Components of maximal dimension in the Noether-Lefschetz locus for Beilinson-Hodge cycles on open surfaces

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Introduction

In the moduli space $M$ of smooth hypersurfaces of degree $d$ in $\mathbb{P}^3$ over $\mathbb{C}$, the locus of those surfaces that possess curves which are not complete intersections of the given surface with another surface is called the Noether-Lefschetz locus and denoted by $M_{NL}$. One can show that $M_{NL}$ is the union of a countable number of closed algebraic subsets of $M$. The classical theorem of Noether-Lefschetz affirms that every component of $M_{NL}$ has positive codimension in $M$ when $d \geq 4$. Note that the theorem is false if $d = 3$ since a smooth cubic surface has the Picard number 7. Since the infinitesimal method in Hodge theory was introduced in [CGGH] as a powerful tool to study $M_{NL}$, fascinating results have been obtained concerning irreducible components of $M_{NL}$. First we have the following.

Theorem 0.1 ([G1]) For every irreducible component $T$ of $M_{NL}$, $\text{codim}(T) \geq d - 3$.

The basic idea of the proof of the result is to translate the problem in the language of the infinitesimal variation of Hodge structures on a family of hypersurfaces. Then, by the Poincaré residue representation of the cohomology of a hypersurface, the result follows from the duality theorem for the Jacobian ring associated to a hypersurface. We note that the inequality is the best possible since the family of hypersurfaces of degree $d \geq 3$ containing a line has codimension exactly $d - 3$. M. Green [G2] and C. Voisin [V] has shown the following striking theorem.

Theorem 0.2 If $d \geq 5$, the only irreducible component of $M_{NL}$ having codimension $d - 3$ is the family of surfaces of degree $d$ containing a line.

In this paper we study an analog of the above problem in the context of Beilinson’s Hodge conjecture. For a quasi-projective smooth variety $U$ over $\mathbb{C}$, the space of Beilinson-Hodge cycles is defined to be

$$F^0H^q(U, \mathbb{Q}(q)) := H^q(U, \mathbb{Q}(q)) \cap F^qH^q(U, \mathbb{C})$$

where $H^q(U, \mathbb{Q}(q))$ is the singular cohomology with coefficient $\mathbb{Q}(q) = (2\pi \sqrt{-1})^q\mathbb{Q}$ and $F^\bullet$ denotes the Hodge filtration of the mixed Hodge structure on the singular cohomology.
defined by Deligne [D]. Beilinson’s conjecture claims the surjectivity of the regulator map
(cf. [Bl] and [Sch])
\[ \text{reg}^q_U : CH^q(U, q) \otimes \mathbb{Q} \to F^0 H^q(U, \mathbb{Q}(q)) \]
where \( CH^q(U, q) \) is Bloch’s higher Chow group. Taking a smooth compactification \( U \subset X \) with \( Z = X \setminus U \), a simple normal crossing divisor on \( X \), we have the following formula for the value of \( \text{reg}^q_U \) on decomposable elements in \( CH^q(U, q) \):
\[ \text{reg}^q_U (\{g_1, \ldots, g_q\}) = d \log g_1 \wedge \cdots \wedge d \log g_q \in H^0(X, \Omega^q_X(\log Z)) = F^0 H^q(U, \mathbb{C}), \]
where \( \{g_1, \ldots, g_q\} \in CH^q(U, q) \) is the products of \( g_j \in CH^1(U, 1) = \Gamma(U, \mathcal{O}_{Z_{tor}}) \). Beilinson’s conjecture is an analog of the Hodge conjecture which claims the surjectivity of cycle class maps from Chow group to space of Hodge cycles on projective smooth varieties. The conjecture is known to hold in case \( q = 1 \) (cf. [J], Th.5.1.3) but open in general in case \( q \geq 2 \).

The main subject of study in this paper is the Noether-Lefschetz locus for Beilinson-Hodge cycles on the complement of the union of a normal crossing divisor in a surface in \( \mathbb{P}^3 \). Let \( X, Y_1, \ldots, Y_s \subset \mathbb{P}^3 \) be smooth surfaces intersecting transversally and put
\[ Z = \bigcup_{1 \leq j \leq s} Z_j \text{ with } Z_j = X \cap Y_j, \quad U = X \setminus Z. \quad (0-1) \]
Let \( H^2(U, \mathbb{Q}(2))_{\text{triv}} \) be the image of the natural restriction map
\[ H^2(\mathbb{P}^3 \setminus \bigcup_{1 \leq j \leq s} Y_j, \mathbb{Q}(2)) \to H^2(U, \mathbb{Q}(2)). \]
One can show ([AS2], Lem.(2-1))
\[ H^2(U, \mathbb{Q}(2))_{\text{triv}} = \text{reg}^2_U(\text{CH}^2(U, 2)_{\text{dec}}), \]
where \( CH^2(U, 2)_{\text{dec}} \subset CH^2(U, 2) \otimes \mathbb{Q} \) is the so-called decomposable part, the subspace generated by the image of the product map \( CH^1(U, 1) \otimes CH^1(U, 1) \to CH^2(U, 2) \). It implies that
\[ H^2(U, \mathbb{Q}(2))_{\text{triv}} \subset \text{Im}(\text{reg}^2_U) \subset F^0 H^2(U, \mathbb{Q}(2)). \]
We define \( F^0 H^2(U, \mathbb{Q}(2))_{\text{prim}} := F^0 H^2(U, \mathbb{Q}(2))/H^2(U, \mathbb{Q}(2))_{\text{triv}} \) called the space of primitive Beilinson-Hodge cycles.

Now fix integers \( d \geq 1 \) and \( e_j \geq 1 \) with \( 1 \leq j \leq s \). Let \( M \) be the moduli space of sets of hypersurfaces \( (X, Y_1, \ldots, Y_s) \) of degree \( (d, e_1, \ldots, e_s) \) which intersect transversally. Let \( (X, Y_1, \ldots, Y_s) \) be the universal family over \( M \) and put
\[ Z = X \cap (\bigcup_{1 \leq j \leq s} Y_j), \quad U = X \setminus Z. \]
For \( t \in M \) let \( U_t \subset X_t \supset Z_t \) be the fibers of \( U \subset X \supset Z \).

**Definition 0.3**  The Noether-Lefschetz locus for Beilinson-Hodge cycles on \( U/M \) is
\[ M_{NL} = \{ t \in M | F^0 H^2(U_t, \mathbb{Q}(2))_{\text{prim}} \neq 0 \} \]
The analogy with the classical Noether-Lefschetz locus is explained as follows. Instead of the map
\[ H^2(\mathbb{P}^3 \setminus \bigcup_{1 \leq j \leq s} Y_j, \mathbb{Q}(2)) \to H^2(U, \mathbb{Q}(2)) \cap F^2 H^2(U, \mathbb{C}) \]
we consider
\[ H^2(\mathbb{P}^3, \mathbb{Q}(1)) \to H^2(X, \mathbb{Q}(1)) \cap F^1 H^2(X, \mathbb{C}). \]
By noting that the space on the left hand side is generated by the cohomology class of a hyperplane, and that that on the right hand side is identified with \( \text{Pic}(X) \otimes \mathbb{Q} \), the space defined in the same way as Def. 0.3 is nothing but the classical Noether-Lefschetz locus.

One can show as before that \( M_{NL} \) is the union of a countable number of closed analytic subsets. By the analogy a series of problems on \( M_{NL} \) arise, the problems to show the counterparts of Th. 0.1 and Th. 0.2 in the new context. In [AS2] the following result is shown.

**Theorem 0.4** Assume \( d \geq 4 \). For every irreducible component \( T \) of \( M_{NL} \),
\[ \text{codim}_M(T) \geq d + \min\{d, e_1, \ldots, e_s\} - 2. \]

We should note that the estimate in Th. 0.4 is far from being optimal to the contrary to the case of Th. 0.1. It is observed in the main theorem of this paper (Th. 0.4 below) that the optimal estimate in some case is given by a quadratic polynomial in \( d \). The basic strategy of the proof of Th. 0.4 is the same as that of Th. 0.1. A new input is the theory of generalized Jacobian rings developed in [AS1], which give an algebraic description of the cohomology of the open surface \( U \). In particular the duality theorem for such rings plays a crucial role.

In order to state the main result of this paper, which is considered a counterpart of Th. 0.2 we need restrict ourselves to the special case that \( s = 3 \) and \( e_j = 1 \) for \( 1 \leq j \leq 3 \).

Let \( P = \mathbb{C}[z_0, z_1, z_2, z_3] \) be the homogeneous coordinate ring of \( \mathbb{P}^3 \). For the rest of the paper we let \( M \) be the moduli space of hypersurfaces of degree \( d \) in \( \mathbb{P}^3 \) which transversally intersects \( Y = \bigcup_{1 \leq j \leq 3} Y_j \) with \( Y_j = \{z_j = 0\} \subset \mathbb{P}^3 \). Let \( \mathcal{X}/M \) be the universal family and \( X_t \) be its fiber over \( t \in M \) and put \( U_t = X_t \setminus (X_t \cap Y) \). Let \( M_{NL} \subset M \) be defined as in Def. 0.3. In order to determine the irreducible components of \( M_{NL} \) of maximal dimension, we need introduce some notations. For an integer \( l > 0 \) let \( P^l \subset P \) be the subspace of homogeneous polynomials of degree \( l \).

**Definition 0.5** For a pair \((p, q)\) of non-negative coprime integers such that \( d = r(p + q) \) with \( r \in \mathbb{Z} \), and \( \underline{c} = [c_{1}]_{1 \leq \nu \leq r} = [c_{1} : \cdots : c_{r}] \in \mathbb{P}^{r}(\mathbb{C}) \), and \( \sigma \in \mathfrak{S}_{3} \), the permutation group on \((1, 2, 3)\), we let \( T_{(p, q)}^{\sigma}(\underline{c}) \subset M \) be the subset of those surfaces defined by an equation
\[ F = wA + \prod_{1 \leq \nu \leq r} (cz_{\sigma(1)}^{p+q} - \nu z_{\sigma(2)}^{p} \cdot z_{\sigma(3)}^{q}) \text{ for some } w \in P^{1}, A \in P^{d-1}, c \in \mathbb{C}^{*}. \]

We will see the following facts (cf. § 1):

1. \( T_{(p, q)}^{\sigma}(\underline{c}) \) is smooth irreducible and \( \text{codim}_M(T_{(p, q)}^{\sigma}(\underline{c})) = \binom{d+2}{2} - 5. \)

2. If \( c_{\nu} \) is a root of unity for \( 1 \leq \nu \leq r \), \( T_{(p, q)}^{\sigma}(\underline{c}) \subset M_{NL} \).
We now state the main theorem in this paper.

**Theorem 0.6** Assume $d \geq 4$.

1. For every irreducible component $T \subset M_{NL}$, $\text{codim}(T) \geq \binom{d+2}{2} - 5$.

2. The equality holds if and only if $T = T_{(p,q)}^\sigma(\mathfrak{c})$ for some $\sigma$, $(p,q)$ and $\mathfrak{c} = [c_\nu]_{1 \leq \nu \leq r}$ such that $c_\nu$ is a root of unity for $1 \leq \forall \nu \leq r$.

