LENS GENERALISATION OF $\tau$-FUNCTIONS FOR THE ELLIPTIC DISCRETE PAINLEVÉ EQUATION

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ABSTRACT. We propose a new bilinear Hirota equation for $\tau$-functions associated with the $E_8$ root lattice, that provides a “lens” generalisation of the $\tau$-functions for the elliptic discrete Painlevé equation. Our equations are characterized by a positive integer $r$ in addition to the usual elliptic parameters, and involve a mixture of continuous variables with additional discrete variables, the latter taking values on the $E_8$ root lattice. We construct explicit $W(E_7)$-invariant hypergeometric solutions of this bilinear Hirota equation, which are given in terms of elliptic hypergeometric sum/integrals.

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1. Introduction

In the literature many variations of the differential and discrete (difference) Painlevé equations have been found. These equations have been classified into rational, trigonometric and elliptic equations. At the top level of the hierarchy is the elliptic discrete Painlevé equation with affine Weyl group symmetry of type \( E_8^{(1)} \). This equation has been obtained from geometric considerations \[1\], and as a discrete system on the \( E_8 \) root lattice \[2\] (see \[3,4\] for relation between the two approaches, and \[5\] for a comprehensive survey).

In a recent work \[6\], Noumi has given details of the construction of ORG \( \tau \)-functions \[2\] on the \( E_8 \) lattice for the elliptic discrete Painlevé equation. The goal of this paper is to present a generalization of Noumi’s \( \tau \)-function, along with solutions of this \( \tau \)-function that are given in terms of elliptic hypergeometric sum/integrals. The latter are generalisations of elliptic hypergeometric integrals, and depend on additional discrete parameters which enter as arguments of the lens elliptic gamma function. Such functions first appeared in the study of supersymmetric gauge theories \[7\], and in recent works several elliptic hypergeometric sum/integral formulas have been studied and proven from a mathematical point of view \[8-10\]. These results motivate the construction of the corresponding lens \( \tau \)-function of this paper, which involves two copies of the \( E_8 \) root lattice and a positive integer parameter \( r \), and the resulting equations depend on the usual continuous variables, as well as additional discrete variables on the \( E_8 \) root lattice. We propose a bilinear Hirota-type equation for the \( \tau \)-function, and construct explicit solutions of the bilinear equation in terms of an elliptic hypergeometric sum/integral for a general value of the integer parameter \( r \), which is fixed throughout the paper. The hypergeometric \( \tau \)-functions of this paper are expected to provide a solution for some (not yet known) generalisation of the elliptic discrete Painlevé equation.

For the case of \( r = 1 \), the elliptic hypergeometric sum/integral used in this paper, reduces to the same elliptic hypergeometric integral which provides the hypergeometric solution of Noumi’s \( \tau \)-function \[6\]. Then it might be expected that for \( r = 1 \), the \( \tau \)-function of this paper will also reduce to Noumi’s \( \tau \)-function. Surprisingly this is not the case, since even for \( r = 1 \) the Hirota equations (and solutions) will be seen to retain the dependence on the discrete variables on the \( E_8 \) root lattice. The \( \tau \)-function of \[6\] would then appear to correspond to a possible degenerate case, where there is no
contribution of the discrete variables on the $E_8$ root lattice, in which case our $\tau$-function will take values in subsets of $\mathbb{C}^8$, as is the case for [6]. This is a rather interesting subtlety that arises here, and appears to be necessary for constructing solutions given in terms of the elliptic hypergeometric sum/integral, which will satisfy the bilinear relations and the invariance under the Weyl group $W(E_7)$.

It is expected that the results of this paper will open up many possible future research directions. For example, it would be interesting to find an explicit Hamiltonian form of the discrete Painlevé equation associated to the $\tau$-function of this paper, and to explore the various degenerations of the equations. It would also be interesting to explore the geometric aspects of these equations along the lines of Sakai’s classification [11]. In another direction, the lens elliptic gamma function, which is a central function for this paper, first appeared in the study of four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories on a circle times the lens space $S^3/Z_r$ [7]. This connection suggests that there exists an interpretation of the results of this paper in terms of supersymmetric gauge theories and associated integrable lattice models [8,11–20].

The rest of this paper is organized as follows. In Section 2 we provide definitions of the “lens” set of special functions, which generalise the special functions that appear in the theory of elliptic hypergeometric integrals. In Section 3 we define an elliptic hypergeometric sum/integral for constructing the hypergeometric $\tau$-function, and present the relevant identities that it satisfies. In Section 4 we formulate the Hirota identities for the $\tau$-function on the $E_8$ lattice, which are then decomposed into the $W(E_7)$-orbits in Section 5. In Section 6 we state the main theorem of this paper (Theorem 1), which provides an explicit $W(E_7)$-invariant, lens elliptic hypergeometric solution of the $\tau$-function. The proof of the main theorem is provided in Section 7. In the Appendices, we respectively present the derivation of the sum/integral transformation for $W(E_7)$ reflection, and provide a brief overview of the multiple Bernoulli polynomials, which are used for the definitions of the lens special functions.

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2. Lens Theta Functions and Lens Elliptic Gamma Function

In this section, the definitions of the special functions are given that play a central role in this paper. Namely, these are the lens theta function, the lens elliptic gamma function, and the lens triple gamma function.

In this paper we use the two complex parameters \( \sigma, \tau \in \mathbb{C} \), that satisfy
\[
\text{Im}(\sigma), \text{Im}(\tau) > 0.
\]

Our equations will also depend on an additional integer parameter
\[
r = 1, 2, \ldots
\]

In this paper, continuous and discrete variables are denoted by a pair \( X = (x, x) \), where \( x, x \) correspond to the continuous and discrete variables respectively.

2.1. Lens Theta Functions. The two lens theta functions \( \theta_{\tau}, \theta_{\sigma} \), are defined by \([8, 10]\)
\[
\begin{align*}
\theta_{\tau}(z, z; \sigma, \tau) &:= e^{\phi_{\tau}(z, z; \sigma, \tau)} \theta(e^{-2\pi i z} e^{2\pi i r |z|}, e^{2\pi i \tau}), \\
\theta_{\sigma}(z, z; \sigma, \tau) &:= e^{\phi_{\sigma}(z, z; \sigma, \tau)} \theta(e^{2\pi i z} e^{2\pi i r |z|}, e^{2\pi i \sigma}),
\end{align*}
\]
where for \( q = e^{2\pi i \tau}, \theta(z | q) \) is the regular theta function
\[
\theta(z | q) = (z; q)_{\infty} (qz^{-1}; q)_{\infty}, \quad (z; q)_{\infty} = \prod_{j=0}^{\infty} (1 - zq^j),
\]
and the normalization factors are given by
\[
\begin{align*}
\phi_{\tau}(z, z; \sigma, \tau) &:= \frac{\pi i}{6r} (3r + 1 - 2z)(2z + 1) - (r^2 - 1)(\sigma - \tau - 1) - 6z(r - z)(\tau + 1), \\
\phi_{\sigma}(z, z; \sigma, \tau) &:= -\frac{\pi i}{6r} (3r - 1 - 2z)(2z - 1) - (r^2 - 1)(\sigma - \tau - 1) + 6z(r - z)(\sigma - 1).
\end{align*}
\]
For \( r = 1 \), the lens theta functions \((3)\) reduce to regular theta functions
\[
\begin{align*}
\theta_{\tau}(z, z; \sigma, \tau) \mid_{r=1} &= \theta(e^{2\pi i z} | e^{2\pi i \tau}), \\
\theta_{\sigma}(z, z; \sigma, \tau) \mid_{r=1} &= \theta(e^{2\pi i z} | e^{2\pi i \sigma}),
\end{align*}
\]
Note that the theta functions \( \theta_{\tau} \) and \( \theta_{\sigma} \) defined in \((3)\), each have non-trivial dependence on both of the parameters \( \sigma, \tau \), through the normalization functions \((5)\).

For brevity, the lens theta functions \((3)\) will typically be denoted by
\[
\theta_{\tau}(z, z) := \theta_{\tau}(z, z; \sigma, \tau), \quad \theta_{\sigma}(z, z) := \theta_{\sigma}(z, z; \sigma, \tau),
\]
with implicit dependence on the two parameters $\sigma$ and $\tau$.

Furthermore, a shorthand notation will be used throughout this paper, where $\pm$ in the argument of a function denotes that respective factors involving $+$ and $-$ should be taken as a product, e.g.

$$\theta_\sigma(x_j \pm x_k, x_j \pm x_k) = \theta_\sigma(x_j + x_k, x_j + x_k) \theta_\sigma(x_j - x_k, x_j - x_k).$$  \hspace{1cm} (8)

**Proposition 1.** The lens theta functions satisfy (here $\theta_{\tau,\sigma}$ indicates that an identity holds for either $\theta_\tau$ or $\theta_\sigma$):

1. **(periodicity)** For $k \in \mathbb{Z}$,
   $$\theta_{\tau,\sigma}(z + 2kr, z) = \theta_{\tau,\sigma}(z, z), \hspace{1cm} \theta_{\tau,\sigma}(z, z + kr) = \theta_{\tau,\sigma}(z, z).$$  \hspace{1cm} (9)

2. **(inversion)**
   $$\theta_{\tau,\sigma}(-z, -z) = -\theta_{\tau,\sigma}(z, z) e^{-\frac{2\pi im}{\tau}(z - z)}.$$  \hspace{1cm} (10)

3. **(recurrence relation)** For $n \in \mathbb{Z}$,
   $$\theta_\tau(z + n\tau, z + n) = \theta_\tau(z, z) e^{-\frac{2\pi in\tau}{\tau}(z + (n-1)\tau + r - 2z - n)};$$
   $$\theta_\sigma(z + n\sigma, z - n) = \theta_\sigma(z, z) e^{-\frac{2\pi in\sigma}{\sigma}(z + (n-1)\sigma + r - 2z + n)}.$$  \hspace{1cm} (11)

4. **(quasi-periodicity)** For $n \in \mathbb{Z}$,
   $$\theta_\tau(z + n\tau, z) = \theta_\tau(z, z) e^{-\frac{2\pi in\tau}{\tau}(2z + (n-1)\tau + r - 2z - n)};$$
   $$\theta_\sigma(z + n\sigma, z) = \theta_\sigma(z, z) e^{-\frac{2\pi in\sigma}{\sigma}(2z + (n-1)\sigma + r - 2z + n)}.$$  \hspace{1cm} (12)

5. **(three-term relation)** For $x_i, x_j, x_k, z \in \mathbb{C}$, and $\xi_i, \xi_j, \xi_k, z \in \mathbb{Z}$, or $\xi_i, \xi_j, \xi_k, z \in \mathbb{Z} + \frac{1}{2}$,
   $$e^{\frac{2\pi i}{\tau}(x_i - x_k)} \theta_{\tau,\sigma}(x_j \pm x_k, x_j \pm x_k) \theta_{\tau,\sigma}(x_i \pm z, x_i \pm z)$$
   $$+ e^{\frac{2\pi i}{\tau}(x_i - x_j)} \theta_{\tau,\sigma}(x_k \pm x_i, x_k \pm x_i) \theta_{\tau,\sigma}(x_j \pm z, x_j \pm z)$$
   $$+ e^{\frac{2\pi i}{\tau}(x_j - x_k)} \theta_{\tau,\sigma}(x_i \pm x_j, x_i \pm x_j) \theta_{\tau,\sigma}(x_k \pm z, x_k \pm z) = 0.$$  \hspace{1cm} (13)

**Proof.** These identities simply follow from the definitions (3), and similar identities that hold for the regular theta function $\theta(z|q)$, defined in (4). \hfill \Box

2.2. **Lens Elliptic Gamma Function.** The lens elliptic gamma function \cite{7,8,18,19} is defined here by

$$\Gamma(z, z; \sigma, \tau) := e^{\phi(z, z; \sigma, \tau)} \gamma_{\sigma}(z, z; \sigma, \tau) \gamma_{\tau}(z, z; \sigma, \tau), \hspace{1cm} z \in \mathbb{C}, \ z \in \mathbb{Z}.$$  \hspace{1cm} (14)
where \( \gamma_{\sigma} \) and \( \gamma_{\tau} \), are the following infinite products

\[
\gamma_{\sigma}(z, z'; \sigma, \tau) := \prod_{j,k=0}^{\infty} \frac{1 - e^{-2\pi i z} e^{-2\pi i z'} e^{2\pi i (\sigma+\tau)(j+1)} e^{2\pi i (\sigma+\tau)(k+1)}}{1 - e^{-2\pi i z} e^{-2\pi i z'} e^{2\pi i (\sigma+\tau)(j+1)} e^{2\pi i (\sigma+\tau)(k+1)}}, \tag{15}
\]

\[
\gamma_{\tau}(z, z'; \sigma, \tau) := \prod_{j,k=0}^{\infty} \frac{1 - e^{-2\pi i z} e^{-2\pi i z'} e^{2\pi i (\sigma+\tau)(j+1)} e^{2\pi i (\sigma+\tau)(k+1)}}{1 - e^{-2\pi i z} e^{-2\pi i z'} e^{2\pi i (\sigma+\tau)(j+1)} e^{2\pi i (\sigma+\tau)(k+1)}}, \tag{16}
\]

and the normalisation function \( \phi_r(z, z'; \sigma, \tau) \) is given by

\[
\phi_r(z, z'; \sigma, \tau) := \pi i \frac{\pi(z-r)(6z-3\sigma-3\tau+(1-\sigma+\tau)(r-2z))}{6r} \tag{17}
\]

Note that the functions (15), (16), are symmetric with respect to the following shifts

\[
\gamma_{\sigma}(z + k\sigma, z - k; \sigma, \tau) = \gamma_{\sigma}(z, z'; \sigma, \tau),
\gamma_{\tau}(z + k\tau, z + k; \sigma, \tau) = \gamma_{\tau}(z, z'; \sigma, \tau),
\tag{18}
\]

for \( k \in \mathbb{Z} \).

