Heat kernel estimates and their stabilities for symmetric jump processes with general mixed polynomial growths on metric measure spaces

Joohak Bae*, Jaehoon Kang†, Panki Kim‡ and Jaehun Lee§

Abstract

In this paper, we consider a symmetric pure jump Markov process $X$ on a general metric measure space that satisfies volume doubling conditions. We study estimates of the transition density $p(t,x,y)$ of $X$ and their stabilities when the jumping kernel for $X$ has general mixed polynomial growths. Unlike [21], in our setting, the rate function which gives growth of jumps of $X$ may not be comparable to the scale function which provides the borderline for $p(t,x,y)$ to have either near-diagonal estimates or off-diagonal estimates. Under the assumption that the lower scaling index of scale function is strictly bigger than 1, we establish stabilities of heat kernel estimates. If underlying metric measure space admits a conservative diffusion process which has a transition density satisfying a general sub-Gaussian bounds, we obtain heat kernel estimates which generalize [2, Theorems 1.2 and 1.4]. In this case, scale function is explicitly given by the rate function and the function $F$ related to walk dimension of underlying space. As an application, we proved that the finite moment condition in terms of $F$ on such symmetric Markov process is equivalent to a generalized version of Khintchine-type law of iterated logarithm at the infinity.

Keywords: Dirichlet form; symmetric Markov process; transition density; heat kernel estimates; metric measure space.

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1 Introduction

Studies on heat kernel estimates for Markov processes have a long history and there are many beautiful results on this topic (see [1, 3, 5, 8, 11, 16, 14, 19, 20, 21, 27, 33, 36, 47, 49, 52] and reference therein). Since there is an intimate interplay between symmetric Markov processes and positive self-adjoint operators, the heat kernel is one of fundamental notions connecting Probability theory and Partial differential equation. Moreover, heat kernel estimates for Markov processes on metric measure spaces provide information on not only the behaviour of the corresponding processes but also intrinsic properties such as walk dimension of underlying space ([3, 5, 33, 47]).

Recently, heat kernel estimates for Markov processes with jumps have been extensively studied due to their importance in theory and applications (see [7, 9, 2, 11, 13, 15, 18, 19, 20, 23, 21, 22, 24, 25, 28, 29, 31, 32, 34, 35, 37, 38, 39, 40, 43, 44, 45, 46, 48, 50, 51, 54, 55] and reference therein). In [20], the authors obtained heat kernel estimates for pure jump symmetric Markov processes on metric measure space $(M,d,\mu)$, where $M$ is a locally compact separable metric measure space with metric $d$ and a positive Radon measure $\mu$ satisfying volume doubling property. The jumping kernel $J(x,y)$ of Markov process in [20] satisfies the following conditions:

$$c^{-1} \frac{V(x,d(x,y))}{V(x,d(x,y))} \phi(d(x,y)) \leq J(x,y) \leq c \frac{V(x,d(x,y))}{V(x,d(x,y))} \phi(d(x,y)),$$

where $V(x,r) = \mu(B(x,r))$ for all $x \in M$ and $r > 0$, and $\phi$ is a strictly increasing function on $[0, \infty)$ satisfying

$$c_1 (R/r)^{\beta_1} \leq \phi(R)/\phi(r) \leq c_2 (R/r)^{\beta_2}, \quad 0 < r < R < \infty$$

with $0 < \beta_1 \leq \beta_2 < 2$. Here, we say $\phi$ is the rate function since $\phi$ gives the growth of jumps. Under the assumptions (1.1), (1.2) and $V(x,r) \asymp \tilde{V}(r)$ for strictly increasing function $\tilde{V}$, the transition density $p(t,x,y)$ of Markov process satisfies the following estimates: for any $t > 0$ and $x, y \in M$,

$$p(t,x,y) \asymp \left( \frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi(d(x,y))} \right).$$

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(See [20 Theorem 1.2]. See also [21] where the extra condition \( V(x, r) \propto \tilde{V}(r) \) is removed). Here and below, we denote \( a \wedge b := \min\{a, b\} \) and \( f \asymp g \) if the quotient \( f/g \) remains bounded between two positive constants. We call a function \( \Phi \) the scale function for \( p(t, x, y) \) if \( \Phi(d(x, y)) = t \) provides the borderline for \( p(t, x, y) \) to have either near-diagonal estimates or off-diagonal estimates. Thus, \( \phi \) is the scale function in (1.3) so that the rate function and the scale function coincide when \( 0 < \beta_1 \leq \beta_2 < 2 \).

Moreover, we see that (1.1) is equivalent to (1.3) since \( p(t, x, y)/t \to J(x, y) \) weakly as \( t \to 0 \). Thus, for a large class of pure jump symmetric Markov processes on metric measure space satisfying volume doubling properties, (1.1) is equivalent to (1.3) under the condition (1.2) with \( 0 < \beta_1 \leq \beta_2 < 2 \).

One of major problems in this field is to obtain heat kernel estimates for jump processes on metric measure space without the restriction \( \beta_2 < 2 \). Recently, several articles have discussed this problem ([2, 21, 29, 37, 48, 51]). In [21], the authors established the stability results on heat kernel estimates of the form in (1.3) for symmetric jump Markov processes on metric measure spaces satisfying volume doubling property. Results in [21] cover metric measure spaces whose walk dimension is bigger than 2 such as Sierpinski gasket and Sierpinski carpet. Without the condition \( \beta_2 < 2 \), (1.1) is no more equivalent to (1.3) in general. In this case, we see from [21 Theorem 1.13] that (1.3) is equivalent to the conjunction of (1.1) and a cut-off Sobolev type inequality (CSJ(\( \phi \)) in [21 Definition 1.5]). Note that CSJ(\( \phi \)) always holds if \( \beta_2 < 2 \) (see [21 Remark 1.7]). As a corollary of main results in [21], it is also shown in [21] that (1.1) is equivalent to (1.3) if underlying space has walk dimension strictly bigger than \( \beta_2 \). In [29], similar equivalence relation was proved for \( \phi(r) = r^{\beta_2} \). In particular, the authors showed that the conjunction of (1.1) and generalized capacity condition is also equivalent to (1.3).

On the other hand, in [2, 48], new forms of heat kernel estimates for symmetric jump Markov processes in \( n \)-dimensional Euclidean spaces were obtained without the condition \( \beta_2 < 2 \). In particular, the results in [2, 48] cover Markov processes with high intensity of small jumps. In this case, the rate function and the scale function may not be comparable and the heat kernel estimates are written in a more general form including (1.3) with \( V(x, d(x, y)) = |x - y|^n \) (see [2 Theorem 1.2 and Theorem 1.4] and [48 Theorem 1.2]).

This paper is a continuation of authors’ journey on investigating the estimates of the transition densities of jump processes whose jumping kernels have general mixed polynomial growths. In this paper we consider a symmetric pure jump Markov process \( X \) on a general metric measure space \( M \) that satisfies volume doubling conditions. The purpose of this paper is two-fold. The first is to establish several versions of heat kernel estimates for a symmetric pure jump Markov process \( X \) whose jumping kernel satisfies mixed polynomial growths, i.e., (1.1) and (1.2) with \( 0 < \beta_1 \leq \beta_2 < \infty \). This will extend the main results in [2]. The second is to investigate conditions equivalent to our heat kernel estimates. This will extend the main results in [21] where the scale function is comparable to the rate function.

Since we will deal with several different types of heat kernel estimates, we will consider different assumptions in each case to obtain our results. First, under the assumption that the lower scaling index of the scale function is strictly bigger than 1, we establish an upper bound of heat kernel and its stability which generalize [21 Theorem 1.15] and [2 Theorem 4.5]. As in [2], the scale function is less than the rate function. Since \( M \) may not satisfy chain condition in general, upper bounds and lower bounds in a generalized version of [2 Theorem 1.4] may have different forms. To obtain sharp two-sided estimates, we further assume that metric measure space satisfies chain condition. Under the same assumption on the scale function and the chain condition, in Theorems 2.13, 2.16
and \[2.17\] we establish a sharp heat kernel estimates and their stability which are generalized version of \[21\] Theorem 1.13 and \[2\] Theorem 1.4.

For the extension of heat kernel estimates in \[2\] Theorem 1.2 and the corresponding stability result, we assume that underlying space admits conservative diffusion process whose transition density has a general sub-Gaussian bounds in terms of an increasing function \(F\) (see Definition \[2.3\]). The function \(F\) serves as a generalization of walk dimension for underlying space. Note that in \[2\] Theorem 1.2, \((d(x,y)/F_1(t))^2\) appears in the exponential term of the off-diagonal part and the order 2 is the walk dimension of Euclidean space. It is shown in \[33\] that the general sub-Gaussian bounds for diffusion is equivalent to the conjunction of elliptic Harnack inequality and estimates of mean exit time for diffusion process if volume double property holds a priori. Diffusion processes on Sierpinski gasket and generalized Sierpinski carpets satisfy our assumption \((5, 10)\). See also \[6, 8, 27, 49, 52\] for studies on stability of (sub-)Gaussian type heat kernel.

Diffusion processes on Sierpinski gasket and generalized Sierpinski carpets satisfy our assumption \(\mu\) and \(F\) estimates for diffusion processes on metric measure spaces. Under the general sub-Gaussian bound assumption on diffusion with \(F\), we can define scale function explicitly by using the rate function and \(F\) (see \[2.20\]). It is worth mentioning that we do not assume neither that the chain condition nor the upper scaling index of the scale function being strictly bigger than 1 in Theorem \[2.19\]. Note that, GHK(\(\Phi, \psi\)) in Theorem \[2.19\] is not sharp in general. Without the chain condition, even the transition density of diffusion may not have the sharp two-sided bounds. However, if the upper scaling index \(\beta_2\) of the rate function is strictly less than the walk dimension, our heat kernel estimates GHK(\(\Phi, \psi\)) is equivalent to \(1.\)

As an application of our heat kernel estimates, we also extend \[2\] Theorem 5.2 to the result on Markov processes on general metric measure spaces. In particular, we show that if the walk dimension of underlying space is \(\gamma > 1\), then \(\gamma\)-th moment condition for Markov process is equivalent to \(\limsup_{t\to\infty} d(x, X_t)/(\log \log t)^{1-1/\gamma} t^{1/\gamma} < (0, \infty)\). See Theorem \[5.4\] for the full version.

Notations: Throughout this paper, the positive constants \(a, A, c_F, c_\mu, C_1, C_2, C_\mu, C_L, C_U, \bar{C}_L, \beta_1, \beta_2, \delta, \gamma_1, \gamma_2, \eta, d_1, d_2\) will remain the same, whereas \(C, c, c_0, a_0, c_1, a_1, c_2, a_2, \ldots\) represent positive constants having insignificant values that may be changed from one appearance to another. The constants \(\alpha_1, \alpha_2, c_L, c_U\) remain the same until Section \[3.4\] and redefined in Section \[4\]. All these constants are positive and finite. The labeling of the constants \(c_0, c_1, c_2, \ldots\) begins anew in the proof of each result. \(c_i = c_i(a, b, c, \ldots), i = 0, 1, 2, \ldots\), denote generic constants depending on \(a, b, c, \ldots\). The constant \(\bar{C}\) in \(2.3\) may not be explicitly mentioned. Recall that we have introduced the notations \(a \land b = \min\{a, b\}\) and \(a \lor b := \max\{a, b\}\), \(\mathbb{R}_+ := \{r \in \mathbb{R} : r > 0\}\), and \(B(x, r) := \{y \in M : d(x, y) < r\}\). We say \(f \asymp g\) if the quotient \(f/g\) remains bounded between two positive constants. Also, let \(\lceil a \rceil := \sup\{n \in \mathbb{Z} : n \leq a\}\).

## 2 Settings and Main results

### 2.1 Settings

Let \((M, d)\) be a locally compact separable metric space, and \(\mu\) be a positive Radon measure on \(M\) with full support and \(\mu(M) = \infty\). We also assume that every ball in \((M, d)\) is relatively compact.

For \(x \in M\) and \(r > 0\), define \(V(x, r) := \mu(B(x, r))\) be the measure of open ball \(B(x, r) = \{y \in M : d(x, y) < r\}\).

**Definition 2.1.** (i) We say that \((M, d, \mu)\) satisfies the volume doubling property \(VD(d_2)\) with index...
$d_2 > 0$ if there exists a constant $C_\mu \geq 1$ such that
\[
\frac{V(x,R)}{V(x,r)} \leq C_\mu \left( \frac{R}{r} \right)^{d_2}
\]
for all $x \in M$ and $0 < r \leq R$.

(ii) We say that $(M, d, \mu)$ satisfies the reverse volume doubling property $\text{RVD}(d_1)$ with index $d_1 > 0$ if there exists a constant $c_\mu > 0$ such that
\[
\frac{V(x,R)}{V(x,r)} \geq c_\mu \left( \frac{R}{r} \right)^{d_1}
\]
for all $x \in M$ and $0 < r \leq R$.

Note that $V(x,r) > 0$ for every $x \in M$ and $r > 0$ since $\mu$ has full support on $M$. Also, under $\text{VD}(d_2)$, we have
\[
\frac{V(x,R)}{V(y,r)} \leq \frac{V(y,d(x,y) + R)}{V(y,r)} \leq C_\mu \left( \frac{d(x,y) + R}{r} \right)^{d_2}
\]
for all $x \in M$ and $0 < r \leq R$. \hfill (2.1)

**Definition 2.2.** We say that a metric space $(M, d)$ satisfies the chain condition $\text{Ch}(A)$ if there exists a constant $A \geq 1$ such that, for any $n \in \mathbb{N}$ and $x, y \in M$, there is a sequence $\{z_k\}_{k=0}^n$ of points in $M$ such that $z_0 = x, z_n = y$ and
\[
d(z_{k-1}, z_k) \leq A \frac{d(x,y)}{n}
\]
for all $k = 1, \ldots, n$.

**Definition 2.3.** For a strictly increasing function $F : (0, \infty) \rightarrow (0, \infty)$, we say that a metric measure space $(M, d, \mu)$ satisfies the condition $\text{Diff}(F)$ if there exists a conservative symmetric diffusion process $Z = (Z_t)_{t \geq 0}$ on $M$ such that the transition density $q(t, x, y)$ of $Z$ with respect to $\mu$ exists and it satisfies the following estimates: there exist constants $c > 0$ and $a_0 > 1$ such that for all $t > 0$ and $x, y \in M$,
\[
\frac{c^{-1}}{V(x,F^{-1}(t))} I_{\{F(d(x,y)) \leq t\}} \leq q(t,x,y) \leq \frac{c}{V(x,F^{-1}(t))} \exp \left( -a_0 F_1(d(x,y),t) \right),
\]
where the function $F_1$ is defined as
\[
F_1(r,t) := \sup_{s > 0} \left[ \frac{r}{s} - \frac{t}{F(s)} \right].
\]

We recall the following definition from [2].

**Definition 2.4.** Let $g : (0, \infty) \rightarrow (0, \infty)$, and $a \in (0, \infty]$, $\beta_1, \beta_2 > 0$, and $0 < c \leq 1 \leq C$.

(1) For $a \in (0, \infty)$, we say that $g$ satisfies $L_a(\beta_1, c)$ (resp. $L^a(\beta_1, c)$) if
\[
\frac{g(R)}{g(r)} \geq c \left( \frac{R}{r} \right)^{\beta_1} \quad \text{for all } r \leq R < a \text{ (resp. } a \leq r \leq R). \]

We also say that $L_a(\beta_1,c,g)$ (resp. $L^a(\beta_1,c,g)$) holds.

(2) We say that $g$ satisfies $U_a(\beta_2, C)$ (resp. $U^a(\beta_2,C)$) if
\[
\frac{g(R)}{g(r)} \leq C \left( \frac{R}{r} \right)^{\beta_2} \quad \text{for all } r \leq R < a \text{ (resp. } a \leq r \leq R). \]

We also say that $U_a(\beta_2, C,g)$ (resp. $U^a(\beta_2,C,g)$) holds.
(3) When \( g \) satisfies \( L_a(\beta_1,c) \) (resp. \( U_a(\beta_2,C) \)) with \( a = \infty \), then we say that \( g \) satisfies the
global weak lower scaling condition \( L(\beta_1,c) \). (resp. the global weak upper scaling condition
\( U(\beta_2,C) \).)

Throughout this paper, we will assume that \( \psi : [0,\infty) \to [0,\infty) \) is a non-decreasing function
satisfying \( L(\beta_1,C_{L}) \) and \( U(\beta_2,C_{U}) \) for some \( 0 < \beta_1 \leq \beta_2 \). Note that \( L(\beta_1,C_{L},\psi) \) implies
\( \lim_{t \to 0} \psi(t) = 0 \).

Denote the diagonal set as \( \text{diag} := \{(x,x) : x \in M\} \). We assume that there exists a regular
Dirichlet form \( (\mathcal{E},\mathcal{F}) \) on \( L^2(M,\mu) \), which is given by

\[
\mathcal{E}(u,v) := \int_{M \times M \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y))J(x,y)\mu(dx)\mu(dy), \quad u,v \in \mathcal{F},
\]

where \( J \) is a symmetric and positive Borel measurable function on \( M \times M \setminus \text{diag} \). In terms of
Beurling-deny formula in [26, Theorem 3.2], the above Dirichlet form has jump part only.

**Definition 2.5.** We say \( J_\psi \) holds if there exists a constant \( \bar{C} > 1 \) so that for every \( x,y \in M \),

\[
\frac{\bar{C}}{V(x,d(x,y))\psi(d(x,y))} \leq J(x,y) \leq \frac{\bar{C}}{V(x,d(x,y))\psi(d(x,y))}.
\]

We say that \( J_{\psi,\leq} \) (resp. \( J_{\psi,\geq} \)) if the upper bound (resp. lower bound) in (2.5) holds.

Associated with the regular Dirichlet form \( (\mathcal{E},\mathcal{F}) \) on \( L^2(M,\mu) \) is a \( \mu \)-symmetric Hunt process
\( X = \{X_t, t \geq 0; \mathbb{P}^x, x \in M \setminus \mathcal{N}\} \). Here \( \mathcal{N} \) is a properly exceptional set for \( (\mathcal{E},\mathcal{F}) \) in the sense that
\( \mu(\mathcal{N}) = 0 \) and \( \mathbb{P}^x(X_t \in \mathcal{N} \text{ for some } t > 0) = 0 \) for all \( x \in M \setminus \mathcal{N} \). This Hunt process is unique up
to a properly exceptional set (see [26, Theorem 4.2.8].) We fix \( X \) and \( \mathcal{N} \), and write \( M_0 = M \setminus \mathcal{N} \).

For a set \( A \subset M \) and process \( X \), define the exit time \( \tau_A = \inf\{t > 0 : X_t \in A^c\} \). Let
\( \mathcal{F}' := \{u+b : u \in \mathcal{F}, b \in \mathbb{R}\} \).

**Definition 2.6.** Let \( U \subset M \) be an open set, \( A \) be any Borel subset of \( U \) and \( \kappa \geq 1 \) be a real
number. A \( \kappa \)-cutoff function of pair \( (A,U) \) is any function \( \varphi \in \mathcal{F} \) such that \( 0 \leq \varphi \leq \kappa \mu \)-a.e. in \( M \),
\( \varphi \geq 1 \mu \)-a.e. in \( A \) and \( \varphi = 0 \mu \)-a.e. in \( U^c \). We denote by \( \kappa \)-cutoff\((A,U)\) the collection of all \( \kappa \)-cutoff
function of pair \( (A,U) \). Any 1-cutoff function will be simply referred to as a cutoff function.

**Definition 2.7** (c.f. [29, Definition 1.11]). For a non-negative function \( \phi \), we say that \( G\text{cap}(\phi) \) holds if there exist constants \( \kappa \geq 1 \) and \( C > 0 \) such that for any \( u \in \mathcal{F}' \cap L^\infty \) and for all \( x_0 \in M \) and \( R, r > 0 \), there exists a function \( \varphi \in \kappa \)-cutoff\((B(x_0,R),B(x_0,R+r))\) such that

\[
\mathcal{E}(u^2\varphi,\varphi) \leq \frac{C}{\phi(r)} \int_{B(x_0,R+r)} u^2 d\mu.
\]

**Definition 2.8.** For a non-negative function \( \phi \), we say that \( E_\phi \) holds if there is a constant \( c > 1 \) such that

\[
c^{-1}\phi(r) \leq \mathbb{E}^x[\tau_{B(x,r)}] \leq c\phi(r) \quad \text{for all } x \in M_0, \ r > 0.
\]

We say that \( E_{\phi,\leq} \) (resp. \( E_{\phi,\geq} \)) holds if the upper bound (resp. lower bound) in the inequality above
holds.
Remark 2.9. Suppose \( \text{VD}(d_2) \), \( \text{RVD}(d_1) \) and \( J_{\psi, \geq} \) hold. Let \( x \in M_0 \) and \( r > 0 \). By the Lévy system in [21, Lemma 7.1] and \( J_{\psi, \geq} \), we have that for \( t > 0 \),

\[
1 \geq \mathbb{P}^x(X_{\tau_{B(x,r)}} \in B(x,2r)^c) \geq \mathbb{E}^x \left[ \int_0^{\tau_{B(x,r)}} \int_{B(x,2r)^c} J(x,y)\mu(dy)ds \right] \\
\geq \mathbb{E}^x[\tau_{B(x,r)}] \inf_{z \in B(x,r)} \int_{B(x,2r)^c} J(z,y)\mu(dy) \\
\geq C^{-1}\mathbb{E}^x[\tau_{B(x,r)}] \int_{B(x,2r)^c} \frac{1}{V(d(x,y))\psi(d(x,y))}\mu(dy).
\]

By \( \text{RVD}(d_1) \), there exists a constant \( c_1 > 1 \) such that \( V(x,c_1r) \geq 2V(x,r) \) for any \( x \in M \) and \( r > 0 \). Using this and \( U(\beta_2, C_U, \psi) \) we obtain

\[
\int_{B(x,2r)^c} \frac{1}{V(d(x,y))\psi(d(x,y))}\mu(dy) \geq \int_{B(x,2c_1r) \setminus B(x,2r)} \frac{1}{V(x,2c_1r) - V(x,2r)} \frac{1}{\psi(2c_1r)} \geq \frac{c_2}{\psi(r)}.
\]

Combining two estimates, we obtain

\[
\mathbb{E}^x[\tau_{B(x,r)}] \leq c\psi(r), \quad x \in M_0, \ r > 0,
\]

which implies \( E_{\psi, \leq} \).

By Remark 2.9, we expect that our scale function with respect to the process \( X \), which is comparable to the exit time \( \mathbb{E}^x[\tau_{B(x,r)}] \), is smaller than \( \psi \).

Let \( \Phi : (0, \infty) \to (0, \infty) \) be a non-decreasing function satisfying \( L(\alpha_1, c_L) \) and \( U(\alpha_2, c_U) \) with some \( 0 < \alpha_1 \leq \alpha_2 \) and \( c_L, c_U > 0 \) and

\[
\Phi(r) < \psi(r), \quad \text{for all } r > 0. \tag{2.6}
\]

By the virtue of Remark 2.9, the assumption (2.6) is quite natural for the scale function. For any \( c > 1, \tag{2.6} \) can be relaxed to the condition \( \Phi(r) \leq c\psi(r) \).

Recall that \( \alpha_2 \) is the global upper scaling index of \( \Phi \). If \( \Phi \) satisfies \( L_0(\delta, \tilde{C}_L) \), then we have \( \alpha_2 \geq \delta \). Indeed, if \( \delta > \alpha_2 \), then for any \( 0 < r \leq R < a \), we have

\[
\tilde{C}_L(R) \frac{R}{r} \delta \leq \frac{\Phi(R)}{\Phi(r)} \leq C_U(R) \frac{R}{r} \alpha_2,
\]

which is contradiction by letting \( r \to 0 \). Also, we define a function \( \Phi_1 : (0, \infty) \times (0, \infty) \to \mathbb{R} \) by

\[
\Phi_1(r,t) := \sup_{s > 0} \left\{ \frac{r}{s} \Phi(s) - \frac{t}{\Phi(s)} \right\}. \tag{2.7}
\]

(c.f., [33].) See Section 3.2 for the properties of \( \Phi_1 \).

For \( a_0, t, r > 0 \) and \( x \in M_0 \), we define

\[
\mathcal{G}(a_0, t, x, r) = \mathcal{G}_{\Phi, \psi}(a_0, t, x, r) := \frac{t}{V(x, r)\psi(r)} + \frac{1}{V(x, \Phi^{-1}(t))} \exp(-a_0 \Phi_1(t, r)), \tag{2.8}
\]

where \( \Phi^{-1} \) is the generalized inverse function of \( \Phi \), i.e., \( \Phi^{-1}(t) := \inf\{s \geq 0 : \Phi(s) > t\} \) (with the convention \( \inf \emptyset = \infty \)).
Definition 2.10. (i) We say that HK($\Phi, \psi$) holds if there exists a heat kernel $p(t, x, y)$ of the semigroup $\{P_t\}$ associated with $(\mathcal{E}, \mathcal{F})$, which has the following estimates: there exist $\eta, a_0 > 0$ and $c \geq 1$ such that for all $t > 0$ and $x, y \in M_0$,

$$c^{-1} \left( \frac{1}{V(x, \Phi^{-1}(t))} 1_{\{d(x, y) \leq \eta \Phi^{-1}(t)\}} + \frac{t}{V(x, d(x, y)) \psi(d(x, y))} 1_{\{d(x, y) > \eta \Phi^{-1}(t)\}} \right)$$

$$\leq p(t, x, y) \leq c \left( \frac{1}{V(x, \Phi^{-1}(t))} \wedge G(a_0, t, x, d(x, y)) \right),$$

where $G = G_{\Phi, \psi}$ is the function in (2.8).

(ii) We say UHK($\Phi, \psi$) holds if the upper bound in (2.9) holds.

(iii) We say UHKD($\Phi$) holds if there is a constant $c > 0$ such that for all $t > 0$ and $x \in M_0$,

$$p(t, x, x) \leq \frac{c}{V(x, \Phi^{-1}(t))}.$$ 

(vi) We say that SHK($\Phi, \psi$) holds if there exists a heat kernel $p(t, x, y)$ of the semigroup $\{P_t\}$ associated with $(\mathcal{E}, \mathcal{F})$, which has the following estimates: there exist $a_L \geq a_U > 0$ and $c \geq 1$ such that for all $t > 0$ and $x, y \in M_0$,

$$\frac{1}{c} \left( \frac{1}{V(x, \Phi^{-1}(t))} \wedge G(a_L, t, x, d(x, y)) \right) \leq p(t, x, y) \leq c \left( \frac{1}{V(x, \Phi^{-1}(t))} \wedge G(a_U, t, x, d(x, y)) \right).$$

(v) Assume that $(M, d, \mu)$ satisfies VD($d_2$), RVD($d_1$) and Diff($F$). We say that GHK($\Phi, \psi$) holds if there exists a heat kernel $p(t, x, y)$ of the semigroup $\{P_t\}$ associated with $(\mathcal{E}, \mathcal{F})$, which has the following estimates: there exists $0 < a_U, 0 < \eta$ and $c \geq 1$ such that for all $t > 0$ and $x, y \in M_0$,

$$c^{-1}V(x, \Phi^{-1}(t))^{-1} 1_{\{d(x, y) \leq \eta \Phi^{-1}(t)\}} + \frac{c^{-1}t}{V(x, d(x, y)) \psi(d(x, y))} 1_{\{d(x, y) \geq \eta \Phi^{-1}(t)\}}$$

$$\leq p(t, x, y) \leq c \left( \frac{ct}{V(x, d(x, y)) \psi(d(x, y))} + \frac{c}{V(x, \Phi^{-1}(t))} e^{-a_U F_1(d(x, y), F(\Phi^{-1}(t)))} \right).$$

(vi) We say GUHK($\Phi, \psi$) holds if the upper bound in (2.10) holds.

Remark 2.11. For strictly increasing and continuous function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ satisfying $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$ and for any $C > 1$, the condition HK($\Phi, C\Phi$) is equivalent to the existence of heat kernel $p(t, x, y)$ and the constant $c \geq 1$ such that for all $t > 0$ and $x, y \in M_0$,

$$c^{-1} \left( \frac{1}{V(x, \Phi^{-1}(t))} \wedge \frac{c^{-1}t}{V(x, d(x, y)) \Phi(d(x, y))} \right) \leq p(t, x, y) \leq c \left( \frac{ct}{V(x, \Phi^{-1}(t))} \wedge \frac{c}{V(x, d(x, y)) \Phi(d(x, y))} \right).$$

This shows that if $\Phi \asymp \psi$, then the condition HK($\Phi, \psi$) is equivalent to (2.11). The proof of the equivalence of (2.11) and HK($\Phi, C\Phi$) is in Appendix A.

From now on, we denote HK($\Phi, C\Phi$) (resp. UHK($\Phi, C\Phi$)) by HK($\Phi$) (resp. UHK($\Phi$)). By Remark 2.11, the condition HK($\Phi$) is equivalent to the condition HK($\Phi$) of [21].

Let $\mathcal{F}_b$ be the collection of bounded functions in $\mathcal{F}$. 

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Definition 2.12. We say that the (weak) Poincaré inequality \( \text{PI}(\Phi) \) holds if there exist constants \( C > 0 \) and \( \kappa \geq 1 \) such that for any ball \( B_r := B(x, r) \) with \( x \in M_0, r > 0 \) and for any \( f \in \mathcal{F}_h, \)

\[
\int_{B_r} (f - \bar{f}_{B_r})^2 d\mu \leq C\Phi(r) \int_{B_{2r} \times B_{2r}} (f(y) - f(z))^2 J(y, z) \mu(dy) \mu(dz),
\]

where \( \bar{f}_{B_r} = \frac{1}{\mu(B_r)} \int_{B_r} f d\mu \) is the average value of \( f \) on \( B_r. \)

2.2 Main results

Recall that we always assume that \( \psi : [0, \infty) \to [0, \infty) \) is a non-decreasing function satisfying \( L(\beta_1, C_L) \) and \( U(\beta_2, C_U) \) for some \( 0 < \beta_1 \leq \beta_2. \)

For the function \( \Phi \) satisfying (2.6) and \( L^a(\delta, \bar{C}_L) \) with \( \delta > 1, \) we define

\[
\tilde{\Phi}(s) := c_U^{-1} \frac{\Phi(a)}{a^{\alpha_2}} s^{\alpha_2} 1_{\{s < a\}} + \Phi(s) 1_{\{s \geq a\}}. \tag{2.12}
\]

Note that for \( s \leq a \) we have, \( \frac{\tilde{\Phi}(s)}{\Phi(s)} = c_U^{-1} \frac{s^{\alpha_2} \Phi(a)}{a^{\alpha_2} \Phi(s)} \leq 1. \) Thus,

\[
\tilde{\Phi}(r) \leq \Phi(r) < \psi(r), \quad r > 0. \tag{2.13}
\]

Also, \( L(\delta, \bar{C}_L, \tilde{\Phi}) \) holds. Indeed, for any \( 0 < r < a \leq R, \)

\[
\frac{\tilde{\Phi}(R)}{\Phi(r)} = \frac{\Phi(R)}{\Phi(a)} \frac{\Phi(a)}{\Phi(r)} \geq \bar{C}_L \left( \frac{R}{a} \right)^{\delta} = \bar{C}_L \left( \frac{R}{r} \right)^{\delta}.
\]

The other cases are straightforward. By the same way as (2.7), let us define

\[
\tilde{\Phi}_1(r, t) := \sup_{s > 0} \left[ \frac{r}{s} - \frac{t}{\Phi(s)} \right]. \tag{2.14}
\]

The following are the main results of this paper.

