Higher-order simple Lie algebras

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Abstract

It is shown that the non-trivial cocycles on simple Lie algebras may be used to introduce antisymmetric multibrackets which lead to higher-order Lie algebras, the definition of which is given. Their generalised Jacobi identities turn out to be satisfied by the antisymmetric tensors (or higher-order ‘structure constants’) which characterise the Lie algebra cocycles. This analysis allows us to present a classification of the higher-order simple Lie algebras as well as a constructive procedure for them. Our results are synthesised by the introduction of a single, complete BRST operator associated with each simple algebra.

1 Introduction

It is well known that, given \([X, Y] := XY – YX\), the standard Jacobi identity (JI) \([[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0\) is automatically satisfied if the product is associative (which will be assumed throughout). For a Lie algebra \(\mathcal{G}\), expressed by the Lie commutators \([X_i, X_j] = C^k_{ij} X_k\) in a certain basis \(\{X_i\}\) \(i = 1, \ldots, r = \dim \mathcal{G}\), the JI implies the Jacobi condition (JC)

\[
\frac{1}{2} \epsilon_{j_1 j_2 j_3} C^\rho_{j_1 j_2} C^\sigma_{j_3} = 0 \quad ,
\]

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on the structure constants. Let \( G \) be simple. Using the Killing metric \( k_{ij} = k(X_i, X_j) \) to lower and raise indices, the fully antisymmetric tensor \( C_{ijk} = C_{ij}^s k_{sk} = k([X_i, X_j], X_k) \) defines a non-trivial Lie algebra three-cocycle. Since it is obtained from \( k \), this three-cocycle is always present \( (H_3^0(G, \mathbb{R}) \neq 0 \) for any \( G \) simple). In fact, it is known since the classical work of Cartan, Pontrjagin, Hopf and others (see in particular [1, 2, 3, 4, 5, 6, 7, 8, 9]) that, from a topological point of view, the group manifolds of all simple compact groups are essentially equivalent to the products of odd spheres [1] that \( S^3 \) is always present in these products and that the simple Lie algebra \( G \)-cocycles are in one-to-one correspondence with bi-invariant de Rham cocycles on the associated compact group manifolds \( G \). The appearance of specific spheres \( S^{2p+1} (p \geq 1) \) other than \( S^3 \) depends on the simple group considered. This is due to the intimate relation between the order of the \( l = \text{rank} \ G \) primitive symmetric polynomials which can be defined on a simple Lie algebra, their \( l \) associated generalised Casimir-Racah invariants [10, 11, 12, 13, 14, 15, 16, 17, 18] and the topology of the associated simple groups, a fact which was found in the eighties to be a key in the understanding of non-abelian anomalies in gauge theories (see [20] for an account of the subject and e.g., [21, 22, 23]).

By looking at the invariant symmetric polynomials on \( G \) we may obtain the higher-order cocycles of the Lie algebra cohomology. These cocycles will turn out to define \( G \)-valued skew-symmetric brackets of even order \( s \) satisfying a generalised Jacobi condition replacing (1.1). Higher-order generalisations of Lie algebras, in the form of the strongly homotopy Lie algebras (SH) of Stasheff [24, 25], have recently appeared in physics. This is the case of the (SH) algebra of products of closed string fields (see [26, 27] and references therein), which involves multilinear, graded-commutative products of \( n \) string fields satisfying certain ‘main identities’ which also generalise the standard Jacobi identity. The higher-order Lie algebras to be discussed in this paper satisfy, however, a generalised Jacobi condition which is a consequence of the assumed associativity of the product of algebra elements and which has also appeared in another, but related context [28]. As a result the definition of the skew-symmetric multibracket to be given in sec. 2 permits, for each even \( s \), the introduction of a

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1 More precisely, if \( G \) is a compact connected Lie group, \( G \) has the (real) cohomology or homology of a product of odd dimensional spheres.

2 In the non-simple case the situation is more involved (see [19]).
coderivation $\partial_s$ of the exterior algebra $\wedge(\mathcal{G})$ constructed on the Lie algebra $\mathcal{G}$. In contrast, the ‘main identities’ of the SH Lie algebras [24, 25] (some detailed expressions can be found in [29]) involve a further extension of our generalised Jacobi identities which in effect describes how the products fail to satisfy them and how the various $\partial_s$ involved in the main identities fail separately to be a coderivation. Our extended higher-order algebras are thus a particular case of the SH Lie algebras in which only one of the $\partial_s$ is non-zero \footnote{Note added: This is also the case of the ‘k-algebras’ [30]. We thank P. Hanlon for sending us this reference.}. We shall now show how introduce them (sec. 2) and present their Cartan–like classification in the simple case (secs. 3,4). In sec. 5 we shall describe our results by introducing the complete BRST operator associated with a simple Lie algebra; some comments concerning applications and extensions will be made in sec. 6.

## 2 Multibrackets and higher-order Lie algebras

Higher-order Lie algebras may be defined by introducing a suitable generalisation of the Lie bracket by means of

**Definition 2.1 (s-bracket)**

Let $s$ be even. A $s$-bracket or skew-symmetric Lie multibracket is a Lie algebra valued $s$-linear skew-symmetric mapping $\mathcal{G} \times \cdots \times \mathcal{G} \to \mathcal{G}$,

$$(X_{i_1}, X_{i_2}, \ldots, X_{i_s}) \mapsto [X_{i_1}, X_{i_2}, \ldots, X_{i_s}] = \omega_{i_1 \ldots i_s} \cdot X_{\sigma}, \quad (2.1)$$

where the constants $\omega_{i_1 \ldots i_s}$ satisfy the condition

$$\epsilon_{i_1 \ldots i_{2s-1}}^{j_1 \ldots j_{2s-1}} \omega_{j_1 \ldots j_s} \cdot \omega_{\rho_{j_{s+1}} \ldots j_{2s-1}} = 0 \quad (2.2)$$

The $\omega_{i_1 \ldots i_s}$ will be called higher-order structure constants, and condition (2.2) will be referred to as the generalised Jacobi condition (GJC); for $s = 2$ it gives the ordinary JC (1.1).

