Comments on Topologically Massive Gravity with Propagating Torsion

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Abstract

We study and discuss some of the consequences of the inclusion of torsion in 3D Einstein-Chern-Simons gravity. Torsion may trigger the excitation of non-physical modes in the spectrum. Higher-derivative terms are then added up and tree-level unitarity is contemplated.
1 Introduction

The concept of torsion as the antisymmetric part of the affine connection was introduced by Cartan, in 1922. He recognized that torsion should be the geometrical object related to the intrinsic angular momentum of matter, and later, with the concept of spin, it was suggested that torsion should mediate a contact interaction between spinning particles without however propagating in matter-free space [1, 2]. After that, a number of authors reassessed the relation between torsion and spin in an attempt to formulate General Relativity as a gauge theory, with the motivation that, at a microscopic level, particles are classified by their mass and spin, according to the Poincaré group [3, 4]. In fact, torsion can be minimally coupled to matter fields: the requirement that the Dirac equation in a gravitational field preserve local invariance under Lorentz transformations yields a direct interaction between torsion and fermions. The observational consequences of a propagating torsion and its interactions with matter have been reported in [5, 6, 7].

Another important reason to consider torsion in gravity theory concerns the discussion of unitarity and renormalizability for the linearized version of the theory. It is well-known that, although standard Einstein gravity in four-dimensional space-time seems satisfactory as a classical theory, its quantum formulation presents a serious drawback: the non-renormalizability problem. In order to circumvent this undesirable aspect, several modifications in the theory have been proposed, such as the insertion of higher-derivative terms, which generally leads to a non-unitary theory. In this context, there appears string theory as the main promising framework to overcome the problems of quantum gravity. As a matter of fact, torsion is naturally incorporated in the string formulation of gravity. It is generally suggested that the rank-3 antisymmetric field of the low-energy effective Lagrangian dictated by string theory might be associated with torsion. Furthermore, the spin-2 component of the torsion field may affect the graviton propagator and can therefore modify the convergence properties of the theory.

On the other hand, the interest in discussing gravity in three-dimensional
space-time has been receiving a great deal of attention [8]. In spite of the fact that planar Einstein theory has no dynamical degrees of freedom, the introduction of a 3D topologically massive term [9, 10] yields a dynamical graviton and leads to a renormalizable model [11, 12, 13]. We believe that the introduction of new degrees of freedom into this theory (here, those coming from the torsion field) could open some trend for a better understanding of 3D quantum gravity in its full potentiality: the metric and torsion quantum fluctuations mix up and lead to a rich mass spectrum. Also, the power-counting renormalizability may be improved once torsion is allowed, although unitarity demands some extra care.

In this paper, we introduce torsion in the framework of topologically massive gravity theory via the usual "minimal" substitution, namely, by replacing the Christoffel symbol, \( \\{ \chi^{\lambda} \}_{\mu\nu} \), by the Cartan affine connection. In the same way as in 4D gravity, the curvature scalar in 3D does not provide any kinetic term for the torsion components, so that their field equations are purely algebraic. However, we see that dynamical contributions for the torsion arise from the Chern-Simons term. The main purpose of our work is to employ the spin-projection operator formalism to find out the propagators for the metric and torsion fields. In view of the results we attain, a ghost-free 3D gravity theory can be formulated once some constraints are imposed on the parameters of the Lagrangians we discuss [14, 15].

Our work is organized as follows: in Section 2, we provide a short review on Riemann-Cartan space-time. 3D torsion is decomposed into its SO(1,2)-irreducible components. Then, one presents the full Lagrangian for the theory in its linearized approximation. Section 3 is devoted to the attainment of the propagators for some specific classes of Lagrangians and to the check of the tree-level unitarity of the theory. Our Concluding Comments are summarized in Section 4.

2 Preliminaries

We propose to carry out our analysis of the propagating torsion by starting off from the following action for topologically massive gravity in three
dimensions:
\[ S = \int d^3x \left[ \sqrt{g} \left( a_1 \mathcal{R} + a_2 \mathcal{R}^2 + a_3 \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} \right) + a_4 \mathcal{L}_{cs} \right], \] (2.1)

where
\[ \mathcal{L}_{cs} = \varepsilon^{\mu\nu\kappa} \Gamma_{\kappa\lambda} \rho \left( \partial_\mu \Gamma_{\rho\lambda} + \frac{2}{3} \Gamma_{\mu\sigma} \lambda \Gamma_{\nu\rho} \sigma \right) \] (2.2)

is the topological Chern-Simons term. \( a_1, a_2 \) and \( a_3 \) are free coefficients, whereas \( a_4 \) is the parameter associated to Chern-Simons. One should notice that the usual action for the cosmological constant and the so-called translational Chern-Simons term [16] have not been included in (2.1).