Theorem 2.13. Assume that the metric measure space \((M, d, \mu)\) satisfies \( \text{VD}(d_2) \), and the process \( X \) satisfies \( \mathcal{L}_{\psi, \leq} \), \( \text{UHKD}(\Phi) \) and \( \mathcal{E}_\Phi \), where \( \psi \) is a non-decreasing function satisfying \( L(\beta_1, C_L) \) and \( U(\beta_2, C_U) \), and \( \Phi \) is a non-decreasing function satisfying (2.6) and \( L(\alpha_1, C_L) \) and \( U(\alpha_2, C_U) \), where \( 0 < \beta_1 \leq \beta_2 \) and \( 0 < \alpha_1 \leq \alpha_2. \)

(i) Suppose that \( \Phi \) satisfies \( L_a(\delta, \bar{C}_L) \) with some \( a > 0 \) and \( \delta > 1 \). Then, for any \( T \in (0, \infty), \)

there exist constants \( a_U > 0 \) and \( c > 0 \) such that for any \( t < T \) and \( x, y \in M_0, \)

\[
p(t, x, y) \leq \frac{ct}{V(x, d(x, y)) \psi(d(x, y))} + \frac{c}{V(x, \Phi^{-1}(t))} \exp \left( -a_U \tilde{\Phi}_1(d(x, y), t) \right). \tag{2.15}
\]

Moreover, if \( \Phi \) satisfies \( L(\delta, \bar{C}_L) \), then (2.15) holds for all \( t < \infty. \)

(ii) Suppose that \( \Phi \) satisfies \( L^a(\delta, \bar{C}_L) \) with some \( a > 0 \) and \( \delta > 1. \) Then, for any \( T \in (0, \infty) \) there exist constants \( a_U > 0 \) and \( c > 0 \) such that for any \( t \geq T \) and \( x, y \in M_0, \)

\[
p(t, x, y) \leq \frac{ct}{V(x, d(x, y)) \psi(d(x, y))} + \frac{c}{V(x, \Phi^{-1}(t))} \exp \left( -a_U \tilde{\Phi}_1(d(x, y), t) \right). \tag{2.16}
\]
Theorem 2.14. Assume that the metric measure space \((M,d,\mu)\) satisfies \(\text{RVD}(d_1)\) and \(\text{VD}(d_2)\). Let \(\psi\) be a non-decreasing function satisfying \(L(\beta_1,C_L)\) and \(U(\beta_2,C_U)\), and \(\Phi\) be a non-decreasing function satisfying (2.6), \(L(\alpha_1,c_L)\) and \(U(\alpha_2,c_U)\) with \(1 < \alpha_1 \leq \alpha_2\). Then the following are equivalent:

1. \(\text{UHK}(\Phi,\psi)\) and \((\mathcal{E},\mathcal{F})\) is conservative.
2. \(\text{J}_{\psi,}\leq, \text{UHK}(\Phi)\) and \((\mathcal{E},\mathcal{F})\) is conservative.
3. \(\text{J}_{\psi,}\leq, \text{UHKD}(\Phi)\) and \(\text{E}_\Phi\).

See [21] Definitions 1.5 and 1.8 for the definitions of \(\text{FK}(\Phi)\), \(\text{CSJ}(\Phi)\) and \(\text{SCSJ}(\Phi)\).

Corollary 2.15. Under the same settings as Theorem 2.14, each equivalent condition in above theorem is also equivalent to the following:

4. \(\text{FK}(\Phi), \text{J}_{\psi,}\leq\) and \(\text{Gcap}(\Phi)\).
5. \(\text{FK}(\Phi), \text{J}_{\psi,}\leq\) and \(\text{CSJ}(\Phi)\).
6. \(\text{FK}(\Phi), \text{J}_{\psi,}\leq\) and \(\text{Gcap}(\Phi)\).

Theorem 2.16. Assume that the metric measure space \((M,d,\mu)\) satisfies \(\text{Ch}(A)\), \(\text{RVD}(d_1)\) and \(\text{VD}(d_2)\). Suppose that the process \(X\) satisfies \(\text{J}_\psi\), \(\text{E}_\Phi\) and \(\text{PI}(\Phi)\), where \(\psi\) is a non-decreasing function satisfying \(L(\beta_1,C_L)\) and \(U(\beta_2,C_U)\), and \(\Phi\) is a non-decreasing function satisfying (2.6), \(L(\alpha_1,c_L)\) and \(U(\alpha_2,c_U)\).

(i) Suppose that \(L_a(\delta,\tilde{C}_L,\Phi)\) holds with \(\delta > 1\). Then, for any \(T \in (0,\infty)\), there exist constants \(c > 0\) and \(a_L > 0\) such that for any \(x,y \in M_0\) and \(t \in (0,T]\),

\[
p(t,x,y) \geq \frac{c}{V(x,\Phi^{-1}(t))} \wedge \left( \frac{ct}{V(x,d(x,y))\psi(d(x,y))} + \frac{c}{V(x,\Phi^{-1}(t))} \exp \left( -a_L\Phi_1(d(x,y),t) \right) \right). \tag{2.17}
\]

Moreover, if \(L(\delta,\tilde{C}_L,\Phi)\) holds, then (2.17) holds for all \(t \in (0,\infty)\).

(ii) Suppose that \(L_a(\delta,\tilde{C}_L,\Phi)\) holds with \(\delta > 1\). Then, for any \(T \in (0,\infty)\), there exist constants \(c > 0\) and \(a_L > 0\) such that for any \(x,y \in M_0\) and \(t \geq T\),

\[
p(t,x,y) \geq \frac{c}{V(x,\Phi^{-1}(t))} \wedge \left( \frac{ct}{V(x,d(x,y))\psi(d(x,y))} + \frac{c}{V(x,\Phi^{-1}(t))} \exp \left( -a_L\tilde{\Phi}_1(d(x,y),t) \right) \right). \tag{2.18}
\]

Theorem 2.17. Under the same settings as Theorem 2.14, the following are equivalent:

1. \(\text{HK}(\Phi,\psi)\).
2. \(\text{J}_\psi, \text{PI}(\Phi), \text{UHK}(\Phi)\) and \((\mathcal{E},\mathcal{F})\) is conservative.
3. \(\text{J}_\psi, \text{PI}(\Phi)\) and \(\text{E}_\Phi\).

If we further assume that \((M,d)\) satisfies \(\text{Ch}(A)\) for some \(A \geq 1\), then the following is also equivalent to others:

4. \(\text{SHK}(\Phi,\psi)\).

By Theorem 2.17 and Corollary 2.15, we also obtain that

Corollary 2.18. Under the same settings as Theorem 2.14, each equivalent condition in Theorem 2.17 is also equivalent to the following:

5. \(\text{J}_\psi, \text{PI}(\Phi)\) and \(\text{SCSJ}(\Phi)\).
6. \(\text{J}_\psi, \text{PI}(\Phi)\) and \(\text{CSJ}(\Phi)\).
7. \(\text{J}_\psi, \text{PI}(\Phi)\) and \(\text{Gcap}(\Phi)\).
In this subsection, we assume that the function $E$ be the Hunt process corresponds to $(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with some $1 < \gamma_1 \leq \gamma_2$, and we assume that $\psi : (0, \infty) \to (0, \infty)$ is a non-increasing function which satisfies $L(\beta_1, C_L), U(\beta_2, C_U)$ and that
\[
\int_0^1 \frac{dF(s)}{\psi(s)} < \infty. \tag{2.19}
\]
Recall that the function $F_1(r, t) = \sup_{s>0} \left[ \frac{r}{s} - F(s) \right]$ has defined in (2.3). Consider
\[
\Phi(r) := \frac{F(r)}{\int_0^r \frac{dF(s)}{\psi(s)} ds}, \quad r > 0. \tag{2.20}
\]
Then $\Phi$ is strictly increasing function satisfying (2.6), $U(\gamma_2, C_U)$ and $L(\alpha_1, \tilde{c})$ for some $\tilde{c} > 0$ (see Section 4).

**Theorem 2.19.** Assume that the metric measure space $(M, d, \mu)$ satisfies RVD($d_1$) and VD($d_2$). Assume further that Diff($F$) holds for a strictly increasing function $F : (0, \infty) \to (0, \infty)$ satisfying $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with $1 < \gamma_1 \leq \gamma_2$. Let $\psi$ be a non-decreasing function satisfying $L(\beta_1, C_L), U(\beta_2, C_U)$ and (2.19), and $\Phi$ be the function defined in (2.20).

(i) $J_{\psi}$ is equivalent to GHK($\Phi, \psi$). Moreover, both equivalent conditions imply PI($\Phi$) and E-$\Phi$.

(ii) If we further assume that $(M, d)$ satisfies Ch($A$) for some $A \geq 1$ and that $\Phi$ satisfies $L(\alpha_1, c_L)$ with $\alpha_1 > 1$, then $J_{\psi}$ is also equivalent to SHK($\Phi, \psi$).

Finally, we now state local estimates of heat kernels.

**Corollary 2.20.** Assume that the metric measure space $(M, d, \mu)$ satisfies RVD($d_1$) and VD($d_2$). Assume further that Diff($F$) holds for a strictly increasing function $F : (0, \infty) \to (0, \infty)$ satisfying $L(\gamma_1, c_F^{-1})$ and $U(\gamma_2, c_F)$ with $1 < \gamma_1 \leq \gamma_2$. Let $\psi$ be a non-decreasing function satisfying $L(\beta_1, C_L), U(\beta_2, C_U)$ and (2.19), and $\Phi$ be the function defined in (2.20). Suppose that the process $X$ satisfies $J_{\psi}$.

(i) Assume that $L_a(\delta, \tilde{C}_L, \Phi)$ holds with some $\delta > 1$ and $a > 0$. Then, for any $T \in (0, \infty)$, there exist constants $0 < a_U \leq a_L$ and $c > 0$ such that (2.15) and (2.17) holds for all $t \in (0, T]$ and $x, y \in M_0$.

(ii) Assume that $L^2(\delta, \tilde{C}_L, \Phi)$ holds with some $\delta > 1$ and $a > 0$. Then, for any $T \in (0, \infty)$, there exist constants $0 < a_U \leq a_L$ and $c > 0$ such that (2.16) and (2.18) holds for all $t \in [T, \infty)$ and $x, y \in M_0$.

3 HKE and stability on general metric measure space

### 3.1 Implications of UHKD($\Phi$), $J_{\psi, \leq}$ and $E-$\$\Phi$

In this subsection, we assume that the function $\psi$ satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$ with $0 < \beta_1 \leq \beta_2$, and $\Phi$ satisfies (2.6), $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$, with $0 < \alpha_1 \leq \alpha_2$.

Assume that there exists regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ defined in (2.4) satisfying $J_{\psi, \leq}$. Let $X$ be the Hunt process corresponds to $(\mathcal{E}, \mathcal{F})$. Fix $\rho > 0$ and define a bilinear form $(\mathcal{E}^\rho, \mathcal{F})$ by
\[
\mathcal{E}^\rho(u, v) = \int_{M \times M \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y)) \mathbf{1}_{\{d(x, y) \leq \rho\}} J(x, y) \mu(dx) \mu(dy).
\]
Clearly, the form $\mathcal{E}(u, v)$ is well defined for $u, v \in \mathcal{F}$, and $\mathcal{E}(u, u) \leq \mathcal{E}(u, u)$ for all $u \in \mathcal{F}$. Since $\psi$ satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$, for all $u \in \mathcal{F}$,

$$\mathcal{E}(u, u) - \mathcal{E}(u, u) = \int (u(x) - u(y))^2 1_{\{d(x,y) > \rho\}} J(x, y) \mu(dx) \mu(dy) \leq 4 \int_M u^2(x) \mu(dx) \int_{B(x, \rho)^c} J(x, y) \mu(dy) \leq \frac{c_0 \|u\|^2}{\psi(\rho)}.$$ 

Thus, $\mathcal{E}_1(u, u) := \mathcal{E}(u, u) + \|u\|^2_2$ is equivalent to $\mathcal{E}_1^*(u, u) := \mathcal{E}(u, u) + \|u\|^2_2$ for every $u \in \mathcal{F}$, which implies that $(\mathcal{E}, \mathcal{F})$ is also a regular Dirichlet form on $L^2(M, d\mu)$. We call $(\mathcal{E}, \mathcal{F})$ the $\rho$-truncated Dirichlet form. The Hunt process associated with $(\mathcal{E}, \mathcal{F})$ which will be denoted by $X^\rho$ can be identified in distribution with the Hunt process of the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ by removing those jumps of size larger than $\rho$. Let $p^\rho(t, x, y)$ and $\tau_D^\rho$ be the transition density and exit time of $X^\rho$ correspond to $(\mathcal{E}, \mathcal{F})$, respectively.

For any open set $D \subset M$, $\mathcal{F}_D$ is defined to be the $\mathcal{E}_1$-closure in $\mathcal{F}$ of $\mathcal{F} \cap C_c(D)$. Let $\{P_t^D\}$ and $\{P_t^{D, D}\}$ be the semigroups of $(\mathcal{E}, \mathcal{F}_D)$ and $(\mathcal{E}, \mathcal{F}_D)$, respectively.

**Lemma 3.1.** Assume $\text{VD}(d_2)$, $J_{\psi, \leq}$ and $\mathbb{E}\Phi$. Then, there is a constant $c > 0$ such that for any $\rho > 0$, $t > 0$ and $x \in M_0$,

$$\mathbb{E}^x \int_0^t \frac{1}{V(X_s^\rho, \rho)} ds \leq \frac{ct}{V(x, \rho)} \left(1 + \frac{t}{\Phi(\rho)}\right)^{d_2+1}.$$ 

**Proof.** Following the proof of [21] Proposition 4.24], using $J_{\psi, \leq}$ we have

$$\mathbb{E}^x \left[\int_0^t \frac{1}{V(X_s^\rho, \rho)} ds\right] = \sum_{k=1}^{\infty} \mathbb{E}^x \left[\int_0^{\tau_{B(x, k\rho)}^\rho} \frac{1}{V(X_s^\rho, \rho)} ds; \tau_{B(x, k\rho)}^\rho > t \geq \tau_{B(x, (k-1)\rho)}^\rho\right] := \sum_{k=1}^{\infty} I_k. \quad (3.1)$$

When $t < \tau_{B(x, k\rho)}^\rho$, we have $d(X_s^\rho, x) \leq k\rho$ for all $s \leq t$. This along with $\text{VD}(d_2)$ yields that for all $k \geq 1$ and $s \leq t < \tau_{B(x, k\rho)}^\rho$,

$$\frac{1}{V(X_s^\rho, \rho)} \leq \frac{c_1 k^{d_2}}{V(X_s^\rho, 2k\rho)} \leq \frac{c_1 k^{d_2}}{\inf_{d(z, x) \leq k\rho} V(z, 2k\rho)} \leq \frac{c_1 k^{d_2}}{V(x, \rho)}. \quad (3.2)$$

On the other hand, by [21] Corollary 4.22], there exist constants $c_i > 0$ with $i = 2, 3, 4$ such that for all $t, \rho > 0$, $k \geq 1$ and $x \in M_0$,

$$\mathbb{P}^x(\tau_{B(x, k\rho)}^\rho \leq t) \leq c_2 \exp \left(-c_3 k + c_4 \frac{t}{\Phi(\rho)}\right). \quad (3.3)$$

Let $k_0 = \lceil \frac{2c_4}{c_3 \Phi(\rho)} \rceil + 1$. Using (3.2) and definition of $k_0$, we have

$$\sum_{k=1}^{k_0} I_k \leq \sum_{k=1}^{k_0} \frac{c_1 k^{d_2} t}{V(x, \rho)} \leq \frac{c_5 k_0^{d_2} t}{V(x, \rho)} \leq \frac{c_6 t}{V(x, \rho)} \left(1 + \frac{t}{\Phi(\rho)}\right)^{d_2+1}. \quad (3.4)$$

On the other hand, for any $k_0 < k$, using (3.2) and (3.3) with $k_0 = \lceil \frac{2c_4}{c_3 \Phi(\rho)} \rceil + 1$ we have

$$I_k \leq \frac{c_1 k^{d_2} t}{V(x, \rho)} \mathbb{P}^x(\tau_{B(x, k\rho)}^\rho \leq t) \leq \frac{c_1 c_2 k^{d_2} t}{V(x, \rho)} \exp \left(-c_3 k + c_4 \frac{t}{\Phi(\rho)}\right) \leq \frac{c_1 c_2 k^{d_2} t}{V(x, \rho)} \exp \left(-\frac{c_3 k}{2}\right).$$

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Thus, we conclude
\[ \sum_{k=k_0+1}^{\infty} I_k \leq \frac{c_1 c_2 t}{V(x, \rho)} \sum_{k=k_0+1}^{\infty} k^{d_2} e^{-\frac{ct}{V(x, \rho)}} := c_3 t. \]

From above two estimates, we obtain \( \sum_{i=1}^{\infty} I_k \leq \frac{ct}{V(x, \rho)} (1 + \frac{t}{\Phi(\rho)})^{d_2+1} \). Combining this with \( \text{(3.1)} \), we obtain the desired estimate.

In the next lemma, we obtain a priori estimate for the upper bound of heat kernel.

**Lemma 3.2.** Assume \( \text{VD}(d_2), J_{\psi, \leq}, \text{UHKD}(\Phi) \) and \( \text{E}_\Phi \). Then, there are constants \( \epsilon > 0 \) and \( C_1, C_2 > 0 \) such that for any \( \rho > 0 \), \( t > 0 \) and \( x, y \in M_0 \),
\[
p(t, x, y) \leq \frac{c}{V(x, \Phi^{-1}(t))} \left( 1 + \frac{d(x, y)}{\Phi^{-1}(t)} \right)^{d_2} \exp \left( C_1 \frac{t}{\Phi(\rho)} - C_2 \frac{d(x, y)}{\rho} \right) + \frac{ct}{V(x, \rho) \psi(\rho)} \left( 1 + \frac{t}{\Phi(\rho)} \right)^{d_2+1}.
\]

**Proof.** Recall that \( X^\rho_t \) and \( p^\rho(t, x, y) \) are the Hunt process and heat kernel correspond to \( (E^\rho, F) \), respectively. Using \( \text{(9, Lemma 3.1, (3.5))} \) and \( \text{(7, Lemma 3.6)} \), we have for \( t > 0 \) and \( x, y \in M_0 \),
\[
p(t, x, y) \leq p^\rho(t, x, y) + \mathbb{E}^x \left[ \int_0^t \int_M J(X^\rho_s, z) \mathbf{1}_{d(z, x^\rho_s) \geq \rho}(z) p(t - s, z, y) \mu(dz) ds \right]. \tag{3.4}
\]

Also, using symmetry of heat kernel, \( J_{\psi, \leq} \) and Lemma \( \text{(3.1)} \) we obtain
\[
\mathbb{E}^x \left[ \int_0^t \int_M J(X^\rho_s, z) \mathbf{1}_{d(z, x^\rho_s) \geq \rho}(z) p(t - s, z, y) \mu(dz) ds \right] \\
\leq c_1 \mathbb{E}^x \left[ \int_0^t \frac{1}{V(x^\rho_s, \rho) \psi(\rho)} ds \right] \leq \frac{c_1 t}{V(x, \rho) \psi(\rho)} \left( 1 + \frac{t}{\Phi(\rho)} \right)^{d_2+1}. \tag{3.5}
\]

Combining the estimates in \( \text{(21, Lemma 5.2)} \) and Lemma \( \text{(3.1)} \) we conclude the proof. Note that since \( J_{\psi, \leq} \) and \( \text{(2.6)} \) imply \( J_{\Phi, \leq} \), the conditions in \( \text{(21, Lemma 5.2)} \) are satisfied.

The proof of next lemma is same as that of \( \text{(21, Lemma 4.2)} \). Thus, we skip the proof.

**Lemma 3.3.** Let \( r, t, \rho > 0 \). Assume that
\[
P^x(\tau^\rho_{B(x_0, r)} \leq t) \leq \phi(r, t) \quad \text{for all } x_0 \in M_0, x \in B(x_0, r/4) \cap M_0,
\]
where \( \phi \) is a non-negative measurable function on \( \mathbb{R}_+ \times \mathbb{R}_+ \). Then, for any \( k \in \mathbb{N} \),
\[
P^x \left( \tau^\rho_{B(x_0, k(\rho + r))} \leq t \right) \leq \phi(r, t)^k \quad \text{for all } x_0 \in M_0, x \in B(x_0, r/4) \cap M_0.
\]

**Lemma 3.4.** Assume \( \text{VD}(d_2), J_{\psi, \leq}, \text{UHKD}(\Phi) \) and \( \text{E}_\Phi \). Let \( T > 0 \) and \( f : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be a measurable function satisfying that \( t \mapsto f(r, t) \) is non-increasing for all \( r > 0 \) and that \( r \mapsto f(r, t) \) is non-decreasing for all \( t > 0 \). Suppose that the following hold: (i) For each \( b > 0 \), \( \sup_{t \leq T} f(b \Phi^{-1}(t), t) < \infty \) (resp., \( \sup_{T \geq t} f(b \Phi^{-1}(t), t) < \infty \)); (ii) there exist \( \eta \in (0, \beta_1], \alpha_1 > 0 \) and \( c > 0 \) such that
\[
P^x \left( d(x, X_t) > r \right) \leq c (\psi^{-1}(t/r))^n + c \exp \left( -\alpha_1 f(r, t) \right) \tag{3.6}
\]
for all \( t \in (0, T] \) (resp. \( t \in [T, \infty) \)), \( r > 0 \) and \( x \in M_0 \).

Then, there exist constants \( k, c_0 > 0 \) such that
\[
p(t, x, y) \leq \frac{c_0 t}{V(x, d(x, y)) \psi(d(x, y))} + \frac{c_0}{V(x, \Phi^{-1}(t))} \left( 1 + \frac{d(x, y)}{\Phi^{-1}(t)} \right)^{d_2} \exp \left( -\alpha_1 k f(d(x, y)/(16k), t) \right)
\]
for all \( t \in (0, T] \) (resp. \( t \in [T, \infty) \)) and \( x, y \in M_0 \).
Proof. Since the proofs for cases \( t \in (0, T) \) and \( t \in [T, \infty) \) are similar, we only prove for \( t \in (0, T) \). For \( x_0 \in M_0 \), let \( B(r) = B(x_0, r) \cap M_0 \). By the strong Markov property, (3.6), and the fact that 
\( t \mapsto f(r, t) \) is non-increasing, we have that for \( x \in B(r/4) \) and \( t \in (0, T/2] \),

\[
P^x(\tau_{B(r)} \leq t) \leq \mathbb{P}^x(X_{2t} \in B(r/2)^c) + \mathbb{P}^x(\tau_{B(r)} \leq t, X_{2t} \in B(r/2)) \leq \mathbb{P}^x(X_{2t} \in B(x, r/4)^c) + \sup_{z \in B(r)^c} \mathbb{P}^z(X_{2t-s} \in B(z, r/4)^c) \leq c(4\psi^{-1}(2t)/r)^\eta + c \exp \left( -a_1 f(r/4, 2t) \right).
\]

From this with \( L(\beta_1, C, \psi) \) and Lemma [A, 2] we have that for \( x \in B(r/4) \) and \( t \in (0, T/2] \),

\[
1 - P_t^{B(r)} 1_{B(r)}(x) = \mathbb{P}^x(\tau_{B(r)} \leq t) \leq c_1 \left( \frac{\psi^{-1}(t)}{r} \right)^\eta + c \exp \left( -a_1 f(r/4, 2t) \right).
\]

By [31] Proposition 4.6 and [21] Lemma 2.1, we have

\[
\left| P_t^{B(r)} 1_{B(r)}(x) - P_t^{r, B(r)} 1_{B(r)}(x) \right| \leq 2t \sup_{z \in M} \int_{B(z, r)^c} J(z, y) dy \leq \frac{c_3 t}{\psi(r)}.
\]

Combining this with (3.7), we see that for all \( x \in B(r/4) \) and \( t \in (0, T/2] \),

\[
\mathbb{P}^x(\tau_{B(r)} \leq t) = 1 - P_t^{r, B(r)} 1_{B(r)}(x) \leq 1 - P_t^{B(r)} 1_{B(r)}(x) + \frac{c_3 t}{\psi(r)} \leq c_2 \left( \frac{\psi^{-1}(t)}{r} \right)^\eta + c_1 \exp \left( -a_1 f(r/4, 2t) \right) + c_3 \frac{t}{\psi(r)} =: \phi(r, t).
\]

Applying Lemma [3.3] with \( r = \rho \) to (3.8), we see that for any \( t \in (0, T/2] \), \( x \in B(r/4) \) and \( k \in \mathbb{N} \),

\[
\int_{B(x, 2kr)^c} p^\rho(t, x, y) \mu(dy) = \int_{B(x, 2kr)^c} 1_{B(x, 2kr)}(x) \leq \phi(r, t)^k.
\]

Let \( k := \lceil \frac{\beta + 2d}{\eta} \rceil + 1 \). For \( t \in (0, T] \) and \( x, y \in M_0 \) satisfying \( 4k\Phi^{-1}(t) \geq d(x, y) \), by using that \( r \mapsto f(r, t) \) is non-decreasing and the assumption (i), we have \( f(d(x, y)/(16k), t) \leq f(\Phi^{-1}(t)/4, t) \leq C < \infty \). Thus, using [21] Lemma 5.1,

\[
p(t, x, y) \leq \frac{c_5 e^{a_1 kC}}{V(x, \Phi^{-1}(t))} \exp \left( -a_1 kf(d(x, y)/(16k), t) \right).
\]

For the remainder of the proof, assume \( t \in (0, T] \) and \( 4k\Phi^{-1}(t) < d(x, y) \). Also, denote \( r = d(x, y) \) and \( \rho = r/(4k) \). Using [21] Lemma 5.2, (3.9) and (2.1), we have

\[
p^\rho(t, x, y) = \int_M p^\rho(t/2, x, z)p^\rho(t/2, z, y) \mu(dz)
\leq \left( \int_{B(x, r/2)^c} + \int_{B(y, r/2)^c} \right) p^\rho(t/2, x, z)p^\rho(t/2, z, y) \mu(dz)
\leq \frac{c_6}{V(y, \Phi^{-1}(t))} \int_{B(x, 2kr)} p^\rho(t/2, x, z) \mu(dz) + \frac{c_6}{V(x, \Phi^{-1}(t))} \int_{B(y, 2kr)} p^\rho(t/2, z, y) \mu(dz)
\leq \frac{c_7}{V(x, \Phi^{-1}(t))} \left( 1 + \frac{r}{\Phi^{-1}(t)} \right)^d \phi(\rho, t/2)^k
\leq \frac{c_8}{V(x, \Phi^{-1}(t))} \left( \frac{r}{\Phi^{-1}(t)} \right)^d \phi(\rho, t/2)^k.
\]
Note that \( k\beta_1 \geq k\eta \geq \beta_2 + 2d_2 \) and \( \rho \geq \Phi^{-1}(t) > \psi^{-1}(t) \). Thus, by \( L(\beta_1, C_L, \psi) \) we obtain
\[
\left( \frac{\psi^{-1}(t)}{\rho} \right)^{\eta k} + \left( \frac{t}{\psi(\rho)} \right)^k \leq c_9 \left( \frac{\psi^{-1}(t)}{\rho} \right)^{\beta_2 + 2d_2} + c_9 \left( \frac{\psi^{-1}(t)}{\rho} \right)^{k\beta_1} \leq c_{10} \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_2 + 2d_2}.
\]
Applying this to (3.11) and using \( VD(d_2) \) and \( U(\beta_2, C_U, \psi) \) we have
\[
p^\eta(t, x, y) \leq \frac{c_{11}}{V(x, \Phi^{-1}(t))} \left( \frac{r}{\Phi^{-1}(t)} \right)^{d_2} \left( \left( \frac{\psi^{-1}(t)}{\rho} \right)^{\eta k} + \exp \left( - a_1 k f(\rho/4, t) \right) + \left( \frac{t}{\psi(\rho)} \right)^k \right) \
\leq \frac{c_{12}}{V(x, \Phi^{-1}(t))} \left( \frac{r}{\Phi^{-1}(t)} \right)^{d_2} \left( \left( \frac{\psi^{-1}(t)}{\rho} \right)^{\beta_2 + 2d_2} + \exp \left( - a_1 k f(\rho/4, t) \right) \right) \
\leq \frac{c_{13} t}{V(x, r) \psi(r)} + \frac{c_{13}}{V(x, \Phi^{-1}(t))} \left( 1 + \frac{r}{\Phi^{-1}(t)} \right)^{d_2} \exp \left( - a_1 k f(r/(16k), t) \right).
\]
Thus, by (3.11), (3.5) and \( U(\beta_2, C_U, \psi) \), we conclude that for any \( t \in (0, T] \) and \( x, y \in M_0 \) with \( 4k\Phi^{-1}(t) < d(x, y) \),
\[
p(t, x, y) \leq p^\eta(t, x, y) + \frac{c_{14} t}{V(x, \psi(\rho))} \left( 1 + \frac{t}{\Phi(\rho)} \right)^{d_2 + 1} \
\leq \frac{c_{15}}{V(x, \Phi^{-1}(t))} \left( 1 + \frac{d(x, y)}{\Phi^{-1}(t)} \right)^{d_2} \exp \left( - a_1 k f(d(x, y)/16k, t) \right) + \frac{c_{15} t}{V(x, d(x, y)) \psi(d(x, y))}.
\]
Here in the second inequality we have used \( \Phi(\rho) \geq t \). Now the conclusion follows from (3.10) and (3.12).

**Lemma 3.5.** Suppose \( VD(d_2), J_\psi, \leq, \text{UHKD}(\Phi) \) and \( E_\Phi \). Then, there exist constants \( a_0, c > 0 \) and \( N \in \mathbb{N} \) such that
\[
p(t, x, y) \leq c t \left( \frac{\psi^{-1}(t)}{V(x, d(x, y)) \psi(d(x, y))} \right) + c V(x, \Phi^{-1}(t))^{-1} \exp \left( - a_0 d(x, y)^{1/N} \right) \frac{\Phi^{-1}(t)^{1/N}}{\Phi^{-1}(t)^{1/N}},
\]
for all \( t > 0 \) and \( x, y \in M_0 \).

