Conical singularities in thin elastic sheets

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Abstract

When one slightly pushes a thin elastic sheet at its center into a hollow cylinder, the sheet forms (to a high degree of approximation) a developable cone, or d-cone for short. Here we investigate one particular aspect of d-cones, namely the scaling of the elastic energy with the sheet thickness \( h \). Following recent work of Brandman, Kohn and Nguyen [2] we study the Dirichlet problem of finding the configuration of minimal elastic energy when the boundary values are given by an exact d-cone. We improve their result for the energy scaling. In particular, we show that the deviation from the logarithmic energy scaling is bounded by a constant times the double logarithm of the thickness.

1 Introduction

Stress and energy focusing in thin elastic sheets has recently attracted a lot of interest in the physics literature [10, 3, 5, 12, 6, 8, 9, 11, 4]. One basic feature are (almost) conical singularities. A conical singularity arises, e.g., in the following experiment. Put an elastic sheet of radius 1 concentrically on top of a hollow cylinder of radius \( R < 1 \) and push the sheet down at its centre. It has been observed that the sheet assumes (to a high degree of approximation) the shape of a developable cone (or d-cone for short). In the physics literature, this has been discussed e.g. in [3, 5, 10, 12]. There are several remarkable features of the d-cone: The angle subtended by the region where the sheet lifts off the rim of the container is a universal constant (approx. \( 139^\circ \)), independent of the indentation, the thickness and the material of the sheet (for small indentations, [5]). The tip of the d-cone consists of a crescent-shaped ridge where curvature and elastic stress focus. In numerical simulations it was found that the radius of the crescent \( R_{\text{cres}} \) scales with the thickness of the sheet \( h \) and the radius of the container \( R_{\text{cont}} \) as \( R_{\text{cres}} \sim h^{1/3}R_{\text{cont}}^{2/3} \). This dependence on the container radius of the shape of the region near the tip is not understood [12]. As argued in this latter reference, it cannot be explained by an analysis of the dominant contributions to the elastic energy, which are: The bending energy from the region far away from the center, that is well captured by modeling the d-cone as a developable surface there; and the bending and stretching energy part from a core region of size \( O(h) \) where elastic strain is not negligible. The result of this (non-rigorous) argument is an energy scaling \( E \sim h^2(C_1|\log h| + C_2) \).

Here, we discuss the scaling of the elastic energy with \( h \) in a rigorous setting.

The natural variational formulation is to minimize the elastic energy of the sheet (which contains stretching and bending contributions) in the class of deformations \( y : B_1 \to \mathbb{R}^3 \) which satisfy the obstacle constraint

\[
\text{Im}(y) \cap \{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 = R_{\text{cont}}^2, x_3 < H_{\text{cont}}, y(0) = 0 \} = \emptyset
\]
where $B_1 = \{ x \in \mathbb{R}^2 : |x| \leq 1 \}$ is the reference configuration of the 2-dimensional sheet, and $H_{\text{cont.}}$ is the height of the container. The derivation of a precise lower bound of the form $E \sim h^2(C_1 \log h + C_2)$ for this model looks very hard. Indeed even, for the much more severe constraint that $y$ maps $B_1$ into a very small ball the only known rigorous lower bound is that $\lim_{h \to \infty} E_h(y)/h^2 = \infty$ while it is conjectured that the correct scaling is $E_h(y) \sim C h^{5/3}$ in this setting (see [7] for a discussion and a rigorous proof of the upper bound). To make progress in rigorously understanding the asymptotic influence of the regularizing effect of the thickness $h$ we follow recent work of Brandman, Kohn and Nguyen [2] and free ourselves from the specific obstacle type constraint and consider instead general Dirichlet problems where the boundary conditions are given by an exact developable cone.

This also partly motivated by results that have been obtained in the physics literature for a simpler model. In [5], the class of allowed deformations $y$ is restricted to isometries with a singularity at the origin. These maps are completely determined by their values on the boundary $\partial B_1$. In order for these maps to be isometric away from the origin, the boundary values have to be unit speed curves

$$y|_{\partial B} = \gamma : \partial B_1 \to S^2.$$ 

This effectively reduces the problem from a 2-dimensional to a 1-dimensional one. After a suitable ad-hoc renormalization of the bending energy (i.e., cutting out a small ball around the origin where the energy density becomes singular), the elastic energy is minimized as a functional of $\gamma$. In this simpler setting, an obstacle of the above type is treatable. In fact, the shape of $\gamma$ in the region where the sheet lifts off the rim of the cylinder is prescribed by an ODE that can be solved more or less explicitly (for small indentations, see [5]).

