A Unified Scheme of Shape Invariant Potentials with Central Symmetry

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Abstract

It is shown that number of shape invariant potentials can be unified to a superpotential in the three dimensional spherical coordinates with central symmetry. In fact a single superpotential generates the well-known, i.e. Coulomb and harmonic oscillator, as well as new potentials in different special conditions. In this formalism, the centrifugal potential plays a key rule that is always appeared in arbitrary dimensions due to non-zero angular momenta in central symmetries. The energy spectrums are obtained by a straightforward analytical procedure.

Key words: Supersymmetry, Central Symmetry, Shape Invariant.

1. Introduction

The ultimate goal of many physicists is achievement to often called theory of everything [1,2]. The basic idea of this theory is that all parts of a system behave consistently and each one reflects a special projection of a general reality. In the nineteenth century, James Clerk Maxwell was the first to demonstrate the electric and magnetic fields are actually two different manifestations of a single underlying electromagnetic field [3]. One of the most important achievements of the twentieth century, known as the "Standard Model", is based on unifying (so far) three of the four fundamental forces in the Universe [4]. Supersymmetry is a theoretical extension of the Standard Model can resolve some
important problems by unifying the description of forces with that of matter [5]. Furthermore, supersymmetric quantum mechanics (SUSY-QM) is based on factorization method and ladder operators explains how two partner potentials derive from a single superpotential [6].

Trying to obtain the exact solutions has been taken into consideration since origin of quantum mechanics and helps to better understanding as well as new achievements about quantum mechanical systems. Exactly solvable potentials referred to the potentials that all the eigenvalues, finite or infinite in number, and the corresponding eigenfunctions can be obtained explicitly. For instance, the exploring has been categorized analytically solvable problems in quantum mechanics into exactly, conditional exactly and quasi-exactly solvable groups [7-9]. These problems have been investigated by different methods such as the transformation of a Schrödinger equation into a hypergeometric-type differential equations by the Nikiforov-Uvarov, asymptotic iteration method and SUSY-QM [10-12]. Among them, SUSY-QM has provided powerful tools to study unified treatments of some exactly solvable of D-dimensional quantum systems in spherical coordinates [13-16]. The algebraically solvable problems in SUSY-QM are known as shape invariance and corresponding potentials called shape invariant potentials [1]. Shape invariant potentials have the same dependence on main variables and differ from each other in the value of parameters. It is demonstrated that few central shape invariant potentials are unified to a superpotential in three dimensions that can they generalized to any arbitrary D-dimensions easily.

2. SUSY-QM for 3-Dimensions in Central Symmetry

In the spherical coordinates, a central potential depends only on the radial variable \( r \). Therefore, by separating the angular variables from the wave function as \( \psi_{n \ell m}(r, \theta, \phi) = R_{n \ell}(r)Y_{\ell m}(\theta, \phi) \) and change of variable \( R_{n \ell}(r) = r^{-\frac{\ell}{2}} u_{n \ell}(r) \), the radial Schrödinger equation is obtained as [11111],

\[
\frac{\hbar^2}{2m} u_{n \ell}''(r) + V_1(r, a, \ell) u_{n \ell}(r) = 0
\]
where subscript "0" refers to the ground state and thus the potential,
\[
V_1(r, a, \ell) = v(r, a, \ell) + V_{Cef}(r, \ell)
\]
where the centrifugal potential,
\[
V_{Cef}(r, \ell) = \frac{\hbar^2 \ell(\ell + 1)}{2m} \frac{1}{r^2}
\]
is always appeared in spherical coordinates for non-zero angular momenta and,
\[
v(r, a, \ell) = V(r, a) - E_{0\ell}
\]
here \(a\) is the set of independent parameters (\(a = \{a_1, a_2, \ldots\}\)) and \(E_{0\ell}\) is the ground state energy of potential \(V + V_{Cef}\). The Eq. (1) implies the potential \(V_1\) has zero ground state energy \(E_{(1)}^{(0\ell)} = 0\) due to unbroken SUSY. In the SUSY-QM, by definition the superpotential as logarithmic derivative of corresponding ground state wave function as,
\[
W_{1\ell}(r) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dr} \ln u_{0\ell}^{(1)}(r)
\]
the Schrödinger equation reduces to the non-linear Riccati equation as,
\[
W_{1\ell}'(r) \pm \frac{\hbar}{\sqrt{2m}} W_{1\ell}(r) = V_1(r, a, \ell)
\]
where the minus and plus signs are related to partner potentials \(V_1\) and \(V_2\), respectively. According to the Eq. (2) for potential \(V_1\), we make a general ansatz for the corresponding superpotential as follows,
\[
W_{1\ell}(r) = \frac{\hbar}{\sqrt{2m}} \left\{ w(r, a, \ell) - \frac{\ell + 1}{r} \right\}
\]
where \(w(r, a, \ell)\) is a function will determined by shape invariance condition in different situations. By replacing this equation in the Eq. (6), partner potentials are obtained as,
\[
\begin{align*}
V_1(r, a, \ell) &= \frac{\hbar^2}{2m} \left\{ w^2(r, a, \ell) - w'(r, a, \ell) - 2w(r, a, \ell) \frac{\ell + 1}{r} \right\} \\
V_2(r, a, \ell) &= \frac{\hbar^2}{2m} \left\{ w^2(r, a, \ell) + w'(r, a, \ell) - 2w(r, a, \ell) \frac{\ell + 1}{r} \right\}
\end{align*}
\]
In the context of SUSY-QM, a potential is said to be shape invariant if its SUSY partner potential has the same spatial dependence as the original potential with possibly altered parameters. Therefore, the $V_1$ and $V_2$ are the shape invariant potentials, if the remainder $R_\ell$ is independent of $r$,