**Remark.** Although we shall only consider here the case of Lie algebras, this definition (as others below) is more general.
The GJC (2.2) is clearly a consistency condition for (2.1). From now on $X_i$ will denote both the algebra basis elements and its representatives in a faithful representation of $G$. Let now $[X_{i_1}, X_{i_2}, \ldots, X_{i_n}]$, $n$ arbitrary, be defined by

$$[X_{i_1}, X_{i_2}, \ldots, X_{i_n}] = \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} X_{i_{\sigma(1)}} X_{i_{\sigma(2)}} \cdots X_{i_{\sigma(n)}} ,$$

(2.3)

where $\pi(\sigma)$ is the parity of the permutation $\sigma$ and the (associative) products on the r.h.s. are well defined as products of matrices or as elements of $U(G)$. Then, the following Lemma holds:

**Lemma 2.1**

Let $[X_1, \ldots, X_n]$, be as in (2.3) above. Then, for $n$ even,

$$\frac{1}{(n-1)! n!} \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} [[X_{\sigma(1)}, \ldots, X_{\sigma(n)}], X_{\sigma(n+1)}, \ldots, X_{\sigma(2n-1)}] = 0$$

(2.4)

is an identity, the *generalised Jacobi identity* (GJI) which for (2.1) implies the GJC (2.2); for $n$ odd, the l.h.s. is proportional to $[X_1, \ldots, X_{2n-1}]$.

**Proof:**

Let $Q_p$ be the antisymmetriser for the symmetric group $S_p$ (*i.e.*, the primitive ‘idempotent’ $[Q^2_p = p!Q_p]$ in the Frobenius algebra of $S_p$, associated with the fully antisymmetric Young tableau). The sum in (2.4) contains $C_{2n-1}^{n-1}$ different terms ($(2n-1)!/(n-1)!n! = C_{2n-1}^{n-1}$). Consider the first of these, $[[X_1, \ldots, X_n], X_{n+1}, \ldots, X_{2n-1}]$. Its full expansion contains $(n!)^2$ terms, which may be written as the sum of $n$ terms

$$[[X_1, \ldots, X_n], X_{n+1}, \ldots, X_{2n-1}] = Q_n(X_1X_2 \cdots X_n)Q_{n-1}(X_{n+1}X_{n+2} \cdots X_{2n-1}) - Q_{n-1}(X_{n+1}Q_n(X_1X_2 \cdots X_n)X_{n+2} \cdots X_{2n-1}) + Q_{n-1}(X_{n+1}X_{n+2}Q_n(X_1X_2 \cdots X_n)X_{n+3} \cdots X_{2n-1}) + \ldots + (-1)^{n-2}Q_{n-1}(X_{n+1}X_{n+2} \cdots X_{2n-2}Q_n(X_1X_2 \cdots X_n)X_{2n-1}) + (-1)^{n-1}Q_{n-1}(X_{n+1}X_{n+2} \cdots X_{2n-1})Q_n(X_1X_2 \cdots X_n)$$

(2.5)

where the antisymmetriser $Q_n [Q_{n-1}]$ acts on the $n \{n-1\}$ indices $(1, \ldots, n)$ $[(n+1, \ldots, 2n-1)]$
only. This sum may be rewritten as

\[
Q_n Q_{n-1} \{ e + (-1)^n(1, n + 1) + (1, n + 1)(2, n + 2) + (-1)^n(1, n + 1)(2, n + 2) \\
\cdot (3, n + 3) + \ldots + (-1)^n(1, n + 1)(2, n + 2) \ldots (n - 2, 2n - 2) + \\
(1, n + 1) \ldots (n - 1, 2n - 1) \} X_1 \ldots X_{2n-1},
\]

where \((i, j)\) indicates the transposition in \(S_{2n-1}\) which interchanges the indices \(i, j\); thus, all the signs in (2.6) are positive for \(n\) even, and they alternate for \(n\) odd according to the parity of the accompanying permutation.

Numerical factors apart, the l.h.s of (2.4) is the result of the action of the \(S_{2n-1}\) antisymmetriser in \((2n-1)\) indices, \(Q_{2n-1}\), on (2.3) or (2.6). Since \(\sigma Q_{2n-1} = (-1)^{\pi(\sigma)} Q_{2n-1} \forall \sigma \in S_{2n-1}\), it turns out that \(Q_{2n-1}(Q_n Q_{n-1}) \propto Q_{2n-1}\). Thus, only the action of \(Q_{2n-1}\) on the curly bracket in (2.6) has to be considered. Since its permutations are half even and half odd, it becomes identically zero for \(n\) even and proportional to \(Q_{2n-1}\) for \(n\) odd, q.e.d.

Lemma 2.1 shows that the higher-order bracket may be defined, as the Lie bracket, by the skew-symmetric product of an (even) number of generators. By analogy with the standard Lie algebra \((s = 2)\) case, we may now give the following

**Definition 2.2** (Higher-order Lie algebra)

Let \(G\) be a Lie algebra. A higher-order Lie algebra on \(G\) is the algebra defined by the \(s\)-bracket (2.1), where the higher-order structure constants satisfy the generalised Jacobi condition (2.2).

Multibrackets appear naturally if we use for the basis \(X_i\) of \(G\) a set of left-invariant vector fields (LIVF) on the group manifold \(G\) of the Lie group \(G\) associated with \(G\). Then, the exterior algebra \(\wedge(G)\) may be identified as the exterior algebra of the LI contravariant, skew-symmetric tensor fields on \(G\) obtained by taking the exterior products of LIVF’s with constant coefficients; this is analogous to the exterior algebra of LI covariant tensor fields (LI forms) on \(G\). Then, in analogy with the exterior derivative of a LI \(q\)-form \(\omega \in \wedge_q(G)\),

\[\text{on } G, \text{ a vector field } X_i \text{ is expressed as } X_i^j(g) \partial/\partial g^j, j = 1, \ldots, r, \] where \(g^j\) are local coordinates of \(G\) at the unity.\footnote{On \(G\), a vector field \(X_i\) is expressed as \(X_i^j(g) \partial/\partial g^j, j = 1, \ldots, r\), where \(g^j\) are local coordinates of \(G\) at the unity.}
an exterior coderivation $\partial : \wedge^q(G) \to \wedge^{q-1}(G)$, $\partial^2 = 0$, may be introduced by taking
\[
\partial(X_1 \wedge \ldots \wedge X_q) = \sum_{1 \leq i < k}^q (-1)^{l+k+1} [X_i, X_k] \wedge X_1 \wedge \ldots \wedge \widehat{X}_i \ldots \wedge \widehat{X}_k \ldots \wedge X_q .
\] (2.7)
For instance, on $X_{i_1} \wedge X_{i_2} \wedge X_{i_3} \in \wedge^3(G)$, the statement $\partial^2(X_{i_1} \wedge X_{i_2} \wedge X_{i_3}) = 0$ is nothing but the standard Jacobi identity.

