Searching Monotone Multi-dimensional Arrays

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Abstract

In this paper we investigate the problem of searching monotone multi-dimensional arrays. We generalize Linial and Saks’ search algorithm [2] for monotone 3-dimensional arrays to $d$-dimensions with $d \geq 4$. Our new search algorithm is asymptotically optimal for $d = 4$.

Key words: search algorithm, complexity, multi-dimensional array, partial order

1 Introduction

In this paper, we investigate the problem of searching monotone multi-dimensional arrays. Suppose we are given a $d$-dimensional array with $n$ entries along each dimension

$$A_{n,d} = \{a_{i_1,i_2,\ldots,i_d} | i_1,i_2,\ldots,i_d = 1,2,\ldots,n\}.$$  

We say that the array $A_{n,d}$ is monotone if its entries are real numbers that are increasing along each dimension. More precisely, if $i_1 \leq j_1$, $i_2 \leq j_2$, $i_d \leq j_d$ then $a_{i_1,i_2,\ldots,i_d} \leq a_{j_1,j_2,\ldots,j_d}$. In other words, if $P = [n]^d$ is the product of $d$ totally ordered sets $\{1,2,\ldots,n\}$, than $A_{n,d}$ is consistent with the partial order $P$.

The search problem is to decide whether a given real number $x$ belongs to the array $A_{n,d}$ by comparing $x$ with a subset of the entries in the array. The complexity of this problem, denoted by $\tau(n,d)$, is defined to be the minimum over all search algorithms for $A_{n,d}$ of the number of comparisons needed in the worst case. Note that for $d = 1$, this problem reduces to searching a totally ordered set. In this case, the binary search algorithm is optimal and requires at most $\lceil \log_2(n+1) \rceil$ comparisons in the worst case.

We first briefly review some previous work. In [3], Linial and Saks presented some general results on the complexity of the above class of search problems, for any finite partially ordered set $P$. In [2] they studied the problems for general finite partially ordered set $P$ and also gave more precise results for the case where $P = [n]^d$, for dimensions $d \geq 2$. They observed that for $d = 2$, it had been known that $\tau(n,2) = 2n - 1$ [1]. For the generalized case $d \geq 2$, they showed that the order of $\tau(n,d)$ is $O(n^{d-1})$. More specifically, they proved that for $d \geq 2$,

$$c_1(d)n^{d-1} \geq \tau(n,d) \geq c_2(d)n^{d-1} + o(n^{d-1}),$$

where $c_1(d)$ is a nonincreasing function of $d$ and upper bounded by 2, and $c_2(d) = \sqrt{(24/\pi)d^{-1/2} + o(d^{-1/2})}$. The upper bound $c_1(d)n^{d-1}$ was obtained by
using a straightforward search algorithm which partitions \( A_{n,d} \) into \( n \) isomorphic copies of \( A_{n,d-1} \) and searches each copy separately. They also described a more efficient algorithm for \( d = 3 \) and proved the following bounds on \( \tau(n, 3) \):

\[
\left\lfloor \frac{3n^2}{2} \right\rfloor \leq \tau(n, 3) \leq \frac{3n^2}{2} + cn \ln n.
\]

In the above inequality, \( c \) is a positive constant, and so the bounds are asymptotically tight. An open problem left is whether their search algorithm for \( d = 3 \) can be generalized to higher dimensions.

In this paper, we present new search algorithms for monotone \( d \)-dimensional arrays for \( d \geq 4 \), by generalizing the techniques in [2] to higher dimensions. For \( d \geq 4 \), the search complexity of our algorithms is

\[
\tau(n, d) \leq \frac{d}{d-1} n^{d-1} + O(n^{d-2}).
\]

The above bound is tight for \( d = 4 \), up to the lower order terms.

The rest of the paper is devoted to the description and analysis of the new algorithms. We start with the case where \( d = 4 \). This special case best illustrates the main idea, and it is also easier to visualize the subspaces that are encountered in the search algorithm. Then we describe the generalized algorithm for \( d \geq 4 \).

Before presenting the technical details, we describe some basic notation and convention that we will follow throughout the paper. In general, we use capital letters to represent sets and small letters to represent numbers. The sets that we need to consider are often subsets of \( A_{n,d} \) for which some of the subscripts are fixed, and we use some simple notation to represent them. For example, we use \( Q = \{a_{i_2, i_3, i_4}\} \) to denote a “surface” of the 4-dimensional array \( A_{n,4} \) for which the first subscript of \( a \) is fixed to be 1. It is understood that all other subscripts range between \([1, n]\), and we often omit the specification “\( i_2, i_3, i_4 = 1, 2, \ldots, n \)” if it is clear from the context.