The normalisation function (17), has a useful factorisation in terms of the multiple Bernoulli polynomial \( B_{3,3} \) [175], as

\[
\phi_r(z, z'; \sigma, \tau) = 2\pi i \left[ R_2(z, 0; \sigma - \frac{1}{2}, \tau + \frac{1}{2}) - R_2(z, z'; \sigma - \frac{1}{2}, \tau + \frac{1}{2}) \right] + 2\pi i \left[ R_2(z, 0; \sigma, \tau) + R_2(0, z; \frac{1}{2}, -\frac{1}{2}) - R_2(z, z'; \sigma, \tau) \right], \tag{19}
\]

where

\[
R_2(z, z'; \sigma, \tau) := R(z + z\sigma; r\sigma, \sigma + \tau) + R(z + (r - z)\tau; r\tau, \sigma + \tau), \tag{20}
\]

and

\[
R(z; \sigma, \tau) := \frac{B_{3,3}(z; \sigma, \tau, -1) + B_{3,3}(z - 1; \sigma, \tau, -1)}{12}. \tag{21}
\]

For \( r = 1 \), the lens elliptic gamma function [14] reduces to the regular elliptic gamma function [21], which is denoted here by \( \Gamma_1(z; \sigma, \tau) \),

\[
\Gamma(z, z'; \sigma, \tau)|_{r=1} = \Gamma_1(z; \sigma, \tau) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i (\sigma+\tau)(j+1)(\sigma+\tau)(k+1)}}{1 - e^{2\pi i (\sigma+\tau)(j+\tau)(k+\tau)}}. \tag{22}
\]

In terms of the regular elliptic gamma function (22), the functions (15), and (16), are simply

\[
\gamma_{\sigma}(z, z; \sigma, \tau) = \Gamma_1(z + \sigma z; r\sigma, \sigma + \tau),
\gamma_{\tau}(z, z; \sigma, \tau) = \Gamma_1(z + \tau (r - z); r\tau, \sigma + \tau). \tag{23}
\]
Similarly to the lens theta functions, the lens elliptic gamma function (14) will typically be denoted as
\[ \Gamma(z, z) := \Gamma(z, z; \sigma, \tau), \]
with implicit dependence on the two parameters \( \sigma \) and \( \tau \).

**Proposition 2.** The lens elliptic gamma function (14) satisfies

1. (periodicity) For \( k \in \mathbb{Z} \),
   \[ \Gamma(z + 2kr, z) = \Gamma(z, z), \quad \Gamma(z, z + kr) = \Gamma(z, z). \]
2. (inversion)
   \[ \Gamma((\sigma + \tau) - z, -z) \Gamma(z, z) = 1. \]
3. (recurrence relation) For \( n = 0, 1, \ldots \),
   \[ \Gamma(z + n\sigma, z - n) = \Gamma(z, z) \prod_{j=0}^{n-1} \theta_\tau(z + j\sigma, z - j), \]
   \[ \Gamma(z + n\tau, z + n) = \Gamma(z, z) \prod_{j=0}^{n-1} \theta_\sigma(z + j\tau, z + j). \]

**Proof.** These identities can be verified by direct computation. A proof of the \( r \)-periodicity in (25) previously appeared in Appendix C of [19]. For (27), (28), the normalisation of the lens theta functions (5) are in fact chosen to satisfy
\[ \phi_e(z + \sigma, z - 1; \sigma, \tau) - \phi_e(z, z; \sigma, \tau) = \phi_\tau(z, z; \sigma, \tau), \]
\[ \phi_e(z + \tau, z + 1; \sigma, \tau) - \phi_e(z, z; \sigma, \tau) = \phi_\sigma(z, z; \sigma, \tau). \]

Due to the relations (18), only the factor of \( \gamma_\tau \) on the left hand side of (27), contributes to the infinite product part of the theta function \( \theta_\tau \), while only the factor of \( \gamma_\sigma \) on the left hand side of (28), contributes to the infinite product part of the theta function \( \theta_\sigma \).

**Remark 1.** Although \( \gamma_{\sigma, \tau}(z + k, z; \sigma, \tau) = \gamma_{\sigma, \tau}(z, z; \sigma, \tau) \) for any \( k \in \mathbb{Z} \), the \( 2r \)-periodicity of \( \Gamma(z, z; \sigma, \tau) \) in (25) comes from the normalisation factor in (19).

2.3. **Lens Triple Gamma Functions.** Here we consider two parameters \( \text{Im}(\omega), \text{Im}(\mu) > 0 \), in addition to the parameters (1).

The lens triple gamma functions \( \Gamma^\tau_{\sigma}(\cdot) \) and \( \Gamma^\sigma_{\tau}(\cdot) \), are defined here by
\[ \Gamma^\tau_{\sigma}(z, z; \sigma, \tau, \omega) := e^{\phi_\tau(z, z; \sigma, \tau, \omega)} \gamma^\tau_{\sigma}(z, z; \sigma, \tau, \omega), \]
\[ \Gamma^\sigma_{\tau}(z, z; \sigma, \tau, \omega, \mu) := \Gamma^\tau_{\sigma}(z, z; \sigma, \tau, \omega) \gamma^\sigma_{\tau}(z, z; \sigma, \tau, \omega), \quad z \in \mathbb{C}, \quad z \in \mathbb{Z}, \]
\[ z \in \mathbb{C}, \quad z \in \mathbb{Z}, \]
\[ (30) \]
where
\[ \gamma^\dagger_\sigma(z; \sigma, \tau, \omega) := g_\sigma(z; \sigma, \tau, \omega) g_\sigma(\sigma + \tau + \omega - z, r + 1 - z; \sigma, \tau, \omega), \]
\[ \gamma^\dagger_\tau(z; \sigma, \tau, \omega) := g_\tau(z; \sigma, \tau, \omega) g_\tau(\sigma + \tau + \omega - z, r - z; \sigma, \tau, \omega), \]
(31)

and
\[ g_\sigma(z; \sigma, \tau, \omega) := \prod_{k_1, k_2, k_3 = 0}^{\infty} \left( 1 - e^{2\pi i x} e^{2\pi i r z} e^{2\pi i r k_1} e^{2\pi i (\sigma + \tau) k_2} e^{2\pi i (\sigma + \omega) k_3} \right), \]
\[ g_\tau(z; \sigma, \tau, \omega) := \prod_{k_1, k_2, k_3 = 0}^{\infty} \left( 1 - e^{2\pi i x} e^{2\pi i (r - z)} e^{2\pi i r k_1} e^{2\pi i (\sigma + \tau) k_2} e^{2\pi i \omega k_3} \right). \]
(32)

The normalisation function \( \phi^+(z; \sigma, \tau, \omega) \) is defined by (c.f. the expression (19) for \( \phi_\gamma \) in terms of \( B_{3,3} \))
\[ \phi^+(z; \sigma, \tau, \omega) := 2\pi i \left( T_2(z, 0; \sigma - \frac{1}{2}, \tau + \frac{1}{2}, \omega) - S_2(z, z; \sigma - \frac{1}{2}, \tau + \frac{1}{2}, \omega) \right), \]
(33)

where
\[ S_2(z, \sigma, \tau, \omega) := S(z + z\sigma; r\sigma, \sigma + \tau, \omega + \sigma) + S(z + (r - z)\tau; r\tau, \sigma + \tau, \omega - \tau), \]
\[ T_2(z, \sigma, \tau, \omega) := S(z + z\sigma; r\sigma, \sigma + \tau, \omega) + S(z + (r - z)\tau; r\tau, \sigma + \tau, \omega), \]
(34)

and
\[ S(z; \sigma, \tau, \omega) := \frac{B_4(z; \sigma, \tau, -1, \omega) + B_4(z - 1; \sigma, \tau, -1, \omega)}{48}. \]
(35)

In the last equation, \( B_4(z; \omega_1, \omega_2, \omega_3, \omega_4) \) is the multiple Bernoulli polynomial (176), defined in Appendix B.

**Proposition 3.** The functions \( \gamma_\sigma, \gamma_\tau \), defined in (31), satisfy

1. **(shift symmetry)**
   \[ \gamma^\dagger_\sigma(z + \sigma, z - 1; \sigma, \tau, \omega) = \gamma^\dagger_\sigma(z, z; \sigma, \tau, \omega), \]
   \[ \gamma^\dagger_\tau(z + \tau, z + 1; \sigma, \tau, \omega) = \gamma^\dagger_\tau(z, z; \sigma, \tau, \omega). \]
   (36)

2. **(inversion)**
   \[ \gamma^\dagger_\sigma(\sigma + \tau + \omega - z, r + 1 - z; \sigma, \tau, \omega) = \gamma^\dagger_\sigma(z, z; \sigma, \tau, \omega), \]
   \[ \gamma^\dagger_\tau(\sigma + \tau + \omega - z, r - z; \sigma, \tau, \omega) = \gamma^\dagger_\tau(z, z; \sigma, \tau, \omega). \]
   (37)
Proof. These relations essentially follow from the definitions given in (31).

**Corollary 1.** The lens triple gamma functions (31) satisfy

1. (inversion)
\[ \Gamma_\sigma^+(\sigma + \tau + \omega - z, r + 1 - z; \sigma, \tau, \omega) = \Gamma_\sigma^+(z, z; \sigma, \tau, \omega), \]
\[ (39) \]
2. (recurrence relation)
\[ \Gamma_\sigma^+(\sigma + r - z, r - z; \sigma, \tau, \omega, \mu) = \Gamma_\sigma^+(z, z; \sigma, \tau, \omega, \mu), \]
\[ (40) \]

Proof. The relations (39), (40) follow from the relations given in Proposition 3, and also the following relations satisfied by the normalisation function (33)

\[ \phi^+(\sigma + \tau + \omega - z, r + 1 - z; \sigma, \tau, \omega) = \phi^+(z, z; \sigma, \tau, \omega), \]
\[ (41) \]
\[ \phi^+(\omega + z, z + 1; \sigma, \tau, \omega) = \phi^+(z, z; \sigma, \tau, \omega) + \phi(z, z; \sigma, \tau), \]

where \( \phi(z, z; \sigma, \tau) \) is the normalisation function for the lens elliptic gamma function given in (19).

**Remark 2.** Note that unlike the lens theta and elliptic gamma functions, the lens triple gamma functions (31) are not \( r \)-periodic in \( z \), and even for \( r = 1 \) there remains a dependence on the integer variable \( z \). This is the reason why the hypergeometric \( \tau \)-function constructed in Section 6 retains the dependence on the discrete variables even for \( r = 1 \).

3. **Elliptic Hypergeometric Sum/Integral and \( W(E_7) \) transformation**

3.1. **Elliptic Hypergeometric Sum/Integral.** A central role in this paper is played by the following sum/integral, defined in terms of the lens elliptic gamma function (14), by

\[ I(x, \tau; \sigma, \tau) = \frac{\lambda(\sigma, \tau)}{2} \sum_{z=0}^{r-1} \int_{[0,1]} dz \prod_{j=0}^{7} \frac{\Gamma(x_j \pm z, \pm \tau \pm \bar{z})}{\Gamma(\pm 2z, \pm 2\bar{z})}, \]
\[ (42) \]

where \( x = (x_0, \ldots, x_7) \in \mathbb{C}^8, \text{Im}(x_i) > 0 \), and \( \tau = (\tau_0, \ldots, \tau_7) \in \mathbb{Z}^8 \cup (\mathbb{Z} + 1/2)^8 \). Notice that in contrast to the previous section, here we allow \( \tau \) to have either integer, or half-integer components. The discrete summation variable
is chosen so that the second argument of each factor of the lens elliptic gamma functions appearing in (42) is an integer, and is defined by
\[
\tilde{z} := z + (r + ((r + 1) \mod 2))(\sigma_i \mod 1)
\]
\[
= \begin{cases} 
z & (\sigma \in \mathbb{Z}^8), \\
\frac{z + r + 1}{2} & (\sigma \in (\mathbb{Z} + \frac{1}{2})^8, r \text{ even}), \\
\frac{z + r}{2} & (\sigma \in (\mathbb{Z} + \frac{1}{2})^8, r \text{ odd}).
\end{cases}
\] (43)
The prefactor \( \lambda(\sigma, \tau) \) in (42) is given by
\[
\lambda(\sigma, \tau) = (e^{2\pi \imath \sigma}; e^{2\pi \imath \tau})_\infty (e^{2\pi \imath \sigma}; e^{2\pi \imath \tau})_\infty.
\] (44)
The condition \( \imath m(x_i) > 0 \) may be relaxed, by deforming the contour connecting the points \( z = 0 \) and \( z = 1 \), such that the respective poles of the integrand of (42) do not cross over the contour [10].

For \( \sigma \in \mathbb{Z}^8 \), the elliptic hypergeometric sum/integral (42) previously appeared as part of a key identity (star-star relation) for the integrability of multi-spin lattice models [10][16], and is a 2-parameter extension of the left hand side of the elliptic beta sum/integral formula that was proven in [8]. It has also previously been studied with respect to \( A_1 \leftrightarrow A_n \), and BC1 \( \leftrightarrow BC_n \) transformations proven by the authors [10], where the BC1 \( \leftrightarrow BC_0 \) transformation was previously proven by Spiridonov [9], and the \( r = 1 \) cases of the transformations were previously proven by Rains [22].

