SYMBOLIC DYNAMICS AND TRANSFER OPERATORS FOR 
WEYL CHAMBER FLOWS: A CLASS OF EXAMPLES

ANKE POHL

Abstract. We provide special cross sections for the Weyl chamber flow on a 
sample class of Riemannian locally symmetric spaces of higher rank, namely 
the direct product spaces of Schottky surfaces. We further present multi-
parameter transfer operator families for the discrete dynamical systems on 
Furstenberg boundary that are related to these cross sections.

1. Introduction

Discretizations of flows on various types of spaces and symbolic dynamics for 
them are useful for many purposes. The applications that motivate this article are 
transfer operator approaches to Laplace eigenfunctions, resonances and dynamical 
zeta functions as, e.g., in [22, 17, 12, 21, 5, 16, 13, 11, 15, 17, 13, 18, 17, 1, 6, 20, 3]. At 
the current state of art, these are limited to hyperbolic spaces, and mostly even 
to hyperbolic surfaces. The necessary discretizations of the geodesic flow on these 
spaces are typically constructed by means of cross sections. For the dynamical 
approaches to Laplace eigenfunctions referred to above it was discovered to be 
crucial to deviate from the classical notion of cross sections. For these applications 
a relaxed notion is used where we require from a cross section that it detects 
all periodic geodesics, but it need not have “time unbounded” intersections with 
all geodesics, in particular not with those that eventually stay in an end of the 
considered space. In contrast, the classical notion would require that a cross 
section has infinitely many intersections with all geodesics in both of their “time 
directions”, past and future. Also transfer operator approaches to dynamical zeta 
functions such as Ruelle and Selberg zeta functions were seen to benefit from this 
more flexible notion of cross section.

In this article we discuss a notion of cross section for the Weyl chamber flow on 
Riemannian locally symmetric spaces that is modelled in analogy to this relaxed 
notion of cross sections for one rank spaces. We construct such cross sections for 
a sample class of spaces. We further provide related discrete dynamical systems on 
the Furstenberg boundary of the Riemannian globally symmetric spaces covering 
these spaces and we present associated multi-parameter transfer operator families. 
This work may be seen as a first step of an attempt towards transfer operator 
approaches for higher rank spaces. However, cross sections for Weyl chamber

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flows are sought for also for other purposes. See, e.g., [8, Question 28]. Our
considerations here might also be of use for such questions.

In a nutshell, we define a cross section for the Weyl chamber flow to be a subset
of the Weyl chamber bundle (or the set of Weyl chambers) that is intersected by
all compact oriented flats and for which each intersection with any oriented flat is
discrete in the regular directions of the flow. The first requirement is motivated by
the fact that compact oriented flats are the Weyl-chamber-flow equivalent of peri-
odic geodesics for the geodesic flow. For spaces of rank one, the compact oriented
flats are precisely the periodic geodesics. The second requirement is motivated
by the idea that intersections should be only momentarily. The restriction of this
requirement to regular directions results from the fact that in singular directions
there are some space dimensions of the considered flat in which no motion happens.
We allow non-discreteness in the space directions without motion.

The sample spaces of higher rank that we consider here are the direct products
of Schottky surfaces. These spaces have a rather simple structure, but nevertheless
their study in regard to our goals is very instructive. For Schottky surfaces, cross
sections for the geodesic flow, induced discrete dynamical systems as well as one-
parameter transfer operator families are well-known. We will take advantage of
this knowledge. As cross section for the geodesic flow on a Schottky surface one
typically takes the set of unit tangent vectors that are based on the boundary of a
standard fundamental domain for the considered surface and that are directed to
the interior of the fundamental domain. We call such a cross section standard for
the moment and refer to Section 3 for details and precise definitions. Our main
results regarding these spaces are essentially as follows.

**Theorem (Coarse statement).** Let \( r \in \mathbb{N} \). For \( j \in \{1, \ldots, r\} \) let \( \mathcal{Y}_j \) be a Schottky
surface and \( \hat{C}_j \) a standard cross section for the geodesic flow on \( \mathcal{Y}_j \). Let
\( \mathcal{Y} := \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_r \)
be the Riemannian locally symmetric space of rank \( r \) which is the direct product of
the Schottky surfaces \( \mathcal{Y}_j \), \( j \in \{1, \ldots, r\} \).

(i) The direct product
\( \hat{C} := \hat{C}_1 \times \cdots \times \hat{C}_r \)
provides a cross section for the Weyl chamber flow on \( \mathcal{Y} \).

(ii) The first return map of \( \hat{C} \) is (semi-)conjugate to a discrete dynamical sys-
tem \( F \) on the Furstenberg boundary of the Riemannian globally symmetric
space \( \mathcal{X} \) covering \( \mathcal{Y} \). The map \( F \) is piecewise given by the action of certain
elements of the fundamental group of \( \mathcal{Y} \).

(iii) The multi-parameter transfer operator family associated to \( F \) can be pro-
vided in explicit form.

The refined version of this theorem features explicit expressions and formulas
at each level. It appears as Theorems 4.1, 4.2 and 5.1 and as Section 6. A few
remarks are in order.

- The choice of \( \hat{C} \) as direct product of the cross sections \( \hat{C}_j \) for the Schottky
surfaces may seem trivial at first glimpse. However, it shows important
properties of cross sections for Weyl chamber flows of spaces of higher
rank. For each \( j \in \{1, \ldots, r\} \), the base point set of \( \hat{C}_j \) is the full boundary of the fundamental domain for \( \mathcal{Y}_j \). The base point set of \( \hat{C} \) is only a small part of the boundary of a fundamental domain for \( \mathcal{Y} \). Thus, the cross section \( \hat{C} \) is a rather sparse subset of the Weyl chamber bundle of \( \mathcal{Y} \), and it does not arise from a Koebe–Morse method (as the \( \hat{C}_j \) do).

- The first return map relies on the notion of “next” intersections. Since the flow is multi-dimensional as soon as the rank \( r \) is larger than 1, the concept of next intersections needs to be discussed with care. It is introduced in Section 2.3.
- The multi-parameter transfer operator family associated to the map \( F \) is not the direct product of the one-parameter transfer operator families for \( \mathcal{Y}_j, j \in \{1, \ldots, r\} \). This family is discussed in detail in Section 6.

This article is structured as follows. In Section 2 we first survey the necessary background on Riemannian globally and locally symmetric spaces. In Section 2.3 we then introduce the notions of cross sections, next intersections, first return map and induced discrete dynamical systems on Furstenberg boundary. Section 3–6 are devoted to the study of our sample class of spaces. In Section 3 we briefly present the well-known cross sections and related objects for Schottky surfaces. In Section 4 we provide and discuss a cross section for the Weyl chamber flow. In Section 5 we introduce the discrete dynamical system on Furstenberg boundary that is induced by a well-chosen set of representatives for the cross section. In the final Section 6 we present the multi-parameter transfer operator family associated to this discrete dynamical system.

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2. Geometry and dynamics of Riemannian locally symmetric spaces

In this section we introduce the elements from the geometry and dynamics of Riemannian globally and locally symmetric spaces that we will use further on. Since there are many excellent treatises of this subject, we will keep our presentation to the bare minimum necessary for our purposes and will omit any proofs. For comprehensive expositions, we refer in particular to [10, 5, 9]. We further present the notion of cross section for the Weyl chamber flow, discuss the concept of next intersections and the existence of the first return map. We also provide the notion of a set of representatives for the cross section and the notion of an associated discrete dynamical system on Furstenberg boundary.

