“Pretty strong” converse for the private capacity of degraded quantum wiretap channels

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Abstract—In the vein of the recent “pretty strong” converse for the quantum and private capacity of degradable quantum channels [Morgan/Winter, IEEE Trans. Inf. Theory 60(1):317-333, 2014], we use the same techniques, in particular the calculus of min-entropies, to show a pretty strong converse for the private capacity of degraded classical-quantum-quantum (cqq-)wiretap channels, which generalize Wyner’s model of the degraded classical wiretap channel.

While the result is not completely tight, leaving some gap between the region of error and privacy parameters for which the converse bound holds, and a larger no-go region, it represents a further step towards an understanding of strong converses of wiretap channels [cf. Hayashi/Tyagi/Watanabe, arXiv:1410.0443 for the classical case].

Index Terms—quantum information, private capacity, strong converse, smooth min-entropies.

I. INTRODUCTION

One of the most outstanding successes of Shannon theory [23] is Shannon’s information theoretic treatment of cryptography [24], and the further development at the hands of Wyner, who introduced the wiretap channel model [25]. While the achievability part for the wiretap channel is a well-understood combination of channel coding and privacy amplification techniques, the converse, even the weak converse, of the generalized wiretap channel required a new idea in Csiszár and Körner’s contribution [3].

Characteristically, for multi-user scenarios strong converses are hard to come by and not known in many instances. The presence of an adversary in the wiretap setting, albeit a passive one, makes the wiretap capacity a multi-user problem, and until recently only weak converses were known for Wyner’s original problem [3]. The same was true for the “static” versions of distillation of shared secret key between Alice and Bob from a prior three-way correlation with Eve [1], [18], where Tyagi and Narayan [28], Tyagi and Watanabe [29], and Watanabe and Hayashi [33] made progress only recently. Most recently, Hayashi, Tyagi and Watanabe [15] (see also their [14]) have given an elegant, very insightful analysis of strong converse rates for general classical wiretap channels, yielding the complete strong converse in the degraded case.

Here, we extend their results somewhat to the quantum case, looking at wiretap channels with classical input but quantum outputs, so-called cqq-wiretap channels. Instead of the elegant hypothesis testing method developed in [15], we use a rather more blunt tool, the min-entropy calculus [22, 27]. Hence, while we can treat channels not amenable to the method of [15], we do not reach a complete understanding of the full tradeoff between decoding error and privacy.

II. CQQ-WIRETAP CHANNEL AND STRONG CONVERSE

The model we consider is that of a discrete memoryless cqq-wiretap channel:

\[ W : X \rightarrow S(B \otimes E) \]
\[ x \rightarrow \rho_x^{BE} \]

with a finite set \( X \) and finite dimensional Hilbert spaces \( B \) and \( E \), of legal user and eavesdropper, respectively. Furthermore, we shall assume most of the time that the channel is degraded, meaning that there exists a quantum channel (cptp map) \( D : \mathcal{L}(B) \rightarrow \mathcal{L}(E) \) such that \( \rho_x^E = D(\rho_x^B) \) for all \( x \in X \). Introducing a Stinespring dilation of \( D \) by an isometry \( V : B \mapsto E' \otimes F \), we have \( \rho_x^E = \operatorname{Tr}_F V \rho_x^B V^\dagger \).

The objective of wiretap coding is for Alice to encode messages in such a way that Bob can decode with small error probability, and that Eve cannot distinguish messages except with small probability. To quantify errors, we use the purified distance

\[ P(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}^2, \]

with the fidelity \( F(\rho, \sigma) = ||\sqrt{\sigma} \sqrt{\rho}||_1 \) between quantum states [16], [30], see [27]. An \( n \)-block code of transmission error \( \epsilon \) and privacy error \( \delta \) for \( W \) consists of a stochastic map \( E : [M] \rightarrow X^n \) and a POVM \( D = (D_u)_{u=1}^M \) on \( B^n \), such that for

\[ \rho^{U^n E^n} = \frac{1}{M} \sum_{u, \hat{u}, x^n} E(x^n_u | u) |u\rangle |U^n \rangle \otimes |\hat{u}\rangle |\hat{U}^n \rangle \]
\[ \otimes \operatorname{Tr}_{B^n} \left[ \rho_x^{BE} (D_u \otimes \mathbb{1}) \right], \]

the following hold:

\[ P(\rho^{U^n}, \Delta^{U^n}) \leq \epsilon, \]

\[ P(\rho^{E^n}, \Delta^{U^n}) \leq \delta. \]

Here, \( \Delta^{U^n} = \frac{1}{M} \sum_u |u\rangle |u\rangle \otimes |u\rangle |\hat{u} \rangle \), so that \( \Delta^{U^n} \) is the maximally mixed state, and \( \rho^{E^n} \) is a suitable state on \( E^n \).