**Proof.** Let \( N := \lceil \frac{\beta_1 + d_2}{\beta_1} \rceil + 1, \) and \( \eta := \beta_1 - (\beta_1 + d_2)/N > 0. \) We first claim that there exist \( a_1 > 0 \) and \( c_1 > 0 \) such that for any \( t, r > 0 \) and \( x \in M_0, \)
\[
\int_{\{y : d(x, y) \geq r\}} p(t, x, y) \mu(dy) \leq c_1 \left( \frac{\psi^{-1}(t)}{r} \right)^{\eta} + c_1 \exp \left( - a_1 r^{1/N} \right) \frac{\Phi^{-1}(t)^{1/N}}{\Phi^{-1}(t)^{1/N}}.
\]
When \( r \leq \Phi^{-1}(t) \), we immediately obtain (3.14) by letting \( c = \exp(a_1) \). Thus, we will only consider the case \( r > \Phi^{-1}(t) \). Fix \( \alpha \in (d_2/(d_2 + \beta_1), 1) \) and define for \( n \in \mathbb{N}, \)
\[
\rho_n = \rho_n(r, t) = 2^{n \alpha} r^{1-1/N} \Phi^{-1}(t)^{1/N}.
\]
Since \( r > \Phi^{-1}(t) \), we have \( \Phi^{-1}(t) < \rho_n \leq 2^n r \). In particular, \( t \leq \Phi(\rho_n) \). Thus, using Lemma 3.2 with \( \rho = \rho_n \), we have that for every \( t > 0 \) and \( x, y \in M_0 \) with \( 2^n r \leq d(x, y) < 2^{n+1} r, \)
\[
p(t, x, y)
\]
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\[ \leq \frac{c_2}{V(x, \Phi^{-1}(t))} \left( \frac{2^{n+1}r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp \left( C_1 \frac{t}{\Phi(\rho_n)} - C_2 \frac{d(x,y)}{\rho_n} \right) + \frac{c_2 t}{V(x, \rho_n)} \left( 1 + \frac{t}{\Phi(\rho_n)} \right)^{d_2+1} \]

\[ \leq \frac{c_3}{V(x, \Phi^{-1}(t))} \left( \frac{2^n r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp \left( -C_2 \frac{2^n r}{\rho_n} \right) + \frac{c_3 t}{V(x, \rho_n)} \left( 1 + \frac{t}{\Phi(\rho_n)} \right)^{d_2+1} \]

\[ = \frac{c_3}{V(x, \Phi^{-1}(t))} \left( \frac{2^n r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp \left( -C_2 \frac{2^n(1-\alpha) r^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) + \frac{c_3 t}{V(x, \rho_n)} \left( 1 + \frac{t}{\Phi(\rho_n)} \right)^{d_2+1} \]

Using the above estimate and VD\( (d_2) \) we get that

\[ \int_{B(x,r)^c} p(t, x, y) \mu(dy) = \sum_{n=0}^\infty \int_{B(x,2^n r) \setminus B(x,2^n r)^c} p(t, x, y) \mu(dy) \]

\[ \leq c_3 \sum_{n=0}^\infty \frac{V(x, 2^{n+1} r)}{V(x, \Phi^{-1}(t))} \left( \frac{2^n r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp \left( -C_2 \frac{2^n(1-\alpha) r^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) + \frac{c_3 t}{V(x, \rho_n)} \left( 1 + \frac{t}{\Phi(\rho_n)} \right)^{d_2+1} \]

\[ =: I_1 + I_2. \]

We first estimate \( I_1 \). Observe that for any \( d_0 \geq 1 \), there exists \( c_1 = c_1(c_0, \alpha) > 0 \) such that

\[ 2^{n d_0} \exp \left( -C_2 \frac{2^n(1-\alpha) \rho^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) \leq 2^{-n d_0} \exp \left( 2 n d_0 - C_2 \frac{2^n(1-\alpha) \rho^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) \]

\[ \leq 2^{-n d_0} \exp \left( \left( \frac{C_2}{2 d_0} 2^n(1-\alpha) + c_1 \right) d_0 - C_2 \frac{2^n(1-\alpha) \rho^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) \]

\[ \leq 2^{-n d_0} \exp \left( \frac{C_2}{2} 2^n(1-\alpha) + c_1 d_0 - C_2 \frac{2^n(1-\alpha) \rho^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) = e^{c_1 d_0} 2^{-n d_0} \exp \left( -\frac{C_2}{2} \rho \right). \] (3.15)

Using \( \Phi^{-1}(t) < r \), VD\( (d_2) \), (3.15), and the fact that

\[ \sup_{1 \leq s} s^{2^{d_2+1}} \exp(-\frac{C_2}{4} s^{1/N}) := c_4 < \infty, \]

we obtain

\[ I_1 = c_3 \sum_{n=0}^\infty \frac{V(x, 2^{n+1} r)}{V(x, \Phi^{-1}(t))} \left( \frac{2^n r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp \left( -C_2 \frac{2^n(1-\alpha) r^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) \]

\[ \leq c_4 \sum_{n=0}^\infty \left( \frac{r}{\Phi^{-1}(t)} \right)^{d_2+1} 2^n 2^{n(2d_2+1)} \exp \left( -C_2 \frac{2^n(1-\alpha) r^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) \]

\[ \leq c_5 \left( \frac{r}{\Phi^{-1}(t)} \right)^{d_2+1} \exp \left( -\frac{C_2}{2} \frac{r^{1/N}}{\Phi^{-1}(t)^{1/N}} \right) \sum_{n=0}^\infty 2^{-n(2d_2+1)} \leq c_6 \exp \left( -\frac{C_2 r^{1/N}}{4 \Phi^{-1}(t)^{1/N}} \right). \] (3.16)

We next estimate \( I_2 \). Note that by (2.6) and \( t < \Phi(\rho_n) \), we have \( \psi^{-1}(t) \leq \Phi^{-1}(t) \leq \rho_n \). Thus, using VD\( (d_2) \) and \( L(\beta_1, C_L, \psi) \) for the first line and using \( \alpha(d_2 + \beta_1) > d_2 \) for the second line, we obtain

\[ I_2 = c_3 \sum_{n=0}^\infty \frac{V(x, 2^{n+1} r) \psi(\psi^{-1}(t))}{V(x, \rho_n)} \leq c_7 \sum_{n=0}^\infty \left( \frac{2^n r}{\rho_n} \right)^{d_2} \left( \frac{\psi^{-1}(t)}{\rho_n} \right)^{\beta_1} \]

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\[ c \left( \frac{\Phi^{-1}(t)}{r} \right)^{-d_2 + \beta_1} \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_1} 2^n(d_2 - \alpha(d_2 + \beta_1)) \]

\[ := c_8 \left( \frac{\Phi^{-1}(t)}{r} \right)^{-d_2 + \beta_1} \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_1} \leq c_8 \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_1 - d_2 + \beta_1} = c_8 \left( \frac{\psi^{-1}(t)}{r} \right)^{\eta}. \]

Thus, by above estimates of \( I_1 \) and \( I_2 \), we obtain (3.14).

By \( \eta < \beta_1 \) and (3.14), assumptions in Lemma 3.4 hold with \( f(r, t) := \left( r / \Phi^{-1}(t) \right)^{1/N} \). Thus, by Lemma 3.4, we have

\[ p(t, x, y) \leq \frac{c_{10} t}{V(x, d(x, y)) \psi(d(x, y))} + \frac{c_{10}}{V(x, \Phi^{-1}(t))} \left( 1 + \frac{d(x, y)}{\Phi^{-1}(t)} \right)^{d_2} \exp \left[ - a_1 k \left( \frac{d(x, y)}{16k \Phi^{-1}(t)} \right)^{1/N} \right]. \]

Using the fact that \( \sup_{s > 0} (1 + s)^{d_2} \exp(-c_{10} s^{1/N}) < \infty \) for every \( c > 0 \), we conclude (3.13). \( \square \)

The next lemma will be used in the next subsection and Section 4.2.

**Lemma 3.6.** Suppose \( \text{VD}(d_2), J, \psi, \leq, \text{UHKD}(\Phi) \) and \( E_\Phi \). Then, for any \( \theta > 0 \) and \( c_0, c_1 \geq 1 \), there exists \( c > 0 \) such that for any \( x \in M_0, t > 0 \) and \( r \geq c_0 \Phi^{-1}(c_1 t)^{1+\theta/s} \),

\[ \int_{B(x, r)^c} p(t, x, y) \mu(dy) \leq c \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_1}. \]

**Proof.** Denote \( t_1 = c_1 t \) and let \( a_0, N \) be the constants in Lemma 3.5. By (2.6) we have that for any \( y \in M_0 \) with \( d(x, y) > r \), there exists \( \theta_0 \in (\theta, \infty) \) satisfying \( d(x, y) = c_0 \Phi^{-1}(t_1)^{1+\theta_0} / \psi^{-1}(t_1)^{\theta_0} \).

Note that there exists a positive constant \( c_2 = c_2(\theta) \) such that for any \( s > 0 \),

\[ s^{-d_2 - \beta_2 - \beta_2 / \theta} \geq c_2 \exp(-a_0 s^{1/N}). \]  

(3.17)

Also, since \( c_0 \geq 1 \) we have

\[ \psi^{-1}(t_1) \leq c_0 \psi^{-1}(t_1) \leq c_0 \Phi^{-1}(t_1) < d(x, y) = c_0 \Phi^{-1}(t_1)^{1+\theta_0} / \psi^{-1}(t_1)^{\theta_0}. \]

Thus, using \( \text{VD}(d_2) \) and \( U(\beta_2, C_U, \psi) \) for the first inequality and (3.17) for the second, we have

\[ \frac{t}{V(x, d(x, y)) \psi(d(x, y))} \leq \frac{c_1}{V(x, c_0 \Phi^{-1}(t_1)) \psi(\psi^{-1}(t_1))} \]

\[ \geq \frac{c_1}{V(x, c_0 \Phi^{-1}(t_1))} \left( \frac{\psi^{-1}(t_1)}{\Phi^{-1}(t_1)} \right)^{d_2 \theta_0} \left( \frac{\psi^{-1}(t_1)}{\Phi^{-1}(t_1)} \right)^{(1+\theta_0)\beta_2} \]

\[ = \frac{c_1}{V(x, c_0 \Phi^{-1}(t_1))} \left( \frac{\Phi^{-1}(t_1)}{\psi^{-1}(t_1)} \right)^{-d_2 - \beta_2 - \beta_2 / \theta} \]

\[ \geq \frac{c_1}{V(x, c_0 \Phi^{-1}(t_1))} \left( \frac{\Phi^{-1}(t_1)}{\psi^{-1}(t_1)} \right)^{-d_2 - \beta_2 - \beta_2 / \theta} \exp \left( -a_0 \Phi^{-1}(t_1)^{\theta_0 / \theta} \right). \]
Applying Lemma \textbf{A.2} for $L(\alpha_1, c_L, \Phi)$, we have $U(1/\alpha_1, c_L^{-1/\alpha_1}, \Phi^{-1})$, which yields $\Phi^{-1}(t) \leq \Phi^{-1}(t_1) \leq c_L^{-1/\alpha_1} c_1^{1/\alpha_1} \Phi^{-1}(t)$. Thus, using this and \text{VD}(d_2) again, we have
\[
\begin{align*}
\frac{1}{V(x, c_0\Phi^{-1}(t_1))} \exp \left( - \frac{a_0 d(x, y)^{1/N}}{\Phi^{-1}(t_1)^{1/N}} \right) & \geq \frac{1}{V(x, c_0 c_L^{-1/\alpha_1} c_1^{1/\alpha_1} \Phi^{-1}(t_1))} \exp \left( - \frac{a_0 d(x, y)^{1/N}}{\Phi^{-1}(t_1)^{1/N}} \right) \\
& \geq C^{-1}_\mu (c_0 c_L^{-1/\alpha_1} c_1^{1/\alpha_1}) d_2 \frac{1}{V(x, \Phi^{-1}(t))} \exp \left( - \frac{a_0 d(x, y)^{1/N}}{\Phi^{-1}(t_1)^{1/N}} \right) .
\end{align*}
\]

Thus, by Lemma \textbf{3.5} and above two estimates, we have that for every $y \in M_0$ with $d(x, y) > r$,
\[
p(t, x, y) \leq \frac{c t}{V(x, d(x, y)) \psi(d(x, y))} + \frac{c}{V(x, \Phi^{-1}(t))} e^{-a_0 d(x, y)^{1/N}} V(x, d(x, y)) \psi(d(x, y)) \leq \frac{c_2 t}{V(x, d(x, y)) \psi(d(x, y))} .
\]

Using this, \textbf{[21] Lemma 2.1} and $L(\beta_1, C_L, \psi)$ with the fact that $r > c_0 \psi^{-1}(c_1 t)$ which follows from \textbf{[2.6]}, we conclude that
\[
\int_{B(x, r)^c} p(t, x, y) \mu(dy) \leq c_2 \int_{B(x, r)^c} \frac{t}{V(x, d(x, y)) \psi(d(x, y))} \mu(dy) \leq c_3 \frac{t}{\psi(r)} \leq c_4 \left( \psi^{-1}(t) \right)^{\beta_1} .
\]
This proves the lemma.

\section{3.2 Basic properties of $T(\phi)$}

In this subsection, we assume that an non-decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$ with some $0 < \alpha_1 \leq \alpha_2$. Recall that $\phi^{-1}(t) := \inf\{ s \geq 0 : \phi(s) > t \}$ is the generalized inverse function of $\phi$.

\textbf{Lemma 3.7.} For any $t > 0$,
\[
c_U^{-1} t \leq \phi(\phi^{-1}(t)) \leq c_U t . \tag{3.18}
\]

\textbf{Proof.} Let $\phi^{-1}(t) = s$. Since $\phi$ is non-decreasing, we have $\phi(u) \leq t$ for all $u < s$. Thus, using $U(\alpha_2, c_U, \phi)$ we have
\[
\phi(s) \leq c_U \left( \frac{s}{u} \right)^{\alpha_2} \phi(u) \leq c_U \left( \frac{s}{u} \right)^{\alpha_2} .
\]

Letting $u \to s$ we obtain $\phi(s) \leq c_U t$. By the similar way, using the fact that $\phi(u) \geq t$ for all $u > s$, we have
\[
\phi(s) \geq c_U^{-1} \left( \frac{s}{u} \right)^{\alpha_2} \phi(u) \geq c_U^{-1} \left( \frac{s}{u} \right)^{\alpha_2} ,
\]
which yields $\phi(s) \geq c_U^{-1} t$ by letting $u \downarrow s$.

For the remainder of this subsection, we further assume that $\phi$ satisfies $L_a(\delta, \tilde{C}_L)$ for some $a > 0$ and $\delta > 1$. For the function $\phi$, we define
\[
T(\phi)(r, t) := \sup_{s > 0} \left[ \frac{r}{s} - \frac{t}{\phi(s)} \right] , \quad r, t > 0 . \tag{3.19}
\]

Note that from $L(\alpha_1, c_L, \phi)$ and $L_a(\delta, \tilde{C}_L)$, we obtain $\lim_{s \to \infty} \phi(s) = \infty$ and $\lim_{s \to 0} \frac{\phi(s)}{s} = 0$, respectively. This concludes that $T(\phi)(r, t) \in [0, \infty)$ for all $r, t > 0$. Also, comparing the definitions in \textbf{[2.7]}. and
Moreover, if $T > 0$, we see that $T(\Phi) = \Phi_1$ for instance. It immediately follows from the definition of $T(\phi)$ that for any $c, r, t > 0$,

$$T(\phi)(cr, ct) = cT(\phi)(r, t).$$

We first observe when the supremum in (3.19) occurs.

**Lemma 3.8.** Let $\delta_1 := \frac{1}{s-1}$. For any $T \in (0, \infty)$, there exists constant $b \in (0, 1)$ such that for any $r > 0, t \in (0, T]$ with $r \geq 2c_U\phi^{-1}(t)$,

$$T(\phi)(r, t) = \sup_{s \in [br^{-\delta_1}(t)\delta_1+1, 2\phi^{-1}(t)]} \left[ \frac{r}{s} - \frac{t}{\phi(s)} \right] \geq \frac{r}{2\phi^{-1}(t)}.$$  

(3.20)

Moreover, if $L(\delta, \bar{C}_L, \phi)$ holds, (3.20) holds for all $t \in (0, \infty)$.

**Proof.** From Remark A.1 we may and do assume $a = \phi^{-1}(T)$ without loss of generality. Denote $b = (c_U^{-1}\bar{C}_L)^{\delta_1} \in (0, 1)$. Fix $r > 0$ and $t \in (0, T]$ with $r \geq 2c_U\phi^{-1}(t)$ and let us define

$$T_1(\phi)(r, t) := \sup_{s \in [br^{-\delta_1}(t)\delta_1+1, 2\phi^{-1}(t)]} \left[ \frac{r}{s} - \frac{t}{\phi(s)} \right].$$

Since $r \geq 2\phi^{-1}(t)$, we have $r^{-\delta_1}\phi^{-1}(t)\delta_1+1 \leq \phi^{-1}(t)$, which yields $\phi^{-1}(t) \in [br^{-\delta_1}\phi^{-1}(t)\delta_1+1, 2\phi^{-1}(t)]$. Now, taking $s = \phi^{-1}(t)$ for (3.20). Using (3.18) and $r \geq 2c_U\phi^{-1}(t)$ we have

$$T_1(\phi)(r, t) \geq \frac{r}{\phi^{-1}(t)} - \frac{t}{\phi^{-1}(t)} \geq \frac{r}{\phi^{-1}(t)} - c_U \geq \frac{r}{2\phi^{-1}(t)}.$$  

(3.21)

Assume $s > 2\phi^{-1}(t)$. Then, we have $\frac{r}{s} - \frac{t}{\phi(s)} \leq \frac{r}{2\phi^{-1}(t)}$. Thus, by (3.21) we obtain $\frac{r}{s} - \frac{t}{\phi(s)} \leq T_1(\phi)(r, t)$ for $s > 2\phi^{-1}(t)$.

Now assume $s < br^{-\delta_1}\phi^{-1}(t)\delta_1+1$. Since $s \leq \phi^{-1}(t) \leq \phi^{-1}(T)$, using (3.18), $L_{\phi^{-1}(T)}(\delta, \bar{C}_L, \phi)$ and $r \geq 2c_U\phi^{-1}(t)$ we have

$$\frac{r}{s} - \frac{t}{\phi(s)} \leq \frac{r}{s} - c_U^{-1}\bar{C}_L\phi^{-1}(t) = \frac{r}{s} - c_U^{-1}\bar{C}_L\phi^{-1}(t)\delta \leq \frac{r}{s} - c_U^{-1}\bar{C}_L\phi^{-1}(t)^{\delta-1} \leq \frac{r}{s} - c_U^{-1}\bar{C}_L(br^{-\delta_1}\phi^{-1}(t)\delta+1)^{\delta-1} = 0,$$

where we have used $c_U^{-1}\bar{C}_Lb^{\delta-1} = 1$ for the last line. Therefore, by (3.21) we obtain $\frac{r}{s} - \frac{t}{\phi(s)} \leq 0 \leq T_1(\phi)(r, t)$. Combining above two cases and the definition of $T_1(\phi)$ we conclude that $\frac{r}{s} - \frac{t}{\phi(s)} \leq T_1(\phi)(r, t)$ for every $s > 0$. This concludes the lemma.

**Lemma 3.9.** (i) For any $T > 0$ and $c_1, c_2 > 0$, there exists a constant $c > 0$ such that for any $r > 0$ and $t \in (0, T]$ with $r \geq 2c_U\phi^{-1}(t)$,

$$T(\phi)(c_1r, c_2t) \leq cT(\phi)(r, t).$$  

(3.22)

(ii) For any $T > 0$ and $c_3 > 0$, there exists a constant $\tilde{c} > 0$ such that for any $t \in (0, T]$ and $r \leq c_3\phi^{-1}(t)$,

$$T(\phi)(r, t) \leq \tilde{c}.$$  

(3.23)

Moreover, if $L(\delta, \bar{C}_L, \phi)$ holds, both (3.22) and (3.23) hold for all $t \in (0, \infty)$.  

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Proof. Without loss of generality we may and do assume $a = 2\phi^{-1}(T)$.

(i) Since

$$T(\phi)(c_1 r, c_2 t) = c_2 T(\phi)(\frac{c_1}{c_2} r, t),$$

it suffices to show that for any $c_4 > 0$, there exists a constant $c(c_4) > 0$ such that for any $r > 0$ and $t \in (0, T]$ with $r \geq 2c_U\phi^{-1}(t)$,

$$T(\phi)(c_4 r, t) \leq cT(\phi)(r, t).$$

(3.24)

Also, since $T(\phi)(r, t)$ is increasing on $r$, we may and do assume that $c_4 \geq 1$. Since $c_4 r \geq r \geq 2c_U\phi^{-1}(t)$, by Lemma 3.8 we have

$$T(\phi)(c_4 r, t) = \sup_{s \in (0, 2\phi^{-1}(t))} \left[ \frac{c_4 r}{s} - \frac{t}{\phi(s)} \right].$$

Let $\theta = (c_4^{-1}C_L)^{1/(\delta-1)} \leq 1$, which satisfies $\theta = c_4 C_L^{-1} \theta^\delta$. Firstly, for any $s \in (0, 2\theta\phi^{-1}(t)]$, we have

$$\frac{r}{s/\theta} - \frac{t}{\phi(s/\theta)} = \frac{\theta r}{s} - \frac{\phi(s)}{\phi(s/\theta)} \phi(s) \geq \frac{\theta r}{s} - \tilde{C}_L^{-1} \theta^\delta \frac{t}{\phi(s)} = \tilde{C}_L^{-1} \theta^\delta \left( \frac{c_4 r}{s} - \frac{t}{\phi(s)} \right),$$

where we have used $L_{2\phi^{-1}(T)}(\delta, \tilde{C}_L, \phi)$ and $s \leq \phi^{-1}(t) \leq 2\phi^{-1}(t)$. Thus, we have

$$\sup_{s \in (0, 2\phi^{-1}(t))} \left[ \frac{c_4 r}{s} - \frac{t}{\phi(s)} \right] \leq \tilde{C}_L \theta^{-\delta} T(\phi)(r, t).$$

Also, using (3.20) and $r \geq 2c_U\phi^{-1}(t)$, for any $s \in (2\theta\phi^{-1}(t), 2\phi^{-1}(t)]$ we have

$$\frac{c_4 r}{s} - \frac{t}{s} \leq \frac{c_4 r}{2\phi^{-1}(t)} \leq \frac{c_4 r}{\theta} T(\phi)(r, t).$$

Combining above two inequalities, we obtain the desired estimate.

(ii) Since $T(\phi)(r, t)$ is increasing on $r$, we may and do assume that $r = c_3\phi^{-1}(t)$ and $c_3 \geq 2c_U$. Observe that by $c_3 \geq 2c_U$, (3.13), Lemma 3.8 and $L_{2\phi^{-1}(T)}(\delta, \tilde{C}_L, \phi)$ we have that for any $t \leq T$,

$$T(\phi)(c_3\phi^{-1}(t), t) = \sup_{s \leq 2\phi^{-1}(t)} \left[ \frac{c_3 \phi^{-1}(t)}{s} - \frac{t}{\phi(s)} \right] \leq \sup_{s \leq 2\phi^{-1}(t)} \left[ \frac{c_3 \phi^{-1}(t)}{s} - c_5 \phi^{-1}(t) \phi(2\phi^{-1}(t)) \phi(s) \right] \leq \sup_{s \leq 2\phi^{-1}(t)} \left[ \frac{c_3 \phi^{-1}(t)}{s} - c_5 \phi(2\phi^{-1}(t)) \phi(s) \right] = \sup_{u > 0} \left( c_3 u - c_5 2^\delta u^\delta \right) := \tilde{c} < \infty.$$

Here in the last line we have used $\delta > 1$ to obtain $\tilde{c} < \infty$. This proves (3.23). \qed

3.3 Proofs of Theorems 2.13 and 2.14

In this subsection, we prove our first main results. We start with the proof of Theorem 2.13.

Proof of Theorem 2.13. Note that under the condition $L^a(\delta, \tilde{C}_L, \Phi)$, we have $\alpha_2 \geq \delta \lor \alpha_1$. Take

$$\theta := \frac{(\delta - 1)\beta_1}{\delta(2d_2 + \beta_1) + (\beta_1 + 2\alpha_2 + 2d_2\alpha_2)} \in (0, \delta - 1)$$

where

$$\beta_1 \equiv \frac{(\delta - 1)\beta_1}{\delta(2d_2 + \beta_1) + (\beta_1 + 2\alpha_2 + 2d_2\alpha_2)} \in (0, \delta - 1)$$

and

$$\beta_2 \equiv \frac{(\delta - 1)\beta_1}{\delta(2d_2 + \beta_1) + (\beta_1 + 2\alpha_2 + 2d_2\alpha_2)} \in (0, \delta - 1).$$
and $C_0 = \frac{4c_U}{C_2}$, where $C_1$ and $C_2$ are the constants in Lemma 3.2. Without loss of generality, we may and do assume that $C_1 \geq 2$ and $C_2 \leq 1$. Let $\alpha$ be a number in $(\frac{d_r}{d_r + \beta_1}, 1)$.

(i) We will show that there exist $a_1 > 0$ and $c_1 > 0$ such that for any $t \leq T$, $x \in M_0$ and $r > 0$,

$$\int_{B(x,r)^c} p(t, x, y) \mu(dy) \leq c_1 \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_1/2} + c_1 \exp(-a_1 \Phi_1(r,t)). \quad (3.25)$$

Firstly, since $\Phi_1(r,t)$ is increasing on $r$, by (3.22) and (3.23) we have that for $r \leq C_0 \Phi^{-1}(C_1 t)$,

$$\Phi_1(r,t) \leq \Phi_1(C_0 \Phi^{-1}(C_1 t), t) \leq c_2 \Phi_1(C_0 \Phi^{-1}(C_1 t), C_1 t) \leq c_3.$$  

Here for the second inequality, $C_0 \geq 2c_U$ yields the condition in (3.22). Thus, for any $x \in M_0$ and $r \leq C_0 \Phi^{-1}(C_1 t)$ we have

$$\int_{B(x,r)^c} p(t, x, y) \mu(dy) \leq 1 \leq e^{a_1 c_3} \exp(-a_1 \Phi_1(r,t)).$$

Also, when $r \geq C_0 \Phi^{-1}(C_1 t)^{1+\theta}/\psi^{-1}(C_1 t)^{\theta}$, (3.25) immediately follows from Lemma 3.6 and the fact that $r > \psi^{-1}(t)$.

Now consider the case $C_0 \Phi^{-1}(C_1 t) < r \leq C_0 \Phi^{-1}(C_1 t)^{1+\theta}/\psi^{-1}(C_1 t)^{\theta}$. In this case, there exists $\theta_0 \in (0, \theta$] such that $r = C_0 \Phi^{-1}(C_1 t)^{1+\theta_0}/\psi^{-1}(C_1 t)^{\theta_0}$ by (2.6). Since $C_0 = \frac{4c_U}{C_2}$, applying Lemma 3.8 with the constant $C_1 T$ we have $r \in [b(C_2 r/2)^{-\delta_1} \Phi^{-1}(C_1 t)^{\delta_1+1}, 2\Phi^{-1}(C_1 t)]$ such that

$$\Phi_1(C_2 r/2, C_1 t) - \frac{C_6 C_2}{8} \leq \frac{C_2 r}{2\rho} \leq \frac{C_t}{\Phi(\rho)} \leq \Phi_1(C_2 r/2, C_1 t),$$

where $\delta_1 = \frac{1}{\alpha - 1}$. Also, let $\rho_n = 2^{n+1} \rho$ for $n \in \mathbb{N}_0$. Then, we have

$$\frac{C_2 r}{\rho_n} = \frac{C_2 r}{2\rho} + \frac{C_2 r}{\rho} (2^{n+1} - \frac{1}{2}) \geq \frac{C_2 r}{2\rho} + \frac{C_2 r}{\rho} 2^{n+1/2} \geq \frac{C_2 r}{2\rho} + \frac{C_2 r}{4\Phi^{-1}(C_1 t)} 2^{n+1/2}.$$  

Using this, $r \geq C_0 \Phi^{-1}(C_1 t)$ and (3.22) with $U(1/\alpha_1, c_L^{-1/\alpha_1}, \Phi^{-1})$, which follows from $L(\alpha_1, c_L, \Phi)$ and Lemma 3.2, yield that for any $n \in \mathbb{N}_0$,

$$\frac{C_t}{\Phi(\rho_n)} - \frac{C_2 2^n r}{\rho_n} \leq \frac{C_t}{\Phi(\rho)} \leq \frac{C_2 r}{2\rho} \leq \frac{C_2 r}{4\Phi^{-1}(C_1 t)} 2^{n+1/2}$$

$$\leq -\Phi_1(C_2 r/2, C_1 t) + \frac{C_6 C_2}{8} - \frac{C_2 r}{4\Phi^{-1}(C_1 t)} 2^{n+1/2}$$

$$\leq -\Phi_1(C_2 r/2, C_1 t) - \frac{C_2 r}{8\Phi^{-1}(C_1 t)} 2^{n+1/2}$$

$$\leq -\Phi_1(C_2 r/2, C_1 t) - \frac{C_2}{8} (c_L / C_1)^{1/\alpha_1} 2^{n+1/2} \frac{r}{\Phi^{-1}(t)}$$

$$\leq -c_4 \Phi_1(r,t) - c_5 2^{n+1/2} \frac{r}{\Phi^{-1}(t)}.$$  

Combining (3.26) and Lemma 3.2 with $\rho = \rho_n$, we have that for $t \in (0, T]$ and $y \in B(x, 2^{n+1} r)/B(x, 2^n r)$,

$$p(t, x, y) \leq \frac{c_6 t}{V(x, \rho_n) \psi(\rho_n)} \left(1 + \frac{t}{\Phi(\rho_n)}\right)^{d_2+1} + \frac{c_7}{V(x, \Phi^{-1}(t))} \left(1 + \frac{2^{n+1} r}{\Phi^{-1}(t)}\right)^{d_2} \exp\left(C_1 \frac{t}{\Phi(\rho_n)} - C_2 2^n r / \rho_n\right).$$
With estimates in (3.27), we get that
\[ r > C \]
Thus, we have
\[ b \psi \]
we obtain
\[ \Phi - c \]
\[ \sup_{n \in \mathbb{N}} \sup_{s > 1} (1 + 2^n s)^d \exp \left[ - \frac{c_5}{2} 2^{n(1-\alpha)} s \right] \leq \sup_{n \in \mathbb{N}} \sup_{s > 1} (1 + 2^n s)^d \exp \left[ - \frac{c_5}{2} 2^{n(1-\alpha)} s \right] \]
\[ \leq \sup_{s > 1} (1 + 2s)^d \exp \left( - \frac{c_5}{2} s^{1-\alpha} \right) < \infty. \]

With estimates in (3.27), we get that
\[
\int_{B(x,r)^c} p(t,x,y)\mu(dy) \leq \sum_{n=0}^\infty \int_{B(x,2^{n+1}r) \setminus B(x,2^n r)} p(t,x,y)\mu(dy) \\
\leq c_8 \sum_{n=0}^\infty \frac{V(x,2^n r)}{V(x,\Phi^{-1}(t))} \exp \left( -c_4 \Phi_1(r,t) - \frac{c_5}{2} \frac{2^{n(1-\alpha)} r}{\Phi^{-1}(t)} \right) \\
+ c_8 \sum_{n=0}^\infty \frac{tV(x,2^n r)}{V(x,\rho_n)\psi(\rho_n)} \left( 1 + \frac{t}{\Phi(\rho_n)} \right)^{d_2+1} \\
:= c_8(I_1 + I_2). 
\]

Using \( r > C_0 \Phi^{-1}(C_1 t) \geq \Phi^{-1}(t) \), we obtain upper bound of \( I_1 \) by following the calculations in (3.16). Thus, we have
\[ I_1 \leq c_9 \exp \left( -a_2 \Phi_1(r,t) \right). \]

Next, we estimate \( I_2 \). Since \( r = C_0 \Phi^{-1}(C_1 t)^{1+\theta_0} / \psi^{-1}(C_1 t)^{\theta_0} \), \( \psi^{-1}(t) < \Phi^{-1}(t) \) and \( \theta_0 \leq \theta < 1/\delta_1 \), we obtain
\[
\frac{\Phi^{-1}(C_1 t)}{\rho} \leq \frac{\Phi^{-1}(C_1 t) b^{-1}(C_2 r/2)^\delta_1}{\Phi^{-1}(C_1 t)^{\delta_1+1}} = b^{-1}(C_0 C_2/2)^\delta_1 \left( \frac{\Phi^{-1}(C_1 t)}{\psi^{-1}(C_1 t)} \right)^{\delta_1 \theta_0} \leq b^{-1}(C_0 C_2/2)^\delta_1 \Phi^{-1}(C_1 t) \psi^{-1}(C_1 t). 
\]

Thus, \( \frac{b \psi^{-1}(C_1 t)}{C_0 C_2/2} \theta_0 \leq \rho \leq 2 \Phi^{-1}(C_1 t) \leq r \). Using this, \( \text{VD}(d_2) \), (3.18), \( L(\beta_1, C_L, \psi) \) and \( U(\alpha_2, c_U, \Phi) \) we have
\[ I_2 = \sum_{n=0}^\infty \frac{V(x,2^n r)}{V(x,\rho_n) \psi(\rho_n)} \left( 1 + \frac{t}{\Phi(\rho_n)} \right)^{d_2+1} \\
\leq c_{10} \sum_{n=0}^\infty \frac{V(x,2^n r)}{V(x,\rho_n) \psi(b^{-1}(C_0 C_2/2)^\delta_1 \rho_n)} \psi(b^{-1}(C_0 C_2/2)^\delta_1 \rho_n) \left( \frac{C_1 t}{\Phi(\rho/2)} \right)^{d_2+1} \\
\leq c_{11} \sum_{n=0}^\infty \left( \frac{2^n r}{\rho_n} \right)^{d_2} \left( \frac{\psi^{-1}(C_1 t)}{\rho_n} \right)^{\delta_1} \left( \frac{\Phi^{-1}(C_1 t)}{\rho} \right)^{\delta_1 \theta_0} \alpha_2(d_2+1) \\
\leq c_{12} \sum_{n=0}^\infty 2^n(d_2 - \alpha(d_2+\beta_1)) \left( \frac{r}{\rho} \right)^{d_2} \left( \frac{\psi^{-1}(C_1 t)}{\rho} \right)^{\delta_1} \left( \frac{\Phi^{-1}(C_1 t)}{\rho} \right)^{\delta_1 \theta_0} \alpha_2(d_2+1) \\
leq 22. \]
By the estimates of
\[ I_2 \leq c_{13} b^{d_2} \psi^{-1}(C_1 t)^{\beta_1} \Phi^{-1}(C_1 t)^{\alpha_2} \rho^{d_2-\beta_1-\alpha_2}. \]

Since \( b^{d_2} \Phi^{-1}(C_1 t)^{\beta_1+1} \leq \rho \), we conclude that
\[ I_2 \leq c_{13} b^{d_2-\beta_1-\alpha_2} \psi^{-1}(C_1 t)^{\beta_1} \Phi^{-1}(C_1 t)^{\alpha_2} (d_2-\beta_1-\alpha_2). \] (3.28)

Using \( r = C_0 \Phi^{-1}(C_1 t)^{1+\theta_0}/\psi^{-1}(C_1 t)^{\theta_0} \), we have \( C_0 \psi^{-1}(C_1 t) < C_0 \Phi^{-1}(C_1 t) < r \). Since \( \theta_0 \leq \theta \), we have
\[ \frac{C_0 \Phi^{-1}(C_1 t)}{r} = \left( \frac{\psi^{-1}(C_1 t)}{\Phi^{-1}(C_1 t)} \right)^{\theta_0} = \left( \frac{C_0}{r} \right)^{\theta_0/(1+\theta_0)} \geq \left( \frac{C_0}{r} \right)^{\theta/(1+\theta)} . \]

By using \( \theta = \frac{(\delta-1)\beta_1}{\delta(2d_2+\beta_1)+(\beta_1+2\alpha_2+2d_2\alpha_2)} > 0 \), we have
\[ \left( \frac{\Phi^{-1}(C_1 t)}{r} \right)^{-\delta_1(2d_2+\alpha_2+\beta_1+d_2)-(d_2+\beta_1)} \leq c_{14} \left( \frac{\psi^{-1}(C_1 t)}{r} \right)^{\frac{\theta}{1+\theta}} \left[ -\delta_1(2d_2+\alpha_2+\beta_1+d_2)-(d_2+\beta_1) \right] = c_{14} \left( \frac{\psi^{-1}(C_1 t)}{r} \right)^{-\beta_1/2} . \]

Therefore, using (3.28) we obtain
\[ I_2 \leq c_{15} \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_1/2} . \]

By the estimates of \( I_1 \) and \( I_2 \), we arrive
\[ \int_{B(x,r)} p(t, x, y) dy \leq c_7 (I_1 + I_2) \leq c_9 \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_1/2} + c_{15} \exp \left( -a_2 \Phi_1(r, t) \right) . \]

Combining all the cases, we obtain (3.29). Thus the assertions on Lemma 3.4 holds with \( f(r, t) := \Phi_1(r, t) \). Thus, using Lemma 3.4, we have constants \( k, c_0 > 0 \) such that
\[ p(t, x, y) \leq \frac{c_0 t}{V(x, d(x, y)) \psi(d(x, y))} + \frac{c_0}{V(x, \Phi^{-1}(t)) \left( 1 + \frac{d(x, y)}{\Phi^{-1}(t)} \right)} \left( 1 + \frac{d(x, y)}{\Phi^{-1}(t)} \right)^{d_2} \exp \left( -a_2 k \Phi_1(d(x, y)/(16k), t) \right) \] (3.29)

for all \( t \in (0, T] \) and \( x, y \in M_0 \). Recall that \( 2cU > 0 \) is the constant in Lemma 3.9 with \( \phi = \Phi \). When \( d(x, y) \leq 32cU k \Phi^{-1}(t) \), using UHKD(\( \Phi \)) and (2.1) we have
\[ p(t, x, y) \leq p(t, x, y)^{1/2} p(t, y, y)^{1/2} \leq \frac{c}{V(x, \Phi^{-1}(t))^{1/2} V(y, \Phi^{-1}(t))^{1/2}} \leq \frac{c_{16}}{V(x, \Phi^{-1}(t))}. \]

Thus, by (3.23) and \( d(x, y)/16k \leq 2cU \Phi^{-1}(t) \) we have
\[ p(t, x, y) \leq \frac{c_{16}}{V(x, \Phi^{-1}(t))} \leq \frac{c_{16}^{a_2 c_{17}}}{V(x, \Phi^{-1}(t))} \exp \left( -a_2 k \Phi_1(d(x, y)/(16k), t) \right) , \]

which yields (2.15) for the case \( d(x, y) \leq 32cU k \Phi^{-1}(t) \). Also, for \( r > 32cU k \Phi^{-1}(t) \) with \( 0 < t \leq T \), using (3.26) with \( n = 0 \) and (3.22) we have
\[ c_4 \Phi_1(r, t) + c_5 \frac{r}{\Phi^{-1}(t)} \leq \Phi_1(C_2 r, C_1 t) \leq c_{17} \Phi_1(r/16k, t) . \]
Therefore, using (3.29) we obtain
\[
p(t, x, y) 
\leq \frac{c_0 t}{V(x, d(x, y))\psi(d(x, y))} + \frac{c_0}{V(x, \Phi^{-1}(t))} \left(1 + \frac{d(x, y)}{\Phi^{-1}(t)}\right)^{d_2} \exp \left(- a_2 k \Phi_1(d(x, y)/(16k), t)\right)
\]
\[
\leq \frac{c_0 t}{V(x, d(x, y))\psi(d(x, y))} + \frac{c_0}{V(x, \Phi^{-1}(t))} \left(1 + \frac{d(x, y)}{\Phi^{-1}(t)}\right)^{d_2} \exp \left(- a_3 \Phi_1(d(x, y), t) - c_{18} \frac{d(x, y)}{\Phi^{-1}(t)}\right).
\]
where we have used \(\sup_{s>0}(1+s)^{d_2}\exp(-c_{18}s) < \infty\) for the last line. Combining two cases, we obtain (3.30).

