Fluctuations of eigenvalues of matrix models and their applications

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Abstract

We study the expectation of linear eigenvalue statistics of matrix models with any $\beta > 0$, assuming that the potential $V$ is a real analytic function and that the corresponding equilibrium measure has a one-interval support. We obtain the first order (with respect to $n^{-1}$) correction terms for the expectation and apply this result to prove bulk universality for real symmetric and symplectic matrix models with the same $V$.

1 Introduction and main results

We consider ensembles of $n \times n$ real symmetric, hermitian or symplectic matrices $M$ with the probability distribution

$$P_n(M)dM = Z_{n,\beta}^{-1} \exp\left\{-\frac{n\beta}{2} \text{Tr} V(M)\right\}dM,$$

where $\beta = 1, 2, 4$ corresponds to real symmetric, Hermitian, and symplectic case respectively, $Z_{n,\beta}$ is a normalization constant, $V: \mathbb{R} \to \mathbb{R}_+$ is a Hölder function satisfying the condition

$$V(\lambda) \geq 2(1 + \epsilon) \log(1 + |\lambda|).$$

The joint eigenvalue distribution which corresponds to (1.1) has the form (see [12])

$$p_{n,\beta}(\lambda_1, \ldots, \lambda_n) = Q_{n,\beta}^{-1} \prod_{i=1}^{n} e^{-n\beta V(\lambda_i)/2} \prod_{1 \leq i<j \leq n} |\lambda_i - \lambda_j|^\beta,$$

where

$$Q_{n,\beta} = \int \prod_{i=1}^{n} e^{-n\beta V(\lambda_i)/2} \prod_{1 \leq i<j \leq n} |\lambda_i - \lambda_j|^\beta d\lambda_1 \ldots d\lambda_n.$$

This distribution can be considered for any $\beta > 0$. We denote

$$\mathbb{E}_\beta\{\ldots\} = \int (\ldots)p_{n,\beta}(\lambda_1, \ldots, \lambda_n)d\lambda_1 \ldots d\lambda_n,$$

and

$$p_{l,\beta}^{(n)}(\lambda_1, \ldots, \lambda_l) = \int_{\mathbb{R}^{n-l}} p_{n,\beta}(\lambda_1, \ldots, \lambda_l, \lambda_{l+1}, \ldots, \lambda_n)d\lambda_{l+1} \ldots d\lambda_n.$$
It is known (see [2] [10]) that if \( V \) is a Hölder function, then the first marginal density \( p_{1,N}^{(n)}(\lambda) \) converges weakly to the density \( \rho(\lambda) \) (equilibrium density) with a compact support \( \sigma \). The support \( \sigma \) and the density \( \rho \) are uniquely defined by the conditions:

\[
v(\lambda) := 2 \int \log |\mu - \lambda| \rho(\mu) d\mu - V(\lambda) = \sup_{\lambda \in \sigma} v(\lambda), \quad \lambda \in \sigma
\]

\[
v(\lambda) \leq \sup_{\lambda \notin \sigma} v(\lambda), \quad \sigma = \text{supp}\{\rho\}.
\]

If we consider the linear eigenvalue statistics of a smooth test function \( f \)

\[
N_n[f] = \sum_{i=1}^{n} f(\lambda_i),
\]

then the above results of [2] [10] mean that

\[
\lim_{n \to \infty} E_{\beta} \left\{ n^{-1} N_n[f] \right\} = \lim_{n \to \infty} \int f(\lambda) p_{1,N}^{(n)}(\lambda) d\lambda = \int f(\lambda) \rho(\lambda) d\lambda,
\]

\[
\lim_{n \to \infty} E_{\beta} \left\{ |n^{-1} N_n[f] - E_{\beta} \{n^{-1} N_n[f]\}|^2 \right\} = 0.
\]

Moreover, in [2] some rather rough bounds on the rate of convergence were found

\[
\left| \int f(\lambda) (p_{1,N}^{(n)}(\lambda) - \rho(\lambda)) d\lambda \right| \leq C||f||_2^{1/2} ||f'||_2^{1/2} n^{-1/2} \log^{1/2} n,
\]

\[
E_{\beta} \left\{ |n^{-1} N_n[f] - E_{\beta} \{n^{-1} N_n[f]\}|^2 \right\} \leq ||f||_2 ||f'||_2 n^{-1} \log n.
\]

Here and below we denote by \( ||| \cdot |||_2 \) a standard \( L^2(\sigma_\varepsilon) \)-norm, with \( \sigma_\varepsilon \) being the \( \varepsilon \)-neighborhood of the support \( \sigma \) with sufficiently small \( \varepsilon \).

In the case of \( \beta = 2 \) these bounds can be improved considerably. It is a simple exercise (see e.g. [13]) to show that for any \( V \) satisfying (1.2) (not necessary Lipshitz) the l.h.s. of the second inequality is \( O(n^{-2}) \), but for other \( \beta \) this fact is not proven yet. With the first inequality of (1.9) the situation is similar. It follows from the results of [4] that for real analytic \( V \) the l.h.s. of the first inequality of (1.9) is \( O(n^{-1}) \) (see also [1] where the asymptotic expansion with respect to \( n^{-1} \) was constructed in the case of even real analytic \( V \) and one or two interval support \( \sigma \)). Unfortunately, similar results are not found for \( \beta \neq 2 \) in the general case of \( \sigma \) till now.

Bounds of the type (1.9) are interesting not only themselves. They have a lot of very important applications, which includes Central Limit Theorem (CLT) for linear eigenvalue statistics, the asymptotic for \( \log Q_{n,\beta} \), etc. One of the most important and interesting applications is that to the universality problem for \( \beta = 1, 4 \). Universality conjecture states that marginal densities (1.6) in the scaling limit, when \( \lambda_i = \lambda_0 + x_i/n^\kappa \) \((i = 1, \ldots , l)\) are universal (i.e. they do not depend on \( V \)). The scaling exponent \( \kappa \) depends on the behavior of the equilibrium density \( \rho(\lambda) \) in a small neighborhood of \( \lambda_0 \). If \( \rho(\lambda_0) \neq 0 \), then \( \kappa = 1 \), if \( \rho(\lambda_0) = 0 \) and \( \rho(\lambda) \sim |\lambda - \lambda_0|^\alpha \), then \( \kappa = 1/(1 + \alpha) \).

For \( \beta = 2 \) universality of local eigenvalue statistics was proved in many cases. For example, in the bulk case \((\rho(\lambda_0) \neq 0)\) it was shown in [13] (see also [14]) that for a general class of \( V \) (the second derivative of \( V \) is Lipshitz in some neighborhood of \( \lambda_0 \)) the scaled reproducing kernel converges uniformly to the sin-kernel. This result for the case of real analytic \( V \) was obtained also in [4]. Universality in the bulk for very general conditions on the potential \( V \) was proved also recently in [11]. Universality near the edge, i.e., the case when \( \lambda_0 \) is the edge point of the spectrum and \( \rho(\lambda) \sim |\lambda - \lambda_0|^{1/2} \), as \( \lambda \sim \lambda_0 \), was studied in [4]. There are also
results on universality near the extreme point, where \( \rho(\lambda) \sim (\lambda - \lambda_0)^2 \), as \( \lambda \sim \lambda_0 \) (see [3] for real analytic \( V \) and [15] for general \( V \)).

The crucial difference between the case \( \beta = 2 \) and other \( \beta \) is that for \( \beta = 2 \) all correlation functions \([1.6]\) can be expressed in terms of the reproducing kernel of the system of normalized polynomials \( p_j^{(n)} = \gamma_j^{(n)} x^j + \ldots, (j = 0, \ldots, n - 1) \) orthogonal on the real line with varying weight

\[
 w^{(n)}(\lambda) := e^{-nV(\lambda)} 
\]

\[
 \int_{\mathbb{R}} p_j^{(n)}(\lambda)p_k^{(n)}(\lambda)w^{(n)}(\lambda) d\lambda = \delta_{j,k} \quad \text{for } j, k \geq 0. 
\]

The orthogonal polynomial machinery, in particular, Christoffel-Darboux formula and Christoffel function simplify considerably the studies of marginal densities \([1.6]\). Moreover, asymptotics of orthogonal polynomials \( p_n^{(n)} \), \( p_n^{(n)} \) are known (see [4] for real analytic \( V \) and the recent paper [9] for non analytic \( V \)) and they can be used to prove bulk and edge universality.

For \( \beta = 1, 4 \) the situation is more complicated. It was shown in [19] that the problem can be reduced to universality of some matrix kernels (see \([1.21], [1.22]\) below), which also can be expressed in terms of orthogonal polynomials \([1.11]\), but to control their behavior one need to control the invertibility of some matrix (see Section 3 for more details). According to Widom [20], if the potential \( V \) is a rational function, then we need to control the inverse of some matrix of fixed size depending of \( V \) (e.g., if \( V \) is polynomial of degree \( 2m \), then we should control some \((2m - 1) \times (2m - 1)\) matrix). Till now this technical problem was solved only in a few cases. In the papers [5, 6] the case \( V(\lambda) = \lambda^{2m}(1 + o(1)) \) (in our notations) was studied. Similar method was used in [7] to prove bulk and edge universality (including the case of hard edge) for the Laguerre type ensembles with monomial \( V \). In [18] universality in the bulk and near the edges were studied for \( V \) being an even quartic polynomial. In [16, 17] bulk and edge universality were studied for \( \beta = 1 \) and real analytic even \( V \) with one interval support \( \sigma \).

But there is also a possibility to prove universality of local eigenvalue statistics by using another technique. In [18] Sojanovich made an important observation (see Remark 5 of [18] or Section 3 of the present paper) which allows one to replace the problem to control the Widom matrix by the problem to control \( E_\beta \{ n^{-1}\mathcal{N}_n[f] \} \) for \( \beta = 1, 2, 4 \). Thus the problem to study the correction terms of the order \( n^{-1} \) for \( E_\beta \{ n^{-1}\mathcal{N}_n[f] \} \) becomes especially important.

In a remarkable paper [10] Johansson studied the expectation and the variance of \( n^{-1}\mathcal{N}_n[f] \) up to the terms \( O(n^{-2}) \). This allows him, in particular, to prove CLT for fluctuations of \( \mathcal{N}_n[f] \). Unfortunately, his method works only in the case of one interval support \( \sigma \) of the equilibrium density \( \rho \) and polynomial \( V \) with some additional assumption.

In the present paper we generalize the idea of [10] to the case of real analytical \( V \) with one interval support of \( \rho \), without any other assumptions. Moreover, we give a more simple proof of this result and apply it to the proof of bulk universality for \( \beta = 1, 4 \).

Let us formulate our main conditions.

**Condition C1.** The support \( \sigma \) of the equilibrium measure density \( \rho \) consists of a single interval: \( \sigma = [a, b], -\infty < a < b < \infty \).

