EXTREMALS OF A LEFT-INvariant SUB-FINSLER QUASIMETRIC ON THE CARTAN GROUP

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Abstract. Using the Pontryagin Maximum Principle for the time-optimal problem in coordinates of the first kind, we find extremals of arbitrary left–invariant sub–Finsler quasimetric on the Cartan group defined by a distribution of rank two.

Keywords and phrases: (ab)normal extremal, extremal, left–invariant sub–Finsler quasimetric, optimal control, polar curve, Pontryagin Maximum Principle.

MSC2010: 49J15, 49K15, 53C17.

INTRODUCTION

In [1], it is indicated that the shortest arcs of any left-invariant (sub-)Finsler metric \(d\) on a Lie group \(G\) are solutions of a left-invariant time-optimal problem with the closed unit ball \(U\) of some arbitrary norm \(F\) on a subspace \(\mathfrak{p}\) of the Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\) of the Lie group \(G\) as a control region. In addition, the subspace \(\mathfrak{p}\) generates \(\mathfrak{g}\). These statements are also valid for (sub-)Finsler quasimetries and the corresponding quasinorms. We explain that quasimetric have all properties of metric, except possibly the symmetry property \(d(p, q) = d(q, p)\). Moreover, \(U\) is an arbitrary convex figure in \(\mathfrak{p}\) with 0 interior to \(U\), perhaps \(U \neq -U\). The Pontryagin Maximum Principle [2] gives the necessary conditions for optimal trajectories of the problem; the curves, satisfying these conditions, are called extremals. Apparently, for the first time the shortest arcs of any left-invariant sub-Finsler metric on Lie group have been found in paper [3] in the case of arbitrary sub-Finsler metric \(d\) on the Heisenberg group \(H\).

In this paper we find extremals of arbitrary left-invariant sub-Finsler quasimetric on the Cartan group, defined by a subspace \(\mathfrak{p}\) of rank two; every extremal is normal for corresponding control. In papers [4] by Sachkov and [5], [6] by Ardentov, Le Donne, Sachkov, they considered special cases in other coordinates.

We apply here classical methods and results from the monograph [2]. Paper [11] uses some new search methods for normal extremals of left-invariant (sub-)Finsler and (sub-)Riemannian metrics.

The authors thank S. K. Vodopyanov for useful discussions.

The work is supported by Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation.
1. The Campbell-Hausdorff Formula for the Cartan Group

Let $X, Y, Z, V, W$ be a basis of the five-dimensional Cartan algebra $\mathfrak{g}$ such that

$$\begin{align*}
[X, Y] &= Z, \\
[X, Z] &= V, \\
[Y, Z] &= W,
\end{align*}$$

all other Lie brackets are equal to zero. Thus $\mathfrak{g}$ is a nilpotent Lie algebra with two generators $X, Y$. Therefore, as it is known, there exists a unique up to isomorphism connected simply connected nilpotent Lie group $G$ with the Lie algebra $\mathfrak{g}$, the Cartan group, and the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism. This diffeomorphism and the Cartesian coordinates $x, y, z, v, w$ in $\mathfrak{g}$ with the basis $X, Y, Z, V, W$ defines coordinates of the first kind on $G$ and thus a diffeomorphism $G \cong \mathbb{R}^5$.

**Proposition 1.** In coordinates of the first kind, the multiplication on the Cartan group $G \cong \mathbb{R}^5$ is given by the following rule

$$\begin{pmatrix}
    x_1 \\
y_1 \\
z_1 \\
v_1 \\
w_1
\end{pmatrix}
\times
\begin{pmatrix}
x_2 \\
y_2 \\
z_2 \\
v_2 \\
w_2
\end{pmatrix}
=
\begin{pmatrix}
x_1 + x_2 \\
y_1 + y_2 \\
z_1 + z_2 + \frac{1}{2}(x_1y_2 - x_2y_1) \\
v_1 + v_2 + \frac{1}{2}(x_1z_2 - x_2z_1) + \frac{1}{12}(x_1^2y_2 - x_1x_2y_1 - x_1x_2y_2 + x_2^2y_1) \\
w_1 + w_2 + \frac{1}{2}(y_1z_2 - y_2z_1) + \frac{1}{12}(x_1y_1y_2 + x_2y_1y_2 - x_2y_1^2 - x_2y_2^1)
\end{pmatrix}.$$  

*Proof.* Set $A_i = x_iX + y_iY + z_iZ + v_iV + w_iW$, $i = 1, 2$. Using (1), we consequently obtain

$$[A_1, A_2] = (x_1y_2 - x_2y_1)Z + (x_1z_2 - x_2z_1)V + (y_1z_2 - y_2z_1)W;$$

$$[A_1, [A_1, A_2]] = x_1(x_1y_2 - x_2y_1)V + y_1(x_1y_2 - x_2y_1)W;$$

$$[A_2, [A_2, A_1]] = [A_1, A_2], A_2 = -x_2(x_1y_2 - x_2y_1)V - y_2(x_1y_2 - x_2y_1)W.$$

Since the Lie algebra $\mathfrak{g}$ is of step three, then it is valid the following Campbell-Hausdorff formula (see [8]):

$$\ln(\exp(A_1) \exp(A_2)) = A_1 + A_2 + \frac{1}{2}[A_1, A_2] + \frac{1}{12}[A_1, [A_1, A_2]] + \frac{1}{12}[A_2, [A_2, A_1]].$$

Therefore

$$\ln(\exp(A_1) \exp(A_2)) = (x_1 + x_2)X + (y_1 + y_2)Y + \left(z_1 + z_2 + \frac{1}{2}(x_1y_2 - x_2y_1)\right)Z +$$

$$\left(v_1 + v_2 + \frac{1}{2}(x_1z_2 - x_2z_1) + \frac{1}{12}(x_1^2y_2 - x_1x_2y_1 - x_1x_2y_2 + x_2^2y_1)\right)V +$$

$$\left(w_1 + w_2 + \frac{1}{2}(y_1z_2 - y_2z_1) + \frac{1}{12}(x_1y_1y_2 + x_2y_1y_2 - x_2y_1^2 - x_2y_2^1)\right)W.$$

The last equality gives (2).
It follows from the applied method to introduce coordinates of the first kind and formulas \cite{2} that in these coordinates, the chosen basis of the Lie algebra \( g \) is realized as left-invariant vector fields on the Lie group \( G \) of the form

\[
X = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} - \frac{z}{2} \frac{\partial}{\partial v} - \frac{xy}{12} \frac{\partial}{\partial w} - \frac{y^2}{12} \frac{\partial}{\partial w}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z} + \frac{x^2}{12} \frac{\partial}{\partial v} + \left( \frac{xy}{12} - \frac{z}{2} \right) \frac{\partial}{\partial w},
\]

\[
Z = \frac{\partial}{\partial z} + \frac{x}{2} \frac{\partial}{\partial v} + \frac{y}{2} \frac{\partial}{\partial w}, \quad V = \frac{\partial}{\partial v}, \quad W = \frac{\partial}{\partial w}.
\]

2. **Left-invariant sub-Finsler quasimetric and the optimal control on the Cartan group**

In \cite{1}, it is said that the shortest arcs of a left-invariant sub-Finsler metric \( d \) on arbitrary connected Lie group \( G \) defined by a left-invariant bracket generating distribution \( D \) and a norm \( F \) on \( D(e) \) coincide with the time-optimal solutions of the following control system

\[
g(t) = dl_{g(t)}(u(t)), \quad u(t) \in U,
\]

with measurable controls \( u = u(t) \). Here \( l_g(h) = gh \), the control region is the unit ball

\[
U = \{ u \in D(e) \mid F(u) \leq 1 \}.
\]

This statement is also true in the case when \( d \) is a quasimetric (respectively, \( F \) is a quasinorm on \( D(e) \)).