3. If $X$ is a general member of $T_{(p,q)}^\sigma(\mathfrak{c})$, $\text{reg}^2_U$ is surjective so that Beilinson’s Hodge conjecture holds for $U = X \setminus (X \cap Y)$.

A key to the proof of Th.0.6 is a result by Otwinowska ([Ot], Th.2) on the Hilbert function of graded algebras of dimension 0.

Finally we discuss an implication of Th.0.6 on the injectivity of the regulator map. Let $X$ be a member of $M$. We are interested in the regulator map to Deligne cohomology

$$\rho_X : CH^2(X,1) \otimes \mathbb{Q} \to H^3_D(X,\mathbb{Q}(2)),$$

where $CH^2(X,1)$ is Bloch’s higher Chow group defined to be the cohomology of the complex

$$K_2(C(X)) \xrightarrow{\partial_{\text{tame}}} \bigoplus_{C \subset X} C(C)^* \xrightarrow{\partial_{\text{div}}} \bigoplus_{x \in X} \mathbb{Z},$$

where the sum on the middle term ranges over all irreducible curves on $X$ and that on the right hand side over all closed points of $X$. The map $\partial_{\text{tame}}$ is the so-called tame symbol and $\partial_{\text{div}}$ is the sum of divisors of rational functions on curves. We have the localization exact sequence

$$CH^2(U,2) \to CH^1(Z,1) \to CH^2(X,1),$$

where

$$CH^1(Z,1) = \text{Ker} \left( \bigoplus_{1 \leq i \leq 3} C(Z_i)^* \xrightarrow{\partial_{\text{div}}} \bigoplus_{x \in Z} \mathbb{Z} \right) \quad \text{with} \ Z_i = X \cap Y_i.$$

By [AS2], Th.(6-1) we get the following.

**Theorem 0.7** For $t \in M \setminus M_{NL}$, $\rho_{X_t}$ is injective on the subspace

$$\Sigma_t := \text{Im}(CH^1(Z_t,1) \to CH^2(X_t,1)) \otimes \mathbb{Q} \subset CH^2(X_t,1) \otimes \mathbb{Q}.$$

In §6 we show there exists $t \in M \setminus M_{NL}$ such that $\Sigma_t \neq 0$ so that Th.0.7 has a non-trivial implication on the injectivity of $\rho_{X_t}$. For this we need introduce some special locus in the moduli space $M$.

**Definition 0.8** Let $T_{12} \subset M$ be the locus of those $X$ defined by an equation

$$F = wA + z_1z_2B + c_1z_1^d + c_2z_2^d$$

for some $w \in P^1$, $A \in P^{d-1}$, $B \in P^{d-2}$, $c_1, c_2 \in \mathbb{C}^*$.

We define $T_{23}$ (resp. $T_{31}$) similarly by replacing $(z_1, z_2)$ by $(z_2, z_3)$ (resp. $(z_3, z_1)$).
Lemma 1.3

Proof Left to the readers as an easy exercise.

It is easy to check \( c_{12}(X) \in CH^1(Z, 1) \). For \( X \) in \( T_{23} \) (resp. \( T_{31} \)) we define an element \( c_{23}(X) \) (resp. \( c_{31}(X) \)) in \( CH^1(Z, 1) \) by the same way. Let \([c_{ij}(X)] \in CH^2(X, 1)\) be the image of \( c_{ij}(X) \in CH^1(Z, 1) \) for \((i, j) = (1, 2)\) or \((2, 3)\) or \((3, 1)\).

Theorem 0.9

(1) If \( d \geq 4 \), \( T_{12} \not\in M_{NL} \) and \( \rho_{X_i}([c_{12}(X_t)]) \neq 0 \) for \( \forall t \in T_{12} \setminus M_{NL} \).

(2) If \( d \geq 6 \), \( T_{12} \cap T_{23} \not\in M_{NL} \) and \( \rho_{X_i}([c_{12}(X_t)]) \), \( \rho_{X_i}([c_{23}(X_t)]) \) are linearly independent for \( \forall t \in (T_{12} \cap T_{23}) \setminus M_{NL} \).

(3) If \( d \geq 10 \), \( T_{12} \cap T_{23} \cap T_{31} \not\in M_{NL} \) and \( \rho_{X_i}([c_{12}(X_t)]) \), \( \rho_{X_i}([c_{23}(X_t)]) \), \( \rho_{X_i}([c_{31}(X_t)]) \) are linearly independent for \( \forall t \in (T_{12} \cap T_{23} \cap T_{31}) \setminus M_{NL} \).

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1 Component of \( M_{NL} \)

Let the notation be as in Thm 0.6. In what follows we fix 0 \( \in M \) and let \( X \) be the fibers over 0 \( \in M \) of the universal family \( \mathcal{X}/M \) and write

\[
U = X \setminus Z, \quad Z = \bigcup_{1 \leq j \leq 3} Z_j \quad \text{with} \quad Z_j = X \cap Y_j.
\]

Definition 1.1 We put

\[
\alpha_U = \left\{ \frac{z_2}{z_1}, \frac{z_3}{z_1} \right\} \in CH^2(U, 2),
\]

\[
\omega_U = d \log \frac{z_2}{z_1} \wedge d \log \frac{z_3}{z_1} \in H^0(X, \Omega^2_X(\log Z)),
\]

where \( z_j/z_i \) is viewed as an element of \( CH^1(U, 1) = \Gamma(U, \mathcal{O}^*_{U_{23}}) \). Note that \( \omega_U = \text{reg}^2_U(\alpha_U) \), under the identification \( F^2H^2(U, \mathbb{C}) \cong H^0(X, \Omega^3_X(\log Z)) \).

Lemma 1.2

(1) \( CH^2(U, 2)_{\text{dec}} \) is generated by \( \alpha_U \) and \( \{ c, z_j/z_i \} \) with \( c \in \mathbb{C} \) and \( 1 \leq i, j \leq 3 \).

(2) \( H^2(U, \mathbb{Q}(2))_{\text{triv}} = \mathbb{Q} \cdot \omega_U \).

Proof The first assertion follows from [AS2] Lem.(2-1). The second assertion follows from the fact \( H^2(U, \mathbb{Q}(2))_{\text{triv}} = \text{reg}^2_U(CH^2(U, 2)_{\text{dec}}) \) by loc.cite.

Let \( T_{(p, q)}^\sigma(\mathbb{C}) \subset M \) be as in Def 1.3.

Lemma 1.3 \( T_{(p, q)}^\sigma(\mathbb{C}) \) is smooth irreducible and \( \text{codim}_M(T_{(p, q)}^\sigma(\mathbb{C})) = \left( \frac{d+2}{2} \right) - 5 \).

Proof Left to the readers as an easy exercise.
Assume $0 \in T_{(p,q)}^\sigma(\zeta)$ and that $X$ is defined by such an equation as in Def.0.5:

$$F = wA + \prod_{1 \leq \nu \leq r} (cz_{\sigma(1)}^p - c_{\nu} z_{\sigma(2)}^p z_{\sigma(3)}^q).$$

We note that $w \not\in \sum_{1 \leq j \leq 3} \mathbb{C} \cdot z_j$ by the assumption that $X$ transversally intersects $Y$.

**Definition 1.4** We define

$$\Sigma(U) := \mathbb{C} \cdot \omega_U \oplus \mathbb{C} \cdot \xi_U \subset H^0(X, \Omega_X^2(\log Z)) = F^0 H^2(U, \mathbb{C}),$$

$$\xi_U = d \log \frac{z_{\sigma(2)}^p z_{\sigma(3)}^q}{z_{\sigma(1)}^{p+q}} \wedge d \log \frac{w}{z_{\sigma(1)}} \in H^0(X, \Omega_X^2(\log Z)).$$

We note that $\xi_U$ is apparently holomorphic only on $U \setminus W$, where $W = U \cap \{w = 0\}$ while it is easy to see that its residue along any irreducible component of $W$ is zero.

Rewriting the equation of $X$ as

$$wA + \prod_{1 \leq \nu \leq r} (cz_{\sigma(1)}^p - c_{\nu} z_{\sigma(2)}^p z_{\sigma(3)}^q) = wA + \prod_{\mu \in I} (cz_{\sigma(1)}^p - c_{\mu} z_{\sigma(2)}^p z_{\sigma(3)}^q)^{e_\mu}, \ (e_\mu \geq 1) \quad (1-1)$$

where $c_\mu \neq c_{\mu'}$ if $\mu \neq \mu' \in I$, $W$ is the disjoint sum of the following smooth irreducible components for $\mu \in I$:

$$W_\mu = U \cap \{w = cz_{\sigma(1)}^p - c_{\mu} z_{\sigma(2)}^p z_{\sigma(3)}^q = 0\}.$$

We consider the condition:

$$\zeta = [c_\nu']_{1 \leq \nu \leq r} \text{ such that } c_\nu' \text{ is a root of unity for } 1 \leq \forall \nu \leq r. \quad (1-2)$$

**Proposition 1.5**

(1) $\Sigma(U) \cap H^2(U, \mathbb{Q}(2)) = \begin{cases} \mathbb{Q} \cdot \omega_U \oplus \mathbb{Q} \cdot \xi_U & \text{if } (1-2) \text{ holds} \\ \mathbb{Q} \cdot \omega_U \text{ otherwise} \end{cases}$

(2) $\Sigma(U) \cap H^2(U, \mathbb{Q}(2)) \subset \text{Im}(\text{reg}^2_U).$

**Corollary 1.6** If the condition $[1-2]$ holds, $T_{(p,q)}^\sigma(\zeta) \subset M_{NL}.$

The corollary follows immediately from Pr.1.5 and Lem.1.2.

Noting $\omega_U \in H^2(U, \mathbb{Q}(2))$, Pr.1.5 follows from the following two claims.