## 2 Setting and statement of the main theorem

Let $B_r = \{ x \in \mathbb{R}^2 : |x| < r \}$, $A_r = B_r \setminus \overline{B_{r/2}}$ and for $y \in W^{2,2}(B_1, \mathbb{R}^3)$ let

$$E_h(y) = \int_{B_1} (|\nabla y|^2 \nabla y - Id|^2 + h^2|\nabla^2 y|^2) \, dx. \quad (1)$$

Furthermore, for a curve $\gamma \in C^3(\partial B_1, \mathbb{R}^3)$ with $|\gamma| = |\gamma'| = 1$, let

$$V_\gamma = \{ y \in W^{2,2}(B_1, \mathbb{R}^3) : y|_{\partial B_1} = \gamma, \ y(0) = 0 \}.$$ 

In the following, consider such a $\gamma$ to be fixed. By $\tilde{y}(x) = |x|\gamma(\tilde{x})$, we denote the 1-homogeneous surface with boundary values prescribed by $\gamma$.

Existence of minimizers of $E_h$ in the class $V_\gamma$ follows easily from the fact that $E_h$ is coercive and convex in the highest derivatives and the compact embedding $W^{2,2} \hookrightarrow W^{1,4}$. Our main result is

**Theorem 1.** Suppose that $\gamma$ does not lie in a plane. Then for sufficiently small $h$ we have

$$C_1 \ln \frac{1}{h} - C_1 \ln \left( \ln \left( \frac{1}{h} \right) \right) - C_2 \leq \ln \frac{1}{h^2} \min_{y \in V_\gamma} E_h(y) \leq C_1 \ln \frac{1}{h} + C_3,$$

where

$$C_1 = C_1(\gamma) = \frac{1}{\ln 2} \int_{B_1 \setminus B_{1/2}} |\nabla^2 \tilde{y}|^2 \, dx$$

and $C_2, C_3$ only depend on $\gamma$.

This improves work of Brandman, Kohn and Nguyen [2] who showed that

$$\liminf_{h \to 0} \frac{1}{h \ln(1/h)} \min E_h \geq C_1/2$$

and

$$\limsup_{h \to 0} \frac{1}{h \ln(1/h)} \min E_h \leq C_1$$

and, very recently in
parallel to our work, that \( \lim_{h \to 0} \frac{1}{h^{1/ln(1/h)}} \min E_h = C_1 \). The result above shows that the deviation from the leading order logarithm is at most a double logarithm. Indeed a natural conjecture is that this error is of order 1, but we have not been able to prove or disprove this so far.

Our proof of the crucial lower bound consists of three steps: First we estimate the \( L^\infty \)-norm of \( y \) in the ball \( B_h \) (see Lemma 4 below). Then we derive the key estimate for the \( L^2 \)-norm of \( e = y - \tilde{y} \) on dyadic rings \( A_{2^{-j}} \) (see Lemma 5 below). Finally we argue that since \( y \) is close to \( \tilde{y} \) in \( L^2 \), the bending energy of \( y \) can be bounded from below by the bending energy of \( \tilde{y} \), up to a small error.

## 3 Upper bound and \( L^2 \) estimate of \( e = y - \tilde{y} \)

The proof of the upper bound is standard and we include it for the convenience of the reader.

**Lemma 1.**

\[
\inf_{y \in V, h} E_h(y) < C_1 h^2 \ln \frac{1}{h} + C_2 h^2.
\]

where \( C_1 = (\ln 2)^{-1} \int_{B_1 \setminus B_{1/2}} |\nabla^2 \tilde{y}|^2. \)

**Proof.** Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a \( C^2 \) function with \( \phi(t) = t \) for \( t \geq 1 \) and \( \phi(t) = 0 \) for \( t \leq 1/2 \) and set

\[
y_h(x) = h \phi \left( \frac{x}{h} \right) \frac{x}{|x|}. \]

Then \( y_h = \tilde{y} \) on \( B_1 \setminus B_h \). Hence \( (\nabla y)^T \nabla y = Id \) in \( B_1 \setminus B_h \) and

\[
\int_{B_1 \setminus B_h} (|\nabla y^T \nabla y - Id|^2 + h^2 |\nabla^2 y|^2) \, dx = C_1 \ln \frac{1}{h}. \tag{2}
\]

Moreover in \( B_h \) one has the estimates \( |\nabla y_h| \leq C \) and \( |\nabla^2 y_h| \leq C/h \). This implies the assertion.

