BPS states on noncommutative tori and duality

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Abstract

We study gauge theories on noncommutative tori. It was proved in [5] that Morita equivalence of noncommutative tori leads to a physical equivalence (SO(d, d|Z)-duality) of the corresponding gauge theories. We calculate the energy spectrum of maximally supersymmetric BPS states in these theories and show that this spectrum agrees with the SO(d, d|Z)-duality. The relation of our results with those of recent calculations is discussed.

1 Introduction

It is shown in [1] that supersymmetric Yang-Mills (SYM) theories on noncommutative tori arise very naturally as compactifications of M-theory (in the Matrix formulation of this theory). They can be interpreted as toroidal compactifications with nonvanishing expectation value of antisymmetric $B$-field in string theory language. It was proven later in [3] that the mathematical notion of Morita equivalence is closely related to duality in physics. The results of [1] and [3] show that Morita equivalence of $d$-dimensional noncommutative tori is governed by the group $SO(d, d|Z)$. Yang-Mills theories on noncommutative tori $T_\theta$ and $T_{\hat{\theta}}$ where $\theta$ and $\hat{\theta}$ are $d \times d$ antisymmetric matrices belonging to the same orbit of the group $SO(d, d|Z)$, are physically equivalent. This equivalence is related to T-duality.
in string theory. The equivalence of different compactifications was proven in [3] at the level of action functionals. Of course, this result implies the coincidence of energy spectra of the corresponding quantum theories. It is necessary to emphasize that this result does not depend on the specific form of action functional and can be applied to any gauge-invariant action functional expressed in terms of gauge fields and fields in adjoint representation of the gauge group (endomorphisms). In particular, it can be applied to Born-Infeld action functional. The energy spectrum cannot be expressed in terms of the action functional in a simple way. However, in supersymmetric case one can analyze the energies of BPS states. We check in this paper that BPS spectra of YM theories on Morita equivalent tori coincide. This coincidence was conjectured at first in [4] and analyzed later in a number of papers ([7], [1], [12], [1]). The paper [12] contains an important observation that quantum fluctuations of BPS fields were not taken into account in [10] and that the proper treatment of fluctuations could essentially change the answer. It follows from the results of our paper that, indeed, the contribution of fluctuations cannot be separated from the contribution of “Kaluza-Klein modes” (such a separation was assumed in [7] and implicitly in [10], v1). However, we will see that in the case when transverse oscillators are in the ground state the energy spectrum agrees with [10], v1 and with the results of our previous calculation mentioned in [10], v1. We will see also that our results agree with the results of [12] and [10], v2 if one takes into account the energy of quantum fluctuations. (The quantum numbers used in these papers are not independent if all transverse oscillators are in the ground state.)

The paper is organized as follows. We start with an explanation of some basic notions of noncommutative geometry and of the results proved in [3]. Our exposition is different from the one given in [3] in one important relation. The paper [3] was based on the theory of C*-algebras and C*-modules. In the present paper we give the main definitions and prove that complete Morita equivalence leads to a physical equivalence of YM action functionals in a more general framework of arbitrary associative algebras. It seems that this modification essentially reduces the amount of mathematical information needed for the understanding of the relation between Morita
equivalence and duality. We give a detailed formulation of the results of [6] and [5] about Morita equivalence of noncommutative tori but do not give new proofs of them (see, however, Appendix C). In this relation we would like to attract attention of the reader to the paper [9]. Along with other results it contains a new proof of the results of [6] and of some of the results of [5] that should be more acceptable for a reader that finds the exposition in [6] and [5] too mathematical. To set a correspondence between the mathematical terminology used in [6], [5] and the one used in [9] one should notice that “adjoint sections on twisted bundles” of [9] are “endomorphisms of modules” in the terminology of [6], [5] and the calculation of the “space of adjoint sections” performed in [9] leads to a description of Morita equivalent torus.

In section 3 we give a semiclassical calculation of the energies of the BPS states. We start with the detailed analysis of the case of two-dimensional noncommutative torus. This analysis permits us to verify the relation between energy spectra of BPS states on Morita equivalent tori that follows from the results of [5]. At the end we analyze the energies of BPS states in the $d$-dimensional case using the results of [6]. (In this paper we consider only maximally supersymmetric BPS states. More precisely, we are studying states that arise from quantization of maximally supersymmetric BPS fields and fluctuations of these fields. Other BPS states on noncommutative tori will be studied in a forthcoming paper [13].) Some information about geometric quantization that is useful (but not necessary) for understanding of our calculations is relegated to Appendix A. Information about spinor representation of $SO(d, d|\mathbb{Z})$ is collected in Appendix B. Using this information we modify proofs given in [6] keeping track of all constant factors (Appendix C). Appendix D contains a description of modules over noncommutative tori that can be equipped with a constant curvature connection.

2 General Theory

Let us consider an associative algebra $A$. We can interpret it as an algebra of functions on a “noncommutative space” and introduce various geometric notions generalizing notions of standard “com-
mutative” geometry. In particular, we can define the notion of a connection on $A$-module $E$. Recall that by definition a linear space $E$ is a left $A$-module if we can multiply elements of $E$ by elements of $A$ from the left and $a(bc) = (ab)c$, $a(e + e') = ae + ae'$, $a(\lambda e) = \lambda(ae)$ (here $a, b \in A$, $e, e' \in E$, $\lambda$ is a number). The definition of a right $A$-module is similar. Direct sum $A^n$ of $n$ copies of $A$ can be considered both as a left and a right module over $A$ in a natural way. Such a module is called a free module. If $A$ is a commutative algebra $C(M)$ (or $C^\infty(M)$) of continuous (or smooth) functions on a compact manifold $M$, we can consider the space $\Gamma(E)$ of (continuous or smooth) sections of vector bundle $E$ over $M$ as a module over $A$. (The distinction between left and right modules disappears for commutative algebras.) One can check that $A$-modules obtained by means of this construction can be characterized as finitely generated projective modules (i.e. direct summands in free modules).

We will give a definition of a connection on an $A$-module $E$, taking as a starting point a Lie algebra $L$ that acts on $A$ by means of infinitesimal automorphisms (derivations). In other words we assume that we fixed operators $\delta_X$ depending linearly on $X \in L$ and obeying the identity $\delta_X(ab) = (\delta_X a)b + a(\delta_X b)$. Then a connection on $E$ is specified by means of linear operators $\nabla_X : E \to E$, $X \in G$, obeying the Leibnitz rule:

$$\nabla_X(ae) = a\nabla_X e + (\delta_X a)e$$

for any $a \in A$, $e \in E$. (We formulated the definition in the case of a left $A$-module $E$. The definition for a right $A$-module is similar.) There exists a more general definition of connection where covariant derivatives $\nabla_X$ are replaced by covariant differentials; we do not use this notion. A curvature of a connection $\nabla_X$ can be defined by the formula

$$F_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

It is easy to check that $aF_{XY} = F_{XY}a$ for any $a \in A$, $X, Y \in L$. A linear operator $\phi : E \to E$ is called an endomorphism of an $A$-module $E$ if it is $A$-linear, i.e. it commutes with multiplication by elements of $A$. We see that a curvature of a connection can be considered as a two-form on the Lie algebra $L$ that takes values in the algebra $\text{End}_AE$ of endomorphisms of the $A$-module $E$. We will
restrict ourselves to the case when the Lie algebra $L$ is abelian. Then the second term in the expression for curvature tensor vanishes. Let us assume that the algebra $\text{End}_A E$ is equipped with a trace (i.e. with a linear functional $\text{Tr}$ obeying $\text{Tr} \alpha \beta = \text{Tr} \beta \alpha$). Then we can construct a Yang-Mills action functional $S_{YM}$ on the set $\text{Conn}$ of connections in the $A$-module $E$ by means of an inner product on the Lie algebra $L$. Namely, we can use the formula

$$S_{YM}(\nabla) = \frac{1}{g_{YM}^2} \sum_{\alpha, \beta} \text{Tr} F_{\alpha \beta} F^{\alpha \beta}$$

Here $F_{\alpha \beta}$ stands for components of curvature tensor in some basis of $L$ and $F^{\alpha \beta} = g^{\alpha \mu} g^{\beta \nu} F_{\mu \nu}$ where $g^{\alpha \mu}$ is the inverse metric tensor on $L$. If $\dim L = 10, 6, 4, 3$ one can construct supersymmetric extension of $S_{YM}$ in the usual way; we do not need the explicit form of this extension. We can also consider the Yang-Mills action in the presence of a background field $\phi$:

$$S_{YM}^{\phi}(\nabla) = \frac{1}{g_{YM}^2} \sum_{\alpha, \beta} \text{Tr}(F_{\alpha \beta} + \phi_{\alpha \beta} \cdot 1)(F^{\alpha \beta} + \phi^{\alpha \beta} \cdot 1).$$

The most important example of a “noncommutative space” is a noncommutative torus. The algebra $A_\theta$ of “smooth functions” on a noncommutative torus can be defined as a linear space $S(Z^d)$ equipped with multiplication given by the formula

$$(f \ast g)(\gamma) = \sum_{\lambda \in Z^d} e^{\pi i \theta \langle \lambda, \gamma - \lambda \rangle} f(\lambda) g(\gamma - \lambda).$$