3.2. Contiguity Relation. Define the shift operator \( T_{r,k} \) \( (k \in \{0, \ldots, 7\}) \), that acts on the continuous variables \( x_k \in \mathbb{C} \), and discrete variables \( \sigma_k \in \mathbb{Z}, (\mathbb{Z} + \frac{1}{2}) \), as
\[
T_{r,k} f(x_0, \ldots, x_7, \sigma_0, \ldots, \sigma_7) = f(x_0, \ldots, x_k + \tau, x_7, \sigma_0, \ldots, \sigma_k + 1, \ldots, \sigma_7).
\] (45)

**Proposition 4.** The elliptic hypergeometric sum/integral (42) satisfies the three-term relation
\[
(e^{2\pi \imath(x_j - x_k)} \theta_{\sigma}(x_j + x_k, \sigma_j \pm \sigma_k) T_{r,i} + e^{2\pi \imath(x_j - x_k)} \theta_{\sigma}(x_j + x_i, \sigma_j \pm \sigma_i) T_{r,j} + e^{2\pi \imath(x_j - x_k)} \theta_{\sigma}(x_j \pm x_i, \sigma_i \pm \sigma_j) T_{r,k}) I(x, \sigma) = 0,
\] (46)
for any triple \( i, j, k \in \{0, \ldots, 7\} \).

**Proof.** By (28), the integrand
\[
\Delta(z, z; x, \sigma) := \frac{\prod_{j=0}^{7} \Gamma(x_j \pm z, \sigma_j \pm \tilde{z})}{\Gamma(\pm 2z, \pm 2\tilde{z})},
\] (47)
satisfies
\[
T_{r,k} \Delta(z, z; x, \sigma) = \theta_{\sigma}(x_k \pm z, \sigma_k \pm \tilde{z}) \Delta(z, z; x, x),
\] (48)
with respect to the shift operator $T_{r,k}$. Note that by the choice of $\tilde{z}$ in (43), $z_j + \tilde{z}$, and $\pm 2\tilde{z}$, are always integers. Next by the three-term relation (13), the equation
\[
\left(e^{\frac{2\pi i}{\vartheta}}(z_i - z_k) \theta_{\sigma}(x_j \pm x_k, x_k \pm x_i) T_{r,j} + e^{\frac{2\pi i}{\vartheta}}(z_i - x_j) \theta_{\sigma}(x_i \pm x_j, x_i \pm x_j) T_{r,j} + e^{\frac{2\pi i}{\vartheta}}(z_j - x_i) \theta_{\sigma}(x_i \pm x_j, x_i \pm x_j) T_{r,k}\right) \Delta(z, z; x, x) = 0,
\]
holds for $i, j, k \in \{0, \ldots, 7\}$. By commuting the shift operator with the sum/integral, we obtain (46).

3.3. $W(E_7)$ Transformation.

**Proposition 5.** For $x = (x_0, \ldots, x_7) \in \mathbb{C}^8$, $\text{Im}(x_i) > 0$, and $\bar{x} = (x_0, \ldots, x_7) \in \mathbb{Z}^8 \cup (\mathbb{Z} + 1/2)^8$ with the restriction
\[
\sum_{i=0}^{7} x_i = 2(\sigma + \tau), \quad \sum_{i=0}^{7} x_i = 2r,
\]
the sum/integral (42) satisfies
\[
I(x, \bar{x}) = I(\tilde{x}, \tilde{\bar{x}}) \prod_{0 \leq i < j \leq 3 \text{ or } 4 \leq i < j \leq 7} \Gamma(x_i + x_j, x_i + x_j),
\]
where the transformed variables $\tilde{x} = (\tilde{x}_0, \ldots, \tilde{x}_7) \in \mathbb{C}^8$ and $\tilde{\bar{x}} = (\tilde{x}_0, \ldots, \tilde{x}_7) \in \mathbb{Z}^8 \cup (\mathbb{Z} + 1/2)^8$ are given by
\[
\tilde{x}_i = \begin{cases} x_i + \frac{\vartheta x_i}{2} - \frac{1}{2} \sum_{j=0}^{3} x_j & (i = 0, 1, 2, 3), \\ x_i + \frac{\vartheta x_i}{2} - \frac{1}{2} \sum_{j=4}^{7} x_j & (i = 4, 5, 6, 7), \end{cases}
\]
\[
\tilde{\bar{x}}_i = \begin{cases} x_i + \frac{\vartheta}{2} - \frac{1}{2} \sum_{j=0}^{3} x_j & (i = 0, 1, 2, 3), \\ x_i + \frac{\vartheta}{2} - \frac{1}{2} \sum_{j=4}^{7} x_j & (i = 4, 5, 6, 7). \end{cases}
\]

Proposition 5 is proven with the use of a variation of the elliptic beta sum/integral formula [8] in Appendix A (a similar proof of this identity first appeared in [9]).

Note that the variables of the elliptic hypergeometric sum/integral (42), essentially transform in the formula (51) under the action of a reflection for an element of the Weyl group $W(E_7)$. This property is particularly important for the construction of the $\tau$-function from the sum/integral (42) (see Section 6).

4. $\tau$-function on the $E_8$ root lattice

In this section we will consider the properties of the root lattice of $E_8$ which are used to define our $\tau$-function. Many of the properties and definitions are essentially based on the work of Noumi [6], which in the following is related to (but not the same as) the $r = 1$ case.
4.1. $E_8$ Root Lattice. We denote the root lattice of $E_8$ by $Q(E_8)$, and the $E_8$ Weyl group by $W(E_8)$. The root lattice is more explicitly given as a $\mathbb{Z}$-span of the vectors

$$\pm v_i \pm v_j, \quad (0 \leq i < j \leq 7),$$

$$\frac{1}{2} (\pm v_0 \pm v_1 \pm \ldots \pm v_7), \quad \text{even number of minus signs},$$

where $\{v_0, \ldots, v_7\}$ is the orthonormal basis with respect to the canonical symmetric bilinear form ($\cdot | \cdot$) on the root lattice $Q(E_8)$, namely $(v_i | v_j) = \delta_{ij}$. Note also that $(a | a) = 2$, for $a \in \Delta(E_8)$, where $\Delta(E_8)$ is the root system for $E_8$.

The following set of vectors in $\mathbb{C}^8$ plays a central role for this paper.

**Definition 1.** A set $\{\pm a_0, \pm a_1, \ldots, \pm a_{l-1}\}$ of $2l$ vectors in $\mathbb{C}^8$ is called a $C_l$-frame if the following two conditions are satisfied:

1. $(a_i | a_j) = \delta_{ij} \quad (0 \leq i, j < l)$,
2. $a_i \pm a_j \in Q(E_8) \quad (0 \leq i < j < l), \quad 2a_i \in Q(E_8) \quad (0 \leq i < l)$.

Notice that this definition implies that the set of $2l^2$ vectors

$$\{\pm a_i \pm a_j | (0 \leq i < j < l)\} \cup \{\pm 2a_i | (0 \leq i < l)\}$$

is contained in the root lattice $Q(E_8)$ and forms a root lattice of type $C_l$. In the following sections we will mostly work with the $C_3$-frame for $l = 3$.

4.2. $E_8$ $\tau$-function. For a pair $Z = (z, z) \in \mathbb{C} \times \mathbb{Z}$, where $z \in \mathbb{C}$, and $z \in \mathbb{Z}$, we define

$$[Z] = [(z, z)] := e^{\pi i (\tau + z \bar{z})} \theta_{\tau}(z, z),$$

where $\theta_{\tau}(z, z)$ is the lens theta function defined in (3).

The function (56) satisfies the following identities (note that these identities are simple corollaries of Proposition 1 but are written here explicitly for convenience)

**Proposition 6.** We have the following identities for the bracket:

1. (periodicity) $[Z + (0, 2r)] = [Z]$.  
2. (reflection) $[-Z] = -[Z]$.
3. (three-term identity) For $Z, Z_i, Z_j, Z_k \in \mathbb{C} \times \mathbb{Z}$, or $Z, Z_i, Z_j, Z_k \in \mathbb{C} \times (\mathbb{Z} + \frac{1}{2})$,

$$[Z_j \pm Z_k] [Z_i \pm Z] + [Z_k \pm Z_i] [Z_j \pm Z] + [Z_i \pm Z_j] [Z_k \pm Z] = 0.$$

where we used the shorthand notation $[X \pm Y] := [X + Y][X - Y]$. 

Due to (6), for \( r = 1 \) there is no dependence on the second argument \( z \), and the bracket may simply be written as \([z]\) with \( z \in \mathbb{C} \). In that case, the three-term identity in Proposition 6 exactly reduces to the standard three-term identity for the theta function, given in (2.1) of [6].

Consider now the space

\[
V := \mathbb{C}^8 \times Q(E_8).
\]

The first (second) factor \( \mathbb{C}^8 (Q(E_8)) \) can be thought of as a \( \mathbb{C} \)-span (\( \mathbb{Z} \)-span) of the root lattice generators (54). We denote an element of this space as \( X = (x, x) \in V \), with \( x \in \mathbb{C}^8 \), and \( x \in Q(E_8) \). A natural addition on this space is defined by

\[
X + Y = (x + y, x + y).
\]

We define \( \tau \) to be a non-zero complex number, and choose a region \( D \subseteq V \), satisfying

\[
D = D + Q(E_8)T,
\]

where \( T \) is the “step size”, defined as

\[
T := (\tau, 1),
\]

and we have used the notation \( vT = (v\tau; v) \in V \), where \( v \in Q(E_8) \).

As an example, \( D \) may be chosen as the whole space \( D = V \), as this will obviously satisfy the condition (61). As another example, we could also minimally choose a completely discrete set for \( D \), as

\[
D = C + Q(E_8)T,
\]

for some point \( C \in V \). Similarly to the situation in [6], the construction of the hypergeometric \( \tau \)-function in Section 5 will in fact involve a combination of discrete and continuous spaces, where \( D \) is chosen as an infinite family of parallel hyperplanes in \( V \), that are indexed by an integer \( n \).

In the following, for \( a \in Q(E_8) \), and \( X = (x, x) \in D \), we define \( (a|X) = ((ax), (a|x)) \in \mathbb{C} \times \mathbb{Z} \).

Our \( \tau \)-function on \( D \subset V \) is defined as follows.

**Definition 2.** A function \( \tau(X) \) defined over the region \( D \) satisfying (61), is called a \( \tau \)-function if it satisfies the non-autonomous bilinear Hirota equations

\[
[(a_1 \pm a_2|X)] \tau(X \pm a_0 T) + [(a_2 \pm a_0|X)] \tau(X \pm a_1 T) + [(a_0 \pm a_1|X)] \tau(X \pm a_2 T) = 0,
\]

for any \( C_3 \)-frame \( (a_0, a_1, a_2) \), and \( X \in D \).

For a general choice of \( D \subset V \) satisfying (61), even for \( r = 1 \) the Hirota equations (64) will have a non-trivial dependence on the discrete variables coming from \( Q(E_8) \). Indeed the hypergeometric solution of (64) obtained in Section 6 will have such a dependence for all \( r = 1, 2, \ldots \). In this respect,
the situation considered here is a different situation than was considered in [6], where \( \tau \)-functions in the latter were defined on subsets of \( V = \mathbb{C}^8 \), and have no dependence on any discrete variables.

As an example, note that the Hirota equations (64) admit the following constant solution:

**Proposition 7.** For \( X = (x, \xi) \in V \), and a constant \( C = (c, \xi) \in V \), the function

\[
\tau(X) = \left( \frac{1}{2\tau}(x|x) + c, \frac{1}{2}(x|\xi) + \xi \right) \quad (X \in V),
\]

(65)
is an example of a \( \tau \)-function associated with the region \( D = V \).

**Proof.** We have

\[
\tau(X \pm a_i T) = \left( \frac{1}{2\tau}(x|x) + \tau + c \pm (a_i|x), \frac{1}{2}(x|\xi) + 1 + c \pm (a_i|\xi) \right)
\]

(66)

where

\[
Z := \left( \frac{1}{2\tau}(x|x) + \tau + c, \frac{1}{2}(x|\xi) + 1 + c \right),
\]

\[
Z_i := (a_i|x) = ((a_i|x), (a_i|\xi)) \quad (i = 0, 1, 2).
\]

The Hirota equations (64) then follow from the three-term identity (59).

For a given \( \tau \)-function, one can also construct a new \( \tau \)-function by an element of the Weyl group \( W(E_8) \). In this sense the Hirota equations are “covariant” with respect to the action of \( W(E_8) \):

**Proposition 8.** For a \( \tau \)-function \( \tau \) on a domain \( D \), and an element \( w \in W(E_8) \), the function \( w \cdot \tau \) defined by

\[
(w \cdot \tau)(X) := \tau(w^{-1} \cdot X) \quad (X \in w \cdot D),
\]

(68)
is also a \( \tau \)-function on the domain \( w \cdot D \).

Note that \( w \cdot \tau \) is in general different from \( \tau \), particularly they will respectively be defined on different domains.

5. Decomposition into \( E_7 \)-Orbits

5.1. Decomposition of \( D \). In the previous section we have considered the domain \( D \) of the \( \tau \)-function, as a general subset of \( V \), satisfying the condition (61). To start to consider hypergeometric solutions, we proceed with a
special choice of $D$, given by
\begin{align}
D &= \bigsqcup_{n \in \mathbb{Z}} D_n, \\
D_n &= H_{n \tau + r} \times (H_{n + r - 1} \cap Q(E_8)),
\end{align}
where the hyperplane $H_k$, is defined by
\begin{equation}
H_k = \left\{ x \in \mathbb{C}^8 \mid (x|\phi) = \kappa \right\},
\end{equation}
and $\phi$ is the highest root, which in the basis $v_0, \ldots, v_7$ of (54), is given by
\begin{equation}
\phi = \frac{v_0 + \cdots + v_7}{2}.
\end{equation}
Thus the coordinates $X = (x, x) \in D_n$ satisfy
\begin{align}
\sum_{i=0}^{7} x_i &= 2(n \tau + \sigma), \\
\sum_{i=0}^{7} x_i &= 2(n + r - 1).
\end{align}

The choice of the highest root $\phi$ breaks the manifest covariance under the $W(E_8)$ symmetry down to the stabilizer of $\phi$, which is the Weyl group $W(E_7)$. Indeed, the $E_7$ root lattice in the basis of (54), is spanned by
\begin{align}
&\pm (v_i - v_j), \quad (0 \leq i < j \leq 7), \\
&\frac{1}{2} (\pm v_0 \pm v_1 \pm \ldots \pm v_7), \quad \text{(total of four minus signs)},
\end{align}
and these vectors together with the highest root $\phi$ of $E_8$, generate the whole $E_8$ root lattice.