**Claim 1** $\Sigma(U) \cap H^2(U, \mathbb{Q}(2)) \subset \mathbb{Q} \cdot \omega_U \oplus \mathbb{Q} \cdot \xi_U.$

**Claim 2** $\xi_U \in H^2(U, \mathbb{Q}(2))$ if and only if $[1-2]$ holds, in which case $\xi_U \in \text{Im}(\text{reg}^2_U).$
We prove Claim 1. For simplicity we assume that $\sigma \in \mathcal{S}_3$ is the identity. The following argument works in general case as well. Define

$$Z_{ij} = X \cap \{z_i = z_j = 0\} \quad (1 \leq i \neq j \leq 3),$$

$$Z_i = X \cap \{z_i = 0\}, \quad V_i = Z_i \cap (\bigcup_{1 \leq j \neq i \leq 3} Z_j).$$

We consider the composite map of the successive residue maps

$$\delta_{ij} : H^0(X, \Omega^2_X(\log Z)) \xrightarrow{\text{Res}_{Z_{ij}}} H^0(Z_i, \Omega_{Z_i}(\log V_i)) \xrightarrow{\text{Res}_{Z_{ij}}^\circ} H^0(Z_{ij}, \mathcal{O}_{Z_{ij}}) \cong \Psi \otimes \mathbb{C},$$

where $\Psi = H^0(Z_{ij}, \mathbb{Q}) = \bigoplus_{x \in Z_{ij}} \mathbb{Q}$. For $\phi \in H^0(X, \Omega^2_X(\log Z))$, $\phi \in H^2(U, \mathbb{Q}(2))$ implies $\delta_{ij}(\phi) \in \Psi \subset \Psi \otimes \mathbb{C}$. Now an easy residue calculation shows

$$\delta_{12}(\omega_U) = \delta_{23}(\omega_U) = \delta_{31}(\omega_U) = -\underline{u},$$

$$\delta_{12}(\xi_U) = -p \cdot \underline{u}, \quad \delta_{31}(\xi_U) = q \cdot \underline{u}, \quad \delta_{23}(\xi_U) = 0,$$

where $\underline{u} = (1, 1, \ldots, 1) \in \Psi$. Thus, if $\phi = a \omega_U + b \xi_U \in H^2(U, \mathbb{Q}(2))$ with $a, b \in \mathbb{C}$, it implies $-(a + bp), -a, -(a - bq) \in \mathbb{Q}$. Noting that at least one of $p$ and $q$ is not zero, it implies $a, b \in \mathbb{Q}$ and the proof of Claim 1 is complete.

Next we prove Claim 2. Consider

$$\beta = \left\{ \frac{z_p^p z_q^q}{z_{p+q}^{p+q}}, \frac{w}{cz \sigma(1)} \right\} \in CH^2(U', 2) \quad (U' := U \setminus W)$$

We have the commutative diagram

$$\begin{array}{ccc}
CH^2(U, 2) \otimes \mathbb{Q} & \xrightarrow{\text{reg}^2_{D,U}} & F^0H^2(U, \mathbb{Q}(2)) & \xrightarrow{t_1} & F^2H^2(U, \mathbb{C}) \\
\downarrow & & \downarrow t_1 & & \\
CH^2(U', 2) \otimes \mathbb{Q} & \xrightarrow{\text{reg}^2_{U'}} & F^0H^2(U', \mathbb{Q}(2)) & \xrightarrow{t_2} & F^2H^2(U', \mathbb{C})
\end{array}$$

and we have $\text{reg}^2_{U'}(\beta) = t_2(\xi_U)$ in $F^2H^2(U', \mathbb{C})$. Since $t_2$ is injective, the first part of the claim follows from the following assertion:

$$\text{reg}^2_{U'}(\beta) \in \text{Im}(t_1) \quad \text{if and only if} \quad (1-2) \quad \text{holds.}$$

To show this we consider the commutative diagram

$$\begin{array}{ccc}
CH^2(U, 2) \otimes \mathbb{Q} & \xrightarrow{\text{reg}^2_{D,U}} & CH^2(U', 2) \otimes \mathbb{Q} & \xrightarrow{\delta_1} & CH^1(W, 1) \otimes \mathbb{Q} \\
\downarrow \text{reg}^2_{D,U} & & \downarrow \text{reg}^2_{U'} & & \downarrow \text{reg}^1_{D,W} \\
H^2_D(U, \mathbb{Q}(2)) & \xrightarrow{\psi} & H^2_D(U', \mathbb{Q}(2)) & \xrightarrow{\pi_U} & H^1_D(W, \mathbb{Q}(1))
\end{array}$$

where $\text{reg}^*_{D,*}$ denotes the regulator map to Deligne cohomology. We have the commutative diagram (cf. [EV])

$$\begin{array}{ccc}
0 \to H^1(U, \mathbb{Q}(1)) \otimes \mathbb{C}/\mathbb{Q}(1) & \to & H^2_D(U, \mathbb{Q}(2)) & \xrightarrow{\pi_U} & F^0H^2(U, \mathbb{Q}(2)) & \to 0 \\
\downarrow & & \downarrow & & \downarrow \psi & & \\
0 \to H^1(U', \mathbb{Q}(1)) \otimes \mathbb{C}/\mathbb{Q}(1) & \to & H^2_D(U', \mathbb{Q}(2)) & \xrightarrow{\pi_{U'}} & F^0H^2(U', \mathbb{Q}(2)) & \to 0 \\
\downarrow \psi & & \downarrow & & \downarrow \partial_2 & & \\
0 \to H^0(W, \mathbb{Q}) \otimes \mathbb{C}/\mathbb{Q}(1) & \to & H^1_D(W, \mathbb{Q}(1)) & \xrightarrow{\pi_{\psi}} & F^0H^1(W, \mathbb{Q}(1)) & \to 0
\end{array}$$
where the composite of \(reg_{D,*}^t\) with \(\pi_*\) coincide with the regulator map to singular cohomology. The horizontal sequences are exact. The vertical sequences are localization sequences and they are exact except the most right one. We have (cf. (1-1))

\[
\text{Coker}(\psi) \cong \mathbb{C}/\mathbb{Q}(1) \otimes \Phi, \quad \Phi := \text{Coker}(\mathbb{Q} \to \bigoplus_{\mu \in I} \mathbb{Q}; \ 1 \to (e_\mu)_{\mu \in I}).
\]

To see this we note the commutative diagram

\[
\begin{array}{ccc}
CH^1(U', 1) \otimes \mathbb{Q} & \xrightarrow{\phi} & \bigoplus_{\mu \in I} \mathbb{Q} \\
\downarrow \text{reg}_{U'}^t & & \downarrow \cong \\
H^1(U', \mathbb{Q}(1)) & \longrightarrow & H^0(W, \mathbb{Q})
\end{array}
\]

where \(\phi\) is given by taking orders of functions along the components of \(W\). One easily sees that \(\text{reg}_{U'}^t\) is surjective and that \(CH^1(U', 1)\) is generated by \(\mathbb{C}^*, w/z_1\) and \(z_j/z_i\) with \(1 \leq i, j \leq 3\) and the desired assertion follows. Thus the above diagram gives rise to the exact sequence

\[
F^0H^2(U, \mathbb{Q}(2)) \xrightarrow{\iota} F^0H^2(U', \mathbb{Q}(2)) \xrightarrow{\delta} \mathbb{C}/\mathbb{Q}(1) \otimes \Phi,
\]

where \(F^0H^2(U', \mathbb{Q}(2))\text{th}_o = \text{Ker}(\partial_2)\). Now an easy calculation shows \(\partial_1(\beta) = (c_\mu e_\mu)_{\mu \in I}\).

Noting the commutative diagram

\[
\begin{array}{ccc}
0 & \to & \mathbb{C}^* \otimes \mathbb{Q} \\
\downarrow \log & & \downarrow \text{reg}_{D,W}\mu^t \\
0 & \to & H^1_D(W_\mu, \mathbb{Q}(1)) \\
& & \xrightarrow{\pi_{W_\mu}} F^0H^1(W_\mu, \mathbb{Q}(1)) \to 0
\end{array}
\]

it implies \(\text{reg}_{U'}^t(\beta) \in F^0H^2(U', \mathbb{Q}(2))\text{th}_o\) and

\[
\delta(\text{reg}_{U'}^t(\beta)) = \text{the class of } (-e_\mu \log c_\mu)_{\mu \in I} \text{ in } \Phi.
\]

This proves (1-3). In order to show the second part of Claim 2, assume that there is \(c' \in \mathbb{C}^*\) such that \(c'_\mu := cc_\mu\) is a root of unity for \(1 \leq \forall \nu \leq r\). Taking

\[
\beta' = \left\{ z_{\sigma(2)}^p z_{\sigma(3)}^q z_{\sigma(1)}^w \right\} \in CH^2(U', 2),
\]

\(reg_{U'}^t(\beta') = reg_{U'}^t(\beta) = \iota_2(\xi_U)\) and \(\partial_1(\beta) = (c_\mu^{-e_\mu})_{\mu \in I} = 0 \in CH^1(W, 1) \otimes \mathbb{Q}\). It implies that \(\beta'\) has a lift \(\beta'' \in CH^2(U, 2) \otimes \mathbb{Q}\). Then \(\text{reg}_{U'}^t(\beta'') = \xi_U \in F^0H^2(U, \mathbb{Q}(2))\) by the injectivity of \(\iota_2\). This completes the proof of Claim 2.

## 2 Infinitesimal interpretation

In this section we take the first step of the proof of Th\([\text{1.4]}\). Let the assumption and the notation be as in §1. Take \(\Delta \subset M\), a simply connected neighbourhood of 0 in \(M\). For \(\lambda \in H^2(U, \mathcal{C})\) and \(t \in \Delta\), let \(\lambda(t) \in H^2(U_t, \mathcal{C})\) be the flat translation of \(\lambda\) with respect to the Gauss-Maninn connection

\[
\nabla : H^2_O(U/M) \to \Omega^1_M \otimes H^2_O(U/M),
\]

8
where \( H^p_\omega(U/M) \) is the sheaf of holomorphic sections of the local system \( H^p_\omega(U/M) := R^p f_* \mathcal{C} \) with \( f : U \to M \), the natural morphism. We sometime consider \( \lambda \) a section over \( \Delta \) of \( H^2_\omega(U/M) \) via \( H^2(U, \mathbb{C}) \cong \Gamma(\Delta, H^2_\omega(U/M)) \). Put
\[
\Delta_\lambda = \{ t \in \Delta | \lambda(t) \in F^2 H^2(U_t, \mathbb{C}) \}.
\]
\( \Delta_\lambda \) is a closed analytic subset of \( \Delta \) since it is defined by the vanishing of the image of \( \lambda \) under the map
\[
\Gamma(\Delta, H^2_\omega(U/M)) \to \Gamma(\Delta, H^2_\omega(U/M)/F^2 H^2_\omega(U/M))
\]
where \( F^q H^p_\omega(U/M) \subset H^p_\omega(U/M) \) is the Hodge subbundle. Taking \( \Delta \) sufficiently small if necessary, we have by Lem.1.2(2)
\[
M_{NL} \cap \Delta = \cup_\lambda \Delta_\lambda \text{ with } \lambda \text{ ranging over } F^0 H^2(U, \mathbb{Q}(2)) \setminus \mathbb{Q} \cdot \omega_U.
\]
(2-1)
By Griffiths transversality, \( \nabla \) induces
\[
\nabla : H^{2,0}(U) \to \Omega^1_{M,0} \otimes H^{1,1}(U),
\]
where for integers \( p, q \) we put \( H^{p,q}(U) = H^q(X, \Omega^p_X(\log Z)) \) and \( \Omega^1_{M,0} \) is the fiber of \( \Omega^1_M \) at 0 \( M \). Let \( T_0(M) \) be the tangent space of \( M \) at 0. Via the natural isomorphism \( T_0(M) \cong \text{Hom}(\Omega^1_{M,0}, \mathbb{C}) \), it induces the pairing
\[
< , > : T_0(M) \otimes H^{2,0}(U) \to H^{1,1}(U).
\]
For \( \lambda \in H^{2,0}(U) \) write
\[
V_\lambda := \{ \partial \in T_0(M) | < \partial, \lambda >= 0 \}.
\]
By the construction we have
\[
T_0(\Delta_\lambda) \subset V_\lambda.
\]
(2-2)
**Theorem 2.1** Let \( \lambda \in H^{2,0}(U) \) and assume \( \lambda \notin \mathbb{C} \cdot \omega_U \).

