Skyrmions and effective model of the resistive electron quadrupling state

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Recent experiments [Grinenko et al. Nature Physics 17, 1254–1259 (2021)] reported the observation of a condensate of four-fermion composites. This is a resistive state that spontaneously breaks the time-reversal symmetry, leading to unconventional magnetic properties, detected in muon spin rotation experiments and by the appearance of a spontaneous Nernst effect. In this work, we derive an effective model that describes the observed spontaneous magnetic fields in this state. We show that this model, which is alike to the Faddeev-Skyrme model for the four-fermion order parameter, hosts skyrmions: magnetic-flux-carrying topological excitations.

Recent experiments [1] reported the observation of a fermion quadrupling state in the multiband material: hole-doped Ba$_{1-x}$K$_x$Fe$_2$As$_2$. This resistive state, coined quartic bosonic metal, is a condensate with an anti-correlated flow of pairs of Cooper pairs belonging to different bands. In contrast to superconductors, which break the U(1) gauge symmetry, this state spontaneously breaks the two-fold (Z$_2$) time-reversal symmetry. This raises the question of the properties of such states.

An effective model can describe the properties of condensates at large length scales. For a pair condensate, the effective model is the celebrated Ginzburg-Landau theory which has been extensively studied since the second half of the last century. The question of effective models describing the fermion quadruplet quartic metal is more subtle. In this paper, we derive an effective long-wave-length model for the resistive quartic state reported in Ba$_{1-x}$K$_x$Fe$_2$As$_2$. Based on this, we report the key properties of that state: Namely its magnetic properties and the nature of the topological excitations it supports.

At low temperatures, the compound is a superconductor characterized by Cooper pair condensates $\Delta_a$, forming in the different bands labelled by $a$. Importantly this superconductor breaks the time-reversal symmetry [2, 3], so that the total symmetry broken by the low-temperature state is U(1) × Z$_2$. The analysis of the magnitude and polarization of spontaneous magnetic fields [3–5], indicates a spin-singlet superconducting state that breaks the time-reversal symmetry. It is the so-called $s+is$ state which has two energetically equivalent locking of the relative phase $\theta_a - \theta_b$ between the superconducting gaps in different components $\Delta_a, \Delta_b$.

The mechanism responsible for the appearance of the quartic metal is the following: The standard assumption of the Bardeen-Cooper-Schrieffer theory is a mean-field approximation for the fields quadratic in fermions: This assumption eliminates, by construction, the possibility for fermion quadrupling. The resulting theory yields a phase diagram of such a superconductor, which is typically a dome of the $s+is$ state between two different superconducting states [6–10]. It was pointed out in [11, 12], that relaxing the mean-field approximation in a multicomponent fermion pairing theory results in a phase diagram with the appearance of fermion quadrupling condensates. The large-scale Monte-Carlo calculations of U(1) × Z$_2$ states demonstrated that, the discrete Z$_2$ transition can exceed the superconducting U(1) transition: $T_c < T_{cZ2}^{12}$ [13–15].

The existence of the spontaneously broken time-reversal symmetry in the resistive state of Ba$_{1-x}$K$_x$Fe$_2$As$_2$, at the doping level $x \approx 0.8$ [1], dictates that the averages of the pairing order parameters $\Delta_a$ are zero, but that there exists a non-zero order parameter which is fourth-order in the fermionic fields. The quadrupling order parameter is proportional to the product of pairing order parameters in different bands $\Delta_a^2 \Delta_b$. Such an order parameter implies an anti-correlation in the flows of the components $a$ and $b$. Crucially, although this type of counter-flows do not represent superconductivity, they are generally coupled to the magnetic field when the densities of the counter-flowing charged components are unequal. An effective model should account for this coupling, and should be different from the Ginzburg-Landau model of a Meissner state.

Below we derive such an effective theory, based on the mean-field approximation for the four-fermion order parameter. We demonstrate that, in an inhomogeneous sample, the model supports spontaneous magnetic fields, consistently with the experimental results [1]. It also predicts the existence of topological excitations carrying a quantized magnetic flux, in the form of skyrmions.

We derive our effective model for a state with composite order, from a generic model of a superconductor with a two-component order parameter $\Psi$, with $\Psi \equiv (\psi_1^*, \psi_2^*)$. The detailed derivation from the microscopic theory can be found in Supplementary Material [16]. The generic Ginzburg-Landau free energy density for a two-component superconductor reads as

$$F(\Psi, A) = \frac{B^2}{2} + \frac{k_{abij}}{2} (D_i \psi_a^*) D_j \psi_b + V(\Psi^\dagger, \Psi), \quad (1)$$

where $V(\Psi^\dagger, \Psi)$ is the potential energy term. The repeated indices are implicitly summed over, and the indices $i, j$ denote the spatial coordinates while $a, b$ label...
the different components. The individual condensates are coupled to the vector potential \( \mathbf{A} \), of the magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} \), via the gauge derivative \( \mathbf{D} = \nabla + ie \mathbf{A} \) in the kinetic term. In this work, we focus on two-component models that break multiple symmetries. The symmetry breaking is encoded in the potential term \( V(\Psi^\dagger, \Psi) \) which explicitly reduces the global SU(2) symmetry of a doublet of complex order parameters down to a smaller symmetry group. For example, the SU(2) symmetry is broken down to \( U(1) \times \mathbb{Z}_2 \), for a superconductor that breaks time-reversal symmetry such as \( s+is, s+id, d+ig, p+ip \), or down to \( U(1) \times \mathbb{Z}_4 \) symmetry for some nematic superconductors [17]. The composite order of interest arises if the fluctuations-driven restoration of the local gauge symmetry occurs without restoring the other broken symmetries. The existence of a composite order was demonstrated in \( U(1) \times U(1) \) [11, 12], and SU(2) systems [18, 19], from these calculations it follows that composite order also exists for \( U(1) \times \mathbb{Z}_n \) symmetries. While most of our results qualitatively apply to all of the above mentioned pairing mechanisms, we focus below on the case of the broken time-reversal symmetry \( U(1) \times \mathbb{Z}_2 \), and in particular on the \( s+is \) state, motivated by the experiment on Ba\(_{1-x}\)K\(_x\)Fe\(_2\)As\(_2\) [1].

At the microscopic level, the minimal model features three distinct superconducting gaps \( \Delta_{1,2,3} \) in three different bands, and the pairing that leads to the time-reversal symmetry breaking states is dominated by the competition between different interband repulsion channels [6, 8, 10]. In the case of an interband-dominated repulsive pairing, only two fields \( \psi_{1,2} \) appear in the effective Ginzburg-Landau model for the superconducting state, see e.g. [8, 20, 21]. When starting from the microscopic three-band model, the relevant two-component Ginzburg-Landau theory features mixed-gradient terms, which can be eliminated by a linear transformation to new fields see e.g. [21, 22], and the Supplementary Material [16]. The resulting Ginzburg-Landau theory is characterized by the free-energy \( \mathcal{F}/\mathcal{F}_0 = \int \mathcal{F} \) whose density reads as:

\[
\mathcal{F}(\Psi, \mathbf{A}) = \frac{B^2}{2} + \frac{1}{2} |D\Psi|^2 + V(\Psi^\dagger, \Psi). \tag{2}
\]

To account for the four-fermion state, the Ginzburg-Landau theory (2) is first mapped onto a model that couples the supercurrent \( \mathbf{J} = \epsilon \text{Im}(\Psi^\dagger D\Psi) \) to a real 3-vector \( \mathbf{m} \). It is defined as the projection of the superconducting degrees of freedom \( \Psi \) onto spin-1/2 Pauli matrices \( \sigma \): \( \mathbf{m} = \Psi^\dagger \sigma \Psi \), hence this is an order parameter which is fourth-order in the fermionic fields. This order parameter depends on the relative phase between the original complex fields, and does not depend on the superconducting degree of freedom: The phase sum. The norm of \( \mathbf{m} \) is related to the total density squared \( \|\mathbf{m}\|^2 = \rho^2 = \Psi^\dagger \Psi \).

