FORMS ON VECTOR BUNDLES OVER HYPERBOLIC
MANIFOLDS AND THE CONFORMAL ANOMALY

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Abstract. We study gauge theories based on abelian $p$–forms on real
compact hyperbolic manifolds. An explicit formula for the conformal
anomaly corresponding to skew–symmetric tensor fields is obtained, by
using zeta–function regularization and the trace tensor kernel formula.
Explicit exact and numerical values of the anomaly for $p$–forms of order
up to $p = 4$ in spaces of dimension up to $n = 10$ are then calculated.

1. Introduction

The conformal deformations of the Riemannian metric and the corre-
sponding conformal anomaly play an important role in quantum theories.
It is well known that evaluation of the conformal anomaly is actually possi-
ble for even dimensional spaces albeit its computation is extremely involved.
The general structure of such anomaly in curved even–dimensional spaces
has been actively studied (see, for example, Ref. [1]). We briefly mention
here an analysis related to this phenomenon for constant curvature spaces.
The calculation of the conformal anomaly for the sphere can be found in Ref.
[2]. Explicit computations of the anomaly (of the stress–energy tensor) for
scalar and spinor quantum fields in compact hyperbolic spaces have been
carried out in Refs. [3, 4] (see also Refs. [5, 6]), using the zeta–function
regularization method [7, 8, 9].

The purpose of this paper is to analyze the conformal anomaly associated
with tensor fields on real hyperbolic spaces. Skew symmetric tensor fields
play an important role in quantum field theory, supergravity, and string
theory, where they naturally couple to two–form connections. Abelian two–
forms are closely related to the theory of gerbes, which plays a key role
in string theory [10]–[16]. Such forms can be understood as a connection
on an abelian gerbe. In the abelian case (which will be considered in this
paper) the self–dual two–form can be easily reduced to the abelian one–form
gauge field. Generally, the covariant quantization of skew–symmetric tensor
fields has met difficulties with ghost counting and BRST–transformations.
In the framework of functional integration, the covariant quantization of free

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generalized gauge fields, \( p \)-forms and the BRST–transformations have been obtained in Ref. [17].

In this paper we present a decomposition of the Hodge Laplacian and the tensor kernel trace formula associated with free generalized gauge fields (\( p \)-forms) on real hyperbolic spaces. The main ingredient required is a type of differential form structure on the physical, auxiliary, or ghost variables. We consider spectral functions and the conformal anomaly associated with physical degrees of freedom of the Hodge–de Rham operators on \( p \)-forms.

2. Quantum Dynamic of Exterior Forms in Hyperbolic Spaces

We shall work with an \( n \)-dimensional compact real hyperbolic space \( X \) with universal covering \( M \) and fundamental group \( \Gamma \). We can represent \( M \) as the symmetric space \( G/K \), where \( G = \text{SO}_1(n, 1) \) and \( K = \text{SO}(n) \) is a maximal compact subgroup of \( G \). Then we regard \( \Gamma \) as a discrete subgroup of \( G \) acting isometrically on \( M \), and we take \( X \) to be the quotient space by that action: \( X = \Gamma \setminus M = \Gamma \setminus G/K \). Let \( \tau \) be an irreducible representation of \( K \) on a complex vector space \( V_{\tau} \), and consider the induced homogeneous vector bundle \( G \times_K V_{\tau} \) (the fiber product of \( G \) with \( V_{\tau} \) over \( K \)) \( \rightarrow M \) over \( M \). Restricting the \( G \) action to \( \Gamma \) we obtain the quotient bundle \( E_{\tau} = \Gamma \setminus (G \times_K V_{\tau}) \rightarrow X = \Gamma \setminus M \) over \( X \). The natural Riemannian structure on \( M \) (therefore on \( X \)) induced by the Killing form \(( , )\) of \( G \) gives rise to a connection Laplacian \( L \) on \( E_{\tau} \). If \( \Omega_K \) denotes the Casimir operator of \( K \), that is

\[
\Omega_K = -\sum_{j} y_j^2, \quad (1)
\]

for a basis \( \{y_j\} \) of the Lie algebra \( k_0 \) of \( K \), where \((y_j , y_{\ell}) = -\delta_{j\ell}\), then \( \tau(\Omega_K) = \lambda_\tau 1 \), for a suitable scalar \( \lambda_\tau \). Moreover, for the Casimir operator \( \Omega \) of \( G \), with \( \Omega \) operating on smooth sections \( \Gamma^\infty E_{\tau} \) of \( E_{\tau} \), one has

\[
\mathfrak{L} = \Omega - \lambda_\tau 1 \quad (2)
\]

