A spinor-like representation of the contact superconformal algebra $K'(4)$

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In this work we construct an embedding of a nontrivial central extension of the contact superconformal algebra $K'(4)$ into the Lie superalgebra of pseudodifferential symbols on the supercircle $S^{1|2}$. Associated with this embedding is a one-parameter family of spinor-like tiny irreducible representations of $K'(4)$ realized just on 4 fields instead of the usual 16.

I. Introduction

Recall that a superconformal algebra is a simple complex Lie superalgebra, such that it contains the centerless Virasoro algebra (i.e. the Witt algebra) $Witt = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n$ as a subalgebra, and has growth 1. The $\mathbb{Z}$-graded superconformal algebras are ones for which $adL_0$ is diagonalizable with finite-dimensional eigenspaces; see Ref. 1. In general, a superconformal algebra is a subalgebra of the Lie superalgebra of all derivations of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $\Lambda(N)$ is the Grassmann algebra in $N$ odd variables.

The Lie superalgebra $K(N)$ of contact vector fields with Laurent polynomials as coefficients is characterized by its action on a contact 1-form (Refs. 1, 2, 3, and 25); it is isomorphic to the $SO(N)$ superconformal algebra (Ref. 4). $K(N)$ is simple except when $N = 4$. In this case $K'(4) = [K(4), K(4)]$ is simple. Note that $K'(N)$ is spanned by $2^N$ fields. It was discovered independently in Ref. 3 and Ref. 5 that the Lie superalgebra of contact vector fields with polynomial coefficients in 1 even and 6 odd variables contains an exceptional simple Lie superalgebra (see also Ref. 2, Refs. 6, 7, and Refs. 23, 24). In Ref. 3 the exceptional superconformal algebra $CK_6$ was discovered as a subalgebra of $K(6)$, and it was shown that the derived Lie superalgebra of divergence-free derivations of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(2)$, which is spanned by 8 fields, can be realized inside $K(4)$ using the construction of $CK_6$ inside $K(6)$.

Note that a Lie algebra of contact vector fields can be realized as a subalgebra of Poisson algebra; see Ref. 8. The Poisson algebra of formal Laurent series on $\tilde{T}^*S^1 = T^*S^1 \setminus S^1$ has
a well-known deformation, that is the Lie algebra $R$ of pseudodifferential symbols on the circle. The Poisson algebra can be considered to be the semiclassical limit of $R$; see Refs. 9, 10, 11, and 12.

In this work we define a family $R_h(N)$ of Lie superalgebras of pseudodifferential symbols on the supercircle $S^1|N$, where $h \in [0, 1]$, which contracts to the Poisson superalgebra.

For each $h$ we construct an embedding of a central extension $\hat{K}'(4)$ into $R_h(2)$. These central extensions are isomorphic to one of 3 independent central extensions, which are known for $K'(4)$ (Refs. 1, 2, 13 and 14). The corresponding central element is $h \in R_h(2)$. The elements of embeddings of $\hat{K}'(4)$ are power series in $h$; considering their limits as $h \to 0$, we obtain an embedding of $K'(4)$ into the Poisson superalgebra.

The idea of our construction is as follows. We consider the Schwimmer-Seiberg’s deformation $S(2, \alpha)$ of the Lie superalgebra of divergence-free derivations of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(2)$ (Refs. 15 and 1) and observe that the exterior derivations of $S'(2, \alpha)$ form an $\mathfrak{sl}(2)$ if $\alpha \in \mathbb{Z}$. The exterior derivations of $S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ generate a subalgebra of the Poisson superalgebra isomorphic to the loop algebra $\tilde{\mathfrak{sl}}(2)$ [$\mathfrak{sl}(2)$ corresponds to $\alpha = 1$]. We prove that the family $S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ and $\tilde{\mathfrak{sl}}(2)$ generate a Lie superalgebra isomorphic to $K'(4)$. The similar construction for each $h \in [0, 1]$ gives an embedding of a nontrivial central extension of $K'(4)$:

$$\hat{K}'(4) \subset R_h(2).$$

(1.1)

It is known that the Lie algebra $R$ has two independent central extensions; see Refs. 9, 10, and 11. Accordingly, there exist, up to equivalence, two nontrivial 2-cocycles on its superanalog $R_{h=1}(N)$. The 2-cocycle on $K'(4)$, which corresponds to the central extension $\hat{K}'(4)$ is equivalent to the restriction of one of the 2-cocycles on $R_{h=1}(2)$.

Finally, the embedding (1.1) for $h = 1$ allows us to define a new one-parameter family of tiny irreducible representations of $\hat{K}'(4)$. Recall that there exists a two-parameter family of representations of $K'(N)$ in the superspace spanned by $2^N$ fields. These representations are defined by the natural action of $K'(N)$ in the spaces of “densities”; see Ref. 1.

We obtain representations of $\hat{K}'(4)$, where the value of the central charge is equal to 1, realized on just 4 fields, instead of the usual 16.

II. Superconformal algebras

In this section we review the notion of a superconformal algebra and give the necessary
A superconformal algebra is a complex Lie superalgebra \( g \) such that

1) \( g \) is simple,

2) \( g \) contains the Witt algebra \( Witt = \text{der} \mathbb{C}[t, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \) with the well-known commutation relations
\[
[L_n, L_m] = (n - m)L_{n+m}
\]
as a subalgebra,

3) \( \text{ad}L_0 \) is diagonalizable with finite-dimensional eigenspaces:
\[
g = \bigoplus j \ g_j, \quad g_j = \{ x \in g \mid [L_0, x] = jx \},
\]
so that \( \text{dim} g_j < C \), where \( C \) is a constant independent of \( j \); see Ref. 1. The main series of superconformal algebras are \( W(N) \) \((N \geq 0)\), \( S'(N, \alpha) \) \((N \geq 2)\), and \( K'(N) \) \((N \geq 1)\); see Refs. 1 and 25. The corresponding central extensions were classified in Ref. 1; see also Refs. 2, 13, 14 and 16.

The superalgebras \( W(N) \). Consider the superalgebra \( \mathbb{C}[t, t^{-1}] \otimes \Lambda(N) \), where \( \Lambda(N) \) is the Grassmann algebra in \( N \) variables \( \theta_1, \ldots, \theta_N \). Let \( p \) be the parity of the homogeneous element. Let \( p(t) = 0 \) and \( p(\theta_i) = 1 \) for \( i = 1, \ldots, N \). By definition \( W(N) \) is the Lie superalgebra of all derivations of \( \mathbb{C}[t, t^{-1}] \otimes \Lambda(N) \). Let \( \partial_i \) stand for \( \partial/\partial \theta_i \) and \( \partial_t \) stand for \( \partial/\partial t \). Every \( D \in W(N) \) is represented by a differential operator,
\[
D = f \partial_t + \sum_{i=1}^{N} f_i \partial_i
\]
where \( f, f_i \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N) \). \( W(N) \) has no nontrivial 2-cocycles if \( N > 2 \). If \( N = 1 \) or \( 2 \), then there exists, up to equivalence, one nontrivial 2-cocycle on \( W(N) \).