**Remark 3.** Recall that there were 2 types of lens theta functions $\theta_\sigma, \theta_\tau$, defined in (3), while the bracket function (56) is defined in terms of $\theta_\sigma$ only. However, in the definition of the bracket function (56), we could also replace $\theta_\sigma$ with $\theta_\tau$, as:
\begin{equation}
\overline{[Z]} = \overline{[(z, z)]} := e^{\frac{\phi}{2}(-z_j + z_j)} \theta_\tau(z_j, z_j).
\end{equation}
This bracket will still satisfy Proposition 6 from which we can build the lens-elliptic $\tau$-function. Then in this case, instead of (69), the definition of a suitable region $D$ would be
\begin{align}
D &= \bigsqcup_{n \in \mathbb{Z}} D_n, \\
D_n &= (H_{n \tau + r} \oplus (H_{n + r - 1} \cap Q(E_8))).
\end{align}
5.2. **Decomposition of \(\tau\)-function.** Let us now analyse the \(\tau\)-function on the domain \(D\) given in \([59]\). Since \([59]\) is a disjoint union, the \(\tau\)-function on the domain \(D\), can be thought of as an infinite sequence of functions \(\tau^{(n)}\) on \(D_n\), which are indexed by the integer \(n\):

\[
\tau^{(n)} := \tau|_{D_n}.
\]  

We wish to write the Hirota equations \((64)\) as a set of conditions for the \(\tau^{(n)}\) defined on \(D_n\).

In the Hirota equation \((64)\), the argument \(X\) is shifted by vectors \(\pm a_0, \pm a_1, \pm a_2\) which come from the particular choice of \(C_3\)-frame. This means that the corresponding Hirota equations on \(D\), will provide relations between \(\tau^{(n)}\)-functions on up to three different hyperplanes, depending on the values of \((\phi|a_i)_{i=0,1,2}\), for the particular \(C_3\)-frame. In terms of the inner product \((\phi|a_i)_{i=0,1,2}\), the \(C_3\)-frames may be classified as one of the following four types:

**Proposition 9** (Proposition 3.2 in \([6]\)). The set of all \(C_3\)-frames may be decomposed into four \(W(E_7)\)-orbits. For \(\{\pm a_0, \pm a_1, \pm a_2\}\), the orbit is classified as one of the following four types \((I), (II_0), (II_1), (II_2)\), according to the pairings with the highest root \(\phi\):

\[
\begin{align*}
(I) : & \quad (\phi|a_0) = (\phi|a_1) = (\phi|a_2) = \frac{1}{2}, \\
(II_0) : & \quad (\phi|a_0) = (\phi|a_1) = (\phi|a_2) = 0, \\
(II_1) : & \quad (\phi|a_0) = 1, \quad (\phi|a_1) = (\phi|a_2) = 0, \\
(II_2) : & \quad (\phi|a_0) = (\phi|a_1) = 1, \quad (\phi|a_2) = 0.
\end{align*}
\]

\(\tag{77}\)

**Remark 4.** The notation \((I), (II_0), (II_1), (II_2)\) is motivated by the facts that (see \([6]\), Propositions 1.4 and 3.1)

1. any \(C_3\)-frame is contained in a unique \(C_8\)-frame.
2. The set of \(C_8\) frames may be decomposed into two \(W(E_7)\)-orbits, which are characterized by

\[
\begin{align*}
(I) : & \quad (\phi|a_i) = \frac{1}{2}, \quad (i = 0, \ldots, 7), \\
(II) : & \quad (\phi|a_0) = (\phi|a_1) = 1, \quad (\phi|a_i) = 0, \quad (i = 2, \ldots, 7).
\end{align*}
\]

\(\tag{78}\) \(\tag{79}\)

Moreover, in the case of \((II)\), we can show that \(a_0 + a_1 = \phi\).

This remark implies that a given \(C_3\)-frame can be enlarged nicely into a \(C_8\)-frame. As an example, suppose that we have a \(C_3\)-frame \(\{\pm a_0, \pm a_1, \pm a_2\}\) of type \((II_2)\). We can then choose a \(C_8\)-frame \(\{\pm a_0, \pm a_1, \ldots, \pm a_7\}\) of type \((II)\) containing the \(C_3\)-frame \(\{\pm a_0, \pm a_1, \pm a_2\}\) that we started with. This \(C_8\)-frame also contains many other \(C_3\)-frames—for example \(\{\pm a_1, \pm a_2, \pm a_3\}\), as...
a $C_3$-frame of type $(\Pi_1)$. This type of manipulation will be useful for some of the proofs below.

Thanks to Proposition \[7\] we find that there are four different types of Hirota identities depending on the different types of $C_3$-frames. These are, for $X \in D_{n+\frac{1}{2}}$,

$$[(a_1 \pm a_2|X)]\tau^n(X-a_0T)\tau^{(n+1)}(X+a_0T)$$

(I)$_{n+\frac{1}{2}}$ : 

$$+ [(a_2 \pm a_0|X)]\tau^n(X-a_1T)\tau^{(n+1)}(X+a_1T)$$

(80)

and for $X \in D_n$,

$$[(a_1 \pm a_2|X)]\tau^n(X \pm a_0T)$$

(II)$_0$ : 

$$+ [(a_2 \pm a_0|X)]\tau^n(X \pm a_1T)$$

(81)

for $n \in \mathbb{Z}$ we can derive (I)$_{n+\frac{1}{2}}$, (II)$_0$ and (II)$_1$ from (II)$_n$.

Thus we have decomposed the Hirota equations (64) for the $\tau$-function of type $E_8$ on $D$, into a set of equations for an infinite sequence of $\tau^n$-functions satisfying Hirota identities depending on the different types of $C_3$-frames.

Furthermore, the four identities above are not independent, and in fact we can focus on (II)$_1$ only, from which all others can be derived:

**Proposition 10.** For $n \in \mathbb{Z}$ we can derive (I)$_{n+\frac{1}{2}}$, (II)$_0$ and (II)$_1$ from

$$\begin{align*}
(\Pi_1)_n & \implies (I)_{n+\frac{1}{2}} \\
(\Pi_1)_n & \implies (\Pi)_0_{n+1} \\
(\Pi_1)_n & \implies (\Pi_2)_n
\end{align*}$$

(84) (85) (86)

**Proof.** Let us here prove only the first statement ($(\Pi_1)_n \implies (\Pi_2)_n$), since the argument is similar for other cases (see also [6 Appendix A] and [23 section 3]).

We wish to show (II)$_1$ for a $C_3$-frame of type $(\Pi_1)$, namely the set \{0, 1, 2\} satisfying (II)$_1$ : $(\phi|a_0) = (\phi|a_1) = 1$, $(\phi|a_2) = 0$. We can choose one more element $a_3$ from the root lattice, such that $(\phi|a_3) = 0$
This vanishes thanks to the three-term identity (59). We have therefore
for \( i = 0, 1 \). We can use these equations to compute
\[
[(a_1 \pm a_2|X])\tau^{(n-1)}(X - a_0T)\tau^{(n)}(X + a_0T)
\]
\[
+ [(a_2 \pm a_0|X])\tau^{(n-1)}(X - a_1T)\tau^{(n+1)}(X + a_1T)
\]
\[
+ [(a_0 \pm a_1|X])\tau^{(n)}(X \pm a_2T)
\]
\[
= - \frac{[(a_1 \pm a_2|X])}{[(a_2 \pm a_3|X])} \left( [(a_3 \pm a_0|X])\tau^{(n)}(X \pm a_2T) + [(a_0 \pm a_2|X])\tau^{(n)}(X \pm a_3T) \right)
\]
\[
- \frac{[(a_2 \pm a_0|X])}{[(a_2 \pm a_3|X])} \left( [(a_3 \pm a_1|X])\tau^{(n)}(X \pm a_2T) + [(a_1 \pm a_2|X])\tau^{(n)}(X \pm a_3T) \right)
\]
\[
+ [(a_0 \pm a_1|X])\tau^{(n)}(X \pm a_2T)
\]
\[
= - \frac{\tau^{(n)}(X \pm a_2T)}{[(a_2 \pm a_3|X])} \left( [(a_1 \pm a_2|X)][(a_3 \pm a_0|X)] + [(a_2 \pm a_0|X)][(a_3 \pm a_1|X)] \right)
\]
\[
- [(a_0 \pm a_1|X)][(a_2 \pm a_3|X]) \right).
\]
(88)
This vanishes thanks to the three-term identity (59). We have therefore proven (II)_n. The cases of (I)_{n+1} and (II)_0 are similar. 

6. Hypergeometric \( \tau \)-Function

6.1. Main Theorem. In this section we give explicit lens elliptic hypergeometric solutions for the \( E_8 \) \( \tau \)-function on \( D \), as an infinite sequence of \( E_7 \) \( \tau^{(n)} \)-functions on \( D_n \).

Definition 3. A \( \tau \)-function on \( D \), with \( \tau^{(n)} = 0 \) for \( n < 0 \), is called hypergeometric.

The hypergeometric solution may be expressed in either a determinant form, and a multi-dimensional sum/integral form, and the latter two forms are equivalent to each other.

To state our main theorem it will be convenient to first define some additional functions. First we define the function \( \psi^{(n)}_{ij}(X) \ (X \in D_n) \), as

\[
\psi^{(n)}_{ij}(X) := \psi \left( X + v^{(n)}_{ij}T \right),
\]
(89)
\[
v^{(n)}_{ij} := (1 - n)a_0 + (n + 1 - i - j)a_1 + (j - i)a_2,
\]
(90)
where \( \{ \pm a_0, \pm a_1, \pm a_2 \} \) is a \( C_3 \)-frame (which we fix for the moment), and \( \psi \) is given in terms of the elliptic hypergeometric sum/integral \( \text{(42)} \), as
\[
\psi(X) = I(\tilde{x}, \tilde{x}; \sigma, \tau), \tag{91}
\]
where the transformed variables \( \tilde{x} \) and \( \tilde{x} \) are defined in \( \text{(52)} \).

Next, we define a function \( \mathcal{G}^{(n)} \) in terms of the lens triple gamma functions \( \text{(31)} \), by
\[
\mathcal{G}^{(n)}(X) := \prod_{0 \leq i \leq 3} \left\{ \Gamma^+_{i \sigma} \left( (1 - n)\tau + x_i + x_j, 1 - n + x_i + x_j; \sigma, \tau, \tau \right) \right. \\
\times \left. \left( \gamma^+_i (x_i + x_j; \sigma, \tau, \tau) \right)^n \right\} \\
\times \prod_{0 \leq i < j \leq 3 \atop 4 \leq i < j \leq 7} \left\{ \Gamma^+_{i \sigma} (\tau + x_i + x_j, 1 + x_i + x_j; \sigma, \tau, \tau) \right. \\
\times \left. \left( \gamma^+_i (x_i + x_j + \mu; \sigma, \tau, \tau) \right)^n \right\}. \tag{92}
\]

We define a function \( d^{(n)} \) by
\[
d^{(n)}(X) := e^{\frac{4\pi i (\tau - 1)}{r}} e^{\frac{2\pi i (\sigma - \tau - x_0 - x_0 + x_1)}{r}} \\
\times \prod_{k=1}^{n} \left[ \theta_{\sigma}(x_0 + x_3 + (1 - n)\tau, x_0 + x_3 + (1 - n))_{k-1} \right. \\
\left. \theta_{\sigma}(x_1 + x_2 + (1 - n)\tau, x_1 + x_2 + (1 - n))_{k-1} \right. \\
\left. \theta_{\sigma}(x_0 - x_3 - (k - 1)\tau, x_0 - x_3 - (k - 1))_{n-k} \right. \\
\left. \theta_{\sigma}(x_1 - x_2 - (k - 1)\tau, x_1 - x_2 - (k - 1))_{n-k} \right], \tag{93}
\]
where
\[
\theta_{\sigma}(x, x) := \prod_{j=0}^{k-1} \theta_{\sigma}(x + j\tau, x + j), \quad (k = 0, 1, \ldots), \tag{94}
\]
and \( \theta_{\sigma} \) is a lens theta function defined in \( \text{(3)} \).

We also define \( e^{(n)} \) by
\[
e^{(n)}(X) := e^{\frac{2\pi i (\sigma + x)}{r}} e^{-nQ(X)}, \tag{95}
\]
where
\[
Q(X) := \frac{2\pi i}{r} \left( \frac{1}{2\tau}(x|x) - \frac{1}{2}(x|x) \right). \tag{96}
\]

Finally, we define a function \( g^{(n)}(X) \), as the following combination of the above three functions
\[
g^{(n)}(X) := \frac{e^{(n)}(X)}{d^{(n)}(X)} \mathcal{G}^{(n)}(X). \tag{97}
\]
We now come to the main theorem of this paper:

**Theorem 1.** For a $C_3$-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type $\Pi_1$, the function $\tau(X)$ on $D = \sqcup_{n \in \mathbb{Z}} D_n \ (69)$, defined on each $D_n$ by

$$\tau(n)(X) = \tau(X)|_{D_n} := \begin{cases} g^{(n)}(X) \det \left( \psi^{(n)}_{i,j}(X) \right)_{i,j=1}^{n} & (n \geq 0), \\ 0 & (n < 0), \end{cases}$$

is a hypergeometric $\tau$-function which satisfies the Hirota equations (64). Moreover each $\tau(n)$ is invariant under the action of $W(E_7)$.