Here $\theta$ is an antisymmetric bilinear form on $Z^d$ and $S(Z^d)$ stands for Schwartz space (the space of complex functions on $Z^d$ that decrease faster than any power function). Instead of the antisymmetric bilinear form we consider the antisymmetric $d \times d$ matrix of its coefficients and use the same notation $\theta$ for it. If $D = Z^d$ is considered as a lattice in $R^d$, then the dual space $R^{*d}$ is an abelian group that acts naturally on $S(Z^d)$: to every $x \in R^{*d}$ we assign a map $\tau_x$ transforming a function $f(\lambda)$ into $e^{2\pi i \langle x, \lambda \rangle} f(\lambda)$. For every $\theta$ the map $\tau_x$ can be considered as an automorphism of $A_\theta$; this automorphism is trivial if $x \in D^*$ where $D^*$ is the lattice dual to $D$. We denote the group $R^{*d}/D^*$ considered as a group of automorphisms of $A_\theta$ by $L_\theta$.
and its Lie algebra as $L_\theta$. If $\theta = 0$ (or, more generally, the matrix $\theta$ has integer entries), then the algebra $A_\theta$ is isomorphic to the algebra of smooth functions on the standard ("commutative") torus. It is easy to check that the notion of connection and the expression for Yang-Mills action functional coincide with the standard ones in this case. We will define a connection on $T_\theta$-module by means of an isomorphism $L \to L_\theta$. Using this isomorphism and a standard basis in the lattice $D^* \subset L_\theta$ we obtain a standard basis in $L$. It consists of elements $X^i$ such that
\[(\delta X_j f)(n^i e_i) \equiv (\delta_j f)(n^i e_i) = i n^j f(n^i e_j), j = 1, \ldots, d.\] (2)

Let us come back to the general case and define the notions of Morita equivalence and complete Morita equivalence of associative algebras. Consider an $(A, \hat{A})$-bimodule $P$ (i.e. we assume that elements of $P$ can be multiplied by elements of $A$ from the left and elements of $\hat{A}$ from the right in such a way that $(ae)\hat{a} = a(e\hat{a})$ for any $a \in A$, $\hat{a} \in \hat{A}$, $e \in P$, and $P$ with this operations can be considered as a left $A$-module and a right $\hat{A}$-module). Notice that every left $A$-module $E$ can be regarded as an $(A, \text{End}_A E)$-bimodule and conversely, for every $(A, \hat{A})$-module $P$ there exists a natural map $\hat{A} \to \text{End}_A P$. For every right $A$-module $E$ and $(A, \hat{A})$-bimodule $P$ we can construct a right $\hat{A}$-module $\hat{E} = E \otimes_A P$. (To define the tensor product over $A$ we identify $ea \otimes p$ with $e \otimes ap$ in standard tensor product of linear spaces $E$ and $P$. This identification respects multiplication by $\hat{a} \in \hat{A}$, therefore, $\hat{E}$ can be considered as a right $\hat{A}$-module.) For every $A$-linear map of $A$-modules $\alpha : E \to E'$ we can construct naturally an $\hat{A}$-linear map $\hat{\alpha} : \hat{E} \to \hat{E}'$. It is easy to check that $\hat{\alpha} \hat{\beta} = \hat{\alpha} \beta$. In particular, we obtain a map from $\text{End}_A E$ into $\text{End}_{\hat{A}} \hat{E}$. We say that an $(A, \hat{A})$-bimodule $P$ is an equivalence bimodule if there exists an $(\hat{A}, A)$-module $P'$ obeying $P \otimes_{\hat{A}} P' = A$ and $P' \otimes_A P = \hat{A}$. (Here $A$ and $\hat{A}$ are considered as $(A, A)$-bimodule and $(\hat{A}, \hat{A})$-bimodule respectively.) Algebras $A$ and $\hat{A}$ are called Morita equivalent if there exists an equivalence $(A, \hat{A})$-bimodule $P$. In mathematical terms every $(A, \hat{A})$-bimodule $P$ determines a functor $E \to \hat{E}$ acting from the category of right $A$-modules into the category of right $\hat{A}$-modules. The $(A, \hat{A})$-bimodule $P$ specifies Morita
equivalence of algebras $A$ and $\hat{A}$ if this functor is an equivalence of categories of modules. (It is easy to derive from our assumptions about $P'$, that $\hat{E} \otimes_A P' = E$ and for every right $\hat{A}$-module $F$ we have $(F \otimes_{\hat{A}} P') \otimes_A P = F$. This means that the existence of $P'$ implies equivalence of categories of modules; the inverse statement is also true.)

Let us assume now that $(A, \hat{A})$-bimodule $P$ is equipped with operators $\nabla^P_X$ that satisfy

$$
\nabla^P_X(ae) = a\nabla^P_Xe + (\delta_X a)e,
$$

$$
\nabla^P_X(e\hat{a}) = (\nabla^P_X e)\hat{a} + e\tilde{\delta}_X \hat{a},
$$

$$
[\nabla^P_X, \nabla^P_Y] = \sigma_{XY} \cdot 1.
$$

Here $\nabla^P_X$ is a linear operator in $P$ that depends linearly on an element $X$ of abelian Lie algebra $L$, $a \in A$, $\hat{a} \in \hat{A}$, $e \in P$; the Lie algebra $L$ acts on $A$ and $\hat{A}$ by means of operators $\delta_X$ and $\tilde{\delta}_X$ correspondingly. The above conditions mean that $\nabla^P_X$ is a connection in $P$ considered as a left $A$-module or as a right $\hat{A}$-module, and that the curvature of this connection is constant. Using the operators $\nabla^P_X$ we can construct a connection $\hat{\nabla}_X$ in a $\hat{A}$-module $\hat{E} = E \otimes_A P$ for every connection $\nabla_X$ in an $A$-module $E$. This construction is based on a remark that the operator $\nabla_X \otimes 1 + 1 \otimes \nabla^P_X$ in $E \otimes P$ descends to $\hat{E} = E \otimes A P$ and determines a connection $\hat{\nabla}_X$ in $\hat{E}$. One can check that the curvature of the connection $\hat{\nabla}_X$ can be expressed in terms of the curvature $F^\nabla_{XY}$ of the connection $\nabla_X$:

$$
F^\nabla_{XY} = \hat{F}^\nabla_{XY} + \sigma_{XY} \cdot 1.
$$

(3)

(Recall that for $X, Y \in L$ we consider $F^\nabla_{XY}$ as an element of $End_A E$ and therefore $\hat{F}^\nabla_{XY}$ is defined as an element of $End_{\hat{A}} \hat{E}$.)

Now we can define a complete Morita equivalence of algebras $A$ and $\hat{A}$ by means of $(A, \hat{A})$-bimodule $P$ equipped with connection $\nabla^P_X$. Namely, we assume that $P$ is an equivalence bimodule and that the above correspondence of connections in $E$ and $\hat{E}$ is bijective. If such a bimodule $P$ exists we say that $A$ and $\hat{A}$ are completely Morita equivalent. It is clear that complete Morita equivalence of algebras
A and $\hat{A}$ implies physical equivalence of Yang-Mills theories on $A$-module $E$ and corresponding to it $A$-module $\hat{E}$. More precisely, for a trace $\text{Tr}$ on $\text{End}_A E$ we define a trace $\hat{\text{Tr}}$ on $\text{End}_{\hat{A}} \hat{E}$ by the formula

$$\hat{\text{Tr}}\alpha = \text{Tr}\alpha. \quad (4)$$

Then

$$S^Y_M(\nabla) = S^Y_M(\hat{\nabla}). \quad (5)$$

We will consider the case when $E$ is a finitely generated projective module. Then for every trace on $A$ one constructs a trace on $\text{End}_A E$. If the trace on $A$ is normalized (i.e. $\text{Tr}\mathbf{1} = 1$), the trace on $\text{End}_A E$ is not necessarily normalized. The dimension $\text{dim} E$ of module $E$ can be defined as the value of the trace on the identity endomorphism: $\text{Tr}\mathbf{1} = \text{dim} E$. If we use this trace we should modify (5) (see below).

In the most interesting cases the algebras at hand are equipped with an involution (complex conjugation). Moreover, these algebras are $C^*$-algebras and modules over them are $C^*$-modules. In these cases it is natural to restrict the attention to Hermitian connections. A theory of Morita equivalence of $C^*$-algebras was developed by Rieffel [2]; the consideration in [5] was based on it. In some relations this theory is easier to apply (for example the $(\hat{A}, A)$-bimodule $P'$ in the definition of Morita equivalence can be obtained from $P$ by means of complex conjugation).

To apply the equivalence (5) in concrete situations we should find the $\hat{A}$-module $\hat{E}$ corresponding to given $A$-module $E$. We can use Chern characters to solve this problem. We consider the situation when the connections on $A$-module $E$ are defined by means of action of Lie algebra $L$. Then the Chern character $\text{ch}(E)$ can be defined by the formula

$$\text{ch}(E) = \text{Tr} \exp \left( \frac{1}{2\pi i} \alpha^k F^\nabla_{kj} \alpha^j \right) = \sum_{k=0}^{\infty} \frac{1}{(2\pi i)^k k!} \text{Tr}(F^\nabla)^k. \quad (6)$$

Here $F^\nabla = \alpha^k F^\nabla_{kj} \alpha^j$ stands for the curvature of connection $\nabla$ on $E$ considered as a two-form on $L$ with values in $\text{End}_A E$ or as an element of $\Lambda^n \otimes \text{End}_A E$ where $\Lambda^d$ is a Grassmann algebra with generators $\alpha^1, \ldots, \alpha^d$, $d = \text{dim} L$. Everywhere below for shortness we drop coefficients $1/2\pi i$ from the formulas involving Chern characters. The Chern character $\text{ch}(E)$ is an element of $\Lambda^n = \Lambda(L^*)$ (an
inhomogeneous form on $L$). It does not depend on the choice of connection $\nabla$.

