ON THE IRREDUCIBILITY OF THE SPACE OF GENUS ZERO STABLE LOG MAPS TO WONDERFUL COMPACTIFICATIONS

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Abstract. In this paper, we prove the moduli spaces of genus zero stable log maps to good wonderful compactifications of sober spherical varieties are irreducible and unirational.

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1. Introduction

Throughout this paper, we work over an algebraically closed field of characteristic zero, denoted by $\mathbb{C}$.

1.1. Background and main result. Let $X$ be a fine and saturated log scheme. A stable log map to $X$ over a log scheme $S$ is given by the following data:

$$(C \to S, f : C \to X, p_1, \cdots, p_n)$$

where

1. $C \to S$ is a family of log curve with markings $p_1, \cdots, p_n$, see [Kat96, Ols07];
2. $f : C \to X$ is a morphism of log schemes;

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the underlying map \( f : C \to X \) obtained by removing log structures of \( f \) is stable in the usual sense.

The arrows between stable log maps are defined by cartesian diagram in the category of log schemes. We thus obtain \( \mathcal{M}_\Gamma(X) \) the category of stable log maps with discrete data \( \Gamma \) fibered over the category of log schemes. Here \( \Gamma = (g, n, \beta, \{c_i\}_{i=1}^n) \) consists of the genus \( g \) of the source curve, the curve class \( \beta \) of the underlying stable map, the number \( n \) of marked points, and the contact orders \( c_i \) for the \( i \)-th marked point. The contact order describes the tangency condition of the stable log map \( f \) with the boundary of \( X \) along the marked point.

The theory of stable log maps has been established in a sequence of papers [GS13, Che14, AC14, ACMW14, Wis]. The fibered category \( \mathcal{M}_\Gamma(X) \) is shown to be represented by a Deligne-Mumford stack with the natural fine and saturated log structure, and is proper when the underlying scheme \( X \) is projective.

In case the target \( X \) is log homogeneous and \( g = 0 \), the stack \( \mathcal{M}_\Gamma(X) \) is log smooth and receives its most beautiful geometry. An example when \( X \) is given by a toric variety with its toric boundary is provided in [CS13], where the stack \( \mathcal{M}_\Gamma(X) \) of two-pointed genus zero stable log maps to \( X \) is shown to have its coarse moduli given by the Chow quotient of \( X \). The goal of the current paper is to provide the following interesting properties in the log homogeneous situation:

**Theorem 1.1.** Let \( G \) be a semi-simple linear algebraic group over \( \mathbb{C} \), and \( H \subset G \) be a sober spherical subgroup. Let \( X \) be a log smooth variety associated to the wonderful compactification of \( G/H \), such that \( X \) is good, see Definition 2.9. For any discrete data \( \Gamma = (g = 0, n, \beta, \{c_i\}_{i=1}^n) \), the coarse moduli space associated to the underlying stack of \( \mathcal{M}_\Gamma(X) \) is irreducible and unirational.

The irreducibility is proved in Proposition 4.3, and the unirationality follows from Proposition 4.8.

**Remark 1.2.** In fact, the above theorem is proved for the curve class \( \beta \) in the discrete data \( \Gamma \) satisfying Assumption 2.7. Note that when \( X \) is good, any curve class satisfies Assumption 2.7. The condition \( X \) is good contains many interesting cases, including wonderful compactifications of semisimple groups, and more generally spherical varieties of minimal rank [Bri07, Res10].

When \( H \) is a parabolic subgroup, \( X \) is a homogeneous variety with the trivial log structure. In this case, the stack of stable log maps \( \mathcal{M}_\Gamma(X) \) is the same as the stack of usual stable maps to \( X \), and is proved to be irreducible and rational by Kim-Pandharipande [KP01].
by considering the maximal degeneration via a general torus action. The above result may be viewed as a generalization of their result in the logarithmic setting.

In this paper, we again consider a general torus action on the stack $\mathcal{M}_\Gamma(X)$ induced via its action on the target $X$. In Section 2, we study the maximal degeneration of underlying stable maps in $\underline{X}$ under the given torus action. This is quite similar to the situation of [KP01].

However, not every underlying stable log map can be lifted to stable log maps, since the canonical morphism $\mathcal{M}_\Gamma(X) \to \mathcal{M}_\Gamma(X)$ to the stack of underlying stable maps is not surjective in general. This is a major technique difficulty of stable log maps. In Section 3, we show that the maximal degenerate underlying stable map in question admits a unique logarithmic lifting. Assumption 2.7 is crucial for the uniqueness of the lifting.

In Section 4, we provide the proof of Theorem 1.1 by investigating the maximal Bialynicki-Birula cell of $\mathcal{M}_\Gamma(X)$ under the given torus action. Note that the stack $\mathcal{M}_\Gamma(X)$ is only log smooth along the fixed locus, and could have toric singularities in general. We thus pass to an equivariant resolution of the Bialynicki-Birula cell. An analysis of the log structure along the fixed locus is given for the desingularization to be unirational.

**Remark 1.3.** A classification of possible curve classes and contact orders for stable log maps to good wonderful compactifications can be found in our previous paper [CZ14]. A further study of genus zero log Gromov-Witten invariants for wonderful compactifications will require a much more detailed analysis of all Bialynicki-Birula cells. We wish to carry this out in our future work.

1.2. **Notations.** All log structures in this paper are assumed to be fine and saturated. We refer to [Kat89] for the basics of logarithmic geometry that will be used in this paper. Capital letters such as $C, S, X, Y$, and $Z$ are reserved for log schemes. The corresponding underlying schemes are denoted by $\underline{C}, \underline{S}, \underline{X}, \underline{Y}$, and $\underline{Z}$ respectively.

Given a log scheme $X$, a stable log map to $X$ is usually denoted by $f : C/S \to X$ where $C \to S$ is the family of source log curves. When $\underline{S}$ is a geometric point with the trivial log structure, we will simply write $f : C \to X$. For each discrete data

\begin{equation}
\Gamma = (g = 0, n, \beta, \{c_i\}_{i=1}^n),
\end{equation}

let $\mathcal{M}_\Gamma(X)$ be the stack of $n$-marked, genus zero stable log maps with curve class $\beta \in H_2(X)$, and contact order $c_i$ along the $i$-th marking. Write $\underline{\Gamma} = (g = 0, n, \beta)$ by removing contact orders from $\Gamma$. 
Throughout this paper, we fix a simply connected semisimple linear algebraic group $G$ over $\mathbb{C}$, and a Borel subgroup $B \subset G$ with the maximal torus $T \subset B$. Denote by $\mathfrak{X}^* := \mathfrak{X}^*(T)$ and $\mathfrak{x}_* = \mathfrak{x}_*(T)$ the character and cocharacter groups of $T$ respectively.