(ii) Again we will show that there exist \(a_1 > 0\) and \(c_1 > 0\) such that for any \(t \geq T\), \(x \in M_0\) and \(r > 0\),
\[
\int_{B(x, r)^c} p(t, x, y) \mu(dy) \leq c_1 \left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1/2} + c_1 \exp \left(- a_1 \Phi_1(r, t)\right).
\]  
(3.30)

Note that using (2.13), the proof of (3.30) for the case \(r \leq C_0 \Phi^{-1}(C_1 t)\) and \(r > C_0 \Phi^{-1}(C_1 t)^{1+\theta}\) are the same as that for (i).

Without loss of generality we may assume \(a = \Phi(T)\). Then for \(t \geq T\), we have \(\Phi^{-1}(t) = \tilde{\Phi}^{-1}(t)\). Applying this and \(2.13\) for Lemma 3.2 we have for any \(t \geq T\),
\[
p(t, x, y) \leq \frac{c_1}{V(x, \tilde{\Phi}^{-1}(t))} (1 + \frac{d(x, y)}{\tilde{\Phi}^{-1}(t)})^{d_2} \exp \left(C_1 \frac{t}{\Phi(r)} - C_2 \frac{d(x, y)}{\rho} \right) + \frac{c_1 t}{V(x, \rho) \psi(\rho)} (1 + \frac{t}{\Phi(\rho)})^{d_2+1}
\]
\[
\leq \frac{c_1}{V(x, \tilde{\Phi}^{-1}(t))} (1 + \frac{d(x, y)}{\tilde{\Phi}^{-1}(t)})^{d_2} \exp \left(C_1 \frac{t}{\Phi(r)} - C_2 \frac{d(x, y)}{\rho} \right) + \frac{c_1 t}{V(x, \rho) \psi(\rho)} (1 + \frac{t}{\Phi(\rho)})^{d_2+1}.
\]
Since \(L(\delta, \tilde{C}_L, \tilde{\Phi})\) holds, following the proof of (i) we have for any \(t > 0\) and \(r > 0\),
\[
\int_{B(x, r)^c} p(t, x, y) \mu(dy) \leq c_1 \left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1/2} + c_1 \exp \left(- a_1 \Phi_1(r, t)\right).
\]
Since the assumptions in Lemma 3.4 follows from (3.23) and the fact that \(\Phi^{-1}(t) = \tilde{\Phi}^{-1}(t)\) for \(t \geq T\), we obtain that for any \(t \geq T\) and \(x, y \in M\),
\[
p(t, x, y) \leq \frac{c_0 t}{V(x, d(x, y))\psi(d(x, y))} + \frac{c_0}{V(x, \Phi^{-1}(t))} (1 + \frac{d(x, y)}{\Phi^{-1}(t)})^{d_2} \exp \left(- a_2 k \Phi_1(d(x, y), t)\right).
\]

Here in the last term we have used (3.22). With the aid of \(L(\delta, \tilde{C}_L, \tilde{\Phi})\), The remainder is same as the proof of (i).

Now we give the proof of Theorem 2.14 and Corollary 2.15.

Proof of Theorem 2.14 Since \(\Psi, \leq \) implies \(\Phi, \leq \), [21] Theorem 1.15 yields that (2) implies (3), and (3) implies the conservativeness of \((E, F)\). Thus, by Theorem 2.13 (3) implies (1). It remains to prove that (1) implies (2). By (2.6) and Remark 2.11 UHK(\(\Phi, \psi\)) implies UHK(\(\Phi\)). Also, following the proof of [21] Proposition 3.3, we easily prove that UHK(\(\Phi, \psi\)) also implies \(\Psi, \leq \).

Proof of Corollary 2.15 By (2.6), \(\Psi, \leq \) implies \(\Phi, \leq \). Thus, [21] Theorem 1.15 implies the equivalence between the condition in Theorem 2.14 (3) and Corollary 2.15 (4) and (5). We now prove the
equivalence between the condition in Theorem \ref{thm:2.14}(3) and Corollary \ref{cor:2.15}(6). To do this, we will use the results in \cite{21, 29}.

Suppose that $J_{\psi, \leq}, \text{UHKD}(\Phi)$ and $E_{\Phi}$ hold. By \cite{21} Proposition 7.6, we have $\text{FK}(\Phi)$. Since we have $E_{\Phi}$, the condition $E_{\Phi_{\leq, \leq}}$ in \cite{21} Definition 1.10 holds by \cite{21} Lemma 4.16. Since $E_{\Phi_{\leq, \leq}}$ implies the condition $(S)$ in \cite{29} Definition 2.7 with $r < \infty$ and $t < \delta(\Phi)(r)$, we can follow the proof of \cite{29} Lemma 2.8 line by line (replace $r^\beta$ to $\Phi(r)$) and obtain $\text{Gcap}(\Phi)$.

Now, suppose that $\text{FK}(\Phi), J_{\psi, \leq}$ and $\text{Gcap}(\Phi)$ hold. Then, by \cite{21} Lemma 4.14, we have $E_{\Phi_{\geq}}$. To obtain $E_{\Phi_{\geq}}$, we first show that \cite{21} Lemma 4.15 holds under our conditions. i.e., by using $\text{Gcap}(\Phi)$ instead of $\text{CSJ}(\Phi)$, we derive the same result in \cite{21} Lemma 4.15. To show \cite{21} Lemma 4.15, we give the main steps of the proof only. Recall that for any $\rho > 0$, $(E^\rho, F)$ is $\rho$-truncated Dirichlet form. For $\rho$-truncated Dirichlet form, we say $AB^\rho(\Phi)$ holds if the inequality \cite{29} (2.1) holds with $R' < \infty$, $\Phi(r \wedge \rho)$ and $J(x, y)1_{\{d(x, y) < \rho\}}$ instead of $R' < R, r^\beta$ and $j$ respectively. Then, by $\text{VD}(d_2), J_{\psi, \leq}$, \cite{21} Lemma 2.1 and \cite{29} (2.3), we can follow the proof of \cite{29} Lemma 2.4 line by line (replace $r^\beta$ to $\Phi(r)$) to obtain $AB^\rho(\Phi)$. To get $AB_{1/\delta}(\Phi)$, we use the proof of \cite{29} Lemma 2.9. Here, we take different $r_n, s_n, b_n, a_n$ from the one in the proof of \cite{29} Lemma 2.9. Let $\lambda > 0$ be a constant which will be chosen later. Take $s_n = c e^{-n\lambda/2a_2}$ for $n \geq 1$, where $c = c(\lambda)$ is chosen so that $\sum_{n=1}^{\infty} s_n = r$ and $c_2$ is the upper scaling index of $\Phi$. Let $r_n = \sum_{k=1}^{n} s_k$ for $n \geq 1$ and $r_0 = 0$. We also take $b_n = e^{-n\lambda}$ for $n \geq 0$ and $a_n = b_{n-1} - b_n$ for $n \geq 1$. (c.f. \cite{21} Proposition 2.4.) With these $r_n, s_n, b_n, a_n$, we can follow the proof of \cite{29} Lemma 2.9 line by line and obtain $AB_{1/\delta}(\Phi)$ by choosing small $\lambda > 0$. Moreover, using the argument in the proof of \cite{21} Proposition 2.3, we also obtain $AB^\rho_{1/\delta}(\Phi)$ which yields \cite{21} (4.8) for $\rho$-truncated Dirichlet form. Thus, we get \cite{21} Corollary 4.12. For open subsets $A, B$ of $M$ with $A \subset B$, and for any $\rho > 0$, define $\text{Cap}^\rho(A, B) = \inf \{E^\rho(\varphi, \varphi) : \varphi \in \text{cutoff}(A, B)\}$. By $\text{Gcap}(\Phi)$ with $u = 1$ (c.f. \cite{29} Definition 1.13 and below), we have

$$\text{Cap}^\rho(B(x, R), B(x, R + r)) \leq \text{Cap}(B(x, R), B(x, R + r)) \leq c e^{-n\lambda} \frac{V(x, R + r)}{\Phi(r)} \leq c e^{-n\lambda} \frac{V(x, R + r)}{\Phi(r \wedge \rho)},$$

which implies the inequalities in \cite{21} Proposition 2.3(5). Having this and \cite{21} Corollary 4.12 at hand, we can follow the proof and get the result of \cite{21} Lemma 4.15. Now $E_{\Phi_{\geq}}$ follows from the proof of \cite{21} Lemma 4.17. Since we have $E_{\Phi}, \text{UHKD}(\Phi)$ holds by \cite{21} Theorem 4.25.

\subsection{Proofs of Theorems \ref{thm:2.16} and \ref{thm:2.17}}

Throughout this subsection, we will assume that the metric measure space $(M, d, \mu)$ satisfies $\text{VD}(d_2)$ and $\text{RVD}(d_1)$, and the regular Dirichlet form $(E, F)$ and the corresponding Hunt process satisfy $J_{\psi}, \text{PI}(\Phi)$ and $E_{\Phi}$, where $\psi$ is non-decreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$, and $\Phi$ is non-decreasing function satisfying \cite{26}, $L(\alpha_1, c_L)$ and $U(\alpha_2, c_U)$.

From $J_{\psi}$ and $\text{VD}(d_2)$, we immediately see that there is a constant $c > 0$ such that for all $x, y \in M_0$ with $x \neq y$,

$$J(x, y) \leq \frac{c}{V(x, r)} \int_{B(x, r)} J(z, y) \mu(dz) \quad \text{for every } 0 < r \leq d(x, y)/2. \quad \text{(UJS)}$$

(See \cite{18} Lemma 2.1). For any open set $D \subset M$, let $\mathcal{F}_D := \{u \in F : u = 0 \text{ q.e. in } D^c\}$. Then, $(E, \mathcal{F}_D)$ is also a regular Dirichlet form. We use $p^D(t, x, y)$ to denote the transition density function corresponding to $(E, \mathcal{F}_D)$.
Note that \((\mathcal{E}, \mathcal{F})\) is a conservative Dirichlet form by [21] Lemma 4.21. Thus, by [21] Theorem 1.15, we see that CSJ(\(\Phi\)) defined in [21] holds. Thus, by \(J_{\psi, \leq}\), PI(\(\Phi\)), CSJ(\(\Phi\)) and [UJS] with [26], we have (7) in [22] Theorem 1.20.

Therefore, by [22] Theorem 1.20, UHK(\(\Phi\)) and the following joint H"older regularity hold for parabolic functions. We refer [22] Definition 1.13 for the definition of parabolic functions. Note that, by a standard argument, we now can take the continuous version of parabolic functions (for example, see [29] Lemma 5.12]). Let \(Q(t, x, r, R) := (t, t + r) \times B(x, R)\).

**Theorem 3.10.** There exist constants \(c > 0\), \(0 < \theta < 1\) and \(0 < \epsilon < 1\) such that for all \(x_0 \in M\), \(t_0 \geq 0\), \(r > 0\) and for every bounded measurable function \(u = u(t, x)\) that is parabolic in \(Q(t_0, x_0, \Phi(r), r)\), the following parabolic H"older regularity holds:

\[
|u(s, x) - u(t, y)| \leq c \left( \frac{\Phi^{-1}(|s - t|) + d(x, y)}{r} \right) \sup_{[t_0, t_0 + \Phi(r)] \times M} |u|
\]

for every \(s, t \in (t_0, t_0 + \Phi(\epsilon r))\) and \(x, y \in B(x_0, \epsilon r)\).

Since \(p^D(t, x, y)\) is parabolic, from now on, we assume \(\mathcal{N} = \emptyset\) and take the joint continuous versions of \(p(t, x, y)\) and \(p^D(t, x, y)\). (c.f., [29] Lemma 5.13).)

Again, by [22] Theorem 1.20 we have the interior near-diagonal lower bound of \(p^B(t, x, y)\) and parabolic Harnack inequality.

**Theorem 3.11.** There exist \(\epsilon \in (0, 1)\) and \(c_1 > 0\) such that for any \(x_0 \in M\), \(r > 0\), \(0 < t \leq \Phi(\epsilon r)\) and \(B = B(x_0, r)\),

\[
p^B(t, x, y) \geq \frac{c_1}{V(x_0, \Phi^{-1}(t))}, \quad x, y \in B(x_0, \epsilon \Phi^{-1}(t)).
\]

**Proposition 3.12.** Suppose \(VD(d_2)\), \(RVD(d_1)\), \(J_{\psi}\), PI(\(\Phi\)) and \(E_{\Phi}\). Then, there exists \(\eta > 0\) and \(C_3 > 0\) such that for any \(t > 0\),

\[
p(t, x, y) \geq C_3 V(x, \Phi^{-1}(t))^{-1}, \quad x, y \in M \text{ with } d(x, y) \leq \eta \Phi^{-1}(t), \quad (3.31)
\]

and

\[
p(t, x, y) \geq \frac{C_3 \lambda t}{V(x, d(x, y))\psi(d(x, y))}, \quad x, y \in M \text{ with } d(x, y) \geq \eta \Phi^{-1}(t).
\]

**Proof.** The proof of the proposition is standard. For example, see [21] Proposition 5.4].

Let \(\eta = \epsilon / 2 < 1 / 2\) where \(\epsilon\) is the constant in Theorem 3.11. Then by Theorem 3.11,

\[
p(t, x, y) \geq p^B(x, \Phi^{-1}(t/\epsilon)(t, x, y) \geq \frac{c_0}{V(x, \Phi^{-1}(t))} \quad \text{for all } d(x, y) \leq \eta \Phi^{-1}(t). \quad (3.32)
\]

Note that in the beginning of this section we have mentioned that UHK(\(\Phi\)) holds under \(J_{\psi, \leq}\), PI(\(\Phi\)) and \(E_{\Phi}\). Thus, by [21] Lemma 2.7] and UHK(\(\Phi\)), we have

\[
\mathbb{P}^x(\tau_{B(x, r)} \leq t) \leq \frac{c_1 t}{\Phi(r)}, \quad r > 0, \ t > 0, \ x \in M.
\]

Let \(\eta_1 := (C_L/2)^{1/\beta_1} \in (0, \eta)\) so that \(\eta \Phi^{-1}((1 - b)t) \geq \eta_1 \Phi^{-1}(t)\) holds for all \(b \in (0, 1/2]\). Then choose \(\lambda \leq c_1^{-1} C_U^{-1}(2\eta_1/3)^{\beta_2} / 2 < 1 / 2\) small enough so that \(\frac{c_1 \lambda t}{\Phi(2\eta_1 \Phi^{-1}(t/\beta))} \leq \lambda c_1 C_U(2\eta_1/3)^{-\beta_2} \leq 1 / 2\). Thus we have \(\lambda \in (0, 1/2]\) and \(\eta_1 \in (0, \eta)\) (independent of \(t\)) such that

\[
\eta \Phi^{-1}((1 - \lambda)t) \geq \eta_1 \Phi^{-1}(t), \quad \text{for all } t > 0, \quad (3.33)
\]
and
\begin{align}
\mathbb{P}^x(\tau_{B(x,2\eta \Phi^{-1}(t)/3)} \leq \lambda t) \leq 1/2, \quad \text{for all } t > 0 \text{ and } x \in M. \tag{3.34}
\end{align}

For the remainder of the proof we assume that \(d(x, y) \geq \eta \Phi^{-1}(t)\). Since, using \(\text{(3.32)}\) and \(\text{(3.33)}\),

\[ p(t, x, y) \geq \int_{B(y, \eta \Phi^{-1}(1-\lambda t)/3)} p(\lambda t, x, z) p((1-\lambda)t, z, y) \mu(dz) \]
\[ \geq \inf_{z \in B(y, \eta \Phi^{-1}(1-\lambda t)/3)} p((1-\lambda)t, z, y) \int_{B(y, \eta \Phi^{-1}(1-\lambda t)/3)} p(\lambda t, x, z) \mu(dz) \]
\[ \geq \frac{c_0}{V(y, \Phi^{-1}(t))} \mathbb{P}^x(X_M \in B(y, \eta \Phi^{-1}(t))), \]

it suffices to prove
\begin{align}
\mathbb{P}^x(X_M \in B(y, \eta \Phi^{-1}(t))) \geq c_2 \frac{t V(y, \Phi^{-1}(t))}{V(x, d(x, y)) \psi(d(x, y))}. \tag{3.35}
\end{align}

For \(A \subset M\), let \(\sigma_A := \inf\{t > 0 : X_t \in A\}\). Using \(\text{(3.31)}\) and the strong Markov property we have

\begin{align}
\mathbb{P}^x&(X_M \in B(y, \eta \Phi^{-1}(t))) \\
\geq\ & \mathbb{P}^x\left(\sigma_{B(y, \eta \Phi^{-1}(t)/3)} \leq \lambda t; \sup_{s \in [\tau_{B(y, \eta \Phi^{-1}(t)/3)}]} d(X_s, X_{\tau_{B(y, \eta \Phi^{-1}(t)/3)}}) \leq 2\eta \Phi^{-1}(t)/3\right) \\
\geq\ & \mathbb{P}^x(\sigma_B(y, \eta \Phi^{-1}(t)/3) \leq \lambda t) \inf_{z \in B(y, \eta \Phi^{-1}(t)/3)} \mathbb{P}^z(\tau_{B(z, 2\eta \Phi^{-1}(t)/3)} > \lambda t) \\
\geq\ & \frac{1}{2} \mathbb{P}^x(\sigma_B(y, \eta \Phi^{-1}(t)/3) \leq \lambda t) \\
\geq\ & \frac{1}{2} \mathbb{P}^x\left(X_{\lambda t} \wedge \tau_{B(x, 2\eta \Phi^{-1}(t)/3)} \in B(y, \eta \Phi^{-1}(t)/3)\right).
\end{align}

Since \(d(x, y) \geq \eta \Phi^{-1}(t) > \eta \Phi^{-1}(t)/3\), clearly \(B(y, \eta \Phi^{-1}(t)/3) \subset \overline{B(x, 2\eta \Phi^{-1}(t)/3)}\). Thus by \(\text{(2.5)}\), Lévy system and \(\text{(3.34)}\), we have

\begin{align}
\mathbb{P}^x\left(X_{\lambda t} \wedge \tau_{B(x, 2\eta \Phi^{-1}(t)/3)} \in B(y, \eta \Phi^{-1}(t)/3)\right) \\
= \mathbb{E}^x\left[ \sum_{s \leq (\lambda t) \wedge \tau_{B(x, 2\eta \Phi^{-1}(t)/3)}} 1_{\{X_s \in B(y, \eta \Phi^{-1}(t)/3)\}} \right] \\
\geq \mathbb{E}^x\left[ \int_{0}^{(\lambda t) \wedge \tau_{B(x, 2\eta \Phi^{-1}(t)/3)}} ds \int_{B(y, \eta \Phi^{-1}(t)/3)} J(X_s, u) \mu(du) \right] \\
\geq c_3 \mathbb{E}^x\left[ \int_{0}^{(\lambda t) \wedge \tau_{B(x, 2\eta \Phi^{-1}(t)/3)}} ds \int_{B(y, \eta \Phi^{-1}(t)/3)} \frac{1}{V(X_s, d(X_s, u)) \psi(d(X_s, u))} \mu(du) \right] \\
\geq c_4 \mathbb{E}^x[(\lambda t) \wedge \tau_{B(x, 2\eta \Phi^{-1}(t)/3)}] V(y, \eta \Phi^{-1}(t)/3) \frac{1}{V(x, d(x, y)) \psi(d(x, y))} \\
\geq c_5 (\lambda t) \frac{\mathbb{P}^x(\tau_{B(x, 2\eta \Phi^{-1}(t)/3)} \geq \lambda t)(\eta / 3)^{d_1}}{V(x, \Phi^{-1}(t))} \frac{1}{V(x, d(x, y)) \psi(d(x, y))} \\
\geq c_6 2^{-1} \lambda(\eta / 3)^{d_1} \frac{t V(y, \Phi^{-1}(t))}{V(x, d(x, y)) \psi(d(x, y))},
\end{align}
Thus, combining the above two inequality, we have proved (3.35).

Recall that for $A \geq 1$, we call that the chain condition (Ch($A$)) holds for the metric measure space $(M, d)$ if for any $n \in \mathbb{N}$ and $x, y \in M$, there is a sequence $\{z_k\}_{k=0}^n$ of points in $M$ such that $z_0 = x, z_n = y$ and

$$d(z_{k-1}, z_k) \leq A \frac{d(x, y)}{n} \quad \text{for all} \quad k = 1, \ldots, n.$$  

Lemma 3.13. Assume Ch($A$), VD($d_2$) and RVD($d_1$). We further assume that there exist $\eta > 0$ and $c > 0$ such that (3.31) holds with the function $\Phi$ satisfying $L_a(\delta, \tilde{C}_L)$ with $a > 0$ and $\delta > 1$. Then, for any $T > 0$ and $C > 0$, there exists a constant $c_1 > 0$ such that for any $t \in (0, T]$ and $x, y \in M$ with $d(x, y) \leq C\Phi^{-1}(t)$,

$$p(t, x, y) \geq \frac{c_1}{V(x, \Phi^{-1}(t))}.$$  

(3.36)

In particular, if $L(\delta, \tilde{C}_L, \Phi)$ holds, then we may take $T = \infty$.

Proof. Without loss of generality we may and do assume $a = \Phi^{-1}(T)$. Fix $t > 0$ and $x, y \in M$ with $d(x, y) \leq C\Phi^{-1}(t)$. Let $N := \lceil (\frac{3AC}{\eta})^{\frac{1}{1+\delta}} \frac{1}{\tilde{C}_L^{-\frac{1}{1+\delta}}} \rceil + 1 \in \mathbb{N}$. Then, by Ch($A$) there exists a sequence $\{z_k\}_{k=0}^N$ of points in $M$ such that $z_0 = x, z_N = y$ and $d(z_k, z_{k+1}) \leq A \frac{d(x, y)}{N}$ for all $k = 0, \ldots, N - 1$. Note that by Lemma A.2 and $L(\delta, \tilde{C}_L, \Phi)$ we have $U_T(1/\delta, \tilde{C}_L^{-1/\delta}, \Phi^{-1})$.

Using this and the definition of $N$, we have

$$A \frac{d(x, y)}{N} \leq \frac{AC\Phi^{-1}(t)}{N} \leq \frac{AC}{N} \tilde{C}_L^{-1/\delta} N^{1/\delta} \Phi^{-1}(t/N) \leq \frac{\eta}{3} \Phi^{-1}(t/N).$$

For $k = 1, \ldots, N$, let $B_k := B(z_k, \eta \Phi^{-1}(t/N)/3)$. Then, for any $0 \leq k \leq N - 1$, $\xi_k \in B_k$ and $\xi_{k+1} \in B_{k+1}$. So we have

$$d(\xi_k, \xi_{k+1}) \leq d(\xi_k, z_k) + d(z_k, z_{k+1}) + d(z_{k+1}, \xi_{k+1}) \leq \frac{2Ad(x, y)}{N} + \frac{\eta \Phi^{-1}(t/N)}{3} \leq \eta \Phi^{-1}(t/N).$$

Thus, by (3.31) and (2.1) with $\xi_{k+1} \in B_{k+1}$, we have for any $k = 0, \ldots, N$, $\xi_k \in B_k$ and $\xi_{k+1} \in B_{k+1}$,

$$p(t/N, \xi_k, \xi_{k+1}) \geq \frac{c_1}{V(\xi_{k+1}, \Phi^{-1}(t/N))} \geq \frac{c_1C_{\mu}^{-1}}{V(z_{k+1}, \Phi^{-1}(t/N))(d(z_k, \xi_{k+1}) + \Phi^{-1}(t/N))} d_2 \geq \frac{c_2}{V(z_{k+1}, \Phi^{-1}(t/N))}.$$

Using above estimates and VD($d_2$), we conclude

$$p(t, x, y) = \int_M \cdots \int_M p(t/N, x, \xi_1)p(t/N, \xi_1, \xi_2) \cdots p(t/N, \xi_{N-1}, y)\mu(d\xi_1)\mu(d\xi_2) \cdots \mu(d\xi_{N-1}) \geq \int_{B_1} \cdots \int_{B_{N-1}} p(t/N, x, \xi_1)p(t/N, \xi_1, \xi_2) \cdots p(t/N, \xi_{N-1}, y)\mu(d\xi_1)\mu(d\xi_2) \cdots \mu(d\xi_{N-1})$$
Proposition 3.14. Assume that the metric measure space $(M, d)$ satisfies Ch($A$), VD($d_2$) and RVD($d_1$). We further assume that there exists $\eta > 0$ and $c > 0$ such that (3.31) holds.

(i) Suppose that $L_{\alpha}(\delta, C_L, \Phi)$ holds with $\delta > 1$. Then, for any $T \in (0, \infty)$, there exist constants $c > 0$ and $a_L > 0$ such that for any $x, y \in M$ and $t \in (0, T]$, 

$$p(t, x, y) \geq cV(x, \Phi^{-1}(t))^{-1} \exp\left(-a_L\Phi_1(d(x, y), t)\right). \quad (3.37)$$

Moreover, if $L(\delta, \tilde{C}_L, \Phi)$ holds, then (3.37) holds for all $t \in (0, \infty)$.