If the algebras $A$ and $\hat{A}$ are completely Morita equivalent one can relate the Chern characters of $E$ and $\hat{E}$ using (3). We obtain

$$ch(\hat{E}) = e^{\alpha_k^i \sigma_{kj} \alpha^j} ch(E).$$

(7)

We should slightly change this formula if the trace on $\text{End}_A E$ is normalized by the condition $\text{Tr} 1 = \dim E$. Namely, we have

$$ch(\hat{E}) = \frac{\dim \hat{E}}{\dim E} e^{\alpha_k^i \sigma_{kj} \alpha^j} ch(E).$$

(8)

Now we restrict ourselves to the case when $E$ is a finitely generated projective module (direct summand in a free module) and the algebra $A$ is the noncommutative torus $T_\theta$. Let us mention first of all that there is a canonical trace on $T_\theta$ given by the formula $\text{Tr}(f) = f(0)$ if $f \in T_\theta$ is considered as a function on a lattice. If $\theta$ is irrational (i.e. has at least one irrational entry) this trace is unique up to a constant factor. Similar statement is correct for the algebra $\text{End}_{T_\theta} E$ provided $\theta$ is irrational. On this algebra we normalize the trace by the condition $\text{Tr} 1 = \dim E$. In what follows we always work with this trace. For the case of a noncommutative torus $T_\theta$ one can prove [3] that the expression

$$\mu(E) = e^{-\frac{1}{2} b_k \theta^{kj} b_j} ch(E)$$

(9)

where $b_k$ stands for the derivative with respect to anticommuting variable $\alpha^k$, is an integral element of the Grassmann algebra $\Lambda = \Lambda(L^*)$, i.e. an element of $\Lambda(D)$. (Recall that the group $\tilde{L}_\theta$ of automorphisms of $T_\theta$ can be represented as $L_\theta/D^* = R^{rd}/D^*$. One can identify $\Lambda(L^*)$ with cohomology algebra of the torus $\tilde{L}_\theta$ and $\Lambda(D)$ with integer cohomology $H(\tilde{L}_\theta, \mathbb{Z})$.) The integer coefficients in the representation of $\mu(E)$ in the basis of $D$ play the role of topological numbers of the module $E$.

Notice that the Grassmann algebras $\Lambda(L^*) = F^*$ and $\Lambda(L) = F$ can be considered as fermionic Fock spaces (irreducible representations of a finite-dimensional Clifford algebra). In particular, in

\footnote{Note that in the corresponding formula in [3] (formula (41)) as well as in other formulas of Sec. 5 one should replace $\phi$ with $\sigma$.}
$F^* = \Lambda(L^*)$ we have operators $a^k$ of multiplication by $\alpha^k$ and operators $b_k = \frac{\partial}{\partial \alpha^k}$ satisfying canonical anticommutation relations

$$\{a^k, b_l\} = \delta^k_l, \quad \{a^k, a^l\} = 0, \quad \{b_k, b_l\} = 0$$

(10)

The group $O(d, d|\mathbb{C})$ can be regarded as a group of automorphisms of Clifford algebra (a group of linear canonical transformations). Operators given by the formulas

$$\tilde{a}^k = M_k^l a^l - N^{kl} b_l, \quad \tilde{b}_k = -R_{kl} a^l + S^l_k b_l$$

obey canonical anticommutation relations (10) iff the matrix

$$g = \begin{pmatrix} M & N \\ R & S \end{pmatrix}$$

(11)

belongs to the group $O(d, d|\mathbb{C})$. Using this remark one can define a projective action of $O(d, d|\mathbb{C})$ on $F^*$ assigning to every $g \in O(d, d|\mathbb{C})$ an operator $V_g : F^* \to F^*$ that satisfies

$$\tilde{a}^k = V_g a^k V_g^{-1}, \quad \tilde{b}_k = V_g b_k V_g^{-1}.$$  

(12)

The projective action is (by definition) the spinor representation of $O(d, d|\mathbb{C})$. We also define an action $\theta \mapsto \hat{\theta}$ of $O(d, d|\mathbb{C})$ on the space of antisymmetric matrices by the formula

$$\hat{\theta} = (M\theta + N)(R\theta + S)^{-1}$$

(13)

where $d \times d$ matrices $M, N, R, S$ correspond to an element $g \in O(d, d|\mathbb{C})$ by formula (11). More precisely, this action is defined on a subset of the space of all antisymmetric matrices where the matrix $R\theta + S$ is invertible. The main results of [5] can now be formulated in the following way.

Tori $T_\theta$ and $T_{\hat{\theta}}$ are completely Morita equivalent iff $\hat{\theta}$ and $\theta$ are related by the formula (13) where the matrix $g$ defined by the formula (11) belongs to the subgroup $SO(d, d|\mathbb{Z})$ of $SO(d, d|\mathbb{C})$ consisting of matrices with integer entries.

(One can say that $\theta$ and $\hat{\theta}$ should belong to the same orbit of $SO(d, d|\mathbb{Z})$.)

If $E$ is a $T_\theta$-module, $\hat{E}$ is the corresponding $T_{\hat{\theta}}$-module, then

$$\mu(\hat{E}) = V_g \mu(E).$$

(14)
Given two Morita equivalent tori $T_\theta$ and $\tilde{T}_\theta$, the connections on modules $E$ and $\tilde{E}$ are defined for fixed isomorphisms:

$$\delta : L \to L_\theta$$

$$\tilde{\delta} : L \to L_{\tilde{\theta}}.$$  

These isomorphisms determine two standard bases (2) in $L$ that we denote as $X^i$ and $\tilde{X}^i$. It is proved in [5] (formulas (45) and (55)) that $\tilde{X}^i = A^i_j X^j$ where the matrix $A$ can be expressed as

$$A = S + R\theta . \quad (15)$$

(See also Appendix C for the proof of this formula and formulas (14) and (19).) The metric tensors in different standard bases are related by the formula

$$\hat{g}_{ij} = A^k_i g_{kl} A^l_j . \quad (16)$$

Therefore, given the transformation from $SO(d,d|\mathbb{Z})$ that relates Morita-equivalent tori $T_\theta$ and $\tilde{T}_\theta$ we can find the metric $\hat{g}_{\alpha\beta}$. To find $\hat{F}$ we may use the expression (3). When written in the basis $\tilde{X}^i$ it reads as

$$A^k_i F_{kl} A^l_j + \sigma_{ij} = F^{\tilde{X}}_{ij} . \quad (17)$$

As shown in [5] (formula (55)) the matrix $\sigma$ is given by the expression

$$\sigma = -RA^t = -R(S + R\theta)^t . \quad (18)$$

Finally, the dimension $\dim \hat{E}$ can be calculated using formula (7):

$$\dim \hat{E} = \dim E |\text{det}(S + R\theta)|^{-1/2} . \quad (19)$$

The relation (19) was given in [9]. It can be obtained from the results of [5] if we take into account that $\dim E$ is equal to the value of Chern character $ch(E)$ at the point $\alpha = 0$. One should use the formula

$$ch(\hat{E}) = e^{\alpha^k \sigma_{kj} \alpha^j} V_3 ch(E) \quad (20)$$

where $V_3$ is a canonical transformation having the form

$$(V_3 f)(\alpha) = c \cdot f((A^t)^{-1} \alpha) .$$
One can check that the operator $V_3$ in (20) preserves an appropriate bilinear form in Fock space. Calculating the constant $c$ from this condition we obtain
\[ c = |\text{det} A|^{-1/2} = |\text{det}(S + R\theta)|^{-1/2}. \]

We mentioned already that Yang-Mills theories on modules $E$ and $\hat{E}$ are equivalent. Now we are able to give more detailed description of this equivalence in the case at hand. The formula (3) is derived for the case when the trace on $\text{End}_{\hat{A}}\hat{E}$ is defined by the formula (4). If we use the canonical normalized trace on $\text{End}_{\hat{T}_\theta}\hat{E}$ and $\text{End}_{\hat{T}_\theta}\hat{E}$ we obtain instead the following relation
\[ (\text{dim}E)^{-1}S_{\phi}^{Y M}(\nabla) = (\text{dim}\hat{E})^{-1}S_{\phi}^{Y M}(\hat{\nabla}). \tag{21} \]

We have shown how one can calculate the quantities related with the module $\hat{E}$ (curvature $\hat{F}$, metric $\hat{g}_{ij}$, $\text{dim}\hat{E}$, etc.) out of the original ones defined on $E$. Now we would like to write down the relations above for the particular case of noncommutative 1+2-to ri and Morita equivalence given by a particular $SO(3, 3|\mathbb{Z})$ transformation to be specified below. We assume that in the standard basis the matrix $\theta_{i,j}$ that defines the torus $T_\theta$ has the form
\[ \theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \vartheta \\ 0 & -\vartheta & 0 \end{pmatrix}. \tag{22} \]

