Mechanism Design for Fair Division: Allocating Divisible Items without Payments

RICHARD COLE, Courant Institute, New York University
VASILIS GKATZELIS, Courant Institute, New York University
GAGAN GOEL, Google Research, New York

We revisit the classic problem of fair division from a mechanism design perspective and provide an elegant truthful mechanism that yields surprisingly good approximation guarantees for the widely used solution of Proportional Fairness. This solution, which is closely related to Nash bargaining and the competitive equilibrium, is known to be not implementable in a truthful fashion, which has been its main drawback. To alleviate this issue, we propose a new mechanism, which we call the Partial Allocation mechanism, that discards a carefully chosen fraction of the allocated resources in order to incentivize the agents to be truthful in reporting their valuations. This mechanism introduces a way to implement interesting truthful outcomes in settings where monetary payments are not an option.

For a multi-dimensional domain with an arbitrary number of agents and items, and for the very large class of homogeneous valuation functions, we prove that our mechanism provides every agent with at least a $1/e \approx 0.368$ fraction of her Proportionally Fair valuation. To the best of our knowledge, this is the first result that gives a constant factor approximation to every agent for the Proportionally Fair solution. To complement this result, we show that no truthful mechanism can guarantee more than 0.5 approximation, even for the restricted class of additive linear valuations. In addition to this, we uncover a connection between the Partial Allocation mechanism and VCG-based mechanism design.

We also ask whether better approximation ratios are possible in more restricted settings. In particular, motivated by the massive privatization auction in the Czech republic in the early 90s we provide another mechanism for additive linear valuations that works really well when all the items are highly demanded.

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1. INTRODUCTION

From inheritance and land dispute resolution to treaty negotiations and divorce settlements, the problem of fair division of diverse resources has troubled man since antiquity. Not surprisingly, it has now also found its way into the highly automated, large scale world of computing. As the leading internet companies guide the paradigm shift into cloud computing, more and more services that used to be run on isolated machines are being migrated to shared computing clusters. Moreover, instead of just human beings bargaining or negotiating, one now also finds programmed strategic agents seeking resources. The goal of the...
resulting multiagent resource allocation problems [Chevaleyre et al. 2006] is to find solutions that are fair to the agents without introducing unnecessary inefficiencies.

One of the most challenging facets of this change is the need for higher quality incentive design in the form of protocols or mechanisms. As the peer-to-peer revolution has taught us, a proper set of incentives can make or break a system as the number of agents grows [Nisan et al. 2007, Chapter 23]. We therefore revisit this classic fair division problem from a purely mechanism design approach, aiming to create simple and efficient mechanisms that are not susceptible to strategic manipulation by the participating agents; in particular, we want to design truthful mechanisms for fair division of heterogenous goods.

One distinguishing property of resource allocation protocols in computing is that, more often than not, they need to eschew monetary transfers completely. This is so because, for instance, agents could represent internal teams in an internet company which are competing for resources. This, of course, severely limits what the mechanism designer can achieve since the collection of payments is the most versatile method for designing truthful mechanisms. In light of this, essentially the only tool left for aligning the agents’ incentives with the objectives of the system is what Hartline and Roughgarden referred to as “money burning” [Hartline and Roughgarden 2008]. That is, the system can choose to intentionally degrade the quality of its services (in our case this will mean discarding resources) in order to influence the preferences of the agents. This degradation of service can often be interpreted as an implicit form of “payment”, but since these payments do not correspond to actual trades, they are essentially burned or used for other purposes.

But even before dealing with the fact that the participating agents may behave strategically, one first needs to ask what is the right objective for fairness. This question alone has been the subject of long debates, in both social science and game theory, leading to a very rich literature. At the time of writing this paper, there are five academic books [Young 1995; Brams and Taylor 1996; Robertson and Webb 1998; Moulin 2003; Barbanel 2004] written on the topic of fair division, providing an overview of various proposed solutions for fairness. In this paper we will be focusing on resources that are divisible; for such settings, the most attractive solution for efficient and fair allocation is the Proportionally Fair solution (PF). In brief, a PF allocation is a Pareto optimal allocation $x^*$ which compares favorably to any other Pareto optimal allocation $x$ in the sense that, when switching from $x$ to $x^*$, the aggregate percentage gain in happiness of the agents outweighs the aggregate percentage loss. The notion of PF was first introduced in the seminal work of Kelly [1997] in the context of TCP congestion control. Since then it has become the de facto solution for bandwidth sharing in the networking community, and is in fact the most widely implemented solution in practice (for instance see [Andrews et al. 2005])$^1$. The wide adoption of PF as the solution for fairness is not a fluke, but is grounded in the fact that PF is equivalent to the Nash bargaining solution [Nash 1950], and to the Competitive Equilibria with Equal Incomes (CEEI) [Varian 1974; Eisenberg 1961] for a large class of valuation functions. Both Nash bargaining and the CEEI are well regarded solutions in microeconomics for bargaining and fairness.

A notable property of the PF solution is that it gives a good tradeoff between fairness and efficiency. One extreme notion of fairness is the Rawlsian notion of the egalitarian social welfare that aims to maximize the quality of service of the least satisfied agent irrespective of how much inefficiency this might be causing. On the other extreme, the utilitarian social welfare approach aims to maximize efficiency while disregarding how unsatisfied some agents might become. The PF allocation lies between these two extremes by providing a significant fairness guarantee without neglecting efficiency. As we showed in a recent work [Cole

\footnote{We note that some of the earlier work on Proportional Fairness such as [Kelly 1997] and [Kelly et al. 1998] have 2000+ and 3900+ citations respectively in google scholar, indicating the importance and usage of this solution.}
et al. 2013], for instances with just two players who have affine valuation functions the PF allocation has a social welfare of at least 0.933 times the optimal one.

Unfortunately, the PF allocation has one significant drawback: it cannot be implemented using truthful mechanisms without the use of payments; even for simple instances involving just two agents and two items, it is not difficult to show that no truthful mechanism can obtain a PF solution. This motivates the following natural question: can one design truthful mechanisms that yield a good approximation to the PF solution? Since our goal is to obtain a fair division, we seek a strong notion of approximation in which every agent gets a good approximation of her PF valuation. One of our main results is to give a truthful mechanism which guarantees that every agent will receive at least a $1/e$ fraction of her PF valuation for a very large class of valuation functions. We note that this is one of the very few positive results in multi-dimensional mechanism design without payments. We demonstrate the hardness of achieving such truthful approximations by providing an almost matching negative result for a restricted class of valuations.

While a $1/e$ approximation factor is quite surprising for such a general setting, in some circumstances one would prefer to restrict the setting in order to achieve a ratio much closer to 1. Our final result concerns such a scenario, which is motivated by the real-world privatization auctions that took place in Czechoslovakia in the early 90s. At that time, the Czech government sought to privatize the state owned firms dating from the then recently ended communist era. The government’s goal was two-fold — first, to distribute shares of these companies to their citizens in a fair manner, and second, to calculate the market prices of these companies so that the shares could be traded in the open market after the initial allocation. To this end, they ran an auction, as described in [Aggarwal and Harper 2000]. Citizens could choose to participate by buying 1000 vouchers at a cost of 1,000 Czech Crowns, about $35, a fifth of the average monthly salary. Over 90% of those eligible participated. These vouchers were then used to bid for shares in the available 1,491 firms. We believe that the PF allocation provides a very appropriate solution for this example, both to calculate a fair allocation and to compute market prices. Our second mechanism solves the problem of finding allocations very close to the PF allocation in a truthful fashion for such natural scenarios where there is high demand for each resource.