Now we prove some auxiliary lemmas which will allow us to estimate \( \sup_{B_h} |y| \). This estimate will be needed in the proof of the \( L^2 \) bound for of \( y - \tilde{y} \) on dyadic rings.

**Lemma 2.** Let \( v \in W^{2,2}(B_h) \). Then

\[
\sup_{x \in B_h} \left| v(x) - v(0) - \left( \int_{B_h(0)} \nabla v(x') \, dx' \right) \cdot x \right| \leq Ch \| \nabla^2 v \|_{L^2(B_h)}.
\]

**Proof.** For \( h = 1 \) this follows from the embedding \( W^{2,2}(B_1) \hookrightarrow C^0(B_1) \) and the Poincaré inequality. For \( h \neq 1 \) the assertion follows by considering the rescaled function \( u(x) = \frac{1}{h} v(hx) \).

**Lemma 3.** Let \( w \in W^{1,2}(B_1) \), \( 0 < \epsilon < 1 \). Then

\[
\left| \int_{B_{2\epsilon}} w \, dx - \int_{B_1} w \, dx \right| \leq C \left( \ln \frac{1}{\epsilon} \right)^{1/2} \left( \int_{B_1} |\nabla w|^2 \, dx \right)^{1/2}.
\]

**Proof.** The Poincaré inequality for \( w - \int_{B_h} w \) implies that for \( R \in [1/2, 1) \)

\[
\left| \int_{B_{R}} w \, dx - \int_{B_1} w \, dx \right| \leq C \int_{B_1} |\nabla w| \, dx \tag{3}
\]

and scaling yields in particular

\[
\left| \int_{B_{r/2}} w \, dx - \int_{B_r} w \, dx \right| \leq C \frac{1}{r} \int_{B_r} |\nabla w| \, dx.
\]

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Apply this with $r = 2^{-k}$ for $k = 0, \ldots, n - 1$ and define
\[ f(x) := \sum_{k=0}^{n-1} 2^k \chi_{B_{2^{-k}}} . \tag{4} \]
Then
\[ \left| \int_{B_{2^{-n}}} w \, dx - \int_{B_1} w \, dx \right| \leq C \int_{B_1} \left| f \right| \delta w \, dx \leq \|f\|_{L^2(B_1)} \left\| \delta w \right\|_{L^2(B_1)} \]
Now we have $f \leq 2^{k+1}$ in $B_{2^{-k}} \setminus B_{2^{-k-1}}$ for $k \leq n - 1$ and $f \leq 2^{n+1}$ in $B_{2^{-n}}$. This implies that $\|f\|_{L^2(B_1)} \leq C \sqrt{n}$. Choose $n$ such that $2^{-n} \geq \epsilon > 2^{-(n+1)}$. Then scaling of (3) and the Cauchy-Schwarz inequality yield
\[ \left| \int_{B_n} w \, dx - \int_{B_{2^{-n}}} w \, dx \right| \leq C2^n \int_{B_{2^{-n}}} |\delta w| \, dx \leq C\|\delta w\|_{L^2(B_{2^{-n}})} \]
which completes the proof since $n \leq \ln_2 \frac{1}{\epsilon}$.

