CHIRAL AND SPIN ORDER IN THE 2D $\pm J$ XY SPIN GLASS:
DOMAIN WALL SCALING ANALYSIS

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This is an analytic study of the two-dimensional XY spin glass with $\pm J$ disorder. The Hamiltonian has a continuous spin symmetry and a discrete chiral symmetry, and therefore possesses, potentially, two different order parameters and correlation lengths. The cost of breaking the symmetries is probed by comparing the ground state energy under periodic (P) boundary conditions with the one under antiperiodic (AP) and under reflecting (R) boundary conditions. Two energy differences (“domain wall energies”) appear, $\Delta E^{AP}$ and $\Delta E^{R}$, whose scaling behavior with system size is nontrivially related to the correlation length exponents.

For a specific distribution of the $\pm J$ disorder we show that the chiral and spin correlation lengths diverge with the same exponent as $T \downarrow 0$. The common exponent has a common cause, viz. the low reversal energy of domains of chiral variables. For general disorder we give a heuristic argumentation in terms of droplet excitations that leads to spin ordering on a longer, or equal, scale than chiral ordering. These results are in contrast with interpretations of Monte Carlo simulations.

I. INTRODUCTION

Due to the rotational symmetry of the XY model, any of its ground states is necessarily part of a continuum of ground states related by global spin rotations. It was first pointed out by Villain [1] that the ground state of an XY spin glass with random $\pm J$ interactions has, in addition, a twofold degeneracy. The two continua can be deduced from one another by a global spin reflection with respect to an arbitrary axis. They are characterized by opposite “chiralities”, that is, by an opposite sense of rotation of the spins as one moves around a plaquette of the lattice. At finite temperature one may have domains belonging to different ground states (“chiral excitations”), so that each plaquette has to be characterized by its own chiral variable.

A number of recent theoretical papers have focused on the rôle of these chiralities [2-6]. The motivation for this interest is the question what the lower critical dimension $d_{c}$ is of the XY spin glass, and by which mechanism this system orders just above $d_{c}$. Ozeki and Nishimori [7-9] have developed arguments for the bound $d_{c} \geq 4$ (see also Schwartz and Young [10]), which is supported by substantial numerical evidence (see references in [9]). However, all these authors consider the conventional Edwards-Anderson order parameter associated with the XY spins. Villain’s [1] discovery of a discrete symmetry therefore naturally led to the idea, due to Kawamura and Tanemura [2, 3], that these authors consider the conventional Edwards-Anderson order parameter associated with the XY spins. Villain’s [1] discovery of a discrete symmetry therefore naturally led to the idea, due to Kawamura and Tanemura [2, 3], that in fact $d_{c}$ might be less than four. We recall briefly how this idea has been investigated in recent years.

One can define two correlation lengths $\xi_{c}$ and $\xi_{s}$, associated with the chiral variables $q_{r}$ (where $r$ denotes the center of a plaquette) and the spin variables $S_{i}$ (where $i$ denotes a lattice site), respectively:

$$\frac{< q_{r} q_{r+R}>^{2}}{< q_{r}^{2}>} \sim e^{-R/\xi_{c}} \quad \text{and} \quad \frac{< S_{i} \cdot S_{i+R}>^{2}}{< S_{i}^{2}>} \sim e^{-R/\xi_{s}} \quad (R \to \infty).$$

Here $< \cdots >$ denotes the thermal and $\cdots$ denotes the disorder average. For the low temperature behavior of these correlation lengths one expects for $d < d_{c}$:

$$\xi_{c}(T) \sim T^{-\nu_{c}} \quad \text{and} \quad \xi_{s}(T) \sim T^{-\nu_{s}} \quad (T \to 0).$$

One method to investigate the relation between $\xi_{c}$ and $\xi_{s}$ is to determine the correlation length exponents $\nu_{c}$ and $\nu_{s}$ via finite-size scaling of “domain wall” energies. In simpler cases where there is a single symmetry and a single correlation length with exponent $\nu$, this is done by finding the ground state energies for two appropriately chosen distinct boundary conditions. The energy difference, $\Delta E$, will then generally scale with the linear system size $N$ as $|\Delta E| \sim N^{-1/\nu}$.

In the present case, with both a continuous and a discrete symmetry, the periodic (P) “reference” boundary conditions can be changed in two ways: to antiperiodic (AP) and to reflecting (R) ones, this latter possibility having been first suggested by Kawamura and Tanemura [3]. Hence there are two energy differences to be considered, $\Delta E^{AP}$ and $\Delta E^{R}$. Across an antiferromagnetic seam two spins interact with each other’s image under reflection in the origin (of spin space). Hence AP boundary conditions probe the spatial rigidity of spin order, and the spin correlation length exponent $\nu_{s}$ is given by the usual relation

$$|\Delta E^{AP}| \sim N^{-1/\nu_{s}}.$$

Across a reflecting seam two spins interact with each other’s image under reflection about a fixed axis (in spin space). Since the chiral variables appear only when the “spin waves” are integrated out - and notwithstanding the obvious link between R boundary conditions and chirality - it is not a priori clear exactly how these boundary conditions probe
the chiral order. In fact the relation $|\Delta E_R| \sim N^{-1/\nu_c}$ that one might naively expect is certainly not correct. The reason is that R boundary conditions, upon closer inspection, appear to also affect the spin wave degrees of freedom. This was first realized with remarkable intuition by Kawamura and Tanemura [3], who then proceeded and extracted the chiral correlation length exponent $\nu_c$ from a heuristic expression involving both $\Delta E^R$ and $\Delta E^\text{AP}$.

On the basis of Monte Carlo determinations of the ground states of two- and three-dimensional systems these authors conclude that in two dimensions $\nu_s = 1.2 \pm 0.15$ and $\nu_c = 2.6 \pm 0.3$. This implies that the chiral variables have their own length scale and order in larger domains than the spin variables. Similar conclusions, namely $\nu_s \approx 1$ and $\nu_c \approx 2$, were reached by Ray and Moore [5], also on the basis of Monte Carlo simulations and a finite size scaling analysis. These studies suggest that as the dimension $d$ is increased, $\nu^{-1}_s$ may vanish before $\nu^{-1}_c$ does, so that just above $d_0$ there would be a phase with long range chiral order but exponentially decaying spin-spin correlations. Indeed this is precisely the scenario that Kawamura and Tanemura [2,3] find in Monte Carlo simulations of the two- and three-dimensional XY spin glass. In recent work on the two- and three-dimensional Heisenberg spin glass Kawamura [11] arrives at fully analogous conclusions. If these are accepted, the XY and Heisenberg vector spin glasses have their lower critical dimensionality between $d = 2$ and $d = 3$.

In order to achieve a better understanding of the interplay between chiral and spin variables, Ney-Nifte, Hilhorst, and Moore [6] recently performed an analytic study of the random $\pm J$ XY spin glass (in its Villain [12] formulation) on the one-dimensional ladder lattice (see also Morris et al. [13]). Their conclusion is that the chiral correlation length $\xi_c$ and spin correlation length $\xi_s$ diverge for $T \downarrow 0$ with the same exponent $\nu_c = \nu_s$ (of which they determine the value $1.8999 \ldots$). The purpose of this work is to extend the methods of Ref. [6] to $d = 2$ and to confront our results with those of Refs. [3] and [5].

Our starting point in Section 2 is the Villain [12] formulation of the XY model with random $\pm J$ interactions. This formulation has the advantage that the chiral variables can easily be defined and can be decoupled from the spin waves. We transform the XY Hamiltonian to a Coulomb gas Hamiltonian with charges $q_r$ that play the role of the chiral variables. The transformation has been known [14-16] since long, including the fact [1, 17] that the charges take half-integer values on the frustrated plaquettes and integer values on the others. Here we take exactly into account all finite size effects, essential for the finite size scaling analysis that follows. Our result for the Coulomb gas partition function in the case of periodic (P) boundary conditions, and for an arbitrary realization of the disorder, is given by Eqs. (2.38). The effective Hamiltonian contains, in addition to the Coulomb interaction, a coupling term between the total electric dipole moment and the boundary conditions. This additional term was obtained via a different approach in a very recent paper by Vallat and Beck [18], and its one-dimensional equivalent had appeared in [6]. In Sections 3 and 4 we go further and discuss how these equations are modified in the case of antiperiodic (AP) and reflecting (R) boundary conditions, respectively. In the limit $T \rightarrow 0$ the equations reduce to expressions for the ground state energy, including all its finite size corrections, provided the ground state itself is known or can be plausibly guessed.