1.1. Our results

In this work we provide some surprising positive results for the problem of multi-dimensional mechanism design without payments. We focus on allocating divisible items and we use the widely accepted solution of proportional fairness as the benchmark regarding the valuation that each participating player deserves. In this setting, we undertake the design of truthful mechanisms that approximate this solution; we consider a strong notion of approximation, requiring that every player receives a good fraction of the valuation that she deserves according to the proportionally fair solution of the instance at hand.

The main contribution of this paper is the Partial Allocation mechanism. In Section 3 we analyze this mechanism and we prove that it is truthful and that it guarantees that every player will receive at least a $1/e$ fraction of her proportionally fair valuation. These results hold for the very general class of instances with players having arbitrary homogeneous valuation functions of degree one. This includes a wide range of well studied valuation functions, from additive linear and Leontief, to Constant Elasticity of Substitution and Cobb-Douglas [Mas-Colell et al. 1995]. We later extend these results to homogeneous valuations of any degree. To complement this positive result, we provide a negative result showing that no truthful mechanism can guarantee to every player an allocation with value greater than 0.5 of the value of the PF allocation, even if the mechanism is restricted to the class of additive linear valuations. In proving the truthfulness of the Partial Allocation mechanism we reveal a connection between the amount of resources that the mechanism discards and the payments in VCG mechanisms. In a nutshell, multiplicative reductions in
allocations are analogous to payments. As a result, we anticipate that this approach may have a significant impact on other problems in mechanism design without money. Indeed, we have already applied this approach to the problem of maximizing social welfare without payments for which a special two-agent version of the Partial Allocation mechanism allowed us to improve upon a setting for which mostly negative results were known [Cole et al. 2013].

In Section 4 we show that, restricting the set of possible instances to ones involving players with additive linear valuations\(^2\) and items with high prices in the competitive equilibrium from equal incomes\(^3\), will actually allow for the design of even more efficient and useful mechanisms. We present the Strong Demand Matching (SDM) mechanism, a truthful mechanism that performs increasingly well as the competitive equilibrium prices increase. More specifically, if \(p^*_j\) is the price of item \(j\), then the approximation factor guaranteed by this mechanism is equal to \(\min_j \left( \frac{p^*_j}{\lceil \frac{p^*_j}{\lceil p^*_j \rceil} \rceil} \right)\). It is interesting to note that scenarios such as the privatization auction mentioned above involve a number of bidders much larger than the number of items; as a rule, we expect this to lead to high prices and a very good approximation of the participants’ PF valuations.

1.2. Related Work

Our setting is closely related to the large topic of fair division or cake-cutting [Young 1995; Brams and Taylor 1996; Robertson and Webb 1998; Moulin 2003; Barbanel 2004], which has been studied since the 1940’s, using the \([0, 1]\) interval as the standard representation of a cake. Each agent’s preferences take the form of a valuation function over this interval, and then the valuations of unions of subintervals are additive. Note that the class of homogeneous valuation functions of degree one takes us beyond this standard cake-cutting model. Leontief valuations for example, allow for complementarities in the valuations, and then the valuations of unions of subintervals need not be additive. On the other hand, the additive linear valuations setting that we focus on in Section 4 is equivalent to cake-cutting with piecewise constant valuation functions over the \([0, 1]\) interval. Other common notions of fairness that have been studied in this literature are, proportionality\(^4\), envy-freeness, and equitability [Young 1995; Brams and Taylor 1996; Robertson and Webb 1998; Moulin 2003; Barbanel 2004].

Despite the extensive work on fair resource allocation, truthfulness considerations have not played a major role in this literature. Most results related to truthfulness were weakened by the assumption that each agent would be truthful in reporting her valuations unless this strategy was dominated. Very recent work [Chen et al. 2010; Mossel and Tamuz 2010; Zivan et al. 2010; Maya and Nisan 2012] studies truthful cake cutting variations using the standard notion of truthfulness according to which an agent need not be truthful unless doing so is a dominant strategy. Chen et al. [2010] study truthful cake-cutting with agents having piecewise uniform valuations and they provide a polynomial-time mechanism that is truthful, proportional, and envy-free. They also design randomized mechanisms for more general families of valuation functions, while Mossel and Tamuz [2010] prove the existence of truthful (in expectation) mechanisms satisfying proportionality in expectation for general valuations. Zivan et al. [2010] aim to achieve envy-free Pareto optimal allocations of multiple divisible goods while reducing, but not eliminating, the agents’ incentives to lie. The extent to which untruthfulness is reduced by their proposed mechanism is only evaluated empirically and depends critically on their assumption that the resource limitations are soft.

\(^2\)Note that our negative results imply that the restriction to additive linear valuations alone would not be enough to allow for significantly better approximation factors.

\(^3\)The prices induced by the market equilibrium when all bidders have a unit of scrip money; also referred to as PF prices.

\(^4\)It is worth distinguishing the notion of PF from that of proportionality by noting that the latter is a much weaker notion, directly implied by the former.
constraints. Very recent work by Maya and Nisan [2012] provides evidence that truthfulness comes at a significant cost in terms of efficiency.

The recent papers of Guo and Conitzer [2010] and of Han et al. [2011] also consider the truthful allocation of multiple divisible goods; they focus on additive linear valuations and their goal is to maximize the social welfare (or efficiency) after scaling every player’s reported valuations so that her total valuation for all items is 1. Guo and Conitzer [2010] study two-agent instances, providing both upper and lower bounds for the achievable approximation; Han et al. [2011] extend these results and also study the multiple agents setting. For problem instances that may involve an arbitrary number of items both papers provide negative results: no non-trivial approximation factor can be achieved by any truthful mechanism when the number of players is also unbounded. For the two-player case, after Guo and Conitzer [2010] studied some classes of dictatorial mechanisms, Han et al. [2011] showed that no dictatorial mechanism can guarantee more than the trivial 0.5 factor. Interestingly, we recently showed [Cole et al. 2013] that combining a special two-player version of the Partial Allocation mechanism with a dictatorial mechanism can actually beat this bound, achieving a 0.622 approximation.

The resource allocation literature has seen a resurgence of work studying fair and efficient allocation for Leontief valuations [Ghodsi et al. 2011; Dolev et al. 2012; Parkes et al. 2012; Gutman and Nisan 2012]. These valuations exhibit perfect complements and they are considered to be natural valuation abstractions for computing settings where jobs need resources in fixed ratios. Ghodsi et al. [2011] defined the notion of Dominant Resource Fairness (DRF), which is a generalization of the egalitarian social welfare to multiple types of resources. This solution has the advantage that it can be implemented truthfully for this specific class of valuations; as the authors acknowledge, the CEEI solution would be the preferred fair division mechanism in that setting as well, and its main drawback is the fact that it cannot be implemented truthfully. Parkes et al. [2012] assessed DRF in terms of the resulting efficiency, showing that it performs poorly. Dolev et al. [2012] proposed an alternate fairness criterion called Bottleneck Based Fairness, which Gutman and Nisan [2012] subsequently showed is satisfied by the proportionally fair allocation. Gutman and Nisan [2012] also posed the study of incentives related to this latter notion as an interesting open problem. Our results could potentially have significant impact on this line of work as we are providing a truthful way to approximate a solution which is recognized as a good benchmark. It would also be interesting to study the extent to which the Partial Allocation mechanism can outperform the existing ones in terms of efficiency.

