Twisted bialgebroids versus bialgebroids from a Drinfeld twist

Andrzej Borowiec\textsuperscript{1} and Anna Pachol\textsuperscript{2,3}

\textsuperscript{1} Institute of Theoretical Physics, University of Wroclaw, pl. M. Borna 9, 50-204 Wroclaw, Poland
\textsuperscript{2} Queen Mary, University of London, School of Mathematics, Mile End Rd, London E1 4NS, UK

E-mail: andrzej.borowiec@ift.uni.wroc.pl and a.pachol@qmul.ac.uk

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Abstract
Bialgebroids (respectively Hopf algebroids) are bialgebras (Hopf algebras) over noncommutative rings. Drinfeld twist techniques are particularly useful in the (deformation) quantization of Lie algebras as well as the underlying module algebras (=quantum spaces). A smash product construction combines both of them into the new algebra which, in fact, does not depend on the twist. However, we can turn it into a bialgebroid in a twist-dependent way. Alternatively, one can use Drinfeld twist techniques in a category of bialgebroids. We show that both the techniques indicated in the title—the twisting of a bialgebroid or constructing a bialgebroid from the twisted bialgebra—give rise to the same result in the case of a normalized cocycle twist. This can be useful for the better description of a quantum deformed phase space. We argue that within this bialgebroid framework one can justify the use of deformed coordinates (i.e. spacetime noncommutativity), which are frequently postulated in order to explain quantum gravity effects.

Keywords: bialgebroids, Drinfeld twist, smash product, deformed phase spaces

1. Introduction

Over the years, the importance of the role of quantum groups and Hopf algebras has been proved in approaches to quantum gravity. They are considered as the tools of noncommutative geometry, which introduces a more general idea of geometry and allows for the natural quantization of manifolds (e.g. spacetime). In some recent works \cite{1, 2}, the idea of the generalization of Hopf algebras into Hopf algebroids \cite{3, 5} in the context of quantum spacetimes has been approached. It is rather well known that the unification of spacetime coordinates with
a Lie algebra of symmetries cannot be done within the category of Lie algebras. The corresponding construction is called a smash (or more generally a cross) product which requires the introduction of the Hopf algebra framework. The algebra of spacetime coordinates is then the Hopf module algebra. A special example of this construction is provided by the so-called phase space algebra (as the Heisenberg algebra in quantum mechanics). The phase space algebra cannot be equipped with a Hopf algebra structure (at most it can be made into unital-non-counital bialgebra). However, such a smash product can be generalized into the bialgebroid. This approach leads to more sophisticated algebraic structures such as Hopf algebroids [3, 5–8]. Recently, quantum (deformed) phase spaces with noncommutative coordinates equipped with bialgebroid (Hopf algebroid) structures have gained some attention in mathematical physics literature.

Deformed quantum phase spaces were considered shortly after the noncommutative spacetimes had been introduced, especially in the context of the κ-deformation [9]. A natural extension of the κ-Poincaré quantum group [9, 10] by the κ-Minkowski commutation relations contains the deformation of the Heisenberg subalgebra (phase space). There have been many constructions of such a deformed phase space, e.g. within the Heisenberg double construction [11] or the smash product construction [12, 13]. Deformed quantum phase spaces are constantly being studied, and a number of interesting papers have appeared recently, like e.g. [14].

Focusing on the special case of deformed quantum phase spaces with noncommutative coordinates satisfying the κ-deformed Minkowski algebra [10, 15], the Hopf algebroid structure was first investigated in [1]. Later on, a more detailed study on the covariance of such space under the action of the κ-deformed Poincaré symmetry within the Hopf algebroid framework was proposed, with the Heisenberg double construction naturally providing the bialgebroid structure [2]. Still, the physical meaning of Hopf algebroids remains a little unclear in this context and requires more studies in this direction. We believe that within the bialgebroid framework one can better justify the deformed Casimir relations, which are frequently used in order to explain some of the quantum gravity effects that come from the spacetime noncommutativity.

Hopf algebroids are Hopf algebras over unital noncommutative rings. One of the oldest definitions of bialgebroids goes back to Sweedler [6] and Takeuchi [7]. Schauenberg [4] also contributed to the topic with his paper on bialgebras over noncommutative rings. However, the concept of Hopf algebroids (bialgebroids with an antipode) was introduced by Lu [3] in 1996. The concept of twisted bialgebroids was first considered by Xu in 2000 [5]. Lu’s definition came as a result of work on grupoids in Poisson geometry, whereas the one by Xu came from the point of view of quantum universal enveloping algebroids (quantum grupoids). It was later shown [16] that these two definitions are equivalent. In 2004, Böhm and Szlachanyi [8] considered pairs of bialgebroids on which the antipode map was defined.

We will be interested in the Drinfeld twist techniques, which are particularly useful in the (deformation) quantization of (complex or real) Lie algebras as well as the underlying module algebras (=quantum spaces). The smash product construction combines both of them into a new algebra, which, in fact, remains isomorphic to itself under twisting. However, if this algebra is turned into a bi- (or Hopf) algebroid, then for the latter this is no longer the case.

Our aim in this note is to show the equivalence between the bialgebroid obtained as a result of the smash product of a twisted triangular bialgebra with the twisted braided commutative module algebra [16] and the one obtained as a result of twisting the smash product in the category of bialgebroids [5].
2. Preliminaries and notations

In this note we shall work in a category of \( K \)-modules, where \( K \) is a base commutative ring with the unit \( 1 \equiv 1_K \). Therefore, all objects are by default \( K \)-modules, and all maps are \( K \)-linear maps. The tensor \( \otimes \) product, if not indicated otherwise, is over the ring \( K \). Particularly interesting cases are when \( K = \mathbb{K} \) is a field (of characteristic 0) or \( K = \mathbb{K}[[h]] \) is a (topological) ring of a formal power series in the (formal) variable \( h \). All rings (algebras) are assumed to be unital. All modules and module maps are assumed to respect the unit. Below, for notational convenience, we shall briefly introduce the main notions involved in our presentation. For deeper study we refer the reader to the literature.

2.1. Smash product construction

Let \( \mathcal{H} = (H, \Delta, \varepsilon, 1_H) \) be a bialgebra and \( A = (A, \ast, 1_A) \) be a left \( H \)-module algebra with the action \( \triangleright : H \otimes A \rightarrow A \) such that: \( 1_H \triangleright a = a, M \triangleright 1_A = \varepsilon(M)1_A \). The smash product algebra \( A \rtimes H \) [17–19] is an algebra determined by the vector space \( A \otimes H \) by the multiplication

\[
(a \otimes b) (b \otimes J) = a(L(1) \triangleright b) \otimes L(2)J,
\]

where \( a, b \in A; L, J \in H \) and \( \Delta(L) = L(1) \otimes L(2) \) in the Sweedler shortcut notation. Obviously, the algebra \( A \rtimes H \) contains algebras \( \ast \otimes \) \( A \rtimes 1 \in A \rtimes H \) and \( H \otimes L \rightarrow 1 \otimes L \in A \rtimes H \) as subalgebras. Later on we shall denote elements from \( A \rtimes H \) of the form \( a \otimes L \) by \( a \triangleright L \). Therefore the previous formula can be rewritten as

\[
(a \triangleright L)(b \triangleright J) = a(L(1) \triangleright b) \times L(2)J. \tag{1}
\]

