SIMILARITY SOLUTIONS AND CONSERVATION LAWS FOR THE BOGOYAVLENSKY-KONOPELCHENKO EQUATION BY LIE POINT SYMMETRIES

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Abstract. The 1 + 2 dimensional Bogoyavlensky-Konopelchenko equation is investigated for its solution and conservation laws using the Lie point symmetry analysis. In the recent past, certain work has been done describing the Lie point symmetries for the equation and this work seems to be incomplete (S.S. Ray, Computers & Mathematics with Applications 74(6) (2017), 1158–1165). We obtained certain new symmetries and corresponding conservation laws. The travelling-wave solution and some other similarity solutions are studied.

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1. Introduction. The 1 + 2-dimensional Bogoyavlensky-Konopelchenko equation is written as

\[ q_t + \alpha q_{xxx} + \beta q_{xyy} + 6\alpha q_{x} + 4\beta q_{y} + 4\beta q_{x}\partial^{-1}_x q_y = 0, \quad (1) \]
where $\alpha$ and $\beta$ are constants and for $\partial_x^{-1}q = u$, (1) is simplified as follows

$$u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx}u_x + 4\beta u_{xy}u_x + 4\beta u_{xx}u_y = 0,$$

(2)

which is a two-dimensional extension of the Korteweg-de Vries (KdV) equation.

In [16, 21] the authors have mentioned various applications of the equation. Briefly, it is mentioned in [14, 16] that the equation signifies the interaction between Riemann wave and long wave. Furthermore certain particular cases were studied in [5, 6, 14, 19].

Various methods such as the inverse scattering method and Darboux transformation are used to study the latter equation, for instance see [5, 6] and references therein. Applications of the equations and the study of the equation for various values of the arbitrary constants are mentioned in [5, 14]. In [10] new position, negaton and complexiton solutions are described for the BK equation using the Darboux transformation. In [17, 18] the author considered the $1 + 2$-dimensional Bogoyavlensky-Konopelchenko equation as a system and obtained the Lie point symmetries and used the condition of quasi self-adjointness to determine the conservation laws. In [1] some of the inaccuracies of [17] are mentioned and conservation laws using the more established Noether’s theorem is presented.

In this work we study the Lie point symmetries of the Bogoyavlensky-Konopelchenko equation and we apply them in order to study certain possible reductions and determine closed-form and analytic solutions. The theory of Lie symmetries of differential equations is the standard technique for computation of solutions and describing the algebra for nonlinear differential equations.

Our study provides new symmetries for the Bogoyavlensky-Konopelchenko equation which have not been reported before in the literature. We apply these symmetries to determine new similarity solutions and also to determine conservation laws [11].

Computation of conservation laws has been a core area of research for decades. The standard technique is the Noether’s theorem for a system with a Lagrangian. Later various researchers developed methods [2, 3, 4, 11, 12] to compute the conservation laws for systems without a Lagrangian by generalising Noether’s theorem. Among them the most widely used and applicable method is the nonlinear self-adjointness method developed by Ibragimov [11, 12].

In this paper, for the determination of the conservation laws for equation (2), we make use of the Ibragimov’s method of nonlinearily self-adjointness. Our analysis extends the previous results of [17] and shows that the conservation laws for the $1 + 2$-dimensional Bogoyavlensky-Konopelchenko equation can be derived directly by the Lie point symmetries. The plan of the paper it follows.

In Section 2 the Lie point symmetries for the $1 + 2$-dimensional Bogoyavlensky-Konopelchenko equation are derived. In particular we find that the equation admits five plus infinity Lie point symmetries. The application of the Lie point symmetries is performed in Section 3 in which similarity solutions are derived for equation (2). More specifically we derive a family of travelling-wave solutions and some scaling solutions. Ibragimov’s method of nonlinear self-adjointness is discussed in Section 4, in which also the generic formula and conditions for the determination of conservation laws are given. The conservation laws (flows) are presented in Section
5 which generalise the results of [17] and includes the results of [1]. Finally in Section 6, we discuss our results and draw our conclusions.

2. Lie point symmetry analysis. For the convenience of the reader we briefly discuss the theory of Lie symmetries of differential equations and the application of the differential invariants for the construction of similarity solutions.