The superalgebras \( S(N, \alpha) \). The Lie superalgebra \( W(N) \) contains a one-parameter family of Lie superalgebras \( S(N, \alpha) \); see Refs. 15 and 1. By definition
\[
S(N, \alpha) = \{ D \in W(N) \mid \text{Div}(t^\alpha D) = 0 \} \text{ for } \alpha \in \mathbb{C}.
\]
Recall that
\[
\text{Div}(D) = \partial_t(f) + \sum_{i=1}^{N} (-1)^{p(f_i)} \partial_i(f_i)
\]
and
\[
\text{Div}(fD) = Df + f\text{Div}D,
\]
where $f$ is an even function. Let $S'(N, \alpha) = [S(N, \alpha), S(N, \alpha)]$ be the derived superalgebra. Assume that $N > 1$. If $\alpha \notin \mathbb{Z}$, then $S(N, \alpha)$ is simple, and if $\alpha \in \mathbb{Z}$, then $S'(N, \alpha)$ is a simple ideal of $S(N, \alpha)$ of codimension one defined from the exact sequence,

$$0 \to S'(N, \alpha) \to S(N, \alpha) \to \mathbb{C} t^{-\alpha} \theta_1 \cdots \theta_N \partial_t \to 0. \quad (2.7)$$

Notice that

$$S(N, \alpha) \cong S(N, \alpha + n) \text{ for } n \in \mathbb{Z}. \quad (2.8)$$

There exists, up to equivalence, one nontrivial 2-cocycle on $S'(N, \alpha)$ if and only if $N = 2$; see Ref. 1. Let $\hat{S}'(2, \alpha)$ be the corresponding central extension of $S'(2, \alpha)$. Note that $S'(2, \alpha)$ is spanned by 4 even fields and 4 odd fields. Sometimes the name “$N = 4$ superconformal algebra” is used for $\hat{S}'(2,0)$; see Refs. 4 and 3.

The superalgebras $K(N)$. By definition

$$K(N) = \{ D \in W(N) \mid D\Omega = f\Omega \text{ for some } f \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N) \}, \quad (2.9)$$

where

$$\Omega = dt - \sum_{i=1}^{N} \theta_i d\theta_i \quad (2.10)$$

is a contact 1-form; see Refs. 1, 2, 3, and 25. (See also Ref. 26, where the contact superalgebra $K(m, n)$ was introduced, and Ref. 24). Every differential operator $D \in K(N)$ can be represented by a single function,

$$f \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N) : f \to D_f. \quad (2.11)$$

Let

$$\Delta(f) = 2f - \sum_{i=1}^{N} \theta_i \partial_i(f). \quad (2.12)$$

Then

$$D_f = \Delta(f) \partial_t + \partial_t(f) \sum_{i=1}^{N} \theta_i \partial_i + (-1)^{p(f)} \sum_{i=1}^{N} \partial_i(f) \partial_i. \quad (2.13)$$

Notice that

$$D_{f+g} = D_f + D_g, \quad (2.14)$$

$$[D_f, D_g] = D_{\{f,g\}},$$

where

$$\{f,g\} = \Delta(f) \partial_t(g) - \partial_t(f) \Delta(g) + (-1)^{p(f)} \sum_{i=1}^{N} \partial_i(f) \partial_i(g). \quad (2.15)$$
The superalgebras $K(N)$ are simple, except when $N = 4$. If $N = 4$, then the derived superalgebra $K'(4) = [K(4), K(4)]$ is a simple ideal in $K(4)$ of codimension one defined from the exact sequence

$$0 \to K'(4) \to K(4) \to CD_{t^{-1} \theta_1 \theta_2 \theta_3 \theta_4} \to 0. \quad (2.16)$$

There exists no nontrivial 2-cocycles on $K(N)$ if $N > 4$. If $N \leq 3$, then there exists, up to equivalence, one nontrivial 2-cocycle. Let $\hat{K}(N)$ be the corresponding central extension of $K(N)$. Notice that $\hat{K}(1)$ is isomorphic to the Neveu-Schwarz algebra (Ref. 17), and $\hat{K}(2) \cong \hat{W}(1)$ is isomorphic to the so-called $N = 2$ superconformal algebra; see Ref. 18. The superalgebra $K'(4)$ has 3 independent central extensions (Refs. 1, 2, 13 and 14), which is important for our task.

III. Lie superalgebras of pseudodifferential symbols

Recall that the ring $R$ of pseudodifferential symbols is the ring of the formal series

$$A(t, \xi) = \sum_{i = -\infty}^{n} a_i(t)\xi^i, \quad (3.1)$$

where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the variable $\xi$ corresponds to $\partial/\partial t$; see Refs. 9, 10, 11, and 12. The multiplication rule in $R$ is determined as follows:

$$A(t, \xi) \circ B(t, \xi) = \sum_{n \geq 0} \frac{1}{n!} \partial_\xi^n A(t, \xi) \partial_t^n B(t, \xi). \quad (3.2)$$

Notice that $R$ is a generalization of the associative algebra of the regular differential operators on the circle, and the multiplication rule in $R$, when restricted to the polynomials in $\xi$, coincides with the multiplication rule for the differential operators. The Lie algebra structure on $R$ is given by

$$[A, B] = A \circ B - B \circ A, \quad (3.3)$$

where $A, B \in R$.