**Remark 1** It is easy to see that changing the variables \( M' = 2(M - \frac{a + b}{2})/(b - a), \) in the case (i) we can always take the support \( \sigma = [-2, 2] \).
**Condition C2.** The equilibrium density $\rho$ can be represented in the form

$$\rho(\lambda) = \frac{1}{2\pi} P(\lambda) \Im X^{1/2}(\lambda + i0), \quad \inf_{\lambda \in [-2, 2]} P(\lambda) > 0, \quad (1.12)$$

where

$$X(z) = z^2 - 4, \quad (1.13)$$

and we choose a branch of $X^{1/2}(z)$ such that $X^{1/2}(z) \sim z$, as $z \to +\infty$. Moreover, the function $v$ defined by (1.7) attains its maximum if and only if $\lambda$ belongs to $\sigma$.

**Condition C3.** $V$ is real analytic on $\sigma$, i.e., there exists an open domain $D \subset \mathbb{C}$ such that $\sigma \subset D$ and $V$ is an analytic function in $D$.

**Remark 2** It is known (see, e.g., [1]) that under conditions C1 and C3 for any $\beta$ the equilibrium density $\rho$ of the ensemble (1.3) has the form (1.12) – (1.13) with $P(\geq 0)$. The analytic function $P$ in (1.12) can be represented in the form

$$P(z) = \int_{\sigma} \frac{V'(z) - V'(\lambda)}{(z - \lambda) \Im X^{1/2}(\lambda + i0)} d\lambda \quad (1.14)$$

Hence, condition C2 states that $P$ has no zeros in $[-2, 2]$. Note also, that in the paper [10] it was assumed additionally that $V$ is a polynomial and $P$ has no zeros on the real line.

The first result of the paper is the theorem which allows us to control the expectation and the variance of linear eigenvalue statistics.

**Theorem 1** Under conditions C1 – C3 for any analytic in $D$ function $f$ we have

$$\mathbb{E}_\beta \{ N_n[f] \} = \int f(\lambda) \rho(\lambda) d\lambda + \frac{1}{n} \left( \frac{2}{\beta} - 1 \right) \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_d} \frac{f(z) dz}{X^{1/2}(z)} \int_{\mathcal{L}_d} \frac{g'(\zeta) d\zeta}{P(\zeta)(z - \zeta)} + n^{-2} r_{n,\beta}(f), \quad (1.15)$$

where the contour $\mathcal{L}_d$ is defined as

$$\mathcal{L}_d = \{ z : \text{dist}\{z, \sigma\} = d \}, \quad (1.16)$$

d is chosen sufficiently small to have all zeros of $P(\zeta)$ outside of $\mathcal{L}_d$,

$$g(z) = \int \frac{\rho(\lambda) d\lambda}{z - \lambda}, \quad (1.17)$$

and $r_{n,\beta}(f)$ satisfies the bound

$$|r_{n,\beta}(f)| \leq C_d \sup_{z: \text{dist}\{z, \sigma\} \leq 2d} |f(z)|,$$

with $C_d$ depending only on $d$.

Moreover,

$$\mathbb{E}_\beta \{ |N_n[f] - \mathbb{E}_\beta \{ N_n[f] \}|^2 \} \leq C_d \sup_{z: \text{dist}\{z, \sigma\} \leq 2d} |f(z)|^2. \quad (1.18)$$
One of the important applications of Theorem 1 (see discussion above) is the asymptotic of \( \log Q_{n,\beta} \). Since the paper [2] it is known that
\[
n^{-2} \log Q_{n,\beta} = \frac{\beta}{2} \mathcal{E}_V + O(\log n/n),
\]
where
\[
\mathcal{E}_V = -\int \log \frac{1}{|\lambda - \mu|} \rho(\lambda) \rho(\mu) d\lambda d\mu - \int V(\lambda) \rho(\lambda) d\lambda.
\]
(1.19)

But for many problems it is important to control the next terms of asymptotic expansion of \( \log Q_{n,\beta} \) (for applications see discussion in [8], where the complete asymptotic expansion with respect to \( n^{-1} \) was constructed for the case \( \beta = 2 \) under assumption that \( V \) is a polynomial close in a certain sense to \( V_0(\lambda) = \lambda^2/2 \).

**Theorem 2** Under conditions C1 – C3 for any \( \beta \)
\[
n^{-2} \log Q_{n,\beta} = n^{-2} \log Q_{n,\beta}^{(0)} + \frac{1}{2} \beta \mathcal{E}_V + \frac{3}{8} \beta
\]
\[
+ \frac{1}{n} \left( 1 - \frac{\beta}{2} \right) \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_2} \frac{(V(z) - z^2/2) dz}{X^{1/2}(z)} \int_{\mathcal{L}_d} \frac{g_1(\zeta) d\zeta}{P_1(\zeta)(z - \zeta)} + O(n^{-2}),
\]
(1.20)

where \( \log Q_{n,\beta}^{(0)} \) corresponds to the Gaussian case \( V_0 = \lambda^2/2 \), \( \mathcal{E}_V \) is defined by (1.19), \( \frac{3}{8} \beta = -\frac{1}{2} \beta \mathcal{E}_{V_0} \), and
\[
P_1(\lambda) = tP(\lambda) + 1 - t, \quad g_1(z) = tg(z) + \frac{1-t}{2} (z - \sqrt{z^2 - 4}).
\]

**Remark 3** By the Selberg formula (see e.g. [12]) for the Gaussian case we have
\[
Q_{n,\beta}^{(0)} = n! \left( \frac{n \beta}{2} \right)^{-n(n-1)/4-n/2} (2\pi)^{n/2} \prod_{j=1}^n \frac{\Gamma(\beta j/2)}{\Gamma(\beta/2)}.
\]

As it was mentioned above, Theorem 1 together with some asymptotic results of [4] for orthogonal polynomials can be used to prove universality of the local eigenvalue statistics of the matrix models (1.1). We restrict our attention to the case when \( V \) is a polynomial of even degree \( 2m \) such that conditions C1–C3 are satisfied. Moreover we consider only even \( n \). It is known (see [19]) that the question of universality is closely related to the large \( n \) behavior of certain matrix kernels
\[
K_{n,1}(\lambda, \mu) := \begin{pmatrix}
S_{n,1}(\lambda, \mu) & -\frac{\partial}{\partial \mu} S_{n,1}(\lambda, \mu) \\
(\epsilon S_{n,1})(\lambda, \mu) - \epsilon(\lambda - \mu) & S_{n,1}(\mu, \lambda)
\end{pmatrix}
\]
for \( \beta = 1, n \) even,
(1.21)

\[
K_{n,4}(\lambda, \mu) := \begin{pmatrix}
S_{n,4}(\lambda, \mu) & -\frac{\partial}{\partial \mu} S_{n,4}(\lambda, \mu) \\
(\epsilon S_{n,4})(\lambda, \mu) & S_{n,4}(\mu, \lambda)
\end{pmatrix}
\]
for \( \beta = 4 \).

(1.22)

Here \( \epsilon(\lambda) = \frac{1}{2} \text{sgn}(\lambda) \), where sgn denotes the standard signum function, and \( (\epsilon S_{n,\beta})(\lambda, \mu) = \int_{\mathbb{R}} \epsilon(x - x') S_{n,\beta}(x', y) d\lambda' \). Some formulæ for the functions \( S_{n,\beta} \) that appear in the definition of \( K_{n,\beta} \) will be introduced in (3.3), (3.4) below. In order to state our theorem we need some more notation. Define
\[
K_\infty(t) := \frac{\sin \pi t}{\pi t},
\]
\[
K^{(1)}(\xi, \eta) := \begin{pmatrix}
K_\infty(\xi - \eta) & K_\infty'(\xi - \eta) \\
\int_0^{\xi-\eta} K_\infty(t) dt - \epsilon(\xi - \eta) & K_\infty(\eta - \xi)
\end{pmatrix},
\]
\[
K^{(4)}(\xi, \eta) := \begin{pmatrix}
K_\infty(\xi - \eta) & K_\infty'(\xi - \eta) \\
\int_0^{\xi-\eta} K_\infty(t) dt & K_\infty(\eta - \xi)
\end{pmatrix}.
\]
Furthermore we denote for a $2 \times 2$ matrix $A$ and $\lambda > 0$
\[
A^{(\lambda)} := \begin{pmatrix} \sqrt{\lambda}^{-1} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} A \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{pmatrix}.
\]

**Theorem 3** Let $V$ be a polynomial of degree $2m$ with positive leading coefficient and such that conditions C1–C2 are satisfied. Then we have for (even) $n \to \infty$, $\lambda_0 \in \mathbb{R}$ with $\rho(\lambda_0) > 0$, and for $\beta \in \{1, 4\}$ that
\[
\frac{1}{q_n} K_{n,1}^{(q_n)} \left( \lambda_0 + \frac{\xi}{q_n}, \lambda_0 + \frac{\eta}{q_n} \right) = K_{\infty}^{(1)}(\xi, \eta) + O(n^{-1/2}),
\]
\[
\frac{1}{q_n} K_{n,2,4}^{(q_n)} \left( \lambda_0 + \frac{\xi}{q_n}, \lambda_0 + \frac{\eta}{q_n} \right) = K_{\infty}^{(4)}(\xi, \eta) + O(n^{-1/2}),
\]
where $q_n = n \rho(\lambda_0)$. The error bound is uniform for bounded $\xi, \eta$ and for $\lambda_0$ contained in some compact subset of $(-2, 2)$ (recall that supp $\rho = [-2, 2]$ by Condition C1).

It is an immediate consequence of Theorem 3 that the corresponding rescaled l-point correlation functions
\[
p_{l,1}^{(n)} \left( \lambda_0 + \frac{\xi_1}{q_n}, \ldots, \lambda_0 + \frac{\xi_l}{q_n} \right), \quad p_{l,4}^{(n/2)} \left( \lambda_0 + \frac{\xi_1}{q_n}, \ldots, \lambda_0 + \frac{\xi_l}{q_n} \right)
\]
converge for $n$ (even) $\to \infty$ to some limit that depends on $\beta$ but not on the choice of $V$.

The paper is organized as follows. In Section 2 we prove Theorems 1 and 2. In Section 3 we prove Theorem 3 modulo some bounds, which we obtain in Section 4. And in Section 5 for the reader’s convenience we give a version of the proof of a priori bound (1.9).

