Field quantization by means of a single harmonic oscillator

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A new scheme of field quantization is proposed. Instead of associating with different frequencies different oscillators we begin with a single oscillator that can exist in a superposition of different frequencies. The idea is applied to the electromagnetic radiation field. Using the standard Dirac-type mode-quantization of the electromagnetic field we obtain several standard properties such as coherent states or spontaneous and stimulated emission. As opposed to the standard approach the vacuum energy is finite and does not have to be removed by any ad hoc procedure.

I. HARMONIC OSCILLATOR IN SUPERPOSITION OF FREQUENCIES

The standard quantization of a harmonic oscillator is based on quantization of \( p \) and \( q \) but \( \omega \) is a parameter. To have, say, two different frequencies one has to consider two independent oscillators. On the other hand, it is evident that there exist oscillators which are in a superposition of different frequencies. The example is an oscillator wave packet associated with distribution of center-of-mass momenta.

This simple observation raises the question of the role of superpositions of frequencies for a description of a single harmonic oscillator. We know that frequency is typically associated with an eigenvalue of some Hamiltonian or, which is basically the same, with boundary conditions. A natural way of incorporating different frequencies into a single harmonic oscillator is by means of the frequency operator

\[
\Omega = \sum_{\omega_k,j_k} \omega_k |\omega_k,j_k\rangle \langle \omega_k,j_k|
\]

where all \( \omega_k \geq 0 \). For simplicity we have limited the discussion to the discrete spectrum but it is useful to include from the outset the possibility of degeneracies. The corresponding Hamiltonian is defined by

\[
H = \hbar \Omega \otimes \frac{1}{2}(a^\dagger a + aa^\dagger)
\]

where \( a = \sum_{n=0}^{\infty} \sqrt{n+1} |n\rangle \langle n+1| \). The eigenstates of \( H \) are \( |\omega_k,j_k,n\rangle \) and satisfy

\[
H |\omega_k,j_k,n\rangle = \hbar \omega_k (n + \frac{1}{2}) |\omega_k,j_k,n\rangle.
\]

The standard case of the oscillator whose frequency is just \( \omega \) coresponds either to \( \Omega = \omega 1 \) or to the subspace spanned by \( |\omega_k,j_k,n\rangle \) with fixed \( \omega_k = \omega \). Introducing the operators

\[
a_{\omega_k,j_k} = |\omega_k,j_k\rangle \langle \omega_k,j_k| \otimes a
\]

we find that

\[
H = \frac{1}{2} \sum_{\omega_k,j_k} \hbar \omega_k \left(a_{\omega_k,j_k}^\dagger a_{\omega_k,j_k} + a_{\omega_k,j_k} a_{\omega_k,j_k}^\dagger \right).
\]

The algebra of the oscillator is

\[
[a_{\omega_k,j_k}, a_{\omega_l,j_l}^\dagger] = \delta_{\omega_k \omega_l} \delta_{j_k j_l} |\omega_k,j_k\rangle \langle \omega_k,j_k| \otimes 1
\]

\[
a_{\omega_k,j_k} a_{\omega_l,j_l} = \delta_{\omega_k \omega_l} \delta_{j_k j_l} (a_{\omega_k,j_k})^2
\]

\[
a_{\omega_k,j_k}^\dagger a_{\omega_l,j_l}^\dagger = \delta_{\omega_k \omega_l} \delta_{j_k j_l} (a_{\omega_k,j_k}^\dagger)^2.
\]

The dynamics in the Schrödinger picture is given by
In the Heisenberg picture we obtain the important formula
\[ a_{\omega_k,j_k}(t) = e^{iHt/\hbar} a_{\omega_k,j_k} e^{-iHt/\hbar} = |\omega_k,j_k\rangle \langle \omega_k,j_k| \otimes e^{-i\omega_k t} a = e^{-i\omega_k t} a_{\omega_k,j_k}(0). \] (10)

Taking a general state
\[ |\psi\rangle = \sum_{\omega_k,j_k,n} \psi(\omega_k,j_k,n)|\omega_k,j_k\rangle|n\rangle \] (12)
we find that the average energy of the oscillator is
\[ \langle H \rangle = \langle \psi|H|\psi\rangle = \sum_{\omega_k,j_k,n} |\psi(\omega_k,j_k,n)|^2\hbar\omega_k \left(n + \frac{1}{2}\right). \] (13)

The average clearly looks as an average energy of an ensemble of different and independent oscillators. The ground state of the ensemble, i.e. the one with \( \psi(\omega_k,j_k,n > 0) = 0 \) has energy
\[ \langle H \rangle = \frac{1}{2} \sum_{\omega_k,j_k} |\psi(\omega_k,j_k,0)|^2\hbar\omega_k < \infty. \] (14)

The result is not surprising but still quite remarkable if one thinks of the problem of field quantization. The very idea of quantizing the electromagnetic field, as put forward by Born, Heisenberg, Jordan \[1\] and Dirac \[2\], is based on the observation that the mode decomposition of the electromagnetic energy is analogous to the energy of an ensemble of independent harmonic oscillators. In 1925, after the work of Heisenberg, it was clear what to do: One had to replace each classical oscillator by a quantum one. But since each oscillator had a definite frequency, to have an infinite number of different frequencies one needed an infinite number of oscillators. The price one paid for this assumption was the infinite energy of the electromagnetic vacuum.