The largest number \( M \) of messages under these conditions is denoted \( M(n, \epsilon, \delta) \). Then, the private capacity is defined as the largest asymptotically achievable rate such that transmission error and privacy error vanish in the limit, i.e.

\[ P(W) := \inf_{\epsilon, \delta > 0} \lim_{n \to \infty} \frac{1}{n} \log M(n, \epsilon, \delta). \]
Theorem 1 (Devetak [6]; Cai/Winter/Yeung [2]) Let $W$ be a cqq-wiretap channel. Then its private capacity is given by $P(W) = \sup_n \frac{1}{n} P^{(1)}(W^\otimes n)$, where $P^{(1)}(W) = \max I(U : B) - I(U : E)$.

Here, the maximum is over joint distributions $P_{UX}$ of the channel input $X$ and an auxiliary variable $U$, and the mutual informations are with respect to the state $\rho^{UXBE} = \sum_{u,x} P_{UX}(u,x) |u\rangle \langle u| \otimes |x\rangle \langle x| \otimes \rho_x^{BE}$.

For degraded channels, it is given by the single-letter formula

$$P(W) = P^{(1)}(W) = \max I(X : F|E'),$$

where the maximum is over distributions $P_X$ of the channel input, and the conditional mutual information is with respect to the state $\rho^{XE'F} = \sum_x P_X(x) |x\rangle \langle x| (V \otimes \mathbb{1}) \rho_x^{BE}(V \otimes \mathbb{1})$.

In other words, w.l.o.g. one may assume $U = X$, and the regularization is not necessary [13, Appendix A].

For completeness, we recall here the definition of the quantum information quantities: For a state $\rho$ on a quantum system $X$, the entropy is $S(\rho) = S(p_X) = -\Tr \rho \log \rho$, the mutual information for a bipartite state $\rho^{XY}$ is $I(X : Y) = S(X) + S(Y) - S(XY)$, and the conditional mutual information for a tripartite state $\rho^{XYZ}$ is $I(X : Y|Z) = S(XZ) + S(YZ) - S(Z) - S(XYZ)$.

It seems to be unknown whether in the cqq-wiretap channel setting the regularization above is necessary, but it is quite clear that the single-letterization in the classical case, by Csiszar and Körner [3], does not work, due to the use of chain rules, we would get information quantities conditioned on quantum registers. Furthermore, the results of Smith, Renes and Smolin [26] suggest that $P(1)$ does not give the private capacity. On the other hand, in the general quantum channel case [2] it is well-known that the regularization is necessary: For an isometry, such as Eve’s channel is the complementary channel to Bob’s, there are instances where $P(1)$ is strictly smaller than $P$ [11, 17, 25, 26]. Also if the eavesdropper’s channel is a degraded version (even trivially) of the authorized channel, and the latter is quantum, $P(1)$ can be strictly smaller than $P$, as observed in [13].

Here we show the following pretty strong converse:

Theorem 2 Let $W : X \rightarrow S(B \otimes E)$ be a degraded cqq-wiretap channel and $\epsilon, \delta > 0$ such that $\epsilon + 2\delta < 1$, then

$$\log M(n, \epsilon, \delta) \leq n P(W) + O(\sqrt{n \log n}),$$

where the implicit constant only depends on $1 - \epsilon - 2\delta$. In particular, under the above assumptions,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log M(n, \epsilon, \delta) = P(W).$$

Its proof relies on the calculus of min- and max-entropies, of which we will briefly review the necessary definitions and properties; cf. [27] for more details.

Definition 3 (Min- and max-entropy) For $\rho^{AB} \in S_{\leq}(AB)$, the min-entropy of $A$ conditioned on $B$ is defined as

$$H_{\min}(A|B)_{\rho} := \max_{\sigma_B \in S(B)} \max_{\lambda \in \mathbb{R}} \{ \lambda : \rho^{AB} \leq 2^{-\lambda} \mathbb{1} \otimes \sigma^B \}.$$

With a purification $|\psi\rangle^{ABC}$ of $\rho$, we define

$$H_{\max}(A|B)_{\rho} := -H_{\min}(A|C)_{\psi^{AC}},$$

with the reduced state $\psi^{AC} = Tr_B \psi$.