(1) \( \text{codim}_{T_0(M)}(V_\lambda) \geq \binom{d+2}{2} - 5. \)

(2) Assume \( d \geq 4 \) and that \( T \subset \Delta_\lambda \) is an irreducible component of \( \text{codim}_\Delta(T) = \binom{d+2}{2} - 5. \)
Then \( T = \Delta_\lambda \), and \( 0 \in T^\sigma_{(p,q)}(\xi) \cap \Delta \) for some \( \sigma, p, q, \xi \) and \( \lambda \in \Sigma(U) \). Moreover, if \( \lambda \in H^2(U, \mathbb{Q}(2)) \), then \( \Delta_\lambda = T^\sigma_{(p,q)}(\xi) \cap \Delta \).

**Theorem 2.2** Assume \( 0 \in T^\sigma_{(p,q)}(\xi) \) for some \( \sigma, p, q, \xi \). Then we have
\[
\Sigma(U) = \{ \omega \in H^{2,0}(U) | < \partial, \omega >= 0 \text{ for } \forall \partial \in T_0(T^\sigma_{(p,q)}(\xi)) \}
\]
In the rest of this section we deduce Th 2.1 from Th 2.1 and Th 2.2. Th 0.6(1) and (2) follow immediately from Th 2.1 and (2-1) and Pr 1.5(1) by noting
\[
\text{codim}_\Delta(\Delta_\lambda) \geq \text{codim}_{T_0(\Delta)}(T_0(\Delta_\lambda)) \geq \text{codim}_{T_0(\Delta)}(V_\lambda)
\]
where the last inequality is due to (2-2). Assume \( 0 \in T^\sigma_{(p,q)}(\xi) \). We shall show that there exists a subset \( E \subset \Delta_T := T^\sigma_{(p,q)}(\xi) \cap \Delta \) which is the union of a countable number of proper
We have the canonical surjective homomorphism
\[ F^0 H^2(U_t, \mathbb{Q}(2)) \subset \Sigma(U_t) \] for all \( t \in \Delta_T \setminus E \). By Pr. \ref{proposition} the last condition implies \( F^2 H^2(U_t, \mathbb{Q}(2)) \subset \text{Im}(\text{reg}^2_{U_t}) \). Hence Th \ref{theorem} in (3) follows.

Write \( H^2(U, \mathbb{Q}(2)) = \{ \lambda_i \}_{i \in I} \) as a set and put
\[ A = \{ i \in I | \Delta_T \subset \Delta_{\lambda_i} \}, \quad B = \{ i \in I | \Delta_T \not\subset \Delta_{\lambda_i} \}, \quad E = \Delta_T \cap \bigcup_{i \in B} \Delta_{\lambda_i}. \]

Note that \( I \) is countable and \( I = A \cup B \) and \( A \cap B = \emptyset \). For every \( t \in \Delta_T - E \), we have \( F^0 H^2(U_t, \mathbb{Q}(2)) = \{ \lambda_i(t) \}_{i \in A} \) so that \( H^2(U_t, \mathbb{C}) \sim \Gamma(\Delta_T, H^2_\mathbb{C}(U/M)) \) induces
\[ F^2 H^2(U_t, \mathbb{Q}(2)) \hookrightarrow \Gamma(\Delta_T, H^2_\mathbb{C}(U/M) \cap F^2 H^2_\mathbb{O}(U/M)), \]
which further implies
\[ F^2 H^2(U_t, \mathbb{Q}(2)) \subset \text{Ker}(H^2,0(U_t) \to \Omega^1_{\Delta_T} \otimes H^{1,1}(U_t)). \]

Th \ref{theorem} implies that the last space is equal to \( \Sigma(U_t) \) and the desired assertion follows. This completes the proof of Th \ref{theorem}.

### 3 Reduction to Jacobian rings

Let the assumption be as in §2. In this section we rephrase the theorems in §2 in terms of Jacobian rings and prove Th \ref{theorem}(1) and Th \ref{theorem}. Let \( P = \mathbb{C}[z_0, z_1, z_2, z_3] \) be the homogeneous coordinate ring of \( \mathbb{P}^3 \). For an integer \( l > 0 \) let \( P^l \subset P \) be the subspace of homogeneous polynomials of degree \( l \). Let the assumption be as in §2 and fix \( F \in P^d \) which defines \( X \subset \mathbb{P}^3 \). Consider the ideal of \( P \);
\[ J_F = \left< \frac{\partial F}{\partial z_0}, \frac{\partial F}{\partial z_1}, \frac{\partial F}{\partial z_2}, \frac{\partial F}{\partial z_3} \right>. \]

The assumption that \( X \) transversally intersects \( Y \) is equivalent to the condition:

**3-1** \( J_F \) is complete intersection of degree \( (d - 1, d, d, d) \).

Write
\[ R_F = P/J_F, \quad J^d_F = J_F \cap P^l, \quad R^l_F = \text{Im}(P^l \to R_F) = P^l/J^l_F. \]

Note \( F \in J^d_F \). We have the following well-known facts:

**3-2** We have the canonical surjective homomorphism
\[ \psi : P^d \longrightarrow T_0(M) ; \quad G \to \{ F + \epsilon G = 0 \} \subset \mathbb{P}^3_{\mathbb{C}[\epsilon]}, \]
where \( \mathbb{C}[\epsilon] \) is the ring of dual numbers. We have \( \text{Ker}(\psi) = \mathbb{C} \cdot F \).

**3-3** We have the isomorphisms
\[ \phi : P^{d-1} \sim H^{2,0}(U), \quad \phi' : R^{2d-1}_F \sim H^{1,1}(U), \]
such that the diagram
\[
\begin{array}{c}
\begin{array}{ccc}
P^d \otimes P^{d-1} & \xrightarrow{\mu} & R^{2d-1}_F \\
\downarrow \psi \otimes \phi & & \downarrow \phi' \\
T_0(M) \otimes H^{2,0}(U) & \xrightarrow{\sim} & H^{1,1}(U)
\end{array}
\end{array}
\]
commutes up to non-zero scalar where \( \mu \) is the multiplication.
We have the following formula

$$\phi(G) = \text{Res}_X \frac{G}{z_1 z_2 z_3} \Omega \quad (G \in P^{d-1}),$$

where \(\Omega = \sum_{i=0}^{3} (-1)^i z_i dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_3 \in H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}^3)\) and

$$\text{Res}_X : H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}^3(\log Y)) \to H^0(X, \Omega_X^2(\log Z)) \quad (Y = \{z_1 z_2 z_3 = 0\} \subset \mathbb{P}^3)$$

is the residue map.

We omit the proof of the following lemma which can be easily shown by using (3-4).

**Lemma 3.1**

1. Putting \(\omega_F = \frac{\partial F}{\partial z_0}\), \(\phi(\omega_F) = \omega_U\) (cf. Def.1.1).

2. Assume \(0 \in \mathcal{T}_{(p,q)}(\sigma)\) and that \(X\) is defined by such an equation as Def.0.5:

\[F = wA + \prod_{1 \leq \nu \leq r} (cz_{\sigma(1)}^p z_{\sigma(2)}^q z_{\sigma(3)}).\]

Put

\[\xi_F = \frac{\partial w}{\partial z_0} (g_{\sigma(2)} \frac{\partial A}{\partial z_{\sigma(2)}} - p z_{\sigma(3)} \frac{\partial A}{\partial z_{\sigma(3)}}) - \frac{\partial A}{\partial z_0} (g_{\sigma(2)} \frac{\partial w}{\partial z_{\sigma(2)}} - p z_{\sigma(3)} \frac{\partial w}{\partial z_{\sigma(3)}}) \in P^{d-1}.

Then we have \(\phi(\xi_F) = \xi_U\) (cf. Def.1.4).

In what follows we identify \(P^{d-1} = H^{2,0}(U)\) via \(\phi\). For \(\lambda \in P^{d-1}\) write

\[I^d_\lambda = \{x \in P^d | \lambda x = 0 \in R^{2d-1}_F\}.

(3-5) \(\psi^{-1}(T_0(\Delta_\lambda)) \subset I^d_\lambda\).

In view of the above lemmas Th.2.1(1) and Th.2.2 follow from the following theorems.

**Theorem 3.2** Assuming \(\lambda \notin J^d_{F} = \mathbb{C} \cdot \omega_F\), \(\dim(P^{d-1}/I^d_\lambda) \geq \left(\frac{d+2}{2}\right) - 5\). The equality holds if and only if \(I^d_\lambda\) is complete intersection of degree \((1, d-1, d, d)\).

**Theorem 3.3** Let the assumption be as in Lem.3.1(2).

1. \(\psi^{-1}(T_{(p,q)}(\sigma))) = wP^{d-1} + J^d_F\).

2. \(I^d_\lambda = wP^{d-1} + J^d_F\) if \(\lambda = a\omega_F + b\xi_F\) with \(b \neq 0\).

3. \(\mathbb{C} \cdot \omega_F \oplus \mathbb{C} \cdot \xi_F = \{\lambda \in P^{d-1} | \lambda x = 0 \in R^{2d-1}_F\text{ for } \forall x \in wP^{d-1} + J^d_F\}\).

In the rest of this section we prove Th.3.2 and Th.3.3. We need the following theorems. The first one is Macaulay’s theorem and we refer [GH], p659, for the proof. The second one is due to A. Otwinowska and is shown by the same method as the proof of [Ot], Th.2.
Theorem 3.4 There exists a natural isomorphism
\[ \tau_F : R_{F}^{4d-5} \sim \to \mathbb{C} \]
and the pairing induced by multiplication
\[ R_{F}^{l} \otimes R_{F}^{d-5-l} \to R_{F}^{d-5} \stackrel{\tau_F}{\to} \mathbb{C} \]
is perfect for \( \forall l \).

Theorem 3.5 Let \( I \subset P \) be a homogeneous ideal satisfying the conditions:

1. \( I \) is Gorenstein of degree \( N > 0 \), namely there exists a non-zero linear map \( \mu : P^N \to \mathbb{C} \) such that \( I^l = \{ x \in P^l | \mu(xy) = 0 \text{ for } \forall y \in P^{N-l} \} \).
2. \( I \) contains a homogeneous ideal \( J \) which is complete intersection of degree \( (e_0, e_1, e_2, e_3) \) with \( e_0 \leq e_1 \leq e_2 \leq e_3 \).
3. There is an integer \( b \) such that \( e_0 \leq b \leq e_1 - 1 \) and \( N + 3 = e_2 + e_3 + b \).

For \( \forall l \geq 1 \) we have
\[ \dim(P^d/I^l) \geq \dim(P^d/ \langle z_0, z_1^b, z_2^{e_2}, z_3^{e_3} \rangle \cap P^d) \].