The infinity is regarded as an “easy” one since one can get rid of it by redefining the Hamiltonian and removing the infinite term. The result looks correct and many properties typical of a quantum harmonic oscillator are indeed observed in electromagnetic field. However, once we remove the infinite term by the procedure of “normal reordering” the resulting Hamiltonian is no longer physically equivalent to the one of the harmonic oscillators. For a single oscillator we can indeed add any finite number and the new Hamiltonian will describe the same physics. But having two or more such oscillators we cannot remove the ground state energies by a single shift of energy: Each oscillator has to be shifted by a different number and, accordingly, we change the energy differences between the levels of the global Hamiltonian describing the multi-oscillator system. And this is not just “shifting the origin of the energy scale”. Alternatively, one can add up all the ground state corrections and remove the overall energy shift by a different choice of the origin of the energy scale. This would have been acceptable if the shift were finite. Subtraction of infinite terms is in mathematics as forbidden as division by zero. (Example: \( 1 + \infty = 2 + \infty \Rightarrow 1 = 2 \) is as justified as \( 1 \cdot 0 = 2 \cdot 0 \Rightarrow 1 = 2. \))

The oscillator which can exist in superpositions of different frequencies is a natural candidate as a starting point for Dirac-type field quantization. We do not need to remove the ground state energy since in the Hilbert space of physical states the correction is finite. The question we have to understand is whether one can obtain the well known quantum properties of the radiation field by this type of quantization.

II. FIELD OPERATORS: FREE MAXWELL FIELDS

The energy and momentum operators of the field are defined in analogy to \( H \) from the previous section
\[ H = \sum_{s,\kappa,\lambda} \hbar\omega_\lambda|s,\kappa_\lambda\rangle \langle s,\kappa_\lambda| \otimes \frac{1}{2}(a^\dagger a + aa^\dagger). \] (15)

\[ = \frac{1}{2} \sum_{s,\kappa,\lambda} \hbar\omega_\lambda \left(a^\dagger a,\kappa_\lambda + a_s,\kappa_\lambda a^\dagger s,\kappa_\lambda \right) \] (16)
\[ \tilde{P} = \sum_{s,\kappa,\lambda} \hbar \kappa_\lambda \langle s, \kappa_\lambda \rangle (s, \kappa_\lambda) \otimes \frac{1}{2} (a^\dagger a + aa^\dagger) \]
\[ = \frac{1}{2} \sum_{s,\kappa,\lambda} \hbar \kappa_\lambda \left( a^\dagger_{s,\kappa,\lambda} a_{s,\kappa,\lambda} + a_{s,\kappa,\lambda} a^\dagger_{s,\kappa,\lambda} \right) \]

where \( s = \pm 1 \) corresponds to circular polarizations. Denote \( P = (H/c, \tilde{P}) \) and \( P \cdot x = Ht - \tilde{P} \cdot \vec{x} \). We employ the standard Dirac-type definitions for mode quantization in volume \( V \)
\[ \hat{A}(t, \vec{x}) = \sum_{s,\kappa,\lambda} \sqrt{\frac{\hbar}{2\omega\lambda V}} \left( a_{s,\kappa,\lambda} e^{-i\omega t \cdot \vec{E}_{s,\kappa,\lambda} e^{i\kappa_\lambda \cdot \vec{x}} + \alpha^\dagger_{s,\kappa,\lambda} e^{i\omega t \cdot \vec{E}_{s,\kappa,\lambda} e^{-i\kappa_\lambda \cdot \vec{x}}} \right) \]
\[ = e^{iP \cdot x / \hbar} \hat{A} e^{-iP \cdot x / \hbar} \]
\[ \hat{E}(t, \vec{x}) = i \sum_{s,\kappa,\lambda} \sqrt{\frac{\hbar \omega \lambda}{2V}} \left( a_{s,\kappa,\lambda} e^{-i\omega t \cdot \vec{E}_{s,\kappa,\lambda} e^{i\kappa_\lambda \cdot \vec{x}}} - \alpha^\dagger_{s,\kappa,\lambda} e^{i\omega t \cdot \vec{E}_{s,\kappa,\lambda} e^{-i\kappa_\lambda \cdot \vec{x}}} \right) \]
\[ = e^{iP \cdot x / \hbar} \hat{E} e^{-iP \cdot x / \hbar} \]
\[ \hat{B}(t, \vec{x}) = i \sum_{s,\kappa,\lambda} \sqrt{\frac{\hbar \omega \lambda}{2V}} \left( a_{s,\kappa,\lambda} e^{-i\omega t \cdot \vec{B}_{s,\kappa,\lambda} e^{i\kappa_\lambda \cdot \vec{x}}} - \alpha^\dagger_{s,\kappa,\lambda} e^{i\omega t \cdot \vec{B}_{s,\kappa,\lambda} e^{-i\kappa_\lambda \cdot \vec{x}}} \right) \]
\[ = e^{iP \cdot x / \hbar} \hat{B} e^{-iP \cdot x / \hbar} \]