In terms of \( \mathbf{J} \) and \( \mathbf{m} \), the free energy reads as [16]:

\[
\mathcal{F} = \frac{1}{2} \left[ \epsilon_{kij} \left( \nabla_i \left( \frac{J_j}{\epsilon^2 g^2} \right) - \frac{1}{4 \epsilon^2 g^2} \mathbf{m} \cdot \partial_j \mathbf{m} \times \partial_j \mathbf{m} \right) \right]^2 + \frac{J^2}{2 \epsilon^2 g^2} + \frac{1}{8 \epsilon^2 g^2} (\nabla \mathbf{m})^2 + V(\mathbf{m}), \tag{3}
\]

where \( \epsilon \) is the rank-3 Levi-Civita symbol. The term in the square brackets in (3) is the magnetic field expressed through gradients of the matter fields. The first term there is the contribution of the Meissner current \( \mathbf{J} \) to the magnetic field, while the second term accounts for the interband counterflow [23, 24]:

\[
\mathbf{B} = \nabla \times \left( \frac{\mathbf{J}}{\epsilon^2 g^2} \right) - \frac{\epsilon_{abc}}{4 \epsilon^2 g^2} m_a \nabla m_b \times \nabla m_c. \tag{4}
\]

The second term is particularly important: It is related to the counterflow of two components, since it has a form of gradients of the composite field \( \psi^* \psi \), i.e. it depends on gradients of the relative phase between components. A counterflow of two identical charged components results in no charge transfer and hence does not couple to the magnetic field. However, if the densities of the components are locally imbalanced, the charge transport occurs. Thus the coupling to the magnetic field involves a dependence of the relative density gradients.

Next, the low-temperature model (3), which microscopic derivation is given in the Supplementary Material [16], is used to obtain an effective model of the fermion quadrupling phase. The fermion quadrupling phase identified in [1, 13, 14] is resistive. This is caused by the disorder of the superconducting phase due to the proliferation of topological defects. The effective model of the resulting fermion quadrupling state is obtained by removing the superconducting degrees of freedom from (3). Indeed, as demonstrated in Monte-Carlo calculations, their prefactors are renormalized to zero [1, 13, 14, 18, 19, 25–28]. It follows that the Meissner current vanishes (\( \mathbf{J} = 0 \)), while the currents associated with gradients of the fermion quadrupling order parameter \( \mathbf{m} \) do not. Assuming that the critical temperatures of the \( \mathbb{Z}_2 \) and \( U(1) \) transitions are well separated, the free energy of the fermion quadrupling state can be written as

\[
\mathcal{F}(\mathbf{m}) = \frac{(\mathbf{m} \cdot \partial_i \mathbf{m} \times \partial_j \mathbf{m})^2}{16 \epsilon^2 \|\mathbf{m}\|^6} + \frac{(\nabla \mathbf{m})^2}{8 \|\mathbf{m}\|^2} + V(\mathbf{m}), \tag{5a}
\]

where \( V(\mathbf{m}) = \sum_{a=0,x,g,z} \alpha_a m_a + \frac{1}{2} \sum_{a,b=0,x,g,z} \beta_{ab} m_a m_b \). \tag{5b}

Here the component \( m_0 \) stands for the magnitude of \( \mathbf{m} \), \( m_0 := \|\mathbf{m}\| \) (see details of the microscopic expressions for the coefficients in [16]). The first term in (5) has to be retained because it depends only on the relative phases and densities of the original of superconducting fields. Hence it cannot vanish at superconducting phase
transition, when $T_c < T_c^\mathbb{Z}_2$ [29]. The fermion quadrupling phase reported in [1] breaks the time-reversal symmetry. Hence the potential term (5b) breaks the symmetry associated with the vector $\mathbf{m}$ down to $\mathbb{Z}_2$. In the original Ginzburg-Landau model (2), the time-reversal operation is the complex conjugation of the superconducting condensates $\Psi$. Correspondingly, for the soft modulus vector it is a reflection of $\mathbf{m}$ on the $x$-$z$-plane of the target space:

$$\mathcal{T}(\Psi) = \Psi^* \Leftrightarrow \mathcal{T}(\mathbf{m}) = (m_x, -m_y, m_z). \quad (6)$$

This means that the states that break the time-reversal symmetry must have $m_y \neq 0$. This is for example enforced by $\beta_{xz}^m > 0$, since it penalizes $m_z^2$. The other details of the analysis of the potential can be found in the Supplementary Material [16]. The essential features can be qualitatively summarized as follows: First, all the coefficients involving a $y$ index vanish: $\alpha_y^m = \beta_{y}^m = \beta_{y}^m = 0$. Moreover, the criterion for the condensation is $\alpha_x^m < \alpha_{x}^m + \alpha_{y}^m$, and also $\beta_{x}^m < \beta_{x}^m > 0$.

The quadrupling phase appears when the mean-field approximation for the pairing fields is relaxed. The model (5) can be viewed as a mean-field approximation for the pairing fields is relaxed. The integrand is obviously ill-defined when $\|\mathbf{m}\| = 0$. However, whenever $|m_x| \neq 0$, the corresponding configuration have an integer topological charge $Q(\mathbf{m}) \in \mathbb{Z}$; this suggest that the model can host skyrmion topological excitations. Note that in three dimensions, the model is characterized by another invariant, the Hopf invariant, which is associated with the maps $S^2 \rightarrow S^2_{\mathbf{m}}$. This suggest the existence of hopfions, but it is beyond the scope of the current discussion.

The model describing the resistive fermion quadrupling state is alike to the Faddeev-Skyrme model. This suggests that it could host nontrivial topological excitation such as skyrmions and hopfions. To investigate the properties of the topological defects of the effective model, the physical degrees of freedom $\mathbf{m}$ are discretized within a finite-element formulation [30], and the free energy (5) is minimized using a nonlinear conjugate gradient algorithm. For details of the numerical procedure, see [16].

The experiments [1] reported spontaneous magnetic fields in the quartic metal state. In the $s+is$ superconducting state, spontaneous magnetic fields can arise due to inhomogeneities such as to thermal gradients [1, 31], hotspot created by a laser pulse [20], the effect of impurities [32, 33], and other inhomogeneous arrays [4, 22]. The effects of inhomogeneities in the quartic state can be modelled by small spatial variations of the coupling constants $\alpha_0^m$ and $\alpha_{s}^m$, reflecting the naturally present weak gradients in doping level. The top panel is the magnetic field $\mathbf{B}$ (7), and the bottom panel displays the associated counter-currents $j_{\text{counter}}$ according to the Ampère’s law. The surface elevation, together with the coloring, represents the magnitude of the $B_z$. The coupling here is $\epsilon = 0.6$, and the other parameters are given in Supplementary Material [16].

$$Q(\mathbf{m}) = \frac{1}{4\pi} \int_{S^2} \frac{\mathbf{m} \cdot \nabla \mathbf{m} \times \nabla \mathbf{m}}{\|\mathbf{m}\|^3} \, dx dy. \quad (8)$$

The integrand is obviously ill-defined when $\|\mathbf{m}\| = 0$. However, whenever $|m_x| \neq 0$, the corresponding configuration have an integer topological charge $Q(\mathbf{m}) \in \mathbb{Z}$; this suggest that the model can host skyrmion topological excitations. Note that in three dimensions, the model is characterized by another invariant, the Hopf invariant, which is associated with the maps $S^2 \rightarrow S^2_{\mathbf{m}}$. This suggest the existence of hopfions, but it is beyond the scope of the current discussion.

