Thermodynamic Volume Product in Spherically Symmetric and Axisymmetric Spacetime

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Abstract

In this Letter, we have examined the thermodynamic volume products for spherically symmetric and axisymmetric spacetimes in the framework of extended phase space. Such volume products usually formulated in terms of the outer horizon ($H^+$) and the inner horizon ($H^-$) of black hole (BH) spacetime. Besides volume product, the other thermodynamic formulations like volume sum, volume minus and volume division are considered for a wide variety of spherically symmetric spacetime and axisymmetric spacetimes. Like area (or entropy) product of multihorizons, the mass-independent (universal) feature of volume products are sometimes also fail. In particular for a spherically symmetric AdS spacetimes the simple thermodynamic volume product of $H^\pm$ is not mass-independent. In this case, more complicated combinations of outer and inner horizon volume products are indeed mass-independent. For a particular class of spherically symmetric cases i.e. Reissner Nordström BH of Einstein gravity and Kehagias-Sfetsos BH of Horava Lifshitz gravity, the thermodynamic volume products of $H^\pm$ is indeed universal. For axisymmetric class of BH spacetime in Einstein gravity all the combinations are mass-dependent. There has been no chance to formulate any combinations of volume product relation is to be mass-independent. Interestingly, only the rotating BTZ black hole in 3D provides the volume product formula is mass-independent i.e. universal and hence it is quantized.

1 Introduction

It has been examined by a number of researchers that the area (or entropy) product of various spherically symmetric and axisymmetric BHs are mass-independent (universal) [1, 2, 3, 4, 5, 6, 7, 8, 9]. For instance, Ansorg and Hennig [1] demonstrated that for a stationary and axisymmetric class of Einstein-Maxwell gravity the area product formula satisfied the universal relation as

$$A_h A_c = (8\pi J)^2 + (4\pi Q)^2.$$  \hspace{1cm} (1)

where $A_h$ and $A_c$ are area of outer horizon (OH) or event horizon (EH) and inner horizon (IH) or Cauchy horizon (CH). The parameters, $J$ and $Q$ are denoted as the angular momentum and charge of the black hole (BH) respectively.

On the other hand, Cvetic et al. [2] extended this work for a higher dimensions spacetime and showed that for multihorizon BHs the area product formula should be quantized by satisfying the following relation

$$A_h A_c = (8\pi \ell_P^2)^2 N, \quad N \in \mathbb{N}.$$ \hspace{1cm} (2)

where $\ell_P$ is the Planck length. This relation indicates that the product relation is indeed universal in nature. This is a very fascinating topic of research since 2009.

Aspects of BH thermodynamic properties have started by the seminal work of Hawking and Page [10], they first proposed that certain type of phase transition occurs between small and large BHs in case of Schwarzschild-AdS BH. This phase transition is now called the famous Hawking-Page phase transition. For a charged AdS BH, the study of thermodynamic properties initiated

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by Chamblin et al. \cite{11,12}, where the authors demonstrated that the critical behaviour of Van-der-Waals liquid-gas phase transitions. This has been brought into a new form by Kubizňák and Mann \cite{13} by examining the thermodynamic properties i.e. $P-V$ criticality of Reissner-Nordström AdS BH in the extended phase space. They determined the BH equation of state and computed the critical exponent by using the mean field theory and also computed the other thermodynamic features.

Motivated by the above mentioned work and our previous investigation \cite{14} in which we have considered the extended phase space framework for a wide variety of spherically symmetric AdS spacetime. In the present work, we would like to extend our study for various classes of spherically symmetric BHs and axisymmetric BHs. In the extended phase formalism, the cosmological constant is treated as thermodynamic pressure $P$ and its conjugate variable as thermodynamic volume $V$ \cite{15,16,17,13}. They are defined as

\begin{equation}
\frac{P}{8\pi} = \frac{\Lambda}{8\pi\ell^2} . \tag{3}
\end{equation}

and

\begin{equation}
V = \left( \frac{\partial M}{\partial P} \right)_{S,Q,J} . \tag{4}
\end{equation}

The extended phase space is more meaningful than conventional phase space due to the following reasons. The conventional phase space allows the physical parameters like temperature, entropy, charge and potential etc. whereas the extended phase space allows the parameters like pressure, volume and enthalpy (rather than internal energy). In addition to that the mass parameter should be considered there as enthalpy of the system, which is useful to study the critical behaviour of the thermodynamic system. The BH equation of state could be used to study for comparisons with the classical thermodynamic equation of state (Van der-Waals equation). Once the BH thermodynamic equation of state is in hand then one may compute different thermodynamic quantities like isothermal compressibility, specific heat at constant pressure etc.

This thermodynamic volume of a spherically symmetric BH and for OH should read

\begin{equation}
V_h = \frac{4}{3} \pi r_h^3 = \frac{A_h r_h}{3} . \tag{5}
\end{equation}

where $r_h$ is OH radius. Similarly, this volume for IH should be

\begin{equation}
V_c = \frac{4}{3} \pi r_c^3 = \frac{A_c r_c}{3} . \tag{6}
\end{equation}

It should be noted that the thermodynamic volume of CH can be obtain by using the symmetric properties \cite{14} of OH radius $r_h$ and IH radius $r_c$ i.e.

\begin{equation}
V_c = V_h |_{r_h \rightarrow r_c} . \tag{7}
\end{equation}

Another important point in the extended phase space is that the ADM mass should be treated as the total enthalpy of the thermodynamic system i.e. $M = H = U + PV$. Where $U$ is thermal energy of the system \cite{15}. Therefore the first law of BH thermodynamics in this phase space for any spherically symmetric spacetime and for OH should be

\begin{equation}
dH = T_h dS_h + V_h dP + \Phi_h dQ . \tag{8}
\end{equation}

where the quantities $T_h$, $S_h$ and $\Phi_h$ are denoted as the BH temperature, entropy and electric potential of OH. The parameter $Q$ is denoted as the charge of a BH.

Analogously, the first law of BH mechanics for IH should be

\begin{equation}
dH = -T_c dS_c + V_c dP + \Phi_c dQ . \tag{9}
\end{equation}

where the quantities $T_c$, $S_c$ and $\Phi_c$ are denoted as the corresponding BH temperature, entropy and electric potential which could be defined on the IH.