Consider a sober spherical subgroup $H \subset G$. This means that $G/H$ is a spherical variety and $N_G(H)/H$ is finite. Let $\underline{X}$ be the wonderful compactification of $G/H$. Denote by $X$ the log smooth varieties associated to the pairs $(\underline{X}, \Delta = \underline{X} \smallsetminus G/H)$ with the natural $G$-action, see [CZ14, Proposition 5.4].

Let $\underline{Y}$ be the unique closed $G$-orbits of $\underline{X}$. Note that $\underline{Y} = G/P$ for some parabolic subgroup $P \subset G$. Denote by $Y \subset X$ the strict closed sub-log schemes with the underlying structure $\underline{Y}$.

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2. Specialization of underlying stable maps

2.1. **Torus invariant curves and divisors in $X$.** Throughout this paper, we fix a general one-dimensional torus $\lambda : \mathbb{G}_m \to T$ given by a regular coweight $\lambda \in \mathfrak{X}_*(T)$. Let $X^\lambda$ be the set of fixed points with the torus action given by $\lambda$. We then have

$$X^\lambda = X^T.$$ 

Note that $X^\lambda$ is a set of finite points with the reduced scheme structure. For any closed subscheme $Z \subset X^\lambda$, we introduce

$$X^+(Z) = \{ x \in X_{\text{reg}} \ | \ \lim_{t \to 0} \lambda(t) \cdot x \in Z \}$$

and

$$X^-(Z) = \{ x \in X_{\text{reg}} \ | \ \lim_{t \to \infty} \lambda(t) \cdot x \in Z \}.$$

By [Bri03, Lemma 2], there is a unique $B$-fixed point $x^- \in X^\lambda$ such that $X^-(x^-) \subset X$ is an open subscheme. We write $X^- := X^-(x^-)$ for short.

**Proposition 2.1.** The closed subset $X \smallsetminus X^-$ consists of irreducible components $D_1, \ldots, D_l$ such that

1. Each $D_i$ is a globally generated cartier divisor. Their linear equivalent class form a basis of the picard group of $X$. 

Any nef divisor on $X$ is given by a non-negative integral linear combination of $D_1, \ldots, D_l$.

Proof. See [Bri03, Theorem 1]. ♠

Note that each $D_j$ has a $B$-action. Let $x_j^- \in X^\lambda \cap D_j$ such that $D_j$ is the closure of $A_j := X^-(x_j^-)$. Denote by $P_j := B \cdot x_j^-$. Then $P_j$ is a $B$-invariant rational curve in $\mathcal{Y}$, intersecting $D_j$ at $x_j^-$.  

**Proposition 2.2.**  
(1) $P_i \cap D_j = \delta_{ij}$ for any $i, j$.  
(2) The curve classes of $P_1, \ldots, P_l$ generate the cone $\text{NE}(X)$, and form a basis of $N_1(X)_{\mathbb{Z}}$.

Proof. The first statement follows the same proof as in [CZ14, Proposition 5.17]. See [Bri03, Theorem 2] for the second statement. ♠

Note that $A_j$ is an open dense subscheme of $D_j$ for all $j$. Consider

\[(2.1.1)\quad U = X^- \cup A_1 \cup \cdots \cup A_r.\]

Then by Bialynicki-Birula decomposition, $U$ is an open subscheme of $X$ whose complement is of codimension at least 2.

### 2.2. A specialization of non-degenerate underlying stable maps.

We consider the following situation:

**Notation 2.3.** Let $\underline{f} : C \to \underline{X}$ be a genus zero usual stable map with discrete data $\Gamma$ induced by $\Gamma$ as in (1.2.1) with the following properties:

1. $C \cong \mathbb{P}^1$, and $f(C) \cap G/H \neq \emptyset$.  
2. $f(C) \subset U$, with $U$ given by (2.1.1).  
3. $\underline{f}(C)$ intersects the divisors $D_i$ transversally at distinct non-marked points.  
4. The images of all markings under $\underline{f}$ lie in $X^-$.  

In particular, the stable map $\underline{f}$ lifts to a unique stable log map $f : C \to X$.

Consider the usual stable map $\underline{f} : C \to \underline{X}$ as in Notation 2.3. We may assume that

\[(2.2.1)\quad \beta = \sum_{i=1}^{l} k_i[P_i],\]

with non-negative integers $k_i$ for all $i$.

Consider the general $\mathbb{G}_m$-action $\lambda$ as in the beginning of Section 2.1. Denote by $\underline{f}_t : \underline{C}_t \to \underline{X}$ the limit of the underlying stable map $\underline{f}$ when $t \to \infty$ under the $\mathbb{G}_m$-action.
Notation 2.4. To construct such \( f_0 \), we introduce a underlying pre-stable map \( \tilde{F} : C_1 \to X \) as follows:

1. \( C_1 = C \cup \bigcup_{i=1}^{l} \bigcup_{j=1}^{k_i} P_{i,j} \), where \( P_{i,j} \) is attached to \( C \) at \( x_{i,j} \);
2. All marked points on \( C_1 \) is given by the markings on \( C \);
3. \( \tilde{F}(C) = x^- \), and \( \tilde{F}|_{P_{i,j}} : P_{i,j} \to P_i \) where \( \tilde{F}[*_{P_{i,j}}] = [P_i] \).

Lemma 2.5. Let \( f \) be the stable map as in Notation 2.3. Then the limit underlying stable map \( f_0 \) is given by the stabilization of \( \tilde{F} \).