Let $\Phi$ describe the map of an one-parameter point transformation such as $\Phi (u (x^i)) = u (x^i)$ with infinitesimal transformation

\begin{align}
  x^{i'} &= x^i + \varepsilon \xi^i (x^i, u) \\
  u' &= u + \varepsilon \eta (x^i, u)
\end{align}

and generator

$$
\Gamma = \frac{\partial t'}{\partial \varepsilon} \partial_t + \frac{\partial x'}{\partial \varepsilon} \partial_x + \frac{\partial u^{A'}}{\partial \varepsilon} \partial_{u^A},
$$

where $\varepsilon$ is the parameter of smallness and $x^i = (t, x, y)$. Assume that the function, $u (x^i)$, is a solution of the partial differential equation $\mathcal{H} (u^A, u^A_t, u^A_x, ... ) = 0$. Then under the map, $\Phi$, the function $u' (x^{i'}) = \Phi (u (x^i))$ remains a solution for the differential equation if and only if the differential equation is also invariant under the action of the map $\Phi$, i.e., $\Phi (\mathcal{H} (u^A, u^A_t, u^A_x, ... )) = 0$.

When the latter is true, $\Gamma$ is called a Lie point symmetry for the differential equation. Mathematically that is formulated with the following condition

$$X^{[n]} (\mathcal{H}) = 0,$$

in which $\Gamma^{[n]}$ describes the $n$–th prolongation/extension of the symmetry vector in the jet-space of variables $\{x^i, u, u_t, u_{tx}, ... \}$.

The importance of the existence of a Lie symmetry for a partial differential equation is that from the associated Lagrange’s system,

$$\frac{dx^i}{\xi^i} = \frac{du}{\eta},$$

zeroth-order invariants, $U^{[0]} (x^i, u)$ can be determined which can be used to reduce the number of the independent variables of the differential equation and lead to the construction of similarity solutions.

As far as the Bogoyavlensky-Konopelchenko equation is concerned (2) the ap-
Application of Lie's theory provides the generic symmetry vector field to be

\[
\Gamma = \left( A_0 + (A_1 - A_2)t \right) \partial_t + \left( A_3 + (A_1 + A_2)y + 4t \left( A_4 - \frac{(A_1 + 2A_2)}{4t\beta} \right) \right) \partial_y + \left( -A_2x + \frac{y(3A_2\alpha + (3A_1 - 2(A_1 + 2A_2) + 3A_2)\alpha)}{\beta} + b(t) \right) \partial_x + \left( A_2u + x \left( A_4 - \frac{(A_1 + 2A_2)}{4t\beta} \right) \right) + a(t) + \frac{5}{2}A_1y^2\alpha + 5A_2y^2\alpha - 12A_4ty\alpha\beta + 2ty\beta b'(t) \right) \partial_u,
\]

where \( A_0-4 \) are arbitrary constants and \( a(t), b(t) \) are arbitrary functions.

Hence the Lie symmetries corresponding to each of the arbitrary constants and functions are

\[
\begin{align*}
\Gamma_{1a} &= \partial_t \\
\Gamma_{2a} &= t\partial_t + \frac{y\alpha}{\beta} \partial_x + \left( \frac{5y^2\alpha}{16t\beta^2} - \frac{xy}{4t\beta} \right) \partial_u \\
\Gamma_{3a} &= -t\partial_t - \left( x - \frac{2y\alpha}{\beta} \right) \partial_x - y \partial_y + \left( u + \frac{5y^2\alpha}{8t\beta^2} - \frac{xy}{2t\beta} \right) \partial_u \\
\Gamma_{4a} &= 4t\beta \partial_y + \left( x - \frac{3y\alpha}{2\beta} \right) \partial_u \\
\Gamma_{5a} &= \partial_y \\
\Gamma_{6a} &= b(t) \partial_x + \left( a(t) + \frac{yb'(t)}{\beta} \right) \partial_u.
\end{align*}
\]

Below we continue with the application of the symmetry vectors and more specifically with the determination of similarity solutions as also with the construction of conservation laws.