Note that in three dimensions terms proportional to \( \mathcal{R}_{\mu\nu\kappa\lambda} \mathcal{R}^{\mu\nu\kappa\lambda} \) are not independent, due to the fact that the curvature tensor can be written in terms of the Ricci tensor (they have the same number of independent components).

We adopt here the Minkowski metric \( \eta_{\mu\nu} = (+; -, -) \) and the Ricci tensor \( \mathcal{R}_{\mu\nu} = \mathcal{R}_{\lambda\mu\nu} \).

Let us consider Riemann-Cartan space-time, where the affine connection is non-symmetric in the first two indices and is no longer completely determined by the metric [1]. The antisymmetric part of the affine connection is the torsion tensor:
\[ T_{\mu\nu}^\lambda = 2 \Gamma_{[\mu|\nu]}^\lambda = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda. \] (2.3)

In this space-time, the coefficients of the affine connection can be expressed in terms of the metric and torsion,
\[ \Gamma_{\mu\nu}^\lambda = \left\{ \lambda_{\mu\nu} \right\} + K_{\mu\nu}^\lambda, \] (2.4)

where \( \left\{ \lambda_{\mu\nu} \right\} \) is the Christoffel symbol, which is completely determined by the metric:
\[ \left\{ \lambda_{\mu\nu} \right\} = \frac{1}{2} g^{\kappa\lambda} \left( \partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu} \right), \] (2.5)

and
\[ K_{\mu\nu}^\lambda = \frac{1}{2} \left( T_{\mu\nu}^\lambda + T_{\nu\mu}^\lambda - T_{\nu\nu}^\lambda \right) \] (2.6)

is the contortion tensor, which is antisymmetric in the last two indices.

In 3D, the torsion with its 9 degrees of freedom can be covariantly split into its SO(1,2)-irreducible components: a trace part, \( t_\mu \equiv T_{\mu\nu}^\nu \), a totally antisymmetric part, \( \varphi \equiv \varepsilon^{\mu\nu\lambda} T_{\mu\nu\lambda} \), and a traceless rank-2 symmetric tensor,
The splitting in the above components is realized according to the following relation:

\[ T_{\mu\nu\lambda} = \varphi \epsilon_{\mu\nu\lambda} + \frac{1}{2} (\eta_{\nu\lambda} t_\mu - \eta_{\mu\lambda} t_\nu) + \epsilon_{\mu\nu\kappa} X^{\kappa}_{\lambda}. \] (2.7)

In order to read off the propagators and, consequently, the particle spectrum of the theory, we linearize the metric-dependent part of the Lagrangian by adopting the weak gravitational field approximation:

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \] (2.8)

where \( \kappa \) stands for the gravitational coupling constant in 3D. \( \kappa h_{\mu\nu} \) represents a small perturbation around flat Minkowski space-time.

The action is invariant under general coordinate transformations,

\[ \delta h_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x), \] (2.9)

where \( \xi_\mu \) is the gauge parameter. Therefore, it is necessary to fix this gauge invariance in order to make the wave operator of the Lagrangian non-singular. This is achieved by adding the De Donder gauge-fixing term,

\[ \mathcal{L}_{g\bar{f}} = \frac{1}{2\alpha} F_{\mu} F^\mu, \] (2.10)

where

\[ F_{\mu} = \partial_\nu \left( h^\nu_{\mu} - \frac{1}{2} \delta^\nu_{\mu} h \right), \] (2.11)

with \( h \equiv h^\mu_{\mu} \). No other gauge-fixing term is required, since the torsion field behaves like a tensor under general coordinate transformations.

First, let us confine ourselves to the study of the topologically massive gravity without the higher derivative terms \( R^2 \) and \( R_{\mu\nu} R^{\mu\nu} \) \( (a_2 = a_3 = 0) \). In this case, the action is the sum of Einstein, Chern-Simons and gauge-fixing terms. So, by decomposing the torsion into its irreducible components (2.7), and making use of the weak field approximation for the metric, the bilinear terms can be collected as below:

\[ \mathcal{L} = \frac{a_1 \kappa^2}{2} \left[ \frac{1}{2} h_{\mu
u} \Box h_{\mu
u} - \frac{1}{2} h \Box h + h \partial_\nu \partial_\mu h_{\mu\nu} - h_{\mu\nu} \partial_\mu \partial_\lambda h_{\lambda\nu} + \right. \] (2.12)