\footnote{There are different types of definitions regarding the volume of a BH in the literature. The idea regarding the BH volume was first introduced by Parikh \cite{31}. For other types of definition like dynamical volume and vector volume (see Ref. \cite{32,33,34}). Here we are particualry interested regarding the thermodynamic volume \cite{18}.}
When we add the rotation parameter the first law of BH thermodynamics in the extended phase space (for axisymmetric spacetime and for OH) becomes
\[
dH = T_h dS_h + V_h dP + \Phi_h dQ + \Omega_h dJ .
\]
where \( \Omega_h \) and \( J \) are the angular velocity defined on the OH and the angular momentum of BH. For IH, the first law becomes
\[
dH = -T_c dS_c + V_c dP + \Phi_c dQ + \Omega_c dJ .
\]
where \( \Omega_c \) is the angular velocity defined on the IH. Using symmetric features of \( r_h \) and \( r_c \), one can determine the following thermodynamic relations for IH
\[
\mathcal{A}_c = \mathcal{A}_h|_{r_c \leftrightarrow r_h}, S_c = S_h|_{r_c \leftrightarrow r_h}, \Omega_c = \Omega_h|_{r_c \leftrightarrow r_h}, \Phi_c = \Phi_h|_{r_c \leftrightarrow r_h}
\]
\[
T_c = -T_h|_{r_c \leftrightarrow r_h}, V_c = V_h|_{r_c \leftrightarrow r_h} .
\]
However in this work, we wish to extend our study by computing the volume product, volume sum, volume minus and volume division in the extended phase space for various spherically symmetric BHs and axisymmetric BHs (including the various AdS spacetimes). By evaluating these quantities we prove that for a spherically symmetric AdS spacetime the simple volume product is \textit{not} mass-independent. In this case, somewhat complicated combination of volume functional relations of OH and IH are indeed mass-independent. For instance, we have derived the mass-independence volume functional relation for RN-AdS BH as
\[
f(V_h, V_c) = \ell^2 ,
\]
where
\[
f(V_h, V_c) = \left( \frac{3}{32\pi} \right)^{\frac{1}{2}} \left( \frac{8\pi\ell^2 Q^2}{3(V_h V_c)^2} \right)^{\frac{1}{2}} - \left( \frac{3}{4\pi} \right)^{\frac{1}{2}} \left[ V_h \mathcal{V} + V_c \mathcal{V} + (V_h V_c)^{\frac{3}{2}} \right] .
\]
For simple Reissner Nordström BH (which is a spherically symmetric solution of Einstein equation) of Einstein gravity and Kehagias-Sfetsos BH of Hořava Lifshitz gravity, the thermodynamic volume products of \( \mathcal{H}^\pm \) is mass-independent. Therefore they behaves as a universal character by its own features. Moreover, we have derived the thermodynamic volume functional relation for Hořava Lifshitz-AdS BH and phantom AdS BH. The phantom fields are exotic because it was produced via negative energy density. Furthermore we have derived volume functional relation for regular BH. Regular BH is a kind of BH which is free from a curvature singularity.

Whereas for axisymmetric class of BHs including AdS spacetimes there has been no chance to formulate any possible combinations of thermodynamic volume product is to be mass-independent. It should be noted that for a KN-AdS BH, there may be a possibility to formulate the area (or entropy) product relations are mass-independent. The reason is that for a simple Kerr BH, the area (or entropy) product is universal i.e. mass-independent while the \textit{volume product} is not! Because the thermodynamic volume is proportional to the spin parameter. That’s why \textit{there has been no chance to produce any combinations of volume product of \( \mathcal{H}^\pm \) is mass-independent. Therefore the axisymmetric BHs showing that \textit{no} universal behaviour for volume products. Interestingly, only rotating BTZ BH shows that the mass-independent feature. Thus \textit{only} axisymmetric BHs in 3D provided that the universal character of thermodynamic volume product.

In our previous investigation\[5, 7\], we computed the BH area (or entropy) products, BH temperature products, Komar energy products and specific heat products for various classes of BHs. Besides the area (or entropy) product, it should be important to study whether the thermodynamic \textit{volume product}, \textit{volume sum}, \textit{volume minus} and \textit{volume division} for all the horizons are universal or not and whether they should be quantized or not. This is the main motivation behind this work.

The structure of the paper is as follows. In Sec. 2, we shall compute the various thermodynamic volume products for spherically symmetric BHs and conclude that the product is mass-independent. In Sec. 3, we compute various thermodynamic volume products for axisymmetric spacetime and conclude that the product is mass-dependent. Interestingly, for the spinning BTZ BH the said volume product is \textit{mass-independent}. 

3
2 Spherically Symmetric BH

In this section we would consider various spherically symmetric BHs.

2.1 Reissner Nordström BH

We begin with charged BH with zero cosmological constant which is a solution of Einstein equation. The metric form is given by

\[ ds^2 = -Z(r)dt^2 + \frac{dr^2}{Z(r)} + r^2d\Omega^2, \]  

(15)

where,

\[ Z(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \]  

(16)

and \(d\Omega^2\) is metric on the unit sphere in two dimensions.

The OH radius and IH radius reads

\[ r_h = M + \sqrt{M^2 - Q^2}, \]  

(17)

\[ r_c = M - \sqrt{M^2 - Q^2}. \]  

(18)

where \(M\) and \(Q\) denotes the mass and charge of BH respectively. When \(M^2 > Q^2\), it describes a BH otherwise it has a naked singularity. The thermodynamic volume for OH and IH should read

\[ V_h = \frac{4}{3}\pi r_h^3, \]  

(19)

\[ V_c = \frac{4}{3}\pi r_c^3. \]  

(20)

The thermodynamic volume product for OH and IH should be

\[ V_h V_c = \frac{16}{9}\pi^2 Q^6. \]  

(21)

It is indeed mass-independent thus it is universal in character and it is also quantized.

The volume sum for OH and IH is calculated to be

\[ V_h + V_c = \frac{32}{3}\pi M^3 \left(1 - \frac{3Q^2}{4M^2}\right). \]  

(22)

Similarly, one can compute the volume minus for OH and IH as

\[ V_h - V_c = \frac{32}{3}\pi M^2 \sqrt{M^2 - Q^2} \left(1 - \frac{Q^2}{4M^2}\right). \]  

(23)

and the volume division should be

\[ \frac{V_h}{V_c} = \left(\frac{M + \sqrt{M^2 - Q^2}}{M - \sqrt{M^2 - Q^2}}\right)^3. \]  

(24)

It follows from the calculation that all these quantities are mass dependent so they are not universal in nature by its own right. From Eq. (23) and Eq. (24), we can easily see that in the extremal limit \(M^2 = Q^2\), one obtains \(V_h = V_c\). This is a new condition of extreme limit in spherically symmetric cases.

\(^2\)There are several definitions of horizons for a static spherically symmetric spacetime. We have used Killing horizons for computations of thermodynamic volume.

\(^3\)In the limit \(Q = 0\), one obtains the thermodynamic volume for Schwarzschild BH. Since in this case the BH has only OH located at \(r_h = 2M\). Therefore the volume should be \(V_h = \frac{32}{3}\pi M^3\). Thus for an isolated Schwarzschild BH the thermodynamic volume should be mass dependent therefore it is not universal and not quantized in nature by its own character.
2.2 Hořava Lifshitz BH

In this section, we would briefly review the UV complete theory of gravity which is a non-relativistic renormalizable theory of gravity known as Hořava Lifshitz [27, 28, 29] gravity. It reduces to Einstein’s gravity at large scales for the value of dynamical coupling constant \( \lambda = 1 \). Using ADM formalism, one could write the metric as

\[
ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt). \tag{25}
\]

In addition for a spacelike hypersurface with a fixed time the extrinsic curvature \( K_{ij} \) is given by

\[
K_{ij} = \frac{1}{2N}(g_{ij} - \nabla_i N_j - \nabla_j N_i). \tag{26}
\]

where a dot represents a derivative with respect to \( t \). The generalized action for Hořava Lifshitz could be written as

\[
S = \int dt d^3x \sqrt{g}N \left[ \frac{\kappa^2}{2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2(\Lambda w R - 3\Lambda^2)}{8(1 - 3\lambda)} + \frac{\kappa^2 \mu^2(1 - 4\lambda)}{32(1 - 3\lambda)}R^2 \right.
\]

\[
- \frac{\kappa^2}{2w^4}(C_{ij} - \frac{\mu w^2}{2}R_{ij})(C^{ij} - \frac{\mu w^2}{2}R^{ij}) + \mu^4R \right]. \tag{27}
\]

Here \( \kappa^2, \lambda, \mu, w \) and \( \Lambda \) are the constant parameters and the cotton tensor, \( C_{ij} \) is defined to be

\[
C^{ij} = \epsilon^{ijkl} \nabla_k (R_l^j - \frac{1}{4} \epsilon^{ikj} \partial_k R). \tag{28}
\]