The Poisson algebra $P$ of pseudodifferential symbols has the same underlying vector space. The multiplication in $P$ is naturally defined. The Poisson bracket is defined as follows:

$$\{A(t, \xi), B(t, \xi)\} = \partial_\xi A(t, \xi) \partial_t B(t, \xi) - \partial_t A(t, \xi) \partial_\xi B(t, \xi). \quad (3.4)$$
(Refs. 12 and 19). One can construct the contraction of the Lie algebra $R$ to $P$ using the linear isomorphisms:

$$\varphi_h : R \rightarrow R$$

(3.5)

defined by

$$\varphi_h(a_i(t)\xi^i) = a_i(t)h^i\xi^i, \text{ where } h \in [0, 1],$$

(3.6)

see Ref. 12. The new multiplication in $R$ is defined by

$$A \circ_h B = \varphi_h^{-1}(\varphi_h(A) \circ \varphi_h(B)).$$

(3.7)

Correspondingly, the commutator is

$$[A, B]_h = A \circ_h B - B \circ_h A.$$  

(3.8)

Thus

$$[A, B]_h = h\{A, B\} + hO(h).$$

(3.9)

Hence

$$\lim_{h \to 0} \frac{1}{h} [A, B]_h = \{A, B\}.$$  

(3.10)

To construct a superanalog of $R$, consider an associative superalgebra $\Theta_h(N)$ with generators $\theta_1, \ldots, \theta_N, \partial_1, \ldots, \partial_N$ and relations

$$\theta_i\theta_j = -\theta_j\theta_i,$$

$$\partial_i\partial_j = -\partial_j\partial_i,$$

$$\partial_i\theta_j = h\delta_{i,j} - \theta_j\partial_i,$$

(3.11)

where $h \in [0, 1]$. Define an associative superalgebra,

$$R_h(N) = R \otimes \Theta_h(N),$$

(3.12)

such that

$$(A \otimes X)(B \otimes Y) = \frac{1}{h}(A \circ_h B) \otimes (XY),$$

(3.13)

where $A, B \in R$, and $X, Y \in \Theta_h(N)$. The product in $R_h(N)$ determines the natural Lie superalgebra structure on this space:

$$[(A \otimes X), (B \otimes Y)]_h = \frac{1}{h}(A \circ_h B) \otimes (XY) - (-1)^{p(X)p(Y)}\frac{1}{h}(B \circ_h A) \otimes (YX).$$

(3.14)
For each $h \in [0, 1]$ there exists an embedding

$$W(N) \subset R_h(N), \quad (3.15)$$

such that the commutation relations in $R_h(N)$, when restricted to $W(N)$, coincide with the commutation relations in $W(N)$. In particular, when $h = 1$, we obtain the superanalog $R(N) := R_{h=1}(N)$ of the Lie algebra of pseudodifferential symbols on the circle.

The Poisson superalgebra $P(N)$ has the underlying vector space $P \otimes \Theta(N)$, where

$$\Theta(N) := \Theta_{h=0}(N)$$

is the Grassman algebra with generators $\theta_1, \ldots, \theta_N, \bar{\theta}_1, \ldots, \bar{\theta}_N$, where $\bar{\theta}_i = \partial_i$ for $i = 1, \ldots, N$. The Poisson bracket is defined as follows:

$$\{A, B\} = \partial_\xi A \partial_t B - \partial_t A \partial_\xi B - (-1)^{p(A)} \sum_{i=1}^{N} \partial_{\theta_i} A \partial_{\bar{\theta}_i} B + \partial_{\bar{\theta}_i} A \partial_{\theta_i} B, \quad (3.16)$$

where $A, B \in P(N)$; cf. Refs. 2, 5. Thus

$$\lim_{h \to 0} [A, B]_h = \{A, B\}. \quad (3.17)$$

Correspondingly, we have the embedding

$$W(N) \subset P(N). \quad (3.18)$$

**Remark 3.1:** Recall that there exist, up to equivalence, two nontrivial 2-cocycles on $R$ (Refs. 9, 10, and 11). Analogously, one can define two 2-cocycles, $c_\xi$ and $c_\tau$, on $R(N)$; cf. Ref. 20. Let $A, B \in R$, and $X, Y \in \Theta_{h=1}(N)$. Then

$$c_\xi (A \otimes X, B \otimes Y) = \text{the coefficient of } t^{-1} \xi^{-1} \theta_1 \ldots \theta_N \partial_1 \ldots \partial_N$$

in $([\log \xi, A] \circ B) \otimes (XY)$, \quad (3.19)

where

$$[\log \xi, A(t, \xi)] = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \partial^k_t A(t, \xi) \xi^{-k}, \quad (3.20)$$

and

$$c_\tau (A \otimes X, B \otimes Y) = \text{the coefficient of } t^{-1} \xi^{-1} \theta_1 \ldots \theta_N \partial_1 \ldots \partial_N$$

in $([\log t, A] \circ B) \otimes (XY)$, \quad (3.21)

where

$$[\log t, A(t, \xi)] = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} t^{-k} \partial^k_\xi A(t, \xi). \quad (3.22)$$
IV. The construction of embedding

Let $\text{Der} S'(2, \alpha)$ be the Lie superalgebra of all derivations of $S'(2, \alpha)$.

Lemma 4.1: The exterior derivations $\text{Der}_{\text{ext}} S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ generate the loop algebra

$$\tilde{\mathfrak{sl}}(2) \subset P(2). \quad (4.1)$$

Proof: In Ref. 21 we observed that the exterior derivations of $S'(2, 0)$ form an $\mathfrak{sl}(2)$. Let

$$\{\mathcal{L}_n^\alpha, E_n, H_n, F_n, h_n^\alpha, p_n^0, x_n^0, y_n^\alpha\}_{n \in \mathbb{Z}} \quad (4.2)$$

be a basis of $S'(2, \alpha)$ defined as follows:

$$\mathcal{L}_n^\alpha = -t^n (t \xi + \frac{1}{2} (n + \alpha + 1) (\theta_1 \partial_1 + \theta_2 \partial_2)), \quad (4.3)$$

$$E_n = t^n \theta_2 \partial_1,$$

$$H_n = t^n (\theta_2 \partial_2 - \theta_1 \partial_1),$$

$$F_n = t^n \theta_1 \partial_2,$$

$$h_n^\alpha = t^n \xi \theta_2 - (n + \alpha) t^{n-1} \theta_1 \theta_2 \partial_1,$$

$$p_n^0 = -t^{n+1} \partial_2,$$

$$x_n^0 = t^{n+1} \partial_1,$$

$$y_n^\alpha = t^n \xi \theta_1 + (n + \alpha) t^{n-1} \theta_1 \theta_2 \partial_2.$$

Let us show that if $\alpha \in \mathbb{Z}$, then $\text{Der}_{\text{ext}} S'(2, \alpha) \cong \mathfrak{sl}(2) = \{\mathcal{E}, \mathcal{H}, \mathcal{F}\}$, where

$$[\mathcal{H}, \mathcal{E}] = 2\mathcal{E}, [\mathcal{H}, \mathcal{F}] = -2\mathcal{F}, [\mathcal{E}, \mathcal{F}] = \mathcal{H}, \quad (4.4)$$

and the action of $\mathfrak{sl}(2)$ is given as follows:

$$[\mathcal{E}, h_n^\alpha] = x_{n+1-\alpha}, [\mathcal{E}, y_n^\alpha] = p_{n+1-\alpha}, [\mathcal{F}, x_n] = h_{n+1-\alpha}, [\mathcal{F}, p_n^0] = y_{n+1-\alpha}, \quad (4.5)$$

$$[\mathcal{H}, x_n^0] = x_n^0, [\mathcal{H}, h_n^\alpha] = -h_n^\alpha, [\mathcal{H}, p_n^0] = p_n^0, [\mathcal{H}, y_n^\alpha] = -y_n^\alpha.$$ 

