A NOTE ON ACTIONS WITH FINITE ORBITS ON DENDRITES

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Abstract. It is shown that the restriction of the action of any group with finite orbit on the minimal sets of dendrites is equicontinuous. Consequently, we obtain that the action of any amenable group and Thompson group on dendrite restricted on minimal sets is equicontinuous. We further provide a class of non-amenable groups whose action on dendrites has finite orbit. We extend also some of our results to dendron. We further give a characterization of the set of invariant probability measures and its extreme points.
1. Introduction.

This paper deals with the action of group on dendrites. It turns out that the minimal action on the nondegenerate dendrite is similar to the minimal action on the circle. For the group action on the circle, Margulis proved that either the group is not amenable or there is a $G$-invariant probability measure [16]. It is also well-known that the minimal sets can only be the whole circle, or a finite set, or a Cantor set [5]. Moreover, the minimal action on the circle is either equicontinuous or strongly proximal, and if the latter case holds then the group is a non-amenable group [16]. Nowadays there are several proofs of this theorem, see [12, Section 5.2], [25, Chap.2, p. 54], [6, Theorem 4.5].

For dendrite, H. Marzougui and the second author classify the minimal sets of the group action on dendrite [17] and local dendrites [18]. This classification is analogue to the case of the circle.

Exploiting result from [17], E. H. Shi and X. D. Ye proved that every amenable group action on dendrites has a fixed point or finite orbit of order 2 [33]. Very recently, E. Glasner and M. Megrelishvili proved that every continuous group action of $G$ on a dendron is tame [13]. They asked

**Question.** Is there an amenable group $G$, an action of $G$ on a dendrite $X$, and a minimal subset $Y \subset X$, such that the system $(G, Y)$ is almost automorphic but not equicontinuous?

The answer to this question was given by E. H. Shi and X. D. Ye in [34]. They proved that the restriction action of any amenable group on any minimal set $K$ is equicontinuous. Their proof is based on their result mentioned above.

In this paper, we will strengthened this result by relaxing the amenability condition.

Indeed, we will established that if the group action admit a finite orbit then its restriction on any minimal set $K$ is equicontinuous. As a consequence, we get that the restriction of Thompson group on any minimal set $K$ of dendrite is equicontinuous. We further obtain the result from [17, Corollary 6.7].
Moreover, we will prove that there is a class of non-amenable groups for which the infinite minimal sets are a Cantor and the action has a finite orbit. Finally, we will describe the set of invariant probability measures and its extreme points.

Our proof is based on the descriptions of the structure of a minimal set and its convex hull and how finite orbits occur on its convex hull.

According to our results, it follows that the equicontinuity of the action on the minimal sets of dendrite can not separate the amenable and not amenable groups. Nevertheless, we hope that the strategy of searching for the good topological dynamical property will serve to enlighten the problem of amenability of Thompson group.

The plan of this paper is as follows. In Section 2, we recall some definitions and tools on dendrites and local dendrites which are useful for the rest of the paper. In Section 3 we state our main results and we give the proof of our main first result. In Section 4, we recall some basic definitions and tools on non-amenable groups and Thompson group \( F \), we further give the proof of our second and third main results. Finally, in Section 5, we describe the set of invariant measures and its extreme points.

2. Set up and tools.

Let \( G \) be a group acting continuously on the topological space \( X \), that is, there is a family \( \{T_g\}_{g \in G} \) of continuous maps from \( X \) to \( X \) such that for any \( g, g' \in G \), we have \( T_g \circ T_{g'} = T_{gg'} \), and \( T_e = \text{Id}_X \) where \( e \) is the identity element of \( G \) and \( \text{Id}_X \) is the identity map on \( X \). Obviously, for each \( g \in G \), \( T_g \) is an element of the group of homeomorphism on \( X \) denoted by \( \text{Hom}(X) \). For any \( x \in X \), the subset \( \text{Orb}_G(x) = \{T_g(x) : g \in G\} \) is called the orbit of \( x \) (under \( G \)). A subset \( A \subset X \) is called \( G \)-invariant (resp. strongly \( G \)-invariant) if \( T_g(A) \subset A \), for each \( g \in G \) (resp., \( T_g(A) = A \)). It is called a minimal set of \( G \) if it is non-empty, closed, \( G \)-invariant and minimal (in the sense of inclusion) for these properties, this is equivalent to say that it is an orbit closure that contains no smaller one; for example a single finite orbit. When \( X \) itself is a minimal set, then we say that \( G \) act minimally. As usual, we denote the closure of any subset \( A \) by \( \overline{A} \). The orbit of a point \( x \in X \) is said to be finite if \( \overline{\text{Orb}_G(x)} = \text{Orb}_G(x) \) is a finite set. Obviously, if a point \( x \) is an atom of an invariant probability measure then its orbit is finite. The action is said to have a finite orbit.
if it admit a finite orbit.