We say that we have a 1+2- torus meaning that we fixed the splitting into $S^1$ part and the part isomorphic to a two-dimensional noncommutative torus. We will consider a $SO(3, 3|\mathbb{Z})$ transformation that preserves the form (22) and maps the parameter $\vartheta$ into $-1/\vartheta$. For this transformation the matrix (11) has (matrix) entries $M = 0$, $R = I$, $N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $R = N$, and $S = I - N$ (where $I$ is the identity matrix. Modules over 1+2 (noncommutative) tori are characterized by an integral element $\mu(E) = p + \frac{1}{2}q_{ij}\alpha^i\alpha^j$ of the Grassmann algebra $\Lambda(L^*)$. We
introduce the notation $E(p; q_{ij})$ for a module having these topological numbers. Using (14) one can calculate $\mu(\hat{E})$ for the transformation at hand: $\mu(\hat{E}) = -q_{23} + p\alpha^2\alpha^3 + q_{13}\alpha^1\alpha^2 - q_{12}\alpha^1\alpha^3$. Thus, the Yang-Mills theory on the modules $E = E(p; q_{23}, q_{12}, q_{13})$ and $\hat{E} = E(-q_{23}; p, q_{13}, -q_{12})$ are equivalent. Assuming that $(g_{ij}) = \text{diag}(R_0^2, R_1^2, R_2^2)$ (i.e. the metric on $T_\theta$ is diagonal with the specified entries), one can easily calculate $(\hat{g}_{i,j}) = \text{diag}(R_0^2, R_2^2\vartheta^2, R_1^2\vartheta^2)$. For the curvature $\hat{\nabla}$ on the module $\hat{E}$ one obtains the expression

$$\hat{F}\hat{\nabla} = \begin{pmatrix} 0 & f_{13}\vartheta & -f_{12}\vartheta \\ -f_{13}\vartheta & 0 & f_{23}\vartheta^2 \\ f_{12}\vartheta & -f_{23}\vartheta & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \vartheta \\ 0 & -\vartheta & 0 \end{pmatrix}. \tag{23}$$

For a connection of constant curvature $f_{ij} = \frac{q_{ij}}{\text{dim}E}$ one can check that formula (23) gives the correct expression for the curvature of the constant curvature connection on $\hat{E}$. To compare the dimensions $\text{dim}E$ and $\text{dim}\hat{E}$ one can calculate $\text{ch}(E)$ and $\text{ch}\hat{E}$ using (4). Comparing the free terms of those expressions one gets $\text{dim}\hat{E} = \vartheta^{-1}\text{dim}E$. As one can easily check, formula (21) holds when explicit expressions for quantities on $\hat{E}$ are substituted to the RHS of the formula.

Let us emphasize that the considerations above can be applied not only to the standard YM action, but also to SUSY YM action, to Born-Infeld action, etc. (We can consider any gauge-invariant action functionals that depend on connections and endomorphisms on a $T_\sigma$-module $E$. In the commutative case this means that we consider gauge fields and fields that transform according to the adjoint representation of the gauge group.)

### 3 Energy of BPS states

The equivalence of SYM action functionals on Morita equivalent tori implies the coincidence of the corresponding energy spectra. In this section we will explicitly calculate the energies of BPS states of the SYM theory on a noncommutative torus and show that they are invariant under Morita equivalence. Our calculation will be semiclassical, but the presence of supersymmetry makes the calculation exact for BPS states. Instead of working with the full supersymmetric action we will consider the $1 + d$ YM action functional and
constant curvature solutions which, by abuse of terminology, we will call BPS fields (they satisfy BPS condition in the supersymmetric theory). It is a valid thing to do because the calculation leads to the same result; the sole role of supersymmetry is to ensure the exactness of the semiclassical approximation. Hence, we start with the following (Euclidean) action functional

\[ S = \frac{1}{4g_Y^2 M} \text{Tr} \left( V \sum_{\alpha\beta} (F_{\alpha\beta} + \phi_{\alpha\beta} \cdot \mathbf{1}) g^{\alpha\mu} g^{\beta\nu} (F_{\mu\nu} + \phi_{\mu\nu} \cdot \mathbf{1}) \right). \]  

Here \( g_Y \) stands for the YM coupling constant, \( \phi_{\alpha\beta} \) plays the role of a background field, \( V = R_0 R_1 \ldots R_d \) is the volume element, the indices \( \alpha, \beta \) take values 0, 1, \ldots, \( d \).

We can fix a connection \( \nabla_0 \) and represent every connection in the form \( \nabla_\alpha = \nabla_0^\alpha + X_\alpha \) where \( X_\alpha \) is an endomorphism. One can represent the functional \( S \) explicitly as a sum over a lattice. By means of Fourier transform we can replace the summation over a lattice with the integration over a commutative torus. However, the action functional is nonlocal in this representation. We would like to calculate the energies of BPS states in Hamiltonian formalism. Therefore, we should single out the time direction \( x_0 \) and perform Wick rotation. To avoid nonlocality in time we should assume that \( \theta_0\alpha = 0 \). In other words, we suppose that the \((1+d)\)-dimensional noncommutative torus \( T_\theta \) is a direct product of a circle and a \( d \)-dimensional noncommutative torus \( T_\vartheta \).

In the gauge \( \nabla_0 = \frac{\partial}{\partial t} \) we obtain a Hamiltonian

\[ H = \text{Tr} \frac{g_Y^2 R_0^2}{2V} P^i g_{ij} P^j + \text{Tr} \frac{V}{4g_Y^2 M} (F_{ij} + \phi_{ij} \cdot \mathbf{1}) g^{ik} g^{jl} (F_{kl} + \phi_{kl} \cdot \mathbf{1}) . \]  

Here \( \nabla_\iota \) is a connection on a \( T_\vartheta \)-module \( E \), \( P^i \in \text{End}_{T_\vartheta} E \), and \( \text{Tr} \) denotes the trace in \( \text{End}_{T_\vartheta} E \). Thus, \( H \) is defined on the space \( \text{Conn} \times (\text{End}_{T_\vartheta} E)^d \). In the derivation of (25) we assumed that the metric \( g_{\alpha\beta} \) obeys \( g_{0i} = 0 \), \( g_{00} = R_0^2 \), \( g_{ij} = \delta_{ij} R_i^2 \) and the antisymmetric tensor \( \phi_{\alpha\beta} \) has only spatial nonzero components \( \phi_{ij} \) (here and everywhere Greek indices \( \alpha, \beta, \ldots \) run from 0 to \( d \) and Latin indices \( i, j, \ldots \) run from 1 to \( d \)). The Hamiltonian (25) should be restricted
to a subspace $\mathcal{N}$ where the constraint

$$[\nabla_i, P^i] = 0$$

is satisfied. More precisely, one should consider $H$ as a function on the space $\mathcal{N}/G = \mathcal{P}$ where $G$ is a group of unitary elements of $\text{End}_{T_\vartheta}E$ (the group of spatial gauge transformations). The symplectic form on the space $\text{Conn} \times (\text{End}_{T_\vartheta}E)^d$ can be written as

$$\omega = \text{Tr}\delta P^i \wedge \delta \nabla_i.$$  \hspace{1cm} (27)

The restriction of this form to $\mathcal{N}$ is degenerate, but it descends to a nondegenerate form on $\mathcal{P} = \mathcal{N}/G$ (on the phase space of our theory). The phase space $\mathcal{P}$ is not simply connected. Its fundamental group is the group of connected components of the gauge group $G$. In other words, $\pi_1(\mathcal{N}/G) = G/G_0 \equiv G_{\text{large}}$ where $G_0$ is the group of “small” gauge transformations (connected component of $G$). One can say that $\pi_1(\mathcal{N}/G)$ is a group of “large” gauge transformations. It is useful to consider the phase space $\mathcal{P}$ as a quotient $\hat{\mathcal{P}}/G_{\text{large}}$ where $\hat{\mathcal{P}} = \mathcal{N}/G_0$ is a symplectic manifold obtained from $\mathcal{N}$ by means of factorization with respect to small gauge transformations.

A Hamiltonian system on a phase space $\mathcal{P}$ with a nontrivial fundamental group can be quantized in several nonequivalent ways (see Appendix A). The freedom is labeled by characters of $\pi_1(\mathcal{P})$ (or, equivalently, by elements of the cohomology group $H^1(\mathcal{P}, \mathbb{R}/\mathbb{Z})$). One can verify that the study of our system in Lagrangian formalism in the presence of topological terms in the action can be reduced to the analysis of the system in Hamiltonian formalism if all possible ways of quantization are taken into account.

Now let us consider the case $d = 2$ in full detail. Let us fix a torus $T_\vartheta$ corresponding to the matrix

$$\begin{pmatrix} 0 & \vartheta \\ -\vartheta & 0 \end{pmatrix}.$$ 

Modules over $T_\vartheta$ are labeled by pairs of integers $(p, q)$ obeying $p - q\vartheta > 0$. An explicit description of these modules $E_{p,q}$ can be found in [4], or [1]. We do not need it. Let us mention only that $\text{dim} E_{p,q} = p - q\vartheta$, $\mu(E_{p,q}) = p + q\alpha_1 \alpha_2$, $\text{ch}(E_{p,q}) = \text{dim} E_{p,q} + q\alpha_1 \alpha_2$, and the curvature of a constant curvature connection is $F_{12} = \frac{q}{\text{dim} E_{p,q}} \cdot 1$.
in the standard basis. (Note that here and everywhere below we omit the $2\pi i$ factor that stands at some integers.) Let us consider at first modules $E_{p,q}$ where $p$ and $q$ are relatively prime. These modules can be called basic modules because every module $E_{p,q}$ can be represented as a direct sum of $D$ copies of identical basic modules $E_{p',q'}$. (Here $D = \text{g.c.d.}(p, q)$ and $Dp' = p, Dq' = q$.) Let us fix a constant curvature connection $\nabla^0$. It follows from the results of [4] that for a basic module any other constant curvature connection can be transformed to the form $\nabla^0_j + iq_j \cdot 1$ by means of small gauge transformations. Using large gauge transformations one can prove that $\nabla^0_j + iq_j \cdot 1$ is gauge equivalent to $\nabla^0_j + i(q_j - \frac{a_i}{\dim E_{p,q}}) \cdot 1$ where $n_j \in \mathbb{Z}$. Therefore, the space of gauge classes of constant curvature connections is a two-dimensional torus. From now on we fix $p, q$ and omit subscripts in the notation of the module $E_{p,q}$.