Notice that

$$\text{Der}_{\text{ext}} S'(2, \alpha) \cong H^1(S'(2, \alpha), S'(2, \alpha)), \quad (4.6)$$

see Ref. 22. Consider the following $\mathbb{Z}$-grading deg of $S'(2, \alpha)$:

$$\deg \mathcal{L}_n^\alpha = n, \deg E_n = n + 1 - \alpha, \deg F_n = n - 1 + \alpha, \deg H_n = n, \quad (4.7)$$

$$\deg h_n^\alpha = n, \deg p_n = n, \deg x_n = n + 1 - \alpha, \deg y_n^\alpha = n - 1 + \alpha.$$
Let
\[ L_0^\alpha = -L_0^\alpha + \frac{1}{2} (1 - \alpha) H_0. \] (4.8)

Then
\[ [L_0^\alpha, s] = (\text{deg } s) s \] (4.9)
for a homogeneous \( s \in S'(2, \alpha) \). Accordingly,
\[ [L_0^\alpha, D] = (\text{deg } D) D \] (4.10)
for a homogeneous \( D \in \text{Der}_{ext} S'(2, \alpha) \). On the other hand, since the action of a Lie superalgebra on its cohomology is trivial, then one must have
\[ [L_0^\alpha, D] = 0. \] (4.11)

Hence the nonzero elements of \( \text{Der}_{ext} S'(2, \alpha) \) have \( \text{deg} = 0 \), and they preserve the superalgebra \( S'(2, \alpha)_{\text{deg}=0} \). One can check that the exterior derivations of \( S'(2, \alpha)_{\text{deg}=0} \) form an \( \mathfrak{sl}(2) \), and extend them to the exterior derivations of \( S'(2, \alpha) \) as in (4.5). One should also note that if the restriction of a derivation of \( S'(2, \alpha) \) to \( S'(2, \alpha)_{\text{deg}=0} \) is zero, then this derivation is inner.

We can identify the exterior derivation \( t^{-\alpha} \xi \theta_1 \theta_2 \) [see (2.7)] with \( -\mathcal{F} \). We cannot realize all the exterior derivations as regular differential operators on the supercircle, but can do this using the symbols of pseudodifferential operators. In fact, let \( \alpha = 1 \). Then
\[ \text{Der}_{ext} S'(2, 1) = \mathfrak{sl}(2) = \langle \mathcal{F}, \mathcal{H}, \mathcal{E} \rangle \subset P(2), \] (4.12)
where
\[ \mathcal{F} = -t^{-1} \xi \theta_1 \theta_2, \mathcal{H} = -\theta_1 \partial_1 - \theta_2 \partial_2, \mathcal{E} = t \xi^{-1} \partial_1 \partial_2. \] (4.13)

One can then construct the loop algebra of \( \mathfrak{sl}(2) \) as follows:
\[ \tilde{\mathfrak{sl}}(2) = \langle \mathcal{F}_n, \mathcal{H}_n, \mathcal{E}_n \rangle_{n \in \mathbb{Z}}, \] (4.14)
where
\[ \mathcal{F}_n = -t^{n-1} \xi \theta_1 \theta_2, \] (4.15)
\[ \mathcal{H}_n = nt^{n-1} \xi^{-1} \theta_1 \theta_2 \partial_1 \partial_2 - t^n (\theta_1 \partial_1 + \theta_2 \partial_2), \]
\[ \mathcal{E}_n = t^{n+1} \xi^{-1} \partial_1 \partial_2. \]

The nonvanishing commutation relations are
\[ [\mathcal{H}_n, \mathcal{E}_k] = 2 \mathcal{E}_{n+k}, [\mathcal{H}_n, \mathcal{F}_k] = -2 \mathcal{F}_{n+k}, [\mathcal{E}_n, \mathcal{F}_k] = \mathcal{H}_{n+k}. \] (4.16)
Let $\alpha \in \mathbb{Z}$. Then
\[
\text{Der}_{\text{ext}} S'(2, \alpha) \cong \langle F_{-\alpha+1}, H_0, E_{\alpha-1} \rangle.
\] (4.17)

**Theorem 4.1:** The superalgebras $S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ together with $\tilde{\mathfrak{sl}}(2)$ generate a Lie superalgebra isomorphic to $K'(4)$.

**Proof:** Let
\[
I_n^0 = t^n(\theta_1 \partial_1 + \theta_2 \partial_2),
\]
\[
v_n = t^{n-1} \theta_1 \theta_2 \partial_1,
\]
\[
s_n = t^{n-1} \theta_1 \theta_2 \partial_2.
\] (4.18)

Then according to (4.3)
\[
\mathcal{L}^\alpha_n = \mathcal{L}^0_n - \frac{1}{2} \alpha I_n^0,
\]
\[
h^\alpha_n = h^0_n - \alpha v_n,
\]
\[
y^\alpha_n = y^0_n + \alpha s_n.
\] (4.19)

One can easily check that the superalgebras $S'(2, \alpha)$, where $\alpha \in \mathbb{Z}$, generate $W(2) \subset P(2)$. In fact, $W(2)$ is spanned by 8 fields defined in Eq. (4.3), where $\alpha = 0$, together with 3 fields defined in Eq. (4.18) and the field $F_n$. If we include two even fields, $E_n$ and $H_n$, into the picture, then from the commutation relations, we obtain two additional odd fields:
\[
q_n = t^n \xi^{-1} \theta_2 \partial_1 \partial_2,
\]
\[
t_n = -t^n \xi^{-1} \theta_1 \partial_1 \partial_2.
\] (4.20)

Let $g \subset P(2)$ be the Lie superalgebra generated by the superalgebras $S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ and $\tilde{\mathfrak{sl}}(2)$. We will show that there exists an isomorphism:
\[
\psi : K'(4) \longrightarrow g.
\] (4.21)

Let
\[
\mathcal{L}_n = \mathcal{L}^0_n + H_n + \frac{1}{2} I_n^0,
\]
\[
I_n = I_n^0 + H_n,
\]
\[
p_n = p_n^0 + t_n,
\]
\[
x_n = x_n^0 - q_n.
\] (4.22)
Set
\[ h_n = h_n^0, y_n = y_n^0. \] (4.23)

Then \( g = g_0 \oplus g_1 \), where
\[
g_0 = \langle \mathcal{L}_n, I_n, E_n, H_n, F_n, \mathcal{E}_n, \mathcal{H}_n, \mathcal{F}_n \rangle, \tag{4.24}
g_1 = \langle h_n, p_n, x_n, y_n, t_n, s_n, q_n, t_n \rangle.
\]

We will describe the nonvanishing commutation relations in \( g \) with respect to this basis.

