ZETA ZERO DEPENDENCE AND THE CRITICAL LINE

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ABSTRACT. On the critical line the conditional distribution of the zeta function’s magnitude around zeta zeros exists and predicts the well-known pair correlation between nontrivial zeta zeros. However, this conditional distribution does not exist at most distances above or below any nontrivial zeta zeros that are off the critical line. This shows that the zeta function’s magnitude cannot have vertical statistical structure at most distances around nontrivial zeta zeros off the critical line. The proofs of these results are straightforward, using only statistical properties of certain prime sums, elementary properties of normal and elliptical random variables, and the pole structure of the zeta function. These results readily generalize to L-functions.

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1. Introduction

The Riemann zeta function \( \zeta(s) \) may be represented by the following sum over integers \( n \) or product over prime numbers \( p \) for \( \text{Re}(s) > 1 \):

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}
\]
The former representation is called the Dirichlet series, while the latter representation is called the Euler product. \( \zeta(s) \) may be extended by analytic continuation to all \( s \) in \( \mathbb{C} \) except \( s = 1 \), as \( \zeta(1) \) is a simple pole. Values of \( s \) such that \( \text{Im}(s) > 0 \) and \( \zeta(s) = 0 \), the nontrivial zeta zeros, are of great importance for number theory, as they are directly relevant for understanding the distribution of the prime numbers. The proof or disproof of the famous Riemann hypothesis (RH), which asserts that all of the zeta function’s nontrivial zeros reside on the critical line, \( \text{Re}(s) = 1/2 \), is likely the most important open problem in number theory.

The importance of the critical line has led to a substantial amount of research on the behavior of the zeta function on this line. This research includes studies of the distribution of the zeta function’s nonzero values on the critical line and the distribution of its zeros on the critical line. A foundational result in the former area is Selberg’s central limit theorem \[44\] \[40\], which states that, for \( t \) uniformly distributed in \([T, 2T]\) with \( T \to \infty \),

\[
\log \left| \zeta \left( \frac{1}{2} + it \right) \right| \overset{d}{\to} \mathcal{N} \left( 0, \frac{1}{2} \log \log T \right).
\]

That is, \( \log |\zeta(1/2 + it)| \) converges to a normally distributed random variable with mean zero and variance \( \frac{1}{2} \log \log T \) under \( t \) uniformly distributed in \([T, 2T]\).

A similarly foundational discovery regarding the distribution of the zeta zeros on the critical line was made by Montgomery [37], who gave the first results describing pair-correlation between zeros, assuming RH. His results predicted a short-range repulsion between consecutive zeros on the critical line. Odlyzko [38] later calculated large numbers of zeta zeros and empirically corroborated Montgomery’s predictions. Dyson [17] famously pointed out that Montgomery’s result was equivalent to that for the pair-correlation of eigenvalues of random unitary matrices. This connection with random matrix theory has been extremely useful for understanding statistical dependence in zeros of the zeta function [45] and its generalizations [30] [42] [43] [32]. This connection has also found many other applications, e.g., by Keating & Snaith [33], who gave influential conjectures on the moments of the zeta function on the critical line, and by Fyodorov, Hiary, & Keating (FHK) [22] [23], who gave a notable conjecture on the distribution of the zeta function’s extreme values on the critical line.

Bourgade [13] provided an important extension to Selberg’s and Montgomery’s results by showing that, for \( t \) uniformly distributed in \([T, 2T]\), the covariance of \( \log |\zeta (1/2 + it)| \) and \( \log |\zeta (1/2 + i(t + \Delta))| \) is

\[
\approx -\frac{1}{2} \log |\Delta|
\]

for mesoscopic distances \( \Delta \) where \( 1/\log T \ll |\Delta| \ll 1 \), showing that \( \log |\zeta(1/2 + it)| \) has logarithmic correlations over such distances. This result then confirmed suggestions by Coram & Diaconis [16] that, along with the microscopic repulsion (\( |\Delta| < 1/\log T \)) between zeta zeros predicted by
Montgomery, there is also a mesoscopic repulsion between zeros. The logarithmic correlation structure of $\log |\zeta(1/2 + it)|$ was an important motivation for FHK’s conjecture, and it served as an important link to the theory of branching random walks. The latter was applied by Arguin, Belius, & Harper [1] to show that the leading terms of FHK’s conjecture hold for a random model of the zeta function, applied by Arguin, Belius, Bourgade, Radziwill, & Soundararajan [2] to verify the leading order of FHK’s conjecture for the zeta function itself, by Arguin, Bourgade, & Radziwill [3] to prove the upper bound in FHK’s conjecture, and by Arguin, Omet, & Radziwill [5] to provide new results describing the moments and maxima of the zeta function on the critical line, proving a related conjecture of Fyodorov & Keating [23], and generalizing the results in [2].