As compared with Einstein’s general relativity, one could obtain the speed of light, Newtonian constant and the cosmological constant as

\[
c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda w}{1 - 3\lambda}} \tag{29}
\]

\[
G = \frac{\kappa^2}{32\pi c} \tag{30}
\]

\[
\Lambda = \frac{3}{2} \Lambda w. \tag{31}
\]

respectively. It should be mentioned here that when \( \lambda = 1 \), the first three terms in Eq.(27) reduces to that one obtains as in Einstein’s gravity. It must also be noted that \( \lambda \) is a dynamic coupling constant and for \( \lambda > \frac{1}{4} \), the cosmological constant should be a negative one. However, it could be made a positive one if one could give a following transformation like \( \mu \rightarrow i\mu \) and \( w^2 \rightarrow -iw^2 \). Here we restrict ourselves that the BH solution in the limit of \( \Lambda w \rightarrow 0 \). That’s why, we have to set \( N^i = 0 \) and to get the spherically symmetric solution we have to choose the metric ansatz as

\[
ds^2 = -N^2(r) dt^2 + \frac{dr^2}{\alpha(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{32}
\]

In order to get the spherically symmetric solution, substituting the metric ansatz (32) into the action and one obtains reduced Lagrangian as

\[
\mathcal{L} = \frac{\kappa^2 \mu^2 N}{8(1 - 3\lambda)\sqrt{g}} \left[(2\lambda - 1)\frac{(g - 1)^2}{r^2} - 2\lambda \frac{g - 1}{r} g' + g - 1 \frac{g^2}{2} - 2\omega(1 - g - r g') \right]. \tag{33}
\]

where \( \omega = \frac{8\alpha^2(3\lambda - 1)}{\kappa^4} \). Here we are interested to investigate the situation \( \lambda = 1 \) i.e. \( \omega = \frac{16\alpha^2}{\kappa^4} \). Then one finds the solution of the metric (20) as

\[
N^2(r) = g = 1 - \sqrt{4M \omega r + \omega^2 r^4} + \omega r^2, \tag{34}
\]

where \( M \) is an integration constant related to the mass parameter. Thus the static, spherically symmetric solution is given by

\[
ds^2 = -g(r) dt^2 + \frac{dr^2}{\alpha(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{35}
\]
For \( r \gg (\frac{\omega}{M})^{\frac{1}{3}} \), one gets the usual behavior of a Schwarzschild BH. The BH horizons correspond to \( g(r) = 0 \). The OH radius and IH radius should read

\[
\begin{align*}
\text{r}_h &= M + \sqrt{M^2 - \frac{1}{2\omega}}, \\
\text{r}_c &= M - \sqrt{M^2 - \frac{1}{2\omega}}.
\end{align*}
\]

where \( M \) and \( \omega \) denotes the mass and coupling constant of BH respectively. When \( M^2 > \frac{1}{2\omega} \), it describes a BH and when \( M^2 < \frac{1}{2\omega} \), it describes a naked singularity.

The thermodynamic volume product for KS BH should be

\[
\text{V}_h \text{V}_c = \frac{2\pi^2}{9\omega^3}.
\]

It indicates that it is mass-independent therefore it is universal in nature and it also be quantized. We do not calculate other possible combination because it is clear that these combinations are surely mass dependent as we have seen in case of RN BH. It should be mentioned that the Smarr formula satisfied in case of Einstein-Aether theory and some variants of infrared HL gravity [21]. It would be interesting if one could examine what is the status of HL gravity when the extended phase space formalism is applied. It could be found elsewhere.

### 2.3 Non-rotating BTZ BH

The non-rotating BTZ BH is a solution of Einstein-Maxwell gravity in three spacetime dimension. The metric form is given by

\[
ds^2 = -\left(\frac{r^2}{\ell^2} - M\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{\ell^2} - M\right)} + r^2d\phi^2.
\]

where \( M \) is the ADM mass of the BH and \( -\Lambda = \frac{1}{\ell^2} = 8\pi G M^3 \) denotes the cosmological constant. Here we have set \( c = h = k = 1 \). The BH OH is located at \( r_h = \sqrt{8G M \ell^3} \). \( G \) is 3D Newtonian constant. Interestingly, the thermodynamic volume for 3D static BTZ BH computed in [17]

\[
\text{V}_h = \frac{\pi r_h^3}{8\pi G M^3} = 8\pi G M \ell^2.
\]

This is an isolated case and the thermodynamic volume is mass dependent thus it is not quantized as well as it is not universal. Where \( \Lambda = -\frac{1}{\ell^2} \) is cosmological constant.

### 2.4 Schwarzschild-AdS BH

This BH is a solution of Einstein equation. The form of the metric function is given by

\[
\mathcal{Z}(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2},
\]

where \( \Lambda = -\frac{3}{\ell^2} \) is cosmological constant. The horizon radii could be calculated from the following equation

\[
r^3 + \ell^2 r - 2M \ell^2 = 0.
\]

Among the three roots, only one root is real. Therefore the BH posess only one physical horizon which is located at

\[
r_h = \left(\frac{\ell}{3}\right)^\frac{1}{3} \left(9M + \sqrt{3\ell^2 + 27M^2}\right)^\frac{1}{3} - \left(\frac{\ell}{3}\right)^\frac{1}{3} \frac{1}{(9M + \sqrt{3\ell^2 + 27M^2})^\frac{1}{3}}.
\]

\[\text{We have already mentioned that in the extended phase space } \ell = \sqrt{\frac{3}{8\pi P^3}} \text{. In the subsequent expression we have to put this condition to obtain the results in terms of thermodynamic pressure.}\]
The thermodynamic volume is computed to be

\[ V_h = \frac{4}{3} \pi r_h^3 = \frac{4}{3} \pi \left( \frac{\ell}{3} \right)^{\frac{3}{2}} \left( 9M + \sqrt{3} \sqrt{\ell^2 + 27M^2} \right)^{\frac{1}{2}} - \left( \frac{\ell^4}{3} \right)^{\frac{1}{2}} \left( \frac{1}{9M + \sqrt{3} \sqrt{\ell^2 + 27M^2}} \right)^{\frac{1}{2}} \right)^3. \]  

(44)

Since it is an isolated case and the thermodynamic volume is mass-dependent therefore it is not universal nor does it quantized. We do not considered the other AdS spacetime because it has already been discussed \[14\].

### 2.5 RN-AdS BH

For this BH, the metric function is given by

\[ Z(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}, \]  

(45)

The horizon radii could be found from the following equation

\[ r^4 + \ell^2 r^2 - 2M\ell^2 r + \ell^4 Q^2 = 0. \]  

(46)

Among the four roots, two roots are real and two roots are imaginary. Thus the OH and IH radii becomes

\[ r_{h,c} = \frac{1}{2} \sqrt{\frac{1}{3} \left( \frac{x}{2} \right)^{\frac{3}{2}} + \left( \frac{2}{x} \right)^{\frac{3}{2}} \ell^2 (\ell^2 + 12Q^2) - \frac{2\ell^2}{3} \pm \frac{1}{2} \sqrt{\frac{4M\ell^2}{\sqrt{\frac{1}{3} \left( \frac{x}{2} \right)^{\frac{3}{2}} + \left( \frac{2}{x} \right)^{\frac{3}{2}} \ell^2 (\ell^2 + 12Q^2) - \frac{2\ell^2}{3}}} - \frac{4\ell^2}{3}. \]  

(47)

where

\[ x = 2\ell^6 + 108M^2\ell^4 - 72\ell^4 Q^2 + \sqrt{(2\ell^6 + 108M^2\ell^4 - 72\ell^4 Q^2)^2 - 4\ell^6 (\ell^2 + 12Q^2)^3}. \]

