DYNAMICS FOR THE DAMPED WAVE EQUATIONS ON TIME-DEPENDENT DOMAINS

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Abstract. We consider the asymptotic dynamics of a damped wave equations on a time-dependent domains with homogeneous Dirichlet boundary condition, the nonlinearity is allowed to have a cubic growth rate which is referred to as the critical exponent. To this end, we establish the existence and uniqueness of strong and weak solutions satisfying energy inequality under the assumption that the spatial domains \(\Omega_t\) in \(\mathbb{R}^3\) are obtained from a bounded base domain \(\Omega\) by a \(C^3\)-diffeomorphism \(r(\cdot, t)\). Furthermore, we establish the pullback attractor under a slightly weaker assumption that the measure of the spatial domains are uniformly bounded above.

1. Introduction. The investigation of differential equations on time-dependent domains (sometimes called non-cylindrical domains or time-varying domains, etc.) has a long and rich history, and the main motivation of this kind of problems arises naturally from applications, such as the Stefan problems [42], the behavior of particles in time-dependent potential well [24], the model of tumor growth [15], the American option pricing problems [16], the image processing [1], and the control problems [30, 41, 45], etc. In addition, the review article [34] shows us lots of interesting concrete examples about the problems on time-varying domains. The second motivation of the study of such issues is theoretical interest, such as [11, 20, 28].

In this work we study the damped wave equations on time-dependent domains. Such equations arise in a wide variety of applications, e.g., waves reflected by a
Let $\mathcal{O} \subset \mathbb{R}^3$ be a nonempty bounded and open set with smooth boundary $\partial \mathcal{O}$, and $r = r(y, t)$ a vector function

$$
r \in C^3(\overline{\mathcal{O}} \times \mathbb{R}; \mathbb{R}^3),
$$

especially satisfying

$$
r(\cdot, t) : \mathcal{O} \to \mathcal{O}_t
$$

is a $C^3$-diffeomorphism for all $t \in \mathbb{R}$.

Define

$$
Q_{\tau, T} := \bigcup_{t \in (\tau, T]} \mathcal{O}_t \times \{t\}
$$

and

$$
Q_{\tau} := \bigcup_{t \in (\tau, +\infty)} \mathcal{O}_t \times \{t\},
$$

$$
\Sigma_{\tau, T} := \bigcup_{t \in (\tau, T]} \partial \mathcal{O}_t \times \{t\},
$$

$$
\Sigma_{\tau} := \bigcup_{t \in (\tau, +\infty)} \partial \mathcal{O}_t \times \{t\}, \forall \tau < T.
$$

For any $\tau < T$, the set $Q_{\tau, T}$ is an open subset of $\mathbb{R}^4$, with boundary

$$
\partial Q_{\tau, T} = \Sigma_{\tau, T} \cup (\mathcal{O}_T \times \{\tau\}) \cup (\mathcal{O}_T \times \{T\}).
$$

We shall assume that the function $\bar{r} = \bar{r}(x, t)$, where $\bar{r}(\cdot, t) = r^{-1}(\cdot, t)$ denotes the inverse of $r(\cdot, t)$, satisfies

$$
\bar{r} \in C^3(\overline{Q_{\tau, T}}; \mathbb{R}^3) \text{ for all } \tau < T.
$$

We also assume that there exists $\gamma^* \geq 0$ such that

$$
g(u)u \geq G(u) - \frac{\gamma}{2} u^2 - \eta \quad \text{and} \quad G(u) \geq -\frac{\gamma}{2} u^2 - \eta, \quad \forall u \in \mathbb{R},
$$

where $G(u) = \int_0^u g(s)ds$, and the constant $\gamma^* \geq 0$ will be characterized later (see (83) for details).

It is well known that the variations of the spatial domains may have a significant effect on the evolution equations, such as equilibrium solutions, eigenvalues and eigenfunctions, especially the asymptotic behavior of the solutions, see, e.g., [3, 4, 5, 6] for the recent significant progress on this subject. On the other hand, when the variation of the spatial domains are time-dependent, which is also called non-cylindrical problems, the situation may be rather complicated, including the...
definition of solutions, well-posedness, regularity and especially the asymptotic behavior of the solutions, etc.

In 1997, Bernardi et al. [8] studied the linear Schrödinger-type partial differential equations on non-cylindrical domains by assuming a monotonicity condition on their section with respect to time and used the method of penalization due to J.L. Lions [36] to establish the existence of the (suitably defined) weak solutions. However, due to the existence of the penalty functions, it is hard to improve the regularity and yet the uniqueness of weak solutions is unknown. In subsequent work [9], they gave a partial answer by singling out a unique solution satisfying the energy equality among all the possible weak ones.

In 2008, Kloeden et al. [32] studied the dynamics of the semilinear heat equations on such non-cylindrical domains with the monotonicity condition relying on the scheme presented in [9]. Because of lacking of strong solutions, they approximate the penalty function by Steklov averages (see [25], Chapter I for details) which are more regular in time to obtain a compact $\mathcal{D}$-pullback absorbing sets. In [22], the authors considered a stochastic partial differential equation of the reaction-diffusion type obtained by the same way as presented in [33]. Due to the variation of the domain, the existence of a family of measure preserving transformation has not been assumed, so the stochastic equations on such domains generate a new type dynamical systems, named “partial-random” dynamical systems. All this being said, we note that, reaction-diffusion equations as well as stochastic reaction-diffusion equations on non-cylindrical domains are essentially different from the ones that on cylindrical domains.

Besides all the aforementioned differences in dealing with parabolic equations on non-cylindrical domains, due to some peculiarities of wave equations, the situation is somewhat more complicated. For example, the uniqueness of weak solutions for the nonlinear wave equations on a non-cylindrical domain only under monotonicity condition, as pointed out by J.L. Lions (see [36], Chapter 3, Problems 11.8), remained open until now.

On the other hand, under some additional assumptions, the existence, uniqueness and long-time dynamics of solutions of equation (4) was studied by several authors. For example, under the assumption that there exists a one-to-one mapping $\phi : \overline{Q} \rightarrow \overline{Q}$, of class $C^3$, with bounded derivatives and inverse $\psi$ of class $C^1$ and satisfying (3.1) in [20], the existence and uniqueness of global weak solutions was proved by Cooper and Bardos [20]. Very recently, under the hypothesis that the lateral boundary is time-like and the domains are expanding, Ma et al. [37] established the pullback attractors of weakly damped wave equations by presented a useful compactness criterion.

In the present paper, we establish the existence of a pullback attractor to the problem (4) under the assumption that the time-varying domains obtained by a temporally continuous dependent diffeomorphic transformation of a bounded reference domain. Our work is in the spirit of [20, 22, 31, 32, 33, 37], and many ideas of this article are taken from these works. However, we develop several generalisations, which can be regarded as the main features of this paper. First of all, with totally different from the heat equations (e.g., [33]), the problem (4) may be ill-posed for a general transformation, so we assume that the transformation is hyperbolic. We also construct a sufficient condition to ensure the transformation is hyperbolic in Appendix which may be of independent interest. Secondly, compared with [37],
our domains are more complicated and general, which brought some intrinsic difficulties in our process, such as obtain the strong solutions. Hence, much of our effort is in proving the existence and uniqueness of strong and weak solutions in appropriate functions spaces as well as in establishing energy inequalities. Thirdly, the nonlinearity $g$ of equation (4) has a critical growth rate, i.e., the mapping $g$ from $H_0^1$ to $L^2$ is continuous, but not compact (see [43] and the references therein). To overcome the difficulties brought by the critical nonlinearity, usually, the so-called energy method developed by Ball [7], a decomposition technique (e.g., see [3, 46]) and the “contractive function” method (see [18], [29] and [37] for further details and some extensions) have been successfully applied to prove the asymptotic compactness of the solutions. Due to the domain is time-dependent, we combine the two ideas presented in [31] (see Section 4.1) and [29] to prove the process is $\mathcal{D}$-pullback asymptotic compact. In addition, this scheme are also helpful to obtain the pullback asymptotic compact of the process generated by the solutions of the equations on cylindrical domains.

The outline of our paper is given below. In the next section, we recall some preliminaries and known results related to pullback attractors for general non-autonomous dynamical systems. Moreover, function space setting and properties of functions are given. Strong and weak solutions are considered in Section 3 and 4, respectively, in particular their existence and uniqueness. In Section 5, based on the conclusions of the previous two sections, we show that the weak solutions generate a process, which is asymptotically compact in the light of established energy estimates in Section 3 and 4. This will lead to the proof of existence of pullback attractor in an appropriate framework in Section 6. Finally, we have included Appendix A, which is devoted to the discussion of the existence of hyperbolic.

2. Preliminaries.

2.1. Pullback attractors. We recall the main points about the theory of pullback attractors which will apply, see Caraballo et al. [12] and Carvalho et al. [13] for more details.

Considering the domain is time-dependent, we prefer to use the language of evolutionary processes rather than the cocycle formalism since the former seems to be more appropriate for our situation.

Let $S$ be a process on a family of metric spaces $\{(X_t, d_t) : t \in \mathbb{R}\}$, i.e., a family $\{S(t, \tau) : -\infty < \tau \leq t < \infty\}$ of continuous mappings $S(t, \tau) : X_\tau \to X_t$ such that $S(\tau, \tau)x = x$ for all $x \in X_\tau$ and

$$S(t, \tau) = S(t, s)S(s, \tau) \text{ for all } \tau \leq s \leq t.$$ 

In addition, suppose $\mathcal{D}$ is a nonempty class of parameterized sets of the form $D = \{D(t) : D(t) \subset X_t, D(t) \neq \emptyset, t \in \mathbb{R}\}$.

**Definition 2.1.** The process $S(\cdot, \cdot)$ is said to be $\mathcal{D}$-pullback asymptotically compact if the sequence $\{S(t, \tau_n)\}$ is relatively compact in $X_t$ for any $t \in \mathbb{R}$, any $D \in \mathcal{D}$, and any sequences $\{\tau_n\}$ and $\{x_n\}$ with $\tau_n \to -\infty$, and $x_n \in D(\tau_n)$.

**Definition 2.2.** A family $B = \{B(t), t \in \mathbb{R}\} \in \mathcal{D}$ is said to be $\mathcal{D}$-pullback absorbing for the process $S(\cdot, \cdot)$ if for any $t \in \mathbb{R}$ and any $D \in \mathcal{D}$, there exists $\tau_0(t, D) \leq t$ such that

$$S(t, \tau)D(\tau) \subset B(t).$$
for all $\tau \leq \tau_0(t, \hat{D})$.

For each $t \in \mathbb{R}$, and $D_1, D_2$ nonempty subsets of $X$, let us denote $\text{dist}_t(D_1, D_2)$ the Hausdorff semi-distance defined as

$$\text{dist}_t(D_1, D_2) := \sup_{u \in D_1} \inf_{v \in D_2} d_t(u, v)$$

**Definition 2.3.** The family $\mathcal{A} = \{\mathcal{A}(t) : \mathcal{A}(t) \subset X, \mathcal{A}(t) \neq \emptyset, t \in \mathbb{R}\}$ is said to be a $\mathcal{D}$-pullback attractor for $S(\cdot, \cdot)$, if:

1. $\mathcal{A}(t)$ is a compact subset of $X$ for all $t \in \mathbb{R}$;
2. $\mathcal{A}$ is $\mathcal{D}$-pullback attracting, i.e.,
   $$\lim_{\tau \to -\infty} \text{dist}_t(S(t, \tau)D(\tau), \mathcal{A}(t)) = 0,$$
   for all $D \in \mathcal{D}, t \in \mathbb{R}$;
3. $\mathcal{A}$ is invariant, i.e.,
   $$S(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t),$$
   for any $-\infty < \tau \leq t < \infty$.

**Theorem 2.4.** Suppose that the process $S(\cdot, \cdot)$ is $\mathcal{D}$-pullback asymptotic compact and that $\hat{B} \in \mathcal{D}$ is a family of $\mathcal{D}$-pullback absorbing sets for $S(\cdot, \cdot)$. Then, the family $\mathcal{A} = \{\mathcal{A}(t)\}$ defined by $\mathcal{A}(t) := \Lambda(\hat{B}, t)$, $\forall t \in \mathbb{R}$, where

$$\Lambda(\hat{B}, t) := \bigcap_{s \leq t} \left( \bigcup_{\tau \leq s} S(t, \tau)D(\tau)^{X_t} \right)$$

is a $\mathcal{D}$-pullback attractor for $S(\cdot, \cdot)$, which in addition satisfies $\forall t \in \mathbb{R}$:

$$\mathcal{A}(t) = \bigcup_{D \in \mathcal{D}} \Lambda(D, t).$$

Furthermore, $\mathcal{A}$ is minimal in the sense that if $\hat{C} = \{C(t); t \in \mathbb{R}\}$ is a family of nonempty sets such $C(t)$ is a closed subset of $X$ and

$$\lim_{\tau \to -\infty} \text{dist}_t(S(t, \tau)B(\tau), C(t)) = 0$$

for any $t \in \mathbb{R}$, then $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

### 2.2. Functional spaces and preliminary results

In this subsection, we recall the functional spaces and notations that we will use throughout this paper.