$$ R_{1\ell} = V_2(r, a, \ell) - V_1(r, \tilde{a}, \ell + 1) $$

$$ = \frac{\hbar^2}{2m} \left\{ w^2(r, a, \ell) - w^2(r, \tilde{a}, \ell + 1) + w'(r, a, \ell) + w'(r, \tilde{a}, \ell + 1) + 2w(r, \tilde{a}, \ell + 1) \frac{\ell + 2}{r} - 2w(r, a, \ell) \frac{\ell + 1}{r} \right\} \quad (9) $$

where $\tilde{a}$ includes a set of changed parameters so that lead to the shape invariance preservation. In the central symmetry, the $r-$ and $\ell-$functional of $w(r, a, \ell)$ are independent, i.e. $w(r, a, \ell) \equiv f(r, a)g(\ell)$, and $g(\ell + 1)$ be a function of $g(\ell)$, i.e, $g(\ell + 1) = G(\ell)g(\ell)$, as a result we have for the reminder,

$$ R_{1\ell} = \frac{\hbar^2}{2m} \left\{ g^2(\ell) \left[ f^2(r, a) - f^2(r, \tilde{a})G^2(\ell) \right] + g(\ell) \left[ f'(r, a) + f'(r, \tilde{a})G(\ell) \right] + 2 \frac{g(\ell)}{r} \left[ f(r, \tilde{a})G(\ell)(\ell + 2) - f(r, a)(\ell + 1) \right] \right\} \quad (10) $$

If the set of independent parameters are unchanged,

$$ \tilde{a} = a \quad (11) $$

then,

$$ R_{1\ell} = \frac{\hbar^2}{2m} \left\{ w^2(r, a, \ell) \left[ 1 - G^2(\ell) \right] + w'(r, a, \ell) \left[ 1 + G(\ell) \right] + 2 \frac{w(r, a, \ell)}{r} \left[ G(\ell)(\ell + 2) - (\ell + 1) \right] \right\} \quad (12) $$

Therefore, the general form of $w(r, a, \ell)$ for $G(\ell) \neq \pm 1$ is,

$$ w(r, a, \ell) = \frac{B_\ell}{G(\ell) - 1} \frac{J(A_\ell + 1, B_\ell r) + CY(A_\ell + 1, B_\ell r)}{J(A_\ell, B_\ell r) + CY(A_\ell, B_\ell r)} \quad (13) $$

where $J$ and $Y$ are the Bessel functions of the first and second kinds, respectively and $C$ is an arbitrary constant. The coefficients $A_\ell$ and $B_\ell$ are,

$$ \left\{ \begin{array}{c} A_\ell = \left( \frac{G(\ell) - 1}{G(\ell) + 1} \right) \frac{2\ell + 3}{2} \\ B_\ell = \sqrt{\left( \frac{G(\ell) - 1}{G(\ell) + 1} \right) \frac{2mB_\ell}{\hbar^2}} \end{array} \right. \quad (14) $$
3. The Special Cases

3.1. $G(\ell) = 1$

If $G(\ell) = 1$, from Eq. (12), $w$ and $R_1$ are independent of $\ell$ and we have,

$$R_1 = \frac{\hbar^2}{m} \left\{ w'(r, a) + \frac{w(r, a)}{r} \right\}$$

solving this first-order differential equation for $w(r, a)$ is yield,

$$w(r, a) = \frac{m}{\hbar^2} R_1 r + \frac{C}{r}$$

where $C$ is the integration constant. By putting result this in Eq. (8) and for $R_1 = 2\hbar\omega$ we have,

$$\begin{cases}
V_1(r, C, \ell) = \frac{1}{2} \omega^2 r^2 + \frac{\hbar^2}{2m} C(\ell + 3/2) - E_{0 \ell} \\
V_2(r, C, \ell) = \frac{1}{2} \omega^2 r^2 + \frac{\hbar^2}{2m} (\ell + 2/2) - E_{1 \ell}
\end{cases}$$

where,

$$\begin{cases}
v(r) = \frac{1}{2} \omega^2 r^2 \\
E_{0 \ell} = \hbar \omega (\ell + 3/2 - C) \\
E_{1 \ell} = \hbar \omega (\ell + 5/2 - C)
\end{cases}$$

in which the results are according to the 3-dimensional harmonic oscillator.