(ii) Suppose that $L^a(\delta, C_L, \Phi)$ holds with $\delta > 1$. Then, for any $T \in (0, \infty)$, there exist constants $c > 0$ and $a_L > 0$ such that for any $x, y \in M$ and $t \geq T$,

$$p(t, x, y) \geq cV(x, \Phi^{-1}(t))^{-1} \exp\left(-a_L\Phi_1(d(x, y), t)\right). \quad (3.38)$$

**Proof.** (i) Without loss of generality we may and do assume that $a = \Phi^{-1}(T)$. Note that by (3.36), we have a constant $c_1 > 0$ such that for any $t \in (0, T]$ and $x, y \in M$ with $d(x, y) \leq 2c_U\Phi^{-1}(t)$,

$$p(t, x, y) \geq \frac{c_1}{V(x, \Phi^{-1}(t))}. \quad (3.39)$$

Note that if $t \in (0, T]$ and $d(x, y) \leq 2c_U\Phi^{-1}(t)$, (3.37) immediately follows from (3.39) since $\Phi_1(d(x, y), t) \geq 0$. Now we consider $x, y \in M$ and $t \in (0, T]$ with $d(x, y) > 2c_U\Phi^{-1}(t)$. Let $r := d(x, y)$ and $\theta := \frac{\delta - c_U}{2c_U} \wedge 2$. Define

$$\varepsilon = \varepsilon(t, r) := \inf\{s > 0 : \frac{\Phi(s)}{s} \geq \frac{t}{r}\}.$$

Note that by (3.18) and $\theta \leq 2$, we have

$$\frac{\Phi(\Phi^{-1}(t))}{\Phi^{-1}(t)} \geq \frac{c_U^{-1}t}{(2c_U)^{-1}r} \geq \frac{t}{r},$$

which implies $\varepsilon(t, r) \leq \Phi^{-1}(t)$. Also, using $\lim_{s \to 0} \frac{\Phi(s)}{s} = 0$ we have $\varepsilon(t, r) > 0$. Observe that by the definition of $\varepsilon$, we have a decreasing sequence $\{s_n\}$ converging to $\varepsilon$ satisfying $\frac{\Phi(s_n)}{s_n} \geq \frac{\theta t}{r}$ for all $n \in \mathbb{N}$. Using $U(\alpha_2, c_U, \Phi)$ we have

$$c_U \left(\frac{\varepsilon}{s_n}\right)^{\alpha_2-1} \Phi(\varepsilon) \geq \frac{\Phi(s_n)}{s_n} \geq \frac{\theta t}{r} \quad \text{for all } n \in \mathbb{N}.$$
Letting $n \to \infty$ we obtain
\[
\frac{\theta t}{c_{ur}} \leq \frac{\Phi(\varepsilon)}{\varepsilon}.
\] (3.40)
By a similar way, using $L_{\Phi^{-1}(T)}(\delta, \tilde{C}_L, \Phi)$ and $\frac{\Phi(s)}{s} \leq \frac{\theta t}{r}$ for any $s < \varepsilon$ we have
\[
\frac{\Phi(\varepsilon)}{\varepsilon} \leq \frac{\theta t}{c_{uL}r}.
\] (3.41)
Also, (3.40) yields that
\[
\Phi_1(2c_{ur}, \theta t) \geq \frac{2c_{ur}}{\varepsilon} - \frac{\theta t}{\Phi(\varepsilon)} \geq \frac{r}{\varepsilon} (2c_{ur} - \frac{\varepsilon}{\Phi(\varepsilon)} r) \geq \frac{c_{ur}r}{\varepsilon}.
\]
Thus, using Lemma 3.9(i) with the fact that $r \geq 2c_{ur}$, we have a constant $c_1 > 0$ satisfying
\[
\frac{r}{\varepsilon} \leq c_{u^{-1}}(2c_{ur}, \theta t) \leq c_1 \Phi_1(r, t).
\] (3.42)
Define $N = N(t, r) := \lceil \frac{3A_r}{2c_{ur}} \rceil + 1$. Since $r \geq 2c_{ur}\Phi^{-1}(t)$, we have $N \geq \lceil 3A \rceil + 1 \geq 4$. Observe that by $\frac{3A_r}{2c_{ur}} \leq N \leq \frac{2A_r}{c_{ur}}$ and (3.41) with $\theta \leq \tilde{C}_{uL}c_{ur}/2A$, we have
\[
\Phi \left( \frac{3A_r}{2c_{ur}N} \right) \leq \Phi(\varepsilon) \leq \frac{\theta t}{rC_L} \leq \frac{2A_r}{c_{ur}C_L N} \leq \frac{t}{N}.
\]
This implies $\frac{4r}{N} \leq \frac{2}{3} c_{ur}\Phi^{-1}(\frac{t}{N})$. On the other hand, since $(M, d)$ satisfies Ch($A$), we have a sequence $\{z_i\}_{i=0}^N$ of points in $M$ such that $z_0 = x$, $z_N = y$ and $d(z_l-1, z_l) \leq A \frac{\theta t}{N}$ for any $l \in \{1, \ldots, N\}$. Thus, for any $\xi_l \in B(z_l, \frac{2}{3} c_{ur}\Phi^{-1}(\frac{t}{N})$) and $\xi_l-1 \in B(z_l-1, \frac{2}{3} c_{ur}\Phi^{-1}(\frac{t}{N})$, we have
\[
d(\xi_l, \xi_l-1) \leq d(\xi_l, z_l) + d(z_l, z_l-1) + d(z_l-1, \xi_l-1)
\leq \frac{2}{3} c_{ur}\Phi^{-1}(t/N) + \frac{A_r}{N} + \frac{2}{3} c_{ur}\Phi^{-1}(t/N)
\leq 2c_{ur}\Phi^{-1}(t/N).
\]
Therefore, using semigroup property and (3.36) with $N \leq \frac{2A_r}{c_{ur}}$ and (3.42) we have
\[
p(t, x, y) \geq \int_{B(\xi_N-1, \frac{2}{3} \Phi^{-1}(t/N))} \cdots \int_{B(z_1, \frac{2}{3} \Phi^{-1}(t/N))} p(x, \xi_1) \cdots p(x, \xi_{N-1}, y) d\xi_1 \cdots d\xi_{N-1}
\geq c_2^N \prod_{l=0}^{N-1} V(z_l, \Phi^{-1}(t/N))^{-1} \prod_{l=1}^{N-1} V(z_l, \Phi^{-1}(t/N)) = c_3c_4^N V(x, \Phi^{-1}(t/N))^{-1}
\geq c_3 \left( \frac{c_4C_{d_1}}{3d_1} \right)^N V(x, \Phi^{-1}(t))^{-1} \geq c_3 V(x, \Phi^{-1}(t))^{-1} \exp (-c_5N)
\geq c_3 V(x, \Phi^{-1}(t))^{-1} \exp (-c_6 \frac{r}{\varepsilon}) \geq c_3 V(x, \Phi^{-1}(t))^{-1} \exp (-c_7 \Phi_1(r, t)).
\] (3.43)
This concludes (3.37). Now assume that $\Phi$ satisfies $L(\delta, \tilde{C}_L)$. Note that the case $d(x, y) \leq 2c_{ur}\Phi^{-1}(t)$ is same, since we have (3.39) for every $t > 0$. Also, by the similar way we obtain $0 < \varepsilon(t, r) \leq \Phi^{-1}(t)$, (3.30) and (3.41) for all $t \in (0, \infty)$ and $r > 2c_{ur}\Phi^{-1}(t)$. Following the calculations in (3.43) again, we conclude (3.37) for every $t > 0$ and $x, y \in M$ with $d(x, y) > 2c_{ur}\Phi^{-1}(t)$. 

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(ii) Without loss of generality, we may assume \( a = \Phi(T) \). Then, it suffices to prove
\[
p(t, x, y) \geq cV(x, \Phi^{-1}(t))^{-1} \exp(-aL\Phi_1(d(x, y), t)), \quad t \geq T, \; x, y \in M.
\]
Indeed, \( \Phi^{-1}(t) = \tilde{\Phi}^{-1}(t) \) for \( t \geq T \). Note that for the proof of (3.37) with \( T = \infty \), we only used near-diagonal estimate in (3.31) and \( L(\delta, \tilde{\mathcal{C}}_L, \Phi) \) with semigroup property. Since \( L(\delta, \tilde{\mathcal{C}}_L, \Phi) \) holds, (3.38) follows from (3.31) and (2.13).

**Proof of Theorem 2.16** Combining Proposition 3.12 and Proposition 3.14 we obtain our desired result. Note that the conditions in Proposition 3.14 follows from Proposition 3.12.

**Proof of Theorem 2.17** First we assume (2). Using Theorem 2.14 we obtain \( \text{UHK}(\Phi, \psi) \). Also, by \( \text{UHK}(\Phi) \), \( J_{\psi, \leq} \) and the conservativeness of \((\mathcal{E}, \mathcal{F})\) with Theorem 2.14 we have \( E_{\Phi} \). Now, the lower bound of \( \text{HK}(\Phi, \psi) \) follows from Proposition 3.12 Therefore, (2) implies both (1) and (3).

Now we assume (1). The implication (1) \( \Rightarrow \) \( J_{\psi} \) is the same as that in the proof of Theorem 2.14. Since \( \text{UHK}(\Phi) \) holds, using [22] Theorem 1.20 (3) \( \Rightarrow \) (7) we obtain \( \text{PI}(\Phi) \). The conservativeness follows from [21] Proposition 3.1.

Applying [22] Theorem 1.20 and [21] Lemma 4.21, respectively, we easily see that (3) with (2.6) implies \( \text{UHK}(\Phi) \) and the conservativeness of \((\mathcal{E}, \mathcal{F})\).

If we further assume \( \text{Ch}(A) \), Theorem 2.16 yields that (3) \( \Rightarrow \) (4). Also, (4) \( \Rightarrow \) (1) is straightforward.

### 4 HKE and stability on metric measure space with sub-Gaussian estimates for diffusion process

In this section, we consider a metric measure space having sub-Gaussian estimates for diffusion process. We will obtain equivalence relation similar to Theorems 2.14 and 2.17 without assuming that the the index of local weak lower scaling conditions is strictly bigger than 1.

Recall that we always assume that \( \psi : (0, \infty) \to (0, \infty) \) is a non-decreasing function which satisfies \( L(\beta_1, C_L) \) and \( U(\beta_2, C_U) \). We also recall that if \( \text{Diff}(F) \) holds, there exists conservative symmetric diffusion process on \( M \) such that the transition density \( q(t, x, y) \) for the symmetric diffusion process \( Z = (Z_t)_{t \geq 0} \) on \( M \) with respect to \( \mu \) exists and satisfies the estimates in [22].

Throughout this section, we assume \( \text{VD}(d_2) \) and \( \text{Diff}(F) \) for the metric measure space \((M, d, \mu)\), where \( F : (0, \infty) \to (0, \infty) \) is strictly increasing function satisfying (2.19), \( L(\gamma_1, c_F^{-1}) \) and \( U(\gamma_2, c_F) \) with \( 1 < \gamma_1 \leq \gamma_2 \), that is,
\[
c_F^{-1} \left( \frac{R}{r} \right)^{\gamma_1} \leq \frac{F(R)}{F(r)} \leq c_F \left( \frac{R}{r} \right)^{\gamma_2}, \quad 0 < r \leq R
\]
with some constants \( 1 < \gamma_1 \leq \gamma_2 \) and \( c_F \geq 1 \). Note that, by Lemma A.2, \( F^{-1} \) satisfies \( L(1/\gamma_2, c_F^{-1/\gamma_2}) \) and \( U(1/\gamma_1, c_F^{1/\gamma_1}) \). Define \( \Phi(r) = F(r)/\int_0^r \frac{dF(s)}{\psi(s)} \) as (2.20).

Since \( \psi \) is non-decreasing and \( \lim_{s \to 0} \psi(s) = 0 \), we easily observe that
\[
\psi(r) = \frac{F(r)}{\int_0^r \frac{dF(s)}{\psi(s)}} > \frac{F(r)}{\int_0^r \frac{dF(s)}{\psi(s)}} = \Phi(r), \quad r > 0,
\]
and
\[
\frac{\Phi(R)}{\Phi(r)} = \frac{F(R)}{F(r)} \frac{\int_0^r \frac{dF(s)}{\psi(s)}}{\int_0^R \frac{dF(s)}{\psi(s)}} \leq \frac{F(R)}{F(r)}, \quad 0 < r \leq R.
\] (4.2)

Thus, \( \Phi \) satisfies \( U(\gamma_2, c_F) \), and (2.28) holds for functions \( \Phi \) and \( \psi \). Recall that \( F_1 = \mathcal{T}(F) \). Note that \( F_1(r, t) \in (0, \infty) \) for every \( r, t > 0 \) under (4.1).

Here we record Lemma 3.19 for the next use. Since \( F \) is strictly increasing and satisfying (4.1), we have that for any \( r, t > 0 \),
\[
F_1(r, t) \geq \left( \frac{F(r)}{t} \right)^{\gamma_1^{-1}} \land \left( \frac{F(r)}{t} \right)^{\gamma_2^{-1}} \geq \left( \frac{F(r)}{t} \right)^{\gamma_2^{-1}} - 1.
\] (4.3)

**Lemma 4.1.** \( \Phi \) is strictly increasing. Moreover, \( L(\alpha_1, c_L, \Phi) \) holds for some \( \alpha_1, c_L > 0 \).

**Proof.** Since \( \psi \) is non-decreasing, we may observe that for any \( 0 \leq a < b \),
\[
\frac{F(b) - F(a)}{\psi(b)} \leq \int_a^b \frac{dF(s)}{\psi(s)} \leq \frac{F(b) - F(a)}{\psi(a)},
\]
regarding \( \frac{1}{\psi(b)} = \infty \). Thus, there exists \( a_* \in (a, b) \) such that \( \int_a^{b} \frac{dF(s)}{\psi(s)} = \frac{F(b) - F(a)}{\psi(a_*)} \). For any \( r < R \), let \( r_* \in (0, r) \) and \( R_* \in (r, R) \) be the constants satisfying
\[
\int_0^r \frac{dF(s)}{\psi(s)} = \frac{F(r)}{\psi(r_*)} \quad \text{and} \quad \int_r^R \frac{dF(s)}{\psi(s)} = \frac{F(R) - F(r)}{\psi(R_*)}.
\]

Then, since \( \psi \) is non-decreasing,
\[
\Phi(R) = \frac{F(R)}{\int_0^r \frac{dF(s)}{\psi(s)}} + \int_r^R \frac{dF(s)}{\psi(s)} = \frac{F(r)}{\psi(r_*)} + \frac{F(R) - F(r)}{\psi(R_*)} \geq \frac{F(r)}{\psi(r_*)} + \frac{F(R) - F(r)}{\psi(r_*)} = \psi(r_*) = \Phi(r).
\]

Thus, \( \Phi \) is also non-decreasing. Now suppose that the equality of above inequality holds. Then, since \( F(R) - F(r) > 0 \), we have \( \psi(r_*) = \psi(R_*) \), which implies that \( \psi(r_*) = \psi(r) \) since \( \psi \) is non-decreasing. Thus, we conclude \( \int_0^r \frac{dF(s)}{\psi(s)} = \frac{F(r)}{\psi(r)} \), which is contradiction since \( \lim_{s \to 0} \psi(s) = 0 \). Therefore, \( \Phi \) is strictly increasing.

Using \( L(\gamma_1, c_F^{-1}, F) \) and \( L(\beta_1, C_L, \psi) \), there is a constant \( C > 1 \) such that
\[
F(Cr) \geq 4F(r) \quad \text{and} \quad \psi(Cr) \geq 4\psi(r), \quad \text{all} \quad r > 0.
\] (4.4)

For \( r > 0 \), let \( r_1 \in (0, r) \), \( r_2 \in (r, Cr) \), \( r_3 \in (Cr, C^2r) \) be the constants satisfying
\[
\int_0^r \frac{dF(s)}{\psi(s)} = \frac{F(r)}{\psi(r_1)}, \quad \int_r^{Cr} \frac{dF(s)}{\psi(s)} = \frac{F(Cr) - F(r)}{\psi(r_2)} \quad \text{and} \quad \int_{Cr}^{C^2r} \frac{dF(s)}{\psi(s)} = \frac{F(C^2r) - F(Cr)}{\psi(r_3)}.
\]
Then,
\[
\Phi(r) = \frac{F(r)}{\int_0^r \frac{dF(s)}{\psi(s)}} = \psi(r_1)
\]
and
\[
\Phi(C^2r) = \frac{F(C^2r)}{\int_0^{C^2r} \frac{dF(s)}{\psi(s)}} = \frac{F(C^2r)}{\frac{F(r)}{\psi(r_1)} + \frac{F(Cr) - F(r)}{\psi(r_2)} + \frac{F(C^2r) - F(Cr)}{\psi(r_3)}} \geq \frac{F(C^2r)}{\frac{F(r)}{\psi(r_2)} + \frac{F(Cr) - F(r)}{\psi(r_3)} + \frac{F(C^2r) - F(Cr)}{\psi(r_3)}}.
\]
By (4.4) and the fact that \( r_1 \leq r \leq Cr \leq r_3 \), we have

\[
\frac{\psi(r_1)}{\psi(r_3)} \leq \frac{1}{4} \quad \text{and} \quad \frac{F(Cr)}{F(C^2r)} \leq \frac{1}{4}.
\]

Therefore, for any \( r > 0 \) we have

\[
\Phi(C^2r) \geq \frac{F(C^2r)}{\Phi(r)} \geq \frac{F(C^2r)}{F(Cr) + \frac{\psi(r_1)}{\psi(r_3)}(F(C^2r) - F(Cr))} \geq \frac{F(C^2r)}{F(Cr) + \frac{1}{2}(F(C^2r) - F(Cr))} \geq 2.
\]

(4.5)

Using (4.5) we easily prove that \( L(\alpha_1, c_L, \Phi) \) holds with \( \alpha_1 = \frac{\log 2}{\log C} \) and \( c_L = \frac{1}{2} \).

\(\Box\)

4.1 \( \text{Pl}(\Phi) \), \( E_\Phi \) and upper heat kernel estimate via subordinate diffusion processes

Let \( \phi \) be the function defined by

\[
\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \frac{dt}{t}\int_0^{F^{-1}(t)}\psi(s)\frac{ds}{t}\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t}.
\]

(4.6)

Note that by (2.19), \( L(\beta_1, C_L, \psi) \) and \( U(\gamma_2, c_F, F) \),

\[
\int_0^\infty (1 + t) \frac{dt}{t}\int_0^{F^{-1}(t)}\psi(s)\frac{ds}{t}\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t} < \infty.
\]

Thus, there exists a subordinator \( S = (S_t, t > 0) \) which is independent of \( Z \) and whose Laplace exponent is \( \phi \). Then, the process \( Y \) defined by \( Y_t := Z_{S_t} \) is a pure jump process whose jump kernel is given by

\[
J_\psi(x, y) = \int_0^\infty q(t, x, y) \frac{1}{t}\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t}.
\]

Also, the transition density \( p^Y(t, x, y) \) of \( Y \) can be written by

\[
p^Y(t, x, y) = \int_0^\infty q(s, x, y)\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t}.
\]

With sub-Gaussian estimates (2.2), we obtain the following lemma.

Lemma 4.2. \( J_\psi \) satisfies (2.5). In other words, \( J_\psi \) holds for \( Y \).

Proof. Fix \( x, y \in M \) and denote \( r := d(x, y) \). We first observe that \( F_1(r, t) \geq 0 \) for any \( r, t > 0 \).

Also, for \( r > 0 \) and \( t \leq F(r) \), we have

\[
F_1(r, t) = \sup_{s > 0} \left[ \frac{r}{s} - \frac{t}{F(s)} \right] \geq \frac{r}{F^{-1}(t)} - \frac{t}{F(F^{-1}(t))} = \frac{r}{F^{-1}(t)} - 1.
\]

By (2.2) and the inequality \( F_1(r, t) \geq 0 \) \( \forall \left( \frac{r}{F^{-1}(t)} - 1 \right) \), we have

\[
J_\psi(x, y) = \int_0^\infty \frac{1}{t}\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t}\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t}q(t, x, y)dt
\]

\[
\leq \int_0^\infty \frac{1}{t}\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t}\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t}\exp \left( -a_0F_1(r, t) \right)dt
\]

\[
\leq \int_0^{F(r)} \frac{c}{t\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t}}dt + \int_0^{F(r)} \frac{c}{t\int_0^{F^{-1}(t)}\psi(s)\frac{dt}{t}}exp \left( -\frac{a_0r}{F^{-1}(t)} + a_0 \right)dt
\]

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\[= I + II.\]

We first consider \(I\). Using \(L(\beta_1, C_L, \psi)\) and \(L(1/\gamma_2, c_F^{-1/\gamma_2}, F^{-1})\) we have

\[
I \leq \frac{c}{F(r)} \int_{F(r)}^{\infty} \frac{1}{t\psi(F^{-1}(t))V(x, F^{-1}(t))} dt \leq \frac{c}{V(x, r)} \int_{F(r)}^{\infty} \frac{1}{t\psi(F^{-1}(t))} dt
\]

\[
\leq \frac{c_1}{V(x, r)\psi(r)} \int_{F(r)}^{\infty} \frac{1}{t\psi(F^{-1}(t))} dt = \frac{c_1}{V(x, r)\psi(r)\beta_1^{1/\gamma_2}},
\]

Now we obtain the upper bound of \(II\). Assume \(t \leq F(r)\). Then, by \(U(\beta_2, C_U, \psi)\), \(VD(d_2)\) and \(U(\gamma_2, c_F, F)\),

\[
\frac{1}{t\psi(F^{-1}(t))V(x, F^{-1}(t))} = \frac{1}{F(r)\psi(r)V(x, r)} \frac{F(r)}{\psi(F^{-1}(t))} \frac{V(x, r)}{V(x, F^{-1}(t))} \leq \frac{c_2}{F(r)\psi(r)V(x, r)} \frac{r}{F^{-1}(t)}^{\gamma_2+\beta_2+d_2}. \tag{4.7}
\]

Since the function \(s \mapsto s^{\gamma_2+\beta_2+d_2}e^{-\alpha_0 s}\) is uniformly bounded on \([1, \infty)\), using \((4.7)\) we have

\[
II \leq c_3 \int_0^{F(r)} \frac{c_3}{F(r)\psi(r)V(x, r)} \left( \frac{r}{F^{-1}(t)} \right)^{\gamma_2+\beta_2+d_2} \exp \left( -\frac{a_0 r}{F^{-1}(t)} \right) dt \leq \frac{c_4}{\psi(r)V(x, r)}.
\]

Thus, \(J_\psi(x, y) \leq I + II \leq \frac{c_5}{\psi(r)V(x, r)}\). For the lower bound, we use \((2.22)\), \(VD(d_2)\) and the fact that \(\psi\) is non-decreasing to obtain that

\[
J_\psi(x, y) \geq \int_{F(r)/2}^{F(r)} \frac{c_3}{t\psi(F^{-1}(t))V(x, F^{-1}(t))} dt \geq \frac{c_3}{2\psi(r)V(x, r)}.
\]

Now the conclusion follows. \(\square\)

**Lemma 4.3.** There exists \(c > 0\) such that for any \(\lambda > 0\),

\[
\frac{1}{2\Phi(F^{-1}(\lambda^{-1}))} \leq \phi(\lambda) \leq \frac{c}{\Phi(F^{-1}(\lambda^{-1}))}. \tag{4.8}
\]

**Proof.** Using \((4.6)\) and \((2.20)\),

\[
\phi(\lambda) = \int_0^{\infty} (1 - e^{-\lambda t}) \frac{dt}{t\psi(F^{-1}(t))} \geq \int_0^{1/\lambda} \frac{\lambda t}{2t\psi(F^{-1}(t))} dt = \int_0^{F^{-1}(1/\lambda)} \frac{\lambda dF(s)}{2\psi(s)} = \frac{1}{2\Phi(F^{-1}(\lambda^{-1}))},
\]

and by \((2.20)\) and \((2.16)\),

\[
\phi(\lambda) = \int_0^{\infty} (1 - e^{-\lambda t}) \frac{dt}{t\psi(F^{-1}(t))} \leq \int_0^{1/\lambda} \frac{\lambda t}{t\psi(F^{-1}(t))} dt + \int_0^{\infty} \frac{dt}{t\psi(F^{-1}(t))} \leq \frac{c}{\Phi(F^{-1}(\lambda^{-1}))}.
\]

From the above two inequalities we conclude the lemma. \(\square\)
Let us define
\[ E_\psi(f, g) := \int_{M \times M} (f(x) - f(y))^2 J_\psi(x, y) \mu(dx) \mu(dy), \quad f \in L^2(M, \mu), \]
and \( \{Q_t, t > 0\} \) be the transition semigroup with respect to \( Z \) on \( L^2(M, \mu) \), thus
\[ Q_t f(x) = \int_M q(t, x, y)f(y)\mu(dy). \]

Following the proof of \([17, \text{Lemma 2.3}]\), we obtain a consequence of \( \text{Diff}(F) \).

**Lemma 4.4.** Assume that the metric measure space \((M, d, \mu)\) satisfies \( \text{VD}(d_2) \), \( \text{RVD}(d_1) \) and \( \text{Diff}(F) \). Then, there exist \( c_1, \theta > 0 \) and \( \varepsilon \in (0, 1) \) such that
\[ q^{B(x_0, r)}(t, x, y) \geq \frac{c_1}{V(x_0, F^{-1}(t))} \text{ for all } x_0 \in M_0, r > 0, x, y \in B(x_0, \varepsilon F^{-1}(t)), t \in (0, \theta F(r)]. \]

**Proof.** Assume \( \theta \in (0, 1] \). Let \( x_0 \in M_0, r > 0 \) and denote \( B_r := B(x_0, r) \). Using \([23]\), for any \( x, y \in B(x_0, \varepsilon F^{-1}(t)) \) and \( t \in (0, \theta F(r)], \)
\[ q^{B_r}(t, x, y) = q(t, x, y) - E^x[q(t - \tau_{B_r}, Z_{\tau_{B_r}}, y) : \tau_{B_r} < t] \]
\[ \geq \frac{c^{-1}}{V(x_0, F^{-1}(t))} - \frac{c^{-1}}{V(x_0, F^{-1}(t))} \exp\left(-a_0 F_1(d(Z_{\tau_{B_r}}, y), t - \tau_{B_r}) : \tau_{B_r} < t\right) \]
\[ \geq \frac{c^{-1}}{V(x_0, F^{-1}(t))} - \frac{c^{-1}}{V(x_0, F^{-1}(t))} \exp\left(-a_0 F_1((1 - \varepsilon)r, t - \tau_{B_r}) : \tau_{B_r} < t\right) \]
\[ \geq \frac{c^{-1}}{V(x_0, F^{-1}(t))} - \frac{1}{\sup_{0 < s \leq t} V(x_0, F^{-1}(s))} \exp\left(-a_0 F_1((1 - \varepsilon)r, s)\right) \]
\[ \geq \frac{c^{-1}}{V(x_0, F^{-1}(t))} - \frac{1}{\sup_{0 < s \leq t} V(x_0, F^{-1}(s))} \exp\left(-a_0 F_1((1 - \varepsilon)r, s)\right). \]

By \([43]\), we also have
\[ \sup_{0 < s \leq t} \frac{1}{V(x_0, F^{-1}(s))} \exp\left(-a_0 F_1((1 - \varepsilon)r, s)\right) \]
\[ \leq \sup_{0 < s \leq t} \frac{e^{a_0}}{V(x_0, F^{-1}(s))} \exp\left(-a_0 \left(\frac{F((1 - \varepsilon)r)}{s}\right)^{\frac{1}{2} - \frac{1}{2}}\right) \]
\[ = \sup_{0 < s \leq t} \frac{e^{a_0}}{V(x_0, F^{-1}(t))} \frac{V(x_0, F^{-1}(t))}{V(x_0, F^{-1}(s))} \exp\left(-a_0 \left(\frac{t}{s}\right)^{\frac{1}{2} - \frac{1}{2}} \left(\frac{F((1 - \varepsilon)r)}{t}\right)^{\frac{1}{2} - \frac{1}{2}}\right) \]
\[ \leq \frac{e^{a_0} C_{\mu} \varepsilon^{d_2/\gamma_1}}{V(x_0, F^{-1}(t))} \sup_{0 < s \leq t} \left(\frac{t}{s}\right)^{d_2/\gamma_1} \exp\left(-a_0 \left(\frac{1}{2} - \frac{1}{2}\right) \left(\frac{F((1 - \varepsilon)r)}{\theta}\right)^{\frac{1}{2} - \frac{1}{2}} \left(\frac{t}{s}\right)^{\frac{1}{2} - \frac{1}{2}}\right) \]
\[ = \frac{e^{a_0} C_{\mu} \varepsilon^{d_2/\gamma_1}}{V(x_0, F^{-1}(t))} \sup_{1 \leq u} \left(\frac{u}{s}\right)^{d_2/\gamma_1} \exp\left(-a_0 \left(\frac{1}{2} - \frac{1}{2}\right) \left(\frac{F((1 - \varepsilon)r)}{\theta}\right)^{\frac{1}{2} - \frac{1}{2}} \left(\frac{1}{u}\right)^{\frac{1}{2} - \frac{1}{2}}\right) \]
\[ \quad := \frac{C(\theta) e^{a_0} C_{\mu} \varepsilon^{d_2/\gamma_1}}{V(x_0, F^{-1}(t))}. \]

Since \( C(\theta) \to 0 \) as \( \theta \downarrow 0 \), there exists a constant \( \theta > 0 \) such that \( C(\theta) \leq \frac{1}{2e^{2a_0} C_{\mu} \varepsilon^{d_2/\gamma_1}}. \) With this, we obtain
\[ q^{B_r}(t, x, y) \geq \frac{c^{-1}}{V(x_0, F^{-1}(t))} - \frac{1}{\sup_{0 < s \leq t} V(x_0, F^{-1}(s))} \exp\left(-a_0 F_1((1 - \varepsilon)r, s)\right) \geq \frac{c^{-1}}{2V(x_0, F^{-1}(t))}. \]
This concludes the lemma. \( \square \)
Lemma 4.5. Suppose that the metric measure space \((M,d,\mu)\) satisfies \(\text{VD}(d_2)\), \(\text{RVD}(d_1)\) and \(\text{Diff}(F)\) where \(F : (0,\infty) \to (0,\infty)\) strictly increasing function satisfying \((2.19)\), \(L(\gamma_1,c_F^{-1})\) and \(U(\gamma_2,c_F)\) with \(1 < \gamma_1 \leq \gamma_2\). There exist constants \(c > 0\) and \(\hat{\epsilon} \in (0,1)\) such that for any \(x_0 \in M_0\), \(r > 0\), \(0 < t \leq \Phi(\hat{\epsilon}r)\) and \(x,y \in B(x_0,\hat{\epsilon}\Phi^{-1}(t))\),

\[
p^{Y,B(x_0,r)}(t,x,y) \geq \frac{c}{V(x_0,\Phi^{-1}(t))}.
\]

Proof. Recall that we have defined \(Y_t = Z_{S_t}\), where \(S_t\) is a subordinator independent of \(Z\) and whose Laplace exponent is the function \(\phi\) in \((4.6)\). Also, by \((4.8)\) we have \(\phi^{-1}(\lambda) \leq \phi^{-1}(\Phi^{-1}(\lambda))\). Take \(\lambda\) by \(F(\Phi^{-1}(c_1t^{-1}))^{-1}\) and \(F(\Phi^{-1}(c_1t^{-1}))^{-1}\), and using the fact that \(\Phi\) and \(F\) are strictly increasing we obtain that for any \(t > 0\)

\[
F(\Phi^{-1}(c_1t)) \leq \phi^{-1}(t^{-1})^{-1} \leq F(\Phi^{-1}(c_1t)).
\]

By \((4.9)\), there exist \(\rho, c_2 > 0\) such that

\[
\mathbb{P}\left(\frac{1}{2\phi^{-1}(t^{-1})} \leq S_t \leq \frac{1}{\phi^{-1}(pt^{-1})} \right) \geq c_2.
\]

Choose \(\hat{\epsilon} > 0\) such that

\[
\hat{\epsilon}\Phi^{-1}(t) \leq \epsilon F^{-1}\left(\frac{1}{2} F(\Phi^{-1}(c_1t))\right) \quad \text{and} \quad F(\Phi^{-1}(c_1\rho^{-1}\Phi(\hat{\epsilon}r))) \leq \theta F(r),
\]

where \(\epsilon \in (0,1)\) and \(\theta\) are the constants in Lemma \((4.4)\). Then, by \((4.9)\), we see that for \(0 < t \leq \Phi(\hat{\epsilon}r)\) and \(s \in \left[\frac{1}{2\phi^{-1}(t^{-1})},\frac{1}{\phi^{-1}(pt^{-1})}\right]\), we have

\[
s \leq \frac{1}{\phi^{-1}(pt^{-1})} \leq F(\Phi^{-1}(c_1\rho^{-1}t)) \leq F(\Phi^{-1}(c_1\rho^{-1}\Phi(\hat{\epsilon}r))) \leq \theta F(r)
\]

and

\[
\hat{\epsilon}\Phi^{-1}(t) \leq \epsilon F^{-1}\left(\frac{1}{2} F(\Phi^{-1}(c_1t))\right) \leq \epsilon F^{-1}\left(\frac{1}{2\phi^{-1}(t^{-1})}\right) \leq \epsilon F^{-1}(s).
\]

Thus, by \((5.3)\), Lemma \((4.4\), \((4.10)\), \((4.9)\), \(\text{VD}(d_2)\), \((4.1)\) and \(U(1/\alpha_1, c_L^{-1/\alpha_1}, \Phi^{-1})\), we see that for \(0 < t \leq \Phi(\hat{\epsilon}r)\) and \(x,y \in B(x_0,\hat{\epsilon}\Phi^{-1}(t))\)

\[
p^{Y,B(x_0,r)}(t,x,y) \geq \int_0^{\infty} q^{B(x_0,r)}(s,x,y)\mathbb{P}(S_t \in ds)
\]

\[
\geq \int_{\frac{1}{2\phi^{-1}(t^{-1})}}^{\frac{c_1}{\phi^{-1}(pt^{-1})}} q^{B(x_0,r)}(s,x,y)\mathbb{P}(S_t \in ds)
\]

\[
\geq \frac{c_2}{V(x_0,F^{-1}(\phi^{-1}(pt^{-1})))} \mathbb{P}\left(\frac{1}{2\phi^{-1}(t^{-1})} \leq S_t \leq \frac{1}{\phi^{-1}(pt^{-1})}\right)
\]

\[
\geq \frac{c_4}{V(x_0,F^{-1}(\Phi^{-1}(c_1\rho^{-1}t)))} \frac{c_3}{V(x_0,\Phi^{-1}(t))}.
\]

This finishes the lemma. \(\square\)
Theorem 4.6. Suppose that the metric measure space \((M,d,\mu)\) satisfies \(\text{VD}(d_2)\), \(\text{RVD}(d_1)\) and \(\text{Diff}(F)\) where \(F : (0,\infty) \to (0,\infty)\) strictly increasing function satisfying (2.19), \(L(\gamma_1,c_F^{-1})\) and \(U(\gamma_2,c_F)\) with \(1 < \gamma_1 \leq \gamma_2\). Assume that \(X\) is a Markov process on \((M,d)\) satisfying \(J_\psi\). Then, there exists a constant \(c > 0\) such that
\[
p(t,x,y) \leq c \left( \frac{1}{V(x,\Phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\Phi(d(x,y))} \right)
\]
for all \(t > 0\) and \(x,y \in M\). Moreover, \(E_\Phi\) and \(\text{PI}(\Phi)\) holds for \(X\).