Since the ground state problem for an arbitrary realization of the disorder cannot be solved, we treat in Section 5 a more restricted two-parameter subset of realizations, in which the frustrated plaquettes are placed in a random rectangular array with infinite range correlation along the $y$ direction. (Disorder of the unidirectionally infinite-ranged type was also the first one considered for the random bond Ising model [19].) We find a regime of main interest in parameter space in which the XY spin glass has a zero-temperature transition with

$$\nu_s = \nu_c \quad (1.4)$$

There is a second regime, with a strongly anisotropic spatial distribution of the frustrated plaquettes, in which the model has a low temperature phase with long range chiral correlation and, very plausibly, power law decay of spin correlations. This is behavior analogous to that of fully frustrated two-dimensional XY models [20-22], on which there exists a large literature.

The result (1.4) in the first regime is different from the scenario proposed by Kawamura and Tanemura [2, 3] and by Ray and Moore [5]. The basic mechanism responsible for (1.4) is that the ground state can accommodate to a changeover from P to AP boundary conditions by means of the formation of a chiral domain wall, which is energetically lower lying than the continuous spin wave deformation that naturally comes to mind. On the basis of this example alone we cannot rule out the possibility that our conclusion, Eq. (1.4), is valid only within the restricted class of disorder realizations.

In order to treat the case of general disorder, we construct in Section 6 a heuristic theory based on the same mechanism. It involves the lowest lying excitations of the standard Coulomb Hamiltonian which are assumed to be collective charges reversals. We find that

$$\nu_s \geq \nu_c \quad (1.5)$$

This is compatible with (1.4) but contradicts the results of Refs.[2,3,5] where the opposite inequality holds. Furthermore, we show that if Eq. (1.5) would hold as a strict inequality, one cannot extract the exponent $\nu_c$ from $\Delta E^R$ and
\[ \Delta E^{AP} \] To find \( \nu \), an appropriate quantity would then be the energy difference between the two ground states of the Coulomb Hamiltonian with periodic and reflecting boundary conditions, without any additional terms.

Our conclusion is that there is no evidence that chiral order extends on a longer length scale than spin order.

**II. PARTITION FUNCTION WITH PERIODIC BOUNDARY CONDITIONS**

**A. The Villain XY model with ± \( J \) interactions**

We consider an XY model on a finite square lattice, periodic in both directions, with sites \( i = (m, n) \), where \( m = 1, \ldots, M \) and \( n = 1, \ldots, N \). Each site \( i \) is occupied by a two-component unit vector or “spin” \( S_i \) whose angle \( \phi_i \) with a reference axis takes values in \( (-\pi, \pi] \). In the ferromagnetic Villain model \[12\] two spins \( \phi_i \) and \( \phi_j \) linked by a nearest-neighbor bond \( <i, j> \) have the Boltzmann weight

\[
e^{-\beta J V(\phi_i - \phi_j)} = \sum_{n=-\infty}^{\infty} e^{-\beta J(\phi_i - \phi_j - 2\pi n)^2}. \tag{2.1}
\]

Here \( J > 0 \) sets the energy scale. The relation (2.1) implies that the Villain interaction \( V(\phi) \) depends on \( \beta J \); when \( \beta \) becomes large, only one term on the RHS of (2.1) will dominate, \( V(\phi) \) will tend to

\[
V(\phi) = \phi^2 \quad \text{for } |\phi| \leq \pi, \quad \beta = \infty \tag{2.2}
\]

and we may interpret \( \beta \) as the inverse temperature \( 1/k_B T \). For all \( \beta \), the sum on \( n \) in (2.1) guarantees the periodicity property \( V(\phi) = V(\phi + 2\pi) \) and the symmetry property \( V(\phi) = V(-\phi) \). The antiferromagnetic Villain model has \( V(\phi_i - \phi_j - \pi) \) instead of \( V(\phi_i - \phi_j) \).

Here we wish to consider an XY spin glass with randomly ferro- or antiferromagnetic Villain interactions, that is, with the Hamiltonian

\[
H = J \sum_{<i,j>} V(\phi_i - \phi_j - \pi_{ij}) \tag{2.3}
\]

where the sum runs over all nearest-neighbor bonds of the periodic lattice, and the \( \pi_{ij} \) are quenched random variables such that

\[
\pi_{ij} = \begin{cases} 
0 & \text{with probability } \frac{1}{2} \\
\pi & \text{with probability } \frac{1}{2} 
\end{cases} \tag{2.4}
\]

The expression for the canonical partition function of this XY spin glass then is

\[
Z_{M,N} = \int_{-\pi}^{\pi} \prod_i d\phi_i \sum_{\{n_{ij}\}} e^{-\beta J \sum_{<i,j>} (\phi_i - \phi_j - \pi_{ij} - 2\pi n_{ij} \pi)^2} \tag{2.5}
\]

in which the \( n_{ij} \) may be seen as additional dynamical variables, and the argument of the exponential as a new effective Hamiltonian. We shall henceforth write

\[
\nu_{ij} = n_{ij} / (2\pi) , \tag{2.6}
\]

which is integer (half-integer) when the interaction between \( \phi_i \) and \( \phi_j \) is ferromagnetic (antiferromagnetic).

**B. Transformation to a Coulomb gas**

1. **From XY spin glass to a SOS model**

The transformation from an XY model to an SOS model is well-known \[14,15\] and has been extended to various types of random XY models. Here we carefully study the finite size effects, that determine the ground state energy
differences under different boundary conditions. In (2.5) we wish to carry out the integrations on the \( MN \) variables \( \varphi_i \). To that end we arbitrarily select a lattice site \( i_0 \) and transform to the new variables of integration

\[
\varphi_0 = \varphi_{i_0} \tag{2.7a}
\]

\[
\varphi_{ij} = \varphi_i - \varphi_j . \tag{2.7b}
\]

These are \( 2MN + 1 \) in number, and there exist \( MN + 1 \) relations between them. In order to formulate these, let \( r, s, \cdots \) be the position vectors of the plaquette centers. We adopt the convention that in \( <i,j> \) the site \( j \) is to the right of \( i \) (for a horizontal bond) or above \( i \) (for a vertical bond), which naturally extends across the periodic boundaries. We define furthermore a sum on the bonds \( <i,i'> \) surrounding a plaquette \( r \) by the diagram of Fig. 1

\[
\hat{\sum}_r \varphi_{ii'} \equiv \varphi_{12} + \varphi_{23} - \varphi_{43} - \varphi_{14} . \tag{2.8}
\]

The hat on the summation sign is a reminder of the sign convention in the RHS of (2.8). The variables \( \varphi_{ij} \) of (2.7b) then satisfy the relations

\[
\hat{\sum}_r \varphi_{ii'} = 0 \text{ mod } 2\pi \quad \text{for all } r . \tag{2.9a}
\]

These are \( MN \) relations of which only \( MN - 1 \) are independent. The two remaining relations correspond to loops around the torus and are

\[
\sum_x \varphi_{ij} \equiv \sum_{m=1}^{M} \varphi_{(m,n_1),(m+1,n_1)} = 0 \text{ mod } 2\pi \tag{2.9b}
\]

\[
\sum_y \varphi_{ij} \equiv \sum_{n=1}^{N} \varphi_{(m_1,n),(m_1,n+1)} = 0 \text{ mod } 2\pi \tag{2.9c}
\]

where \( i_1 \equiv (m_1,n_1) \) is another arbitrarily selected lattice site. Upon introducing the variables of integration (2.7) in Eq. (2.5) we must represent the conditions (2.9) by delta functions and find
\[
Z_{M,N} = 2\pi \int_{-\infty}^{\infty} \prod_{\langle i,j \rangle} d\varphi_{ij} \prod_{r \neq r_0} \delta \left( \sum_{r} \varphi_{ij} \mod 2\pi \right) \\
\times \delta \left( \sum^{x} \varphi_{ij} \mod 2\pi \right) \delta \left( \sum^{y} \varphi_{ij} \mod 2\pi \right) \\
\times \exp \left[ -\beta J \sum_{\langle i,j \rangle} \left( \varphi_{ij} - \pi_{ij} \right)^2 \right] 
\]