Moreover, if the equality holds for some \( l_0 \leq N - b \), then there is a complete intersection ideal \( I_0 \) of degree \( (1, b, e_2, e_3) \) such that \( I^l = I_0^l \) for all \( l \leq l_0 \).

Now we start the proof of Th.3.2. For \( \lambda \in P^{d-1} \) consider the linear map
\[ \lambda^* : P^{3d-4} \to \mathbb{C} ; x \to \tau_F(\lambda x). \]

For an integer \( l \geq 0 \) define
\[ I^l_\lambda = \{ x \in P^d | \lambda^* (xy) = 0 \text{ for } \forall y \in P^{3d-4-l} \}. \]

By Th.3.4 \( \lambda^* \neq 0 \) if and only if \( \lambda \notin J_F^{d-1} \) and \( I^l_\lambda \) in case \( l = d \) coincides with \( I^d_\lambda \) defined before. Define a homogeneous ideal of \( P \) by
\[ I_\lambda = \bigoplus_{l \geq 0} I^l_\lambda \subset P. \]

We take \( N = 3d - 4, (e_0, e_1, e_2, e_3, b) = (d - 1, d, d, d, d - 1) \) and apply Th.3.4 to \( I = I_\lambda \) and \( J = J_F \) noting (3-1). Since
\[ \dim(P^d/ \langle z_0, z_1^{d-1}, z_2^d, z_3^d \rangle \cap P^d) = \binom{d + 2}{2} - 5, \]
it implies Th.3.2.
Next we show Th.3.3 Let PGL$_4$ be the group of projective transformations on $\mathbb{P}^3$ and let $G \subseteq$ PGL$_4$ be the subgroup of such $g \in$ PGL$_4$ that $g(Y_j) = Y_j$ for $1 \leq j \leq 3$. It is evident that $G$ naturally acts on $M$ and $T^\sigma_{(p,q)}(\mathcal{C}) \subset M$ is stable under the action. Let $T^\sigma_{(p,q)}(\mathcal{C})_{(w,c)} \subset T^\sigma_{(p,q)}(\mathcal{C})$ be the closed subset of those surfaces defined by equations of the form

$$wB + \prod_{1 \leq \nu \leq r} (c_\nu z^{p+q}_\sigma(1) - c_\nu z^{p}_\sigma(2)z^{q}_\sigma(3)) \quad \text{for some } B \in P^{d-1}.$$

It is easy to see that the natural map $G \times T^\sigma_{(p,q)}(\mathcal{C})_{(w,c)} \to T^\sigma_{(p,q)}(\mathcal{C})$ is smooth and surjective and that $\psi^{-1}(T_0(T^\sigma_{(p,q)}(\mathcal{C})_{(w,c)})) = wP^{d-1}$. The map $T_0(M) \xrightarrow{\psi^{-1}} P^d/C \cdot F \to R^d_F$ identifies $R^d_F$ with the quotient of $T_0(M)$ by the infinitesimal action of the tangent space at the identity of $G$. It implies

$$\psi^{-1}(T_0(T^\sigma_{(p,q)}(\mathcal{C}))) = \pi^{-1}\pi^{-1}(T_0(T^\sigma_{(p,q)}(\mathcal{C})_{(w,c)})) = wP^{d-1} + J^d_F.$$

This completes the proof of Th.3.3(1).

For $\lambda = a\omega_F + b\xi_F$, an easy calculation shows $\lambda w \in J^d_F$ so that $I^d_\lambda \supset wP^{d-1} + J^d_F$. Assuming $b \neq 0$ we have

$$\left(\frac{d+2}{2}\right) - 5 \leq \dim(P^d/I^d_\lambda) \leq \dim(P^d/wP^{d-1} + J^d_F)$$

$$\leq \text{codim}_{T_0(M)}(T_0(T^\sigma_{(p,q)}(\mathcal{C}))) \leq \text{codim}_M(T^\sigma_{(p,q)}(\mathcal{C})) \leq \left(\frac{d+2}{2}\right) - 5,$$

where $(*)$ follows from Th.3.2, $(**)$ from Th.3.3(1), and $(***)$ from Lem.1.3. Thus the above inequalities are all equalities so that $I^d_\lambda = wP^{d-1} + J^d_F$. This completes the proof of Th.3.3(2).

For $\lambda \in P^{d-1}$ we have the following equivalences

$$\lambda wy \equiv 0 \in R^d_F \text{ for } \forall y \in P^{d-1} \iff \lambda wyz \equiv 0 \in R^d_{F} \text{ for } \forall y \in P^{d-1}, \forall z \in P^{2d-4}$$

$$\iff \lambda wx \equiv 0 \in R^d_{F} \text{ for } \forall x \in P^{3d-5}$$

$$\iff \lambda w \equiv 0 \in R^d_{F}$$

where the first and the last equivalences follows from Th.3.3. Hence it suffices to show

$$C \cdot \omega_F \oplus C \cdot \xi_F = \text{Ker}(P^{d-1} \xrightarrow{w} R^d_F).$$

We have already seen that the left hand side is contained in the right hand side. Thus it suffices to show $\dim(\text{Ker}(P^{d-1} \xrightarrow{w} R^d_F)) = 2$. We have $\dim(P^{d-1}) = \left(\frac{d-1+3}{3}\right)$. By (3-1) we have

$$\dim(R^d_F) = \dim(P^d/P^d \cap <z_0^{d-1}, z_1^{d}, z_2^{d}, z_3^{d}>) = \left(\frac{d+3}{3}\right) - \left(\frac{1+3}{3}\right) + 3\left(\frac{0+3}{3}\right).$$

We easily see that $<w> + J_F$ is complete intersection of degree $(1, d-1, d, d)$ so that

$$\dim(\text{Coker}(P^{d-1} \xrightarrow{w} R^d_F)) = \dim(P^d/P^d \cap <z_0^{d-1}, z_1^{d}, z_2^{d}, z_3^{d}>)$$

$$= \left(\frac{d+3}{3}\right) - \left(\frac{d-1+3}{3}\right) + \left(1+3\right) + 2\left(\frac{0+3}{3}\right) + \left(0+3\right).$$

These imply the desired assertion and the proof of Th.3.3 is complete.
4 Proof of key theorem

In this and next sections we prove Th.2.1(2) to complete the proof of Th.0.6. Let the assumption be as in Th.2.1(2). For \( t \in \Delta \) let \( F_t \in P^d \) define \( X_t \subset \mathbb{P}^3 \), \( R_{F_t} \) be the corresponding Jacobian ring. For \( t \in \Delta \), \( \lambda(t) \subset P \) be defined in the same manner as \( I_\lambda \) with \( \lambda \) replaced by \( \lambda(t) \in H^2(0)(U_t) \), the flat translation of \( \lambda \). For \( \forall t \in T \) we have

\[
\text{codim}_\Delta(T) \geq \text{codim}_{T_t(\Delta)}(T_t(\Delta)) \geq \dim(P^d/I_{\lambda t}^d) \geq \left(\frac{d+2}{2}\right) - 5,
\]

where \( T_t(*) \) denotes the tangent space at \( t \). The second inequality follows from (3-5) and the last from Th.3.2. Hence the assumption implies that the above inequalities are all equalities, which implies \( T = \Delta \) and \( \psi^{-1}(T_t(\Delta)) = I_{\lambda t}^d \). It also implies that \( I_{\lambda t} \) is complete intersection of degree \((1,d-1,d,d)\) so that \( I_{\lambda t}^1 = \mathbb{C} \cdot w_t \) for some \( w_t \in P^1 \) determined up to non-zero scalar. It gives rise to the morphism

\[
h : \Delta \to \mathbb{P}(P^1) = \mathbb{P}^3 ; t \to [w_t] := \mathbb{C} \cdot w_t.
\]

(4-1)

For \([w] \in \mathbb{P}(P^1)\) define the closed subset \( \Delta_{\lambda,w} = h^{-1}(w) \subset \Delta \). Note \( \text{codim}_\Delta(\Delta_{\lambda,w}) \leq 3 \).

In what follows we put \([w] = h(0) \in \mathbb{P}(P^1)\). For an integer \( l \geq 0 \) put \( P^l_t = P/wP^{l-1} \).

For \( t \in M \) put

\[
\Phi_t = \text{Im}(J_{F_t}^d \to P^d_{w_t}) = \text{Im}(\sum_{i=1}^3 C \cdot z_i \frac{\partial F_t}{\partial z_i} + P^1 \cdot \frac{\partial F_t}{\partial z_0} \to P^d_t).
\]

If \( t \in \Delta_{\lambda,w} \), \( I_{\lambda(t)} \) is complete intersection of degree \((1,d-1,d,d)\) with \( I_{\lambda t}^1 = \mathbb{C} \cdot w_t \). Noting \( J_{F_t}^d \subset I_{\lambda t}^d \), it implies

\[
\dim(\Phi_t) \leq \dim(I_{\lambda t}^d/wP^{d-1}) = \dim(P^1_t) + 2 \dim(P^0_t) = 5.
\]

Thus we have \( \Delta_{\lambda,w} \subset M_w \) where \( M_w = \{ t \in M | \dim(\Phi_t) \leq 5 \} \), which is a closed algebraic subset of \( M \). Put \( E_w = \mathbb{C}^3 \oplus P^1_{w_t} \). For \( \Gamma = [\gamma_1 : \gamma_2 : \gamma_3 : L] \in \mathbb{P}(E_w) \) put

\[
M_{w,\Gamma} = \{ t \in M | \sum_{i=1}^3 \gamma_i z_i \frac{\partial F_t}{\partial z_i} + L \frac{\partial F_t}{\partial z_0} \in wP^{d-1} \}.
\]

We note \( M_w = \cup_{\Gamma} M_{w,\Gamma} \) with \( \Gamma \) ranging over \( \mathbb{P}(E_w) \).

Lemma 4.1

(1) If \( M_{w,\Gamma} \neq \emptyset \), \( \text{codim}_{M_w}(M_{w,\Gamma}) \leq 5 \).

(2) For \( t \in M_{w,\Gamma} \), \( \psi^{-1}(T_t(M_{w,\Gamma})) = \{ G \in P^d_t | \sum_{i=1}^3 \gamma_i z_i \frac{\partial G}{\partial z_i} + L \frac{\partial G}{\partial z_0} \in wP^{d-1} \} \).