Most of the papers mentioned above contribute to our understanding of the trade-offs between either truthfulness and fairness, or truthfulness and social welfare. Another direction that has been actively pursued is to understand and quantify the interplay between fairness and social welfare. Caragiannis et al. [2012] measured the deterioration of the social welfare caused by different fairness restrictions, the price of fairness. More recently, Cohler et al. [2011] designed algorithms for computing allocations that (approximately) maximize social welfare while satisfying envy-freeness.

Our results fit into the general agenda of approximate mechanism design without money, explicitly initiated by Procaccia and Tennenholtz [2009]. More interestingly, the underlying connection with VCG payments proposes a framework for designing truthful mechanisms without money and we anticipate that this might have a significant impact on this literature.

2. PRELIMINARIES

Let \( M \) denote the set of \( m \) items and \( N \) the set of \( n \) bidders. Each item is divisible, meaning that it can be divided into arbitrarily small pieces, which are then allocated to different bidders. An allocation \( x \) of these items to the bidders defines the fraction \( x_{ij} \) of each item \( j \) that each bidder \( i \) will be receiving; let \( \mathcal{F} = \{ x \mid x_{ij} \geq 0 \text{ and } \sum_i x_{ij} \leq 1 \} \) denote the set of feasible allocations. Each bidder is assigned a weight \( b_i \geq 1 \) which allows for
interpersonal comparison of valuations, and can serve as priority in computing applications, as clout in bargaining applications, or as a budget for the market equilibrium interpretation of our results. We assume that $b_i$ is defined by the mechanism as it cannot be truthfully elicited from the bidders. The preferences of each bidder $i \in N$ take the form of a valuation function $v_i(\cdot)$, that assigns nonnegative values to every allocation in $\mathcal{F}$. We assume that every player’s valuation for a given allocation $x$ only depends on the bundle of items that she will be receiving.

We will present our results assuming that the valuation functions are homogeneous of degree one, i.e. player $i$’s valuation for an allocation $x' = f \cdot x$ satisfies $v_i(x') = f \cdot v_i(x)$, for any scalar $f > 0$. We later discuss how to extend these results to general homogeneous valuations of degree $d$ for which $v_i(x') = f^d \cdot v_i(x)$. A couple of interesting examples of homogeneous valuations functions of degree one are additive linear valuations and Leontief valuations; according to the former, every player has a valuation $v_{ij}$ for each item $j$ and $v_i(x) = \sum_j x_{ij} v_{ij}$, and according to the latter, each player $i$’s type corresponds to a set of values $a_{ij}$, one for each item, and $v_i(x) = \min_j \{ x_{ij}/a_{ij} \}$. (i.e. player $i$ desires the items in the ratio $a_{i1} : a_{i2} : \ldots : a_{im}$.)

An allocation $x^* \in \mathcal{F}$ is Proportionally Fair (PF) if, for any other allocation $x' \in \mathcal{F}$ the (weighted) aggregate proportional change to the valuations after replacing $x^*$ with $x'$ is not positive, i.e.:

$$\sum_{i \in N} \frac{b_i[v_i(x') - v_i(x^*)]}{v_i(x^*)} \leq 0. \quad (1)$$

This allocation rule is a strong refinement of Pareto efficiency, since Pareto efficiency only guarantees that if some player’s proportional change is strictly positive, then there must be some player whose proportional change is negative. The Proportionally Fair solution can also be defined as an allocation $x \in \mathcal{F}$ that maximizes $\prod_i [v_i(x)]^{b_i}$ or equivalently (after taking a logarithm), that maximizes $\sum_i b_i \log v_i(x)$; we will refer to these two equivalent objectives as the PF objectives. Note that, although the PF allocation need not be unique for a given instance, it does provide unique achieved bidder valuations [Eisenberg and Gale 1959].

We also note that the PF solution is equivalent to the Nash bargaining solution. John Nash in his seminal paper [Nash 1950] considered an axiomatic approach to bargaining and gave four axioms that any bargaining solution must satisfy. He showed that these four axioms yield a unique solution which is captured by a convex program; this convex program is equivalent to the one defined above for the PF solution. Another well-studied allocation rule which is equivalent to the PF allocation is the Competitive Equilibrium. Eisenberg [1961] showed that if all agents have valuation functions that are quasi-concave and homogeneous of degree 1, then the competitive equilibrium is also captured by the same convex program as the one for the PF solution. The Competitive Equilibrium with Equal Incomes (CEEI) has been proposed as the ideal allocation rule for fairness in microeconomics [Varian 1974; Budish 2010; Othman et al. 2010].

Given a valuation function reported from each bidder, we want to design mechanisms that output an allocation of items to bidders. We restrict ourselves to truthful mechanisms, i.e. mechanisms such that any false report from a bidder will never return her a more valuable allocation. Since proportional fairness cannot be implemented via truthful mechanisms, we will measure the performance of our mechanisms based on the extent to which they approximate this benchmark. More specifically, the approximation factor, or competitive factor of a mechanism will correspond to the minimum value of $\rho(\mathcal{I})$ across all relevant instances $\mathcal{I}$, where

$$\rho(\mathcal{I}) = \min_{i \in N} \left\{ \frac{v_i(x)}{v_i(x^*)} \right\},$$

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and $x, x^*$ are the allocation generated by the mechanism for instance $I$ and the PF allocation of $I$ respectively.

3. PARTIAL ALLOCATION MECHANISM

In this section, we define the Partial Allocation (PA) mechanism as a novel way to allocate divisible items to bidders with homogeneous valuation functions of degree one. We subsequently prove that this non-dictatorial mechanism not only achieves truthfulness, but also guarantees that every bidder will be receiving at least a $1/e$ fraction of the valuation that she deserves, as dictated by the PF solution. This mechanism depends on a subroutine that computes the PF allocation for the problem instance at hand; we therefore later study the running time of this subroutine, as well as the robustness of our results in case this subroutine returns only approximate solutions.

The PA mechanism elicits the valuation function $v_i(\cdot)$ from each player $i$ and it computes the PF allocation $x^*$ considering all the players’ valuations. The final allocation $x$ output by the mechanism gives each player $i$ only a fraction $f_i$ of her PF bundle $x^*_i$, i.e. for every item $j$ of which the PF allocation assigned to her a portion of size $x^*_{ij}$, the PA mechanism instead assigns to her a portion of size $f_i \cdot x^*_{ij}$, where $f_i \in [0, 1]$ depends on the extent to which the presence of player $i$ inconveniences the other players; the value of $f_i$ may therefore vary across different players. The following steps give a more precise description of the mechanism.