(see Lemma 3.1 of [18]). For \( \lambda \geq 0 \), let

\[
\Gamma^\infty (X, E_{\tau})_\lambda = \{ s \in \Gamma^\infty E_{\tau} \mid -\mathfrak{L}s = \lambda s \} \quad (3)
\]

be the space of eigensections of \( \mathfrak{L} \) corresponding to \( \lambda \). Here we note that since \( X \) is compact we can order the spectrum of \( -\mathfrak{L} \) by taking \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \), with \( \lim_{j \to \infty} \lambda_j = \infty \). We shall focus on the more difficult (and interesting) case when \( n = 2k \) is even, and we specialize \( \tau \) to be the representation \( \tau^{(p)} \) of \( K = \text{SO}(2k) \) on \( \Lambda^p \mathbb{C}^{2k} \), say \( p \neq k \). The case when \( n \) is odd will be dealt with later. It is convenient, moreover, to work with the normalized Laplacian \( \mathfrak{L}_p = -c(n)\mathfrak{L} \), where \( c(n) = 2(n-1) = 2(2k-1) \). \( \mathfrak{L}_p \) has spectrum \( \{c(n)\lambda_j , m_j\}_{j=0}^{\infty} \), where the multiplicity \( m_j \) of the eigenvalue \( c(n)\lambda_j \) is given by

\[
m_j = \dim \Gamma^\infty (X, E_{\tau^{(p)}})_{\lambda_j} \quad (4)
\]
Let $T_{j_1j_2\ldots j_k}$ be a skew–symmetric tensor of $(0,k)$–type, i.e. $T_{(j_1\ldots j_k)} \overset{\text{def}}{=} \text{sgn}(\sigma)T_{j_1j_2\ldots j_k}$, where $\text{sgn}(\sigma) = \pm 1$ is the sign of the permutation $\sigma$. The exterior differential $p$–form is
d\omega_p = \frac{1}{p!} \sum_{j_1,\ldots, j_p} T_{j_1j_2\ldots j_p} dx^{j_1} \wedge \ldots \wedge dx^{j_p}.
(5)\
Here $\wedge$ is the exterior product, $dx^j$ are the basis one–forms, and $j = 1,2,\ldots,n$ $(\dim M = n)$. Let $\Lambda^* (M) \equiv \bigoplus_{p=0}^{\infty} \Lambda^p$ be the graded Cartan exterior algebra of differential forms, where $\Lambda^p$ is the space of all $p$–forms on $M$. Let $(\ast T)$ denote a skew symmetric tensor of type $(0,-k)$, i.e.
\[ (\ast T)_{j_{k+1}\ldots j_n} = \frac{1}{k!} \sqrt{|g|} \varepsilon_{j_1\ldots j_n} T^{j_1\ldots j_k}, \quad T^{j_1\ldots j_k} = g^{j_1\ell_1} \ldots g^{j_k\ell_k} T_{\ell_1\ldots \ell_k}, \]
where $\varepsilon_{j_1\ldots j_n} = \pm 1$ for $\text{sgn}(j_1\ldots j_n) = \pm 1$ is the Levi–Civita tensor density, and the metric $g_{j\ell}$ (external gravitational field) has the signature $(+,+,\ldots,+)$. In local coordinates the exterior differential, $d : \Lambda^p \rightarrow \Lambda^{p+1}$, and the co–differential, $\delta : \Lambda^p \rightarrow \Lambda^{p-1}$, take respectively the forms
\[ d\omega = \frac{1}{p!} \sum_{j_1,\ldots, j_{p+1}} \frac{\partial T_{j_2\ldots j_{p+1}}}{\partial x^{j_1}} dx^{j_1} \wedge \ldots \wedge dx^{j_{p+1}}, \]
\[ \delta \omega = - \frac{1}{(p-1)!} \sum_{j_1,\ldots, j_{p-1}} \frac{\partial T_{j_2\ldots j_p}}{\partial x^{j_1}} dx^{j_2} \wedge \ldots \wedge dx^{j_p}. \]
From the last equations it is easy to prove the following properties for operators and forms: $dd = \delta \delta = 0$, $\delta = (-1)^{np+n+1} * d *$, $**\omega_p = (-1)^{p(n-p)}\omega_p$.
Let $\alpha_p, \beta_p$ be $p$–forms; then, the invariant inner product is defined by $(\alpha_p, \beta_p) \overset{\text{def}}{=} \int_M \alpha_p \wedge \ast \beta_p$. The operators $d$ and $\delta$ are adjoint to each other with respect to this inner product for $p$–forms: $(\delta \alpha_p, \beta_p) = (\alpha_p, d\beta_p)$.
The following result holds.