## 2 Proof of Theorems 1, 2

**Proof of Theorem 1** Take an $n$-independent $\varepsilon$, small enough to provide that $\sigma_\varepsilon \subset D$, where $\sigma_\varepsilon \subset \mathbb{R}$ means the $\varepsilon$-neighborhood of $\sigma$. It is known (see e.g. [14]) that if we replace in (1.3), (1.5) and (1.6) the integration over $\mathbb{R}$ by the integration $\sigma_\varepsilon$, then the new marginal densities will differ from the initial ones by the terms $O(e^{-nc})$ with some $c$ depending on $\varepsilon$, but independent of $n$. Since for our purposes it is more convenient to consider the integration with respect to $\sigma_\varepsilon$, we assume from this moment that this replacement is made, so everywhere below the integration without limits means the integration over $\sigma_\varepsilon$.

Following the idea of [10], we will study a little bit modified form of the joint eigenvalue distribution, than in (1.3). Namely, consider any real on $\sigma$ and analytic in $D$ function $h(\zeta)$ and denote
\[
V_h(\zeta) = V(\zeta) + \frac{1}{n} h(\zeta).
\]
Let $p_{n,\beta,h}, \mathcal{E}_{\beta,h}\{\ldots\}, \mathcal{P}_{l,\beta,h}^{(n)}$ be the distribution density, the expectation, and the marginal densities defined by (1.3), (1.5) and (1.6) with $V$ replaced by $V_h$.

By (1.3) the first marginal density can be represented in the form
\[
p_{1,\beta,h}(\lambda) = Q_{1,\beta,h}^{-1} \int e^{-n\beta V_h(\lambda)/2} \prod_{i=2}^n (|\lambda - \lambda_i|^{\beta} e^{-n\beta V_h(\lambda_i)/2} \prod_{2 \leq i < j \leq n} |\lambda_i - \lambda_j|^{\beta} d\lambda_2 \ldots d\lambda_n. \quad (2.1)
\]
Using the representation and integrating by parts, we obtain

$$
\int \frac{V''_h(\lambda)p^{(n)}_{1,\beta,h}(\lambda)}{z-\lambda}d\lambda = \frac{2}{\beta n} \int \frac{p^{(n)}_{1,\beta,h}(\lambda)}{(z-\lambda)^2}d\lambda + \frac{2(n-1)}{n} \int \frac{p^{(n)}_{2,\beta,h}(\lambda,\mu)d\lambda d\mu}{(z-\lambda)(\lambda-\mu)} + O(e^{-nc}). \quad (2.2)
$$

Here $O(e^{-nc})$ is the contribution of the integrated term. In fact all equations below should contain $O(e^{-mc})$, but in order to simplify formula below we omit it.

Since the function $p^{(n)}_{2,\beta,h}(\lambda,\mu)$ is symmetric with respect to $\lambda,\mu$, we have

$$
2 \int \frac{p^{(n)}_{2,\beta,h}(\lambda,\mu)d\lambda d\mu}{(z-\lambda)(\lambda-\mu)} = \int \frac{p^{(n)}_{2,\beta,h}(\lambda,\mu)d\lambda d\mu}{(z-\lambda)(\lambda-\mu)} + \int \frac{p^{(n)}_{2,\beta,h}(\lambda,\mu)d\lambda d\mu}{(z-\lambda)(\lambda-\mu)} = \int \frac{p^{(n)}_{2,\beta,h}(\lambda,\mu)d\lambda d\mu}{(z-\lambda)(\lambda-\mu)}.
$$

Hence, equation (2.2) can be written in the form

$$
\int \frac{V''_h(\lambda)p^{(n)}_{1,\beta,h}(\lambda)}{z-\lambda}d\lambda = \frac{2}{\beta n} \int \frac{p^{(n)}_{1,\beta,h}(\lambda)}{(z-\lambda)^2}d\lambda + \frac{2(n-1)}{n} \int \frac{p^{(n)}_{2,\beta,h}(\lambda,\mu)d\lambda d\mu}{(z-\lambda)(\lambda-\mu)}. \quad (2.3)
$$

Let us introduce notations:

$$
\delta_{n,\beta,h}(z) = n(n-1) \int \frac{p^{(n)}_{2,\beta,h}(\lambda,\mu)d\lambda d\mu}{(z-\lambda)(\lambda-\mu)} - n^2 \left( \int \frac{p^{(n)}_{1,\beta,h}(\lambda)d\lambda}{z-\lambda} \right)^2 + n \int \frac{p^{(n)}_{1,\beta,h}(\lambda)d\lambda}{(z-\lambda)^2}d\lambda = \int \frac{k_{n,\beta,h}(\lambda,\mu)d\lambda d\mu}{(z-\lambda)(\lambda-\mu)}, \quad (2.4)
$$

where

$$
k_{n,\beta,h}(\lambda,\mu) = n(n-1)p^{(n)}_{2,\beta,h}(\lambda,\mu) - n^2 p^{(n)}_{1,\beta,h}(\lambda)p^{(n)}_{1,\beta,h}(\mu) + n(\lambda-\mu)p^{(n)}_{1,\beta,h}(\lambda). \quad (2.5)
$$

Moreover, we denote

$$
g_{n,\beta,h}(z) = \int \frac{p^{(n)}_{1,\beta,h}(\lambda)d\lambda}{z-\lambda}, \quad V(z,\lambda) = \frac{V'(z)-V'(\lambda)}{z-\lambda}. \quad (2.6)
$$

Then equation (2.2) takes the form

$$
g_{n,\beta,h}(z) - V'(z)g_{n,\beta,h}(z) + \int V(z,\lambda)p^{(n)}_{1,\beta,h}(\lambda)d\lambda

= \frac{1}{n} \int \frac{h'(\lambda)p^{(n)}_{1,\beta,h}(\lambda)}{z-\lambda}d\lambda - \frac{1}{n} \left( \frac{2}{\beta} - 1 \right) \int \frac{p^{(n)}_{1,\beta,h}(\lambda)}{(z-\lambda)^2}d\lambda - \frac{1}{n^2} \delta_{n,\beta,h}(z). \quad (2.7)
$$

Using that $V(z,\zeta)$ is an analytic function of $\zeta$ in $\mathcal{D}$, we obtain by the Cauchy theorem that for any $z$ outside of $\mathcal{L}_d$

$$
\int V(z,\lambda)p^{(n)}_{1,\beta,h}(\lambda)d\lambda = \frac{1}{2\pi i} \oint_{\mathcal{L}_d} V(z,\zeta)g_{n,\beta,h}(\zeta)d\zeta.
$$

Thus, (2.7) takes the form

$$
g_{n,\beta,h}(z) - V'(z)g_{n,\beta,h}(z) + \frac{1}{2\pi i} \oint_{\mathcal{L}_d} V(z,\zeta)g_{n,\beta,h}(\zeta)d\zeta

= \frac{1}{n} \int \frac{h'(\lambda)p^{(n)}_{1,\beta,h}(\lambda)}{z-\lambda}d\lambda - \frac{1}{n} \left( \frac{2}{\beta} - 1 \right) \int \frac{p^{(n)}_{1,\beta,h}(\lambda)}{(z-\lambda)^2}d\lambda - \frac{1}{n^2} \delta_{n,\beta,h}(z). \quad (2.8)
$$
Passing to the limit $n \to \infty$, we obtain for any fixed $z$ the quadratic equation

$$g^2(z) - V'(z)g(z) + Q(z) = 0, \quad Q(z) = \frac{1}{2\pi i} \oint_{\mathcal{L}_d} V(z, \zeta)g(\zeta)d\zeta,$$  \hspace{1cm} (2.9)

where $g$ is defined by (1.17). Hence,

$$g(z) = \frac{1}{2}V'(z) - \frac{1}{2}\sqrt{V'(z)^2 - 4Q(z)}.$$

Using the inverse Stieltjes transform and comparing with (1.12), we get that

$$2g(z) - V'(z) = P(z)X^{1/2}(z),$$  \hspace{1cm} (2.10)

where $X(z)$ is defined by (1.13).

Denote

$$u_{n,\beta,h}(z) = n(g_{n,\beta,h}(z) - g(z)) \iff g_{n,\beta,h}(z) = g(z) + \frac{1}{n}u_{n,\beta,h}(z).$$  \hspace{1cm} (2.11)

Then, subtracting (2.9) from (2.8) and multiplying the result by $n$, we get

$$(2g(z) - V'(z))u_{n,\beta,h}(z) + \frac{1}{2\pi i} \oint_{\mathcal{L}_d} V(z, \zeta)u_{n,\beta,h}(\zeta)d\zeta = F(z),$$  \hspace{1cm} (2.12)

where

$$F(z) = \int \frac{h'(\lambda)p_{1,\beta,h}^{(n)}(\lambda)}{z - \lambda}d\lambda + \left(\frac{2}{\beta} - 1\right)\left(g'(z) + \frac{1}{n}u'_{n,\beta,h}(z)\right) - \frac{1}{n}u_{n,\beta,h}^2(z) - \frac{1}{n}\delta_{n,\beta,h}(z).$$  \hspace{1cm} (2.13)

Using (2.10), we obtain from (2.12)

$$P(z)X^{1/2}(z)u_{n,\beta,h}(z) + Q_n(z) = F(z), \quad Q_n(z) = \frac{1}{2\pi i} \oint_{\mathcal{L}_d} V(z, \zeta)u_{n,\beta,h}(\zeta)d\zeta.$$  \hspace{1cm} (2.14)

Then, choosing $d$ such that the contour $\mathcal{L}_d$ defined by (1.16) does not contain zeros of $P(\zeta)$, we get for any $z$ outside of $\mathcal{L}_d$

$$\frac{1}{2\pi i} \oint_{\mathcal{L}_d} \left(P(\zeta)X^{1/2}(\zeta)u_{n,\beta,h}(\zeta) + Q_n(\zeta) - F(\zeta)\right) \frac{d\zeta}{P(\zeta)(z - \zeta)} = 0.$$  \hspace{1cm} (2.15)

Since, by definition (2.14), $Q_n(\zeta)$ is an analytic function in $\mathcal{D}$, and $z$ and all zeros of $P$ are outside of $\mathcal{L}_d$, the Cauchy theorem yields

$$\frac{1}{2\pi i} \oint_{\mathcal{L}_d} \frac{Q_n(\zeta)d\zeta}{P(\zeta)(z - \zeta)} = 0.$$

Moreover, since

$$u_{n,\beta,h}(z) = \frac{n}{z} \left(\int d\lambda p_{1,\beta,h}^{(n)}(\lambda) - \int d\lambda \rho(\lambda)\right) + n\mathcal{O}(z^{-2}) = n\mathcal{O}(z^{-2}), \quad z \to \infty$$

we have

$$X^{1/2}(z)u_{n,\beta,h}(z) = n\mathcal{O}(z^{-1}).$$  \hspace{1cm} (2.16)
Then the Cauchy theorem yields
\[
\frac{1}{2\pi i} \oint_{\mathcal{L}} X^{1/2}(\zeta) u_{n,\beta,h}(\zeta) d\zeta = X^{1/2}(z) u_{n,\beta,h}(z).
\]

Finally, we obtain from (2.15)
\[
\begin{align*}
u_{n,\beta,h}(z) & = \frac{1}{2\pi i X^{1/2}(z)} \oint_{\mathcal{L}_d} F(\zeta) d\zeta \quad \text{for all } z \in \mathbb{C},
\end{align*}
\] (2.17)

Now take \(d\) small enough to have all zeros of \(P\) outside of \(\mathcal{L}_d\). Then for any \(z: \text{dist}\{z,\sigma\} = 2d\) equation (2.17) implies
\[
u_{n,\beta,h}(z) = \frac{F(z)}{X^{1/2}(z)P(z)} + \frac{1}{2\pi i X^{1/2}(z)} \oint_{\mathcal{L}_d} F(\zeta) d\zeta.
\] (2.18)

According to the result of [2] for any \(\beta\) we have a priory bound
\[
|\delta_{n,\beta,h}| \leq \frac{Cn \log n}{\text{dist}\{z,\sigma\}}, \quad |u_{n,\beta,h}(z)| \leq \frac{Cn^{1/2} \log n}{\text{dist}^2\{z,\sigma\}}, \quad |u'_{n,\beta,h}(z)| \leq \frac{Cn^{1/2} \log n}{\text{dist}^4\{z,\sigma\}},
\] (2.19)

where \(C\) is an absolute constant.