Definition 4 (Smooth min- and max-entropy) Let $\epsilon \geq 0$ and $\rho_{AB} \in S(AB)$. The $\epsilon$-smooth min-entropy of $A$ conditioned on $B$ is defined as

$$H_{\min}^{\epsilon}(A|B)_{\rho} := \max_{\rho' \approx_{\epsilon,\rho} \rho} H_{\min}(A|B)_{\rho'}.$$

Similarly,

$$H_{\max}^{\epsilon}(A|B)_{\rho} := \min_{\rho' \approx_{\epsilon,\rho} \rho} H_{\max}(A|B)_{\rho'} = -H_{\min}^{\epsilon}(A|C)_{\psi},$$

where $\rho' \approx_{\epsilon,\rho} \rho$ means $P(\rho', \rho) \leq \epsilon$ for $\rho' \in S_{\leq}(AB)$.

All min- and max-entropies, smoothed or not, are invariant under local unitaries and local isometries. The following two lemmas show that min- and max-entropies have properties close to those of the von Neumann entropy.

Lemma 5 (Data processing) For $\rho \in S(ABC)$ and $\epsilon \geq 0$,

$$H_{\min}^{\epsilon}(A|BC) \leq H_{\min}^{\epsilon}(A|B), \quad H_{\max}^{\epsilon}(A|BC) \leq H_{\max}^{\epsilon}(A|B).$$

Lemma 6 (Chain rules) Let $\epsilon, \delta \geq 0$, $\eta > 0$. Then, with respect to the same state $\rho \in S(ABC)$,

$$H_{\max}^{\epsilon+2\delta+\eta}(AB|C) \leq H_{\max}^{\delta}(B|C) + H_{\max}^{\epsilon}(A|BC) + \log \frac{2}{\eta^2},$$

and

$$H_{\max}^{\epsilon+2\delta+2\eta}(AB|C) \geq H_{\min}^{\epsilon}(B|C) + H_{\max}^{\epsilon+2\delta+2\eta}(A|BC) - 3 \log \frac{2}{\eta^2}.$$

Proof of Theorem 2 We follow closely the initial steps of the analysis in [19, Thm. 14]. Consider an $n$-block code with $M$ messages, and transmission and privacy error $\epsilon$ and is $\delta$: message $u$ (chosen uniformly) is encoded by a distribution $E(x^n|u)$ and sent through the channel, giving rise to a cqq-state between message $U$, input $X^n$, output $B^n$ and environment $E^n$:

$$\rho^{UX^nB^nE^n} = \frac{1}{M} \sum_{u,x^n} E(x^n|u) |u\rangle \langle u| \otimes |x^n\rangle \langle x^n| \otimes \rho_x^{BE}|E^n).$$
The “trivial” converse shows that
\[
\log M \leq H_{\min}^\delta (U \mid E^n) - H_{\max}^\epsilon (U \mid B^n),
\]
cf. Renes and Renner [23]. Namely, according to the definition of privacy given above, the reduced state \( \rho_{UB^n}^\epsilon \) is within purified distance \( \epsilon \) of a product state of the form
\[
\frac{1}{\sqrt{\sum_n |u|u \rangle \langle u| \otimes |\tau_n\rangle \langle \tau_n|}},
\]
hence \( H_{\min}^\delta (U \mid E^n) \geq \log M \). Likewise, there exists a decoding ctp map \( D : L(B^n) \rightarrow \mathcal{U} \) such that (id \( \otimes D \))\( \rho_{UB^n}^\epsilon \) is within \( \epsilon \) purified distance from the perfectly correlated state \( \frac{1}{\sqrt{\sum_n |u|u \rangle \langle u|}} \otimes |\tau_n\rangle \langle \tau_n| \), hence \( H_{\max}^\epsilon (U \mid B^n) \leq 0 \).

We apply the Stinespring dilation of the degrading map to \( \rho \)
yielding
\[
\omega_{UX^nE^nF^nE^n} = \frac{1}{M} \sum_{u,x^n} E(x^n|u) |u\rangle \langle u| \otimes |x^n\rangle \langle x^n| V \otimes \rho_{UB^nE^n}^\epsilon V^\dagger \otimes n.
\]
With respect to this state, we now have (cf. Eq. (18) of [19]),
\[
\log M \leq H_{\min}^\delta (U \mid E^n) - H_{\max}^\epsilon (U \mid E^n F^n)
= H_{\min}^\delta (U \mid E^n) - H_{\max}^\epsilon (U \mid E^n)
\leq H_{\max}^\delta (F^n \mid E^n) - H_{\max}^\epsilon (F^n \mid E^n + U) + 4 \log \frac{2}{\eta^2}
\leq H_{\max}^\delta (F^n \mid E^n) - H_{\max}^\epsilon (F^n \mid E^n X^n) + 4 \log \frac{2}{\eta^2},
\]
where we have once more invoked the AEP, Proposition 8.