3. Similarity solutions. We compute the travelling-wave solution for the Bogoyavlensky-Konopelchenko equation (2) with respect to the generators \( \Gamma_{6a} + \Gamma_{5a} - c\Gamma_{1a} \), for \( a(t) \) and \( b(t) \) being constant with the similarity variables defined as

\[
\begin{align*}
x + y - ct &= s \\
u(t, x, y) &= w(s).
\end{align*}
\]

The fourth-order equation (2) is reduced to the ordinary differential equation,

\[
(\alpha + \beta)w''''(s) + 8\beta w'(s)w''(s) + 6\alpha w'(s)w''(s) - cw''(s) = 0.
\]
The Lie point symmetries of this equation are

\[
\Gamma_{1b} = \partial_s, \\
\Gamma_{2b} = \partial_w, \\
\Gamma_{3b} = \partial_s + \left( \frac{cs}{t} - w \right) \partial_w.
\]

The application of \( \Gamma_{1b} \), reduces the fourth-order equation to one of third order

\[
v'''(r) = \frac{10v'(r)v''(r)}{v(r)} - \frac{15v^3}{v(r)^2} + \left( \frac{cv(r)^2}{\alpha + \beta} - \frac{14v(r)}{\alpha + \beta} \right) v'(r),
\]

where the variables are defined as

\[
r = w(s) \\
v(r) = \frac{1}{w'(s)}.
\]

The third-order equation, (11), has a lone point symmetry which is \( \partial_r \), because it is autonomous. The new similarity variables are

\[
h = v(r) \\
g(h) = \frac{1}{v(r)},
\]

and enables the reduction of (11) to the second-order equation

\[
g''(h) = \frac{3g^2}{g(h)} + \frac{10g'(h)}{h} + \left( \frac{14h - h^2 c}{\alpha + \beta} \right) g(h)^3 + \frac{15g(h)}{h^2}
\]

which is maximally symmetric and is easily integrated.

Now we consider reduction with respect to \( \Gamma_{2b} \) with similarity variables

\[
n = s \\
m(n) = w'(s).
\]

The reduced third-order equation is

\[
m'''(n) = m'(n) \left( \frac{-14m(n)}{\alpha + \beta} + \frac{c}{\alpha + \beta} \right).
\]

This equation has two symmetries which are

\[
\Gamma_{1c} = \partial_n \\
\Gamma_{2c} = n \partial_n + \left( \frac{c}{t} - 2m \right) \partial_m.
\]

The \( \Gamma_{1c} \), with the variables \( j = m(n) \) and \( p(j) = \frac{1}{m'(n)} \) leads to the second-order equation which is also maximally symmetric,

\[
p''(j) = \frac{3p^2}{p(j)} - \frac{(c - 14q)p(j)^3}{\alpha + \beta}.
\]
Interestingly $\Gamma_2c$ with the variables $l = \frac{(14m(n) - c)n^2}{14}$ and

$$k(l) = \frac{7}{n^27nm'(n) - c + 14m(n)}$$

leads to the second-order equation with no point symmetries, namely

$$k''(l) = \frac{3k'^2}{k(l)} + 9k(l)k'(l) + \frac{(26\alpha + 26\beta + 14l)k(l)^3}{\alpha + \beta} - \frac{(24\alpha l + 24\beta l + 28l^2)k(l)^4}{\alpha + \beta}.$$  \hspace{1cm} (15)

The reduction with respect to $\Gamma_{3b}$, leads to a third-order equation with zero point symmetries.

The solutions provided by the set of equations (12) and (14) describe the travelling-wave solutions for the mother equation (2).

3.1. **Further reductions based on other symmetries.** We study the equation (2) with respect to $\Gamma_{2a} - \Gamma_{2a}^{3/2}$, which can be easily verified to be a symmetry. The similarity variables are

$$d = \frac{x}{y},$$

$$e = \frac{y^3}{t} \text{ and}$$

$$p(d, e) = yu(t, x, y).$$

The reduced PDE is

$$d\beta p_{dddd} - 3e\beta p_{dde} - \alpha p_{dddd} + 4\beta p_{ddd} + cA^2p_{de} + 4\beta pp_{dd} - 12\alpha p_erp_{dd}$$

$$-2p_d(6e\beta p_{de} + (3\alpha - 4d\beta)p_{dd}) + 8\beta p_d^2 = 0.$$  \hspace{1cm} (16)