For the case of action on dendrite, it is straightforward to see that the action has a finite orbit if and only if there is an invariant measure on $X$. Notice further that in this case the action fixed an arc. We shall strengthened this result to the action of group on dendron.

The point $x$ is said to be a recurrent point if there exists a sequence $(g_n)_{n \in \mathbb{N}}$ of elements in $G$ such that $g_n.x \to x$ when $n$ goes to $\infty$ and $g_n(x) \neq x$ for all $n$.

The action is proximal if for every $(x, y) \in X^2$ there is a sequence $(s_n)$ in $G$ and point $z \in X$ such that such that $\lim s_n. x = \lim s_n. y = z$. It is well known that if the action is proximal then there is unique minimal set.

Let $C(X)$ be the space of continuous functions on $X$ equipped with the weak-star topology. By the classical representation theorem of Riesz the dual of $C(x)$ denoted by $C^*(X)$ is the space of the Borel bounded measures on $X$. Let $P(X)$ be the set of probabilities measures on $X$. The elementary probability measures are given the Dirac measure on a point. We denote the Dirac measure on a point $x$ by $\delta_x$. By the theorem of Banach-Alaoglu-Bourbaki, $P(X)$ is a compact convex set of the space of measure on $X$. The action of $G$ on $X$ induce an affine action on the $P(X)$ as follows:

$$(g, \mu) \mapsto g\mu,$$

where $g\mu$ is the push-forward measure given by

$$g\mu(f) = \int f(g.x)d\mu(x), \forall f \in C$$

If the action of $G$ on $P(X)$ is proximal we say that the action $G$ on $X$ is strongly proximal. The compact set of $G$-invariant measures is denoted by $P(X \leftrightarrow G)$. We recall that $\mu$ is in $P(X \leftrightarrow G)$ if for any Borel set $A$, $\mu(g.A) = \mu(A)$.

Let $G$ acts on two compact space $X, Y$ and assume that there is a subjective homeomorphism $X$ to $Y$ such that

$$\phi \circ T_g = T_g \circ \phi,$$

then $Y$ is said to be a factor of $X$. By Lemma 3.7 from [6], the factor of proximal action is proximal, so is for the factor of strongly proximal...
action.

Let us recall now the definitions and some basic properties of dendrites and local dendrites.

A continuum is a compact connected space. Following [35], for any topological property $P$, a continuum is said to be rim-$P$ if it has a basis of open sets with boundaries enjoy property $P$. For the rim-finite case the space are also called regular space [30, Chap. VI, §51, p. 274]. Please, notice that therein the notion of order of the point is given with respect to the cardinality of the boundaries. An arc is any space homeomorphic to the compact interval $[0,1]$. A topological space is arcwise connected if any two of its points can be joined by an arc.

Following the terminology of Nadler, by a dendrite $X$, we mean a locally connected metrizable continuum containing no homeomorphic copy to a circle. Every sub-continuum of a dendrite is a dendrite ([22], Theorem 10.10) and every connected subset of $X$ is arc-wise connected ([22], Proposition 10.9). In addition, any two distinct points $x, y$ of a dendrite $X$ can be joined by a unique arc with endpoints $x$ and $y$, denote this arc by $[x, y]$ and let denote by $[x, y) = [x, y] \setminus \{y\}$ (resp. $(x, y] = [x, y] \setminus \{x\}$ and $(x, y) = [x, y] \setminus \{x, y\}$). A point $x \in X$ is called an endpoint if $X \setminus \{x\}$ is connected. It is called a branch point if $X \setminus \{x\}$ has more than two connected components. The number of connected components of $X \setminus \{x\}$ is called the order of $x$ and denoted by $\text{ord}(x)$. The order of $x$ relatively to a subdendrite $Y$ of $X$ is denoted by $\text{ord}_Y(x)$. Denote by $E(X)$ and $B(X)$ the sets of endpoints, and branch points of $X$, respectively. By ([30], Theorem 6, 304 and Theorem 7, 302), $B(X)$ is at most countable. A point $x \in X \setminus E(X)$ is called a cut point. It is known that the set of cut points of $X$ is dense in $X$ ([30], VI, Theorem 8, p. 302). Following ([3], Corollary 3.6), for any dendrite $X$, we have $B(X)$ is discrete whenever $E(X)$ is closed. An arc $I$ of $X$ is called free if $I \cap B(X) = \emptyset$. For a subset $A$ of $X$, we call the convex hull of $A$, denoted by $[A]$, the intersection of all sub-continua of $X$ containing $A$, one can write $[A] = \cup_{x,y \in A}[x,y]$.