If we now define $\partial(X_{i_1} \wedge X_{i_2}) = \epsilon^j_{i_1 i_2} X_{j_1} X_{j_2} = [X_{i_1}, X_{i_2}]$, the coderivation $\partial$ above corresponds to $\partial : \wedge^q(G) \to \wedge^{q-1}(G)$. This may now be extended to a general even coderivation $\partial_s$, $\partial_s : \wedge^q(G) \to \wedge^{q-(s-1)}(G)$, $\partial_s^2 = 0$:

**Definition 2.3 (coderivation $\partial_s$)**

Let $s$ be even. The mapping $\partial_s : \wedge^s(G) \to \wedge^1(G) \sim G$ given by $\partial_s : X_1 \wedge \ldots \wedge X_s \mapsto [X_1, \ldots, X_s]$, where the $s$-bracket is given by Def. 2.1, may be extended to a higher-order coderivation $\partial_s : \wedge^n(G) \to \wedge^{n-s+1}$ by
\[
\partial_s(X_1 \wedge \ldots \wedge X_n) = \frac{1}{s!(n-s)!} \epsilon^i_{1 \ldots n} \partial_s(X_{i_1} \wedge \ldots \wedge X_{i_s}) \wedge X_{i_{s+1}} \wedge \ldots \wedge X_{i_n} ,
\] (2.8)
with $\partial_s : \wedge^n(G) = 0$ for $s > n$. It follows from (2.2) that $\partial_s^2 = 0$.

For $s = 2$, eq. (2.8) reduces to (2.7). On $X_{i_1} \wedge \ldots \wedge X_{i_7} \in \wedge^7(G)$, for instance, $\partial_s^2 = 0$ leads to the GJI which must be satisfied by a 4-th order Lie algebra. As mentioned, these higher-order algebras are particular cases of the strongly homotopy algebras [24, 25] of recent relevance in string field theory (see [27]). We shall now give explicit examples of higher-order algebras and, as a result, provide the classification of all higher-order simple Lie algebras.

### 3 Higher-order simple Lie algebras. The case of $su(n)$

Let $G$ be now a simple Lie algebra. In what follows, we shall also assume $G$ to be compact (although compactness is not essential in many reasonings below) so that the non-degenerate Killing matrix $k_{ij}$ may be taken as the unity $\delta_{ij}$ after suitable normalization of the generators.

As mentioned, there are $l$ primitive invariant polynomials for each simple algebra of rank $l$ which are in turn related to the Casimir-Racah operators of the algebra [31, 10, 12, 13, 14, 16, 17, 18, 32, 33], to the Lie algebra cohomology for the trivial action and to the topology and
de Rham cohomology of the associated simple compact Lie group $\{1, 2, 3, ..., 7, 8\}$. We now use this fact to provide a classification of the possible higher-order simple Lie algebras. Given a simple Lie algebra $G$, the orders $m_i$ of the $l$ invariant polynomials (or of the generalised Casimir invariants) and of the $l$ cocycles (or bi-invariant forms on the corresponding compact group $G$) are given by the following table.

| $G$  | algebra dimension $r = \text{dim} G$ | order of invariants $m_1, \ldots, m_l$ | order of $G$-cocycles $(2m_1 - 1), \ldots, (2m_l - 1)$ |
|------|-------------------------------------|--------------------------------------|--------------------------------------------------|
| $A_l$ | $(l + 1)^2 - 1 \ [l \geq 1]$       | $2, 3, \ldots, l + 1$           | $3, 5, \ldots, 2l + 1$                               |
| $B_l$ | $l(2l + 1) \ [l \geq 2]$             | $2, 4, \ldots, 2l$            | $3, 7, \ldots, 4l - 1$                                |
| $C_l$ | $l(2l + 1) \ [l \geq 3]$             | $2, 4, \ldots, 2l$            | $3, 7, \ldots, 4l - 1$                                |
| $D_l$ | $l(2l - 1) \ [l \geq 4]$             | $2, 4, \ldots, 2l - 2, l$       | $3, 7, \ldots, 4l - 5, 2l - 1$                        |
| $G_2$ | 14                              | 2, 6                       | 3, 11                                              |
| $F_4$ | 52                              | 2, 6, 8, 12               | 3, 11, 15, 23                                      |
| $E_6$ | 78                              | 2, 5, 6, 8, 9, 12         | 3, 9, 11, 15, 17, 23                              |
| $E_7$ | 133                             | 2, 6, 8, 10, 12, 14, 18   | 3, 11, 15, 19, 23, 27, 35                         |
| $E_8$ | 248                             | 2, 8, 12, 14, 18, 20, 24, 30 | 3, 15, 23, 27, 35, 39, 47, 59                      |

**Dimension of the Casimir-Racah invariants and Lie algebra cocycles for $G$ simple.**

We see that $\sum_{i=1}^{l}(2m_i - 1) = r$.

**Definition 3.1 (Higher-order simple Lie algebras)**

A higher-order simple Lie algebra associated with a simple Lie algebra $G$ is the higher-order algebra defined by a primitive $G$-cocycle (of order $> 3$) on $G$.

Thus, to find the higher-order simple Lie algebras one has to look for the invariant polynomials on them. For the compact forms on these groups, the cocycle orders are also the dimensions of the primitive de Rham cycles (odd spheres) to which the group manifolds are essentially equivalent. We shall now find explicit realizations of these algebras.