Proof Lem.4.1(2) follows directly from (3-2). We prove Lem.4.1(1). For \( t \in M \) put

\[
\Gamma_t = \{ (\gamma_1, \gamma_2, \gamma_3, L) \in E_w \mid \sum_{i=1}^3 \gamma_i z_i \frac{\partial F_t}{\partial z_i} + L \frac{\partial F_t}{\partial z_0} \in wP^{d-1} \}.
\]

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It is a linear subspace of $E_w$ and $\dim(\Gamma_t) = 6 - \dim(\Phi_t)$. For an integer $0 \leq e \leq 5$ put $M_w^{(e)} = \{ t \in M_w | \dim(\Phi_t) = 5 - e \}$ which is a locally closed subset of $M_w$. Letting $G^{(e)}$ be the Grassman variety of $(e + 1)$-dimensional linear subspaces in $E_w$, we define the morphism $\pi^{(e)} : M_w^{(e)} \to G^{(e)}$ by $\pi^{(e)}(t) = \Gamma_t$. For $\Gamma \in G^{(0)} = \mathbb{P}(E_w)$ put $G^{(e)}_\Gamma = \{ \Gamma' \in G^{(e)} | \Gamma' \supset \Gamma \}$. By definition we have set-theoretically

$$M_{w,\Gamma} = \prod_{0 \leq e \leq 5} M_w^{(e)} \times G^{(e)}_{\Gamma}.$$ 

It proves Lem.4.1(1) by noting that $G^{(e)}_{\Gamma}$ is smooth of codim$_{G^{(e)}_\Gamma}(G^{(e)}_{\Gamma}) = 5 - e$.

Now we fix $\Gamma = [\gamma_1 : \gamma_2 : \gamma_3 : L] \in \mathbb{P}(E_w)$ such that $0 \in M_{w,\Gamma}$. It means

\begin{equation}
(4-2) \sum_{i=1}^3 \gamma_i z_i \frac{\partial F}{\partial z_i} + L \frac{\partial F}{\partial z_0} \in w^{d-1}. \tag{4-2}
\end{equation}

Put $\Delta_{\lambda,w,\Gamma} = \Delta_{\lambda,w} \cap M_{w,\Gamma}$. By Lem.4.1(1) and the fact codim$_\Delta(\Delta_{\lambda,w}) \leq 3$ (cf. (4-1)), we have

$$8 = 3 + 5 \geq \text{codim}_\Delta(\Delta_{\lambda,w,\Gamma}) \geq \text{codim}_T(\Delta_{\lambda,w,\Gamma}).$$

It implies that $T_0(M_{w,\Gamma})$ contains a subspace of codimension $\leq 8$ in $T_0(\Delta)$. Recall that we have shown $\psi^{-1}(T_0(\Delta)) = I^d_\lambda \supset w P^{d-1}$. Hence Lem.4.1(2) implies that there exists a subspace $Q$ of codimension $\leq 8$ in $P^{d-1}$ such that

$$G(\sum_{i=1}^3 \gamma_i z_i \frac{\partial w}{\partial z_i} + L \frac{\partial w}{\partial z_0}) \in w^{d-1} \text{ for } \forall G \in Q.$$ 

If $\sum_{i=1}^3 \gamma_i z_i \frac{\partial w}{\partial z_i} + L \frac{\partial w}{\partial z_0} \notin \mathbb{C} \cdot w$, it implies $G \notin w^{d-2}$. Since codim$_{P^{d-1}}(w P^{d-2}) = \left(\begin{array}{c} d+1 \\ 2 \end{array}\right)$, this is a contradiction if $\left(\begin{array}{c} d+1 \\ 2 \end{array}\right) > 8$ which holds when $d \geq 4$. Thus we get the condition:

\begin{equation}
(4-3) \sum_{i=1}^3 \gamma_i z_i \frac{\partial w}{\partial z_i} + L \frac{\partial w}{\partial z_0} \in \mathbb{C} \cdot w. \tag{4-3}
\end{equation}

Now a key lemma is the following.

**Lemma 4.2** There exists $t \in \Delta_\lambda$ such that $w_t \notin \sum_{i=1}^3 \mathbb{C} \cdot z_i$.

We will prove Lem.4.2 in the next section. Admitting Lem.4.2 we finish the proof of Th.2.1(2). Let

$$\Delta_\lambda^0 = \{ t \in \Delta_\lambda | w_t \notin \sum_{i=1}^3 \mathbb{C} \cdot z_i \}.$$ 

By Lem.4.2 it is a non-empty open subset of $\Delta_\lambda$. We may assume $0 \in \Delta_\lambda^0$. By transforming by an element of $G$ (cf. the proof of Th.3.2), we may suppose $w = z_0$. The condition (4-3) now reads $L \in \mathbb{C} \cdot z_0$. Then $\gamma_1, \gamma_2, \gamma_3$ are not all zero and the condition (4-2) implies

$$\sum_{i=1}^3 \gamma_i z_i \frac{\partial F}{\partial z_i} \in z_0 P^{d-1}.$$ 

Writing $F = z_0 B + C$ with $C$, a homogeneous polynomial of degree $d$ in $\mathbb{C}[z_1, z_2, z_3]$, the above condition is equivalent to

$$\sum_{i=1}^3 \gamma_i z_i \frac{\partial C}{\partial z_i} = 0.$$
Write
\[ C = \sum_{\alpha=(\alpha_1, \alpha_2, \alpha_3)} c_\alpha z^\alpha, \quad (z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}, \, c_\alpha \in \mathbb{C}) \]
and take \( \alpha \) with \( c_\alpha \neq 0 \). The above condition implies that \( \alpha \) is an integral point lying on the sectional line \( \ell \) in \( x_1 x_2 x_3 \)-space defined by
\[ \ell : \sum_{i=1}^{3} x_i - d = \sum_{i=1}^{3} y_i x_i = 0, \, x_i \geq 0 \quad (i = 1, 2, 3). \]
Furthermore the condition (3-1) implies that \( C \) is divisible by neither of \( z_1, z_2, z_3 \). Writing \( \pi_i : x_i = 0 \), it implies that \( \ell \) and \( \pi_i \) intersect at an integral point for \( 1 \leq \forall i \leq 3 \). This implies that \( \ell \) passes through one of the points \((d, 0, 0), (0, d, 0), (0, 0, d)\). Assuming that \( \ell \) passes through the first point, we get \( \gamma_1 = 0 \) and hence \( \alpha_2 : \alpha_3 = -\gamma_2 : \gamma_3 = p : q \) for some coprime non-negative integer \( p, q \). Writing \( \alpha_2 = pj, \alpha_3 = qj \) with \( j \in \mathbb{Z} \), we get \( \alpha_1 = d - (p + q)j \) since \( \sum_{i=1}^{3} \alpha_i = d \). The condition that \( \ell \) and \( \pi_1 \) intersect at an integral point implies that \( r := d/(p + q) \) is an integer and hence \( \alpha_1 = (p + q)(r - j) \). Thus we can write
\[ C = \sum_{j=0}^{r} b_j (z_1^{p+q})^{-j} (z_2 z_3^q)^j = \prod_{1 \leq \nu \leq r} (c_\nu z_1^{p+q} - c_\nu z_2 z_3^q) \quad \text{for some } b_j, c_\nu, c \in \mathbb{C}. \]
Hence \( X \in T^\sigma_{(p,q)}(\mathcal{C}) \) for \( \sigma \), the identity, and \( \mathcal{C} = \{ c_\nu \}_{1 \leq \nu \leq r} \). By definition \( \lambda x = 0 \in R_{F_r}^{2d-1} \) for \( \forall x \in I_{\mathcal{A}}^d \supset wP^{d-1} + J^d_F \). Thus Th.3.3(3) shows \( \lambda \in \Sigma(U) \). This proves the first assertion of Th.2.1.2. To show the second assertion, we note that \( \mathcal{C} \in \mathbb{P}^r \) has been uniquely determined by \( 0 \in \Delta_\mathcal{A}^j \). Applying the same argument to any \( t \in \Delta_\mathcal{A}^j \), we get a holomorphic map \( g : \Delta_\mathcal{A}^j \to \mathbb{P}^3 \) defined by the condition:
\[ g(t) = t \mathcal{C} := [c_\nu, t]_{1 \leq \nu \leq r} \text{ with } F_t \in w_t P^{d-1} + \prod_{1 \leq \nu \leq r} (c_\nu z_1^{p+q} - c_\nu z_2 z_3^q). \]
If \( \lambda \in H^2(U, \mathbb{Q}(2)) \), then \( \lambda(t) \in H^2(U_t, \mathbb{Q}(2)) \) for any \( t \in \Delta \) and the assumption \( \lambda \not\in J^d_F \) implies \( \lambda(t) \not\in J^d_F \) in view of Lem.3.1.1. By Pr.1.3 it implies \( \mathcal{C} \in \mathbb{P}^r(\mathbb{Q}) \) and hence that \( g \) is constant. Therefore \( \Delta_\mathcal{A}^j \subset T^\sigma_{(p,q)}(\mathcal{C}) \cap \Delta \) and hence \( \Delta_\mathcal{A}^j \subset \Delta \mathcal{A}^j \subset T^\sigma_{(p,q)}(\mathcal{C}) \cap \Delta \) by taking the closure in \( \Delta \). Finally, comparing the codimension in \( \Delta \), we conclude that the last inclusion is the equality and the proof of Th.3.3 is complete.

5 Proof of key lemma

In this section we prove Lem.12. Assume that \( w_t \in \sum_{i=1}^{3} \mathbb{C} \cdot z_i \) for \( \forall t \in \Delta_\mathcal{A}^j \). Recall that we have fixed \( \Gamma = [\gamma_1 : \gamma_2 : \gamma_3 : L] \in \mathbb{P}(E_w) \) such that \( 0 \in M_{w,r} \). If \( L \not\in \sum_{i=1}^{3} \mathbb{C} \cdot z_i \), (4-2) implies \( \frac{\partial F}{\partial z_0} = 0 \) on \([1 : 0 : 0 : 0] \in \mathbb{P}^3 \), which contradicts (3-1). Hence we have

(5-1) \( L \in \sum_{i=1}^{3} \mathbb{C} \cdot z_i \)

We may write
\[ w_t = \sum_{i=1}^{3} a_i(t) z_i \text{ and } w = w_0 = \sum_{i=1}^{3} a_i z_i, \]
where \( a_i(t) \) is a holomorphic function on \( \Delta_\lambda \) with \( a_i = a_i(0) \). The condition (4-3) now reads \( \sum_{i=1}^{3} \gamma_i a_i z_i \in \mathbb{C} \cdot \sum_{i=1}^{3} a_i z_i \), which implies the condition:

$$ (5-2) \quad \gamma_1 a_1 : \gamma_2 a_2 : \gamma_3 a_3 = a_1 : a_2 : a_3. $$

The proof is now divided into some cases. Frist we suppose that we are in:

**Case (1)** There exists \( t \in \Delta_\lambda \) such that \( a_i(t) \neq 0 \) for \( 1 \leq i \leq 3 \).

Without loss of generality we may suppose that \( t = 0 \) satisfies the above condition. (5-2) implies \( \gamma_1 = \gamma_2 = \gamma_3 \). If \( \gamma_i = 0 \) for \( 1 \leq i \leq 3 \), \( L \notin \mathbb{C} \cdot w \). Then (4-2) implies \( \partial F \frac{\partial F}{\partial z_0} \in wP^{d-1} \) so that \( \frac{\partial F}{\partial z_0} = 0 \) on \( [1 : 0 : 0 : 0] \), which contradicts (3-1). Thus we may assume \( \gamma_i = 1 \) for \( 1 \leq i \leq 3 \). By noting \( dF = \sum_{i=0}^{3} z_i \frac{\partial F}{\partial z_i} \), (4-2) now reads:

$$ (5-3) \quad dF + (L - z_0) \frac{\partial F}{\partial z_0} \in wP^{d-1}. $$

**Claim 1** \( L \notin \mathbb{C} \cdot w \).

**Proof** Assume \( L \in \mathbb{C} \cdot w \). (5-3) implies \( dF - z_0 \frac{\partial F}{\partial z_0} \in wP^{d-1} \). By the assumption \( a_1 \neq 0 \) we can write

$$ F = wA + z_0 B + C \quad \text{with} \quad B \in \mathbb{C}[z_0, z_2, z_3] \cap P^{d-1}, \quad C \in \mathbb{C}[z_2, z_3] \cap P^d. $$

Then \( \frac{\partial F}{\partial z_0} = w \frac{\partial A}{\partial z_0} + z_0 \frac{\partial B}{\partial z_0} + B \) and hence \( d(z_0 B + C) - z_0 (z_0 \frac{\partial B}{\partial z_0} + B) = 0 \) by noting \( wP^{d-1} \cap \mathbb{C}[z_0, z_2, z_3] = 0 \). It implies \( C = 0 \) and \( (d-1)B = z_0 \frac{\partial B}{\partial z_0} \). From the last equation we immediately deduce \( B = cz_0^{d-1} \) with some \( c \in \mathbb{C} \). Hence \( F = wA + cz_0^d \), which is singular on \( \{w = A = z_0 = 0\} \). It contradicts (3-1) and completes the proof of Claim 1.