**Theorem.** For every $p = 0,1,2,\ldots,n$, $\Lambda^n(M)$ admits the orthogonal direct sum decomposition
\[ \Lambda^p (M) = d\Lambda^{p-1}(M) \oplus \delta \Lambda^{p+1}(M) \oplus \mathcal{H}^p(M), \]
where $\mathcal{H}^p$ is the space of all harmonic $p$–forms. That means, every form $\omega_p \in \Lambda^p(M)$ can be written as $\omega_p = \delta \omega_{p+1} + d\omega_{p-1} + h_p$ with $h_p$ being a harmonic $p$–form, $\Delta h_p = 0$.

This theorem is implied by the existence of the orthogonal sum decomposition
\[ \Lambda^p (M) = \mathcal{H}^p(M) \oplus \Delta \Lambda^p(M). \]
(10)
It is known that the $L^2$ harmonic $p$–form $h_p^{(2)}$ appears on even real hyperbolic manifolds only. In fact, the following result holds: The manifold $\mathbb{H}^n$ admits $L^2$ harmonic $p$–forms if and only if $n = 2p$; for even dimensional
real hyperbolic manifolds the space of $L^2$ harmonic $p$--forms is infinite dimensional (Ref. [19], p. 373). One can consider the $L^2$--de Rham complex:

$$0 \to \Lambda^0(M) \xrightarrow{d_0} \Lambda^1(M) \xrightarrow{d_1} \cdots \to \Lambda^p(M) \xrightarrow{d_p} \Lambda^{p+1}(M) \to \cdots \to \Lambda^n(M) \to 0,$$

and its associated $L^2$--cohomology

$$H^p(M) = \frac{\ker (\Lambda^p(M) \xrightarrow{d_p} \Lambda^{p+1}(M))}{\text{range } (d_{p-1} \Lambda^{p-1}(M))}.$$  

(11)

A theorem of Kodaira (Ref. [20], p. 165) gives the following injection:

$$h^{(2)}_p(M) \to H^p(M).$$

(13)

The map (injection) $j$ is an isomorphism if and only if $d_{p-1}$ has closed range. If $j$ is not an isomorphism, then $j$ has infinite dimensional co--kernel. The associated Laplacian $\mathcal{L}_p$ has closed range if and only if $d_p$ and $d_{p-1}$ have closed range (Ref. [21], p. 446).

In quantum field theory the Lagrangian associated with $\omega_p$ takes the form:

$$L = d\omega_p \wedge *d\omega_p \text{ (gauge field)}, \quad L = \delta\omega_p \wedge *\delta\omega_p \text{ (co--gauge field)}.$$  

The Euler--Lagrange equations, supplied with the gauge, give: $\mathcal{L}_p\omega_p = 0$, $\delta\omega_p = 0$ (Lorentz gauge); $\mathcal{L}_p\omega_p = 0$, $d\omega_p = 0$ (co--Lorentz gauge). These Lagrangians give possible representation of tensor fields or generalized abelian gauge fields. The two representations of tensor fields are not completely independent, because of the well--known duality property of exterior calculus which gives a connection between star--conjugated gauge and co--gauge tensor fields. The gauge $p$--forms are mapped into the co--gauge $(n-p)$--forms under the action of the Hodge $*$ operator. The vacuum--to--vacuum amplitude for the gauge $p$--form $\omega_p$ becomes [17]:

$$Z = N \int D\omega \exp \left[ -(\omega, \mathcal{L}_p\omega) \right] \prod_{j=1}^p \left( \text{Vol}_{p-j} (\det \mathcal{L}_{p-j})^{(j+1)/2} \right)^{(-1)^{j+1}},$$

where we need to factorize the divergent gauge group volume and integrate over the classes of gauge transformations ($\omega \to \omega + d\phi$).