**Remark 5.** The definition of the $\tau$-function in Theorem [1] a priori depends on the choice of the $C_3$-frame of type $\Pi_1$. However, the $W(E_7)$-invariance, together with the fact that any two $C_3$-frames of type $\Pi_1$ are related by an element of $W(E_7)$ (Proposition [9]), means that the $\tau$-function is actually independent of such a choice.

The $n = 0$, and $n = 1$ cases of Theorem [1] are explicitly given by:

$$\tau(0)(X) = g^{(0)}(X) = \prod_{0 \leq i < j \leq 7} \Gamma_{x,\tau}(x_i + x_j, 1 + x_i + x_j; \sigma, \tau), \quad (99)$$

and

$$\tau(1)(X) = g^{(1)}(X) \psi(X) = e^{-Q(X)} I(x, \tilde{x}; \sigma, \tau) \prod_{0 \leq i \leq 3, 4 \leq j \leq 7} \Gamma_{x,\tau}(x_i + x_j, 1 + x_i + x_j; \sigma, \tau, \mu) \times \prod_{0 \leq i < j \leq 3 \text{ or } 4 \leq i \leq j \leq 7} \Gamma_{x,\tau}(x_i + x_j, 1 + x_i + x_j; \sigma, \tau, \mu) \psi(x_i + x_j; \sigma, \tau) \prod_{0 \leq i < j \leq 7} \Gamma_{x,\tau}(x_i + x_j, 1 + x_i + x_j; \sigma, \tau, \mu), \quad (100)$$

where in the last line we have used the transformation (51). This last equality of (100) gives a manifestly $\mathcal{E}_8$-symmetric expression for $\tau^{(1)}$.

**Proposition 11.** The $\tau$-function given by Theorem [1] is non-zero for all $n \geq 0$. It is also the unique hypergeometric $\tau$-function for the initial conditions (99), (100).

**Proof.** Suppose that $\tau^{(n+1)} \equiv 0$ for some $n > 0$. Then from the recurrence relation (83) we find $\tau^{(n)}(X \pm a_2 T) = 0$. By repeating this recurrence we arrive at $\tau^{(1)} = 0$, which contradicts the expression for $\tau^{(1)}$ given in (100). Therefore $\tau^{(n)} \neq 0$ for all $n \geq 0$. 
Next, the recurrence relation (82) means that we can recursively determine $\tau^{(n+1)} (n > 1)$ from $\tau^{(n)}$ and $\tau^{(n-1)}$. This in turn means that the $\tau$-function in Theorem 1 is a unique hypergeometric function with given $\tau^{(0)}$ and $\tau^{(1)}$ (cf. Theorem 4.2 of [6]). □

6.2. Multi-Dimensional Sum/Integral Expression. In the definition (98), the $\tau$-function $\tau^{(n)}$ was expressed in terms of an $n \times n$ determinant. The $\tau$-function also has the following equivalent expression given in terms of an $n$-dimensional elliptic hypergeometric sum/integral:

**Theorem 2.** The $\tau$-function $\tau^{(n)}$ on $D_n$, can be written in terms of a multi-dimensional elliptic hypergeometric sum/integral, as

$$\tau^{(n)}(X) = g^{(n)}(X) d^{(n)}(X) I_n(X) = e^{(n)}(X) G^{(n)}(X) I_n(X),$$

where

$$I_n(X) = \frac{A^n}{n!} \sum_{z_1, \ldots, z_n=0}^{r-1} \prod_{1 \leq i < j \leq n} \theta_\sigma(\pm z_i \pm z_j, \pm \tilde{z}_i \pm \tilde{z}_j) \prod_{k=1}^{n} \Delta(z_k, \tilde{z}_k; \tilde{x}, \tilde{z}) \, dz_k,$$

(102)

$\Delta(z, z; \tilde{x}, \tilde{z})$ is the integrand defined in (47) with transformed variables $(\tilde{x}, \tilde{z})$ (52), $\lambda$ is given by (44), and

$$\tilde{z}_i := z_i + (r + (r + 1 \text{ mod } 2))(\tilde{x}_1 \text{ mod } 1), \quad i = 1, \ldots, n. \quad (103)$$

Note that (103) takes the following values,

$$\tilde{z}_i = \begin{cases} z_i, & (\tilde{x} \in \mathbb{Z}^8), \\ z_i + \frac{r}{2}, & (\tilde{x} \in \left(\mathbb{Z} + \frac{1}{2}\right)^8, \ r \text{ odd}), \\ z_i + \frac{r+1}{2}, & (\tilde{x} \in \left(\mathbb{Z} + \frac{1}{2}\right)^8, \ r \text{ even}). \end{cases} \quad (104)$$

To prove Theorem 2 we will use the following $r \geq 1$ analogue of War- naar's elliptic Krattenthaler determinant formula (24).

**Lemma 1.** For complex $z_i, i = 1, \ldots, n$, integer $z_i, i = 1, \ldots, n$, complex parameters $x_1, x_2$, and integer parameters $x_1, x_2$,

$$\det \left( \theta_\sigma(x_1 \pm z_i, x_1 \pm z_i)_{-1} \theta_\sigma(x_2 \pm z_i, x_2 \pm z_i)_{n-1} \right)_{i,j=1}^{n} = e^{\frac{2\pi i}{r}(\tau^{-1}(\tau))} e^{\frac{2\pi i}{p}(x_1-x_1)}$$

$$\times \prod_{k=1}^{n} \theta_\sigma(x_2 \pm (x_1 + (k-1)\tau), x_2 \pm (x_1 + (k-1)))_{n-k} \quad (105)$$

$$\times \prod_{1 \leq i < j \leq n} \theta_\sigma(z_i \pm z_j, z_i \pm z_j).$$
Proof. This follows from the analogous identity with the regular theta functions for \(r = 1\) given in [24].

Proof of Theorem 2. Let us compute the determinant of \(\psi_{ij}^{(n)}(X) = \psi(X + \nu_{ij}^{(n)}T)\). We choose a \(C_3\)-frame of type II, as

\[
a_0 = \frac{1}{2}(v_0 + v_1 + v_2 + v_3),
\]

\[
a_1 = \frac{1}{2}(v_0 + v_1 - v_2 - v_3),
\]

\[
a_2 = \frac{1}{2}(v_0 - v_1 + v_2 - v_3),
\]

so that

\[
\psi_{ij}^{(n)} = (1 - i)v_0 + (1 - j)v_1 + (j - n)v_2 + (i - n)v_3.
\]

The dependence on \(i, j\), shifts the variable \(X = (x, \xi)\), and when converted into the transformed variables \(\tilde{X} = (\tilde{x}, \tilde{\xi})\) (52), this amounts to the shift \(\tilde{X} \rightarrow \tilde{X} + \nu_{ij}^{(n)}T\) with

\[
\tilde{v}_{ij}^{(n)} = \psi_{ij}^{(n)} + (n - 1) \sum_{i=0}^{3} v_i = (n - i)v_0 + (n - j)v_1 + (j - 1)v_2 + (i - 1)v_3.
\]

From the definition (89) it follows that

\[
\psi_{ij}^{(n)}(X) = \psi(X + \nu_{ij}^{(n)}T) = \lambda \sum_{x=0}^{\ell-1} \int_{[0,1]} dz \Delta(z; \hat{x}, \hat{\xi}) \hat{f}_i(z, \hat{\xi}) \hat{g}_j(z, \hat{\xi}),
\]

where

\[
f_i(z, \xi) := \theta_{\sigma}(\tilde{x}_0 \pm z, x_0 \pm \xi)_{n-i} \theta_{\sigma}(\tilde{x}_3 \pm z, x_3 \pm \xi),
\]

\[
g_j(z, \xi) := \theta_{\sigma}(\tilde{x}_1 \pm z, x_1 \pm \xi)_{n-j} \theta_{\sigma}(\tilde{x}_2 \pm z, x_2 \pm \xi),
\]

for \(i, j = 1, 2, \ldots, n\), where \(\theta_{\sigma}(z, \xi)\) and \(\lambda\) are defined in (94) and (44), respectively.

The determinant of \(\psi_{ij}^{(n)}(X)\) (109), may be written as

\[
\det(\psi_{ij}^{(n)}(X))_{i,j=1}^n = \frac{\lambda^n}{n!} \sum_{z_1, \ldots, z_n = 0}^{\ell-1} \int_{[0,1]^n} \det(f_i(z_j, \hat{\xi}_j))_{i,j=1}^n \det(g_i(z_j, \hat{\xi}_j))_{i,j=1}^n \Delta(z_k, \hat{\xi}_k) \hat{z}_k dz_k.
\]
This determinant may be evaluated using Lemma (1), which results in the expression (101), where \(d^{(n)}(X)\) defined in terms of transformed variables \((\tilde{x}, \tilde{x})\), is given by (equivalent to previous definition given in (93)),

\[
d^{(n)}(X) = e^{\frac{4\pi i}{7}(\tau - 1)(\tau)} e^{\frac{2\pi i}{7}(\tilde{x}_0 + \tilde{x}_1 - 2\tilde{x}_2)} \prod_{k=1}^{n} \theta_{x}((\tilde{x}_0 \pm (\tilde{x}_3 + (k - 1)\tau), (\tilde{x}_0 \pm (\tilde{x}_3 + (k - 1)\tau))_{n-k}) \quad (113)
\]

Here we have used the properties of the lens theta function \(\theta_{x}\) which appear in (10) and (11).

\[\square\]

7. Proof of Theorem [I]

In this final section we will prove Theorem [I]. We begin by proving the \(W(E_7)\)-invariance of the \(\tau\)-function of Theorem [I].

7.1. \(W(E_7)\) Invariance.

**Proposition 12.** \(\tau^{(0)}\) is \(W(E_7)\)-invariant.

**Proof.** The Weyl group \(W(E_7)\) is generated by Weyl reflection with respect to the roots listed in (73). The Weyl reflections with respect to the roots \(\pm (\nu_i - \nu_j)\), generate the symmetric group \(S_8\), under which the \(\tau^{(0)}\) given in (99) is manifestly invariant. The remaining roots \((\pm \nu_0 \pm \nu_1 \pm \ldots \pm \nu_7) / 2\) (with four minus signs), are mapped to each other under the symmetric group \(S_8\), and we conclude the \(W(E_7)\) is generated by \(S_8\) together with an extra element \(w_0 \in W(E_7)\), representing the Weyl reflection with respect to \((-\nu_0 - \nu_1 - \nu_2 - \nu_3 + \nu_4 + \nu_5 + \nu_6 + \nu_7) / 2\).

Let us consider \(\tau^{(0)}\). The Weyl reflection \(w_0 \in W(E_7)\) acts on the coordinate \(X = (x, \bar{x}) \in D_0\) as

\[
w_0(x_i + x_j) = \begin{cases} 
\sigma - x_i - x_j & (\{i, j, k, l\} = \{0, 1, 2, 3\} \text{ or } \{4, 5, 6, 7\}), \\
x_i + x_j & (i \in \{0, 1, 2, 3\}, j \in \{4, 5, 6, 7\}),
\end{cases} \quad (114)
\]

for continuous variables and

\[
w_0(\bar{x}_i + \bar{x}_j) = \begin{cases} 
\bar{r} - 1 - \bar{x}_i - \bar{x}_j & (\{i, j, k, l\} = \{0, 1, 2, 3\} \text{ or } \{4, 5, 6, 7\}), \\
\bar{x}_i + \bar{x}_j & (i \in \{0, 1, 2, 3\}, j \in \{4, 5, 6, 7\}),
\end{cases} \quad (115)
\]

for discrete variables. Note that the constants \(\sigma\), and \(r - 1\), in the two equations, come from the fact that we are considering the specific hyperplane \(D_0 \subset H_\sigma \times H_{r-1}\).
Thus the action of \( w \) under the Weyl reflection \( w \) is manifestly invariant under \( \Gamma \). Since we find for \( \{i, j, k, l\} = \{0, 1, 2, 3\} \) or \( \{4, 5, 6, 7\} \), that
\[
\begin{align*}
    w_0 \left( \Gamma^+ \left( \tau + x_i + x_j, 1 + x_i + x_j; \sigma, \tau, \tau \right) \right) &= \Gamma^+ \left( \tau + x_i - x_j, r - x_i - x_j; \sigma, \tau, \tau \right) \\
    &= \Gamma^+ \left( \tau + x_i + x_j, 1 + x_i + x_j; \sigma, \tau, \tau \right),
\end{align*}
\]
where in the last line we used (100). This shows the invariance of \( \tau^{(0)} \) in (99) under the Weyl reflection \( w_0 \).

\[ \square \]

**Proposition 13.** \( \tau^{(1)} \) is \( W(E_7) \)-invariant.

**Proof.** Let us start with the manifestly \( \mathcal{E}_8 \)-symmetric expression for \( \tau^{(1)} \) previously given in (100). Since \( W(E_7) \) is generated by \( w_0 \) and \( \mathcal{E}_8 \) we only need to check invariance under \( w_0 \), as in the proof of Proposition 12. The difference from the proof there is that now we have \( X \in D_1 \subset H_{\tau+\sigma} \times H_r \).

