MHD Effect on Unsteady Mixed Convection Boundary Layer Flow past a Circular Cylinder with Constant Wall Temperature

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Abstract. Magnetohydrodynamic (MHD) effect is a study on motion of electrical-conducting fluid under magnetic fields. This effect has great intention due to its applications such as design of heat exchanger and nuclear reactor. In the problem in fluid motion, flow of separation can reduce the effectiveness of the system as well as can increase the energy lost. This study will present the results on reducing the flow separation by considering magnetic effect. In this study, unsteady mixed convection boundary layer flow past a circular cylinder is given attention. Focus of study is on the separation times that affected by the magnetic fields. The mathematical models in the form of partial differential equations are transformed into nonlinear coupled ordinary differential equations and solved numerically using an implicit finite-difference scheme known as Keller-box method. The effect of magnetic parameter on velocity and temperature profiles as well as skin friction and Nusselt number are studied.

1. Introduction
MHD convection flow has many important engineering applications in the design of heat exchangers, pumps, and flow meters for solving space vehicle propulsion, control, and reentry problems. In designing communications and radar systems and creating novel power-generating systems MHD has great influences in this fields. Other than that, developing confinement schemes for controlled fusion and nuclear engineering also used MHD to increase the cooling function and connection for the system. A considerable amount of work has been devoted to the study of unsteady, mixed convection flow on circular cylinder imposed on a magnetic field [1-4].

Ram and Jain [1] observed the effect of strong magnetic field on the velocity and temperature distributions of an electrically conducting liquid flowing past an infinite vertical porous plate through a porous medium in a rotating frame of reference. Sacheti et al. [2] performed the exact solution for unsteady magnetohydrodynamic free convection flow of a viscous incompressible and electrical conducting fluid generated by an impulsively moving vertical plate subjected to constant heat flux. The exact solution is obtained using Laplace transform. The authors concluded that the magnetic field has a retarding effect on velocity while increasing the skin friction at the plate. Cramer and Pai [3] presented a similarity solution for the natural convection boundary layer flow of an electrically conducting fluid on a hot vertical wall in the presence of a strong magnetic field for varying surface temperature.
Recently, Hossain and Ahmed [4] studied the combined effect of forced and free convection boundary layer flow near the leading edge of a vertical plate with a uniform surface heat flux and a magnetic field applied parallel to the plate. The equations governing the flow are solved numerically using the method of superposition for small values of \( g \), the buoyancy parameter. They observed that the increase of the magnetic field leads to a decrease in the velocity and a rise in the values of the temperature in the flow field. Next, Aldoss et al. [5] studied mixed convection on cylinder with the present of magnetic fields. Javed et al. [6] studied the magnetic fields effect on circular cylinder with the present of magnetic fields. The result shows the magnetic fields give more resistance to the surface of the cylinder. Other than that, Merkin [7] studied unsteady mixed convection boundary layer flow on an isothermal horizontal circular cylinder. Ingham [8] examined free convection boundary layer flow on an isothermal horizontal cylinder.

Although convective heat transfer about heated bodies with different geometries has been studied by many authors, few works are available in literature dealing with the effect of the magnetic field on these bodies except those directed toward the effect of the magnetic field on flow on circular cylinder. In the present article, MHD mixed convection flow from a horizontal circular cylinder is investigated. The focuses for this paper is on separation times where many authors did not go through yet. A variable surface temperature boundary condition is considered, and a non-similarity solution is obtained. The results are obtained using local non-similarity and verified using a coordinate perturbation method. The influences of the magnetic field on local Nusselt number and skin friction are present.

2. Mathematical Model

In this study, we consider unsteady two-dimensional mixed convection boundary layer flow past a circular cylinder of a radius \( a \), which is placed in an incoming stream of viscous fluid with a constant free-stream velocity of \( U_\infty \) and constant temperature \( T_\infty \). It is assumed that the force convection is moving upward, while the gravity sector is acts downward. It also assume that the cylinder and fluid is maintained at constant temperature, \( T_w \), with \( T_w > T_\infty \) for heated cylinder (assisting flow) and \( T_w < T_\infty \) for cooled cylinder (opposing flow) and ensuring no slip velocity between them.

