A GELFAND MODEL FOR WREATH PRODUCTS

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Abstract. A Gelafand model for wreath products \( Z_r \wr S_n \) is constructed. The proof relies on a combinatorial interpretation of the characters of the model, extending a classical result of Frobenius and Schur.

1. Introduction

A complex representation of a group \( G \) is called a Gelfand model for \( G \), or simply a model, if it is equivalent to the multiplicity-free direct sum of all the irreducible representations of \( G \). The problem of constructing models was introduced by Bernstein, Gelfand and Gelfand \( [6] \). Constructions of models for the symmetric group, using representations induced from centralizers, were found by Klyachko \( [13,14] \) and by Inglis, Richardson and Saxl \( [10] \); see also \( [5,17,3,2,4] \).

In this paper we determine an explicit and simple combinatorial action which gives a model for wreath products \( Z_r \wr S_n \), and in particular for the Weyl groups of type \( B \). For \( r = 1 \) (i.e., for the symmetric group) the construction is identical with the one given in \( [15,1] \). The proof relies on a combinatorial interpretation of the characters, extending a classical result of Frobenius and Schur.

If all the (irreducible) representations of a finite group are real then, by a result of Frobenius and Schur, the character-value of a model at a group element is the number of square roots of this element in the group. We are concerned in this paper with \( G(r,n) = Z_r \wr S_n \), the wreath product of a cyclic group \( Z_r \) with a symmetric group \( S_n \). For \( r > 2 \) this group is not real, and Frobenius’ theorem does not apply. It will be shown that the character-value of a model at an element of \( G(r,n) \) is the number of “absolute square roots” of this element in the group; see Theorem \( [5,3] \) below.

The rest of the paper is organized as follows. The construction of the model is described in Subsection \( [1.1] \). Necessary preliminaries and notation are given in Section \( 2 \). The combinatorial interpretation of the characters of the model is described in Section \( 3 \) (Theorem \( [3,3] \)). Two proofs for this interpretation are given. A direct combinatorial proof, using the Murnaghan-Nakayama rule, is given in Section \( 4 \). The second proof combines the properties of the generalized Robinson-Schensted algorithm for wreath products, due to Stanton and White,

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with a generalized Frobenius-Schur formula due to Bump and Ginzburg; see Section 5. The main theorem (Theorem 1.2) is proved in Section 6. The proof applies generalized Frobenius-Schur character formula (Theorem 3.4) together with Corollary 4.3. Section 7 ends the paper with final remarks and open problems.

1.1. Main Result.

Definition 1.1. Consider the natural representation $\varphi : \mathbb{Z}_r \wr S_n \to GL_n(\mathbb{C})$. An element $\pi \in \mathbb{Z}_r \wr S_n$ is called symmetric if $\varphi(\pi)$ is a (complex) symmetric matrix, that is $\varphi(\pi)^t = \varphi(\pi)$.

Denote by $I_{r,n}$ - the set of symmetric elements in $\mathbb{Z}_r \wr S_n$, and let $V_{r,n} := \text{span}_\mathbb{Q}\{C_v \mid v \in I_{r,n}\}$ be a vector space over $\mathbb{Q}$ with a basis indexed by the symmetric elements.

Recall that that each element $v \in \mathbb{Z}_r \wr S_n$ may be represented by a pair $(\sigma, z)$ where $\sigma \in S_n$ and $z \in \mathbb{Z}_n$. Denote $|v| := \sigma$ and $\omega := e^{2\pi i/r}$. Let $S$ be the standard generating set of simple complex reflections in $\mathbb{Z}_r \wr S_n$: namely, $S = \{s_0, s_1, \ldots, s_{n-1}\}$ where $s_0 = ([1, 2, \ldots, n], (1, 0, \ldots, 0))$ and $s_i = ([1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n], (0, \ldots, 0))$ ($i > 0$). Note that for $r > 2$ $s_0$ is not an involution.

Define a map $\rho : S \to GL(V_{r,n})$ by

$$\rho(s_i)C_v := \text{sign}(i; v) \cdot C_{s_i vs_i}, \quad (0 \leq i \leq n-1, v \in I_{r,n}),$$

where

$$\text{sign}(i; v) := \begin{cases} -1, & \text{if } s_i vs_i = v \text{ and } s_i \in \text{Des}(|v|); \\ 1, & \text{otherwise} \end{cases} \quad (\forall i > 0)$$

(namely, sign$(i, v) = -1$ iff $|v| \in S_n$ permutes $i$ and $i+1$), while

$$\text{sign}(0; v) := \begin{cases} -1, & \text{if } v(1) = 1 \cdot \omega^{-1} \text{ and } r \text{ is even}; \\ 1, & \text{otherwise}, \end{cases}$$

In particular, if $r$ is odd then $\text{sign}(0; v)$ is always 1.

Using a generalized Frobenius-Schur character formula (Theorem 3.4) together with Corollary 4.3 we prove

Theorem 1.2. $\rho$ extends to a Gelfand model for $\mathbb{Z}_r \wr S_n$.

Remark 1.3. Note that for $r = 2$ (i.e., for the Weyl group of type $B$)

$$\text{sign}(0; v) := \begin{cases} -1, & \text{if } s_0 vs_0 = v \text{ and } s_0 \in \text{Des}(v); \\ 1, & \text{otherwise}. \end{cases}$$

Thus Theorem 1.2 implies [1, Theorem 5.1.1], which was stated there without proof.
2. Preliminaries and Notation

Let $S_n$ be the symmetric group on $n$ letters, $\mathbb{Z}_r$ the cyclic group of order $r$ (realized as, the additive group of integers modulo $r$), and $G(r, n) = \mathbb{Z}_r \wr S_n$ their wreath product:

$$G(r, n) := \{ g = (\sigma, (c_1, \ldots, c_n)) \mid \sigma \in S_n, c_i \in \mathbb{Z}_r \ (\forall i) \}$$

with the group operation

$$(\sigma, (c_1, \ldots, c_n)) \cdot (\tau, (d_1, \ldots, d_n)) := (\sigma \tau, (c_{\tau^{-1}(1)} + d_1, \ldots, c_{\tau^{-1}(n)} + d_n)).$$

The Murnaghan-Nakayama rule is an explicit formula for the character values of irreducible representations of $S_n$ (and of $G(r, n)$). We shall first describe the formula for $S_n$ (for later use in our proofs), and then give its generalization to $G(r, n)$.

A rim hook tableau of shape $\lambda$ is a sequence

$$\emptyset = \lambda^{(0)} \subset \ldots \subset \lambda^{(t)} = \lambda$$

of Young diagrams such that each consecutive difference $rh_i := \lambda^{(i)} \setminus \lambda^{(i-1)}$ ($1 \leq i \leq t$) is a non-empty rim hook (or border strip), namely a connected skew diagram “of width 1” (i.e., containing no $2 \times 2$ square). The sequence can be described by one tableau $T$ of shape $\lambda$ in which the cells of each rim hook $rh_i$ are marked $i$. The length $l(rh_i)$ of a rim hook $rh_i$ is the number of cells it contains; its height $ht(rh_i)$ is the height difference between its two extreme cells.