The thermodynamic volume product of RN-AdS BH for OH and IH is computed to be

\[ V_h V_c = \frac{\pi^2}{36} \left[ \frac{2}{3} \left( \frac{x}{2} \right)^{\frac{3}{2}} + \left( \frac{2}{x} \right)^{\frac{3}{2}} \ell^2 (\ell^2 + 12Q^2) - \frac{2\ell^2}{3} \right]^3 + \frac{4M\ell^2}{\sqrt{\frac{1}{3} \left( \frac{x}{2} \right)^{\frac{3}{2}} + \left( \frac{2}{x} \right)^{\frac{3}{2}} \ell^2 (\ell^2 + 12Q^2) - \frac{2\ell^2}{3}}} \right]^{\frac{1}{2}} - \frac{4\ell^2}{3}. \]  

(48)

It is clearly evident from the above expression that the product is strictly mass-dependent. Thus the product is not universal. But below we would like to determine that somewhat complicated function of inner and outer horizon volume is indeed mass-independent. To proceed it we would like to use the Vieta’s theorem. Therefore from Eq. \[10\], we get

\[ \sum_{i=1}^{4} r_i = 0. \]  

(49)

\[ \sum_{1 \leq i < j \leq 4} r_i r_j = \ell^2. \]  

(50)

\[ \sum_{1 \leq i < j < k \leq 4} r_i r_j r_k = 2M\ell^2. \]  

(51)

\[ \prod_{i=1}^{4} r_i = \ell^2 Q^2. \]  

(52)

Hence the mass-independent volume sum and volume product relations are

\[ \sum_{i=1}^{4} V_i^{\frac{1}{2}} = 0. \]  

(53)

\[ \sum_{1 \leq i < j \leq 4} (V_i V_j)^{\frac{1}{3}} = \left( \frac{3}{8\pi} \right)^{\frac{1}{2}} \frac{8\pi \ell^2}{3}. \]  

(54)

\[ \prod_{i=1}^{4} (V_i)^{\frac{1}{3}} = \left( \frac{\pi}{6} \right)^{\frac{1}{2}} \frac{8\pi \ell^2 Q^2}{3}. \]  

(55)
The mass independent volume functional relation in terms of two horizons are

\[
f(V_h, V_c) = \ell^2. \tag{56}
\]

where

\[
f(V_h, V_c) = \left(\frac{3}{32\pi}\right)^\frac{3}{2} \frac{\left(\frac{4\pi^2\mathcal{Q}^2}{3}\right)}{(V_h V_c)^{\frac{3}{4}}} - \left(\frac{3}{4\pi}\right)^\frac{3}{2} \left[V_h \hat{V} + V_c \hat{V} + (V_h V_c) \hat{V}\right]. \tag{57}
\]

These are explicitly mass-independent volume functional relations in the extended phase space.

### 2.6 Horava Lifshitz-AdS BH

The metric function for Hořava Lifshitz BH in AdS space [35, 38] is given by

\[
\mathcal{Z}(r) = 1 + \left(1 - \frac{2\Lambda}{3\omega}\right)\omega r^2 - \omega r^2 \sqrt{1 - \frac{4\Lambda}{3\omega} + \frac{4M}{\omega r^3}}. \tag{58}
\]

The horizon radii could be calculated from the following equation

\[
4r^4 + 2(\omega\ell^2 + 2)\ell^2 r^2 - 4M\omega\ell^4 r + \ell^4 = 0. \tag{59}
\]

Similarly, among the four roots, two roots are real and two roots are imaginary. Thus the OH and IH radii becomes

\[
r_h = \frac{(a+b)}{2}, \quad r_c = \frac{(a-b)}{2}. \tag{60}
\]

where

\[
a = \sqrt{2\frac{\omega^2\ell^8 + 4\omega\ell^6 + 16\ell^4 - \frac{4\pi^2\mathcal{Q}^2}{3}}{27y^4 - \frac{12}{\omega r^3}} + \frac{1}{12} \left(\frac{y}{2}\right)^\frac{3}{2}}
\]

\[
b = \sqrt{\frac{2M\omega\ell^4}{a} - \frac{1}{12} \left(\frac{y}{2}\right)^\frac{3}{2} - \frac{2\frac{\omega^2\ell^8 + 4\omega\ell^6 + 16\ell^4 + \frac{4\pi^2\mathcal{Q}^2}{3}}{27y^4 - \frac{12}{\omega r^3}}}{3y^4}
\]

and

\[
y = 1728M^2\omega^2\ell^8 - 576\omega^2(\omega^2\ell^2 + 2) + 16\ell^6(\omega^2\ell^2 + 2)^3 + \sqrt{\left[1728M^2\omega^2\ell^8 - 576\omega^2(\omega^2\ell^2 + 2) + 16\ell^6(\omega^2\ell^2 + 2)^3\right]^2 - 256\ell^4(\omega^2\ell^4 + 4\omega\ell^2 + 16)^3}. \tag{61}
\]

The thermodynamic volume for this BH is quite different from RN-AdS spacetime and it has been calculated in [38]

\[
V_h = \frac{4}{3}\pi r_h^3 \left[\frac{4}{\ell^2} + \frac{1}{\omega r_h^3}\right]. \tag{62}
\]

and

\[
V_c = \frac{4}{3}\pi r_c^3 \left[\frac{4}{\ell^2} + \frac{1}{\omega r_c^3}\right]. \tag{63}
\]

The volume product is calculated to be

\[
V_h V_c = \frac{16\pi^2}{9} \left[\frac{2\frac{\omega^2\ell^8 + 4\omega\ell^6 + 16\ell^4 + \frac{4\pi^2\mathcal{Q}^2}{3}}{27y^4 - \frac{12}{\omega r^3}}}{3y^4} + \frac{1}{6} \left(\frac{y}{2}\right)^\frac{3}{2} - \frac{2M\omega\ell^4}{a}\right] \times \left[\frac{1}{4\ell^2} \left[\frac{2\frac{\omega^2\ell^8 + 4\omega\ell^6 + 16\ell^4 + \frac{4\pi^2\mathcal{Q}^2}{3}}{27y^4 - \frac{12}{\omega r^3}}}{3y^4} + \frac{1}{6} \left(\frac{y}{2}\right)^\frac{3}{2} - \frac{2M\omega\ell^4}{a}\right] + \frac{1}{\omega^2} + \frac{1}{\omega r^3} \left[\frac{2M\omega\ell^4}{a} - \frac{2\frac{4\pi^2\mathcal{Q}^2}{3}}{3y^4}\right]\right]. \tag{64}
\]
From the above expression we can conclude that the volume product for Hořava Lifshitz BH in AdS space is strictly mass dependent. Thus this product is not a universal quantity. Below we will derive more complicated function of inner and outer horizon volume is indeed mass-independent. To compute it we should apply the Vieta’s theorem. Thus from Eq. (69), we find

\[ \sum_{i=1}^{4} r_i = 0 \quad (65) \]

\[ \sum_{1 \leq i < j \leq 4} r_i r_j = \frac{\omega \ell^4}{2} \left( 1 + \frac{2}{\omega \ell^2} \right) \quad (66) \]

\[ \sum_{1 \leq i < j < k \leq 4} r_i r_j r_k = M \omega \ell^4 \quad (67) \]

\[ \sum_{1 \leq i < j < k < l \leq 4} r_i r_j r_k r_l = \frac{\ell^4}{4} \quad (68) \]