We further have that $X$ is a dendrite if and only if any two points of $X$ are separated in $X$ by a third point of $X$ ([22], Theorem 10.2). We recall that a point $z$ separates $x$ and $y$ in $X$ if there exist in $X$ open disjoint neighborhoods $U, V$ of $x$ and $y$ respectively such that $X \setminus \{z\} = U \cup V$. 
More generally, a continuum $X$ is said to be a *dendron* if every pair of distinct points $x, y$ can be separated in $X$ by a third point $z$. Obviously, a dendrite is a metrizable dendron. A local dendron is a continuum having the property that every of its points has a neighborhood which is a dendron. It is easy to see that the dendron is rim-finite. More generally, we have that any local dendron is rim-finite [35]). Whence any local dendrite is rim-finite (for a direct proof, we refer to [30, Theorem 1. page 303]). If $Y$ is a sub-continuum of a dendron $X$, then the canonical retraction of $X$ onto $Y$ is denoted by $r_Y$, for more details see [20] (see also Theorem 3.12 in [13]). If $a$ and $b$ are two distinct points in a dendron $X$, then we define the generalized arc as follows:

$$[a,b] = \{x \in X : x \text{ separated } a \text{ from } b \text{ in } X\} \cup \{u,v\}.$$

The convex hull of a given subset in a dendron is defined and denoted similarly as in the case of dendrite.

Let us also recall some tools about the Vietoris topology defined on the non-empty closed sets of $X$ denoted by $2^X$. Since $X$ is a compact metric space, the Vietoris topology is compact and metrizable [19]. Furthermore, the metric is given by

$$D(F_1,F_2) = \max \left\{ \sup_{x \in F_1} d(x,F_2), \sup_{y \in F_2} d(y,F_1) \right\},$$

where $d(x,F) = \inf_{y \in F} d(x,y)$. The metric $D$ is known as Hausdorff metric.

We notice further that the map $A \mapsto [A]$ is continuous with respect to the Vietoris topology [9].

For a family of open set $\mathcal{U}$, we define $\text{mesh}(\mathcal{U})$ by

$$\text{mesh}(\mathcal{U}) = \sup \left\{ \frac{\text{diam}(U)}{U \in \mathcal{U}} \right\}.$$

Using the compactness of the Vietoris topology, it can be seen without using Zorn lemma that $X$ admit a minimal set [10, p.30]. On the other hand, by applying Zorn lemma, it is easy to see the following.

**Proposition 2.1.** Let $G$ a group acting on the dendrite $X$. Then, there exist a minimal $G$-invariant subdendrite $Y$.

We warn the reader here that the minimality is used in the sense of inclusion in the family of all $G$-invariant subdendrites. Indeed, it is well known that there is example of minimal subdendrite which is not minimal in the usual sense.
We need also the following lemmas from [22] and [20].

**Lemma 2.2.** Let $X$ be a dendrite with metric $d$. Then, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\text{diam}([x, y]) < \varepsilon$ whenever $d(x, y) < \delta$.

**Lemma 2.3 ([20]).** Let $X$ be a dendron. Then

(i) $X$ is locally connected.

(ii) If $(Y_i)_{i \in I}$ is a family of subcontinua of $X$ such that for any $i \neq j$, $Y_i \cap Y_j \neq \emptyset$. Then, $\bigcap_{i \in I} Y_i \neq \emptyset$.

(iii) For any $x, y \in X$,

$$[x, y] = \bigcap \{C \text{ subcontinuum of } X \text{ which contains } x, y\}.$$

We notice that (i) and (ii) are stated in [20] as Corollary 2.15.1, and the proof (iii), as noted by Malyutin [14], can be obtained as a consequence of Lemma 2.7 and Corollary 2.14 of [20].

3. Main Results on Equicontinuity

We start by stating our first main result.

**Theorem 3.1.** Let $X$ be a dendrite, a group $G$ act on $X$ with a finite orbit. If $M$ is an infinite minimal set of $G$ then the action of $G$ restricted to $M$ is equicontinuous.

Our second main result is as follows.

**Theorem 3.2.** There exists a class of non-amenable groups for which the action on dendrite has a finite orbit. Therefore, the restricted action on any minimal set is equicontinuous and infinite minimal sets are Cantor sets.

For the proof of our main results, we need the following.

**Theorem 3.3 ([17]).** Let $X$ be a dendrite, a group $G$ act on $X$ and $M$ be a minimal set of $G$. Let $[M]$ be the convex hull of $M$. Assume that the action of $G$ has a finite orbit. Then,

(i) $M$ is the set of endpoints of $[M]$.

(ii) For any point $a \in X$ in a finite orbit, the point $r_{[M]}(a)$ is in a finite orbit.

(iii) $[M]$ contains at least one finite orbit consisting of one or two points.

(iv) If $M$ is infinite, then it is the only infinite minimal subset in $[M]$.

Moreover, we have the following.
Proposition 3.4 (17). Let $M$ be a minimal set of $G$. If $M$ is infinite, then there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of sub-trees in $D$ satisfying the following properties:

(i) $T_n \subset T_{n+1}, \forall n \in \mathbb{N}$.
(ii) $E(T_n) \subset B(D)$ is a finite orbit, $\forall n \in \mathbb{N}$.
(iii) $D = \bigcup_{n \in \mathbb{N}} T_n$.
(iv) $\lim_{n \to +\infty} E(T_n) = M$ (in the Hausdorff metric).