For \([g_0, g_0]\) the relations are:
\[
[\mathcal{L}_n, \mathcal{L}_k] = (n - k)\mathcal{L}_{n+k}; \\
[H_n, E_k] = 2E_{n+k}, [H_n, F_k] = -2F_{n+k}, [E_n, F_k] = H_{n+k}; \\
[\mathcal{H}_n, \mathcal{E}_k] = 2\mathcal{E}_{n+k}, [\mathcal{H}_n, \mathcal{F}_k] = -2\mathcal{F}_{n+k}, [\mathcal{E}_n, \mathcal{F}_k] = \mathcal{H}_{n+k}; \\
[\mathcal{L}_n, X_k] = -kX_{n+k}, \text{ where } X_k = I_k, E_k, H_k, F_k, \mathcal{E}_k, \mathcal{H}_k, \mathcal{F}_k.
\]

For \([g_0, g_1]\) the relations are:
\[
[\mathcal{L}_n, X_k] = (-k + \frac{n}{2})X_{n+k}, \text{ where } X_k = h_k, p_k, x_k, y_k; \tag{4.26}
[\mathcal{L}_n, X_k] = (-k - \frac{n}{2})X_{n+k}, \text{ where } X_k = r_k, s_k, q_k, t_k; \\
[I_n, X_k] = nY_{n+k}, \text{ where } X_k = h_k, p_k, x_k, y_k, \text{ and } Y_k = r_k, t_k, -q_k, -s_k, \text{ respectively}; \\
[H_n, X_k] = X_{n+k}, \text{ where } X_k = h_k, x_k, q_k, t_k; \\
[H_n, X_k] = -X_{n+k}, \text{ where } X_k = y_k, p_k, s_k, t_k; \\
[E_n, X_k] = Y_{n+k}, [F_n, Y_k] = X_{n+k},
\]
where \( X_k = y_k, p_k, s_k, t_k, \text{ and } Y_k = h_k, x_k, -r_k, -q_k, \text{ respectively}; \\
[\mathcal{H}_n, X_k] = X_{n+k} + nY_{n+k}, \text{ where } X_k = p_k, x_k, q_k, t_k, \text{ and } Y_k = t_k, -q_k, 0, 0, \text{ respectively}; \\
[\mathcal{H}_n, X_k] = -X_{n+k} - nY_{n+k}, \text{ where } X_k = h_k, y_k, r_k, s_k, \text{ and } Y_k = r_k, -s_k, 0, 0, \text{ respectively}; \\
[\mathcal{E}_n, X_k] = Y_{n+k} - nZ_{n+k}, [\mathcal{F}_n, Y_k] = X_{n+k} - n\bar{Z}_{n+k}, \text{ where } X_k = h_k, y_k, r_k, s_k,
\]
\[ Y_k = x_k, p_k, -q_k, -t_k, Z_k = q_k, -t_k, 0, 0, \text{ and } \bar{Z}_k = -r_k, s_k, 0, 0, \text{ respectively}. \]

Finally, for \([g_1, g_1]\) the relations are:
\[
[h_n, x_k] = (k - n)E_{n+k}, [p_n, y_k] = (k - n)F_{n+k}, \tag{4.27}
[h_n, p_k] = \mathcal{L}_{n+k} - \frac{1}{2}(k - n)H_{n+k}, [x_n, y_k] = -\mathcal{L}_{n+k} + \frac{1}{2}(k - n)H_{n+k}, \\
[h_n, q_k] = E_{n+k}, [x_n, r_k] = E_{n+k}, [p_n, s_k] = F_{n+k}, [y_n, t_k] = F_{n+k}, \\
[p_n, q_k] = -\mathcal{E}_{n+k}, [x_n, t_k] = -\mathcal{E}_{n+k}, [h_n, s_k] = -\mathcal{F}_{n+k}, [y_n, r_k] = -\mathcal{F}_{n+k}, \\
[p_n, r_k] = \frac{1}{2}I_{n+k} - \frac{1}{2}(H_{n+k} + H_{n+k}), [x_n, s_k] = \frac{1}{2}I_{n+k} + \frac{1}{2}(H_{n+k} - H_{n+k}), \\
[h_n, t_k] = \frac{1}{2}I_{n+k} + \frac{1}{2}(H_{n+k} + H_{n+k}), [y_n, q_k] = \frac{1}{2}I_{n+k} - \frac{1}{2}(H_{n+k} - H_{n+k}).
\]
Recall that the elements of $K(4)$ can be identified with the functions from $\mathbb{C}[t, t^{-1}] \otimes \Lambda(4)$. Let
\[
\tilde{\theta}_1 = \theta_2 \theta_3 \theta_4, \tilde{\theta}_2 = \theta_1 \theta_3 \theta_4, \tilde{\theta}_3 = \theta_1 \theta_2 \theta_4, \tilde{\theta}_4 = \theta_1 \theta_2 \theta_3.
\] (4.28)

The following 16 series of functions together with $t^{-1} \theta_1 \theta_2 \theta_3 \theta_4$ span $\mathbb{C}[t, t^{-1}] \otimes \Lambda(4)$:

\[
f^1_n = 2nt^{n-1} \theta_1 \theta_2 \theta_3 \theta_4, \quad f^2_n = -\frac{1}{2}t^{n+1} + \frac{1}{2}it^n (\theta_2 \theta_3 - \theta_1 \theta_4) - \frac{1}{2}n(n + 1)t^{n-1} \theta_1 \theta_2 \theta_3 \theta_4,
\]
\[
f^k_n = \frac{1}{2}t^{n+1}(\pm \theta_1 \theta_2 \mp \theta_3 \theta_4 - i\theta_1 \theta_3 - i\theta_2 \theta_4), k = 3, 4,
\]
\[
f^5_n = it^n(\theta_1 \theta_4 - \theta_2 \theta_3),
\]
\[
f^6_n = \frac{1}{2}t^n(\mp \theta_1 \theta_4 \mp \theta_2 \theta_3 + i\theta_2 \theta_4 - i\theta_1 \theta_3), k = 6, 7,
\]
\[
f^7_n = -it^n(\theta_1 \theta_2 + \theta_3 \theta_4),
\]
\[
f^k_n = \frac{(i)^{p(k)}}{\sqrt{8}} (t^n(\theta_1 \mp i\theta_2 \mp \theta_3 + i\theta_4) - nt^{n-1}(\tilde{\theta}_1 \pm i\tilde{\theta}_2 \mp \tilde{\theta}_3 - i\tilde{\theta}_4)), k = 9, 10,
\]
\[
f^k_n = \frac{(-i)^{p(k)}}{\sqrt{8}} (t^{n+1}(\theta_1 \pm i\theta_2 \mp \theta_3 - i\theta_4) - (n + 1)t^n(\tilde{\theta}_1 \mp i\tilde{\theta}_2 \mp \tilde{\theta}_3 + i\tilde{\theta}_4)), k = 11, 12,
\]
\[
f^k_n = \frac{(-i)^{p(k)}}{\sqrt{2}} t^{n-1}(\tilde{\theta}_1 \pm i\tilde{\theta}_2 \mp \tilde{\theta}_3 - i\tilde{\theta}_4), k = 13, 14,
\]
\[
f^k_n = \frac{(-i)^{p(k)}}{\sqrt{2}} t^n(\tilde{\theta}_1 \mp i\tilde{\theta}_2 \mp \tilde{\theta}_3 + i\tilde{\theta}_4), k = 15, 16,
\]