2.1. Globally symmetric spaces. Let \( X \) be a Riemannian symmetric space of noncompact type. A flat of \( X \) is any totally geodesic, flat submanifold of \( X \) of maximal dimension among all such submanifolds. The common dimension of all flats is the rank of \( X \). Let \( F \) be a flat of \( X \), pick a point \( x \in F \) and consider the union of all flats that are distinct from \( F \) but contain the point \( x \). This union intersects \( F \) in a finite union of hyperplanes of \( F \). The connected components of the (open) complement of this union of hyperplanes in \( F \) are the Weyl chambers
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in $F$ with base point $x$. We let

$$C_X := \{ \text{Weyl chamber in } F \text{ with base point } x \mid F \text{ flat, } x \in F \}$$

denote the set of all Weyl chambers of $X$, i.e., the family of all Weyl chambers of all flats with all base points. We note that each Weyl chamber $c \in C_X$ is contained in a unique flat $F$ and has a unique base point $x$.

For the introduction of further objects, in particular the Weyl chamber flow, we take advantage of some elements of the structure theory of $X$. We fix an arbitrary point $x_0$ in $X$, an origin or reference point, let $G$ be the identity component of the group of Riemannian isometries of $X$, and let $K := \text{Stab}_G(x_0)$ denote the stabilizer group of $x_0$ in $G$. Then $G$ is a real semisimple Lie group with finite center, and $K$ is a maximal compact subgroup of $G$. We may identify the symmetric space $X$ with the homogeneous space $G/K$ via the isomorphism

$$G/K \rightarrow X, \quad gK \mapsto g(x_0).$$

We fix a flat $F_0$ through $x_0$, a reference flat. Then there exists a unique maximal abelian subgroup $A$ of $G$ such that $F_0 = A(x_0)$. Indeed, $F_0$ may be identified with $A$ via the isomorphism

$$A \rightarrow F_0, \quad a \mapsto a(x_0).$$

Let

$$\pi_B: C_X \rightarrow X, \quad c \mapsto x_c,$$

denote the map from a Weyl chamber $c$ to its base point $x_c$. Under the isomorphisms from (1) and (3), the map $\pi_B$ becomes the quotient map

$$G/M \rightarrow G/K, \quad gM \mapsto gK,$$

which also turns $C_X$ into a bundle over $X$. We further let $M := Z_K(A)$ denote the centralizer of $A$ in $K$. Then we may interpret the set $C_X$ as the homogeneous space $G/M$ via the isomorphism

$$G/M \rightarrow C_X, \quad gM \mapsto g(c_0).$$

Let

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denote the map from a Weyl chamber $c$ to its base point $x_c$. Under the isomorphisms from (1) and (3), the map $\pi_B$ becomes the quotient map

$$G/M \rightarrow G/K, \quad gM \mapsto gK,$$

which also turns $C_X$ into a bundle over $X$. We further let $M := Z_K(A)$ denote the normalizer of $A$ in $K$, and let $W := M'/M$ denote the Weyl group. With respect to the isomorphism in (3), the action of $G$ on $C_X$ becomes

$$G \times G/M \rightarrow G/M, \quad (h, gM) \mapsto hgM,$$

and the action of the Weyl group $W$ on $C_X$ becomes

$$W \times G/M \rightarrow G/M, \quad (m'M, gM) \mapsto gm'M.$$

We immediately see that the set of all Weyl chambers in $F_0$ with base point $x_0$ are the (finitely many and pairwise distinct) elements of the orbit of the reference Weyl chamber $c_0$ under the Weyl group $W$. Further, the set of all flats of $X$ consists of all $G$-translates of the reference flat $F_0$. The Weyl chamber flow on $X$ is the action of $A$ on $C_X$ which under the isomorphism in (3) becomes

$$A \times G/M \rightarrow G/M, \quad (a, gM) \mapsto gaM.$$

The isomorphism in (2) implies that the Lie group $A$ is isomorphic to $\mathbb{R}^r$, where $r$ is the range of $X$. Thus, the Weyl chamber flow is a flow on $C_X$ with $r$ “time”

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dimensions. The (unoriented) flow “lines” of the Weyl chamber flow on $\mathbb{X}$ are the flats. More precisely, for each each Weyl chamber $gM$, the base point set of its $A$-orbit is

$$\pi_B(gAM) = gAK,$$

which is the unique flat that contains $gM$. For Riemannian symmetric spaces of rang one, the Weyl chamber flow coincides with the geodesic flow.

In what follows we introduce a rather coarse notion of orientation of flats (following [10]), which provides an appropriate notion of directions for the Weyl chamber flow and will be crucial for our notion of cross sections. We further introduce the Furstenberg boundary, which provides the appropriate geometry at infinity for our codings in Section 5. To that end we note that the choice of the reference Weyl chamber $c_0$ distinguishes an open subset, $A^+$, of $A$ by means of the isomorphism in (2). The subset $A^+$ is often called the open positive Weyl chamber in $A$, subject to the choice of $c_0$. We let $\mathfrak{g}$ and $\mathfrak{a}$ denote the Lie algebra of $G$ and $A$, respectively, and let $\Lambda$ be the set of roots of $(\mathfrak{g}, \mathfrak{a})$. We let $\mathfrak{a}^+$ denote the subset of $\mathfrak{a}$ that corresponds to $A^+$ under the exponential map. Then

$$\Lambda^+ := \{\lambda \in \Lambda | \forall H \in \mathfrak{a}^+: \lambda(H) > 0\}$$

is a choice of positive roots of $(\mathfrak{g}, \mathfrak{a})$. We set

$$n := \bigoplus_{\lambda \in \Lambda^+} \mathfrak{g}_\lambda,$$

where

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} | \forall H \in \mathfrak{a}: \text{ad}_H(X) = \lambda(H)X\}$$

is the root space of $\lambda$, and ad is the adjoint representation of $\mathfrak{g}$. We let $N := \exp n$ be the (unipotent) subgroup associated to $n$. We note that $N$ completes the pair $(K, A)$ to an Iwasawa decomposition of $G$. Thus, the map

$$N \times A \times K \to G, \quad (n, a, k) \mapsto nak,$$

is an isomorphism of Lie groups. We let $P := NAM$ be the associated minimal parabolic subgroup of $G$. The Furstenberg boundary of $\mathbb{X}$ is the homogeneous space $G/P$.

We let

$$\nu: G/M \to G/P, \quad gM \mapsto gP,$$

denote the canonical projection from the Weyl chamber bundle to the Furstenberg boundary. In order to define a notion of orientation of flats, we call two Weyl chambers $gM$ and $hM$ asymptotic if they project to the same point in the Furstenberg boundary, thus if

$$\nu(gM) = \nu(hM).$$

This property induces an equivalence relation on the set of all Weyl chambers. The combined map

$$\alpha := (\pi_B, \nu): G/M \to G/K \times G/P$$

is an isomorphism. Therefore, the Furstenberg boundary $G/P$ can be interpreted as the set of equivalence classes of asymptotic Weyl chambers.
An orientation or direction of a flat $F$ is an equivalence class of asymptotic Weyl chambers that has a representative in $F$. An oriented flat is a flat endowed with a distinguished orientation. The set of oriented flats can be identified with the homogeneous space $G/AM$ via the isomorphism
\[ G/AM \rightarrow \{\text{oriented flats}\}, \quad gAM \mapsto (g(F_0), gP). \]
The map $\nu$ in (9) splits into the two canonical maps
\[ \nu_1: G/M \rightarrow G/AM \quad \text{and} \quad \nu_2: G/AM \rightarrow G/P, \]
where $\nu_1$ maps a Weyl chamber $gM$ to the flat that contains it and endows this flat with the equivalence class of $gM$ as orientation, and $\nu_2$ projects an oriented flat to its orientation, identified with the point in the Furstenberg boundary $G/P$.

The actions of $G$ and $W$ on $G/M$ from (6) and (7) descend to actions on $G/AM$ and $G/P$, turning $\nu_1$ and $\nu_2$ into $G$-equivariant as well as $W$-equivariant maps. In particular, we have
\[ W \times G/AM \rightarrow G/MA, \quad (m'M, gMA) \mapsto gm'MA, \]
and
\[ W \times G/P \rightarrow G/P, \quad (m'M, gP) \mapsto gm'P. \]
For any flat $gF_0$, the possible orientations are therefore characterized by the finitely many points $gm'P$, where $m'$ runs through a representative set of $W$ in $M'$.