Proof. The proof is similar to that of [KP01, Proposition 2]. For completeness, we record it below. Consider the family of underlying stable maps

\[ h : \mathbb{G}_m \times C \to X \]

given by the \( \mathbb{G}_m \)-action. By the local structure theorem of wonderful compactifications, see for example [CZ14, Theorem 5.5], we could extend \( h \) to a morphism of usual schemes:

\[ h : \mathbb{A}^1 \times (\overline{C} \setminus \{x_{i,j}\}) \to X. \]

Let \( S \to \mathbb{A}^1 \times C \) be a suitable blow-up along the isolated nonsingular points \( \{0 \times x_{i,j}\} \), we obtain a morphism

\[ h' : S \to X. \]

Denote by \( E_{i,j} \) the exceptional divisor over \( x_{i,j} \). Since the limits of \( f(x_{i,j}) \) at \( t \to \infty \) under the fixed \( \mathbb{G}_m \)-action is \( x^- \), the image \( h_0(E_{i,j}) \) contains both \( x^- \) and \( x^- \). Note that the torus action does not change the cross-ratio of the set of points \( \{x_{i,j}\} \). Since \( f \) intersects transversally with \( D_i \) at \( x_{i,j} \), we conclude that \( h'(C_{i,j}) = P_i \). Thus, after further birational transformations of \( S \), we may assume that firstly \( S \to \mathbb{A}^1 \) is a family of nodal curves; and secondly each \( C_{i,j} \) is a irreducible rational component. This proves the statement.

We then summarize our discussion as follows:

Proposition 2.6. Let \( f \) be the underlying stable map as in Notation 2.3. Then the limiting underlying stable map \( f_0 \) is given by one of the following situations:

1. \( \tilde{F} \) is stable, and \( f_0 = \tilde{F} \); or
2. \( \sum_i k_i = 1, n \leq 1 \), and \( f_0 \) is given by \( P^1 \to P_i \) with the marking mapped to \( x^- \); or
3. \( \sum_i k_i = 2, n = 0 \), and \( f_0 : P^1 \cup P^1 \to P_i \cup P_j \) for some \( i,j \), with the unique node mapped to \( x^- \).
Proof. This follows from Lemma 2.5, and taking the stabilization of $\tilde{F}$.

2.3. Curves supported on the center. In general, the torus fixed points $x_j^-$ for $j = 1, \cdots, l$ as in Section 2.1 do not necessarily lie in the center $\overline{Y}$. Thus, the limit underlying stable map $f_0$ does not necessarily factors through the center $\overline{Y}$. In this case, the analysis of the possible stable log maps over $f_0$ becomes quite complicated. In fact, the uniqueness of Proposition 3.1 could fail when $f_0$ does not factor through $\overline{Y}$. We therefore introduce the following condition for the curve class $\beta$ as in (2.2.1):

Assumption 2.7. $k_i = 0$ for each $x_i^- \not\in \overline{Y}$.

Lemma 2.8. Under Assumption 2.7, the underlying stable map $f_0$ factors through the center $\overline{Y}$.

Proof. This follows from the fact that the curve $P_i = B \cdot x_i^-$ lies in $\overline{Y}$ if and only if $x_i^- \in \overline{Y}$.

We recall that the following conception introduced in [CZ14]:

Definition 2.9. $\overline{X}$ is called good if there exists a regular coweight $\lambda: \mathbb{G}_m \to T$ such that $x_i^- \in \overline{Y}$ for all $i = 1, \cdots, l$.

Note that any curve class satisfies Assumption 2.7 if $\overline{X}$ is good. Assumption 2.7 will be assumed throughout the rest of this paper.

3. Specialization of stable log maps

3.1. Uniqueness of the lifting. We have analyzed the specialization of the underlying stable map $\tilde{f}$ in Notation 2.3 under the given $\mathbb{G}_m$-action. Note that the underlying stable map $f$ uniquely determines a stable log map $f$. Thus by the properness of $\mathcal{M}_F(X)$, such limit as stable log maps exists, and is unique. But for our purposes, we need to understand all possible liftings over the limiting underlying stable map in Notation 2.4 as a stable log maps. We first show that

Proposition 3.1. Let $f_0$ be a underlying stable map given by the stabilization of $\tilde{F}$ in Notation 2.4. Under Assumption 2.7, there exists up to a unique isomorphism at most one minimal stable log map $f_0$, whose underlying stable map is given by $f_0$.

Assume such lifting $f_0$ is given. We first calculate the possible characteristic monoid $\mathcal{M}_S$ with the given underlying stable map $f_0$. Denote
by \( \overline{M} := \overline{M}_{Y, y} \) for any point \( y \in Y \subset X \). Recall from [CZ14, Proposition 5.8] that there is a global morphism from the globally constant sheaf of monoids:

\[
\gamma : \overline{M} \to \overline{M}_X
\]

such that

1. \( \gamma \) lifts to a chart of \( M_X \) Zariski locally on \( X \);
2. the restriction \( \gamma|_Y : \overline{M} \to \overline{M}_Y \) is an isomorphism of sheaves of monoids.

The minimal monoid is defined in the Deligne-Faltings case in [AC14, Section 4.1] and for Zariski log structures in [GS13, Construction 1.16]. In what follows, we will mimic [AC14, Section 4.1], and reformulate the minimal monoid in a slightly different manner for the convenience of our argument. It should be straight forward to verify that the following is equivalent to the definitions in [AC14, GS13] in our particular case.

**Construction 3.2.** Denote by \( \Phi \) the dual graph of the underlying curve \( C_0 \). This means that \( \Phi \) is a connected graph with the set of vertices \( V(\Phi) \) given by the irreducible components of \( C_0 \) and the set of edges \( E(\Phi) \) given by the nodes of \( C_0 \). For each \( v \in V(\Phi) \) and \( e \in E(\Phi) \), denote by \( Z_v \) and \( p_v \) the corresponding irreducible component and node respectively. Denote by \( \Phi \) the marked graph obtained by decorating \( \Phi \) with the following data:

1. For each \( v \in V(\Phi) \) we associate the monoid \( \overline{M}_v := \overline{M} \).
2. For each \( e \in E(\Phi) \) we associate the free monoid \( N_e \cong \mathbb{N} \). We use \( e \) to denote the generator of \( N_e \).
3. We fix an orientation on the graph as follows. If \( f_0 = \tilde{F} \), then each edge \( e \) is oriented from the teeth \( \mathbb{P}_{ij} \) to the handle \( C \). Otherwise, \( C_0 \) has at most two components, in which case we fix an arbitrary orientation of the unique edge.
4. For each edge \( e \) orienting from \( v_1 \) to \( v_2 \), denote by \( c_e \in (\overline{M})^{gp} \) the \( \beta \)-contact order of the one cycle \( (f_0)_* Z_{v_1} \) as in [CZ14, Definition 3.4]. In particular, \( c_e \) defines a group morphism \( \overline{M}^{gp} \to \mathbb{Z} \).