**Proposition 2.1.** (Murnaghan-Nakayama rule for $S_n$) Fix an ordering $c = (c_1, \ldots, c_m)$ of the disjoint cycles of a permutation $\sigma \in S_n$, and let $l(c_i)$ be the length of $c_i$. For any partition $\lambda$ of $n$ let $\chi^{\lambda}$ be the corresponding irreducible character of $S_n$. Then

$$\chi^{\lambda}(\sigma) = \sum_{T \in RHT_c(\lambda)} \prod_{i=1}^{m} (-1)^{ht(rh_i)},$$

where $RHT_c(\lambda)$ is the set of all rim hook tableaux of shape $\lambda$ with $l(rh_i) = l(c_i)$ ($\forall i$).

In order to describe the characters of $G(r, n)$ let us recall the notions of $r$-partite partitions and tableaux. An $r$-partite partition of $n$ is an $r$-tuple $\lambda = (\lambda_0, \ldots, \lambda_{r-1})$ such that each $\lambda_i$ is a partition of a nonnegative integer $n_i$ and $n_0 + \ldots + n_{r-1} = n$. (We shall usually use boldface to denote $r$-partite concepts.) An $r$-partite standard Young tableau of shape $\lambda$ is obtained by inserting the integers $1, \ldots, n$ bijectively into the cells of the corresponding diagrams such that entries increase along each row and column of each diagram. An $r$-partite rim hook tableau of shape $\lambda$ is a sequence

$$\emptyset = \lambda^{(0)} \subset \ldots \subset \lambda^{(t)} = \lambda$$
of \( r \)-partite partitions (diagrams) such that each consecutive difference \( r\mathbf{h}_i := \lambda^{(i)} \setminus \lambda^{(i-1)} \) \((1 \leq i \leq t)\), as an \( r \)-tuple of skew shapes, has \( r-1 \) empty parts and one non-empty part which is a rim hook \( \mathbf{r}\mathbf{h}_i \): \( \mathbf{r}\mathbf{h}_i = (\ldots, \emptyset, r\mathbf{h}_i, \emptyset, \ldots) \). Let \( f(i) \in [0, r-1] \) be the index of the non-empty part of \( r\mathbf{h}_i \). Again, an \( r \)-partite rim hook tableau can be described by an \( r \)-partite tableau in which the cells of each rim hook \( r\mathbf{h}_i \) are marked \( i \) \((1 \leq i \leq t)\).

The conjugacy classes of \( G(r, n) \) are described by the cycle structure of the underlying permutations in \( S_n \), sub-classified by the sum of colors (in \( \mathbb{Z}_r \)) in each cycle. These correspond to \( r \)-partite partitions. The irreducible representations of \( G(r, n) \) may be indexed by the same combinatorial objects. A construction of the irreducible representation \( S^\lambda \) indexed by each \( r \)-partite partition \( \lambda \) was given by Specht in the thirties.

The dimension of \( S^\lambda \) is equal to the number of \( r \)-partite standard Young tableaux of shape \( \lambda \). It follows that the number of pairs of \( r \)-partite standard Young tableaux of the same shape is equal to the cardinality of \( G(r, n) \). A bijective proof was given by Stanton and White [18], using a generalized Robinson-Schensted algorithm; see Section 5.

**Proposition 2.2.** (Murnaghan-Nakayama rule for \( \mathbb{Z}_r \wr S_n \)) Fix an ordering \( c = (c_1, \ldots, c_m) \) of the disjoint cycles of a colored permutation \( g \in \mathbb{Z}_r \wr S_n \). Let \( \ell(c_i) \) be the length of \( c_i \) and let \( z(c_i) \in \mathbb{Z}_r \) be its color (the sum of colors of its elements). For any \( r \)-partite partition \( \lambda \) of \( n \), let \( \chi^\lambda \) be the corresponding irreducible character of \( \mathbb{Z}_r \wr S_n \). Then

\[
\chi^\lambda(g) = \sum_{T \in RHT_{e}(\lambda)} \prod_{i=1}^{m} (-1)^{\ell(\mathbf{r}\mathbf{h}_i)} \omega^{f(i) z(c_i)},
\]

where \( RHT_{e}(\lambda) \) is the set of all \( r \)-partite rim hook tableaux of shape \( \lambda \) such that \( \ell(\mathbf{r}\mathbf{h}_i) = \ell(c_i) \) \((\forall i)\); \( f(i) \in [0, r-1] \) is the index of the nonempty part \( r\mathbf{h}_i \) of \( \mathbf{r}\mathbf{h}_i \), as above; and \( \omega := e^{2\pi i/r} \).

### 3. A Character Formula

Let \( \text{Irr}(G) \) be the set of irreducible complex characters of a finite group \( G \). A classical result of Frobenius and Schur [8] (see, e.g., [11]) is

**Proposition 3.1.** (Frobenius-Schur) For any finite group \( G \) and any \( g \in G \),

\[
\#\{v \in G \mid v^2 = g\} = \sum_{\chi \in \text{Irr}(G)} \epsilon(\chi) \chi(g),
\]

where

\[
\epsilon(\chi) := \begin{cases} 
1, & \text{if } \chi \text{ is afforded by a real representation;} \\
-1, & \text{if } \chi \text{ is real-valued, but is not afforded by a real representation;} \\
0, & \text{if } \chi \text{ is not real-valued.}
\end{cases}
\]
In particular, if every character of $G$ is afforded by a real representation then
\[ \sum_{\chi \in \text{Irr}(G)} \chi(g) = \# \{ v \in G \mid v^2 = g \} \quad (\forall g \in G). \]

For $r > 2$ the group $G(r, n)$ has non-real representations, so that a Gelfand model for it does not give the number of square roots of an element. What does it give?

**Definition 3.2.**
For $v = (\sigma, (z_1, \ldots, z_n)) \in G(r, n)$ ($\sigma \in S_n$, $z_i \in \mathbb{Z}_r \, (\forall i))$, define the bar operation
\[ \bar{v} := (\sigma, (-z_1, \ldots, -z_n)). \]

An element $v \in G(r, n)$ is an absolute square root of $g \in G(r, n)$ if $v \cdot \bar{v} = g$. An element $v \in G(r, n)$ is an absolute involution if $v \cdot \bar{v} = \text{id}$.

**Remark 3.3.** Elements of $G(r, n)$ may also be represented by monomial matrices: $v = (\sigma, (z_1, \ldots, z_n))$ corresponds to $M = (m_{ij})$, where
\[ m_{ij} = \begin{cases} \omega^{z_j}, & \text{if } i = \sigma(j); \\ 0, & \text{otherwise.} \end{cases} \]
Then $\bar{v}$ corresponds to $\bar{M} = (\bar{m}_{ij})$, the (entry-wise) complex conjugate of $M$.

**Theorem 3.4.**
For any $g \in G(r, n) = \mathbb{Z}_r \wr S_n$,
\[ \sum_{\chi \in \text{Irr}(G)} \chi(g) = \# \{ v \in G \mid v \cdot \bar{v} = g \}. \]

In particular, these sums are nonnegative integers.