The Lie point symmetries are

$$\Gamma_{1d} = e^{\frac{1}{2}}p_{p} \text{ and}$$

$$\Gamma_{2d} = \frac{\partial_d}{e^{\frac{1}{2}}} + \frac{e^{\frac{3}{2}}}{12\beta}p_{p}.$$

Reduction with respect to $\Gamma_{2d}$ leads to the solvable first- order ode

$$r\beta^3 v(r)v'(r) - r\beta v'(r) - \beta v(r) + \frac{\beta^3 v(r)^2}{3} = 0,$$  \hspace{1cm} (16)

in which the variables are

$$v(r) = \frac{12\beta p(e, d)}{ed},$$

$$r = e.$$  

Further reduction based on $\Gamma_{4a}$, leads to the PDE

$$\alpha p_{dd} + 6\alpha p_d p_{dd} + \frac{p_d}{e} + p_{de} + \frac{dp_{dd}}{e} = 0,$$  \hspace{1cm} (17)
Similarity solutions and conservation laws

where the similarity variables are \( d = x, \ t = e \) and

\[
    u(t, x, y) = p(x, t) + \frac{(xy - \frac{3αy^2}{4b\beta})}{4tβ}.
\]

The Lie point symmetries for the latter equation (17) are determined to be

\[
    \begin{align*}
        \Gamma_{1e} &= \frac{d}{3} \frac{\partial}{\partial d} + e \frac{\partial}{\partial e} - \frac{p}{3} \frac{\partial}{\partial p} \\
        \Gamma_{2e} &= \frac{d^2}{12e^2} \frac{\partial}{\partial p} \\
        \Gamma_{3e} &= \frac{4e^3}{3} \frac{\partial}{\partial e} - 2\sqrt{ep} \frac{\partial}{\partial p} \\
        \Gamma_{4e} &= \frac{d}{e} \frac{\partial}{\partial p} - 6α\frac{\partial}{\partial d} \quad \text{and} \\
        \Gamma_{5e} &= \frac{e}{d} \frac{\partial}{\partial d}.
    \end{align*}
\]

The reduction with respect to \( \Gamma_{1e} \) leads to the fourth-order ode

\[
    9α[3rv''''(r) + 8v''''(r) - 54r^2αv''(r)] + r[v'(r)(r - 24α + 162r^2αv''(r))] + 3r[2(r + 12α)v''(r) - 12αe(r)] + v(r)[r + 24α + 36αrv'(r)] = 0,
\]

where \( r = \frac{d^3}{e} \) and \( p(d, e) = \frac{v(r)}{d} \). The sole Lie point symmetry is \( r^{\frac{1}{2}} \frac{\partial}{\partial r} \). Accordingly the third-order ode with zero Lie point symmetry is

\[
    m'''(n) = -\frac{4m''(n)}{n} - \left( 2m(n) + \frac{6n^\frac{4}{3} + 180n^\frac{5}{3}α}{n^{\frac{2}{3} + 81n^2α}} \right) m'(n) - \frac{4m(n)^2}{3n^\frac{2}{3}} - \frac{5m(n)}{81n^2α}. \tag{20}
\]

Similarly with respect to \( \Gamma_{2e} \), the reduced ODE is

\[
    αv''''(r) + 6αv'(r)v''(r) = 0, \tag{21}
\]

where the similarity variables are \( d = r \) and \( p(d, e) = v(r) - \frac{d^2}{12eα} \). The Lie point symmetries are

\[
    \begin{align*}
        \Gamma_{1f} &= \frac{\partial}{\partial r} \\
        \Gamma_{2f} &= \frac{\partial}{\partial v} \quad \text{and} \\
        \Gamma_{3f} &= r\frac{\partial}{\partial r} - v\frac{\partial}{\partial v}.
    \end{align*}
\]