Denote
\[
M_n(d) = \sup_{z: \text{dist}(z,\sigma) \geq 2d} |u_{n,\beta,h}(z)|
\]

By (2.16) and the maximum principle, there exists a point \(z: \text{dist}\{z,\sigma\} = 2d\) such that
\[
M_n(d) = |u_{n,\beta,h}(z)|.
\]

Then, using (2.18), the definition of \(F\) (see (2.13)), and (2.19), we obtain the inequality
\[
M_n(d) \leq \frac{1}{n} C_1 M_n^2(d) + C_2 \log n,
\]

where \(C_1\) and \(C_2\) depend only on \(d\), \(\sup_{\text{dist}\{z,\sigma\} \leq 3d} |P^{-1}(z)|\), \(\sup_{\text{dist}\{z,\sigma\} \leq d/2} |n^{-1}h(z)|\), and from \(C\) of (2.19). Solving the above quadratic inequality, we get
\[
\begin{cases}
M_n(d) \geq (2C_1)^{-1} (n + \sqrt{n^2 - 4C_1 C_2 n \log n}) \\
M_n(d) \leq (2C_1)^{-1} (n - \sqrt{n^2 - 4C_1 C_2 n \log n})
\end{cases}
\]

Since the first inequality contradicts to (2.19), we conclude that the second inequality holds. Hence, we get
\[
\sup_{z: \text{dist}(z,\sigma) \geq 2d} |u_{n,\beta,h}(z)| \leq 2C_2 \log n + C(\sup_{\lambda \in \sigma_{\epsilon}} |h'(\lambda)| + \text{dist}^{-2}\{z,\sigma\}).
\]

Note that the bound gives us that for any real analytic \(\varphi(\zeta)\)
\[
\left| \int \varphi(\lambda)(p_{1,\beta,h}(\lambda) - \rho(\lambda)) d\lambda \right| = \left| \frac{1}{2\pi i} \oint_{\mathcal{L}_d} \varphi(\zeta) u_{n,\beta,h}(\zeta) d\zeta \right| \leq w_n \left( \sup_{z \in \mathcal{L}_d} |\varphi(z)| + \sup_{\lambda \in \sigma_{\epsilon}} |h'(\lambda)| \right),
\] (2.20)

where
\[
w_n = 2C_2 \log n
\]

Now we are going to use the following lemma, which is an analog of Lemma 3.11 of [10].
Lemma 1  If \((2.20)\) holds for any real \(h\), and some \(\varphi\) which is analytic in \(D_1 \subset D\) \((\sigma_\varepsilon \subset D_1)\), then there exists an \(n\)-independent constant \(C_*\) such that

\[
\int k_{n,\beta,h}(\lambda, \mu)\varphi(\lambda)\varphi(\mu)d\lambda d\mu \leq C_*w_n^2 \sup |\varphi^2|
\]  

(2.21)

The lemma was proved in [10], but for convenience of readers we give its proof at the end of the proof of Theorem 1.

Applying the lemma to \(\varphi^{(1)}_\varepsilon(z) = \Re(z - \lambda)\) and \(\varphi^{(2)}_\varepsilon(z) = \Im(z - \lambda)\) with dist\{\(z, \sigma\}\} \(\geq d\), and using \((2.20)\), we obtain that \(|\delta_{n,\beta,h}| \leq C_d'\log^2 n, |u_{n,\beta,h}(z)|, |u'_{n,\beta,h}(z)| \leq C_d'\log n.\) (2.22)

Then, using this bound in \((2.18)\) instead of \((2.19)\), by the same way as above we get \((2.20)\) with \(w_n = C_1(\sup_{\lambda \in \sigma_\varepsilon} |h'(\lambda)| + C_d).\) Then, applying Lemma 1 once more, we obtain that

\[
|\delta_{n,\beta,h}| \leq C''_d, |u_{n,\beta,h}(z)|, |u'_{n,\beta,h}(z)| \leq C''_d.
\]  

(2.23)

Using these final bounds in \((2.17)\), we obtain that

\[
u_{n,\beta,h}(z) = \frac{1}{2\pi} \int_{\delta_d} \frac{g'(\zeta) d\zeta}{P(z - \zeta)} + r_n(z),
\]  

(2.24)

where

\[
|r_n(z)| \leq n^{-1} C_d.
\]

\(\Box\)

Proof of Lemma 1 Take any real analytic \(\varphi\) such that \(\sup_{z \in \mathbb{C}} |\varphi(z)| \leq 1.\) Using the method of [10], consider the function

\[
F_n(t) = E_{\beta,h} \left\{ \exp \left[ -\frac{t}{2w_n} \sum_{i=1}^{n} (\varphi(\lambda_i) - \int \varphi(\lambda)\rho(\lambda)d\lambda) \right] \right\}.
\]

It is easy to see that

\[
\frac{d^2}{dt^2} \log F_n(t) = (2w_n)^{-2} E_{\beta,h+t\varphi/2w_n} \left( \left\{ \sum_{i=1}^{n} (\varphi(\lambda_i) - E_{\beta,h+t\varphi/2w_n} \{\varphi(\lambda_i)\}) \right\}^2 \right) \geq 0.
\]  

(2.25)

Hence, by \((2.20)\), for \(t \in [-1, 1]\)

\[
\log F_n(t) = \log F_n(t) - \log F_n(0) = \int_{0}^{t} \frac{d}{d\tau} \log F_n(\tau) d\tau \leq |t| \frac{d}{dt} \log F_n(t)
\]

\[
= |t|(2w_n)^{-1} E_{\beta,h+t\varphi/2w_n} \left\{ \sum_{i=1}^{n} \left( \varphi(\lambda_i) - \int \varphi(\lambda)\rho(\lambda)d\lambda \right) \right\}
\]

\[
= |t|n \int \varphi(\lambda) \left( p_{1,\beta,h+t\varphi/2w_n}^{(n)}(\lambda) - \rho(\lambda) \right) d\lambda \leq |t|
\]

Thus, for \(t \in [-1, 1]\)

\[
F_n(t) \leq e^{|t|} \leq 3,
\]

and for any \(t \in \mathbb{C}, |t| < 1\)

\[
|F_n(t)| \leq F_n(|t|) < 3.
\]  

(2.26)
Then, we have by the Cauchy theorem, for $|t| \leq \frac{1}{2}$

$$|F_n'(t)| = \left| \frac{1}{2\pi} \oint_{|t'|=1} \frac{F_n(t')dt'}{(t'-t)^2} \right| \leq 6,$$

and therefore for $|t| \leq \frac{1}{12}$

$$|F_n(t)| = |F(0) - \int_0^t F_n'(t)dt| \geq \frac{1}{2}.$$

Hence, $\log F_n(t)$ is an analytic function for $|t| \leq \frac{1}{12}$ and so, using the above bounds, we have

$$\frac{d^2}{dt^2} \log F_n(0) = \frac{1}{2\pi i} \oint_{|t|=1/12} \frac{\log F_n(t)}{t^3}dt \leq C.$$

Finally, using (2.25), we get

$$\int k_{n,\beta,h}(\lambda, \mu)\varphi(\lambda)\varphi(\mu)d\lambda d\mu = E_{\beta,h} \left\{ \left( \sum_{i=1}^n (\varphi(\lambda_i) - E_{\beta,h}\{\varphi(\lambda_i)\}) \right)^2 \right\} \leq 4Cw_n^2.$$

□

**Proof of Theorem 2** Consider the functions $V_t$ of the form

$$V_t(\lambda) = tV(\lambda) + (1-t)V_0(\lambda),$$

(2.27)

where $V_0(\lambda) = \lambda^2/2$. Let $Q_{n,\beta}(t)$ be defined by (1.4) with $V$ replaced by $V_t$. Then evidently $Q_{n,\beta}(1) = Q_{n,\beta}$ and $Q_{n,\beta}(0)$ corresponds to the Gaussian case $V_0(\lambda) = \lambda^2/2$. Hence,

$$\frac{1}{n^2} \log Q_{n,\beta}(1) - \frac{1}{n^2} \log Q_{n,\beta}(0) = \frac{1}{n^2} \int_0^1 dt \frac{d}{dt} \log Q_{n,\beta}(t)$$

$$= \frac{\beta}{2} \int_0^1 dt \int d\lambda (V(\lambda) - V_0(\lambda))p_{1,\beta}(\lambda; t),$$

(2.28)

where $p_{1,\beta}(\lambda; t)$ is the first marginal density corresponding to $V_t$. Using (1.7) one can check that if we consider the distribution (1.3) with $V$ replaced by $V_t$, then the limiting DOS $\rho_t$ has the form

$$\rho_t(\lambda) = t\rho(\lambda) + (1-t)\rho_0(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2} \left[ tP(\lambda) + (1-t)P_0(\lambda) \right],$$

(2.29)

with $X$ defined by (1.13) and $P_0(\lambda) = 1$. Hence, using (1.15) for the last integral in (2.28), we get

$$\frac{1}{n^2} \log Q_{n,\beta} = \frac{1}{n^2} \log Q_{n,\beta}(0) - \frac{\beta}{2}E_0 + \frac{\beta}{2}E_V$$

$$+ \frac{1}{n} \left( 1 - \frac{\beta}{2} \right) \frac{1}{(2\pi i)^2} \oint_{\mathcal{L}_{2\lambda}} \frac{(V(z) - \sqrt{z}/2)dz}{X^{1/2}(z)} \oint_{\mathcal{L}_{2\lambda}} g_t(\zeta)d\zeta + O(n^{-2}),$$

where $E_V$ is defined by (1.19), $P_t$ and $g_t$ are defined in (1.20) and $E_0 = -\frac{3}{4}$ (see, e.g., [12]). □
3 Bulk universality for orthogonal and symplectic ensembles