A special case of this construction provides the algebra of canonical commutation relations between commuting coordinates and momenta generators (see e.g. [20] and the references therein), which are fundamental from the point of view of quantum mechanics (quantum phase space). In the physically motivated examples, this algebra is further extended by the presence of symmetry, e.g. Lorentz generators, which together with the position and momentum generators form the so-called extended spacetime-Poincaré algebra (also called the extended phase space, see e.g. [13] and the references therein). Various applications of the description of quantum gravity effects rely on the suitable (quantum) deformation of both a coordinate algebra as well as a corresponding symmetry (Hopf) algebra (see e.g. [21]). In such cases, besides the traditional position-momentum noncommutativity one also postulates, following [23, 24], the noncommutativity between the position variables and/or, less frequently, between the momentum variables. Such kinds of theories can also be considered as noncommutative versions of quantum mechanics [25, 26]. Drinfeld twist techniques turn out to be useful tools in their construction. This point will be a subject of the present note.

2.2. Quasitriangular Hopf algebras and Drinfeld twist techniques

Let \( (\mathcal{H}, R) \) be a quasitriangular bialgebra [27, 28] with the universal quantum \( R \)-matrix \( R = R_1 \otimes R_2 \in H \otimes H \) satisfying

\[
R \Delta(X) R^{-1} = \Delta^\text{op}(X), \quad (\Delta \otimes \text{id}) R = R_1 R_2, \quad (\text{id} \otimes \Delta) R = R_1 R_2, \quad (\varepsilon \otimes \text{id}) R = (\text{id} \otimes \varepsilon) R = 1
\]

which implies the quantum Yang–Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{3}
\]
As is known [30, 31], any (left) module $A$ over $(H, R)$ automatically becomes a (left–right)
Yetter–Drinfeld module with the right coaction $\delta_R(a) = (R_2 \bowtie a) \otimes R_1$ for all $a \in A^4$.

The category of all (left–right) Yetter–Drinfeld modules $\mathcal{YD}^H$ is a prebraided (and
braided if $H$ is a Hopf algebra) monoidal category [32]. In particular, a left $H$- module algebra
$A = (A, *, I_A)$ is an algebra in $\mathcal{YD}^H$ if and only if it is a braided commutative, i.e.

$$a * b = (R_2 \bowtie b) * (R_1 \bowtie a).$$

Let $F \in H \otimes H$ be a normalized cocycle twist [29] in $(H, R)$, i.e an invertible element which
satisfies the following conditions:

$$F_{12}(\Delta \otimes id)(F) = F_{23}(id \otimes \Delta)(F) \Leftrightarrow F_{12}(F_{(1)}) \otimes F_{23}(F_{(2)}) \otimes F_2 = F_1 \otimes F_1(F_{(2)}) \otimes F_2(F_{(3)})$$

(5)

$$\epsilon \otimes (id)(F) = 1_H \otimes 1_H = (id \otimes \epsilon)(F).$$

(6)

Its inverse satisfies similar conditions (according to our notation $F = F_1 \otimes F_2, \quad F^{-1} = F_1 \otimes F_2$):

$$((\Delta \otimes id)F^{-1})F_{12} = ((id \otimes \Delta)F^{-1})F_{23} \Leftrightarrow (F_{(1)})_1 \otimes (F_{(2)})_2 \otimes F_2
\quad = F_1 \otimes (F_2)_1 \otimes (F_2)_2 \otimes F_2$$

(7)

$$\epsilon \otimes (id)F^{-1} = 1_H \otimes 1_H = (id \otimes \epsilon)F^{-1}.$$  

(8)

The twisting element serves the purpose of deforming both the structure of the bialgebra $\Delta \mapsto \Delta^F = F \Delta F^{-1}$ as well the corresponding module algebra structure $* \mapsto *_{F} = * \circ (F_1 \bowtie F_2 \bowtie)$. We shall denote these new algebras as $\mathcal{H}^F = (H, \Delta^F, \epsilon)$ and $A_F = (A, *_{F})$. Moreover, $(\mathcal{H}^F, R^F \equiv F_2 R F^{-1})$ is quasi-triangular and the module algebra $(A, *_{F}) \in \mathcal{YD}^{H^F}$ if and only if $(A, *) \in \mathcal{YD}^{H}$. Drinfeld twisting techniques are very useful in mathematical physics and noncommutative geometry (see e.g [33]) when looking for new quantum spaces and their quantum symmetries (e.g. [34]).

2.3. Bialgebroids

The bialgebroid $\mathcal{M} = (M, A, s, t, \Delta, \epsilon)$ [3, 5, 8] consists of a total algebra $M$ and a base alge-
bra $A$ as well as the following data:

(B1) Two mappings: an algebra homomorphism $s : A \to M$ called a source map and an
algebra anti-homomorphism $t : A \to M$ called a target map such that: $s(a)t(b) = t(b)s(a)$ is satisfied for all $a, b \in A$.

We consider a left bialgebroid $\mathcal{M}$ as an $A-\text{bimodule}$ (with a bimodule structure which prefers the left-hand side) as follows: $a.m.b = s(a)t(b)m$ for all $a, b \in A, m \in M$.

(B2) Additionally, it is equipped with coproduct and counit maps. The coproduct and counit make $M$ an $A$-coring [16] (with axioms like that of a coalgebra such that all mappings are $A$-bimodule homomorphisms and all tensors are over $A$).

More exactly, the bialgebroid coproduct map $\Delta : M \to M \otimes_A M$ is an $A$-bimodule map,
where $M \otimes_A M$ is constructed in such a way that $(t(a)m) \otimes_A n = m \otimes_A (s(a)n)$; simplifying the notation one can write $(m.a) \otimes_A n = m \otimes_A (a.n)$. This is due to the fact that as an Abelian (additive) group $M \otimes_A M$ is a quotient group of $M \otimes M$ by a subgroup generated by the elements $\{ (t(a) \otimes 1 - 1 \otimes s(a))m \otimes n : a \in A, m, n \in M \}$. This subgroup is, in fact, a left ideal in the

$\Delta(A) = L(1) \otimes L(2)$. For elements $R \in H \otimes H$ we write $R = R_1 \otimes R_2$. If $R$ is invertible we write $R^{-1} = \bar{R}_1 \otimes \bar{R}_2$; $\bar{R}_1 \otimes \bar{R}_2 = \bar{R}_1 \bar{R}_2 \otimes \bar{R}_2 \bar{R}_1 = a_H \otimes a_H$. 