where $p(k) = 0$ if $k$ is even, and $p(k) = 1$ if $k$ is odd.

The 16 series of the corresponding differential operators $\{D_{f^k_n}\}_{i=1,\ldots,16}$ span $K'(4)$. Set
\[
\psi(D_{f^1_n}) = I_n, \psi(D_{f^2_n}) = \mathcal{L}_n,
\]
\[
\psi(D_{f^3_n}) = E_n, \psi(D_{f^4_n}) = F_n, \psi(D_{f^5_n}) = H_n,
\]
\[
\psi(D_{f^6_n}) = \mathcal{E}_n, \psi(D_{f^7_n}) = \mathcal{F}_n, \psi(D_{f^8_n}) = \mathcal{H}_n,
\]
\[
\psi(D_{f^9_n}) = x_n, \psi(D_{f^{10}_n}) = h_n, \psi(D_{f^{11}_n}) = y_n, \psi(D_{f^{12}_n}) = p_n,
\]
\[
\psi(D_{f^{13}_n}) = q_n, \psi(D_{f^{14}_n}) = r_n, \psi(D_{f^{15}_n}) = s_n, \psi(D_{f^{16}_n}) = t_n.
\]

Notice that $f^1_n = 0$, if $n = 0$. This corresponds to the fact that $D_{t^{-1} \theta_1 \theta_2 \theta_3 \theta_4} \not\in K'(4)$. One can verify that $\psi$ is an isomorphism from $K'(4)$ onto $\mathfrak{g}$. □
Remark 4.2: We have obtained an embedding

\[ K'(4) \subset P(2). \]  

(4.31)

In general, a Lie algebra of contact vector fields can be realized as a subalgebra of Poisson algebra; see Ref. 8. We will explain this from the geometrical point of view in application to our case. Recall that the Lie algebra \( \text{Vect}(S^1) \) of smooth vector fields on the circle has a natural embedding into the Poisson algebra of functions on the cylinder \( \hat{T}^*S^1 = T^*S^1 \setminus S^1 \) with the removed zero section; see Refs. 11, 12 and 19. One can introduce the Darboux coordinates \((q, p) = (t, \xi)\) on this manifold. The symbols of differential operators are functions on \( \hat{T}^*S^1 \) which are formal Laurent series in \( p \) with coefficients periodic in \( q \). Correspondingly, they define Hamiltonian vector fields on \( \hat{T}^*S^1 \):

\[ A(q, p) \mapsto H_A = \partial_p A \partial_q - \partial_q A \partial_p. \]  

(4.32)

The embedding of \( \text{Vect}(S^1) \) into the Lie algebra of Hamiltonian vector fields on \( \hat{T}^*S^1 \) is given by

\[ f(q) \partial_q \mapsto H_{f(q)}p. \]  

(4.33)

Notice that we obtain a subalgebra of Hamiltonian vector fields with Hamiltonians which are homogeneous of degree 1. (This condition holds in general, if one considers the symplectification of a contact manifold; see Ref. 8.) In other words, we obtain a subalgebra of Hamiltonian vector fields, which commute with the (semi-) Euler vector field:

\[ [H_A, p \partial_p] = 0. \]  

(4.34)

We will show that for \( N \geq 0 \) there exists the analogous embedding:

\[ K(2N) \subset P(N). \]  

(4.35)

The analog of the formula (4.32) in the supercase is as follows (Refs. 2, 5):

\[ A(q, p, \theta_i, \bar{\theta}_i) \mapsto H_A = \partial_p A \partial_q - \partial_q A \partial_p - (-1)^p(A) \sum_{i=1}^{N} (\partial_{\theta_i} A \partial_{\bar{\theta}_i} + \partial_{\bar{\theta}_i} A \partial_{\theta_i}). \]  

(4.36)

Then \( K(2N) \) is defined as the set of all (Hamiltonian) functions \( A(q, p, \theta_i, \bar{\theta}_i) \in P(N) \) such that

\[ [H_A, p \partial_p + \sum_{i=1}^{N} \theta_i \partial_{\theta_i}] = 0. \]  

(4.37)
Equivalently, we have the following characterization of the embedding (4.35). Consider a $\mathbb{Z}$-grading of the (associative) superalgebra $P(N) = \bigoplus_{j \in \mathbb{Z}} P_j(N)$ defined by
\begin{align*}
deg p = \deg \tilde{\theta}_i = 1 & \quad \text{for } i = 1, \ldots, N, \tag{4.38} \\
deg q = \deg \theta_i = 0 & \quad \text{for } i = 1, \ldots, N.
\end{align*}
Thus with respect to the Poisson bracket,
\begin{equation}
\{P_j(N), P_k(N)\} \subset P_{j+k-1}(N). \tag{4.39}
\end{equation}
Then
\begin{equation}
K(2N) = P_1(N). \tag{4.40}
\end{equation}

**Theorem 4.2:** There exists an embedding,
\begin{equation}
\hat{K}'(4) \subset R_h(2), \tag{4.41}
\end{equation}
for each $h \in [0, 1]$, such that the central element in $\hat{K}'(4)$ is $h \in R_h(2)$, and
\begin{equation}
\lim_{h \to 0} \hat{K}'(4) = K'(4) \subset P(2). \tag{4.42}
\end{equation}