We will say that a set $(\nabla_i, P^j)$ is a BPS field if $\nabla_i$ is a constant curvature connection and $P^j = p^j \cdot 1$. (This terminology is prompted by the fact that after supersymmetrization these fields satisfy BPS condition). We will obtain the energies of quantum BPS fields restricting our Hamiltonian to the neighborhood of the space of BPS states and quantizing the restricted Hamiltonian. Let us consider the fields of the form

\begin{align}
\nabla_j &= \nabla^0_j + iq_j \cdot 1 + x_j, \\
P^i &= p^i \cdot 1 + \pi^i
\end{align}

where $x_i, \pi^j \in \text{End}_{T_\eta} E$, $\text{Tr} x_i = \text{Tr} \pi^j = 0$. For a basic module we can identify $\text{End}_{T_\eta} E$ with a noncommutative torus $T_{\tilde{\vartheta}}$ where

\[ \tilde{\vartheta} = (b - a\vartheta)(\dim E)^{-1}, \text{ and } a, b \text{ satisfy } qb - ap = 1 \]

as it was shown in [4]. Hence, we can consider $x_i$ and $\pi^j$ as functions on a lattice: $x_i = \sum_k x_i(k) Z_k$, $\pi^j = \sum_k \pi^j(k) Z_k$ where $Z_k$ are elements of $T_{\tilde{\vartheta}}$ satisfying

\[ Z_k Z_n = \exp(2\pi i \tilde{\vartheta}(k_2 n_1 - k_1 n_2)) Z_n Z_k. \]

Substituting expressions (28) and (29) into the Hamiltonian (25), keeping the terms up to the second order in fluctuations $x_i, p^i$ we
obtain

\[
H_{\text{fluct}} = \frac{R_0 g_Y^2 \dim E}{2 R_1 R_2} (p^i)^2 R_i^2 + \frac{R_0}{2 g_Y^2 R_1 R_2 \dim E} (q + \phi \dim E)^2 +
\]

\[
+ \frac{R_0 g_Y^2 \dim E}{2 R_1 R_2} \sum_k \pi^i(k) \pi^i(-k) R_i^2 + \frac{R_0}{2 g_Y^2 R_1 R_2 \dim E} \sum_k (k_1 x_2(k) - k_2 x_1(k))(k_1 x_2(-k) - k_2 x_1(-k)).
\]

(32)

In the derivation of this formula we used the relation

\[
[\nabla_j, x_l](k_1, k_2) = \frac{ik_j}{\dim E} x_l(k_1, k_2).
\]

(33)

The constraint (26) in the approximation at hand now takes the form

\[
k_j \pi^j(k) = 0.
\]

(34)

In a neighborhood of the space of BPS fields every field satisfying (34) can be transformed by means of a small gauge transformation into a field obeying

\[
k_i x_i(k) R_i^{-2} = 0.
\]

(35)

This means that in our approximation the conditions (34), (33) single out a symplectic manifold \( \tilde{\mathcal{P}} \) that can be identified with \( \mathcal{P} = \mathcal{N}/G_0 \). It remains to factorize with respect to large gauge transformations to obtain the phase space \( \mathcal{P} \). The group \( G = G/G_0 \) of large gauge transformations can be identified with the subgroup \( G_{\text{mon}} \) of \( G \) consisting of the elements \( Z_k \) (more precisely, every coset in \( G/G_0 \) has a unique representative of the form \( Z_k \)). It is easy to check that \( \tilde{\mathcal{P}} \) is invariant under the action of \( G_{\text{mon}} \). This observation permits us to identify \( \mathcal{P} = \tilde{\mathcal{P}}/G_{\text{large}} \) with \( \tilde{\mathcal{P}}/G_{\text{mon}} \). The Hamiltonian \( H \) on \( \tilde{\mathcal{P}} \) describes a free motion on a plane and an infinite system of harmonic oscillators with frequencies

\[
\omega(k) = \frac{R_0}{\dim E} \sqrt{\frac{k_1^2}{R_1^2} + \frac{k_2^2}{R_2^2}}.
\]
More precisely, the fields under consideration can be represented in the form

$$\nabla_j = \nabla_j^0 + iq_j \cdot 1 + \sum_k \mu(k) x_j^+(k) 2^{-1/2}(a(k) + a^*(-k)) Z_k \quad (36)$$

$$P^j = p^j \cdot 1 + \sum_k \pi_j^+(k) \mu(k)^{-1}(\text{dim}E)^{-1/2}a(-k) - a^*(k)) Z_k \quad (37)$$

where $a^*(k), a(k)$ are classical counterparts of creation and annihilation operators obeying the canonical commutation relations, $x_j^+(k)$ is a unit vector satisfying (35), $\pi_j^+(k)$ is a unit vector satisfying (34), and

$$\mu(k) = \left( \frac{R_0 g^2_{YM}}{R_1 R_2 \text{dim}E \omega(k)} \right)^{1/2}. \quad (39)$$

The Hamiltonian now reads as

$$H = \frac{R_0 g^2_{YM} \text{dim}E}{2R_1 R_2} (p^j)^2 R_i^2 + \frac{R_0}{2g^2_{YM} R_1 R_2 \text{dim}E} (q + \phi \text{dim}E)^2 + \sum_k \omega(k) a^*(k)a(k). \quad (38)$$

The action of the group $G^{mon}$ on the coordinates $q_j, p^j, a^+(k), a(k)$ can be expressed by the formulas

$$q_j \mapsto q_j - \frac{n_j}{\text{dim}E}$$

$$p^j \mapsto p^j$$

$$a(k) \mapsto \exp(i\tilde{\vartheta}(n_2 k_1 - n_1 k_2)) a(k)$$

$$a^*(k) \mapsto \exp(-i\tilde{\vartheta}(n_2 k_1 - n_1 k_2)) a^*(k) \quad (39)$$

These formulas follow immediately from the relations (31), (33).

The quantization of the system with Hamiltonian (38) is straightforward. The corresponding space of states is spanned by the wave functions

$$\Psi_{\mu; k_1^{i_1} \ldots k_l^{i_l};} = \exp(ip^m q_m \text{dim}E) \prod_{j=1}^l (a_j^+(k_j))^{N_j} |0 \rangle \quad (40)$$

where $|0 \rangle$ is the oscillators ground state. Here $a_j^+(k)$ are creation operators and $p^i$ are eigenvalues of the quantum operator corresponding to the coordinate $p^i$ (which by abuse of notation we denote by
the same letter). The group $G^{\text{mon}}$ acts on the space of states. Under this action the state $\ket{40}$ gets multiplied by the exponential factor

$$\exp \left(i(-n_j p^j + n_j \lambda^j + \tilde{\vartheta} \sum_{j=1}^{l} N_j (k_1^j n_2 - k_2^j n_1))\right),$$

where the parameters $\lambda^j$ have the meaning of topological “theta-angles” (see Appendix A). Thus, the invariance of state vectors under the gauge transformations leads to the following quantization law of the $p^j$-values:

$$p^1 = e^1 + \lambda^1 + \tilde{\vartheta} \sum_{j=1}^{l} N_j k_2^j,$$

$$p^2 = e^2 + \lambda^2 - \tilde{\vartheta} \sum_{j=1}^{l} N_j k_1^j$$

where $e^1$ and $e^2$ are integers. Substituting this quantization condition into the Hamiltonian (38) we get the energy spectrum

$$E = \frac{R_0 g^2_{\text{YM}} \text{dim}E}{2 R_1 R_2} (e^1 + \lambda^1 + (b - a \vartheta)(\text{dim}E)^{-1} \sum_{j=1}^{l} N_j k_2^j)^2 R_1^2 +$$

$$+ \frac{R_0 g^2_{\text{YM}} \text{dim}E}{2 R_1 R_2} (e^2 + \lambda^2 - (b - a \vartheta)(\text{dim}E)^{-1} \sum_{j=1}^{l} N_j k_1^j)^2 R_2^2 +$$

$$+ \frac{R_0}{2 g^2_{\text{YM}} R_1 R_2 \text{dim}E} (q + \phi \text{dim}E)^2 + \frac{R_0}{\text{dim}E} \sum_{j=1}^{l} N_j \sqrt{(k_1^j)^2 \frac{R_1^2}{R_1^2} + (k_2^j)^2 \frac{R_1^2}{R_2^2}}$$

(42)

Note that to compare this expression with the analogous formulas from [7], [10] one should use the relation between the Yang-Mills and the M-theory coupling constants:

$$g_{\text{YM}} = g_M R_1 R_2$$

(see [14] for example). Now we would like to show that the expression (42) is invariant under $SO(2,2|\mathbb{Z})$ transformations that govern the Morita equivalence. It suffices to check that (42) is invariant under the transformation $\vartheta \mapsto -\vartheta^{-1}$. Using the results of the previous
section we write the following transformation laws for the quantities constituting expression (42)

\[ q \mapsto p, \quad p \mapsto -q, \quad \text{dim}E \mapsto \vartheta^{-1} \text{dim}E \]

\[ R_2 \mapsto R_1 \vartheta, \quad R_1 \mapsto R_2 \vartheta, \quad R_0 \mapsto R_0 \]

\[ \phi \mapsto \phi \vartheta^2 - \vartheta, \quad \lambda^1 \mapsto \lambda^2, \quad \lambda^2 \mapsto -\lambda^1 \]

\[ e^1 \mapsto e^2, \quad e^2 \mapsto -e^1, \quad k^i_1 \mapsto k^i_2, \quad k^i_2 \mapsto -k^i_1 \]

\[ \tilde{\vartheta} \mapsto \tilde{\vartheta}, \quad g_{YM} \mapsto g_{YM} \vartheta \]

where the change of the coupling constant \( g_{YM} \) can be interpreted as an adjustment of different trace normalizations on our tori. As one can easily check, the expression (42) is invariant under transformations (43).

