WEAK SEPARATION PROPERTIES FOR CLOSED SUBGROUPS OF LOCALLY COMPACT GROUPS

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Abstract. Three separation properties for a closed subgroup $H$ of a locally compact group $G$ are studied: (1) the existence of a bounded approximate indicator for $H$, (2) the existence of a completely bounded invariant projection $VN(G) \rightarrow VN_H(G)$, and (3) the approximability of the characteristic function $\chi_H$ by functions in $M_{cb}A(G)$ with respect to the weak* topology of $M_{cb}A(G_d)$. We show that the $H$-separation property of Kaniuth and Lau is characterized by the existence of certain bounded approximate indicators for $H$ and that a discretized analogue of the $H$-separation property is equivalent to (3). Moreover, we give a related characterization of amenability of $H$ in terms of any group $G$ containing $H$ as a closed subgroup. The weak amenability of $G$ or that $G_d$ satisfies the approximation property, in combination with the existence of a natural projection (in the sense of Lau and Ülger), are shown to suffice to conclude (3). Several consequences of (2) involving the cb-multiplier completion of $A(G)$ are given. Finally, a convolution technique for averaging over the closed subgroup $H$ is developed and used to weaken a condition for the existence of a bounded approximate indicator for $H$.

1. Introduction

Our objective is to study connections between various forms of amenability for a locally compact group $G$ and certain separation properties for closed subgroups, and moreover to establish relationships between these separation properties. Following the influential work of Ruan [34], much work on the homology of the Fourier algebra $A(G)$ as a completely contractive Banach algebra has affirmed this as the appropriate category in which to consider $A(G)$ and the related algebras of abstract harmonic analysis (e.g. [1, 9, 17, 18]). Motivated by the success of this perspective, we focus on completely bounded projections, operator amenability, and the completely bounded multiplier algebra of $A(G)$. We consider the following separation properties for a closed subgroup $H$ of $G$:

(1) The existence of a bounded approximate indicator for $H$.
(2) The existence of a completely bounded $A(G)$-bimodule projection of $VN(G)$ onto $VN_H(G)$.

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(3) When the characteristic function $\chi_H$ may be approximated by functions in $B(G)$ or $M_{cb}A(G)$ in the weak* topology on the corresponding algebra of the discretized group.

Condition (3) is also considered for subsets of $G$ that are not necessarily closed subgroups.

Bounded approximate indicators for closed subgroups were introduced in [1] as a means of obtaining invariant projections. They have subsequently been shown to have an intimate connection with homological properties of $A(G)$ and its completion in the cb-multipliers $A_{cb}(G)$ [9]. A result of Granirer and Leinert [19, Theorem B2] yields bounded approximate indicators in $B(G)$ from weaker conditions than given in Definition 2.2. This useful tool is unavailable for nets in $M_{cb}A(G)$ and Section 6 develops a convolution technique relative to the closed subgroup $H$ that recovers the weakened condition in the cb-multiplier setting.

The existence of invariant projections has been studied by several authors in connection with other separation properties and the existence of approximate identities for ideals in $A(G)$ [1, 9, 10, 15, 17, 24]. In particular, in [24] it is shown that if $G$ has the $H$-separation property, then an invariant projection $VN(G) \to VN_H(G)$ exists. We show in Section 4 that, in fact, the $H$-separation property is equivalent to the existence of a bounded approximate indicator for $H$ consisting of positive definite functions that are identically one on $H$. An analogue of the $H$-separation property is moreover shown to characterize the $B(G)$-approximability of $\chi_H$. Condition (3) was first studied in [1], where it was claimed that a bounded approximate indicator for $H$ exists whenever $\chi_H$ is $B(G)$-approximable. This argument was later found to contain a gap [2]. We give examples in Section 3 showing that the cb-multiplier analogue is false.

Conditions (1) to (3) are related to amenability properties of $G$ and to homological properties of $A(G)$, and it is this connection that is the main focus of the present article. We show in Section 4 that $H$ is already amenable when $\chi_H$ is $A(G)$-approximable for any locally compact group $G$ containing $H$ as a closed subgroup. In the case that $G$ is amenable, the algebra $A(G)$ has a bounded approximate identity and [15, Proposition 6.4] then asserts that an invariant projection $VN(G) \to VN_H(G)$ exists exactly when the ideal $I_{A(G)}(H)$ in $A(G)$ has a bounded approximate identity. By [17], the latter occurs for every closed subgroup of $G$ and it follows that an approximate indicator exists for every closed subgroup, since $(1_G - e_a)_a$ is an approximate indicator for $H$ when $(e_a)_a$ is a bounded approximate identity for $I_{A(G)}(H)$. Thus all closed subgroups of an amenable locally compact group are separated in the strongest sense that we consider. For generic locally compact groups the situation is more complicated, although some strong connections are known to hold in general. For example, using the identity $A(G) \hat{\otimes} A(G) = A(G \times G)$, it is routine to show that an approximate indicator for the diagonal $G_\Delta$ in $A(G \times G)$ (and bounded there) is exactly a bounded approximate diagonal for $A(G)$, the existence of which
characterizes amenability of $G$ \[34\]. Moreover, the existence of an invariant projection $VN(G \times G) \to VNG_\Delta(G \times G)$ characterizes the operator biflatness of $A(G)$, a weaker homological condition than operator amenability. Contractive operator biflatness, which asks that this invariant projection to be a complete contraction, was recently shown to be equivalent to the existence of a contractive approximate indicator for $G_\Delta$ in $B(G \times G)$ \[9\].

A bounded approximate indicator for $H$ always yields the approximability of $\chi_H$ in the corresponding algebra, however it is unclear when the latter follows from the existence of an invariant projection $VN(G) \to VN_H(G)$ alone. In Section 3 we show that if $H$ satisfies certain weak forms of amenability, then the existence of a bounded map $VN(G) \to VN_H(G)$ satisfying $\lambda(s) \mapsto \chi_H(s)\lambda(s)$ — a weaker condition than the existence of an invariant projection onto $VN_H(G)$ — implies the $M_dA(G)$-approximability of $\chi_H$. We give an example in which the former condition fails while the latter holds. Establishing relations amongst the conditions (1) to (3) in the article.

## 2. Preliminaries

For a locally compact group $G$, the following algebras were defined by Eymard in \[13\], who established the basic properties we outline below. The space of coefficient functions of strongly continuous unitary representations of $G$,

\[ (2.1) \quad B(G) = \{ \langle \pi(\cdot) \xi | \eta \rangle : \pi : G \to B(\mathcal{H}) \text{ is a representation, } \xi, \eta \in \mathcal{H} \} , \]

forms a commutative completely contractive Banach algebra, the Fourier–Stieltjes algebra of $G$, under pointwise multiplication and norm

\[ \|u\|_{B(G)} = \inf \{ \|\xi\| \|\eta\| : u = \langle \pi(\cdot) \xi | \eta \rangle \text{ with } \pi, \xi, \eta \text{ as in (2.1)} \} . \]

The operator space structure on $B(G)$ arises from its identification with the dual space of the universal enveloping $C^*$-algebra of $L^1(G)$ — the group $C^*$-algebra $C^*(G)$ of $G$ — via the duality

\[ \langle u, \pi(f) \rangle_{B(G), C^*(G)} = \int_G fu \quad (u \in B(G), f \in L^1(G)) . \]

We refer to \[12\] for the theory of operator spaces and completely contractive Banach algebras. The positive definite functions in $B(G)$ are those that correspond to positive functionals on $C^*(G)$ and are denoted $P(G)$. For $u \in P(G)$ we have $\|u\|_{B(G)} = \|u\|_{L^\infty(G)} = u(e)$. The adjoint on $B(G)$ is given by

\[ u^*(s) = u(s^{-1}) \quad (u \in B(G), s \in G) \]

and the self-adjoint functions in $B(G)$ correspond to the self-adjoint bounded functionals on $C^*(G)$. Consequently, given $u \in B(G)$ self-adjoint, there exist $u^+ \in P(G)$ such that $u = u^+ - u^-$ and $\|u\|_{B(G)} = \|u^+\|_{B(G)} + \|u^-\|_{B(G)}$. 

The Fourier algebra of $G$ is the closed ideal $A(G)$ of $B(G)$ given by the coefficients of the left regular representation $\lambda: G \to B(L^2(G))$, which is defined by

$$\lambda(s) \xi(t) = \xi(s^{-1}t) \quad (s, t \in G, \xi \in L^2(G)).$$

When necessary, we denote this representation of $G$ by $\lambda_G$. The Fourier algebra coincides with the closure of the compactly supported functions in $B(G)$, is closed under the adjoint, and is regular in the sense that for any $K \subset G$ compact and $U \supset K$ open, there is a function $v \in A(G)$ with $v(K) = 1$ and $v(G \setminus U) = 0$. The group von Neumann algebra of $G$ is the weak operator topology closure $VN(G)$ of span$\lambda(G)$ in $B(L^2(G))$ and is identified with the dual of the Fourier algebra via

$$\langle \lambda(s), u \rangle_{VN(G), A(G)} = u(s) \quad (s \in G, u \in A(G)).$$

Given an algebra $A$ of functions on $G$ and a subset $E$ of $G$, we denote by $I_A(E)$ the ideal of functions in $A$ vanishing on $E$. For a closed subgroup $H$ of $G$, the annihilator $I_{A(G)}(H)^\perp$ coincides with the von Neumann algebra $VN_H(G)$ generated by $\lambda_H$ [39, Theorem 6], which is identified with $VN(H)$ via the normal $*$-isomorphism defined by $\lambda_H(h) \mapsto \lambda_G(h)$ for $h \in H$. The preadjoint of this normal $*$-isomorphism is the restriction map $r_H: A(G) \to A(H)$, which is thus a complete quotient. The closed subgroups of $G$ are sets of spectral synthesis for $A(G)$ [20], meaning that the ideal $I_{A(G)}(H)$ of $A(G)$ coincides with the closure of the functions in $A(G)$ that have compact support disjoint from $H$.