1. Compute the PF allocation $x^*$ based on the reported bids.
2. For each player $i$, remove this player and compute the PF allocation $x^*_{-i}$ that would arise in her absence.
3. Allocate to each player $i$ a fraction $f_i$ of everything that she receives according to $x^*$, where

$$f_i = \left( \prod_{i' \neq i} \frac{v_{i'}(x^*)^{b_{i'}}}{v_{i'}(x^*_{-i})^{b_{i'}}} \right)^{1/b_i}.$$  

**Lemma 3.1.** The allocation $x$ produced by the PA mechanism is feasible.

**Proof.** Since the PF allocation $x^*$ is feasible, to verify that the allocation produced by the PA mechanism is also feasible, it suffices to show that $f_i \in [0, 1]$ for every bidder $i$. The fact that $f_i \geq 0$ is clear since both the numerator and the denominator are non-negative. To show that $f_i \leq 1$, note that

$$x^*_{-i} = \arg \max_{x' \in \mathcal{F}} \left\{ \prod_{i' \neq i} v_{i'}(x') \right\}.$$  

Since $x^*$ remains a feasible allocation ($x^* \in \mathcal{F}$) after removing bidder $i$ (we can just discard bidder $i$’s share), this implies

$$\prod_{i' \neq i} v_{i'}(x^*) \leq \prod_{i' \neq i} v_{i'}(x^*_{-i}).$$

**3.1. Truthfulness**

We now show that, despite the fact that this mechanism is not dictatorial and does not use monetary payments, it is still in the best interest of every player to report her true valuation function, irrespective of what the other players do.

**Theorem 3.2.** The PA mechanism is truthful.
Proof. In order to prove this theorem, we approach the PA mechanism from the perspective of some arbitrary player $i$. Let $\bar{v}_i(\cdot)$ denote the valuation function that each player $i' \neq i$ reports to the PA mechanism. We assume that the valuation functions reported by these players may differ from their true ones, $v_i(\cdot)$. Player $i$ is faced with the options of either reporting her true valuation function $v_i(\cdot)$, or reporting some false valuation function $\bar{v}_i(\cdot)$. After every player has reported some valuation function, the PA mechanism computes the PF allocation with respect to these valuation functions; let $x_T$ denote the PF allocation that arises if player $i$ reports the truth and $x_L$ otherwise. Finally, player $i$ receives a fraction of what the computed PF allocation assigned to her, and how big or small this fraction will be depends on the computed PF allocation. Let $f_T$ denote the fraction of her allocation that player $i$ will receive if $x_T$ is the computed PF allocation and $f_L$ otherwise. Since the players have homogeneous valuation functions of degree one, what we need to show is that $f_T v_i(x_T) \geq f_L v_i(x_L)$, or equivalently that

$$[f_T v_i(x_T)]^{b_i} \geq [f_L v_i(x_L)]^{b_i}.$$  

Note that the denominators of both fractions $f_T$ and $f_L$, as given by Equation (2), will be the same since they are independent of the valuation function reported by player $i$. Our problem therefore reduces to proving that

$$[v_i(x_T)]^{b_i} \prod_{i' \neq i} [\bar{v}_{i'}(x_T)]^{b_{i'}} \geq [v_i(x_L)]^{b_i} \prod_{i' \neq i} [\bar{v}_{i'}(x_L)]^{b_{i'}}. \tag{3}$$

To verify that this inequality holds we use the fact that the PF allocation is the one that maximizes the product of the corresponding reported valuations. This means that

$$x_T = \arg \max_{x \in \mathcal{F}} \left\{ [v_i(x)]^{b_i} \prod_{i' \neq i} [\bar{v}_{i'}(x)]^{b_{i'}} \right\},$$

and since $x_i \in \mathcal{F}$, this implies that Inequality (3) holds, and therefore reporting her true valuation function is a dominant strategy for every player $i$. ⌣

The arguments used in the proof of Theorem 3.2 imply that, given the valuation functions reported by all the other players $i' \neq i$, player $i$ can effectively choose any bundle that she wishes, but for each bundle the mechanism defines what fraction player $i$ can keep. One can therefore think of the fraction of the bundle thrown away as a form of non-monetary “payment” that the player must suffer in exchange for that bundle, with different bundles matched to different payments. The fact that the PA mechanism is truthful implies that these payments, in the form of fractions, make the bundle allocated to her by allocation $x^*$ the most desirable one. We revisit this interpretation later on in this section.

3.2. Approximation

Before studying the approximation factor of the PA mechanism, we first state a lemma which will be useful for proving Theorem 3.4 (its proof is deferred to the full version of the paper).

Lemma 3.3. For any set of pairs $(\delta_i, \beta_i)$ with $\beta_i \geq 1$ and $\sum_i \beta_i \cdot \delta_i \leq b$ the following holds (where $B = \sum_i \beta_i$)

$$\prod_i (1 + \delta_i)^{\beta_i} \leq \left( 1 + \frac{b}{B} \right)^B.$$ 

Using this lemma we can now prove tight bounds for the approximation factor of the Partial Allocation mechanism. As we show in this proof, the approximation factor depends
directly on the relative weights of the players. For simplicity in expressing the approximation factor, let \( b_{\text{min}} \) denote the smallest value of \( b_i \) across all bidders of an instance and let \( B = (\sum_{i \in N} b_i) - b_{\text{min}} \) be the sum of the \( b_i \) values of all the other bidders. Finally, let \( \psi = B/b_{\text{min}} \) denote the ratio of these two values.

**Theorem 3.4.** The approximation factor of the Partial Allocation mechanism for the class of problem instances of some given \( \psi \) value is exactly

\[
\left(1 + \frac{1}{\psi}\right)^{-\psi}.
\]

**Proof.** The PA mechanism allocates to each player \( i \) a fraction \( f_i \) of her PF allocation, and for the class of homogeneous valuation functions of degree one this means that the final valuation of player \( i \) will be \( v_i(x) = f_i \cdot v_i(x^*) \). The approximation factor guaranteed by the mechanism is therefore equal to \( \min_i \{f_i\} \). Without loss of generality, let player \( i \) be the one with the minimum value of \( f_i \). In the PF allocation \( x_{-i}^* \) that the PA mechanism computes after removing player \( i \), every other player \( i' \) experiences a value of \( v_{i'}(x_{-i}^*) \). Let \( d_{i'} \) denote the proportional change between the valuation of player \( i' \) for allocation \( x_{-i}^* \) and allocation \( x^* \), i.e.

\[
v_{i'}(x_{-i}^*) = (1 + d_{i'})v_{i'}(x^*).
\]

Substituting for \( v_{i'}(x_{-i}^*) \) in Equation (2) yields:

\[
f_i = \left(\frac{1}{\prod_{i' \neq i} (1 + d_{i'})^{b_{i'}}}\right)^{1/b_i}.
\]

Since \( x^* \) is a PF allocation, Inequality (1) implies that

\[
\sum_{i' \in N} \frac{b_{i'} [v_{i'}(x_{-i}^*) - v_{i'}(x^*)]}{v_{i'}(x^*)} \leq 0 \iff \sum_{i' \neq i} b_{i'}d_{i'} + \frac{b_i [v_{i}(x_{-i}^*) - v_{i}(x^*)]}{v_{i}(x^*)} \leq 0 \iff \sum_{i' \neq i} b_{i'}d_{i'} \leq b_i.
\]

The last equivalence holds due to the fact that \( v_i(x_{-i}^*) = 0 \), since allocation \( x_{-i}^* \) clearly assigns nothing to player \( i \).