(2.10)

in which \( r_0 \) is an arbitrarily selected plaquette, and where a trivial factor 2\( \pi \) comes from the integration on \( \varphi_0 \). The delta functions may be represented by a Fourier sum with the aid of

\[
\delta (x \mod 2\pi) = \sum_{\ell = -\infty}^{\infty} \delta (x - 2\pi \ell) = (2\pi)^{-1} \sum_{\ell = -\infty}^{\infty} e^{i \ell x} 
\]

(2.11)

which requires the introduction of a set \( \{n_r\} \) of summation variables for the plaquettes, and of two summation variables \( n_x \) and \( n_y \) for the loops around the torus. One finds

\[
Z_{M,N} = (2\pi)^{-MN} \sum_{\{n_r\}} \sum_{n_x} \sum_{n_y} \int_{-\infty}^{\infty} \prod_{\langle i,j \rangle} d\varphi_{ij} \exp \left[ i \sum_{r} n_r \sum_{x} \varphi_{ij} \right] \\
\times \exp \left[ i n_x \sum_{x} \varphi_{ij} + i n_y \sum_{y} \varphi_{ij} \right] \\
\times \exp \left[ -\beta J \sum_{\langle i,j \rangle} \left( \varphi_{ij} - \pi_{ij} \right)^2 \right] 
\]

(2.12)

where the prime on the summation sign indicates the restriction to \( n_{r_0} = 0 \). The integrations on the \( \varphi_{ij} \) can now be performed.

Let the geometric relation between the pair of lattice sites \( \langle i,j \rangle \) and the pair of plaquette centers \( \langle r,s \rangle \) be as in Fig. 2.

Figure 2: Geometric relation between the pair of lattice sites \( \langle i,j \rangle \) and the pair of plaquette centers \( \langle r,s \rangle \).

It is furthermore useful to define

\[
\tau^{x(y)}_{rs} = \begin{cases} 
1 & \text{if } \langle i,j \rangle \text{ is part of the loop } (2.9b) \text{ (the loop } (2.9c) \rangle \\
0 & \text{otherwise}
\end{cases} 
\]

(2.13)

Henceforth we shall generally write \( \pi_{rs} \) instead of \( \pi_{ij} \) when \( \langle i,j \rangle \) and \( \langle r,s \rangle \) are related as in Fig. 2. With this notation the result of the \( \varphi_{ij} \) integrations in (2.12) is
\[ Z_{M,N} = (2\beta J)^{-MN} \sum_{n_x} \sum_{n_y} \left[ \frac{1}{2} \exp \left[ i \sum_{<r,s>} \pi_{rs} \left( n_r - n_s + \sum_{\alpha=x,y} \tau_{rs}^\alpha n_\alpha \right) \right] \right] \]

\[ \times \exp \left[ -\left( 4\beta J \right)^{-1} \sum_{<r,s>} \left( n_r - n_s + \sum_{\alpha=x,y} \tau_{rs}^x n_\alpha \right)^2 \right] . \]  

Equation (2.14) represents the partition function of a solid-on-solid model (or “column model”) with Gaussian interaction, in which, due to the randomness in the original XY model, some terms in occur with negative sign. In the nonrandom case with all \( \pi_{rs} = 0 \), we recover the partition function of the well-known discrete Gaussian model, summed on different step boundary conditions represented by the variables \( n_x \) and \( n_y \).

2. From SOS model to Coulomb gas

The transformation from an SOS model to a Coulomb gas is also well-known [16]. We have to apply it here to the random SOS model of Eq. (2.14), taking properly into account again all finite size effects. For a function \( f(n) \) of an integer variable \( n \) one has

\[ \sum_{n=-\infty}^{\infty} f(n) = \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu \ e^{2\pi i q \nu} f(\nu) . \]  

(2.15)

Applying this identity to the sum on the \( \{n_r\} \) in (2.14) we get

\[ Z_{M,N} = (2\beta J)^{-MN} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda \sum_{n_x} \sum_{n_y} \sum_{\{q_{\nu}\}} \int_{-\infty}^{\infty} \prod_{r} d\nu_r \times \]

\[ \times \exp \left[ 2\pi i \sum_{\nu_{\nu}} q_{\nu} \nu_r + 2\pi i \lambda \nu_0 + i \sum_{<r,s>} \pi_{rs} \left( \nu_r - \nu_s + \sum_{\alpha=x,y} \tau_{rs}^x n_\alpha \right) - \left( 4\beta J \right)^{-1} \sum_{<r,s>} \left( \nu_r - \nu_s + \sum_{\alpha=x,y} \tau_{rs}^x n_\alpha \right)^2 \right] \]

(2.16)

in which \( \nu_0 \equiv \nu_{r_0} \) and the integral on \( \lambda \) takes care of the condition \( n_{r_0} = 0 \) in (2.14). The integrations on the \( \nu_r \) are Gaussian. The most convenient way to carry them out requires some preliminaries. We introduce the coordinate representation \( r = (x, y) \) with \( x = 1, \cdots, M \) and \( y = 1, \cdots, N \), and the periodicity condition that \( (x+M, y) \) and \( (x, y+N) \) also denote the plaquette center \( r \). For horizontal and vertical pairs \( <r, s> \) we shall have, by convention, \( s = (x+1, y) \) and \( s = (x, y+1) \), respectively. For each plaquette center \( r \) we define a frustration variable \( p_r \) by

\[ 2\pi \ p_r = \sum_{i=1}^{R} \pi_{ii'} = \pi_{01} - \pi_{20} - \pi_{30} + \pi_{04} \]  

(2.17)

where the notation is as in Eq. (2.8), together with Fig. 3.
Let furthermore

\[ \Pi_\alpha = \sum_{<r,s>} \alpha \pi_{rs} \quad , \]

(2.18)

where the subscript \( \alpha = x \) (or \( \alpha = y \)) denotes restriction to vertical (or horizontal) bonds (recall that a horizontal pair \( <i,j> \) corresponds to a vertical pair \( <r,s> \) and vice versa). At this point it is useful to fix the positions of the two loops around the torus by the specific choice

\[ \tau_{rs}^{x} = \delta_{y,N} \quad \text{for} \quad <r,s> = <(x,y),(x,y+1)> \]

\[ \tau_{rs}^{y} = \delta_{x,M} \quad \text{for} \quad <r,s> = <(x,y),(x+1,y)> \quad (2.19) \]

We pass in Eq. (2.16) from the \( \nu_r \) to new variables of integration \( \nu'_r \) defined by

\[ \nu'_r = \nu_r + M^{-1}xn_y + N^{-1}yn_x \]

(2.20)

where as before \( r \) runs through \( (x,y) \) with \( x = 1, \cdots, M \) and \( y = 1, \cdots, N \). After rearranging terms in the exponential one finds for the partition function (2.16)

\[ Z_{M,N} = (2\beta J)^{-MN} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\lambda \sum_{n_x} \sum_{n_y} \exp \left[ -2\pi i \lambda (n_x + n_y) + i \left( M^{-1} \Pi_x n_y + N^{-1} \Pi_y n_x \right) - (4\beta J)^{-1} \left( N^{-1} M n_x^2 + M^{-1} N n_y^2 \right) \right] \]

\[ \times \sum_{\{qr\}} \exp \left[ -2\pi i \sum_r q_r \left( M^{-1} x n_y + N^{-1} y n_x \right) \right] \times Z'_{M,N} \]

(2.21a)

with

\[ Z'_{M,N} = \int_{-\infty}^{\infty} \prod_r d\nu'_r \exp \left[ 2\pi i \sum_r \nu'_r (q_r + p_r + \lambda \delta_{r,r_0}) - (4\beta J)^{-1} \sum_{<r,s>} (\nu'_r - \nu'_s)^2 \right] . \]

(2.21b)

We shall abbreviate

\[ Q_r = q_r + p_r + \lambda \delta_{r,r_0} \quad . \]

(2.22)
The integrals on the $\nu'$ in Eq. (2.21b) are now easily carried out with the aid of the Fourier variables

$$\tilde{\nu}'_k = (MN)^{-\frac{1}{2}} \sum_r e^{-ik\cdot r'} \nu'_r,$$

(2.23)

and analogously defined $\tilde{Q}_k$, where

$$k = (k_x, k_y) = 2\pi \left( M^{-1} \kappa_x, N^{-1} \kappa_y \right),$$

(2.24a)

$$\kappa_x = 0, 1, \ldots, M - 1, \quad \kappa_y = 0, 1, \ldots, N - 1.$$

(2.24b)