## 2 Searching 4-Dimensional Arrays

In this section, we present a \( \frac{4}{3} n^3 + O(n^2) \) algorithm for searching monotone 4-dimensional arrays. The algorithm is optimal up to the lower order terms.

We start with a lower bound on \( \tau(n, 4) \) which will be seen asymptotically tight later, followed by the description of an algorithm for partitioning monotone two-dimensional arrays, which will be a useful subroutine for our searching
algorithm. Then, we will present the main idea and the details of our search algorithm for 4-dimensional arrays.

2.1 A lower bound on $\tau(n, 4)$

Using the method in [2], we can calculate a lower bound on $\tau(n, 4)$. Let $[n]$ denote the totally ordered set $\{1, 2, \ldots, n\}$, and let

$$
D_1(n, 4) = \{(i_1, i_2, i_3, i_4) \in [n]^4 | i_1 + i_2 + i_3 + i_4 = 2n + 1\},
$$

$$
D_2(n, 4) = \{(i_1, i_2, i_3, i_4) \in [n]^4 | i_1 + i_2 + i_3 + i_4 = 2n + 2\}.
$$

Define $D(n, 4) = D_1(n, 4) \cup D_2(n, 4)$. Then $D(n, 4)$ is a section (see [2]) of $[n]^4$, and there is no ordered chain having length more than 2 in $D(n, 4)$. Therefore, $\tau(n, 4)$ is lower bounded by $|D(n, 4)|$. Let

$$
X = \{(i_1, i_2, i_3, i_4) \in [2n + 1]^4 | i_1 + i_2 + i_3 + i_4 = 2n + 1\},
$$

$$
Y_k = \{(i_1, i_2, i_3, i_4) \in X | i_k > n\} \text{ for } k = 1, 2, 3, 4,
$$

$$
Z = \{(i_1, i_2, i_3, i_4) \in [n + 1]^4 | i_1 + i_2 + i_3 + i_4 = n + 1\}.
$$

It is easy to see that $|Y_k| = |Z| = \binom{n}{3}$ for $k = 1, 2, 3, 4$. Thus, $|D_1(n, 4)| = |X| - \sum_{k=1}^4 |Y_k| = \binom{2n}{3} - 4\binom{n}{3} = \frac{4}{3}(2n^3 - 2n)$. Similarly, $|D_2(n, 4)| = \binom{2n+1}{3} - 4\binom{n+1}{3} = \frac{4}{3}(2n^3 + n)$. Therefore,

$$
\tau(n, 4) \geq |D(n, 4)| = |D_1(n, 4)| + |D_2(n, 4)| = \frac{4}{3}n^3 - \frac{n}{3}.
$$

2.2 Partitioning 2-dimensional arrays

In [2], Linial and Saks gave a simple search algorithm for an $m \times n$ matrix $(m, n \geq 1)$ with entries increasing along each row and column. The algorithm needs at most $m + n - 1$ comparisons. We will refer to this algorithm as the Matrix Search Algorithm. Since $A_{n,2}$ is isomorphic to an $n \times n$ matrix, we can adapt the Matrix Search Algorithm to partition $A_{n,2}$ into two subsets $S$ and $L$ given an input $x$, such that $S$ contains entries smaller than $x$ and $L$ contains entries larger than $x$, using at most $2n - 1$ comparisons. Below, we provide the detailed description of the partition algorithm for the sake of completeness.

**Algorithm:** Partition 2-Dimensional Array

**Input.**
Fig. 1. u, v, S, L: partition of the monotone 2-dimensional array $A_{n,2}$

- A real number $x$.
- A monotone 2-dimensional array $A_{n,2} = \{a_{i_1, i_2}\}$.

Output.
- If $x \in A_{n,2}$, output $(i_1, i_2)$ such that $a_{i_1, i_2} = x$.
- If $x \notin A_{n,2}$, output a partition $\{u, v, S, L\}$ of $A_{n,2}$ with the following properties:
  - $u$ and $v$ are two arrays each contains $n$ integers such that $i_1 \leq u[i_2]$ iff $a_{i_1, i_2} < x$ and $i_2 \leq v[i_1]$ iff $a_{i_1, i_2} < x$.
  - $S$ and $L$ form a partition of $\{(i_1, i_2)|i_1, i_2 \in [n]\}$ such that if $(i_1, i_2) \in S$ then $a_{i_1, i_2} < x$, and if $(i_1, i_2) \in L$ then $a_{i_1, i_2} > x$.