\[ \left. + \frac{1}{\kappa^2} (-3 \varphi^2 - t_\mu t^\mu + 2 X_{\mu\nu} X^{\mu\nu}) \right] + \]
\[+\frac{1}{2\alpha} \left[ -h^{\mu\nu} \partial_{\mu} \partial_{\nu} h^{\lambda}_{\nu} + h^{\mu\nu} \partial_{\mu} \partial_{\nu} h - \frac{1}{4} h \square h \right] +
\]
\[+ a_4 \left[ \frac{\kappa^2}{2} \varepsilon^{\mu\nu\lambda} (h_{\lambda}^{\kappa} \square \partial_{\mu} h_{\kappa\nu} - h_{\lambda}^{\kappa} \partial_{\mu} \partial_{\kappa} \partial_{\rho} h^{\rho}_{\nu}) +
\]
\[- \frac{3\kappa}{2} \left( X^{\mu\nu} \square h_{\mu\nu} + X^{\mu\nu} \partial_{\mu} \partial_{\nu} h - 2X^{\mu\nu} \partial_{\mu} \partial_{\lambda} h^{\lambda}_{\nu} \right) +
\]
\[- \frac{1}{2} \phi \partial_{\mu} t^\mu + \frac{1}{4} \varepsilon^{\mu\nu\lambda} t_{\nu} \partial_{\mu} t_{\lambda} + X^{\mu\nu} \partial_{\mu} t_{\nu} - \varepsilon^{\mu\nu\lambda} X_{\lambda}^{\kappa} \partial_{\mu} X_{\kappa\nu} \right], \]

where \( a_1 \) has dimension of mass, while the coefficient of the topological term, \( a_4 \), is dimensionless. Later on, suitable choice of these parameters will be adopted in order to avoid tachyon and ghost modes in the spectrum.

### 3 The Propagators

We now rewrite the linearized Lagrangian \( \mathcal{L} \) in a more convenient form, namely

\[
\mathcal{L} = \frac{1}{2} \sum_{\alpha \beta} \phi_{\alpha} \mathcal{O}_{\alpha\beta} \phi_{\beta},
\]

where \( \phi_{\alpha} = (h^{\mu\nu}, X^{\mu\nu}, t^\mu, \varphi) \) and \( \mathcal{O}_{\alpha\beta} \) is the wave operator. The propagators are given by

\[
\langle 0 | T [\phi_{\alpha} (x) \phi_{\beta} (y)] | 0 \rangle = i (\mathcal{O}_{\alpha\beta})^{-1} \delta^3 (x - y). \]

In order to invert the wave operator, we shall use an extension of the spin-projection operator formalism introduced in [17, 18], where one needs to add now other four new operators coming from the Chern-Simons and torsion terms. The operators for rank-2 symmetric tensors in 3D are given by:

\[
P^{(2)}_{\mu\nu,\kappa\lambda} = \frac{1}{2} (\theta_{\mu\kappa} \theta_{\nu\lambda} + \theta_{\mu\lambda} \theta_{\nu\kappa}) - \frac{1}{2} \theta_{\mu\nu} \theta_{\kappa\lambda},
\]
\[
P^{(1)}_{\mu\nu,\kappa\lambda} = \frac{1}{2} (\theta_{\mu\kappa} \omega_{\nu\lambda} + \theta_{\mu\lambda} \omega_{\nu\kappa} + \theta_{\nu\kappa} \omega_{\mu\lambda} + \theta_{\nu\lambda} \omega_{\mu\kappa}),
\]
\[
P^{(0-s)}_{\mu\nu,\kappa\lambda} = \frac{1}{2} \theta_{\mu\nu} \theta_{\kappa\lambda},
\]

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\[ P_{\mu\nu,\kappa\lambda}^{(0-w)} = \omega_{\mu\nu}\omega_{\kappa\lambda}, \]
\[ P_{\mu\nu,\kappa\lambda}^{(0-sw)} = \frac{1}{\sqrt{2}}\theta_{\mu\nu}\omega_{\kappa\lambda}, \]
\[ P_{\mu\nu,\kappa\lambda}^{(0-ws)} = \frac{1}{\sqrt{2}}\theta_{\kappa\lambda}\omega_{\mu\nu}, \]

where \( \theta_{\mu\nu} \) and \( \omega_{\mu\nu} \) are respectively the transverse and longitudinal projector operators for vectors. The other four operators coming from the bilinear terms are

\begin{align*}
S_{\mu\nu,\kappa\lambda} &= (\varepsilon_{\alpha\mu}\theta_{\nu\lambda} + \varepsilon_{\alpha\lambda}\theta_{\nu\mu} + \varepsilon_{\alpha\nu}\theta_{\mu\lambda} + \varepsilon_{\alpha\lambda\nu}\theta_{\mu\kappa}) \partial^\alpha, \\
R_{\mu\nu,\kappa\lambda} &= (\varepsilon_{\alpha\mu}\omega_{\nu\lambda} + \varepsilon_{\alpha\lambda}\omega_{\nu\mu} + \varepsilon_{\alpha\nu}\omega_{\mu\lambda} + \varepsilon_{\alpha\lambda\nu}\omega_{\mu\kappa}) \partial^\alpha, \\
A_{\mu\nu} &= \varepsilon_{\mu\nu\kappa}\partial^\kappa, \\
E_{\mu\nu,\kappa} &= \eta_{\mu\nu}\partial_\kappa + \eta_{\nu\kappa}\partial_\mu;
\end{align*}

\( A, E \) and \( R \) appear exclusively due to the inclusion of torsion.