Now, we face the case of general encodings, and reduce it to the above form of constant type. Introduce another register \( T \) holding the type \( t(x^n) \) of \( x^n \), of dimension \( |T| \leq (n+1)^{|X|} \), so that we have an extended joint state
\[
\rho_{UX^nB^nE^nT} = \frac{1}{M} \sum_{u,x^n} E(x^n|u) |u\rangle \langle u| \otimes |x^n\rangle \langle x^n| x^n \otimes \rho_{UB^nE^n}^\epsilon \otimes |t(x^n)\rangle \langle t(x^n)| T.
\]
Imagine that \( T \) is handled by the eavesdropper; this clearly doesn’t increase Bob’s decoding error, but it can affect the privacy of the code.

so that we obtain a joint state
\[
\tilde{\rho}^{U'X^nB^nE^mT} = \frac{1}{N} \sum_{u',v,x} \frac{1}{N} E(x^n|u',v)\rho'(u'\rho|x^n|x^n)^{X^n} \otimes \rho(\tilde{X}|x^n|x^n)^T
\]
where \(\tilde{E}(x^n|u') = \frac{1}{2} \sum_i E(x^n|u',v).\) This is a new code: By the properties of random mixing, with high probability, Bob can apply the same decoding as in the original code to obtain an error \(\leq \epsilon + \vartheta,\) and the privacy error for the combined register \(E^nT\) is \(\leq \delta + \vartheta.\) Furthermore, the privacy error of the register \(T\) alone is \(\leq \vartheta.\) This has the important consequence that we can modify the encoding \(\tilde{E}(x^n|u')\) to a slightly different one \(E'(x^n|u') = Q(t(x^n))E'(x^n|u',t(x^n)),\) with a universal distribution \(Q\) over the types, such that
\[
\rho^{U'X^nB^nE^mT} = \frac{1}{N} \sum_{u',v,x} E'(x^n|u')\rho'(u'|u'\rho|x^n|x^n)^{X^n} \otimes \rho(\tilde{X}|x^n|x^n)^T
\]
fulfills the decoding and eavesdropper constraints with transmission error \(\epsilon'\) and privacy error \(\delta',\) and has a perfectly independent type-register \(T.\)

Consider now the codes obtained by using \(E'(\cdot|\cdot, P_0)\) for a fixed \(P_0\) (but always the same decoder for Bob). These have transmission errors \(\epsilon(P_0)\) and privacy errors \(\delta(P_0).\) By the direct sum types \(P_0 -\) with probability \(Q(P_0)\), and the concavity of \(\sqrt{1-x^2}\), one can see that
\[
\epsilon' \geq \sum_{P_0 \text{ type}} Q(P_0)\epsilon(P_0), \quad \delta' \geq \sum_{P_0 \text{ type}} Q(P_0)\delta(P_0),
\]
and so
\[
\sum_{P_0 \text{ type}} Q(P_0)[\epsilon(P_0) + 2\delta(P_0)] \leq \epsilon' + 2\delta' < 1,
\]
and so there must exist a type \(P_0\) such that the encoding \(E'(\cdot|\cdot, P_0)\) has \(\epsilon(P_0) + 2\delta(P_0) < \epsilon' + 2\delta' < 1.\) But this code has only \(O(\log n)\) less information in the message, and has the property that the encoder maps only into the type class \(\tau(P_0),\) hence can use the previous bound:
\[
\log M \leq \log N + O(\log n) \leq nI(X:F[E']) + O(\sqrt{n \log n}),
\]
concluding the proof.

**Proposition 8 (Min- and max-entropy AE for cq-channels)** Let \(\rho \in \mathcal{S}(\mathcal{H}_{AB})\) and \(0 < \epsilon < 1.\) Then,
\[
\lim_{n \to \infty} \frac{1}{n} H^c_{\min}(A^n|B^n)_{\rho^n} = S(A|B)_{\rho}
\]
where
\[
H^c_{\min}(A^n|B^n)_{\rho^n} = \lim_{n \to \infty} \frac{1}{n} H^c_{\max}(A^n|B^n)_{\rho^n}.
\]
More precisely, for a purification \(|\psi\rangle \in \mathcal{A}(\rho)\) of \(\rho\), denote \(\mu_X := \log \|\langle \psi |X^{-1}\|\), where the inverse is the generalised inverse (restricted to the support), for \(X = B, C.\) Then, for every \(n,\)
\[
H^c_{\min}(A^n|B^n) \geq nS(A|B) - (\mu_B + \mu_C)\sqrt{n \log 2 \epsilon},
\]
\[
H^c_{\max}(A^n|B^n) \leq nS(A|B) + (\mu_B + \mu_C)\sqrt{n \log 2 \epsilon},
\]
and similar opposite bounds via Lemma 9.