**Lemma 4.** There exists constants such that for all $0 < h \leq 1/4$ and all $y \in V_\gamma$ we have
\[ \sup_{B_{\frac{1}{4}h}} |y| \leq Ch + Ch \left( \ln \frac{\ln h}{h} \right)^{1/2} \left\| \nabla^2 y \right\|_{L^2(B_{\frac{1}{4}h})} \leq C - h \left( \ln \frac{\ln h}{h} \right)^{1/2} E_{h^{1/2}}(y) . \tag{5} \]

**Proof.** This follows from Lemma 2 and Lemma 3 (applied with $w = \nabla v$) and the following calculation
\[ \int_{B_1} \nabla y \, dx = \int_{\partial B_1} y \otimes \nu \, dS = \int_{\partial B_1} \gamma \otimes \nu \, dS \]
which yields
\[ \left| \int_{B_1} \nabla y \, dx \right| \leq 2\pi = 2. \tag*{□} \]

We now come to the key estimate for the difference between a low energy map $y$ and $\tilde{y}$ in the $L^2$ norm on annuli. The idea is to look at fibres in radial direction in the domain, i.e. at line segments connecting the origin with $\partial B_1$. By the upper bound on the elastic energy, the “stretching” of $y$ on such a line segment is small. The boundary values of $y$ on the line segment are fixed as well, and so the deviation of $y$ from the straight line connecting the boundary points cannot be large.

**Lemma 5.** Let $h$ be small enough, $2h \leq r_0 \leq 1$, and assume that $E_h(y) \leq 2C_1 \ln \frac{1}{h}$. Then
\[ \int_{B_{r_0} \setminus B_{r_0/2}} |y - \tilde{y}|^2 \, dx \leq C r_0^3 h \ln \frac{1}{h} + C r_0^2 h^2 \left( \ln \frac{1}{h} \right)^2 , \]
where $C$ is a constant that only depends on $\gamma$.

**Remark** Note that the second term on the right hand side of the estimate is controlled by the first as long as $r_0 \geq h \ln \frac{1}{h}$.

**Proof.** We consider polar coordinates and set
\[ \eta(r, \theta) := y(r \cos \theta, r \sin \theta), \quad e(r, \theta) := (y - \tilde{y})(r \cos \theta, r \sin \theta) \tag{6} \]
and we write $e' = (\partial / \partial r) e$ and $\eta' = (\partial / \partial r) \eta$. By Fubini’s theorem the maps $r \mapsto \eta(r, \theta)$ and $r \mapsto e(r, \theta)$ are weakly differentiable for a.e. $\theta$.

1. The key estimate is
\[ \int_0^{2\pi} \int_0^1 |e'(\rho, \theta)|^2 \, d\rho \, d\theta \leq C h \ln \frac{1}{h} . \tag{7} \]
To prove this note that $|\eta'|^2 = |\gamma + e'|^2 = 1 + 2\gamma \cdot e' + |e'|^2$ which yields
\[ |e'|^2 = |\eta'|^2 - 1 - (2\gamma \cdot e') \]
since $\gamma$ depends only on $\theta$ but not on $r$. Using the orthonormal basis $x/|x|$, $x^\perp/|x|$ we get the pointwise estimate
\[ |(\nabla y)^T \nabla y - \text{Id}|^2 \geq (|\eta'|^2 - 1)^2. \]
By the boundary condition we have $e(1, \theta) = 0$ and Lemma 4 yields $|e(h, \theta)| \leq Ch \ln \frac{1}{h}$ (since $|\tilde{g}(x)| = |x|$). Thus an application of the Cauchy-Schwarz inequality gives
\[
\begin{align*}
\int_0^{2\pi} \int_h^1 |e'(\rho, \theta)|^2 \, d\rho \, d\theta \\
\leq \int_0^{2\pi} \int_h^1 (|\eta'|^2 - 1) \, d\rho \, d\theta + Ch \ln \frac{1}{h} \\
\leq \left( \int_0^{2\pi} \int_h^1 (|\eta'|^2 - 1)^2 \, d\rho \, d\theta \right)^{1/2} \left( \int_0^{2\pi} \int_h^1 \frac{1}{\rho} \, d\rho \, d\theta \right)^{1/2} + Ch \ln \frac{1}{h} \\
\leq E_h(y)^{1/2} \left( 2\pi \ln \frac{1}{h} \right)^{1/2} + Ch \ln \frac{1}{h}
\end{align*}
\]
and (7) follows from the assumption on $E_h(y)$.