Consider first the case of $su(n)$, $n \geq 3$ with $k_{ij} \sim \delta_{ij}$ (there are no higher-order simple Lie algebras on $su(2)$). In terms of its structure constants (for hermitian generators $T_i$) $[T_i, T_j] = iC_{ijk}T_k$, the anticommutator of two $n \times n$ $su(l + 1)$ matrices may be expressed...
as \( \{T_i, T_j\} = c\delta_{ij} + d_{ijk} T_k \) (with \( c = 1/n \), \( \text{Tr}(T_i T_j) = \frac{1}{2}\delta_{ij} \)). The \( d_{ijk} \propto \text{Tr}(T_i \{T_j, T_k\}) \) term (absent for \( su(2) \)) is the first example of a symmetric invariant polynomial (of 3rd order\(^5\)) beyond the Killing tensor \( k_{ij} \) (see Table). Invariant, symmetric polynomials are given by the symmetric traces (sTr) of products of \( su(n) \) generators (cf. the theory of characteristic classes). Let us then consider the next case, \( m_3 = 4 \). The coordinates of this fourth-order polynomial \( k_{i_1 i_2 i_3 i_4} \) are given by \( s\text{Tr}(T_i \{T_j, T_k\} T_l) \) or (ignoring numerical factors) by \( \text{Tr}(s(T_{i_1} T_{i_2} T_{i_3}) T_{i_4}) \propto \text{Tr}(s(\{T_{i_1}, T_{i_2}\}, T_{i_3}) T_{i_4}) \propto (d_{i_1 i_2 i_3} d_{i_1 i_3}) \) where \( s \) symmetrises the \( i_1, i_2, i_3 \) indices. Thus, we may take

\[
k_{i_1 i_2 i_3 i_4} = d_{i_1 i_2} d_{i_3 i_4} + d_{i_1 i_3} d_{i_2 i_4} + d_{i_1 i_4} d_{i_2 i_3} + 2c(\delta_{i_1 i_2} \delta_{i_3 i_4} + \delta_{i_1 i_3} \delta_{i_2 i_4} + \delta_{i_1 i_4} \delta_{i_2 i_3}) \quad (3.1)
\]

Clearly, the last term will not generate a primitive 4-th order Casimir operator\(^6\) since it is proportional to the square of the second order one, \((I_2)^2\). Eq. (3.1) reflects the well known ambiguity in the selection of the higher-order Casimirs for the simple Lie algebras (see, e.g., \[12, 14, 18, 30\]). The first part, which generalises easily up to \( k_{i_1 \ldots i_n} \) leads to the form of the Casimir-Racah operator \( I_n \) given in \[33\].

We are now in a position to introduce all \( A_l \) higher-order simple Lie algebras

**Theorem 3.1 (Higher-order \( A_l \) Lie algebras)**

Let \( X_i \) a basis of \( A_l, i = 1, \ldots, (l + 1)^2 - 1 \). Then, the even multibracket

\[
[X_{i_1}, \ldots, X_{i_{2m-2}}] := \epsilon^{j_1 \ldots j_{2m-2}}_{i_1 \ldots i_{2m-2}} X_{j_1} \ldots X_{j_{2m-2}} \quad (3.2)
\]

is \( G \)-valued and defines a higher-order simple Lie algebra

\[
[X_{i_1}, \ldots, X_{i_{2m-2}}] = \omega_{i_1 \ldots i_{2m-2}, \sigma} X_\sigma \quad ,
\]

(3.3)

where the higher-order structure constants \( \omega_{i_1 \ldots i_{2m-2}, \sigma} \) associated to the invariant polynomial \( k_{i_1 \ldots i_m} \) are given by the skew-symmetric tensor

\[
\omega_{i_1 \ldots i_{2m-2}, \sigma} = \epsilon^{j_2 \ldots j_{2m-2}}_{i_1 \ldots i_{2m-2}} C_{i_1 j_1} \ldots C_{i_{2m-3} j_{2m-2}} k_{i_1 \ldots i_{2m-2}, \sigma} \quad ,
\]

\[(3.4)\]

---

\(^5\) For the properties of the \( d \)-tensors see \[33\].

\(^6\) Notice that \( l \geq 3 \) for \( k_{i_1 i_2 i_3 i_4} \) to be primitive. For \( su(3) \) the identity \( d_{i_1 i_2} d_{i_1 i_3} = \frac{1}{2}\delta_{i_1 i_2} \delta_{i_1 i_3} \) is not a primitive invariant, since it is produced from \( k_{i_1 i_2 i_3} \). Similar type relations hold for higher ranks \[34\] (see also \[37\], where higher order Casimir operators were introduced as \( W \)-algebras).
which defines a non-trivial \((2m - 1)\)-cocycle for \(su(l + 1)\), \(3 \leq m \leq l + 1\) \((m = 2\) is the standard Lie algebra).

Before presenting a general proof, let us illustrate the theorem in the two simplest cases. For \(m = 2\) eq. (3.4) reads

\[
\omega_{i_1 i_2 \sigma} = \delta_{i_2}^{i_1} C_{i_1 j_2}^{i_3} k_{i_1 \sigma} = k([X_{i_1}, X_{i_2}], X_{i_3}) \, ,
\]

and the \(\omega_{i_1 i_2} \sigma\) are the standard structure constants of \(\mathcal{G}\). Thus, the \(m = 2\) (lowest) polynomial corresponds to the ordinary \((su(n)\), in this case) Lie algebra commutators.