**Lemma 9 (Proposition 5.5 in [27])** Let \(\rho \in \mathcal{S}(\mathcal{A}B)\) and \(\alpha, \beta \geq 0\) such that \(\alpha + \beta < \frac{\pi}{2}.\) Then,
\[
H^\alpha_{\min}(A|B)_{\rho} \leq H^\beta_{\max}(A|B)_{\rho} + \log \frac{1}{\cos^2(\alpha + \beta)}.
\]
For \(\epsilon, \delta \geq 0, \epsilon + \delta < 1\) this can be relaxed to the simpler form
\[
H^\epsilon_{\min}(A|B)_{\rho} \leq H^\delta_{\max}(A|B)_{\rho} + \log \frac{1}{1 - (\epsilon + \delta)^2}.
\]

**Lemma 10 (Dupuis [9])** Let \(\rho \in \mathcal{S}(\mathcal{A}B)\) and \(0 \leq \epsilon \leq 1.\) Then,
\[
H^{\frac{1}{1-\epsilon}}(A|B)_{\rho} \leq H^\epsilon_{\max}(A|B)_{\rho},
\]
which can be rewritten and relaxed into the form \((0 \leq \delta \leq 1)\)
\[
H^\delta_{\max}(A|B)_{\rho} \leq H^{\frac{1}{1-\epsilon^2}}(A|B)_{\rho} \leq H^{1-\frac{\epsilon^2}{2}}_{\min}(A|B)_{\rho}.
\]

**III. Discussion**

We showed that the min-entropic machinery employed in the analysis of degradable quantum channels [4, 7, 19] can be used equally, if not more easily, to obtain a pretty strong converse for degraded quantum wiretap channels of the cqq kind; the reason for focusing on this class of channels lies in the availability of a single-letter formula.

For degraded ccc-wiretap channels, i.e., the original Wyner model, Hayashi, Tyagi and Watanabe [15] have found a very elegant argument, via hypothesis testing, to give the tighter result that if \(\epsilon + \delta < 1,\) then \(\lim_{n \to \infty} \frac{1}{n} \log M(n,\epsilon,\delta) \geq P(W).\) In fact, by phrasing error and privacy in terms of the trace distance \(D(\rho,\sigma) = \frac{1}{2}\|\rho - \sigma\|_1 \leq P(\rho,\sigma),\) they show that this holds if and only if: For \(1 - \delta \leq \epsilon < 1\) the above limit gives the classical capacity \(C(W)\) of the channel from Alice to Bob. It seems however that their technique does not easily generalize to cqq-channels, as it exploits the classical nature of the output signals.

Similarly, if \(\epsilon\) and \(\delta\) are “too big”, namely \(\sqrt{1 - \delta^2} \leq \epsilon < 1,\) we can easily see that Theorem 2 breaks: In that case,
\[
\lim_{n \to \infty} \frac{1}{n} \log M(n,\epsilon,\delta) = C(W),
\]
where $C(W)$ is the classical capacity of the cq-channel $W : x \mapsto \rho_x^n$. Namely, given any fixed $x^n_0$, then for an asymptotically error-free and capacity-achieving code $x^n(m)$, with $m = 1, \ldots, N = 2^{nC(W) - o(n)}$, consider encoding message $m$ by the mixture $\varepsilon^2 |x(m)|_n |x^n(m)| + (1 - \varepsilon^2) |x^n_0|_n$. This scheme has transmission error arbitrarily close to $\varepsilon$ and privacy error $\leq \sqrt{1 - \varepsilon^2} \leq \delta$.

By ignoring the privacy constraint, our Theorem 2 includes a proof of the strong converse for the classical capacity of cq-channels [20], [35]: simply consider a trivial eavesdropper and a proof of the strong converse for the classical capacity of cq-channels.

![Fig. 1. Transmission error $\epsilon$ vs. privacy error $\delta$. Below the straight line we have a strong converse, above the circle the strong converse cannot hold.](image)

We leave it as an open problem to try and close the gap between the two regimes (see Fig. 1).

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