Procedure.
- Initially set $S = L = \emptyset$.
- View $A_{n,2}$ as an $n \times n$ matrix and repeat comparing $x$ with the element $e$ at the top right corner of the current matrix.
  - If $x > e$, then eliminate the first row of the current matrix and put their entries into $S$;
  - If $x < e$, then eliminate the last column of the current matrix and put their entries into $L$;
  - If $x = e$, then return this entry and exit.
- Stop when the partition is finished, thus also obtain $u$ and $v$ (see Fig. 1).

We will use the notation $u, v, S, L$ throughout the paper. Sometimes we will introduce subscripts to them to represent the dimension indices to be considered. Ignoring the indices, these four variables have the following useful relations:

$$S = \{(i_1, i_2)|1 \leq i_1 \leq u[i_2]\} = \{(i_1, i_2)|1 \leq i_2 \leq v[i_1]\}$$

$$L = \{(i_1, i_2)|u[i_2] < i_1 \leq n\} = \{(i_1, i_2)|v[i_1] < i_2 \leq n\}.$$
Obviously, \( S \cap L = \emptyset \) and \( S \cup L = [n]^2 \), hence \(|S| + |L| = n^2\). In addition, \(|S| = u[1] + \ldots + u[n] = v[1] + \ldots + v[n] \) and \(|L| = (n-u[1]) + \ldots + (n-u[n]) = (n-v[1]) + \ldots + (n-v[n])\).

Notice that when \( m = 0 \) or \( n = 0 \), we can “search” an \( m \times n \) matrix using 0 comparisons. Therefore, based on the Matrix Search Algorithm in [2], we have the following lemma that will be useful later.

**Lemma 2.1.** For \( m, n \geq 0 \), any \( m \times n \) matrix with entries increasing along each dimension can be searched using at most \( m + n \) comparisons.

**Proof.** If \( m = 0 \) or \( n = 0 \), the matrix is empty, thus needs no comparison. If \( m, n > 0 \), using the Matrix Search Algorithm, we can search the matrix using at most \( m + n - 1 \) comparisons. Therefore, the lemma holds. 

### 2.3 Main idea of the search algorithm

The main idea of our algorithm for \( d = 4 \) is to first search the “surfaces” (3-dimensional arrays) of \( A_{n,4} \) and then the problem reduces to searching a “smaller” array \( A_{n-2,4} \). At a high level, searching the surfaces consists of two major steps:

- **Step 1:** Select 8 special 2-dimensional arrays and partition each into two subsets \( L \) and \( S \), where elements in \( L \) are larger than or equal to \( x \), and elements in \( S \) are smaller than \( x \), using the algorithm Partition 2-Dimensional Array.
- **Step 2:** Search the 8 “surfaces” of \( A_{n,4} \). The subsets \( S, L \) obtained in Step 1 help to “cut” each surface into a sequence of 2-dimensional matrices that allows searching with less comparisons.

### 2.4 Description and analysis of the search algorithm

Now we are ready to present our search algorithm for \( d = 4 \). As explained in Section 2.3, the algorithm is recursive, which reduces \( n \) by two for each recursion. Without loss of generality, we consider the case where \( x \notin A_{n,4} \). We first describe the algorithm and then analyze the number of comparisons needed.

**Step 1.** Apply the algorithm Partitioning 2-Dimensional Array to divide each of the following eight 2-dimensional arrays into two subsets (the eight arrays are defined by fixing two of the subscripts to either 1 or \( n \), thus reducing the number of dimensions by two).
In addition to the eight pairs of subsets, the algorithm also outputs $S_u$ and $S_v$ (for each $i_3 = 1, 2, 3, 4$) have the following properties:

$$a_{i_1,i_2,n} < x < a_{j_1,j_2,n} \quad \text{for } (i_1,i_2) \in S_1 \text{ and } (j_1,j_2) \in L_1$$

$$a_{n,i_2,i_3,1} < x < a_{n,j_2,j_3,1} \quad \text{for } (i_2,i_3) \in S_2 \text{ and } (j_2,j_3) \in L_2$$