It is useful to make the specific choice $r_0 = (M, N)$. Taking properly care of the integration on $\tilde{\nu}'_0$, which is exceptional and gives a factor $\delta(Q_0)$, one finds from (2.21b)

$$Z'_{M,N} = (MN)^{\frac{1}{2}} \delta \left( \sum_r q_r + \lambda \right) C_{M,N} \exp \left[ -4\pi^2 \beta J \sum_{k \neq 0} \lambda_k^{-1} |Q_k|^2 \right],$$

(2.25)

with

$$\lambda_k = 4 \left( \sin^2 \frac{k_x}{2} + \sin^2 \frac{k_y}{2} \right),$$

(2.26)

$$C_{M,N} = \prod_{k \neq 0} \left( 4\pi \beta J / \lambda_k \right)^{\frac{1}{2}}.$$

(2.27)

Upon inserting (2.25) in (2.21a) one may carry out the $\lambda$ integration with the result

$$Z_{M,N} = (2\beta J)^{-MN} (MN)^{\frac{1}{2}} C_{M,N} \sum_{n_x} \sum_{n_y} \exp \left[ i \left( M^{-1} \Pi_y n_y + N^{-1} \Pi_x n_x \right) - (4\beta J)^{-1} \left( N^{-1} M n_x^2 + M^{-1} N n_y^2 \right) \right]$$

$$\times \sum_{\{q_r\}'} \exp \left[ -2\pi i \sum_r \left( M^{-1} x n_y + N^{-1} y n_x \right) - \beta H_C \left( \{q_r + p_r\} \right) \right].$$

(2.28)

Here the prime restricts the summation to neutral “charge” configurations $\{q_n\}$, i.e. satisfying

$$\sum_r q_r = 0,$$

(2.29)

and $H_C$ is the Coulomb Hamiltonian

$$H_C \left( \{q_r + p_r\} \right) = 8\pi^2 J \sum_r \sum_{r'} U_{M,N}(r - r') (q_r + p_r) (q_{r'} + p_{r'}).$$

(2.30)

where $U_{M,N}$ is the Coulomb potential on a periodic lattice,

$$U_{M,N}(r) = \frac{1}{2MN} \sum_{k \neq 0} \frac{e^{-ik\cdot r} - 1}{\lambda_k}.$$

(2.31)
This function has the periodicity properties

\[ U_{M,N}(r) = U_{M,N}(r + Me_1) = U_{M,N}(r + Ne_2) \]  

where \( e_1 \equiv (1,0) \) and \( e_2 \equiv (0,1) \).

The last step needed is to transform the sums on \( n_x \) and \( n_y \) in (2.28) according to Eq. (2.15), by introducing continuous variables \( \nu_x \) and \( \nu_y \), and new summation variables \( q_x \) and \( q_y \). The integrations on \( \nu_x \) and \( \nu_y \) are again Gaussian and easily carried out. The result can be slightly rewritten with the aid of the relations

\[ \begin{align*}
  M^{-1} \left( \Pi_y - 2\pi \sum_r x q_r \right) &= -2\pi M^{-1} P_x + \pi_y \\
  N^{-1} \left( \Pi_x - 2\pi \sum_r y q_r \right) &= -2\pi N^{-1} P_y + \pi_x
\end{align*} \]

where \( \pi_y \) and \( \pi_x \) are sums along loops around the torus,

\[ \begin{align*}
  \pi_y &= \sum_{y=1}^{N} \pi_{(M,y),(1,y)} \\
  \pi_x &= \sum_{x=1}^{M} \pi_{(x,N),(x,1)}
\end{align*} \]

and \( \mathbf{P} \) is the electric dipole moment

\[ \mathbf{P} = (P_x, P_y) = \sum_r r(q_r + p_r) \]  

The final result for \( Z_{M,N} \) then becomes

\[ Z_{M,N} = (2\beta J)^{-MN} (MN)^{1/2} C_{M,N} \sum_{\{q_r\}} \Theta_{M/N} \left( N^{-1} P_y - \pi_y/(2\pi) \right) \Theta_{M/N} \left( M^{-1} P_x - \pi_x/(2\pi) \right) e^{-\beta \mathcal{H}_C} \]  

where

\[ \Theta_a(u) = \left( \frac{4\pi \beta a}{2} \right)^{1/2} \sum_{q=-\infty}^{\infty} \exp \left[ -4\pi^2 \beta a (q - u)^2 \right] \]

has the periodicity property \( \Theta_a(u) = \Theta_a(u + 1) \), and the prime refers to the charge neutrality condition (2.29).

C. Summary

In view of the relatively technical character of the above transformation, and for easy later reference, we summarize here the result. The partition function \( Z_{M,N} \) (see Eq. (2.5)) of the ±J Villain XY model (Eqs. (2.3) and (2.4)) can be expressed as the partition function of a system of Coulomb charges \( \{q_r\} \) on the dual lattice,

\[ Z_{M,N} = 2\pi(2\beta J)^{-MN+1} (MN)^{1/2} C_{M,N} \sum_{q_x,q_y=-\infty}^{\infty} \sum_{\{q_r\}} e^{-\beta \hat{\mathcal{H}}(q_x,q_y,\{q_r\})} \]

where

\[ \hat{\mathcal{H}}(q_x,q_y,\{q_r\}) = 4\pi^2 JNM^{-1} \left[ q_y + N^{-1} \sum_r y(q_r + p_r) + (2\pi)^{-1} \sum_{x=1}^{M} \pi_{(x,N),(x,1)} \right]^2 \]
\[+4\pi^2 JN^{-1}M[q_x + M^{-1}\sum_r x(qr + pr) + (2\pi)^{-1}\sum_{y=1}^N \pi_{(M,y),(1,y)}]^2 \]

\[+8\pi^2 J\sum_r \sum_{r'} U_{MN}(r - r')(qr + pr)(qp' + p_{r'}) \quad (2.38b)\]

The Coulomb potential \(U_{MN}\) and the constant \(C_{M,N}\) are given by (2.31) and (2.27), respectively, together with (2.26) and (2.24). The \(qr\) run through all integer values subject to the charge neutrality condition (2.29). The sums on \(r\) and \(r'\) run through \((x,y)\) with \(x = 1, \ldots, M\) and \(y = 1, \ldots, N\). Each nearest neighbor bond \((r,s)\) on the charge lattice is dual to a nearest neighbor bond \((i,j)\) of the original spin lattice, and the disorder variables \(\pi_{rs}\) that appear in (2.38) are equal to the corresponding \(\pi_{ij}\). The frustration variables \(p_r\) are defined in terms of the \(\pi_{rs}\) by the diagram of Fig. 3; they are half-integer for a frustrated plaquette, and integer for an unfrustrated plaquette. Even though the first two terms in (2.38b) seem to favor a definite coordinate representation of the lattice, one may verify with the aid of some algebra that \(Z_{M,N}\) is invariant under translation of the origin, as of course it should be.

The transformation from the ferromagnetic 2D XY model to the Coulomb gas Hamiltonian \(H_C\), including the charge neutrality condition, has been known since long [15]. Similarly the replacement of the charges \(qr\) by \(qr + pr\) in the random case has been known since Villain [1] and was treated in a more general context in Ref. [17]. Eqs. (2.38) show how on a finite lattice the charge configurations receive a supplementary weight, represented by the first two terms in (2.38b), that depend on the electric dipole moment, in agreement with [18]. Whereas this weight plays no rôle for a bulk system in the thermodynamic limit, it is essential for finite size scaling considerations, and therefore (see Eqs. (1.2) and (1.3)) for the determination of the low temperature behavior of the correlation lengths. This fact will be exploited in the example of Section 5 and in the general heuristic theory of Section 6.