Therein the Pontryagin Maximum Principle \cite{2} for (local) time optimal control \( u(t) \) and corresponding trajectory \( g(t) \), \( t \in \mathbb{R} \) implies the existence of a non-vanishing absolutely continuous vector-function \( \psi(t) \in T^*_g G \) such that for almost all \( t \in \mathbb{R} \) the function \( \mathcal{H}(g(t); \psi(t); u) = \psi(t)(dl_{g(t)}(u)) \) of the variable \( u \in U \) attains a maximum at the point \( u(t) \):

\[
M(t) = \psi(t)(dl_{g(t)}(u(t))) = \max_{u \in U} \psi(t)(dl_{g(t)}(u)).
\]

In addition, the function \( M(t) \), \( t \in \mathbb{R} \), is constant and non-negative, \( M(t) \equiv M \geq 0 \). In case when \( M = 0 \) (respectively, \( M > 0 \)) the corresponding **extremal**, i.e. the curve, satisfying the Pontryagin Maximum Principle, is called **abnormal** (respectively, **normal**).

If \( x = (x^1, \ldots, x^n) \) is a global coordinate system on \( G \),

\[
x(t) = (x^1(t), \ldots, x^n(t)) := (x^1(g(t)), \ldots, x^n(g(t))),
\]

\[
\psi_j = \psi_j(t) = \psi(t) \left( \frac{\partial}{\partial x^j} \right)(x(t)), \quad j = 1, \ldots, n, \quad \psi(t) := (\psi_1(t), \ldots, \psi_n(t)),
\]

then according to \cite{2}, the pair \((g(t), \psi(t))\) satisfies *the Hamiltonian system* in a symbolic notation

\[
\dot{x}(t) = \frac{\partial \mathcal{H}}{\partial \psi}(x(t), \psi(t), u(t)), \quad \dot{\psi}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), \psi(t), u(t)).
\]

It follows from \cite{1} that the left-invariant distribution \( D \) on \( G \) with the basis \( X, Y \) for \( D(e) \) is bracket generating. Let \( F \) be an arbitrary quasimetric on \( D(e) \). Then
the pair \((D(e), F)\) defines a left-invariant sub-Finsler quasimetric \(d\) on \(G\); therein \(u_1 X(e) + u_2 Y(e)\) is identified with \(u = (u_1, u_2)\), where \(u_i \in \mathbb{R}, i = 1, 2\).

Let \(\psi_k, k = 1, \ldots, 5,\) be covector components of \(\psi = \psi(t)\) relative to the coordinate system \((x, y, z, v, w)\), i.e. \[(8) \quad \psi_1 = \psi \left( \frac{\partial}{\partial x} \right), \quad \psi_2 = \psi \left( \frac{\partial}{\partial y} \right), \quad \psi_3 = \psi \left( \frac{\partial}{\partial z} \right), \quad \psi_4 = \psi \left( \frac{\partial}{\partial v} \right), \quad \psi_5 = \psi \left( \frac{\partial}{\partial w} \right),\]

\[(9) \quad h_1 = \psi(X), \quad h_2 = \psi(Y), \quad h_3 = \psi(Z), \quad h_4 = \psi(V), \quad h_5 = \psi(W).\]

Using (3), (4), (8), (9), we obtain

\[(10) \quad h_1 = \psi_1 - \frac{1}{2} \psi_3 y - \frac{1}{12} \psi_4 x y - \frac{1}{12} \psi_5 y^2 - \frac{1}{2} \psi_4 z, \quad h_2 = \psi_2 + \frac{1}{2} \psi_3 x + \frac{1}{12} \psi_4 x^2 + \frac{1}{12} \psi_5 x y - \frac{1}{2} \psi_5 z,\]

\[(11) \quad h_3 = \psi_3 + \frac{1}{2} \psi_4 x + \psi_5 y, \quad h_4 = \psi_4, \quad h_5 = \psi_5.\]

Then the function \(H(x, y, z, v, w; \psi_1, \psi_2, \psi_3, \psi_4, \psi_5; u_1, u_2)\) can be written as

\[(12) \quad H = \psi(u_1 X + u_2 Y) = u_1 \psi(X) + u_2 \psi(Y) = h_1 u_1 + h_2 u_2.\]

With regard to the first equality in (7), (12) and (10) system (1) takes a form

\[(13) \quad \dot{x}(t) = u_1, \quad \dot{y}(t) = u_2, \quad \dot{z}(t) = \frac{1}{2}(x u_2 - y u_1),\]

\[(14) \quad \dot{v}(t) = -\frac{1}{2} \left( \frac{1}{6} x y \right) u_1 + \frac{1}{12} x^2 u_2, \quad \dot{w}(t) = -\frac{1}{12} y^2 u_1 - \frac{1}{2} \left( z - \frac{1}{6} x y \right) u_2,\]

where \((u_1, u_2) = (u_1(t), u_2(t)) \in U\).

In consequence of left-invariance of the metric \(d\) we can assume that the trajectories initiate at the unit \(e \in G\), i.e. \(x(0) = y(0) = z(0) = \psi(0) = u(0) = 0\).

The control \(u = u(t) = (u_1(t), u_2(t)) \in U, t \in \mathbb{R},\) defined by the Pontryagin Maximum Principle is bounded and measurable \([2]\), therefore integrable. Then the functions \(x(t), y(t), u, t \in \mathbb{R},\) defined by the first two equations in (13) are Lipschitz, the product of any finite number of these functions is Lipschitz, and its derivative is bounded and measurable on each compact segment of \(\mathbb{R}\). So this derivative can be computed by the usual differentiation rule of a product from differential calculus for functions of one variable. Therefore, the last equation of the system (13) and equations of (14) can be integrated by parts, using the first two equations in (13) (see ss. 2.9.21, 2.9.24 in [2]). By \(x(0) = y(0) = z(0) = \psi(0) = u(0) = 0\) we get successively

\[(15) \quad z(t) = -\frac{1}{2} x(t) y(t) + \int_0^t x(\tau) u_2(\tau) d\tau,\]

\[(16) \quad v(t) = \frac{1}{12} x^2(t) y(t) - \frac{1}{2} x(t) \int_0^t x(\tau) u_2(\tau) d\tau + \frac{1}{2} \int_0^t x^2(\tau) u_2(\tau) d\tau,\]
Therefore, by the initial data of systems (13) and (18), we get

\begin{equation}
\begin{aligned}
\dot{\psi}_1 &= \frac{1}{12} \psi_4 y u_1 - \left( \frac{1}{2} \psi_3 + \frac{1}{6} \psi_4 x + \frac{1}{12} \psi_5 y \right) u_2, \\
\dot{\psi}_2 &= \left( \frac{1}{2} \psi_3 + \frac{1}{6} \psi_4 x + \frac{1}{12} \psi_5 y \right) u_1 - \frac{1}{12} \psi_5 x u_2, \\
\dot{\psi}_3 &= \frac{1}{2} \psi_4 u_1 + \frac{1}{2} \psi_5 u_2, \\
\dot{\psi}_4 &= 0, \\
\dot{\psi}_5 &= 0.
\end{aligned}
\end{equation}