Figure 1. Spontaneous magnetic field in the quartic phase generated by inhomogeneities. The inhomogeneities are modelled by random spatial modulation of the parameters $\alpha_0^m$ and $\alpha_{s}^m$, reflecting the naturally present weak gradients in doping level. The top panel is the magnetic field $\mathbf{B}$ (7), and the bottom panel displays the associated counter-currents $j_{\text{counter}}$ according to the Ampère’s law. The surface elevation, together with the coloring, represents the magnitude of the $B_z$. The coupling here is $\epsilon = 0.6$, and the other parameters are given in Supplementary Material [16].

First note that because the time-reversal symmetry ($\mathbb{Z}_2$) is broken, the model has domain-wall excitations. These are similar, in a way, to the domain-walls found in a three-component model [1, 34]. They are thus discussed in the Supplementary Material [16]. However the quantization of $Q(\mathbf{m})$ suggests that the model has more nontrivial topological excitations with quantized magnetic flux according to $\int B_z = \Phi_0 Q$, where $\Phi_0 = -2\pi/\epsilon$ is the flux quantum. If a model breaks the $\mathbb{Z}_2$ symmetry and has only gradient terms which are second order in derivatives, according to the Hobart-Derrick theorem [35, 36],

$$\mathcal{T}(\Psi) = \Psi^* \Leftrightarrow \mathcal{T}(\mathbf{m}) = (m_x, -m_y, m_z). \quad (6)$$
Figure 2. Skyrmion solutions in a time-reversal symmetry broken state, for increasing values of the topological charge $Q(m)$. The panels on the top row display the texture of the four-fermion order parameter $m$. The panels on the middle row show the associated magnetic field $B$ (7), and the bottom row shows the corresponding charge transferring counter-currents $j_{\text{counter}}$ according to the Ampère’s law. The parameters are the same as in Fig. 1, while the coupling $\epsilon = 0.25$.

Skyrmions cannot exist. In our case, the presence of the Skyrme term, in the effective model (5), allows for non-trivial configurations that evade the Hobart-Derrick theorem. Indeed, in two-dimensions, the Skyrme term in the effective model scales as $1/R^2$ (where $R$ is a texture size), and therefore stable skyrmions may exist due to the competition between the Skyrme and potential terms.

We performed numerical simulation by minimizing the energy (5) from various initial states. When the initial guess has a nontrivial topological charge, the minimization procedure leads, after convergence of the algorithm, to stable skyrmion configurations. Fig. 2 shows these skyrmion solutions for increasing values of the topological charge $Q(m)$, which is integer with an accuracy around $10^{-4}$. As shown on the middle row of Fig. 2, the skyrmions carry a nonzero magnetic field. Moreover, since the topological charge (8) is quantized, the skyrmions carry integer quanta of magnetic flux. The circulating current pattern that induces this magnetic field is illustrated in the bottom row. This current, defined according to the Ampère’s law for the magnetic field (7) corresponds to the charge-carrying counterflow between the different components.

Furthermore we find that the inter-skyrmion forces are attractive. Hence, single quanta skyrmions attract each other to form skyrmions with higher topological charge. Thus in general one would not expect the formation of regular skyrmion lattices but rather skyrmion lumps formed by the competition between the attractive forces and pinning landscape. Interestingly, in a single quantum skyrmion, the time-reversed state is realized at a zero measure area inside the skyrmion. On the other hand, skyrmions carrying more than one quantum feature inner regions of the time-reversed state. The enclosed area of the time-reversed state increases with the topological charge. This suggest that in case where $\mathbb{Z}_2$ symmetry is strongly broken the formation of skyrmions is strongly inhibited compared to domain-walls.

In summary, the recent experiment reported a fermion quadrupling phase in $\text{Ba}_{1-x}\text{K}_x\text{Fe}_2\text{As}_2$ [1]. In this resistive phase, there is no condensate of Cooper pairs, but a four-fermion condensate which breaks the $\mathbb{Z}_2$ time-reversal symmetry. We derived an effective model of that resistive state, starting from a microscopic three-band model with dominant inter-band interaction for $\text{Ba}_{1-x}\text{K}_x\text{Fe}_2\text{As}_2$ and by implementing a mean-field approximation for the fields that are fourth-order in fermions.

The effective field theory has a structure similar to the Faddeev-Skyrme model, but for a soft modulus vector field that represents the fermion quadrupling order parameter. If spatial inhomogeneities are present the
model accounts for spontaneous magnetic fields, consistently with the experimental observations [1]. We report that despite the lack of Meissner effect and the lack of conserved U(1) topological charge, the model has stable topological excitations in the form of skyrmions with conserved topological charge (8). Finally, we would like to remind the reader that, similarly to skyrmions that appear in other contexts, such as magnetism, their existence also depends on factors that are beyond the effective long-wave-length field-theoretic models. Namely, in contrast to vortices, the skyrmionic topological charge is obtained through a surface integral. Consequently, if the terms that break the O(3) symmetry are very strong, the localization of the skyrmionic topological charge can shrink down to scales where the effective theory is ill-defined, thereby destroying the topological protection. Such a collapse is similar to the collapse of magnetic skyrmions in a sufficiently strong magnetic field.

Within the range of applicability of the effective theory, the skyrmions can be induced by taking advantage of the Kibble-Zurek mechanism [37, 38], by quenching the material trough the Z_2 phase transition where the time-reversal symmetry is broken.

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Supplementary Material: Skyrmions and magnetic properties of the resistive electron quadrupling state

In the Supplementary Information, we discuss the details of the derivation of the effective theory for the fermion quadrupling state. In particular, we start from the microscopic model of a three band superconductor with interband dominated pairing. This yields a two-component Ginzburg-Landau theory with inter-component mixed gradient terms which are eliminated by reparametrization of the superconducting degrees of freedom. Next, the theory is mapped to a model that couples the fermion quadrupling order parameter to the Meissner current. In the resistive state, the Meissner screening is absent, and the theory reduces to a Faddeev-Skyrme model that depends only on the four-fermion order parameter. We also discuss details of the numerical methods, and present additional results.

I. MICROSCOPIC DERIVATION OF THE EFFECTIVE MODEL

The first part starts with the microscopic derivation of the two-component Ginzburg-Landau theory that is relevant to describe a three-band superconductor with interband dominated repulsive pairing. See [S1] for a more detailed derivation. We are interested in values of coupling constants that can result in superconducting states that spontaneously break the time-reversal symmetry, aiming in particular to describe iron pnictides.

The band structure of iron pnictides typically consists of two electron pockets at (0, π) and (π, 0) and of two hole pockets at the Γ point. This structure is sketched on Fig. S1, where the dominating pairing channels are the interband repulsion between the two hole pockets at Γ, as well as between the electron and the hole bands. Note that, the order parameter is the same in both electron and hole Fermi surfaces.

Figure S1. Schematic view of the band structure for the hole-doped iron pnictide compound Ba$_{1-x}$K$_x$Fe$_2$As$_2$. It consists of two hole pockets at the Γ point shown by circles and two electron pockets at (0; π) and (π; 0) displayed by ellipses.

As discussed in the text, the s+$i$s state is favoured by the superconducting coupling that is dominated by the interband repulsion between the electron and the hole Fermi surfaces $u_{he}$, as well as between the two hole pockets $u_{hh}$.

A. Generic three-component expansion

We consider the microscopic model of a clean superconductor with three overlapping bands at the Fermi level. Within the quasiclassical approximation, the band parameters that characterize the different cylindrical sheets of the Fermi surface are the partial densities of states (DOS) $\nu_a$, and the Fermi velocities $v_F^{(a)}$; here the index $a = 1, 2, 3$ labels the different bands. The Eilenberger equations for the quasiclassical propagators read as

$$\hbar v_F^{(a)} D f_a + 2\omega_n f_a - 2\Delta_a g_a = 0,$$  \hspace{1cm} (S1a)

$$\hbar v_F^{(a)} \nabla^2 f_a^+ - 2\omega_n f_a^+ + 2\Delta_a^* g_a = 0,$$  \hspace{1cm} (S1b)

where $\omega_n = (2n + 1)\pi T$, with $n \in \mathbb{Z}$, are the fermionic Matsubara frequencies and $T$ is the temperature. The gauge derivative is $D \equiv \nabla + ieA$, where $A$ is the vector potential, and the gauge coupling is related to the flux quantum $\Phi_0$ by $e = -2\pi/\Phi_0$.