3. The Trace Formula Applied to the Tensor Kernel

Since $\Gamma$ is torsion free, each $\gamma \in \Gamma - \{1\}$ can be represented uniquely as some power of a primitive element $\rho : \gamma = \rho^j(\gamma)$ where $j(\gamma) \geq 1$ is an integer and $\delta$ cannot be written as $\gamma_1^j$ for $\gamma_1 \in \Gamma$, with $j > 1$ an integer. Taking $\gamma \in \Gamma$, $\gamma \neq 1$, one can find $t_\gamma > 0$ and $m_\gamma \in \mathfrak{M}$ defined by $m_\gamma \in K| m_\gamma a = am_\gamma, \forall a \in A|$ such that $\gamma$ is $G$--conjugate to $m_\gamma \exp (t_\gamma H_0)$, namely, for some $g \in G$, one has $g^\gamma g^{-1} = m_\gamma \exp (t_\gamma H_0)$; that is, $\gamma$ is $G$--conjugate to $m_\gamma \exp (t_\gamma H_0)$ and $m_\gamma \in SO(n-1)$. For $\text{Ad}$ denoting the
adjoint representation of $G$ on its complexified Lie algebra, one can compute $t_\gamma$ as follows [22]:

$$e^{t_\gamma} = \max \{|c| \mid c \text{ is an eigenvalue of } \text{Ad}(\gamma) : g \to g\}.$$  \hspace{1cm} (15)

Also, $\gamma = \delta j(r)$, where $j(r) \geq 1$ is a whole number and $\delta \in \Gamma - \{1\}$ is a primitive element, i.e., $\delta$ can not be expressed as $\gamma_1^j$ for some $\gamma_1 \in \Gamma$ and some integer $j > 1$. The pair $(j(\gamma), \delta)$ is uniquely determined by $\gamma \in \Gamma - \{1\}$. These facts are known to follow since $\Gamma$ is torsion free.

Let $a_0, n_0$ denote the Lie algebras of $A, N$ in an Iwasawa decomposition $G = KAN$. The complexified Lie algebra $g = g^C_0 = so(2k + 1, \mathbb{C})$ of $G$ is of Cartan type $B_k$ with the Dynkin diagram

$$\circ - \circ - \circ \cdots \circ - \circ = \circ.$$  \hspace{1cm} (16)

Since the rank of $G$ is one, $\dim a_0 = 1$ by definition, say $a_0 = \mathbb{R}H_0$ for a suitable basis vector $H_0$:

$$H_0 = \begin{bmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 \\
\vdots & \ddots & \ddots \\
0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{bmatrix},$$  \hspace{1cm} (17)

is a $(k + 1) \times (k + 1)$ matrix. With this choice we have the normalization $\beta(H_0) = 1$, where $\beta : a_0 \to \mathbb{R}$ is the positive root which defines $n_0$ (for more details see Ref. 15). Define now $C(\gamma)$ on $\Gamma - \{1\}$ by

$$C(\gamma) \overset{\text{def}}{=} e^{-\rho_0 t_\gamma} |\text{det}_{n_0}(\text{Ad}(m_\gamma e^{t_\gamma H_0})^{-1} - 1)|^{-1}.$$  \hspace{1cm} (18)

Finally, let $C_\Gamma \subset \Gamma$ be a complete set of representations in $\Gamma$ of its conjugacy classes. This means that any two elements in $C_\Gamma$ are non-conjugate, and any $\gamma \in \Gamma$ is $\Gamma$-conjugate to some element $\gamma_1 \in C_\Gamma : x\gamma x^{-1} = \gamma_1$ for some $x \in \Gamma$. The reader may consult the appendix of [24] for further structural data concerning the Lie group $SO_1(n, 1)$ (and other rank one groups).

Let $\tau = \tau_k = \text{representation of } K$ on $\Lambda^j \mathbb{C}^{2k}$. The space of smooth sections $\Gamma^\infty E_\tau$ of $E_\tau$ is just the space of smooth $p$–forms on $X$. We can therefore apply the version of the trace formula developed by Fried in Ref. [23]. First we set up some additional notation. For $\sigma_j$ the natural representation of $SO(2k - 1)$ on $\Lambda^j \mathbb{C}^{2k - 1}$, one has the corresponding Harish–Chandra–Plancherel density given —for a suitable normalization of the Haar measure $dx$ on $G$— by