Assign an arbitrary set of initial data \( \psi_i(0) = \varphi_i, i = 1, \ldots, 5 \), of the system (18). It follows from (18), (13) and the initial condition \( x(0) = y(0) = 0 \) that

\begin{equation}
\psi_5 \equiv \varphi_5, \quad \psi_4 \equiv \varphi_4, \quad \psi_3 = \varphi_3 + \frac{1}{2} \varphi_4 x + \frac{1}{2} \varphi_5 y.
\end{equation}

Notice that \( \left( \frac{3}{2} x y - z \right) = x u_2, \left( \frac{3}{2} x y - z \right) = y u_1 \) on the ground of (13). With regard to (19) the first and the second equations in (18) take a form

\begin{equation}
\begin{aligned}
\dot{\psi}_1 &= \frac{1}{12} \varphi_4 \left( \frac{3}{2} x y - z \right) - \frac{1}{2} \varphi_3 \dot{y} - \frac{5}{12} \varphi_4 \left( \frac{3}{2} x y + z \right) - \frac{1}{3} \varphi_5 y \dot{x}, \\
\dot{\psi}_2 &= \frac{5}{12} \varphi_5 \left( \frac{3}{2} x y - z \right) + \frac{1}{2} \varphi_3 \dot{x} - \frac{1}{12} \varphi_5 \left( \frac{3}{2} x y + z \right) + \frac{1}{3} \varphi_4 \dot{x}.
\end{aligned}
\end{equation}

Therefore, by the initial data of systems (13) and (18), we get

\begin{equation}
\psi_1 = \varphi_1 - \frac{1}{2} \varphi_3 y - \frac{1}{6} \varphi_5 y^2 - \frac{1}{6} \varphi_4 x y - \frac{1}{2} \varphi_4 z, \quad \psi_2 = \varphi_2 + \frac{1}{2} \varphi_3 x + \frac{1}{6} \varphi_4 x^2 + \frac{1}{6} \varphi_5 x y - \frac{1}{2} \varphi_5 z.
\end{equation}

Inserting (19) and (20) into (10), (11), we find

\begin{equation}
h_1 = \varphi_1 - \left( \varphi_3 + \frac{1}{2} \varphi_4 x + \frac{1}{2} \varphi_5 y \right) y - \varphi_4 z, \quad h_2 = \varphi_2 + \left( \varphi_3 + \frac{1}{2} \varphi_4 x + \frac{1}{2} \varphi_5 y \right) x - \varphi_5 z,
\end{equation}

\begin{equation}
h_3 = \varphi_3 + \varphi_4 x + \varphi_5 y, \quad h_4 = \varphi_4, \quad h_5 = \varphi_5.
\end{equation}

From (21) and (22) we obtain an integral of the Hamiltonian system (13) – (14), (18):

\begin{equation}
\mathcal{E} = \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4 \equiv \frac{\varphi_3^2}{2} + \varphi_1 \varphi_5 - \frac{\varphi_2^2}{2}.
\end{equation}

Thus the functions \( \mathcal{H}(t) = M(t) \) and three the so-called Casimir functions \( h_4 = \varphi_4, h_5 = \varphi_5, \) and \( \mathcal{E} \) are integrals of this Hamiltonian system.

Now, using (13), (21) and (22), we compute

\begin{equation}
\dot{h}_1 = -h_3 u_2, \quad \dot{h}_2 = h_3 u_1.
\end{equation}
For an extremal \((x(t), y(t), z(t), v(t), w(t))\), a bounded measurable control \(u(t)\) and a non-vanishing absolutely continuous vector-function \(\psi(t)\), the function \(\mathcal{H}(x(t), y(t), z(t), v(t), w(t); \psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t), \psi_5(t); u_1, u_2)\) of \(u \in U\) attains the maximum at the point \(u = u(t)\):

\[
M(t) = h_1(t)u_1(t) + h_2(t)u_2(t) = \max_{u \in U}(h_1(t)u_1 + h_2(t)u_2) \equiv M \geq 0.
\]

Relations \((13), (21)\) and \((25)\) imply that under multiplication of functions \(\psi_i(t), i = 1, \ldots, 5\), by a positive constant \(k\) the trajectory \((x(t), y(t), z(t), v(t), w(t))\) does not change, while \(M\) is multiplied by \(k\). Therefore in case when \(M > 0\) we shall assume that \(M = 1\). Further in this section we consider this case.

It follows from \((25)\) that \((h_1(t), h_2(t))\) in \((21)\) and \((\varphi_1, \varphi_2) = (h_1(0), h_2(0))\) lie on the boundary \(\partial U^*\) of the polar figure \(U^* = \{h | F_U(h) \leq 1\}\) to \(U\), where \(F_U\) is a quasinorm on \(H = \{h\}\), is equal to the support Minkowski function of the body \(U\):

\[
F_U(h) = \max_{u \in U} h \cdot u.
\]

In addition, \((H, F_U)\) is the conjugate quasinormed vector space to \((D(e), F)\) and \((U^*)^* = U\) (see Theorem 14.5 in \([10]\)). Moreover, using \((24)\) and \((25)\), we get

\[
h_1(t)h_2(t) - h_1'(t)h_2(t) = h_3(t)(h_1(t)u_1(t) + h_2(t)u_2(t)) = h_3(t).
\]

Let \(r = r(\theta), \theta \in \mathbb{R}\), be a polar equation of the curve \(F_U(x, y) = 1\). At every point \(\theta \in \mathbb{R}\) there exist one-sided derivatives of \(r = r(\theta)\) (and with possible exclusion of no more than countable number of values \(\theta\) there exists the usual derivative \(r'(\theta)\)). For simplicity we shall denote every value between these derivatives by \(r'(\theta)\). Then

\[
h_1(t) = h_1(\theta) = r(\theta) \cos \theta, \quad h_2(t) = h_2(\theta) = r(\theta) \sin \theta, \quad \theta = \theta(t),
\]

\[
h_1'(t) = -r'(\theta) \sin \theta - r'(\theta) \cos \theta, \quad h_2'(t) = (r'(\theta) \sin \theta + r(\theta) \cos \theta).
\]

Independently on the existence of usual derivative \((28), (26)\) implies the existence of usual derivative for the doubled oriented area

\[
\sigma(t) = 2S(\theta(t)) = \int_0^{\theta(t)} r^2(\theta)d\theta
\]

of the sector, counted from 0. In addition, by \((11)\) and \((26)\)

\[
\dot{\sigma}(t) = \varphi_3 + \varphi_4x(t) + \varphi_5y(t) = r^2(\theta(t))\dot{\theta}(t), \quad \dot{\theta}(t) = \frac{\dot{\sigma}(t)}{r^2(\theta(t))}.
\]

If we square the second equality in \((29)\), we get by \((21)\)

\[
r^4(\theta)\dot{\theta}^2 = \varphi_3^2 + \frac{1}{2}\varphi_4x + \frac{1}{2}\varphi_5y = (\varphi_3 + \frac{1}{2}\varphi_4x + \frac{1}{2}\varphi_5y) (2\varphi_4 + 2\varphi_5y) = \varphi_3^2 + 2\varphi_4(h_2 - \varphi_2) - 2\varphi_5(h_1 - \varphi_1),
\]

\[
\dot{\theta}^2 = \frac{\varphi_3^2 + 2\varphi_4(h_2 - \varphi_2) - 2\varphi_5(h_1 - \varphi_1)}{r^4(\theta)}.
\]

On the ground of \((23), (26)\), and \((29)\),

\[
\dot{\sigma}(t) = \varphi_4u_1(t) + \varphi_5u_2(t),
\]
so on the base of (13), (21), and (25) we get, omitting for brevity the variable $(36)$ starting at the unit is one of the following one-parameter subgroups $(35)$

$\phi$

Taking into account of the initial data of systems (13) and (14), we obtain

$h = x(0) = 0$ $(37)$

An abnormal extremal

Proposition 2. Let $\phi$ and is not strongly abnormal.