### 2.2. Locally symmetric spaces.

We continue to use the notation from the previous section and now apply the identifications discussed there without mentioning the isomorphisms. In particular, we allow ourselves to write $X = G/K$, and analogously for other objects. We let $\Gamma$ be a discrete subgroup of $G$. The quotient space
\[ Y := \Gamma \backslash X = \Gamma \backslash G/K \]
is a locally symmetric space or more precisely, if $\Gamma$ has torsion, an orbifold. We let $\pi^\Gamma$ denote the canonical quotient map. Most of the objects defined for $X$ in the previous section descend to analogous objects for $Y$, via $\pi^\Gamma$. The rank, $r$, of $Y$ is the rank of $X$. The $\pi^\Gamma$-images of the flats of $X$ are called the flats of $Y$. We remark that flats of $Y$ are not necessarily isometric to $\mathbb{R}^r$, much in contrast to flats of $X$. In particular, a flat of $Y$ might be compact (as a subset of $Y$). For locally symmetric spaces of rank one, compact flats coincide with periodic geodesics, more precisely, with the subsets of $Y$ traced out by periodic geodesics.

The set of Weyl chambers of $Y$ is
\[ CY = \Gamma \backslash G/M, \]
the Weyl chamber flow on $Y$ is
\[ A \times \Gamma \backslash G/M \rightarrow \Gamma \backslash G/M, \quad (a, \Gamma gM) \mapsto \Gamma gaM, \]
and the set of oriented flats of $Y$ is
\[ \Gamma \backslash G/AM. \]
Since the projection maps $\pi_B$ and $\nu_1$ in (4) and (12) are $\Gamma$-equivariant, they induce the analogous maps on $\mathcal{Y}$:

$$\pi_B^\Gamma: C\mathcal{Y} \to \mathcal{Y}, \quad \Gamma gM \mapsto \Gamma gK,$$

and

$$\nu_1^\Gamma: \Gamma\backslash G/M \to \Gamma\backslash G/AM, \quad \Gamma gM \mapsto \Gamma gAM.$$

2.3. Cross sections and induced discrete dynamical systems. We resume the notation from the previous two sections. We let

$$A^{\text{reg}} := W(A^+),$$

denote the set of regular elements in $A$. For any subset $S \subseteq G/M$, we say that an oriented flat $gAM$ intersects $S$ in $hM$ if

$$\nu_1(hM) = gAM.$$ We say that the intersection is discrete if there exists a neighborhood $U$ of the identity element $id$ in $A$ such that for all $a \in U \cap A^{\text{reg}}$ we have

$$haM \notin S.$$ We note that in this case, the Weyl chamber $hM$ is contained in the oriented flat $gAM$ and determines its orientation. Analogously, for any subset $\hat{S} \subseteq \Gamma\backslash G/M$ we say that an oriented flat $\Gamma gAM$ of $\mathcal{Y}$ intersects $\hat{S}$ in the Weyl chamber $\Gamma hM$ of $\mathcal{Y}$ if

$$\nu_1^\Gamma(\Gamma hM) = \Gamma gAM,$$

and we call the intersection discrete if there exists a neighborhood $U$ of id in $A$ such that for all $a \in U \cap A^{\text{reg}}$ we have

$$\Gamma haM \notin \hat{S}.$$ With these preparations we can now propose the following notion of cross section.

**Definition 2.1.** We call a subset $\hat{C}$ of $\Gamma\backslash G/M$ a cross section for the Weyl chamber flow on $\mathcal{Y}$ if

(C1) every compact oriented flat of $\mathcal{Y}$ intersects $\hat{C}$, and (C2) each intersection of any flat of $\mathcal{Y}$ with $\hat{C}$ is discrete.

We emphasize the following aspects of this definition.

- We do not request that every oriented flat shall intersect $\hat{C}$. Definition 2.1 is motivated by presumed properties necessary for transfer-operator-based investigations of the spectral theory of $\mathcal{Y}$. For rank one spaces, our previous investigations showed that for such applications, we only needed to request that all periodic geodesics intersect a cross section for the geodesic flow. By density properties of these geodesics and a certain smoothness of the cross sections, it automatically meant that all geodesics that returned infinitely often to the compact core of the considered space intersect the cross section. For several results, it was crucial that those geodesics that eventually stay in the ends of the space, do not need to intersect the cross section at all or, when travelling along these geodesics, eventually stop intersecting it. For this reason, also for higher rank spaces, we only request that at least all...
compact oriented flats are detected by the cross section. In our examples in the following sections, we will see that all flats that are “returning” intersect the cross sections constructed there, but that flats “vanishing to infinity” eventually will not intersect anymore.

- We require discreteness of intersections only in the regular directions of the Weyl chamber flow, thus, for the action of $A^{reg}$. The application of $a \in A^{reg}$ on a Weyl chamber $\Gamma gM$ causes motion in each space dimension of the flat that contains $\Gamma gM$. In stark contrast, for $a \in A \setminus A^{reg}$, there is no motion in some space dimensions. We allow non-discrete intersections in these dimensions.

- Since a set $\hat{C}$ as in Definition 2.1 is not a cross section in the classical sense, one might want to call it a “cross section for the returning parts of the Weyl chamber flow in the regular time directions.”

We now turn to the definition of a first return map for a cross section of the Weyl chamber flow, where we aim to preserve the idea that this map should be given as follows. We pick a Weyl chamber $\Gamma gM$ in $\hat{C}$ and consider the oriented flat $F = \Gamma gAM$ that contains $\Gamma gM$. We move along $F$ in the direction given by $gM$, starting at $\Gamma gM$ and ask for the “next” intersection of $F$ with $\hat{C}$, say $\Gamma g'M$. The first return map should map $\Gamma gM$ to $\Gamma g'M$. Moving in the direction $\nu(gM)$ is the same as restricting the flow to the positive Weyl chamber $A^+$ (that deduces with the choice of the reference Weyl chamber $c_0$, see Section 2.1). However, if the rank $r$ of $Y$ is larger than 1, then $A^+$ has $r$ time parameters and hence there does not need to be a well-defined “first” next intersection. We overcome this issue with Definition 2.2 below, for which we start with a brief preparation.

The positive Weyl chamber $A^+$ can be parametrized by an open cone in $(\mathbb{R}_{>0})^r$, that is, by a convex subset $\tau^+$ of $(\mathbb{R}_{>0})^r$ such that for each $t = (t_1, \ldots, t_r) \in \tau^+$ the whole open ray $\mathbb{R}_{>0} \cdot t = \{(ct_1, \ldots, ct_r) \mid c > 0\}$ is contained in $\tau^+$. We fix such a parametrization

$$\tau^+ \to A^+, \ t \mapsto a_t. \tag{18}$$

**Definition 2.2.** Let $\hat{C} \subseteq \Gamma \setminus G/M$ be a cross section for the Weyl chamber flow on $Y$.

(i) We say that $\Gamma gM \in \hat{C}$ has a future intersection with $\hat{C}$ if $\Gamma gA^+M \cap \hat{C} \neq \emptyset$.

In this case, let

$$T := \{t = (t_1, \ldots, t_r) \in \tau^+ \mid \Gamma gatM \in \hat{C}\}$$

be the set of time vectors of the future intersections. For $j \in \{1, \ldots, r\}$, let

$$\text{pr}_j: \mathbb{R}^r \to \mathbb{R}, (t_1, \ldots, t_r) \mapsto t_j,$$

be the projection on the $j$-th component, and set

$$T_j := \text{pr}_j(T) = \{t_j \mid \exists t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_r: (t_1, \ldots, t_r) \in T\}.$$
We say that $\Gamma gM$ has a next intersection with $\tilde{C}$ if for all $j \in \{1, \ldots, r\}$,
\[ t_{0,j} := \min T_j \]
exists,
\[ t_0 := (t_{0,1}, \ldots, t_{0,r}) \in \tau^+ \]
and
\[ \Gamma g_{a_0}M \in \tilde{C}. \]
In this case, we call $t_0$ the first return time vector of $\Gamma gM$.