We need also the following results.

Lemma 3.5 (17). Let a group $G$ act on a dendrite $X$. Suppose that the action has a finite orbit. Then the minimal set $M$ is either a finite orbit or a Cantor set.

Lemma 3.6 (17). Let $G$ be a group acting on a continuum $X$ and let $M$ be a minimal set. Then, $M$ is

(1) a finite orbit, or
(2) $X$, that is, the action is minimal, or
(3) a $G$-invariant, compact perfect nowhere dense subset of $X$.

For the case of the dendrites, we need the following results from 19, and 13 which we state in the more general setting. For sake of completeness, we present the proof.

Lemma 3.7. Let $G$ acts on a dendron $X$ and assume that the action has no finite orbit. Then

(1) There is a unique infinite minimal set $M \subset X$.
(2) The action of $G$ on $M$ is strongly proximal.
(3) $G$ contains a free group on two generators.

For the proof, we need the following lemma from 38.

Lemma 3.8. Let $G$ acts on a compact space $X$ and $M' = \{x \in X/\forall \mu \in \mathcal{P}(X), \delta_x \in \text{Orb}_G(\mu)\}$. Then, the following are equivalent.

(1) The action is strongly proximal.
(2) $M'$ is the unique minimal subset.
(3) $M'$ is nonempty.

Proof of Lemma 3.7. For the proof of (1), We proceed by contradiction. Let $(M_\alpha)_{\alpha \in I}$ be the family of minimal sets of $X$. Then, for any $\alpha, \beta \in I$, if $M_\alpha \cap M_\beta = \emptyset$ then $[M_\alpha] \cap [M_\beta] \neq \emptyset$, since otherwise

1Therein, the results are stated for the action of semi-group. This leads us to ask whether it is possible to extend the results of this paper to the action of semi-group.
the endpoints of the arc \([r_{M_\beta}](x), r_{M_\beta}(y)]\) (where \((x, y)\) is any pair in \([M_\beta] \times [M_\alpha]\)) would form a finite orbit. Put
\[
X_\infty = \bigcap_{\alpha} [M_\alpha].
\]
Then, by Lemma 3.8, we get that \(X_\infty\) is nonempty. We further have that the restriction action of \(G\) to it has no finite orbit. Now, let \(M_1, M_2\) be two minimal sets of \(X_\infty\), then \([M_1] = [M_2] = X_\infty\), and so \(M_1 = M_2\). At this stage, we have proved that in \(X_\infty\) there is a unique minimal set \(M\) and its convex hull \([M] = X_\infty \subset [M_\alpha]\) for any \(\alpha\). We further have, by Lemma 3.8
\[
M = \{m \in X_\infty : \forall \mu \in \mathcal{P}(X_\infty), \delta_x \in \text{Orb}_G(\mu)\}.
\]
Fix an \(\alpha \in I\) such that \(M_\alpha \cap M = \emptyset\), and put \(Y = [M_\alpha]\). Let \(x \in M_\alpha\) and suppose \(Z\) is the connected component in \(Y\) of \(Y \setminus X_\infty\) that contains \(x\) and \(a = r_{X_\infty}(x)\). It follows that for any \(g \in G\), \(g(Z) \cap X_\infty = \{g(a)\}\). By assumption, there is no finite orbit, therefore the orbit of \(a\) is infinite and so is the orbit of the set \(Z\). By taking a suitable net \((g_i)_{i \in I}\) in \(G\), we may assume that \(\lim g_i(a) = b \in X_\infty\) and \(g_i(a) \neq b\) for all \(i\). Therefore \(M\) is a subset of \(\text{Orb}_G(b)\), and the point \(b\) is obviously outside the set \(M_\alpha\). But \(X\) is rim-finite, we can thus find a neighborhood \(U\) of \(b\) in \(X\) with finite boundary and such that \(U \cap M_\alpha = \emptyset\). It turns out that for infinitely of \(i\), \(\{g_i(x)\}\) meets \(U\) however \(\{g_i(x)\}\) meets \(X \setminus U\) and this yields that the boundary of \(U\) is infinite, which is inconsistent with the rime-finiteness.