The basic equations of the problem consist of continuity, momentum, and energy equation in Cartesian coordinates \( x \) and \( y \) are [9]

\[
\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0
\]

\[
\rho \left( \frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial x} + \hat{v} \frac{\partial \hat{u}}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} \right] + (\rho \beta) g \left( \bar{T} - T_\infty \right) \sin \left( \frac{x}{a} \right) - \sigma B_0^2 \hat{u}
\]

\[
\rho \left( \frac{\partial \hat{v}}{\partial t} + \hat{u} \frac{\partial \hat{v}}{\partial x} + \hat{v} \frac{\partial \hat{v}}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left[ \frac{\partial^2 \hat{v}}{\partial x^2} + \frac{\partial^2 \hat{v}}{\partial y^2} \right] - (\rho \beta) g \left( \bar{T} - T_\infty \right) \cos \left( \frac{x}{a} \right) - \sigma B_0^2 \hat{v}
\]

\[
\frac{\partial \bar{T}}{\partial t} + \hat{u} \frac{\partial \bar{T}}{\partial x} + \hat{v} \frac{\partial \bar{T}}{\partial y} = \frac{k}{\rho c_p} \left[ \frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} \right]
\]

subject to boundary condition
\[ T < 0 : \quad \bar{u} = \bar{v} = 0, \bar{T} = T_\infty \quad \text{for any } \bar{x}, \bar{y}, \]

\[ T \geq 0 : \quad \bar{u} = \bar{v} = 0, \bar{T} = T_\infty \quad \text{at } \bar{y} = 0 \]

\[ \bar{u} = \bar{u}_e(x), \bar{T} = T_\infty \quad \text{as } \bar{y} \to \infty \]  \hspace{1cm} (5)

Figure 1. Physical model of the coordinate system.

Where in this study \( \bar{u}_e = U_\infty \sin(\bar{x}/a) \). Here \( \bar{u} \) and \( \bar{v} \) are the velocity components along the \( \bar{x} \) and \( \bar{y} \) axes, respectively, \( \bar{u}_e(x) \) is the local free-stream velocity, \( T \) is the fluid temperature, \( \bar{p} \) is the fluid pressure, \( \beta \) is the thermal expansion coefficient of the fluid, \( \rho \) is the density of the fluid, \( \mu \) is the viscosity of the fluid, \( k \) is thermal conductivity of the fluid, and \( c_p \) is heat capacitance of the fluid.

The dimensionless variables [10] are introduced to simplified all equations (1) -(5)

\[ x = \bar{x}/a, \quad y = \text{Re}^{1/2}(\bar{y}/a), \quad t = U_\infty \bar{T}/a, \quad u = \bar{u}/U_\infty, \]

\[ v = \text{Re}^{1/2}(\bar{v}/U_\infty), \quad T = (\bar{T} - T_\infty)/(T_\infty - T_\infty), \]

\[ u_e(x) = \bar{u}_e(x)/U_\infty, \quad p = (\bar{p} - p_\infty)/(\rho_\infty U_\infty^2). \]  \hspace{1cm} (6)

Where \( \text{Re} = \frac{U_\infty a}{\nu} \) is Reynold number, \( \nu \) is kinematic viscosity and \( p_\infty \) is free stream pressure. Then substitute all variables into equation (1) -(4) and applied boundary layer approximation, those equation become

\[ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \]  \hspace{1cm} (7)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial^2 u}{\partial y^2} \]  \hspace{1cm} (8)

\[ \lambda T \sin x - Mu \]

\[ -\frac{\partial \bar{p}}{\partial y} = 0 \]  \hspace{1cm} (9)

\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{Pr} \frac{\partial^2 T}{\partial y^2} \]  \hspace{1cm} (10)

