrt{\varepsilon} |\nabla u_\varepsilon| \right|^2 \, dx.
$$

Inserting the estimates (6c) and (6d) and using the fact that $H_I = (H_I \cdot \xi) + O(\text{dist}^2(x, I))$, we obtain

$$
\int_{\mathbb{R}^d} |H_I|^2 \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + |\nabla \cdot \xi|^2 \frac{1}{\varepsilon} W(u_\varepsilon) + H_I \cdot n_\varepsilon \nabla \cdot \xi |\nabla \psi_\varepsilon| \, dx
$$

$$
\leq C \int_{\mathbb{R}^d} \left| \sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right|^2 \, dx
$$

$$
+ C \int_{\mathbb{R}^d} \min\{\text{dist}^2(x, I), 1\} (\varepsilon |\nabla u_\varepsilon|^2 + |n_\varepsilon - \xi|^2 \varepsilon |\nabla u_\varepsilon|^2) \, dx.
$$

By Lemma 3, we see that these terms are estimated by $CE[u_\varepsilon |I]$.

\[\square\]
3. Estimate for the Interface Error

Proof of Theorem 4. Step 1: Estimate for the relative energy. In view of Theorem 2, in order to prove

\begin{equation}
\sup_{t \in [0,T]} E[u_\varepsilon | I] + \int_0^T \int_{\mathbb{R}^d} \frac{1}{\varepsilon} |H_\varepsilon - H_I \varepsilon [\nabla u_\varepsilon]|^2 + \frac{1}{\varepsilon} |n_\varepsilon \cdot H_\varepsilon - (\nabla \cdot \xi) \sqrt{2W(u_\varepsilon)}|^2 \, dx \, dt \\
\leq C(d,T, (I(t))_{t \in [0,T]}) \varepsilon^2
\end{equation}

it only remains to show that the initial relative energy satisfies \( E[u_\varepsilon | I](0) \leq C(d,I(0)) \varepsilon^2 \). To this aim, we compute using \( u_\varepsilon(x,0) = \theta(\varepsilon^{-1} \text{dist}^\pm(x,I(0))) \) and the fact that \( \nabla \text{dist}^\pm(x,I(0)) \cdot \xi = |\nabla \text{dist}^\pm(x,I(0))| \xi | \geq |\xi|^2 \)

\begin{align*}
E[u_\varepsilon | I](0) & \leq \int_{\mathbb{R}^d} \frac{|\xi|^2}{2\varepsilon} |\theta'(\varepsilon^{-1} \text{dist}^\pm(x,I(0)))|^2 + \frac{|\xi|^2}{\varepsilon} W(\theta(\varepsilon^{-1} \text{dist}^\pm(x,I(0)))) \\
& \quad - \frac{1}{\varepsilon} \sqrt{2W(\theta(\varepsilon^{-1} \text{dist}^\pm(x,I(0))))} \theta'(\varepsilon^{-1} \text{dist}^\pm(x,I(0))) |\xi|^2 \, dx \\
& \quad + \int_{\mathbb{R}^d} \frac{1}{2\varepsilon}(1 - |\xi|^2) \left( |\theta'(\varepsilon^{-1} \text{dist}^\pm(x,I(0)))|^2 + W(\theta(\varepsilon^{-1} \text{dist}^\pm(x,I(0)))) \right) \, dx.
\end{align*}

Using the defining equation \( \theta'(s) = \sqrt{2W(\theta(s))} \) as well as the fact that \( |\theta'(s)| \) decays exponentially in \( s \) and that \( |\xi|^2 \geq 1 - c \text{dist}^2(x,I) \), we deduce \( E[u_\varepsilon | \chi](0) \leq C(d,I(0)) \varepsilon^2 \).

Step 2: Interface error estimate. We now perform an additional computation to obtain a more explicit control on the interface error. We may write

\( \partial_t \psi_\varepsilon = \sqrt{2W(u_\varepsilon)} \partial_t u_\varepsilon = -\varepsilon^{-1} \sqrt{2W(u_\varepsilon)} H_\varepsilon \cdot n_\varepsilon. \)

Choosing \( \tau : \mathbb{R} \to [-1,1] \) to be a smooth monotone truncation of the identity map (with \( \tau(s) \geq \min\{s, \frac{1}{2}\} \) for \( s > 0 \) and \( \tau(s) \leq \max\{s, -\frac{1}{2}\} \) for \( s < 0 \) and fixing \( s_0 > 0 \) to be determined later, we obtain