For a fixed finite time interval $[\tau, T]$, let $\{(X_t, \| \cdot \|_{X_t}) : t \in [\tau, T]\}$ be a family of Banach spaces such that $X_t \subset L^1_{\text{loc}}(O_t)$ for all $t \in [\tau, T]$. For any $1 \leq q \leq \infty$, we denote by $L^q(\tau, T; X_t)$ the vector space of all functions $u \in L^q_{\text{loc}}(Q_{\tau, T})$ such that $u(t) = u(\cdot, t) \in X$ a.e. $t \in (\tau, T)$, and the function $\|u(\cdot)\|_{X}$ defined by $t \mapsto \|u(t)\|_{X}$ belongs to $L^q(\tau, T)$.

We consider on $L^q(\tau, T; X_t)$ the norm given by

$$\|u\|_{L^q(\tau, T; X_t)} := \|\|u(\cdot)\|_{X}\|_{L^q(\tau, T)}.$$ 

For each $u \in L^1_{\text{loc}}(Q_{\tau, T})$, we can extend $u$ trivially to $\mathbb{R}^3 \times (\tau, T)$ by

$$\hat{u}(x, t) = \begin{cases} u(x, t), & (x, t) \in O_t \times (\tau, T), \\ 0, & (x, t) \in (\mathbb{R}^3 \setminus O_t) \times (\tau, T). \end{cases}$$

For any $u \in L^1_{\text{loc}}(Q_{\tau, T})$, we will denote $u' = u_t$ the derivative of $u$ with respect to time $t$ in the sense of distributions in $Q_{\tau, T}$, defined by
\[ \langle u', \phi \rangle := - \int_{\tau}^{T} \int_{\Omega} \phi'(x,t)u(x,t)dxdt, \] for all functions \( \phi \in C_c^\infty(Q_{\tau,T}) \),

where \( \phi' = \frac{\partial \phi}{\partial t} \) is the classical partial derivative.

The following result was proved in [33]:

**Lemma 2.5.** ([33]) If \( u \in L^2(\tau, T; H^1_0(\mathcal{O}_t)) \) and \( u' \in L^2(\tau, T; L^2(\mathcal{O}_t)) \), and then the trivial extension \( \hat{u} \) belongs to \( H^1(\mathbb{R}^3 \times (\tau, T)) \), satisfies \( \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \) (1 \( \leq i \leq 3 \)), and its derivative with respect to time is given by

\[ \dot{u}' = \hat{u}' \quad (7) \]

As that in Kloeden, Real and Sun [33], we say that a function \( u \in L^1_{loc}(Q_{\tau,T}) \) belongs to \( C([\tau, T]; L^2(\mathcal{O}_t)) \) if its trivial extension \( \hat{u} \) belongs to \( C([\tau, T]; L^2(\mathbb{R}^3)) \) and we say that a sequence \( \{u_m\} \) converges to \( u \) in \( C([\tau, T]; L^2(\mathcal{O}_t)) \) as \( m \rightarrow \infty \), if the sequence \( \{\hat{u}_m\} \) converges to \( \hat{u} \) in \( C([\tau, T]; L^2(\mathbb{R}^3)) \) as \( m \rightarrow \infty \).

Similarly, we say that a function \( u \in L^1_{loc}(Q_{\tau,T}) \) belongs to \( C([\tau, T]; H^1(\mathbb{R}^3)) \) if its trivial extension \( \hat{u} \) belongs to \( C([\tau, T]; H^1(\mathbb{R}^3)) \) and we say that a sequence \( \{u_m\} \) converges to \( u \) in \( C([\tau, T]; H^1(\mathcal{O}_t)) \) as \( m \rightarrow \infty \), if the sequence \( \{\hat{u}_m\} \) converges to \( \hat{u} \) in \( C([\tau, T]; H^1(\mathbb{R}^3)) \) as \( m \rightarrow \infty \).

From now on, we will use \( \langle \cdot, \cdot \rangle, \| \cdot \|_t \) to denote the usual inner product and associated norm in \( L^2(\mathcal{O}_t) \).

**Lemma 2.6.** If \( u \in L^2(\tau, T : H^1_0(\mathcal{O}_t) \cap H^2(\mathcal{O}_t)), \ u' \in L^2(\tau, T : H^1_0(\mathcal{O}_t)) \), then

\[ 2 \int_{\mathbb{R}^3} \left( -\Delta u(r), u'(r) \right) dr = \| \nabla u(t) \|^2_2 - \| \nabla u(s) \|^2_2. \quad (8) \]

**Proof.** Using Lemma 2.5,

\[
2 \int_{\mathbb{R}^3} \left(-\Delta u(r), u'(r)\right) dr = 2 \int_{\mathbb{R}^3} \left(\nabla u(r), \nabla u'(r)\right) dr = 2 \int_{\mathbb{R}^3} \left(\nabla \hat{u}(r), \nabla \hat{u}'(r)\right) dr = \| \nabla \hat{u}(t) \|^2_2 - \| \nabla \hat{u}(s) \|^2_2. \]

\[ \square \]

We consider a finite time interval \([\tau, T]\), and set

\[ v(y, t) = u(r(y, t), t) \quad \text{for} \ y \in \mathcal{O}, t \in [\tau, T], \]

or, equivalently,

\[ u(x, t) = v(r(x, t), t) \quad \text{for} \ x \in \mathcal{O}, t \in [\tau, T]. \]

3. **Strong solutions.** For each \( \tau < T \), we consider the auxiliary problem

\[
\begin{align*}
& u_{tt} + u_t - \Delta u + g(u) = f(t, x) \quad \text{in} \ Q_{\tau,T}, \\
& u = 0 \quad \text{on} \ \Sigma_{\tau,T}, \\
& u(\tau, x) = u_0, \ u'(\tau, x) = u_1, \ x \in \mathcal{O}. 
\end{align*}
\]
has a more complicated structure defined over a cylindrical domain.

The method that we use to prove the result of existence and uniqueness is based on the transformation of our problem into another initial boundary problem which has a more complicated structure defined over a cylindrical domain.

Using a suitable change of variables, consider the problem

\[
\begin{align*}
\frac{\partial^2 v(y,t)}{\partial t^2} + \frac{\partial v(y,t)}{\partial t} - \sum_{i,j=1}^{3} \frac{\partial}{\partial y_j} \left( a_{ij}(y,t) \frac{\partial v}{\partial y_i}(y,t) \right) + \sum_{i=1}^{3} b_i(y,t) \frac{\partial v(y,t)}{\partial y_i} \\
+ 2 \sum_{i=1}^{3} \frac{\partial \bar{r}_i(y,t)}{\partial t} \frac{\partial^2 v(y,t)}{\partial y_i \partial t} + g(v(y,t)) = \bar{f}(y,t) \text{ in } \mathcal{O} \times (\tau,T), \\
v = 0 \text{ on } \partial \mathcal{O} \times (\tau,T), \\
v(y,\tau) = v_0, \quad v'(y,\tau) = v_1, \quad y \in \mathcal{O},
\end{align*}
\]

where

\[
\bar{f}(y,t) = f(r(y,t), t), \quad v_0 = u_0(r(y,\tau)), \quad v_1 = u_1(r(y,\tau)) + \sum_{i=1}^{3} \frac{\partial u_{0i}}{\partial x_i}(r(y,\tau)) \frac{\partial r_i}{\partial t}(y,\tau),
\]

\[
a_{ij}(y,t) = \sum_{k=1}^{3} \frac{\partial \bar{r}_i}{\partial x_k}(r(y,t), t) \frac{\partial \bar{r}_j}{\partial x_k}(r(y,t), t) - \frac{\partial \bar{r}_i}{\partial t}(r(y,t), t) \frac{\partial \bar{r}_j}{\partial t}(r(y,t), t),
\]

and \( b = (b_1,b_2,b_3) \) is defined by

\[
b_i(t) = \frac{\partial^2 \bar{r}_i}{\partial t^2}(r(y,t), t) + \frac{\partial \bar{r}_i}{\partial t}(r(y,t), t) + \sum_{j=1}^{3} \frac{\partial a_{ij}(y,t)}{\partial y_j} - \Delta_x \bar{r}_i(r(y,t), t), \quad 1 \leq i,j \leq 3.
\]

Similarly, one can define the strong solution of (10). From the results in the previous sections and Section 3 in [33], by Proposition IX.18 in [10], one obtain that \( u = u(x,t) \) is a strong solution for problem (9) if and only if the function \( v(y,t) = u(r(y,t), t) \) is a strong solution of the problem (10).

**Definition 3.2.** We call \( r \) is hyperbolic if for any \(-\infty < \tau \leq T < +\infty\), there exists a constant \( \theta = \theta(r) > 0 \) such that, for all \((y,t) \in \mathcal{O} \times (\tau,T), \)

\[
\sum_{i,j=1}^{3} a_{ij}(y,t) \xi_i \xi_j \geq \theta |\xi|^2 \quad \forall \xi \in \mathbb{R}^3,
\]

where \( a_{ij}(y,t) = \sum_{k=1}^{3} \frac{\partial \bar{r}_i}{\partial x_k}(r(y,t), t) \frac{\partial \bar{r}_j}{\partial x_k}(r(y,t), t) - \frac{\partial \bar{r}_i}{\partial t}(r(y,t), t) \frac{\partial \bar{r}_j}{\partial t}(r(y,t), t), \quad 1 \leq i,j \leq 3. \)

In order to ensure the well-posedness for our problems, from now on, we shall suppose \( r \) is hyperbolic (see e.g. [20] for some extensions). In Appendix A, we construct a criteria for the existence of hyperbolic.

Due to the assumption on \( \partial \mathcal{O}, r \) and \( \bar{r} \), we have (e.g., see [35], Chapter II)
Lemma 3.3. ([33]) For any $-\infty \leq \tau \leq T + \infty$, there exists two positive constants $C_1$ and $C_2$, such that for any $u \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$, the following estimate holds

$$C_1 \int_{\mathcal{O}} |\Delta u(y)|^2 dy \leq \int_{\mathcal{O}} \sum_{i,j=1}^{3} a_{ij}(y, t) a_{ij}(\partial u, \partial v, \partial y_i, \partial y_j) dy + C_2 \int_{\mathcal{O}} |u(y)|^2 dy \text{ for all } t \in [\tau, T].$$

Theorem 3.4. Let $f \in H^1_0(Q_{\tau,T})$, $r$ and $\bar{r}$ satisfy assumptions (1), (2) and (3). Then, for any $u_0 \in H^2(\mathcal{O}_\tau) \cap H^1_0(\mathcal{O}_\tau)$, $u_1 \in H^1_0(\mathcal{O}_\tau)$ and any $-\infty < \tau \leq T < +\infty$, we have the following:

(i) There exists a unique strong solution $u(t)$ of (9).

(ii) If $u(t)$ is the strong solution of (9), then $\forall t \in [\tau, T]$, $(u(t), u'(t))$ satisfies the inequality of energy

$$\|u'(t)\|^2 + \|\nabla u(t)\|^2 \leq 8e^{-\lambda_{\tau,t}(t-\tau)} \left(\|u'(\tau)\|^2 + \|\nabla u(\tau)\|^2 + \|\nabla u(\tau)\|^{p+1}\right)$$

$$+ 4e^{-\lambda_{\tau,t}} \left(4 \int_{\tau}^{t} e^{\lambda_{\tau,t}} \|f(\xi)\|^2 d\xi + \int_{\tau}^{t} e^{\lambda_{\tau,t}} |\xi| |\xi| d\xi\right) + 8\eta |\mathcal{O}_\tau|,$$

where $\lambda_{\tau,t} = \sup_{\xi \in [\tau,t]} \{2C_0^2 |\mathcal{O}_\tau|^2\}$, $\eta = \min\{1/2, 1/\kappa_{\tau,T}\}$ and $\vartheta = \vartheta\left(1 - \vartheta(1 - \eta)\right).$

(iii) If $\bar{u}_1$ and $\bar{u}_2$ are two strong solutions of equation (9), denote $w \equiv \bar{u}_1 - \bar{u}_2$, we have the following estimate

$$\|w'(t)\|^2 + \|\nabla w(t)\|^2 \leq 4e^{-\lambda_{\tau,t}(t-\tau)} \left(\|w'(\tau)\|^2 + \|\nabla w(\tau)\|^2\right)$$

$$- 2e^{-\lambda_{\tau,t}} \int_{\tau}^{t} e^{\lambda_{\tau,t}} \left(g(\bar{u}_1(\xi)) - g(\bar{u}_2(\xi)), \bar{u}_1'(\xi) - \bar{u}_2'(\xi)\right) d\xi$$

$$- 2\vartheta e^{-\lambda_{\tau,t}} \int_{\tau}^{t} e^{\lambda_{\tau,t}} \left(g(\bar{u}_1(\xi)) - g(\bar{u}_2(\xi)), \bar{u}_1(\xi) - \bar{u}_2(\xi)\right) d\xi.$$

Proof. (i) Let us prove the existence and uniqueness of strong solution.