Two proofs of Theorem 3.4 will be given in the next two sections.

### 4. A Combinatorial Proof of Theorem 3.4

A direct proof, using the Murnaghan-Nakayama rule, is given in this section.

**Lemma 4.1.** Let $g \in G = G(r, n)$, and fix an ordering $c = (c_1, \ldots, c_m)$ of the disjoint cycles of $g$. Then
\[ \sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_f \omega^{\alpha(f)} \prod_{j=0}^{r-1} \sum_{\lambda_j = n_j} \chi_{\lambda_j}(\sigma_j), \]
where the sum is over all functions $f : [m] \to [0, r - 1]$, $z(c_i)$ and $l(c_i)$ are as in Proposition 2.2,
\[ \alpha(f) := \sum_{i=1}^{m} f(i) \cdot z(c_i) \in \mathbb{Z}_r, \]
\[ n_j := \sum_{i \in f^{-1}(j)} l(c_i) \quad (0 \leq j \leq r - 1), \]
and $\sigma_j \in S_{n_j}$ is the product of all disjoint cycles $|c_i|$ with $f(i) = j$ ($0 \leq j \leq r - 1$).
Proof. By Proposition 2.2 (the Murnaghan-Nakayama rule for $G(r,n)$),
\[
\sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{\lambda} \chi^\lambda(g) = \sum_{\lambda} \prod_{T \in \text{RHT}_r(\lambda)} \prod_{i=1}^{m} (-1)^{\text{ht}(rh_i)} \omega^f(i) \cdot z(c_i),
\]
Recall that each $r$-partite rim hook tableau $T$ determines a function $f : [m] \to [0, r-1]$, where $f(i)$ is the index of the tableau to which a rim hook is added in step $i$, corresponding to cycle $c_i$ ($1 \leq i \leq m$). We shall change the order of summation by first summing over the possible functions $f$. The function $f$ determines which cycles $c_i$ “go” to each tableau $T_j$ ($0 \leq j \leq r-1$), and therefore also the size
\[
n_j := \sum_{i \in f^{-1}(j)} \ell(c_i)
\]
of this tableau (but not its shape $\lambda_j$). Also, the product
\[
\omega^\alpha(f) := \prod_{i=1}^{m} \omega^{f(i) \cdot z(c_i)}
\]
depends only on the function $f$. Thus
\[
\sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{f} \omega^\alpha(f) \prod_{j=0}^{r-1} \sum_{T_j \in \text{RHT}_{r\ell(j)}(\lambda_j)} \prod_{i \in f^{-1}(j)} (-1)^{\text{ht}(rh_i)},
\]
where $|c(j)|$ is the sequence of cycles $|c_i|$ with $f(i) = j$, ordered by increasing $i$.

The expression after $\omega^\alpha(f)$ is clearly color-free, and depends only on $\sigma := |g| \in S_n$. Given $\sigma$ and $f$, define
\[
\sigma_j := \text{product of all disjoint cycles } |c_i| \text{ with } f(i) = j \quad (1 \leq j \leq r-1).
\]
Then $\sigma_j$ permutes a set of size $n_j$, and by abuse of language we can write $\sigma_j \in S_{n_j}$.

By Proposition 2.1 (the Murnaghan-Nakayama rule for $S_n$) we conclude
\[
\sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{f} \omega^\alpha(f) \prod_{j=0}^{r-1} \sum_{\lambda_j \vdash n_j} \chi^\lambda(\sigma_j),
\]
as required. \qed

Observation 4.2. Let $v \in G = G(r, d)$ be a single colored cycle, and let $w := v \cdot \bar{v}$.

1. If $d$ is odd then $w$ is a single colored cycle (of length $d$) with $z(w) = 0$. Any such $w$ is obtained from $r$ possible cycles $v$.

2. If $d$ is even then $w$ is a product of two disjoint colored cycles $w_1$ and $w_2$, each of length $d/2$, and $z(w_1) + z(w_2) = 0$. Any such $w$ is obtained from $rd/2$ possible cycles $v$.

Corollary 4.3. Let $g \in G = G(r, n)$, and fix an ordering $c = (c_1, \ldots, c_m)$ of the disjoint cycles of $g$. Then
\[
\# \{ v \in G \mid v \cdot \bar{v} = g \} = \prod_{d=1}^{\infty} N_d(c),
\]
where
\[ N_d(c) := \begin{cases} \sum_{P \in \Pi^1_d(c)} (dr)^{n_2(P)} r^{n_1(P)}, & \text{if } d \text{ is odd;} \\
\sum_{P \in \Pi^2_d(c)} (dr)^{n_2(P)}, & \text{if } d \text{ is even.} \end{cases} \]
Here \( \Pi^1_d(c) \) is the set of all partitions \( P \) of the set \( \{ i \mid l(c_i) = d \} \) into \( n_2(P) \) pairs and \( n_1(P) \) singletons such that \( z(c_i) + z(c_j) = 0 \) for each pair \( \{i, j\} \) in \( P \) and \( z(c_i) = 0 \) for each singleton \( \{i\} \) in \( P \). \( \Pi^2_d(c) \) is defined similarly, where only pairs are allowed.

**Corollary 4.4.** \((r = 1)\) Let \( \sigma \in S_n \), and fix an ordering \( c = (c_1, \ldots, c_m) \) of the disjoint cycles of \( \sigma \). Then
\[ \# \{ v \in S_n \mid v^2 = \sigma \} = \prod_{d=1}^{\infty} N_d(c), \]
where
\[ N_d(c) := \begin{cases} \sum_{P \in \Pi^1_d(c)} d^{n_2(P)}, & \text{if } d \text{ is odd;} \\
\sum_{P \in \Pi^2_d(c)} d^{n_2(P)}, & \text{if } d \text{ is even.} \end{cases} \]
Here \( \Pi^1_d(c) \) is the set of all partitions \( P \) of the set \( \{ i \mid l(c_i) = d \} \) into \( n_2(P) \) pairs and \( n_1(P) \) singletons, and \( \Pi^2_d(c) \) is the set of all partitions of this set into \( n_2(P) \) pairs (with no singletons).