algebra $M \otimes M$. However, $M \otimes_A M$ (unlike $M \otimes M$) is not an algebra in general. To fix this problem one introduces the so-called Takeuchi product $M \times_A M$ [7]. This is defined as a subgroup of the invariant elements $M \times_A M = \{m \otimes_A n \in M \otimes_A M : (mt(a)) \otimes_A n = m \otimes_A n(s(a)); \forall a \in A\}$ which have natural (component-wise) multiplication $((m \otimes_A n)(p \otimes_A q) = mp \otimes_A nq)$. Both $M \otimes_A M$ and $M \times_A M$ inherit the $A$-bimodule structure determined by the action $m \otimes_A n \mapsto (s(a)m) \otimes_A (t(b)n)$, or $a.(m \otimes_A n) \cdot b = (a.m) \otimes_A (n.b)$. Now we can additionally request that the image of the coproduct map is in $M \times_A M$, i.e. it deals, in fact, with the algebra map: 

$$\Delta(mn) = \Delta(m)\Delta(n) \equiv m_{(1)}n_{(1)} \otimes_A m_{(2)}n_{(2)}$$

The counit map $\epsilon : M \to A$ has to satisfy:

$$\epsilon(1_M) = 1_A, \quad \epsilon(mn) = \epsilon(ms(\epsilon(n))) = \epsilon(mt(\epsilon(n))), \quad s(\epsilon(m_{(1)}))m_{(2)} = t(\epsilon(m_{(2)}))m_{(1)} = m. \quad (9)$$

The axioms [3] are similar to those of a bialgebra but are complicated by the possibility of $A$ being a noncommutative algebra, instead of a commutative ring $K$, or its images under $s$ and $t$ are not in the center of $M$.

In the case of Hopf algebroids, one additionally assumes that an antipode $\tau : M \to M$ is to be an algebra anti-automorphism satisfying conditions exchanging the source and target maps and satisfying two axioms similar to the Hopf algebra antipode axioms. Different ways of introducing the antipode map are possible (see e.g. the second reference in [8]).

We recall that a morphism between two bialgebroids: $(M, A, s, t, \Delta, \epsilon)$ and $(M', A', s', t', \Delta', \epsilon')$ over the same algebra $A$ consists of an algebra map $\phi : M \to M'$ such that $\phi \circ s = s', \phi \circ t = t'$, $\epsilon = \epsilon' \circ \phi$ and the following diagram commutes

$$\begin{array}{ccc}
M & \xrightarrow{\phi} & M' \\
\Delta & \downarrow & \Delta' \\
M \otimes_A M & \xrightarrow{\phi \otimes \phi} & M' \otimes_A M'
\end{array} \quad (10)$$

i.e. $\Delta' \circ \phi = (\phi \otimes_A \phi) \circ \Delta$. For the case of Hopf algebroids with an antipode $\tau$ one should also assume that $\tau' \circ \phi = \phi \circ \tau$.

2.4. Smash product algebras as bialgebroids

In [16] it was shown how a Hopf algebroid structure can be associated with a smash product of the Hopf algebra with a braided commutative algebra in the Yetter–Drinfeld $\mathcal{YD}$ category [16]. The theorem (theorem 4.1 in [16]) adapted to our needs reads as follows:

**Theorem 2.1.** Let $H = (H, \Delta, \epsilon)$ be a bialgebra, $A = (A, \ast)$ is a left $H$-module algebra and $(A, \rho)$ a right $H$-comodule.

If $(A, \ast, \rho)^H$ is a braided commutative algebra in $\mathcal{YD}^H$, then $(A \rtimes_H s, t, \Delta, \epsilon, \tau)$ is an $A$-bialgebroid with the source, target, coproduct and the counit given by the following maps:

$$s(a) = a \rtimes 1_H, \quad t(a) \equiv \rho(a) = a_{(0)} \rtimes a_{(1)} \quad (11)$$

$$\Delta(a \rtimes L) = (a \rtimes L_{(1)}) \otimes_A (1_A \rtimes L_{(2)}) \quad (12)$$

5 We also follow the notational convention introduced in the previous section for elements of $M \otimes_A M$.

6 $\rho$ is an algebra map called a coaction: $\rho : A \to A \otimes H^\text{op}$, $\rho(a) = a_{(0)} \otimes a_{(1)}$ in Sweedler notation.
for all \( a \in A \) and \( L \in H \).

Thus, \((a \triangleright L) \triangleright b = a \triangleright (L \triangleright b)\). In particular, \((1_L \triangleright L) \triangleright b = a \triangleright b\) acts by multiplication from the left, while \((1_L \triangleright L) \triangleright b = L \triangleright b\) preserves the initial action.

2.5. Twisted bialgebroids

The category of bialgebroids (Hopf algebroids) was introduced by Xu in [5]. Let us recall the relevant results from [5] on the twist deformation of bialgebroids. Before proceeding further one should remember that the bialgebroid definition provides a canonical action \( \triangleright : M \otimes A \to A^k \):

\[
m \triangleright a = \epsilon(ms(a)) = \epsilon(mt(a)),
\]

induced by the counit \(\epsilon\) (see (9)). We should point out that the multiplication in \(M\) does not change, like in the case of the Drinfeld theory, where twist deformation modifies the coalgebraic sector only. Nevertheless, the one in \(A\) changes (\(\rightarrow\)):

\[
\bar{\Delta} \otimes \bar{\epsilon} \circ \bar{F}.
\]

We are now in position to present the simplified version of (theorem 4.14 in [5]), skipping some details which are not relevant for our considerations (following the Drinfeld convention our twist is inverse with respect to the one considered by Xu in [5].):

**Theorem 2.2.** Assume that \((M, A, s, t, \Delta, \epsilon)\) is a bialgebroid over the algebra \(A\) and \(F = F_1 \otimes_A F_2 \in M \otimes_A M\) is a ‘twistor’ (a Hopf algebroid twist). Then \((M, A_F, s_F, t_F, \Delta_F, \epsilon)\) is a bialgebroid over the algebra \(A_F\), where

\[
s_F(a) = s(F_1 \triangleright a)F_2 \quad ; \quad t_F(a) = t(F_2 \triangleright a)F_1 \quad \forall \ a \in A.
\]

and a new twisted coproduct \(\Delta_F : M \to M \otimes_{A_F} M\):

\[
\Delta_F(m) = F^*(\Delta(m)F^{-1}) \quad \forall \ m \in M.
\]

The map \(F^* : M \otimes_A M \to M \otimes_{A_F} M\) is defined by (see corollary 4.4 in [5]):

\[
F^*(m \otimes_A n) = (F_1 m) \otimes_{A_F} (F_2 n).
\]

3. Main result

In this section we are going to revisit bialgebroids in the context of the twist deformation of smash product algebras. It appears, according to the construction of Brzeziński–Militaru from [16], that smash product algebras, under suitable assumptions, can be equipped with bialgebroid structures. Moreover, we are going to show that the bialgebroid obtained by the bialgebroid twisting [5] of the smash product algebra and bialgebroid obtained from the smash product algebra of the twisted bialgebra with its twisted module algebra are equivalent (isomorphic).

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7 It should be noted that Xu’s paper concerns, in fact, bialgebroids (his definition does not include the antipode map).