Remark 1. $(32)$ is equivalent to $(30)$.

It follows from (13) and (14) that

$(33)$

$(34)$

so on the base of (13), (21), and (25) we get, omitting for brevity the variable $t$,

$h_1u_1 + h_2u_2 = \varphi_1\dot{x} + \varphi_2\dot{y} + 2\varphi_3\dot{z} + \varphi_4(x\dot{z} - z\dot{x}) + \varphi_5(y\dot{z} - z\dot{y}) =

\left(\varphi_1x + \varphi_2y + 2\varphi_3z + 3\varphi_4v + 3\varphi_5w + \frac{\varphi_4}{2}xz + \frac{\varphi_5}{2}yz\right) = 1.$

Taking into account of the initial data of systems (13) and (14), we obtain

$(35)$

$\varphi_1x(t) + \varphi_2y(t) + 2\varphi_3z(t) + 3\varphi_4v(t) + 3\varphi_5w(t) + \frac{\varphi_4}{2}x(t)z(t) + \frac{\varphi_5}{2}y(t)z(t) = t.$

3. Search for Sub-Finsler extremals

1. Let us consider an abnormal case. The following proposition is valid.

Proposition 2. An abnormal extremal $(x, y, z, v, w)(t), t \in \mathbb{R}$, on the Cartan group starting at the unit is one of the following one-parameter subgroups

$(36)$

$(37)$

$(38)$

and is not strongly abnormal.

Proof. Assume that $M = 0$. Then we obtain from the maximum condition that $h_1(t) = h_2(t) \equiv 0$ and $\varphi_1 = \varphi_2 = 0$. Since $u_1(t)$ and $u_2(t)$ could not simultaneously vanish at any $t \in \mathbb{R}$, then $\varphi_3 + \varphi_4x(t) + \varphi_5y(t) \equiv 0$ on the base of (22) and (24).

This implies that $\varphi_3 = 0$ and $\varphi_4x(t) + \varphi_5y(t) \equiv 0$ because $x(0) = y(0) = 0$. Hence in consequence of (13) and (14) we get $\varphi_4 \neq 0$ or/and $\varphi_5 \neq 0$ because $\psi(t)$ does not vanish. It follows from this and (21) that $z(t) \equiv 0$.

Let $\varphi_4 \neq 0$, $\varphi_5 = 0$. Then $x(t) \equiv 0$ and $u_1(t) \equiv 0$ according to the first equation (13). Hence in consequence of (14) and the initial condition $v(0) = w(0) = 0$ we successively get $v(t) = w(t) \equiv 0$. Further, since $u_1(t) \equiv 0$ and $F(u_1(t), u_2(t)) \equiv 1$,
then \( u_2(t) \equiv \frac{s}{F(0,s)} \), \( s = \pm 1 \). This, the second equation in (13), and the initial condition \( y(0) = 0 \) imply that \( y(t) = \frac{st}{F(0,s)} \), \( s = \pm 1 \), and we get (36). In consequence of (2), the extremal is one of two one-parameter subgroups

\[
g(t) = \exp \left( \frac{stY}{F(0,s)} \right), \quad s = \pm 1, \quad t \in \mathbb{R},
\]

satisfies (25) with \( M(t) \equiv 1 \) for constant covector function

\[
\psi(t) = (0, \varphi_2, 0, 0, 0) = (0, sF(0,s), 0, 0, 0) = (0, h_2(t), 0, 0, 0), \quad s = \pm 1,
\]

subject to differential equations (18) and (24); therefore it is normal relative to this covector function and is not strongly abnormal.

Let \( \varphi_4 = 0, \varphi_5 \neq 0 \). Then \( y(t) \equiv 0 \) and \( u_2(t) \equiv 0 \) by the second equation (13). Then from (13), (14) and the initial condition \( v(0) = w(0) = 0 \) we successively get \( v(t) = w(t) \equiv 0 \). Also, since \( u_2(t) \equiv 0 \) and \( F(u_1(t), u_2(t)) \equiv 1 \) then \( u_1(t) \equiv \frac{s}{F(s,0)} \), \( s = \pm 1 \). This, the first equation in (13), and the initial condition \( x(0) = 0 \) imply that \( x(t) = \frac{st}{F(s,0)} \), \( s = \pm 1 \), and we get (37). In consequence of (2), the extremal is one of two one-parameter subgroups

\[
g(t) = \exp \left( \frac{stX}{F(s,0)} \right), \quad s = \pm 1, \quad t \in \mathbb{R},
\]

satisfies (25) with \( M(t) \equiv 1 \) for constant covector function

\[
\psi(t) = (\varphi_1, 0, 0, 0, 0) = (sF(s,0), 0, 0, 0, 0) = (h_1(t), 0, 0, 0, 0), \quad s = \pm 1,
\]

subject to differential equations (18) and (24); therefore it is normal relative to this covector function and is not strongly abnormal.

Let \( \varphi_4 \neq 0 \) and \( \varphi_5 \neq 0 \). Then \( u_2(t) = -\frac{s}{\varphi_5}u_1(t) \) on the ground of (13) and the equality \( \varphi_4x(t) + \varphi_5y(t) \equiv 0 \). Since \( F(u_1(t), u_2(t)) \equiv 1 \) then \( u_1(t) \equiv \frac{s\varphi_5}{F(s\varphi_5, -s\varphi_4)} \), \( s = \pm 1 \). This, (13), and the initial condition \( x(0) = z(0) = 0 \) imply that \( x(t) = \frac{s\varphi_5}{F(s\varphi_5, -s\varphi_4)} \), \( z(t) \equiv 0 \). By substitution the equalities \( y(t) = -\frac{s}{\varphi_5}x(t), \quad u_2(t) = -\frac{s}{\varphi_5}u_1(t), \) and \( z(t) \equiv 0 \) to the equations (14), we get \( \dot{v}(t) = w(t) \equiv 0 \), whence \( v(t) = w(t) \equiv 0 \) because of \( v(0) = w(0) = 0 \). In consequence of (2), the extremal is one of two one-parameter subgroups

\[
g(t) = \exp \left( \frac{st(\varphi_5X - \varphi_4Y)}{F(s\varphi_5, -s\varphi_4)} \right), \quad s = \pm 1, \quad t \in \mathbb{R},
\]

satisfies (25) with \( M(t) \equiv 1 \) for constant covector function

\[
\psi(t) = (\varphi_1, \varphi_2, 0, 0, 0) = \left( \frac{F(s\varphi_5, -s\varphi_4)}{2\varphi_5}, \frac{F(s\varphi_5, -s\varphi_4)}{2\varphi_4}, 0, 0, 0 \right) = (h_1(t), h_2(t), 0, 0, 0), \quad s = \pm 1,
\]

subject to differential equations (18) and (24); therefore it is normal relative to this covector function and is not strongly abnormal.

\[\square\]

2. Set \( M = 1 \).
Theorem 1. For every extremal on the Cartan group starting at the unit,

\[
\begin{align*}
(39) \quad x(t) &= \int_0^t \left[ r'(\theta(\tau)) \sin \theta(\tau) + r(\theta(\tau)) \cos \theta(\tau) \right] d\tau, \\
(40) \quad y(t) &= \int_0^t \left[ r(\theta(\tau)) \sin \theta(\tau) - r'(\theta(\tau)) \cos \theta(\tau) \right] d\tau
\end{align*}
\]

with arbitrary measurable integrands of indicated view and continuously differentiable function \( \theta = \theta(t) \), satisfying (29), (30). The functions \( z(t) \), \( v(t) \), \( w(t) \) are defined by formulae (15), (16), (17) or

\[
(41) \quad z(t) = \frac{1}{2} \int_0^t (x'y - y'x) d\tau,
\]

\[
(42) \quad v(t) = \frac{1}{3} \int_0^t (x'z - z'x) d\tau - \frac{1}{6} x(t) z(t), \quad w(t) = \frac{1}{3} \int_0^t (y'z - z'y) d\tau - \frac{1}{6} y(t) z(t).
\]

Proof. By Proposition 2, every extremal is normal for corresponding control. The proof of the first statement is completed as in the theorem 1 in [11]. The equalities (41), (42) are consequences of (13), (33), (34) and the initial condition \( z(0) = v(0) = w(0) = 0 \).