Let \( B_{-i} = \sum_{i' \neq i} b_{i'} \); using Inequality (5) and Lemma 3.3 (on substituting \( b_i \) for \( b \), \( d_{i'} \) for \( \delta_{i} \), \( b_{i'} \) for \( \beta_{i} \), and \( B_{-i} \) for \( B \)), it follows from Equation (4) that

\[
f_i \geq \left(1 + \frac{b_i}{B_{-i}}\right)^{-\frac{b_{\text{min}}}{b_i}}.
\]

To verify that this bound is tight, consider any instance with just one item and the given \( \psi \) value. The PF solution dictates that each player should be receiving a fraction of the item proportional to the player’s \( b_i \) value. The removal of a player \( i \) therefore leads to a proportional increase of exactly \( b_i/B_{-i} \) for each of the other players’ PF valuation. The PA mechanism therefore assigns to every player \( i \) a fraction of her PF allocation which is equal to the right hand side of Inequality (6). The player with the smallest \( b_i \) value receives the smallest fraction. \( \square \)
The approximation factor of Theorem 3.4 implies that \( f_i \geq 1/2 \) for instances with two players having equal \( b_i \) values, and \( f_i \geq 1/e \) even when \( \psi \) goes to infinity; we therefore get the following corollary.

**Corollary 3.5.** The Partial Allocation mechanism always yields an allocation \( x \) such that for every participating player \( i \)

\[
v_i(x) \geq \frac{1}{e} \cdot v_i(x^*).
\]

To complement this approximation factor, we now provide a negative result showing that, even for the special case of additive linear valuations, no truthful mechanism can guarantee an approximation factor better than \( \frac{n+1}{2n} \).

**Theorem 3.6.** There is no truthful mechanism that can guarantee an approximation factor greater than \( \frac{n+1}{2n} + \epsilon \) for any constant \( \epsilon > 0 \) for all \( n \)-player problem instances, even if the valuations are restricted to being additive linear.

**Proof.** For an arbitrary real value of \( n > 1 \), let \( \rho = \frac{n+1}{2n} \), and assume that \( Q \) is a truthful resource allocation mechanism that guarantees a \( (\rho + \epsilon) \) approximation for all \( n \)-player problem instances, where \( \epsilon \) is a positive constant. This mechanism receives as input the bidders’ valuations and it returns a valid (fractional) allocation of the items. We will define \( n + 1 \) different input instances for this mechanism, each of which will consist of \( n \) bidders and \( m = (k + 1)n \) items, where \( k > \frac{\epsilon}{2} \) will take very large values. In order to prove the theorem, we will then show that \( Q \) cannot simultaneously achieve this approximation guarantee for all these instances, leading to a contradiction. For simplicity we will refer to each bidder with a number from 1 to \( n \), to each item with a number from 1 to \( (k + 1)n \), and to each problem instance with a number from 1 to \( n + 1 \).

We start by defining the first \( n \) problem instances. For \( i \leq n \), let problem instance \( i \) be as follows: Every bidder \( i' \neq i \) has a valuation of \( kn + 1 \) for item \( i' \) and a valuation of 1 for every other item; bidder \( i \) has a valuation of 1 for all items. In other words, all bidders except bidder \( i \) have a strong preference for just one item, which is different for each one of them. The PF allocation for such additive linear valuations dictates that every bidder \( i' \neq i \) is allocated only item \( i' \), while bidder \( i \) is allocated all the remaining \( kn + 1 \) items. Since \( Q \) achieves a \( \rho + \epsilon \) approximation for this instance, then it needs to provide bidder \( i \) with an allocation which the bidder values at least at \( (\rho + \epsilon)(kn + 1) \). In order to achieve this, mechanism \( Q \) can assign to this bidder fractions of the set \( M_i \) of the \( n - 1 \) items that the PF solution allocates to the other bidders as well as fractions of the set \( M_i \) of the \( kn + 1 \) items that the PF allocation allocates to bidder \( i \). Even if all of the \( n - 1 \) items of \( M_i \) were fully allocated to bidder \( i \), the mechanism would still need to assign to this bidder an allocation of value at least \( (\rho + \epsilon)(kn + 1) - (n - 1) \) using items from \( M_i \). Since \( k > \frac{\epsilon}{2} \), \( n - 1 < \frac{\epsilon}{2} (kn + 1) \), and therefore mechanism \( Q \) will need to allocate to bidder \( i \) a fractional assignment of items in \( M_i \) that the bidder values at least at \( (\rho + \frac{\epsilon}{2}) (kn + 1) \). This implies that there must exist at least one item in \( M_i \) of which bidder \( i \) is allocated a fraction of size at least \( \rho + \frac{\epsilon}{2} \). Since all the items in \( M_i \) are identical and the numbering of the items is arbitrary, we can, without loss of generality, assume that this item is item \( i \). We have therefore shown that, for every instance \( i \leq n \) mechanism \( Q \) will have to assign to bidder \( i \) at least \( \rho + \frac{\epsilon}{2} \) of item \( i \), and an allocation of items in \( M_i \) that guarantees her a valuation of at least \( \rho + \frac{\epsilon}{2} (kn + 1) \).

We now define problem instance \( n + 1 \), in which every bidder \( i \) has a valuation of \( kn + 1 \) for item \( i \) and a valuation of 1 for all other items. The PF solution for this instance would allocate to each bidder \( i \) all of item \( i \), as well as \( k \) items from the set \( \{n + 1, \ldots, (k + 1)n\} \) (or more generally, fractions of these items that add up to \( k \)). Clearly, every bidder \( i \) can unilaterally misreport her valuation leading to problem instance \( i \) instead of this instance;
so, in order to maintain truthfulness, mechanism $Q$ will have to provide every bidder $i$ of problem instance $n+1$ with at least the value that such a deviation would provide her with. One can quickly verify that, even if mechanism $Q$ when faced with problem instance $i$ provided bidder $i$ with no more than a $(\rho + \frac{\epsilon}{2})$ fraction of item $i$, still such a deviation would provide bidder $i$ with a valuation of at least

\[(\rho + \frac{\epsilon}{2})(kn + 1) + (\rho + \frac{\epsilon}{2})kn \geq (\rho + \frac{\epsilon}{2})2kn.\]

The first term of the left hand side comes from the fraction of item $i$ that the bidder receives and the second term comes from the average fraction of the remaining items. If we substitute $\rho = \frac{n+1}{2n}$, we get that the truthfulness of $Q$ implies that every bidder $i$ of problem instance $n+1$ will have to receive an allocation of value at least

\[\left(\frac{n+1}{2n} + \frac{\epsilon}{2}\right)2kn = kn + k + \epsilon kn.\]

For any given constant value of $\epsilon$ though, since $k > \frac{3}{2}$ and $n > 1$, every bidder will need to be assigned an allocation that she values at more than $kn + k + 2$, which is greater than the valuation of $kn + k + 1$ that the player receives in the PF solution. This is obviously a contradiction since the PF solution is Pareto efficient and there cannot exist any other allocation for which all bidders receive a strictly greater valuation. □

Theorem 3.6 implies that, even if all the players have equal $b_i$ values, no truthful mechanism can guarantee a greater than $3/4$ approximation even for instances with just two bidders, and this bound drops further as the number of bidders increases, finally converging to $1/2$. To complement the statement of Corollary 3.5, we therefore get the following corollary.