The quasi-classical propagators $f_a$ and $g_a$ are, respectively, the anomalous and the normal Green’s functions in each band; they obey the normalization condition $|f_a|^2 + g_a^2 = 1$. The components $\Delta_a$ of the order parameter are determined by the self-consistency equations

$$\Delta_a(p, r) = 2\pi T \sum_{n, p', b} \lambda_{ab}(p, p') f_b(p, r, \omega_n).$$  \hspace{1cm} (S2)

Here, the parameters $p$ run over the Fermi surfaces, and $\lambda_{ab}$ are the components of the coupling potential matrix. For simplicity the pairing states are assumed to be isotropic on each of the Fermi surfaces, so that $\lambda_{ab}(p, p') = \text{const}$, see details in [S1]. Finally, the self-consistent electric current is

$$j(r) = 2\pi e T \sum_{n, p, a} \nu_a v_F^{(a)} \text{Im} g_a(p, r, \omega_n).$$  \hspace{1cm} (S3)

where $\nu_a$ is the partial density of state, and $g_a = \text{sign}(\omega_n)\sqrt{1 - f_a f_a^+}$.

The Ginzburg-Landau functional, is obtained by expressing the solutions of the Eilenberger equations (S1) as an expansion by powers of the gap functions ampli-
tudes $\Delta_a$ and of their gradients:

$$f_a(p, r, \omega_n) = \frac{\Delta_a}{\omega_n} - \frac{|\Delta_a|^2 \Delta_a}{2\omega_n^3} - \frac{\hbar v_F^{(a)} D \Delta_a}{2\omega_n^2} + \frac{\hbar^2 v_F^{(a)} D(v_F^{(a)} D) \Delta_a}{4\omega_n^3}.$$ (S4)

The summation over the Matsubara frequencies gives

$$2\pi T \sum_{n=0}^{N_d} \omega_n^{-1} = G_0 + \tau, \text{ with } \tau = (1 - T/T_c).$$ (S5)

Note that, $f_a^+(p, r, \omega_n) = f_a^-(p, r, \omega_n)$.

The Ginzburg-Landau equations are determined by substituting the expansion (S4) into the self-consistency equation (S2). After normalizing the gaps functions by $T_c/\sqrt{\rho}$ (where $\rho = \sum_n \pi T_c^3 \omega_n^{-3} \approx 0.1$), the Ginzburg-Landau equations read as

$$[(G_0 + \tau - \tilde{\Lambda}^{-1}) \Delta]_a = -K_{ij}^{(a)} D_i D_j \Delta_a + |\Delta_a|^2 \Delta_a,$$ (S6)

where $\Delta = (\Delta_1, \Delta_2, \Delta_3)^T$, and the anisotropy tensor is $K_{ij}^{(a)} = \hbar^2 \rho \left(v_F^{(a)} D(v_F^{(a)} D)\right)/2T_c^2$. The indices $i, j$ stand for the $x, y$ coordinates, and the average is taken over the $a$-th Fermi surface. The current reads as

$$J(r) = \frac{4e T_c^2}{\hbar} \sum_{a=1}^{3} \nu_a \text{Im} \Delta_a^{*} \tilde{K}^{(a)} D \Delta_a.$$ (S7)

The critical temperature is given by the smallest positive eigenvalue of the inverse coupling matrix $\tilde{\Lambda}^{-1}$. Namely, if $\lambda_a^{-1}$ denote the positive eigenvalues of the inverse coupling matrix $\tilde{\Lambda}^{-1}$, the critical temperature is determined by the equation $G_0 = \min_a (\lambda_a^{-1})$. Provided that all the eigenvalues are positive, the number of components of the effective field theory coincide with the number of bands. In this case, the Ginzburg-Landau equations for the three-component system read as

$$-K_{ij}^{(a)} D_i D_j \Delta_a + \alpha_{aa} \Delta_a + \alpha_{ab} \Delta_b + \beta_a |\Delta_a|^2 \Delta_a = 0,$$ (S8)

where

$$\alpha_{aa} = (\tilde{\lambda}_a^{-1} - G_0 - \tau) \delta_{ab},$$

$$\alpha_{ab} = (1 - \delta_{ab}) \tilde{\lambda}_a^{-1} \quad \text{and} \quad \beta_a = 1.$$ (S9a, S9b)

While the precise microscopic physics behind the superconductivity in Ba$_{1-x}$K$_x$Fe$_2$AsO is still unknown, we focus on the scenario of a three-band model with interband dominated repulsive pairing. In this case, the eigenvalues of the inverse coupling matrix are not all positive. This implies, as detailed below, that the three-band theory is described by a two-component order parameter.

B. Two-component Ginzburg-Landau theory for the $s+i$ superconducting state

Our principal interest here, is the time-reversal symmetry breaking $s+i$ state in a three-band superconductor. We consider an interband dominated repulsive pairing, suggested to be relevant for iron-based superconductors [S2]. The corresponding coupling matrix $\tilde{\Lambda}$ is parametrized as

$$\tilde{\Lambda} = \begin{pmatrix} 0 & u_{hh} & u_{eh} \\ u_{hh} & 0 & u_{eh} \\ u_{eh} & u_{eh} & 0 \end{pmatrix}.$$ (S10)

Thus the fields $\Delta_{1,2}$ correspond to the gap functions at the hole Fermi surfaces while $\Delta_3$ is the gap at the electron pockets sketched in Fig. S1. The coefficients $u_{hh}$ and $u_{eh}$ are respectively the hole-hole and electron-hole interactions. The linear equation that determines the critical temperature $G_0 = \min(G_1, G_2)$ is obtained by neglecting the r.h.s. of (S6). Here $G_1$ and $G_2$ are the only two positive eigenvalues of the inverse coupling matrix

$$\tilde{\Lambda}^{-1} = \frac{1}{2u_{eh}^2 u_{hh}} \begin{pmatrix} u_{hh}^2 & -u_{eh}^2 & -u_{eh} u_{hh} \\ -u_{eh}^2 & u_{eh}^2 & -u_{eh} u_{hh} \\ -u_{eh} u_{hh} & -u_{eh} u_{hh} & u_{hh}^2 \end{pmatrix}. $$ (S11)

They explicitly reads as $G_1 = 1/u_{hh}$ and $G_2 = (u_{hh} + \sqrt{u_{hh}^2 + 8u_{eh}^2})/4u_{eh}$. The associated eigenvectors are $\Delta_1 = (-1, 1, 0)^T$ and $\Delta_2 = (x, x, 1)^T$, where $x = (u_{hh} - \sqrt{u_{hh}^2 + 8u_{eh}^2})/4u_{eh}$. Since the only fields that can nucleate are those associated with positive eigenvalues, the Ginzburg-Landau theory (S6) has to be reduced to a two-component one. This reduction is obtained by expressing the general order parameter as the linear combination

$$\Delta = \eta_1 \Delta_1 + \eta_2 \Delta_2,$$

and $(\Delta_1, \Delta_2, \Delta_3) = (x \eta_2 - \eta_1, x \eta_2 + \eta_1, \eta_2)$. (S12)

Here $\eta_1$ and $\eta_2$ are the order parameter of the $s+i$ pairing channels respectively between the two concentric hole surfaces and between the hole and electron surfaces.