Proof. Note that from Lemma 4.2 and Lemma 4.5 the condition (4) in [22, Theorem 1.20] holds for the process \(Y\). In particular, using [22, Theorem 1.20], the conditions \(\text{CSJ}(\Phi)\) and \(\text{PI}(\Phi)\) holds for the process \(Y\). Since the jump kernel of \(X\) and \(Y\) are comparable by Lemma 4.2, the conditions \(\text{PI}(\Phi)\) and \(\text{CSJ}(\Phi)\) also hold for \(X\). In particular, the process \(X\) satisfies condition (7) in [22, Theorem 1.20]. Now, using [22, Theorem 1.20] again we obtain \(E_\Phi\) and \(\text{UHK}(\Phi)\). This completes the proof. \(\square\)

4.2 Proofs of Theorem 2.19

In this section, we prove Theorem 2.19.

Theorem 4.7. Assume that the metric measure space \((M,d,\mu)\) satisfies \(\text{RVD}(d_1)\) and \(\text{VD}(d_2)\). Also, assume further that \(\text{Diff}(F)\) holds for a strictly increasing function \(F : (0,\infty) \to (0,\infty)\) satisfying \(L(\gamma_1,c_F^{-1})\) and \(U(\gamma_2,c_F)\) with \(1 < \gamma_1 \leq \gamma_2\). Suppose that the process \(X\) satisfies \(J_\psi\), where \(\psi\) is a non-decreasing function satisfying \(L(\beta_1,C_1), U(\beta_2,C_2)\) and (2.19). Then, there exist constants \(c > 0\) and \(a_1 > 0\) such that for all \(t > 0\) and \(x,y \in M\),
\[
p(t,x,y) \leq \frac{c}{V(x,\Phi^{-1}(t))} \wedge \left( \frac{ct}{V(x,d(x,y))\psi(d(x,y))} + \frac{c}{V(x,\Phi^{-1}(t))} e^{-a_1 F_1(d(x,y),F(\Phi^{-1}(t)))} \right).
\]

Proof. Since the proof is similar to that of Theorem 2.13, we just give the difference. As in the proof of Theorem 2.13, we will show that there exist \(a_1 > 0\) and \(c_1 > 0\) such that for any \(t > 0\) and \(r > 0\),
\[
\int_{B(x,r)^c} p(t,x,y) \mu(dy) \leq c_1 \left( \frac{\psi^{-1}(t)}{r} \right)^{\beta_1/2} + c_1 \exp \left( -a_1 F_1(r,F(\Phi^{-1}(t))) \right). \tag{4.11}
\]
Let \(\gamma := \frac{1}{\gamma_1 - 1}\),
\[
\theta := \frac{(\gamma_1 - 1)\beta_1}{\gamma_1(2d_2 + \beta_1) + (\beta_1 + 2\beta_2 + 2d_2\beta_2)} \in (0,\gamma_1 - 1)
\]
and \(C_0 = \frac{4\theta}{C_2}\), where \(C_1\) and \(C_2\) are the constants in Lemma 3.3. Note that we may and do assume that \(C_1 \geq 1\) and \(C_2 \leq 1\) without loss of generality. Assume \(r \leq C_0 \Phi^{-1}(C_1 t)\). Then, using \(L(\alpha_1,c_L,\Phi)\) and Lemma A.2 we have
\[
r \leq C_0 \Phi^{-1}(C_1 t) \leq C_0 C_1^{1/\alpha_1} C_2^{-1/\alpha_1} \Phi^{-1}(t) \leq c_1 F^{-1}(F(\Phi^{-1}(t))).
\]
Thus, by (3.23), there is a constant \(c_2 \geq 0\) such that
\[
F_1(r,F(\Phi^{-1}(t))) \leq c_2
\]
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for any \( r, t > 0 \) with \( r \leq C_0 \Phi^{-1}(t) \). Thus, we have
\[
\int_{B(x,r)^c} p(t,x,y)\mu(dy) \leq 1 \leq e^{a_1 c_2} \exp\left(-a_1 F_1(r,t)\right).
\]

This implies (4.11) for \( r \leq C_0 \Phi^{-1}(C_1 t) \). Also, (4.11) for \( r \geq C_0 \Phi^{-1}(C_1 t)^{1+\theta}/\psi^{-1}(C_1 t)^{\theta} \) follows from Lemma 3.6.

Now consider the case \( C_0 \Phi^{-1}(C_1 t) < r \leq C_0 \Phi^{-1}(C_1 t)^{1+\theta}/\psi^{-1}(C_1 t)^{\theta} \). Since (2.8) holds, there exists \( \theta_0 \in (0, \theta] \) satisfying \( r = C_0 \Phi^{-1}(C_1 t)^{1+\theta_0}/\psi^{-1}(C_1 t)^{\theta_0} \). By Lemma 3.8 and \( C_0 \geq 2c_F > 0 \) with \( L(\gamma_1, c_F^{-1}, F) \), there is \( \rho \in [b(C_2 r/2c_F)^{-\gamma} \Phi^{-1}(C_1 t)^{\gamma+1}, 2 \Phi^{-1}(C_1 t)] \) such that
\[
\frac{C_2 r}{2c_F \rho} - \frac{F(\Phi^{-1}(C_1 t))}{F(\rho)} + 1 \geq F_1(\frac{C_2 r}{2c_F \rho}, F(\Phi^{-1}(C_1 t))).
\]

Let us define \( r_n = 2^nr \) and \( \rho_n = 2^{n\alpha} \rho \) for \( n \in \mathbb{N}_0 \), with some \( \alpha \in (d_2/(d_2 + \beta_1), 1) \). Since \( \rho \in [b(C_2 r/2c_F)^{-\gamma} \Phi^{-1}(C_1 t)^{\gamma+1}, 2 \Phi^{-1}(C_1 t)] \), using (3.18), (1.2) and (3.22) we have
\[
\begin{align*}
\frac{C_1 t}{\Phi(\rho_n)} - \frac{C_2 2^nr}{\rho_n} &\leq \left( \frac{c_F \Phi(\Phi^{-1}(C_1 t))}{\Phi(\rho)} - \frac{C_2 r}{2\rho} \right) + \frac{C_2 r}{2\rho} = \frac{2^n(1-\alpha)C_2 r}{\rho} \\
&\leq \left( \frac{c_F F(\Phi^{-1}(C_1 t))}{F(\rho)} - \frac{C_2 r}{2\rho} \right) + \left( \frac{1}{2} - \frac{2^n(1-\alpha)}{2^n(1-\alpha)} \right) \frac{C_2 r}{\rho} \\
&\leq -c_F F_1(\frac{C_2 r}{2c_F}, F(\Phi^{-1}(C_1 t))) + c_F + \left( \frac{1}{2} - 2^n(1-\alpha) \right) \frac{C_2 r}{\rho} \\
&\leq -c_3 F_1(r, F(\Phi^{-1}(t))) - \frac{1}{2} 2^n(1-\alpha) \frac{C_2 r}{2\Phi^{-1}(C_1 t)} + c_F \\
&\leq -c_3 F_1(r, F(\Phi^{-1}(t))) - c_4 2^n(1-\alpha) \frac{r}{\Phi^{-1}(t)} + c_F
\end{align*}
\]

which is the counterpart of (3.26). Since we have Lemma 3.2 and (4.12), following the proof of Theorem 2.13 we obtain (4.11). Thus, we can apply Lemma 3.4 with \( f(r,t) := F_1(r,F(\Phi^{-1}(t))) \). Note that condition (i) in Lemma 3.4 follows from (3.23) and (3.18). Thus,
\[
p(t,x,y) \leq \frac{c_0 t}{V(x,d(x,y))\psi(d(x,y))} + \frac{c_0}{V(x,\Phi^{-1}(t))}\left(1 + \frac{d(x,y)}{\Phi^{-1}(t)}\right)^{d_2} e^{-a_2 k F_1(d(x,y)/(16k),F(\Phi^{-1}(t)))}.
\]

The remainder is the same as the proof of Theorem 2.13.

\[\square\]

\textbf{Proof of Theorem 2.19} Assume \( J_\psi \). Then, the upper bound in \( \text{GHK}(\Phi, \psi) \) follows from Theorem 4.7. Also, using Theorem 4.6 and Lemma 4.5 we have \( E_\Phi \) and \( \text{PI}(\Phi) \). Since all conditions in Proposition 3.12 hold, we obtain the lower bound of \( \text{GHK}(\Phi, \psi) \).

Also, following the proof of [21, Proposition 3.3], we obtain that \( \text{GHK}(\Phi, \psi) \) implies \( J_\psi \).

\[\square\]

\textbf{Proof of Corollary 2.20} Using Theorem 2.19 \( X \) satisfies \( J_\psi, \text{PI}(\Phi) \) and \( E_\Phi \). Thus, the conclusion follows from Theorems 2.13 and 2.16.

\[\square\]
5 Generalized Khintchine-type law of iterated logarithm

In this section, using our main result, we establish a generalized version of the law of iterated logarithm on metric measure space. Throughout this section, we assume that \((M, d, \mu)\) satisfies Ch(A), RVD\((d_1)\), VD\((d_2)\) and Diff\((F)\), where \(F\) is strictly increasing and satisfies (4.1) with \(1 < \gamma_1 \leq \gamma_2\).

Let \((\mathcal{E}, \mathcal{F})\) be a regular Dirichlet form given by (2.4), which satisfies \(J_\psi\) with a non-decreasing function \(\psi\) satisfying (2.19), \(L(\beta_1, C_L)\) and \(U(\beta_2, C_U)\). Recall that \(X = \{X_t, t \geq 0; \mathbb{P}^x, x \in M\}\) is the \(\mu\)-symmetric Hunt process associated with \((\mathcal{E}, \mathcal{F})\). Recall that \(\Phi\) is the function defined in (2.20).

We first establish the zero-one law for tail events.

**Lemma 5.1.** Let \(A\) be a tail event (with respect to the natural filtration of \(X\)). Then, either \(\mathbb{P}^x(A) = 0\) for all \(x \in M\) or else \(\mathbb{P}^x(A) = 1\) for all \(x \in M\).

**Proof.** By [21, Lemma 2.7], we have the constant \(c > 0\) such that for any \(x \in M\) and \(r, t > 0\),

\[
\mathbb{P}^x(\tau_{B(x, r)} \leq t) \leq \frac{ct}{\Phi(r)}.
\]

Let us fix \(t_0, \varepsilon > 0\) and \(x_0 \in M\). Then, by the above inequality and \(L(\alpha_1, c_L, \Phi)\) from Lemma 4.1, there is \(c_1 > 0\) such that

\[
\mathbb{P}^{x_0}(\sup_{s \leq t_0} d(x_0, X_s) > c_1 \Phi^{-1}(t_0)) \leq \mathbb{P}^{x_0}(\tau_{B(x_0, c_1 \Phi^{-1}(t_0))} \leq t_0) \leq \varepsilon. \tag{5.1}
\]

Using (5.1) and Theorem 3.10, the remainder part of the proof is the same as those of [42, Theorem 2.10] and [21, Theorem 5.1]. Thus, we skip it. \(\square\)

From (2.20) and (2.19) with VD\((d_2)\), we easily see that the following three conditions are equivalent:

\[
\sup_{x \in M} \left( \text{or} \inf_{x \in M} \right) \int_M F(d(x, y))J(x, dy) < \infty; \tag{5.2}
\]

\[
\exists c > 0 \text{ such that } c^{-1}F(r) \leq \Phi(r) \leq cF(r), \text{ for all } r > 1; \tag{5.3}
\]

\[
\int_1^\infty \frac{dF(s)}{\psi(s)} < \infty. \tag{5.4}
\]

We will show that from GHK\((\Phi, \psi)\), the above conditions (5.2)-(5.4) are also equivalent to the following moment condition.

**Lemma 5.2.** Suppose that \((M, d, \mu)\) satisfies RVD\((d_1)\), VD\((d_2)\) and Diff\((F)\) where \(F\) is strictly increasing and satisfies (2.19), \(L(\gamma_1, c_F^{-1})\) and \(U(\gamma_2, c_F)\) with \(1 < \gamma_1 \leq \gamma_2\). Let \((\mathcal{E}, \mathcal{F})\) be a regular Dirichlet form given by (2.4), which satisfies \(J_\psi\) with a non-decreasing function \(\psi\) satisfying (2.19). Then the following is also equivalent to (5.2)-(5.4):

\[
\sup_{x \in M} \left( \text{or} \inf_{x \in M} \right) \mathbb{E}^x[F(d(x, X_t))] < \infty, \quad \forall (\text{or} \exists) \ t > 0.
\]

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Proof. (i) Fix \( t > 0 \) and assume (5.4). Using GHK(\( \Phi, \psi \)), we have for all \( x \in M \),

\[
\mathbb{E}^x[F(d(x, X_t))] = \int_{M} p(t, x, y) F(d(x, y)) \mu(dy)
\]

\[
= \int_{d(x, y) \leq F^{-1}(t)} p(t, x, y) F(d(x, y)) \mu(dy) + \int_{d(x, y) > F^{-1}(t)} p(t, x, y) F(d(x, y)) \mu(dy)
\]

\[
\leq c_2 \int_{d(x, y) \leq F^{-1}(t)} \frac{V(x, \Phi^{-1}(t))}{V(x, \Phi^{-1}(t))} \mu(dy) + \int_{d(x, y) > F^{-1}(t)} \frac{c_2 t F(d(x, y))}{V(x, \Phi^{-1}(t))} \mu(dy)
\]

\[
+ \int_{d(x, y) > F^{-1}(t)} \frac{c_2}{V(x, \Phi^{-1}(t))} e^{-a_0 F_1(d(x, y), F(\Phi^{-1}(t)))} \mu(dy) := c_2 (I + II + III).
\]

Using VD(\( d_2 \)) we have

\[
I = \int_{d(x, y) \leq F^{-1}(t)} \frac{1}{V(x, \Phi^{-1}(t))} F(d(x, y)) \mu(dy)
\]

\[
\leq \frac{V(x, F^{-1}(t))}{V(x, \Phi^{-1}(t))} F^{-1}(t) \leq C \mu t \left( \frac{F^{-1}(t)}{\Phi^{-1}(t)} \vee 1 \right)^{d_2} = c_3(t) < \infty.
\]

For II, we first observe that by \( L(\gamma_1, c_1, F) \), there exists \( c_4 > 1 \) such that \( F(c_4 r) \geq 2F(r) \) for any \( r > 0 \). Using this, VD(\( d_2 \)) and (5.4) we obtain

\[
II = \int_{d(x, y) > F^{-1}(t)} \frac{t}{V(x, d(x, y))} \psi(d(x, y)) F(d(x, y)) \mu(dy)
\]

\[
= t \sum_{i=0}^{\infty} \int_{c_4^{i+1} F^{-1}(t) < d(x, y) \leq c_4^{i+1} F^{-1}(t)} \frac{1}{V(x, d(x, y))} \psi(d(x, y)) F(d(x, y)) \mu(dy)
\]

\[
\leq t \sum_{i=0}^{\infty} \frac{V(x, c_4^{i+1} F^{-1}(t))}{V(x, c_4^{i+1} F^{-1}(t))} \psi(c_4^{i+1} F^{-1}(t)) \leq c_5 \sum_{i=0}^{\infty} \frac{F(c_4^{i+1} F^{-1}(t))}{\psi(c_4^{i+1} F^{-1}(t))}
\]

\[
\leq 2 c_5 \sum_{i=0}^{\infty} \frac{F(c_4^{i+1} F^{-1}(t)) - F(c_4^{i} F^{-1}(t))}{\psi(c_4^{i+1} F^{-1}(t))} \leq 2 c_5 \sum_{i=0}^{\infty} \int_{c_4^{i+1} F^{-1}(t)}^{c_4^{i+1} F^{-1}(t)} \frac{dF(s)}{\psi(s)}
\]

\[
= c_6(t) < \infty.
\]

For III, using (4.3) for the fourth line, and VD(\( d_2 \)) for the fifth line we have

\[
III = \int_{d(x, y) > F^{-1}(t)} \frac{1}{V(x, \Phi^{-1}(t))} \exp \left( - a_0 F_1(d(x, y), F(\Phi^{-1}(t))) \right) \mu(dy)
\]

\[
= \sum_{i=0}^{\infty} \int_{2^i F^{-1}(t) < d(x, y) \leq 2^{i+1} F^{-1}(t)} \frac{1}{V(x, \Phi^{-1}(t))} \exp \left( - a_0 F_1(d(x, y), F(\Phi^{-1}(t))) \right) \mu(dy)
\]

\[
\leq \sum_{i=0}^{\infty} \frac{V(x, 2^{i+1} F^{-1}(t))}{V(x, \Phi^{-1}(t))} \exp \left( - a_0 F_1(2^{i} F^{-1}(t), F(\Phi^{-1}(t))) \right)
\]

\[
\leq \sum_{i=0}^{\infty} \frac{V(x, 2^{i+1} F^{-1}(t))}{V(x, \Phi^{-1}(t))} \exp \left( - a_0 \left( \frac{F(2^{i} F^{-1}(t))}{F(\Phi^{-1}(t))} \right)^{\frac{1}{a_0}} + a_0 \right)
\]

\[
\leq (c_7(t))^{d_2} \sum_{i=0}^{\infty} (2^{i+1})^{d_2} \exp \left( -(a(t) 2^{i})^{\frac{1}{a_0}} \right) = c_8(t) < \infty.
\]
Combining all the estimates above, we conclude that for any $t > 0$,

$$\sup_{x \in M} \mathbb{E}^x[F(d(x, X_t))] \leq c_2(I + II + III) \leq c_3(t) < \infty.$$  

(ii) Assume that there exist $x \in M$ and $t > 0$ such that $\mathbb{E}^x[F(d(x, X_t))] < \infty$. Note that by RVD($d_1$) and $L(\gamma_1, c_1, F)$, we have constants $\theta, c > 1$ such that

$$V(x, \theta r) \geq cV(x, r) \text{ and } F(\theta r) \geq cF(r), \quad x \in M, r > 0. \tag{5.5}$$

Then using GHK($\Phi, \psi$) for the first inequality and $\text{(5.5)}$ for the third one, we have

$$\mathbb{E}^x[F(d(x, X_t))] = \int_M p(t, x, y)F(d(x, y))\mu(dy)$$

$$\geq c_{10}t \int_{d(x, y) > \eta \Phi^{-1}(t)} F(d(x, y)) V(x, d(x, y)) \psi(d(x, y)) \mu(dy)$$

$$= c_{10}t \int_{1}^{\infty} \sum_{i=1}^{\infty} \left\{ \int_{\eta \Phi^{-1}(t) < d(x, y) \leq \eta \Phi^{-1}(t) + 1} F(d(x, y)) V(x, d(x, y)) \psi(d(x, y)) \mu(dy) \right\}$$

$$\geq c_{10}t \sum_{i=0}^{\infty} \left( V(x, \eta \Phi^{-1}(t) - 1) - V(x, \eta \Phi^{-1}(t)) \right) \frac{F(\eta \Phi^{-1}(t))}{V(x, \eta \Phi^{-1}(t))\psi(\eta \Phi^{-1}(t))}$$

$$\geq c_{11}t \sum_{i=0}^{\infty} \frac{F(\eta \Phi^{-1}(t) - 1)}{\psi(\eta \Phi^{-1}(t))} \geq c_{11}t \int_{0}^{\infty} \frac{dF(s)}{\psi(s)} = c_{11}t \int_{0}^{\infty} \frac{dF(s)}{\psi(s)}.$$ 

Here we have used that $\psi, F$ are non-decreasing for the last two lines. This implies $\text{(5.4)}$.

Combining the above results in (i) and (ii), we conclude the lemma.  \hfill \Box

Let us define an increasing function $h(t)$ on $[16, \infty)$ by

$$h(t) := (\log \log t) F^{-1}(\frac{t}{\log \log t}).$$

**Lemma 5.3.** For any $c_1 > 0$, $c_2 \in (0, 1]$ and $t \in [16, \infty)$,

$$F_1((c_1 + 1)h(t), t) \geq c_1 \log t \tag{5.6}$$

and

$$F_1(c_2 h(t), t) \leq c_F^{1/(\gamma_1 - 1)} c_2 \log t. \tag{5.7}$$

**Proof.** By the definition of $F_1$, letting $s = h(t)(\log \log t)^{-1}$ we have that for $t \geq 16$,

$$F_1((c_1 + 1)h(t), t) = \sup_{s > 0} \left( \frac{(c_1 + 1)h(t)}{s} - \frac{t}{F(s)} \right) \geq \frac{(c_1 + 1)h(t)}{h(t)(\log \log t)^{-1}} - \frac{t}{F(h(t)(\log \log t)^{-1})}$$

$$= c_1 \log t.$$

For $\text{(5.7)}$, we fix $t > 0$ and let $s_0 := c_F^{1/(\gamma_1 - 1)} h(t)(\log \log t)^{-1} \leq h(t)(\log \log t)^{-1}$. If $s \leq s_0$, using $L(\gamma_1, c_F^{-1}, F)$ we have

$$\frac{s}{F(s)} \geq c_F^{-1} \left( \frac{h(t)(\log \log t)^{-1}}{s} \right)^{\gamma_1 - 1} \frac{h(t)(\log \log t)^{-1}}{F(h(t)(\log \log t)^{-1})} \geq c_F^{-1} \frac{h(t)}{F(h(t)(\log \log t)^{-1})} \frac{h(t)}{t} = \frac{h(t)}{t} \geq \frac{c_2 h(t)}{t}.$$ 

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Thus, we obtain \( \frac{c_2 h(t)}{s} - \frac{t}{F(s)} \leq 0 \) for \( s \leq s_0 \). Since \( F_1(r, t) > 0 \) for all \( r, t > 0 \), we have
\[
F_1(c_2 h(t), t) = \sup_{s > 0} \left( \frac{c_2 h(t)}{s} - \frac{t}{F(s)} \right) = \sup_{s \geq s_0} \left( \frac{c_2 h(t)}{s} - \frac{t}{F(s)} \right) \leq \frac{c_2 h(t)}{s_0} = c_F^{1/(\gamma_1-1)} c_2 \log \log t.
\]

Note that if \( (M, d, \mu) = (\mathbb{R}^d, \cdot, \cdot, dm) \), we have \( F(r) = r^2 \) and so \( h(t) = (t \log \log t)^{1/2} \). Thus, the next theorem is the counterpart of [2, Theorem 5.2].

**Theorem 5.4.** Suppose that \( (M, d, \mu) \) satisfies Ch(A), RVD\((d_1)\), VD\((d_2)\) and Diff\((F)\) where \( F \) is strictly increasing and satisfies (2.19), \( L(\gamma_1, \gamma_F^{-1}) \) and \( U(\gamma_2, \gamma_F) \) with \( 1 < \gamma_1 \leq \gamma_2 \). Let \((\mathcal{E}, \mathcal{F})\) be a regular Dirichlet form given by (2.24), which satisfies \( J_\psi \) with a non-decreasing function \( \psi \) satisfying (2.19). (i) Assume that (5.2) holds. Then there exists a constant \( c \in (0, \infty) \) such that for all \( x \in M \),
\[
\limsup_{t \to \infty} \frac{d(x, X_t)}{h(t)} = c \quad \text{for} \quad \mathbb{P}^x \text{-a.e.} \tag{5.8}
\]
(ii) Suppose that (5.2) does not hold. Then for all \( x \in M \), (5.8) holds with \( c = \infty \).

**Proof.** Fix \( x \in M \). We first observe that by (4.1), there exist constants \( a > 16 \) and \( c_1(a) > 1 \) such that for any \( t \geq 16 \\
(2c_F)^{1/\gamma_1} t \leq h(at) \leq c_1 h(t). \tag{5.9}
\]
In particular, combining (5.9) and \( L(\gamma_1, \gamma_F^{-1}, F) \) we have
\[
2F(h(t)) \leq F(h(at)). \tag{5.10}
\]
Also, using \( L(\gamma_1, \gamma_F^{-1}, F) \), we have for \( t \geq 16 \)
\[
\frac{F(h(t))}{t/\log \log t} = \frac{F(h(t))}{F(h(t)/\log \log t)} \geq c_F^{-1}(\log \log t)^{\gamma_1}. \quad \text{Thus,}
\]
\[
c_F^{-1} t (\log \log t)^{\gamma_1-1} \leq F(h(t)), \quad t \geq 16. \tag{5.11}
\]
Using (5.10), (5.9) and \( U(\beta_2, C_U, \psi) \) we obtain that for \( n \geq 1 \\
\int_{h(a^n)}^{h(a^{n+1})} \frac{dF(s)}{\psi(s)} \geq (F(h(a^{n+1})) - F(h(a^n))) \frac{1}{\psi(h(a^{n+1}))} \geq c_2 \frac{F(h(a^{n+1}))}{\psi(h(a^n))} \]
\[
\geq c_3 a^{n+1} (\log \log a^{n+1})^{\gamma_1-1} \psi(h(a^n)) \geq c_3 \int_{a^n}^{a^{n+1}} \frac{(\log \log t)^{\gamma_1-1}}{\psi(h(t))} dt.
\]
In particular, this and \( a > 16 \) imply that
\[
\int_{h(a)}^{\infty} \frac{dF(s)}{\psi(s)} ds \geq c_3 \int_a^{\infty} \frac{1}{\psi(h(t))} dt. \tag{5.12}
\]
(i) Let \( k_0 \in \mathbb{N} \) be a natural number satisfying \( 2^{k_0} \geq a \). By (5.4) and (5.12),
\[
\sum_{k=k_0}^{\infty} \frac{2^k}{\psi(h(2^k))} \leq c_4 \sum_{k=k_0}^{\infty} \int_{2^k}^{2^{k+1}} \frac{dt}{\psi(h(t))} \leq c_4 \int_a^{\infty} \frac{dt}{\psi(h(t))} < \infty. \tag{5.13}
\]
By (5.3), we have \( c_8^{-1} t \geq F(\Phi^{-1}(u)) \) for any \( u \geq 16 \) and \( t \leq u \leq 4t \). Thus, we have
\[
F_1(d(x, y), F(\Phi^{-1}(u))) \geq F_1(d(x, y), c_8^{-1} t) = c_8^{-1} F_1(c_8 d(x, y), t). \tag{5.14}
\]
Using GUHK($\Phi$), VD($d_2$) and (5.14) we have
\[
\mathbb{P}^x(d(x, X_u) > C h(t)) = \int_{\{y: d(x, y) > Ch(t)\}} p(u, x, y) \mu(dy)
\]
\[
\leq c_5 t \int_{\{d(x, y) > Ch(t)\}} \frac{\mu(dy)}{V(x, d(x, y))} + \frac{c_5}{V(x, F^{-1}(t))} \int_{\{d(x, y) > Ch(t)\}} e^{-c_7 F_1(c_8 d(x, y), t)} \mu(dy)
\]
\[
:= c_5 (I + II).
\]
(5.15)

Let us choose $C = c_8^{-1} (1 + 4 c_7^{-1})$ for (5.15). By [21 Lemma 2.1], we have $I \leq c_{11} \frac{t}{\psi(h(t))}$. For $II$, using VD($d_2$) and (4.3) we have
\[
II = \frac{1}{V(x, F^{-1}(t))} \int_{\{d(x, y) > Ch(t)\}} \exp(-c_7 F_1(c_8 d(x, y), t)) \mu(dy)
\]
\[
= \frac{1}{V(x, F^{-1}(t))} \sum_{i=0}^{\infty} \int_{\{C^{2^i} h(t) < d(x, y) \leq C^{2^{i+1}} h(t)\}} \exp(-c_7 F_1(c_8 d(x, y), t)) \mu(dy)
\]
\[
\leq \sum_{i=0}^{\infty} \frac{V(x, C^{2^{i+1}} h(t))}{V(x, F^{-1}(t))} \exp\left(-c_7 F_1((1 + 4 c_7^{-1}) 2^i h(t), t)\right)
\]
\[
\leq e^{c_7/2} c_9 \exp\left(-\frac{c_7}{2} F_1((1 + 4 c_7^{-1}) h(t), t)\right) \sum_{i=0}^{\infty} \left(\frac{2^i h(t)}{F^{-1}(t)}\right)^{d_2} \exp\left(-c_{10} \left(\frac{F(2^i h(t))}{t}\right)^{\frac{1}{12} - 1}\right)
\]
\[
\leq e^{c_7/2} c_9 \exp\left(-\frac{c_7}{2} F_1((1 + 4 c_7^{-1}) h(t), t)\right) \sum_{i=0}^{\infty} \left(\frac{2^i h(t)}{F^{-1}(t)}\right)^{d_2} \exp\left(-c_{11} \left(\frac{2^i h(t)}{F^{-1}(t)}\right)^{\frac{1}{12} - 1}\right)
\]
\[
\leq e^{c_7/2} c_9 \exp\left(-\frac{c_7}{2} F_1((1 + 4 c_7^{-1}) h(t), t)\right) \sum_{s \geq 1} \sum_{i=0}^{\infty} (2^i s)^{d_2} \exp\left(-c_{11} (2^i s)^{\frac{1}{12} - 1}\right)
\]
\[
\leq c_{12} \exp\left(-\frac{c_7}{2} F_1((1 + 4 c_7^{-1}) h(t), t)\right)
\]
Note that by (5.11), we have $h(t) \geq c F^{-1}(t)$. Using (5.6), we obtain
\[
II \leq c_{13} \exp\left(-\frac{c_7}{2} F_1((1 + 4 c_7^{-1}) h(t), t)\right) \leq c_{13} \exp\left(-2 \log \log t\right) \leq c_{13} (\log t)^{-2}.
\]
Thus, for $C = c_8^{-1} (1 + 4 c_7^{-1})$ and any $t \geq 16$ and $t \leq u \leq 4t$, we have
\[
\mathbb{P}^x(d(x, X_u) > C h(t)) \leq c_{14} \left(\frac{t}{\psi(h(t))} + (\log t)^{-2}\right).
\]
Using this and the strong Markov property, for $t_k = 2^k$ with $k \geq k_0 + 1$ we get
\[
\mathbb{P}^x(d(x, X_s) > 2 C h(s) \text{ for some } s \in [t_{k-1}, t_k]) \leq \mathbb{P}^x(\tau_{B(x, Ch(t_{k-1}))} \leq t_k)
\]
\[
\leq 2 \sup_{s \leq t_k, z \in M} \mathbb{P}^z(d(z, X_{t_{k+1}-s}) > Ch(t_{k-1})) \leq c_{15} \left(\frac{1}{k^2} + \frac{2^k}{\psi(h(2^k))}\right).
\]
Thus, by (5.13) and the Borel-Cantelli lemma, the above inequality implies that
\[
\mathbb{P}^x(d(x, X_t) \leq 2 C h(t) \text{ for all sufficiently large } t) = 1.
\]
Thus, $\limsup_{t \to \infty} \frac{d(x, X_t)}{h(t)} \leq 2C$. 