Random matrix theory is closely connected to the field of quantum chaos [6] [12], and it was ideas from this field that first illuminated long-range or macroscopic ($|\Delta| \geq 1$) statistical dependence in the zeta zeros. In particular, Bogomolny & Keating [9] [10] [11] gave the first predictions for such dependence in the Riemann zeros by utilizing semi-classical techniques from the field of quantum chaos along with the Hardy-Littlewood conjecture from number theory. Their results describe an effect where differences between zeros tend to avoid the imaginary parts of the low-lying zeta zeros themselves. Conrey & Snaith [15] showed similar results from a conjecture on the ratios of L-functions given by Conrey, Farmer, & Zirnbauer [14] and Rodgers [41] proved the effect under RH. Ford & Zaharescu [20] then used uniform versions of Landau’s formula [34] [25] [19] to give unconditional proof of the effect in zeta and L-function zeros on the critical line, explaining numerical observations of the effect in L-function zeros by Perez-Marco [39].

The formal results above concerning the zeta zeros either only consider zeros on the critical line, assume RH, or include contributions from zeros off the critical line in an error term. A question then arises: would nontrivial zeta or L-function zeros off the critical line have the same vertical statistical dependence structure as that described by the results above? There is little research on this topic. Gonek [28] applied a finite Euler product approximation of the zeta function [26] [27] to predict that any such zeros off the critical line have a structurally different generating process from those on the critical line, which suggests the answer is no. In this paper we apply probabilistic methods similar to those used in much of the research described above as well as other, related work [36] [29] [4], to show that such zeros likely would not have any vertical statistical structure.

On the critical line we can prove the following result describing the conditional distribution of $\log |\zeta(1/2 + i(t + \Delta))|\) when $\zeta(1/2 + it) = 0$:\

\[\text{For a random variable } X \text{ and event } Y, \text{ we denote by } X|Y \text{ the random variable } X \text{ given that the event } Y \text{ has occurred.}\]
Theorem 1.1. If $t$ is uniformly distributed in $[T, 2T]$ with $\Delta \in [T-t, 2T-t]$, then as $T \to \infty$,

$$\log \left| \zeta \left( \frac{1}{2} + it + \Delta \right) \right| \left| \zeta \left( \frac{1}{2} + it \right) \right| = 0$$

(1.3)

$$\frac{d}{\Delta} \to \mathcal{N} \left( - \sum_{k=1}^{\infty} \frac{\text{Re}\mathcal{P}(k (1 + it))}{k^2}, \frac{1}{2} \log \log T \right)$$

where $\mathcal{P}(s)$ denotes the prime zeta function.

This limit theorem predicts the statistical dependence between zeta zeros described above. In particular, it predicts an "average gap" around zeta zeros, which agrees with the zeros’ known short-range dependence, and, since the conditional mean in (1.3) has local maxima at values of $\Delta$ approximately equal to the Riemann zeros’ imaginary parts, it predicts the long-range repulsion between zeta zero differences and the imaginary parts of the low-lying zeta zeros (see Figure 1). We will give further details on this result below.

Perhaps most notably, we show that statistical dependence like that described by (1.3) for zeta zeros on the critical line cannot exist for zeros that are off the critical line. In particular, for $\sigma > 1/2$, we show that the conditional distribution of $\log |\zeta (\sigma + it + \Delta)|$ does not exist when $\zeta (\sigma + it) = 0$ for most values of $\Delta$, i.e., the conditional moments are infinite and the conditional probability density function is identically zero. This shows that the zeta function’s magnitude cannot have vertical statistical structure at most distances around nontrivial zeta zeros that are off the critical line.

These results can be proven relatively easily, using only statistical properties of certain prime sums, elementary properties of normal and elliptical random variables, and the pole structure of the zeta function. For several related results we additionally apply elementary properties of generalized hyperbolic random variables. All of these results readily generalize to L-functions.