Since \( Q(X) \) is generated by \( \mathcal{E}_7 \)-invariant bilinear form, we only need to check the \( W(E_7) \)-invariance of the factors
\[
I(x, \mathcal{S} \sigma, \mathcal{S} \tau) \prod_{0 \leq i < j \leq 7} \Gamma^+_{\sigma \tau}(x_i + x_j, x_i + x_j; \sigma, \tau, \mu).
\]

The Weyl reflection \( w_0 \in W(E_7) \) acts on the coordinates \( X \in D_1 \subset H_{\tau+\sigma} \times H_r \) as
\[
\begin{align*}
    w_0(x_i) &= \begin{cases} 
        x_i + \frac{\sigma + \tau}{2} - \frac{1}{2} \sum_{i=0}^{3} x_i & i \in \{0, 1, 2, 3\}, \\
        x_i + \frac{\sigma + \tau}{2} - \frac{1}{2} \sum_{i=4}^{7} x_i & i \in \{4, 5, 6, 7\}, 
    \end{cases} \\
    w_0(x_i) &= \begin{cases} 
        x_i + \frac{\sigma + \tau}{2} - \frac{1}{2} \sum_{i=0}^{3} x_i & i \in \{0, 1, 2, 3\}, \\
        x_i + \frac{\sigma + \tau}{2} - \frac{1}{2} \sum_{i=4}^{7} x_i & i \in \{4, 5, 6, 7\}. 
    \end{cases}
\end{align*}
\]

Note that this \( w_0 \) transformation takes the same form as the transformation rule of \( I(x, \mathcal{S} \sigma, \mathcal{S} \tau) \), given in (52), where the coordinates of \( X \in D_1 \subset H_{\tau+\sigma} \times H_r \), also exactly satisfy the balancing condition
\[
\sum_{i=0}^{7} x_i = 2(\sigma + \tau), \quad \sum_{i=0}^{7} x_i = 2r.
\]

Thus the action of \( w_0 \), followed by the transformation (51), gives
\[
w_0(I(x, \mathcal{S} \sigma, \mathcal{S} \tau)) = I(x, \mathcal{S} \sigma, \mathcal{S} \tau) \prod_{0 \leq i < j \leq 7} \left( \Gamma(x_i + x_j, x_i + x_j; \sigma, \tau) \right)^{-1}.
\]

We also have
\[
w_0(x_i + x_j) = \begin{cases} 
    \sigma + \tau - x_i - x_j & (i, j, k, l) = \{0, 1, 2, 3\} \text{ or } \{4, 5, 6, 7\}, \\
    x_i + x_j & (i \in \{0, 1, 2, 3\}, j \in \{4, 5, 6, 7\}),
\end{cases}
\]
and
\[ w_0(x_i + x_j) = \begin{cases} 
  r - x_i - x_j & \text{if } \{i, j, k, l\} = \{0, 1, 2, 3\} \text{ or } \{4, 5, 6, 7\}, \\
  x_i + x_j & \text{if } \{i, j\} = \{0, 1, 2, 3\}, \{4, 5, 6, 7\},
\end{cases} \]
(123)

which for \( i, j \in \{0, 1, 2, 3\} \) or \( i, j \in \{4, 5, 6, 7\} \), results in
\[
\begin{align*}
  w_0 & \left( \Gamma_{\sigma \tau}(x_i + x_j, x_i + x_j; \sigma, \tau, \mu) \right) \\
  &= \Gamma_{\sigma \tau}(\sigma + \tau - (x_k + x_l), r - (x_k + x_l); \sigma, \tau, \mu) \\
  &= \Gamma_{\sigma \tau}(x_k + x_l, x_k + x_l; \sigma, \tau, \mu) \Gamma(x_k + x_l, x_k + x_l; \sigma, \tau),
\end{align*}
\]
(124)

where in the last line we have used (40). The contributions of the type \( \Gamma \) in the last line exactly cancel the contribution coming from the factors in the product of (121), and thus (117) is invariant under the Weyl reflection \( w_0 \).

**Proposition 14.** \( \tau^{(n)} \) is \( W(E_7) \)-invariant for \( n = 0, 1, \ldots \).

**Proof.** As stated in the proof of Proposition 11, \( \tau^{(n)} \) is defined recursively from \( \tau^{(0)} \) and \( \tau^{(1)} \), which we have shown already to be \( W(E_7) \)-invariant. This proves the \( W(E_7) \)-invariance of \( \tau^{(n)} \).

**Corollary 2.** The product \( \mathcal{G}^{(n)}(X)I_n(X) \), with \( \mathcal{G}^{(n)}(X) \) given in (92) and \( I_n \) given in (102), is \( W(E_7) \)-invariant.

**Proof.** This immediately follows from Proposition 14, the expression of \( \tau^{(n)} \) given in (101), and the \( W(E_7) \)-invariance of \( e^{(n)}(X) \).

**7.2. Bilinear Identities.** Having proven the \( W(E_7) \)-invariance of the \( \tau^{(n)} \), we will now prove that the Hirota equations are satisfied. For this purpose we start with a few lemmas concerning the function \( g^{(n)} \).

**Lemma 2.** For a \( C_3 \)-frame \( \{\pm a_0, \pm a_1, \pm a_2\} \) of type \( \Pi_1 \), we have
\[
g^{(n-1)}(X - a_0T)g^{(n+1)}(X + a_0T) = \frac{[(a_0 \pm a_2|X)]}{[(a_1 \pm a_2|X)]} g^{(n)}(X \pm a_1T),
\]
(125)

\( (X \in D_n, \quad n = 1, 2, \ldots) \).

**Proof.** Since each \( \tau^{(n)} \) has manifest \( W(E_7) \)-symmetry (Proposition 14), it is sufficient to write down the Hirota equations (64) for a special example of \( C_3 \)-frame of type \( \Pi_1 \). Let us choose the \( C_3 \)-frame to be as given in (106).

We compute the ratio \( g^{(n-1)}(X - a_0T)g^{(n+1)}(X + a_0T)/g^{(n)}(X \pm a_1T) \) for each factor \( e^{(n)}, d^{(n)}, \mathcal{G}^{(n)} \), of the function \( g^{(n)} \) (97). For \( e^{(n)} \) we find from the
\[ e^{(n-1)}(X - a_0 T) e^{(n+1)}(X + a_0 T) = e^{2\pi i (\sigma + 1)} e^{-Q(X + a_0 T) - Q(X - a_0 T)} \]
\[ = e^{2\pi i (\sigma + 1 - 2(a_0 x) + 2(a_0 x))} \]
\[ = e^{2\pi i (\sigma + 1 - x_0 - x_2 - x_1 + x_0 + x_1 + x_2 + x_3)}. \]

For \( d^{(n)} \) we compute from the definition \((93)\), after many cancellations,
\[ \frac{d^{(n-1)}(X - a_0 T) d^{(n+1)}(X + a_0 T)}{d^{(n)}(X \pm a_1 T)} = e^{2\pi i (\sigma + 1 - x_0 - x_2 + x_1 + x_3)} \theta_{\nu}(X_0 \pm X_3) \theta_{\nu}(X_1 \pm X_2), \]

Let us next compute the ratio
\[ \frac{\mathcal{G}^{(n-1)}(X - a_0 T) \mathcal{G}^{(n+1)}(X + a_0 T)}{\mathcal{G}^{(n)}(X \pm a_1 T)} , \]
for \( \mathcal{G}^{(n)} \). In this computation most of the gamma function factors in the definition of \( \mathcal{G}^{(n)} \) in \((92)\) cancel out; the exceptions are the cross terms involving \( \Gamma^+_\nu \), for \( 0 \leq i < j \leq 3 \) (hence the result is independent of the value of \( n \)). For example, for the term with \( i = 0, j = 2 \), the expression \( X \pm a_0 T \) gives \((x_0 + x_2 \pm \tau, x_0 + x_2 \pm 3)\) while \( X \pm a_1 T \) gives \((x_0 + x_2, x_0 + x_2)\) twice, so that we have
\[ \Gamma^+_\nu(x_0 + x_2 + 2\tau, x_0 + x_2 + 2; \sigma, \tau, \tau) \Gamma^+_\nu(x_0 + x_2, x_0 + x_2; \sigma, \tau, \tau) \]
\[ = \frac{e^{2\pi i (x_0 + x_2 + \tau, x_0 + x_2 + 1; \sigma, \tau, \tau)}}{e^{2\pi i (x_0 + x_2 + x_0 + x_2; \sigma, \tau, \tau)}} \gamma_{\sigma}(x_0 + x_2 + \tau, x_0 + x_2 + 1; \sigma, \tau, \tau) \gamma_{\sigma}(x_0 + x_2, x_0 + x_2; \sigma, \tau, \tau) \]
\[ = \frac{\Gamma(x_0 + x_2 + \tau, x_0 + x_2 + 1; \sigma, \tau)}{\Gamma(x_0 + x_2, x_0 + x_2; \sigma, \tau)} \theta_{\nu}(x_0 + x_2, x_0 + x_2), \]
where we used \((40), (38), (14)\) and then \((28)\). By repeating this manipulation we obtain
\[ \frac{\mathcal{G}^{(n-1)}(X - a_0 T) \mathcal{G}^{(n+1)}(X + a_0 T)}{\mathcal{G}^{(n)}(X \pm a_1 T)} = \theta_{\nu}(X_0 + X_2) \theta_{\nu}(X_0 + X_3) \theta_{\nu}(X_1 + X_2) \theta_{\nu}(X_1 + X_3). \]
Finally by combining all of the above, we obtain
\[
g^{(n-1)}(X - a_0 T)g^{(n+1)}(X + a_0 T) = e^{\sum_{j_2 + x_3 - x_2 - x_3}(X_0 + X_2)\theta_\sigma(X_1 + X_3)} \frac{\theta_\tau(X_0 - X_3)\theta_\sigma(X_1 - X_2)}{[(a_0 \pm a_2|X)]}.
\]  
(131)

**Lemma 3.** For a C3-frame \(\{\pm a_0, \pm a_1, \pm a_2\}\) of type \(\Pi_1\), we have
\[
g^{(n)}(X \pm a_1 T) = \frac{[(a_0 \pm a_1|X)]}{[(a_0 \pm a_2|X)]}, \quad (X \in D_n), \quad n = 0, 1, 2, \ldots.
\]  
(132)

**Proof.** Let us first consider the case \(n = 0\). Let us choose the C3-frame of type \(\Pi_1\) to be (106) as before. In the previous computation of the ratio \(g^{(0)}(X \pm a_1 T)/g^{(0)}(X \pm a_2 T)\) most of the gamma function factors cancels out; the exceptions are the cross terms involving \(\gamma^+\), for \(0 \leq i < j \leq 3\).

For example, for the term with \(i = 0, j = 1\), the expression \(X \pm a_1 T\) gives \((x_0 + x_1 \pm \tau, x_0 \pm x_1 \pm 1)\) while \(X \pm a_2 T\) gives \((x_0 + x_1, x_0 + x_1 \pm 1)\) twice. We can then appeal to the manipulations (129), to obtain
\[
g^{(0)}(X \pm a_1 T) = \frac{\theta_\sigma(x_0 + x_1, x_0 + x_1; \tau) \theta_\tau(x_2 + x_3, x_2 + x_3; \tau)}{\theta_\sigma(x_0 + x_2, x_0 + x_2; \tau) \theta_\tau(x_1 + x_3, x_1 + x_3; \tau)} = \frac{[X_0 + X_1][X_2 + X_3]}{[X_0 + X_2][X_1 + X_3]} = \frac{[(a_0 \pm a_1|X)]}{[(a_0 \pm a_2|X)]}.
\]  
(133)

The case of \(n = 1\) is similar. We can use the expression for \(g^{(1)}\) coming from (97),
\[
g^{(1)}(X) = e^{-Q(X)} \prod_{0 \leq i \leq 3} \frac{\Gamma^+(x_i + x_j, x_i + x_j; \sigma, \tau, \mu)}{\Gamma^+(x_i + x_j, x_i + x_j; \sigma, \tau, \mu)} \times \prod_{4 \leq i \leq j \leq 7} \frac{\Gamma^+(x_i + x_j, 0; \sigma, \tau, \mu)}{\Gamma^+(x_i + x_j, 0; \sigma, \tau, \mu)}.
\]  
(134)

In the computation of the ratio \(g^{(0)}(X \pm a_1 T)/g^{(0)}(X \pm a_2 T)\) the contribution from the prefactor \(e^{-Q(X)}\) cancels out, and the only relevant part which remains after taking the ratio is
\[
\prod_{0 \leq i < j \leq 3} \frac{\Gamma^+(x_i + x_j, 1; \sigma, \tau, \mu)}{\Gamma^+(x_i + x_j, 1; \sigma, \tau, \mu)}.
\]  
(135)

This is exactly the same factor from the definition of \(g^{(0)}\) that contributes to the case of \(n = 0\), and hence the computation will goes through exactly the same as for the \(n = 0\) case, to obtain (132) for \(n = 1\).
For the case $n > 1$ we use the induction with respect to the integer $n$. From (125) we have

\[
g^{(n+1)}(X + a_0 T) = \frac{g^{(n)}(X + a_1 T)}{g^{(n)}(X - a_0 T)} \frac{[(a_0 + a_2|X)]}{[(a_1 + a_2|X)]} = \frac{g^{(n)}(X + a_2 T)}{g^{(n)}(X - a_0 T)} \frac{[(a_0 + a_1|X)]}{[(a_1 + a_2|X)]},
\]

where in the last line we used the assumption for $n$. Shifting the value of $X$, and using one of the two expressions above, gives

\[
g^{(n+1)}(X + a_1 T) = \frac{g^{(n)}(X - a_0 T + a_1 T + a_2 T)}{g^{(n-1)}(X - 2a_0 T + a_1 T)} \frac{[(a_0 + a_1|X) - T + T]}{[(a_1 + a_2|X) + T]},
\]

\[
g^{(n+1)}(X + a_2 T) = \frac{g^{(n)}(X - a_0 T + a_1 T + a_2 T)}{g^{(n-1)}(X - 2a_0 T + a_2 T)} \frac{[(a_0 + a_2|X) - T + T]}{[(a_1 + a_2|X) + T]},
\]

where we used $(a_i|a_j) = \delta_{ij}$. We thus obtain

\[
\frac{g^{(n+1)}(X + a_1 T)}{g^{(n+1)}(X + a_2 T)} = \frac{g^{(n-1)}(X - 2a_0 T + a_2 T)}{g^{(n-1)}(X - 2a_0 T + a_1 T)} \frac{[(a_0 + a_1|X) - T + T]}{[(a_0 + a_2|X) - T + T]} \frac{[(a_0 + a_1|X)]}{[(a_0 + a_2|X)]} = \frac{[(a_0 + a_1|X)]}{[(a_0 + a_2|X)]},
\]

where in the last line we used the assumption for $n - 1$. This is what we wanted to show. \hfill \Box

We finally come to the proof that the $\tau^{(n)}$-functions satisfy the desired bilinear identities. We first prove the special cases $(\Pi_2)_0, (\Pi_1)_0, (\Pi_0)_0, (\Pi_{1/2})_0$. We then prove $(\Pi_1)_{n=1,2,...}$, from which the remaining cases will follow, by Proposition 10.

