Vibration characteristics of cracked functionally graded structures using XFEM

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Abstract. Crack is critical for structural health and their service life under dynamic condition. Vibration characteristics of cracked functionally graded structures have been investigated using extended finite element method (XFEM). A small crack in structures can lead to catastrophic failure. Vibrations of structures may initiate cracks and also causes opening and closing of cracks. Therefore, the study of cracked geometry especially in dynamic condition becomes more important. The location of cracks is tracked by level set functions. Numerical results for both crack and un-crack plates have been presented. The effect of crack on natural frequency has been studied in detail. Natural frequency for in-plane free vibration of functionally graded structures with various crack length and crack orientation have been presented for different crack position. Further, different cases are considered with multiple crack systems to understand dynamic response of cracked structures.

1. Introduction
Development of science and technology raises challenging task for engineers to use hi-tech materials for versatile environment conditions. Requirement of the hi-tech materials leads to the invention of functionally graded materials (FGM). First of all Bever and Duwez [1] gave the concept of gradation of properties in composite materials. Composition of FGM is varied continuous and smoothly in preferred direction [2] to get desired local material properties to withstand high mechanical and thermal load. FGM provides an excellent heat and corrosion resistance capability which enable to withstand very high temperature gradients. Functionally graded materials can be widely used in aerospace industry, nuclear reactor, wear resistance coating, and biomedical implant [3]. Engineering components for above mentioned applications may experience loads in the range of static to dynamic environment. Exposure of these components in dynamic loading conditions results in vibration of structure. This vibration may also initiate voids and cracks at maximum stress region. The natural frequency of the crack domain varies with change in crack length and its orientation [4, 5]. The variations of natural frequency for various crack length and various angles may meet unforeseen phenomena such as resonance or failure of structures. Earlier finite element method (FEM) was implemented to investigate the crack structures for dynamic conditions. Stahl and Keer [6] investigated the vibrational behavior of the crack structures and illustrated the natural frequency and buckling mode shapes. FEM as it needs very fine mesh near the crack and it also needs continuous remeshing for the crack growth problem. It is also difficult to capture tip singularity through FEM, hence it is time consuming and tedious for crack problem. To deal with this type of problem in fracture mechanics, XFEM [7, 8, 9] has been evolved. The crack problem of fracture mechanics can be efficiently, accurately and easily handle with XFEM. In XFEM, geometric discontinuities are not the part of mesh topology and are modeled by augmentation of primary variable approximation through partition of unity method. Cracks are tracked by level set function; hence complex geometries with
moving crack front can be easily handled. Classical FEM can be used as XFEM after augmentation of primary variable approximation. In XFEM, even coarse mesh can give high convergence rate and good accuracy.

The robustness of XFEM methodology to handle fracture mechanics problem has motivated to investigate vibration behavior of cracked FGM geometry. Numerical results for both crack and un-crack plates have been presented. The effect of crack on natural frequency has been studied in detail. Natural frequency for in-plane free vibration of functionally graded structures with various crack length and crack orientation has been presented for different crack position. Further, different cases are considered with multiple crack systems to understand dynamic response of cracked structures.

2. Formulation

2.1. Governing Equation

Consider a two dimensional domain (Ω) with strong discontinuity shown in Figure 1. The domain boundary is divided into three parts: \(\Gamma_u\), \(\Gamma_t\) and \(\Gamma_c\).

Where, \(\Gamma_u\) is displacement boundary, \(\Gamma_t\) is traction boundary and \(\Gamma_c\) is crack boundary.

The governing equation [9] for the given problem is:

\[
\nabla \cdot \sigma + b = \rho \ddot{u}
\]

(1a)

Here, \(\sigma\) is Cauchy stress tensor, \(b\) is body force vector, \(\rho\) is mass density of materials and \(\ddot{u}\) is second derivative of displacement field vector. The primary and secondary boundary conditions for considered crack problem are given as:

\[
\sigma \cdot n = t \quad \text{on} \quad \Gamma_t; \quad \sigma \cdot n = t \quad \text{on} \quad \Gamma_c; \quad u = \ddot{u} \quad \text{on} \quad \Gamma_u
\]

(1b)

The constitutive relation for material is given by Hook’s law as below:

\[
\sigma = C \varepsilon
\]

(2)

Here, \(C\) is material stiffness matrix which depends on elastic modulus and Poisson’s ratio of materials and \(\varepsilon\) is strain tensor which is \(\varepsilon = Bu\), whereas \(B\) is strain-displacement matrix and \(u\) is nodal displacement vector. Material stiffness matrix \(C\) for plane stress homogeneous condition is given as:

\[
C = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}
\]