The substitution of the linear combination (S12) into the Ginzburg-Landau equations (S6), after projection onto the eigenvectors $\Delta_{1,2}$, yields the system of two Ginzburg-Landau equations [S21]:

$$a_{11} \eta_1 + b_{11} |\eta_1|^2 \eta_1 + c_{12} \eta_1^* \eta_2 = \frac{k_{12}}{2} D D \eta_j,$$ (S13a)

$$a_{22} \eta_2 + b_{22} |\eta_2|^2 \eta_2 + c_{12} \eta_1^* \eta_2 = \frac{k_{22}}{2} D D \eta_j.$$ (S13b)

The parameters on the left hand side of the Ginzburg-Landau equations (S13) are expressed, in terms of the coefficients of the coupling matrix (S10) as

$$a_{jj} = -|\Delta_j|^2(G_0 - G_j + \tau), \quad a_{12} = 0$$ (S14)

$b_{11} = 2, \quad b_{22} = (2x^4 + 1), \quad b_{12} = 4x^2, \quad c_{12} = 2x^2$, where $x = (u_{hh} - \sqrt{u_{hh}^2 + 8u_{eh}^2})/4u_{eh}$.\]
where $|\Delta_1|^2 = 2$ and $|\Delta_2|^2 = 2x^2 + 1$. The $s + is$ state is symmetric under the $C_4$ transformations, thus the coefficients satisfy $k^{(j)}_{x^2} = K^{(j)}_{y^2} = K^{(j)}$. As a results, the coefficients of the gradient terms in (S13) read as

\begin{align}
    k_{11} &= 2\xi_0^2 \left[K^{(1)} + K^{(2)}\right] \\
    k_{22} &= 2\xi_0^2 \left[K^{(1)} + K^{(2)}\right] + 2K^{(3)} \\
    k_{12} &= 2\xi_0^2 \left[K^{(2)} - K^{(1)}\right].
\end{align}

(S15a) (S15b) (S15c)

The total superconducting current (S7), is the superposition of the partial currents $J^{(a)}$ of the different components of the order parameters, as $J = \sum_n J^{(a)}$, and the partial currents read as

$$J^{(a)} = e\text{Im}\left(\eta^*_a \sum_b [k_{ab} D\eta_b]\right).$$

(S16)

The two-component free energy functional that corresponds to the Ginzburg-Landau equations (S13), and whose variations with respect to $A$ give the supercurrent (S16), reads as (in dimensionless units):

$$F = \frac{B^2}{2} + \frac{1}{2} \sum_{a,b=1}^2 k_{ab} (D\eta_a)^* D\eta_b + V(\eta),$$

(S17a)

where $V(\eta) = \sum_{a,b=1}^2 a_{ab} \eta^*_a \eta_b + \frac{b_{ab}}{2} |\eta_a|^2 |\eta_b|^2 + \frac{C_{12}}{2} (\eta^*_1 \eta^*_2 + c.c.)$. (S17b) (S17c)

Here, the complex fields $\eta_{1,2}$ are the components of the superconducting order parameter. They are electromagnetically coupled by the vector potential $A$ of the magnetic field $B = \nabla \times A$, through the gauge derivative $D \equiv \nabla + ieA$. The original superconducting degrees of freedom, where the kinetic term (S18) is diagonal:

$$F_k = \frac{1}{2} \langle D\Psi^\dagger D\Psi \rangle, \quad \Psi = \eta + R\eta,$$

(S20)

and $\Psi^\dagger = (\psi^*_1, \psi^*_2)$. The original superconducting degrees of freedom $\eta$ are restored via the reverse transformation $\eta = R^{-1}\Psi$, where

$$R^{-1} = \frac{\sqrt{\text{tr} k + 2\sqrt{\det k}}}{\det(\hat{k} + 1\sqrt{\det k})}. \quad (S21)$$

Using the relations (S19) and (S21) to parametrize the superconducting degrees of freedom with $\Psi$ instead of $\eta$, greatly simplifies the kinetic term (S17) has to be rewritten in terms of $\Psi$. In all generality, the potential energy reads as

$$F_p := V(\Psi) = a_{ij} \eta^*_i \eta^*_j + b_{ijkl} \eta^*_i \eta^*_j \eta^*_k \eta^*_l,$$

(S22)

with the summation over the repeated indices. Note that for the energy to be a real quantity, the tensor coefficients $a_{ij}$, and $b_{ijkl}$ should obey some symmetry relations:

$$a_{ij} = a_{ji}, \quad b_{ijkl} = b_{ijlk} = b_{klij}.$$ (S23a) (S23b)

Similarly, in terms of $\Psi$ the potential energy reads as

$$F_p := V(\Psi) = a_{ij} \psi^*_i \psi^*_j + \frac{b_{ijkl}}{2} \psi^*_i \psi^*_j \psi^*_k \psi^*_l,$$

(S24)

where the tensor coefficients $a_{ij}$ and $b_{ijkl}$ obey the same symmetry relations (S23) as $a_{ij}$, and $b_{ijkl}$. They are obtained via the transformation $\eta = R^{-1}\Psi$, and the relations are

$$a_{ij} = a_{ab} R_a^{-1} R_b^{-1}, \quad \beta_{ijkl} = b_{abcd} R_a^{-1} R_b^{-1} R_c^{-1} R_d^{-1}.$$ (S25a) (S25b)

Simple, yet lengthy algebraic manipulations thus yield the free energy in terms of the superconducting degrees of freedom $\Psi$

$$F = \frac{B^2}{2} + \frac{1}{2} \langle D\Psi^\dagger D\Psi \rangle + V(\Psi),$$

(S26a)

where $V(\Psi) = \sum_{i,j=1}^2 \alpha_{ij} \psi^*_i \psi^*_j + \frac{\beta_{ij}}{2} |\psi_i|^2 |\psi_j|^2 + (\gamma_{11} |\psi_1|^2 + \gamma_{22} |\psi_2|^2) (\psi^*_1 \psi_2 + c.c.).$ (S26c)

(S26d)
\[ a_{11} = a_{11}(R_{11}^{-1})^2 + a_{22}(R_{21}^{-1})^2 + 2a_{12}R_{11}^{-1}R_{21}^{-1} \]  
\[ a_{22} = a_{11}(R_{12}^{-1})^2 + a_{22}(R_{22}^{-1})^2 + 2a_{12}R_{12}^{-1}R_{22}^{-1} \]  
\[ a_{12} = a_{11}R_{11}^{-1}R_{12}^{-1} + a_{22}R_{21}^{-1}R_{22}^{-1} + a_{12}(R_{11}^{-1}R_{22}^{-1} + R_{12}^{-1}R_{21}^{-1}) . \]  
\[ \beta_{11} = \frac{b_{11}}{2}(R_{11}^{-1})^4 + \frac{b_{22}}{2}(R_{21}^{-1})^4 + (b_{12} + c_{12})(R_{11}^{-1})^2(R_{21}^{-1})^2 + 2R_{11}^{-1}R_{21}^{-1}(c_{11}(R_{11}^{-1})^2 + c_{22}(R_{21}^{-1})^2) \]  
\[ \beta_{22} = \frac{b_{11}}{2}(R_{12}^{-1})^4 + \frac{b_{22}}{2}(R_{22}^{-1})^4 + (b_{12} + c_{12})(R_{12}^{-1})^2(R_{22}^{-1})^2 + 2R_{12}^{-1}R_{22}^{-1}(c_{11}(R_{12}^{-1})^2 + c_{22}(R_{22}^{-1})^2) \]  
\[ \beta_{12} = 2b_{11}(R_{11}^{-1})^2(R_{12}^{-1})^2 + 2b_{22}(R_{21}^{-1})^2(R_{22}^{-1})^2 + b_{12}(R_{11}^{-1}R_{12}^{-1} + R_{12}^{-1}R_{21}^{-1}) + 4c_{11}R_{11}^{-1}R_{22}^{-1}R_{21}^{-1} + 4(c_{11}R_{11}^{-1}R_{12}^{-1} + c_{22}R_{21}^{-1}R_{22}^{-1})(R_{11}^{-1}R_{22}^{-1} + R_{12}^{-1}R_{21}^{-1}) . \]

and \( \gamma_{i} \beta_{ijk} \delta_{ik} \delta_{ij} \), \( \gamma_{ik} = \beta_{ijk} \delta_{ij} \). Moreover, all the coefficients are symmetric, for example \( \beta_{21} = \beta_{12} \). Using the relations (S25), and collecting the various terms yields the relation for the coefficients of the bilinear terms

\[ B_k = \epsilon_{kij} \left\{ \nabla_i \left( \frac{J_j}{e^2 q^2} \right) + \frac{i}{e q^2} Z_{ij} \right\} \]  
where \( Z_{ij} = q^2 \nabla_i \Psi \nabla_j \Psi + (\Psi^i \nabla_j \Psi)(\nabla_j \Psi^i \Psi) \).