**Proposition 15.** $(\Pi_2)_0$ holds.

**Proof.** The identity $(\Pi_2)_0$ reads

\[
[(a_0 + a_1|X)]\tau^{(0)}(X + a_2) = 0,
\]

for a $C_3$-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ with $(\phi|a_0) = (\phi|a_1) = 1, (\phi|a_2) = 0$ (recall (77)). Since $a_0 + a_1 = \phi$ (see Remark 4), one finds for $X \in D_0$ that

\[
[(a_0 + a_1|X)] = [(\phi|X)] = [(\sigma, -1)] = e^{\phi_{\sigma(-1)}}e^{2\pi i r} = 0.
\]

\hfill \Box
Proposition 16. (II)_{10} holds.

Proof. This is an immediate consequence of (132), since \( \tau^{(0)} = g^{(0)} \).

Proposition 17. (II)_{00} holds.

Proof. For a \( C_{3} \)-frame \( \{ \pm a_{0}, \pm a_{1}, \pm a_{2} \} \) of type (II)_{0}, we can choose an extra vector \( a_{3} \) such that \( \{ \pm a_{i}, \pm a_{j}, \pm a_{3} \} \) for \( 0 \leq i < j \leq 2 \), are all \( C_{3} \)-frames of type (II)_{1} (see Remark 4). This implies that by using (132), for \( \tau^{(0)} = g^{(0)} \)

\[
\tau^{(0)}(X \pm a_{i}T) = \frac{[(a_{3} \pm a_{i}|X)]}{[(a_{3} \pm a_{j}|X)]} \tau^{(0)}(X \pm a_{j}T) \quad (0 \leq i < j \leq 2).
\]

The identity in question, namely (II)_{00} (81) with \( n = 0 \), now reduces to

\[
\begin{align*}
[(a_{1} \pm a_{2}|X)][(a_{3} \pm a_{0}|X)] + [(a_{2} \pm a_{0}|X)][(a_{3} \pm a_{1}|X)] \\
+ [(a_{0} \pm a_{1}|X)][(a_{3} \pm a_{2}|X)] = 0.
\end{align*}
\]

This holds due to the three-term identity (59).

Proposition 18. (I)_{1/2} holds.

Proof. As a \( C_{3} \)-frame of type (I)_{1/2} we can choose from the \( W(E_{7}) \)-orbit a representative \( \{ \pm a_{0}, \pm a_{1}, \pm a_{2} \} = \{ \pm v_{1}, \pm v_{2}, \pm v_{3} \} \). We wish to show

\[
[X_{2} \pm X_{3}] \tau^{(0)}(X - v_{0}T) \tau^{(1)}(X + v_{0}T) + [X_{3} \pm X_{1}] \tau^{(0)}(X - v_{0}T) \tau^{(1)}(X + v_{1}T) \\
+ [X_{1} \pm X_{2}] \tau^{(0)}(X - v_{2}T) \tau^{(1)}(X + v_{2}T) = 0.
\]

Let us define

\[
\mathcal{F}(X) := \prod_{0 \leq i < j \leq 7} \Gamma_{\sigma}^{+}(x_{i} + x_{j}, x_{i} + x_{j}; \sigma, \tau, \tau),
\]

\[
J(X) := e^{-Q(X)}I(X) \prod_{0 \leq i < j \leq 7} \gamma_{\sigma}^{+}(x_{i} + x_{j}, x_{i} + x_{j}; \sigma, \tau, \mu),
\]

so that

\[
\tau^{(0)} = \mathcal{F}(X + T), \quad \tau^{(1)} = \mathcal{F}(X)J(X).
\]

We see that the following two ratios

\[
\begin{align*}
\frac{\mathcal{F}(X + T - v_{0}T)}{\mathcal{F}(X + T)} &= \prod_{0 \leq i \leq 7} \frac{\Gamma_{\sigma}^{+}(x_{0} + x_{j}, x_{0} + x_{j}; \sigma, \tau, \tau)}{\Gamma_{\sigma}^{+}(\tau + x_{0} + x_{j}, 1 + x_{0} + x_{j}; \sigma, \tau, \tau)}, \\
\frac{\mathcal{F}(X + v_{0}T)}{\mathcal{F}(X)} &= \prod_{0 \leq i \leq 7} \frac{\Gamma_{\sigma}^{+}(\tau + x_{0} + x_{j}, 1 + x_{0} + x_{j}; \sigma, \tau, \tau)}{\Gamma_{\sigma}^{+}(x_{0} + x_{j}, x_{0} + x_{j}; \sigma, \tau, \tau)}.
\end{align*}
\]
are the inverse of each other. This means that
\[
\tau^{(0)}(X - v_0 T)\tau^{(1)}(X + v_0 T) = \mathcal{F}(X + T - v_0 T)\mathcal{F}(X + v_0 T)J(X + v_0 T)
\]
\[
= \mathcal{F}(X + T)\mathcal{F}(X)e^{-Q(X + v_0 T)}I(X + v_0 T),
\]
(147)
Substituting (147) into (143), (143), gives
\[
[X_1 \pm X_2]e^{-Q(X + v_0 T)}I(X + v_0 \tau) + [X_2 \pm X_0]e^{-Q(X + v_1 T)}I(X + v_1 T)
+ [X_0 \pm X_1]e^{-Q(X + v_2 T)}I(X + v_2 T) = 0.
\]
(148)
Next from the definition of \(Q(X)\) in (96), we obtain
\[
Q(X + v_1 T) = Q(X) + \frac{2\pi i}{r} \left( x_j - x_i + \frac{\tau - 1}{2} \right),
\]
(149)
and consequently (148) reduces to
\[
[X_1 \pm X_2] e^{-\frac{2\pi i}{r}(x_i - x_j)}T_{\tau,j}I(t, a)
+ [X_2 \pm X_0] e^{-\frac{2\pi i}{r}(x_j - x_k)}T_{\tau,k}I(t, a)
+ [X_0 \pm X_1] e^{-\frac{2\pi i}{r}(x_k - x_l)}T_{\tau,l}I(t, a) = 0,
\]
(150)
where the shift operator \(T_{\tau,k}\) is defined in (45). This is exactly the contiguity relation (46).

\[\square\]

**Proposition 19.** (II)\(_n\) holds for \(n = 1, 2, \ldots\).

**Proof.** For a \(C_2\)-frame \{\(\pm a_0, \pm a_1, \pm a_2\)\} of type II, we obtain from Lemma 2 and Lemma 3 that for \(n = 0, 1, \ldots\),
\[
[(a_1 \pm a_2|X)]g^{(n-1)}(X - a_0 T)g^{(n+1)}(X + a_0 T)
= [(a_2 \pm a_0|X)]g^{(n)}(X \pm a_1 T)
= [(a_0 \pm a_1|X)]g^{(n)}(X \pm a_2 T).
\]
(151)
This means that each of the above factors may be cancelled out of (II)\(_n\), resulting in
\[
K^{(n-1)}(X - a_0 T)K^{(n+1)}(X + a_0 T)
+ K^{(n)}(X \pm a_1 T) + K^{(n)}(X \pm a_2 T) = 0, \quad (n = 1, 2, \ldots),
\]
(152)
where \(K^{(n)}(x) := \det \left( \psi^{(n)}_{ij}(x) \right)_{i,j=1}^n\) is the Casorati determinant. The last equation is satisfied as a consequence of the Lewis Carroll formula. \[\square\]
APPENDIX A. DERIVATION OF W(E₇) SUM/INTEGRAL TRANSFORMATION

In this Appendix a proof will be given of Proposition 20. Note that in the following, the function $\Gamma(z, z)$ denotes the lens elliptic gamma function defined in [10]. First consider the following $A_1 \leftrightarrow A_0$ transformation of [10].

**Proposition 20.** Suppose that $t = (t_0, \ldots, t_5) \in \mathbb{C}^6$, $\text{Im}(t_i) > 0$, and $\ell = (\ell_0, \ldots, \ell_5) \in \mathbb{Z}^6$, satisfy

$$
\sum_{i=0}^{5} t_i \equiv \sigma + \tau \pmod{2r}, \quad \sum_{i=0}^{5} \ell_i \equiv 0 \pmod{r}.
$$

Then the sum/integral

$$
I_0(S|t, \ell) = \frac{\lambda}{2} \sum_{z_0, z_1 = 0}^{1} \prod_{0 \leq i < j \leq 3} \Gamma(t_i + t_j, \ell_i + \ell_j) \prod_{0 \leq i < j \leq 5} \Gamma(t_i + t_j, \ell_i + \ell_j + S)
$$

$$
\int_{z_0}^{z_1} dz_0 dz_1 \prod_{i=0}^{5} \Gamma(t_i + z_i, \ell_i + z_i) \prod_{j=3}^{5} \Gamma(t_j - z_i, \ell_j - z_i) \prod_{j=0}^{2} \Gamma(\pm(z_0 - z_1), \pm(z_0 - z_1)),
$$

(154)

where $S \in \mathbb{Z}$, and $\lambda$ is defined in [44], may be evaluated as

$$
I_0(S|t, \ell) = \prod_{0 \leq i < j \leq 2} \Gamma(t_i + t_j, \ell_i + \ell_j) \prod_{0 \leq i < j \leq 3} \Gamma(t_i + t_j, \ell_i + \ell_j + S)
$$

$$
\times \prod_{3 \leq i < j \leq 5} \Gamma(t_i + t_j, \ell_i + \ell_j - S).
$$

(155)

Note that $z_0, z_1$, in the sum (154), are regarded as elements of $\mathbb{Z}/r\mathbb{Z}$. For $S = 0$, (155) is the elliptic beta sum/integral formula [8].

We wish to extend the above formula to the case of $\ell \in \mathbb{Z}^8 \cup (\mathbb{Z} + 1/2)^8$. For this purpose, we set

$$
S = \begin{cases} 0 & (\ell \in \mathbb{Z}^8), \\ r & (\ell \in (\mathbb{Z} + 1/2)^8). \end{cases}
$$

(156)

The sum over $z_0, z_1$, satisfying $z_0 + z_1 = S$ can be exchanged for a sum over a new variable $z$, where

$$
\begin{cases}
z_0 = z, & z_1 = -z, \\
z_0 = z + \frac{r}{2}, & z_1 = -z + \frac{r}{2}, \\
z_0 = z + \frac{r+1}{2}, & z_1 = -z + \frac{r-1}{2},
\end{cases}
$$

(157)

(\ell \in \mathbb{Z}^8, S = 0),

(\ell \in (\mathbb{Z} + 1/2)^8, S = r; \ r \text{ even}),

(\ell \in (\mathbb{Z} + 1/2)^8, S = r; \ r \text{ odd}).
This may be concisely written as
\begin{align}
    z_0 &= +z + (r + ((r + 1) \mod 2))(\ell_1 \mod 1), \\
    z_1 &= -z + (r - ((r + 1) \mod 2))(\ell_1 \mod 1),
\end{align}
(158)

Using the \( r \)-periodicity of the lens elliptic gamma function, with the choice (156), (158), the sum/integral (154) may be written as
\[
I_0(t, \ell) := I_0(\mathbb{S}|t, \ell) = \lambda_{2r - 1} \sum_{z=0}^{r-1} \int_{[0,1]} dz \prod_{j=0}^{5} \frac{\Gamma(t_j \pm z, \ell_j \pm z_0)}{\Gamma(\pm 2z, \pm 2z_0)},
\]
(159)
while the formula (155) becomes
\[
I_0(t, \ell) = \prod_{0 \leq i \leq 5} \Gamma(t_i + t_j, \ell_i + \ell_j).
\]
(160)

Note that (160) is valid for both \( \ell \in \mathbb{Z}^8 \) and \( \ell \in (\mathbb{Z} + \frac{1}{2})^8 \).

Proposition 5 may be proven with the use of (160), analogously to a derivation given by Spiridonov [9].