Now choose \( u \in \sum_{i=1}^{3} a_i z_i \) such that \( w, L, u \) are linearly independent and write

$$ F = wA + \sum_{\nu=0}^{d} L^\nu B_\nu, \quad \text{with} \quad B_\nu \in \mathbb{C}[z_0, u] \cap P^{d-\nu}. $$

Then the condition (5-2) implies

$$ d(\sum_{\nu=0}^{d} L^\nu B_\nu) + (L - z_0) \sum_{\nu=0}^{d} L^\nu \frac{\partial B_\nu}{\partial z_0} = \sum_{\nu=0}^{d} L^\nu (dB_\nu - z_0 \frac{\partial B_\nu}{\partial z_0} + \frac{\partial B_{\nu-1}}{\partial z_0}) \in wP^{d-1} $$

where \( B_{-1} = 0 \) by convention. Hence we get \( dB_\nu - z_0 \frac{\partial B_\nu}{\partial z_0} + \frac{\partial B_{\nu-1}}{\partial z_0} = 0 \) for \( 0 \leq \nu \leq d \). We easily solve the equations to get \( B_\nu = c(-1)^\nu \frac{d}{\nu} z_0^{d-\nu} \) for some \( c \in \mathbb{C} \) and hence

$$ F = wA + c \sum_{\nu=0}^{d} (-1)^\nu \frac{d}{\nu} z_0^{d-\nu} = wA + c(z_0 - L)^d. $$

The equation is singular on \( \{w = A = z_0 - L = 0\} \subset \mathbb{P}^3 \), which contradicts (3-1). This completes the proof in Case (1).
By Case (1) we may suppose $\Delta_\lambda \subset \cup_{1 \leq i \leq 3}\{t \in \Delta \mid a_i(t) = 0\}$. Since we have shown that $\Delta_\lambda$ is irreducible, we may suppose $a_3(t) = 0$ for $\forall t \in \Delta_\lambda$. Now we assume that we are in:

**Case (2)** There exists $t \in \Delta_\lambda$ such that $a_1(t)a_2(t) \neq 0$.

Put $\Delta_1^1 = \{t \in \Delta_\lambda \mid a_1(t)a_2(t) \neq 0\}$. It is a non-empty open subset of $\Delta_\lambda$. Without loss of generality we may assume $0 \in \Delta_1^1$. Thus $a_3 = 0$ and $a_1a_2 \neq 0$. (5-2) implies $\gamma_1 = \gamma_2$. Assuming $\gamma_3 \neq 0$, (4-2) implies $z_3\frac{\partial F}{\partial z_3} = 0$ on $\{z_1 = z_2 = \frac{\partial F}{\partial z_0} = 0\}$, which contradicts (3-1). Thus $\gamma_3 = 0$. If $\gamma_1 = \gamma_2 = 0$, the same argument as in the beginning of Case (1) induces a contradiction. Thus we may assume $\gamma_1 = \gamma_2 = 1$. Hence (4-2) now reads:

\[(5-4) \sum_{i=1}^{2} z_i \frac{\partial F}{\partial z_i} + L \frac{\partial F}{\partial z_0} \in wP^{d-1}.\]

**Claim 2** $L \in \sum_{i=1}^{2} \mathbb{C} \cdot z_i$ and $L \notin \mathbb{C} \cdot w$.

**Proof** Assume $L \notin \sum_{i=1}^{2} \mathbb{C} \cdot z_i$. By (5-1) we may suppose $L = z_3 + l_1 z_1 + l_2 z_2$. Then (5-4) implies $\frac{\partial F}{\partial z_0} \notin <z_1, z_2>$, which contradicts (3-1). The proof of the second assertion is similar to that of Claim 1 and omitted.

Noting $\mathbb{C}[z_0, z_1, z_2, z_3] = \mathbb{C}[z_0, w, L, z_3]$, we may write

$$F = wA + \sum_{\mu=0}^{d} z_3^\mu G_\mu \text{ with } A \in P^{d-1}, \ G_\mu \in \mathbb{C}[z_0, L] \cap P^{d-\mu}.$$ 

Noting $\sum_{i=1}^{2} z_i \frac{\partial w}{\partial z_i} = w$, (5-4) implies

$$\sum_{\mu=0}^{d} z_3^\mu (\sum_{i=1}^{2} z_i \frac{\partial G_\mu}{\partial z_i} + L \frac{\partial G_\mu}{\partial z_0}) \in wP^{d-1}.$$ 

Noting $(d - \mu)G_\mu = \sum_{i=0}^{3} \frac{\partial G_\mu}{\partial z_i}$ and $\frac{\partial G_\mu}{\partial z_3} = 0$, we get

$$0 = \sum_{i=1}^{2} z_i \frac{\partial G_\mu}{\partial z_i} + L \frac{\partial G_\mu}{\partial z_0} = (d - \mu)G_\mu + (L - z_0) \frac{\partial G_\mu}{\partial z_0} \text{ for } \forall 1 \leq \mu \leq d.$$ 

We solve the last equation in the same manner as Case (1) to get $G_\mu = b_\mu (L - z_0)^{d-\mu}$ with $b_\mu \in \mathbb{C}$ and hence

\[(5-5) F = wA + \sum_{\mu=0}^{d} b_\mu z_3^\mu (L - z_0)^{d-\mu}.\]

**Claim 3** Put $\eta_F = A + \sum_{i=1}^{2} z_i \frac{\partial A}{\partial z_i} + L \frac{\partial A}{\partial z_0}$.

1. $\phi(\eta_F) = \frac{\omega}{w} d(\frac{\omega - L}{z_3}) \wedge dlog \frac{\omega}{z_3}$.

2. $\mathbb{C} \cdot \omega_F \oplus \mathbb{C} \cdot \eta_F = \{y \in P^{d-1} \mid yx = 0 \in R^{2d-1}_F \text{ for } \forall x \in wP^{d-1} + J_F^d\} \ (cf. \ Lem.3.3(1)).$

Claim 3(1) is easily proven by using (3-3) and Claim 3(2) is proven by the same argument as the proof of Th 3.3(3). We omit the details.
By Claim 3 $\lambda(t) \in H^{2,0}(U_t)$, the flat translation of $\lambda$ for $t \in \Delta_\lambda$, is written as

$$\lambda(t) = f_1(t)\eta(t) + f_2(t)\omega(t) \text{ for } t \in \Delta^1_\lambda.$$ 

Here $f_1(t)$ and $f_2(t)$ are holomorphic functions on $\Delta^1_\lambda$ and

$$\omega(t) = d\log\frac{z_2}{z_1} \land d\log\frac{z_3}{z_1}, \quad \eta(t) = \frac{z_3}{w_t}d\left(\frac{z_0 - L_t}{z_3}\right) \land d\log\frac{z_1}{z_2},$$

where $w_t$ is as in the beginning of this section and

$$F_t = w_tA_t + \sum_{\mu=0}^{d} b_{\mu,t}z_\mu^3(L_t - z_0)^{d-\mu}, \quad L_t = l_1(t)z_1 + l_2(t)z_2$$

is the defining equation of $X_t$ such as (5-5), which varies holomorphically with $t \in \Delta^1_\lambda$. Recalling $Y = \cup_{1 \leq j \leq 3} Y_j$ with $Y_j = \{z_j = 0\} \subset \mathbb{P}^3$, write

$$Z_t = X_t \cap Y \supset Z_{3t} = X_t \cap Y_3 \supset V_t = Z_{3t} \cap (Y_1 \cup Y_2) \supset S_t = Z_{3t} \cap Y_2.$$ 

We consider the composite of the residue maps

$$\theta_t : H^{2,0}(U_t) = H^0(X_t, \Omega_{X_t}^2(\log Z_t)) \xrightarrow{\text{Res}_{Z_t}} H^0(Z_{3t}, \Omega_{Z_{3t}}^1(\log V_t)) \xrightarrow{\text{Res}_{S_t}} \mathbb{C}^d \sim \mathbb{C}^d,$$

where the last isomorphism is obtained by choosing $\epsilon_t : \{1, 2, \ldots, d\} \sim S_t$, an isomorphism of local systems of sets over $\Delta$. Since $\lambda(t)$ is flat, we get the condition:

$$(5-6) \quad \theta_t(\lambda(t)) \in \mathbb{C}^d \text{ is constant with } t \in \Delta^1_\lambda.$$ 

We shall show that the condition induces a contradiction, which completes the proof of Lem 4.2 in Case (2). In order to calculate $\theta_t(\lambda(t))$ we introduce some notations. Let $\mathbb{A} = \{z_2 = z_3 = 0\} - \{0 : 1 : 0 : 0\} \subset \mathbb{P}^3$ be identified with $\mathbb{C}$ via $[z_0 : z_1] \rightarrow z_0/z_1$. Let

$$\Sigma = \{(s_1, \ldots, s_d) | s_\nu \in \mathbb{A}, s_\nu \neq s_\mu \text{ for } 1 \leq \nu \neq \mu \leq d\}.$$ 

We define a holomorphic map

$$\pi : \Delta \rightarrow \Sigma; \quad t \rightarrow (s_\nu(t))_{1 \leq \nu \leq d} \text{ with } \epsilon_t(\nu) = s_\nu(t).$$

Now an easy residue calculation shows

$$\theta_t(\omega(t)) = (1, \ldots, 1), \quad \theta_t(\eta(t)) = \left(\frac{l_1(t) - s_\nu(t)}{a_1(t)}\right)_{1 \leq \nu \leq d}$$

and hence

$$\theta_t(\lambda(t)) = (p(t)s_\nu(t) + q(t))_{1 \leq \nu \leq d} \text{ with } p(t) = -\frac{f_1(t)}{a_1(t)}, \quad q(t) = f_1(t)\frac{l_1(t)}{a_1(t)} + f_2(t).$$

Therefore (5-6) implies that for $\forall \partial \in T_0(\Delta_\lambda)$, we have

$$0 = \partial(p(t)s_\nu(t) + q(t)) = p(0)\partial s_\nu(t) + s_\nu(0)\partial p(t) + \partial q(t) \text{ for } 1 \leq \forall \nu \leq d.$$
Letting $\pi_* : T_0(\Delta) \to T_{\pi(0)}(\Sigma) \cong \mathbb{C}^d$ be the differential of $\pi$, we get
\[ p(0) \cdot \pi_* (\partial) = p(0) \cdot (\partial s_\nu(t))_{1 \leq \nu \leq d} = p(0) \cdot (s_\nu(0))_{1 \leq \nu \leq d} + \partial q(t) \cdot (1, \ldots, 1). \]

Since $\lambda \not\in \mathbb{C} \cdot \omega_F$, $f_1(0) \neq 0$ and hence $p(0) \neq 0$. Thus it implies $\dim (\pi_*(T_0(\Delta))) \leq 2$. Therefore we get a contradiction if we show the following.