- Existence.

The existence of strong solution for the equation (10) can be obtained by the Faedo-Galerkin method (see [26, 36, 40]), we sketch a proof for the reader’s convenience.

Defining the time-dependent bilinear form

$$B[u, v; t] = \int_{\mathcal{O}} \left(\sum_{i,j=1}^{3} a_{ij}(y, t) \frac{\partial u}{\partial y_i} \frac{\partial v}{\partial y_j} + \sum_{i=1}^{3} b_i(y, t) \frac{\partial u}{\partial y_i} v\right) dy$$

for $u, v \in H^1_0(\mathcal{O})$ and $\tau \leq t \leq T$.

Let $e_k \equiv e_k(y) \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$ be the eigenfunction of $-\Delta$ on $H^1_0(\mathcal{O})$, and let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the corresponding eigenvalues. Then,

$$\lambda_n \to \infty \text{ as } n \to \infty,$$

and we can assume that
\{e_k\}_{k=1}^\infty \text{ is an orthogonal basis of } H_0^1(O) \text{ and an orthonormal basis of } L^2(O).

For each fixed integer \(m\), set
\[ v_m(t) := \sum_{k=1}^m d_m^k(t)e_k, \quad (15) \]
and consider the finite dimensional approximate system
\[ (v_m^\prime(t), e_k) + (v_m(t), e_k) + B[v_m, e_k; t] \]
\[ + 2 \int_{\Omega} \sum_{i=1}^3 \frac{\partial \bar{r}_i}{\partial t} \frac{\partial v_m'(t)}{\partial y_i} e_k dy + (g(v_m(t)), e_k) = (\tilde{f}(t), e_k), \quad (16) \]
and
\[ d_m^k(\tau) = (v_0, e_k), \quad d_m^k'(\tau) = (v_1, e_k), \quad k = 1, \cdots, m \text{ and } \tau \leq t \leq T. \quad (17) \]

Noticing that \(g \in C^1(\mathbb{R})\), then as a direct consequence of the existence and uniqueness result for ODEs, we have that for each integer \(m = 1, 2, \cdots\) there exists a unique local solution \(v_m\) of the form \((15)\) satisfying equation \((17)\) and solving \((16)\), defined in an interval \([\tau, T_m]\), with \(\tau < T_m \leq T\).

To show the existence of strong solution, we need some a priori estimates about \(v_m\), and we divide the proof into several steps.

**Step 1.** Multiply identity \((16)\) by \(d_m^k'(t)\), sum \(k = 1, \cdots, m\), and recall \((15)\) to discover
\[ \frac{1}{2} \frac{d}{dt} \left( \|v_m'(t)\|^2 + A[v_m, v_m; t] \right) + \|v_m'(t)\|^2 \]
\[ - \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^3 a_{ij}'(y, t) \frac{\partial v_m(t)}{\partial y_i} \frac{\partial v_m(t)}{\partial y_j} dy + \int_{\Omega} \sum_{i=1}^3 b_i(y, t) \frac{\partial v_m(t)}{\partial y_i} v_m'(t) dy \]
\[ + 2 \int_{\Omega} \sum_{i=1}^3 \frac{\partial \bar{r}_i}{\partial t} \frac{\partial v_m'(t)}{\partial y_i} v_m'(t) dy + (g(v_m(t)), v_m'(t)) = \left( \tilde{f}(t), v_m'(t) \right), \quad t \in [\tau, T_m], \]
where
\[ A[v_m, v_m; t] = \int_{\Omega} \sum_{i,j=1}^3 a_{ij}(y, t) \frac{\partial v_m(t)}{\partial y_i} \frac{\partial v_m(t)}{\partial y_j} dy. \quad (18) \]

Employing integration by parts formula, we get
\[ \left| 2 \sum_{i=1}^3 \int_{\Omega} \frac{\partial \bar{r}_i}{\partial t} \frac{\partial v_m'(t)}{\partial y_i} v_m'(t) dy \right| \]
\[ = - \int_{\Omega} \sum_{i=1}^3 \frac{\partial}{\partial y_i} \left( \frac{\partial \bar{r}_i}{\partial t} \right) (v_m'(t))^2 dy \leq C_r \|v_m'(t)\|^2. \]

Performing simple calculations, using the condition on \(r\), \(6\) and Young inequality, we deduce
\[ \frac{d}{dt} \left( \|v_m'(t)\|^2 + A[v_m, v_m; t] + 2 \int_{\Omega} G(v_m(t)) dy \right) \]
where \( C_3 \) only depend on \( r, \eta \) and \( \gamma \). Applying Gronwall inequality, which along with (5) shows that
\[
\|v'_m(t)\|^2 + A[v_m, v_m; t] + 2 \int_\mathcal{O} G(v_m(t))dy \\
\leq C_3 e^{C_3(t-\tau)} \left( \|v'_m(\tau)\|^2 + \|\nabla v_m(\tau)\|^p + \int_\tau^t \|\tilde{f}(s)\|^2 ds + 1 \right).
\] (19)

Denote \( \kappa_1 = 2C_0^2|\mathcal{O}|^{\frac{2}{p}} \), and suppose
\[
\gamma_1^2 = \frac{\theta}{\kappa_1},
\] (20)
where \( C_0 \) (independent of domain \( \mathcal{O} \), see [26] Chapter 5 for more details) satisfying
\[
\|u(t)\|^2 \leq C_0^2|\mathcal{O}|^{\frac{2}{p}}\|\nabla u(t)\|^2.
\] (21)

By (6) and \( r \) is hyperbolic, we deduce
\[
\|v'_m(t)\|^2 + A[v_m, v_m; t] + 2 \int_\mathcal{O} G(v_m(t))dy \\
\geq \|v'_m(t)\|^2 + \theta \|\nabla v_m(t)\|^2 - \gamma C_0^2|\mathcal{O}|^{\frac{2}{p}}\|\nabla v_m(t)\|^2 - 2\eta|\mathcal{O}|
\]
\[
\geq \|v'_m(t)\|^2 + \frac{\theta}{2}\|\nabla v_m(t)\|^2 - 2\eta|\mathcal{O}|.
\] (22)

Combining (19) and (22), we can easily deduce that
\[
\sup_{t \in [\tau, T]} (\|v'_m(t)\|^2 + \|\nabla v_m(t)\|^2)
\leq C_4 e^{C_4(t-\tau)} \left( \|v'_m(\tau)\|^2 + \|\nabla v_m(\tau)\|^p + \int_\tau^t \|\tilde{f}(s)\|^2 ds + |\mathcal{O}| \right)
\]
and
\[
\sup_{t \in [\tau, T]} (\|v'_m(t)\|^2 + \|\nabla v_m(t)\|^2) \leq C_5,
\] (23)
for some positive constants \( C_4 \) and \( C_5 \), which are independent of \( m \).

**Step 2.** Differentiate the identity (16) with respect to \( t \), we obtain
\[
(v''_m(t), e_k) + (v'_m(t), e_k) + B[v'_m, e_k; t] + \sum_{i,j=1}^3 \int_\mathcal{O} a'_{ij}(y, t) \frac{\partial v_m(t)}{\partial y_i} \frac{\partial e_k}{\partial y_j} dy
\]
\[
+ \sum_{i=1}^3 \int_\mathcal{O} b'_i(y, t) \frac{\partial v_m(t)}{\partial y_i} e_k dy + 2 \sum_{i=1}^3 \int_\mathcal{O} \bar{r}_i \frac{\partial v'_m(t)}{\partial y_i} e_k dy
\]
\[
+ 2 \sum_{i=1}^3 \int_\mathcal{O} \frac{\partial^2 \bar{r}_i}{\partial t} \frac{\partial v'_m(t)}{\partial y_i} e_k dy + 2 \sum_{i=1}^3 \sum_{j=1}^3 \int_\mathcal{O} \frac{\partial^2 \bar{r}_i}{\partial y_j \partial t} \frac{\partial r_j}{\partial t} \frac{\partial v'_m(t)}{\partial y_i} e_k dy
\]
\[
+ \left( g'(v_m(t))v'_m(t), e_k \right) = (f_1(t), e_k),
\] (24)
where

\[ f_1(t) = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial r_i}{\partial t} + f'(t). \]  

(25)

Multiply (24) by \( d_{m}^{\nu}(t) \), sum \( k = 1, \cdots, m \), we discover

\[ \frac{1}{2} \frac{d}{dt} \left( \|v_m^{\nu}(t)\|^2 + A[v_m, v_v, t] \right) + \|v_m^{\nu}(t)\|^2 \]

\[ = \frac{1}{2} \sum_{i,j=1}^{3} \int_{\mathcal{O}} a'_{ij} \frac{\partial v_m^{\nu}(t)}{\partial y_i} \frac{\partial v_m^{\nu}(t)}{\partial y_j} dy - \sum_{i=1}^{3} \int_{\mathcal{O}} b_i \frac{\partial v_m^{\nu}(t)}{\partial y_i} v_m^{\nu}(t)dy \]

\[ - \sum_{i=1}^{3} \int_{\mathcal{O}} b_i \frac{\partial v_m^{\nu}(t)}{\partial y_i} v_m^{\nu}(t)dy - 2 \sum_{i=1}^{3} \int_{\mathcal{O}} \frac{\partial^2 r_i}{\partial t^2} \frac{\partial v_m^{\nu}(t)}{\partial y_i} v_m^{\nu}(t)dy \]

\[ - 2 \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\mathcal{O}} \frac{\partial^2 r_i}{\partial x_j \partial t} \frac{\partial v_m^{\nu}(t)}{\partial y_i} \frac{\partial v_m^{\nu}(t)}{\partial y_j} dy - \sum_{i,j=1}^{3} \int_{\mathcal{O}} a'_{ij} \frac{\partial v_m^{\nu}(t)}{\partial y_i} \frac{\partial v_m^{\nu}(t)}{\partial y_j} dy \]

\[ - 2 \sum_{i=1}^{3} \int_{\mathcal{O}} \frac{\partial^2 r_i}{\partial t} \frac{\partial v_m^{\nu}(t)}{\partial y_i} v_m^{\nu}(t)dy - (g'(v_m(t)v'_v(t), v''_v(t)) + (f_1(t), v''_v(t)). \]  

(26)

We now estimate every term on the right-hand side of (26).