Let \(m = 3\). If \(d\) denotes the symmetric polynomial, eq. (3.4) gives

\[
\omega_{i_1 i_2 i_3 \sigma} = \epsilon_{i_1 j_2 j_3}^{i_4} C_{i_1 j_2}^{i_3} C_{j_3 j_4}^{i_4} d_{i_1 i_2 \sigma} =
\]

\[
= \epsilon_{i_1 j_2 j_3}^{i_4} d([X_{i_1}, X_{j_2}], [X_{j_3}, X_{j_4}], X_{i_3}) \, ,
\]

which is the expression of the fully antisymmetric five-cocycle. On the other hand,

\[
[X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}] = \epsilon_{i_1 j_2 j_3 j_4}^{i_5} X_{j_1} \ldots X_{j_4} = \frac{1}{2^2} \epsilon_{i_1 j_2 j_3}^{i_4 j_4} [X_{j_1}, X_{j_2}, [X_{j_3}, X_{j_4}]
\]

\[
= \frac{1}{2^2} \epsilon_{i_1 j_2 j_3}^{i_4 j_4} C_{j_1 j_2}^{i_3} C_{j_3 j_4}^{i_4} d_{i_1 i_2 \sigma} X_{i_3} X_{i_4} \, .
\]

Taking into account that \(\epsilon_{i_1 j_2 j_3}^{i_4 j_4} C_{j_1 j_2}^{i_3} C_{j_3 j_4}^{i_4}\) is symmetric in \(i, l\) this is equal to

\[
\frac{1}{2^2} \epsilon_{i_1 j_2 j_3}^{i_4 j_4} C_{j_1 j_2}^{i_3} C_{j_3 j_4}^{i_4} (d_{i_1 i_2 \sigma} X_{i_3} + c\delta_{i_1 i_2}) \, .
\]

The term in \(c\) may be dropped since, for each \(j_4\), it is proportional to the antisymmetrised sum \(C_{j_1 j_2}^{i_3}\) \(C_{j_3 j_4}^{i_4}\) in \(j_1, j_2, j_3\) which is zero by the Jacobi identity. Using now that

\[
\epsilon_{i_1 j_2 j_3}^{i_4 j_4} = \sum_{s=1}^{4} (-1)^{s+1} \delta_{i_1}^{j_1} \epsilon_{j_2 \cdots j_4}^{i_4 \cdots i_3} \, ,
\]

it is easy to see that all the terms in (3.3) give the same contribution for the remaining \(d\) term in (3.8). Hence, the fourth-commutator

\[
[X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}] = \frac{1}{2} \epsilon_{i_1 j_2 j_3}^{i_4 j_4} C_{j_1 j_2}^{i_3} C_{j_3 j_4}^{i_4} d_{i_1 i_2 \sigma} X_{i_3} \, X_{i_4} \, .
\]

is indeed of the form (3.6), and it may be checked explicitly that it is in \(su(3)\).

The proof of Theorem 3.1 requires now the following simple
**Lemma 3.1**

If $k_{l_1...l_m}$ is an ad-invariant, symmetric polynomial on a simple Lie algebra $G$,

$$\epsilon^{j_1...j_{2m}} C_{j_1j_2}^{l_1} \cdots C_{j_{2m-1}j_{2m}}^{l_m} k_{l_1...l_m} = 0 \quad (3.11)$$

**Proof:**

First, we note that the ad-invariance condition of the $m$-tensor $k$ may be expressed in coordinates by

$$\sum_{s=1}^{m} C_{j_{2m-ls}l_s}^{l_{s-1}kl_{s+1}...l_m} k_{l_1...l_m} = 0 \quad (3.12)$$

Hence, replacing $C_{j_{2m-ls}l_s}^{l_{s-1}kl_{s+1}...l_m}$ in the l.h.s. of (3.11) by the other terms in (3.12) we get

$$\epsilon^{j_1...j_{2m}} C_{j_1j_2}^{l_1} \cdots C_{j_{2m-1}j_{2m}}^{l_m-1} \left( \sum_{s=1}^{m-1} C_{j_{2m-ls}l_s}^{l_{s-1}kl_{s+1}...l_m} k_{l_1...l_m} j_{2m-ls}l_s \right) = 0 \quad (3.13)$$

which vanishes since all terms in the sum include products of the form $C_{j'j''}^k C_{j}^{k'}$ antisymmetrised in $j, j', j''$, which are zero due to the standard JC (1.1), q.e.d.

To prove now Theorem 3.1, we write the $(2m-2)$ bracket as

$$[X_{l_1}, \ldots, X_{l_{2m-2}}] = \frac{1}{2m-1} \epsilon^{j_1...j_{2m-2}} [X_{j_1}, X_{j_2}] \cdots [X_{j_{2m-3}}, X_{j_{2m-2}}] = \frac{1}{2m-1} \epsilon^{j_1...j_{2m-2}} C_{j_1j_2}^{l_1} \cdots C_{j_{2m-3}j_{2m-2}}^{l_m-1} \frac{1}{(m-1)!} s(X_{l_1}, \ldots, X_{l_{m-1}}) \quad (3.14)$$

where we have used (cf. the $m = 3$ case) that $\epsilon^{j_1...j_{2m-2}} C_{j_1j_2}^{l_1} \cdots C_{j_{2m-3}j_{2m-2}}^{l_m-1}$ is symmetric in $l_1, \ldots, l_{m-1}$ to introduce the symmetrised product of generators, which in turn may be replaced, adding the appropriate factors, by $s(\{ \ldots \{ X_{l_1}, X_{l_2} \}, X_{l_3} \}, \ldots, X_{l_{m-1}} \})$. Using that $\{ X_i, X_j \} = c\delta_{ij} + d_{ijk} X_k$ in the expression of the nested anticommutators, we then conclude that it has the form

$$(\text{factors}) s(X_{l_1}, \ldots, X_{l_{m-1}}) = \hat{k}_{l_1...l_{m-1}1} X_{\sigma} + \hat{k}_{l_1...l_{m-1}} X_{\sigma} \quad (3.15)$$

By Lemma 3.1, the second term does not contribute to (3.14) because $\hat{k}$ is an invariant polynomial of $(m-1)$-order. On the other hand since $\text{Tr}(s(X_{l_1}, \ldots, X_{l_{m-1}}) X_{\sigma}) \propto s \text{Tr}(X_{l_1} \cdots X_{l_{m-1}} X_{\sigma})$, we conclude that $\hat{k}_{l_1...l_{m-1}1}$ is an invariant symmetric $m$-th order polynomial. Absorbing all
numerical factors in $\tilde{k}$ and renaming it as $k$, we find that the $(2m - 2)$-commutator in (3.14) is given by

$$
\frac{1}{(2m - 2)} \epsilon_{i_1 \ldots i_{2m-2}}^j k_{j_1 j_2} \ldots C_{j_{2m-3} j_{2m-2}}^{j_{m-1}} k_{i_1 \ldots i_{m-1}} \sigma X_\sigma = \omega_{i_1 \ldots i_{2m-2}} \sigma X_\sigma
$$

(3.16)
i.e., by the $(2m - 1)$-cocycle (3.4). Since (3.2) is given by the product of associative operators, the GJC (2.2) follows from the GJI (2.4), q.e.d. Equivalently, one may show that the cocycle condition for $\omega_{i_1 \ldots i_{2m-2}} \sigma$ guarantees that the GJC is satisfied (see the Remark after Th. 5.1 below). This establishes the connection between Lie algebra cohomology cocycles and higher-order Lie algebras.