2. The Cauchy-Schwarz inequality yields
\[ |e(r, \theta) - e(h, \theta)| \leq r^{1/2} \left( \int_h^r |e'(\rho, \theta)|^2 \, d\rho \right)^{1/2}. \]
for a.e. $\theta$. Taking the square, integrating over $\theta$ and using (7) we get
\[
\int_0^{2\pi} |e(r, \theta) - e(h, \theta)|^2 \, d\theta \leq Crh \ln \frac{1}{h}.
\]
By Lemma 4 we have
\[ |e(h, \theta)|^2 \leq Ch^2 \left( \ln \frac{1}{h} \right)^2 \]
and the assertion follows by integrating these two inequalities from $r_0/2$ to $r_0$ with respect to the measure $r \, dr$.

\[ \square \]

4 Proof of Theorem 1

We first recall two interpolation inequalities for Sobolev functions.

**Lemma 6.** Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $v \in W^{1,2}(\Omega)$, $u \in W^{2,2}(\Omega)$. Then
\[
\begin{align*}
\|v\|_{L^2(\partial \Omega)}^2 &\leq C\|v\|_{L^2(\Omega)}\|v\|_{W^{1,2}(\Omega)}, \\
\|\nabla u\|_{L^2(\Omega)}^2 &\leq C\|u\|_{L^2(\Omega)}^2 + C\|u\|_{L^2(\Omega)}\|\nabla^2 u\|_{L^2(\Omega)}
\end{align*}
\]
where all constants $C$ only depend on $\Omega$.

**Proof.** The first inequality follows from the continuity of the trace operator $W^{1/2,2}(\Omega) \to L^2(\partial \Omega)$, and the fact that $W^{1/2,2}(\Omega)$ is a real interpolation space of the pair $(L^2(\Omega), W^{1,2}(\Omega))$ which yields
\[
\|v\|_{L^2(\partial \Omega)}^2 \leq C\|v\|_{W^{1/2,2}(\Omega)}^2 \leq C\|v\|_{L^2(\Omega)}\|v\|_{W^{1,2}(\Omega)},
\]
see [1]. The second interpolation inequality follows directly from Theorem 5.2. in [1] which states that for all \( \epsilon \leq 1 \)

\[
\| \nabla u \|_{L^2(\Omega)} \leq K \left( \frac{1}{\epsilon} \| u \|_{L^2(\Omega)} + \epsilon \| \nabla^2 u \|_{L^2(\Omega)} \right)
\]

If \( \| \nabla^2 u \|_{L^2(\Omega)} \leq \| u \|_{L^2(\Omega)} \) one can take \( \epsilon = 1 \) to obtain (9), otherwise one takes \( \epsilon^2 = \| u \|_{L^2(\Omega)}/\| \nabla^2 u \|_{L^2(\Omega)} \).

**Proof of Theorem 1.** Let \( M \in \mathbb{N} \) (to be chosen later). Recall that \( e = y - \tilde{y} \). We have

\[
\frac{1}{h^2} E_h(y) \geq \int_{B_1 \setminus B_{2-M}} |\nabla^2 y|^2 dx = \int_{B_1 \setminus B_{2-M}} |\nabla^2 \tilde{y}|^2 dx - 2 \int_{B_1 \setminus B_{2-M}} \nabla^2 e : \nabla^2 \tilde{y} dx + \int_{B_1 \setminus B_{2-M}} |\nabla^2 e|^2 dx
\]

Consider the second term on the right hand side with the integration restricted to the annulus \( A_{r_0} \).

We define \( \hat{e} : A_1 \rightarrow \mathbb{R}^3 \) by \( \hat{e}(x) = r_0^{-1} e(r_0 x) \). Observe that

\[
\nabla^2 \hat{e}(x) = r_0 (\nabla^2 e)(r_0 x), \quad \nabla^2 \tilde{y}(x) = r_0 (\nabla^2 \tilde{y})(r_0 x)
\]

Since \( \tilde{y} \) is 1-homogeneous we get

\[
\int_{A_1} \nabla^2 \hat{e} : \nabla^2 \tilde{y} dx = \int_{A_1} r_0^2 (\nabla^2 e)(r_0 x) : (\nabla^2 \tilde{y})(r_0 x) dx = \int_{A_{r_0}} \nabla^2 e : \nabla^2 \tilde{y} dx.
\]