**Corollary 3.7.** No truthful mechanism can guarantee that it will always yield an allocation $x$ such that for any $\epsilon > 0$ and for every participating player $i$

\[v_i(x) \geq \left(\frac{1}{2} + \epsilon\right) \cdot v_i(x^*).\]

### 3.3. Running Time and Robustness

The PA mechanism has reduced the problem of truthfully implementing a constant factor approximation of the PF allocation to computing exact PF allocations for several different problem instances, as this is the only subroutine that the mechanism calls. If the valuation functions of the players are affine, then there is a polynomial time algorithm to compute the exact PF allocation [Devanur et al. 2008; Jain and Vazirani 2007].

We now show that, even if the PF solution can be only approximately computed in polynomial time, our truthfulness and approximation related statements are robust with respect to such approximations (all the proofs of this subsection are deferred to the full version of the paper). More specifically, we assume that the PA mechanism uses a polynomial time algorithm that computes a feasible allocation $\tilde{x}$ instead of $x^*$ such that

\[\left[\prod_i [v_i(\tilde{x})]^{b_i}\right]^{1/B} \geq \left[ (1 - \epsilon) \prod_i [v_i(x^*)]^{b_i}\right]^{1/B}, \text{ where } B = \sum_{i=1}^{n} b_i.\]

Using this algorithm, the PA mechanism can be adapted as follows:

1. Compute the approximate PF allocation $\tilde{x}$ based on the reported bids.
2. For each player $i$, remove this player and compute the approximate PF allocation $\tilde{x}_{-i}$ that would arise in her absence.
(3) Allocate to each player $i$ a fraction $\tilde{f}_i$ of everything that she receives according to $\tilde{x}$ where

$$
\tilde{f}_i = \min \left\{ 1, \left( \frac{\prod_{i' \neq i} [v_{i'}(\tilde{x})]^{b_{i'}}}{\prod_{i' \neq i} [v_{i'}(x_{-i})]^{b_{i'}}} \right)^{1/b_i} \right\}.
$$

For this adapted version of the PA mechanism to remain feasible, we need to make sure that $\tilde{f}_i$ remains less or equal to 1. Even if, for some reason, the allocation $\tilde{x}_{-i}$ computed by the approximation algorithm does not satisfy this property, the adapted mechanism will then choose $\tilde{f}_i = 1$ instead.

We start by showing two lemmas verifying that this adapted version of the PA mechanism is robust both with respect to the approximation factor it guarantees and with respect to the truthfulness guarantee.

**Lemma 3.8.** The approximation factor of the adapted PA mechanism for the class of problem instances of some given $\psi$ value is at least 

$$(1 - \epsilon) \left( 1 + \frac{1}{\psi} \right)^{-\psi}.$$ 

**Lemma 3.9.** If a player misreports her preferences to the adapted PA mechanism, she may increase her valuation by at most a factor $(1 - \epsilon)^{-2}$.

Finally, we show that if the valuation functions are, for example, concave and homogeneous of degree one, then a feasible approximate PF allocation can indeed be computed in polynomial time.

**Lemma 3.10.** For concave homogeneous valuation functions of degree one, there exists an algorithm that computes a feasible allocation $\tilde{x}$ in time polynomial in $\log 1/\epsilon$ and the problem size, such that

$$\prod_i [v_i(\tilde{x})]^{b_i} \geq (1 - \epsilon) \prod_i [v_i(x^*)]^{b_i}.$$ 

### 3.4. Extension to General Homogeneous Valuations

We can actually extend most of the results that we have shown for homogeneous valuation functions of degree one to any valuation function that can be expressed as $v_i(f \cdot x) = g_i(f) \cdot v_i(x)$, where $g_i(\cdot)$ is some increasing invertible function; for homogeneous valuation functions of degree $d$, this function is $g_i(f) = f^d$. If this function is known for each bidder, we can then adapt the PA mechanism as follows: instead of allocating to bidder $i$ a fraction $f_i$ of her allocation according to $x^*$ as defined in Equation (2), we instead allocate to this bidder a fraction $g_i^{-1}(f_i)$, where $g_i^{-1}(\cdot)$ is the inverse function of $g_i(\cdot)$. If, for example, some bidder has a homogeneous valuation function of degree $d$, then allocating her a fraction $f_i^{1/d}$ of her PF allocation has the desired effect and both truthfulness and the same approximation factor guarantees still hold. The idea behind this transformation is that all that we need in order to achieve truthfulness and the approximation factor is to be able to discard some fraction of a bidder’s allocation knowing exactly what fraction of her valuation this will correspond to.

### 3.5. Connection with VCG-based Mechanism Design

In hindsight, a closer look at the PA mechanism reveals an interesting connection with the well known VCG-mechanism: The valuation of player $i$ for the output of the PA mechanism

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is \( v_i(x) = f_i \cdot v_i(x^*) \), or

\[
v_i(x) = \left( \prod_{i' \neq i} [v_{i'}(x^*)]^{b_{i'}} \right)^{1/b_i} \cdot v_i(x^*). \tag{8}
\]

The connection is revealed after considering the surrogate valuation \( u_i(\cdot) = b_i \log v_i(\cdot) \) for each player \( i \). After taking a logarithm on both sides of Equation (8) and then multiplying them by \( b_i \), using surrogate valuation function \( u_i(\cdot) \), yields

\[
u_i(x) = u_i(x^*) - \left( \sum_{i' \neq i} u_i'(x^*_{-i}) - \sum_{i' \neq i} u_i'(x^*) \right).
\]

This equation shows that the surrogate valuation of a player for the output of the PA mechanism equals her surrogate valuation for the PF allocation minus a payment which corresponds to the externalities with respect again to the surrogate valuations. The connection is complete if one notices that the PF objective aims to compute an allocation maximizing the social welfare with respect to these surrogate valuations. Therefore, the impact of the fraction that is removed from each player’s PF allocation is analogous to that of a VCG payment in the surrogate valuations’ space.

4. STRONG DEMAND MATCHING MECHANISM

The main result of the previous section shows that one can guarantee a good constant factor approximation for any problem instance within a very large class of bidder valuations. The subsequent impossibility result shows that, even if we restrict ourselves to problem instances with additive linear bidder valuations, no truthful mechanism can guarantee more than a 1/2 approximation.

In this section we study the question of whether one can achieve even better factors when restricted to some well-motivated class of instances. We focus on additive linear valuations, and we provide a positive answer to this question for problem instances where every item is highly demanded. More formally, we consider problem instances for which the PF price (or equivalently the competitive equilibrium price) of every item is large when the budget of every player is fixed to one unit of scrip money\(^5\). The motivation behind this class of instances comes from problems such as the one that arose with the Czech privatization auctions [Aggarwal and Harper 2000]. For such instances, where the number of players is much higher than the number of items, one naturally anticipates that all item prices will be high in equilibrium.