$$a_{1,n,i_3,i_4} < x < a_{1,n,j_3,j_4} \quad \text{for } (i_3,i_4) \in S_3 \text{ and } (j_3,j_4) \in L_3$$

$$a_{i_1,1,n,i_4} < x < a_{j_1,1,n,j_4} \quad \text{for } (i_1,i_4) \in S_4 \text{ and } (j_1,j_4) \in L_4$$

$$a_{i_1,i_2,n,1} < x < a_{j_1,j_2,n,1} \quad \text{for } (i_1,i_2) \in S_1^* \text{ and } (j_1,j_2) \in L_1^*$$

$$a_{1,i_2,i_3,n} < x < a_{1,j_2,j_3,n} \quad \text{for } (i_2,i_3) \in S_2^* \text{ and } (j_2,j_3) \in L_2^*$$

$$a_{n,1,i_3,i_4} < x < a_{n,1,j_3,j_4} \quad \text{for } (i_3,i_4) \in S_3^* \text{ and } (j_3,j_4) \in L_3^*$$

$$a_{i_1,1,n,i_4} < x < a_{j_1,1,n,j_4} \quad \text{for } (i_1,i_4) \in S_4^* \text{ and } (j_1,j_4) \in L_4^*$$

In addition to the eight pairs of subsets, the algorithm also outputs $u_k, v_k$ and $u_k^*, v_k^*$, corresponding to $S_k, L_k$ and $S_k^*, L_k^*$ respectively, with the properties given in Equation 1 and 2. For each $k$, at most $2n - 1$ comparisons are needed to partition $M_k$ ($M_k^*$). Thus, at most $8 \times (2n - 1)$ comparisons are needed in this step.

**Step 2.** Search the following eight 3-dimensional surfaces of $A_{n,4}$ (each surface is defined by setting one of the subscripts to either 1 or $n$, thus reducing the number of dimensions by one).

$$Q_1 = \{a_{i_1,i_2,i_3,i_4}\} \quad Q_1^* = \{a_{n,i_2,i_3,i_4}\}$$

$$Q_2 = \{a_{i_1,1,i_3,i_4}\} \quad Q_2^* = \{a_{i_1,n,i_3,i_4}\}$$

$$Q_3 = \{a_{i_1,i_2,1,i_4}\} \quad Q_3^* = \{a_{i_1,i_2,n,i_4}\}$$

$$Q_4 = \{a_{i_1,i_2,i_3,1}\} \quad Q_4^* = \{a_{i_1,i_2,i_3,n}\}$$

By symmetry, we only need to show how to search $Q_1$. The algorithm proceeds by fixing $i_3 = i_3^*$ for $i_3^* = 1, 2, ..., n$ and searching each of the 2-dimensional
Fig. 2. Searching the 3-dimensional surface $Q_1 = \{a_{1,i_2,i_3,i_4}\}$ of $A_{n,4}$. (a): partition of $M_1^* = \{a_{1,i_2,i_3,i_4}\}$ into $S_2^*$ and $L_2^*$; (b): partition of $M_3 = \{a_{1,n,i_3,i_4}\}$ into $S_3$ and $L_3$; (c): the ‘pyramid’ composed of a sequence of 2-dimensional matrices to be searched.

Array $\{a_{1,i_3,i_4}\}$. A useful observation is that for each $i_3$, we can restrict the search to a smaller matrix (in contrast to an $n \times n$ matrix) by leveraging on information obtained in Step 1.

Below, we explain the above observation and Step 2 in more details. Consider an element $a_{1,i_2,i_3,i_4} \in Q_1$. If $(i_2, i_3) \in S_2^*$, then we know that $a_{1,i_2,i_3,i_4} \leq a_{1,i_2,i_3,n} < x$. Hence, in order for $a_{1,i_2,i_3,i_4} = x$, it must be the case that $(i_2, i_3) \in L_2^*$, or equivalently, $u_2[i_3'] < i_2 \leq n$. Similarly, we can conclude that in order for $a_{1,i_2,i_3,i_4} = x$, it must be the case that $(i_3, i_4) \in L_3$, or equivalently, $v_3[i_3'] < i_4 \leq n$. Hence, we obtain a constraint on the indices $(i_2, i_4)$. By lemma 2.1, searching this restricted $(n - u_2[i_3']) \times (n - v_3[i_3'])$ matrix needs at most $(n - u_2[i_3']) + (n - v_3[i_3'])$ comparisons. Notice that $n - u_2[i_3']$ is the number of entries $(i_2, i_3')$'s in $L_2^*$ with $i_3 = i_3'$, and $n - v_3[i_3']$ is the number of entries $(i_3, i_4)$'s in $L_3$ with $i_3 = i_3'$. Thus,

$$(n - u_2[i_3']) + (n - v_3[i_3']) = \{|(i_2, i_3) \in L_2^*| i_3 = i_3'|\} + \{|(i_3, i_4) \in L_3| i_3 = i_3'|\}.$$