8 This action is sometimes referred to as an anchor \(M \triangleright m \mapsto m \in \text{End} A\), see [16].

9 This satisfies the same Drinfeld conditions (5)–(8) with \(
\otimes\) replaced by \(\otimes_A\).
Let $\mathcal{H} \equiv (H, \Delta, \epsilon)$ be a bialgebra and $A \equiv (A, \ast)$ be a (left) module algebra over $\mathcal{H}$. We denote the corresponding (left) action as $\triangleright : A \otimes H \to A$:

$$L \triangleright (a \ast b) = (L_{(1)} \triangleright a) \ast (L_{(2)} \triangleright b).$$

Assume that $F = F_1 \otimes F_2 \in H \otimes H$ is a normalized cocycle twist for $\mathcal{H}$. This allows us to construct new bialgebra $H^F \equiv (H, \Delta^F, \epsilon)$ and new module algebra $A^F \equiv (A, \ast_F)$ with the same action, where $\Delta^F = F \Delta F^{-1}$ and $\ast_F = \ast(F^{-1} \circ (\triangleright \otimes \triangleright))$ with $F^{-1} = F_1 \otimes F_2$. Actually, as far as the smash product is concerned, it turns out that:

**Proposition 3.1.** For any Drinfeld twist $F$, two smash product algebras $A \bowtie H$ and $A \bowtie H^F$ are isomorphic, even though the algebras $A$ and $A^F$ are not isomorphic, and $H$ and $H^F$ are not isomorphic as bialgebras (see e.g. [13]).

Although this fact seems to be known, e.g. [35], we provide the proof for completeness.

**Proof.** Firstly, we recall that both algebras are determined on the same $K$-module $A \otimes H$ but differ by the multiplications (see (1)):

$$(a \times L) \ast_F (b \times J) = a \ast (L_{(1)} \triangleright b) \times L_{(2)} J, \quad (a \times L) \ast_F (b \times J) = a \ast_F (L_{(1)} \triangleright b) \times L_{(2)} J$$

(18)

where $\Delta^F(L) = F \Delta(L) F^{-1} = L_{(1)} F_{(2)} \otimes L_{(2)} F_{(2)}$ is the twisted coproduct of the bialgebra $H^F$. Both algebras are generated by simpler elements: $a \times l_H, a \in A$ and $l_H \times l_H, l \in H$, i.e. $a \times L = (a \times l_H) \ast (l_H \times L) = (a \times l_H) \ast_F (l_H \times L)$. Of course, the unit $1_H \times 1_H$ is the same for both multiplications.

The isomorphism $\varphi : A \bowtie H^F \to A \times \mathcal{H}$ can be defined by the formula

$$\varphi(a \times L) = (F_1 \triangleright a) \times F_2 L$$

(19)

such that:

$$\varphi((a \times L) \ast_F (b \times J)) = \varphi(a \times L) \ast \varphi(b \times J)$$

(20)

for all $a, b \in A$ and $l_H, J \in H$. One notices that due to the normalization condition $\varphi(1_H \times L) = 1_H \times L$. The inverse map $\varphi^{-1} : A \times \mathcal{H} \to A \bowtie H^F$ is, of course, given by $\varphi^{-1}(a \times L) = (F_1 \triangleright a) \times F_2 L$.

We begin by checking the equality (20) for some special cases. Firstly, we take

(i) $\varphi((a \times l_H) \ast_F (b \times J)) = \varphi(a \times 1) \ast \varphi(b \times J)$.

In $A \bowtie H^F$: $(a \times 1) \ast_F (b \times J) = (a \ast_F b) \times J$ and in $A \times \mathcal{H}$: $(a \times l_H) \ast (b \times J) = (a \times b) \times J$.

On the one hand, from the above and from (20) we have the following series of equalities:

$$\varphi((a \times l_H) \ast_F (b \times J)) = \varphi((a \ast_F b) \times J) = (F_1 \triangleright (a \ast_F b)) \times F_2 J = F_1 \triangleright (F_2 \triangleright a) \times (F_2 \triangleright b) \times F_2 J = (F_1 \triangleright a) \ast ((F_2)_{(1)} F_1 \triangleright b) \times (F_2)_{(2)} F_2 J$$

where we used the cocycle identity for the inverse twist (7).

On the other hand we have from (20) and the above:
\[ \varphi(a \times 1_H) \ast \varphi(b \times J) = [(\tilde{F}_1 \triangleright a) \times \tilde{F}_2] \ast [(\tilde{F}_1 \triangleright b) \times \tilde{F}_2 J] = (\tilde{F}_1 \triangleright a) \ast ((\tilde{F}_2, 1) \langle \tilde{F}_1 \triangleright b \rangle) \times (\tilde{F}_2)_{(2)} \tilde{F}_2 J = \varphi ((a \times 1_H) \ast_F (b \times J)). \]

As a next step we consider

(ii) \( \varphi ((1_A \times L) \ast_F (b \times J)) = \varphi (1_A \times L) \ast \varphi (b \times J). \)

In \( A_F \otimes \mathcal{H}^F \) one calculates:

\[
(1_A \times L) \ast_F (b \times J) = \left( L_{(1)} \langle \tilde{F}_1 \triangleright b \rangle \right) \times L_{(2)} \tilde{F}_2 J = \varphi^{-1} ((\tilde{L}_{(1)} \langle \tilde{F}_1 \triangleright b \rangle) \times \tilde{L}_{(2)} \tilde{F}_2 J)
\]

where the first equality is due to the normalization condition \( \varepsilon(F_1)F_2 = 1_H = \tilde{F}_1 \varepsilon(\tilde{F}_2). \)

Therefore, one has the following equalities:

\[
\varphi ((1_A \times L) \ast_F (b \times J)) = \varphi ((1_A \times L) \ast ((\tilde{L}_{(1)} \langle \tilde{F}_1 \triangleright b \rangle) \times \tilde{L}_{(2)} \tilde{F}_2 J)) = \varphi (1_A \times L) \ast \varphi (b \times J).
\]

Verification on the remaining pair of generators: \( (a \times 1_H) \ast_F (1_A \times J) = a \times J \) and \( (1_A \times L) \ast_F (1_A \times J) = 1_A \times LJ \) is rather straightforward.

To finish the proof one checks (20) using the above partial results:

\[
\varphi ((a \times L) \ast_F (b \times J)) = \varphi ((a \times 1_H) \ast_F ((\tilde{L}_{(1)} \langle \tilde{F}_1 \triangleright b \rangle) \times \tilde{L}_{(2)} J)) = \varphi (a \times 1_H) \\
\ast \varphi ((\tilde{L}_{(1)} \langle \tilde{F}_1 \triangleright b \rangle) \times \tilde{L}_{(2)} J) \\
= \varphi (a \times 1_H) \ast \varphi ((1_A \times L) \ast_F (b \times J)) = \varphi (a \times 1_H) \ast \varphi (1_A \times L) \\
\ast \varphi (b \times J) \\
= \varphi (a \times L) \ast \varphi (b \times J).
\]

The proof is done. \( \square \)