The completely bounded (cb-) multiplier algebra $M_{cb}A(G)$ of $A(G)$ is the algebra of functions $m$ on $G$ for which $mA(G) \subset A(G)$ and the map $M_m: VN(G) \to VN(G)$ is completely bounded, where $M_m$ is the adjoint of the multiplication map $u \mapsto mu$ on $A(G)$. Such functions are continuous and bounded, and form a completely contractive Banach algebra under pointwise multiplication and norm $\|m\|_{M_{cb}A(G)} = \|M_m\|_{cb}$. The cb-multipliers of $A(G)$ admit the following representation theorem [22].

**Theorem 2.1.** (Gilbert’s representation theorem) Let $G$ be a locally compact group. A function $m$ on $G$ is in $M_{cb}A(G)$ if and only if there exists a Hilbert space $H$ and bounded continuous maps $P, Q: G \to H$ such that

$$m(s^{-1}t) = \langle P(t), Q(s) \rangle \quad (s, t \in G).$$

The norm $\|m\|_{M_{cb}A(G)}$ is the infimum of the quantities $\|P\|_{\infty} \|Q\|_{\infty}$ taken over all such maps $P$ and $Q$ and Hilbert spaces $H$.

It follows from Gilbert’s representation theorem that $B(G) \subset M_{cb}A(G)$ and that $\|m\|_{M_{cb}A(G)} \leq \|m\|_{B(G)}$ on $B(G)$. The restriction $r_H: M_{cb}A(G) \to M_{cb}A(H)$ is a well defined complete contraction [11, Proposition 1.12]. The norm closure of $A(G)$ in the potentially smaller cb-multiplier norm is denoted $A_{cb}(G)$ and forms a completely contractive Banach subalgebra of the cb-multipliers with spectrum $G$ [13, Proposition 2.2].
Since cb-multipliers of $A(G)$ lie in $L^\infty(G)$, we may consider $L^1(G)$ as a subspace of the dual of $M_{cb}A(G)$. Taking the completion of $L^1(G)$ with respect to the norm given, for $f \in L^1(G)$, by

$$\|f\|_{Q(G)} = \sup \left\{ \left| \int_G fm \right| : m \in M_{cb}A(G) \text{ with } \|m\|_{M_{cb}A(G)} \leq 1 \right\}$$

yields a predual $Q(G)$ for $M_{cb}A(G)$ \cite{pass} Proposition 1.10]. With this predual, the cb-multipliers $M_{cb}A(G)$ form a completely contractive dual Banach algebra in the sense of Runde \cite{ru}. It follows from Theorem 2.1 that $\|\cdot\|_{L^\infty(G)} \leq \|\cdot\|_{M_{cb}A(G)}$ and consequently $\|\cdot\|_{Q(G)} \leq \|\cdot\|_{L^1(G)}$ on $L^1(G)$. We let $C_c(G)$ and $C_b(G)$ denote respectively the continuous compactly supported and continuous bounded functions of $G$. We have

$$A(G) \subset A_{cb}(G) \quad \text{and} \quad A(G) \subset B(G) \subset M_{cb}A(G) \subset C_b(G).$$

The second containment is strict unless $G$ is compact. An unpublished result of Ruan asserts that $A(G)$ is closed in $M_{cb}A(G)$ exactly when $G$ is amenable, so that that the first and third containments are strict unless $G$ is amenable. For $u \in A(G)$ and $v \in B(G)$,

$$\|u\|_\infty \leq \|u\|_{A_{cb}(G)} = \|u\|_{M_{cb}A(G)} \leq \|u\|_{A(G)} = \|u\|_{B(G)},$$

$$\|v\|_\infty \leq \|v\|_{M_{cb}A(G)} \leq \|v\|_{B(G)},$$

so the first, third, and fourth inclusions are in general contractive while the second is isometric.

The locally compact group $G$ equipped with the discrete topology is denoted $G_d$. The inclusions $B(G) \subset B(G_d)$ and $M_{cb}A(G) \subset M_{cb}A(G_d)$ are complete isometries \cite{ru} and \cite[Corollary 6.3]{pass}, respectively. For each $s \in G$ the point mass $\delta_s \in \ell^1(G_d)$ is contained in $\ell^1(G_d)$ and in $Q(G_d)$ as the evaluation functional at $s$, from which it follows that convergence in the weak* topology of $B(G_d)$ or $M_{cb}A(G_d)$ implies pointwise convergence. It will be important for us that, on bounded sets, the converse holds (see \cite{ru} regarding $B(G_d)$ and \cite[Lemma 2.6]{pass} or the useful Appendix A of \cite{we} regarding $M_{cb}A(G_d)$).

The separation properties for closed subgroups that we discuss are defined as follows.

**Definition 2.2.** Let $G$ be a locally compact group and $H$ a closed subgroup.

1. A **bounded approximate indicator** for $H$ is a bounded net $(m_\alpha)_{\alpha}$ in $M_{cb}A(G)$ satisfying
   (a) $\|ur_H(m_\alpha) - u\|_{A(H)} \to 0$ for all $u \in A(H)$, and
   (b) $\|um_\alpha\|_{A(G)} \to 0$ for all $u \in I_{A(G)}(H)$.

   If $(m_\alpha)_{\alpha}$ is in $B(G)$ and is also bounded there, then we refer to a bounded approximate indicator in $B(G)$.

2. A completely bounded projection $VN(G) \to VN_H(G)$ is **invariant** if it is an $A(G)$-bimodule map.
(3) Given a norm closed subalgebra \( A \) of \( B(G) \) or \( M_{cb}A(G) \) and \( E \subset G \), the characteristic function \( \chi_E \) is called \( A \)-approximable if \( \chi_E \) is in the weak* closure of \( A \) in \( B(G_d) \) or \( M_{cb}A(G_d) \), respectively.

The operator amenability of the Fourier algebra asserts the existence of a bounded approximate diagonal in the operator space projective tensor product \( A(G) \hat{\otimes} A(G) \). This is a bounded net \((d_\alpha)_\alpha\) in the tensor product which satisfies the norm convergence

\[
u \cdot d_\alpha - d_\alpha \cdot u \rightarrow 0 \quad \text{and} \quad \Delta(d_\alpha) u \rightarrow u \quad (u \in A(G)),\]

where \( \Delta : A(G) \otimes A(G) \rightarrow A(G) \) is the completely bounded linearization of the multiplication and the \( A(G) \) action on the tensor product is given by \( u \cdot (v \otimes w) = uv \otimes w \) and \( (v \otimes w) \cdot u = v \otimes wu \), for \( u, v, w \in A(G) \). The product map \( \Delta \) is a complete quotient and the operator biflatness of \( A(G) \) asserts the existence of a completely bounded \( A(G) \)-bimodule left inverse to its adjoint \( \Delta^* : VN(G) \rightarrow VN(G) \hat{\otimes} VN(G) \), where the \( A(G) \) action on \( VN(G) \) is the dual action. The identification \( A(G) \hat{\otimes} A(G) = A(G \times G) \) (see [12, Theorem 7.2.4]) yields \( \ker \Delta = I_{A(G \times G)}(G_\Delta) \), from which it follows that such a left inverse is exactly an invariant projection \( VN(G \times G) \rightarrow VN_{G_\Delta}(G \times G) \). A thorough account of the homological conditions we consider is given in [36].

Leptin’s classical result states that amenability of a locally compact group \( G \) is characterized by the existence of a bounded approximate identity in \( A(G) \) [29]. The locally compact group \( G \) is called weakly amenable when \( A_{cb}(G) \) has a bounded approximate identity. This weaker notion was introduced by de Cannière and Haagerup [11], who showed that the free group on two generators is weakly amenable. The weak amenability of \( G \) is equivalent to the assertion that every cb-multiplier is the weak* limit of a bounded net in \( A(G) \). When \( A(G) \) is merely weak* dense in \( M_{cb}A(G) \), the group \( G \) is said to have the approximation property (see [21]).

3. Approximability of characteristic functions

In this section, we investigate when characteristic functions of subsets of a locally compact group \( G \) are approximable. For a closed subgroup \( H \) of \( G \), the assertion that \( \chi_H \) is approximable may be viewed as a very weak form of subgroup separation. In [4], the discretized Fourier–Stieltjes algebra \( B^d(G) \) is defined to be the weak* closure of \( B(G) \) in \( B(G_\Delta) \). We make the analogous definition for the cb-multipliers of \( G \).

**Definition 3.1.** The discretized cb-multiplier algebra \( M_{cb}^dA(G) \) of a locally compact group \( G \) is the weak* closure of \( M_{cb}A(G) \) in \( M_{cb}A(G_d) \). The predual \( Q(G_d)_d / M_{cb}^d(G)_d \) of \( M_{cb}^dA(G) \) is denoted \( Q^d(G) \). Let \( \hat{A}^d(G) \) and \( A_{cb}^d(G) \) denote the weak* closures of \( A(G) \) in \( B(G_d) \) and in \( M_{cb}A(G_d) \), respectively.