### III. ANTI-PERIODIC BOUNDARY CONDITIONS

In view of the remark preceding Eq. (2.3), it is very simple to change the periodic boundary conditions in the system discussed above into antiperiodic boundary conditions (we shall always take these along the seam joining the \(M\)th and the 1st column). It amounts to changing \(\pi_{ij}\) into \(\pi_{ij} + \pi\) on the antiferromagnetic seam. This just means drawing another member of the class of random systems under consideration. In particular, frustrated (unfrustrated) plaquettes remain frustrated (unfrustrated). The corresponding antiperiodic partition function \(Z_{M,N}^{AP}\) differs from Eq. (2.36) only in that \(\pi_x/(2\pi)\) is replaced by \(\pi_x/(2\pi) + \frac{1}{2}\). This difference is at the origin of the finite size effect of interest.

The following example, although trivial, shows how ground state energy differences are extracted from the final equations of Section 2. We compare the ground states of an XY ferromagnet with periodic and antiperiodic boundary conditions. The energy difference \(\Delta E^{AP}\) is that of a spin wave of wavelength \(2M\), which can be written down immediately:

\[\Delta E^{AP} = NMJ \left(\frac{2\pi}{2M}\right)^2 = \pi^2 N J.\]  \( (3.1)\)

To see how the same result can be obtained from Eq. (2.36) we use that for both types of boundary conditions all \(pr\) vanish, and in the ground state all \(qr\) as well; furthermore that the periodic boundary conditions have \(\pi_x = 0\) and the antiperiodic ones \(\pi_x = \frac{1}{2}\). This gives

\[\Delta E^{AP} = -\lim_{\beta \to \infty} \frac{1}{\beta} \log \Theta_{N/M}(\frac{1}{2})\]  \( (3.2)\)

which in view of (2.37) exactly coincides with (3.1).

The \(\Theta\) functions in Eq.(2.36), or equivalently, the first two terms in Eq.(2.38b), can be traced back mathematically to the global constraint on the spin variable differences \(\varphi_{ij}\). This example shows that they represent the energy of a continuous spin wave deformation forced into the system by the boundary conditions. We shall extend this interpretation to the case of general disorder, and speak of a “global spin wave”.

Ground state energy differences between P and AP boundary conditions for more complicated situations can also be obtained from (2.36), or (2.38), at least in those cases where the ground state of the charge system is known or can be plausibly guessed. A nontrivial example is discussed in Section 5.
IV. REFLECTING BOUNDARY CONDITIONS

Reflecting boundary conditions are introduced into the XY Hamiltonian by letting the spins \( \mathbf{S}_{(1,n)} \) in the first lattice column interact with the images of the \( \mathbf{S}_{(M,n)} \) in the last column under reflection about an axis in spin space. The counterpart of the periodic Hamiltonian (2.3) is

\[
\mathcal{H}^R = J \sum_{<i,j>} \text{reg} V(\varphi_i - \varphi_j - \pi_{ij}) + J \sum_{<i,j>} \text{exc} V(\varphi_i + \varphi_j - \pi_{ij}^\prime). \tag{4.1}
\]

Its partition function will be denoted by \( Z^R_{M,N} \), the upper index \( R \) indicating, here and henceforth, reflecting boundary conditions. The subscript “reg” in (4.1) refers to the regular terms and the subscript “exc” to the exceptional ones, modified by the reflecting boundary conditions.

Transformation to a Coulomb gas

The conversion of the Hamiltonian (4.1) into a Coulomb gas Hamiltonian proceeds via the same succession of transformations as in the case of periodic boundary conditions. However, several differences occur, and we shall indicate the main modifications below.

(i) When transforming to the variables of Eq. (2.7), it is convenient to choose \( i_0 = (1,1) \). The expressions for \( \varphi_i + \varphi_j \) in the exceptional terms in (4.1) then become

\[
\varphi(M,n) + \varphi(1,n) = \varphi(M,n),(1,n) - 2 \sum_{\ell = 2}^{n} \varphi(1,\ell - 1),(1,\ell) + 2\varphi_0 \tag{4.2}
\]

\[ (n = 1,2,\ldots,N) \]

(ii) The integration on \( \varphi_0 \) can be carried out only after the introduction of the plaquette and loop variables \( \{n_r\}, n_x, \) and \( n_y \), and leads to the result \( 2\pi \delta_{n_x,0} \). The integrations on the \( \varphi(M,n),(1,n) \) and \( \varphi,(1,n-1),(1,n) \) are exceptional.

(iii) The discrete Gaussian partition function, obtained as an intermediate result, now becomes

\[
Z^R_{M,N} = (2\beta J)^{-MN} \sum_{n_y} \sum_{\{n_r\}}' \exp \left[ i \sum_{<r,s>} \text{reg} \pi_{rs}(n_r - n_s) - (4\beta J)^{-1} \sum_{<r,s>} \text{reg} (n_r - n_s)^2 \right] \]

\[
\times \exp \left[ i \sum_{<r,s>} \text{exc} \pi_{rs}(-n_r - n_s + n_y) - (4\beta J)^{-1} \sum_{<r,s>} \text{exc} (n_r - n_s - n_y)^2 \right] \tag{4.3}
\]

in which, as before \( r = (x,y) \) with \( x = 1,\cdots,M \) and \( y = 1,\cdots,N, \) and the exceptional terms are the horizontal bonds linking the \( M \)th to the first column; we have chosen \( \pi_{rs}^\prime = 1 \) if \( \{r,s\} \) is exceptional (and hence \( \pi_{rs}^\prime = 0 \) if \( \{r,s\} \) is regular) ; and the prime restricts the summation to configurations \( \{n_r\} \) with \( n_{(M,N)} = 0 \). The exceptional terms show that between the columns \( M \) and 1, in addition to the step boundary condition represented by the variable \( n_y \), a reflection is imposed with respect to the zero level of the column heights.

(iv) Upon continuing the succession of transformations, one finds that the integral on \( n_y \) gives \( 2\delta(2q_y + \sum_r qr + \lambda) \). After integration over \( \lambda \) this becomes \( 2\delta(2q_y - \sum_r qr,0) \) where \( \delta(a,b) \equiv \delta_{a,b} \), and after summation on \( q_y \) one gets \( 2\delta(\sum_r qr \mod 2,0) \), i.e. the sum of the charges \( qr \) should be even.

(v) The relevant wavevectors are now

\[
k = (k_x,k_y) = 2\pi \left( M^{-1} \left( \kappa_x + \frac{1}{2} \right), N^{-1} \kappa_y \right) \tag{4.4}
\]
with $\kappa_x$ and $\kappa_y$ as in (2.24b); this set of wavevectors will be referred to by the upper index $R$ on summation and product signs. We define in particular

$$C_{M,N}^{R} = \prod_{k}^{R} \left(4\pi\beta J/\lambda_{k}\right)^{\frac{1}{2}} .$$ (4.5)

(vi) The variable $q_{r} - \delta_{x,M} \pi \mathbf{r} / \pi$, which appears in the Coulomb Hamiltonian, may be renamed $q_{r}$ by a shift of the variables $q_{(M,y)}$.

The final result is

$$Z_{M,N}^{R} = 2(2\beta J)^{-MN} C_{M,N}^{R} \sum_{\{q_{r}\}} \text{par} e^{-\beta \mathcal{H}_{C}^{R}}$$ (4.6)

where the subscript “par” refers to the parity condition

$$\left(\sum_{r} q_{r} + \pi y / \pi\right) \mod 2 = 0$$ ,

and where

$$\mathcal{H}_{C}^{R} (\{q_{r} + p_{r}\}) = 8\pi^{2} J \sum_{r} \sum_{r'} U_{M,N}^{R}(r - r') (q_{r} + p_{r}) (q_{r'} + p_{r'})$$ (4.8)

with $r$ and $r'$ in the range $(x,y), x = 1, \ldots, M, y = 1, \ldots, N$, and with

$$U_{M,N}^{R}(r) = \frac{1}{2 MN} \sum_{k}^{R} e^{\bar{i} k r} \lambda_{k} .$$ (4.9)

This function has the (anti-)periodicity properties

$$U_{M,N}^{R}(r) = U_{M,N}^{R}(r + Ne_{2}) = -U_{M,N}^{R}(r + Me_{1}) .$$ (4.10)

Hence two charges at a fixed finite distance on opposite sides of the reflecting boundary interact with each other’s charge conjugated image: reflecting boundary conditions for the $XY$ spins lead to charge conjugating boundary conditions for the Coulomb charges.