(ii) Let $\tilde{C}_1$ be the subset of $\tilde{C}$ for which the first return vector exists. The first return map is the map
\[ \tilde{R} : \tilde{C}_1 \to \tilde{C}, \quad \tilde{v} = \Gamma gM \mapsto \Gamma g_{a_0(\tilde{v})}M, \]
where $t(\tilde{v})$ is the first return time vector of $\tilde{v}$.

For the applications that motivate this article we need to be able to (semi-)conjugate the first return map to a function on the Furstenberg boundary of $\mathbb{X}$. In what follows we present the necessary structures.

Let $\tilde{C} \subseteq \Gamma \backslash G/M$ be a cross section for the Weyl chamber flow of $\mathbb{Y}$. To determine the subset of $\tilde{C}$ on which the first return map $\tilde{R}$ becomes a self-map, we define iteratively for $n \in \mathbb{N}$, $n \geq 2$, the sets
\[ \tilde{C}_2 := R(\tilde{C}_1) \cap \tilde{C}_1, \quad \tilde{C}_3 := R(\tilde{C}_2) \cap \tilde{C}_1, \quad \ldots, \]
hence
\[ \tilde{C}_n := R(\tilde{C}_{n-1}) \cap \tilde{C}_1 \quad \text{for } n \in \mathbb{N}, \ n \geq 2, \]
where $\tilde{C}_1$ was defined in Definition 2.2. Then
(19) \[ \tilde{C}_{st} := \bigcap_{n \in \mathbb{N}} \tilde{C}_n \]
is the subset of $\tilde{C}$ that consists of all those Weyl chambers in $\tilde{C}$ that yield an infinite sequence of successive next intersections with $\tilde{C}$. For each element in $\tilde{C}_{st}$, the next intersection is obviously also contained in $\tilde{C}_{st}$. Thus, if $\tilde{C}_{st}$ is nonempty, then the first return map $\tilde{R}$ restricts to a self-map of $\tilde{C}_{st}$:
\[ \tilde{R} : \tilde{C}_{st} \to \tilde{C}_{st}. \]

The set $\tilde{C}_{st}$ may constitute a cross section on its own, in which case we call it the strong cross section contained in $\tilde{C}$. (The subscript $st$ refers to “strong” and is used here for the same motivation as in [19].)

We say that $C \subseteq G/M$ is a set of representatives for $\tilde{C}$ if the quotient map
(20) \[ \pi^\Gamma : G/M \to \Gamma \backslash G/M, \quad gM \mapsto \Gamma gM, \]
restricts to a bijection between $C$ and $\tilde{C}$. (We use $\pi^\Gamma$ to denote both the map in (13) and the map in (20). This double use is motivated by the joint property of these maps to project to $\Gamma$-equivalence classes. The context will always clarify which instance of $\pi^\Gamma$ is used.) If $C$ is any set of representatives for $\tilde{C}$ and $C_1$ is the
subset of \( C \) that corresponds to \( \hat{C}_1 \), then the first return map \( \hat{R} \) induces a map \( R: C_1 \to C \) which makes the diagram

\[
\begin{array}{ccc}
C_1 & \xrightarrow{R} & C \\
\pi^r & \downarrow & \pi^r \\
\hat{C}_1 & \xrightarrow{\hat{R}} & \hat{C}
\end{array}
\]

commutative. If \( C_{st} \) denotes the subset of \( C \) that corresponds to \( \hat{C}_{st} \), then \( R \) restricts to a self-map of \( C_{st} \).

We recall the map \( \nu: G/M \to G/P \) from (9) that projects the Weyl chambers of \( X \) to the points in the Furstenberg boundary that are identified with their equivalence class of asymptotic Weyl chambers (and hence, in a certain sense, the direction of the Weyl chamber). For a well-chosen pair \( (\hat{C}, C) \) one may find a (unique) map \( F: \nu(C_1) \to \nu(C) \) such that the diagram

\[
\begin{array}{ccc}
C_1 & \xrightarrow{R} & C \\
\nu & \downarrow & \nu \\
\nu(C_1) & \xrightarrow{F} & \nu(C)
\end{array}
\]

commutes. In this case and if \( C_{st} \neq \emptyset \), \( F \) restricts to a self-map of \( \nu(C_{st}) \). If \( \hat{C}_{st} \) is a cross section, we call \( F: \nu(C_{st}) \to \nu(C_{st}) \) the \textit{discrete dynamical system on the Furstenberg boundary} induced by \( (\hat{C}, C) \).

Typically, the map \( F \) is piecewise given by the action of certain elements from \( \Gamma \) on subsets of \( G/P \). The orbits of Weyl chambers \( \Gamma gM \) under the first return map \( \hat{R} \) relate to orbits of \( F \) and hence to sequences of the acting elements from \( \Gamma \). These sequences are often called \textit{coding sequences} for the oriented flats or Weyl chambers, and the shift along coding sequences provides a \textit{symbolic dynamics} for the Weyl chamber flow.

We remark that the set \( C \) completely determines the cross section \( \hat{C} \). Therefore, for constructions of cross section we may start by finding a “nice” set \( C \) and define a cross section as \( \pi^r(C) \). We also note that the knowledge of \( C \) is sufficient to determine the discrete dynamical system on the Furstenberg boundary. Thus, also this is induced by \( C \) alone.

In Sections 4 and 5 we will present for a certain class of locally symmetric spaces cross sections for their Weyl chamber flows as well as codings and associated discrete dynamical systems

3. \textbf{Schottky surfaces}

The locally symmetric spaces for which we will demonstrate the existence of cross sections for the Weyl chamber flow and induced discrete dynamical systems in the sense of Section 2.3 are product spaces of Schottky surfaces. Schottky surfaces are certain hyperbolic surfaces, hence locally symmetric spaces of rank one. Therefore the Weyl chamber flow on Schottky surfaces coincides with the geodesic
flow, and is a well-studied object. The classical Koebe–Morse method gives rise to cross sections and codings for the geodesic flow on Schottky surfaces, which have already been used for many different purposes. Also we will take advantage of these results for our constructions in Sections 4 and 5. In this section, we briefly present these classical results, with an emphasis on the dynamical aspects. We refer to [2] for details and proofs.

The Riemannian symmetric space we consider in this section is the hyperbolic plane. We will use throughout the upper half-plane model

\[ H \equiv \mathbb{H}^2 := \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}, \quad ds^2 := \frac{dz d\overline{z}}{\text{Im } z^2}, \]

where \( \text{Im } z \) denotes the imaginary part of \( z \in \mathbb{C} \). We identify the identity component \( G \) of the group of Riemannian isometries of \( \mathbb{H} \) with the Lie group \( \text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/\{ \pm I \} \), where \( I \) denotes the identity matrix in \( \text{SL}_2(\mathbb{R}) \). We denote an element of \( G = \text{PSL}_2(\mathbb{R}) \) by

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

if it is represented by the matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{R}) \). With respect to this identification, \( G \) acts on \( \mathbb{H} \) by fractional linear transformations. Thus,

\[ g(z) = \frac{az + b}{cz + d} \]

for all \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G, \ z \in \mathbb{H} \). As origin of \( \mathbb{H} \) we pick \( i \), as reference flat the imaginary axis \( i\mathbb{R}_{>0} \) and as reference Weyl chamber the upper half of the reference flat, thus \( i(1, \infty) \). The stabilizer group of \( i \) is

\[ K = \text{PSO}(2) = \text{SO}(2)/\{ \pm I \}, \]

the maximal abelian subgroup \( A \) of \( G \) is

\[ A = \left\{ a_t := \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \mid t \in \mathbb{R} \right\}, \]

and the positive Weyl chamber in \( A \) is

\[ A^+ = \left\{ a_t \mid t > 0 \right\}. \]