Now for each edge \( e \) orienting from \( v_1 \) to \( v_2 \), and each element \( \delta \in \overline{M}^{gp} \), we introduce the relation

\[
\delta_{v_1} + c_e(\delta) \cdot e = \delta_{v_2}
\]

where \( \delta_{v_i} \) denotes the element in \( \overline{M}_{v_i}^{gp} \) given by \( \delta \). Denote by \( \overline{M}(\Phi)^{gp} \) the lattice given by \( \sum_e N_e^{gp} + \sum_v \overline{M}_v^{gp} \) modulo the relations (3.1.2) for
all $e$. We thus have a natural morphism
\[(3.1.3) \quad \psi : \sum_e \mathbb{N}_e \oplus \sum_v \mathcal{M}_v \to \overline{\mathcal{M}}(\Phi).\]

Denote by $\overline{\mathcal{M}}(\Phi)_Q^+$ the rational cone generated by the image of $\phi$ in $\overline{\mathcal{M}}(\Phi)_Q := \overline{\mathcal{M}}(\Phi) \otimes \mathbb{Z} \mathbb{Q}$. Then we write
\[(3.1.4) \quad \overline{\mathcal{M}}(\Phi) := \overline{\mathcal{M}}(\Phi)_Q^+ \cap \overline{\mathcal{M}}(\Phi)_Q^+.\]

**Lemma 3.3.** Notations and assumptions as above, assume $f_0$ lifts to a minimal stable log map $f_0$. Then we have

1. $\overline{\mathcal{M}}_S = \overline{\mathcal{M}}(\Phi)$, and
2. a natural splitting $\overline{\mathcal{M}}(\Phi)^{op} = \overline{\mathcal{M}}^{op} \oplus \sum_e \mathbb{N}^{op}_e$.

**Proof.** Notice that the possibilities of $f_0$ are listed in Proposition 2.6. For an edge $e$ orienting from $v_1$ to $v_2$, there is no marking on $Z_{v_1}$. Thus, the contact order of the node $p_e$ is given by $c_e$. Now the first statement follows from the definition of minimality in [AC14, Section 4] or [GS13, Construction 1.16].

Consider the second statement. We first assume that $f_0 = \tilde{F}$. Consider the vertex $v_0$ associated to the handle of $C_0$. Then for any other vertex $v$, there is a unique edge $e$ orienting from $v$ to $v_0$. Then the relation (3.1.2) uniquely expresses $\delta_v = \delta_{v_0} - c_e(\delta) \cdot e$ for any $\delta$, which proves (2) in this case. The two other cases in Proposition 2.6 can be proved similarly.

\[\blacksquare\]

**Proof of Proposition 3.1.** By Lemma 3.3 and (3.1.3), we have the canonical map of monoids:

\[\psi_e : \sum_e \mathbb{N}_e \to \overline{\mathcal{M}}(\Phi).\]

Denote by $C_0^s \to S^s$ the canonical log curve associated to the underlying curve $C_0$. We then fix a chart $\gamma^s : \sum_e \mathbb{N}_e \to \mathcal{M}_{s_0}$. Since $C_0 \to S$ is obtained by pulling back $C_0^s \to S^s$, and $S$ is a geometric point, we may assume that
\[(3.1.5) \quad \mathcal{M}_S = \overline{\mathcal{M}}(\Phi) \oplus \psi_e, \sum_e \mathbb{N}_e, \gamma^s \mathcal{M}_{s_0}.\]

This naturally associated with a morphism of log structures $\mathcal{M}_{s_0} \to \mathcal{M}_S$, which is unique up to a unique isomorphism by Lemma 3.3(2). This defines the log curve $C_0 \to S$ up to a unique isomorphism together with a chart $\gamma_S : \overline{\mathcal{M}}(\Phi) \to \mathcal{M}_S$.

Now assume that we have two liftings $f_{01} : C_0/S \to X$ and $f_{02} : C_0/S \to X$ over $f_0$. We need to verify that $f_{01}$ and $f_{02}$ is differ by an isomorphism of $C_0/S$. We first notice that the two morphisms $f_{01}^\#: 

\( f_{02} : f_{02}^* M_X \to M_{C_0} \) on the level of characteristic monoids coincide. This is because the discrete data of \( f_{01} \) and \( f_{02} \) are given by the same marked graph \( \Phi \). We may denote both \( f_{01}^\flat \) and \( f_{02}^\flat \) by \( f^\flat \).

Next consider the quotients \( q_X : M_X \to M_X \) and \( q_{C_0} : M_{C_0} \to M_{X_0} \). For any \( \delta \in M \), we obtain two \( \mathcal{O}^* \)-torsors
\[
T_X(\delta) := q_X^{-1}(\gamma(\delta)) \quad \text{and} \quad T_{C_0}(\delta) := q_{C_0}^{-1}(\bar{f}_0 \circ \gamma(\delta))
\]
over \( C_0 \). For each \( i \), we obtain an isomorphism of torsors:
\[
(3.1.6) \quad f^\flat_{0i}|_{T_X(\delta)} : T_X(\delta) \to T_{C_0}(\delta).
\]
Since those are isomorphisms of the same torsors over a proper curve, we have
\[
(3.1.7) \quad f^\flat_{01}|_{T_X(\delta)} = \log u_\delta + f^\flat_{02}|_{T_X(\delta)}
\]
for a non-zero constant \( u_\delta \) uniquely determined by \( \delta, f_{01}, \) and \( f_{02} \). This defines a map of sets:
\[
h^{gp} : M^{gp} \to \mathbb{C}^*.
\]
In fact, it is not hard to see that \( h \) is a morphism of groups.