The reduction with respect to \( \Gamma_{1f} \), with similarity variables \( n = v(r) \) and \( m(n) = \frac{1}{v'(r)} \), leads to a third-order equation with two symmetries. It is

\[
    m'''(n) = -6m'(n)m(n) + \frac{10m'(n)m''(n)}{m(n)} - \frac{15m^3}{m(n)^2}. \tag{23}
\]

and the two symmetries are \( \frac{\partial}{\partial n} \) and \( n\frac{\partial}{\partial m} - 2m\frac{\partial}{\partial m} \). With respect to \( \frac{\partial}{\partial n} \) the equation is reduced to a maximally symmetric second-order equation, which is

\[
    b'' + 3b'^2 \frac{b'(a)}{a} + \frac{10b'(a)}{a} + \frac{15b(a)}{a^2} = 0. \tag{24}
\]
With respect to the other symmetry, (23) is reduced to a second-order equation with no Lie point symmetries, namely,

\[ b''(a) = \frac{3b'^2}{b(a)} + \left( \frac{10}{a} - 11b(a) \right) b'(a) + \frac{15b(a)}{a^2} \frac{40b(a)^2}{a} + \frac{(6a^3 + 46a^2)b(a)^3}{a^2} - \frac{(12a^4 + 24a^3)b(a)^4}{a^2}. \]

Reduction with respect to \( \Gamma_{2f} \) leads to the equation

\[ m''''(n) = -6m'(n)m(n), \] (25)

where \( n = r \) and \( m(n) = v'(r) \). The symmetries are \( \partial_n \) and \( n\partial_n - 2m\partial_m \). The \( \partial_n \) symmetry reduces it to

\[ b + \frac{3b'^2}{b(a)} = 0 \] (26)

which is linearisable. As in the case of \( \Gamma_{1f} \) the \( n\partial_n - 2m\partial_m \) symmetry leads to an equation with no Lie point symmetries.

The last possible reduction with respect to \( \Gamma_{3f} \) leads to a third-order equation with no Lie point symmetries. The other possible reductions with respect to other symmetries do not have a concrete result and hence we omit those.

However, the interesting part of our analysis is the determination of the conservation laws for the Bogoyavlensky-Konopelchenko equation with the use of Ibragimov’s method of nonlinear self-adjointness.

4. **Nonlinear self-adjointness and conservation laws.** In this section we mention the preliminaries of Ibragimov’s method of nonlinear self-adjointness. Let

\[ E(x, u, u_i, u_{ij}, \ldots) = 0 \] (27)

be a scalar PDE, where \( x = (x^i, x^j, x^k, \ldots) \) are the independent variables and \( u_i, u_{ij} \) are defined as \( u_i = \frac{\partial u}{\partial x^i}, u_{ij} = \frac{\partial u}{\partial x^i \partial x^j} \). Similarly we can define \( u_{ijk} \) and other terms. The adjoint system to the scalar PDE can be defined as

\[ E^*(x, u, u_i, u_{ij}, \ldots) = \frac{\delta L}{\delta u}, \]

where \( L \) can be defined as

\[ L = v(x)E(x, u, u_i, u_{ij}, \ldots). \]

\( v(x) \) is the new dependent variable and \( x \) is defined as \( x = (x^i, x^j, x^k, \ldots) \). The variational derivative \( \frac{\delta}{\delta u} \) is

\[ \frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} D_{i_2} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1i_2\ldots i_s}}. \] (28)
Now to verify the nonlinear self-adjointness, the substitution of \( v = \phi(x, u) \), where \( \phi(x, u) \neq 0 \), to the adjoint equation of the scalar PDE must satisfy for all solutions, \( u \), of equation (27). In other words

\[
E^*(x, u, u_i, u_{ij}, \ldots) = \lambda E(x, u, u_i, u_{ij}, \ldots),
\]

where \( \lambda \) is the undetermined coefficient.

4.1. Conservation laws. Let the scalar PDE admit the following generators of the infinitesimal transformation

\[
V = \xi^i(x, u, u_i, \ldots) \frac{\partial}{\partial x^i} + \eta(x, u, u_i, \ldots) \frac{\partial}{\partial u}.
\]

Then the scalar PDE and its adjoint equation as defined above admits the conservation law

\[
C^i = \xi^i L + W \left[ \frac{\partial L}{\partial u_i} - D_j \left( \frac{\partial L}{\partial u_{ij}} \right) + D_j D_k \left( \frac{\partial L}{\partial u_{ijk}} \right) - \ldots \right]
\]

\[
+ D_j (W) \left[ \frac{\partial L}{\partial u_{ij}} - D_k \left( \frac{\partial L}{\partial u_{ij}} \right) + \ldots \right] + D_j D_k (W) \left[ \frac{\partial L}{\partial u_{ijk}} - \ldots \right],
\]

where \( W = \eta - \xi^i u_i \).