2.1. Let us assume that \( \varphi_3 = \varphi_4 = \varphi_5 = 0 \). The following proposition is true.

Proposition 3. For any extremal on the Cartan group with conditions \( \varphi_3 = \varphi_4 = \varphi_5 = 0 \) and the unit origin, \( \theta(t) \equiv \theta_0 \), \( t \in \mathbb{R} \), for some \( \theta_0 \). In addition, every such extremal is a one-parameter subgroup if and only if there exists the usual derivative \( r'(\theta_0) \). In general case, any extremal with conditions \( \varphi_3 = \varphi_4 = \varphi_5 = 0 \) is a metric straight line.

Proof. The first statement follows from (29).

In addition, by Theorem 1, every admissible control \((u_1(t), u_2(t)) = (u_1(\theta_0), u_2(\theta_0))\), with components equal to the integrands in (39), (40), is constant if and only if there exists the usual derivative \( r'(\theta_0) \), what is equivalent to condition that the system (13)–(14) has unique solution, a one-parameter subgroup

\[
\begin{align*}
x(t) &= u_1(\theta_0)t, \quad y(t) = u_2(\theta_0)t, \quad z(t) = v(t) = w(t) \equiv 0.
\end{align*}
\]

Notice that there exists at most countable number of values \( \theta_0 \) for which the second statement is false. For any such \( \theta_0 \), \( x(t), y(t), t \in \mathbb{R} \), are as in (39), (40) with \( \theta(\tau) \equiv \theta_0 \) and arbitrary measurable integrands \( u_1(\tau), u_2(\tau) \) of the type, indicated in Theorem 1 and the functions \( z(t), v(t) \) and \( w(t) \) are defined by formulas (15), (16) and (17) respectively.

It follows from (13) that the length of any arc for the curve \((x(t), y(t), z(t), v(t), w(t))\) in \((G, d)\) is equal to the length of corresponding arc for its projection \((x(t), y(t))\) on the Minkowski plane. \( z = v = w = 0 \). One can easily see that projections of indicated curves are metric straight lines on the Minkowski plane. Therefore the curves itself are metric straight lines. \( \square \)
Remark 2. The metric straight lines are obtained only in the case of Proposition 3 in particular, Proposition 2.

2.2. Let us consider the case \( \varphi_4 = \varphi_5 = 0, \varphi_3 \neq 0 \).

Proposition 4. Let \((x, y, z, v, w)(t), t \in \mathbb{R}, \) be an extremal with conditions \( x(0) = y(0) = z(0) = v(0) = w(0) = 0 \) on the Cartan group such that \( \varphi_4 = \varphi_5 = 0, \varphi_3 \neq 0 \). Then the functions \( \theta(t), h(t) = (h_1(t), h_2(t)), x(t), y(t) \) are periodic with joint period \( L = 2S_0/|\varphi_3| \), where \( S_0 \) is the area of the figure \( U^* \). The projection \((x, y)(t)\) of the extremal onto the Minkowski plane \( z = v = w = 0 \) with the quasinorm \( F \) has a form

\[
x(t) = \frac{h_2(t) - \varphi_2}{\varphi_3}, \quad y(t) = -\frac{h_1(t) - \varphi_1}{\varphi_3},
\]

and it is a parametrized by the arc length periodic curve on an isoperimetrix. In addition, \( h_1 = h_1(\theta(t)), h_2 = h_2(\theta(t)) \) are given by formulas \((22)\), \( \theta = \theta(t) \) is the inverse function to the function \( t(\theta) = \int_{\theta_0}^{\theta}(r^2(\xi)/\varphi_3)d\xi \), and

\[
z(t) = \frac{t - \varphi_1x(t) - \varphi_2y(t)}{2\varphi_3}
\]

is equal to oriented area on the Euclidean plane with the Cartesian coordinates \( x, y \), traced by rectilinear segment connecting the origin with the point \((x(\tau), y(\tau))\), \( \tau \in [0, t] \). The functions \( v(t), w(t) \) are defined by formulas \((10), (17) \) or \((43)\).

Proof. The statements on the function \( \theta(t) \) follow from \((22)\). It follows from \((26)\) and \((22)\) that analogously to the second Kepler law the radius-vector-function \( h(\tau) = (h_1(\tau), h_2(\tau)) \in U^*, t_1 \leq \tau \leq t_2 \), traces in the plane \( h_1, h_2 \) (or, if it is desired, \( u_1, u_2 \) or \( x, y \)) with the standard Euclidean metric the oriented area \((\varphi_3/2)(t_2 - t_1)\). Consequently, \( h(t), t \in \mathbb{R}, \) is a periodic function with period \( L = 2S_0/|\varphi_3|, \) where \( S_0 \) is the area of the figure \( U^* \). Moreover, \((22), (24)\) and \((13)\) imply formulas \((43)\), i.e. the projection \((x, y)(t)\) of the curve \((x, y, z, v, w)(t)\) lies on the boundary \( I(\varphi_1, \varphi_2, \varphi_3) \) of the figure obtained by rotation of \( U^*/|\varphi_3| \) by the angle \( \pi/2 \) around the center (origin of coordinates) with subsequent shift by vector \((-\varphi_1\varphi_3, -\varphi_2\varphi_3, -\varphi_3)\). Thus, analogously to the case of the Heisenberg group with left-invariant sub-Finsler metric, considered in \[3\], \( I(\varphi_1, \varphi_2, \varphi_3) \) is an isoperimetrix of the Minkowski plane with the quasinorm \( F \) \[12\].

Analogously to \[3\], it follows from \((43)\) that \((x(t), y(t))\) is a periodic curve on \( I(\varphi_1, \varphi_2, \varphi_3) \) with period \( L \) indicated above. It follows from \((35)\) and \((13)\) that

\[
z(t) = \frac{t - \varphi_1x(t) - \varphi_2y(t)}{2\varphi_3} = \frac{1}{2\varphi_3}(\varphi_3t - \varphi_1h_2(t) + \varphi_2h_1(t)),
\]

\[
z(L) = \frac{L}{2\varphi_3} = \frac{S_0}{|\varphi_3|}\varphi_3.
\]

The statement of Proposition 4 on the function \( z(t) \) follows from \((13)\). Since \((x(t), y(t))\) lies on the isoperimetrix passing clockwise (counterclockwise) if \( \varphi_3 < 0 \) (\( \varphi_3 > 0 \)), then \( z(t) \) is a monotone function. In particular, \( z(L) \) is the oriented area of the figure spanned by the isoperimetrix \( I(\varphi_1, \varphi_2, \varphi_3) \), or, what is the same, the area of \( U^*/|\varphi_3| \) taken with the sign equal to the sign of \( z(L) \).
The last statement was proved in Theorem 1.