We now present the relations between the above spin-projector operators. The products between the usual Barnes-Rivers operators in \( D=3 \) are given by

\begin{align*}
{P^i} - a {P^j} - b &= \delta^i_j \delta^{ab} {P^j} - b, \\
{P^i} - ab {P^j} - cd &= \delta^i_j \delta^{bc} {P^j} - a, \\
{P^i} - a {P^j} - bc &= \delta^i_j \delta^{ab} {P^j} - ac, \\
{P^i} - ab {P^j} - c &= \delta^i_j \delta^{bc} {P^j} - ac,
\end{align*}

and satisfy the following tensorial identity:

\[ \left( P^{(2)} + P^{(1)} + P^{(0-s)} + P^{(0-w)} \right)_{\mu\nu,\kappa\lambda} = \frac{1}{2} (\eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\kappa}). \]  

We list below some of the non-trivial relations involving the new projectors

\begin{align*}
SS &= -16\Box P^{(2)}, \\
RR &= -4\Box P^{(1)}, \\
SP^{(2)} &= S, \\
RP^{(1)} &= R.
\end{align*}
Thus, the wave operator can be decomposed in four sectors, namely

\[
O = \begin{pmatrix} M & N \\ P & Q \end{pmatrix},
\]

(3.20)

where the blocks \( M, N, P \) and \( Q \) are all \( 2 \times 2 \)-matrices. Their non-trivial elements, expanded in terms of the spin projection operators, are listed below:

\[
\begin{align*}
(M_{11})_{\mu\nu,\kappa\lambda} &= \frac{a_1\kappa^2}{2} P_{\mu\nu,\kappa\lambda}^{(2)} - \frac{\Box}{2\alpha} P_{\mu\nu,\kappa\lambda}^{(1)} - \frac{(a_1\kappa^2 + 1)\Box}{2\alpha} P_{\mu\nu,\kappa\lambda}^{(0-s)} + \\
&+ \frac{\sqrt{2}}{4\alpha} (P_{\mu\nu,\kappa\lambda}^{(0-s)} + P_{\mu\nu,\kappa\lambda}^{(0-\omega s)}) - \frac{\Box}{4\alpha} P_{\mu\nu,\kappa\lambda}^{(0-\omega)} + \frac{a_4\kappa^2}{4} S_{\mu\nu,\kappa\lambda}, \\
(M_{12})_{\mu\nu,\kappa\lambda} &= (M_{21})_{\mu\nu,\kappa\lambda} = -\frac{3a_4\kappa\Box}{2} (P_{\mu\nu,\kappa\lambda}^{(2)} - P_{\mu\nu,\kappa\lambda}^{(0-s)}), \\
(M_{22})_{\mu\nu,\kappa\lambda} &= 2a_1 (P_{\mu\nu,\kappa\lambda}^{(2)} + P_{\mu\nu,\kappa\lambda}^{(1)} + P_{\mu\nu,\kappa\lambda}^{(0-s)} + P_{\mu\nu,\kappa\lambda}^{(0-\omega)}) - \frac{a_4^2}{2} (S_{\mu\nu,\kappa\lambda} + R_{\mu\nu,\kappa\lambda}), \\
(N_{21})_{\mu\nu,\kappa} &= \frac{a_4}{2} E_{\mu\nu,\kappa}, \\
(P_{12})_{\mu,\nu\kappa} &= -\frac{a_4}{2} E_{\mu,\nu\kappa}, \\
(Q_{11})_{\mu\nu} &= -a_1 (\theta_{\mu\nu} + \omega_{\mu\nu}) + \frac{a_4}{2} A_{\mu\nu}, \\
(Q_{12})_{\mu} &= -(Q_{21})_{\mu} = \frac{a_4}{2} \partial_{\mu}, \\
(Q_{22}) &= -3a_1.
\end{align*}
\]

(3.21)

(3.22)

Before computing the inverse of the operator \( O \), it is perhaps worthwhile to discuss an interesting specific case. Note that, as already remarked, the massless Einstein theory in three-dimensional space-time is not dynamical; however, the introduction of the topological mass term leads to a dynamical theory with a physical propagating massive spin-two particle (if we take \( a_1 > 0 \)) [10]. In this context, we expect that the Chern-Simons term provides dynamics for the torsion components too. Then, let us now restrict the analysis to the Lagrangian of torsion in a flat space-time background, in order to get a first idea on the behaviour of torsion components from the point of view of the topological theory. Eliminating the graviton terms (due to our choice of a flat space-time) yields the Lagrangian density:

\[
\text{7}
\]
\[
\mathcal{L} = \frac{a_1}{2}(-3\varphi^2 - t_\mu t^\mu + 2X_{\mu\nu}X^{\mu\nu}) + \\
+ \frac{a_4}{4}(-2\varphi \partial_\mu t^\mu + \varepsilon^{\mu\nu\lambda\kappa}t_\nu \partial_\mu t_\lambda + \\
+ 4X^{\mu\nu} \partial_\mu t_\nu - 4\varepsilon^{\mu\nu\lambda}X^\kappa_\lambda \partial_\mu X_{\kappa\nu}).
\]

Here, one notices that the curvature scalar, \( R \), provides only mass terms for the fields, while the kinetic terms are all coming from the Chern-Simons sector.