4 Higher-order orthogonal and symplectic algebras

We now extend to the $B_l$ ($l \geq 2$), $C_l$ ($l \geq 3$), $D_l$ ($l \geq 4$) series the considerations in sec. 3 for $A_l$. First we notice that for all of them the third-order symmetric polynomial is absent and that only for the even orthogonal algebra $D_l$ (and odd $l$) we may have an odd-order invariant polynomial. We shall ignore this case for a moment, and look first for the even-order symmetric polynomials. Let us realise the generators of the above algebras in terms of the $n \times n$ matrices of the defining representation, where $n = (2l + 1, 2l, 2l)$ for ($B_l, C_l, D_l$) respectively. These matrices $T$ have all in common the metric preserving defining property $T g = -g T^t$, where $g$ is the $n \times n$ unit matrix for the orthogonal algebras and the symplectic metric for $C_l$. If we define the symmetric third-order anticommutator by

$$\{T_1, T_2, T_3\} = \sum_{\sigma \in S_3} T_{\sigma(1)} T_{\sigma(2)} T_{\sigma(3)} \equiv s(T_1 T_2 T_3)$$

(4.1)
it is trivial to check that $\{T_1, T_2, T_3\} g = -g \{T_1, T_2, T_3\}^t$ so that $\{T_1, T_2, T_3\} \in \mathcal{G}$ ($\mathcal{G} = so(2l + 1), sp(2l)$ or $so(2l)$). Notice that such a relation cannot be satisfied for the ordinary anticommutator, and that in general requires odd-order anticommutators in order to preserve the minus sign in the r.h.s. Note also the absence of the identity matrix in the r.h.s. of the odd-order anticommutator, which was allowed for $A_l$. Let then $\{T_{i_1}, T_{i_2}, T_{i_3}\} = k_{i_1 i_2 i_3} \sigma T_\sigma$. Extending this result to the arbitrary odd case, we find

**Lemma 4.1**
The symmetrised product of an odd number of \( n \times n \) matrix generators of \( so(2l+1) \), \( sp(2l) \) or \( so(2l) \) is also an element of these algebras which is determined by the associated invariant symmetric polynomial.

**Proof:**

\[
s(T_{i_1}T_{i_2}T_{i_3}T_{i_4} \ldots T_{i_{2p-1}}) = \frac{1}{6^{p-1}} s\left( \ldots \{ \{ T_{i_{11}}, T_{i_{12}}, T_{i_{13}} \} T_{i_{14}}, T_{i_{15}} \}, \ldots , T_{i_{2p-2}}, T_{i_{2p-1}} \right)
\]

(4.2)

Since \( s \) symmetrises all \( i_1, i_2, \ldots , i_{2p-1} \) indices we may write this as

\[
\{ T_{i_1}, \ldots , T_{i_{2p-1}} \} = k_{i_1 \ldots i_{2p-1}} \sigma T_\sigma
\]

(4.3)

and identify \( k \) with the invariant symmetric polynomial of even \( 2p \) (see Table) since \( \text{Tr}(\{ T_{i_1}, \ldots , T_{i_{2p-1}} \} T_\sigma) \) is equal to

\[
s\text{Tr}(T_{i_1} \ldots T_{i_{2p-1}} T_\sigma) = k_{i_1 \ldots i_{2p-1}} \sigma
\]

(4.4)

q.e.d.

This now leads to the following

**Theorem 4.1**

Let \( G \) be a simple orthogonal or symplectic algebra. Let \( k_{i_1 \ldots i_{2p}} \) be as in (4.4) for \( 2 \leq p \leq l \) (\( B_l, C_l \)) and \( 2 \leq p \leq l - 1 \) (\( D_l \)). Then, the even \( (4p - 2) \) bracket defined as in (3.2) defines a higher-order orthogonal or symplectic algebra, the structure constants of which are given by the Lie algebra \((4p - 1)\)-cocycles associated with the symmetric invariant polynomials on \( G \).

**Proof:**

It suffices to use Lemma 4.1 and to insert (4.3) in expression (3.14). As a result, the \((4p - 1)\)-cocycle is given again by (3.4) where the \( k_{i_1 \ldots i_{2p-1}} \sigma \) is now found in (4.4), q.e.d.

Let us consider now the order \( l \) invariant for \( so(2l) \). The reasonings before Lemma 4.1 show that, for \( l \) odd, the order \( l \) invariant polynomial cannot be obtained from the symmetric trace of \((l - 1) 2l \times 2l T'\)s, since the symmetrised bracket of an even number of \( T'\)s cannot be expressed as a linear combination of the \( 2l \times 2l \) matrix generators of \( so(2l) \). It is well known, however, that for \( so(2l) \) there is an order \( l \) (even or odd) invariant polynomial (which gives the Euler class of a real oriented vector bundle with even-dimensional fibre) which comes from
the Pfaffian, since \( Pf(ATA^t) = Pf(T) \) for \( A \in SO(2l) \). Using pairs of indices to relabel the generators \( T_i \, i = 1, \ldots, (\frac{2l}{2}) \) as \( T_{\mu\nu} = -T_{\nu\mu} \), \( \mu, \nu = 1, \ldots, 2l \), the order \( l \) invariant (corresponding to the last one in the Table for \( D_l \)) is given by

\[
Pf(T) = \frac{(-1)^l}{2^l l!} \epsilon_{\mu_1\nu_1 \cdots \mu_{2l}} T_{\mu_1\nu_1} T_{\mu_2\nu_2} \cdots T_{\mu_{2l}\nu_{2l}} .
\] (4.5)

The antisymmetric tensor \( \epsilon_{\mu_1\nu_1\mu_2\nu_2 \cdots \mu_{2l}\nu_{2l}} \) defining the invariant is symmetric under the exchange of pairs of indices \( (\mu_i\nu_i) \, i = 1, \ldots, l \). Although it cannot be obtained as the symmetric trace of a product of \( 2l \times 2l \) generators it may be obtained again in the standard way if we use an appropriate spinorial representation for \( so(2l) \). This means that the previous arguments may be also carried through to the \( l \)-th order invariant of the \( D_l \) algebra. To see it explicitly, consider the \( 2^l \)-dimensional Clifford algebra \( \{ \Gamma_\mu, \Gamma_\nu \} = 2\delta_{\mu\nu} \, (\mu, \nu = 1, \ldots, 2l) \).