**Proof.** (of Theorem 3.4) For \( 0 \leq j \leq r - 1 \) let \( c(j) \) be the sequence of cycles \( c_i \) with \( f(i) = j \), ordered by increasing \( i \). By Lemma 4.1, Proposition 3.1 for the symmetric groups \( G = S_{n_j} \) (all of whose representations are real), and Corollary 4.4 for these groups:
\[ \sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_f \omega^a(f) \prod_{j=0}^{r-1} \sum_{\lambda_j} \chi^\lambda(\sigma_j) \]
\[ = \sum_f \omega^a(f) \prod_{j=0}^{r-1} \# \{ v_j \in S_{n_j} \mid v_j^2 = \sigma_j \} \]
\[ = \sum_f \omega^a(f) \prod_{j=0}^{r-1} \prod_{d=1}^{\infty} \sum_{P \in \Pi^2_d(c(j))} d^{n_2(P, d)}. \]
Here we used the notations of the previous lemmas and corollaries, as well as the shorthand notation
\[ \Pi^{2,(1)}_d(|c(j)|) := \begin{cases} \Pi^2_d(|c(j)|), & \text{if } d \text{ is odd;} \\
\Pi^2_d(|c(j)|), & \text{if } d \text{ is even.} \end{cases} \]
A more succinct expression is
\[ \sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_f \omega^a(f) \sum_{P \in \Pi^{2,(1)}_d(|c|)} \beta(P), \]
where \( \Pi_f^{2,(1)}(|c|) \) is the set of all partitions \( P \) of the set \([m]\) into pairs and singletons such that, for each pair \( \{i,j\} \) in \( P \), \( l(c_i) = l(c_j) \) and \( f(i) = f(j) \), and for each singleton \( \{i\} \) in \( P \) the length \( l(c_i) \) is odd; and for any such partition \( P \)

\[
\beta(P) := \prod_{d=1}^{\infty} d^{n_{2,d}(P)},
\]

where \( n_{2,d}(P) \) is the number of pairs \( \{i,j\} \) in \( P \) such that \( l(c_i) = l(c_j) = d \).

The next step is to change the order of summation:

\[
\sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{P \in \Pi_f^{2,(1)}(|c|)} \beta(P) \sum_{f \in F_P} \omega^\alpha(f),
\]

where \( \Pi_f^{2,(1)}(|c|) \) is the set of all partitions as above but \textit{without} the restriction \( f(i) = f(j) \); and where \( F_P \) is the set of all functions \( f : [m] \rightarrow [0,r-1] \) such that \( f(i) = f(j) \) whenever \( \{i,j\} \) is a pair in \( P \). This requirement means that \( f \) is constant on each part of \( P \), and therefore determines a unique function \( f' : P \rightarrow [0,r-1] \), where \( P \) is viewed as a set of pairs and singletons. For each part \( p \in P \) let \( l(p) \) be its length \((= l(c_i) = l(c_j) \) if \( p = \{i,j\} \), \( = l(c_i) \) if \( p = \{i\}\)) and \( z(p) \) its color \((= z(c_i) + z(c_j) \) if \( p = \{i,j\} \), \( = z(c_i) \) if \( p = \{i\}\)). Recalling the definition of \( \alpha(f) \) from Lemma 1.1

\[
\sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{P \in \Pi_f^{2,(1)}(|c|)} \beta(P) \sum_{f' \in P \rightarrow [0,r-1]} \prod_{p \in P} \omega^{f'(p) \cdot z(p)}
\]

\[
= \sum_{P \in \Pi_f^{2,(1)}(|c|)} \beta(P) \prod_{p \in P} \sum_{j=0}^{r-1} \omega^{j \cdot z(p)}.
\]

Now use the observation

\[
\sum_{j=0}^{r-1} \omega^{j \cdot a} = \begin{cases} r, & \text{if } 0 = a \in \mathbb{Z}_r; \\ 0, & \text{if } 0 \neq a \in \mathbb{Z}_r \end{cases}
\]

to simplify:

\[
\sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{P \in \Pi_f^{2,(1)}(|c|)} \beta(P) \cdot r^{n_2(P) + n_1(P)}.
\]

Denote by \( \Pi_f^{2,(1)}(c) \) the set of partitions \( P \in \Pi_f^{2,(1)}(|c|) \) such that \( z(p) = 0 \) for all \( p \in P \), in accordance with the notation in Corollary 1.3. Recalling the definition of \( \beta(P) \),

\[
\sum_{\chi \in \text{Irr}(G)} \chi(g) = \sum_{P \in \Pi_f^{2,(1)}(c)} \beta(P) \cdot r^{n_2(P) + n_1(P)}
\]

\[
= \prod_{d=1}^{\infty} \sum_{P_d \in \Pi_f^{2,(1)}(c)} (dr)^{n_2(P_d) \cdot r^{n_1(P_d)}}.
\]
Comparison to Corollary 4.3 completes the proof:

$$
\sum_{\chi \in \text{Irr}(G)} \chi(g) = \# \{ v \in G \mid v \cdot \bar{v} = g \}.
$$

\[\square\]

5. A Second Proof of Theorem 3.4

For a complex matrix \( A \) let \( \bar{A} \) be the matrix obtained from \( A \) by complex conjugation of each entry, and let \( A^t \) be the transposed matrix. Consider the \( n \)-dimensional natural representation \( \varphi \) of \( G(r, n) \). An element \( \pi \in G(r, n) \) is called \textit{symmetric} if \( \varphi(\pi) \) is a symmetric matrix, that is \( \varphi(\pi)^t = \varphi(\pi) \).

\textbf{Observation 5.1.} Let \( \pi = (\sigma, (z_1, \ldots, z_n)) \in G(r, n) \). Then:

1. \( \varphi(\pi) = \varphi(\bar{\pi}) \), where \( \bar{\pi} := (\sigma, (-z_1, \ldots, -z_n)) \).
2. \( \varphi(\pi)^t = \varphi(\pi^t) \), where \( \pi^t := (\bar{\pi})^{-1} \).
3. \( \pi \in G(r, n) \) is symmetric if and only if it is an absolute involution: \( \pi \cdot \bar{\pi} = \text{id} \).

Recall the notions of \( r \)-partite partitions and tableaux from Section 2. Stanton and White [18] described and studied the following generalization of the Robinson-Schensted algorithm to \( G(r, n) \). Given \( \pi \in G(r, n) \) produce a pair \((P, Q)\) of \( r \)-partite standard Young tableaux, where \( P = (P_0, \ldots, P_{r-1}) \) and \( Q = (Q_0, \ldots, Q_{r-1}) \), by mapping the letters colored by \( i \) to the \( i \)-th tableaux \( P_i \) according to the usual Robinson-Schensted algorithm; their positions are recorded in the tableaux \( Q_i \). This gives a bijection from the set of all elements in \( G(r, n) \) to the set of all pairs of \( r \)-partite standard Young tableaux of same shape.

The following lemma is a reformulation of [18, Corollary 29].

\textbf{Lemma 5.2.} For every \( \pi \in G(r, n) \)

\[ \pi \xrightarrow{\text{RS}} (P, Q) \iff \pi^t \xrightarrow{\text{RS}} (Q, P). \]

We deduce

\textbf{Corollary 5.3.} The dimension of the model of \( G(r, n) \) is equal to the number of absolute involutions in \( G(r, n) \).

\textbf{Proof.} It is well known that the dimension of the irreducible \( G(r, n) \) representation indexed by an \( r \)-partite partition \( \lambda \) is the number of \( r \)-partite SYT of shape \( \lambda \). By Lemma 5.2 the number of symmetric elements \( \pi \in G(r, n) \) is equal to the number of \( r \)-partite SYT. Observation 5.1(3) completes the proof. \[\square\]

It should be noted that corollary 5.3 is analogous to a remarkable theorem of Klyachko [13, 14] and Gow [9] regarding the dimension of the model for \( GL_n(\mathbb{F}_q) \).