Let us now denoted by \(M\) this unique minimal set. It follows again by Lemma 3.8 that \(M = M'\). Whence \(M'\) is not empty. Therefore, the action is strongly proximal by appealing to Lemma 3.8 but the factor of strongly proximal action is strongly proximal. Whence, the action of \(G\) on \(M\) is strongly proximal. To conclude, we need to prove (3). For that we follows [13]. The proof of the lemma is complete. \(\square\)

**Proof of Theorem 3.7** Suppose that \(M\) is an infinite minimal subset. Let \((T_n)\) be the sequence of trees defined in Proposition 3.4. For each \(n \in \mathbb{N}\) and for any \(a \in E(T_n)\), let \(F^n(a)\) be the closure of the union of all connected components of \([M] \setminus \{a\}\) which are disjoint from \(T_n\). In this way, \(F^n(a) = \bigcup [a, x]\) where the union is taken over all arcs \([a, x]\) such that \(x \in M\) and \([a, x] \cap T_n = \{a\}\). we let \(F^n = \{F^n(a) : a \in E(T_n)\}\).

Claim 1. \(\lim_{n \to +\infty} \text{Mesh}(F^n) = 0\). Let \(\varepsilon > 0\) and \(\delta\) be as in Lemma 2.2. By assertion (iv) in Proposition 3.4 there is \(N \in \mathbb{N}\) such that
\[ d_H(E(T_n), M) < \delta \] for any \( n \geq N \). Take an integer \( n \geq N \) and any \( a \in E(T_n) \), if \( x \in M \) is such that \([a, x] \cap T_n = \{a\}\) then there is \( b \in E(T_n) \) such that \( d(x, b) < \delta \), it follows from Lemma 2.2 that \( \text{diam}([b, x]) < \varepsilon \). It turns out that \( \text{diam}(F^n(a)) < 2\varepsilon \). We notice that an alternative proof can be given using the continuity of the map \( F \mapsto [F] \).

Claim 2. For any \( g \in G \), \( n \in \mathbb{N} \) and \( a \in E(T_n) \),

\[ g(F^n(a)) = F^n(g(a)). \]

Indeed, let \( x \in M \) be such that \([a, x] \cap T_n = \{a\}\) then \( g([a, x]) = [g(a), g(x)] \). Moreover, as \( g(T_n) = T_n \), \([g(a), g(x)] \cap T_n = \{g(a)\}\). It follows that \( g(F^n(a)) = F^n(g(a)) \).

Now, let \( \varepsilon > 0 \) and \( x \in M \). Then by claim 1, there is \( n \in \mathbb{N} \) such that \( \text{Mesh}(F^n) < \varepsilon \). Take \( a \in E(T_n) \) such that \( x \in F^n(a) \) then \( F^n(a) \) is a neighborhood of \( x \) in \([M]\). For any \( y \in F^n(a) \) and for any \( g \in G \), we have \( g(y) \in F^n(g(a)) \). However, \( \text{diam}(F^n(g(a)) < \varepsilon \). This shows the equicontinuity of the action of \( G \) restricted to \( M \). \( \Box \)

4. ON THE ACTION OF NON-AMENABLE GROUP AND THOMPSON GROUPS.

The notion of amenability was introduced by von Neumann to shed some light on the Banach-Tarski paradox \([26]\). Roughly speaking, the Banach-Tarski paradox \([4]\) say that in \( \mathbb{R}^3 \), every two bounded sets \( A \) and \( B \) with non-empty interior can be decomposed in finite pieces say \( n \) such that \( A = \bigcup_{i=1}^{n} A_i \) and \( B = \bigcup_{i=1}^{n} B_i \) so that \( A_i \) can be rotated to \( B_i \), \( i = 1, \ldots, n \). As, we will see, von Neumann noticed that this paradox is due to the fact that the group of rotation \( SO(3) \) is not amenable.

We recall that if \( G \) is a group, then, obviously, \( G \) acts from the left on the space of all bounded complex-valued function \( \mathcal{B}(G) \) equipped with the uniform norm \( \|\cdot\|_\infty \). \( G \) is said to be amenable if there exist a non-negative bounded linear functional \( \mu \) on \( \mathcal{B}(G) \) left invariant and such that \( \mu(1) = 1 \). \( \mu \) is called left mean. In the case of discrete group (group equipped with discrete topology) this equivalent to the existence of finitely additive measure. von Neumann proved that the class of amenable groups is closed under subgroups, factor groups, extensions and direct groups. He further proved that all abelian groups are amenable. It follows that the solvable groups are amenable. By applying Følner criterion, it is easy to see that the finitely generated groups of subexponential growth are amenable. It follows that the
finitely generated nilpotent groups are amenable. We recall that the group $G$ is amenable if and only if it has a Følner sequence, that is, a sequence $\{F_n\}$ such that
\[ \frac{|gF_n \Delta F_n|}{|F_n|} \to 0 \quad \text{as} \quad n \to +\infty. \]
von Neumann proved also the following.

\textbf{Lemma 4.1.} Let $G$ be a group which contain a free non-abelian group then $G$ is not amenable.

We recall that the free group $F_S$ generated by the alphabet $S$ is the set of all the class words generated by $S$. For a nice account we refer to [15, Chap. 7]. Let $s \in S$, we denoted by $W(s)$ the set of reduced words beginning with $s$. Now, let us observe that the proof of Lemma 4.1 follows from the following.