Let us emphasize that the energy spectrum (42) is obtained in the assumption that \( E \) is a basic module (i.e. in the case when the topological numbers \( p \) and \( q \) are relatively prime). If \( D = \text{g.c.d.}(p, q) \neq 1 \), then the factor \( D^2 \) appears in the denominator of the first term in (42). This reconciles formula (42) with calculations in the commutative case.

The method used above to obtain the energy spectrum can be applied to calculate eigenvalues of a momentum operator. Classical momentum functional has the form \( P_i = \text{Tr} F_{ij} P^j \). In the vicinity of a BPS field it takes the form

\[ P_i = q \epsilon_{ij} p^j + \sum_k \frac{k_i}{\text{dim}E} a^*(k) a(k) \]

The corresponding operator has the following eigenvalues

\[ m_i = q \epsilon_{ij} e^j - a \sum_j k^j_i N_j \]

where \( a \in \mathbb{Z} \) is the integer that enters the expression (30) for \( \tilde{\theta} \) and the parameters \( \lambda^i \) are assumed to be equal to zero. Thus, we see that the total momentum is quantized in the usual way (provided \( \lambda^i = 0 \)). This is not surprising, the integrality of eigenvalues is related to the periodicity of the torus. One can rewrite the first two terms of the spectrum (42) (contribution of “electric charges”) using the numbers (44):

\[ E = \frac{R_0 g_{YM}^2}{2R_1 R_2 \text{dim}E} (n^1 + p \lambda^1 + \vartheta (m_2 - q \lambda^1))^2 R_1^2 + \]
\[ E = \frac{R_0g_\gamma^2}{2R_1R_2\dim E} (n^1 + p\lambda^1 + \vartheta(m_2 - q\lambda^2))^2 R_1^2 + \]
\[ + \frac{R_0g_\gamma^2}{2R_1R_2\dim E} (n^2 + p\lambda^2 + \vartheta(m_1 + q\lambda^2))^2 R_2^2 + \]
\[ + \frac{R_0}{2g_\gamma^2 R_1R_2\dim E} (q + \phi \dim E)^2 + \]
\[ + \frac{R_0}{\dim E} \sqrt{\frac{(m_1p - n^2q)^2}{R_1^2} + \frac{(m_2p + n^1)^2q}{R_2^2}} \]  
(47)

(When minimizing it is convenient to use the simple fact that a norm of a sum of vectors is always larger or equal then the corresponding sum of norms.) The last formula agrees with the results of [12] and contradicts to the results of [14, v2] (the last term is missing in [14, v2]). It is easy to check that (47) gives energies of 1/4 BPS states; we obtain 1/2 BPS states when all oscillators are in the ground states (i.e. \( m_ip = q\epsilon_{ij}n^j \)).

Let us come back to the consideration of \((1+d)\)-dimensional case. We will restrict ourselves to the consideration of modules generalizing the basic modules studied in the two-dimensional case. We start with the general definition of a basic module. We say that a module
$E$ over a noncommutative torus $T_\theta$ is a basic module if the algebra $\text{End}_{T_\theta}E$ is again a noncommutative torus $\tilde{T}_\theta$ and the module $E$ is equipped with a constant curvature connection $\nabla_\alpha$ satisfying the condition $[\nabla_\alpha, \phi] = \tilde{\delta}_\alpha \phi$ for every $\phi \in T_\theta$ (here $\tilde{\delta}_1, \ldots, \tilde{\delta}_d$ is a basis of the Lie algebra $L_\theta$ of infinitesimal automorphisms of $T_\theta$). A $T_\theta$-module $E$ can be considered as a $(T_\theta, T_\theta)$-bimodule. The conditions defining a basic module are equivalent to the condition that this bimodule is a complete Morita equivalence bimodule. (A complete description of basic modules is given in Appendix D.) Notice that every Heisenberg\footnote{We use here the term “Heisenberg module” for the modules described in Sec. 3 of \cite{5}.} module $E$ such that $\text{End}_{T_\theta}E$ is a noncommutative torus, is a basic module. (Modules studied in \cite{12} are of this kind.) Let us consider a free one-dimensional $T_\tilde{\theta}$-module $T_\tilde{\theta}$ (i.e. $T_\tilde{\theta}$ considered as a right $T_\tilde{\theta}$-module). It is easy to check that this module transforms into $E$ by a complete Morita equivalence between $T_\tilde{\theta}$ and $T_\theta$. Using the results of \cite{5} we arrive at the conclusion that we can express energies of the BPS states corresponding to $E$ in terms of the BPS states corresponding to $T_\tilde{\theta}$. Let us give the expression of BPS energies in terms of topological numbers of the basic module $E$. Without loss of generality we can write

$$\mu(E) = K \cdot \exp\left(-\frac{1}{2} \alpha^i Q_{ij} \alpha^j\right)$$

where $K$ is a constant and $Q_{ij}$ is a $d \times d$ matrix. It follows from the definition of a basic module that there exists a matrix

$$\left(\begin{array}{cc} M & N \\ R & S \end{array}\right) \in SO(d, d|\mathbb{Z})$$

such that $Q = S^{-1} R$. This matrix establishes the Morita equivalence between $T_\theta$ and $T_\tilde{\theta}$:

$$\tilde{\theta} = (M \theta + N)(R \theta + S)^{-1}.$$ 

The answer for the energies of 1/2 BPS states is as follows

$$E = \frac{g_{YM}^2 R_0^2}{2 V \dim E} e^i A_i^l g_{lm} (A^l)_j^m e^j +$$

$$+ \frac{V \dim E}{4 g_{YM}^2} ((A^{-1} R)_{ij} + \phi_{ij}) g^{ik} g^{jl} ((A^{-1} R)_{kl} + \phi_{kl})$$

\footnote{We use here the term “Heisenberg module” for the modules described in Sec. 3 of \cite{5}.}
where $A = R \theta + S$, $\text{dim} E = |\text{det}A|^{1/2}$, $e^i$ are integers. It is easy to check that this answer is $SO(d,d|\mathbb{Z})$-invariant. Therefore, it suffices to verify it for a one-dimensional free module. The calculation for a free module is based on the same ideas as in the two-dimensional case, but technically it is even simpler. One should mention, however, that for $d \geq 4$ the group $G^{mon}$ is a proper subgroup of the group $G/G_0 = \mathcal{G}_{\text{large}}$ of connected components of the group of gauge transformations. This remark does not influence the calculation of energies of 1/2 BPS states.

In the $(1 + d)$-dimensional case we omitted parameters generalizing the topological “theta-angles” $\lambda^i$ considered for $d = 1$. It is easy to take them into account.

**Appendix A. Geometric quantization.**

It is convenient to use geometric quantization to derive the formulas for energies of BPS states. Let us remind the scheme of geometric quantization approach. Consider a manifold $\mathcal{X}$ with a symplectic structure specified by means of a closed form $\omega = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j$. If this form is exact ($\omega_{ij} = \partial_i \alpha_j - \partial_j \alpha_i$), then to every function $F$ on $\mathcal{X}$ one can assign an operator $\hat{F}$ acting on the space of functions on $\mathcal{X}$ by the formula

$$\hat{F} \phi = F \phi + \omega^{ij} \frac{\partial F}{\partial x^j} \nabla_i \phi$$

where $\omega^{ij}$ is the inverse matrix to $\omega_{ij}$ and $\nabla_i \phi = \hbar \partial_i + \alpha_i$ can be considered as a covariant derivative with respect to $U(1)$-gauge field having the curvature $\omega_{ij}$. It is easy to check that

$$[\hat{F}, \hat{G}] = i\hbar \{ F, G \}$$

where $\{ F, G \} = \frac{\partial F}{\partial x^i} \omega^{ij} \frac{\partial G}{\partial x^j}$ is the Poisson bracket. However, the correspondence $F \mapsto \hat{F}$ cannot be considered as quantization because the operators $\hat{F}$ act on the functions depending on coordinates of the phase space $\mathcal{X}$. The construction of operators $\hat{F}$ is called prequantization. In quantization procedure we should construct operators $\hat{F}$ acting on the space of functions depending on $\text{dim} \mathcal{X}/2$ variables and
obeying \([\hat{F},\hat{G}] \approx i\hbar[\tilde{F},\tilde{G}]\) as \(\hbar \to 0\). This can be done if we can construct an appropriate polarization of \(\mathcal{X}\) (see [15]). If \(\mathcal{X}\) is a Kähler manifold we can use holomorphic polarization (i.e. we can define \(\hat{F}\) as an operator acting on holomorphic sections of an appropriate holomorphic line bundle over \(\mathcal{X}\)).