From now on we assume that \((\mathcal{H}, R)\) is quasi-triangular bialgebra and the algebra \( \mathcal{A} \) is a braided commutative in the category \( \mathcal{YD}_{\mathcal{H}} \), i.e. \( a \ast b = (R_2 \triangleright b) \ast (R_1 \triangleright a) \). Then, according to the Brzezinski–Militaru construction (theorem 2.1, \cite{16}) \( \mathcal{A} \times \mathcal{H} \) is a bialgebroid over the algebra \( \mathcal{A} \) if we define (shifting \( \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \) to \( \tilde{\Delta} : \mathcal{A} \times \mathcal{H} \to (\mathcal{A} \times \mathcal{H}) \otimes (\mathcal{A} \times \mathcal{H}) \))

\[
\tilde{\Delta}(a \times L) = (a \times L_{(1)}) \otimes_A (1_A \times L_{(2)}), \quad s(a) = a \times 1_H, \quad t(a) = (R_2 \triangleright a) \times R_1, \quad \tau(a \times L) = \varepsilon(L)a.
\]

\[
(21)
\]

The bialgebroid counit map \( \tilde{\varepsilon} \) will not change throughout the rest of this note. Following the same idea we can shift the quantum \( R \)-matrix from the bialgebra \( \mathcal{H} \) to the bialgebroid \( \mathcal{A} \times \mathcal{H} \)

\[
R \to \tilde{R} = (1_A \times R_1) \otimes_A (1_A \times R_2) \in (\mathcal{A} \times \mathcal{H}) \otimes_A (\mathcal{A} \times \mathcal{H}).
\]

\[
(22)
\]

One can easily check, using direct calculations, that the properties of the cocycle type will be preserved (see (2)).
\[
(\tilde{\Delta} \otimes \text{id}) \tilde{R} = \tilde{R}_1 \tilde{R}_2, \quad (\text{id} \otimes \tilde{\Delta}) \tilde{R} = \tilde{R}_1 \tilde{R}_2, \quad (\tilde{\varepsilon} \otimes \text{id}) \tilde{R} = (\text{id} \otimes \tilde{\varepsilon}) \tilde{R} = (1_A \times 1_{\eta_1}) \otimes_A (1_A \times 1_{\eta_1})
\]
while the remainder is lost
\[
\tilde{R} \Delta(a \otimes L) \tilde{R}^{-1} = ((R \triangleright a) \otimes L_{(2)}) \otimes_A (1_A \otimes R_2L_{(1)}) = \tilde{\Delta}^q(a \otimes L) = ((R_2 \triangleright a) \otimes R_1L_{(2)}) \otimes_A (1_A \otimes L_{(1)}),
\]
where in the last equation we have used the property \(X \otimes_A s(a)Y = t(a)X \otimes_A Y\). Therefore, any Drinfeld twist \(F = F_1 \otimes F_2 \in H \otimes H\) in the bialgebra \(H\) can also be shifted to the bialgebroid twist \(\tilde{F} \in (\tilde{A} \times \tilde{H}) \otimes_A (\tilde{A} \times \tilde{H})\) by\(^{10}\)
\[
F \mapsto \tilde{F} = (1_A \times F_1) \otimes_A (1_A \times F_2)
\]
which automatically satisfies the bialgebroid cocycle and normalization conditions.

Similarly, the construction of Brzezinski–Militaru (theorem 2.1, [16]) makes \(?A_F \times \tilde{H}^F\) a bialgebroid over the algebra \(A_F\) if we set
\[
\widetilde{\Delta}^F(a \times L) = (a \times L_{(2)}) \otimes_{A_F} (1_A \otimes L_{(1)}), \quad \tilde{s}^F(a) = a \times 1_H, \quad \tilde{t}^F(a) = (R_2^F \triangleright a) \times R_1^F
\]
where \(R^F_2 = F_2R^F_1F_2 \otimes F_1F_2F_2^0\) and the algebra \(A_F\) is a braided commutative as well: \(a \star_F b = (R_2^F \triangleright a) \star_F (R_1^F \triangleright b)\). More explicitly
\[
\widetilde{\Delta}^F(a \times L) = (a \times F_1L_{(1)}F_1) \otimes_{A_F} (1_A \times F_2L_{(2)}F_2).
\]

As a next task, according to Xu (theorem 2.2), one applies bialgebroid twisting in order to obtain a new twisted bialgebroid \((\tilde{A} \times \tilde{H})^\tilde{F}\) by making use of the twist (25):
\[
\tilde{\Delta}_F(a \times J) = \tilde{F}^\#(\tilde{\Delta}(a \times J)F), \quad \tilde{s}_F(a) = (\tilde{F}_1 \triangleright a) \times \tilde{F}_2, \quad \tilde{t}_F(a) = (R_1^F \triangleright a) \times R_2^F\tilde{F}_1
\]
where, in our case, \(\tilde{F}^\#: (\tilde{A} \times \tilde{H}) \otimes_A (\tilde{A} \times \tilde{H}) \rightarrow (\tilde{A} \times \tilde{H}) \otimes_{A_F} (\tilde{A} \times \tilde{H})\) is determined by the formula (see(17))
\[
\tilde{F}^\#((a \times L) \otimes_A (b \times J)) = ((F_1)_{(1)} \triangleright a) \times (F_1)_{(2)}L \otimes_{A_F} (F_2)_{(1)} \triangleright b \times (F_2)_{(2)}J).
\]

Since \((1_A \times M) \triangleright a = M \triangleright a\) for any \(M \in H\), the base algebra is just \(A_F\).

Our goal is to compare the bialgebroids \(A_F \times \tilde{H}^F\) and \((\tilde{A} \times \tilde{H})^\tilde{F}\). In fact we are going to prove the following:

**Theorem 3.1.** Let \((\tilde{H}, R)\) be a quasi-triangular bialgebra; \(A\) stands for a braided commutative module algebra w.r.t. \((\tilde{H}, R)\). Assume that \(F = F_1 \otimes F_2 \in H \otimes H\) is a normalized cocycle twist in \(H\). Then
\[
A_F \times \tilde{H}^F \cong (\tilde{A} \times \tilde{H})^\tilde{F}
\]
are isomorphic bialgebroids, where \(\tilde{F}\) denotes the bialgebroid cocycle twist (25) obtained from \(F\).

**Proof.** All properties of the bialgebroid are fulfilled according to theorems 2.1 and 2.2. The base algebra \(A_F\) is the same on both sides. Therefore, we can use the isomorphism (20) \(\varphi: A_F \times \tilde{H}^F \rightarrow A \times \tilde{H}\) of the total algebras, and at the same time, demonstrate that the following diagram commutes (see (10))

\(^{10}\) Our twist is inverse with respect to the one considered by Xu in [5].
\[ \mathcal{A}_\mathcal{F} \ltimes \mathcal{H}^\mathcal{F} \xrightarrow{\varphi} \mathcal{A} \times \mathcal{H} \]
\[ \Delta^\mathcal{F} \downarrow \quad \downarrow \Delta_{\mathcal{F}} \] (31)

\[ (\mathcal{A}_\mathcal{F} \ltimes \mathcal{H}^\mathcal{F}) \otimes_{\mathcal{A}_\mathcal{F}} (\mathcal{A}_\mathcal{F} \ltimes \mathcal{H}^\mathcal{F}) \xrightarrow{\varphi \otimes \varphi} (\mathcal{A} \times \mathcal{H}) \otimes_{\mathcal{A}_\mathcal{F}} (\mathcal{A} \times \mathcal{H}) \]

i.e. \( \Delta_{\mathcal{F}} \circ \varphi = (\varphi \otimes_{\mathcal{A}_\mathcal{F}} \varphi) \circ \Delta_{\mathcal{F}} \).