Firstly, by the assumption on \( r \), we have

\[ \left| \frac{1}{2} \sum_{i,j=1}^{3} \int_{\mathcal{O}} a'_{ij} \frac{\partial v_m^{\nu}(t)}{\partial y_i} \frac{\partial v_m^{\nu}(t)}{\partial y_j} dy \right| \leq C_r \|\nabla v_m^{\nu}(t)\|^2, \]

(27)

\[ \left| \sum_{i=1}^{3} \int_{\mathcal{O}} b_i \frac{\partial v_m^{\nu}(t)}{\partial y_i} v_m^{\nu}(t)dy \right| \leq C_r \left( \|\nabla v_m^{\nu}(t)\|^2 + \|v_m^{\nu}(t)\|^2 \right), \]

(28)

\[ \left| \sum_{i=1}^{3} \int_{\mathcal{O}} b_i \frac{\partial v_m^{\nu}(t)}{\partial y_i} v_m^{\nu}(t)dy \right| \leq C_r \left( \|\nabla v_m(t)\|^2 + \|v_m^{\nu}(t)\|^2 \right), \]

(29)

\[ \left| 2 \sum_{i=1}^{3} \int_{\mathcal{O}} \frac{\partial^2 r_i}{\partial t^2} \frac{\partial v_m^{\nu}(t)}{\partial y_i} v_m^{\nu}(t)dy \right| \leq C_r \left( \|\nabla v_m^{\nu}(t)\|^2 + \|v_m^{\nu}(t)\|^2 \right) \]

(30)

and

\[ \left| 2 \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\mathcal{O}} \frac{\partial^2 r_i}{\partial x_j \partial t} \frac{\partial v_m^{\nu}(t)}{\partial y_i} \frac{\partial v_m^{\nu}(t)}{\partial y_j} dy \right| \leq C_r \left( \|\nabla v_m^{\nu}(t)\|^2 + \|v_m^{\nu}(t)\|^2 \right). \]

(31)

Integrating by parts, we find that

\[ \sum_{i,j=1}^{3} \int_{\mathcal{O}} a'_{ij}(y,t) \frac{\partial v_m(t)}{\partial y_i} \frac{\partial v_m(t)}{\partial y_j} dy \]

\[ = \sum_{i,j=1}^{3} a'_{ij}(y,t) \frac{\partial v_m(t)}{\partial y_i} v_m(t) \bigg|_{\partial \mathcal{O}} - \sum_{i,j=1}^{3} \int_{\mathcal{O}} \frac{\partial}{\partial y_j} \left( a'_{ij}(y,t) \frac{\partial v_m(t)}{\partial y_i} \right) v_m(t) dy \]

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and consequently
\[
\left| \sum_{i,j=1}^{3} \int_{\mathcal{O}} a'_{ij}(y,t) \frac{\partial v_m(t)}{\partial y_i} \frac{\partial v''_m(t)}{\partial y_j} \right| \leq C_r \left( \| \Delta v_m(t) \|^2 + \| \nabla v_m(t) \|^2 + \| v''_m(t) \|^2 \right).
\]  

(32)

Similarly, the seventh term on the right-hand side of (26) are bounded by
\[
\left| 2 \sum_{i=1}^{3} \int_{\mathcal{O}} \frac{\partial r_i}{\partial t} \frac{\partial v''_m(t)}{\partial y_i} v''_m(t) dy \right| \leq C_r \| v''_m(t) \|^2.
\]  

(33)

Furthermore, we also have
\[
| (f_1(t), v''_m(t)) | \leq C \left( \| f_1(t) \|^2 + \| v''_m(t) \|^2 \right).
\]  

(34)

Recalling (5), (23) and applying H"older inequality, we deduce
\[
| (g'(v_m(t))v'_m(t), v''_m(t)) |
\leq C \int_{\mathcal{O}} (|v_m(t)|^{p-1} + 1) |v'_m(t)| |v''_m(t)| dy
\leq C \left( \int_{\mathcal{O}} (|v_m(t)|^{p-1} + 1) \frac{2p}{2p} dy \right)^{\frac{p}{2p}} \left( \int_{\mathcal{O}} |v'_m(t)|^{2p} dy \right)^{\frac{1}{2p}} \left( \int_{\mathcal{O}} |v''_m(t)|^{2p} dy \right)^{\frac{1}{2p}}
\leq C(\| \nabla v'_m(t) \|^2 + \| v''_m(t) \|^2).
\]  

(35)

By (27)-(35), recalling \( r \) is hyperbolic, we conclude
\[
\frac{d}{dt} \left( \| v''_m(t) \|^2 + A[v'_m, v''_m; t] \right) + 2 \| v''_m(t) \|^2
\leq C_6 \left( \| v'_m(t) \|^2 + A[v'_m, v''_m; t] + \| \Delta v_m(t) \|^2 + \| \nabla v_m(t) \|^2 + \| v'_m(t) \|^2 + \| f_1(t) \|^2 \right),
\]  

(36)

for some positive constant \( C_6 \) independent of \( m \).

**Step 3.** Recall we are taking \( \{e_k\}_{k=1}^{\infty} \) to be the complete collection of eigenfunctions for \( -\Delta \) on \( H_0^1(\mathcal{O}) \). Multiplying (16) by \( \lambda_k d^k_m(t) \) and summing \( k = 1, \cdots, m \), we deduce
\[
\left( v''_m(t), -\Delta v_m(t) \right) + \left( v'_m(t), -\Delta v_m(t) \right) + B[v_m, -\Delta v_m; t]
+ 2 \sum_{i=1}^{3} \int_{\mathcal{O}} \frac{\partial r_i}{\partial t} \frac{\partial v'_m(t)}{\partial y_i} (-\Delta v_m(t)) dy + (g(v_m(t)), -\Delta v_m(t)) = (\bar{f}(t), -\Delta v_m(t)).
\]  

(37)

Now, we hand each term as following.

It is obviously
\[
\left| \left( v''_m(t), -\Delta v_m(t) \right) \right| \leq \delta \| \Delta v_m(t) \|^2 + C_6 \| v''_m(t) \|^2
\]  

(38)

and
\[
\left| \left( v'_m(t), -\Delta v_m(t) \right) \right| \leq \delta \| \Delta v_m(t) \|^2 + C_6 \| v'_m(t) \|^2.
\]  

(39)
On the other hand, employing integration by parts formula,

\[ B[v_m, -\Delta v_m; t] = \int_\Omega \sum_{i,j=1}^3 a_{ij}(y,t) \frac{\partial^2 v_m(t)}{\partial y_i \partial y_j} \Delta v_m(t) dy \]

\[ + \int_\Omega c_i(y,t) \frac{\partial v_m(t)}{\partial y_i} (-\Delta v_m(t)) dy, \]

where

\[ c_i(t) = \frac{\partial^2 \bar{f}_i(r(y,t), t) + \frac{\partial \bar{f}_i}{\partial t}(r(y,t), t) - \Delta x \bar{f}_i(r(y,t), t) \cdot 1 \leq i, j \leq 3. \]

Using Lemma 3.3 and Young inequality, we arrive at

\[ C_1' \|\Delta v_m(t)\|^2 \leq B[v_m, -\Delta v_m; t] + C_2' \|v_m(t)\|^2, \]

where \( C_1' \) and \( C_2' \) are positive constants. By Young inequality, we get

\[ \left| \int_\Omega \left( \frac{\partial \bar{f}_i}{\partial t} \frac{\partial v_m(t)}{\partial y_i} (-\Delta v_m(t)) dy \right) \leq \delta \|\Delta v_m(t)\|^2 + C_{\delta,r} A[v_m', v_m'; t], \]

\[ \left| (\tilde{f}(t), -\Delta v_m(t)) \right| \leq \delta \|\Delta v_m(t)\|^2 + C_\delta \|\tilde{f}(t)\|^2, \]

and

\[ \left| (g(v_m(t)), -\Delta v_m(t)) \right| \leq \delta \|\Delta v_m(t)\|^2 + C_\delta \int_\Omega |g(v_m(t))|^2 dy \]

\[ \leq \delta \|\Delta v_m(t)\|^2 + C_\delta \left( \|\nabla v_m(t)\|^{2p} + 1 \right). \]

Combining the above estimates, i.e., (37)-(43), we can obtain

\[ \|\Delta v_m(t)\|^2 \leq C_7 \left( A[v_m', v_m'; t] + \|v_m'(t)\|^2 + \|v_m''(t)\|^2 + \|\nabla v_m(t)\|^{2p} + \|\tilde{f}(t)\|^2 + 1 \right). \]

Using this estimate in (36), we have the following

\[ \frac{d}{dt} \left( \|v_m''(t)\|^2 + A[v_m', v_m'; t] + 2\|v_m''(t)\|^2 \right) \]

\[ \leq C_8 \left( A[v_m', v_m'; t] + \|v_m'(t)\|^2 + \|v_m''(t)\|^2 + \|\nabla v_m(t)\|^{2p} + \|\tilde{f}(t)\|^2 + 2f_t(t) \right), \]

by Gronwall inequality, which along with (23), we discover

\[ \sup_{t \in [\tau, T]} (\|v_m'(t)\|^2 + \|\nabla v_m(t)\|^{2p} + \|\Delta v_m(t)\|^2) \leq C_9, \]

where \( C_7, C_8 \) and \( C_9 \) are positive constants (independent of \( m \)).

**Step 4.** Under the above estimates (45), we conclude that

\[ \{v_m\} \text{ is bounded in } L^\infty(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)), \]

\[ \{v'_m\} \text{ is bounded in } L^\infty(\tau, T; H_0^1(\Omega)), \]

\[ \{v''_m\} \text{ is bounded in } L^\infty(\tau, T; L^2(\Omega)). \]

It is now a standard matter to prove that a subsequence of \( \{v_m\}_{m=1}^\infty \), still denote \( \{v_m\}_{m=1}^\infty \), satisfying
where

\[ v_m \to v \text{ weakly star in } L^\infty(\tau,T; H^2(\mathcal{O})), \]

weakly in \( L^2(\tau,T; H^2(\mathcal{O})); \)

\[ v'_m \to v' \text{ weakly star in } L^\infty(\tau,T; H^1_0(\mathcal{O})), \]

weakly in \( L^2(\tau,T; H^1_0(\mathcal{O})); \)

\[ v''_m \to v'' \text{ weakly star in } L^\infty(\tau,T; L^2(\mathcal{O})), \]

weakly in \( L^2(\tau,T; L^2(\mathcal{O})). \)

Obviously, \( v \) is the strong solution of (10).

Uniqueness.

Let \((u_0, u_1) \in H^1_0(\Omega_T) \times L^2(\Omega_T)\) and \(u_i (i = 1, 2)\) be the corresponding strong solutions. Set \( \Gamma(y,t) = \Gamma_i(y,t), t)\) and \( \Gamma = \Gamma_1 - \Gamma_2,\) then \( \Gamma \) is a strong solution of the following equation

\[
\frac{\partial^2 \Gamma(y,t)}{\partial t^2} + \frac{\partial \Gamma(y,t)}{\partial t} - \sum_{i,j=1}^{3} \frac{\partial (a_{ij}(y,t)) \partial \Gamma}{\partial y_i} + \sum_{i=1}^{3} b_i(y,t) \frac{\partial \Gamma}{\partial y_i} \\
+ 2 \sum_{i=1}^{3} \frac{\partial \Gamma_i(y,t)}{\partial t} \frac{\partial \Gamma}{\partial y_i} + g(\Gamma_1(y,t)) - g(\Gamma_2(y,t)) = 0,
\]

where

\[ \Gamma(y, \tau) = 0, \quad \Gamma'(y, \tau) = 0. \]

Multiplying the above equation (46) by \( \Gamma', \) to find

\[
\frac{1}{2} \frac{d}{dt} \left( ||\Gamma'(t)||^2 + A||\Gamma, \Gamma'; t|| \right) + ||\Gamma'(t)||^2 \\
- \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{3} a_{ij} \frac{\partial \Gamma}{\partial y_i} \frac{\partial \Gamma}{\partial y_j} dy + \int_{\Omega} \sum_{i=1}^{3} b_i \frac{\partial \Gamma}{\partial y_i} \Gamma(t) dy + 2 \int_{\Omega} \sum_{i=1}^{3} \frac{\partial \Gamma}{\partial y_i} \frac{\partial \Gamma}{\partial y_i} dy \\
+ \int_{\Omega} (g(\Gamma_1(t)) - g(\Gamma_2(t))) \Gamma'(t) dy = 0.
\]

(47)

Similar to the proof of Step 1, we have the following estimates

\[
\left| -\frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{3} a_{ij} \frac{\partial \Gamma}{\partial y_i} \frac{\partial \Gamma}{\partial y_j} dy \right| \leq C_r ||\nabla \Gamma(t)||^2,
\]

(48)

\[
\left| \int_{\Omega} \sum_{i=1}^{3} b_i \frac{\partial \Gamma}{\partial y_i} \Gamma(t) dy \right| \leq C_r (||\nabla \Gamma(t)||^2 + ||\Gamma'(t)||^2)
\]

(49)

and

\[
2 \int_{\Omega} \sum_{i=1}^{3} \frac{\partial^2 \Gamma}{\partial y_i \partial t} \frac{\partial \Gamma_i}{\partial t} dy \\
= \left| \int_{\Omega} \sum_{i=1}^{3} \frac{\partial \Gamma(t)}{\partial y_i} \frac{\partial \Gamma_i}{\partial t} dy \right| \\
= \left| \int_{\Omega} \sum_{i=1}^{3} (\Gamma(t))^2 \frac{\partial \Gamma_i}{\partial t} dy \right| \leq C_r ||\Gamma'(t)||^2.
\]

(50)
As usual, we deal with the nonlinear term. To do so, using Hölder inequality, recalling $v_1, v_2 \in L^\infty(\tau,T;H^1_0(\mathcal{O}))$, we find that
\[
\int_\mathcal{O} (g(\varpi_1(t)) - g(\varpi_2(t))) \varpi'(t) dy
\]
\[
\leq C \int_\mathcal{O} (\varpi_1(t)^{p-1} + \varpi_2(t)^{p-1} + 1) \varpi_1(t) - \varpi_2(t) \|\varpi'(t)\| dy
\]
\[
\leq C \left( \int_\mathcal{O} (\varpi_1(t)^{p-1} + \varpi_2(t)^{p-1} + 1) \frac{2}{p+1} dy \right)^{\frac{p-1}{p}} \left( \int_\mathcal{O} \varpi'(t)^2 dy \right)^{\frac{1}{2}}
\]
\[
\leq C (\|\nabla u(t)\|^2 + \|\varpi'(t)\|^2).
\] (51)

Taking into account of (47)-(51), we conclude that
\[
\frac{d}{dt} \left( \|\varpi'(t)\|^2 + A(\varpi, \varpi; t) \right) \leq C \left( \|\varpi'(t)\|^2 + A(\varpi, \varpi; t) \right),
\]
which implies the uniqueness on account of Gronwall inequality immediately.