As there are no derivatives of the scalar field, \( \varphi \), in the Lagrangian, its equation of motion leads to

\[
\varphi = -\frac{a_4}{6a_1} \partial_\mu t^\mu.
\]

Inserting (3.24) back into (3.23), one obtains a Lagrangian written exclusively in terms of \( t_\mu \) and \( X_{\mu\nu} \). Using the spin-projector algebra, the propagators are readily obtained. Their explicit form in momentum space are as follows:

\[
\langle XX \rangle = i \left\{ -\frac{a_1}{2(a_1^2p^2 - a_1^2)} P^{(2)} + \frac{1}{2a_1} \left( P^{(1)} + P^{(0-s)} \right) + \\
- \frac{(a_1^2p^2 - 12a_1^2)}{a_1(5a_1^2p^2 + 12a_1^2)} P^{(0-\omega)} - \frac{a_4}{8(a_1^2p^2 - a_1^2)} S + \frac{a_4}{8a_1^2} R \right\},
\]

\[
\langle t_\mu t_\nu \rangle = i \left\{ \frac{a_1}{a_1} \theta_{\mu\nu} - \frac{ia_1}{2a_1^2} A_{\mu\nu} - \frac{12ia_1}{5a_1^2p^2 + 12a_1^2} \omega_{\mu\nu} \right\},
\]

where we have suppressed the indices in the field \( X_{\mu\nu} \) and in the projectors \( P_{\mu\nu,\kappa\lambda}, S_{\mu\nu,\kappa\lambda} \) and \( R_{\mu\nu,\kappa\lambda} \).

From the above results, we see that this theory contains a massive non-tachyonic pole in the spin-2 sector of \( \langle X_{\mu\nu}X_{\kappa\lambda} \rangle \). On the other hand, (3.24) shows that the \( t_\mu \)-field describes only the propagation of a longitudinal tachyonic mode that does not couple to external conserved sources. The same holds true for the mixed propagator \( \langle X_{\mu\nu}t_\lambda \rangle \), which we do not quote above.

The next step is to check the tree-level unitarity of the theory. This is done by analysing the residues of the saturated propagator. The source with which we are going to saturate the \( \langle X_{\mu\nu}X_{\kappa\lambda} \rangle \)-propagator is compatible with
the symmetries of the theory and can be expanded in terms of a complete basis as follows:

\[
\tau_{\mu\nu}(p) = a(p)p_{\mu}p_{\nu} + b(p)p_{\mu}\tilde{p}_{\nu} + c(p)p_{(\mu}\varepsilon_{\nu)} + \\
+ d(p)\tilde{p}_{\mu}\tilde{p}_{\nu} + e(p)\tilde{p}_{(\mu}\varepsilon_{\nu)} + f(p)\varepsilon_{(\mu}\varepsilon_{\nu)},
\]

where \( p^\mu = (p^0, \vec{p}) \), \( \tilde{p}^\mu = (p^0, -\vec{p}) \), and \( \varepsilon^\mu = (0, \vec{\epsilon}) \) are linearly independent vectors. They satisfy the conditions:

\[
p_{\mu}\tilde{p}_{\mu} = (p^0)^2 + (\vec{p})^2 \neq 0,
\]

\[
p^\mu\varepsilon_{\mu} = 0,
\]

\[
\varepsilon^\mu\varepsilon_{\mu} = -1.
\]

The current-current transition amplitude in momentum space is the saturated propagator

\[
A = \tau^{\mu\nu}(p)\langle X_{\mu\nu}(p)X_{\kappa\lambda}(p)\rangle \tau_{\kappa\lambda}(p).
\]

Due to the source constraint \( p_\mu\tau^{\mu\nu} = 0 \), only the transverse projectors \( P^{(2)} \), \( P^{(0-s)} \) and \( S \) can give non-vanishing contributions to the amplitude. However, when we take the imaginary part of the residue of the amplitude at the massive pole, \( \mu^2 \equiv \left( \frac{a_1}{a_2} \right)^2 \), only the spin-2 sector of the propagator will contribute. Then, we obtain the following result:

\[
I m(\text{Res}A) = \lim_{p^2 \to \mu^2} -\frac{1}{2a_1}\left( |\tau_{\mu\nu}|^2 - \frac{1}{2} |\tau_{\mu\mu}|^2 \right).
\]

For a massive pole, the analysis can be done in the rest frame. In this case

\[
p^\mu = (\mu, 0, 0),
\]

\[
\tilde{p}^\mu = (\mu, 0, 0),
\]

\[
\varepsilon^\mu = (0, \vec{\epsilon}).
\]

Thus, by expressing the sources in terms of this basis, (3.32) becomes

\[
I m(\text{Res}A) = -\frac{1}{4a_1} |f|^2.
\]
Therefore, unlike the torsion-free theory, the sign of the Einstein part must be negative, namely

$$a_1 = -\frac{1}{\kappa^2},$$

otherwise the massive mode would become ghost-like.