In a remarkable paper [14], Tracy and Widom showed how to express the functions $S_{n,\beta}$ that appear in the definitions (1.21), (1.22) in terms of orthogonal polynomials defined by (1.10) – (1.11). Set $\psi^{(n)}_j := p_j^{(n)} \sqrt{w^{(n)}}$, $j \geq 0$. Then the system $\{\psi^{(n)}_j\}_{j \geq 0}$ defines an orthogonal basis in $L^2(\mathbb{R})$ with respect to the standard inner product $(f, g) := \int_{\mathbb{R}} f(\lambda)g(\lambda)\,d\lambda$. Moreover, they satisfy the recursion relations

$$\lambda \psi^{(n)}_k(\lambda) = a_k^{(n)} \psi^{(n)}_{k+1}(\lambda) + b_k^{(n)} \psi^{(n)}_k(\lambda) + a_k^{(n)} \psi^{(n)}_{k-1}(\lambda),$$

which define a semi-infinite Jacobi matrix $J^{(n)}$. It is known (see, e.g. [13]) that

$$|a_k^{(n)}| \leq C, \quad |b_k^{(n)}| \leq C, \quad |n - k| \leq \varepsilon n.$$  \hfill (3.1)

In order to state the formulae for $S_{n,\beta}$ we need to introduce more notation. Let $D^{(n)}_\infty$ and $M^{(n)}_\infty$ be semi-infinite matrices that correspond to the differentiation operator and to some integration operator respectively.

$$D^{(n)}_\infty := \left(\begin{pmatrix} \psi^{(n)}_j \\ \psi^{(n)}_k \end{pmatrix} \right)_{j,k \geq 0},$$

$$M^{(n)}_\infty := \left(\epsilon \psi^{(n)}_j, \psi^{(n)}_k \right)_{j,k \geq 0},$$

with $(\epsilon f)(\lambda) := \int_{\mathbb{R}} \epsilon(\lambda - \mu)f(\mu)\,d\mu$.

Both matrices $D^{(n)}_\infty$ and $M^{(n)}_\infty$ are skew-symmetric. Using in addition that for $j < k$

$$(D^{(n)}_\infty)_{jk} = \int_{\mathbb{R}} \left( (p_j^{(n)}(\lambda))' - \frac{n}{2} V'(|\lambda|) p_j^{(n)}(\lambda) \right) p_k^{(n)}(\lambda) w^{(n)}(\lambda) d\lambda$$

$$= \int_{\mathbb{R}} V'(\lambda) p_j^{(n)}(\lambda) p_k^{(n)}(\lambda) w^{(n)}(\lambda) d\lambda = \frac{n}{2} V'(J^{(n)})_{jk}$$

by orthogonality and the spectral theorem, we see $(D^{(n)}_\infty)_{j,k} = 0$ for $|j - k| \geq 2m$ and

$$|(D^{(n)}_\infty)_{j,k}| \leq nC, \quad |j - n|, |k - n| \leq \varepsilon n.$$

In particular, we may write

$$(\psi^{(n)}_j)' = \sum_{|k-j| < 2m} (D^{(n)}_\infty)_{jk} \psi^{(n)}_k$$

as a finite sum. Since $\epsilon(\psi^{(n)}_j)' = \psi^{(n)}_j$ we have for any $j, l \geq 0$ that

$$\delta_{jl} = \epsilon(\psi^{(n)}_j)', \psi_l) = \sum_{|k-j|<2m} (D^{(n)}_\infty)_{jk} (M^{(n)}_\infty)_{kl}.$$  \hfill (3.2)

This relation together with the skew-symmetry of $M^{(n)}_\infty$ and $D^{(n)}_\infty$ proves

$$D^{(n)}_\infty M^{(n)}_\infty = 1 = M^{(n)}_\infty D^{(n)}_\infty.$$  \hfill (3.3)

Next we denote by $M^{(n)}_n$, $D^{(n)}_n$ the principal $n \times n$ submatrices of $M^{(n)}_\infty$ and $D^{(n)}_\infty$, i.e.

$$M^{(n)}_n := ((M^{(n)}_\infty)_{jk})_{0 \leq j,k \leq n-1} \quad D^{(n)}_n := ((D^{(n)}_\infty)_{jk})_{0 \leq j,k \leq n-1}.$$
The formula of Tracy-Widom for $S_{n,\beta}$ now reads

\[ S_{n,1}(\lambda, \mu) = -\sum_{j,k=0}^{n-1} \psi_j^{(n)}(\lambda)(M_n^{(n)})^{-1}_{jk}(\epsilon \psi_k^{(n)})(\mu) \] (3.3)

\[ S_{n/2,4}(\lambda, \mu) = -\sum_{j,k=0}^{n-1} (\psi_j^{(n)})'(\lambda)(D_n^{(n)})^{-1}_{jk}\psi_k^{(n)}(\mu) \] (3.4)

As a by product of the calculation in [19] one also obtains relations between the partition functions $Q_{n,\beta}$ and the determinants of $M_n^{(n)}$ and $D_n^{(n)}$.

\[ \det M_n^{(n)} = \left( \frac{Q_{n,1}\Gamma_n}{n!2^{n/2}} \right)^2, \quad \det D_n^{(n)} = \left( \frac{Q_{n/2,4}\Gamma_n}{(n/2)!2^{n/2}} \right)^2, \]

where $\Gamma_n := \prod_{j=0}^{n-1} \gamma_j^{(n)}$ and $\gamma_j^{(n)}$ is the leading coefficient of $p_j^{(n)}$. It is also known (see [12]) that $Q_{n,2} = \Gamma_n^2/n!$ and we arrive at

\[ \det(D_n^{(n)} M_n^{(n)}) = \left( \frac{Q_{n,1}Q_{n/2,4}}{Q_{n,2}(n/2)!2^{n}} \right)^2. \]

Since $D_\infty^{(n)} M_\infty^{(n)} = 1$ and $(D_\infty^{(n)})_{jk} = 0$ for $|j - k| > 2m - 1$ we have $D_n^{(n)} M_n^{(n)} = 1 + \Delta_n$ with $\Delta_n$ being zero except for the bottom $2m - 1$ rows. Define $T_n$ to be the $(2m - 1) \times (2m - 1)$ block in the bottom right corner of $D_n^{(n)} M_n^{(n)}$, i.e.

\[ (T_n)_{jk} := (D_n^{(n)} M_n^{(n)})_{n-2m+j,n-2m+k}, \quad 1 \leq j, k \leq 2m - 1. \]

Then we have that $\det(T_n)$ equals $\det(M_n^{(n)} D_n^{(n)})$ and we arrive at a formula, first observed by Stojanovic in [18]:

\[ \det(T_n) = \left( \frac{Q_{n,1}Q_{n/2,4}}{Q_{n,2}(n/2)!2^{n}} \right)^2. \] (3.5)

Since $D_\infty^{(n)} M_\infty^{(n)}$ equals 1 up to the matrix $\Delta_n$ of rank $2m - 1$ (independent of $n$) it is conceivable that one may express $(M_n^{(n)})^{-1}$ and $(D_n^{(n)})^{-1}$ that appear in (3.3), (3.4) by $D_n^{(n)}$ and $B_n^{(n)}$ respectively up to some correction terms that involves the inverse of $T_n^{-1}$. Using this idea Widom provided in [20] a useful formula for $S_{n,\beta}$ that was later refined in [7]. In order to present this formula introduce some more notation:

\[ \Phi_1^{(n)} := (\psi_{n-2m+1}^{(n)}, \psi_{n-2m+2}^{(n)}, \ldots, \psi_{n-1}^{(n)})^T, \]

\[ \Phi_2^{(n)} := (\psi_n^{(n)}, \psi_{n+1}^{(n)}, \ldots, \psi_{n+2m-2}^{(n)})^T \]

and

\[ M_{rs} := (\epsilon \Phi_r^{(n)}, \Phi_s^{(n)})^T, \quad D_{rs} := ((\Phi_r^{(n)})', (\Phi_s^{(n)})^T), \quad 1 \leq r, s \leq 2 \]

define some $(2m - 1) \times (2m - 1)$ submatrices of $M_n^{(n)}$ and $D_n^{(n)}$. Observe that $M_\infty^{(n)} D_\infty^{(n)} = 1$ together with $(D_\infty^{(n)})_{jk} = 0$ for $|j - k| \geq 2m$ implies

\[ T_n = 1 - D_{12}M_{21}. \]
Finally we denote by $K_n(\lambda, \mu) := \sum_{j=0}^{n-1} \psi_j^{(n)}(\lambda)\psi_j^{(n)}(\mu)$ the reproducing kernel. We then have
\[ S_{n,1}(\lambda, \mu) = K_n(\lambda, \mu) + \Phi_1(\lambda)^T D_{12} \epsilon \Phi_2(\mu) - \Phi_1(\lambda)^T G \epsilon \Phi_1(\mu), \]
\[ G := D_{12} M_{22}(1 - D_{21} M_{12})^{-1} D_{21} \]
\[ S_{n/2,4}(\lambda, \mu) = K_n(\lambda, \mu) + \Phi_2(\lambda)^T D_{12} \epsilon \Phi_1(\mu) - \Phi_2(\lambda)^T G \epsilon \Phi_2(\mu), \]
\[ G := -D_{21}(1 - M_{12} D_{21})^{-1} M_{11} D_{12} \]
Since $S_{n,\beta}(\lambda, \mu) = -S_{n,\beta}(\mu, \lambda)$ one has for even $n$
\[ (\epsilon S_{n,1})(\lambda, \mu) = -\int_{\lambda}^{\mu} S_{n,1}(t, \mu) \, dt, \quad (\epsilon S_{n/2,4})(\lambda, \mu) = -\int_{\lambda}^{\mu} S_{n/2,4}(t, \mu) \, dt. \]
Using this representation in the 21-entry of $K_n,\beta$ together with $\det T_n = \det(1 - D_{12} M_{12}) = \det(1 - D_{21} M_{21})$ its straightforward to see that Theorem 3 follows from the following Lemma.

**Lemma 2** Given any compact set $K \subset (a, b)$ there exists a $C > 0$ such that for all $n \geq 2m$ and all $j, k \in \{n - 2m + 1, \ldots, n + 2m - 2\}$ one has
\[ (a) \sup_{x \in K} |\psi_j^{(n)}(\lambda)| \leq \frac{C}{\sqrt{n}}; \quad (b) \quad |(M_{\infty}^{(n)})_{jk}| \leq \frac{C}{n}; \quad (c) \quad |\log \det(T_n)| \leq C. \]

Statements (a) and (b) will be derived from the asymptotics of the orthogonal polynomials in Appendix 4. We now prove statement (c) using Theorem 1.