Given \( E \subset G \), for \( \chi_E \) to be approximable, it must already be that \( \chi_E \in M_{cb}A(G_d) \), and the subsets of \( G \) for which this occurs are not well understood when \( G_d \) is not amenable. In the amenable case, the algebras
M_{cb}A(G_d) and B(G_d) coincide \[31] and the Cohen-Host idempotent theorem provides a complete description of the subsets of G with characteristic function in B(G_d). For discrete groups G, determining the approximable characteristic functions is exactly the problem of determining the sets with characteristic function in the cb-multipliers. When G is moreover weakly amenable, Corollary 5.4 of \[9\] together with Corollary 3.5 of \[18\] implies that such sets E are exactly those for which the ideal I_{A(G)}(E) has a cb-multiplier bounded approximate identity.

**Example 3.2.** Let G be a locally compact group and H a closed subgroup for which a bounded approximate indicator \((m_\alpha)_\alpha\) exists. We show that \(\chi_H\) is approximable. If \(s \in H\), then we may find \(u \in A(H)\) with \(u(s) = 1\), in which case
\[
|m_\alpha(s) - 1| \leq \|ur_H(m_\alpha) - u\|_{L^\infty(H)} \leq \|ur_H(m_\alpha) - u\|_{A(H)} \to 0.
\]
If \(s \in G \setminus H\), then \[13, Lemme 3.2\] asserts that we may find \(w \in I_{A(G)}(H)\) with \(w(s) = 1\), and
\[
|m_\alpha(s)| \leq \|wm_\alpha\|_{L^\infty(G)} \leq \|wm_\alpha\|_{A(G)} \to 0.
\]
Since \(M_{cb}A(G)\) is contained in \(M_{cb}A(G_d)\) and weak* and pointwise convergence coincide on bounded subsets of \(M_{cb}A(G_d)\), it follows that \((m_\alpha)_\alpha\) has weak* limit \(\chi_H\) in \(M_{cb}A(G_d)\).

The algebra \(M^d_{cb}A(G)\) is a weak* closed subalgebra of \(M_{cb}A(G_d)\) and thus has separately weak* continuous multiplication. If we impose a rather weak condition on the discrete group \(G_d\), then every function in \(M_{cb}A(G_d)\) may be approximated in the weak* topology by functions in \(A(G)\).

**Proposition 3.3.** Let G be a locally compact group. The inclusion \(A_{cb}(G_d) \subset A^d_{cb}(G)\) always holds. Consequently, if \(G_d\) has the approximation property, then \(M_{cb}A(G_d) = A^d_{cb}(G)\).

**Proof.** By Proposition 1 of \[1\] we have \(A(G_d) \subset A^d(G)\) and, because weak* convergence in \(B(G_d)\) implies weak* convergence in \(M_{cb}A(G_d)\), it follows that \(A(G_d) \subset A^d_{cb}(G)\). Taking norm closures in \(M_{cb}A(G_d)\) yields \(A_{cb}(G_d) \subset A^d_{cb}(G)\). If \((e_\alpha)_\alpha\) is a net in \(A(G_d)\) converging weak* to \(1_G\) in \(M_{cb}A(G_d)\) and if \(m \in M_{cb}A(G_d)\), then \(me_\alpha \in A(G_d)\) for each \(\alpha\) and \(me_\alpha \xrightarrow{w^*} m\) in \(M_{cb}A(G_d)\), implying that \(m \in A^d_{cb}(G)\). \(\square\)

In Section 3 of \[1\], that the Fourier–Stieltjes algebra is the dual of a \(C^*\)-algebra is noted to imply that the unique weak* continuous extension of the inclusion \(B(G) \subset B^d(G)\) is a quotient map \(B(G)^{**} \to B^d(G)\). We provide a more concrete construction of the analogous canonical map \(M_{cb}A(G)^{**} \to M^d_{cb}A(G)\) that exploits the relation between \(M_{cb}A(G)\) and \(M^d_{cb}A(G)\). Let \(i_d : M_{cb}A(G) \to M_{cb}A(G_d)\) and \(\kappa_Q : Q(G_d) \to Q(G_d)^{**}\) denote the inclusion maps. By the bipolar theorem \(M^d_{cb}A(G) = M_{cb}A(G)_{**}\) and we have
\[
\kappa_Q(M_{cb}A(G)_{**}) \subset M_{cb}A(G)_{**} = \text{im}(i_d)^\perp = \ker(i_d^*),
\]
which together imply that the composition
\[ Q(G_d) \xrightarrow{\kappa} Q(G_d)^\ast\ast \xrightarrow{\iota^*} Q(G)^\ast\ast \]
induces a map \( Q^d(G) \to Q(G)^\ast\ast \). Denote the adjoint of this induced map by \( \tau : M_{cb}A(G)^\ast\ast \to M_{cb}^dA(G) \). It is straightforward to verify that \( \tau \) extends inclusion \( M_{cb}A(G) \subset M_{cb}^dA(G) \), so that \( \tau(m)(s) = m(s) \) for \( m \in M_{cb}A(G) \) and \( s \in G \). It follows that if a net \( (m_\alpha)_\alpha \) in \( M_{cb}A(G) \) converges weak* to \( \omega \in M_{cb}A(G)^\ast\ast \), then
\[ \tau(\omega)(s) = \lim_{\alpha} m_\alpha(s) \quad (s \in G), \]
so that the map \( \tau \) extracts pointwise limits from nets in \( M_{cb}A(G) \) that are weak* convergent in the bidual. Moreover, the range of \( \tau \) consists of exactly those functions in \( M_{cb}^dA(G) \) that are limits of bounded nets in \( M_{cb}A(G) \).

**Proposition 3.4.** Let \( G \) be a locally compact group and \( E \subset G \). If there is a bounded map \( \Psi : VN(G) \to VN(G) \) satisfying \( \Psi(\lambda(s)) = \chi_E(s)\lambda(s) \) for all \( s \in G \), then \( \chi_EA(G) \subset A_{cb}^d(G) \). If, moreover, \( \chi_E \in M_{cb}A(G_d) \) and \( 1_G \) is \( A_{cb}(G) \)-approximable, then \( \chi_E \) is \( A_{cb}(G) \)-approximable.

**Proof.** Let \( \kappa_A : A(G) \to A(G)^\ast\ast \) and \( \iota_A : A(G) \to M_{cb}A(G) \) be the inclusions and let \( \sigma \) denote the composition
\[ A(G)^\ast\ast \xrightarrow{\Psi} A(G)^\ast\ast \xrightarrow{\iota^*} M_{cb}A(G)^\ast\ast \xrightarrow{\tau} M_{cb}^dA(G). \]
For \( u \in A(G) \) and \( s \in G \), with \( \delta_s \) denoting the point mass at \( s \) in \( \ell^1(G_d) \subset Q(G_d) \subset Q^d(G) \),
\[ \sigma(u)(s) = \langle \sigma(u), \delta_s \rangle_{M_{cb}^dA(G), Q^d(G)} = \langle \iota^*\Psi\kappa_A(u), \iota^*\kappa_Q(\delta_s) \rangle_{M_{cb}A(G)^\ast\ast, M_{cb}A(G)^\ast} = \langle \Psi\kappa_A(u), \lambda(s) \rangle_{A(G)^\ast\ast, VN(G)} = \langle \Psi(\lambda(s)), u \rangle_{VN(G), A(G)} = \chi_E(s)u(s). \]
Thus \( \chi_Eu = \sigma(u) \in M_{cb}^dA(G) \) for all \( u \in A(G) \). Since \( \tau \) extends the inclusion of \( M_{cb}A(G) \) into \( M_{cb}^dA(G) \), it maps \( A(G) \) into \( A_{cb}^d(G) \), which together with the weak* continuity of \( \tau\iota_A^* \) implies that \( \sigma \) in fact has range in \( A_{cb}^d(G) \). If \( (e_{\alpha})_\alpha \) is a net in \( A(G) \) converging weak* to \( 1_G \) in \( M_{cb}A(G_d) \), then that \( \chi_E \in M_{cb}A(G_d) \) implies \( \chi_Ee_{\alpha} \xrightarrow{\text{w*}} \chi_E \), by weak* continuity of multiplication in \( M_{cb}A(G_d) \). Since \( \chi_Ee_{\alpha} \in A_{cb}^d(G) \), we conclude that \( \chi_E \) is \( A_{cb}(G) \)-approximable.

For a subgroup \( H \) of a locally compact group \( G \), it is straightforward to verify that \( \chi_H \) is a positive definite function on \( G_d \), so that \( \chi_H \in B(G_d) \subset M_{cb}A(G_d) \) and the second part of Proposition 3.4 is applicable to characteristic functions of subgroups. It is shown in Lemma 3.8 below that, when \( \chi_HA(G) \subset A_{cb}^d(G) \), we need only require \( 1_H \) to be \( A_{cb}(H) \)-approximable to deduce that \( \chi_H \) is \( A_{cb}(G) \)-approximable.
In [30], Lau and Ülger define a projection \( \Psi \) on \( VN(G) \) to be natural if \( \Psi(\lambda(s)) = \chi_E(s)\lambda(s) \) for some subset \( E \) of \( G \). We may interpret Proposition 3.4 as imposing restrictions on which subsets of \( G \) can arise from a natural projection.