Eliminating the mass parameter in terms of two horizons one could obtain the following mass-independent volume functional relation

\[ g(V_h, V_c) = \frac{\omega \ell^4}{2} \left( 1 + \frac{2}{\omega \ell^2} \right) \quad (69) \]

where

\[ g(V_h, V_c) = \left( \frac{\ell}{r_h r_c} \right)^4 - \left( r_h^2 + r_c^2 + r_h r_c \right) \quad (70) \]

where the parameters \( r_h \) and \( r_c \) could be obtain by solving the Eq. (62) and Eq. (63) in terms of thermodynamic volume as

\[ r_h = \left( \frac{u_h}{9} \right)^{\frac{1}{3}} \frac{1}{\omega} - \frac{2\ell^2}{(3u_h)^{\frac{2}{3}}} \quad (71) \]

\[ r_c = \left( \frac{u_c}{9} \right)^{\frac{1}{3}} \frac{1}{\omega} - \frac{2\ell^2}{(3u_c)^{\frac{2}{3}}} \quad (72) \]

and

\[ u_h = \sqrt{3} \sqrt{8\omega^3 \ell^6 + 27\ell^4 \omega^6 + 27\ell^4 \omega^6 \left( \frac{3V_h}{4\pi} \right)^3} - 9\ell^2 \omega \left( \frac{3V_h}{4\pi} \right) \quad (73) \]

\[ u_c = \sqrt{3} \sqrt{8\omega^3 \ell^6 + 27\ell^4 \omega^6 + 27\ell^4 \omega^6 \left( \frac{3V_c}{4\pi} \right)^3} - 9\ell^2 \omega \left( \frac{3V_c}{4\pi} \right) \quad (74) \]

Now the Eq. (69) is completely mass independent volume functional relation.

### 2.7 Thermodynamic Volume Products for Phantom BHs

In this section, we would like to discuss the thermodynamic volume products for phantom AdS BH [36]. The fact that phantom fields are exotic fields in BH physics. It could be generated via negative energy density. It could explain the acceleration of our universe. Thus one could expected that this exotic fields might have an important role in BH thermodynamics. We want to study here what is the key role of this phantom fields in thermodynamic volume functional relation? This is the main motivation behind this work. For phantom BH, the metric function is given by

\[ Z(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \eta \frac{Q^2}{r^2} \quad (75) \]

where the parameter \( \eta \) determines the nature of electro-magnetic (EM) field. For \( \eta = 1 \), one obtains the classical EM theory but when \( \eta = -1 \), one obtains the Maxwell field which is phantom.

Therefore for phantom BH, the horizon radii could be found from the following equation

\[ r^4 + \ell^2 r^2 - 2M \ell^2 r - \ell^2 Q^2 = 0 \quad (76) \]
The above equation has four roots, among them the two roots are real and other two roots are imaginary. Thus the OH and IH radii are

\[ r_{h,c} = \frac{1}{2} \sqrt{\frac{1}{3} \left( \frac{z}{2} \right)^{\frac{1}{3}} + \left( \frac{2}{z} \right)^{\frac{1}{3}} \frac{\ell^2(\ell^2 - 12Q^2)}{3} - \frac{2\ell^2}{3} \pm \frac{1}{3} \left( \frac{z}{2} \right)^{\frac{1}{3}} - \left( \frac{2}{z} \right)^{\frac{1}{3}} \frac{\ell^2(\ell^2 - 12Q^2)}{3} - \frac{4\ell^2}{3} } . \]  

(77)

where

\[ z = 2\ell^6 + 108M^2\ell^4 + 72\ell^4Q^2 + \sqrt{(2\ell^6 + 108M^2\ell^4 + 72\ell^4Q^2)^2 - 4\ell^6(\ell^2 - 12Q^2)^3} \]

Now we compute the thermodynamic volume product which turns out to be

\[ V_hV_c = \frac{\pi^2}{36} \left[ \frac{2}{3} \left( \frac{z}{2} \right)^{\frac{1}{3}} + \left( \frac{2}{z} \right)^{\frac{1}{3}} \frac{2\ell^2(\ell^2 - 12Q^2)}{3} + \frac{2\ell^2}{3} \frac{4M^2}{\sqrt{\frac{\frac{1}{3}}{\left( \frac{z}{2} \right)^{\frac{1}{3}} + \left( \frac{2}{z} \right)^{\frac{1}{3}} \frac{2\ell^2(\ell^2 - 12Q^2)}{3} - \frac{2\ell^2}{3} } . \right)^3 } . \]

(78)

The above product indicates that it is strictly mass-dependent. Therefore the product is not universal. Below we would like to prove that more complicated function of inner and outer horizon volume is indeed mass-independent.

To do this we would like to use the Vieta’s theorem. Thus from Eq. (76), we find

\[ \sum_{i=1}^{4} r_i = 0 . \]  

(79)

\[ \sum_{1 \leq i < j \leq 4} r_ir_j = \ell^2 . \]  

(80)

\[ \sum_{1 \leq i < j < k \leq 4} r_ir_jr_k = 2M\ell^2 . \]  

(81)

\[ \prod_{i=1}^{4} r_i = -\ell^2Q^2 . \]  

(82)

Thus the mass-independent volume sum and volume product relations should read

\[ \sum_{i=1}^{4} V_i^{\frac{1}{3}} = 0 . \]  

(83)

\[ \sum_{1 \leq i < j \leq 4} (V_iV_j)^{\frac{1}{3}} = \left( \frac{3}{32\pi} \right)^{\frac{1}{3}} \frac{8\pi\ell^2}{3} . \]  

(84)

\[ \prod_{i=1}^{4} (V_i)^{\frac{1}{3}} = \left( \frac{\pi}{6} \right)^{\frac{1}{3}} \frac{8\pi\ell^2Q^2}{3} . \]  

(85)

Therefore the mass independent volume functional relations in terms of two horizons are

\[ f(V_h, V_c) = -\ell^2 . \]  

(86)

where

\[ f(V_h, V_c) = \left( \frac{3}{32\pi} \right)^{\frac{1}{3}} \frac{8\pi\ell^2Q^2}{3} \left[ V_h^{\frac{1}{3}} + V_c^{\frac{1}{3}} \right] + \left( \frac{3}{4\pi} \right)^{\frac{1}{3}} \left[ V_h^{\frac{1}{3}} + V_c^{\frac{1}{3}} + (V_hV_c)^{\frac{1}{3}} \right] . \]  

(87)

These are explicitly mass-independent volume functional relations in the extended phase space.
2.8 Thermodynamic Volume Products for AdS BH in $f(R)$ Gravity

In this section we are interested to derive the thermodynamic volume products for a static, spherically symmetric AdS BH in $f(R)$ gravity. To some extent it is called modified gravity. It is a very crucial tool for explaining the current and future status of the accelerating universe. Thus it is very important to investigate the thermodynamic volume products for this gravity. The metric function for this kind of gravity can be written as

$$Z(r) = 1 - \frac{2m}{r} + \frac{q^2}{\alpha r^2} - \frac{R_0}{12} r^2 .$$

(88)

where $\alpha = 1 + f'(R_0)$. The parameters $m$ and $q$ are related to the ADM mass, $M$ and electric charge, $Q$ by the following expression

$$m = \frac{M}{\alpha}, \quad q = \sqrt{\alpha Q} .$$

(89)

In this gravity, the thermodynamic pressure could be written as $P = \frac{\alpha}{3} \frac{\alpha}{\sqrt{\alpha Q}}$ and the scalar curvature constant as $R_0 = -\frac{12}{\alpha^2} = 4\Lambda$. Thus the horizon equation for $f(R)$ gravity becomes

$$r^4 + \ell^2 r^2 - 2m\ell^2 r + \frac{\ell^2 q^2}{\alpha} = 0 .$$

(90)