The \( (\frac{2l}{2}) \) \( Spin(2l) \) generators are given by \( \Sigma_{\mu\nu} = \frac{i}{2}[\Gamma_\mu, \Gamma_\nu] \), and the \( 2l+1 \) matrix by \( \Gamma_{2l+1} = \frac{i}{2l}[\Gamma_{\mu_1}, \Gamma_{\mu_2}, \ldots, \Gamma_{\mu_{2l}} \Gamma_{\mu} \uparrow = \Gamma_\mu, \Gamma_{2l+1} \uparrow = \Gamma_{2l+1} \uparrow \). Thus, we may write with all indices different \( \mu_1 \neq \nu_1 \neq \ldots \neq \mu_{2l-1} \neq \nu_{2l-1} \neq \alpha \neq \beta \),

\[
\Gamma_{\mu_1} \Gamma_{\nu_1} \cdots \Gamma_{\mu_{2l-1}} \Gamma_{\nu_{2l-1}} \propto \epsilon_{\mu_1\nu_1 \cdots \mu_{2l-1}\nu_{2l-1} \alpha \beta} \Gamma_{2l+1} \Gamma_{\alpha} \Gamma_{\beta} .
\] (4.6)

Antisymmetrising the \((l - 1)\) pairs of gammas this leads to

\[
\Sigma_{\mu_1\nu_1} \Sigma_{\mu_2\nu_2} \cdots \Sigma_{\mu_{2l-1}\nu_{2l-1}} \propto \epsilon_{\mu_1\nu_1 \cdots \mu_{2l-1}\nu_{2l-1} \alpha \beta} \Gamma_{2l+1} \Sigma_{\alpha\beta},
\] (4.7)

an expression which is symmetric in the \((\mu\nu)\) pairs which are all different. To check that the definition \((2.3)\) for the \((2l - 2)\) bracket is indeed \( so(2l) \)-valued, we notice that the \( so(2l) \) commutators \( [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] \equiv iC_{(\mu\nu)(\rho\sigma)}^{\lambda\kappa}\Sigma_{\lambda\kappa} \) are non-zero only if the pairs \((\mu\nu), (\rho\sigma)\) have one (and only one) index in common. Thus, the only non-zero \((2l - 2)\)-brackets have the form \( [\Sigma_{i_1k}, \Sigma_{i_2k}, \ldots, \Sigma_{i_{2l-2}k}] \) where all indices \( i \) are different. Since the ordinary product of such \( \Sigma \)'s sharing an index is already antisymmetric, we find that (cf. \((3.14)\))

\[
[S_{i_1k}, S_{i_2k}, \ldots, S_{i_{2l-2}k}] \propto C_{(i_1k)(i_2k)}^{j_1j_2} \cdots C_{(i_{2l-3}k)(i_{2l-2}k)}^{j_{2l-3}j_{2l-2}} \{\Sigma_{j_1j_2}, \ldots, \Sigma_{j_{2l-3}j_{2l-2}}\}
\]

\[
\propto \omega_{i_1k, \ldots, i_{2l-2}k}^{\alpha\beta} \Gamma_{2l+1} \Sigma_{\alpha\beta} ,
\] (4.8)

and we may now use the chiral projectors \( \frac{1}{2}(1 \pm \Gamma_{2l+1}) \) to extract from the reducible \( 2^l \times 2^l \) representation \( \Sigma_{\mu\nu} \) its two irreducible \( 2^{l-1} \times 2^{l-1} \) components.
5 Higher-order simple Lie algebras and their complete BRST operator

The case of the exceptional algebras requires more care, and we shall not discuss here their realization. We may nevertheless state the following

**Theorem 5.1**  
(Classification theorem for higher-order simple algebras)

Given a simple Lie algebra $G$ of rank $l$, there are $(l - 1) (2m_1 - 2)$-higher-order simple algebras associated with $G$. They are given by the $(l - 1)$ Lie algebra cocycles of order $(2m_1 - 1) > 3$ which may be obtained from the $(l - 1)$ symmetric invariant polynomials on $G$ of order $m_1 > m_1 = 2$. The $m_1 = 2$ case (Killing metric) reproduces the original simple Lie algebra $G$; for the other $(l - 1)$ cases, the skew-symmetric $(2m_1 - 2)$-commutators define an element of $G$ by means of the $(2m_1 - 1)$-cocycles. These higher-order structure constants (as the ordinary structure constants with all indices are written down) are fully antisymmetric and satisfy, by virtue of being Lie algebra cocycles, the generalised Jacobi condition (2.2).

**Remark.** It may be checked explicitly that the coordinate definition of the cocycles $\omega_{i_1 \ldots i_{2m-2}}$ and the invariance condition (3.12) for their associated invariant polynomials entail the GJI. Indeed, the l.h.s of (2.2) (for $s = 2m - 2$) is, using (3.4), equal to

$$
\varepsilon^{j_1 \ldots j_{2m-5}} \omega_{j_1 \ldots j_{2m-2}} \epsilon^{l_1 \ldots l_{2m-3}} \rho_{l_2 \ldots l_{2m-5}} C_{j_1 \ldots j_{2m-2}}^{p_1 \ldots p_{2m-3}} = 0 ,
$$

which is zero since if $\omega_{j_1 \ldots j_{2m-2}}$ is a $(2m - 1)$-cocycle (3.4)

$$
\varepsilon^{j_1 \ldots j_{2m-1}} C_{j_1 \ldots j_{2m-1}}^{\rho} = 0 ,
$$

which follows from Lemma 3.1.