Where \( \text{Pr} \) is Prandtl number, \( \lambda \) is mixed convection parameter, and \( M \) is magnetic parameter which define as
\[ \Pr = \frac{\nu (\rho C_v)}{k}, \quad \lambda = \frac{Gr}{Re^2}, \quad M = \frac{\sigma B_0^2 a}{\rho U_w} \]  

with \( Gr = \frac{g \beta (T_w - T_{x0}) a^3}{\nu^2} \), being Grashof number also \( \lambda > 0 \) is for heated cylinder and \( \lambda < 0 \) is for cooled cylinder. Then the boundary condition (5) become

\[
\begin{align*}
    t < 0: & \quad u = v = 0, \quad T = 0 \quad \text{for any } x, y, \\
    t \geq 0: & \quad u = v = 0, \quad T = 1 \quad \text{at } y = 0 \\
    & \quad u = u_e(x), \quad T = 0 \quad \text{as } y \to \infty
\end{align*}
\]

From equation (9) shows that \( p = p(x) \) therefore

\[ -\frac{\partial p}{\partial x} = u_e \frac{\partial u_e}{\partial x} + Mu_e \]  

Substitute equation (13) into equation (8) and get

\[
\begin{align*}
    \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\
    + u \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} = u_e \frac{du_e}{dx} + \frac{\partial^2 u}{\partial y^2} + \\
    \lambda T \sin x - M (u - u_e) \\
    + u \frac{\partial T}{\partial t} + v \frac{\partial T}{\partial y} = \frac{1}{\Pr} \frac{\partial^2 T}{\partial y^2}
\end{align*}
\]

where \( u_e = \sin x \) and \( \frac{du_e}{dx} = 1 \) for forward stagnation and \( \frac{du_e}{dx} = -1 \) for rear stagnation. Now we introduce similarity transformation to solve equation (14) to (16) subject to boundary condition (12). Therefore, the similarity variables are defined as [11]

\[ u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \]  

For small time case, the stream function is defined as

\[ \psi = \frac{1}{\tau^2} u_e(x) f(x, \eta, t), \quad T = s(x, \eta, t), \quad \eta = y / \tau^2 \]  

Then the equation (15) – (16) becomes
\[ \frac{\partial^3 f}{\partial \eta^3} + \eta \frac{\partial^2 f}{\partial \eta^2} + \frac{du_c}{dx}t \left[ 1 - \left( \frac{\partial f}{\partial \eta} \right)^2 + f \frac{\partial^2 f}{\partial \eta^2} \right] \]
\[ + tM \left( 1 - \frac{\partial f}{\partial \eta} \right) + t\lambda s = t \frac{\partial^2 f}{\partial \eta \partial t} + tu_c \left[ \frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \eta \partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} \right] \]
\[ \frac{\partial^3 s}{\partial \eta^3} + \frac{Pr}{2} \frac{\partial s}{\partial \eta} + \frac{du_c}{dx} Pr t \frac{\partial s}{\partial \eta} = Pr t \left[ \frac{\partial s}{\partial t} + u_c \left( \frac{\partial f}{\partial \eta} \frac{\partial s}{\partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial s}{\partial \eta} \right) \right] \]  

(19)  

Subject to boundary condition

\[ t < 0: \quad f = 0, \quad \frac{\partial f}{\partial \eta} = 0, \quad s = 0 \quad \text{for any } x, y, \]
\[ t \geq 0: \quad f = \frac{\partial f}{\partial \eta} = 0, \quad s = 1 \quad \text{at } y = 0 \]
\[ \frac{\partial f}{\partial \eta} = 1, \quad s = 0 \quad \text{as } y \to \infty \]  

(20)  

For large time case, consider

\[ \psi = u_c(x)F(x,Y,t), \quad T = S(x,Y,t), \quad Y = y \]  

(22)  

Therefore equation (15) to (16) becomes

\[ \frac{\partial^3 F}{\partial Y^3} + \frac{du_c}{dx} \left[ 1 - \left( \frac{\partial F}{\partial Y} \right)^2 + F \frac{\partial^2 F}{\partial Y^2} \right] + M \left( 1 - \frac{\partial F}{\partial Y} \right) \]
\[ + \lambda S = \frac{\partial^2 F}{\partial Y \partial t} + \left[ \frac{\partial F}{\partial Y} \frac{\partial^2 F}{\partial Y \partial \xi} - \frac{\partial F}{\partial \xi} \frac{\partial^2 F}{\partial Y^2} \right] \]
\[ \frac{\partial^3 S}{\partial Y^3} + Pr \frac{du_c}{dx} \frac{\partial S}{\partial Y} = Pr t \left[ \frac{\partial S}{\partial t} + u_c \left( \frac{\partial F}{\partial Y} \frac{\partial S}{\partial \xi} - \frac{\partial F}{\partial \xi} \frac{\partial S}{\partial Y} \right) \right] \]  