\textbf{Lemma 4.2.} Let $F_2$ be a free non-abelian group generated by two element $s, t$ then $F_2$ is not amenable.

\textit{Proof.} Suppose, per impossible, that there is a left invariant finitely additive measure $\mu$ on $F_2$. We start by noticing that we have
\begin{equation}
F_2 = \bigcup_{g \in E} W(g), \quad \text{where} \quad E = \{e, s, s^{-1}, t, t^{-1}\}
\end{equation}
and
\begin{equation}
F_2 = W(s) \bigcup s^{-1}W(s) = W(t) \bigcup t^{-1}W(t).
\end{equation}
Therefore, $\mu\{e\} = 0$. Moreover, since $\mu$ is additive, by (4.1), we get,
\[ \mu(F_2) = \mu(\{e\}) + \mu(W(s)) + \mu(W(s^{-1})) + \mu(W(t)) + \mu(W(t^{-1})). \]
We further have, by (4.2),
\[ \mu(F_2) = \mu(W(s)) + \mu(W(s^{-1})) = \mu(W(t)) + \mu(W(t^{-1})). \]
This yields a contradiction since $\mu(F_2) = 1$, and thus the proof of the theorem is complete. \hfill \QED

Let us notice that the equation (4.1) combined with (4.2) is exactly what is called Banach-Tarski paradox.

According to Day [8], it is seems that von Neumann conjectured that any non-amenable group contain a non-abelian free group. It is turns out that this conjecture is false. Indeed, Ol’shanskii proved that the so-called “Tarski monsters” groups are not non-amenable groups [27], [28], [29]. Later, Adian showed [2] that the free Burnside group
of odd exponent $> 665$ with at least two generator is non amenable. Very recently, N. Monod \cite{21} gives a simple counterexample to von Neumann conjecture. Indeed, he proved that if $A$ is a subring of $\mathbb{R}$ and $A \neq \mathbb{Z}$, then the subgroups $H(A)$ of the group $G(A)$ of piecewise projective transformations contain finitely generated subgroups that are non-amenable without non-abelian free subgroups.

For the proof of Theorem 3.2, we start by proving the following.

**Lemma 4.3.** Let $G$ be a non-amenable group without non-abelian free group acting on the nondegenerate dendrite $X$. Then the action has a finite orbit.

**Proof.** Suppose, *per impossible*, that the action has no finite orbit. Then, by Lemma 3.7, the action on its minimal set is strongly proximal. We further have that the group $G$ contains a free group which contradicts our assumption. The proof of the lemma is complete. \qed

**Lemma 4.4.** Let $G$ be a non-amenable group without non-abelian free group acting on the nondegenerate dendrite $X$. Then the infinite minimal sets of the action of $G$ are Cantor sets.

**Proof.** By Lemma 4.3, the action has a finite orbit. Therefore, by appealing to Lemma 3.5, it follows that the infinite minimal sets are Cantor sets. The proof of the lemma is complete. \qed

**Proof of Theorem 3.2.** Take $G$ to be any non-amenable group without non-abelian free group. For example, “Tarski monsters” groups, free Burnside group of odd exponent $> 665$ with at least two generator or the subgroups of $H(A)$. Suppose $X$ is a non-degenerate dendrite then by Lemma 4.3 any action of $G$ on $X$ has a finite orbit. By Theorem 3.1, the restriction of this action to any minimal subset is equicontinuous. Moreover, by Lemma 4.4 any infinite minimal set is a Cantor set. \qed

4.1. **Thompson group $F$.** Consider the following two homeomorphisms of $[0, 1]$:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \text{if } \frac{1}{2} < x < \frac{3}{4} \\ 2x - 1 & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}, \quad g(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \text{if } \frac{1}{2} < x < \frac{3}{4} \\ x - \frac{1}{8} & \text{if } \frac{3}{4} < x < \frac{7}{8} \\ 2x - 1 & \text{if } \frac{7}{8} \leq x \leq 1. \end{cases}$$

2The representation of the group is given by $\langle a_1, \cdots, a_m/w^n = 1 \rangle$, where $w$ is in the set of all words of the alphabet $\{a_1, \cdots, a_m\}$. 

The group generated by $f$ and $g$ is called Thompson group. We denoted by $F$, so $F = \langle f, g \rangle$. For a nice account on Thompson groups, we referee to [7].

B. Duchesne and N. Monod established that the action of Thompson group has a finite orbit [9]. From this we deduce the following.

**Corollary 4.5.** Any action of the Thompson group $F$ on a dendrite $X$ restricted to minimal sets is equicontinuous.

**Remarks.** It is seems not known whether Thompson group is amenable. But, according to our main results combined with Corollary 4.5, it can be deduced that the strategy of the study of the dynamical properties of the action of $F$ on dendrite may not be the right tool to highlight the question of amenability of $F$.