We integrate by parts to obtain

\[
\int_{A_1} \nabla^2 \hat{e} : \nabla^2 \tilde{y} dx = \int_{A_1} \hat{e}_{i,j} \cdot \tilde{y}_{,ij} dx - \int_{A_1} \hat{e}_{,i} \cdot \tilde{y}_{,ij} dx + \int_{\partial A_1} \hat{e}_{,i} \cdot \tilde{y}_{,ij} dS,
\]

where \( \nu \) is the unit outer normal on \( \partial A_1 \). Hence

\[
\left| \int_{A_1} \nabla^2 \hat{e} : \nabla^2 \tilde{y} dx \right| \leq \| \nabla^3 \tilde{y} \|_{L^2(A_1)} \| \nabla \hat{e} \|_{L^2(\partial A_1)} \| \nabla^2 \tilde{y} \|_{L^2(\partial A_1)}
\]

\[
\leq C \left( \| \nabla \hat{e} \|_{L^q(\Omega)} + \| \nabla \hat{e} \|_{H^{-1,2}(\Omega)} \right)^{1/2}
\]

\[
\leq C \left( \| \nabla \hat{e} \|_{L^q(\Omega)} + \| \nabla \hat{e} \|_{H^{-1,2}(\Omega)} \right)^{1/2}
\]

where we used Lemma 6. Applying again the second inequality in that lemma, we get

\[
\left| \int_{A_1} \nabla^2 \hat{e} : \nabla^2 \tilde{y} dx \right| \leq C \left( \| \hat{e} \|_{L^2(\Omega)} + \| \hat{e} \|_{L^2(\Omega)}^{1/2} \| \nabla^2 \hat{e} \|_{L^2(\Omega)}^{1/2} + \| \hat{e} \|_{L^2(\Omega)}^{1/4} \| \nabla^2 \hat{e} \|_{L^2(\Omega)}^{3/4} \right)
\]

We apply Young’s inequality \( ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'} \) with the pairs \( p = \frac{4}{3}, p' = 4 \) and \( p = \frac{8}{5}, p' = \frac{5}{2} \), and \( \delta - p' = \frac{1}{h^2} \) to obtain

\[
\left| \int_{A_1} \nabla^2 \hat{e} : \nabla^2 \tilde{y} dx \right| \leq C \left( \| \hat{e} \|_{L^2(\Omega)} + \| \hat{e} \|_{L^2(\Omega)}^{2/3} \| \nabla^2 \hat{e} \|_{L^2(\Omega)}^{2/3} + \| \hat{e} \|_{L^2(\Omega)}^{2/5} \right)
\]

Now we undo the rescaling. We have

\[
\| \hat{e} \|_{L^2(\Omega)} = r_0^{-2} \| e \|_{L^2(A_{r_0})}, \quad \| \nabla^2 \hat{e} \|_{L^2(\Omega)} = \| \nabla^2 e \|_{L^2(A_{r_0})}.
\]

By Lemma 5

\[
r_0^{-2} \| e \|_{L^2(A_{r_0})} \leq C \left( \frac{h}{r_0} \right)^{1/2} \left( \ln \frac{1}{h} \right)^{1/2}
\]
as long as
\[ r_0 \geq h \ln \frac{1}{h}. \]  
(13)

Thus
\[ \left| \int_{A_0} \nabla^2 e : \nabla^2 \tilde{y} \, dx \right| \leq C \left( \frac{h}{r_0} \right)^{1/5} \left( \ln \frac{1}{h} \right)^{1/5} + \frac{1}{2} \| \nabla^2 e \|^2_{L^2(A_{r_0})} \]  
(14)
as long as (13) holds.

Choose \( M \) such that
\[ \log_2 \frac{1}{h} - \log_2 \ln \frac{1}{h} \leq M \leq \log_2 \frac{1}{h} - \log_2 \ln \frac{1}{h} + 1. \]  
(15)
This implies that
\[ 2^{-(M-1)} \geq h \ln \frac{1}{h}. \]