3.2. $G(\ell) = -1$

On the other hand, for $G(\ell) = -1$, $w(r, a, \ell)$ is obtained from Eq. (12) as,

$$w(r, a, \ell) = \frac{m}{\hbar^2} (2\ell + 3) R_{1 \ell} r$$

similarly, putting result in Eq. (8) and for $R_{1 \ell} = \frac{\hbar \omega}{\ell + 3}$ we have,

$$\begin{cases}
V_1(r, \ell) = \frac{1}{2} \omega^2 r^2 + \frac{\hbar^2}{2m} \ell(\ell + 1) - E_{0 \ell} \\
V_2(r, \ell) = \frac{1}{2} \omega^2 r^2 + \frac{\hbar^2}{2m} (\ell + 2)(\ell + 1) - E_{1 \ell}
\end{cases}$$

where,

$$\begin{cases}
v(r) = \frac{1}{2} \omega^2 r^2 \\
E_{0 \ell} = -\hbar \omega (\ell + 3/2) \\
E_{1 \ell} = -\hbar \omega (\ell + 5/2)
\end{cases}$$

in which the results are according to the 3-dimensional harmonic oscillator.
3.3. \( G(\ell) = \frac{\ell + \frac{1}{2}}{\ell + \frac{3}{2}} \)

In this case the reminder is,

\[
R_{1\ell} = \frac{\hbar^2 (2\ell + 3)}{2m(\ell + 2)} \left\{ \frac{w^2(r, \ell)}{\ell + 2} + w'(r, \ell) \right\}
\]

(22)

w(r, \ell) is,

\[
w(r, \ell) = k_{0\ell}(\ell + 1) \tanh \{k_{0\ell}(r + C)\}
\]

(23)

where,

\[
k_{0\ell}^2 = \frac{2mR_{1\ell}}{\hbar^2 (2\ell + 3)}
\]

(24)

potential \( V_{1\ell} \) is,

\[
\begin{align*}
V_1(r, \ell) &= \frac{\hbar^2 k_{0\ell}^2}{2m}(\ell + 2)(\ell + 3) \tanh^2 \{k_{0\ell}(r + C)\} \\
&- \frac{\hbar^2 k_{0\ell}^2}{2m}(\ell + 1)(\ell + 2) \tanh \left(\frac{k_{0\ell}(r + C)}{r}\right) + \frac{\hbar^2}{2m} \frac{\ell(\ell + 1)}{r^2} - E_{0\ell}
\end{align*}
\]

(25)

where,

\[
\begin{align*}
v(r, \ell) &= \frac{\hbar^2 k_{0\ell}^2}{2m}(\ell + 2)(\ell + 3) \tanh^2 \{k_{0\ell}(r + C)\} \\
&- \frac{\hbar^2 k_{0\ell}^2}{2m}(\ell + 1)(\ell + 2) \tanh \left(\frac{k_{0\ell}(r + C)}{r}\right)
\end{align*}
\]

(26)

3.4. \( w(r, a, \ell) \) is independent of \( r \)

If \( w(r, a, \ell) \equiv w(a, \ell) \) to be independent of \( r \), from Eq. (10) we have,

\[
R_{1\ell} = \frac{\hbar^2}{2m} \left\{ w^2(\ell) - w^2(\ell + 1) + 2w(r, \ell + 1)\frac{\ell + 2}{r} - 2w(r, \ell)\frac{\ell + 1}{r} \right\}
\]

(27)

In this case, \( R_{1\ell} \) is a constant only when,

\[
2w(\ell + 1)\frac{\ell + 2}{r} - 2w(\ell)\frac{\ell + 1}{r} = 0
\]

(28)

As a result,

\[
w(\ell + 1) = w(\ell)\frac{\ell + 1}{\ell + 2}
\]

(29)

thus,

\[
w(\ell) = (\ell + 2)\sqrt{\frac{2mR_{1\ell}}{\hbar^2}}
\]

(30)
where,
\[
\begin{align*}
R_{1\ell} &= \frac{m}{2\hbar^2} \left( \frac{Z_1 Z_2 e^2}{4\pi \varepsilon_0 (\ell+1)(\ell+2)} \right)^2 \\
E_{0\ell} &= \frac{m}{2\hbar^2} \left( \frac{Z_1 Z_2 e^2}{4\pi \varepsilon_0 (\ell+1)} \right)^2
\end{align*}
\]

4. Conclusion

A superpotential has been introduced can generate number of shape invariant partner potentials in different conditions. Although it is performed in three dimensional spherical coordinates with central symmetry it can easily generalized to any arbitrary D-dimensions.

One can seen from Eqs. (13) and (14), if the reminder is vanished \( R_{1\ell} = 0 \) then \( B_\ell = 0 \) and then \( w(r, a, \ell) = 0 \) and hence supersymmetry is broken. This implies there is not any partner potentials (and then bound system) can derived from superpotential \( W_{1\ell} \) (Eq. (7)).
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