The coproduct \( \Delta_{\mathcal{F}} \) can be found in a more explicit form as
\[ \Delta_{\mathcal{F}}(a \times J) = \Delta_{\mathcal{F}}(1_A \times J) = (((F_1)_{(1)} \triangleright a) \times (F_1)_{(2)}A_\mathcal{F}1 \triangleright \tilde{F}_1 \otimes_{\mathcal{A}_\mathcal{F}} (1_A \times F_2J_2\tilde{F}_2) \]
\[ = (1_A \times F_1J_1\tilde{F}_1 \otimes_{\mathcal{A}_\mathcal{F}} (1_A \times F_2J_2\tilde{F}_2)) = \Delta_{\mathcal{F}}(1_A \times J). \]

Next we check
\[ \Delta_{\mathcal{F}}(\varphi(a \times 1_H)) = \Delta_{\mathcal{F}}((F_1 \triangleright a) \times \tilde{F}_2) = \tilde{F}^\#(((F_1)_{(1)} \triangleright a) \times (F_2)_{(1)}\tilde{F}_1 \otimes_{\mathcal{A}_\mathcal{F}} (1_A \times (F_2)_{(2)}\tilde{F}_2)) = \]
\[ \tilde{F}^\#(((\tilde{F}_1)_{(1)}\tilde{F}_1 \triangleright a) \times (\tilde{F}_1)_{(2)}\tilde{F}_2) \otimes_{\mathcal{A}_\mathcal{F}} (1_A \times \tilde{F}_2) = (((F_1)_{(1)}\tilde{F}_1)_{(1)} \triangleright a) \times (F_2)_{(2)}(\tilde{F}_1)_{(2)}\tilde{F}_2) \otimes_{\mathcal{A}_\mathcal{F}} (1_A \times \tilde{F}_2). \]

Since \((\Delta \otimes \text{id})(F)(\Delta \otimes \text{id})(F^{-1}) = 1_H \otimes 1_H\), the last expression simplifies to
\[ (((\tilde{F}_1)_{(1)}\tilde{F}_1 \triangleright a) \otimes_{\mathcal{A}_\mathcal{F}} (\tilde{F}_2)\otimes_{\mathcal{A}_\mathcal{F}} (1_A \times 1_H)) = \varphi(a \times 1_H) \otimes_{\mathcal{A}_\mathcal{F}} \varphi(1_A \times 1_H) = (\varphi \otimes_{\mathcal{A}_\mathcal{F}} \varphi)(\Delta_{\mathcal{F}}(a \times 1_H)). \]

In order to complete the proof one has to check that \( \varphi(t_F(a)) = s_F(a), \varphi(t_F^{-1}(a)) = t_F(a) \) and \( \overline{\tau} \circ \varphi = \overline{\tau} \). It is not difficult to get these equalities.

For example, for the target maps we have
\[ \varphi(t_F(a)) = (\tilde{F}_1)_{(1)}\tilde{F}_1 \triangleright a) \times F_2R_1^\mathcal{F} = (\tilde{F}_1)_{(1)}\tilde{F}_1 \triangleright F_2R_1^\mathcal{F} = (R_2F_2^\mathcal{F} \triangleright a) \times R_1\tilde{F}_1^\mathcal{F} \]
\[ = t_F(a) \]

using \( R^\mathcal{F} = R_1^\mathcal{F} \otimes R_2^\mathcal{F} = F_2R_1^\mathcal{F} \otimes F_1R_2^\mathcal{F} \) and \( \tilde{F}_1\tilde{F}_2 \otimes \tilde{F}_2\tilde{F}_2 = F^{-1} \).

Therefore the proof is completed. □
3.1. Comments on the crossed product and the Hopf–Galois extension

A smash product \( \mathcal{A} \rtimes \mathcal{H} \) is a particular kind of crossed product algebra \( \mathcal{A} \times \mathcal{H} \), where a convolution invertible map \( \sigma : \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A} \) has to satisfy (in \( \mathcal{A} \)) the so-called 2-cocycle
\[
\sigma_{L(1)} \sigma_{L(2)} = \sigma_{L(2)} \sigma_{L(1)} \quad \text{for any } a \in \mathcal{A} \text{ and } L, J, K \in \mathcal{H}.
\]
These properties allow us to establish the structure of the unital, associative algebra on the vector space \( \mathcal{A} \otimes \mathcal{H} \) with the multiplication
\[
(a \otimes L)(b \otimes J) = a(L(1) \triangleright a) b(L(2), J(1)) \otimes L(3) J(2),
\]
This algebra is denoted as \( \mathcal{A} \rtimes \mathcal{H} \) [36]. It has a natural left \( \mathcal{A} \)-module and right \( \mathcal{H} \)-comodule structure (the so-called normal basis property), which makes it an \( \mathcal{H} \)-comodule algebra (a coring) with the subalgebra \( \mathcal{A} \otimes 1_{\mathcal{H}} = (\mathcal{A} \times \mathcal{H})^{\text{coalg}} \) composed of coinvariants of the coaction.\(^{11}\)

Due to this, a canonical example of the Hopf–Galois extension is provided [36], which, in turn, is an algebraic counterpart of the quantum principal bundle [36–38]. Various twist deformations of such principal bundles have been proposed recently in [38]. Taking the trivial cocycle \( \sigma_{0}(L, J) = \epsilon(L) \epsilon(J) \) one reconstructs the smash product. A natural question which appears now is whether the result of the present section can be extended to the case of the nontrivial cocycle \( \sigma : \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A} \)?

4. The Lie algebra case

A nice, simple illustration of the framework presented in this paper is provided by the Lie algebra \( \mathfrak{g} \) itself. This is also important from the physical point of view as symmetries in physics are described by the Lie algebras. The Lie algebra \( \mathfrak{g} \) can be generalized as \( \mathcal{U}_{\mathfrak{g}} \) — a universal enveloping algebra to a Hopf algebra (with primitive Hopf algebra maps) \( \mathcal{U}_{\mathfrak{g}} = (U_{\mathfrak{g}}, \Delta, \epsilon, S_{0}) \) over the field \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{R} \) of complex or real numbers. Through the deformation procedure (see section 2.1), which requires an extension to \( \mathbb{K}[[\hbar]] \), it becomes a \( \mathcal{U}_{\mathfrak{g}, \hbar} = (U_{\mathfrak{g}}[[\hbar]], \Delta, \epsilon, S) \) deformed quantum symmetry algebra (quantum group) of the corresponding noncommutative quantum spacetime (=Hopf module algebra). In the case of the Lie algebras, there is well known correspondence between the classical and quantum \( r \)-matrices
\[
R = 1 + \hbar r + \mathcal{O}(\hbar^2)
\]
where \( r \in \mathfrak{g} \otimes \mathfrak{g} \) denotes the classical \( r \)-matrix satisfying, due to (3), the classical Yang–Baxter equation (CYYBE). Its skew symmetric part \( r - r_{21} \) describes the Poisson–Lie structure on the corresponding Lie group. In fact, there are two types of quantum deformations of Lie algebras: triangular (nonstandard) and quasi-triangular (standard). The former corresponds to the situation in which \( r \) is skew-symmetric, i.e. \( r \in \mathfrak{g} \otimes \mathfrak{g} \). In this case, the existence of a cocycle \( \sigma_{0}(L, J) = \epsilon(L) \epsilon(J) \) one reconstructs the smash product. A natural question which appears now is whether the result of the present section can be extended to the case of the nontrivial cocycle \( \sigma : \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A} \).