The centralizer of \( A \) in \( K \) is the trivial group \( M = \{ \text{id} \} \), and the normalizer of \( A \) in \( K \) is

\[ M' = \left\{ \text{id}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}. \]

We may identify the reference Weyl chamber with the unit tangent vector at \( i \) that is tangent to \( i(1, \infty) \) (i.e., the vector at \( i \) that points upwards). The unipotent subgroup is

\[ N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{R} \right\}. \]

Hence, the associated minimal parabolic subgroup \( P = \text{NAM} \) is the subgroup of \( G \) that is represented by upper triangular matrices in \( \text{SL}_2(\mathbb{R}) \). The Furstenberg boundary \( G/P \) coincides with the geodesic boundary of \( \mathbb{H} \), and the action of \( G \)
on $H = G/K$ extends continuously to $G/P$. We may identify the Furstenberg boundary $G/P$ with $P^1_\mathbb{R} = \mathbb{R} \cup \{\infty\}$ by means of the isomorphism

$$G/P \rightarrow P^1_\mathbb{R}, \quad gP \mapsto g(\infty),$$

where

$$g(\infty) = \frac{a}{c}$$

for $g = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \in G$, using the convention $a/0 = \infty$.

A Schottky surface is a hyperbolic surface of infinite area without cusps and conical singularities. Each Schottky surface (and only those) arises from the following construction. We pick $q \in \mathbb{N}$ and fix $2q$ closed Euclidean disks in $\mathbb{C}$ that are centered in $\mathbb{R}$ and that are pairwise disjoint. We fix a pairing of these disks, that we shall indicate by indices with opposite signs. Let $D_1, D_{-1}, \ldots, D_q, D_{-q}$ be the chosen disks. For $k \in \{1, \ldots, q\}$ we pick an element $g_k \in G$ that maps the exterior of the disk $D_k$ to the interior of the disk $D_{-k}$. The subgroup $\Gamma$ of $G$ generated by the elements $g_1, \ldots, g_q$,

$$\Gamma = \langle g_1, \ldots, g_q \rangle,$$

is a Schottky group, and the hyperbolic surface $Y := \Gamma \backslash H$ is a Schottky surface.

The complement of the union of the disks in $H$,

$$\mathcal{F} := H \setminus \bigcup_{k=1}^{q} (D_k \cup D_{-k}),$$

is an open fundamental domain for $Y$, all of whose sides in $H$ are given by geodesics (namely the part of the boundary of the disks that is in $H$). Its side-pairings are given by the elements $g_1, \ldots, g_q$.

In order to provide a cross section for the geodesic flow on $Y$, we set $\mathcal{I} := \{\pm 1, \ldots, \pm q\}$ and, for any $k \in \mathcal{I}$, let $s_k$ be the boundary of $D_k$ in $\mathbb{H}$, and $I_k$ denote the part of the Furstenberg boundary $G/P = \mathbb{R} \cup \{\infty\}$ that is exterior to $D_k$. We refer to $I_k$ as forward interval. We recall the map $\alpha$ from (10) and set

$$C_k := \alpha^{-1}(s_k \times I_k).$$

The set $C_k$ may be identified with the set of unit tangent vectors of $\mathbb{H}$ that are based on $s_k$ and “point into” the fundamental domain $\mathcal{F}$. We set

$$C := \bigcup_{k \in \mathcal{I}} C_k \quad \text{and} \quad \tilde{C} := \pi^\Gamma(C).$$

The following statement is immediately implied from the definitions of $\tilde{C}$ and $C$ and the property of $\mathcal{F}$ to be geodesically convex. It is also an immediate consequence of the Koebe–Morse method, which discusses instead the neighboring $\Gamma$-translates of $\mathcal{F}$.

**Proposition 3.1.** The set $\tilde{C}$ is a cross section for the geodesic flow on the Schottky surface $\Gamma \backslash \mathbb{H}$, and $C$ is a set of representatives for $\tilde{C}$. 
For the presentation of the induced discrete dynamical system, we restrict the consideration to the strict cross section contained to $\hat{C}$. To that end we denote by $L$ the limit set of $\Gamma$, that is the set of limit points of the $\Gamma$-orbit $\Gamma(z)$ in the Furstenberg boundary $P^1_\mathbb{R}$, where $z$ is any point of $\mathbb{H}$. We let
\begin{equation}
C_{st} := \{ c \in C \mid \nu(c) \in L \}
\end{equation}
be the subset of Weyl chambers (or unit tangent vectors) in $C$ that project to the limit set $L$, and we set
\begin{equation}
\hat{C}_{st} := \pi^\Gamma(C_{st}).
\end{equation}
(For convenience we allow ourselves here this slight abuse of notation: we have not yet shown that $\hat{C}_{st}$ coincides with the set defined in \text{[19]}. This will be done in Proposition 3.1.) For $k \in I$ we set
\begin{equation}
I_{st,k}^c := L \cap (P^1_\mathbb{R} \setminus I_k),
\end{equation}
which is the part of the limit set $L$ contained in the interior of the disk $D_k$. We call $I_{st,k}^c$ a strong coding set, a wording whose meaning will become clear in Section 5. Then $L$ is the disjoint union of these sets:
\begin{equation}
L = \bigcup_{k \in I} I_{st,k}^c.
\end{equation}
We define a self-map $F: L \to L$ by
\begin{equation}
F|_{I_{st,k}^c} : I_{st,k}^c \to L, \quad x \mapsto g_k(x),
\end{equation}
for $k \in I$.

As above in Proposition 3.1, the following statements follow immediately from the definitions of $\hat{C}_{st}$, $C_{st}$ and $F$ as well as the properties of limit sets of Schottky groups. For the convenience of the reader, we provide a sketch of the proof.

**Proposition 3.2.** The set $\hat{C}_{st}$ is a cross section for the geodesic flow on the Schottky surface $\Gamma \setminus \mathbb{H}$. It is intersected by every geodesic on $\Gamma \setminus \mathbb{H}$ that is contained in a compact subset of $\Gamma \setminus \mathbb{H}$, which may depend on the considered geodesic. It is the strong cross section contained in $\hat{C}$. The set $C_{st}$ is a set of representatives for $\hat{C}_{st}$, and the map $F$ is the discrete dynamical system on the Furstenberg boundary that is induced by $C_{st}$.

**Sketch of proof.** We recall that oriented flats on $\Gamma \setminus \mathbb{H}$ are precisely the geodesics on $\Gamma \setminus \mathbb{H}$. Let $\hat{\gamma}$ be such a geodesic and suppose that $\gamma$ is one of its representative geodesics on $\mathbb{H}$. If $\gamma$ is directed towards a point in the limit set $L$, then while moving towards this points, $\gamma$ intersects infinitely many $\Gamma$-translates of the fundamental domain $F$. In turn, $\hat{\gamma}$ stays “far away” from the ends of $\Gamma \setminus \mathbb{H}$ and intersects $\hat{C}$ in an unbounded set of times. However, if $\gamma$ is directed towards a point not in $L$, then eventually $\gamma$ will stay in a single $\Gamma$-translate of $F$. In turn, $\hat{\gamma}$ will travel to an end of $\Gamma \setminus \mathbb{H}$, and will intersect $\hat{C}$ only finitely many times in this direction. From this dichotomy, one can easily deduce that $\hat{C}_{st}$ as defined in \text{[22]} coincides with the set

\begin{equation}
\text{[19]} \text{Due to the special structure of Schottky surfaces, we may choose a uniform compact set, namely the compact core of $\Gamma \setminus \mathbb{H}$.}
\end{equation}
defined in [19]. We remark that this dichotomy takes advantage of properties of limit sets that are rather specific to Schottky groups.