(3a)

For the case of inhomogeneous FGM domain consecutive matrix would be as follows:
\[ C(x) = \frac{E(x)}{(1 - v(x))^2} \begin{bmatrix} 1 & v(x) & 0 \\ v(x) & 1 & 0 \\ 0 & 0 & 0.5(1 - v(x)) \end{bmatrix} \] (3b)

2.2. Weak formulation

The weak form of the governing equation can be derived as

\[
\int_\Omega [\rho \dot{u} \cdot \dot{u} d\Omega + \int_\Omega \sigma : \dot{\varepsilon} d\Omega = \int_\Omega t \cdot \dot{u} d\Gamma + \int_\Omega b \cdot \dot{u} d\Omega \] (4)

After putting the trial and test function in equation (5) and using the arbitrariness of the nodal variations, following sets of discrete system of equations are obtained:

\[ [M][\dot{d}] + [K][d] = \{f\} \] (5)

For free vibration above equation becomes;

\[ [K] = \lambda[M] \] (6)

Where \( M \) is global mass matrix, \( K \) is global stiffness matrix and \( \lambda \) is eigenvalue (natural frequency). The above equation is eigenvalue problem hence from above equation eigenvalue. \( \lambda = \omega^2 \) where \( \omega \) in rad/s.

2.3. Enrichment Approximation

To model crack discontinuity in XFEM formulation, primary variable approximation can be augmented by enrichment functions. The enriched displacement approximation \([7, 9]\) for two dimensional cracked bodies is given as:

\[
u(x) = \sum_{i=1}^{n} N_i(x)u_j + \sum_{i=1}^{n} N_i(x)H(x)\alpha_j + \sum_{i=1}^{n} N_i(x)\sum_{\alpha=1}^{4} \psi_{\alpha}(x)\beta_j \] (7)

Here, \( N_i \) is Lagrange shape function, \( H \) is Heaviside function to model strong discontinuity and \( \psi \) is branch function to model the crack tip stress singularity. Heaviside \( H(x) \) is +1 for above the crack and -1 for below the crack. In Fig. 2, crack tip element (partially cut by crack) and split element (element is completely cut by crack) can be seen. One additional node will be created at split nodes and four additional at tip nodes as shown in Figure 2.

![Figure 2. Enrichments in tip element and split element](image)
2.4. FGM Modeling

In the present work, FGM is composed of aluminium alloy and ceramic (alumina). It is considered that constituent of FGM at \( x = 0 \) is aluminium alloy and at \( x = L \) is ceramic. The gradation in material properties for FGM [10] is modeled as:

\[
E(x) = E_a e^{\beta x}
\]  
(8)

\[
\beta = \frac{1}{L} \ln \left( \frac{E_c}{E_a} \right)
\]  
(9)

Whereas \( E(x) \) is variation of elastic modulus in \( x \) direction. Gradation of FGM is shown in Figure 3. \( E_a \) is Young’s modulus of aluminium alloy and \( E_c \) is Young’s modulus of ceramic. Material properties of metal and ceramic are given in Table 1. The Volume fraction of ceramic and alloy calculated from equation (10) and (11) respectively. Poisson’s ratio for given FGM is obtained from equation (12). The values vary from 0.21 to 0.33 and calculated in material constituent varying direction by Halpin–Tsai equation.

\[
V_c(x) = \frac{E(x) - E_a}{E_c - E_a}
\]  
(10)

\[
V_a(x) = 1 - V_c(x)
\]  
(11)

\[
v(x) = \frac{v_a V_a(x) E_c + v_c V_c(x) E_a}{V_a(x) E_c + V_c(x) E_a}
\]  
(12)

| Sr. NO. | Material Properties | Aluminium Alloy | Alumina |
|--------|-------------------|-----------------|---------|
| 1      | Young’s Modulus (GPa) | 70              | 300     |
| 2      | Poisson’s ratio    | 0.33            | 0.21    |
| 3      | Density (Kg/m³)    | 2700            | 3800    |

Table 1. Material properties for given alloy and ceramic

![Figure 3. 2D Cracked FGM domain](image)
3. Numerical Results and Discussion
The modal analysis of 2D FGM crack domain has been done using XFEM methodology. MATLAB code has been developed to obtain natural frequency of 2D structures using extended finite element method. The given numerical examples for isotropic materials are compared and validated with available literature and ABAQUS solution. The differences of the results obtained from both have been reported. A plane stress condition has been considered throughout the analysis. A four node quadrilateral element has been used in uniformly discretized domain.