5. **Invariant measures of group action on dendrites.**

We start by pointing out that, by the classical Krylov-Bogolyobov, any amenable group action has an invariant probability measure on dendrite. Indeed, Krylov-Bogolyobov procedure allows us to construct an invariant measure as limit of sequence of Dirac measures. This later sequence has a weak limit if the space $X$ is compact since the set of the probabilities measures is compact convex space with respect to the weak star topology. The limit is invariant since the group is amenable. We further have by the classical Krein-Milan theorem that the set of extremal measures, that is, ergodic probabilities measures is not empty. We recall that the probability measure $\mu$ is ergodic if the measure of any Borel invariant set is 0 or 1. We need here to explore the case of general groups. In this case, such invariant measure may not exist.

But, as for the case of the action of the group of homeomorphism on the real line and circle, on may expect to give a characterization of the support of such measure when it is exist. The endpoints will play a great role in this characterization.

It follows from the previous discussion that the action of any amenable group has a finite orbit. We thus get that the set of invariant probability measures under the action of any amenable group is not empty. Since the atomic probability measure supported on the finite orbit is in $P(X \leftrightarrow \mathcal{P} G)$. On can produce also an invariant probability measure. Indeed, we have the following.
**Theorem 5.1.** Let $G$ be an amenable group, and suppose that $G$ acts continuously on a non-degenerate dendrite $X$. Then the compact set $\mathcal{P}(X \leftrightarrow G)$ contain an invariant probability measure associated to $\mu$.

It follows, from Theorem 5.1 combined with the fact that the action of any amenable group has a finite orbit, that the set of ergodic invariant probability measures contains a Dirac measures or a continuous ergodic measures.

More precisely, by appealing to Lemma 3.5, we have the following result due essentially to Glasner and Megrelishvili [13].

**Theorem 5.2** ([13]). Let $G$ be an amenable group acts on a dendrite $X$, then the set $\mathcal{P}(X \leftrightarrow G)$ contains a uniform distribution on finite set. Moreover, the ergodic invariant measure either it is a Dirac measure, or, the unique ergodic measure on the infinite minimal set.

Let us notice that Theorem 5.2 is stated differently in [13].

We further have the following.

**Theorem 5.3.** Let $G$ act on a dendrite $X$. Then either the set $\mathcal{P}(X \leftrightarrow X)$ contains a uniform distribution of a finite set or it is empty.

We proceed now to the proof of Theorem 5.3 and Theorem 5.1.

**Proof of Theorem 5.3.** Obviously, we have only two alternative, either the action has a finite orbit or not. If the action has a finite orbit. Then, by taking the probability measure

$$\mu = \frac{1}{|M|} \sum_{x \in M} \delta_x,$$

where $\delta_x$ is a Dirac measure on $x$, it is easy to see that $\mu$ is an invariant probability measure. Now, suppose that the action has no finite orbit. Then, by Lemma 3.8 the action is strongly proximal and the minimal set is given by

$$M = \{x \in X/\forall \mu \in \mathcal{P}(X), \delta_x \in \text{Orb}_G(\mu)\}$$

Assume further that there is a $G$-invariant measure $\mu$. Then, for $x \in M$, we get $\delta_x = \text{Orbit}_G(\mu) = \mu$. This contradicts our assumption. The proof of the theorem is complete.

The previous result was announced in [14] without proof.
For the proof of Theorem 5.1, we need to introduce some definitions and tools.

Let $\mu$ be a measure on $X$ and $\nu$ a measure on $G$, we define the convolution of $\mu$ and $\nu$ denoted by $\nu \ast \mu$ by

$$\nu \ast \mu(A) = \int_G \mu(g^{-1}.A)d\nu(g),$$

for any Borel set $A$. $\mu$ is said to be stationary measure if $\nu \ast \mu = \mu$.

By appealing to the classical fixed-point theorem [11], it is easy to see that there is a stationary measure. If, moreover, $G$ admit a Haar measure and the measure $\nu$ is absolutely continuous with respect to the Haar measure, symmetric and its support is $G$, then the stationary measure is quasi-invariant (i.e., the push-forward measures of $\xi$ under the action of $G$ are equivalent to $\xi$.)

We need also the following classical result from [22, Theorem 10.28].

Lemma 5.4. If $X$ is a nondegenerate dendrite, then $X$ can be written as follows:

$$X = E(X) \cup \bigcup_{i=0}^{+\infty} A_i,$$

where $E(X)$ is is the endpoint set of $X$ and each $A_i$ is an arc with endpoints $p_i$ and $q_i$ such that $A_{i+1} \cap \left( \bigcup_{j=1}^{i} A_j \right) = \{p_{i+1}\}$, for each $i = 1, 2, \cdots$ and $diam(A_i) \longrightarrow 0$ as $i \longrightarrow +\infty$.