The following generalization of the Frobenius-Schur theorem (Proposition 3.1) was proved by Bump and Ginzburg.
Proposition 5.4. [7, Theorem 3] Let $G$ be a finite group, let $\tau$ be an automorphism of $G$ satisfying $\tau^2 = 1$, and let $z \in G$ such that $z^2 = \text{id}$. If
\[
\sum_{\rho \in \text{Irr}(g)} \chi(id) = \# \{ w \in G : w \cdot \tau(w) = z \}
\]
then
\[
\sum_{\rho \in \text{Irr}(g)} \chi(g) = \# \{ w \in G : w \cdot \tau(w) = g \cdot z \} \quad (\forall g \in G).
\]

Now let $G := G(r,n)$, $\tau$ be the bar operation from Definition 3.2, and $z := \text{id}$. By Corollary 5.3, the assumptions of Theorem 5.4 are satisfied, implying Theorem 3.4.

\[\square\]

6. Proof of Theorem 1.2

We will first prove the theorem for odd $r$. The necessary modifications for the more complicated case of even $r$ will be indicated afterwards.

6.1. The Case of Odd $r$.

6.1.1. Part 1. We start by showing that $\rho$ can be extended to a group homomorphism. Recall the definition of the inversion set of a permutation $\sigma \in S_n$,
\[
\text{Inv}(\sigma) := \{ \{ i, j \} : (j - i) \cdot (\sigma(j) - \sigma(i)) < 0 \}.
\]
For each involution $v \in S_n$ (including $v = \text{id}$), let $\text{Pair}(v)$ be the set of pairs $\{ i, j \}$ such that $(i, j)$ is a 2-cycle of $v$.

Definition 6.1. For an element $\pi \in G(r,n)$ and an absolute involution $w \in I_{r,n}$ let
\[
\text{sign}_w(\pi, w) := (-1)^{\#(\text{Inv}(\pi) \cap \text{Pair}(|w|))}.
\]
Define a map $\rho : G(r,n) \to GL(V_{r,n})$ by
\[
\rho(\pi) C_w := \text{sign}_w(\pi, w) \cdot C_{\pi w \pi^t}, \quad (\forall \pi \in G(r,n), w \in I_{r,n}).
\]

One can verify that this definition of $\rho$ coincides, on the set $S$ of generators of $G(r,n)$, with the previous definition [11]. It thus suffices to show that $\rho$ is a group homomorphism. By definition of $\rho$, it suffices to prove that
\begin{equation}
(2) \quad \text{sign}_w(\pi_1 \pi_2, w) = \text{sign}_w(\pi_1, w) \cdot \text{sign}_w(\pi_2, \pi_1 \pi_2^t).
\end{equation}
Indeed, let $X[\text{condition}]$ be $-1$ if the condition holds, and $1$ otherwise. Then, for any $\pi_1, \pi_2 \in G(r,n)$, $w \in I_{r,n}$ and $i \neq j$, denoting $\sigma_1 := |\pi_1|$, $\sigma_2 := |\pi_2|$ and $v := |w|$: \[\{ i, j \} \in \text{Pair}(v) \iff \{ \sigma_1(i), \sigma_1(j) \} \in \text{Pair}(\sigma_1 v \sigma_1^{-1})\]
and
\[
X[\{ i, j \} \in \text{Inv}(\sigma_2 \sigma_1)] = X[\{ i, j \} \in \text{Inv}(\sigma_1)] \cdot X[\{ \sigma_1(i), \sigma_1(j) \} \in \text{Inv}(\sigma_2)].
\]
Thus
\[ X[\{i, j\} \in \text{Inv}(\pi_2 \pi_1)] \cap \text{Pair}(\pi) = X[\{i, j\} \in \text{Inv}(\pi_1)] \cap \text{Pair}(\pi) \cdot X[\{\pi_1(i), |\pi_1(j)| \} \in \text{Inv}(\pi_2)] \cap \text{Pair}(\pi_1 \pi_1')] \]
which implies (2) by taking a product over all possible pairs \(\{i, j\}\).

6.1.2. Part 2. For an arbitrary element \(\pi \in G(r, n)\) let
\[ \text{Fix}(\pi) := \{w \in I_{r,n} : \pi w \pi^t = w\} = \{w \in I_{r,n} : \pi w = w \bar{\pi}\}. \]
We shall prove that \(\rho\) is a model for \(G(r, n)\) by showing that
\[ \sum_{w \in \text{Fix}(\pi)} \text{sign}_o(\pi, w) = \prod_{\chi \in \text{Irr}(G(r, n))} \chi(\pi) \quad (\forall \pi \in G(r, n)). \]
Let \(\pi = (\sigma, z) \in G(r, n)\). For each \(d \geq 1\) let \(\text{Supp}_d(\pi)\) be the set of all \(i \in [n]\) that belong to a cycle of length \(d\) in \(\sigma\), and let \(\sigma_d, z_d\) and \(\pi_d = (\sigma_d, z_d)\) be the restrictions to \(\text{Supp}_d(\pi)\) of \(\sigma, z\) and \(\pi\). It is clear that if \(w \in \text{Fix}(\pi)\) and \(\{i, j\} \in \text{Pair}(\pi)\) then, since \(|w|\) and \(\sigma\) commute, \(i\) and \(j\) belong to cycles of the same length in \(\sigma\). Thus we can write (uniquely) \(w = w_1 \cdots w_n\), where each \(w_d\) is supported within \(\text{Supp}_d(\pi)\). It is also clear that
\[ w_d \in \text{Fix}(\pi_d) \quad (\forall d) \]
and
\[ \text{sign}_o(\pi, w) = \prod_{d=1}^{n} \text{sign}_o(\pi_d, w_d), \]
so that
\[ \sum_{w \in \text{Fix}(\pi)} \text{sign}_o(\pi, w) = \prod_{d=1}^{n} \sum_{w_d \in \text{Fix}(\pi_d)} \text{sign}_o(\pi_d, w_d). \]
On the other hand, by Theorem 3.4 and Corollary 4.3
\[ \sum_{\chi \in \text{Irr}(G(r, n))} \chi(\pi) = \#\{g \in G(r, n) \mid g \cdot \bar{g} = \pi\} = \prod_{d=1}^{n} \#\{g_d \in G(r, n_d) \mid g_d \cdot \bar{g_d} = \pi_d\}, \]
where \(n_d\) is the size of \(\text{Supp}_d(\pi)\).