2.3. Assume that $\varphi_3^2 + \varphi_5^2 \neq 0$.

**Lemma 1.** If $\varphi_5 \neq 0$ and the function $\theta(t)$ is constant on some non-degenerate interval $J \subset \mathbb{R}$, then on $J$

\begin{align*}
x(t) &= x_0 + \frac{\varphi_5}{E}(t-t_0), \quad y(t) = -\frac{\varphi_4}{E}(t-t_0) - \frac{1}{\varphi_5}(\varphi_3 + \varphi_4x_0), \quad z(t) = z_0 + \frac{\varphi_3}{2E}(t-t_0), \\
v(t) &= v_0 - \frac{\varphi_3\varphi_5}{12E^2}(t-t_0)^2 + \frac{1}{12E}(\varphi_3x_0 - 6\varphi_5z_0)(t-t_0), \\
w(t) &= w_0 + \frac{\varphi_3\varphi_5}{12E^2}(t-t_0)^2 + \frac{1}{12\varphi_5E}(6\varphi_4\varphi_5z_0 - 3\varphi_3\varphi_4x_0 - \varphi_3^2)(t-t_0),
\end{align*}

where $x_0 = x(t_0)$, $z_0 = z(t_0)$, $v_0 = v(t_0)$, $w_0 = w(t_0)$, $t_0$ is a point of the interval $J$ closest to zero, $E$ is the Casimir function \([23]\). Moreover, $E \neq 0$, $F(\varphi_5/E, -\varphi_4/E) = 1$, $w_0$ is calculated by $x_0$, $y_0 = -\left((\varphi_3 + \varphi_4x_0)/\varphi_5, z_0, v_0 \text{ and (35) for } t = t_0\right)$.

In particular, for $\varphi_3 = 0$, we have $E = \varphi_1\varphi_5 - \varphi_2\varphi_4$ and

\begin{align*}
x(t) &= x_0 + \frac{\varphi_5}{E}(t-t_0), \quad y(t) = -\frac{\varphi_4}{E}(t-t_0) - \frac{\varphi_4x_0}{\varphi_5}, \quad z(t) = z_0, \\
v(t) &= v_0 - \frac{\varphi_5z_0}{2E}(t-t_0), \quad w(t) = w_0 + \frac{\varphi_4z_0}{2E}(t-t_0).
\end{align*}

**Proof.** If $\varphi_5 \neq 0$ and $\theta(t) \equiv \theta_0$ on some non-degenerate interval $J$, then $\dot{\theta}(t) \equiv 0$ and, in consequence of \([29]\) and \([13]\),

\begin{equation}
y(t) = -\frac{1}{\varphi_5}(\varphi_3 + \varphi_4x(t)), \quad z(t) = \frac{\varphi_3}{2\varphi_5}(x(t) - x_0) + z_0, \quad t \in J.
\end{equation}

It follows from the first equation \([14]\) and \([13]\) that

\begin{equation}
\dot{v}(t) = -\frac{\varphi_3}{6\varphi_5}x(t)u_1(t) + \left(\frac{\varphi_3x_0}{4\varphi_5} - \frac{z_0}{2}\right)u_1(t),
\end{equation}

\begin{equation}
v(t) = -\frac{\varphi_3}{12\varphi_5}\left(x^2(t) - x_0^2\right) + \left(\frac{\varphi_3x_0}{4\varphi_5} - \frac{z_0}{2}\right)(x(t) - x_0) + v_0, \quad t \in J.
\end{equation}

The second equation \([14]\), \([13]\), and \([31]\) imply that

\begin{equation}
\dot{w}(t) = \frac{\varphi_3\varphi_4}{6\varphi_5^2}x(t)u_1(t) + \left(\frac{\varphi_3z_0}{2\varphi_5} - \frac{\varphi_3^2}{12\varphi_5^2} - \frac{\varphi_3\varphi_4}{4\varphi_5^2}x_0\right)u_1(t),
\end{equation}

\begin{equation}
w(t) = \frac{\varphi_3\varphi_4}{12\varphi_5^2}\left(x^2(t) - x_0^2\right) + \left(\frac{\varphi_3z_0}{2\varphi_5} - \frac{\varphi_3^2}{12\varphi_5^2} - \frac{\varphi_3\varphi_4}{4\varphi_5^2}x_0\right)(x(t) - x_0) + w_0, \quad t \in J.
\end{equation}

Inserting \([31]\) – \([33]\) into the equality \([33]\), we obtain

\begin{equation}
(\varphi_3^2 + 2\varphi_1\varphi_5 - 2\varphi_2\varphi_4)x(t) + 3\varphi_3\varphi_5z_0 + 6\varphi_4\varphi_5v_0 + 6\varphi_5^2w_0 - \varphi_3^2x_0 - 2\varphi_2\varphi_3 = 2\varphi_5t, \quad t \in J.
\end{equation}
This, (23), and the first equality in (13) imply that \( E \neq 0 \) and \( u_1(t) = \varphi_5/E \), thus \( x(t) = \frac{\varphi_3}{\varphi_4}(t-t_0) + x_0 \). Inserting the equality into (51) – (53), we get (46) – (48). Now it is easy to obtain the remaining statements.

**Corollary 1.** If \( \varphi_5 \neq 0 \), the function \( \theta(t) \) is constant on some non-degenerate interval \( J \subset \mathbb{R} \), and \( 0 \in J \), then on \( J \), \((x,y,z,v,w)(t)\) is an extremal (57) if \( \varphi_4 = 0 \), or an extremal (55) if \( \varphi_4 \neq 0 \).

**Proof.** In this case, it is more convenient to assume that \( t_0 = 0 \) under conditions of Lemma 1. Then \( x_0 = y_0 = z_0 = v_0 = w_0 = 0 \) and \( \varphi_3 = 0 \) on the ground of (29). Inserting these equalities and the equality \( E = sF(s\varphi_5, -s\varphi_4), s = \text{sgn}(E) \), into (49) and (50), we get the required statement.

**Lemma 2.** If \( \varphi_5 = 0 \), \( \varphi_4 \neq 0 \) and the function \( \theta(t) \) is constant on some non-degenerate interval \( J \subset \mathbb{R} \), then on \( J \)

\[(54) \quad x(t) = -\frac{\varphi_3}{\varphi_4}, \quad y(t) = y_0 - \frac{\varphi_4}{E}(t-t_0), \quad z(t) = z_0 + \frac{\varphi_3}{2E}(t-t_0), \]

\[(55) \quad v(t) = v_0 - \frac{\varphi_3^2}{12\varphi_4E}(t-t_0), \quad w(t) = w_0 + \frac{\varphi_4^2}{12E^2}(t-t_0)^2 + \frac{1}{12E}(\varphi_3y_0 + 6\varphi_4z_0)(t-t_0), \]

where \( y_0 = y(t_0), z_0 = z(t_0), v_0 = v(t_0), w_0 = w(t_0) \), \( t_0 \) is a point of the interval \( J \) closest to zero, \( E = \frac{\varphi_3^2}{2} - \varphi_2\varphi_4 \) is the Casimir function (23). Moreover, \( w_0 \) is calculated by \( x_0 = -\frac{\varphi_3}{\varphi_4}, y_0, z_0, v_0 \) and (33) for \( t = t_0 \).