Let $X$ be a dendrite with a metric $\rho$. For any arc $[a, b]$ in $X$, put

$$d(a, b) = \sum_{i=1}^{+\infty} \frac{1}{2^i} |h_i^{-1}([a, b] \cap A_i)|,$$

Where $(h_i)$ is a family a homeomorphism defined for each $i$ from $[0, 1]$ to $A_i$, $|.|$ denote the Euclidean metric on the interval $[0, 1]$. Then, it is observed in [32] and it is easy to see, that $d$ is equivalent to $\rho$. 
In the same manner, we define a probability measure $\mu$ on $X$ by putting, for any $f \in C(X)$,

$$
\mu(f) = \sum_{i=1}^{+\infty} \frac{1}{2^i} \int_{h_i^{-1}(A_i)} f \circ h_i(x) dx
$$

(5.1)

$$
= \sum_{i=1}^{+\infty} \frac{1}{2^i} \int_0^1 f \circ h_i(x) dx,
$$

denote the Lebesgue measure on the interval $[0, 1]$. We further have $\mu([a, b]) = d(a, b)$.

We are now able to proof Theorem 5.1.

**Proof of Theorem 5.1** By our assumption, let $\nu$ be a mean on $G$ and $\mu$ the measure defined by (5.1). We further assume without loss of generality, that $G$ is a discrete group. Put $\tau_g(f)(x) = f(g.x)$, for $f \in C(X)$, and $\tau_h(F)(g) = F(g.h)$, for $F \in \mathcal{B}(G)$. Therefore, for any $f \in C(X)$ and $g \in G$, let us put

$$
(\Phi(f))(g) = \int f(g.x) d\mu(x).
$$

Obviously $\Phi$ is a positive linear application from $C(X)$ to $\mathcal{B}(G)$, that is,

- $\Phi(1_X) = 1_G$;
- $\Phi(f) \geq 0$ if $f \geq 0$, and
- $\Phi(f + c.g) = \Phi(f) + c.\Phi(g)$, $f, g \in C(X)$ and $c \in \mathbb{C}$.

This, combined with the Riesz representation theorem, allows us to define a positive probability measure by putting,

$$
\tilde{\mu}(f) = \nu(\Phi(f)).
$$

We further have

$$
g\tilde{\mu}(f) = \tilde{\mu}(\tau_g(f))
$$

$$
= \nu(\Phi(\tau_g(f))).
$$

But,

$$
\Phi(\tau_g(f))(h) = \int_X \tau_g(f)(h.x) d\mu(x),
$$

$$
= \int_X f(hg.x) d\mu(x)
$$

$$
= \Phi(f)(gh).\]
Whence
\[ g\tilde{\mu}(f) = \int_G \Phi(f)(gh)d\nu(h) \]
\[ = \int_G \Phi(f)(h)d\nu \]
\[ = \tilde{\mu}(f), \]
since, for any bounded function \( F : G \to \mathbb{C} \), we have
\[ \int_G F(g.h)d\nu(h) = \int_G F(h)d\nu(h). \]
The proof of the theorem is complete. \( \square \)

**Remarks.** In the proof of Theorem 5.1, we assume that \( G \) is a discrete group. Therefore, the mean is defined on \( B(G) \). For the more general case of locally compact group \( G \), the mean is defined on \( L^\infty(G) \) or \( CB(G) \) the space of bounded continuous functions.

At this point, let us further observe that we have the following result which we hope can be used in future studies and research.

**Theorem 5.5.** Let \( G \) acts on the non-degenerate dendrite \( X \) and assume that the set of recurrent points is not empty. Then there exists a continuous probability measure with gives one or zero to any invariant Borel set.

**Proof.** By Glimm-Effros theorem (see [23, p. 91]), if \( x \) is a recurrent point then there is a continuous probability measure on \( \text{Orb}_G(x) \) which gives mass zero or one to \( G \)-invariant sets. Since the set of recurrent points is not empty, the result follows. \( \square \)

**Corollary 5.6.** Let \( G \) acts on the non-degenerate dendrite \( X \). Suppose that \( G \) is an amenable group and the set of recurrent points is not empty. Then, there exists an invariant probability measure which is ergodic.

**Proof.** By Theorem 5.5 there exists a continuous probability measure with gives one or zero to any invariant Borel set. Let \( \sigma \) be such probability measure. By applying the same procedure as in the proof of Theorem 5.1 we get an invariant probability measure which is ergodic. This finish the proof of the corollary. \( \square \)

**Questions.**

1. Let \( G \) be a semi-group and \( X \) a local dendron. It is natural to ask if our results can be extended to the action of \( G \) to \( X \).
(2) Let $G$ be a group and $X$ a dendrite. One may ask on the classification of invariant $\sigma$-finite measures, ergodic invariant $\sigma$-finite measures, stationary measures (Quasi-invariant probability measures) for amenable and non-amenable group and their ergodic properties.

Let us point out that a characterization of discrete spectrum for action of amenable group on compact space is given in [1].

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