Proof of Proposition 5. Consider \( \alpha \in \mathbb{C}, x, y \in \mathbb{C}^4, \) and \( x, y \in \mathbb{Z}^4 \cup (\mathbb{Z} + \frac{1}{2})^4, \) where \( \text{Im}(\alpha), \text{Im}(x_i), \text{Im}(y_i) > 0, \) and
\[
2\alpha + \sum_{j=0}^{3} x_j = 2\alpha + \sum_{j=0}^{3} y_j = \sigma + \tau, \quad 2\psi + \sum_{j=0}^{3} x_j = 2\psi + \sum_{j=0}^{3} y_j = kr,
\]
(161)
for some integer \( k. \) If both \( x, y \in \mathbb{Z}^4 \) or both \( x, y \in (\mathbb{Z} + \frac{1}{2})^4, \) then we choose \( \psi \in \mathbb{Z}, \) otherwise we choose \( \psi \in (\mathbb{Z} + \frac{1}{2}). \)

In terms of the above variables, consider the following sum/integral
\[
\sum_{z=0}^{r-1} \sum_{w=0}^{r-1} \int_{[0,1]^2} dwdz \frac{\Gamma(\alpha \pm z \pm w, \alpha \pm \hat{z} \pm \hat{w}) \Gamma(\pm 2z, \pm 2\hat{z}) \Gamma(\pm 2w, \pm 2\hat{w})}{\Gamma(t_j \pm z, \ell_j \pm z_0)} \prod_{j=0}^{3} \Gamma(x_j \pm z, x_j \pm \hat{z}) \Gamma(y_j \pm w, y_j \pm \hat{w})}
\]
(162)
where
\[
\hat{z} = +z + (r + ((r + 1) \mod 2))(\ell_1 \mod 1), \quad \hat{w} = +w + (r + ((r + 1) \mod 2))(\ell_1 \mod 1)
\]
(163)

The derivation appearing in [9] uses a different notation than is used here, but after a change of variables, both derivations are seen to be based on the same elliptic beta sum/integral formula (155) that was first proven by the authors [8,10]. Thus the derivations are equivalent. The authors thank V.P. Spiridonov for pointing this out.
The expression (162) may be sum/integrated in two different ways. First using (160) to sum/integrate (162) over $z, z$, gives

$$
\Gamma(2\alpha, 2\varpi) \prod_{0 \leq i < j \leq 3} \Gamma(x_i + x_j, x_i + x_j)
$$

$$
\times \sum_{w=0}^{r-1} \int_{[0,1]} dw \frac{\prod_{j=0}^{3} \Gamma(\alpha \pm w + x_j, \varpi \pm \hat{\varpi} + x_j) \Gamma(y_j \pm w, y_j \pm \hat{\varpi})}{\Gamma(\pm 2w, \pm 2\hat{\varpi})}. \tag{164}
$$

Next using (160) to sum/integrate (162) over $w, \varpi$, gives

$$
\Gamma(2\alpha, 2\varpi) \prod_{0 \leq i < j \leq 3} \Gamma(y_i + y_j, y_i + y_j)
$$

$$
\times \sum_{z=0}^{r-1} \int_{[0,1]} dz \frac{\prod_{j=0}^{3} \Gamma(\alpha \pm z + y_j, \varpi \pm \hat{\varpi} + y_j) \Gamma(x_j \pm z, x_j \pm \hat{\varpi})}{\Gamma(\pm 2z, \pm 2\hat{\varpi})}. \tag{165}
$$

Define the variables $t = (t_0, \ldots, t_7) \in \mathbb{C}^8$, and $\ell = (\ell_0, \ldots, \ell_7) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8$, as

$$
t_i = \alpha + x_i, \quad t_{i+4} = y_i, \quad \ell_i = \varpi + x_i, \quad \ell_{i+4} = y_i, \quad (i = 0, \ldots, 3), \tag{166}
$$

and $I(x, \varpi)$ as the following sum/integral

$$
I(x, \varpi) = \frac{A}{2} \sum_{z=0}^{r-1} \int_{[0,1]} dz \frac{\prod_{j=0}^{7} \Gamma(x_j \pm z, x_j \pm \hat{z})}{\Gamma(\pm 2z, \pm 2\hat{z})}. \tag{167}
$$

With these variables, the sum/integral appearing in (164) is given by

$$
I(t, \ell), \tag{168}
$$

while the sum/integral appearing in (165) is given by

$$
I(t', \ell'), \tag{169}
$$

where

$$
t'_i = t_i - \alpha = t_i + \frac{\sigma + \tau}{2} - \frac{1}{2} \sum_{i=0}^{3} t_i, \quad i = 0, 1, 2, 3,
$$

$$
t'_i = t_i + \alpha = t_i + \frac{\sigma + \tau}{2} - \frac{1}{2} \sum_{i=4}^{7} t_i, \quad i = 4, 5, 6, 7,
$$

$$
\ell'_i = \ell_i - \varpi = \ell_i + \frac{k\varpi}{2} - \frac{1}{2} \sum_{i=0}^{3} \ell_i, \quad i = 0, 1, 2, 3,
$$

$$
\ell'_i = \ell_i + \varpi = \ell_i + \frac{k\varpi}{2} - \frac{1}{2} \sum_{i=4}^{7} \ell_i, \quad i = 4, 5, 6, 7. \tag{170}
$$
For the factors of the lens elliptic gamma functions that appear outside the sum/integrals in (164), and (165), we have for distinct \( i, j \in \{0, 1, 2, 3\} \),

\[
x_i + x_j = t_i + t_j - 2\alpha, \quad x_i + x_j = \xi_i + \xi_j - 2\alpha,
\]

(171)

leading to, for \( i, j, k, l \in \{0, 1, 2, 3\} \),

\[
x_k + x_l = \sigma + \tau - t_i - t_j, \quad x_k + x_l = k\tau - \xi_i - \xi_j.
\]

(172)

Equating (164), with (165), and collecting all factors, we finally obtain

\[
I(t, \xi) = I(t', \xi') \prod_{0 \leq i < j \leq 3} \Gamma(t_i + t_j, \xi_i + \xi_j) \prod_{4 \leq i < j \leq 7} \Gamma(t_i + t_j, \xi_i + \xi_j),
\]

(173)

where \( t' \), and \( \xi' \) are given in (170).

\[\square\]

APPENDIX B. MULTIPLE BERNOULLI POLYNOMIALS

Let us define the multiple Bernoulli polynomials \( B_{n,k}(z; \omega_1, \ldots, \omega_n) \) via the generating function

\[
\frac{x^n e^{zx}}{\prod_{j=1}^{n} (e^{\omega_j x} - 1)} = \sum_{k=0}^{\infty} B_{n,k}(z; \omega_1, \ldots, \omega_n) \frac{x^k}{k!},
\]

(174)

where \( z \in \mathbb{C} \), and \( \omega_1, \ldots, \omega_n \in \mathbb{C} - \{0\} \). These functions previously appeared in relation to the modular properties of multiple gamma functions [25]. For this paper, only two particular multiple Bernoulli polynomials are needed.

One of these is \( B_{3,3}(z; \omega_1, \omega_2, \omega_3) \), which is given explicitly by

\[
B_{3,3}(z; \omega_1, \omega_2, \omega_3) = \frac{z^3}{\omega_1 \omega_2 \omega_3} - \frac{3z^2 \sum_{i=1}^{3} \omega_i}{2\omega_1 \omega_2 \omega_3} + \frac{z \left( \sum_{i=1}^{3} \omega_i^2 + 3 \sum_{1 \leq i < j \leq 3} \omega_i \omega_j \right)}{2\omega_1 \omega_2 \omega_3} - \frac{\left( \sum_{i=1}^{3} \omega_i \right) \left( \sum_{1 \leq i < j \leq 3} \omega_i \omega_j \right)}{4\omega_1 \omega_2 \omega_3}.
\]

(175)
The other is \( B_{4,4}(z; \omega_1, \omega_2, \omega_3, \omega_4) \), which is given by

\[
B_{4,4}(z; \omega_1, \omega_2, \omega_3, \omega_4) = \frac{z^4}{\prod_{i=1}^{4} \omega_i} - \frac{2z^3}{\prod_{i=1}^{4} \omega_i} \left( \sum_{i=1}^{4} \omega_i + 3 \sum_{1 \leq i < j \leq 4} \omega_i \omega_j \right) + \frac{z^2}{\prod_{i=1}^{4} \omega_i} \left( \sum_{i=1}^{4} \omega_i \right) \left( \sum_{1 \leq i < j \leq 3} \omega_i \omega_j \right) - \frac{z}{\prod_{i=1}^{4} \omega_i} \left( \sum_{i=1}^{4} \omega_i \right) \left( \sum_{1 \leq i < j < k \leq 4} \omega_i \omega_j \omega_k \right) - \frac{5}{30 \prod_{i=1}^{4} \omega_i} \sum_{1 \leq i < j \leq 4} (\omega_i \omega_j)^2 - \frac{15}{4} \sum_{i=1}^{4} \sum_{1 \leq j < k \leq 4} \omega_i^2 \omega_j \omega_k - \frac{45}{4} \prod_{i=1}^{4} \omega_i.
\]

(176)

The above two multiple Bernoulli polynomials are related by

\[
B_{4,4}(z + \omega_4; \omega_1, \omega_2, \omega_3, \omega_4) - B_{4,4}(z; \omega_1, \omega_2, \omega_3, \omega_4) = 4B_{3,3}(z; \omega_1, \omega_2, \omega_3).
\]

(177)

References

[1] H. Sakai, “Rational surfaces associated with affine root systems and geometry of the Painlevé equations,” Comm. Math. Phys. 220 no. 1, (2001) 165–229, [https://doi.org/10.1007/s002200100446](https://doi.org/10.1007/s002200100446).

[2] Y. Ohta, A. Ramani, and B. Grammaticos, “An affine Weyl group approach to the eight-parameter discrete Painlevé equation,” J. Phys. A 34 no. 48, (2001) 10523–10532, Symmetries and integrability of difference equations (Tokyo, 2000).

[3] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, and Y. Yamada, “\(E_9\) solution to the elliptic Painlevé equation,” J. Phys. A 36 no. 17, (2003) L263–L272.

[4] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, and Y. Yamada, “Point configurations, Cremona transformations and the elliptic difference Painlevé equation,” in Théories asymptotiques et équations de Painlevé, vol. 14 of Sémin. Congr., pp. 169–198. Soc. Math. France, Paris, 2006.

[5] K. Kajiwara, M. Noumi, and Y. Yamada, “Geometric aspects of painlevé equations,” J. Phys. A 50 no. 7, (Jan, 2017) 073001.

[6] M. Noumi, “Remarks on \(\tau\)-functions for the difference Painlevé equations of type \(E_8\),” [arXiv:math.CA/1604.06869 [math.CA]](https://arxiv.org/abs/1604.06869).

[7] F. Benini, T. Nishioka, and M. Yamazaki, “4d Index to 3d Index and 2d TQFT,” Phys. Rev. D86 (2012) 065015, [arXiv:1109.0283 [hep-th]](https://arxiv.org/abs/1109.0283).

[8] A. P. Kels, “New solutions of the star–triangle relation with discrete and continuous spin variables,” J. Phys. A48 no. 43, (2015) 435201, [arXiv:1504.07074 [math-ph]](https://arxiv.org/abs/1504.07074).

[9] V. P. Spiridonov, “Rarefied elliptic hypergeometric functions,” Adv. Math. 331 (2018) 830–873, [arXiv:1609.00715 [math.CA]](https://arxiv.org/abs/1609.00715).
[10] A. P. Kels and M. Yamazaki, “Elliptic hypergeometric sum/integral transformations and supersymmetric lens index,” SIGMA 14 (2018) 013, arXiv:1704.03159 [math-ph].

[11] V. V. Bazhanov and S. M. Sergeev, “A Master solution of the quantum Yang-Baxter equation and classical discrete integrable equations,” Adv. Theor. Math. Phys. 16 no. 1, (2012) 65–95, arXiv:1006.0651 [math-ph].

[12] V. P. Spiridonov, “Elliptic beta integrals and solvable models of statistical mechanics,” Contemp. Math. 563 (2012) 181–211, arXiv:1011.3798 [hep-th].

[13] V. V. Bazhanov and S. M. Sergeev, “Elliptic gamma-function and multi-spin solutions of the Yang-Baxter equation,” Nucl. Phys. B856 (2012) 475–496, arXiv:1106.5874 [math-ph].

[14] M. Yamazaki, “Quivers, YBE and 3-manifolds,” JHEP 05 (2012) 147, arXiv:1203.5784 [hep-th].

[15] Y. Terashima and M. Yamazaki, “Emergent 3-manifolds from 4d Superconformal Indices,” Phys. Rev. Lett. 109 (2012) 091602, arXiv:1203.5792 [hep-th].

[16] M. Yamazaki, “New Integrable Models from the Gauge/YBE Correspondence,” J. Statist. Phys. 154 (2014) 895, arXiv:1307.1128 [hep-th].

[17] V. V. Bazhanov, A. P. Kels, and S. M. Sergeev, “Comment on star-star relations in statistical mechanics and elliptic gamma-function identities,” J. Phys. A46 (2013) 152001, arXiv:1301.5775 [math-ph].

[18] S. S. Razamat and B. Willett, “Global Properties of Supersymmetric Theories and the Lens Space,” Commun. Math. Phys. 334 no. 2, (2015) 661–696, arXiv:1307.1128 [hep-th].

[19] I. Gahramanov and A. P. Kels, “The star-triangle relation, lens partition function, and hypergeometric sum/integrals,” JHEP 02 (2017) 040, arXiv:1610.09229 [math-ph].

[20] M. Yamazaki, “Integrability as Duality: the Gauge/YBE Correspondence,” arXiv:1806.04374 [hep-th].

[21] S. N. M. Ruijsenaars, “First order analytic difference equations and integrable quantum systems,” J. Math. Phys. 38 no. 2, (1997) 1069–1146, https://doi.org/10.1063/1.531809.

[22] E. M. Rains, “Transformations of elliptic hypergeometric integrals,” Ann. of Math. 171 (2010) 169–243, arXiv:math.QA/0309252 [math.QA].

[23] T. Masuda, “Hypergeometric r-functions of the q-Painlevé system of type E_8^{(1)},” Ramanujan J. 24 no. 1, (2011) 1–31, https://doi.org/10.1007/s11139-010-9262-1.

[24] S. O. Warnaar, “Summation formulae for elliptic hypergeometric series,” Proc. Amer. Math. Soc. 133 no. 2, (2005) 519–527.

[25] A. Narukawa, “The modular properties and the integral representations of the multiple elliptic gamma functions,” Adv. Math. 189 no. 2, (2004) 247–267, https://doi.org/10.1016/j.aim.2003.11.009.

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