(23)  

(24)  

Subject to boundary condition

\[ F = 0, \quad \frac{\partial F}{\partial Y} = 0, \quad S = 1 \quad \text{at } Y = 0 \]
\[ \frac{\partial F}{\partial Y} = 1, \quad S = 0 \quad \text{as } Y \to \infty \]  

(25)  

3. Result and Discussion

The equation (19) – (20) for small time cases and (23) – (24) subject to its boundary condition then being solved numerically by using implicit finite difference method known as Keller-box method. The computational process is being completed in MATLAB environment. The result such as separation times and other physical quantities are presented in tabular form. The value of mixed convection parameter is chosen as \( \lambda = 1 \) (assisting flow) and \( \lambda = -1 \) (opposing flow). To verify the computational
programming, we compare the result with Anati [9] for $Pr = 1$ and $\lambda = -3$ as showed in Table 1. The results show an absolute agreement.

Table 1. Comparison of separation time at the surface of a cylinder when $Pr = 1$ and mixed convection, $\lambda = -3$ and $M=0$.

| Author       | Separation time, $t_s$ for $\lambda = -3$ |
|--------------|------------------------------------------|
|              | $180^\circ$ | $171^\circ$ | $162^\circ$ | $135^\circ$ | $126^\circ$ | $108^\circ$ |
| Anati [9]    | 0.2194      | 0.2205      | 0.2234      | 0.2439      | 0.2557      | 0.2873      |
| Present Study| 0.2120      | 0.2204      | 0.2231      | 0.2440      | 0.2554      | 0.2875      |

Table 2. Separation Times for various value of $x$ and $M$ when $Pr = 0.7$ and $\lambda = -1$ (opposing flow).

| $x/M$ | M=0    | M=0.1   | M=0.5   | M=1.0   | M=1.5   |
|-------|--------|---------|---------|---------|---------|
| $180^\circ$ | 0.3577 | 0.3737  | 0.4552  | 0.6304  | 1.0625  |
| $171^\circ$ | 0.3800 | 0.4050  | 0.5524  | 1.0017  | -       |
| $162^\circ$ | 0.3890 | 0.4145  | 0.5593  | 0.9583  | -       |
| $153^\circ$ | 0.4032 | 0.4315  | 0.6016  | 1.1731  | -       |
| $144^\circ$ | 0.4260 | 0.4567  | 0.6381  | 1.1924  | -       |
| $135^\circ$ | 0.4565 | 0.4929  | 0.7246  | -       | -       |
| $126^\circ$ | 0.5002 | 0.5429  | 0.8148  | -       | -       |
| $117^\circ$ | 0.5580 | 0.6130  | 1.0033  | -       | -       |
| $108^\circ$ | 0.6410 | 0.7121  | 1.2406  | -       | -       |
| $099^\circ$ | 0.7756 | 0.8585  | 1.7850  | -       | -       |
| $090^\circ$ | 0.9325 | 1.0880  | -       | -       | -       |

Table 3. Separation Times for various value of $x$ and $M$ when $Pr = 7$ and $\lambda = 1$ (assisting flow).