To show that $\hat{\mathcal{C}}_{\text{st}}$ is indeed a cross section, we first note that the compact oriented flats on $\Gamma \setminus \mathbb{H}$ are precisely the periodic geodesics of $\Gamma \setminus \mathbb{H}$. Let $\hat{\gamma} = \Gamma gAM$ be such a periodic geodesic. By Proposition 3.1, $\gamma$ intersects $\hat{C}$ at least once, say in $\hat{v}$. Since all intersections of $\hat{\gamma}$ and $\hat{C}$ are discrete by Proposition 3.1 and the group $A$ is one-dimensional, every future intersection with $\hat{C}$ is a next intersection. Since $\hat{\gamma}$ is periodic, the intersection in $\hat{v}$ will repeatedly be among the future intersections, showing that these exist unboundedly. Thus, $\hat{v}$ is in $\hat{C}_{\text{st}}$ and $\hat{\gamma}$ intersects $\hat{C}_{\text{st}}$. Obviously, $C_{\text{st}}$ is a set of representatives of $\hat{C}_{\text{st}}$.

It remains to indicate why $F$ is the induced discrete dynamical system. Let $\hat{v} = \Gamma gM \in \hat{C}_{\text{st}}$. The geodesic $\hat{\gamma}$ determined by $\hat{v}$ is $\Gamma gAM$. Without loss of generality, we may assume that $g \in G$ is chosen such that $v := gM$ is the unique representative of $\hat{v}$ in $C_{\text{st}}$. Then the geodesic $\gamma$ on $\mathbb{H}$ determined by $v$ is $gAM$, or, from a more dynamical point of view, the trajectory $t \mapsto ga_t M$.

Let $b_0 := \lim_{t \to \infty} ga_t M = g(\infty)$ be the point in the Furstenberg boundary of $\mathbb{H}$ to which $\gamma$ is oriented (or projects). Due to the relation between $C$ and $F$, the time-minimal intersection between $\gamma^+ := \{ ga_t M | t > 0 \}$ and the $\Gamma$-translates of $C$ is located at the (unique) boundary component of $F$ through which $\gamma^+$ passes. The structure of $F$ implies that this intersection is in $g_{-k}(C_{-k})$ for some $k \in \mathcal{I}$ if and only if $b_0 \in F_{c_{-k}}$. Suppose that the intersection is in $g_{-k_0}(C_{-k_0})$ at time $t_0 > 0$, with $k_0 \in \mathcal{I}$. Then the next intersection of $\hat{v}$ with $\hat{C}_{\text{st}}$ is in $\Gamma ga_{t_0} M$, which corresponds via $(\pi_{\Gamma | C})^{-1}$ to the element $g_{-k_0}^{-1}(ga_{t_0} M) = g_{k_0}(ga_{t_0} M)$ of $C_{\text{st}}$. In turn, $F(b_0) = F(\nu(gM)) = \nu(R(gM)) = \nu(g_{k_0}(ga_{t_0} M)) = g_{k_0} \nu(ga_{t_0} M) = g_{k_0}(b_0)$.

This shows that $F$ is indeed the induced discrete dynamical system. \qed

4. Cross sections for the Weyl chamber flow on product spaces

Let $r \in \mathbb{N}$. We recall that $\mathbb{H}^2 = \mathbb{H}$ denotes the hyperbolic plane and consider the Riemannian symmetric space $\mathbb{X} := (\mathbb{H}^2)^r = \mathbb{H} \times \cdots \times \mathbb{H}$ of rank $r$, given by the direct product of $r$ copies of $\mathbb{H}$. We identify the identity component of the group of Riemannian isometries of $\mathbb{X}$ with $G := \text{PSL}_2(\mathbb{R})^r$.

Then the action of $G$ on $\mathbb{X}$ is $g(z) = (g_1(z_1), \ldots, g_r(z_r))$.
for all \( g = (g_1, \ldots, g_r) \in G \) and \( z = (z_1, \ldots, z_r) \in X \). For \( j \in \{1, \ldots, r\} \) we choose a (Fuchsian) Schottky group \( \Gamma_j \) in \( \text{PSL}_2(\mathbb{R}) \) and set

\[
\Gamma := \Gamma_1 \times \cdots \times \Gamma_r.
\]

In this section we construct a cross section for the Weyl chamber flow on the locally symmetric space

\[
\mathcal{Y} := \Gamma \backslash X.
\]

We start with some preparatory considerations. Since \( X \) as well as \( Y = \Gamma_1 \backslash \mathbb{H} \times \cdots \times \Gamma_r \backslash \mathbb{H} \) enjoy clear product structures, several of the necessary objects are the direct products of the analogous objects of the single factors. As origin of \( X \) we choose \( x_0 := (i, \ldots, i) \in X \).

The flats and Weyl chambers of \( X \) are the direct products of the flats and Weyl chambers of \( H \). Therefore we choose \( F_0 := (i(\mathbb{R} > 0))^r \) as reference flat of \( X \), and \( c_0 := (i(1, \infty))^r \) as reference Weyl chamber of \( X \), which are the direct products of our chosen reference flat and reference Weyl chamber of \( \mathbb{H} \).

In Section 3 we discussed the groups and maps associated to our choices of reference objects of \( H \). In what follows we will use for these groups and maps the notation from Section 3 but with the additional subscript “1”. Thus, \( G_{1u} = \text{PSL}_2(\mathbb{R}) \), \( K_{1u} = \text{PSO}(2) \), etc. The subscript-free notation is preserved for the objects related to \( X \) and \( Y \). The stabilizer group of \( x_0 \) in \( G \) is

\[
K = \text{Stab}_G(x_0) = K_{1u}^r = \text{PSO}(2)^r,
\]

the maximal abelian subgroup of \( G \) determined by \( F_0 \) is \( A = A_{1u}^r \), the positive Weyl chamber in \( A \) determined by \( c_0 \) is \( A^+ = (A_{1u}^+)^r \). The centralizer group and normalizer group of \( A \) in \( K \) are \( M = M_{1u}^r \) and \( M' = (M_{1u}^+)^r \), respectively, and the Weyl group is \( W = W_{1u}^r \). The unipotent subgroup is \( N = N_{1u}^r \), and the minimal parabolic subgroup is \( P = P_{1u}^r \). Thus, the Furstenberg boundary of \( X \) is

\[
G/P = G_{1u}/P_{1u} \times \cdots \times G_{1u}/P_{1u},
\]

the \( r \)-times direct product of the Furstenberg boundary of \( \mathbb{H} \), which we identify with \( (P(\mathbb{R}))^r \) via the isomorphism

\[
gP = (g_1P_{1u}, \ldots, g_rP_{1u}) \mapsto (g_1(\infty), \ldots, g_r(\infty)).
\]

For \( j \in \{1, \ldots, r\} \) we fix a fundamental domain \( \mathcal{F}_j \) for the Fuchsian Schottky group \( \Gamma_j \) in \( \mathbb{H} \) of the form as in Section 3 arising from the choice of \( q_j \) Euclidean disks. We set \( I_j = \{ \pm 1, \ldots, \pm q_j \} \), let

\[
g_{j,k}, \ s_{j,k}, \ I_{j,k} \quad \text{for} \ k \in I_j
\]

(23)

denote the side-pairing elements in \( \Gamma_j \), the geodesic sides of \( \mathcal{F}_j \), and the forward intervals, respectively. We recall the map \( \alpha \) from (10) and set

\[
\mathcal{J} := \prod_{j=1}^r I_j.
\]
For each $m = (m_1, \ldots, m_r) \in J$ we set
\[ Q_m := \prod_{j=1}^r (s_j, m_j, I_{j,m_j}) , \quad C_m := \alpha^{-1}(Q_m) , \]
and
\[ C := \bigcup_{m \in J} C_m \quad \text{and} \quad \hat{C} := \pi^\Gamma(C) , \]
where $\pi^\Gamma$ is the map in (20).

**Theorem 4.1.** The set $\hat{C}$ is a cross section for the Weyl chamber flow on $\mathcal{Y}$, and $C$ is a set of representatives for $\hat{C}$.