Hence, the magnetic field features a contribution from the Meissner current \( J \), together with a contribution from the interband counterflow \( Z_{ij} \). Note that since \( Z_{ij}^* = Z_{ji} \), the magnetic field can be written as

\[ B_k = \epsilon_{kij} \left\{ \nabla_i \left( \frac{J_j}{e^2 q^2} \right) - \frac{\Im Z_{ij}}{e q^2} \right\} . \]  

It follows that the free energy (S26) can be rewritten as

\[ \mathcal{F} = \frac{1}{2} \left[ \epsilon_{kij} \left\{ \nabla_i \left( \frac{J_j}{e^2 q^2} \right) + \frac{i}{e q^2} Z_{ij} \right\} \right]^2 + \frac{J^2}{2e^2 q^2} + \nabla \Psi^i \nabla \Psi + \frac{1}{4q^2} (\Psi^i \nabla \Psi - \nabla \Psi^i \Psi)^2 + V(\Psi), \]  
where \( Z_{ij} \) is defined in (S33).
E. Mapping to the effective model

Two-component Ginzburg-Landau models can often be mapped onto a version of the nonlinear O(3) σ-model [S1, S4–S6]. In these mappings the O(3) symmetry is explicitly broken by the potential terms, consistently with the symmetry of the superconducting state. The mapping couples the massive U(1) vector field (the current \( J \)) to a compact O(3) unit vector (the pseudo-spin \( n \)) and a real scalar (the density \( \rho \)). The pseudo-spin unit is defined by projecting the superconducting degrees of freedom onto the spin-1/2 Pauli matrices \( \sigma \). For derivation of this mapping for different two-component Ginzburg-Landau models, see e.g. [S1, S6].

Here, we use an alternative mapping to a model that couples the massive U(1) vector field (the current \( J \)) to the fermion quadrupling order parameter in the form of a 3-vector \( m \). The fermion quadrupling field is defined as the projection of the superconducting degrees of freedom \( \Psi \) onto the spin-1/2 Pauli matrices \( \sigma \):

\[
m \equiv (m_x, m_y, m_z) = \Psi \gamma^\dagger \Psi . \tag{S36}
\]

Unlike the pseudo-spin \( n \), which is a unit vector, the norm of \( m \) is not fixed. Thus \( n \) and \( m \) are related to each other according to \( m = \rho^2 n \), and we sometime refer to \( m \) as a soft modulus vector field. The projection (S36) determines the following relations

\[
\partial_i m \cdot \partial_j m = 2 \rho^2 \left( \partial_i \Psi \gamma^\dagger \partial_j \Psi + \partial_j \Psi \gamma^\dagger \partial_i \Psi \right) + \left( \Psi \gamma^\dagger \partial_i \Psi - \partial_i \Psi \gamma^\dagger \Psi \right) \left( \Psi \gamma^\dagger \partial_j \Psi - \partial_j \Psi \gamma^\dagger \Psi \right), \tag{S37}
\]

where, for the product of Pauli matrices, we used the Fierz identity

\[
\sigma_{ab} \sigma_{cd} = 2 \delta_{ad} \delta_{bc} - \delta_{ab} \delta_{cd}, \tag{S38}
\]

where \( \delta_{ab} \) is the Kronecker symbol. It follows that

\[
(\nabla m)^2 = 4 \rho^2 \nabla \Psi \cdot \gamma^\dagger \nabla \Psi + (\Psi \gamma^\dagger \nabla \Psi - \nabla \Psi \gamma^\dagger \Psi)^2 . \tag{S39}
\]

The kinetic term (S32) can thus be written as

\[
|D\Psi|^2 = \frac{J^2}{e^2 \rho^2} + \frac{1}{4 \rho^2} (\nabla m)^2 . \tag{S40}
\]

Similarly, the projection (S36) determines the relation

\[
m \cdot \partial_i m \times \partial_j m = -2 i \rho^2 \left( \partial_i \Psi \gamma^\dagger \partial_j \Psi - \partial_j \Psi \gamma^\dagger \partial_i \Psi \right) - 2 i \rho^2 \left[ (\Psi \gamma^\dagger \partial_i \Psi) (\partial_j \Psi \gamma^\dagger \Psi) - (\partial_j \Psi \gamma^\dagger \Psi) (\Psi \gamma^\dagger \partial_i \Psi) \right] , \tag{S41}
\]

where, for the triple product of Pauli matrices, we used the identity

\[
\epsilon_{abc} \sigma_{a} \sigma_{b} \sigma_{c} \gamma_{f} = 2i \left( \delta_{af} \delta_{bc} \delta_{de} - \delta_{ad} \delta_{bf} \delta_{ce} \right) . \tag{S42}
\]

There \( \delta_{ab} \) is the Kronecker symbol, and \( \epsilon_{abc} \) is the rank-3 Levi-Civita symbol. It follows that

\[
\epsilon_{ijk} m \cdot \partial_i m \times \partial_j m = -4 i \rho^2 \epsilon_{ijk} \left( \rho^2 \partial_i \Psi \gamma^\dagger \partial_j \Psi + (\Psi \gamma^\dagger \partial_i \Psi) (\partial_j \Psi \gamma^\dagger \Psi) \right) . \tag{S43}
\]

Hence, the magnetic field reads as

\[
B = \nabla \times \left( \frac{J}{e^2 \rho^2} \right) - \frac{\epsilon_{abc}}{4 e \rho^6} m_a \nabla m_b \times \nabla m_c . \tag{S44}
\]

As a result,

\[
\mathcal{F} = \frac{1}{2} \left[ \epsilon_{ijk} \left\{ \nabla_i \left( \frac{J}{e^2 \rho^2} \right) - \frac{1}{4 e \rho^6} m \cdot \partial_i m \times \partial_j m \right\} \right]^2 + \frac{J^2}{2 e^2 \rho^2} + \frac{1}{8 \rho^2} (\nabla m)^2 + V(m) , \tag{S45}
\]

where the density is \( \rho^2 \equiv \| m \| = \sqrt{\nabla \cdot m} . \)

The effective model of the fermion quadrupling resistive state is deduced by removing the superconducting degrees of freedom from (S45), since their prefactors are renormalized to zero. Namely, in that resistive state the Meissner current vanishes (\( J = 0 \)), while the currents associated with the gradients of the fermion quadrupling order parameter \( m \) do not. Assuming that the critical temperatures of \( \mathbb{Z}_2 \) and U(1) transitions are well separated, and assuming a mean-field approximation for the fields that are fourth-order in fermions the free energy of the fermion quadrupling state reads as

\[
\mathcal{F} = \frac{(m \cdot \partial_i m \times \partial_j m)^2}{16 e^2 \| m \|^6} + \frac{(\nabla m)^2}{8 \| m \|^2} + V(m) . \tag{S46}
\]