**Proof:** For each $h \in [0, 1]$ and $\alpha \in \mathbb{Z}$ we have an embedding,
\begin{equation}
\text{Der}_{S'}(2, \alpha) \subset R_h(2). \tag{4.43}
\end{equation}
The exterior derivations $\text{Der}_{\text{ext}} S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ generate the loop algebra,
\begin{equation}
\hat{\mathfrak{sl}}(2) = \langle \mathcal{F}_n, \mathcal{H}_n, \mathcal{E}_n \rangle_{n \in \mathbb{Z}} \subset R_h(2), \tag{4.44}
\end{equation}
where
\begin{align*}
\mathcal{F}_n &= -t^{n-1} \xi \theta_1 \theta_2, \tag{4.45} \\
\mathcal{H}_n &= \frac{1}{h}((\xi^{-1} \circ_h t^n \xi)(h^2 - h \theta_1 \partial_1 - h \theta_2 \partial_2 - \theta_1 \theta_2 \partial_1 \partial_2) + t^n \theta_1 \theta_2 \partial_1 \partial_2), \\
\mathcal{E}_n &= (\xi^{-1} \circ_h t^{n+1}) \partial_1 \partial_2.
\end{align*}
so that Eqs. (4.16)-(4.17) hold. Let $\mathfrak{g} \subset R_h(2)$ be the Lie superalgebra generated by $S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ and $\hat{\mathfrak{sl}}(2)$. Set
\begin{align*}
q_n &= (\xi^{-1} \circ_h t^n)(h \partial_1 + \theta_2 \partial_1 \partial_2), \tag{4.46} \\
t_n &= (\xi^{-1} \circ_h t^n)(h \partial_2 - \theta_1 \partial_1 \partial_2).
\end{align*}
The basis (4.24) in \( \mathfrak{g} \) is defined by Eqs. (4.3), (4.18), (4.22)-(4.23) and (4.45)-(4.46). The commutation relations in \( \mathfrak{g} \) with respect to this basis are given by Eqs. (4.25)-(4.27). The Lie superalgebra \( \mathfrak{g} \) is isomorphic to a central extension,

\[
\hat{K}'(4) = K'(4) \oplus \mathbb{C}C
\]

of \( K'(4) \). The corresponding 2-cocycle (up to equivalence) is

\[
c(t^{n+1}, t^{k+1} \theta_1 \theta_2 \theta_3 \theta_4) = \delta_{n+k+2,0}, \\
c(t^{n+1} \theta_i, t^{k+1} \partial_i (\theta_1 \theta_2 \theta_3 \theta_4)) = \frac{1}{2} \delta_{n+k+2,0} \text{ for } i = 1, \ldots, 4.
\]

The isomorphism,

\[
\psi : \hat{K}'(4) \longrightarrow \mathfrak{g}
\]

is defined by Eq. (4.30) and the equation

\[
\psi(C) = I_0 = h \in R_h(2).
\]

The corresponding 2-cocycle in the basis (4.24) is

\[
c(p_n, r_k) = \frac{1}{2} \delta_{n,-k}, \\
c(x_n, s_k) = \frac{1}{2} \delta_{n,-k}, \\
c(h_n, t_k) = \frac{1}{2} \delta_{n,-k}, \\
c(y_n, q_k) = \frac{1}{2} \delta_{n,-k}, \\
c(L_n, I_k) = n \delta_{n,-k}.
\]

Note that in the realization of \( K'(4) \) inside \( P(2) \), obtained in Theorem 4.1, we have \( I_0 = 0 \).

\[\square\]

Remark 4.3: The 2-cocycle \( c \) is one of three nontrivial 2-cocycles on \( K'(4) \); see Refs. 1 and 2. [In Ref. 1 this cocycle is defined by Eq. (4.22), where \( d = 0, e = 1 \)]. Note that the cocycle \( c \) is equivalent to the restriction of the 2-cocycle \( c_t \) on \( R(2) \); see Eqs. (3.21), (3.22).

V. One-parameter family of representations of \( \hat{K}'(4) \)
Theorem 5.1: There exists a one-parameter family of irreducible representations of $\hat{K}'(4)$ depending on parameter $\mu \in \mathbb{C}$ in the superspace spanned by 2 even fields and 2 odd fields where the value of the central charge is equal to one.

Proof: Let $g \in t^\mu \mathbb{C}[t, t^{-1}]$, where $\mu \in \mathbb{R} \setminus \mathbb{Z}$. One can think of $\xi^{-1}$ as the anti-derivative,

$$\xi^{-1} g(t) = \int g(t) dt. \quad (5.1)$$

Let $f(t) \in \mathbb{C}[t, t^{-1}]$. According to (3.2),

$$\xi^{-1} \circ f = \sum_{n=0}^{\infty} (-1)^n (\xi^n f) \xi^{-n-1}. \quad (5.2)$$

Notice that this formula, when applied to a function $g$, corresponds to the formula of integration by parts. Let

$$V^\mu = t^\mu \mathbb{C}[t, t^{-1}] \otimes \Lambda(2) = t^\mu \mathbb{C}[t, t^{-1}] \otimes (1, \theta_1, \theta_2, \theta_1 \theta_2), \; \mu \in \mathbb{R} \setminus \mathbb{Z}. \quad (5.3)$$

Using the realization of $\hat{K}'(4)$ inside $R(2)$ (see Theorem 4.2 for $h = 1$) we obtain a representation of $\hat{K}'(4)$ in $V^\mu$. A central element in $\hat{K}'(4)$ is $I_0 = 1 \in R(2)$; the 2-cocycle is defined by Eq. (4.51). Let $\{v^i_m\}$, where $m \in \mathbb{Z}$ and $i = 0, 1, 2, 3$, be the following basis in $V^\mu$:

$$v^0_m = \frac{1}{m + \mu} t^{m+\mu},$$

$$v^1_m = t^{m+\mu} \theta_1,$$

$$v^2_m = t^{m+\mu} \theta_2,$$

$$v^3_m = t^{m+\mu} \theta_1 \theta_2.$$

The action of $\hat{K}'(4)$ is given as follows:

$$\mathcal{L}_n(v^0_m) = -(m + n + \mu - 1) v^0_{m+n},$$

$$\mathcal{L}_n(v^i_m) = -(m + \frac{1}{2} n + \mu) v^i_{m+n}, \; i = 1, 2,$$

$$\mathcal{L}_n(v^3_m) = -(m + n + \mu + 1) v^3_{m+n},$$

$$E_n(v^1_m) = v^2_{m+n}, \; F_n(v^2_m) = v^1_{m+n},$$

$$\mathcal{E}_n(v^3_m) = v^0_{m+n+2}, \; \mathcal{F}_n(v^0_m) = -v^3_{m+n-2},$$

$$H_n(v^i_m) = \mp v^i_{m+n}, \; i = 1, 2,$$

$$\mathcal{H}_n(v^i_m) = \pm v^i_{m+n}, \; i = 0, 3.$$
\[ h_n(v^1_m) = -(m + n + \mu)v^3_{m+n-1}, \quad y_n(v^2_m) = (m + n + \mu)v^3_{m+n-1}, \]
\[ h_n(v^0_m) = v^2_{m+n-1}, \quad y_n(v^0_m) = v^1_{m+n-1}, \]
\[ x_n(v^1_m) = (m + n + \mu)v^0_{m+n+1}, \quad p_n(v^2_m) = -(m + n + \mu)v^0_{m+n+1}, \]
\[ x_n(v^3_m) = v^2_{m+n+1}, \quad p_n(v^3_m) = v^1_{m+n+1}, \]
\[ r_n(v^1_m) = v^3_{m+n-1}, \quad s_n(v^2_m) = v^3_{m+n-1}, \]
\[ q_n(v^1_m) = v^0_{m+n+1}, \quad t_n(v^2_m) = v^0_{m+n+1}, \]
\[ I_n(v^i_m) = v^i_{m+n}, \quad i = 0, 1, 2, 3. \]