Consider the functions $V_t$ defined in (2.27) Then, as it was mentioned above (see the proof of Theorem 2) the limiting equilibrium density $\rho_t$ has the form (2.29). Hence, for any $t \in [0, 1]$ $V_t$ satisfies conditions C1-C3 and if we introduce the matrix $T_n(t)$ by the same way as above for the potential $V_t$, then $T_n(0)$ corresponds to the GOE and GSE. Consider the function
\[ L(t) = \log \det T_n(t). \]
To prove that $|L(1)| \leq C$ it is enough to prove that
\[ |L(0)| \leq C, \quad |L(t)| \leq C, \quad t \in [0, 1] \]
(3.7)
The first inequality here follows from the results of [19]. To prove the second inequality we use (3.5) for $V$ replaced by $V_t$. Then we get
\[ L'(t) = n^2 \int \Delta V(\lambda) p_{1,4,t}^{(n/2)}(\lambda) d\lambda + n^2 \int \Delta V(\lambda) p_{1,1,t}^{(n)}(\lambda) d\lambda - 2n^2 \int \Delta V(\lambda) p_{1,2,t}^{(n)}(\lambda) d\lambda. \]
It is easy to see that
\[ \lim_{n \to \infty} p_{1,4,t}^{(n/2)} = \lim_{n \to \infty} p_{1,1,t}^{(n)} = \lim_{n \to \infty} p_{1,2,t}^{(n)} = \rho_t(\lambda) \]
with $\rho_t$ defined by (2.29). Hence, using (1.15), we obtain that the first and the second terms of (1.15) give zero contributions in $L'(t)$, and therefore
\[ L'(t) = 2(r_{n/2,4,t}(\Delta V) + r_{n,1,t}(\Delta V) - r_{n,2,t}(\Delta V)). \]
But, according to Theorem 1, all terms here are bounded uniformly in $n$. Thus, we have proved the second inequality in (3.7) and so statement (c) of Lemma 2.

□
4 Appendix: uniform bounds for \((M_{\infty}^{(n)})_{ij}\)

Set

\[
\delta_n = n^{-2/3 + \kappa}, \quad 0 < \kappa < 1/3. \tag{4.1}
\]

Then, according to [3], we have

\[
\psi_n^{(\pm)}(\lambda) = \frac{\cos nF_n(\lambda)}{(4 - \lambda^2)^{1/4}} (1 + O(n^{-1})), \quad |\lambda| \leq 2 - \delta_n;
\]

\[
\psi_{n-1}^{(\pm)}(\lambda) = \frac{\cos nF_{n-1}(\lambda)}{(4 - \lambda^2)^{1/4}} (1 + O(n^{-1})), \quad |\lambda| \leq 2 - \delta_n;
\]

\[
\psi_n^{(\pm)}(\lambda) = n^{1/6}B_{11}^{(\pm)}Ai\left(\pm n^{2/3}\Phi_{\pm}(\lambda + 2)\right) (1 + O(\|\lambda + 2\|)) + \int \frac{P(\lambda)\sqrt{4 - \lambda^2}d\lambda + \frac{1}{n} \arccos(\lambda/2)}{n} \quad F_{n-1}(\lambda) = F_n(\lambda) - \frac{2}{n} \arccos(\lambda/2) \tag{4.3}
\]

with \(P\) defined in \((1.14)\). Functions \(\Phi_{\pm}(\lambda)\) in \((4.2)\) are analytic in some neighborhood of 0 and such that \(\Phi_{\pm}(\lambda) = a_{\pm} x + O(x^2)\) with some positive \(a_{\pm}\).

Denote

\[
A(\lambda) = \sin nF_n(\lambda) \frac{1}{n}|_{|\lambda| \leq 2 - \delta_n} + \frac{\sin nF_n(2 - \delta_n)}{n} \frac{1}{n}|_{|\lambda - 2| \leq \delta_n} + \frac{\sin nF_n(-2 + \delta_n)}{n} \frac{1}{n}|_{|\lambda + 2| \leq \delta_n} - n^{-1/2} \Psi \left(n^{2/3}\Phi_+(\lambda - 2)\right) - \Psi \left(n^{2/3}\Phi_+(-\delta_n)\right) \frac{1}{n}|_{|\lambda - 2| \leq \delta_n}.
\]

\[
B(\lambda) = \frac{1}{F_n(\lambda)X^{1/4}(\lambda)} \frac{1}{n}|_{|\lambda| \leq 2 - \delta_n} + \frac{B^{(+)\pm}_{11}}{\Phi_+(\lambda - 2)} \frac{1}{n}|_{|\lambda - 2| \leq \delta_n} + \frac{B^{(-)\pm}_{11}}{\Phi_-(\lambda + 2)} \frac{1}{n}|_{|\lambda + 2| \leq \delta_n}
\]

with

\[
\Psi(x) := \int_{-\infty}^{x} Ai(t)dt. \tag{4.5}
\]

**Proposition 1** Under conditions of C1-C3 for any smooth function \(f\) we have uniformly in \([-\delta_n - 2, \delta_n + 2]\]

\[
\epsilon(f\psi_n^{(\pm)})(\lambda) = A(\lambda)B(\lambda)f(\lambda) + \epsilon r_n(\lambda) + O(n^{-1}) + 1|_{|\lambda + 2| \leq \delta_n}O(n^{-5/6}), \tag{4.6}
\]

where

\[
\frac{1}{4} + \frac{2\delta_n + 1}{2-\delta_n} |r_n(\lambda)|d\lambda \leq Cn^{-1/2-3\kappa/4}.
\]

Similar representation is valid for \(\epsilon(f\psi_n^{(\pm)}\psi_{n-1})\) if we replace in \((4.3)\) \(F_n\) by \(F_{n-1}\) and \(B^{(+)\pm}_{11}\) by \(B^{(+)\pm}_{21}\). Moreover, it follows from \((4.6)\) that

\[
|\epsilon(f\psi_n^{(\pm)})(\lambda)| \leq Cn^{-1/2}, \quad |\epsilon(f\psi_{n-1})(\lambda)| \leq Cn^{-1/2}. \tag{4.7}
\]


Proof. We use the following simple relation, valid for any continuous piecewise differentiable functions $A$, $f$, and any piecewise differentiable $B$, if $A(\lambda)B(\lambda)f(\lambda) \to 0$, as $\lambda \to \pm \infty$:

$$
\epsilon(A^\prime B f)(\lambda) = A(\lambda)B(\lambda)f(\lambda) - \epsilon(A B f')(\lambda),
$$

where $(B f)'$ may contain $\delta$-functions at the points of jumps of $B$. By the choice of $A, B$ (cf. (4.1) and (4.2)) $\epsilon(A^\prime B f)$ corresponds to the principal part of $\epsilon(f \psi_n(\lambda))$. The terms $O(n^{-1})$ and $1_{\lambda \pm 2} \leq \delta_n O(n^{-5/6})$ in (4.6) appear because of the integrals of $O(n^{-1})$ in the second line of (4.2) and the terms in the forth line of (4.2) respectively. Hence we need only to prove the bound for $r_n = A(\lambda)(B f)'$. Observe that

$$
\int |r_n(\lambda)| d\lambda \leq \frac{C}{n^{1/2}} \left( \int_{2-\delta_n}^{2+\delta_n} + \int_{-2-\delta_n}^{-2+\delta_n} \right) |(B f)'(\lambda)| d\lambda
$$

where $\psi^{(n)}(\lambda) = f_{0j}(\lambda)\psi^{(n)}(\lambda) + f_{1j}(\lambda)\psi^{(n)}(\lambda)$, where $f_{0j}$ and $f_{1j}$ are polynomials of degree at most $|j|$. Note that since it is known that $a_k^{(n)}$ and $b_k^{(n)}$ for $k - n = o(n)$ are bounded uniformly in $n$, $f_{0j}$ and $f_{1j}$ have coefficients, bounded uniformly in $n$. Hence for our purposes it is enough to estimate

$$
I_1 := (f_{1j}\psi^{(n)}_{n-1} , \epsilon(f_{0k}\psi^{(n)}_{n})) , I_2 := (f_{1j}\psi^{(n)}_{n-1} , \epsilon(f_{1k}\psi^{(n)}_{n-1})) , I_3 := (f_{0j}\psi^{(n)}_{n} , \epsilon(f_{0k}\psi^{(n)}_{n})).
$$

(4.8)

It follows from Proposition $\square$ that

$$
I_1 = I_{11} + I_{12} + I_{13} + (f_{1j}\psi^{(n)}_{n-1} , e_r n) + O(n^{-1}),
$$

where

$$
I_{11} = n^{-1} \int_{2-\delta_n}^{2+\delta_n} f_{0j}(\lambda)f_{1k}(\lambda) \sin nF_n(\lambda) \cos nF_{n-1}(\lambda) d\lambda,
$$

$$
I_{12} = n^{-1/3} B_{11}^{(+)} B_{21}^{(-)} \int_{2-\delta_n}^{2+\delta_n} \left( \Psi n^{2/3} F_+ (\lambda - 2) - \Psi n^{2/3} F_+ (-\delta_n) \right) \frac{Ai n^{2/3} F_+ (\lambda - 2)}{n^{2/3} F_+ (\lambda - 2)} f_{0j}(\lambda)f_{1k}(\lambda) d\lambda,
$$

and $I_{13}$ is the integral similar to $I_{12}$ for the region $|\lambda + 2| \leq \delta_n$. It is easy to see that

$$
I_{12} = B_{11}^{(+)} B_{21}^{(+)} \frac{f_{0j}(\lambda)f_{1k}(\lambda)}{2n(F_+(0))^2} (1 + o(1)),
$$

$$
I_{13} = B_{11}^{(-)} B_{21}^{(-)} \frac{f_{0j}(\lambda)f_{1k}(\lambda)}{2n(F_+(0))^2} (1 + o(1)).
$$

Moreover, using the bound for $r_n$ from (4.6) and (4.7), we get

$$
(f_{1j}\psi^{(n)}_{n-1} , e_r n) = -(\epsilon(f_{1j}\psi^{(n)}_{n-1} , r_n)) \leq \int |\epsilon(f_{1j}\psi^{(n)}_{n-1})| |r_n| d\lambda = O(n^{-1-3\epsilon/4}).
$$
Hence we are left to find the bound for $I_{11}$.

$$I_{11} = (2n)^{-1} \int_{-2+\delta_n}^{2-\delta_n} f_0j(\lambda)f_{1k}(\lambda) \sin n(F_n(\lambda) - F_{n-1}(\lambda)) \frac{d\lambda}{F_n'(\lambda)(4 - \lambda^2)^{1/2}}$$

$+(2n)^{-1} \int_{-2+\delta_n}^{2-\delta_n} f_0j(\lambda)f_{1k}(\lambda) \sin n(F_n(\lambda) + F_{n-1}(\lambda)) \frac{d\lambda}{F_n'(\lambda)(4 - \lambda^2)^{1/2}} = I_1' + I_1''$

By the definition of $F_n$ and $F_{n-1}$ (4.3), we obtain

$$I_1' = \frac{1 + o(1)}{2n} \int_{-2}^{2} \frac{f_0j(\lambda)f_{1k}(\lambda)}{P(\lambda)(4 - \lambda^2)^{1/2}} d\lambda.$$

Moreover, integrating by parts one can get easily that $I_{11}'' = O(n^{-2}\delta^{-3/2}) = O(n^{-1-3\kappa/2})$.