Similarly, the magnetic field in the fermion quadrupling resistive state becomes

\[
B = -\frac{\epsilon_{abc} m_a \nabla m_b \times \nabla m_c}{4 e \| m \|^3} . \tag{S47}
\]

Equivalently, component-wise, the magnetic field is

\[
B_k = -\frac{\epsilon_{ijk} m \cdot \nabla_j m \times \nabla_k m}{4 e \| m \|^3} . \tag{S48}
\]

Finally, the potential term reads as

\[
V(m) = \sum_{a=0,x,y,z} \left( \alpha^a_m + \sum_{b=0,x,y,z} \frac{\beta^a_{mb}}{2} m_b \right) m_a , \tag{S49}
\]

where the component \( m_0 := \| m \| \), and the coefficients depend on the coefficients (S27), (S28) and (S29) of the diagonalized free-energy (S26). All the coefficients involving a \( y \) index vanish: \( \alpha^y_m = \beta^{ay}_m = \beta^{ya}_m = 0 \). The non-zero coefficients \( \alpha^a_m \) of the linear term in \( m_a \) are

\[
\alpha^0_m = \alpha^{11}_m + \alpha^{22}_m , \quad \alpha^x_m = \alpha^{12}_m , \quad \alpha^z_m = \frac{\alpha^{11}_m - \alpha^{22}_m}{2} . \tag{S50}
\]

Next, the non-zero coefficients \( \beta^a_{mb} \) of the bilinear term in \( m_a \) are

\[
\beta^0_{00} = \frac{\beta^{11}_m + \beta^{22}_m + 2(\beta^{12}_m - \gamma^{12}_m)}{4} , \tag{S51a}
\]

\[
\beta^0_{0x} = \frac{\beta^{12}_m}{2} , \tag{S51b}
\]

\[
\beta^0_{0z} = \frac{\beta^{11}_m - \beta^{22}_m}{2} . \tag{S51c}
\]
| Parameters of the effective model | $\alpha_0^m$ $(\times 10^{-1})$ | $\alpha_x^m$ $(\times 10^{-2})$ | $\alpha_z^m$ $(\times 10^{-2})$ | $\beta_{00}^m$ $(\times 10^{-1})$ | $\beta_{02}^m$ $(\times 10^{-2})$ | $\beta_{0z}^m$ $(\times 10^{-2})$ | $\beta_{12}^m$ $(\times 10^{-1})$ | $\beta_{1z}^m$ $(\times 10^{-2})$ | $\beta_{2z}^m$ $(\times 10^{-2})$ |
|----------------------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
|                                 | -8.2605          | 8.0923           | -6.3919          | 4.5431           | -6.5085          | 1.4987           | 2.1940           | -0.0875          | 2.3491           |

Table I. Coefficients of the Ginzburg-Landau free energy functional that correspond in the main body of the text. Starting from the microscopic model (S10), the parameters of the coupling matrix are $u_{hh} = 0.45$ and $u_{hh} = 0.5$, and the coefficients of the gradient term are $K^{(1)} = 0.5$, $K^{(2)} = 0.35$, $K^{(3)} = 0.45$ and the temperature parameter is $T/T_c = 0.25$. Next, the coefficients of the diagonalized Ginzburg-Landau model (S26) are evaluated using the formulas (S27), (S28) and (S29). Finally the coefficients of the soft modulus effective model are obtained with the formulas (S50), (S51), and (S52).

and

$$\beta_{xx}^m = \gamma_{12}, \quad (S52a)$$

$$\beta_{xz}^m = \beta_{zx}^m = \gamma_{11} - \gamma_{22}, \quad (S52b)$$

$$\beta_{zz}^m = \frac{\beta_{11} + \beta_{22} - 2(\beta_{12} - \gamma_{12})}{4}. \quad (S52c)$$

**Topological properties in two-dimensions**

The soft modulus vector field $m$ (S36) can be associated with non-trivial topological properties, by considering the properties of the corresponding unit vector field $n := m/\|m\|$. Indeed, the unit vector $n$ is a map from the one-point compactification of the plane ($\mathbb{R}^2 \cup \{\infty\} \cong \mathbb{S}^2$) onto the two-sphere target space spanned by $n$. That is $n : \mathbb{S}^2 \to \mathbb{S}^2_n$, which is classified by the homotopy class $\pi_2(S^2_n) \in \mathbb{Z}$, thus defining the topological invariant, i.e. the degree of the map, as

$$Q(n) = \frac{1}{4\pi} \int_{\mathbb{R}^2} n \cdot \partial_x n \times \partial_y n \; dx dy. \quad (S53)$$

Note that $n$ is ill-defined when $\|m\| = 0$. On the other hand, whenever $\|m\| \neq 0$, the corresponding configuration have an integer topological charge $Q(n) \in \mathbb{Z}$. In a way, $Q(n)$ counts the number of times the pseudo-spin texture of $n$ wraps the target two-sphere. The topological invariant (S53), the index of the map $n$, can be expressed directly in terms of the soft modulus vector field $m$. This is easily done by replacing $n$ with its actual definition $n := m/\|m\|$. The topological invariant thus reads as

$$Q(m) = \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot \partial_x m \times \partial_y m \; dx dy. \quad (S54)$$

Here again, the integrand is obviously ill-defined when $\|m\| = 0$. Whenever $\|m\| \neq 0$, the corresponding configuration have an integer topological charge $Q(m) \in \mathbb{Z}$. The quantization of $Q(m)$ implies that the magnetic flux is quantized as well according to $\int B_z = \Phi_0 Q$, where $\Phi_0 = -2\pi/e$ is the flux quantum.

It should be emphasized that unlike the flux quantization condition for the superconducting states, which is related to the $U(1)$ topological invariant, the condition (S54) is also valid in the non-superconducting phase. Indeed, the quantization in the superconducting state is given by the $U(1)$ topological invariant, which is related to the total phase winding at spatial infinity (the usual winding number). In the fermion quadrupling resistive state, the total phase of $\Psi$ is disordered and the $U(1)$ invariant does not exist. On the other, since it is associated only with the relative phases, $Q(m)$ is the quantity that defines the flux quantization.

It is worth emphasizing that the topological charge (S53) is an integer, when integrated over the infinite plane $\mathbb{R}^2$, or at least an large enough domain $\Omega \subset \mathbb{R}^2$.

**F. Parameter sets**

The essential features can be qualitatively summarized as follows: First, all the coefficients involving a $y$ index vanish: $a_{yy}^m = \beta_{yy}^m = \beta_{zm}^m = 0$. Moreover, the criterion for the condensation is $\alpha_{m}^0 > \alpha_{m}^2 + \alpha_{m}^{m2}$, and also $\beta_{00}^m, \beta_{zz}^m > 0$. The effect of the time-reversal symmetry operation for the soft modulus vector is a reflection of $m$ on the $xz$-plane of the target space:

$$T(\Psi) = \Psi^* \leftrightarrow T(m) = (m_x, -m_y, m_z). \quad (S55)$$

A typical value of the parameter set, when obtained from the microscopic model, is given in the Table I.

**G. Modulation of the parameters**

Inhomogeneities in a sample typically result in spatially varying parameters of the Ginzburg-Landau model. In the system with broken time-reversal symmetry this can result in gradients of both densities and relative phases. This can in principle produce spontaneous magnetic fields. For example, the parameters of the quadratic term of the original Ginzburg-Landau theory (S17) formally depend on the temperature, $a_{ii} \equiv a_{ii}(\tau) = a_{ii}^0(\tau - 1)$. Hence, it may be possible to model the effect of temperature inhomogeneities by requiring a spatial dependence of the parameters $a_{ii} \equiv a_{ii}(\tau, \mathbf{x}) = a_{ii}^0(\tau - 1)$. Different areas of an inhomogeneous sample indeed can have different local critical temperatures. Such a local modification of the parameter was demonstrated to be responsible for the existence of spontaneous magnetic
fields, in different models with time-reversal symmetry breaking states, in various situations in superconducting state. These include the responses to linear thermal gradients [S7, S8], hotspot created by a laser pulse [S9] but also the effect of impurities [S10, S11], and other inhomogeneous arrays [S3, S12].