When $i_3$ ranges over $1, 2, \ldots, n$, we obtain that the total number of comparisons needed to search $Q_1$ is at most $N(Q_1) = |L_2^*| + |L_3|$ (see Fig. 2).

Similarly, if $a_{n,i_2,i_3,i_4} \in Q_1^*$ equals to $x$, it must be the case that $(i_2, i_3) \in S_2$ and $(i_3, i_4) \in S_3^*$, it follows that the total number of comparisons needed to search $Q_1^*$ is at most $N(Q_1^*) = |S_2| + |S_3^*|$. Using similar arguments, the numbers of comparisons needed for searching the above eight 3-dimensional arrays are:

$$N(Q_1) = |L_3| + |L_2^*|, \quad N(Q_1^*) = |S_2| + |S_3^*|,$$
$$N(Q_2) = |L_4| + |L_3^*|, \quad N(Q_2^*) = |S_3| + |S_4^*|,$$
$$N(Q_3) = |L_1| + |L_4^*|, \quad N(Q_3^*) = |S_4| + |S_1^*|,$$
$$N(Q_4) = |L_2| + |L_1^*|, \quad N(Q_4^*) = |S_1| + |S_2^*|.$$
Therefore, the total number of comparisons needed for searching these eight arrays is at most
\[ 4 \sum_{k=1}^{4} (|S_k| + |L_k| + |S^*_k| + |L^*_k|) = 4 \times 2n^2 = 8n^2. \]

Steps 1 and 2 leave an \((n - 2)^4\) array
\[ A_{n-2,4} = \{a_{i_1, i_2, i_3, i_4} | i_1, i_2, i_3, i_4 = 2, \ldots, n-1\}. \]

Hence, we have for \(n > 2\),
\[ \tau(n, 4) \leq \tau(n - 2, 4) + 8n^2 + 8(2n - 1) \]

From this recursion we can get (see Equation 4 for the derivation)
\[ \tau(n, 4) \leq \frac{4}{3}n^3 + O(n^2). \] (3)

3 Searching \(d\)-Dimensional Arrays

The algorithm for 4-dimensional arrays can be generalized to higher dimensions \((d \geq 4)\). The main idea is quite similar: the \(2d\) “surfaces” \(((d - 1)\)-dimensional arrays) of \(A_{n,d}\) can be searched using \(2dn^{d-2} + O(n^{d-3})\) comparisons. We achieve this in two steps. First, select \(2d\) special \((d - 2)\)-dimensional arrays and partition each of them into two subsets \(S\) and \(L\). Second, we search the \(2d\) “surfaces”. The subsets \(\{S, L\}\) will help cut some part of each surface, i.e., reduce the comparison number. In particular, if we fix \((d - 3)\) subscripts, the resulting part is a smaller matrix (in contrast to an \(n \times n\) matrix). An \(a \times b\) matrix can be searched using at most \(a + b\) comparisons (Lemma 2.1), adding them up for all the \(2d\) “surfaces”, we can get the desired upper bound.

First we describe how to select and partition the \((d - 2)\)-dimensional arrays. Define \(M_1 = \{a_{i_1, i_2, \ldots, i_d} \in A_{n,d} | i_{d-1} = 1, i_d = n\}\). Consider the case where \(x \not\in M_1\). For fixed \(i_2 = i'_2, i_3 = i'_3, \ldots, i_{d-3} = i'_{d-3}\) (where \(i'_2, i'_3, \ldots, i'_{d-3} \in \{1, 2, \ldots, n\}\) are constants) we can get two integer arrays \(u[n]\) and \(v[n]\) such
that
\[
\begin{align*}
  a_{i_1,i_2,\ldots,i_d} |_{i_2=i_2',\ldots,i_{d-3}=i_{d-3}'} &< x \quad \text{for } i_1 \leq u[i_{d-2}]; \\
  a_{i_1,i_2,\ldots,i_d} |_{i_2=i_2',\ldots,i_{d-3}=i_{d-3}'} &> x \quad \text{for } i_1 > u[i_{d-2}]; \\
  a_{i_1,i_2,\ldots,i_d} |_{i_2=i_2',\ldots,i_{d-3}=i_{d-3}'} &< x \quad \text{for } i_{d-2} < v[i_1]; \\
  a_{i_1,i_2,\ldots,i_d} |_{i_2=i_2',\ldots,i_{d-3}=i_{d-3}'} &> x \quad \text{for } i_{d-2} > v[i_1]; \\
\end{align*}
\]