From the results (3.25) and (3.34), it follows that the Lagrangian (3.23) describes the propagation of a ghost-free and non-tachyonic torsion with spin-2 and mass $\mu^2$. In fact, as we shall see next, the spin-2 component of torsion plays a central role here, in the sense that only this component can affect the spin-2 sector of the graviton propagator.

We now turn back to invert the complete wave operator (3.20). The procedure is straightforward, but tedious. We have chosen to invert the sectors $M$ and $Q$, using the algebra of the projectors listed in (3.17-3.19). Afterwards, one must invert $(M - NQ^{-1}P)$ and $(Q - PM^{-1}N)$. In so doing, the propagators are classified in four sectors, namely

$$O^{-1} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}, \quad (3.35)$$

where

$$X = (M - NQ^{-1}P)^{-1},$$

$$Z = -Q^{-1}PX,$$

$$W = (Q - PM^{-1}N)^{-1},$$

$$Y = -M^{-1}NW,$$

or explicitly, in terms of matrix elements $iX_{11} = \langle hh \rangle$, $iX_{12} = \langle hX \rangle$, ..., $iW_{22} = \langle \phi \phi \rangle$. Now, let us present the non-vanishing propagators in momentum space:

$$\langle hh \rangle = i \left\{ -\frac{8a_1(5a_4^2p^2 + 4a_1^2)}{\kappa^2p^2(a_4^2p^2 - 4a_1^2)^2} P^{(2)} + \frac{2\alpha}{p^2} P^{(1)} + \frac{8a_1}{\kappa^2p^2(9a_4^2p^2 - 4a_1^2)} P^{(0-s)} + \frac{4[k^2\alpha(9a_4^2p^2 - 4a_1^2) - 4a_1]}{\kappa^2p^2(9a_4^2p^2 - 4a_1^2)} P^{(0-w)} + \frac{8\sqrt{2}a_1}{\kappa^2p^2(9a_4^2p^2 - 4a_1^2)} (P^{(0-sw)} + P^{(0-ws)}) \right\} + \ldots \quad (3.37)$$
\( \langle X h \rangle = i \left\{ \frac{6a_4(a_2^4p^2 + 4a_1^2)}{k(a_4^2p^2 - 4a_1^2)^2} P^{(2)} - \frac{6a_4}{k(9a_4^2p^2 - 4a_1^2)} P^{(0-s)} + \frac{6\sqrt{2}a_4}{k(9a_4^2p^2 - 4a_1^2)} P^{(0-w)} - \frac{6a_4^2a_1}{k(a_4^2p^2 - 4a_1^2)^2} S \right\}, \) 

\( \langle XX \rangle = i \left\{ -\frac{2a_1(7a_4^2p^2 - 4a_1^2)}{(a_4^2p^2 - 4a_1^2)^2} P^{(2)} + \frac{1}{2a_1} P^{(1)} + \frac{2a_1}{(9a_4^2p^2 - 4a_1^2)} P^{(0-s)} - \frac{(a_4^2p^2 - 12a_1^2)}{2a_1(5a_4^2p^2 + 12a_1^2)} P^{(0-w)} + \frac{a_4(a_2^4p^2 + 2a_1^2)}{(a_4^2p^2 - 4a_1^2)^2} S + \frac{a_4}{8a_1^2} R \right\}, \)

\( \langle X_{\mu\nu}t_{\kappa} \rangle = \frac{a_4}{4a_1^2}(\theta_{\mu\nu}p_{\nu} + \theta_{\nu\mu}p_{\mu}) + \frac{6a_4}{5a_4^2p^2 + 12a_1^2} \omega_{\mu\nu} p_{\kappa}, \) 

\( \langle \varphi X_{\mu\nu} \rangle = i \frac{a_4p^2(a_3^2p^2 - 2a_1^2)}{5a_4^2p^2 + 12a_1^2} \omega_{\mu\nu}, \) 

\( \langle t_{\mu}t_{\nu} \rangle = i \frac{\theta_{\mu\nu} - \frac{i a_4}{a_1} A_{\mu\nu}}{2a_1^2} A_{\mu\nu} - \frac{12ia_1}{5a_4^2p^2 + 12a_1^2} \omega_{\mu\nu}, \) 

\( \langle t_{\mu}\varphi \rangle = \frac{2ia_4}{5a_4^2p^2 + 12a_1^2} p_{\mu}, \) 