In particular, for \( \varphi_3 = 0 \) we have \( E = -\varphi_2\varphi_4 \) and

\[(56) \quad x(t) \equiv 0, \quad y(t) = y_0 - \frac{\varphi_4}{E}(t-t_0), \quad z(t) \equiv z_0, \quad v(t) \equiv v_0, \quad w(t) = w_0 + \frac{\varphi_4z_0}{2E}(t-t_0). \]

**Proof.** If \( \varphi_5 = 0 \), \( \varphi_4 \neq 0 \) and \( \theta(t) \equiv \theta_0 \) on some non-degenerate interval \( J \), then \( \dot{\theta}(t) \equiv 0 \) and, due to (29) and (13),

\[(57) \quad x(t) = -\frac{\varphi_3}{\varphi_4}, \quad z(t) = -\frac{\varphi_3}{2\varphi_4}(y(t) - y_0) + z_0, \quad t \in J. \]

It follows from the first equation (14) and (13) that

\[
\dot{v}(t) = \frac{\varphi_3^2}{12\varphi_4^2}u_2(t), \quad \dot{w}(t) = \frac{\varphi_3}{6\varphi_4}y(t)u_2(t) - \left( \frac{z_0}{2} + \frac{\varphi_3y_0}{4\varphi_4} \right)u_2(t),
\]

hence for \( t \in J \),

\[(58) \quad v(t) = \frac{\varphi_3^2}{12\varphi_4^2}y(t)y_0y_0 + v_0, \quad w(t) = \frac{\varphi_3}{12\varphi_4} \left( y^2(t) - y_0^2 \right) - \left( \frac{z_0}{2} + \frac{\varphi_3y_0}{4\varphi_4} \right)(y(t) - y_0) + w_0. \]

Inserting the equalities (54), (58) into (35), we obtain

\[-(\varphi_3^2 - 2\varphi_2\varphi_4)y(t) - 2\varphi_1\varphi_3 + 6\varphi_1^2v_0 + \varphi_3^2y_0 + 3\varphi_3\varphi_4z_0 = 2\varphi_4t. \]

This, (23), and the second equation in (13) imply that \( E \neq 0 \) and \( u_2(t) = -\varphi_4/E \), thus \( y(t) = y_0 - \frac{\varphi_4}{E}(t-t_0) \). Inserting the equality into (57) and (58), we get the third equality (54) and (55). Now it is easy to obtain the remaining statements. \( \square \)
Corollary 2. If \( \varphi_5 = 0, \varphi_4 \neq 0 \), the function \( \theta(t) \) is constant on some non-degenerate interval \( J \subset \mathbb{R} \), and \( 0 \in J \), then the trajectory \( (x,y,z,v,w)(t) \) on the interval \( J \) is the extremal \( \text{(36)} \).

Proof. In this case, it is more convenient to assume that \( t_0 = 0 \) under conditions of Lemma 2. Then \( x_0 = y_0 = v_0 = w_0 = 0 \) and \( \varphi_3 = 0 \) on the ground of \( \text{(29)} \). Inserting these equalities and the equality \( \mathcal{E} = sF(0, -s\varphi_4), s = \text{sgn}(\mathcal{E}) \), into \( \text{(56)} \), we get the required statement.

Using \( \text{(27)} \), the equality \( \text{(30)} \) can be rewritten as

\[
(59) \quad r^4(\theta)^2 = \varphi_3^2 + 2\sqrt{\varphi_4^2 + \varphi_5^2} (r(\theta) \sin(\theta + \theta^*) - r(\theta_0) \sin(\theta_0 + \theta^*)) ,
\]

where

\[
(60) \quad \cos \theta^* = \frac{\varphi_4}{\sqrt{\varphi_4^2 + \varphi_5^2}}, \quad \sin \theta^* = -\frac{\varphi_5}{\sqrt{\varphi_4^2 + \varphi_5^2}}.
\]

Let \( \tilde{F} \) be a quasinorm on \( D(e) \), defined by the equality

\[
\tilde{F}(u_1, u_2) = F(u_1 \cos \theta^* + u_2 \sin \theta^*, -u_1 \sin \theta^* + u_2 \cos \theta^*), \quad (u_1, u_2) \in D(e),
\]

\( \hat{U} = \{ u \in D(e) | \tilde{F}(u) \leq 1 \} \) is the unit ball of the quasinorm \( \tilde{F} \). It follows from the definitions of \( \tilde{F} \) and its support Minkowski function that the curve \( F_{\hat{U}}(x, y) = 1 \), that is the polar boundary for the body \( \hat{U} \), is obtained from the curve \( F_U(x, y) = 1 \) with the rotation by angle \( \theta^* \) around the origin. Then \( \tilde{r}(\theta) = r(\theta - \theta^*), \theta \in \mathbb{R} \), is a polar equation of the curve \( F_{\hat{U}}(x, y) = 1 \). Set \( \tilde{\theta}(t) = \theta(t) + \theta^* \) and \( \tilde{\theta}_0 = \tilde{\theta}(0) = \theta_0 + \theta^* \). Then the equation \( \text{(59)} \) can be rewritten as

\[
(61) \quad \left( \frac{\dot{\tilde{\theta}}}{\tilde{r}(\theta)} \right)^2 = \frac{\varphi_3^2 + 2\sqrt{\varphi_4^2 + \varphi_5^2} (\tilde{r}(\tilde{\theta}) \sin(\tilde{\theta}) - \tilde{r}(\tilde{\theta}_0) \sin(\tilde{\theta}_0))}{\tilde{r}(\theta)}.
\]

Theorem 2. If \( \varphi_4^2 + \varphi_5^2 \neq 0 \) then any extremal on the Cartan group starting at the unit is defined by the equations \( \text{(30)}, \text{(41)} \) (with arbitrary measurable integrands of indicated view and continuously differentiable function \( \theta(t) \) satisfying \( \text{(30)}, \text{(56)} \)).

Moreover, if \( \varphi_5 \neq 0 \) then

\[
(62) \quad z(t) = -\frac{1}{\varphi_5} \left( \varphi_2 + \left( \varphi_3 + \frac{1}{2} \varphi_4 x(t) + \frac{1}{2} \varphi_5 y(t) \right) x(t) - r(\theta(t)) \sin \theta(t) \right),
\]

the function \( v(t) \) is given by the second formula \( \text{(42)} \), and

\[
(63) \quad w(t) = \frac{1}{3\varphi_5} \left( t - \varphi_1 x(t) - \varphi_2 y(t) - 2\varphi_3 z(t) - 3\varphi_4 v(t) - \frac{\varphi_4}{2} x(t) z(t) - \frac{\varphi_5}{2} y(t) z(t) \right).
\]

If \( \varphi_5 = 0 \) and \( \varphi_4 \neq 0 \) then

\[
(62) \quad z(t) = \frac{1}{\varphi_4} \left( \varphi_1 - \left( \varphi_3 + \frac{1}{2} \varphi_4 x(t) \right) y(t) - r(\theta(t)) \cos \theta(t) \right),
\]

\[
(63) \quad v(t) = \frac{1}{3\varphi_4} \left( t - \varphi_1 x(t) - \varphi_2 y(t) - 2\varphi_3 z(t) - \frac{\varphi_4}{2} x(t) z(t) \right).
\]
and the function $w(t)$ is given by the second formula (42).

Let us set

$$\theta_0 := \theta(0), \quad E_0 = \max_{h \in U^*} (\varphi_5 h_1 - \varphi_4 h_2), \quad E_{-1} = \min_{h \in U^*} (\varphi_5 h_1 - \varphi_4 h_2).$$

The following cases are possible.