\(^{11}\) For a coring \( M \) over the Hopf algebra \( \mathcal{H} \) with the right coaction \( \rho : M \rightarrow M \otimes \mathcal{H} \) one defines a subalgebra of coinvariant elements, \( M^{\text{coalg}} = \{ h \in M : \rho(m) = m \otimes 1_{\mathcal{H}} \} \). We say that the extension \( M^{\text{coalg}} \subset M \) is an \( \mathcal{H} \)-Hopf–Galois if the map \( M \otimes M^{\text{coalg}} \rightarrow M \otimes H \), given by \( m \otimes n \mapsto (m \otimes 1_{\mathcal{H}}) \rho(n) \), is bijective [36].
which apply to semi-simple Lie algebras, are related to the classical \( r \)-matrices with the skew-symmetric part satisfying the modified classical Yang–Baxter equation (MCYBE).

Representations of the Lie algebra \( g \) provide examples of the module algebras via deformation of the (commutative) algebra of the smooth functions \( \mathcal{A} = C^\infty(V) (= \text{the algebra of spacetime coordinates}) \) on the corresponding vector space \( V \) in the following way. Given that the representation \( \rho \) induces the action on the vector space \( V \):

\[
\rho : g \to \text{End}_k V \quad \Leftrightarrow \quad \triangleright : g \otimes V \to V
\]  

\( \triangleright \) \( v \equiv \rho(L)(v) \). This action can be uniquely extended to the action of the entire universal enveloping algebra \( \triangleright : U_g \otimes V \to V \) (and eventually to its topological extension \( U_g[[h]] \)).

Further extension relies on the possibility of replacing \( V \) by the commutative algebra of smooth functions on \( V \) in the case of finite dimensional representation. Assume \( \{ e_i \}^m_{m = \dim_k V} \) is some basis providing coordinates for the vectors: \( v = x e_i \). Let us denote \( \rho(L) = [L_i] \) as the corresponding matrix. Then we are in position to construct the first order differential operators acting on the manifold \( V \), or more exactly on the algebra of its smooth functions \( C^\infty(V) \):

\[
\tilde{\rho}(L) = -L^\alpha_i x^\alpha \partial_\beta
\]  

(35)

which is in fact a coordinate-independent object. Therefore, this defines an extended action \( \triangleright : U_g \otimes C^\infty(V) \to C^\infty(V) \).

Note that the Leibniz rule

\[
\tilde{\rho}(L)(a \cdot b) = \tilde{\rho}(L)a \cdot b + a \cdot \tilde{\rho}(L)b
\]  

(36)

for all \( a, b \in C^\infty(V) \) is automatically satisfied. And since \( \tilde{\rho}(L) \) is a vector field on \( V \) one can make use of the primitive Hopf algebra structure \( (\Delta(L) = L \otimes 1 + 1 \otimes L) \) and rewrite (36) as

\[
L \triangleright (a \cdot b) = (L_{a_1} \triangleright a) \cdot (L_{a_2} \triangleright b),
\]

which provides the module algebra condition over \( U_g \).

The realization (35) allows us to merge the initial Lie algebra \( g \) with a canonical Heisenberg algebra. The resulting algebra can be represented by the following commutation relations

\[
\begin{align*}
[L_{a_1}, L_{a_0}] &= \gamma^\alpha_{a_0 b} L_{b_\alpha}, \\
[L_{b_0}, p_{\mu}] &= (L_{b_0})_p^{p_\mu} p_{\mu}, \\
[p_{\mu}, p_{\nu}] &= 0
\end{align*}
\]

(37)

\[
\begin{align*}
[L_{a_1}, x^{\mu}] &= -m \delta_\alpha^{\mu} x^{\alpha}, \\
[p_{\mu}, x^{\nu}] &= 1 \delta_\mu^{\nu}, \\
[x^{\mu}, x^{\nu}] &= 0.
\end{align*}
\]

(38)

The first line (37) represents a Lie subalgebra which can be identified as an inhomogeneous extension \( \mathfrak{i}_g \) of the initial Lie algebra \( g \) with respect to the representation \( \rho \). Thus, the unital associative algebra generated by the relations (37) and (38) can be introduced as a smash product \( \mathfrak{Pol}[x^1, \ldots, x^m] \rtimes U_{\mathfrak{i}_g} \), where \( \mathfrak{Pol}[x^1, \ldots, x^m] = U_{\mathfrak{ab}}[x^1, \ldots, x^m] \) is the same as an enveloping algebra of the Abelian Lie algebra \( [x^\mu, x^\nu] = 0 \). This former algebra can be interpreted as an extended (quantum) phase space. The natural Hopf action of \( U_{\mathfrak{i}_g} \) on the module algebra \( \mathfrak{Pol}[x^1, \ldots, x^m] \) is given by the commutators (38): \( L_{a_1} \triangleright x^{\mu} = -(L_{a_1})_p^{p_\mu} p_{\mu} \), \( \triangleright x^{\mu} = 1 \delta_\mu^{\nu} \).

We would like to point out that while extending the Lie algebra (37) by adding the vector space (38) of some representation, we exit beyond the category of Lie algebras. However, we remain in a category of associative unital algebras which include Lie algebras as a subcategory. To be more precise, the obtained algebra is not an enveloping algebra of some Lie algebra. In other words, the unit as a group-like element cannot belong to any Lie algebra. Consequently, instead of bi- (Hopf) algebras, one gets bi- (Hopf) algebroids. Replacing the unit by a central (primitive) Lie-algebraic element, we change the structure in such a way

12 More generally we can assume that \( X \) is a (smooth) \( G \)-manifold, where \( g \) denotes the Lie algebra of \( G \). Then we have the action \( U_g \otimes C^\infty(X) \to C^\infty(X) \) provided by the so-called Killing vector fields.

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that the Lie algebra—and therefore the Hopf algebra—are possible. But neither structure is equivalent (isomorphic) in the algebraic sense. We claim that our construction is more natural for physics, since in the undeformed case it is related to quantum mechanics and representations (infinitesimal version) of the so-called Mackey’s imprimitivity systems.