(ii) Let $u(t)$ a strong solution, then it satisfies
\[
u_{tt} + u_t - \Delta u + g(u) = f(t)
\] (52)
a.e. on time-dependent domain $Q_{\tau,T}$.

Set $\tilde{u} = u' + \vartheta u$ (\(\vartheta > 0\)), whereupon (52) becomes
\[
\tilde{u}'(t) + (1 - \vartheta)\tilde{u}(t) - (1 - \vartheta)\vartheta u(t) - \Delta u(t) + g(u(t)) = f(t).
\] (53)

Multiplying (53) by $\tilde{u} = u' + \vartheta u$, we deduce
\[
(\tilde{u}'(t), \tilde{u}(t)) + (1 - \vartheta)(\tilde{u}(t), \tilde{u}(t)) + \vartheta(-\Delta u(t), u'(t)) + \vartheta \int_\mathcal{O} (\nabla u(t), \nabla u'(t)) + g(u(t), u'(t)) + g(u(t), \vartheta u(t)) = (f(t), \tilde{u}(t)).
\] (54)

So, integrating (54) over $[\tau, s]$, by (8), Lemma 2.5 and Corollary 3.4 in [33], we observe
\[
\left( \|\tilde{u}(s)\|^2 + \|\nabla u(s)\|^2 + 2 \int_\mathcal{O}_s G(u(s)) dx \right)
\]
\[
- \left( \|\tilde{u}(\tau)\|^2 + \|\nabla u(\tau)\|^2 + 2 \int_\mathcal{O}_s G(u(\tau)) dx \right)
\]
\[
+ 2(1 - \vartheta) \int_\mathcal{O}_s \|\tilde{u}(\xi)\|^2 d\xi + 2\vartheta \int_\mathcal{O}_s \|\nabla u(\xi)\|^2 d\xi + 2\vartheta \int_\mathcal{O}_s (g(u(\xi)), u(\xi)) d\xi
\]
\[
- 2\vartheta(1 - \vartheta) \int_\mathcal{O}_s (u(\xi), \tilde{u}(\xi)) d\xi = 2 \int_\mathcal{O}_s (f(\xi), \tilde{u}(\xi)) d\xi.
\] (55)

Moreover, using Young inequality, we obtain the estimates
\[
|2\vartheta(1 - \vartheta)(u(\xi), \tilde{u}(\xi))| \leq \frac{1 - \vartheta}{2} \|\tilde{u}(\xi)\|^2 + 2\vartheta^2(1 - \vartheta)\|u(\xi)\|^2
\] (56)

and
\[
|2f(\xi, \tilde{u}(\xi))| \leq \frac{1 - \vartheta}{2} \|\tilde{u}(\xi)\|^2 + \frac{2}{1 - \vartheta} \|f(\xi)\|^2.
\] (57)

Recalling (21), similarly, we have
\[
\|u(t)\|^2 \leq C_n^2 \|\tilde{u}(t)\|^2.
\] (58)
Following (55)-(57), by (58) and conditions on \( r \), after simplification, we obtain

\[
\left( \| \tilde{u}(s) \|_r^2 + \| \nabla u(s) \|_r^2 + 2 \int_{\Omega} G(u(s)) dx \right) + (1 - \vartheta) \int_{\tau}^{s} \| \tilde{u}(\xi) \|_r^2 d\xi \\
+ 2\vartheta \int_{\tau}^{s} \| \nabla u(\xi) \|_r^2 d\xi + \vartheta(1 - \vartheta) + \gamma) C^2 \int_{\tau}^{s} \| \nabla u(\xi) \|_r^2 d\xi \\
+ 2\vartheta \int_{\tau}^{s} \int_{\Omega} G(u(\xi)) d\xi \\
\leq \left( \| \tilde{u}(\tau) \|_r^2 + \| \nabla u(\tau) \|_r^2 + 2 \int_{\Omega} G(u(\tau)) dx \right) \\
+ 2(1 - \vartheta)^{-1} \int_{\tau}^{s} \| f(\xi) \|_r^2 d\xi + 2\vartheta \int_{\tau}^{s} |\Omega_\xi| d\xi.
\]

Denote \( \kappa_{\tau,s} = \sup_{\xi \in [\tau,s]} \{ 2C^2_0 |\Omega_\xi|^{\frac{2}{3}} \} \), supposing

\[
\gamma_2^* = \min\{ \gamma_1^*, 1/\kappa_{\tau,T} \}.
\]

we get

\[
\left( \| \tilde{u}(s) \|_r^2 + \| \nabla u(s) \|_r^2 + 2 \int_{\Omega} G(u(s)) dx \right) + (1 - \vartheta) \int_{\tau}^{s} \| \tilde{u}(\xi) \|_r^2 d\xi \\
+ \vartheta(1 - \vartheta)(1 - \vartheta)\kappa_{\tau,s} \int_{\tau}^{s} \| \nabla u(\xi) \|_r^2 d\xi + 2\vartheta \int_{\tau}^{s} \int_{\Omega} G(u(\xi)) d\xi d\xi \\
\leq \left( \| \tilde{u}(\tau) \|_r^2 + \| \nabla u(\tau) \|_r^2 + 2 \int_{\Omega} G(u(\tau)) dx \right) \\
+ 2(1 - \vartheta)^{-1} \int_{\tau}^{s} \| f(\xi) \|_r^2 d\xi + 2\vartheta \int_{\tau}^{s} |\Omega_\xi| d\xi.
\]

For \( \tau \leq s \leq t \leq T \), we get

\[
\kappa_{\tau,s} \leq \kappa_{\tau,t}.
\]

Denote

\[
\lambda_{r,t} = \min\{ 1 - \vartheta, \vartheta \left( 1 - \vartheta(1 - \vartheta)\kappa_{\tau,t} \right) \},
\]

and suppose

\[
\vartheta = \min\{ 1/2, 1/\kappa_{r,T} \},
\]

then

\[
\lambda_{r,t} = \vartheta \left( 1 - \vartheta(1 - \vartheta)\kappa_{r,t} \right)
\]
and

\[
\left(\|\tilde{u}(s)\|^2 + \|\nabla u(s)\|^2 + 2\int_{\mathcal{O}_s} G(u(s))\,dx\right)
+ \lambda_{r,t} \int_{\tau}^{s} \left(\|\tilde{u}(\xi)\|^2 + \|\nabla u(\xi)\|^2 + 2\int_{\mathcal{O}_\xi} G(u(\xi))\,d\xi\right)\,d\xi
\leq \left(\|\tilde{u}(\tau)\|^2 + \|\nabla u(\tau)\|^2 + 2\int_{\mathcal{O}_\tau} G(u(\tau))\,dx\right)
+ 4\int_{\tau}^{s} \|f(\xi)\|^2\,d\xi + \int_{\tau}^{s} |\mathcal{O}_\xi|\,d\xi.
\] (66)

Utilizing Gronwall inequality, we can deduce the estimate

\[
\|\tilde{u}(t)\|^2 + \|\nabla u(t)\|^2 + 2\int_{\mathcal{O}_t} G(u(t))\,dx
\leq e^{-\lambda_{r,t}(s-r)} \left(\|\tilde{u}(\tau)\|^2 + \|\nabla u(\tau)\|^2 + \|\nabla u(\tau)\|^2 + 1\right)
+ e^{-\lambda_{r,t}t} \left(4\int_{\tau}^{t} e^{\lambda_{r,t}\xi} \|f(\xi)\|^2\,d\xi + \int_{\tau}^{t} e^{\lambda_{r,t}\xi} |\mathcal{O}_\xi|\,d\xi\right).
\]

Applying (6) and (58), we find that

\[
\|\tilde{u}(t)\|^2 + \|\nabla u(t)\|^2
\leq e^{-\lambda_{r,t}(s-r)} \left(\|\tilde{u}(\tau)\|^2 + \|\nabla u(\tau)\|^2 + \|\nabla u(\tau)\|^2 + 1\right)
+ e^{-\lambda_{r,t}t} \left(4\int_{\tau}^{t} e^{\lambda_{r,t}\xi} \|f(\xi)\|^2\,d\xi + \int_{\tau}^{t} e^{\lambda_{r,t}\xi} |\mathcal{O}_\xi|\,d\xi\right) + 2\eta|\mathcal{O}_1|.
\]

Recalling (60), we can deduce that

\[
\|\tilde{u}(t)\|^2 + \|\nabla u(t)\|^2
\leq 2e^{-\lambda_{r,t}(s-r)} \left(\|\tilde{u}(\tau)\|^2 + \|\nabla u(\tau)\|^2 + \|\nabla u(\tau)\|^2 + 1\right)
+ 2e^{-\lambda_{r,t}t} \left(4\int_{\tau}^{t} e^{\lambda_{r,t}\xi} \|f(\xi)\|^2\,d\xi + \int_{\tau}^{t} e^{\lambda_{r,t}\xi} |\mathcal{O}_\xi|\,d\xi\right) + 4\eta|\mathcal{O}_1|.
\] (67)

By (58) and (64), we note that

\[
\|u'(t)\|^2 + \|\nabla u(t)\|^2
= \|u'(t) + \vartheta u(t) - \vartheta u(t)\|^2 + \|\nabla u(t)\|^2
\leq 2\|u'(t) + \vartheta u(t)\|^2 + 2\vartheta^2 \|u(t)\|^2 + \|\nabla u(t)\|^2
\leq 2\left(\|\tilde{u}(t)\|^2 + \|\nabla u(t)\|^2\right).
\] (68)

Similarly, we get

\[
\|\tilde{u}(t)\|^2 + \|\nabla u(t)\|^2 \leq 2\left(\|u'(t)\|^2 + \|\nabla u(t)\|^2\right).
\] (69)

Combing (67), (68) and (69), we obtain (12).

(iii) Let \((\mathfrak{p}_1, \mathfrak{p}_2), (i = 1, 2)\) be the corresponding strong solutions to \((u_0^i, u^i_1)\), now write \(w = \mathfrak{p}_1 - \mathfrak{p}_2\) and \(\tilde{w} = w' + \vartheta u\), we discover that

\[
\tilde{w}'(t) + (1 - \vartheta)\tilde{w}(t) + (\vartheta^2 - \vartheta)w(t) - \Delta w(t) + g(\mathfrak{p}_1(t)) - g(\mathfrak{p}_2(t)) = 0.
\] (70)
Let us denote

\[ \Phi_{\tau,T} := \{ \phi | \phi \in L^2(\tau,T; H^1_0(\Omega)), \phi' \in L^2(\tau,T; L^2(\Omega)), \phi(\tau) = \phi(T) = 0 \}. \]

**Definition 4.1.** Let \( u_0 \in H^1_0(\Omega) \), \( u_1 \in L^2(\Omega) \), \( f \in L^2(\Omega) \), and \(-\infty < \tau \leq T < \infty \) be given. We say that a function \( u \) is a weak solution of (9) if

1. \( u \in C([\tau,T]; H^1_0(\Omega)) \) with \( u(\tau) = u_0 \);
2. \( u' \in C([\tau,T]; L^2(\Omega)) \) with \( u'(\tau) = u_1 \);
3. there exists a sequence of regular data \( u_{0m} \in H^1_0(\Omega) \cap H^2(\Omega), u_{1m} \in H^1_0(\Omega) \) and \( f_m \in H^1_0(Q_{\tau,T}), m = 1,2,\ldots \), such that

\[ u_{0m} \to u_0 \quad \text{in} \quad H^1_0(\Omega), \quad u_{1m} \to u_1 \quad \text{in} \quad L^2(\Omega), \quad f_m \to f \quad \text{in} \quad L^2(\Omega), \]

and

\[ u_m \to u \quad \text{in} \quad C([\tau,T]; H^1_0(\Omega)), \quad u'_m \to u' \quad \text{in} \quad C([\tau,T]; L^2(\Omega)), \]

where \( u_m \) is the unique strong solution of (9) corresponding to \((u_{0m}, u_{1m}, f_m)\);

Multiplying the above equation (70) by \( \widetilde{w} \), and repeating the argument used in the proof of (12) we can establish that

\[
\|\tilde{w}(t)\|_t^2 + \|\nabla w(t)\|_t^2 \leq e^{-\lambda_r,t(-\tau)}(\|\tilde{w}(\tau)\|_t^2 + \|\nabla w(\tau)\|_t^2)
- e^{-\lambda_r,t} \int_\tau^t e^{\lambda_r,\xi} (g(\varpi_1(\xi)) - g(\varpi_2(\xi)), \varpi_1(\xi) - \varpi_2(\xi))_\xi d\xi
- \theta e^{-\lambda_r,t} \int_\tau^t e^{\lambda_r,\xi} (g(\varpi_1(\xi)) - g(\varpi_2(\xi)), \varpi_1(\xi) - \varpi_2(\xi))_\xi d\xi.
\]

Applying (68) and (69), we deduce (13).