$$\mu_{\sigma_p}(r) = \frac{\pi}{2^{4k-4} [\Gamma(k)]^2} \left( \frac{2k - 1}{p} \right) r P_{\sigma_p}(r) \tanh(\pi r),$$  \hspace{1cm} (19)
for $0 \leq p \leq k - 1$, where
\[
P_{\sigma_p}(r) = \prod_{\ell=0}^{p+1} \left[ r^2 + \left( k - \ell + \frac{3}{2} \right)^2 \right] \prod_{\ell=p+2}^{k} \left[ r^2 + \left( k - \ell + \frac{1}{2} \right)^2 \right]
\] (20)
is an even polynomial of degree $2k - 2$. One has that $P_{\sigma_p}(r) = P_{\sigma_{2k-1-p}}(r)$ and $\mu_{\sigma_p}(r) = \mu_{\sigma_{2k-1-p}}(r)$ for $k \leq p \leq 2k - 1$. Define the Miatello coefficients \[25, 26] a_{2\ell}^{(p)} for $G = SO_1(2k+1,1)$ by $P_{\sigma_p}(r) = \sum_{\ell=0}^{k} a_{2\ell}^{(p)} r^{2\ell}$, $0 \leq p \leq 2k - 1$.

Let $\text{Vol}(\Gamma \backslash G)$ denote the integral of the constant function $1$ on $\Gamma \backslash G$ with respect to the $G$–invariant measure on $\Gamma \backslash G$ induced by $dx$. For $0 \leq p \leq n - 1$ the Fried trace formula applied to kernel $\mathcal{K}_t$ holds \[23]:
\[
\text{Tr} \left( e^{-t \mathcal{K}_t} \right) = I_{\Gamma}^{(p)}(\mathcal{K}_t) + I_{\Gamma}^{(p-1)}(\mathcal{K}_t) + H_{\Gamma}^{(p)}(\mathcal{K}_t) + H_{\Gamma}^{(p-1)}(\mathcal{K}_t),
\] (21)
where $I_{\Gamma}^{(p)}(\mathcal{K}_t)$, $H_{\Gamma}^{(p)}(\mathcal{K}_t)$ are the identity and hyperbolic orbital integrals, respectively,
\[
I_{\Gamma}^{(p)}(\mathcal{K}_t) \overset{def}{=} \frac{\chi(1)\text{Vol}(\Gamma \backslash G)}{4\pi} \int_\mathbb{R} dr \mu_{\sigma_p}(r)e^{-t(r^2+p+\rho_0^2)},
\] (22)
\[
H_{\Gamma}^{(p)}(\mathcal{K}_t) \overset{def}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in \Gamma - \{1\}} \frac{\chi(\gamma)}{j(\gamma)} t_\gamma C(\gamma) \chi_{\sigma_p}(m_\gamma) \exp \left\{ -t(\rho_0^2 + p) - t_\gamma^2/(4t) \right\},
\] (23)
with $\rho_0 = (n - 1)/2$, and $\chi_{\sigma}(m) = \text{Tr} \sigma(m)$ for $m \in SO(2n - 1)$. For $p \geq 1$ there is a measure $\mu_{\sigma_p}(r)$ corresponding to a general irreducible representation $\sigma$ of $\mathfrak{M}$. Let $\sigma_p$ be the standard representation of $\mathfrak{M} = SO(n - 1)$ on $\Lambda^p \mathbb{C}^{n-1}$. If $n = 2k$ is even then $\sigma_p \ (0 \leq p \leq n - 1)$ is always irreducible; if $n = 2k + 1$ then every $\sigma_p$ is irreducible except for $p = (n - 1)/2 = k$, in which case $\sigma_k$ is the direct sum of two spin–(1/2) representations $\sigma^\pm$: $\sigma_k = \sigma^+ \oplus \sigma^-$. For $p = k$ the representation $\tau_k$ of $K = SO(2k)$ on $\Lambda^k \mathbb{C}^{2k}$ is not irreducible: $\tau_k = \tau_k^+ \oplus \tau_k^-$ is the direct sum of two spin–(1/2) representations.

**The case of the trivial representation.** In the case of the trivial representation ($p = 0$, i.e. for smooth functions or smooth vector bundle sections) the measure $\mu(r) = \mu_0(r)$ corresponds to the trivial representation of $\mathfrak{M}$. Therefore, we take $I_{\Gamma}^{(-1)}(\mathcal{K}_t) = H_{\Gamma}^{(-1)}(\mathcal{K}_t) = 0$. Let $\chi_{\sigma}(m) = \text{trace}(\sigma(m))$ be the character of $\sigma$, for $\sigma$ a finite–dimensional representation of $\mathfrak{M}$. Since $\sigma_0$ is the trivial representation, one has $\chi_{\sigma_0}(m_\gamma) = 1$. In this case, formula
(15) reduces exactly to the trace formula for \( p = 0 \) [18, 5, 6, 24, 27],

\[
I^{(0)}_x(K) = \frac{\chi(0)\text{vol}(\Gamma \backslash G)}{4\pi} \int_\mathbb{R} dr \mu_{\sigma_0}(r) e^{-t(r^2 + \rho^2_0)}dr
\]