\( \frac{d}{dt} \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \, dx \)

\begin{align*}
&= - \int_{\mathbb{R}^d} \varepsilon^{-1} \sqrt{2W(u_\varepsilon)} H_\varepsilon \cdot n_\varepsilon \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \, dx \\
& \quad + \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) \frac{1}{s_0} \tau' \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \partial_t \text{dist}^\pm(x,I) \, dx \\
&= - \int_{\mathbb{R}^d} \varepsilon^{-1} \sqrt{2W(u_\varepsilon)} H_\varepsilon \cdot n_\varepsilon \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \, dx \\
& \quad - \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) H_I \cdot \nabla \left( \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \right) \, dx \\
& \quad + \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) \frac{1}{s_0} \tau' \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \left( \partial_t \text{dist}^\pm(x,I) + H_I \cdot \nabla \text{dist}^\pm(x,I) \right) \, dx \\
&= - \int_{\mathbb{R}^d} \varepsilon^{-1} \sqrt{2W(u_\varepsilon)} H_\varepsilon \cdot n_\varepsilon - \nabla \psi_\varepsilon \cdot H_I \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \, dx.
\end{align*}
\[
+ \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \nabla \cdot H_I \, dx \\
+ \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) \frac{1}{s_0} \tau' \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) (\partial_t \text{dist}^\pm(x,I) + H_I \cdot \nabla \text{dist}^\pm(x,I)) \, dx
\]

where in the last step we have used integration by parts and \( \tau(\text{dist}^\pm(x,I(t)) = 0 \) on supp \( \nabla \chi(\cdot,t) \).

This may be rewritten using the definition of \( \psi_\varepsilon \) and \( n_\varepsilon \) as

\[
\frac{d}{dt} \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \, dx \\
= - \int_{\mathbb{R}^d} \varepsilon^{-1} \sqrt{2W(u_\varepsilon)(H_\varepsilon - H_I \varepsilon |\nabla u_\varepsilon|)} \cdot n_\varepsilon \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \, dx \\
+ \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \nabla \cdot H_I \, dx \\
+ \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) \frac{1}{s_0} \tau' \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) (\partial_t \text{dist}^\pm(x,I) + H_I \cdot \nabla \text{dist}^\pm(x,I)) \, dx.
\]

Since \( \partial_t \text{dist}^\pm(x,I) = -H_I \cdot \nabla \text{dist}^\pm(x,I) \) holds in a neighborhood of the interface, the last integral vanishes identically if we chose \( s_0 > 0 \) sufficiently small. Using Cauchy-Schwarz we deduce

\[
\frac{d}{dt} \int_{\mathbb{R}^d} (\psi_\varepsilon - \chi) \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \, dx \\
\leq \int_{\mathbb{R}^d} \varepsilon^{-1} |H_\varepsilon - H_I \varepsilon |\nabla u_\varepsilon| \right)^2 \, dx + \int_{\mathbb{R}^d} \varepsilon^{-1} 2W(u_\varepsilon) \left| \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \right|^2 \, dx \\
+ \| (\nabla \cdot H_I - ) \|_{L^\infty} \int_{\mathbb{R}^d} |\psi_\varepsilon - \chi| \left| \tau \left( \frac{1}{s_0} \text{dist}^\pm(x,I) \right) \right| \, dx.
\]

By the Gronwall inequality and (16) (note that the relative entropy \( E[u_\varepsilon|I](t) \) controls \( c \int_{\mathbb{R}^d} \varepsilon^{-1} W(u_\varepsilon) \text{dist}^2(x,I) \, dx \)), this shows that

(17) \[ \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |\psi_\varepsilon - \chi| \min\{\text{dist}(x,I),1\} \, dx \leq C(d,T,(I(t))_{t \in [0,T]}) \varepsilon^2. \]

Note that Fubini’s theorem for the square \([0,\delta]^2\) yields

\[
\left( \int_0^\delta |\psi_\varepsilon(w + yn_I(w),t) - \chi(w + yn_I(w),t)| \, dy \right)^2 \\
\leq 2 \int_0^\delta |\psi_\varepsilon(w + yn_I(w),t) - \chi(w + yn_I(w),t)| \int_0^y 2 \, ds \, dy.
\]
This allows to estimate for a small $\delta$-neighborhood of $I(t)$
\[
\left(\int_{I(t)+B_\delta} |\psi_\varepsilon(x,t) - \chi(x,t)| \, dx\right)^2 \\
\leq C(d, I(t)) \left( \int_{I(t)} \int_0^\delta |\psi_\varepsilon(w + yn_I(w),t) - \chi(w + yn_I(w),t)| \, dy \\
+ \int_0^\delta |\psi_\varepsilon(w - yn_I(w),t) - \chi(w - yn_I(w),t)| \, dy \, dS(w) \right)^2 \\
\leq C(d, I(t)) \left( \int_{I(t)} \int_{-\delta}^\delta |\psi_\varepsilon(w + yn_I(w),t) - \chi(w + yn_I(w),t)| \\
\times \text{dist}(w + yn_I(w), I(t)) \, dy \, dS(w) \right) \\
\leq C(d, I(t)) \int_{I(t)+B_\delta} |\psi_\varepsilon(x,t) - \chi(x,t)| \text{dist}(x, I) \, dx,
\]
which in view of (17) yields Theorem 1. \qed

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