By Lemma 3.3(2), the chart \( \gamma_S \) induces a morphism of groups
\[
\gamma_S^{gp} : M^{gp} \oplus \sum_e \mathbb{N}^{gp}_e \to M_S
\]
Then we obtain a morphism of groups
\[
\tilde{h}^{gp} : M^{gp}_S \to M^{gp}_S
\]
such that \( \tilde{h}^{gp}|_{\mathcal{O}^*} = id_{\mathcal{O}^*}, \tilde{h}^{gp}|_{\gamma_S^{gp}(\sum N_e)} = id_{\gamma_S^{gp}(\sum N_e)}, \) and \( \tilde{h}^{gp}(\gamma_S^{gp}(\delta)) = \log h(\delta) + \gamma_S^{gp}(\delta) \) for any \( \delta \in M \). Thus the restriction \( \tilde{h} := \tilde{h}^{gp}|_{M_S} \) defines an isomorphism of the log structure \( M_S \). Furthermore, since \( \tilde{h} \) fixes the image \( \gamma_S(\sum N_e) \), it induces an isomorphism of the log curve \( C_0/S \). Denote by \( h_{C_0} : M_{C_0} \to M_{C_0} \) the isomorphism induced by \( \tilde{h} \). Then (3.1.7) and the construction of \( \tilde{h} \) imply that
\[
f^\flat_{01} = f^\flat_{02} \circ h_{C_0}.
\]
Thus the log maps \( f_{01} \) and \( f_{02} \) differ by an isomorphism of \( C_0/S \). The uniqueness of \( \tilde{h} \) follows from the uniqueness of \( u_\delta \) as in (3.1.7).

3.2. Existence of the lifting. The proof of Proposition 3.1 can be modified to show the existence of lifting as follows:

Lemma 3.4. Assume \( \mathcal{M}_\Gamma(X) \neq \emptyset \), and the curve class \( \beta \) satisfies Assumption 2.7. For each underlying stable map \( f_0 \) given by the stabilization of \( \bar{F} \) in Notation 2.4 with discrete data \( \Gamma \), there exist a unique stable log map \( [f_0] \in \mathcal{M}_\Gamma(X) \) above \( f_0 \) with discrete data \( \Gamma \).
Proof. The proof of this statement is similar to the construction in [CZ14, Section 4.2].

First notice that the discrete data of \( f_0 \) is uniquely determined by the marked graph \( \Phi \) associated to the underlying stable map \( f_0 \) as in Construction 3.2. Since \( M_\Gamma(X) \neq \emptyset \), there is a stable log map with the marked graph \( \Phi \). This implies that the monoid in (3.1.4) is sharp. The same construction in (3.1.5) uniquely determines the log curve \( C_0/S \) up to a unique isomorphism. Since the underlying stable map is given, it remains to construct morphism of log structures \( f_0^\flat : \mathcal{M}_X^\flat \to \mathcal{M}_{C_0}^\flat \).

Since on the level of characteristic sheaves monoids, the morphism \( \bar{f}_0^\flat : \mathcal{M}_X^\flat \to \mathcal{M}_{C_0}^\flat \) has been determined by the graph \( \Phi \), it suffices to construct \((f_0^\flat)^{gp} : \mathcal{M}_X^{gp} \to \mathcal{M}_{C_0}^{gp}\).

Similarly as in (3.1.6), given \((f_0^\flat)^{gp}\) is equivalent to construct isomorphisms of torsors for each \( \delta \in \mathcal{M}^{gp} \). Hence to construct \((f_0^\flat)^{gp}\), it suffices to find a collection of isomorphisms of torsors for each \( \delta \) in a basis of \( \mathcal{M}^{gp} \). But such isomorphisms of torsors always exists for genus zero curves, since the degree of the torsors are compatible by the discrete data in Construction 3.2. This proves the existence. ♠

4. Irreducibility and Unirationality

4.1. The irreducibility.

Lemma 4.1. Let \( f : C \to X \) be a genus zero stable log map with discrete data \( \Gamma \). Assume that \( C \) is irreducible, and \( f(C) \cap G/H \neq \emptyset \). Then there is a nonempty open subset \( V \subset G \) such that for any \( g \in V \),

the composition \( g \circ f \) is a \( \Gamma \)-stable log map with the underlying map \( g \circ f \) satisfying the conditions in Notation 2.3.

Proof. First consider the open curve \( C^o = C \setminus \{q_j\} \) by removing markings with non-trivial contact orders. Then the restriction \( f|_{C^o} \) factors through \( G/H \subset X \). By Kleiman-Bertini Theorem [Har77, Theorem 10.8], there is an open dense \( V \subset G \), such that for any \( g \in V \) the restriction \( g \circ f|_{C^o} \) satisfies the conditions (2) and (3) in Notation 2.3.

We notice that for each contact marking \( q_j \), its image lies in a \( G \)-orbit, say \( O_j \). Since the complement of \( X^- \cap O_j \) in \( O_j \) is of codimension greater or equal than one. Applying Kleiman-Bertini Theorem again, condition (4) can be achieved for contact markings by further shrinking \( V \). ♠

Consider the following condition

(4.1.1) \[ n + \sum k_i \geq 4. \]
Under this assumption, the map $\tilde{F}$ in Notation 2.4 is stable. Consider the quasi-projective variety

\[(4.1.2) \quad M = M_{0,n+\sum_i k_i}\]

parameterizing $(n + \sum_i k_i)$-distinct points over a smooth genus zero curves. The markings are labeled by

$p_1, \cdots, p_n; x_{1,1}, \cdots, x_{1,k_1}; \cdots; x_{l,1}, \cdots, x_{l,k_l}$.

Let $\mathcal{S}_{k_i}$ be the symmetric group acting on $M$ by permuting the markings $x_{i,1}, \cdots, x_{i,k_i}$. Denote by $B = M/\mathcal{S}$ the quotient with $\mathcal{S}$ given by the product

\[(4.1.3) \quad \mathcal{S} = \mathcal{S}_{k_1} \times \cdots \times \mathcal{S}_{k_l}.

As observed in \[KP01, \text{Section 4}\], when $\sum_i k_i \geq 3$ the quotient $B$ is birational to the quotient

$$\mathbb{P}(\text{Sym}^{k_1} V^*) \times \cdots \times \mathbb{P}(\text{Sym}^{k_1} V^*) // \text{PGL}(V),$$

where the latter is rational by \[Bog86, Kat84\].