1. Let $\varphi_3 \neq 0$ and $E > E_0$. Then the function $\theta(t)$, $t \in \mathbb{R}$, is inverse to the function $t(\theta)$, defined by formula

$$t(\theta) = \int_{\theta_0}^{\theta} \frac{r^2(\xi) d\xi}{\varphi_3 \sqrt{1 + (2\varphi_4/\varphi_3^2)(r(\xi) \sin \xi - \varphi_2) - (2\varphi_5/\varphi_3^2)(r(\xi) \cos \xi - \varphi_1)},$$

where

$$r(\theta_0) \cos \theta_0 = \varphi_1, \quad r(\theta_0) \sin \theta_0 = \varphi_2.$$  

2. Let $\varphi_3 = 0$ and $E = E_{-1}$. Then $\theta(t) \equiv \theta_0$ and the desired extremal is the metric straight line (36), (37) or (38).

3. Let $E_{-1} < E < E_0$. Then we have for some numbers $t_1$, $t_2$, $t_1 < t_2$, for any $t \in \mathbb{R}$ and $k \in \mathbb{Z}$

$$\theta(t + 2k(t_2 - t_1)) = \theta(t), \quad \dot{\theta}(t_i + t) = -\dot{\theta}(t_i - t), \quad t(t_i + t) = \theta(t_i - t), \quad i = 1, 2.$$

3.1. If $\varphi_3 \neq 0$ then $t_i = t(\theta_i)$, $i = 1, 2$, in equalities (66) are calculated by (64), where $\theta_1 \neq \theta_2$ are the nearest to $\theta_0$ values such that the right-hand side in (30) vanishes and $\varphi_3(\theta_2 - \theta_1) > 0$.

3.2. If $\varphi_3 = 0$ then $\theta_2 \neq \theta_1 = \theta_0$, $t_1 = 0$, $t_2 = t(\theta_2)$ in equalities (66), where

$$t(\theta) = \pm \frac{r^2(\xi) d\xi}{\sqrt{2\varphi_4(r(\xi) \sin \xi - \varphi_2) - 2\varphi_5(r(\xi) \cos \xi - \varphi_1)},$$

on the right-hand side stands + (respectively, −) if $\theta_2 > \theta_0$ ($\theta_2 < \theta_0$) and (63) holds. Here $\theta_2 \neq \theta_0$ is a number such that $\varphi_4(h_2(\theta) - h_2(\theta_0)) \geq \varphi_5(h_1(\theta) - h_1(\theta_0))$ for any $\theta$ from interval $I = (\min(\theta_0, \theta_1), \max(\theta_0, \theta_1))$, and the equality holds only for $\theta = \theta_0$ and $\theta = \theta_2$.

4. Let $\varphi_3 \neq 0$ and $E = E_0$. Then there exist the nearest to $\theta_0$ values $\theta_1$, $\theta_2$ such that $\theta_1 < \theta_0 < \theta_2$ and the right-hand side in (30) vanishes for $\theta = \theta_i$, $i = 1, 2$.

If improper integrals (64) diverge for $\theta = \theta_1$ and $\theta = \theta_2$, then $t(\theta(t))$ is the inverse function to the function $t(\theta)$ defined by (64).

If improper integral (64) is finite for $\theta = \theta_i$, $i \in \{1, 2\}$, then the function $t(\theta(t))$ is not unique and can take constant values equal to $\theta_1 + 2\pi k$ for some $k \in \mathbb{Z}$ (and with an arbitrary alternation of increase’s and decrease’s intervals) if $i = 1, 2$ and $\theta_2 = \theta_1 + 2\pi$, and equal to $\theta_1$ in other cases, on some non-degenerate closed intervals of arbitrary length, on which (40) – (43) are valid if $\varphi_5 \neq 0$ or (44), (45) are valid if $\varphi_5 = 0$.

5. Let $\varphi_3 = 0$ and $E = E_0$. Then there exists the largest segment $[\theta_1, \theta_2]$, $\theta_1 \leq \theta_2$, such that $\theta_0 \in [\theta_1, \theta_2]$ and $\varphi_5(h_1(\theta_1) - \varphi_1) = \varphi_4(h_2(\theta_1) - \varphi_2)$ for any $\theta \in [\theta_1, \theta_2]$. If $\theta_0 = \theta_2$ (respectively, $\theta_0 = \theta_1$) we will assume that $t(\theta)$ is an improper integral (64) for $\theta \in [\theta_0, \theta_1 + 2\pi]$ (respectively, $\theta \in [\theta_1 - 2\pi, \theta_0]$) without + and −.
Then $\theta(t) \equiv \theta_0$ and the desired extremal is the metric straight line (36), (37) or (38) in the following cases:

5.1. $\theta_1 = \theta_0 = \theta_2$ and $t(\theta) = \infty$ for $\theta \nearrow \theta_0$ and for $\theta \searrow \theta_0$;

5.2. $\theta_1 < \theta_0 < \theta_2$;

5.3. $\theta_0 = \theta_1 < \theta_2$ and $t(\theta) = \infty$ for $\theta \nearrow \theta_0$;

5.4. $\theta_0 = \theta_2 > \theta_1$ and $t(\theta) = \infty$ for $\theta \searrow \theta_0$.

In all other cases, the function $\theta(t)$ is not unique and can take constant values on some closed intervals of arbitrary length, on which (49), (50) are valid if $\varphi_5 \neq 0$, or (56) is valid if $\varphi_5 = 0$. These constant values may be equal to 1) $\theta_0$ and $\theta_1 + 2\pi$ if $\theta_1 < \theta_0 = \theta_2$, $t(\theta)$ is finite for $\theta \searrow \theta_0$, and $t(\theta_1 + 2\pi)$ is finite; 2) $\theta_0$ and $\theta_2 - 2\pi$ if $\theta_0 = \theta_1 < \theta_2$, $t(\theta)$ is finite for $\theta \nearrow \theta_0$, and $t(\theta_2 - 2\pi)$ is finite; 3) $\theta_0 + 2\pi k$ for some $k \in \mathbb{Z}$ (and with an arbitrary alternation of increase’s and decrease’s intervals) if $\theta_0 = \theta_1 = \theta_2$, $t(\theta)$ is finite for $\theta \nearrow \theta_0$ and for $\theta \searrow \theta_0$; 4) $\theta_0$ in all other cases.

Proof. The first statement of this Theorem follows from Theorem 1. Moreover, formulae for $z(t)$ are consequences of equalities (21) and (27), formulae for $w(t)$ in the case $\varphi_5 \neq 0$ and for $v(t)$ in the case $\varphi_5 = 0$ follow directly from (35). Formulae (64) and (67) follow from the equality (30), which can be written in the form (61).

All other statements of Theorem 2 follow from Lemmas 1 and 2 of our paper and from Theorem 2 and its proof in paper [13] for the following replacements in the last theorem:

$$\varphi_1 \Rightarrow \frac{\varphi_1 \varphi_4 + \varphi_2 \varphi_5}{\sqrt{\varphi_4^2 + \varphi_5^2}}, \quad \varphi_2 \Rightarrow \frac{\varphi_2 \varphi_4 - \varphi_1 \varphi_5}{\sqrt{\varphi_4^2 + \varphi_5^2}}, \quad \varphi_3 \Rightarrow \varphi_3, \quad \varphi_4 \Rightarrow \sqrt{\varphi_4^2 + \varphi_5^2},$$

$$h_1(\theta) \Rightarrow \frac{\varphi_4 h_1(\theta) + \varphi_5 h_2(\theta)}{\sqrt{\varphi_4^2 + \varphi_5^2}}, \quad h_2(\theta) \Rightarrow \frac{\varphi_4 h_2(\theta) - \varphi_5 h_1(\theta)}{\sqrt{\varphi_4^2 + \varphi_5^2}}, \quad \mathcal{E} \Rightarrow \mathcal{E}.$$

\[ \blacksquare \]

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