(24)

The function \( I^{(0)}_x(K) \) has the form

\[
I^{(0)}_x(K) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C_{\Gamma}^{-1}} \chi(\gamma) t \gamma j(\gamma)^{-1} C(\gamma) \exp\{-t \rho^2_0 - t^2/(4t)\}.
\]

(25)

4. Spectral Functions on \( p \)-Forms and the Conformal Anomaly

The transverse part of the skew symmetric tensor is represented by the co–exact \( p \)-form \( \omega^{(CE)}_p = \delta \omega_{p+1} \), which trivially satisfies \( \delta \omega^{(CE)}_p = 0 \), and we denote by \( \omega^{(CE)}_p \) the restriction of the Laplacian on the co–exact \( p \)-form. The goal now is to extract the co–exact \( p \)-form on the manifold which describes the physical degrees of freedom of the system. Choosing a basis \( \{\omega^i_p\} \) of \( p \)-forms (eigenfunctions of the Laplacian), we get [28, 29, 30]

\[
\text{Tr} \left(e^{-t\omega^{(CE)}_p}\right) = \sum_{j=0}^{p} (-1)^j \left(I^{(p-j)}_x(K) + I^{(p-j-1)}_x(K)\right)
+ H^{(p-j)}_x(K) + H^{(p-j-1)}_x(K) - b_{p-j},
\]

(26)

where \( b_j \) are the Betti numbers, \( b_j \equiv b_j(M) = \text{rank}_Z H_j(M; Z) \).

For constant conformal deformations of the Riemannian metric \( g^{\mu\nu} \) the variation of the connected vacuum functional \( \mathcal{W} \) can be expressed in terms of the generalized zeta function \( \zeta(s|\mathfrak{A}) \) [31, 32] associated with the Laplace–Beltrami type operator \( \mathfrak{A} \),

\[
\delta \mathcal{W} = -\zeta(0|\mathfrak{A}) \log \mu^2 = (1/2) \int d(\mathfrak{W} \mathfrak{v}) < T_{\mu\nu}(x) > \delta g^{\mu\nu}(x),
\]

(27)

where \( \mu \) is a renormalization mass parameter and \( < T_{\mu\nu}(x) > \) means that all connected vacuum graphs of the stress–energy tensor \( T_{\mu\nu}(x) \) are to be included. Then Eq. (27) leads to the result

\[
< T_{\mu}^\mu(x) > = \mathfrak{W} \mathfrak{v}^{-1} \zeta(0|\mathfrak{A}),
\]

(28)

where for \( S^n \): \( \mathfrak{W} \mathfrak{v} = 2\pi^{(n+1)/2} R^n / \Gamma((n + 1)/2) \), while for the compact manifold \( \Gamma \backslash \mathbb{H}^n \): \( \mathfrak{W} \mathfrak{v} = \text{Vol}(\Gamma \backslash G) R^n \), where \( R \) is the radius corresponding to the compact space.

Our goal now is to calculate the value of generalized zeta function

\[
\zeta(s|\mathcal{L}^{(CE)}_p) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} e^{-t\mathcal{L}^{(CE)}_p} = \sum_{j=0}^{p} \frac{(-1)^j}{\Gamma(s)} \int_0^\infty dt t^{s-1}
\]

(29)
\[ \times \left( I_{\Gamma}^{(p-j)}(K_t) + I_{\Gamma}^{(p-j-1)}(K_t) + H_{\Gamma}^{(p-j)}(K_t) + H_{\Gamma}^{(p-j-1)}(K_t) - b_{p-j} \right). \] (29)