**Proposition 4.2.** Assume $\mathcal{M}_\Gamma(X) \neq \emptyset$ and (4.1.1). Under Assumption 2.7, there is a unique locally closed embedding $F : B \to \mathcal{M}_\Gamma(X)$ which sends a marked genus zero curve $C$ to the stable log map $\tilde{F}$ over the underlying stable map $\tilde{F}$ as in Notation 2.4 with handle $C$.

**Proof.** Let $\mathcal{M}(\Phi) \subset \mathcal{M}_\Gamma(X)$ be locally closed substack consisting of stable log maps with the marked graph $\Phi$ as in Construction 3.2. Since $\mathcal{M}_\Gamma(X)$ is log smooth, the stack $\mathcal{M}(\Phi)$ has smooth underlying structure. Let $\mathcal{M}_{\lambda,\infty} \subset \mathcal{M}(\Phi)$ be the torus fixed locus consisting of the limits of general stable log maps as in Notation 2.3 under the torus action. Thus, $\mathcal{M}_{\lambda,\infty}$ also has smooth underlying structure. Lemma 3.4 implies the tautological morphism

$$G : \mathcal{M}_{\lambda,\infty} \to B$$

between two smooth stacks are one-to-one on the level of closed points. Since $\mathcal{M}_\Gamma(X)$ is proper, $\mathcal{M}_{\lambda,\infty}$ is irreducible.

Denote by $M \subset \mathcal{M}_{\lambda,\infty}$ the fiber over the automorphism free locus of $B$. Then the restriction $G|_M$ is representable by the representability of \[AC14, \text{Corollary 3.13}\], since in this case the underlying stable maps of the log maps in $\mathcal{M}_{\lambda,\infty}$ is automorphism free.

The cases when $B$ has stacky locus only occur when the equality in (4.1.1) holds. Since $B$ parametrizes the underlying stable maps of the log maps in $\mathcal{M}_{\lambda,\infty}$, the stackyness of $B$ corresponding to the automorphism of the underlying stable maps. Note that this automorphism comes from subgroups of (4.1.3), hence perserves the discrete data of
Φ. Thus we could lift the automorphism of the underlying stable maps to the corresponding log stable maps. One could also see the existence of such lifting of automorphisms from the quotient construction using the rigidification (4.2.1) in the next section. Thus the morphism $G$ induces an isomorphism of the automorphism groups. This implies that $G$ is an isomorphism, whose inverse is the embedding $F$ as in the statement.

Proposition 4.3. Under Assumption 2.7 for the curve class $\beta$, the moduli space $M_{Γ}(X)$ is irreducible.

Proof. Since $M_{Γ}(X)$ is log smooth, the irreducibility of $M_{Γ}(X)$ is equivalent to the connectedness.

Notice that the open substack $M_{Γ}^{\circ}(X) \subset M_{Γ}(X)$ consisting of the points with trivial log structure is dense in $M_{Γ}(X)$. Thus, by Lemma 4.1, any stable log map $[f'] \in M_{Γ}(X)$ deforms to a stable log map $[f]$ satisfying the conditions in Notation 2.3. When $n + \sum_i k_i \leq 3$, the log map $[f]$ flows into a unique (possibly stacky) point under the $\mathbb{G}_m$-action. When $n + \sum_i k_i \geq 4$, the log map $[f]$ flows into a point in the connected locus $F(B) \subset M_{Γ}(X)$ by Proposition 4.2. This proves the irreducibility.

4.2. The unirationality. Denote by $M_{Γ,∞} \subset M_{Γ}(X)$ the locally closed substack with underlying stable maps given by Proposition 2.6. Then $M_{Γ,∞}$ is in the torus fixed locus. Consider the open substack $M_{Γ}^{\circ}(X) \subset M_{Γ}(X)$ such that any maps $f$ in $M_{Γ}^{\circ}(X)$ intersects $D_i$ transversally at the markings $x_{i,1}, \ldots, x_{i,k_i}$.

Then $M_{Γ}^{\circ}$ is an irreducible log smooth stack. This means that the underlying stack of $M_{Γ}^{\circ}$ has possibly toric singularities. To analyze the toric singularities, we first replace $M_{Γ}^{\circ}$ by an étale cover to remove the monodromy of the log structure along the fixed locus $M_{Γ,∞}$ as follows.

Consider the discrete data $Γ'$ obtained by adding extra markings

(4.2.1) $x_{1,1}, \ldots, x_{1,k_1}; \ldots; x_{l,1}, \ldots, x_{l,k_l}$

to $Γ$ with trivial contact orders. Denote by $M_{Γ'}(X)_0 \subset M_{Γ'}(X)$ the dense open substack of automorphism free locus. We consider the locally closed substack

(4.2.2) $M' \subset M_{Γ'}(X)_0$

such that any maps $f$ in $M'$ intersects $D_i$ transversally at the markings $x_{i,1}, \ldots, x_{i,k_i}$. 
We observe that $\mathcal{M}'$ is log smooth, and the stack quotient $[\mathcal{M}'/\mathcal{S}]$ is birational to $\mathcal{M}_\lambda$, where the symmetry group $\mathcal{S}$ acts by permuting the markings (4.2.1) as in (4.1.3).

Consider the torus action on $\mathcal{M}'$ induced by $\lambda$. Denote by $\mathcal{M}_{\lambda,\infty}' \subset \mathcal{M}'$ the closed substack consisting of underlying stable maps as in Proposition 2.6 with the extra markings (4.2.1) given by the intersection points with $D_i$ for all $i$. It is not hard to check that $\mathcal{M}_{\lambda,\infty}'$ is an open substack of a component of the fixed loci of $\mathcal{M}'$ under the torus action $\lambda$. A similar proof as in the case of $\mathcal{M}_{\lambda,\infty}$ shows that

**Lemma 4.4.** When (4.1.1) holds, the underlying structure of $\mathcal{M}_{\lambda,\infty}'$ is given by $M$ as in (4.1.2). Otherwise, $\mathcal{M}_{\lambda,\infty}'$ is a single non-stacky point.