Now take a state (say, in the Heisenberg picture)
\[ |\Psi\rangle = \sum_{s,\kappa_\lambda, n} \Psi_{s,\kappa_\lambda, n} |s, \kappa_\lambda, n\rangle \]
\[ = \sum_{s,\kappa_\lambda} \Phi_{s,\kappa_\lambda} |s, \kappa_\lambda \rangle |\alpha_{s,\kappa_\lambda}\rangle \]
where \( |\alpha_{s,\kappa_\lambda}\rangle \) form a family of coherent states:
\[ a |\alpha_{s,\kappa_\lambda}\rangle = \alpha_{s,\kappa_\lambda} |\alpha_{s,\kappa_\lambda}\rangle \]

The averages of the field operators are
\[ \langle \Psi | \hat{A}(t, \vec{x}) | \Psi \rangle = \sum_{s,\kappa_\lambda} | \Phi_{s,\kappa_\lambda} |^2 \sqrt{\frac{\hbar}{2\omega\lambda V}} \left( \alpha_{s,\kappa_\lambda} e^{-i\kappa_\lambda \cdot \vec{x}} \bar{e}_{s,\kappa_\lambda} + \alpha^\dagger_{s,\kappa_\lambda} e^{i\kappa_\lambda \cdot \vec{x}} e_{s,\kappa_\lambda} \right) \]
\[ \langle \Psi | \hat{E}(t, \vec{x}) | \Psi \rangle = \sum_{s,\kappa_\lambda} | \Phi_{s,\kappa_\lambda} |^2 \sqrt{\frac{\hbar \omega \lambda}{2V}} \left( \alpha_{s,\kappa_\lambda} (0) e^{-i\kappa_\lambda \cdot \vec{x}} \bar{e}_{s,\kappa_\lambda} - \alpha^\dagger_{s,\kappa_\lambda} (0) e^{i\kappa_\lambda \cdot \vec{x}} e_{s,\kappa_\lambda} \right) \]
\[ \langle \Psi | \hat{B}(t, \vec{x}) | \Psi \rangle = i \sum_{s,\kappa_\lambda} | \Phi_{s,\kappa_\lambda} |^2 \sqrt{\frac{\hbar \omega \lambda}{2V}} \left( \alpha_{s,\kappa_\lambda} e^{-i\kappa_\lambda \cdot \vec{n}_{s,\kappa_\lambda} \times \bar{e}_{s,\kappa_\lambda} - \alpha^\dagger_{s,\kappa_\lambda} e^{i\kappa_\lambda \cdot \vec{n}_{s,\kappa_\lambda} \times e_{s,\kappa_\lambda}} \right) \]

These are just the classical fields. More precisely, the fields look like averages of monochromatic coherent states with probabilities \( |\Phi_{s,\kappa_\lambda}|^2 \). The energy-momentum operators satisfy also the standard relations
\[ H = \frac{1}{2} \int_V d^3x \left( \hat{E}(t, \vec{x}) \cdot \hat{E}(t, \vec{x}) + \hat{B}(t, \vec{x}) \cdot \hat{B}(t, \vec{x}) \right) \]
\[ \tilde{P} = \int_V d^3x \hat{E}(t, \vec{x}) \times \hat{B}(t, \vec{x}) \]

It should be stressed, however, that these relations have a completely different mathematical origin than in the usual formalism where the integrals are necessary in order to make plane waves into an orthonormal basis. Here orthogonality follows from the presence of the projectors in the definition of \( a_{s,\kappa_\lambda} \) and the integration in itself is trivial since
\[ \hat{E}(t, \vec{x}) \cdot \hat{E}(t, \vec{x}) + \hat{B}(t, \vec{x}) \cdot \hat{B}(t, \vec{x}) = \hat{E} \cdot \hat{E} + \hat{B} \cdot \hat{B} \] 
\[ \hat{E}(t, \vec{x}) \times \hat{B}(t, \vec{x}) = \hat{E} \times \hat{B}. \]