For the rest of the paper we assume that the weights of all players are equal and that their valuations are additive linear. Let \( p_j^* \) denote the PF price of item \( j \) when every bidder \( i \)’s budget \( b_i \) is equal to 1. Our main result in this section is the following:

**Theorem 4.1.** For additive linear valuations there exists a truthful mechanism that achieves an approximation factor of \( \min_j \{ p_j^* / \lceil p_j^* \rceil \} \).

Note that if \( k = \min_j p_j^* \), this is an approximation factor of at least \( k/(k + 1) \).

We now describe our solution which we call the Strong Demand Matching mechanism (SDM). Informally speaking, SDM starts by giving every bidder a unit amount of scrip money. It then aims to discover minimal item prices such that the demand of each bidder at these prices can be satisfied using (a fraction of) just one item. In essence, our mechanism is restricted to computing allocations that assign each bidder to just one item, and this

\(^5\)Remark: Our mechanism does not make this assumption, but the approximation guarantees are much better with this assumption.
restriction of the output space renders the mechanism truthful and gives an approximation guarantee much better than that of the PA mechanism for instances where every item is highly demanded.

The procedure used by our mechanism is reminiscent of the method utilized by Demange et al. for multi-unit auctions [Demange et al. 1986]. Recall that this method increases the prices of all over-demanded items uniformly until the set $R$ of over-demanded items changes, iterating this process until $R$ becomes empty. At that point, bidders are matched to preferred items. For our setting, each bidder will seek to spend all her money, and we employ an analogous rising price methodology, again making allocations when the set of over-demanded items is empty. In our setting, the price increases are multiplicative rather than additive, however. This approach also has some commonality with the algorithm of Devanur et al. [2008] for computing the competitive equilibrium for divisible items and bidders with additive linear valuations. Their algorithm also proceeds by increasing the prices of over-demanded items multiplicatively. Of course, their algorithm does not yield a truthful mechanism. Also, in order to achieve polynomial running time in computing the competitive equilibrium, their algorithm needs, at any one time, to be increasing the prices of a carefully selected subset of these items; this appears to make their algorithm quite dissimilar to ours. Next we specify our mechanism in more detail.

Let $p_j$ denote the price of item $j$, and let the bang per buck that bidder $i$ gets from item $j$ equal $v_{ij}/p_j$. We say that item $j$ is an MBB item of bidder $i$ if she gets the maximum bang per buck from that item. For a given price vector $p$, let the demand graph $D(p)$ be a bipartite graph with bidders on one side and items on the other, such that an edge between bidder $i$ and item $j$ exists if and only if $j$ is an MBB item of bidder $i$. We call $c_j = \lceil p_j \rceil$ the capacity of item $j$ when its price is $p_j$, and we say an assignment of bidders to items is valid if it matches each bidder to one of her MBB items and no item $j$ is matched to more than $c_j$ bidders. Given a valid assignment $A$, we say an item $j$ is reachable from bidder $i$ if there exists an alternating path $(i, j_1, j_2, \ldots, j_k, j)$ in the graph $D(p)$ such that edges $(i, j_1), \ldots, (i_k, j_k)$ lie in the assignment $A$. Finally, let $d(R)$ be the collection of bidders with all their MBB items in set $R$.

The SDM mechanism initializes all item prices to $p_j = 1$ and iterates as follows:

1. Find a valid assignment that maximizes the number of matched bidders. If all the bidders are matched, conclude with Step 3.
2. Let $U$ be the set of bidders who are not matched in Step 1. Let $R$ be the set of all items reachable from bidders in the set $U$. Increase the price of each item $j$ in $R$ from $p_j$ to $r \cdot p_j$, where $r \geq 1$ is the minimum value for which one of the following events takes place:
   (a) The price of an item in $R$ reaches an integral value. Then repeat Step 1.
   (b) For some bidder $i \in d(R)$, her set of MBB items increases, causing $R$ to grow:
      i. If for each item $j$ added to $R$, the number of bidders matched to it equals $c_j$, continue with Step 2.
      ii. If some item $j$ added to $R$ has $c_j$ greater than the number of bidders matched to it, continue with Step 1.
3. Every bidder matched to some item $j$ is allocated a fraction $1/p_j$ of that item.

It remains to explain how to carry out Step 2. Set $R$ can be found using a breadth-first-search like algorithm. To determine when (a) is reached, we just need to know the smallest $\lceil p_j \rceil/p_j$ ratio over all items whose price is being increased. For (b), we need to calculate,

---

6Note that for each bidder there could be multiple MBB items and that in the PF solution bidders are only allocated such MBB items.
for each bidder in \( d(R) \), the ratio of the bang per buck for her MBB items and for the items outside the set \( R \).

4.1. Running time

If \( c(R) = \sum_{j \in R} c_j \) denotes the total capacity in \( R \), it is not difficult to see that if \( U \) is non-empty, \(|d(R)| > c(R)\). Note that each time either event (a) or event (b)-ii occurs, \( c(R) \) increases by at least 1, and thus, using the alternating path from a bidder in the set \( U \) to the corresponding item, we can increase the number of matched bidders by at least 1; this means that this can occur at most \( n \) times. The only other events are the unions (of connected components in graph \( D(p) \)) resulting from (b)-i. There can be at most \( \min(n, m) \) of these, and they are followed by either Step (a) or (b)-ii. Thus there are \( O(n \cdot \min(n, m)) \) iterations of Step (b)-i and \( O(n) \) iterations of Steps 1 and (b)-ii.

4.2. Truthfulness and Approximation

The proofs of the truthfulness and the approximation of the SDM mechanism use the following lemma which states that the prices computed by the mechanism are the minimum prices supporting a valid assignment. An analogous result was shown in [Demange et al. 1986] for a multi-unit auction of non-divisible items. We provide an algorithmic argument.

**Lemma 4.2.** For any problem instance, if \( p \geq 1 \) is a set of prices for which there exists a valid assignment, then the prices \( q \) computed by the SDM mechanism will satisfy \( q \leq p \).

**Proof.** Aiming for a contradiction, assume \( q_j > p_j \) for some item \( j \), and let \( \bar{q} \) be the maximal price vector that the SDM mechanism reaches before increasing the price of some item \( j' \) beyond \( p_{j'} \) for the first time. In other words, \( \bar{q} \leq p \) and \( \bar{q}_{j'} = p_{j'} \). Also, let \( S = \{ j \in M \mid \bar{q}_j = p_j \} \), which implies that \( \bar{q}_j < p_j \) for all \( j \notin S \). Clearly, any bidder \( i \) who has MBB items in \( S \) at prices \( \bar{q} \) will not be interested in any other item at prices \( p \). This implies that the valid assignment that exists for prices \( p \) assigns every such bidder to one of her MBB items \( j \in S \). Therefore, the total capacity of items in \( S \) at prices \( \bar{q} \) is large enough to support all these bidders and hence no item in \( S \) will be over-demanded at prices \( \bar{q} \). As a result, the SDM mechanism will not increase the price of any item in \( S \), which leads us to a contradiction.

Using this lemma we can now prove the statements regarding the truthfulness and the approximation factor of SDM; the following two lemmata imply Theorem 4.1.