**Example 3.5.** Let \( G \) be a locally compact group and \( H \) a closed subgroup. We show that an invariant projection \( \Psi : VN(G) \to VN_H(G) \) is natural. It is clear that \( \Psi(\lambda(s)) = \lambda(s) \) for \( s \in H \). Let \( s \in G \setminus H \) and let \( T_\alpha \in \text{span}\lambda(H) \) converge weak* to \( \Psi(\lambda(s)) \) in \( VN(G) \). If \( u \in A(G) \) with \( u(s) = 1 \) and \( u|_H = 0 \), then
\[
\langle u \cdot \lambda(s), v \rangle = \langle \lambda(s), vu \rangle = v(s)u(s) = v(s) = \langle \lambda(s), v \rangle \quad (v \in A(G)),
\]
so \( u \cdot \lambda(s) = \lambda(s) \). If \( S = \sum_j \alpha_j \lambda(s_j) \in \text{span}\lambda(H) \), then
\[
\langle u \cdot S, v \rangle = \sum_j \alpha_j v(s_j)u(s_j) = 0 \quad (v \in A(G)),
\]
so that \( u \cdot S = 0 \). Thus \( 0 = u \cdot T_\alpha \xrightarrow{w^*} u \cdot \Psi(\lambda(s)) = \Psi(u \cdot \lambda(s)) = \Psi(\lambda(s)) \) and \( \Psi(\lambda(s)) = 0 \).

Let \( G \) be a locally compact group. A bounded net \( (e_\alpha)_\alpha \) in \( A(G) \) or \( A_{cb}(G) \) is called a \( \Delta \)-weak bounded approximate identity if it converges pointwise to \( 1_G \). This notion was introduced in [23] and shown in [30] to be closely related to the existence of natural projections. Reasoning as in Example 3.2 shows that a bounded approximate identity for \( A_{cb}(G) \) is a \( \Delta \)-weak one, so that \( A_{cb}(G) \) has \( \Delta \)-weak bounded approximate identity whenever \( G \) is weakly amenable. The function \( 1_G \) is \( A_{cb}(G) \)-approximable when \( A_{cb}(G) \) has a \( \Delta \)-weak bounded approximate identity.

The construction of the map \( \tau \) above and the proof of Proposition 3.4 may be carried out with \( M_{cb}A(G) \) replaced by \( B(G) \), but, to conclude that \( \chi_H \) is \( B(G) \)-approximable using this result, we require \( 1_G \) to be in the weak* closure of \( A(G) \) in \( B(G_d) \). The proof of Theorem 1.6 shows that the canonical map \( A(G)^{**} \rightarrow A^d(G) \) extending inclusion \( A(G) \subset A^d(G) \) is surjective, so that \( 1_G \) is then the weak* limit of a bounded net, which is then a \( \Delta \)-weak bounded approximate identity for \( A(G) \), implying that \( G \) is already amenable [25, Theorem 5.1]. It is the availability of \( \Delta \)-weak bounded approximate identities in \( A_{cb}(G) \) for a larger class of groups — containing at least the weakly amenable groups — that is responsible for the utility of Proposition 3.4. Whether the existence of a \( \Delta \)-weak bounded approximate identity for \( A_{cb}(G) \) implies weak amenability of \( G \) appears to be an open question.

For a locally compact group \( G \) and closed subgroup \( H \), let
\[
r_H : M_{cb}A(G_d) \rightarrow M_{cb}A(H_d) \quad \text{and} \quad e_H : M_{cb}A(H_d) \rightarrow M_{cb}A(G_d)
\]
denote respectively the restriction map and the extension by zero. For a function \( f \) on \( H \), let \( \hat{f} \) denote its extension by zero to \( G \). The restriction \( r_H \) is a complete quotient and extension \( e_H \) a complete isometry ([38, Corollary 6.3] or [37, Proposition 4.1]), and it is clear that \( r_H e_H = \text{id}_{M_{cb}A(H_d)} \) and \( e_H r_H = M_{\chi_H} \), the multiplication by \( \chi_H \).
Lemma 3.6. Let $G$ be a locally compact group and $H$ a closed subgroup. The maps $r_H$ and $e_H$ are weak* continuous.

Proof. If $f = \sum_{j=1}^{n} \alpha_j \delta_{x_j} \in \mathcal{C}_c(H_d)$, then $\hat{f} \in \mathcal{C}_c(G_d)$ and

$$\langle r_H^*(\hat{f}), m \rangle = \sum_{j=1}^{n} \alpha_j m(x_j) = \langle \hat{f}, m \rangle \quad (m \in M_{cb}A(G_d)),$$

showing that $r_H^*(\mathcal{C}_c(H_d)) \subset Q(G_d)$. Since $\mathcal{C}_c(H_d)$ is dense in $Q(H_d)$, it follows that $r_H$ is weak* continuous.

Now if $f = \sum_{j=1}^{n} \alpha_j \delta_{x_j} \in \mathcal{C}_c(G_d)$, then, for $m \in M_{cb}A(H_d)$,

$$\langle e_H^*(f), m \rangle = \langle \hat{m}, f \rangle = \sum_{j=1}^{n} \alpha_j \chi_H(x_j) m(x_j) = \langle \sum_{j=1}^{n} \alpha_j \chi_H(x_j) \delta_{x_j}, m \rangle,$$

and so $\sum_{j=1}^{n} \alpha_j \chi_H(x_j) \delta_{x_j} \in \mathcal{C}_c(H_d)$. Therefore $e_H^*(\mathcal{C}_c(G_d)) \subset Q(H_d)$ and the claim follows by density, as above. $\square$

Lemma 3.7. Let $G$ be a locally compact group and $H$ a closed subgroup. The restriction $r_H$ maps $M_{cb}(G)$ into $M_{cb}(H)$. In addition, the following are equivalent:

1. $\chi_H A(G) \subset A_{cb}^d(G)$.
2. $e_H \left( A_{cb}^d(H) \right) \subset A_{cb}^d(G)$.
3. $A_{cb}^d(G) = I_{A_{cb}^d(G)}(G \setminus H) \oplus I_{A_{cb}^d(G)}(H)$ (algebraic direct sum).

Proof. Since the restriction of a cb-multiplier of $G$ to the closed subgroup $H$ yields a cb-multiplier of $H$ [111 Proposition 1.12], the first claim follows from weak* continuity of $r_H$.

1 implies 2: If $\chi_H A(G) \subset A_{cb}^d(G)$, then, because $A(H) = r_H(A(G))$,

$$e_H(A(H)) = e_H(r_H(A(G))) = \chi_H A(G) \subset A_{cb}^d(G)$$

and 2 follows by weak* continuity of $e_H$.

2 implies 3: If $e_H \left( A_{cb}^d(H) \right) \subset A_{cb}^d(G)$, then given $m \in A_{cb}^d(G)$, the weak* continuity of $r_H$ implies $r_H(m) \in A_{cb}^d(H)$ and it follows that $\chi_H m = e_H r_H(m) \in A_{cb}^d(G)$, whence $\chi_H m = m - \chi_H m \in A_{cb}^d(G)$ and therefore $m = \chi_H m + \chi_H m \in I_{A_{cb}^d(G)}(G \setminus H) + I_{A_{cb}^d(G)}(H)$. These ideals clearly have trivial intersection.

3 implies 1: Write $m \in A(G)$ as $m = m_1 + m_2$ for $m_1 \in I_{A_{cb}^d(G)}(G \setminus H)$ and $m_2 \in I_{A_{cb}^d(G)}(H)$, in which case $\chi_H m = m_1 \in A_{cb}^d(G)$. $\square$

When the equivalent conditions of Lemma 3.7 hold, condition 2 implies that $e_H \left( A_{cb}^d(H) \right) = I_{A_{cb}^d(G)}(G \setminus H)$ and, because $e_H$ is isometric, condition 3 asserts that $A_{cb}^d(G) = A_{cb}^d(H) \oplus I_{A_{cb}^d(G)}(H)$.

Lemma 3.8. Let $G$ be a locally compact group and $H$ a closed subgroup. If $\chi_H A(G) \subset A_{cb}^d(G)$ and $1_H$ is $A_{cb}(H)$-approximable, then $\chi_H$ is $A_{cb}(G)$-approximable.

Proof. It follows from Lemma 3.7(2) that $\chi_H = e_H(1_H) \in A_{cb}^d(G)$. $\square$
Combining the results of this section, we obtain the following.

**Theorem 3.9.** The characteristic function of a closed subgroup $H$ of a locally compact group $G$ is $A_{cb} (G)$-approximable when either of the following conditions is satisfied:

1. $G_d$ has the approximation property.
2. There is a bounded map $\Psi : VN (G) \to VN (G)$ such that $\Psi (\lambda (s)) = \chi_H (s) \lambda (s)$ for $s \in G$, which is satisfied if $\Psi$ is a natural or invariant projection onto $VN_H (G)$, and $1_H$ is $A_{cb} (H)$-approximable, which occurs when $H$ is weakly amenable or $H_d$ has the approximation property.

**Proof.** (1) When $G_d$ has the approximation property, Proposition 3.3 asserts that $A_{cb}^d (G) = M_{cb} A (G_d)$. Since $M_{cb} A (G_d)$ contains the characteristic functions of all subgroups of $G$, we have $\chi_H \in A_{cb}^d (G)$.

(2) If $\Psi : VN (G) \to VN_H (G)$ is a bounded map such that $\Psi (\lambda (s)) = \chi_H (s) \lambda (s)$ for $s \in G$ and $1_H$ is $A_{cb} (H)$-approximable, then $\chi_H A (G) \subset A_{cb}^d (G)$ by Proposition 3.4. Lemma 3.8 then asserts that $\chi_H$ is $A_{cb} (G)$-approximable. Example 3.5 shows that invariant projections $VN (G) \to VN_H (G)$ are natural, and the latter are by definition bounded map satisfying the condition of (2). It was noted above that weak amenability of $H$ implies $1_H \in A_{cb}^d (H)$, while Proposition 3.3 implies $1_H \in A_{cb}^d (H)$ when $H_d$ has the approximation property. $\square$

Theorem 3.7 of [1] claims that a bounded approximate indicator in $B (G)$ for a closed subgroup $H$ of the locally compact group $G$ exists whenever $\chi_H$ is $B (G)$-approximable. The argument establishing this result was found to contain an error [2] and it is not known whether the claim holds. The following examples show that $\chi_H$ may be $M_{cb} A (G)$-approximable even when no invariant projection onto $VN_H (G)$ exists, in which case no bounded approximate indicator for $H$ exists, either, by Proposition 5.1.