We now comment on this result and compare it to its counterpart for P and AP boundary conditions Eq. (2.36). The fact that in the partition function (4.6) no charge neutrality is imposed becomes understandable if one writes the interaction $U^{R}$ as

$$U_{M,N}^{R}(r) = U_{2M,N}(r) - U_{2M,N}(r + Me_{1})$$ (4.11)

which is easily checked. It is as though we have a double system, of size $2M \times N$ and with periodic boundary conditions, in which every charge at a site $r$ of the original system is paired up with an image charge, equal but of opposite sign, at the corresponding site $r + Me_{1}$. Using Eq. (4.11) and obvious symmetry properties one can make this explicit by writing the Hamiltonian (4.8) as

$$\mathcal{H}_{C}^{R} (\{q_{\rho} + p_{\rho}\}) = 4\pi^{2} J \sum_{\rho} \sum_{\rho'} U_{2M,N}(\rho - \rho')(q_{\rho} + p_{\rho})(q_{\rho'} + p_{\rho'})$$ , (4.12)

where $\rho$ and $\rho'$ run through the $2M \times N$ lattice and it is understood that corresponding sites carry opposite charges. The extra prefactor $\frac{1}{2}$ that (4.12) has with respect to (4.8) indicates that the energy density has been smeared out over twice the volume. The self-interaction present in (4.8) has become the interaction energy between a charge and its image in (4.12).
V. AN EXAMPLE

A. Specific expressions for the disorder. Ground state and ground state energy with \( P \) boundary conditions.

As an example we consider an \( M \times N \) lattice containing a rectangular array of \( M' \times N' \) frustrated plaquettes, consisting of the points \( r_{a,b} \equiv (x_a,y_b) \) with \( a = 1, 2, \ldots, M' \) and \( b = 1, 2, \ldots, N' \), where \( y_b = bN/N' = bt_y \) with \( t_y \) an even integer, and the \( x_a \) are drawn randomly and independently with a density \( M'/M \equiv 1/\ell_x \). The density of frustrated plaquettes is therefore \( 1/\ell_x \ell_y \). (Note that drawing the frustrated plaquettes randomly is not the same thing as drawing the negative bonds randomly \([23, 24]\); the latter procedure creates pairs of frustrated plaquettes, leading to a Coulomb gas with quenched dipoles rather than quenched charges). We let \( Q^0_{\nu} \) denote the ground state value of \( q_r + pr \). The arguments of this section rest on the hypothesis that we can find the true ground state. For \( M' \) and \( N' \) even we may safely assume that the ground state has the chessboard array of charges

\[
Q^0_{(x_a,y_b)} = \pm \frac{1}{2} (-1)^{a+b}
\]

and zero charges on all the other plaquettes. The ground state value of \( \mathcal{H}_c \) therefore is, from (5.1) and (2.30),

\[
\mathcal{H}_c(\{Q^0_{\nu}\}) = 2\pi^2 J \sum_{a,b} \sum_{a',b'} U_{M,N}(r_{a,b} - r_{a',b'})(-1)^{a+a'+b+b'}
\]

Upon using the explicit form of the lattice Coulomb potential, Eqs. (2.31) and (2.26), we can rewrite (5.2) as a sum of interaction energies between the charge carrying lattice columns,

\[
\mathcal{H}_c(\{Q^0_{\nu}\}) = \pi^2 J \ell_y^{-2} N \sum_{a,a'} (-1)^{a+a'} \sum_{\nu} V_{\nu}(x_a - x_a')
\]

where \( \nu \) runs through the values \( \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots, \pm \left( \frac{1}{2} \ell_y - \frac{1}{2} \right) \), and in which, with \( c_{\nu} \equiv \cos 2\pi\nu \ell_y^{-1} \),

\[
V_{\nu}(x) = \frac{1}{M} \sum_{k_x} \frac{e^{ik_xx}}{4 - 2\cos k_x - 2c_{\nu}}
\]

\[
\simeq \frac{1}{2} \left[ (2 - c_{\nu})^2 - 1 \right]^{-1/2} \left[ (2 - c_{\nu})^2 - 1 \right]^{1/2} |x|^{2}\]

the last step involving the limit \( M \to \infty \) at finite \( |x| \). This expression takes the much simpler form

\[
V_{\nu}(x) \sim (2\pi|\nu|)^{-1} \ell_y \exp(-2\pi|\nu| \ell_y^{-1})
\]

in the limit \( \ell_x, \ell_y \to \infty \) with \( \ell_x/\ell_y \) fixed and \( x \) of order \( \ell_x \). In this limit, also, the sum on \( \nu \) in (5.3) can be evaluated exactly and one finds

\[
\mathcal{H}_c(\{Q^0_{\nu}\}) \simeq -\pi J \ell_y^{-1} N \sum_{a,a'} (-1)^{a+a'} \log \tanh(\pi|x_a - x_a'|/\ell_y^{-1})
\]

This shows that the interaction energy is negative between two neighboring columns \( (a \text{ and } a+1) \), that it diverges logarithmically at small distances \( |x_a - x_{a'}| \), as expected, and tends to zero as \( \exp(-\pi|x_a - x_{a'}|/\ell_y^{-1}) \) for large \( |x_a - x_{a'}| \), so that it is short-ranged.

One remark is in place. The limit \( \ell_x, \ell_y \to \infty \) corresponds to a dilute array of frustrated plaquettes. These must be joined pairwise by long ladders of negative bonds – for which there is no physical reason. The analysis that follows will use Eq. (5.6); however qualitatively similar results can be obtained from Eqs. (5.3) and (5.4), where \( \ell_x \) and \( \ell_y \) are arbitrary, the key point being that the large distance behavior of the column-column interaction is still determined by the \( \nu = \pm \frac{1}{2} \) terms in Eq. (5.3). We shall come back to the case of general \( \ell_x \) and \( \ell_y \) in the discussion in Sec. 6.

B. Comparison of \( P, R, \) and \( AP \) boundary conditions

We now wish to consider the ground state energies and their differences for the three different boundary conditions, \( P \), \( AP \), and \( R \).

(i) Periodic boundary conditions
For the frustrated plaquettes located on the array defined in the beginning of this section, we can always choose the negative bonds such that \( \pi_x = \pi_y = 0 \). Since the ground state \( \{Q^0_T\} \) has \( P_z = P_y = 0 \) (see Eq. (2.35)), the first two terms in (2.38b) are minimized by \( q_x = q_y = 0 \) and so do not contribute to the ground state energy, which is therefore entirely given by (5.6).

One possible type of excitations from the ground state are those consisting of reversing all charges \( Q^0_T \) in one column \( a \). The excitation energy is of the order of

\[
\Delta E(a) \equiv 2 \pi J \xi_y^{-1} N \log \tanh(\pi \xi_y^{-1})
\]

where \( \xi_a \) is the distance between \( x_a \) and the closer one of \( x_{a-1} \) and \( x_{a+1} \). Because of its exponential decay with \( |x_a - x_{a'}| \), the interaction between \( a \) and more distant columns does not change this estimate.

(ii) Reflecting boundary conditions

We first of all have to find the a priori unknown ground state. To this end we consider again the chessboard configuration of charges of Eq. (5.1) and calculate its energy \( H^R (\{Q^0_T\}) \) under reflecting boundary conditions, starting from Eqs. (4.8) and (4.9). The result is again Eq. (5.6), but with charge conjugating boundary conditions, the interaction terms between columns \( a \) and \( a' \) on different sides of the reflecting boundary have an extra minus sign.

Since the number \( M \) of columns was supposed even, this means that there is somewhere a mismatch consisting of two neighboring columns \( a \) and \( a+1 \) that are identically instead of oppositely charged. The energy cost is \( \sim \Delta E(\xi) \), with \( \xi \) the distance between these columns, and can be minimized by shifting the mismatch to the largest interneighbor distance. The typical maximum distance that can be found is

\[
\xi_{\text{max}} \simeq \ell_x \log \left( \frac{M}{\ell_x} \right) + O(1) \quad (M \to \infty)
\]

and therefore the ground state energy difference \( \Delta E^R \) between reflecting and periodic boundary conditions is of the order of \( \Delta E(\xi_{\text{max}}) \), which leads to

\[
\Delta E^R \sim \left( \frac{N}{\xi_y} \right) \left( \frac{M}{\ell_x} \right)^{-\pi \xi_x/\ell_y}
\]

We can now link the finite size scaling of the ground state to the low temperature behavior of the correlation length as indicated in the introduction. Since the mechanism leading to Eq.(5.9) is the reversal of large domains of chiral variables, one deduces that in case \( \pi \xi_x/\ell_y > 1 \), there is in the thermodynamic limit \( M \to N \to \infty \) no chiral order at any finite temperature \( T \), and that for \( T \downarrow 0 \) the chiral correlation length \( \xi_c \) diverges with an exponent \( \nu_c = 1/(1 - \pi \xi_x/\ell_y) \). In case \( \pi \xi_x/\ell_y < 1 \) (which corresponds to relatively strong spatial anisotropy), one deduces similarly that the system has a low temperature phase with long range chiral order. We shall come back to both cases after discussing AP boundary conditions.