In the main body, we considered random modulations \( \tau(x) = \tau_0[1 + \delta\tau \, \text{ran}(x)] \) where \( \text{ran}(x) \) is a smooth random surface. Here \( \tau_0 \) is the nominal reduced temperature, \( \delta\tau \) is the amplitude of the thermal variation. The idea to construct a random, smoothly varying quantity is to represent it as a Fourier series with random coefficients:

\[
f(x) = -c_0 + \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} c_{ij} \cos 2\pi \left( \frac{ix}{L_x} + \frac{jy}{L_y} \right) + s_{ij} \sin 2\pi \left( \frac{ix}{L_x} + \frac{jy}{L_y} \right),
\]

were \( L_x \) and \( L_y \) are the dimensions of the box that bounds the numerical domain. \( N_x \) and \( N_y \) are cut-off in the Fourier expansion, and the coefficients \( c_{ij} \) and \( s_{ij} \) are random numbers \( \in [-0.5 : 0.5] \).

II. NUMERICAL METHODS

In the numerical investigations in the main body of the paper, we use Finite-Element Methods (FEM) (see e.g. [S13, S14]) to handle the spatial discretization of the problem. In practice we use the finite-element framework provided by the FreeFEM library [S15]. Within this finite-element framework, the minimization of the free energy is addressed using a non-linear conjugate gradient algorithm [S16–S19].

A. Finite-element formulation

We consider the domain \( \Omega \) which a bounded open subset of \( \mathbb{R}^2 \) and denote \( \partial \Omega \) its boundary. \( H(\Omega) \) stands for the Hilbert space, such that a function belonging to \( H(\Omega) \), and its weak derivatives have a finite \( L^2 \)-norm. Furthermore, the Hilbert spaces of real-valued functions is equipped with the inner product \( \langle \cdot, \cdot \rangle \), defined as:

\[
\langle u, v \rangle = \int_{\Omega} uv, \text{ for } u, v \in H(\Omega).
\]

The spatial domain \( \Omega \) is discretized as a mesh of triangles using for the Delaunay-Voronoi algorithm, and the regular partition \( T_h \) of \( \Omega \) refers to the family of the triangles that compose the mesh. Given a spatial discretization, the functions are approximated to belong to a finite-element space whose properties correspond to the details of the Hilbert spaces to which the functions belong. We define \( P_h^{(2)} \) as the 2-norm order Lagrange finite-element subspace of \( H(\Omega) \). Now, the physical degrees of freedom can be discretized in their finite element subspaces. And we define the finite-element description of the degrees of freedom as \( n_i \mapsto n_i^{(h)} \in P_h^{(2)}. \) This describes a linear vector space of finite dimension, for which a basis can be found. The canonical basis consists of the shape functions \( \phi_k(x) \), and thus

\[
V_h(T_h, P_h^{(2)}) = \left\{ w(x) = \sum_{k=1}^{M} w_k \phi_k(x), \phi_k(x) \in P_h^{(2)} \right\}.
\]

Here \( M \) is the dimension of \( V_h \) (the number of vertices), the \( w_k \) are called the degrees of freedom of \( w \) and \( M \) the number of the degrees of freedom. To summarize, a given function is approximated as its decomposition: \( w(x) = \sum_{k=1}^{M} w_k \phi_k(x) \), on a given basis of shape functions \( \phi_k(x) \) of the polynomial functions \( P_2 \) for the triangle \( T_i \). The finite element space \( V_h(T_h, P_2) \) hence denotes the space of continuous, piecewise quadratic functions of \( x, y \) on each triangle of \( T_h \).

B. Initial guess: Skyrmions and domain-walls

The skyrmions and the domain-walls are field configurations for the two-dimensional system. More precisely, either for the true two-dimensional system, or for a three-dimensional system with a translational invariance along the third direction \( z \). The initial guess is defined, such that the ground state would be \( m_i = \pm 1 \). The configuration is then rotated using the rotation matrices \( R_i \) and scaled to match the actual ground state \( \tilde{m} \):

\[
m = r_0R_z(-\varphi_0)R_y(-\theta_0)\text{e}^{\iota(n)}/n^{(dw)}.
\]

Here \( r_0 = \|\tilde{m}\|, \theta_0 = \arccos(m_z/r_0) \), and \( \varphi_0 = \arctan(m_y/m_x) \) are the spherical coordinates of the ground state \( \tilde{m} = \arg\min V(m) \), for the potential (S49). Here \( n^{(dw)} \) is unit 3-vectors that encode the information about domain-walls (S60), and \( S(n^{(sk)}) \) (S61) is the function that imprints skyrmions on \( n^{(dw)} \).

The configuration that interpolates between the north and south pole of the unit sphere can be parametrized as follow:

\[
n^{(dw)} = \left( \frac{1 - \Upsilon(x)^2}{2}, \frac{1 - \Upsilon(x)^2}{2}, \Upsilon(x) \right),
\]

\[
\Upsilon(x) = \tanh \left( \frac{x_\perp - x_0}{\xi_{dw}} \right),
\]

where \( \xi_{dw} \) determines the width of the domain-wall. In (S60), \( x_0 \) is the curvilinear abscissa that determines the position of the domain-wall, and \( x_\perp \) is the coordinate perpendicular to the domain-wall. In the absence of domain-walls, then \( n^{(dw)} = (0, 0, 1) \), simply points to the north pole.

The skyrmions are implemented by successively rotating the vector \( n^{(dw)} \). Namely, a set of \( N_{sk} \) skyrmions is
Figure S2. Skyrmion solutions in a time-reversal symmetry broken state, for increasing values of the topological charge $Q(m)$. On each block, the panels on the top row display the texture of the four-fermion order parameter $m$. The panels on the middle row show the associated magnetic field $B$ (S47), and the bottom row display the corresponding charge transferring counter-currents $j_{\text{counter}}$ according to the Ampère’s law. The parameters are the same as in the main body. Note that as can be seen from the densities of arrows, the configurations for $Q(m)=7–10$ are zoomed out, as compared to those for $Q(m)=1–4$. 
realized by successfully composing the rotations according to
\[ S(sk) = \prod_{k=1}^{Nsk} R_z(\Phi_k(x)) R_y(\Theta_k(x)). \quad (S61) \]

Here again, \( R_i \) are the rotation matrices, and the angles defining a given skyrmion are
\[
\begin{align*}
\Phi_k(x) &= Q_k \arctan \left( \frac{y - y_k}{x - x_k} \right), \quad (S62a) \\
\Theta_k(x) &= \pi \exp \left\{ -\frac{(x - x_k)^2 + (y - y_k)^2}{2\xi_{sk}^2} \right\}. \quad (S62b)
\end{align*}
\]

The parameters \((x_k, y_k)\) determine the position of the core of the \(k\)-th skyrmion, of charge \(Q_k\), and \(\xi_{sk}\) determines the size of the skyrmions.

III. ADDITIONAL RESULTS

A. Skyrmions

The model has a great variety of skyrmion solutions with different different topological charges. This can be seen from Fig. S2 that displays several examples of stable skyrmions with topological charges \(Q(m)=1–10\).

B. Domain-walls

The fermion quadrupling resistive state, which precedes the \(s+is\) state, spontaneously breaks the time-reversal symmetry. For the fermion quadrupling order parameter \(m\), the time-reversal operation \((S55)\) implies that a state that breaks the time-reversal symmetry has \(m_y \neq 0\). In such a situation, there exist domain-wall excitations between both \(s \pm is\) states, as illustrated in Fig. S3.

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