2. Preliminaries

We first describe the prime zeta function, $\mathcal{P}(s)$, which has the expressions

(2.1) $$\mathcal{P}(s) = \sum_p \frac{1}{p^s}$$

and

(2.2) $$\mathcal{P}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns),$$

where

(2.3) $$\mu(n) = \begin{cases} 0; & n \text{ has a repeated prime factor} \\ (-1)^{\omega(n)}; & n \text{ has } \omega(n) \text{ distinct prime factors} \end{cases}$$
Figure 1. A plot (Red) of the conditional probability $P \{ \log |\zeta(1/2 + it + \Delta)| \leq -3\sigma |\zeta(1/2 + it) = 0 \}$ approximated using the distribution described by (1.3) with $\sigma = \frac{1}{2} \log \log t$, where $t$ is the imaginary part of the 100,000th Riemann zero. Vertical lines (Blue) are positioned at values of $\Delta$ such that $\zeta(1/2 + i\Delta) = 0$. Since large negative values of $\log |\zeta(s)|$ are a necessary condition for $s$ to be the position of a zeta zero, this result demonstrates the statistical dependence between zeta zeros on the critical line described above.

is the well-known M"obius function. The sum over primes (2.1) is absolutely convergent for $\text{Re}(s) > 1$ and the analytic continuation (2.2), which is derived using the technique of M"obius inversion, is convergent everywhere in $\text{Re}(s) > 0$ except at the points $ns = 1$, which arise from the simple pole $\zeta(1)$, and at the points $ns$ where $\zeta(ns) = 0$ [24] [21]. Note that, since $\zeta(s)$
has no zeros in $\text{Re}(s) \geq 1$, the singularities of (2.2) in $1/2 \leq \text{Re}(s) < 1$ with $\text{Im}(s) \neq 0$ unambiguously define the positions of the nontrivial zeta zeros.

We make some important notes about the statistical properties of (2.1). Under $t$ uniformly distributed in $[a, b]$ with $b - a \to \infty$, the summands of (2.1)'s $\text{Re} \mathcal{P}(\sigma + it)$ are statistically independent\(^2\). This is directly related to the fact that, due to the fundamental theorem of arithmetic, the characteristic function of (2.1)'s $\text{Re} \mathcal{P}(\sigma + it)$ has the factorizable form

\[
\varphi(\lambda) = \prod_p J_0 \left( \frac{\lambda \sigma}{p^\sigma} \right) = \prod_p \varphi_p(\lambda),
\]

where $\varphi_p(\lambda)$ here denotes the characteristic function of $\cos(t \log p)/p^\sigma$ and

\[
J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left( \frac{z}{2} \right)^{2m},
\]

is the 0th order Bessel function of the first kind [35]. We apply this statistical independence property first here by multiplying $\text{Re} \mathcal{P}(k(\sigma + (t + \Delta)))$ and $\text{Re} \mathcal{P}(j(\sigma + it))$ and applying the expected value\(^3\) to give the following useful result describing the covariance of $\text{Re} \mathcal{P}(j(\sigma + it))$ and $\text{Re} \mathcal{P}(k(\sigma + (t + \Delta)))$:

**Lemma 2.1.** If $t$ is uniformly distributed in $[T, 2T]$, then as $T \to \infty$,

\[
E \{ \text{Re} \mathcal{P}(k(\sigma + (t + \Delta))) \text{Re} \mathcal{P}(j(\sigma + it)) \} \to \begin{cases} 
\frac{1}{2} \text{Re} \mathcal{P}(k(2\sigma + i\Delta)) ; & k = j \\
0 ; & k \neq j
\end{cases}
\]

for $\Delta \in \mathbb{R}$ and $k, j \in \mathbb{N}$.

**Proof.** We begin by noting from the Fubini and Tonelli theorems that if

\[
\sum_n \frac{1}{b - a} \int_a^b |f_n(t)| \, dt < \infty
\]

for general functions $f_n(t)$, then

\[
\frac{1}{b - a} \int_a^b \sum_n f_n(t) \, dt = \sum_n \frac{1}{b - a} \int_a^b f_n(t) \, dt.
\]

We then apply (2.1) to note that, for any $t, \Delta \in \mathbb{R}$, $k \geq j \in \mathbb{N}$, and $j\sigma > 1$,

\[
\text{Re} \mathcal{P}(k(\sigma + (t + \Delta))) \text{Re} \mathcal{P}(j(\sigma + it)) = \sum_p \frac{\cos(k(t + \Delta) \log p) \cos(jt \log p)}{p^{k+j}\sigma} + 2 \sum_p \frac{\cos(k(t + \Delta) \log p)}{p^{k\sigma}} \sum_{q<p} \cos(qj \log q) q^{j\sigma}.
\]

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\(^2\)See Appendix A for sketch of proof.