Preparatory for the proof we briefly discuss the product structure of $\hat{C}$ and $C$. To that end, for $j \in \{1, \ldots, r\}$ and $m_j \in I_{j,m_j}$, we set
\[ C_{j,m_j} := \alpha^{-1}_u(s_j, m_j, I_{j,m_j}) , \]
where $\alpha_u$ denotes the map in (10) for $G_u$. Further, we set
\[ C_j := \bigcup_{m_j \in I_{j,m_j}} C_{j,m_j} \quad \text{and} \quad \hat{C}_j := \pi^\Gamma_u(C_j) . \]
Then, for any $m = (m_1, \ldots, m_r) \in J$, we have
\[ C_m = \prod_{j=1}^r C_{j,m_j} , \]
and further
\[ \hat{C} = \prod_{j=1}^r \hat{C}_j . \]
For each $j \in \{1, \ldots, r\}$, the set $\hat{C}_j$ is a cross section for the Weyl chamber flow (geodesic flow) on the Schottky surface $\Gamma_j\setminus \mathbb{H}$ with $C_j$ as set of representatives by Proposition 3.1.

**Proof of Theorem 4.1.** Let $\Gamma gA M$ be an oriented compact flat of $\mathcal{Y}$. With $g = (g_1, \ldots, g_r)$ we have
\[ \Gamma gA M = (\Gamma_1 g_1 A_u M_u, \ldots, \Gamma_r g_r A_u M_u) . \]
Thus, for each $j \in \{1, \ldots, r\}$,
\[ \gamma_j := \Gamma_{j,g_j A_u} M_u \]
is an oriented flat (oriented geodesic) on $\Gamma_j\setminus \mathbb{H}$ that is contained in a compact subset of $\Gamma_j\setminus \mathbb{H}$. It is even a periodic geodesic. Thus, $\gamma_j$ intersects $\hat{C}_j$ by Proposition 3.1, say in $\Gamma_{j,g_j a_j} M_u$. Then $\Gamma gA M$ intersects $\hat{C}$ in $\Gamma g(a_1, \ldots, a_r) M$. This establishes (C1) for $\hat{C}$. In order to show (C2) let $\Gamma gA M \in \bar{C}$ and suppose that $g = (g_1, \ldots, g_r)$. Then
\[ \Gamma gA M = (\Gamma_1 g_1 M_u, \ldots, \Gamma_r g_r M_u) \in \hat{C}_1 \times \cdots \times \hat{C}_r . \]
Thus, for each $j \in \{1, \ldots, r\}$, the Weyl chamber (unit tangent vector) $\Gamma_j g_j M_u$ is in the cross section $\tilde{C}_j$ for the geodesic flow on $\Gamma_j \backslash \mathbb{H}$. Hence we find $\varepsilon_j > 0$ such that for all $t \in (-\varepsilon_j, \varepsilon_j)$, $t \neq 0$,

$$\Gamma_j g_j a_t M_u \notin \tilde{C}_j,$$

where

$$a_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \in G_\Delta = \text{PSL}_2(\mathbb{R}).$$

For

$$U := (-\varepsilon_1, \varepsilon_1) \times \cdots \times (-\varepsilon_r, \varepsilon_r)$$

we have that each element $a \in U \cap A^\text{reg}$ is of the form

$$a = (a_1, \ldots, a_r)$$

with $t_j \in (-\varepsilon_j, \varepsilon_j)$, $t_j \neq 0$, for $j \in \{1, \ldots, r\}$, and hence

$$\Gamma g a M = (\Gamma_1 g_1 a_1 M_u, \ldots, \Gamma_r g_r a_r M_u) \notin \tilde{C}.$$

This establishes ((C2)) for $\tilde{C}$ and finishes the proof that $\tilde{C}$ is a cross section for the Weyl chamber flow on $Y$. Finally for each $j \in \{1, \ldots, r\}$, the set $C_j$ is a set of representatives for $\tilde{C}_j$. This property is stable under direct products, and hence $C$ is indeed a set of representatives for $\tilde{C}$. \qed

We briefly discuss an aspect of the structure and thickness of $\tilde{C}$. For each $j \in \{1, \ldots, r\}$ the set of base points of the cross section $\tilde{C}_j$ is the full boundary of the fundamental domain $F_j$. A fundamental domain for $Y = \Gamma \backslash X$ is given by

$$F := \bigtimes_{j=1}^r F_j.$$

The set of base points of $\tilde{C}$, however, is only a rather sparse subset of the boundary of $F$, getting sparser if the rank becomes larger. This shows that the cross section $\tilde{C}$ is not implied by a Koebe–Morse method, but is a genuinely related to the Weyl chamber bundle.

In a way similar to the construction of $\tilde{C}$ we can find the strong cross section contained in $\tilde{C}$. To that end, for $j \in \{1, \ldots, r\}$, we let $L_j$ denote the limit set of $\Gamma_j$ (in the rank one situation), set

$$C_{st,j} := \{c \in C_j \mid \nu_1(c) \in L_j\}$$

as well as

$$\tilde{C}_{st,j} := \pi_u(\nu_1(C_{st,j})).$$

As stated in Proposition 3.2, $\tilde{C}_{st,j}$ is the strong cross section contained in $\tilde{C}_j$ for the geodesic flow on $\Gamma_j \backslash \mathbb{H}$. We further define

$$C_{st} := \bigtimes_{j=1}^k C_{st,j} \quad \text{and} \quad \tilde{C}_{st} := \bigtimes_{j=1}^r \tilde{C}_{st,j}.$$

Then $\tilde{C}_{st} = \pi^\Gamma(C_{st})$. 

Theorem 4.2. The set $\hat{C}_{st}$ is the strong cross section contained in $\hat{C}$, and $C_{st}$ is a set of representatives.

Proof. These statements can be proven analogously to those in Theorem 4.1, using Proposition 3.2 instead of Proposition 3.1. □

5. Codings and discrete dynamical systems on Furstenberg boundary

In this section we present the discrete dynamical system on the Furstenberg boundary that is induced by the set of representatives $C_{st}$ for the strong cross section $\hat{C}_{st}$ from Section 4. We resume the notation from Section 4.

For $j \in \{1, \ldots, r\}$, we denote the strong coding sets determined by the choice of the fundamental domain $F_j$ for $\Gamma_j$ by $I_{st,j,k}$ for $k \in I_j$.

For $m \in J = \bigotimes_{j=1}^r I_j$, $m = (m_1, \ldots, m_r)$, we set $I_{st,m} := \bigotimes_{j=1}^r I_{st,j,m_j}$ and $D := \bigcup_{m \in J} I_{st,m}$.

We define the map $F: D \to D$ as follows: for $m \in J$, $m = (m_1, \ldots, m_r)$, we set $g_m := (g_{1,m_1}, \ldots, g_{r,m_r})$, where $g_{j,m_j}$ are the side-pairing elements of $F_j$ (see (23)). Restricted to the subset $I_{st,m}$ of $D$, the map $F$ is $F|_{I_{st,m}}: I_{st,m} \to D$, $x \mapsto g_m(x)$.

Further, for $j \in \{1, \ldots, r\}$ let $F_j: L_j \to L_j$ denote the discrete dynamical system induced on the Furstenberg boundary $P_\mathbb{H}$ of $\mathbb{H}$ induced by the cross section $\hat{C}_{st,j}$ and its set of representatives $C_{st,j}$ for the geodesic flow on $\Gamma_j \setminus \mathbb{H}$.

Theorem 5.1. (i) The map $F$ is the discrete dynamical system on Furstenberg boundary that is induced by $C_{st}$.

(ii) For any $m = (m_1, \ldots, m_r) \in J$ we have $F|_{I_{st,m}} = \left( F_1|_{I_{st,1,m_1}}, \ldots, F_r|_{I_{st,r,m_r}} \right)$.