The integrals related to the identity orbital integrals can be written as follows

\[ \int_0^\infty dt \, t^{s-1} I_{\Gamma}^{(p-j)}(K_t) = \frac{\chi(1) \text{Vol}(\Gamma \backslash G)}{2^{2(n-1)} \Gamma(n/2)^2} \left( \frac{n-1}{p-j} \right) \sum_{\ell=0}^{n/2-1} a_{2\ell}^{(p-j)} \times \int_0^\infty dt \, t^{s-1} e^{-t^2} \int_\mathbb{R} dr \, r \, e^{-t^2} \tanh(\pi r), \] (30)

where \( \alpha \equiv p + \rho_0^2 \). Using the identities

\[ 1 - \tanh(\pi r) = \frac{2}{1 + \exp(2\pi r)}, \]
\[ \int_0^\infty \frac{r^{2\ell-1} dr}{1 + e^{2\pi r}} = \frac{(-1)^{\ell-1}}{4\ell} (1 - 2^{1-2\ell}) B_{2\ell}, \] (31)

where \( B_{\ell} \) are the Bernoulli numbers, we obtain:

\[ \int_\mathbb{R} dr \, r^{2\ell+1} e^{-tr^2} \tanh(\pi r) = \ell! t^{-\ell-1} - \sum_{k=0}^{\infty} \frac{(-1)^{\ell}(1 - 2^{1-2\ell-2k-1}) k^k}{k!(\ell + k + 1)} B_{2(\ell+k+1)} \] (32)

and

\[ \int_0^\infty dt \, t^{s-1} I_{\Gamma}^{(p-j)}(K_t) = \frac{\chi(1) \text{Vol}(\Gamma \backslash G)}{2^{2(n-1)} \Gamma(n/2)^2} \left( \frac{n-1}{p-j} \right) \sum_{\ell=0}^{n/2-1} a_{2\ell}^{(p-j)} \times \left\{ \ell! \left( s - \ell - 1 \right) - \sum_{k=0}^{\infty} \frac{(-1)^{\ell}(1 - 2^{1-2\ell-2k-1}) \Gamma(k + s)}{k!(\ell + k + 1)} \frac{B_{2(\ell+k+1)}}{\alpha - j)^{k+s}} \right\}. \] (33)

The contribution associated with the identity integral at the point \( s = 0 \) becomes

\[ \lim_{s \to 0} \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} I_{\Gamma}^{(p-j)}(K_t) = \frac{\chi(1) \text{Vol}(\Gamma \backslash G)}{2^{2(n-1)} \Gamma(n/2)^2} \left( \frac{n-1}{p-j} \right) \times \sum_{\ell=0}^{n/2-1} a_{2\ell}^{(p-j-1)} \frac{(-1)^{\ell+1}}{\ell+1} \left( (1 - 2^{-2\ell-1}) B_{2(\ell+1)} + [\alpha + j + 1]^{\ell+1} \right). \] (34)

The hyperbolic orbital integrals can be rewritten in terms of McDonald functions \( K_\nu(z) \),

\[ K_\nu(z) = 2^{-\nu-1} z^\nu \int_0^\infty dt - t^{-\nu-1} z^2/(4t), \quad |\arg z| < \pi/2, \quad \Re z^2 > 0. \] (35)

The result is:
\[
\int_0^\infty dt \ t^{s-1} H_{\Gamma}^{(p-j)}(K_t)
= \sum_{\gamma \in \mathcal{C}_{\Gamma-1}} \frac{\chi(\gamma)}{\sqrt{\pi j(\gamma)}} t_\gamma C(\gamma) \chi_{\sigma_{p-j}}(m_\gamma) \left( \frac{2\sqrt{\alpha + j}}{t_\gamma} \right)^{-s+1/2} K_{-s+1/2} \left( t_\gamma \sqrt{\alpha - j} \right).
\]

The analysis of the integral Eq. (36) gives the following result (see also Refs. [5, 3, 6, 4]): the terms associated with the hyperbolic orbital integrals vanish when \( s = 0 \). Finally, using Eqs. (28), (29) and (34), we get for the conformal anomaly the following explicit formula, in terms of the dimension \( n \) of the hyperbolic space, the order \( p \) of the form, the radius \( R \) of the compact spatial section, the value of \( \alpha \) (which will depend on the nature of the field, see below), Miatello coefficients, and Bernoulli numbers:

\[
<T^{\mu}_\mu(x)> = \frac{1}{(4\pi)^{n/2} \Gamma(n/2) R^n} \sum_{j=0}^p (-1)^j \sum_{\ell=0}^{n/2-1} \left( \frac{(-1)^{\ell+1}}{\ell + 1} \left( \frac{n-1}{p-j} \right) \right) \times \left\{ a_{2\ell}^{(p-j)} \left[ (1 - 2^{-2\ell-1}) B_{2(\ell+1)} + (\alpha - j)^{\ell+1} \right] + a_{2\ell}^{(p-j-1)} \frac{p-j}{n-p} \left[ (1 - 2^{-2\ell+1}) B_{2(\ell+1)} + (\alpha - j-1)^{\ell+1} \right] \right\} + \text{B} \tag{37}
\]

This constitutes the main result of the present paper.