Note that \( I_0 \) acts by the identity operator. One can then define a one-parameter family of representations of \( \hat{K}'(4) \) depending on parameter \( \mu \in \mathbb{C} \) in the superspace \( V = \langle v^0_m, v^3_m, v^1_m, v^2_m \rangle_{m \in \mathbb{Z}} \), where \( p(v^i_m) = 0 \), for \( i = 0, 3 \), and \( p(v^i_m) = 1 \) for \( i = 1, 2 \), according to the formulas (5.5).

\[ \Box \]

**Remark 5.1:** The elements \( \{ \mathcal{L}_n, H_n, h_n, p_n \} \) span a subalgebra of \( K'(4) \) isomorphic to \( K(2) \). Note that \( V \) decomposes into the direct sum of two submodules over this superalgebra:

\[ V = \langle v^0_m, v^2_m \rangle_{m \in \mathbb{Z}} \oplus \langle v^3_m, v^1_m \rangle_{m \in \mathbb{Z}}. \]  

(5.6)

**Remark 5.2:** We conjecture that there exists a two-parameter family of representations of \( \hat{K}'(4) \) in the superspace spanned by 4 fields. In order to define it, instead of the superspace of functions, \( V^\mu \), one should consider the superspace of “densities”.

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[1] V. G. Kac and J. W. van de Leur, “On classification of superconformal algebras”, in Strings-88, edited by S. J. Gates et al. (World Scientific, Singapore, 1989), pp. 77-106.
[2] P. Grozman, D. Leites, and I. Shchepochkina, “Lie superalgebras of string theories”,
[3] S.-J. Cheng and V. G. Kac, “A new \( N = 6 \) superconformal algebra”, Commun. Math. Phys. 186, 219-231 (1997).

[4] M. Ademollo, L. Brink, A. D’Adda et al., “Dual strings with \( U(1) \) colour symmetry”, Nucl. Phys. B 111, 77-110 (1976).

[5] I. Shchepochkina, “The five exceptional simple Lie superalgebras of vector fields”, hep-th/9702121.

[6] I. Shchepochkina, “The five exceptional simple Lie superalgebras of vector fields”, Funkt. Anal. i Prilozhen. 33, 59-72 (1999). [Funct. Anal. Appl. 33, 208-219 (1999)].

[7] I. Shchepochkina, “The five exceptional simple Lie superalgebras of vector fields and their fourteen regradings”, Represent. Theory 3, 373-415 (1999).

[8] V. I. Arnold, Mathematical Methods of Classical Mechanics (Springer-Verlag, New York, 1989).

[9] O. S. Kravchenko and B. A. Khesin, “Central extension of the algebra of pseudodifferential symbols”, Funct. Anal. Appl. 25, 83-85 (1991).

[10] B. Khesin and I. Zakharevich, “Poisson-Lie group of pseudodifferential symbols”, Commun. Math. Phys. 171, 475-530 (1995).

[11] B. Khesin, V. Lyubashenko, and C. Roger, “Extensions and contractions of the Lie algebra of q-pseudodifferential symbols on the circle”, J. Funct. Anal. 143, 55-97 (1997).

[12] V. Ovsienko and C. Roger, “Deforming the Lie algebra of vector fields on \( S^1 \) inside the Lie algebra of pseudodifferential symbols on \( S^1 \)”, Am. Math. Soc. Trans. 194, 211-226 (1999).

[13] K. Schoutens, “A non-linear representation of the \( d = 2 \) \( so(4) \)-extended superconformal algebra”, Phys. Lett. B 194, 75-80 (1987).

[14] K. Schoutens, “\( O(N) \)-extended superconformal field theory in superspace”, Nucl. Phys. B 295, 634-652 (1988).

[15] A. Schwimmer and N. Seiberg, “Comments on the \( N = 2,3,4 \) superconformal algebras in two dimensions”, Phys. Lett. B 184, 191-196 (1987).

[16] B. Feigin and D. Leites, “New Lie superalgebras of string theories”, in Group-Theoretical Methods in Physics, edited by M. Markov et al., (Nauka, Moscow, 1983), Vol. 1, pp. 269-273. [English translation Gordon and Breach, New York, 1984].

[17] A. Neveu and J. H. Schwarz, “Factorizable dual models of pions”, Nucl. Phys. B 31, 86-112 (1971).

[18] B. L. Feigin, A. M. Semikhatov, and I. Yu. Tipunin, “Equivalence between chain categories of representations of affine \( sl(2) \) and \( N = 2 \) superconformal algebras”, J. Math. Phys. 39,
3865-3905 (1998).

[19] V. Ovsienko and C. Roger, “Deforming the Lie algebra of vector fields on $S^1$ inside the Poisson algebra on $\hat{T}^*S^1$”, Commun. Math. Phys. 198, 97-110 (1998).

[20] A. O. Radul, “Non-trivial central extensions of Lie algebra of differential operators in two and higher dimensions”, Phys. Lett. B 265, 86-91 (1991).

[21] E. Poletaeva, “Semi-infinite cohomology and superconformal algebras”, C. R. Acad. Sci., Ser. I: Math 326, 533-538 (1998).

[22] D. Fuks, Cohomology of Infinite-Dimensional Lie Algebras (Consultants Bureau, New York, 1986).

[23] V. G. Kac, “Classification of infinite-dimensional simple linearly compact Lie superalgebras”, Adv. Math. 139, 1-55 (1998).

[24] V. G. Kac, “Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields”, Transform. Groups 4, 219-272 (1999).

[25] V. G. Kac, “Superconformal algebras and transitive group actions on quadrics”, Commun. Math. Phys. 186, 233-252 (1997).

[26] V. G. Kac, “Lie superalgebras”, Adv. Math. 26, 8-96 (1977).

[27] E. Poletaeva, “Superconformal algebras and Lie superalgebras of the Hodge theory”, preprint MPI 99-136.