The proof of Theorem 3.4 is now complete. \( \square \)

**Remark 1.** In fact, as a consequence of Lemma 3.8 and Lemma 3.9 in [33], we can easily get \( u \in C([\tau,T]; H^1_0(\Omega)) \) and \( u' \in C([\tau,T]; L^2(\Omega)) \).

**Remark 2.** With the help of inequality (58), we can obtain the \( \mathcal{Q} \)-pullback absorbing sets (see Section 5), avoid discussing eigenvalues (see [33] and [44], also [37] for more details), and only assuming the measure of every spatial domains are uniformly bounded above. Furthermore, this method can improve the restriction on varying domains in the aforementioned works on the topic, e.g., [33], Section 7.

**Remark 3.** In order to ensure the existence of strong solutions, we assume the transformation \( \tau \) is \( C^1 \)-diffeomorphism. As a matter of fact, this condition is slightly strong, however, this matter will be pursued elsewhere.

4. Weak solutions. In this section, combining the ideas of [33], we will give a proper definition for weak solution about the wave equations on time-dependent domains.

Let us denote

\[ \Phi_{\tau,T} := \{ \phi | \phi \in L^2(\tau,T; H^1_0(\Omega)), \phi' \in L^2(\tau,T; L^2(\Omega)), \phi(\tau) = \phi(T) = 0 \}. \]
4. for all $\phi \in \Phi_{\tau,T}$,

$$
\begin{aligned}
&\int_\tau^T \int_{\mathcal{O}_t} u'(x,t)\phi'(t,x)dxdt - \int_\tau^T \int_{\mathcal{O}_t} u(x,t)\phi'(t,x)dxdt \\
+ &\int_\tau^T \int_{\mathcal{O}_t} \nabla u(x,t)\nabla \phi(t,x)dxdt + \int_\tau^T \int_{\mathcal{O}_t} g(u(x,t))\phi(x,t)dxdt \\
= &\int_\tau^T \int_{\mathcal{O}_t} f(x,t)\phi(x,t)dxdt.
\end{aligned}
$$

(71)

**Remark 4.** The strong solution $u$ is obviously a weak solution.

**Theorem 4.2.** Let $f \in L^{2}_{loc}(\mathbb{R}; L^{2}(\mathcal{O}_1))$, then for any $u_{0} \in H^{1}_{0}(\mathcal{O}_{\tau})$, $u_{1} \in L^{2}(\mathcal{O}_{\tau})$, and any $-\infty < \tau \leq T < \infty$, there exists a unique weak solution $u(t)$ for Eq. (9). Moreover, $(u(t), u'(t))$ satisfies the following estimates for all $t \in [\tau, T]$,

$$
\begin{aligned}
&\|u'(t)\|_{w}^{2} + \|\nabla u(t)\|_{w}^{2} \\
\leq & 8e^{-\lambda_{\tau,t}(t-\tau)} (\|u'(\tau)\|_{\tau}^{2} + \|\nabla u(\tau)\|_{\tau}^{2} + \|\nabla u(\tau)\|_{\tau}^{p+1}) \\
+ &4e^{-\lambda_{\tau,t}t} \left( 4 \int_{\tau}^{t} e^{\lambda_{\tau,t}\xi} \|f(\xi)\|_{\xi}^{2}d\xi + \int_{\tau}^{t} e^{\lambda_{\tau,t}\xi} \|O_{\xi}|d\xi \right) + 8\eta|\mathcal{O}_1|.
\end{aligned}
$$

(72)

and

$$
\begin{aligned}
&\|u'(t)\|_{w}^{2} + \|\nabla w(t)\|_{w}^{2} \leq 4e^{-\lambda_{\tau,t}(t-\tau)} (\|u'(\tau)\|_{\tau}^{2} + \|\nabla w(\tau)\|_{\tau}^{2}) \\
- &2e^{-\lambda_{\tau,t}t} \int_{\tau}^{t} e^{\lambda_{\tau,t}\xi}(g(\xi_{1}(\xi)) - g(\xi_{2}(\xi)), \xi_{1}(\xi) - \xi_{2}(\xi))\xi d\xi \\
- &2\partial e^{-\lambda_{\tau,t}t} \int_{\tau}^{t} e^{\lambda_{\tau,t}\xi}(g(\xi_{1}(\xi)) - g(\xi_{2}(\xi)), \xi_{1}(\xi) - \xi_{2}(\xi))\xi d\xi.
\end{aligned}
$$

(73)

where $\lambda_{\tau,t}$, $\partial$ and $w$ are the same as presented in Theorem 3.4.

**Proof.** Let $u_{0m} \in H^{1}_{0}(\mathcal{O}_{\tau}) \cap H^{2}(\mathcal{O}_{\tau})$, $u_{1m} \in H^{1}_{0}(\mathcal{O}_{\tau})$, and $f_{m} \in H^{1}_{0}(Q_{\tau,T})$ such that

$$
\begin{aligned}
u_{0m} \rightarrow u_{0} \text{ in } H^{1}_{0}(\mathcal{O}_{\tau}), & \quad u_{1m} \rightarrow u_{1} \text{ in } L^{2}(\mathcal{O}_{\tau}), \quad f_{m} \rightarrow f \text{ in } L^{2}(\mathcal{O}_{\tau}).
\end{aligned}
$$

(74)

Then for each $(u_{0m}, u_{1m}, f_{m})$, $m = 1, 2, \cdots$, there exists a unique strong solution $u_{m}$ for the following problem:

$$
\begin{aligned}
u_{m}^{\prime\prime} + u_{m}^{\prime} - \Delta u_{m} + g(u_{m}) = f_{m}(t) \ \text{in} \ \mathcal{Q}_{\tau,T}, \\
u_{m} = 0 \ \text{on} \ \Sigma_{\tau,T}, \\
u_{m}(\tau) = u_{0m}, \ \nu_{m}'(\tau) = u_{1m}, \ \text{in} \ \mathcal{O}_{\tau}.
\end{aligned}
$$

Moreover, from (12), it follows

$$
\begin{aligned}
\{u_{m}\} \text{ is bounded in } L^{2}(\tau,T; H^{1}_{0}(\mathcal{O}_{1})), \\
\{u_{m}'\} \text{ is bounded in } L^{2}(\tau,T; L^{2}(\mathcal{O}_{1})).
\end{aligned}
$$
Therefore, taking into account Lemma 3.5 and 3.6 in [33], we can extract a subsequence (denoted also by \( \{u_m\} \)) such that

\[
\begin{align*}
&u_m \rightharpoonup u \text{ weakly in } L^2(\tau, T; H^1_0(Ω)), \\
u'_m \rightharpoonup u' \text{ weakly in } L^2(\tau, T; L^2(Ω)), \\
g(u_m) \rightharpoonup \chi \text{ weakly in } L^2(\tau, T; L^2(Ω)).
\end{align*}
\]

(75)

(76)

(77)

At the same time, we have

\[
(u_m - u_n)'' + (u_m - u_n)' - \Delta(u_m - u_n) + g(u_m) - g(u_n) = f_m - f_n.
\]

(78)

Multiplying (78) by \((u_m - u_n)'\), repeating the argument used in the proof of (35), we discover

\[
\|u'_m(t) - u'_n(t)\|^2_t + \|\nabla(u_m(t) - u_n(t))\|^2_t \\
\leq e^{C(t-\tau)} \left(\|u'_m(\tau) - u'_n(\tau)\|^2_\tau + \|\nabla(u_m(\tau) - u_n(\tau))\|^2_\tau\right) \\
+ C e^{C(t-\tau)} \int_\tau^t \|f_m(s) - f_n(s)\|^2_s ds,
\]

(79)

and therefore

\[
\{u_m\} \text{ is a Cauchy sequence in } C([\tau, T]); H^1_0(Ω)),
\]

(80)

\[
\{u'_m\} \text{ is a Cauchy sequence in } C([\tau, T]; L^2(Ω)).
\]

(81)

So by the uniqueness of the limit, (75) and (76), we find that

\[
u_m \to u \text{ in } C([\tau, T]; H^1_0(Ω)), \quad u'_m \to u' \text{ in } C([\tau, T]; L^2(Ω)).
\]

(82)

Therefore, we can assume that

\[
g(u_m) \to g(u), \text{ a.e. in } Q_{\tau, T},
\]

and then from (77), we can find \( \chi = g(u) \).

Finally, we will show that \( u \) is the unique weak solution of the equation (9).

For any test function \( \phi \in \mathcal{F}_{\tau, T} \), we know that \( u_m \) satisfies (71). Then, using (75)-(77), and (80)-(81), by passing to the limit, we obtain that \( u \) also satisfies (71). So \( u \) is a weak solution of (9) with initial data \( u_0, u_1 \).

The estimates (72) and (73) follow from (12), (13), (74) and (82) directly. The uniqueness follows from easily from (79). The proof is complete.

5. Process \( S(\cdot, \cdot) \) generated by the weak solutions.

Definition 5.1. A function \( u : \bigcup_{t \in [\tau, \infty]} \Omega_t \times \{t\} \to \mathbb{R} \) is called a weak solution of (4) if for any \( \tau \leq T \), the restriction of \( u \) on \( \bigcup_{t \in [\tau, T]} \Omega_t \times \{t\} \) is a weak solution of (9).

Recalling (20) and (60), we suppose

\[
\gamma^* = \inf_{-\infty < \tau \leq T < +\infty} \left\{ \theta_{\kappa_{\tau, T}}, \frac{1}{\kappa_{\tau, T}} \right\},
\]

(83)

where \( \kappa_{\tau, T} = \sup_{s \in [\tau, T]} \{2C_0^2|\Omega_s|^{\frac{1}{2}}\} \). By Theorem 4.2, we have:
Theorem 5.2. Under the assumption of Theorem 4.2 and (83), for any \( u_0 \in H_0^1(\mathcal{O}_t), u_1 \in L^2(\mathcal{O}_t) \) and \( f \in L^2_{loc}(\mathbb{R}; L^2(\mathcal{O}_t)) \), (4) has a unique weak solution. This weak solution satisfies (72) and (73) for all \( t \in [\tau, \infty) \).

Let \( f \in L^2_{loc}(\mathbb{R}; L^2(\mathcal{O}_t)) \), from Theorem 5.2 above, we can define the operators

\[
S(t, \tau) : X_\tau \rightarrow X_t, -\infty < \tau < t < +\infty,
\]

by

\[
S(t, \tau)(u_0, u_1) := u(t; \tau, (u_0, u_1)) = (u(t), u'(t)), \forall (u_0, u_1) \in X_\tau
\]

where \( X_\tau = H_0^1(\mathcal{O}_t) \times L^2(\mathcal{O}_t), t \in \mathbb{R} \) and \( u(\cdot; \tau, (u_0, u_1)) \) is the unique weak solution of (4). Then by the existence and uniqueness again, we know that the family operators \( \{S(t, \tau) : -\infty < \tau < t < +\infty\} \) forms a process, that is:

\[
S(\tau, \tau) = Id \forall \tau \in \mathbb{R},
\]

\[
S(t, s)S(s, \tau) = S(t, \tau) - \infty < \tau < t < +\infty.
\]

In the following, we will give some properties of the process \( \{S(t, \tau) : -\infty < \tau < t < +\infty\} \) defined above.