The other two integrals from (4.8) can be estimated similarly.

\[\square\]

5 Appendix: proof of the bounds (2.19)

Let us introduce a function $H$ which we call Hamiltonian to stress the analogy with statistical mechanics.

$$H(\Lambda) = -\sum_{i=1}^{n} (V(\lambda_i) + n^{-1}h(\lambda_i)) + 2 \sum_{1 \leq i < j \leq n} \log |\lambda_i - \lambda_j|, \quad \Lambda = (\lambda_1, \ldots, \lambda_n).$$

It is evident that for any continuous $f(\lambda_1, \ldots, \lambda_k)$

$$\int f(\lambda_1, \ldots, \lambda_k)p^{(n)}_{k,\beta,h}(\lambda_1, \ldots, \lambda_k)d\lambda_1 \ldots d\lambda_k = \frac{\int f(\lambda_1, \ldots, \lambda_k)e^{n\beta H(\Lambda)}d\Lambda}{\int e^{n\beta H(\Lambda)}d\Lambda} = \langle f \rangle_{\beta H}$$

Moreover we introduce the ”approximating” Hamiltonian, depending on a functional parameter $m : \text{supp } m \subset [-2, 2]$

$$H_a(\Lambda; m) = \sum_{i=1}^{n} v_n(\lambda_i; m) + (n - 1)\mathcal{L}[m, m].$$

Here

$$v_n(\lambda; m) = -V(\lambda) - \frac{1}{n}h(\lambda) + 2\frac{n-1}{n}\mathcal{L}(\lambda; m), \quad (5.1)$$

$$\mathcal{L}(\lambda; m) = \int \log |\lambda - \mu|m(\mu)d\mu,$$

$$\mathcal{L}[m, m] = \int \log |\lambda - \mu|^{-1}m(\lambda)m(\mu)d\lambda d\mu.$$

By the Jensen inequality for any two real functions $\mathcal{H}_1(\Lambda), \mathcal{H}_2(\Lambda)$ we have

$$\frac{\int e^{n\beta \mathcal{H}_1(\Lambda)/2}d\Lambda}{\int e^{n\beta \mathcal{H}_2(\Lambda)/2}d\Lambda} \geq e^{n\beta/2(\mathcal{H}_1 - \mathcal{H}_2)_{\beta \mathcal{H}_2}}, \quad \frac{\int e^{n\beta \mathcal{H}_2(\Lambda)/2}d\Lambda}{\int e^{n\beta \mathcal{H}_1(\Lambda)/2}d\Lambda} \geq e^{n\beta/2(\mathcal{H}_2 - \mathcal{H}_1)_{\beta \mathcal{H}_2}}$$

where we denote $\langle \ldots \rangle_{\beta \mathcal{H}_5} = \int (\ldots)e^{n\beta \mathcal{H}_5(\Lambda)/2}d\Lambda/\int e^{n\beta \mathcal{H}_5(\Lambda)/2}d\Lambda$ ($\delta = 1, 2$). Then we get

$$\langle \mathcal{H}_2 - \mathcal{H}_1 \rangle_{\beta \mathcal{H}_2} \leq \langle \mathcal{H}_2 - \mathcal{H}_1 \rangle_{\beta \mathcal{H}_2}.$$
Taking here \( H_1 = H, \ H_2 = H_a \), we obtain
\[
R[m] := \frac{\int (H_a - H)e^{-\beta nH/2}d\Lambda}{(n-1)\int e^{-\beta nH/2}d\Lambda} \leq \frac{\int (H_a - H)e^{-\beta nH_a(\Lambda;m)/2}d\Lambda}{(n-1)\int e^{-\beta nH_a(\Lambda;m)/2}d\Lambda} =: R_a[m], \tag{5.2}
\]

Since \( H \) and \( H_a \) are symmetric, we can rewrite the l.h.s. of (5.2) as
\[
R[m] = \int \log \left| \frac{1}{\lambda - \mu} \right| p_{2,\beta,h}^{(n)}(\lambda,\mu) - p_{1,\beta,h}^{(n)}(\lambda)p_{1,\beta,h}^{(n)}(\mu) \right) d\lambda d\mu + \mathcal{L}[p_{1,\beta,h}^{(n)} - m, p_{1,\beta,h}^{(n)} - m], \tag{5.3}
\]
where \( p_{1,\beta,h}^{(n)} \) and \( p_{2,\beta,h}^{(n)} \) are defined by (1.6) if we replace \( V \) by \( V_h \). To obtain the expression for the r.h.s. of (5.2) we need to replace \( p_{2,\beta}^{(n)}(\lambda) \) and \( p_{2,\beta}^{(n)}(\lambda,\mu) \) in (5.3) by \( p_{1,\beta,h}^{(n,a)}(\lambda;m) \) and \( p_{1,\beta,h}^{(n,a)}(\lambda;m)p_{1,\beta,h}^{(n,a)}(\mu;m) \), correlation functions of the approximating Hamiltonian (5), where
\[
p_{1,\beta,h}^{(n,a)}(\lambda;m) = e^{\beta n v_n(\lambda,m)/2} \left( \int d\lambda e^{\beta n v_n(\lambda,m)/2} \right)^{-1}. \tag{5.4}
\]
This yields:
\[
R_a[m] = \mathcal{L}[p_{1,\beta,h}^{(n,a)} - m, p_{1,\beta,h}^{(n,a)} - m], \tag{5.5}
\]
Now let us choose the function \( m \). Set
\[
m_n(\lambda) = \frac{n}{n-1} \left( \rho(\lambda) + \frac{1}{\beta_n} \nu_n(\lambda) \right) 1_{|\lambda| \leq 2}, \tag{5.6}
\]
where
\[
\nu_n(\lambda) = \frac{\sqrt{4 - \lambda^2}}{\pi} \int_{-2}^{2} d\mu \frac{((log \rho)^{\prime}_n(\mu) + \beta h^{\prime}(\mu)/2)}{(\mu - \lambda)\sqrt{4 - \mu^2}} + \frac{\alpha_n}{\pi \sqrt{4 - \lambda^2}} = \nu_n^{(1)}(\lambda) + \alpha_n v^{(0)}(\lambda), \tag{5.7}
\]
the function \((log \rho)_n(\lambda)\) coincides with \( log \rho(\lambda) \) on the interval \( \sigma_n = [-2 + n^{-1/2}, 2 - n^{-1/2}] \), and \((log \rho)_n(\lambda)\) is a linear function for \( \lambda \in \sigma \setminus \sigma_n \), chosen so that \((log \rho)_n(\lambda)\) has continuous derivative on \( \sigma \). The constant \( \alpha_n \) here is chosen to provide the condition
\[
\int m_n(\lambda)d\lambda = 1 \iff \alpha_n = -\beta - \int \nu_n^{(1)}(\lambda)d\lambda = -\beta - \int_{-2}^{2} \frac{((log \rho)^{\prime}_n(\mu) + \beta h^{\prime}(\mu)/2)\mu d\mu}{\sqrt{4 - \lambda^2}} = \mathcal{O}(n^{1/4}).
\]
Since \( \rho \) has the form (1.12), \( \nu_n^{(1)}(\lambda) \) is a sum of a bounded function which comes from \( P \) and of a negative function which comes from the integral of \((log \sqrt{4 - \lambda^2})^{\prime}\). Hence,
\[
\int |\nu_n^{(1)}(\lambda)|d\lambda \leq C - \int \nu_n^{(1)}(\lambda)d\lambda = \mathcal{O}(n^{1/4}). \tag{5.8}
\]
It is easy to see that \( \nu_n(\lambda) \) is chosen to satisfy the equation
\[
\int_{-2}^{2} \frac{\nu_n(\mu)d\mu}{\lambda - \mu} = (log \rho)^{\prime}_n(\lambda) + \beta h^{\prime}(\lambda)/2.
\]
Therefore
\[
\mathcal{L}(\lambda, \nu_n) = (log \rho)_n(\lambda) + \beta h(\lambda)/2 + r_n(\lambda), \tag{5.9}
\]
where for $|\lambda| \leq 2$ $r_n(\lambda) = C_n$ and $C_n$ is a constant independent of $\lambda$, but depending on $n$. One can find $C_n$ as

$$C_n = \mathcal{L}(0, \nu_n) - (\log \rho)_n(0) - \beta h(0)/2 = \int \log |\nu_n(\lambda)|d\lambda - (\log \rho)_n(0) - \beta h(0)/2 = O(n^{1/4})$$

(5.10)

(here we used that $\nu_n^{(1)}$ for $|\lambda| \leq 1$ is bounded uniformly in $n$). Hence,

$$\beta n\nu_n(\lambda, m_n)/2 = \beta n \left( \mathcal{L}(\lambda, \rho) - V(\lambda) \right)/2 + (\log \rho)_n(\lambda) + C_n, \quad |\lambda| \leq 2$$

(5.11)

Let us estimate \(d/d\lambda \mathcal{L}(\lambda, \nu_n^{(1)})\) for $\lambda > 2$. From (5.7) we get

$$\left| \frac{d}{d\lambda} \mathcal{L}(\lambda, \nu_n^{(1)}) \right| = \left| \int \frac{\nu_n^{(1)}(\lambda_1)d\lambda_1}{\lambda - \lambda_1} \right| = \frac{1}{\pi} \int_{-2}^{2} d\lambda_1 \int_{-2}^{2} d\mu \frac{\sqrt{4 - \lambda_1^2}}{(\lambda - \lambda_1)(\mu - \lambda_1)\sqrt{4 - \mu^2}} \left| (\log \rho)'_n(\mu) + \beta h'(\mu)/2 \right| \leq \sup \{(\log \rho)'_n(\mu) + \beta h'(\mu)/2\} \int_{-2}^{2} \frac{d\mu}{\lambda - \mu} \left( 1 + \frac{\sqrt{\lambda^2 - 4}}{\lambda - \mu} \right) (4 - \mu^2)^{-1/2} \mu \leq n^{1/4} C_1.$$