These observations about the multiplicative property of both sides of (3) make it sufficient to prove (4) for \(\pi\) with all cycles of the same length. Indeed, let \(\pi \in G(r, md)\) have \(m\) cycles, each of length \(d\), ordered arbitrarily \(c = (c_1, \ldots, c_m)\). Again, by Theorem 3.4 and Corollary 4.3
\[ \sum_{\chi \in \text{Irr}(G)} \chi(\pi) = \sum_{P \in \Pi_2^{(1)}(c)} (dr)^{n_2(P)} r^{n_1(P)} \]
\[ = \begin{cases} \sum_{P \in \Pi_2^{(1)}(c)} (dr)^{n_2(P)} r^{n_1(P)}, & \text{if } d \text{ is odd;} \\ \sum_{P \in \Pi_2^{(1)}(c)} (dr)^{n_2(P)}, & \text{if } d \text{ is even.} \end{cases} \]
Here $\Pi^{(1)}(c)$ is the set of all partitions $P$ of the set $[m]$ into $n_2(P)$ pairs and $n_1(P)$ singletons such that $z(c_i) + z(c_j) = 0$ for each pair $\{i, j\}$ in $P$ and $z(c_i) = 0$ for each singleton $\{i\}$ in $P$, and where we require $n_1(P) = 0$ if $d$ is even.

It remains to show that for every $\pi \in G(r, md)$ of cycle type $d^m$ as above,

$$
\sum_{w \in \text{Fix}(\pi)} \text{sign}_o(\pi, w) = \sum_{P \in \Pi^{(1)}(c)} (dr)^{n_2(P)} r^{n_1(P)}.
$$

Let $w \in \text{Fix}(\pi)$. Then $|w| \in S_{md}$ is an involution. Choosing $i_0 \in [md]$ there are three cases to analyze:

**Case (1):** $|w(i_0)| = 0$.

Then there exists a cycle $c = (i_0, \ldots, i_{d-1})$ of $|\pi|$ such that $|w(i_t)| = i_t$ for all $0 \leq t \leq d-1$ and

$$
\pi w = w \pi \iff z_\pi(i_t) + z_\pi(i_t) = -z_\pi(i_t) + z_\pi(i_{t+1}) \quad (0 \leq t \leq d-1),
$$

where $t+1$ is computed mod $d$. Summing over $0 \leq t \leq d-1$ gives

$$
2z_\pi(c) = 2 \sum_{t=0}^{d-1} z_\pi(i_t) = 0
$$

(in $\mathbb{Z}_r$). Since $r$ is odd, this implies that

$$
z_\pi(c) = 0.
$$

Given $\pi$, a choice of $z_w(i_0) \in \mathbb{Z}_r$ determines uniquely $z_w(i_t)$ for all $0 \leq t \leq d-1$.

**Case (2):** $i_0$ and $|w(i_0)|$ are distinct and belong to different cycles of $|\pi|$.

Then there are disjoint cycles $c_1 = (i_0, \ldots, i_{d-1})$ and $c_2 = (j_0, \ldots, j_{d-1})$ of $|\pi|$ such that $|w(i_t)| = j_t$ (and vice versa) for every $0 \leq t \leq d-1$. In this case

$$
\pi w = w \pi \iff z_\pi(i_t) + z_\pi(j_t) = -z_\pi(i_t) + z_\pi(i_{t+1}) \quad \text{and}
$$

$$
z_\pi(j_t) + z_\pi(i_t) = -z_\pi(j_t) + z_\pi(j_{t+1}) \quad (0 \leq t \leq d-1),
$$

where $t+1$ is computed mod $d$. Summing any of these over $0 \leq t \leq d-1$ gives

$$
z_\pi(c_1) + z_\pi(c_2) = \sum_{t=0}^{d-1} [z_\pi(i_t) + z_\pi(j_t)] = 0.
$$

Given $\pi$, a choice of $z_w(i_0) \in \mathbb{Z}_r$ determines uniquely $z_w(i_t)$ and $z_w(j_t)$ for all $0 \leq t \leq d-1$ (since $z_w(j_t) = z_w(i_t)$ by the condition $w = w^t$).

**Case (3):** $i_0$ and $|w(i_0)|$ are distinct but belong to the same cycle of $|\pi|$.

This is possible only if $d = 2e$ is even. Then there is a cycle $c = (i_0, \ldots, i_{2e-1})$ of $|\pi|$ such that $|w(i_t)| = i_{t+e}$ for every $0 \leq t \leq e-1$. In this case

$$
\pi w = w \pi \iff z_\pi(i_t) + z_\pi(i_{t+e}) = -z_\pi(i_t) + z_\pi(i_{t+1}) \quad (0 \leq t \leq 2e-1),
$$
where \( t + e \) and \( t + 1 \) are computed mod \( d \). Summing over \( 0 \leq t \leq e - 1 \), remembering that \( z_w(i_{t+e}) = z_w(i_t) \) since \( w = w^t \), yields

\[
z_\pi(e) = \sum_{t=0}^{2e-1} z_\pi(i_t) = 0,
\]

and a choice of \( z_w(i_0) \in \mathbb{Z}_r \) determines uniquely \( z_w(i_t) \) for all \( 0 \leq t \leq 2e - 1 \).

Summing up, each \( w \in \text{Fix}(\pi) \) defines a partition \( P \) of the set of cycles of \( \pi \) into pairs (Cases (2) and (3)) and singletons (Case (1)), with the same color restrictions as in the definition of \( \Pi^{2,1}(c) \); but, as we shall see, equality (4) holds on the level of a single partition \( P \) only if both \( r \) and \( d \) are odd. Further considerations will be needed in the other cases.

Approaching the actual computation of signs, let us make one further simplification. Since both sides of (4) are class functions of \( \pi \) (the left-hand-side being the trace of \( \rho(\pi) \)), we may assume that

\[
|\pi| = (1,2,\ldots,d)(d+1,\ldots,2d)\cdots((m-1)d+1,\ldots,md).
\]

Note that \( \text{sign}_o(\pi, w) \) depends only on \(|\pi|\) and \(|w|\).

**Observation 6.2.** Consider the above 3 cases.

1. If \(|\pi| = (1,2,\ldots,d)\) and \(|w| = (1)(2)\cdots(d)\) then

\[
\text{sign}_o(\pi, w) = 1.
\]

2. If \(|\pi| = (1,2,\ldots,d)(d+1,\ldots,2d), \ 0 \leq i \leq d - 1, \ \text{and} \ |w| = (1,d+1)(2,d+2)\cdots(d,2d)\ (\text{for } i = 0) \ \text{or} \ |w| = (1,d+1+i)(2,d+2+i)\cdots(d-i,2d)(d-i+1,d+1)\cdots(d,d+i)\ (\text{for } i > 0)\) then

\[
\text{sign}_o(\pi, w) = 1.
\]

3. If \( d = 2e \) is even, \(|\pi| = (1,2,\ldots,2e)\) and \(|w| = (1,e+1)(2,e+2)\cdots(e,2e)\) then

\[
\text{sign}_o(\pi, w) = -1.
\]

If \( d \) is odd then Case (3) does not occur. It then follows from Observation 6.2(1)(2) that \( \text{sign}_o(\pi, w) = 1 \) for all \( w \in \text{Fix}(\pi) \). Thus

\[
\sum_{w \in \text{Fix}(\pi)} \text{sign}_o(\pi, w) = \#\text{Fix}(\pi) = \sum_{P \in \Pi^{2,1}(c)} (dr)^{n_2(P)}r^{n_1(P)},
\]

where the second equality follows from enumeration of all possible \( w \) according to Cases (1) and (2) above.