Lemma 5.3. Let \((u_0^i, u_1^i) \in X_\tau \) and \( (u^i(s), \partial_t u^i(s)) = S(s, \tau)(u_0^i, u_1^i), (i = 1, 2) \). Then, we have

\[
\| \partial_t u^1(t) - \partial_t u^2(t) \|^2_t + \| \nabla(u^1(t) - u^2(t)) \|^2_t
\]

\[
\leq e^{C(t-s)} \left( \| \partial_t u_1^1(s) - \partial_t u_2^1(s) \|^2_s + \| \nabla(u_1^1(s) - u_2^1(s)) \|^2_s \right).
\]

(84)

Proof. Let us fix \( \tau < t \). By definition we know that there are two sequences \( \{(u_{0m}^i, u_{1m}^i, f_{m}^i)\} \) satisfying

\[
u_{0m}^i \in H_0^1(\mathcal{O}_\tau) \cap H^2(\mathcal{O}_\tau), \quad u_{1m}^i \in L^2(\mathcal{O}_\tau), \quad f_{m}^i \in H_0^1(\mathcal{O}_\tau),
\]

such that

\[
u_{0m}^i \rightarrow u_0^i \quad \text{in} \quad H_0^1(\mathcal{O}_\tau) \cap L^p(\mathcal{O}_\tau), \quad (i = 1, 2),
\]

\[
u_{1m}^i \rightarrow u_1^i \quad \text{in} \quad L^2(\mathcal{O}_\tau), \quad (i = 1, 2),
\]

\[
f_{m}^i \rightarrow f^i \quad \text{in} \quad L^2(\tau, t; L^2(\mathcal{O}_\tau)),
\]

(85)

and

\[
u_{m}^i \rightarrow u^i \quad \text{in} \quad C([\tau, t]; H_0^1(\mathcal{O}_\tau)), \quad (i = 1, 2),
\]

\[
\partial_t u_{m}^i \rightarrow \partial_t u^i \quad \text{in} \quad C([\tau, t]; L^2(\mathcal{O}_\tau)), \quad \text{as} \quad m \rightarrow \infty,
\]

(86)

where \( u_m^i \) is the unique strong solution corresponding to the regular data \( \{(u_{0m}^i, u_{1m}^i, f_{m}^i)\} \).

Then, similarly to (35), we have

\[
\| \partial_t u_1^m(t) - \partial_t u_2^m(t) \|^2_t + \| \nabla(u_1^m(t) - u_2^m(t)) \|^2_t
\]

\[
\leq e^{C(t-s)} \left( \| \partial_t u_1^m(s) - \partial_t u_2^m(s) \|^2_s + \| \nabla(u_1^m(s) - u_2^m(s)) \|^2_s \right)
\]

\[
+ e^{C(t-s)} \int_s^t \| f_{m}^1(r) - f_{m}^2(r) \|^2 dr.
\]

(87)

Therefore, we get (84) immediately from (85), (86) and (87). \( \Box \)

As a direct consequence of (84), we have the following continuity result:
Lemma 5.4. For any $-\infty < \tau \leq t < +\infty$, the process $S(t, \tau) : X_t \to X_\tau$ is continuous.

6. $\mathcal{D}_\lambda$-pullback attractor. Throughout this section, we assume that
\[ \kappa_0 := \sup_{s \in \mathbb{R}} |O_s| < \infty. \] 
(88)

6.1. $\mathcal{D}_\lambda$-pullback absorbing set. Let $(u_0, u_1) \in X_\tau$, and $u(t) = S(t, \tau)(u_0, u_1)$. We denote
\[ \lambda_0 = \min\{1 - \vartheta, \vartheta(1 - \vartheta(1 - \vartheta\kappa_0)), \vartheta\}. \]
For sufficiently small $\vartheta > 0$, from (65) and (88), we observe that
\[ \lambda_0 \leq \lambda_{\tau, t}, \text{ for all } \tau \leq t. \] 
(89)

Combining with (72), we discover
\[
\|u'(t)\|^2_t + \|\nabla u(t)\|^2_t \\
\leq 8e^{-\lambda_0(t-\tau)} \left( \|u'(\tau)\|^2_t + \|\nabla u(\tau)\|^2_t + \|\nabla u(\tau)\|_{\mathbb{X}}^{p+1} \right) \\
+ 4e^{-\lambda_0 t} \left( 4 \int_\tau^t e^{\lambda_0 \xi} \|f(\xi)\|^2 d\xi + \int_\tau^t e^{\lambda_0 \xi} |O_\xi| d\xi \right) + 8\eta|O_t|. 
\]
(90)

Let $R_\lambda$ be the set of all the functions $\rho : \mathbb{R} \to [0, \infty)$ such that
\[ e^{\lambda_0 \tau} \rho^{p+1}(\tau) \to 0, \text{ as } \tau \to -\infty, \]
and $\mathcal{D}_\lambda$ the class of all the families $\hat{D} := \{D(t) : t \in \mathbb{R}, D(t) \subset X_t\}$, such that $D(t) \subset \{u \in X_t : \|u\|_{X_t} \leq \rho_D(t)\}$ for some $\rho_D \in R_\lambda$.

Suppose that $f \in L^2_{\text{loc}}(\mathbb{R} ; L^2(O))$ satisfies
\[ \int_{-\infty}^t e^{\lambda_0 \xi} \|f(\xi)\|^2 d\xi < \infty, \] 
(91)
where $\lambda_0^*$ is a positive constant satisfying
\[ \lambda_0^* < \frac{2}{p+1} \lambda_0. \] 
(92)

For each $t \in \mathbb{R}$, we set $R(t)$ the positive number given by
\[
R^2(t) = 4e^{-\lambda_0^* t} \left( 4 \int_{-\infty}^t e^{\lambda_0^* \xi} \|f(\xi)\|^2 d\xi + \int_{-\infty}^t e^{\lambda_0^* \xi} |O_\xi| d\xi \right) + 8\eta|O_t| + 1. 
\]
(93)

Lemma 6.1. The family of sets
\[
\hat{B}_0 := \{u \in X_t : \|u\|_{X_t} \leq R(t)\} : t \in \mathbb{R}
\]
is a pullback $\mathcal{D}_\lambda$-absorbing family for the solution process $S(t, \tau)$. Moreover, $\hat{B}_0$ belongs to $\mathcal{D}_\lambda$. 

(94)
We can of course combine (88), (92) and (98) to discover that
\[ D_{\text{6.2}}. \]
Applying (91) and (92) in (90), we arrive at
\[ \text{Proof.} \]
\[ \text{The process} \]
\[ \text{Lemma 6.2.} \]
\[ \varepsilon \]
\[ f \]
\[ X \]
\[ \tau \]
\[ \leq \]
\[ \pi \]
\[ R \]
\[ t \]
\[ \lambda \]
\[ \text{Let} \]
\[ \text{It remains to prove that the pullback absorbing} \]
\[ \text{We must show that} \]
\[ \lim_{t \to -\infty} e^{\lambda_{0}t} |R(t)|^{p+1} = 0. \]
\[ \text{Consequently, (93) yields the inequality} \]
\[ e^{\frac{2\lambda_{0}}{p+1} t} R^{2}(t) \]
\[ = e^{\left(\frac{2\lambda_{0}}{p+1} - \lambda_{0}\right) t} \left( 16 \int_{-\infty}^{t} \left| f(\xi) \right|^{2} d\xi + 4 \int_{-\infty}^{t} \left| \xi \right| |O_{\xi}| d\xi \right) + 8e^{\frac{2\lambda_{0}}{p+1} t} \eta |O_{t}| + e^{\frac{2\lambda_{0}}{p+1} t}. \]
\[ \text{We can of course combine (88), (92) and (98) to discover that} \]
\[ \lim_{t \to -\infty} e^{\frac{2\lambda_{0}}{p+1} t} R^{2}(t) = 0. \]
\[ \text{Therefore (97) holds and} \]
\[ \text{The proof is finished.} \]
\[ 6.2. \]
\[ \mathcal{D}_{\lambda}-\text{pullback asymptotic compactness.} \]
\[ \text{Lemma 6.2.} \]
\[ \text{The process} \]
\[ \text{Proof.} \]
\[ \text{Now fix} \]
\[ \text{Let} \]
\[ \psi_{t,\tau}(u_{0}^{1}, u_{1}^{1}, u_{0}^{2}, u_{1}^{2}) = -2e^{-\lambda_{0}t} \int_{\tau}^{t} e^{\lambda_{0}\xi} \left( g(\pi_{1}(\xi)) - g(\pi_{2}(\xi)), \pi_{1}(\xi) - \pi_{2}(\xi) \right)_{\xi} d\xi \]
\[ - 2\vartheta_{0} e^{-\lambda_{0}t} \int_{\tau}^{t} e^{\lambda_{0}\xi} \left( g(\pi_{1}(\xi)) - g(\pi_{2}(\xi)), \pi_{1}(\xi) - \pi_{2}(\xi) \right)_{\xi} d\xi, \]
where \( \vartheta_0 = \min\{1/2, 1/\kappa_0\} \). Combining (73), (89) and (99), we conclude that
\[
\|w'(t)\|_1^2 + \|\nabla w(t)\|_1^2 \\
\leq 2e^{-\lambda_0(t-\tau)} (\|w'(\tau)\|_1^2 + \|\nabla w(\tau)\|_1^2) + \psi_{t,\tau},
\]
where \( w = \bar{\pi}_1 - \bar{\pi}_2 \). Since \( \lim_{t\to-\infty} e^{\lambda_0t}|R(t)|^{p+1} \to 0 \), for any \( \varepsilon > 0 \), there exists an index \( \tau_k = \tau_k(\varepsilon, B_0, t) \) such that
\[
\|w'(t)\|_1^2 + \|\nabla w(t)\|_1^2 \\
\leq \varepsilon + \psi_{t,\tau_k}(u_0^i, u_1^i; \tau_k) \text{ for all } (u_0^i, u_1^i) \in B_0(\tau_k), i = 1, 2.
\] (100)

Denote
\[
 u_k(s) := S(s, \tau_k)(u_{\tau_k}, u'_{\tau_k}), \quad (u_{\tau_k}, u'_{\tau_k}) \in B_0(\tau_k), \quad s \in [\tau_k, t].
\]

Since \( B_0(\tau_k) \) is a bounded subset in \( X_{\tau_k} \), by (90), we obtain that
\[
\{u_k\} \text{ is bounded in } L^\infty(\tau_k, t; H_0^1(O_s)), \\
\{u'_k\} \text{ is bounded in } L^\infty(\tau_k, t; L^2(O_s)), \\
\{u_k(s)\} \text{ is bounded in } H_0^1(O_s), \forall s \in [\tau_k, t].
\]

Also, recalling (5), we see that
\[
g(u_k) \text{ is bounded in } L^2(\tau_k, t; L^2(O_s)).
\]

Applying Sobolev compact embedding and Lemma 2.1 [37], there is an index \( j = j(\tau_k, \varepsilon) \in J_{\varepsilon} \), where \( J_{\varepsilon} \subset \mathbb{N} \) is a finite index, such that
\[
\int_{\tau_k}^t (g(u_k(\xi)) - g(u^i(\xi)), \psi(\xi))_\xi d\xi \leq \varepsilon, \quad \forall \psi \in L(\tau_k, t; L^2(O_s));
\] (101)
\[
\int_{\tau_k}^t (g(u_k(\xi)) - g(u^i(\xi)), \psi(\xi))_\xi d\xi \leq \varepsilon, \quad \forall \psi \in L^2(\tau_k, t; L^2(O_s));
\] (102)
\[
\|u_k - u^i\|_{L^2(\tau_k, t; L^2(O_s))} \leq \varepsilon;
\] (103)
\[
\|u_k(\tau_k) - u^i(\tau_k)\|_{L^{p+1}(O_{\tau_k})} \leq \varepsilon;
\] (104)
\[
\|u_k(t) - u^i(t)\|_{L^{p+1}(O_t)} \leq \varepsilon.
\] (105)

Firstly, by the Hölder inequality and the growth condition (5), we have
\[
\int_{\tau_k}^t e^{-\lambda_0(t-\xi)} (g(u_k(\xi)) - g(u^i(\xi)), u_k(\xi) - u^i(\xi))_\xi d\xi \\
\leq \int_{\tau_k}^t \left| (g(u_k(\xi)) - g(u^i(\xi)), u_k(\xi) - u^i(\xi))_\xi \right| d\xi \\
\leq \|g(u_k) - g(u^i)\|_{L^2(\tau_k, t; L^2(O_s))} \times \|u_k - u^i\|_{L^2(\tau_k, t; L^2(O_s))} \\
\leq C \left( \int_{\tau_k}^t (1 + |u_k|^{2p} + |u^i|^{2p}) dx d\xi \right)^{\frac{1}{2}} \times \|u_k - u^i\|_{L^2(\tau_k, t; L^2(O_s))},
\]
by (90) and (103), using continuous embedding $H_0^1(\mathcal{O}_t) \hookrightarrow L^{2p}(\mathcal{O}_t)$, we can easily get that
\begin{align*}
\left| \int_{\tau_k}^t e^{-\lambda_0(t-\xi)} \left( g(u_k(\xi)) - g(u^j(\xi)), u_k(\xi) - u^j(\xi) \right) d\xi \right| \leq \varepsilon. \quad (106)
\end{align*}