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On the other hand, by \([5.3]\) and \(L(\gamma_1, c_F^{-1}, F)\), we have \(L^1(\gamma_1, c_L, \Phi)\) with some \(c_L > 0\). Also, by \([4.2]\) we have \(U(\gamma_2, c_F, \Phi)\). Since \(\gamma_1 > 1\), using \([3.38]\) we have for any \(x, y \in M\) and \(t \geq T\),

\[
p(t, x, y) \geq c_{16} V(x, \Phi^{-1}(t))^{-1} \exp \left( - a_L \bar{\Phi}_1(d(x, y), t) \right),
\]

where \(\bar{\Phi}(r) = r^{\gamma_2} \Phi(1) \mathbf{1}_{\{r < 1\}} + \Phi(r) \mathbf{1}_{\{r \geq 1\}}\). Note that by \(RVD(d)\), we have \(\bar{\Phi}(r) = r^{\gamma_2} \Phi(1) \leq c_F \frac{\Phi(1) F(r)}{F(1)}\) for \(r < 1\). Using this and \([5.3]\), we obtain that \(\bar{\Phi}(r) \leq c F(r)\) for all \(r > 0\). Thus, by the definitions of \(\bar{\Phi}_1\) and \(F_1\) we obtain

\[
\bar{\Phi}_1(r, t) \leq F_1(\frac{t}{c}), \quad r, t > 0.
\]

Combining \([5.16]\) and \([5.17]\), we have that for all \(c_0 \in (0, 1), t \geq 16\) and \(t \leq u \leq 4t\),

\[
\mathbb{P}^x(d(x, X_u) > c_0 h(t)) = \int_{\{d(x, y) > c_0 h(t)\}} p(u, x, y) \mu(dy)
\]

\[
\geq \frac{c_{16}}{V(x, \Phi^{-1}(u))} \int_{\{d(x, y) > c_0 h(t)\}} e^{-a_L \bar{\Phi}_1(d(x, y), u)} \mu(dy)
\]

\[
\geq \frac{c_{16}}{V(x, F^{-1}(t))} \int_{\{d(x, y) > c_0 h(t)\}} e^{-a_L F_1(d(x, y), \frac{u}{c})} \mu(dy).
\]

Note that by \(RVD(d_1)\), we have a constant \(c_{17} > 0\) such that

\[
V(x, c_1 r) \geq 2 V(x, r), \quad \text{for all } x \in M, r > 0.
\]

Thus, using this and \([5.11]\) we have that for \(u \geq t\),

\[
\frac{1}{V(x, F^{-1}(t))} \int_{\{d(x, y) > c_0 h(t)\}} e^{-a_L F_1(d(x, y), \frac{u}{c})} \mu(dy)
\]

\[
\geq \frac{1}{V(x, F^{-1}(t))} \int_{\{c_0 h(t) < d(x, y) \leq c_0 c_1 h(t)\}} e^{-a_L F_1(d(x, y), \frac{u}{c})} \mu(dy)
\]

\[
\geq \frac{V(x, c_0 h(t))}{V(x, F^{-1}(t))} \exp \left( - a_L F_1(c_0 c_1 h(t), t c^{-1}) \right)
\]

\[
\geq c_{18}^{d_2} c_{19}^{-1} \exp \left( - c_18 F_1(c_0 c_19 h(t), t) \right).
\]

Since the constants \(c_{16}, c_{18}, c_{19}\) are independent of \(c_0\), provided \(c_0 > 0\) small and using \([5.7]\), we have

\[
\mathbb{P}^x(d(x, X_u) > c_0 h(t)) \geq c_{16} \exp \left( - c_18 F_1(c_0 c_19 h(t), t) \right)
\]

\[
\geq c_{18}^{d_2} c_{16} C_{19}^{-1} \exp \left( - c_{20} c_0 \log \log t \right) \geq c_{18}^{d_2} c_{16} C_{19}^{-1} (\log t)^{-1/2}.
\]

Thus, by the strong Markov property and \([5.18]\), we have

\[
\sum_{k=1}^{\infty} \mathbb{P}^x(d(X_{t_k}, X_{t_{k+1}}) \geq c_0 h(t_k) | F_{t_k}) \geq \sum_{k=4}^{\infty} c_{16}^{d_2} C_{19}^{-1} (\log t_k)^{-1/2} = \infty.
\]

Thus, by the second Borel-Cantelli lemma, \(\mathbb{P}^x(\limsup \{d(X_{t_k}, X_{t_{k+1}}) \geq c_0 h(t_k)\}) = 1\). Whence, for infinitely many \(k \geq 1\), \(d(x, X_{t_{k+1}}) \geq \frac{c_0 h(t_k)}{2}\) or \(d(x, X_{t_k}) \geq \frac{c_0 h(t_k)}{2}\). Therefore, for all \(x \in M\),

\[
\limsup_{t \to \infty} \frac{d(x, X_t)}{h(t)} \geq \limsup_{k \to \infty} \frac{d(x, X_{t_k})}{h(t_k)} \geq c_{21}, \quad \mathbb{P}^x\text{-a.e.,}
\]

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where \( c_{21} > 0 \) is the constant satisfying \( c_{21} h(2t) \leq \frac{C}{2} h(t) \) for any \( t \geq 16 \). Since
\[
P^x(c_{21}) \leq \limsup_{t \to \infty} \frac{d(x, X_t)}{h(t)} \leq 2C = 1,
\]
by Lemma 5.1 there exists a constant \( c > 0 \) satisfying (5.8).

(ii) Recall that by Lemma 4.1, \( L(\alpha_1, c, L, \Phi) \) holds for some \( \alpha_1, c_L > 0 \). Let \( t_k = 2^k \). Note that using the assumption \( \int_0^\infty \frac{dF(s)}{\psi(s)} = \infty \) to (2.20) we obtain \( \lim_{t \to \infty} F(t) = \infty \), which implies
\[
\lim_{s \to \infty} \frac{\Phi^{-1}(s)}{F^{-1}(s)} = \infty. \tag{5.19}
\]

Let \( \eta > 0 \) be the constants in (3.31). Let \( C_0 \in (0, 1) \) be a constant which will be determined later. Define \( N := N(k) := [C_0 \log \log t_k] + 1 \). Then, by (5.19) we have \( \lim_{k \to \infty} \frac{\Phi^{-1}(t_k/N)}{F^{-1}(t_k/N)} = \infty \). Thus, there exists \( k_1 \in \mathbb{N} \) such that for any \( k \geq k_1 \), we have \( N(k) \geq 3 \) and \( \frac{\Phi^{-1}(t_k/N)}{F^{-1}(t_k/N)} \geq \frac{\theta A_c}{C_0} (2C_0c_F)^{1/2} \), where \( \theta \) is the constant in (5.5).

Using this and \( L(1/\gamma_2, c_F^{-1/\gamma_2}, F^{-1}) \), we have for \( k \geq k_1 \),
\[
\frac{\eta}{3} \Phi^{-1}(t_k/N) = \frac{\eta}{3} \Phi^{-1}(t_k/N) F^{-1}(t_k/N) \geq \frac{2\theta A_c}{C_0} (2C_0c_F)^{1/2} F^{-1}(t_k/N)
\]
\[
\geq \frac{2\theta A_c}{C_0} (2C_0c_F)^{1/2} F^{-1}(t_k/N) \geq \frac{2\theta A_c}{C_0} (2C_0c_F)^{1/2} F^{-1}(t_k/N)
\]
\[
\geq \frac{2\theta A_c}{C_0} (2C_0c_F)^{1/2} (2C_0c_F)^{-1/2} F^{-1}(t_k/N) \geq \frac{2\theta A_c}{C_0} (2C_0c_F)^{1/2} F^{-1}(t_k/N)
\]
\[
\geq \frac{2\theta A_c}{C_0} (2C_0c_F)^{1/2} F^{-1}(t_k/N) \geq \frac{2\theta A_c}{C_0} (2C_0c_F)^{1/2} F^{-1}(t_k/N) = \frac{2\theta A_c}{C_0} f(t_k) \geq \frac{2\theta A_c}{C_0} f(t_k).
\]

Note that by \( \text{Ch}(A) \), we have a sequence \( \{z_l \}_{l=0}^N \) of points in \( M \) such that \( z_0 = x, z_N = y \) and \( d(z_l-1, z_l) \leq A(d(x,y)/N) \) for any \( l \in \{1, \ldots, N\} \). Thus, following the chain argument in (3.33) equipped with (3.31) and \( \text{RVD}(d_1) \), we have for \( k \geq k_1 \) and \( 2c_F h(t_k) \leq d(x,y) \leq 2\theta c_F h(t_k) \),
\[
p(t_k, x, y) \geq \prod_{l=0}^{N-1} C_3 V(z_l, \Phi^{-1}(t_k/N))^{-1} \prod_{l=1}^{N-1} \frac{d}{d \xi_1} V(z_l, \Phi^{-1}(t_k/N))
\]
\[
\geq c_1 \left( \frac{C_3 \eta d_1}{3 d_1} \right)^N V(x, \Phi^{-1}(t_k))^{-1} \geq c_1 V(x, \Phi^{-1}(t_k))^{-1} \exp \left( -N \log \left( \frac{3 d_1}{C_3 \eta d_1} \right) \right).
\]

Since \( C_3 \) and \( \eta \) are the constants in (3.31) which are independent of \( N \), letting \( C_0 = \frac{1}{4} (\log (C_3 \eta d_1))^{-1} \) we obtain that for any \( k \geq k_1 \) and \( x, y \in M \) with \( 2c_F h(t_k) \leq d(x,y) \leq 2\theta c_F h(t_k) \),
\[
p(t_k, x, y) \geq \frac{c_1}{V(x, \Phi^{-1}(t_k))} \exp \left( -N \log \left( \frac{3 d_1}{C_3 \eta d_1} \right) \right)
\]
\[
\geq \frac{c_1}{V(x, \Phi^{-1}(t_k))} \exp \left( -\frac{1}{2} \log \log t_k \right) = \frac{c_1}{V(x, \Phi^{-1}(t_k))} k^{-1/2}. \tag{5.20}
\]

Now we claim that for every \( C > 1 \),
\[
\sum_{k=1}^\infty \inf_{x \in M} \int_{d(x,y) \geq C h(t_k)} p(t_k, x, y) \mu(dy) = \infty, \tag{5.21}
\]
which implies the theorem. Indeed, the strong Markov property and (5.21) imply that for all \( C > 0 \),
\[
\sum_{k=1}^{\infty} \mathbb{P}^x (d(X_{t_k}, X_{t_{k+1}}) \geq Ch(t_{k+1}) \mid \mathcal{F}_{t_k}) \geq \sum_{k=1}^{\infty} \inf_{x \in M} \int_{d(x, y) \geq Ch(t_{k+1})} p(t_k, x, y) \mu(dy) = \infty.
\]
Thus, by the second Borel-Cantelli lemma, \( \mathbb{P}^x (\limsup \{d(X_{t_k}, X_{t_{k+1}}) \geq Ch(t_{k+1})\}) = 1 \). Whence, for infinitely many \( k \geq 1 \), \( d(x, X_{t_{k+1}}) \geq \frac{Ch(t_{k+1})}{2} \) or \( d(x, X_{t_k}) \geq \frac{Ch(t_{k+1})}{2} \). Therefore, for all \( x \in M \),
\[
\limsup_{t \to \infty} \frac{d(x, X_t)}{h(t)} \geq \limsup_{k \to \infty} \frac{d(x, X_{t_k})}{h(t_k)} \geq \frac{C}{2} \quad \mathbb{P}^x \text{-a.e.}
\]
Since the above holds for every \( C > 1 \), the theorem follows.

We now prove the claim (5.21) by considering two cases separately. Let \( \eta > 0 \) be the constant in Proposition 3.12. Using GUHK(\( \Phi \)) and RVD(\( d_1 \)), there is \( \lambda \in (0, 1) \) such that
\[
\sup_{t \geq 1} \sup_{x \in M} \int_{d(x, y) < \lambda \Phi^{-1}(t)} p(t, x, y) \mu(dy) \leq c_2 \sup_{t \geq 1} \sup_{x \in M} \frac{V(x, \lambda \Phi^{-1}(t))}{V(x, \Phi^{-1}(t))} < \frac{1}{2}. \tag{5.22}
\]

**Case 1**: If there exist infinitely many \( k \geq 1 \) such that \( Ch(t_{k+1}) \leq \lambda \Phi^{-1}(t_k) \), then, by (5.22), for infinitely many \( k \geq 1 \),
\[
\inf_{x \in M} \int_{d(x, y) \geq \lambda \Phi^{-1}(t_k)} p(t_k, x, y) \mu(dy) = 1 - \sup_{x \in M} \int_{d(x, y) < \lambda \Phi^{-1}(t_k)} p(t_k, x, y) \mu(dy) > 1/2.
\]
Thus we get (5.21).

**Case 2**: Assume that there is \( k_2 \geq 3 \) satisfying that \( Ch(t_{k+1}) \geq \lambda \Phi^{-1}(t_k) \) for all \( k \geq k_2 \). Then, using (5.20), (5.5) and VD(\( d_2 \)) we have that for every \( k \geq k_1 \cap k_2 \),
\[
\inf_{y \in M} \int_{\{z : Ch(t_{k+1}) \leq d(y, z) < \theta Ch(t_{k+1})\}} p(t_k, y, z) \mu(dz) \\
\geq c_3 \frac{(\varepsilon - 1) V(x, Ch(t_{k+1}))}{V(x, \Phi^{-1}(t_k))^{1/2}} \geq c_3 \frac{V(x, \eta \Phi^{-1}(t_k))}{V(x, \Phi^{-1}(t_k))^{1/2}} \geq \frac{c_4}{k^{1/2}}.
\]
This proves (5.21). \( \Box \)

### 6 Examples

In this section, we give some examples which are covered by our results. Throughout this section, \((M, d, \mu)\) is a metric measure space satisfying Ch(\( A \)), VD(\( d_2 \)), RVD(\( d_1 \)) and Diff(\( F \)), where the function \( F : (0, \infty) \to (0, \infty) \) is strictly increasing function satisfying \( L(\gamma_1, c_F^{-1}) \) and \( U(\gamma_2, c_F) \) with some constants \( 1 < \gamma_1 \leq \gamma_2 \).

Typical examples of metric measure spaces satisfying the above conditions are unbounded Sierpinski gasket and unbounded Sierpinski carpet in \( \mathbb{R}^n \) with \( n \geq 2 \). First, let us check that unbounded Sierpinski gasket in \( \mathbb{R}^n \) satisfies the above conditions. Let \((M_{SG}, d_{SG}, \mu_{SG})\) be the unbounded Sierpinski gasket in \( \mathbb{R}^2 \), which was introduced in [10]. Here \( d_{SG}(x, y) \) denotes the length of the shortest path in \( M_{SG} \) from \( x \) to \( y \), and \( \mu_{SG} \) is a multiple of the \( d_f \)-dimensional Hausdorff measure on \( M_{SG} \) with \( d_f = \log 3/\log 2 \) (see [10] Lemma 1.1). By [10] (1.13), \( d_{SG}(x, y) \) is comparable to \(|x - y|\) which is the Euclidean distance, which implies Ch(\( A \)). Also, by [10] Theorem 1.5, Diff(\( F \)) holds
for $F(r) = r^{d_w}$ with $d_w = \log 5/\log 2 > 2$. Since $d_{SC}(x,y) \asymp |x-y|$ and $M_{SG}$ is subset of $\mathbb{R}^2$, all metric balls in $(M_{SG}, d_{SG})$ are precompact. Since $M_{SG}$ is unbounded, by [33, Corollary 7.6], we have $VD(d_2)$. Moreover, by [30, Corollary 5.3], we also have $RVD(d_1)$ since $M_{SG}$ is connected. Thus, we see that $(M_{SG}, d_{SG}, \mu_{SG})$ satisfies $\text{Ch}(A)$, $VD(d_2)$, $RVD(d_1)$ and $\text{Diff}(F)$. This result also holds for unbounded Sierpinski gaskets constructed in $n$-dimensional $(n \geq 3)$ Euclidean space with different $d_f(n) > 0$ and $d_w(n) > 1$ (see [10, Section 10]). Now, let $(M_{SC}, \cdot, |\cdot|, H(d_f))$ be the unbounded generalized Sierpinski carpet constructed in $\mathbb{R}^n$, which was introduced in [5]. Then, by [5, Hypotheses 2.1], $(M_{SC}, |\cdot|)$ satisfies $\text{Ch}(A)$ and connected. Also, by [5, Theorem 1.3], $\text{Diff}(F)$ holds for $F(r) = r^{d_w}$ with $d_w \geq 2$. Moreover, by [5, Remark 2.2], $|x-y| \asymp d_{SC}(x,y)$ for all $x, y \in M_{SC}$, where $d_{SC}(x,y)$ is the length of the shortest path in $M_{SC}$ from $x$ to $y$. Thus, by the same argument as in the unbounded Sierpinski gasket case, we see that $(M_{SC}, |\cdot|, H(d_f))$ satisfies $\text{Ch}(A)$, $VD(d_2)$, $RVD(d_1)$ and $\text{Diff}(F)$.

Let $X$ be the symmetric pure-jump Hunt process on $(M, d, \mu)$, which is associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in (2.3) satisfying $\mathcal{J}_\psi$ and $p(t,x,y)$ be the transition density of $X$. In this section, we will use the notation $f(\cdot) \asymp g(\cdot)$ at $\infty$ (resp. 0) if $\int_0^t f(t) \to 1$ as $t \to \infty$ (resp. $t \to 0$).

**Definition 6.1.** (i) We denote $\mathcal{R}_0^\infty$ (resp. $\mathcal{R}_0^0$) by the class of slowly varying functions at $\infty$ (resp. 0).

(ii) For $\ell \in \mathcal{R}_0^\infty$ (resp. $\mathcal{R}_0^0$) we denote $\Pi_\ell^\infty$ (resp. $\Pi_\ell^0$) by the class of real-valued measurable function $f$ on $[c, \infty)$ (resp. $(0,c]$) such that for all $\lambda > 0$, $f(\lambda \cdot) - f(\cdot) \asymp \log \lambda \ell(\cdot)$ at $\infty$ (resp. 0).

(iii) $\Pi_\ell^\infty$ (resp. $\Pi_\ell^0$) is called de Haan class at $\infty$ (resp. 0) determined by $\ell$.

(iv) For $\ell \in \mathcal{R}_0^\infty$ (resp. $\mathcal{R}_0^0$), we say $\ell_#$ is de Bruijn conjugate of $\ell$ if both $\ell(t)\ell_#(t\ell(t)) \asymp 1$ and $\ell_#(t)\ell(t\ell_#(t)) \asymp 1$ at $\infty$ (resp. 0).

Note that $|f| \in \mathcal{R}_0^\infty$ if $f \in \Pi_\ell^\infty$ (see [12, Theorem 3.7.4]). In the following corollaries and examples, $a_i = a_{1,i}$ or $a_i = a_{2,i}$ depending on whether we consider lower or upper bound.

**Corollary 6.2.** Suppose $F$ is differentiable function satisfying $F(s) \asymp s^{F'(s)}$. Let $T \in (0, \infty)$ and $\psi$ be a non-decreasing function that satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Suppose that the Hunt process $X$ on $(M, d, \mu)$ satisfies $\mathcal{J}_\psi$. Let $p(t,x,y)$ be the transition density of $X$.

(1) Let $\ell \in \mathcal{R}_0^0$ be such that $\int_0^1 \frac{s}{\ell(s)} ds < \infty$ and $f(s) := \int_0^s \frac{\ell(t)}{t} dt \in \mathcal{R}_0^0$. Suppose $\psi(s) \asymp \frac{F(s)}{\ell(s)}$ for $s < 1$. Then, for $t < T$,

$$p(t,x,y) \asymp \frac{1}{V(x, (F/f)^{-1}(t))} \wedge \left( \frac{t}{V(x, d(x,y)) \psi(d(x,y))} + \frac{1}{V(x, (F/f)^{-1}(t)) \exp \left( \frac{-a_1 d(x,y)}{(F/f)^{-1}(t/d(x,y))} \right)} \right).$$

(2) Assume that $\ell \in \mathcal{R}_0^\infty$ satisfies $\int_1^\infty \frac{\ell(t)}{t} dt = \infty$. Suppose that $\psi(s) \asymp \frac{F(s)}{\ell(s)}$ for $s > 1$ and $f \in \Pi_\ell^\infty$. Then for $t > T$,

$$p(t,x,y) \asymp \frac{1}{V(x, (F/f)^{-1}(t))} \wedge \left( \frac{\ell(d(x,y))}{V(x, d(x,y)) \psi(d(x,y))} + \frac{1}{V(x, (F/f)^{-1}(t)) \exp \left( \frac{-a_2 d(x,y)}{(F/f)^{-1}(t/d(x,y))} \right)} \right).$$

**Proof.** Let $r = d(x,y)$ and $\Phi(u) := F(u) / \int_0^u \frac{F(s)}{\psi(s)} ds$. By [12] and Lemma 4.1, we see that $\Phi$ satisfies $U(\gamma_2, C_F)$ and $L(\alpha_1, C_L)$ for some $\alpha_1, c_L > 0$. Without loss of generality, we assume that $T = 1$. 

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(1) Since $\Phi(s) \asymp F(s)/f(s)$ for $s < 1$ and $f \in R_0^0$, we observe that $\Phi$ satisfies $L_1(\delta, \tilde{C}_L)$ for some $\delta > 1$. Thus, we can apply Corollary 2.20(i) and obtain that for $t < 1$,

$$p(t, x, y) \asymp \frac{1}{V(x, (F/f)^{-1}(t))} \wedge \left(\frac{t}{V(x, (F/f)^{-1}(t))} + \frac{1}{V(x, (F/f)^{-1}(t))} \exp (-a_1\Phi_1(r, t))\right).$$

By Lemma A.2 and $L_1(\delta, \tilde{C}_L, \Phi)$, there exists $c_1 > 0$ such that $\Phi^{-1}(t) > c_1 t^{1/\delta}$ holds for all $t < 1$. Thus, for $r > 2c_F^2\Phi^{-1}(t)$ and $t < 1$, we have $r > 2c_F^2c_1 t^{1/\delta} > 2c_F^2c_1 t$ which shows $t/r < 1/2c_F^2c_1^{-2}$. Note that by (A.2), $\mathcal{K}(s) \asymp \Phi(s)/s \asymp (F/f)(s)$ for $s < 1/2c_F^2c_1^{-2}$. From this and Lemma A.3 we see that $\Phi_1(r, t) \asymp r/\mathcal{K}^{-1}(t/r) \asymp r/(F/f)^{-1}(t/r)$ for $r > 2c_F^2\Phi^{-1}(t)$ and $t < 1$. This, combined with Lemma A.9(ii), completes the proof of (1).

(2) Since $\Phi(s) \asymp F(s)/f(s)$ for $s > 1$ and $f \in R_0^\infty$, we observe that $\Phi$ satisfies $L_1(\delta, \tilde{C}_L)$ for some $\delta > 1$. Let $\tilde{\Phi}$ be the function defined in (2.12). Then, $U(\gamma_2, c_F, \tilde{\Phi})$, $L(\delta, \tilde{C}_L, \tilde{\Phi})$ hold and $\tilde{\Phi}(s) = \Phi(s)$ for $s > 1$. By applying Corollary 2.20(ii), we obtain that for $t \geq 1$,

$$p(t, x, y) \asymp \frac{1}{V(x, (F/f)^{-1}(t))} \wedge \left(\frac{t\ell(r)}{V(x, t/r)} + \frac{1}{V(x, (F/f)^{-1}(t))} \exp (-a_2\tilde{\Phi}_1(r, t))\right).$$

Choose small $\theta_0 > 0$ such that $1/\delta + \theta_0(1/\delta - 1/2) =: \varepsilon_0 < 1$ and let $c_2 = c_2(\theta_0) > 0$ be a constant satisfying $s^{-d_2 - \beta_2 - \beta_2/\theta_0} \geq c_2 \exp (-a_2 s)$ for all $s > 0$. For $r > 2c_F^2\Phi^{-1}(t)^{1+\theta}$, there exists $\theta > \theta_0$ such that $r = 2c_F^2\Phi^{-1}(t)^{1+\theta}$ since $\Phi^{-1}(t) > \psi^{-1}(t)$. Then, by using $\text{VD}(d_2)$, $U(\beta_2, C_U, \psi)$ and the same argument in the proof of Lemma 3.6,

$$\frac{t}{V(x, \Phi^{-1}(t))} \geq \frac{c_3}{V(x, \Phi^{-1}(t))} \left(\Phi^{-1}(t) / \psi^{-1}(t)\right)^{\theta(-d_2 - \beta_2 - \beta_2/\theta_0)} \geq \frac{c_3}{V(x, \Phi^{-1}(t))} \left(\Phi^{-1}(t) / \psi^{-1}(t)\right)^{\theta(-d_2 - \beta_2 - \beta_2/\theta_0)} \geq \frac{c_2c_3}{V(x, \Phi^{-1}(t))} \exp (-a_2(\Phi^{-1}(t)^{1+\theta}))$$

where the last inequality follows from the definition of $\tilde{\Phi}_1(r, t)$. Thus, by this and Lemma A.9(ii), it suffices to estimate $\tilde{\Phi}_1(r, t)$ for $2c_F^2\Phi^{-1}(t)^{1+\theta} < r \leq 2c_F^2\Phi^{-1}(t)^{1+\theta}_0$ and $t \geq 1$. By Lemma A.2 $L(\delta, \tilde{C}_L, \tilde{\Phi})$ and $U(\beta_2, C_U, \psi)$, we see that there exists $c_5 > 0$ such that $r \leq c_5 t^\varepsilon_0$ holds for $r \leq 2c_F^2\Phi^{-1}(t)^{1+\theta}_0$ and $t \geq 1$. Thus, we have $t/r \geq c_5^{-1} t^{1-\varepsilon_0} \geq c_5^{-1}$. Define $\mathcal{K}(r) := \sup_{0<s<r} \Phi(s)/s$. Then, by (A.2), $\mathcal{K}(s) \asymp \Phi(s)/s \asymp (F/f)(s)$ for $s > c_5^{-1}$. Thus, by Lemma A.3 we have $\tilde{\Phi}_1(r, t) \asymp r/\mathcal{K}^{-1}(t/r) \asymp r/(F/f)^{-1}(t/r)$ for $2c_F^2\Phi^{-1}(t)^{1+\theta} < r \leq 2c_F^2\Phi^{-1}(t)^{1+\theta}_0$ and $t \geq 1$. This completes the proof.

\[\Box\]

**Remark 6.3.** (1) Let $H$ be a strictly increasing function with weak scaling property. Suppose that $f \in R_0^0$ and there exists $\tilde{f} \in R_0^0$ such that $H(r)/f(r) \asymp H(r/\tilde{f}(r)) =: h(r)$. Note that $h$ also satisfies weak scaling property at 0. By the weak scaling property of $H$, we see that for $r < 1$,

$$h\left(H^{-1}(r)\tilde{f}(H^{-1}(r))\right) = H\left(H^{-1}(r)\tilde{f}(H^{-1}(r))\right) \asymp H\left(H^{-1}(r)\tilde{f}(H^{-1}(r))\right) \asymp r.$$
Thus, by using the weak scaling property of $h$,
\[
(H/f)^{-1}(r) \asymp H^{-1}(r)f_\#(H^{-1}(r)) \quad \text{for } r < 1.
\]

Similarly, suppose that $g \in \mathcal{R}_0^\infty$ and there exists $\tilde{g} \in \mathcal{R}_0^\infty$ such that $H(r)/g(r) \asymp H(r\tilde{g}(r))$. Then,
\[
(H/g)^{-1}(r) \asymp H^{-1}(r)\tilde{g}_\#(H^{-1}(r)) \quad \text{for } r \geq 1.
\]

(C.f. \cite{12} Proposition 1.5.15.)

(2) Let $\ell \in \mathcal{R}_0^\infty$ (resp. $\mathcal{R}_0^0$). Suppose that $\ell$ satisfies
\[
\lim_{s \to \infty (\text{resp. } s \to 0)} \frac{\ell(s\ell^n(s))}{\ell(s)} = 1 \quad \text{for some } \eta \in \mathbb{R} \setminus \{0\}.
\]

Then by \cite{12} Corollary 2.3.4, $(\ell^n)_\# \asymp 1/\ell^n$ at $\infty$ (resp. 0).

Combining Corollary \cite{6.2} and Remark \cite{6.3}, we have the following:

**Corollary 6.4.** Suppose $F$ is differentiable function satisfying $F(s) \asymp sF'(s)$. Let $T \in (0, \infty)$, $\gamma_3, \gamma_4 > 1$ and $\psi$ be a non-decreasing function that satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$.

(1) Let $\ell_0, \ell_1 \in \mathcal{R}_0^\infty$ be such that $\int_1^t \ell_0(u)u^{-1}du < \infty$ and $f_1(s) := \ell_1(s)^{-1}\int_0^s \ell_0(u)u^{-1}du \in \mathcal{R}_0^\infty$ satisfies \cite{5.1} for $s \in \{1/\gamma_3, 1/(\gamma_3 - 1)\}$. Suppose that for $s < 1$, $F(s) \asymp s^{\gamma_3}\ell_1(s)$ and $\psi(s) \asymp F(s)/\ell_0(s)$. Then for $t < T$,
\[
\begin{align*}
\rho(t, x, y) & \asymp \frac{1}{V(x, t^{1/\gamma_3}f_1^{1/\gamma_3}(t^{1/\gamma_3}))} \\
& \wedge \left(\frac{t}{V(x, d(x, y))\psi(d(x, y))} + \frac{1}{V(x, t^{1/\gamma_3}f_1^{1/\gamma_3}(t^{1/\gamma_3}))} \exp \left(-a_1 \left(\frac{d(x, y)^{\gamma_3}}{tf_1((\frac{t}{d(x, y)})^{\gamma_3-1})} \right)^{\frac{1}{\gamma_3-1}} \right)\right). 
\end{align*}
\]

Furthermore, if $f_1$ is monotone and satisfies $f_1(s^{\gamma_3}) \asymp f_1(s)$ for $s < 1$, then for $t < T$,
\[
\begin{align*}
\rho(t, x, y) & \asymp \frac{1}{V(x, t^{1/\gamma_3}f_1^{1/\gamma_3}(t))} \\
& \wedge \left(\frac{t}{V(x, d(x, y))\psi(d(x, y))} + \frac{1}{V(x, t^{1/\gamma_3}f_1^{1/\gamma_3}(t))} \exp \left(-a_2 \left(\frac{d(x, y)^{\gamma_3}}{tf_1(t)} \right)^{\frac{1}{\gamma_3-1}} \right)\right). 
\end{align*}
\]

(2) Let $\ell_3, \ell_2 \in \mathcal{R}_0^\infty$ be such that $\int_1^\infty \ell_3(u)u^{-1}du = \infty$ and $f_2(s) := \ell_2(s)^{-1}\int_0^s \ell_3(u)u^{-1}du \in \mathcal{R}_0^\infty$ satisfies \cite{5.1} for $s \in \{1/\gamma_4, 1/(\gamma_4 - 1)\}$. Suppose that for $s > 1$, $F(s) \asymp s^{\gamma_4}\ell_2(s)$ and $\psi(s) \asymp F(s)/\ell_3(s)$. Then for $t > T$,
\[
\begin{align*}
\rho(t, x, y) & \asymp \frac{1}{V(x, t^{1/\gamma_4}f_2^{1/\gamma_4}(t^{1/\gamma_4}))} \\
& \wedge \left(\frac{t\ell(d(x, y))}{V(x, d(x, y))\psi(d(x, y))d(x, y)^{\gamma_4}} + \frac{1}{V(x, t^{1/\gamma_4}f_2^{1/\gamma_4}(t^{1/\gamma_4}))} \exp \left(-a_3 \left(\frac{d(x, y)^{\gamma_4}}{tf_2((\frac{t}{d(x,y)})^{\gamma_4-1})} \right)^{\frac{1}{\gamma_4-1}} \right)\right). 
\end{align*}
\]

Furthermore, if $f_2$ is monotone and satisfies $f_2(s^{\gamma_4}) \asymp f_2(s)$ for $s > 1$, then for $t > T$,
\[ p(t, x, y) \propto \frac{1}{V(x, t^{1/\gamma_2} f_2^{1/\gamma_2}(t))} \]
\[ \land \left( \frac{t\ell(d(x, y))}{V(x, t^{1/\gamma_2} f_2^{1/\gamma_2}(t))} + \frac{1}{V(x, t^{1/\gamma_2} f_2^{1/\gamma_2}(t))} \exp \left( -a_4 \left( \frac{d(x, y)^{\gamma_2}}{tf_2(t)} \right)^{\frac{1}{\gamma_2-1}} \right) \right). \] (6.3)

**Proof.** Let \( r = d(x, y) \), \( \Phi(u) := F(u)/\int_0^u F'(s) \psi(s) ds \). Without loss of generality, we assume that \( T = 1 \).

(1) We will apply Corollary 6.2(1) to prove the claim. Since \( \Phi(s) \propto s^{\gamma_3}/f_1(s) \) for \( s < 1 \) and \( f_1 \in \mathcal{R}_0^0 \), we observe that \( \Phi \) satisfies \( L_1(\delta, \overline{C}_L) \) for some \( 1 < \delta < \gamma_3 + 1 \) and \( U_1(\gamma_3 + 1, \psi_U) \). By Lemma A.2 and \( L_1(\delta, \overline{C}_L, \Phi) \), there exists \( c_1 > 0 \) such that \( \Phi^{-1}(t) > c_1 t^{1/\delta} \) holds for all \( t < 1 \). Thus, for \( r > \Phi^{-1}(t) \) and \( t < 1 \), we have \( r > c_1 t^{1/\delta} > c_1 t \) which shows \( t/r \leq c_1^{-1} \).