Assume now that \( d = 2e \) is even. In this case \( \text{sign}_o(\pi, w) \) may be negative, and cancellations will occur. Define a function \( \varphi: \text{Fix}(\pi) \rightarrow \text{Fix}(\pi) \) as follows: let \( w \in \text{Fix}(\pi) \). Each of the \( m \) cycles of \(|\pi|\) belongs, with respect to \(|w|\), to one of the Cases (1), (2) and (3). If all the cycles belong to Case (2), define \( \varphi(w) := w \).

Otherwise, let \( c_i = ((i-1)d+1,\ldots,(i-1)d+d) \) be the first cycle of \(|\pi|\) that
belongs to Cases (1) or (3). If it belongs to Case (1), i.e., if $|w(j)| = j$ for all $(i - 1)d + 1 \leq j \leq (i - 1)d + d$, define $w' = \varphi(w)$ by

$$z_w(j) := z_w(j) \quad (\forall j)$$

and

$$|w'(j)| := \begin{cases} j + e, & \text{if } (i - 1)d + 1 \leq j \leq (i - 1)d + e; \\ j - e, & \text{if } (i - 1)d + e + 1 \leq j \leq (i - 1)d + 2e; \\ |w(j)|, & \text{otherwise}. \end{cases}$$

If the first cycle belongs to Case (3), i.e., if $|w(j)| = j \pm e$ for all $(i - 1)d + 1 \leq j \leq (i - 1)d + d$, define $w' = \varphi(w)$ by

$$z_w(j) := z_w(j) \quad (\forall j)$$

and

$$|w'(j)| := \begin{cases} j, & \text{if } (i - 1)d + 1 \leq j \leq (i - 1)d + d; \\ |w(j)|, & \text{otherwise}. \end{cases}$$

Thus $\varphi$ toggles one of the cycles of $\pi$ between Cases (1) and (3). It is easy to see that $\varphi$ is a “sign-reversing involution” on $\text{Fix}(\pi)$, i.e., an involution satisfying

$$\text{sign}_w(\pi, \varphi(w)) = \begin{cases} \text{sign}_w(\pi, w), & \text{if } \varphi(w) = w; \\ -\text{sign}_w(\pi, w), & \text{otherwise} \end{cases} \quad (\forall w \in \text{Fix}(\pi)).$$

Thus the signs of elements $w \in \text{Fix}(\pi)$ with $\varphi(w) \neq w$ cancel each other, whereas $\text{sign}_w(\pi, w) = 1$ when $\varphi(w) = w$. It follows that, for even $d$,

$$\sum_{w \in \text{Fix}(\pi)} \text{sign}_w(\pi, w) = \# \{w \in \text{Fix}(\pi) \mid \varphi(w) = w\} = \sum_{P \in \Pi^2(e)} (dr)^{n_2(P)}.$$

This completes the proof of Theorem 1.2 for odd $r$.

6.2. The Case of Even $r$. The proof is in general similar to the proof for odd $r$, described above. We shall focus on the differences.

6.2.1. Part 1. Again, we first prove that $\rho$ extends to a group homomorphism. Partition the set $Z_r = [0, r - 1]$ into two complementary “arcs” (or “intervals”) $[0, r/2 - 1]$ and $[r/2, r - 1]$.

**Definition 6.3.** For an element $\pi \in G(r, n)$ and an absolute involution $w \in I_{r,n}$ let

$$B(\pi, w) := \{i : |w(i)| = i, z_w(i) = 2k_w(i) + 1 \text{ is odd with}$$

$$k_w(i) \in [0, r/2 - 1] \text{ and } k_w(i) + z_w(i) \in [r/2, r - 1]\},$$

and define

$$\text{sign}_w(\pi, w) := (-1)^{\#B(\pi, w)} \cdot (-1)^{\#(\text{Inv}(\pi) \cap \text{Pair}(|w|))}.$$

The second factor is the same as in Definition 6.1.
Define a map $\rho : G(r, n) \to GL(V_{r, n})$ by
$$\rho(\pi)C_w := \text{sign}_e(\pi, w) \cdot C_{w\pi} \quad (\forall \pi \in G(r, n), w \in I_{r, n}).$$

Again, one can verify that this definition of $\rho$ coincides, on the set $S$ of generators of $G(r, n)$, with the previous definition (1). It thus suffices to show that $\rho$ is a group homomorphism, namely that
$$\text{sign}_e(\pi_2\pi_1, w) = \text{sign}_e(\pi_1, w) \cdot \text{sign}_e(\pi_2, \pi_1w\pi_1^i).$$

By the proof of Part 1 for odd $r$, it suffices to prove that
$$(-1)^{|B(\pi_2\pi_1, w)|} = (-1)^{|B(\pi_1, w)|} \cdot (-1)^{|B(\pi_2, \pi_1w\pi_1^i)|}.$$  

Indeed, letting again $X[\text{condition}]$ be $-1$ if the condition holds and $1$ otherwise, it suffices to prove that for every $1 \leq i \leq n$

$$(5) \quad X[i \in B(\pi_2, \pi_1w\pi_1^i)] = X[i \in B(\pi_1, w)] \cdot X[[\pi_1(i)] \in B(\pi_2, \pi_1w\pi_1^i)].$$

Note that
$$|w(i)| = i \iff |\pi_1w\pi_1^i(\pi_1(i))| = |\pi_1(i)|$$
and
$$z_w(i) \text{ is odd } \iff z_{\pi_1w\pi_1^i}(\pi_1(i)) \text{ is odd.}$$

Hence, in order to prove (5), we can assume that $|w(i)| = i$ and that $z_w(i)$ is odd. Denote $z_w(i) = 2k + 1$ with $k = k_w(i) \in [0, r/2 - 1] \subseteq \mathbb{Z}_r$, $m_1 := z_{\pi_1}(i) \in \mathbb{Z}_r$ and $m_2 := z_{\pi_1w\pi_1^i}(\pi_1(i)) \in \mathbb{Z}_r$. Note that $z_{\pi_1w\pi_1^i}(\pi_1(i)) = 2(k + m_1) + 1$. Define
$$k_1 := \begin{cases} k + m_1, & \text{if } k + m_1 \in [0, r/2 - 1]; \\ k + m_1 - r/2, & \text{otherwise}, \end{cases}$$
where all operations are in $\mathbb{Z}_r$. Equation (5) now reduces to
$$X[k + m_1 + m_2 \in [r/2, r - 1]] = X[k + m_1 \in [r/2, r - 1]] \cdot X[k_1 + m_2 \in [r/2, r - 1]].$$
This is obviously true, completing the first part of the proof.

6.2.2. Part 2. By the arguments used for odd $r$, in order to show that $\rho$ is a model for $G(r, n)$ it suffices to show that, for every $\pi \in G(r, md)$ with $|\pi|$ of cycle type $d^m$,

$$(6) \quad \sum_{w \in \text{Fix}(\pi)} \text{sign}_e(\pi, w) = \sum_{P \in \Sigma_{d^m}(\pi)} (dr)^{n_2(P)}r^{n_1(P)}.$$

Again, for $w \in \text{Fix}(\pi)$ and $i_0 \in [md]$, there are 3 cases to consider.