Secondly, note that
\begin{align*}
\int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_\xi} \left( g(u_k(\xi)) - g(u^j(\xi)) \right) \left( u'_k(\xi) - u'^j(\xi) \right) dx d\xi \\
= \int_{\tau_k}^t e^{\lambda_0 t} G(u_k(t)) dx - \int_{\tau_k}^t e^{\lambda_0 \tau_k} G(u_k(\tau_k)) dx \\
+ \int_{\tau_k}^t e^{\lambda_0 t} G(u^j(t)) dx - \int_{\tau_k}^t e^{\lambda_0 \tau_k} G(u^j(\tau_k)) dx \\
- \int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} g(u_k(\xi)) u'^j(\xi) dx d\xi - \int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} g(u^j(\xi)) u'_k(\xi) dx d\xi \\
- \lambda_0 \int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} G(u_k(\xi)) dx d\xi - \lambda_0 \int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} G(u^j(\xi)) dx d\xi. \quad (107)
\end{align*}

By (101) and (102), we obtain
\begin{align*}
\left| \int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} g(u_k(\xi)) u'^j(\xi) dx d\xi - \int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} g(u_k(\xi)) u'_k(\xi) dx d\xi \right| \leq \varepsilon \quad (108)
\end{align*}
and
\begin{align*}
\left| \int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} g(u^j(\xi)) u'_k(\xi) dx d\xi - \int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} g(u^j(\xi)) u'_k(\xi) dx d\xi \right| \leq \varepsilon. \quad (109)
\end{align*}
Combining (107), (108) and (109), we deduce that
\begin{align*}
\int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} \left( g(u_k(\xi)) - g(u^j(\xi)) \right) \left( u'_k(\xi) - u'^j(\xi) \right) dx d\xi \\
\leq 2\varepsilon + e^{\lambda_0 t} \int_{\mathcal{O}_u} |G(u_k(t)) - G(u^j(t))| dx + e^{\lambda_0 \tau_k} \int_{\mathcal{O}_u} |G(u_k(\tau_k)) - G(u^j(\tau_k))| dx \\
+ \lambda_0 \int_{\tau_k}^t e^{\lambda_0 \xi} \int_{\mathcal{O}_u} |G(u_k(\xi)) - G(u^j(\xi))| dx d\xi. \quad (110)
\end{align*}

By (5), we have
\begin{align*}
|G(u) - G(v)| \leq C(1 + |u|^p + |v|^p)|u - v|, \quad \forall u, v \in \mathbb{R}. \quad (111)
\end{align*}
Applying Hölder inequality, we find that
\[ e^{\lambda_0 t} \int_{\Omega} |G(u_k(t)) - G(u^j(t))| \, dx \]
\[ \leq C e^{\lambda_0 t} \left( \int_{\Omega} \left( 1 + |u_k(t)|^{2p} + |u^j(t)|^{2p} \right) \, dx \right)^{\frac{1}{p}} \times \left( \int_{\Omega} |u_k(t) - u^j(t)|^2 \, dx \right)^{\frac{1}{2}}. \]

Then, by (90), (105) and embedding theorem, we can easily have
\[ e^{\lambda_0 t} \int_{\Omega} |G(u_k(t)) - G(u^j(t))| \, dx \leq \varepsilon. \tag{112} \]

Similarly, we can get
\[ e^{\lambda_0 \tau_k} \int_{\Omega} |G(u_k(\tau_k)) - G(u^j(\tau_k))| \, dx \leq \varepsilon. \tag{113} \]

Now, by (90), (111), (103) and embedding theorem, we have
\[ \lambda_0 \int_{\tau_k}^{t} e^{\lambda_0 \xi} \int_{\Omega} |G(u_k(\xi)) - G(u^j(\xi))| \, dx \, d\xi \]
\[ \leq C \lambda_0 \int_{\tau_k}^{t} e^{\lambda_0 \xi} \left( 1 + |u_k(\xi)|^3 + |u^j(\xi)|^3 \right) |u_k(\xi) - u^j(\xi)| \, dx \, d\xi \]
\[ \leq C \tau_k t \left( \int_{\tau_k}^{t} \|u(\xi) - u_n(\xi)\|_2^2 \, d\xi \right)^{1/2} \leq \varepsilon. \tag{114} \]

Combining (110), (112), (113) and (114), we arrive at
\[ \left| \int_{\tau_k}^{t} e^{\lambda_0 \xi} \left( g(u_k(\xi)) - g(u^j(\xi)) \right) \left( u'_k(\xi) - u'^j(\xi) \right) \, dx \, d\xi \right| \leq 5 \varepsilon. \tag{115} \]

Finally, applying (100), (106) and (115), we conclude that the set \( \{ S(t, \tau_k)(u_{\tau_k}, u'_{\tau_k}) \}_{k \in \mathbb{N}} \), \( (u_{\tau_k}, u'_{\tau_k}) \in B_0(\tau_k) \) is totally bounded. So, the proof is complete.

6.3. \( \mathcal{D}_\lambda \)-pullback attractor. From Lemma 6.2, and the fact that the sets in \( \hat{B}_0 \) are closed, and the family \( \mathcal{D}_\lambda \) is inclusion closed, we obtain that \( S(t, \tau) \) has a \( \mathcal{D}_\lambda \)-pullback attractor, and more exactly:

**Theorem 6.3.** Under the assumption of Theorem 4.2, furthermore, assume (88) and \( f \) satisfies (91). Then the family \( \mathcal{A} = \{ \mathcal{A}(t) : t \in \mathbb{R} \} \) defined by
\[ \mathcal{A}(t) = \Lambda(\hat{B}, t), \quad t \in \mathbb{R}, \]
where \( \hat{B} \) is given by (94), and for any \( \hat{D} \in \mathcal{D}_\lambda \),
\[ \Lambda(\hat{D}, t) := \bigcap_{s \leq t} \left( \bigcup_{\tau \leq s} S(t, \tau) D(\tau) \right)^{X_\lambda}, \quad t \in \mathbb{R} \text{ (closure in } X_\lambda), \]

Finally, applying (100), (106) and (115), we conclude that the set \( \{ S(t, \tau_k)(u_{\tau_k}, u'_{\tau_k}) \}_{k \in \mathbb{N}} \), \( (u_{\tau_k}, u'_{\tau_k}) \in B_0(\tau_k) \) is totally bounded. So, the proof is complete. \( \square \)
is the unique $\mathcal{D}_\lambda$-pullback attractor for the process $S(t, \tau)$ belonging to $\mathcal{D}_\lambda$. In addition, $\mathcal{A}$ satisfies
\[
\mathcal{A}(t) = \bigcup_{\hat{D} \in \hat{D}_\lambda} \Lambda(\hat{D}, t) X_t \quad \forall t \in \mathbb{R}.
\]
Furthermore, $\mathcal{A}$ is minimal in the sense that if $\hat{C} = \{C(t) : t \in \mathbb{R}\}$ is a family of nonempty sets such that $C(t)$ is a closed subset of $X_t$ and
\[
\lim_{\tau \to -\infty} \text{dist}_t (S(t, \tau)D_0(\tau), C(t)) = 0
\]
for all $t \in \mathbb{R}$, then $\mathcal{A}(t) \subset C(t)$ for any $t \in \mathbb{R}$.

**Remark 5.** The above process and corresponding results can be applied to a variety of other dissipative equations defined on non-cylindrical domains, such as complex Ginzburg-Landau equation and Schrödinger equation, which are with a widely background in Physics (see [34] for further details, and additional references). Furthermore, the stochastic wave equations with multiplicative white noise (or additive white noise) on non-cylindrical domains, could also be treated by the presented scheme. All these problems will be reported in our subsequent papers.

**Remark 6.** As mentioned in our Introduction, the nonlinear wave equations on non-cylindrical domains with expanding spatial domains are difficult to deal with. Here, our results give some useful information about this issue, e.g., if the domain satisfying some regularity condition, the existence of the strong solution is objective. However, to obtain the strong solution or uniqueness of weak solutions, a new method should be developed, and a possible approach may be the one taken by Cannarsa et al. in [11]. If not, we could consider the asymptotic behavior of the solutions under the other frameworks, such as generalized semiflows [7] or evolution systems [17].

**Remark 7.** With the discussion in the present paper, combing the situation and problems in [4] (see also [2, 5, 6]), we can assume that the perturbation of the domain is time-dependent. Hence, a natural question is: is it possible to achieve the similar results, such as spectral convergence and continuity of attractors, which are constructed in [2, 4, 5, 6]? Recently, Pereira & Silva [39] considered the reaction-diffusion equations in a time-dependent thin domain, which is a good try for further research in this field. However, when referred to wave equations, to answer these questions, new methods and theories may be needed.

**Appendix A.**

**A.1. A sufficient condition.** Here we construct a sufficient condition to insure that $r$ is hyperbolic.

**Theorem A.1.** For any $-\infty < \tau \leq T < \infty$, $r$ is hyperbolic if
\[
\lambda_k(y, t) - \lambda(y, t) \geq \theta_0, \quad \forall (y, t) \in \bar{O} \times [\tau, T],
\]
where $\theta_0$ is a positive constant, $\lambda_k(y, t) \ (k = 1, 2, 3)$ is eigenvalue of real positive definite matrix $B = (b_{ij})_{3 \times 3}$, $b_{ij} = \sum_{k=1}^{3} \frac{\partial r_k}{\partial x_k} (r(y, t), t) \frac{\partial r_i}{\partial x_i} (r(y, t), t)$, $\lambda(y, t) = \sum_{i=1}^{3} (\frac{\partial r_i}{\partial t} (y, t))^2$. 

Lemma 4.2 in [33]) and Remark 9. It is easy to see that Theorem A.1 can also be applied to the case that $d$ where $t \in O \times [\tau, T]$, we arrive at
\[
\sum_{i,j=1}^{3} a_{ij} \xi_i \xi_j = \xi^T A(y_0, t_0) \xi = \xi^T B(y_0, t_0) \xi - \xi^T C(y_0, t_0) \xi
\]
\[
= \sum_{k=1}^{3} (\lambda_k(y_0, t_0) - \lambda(y_0, t_0)) |\xi_k|^2 \geq \theta_0 |\xi|^2,
\]
recalling Definition 3.2, we discover that $r$ is hyperbolic. \hfill \square

**Remark 8.** We can easily obtain that the result is also reasonable if the space dimension $n > 3$.

**A.2. An application.** To better understand the larger scope of our results, we illustrate their application to the following case.

**Example A.2.** Let $r(y, t) = h(t)y$ for any $t \in \mathbb{R}$ and $y = (y_1, y_2, y_3) \in O \subset \mathbb{R}^3$. Suppose $h(t) \in C^2(\mathbb{R})$ and satisfies
\[
h(t) \neq 0 \quad \forall t \in \mathbb{R} \quad \text{and} \quad \sup_{t \in \mathbb{R}} |h(t)| = h_M < +\infty.
\]
For $u(x, t) = v(\bar{r}(x, t), t) = v(y, t)$ and $\bar{r}(x, t) = \frac{1}{h(t)} x$ we have
\[
B(y, t) = \begin{pmatrix}
\frac{1}{h^2(t)} & 0 & 0 \\
0 & \frac{1}{h^2(t)} & 0 \\
0 & 0 & \frac{1}{h^2(t)}
\end{pmatrix}
\]
and
\[
C(y, t) = \left( \frac{h'(t)}{h(t)} \right)^2 \times \begin{pmatrix}
y_1 y_1 & y_1 y_2 & y_1 y_3 \\
y_2 y_1 & y_2 y_2 & y_2 y_3 \\
y_3 y_1 & y_3 y_2 & y_3 y_3
\end{pmatrix},
\]
so $\lambda_k(y, t) = \frac{1}{h^2(t)}$ and $\lambda(y, t) = \frac{(h'(t))^2}{h^2(t)} |y|^2$. Applying Theorem A.1, we get if for all $t \in \mathbb{R}$,
\[
|h'(t)| < \frac{1}{\max_{y \in O} |y|},
\]
then $r$ is hyperbolic. Especially, if $O$ containing the origin, we can suppose
\[
|h'(t)| < \frac{1}{d} \quad \text{for all} \ t \in \mathbb{R},
\]
where $d = \text{diam}(O)$.

**Remark 9.** It is easy to see that Theorem A.1 can also be applied to the case that $r(y, t) = (h_1(t)y_1, h_2(t)y_2, h_3(t)y_3)$, $t \in \mathbb{R}$. E.g., see Section 2 in [37] for more details.

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