Let us emphasize that the construction of operators \(\tilde{F}\) in the prequantization procedure depends on the choice of the one-form \(\alpha\) or, better, on the choice of a \(U(1)\)-gauge field having the curvature \(\omega\). It is easy to check that replacing a \(U(1)\)-gauge field with a gauge-equivalent field we obtain an equivalent prequantization. Different prequantization constructions are labeled by the elements of one-dimensional cohomology group \(H^1(\mathcal{X},\mathbb{R}/\mathbb{Z})\).

We can represent the symplectic manifold \(\mathcal{X}\) as a quotient space \(\tilde{\mathcal{X}}/\Gamma\) where \(\tilde{\mathcal{X}}\) is a simply connected space (the universal covering of \(\mathcal{X}\)) and the group \(\Gamma\), that acts freely on \(\mathcal{X}\), is isomorphic to \(\pi_1(\mathcal{X})\). Consider a character \(\chi\) of the group \(\Gamma\) and the set \(E_\chi\) of functions on \(\tilde{\mathcal{X}}\) satisfying the relation \(F(gx) = \chi(g)F(x)\) for any \(g \in \Gamma\). We represent the symplectic form \(\tilde{\omega}\) on \(\tilde{\mathcal{X}}\) as \(\tilde{\omega} = d\alpha\) where \(\alpha\) is a \(\Gamma\)-invariant one-form. Then, for every \(\Gamma\)-invariant function on \(\tilde{\mathcal{X}}\) the prequantization construction gives us an operator acting on \(E_\chi\). Taking into account that \(\Gamma\)-invariant functions on \(\tilde{\mathcal{X}}\) can be identified with functions on \(\mathcal{X}\) and characters of \(\Gamma = \pi_1(\mathcal{X})\) can be identified with elements of the cohomology group \(H^1(\mathcal{X},\mathbb{R}/\mathbb{Z})\), it is easy to check that the construction of prequantization in terms of the spaces \(E_\chi\) is equivalent to prequantization on \(\mathcal{X}\). Using this observation we can quantize the theory on \(\mathcal{X}\) in the following way. We quantize \(\tilde{\mathcal{X}}\) and assume that the group \(\Gamma\) of symplectomorphisms of \(\tilde{\mathcal{X}}\) can be lifted to a group of unitary transformations acting in the space of wave functions. Then it is natural to suppose that the quantum space corresponding to \(E_\chi\) consists of the wave functions satisfying \(g\psi = \chi(g)\psi\) for all \(g \in \Gamma\).

Let us consider some examples. We start with a free motion on a circle \(S^1\). Representing \(S^1\) as \(\mathbb{R}/\mathbb{Z}\) we interpret \(\pi_1(S^1) = \mathbb{Z}\) as a group of translations of wave functions \(\psi(x) \mapsto \psi(x + n)\) where \(n \in \mathbb{Z}\). The space of wave functions corresponding to a character \(\chi\) consists of functions obeying \(\psi(x + n) = e^{2\pi in\theta} \psi(x)\) where \(0 \leq \theta < 1\).
The spectrum of the Hamiltonian reads

\[ E_k = \frac{(k - \theta)^2}{2m} \]

where \( k \in \mathbb{Z} \). Now let us consider a more complicated example where \( \tilde{\mathcal{X}} = \mathbb{R}^1 \times \mathbb{R}^1 \times (\mathbb{R}^2)^n \) is a symplectic manifold with coordinates \( q, p, a_1^*, a_1, \ldots, a_n^*, a_n \) and symplectic form

\[ \omega = dp \wedge dq + i \sum_{j=1}^{n} da_j \wedge da_j^* . \]

Here \( q, p \in \mathbb{R}^1, a_j \in \mathbb{C} \). Define \( \mathcal{X} \) as \( \tilde{\mathcal{X}}/\Gamma \) where \( \Gamma \) is a cyclic group generated by the transformation

\[ (q, p, a_1^*, a_1, \ldots, a_n^*, a_n) \mapsto (q+1, p, e^{-i\rho_1}a_1^*, e^{i\rho_1}a_1, \ldots, e^{-i\rho_n}a_n^*, e^{i\rho_n}a_n) \]

We will consider a Hamiltonian

\[ H = \frac{p^2}{2m} + \sum_{k=1}^{n} \omega_k a_k^* a_k \]  

(50)

on \( \tilde{\mathcal{X}} \) (a free particle and \( n \) independent oscillators). This Hamiltonian is \( \Gamma \)-invariant and therefore generates a Hamiltonian on \( \mathcal{X} \). Wave functions of the theory on \( \tilde{\mathcal{X}} \) can be considered as functions of \( q, a_1^*, \ldots, a_n^* \) (we use the holomorphic representation of oscillator wave function). The group \( \Gamma \) generates a group of unitary operators on these wave functions. The spectrum of Hamiltonian (50) reads

\[ E = (k - \theta + \sum_{j=1}^{n} N_j \rho_j)^2 / 2m + \sum_{j=1}^{n} \omega_j N_j \]  

(51)

where \( k \in \mathbb{Z} \).

**Appendix B. Spinor representation of \( SO(d, d|\mathbb{Z}) \).**

Formula (12) determines the spinor representation of \( SO(d, d|\mathbb{C}) \) as a projective action of \( SO(d, d|\mathbb{C}) \) on the Fock space \( F^* \). This means that the operators \( V_g \) are defined only up to a constant factor. It is possible however to define the spinor representation as a two-valued...
representation of $SO(d, d|\mathbb{C})$. This can be done in the following day. We introduce a bilinear form on $F^*$ defined by the formula

$$< f, g > = \sum_k (-1)^{\epsilon_k} f_k(\alpha) g_{d-k}(\alpha) d\alpha_1 \ldots d\alpha_d.$$  \hfill (52)

Here an element of $F^*$ is considered as a function of anticommuting variables $\alpha_1, \ldots, \alpha_d$ and $f_k(\alpha)$ stands for the $k$-th homogeneous component of $f(\alpha)$. It is easy to check that the numbers $\epsilon_k$ can be chosen in such a way that linear canonical transformations (operators $V_g$) preserve the form (52) up to a constant factor. (The verification of this fact can be based on the remark that the Lie algebra $so(d, d|\mathbb{C})$ is represented by operators that are quadratic with respect to $a^k, b_k$.) This means that we can impose the requirement that operators $V_g$ preserve the bilinear form (52). This requirement specifies an operator $V_g$ up to a sign; in what follows we use this choice of $V_g$.

One can define integral elements of the Fock space $F^*$ (integral elements of a Grassmann algebra) as linear combinations of monomials $\alpha_1 \ldots \alpha_k$ with integer coefficients. An integrality preserving operator on $F^*$ can be defined as a linear operator transforming the set of integral elements into itself, or, equivalently, as a linear combination with integer coefficients of monomials composed of $a^k, b_k$. For every $g \in SO(d, d|\mathbb{Z})$ the corresponding operator $V_g$ is integrality preserving. It is sufficient to verify this fact only for generators of $SO(d, d|\mathbb{Z})$. The group $SO(d, d|\mathbb{Z})$ is generated by the transformations

$$(x, y) \mapsto ((A^t)^{-1} x, A y)$$  \hfill (53)

$$(x^1, \ldots, x^d, y_1, \ldots, y_d) \mapsto$$

$$(x^1, \ldots, y_i, \ldots, y_j, \ldots, x^d, y_1, \ldots, x^i, \ldots, x^j, \ldots, y_d)$$  \hfill (54)

$$(x, y) \mapsto (x, y + N x).$$  \hfill (55)

Here $(x, y) = (x^1, \ldots, x^d, y_1, \ldots, y_d)$ is a point of $\mathbb{R}^{2d}$, $A \in SL(d, \mathbb{Z})$, $N$ is an arbitrary antisymmetric matrix with integer entries. The inner product in $\mathbb{R}^{2d}$ is defined by the formula $<(x, y), (x', y')> = x^i y'_i + y_i (x')^i$. The linear canonical transformations corresponding to transformations of the first kind are integrality preserving because
they are given by the formula \( f(\alpha) \mapsto f((A^t)^{-1}\alpha) \). They preserve the bilinear form \( \text{(52)} \) due to the fact that the superdeterminant of the change of variables \( \alpha \mapsto A^t\alpha \) is equal to 1. Transformations of the third kind generate linear canonical transformations of the form \( \exp(\frac{1}{2}b_iN^{ij}b_j) \). It is easy to check that these transformations, as well as transformations corresponding to generators of the second kind, are integrality preserving and preserve \( \text{(52)} \). If the operator \( V_g \) satisfies \( \text{(53)} \) and both \( V_g \) and \( V_g^{-1} \) are integrality preserving then \( g \) is also integrality preserving, i.e. \( g \in SO(d,d|\mathbb{Z}) \) (we use the fact that a product of integrality preserving operators is again an integrality preserving operator). Notice that in this case \( V_g \) automatically preserves the bilinear form \( \text{(52)} \).