**Lemma 4.3.** The SDM mechanism is truthful.

**Proof.** Given a problem instance, fix some bidder \( i \) and let \( x' \) and \( q' \) denote the assignment and the prices that the SDM mechanism outputs instead of \( x \) and \( q \) when this bidder reports a valuation vector \( v'_i \) instead of her true valuation vector \( v_i \).

If the item \( j \) to which bidder \( i \) is assigned in \( x' \) is one of her MBB items w.r.t. her true valuations \( v_i \) and prices \( q' \), then \( x' \) would be a valid assignment for prices \( q' \) even if the bidder had not lied. Lemma 4.2 therefore implies that \( q \leq q' \). Since the item to which bidder \( i \) is assigned by \( x \) is an MBB item and \( q \leq q' \), we can conclude that \( v_i(x) \geq v_i(x') \).

If on the other hand item \( j \) is not an MBB item w.r.t. the true valuations of bidder \( i \) and prices \( q' \), we consider an alternative valid assignment and prices. Starting from prices \( q' \), we run the steps of the SDM mechanism assuming bidder \( i \) has reported her true valuations \( v_i \), and we consider the assignment \( \bar{x} \) and the prices \( \bar{q} \) that the mechanism would yield upon termination. Assignment \( \bar{x} \) would clearly be valid for prices \( \bar{q} \) if bidder \( i \) had reported the truth; therefore Lemma 4.2 implies \( q \leq \bar{q} \) and thus \( v_i(x) \geq v_i(\bar{x}) \). As a result, to conclude the proof it suffices to show that \( v_i(\bar{x}) \geq v_i(x') \). To verify this fact, we show that \( q'_{ij} = \bar{q}_{ij} \), implying that \( \bar{x} \) allocates to \( i \) (a fraction of) some item which she values at least as much as a \( 1/q'_{ij} \) fraction of item \( j \).
Consider the assignment \( x'_{-i} \) that matches all bidders \( i' \neq i \) according to \( x' \) and leaves bidder \( i \) unmatched. In the graph \( D(q') \), if item \( j \) is reachable from bidder \( i \) given the valid assignment \( x'_{-i} \), then all bidders would be matched by the very first execution of Step 1 of the mechanism. This is true because the capacity of item \( j \) according to prices \( q' \) is greater than the number of bidders matched to it in \( x'_{-i} \). The alternating path \((i, j_1, i_1, j_2, i_2, \ldots, j_k, i_k, j)\) implied by the reachability can therefore be used to ensure that bidder \( i \) is matched to an MBB item as well; this is achieved by matching \( i \) to \( j_1, i_1 \) to \( j_2 \) and so on. Otherwise, if not all bidders can be matched in that very first step of the SDM mechanism, the mechanism can instead match the bidders according to \( x'_{-i} \) and set \( U = \{i\}. \) Before the price of item \( j \) can be increased, event (b)-ii must add this item to the set \( R \). If this happens though, item \( j \) becomes reachable from bidder \( i \) thus causing an alternating path to form, and the next execution of Step 1 of the mechanism yields a valid assignment before \( q'_j \) is ever increased.

### Lemma 4.4.

The SDM mechanism achieves an approximation of \( \min_i \{p_j^*/[p_j^*]\} \).

**Proof.** We start by showing that there must exist a valid assignment at prices \( fp^* \), where \( p^* \) corresponds to the PF prices and \( f = \max_j \{p_j^*/p^*_j\} \). Given any PF allocation \( x^* \), we consider the bipartite graph on items and bidders that has an edge between a bidder and an item if and only if \( x^* \) assigns a portion of the item to that bidder. If there exists a cycle in this graph, one can remove an edge in this cycle by reallocating along the cycle while maintaining the valuation of every bidder. To verify this is possible, note that all the items that a bidder is connected to with an edge are MBB items for this bidder, and therefore the bidder is indifferent regarding how her spending is distributed among them. Hence w.l.o.g. we can assume that the graph of \( x^* \) is a forest.

For a given tree in this forest, root it at an arbitrary bidder. For each bidder in this tree, assign her to one of her child items, if any, and otherwise to her parent item. Note that the MBB items for each bidder at prices \( fp^* \) are the same as at prices \( p^* \), so every bidder is assigned to one of her MBB items. Therefore, in order to conclude that this assignment is valid at prices \( fp^* \) it is sufficient to show that the capacity constraints are satisfied. The fact that \( fp^*_j \geq [p_j^*] \) implies that \( fp^*_j \geq [p_j^*] \), so we just need to show that, for each item \( j \), at most \( [p_j^*] \) bidders are assigned to it. To verify this fact, note that any bidder who is assigned to her parent item does not have child items so, in \( x^* \), she is spending all of her unit of scrip money on that parent item. In other words, for any item \( j \), the only bidder that may be assigned to it without having contributed to an increase of \( j \)'s PF price by 1 is the parent bidder of \( j \) in the tree; thus, the total number of bidders is at most \( [p_j^*] \).

Now, let \( q \) and \( x \) denote the prices and the assignment computed by the SDM mechanism; by Lemma 4.2, since there exists a valid assignment at prices \( fp^* \), this implies that \( q \leq fp^* \). The fact that the SDM mechanism assigns each bidder to one of her MBB items at prices \( q \) implies that \( v_i(x) = \max_j \{v_{ij}/q_j\} \). On the other hand, let \( r \) be an MBB item of bidder \( i \) at the PF prices \( p^* \). If bidder \( i \) had \( b_i \) units of scrip money to spend on such MBB items, this would mean that \( v_i(x^*) = b_i(v_{ir}/p^*_{r}) \) so, since \( b_i = 1 \), this implies that \( v_i(x^*) = v_{ir}/p^*_{r} \). Using this inequality along with the fact that \( q_j \leq fp^*_j \) for all items \( j \), we can show that

\[
v_i(x) = \max_j \left\{ \frac{v_{ij}}{q_j} \right\} \geq \frac{v_{ir}}{q_r} \geq \frac{v_{ir}}{fp^*_r} = \frac{1}{f} \cdot v_i(x^*),
\]

from which we conclude that \( v_i(x) \geq \min_j \{p_j^*/[p_j^*]\} \cdot v_i(x^*) \) for any bidder \( i \).

---

\(^7\)Note that this may not be the only way in which the SDM mechanism can proceed but, since the bidders’ valuations for the final outcome are unique, this is without loss of generality.
5. CONCLUSION

Our work was motivated by the fact that no incentive compatible mechanisms were known for the natural and widely used fairness concept of Proportional Fairness. In hindsight our work provides several new contributions. Firstly, the class of bidder valuation functions for which our results apply is surprisingly large and it contains several well studied functions; previous truthful mechanisms for fairness were studied for much more restricted classes of valuation functions. Secondly, to the best of our knowledge, this is first work that defines and gives guarantees for a strong notion of approximation for fairness, where one desires to approximate the valuation of every bidder. Lastly, our Partial Allocation mechanism can be seen as a framework for designing truthful mechanisms without money. This mechanism can be generalized further by restricting the range of the outcomes (similar to maximal-in-range mechanisms when one can use money). We believe that this generalization is a powerful one, and might allow for new solutions to other mechanism design problems without money. We plan to explore this in our future research.

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