In particular for sufficiently small \( h \) and for \( r_0 \geq 2^{-(M-1)} \) the inequality (13) holds and we also have \( r_0 \geq 2h \). We thus get
\[ \int_{B_1 \setminus B_{2^{-M}}} |\nabla^2 y|^2 \, dx \]
\[ \geq \int_{B_1 \setminus B_{2^{-M}}} |\nabla^2 \tilde{y}|^2 \, dx - 2 \int_{B_1 \setminus B_{2^{-M}}} \nabla^2 \tilde{y} : \nabla^2 e \, dx + \int_{B_1 \setminus B_{2^{-M}}} |\nabla^2 e|^2 \, dx \]
\[ \geq M \left( \int_{A_1} |\nabla^2 \tilde{y}|^2 \, dx \right) - 2C \sum_{j=0}^{M-1} 2^{j/5} h^{1/5} \left( \ln \frac{1}{h} \right)^{1/5} \]
\[ \geq M \left( \int_{A_1} |\nabla^2 \tilde{y}|^2 \, dx \right) - 20C 2^{(M-1)/5} h^{1/5} \left( \ln \frac{1}{h} \right)^{1/5} \]
\[ \geq \frac{1}{\ln 2} \left( \ln \frac{1}{h} - \ln \left( \ln \frac{1}{h} \right) \right) \left( \int_{A_1} |\nabla^2 \tilde{y}|^2 \, dx \right) - 20C \]  
(16)
This completes the proof of Theorem 1.

\[ \square \]

**Remark.** The estimate (16) in connection with the upper bound on \( E_h \) shows that for any \( y \) with \( E_h(y) \leq C_1 \ln \frac{1}{h} + C_3 \) (and in particular for any minimizer of \( E_h \)) we have
\[ \int_{B_1} |\nabla y^T \nabla y - Id|^2 \, dx \leq C_1 \ln \left( \frac{1}{h} \right) + C_4, \]  
(17)
\[ \int_{B_2} |\nabla^2 y|^2 \, dx \leq C_1 \ln \left( \frac{1}{h} \right) + C_4, \]  
(18)
where \( M \) is as in (15) and (16). In addition we may assume that (14) holds with the term \( \frac{1}{2} \| \nabla^2 e \|^2_{L^2(A_{r_0})} \) instead of \( \frac{1}{2} \| \nabla^2 e \|^2_{L^2(A_{r_0})} \) on the right hand side. Then the same argument as in (16) yields in addition that
\[ \frac{1}{2} \int_{B_1 \setminus B_{2^{-M}}} |\nabla^2 e|^2 \, dx \leq C_1 \ln \left( \frac{1}{h} \right) + C_4 \]  
(19)
These additional estimates can be used to improve the prefactor of \( \ln \left( \ln \frac{1}{h} \right) \) in the lower bound from 1 to \( \frac{1}{2} + \varepsilon. \)
References

[1] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] Jeremy Brandman, Robert V. Kohn, and Hoai-Minh Nguyen. Energy scaling laws for conically constrained thin elastic sheets. *Preprint*, 2012.

[3] E. Cerda and L. Mahadevan. Conical surfaces and crescent singularities in crumpled sheets. *Phys. Rev. Lett.*, 80:2358–2361, Mar 1998.

[4] E. Cerda and L. Mahadevan. Geometry and physics of wrinkling. *Phys. Rev. Lett.*, 90:074302, Feb 2003.

[5] E. Cerda and L. Mahadevan. Confined developable elastic surfaces: cylinders, cones and the elastica. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 461(2055):671–700, 2005.

[6] Enrique Cerda, Sahraoui Chaieb, Francisco Melo, and L. Mahadevan. Conical dislocations in crumpling. *Nature*, 401:46–49, 1999.

[7] Sergio Conti and Francesco Maggi. Confining thin elastic sheets and folding paper. *Arch. Ration. Mech. Anal.*, 187(1):1–48, 2008.

[8] B. A. DiDonna and T. A. Witten. Anomalous strength of membranes with elastic ridges. *Phys. Rev. Lett.*, 87:206105, Oct 2001.

[9] Eric M. Kramer and Thomas A. Witten. Stress condensation in crushed elastic manifolds. *Phys. Rev. Lett.*, 78:1303–1306, Feb 1997.

[10] Tao Liang and Thomas A. Witten. Crescent singularities in crumpled sheets. *Phys. Rev. E*, 71:016612, Jan 2005.

[11] Alex Lobkovsky, Sharon Gentges, Hao Li, David Morse, and T. A. Witten. Scaling properties of stretching ridges in a crumpled elastic sheet. *Science*, 270(5241):1482–1485, 1995.

[12] T. A. Witten. Stress focusing in elastic sheets. *Rev. Mod. Phys.*, 79:643–675, Apr 2007.