| $x/M$ | M=0    | M=0.1   | M=0.5   | M=1.0   | M=1.5   |
|-------|--------|---------|---------|---------|---------|
| $180^\circ$ | 0.3577 | 0.3737  | 0.4552  | 0.6304  | 1.0625  |
| $171^\circ$ | 0.3800 | 0.4050  | 0.5524  | 1.0017  | -       |
| $162^\circ$ | 0.3890 | 0.4145  | 0.5593  | 0.9583  | -       |
| $153^\circ$ | 0.4032 | 0.4315  | 0.6016  | 1.1731  | -       |
| $144^\circ$ | 0.4260 | 0.4567  | 0.6381  | 1.1924  | -       |
| $135^\circ$ | 0.4565 | 0.4929  | 0.7246  | -       | -       |
| $126^\circ$ | 0.5002 | 0.5429  | 0.8148  | -       | -       |
| $117^\circ$ | 0.5580 | 0.6130  | 1.0033  | -       | -       |
| $108^\circ$ | 0.6410 | 0.7121  | 1.2406  | -       | -       |
| $099^\circ$ | 0.7756 | 0.8585  | 1.7850  | -       | -       |
| $090^\circ$ | 0.9325 | 1.0880  | -       | -       | -       |
Tables 2 and 3 present separation time around a circular cylinder surface for some values of $M$ when $\Pr = 0.7, 7$ and $\lambda = -1, 1$ for both cases. From these tables, it shows that separation occur everywhere along the cylinder surface. The separation time increases with the increased values of magnetic parameter, $M$. For each values of $M$ and at any $x$ location, the separation times is higher when the values of magnetic parameter are increased. The same behavior as stated above are observed at both cases. We also observed that on a given $x$ location, the separation times for the case of $\Pr = 7$ is higher compared to that of the case $\Pr = 0.7$. There are also $x$ locations where flow do not separate for both cases. At these point, the separation has stop separates because of resistance from the magnetic fields to the fluid flows. In the case of $M = 0$, leads to an earlier location compare to other values. As $t$ increased, the separation point moves form rear stagnation point up to $x = 144^\circ$ for $M = 1$ in both cases. For $M = 0$ the separation point could move to forward stagnation.

![Figure 2](image1.png)

**Figure 2.** (a,b) Variation with $x$ of skin friction coefficient, $C_f \Re^\frac{1}{2}$ around circular cylinder when $M = 0, 0.2$, $\Pr = 1$ and $\lambda = 1$.

![Figure 3](image2.png)

**Figure 3.** (a,b) Variation with $x$ of skin friction coefficient, $C_f \Re^\frac{1}{2}$ around circular cylinder when $M = 0.0, 1.0$, $\Pr = 0.7$ and $\lambda = -1$. 
Figure 4. (a, b) Variation with $x$ of Nusselt number, $Nu \cdot Re^{1/2}$ around circular cylinder when $M = 0.0$, $0.5$, $Pr = 0.7$ and $\lambda = -1$.

Figure 5. (a, b) Variation with $x$ of Nusselt number, $Nu \cdot Re^{1/2}$ around circular cylinder when $M = 0.0$, $0.2$, $Pr = 7$ and $\lambda = 1$.

The effects of magnetic parameter $M$ on skin-friction coefficient $C_f \cdot Re^{1/2}$ and the Nusselt number $Nu \cdot Re^{1/2}$ are plotted in Figures (a, b) respectively. One can see prominent difference between the absence of magnetic parameter ($M = 0.0$) and in the presence of magnetic parameter. As we discussed above magnetic parameter produces resistance in fluid motion and these effects can give effect to skin friction. So, skin friction will increase in presence of magnetic parameter. It is concluded that skin friction at the surface for $M = 0.0$ remains lower as compare to present magnetic fields for both Figures 2 and 3. By including the effects of magnetic parameter the thermal energy within boundary layer reduces the temperature difference between surface of cylinder and thermal boundary layer. Hence rate of heat transfer will increase in presence of magnetic parameter $M$ (Figures 4 and 5). This happens due to the physical fact that the increment in the value of $M$ implies more resistance to momentum and thermal boundary layers. As $t$ increased, the value of $C_f \cdot Re^{1/2}$ are decreased at $x$. there are negative values of $C_f \cdot Re^{1/2}$ detected around $100^\circ$ to $180^\circ$ for opposing cases ($\lambda = -1$ ). For $t \geq 0.5$, the maximum value of $C_f \cdot Re^{1/2}$ is about $C_f \cdot Re^{1/2} = 1.5$, around $x = 60^\circ$ which means at this point, the fluid is accelerating at its best. Then the values of $C_f \cdot Re^{1/2}$ decreases and turning into negative values which signify the reversal of flow occurred and can be detected at these points.
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