by using the algorithm *Partitioning 2-Dimensional Array*, at most \(2n - 1\) comparisons are needed for each fixed \(i_2, \ldots, i_{d-3}\). Thus using at most \(n^{d-4}(2n - 1)\) comparisons we can get two integer arrays \(u_1\) and \(v_1\) of sizes \(n^{d-3}\) such that
- If \(i_1 \leq u_1[i_2, \ldots, i_{d-2}]\), then \(a_{i_1,i_2,\ldots,i_d} |_{i_{d-1}=1,i_d=n} < x\).
- Otherwise, \(a_{i_1,i_2,\ldots,i_d} |_{i_{d-1}=1,i_d=n} > x\).
- If \(i_{d-2} \leq v_1[i_1, \ldots, i_{d-3}]\), then \(a_{i_1,i_2,\ldots,i_d} |_{i_{d-1}=1,i_d=n} < x\).
- Otherwise, \(a_{i_1,i_2,\ldots,i_d} |_{i_{d-1}=1,i_d=n} > x\).

Thus, we can partition \([n]^{d-2}\) into two subsets \(S_1\) and \(L_1\) such that
- \(a_{i_1,i_2,\ldots,i_d} |_{i_{d-1}=1,i_d=n} < x\) for \((i_1, \ldots, i_{d-2}) \in S_1\).
- \(a_{i_1,i_2,\ldots,i_d} |_{i_{d-1}=1,i_d=n} > x\) for \((i_1, \ldots, i_{d-2}) \in L_1\).

Obviously, we have
- \((i_1, \ldots, i_{d-2}) \in S_1\) if and only if \(i_1 \leq u_1[i_2, \ldots, i_{d-2}]\) (also \(i_{d-2} \leq v_1[i_1, \ldots, i_{d-3}]\)).
- \((i_1, \ldots, i_{d-2}) \in L_1\) if and only if \(i_1 > u_1[i_2, \ldots, i_{d-2}]\) (also \(i_{d-2} > v_1[i_1, \ldots, i_{d-3}]\)).

Next we describe the algorithm for searching \(d\)-dimensional arrays \(A_{n,d}\), for \(d \geq 4\). Without loss of generality, we consider the case where \(x \notin A_{n,d}\).

**Step 1.** Partition each of the following \(2d\) \((d-2)\)-dimensional arrays into two subsets.
\[
\begin{align*}
  M_k &= \{ a_{i_1,i_2,\ldots,i_d} | i_{k-2} = 1, i_{k-1} = n \} : S_k, L_k \\
  M'_k &= \{ a_{i_1,i_2,\ldots,i_d} | i_{k-2} = n, i_{k-1} = 1 \} : S'_k, L'_k
\end{align*}
\]

\(k = 1, 2, \ldots, d\) (here \(i_{k-2}\) means \(i_{(k-2) \mod d}\), and \(i_{k-1}\) means \(i_{(k-1) \mod d}\)).

We get \(2d\) pairs of mutually complementary subsets \(S_k, L_k\) and \(S'_k, L'_k\) with the following properties:
For the pair \( (i_k, \ldots, i_{k+d-3}) \in S_k \) and \( (j_k, \ldots, j_{k+d-3}) \in L_k \):

\[
a_{i_1, \ldots, i_d} |_{i_{k-2}=1, i_{k-1}=n} < x < a_{j_1, \ldots, j_d} |_{j_{k-2}=1, j_{k-1}=n}
\]

for \( (i_k, \ldots, i_{k+d-3}) \in S_k \) and \( (j_k, \ldots, j_{k+d-3}) \in L_k \);

\[
a_{i_1, \ldots, i_d} |_{i_{k-2}=n, i_{k-1}=1} < x < a_{j_1, \ldots, j_d} |_{j_{k-2}=n, j_{k-1}=1}
\]

for \( (i_k, \ldots, i_{k+d-3}) \in S_k^* \) and \( (j_k, \ldots, j_{k+d-3}) \in L_k^* \)

\( k = 1, 2, \ldots, d \) (here \( i_{k+d-3} \) means \( i_{(k+d-3)} \mod d \) etc.).