4.2. Condition for nonlinear self-adjointness. We employ Ibragimov’s method of nonlinear self-adjointness to construct the conservation laws for (2). We firstly prove that equation (2) is indeed nonlinearly self-adjoint by considering the Lagrangian as

\[
L = v(t, x, y)(u_{xt} + \alpha u_{xxx} + \beta u_{xxy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xy} u_y),
\]

where \( v(t, x, y) \) is the new dependent variable. To verify the nonlinear self-adjointness, the substitution of \( v = \phi(t, x, y, u) \) into the adjoint equation of (2) must satisfy for all solutions, \( u \), of equation (2). The adjoint equation for (2) is

\[
\frac{\delta L}{\delta u} = 6\alpha u_{x} v_{xx} + 4\beta u_{y} v_{xx} + 6\alpha u - xxv_{x} + 4\beta u_{yy} v_{x} + v_{xt} + 4\beta u_{x} v_{xy} + 4\beta v_{x} u_{xy} + \alpha v_{xxx} + \beta v_{xxy}.
\]

The corresponding derivative terms are

\[
\begin{align*}
v_x &= \phi_x + \phi_u u_x, \\
v_y &= \phi_y + \phi_u u_y, \\
v_{xx} &= \phi_{xx} + u_x (\phi_{uu} u_x) + \phi_u u_{xx}, \\
v_{xy} &= \phi_{xy} + u_x (\phi_{uu} u_y) + \phi_u u_{xy}, \\
v_{xt} &= \phi_{xt} + u_x (\phi_{uu} u_t) + \phi_u u_{xt}, \\
v_{xxx} &= \phi_{xxx} + u_x^2 (\phi_{uuu} u_x) + 2\phi_{uuu} u_x u_{xx} + u_{xx} \phi_u u_x + \phi_u u_{xxx}, \\
v_{xxxx} &= \phi_{xxxx} + u_x^3 (\phi_{uuuu} u_x) + 3\phi_{uuuu} u_x^2 u_{xx} + \ldots \\
v_{xxy} &= \phi_{xxy} + \ldots.
\end{align*}
\]
We substitute the values of $v$ into the adjoint equation and look for the possible values for $\phi(t, x, y, u)$.

\[
\begin{align*}
6\alpha u_x v_{xx} + 4\beta u_y v_{xx} + 6\alpha u - xxv_x + 4\beta u_{yy} + \\
+v_{xt} + 4\beta u_x v_{xy} + 4\beta v_x u_{xy} + \alpha v_{xxxx} + \beta v_{xxyy}
\end{align*}
\]

\[= \lambda(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y). \tag{33}\]

Substituting (32) into (33), we obtain the undetermined coefficient $\lambda = 0$ and $\phi(t, x, y, u) = A_5$, where $A_5$ is a constant. Hence equation (2) satisfies the nonlinear self-adjointness condition.

5. The conservation laws. Corresponding to each of the symmetries for equation (2), we obtain the fluxes for the corresponding density. These conservation laws extend and complete previous results in the literature.

For $\Gamma_{1a}$ the conserved fluxes are

\[
\begin{align*}
c^t &= \phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y), \\
c^x &= -\phi(t, x, y, u)u_{xt}(6\alpha u_x + 4\beta u_y) \quad \text{and} \\
c^y &= -\phi(t, x, y, u)u_t 4\beta u_{xx}.
\end{align*}
\]

For $\Gamma_{2a}$ the conserved fluxes are

\[
\begin{align*}
c^t &= t\phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y), \\
c^x &= \frac{y\alpha}{\beta} \phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y) - \\
&\quad \phi(t, x, y, u)\left(\frac{y}{4t\beta} + tu_{xt} + \frac{y\alpha}{\beta} u_{xx}\right)(6\alpha u_x + 4\beta u_y) \quad \text{and} \\
c^y &= 4\phi(t, x, y, u)\beta u_{xx} \left(\frac{5y^2\alpha}{16t\beta^2} - \frac{xy}{4t\beta} - tu_t - \frac{y\alpha}{\beta} u_x\right).
\end{align*}
\]