\( \langle \varphi\varphi \rangle = -\frac{2i(a_2^2p^2 + 2a_1^2)}{a_1(5a_4^2p^2 + 12a_1^2)}, \)

Upon inspection of the poles of the propagators quoted above, we can state the following results:

(i) there appears an undesirable double pole at \( p^2 = \left( \frac{2a_1}{a_4} \right)^2 \) located in the spin-2 sector of the \( h_{\mu\nu} - X_{\kappa\lambda} \) propagators (we shall handle this situation below);

(ii) there is a massive mode at \( p^2 = \left( \frac{2a_1}{3a_4} \right)^2 \) in the spin-0 sector of the \( h_{\mu\nu} - X_{\kappa\lambda} \) propagators;
(iii) a tachyonic pole shows up at \( p^2 = -\frac{12a_1^2}{5a_4^2} \) in the longitudinal sector, \( P^{(0-w)} \), of the \( X_{\mu\nu} - t_\kappa \) propagators (it does not however contribute to the residue of the current-current amplitude whenever the sources are transverse).

We now search for the constraints on the parameters of the theory that follow from the requirement of having a positive-definite residue matrix at the pole. Using (3.37-3.39), one obtains the following residue of the saturated propagator at the pole \( p^2 = \left( \frac{2a_1}{3a_4} \right)^2 \equiv m_1^2 : 

\[
\text{Im}(\text{Res}\mathcal{A}) = \lim_{p^2 \to m_1^2} \left( \begin{array}{cc}
\tau^* & \sigma^*
\end{array} \right) \left( \begin{array}{cc}
-\frac{8a_1}{p^2a_4^2} & -6a_4 \\
-6a_4 & -2a_1a_4^2
\end{array} \right) P^{(0-s)} \left( \begin{array}{c}
\tau \\
\sigma
\end{array} \right),
\]

(3.45) where we have redefined the \( X_{\mu\nu} \)-field, \( X'_{\mu\nu} = \kappa X_{\mu\nu} \), in order that \( X_{\mu\nu} \) and \( h_{\mu\nu} \) have the same dimension. \( \tau \) is the external current associated to the graviton field and \( \sigma \) the one associated to the torsion. The above residue matrix has one non-trivial positive eigenvalue (non-ghost) by choosing \( a_1 < 0 \), which corresponds to a physical spin-0 mode.

Furthermore, we see that the inclusion of torsion into the Einstein-Chern-Simons theory, by replacing the Christoffel symbol by the Cartan connection, leads to a theory in which the spin-2 propagators contain second order poles, and unitarity is consequently violated [19]. To overcome this undesirable situation, we propose the introduction of the following higher-derivative terms:

\[ a_2 R^2 \] and \( a_3 R^\mu R^\nu \), as well as the new coupling term, \( a_5 X^\mu R^\nu \), into the original Lagrangian, where the tilde in \( \tilde{R}^\mu \) means that we are considering only the Riemannian part of this tensor. The meaning of the last term will become clear later.

Confining ourselves, for simplicity, only to the \( h_{\mu\nu} - X_{\kappa\lambda} \) sector of the Lagrangian, since the other blocks do not affect the spin-2 sector of the propagators, we can be concerned only with the inversion of the M-sector of the wave operator, including of course the contributions from the new coupling terms into its matrix elements. In this case, the operator we shall deal with reads as follows:

\[
M = \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix},
\]

(3.46)
where

\[
M_{11} = M_{11,0} + \frac{a_3}{2} \kappa^2 \Box^2 P^{(2)} + \left(\frac{3}{2} a_3 + 4 a_2 \right) \kappa^2 \Box^2 P^{(0-s)},
\]

\[
M_{12} = M_{21} = M_{12,0} + \frac{a_3 \kappa}{4} \Box S + \frac{a_5 \kappa}{4} \Box \left( P^{(2)} - P^{(0-s)} \right),
\]

\[
M_{22} = M_{22,0} - a_3 \Box \left( 2 P^{(2)} + P^{(1)} + P^{(0-s)} + \frac{1}{2} P^{(0-w)} \right),
\]

where the \(M_{\ldots,0}\)’s are the previous matrix elements coming from the curvature scalar and Chern-Simons terms, listed in (3.21). Proceeding like in (3.36), we may work out the \(h_{\mu
u} - \Sigma_{\kappa\lambda}\) propagators.