3.1. Two dimensional uncrack and crack Geometry (Isotropic)
Modal analysis of 2D uncrack geometry has been done and validated with available literatures before going to analyse crack domain. Natural frequency of 2D domain for first mode has been obtained and compared with references as given in Table 2. For uncrack domain FEM has been implemented. Throughout this study, geometry dimension and boundary condition are same. In this section, a plate with dimension of 1m x 1.2m has been considered. Material properties i.e. Young’s modulus of 70GPa, Poisson’s ratio of 0.33 and mass density is 2700 Kg/m$^3$ have been taken.

| Mode No. | Present | Du et al | Wang and Wereley | Farag and Pan |
|----------|---------|----------|------------------|---------------|
| 1        | 1848    | 1802     | 1811             | 1892          |

Difference (%) 2.4 2 2.38

Further, the work is extended to investigate modal analysis of cracked geometry. In this section, a plate of isotropic material with dimension 1m x 1.2m and crack length 0.3 is considered. Material properties are taken same as the previous problem. For the crack domain XFEM is implemented and Natural frequency for different boundary condition has been obtained as given in Table 3. Results obtained from MATLAB (XFEM) are compared with ABAQUS (FEM) shown in Table 3.

| Methodology | CCCC Natural Frequency | CCCF Natural Frequency | CFCF Natural Frequency | CFFF Natural Frequency |
|-------------|------------------------|------------------------|------------------------|------------------------|
| MATLAB (XFEM) | 2623                   | 1761                   | 1406                   | 567                    |
| ABAQUS (FEM) | 2625                   | 1752                   | 1405                   | 566                    |
| Difference (%) | 0.06                   | 0.5                    | 0.06                   | 0.01                   |

3.2. Crack 2D geometry (FGM)
The work is further extended to do the modal analysis of 2D crack FGM domain using in-house developed XFEM code. In this section, a plate with dimension 1m x 1.2m with different crack length
and various crack angles is considered. The material properties of FGM domain has been used as in given Table 1 to obtained natural frequency. Three sides clamped and one side free is considered as boundary condition for given problem. Discretized 2D domain with different crack location and multiple crack system are shown in Figure 4. The variation of natural frequency with crack length and various crack angles for edge crack, center crack and multiple cracks are shown in Figure 6.

![Figure 4](image1.png)

Figure 4. Discretized 2D domain (a) edge crack, (b) centre crack, (c) double crack one at right side edge second is at left side edge

![Figure 5](image2.png)

Figure 5 (a). Gauss point distribution for numerical integration, (b) Enlarge view of previous figure.

Sub-triangulation of enriched element has been done to get better numerical results within discontinuous elements as shown in Figure 5.

For edge crack problem the crack length to width ratio \((a/w)\) 0.1, 0.2, 0.3, 0.4 and 0.5 has been considered and crack orientation varies from 0° to 75° and is plotted as shown in Figure 6 (a). The value of natural frequency is decreasing with increase in crack length. With increase in crack angle, it is first increasing up to 30° and then decreasing. For center crack problem the crack length to width ratio \((a/w)\) 0.2, 0.4, 0.6 and 0.8 has been considered and crack orientation varies from 0° to 90° and is plotted as shown in Figure 6 (b). The value of natural frequency is decreasing with increase in crack angle. With increase in crack angle the value of natural frequency is increasing. In multiple crack system, two cracks have been considered one is at the mid of left side edge and other is at mid of right side edge. Crack length to width ratio \((a/w)\) at right edge is constant with value 0.1 and at left edge crack length varies. At left edge, crack length to width ratio \((a/w)\) 0.1, 0.2, 0.3 and 0.4 has been
considered and crack angle varies from 0° to 75° and is plotted as shown in Figure 6 (c). The value of natural frequency is decreasing with increase in crack length. With increase in crack angle, it is first increasing up to 45° and then decreasing.

**Figure 6 (a).** Natural Frequency (Hz) for 2D crack FGM domain (a) edge crack  
**Figure 6 (b).** centre crack  
**Figure 6 (c).** double crack one at right side edge second is at left side edge

4. **Conclusions**

The presented work deals with modal analysis of 2D FGM cracked structure. The extended finite element approach has been efficiently implemented to investigate 2D cracked domain. The numerical results are obtained from in-house developed MATLAB code and compared with ABAQUS (FEM) solutions. Effect of crack and its orientation on the natural frequency of the 2D domain has been studied. The salient features of obtained results are as follows:

- Stiffness of cracked structures has been decreased with increase in crack length, consequently natural frequency of cracked structures are in decreasing mode.
- Natural frequency for cracked 2D domain is decreasing with boundary condition from CCCC to CFFF.
- With changing crack orientation, the value of natural frequency is also changing.
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