(iii) Antiperiodic boundary conditions

Under AP boundary conditions the energy of a charge configuration is again given by (2.38b). In this case, \( H_c (\{Q^0_T\}) \) is as given by (5.6), but since now \( \pi_y = 0 \) and \( \pi_x = \pi \), the energy of the charge configuration \( \{Q^0_T\} \) has an extra contribution \( \pi^2 J N \ell^M \) from the first two terms in (2.38b), which is exactly the spin wave energy encountered in the discussion of the ferromagnet in Section 3. One cannot, however, without further inspection, identify this contribution as the ground state energy difference \( \Delta E^{AP} \), since \( \{Q^0_T\} \) is not necessarily the ground state any more. The remainder of the discussion depends on the value of \( \ell_x/\ell_y \). We discuss first the case \( \pi \xi_x/\ell_y > 1 \). Curiously, in this case the mechanism to construct a lower lying state is the same as it was for \( R \) boundary conditions, viz. columnwise change reversal.

Reversing a single column leaves \( M^{-1} P_z = 0 \) but changes \( N^{-1} P_y \) from 0 to \( \frac{1}{2} \), which annihilates the effect of \( \pi_x = \pi \) and therefore cancels the spin wave contribution \( \pi^2 J N \ell^M \). The same cancellation can be obtained by reversing the charges in an odd number of otherwise arbitrary columns. The energy cost is the excitation energy of the columns. It can be minimized by choosing an odd number of consecutive columns \( a, a+1, \ldots a' \), such that the distances \( |x_{a-1} - x_a| \) and \( |x_{a'+1} - x_{a'}| \) are as large as possible. Typically, they will again be of order \( \ell_x \log (M/\ell_x) \). Hence we arrive for \( \Delta E^{AP} \) at the estimate

\[
\Delta E^{AP} \sim \left( \frac{N}{\xi_y} \right) \left( \frac{M}{\ell_x} \right)^{-\pi \xi_x/\ell_y} \quad (\pi \xi_x/\ell_y > 1)
\]

identical to \( \Delta E^{R} \). This energy difference is less than the spin wave energy \( \pi^2 J N \ell^M \), and therefore the ground state will adjust to AP boundary conditions by a chiral excitation, and not by a spin wave excitation. In view of Eq.(1.3)
and our result for \( \nu_c \) found above we now conclude that \( \nu_s = \nu_c = 1/(1 - \pi \ell_x/\ell_y) \). Hence at low temperature the spin and chiral correlation lengths, \( \xi_s \) and \( \xi_c \), behave as

\[
\xi_s \sim \xi_c \sim T^{-1/(1-\pi \ell_x/\ell_y)} \quad (\pi \ell_x/\ell_y > 1)
\]

(5.11)

In case \( 0 < \pi \ell_x/\ell_y < 1 \) (large spatial anisotropy, and, as seen above, a low temperature phase with long range chiral order), the situation is different. The minimum excitation energy of the columns is still \((N/\ell_y)(M/\ell_x)^{-\pi \ell_x/\ell_y}\) but this is higher than the spin wave contribution which it cancels. Therefore in this case the ground state \( \{Q_{P}^0\} \) remains unchanged (and in the spin representation acquires only an additional global spin wave), and

\[
\Delta E_{AP} = \pi^2 JN M^{-1} \quad (0 < \pi \ell_x/\ell_y < 1)
\]

(5.12)

In the thermodynamic limit \( M \sim N \to \infty \) it follows (see Eqs. (1.2) and (1.3)) that \( \nu_s = \infty \), which is most naturally (although not strictly necessarily) associated with a low temperature phase with power law decay of the spin-spin correlation. In this case the energy \( \Delta E_{AP} \) is not the result of a domain wall, but smeared out across the system.

All of the above discussion is for an even number \( M' \) of charge carrying columns. We now comment on the differences that intervene when \( M' \) is odd. In that case the value of \( N^{-1} P_y \) in (2.36) is \( \pm \frac{1}{4} \) and is changed to \( \mp \frac{1}{4} \) (modulo 1) under replacement of \( P \) by \( AP \) boundary conditions. Hence the value of \( \Theta_{M/N} \) does not change and (since \( M^{-1} P_x = 0 \) both before and after) we have \( \Delta E_{AP} = 0 \). The interpretation is that a global spin wave is “caught” in the system and cannot release its energy. When we replace \( P \) by \( R \) boundary conditions, the global spin wave terms disappear from the Hamiltonian and release their energy so that \( \Delta E_{R} \) is negative. The phenomenon of spin wave energy release was first observed by Kawamura and Tanemura [3] in their Monte Carlo simulations of an \( XY \) model with general disorder. It appears here analytically, but seemingly as a consequence of the distinction between even and odd \( M' \) and hence closely tied to the present example. But we shall see in the next section that it also appears in the general case.

VI. GENERAL CASE

On the basis of the experience gained we now attempt a theory, admittedly heuristic, for the general case. Let, as before, \( \mathcal{H}_c \) denote the standard lattice Coulomb Hamiltonian (2.30), and \( \mathcal{H}_c^R \) its counterpart (4.8) with \( R \) boundary conditions. Let in this section \( \mathcal{H}_P \) denote the Hamiltonian \( \mathcal{H} \) of Eq. (2.38b), and \( \mathcal{H}_{AP} \) its counterpart with \( AP \) boundary conditions. We shall take the unknown ground state \( \mathcal{Q}^0 \) of \( \mathcal{H}_c \) as our reference state and denote its energy by \( E_0 \). Our strategy will be to try to determine how the ground states of \( \mathcal{H}_c^R \) and \( \mathcal{H}_{P,AP}^R \) differ from \( \mathcal{Q}^0 \).

Since the remaining discussion concerns the low-temperature regime, it is convenient to work with charges \( \pm \frac{1}{4} \) on the frustrated plaquettes and charges zero on all other plaquettes. This simplification, that was also made by Villain [12], leads to the problem of an Ising model on a random lattice (namely, the one composed of centers of the frustrated plaquettes), with logarithmically increasing antiferromagnetic interactions and zero magnetization. The logarithmic interaction will be screened in a way that we cannot precisely describe [it has to be if the energy per charge is to be finite] and it is reasonable to imagine that there exist some effective interaction decaying as a power law between spatially separated neutral sets of charges.

Assuming that we know the ground state \( \mathcal{Q}^0 \), the next relevant question is what the lowest lying excitations are. Clearly all excitations can be described in terms of charge reversals with respect to the ground state \( \mathcal{Q}^0 \), or alternatively, in terms of contours on the dual lattice (given any reasonable planar graph representing the nearest neighbor relations on the lattice of charges). In the spirit of Fisher and Huse [26] we shall consider the lowest lying excitations involving the reversal of order \( \sim L^2 \) charges and localized in an area of linear size \( \sim 2L \) around a preassigned point in space. Such excitations will be called droplets. Let their excitation energy scale as \( L^{-1/\nu_c} \), where, since we are below \( d_c \), the exponent \( \nu_c \) is positive. Due to the long-range forces, the droplets so defined need not constitute single domains. [For short-range forces they should and reduce to the droplets of Fisher and Huse [26] theory.] A typical scale-\( L \) droplet may have to be represented by a set of disconnected contours. We shall assume, nevertheless, that it is compact enough so that there is always, typically, one main contour enclosing \( \sim L^2 \) charges. This amounts to assuming that a domain wall can be defined, whatever its width, around the reversed area. Excitations that would not fall in this category are, for example, those that consist of many small-size domains dispersed in the volume \( L^2 \), or fractal excitations extending throughout the volume \( L^2 \) but having fractal dimension less than 2. Our hypothesis is that such reversals have excitation energies at least as high as those of the scale-L droplets.