**Claim 4** $\dim (\pi_*(T_0(\Delta))) \geq d$.

**Proof** Let $Q = \mathbb{C}[z_0, z_1]$ and $Q^l = P^l \cap Q$ for an integer $l \geq 0$. For $G \in P$ write $\overline{G} = G \mod <z_2, z_3> \in Q$. Consider the morphism
\[ \rho : \Sigma \to N := \mathbb{P}(Q^d); \ z = (s_\nu)_{1 \leq \nu \leq d} \to [F_\lambda] \text{ with } F_\lambda = \prod_{1 \leq \nu \leq d} (z_0 - s_\nu z_1). \]

It is finite etale and induces an isomorphism on the tangent spaces. Hence it suffices to show Claim 4 by replacing $\pi$ with $\tilde{\pi} := \rho \circ \pi$. We have $\tilde{\pi}(t) = [F_\lambda]$ and we have the commutative diagram
\[ \begin{array}{ccc} P^d & \xrightarrow{\text{mod } <z_2, z_3>} & Q^d \\ \downarrow \psi & & \downarrow \psi' \\ T_0(\Delta) & \xrightarrow{\tilde{\pi}} & T_{\tilde{\pi}(0)}(N) \end{array} \]

where $\psi'$ is defined in the same way as $\psi$ in (3-1) and Ker($\psi'$) = $\mathbb{C} \cdot \overline{F}$. We have shown that $\psi^{-1}(T_0(\Delta)) = I_\lambda^d \supset w P^{d-1} + J_F^d$. Hence $\tilde{\pi}_*(T_0(\Delta)) \supset \psi'(z_1 Q^{d-1} + \mathbb{C} \cdot \overline{F})$. Noting $F \not\in <z_1, z_2, z_3>$ so that $\overline{F} \not\in z_1 Q^{d-1}$, this implies
\[ \dim (\tilde{\pi}_*(T_0(\Delta))) \geq \dim z_1 Q^{d-1} = d. \]

This completes the proof of Claim 4.

By Case (2) we may assume now that we are in:

**Case (3)** $a_2(t) = a_3(t) = 0$ for $\forall t \in \Delta$.

In this case we may assume $w = z_1$. We have
\[ I_\lambda \supset I := <z_1> + J_F = <z_1, \frac{\partial F}{\partial z_0}, \frac{\partial F}{\partial z_2}, \frac{\partial F}{\partial z_3}> \]
so that $I$ is complete intersection of degree $(1, d - 1, d, d)$. Hence $I = I_\lambda$ and $I_\lambda^d = z_1 P^{d-1} + J_F^d$. As before we can show the following.

**Claim 5** Put $\kappa_F = \frac{\partial F}{\partial z_1}$.

(1) $\phi(\kappa_F) = \frac{\partial F}{\partial z_1} \text{mod } \mathbb{Z}_0 \text{ and } d \log \frac{z_0}{z_1} \wedge d \log \frac{z_2}{z_0} \wedge d \log \frac{z_3}{z_0}$.

(2) $\mathbb{C} \cdot \omega_F \oplus \mathbb{C} \cdot \kappa_F = \{ y \in P^{d-1} \mid y x = 0 \in R_F^{2d-1} \text{ for } \forall x \in z_1 P^{d-1} + J_F^d \}$.

As before Claim 5 implies
\[ \lambda(t) = f_1(t) \kappa(t) + f_2(t) \omega(t) \text{ for } t \in \Delta, \]
where $f_1(t), f_2(t)$ and $\omega(t)$ are as before and
\[ \kappa(t) = \frac{z_0}{z_1} d \log \frac{z_2}{z_0} \wedge d \log \frac{z_3}{z_0} \in H^{2,0}(U_t). \]

An easy residue calculation shows $\theta_t(\lambda(t)) = (f_1(t) s_\nu(t) + f_2(t))_{1 \leq \nu \leq d}$ and the same argument as Case (2) induces a contradiction. This completes the proof of Lemma 1.
6 Injectivity of regulator map

In this section we prove Th.0.9. Fix $0 \in M$ and let the notation be as in the beginning of §1. By Lem.1.2, if $0 \in M \setminus M_{NL}$, we have

$$F^0 H^2(U, \mathbb{Q}(2)) = \mathbb{Q} \cdot \text{reg}^2_{U}(\alpha_U)$$

with $\alpha_U = \{\frac{z_2}{z_1}, \frac{z_3}{z_1}\} \in CH^2(U, 2)$.

By [AS2], Th.(6-1), it implies that the kernel of the composite map

$$CH^1(Z, 1) \otimes \mathbb{Q} \to CH^2(X, 1) \otimes \mathbb{Q} \xrightarrow{\rho_X} H^3_D(X, \mathbb{Q}(2))$$

is generated by

$$\partial_U(\alpha_U) = \delta := ((\frac{z_3}{z_2})|_{Z_1}, (\frac{z_1}{z_3})|_{Z_2}, (\frac{z_2}{z_1})|_{Z_2}) \in CH^1(Z, 1),$$

where $\partial_U : CH^2(U, 2) \to CH^1(Z, 1)$.

Claim 1 Write $\Lambda = \bigoplus_{1 \leq j \leq 3} C(Z_j)^*$. 

(1) Assume $0 \in T_{12}$ and that $X$ is defined by an equation as Def.0.8:

$$F = wA + z_1z_2B + c_1z_1^d + c_2z_2^d.$$  

Then the following elements are linearly independent in $\Lambda \otimes \mathbb{Q}$:

$$\delta, ((\frac{z_2}{w})|_{Z_1}, (\frac{w}{z_1})|_{Z_2}, 1).$$

(2) Assume $0 \in T_{12} \cap T_{23}$ and that $X$ is defined by an equation as Def.0.8:

$$F = wA + z_1z_2B + c_1z_1^d + c_2z_2^d$$

$$= vA + z_2z_3B' + c_2z_2^d + c_3z_3^d$$

Then the following elements are linearly independent in $\Lambda \otimes \mathbb{Q}$:

$$\delta, ((\frac{z_2}{w})|_{Z_1}, (\frac{w}{z_1})|_{Z_2}, 1), (1, (\frac{z_3}{v})|_{Z_2}, (\frac{v}{z_2})|_{Z_3}).$$

(3) Assume $0 \in T_{12} \cap T_{23}$ and that $X$ is defined by an equation as Def.0.8:

$$F = wA + z_1z_2B + c_1z_1^d + c_2z_2^d$$

$$= vA + z_2z_3B' + c_2z_2^d + c_3z_3^d$$

$$= uA + z_3z_1B'' + c_3z_3^d + c_1z_1^d$$

Then the following elements are linearly independent in $\Lambda \otimes \mathbb{Q}$:

$$\delta, ((\frac{z_2}{w})|_{Z_1}, (\frac{w}{z_1})|_{Z_2}, 1), (1, (\frac{z_3}{v})|_{Z_2}, (\frac{v}{z_2})|_{Z_3}), ((\frac{u}{z_3})|_{Z_1}, 1, (\frac{z_2}{w})|_{Z_2}).$$

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Proof We only show Claim 1(3). The other are easier and left to the readers. Assume the contrary. Then there are integers $e,l,m,n$ not all zero such that

\[
\begin{align*}
\left(\frac{z_2}{w}\right)^l\left(\frac{u}{z_3}\right)^m\left(\frac{z_3}{z_2}\right)^{e} & \equiv 1 \mod z_1, \\
\left(\frac{w}{z_1}\right)^l\left(\frac{z_2}{v}\right)^m\left(\frac{z_3}{z_2}\right)^{e} & \equiv 1 \mod z_2, \\
\left(\frac{v}{z_2}\right)^m\left(\frac{z_1}{u}\right)^n\left(\frac{z_2}{z_1}\right)^{e} & \equiv 1 \mod z_3.
\end{align*}
\]

We note $u,v,w \not\in \sum_{1 \leq j \leq 3} C \cdot z_j$ since otherwise it would contradicts (3-1). Hence the condition implies $l = m = n = e$ and $u,v,w$ coincides up to non-zero constant. Thus we get

\[
F \equiv z_1z_2B + c_1z_1^d + c_2z_2^d \equiv z_2z_3B' + c_2z_2^d + c_3z_3^d \equiv z_3z_1B'' + c_3z_3^d + c_1z_1^d \mod w,
\]

which is absurd. This completes the proof of Claim 1.

By Claim 1, the proof of Th.0.9 is complete if we show that $T_{12} \not\subset M_{NL}$ (resp. $T_{12} \cap T_{23} \not\subset M_{NL}$, resp. $T_{12} \cap T_{23} \cap T_{31} \not\subset M_{NL}$) if $d \geq 4$ (resp. $d \geq 6$, resp. $d \geq 10$). Indeed we have

\[
\text{codim}_M(T_{12}) = \left(\frac{d+3}{3}\right) - \left(\frac{d+2}{3}\right) + \left(\frac{d}{2}\right) + 2 = 2d - 1.
\]

One note that $T_{12} \cap T_{23} \cap T_{31} \neq \emptyset$ since the Fermat surface $z_0^d + z_1^d + z_2^d + z_3^d = 0$ belongs to it. Hence, for any irreducible component $T$ of $T_{12} \cap T_{23}$ (resp. $T_{12} \cap T_{23} \cap T_{31}$), $\text{codim}_M(T) \leq 2(2d - 1)$ (resp. $\text{codim}_M(T) \leq 3(2d - 1)$). By Th.[L.31) it suffices to check $(\frac{d+2}{2}) - 5$ is greater than $2d - 1$ (resp. $2(2d - 1)$, resp. $3(2d - 1)$) if $d \geq 4$ (resp. $d \geq 6$, resp. $d \geq 10$). This completes the proof of Th.0.9.

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