The case of a conformally invariant scalar field. Restoring now the dependence on the radius \( R \), for the specific case of a minimally coupled scalar field of mass \( m \), we have: \( p = j = 0, \alpha \Rightarrow \alpha + R^2 m^2 \), and \( \alpha = \rho_0^2 \).

For the case of a conformally invariant scalar field, we have: \( \alpha = \rho_0^2 + (n - 2) R^2 R(x)/[4(n - 1)] \), where \( R(x) = -n(n - 1)R^{-2} \) is the scalar curvature. Therefore, the final result is in this case

\[
<T^{\mu}_\mu(x)> = \frac{1}{(4\pi)^{n/2} \Gamma(n/2) R^n} \sum_{\ell=0}^{n/2-1} \frac{(-1)^{\ell+1}}{\ell + 1} a_{2\ell} \times \left[ 2^{-2\ell-2} + (1 - 2^{-2\ell+1}) B_{2\ell+2} \right]. \tag{38}
\]

This formula is in full agreement with a previous result obtained in Ref. [3] and constitutes a check of our main formula Eq. (37). In fact, we obtain from this expression Table 1 [note a small missprint in the denominator of the last value given in the table in Ref. [3]].

Explicit and numerical values of the conformal anomaly for \( p \)-forms. Using our Eq. (37), exact explicit values and also numerical values of the conformal anomaly corresponding to spaces of arbitrary dimension \( n \) and forms of any order \( p \) are easily obtained, with the help of any standard program as Matlab, Maple or Mathematica. Using Mathematica 5.0 on a laptop, in a question of seconds we have obtained the following table (Table
Table 1. Exact and numerical values of the conformal anomaly for the conformally invariant scalar field, in dimensions $n = 2$ to $n = 14$ (we have set $R = 1$).

| $< T_\mu^\nu(x) >_{c.i.s.}$ | exact | numerical |
|-----------------------------|-------|-----------|
| $n = 2$                     | $-\frac{1}{12 \pi}$ | $-0.0265258$ |
| $n = 4$                     | $-\frac{1}{240 \pi^2}$ | $-4.22172 \times 10^{-4}$ |
| $n = 6$                     | $-\frac{5}{4032 \pi^3}$ | $-3.99945 \times 10^{-5}$ |
| $n = 8$                     | $-\frac{23}{34560 \pi^4}$ | $-6.83210 \times 10^{-6}$ |
| $n = 10$                    | $-\frac{263}{506880 \pi^5}$ | $-1.69551 \times 10^{-6}$ |
| $n = 12$                    | $-\frac{133787}{251596800 \pi^6}$ | $-5.53107 \times 10^{-7}$ |
| $n = 14$                    | $-\frac{157009}{232243200 \pi^7}$ | $-2.23837 \times 10^{-7}$ |

2) for the conformal anomaly, where we have set $R = 1$ and $\alpha = p + \rho_0^2$, with $\rho_0 = (n - 1)/2$.

5. Conclusions

We have here evaluated the conformal anomaly for the family of space-times of arbitrary dimension which possess a compact spatial section of the form $\Gamma \backslash \mathbb{H}^n$. We have restricted ourselves to the situation where the manifold is smooth and $\Gamma$ is a discrete subgroup of $SO_1(n, 1)$, acting freely and properly discontinuously on $\mathbb{H}^n$. The terms associated with hyperbolic orbital integrals do not contribute to the conformal anomaly, as we have shown above.
Table 2. Exact and numerical values of the conformal anomaly for the family of spacetimes of dimension $n = 2$ to $n = 10$ which possess a compact spatial section, corresponding to forms of order up to $p = 4$.

Explicit exact and numerical results for the conformal anomaly corresponding to $p$–forms of orders $p = 0$ to $p = 4$ in spaces of dimension $n = 2$ to $n = 10$ have been given in Table 2. Both the sign and the magnitude of the anomaly seem to change in a rather non–uniform way in the cases considered, their absolute value being always less than 1 for the calculated cases (but this can be shown to be not a bound for forms of higher order). In fact, one sees clearly, that the absolute value of the conformal anomaly for
p–forms definitely increases with the order of the form in hyperbolic spaces of higher dimensionality, of the class considered.

As a particular case, we recover the formula for the conformal invariant scalar field in any dimension and, in a similar way, a number of more general situations can be treated with the same techniques as described in this paper.

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