For the pair \( S_k \) and \( L_k \), we have two \((d-3)\)-dimensional arrays \( u_k \) and \( v_k \) such that, if \( i_k \leq u_k[i_{k+1}, \ldots, i_{k+d-3}] \) then \( (i_k, \ldots, i_{k+d-3}) \in S_k \) else \( (i_k, \ldots, i_{k+d-3}) \in L_k \); if \( i_k \leq v_k[i_{k+1}, \ldots, i_{k+d-3}] \) then \( (i_k, \ldots, i_{k+d-3}) \in S_k \) else \( (i_k, \ldots, i_{k+d-3}) \in L_k \), \( k = 1, 2, \ldots, d \). Similarly, we have \( u_k^* \) and \( v_k^* \) for the pair \( S_k^* \) and \( L_k^* \), for \( k = 1, 2, \ldots, d \).

In this step, we obtain \( 4d \) \((d-3)\)-dimensional arrays \( u_k, v_k, u_k^*, v_k^* \) \((k = 1, 2, \ldots, d)\), using at most \( 2dn^{d-4}(2n-1) \) comparisons.

**Step 2.** Search the following \( 2d \) \((d-1)\)-dimensional surfaces of \( A_{n,d} \), which are defined by fixing one of the subscripts to either 1 or \( n \).

\[
Q_k = \{a_{i_1, \ldots, i_d} | i_k = 1\} \quad Q_k^* = \{a_{i_1, \ldots, i_d} | i_k = n\}
\]

\( k = 1, 2, \ldots, d \).

By symmetry, we only need to consider searching \( Q_1 = \{a_{i_1, i_2, \ldots, i_d}\} \).

If \( a_{i_1, \ldots, i_d} \in Q_1 \) equals to \( x \), we have \((i_3, \ldots, i_d) \in L_3 \) and \((i_2, \ldots, i_{d-1}) \in L_2^* \). For fixed \( i_3 = i'_3, \ldots, i_{d-1} = i'_{d-1} \) (where \( i'_3, \ldots, i'_{d-1} \in \{1, 2, \ldots, n\} \)), there exist two integers \( u = u_2[i'_3, \ldots, i'_{d-1}] \) and \( v = v_3[u'_3, \ldots, u'_{d-1}] \) such that only when \( u < i_2 \leq n, (i'_3, \ldots, i'_d) \in L_2^* \); only when \( v < i_d \leq n, (u'_3, \ldots, u'_d) \in L_3 \). Thus for fixed \( i_3 = i'_3, \ldots, i_{d-1} = i'_{d-1} \), only when \( u < i_2 \leq n \) and \( v < i_d \leq n \), \( a_{i_1, i_2, i'_3, \ldots, i'_{d-1}, i_d} \) possibly equal to \( x \). Searching this \((n-u) \times (n-v)\) matrix needs at most \((n-u) + (n-v)\) comparisons. Notice that \( n-u \) is the number of elements \((i_2, \ldots, i_{d-1})\)'s in \( L_2^* \) with \( i_3 = i'_3, \ldots, i_{d-1} = i'_{d-1} \), and \( n-v \) is the number of elements \((i_3, \ldots, i_d)\)'s in \( L_3 \) with \( i_3 = i'_3, \ldots, i_{d-1} = i'_{d-1} \). Thus \((n-u) + (n-v) = |\{(i'_2, \ldots, i'_{d-1}) \in L_2^* | i'_3 = i'_3, \ldots, i'_{d-1} = i'_{d-1}\}| + |\{(i_3, \ldots, i_d) \in L_3 | i_3 = i'_3, \ldots, i_{d-1} = i'_{d-1}\}| \). When \((i_3, \ldots, i_{d-1})\) ranges over all elements in \([n]^{d-3}\), we obtain that the total number of comparisons needed to search \( Q_1 \) is at most \( N(Q_1) = |L_3| + |L_2^*| \).

Similarly for \( Q_1^* = \{a_{n,i_2, \ldots, i_d}\} \), we have \( N(Q_1^*) = |S_2^*| + |S_3^*| \).