Any Drinfeld twist in the Hopf algebra $U_{ig}$ can be used to deform the smash product algebra $\mathrm{Pol}[x^1, \ldots, x^n] \rtimes U_{ig}$ in two equivalent ways, as described by theorem 3.1, in order to obtain new quantum phase space. In the process of twist deformation (section 2.2) (requiring the extension of all objects and morphisms to the category of modules over the $K[[\hbar]]$ ring) the Hopf algebra $U_{ig}$ gets a new coproduct and antipods while the underlying module algebra $\mathrm{Pol}[x^1, \ldots, x^n]$ gets a new twist-deformed (noncommutative) star product:

$$a \ast_F b = m \circ F^{-1} \triangleright (a \otimes b) = (\tilde{F}_1 \triangleright a) \cdot (\tilde{F}_2 \triangleright b)$$  \hspace{1cm} (39)

replacing the ordinary (commutative) multiplication of scalar-valued functions. The 2-cocycle condition guarantees the associativity of the corresponding twisted star product (39). Note that the twisted star product $\ast_F$ is a braided commutative. One can check this using the relation: $R = F_1 F^{-1} = \tilde{F}_2 \tilde{F}_1 \otimes \tilde{F}_2 \tilde{F}_1 = R_1 \otimes R_1$. Starting from the definition of the braided commutativity (4), then from a definition of the star product (39)

$$(R_1 \triangleright b) \ast_F (R_1 \triangleright a) = (\tilde{F}_1 \triangleright R_2 \triangleright b) \cdot (\tilde{F}_2 \triangleright R_1 \triangleright a)$$

$$= (\tilde{F}_1 \tilde{F}_2 \triangleright b) \cdot (\tilde{F}_1 \tilde{F}_2 \triangleright a) = (\tilde{F}_2 \triangleright b) \cdot (\tilde{F}_1 \triangleright a)$$

$$= (\tilde{F}_1 \triangleright a) \cdot (\tilde{F}_2 \triangleright b) = a \ast_F b$$

the second line equality is due to the identity: $1 \otimes 1 = F^{-1}F = \tilde{F}_1 \tilde{F}_1 \otimes \tilde{F}_1 \tilde{F}_1$ and the third uses the commutativity and definition of the $\ast_F$ product.

Therefore, $\mathcal{A}_F = (A, \ast_F) \in U_{gh}^2 [\mathbb{D}]^J[h]$ is bi-algebroid with the bialgebroid structure (with all the maps as defined in theorem 2.1). Besides the triangular deformations, one can consider smash product algebras based on the quasi-triangular ones. However, in this case we may not know the quantum $r$-matrix $R$ explicitly (e.g. in the case of non-semi-simple Lie algebras), meaning we are unable to check the braided commutativity of the corresponding module algebra. This happens, e.g., in the case of the celebrated $\kappa$-Poincaré symmetry (see e.g. [39] and the references therein). In this case, it has recently been proposed to construct a Hopf algebroid, which is based on a Heisenberg double instead of the smash product construction [2].

Many examples of the smash product algebras for specific Lie algebras have already been investigated. For example, the $\kappa$-deformation by twists providing the $\kappa$-Minkowski algebra:

$$[\hat{x}^i, \hat{x}^j] = 0, \quad [\hat{\theta}^\mu, \hat{\theta}^\nu] = \frac{i}{\kappa} \hat{\theta}^\mu,$$  \hspace{1cm} (40)

as a covariant quantum space of the extended symmetries and their smash products was investigated in [13] for $U_{\mathrm{gln}h}^{\kappa}$-inhomogeneous general linear algebra, in [40] for $\mathcal{U}_{\kappa}\omega\omega - \kappa$-Poincaré–Weyl algebra (one generator extension of the Poincaré algebra) or in [41] for the case of $U_{\text{kern2}A,h}^{\kappa}$—the conformal algebra.

Such smash product algebras, called extended phase spaces, contain deformed quantum-mechanical phase space (i.e. $\{\hat{P}_\mu, \hat{\theta}^\nu\}$ commutators). The deformation of the momenta-coordinate sector leads to the deformation of the Poincaré Casimir operator and therefore to the deformation of the dispersion relations. This is due to the fact that the standard Casimir operator $P^2$ of the Poincaré algebra no longer satisfies:
once the phase space is deformed. One then looks for another invariant operator—the deformed Casimir operator $C_\kappa$—for which:

$$[M_{\mu\nu}, C_\kappa] = [C_\kappa, P_\mu] = 0; \quad [C_\kappa, \xi_\mu] = 2P_\mu. \quad (42)$$

This will lead to a deformed dispersion relation of the form:

$$C_\kappa + m_\kappa^2 = 0 \quad (43)$$

(for the consequences of this effect, see e.g. [21]; for discussion of the triangular case, see e.g. [22] and the references therein).

In this framework the deformation of the Poincaré Casimir operator (and the corresponding deformation of the dispersion relations) is motivated by the use of the noncommutative coordinates, assuming that the relations (42) are preserved. However, from a purely algebraic point of view one deals with the isomorphic algebraic structures $A_F \rtimes U_{F,h}^F \equiv A \rtimes U_h$; therefore, both commuting and noncommuting coordinates are equally justified.

We argue in this paper that distinguishing between noncommutative coordinates can be in turn dictated by the choice of bialgebroid structure. In such an approach, the deformed extended phase space $A_F \rtimes U_{F,h}^F$ is not isomorphic to the undeformed one $A \rtimes U_h$ as bialgebroids; therefore, the corresponding deformed coordinates are preferred.

### 5. Conclusions

The Hopf algebroids (bialgebroids) have only recently gained attention from a mathematical physics point of view. In this note we focus on one of the physically important cases where the bialgebroid structure arises, namely the smash product construction. We focus on the smash product of the triangular Hopf algebra with the Yetter–Drinfeld module algebra, and then on the smash product of their twist-deformed counterparts. In addition, we investigate the Drinfeld twist techniques in a category of bialgebroids. We prove that these two approaches—the twisting of a bialgebroid or the construction of a bialgebroid from twisted bialgebra—are isomorphic in the case of a normalized cocycle twist.

As a special example, we have presented how to obtain Hopf algebroid from Lie algebra. As it is known, Lie algebras play a special role in physics, and many of them have already been considered in the Hopf algebras framework. The smash product algebras of symmetry algebra (i.e. their corresponding Hopf algebra) with the noncommutative coordinates (Hopf) module algebra are called extended phase spaces. In the case of the twist deformation, such extended phase spaces can easily be equipped with the bialgebroid structure, as shown in section 2.5. However, whether the $\kappa$-Minkowski spacetime (40) and $\kappa$-Poincaré (quasi-triangular nontriangular case) smash product algebra (true $\kappa$-extended phase space) can be equipped with Hopf algebroid maps is still an open issue. Some approaches have been made in this direction in [1, 2]. Another two points to consider in the future would be the accommodation of the antipode map within this formalism (knowing it is possible in the case presented in section 2.4, and that it has already been done in [16]), as well as an extension of the present formalism to a more general cross product construction.

Moreover, there are plenty of other examples of particular interest in physics where the braided commutativity appears naturally, like for example commutative superalgebras (see
e.g. [42]). Also, the very well-known Drinfeld doubles can provide examples of Hopf algebroids (see e.g. [3], theorem 5.1), i.e. the smash product of the Hopf algebra and the module of its Drinfeld double algebra is a Hopf algebroid. A more complete presentation of the physically motivated Hopf algebroids, as well as other more interesting examples have been postponed for another paper.

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