Case (1): $|w(i_0)| = i_0$.
There exists a cycle $c = (i_0, \ldots, i_{d-1})$ of $|\pi|$ such that $|w(i_t)| = i_t$ for all $0 \leq t \leq d - 1$, but the equation
$$2z_\pi(c) = 2 \sum_{i=0}^{d-1} z_\pi(i_t) = 0$$
(in $\mathbb{Z}_r$) implies, for even $r$, only that
\[ z_\pi(c) \in \{0, r/2\}. \]
The case $z_\pi(c) = r/2$ does not fit the color restrictions for $P \in \Pi^{2,(1)}(c)$, if the cycle $c$ is taken as a singleton.

**Cases (2), (3):** $|w(i_0)| \neq i_0$.

The analysis here is exactly the same as for odd $r$.

Again, since both sides of $\pi$ are class functions, we can choose $\pi$ to be any representative of its conjugacy class in $G(r, n)$. We shall require that
\[ |\pi| = (1, 2, \ldots, d)(d + 1, \ldots, 2d) \cdots ((m - 1)d + 1, \ldots, md) \]
and, moreover, that $z_\pi(i) = 0$ unless $i \equiv 1 \pmod{d}$; i.e., that $z_\pi$ of each cycle is concentrated in its smallest element.

The analogues of Observation 6.2(2)(3) are exactly as for odd $r$, since $|w|$ has no fixed points in these cases, and therefore $\#B(\pi, w) = 0$.

Consider now the situation in Observation 6.2(1). By the above assumptions on $\pi$, the only possible member of $B(\pi, w)$ is $i = 1$ (since $z_\pi(i) = 0$ for $i \neq 1$). Also, $z_\pi(1)$ is either 0 or $r/2$. If $z_\pi(1) = 0$ then $\#B(\pi, w) = 0$ for any $w \in \text{Fix}(\pi)$. If $z_\pi(1) = r/2$ then $\#B(\pi, w) = 1$ for any $w \in \text{Fix}(\pi)$ with $z_w(1)$ odd, but $\#B(\pi, w) = 0$ for any $w \in \text{Fix}(\pi)$ with $z_w(1)$ even. Since $z_w(1)$ uniquely determines $w$, we conclude that for $|\pi|$ with a single cycle, $|w|$ of Case (1),
\[ \sum_{w \in \text{Fix}(\pi)} \text{sign}_w(\pi, w) = \begin{cases} r, & \text{if } z_\pi(1) = 0; \\ 0, & \text{if } z_\pi(1) = r/2. \end{cases} \]

Thus, for $|\pi|$ of cycle type $d^n$, if $z_\pi(c) = r/2$ for some cycle $c$ then the total sign contribution of all $w \in \text{Fix}(\pi)$ which are of Case (1) on $c$ is zero, and this set of $w$-s can be discarded. The rest of the proof is exactly as for odd $r$. 

\[ \square \]

## 7. Remarks and Questions

### 7.1. RSK for wreath products.

If two absolute involutions are mapped to the same shape then their corresponding uncolored involutions in $S_n$ are conjugate. It results that all absolute involutions with fixed number of cycles form an invariant submodule of $V_{r,n}$.

We conjecture that the RSK on absolute involutions is compatible with the decomposition of the Gelfand model, namely

**Conjecture 7.1.** The submodule spanned by all absolute involutions with fixed number of cycles is a multiplicity free sum of all irreducible $\mathbb{Z}_r \wr S_n$ representations whose shape is obtained from these absolute involutions via the colored RSK.
7.2. Construction of Models. It should be noted that the construction for type $B$ may be adapted to $D_{2n+1}$.

**Question 7.2.** Give a construction of a Gelfand model for $D_{2n}$; for general complex reflection groups; for affine Weyl groups.

Finally, a $q$-deformation of the model for $S_n = Z_1 \wr S_n$, which gives a model for the Iwahori Hecke algebra of type $A$, was described in [1]. A construction of a model for the Iwahori Hecke algebra of $Z_r \wr S_n$ is desired.

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References

[1] R. M. Adin, A. Postnikov and Y. Roichman, Combinatorial Gelfand Models, J. Algebra, to appear.
[2] J. L. Aguado and J. O. Araujo, A Gelfand model for the symmetric group, Communications in Algebra 29 (2001), 1841–1851.
[3] J. O. Araujo, A Gelfand model for a Weyl group of type $B_n$, Beiträge Algebra Geom. 44 (2003), 359–373.
[4] J. O. Araujo and J. J. Bigeón, A Gelfand model for a Weyl group of type $D_n$ and the branching rules $D_n \twoheadrightarrow B_n$, J. Algebra 294 (2005), 97–116.
[5] R. W. Baddeley, Models and involution models for wreath products and certain Weyl groups, J. London Math. Soc. (2) 44 (1991), 55–74.
[6] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Models of representations of compact Lie groups (Russian), Funkcional. Anal. i Prilozhen. 9 (1975), 61–62.
[7] D. Bump and D. Ginzburg, Generalized Frobenius-Schur numbers, J. Algebra 278 (2004), 294–313.
[8] G. Frobenius and I. Schur, Über die reellen Darstellungen de rendlichen Gruppen, S’ber. Akad. Wiss. Berlin (1906), 186–208.
[9] R. Gow, Real representations of the finite orthogonal and symplectic groups of odd characteristic, J. Algebra 96 (1985), 249–274.
[10] N. F. J. Inglis, R. W. Richardson and J. Saxl, An explicit model for the complex representations of $S_n$, Arch. Math. (Basel) 54 (1990), 258–259.
[11] I. M. Isaacs, Character Theory of Finite Groups, Dover, New York, 1994.
[12] N. Kawanaka and H. Matsuyama, A twisted version of the Frobenius-Schur indicator and multiplicity-free permutation representation, Hokkaido Math. J. 19 (1990), 495–508.
[13] A. A. Klyachko, Models for complex representations of the groups $GL(n, q)$ and Weyl groups (Russian), Dokl. Akad. Nauk SSSR 261 (1981), 275–278.
[14] A. A. Klyachko, Models for complex representations of groups $GL(n, q)$ (Russian), Mat. Sb. (N.S.) 120(162) (1983), 371–386.
[15] V. Kodiyalam and D.-N. Verma, A natural representation model for symmetric groups, preprint, 2004.
[16] Y. Roichman, A recursive rule for Kazhdan-Lusztig characters, Adv. in Math. 129 (1997), 24–45.
[17] P. D. Ryan, Representations of Weyl groups of type $B$ induced from centralisers of involutions, Bull. Austral. Math. Soc. 44 (1991), 337–344.
[18] D. Stanton and D. E. White, A Schensted algorithm for rim hook tableaux, J. Combin. Theory A 40 (1985), 211–247.
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