The ground states \( \mathcal{Q}_P^0 \) and \( \mathcal{Q}_c^0 \) of \( \mathcal{H}_c^R \) and \( \mathcal{H}_c \), respectively, must differ at least by a contour going around the torus in the \( y \) direction, that is, by a scale-\( N \) droplet. For the energy difference between the two ground states we
therefore have
\[ E_0^R - E_0 \simeq c N^{-1/\nu_c} \] (6.1)
where \( c \) is a (positive or negative) random constant, and \( \nu_c \) is as before the exponent of the chirality-chirality correlation function, see Eq. (1.2). This gives a way to determine this exponent.

We now turn to the boundary conditions P and AP. Let \( Q^{P,AP} (\hat{Q}^{0,AP}) \) be the ground state of \( \hat{H}^P (\hat{H}^{AP}) \) and let \( E_0^P (E_0^P) \) be its energy. Since \( \hat{H}^{P,AP} \) differ from \( \hat{H}_c \) by the addition of two quadratic terms, we have necessarily \( E_0^P - E_0^P \geq 0 \). In the ground state \( Q^{0} \) these additional terms take a random positive value of order 1 (at least, if we assume that \( M^{-1}P_x \) and \( N^{-1}P_y \) do so in the ground state). This value can be reduced by reversing domains of charges in \( Q^0 \). The ground state \( Q^{0,P} (Q^{0,AP}) \) is now determined by a compromise between the minimization of \( \hat{H}_c \) and the minimization of the additional terms, which drive the total electric dipole moment to a specific value. Since this is a global biasing force, and since the excitation energies go down with increasing length scale, we expect that the compromise leads to a \( Q^{0,P} (Q^{0,AP}) \) that differs from \( Q^0 \) by an excitation on the scale \( N \) of the system. This excitation might not be of the same type as the droplet excitation discussed above, and therefore we put
\[ E_0^{P,AP} - E_0 \simeq c^{P,AP} N^{-1/\nu_s} \] (6.2)
where \( c^P \) and \( c^{AP} \) are positive random constants and the exponent \( \nu_s \) is also positive, it is the spin correlation length exponent of Eq. (1.2). But since, by hypothesis, the droplet excitations are the lowest lying ones that exist at each given scale, the excitation that leads to (6.2) is, at best, also a scale-\( N \) droplet excitation in which case \( \nu_s = \nu_c \), or is possibly a combination of droplet excitations on smaller scales in which case one might have \( \nu_s^{-1} \leq \nu_c^{-1} \). Hence
\[ \nu_s^{-1} < \nu_c^{-1}. \] (6.3)

Upon combining these considerations we arrive at the conclusion
\[ \Delta E^{AP} \simeq (c^{AP} - c^P) N^{-1/\nu_s} \]
\[ \Delta E^R \simeq -c^P N^{-1/\nu_s} - c N^{-1/\nu_c} \] (6.4)
in which \( c^{P,AP} \geq 0 \) and \( c \) may be of either sign with \( \bar{c} = 0 \). It follows that \( \Delta E^R \simeq -c^P N^{-1/\nu_s} \), which is negative. This is the phenomenon of the release of spin wave energy as one passes from P to R boundary conditions. One sees furthermore that
\[ \frac{\Delta E^{AP}}{\Delta E^R} \simeq N^{-1/\nu_c} \]
i.e. both energy differences scale with the same power of \( N \).

We now discuss the two possibilities implied by Eq. (6.3). If it holds with the equality sign, we are in the case of a single correlation length for the spin and the chiral variables, as found in the example studied. This case needs no further comment. If Eq. (6.3) were to hold as a strict inequality, then there are two different exponents \( \nu_s \) and \( \nu_c \). This would imply a longer correlation length for the spin variables than for the chiral variables. Moreover, \( \Delta E^R \) is then not the appropriate quantity to determine \( \nu_c \). In fact, in this case \( \nu_c \) cannot be found from a comparison of P, AP and R boundary conditions, but should result from a comparison of \( \hat{H}_c \) and \( \hat{H}_c^P \) according to Eq. (6.1). Kawamura and Tanemura [3] extracted the exponent \( \nu_s \) from a numerical determination of \( \Delta E^{AP} \) and \( \nu_c \) from a postulated expression involving both \( \Delta E^{AP} \) and \( \Delta E^R \). In this work there is a strict inequality that goes in the sense opposite to (6.3).

The conclusion is that we find no evidence for chiral order extending on a longer length scale than spin order.

**VII. FURTHER COMMENTS AND CONCLUSIONS**

We have considered the two-dimensional XY spin glass with ±J interactions. In sections 2 through 4 general formulas are presented that exhibit the interplay between chiral and spin variables in determining domain wall energies on finite \( M \times N \) lattices with various types of boundary conditions. In Section 5 we have considered, as an example, a specific type of disorder with infinite ranged correlations in the \( y \) direction. The spatial distribution of the frustrated plaquettes depends on two parameters \( \ell_x \) and \( \ell_y \), whose ratio is an anisotropy parameter. The scaling properties of the domain wall energies with system size lead us to the following conclusions.
(i) $XY$ spin glasses with critical temperature $T = 0$

For $\ell_x, \ell_y \to \infty$ and for $\pi \ell_x/\ell_y > 1$, the system of Sec. 5 has only a $T = 0$ critical point at which the chiral and spin correlation lengths diverge with the same exponent $\nu_s = \nu_c = 1/(1 - \pi \ell_x/\ell_y)$. We add here without proof that the phenomenon of a $T = 0$ transition with $\nu_s = \nu_c$ holds in the entire region of the $\ell_x, \ell_y$ plane determined by

$$4\ell_x(\ell_x - 1) \sin^2 \frac{\pi}{2\ell_y} > 1 \quad (\ell_x > 1 \quad \text{and} \quad \ell_y = 1, 2, \ldots)$$

(7.1)

This phenomenon is analogous to what we found on the one-dimensional ladder [6] and tube [25] lattices. It is distinct, however, from the scenario proposed by Kawamura and Tanemura [2, 3] and by Ray and Moore [5] for the two-dimensional uncorrelated random $\pm J$ $XY$ spin glass, according to which one would have $0 < \nu_c^{-1} < \nu_s^{-1}$. In order to rule out the possibility that our conclusions are restricted to a specific class of correlated disorder, we consider in Section 6 the general case of random disorder. The heuristic theory presented yields a different inequality, namely $0 \leq \nu_c^{-1} \leq \nu_s^{-1}$. We therefore find no evidence for chiral order extending on a longer scale than spin order in two dimensions.

One may now go one step further and speculate that since we had $\nu_s = \nu_c$ for $d = 1$, the mechanism uncovered above, which enables the ground state to accommodate to a change from P to AP boundary conditions by a low energy chiral excitation, is general for uncorrelated random $\pm J$ $XY$ model, and that $\nu_s = \nu_c$ for all $d < d_c$.

(ii) random $XY$ models with nonzero critical temperature

This class encompasses the model of Section 5 when Eq. (7.1) is not satisfied. In this case there is a low temperature phase with long range chiral order and, very probably, power law decay of the spin-spin correlations. This is the same scenario that is believed to hold for a fully frustrated two-dimensional $XY$ model [20-22], which is, as a matter of fact, recovered in the special limit $\ell_x = \ell_y = 1$, and which is characterized by $\nu_c^{-1} < 0 = \nu_s^{-1}$. It cannot, therefore, be identified with the scenario $\nu_c^{-1} < 0 < \nu_s^{-1}$ that Kawamura and Tanemura [3] propose for the three-dimensional random $\pm J$ $XY$ model.

Finally we comment on the transition line $4\ell_x(\ell_x - 1) \sin^2 (\pi/2\ell_y) = 1$ between the regions (i) and (ii). Clearly the distinction between these two types of ground state behavior is not due to the density of frustrated plaquettes, which is $1/\ell_x\ell_y$, but to the nature of correlations in their spatial arrangement. The different behaviors can be distinguished by the ground state response to a change between P and AP boundary conditions. In region (ii), the ground state is elastic (it deforms continuously), whereas in region (ii) it is “brittle” (it responds by the formation of a chiral domain wall). This transition appears in this work somewhat as a byproduct of the analysis, but it is interesting in itself and merits a separate study.

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