For $\Gamma_{3a}$ the conserved fluxes are

\[
\begin{align*}
c^t &= -t\phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y), \\
c^x &= \left(\frac{2y\alpha}{\beta} - x\right) \phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y) \\
&\quad + \left(yu_{xy} + tu_{xt} + xu_{xx} + 2u_x - \frac{y}{2t\beta}\right) \phi(t, x, y, u)(6\alpha u_x + 4\beta u_y) \quad \text{and} \\
c^y &= -y\phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y + \\
&\quad 4\beta u_{xy} u_x + 4\beta u_{xx} u_y) + 4\beta \phi(t, x, y, u)u_{xx}(u + \frac{5y^2\alpha}{8t\beta^2} - \frac{xy}{2t\beta} + tu_t + xu_x - \frac{2y\alpha}{\beta} u_x + yu_y).
\end{align*}
\]
For $\Gamma_{4a}$ the nonzero conserved fluxes are
\[ c^x = \phi(t, x, y, u)(1 - 4t\beta u_{xy})(6\alpha u_x + 4\beta u_y) \quad \text{and} \]
\[ c^y = 4t\beta \phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y) \]
\[ + (x - \frac{3y\alpha}{2\beta} - 4t\beta u_y)4\beta \phi(t, x, y, u)u_{xx}. \]

For $\Gamma_{5a}$ the nonzero conserved fluxes are
\[ c^x = -u_{xy} \phi(t, x, y, u)(6\alpha u_x + 4\beta u_y) \quad \text{and} \]
\[ c^y = \phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y) \]
\[ - u - y\phi(t, x, y, u)4\beta u_{xx}. \]

For $\Gamma_{6a}$ the nonzero conserved fluxes are
\[ c^x = b(t)\phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y) \]
\[ - b(t)\phi(t, x, y, u)u_{xx}(6\alpha u_x + 4\beta u_y) \]
\[ c^y = 4\beta \phi(t, x, y, u)u_{xx}(a(t) + \frac{y b'(t)}{4\beta} - b(t)u_x) \]

For $\Gamma_{7a}$, the nonzero conserved vectors are
\[ c^t = -\frac{3t}{2} \phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y), \]
\[ c^x = -\frac{x}{2} \phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y) + \]
\[ \left( u_x + \frac{x u_{xx}}{2} + \frac{y u_{xy}}{2} + \frac{3t u_{xt}}{2} \right) \phi(t, x, y, u)(6\alpha u_x + 4\beta u_y) \quad \text{and} \]
\[ c^y = -\frac{y}{2} \phi(t, x, y, u)(u_{xt} + \alpha u_{xxxx} + \beta u_{xxyy} + 6\alpha u_{xx} u_x + 4\beta u_{xy} u_x + 4\beta u_{xx} u_y) + \]
\[ 4\beta \phi(t, x, y, u)u_{xx} \left( \frac{u}{2} + \frac{x u_{xx}}{2} + \frac{y u_{xy}}{2} + \frac{3t u_{t}}{2} \right). \]

6. Conclusions. In this work we applied symmetry analysis to determine similarity solutions and conservation laws for the $1+2$-dimensional Bogoyavlensky-Konopelchenko equation. More specifically we considered the Lie point symmetries and we studied the reduction process in order to find similarity solutions and also determined conservation laws.

The application of Lie’s theory provides us that equation (2) admits five plus infinity Lie symmetries. The infinity symmetries are directed related with the linearity of the Bogoyavlensky-Konopelchenko equation. We apply the five finite Lie symmetries to reduce the differential equation to an ordinary differential equation, while for the latter one we apply again the Lie symmetries to reduce the order of the ordinary differential equation and consequently determine similarity solutions, or reduce the equation to well-known integrable differential equations. This analysis extends previously published results in the literature [17, 18]. Our future work
is to study the integrability of the reduced ODEs with zero point symmetries using Singularity Analysis.

Furthermore we apply Ibragimov’s approach to determine conservation laws by constructing self-adjoint operators from the admitted Lie point symmetries.

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