Define \( f(s) := \int_s^\delta \ell(u) u^{-1} du \). By Remark 6.3 and the condition that \( f_1 \) satisfies (6.1) for \( \eta \in \{1/\gamma_3, 1/(\gamma_3 - 1)\} \), we see that for \( s < 1 \),
\[ (F/f)^{-1}(s) \propto (s^{\gamma_3}/f_1)^{-1}(s) \propto s^{1/\gamma_3} (1/f_1^{1/\gamma_3}) \propto (s^{1/\gamma_3}) \propto s^{1/\gamma_3} f_1^{1/\gamma_3} (s^{1/\gamma_3}) \]

and
\[ (F'/f)^{-1}(s) \propto (s^{\gamma_3-1}/f_1)^{-1}(s) \propto s^{1/(\gamma_3-1)} (1/f_1^{1/(\gamma_3-1)}) \propto (s^{1/(\gamma_3-1)}) \propto s^{1/(\gamma_3-1)} f_1^{1/(\gamma_3-1)} (s^{1/(\gamma_3-1)}) \]

Using this, volume doubling property and the fact that \( t/r \leq c_1^{-1} \) for \( r > \Phi^{-1}(t) \) and \( t < 1 \), we can apply Corollary 6.2(1) to obtain the first claim.

Now we prove (6.2). By using \( f_1(s^{\gamma_3}) \propto f_1(s) \) for \( s < 1 \) and volume doubling property, we have \( V(x, t^{1/\gamma_2} f_1^{1/\gamma_2}(1/\gamma_3)) \propto V(x, t^{1/\gamma_2} f_1^{1/\gamma_2}(t)) \) for \( t < 1 \). Thus, it is enough to show that \( f_1((t/r)^{1/(\gamma_3-1)}) \) in the exponential term is comparable to \( f_1(t) \). To show this, we choose small \( \theta_1 > 0 \) such that \( \frac{1}{\gamma_3+1} + \theta_1 (\frac{1}{\gamma_3+1} - \frac{1}{\beta_3}) = \varepsilon_1 \in (0, \delta^{-1}) \). Let \( c_2 = c_2(\theta_1) > 0 \) be a constant satisfying \( s^{-d_2-\beta_2-\beta_2/\theta_1} \geq c_2 \exp(-a_1 s) \) for all \( s > 0 \). For \( r > \frac{\Phi^{-1}(t)^{1+\theta_1}}{\psi^{-1}(t)^{1+\theta_1}} \), there exists \( \theta > \theta_1 \) such that \( r = \frac{\Phi^{-1}(t)^{1+\theta}}{\psi^{-1}(t)^{1+\theta}} \) since \( \Phi^{-1}(t) < \psi^{-1}(t) \). Then, by using \( VD(d_2), U(\beta_2, C_U, \psi) \) and the same argument in the proof of Lemma 3.6
\[ \frac{t}{V(x, r) \psi(r)} \geq \frac{c_3}{V(x, \Phi^{-1}(t))} \left( \frac{\Phi^{-1}(t)}{\psi^{-1}(t)} \right)^{\theta (-d_2 - \beta_2 - \beta_2/\theta)} \geq \frac{c_3}{V(x, \Phi^{-1}(t))} \left( \frac{\Phi^{-1}(t)}{\psi^{-1}(t)} \right)^{\theta (-d_2 - \beta_2 - \beta_2/\theta_1)} \]
\[ \geq \frac{c_2 c_3}{V(x, \Phi^{-1}(t))} \exp (-a_1 \left( \frac{\Phi^{-1}(t)}{\psi^{-1}(t)} \right)^{\theta}) = \frac{c_2 c_3}{V(x, \Phi^{-1}(t))} \exp (-a_1 \left( r/\Phi^{-1}(t) \right)) \]
\[ \geq \frac{c_4}{V(x, \Phi^{-1}(t))} \exp (-a_1 \Phi^{-1}(r)), \]
where the last inequality follows from the definition of \( \Phi_1(r, t) \). Thus, it is enough to show
\[ f_1((t/r)^{1/(\gamma_3-1)}) \propto f_1(t) \]
for \( \Phi^{-1}(t) < r < \frac{\Phi^{-1}(t)^{1+\theta_1}}{\psi^{-1}(t)^{1+\theta_1}} \). By Lemma A.2, \( L_1(\delta, \overline{C}_L, \Phi), U_1(\gamma_3 + 1, \psi_U, \Phi) \) and \( L(\beta_1, C_L, \psi) \), we have \( \frac{\Phi^{-1}(t)^{1+\theta_1}}{\psi^{-1}(t)^{1+\theta_1}} \leq c_5 t^{1/(\gamma_3 + 1 + \theta_1)} = c_5 t^{\varepsilon_1} \) and \( \Phi^{-1}(t) \geq c_1 t^{1/\delta} \) for \( t < 1 \). Thus, \( c_1 t^{1/\delta} < r < c_5 t^{\varepsilon_1} \) for \( t < 1 \) and \( \Phi^{-1}(t) < r \leq \frac{\Phi^{-1}(t)^{1+\theta_1}}{\psi^{-1}(t)^{1+\theta_1}} \). Note that \( 0 < \varepsilon_1 < \delta^{-1} < 1 \). Using that \( f_1 \) is monotone and \( c_1 t^{1/\delta} < r \leq c_5 t^{\varepsilon_1} \), we see that
\[ f_1(c_5 t^{(1-\varepsilon_1)/(\gamma_3-1)}) \leq f_1((t/r)^{1/(\gamma_3-1)}) \leq f_1(c_1 t^{(1-1/\delta)/(\gamma_3-1)}) \]
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or

\[ f_1(c_5 t^{1-(\varepsilon_1)/(\gamma_3-1)}) \geq f_1((t/r)^{1/(\gamma_3-1)}) \geq f_1(c_1 t^{1-(1-\delta)/(\gamma_3-1)}). \]

By Lemma A.2 and the fact that \( f_1 \) is slowly varying at 0, we have \( f_1(c_5 s^{\varepsilon_2}) \asymp f_1(s) \) for \( s < 1 \). Thus, the above inequalities give that \( f_1((t/r)^{1/(\gamma_3-1)}) \asymp f_1(t) \) for \( t < 1 \).

(2) The proof is similar to that of (1). We will apply Corollary 6.2(2) to prove the claim. Since \( \Phi(s) \asymp s^{\gamma_4}/f_2(s) \) for \( s > 1 \) and \( f_2 \in R_0^\infty \), we observe that \( \Phi \) satisfies \( L^1(\delta, C_L) \) for some \( 1 < \delta < \gamma_4 + 1 \) and \( U^{1}(\gamma_4 + 1, c_U) \). Choose small \( \theta_2 > 0 \) such that \( \frac{1}{\delta} + \theta_2(\frac{1}{\gamma_4} - \frac{1}{\beta_2}) =: \varepsilon_2 \in ((1 + \gamma_4)^{-1}, 1) \), where \( \beta_2 > 0 \) is the upper scaling index of \( \psi \). Then, by Lemma A.2 \( L^1(\delta, C_L, \Phi), U^{1}(\gamma_4 + 1, c_U, \Phi) \) and \( U(\beta_2, C_U, \psi) \), we see that there exist \( c_1, c_2 > 0 \) such that for \( 2c_2^2 \Phi^{-1}(t) < r \leq 2c_2^2 \Phi^{-1}(t)^{1+\theta_2} \) and \( t \geq 1 \),

\[ c_1 t^{1/(\gamma_4 + 1)} < r \leq c_2 t^{\varepsilon_2}. \]  \( (6.4) \)

As we have seen in the proof of Corollary 6.2(2), for the estimates of \( (F'/f)^{-1}(t/r) \), it suffices to consider the case of \( 2c_2^2 \Phi^{-1}(t) < r \leq 2c_2^2 \Phi^{-1}(t)^{1+\theta_2} \) and \( t \geq 1 \) since \( \Phi(s) \) defined in (2.12) is equal to \( \Phi(s) \) for \( s > 1 \). In this case, we have \( t/r \geq c_2^{-1} t^{1-\varepsilon_2} \geq c_2^{-1} \) by (6.4).

Define \( f(s) := \int_{0}^{s} \ell(u) u^{-1} du \). By Remark 6.3 and the condition that \( f_2 \) satisfies \( (6.1) \) for \( \eta \in \{1/\gamma_4, 1/(\gamma_4 - 1)\} \), we have for \( s \geq 1 \),

\[ (F/f)^{-1}(s) \asymp (s^{\gamma_4}/f_2)^{-1}(s) \asymp s^{1/\gamma_4} (1/f_2^{1/\gamma_4}) \# (s^{1/\gamma_4}) \asymp s^{1/\gamma_4} f_2^{1/\gamma_4} (s^{1/\gamma_4}) \]

and

\[ (F'/f)^{-1}(s) \asymp (s^{\gamma_4-1}/f_2)^{-1}(s) \asymp s^{1/(\gamma_4-1)} (1/f_2^{1/(\gamma_4-1)}) \# (s^{1/(\gamma_4-1)}) \asymp s^{1/(\gamma_4-1)} f_2^{1/(\gamma_4-1)} (s^{1/(\gamma_4-1)}). \]

Using this, volume doubling property and the fact that \( t/r \geq c_2^{-1} \) for \( 2c_2^2 \Phi^{-1}(t) < r \leq 2c_2^2 \Phi^{-1}(t)^{1+\theta_2} \) and \( t \geq 1 \), we can apply Corollary 6.2(2) to obtain the first claim.

For \( (6.3) \), it is enough to show that \( f_2((t/r)^{1/(\gamma_4-1)}) \asymp f_2(t) \) for \( 2c_2^2 \Phi^{-1}(t) < r \leq 2c_2^2 \Phi^{-1}(t)^{1+\theta_2} \) and \( t \geq 1 \). Using that \( f_2 \) is monotone and \( (6.4) \), we see that

\[ f_2(c_2 t^{1-(\varepsilon_2)/(\gamma_4-1)}) \leq f_2((t/r)^{1/(\gamma_4-1)}) \leq f_2(c_1 t^{1-(1-\varepsilon_1)/(\gamma_4-1)}) \]

or

\[ f_2(c_2 t^{1-(\varepsilon_2)/(\gamma_4-1)}) \geq f_2((t/r)^{1/(\gamma_4-1)}) \geq f_2(c_1 t^{1-(1-\varepsilon_1)/(\gamma_4-1)}) \]

By Lemma A.2 and the fact that \( f_2 \) is slowly varying at \( \infty \), we have \( f_2(c_3 s^{\varepsilon_4}) \asymp f_2(s) \) for \( s > 1 \). Thus, the above inequalities give that \( f_2((t/r)^{1/(\gamma_4-1)}) \asymp f_2(t) \) for \( t \geq 1 \).

\( \square \)

**Example 6.5.** Suppose \( F \) is differentiable function satisfying \( F(s) \asymp s F'(s) \) and \( F(s) 1_{\{s < 1\}} \asymp s^{\gamma}(\log s)^{\kappa} 1_{\{s < 1\}} \) for \( \gamma > 1 \) and \( \kappa \in \mathbb{R} \). Suppose further that \( \psi : (0, \infty) \to (0, \infty) \) is a non-decreasing function which satisfies \( L(\beta_1, C_L) \) and \( U(\beta_2, C_U) \). Define \( f_{a,\beta}(s) := (\log s)^{1-\alpha}(\log \log s)^{-\beta} \) and \( D := \{(a, b) \in \mathbb{R}^2 : a > 1, b \in \mathbb{R}\} \cup \{(1, b) \in \mathbb{R}^2 : b > 1\} \). Then, we observe that for \( (a, \beta) \in D \), \( \ell_{a,\beta}(s) := sf_{a,\beta}'(s) \asymp (\log s)^{1-\alpha}(\log \log s)^{-\beta} \). In particular, \( \ell_{a,\beta} \in R_0^\infty \) and \( f_{a,\beta}(s) = \int_{0}^{s} \ell_{a,\beta}(u) u^{-1} du \). Assume that for \( (a, \beta) \in D \)

\[ \psi(\lambda) \asymp F(\lambda)(\log \frac{1}{\lambda})^{-\alpha}(\log \frac{1}{\lambda})^{-\beta}, \quad 0 < \lambda < 2^{-4}. \]
Then, \( f_{\alpha - \kappa, \beta} \) satisfies (6.1) for all \( \eta \in \mathbb{R} \setminus \{0\} \). Moreover, there exist \( T = T(\alpha - \kappa, \beta) \leq 2^{-4} \) such that for \( s \leq T \), \( f_{\alpha - \kappa, \beta} \) is monotone and satisfies \((f_{\alpha - \kappa, \beta})(s^\gamma) \prec (f_{\alpha - \kappa, \beta})(s)\). Thus, by the above observation and (6.2), we have the following heat kernel estimates for \( t < T \):

\[
p(t, x, y) \prec \frac{1}{V(x, t^{1/\gamma}(f_{\alpha - \kappa, \beta})(t)^{1/\gamma})} \wedge \left( \frac{t}{V(x, d(x, y))\psi(d(x, y)) + \frac{1}{V(x, t^{1/\gamma}(f_{\alpha - \kappa, \beta})(t)^{1/\gamma}} \exp\left(-a_1 \left( \frac{d(x,y)^\gamma}{(f_{\alpha - \kappa, \beta})(t)^{1/(\gamma - 1)}} \right)^{1/(\gamma - 1)} \right) \right).
\]

**Example 6.6.** Suppose \( F \) is differentiable function satisfying \( F(s) \prec sF'(s) \) and \( F(s)1_{\{s > 1\}} \prec s^\gamma (\log s)^\kappa 1_{\{s > 1\}} \) for \( \gamma > 1 \) and \( \kappa \in \mathbb{R} \). Suppose further that \( \psi : (0, \infty) \to (0, \infty) \) is a non-decreasing function which satisfies (2.19), \( L(\beta_1, C_L) \), \( U(\beta_2, C_U) \) and \( \psi(r)1_{\{r > 1\}} \prec F(r)(\log r)^\beta 1_{\{r > 16\}} \) for \( \beta \in \mathbb{R} \). Let \( \ell(s) = (\log s)^{-\beta} \). Then for \( \beta \leq 1 \), \( \int_1^\infty \frac{\ell(s)}{s}ds = \infty \). For \( s > 16 \), let

\[
f(s) = \begin{cases} 
\frac{1}{1 - \beta}(\log s)^{1 - \beta} & \text{if } \beta < 1, \\
\log \log s & \text{if } \beta = 1.
\end{cases}
\]

Then, \( f \in \Pi^{\infty} \) and \( f(s)/(\log s)^\kappa \) satisfies (6.1) for all \( \eta \in \mathbb{R} \setminus \{0\} \). Moreover, there exists \( T = T(\beta, \kappa) \geq 16 \) such that for \( s \geq T \), \( f(s)/(\log s)^\kappa \) is monotone and \( f(s)/(\log s)^\kappa \prec f(s^\gamma)/(\log s^\gamma)^\kappa \). Thus, by (6.3), we have the following heat kernel estimates for \( t \geq T \):

(i) If \( \beta < 1 \):

\[
p(t, x, y) \prec \frac{1}{V(x, t^{1/\gamma'}(\log t)^{(1 - \beta - \kappa)/\gamma'})} \wedge \left( \frac{t}{V(x, d(x, y))d(x, y)^{\gamma'}(\log(1 + d(x, y)))^{\beta + \kappa}} + \frac{1}{V(x, t^{1/\gamma'}(\log t)^{(1 - \beta - \kappa)/\gamma'})} \exp\left(-a_2 \left( \frac{d(x,y)^{\gamma'}}{t(\log t)^{1 - \beta - \kappa}} \right)^{\frac{1}{\gamma - 1}} \right) \right),
\]

(ii) If \( \beta = 1 \):

\[
p(t, x, y) \prec \frac{1}{V(x, t^{1/\gamma'}(\log t)^{-\kappa/\gamma'}(\log \log t)^{1/\gamma'})} \wedge \left( \frac{t}{V(x, d(x, y))d(x, y)^{\gamma'}(\log(1 + d(x, y)))^{1 + \kappa}} + \frac{1}{V(x, t^{1/\gamma'}(\log t)^{-\kappa/\gamma'}(\log \log t)^{1/\gamma'})} \exp\left(-a_3 \left( \frac{d(x,y)^{\gamma'}}{t(\log t)^{-\kappa}} \right)^{\frac{1}{\gamma - 1}} \right) \right).
\]

**Example 6.7.** Recall that \( \gamma_1, \gamma_2 > 1 \) are the constants in (4.1). Suppose \( F \) is differentiable function such that there exists \( c > 0 \) satisfying \( \gamma_1 F(s) \leq sF'(s) \leq cF(s) \) for all \( s > 0 \). Let \( T > 0 \) and \( \psi(r) = r^\alpha 1_{\{r \leq 1\}} + r^\beta 1_{\{r > 1\}} \), where \( \alpha < \gamma_1 \leq \gamma_2 < \beta \). Then, by Corollary 2.19 we see that for \( t \leq T \),

\[
p(t, x, y) \prec \frac{1}{V(x, t^{1/\alpha})} \wedge \frac{t}{V(x, d(x, y))\psi(d(x, y))}, \tag{6.5}
\]

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Indeed, for \(d(x, y) < 1\), \([6.5]\) follows from Theorem \([4.6]\). If \(d(x, y) \geq 1\), then \(\frac{t}{V(x, d(x, y))\psi(d(x, y))}\) dominates the upper bound of off-diagonal term in \([2.10]\).

On the other hand, by the condition \(\gamma_2 < \beta\), we have \(\int_0^\infty \frac{dF(s)}{V(s)} \leq c + c\int_1^\infty \frac{s^{-\gamma_2}}{s^{\alpha+\beta}} ds < \infty\). Thus, for \(r > 1\), \(\Phi(r)\) defined in \([2.20]\) is comparable to \(F(r)\) and \(\Phi(r)/r \propto F(r)/r \propto F'(r)\). Now, by the same argument as in the proof of Corollary \([6.2]\), we see that for \(t > T\),

\[
p(t, x, y) \approx \frac{1}{V(x, F^{-1}(t))} \wedge \left(\frac{t}{V(x, d(x, y))d(x, y)^\beta} + \frac{1}{V(x, F^{-1}(t))} \exp \left(-a_5 \frac{d(x, y)}{F'(t/d(x, y))}\right)\right).
\]

\[\text{A Appendix}\]

**Remark A.1.** Suppose \(g : (0, \infty) \rightarrow (0, \infty)\) is non-decreasing. If \(g\) satisfies \(L_\alpha(\beta, c)\), then \(g\) satisfies \(L_b(\beta, c(ab^{-1})^\beta)\) for any \(b > a\). Indeed, for \(a \leq a \leq R \leq b\),

\[
g(R) \geq g(a) \geq c \left(\frac{a}{R}\right)^\beta g(r) \geq c \left(\frac{a}{b}\right)^\beta \left(\frac{R}{r}\right)^\beta g(r)
\]

and for \(a \leq r \leq R \leq b\),

\[
g(R) \geq g(r) \geq c \left(\frac{a}{b}\right)^\beta \left(\frac{R}{r}\right)^\beta g(r).
\]

Similarly, if \(g\) satisfies \(L^a(\beta, c)\), then \(g\) satisfies \(L^b(\beta, c(a^{-1}b)^\beta)\) for \(b < a\).

**Lemma A.2.** Let \(g : (0, \infty) \rightarrow (0, \infty)\) be a non-decreasing function with \(g(\infty) = \infty\).

1. If \(g\) satisfies \(L_\alpha(\beta, c)\) (resp. \(U_\alpha(\beta, C)\)), then \(g^{-1}\) satisfies \(U_{g(a)}(1/\beta, c^{-1/\beta})\) (resp. \(L_{g(a)}(1/\beta, C^{-1/\beta})\)).

2. If \(g\) satisfies \(L^a(\beta, c)\) (resp. \(U^a(\beta, C)\)), then \(g^{-1}\) satisfies \(U^{g(a)}(1/\beta, c^{-1/\beta})\) (resp. \(L^{g(a)}(1/\beta, C^{-1/\beta})\)).

**Proof of Remark 2.11.** Assume that the condition \(HK(\Phi, C\Phi)\) holds. If \(d(x, y) \leq \eta \Phi^{-1}(t)\), by \(VD(d_2)\) and \(L(\alpha_1, c_L, \Phi)\) we have

\[
\frac{1}{V(x, \Phi^{-1}(t))} \leq \frac{c_1}{V(x, \eta \Phi^{-1}(t))} \leq \frac{c_1}{V(x, d(x, y))} \leq \frac{c_2 t}{c_1 t} \leq \frac{c_2 t}{V(x, d(x, y))\Phi(\eta^{-1}d(x, y))} \leq \frac{c_2 t}{V(x, d(x, y))\Phi(d(x, y))}.
\]

Also, if \(d(x, y) \geq \eta \Phi^{-1}(t)\), using \(U(\alpha_2, c_U, \Phi)\), we obtain that

\[
\frac{1}{V(x, \Phi^{-1}(t))} \geq \frac{c_3}{V(x, \eta \Phi^{-1}(t))} \geq \frac{c_3}{V(x, d(x, y))} \geq \frac{c_4 t}{c_3 t} \geq \frac{c_4 t}{V(x, d(x, y))\Phi(\eta^{-1}d(x, y))} \geq \frac{c_4 t}{V(x, d(x, y))\Phi(d(x, y))}.
\]

By above two inequalities, we have that lower bounds in \([2.9]\) and \([2.11]\) are equivalent. For the upper bound, it suffices to verify the existence of constant \(c > 0\) satisfying

\[
c^{-1} \frac{t}{V(x, d(x, y))\Phi(d(x, y))} \leq G(a_0, t, x, d(x, y)) \leq c \frac{t}{V(x, d(x, y))\Phi(d(x, y))}
\]
for all $t > 0$ and $x, y \in M$ with $d(x, y) \geq 2c_U \Phi^{-1}(t)$. Indeed, when $d(x, y) \leq 2c_U \Phi^{-1}(t)$, following the calculations in \[A.1\] we have

$$\frac{c_5}{V(x, \Phi^{-1}(t))} \leq \frac{t}{2V(x, d(x, y))\Phi(d(x, y))} \leq G(a_0, t, x, d(x, y)).$$

Note that the second inequality immediately follows from the definition of $G$. Now we observe that from $\Phi_1(r, t) = \sup_{s > 0} \left[ \frac{r}{s} - \frac{t}{\Phi(s)} \right]$ and $r \geq 2c_U \Phi^{-1}(t)$,

$$\Phi_1(r, t) \geq \frac{r}{\Phi^{-1}(t)} - \frac{t}{\Phi(\Phi^{-1}(t))} \geq \frac{r}{2\Phi^{-1}(t)},$$

where we have used \[B.18\] for the second inequality. Thus,

$$\frac{1}{V(x, \Phi^{-1}(t))} \exp(-a_0 \Phi_1(d(x, y), t)) \leq \frac{c_6}{V(x, \Phi^{-1}(t))} \exp(-a_0 \frac{d(x, y)}{2\Phi^{-1}(t)}) \leq \frac{c_7}{V(x, \Phi^{-1}(t))} (\Phi^{-1}(t))^{d_2 + \beta_2} \leq \frac{c_8}{V(x, \Phi^{-1}(t))} \Phi(\Phi^{-1}(t)) \Phi(d(x, y)) = \frac{c_{st}}{V(x, d(x, y))\Phi(d(x, y))},$$

where we have used $VD(d_2)$ and $U(\beta_2, C_U, \Phi)$. This finishes the proof. \hfill \Box

Suppose that the function $\Phi$ satisfies $L_a(\delta, \tilde{C}_L)$ for some $a \in (0, \infty]$, $\tilde{C}_L > 0$ and $\delta > 1$. Now, define

$$\mathcal{K}(r) := \sup_{0 < s \leq r} \frac{\Phi(s)}{s}.$$  

Then, for any $R_0 > 0$, letting $c_1 = \tilde{C}_L^{-1}(aR_0^{-1} \wedge 1) - \delta \geq 1$ we have for any $r \in (0, R_0]$,

$$\frac{\Phi(r)}{r} \leq \mathcal{K}(r) \leq c_1 \frac{\Phi(r)}{r}, \quad (A.2)$$

(c.f. \[2\] Lemma 2.5). Note that if $a = \infty$, \[A.2\] holds for every $r > 0$. Let $\mathcal{K}^{-1}$ be the generalized inverse of non-decreasing function $\mathcal{K}$.

The following lemma yields that \[2\] Theorem 1.4 is the special case of Corollary \[2.24\].

**Lemma A.3.** Suppose $\Phi$ is non-decreasing function satisfying $L(\alpha_1, c_L)$, $U(\alpha_2, c_U)$ and $L_a(\delta, \tilde{C}_L)$ for $\delta > 1$. Let $T \in (0, \infty]$. Then, there exists a constant $c > 1$ such that for any $t \in (0, T]$ and $r \geq 2c_U \Phi^{-1}(t)$,

$$c^{-1} \Phi_1(r, t) \leq \frac{r}{\mathcal{K}^{-1}(t/r)} \leq c \Phi_1(r, t). \quad (A.3)$$

Moreover, if $L(\delta, \tilde{C}_L)$ holds, then \[A.3\] holds for any $t \in (0, \infty)$ and $r \geq 2c_U \Phi^{-1}(t)$.

**Proof.** Without loss of generality we may and do assume $a = \Phi^{-1}(T)$. Note that $\alpha_2 > 1$. Let $R_0 := \Phi^{-1}(T)$ and $c_1 = \tilde{C}_L^{-1}$ so that $L_{R_0}(\delta, c_1^{-1}, \Phi)$ and \[A.2\] hold. Denote $\varepsilon := \frac{1}{\alpha_2 - 1}$. Since $r \geq 2c_U \Phi^{-1}(t)$, we have

$$c_2 c_U \Phi^{-1}(t) \frac{1 + \varepsilon}{r^\varepsilon} \leq \Phi^{-1}(t) \leq R_0.$$

It follows from \[A.2\], Lemma \[3.7\] and $U(\alpha_2, c_U, \Phi)$ that

$$\mathcal{K} \left( c_2 c_U \Phi^{-1}(t) \frac{1 + \varepsilon}{r^\varepsilon} \right) \geq c_2^{-2 \varepsilon} \Phi^{-1}(t) c_2 \frac{1 + \varepsilon}{r^\varepsilon} \Phi \left( \Phi^{-1}(t) c_2 \frac{1 + \varepsilon}{r^\varepsilon} \right).$$

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\[
\geq c_U^{-1-2\varepsilon} \frac{r^\varepsilon t}{\Phi^{-1}(t)^{1+\varepsilon}} \frac{\Phi^{-1}(t) c_U^2 \Phi^{-1}(t)^\varepsilon}{\Phi(\Phi^{-1}(t))} \\
\geq c_U^{-2-2\varepsilon} \frac{t}{r} \frac{r^{1+\varepsilon}}{\Phi^{-1}(t)^{1+\varepsilon}} \left( \frac{\Phi^{-1}(t)^\varepsilon}{r^\varepsilon} \right)^{\alpha_2} = \frac{t}{r}.
\]
Thus,
\[
\rho := \mathcal{K}^{-1} \left( \frac{t}{r} \right) \leq c_U^2 \frac{\Phi^{-1}(t)^{1+\varepsilon}}{r^\varepsilon} \leq 2^{-\varepsilon} \Phi^{-1}(t) \leq R_0.
\]
By (A.2), \( \mathcal{K} \) satisfies \( U_{R_0}(\alpha_2 - 1, c_1 c_U) \) and \( L_{R_0}(\delta - 1, c_1^{-1} C_L) \). Thus, using Lemma 3.7 we have
\[
(c_1 c_U)^{-1} \frac{t}{r} \leq \mathcal{K}(\rho) \leq c_1 c_U \frac{t}{r}.
\]
(A.4)
Using (A.4) and (A.2), we have
\[
(c_1 c_U)^{-1} \frac{t}{r} \leq \mathcal{K}(\rho) \leq c_1 \frac{\Phi(\rho)}{\rho}.
\]
Then, letting \( c_2 = c_1^2 c_U \), the above inequality and (3.24) imply that there exists \( c_3 > 0 \) such that
\[
c_3 \Phi_1(r, t) \geq \Phi_1(2c_2r, t) \geq \frac{2c_2r}{\rho} - \frac{t}{\Phi(\rho)} \geq \frac{r}{\rho} \left( 2c_2 - \frac{t}{r} \frac{\rho}{\Phi(\rho)} \right) \geq \frac{c_2r}{\rho} = \frac{c_2r}{\mathcal{K}^{-1}(t/r)}.
\]
This proves the second inequality in (A.3). For the first one, we take a \( s > 0 \) such that
\[
0 \leq \frac{r}{s} - \frac{t}{\Phi(s)} \leq \Phi_1(r, t) \leq 2 \left( \frac{r}{s} - \frac{t}{\Phi(s)} \right).
\]
(A.5)
Since \( \Phi_1(r, t) \geq 0 \), we have \( \Phi(s)/s \geq t/r \). Using this, (A.2) and (A.4) we have
\[
\frac{\Phi(\rho)}{\rho} \leq \mathcal{K}(\rho) \leq c_1 c_U \frac{t}{r} \leq c_1 c_U \frac{\Phi(s)}{s}.
\]
(A.6)
Thus, if \( s < \rho \leq R_0 \), using \( L_{R_0}(\delta, c_1^{-1}, \Phi) \) and (A.6)
\[
c_1^{-1} \frac{\Phi(\rho)}{\rho} \leq \mathcal{K}(\rho) \leq c_1 c_U \frac{t}{r} \leq c_1 c_U \frac{\Phi(s)}{s} \leq c_1 c_U.
\]
Thus, we conclude that there is \( c_4 > 0 \) such that \( s > c_4 \rho \). Using this and (A.5), we have
\[
\Phi_1(r, t) \leq 2 \frac{r}{s} \leq 2c_4^{-1} \frac{r}{\rho} = 2c_4^{-1} \frac{r}{\mathcal{K}^{-1}(t/r)}.
\]
When \( L(\delta, C_L, \Phi) \) holds, we may take \( R_0 = \infty \) and \( c_1 = C_L^{-1} \). Then, the proof is the same with the above since (A.2) holds for all \( r > 0 \) and (3.24) holds for all \( t > 0 \) and \( r \geq 2c_2^2 \Phi^{-1}(t) \). This completes the proof. 

\( \square \)

Lemma A.4. Let \( h : (0, \infty) \to (0, \infty) \) be a monotone function. Suppose that there exist \( k > 1 \) and \( c_1 > 1 \) such that \( c_1^{-1} h(r) \leq h(r^k) \leq c_1 h(r) \) for all \( r < 1 \) (resp. \( r > 1 \)). Then, for any \( m > 0 \), there exists \( c_2 = c_2(k, c_1, m) > 1 \) such that
\[
c_2^{-1} h(r) \leq h(r^m) \leq c_2 h(r) \quad \text{for all } r < 1 \text{ (resp. } r > 1 \).
Proof. Since proofs for other cases are all similar, we only prove the case that $h$ is non-decreasing and the comparability condition holds for $r < 1$. When $1 \leq m \leq k$, using the comparability condition, clearly $h(r^m) \leq h(r) \leq c_1 h(r^k) \leq c_1 h(r^m)$.

If $m < 1$, we let $n \in \mathbb{N}$ be a constant satisfying $mk^n \geq 1$. Then, we get $h(r) \leq h(r^m) \leq c_1 h(r^{mk^n}) \leq \cdots \leq c^n_1 h(r)$.

For $m > k$, let $l \in \mathbb{N}$ be a constant satisfying $k^l \geq m$. Then, $h(r^m) \leq h(r) \leq c_1 h(r^k) \leq c_1^2 h(r^{mk^l}) \leq \cdots \leq c_1^l h(r^m)$. \hfill \Box

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Joohak Bae
Samsung Fire & Marine Insurance, Seoul 06620, Republic of Korea
E-mail: juhak88@snu.ac.kr

Jaehoon Kang
Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany
E-mail: jkang@math.uni-bielefeld.de

Panki Kim
Department of Mathematical Sciences and Research Institute of Mathematics,
Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu Seoul 08826, Republic of Korea
E-mail: pkim@snu.ac.kr

Jaehun Lee
Department of Mathematical Sciences,
Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu Seoul 08826, Republic of Korea
E-mail: hun618@snu.ac.kr