Note that we have an étale strict morphism

$$\mathcal{M}_{\lambda,\infty}' \to \mathcal{M}_{\lambda,\infty}$$

by forgetting the extra marking (4.2.1). Furthermore, this morphism removes the monodromy of the log structure as follows:

**Lemma 4.5.** The minimal log structure on $\mathcal{M}_{\lambda,\infty}'$ is defined over Zariski site.

**Proof.** Since any log structure on a geometric point is Zariski, it suffices to prove the statement under the assumption (4.1.1). For simplicity, we write $S = \mathcal{M}_{\lambda,\infty}'$.

By Lemma 3.3 and the strictness of (4.2.3), the sheaf of groups $\mathcal{M}^{gp}_S$ is a locally constant sheaf on $S$. To prove the statement, it suffices to verify that $\mathcal{M}^{gp}_S$, hence $\mathcal{M}_S$ is globally constant. Denote by $f_S : \mathcal{C}_S/S \to X$ the universal family of stable log maps over $S$. Let $\sigma \subset \mathcal{C}_S$ be the section over $S$ given by a fixed special point on the handle of each fiber. Note that $\sigma$ could be either node or marking. We next assume $\sigma$ is a node. The case $\sigma$ is a marking can be proved similarly.

Let $\mathcal{C}_S^* \to S^*$ the canonical log structure associated to the underlying family of $\mathcal{C}_S \to S$. By Lemma 4.4, both $\mathcal{M}_{\mathcal{C}_S^*}$ and $\mathcal{M}_{S^*}$ are Zariski. In fact, $\mathcal{M}_{S^*} = \sum_e N_e$ is a globally constant sheaf given by the product of the canonical log structure smoothing each node of the underlying family $\mathcal{C}_S \to S$ by [Ols03], Hence $\mathcal{M}_{\mathcal{C}_S^*}|_{\sigma}$ is a globally constant sheaf of monoids of the form:

$$\mathcal{M}_{\mathcal{C}_S^*}|_{\sigma} = \sum_{e \neq \sigma} N_e \oplus \mathbb{N}^2.$$
Then we have
\[(4.2.4) \quad \mathcal{M}_{cs}|_{\sigma} = \mathcal{M}_S \oplus_{\mathbb{N}_a} \mathbb{N}^2.\]

Now the morphism of sheaves of groups:
\[(\bar{f}_S^\flat|_{\sigma})_{gp} : f^*_S\mathcal{M}_X^{gp}|_{\sigma} \to \mathcal{M}_{cs}^{gp}|_{\sigma}\]
is given by \((\bar{f}_S^\flat|_{\sigma})(\delta) = a_\delta + b_\delta\) where \(a_\delta \in \mathcal{M}_S\) and \(b_\delta \in \mathbb{N}^2\) such that \(b_\delta\) is a globally constant section of the form \((b, 0)\) or \((0, b)\) in \(\mathbb{N}^2\).

Consider the composition
\[(4.2.5) \quad f^*_S\mathcal{M}_X^{gp}|_{\sigma} \oplus \mathcal{M}_S^{gp} \to \mathcal{M}_{cs}^{gp}|_{\sigma} \to \mathcal{M}_S^{gp}\]
where the second arrow is given by the projection induced by \(a_\delta + b_\delta \mapsto a_\delta\). Using Lemma 3.3(2) and (3.1.2), we verify that (4.2.5) is an isomorphism over each fiber, hence is an isomorphism of sheaves of groups. On the other hand, since both \(\mathcal{M}_X^{gp}\) and \(\mathcal{M}_S^{gp}\) are globally constant, this implies that \(\mathcal{M}_S^{gp}\) is also globally constant. This finishes the proof. ♠

Choose the open substack of \(\mathcal{M}'\):
\[\mathcal{M}'_\lambda := \{ f \in \mathcal{M}' \mid \lim_{t \to \infty} t[f] \in \mathcal{M}'_{\lambda, \infty}\}.
\]
Since \(\mathcal{M}'_\lambda\) is an irreducible, log smooth variety, by [Niz06, ACMW14], we may take the projective resolution
\[(4.2.6) \quad \phi : \mathcal{M}_{res}^\infty \to \mathcal{M}'_\lambda\]
by a sequence of log étale blow-ups.

**Lemma 4.6.** The resolution \(\phi\) can be chosen to be \(\mathbb{G}_m\)-equivariant.

**Proof.** This can be seen from the construction of [ACMW14]. In fact, we may construct \(\phi\) by first taking a barycentric subdivision as in [ACMW14, Section 4.3], which is \(\mathbb{G}_m\)-equivariant since the log structure of \(\mathcal{M}'_\lambda\) is stable under the torus action. Then by [ACMW14, Lemma 4.4.1], we may further construct \(\phi\) by resolving the toric singularities Zariski locally over the barycentric subdivision as above, which is again \(\mathbb{G}_m\)-equivariant. ♠

Let \(\mathcal{M}_{\infty}^{res} \subset \mathcal{M}_{res}^{res}\) be the irreducible subvariety consisting of the \(\mathbb{G}_m\)-limits of general points in \(\mathcal{M}_{res}^{res}\) as \(t \to \infty\). Then \(\phi(\mathcal{M}_{\infty}^{res}) \subset \mathcal{M}_{\lambda, \infty}\). By [BB73], \(\mathcal{M}_{res}\) hence \(\mathcal{M}'\) is birational to a vector bundle over \(\mathcal{M}_{\infty}^{res}\).

**Lemma 4.7.** \(\mathcal{M}_{\infty}^{res}\) is rational.
\textit{Proof.} Consider any reduced irreducible stratum $S$ of 
\[ M_{\lambda, \infty}^{res} \times m_{\lambda}^t M_{\lambda, \infty}^{t} \]
with $\mathcal{M}_S$ a locally constant sheaf of monoids on $S$. Observe that $S$ is stable under the torus action. In fact, consider the projection 
\[ \pi : S \to M_{\lambda, \infty}^t. \]
Then for any point $y \in M_{\lambda, \infty}^t$, the fiber $\pi^{-1}(y)$ is a toric variety with its toric boundary given by the strata defined by the log structure on $S$. The torus action given by $\lambda$ on $\pi^{-1}(y)$ is compatible with the torus action of the toric variety, since the resolution (4.2.6) is locally given by toric blow-ups. Thus $M_{\lambda, \infty}^{res}$ is given by some stratum $S$ as above.