Therefore the role of the integral is simply to produce the factor \( V \) which cancels with \( 1/V \) arising from the term \( 1/\sqrt{V} \) occurring in the mode decomposition of the fields. To end this section let us note that

\[ \langle \Psi | H | \Psi \rangle = \sum_{s, \kappa} \hbar \omega \lambda | \Phi_{s, \kappa} \rangle^2 \left( | \alpha_{s, \kappa} \rangle |^2 + \frac{1}{2} \right) \] 
\[ \langle \Psi | \hat{P} | \Psi \rangle = \sum_{s, \kappa} \hbar \kappa \lambda | \Phi_{s, \kappa} \rangle^2 \left( | \alpha_{s, \kappa} \rangle |^2 + \frac{1}{2} \right). \]

The contribution from the vacuum fluctuations is nonzero but finite.

### III. SPONTANEOUS AND STIMULATED EMISSION

The next test we have to perform is to check the examples that were responsible for the success of Dirac’s quantization in atomic physics. It is clear that no differences are expected to occur for single-mode problems such as the Jaynes-Cummings model. In what follows we will therefore concentrate on spontaneous and stimulated emission from two-level atoms.

Beginning with the dipole and rotating wave approximations we arrive at the Hamiltonian

\[ H = \frac{1}{2} \hbar \omega_0 \sigma_3 + \frac{1}{2} \sum_{s, \kappa} \hbar \omega \lambda \left( a_{s, \kappa}^{\dagger} a_{s, \kappa} + a_{s, \kappa} a_{s, \kappa}^{\dagger} \right) + \hbar \omega_0 \imath \sum_{s, \kappa} \left( g_{s, \kappa} a_{s, \kappa} \sigma_+ + g_{s, \kappa}^{*} a_{s, \kappa}^{\dagger} \sigma_- \right) \]

where \( \langle d \rangle = \langle d \rangle^* \) is the matrix element of the dipole moment evaluated between the excited and ground states, and \( g_{s, \kappa} = \imath \sqrt{2 \omega_0 / E_{s, \kappa}} \). The Hamiltonian represents a two-level atom located at \( \vec{x}_0 = 0 \).

The Hamiltonian in the interaction picture has the well known form

\[ H_I = \hbar \omega_0 \imath \sum_{s, \kappa} \left( g_{s, \kappa} e^{i(\omega_0 - \omega) t} a_{s, \kappa} \sigma_+ + g_{s, \kappa}^{*} e^{-i(\omega_0 - \omega) t} a_{s, \kappa}^{\dagger} \sigma_- \right). \]

Consider the initial state

\[ |\Psi(0)\rangle = \sum_{s, \kappa, m} \Psi_{s', \kappa', m} |s', \kappa', m, +\rangle = \sum_{s', \kappa'} \Psi_{s', \kappa', 0} |s', \kappa', 0, +\rangle + \sum_{s', \kappa'} \Psi_{s', \kappa', n} |s', \kappa', n, +\rangle. \]

The states corresponding to \( n = 0 \) play a role of a vacuum. As a consequence the vacuum is not represented here by a unique vector, but rather by a subspace of the Hilbert space of states. It is also clear that the energy of this vacuum may be nonzero since no normal ordering of observables is necessary.

Using the first-order time-dependent perturbative expansion we arrive at

\[ |\Psi(t)\rangle = |\Psi(0)\rangle + \omega_0 \sum_{s, \kappa} \frac{1}{\omega_0 - \omega} \Psi_{s, \kappa, 0, 0} g_{s, \kappa}^{*} |s, \kappa, 0, -\rangle \]
\[ + \omega_0 \sum_{s, \kappa} \frac{1}{\omega_0 - \omega} \Psi_{s, \kappa, n, 1} g_{s, \kappa}^{*} |s, \kappa, n, 1, -\rangle. \]

One recognizes here the well known contributions from spontaneous and stimulated emissions. It should be stressed that although the final result looks familiar, the mathematical details behind the calculation are different from what we are accustomed to. For example, instead of
\[ a_{s_1, \vec{k}_1}^\dagger |s, \vec{k}, m\rangle \sim |s_1, \vec{k}_1, 1; s, \vec{k}, m\rangle, \]  

which would hold in the standard formalism for \( \vec{k}_1 \neq \vec{k} \), we get simply

\[ a_{s_1, \vec{k}_1}^\dagger |s, \vec{k}, m\rangle = 0, \]

a consequence of \( a_{s_1, \vec{k}_1}^\dagger a_{s, \vec{k}}^\dagger = 0 \).

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[1] M. Born, W. Heisenberg, and P. Jordan, Z. Phys. 35, 557 (1925)
[2] P. A. M. Dirac, Proc. Roy. Soc. A 112, 661 (1926); ibid. 114, 243 (1927).