Using similar arguments, the numbers of comparisons needed for searching the
above 2d surfaces are

\[ N(Q_k) = |L_{k+2}| + |L_{k+1}^*| \quad N(Q_k^*) = |S_{k+1}| + |S_{k+2}^*| \]

\( k = 1, 2, \ldots, d \) (here \( L_{k+2} \) means \( L_{(k+2) \mod d} \) etc.).

Thus, searching these 2d \( (d-1) \)-dimensional surfaces needs at most

\[
\sum_{k=1}^{d} \{ |L_{k+2}| + |L_{k+1}^*| + |S_{k+1}| + |S_{k+2}^*| \} = d \times 2n^{d-2} = 2dn^{d-2}
\]

comparisons.

Steps 1 and 2 leave an \((n-2)^d\) \( d \)-dimensional array

\[ A_{n-2,d} = \{a_{i_1, \ldots, i_d} | i_1, \ldots, i_d = 2, \ldots, n - 1 \} \].

Hence the generalized recursion is

\[
\tau(1, d) = 1 \\
\tau(2, d) \leq 2^d \\
\tau(n, d) \leq \tau(n - 2, d) + 2dn^{d-2} + 2dn^{d-4}(2n - 1) \quad \text{for } n > 2
\]

From the recursion, for fixed \( d \) there exist a constant \( C \) and \( \varepsilon \in \{1, 2\} \) such that

\[
\tau(n, d) \leq \tau(n - 2, d) + 2dn^{d-2} + 4dn^{d-3} \\
\leq \tau(n - 4, d) + 2d(n^{d-2} + (n - 2)^{d-2}) + 4d(n^{d-3} + (n - 2)^{d-3}) \\
\leq \tau(n - 4, d) + 2d(n^{d-2} + (n - 2)^{d-2} + \ldots + \varepsilon^{d-2}) + 4d(n^{d-3} + (n - 2)^{d-3} + \ldots + \varepsilon^{d-2}) \\
\leq C + d((n + 1)^{d-2} + n^{d-2} + (n - 1)^{d-2} + \ldots + 1^{d-2}) + 4dn^{d-2} \\
\leq C + d \times \int_{1}^{n+2} t^{d-2} \; dt + 4dn^{d-2} \\
= C + \frac{d}{d - 1} (n + 2)^{d-1} - \frac{d}{d - 1} + 4dn^{d-2} \\
= \frac{d}{d - 1} n^{d-1} + O(n^{d-2})
\]

Therefore,

\[
\tau(n, d) \leq \frac{d}{d - 1} n^{d-1} + O(n^{d-2}) \quad d = 4, 5, \ldots
\] (4)

The following theorem summarizes our main results.
Theorem 3.1 For \( n \geq 1 \) and \( d \geq 4 \), \( \tau(n, d) \leq \frac{d}{d-1}n^{d-1} + O(n^{d-2}) \).
Specially for \( d = 4 \), \( \frac{4}{3}n^3 - \frac{2}{3} \leq \tau(n, 4) \leq \frac{4}{3}n^3 + O(n^2) \).

4 Discussions

In this paper we give an algorithm searching monotone \( d \)-dimensional \((d \geq 4)\) arrays \( A_{n,d} \), which requires at most \( \frac{d}{d-1}n^{d-1} + O(n^{d-2}) \) comparisons. For \( d = 4 \), it is optimal up to the lower order terms.

For \( d = 5 \), let \( D(n, 5) = \{(i_1, i_2, i_3, i_4, i_5) \in [n]^5 | i_1 + i_2 + i_3 + i_4 + i_5 = \left\lfloor \frac{5}{2}(n+1) \right\rfloor \} \cup \{(i_1, i_2, i_3, i_4, i_5) \in [n]^5 | i_1 + i_2 + i_3 + i_4 + i_5 = \left\lfloor \frac{5}{2}(n+1) \right\rfloor + 1 \} \), then \( |D(n, 5)| \) can be calculated to be \( \frac{115}{96}n^4 + O(n^3) \), which is the best lower bound on \( \tau(n, 5) \) currently known. However, applying the techniques in this paper, a \( \frac{115}{96}n^4 + O(n^3) \) search algorithm for \( A_{n,5} \) hasn’t been found (our algorithm requires \( \frac{5}{4}n^4 + O(n^3) \) comparisons in the worst case). So it may be interesting to tighten the bounds for \( d > 4 \).

References

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