Consider the projection $\varphi : M_{\lambda, \infty}^{res} \to M_{\lambda, \infty}^t$. By further restricting to a Zariski open set of $M_{\lambda, \infty}^t$, we may assume $\varphi$ is smooth. We next verify that $\varphi$ defines a Zariski locally trivial family of toric varieties.

Let $A_1$ and $A_2$ be the Artin fan of $M_{\lambda}^t$ and $M_{\lambda, \infty}^t$ respectively [ACMW14, Section 3.2]. Since Artin fans in the initial factorization [ACMW14, Proposition 3.1.1], we have the following commutative diagram:

\[
\begin{array}{ccc}
M_{\lambda, \infty}^{t} & \longrightarrow & M_{\lambda}^{t} \\
\downarrow & & \downarrow \\
A_2 & \longrightarrow & A_1.
\end{array}
\]

The resolution (4.2.6) is obtained via the subdivision $A'_1 \to A_1$ as in [ACMW14, Section 3.17]. This induces the subdivision $A'_2 := A'_1 \times A_1 A_2 \to A_2$ by [ACMW14, Proposition 3.16]. Since the log structure on $M_{\lambda, \infty}^{t}$ is Zariski by Lemma 4.5, the Artin fan is of the form

\[ A_2 = [\text{Spec } \mathbb{C}[N] / \text{Spec } \mathbb{C}[N^{gp}]], \]

where $N$ is the characteristic monoid over $M_{\lambda, \infty}^{t}$. Thus the subdivision $A'_2$ is obtained by a sequence of toric blow-ups of $A_2$. Since

\[ M_{\lambda, \infty}^{res} \times m_{\lambda}^t M_{\lambda, \infty}^{t} = A'_2 \times A_2 M_{\lambda, \infty}^{t}, \]

by Lemma 4.5 we find $\phi$ is a family of toric varieties which admits a Zariski local trivialization. Finally, the statement follows from the rationality of $M$ and Lemma 4.4. \hfill \blackdiamond

Summing up the above argument, we have

**Proposition 4.8.** Under Assumption 2.7 for the curve class $\beta$, the stack $\mathcal{M}_{\Gamma}(X)$ is birational to a quotient of the rational variety $M^{t}$ by the product of symmetric groups $\mathcal{S}$ as in (4.1.3).

This concludes the proof of unirationality in Theorem 1.1. \hfill \blackdiamond
Corollary 4.9. Assume that \( k_i \leq 1 \) for all \( i \) in (2.2.1), and Assumption 2.7 holds. Then \( \mathcal{M}_X \) is rational.

Proof. Under the assumption of the statement, the group \( \mathcal{G} \) is trivial.

References

[AC14] Dan Abramovich and Qile Chen, Stable logarithmic maps to Deligne-Faltings pairs II, Asian Journal of Mathematics 18 (2014), no. 3, 465–488, arXiv:1102.4531.

[ACMW14] Dan Abrmovich, Qile Chen, Steffen Marcus, and Jonathan Wise, Boundedness of the space of stable log maps, arXiv:1408.0869.

[BB73] A. Bialynicki-Birula, Some theorems on actions of algebraic groups, Ann. of Math. (2) 98 (1973), 480–497. MR 0366940 (51 #3186)

[Bog86] F. A. Bogomolov, Rationality of the moduli of hyperelliptic curves of arbitrary genus, Proceedings of the 1984 Vancouver conference in algebraic geometry, CMS Conf. Proc., vol. 6, Amer. Math. Soc., Providence, RI, 1986, pp. 17–37. MR 846014 (87k:14022)

[Bri03] Michel Brion, The cone of effective one-cycles of certain \( G \)-varieties, A tribute to C. S. Seshadri (Chennai, 2002), Trends Math., Birkhäuser, Basel, 2003, pp. 180–198. MR 2017584 (2004m:14008)

[Bri07] ______, Construction of equivariant vector bundles, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007, pp. 83–111. MR 2348903 (2008k:14101)

[Che14] Qile Chen, Stable logarithmic maps to Deligne-Faltings pairs I, Ann. of Math. (2) 180 (2014), no. 2, 455–521, arXiv:1008.3090. MR 3224717

[CS13] Qile Chen and Matthew Satriano, Chow quotients of toric varieties as moduli of stable log maps, Algebra Number Theory 7 (2013), no. 9, 2313–2329. MR 3152015

[CZ14] Qile Chen and Yi Zhu, \( \mathbb{A}^1 \)-curves on log smooth varieties, arXiv:1407.5476.

[GS13] Mark Gross and Bernd Siebert, Logarithmic Gromov-Witten invariants, J. Amer. Math. Soc. 26 (2013), no. 2, 451–510. MR 3011419

[Har77] Robin Hartshorne, Algebraic geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[Kat84] P. I. Katsylo, Rationality of fields of invariants of reducible representations of the group \( SL_2 \), Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1984), no. 5, 77–79. MR 764040 (86c:14009)

[Kat89] Kazuya Kato, Logarithmic structures of Fontaine-Illusie, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224. MR 1046370 (99b:14020)

[Kat96] Fumiharu Kato, Log smooth deformation theory, Tohoku Math. J. (2) 48 (1996), no. 3, 317–354. MR 1404507 (99a:14012)
B. Kim and R. Pandharipande, *The connectedness of the moduli space of maps to homogeneous spaces*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 187–201. MR 1882330 (2002k:14021)

Wiesława Nizioł, *Toric singularities: log-blow-ups and global resolutions*, J. Algebraic Geom. 15 (2006), no. 1, 1–29. MR 2177194 (2006i:14015)

Martin C. Olsson, *Universal log structures on semi-stable varieties*, Tohoku Math. J. (2) 55 (2003), no. 3, 397–438. MR MR1993863 (2004f:14025)

Martin C. Olsson, *(Log) twisted curves*, Compos. Math. 143 (2007), no. 2, 476–494. MR MR2309994 (2008d:14021)

N. Ressayre, *Spherical homogeneous spaces of minimal rank*, Adv. Math. 224 (2010), no. 5, 1784–1800. MR 2646110 (2011h:14071)

Jonathan Wise, *Moduli of morphisms of logarithmic schemes*, arXiv:1408.0037.