The EH radius and CH radius are

$$r_h = \frac{(a + b)}{2}, \quad r_c = \frac{(a - b)}{2} .$$

(91)

where

$$a = \sqrt{\left(\frac{1}{3\alpha} \left(\mu \right)^\frac{2}{3} \right) + \left(\frac{2}{\mu} \right) \frac{\ell^2 (\alpha \ell^2 + 12q^2) - 2\ell^2}{3} }$$

(92)

and

$$b = \sqrt{\left(\frac{4m\ell^2}{a} - \frac{1}{3\alpha} \left(\mu \right)^\frac{2}{3} \right) - \left(\frac{2}{\mu} \right) \frac{\ell^2 (\alpha \ell^2 + 12q^2) - 4\ell^2}{3} } .$$

(93)

As is the volume products for $f(R)$ gravity derived to be

$$V_h V_c = \frac{\pi^2}{36} \left[ \frac{2}{3\alpha} \left(\mu \right)^\frac{2}{3} + \left(\frac{2}{\mu} \right) \frac{2\ell^2 (\alpha \ell^2 + 12q^2) - 4m\ell^2}{3} - \frac{4m\ell^2}{a} \right] .$$

(94)

It indicates that the volume product is not mass-independent. Now we shall give an alternative approach where we would see that more complicated function of volume functional relation is quite mass-independent. To derive it, we should use the Vieta’s theorem then one could find

$$\sum_{i=1}^{4} r_i = 0 .$$

(95)

$$\sum_{1 \leq i < j \leq 4} r_i r_j = \ell^2 .$$

(96)

$$\sum_{1 \leq i < j < k \leq 4} r_i r_j r_k = 2m\ell^2 .$$

(97)

Eliminating third and fourth roots, the mass independent volume functional relation derived as

$$f(V_h, V_c) = \ell^2 .$$

(98)

where

$$f(V_h, V_c) = \left(\frac{3}{32\pi} \right)^\frac{2}{3} \frac{8\pi\ell^2 q^2}{3} - \left(\frac{3}{4\pi} \right)^\frac{2}{3} \left[ V_h \frac{2}{3} + V_c \frac{2}{3} + (V_h V_c) \frac{2}{3} \right] .$$

(99)

This equation is explicitly mass-independent.
2.9 Thermodynamic Volume Products for Regular BH

In this section we compute the thermodynamic volume products for a regular BH derived by Ayón-Beato and García (ABG) [37, 23]. It is a spherically symmetric solution of Einstein’s general relativity and it is a curvature singularity free solution. The metric function form of ABG BH is given by

\[ Z(r) = 1 - \frac{2m^2}{(r^2 + q^2)\frac{2}{3}} + \frac{q^2 r^2}{(r^2 + q^2)^2}. \]  

(100)

where \( m \) is the mass of the BH and \( q \) is the monopole charge. The horizon radii could be found from the following equation

\[ r^8 + (6q^2 - 4m^2)r^6 + (11q^4 - 4m^2q^2)r^4 + 6q^6r^2 + q^8 = 0. \]  

(101)

This is a polynomial equation of order 8th. This could be reduced to fourth order polynomial equation by putting \( r^2 = z \) then one obtains [23]

\[ z^4 + (6q^2 - 4m^2)z^3 + (11q^4 - 4m^2q^2)z^2 + 6q^6z + q^8 = 0. \]  

(102)

The EH and CH are located at

\[ r_h = \sqrt{\frac{(2m^2 - 3q^2)}{2} + \frac{a}{2} + \frac{b}{2}} \] 

(103)

\[ r_c = \sqrt{\frac{(2m^2 - 3q^2)}{2} - \frac{a}{2} - \frac{b}{2}} \]  

(104)

and the other horizons [5] are located at

\[ r_{hc} = \sqrt{\frac{(2m^2 - 3q^2)}{2} + \frac{a}{2} - \frac{b}{2}} \]  

(105)

\[ r_{ch} = \sqrt{\frac{(2m^2 - 3q^2)}{2} - \frac{a}{2} + \frac{b}{2}} \]  

(106)

where

\[ a = \sqrt{4m^2q^2 - 11q^4 + (2m^2 - 3q^2)^2 + \frac{(11q^4 - 4m^2q^2)}{3}} + \frac{2}{3}\frac{\left(16m^4q^4 - 16m^2q^6 + 25q^8\right)}{3} + \frac{1}{3}\frac{2}{3}\frac{\left(\frac{\delta}{3}\right)}{1} \] 

(107)

and

\[ b = \sqrt{c} \]  

(108)

where

\[ c = 4m^2q^2 - 11q^4 + 2(2m^2 - 3q^2)^2 + \frac{(4m^2q^2 - 11q^4)}{3} - \frac{2}{3}\frac{\left(16m^4q^4 - 16m^2q^6 + 25q^8\right)}{3} - \frac{1}{3}\frac{2}{3}\frac{\left(\frac{\delta}{3}\right)}{1} + \frac{48q^6 - 8(2m^2 - 3q^2)^3 + 8(2m^2 - 3q^2)(11q^4 - 4m^2q^2)}{4a} \] 

and

\[ \delta = 624m^4q^8 - 128m^5q^6 - 240m^2q^10 + 250q^{12} - \sqrt{324864m^8q^{16} - 110592m^{10}q^{14} - 193536m^6q^{18} + 172800m^4q^{20}}. \]  

(109)

The volume product of \( \mathcal{H}^\pm \) is evaluated to be

\[ V_h V_c = \frac{\pi^2}{36} \left[ (2m^2 - 3q^2)^2 - (a + b)^2 \right] \]  

(110)

\(^5\)We have considered only here EH and CH. The other horizons are discarded.
As usual, the volume product is not mass-independent. Now we would see below somewhat more complicated function of inner and outer horizon volume is indeed mass-independent. To derive it we have to apply Vieta’s theorem in Eq. (102) thus one obtains

\[
\sum_{i=1}^{4} z_i = 4m^2 - 6q^2. \quad (111)
\]

\[
\sum_{1 \leq i < j \leq 4} z_i z_j = 11q^4 - 4m^2q^2. \quad (112)
\]

\[
\sum_{1 \leq i < j < k \leq 4} z_i z_j z_k = -6q^2. \quad (113)
\]

\[
\prod_{i=1}^{4} z_i = q^8. \quad (114)
\]

Eliminating the mass parameter one obtains the mass independent equation in terms of two horizons

\[
z_1 z_2 (z_1 + z_2) + 6q^2 z_1 z_2 - q^4 \left( \frac{z_1 + z_2}{z_1 z_2} \right) - \frac{1}{z_1 + z_2 + q^2} \left[ (z_1 + z_2)^2 + 6q^2(z_1 + z_2) - z_1 z_2 - \frac{q^8}{z_1 z_2} + 11q^4 \right] = 6q^6. \quad (115)
\]

It should be noted that the symbols \((h,c)\) and \((1,2)\) both have same meaning. Now in terms of volume of \(\mathcal{H}\) the mass-independent volume functional relation becomes

\[f(V_h, V_c) = 6q^6. \quad (116)\]

where

\[
f(V_h, V_c) = \left( \frac{3}{4\pi} \right)^2 (V_h V_c) \left( \frac{V_h^2 + V_c^2}{2} \right) + 6q^2 \left( \frac{3}{4\pi} \right)^2 (V_h V_c) - q^4 \left( \frac{4\pi}{3} \right)^\frac{4}{3} \left( \frac{(V_h^2 + V_c^2)}{(V_h V_c)} \right)^\frac{4}{3} \times \left( \frac{3}{4\pi} \right)^\frac{4}{3} \left( V_h^2 + V_c^2 \right)^{\frac{4}{3}} + 6q^2 \left( \frac{3}{4\pi} \right)^\frac{4}{3} \left( V_h^2 + V_c^2 \right)^{\frac{4}{3}} - \left( \frac{3}{4\pi} \right)^\frac{4}{3} \left( V_h^2 + V_c^2 \right)^{\frac{4}{3}} - \left( \frac{4\pi}{3} \right)^\frac{4}{3} \left( V_h^2 + V_c^2 \right)^{\frac{4}{3}} + 11q^4 \right) \quad (117)
\]

Now we are moving to axisymmetric spacetime. To see what happens there?