**Example 3.10.** The locally compact group $G = SL (2, \mathbb{R})$ contains $H = \mathbb{R}_2$ as a closed subgroup. It has recently been shown that $G_d$ is weakly amenable [28], so that $\chi_H$ is $A_{cb} (G)$-approximable by Proposition 3.3. Since $G$ is connected, its group von Neumann algebra is injective [6, Corollary 6.9(c)], so there exists a completely bounded projection $B (L^2 (G)) \to VN (G)$. If a completely bounded projection $VN (G) \to VN_H (G)$ existed, then composition would yield a completely bounded projection $B (L^2 (G)) \to VN_H (G)$, implying that $VN_H (G) = VN (H)$ is an injective von Neumann algebra [5]. It would follow that the discrete group $H$ is amenable [32, (2.35)], which is false.

**Example 3.11.** Let $G = SL (2, \mathbb{R})$ and consider the diagonal subgroup $G_\Delta$ of $G \times G$. The Fourier algebra $A (G)$ is not operator biflat by Corollary 3.7 of [2], meaning that no invariant projection $VN (G \times G) \to VN_{G_\Delta} (G \times G)$ exists. But the weak amenability of $G_d$, noted in the preceding example,
implies the weak amenability of \((G \times G)_d\), so that \(\chi_{G_\Delta}\) is \(A_\text{cb}(G \times G)\)-approximable, again by Proposition 3.3.

4. The discretized H-separation property

In this section, we characterize the approximability of the characteristic function of a closed subgroup \(H\) of a locally compact group \(G\) in the spirit of the \(H\)-separation property of Kaniuth and Lau. For a closed subgroup \(H\) of \(G\), let \(P_H(G)\) denote the norm closed convex set \(\{u \in P(G) : u(H) = 1\}\).

**Definition 4.1.** ([24]) A locally compact group \(G\) is said to have the \(H\)-separation property for a closed subgroup \(H\) if, for each \(s \in G \setminus H\), there exists \(u \in P_H(G)\) such that \(u(s) \neq 1\).

It is routine to verify that \(G\) has the \(H\)-separation property for any open, compact, or normal subgroup \(H\), and it was shown by Forrest [16] that if \(G\) is a SIN group, then \(G\) has the \(H\)-separation property for every closed subgroup \(H\). In [24], a fixed point argument is used to show that an invariant projection \(VN(G) \to VN_H(G)\) exists when the locally compact group \(G\) has the \(H\)-separation property (it is noted in Proposition 5.1 below that the projections arising this way are completely positive, in particular completely bounded). In fact, the following stronger result holds.

**Proposition 4.2.** Let \(G\) be a locally compact group and \(H\) a closed subgroup. Then \(G\) has the \(H\)-separation property if and only if there exists a bounded approximate indicator for \(H\) in \(P_H(G)\).

**Proof.** Suppose that \(G\) has the \(H\)-separation property. The proof of [24, Proposition 3.1] constructs an invariant projection \(P : VN(G) \to VN_H(G)\) that is the weak* operator topology limit of a net \((M_{u_\alpha})_\alpha\), where \(u_\alpha \in P_H(G)\) and \(M_{u_\alpha} : VN(G) \to VN(G)\) is the adjoint of the multiplication map \(u \mapsto u_\alpha u\) on \(A(G)\). Let \(r_H : A(G) \to A(H)\) be the restriction map, which is a surjection satisfying \(r_H^* (VN(H)) \subset VN_H(G)\). Given \(u \in A(H)\), let \(\tilde{u} \in A(G)\) with \(r_H(\tilde{u}) = u\), so that for \(T \in VN(H)\),

\[
\langle T, ur_H(u_\alpha) \rangle = \langle T, r_H(\tilde{u}u_\alpha) \rangle = \langle M_{u_\alpha} (r^*_H(T)) , \tilde{u} \rangle = \langle P (r^*_H(T)) , \tilde{u} \rangle = \langle r^*_H(T) , \tilde{u} \rangle = \langle T, u \rangle.
\]

If \(w \in I_{A(G)}(H)\), then

\[
\langle T, wu_\alpha \rangle = \langle M_{u_\alpha} (T) , w \rangle \rightarrow \langle P(T) , w \rangle = 0 \quad (T \in VN(G))
\]

since \(P(T) \in VN_H(G) = I_{A(G)}(H)^\perp\). Therefore \(ur_H(u_\alpha) \to u\) weakly in \(A(H)\) for all \(u \in A(H)\) and \(wu_\alpha \to 0\) weakly in \(A(G)\) for all \(w \in I_{A(G)}(H)\). Passing to convex combinations yields a bounded approximate indicator for \(H\) which remains in the convex set \(P_H(G)\).
Conversely, if \((u_\alpha)_\alpha\) is a bounded approximate indicator for \(H\) in \(P_H(G)\), then, given \(s \in G \setminus H\), choose \(w \in I_{A(G)}(H)\) with \(w(s) = 1\), in which case
\[
|u_\alpha(s)| = |u_\alpha(s)w(s)| \leq \|u_\alpha w\|_{L^\infty(G)} \leq \|u_\alpha w\|_{A(G)} \to 0
\]
implies \(u_\alpha(s) \neq 1\) eventually. \(\Box\)

For a closed subgroup \(H\) of a locally compact group \(G\), we now show that a weaker form of the \(H\)-separation property, replacing the algebra \(B(G)\) with \(B^d(G)\), characterizes when \(\chi_H\) is \(B(G)\)-approximable.

**Definition 4.3.** Let \(G\) be a locally compact group and \(H\) a closed subgroup. The group \(G\) is said to have the **discretized \(H\)-separation property** if, for any \(s \in G \setminus H\), there exists \(u \in B^d(G) \cap P_H(G_d)\) such that \(u(s) \neq 1\).

**Proposition 4.4.** Let \(G\) be a locally compact group and \(H\) a closed subgroup. Then \(G\) has the discretized \(H\)-separation property if and only if \(\chi_H\) is \(B(G)\)-approximable.

**Proof.** Suppose that \(G\) has the discretized \(H\)-separation property and for each \(s \in G \setminus H\) let \(u_s \in B^d(G) \cap P_H(G_d)\) with \(u_s(s) \neq 1\). Replacing \(u_s\) by \(\frac{1}{2} (1_G + u_s)\), which remains in \(B^d(G) \cap P_H(G_d)\), we may assume that \(|u_s(s)| < 1\). Then the sequence \((u^n_s)_{n \geq 1}\) is in \(B^d(G) \cap P_H(G_d)\) with 
\[
\|u^n_s\|_{B(G_d)} = u^n_s(e) = 1 \quad \text{and thus has a weak* cluster point } u^0_s \text{ in the unit ball of } B^d(G).
\]
Then \(u^n_s|_H = 1\) and \(|u^n_s(s)| \leq \limsup_n |u^n_s(s)| = 0\), so \(u^0_s(s) = 0\). Let \(\mathcal{F}\) be the collection of finite subsets of \(G\) and for each \(F \in \mathcal{F}\) let \(u_F = \prod_{s \in F} u^0_s\). Ordering \(\mathcal{F}\) by inclusion, we have \(u_F|_H = 1\) and \(u_F(s) = 0\) eventually for each \(s \in G \setminus H\), so that \(u_F \xrightarrow{pw} \chi_H\) and by boundedness \(u_F \xrightarrow{w^*} \chi_H\) in \(B(G_d)\), whence \(\chi_H \in B^d(G)\). The converse is clear, given that the characteristic function of a subgroup is always in \(P_H(G_d)\). \(\Box\)

When the locally compact group \(G\) is second countable, the \(H\)-separation property may also be characterized in terms of a single function on \(G\).

**Theorem 4.5.** Let \(G\) be a second countable locally compact group. For a closed subgroup \(H\), the following are equivalent:

1. \(G\) has the \(H\)-separation property.
2. There is \(u \in B(G)\) of norm one with \(\{s \in G : u(s) = 1\} = H\).
3. There is \(u \in P(G)\) with \(\{s \in G : u(s) = 1\} = H\).

**Proof.** (1) implies (2): For \(s \in G \setminus H\), let \(u_s \in P_H(G)\) with \(u_s(s) \neq 1\) and choose an open neighborhood \(U_s\) of \(s\) with \(1 \notin u_s(U_s)\). Then \((U_s)_{s \in G \setminus H}\) is an open cover of \(G \setminus H\), so has a countable subcover \((U_{s_n})_{n \geq 1}\) by \(\sigma\)-compactness of the open set \(G \setminus H\) in \(G\). The function \(u = \sum_{n \geq 1} 2^{-n} u_{s_n}\) is in the norm closed convex set \(P_H(G)\), so that \(\|u\|_{B(G)} = u(e) = 1\). Given \(s \in G \setminus H\), choose \(n\) such that \(s \in U_{s_n}\), so \(u_{s_n}(s) \neq 1\). Since \(\|u_{s_n}\|_{L^\infty(G)} = 1\), we have \(\text{Re} u_{s_n}(s) < 1\), implying that \(\text{Re} u(s) = \sum_{n \geq 1} 2^{-n} \text{Re} u_{s_n}(s) < 1\) and hence that \(u(s) \neq 1\).
A locally compact group

Theorem 4.6. Let $\langle e, \chi_H(s) \rangle$ be a weak* cluster point of the bounded net $(\Psi^*(\tau^*)^{-1}(e_\alpha), \lambda_H(s))$ in $V\overline{N}(G)^*$, so that $\langle E, \lambda_G(s) \rangle = \chi_H(s)$ for all $s \in G$ by the above computation. Letting $(u_\alpha)_{\alpha}$ be a bounded net in $A(G)$ converging weak* to $E$, we have $u_\alpha \overset{ptw}{\to} \chi_H$ and therefore $u_\alpha \overset{w^*}{\to} \chi_H$ in $B(G_d)$ by boundedness.