Here we used the identities (valid for $\lambda \notin [-2, 2]$)

$$\frac{1}{\pi} \int_{-2}^{2} d\lambda_1 \frac{\sqrt{4 - \lambda_1^2}}{(\lambda - \lambda_1)(\mu - \lambda_1)} = 1 - \frac{\sqrt{\lambda^2 - 4}}{\lambda - \mu}, \quad \frac{1}{\pi} \int_{-2}^{2} \frac{(4 - \mu^2)^{-1/2}d\mu}{\lambda - \mu} = \frac{1}{\sqrt{\lambda^2 - 4}},$$

and the bound $\sup |(\log \rho)'_n(\mu)| \leq C n^{1/4}$. Therefore

$$\mathcal{L}(\lambda, \nu_n) - \mathcal{L}(2, \nu_n) \leq C_1 n^{1/4} |\lambda - 2| + C_2 n^{1/4} |\lambda - 2|^{1/2},$$

where the second term in the l.h.s. comes from $\alpha_n(\nu(0))$ in (5.6). Moreover, since under conditions C2, C3 there exists $C^*$ such that

$$\mathcal{L}(\lambda, \rho) - V(\lambda) = C^*, \quad |\lambda| \leq 2, \quad \mathcal{L}(\lambda, \rho) - V(\lambda) - C^* \leq -C_0 |\lambda^2 - 4|^{3/2}, \quad |\lambda| \geq 2,$$

we obtain

$$n\beta \nu_n(\lambda, m_n)/2 - n\beta C^*/2 - C_n = (\log \rho)_n(1)_{|\lambda| \leq 2} + \tilde{r}_n(\lambda)(1)_{|\lambda| > 2}, \quad \tilde{r}_n(\lambda) \leq C n^{1/4} |\lambda - 2|^{1/2} - nC_0 |\lambda^2 - 4|^{3/2}.$$

Then we have

$$\int_{\mathbb{R}\setminus]\sigma} e^{n\beta \nu_n(\lambda, m_n)/2 - n\beta C^*/2 - C_n} d\lambda \leq 2 \int_{0}^{\infty} e^{C_n^{1/4} x^{1/2} - C_0 x^{3/2}} dx \leq C n^{-2/3}.$$

(5.12)

The last bound can be obtained by splitting the interval $[0, \infty)$ in two parts: $[0, n^{-2/3})$ and $[n^{-2/3}, \infty)$. Then in the first interval we used the fact that $\sup \{C n^{1/4} x^{1/2} - n x^{3/2}\} \leq c$ and in the second interval we used that this function is negative, its derivative is a negative decreasing function, bounded from above by $(-C n^{2/3})$.

For $|\lambda| \leq 2$, since $|e^{(\log \rho)_n} - \rho| \leq C n^{-1/4} 1_{\sigma\setminus\sigma_n}$ we get

$$\int_{-2}^{2} e^{(\log \rho)_n} d\lambda = \int_{\sigma_n} \rho(\lambda) d\lambda + O(n^{-3/4}) = 1 + O(n^{-3/4}).$$
Hence, using the above inequality and (5.12), we obtain
\[ p_{1,\beta,h}^{(n,a)}(\lambda; m_n) = \frac{\rho(\lambda) + O(n^{-1/4}) 1_{\sigma|\sigma_n}}{1 + O(n^{-2/3})} = \rho(\lambda) + O(n^{-1/4}) 1_{\sigma|\sigma_n} + O(n^{-2/3}). \]

Thus
\[ p_{1,\beta,h}^{(n,a)}(\lambda; m_n) - m_n = -\frac{\beta}{n} \nu_n(\lambda) + O(n^{-1/4}) 1_{\sigma|\sigma_n} + O(n^{-2/3}) \]
and
\[ \mathcal{L}[p_{1,\beta,h}^{(n,a)}(\lambda; m_n) - m_n, p_{1,\beta,h}^{(n,a)}(\lambda; m_n) - m_n] \leq C \left( n^{-2} \mathcal{L}[\nu_n, \nu_n] + n^{-4/3} + n^{-3/2} \log n \right). \]

Moreover, using (5.9) and (5.8), we write
\[ \mathcal{L}[\nu_n, \nu_n] = -\int \left( (\log \rho)_n(\lambda) + \beta h(\lambda)/2 + C_n \right) \nu_n(\lambda) d\lambda \leq C n^{1/2}. \quad (5.13) \]

Finally we get
\[ \mathcal{L}[p_{1,\beta,h}^{(n,a)}(\lambda; m_n) - m_n, p_{1,\beta,h}^{(n,a)}(\lambda; m_n) - m_n] \leq C n^{-4/3}. \]

The inequality combined with (5.2) gives us
\[ \int \log \frac{1}{|\lambda - \mu|} \left( p_{2,\beta,h}^{(n)}(\lambda, \mu) - p_{1,\beta,h}^{(n)}(\lambda) p_{1,\beta,h}^{(n)}(\mu) \right) d\lambda d\mu \]
\[ + \mathcal{L}[p_{1,\beta,h}^{(n,a)} - m_n, p_{1,\beta,h}^{(n,a)} - m_n] \leq C n^{-4/3}. \quad (5.14) \]

Let us prove that
\[ \int \log \frac{1}{|\lambda - \mu|} \left( p_{2,\beta,h}^{(n)}(\lambda, \mu) - p_{1,\beta,h}^{(n)}(\lambda) p_{1,\beta,h}^{(n)}(\mu) \right) d\lambda d\mu \geq -C \log n/n. \quad (5.15) \]

Introduce the function
\[ l_n(\lambda) = \log |\lambda|^{-1} 1_{|\lambda| > n^{-\gamma}} + (\log n^7 + n^7 (n^{-7} - |\lambda|)) 1_{|\lambda| < n^{-\gamma}}. \]

It is easy to check that for any \( k \neq 0 \) the Fourier transform \( \hat{f}(k) \geq 0 \) and hence, for any positive operator \( K : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R}) \) and such that \( \int K(\lambda, \mu) d\lambda d\mu = 0 \), we have
\[ \int l_n(\lambda - \mu) K(\lambda, \mu) d\lambda d\mu \geq 0 \quad (5.16) \]

From (1.6) it is easy to obtain that for any \(|x| \leq n^{-3}\)
\[ |p_{1,\beta,h}^{(n)}(\lambda + x) - p_{1,\beta,h}^{(n)}(\lambda)| \leq C n^{-1} (\sup |V'| + \sup |h'|), \]
therefore
\[ 1 \geq \int p_{1,\beta,h}^{(n)}(\lambda) d\lambda \geq (1 - C n^{-1} (\sup |V'| + n^{-1} \sup |h'|)) \max p_{1,\beta,h}^{(n)}(\lambda) n^{-3}. \]

Hence
\[ \max p_{1,\beta,h}^{(n)}(\lambda) \leq n^3 (1 + C n^{-1} (\sup |V'| + n^{-1} \sup |h'|)). \]

Similarly
\[ \max p_{2,\beta,h}^{(n)}(\lambda, \mu) \leq n^6 (1 + C n^{-1} (\sup |V'| + n^{-1} \sup |h'|)). \]
Then, the above bounds and the inequality

\[ \int | \log \frac{1}{|\lambda - \mu|} - l_n(\lambda - \mu)| d\lambda \leq C n^{-7} \log n, \]

imply

\[ \int \log \frac{1}{|\lambda - \mu|} \left( p_{2,\beta,h}^{(n)}(\lambda, \mu) - p_{1,\beta,h}^{(n)}(\lambda) p_{1,\beta,h}^{(n)}(\mu) \right) d\lambda d\mu \]

\[ \geq \int l_n(\lambda - \mu) \left( p_{2,\beta,h}^{(n)}(\lambda, \mu) - p_{1,\beta,h}^{(n)}(\lambda) p_{1,\beta,h}^{(n)}(\mu) \right) d\lambda d\mu - O(n^{-1} \log n) \]

\[ \geq \int l_n(\lambda - \mu) \left( p_{2,\beta,h}^{(n)}(\lambda, \mu) - \frac{1}{n-1} p_{1,\beta,h}^{(n)}(\lambda) p_{1,\beta,h}^{(n)}(\mu) \right) d\lambda d\mu - O(n^{-1} \log n) \]

\[ = \frac{1}{n(n-1)} \int l_n(\lambda - \mu) k_n(\lambda, \mu) d\lambda d\mu - \frac{1}{n-1} \int l_n(0) p_{1,\beta,h}^{(n)}(\lambda) d\lambda - O(n^{-1} \log n), \]

where the kernel \( k_n \) is defined by (5.5). Since \( k_n \) is positively definite, and \( \int k(\lambda, \mu) d\lambda d\mu = 0 \), we can use (5.16), and taking into account that \( l_n(0) = O(\log n) \), obtain (5.15).

Then

\[ \left( L^{1/2} [ p_{1,\beta,h}^{(n)}(\lambda - \rho, p_{1,\beta,h}^{(n)} - \rho) - L^{1/2} [ m_n - \rho, m_n - \rho] \right)^2 \leq L [ p_{1,\beta,h}^{(n)} - m_n, p_{1,\beta,h}^{(n)} - m_n] \leq C \log n / n \]

And since it follows from (5.6) and (5.13) that

\[ L [ m_n - \rho, m_n - \rho] \leq C n^{-4/3}, \]

we have

\[ L [ p_{1,\beta,h}^{(n)} - \rho, p_{1,\beta,h}^{(n)} - \rho] \leq C n^{-1} \log n. \]  (5.17)

For any \( \Im z \neq y \), taking the Fourier transforms \( \hat{p}_{1,\beta,h}^{(n)} \) and \( \hat{\rho} \) of the functions \( p_{1,\beta,h}^{(n)} \) and \( \rho \), we get

\[ \left| \int \frac{p_{1,\beta,h}^{(n)}(\lambda) - \rho(\lambda)}{\lambda - z} d\lambda \right| \leq 2 \int_0^\infty | p_{1,\beta,h}^{(n)}(k) - \hat{\rho}(k) | e^{-|k||y|} dk \]

\[ \leq 2 \left( \int |k| e^{-|k||y|} dk \right)^{1/2} \left( \int |p_{1,\beta,h}^{(n)}(k) - \hat{\rho}(k)|^2 dk \right)^{1/2} = 2 |y|^{-1} L^{1/2} [ p_{1,\beta,h}^{(n)} - \rho, p_{1,\beta,h}^{(n)} - \rho]. \]

Then (5.17) yields the second and the third bounds of (5.19). The first bound follows from Lemma 1.

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