### 3 Axisymmetric Spacetime

In this section we have considered only the various axisymmetric BHs. It is easy to compute volume products for spherically symmetric cases because of \(V_h \propto A_h r_h\) for OH and \(V_c \propto A_c r_c\) for IH. While for axisymmetric spacetime this proportionality is quite different because here the spin parameter is present. Now see what happens in this case by starting with Kerr BH.

#### 3.1 Kerr BH

The Kerr BH is a solution of Einstein equation. The OH radius and IH radius reads for this BH should read

\[
r_h = M + \sqrt{M^2 - a^2} \quad (118)
\]

\[
r_c = M - \sqrt{M^2 - a^2}. \quad (119)
\]
where \( a = \frac{J}{M} \). \( J \) is angular momentum of the BH. When \( M^2 > a^2 \), it describes a BH while \( M^2 < a^2 \) it describes a naked singularity. The thermodynamic volume for OH and IH becomes

\[
V_h = \frac{A_h r_h}{3} \left[ 1 + \frac{a^2}{2r_h^2} \right], \quad V_c = \frac{A_c r_c}{3} \left[ 1 + \frac{a^2}{2r_c^2} \right].
\]

The thermodynamic volume product of Kerr BH for OH and IH is calculated to be

\[
V_h V_c = \frac{128}{9} \pi^2 J^2 M^2 \left( 1 + \frac{a^2}{8M^2} \right).
\]

The thermodynamic volume sum for OH and IH is

\[
V_h + V_c = \frac{32}{3} \pi M^3 \left( 1 - \frac{a^2}{4M^2} \right).
\]

Similarly, the volume minus for OH and IH is

\[
V_h - V_c = \frac{32}{3} \pi M^2 \sqrt{M^2 - a^2}.
\]

and the volume division is

\[
\frac{V_h}{V_c} = \left( \frac{4M^2 - a^2 + 4M \sqrt{M^2 - a^2}}{4M^2 - a^2 - 4M \sqrt{M^2 - a^2}} \right).
\]

It indicates that the volume product, volume sum, volume minus and volume division for Kerr BH is mass dependent. Therefore the product, the sum, the minus and the division all are not universal.

### 3.2 Kerr-AdS BH

The horizon function for Kerr-AdS BH is given by

\[
\Delta_r = \left( r^2 + a^2 \right) \left( 1 + \frac{r^2}{\ell^2} \right) - 2Mr = 0.
\]

which gives the quartic order of horizon equation

\[
\frac{r^4}{\ell^2} + \left( 1 + \frac{a^2}{\ell^2} \right) r^2 - 2mr + a^2 = 0.
\]

The quantities \( m \) and \( a \) are related to the parameters mass \( M \) and angular momentum \( J \) as follows

\[
m = M \Xi^2, \quad a = \frac{J}{m} \Xi^2 \]

where \( \Xi = 1 - \frac{a^2}{\ell^2} \). To obtain the roots of Eq. \( 127 \) we apply the Vieta’s theorem, we find

\[
\sum_{i=1}^{4} r_i = 0, \quad \sum_{1 \leq i < j \leq 4} r_ir_j = \ell^2 \left( 1 + \frac{a^2}{\ell^2} \right), \quad \sum_{1 \leq i < j < k \leq 4} r_ir_jr_k = 2m\ell^2, \quad \prod_{i=1}^{4} r_i = a^2\ell^2.
\]
There is at least two real zeros of the Eq. (127) which is OH radius and IH radius. After some algebraic computation we have

\[ r_h + r_c = \frac{2m\ell^2}{a^2 + \ell^2 + r_h^2 + r_c^2}, \]  
(133)

\[ r_hr_c = \frac{a^2\ell^2 - (r_hr_c)^2}{a^2 + \ell^2 + r_h^2 + r_c^2}. \]  
(134)

The area of the BH for OH is

\[ A_h = \frac{4\pi (r_h^2 + a^2)}{\Xi}. \]  
(135)

and for IH is

\[ A_c = \frac{4\pi (r_c^2 + a^2)}{\Xi}. \]  
(136)

The thermodynamic volume for OH \[17, 18\] becomes

\[ V_h = \frac{2\pi [(r_h^2 + a^2)(2r_h^2 + a^2\ell^2 - r_h^2a^2)]}{3r_h\ell^2\Xi^2}. \]  
(137)

and we derive the thermodynamic volume for IH becomes

\[ V_c = \frac{2\pi [(r_c^2 + a^2)(2r_c^2 + a^2\ell^2 - r_c^2a^2)]}{3r_c\ell^2\Xi^2}. \]  
(138)

The thermodynamic volume product for Kerr-AdS BH is computed to be

\[ V_hV_c = \frac{16\pi^2 (r_h^2 + a^2)(r_c^2 + a^2) + a^4}{9\Xi^4 r_hr_c} \times \left[ 3r_h^2 r_c^2 + 2a^2 (r_h^2 + r_c^2) + \Xi^2 r_h^2 r_c^2 \right]. \]  
(139)

Using Eq. (139), Eq. (134), Eq. (135) and Eq. (136), we observe that there is no way to eliminate the mass parameter from Eq. (139) therefore the volume product for Kerr-AdS BH is not mass-independent thus it is not universal and not quantized.

### 3.3 Kerr-Newman BH

It is an axisymmetric solution of Einstein-Maxwell equations. The OH radius and IH radius for this BH becomes

\[ r_h = M + \sqrt{M^2 - a^2 - Q^2} \]  
(140)

\[ r_c = M - \sqrt{M^2 - a^2 - Q^2}. \]  
(141)

The thermodynamic volume for OH \[17\] is

\[ V_h = \frac{2\pi [(r_h^2 + a^2)(2r_h^2 + a^2\ell^2) + a^2Q^2]}{3r_h}. \]  
(142)

and we derive that the thermodynamic volume for IH is

\[ V_c = \frac{2\pi [(r_c^2 + a^2)(2r_c^2 + a^2\ell^2) + a^2Q^2]}{3r_c}. \]  
(143)

The thermodynamic volume product for KN BH is computed to be

\[ V_hV_c = \frac{16\pi^2}{9} \times \frac{[J^2(8J^2 + a^4 - a^2Q^2 - 2Q^4 + 8M^2Q^2) + Q^4(a^2 + Q^2)^2]}{a^2 + Q^2}. \]  
(144)

It also indicates that the thermodynamic volume for KN BH is mass dependent. Thus the volume product is not universal for any axisymmetric spacetime. In the appropriate limit i.e. when \( a = J = 0 \), one obtains the thermodynamic volume product for Reissner Nordström BH and when \( Q = 0 \), one obtains the volume product for Kerr BH.
3.4 Kerr-Newman-AdS BH