It now becomes clear the importance of the new coupling term \(a_5 X^{\mu\nu} \tilde{R}_{\mu\nu}\): it has the same operator structure as \(M_{12,0}\); therefore, if we set \(a_5 = 6 a_4\), and \(a_3 = 0\), the torsion field decouples from the graviton and consequently they can be treated independently. On the other hand, if \(a_3 \neq 0\), \(a_5 = 6 a_4\), and \(a_1 = 0\), one obtains the following results

\[
\langle hh \rangle = i \left\{ \frac{2 a_3}{p^2 \kappa^2 (9 a_3^2 p^2 - 4 a_4^2)} P^{(2)} + \frac{2 a_3}{p^2} P^{(1)} + \frac{2}{p^4 \kappa^2 (3 a_3 + 8 a_2)} P^{(0-s)} + \right. \\
+ \frac{4 \left[ \alpha (3 a_3 + 8 a_2) \kappa^2 p^2 + 1 \right]}{p^4 \kappa^2 (3 a_3 + 8 a_2)} P^{(0-w)} + \right.
\]

\[
+ \frac{2 \sqrt{2}}{p^4 \kappa^2 (3 a_3 + 8 a_2)} \left( P^{0-s w} + P^{0-w s} \right) - \frac{(3 a_3^2 p^2 - 2 a_4^2)}{2 a_4 p^4 \kappa^2 (9 a_3^2 p^2 - 4 a_4^2)} S \right\},
\]

\[
\langle Xh \rangle = -i \left\{ \frac{a_3^2}{a_4 \kappa (9 a_3^2 p^2 - 4 a_4^2)} P^{(2)} + \frac{a_3^2}{2 p^2 \kappa (9 a_3^2 p^2 - 4 a_4^2)} S \right\},
\]

\[
\langle XX \rangle = i \left\{ \frac{2 a_3}{(9 a_3^2 p^2 - 4 a_4^2)} P^{(2)} + \frac{a_3^2}{(9 a_3^2 p^2 - 4 a_4^2)} P^{(1)} + \right. \\
+ \frac{1}{a_3 p^2} P^{(0-s)} + \frac{2}{a_3 p^2} P^{(0-w)} + \right.
\]

\[
+ \frac{a_4}{2 p^2 (a_3^2 p^2 - a_4^2)} R - \frac{(3 a_3^2 p^2 - 4 a_4^2)}{a_4 p^2 (9 a_3^2 p^2 - 4 a_4^2)} S \right\}.
\]

Using (3.48-3.50), one obtains the following residue of the saturated propagator at the pole \(p^2 = (\frac{2 a_4}{3 a_3})^2 \equiv m_2^2\).
\[ I m(\text{Res} \mathcal{A}) = \lim_{p^2 \to m_2^2} \left( \tau^* \sigma^* \right) \left( \frac{2a_3}{p^2 \kappa^2} - \frac{a_2^2}{a_4} \right) P^{(2)} \left( \tau \sigma \right), \quad (3.51) \]

From (3.51), we obtain the following condition on the parameters of the Lagrangian for not having a ghost at the massive pole:

\[ a_3 = a_4 \kappa^2, \quad a_4 > 0. \quad (3.52) \]

Furthermore, we remark that the residue matrix of eq. (3.51) provides two positive eigenvalues, so that this theory propagates a massive spin-2 graviton along with a massive spin-2 quantum of the torsion.

As for the massless pole present in the \( h_{\mu\nu} - X_{\kappa\lambda} \) propagators, it is harmless: indeed, like the case where torsion is absent, it is non-dynamical, yielding a vanishing residue for the current-current amplitude. So, the physical excitations propagated as gravitational degrees of freedom are all massive.

## 4 Concluding Comments

The main purpose of our investigation was the assessment of the rôle torsion may play in the framework of three-dimensional gravity in the presence of the topological mass term.

Several peculiarities have been found out. For instance, the appearance of a (gauge-independent) double pole in the spin-2 sector of the graviton and the torsion propagators, which spoils the unitarity of the model. However, as we have checked, the inclusion of higher powers of the curvature \( (\mathcal{R}^2 \text{ and } \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}) \) and a mixing term between the Ricci tensor and the spin-2 part of the torsion may restore unitarity, for the parameters may be so chosen that the double pole is suppressed.

We have contemplated the possibility of switching off the graviton degrees of freedom and we have considered the propagation of torsion on a flat background. In such a situation, no double pole shows up and we have checked that the tree-level unitarity is ensured.

The three different spin sectors carried by the torsion tensor in 3D gravity do not share common characteristics as long as the dynamics is concerned:
in the case in which only the Einstein-Hilbert and the Chern-Simons terms are present, the scalar part, \( \varphi \), behaves like an auxiliary field, whereas the spin-1 sector, \( t_\mu \), propagates only its longitudinal component and the spin-2 sector, \( X_{\mu \nu} \), is dynamical. We have not here coupled matter to gravity with torsion. Nevertheless, in so doing, minimal coupling to fermions selects only the scalar component of the torsion to couple directly to fermion bilinears \cite{21}.

With the results we have got here, it might be worthwhile to extend our analysis to 3D supergravity and try to understand how the dynamics of torsion and its fermionic counterparts comes out.

Also, in view of the finiteness of pure gravity described by the Chern-Simons term \( \mathcal{L}_C \), we should inquire, once we know how torsion propagates and interacts, whether or not finiteness is kept whenever torsion effects are included. This matter is now being considered and the results shall be reported elsewhere \cite{21}.

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