Conversely, suppose that $(u_\alpha)_{\alpha}$ is a net in $A(G)$ such that $u_\alpha \overset{w^*}{\to} \chi_H$ in $B(G_d)$. Analogously to the proof of Lemma 3.3, the restriction $r_H : B(G_d) \to B(H_d)$ is weak* continuous, so that $r_H(u_\alpha) \overset{w^*}{\to} 1_H$ and $1_H$ is in the weak* closure of $A(H)$ in $B(H_d)$. Viewing $\lambda_H$ as a representation of $H_d$, the universal
property of \( C^*(H_d) \) yields a quotient \(*\)-homomorphism \( C^*(H_d) \to C^*_\delta(H) : \omega_{H_d}(s) \mapsto \lambda_H(s) \), where \( \omega_{H_d} \) denotes the universal representation of \( H_d \) and \( C^*_\delta(H) \) the \(*\)-algebra generated by \( \lambda_H(H) \) in \( B(L^2(H)) \), a subalgebra of \( VN(H) \). Composing with the inclusion, we obtain a \(*\)-homomorphism \( \Psi : C^*(H_d) \to VN(H) \). The adjoint \( \Psi^* : A(H)^{**} \to B(H_d) \) is the weak* continuous extension of the inclusion \( A(H) \subset B(H_d) \), so that \( \ker \Psi = A(H)_\perp \) (preannihilator taken with respect to the \( B(H_d)\)-\( C^*(H_d) \) duality). Then \( A(H)_\perp = A^d(H)_\perp \) is an ideal in \( C^*(H_d) \) and \( \Psi \) drops to an injective \(*\)-homomorphism \( C^*(H_d)/A^d(H)_\perp \to VN(H) \). This injective \(*\)-homomorphism is isometric, so that its adjoint \( A(H)^{**} \to A^d(H) \) is a quotient map and given \( \epsilon > 0 \), the function \( 1_H \) is the weak* limit in \( B(H_d) \) of a net \( (v_\alpha)_\alpha \) in \( A(H) \) of bound \( 1 + \epsilon \). The adjoint on \( B(H_d) \) is a weak* continuous isometry preserving \( A(H) \), so that we may replace \( v_\alpha \) by \( \frac{1}{2}(v_\alpha + v_\alpha^*) \) and assume that \( v_\alpha = v_\alpha^+ - v_\alpha^- \) for \( v_\alpha^+ \in P(H) \cap A(H) \) with \( \|v_\alpha\|_{\overline{B}(H)} = \|v_\alpha^+\|_{\overline{B}(H)} + \|v_\alpha^-\|_{\overline{B}(H)} \). Passing to subnets, let \( v_\pm \) be weak* limits of \( (v_\alpha^\pm)_\alpha \) in \( B(H_d) \). Then

\[
1 = \lim_\alpha v_\alpha(e) = \lim_\alpha v_\alpha^+(e) - \lim_\alpha v_\alpha^-(e) = v^+(e) - v^-(e)
\]

and so

\[
v^-(e) = v^+(e) - 1 = \lim_\alpha v_\alpha^+(e) - 1 = \lim_\alpha \|v_\alpha^+\|_{\overline{B}(H)} - 1 \leq \epsilon.
\]

Let \( \omega_\epsilon \) be a weak* cluster point of \( (v_\alpha^\pm)_\alpha \) in \( A(H)^{**} \), so that \( \omega_\epsilon \) is a state on \( VN(H) \) satisfying, for \( s \in H \),

\[
|\langle \omega_\epsilon, \lambda_H(s) \rangle - 1| = \left| \lim_\alpha v_\alpha^+(s) - \lim_\alpha v_\alpha(s) \right| = \lim_\alpha |v_\alpha^-(s)| \leq \lim_\alpha v_\alpha^-(e) = v^-(e) \leq \epsilon.
\]

Letting \( \omega \) be a weak* cluster point of the states \( (\omega_\epsilon)_{\epsilon>0} \) on \( VN(H) \), we have \( \langle \omega, \lambda_H(s) \rangle = 1 \) for all \( s \in H \), and any extension of \( \omega \) to a state on \( B(L^2(H)) \) still takes the value 1 on the unitaries \( \lambda_H(s) \) for \( s \in H \). The amenability of \( H \) follows: by a Cauchy-Schwarz argument, any extension of \( \omega \) to a state on \( B(L^2(H)) \) is invariant under the conjugation action of the unitaries \( \lambda_H(s) \) for \( s \in H \), whence \( \lambda_H \) is an amenable representation of \( H \) (see [3]).

**Corollary 4.7.** Let \( G \) be a locally compact group and \( H \) a closed subgroup. Then \( H \) is amenable if and only if \( G \) has the discretized \( H \)-separation property witnessed by functions in \( A^d(G) \cap P_H(G_d) \).

**Proof.** In the argument establishing Proposition 4.4 substituting \( \frac{1}{2}(u_s^2 + u_s) \) for the function \( \frac{1}{2}(1_G + u_s) \) yields a proof that \( G \) has the desired property if and only if \( \chi_H \) is \( A(G) \)-approximable.

5. **Invariant projections and bounded approximate indicators**

In this section we establish some consequences of the existence of a bounded approximate indicator for a closed subgroup of a locally compact
group. We first provide the well known argument that this stronger separation property indeed yields invariant projections. For a commutative completely contractive Banach algebra $A$, let $CB_A(A^*)$ denote the completely bounded $A$-bimodule maps on $A^*$. This space has compact unit ball when given the weak$^*$ operator topology, which is determined by the seminorms

$$\Psi \mapsto |\langle \psi(\varphi), a \rangle| \quad (\varphi \in A^*, a \in A).$$

**Proposition 5.1.** Let $G$ be a locally compact group and $H$ a closed subgroup. If there is a bounded approximate indicator for $H$, then there is a completely bounded invariant projection $VN(G) \to VN_H(G)$. If there is a bounded approximate indicator for $H$ consisting of positive definite functions, then there is a completely positive invariant projection $VN(G) \to VN_H(G)$.

**Proof.** Let $(m_\alpha)_\alpha$ a bounded approximate indicator for $H$, so that the net of multiplication maps $(M_{m_\alpha})_\alpha$ in $CB_{A(G)}(VN(G))$ is then bounded and thus has a weak$^*$ operator topology cluster point $\Psi \in CB(VN(G))$. Passing to a subnet if necessary, we may assume that $\Psi$ is the limit of this net. For $u, v \in A(G)$ and $T \in VN(G)$,

$$\langle \Psi(v \cdot T), u \rangle = \lim_\alpha \langle T, m_\alpha uv \rangle = \langle \Psi(T), uv \rangle = \langle v \cdot \Psi(T), u \rangle,$$

showing that $\Psi$ is invariant. Given $T \in VN_H(G)$, so that $T = r_H^*(S)$ for some $S \in VN(H)$, we have for $u \in A(G)$ that

$$\langle \Psi(T), u \rangle = \lim_\alpha \langle S, r_H(uu_\alpha) \rangle = \lim_\alpha \langle S, r_H(u)r_H(u_\alpha) \rangle = \langle S, r_H(u) \rangle = \langle T, u \rangle,$$

whence $\Psi$ is the identity on $VN_H(G)$. For $T \in VN(G)$ and $u \in I_{A(G)}(H)$ we have $\langle \Psi(T), u \rangle = \lim_\alpha \langle S, uu_\alpha \rangle = 0$, so that $\Psi$ maps into $I_{A(G)}(H) = VN_H(G)$ and is thus a projection onto $VN_H(G)$.

If the functions $m_\alpha$ are in $P(G)$, then the maps $M_{m_\alpha}$ are completely positive [33, Proposition 4.2] and by [33, Theorem 7.4] their weak$^*$ operator topology cluster point $\Psi$ is also completely positive. \qed

From the preceding we obtain an analogous result for $A_{cb}(G)$, at least when $G$ is a weakly amenable locally compact group.

**Proposition 5.2.** Let $G$ be a weakly amenable locally compact group and $H$ a closed subgroup. If there is a bounded approximate indicator for $H$, then there is a completely bounded invariant projection $A_{cb}(G)^* \to I_{A_{cb}(G)}(H)^\perp$.