Proof. The statement in (i) follows immediately from the definition of the map $F$. To establish (ii), we recall that the $r$-dimensional Weyl chamber flow on $\mathbb{Y}$ is the direct product of the $1$-dimensional geodesic flows on the factors $\Gamma_j \setminus \mathbb{H}$, $j \in \{1, \ldots, r\}$, of $\mathbb{Y}$. We recall further that the cross section $\hat{C}_{st}$ and its set of representatives $C_{st}$ are direct products of the cross sections and sets of representatives for these factors. Therefore, this product structure descends to the induced discrete dynamical systems on Furstenberg boundary, which yields (ii) due to the identity in (i). □
6. Transfer operators

In this final section we propose a definition of a transfer operator family for the multi-dimensional discrete dynamical system $F$ in Section 5. We continue to use the notation from Sections 4 and 5 and start by presenting the well-known definition of transfer operator families for one-dimensional flows, specialized to our setup.

Let $j \in \{1, \ldots, r\}$ and recall from (25) and Section 3 the discrete dynamical system $F_j : L_j \to L_j$ induced by the set of representatives $C_j$ of the cross section $\hat{C}_j$ for the geodesic flow on the Schottky surface $\Gamma_j \setminus \mathbb{H}$. The Ruelle-type transfer operator $L_{j,s}$ with parameter $s \in \mathbb{C}$ associated to $F_j$ is (at least initially) an operator on the space of functions $\psi : L_j \to \mathbb{C}$, given by

$$L_{j,s} \psi(x) := \sum_{y \in F_j^{-1}(x)} |F_j'(y)|^{-s} \psi(y) \quad (x \in L_j).$$

Here, the derivative of $F_j'$ at $y$ is understood as follows. The point $y$ is contained in $I_{st,j,k}^-$ for a unique $k \in \mathcal{I}_j$. Then $F_j$ acts on a small neighborhood of $y$ in $L_j$ by the fractional linear transformation $g_{j,k}$, which extends to an analytic map in a small neighborhood of $y$ in $\mathbb{R}$. We use the derivative of this extended map for $F_j'(y)$.

The function space which one uses as domain for the transfer operator $L_{j,s}$ depends on its further applications. One may choose spaces of functions with larger domain or with some regularity properties. We will refrain here from these discussions and will use the space of functions on the limit sets as place holder.

We shall now provide another presentation of $L_{j,s}$ that takes advantage of the explicit description of $F_j$. To that end we note that $F_j$ restricts to the bijections

$$I_{st,j,k}^+ \to L_j \setminus I_{st,j,-k}, \quad x \mapsto g_{j,k}(x),$$

for each $k \in \mathcal{I}_j$. Thus, for each $k \in \mathcal{I}_j$, each $x \in I_{st,j,k}^+$ has the $|\mathcal{I}_j| - 1$ preimages

$$\{g_{j,\ell}^{-1}(x) \mid \ell \in \mathcal{I}_j, \ell \neq -k\}.$$

For any function $\psi : L_j \to \mathbb{C}$, we set

$$\psi_k := \psi \cdot 1_{I_{st,j,k}^+} \quad \text{for } k \in \mathcal{I}_j,$$

where $1_A$ denotes the characteristic function of the set $A$. Then

$$\psi = \sum_{k \in \mathcal{I}_j} \psi_k.$$

Further, for $h \in \Gamma_j$, $s \in \mathbb{C}$, any subset $A \subseteq \mathbb{R}$ and any function $\varphi : A \to \mathbb{C}$ we set

$$\tau_s(h^{-1})\varphi(x) := \left(h'(x)\right)^s \varphi(h(x)) \quad (x \in A),$$

whenever it is well-defined (as it will be in all our applications). For each $k \in \mathcal{I}_j$ we have then

$$(L_{j,s} \psi)_k = \sum_{\ell \in \mathcal{I}_j, \ell \neq -k} \tau_s(g_{j,\ell}) \psi_\ell$$
or, in a more compact form,

\[ L_{j,s} = \sum_{k \in J_j} 1_{\tau_{j,k}} \cdot \sum_{\ell \in J_{\ell \neq -k}} \tau_{s}(g_{j,\ell}) \cdot \]

For the transfer operator family associated to the multi-dimensional map \( F \) we propose a definition analogous to those in (26) but allowing a multi-dimensional parameter \( s \in \mathbb{C}^r \). (We recall that the rank of the considered Riemannian locally symmetric space \( Y \) is \( r \).) We first consider the parameter-free transfer operator

\[ (28) \quad L f(x) := \sum_{y \in F^{-1}(x)} |F'(y)|^{-1} f(y), \]

acting on functions \( f: D \to \mathbb{C} \), where the derivative \( F' \) is understood analogously to above and \( |F'(y)| \) is the absolute value of the determinant of the linear map \( F'(y) \). For any \( y \in D \) we find a unique element \( m \in J \), \( m = (m_1, \ldots, m_r) \), such that \( y \in I_{start,m} \). Letting \( y = (y_1, \ldots, y_r) \) and \( g_m = (g_{1,m_1}, \ldots, g_{r,m_r}) \) we then have \( F = g_m \) is a small neighborhood of \( y \) in \( D \). Thus,

\[ F(y) = g_m(y) = (g_{1,m_1}(y_1), \ldots, g_{r,m_r}(y_r)) \]

and the Jacobi matrix of \( F \) at \( y \) is the diagonal matrix

\[ (29) \quad J_F(y) = \begin{pmatrix} g_{1,m_1}'(y_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{r,m_r}'(y_r) \end{pmatrix} . \]

Thus,

\[ (30) \quad |F'(y)|^{-1} = |\det J_F(y)|^{-1} = \prod_{j=1}^r (g_{j,m_j}'(y_j))^{-1}. \]

Motivated by the diagonal structure of the Jacobi matrix in (29), we propose to endow each non-zero entry separately with a weight. Thus, the parametrized transfer operator \( L_s \) with \( s = (s_1, \ldots, s_r) \in \mathbb{C}^r \) is defined as

\[ (31) \quad L_s f(x) = \sum_{y \in F^{-1}(x)} |F'(y)|^{-s} f(y), \]

where

\[ |F'(y)|^{-s} := \prod_{j=1}^r (g_{j,m_j}'(y_j))^{-s_j} \]

in the notation from (30). This use of the parameter also reflects well the independence of the \( r \) dimensions of the oriented flats of \( Y \).

The analogy between the transfer operators in (31) and (26) goes further. For \( m = (m_1, \ldots, m_r) \in J \) we set

\[ B(m) := \{ n = (n_1, \ldots, n_r) \in J \mid \exists j \in \{1, \ldots, r\}: n_j = -m_j \} . \]
The map $F: D \to D$ restricts to the bijections

$$I_{s,t,m}^c \to D \setminus \bigcup_{n \in B(m)} I_{s,t,n}, \quad x \mapsto g_m(x).$$

In analogy to (27), we define for $h = (h_1, \ldots, h_n) \in \Gamma$, $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$, any subset $A \subseteq \mathbb{R}$, any function $\varphi: A \to \mathbb{C}$,

$$\omega_s(h^{-1}) \varphi(x) = \left| h_1'(y_1) \right|^{-s_1} \cdots \left| h_r'(y_r) \right|^{-s_r} \varphi(h(y)).$$

Then

$$L_s = \sum_{m \in J} 1_{I_{s,t,m}} \cdot \sum_{n \in J, n \notin B(m)} \omega_s(g_n),$$

or, if we set

$$f_m := f \cdot 1_{I_{s,t,m}} \quad (m \in J)$$

for any function $f: D \to \mathbb{C}$, then

$$(L_s f)_m = \sum_{n \in J, n \notin B(m)} \omega_s(g_n) f_n.$$
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Anke Pohl, University of Bremen, Department 3 – Mathematics, Bibliothekstr. 5, 28359 Bremen, Germany
Email address: apohl@uni-bremen.de