The horizon function for Kerr-Newman-AdS BH \[24\] reads

\[
\Delta_r = (r^2 + a^2) \left(1 + \frac{r^2}{\ell^2}\right) - 2Mr + q^2 = 0 .
\] (145)

which has the quartic order of horizon equation

\[
\frac{r^4}{\ell^2} + \left(1 + \frac{q^2}{\ell^2}\right) r^2 - 2mr + a^2 + q^2 = 0 .
\] (146)

The quantity \(q\) is related to the charge parameter \(Q\) as

\[
q = Q\Xi
\] (147)

To determine the roots of Eq. (146) again we apply the Vieta’s rule then one obtains

\[
\sum_{i=1}^{4} r_i = 0 . \tag{148}
\]

\[
\sum_{1 \leq i < j \leq 4} r_i r_j = \ell^2 \left(1 + \frac{a^2}{\ell^2}\right) . \tag{149}
\]

\[
\sum_{1 \leq i < j < k \leq 4} r_i r_j r_k = 2m\ell^2 . \tag{150}
\]

\[
\prod_{i=1}^{4} r_i = (a^2 + q^2)\ell^2 . \tag{151}
\]

Similarly, there is at least two real zeros of the Eq. (146) which is OH radius and IH radius. After some algebraic derivation one get

\[
r_h + r_c = \frac{2m\ell^2}{a^2 + \ell^2 + r_h^2 + r_c^2} , \tag{152}
\]

\[
r_h r_c = \frac{(a^2 + q^2)\ell^2 - (r_h r_c)^2}{a^2 + \ell^2 + r_h^2 + r_c^2} . \tag{153}
\]

The area of this BH for OH is

\[
\mathcal{A}_h = \frac{4\pi (r_h^2 + a^2)}{\Xi} . \tag{154}
\]

and for IH is

\[
\mathcal{A}_c = \frac{4\pi (r_c^2 + a^2)}{\Xi} . \tag{155}
\]

The thermodynamic volume for OH \[17, 18\] becomes

\[
\mathcal{V}_h = \frac{2\pi \left((r_h^2 + a^2)(2r_h^2\ell^2 + a^2\ell^2 - r_h^2a^2) + \ell^2q^2a^2\right)}{3r_h\ell^2\Xi^2} .
\]

and we derive the thermodynamic volume for IH becomes

\[
\mathcal{V}_c = \frac{2\pi \left((r_c^2 + a^2)(2r_c^2\ell^2 + a^2\ell^2 - r_c^2a^2) + \ell^2q^2a^2\right)}{3r_c\ell^2\Xi^2} .
\]

The thermodynamic volume product for Kerr-Newman-AdS BH is computed in Eq. (156).

\[
\mathcal{V}_h \mathcal{V}_c = \frac{4\pi^2}{9\Xi^4r_h r_c} \times \frac{[r_h^2 r_c^2 + a^2 (r_h^2 + r_c^2) + a^4]^2 + [r_h^2 r_c^2 + a^2 (r_h^2 + r_c^2) + a^4] \{2r_h^2 r_c^2 + a^2 (r_h^2 + r_c^2)\}]}{[r_h^2 r_c^2 + a^2 (r_h^2 + r_c^2) + a^4]^2 + [r_h^2 r_c^2 + a^2 (r_h^2 + r_c^2) + a^4] \{2r_h^2 r_c^2 + a^2 (r_h^2 + r_c^2)\}]}
\]
Again using Eq. (152), Eq. (153), Eq. (154) and Eq. (155), we speculate that there has been no chance to eliminate the mass parameter from Eq. (156) thus the volume product for Kerr-Newman-AdS BH is not mass-independent therefore it is not universal and not quantized.

### 3.5 Spinning BTZ BH

The metric for rotating BTZ BH [30] in $2+1$ dimension is given by

$$ds^2 = -\left(\frac{r^2}{\ell^2} + \frac{J^2}{4r^2} - M\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{\ell^2} + \frac{J^2}{4r^2} - M\right)} + r^2 \left(\frac{J}{2r^2} dt + d\phi\right)^2.$$  \hspace{1cm} (157)

where $M$ and $J$ represents the ADM mass, and the angular momentum of the BH. $-\Lambda = \frac{1}{\ell^2} = 8\pi P G_3$ denotes the cosmological constant. Here we have set $8G_3 = 1 = c = h = k$. When $J = 0$, one obtains the static BTZ BH.

The BH OH radius and IH radius are [30, 20]

$$r_h = \sqrt{-\frac{M\ell^2}{2} \left(1 + \sqrt{1 + \frac{J^2}{M^2\ell^2}}\right)},$$ \hspace{1cm} (158)

$$r_c = \sqrt{-\frac{M\ell^2}{2} \left(1 - \sqrt{1 + \frac{J^2}{M^2\ell^2}}\right)}.$$ \hspace{1cm} (159)

The thermodynamic volume for 3D spinning BTZ BH for OH and IH:

$$V_h = \left(\frac{\partial M}{\partial P}\right)_j = \pi r_h^2$$ \hspace{1cm} (160)

$$V_c = \left(\frac{\partial M}{\partial P}\right)_j = \pi r_c^2$$ \hspace{1cm} (161)

The thermodynamic volume product is computed to be

$$V_h V_c = \frac{\pi^2 J^2 \ell^2}{4}.$$ \hspace{1cm} (162)

Interestingly, the thermodynamic volume product for rotating BTZ BH is mass-independent i.e. universal and it is also quantized. This is the only example for rotating cases, the volume product is universal. This is an interesting result of this work.

### 4 Discussion

In this work, we have demonstrated that the thermodynamic products in particular thermodynamic volume products of spherically symmetric spacetime and axisymmetric spacetime by incorporating the extended phase-space formalism. In this formalism, the cosmological constant should be considered as a thermodynamic pressure and its conjugate parameter as thermodynamic volume. In addition to that the mass parameter should be treated as enthalpy of the system rather than internal energy. Then in this phase space the first law of BH thermodynamics should be satisfied for both the OH and IH.

We explicitly computed the thermodynamic volume products both for OH and IH of several classes of spherically symmetric and axisymmetric BHs including the AdS spacetime. In this cases,
the simple volume product of $H^\pm$ is not mass independent. Rather slightly more complicated volume functional relations are indeed mass-independent. We have proved that for simple Reissner Nordström BH of Einstein gravity and Kehagias-Sfetsos BH of Hořava Lifshitz gravity, the thermodynamic volume product of $H^\pm$ is indeed universal. Such products are mass independent for spherically symmetric cases because of $V_h \propto A_h r_h$ for OH and $V_c \propto A_c r_c$ for IH.

Axisymmetric spacetime does not satisfy this proportionality due to presence of the spin parameter thus such spacetime shows no mass-independent features except the rotating BTZ BH, the only axisymmetric spacetime in 3D showed that universal features thus it has been quantized in this sense. We also computed thermodynamic volume sum but they are always mass dependent so they are not universal as well as they are not quantized.

Like area (or entropy) products, the simple thermodynamic volume product of $H^\pm$ is not mass-independent rather more complicated function of volume functional relation is indeed mass independent. This is often true for spherically symmetric BHs including AdS spacetime. This scenario for axisymmetric spacetime (except 3D BTZ BH) is quite different. In this cases, the area functional relation becomes mass-independent whereas the volume functional relation is not mass-independent. For volume products, this is the main differences between spherically symmetric spacetime and axisymmetric spacetime. To sum up, the volume functional relation that we have studied in this work in spherically symmetric cases (but not for axisymmetric cases) further provides some universal properties of the BH which gives some insight of microscopic origin of BH entropy both outer and inner.

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