**Proof.** Let $(m_\alpha)_\alpha$ an approximate indicator for $H$. Since $A(G)$ is an ideal in $M_{cb}A(G)$, so too is its closure $A_{cb}(G)$, so that multiplication by $m_\alpha$ is a completely bounded $A_{cb}(G)$-module map on $A_{cb}(G)$. Denote its adjoint by $M_{m_\alpha}$. Passing to a subnet, we may assume that $(M_{m_\alpha})_\alpha$ has a weak$^*$ operator topology limit $\Psi \in CB(A_{cb}(G)^*)$, and passing to a further subnet we may assume that the net of maps $(M_{m_\alpha})_\alpha$ in $CB_{A(G)}(VN(G))$ also has a weak$^*$
operator topology limit $\Psi_A$, which is an invariant projection $VN(G) \to VN_H(G)$ by the argument of Proposition 5.1. For $u, v \in A_{cb}(G)$ and $T \in A_{cb}(G)^*$, we have

$$\langle \Psi(v \cdot T), u \rangle = \lim_{\alpha} \langle T, m_{\alpha}uv \rangle = \langle \Psi(T), uv \rangle = \langle v \cdot \Psi(T), u \rangle,$$

so $\Psi$ is invariant. Let $\iota: A(G) \to A_{cb}(G)$ be the inclusion. If $T \in A_{cb}(G)^*$ and $u \in A(G)$, then

$$\langle \Psi_{A_{cb}}(T), u \rangle = \lim_{\alpha} \langle \iota^*(T), um_{\alpha} \rangle = \lim_{\alpha} \langle T, \iota(u) m_{\alpha} \rangle = \langle \Psi(T), \iota(u) \rangle = \langle \iota^* \Psi(T), u \rangle$$

and $\Psi_{A_{cb}} = \iota^* \Psi$ by density of $A(G)$ in $A_{cb}(G)$. It follows that

$$\iota^* \Psi^2 = \Psi_{A_{cb}} \Psi = \Psi^2 = \Psi.$$

which, together with injectivity of $\iota^*$, implies that $\Psi^2 = \Psi$. If $T \in I_{A_{cb}(G)}(H)^\perp$, then $\iota^*(T) \in I_{A(G)}(H)^\perp$ and

$$\langle \Psi(T), \iota(u) \rangle = \langle \Psi_{A_{cb}}(T), u \rangle = \langle \iota^*(T), u \rangle = \langle T, \iota(u) \rangle \quad (u \in A(G)),$$

whence $\Psi(T) = T$, again by density of $A(G)$. Therefore $I_{A_{cb}(G)}(H)^\perp$ is contained in the range of $\Psi$. Finally, for any $T \in A_{cb}(G)^*$, if $u \in I_{A(G)}(H)$, then $\langle \Psi(T), \iota(u) \rangle = \langle \Psi_{A_{cb}}(T), u \rangle = 0$, so $\Psi(T) \in I_{A(G)}(H)^\perp = \left(\overline{I_{A(G)}(H)_{A_{cb}(G)}}\right)^\perp$. That $A_{cb}(G)$ has bounded approximate identity implies every set of synthesis for $A(G)$ is one for $A_{cb}(G)$ [18, Proposition 3.1] and, because compactly supported functions in $A_{cb}(G)$ are in $A(G)$, that $H$ is of spectral synthesis for $A_{cb}(G)$ is exactly the assertion that $\overline{I_{A(G)}(H)_{A_{cb}(G)}} = I_{A_{cb}(G)}(H)$. Therefore $\Psi$ has range in $I_{A_{cb}(G)}(H)^\perp$.

Note that the arguments of the preceding two propositions yield projections of completely bounded norm at most the bound on an approximate indicator for the subgroup. Let $\Lambda_G$ denotes the Cowling–Haagerup constant, that is, the infimum of bounds on approximate identities for $A_{cb}(G)$.

**Corollary 5.3.** Let $G$ be a weakly amenable locally compact group and $H$ a closed subgroup for which an approximate indicator of bound $C$ exists. The following hold:

1. $I_{A_{cb}(G)}(H)$ has an approximate identity of bound $(1 + C) \Lambda_G$.
2. An approximate indicator for $H$ of bound $1 + (1 + C) \Lambda_G$ exists that takes the value one on $H$.
3. An approximate indicator for $H$ of bound $C \Lambda_G$ exists in $A_{cb}(G)$.

**Proof.** (1) The argument of Proposition 5.2 yields an invariant projection $A_{cb}(G)^* \to I_{A_{cb}(G)}(H)^\perp$ of norm at most $C$ and, because the Banach algebra $A_{cb}(G)$ has an approximate identity of bound $\Lambda_G$, it follows from [15]...
Proposition 6.4] and its proof that the ideal $I_{A_b(G)}(H)$ has an approximate identity of bound $(1 + C) \Lambda_G$.

(2) If $(e_\alpha)_\alpha$ is an approximate identity for $I_{A_b(G)}(H)$ of bound $(1 + C) \Lambda_G$, then $(1_G - e_\alpha)_\alpha$ is an approximate indicator for $H$ with the claimed norm bound.

(3) Let $(e_\alpha)_{\alpha \in A}$ be a bounded approximate identity for $A_b(G)$, let $(m_\beta)_{\beta \in B}$ a bounded approximate indicator for $H$, and for $\gamma = (\beta, (\alpha_\beta'_{\beta \in B})) \in B \times A^B$ set $u_\gamma = m_\beta e_\alpha_\beta$, which is in the ideal $A_b(G)$ of $M_{cb}A(G)$.  Giving $B \times A^B$ the product order, for $u \in A(H)$ and $w \in I_A(G)(H)$ we have the norm limits

$$\lim_{\gamma \in B \times A^B} r_H(u_\gamma) u = \lim_{\beta} \lim_{\alpha} r_H(m_\beta e_\alpha) u = \lim_{\beta} r_H(m_\beta) u = u$$

and

$$\lim_{\gamma \in B \times A^B} u_\gamma w = \lim_{\beta} \lim_{\alpha} m_\beta e_\alpha w = \lim_{\beta} m_\beta w = 0$$

by [26 p. 69], hence $(u_\gamma)_{\gamma \in B \times A^B}$ is a bounded approximate indicator for $H$ of norm bound $\sup_\alpha \|e_\alpha\|_{M_{cb}A(G)} \sup_\beta \|m_\beta\|_{M_{cb}A(G)}$. 

6. CONVERGENCE OF CB-MULTIPLIERS AND AVERAGING OVER CLOSED SUBGROUPS

Fix a locally compact group $G$ and a closed subgroup $H$. It is folklore that the convergence properties of nets of cb-multipliers can be improved by convolving them with probability measures in $C_c(G)$. For example, Knudby recently recorded the following, the second part of which originates in an argument of Cowling and Haagerup [7, Proposition 1.1]. If a net $(m_\alpha)_\alpha$ of functions on a topological space converges uniformly on compact sets to a function $m$, we write $m_\alpha \ucas m$.

**Theorem 6.1.** ([27 Lemma B.2]) Let $(m_\alpha)_\alpha$ be a bounded net in $M_{cb}A(G)$, $m \in M_{cb}A(G)$, and let $f \in C_c(G)$ be such that $f \geq 0$ and $\int_G f = 1$. Convolution on the left with $f$ is a contraction on $M_{cb}A(G)$ and the following hold:

1. If $m_\alpha \ucas m$ in $M_{cb}A(G)$, then $f \ast m_\alpha \ucas f \ast m$.
2. If $m_\alpha \ucas m$, then $\|(f \ast m_\alpha) u - (f \ast m) u\|_{A(G)} \rightarrow 0$ for all $u \in A(G)$.

In this section, we develop an analogue of the convolution technique relative to a closed subgroup. Fix a function $f \in C_c(H)$ such that $f \geq 0$ and $\int_H f = 1$. For any function $f$ on $G$ and $s, t \in G$, let $s f(t) = f(st)$.

**Definition 6.2.** For $u \in C_b(G)$, define a function $\Omega_f(u)$ on $G$ by the formula

$$\Omega_f(u)(s) = \int_H f(h) u(h^{-1}s) dh \quad (s \in G).$$

We will show that $\Omega_f$ defines a bounded map on $M_{cb}A(G)$. For a Hilbert space $H$, let $C_c(G, H)$ and $C_b(G, H)$ denote the continuous functions $G \rightarrow H$ that are compactly supported and bounded, respectively.
Lemma 6.3. Let $\mathcal{H}$ be a Hilbert space. If $u \in C_c (G, \mathcal{H})$ then for any $\epsilon > 0$ there is an open neighborhood $U$ of the identity $e$ such that
\[ \sup_{t \in G} \|u (st) - u (t)\| < \epsilon \quad \text{for all } s \in U. \]

Proof. The standard proof in the case that $\mathcal{H} = \mathbb{C}$, for example [14, Proposition 2.6], works for any Hilbert space.

Lemma 6.4. Let $\mathcal{H}$ be a Hilbert space. If $u \in C_b (G, \mathcal{H})$, $s_0 \in G$, and $\epsilon > 0$, then there is an open neighborhood $U$ of $s_0$ in $G$ such that
\[ \sup_{h \in H} \|f (h) u (sh) - f (h) u (s_0 h)\| < \epsilon \quad \text{for all } s \in U. \]

Proof. If $u = 0$, then the claim trivially holds, so assume $u \neq 0$. Since $H$ is closed in $G$, the function $f$ extends to a continuous compactly supported function $f'$ on $G$. Assume that $s_0 = e$. Since $f'$ is compactly supported, Lemma 6.3 yields an open neighborhood $U$ of $e$ such that
\[ \sup_{t \in G} \|f' u (st) - f' u (t)\| < \frac{\epsilon}{2} \quad \text{and} \quad \sup_{t \in G} \|f' (st) - f' (t)\| < \frac{\epsilon}{2 \|u\|_\infty} \]
for all $s \in U$. Then
\[ \sup_{h \in H} \|f (h) u (sh) - f (h) u (h)\| \leq \sup_{t \in G} \|f' (t) u (st) - f' (t) u (t)\| \leq \sup_{t \in G} (\|f' (t) u (st) - f' u (st)\| + \|f' u (st) - f' (t) u (t)\|) \leq \|u\|_\infty \sup_{t \in G} \|f' (st) - f' (t)\| + \frac{\epsilon}{2} < \epsilon, \]
for all $s \in U$. For $s_0 \neq e$, the above argument with $u$ replaced by $s_0 u$ yields a neighborhood $U$ of $e$ and $s_0 U$ is then the desired neighborhood of $s_0$.