There is a simple way of expressing the above results making use of the Chevalley-Eilenberg formulation of the Lie algebra cohomology. For the standard case, we may introduce the BRST operator

$$
s = -\frac{1}{2} \varepsilon^i \varepsilon^j C_{ij}^k \frac{\partial}{\partial c^k} , \quad s^2 = 0 ,
$$

14
with \( c^i c^j = -c^j c^i \) (in a graded algebra case, the \( c \)'s would have a grading opposite to that of the associated generators). Then, \( sc^k = -\frac{1}{2} C^k_{ij} c^i c^j \) (Maurer-Cartan eqs.) and the nilpotency of \( s \) is equivalent to the JC \((1.1)\). In the present case, we may describe all the previous results by introducing the following generalisation:

**Theorem 5.2** (Complete BRST operator for a simple Lie algebra)

Let \( \mathcal{G} \) be a simple Lie algebra. Then, there exists a nilpotent associated operator given by the odd vector field

\[
s = -\frac{1}{2} c^j_1 c^j_2 \sigma \frac{\partial}{\partial \sigma} - \ldots - \frac{1}{(2m_i - 2)!} c^j_1 \ldots c^{j_{2m_i - 2}} \omega_{j_1 \ldots j_{2m_i - 2}} \sigma \frac{\partial}{\partial \sigma} - \ldots
\]

\[
- \frac{1}{(2m_l - 2)!} c^j_1 \ldots c^{j_{2m_l - 2}} \omega_{j_1 \ldots j_{2m_l - 2}} \sigma \frac{\partial}{\partial \sigma} \equiv s_2 + \ldots + s_{2m_i - 2} + \ldots + s_{2m_l - 2},
\]

(5.4)

where \( i = 1, \ldots, l, \omega_{j_1 j_2} \sigma \equiv C^\rho_{j_1 j_2} \sigma \) and \( \omega_{j_1 \ldots j_{2m_l - 2}} \sigma \) are the corresponding \( l \) \((c\text{-number})\) higher-order cocycles. The operator \( s \) will be called the complete BRST operator associated with \( \mathcal{G} \).

**Proof:**

The nilpotency of \( s \) encompasses, in fact, the JC and the \((l - 1)\) GJC’s which have to be satisfied, respectively, by the \( \omega \)'s which determine the standard BRST operator \((5.3)\) and the \((l - 1)\) higher-order BRST operators; all the cohomological information on \( \mathcal{G} \) is contained in the complete BRST operator. The GJC’s come from the squares of the individual terms \( s^2_p \), the crossed products \( s_p s_q \) not contributing since the terms \( s_{2m_i - 2} \) are given by Lie algebra \((2m_i - 1)\)-cocycles. To see this, we first notice that there are no \( \omega \)'s with an even number of indices (\( s \) is an odd operator). Consider now a mixed product \( s_p s_q \) \((p \) and \( q \) even). This is given by

\[
\begin{align*}
    s_p s_q &\propto \omega_{i_1 \ldots i_p} \sigma \rho_{j_1 \ldots j_q} \sigma \sum_{l=1}^{q} (-1)^{l+1} \delta^j_\rho c^{i_1} \ldots \delta^j_\rho c^{i_l} \partial \sigma \\
    &= q \omega_{i_1 \ldots i_p} \sigma \rho_{j_1 \ldots j_q} \sigma \delta^j_\rho c^{i_1} \ldots \delta^j_\rho c^{i_l} \partial \sigma \\
    &= q \omega_{i_1 \ldots i_p} \sigma \rho_{j_1 \ldots j_q} \sigma \rho c^{i_1} \ldots \rho c^{i_l} \partial \sigma
\end{align*}
\]

(5.5)

where the term in \( \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \sigma} \) has been omitted since \((p \) and \( q \) being even\) it cancels with the one coming from \( s_q s_p \). Recalling now expression \((3.4)\) it is found that \((5.5)\) is zero because of \((5.2)\),
which in the present language reads $s_p s_q = 0$. Thus, $s^2 = s_2^2 + \ldots + s_{2m_1-2}^2 + \ldots + s_{2m_l-2}^2 = 0$, each of the $l$ terms being zero separately as a result of the GJC $[2,3]$, q.e.d.

6 Concluding remarks

Many questions arise now that require further study. From a physical point of view it would be interesting to find applications of these higher-order Lie algebras to know whether the cohomological restrictions which determine and condition their existence have a physical significance. Lie algebra cohomology arguments have already been very useful in various physical problems as e.g., in the description of anomalies [20] or in the construction of the Wess-Zumino terms required in the action of extended supersymmetric objects [37]. In the form (3.4), the above formulation of the higher algebras has a resemblance with the closed string BRST cohomology and the SH algebras [24, 25] relevant in the theory of graded string field products [26, 27] (see also [38]). Note, however, that because of the cocycle form of the $\omega$’s, the GJI’s are not modified as already mentioned in the introduction. In the SH algebras such a modification is the result of having, for instance, terms lower than quadratic in (5.4) (with the appropriate change in ghost grading).

Due to their underlying BRST symmetry, similar structures appear in the determination of the different gauge structure tensors through the antibrackets and the master equation in the Batalin-Vilkovisky formalism (for a review, see [39, 40]), where violations of the JI are also present (the Batalin-Vilkovisky antibracket is a two-bracket, but higher-order ones may also be considered [41]).

Other questions may be posed from a purely mathematical point of view. As the discussion in sec. 4 shows, a representation of a simple Lie algebra may not be a representation for the associated higher-order Lie algebras. Thus, the representation theory of higher-order algebras requires a separate analysis. Other problems may be more interesting from a structural point of view as, for instance, the contraction theory of higher-order Lie algebras (which will take us outside the domain of the simple ones), as well as the study of the non-simple higher-order algebras themselves and their cohomology. These, and the generalisation of these ideas to superalgebras (for which there exist simple finite dimensional ones with zero
Killing form) are problems for further research.

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