**Appendix C. Morita equivalent tori.**

In the proof of the results of section 5 of [5] the spinor representation was considered as a projective representation of \( SO(d,d|\mathbb{Z}) \). Therefore, all equations were written up to a constant factor. Now we will repeat the proof of these results considering the spinor representation as a two-valued representation of \( SO(d,d|\mathbb{C}) \) and keeping track of all constant factors. We consider completely Morita equivalent multidimensional tori \( T_\theta \) and \( \hat{T}_\theta \) and study the relation between \( \mu = \mu(E) \) and \( \hat{\mu} = \mu(\hat{E}) \) where \( T_\theta \)-module \( \hat{E} \) corresponds to a \( T_\theta \)-module \( E \). Using formulas \( \text{(8)} \) and \( \text{(9)} \) we obtain

\[
\hat{\mu} = W\mu \equiv W_1W_2W_3W_4\mu \tag{56}
\]

where
\[
W_1f = \exp(-\frac{1}{2}b_k\hat{\theta}^{kj}b_j)f \\
W_2f = \exp(d^k\sigma_{kj}a^j)f \\
W_3f = \frac{\dim E}{\dim \hat{E}}f((A^t)^{-1}\alpha) \\
W_4f = \exp(\frac{1}{2}b_k\theta^{kj}b_j)f . \tag{57}
\]

The operator \( W_1 \) relates \( \hat{\mu} \) and \( \text{ch}\hat{E} \), the operator \( W_4 \) relates \( \mu \) and \( \text{ch}(E) \) (see [5]). The operator \( W_2W_3 \) relates \( \text{ch}(\hat{E}) \) and \( \text{ch}(E) \). This
relation follows from (8) if we take into account that we should identify $L_\theta$ and $\hat{L}_\theta$ by means of a linear operator $A$. It is clear from the formulas above that the operators $W_1, W_2, W_3, W_4$ and hence their product that we denoted by $W$ are linear canonical transformations. We know that $\hat{\mu}$ and $\mu$ are integral elements of $F^*$, therefore, the operator $W$ transforms integral elements of $F^*$ into integral elements (i.e. $W$ is an integrality preserving operator). The same is true for the inverse operator $W^{-1}$ because $\mu$ and $\hat{\mu}$ are on equal footing. Thus, we can say that the linear canonical transformation $W$ corresponds to an element of $SO(d, d|\mathbf{Z})$. We proved that $\hat{\mu}$ and $\mu$ are related by a linear canonical transformation corresponding to an element of $SO(d, d|\mathbf{Z})$. We denote this element by

$$g = \begin{pmatrix} M & N \\ R & S \end{pmatrix}.$$ 

This transformation, as well as transformations $W_1, W_2, W_4$, preserves the bilinear form (52) (for $W$ this follows from integrality of $W$ and $W^{-1}$, and for $W_1, W_2, W_4$ it can be checked directly). This means that $W_3$ also preserves (52) and therefore

$$\frac{\dim \hat{E}}{\dim E} = |\det(A)|^{-1/2}.$$ 

(58)

Going from $W_1, W_2, W_4$ to the corresponding elements of $SO(d, d|\mathbf{C})$ we obtain

$$\begin{pmatrix} M & N \\ R & S \end{pmatrix} = \begin{pmatrix} 1 & \hat{\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\sigma & 1 \end{pmatrix} \begin{pmatrix} (A^t)^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (A^t)^{-1} - \hat{\theta}\sigma(A^t)^{-1} & -(A^t)^{-1}\theta - \hat{\theta}(\sigma(A^t)^{-1} - A) \\ -\sigma(A^t)^{-1} & \sigma(A^t)^{-1}\theta + A \end{pmatrix}.$$ 

(59)

From the last formula one readily derives formulas (13), (18) and (13).

**Appendix D. Modules with constant curvature connection.**

One can give a complete description of modules over noncommutative tori $T_\theta$ that can be equipped with a constant curvature con-
nection ([10]). In this appendix we will give this description for
the case when the matrix $\theta$ is irrational and, moreover, any linear
combination of its entries is irrational provided all coefficients are
integers.

For every element $x \in F^*$ we can define a subset $T_x$ of $\mathbb{R}^{2d}$
consisting of vectors $(u,v) \in \mathbb{R}^{2d}$ that obey

$$(u^i \frac{\partial}{\partial \alpha^i} + v_i \alpha^i)x = 0.$$  

It is easy to check that $T_x$ is an isotropic linear subspace of $\mathbb{R}^{2d}$.
(We equip $\mathbb{R}^{2d}$ with the same bilinear product as in Appendix B).
We will say that an even element $x \in F^*$ is a generalized quadratic
exponent (GQE) if $T_x$ is a maximal isotropic subspace of $\mathbb{R}^{2d}$ (i.e.
$\dim T_x = d$). One can say that GQE $x$ satisfies

$$(U^i_j \frac{\partial}{\partial \alpha^i} + V_{ij} \alpha^j)x = 0 \quad (60)$$

where $(U, V) = (U^i_j, V_{ij})$ is a $d \times 2d$ matrix of rank $d$. Let us represent
$x$ in the form

$$x = N + \frac{1}{2} \alpha^i m_{ij} \alpha^j + \ldots = N + \frac{1}{2} \alpha M \alpha + \ldots$$

where the omitted terms have higher order with respect to $\alpha$. It is
easy to check that $VN + UM = 0$. If the matrix $U$ is nondegener-
ete, we can express $x$ in terms of $N$ and $M$ solving the differential
equation (60). We obtain

$$x = N \cdot \exp(-\frac{1}{2} \alpha N^{-1} M \alpha)$$

where $M = -U^{-1} VN$. We see that $x$ is a quadratic exponent; it is
easy to check that the set of quadratic exponents is a dense subset
of the set of all GQE. Now we can formulate the following theorem.
A module $E$ over a noncommutative torus $T_\theta$ can be equipped with
a constant curvature connection iff $\mu(E)$ is a GQE. The module $E$
is basic iff $\mu(E)$ is a GQE without nontrivial divisors (i.e. $\mu(E)$
cannot be represented in the form $\mu(E) = D \nu$ where $D \in \mathbb{Z}$, $D > 1$,
and $\nu$ is an integral element of $F^*$).
To prove this theorem we notice first of all that the set of all GQE is invariant under projective action of $SO(d, d|\mathbb{Z})$ in $F^*$. It follows immediately from (6) that if module $E$ admits a constant curvature connection, then $ch(E)$ is a quadratic exponent. Taking into account that $\mu(E)$ is related to $ch(E)$ by means of a linear canonical transformation, we obtain that $\mu(E)$ is a GQE.

Now we should assume that $\mu(E)$ is a GQE and prove that $E$ can be equipped with a constant curvature connection. It is sufficient to find such an element $g \in SO(d, d|\mathbb{Z})$ that

$$V_g \mu(E) = \text{const.} \quad (61)$$

Then we can use this element $g$ to transform $T_\theta$ into a completely Morita equivalent torus $T_\hat{\theta}$. The module $\hat{E}$ over $T_\hat{\theta}$ corresponding to the module $E$ by means of this construction is a free module. The results of [3] give a correspondence between constant curvature connections in $E$ and in $\hat{E}$. A free module has a zero curvature connection. Therefore, (61) implies the existence of constant curvature connection on $E$. As the first step in the construction of $g$ satisfying (61) we construct such an element $h \in SO(d, d|\mathbb{Z})$ that $V_h \mu(E)$ is a quadratic exponent. This is easy to do applying generators (54) of $SO(d, d|\mathbb{Z})$ and taking into account that the rank of the matrix $(U, V)$ is equal to $d$. The next step is to simplify the quadratic exponent by means of transformations (53). Our consideration will be similar to the arguments applied in [12] to a different problem.

It is known (see [17]) that antisymmetric bilinear form with integer entries can be reduced to a block-diagonal form with $2 \times 2$ blocks by means of $SL(d, \mathbb{Z})$ transformations. This means that we can restrict ourselves to the case when $\mu(E)$ is an integral element of $F^*$ having the form

$$\mu(E) = N \cdot \exp\left(\frac{1}{N}(m_1 \alpha_1 \alpha_2 + m_2 \alpha_3 \alpha_4 + \ldots + m_k \alpha_{2k-1} \alpha_{2k})\right)$$

where $m_1, \ldots, m_k$ are non-zero integers. Now we will use an element of $SO(d, d|\mathbb{Z})$ that can be represented by means of block-diagonal
transformation with blocks

\[
\begin{pmatrix}
\tilde{x}_{2i-1}^{2i-1} \\
\tilde{x}_{2i} \\
\tilde{y}_{2i-1}^{2i-1} \\
\tilde{y}_{2i}
\end{pmatrix}
= \begin{pmatrix}
N/n_i & 0 & 0 & k_i \\
0 & N/n_i & -k_i & 0 \\
0 & -m_i/n_i & l_i & 0 \\
m_i/n_i & 0 & 0 & l_i
\end{pmatrix}
\begin{pmatrix}
x_{2i-1}^{2i-1} \\
x_{2i} \\
y_{2i-1}^{2i-1} \\
y_{2i}
\end{pmatrix}
\]

where \( n_i = \text{g.c.d.}(m_i, N) \), and \( k_i, l_i \) are integers satisfying \( l_iN - m_i k_i = n_i \). It is easy to check that after this transformation (61) is satisfied.

Notice that to consider Morita equivalence with a given \( g \in SO(d, d|\mathbb{Z}) \) we should require that the corresponding fractional linear transformation of \( \theta \) is well defined. This condition is satisfied due to the assumptions imposed on \( \theta \).

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