Proposition 6.5. If $u \in M_{cb} A (G)$, then $\Omega_f (u) \in M_{cb} A (G)$ with $\|\Omega_f (u)\|_{M_{cb} A (G)} \leq \|u\|_{M_{cb} A (G)}$ and $r_H \Omega_f (u) = f * r_H (u)$. 

Proof. Let $u \in M_{cb} A (G)$ and apply Gilbert’s theorem to obtain a Hilbert space $\mathcal{H}$ and functions $P, Q \in C_b (G, \mathcal{H})$ such that $u (s^{-1} t) = \langle P (t) \| Q (s) \rangle$ for all $s, t \in G$. Then, for $s, t \in G$,
\[ \Omega_f (u) (s^{-1} t) = \int_H f (h) u (h^{-1} s^{-1} t) \, dh = \left\langle P (t) \left| \int_H f (h) Q (sh) \, dh \right. \right\rangle. \]

We show that $q (s) = \int_H f (h) Q (sh) \, dh$ defines a bounded continuous function on $G$, from which it will follow that $\Omega_f (u)$ is in $M_{cb} A (G)$, again by Gilbert’s theorem. Define $Q' : G \to L^1 (H, \mathcal{H})$ by $Q' (s) = f (s) Q$, which maps into $L^1 (H, \mathcal{H})$ since $f$ has compact support. Set $K = \text{supp } f$ and let $|K|$ denote the Haar measure of $K$, which is nonzero and finite by continuity of the nonzero, compactly supported function $f$. Given $s_0 \in G$ and $\epsilon > 0$, Lemma 6.4 yields an open neighborhood $U$ of $s_0$ in $G$ such that
\[ \|Q' (s) - Q' (s_0)\|_{L^\infty (H, \mathcal{H})} = \sup_{h \in H} \|f (h) Q (sh) - f (h) Q (s_0 h)\| < \frac{\epsilon}{|K|}. \]
for all $s \in U$. Since $Q'(s)$ is supported in $K$ for every $s \in G$, it follows that
\[
\|Q'(s) - Q'(s_0)\|_{L^1(H,\mathcal{H})} = \|\chi_K (Q'(s) - Q'(s_0))\|_{L^1(H,\mathcal{H})} \\
\leq \|\chi_K\|_{L^1(H,\mathcal{H})} \|Q'(s) - Q'(s_0)\|_{L^\infty(H,\mathcal{H})} \\
< \epsilon
\]
for all $s \in U$. Thus $Q'$ is continuous and so too is $q$, the latter being the composition of $Q'$ with the bounded map $L^1(H,\mathcal{H}) \to \mathcal{H} : g \mapsto \int_H g$.

Using that $f$ is nonnegative with mass one, if $s \in G$, then $\|q(s)\| \leq \int_H f(h) \|Q(sh)\|dh \leq \|Q\|_\infty$, so $q$ is bounded with $\|q\|_\infty \leq \|Q\|_\infty$. By the norm characterization of Gilbert’s theorem, $\|\Omega_f(u)\|_{M_{cb}A(G)} \leq \|P\|_\infty \|q\|_\infty \leq \|P\|_\infty \|Q\|_\infty$ and since $P, Q,$ and $\mathcal{H}$ are an arbitrary representation of $u$, we conclude that $\|\Omega_f(u)\|_{M_{cb}A(G)} \leq \|u\|_{M_{cb}A(G)}$.

Finally, we have for $s \in H$ that
\[
r_H \Omega_f(u)(s) = \int_H f(h) u(h^{-1}s) dh \\
= \int_H f(h) r_H(u)(h^{-1}s) dh \\
= f \ast r_h(u)(s).
\]

**Theorem 6.6.** Let $(m_\alpha)_\alpha$ be a bounded net in $M_{cb}A(G)$ and let $m \in M_{cb}A(H)$. The following hold:

1. If $r_H(m_\alpha) \xrightarrow{w^*} m$ in $M_{cb}A(H)$, then $r_H \Omega_f(m_\alpha) \xrightarrow{w^*} f \ast m$.
2. If $r_H(m_\alpha) \xrightarrow{w^*} m$, then $\|r_H \Omega_f(m_\alpha) u - (f \ast m) u\|_{A(H)} \to 0$ for all $u \in A(H)$.

**Proof.** These follow immediately from Theorem 6.1 and Proposition 6.5. 

In our applications, the preceding theorem will be applied with $m = 1_H$, which is fixed under convolution with $f$ on the left. We list some additional properties that the map $\Omega_f$ enjoys.

1. An argument very similar to that establishing Proposition 6.5 shows that $\Omega_f(u)$ is bounded and continuous for any bounded continuous function $u$ on $G$.
2. If $u = \langle \pi(\cdot) \xi | \eta \rangle$ is a coefficient of the unitary representation $\pi$ of $G$, then

\[
\Omega_f(u)(s) = \int_H f(h) \langle \pi(s) \xi | \pi(h) \eta \rangle dh \\
= \left\langle \pi(s) \xi \left| \left( \int_H f(h) \pi(h) dh \right) \eta \right. \right\rangle,
\]
so $\Omega_f(u) \in B(G)$, and from $\|\int_H f(h) \pi(h) dh\| \leq \int_H f(h) \|\pi(h)\| dh = 1$ it follows that $\|\Omega_f(u)\|_{B(G)} \leq \|u\|_{B(G)}$. Thus $\Omega_f$ restricts to a contraction on $B(G)$ and moreover restricts to a contraction on $A(G)$, since $\Omega_f(u)$ is a coefficient of the same representation as $u$. 
(3) An argument similar to that establishing the weak* continuity of the map \( \Phi_f \) in the proof of [21] Lemma 1.16 shows that \( \Omega_f \) is weak* continuous on \( M_{cb}(G) \) with preadjoint mapping \( g \in L^1(G) \) to the \( L^1(G) \) function \( s \mapsto \int_G f(h)g(hs) \, dh \).

Say that a net \( (m_\alpha)_\alpha \) of functions on a topological space \( X \) **converges locally eventually to zero on \( A \subset X \)** and write \( m_\alpha \overset{w}{\to} 0 \) if for any compact subset \( K \) of \( A \) there is an index \( \alpha_0 \) such that \( m_\alpha|_K = 0 \) for all \( \alpha \geq \alpha_0 \).

**Proposition 6.7.** If \( (m_\alpha)_\alpha \) is a bounded net in \( M_{cb}(G) \) and \( m'_\alpha = \Omega_f(m_\alpha) \), then the net \( (m'_\alpha)_\alpha \) has the same norm bound as \( (m_\alpha)_\alpha \) and

1. if \( r_H(m_\alpha) \overset{w}{\to} 1_H \), then \( \|u \cdot r_H(m'_\alpha) - u\|_{A(H)} \to 0 \) for all \( u \in A(H) \), and
2. if \( m_\alpha \overset{w}{\to} 0 \) on \( G \setminus H \), then \( m'_\alpha \overset{w}{\to} 0 \) on \( G \setminus H \).

If the boundedness of \( \Omega_f \) is satisfied by both (1) and (2), then \( (m'_\alpha)_\alpha \) is a bounded approximate indicator for \( H \).

**Proof.** The claim regarding norm bounds holds since the map \( \Omega_f \) of Section 6 is a contraction on \( M_{cb}(G) \).

(1) If \( r_H(m_\alpha) \overset{w}{\to} 1_H \), then, since restriction is a contraction from \( M_{cb}(G) \) into \( M_{cb}(H) \), the net \( (r_H(m_\alpha))_\alpha \) is bounded and (1) follows from Theorem 6.6.

(2) Suppose that \( m_\alpha \overset{w}{\to} 0 \) on \( G \setminus H \). Let \( K \subset G \setminus H \) be compact and choose \( \alpha_0 \) such that \( \alpha \geq \alpha_0 \) implies \( m_\alpha = 0 \) on the compact set \((\text{supp}(f))^{-1}K\). For \( \alpha \geq \alpha_0 \), if \( s \in K \) and \( h \in H \), then \( f(h)m_\alpha(h^{-1}s) = 0 \) since either \( h \notin \text{supp}(f) \) or \( h^{-1}s \in ((\text{supp}(f))^{-1}K) \), implying that \( m'_\alpha(s) = \int_H f(h)m_\alpha(h^{-1}s) \, dh = 0 \). Therefore \( m'_\alpha = 0 \) on \( K \), for all \( \alpha \geq \alpha_0 \).

If \( (m_\alpha)_\alpha \) satisfies the hypotheses of both (1) and (2), then \( (m'_\alpha)_\alpha \) satisfies the first condition of Definition 2.2. If \( u \in I_{A(G)}(H) \) has compact support, then \( m'_\alpha u = 0 \) eventually by (2), so certainly \( \|um'_\alpha\|_{A(G)} \to 0 \). Using that \( H \) is of spectral synthesis in \( A(G) \), if \( u \in I_{A(G)}(H) \) is arbitrary, then given \( \epsilon > 0 \) choose \( u_0 \in I_{A(G)}(H) \) of compact support with \( \|u - u_0\|_{A(G)} < \epsilon \). For sufficiently large \( \alpha \),

\[
\|um'_\alpha\|_{A(G)} \leq \|u_0m'_\alpha\|_{A(G)} + \|u_0m'_\alpha - um'_\alpha\|_{A(G)} < \epsilon \|m'_\alpha\|_{M_{cb}(A(G))},
\]

and thus \( \|um'_\alpha\|_{A(G)} \to 0 \) by boundedness of \( (m'_\alpha)_\alpha \). \( \square \)

Proposition 6.7 allows one to obtain approximate indicators consisting of cb-multipliers by verifying the same conditions that yielded approximate indicators in [1].

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