\(^3\)For $t$ uniformly distributed in $[a, b]$, the expected value of a given function $f(.)$ is defined $E\{f(t)\} = \frac{1}{b-a} \int_a^b f(x) \, dx$. 
We then note that the series on the right-hand side of (2.9) satisfy the absolute convergence property

\[
\sum_p \frac{|\cos (k(t + \Delta) \log p) \cos (jt \log p)|}{p^{(k+j)\sigma}} + 2 \sum_p \frac{|\cos (k(t + \Delta) \log p)|}{p^{k\sigma}} \sum_{q<p} \frac{1}{q^{j\sigma}} < \mathcal{P}((k + j)\sigma) + 2\mathcal{P}(k\sigma)\mathcal{P}(j\sigma).
\]

(2.10)

Note that (2.10) shows that (2.9) satisfies (2.7). Therefore the expected value can be applied term-by-term to (2.9).

We then apply the expected value with \( a = T, \ b = 2T, \) and \( T \to \infty \) and note that, by the statistical independence shown in Appendix A, the expected value of the double-summation in (2.9) always vanishes because \( E\{\cos (\omega t + \theta)\} = 0 \) for all \( \omega, \theta \in \mathbb{R} \). This latter vanishing property combined with the fact that

\[
\cos (k(t + \Delta) \log p) \cos (jt \log p)
\]

(2.11)

shows that the expected value of (2.9)’s first summation vanishes unless \( k = j \). This gives

\[
E \{ \Re \mathcal{P}(k(\sigma + i(t + \Delta))) \Re \mathcal{P}(j(\sigma + it)) \} \to \begin{cases} \frac{1}{2} \sum_p \frac{\cos(k\Delta \log p)}{p^{2k\sigma}}; \quad k = j \\
0; \quad k \neq j \end{cases}
\]

(2.12)

which completes the proof.

Note that the prime sum in (2.12) is convergent for \( 2k\sigma > 1 \), which is a much weaker constraint than \( k\sigma > 1 \). Therefore applying the expected value as above serves as an analytic continuation to a larger domain for \( \sigma \). Furthermore, (2.6) may be extended to the still larger domain in \( \sigma > 0 \) allowed by the analytic continuation (2.2). We lastly note that (2.6) with \( k = j = 1 \) gives the following result for \( \Re \mathcal{P}(\sigma + it) \)’s autocovariance function:

\[
R_{\mathcal{P}}(\sigma, \Delta) = E \{ \Re \mathcal{P}(\sigma + i(t + \Delta)) \Re \mathcal{P}(\sigma + it) \} \to \frac{1}{2} \Re \mathcal{P}(2\sigma + i\Delta).
\]

(2.13)

2.1. Elliptical distributions. We next note that (2.4) is an even function since \( J_0(x) \) is an even function. This implies that \( \Re \mathcal{P}(\sigma + it) \)’s probability density function \( p(x) \) is even as well. Relatedly, it is clear from (2.4)-(2.5) that there exists a function \( \varphi(.) \) such that

\[
\varphi(\lambda) = \phi\left(\lambda^2\right),
\]
which, with (2.13), shows that \( \text{Re}\mathcal{P}(\sigma + it) \) and \( \text{Re}\mathcal{P}(\sigma + i(t + \Delta)) \) have an elliptical joint distribution, which is a large class of distributions that includes the multivariate normal distribution. From the general results of Fang, Kotz, & Ng [18] (Theorem 2.18 pg. 45, Eqn. 2.43 pg. 46) the conditional probability density function of an elliptical random variable \( X(t + \Delta) \) given the value of \( X(t) \) has the form

\[
 p_{t+\Delta|t}(x) = \frac{1}{\sqrt{h(\Delta)}} g \left( \frac{(x - \mu_{t+\Delta|t})^2}{h(\Delta)} \right) 
\]

where \( g(.) \) is a non-negative function,

\[
 h(\Delta) = \frac{R(\Delta)}{R(0)} \left( 1 - \left( \frac{R(\Delta)}{R(0)} \right)^2 \right),
\]

\( R(\Delta) = E\{X(t + \Delta)X(t)\} \), \( R(0) \) is thus the unconditional variance, and the conditional expectation\(^4\) is given by

\[
 \mu_{t+\Delta|t} = E\{X(t + \Delta)|X(t)\} = \frac{R(\Delta)}{R(0)} X(t).
\]

Additionally, from [18] (Theorem 2.18 pg. 45), one can show that the conditional variance has the form

\[
 \text{var}\{X(t + \Delta)|X(t)\} = h(\Delta) V(X(t)),
\]

where \( V(.) \) is a function that depends on the specific distribution of \( X(t) \) and \( X(t + \Delta) \). In the case of \( X(t) \) and \( X(t + \Delta) \) normally distributed, \( V(.) = 1 \). In general, however, this is not the case and (2.17) is dependent on the value of \( X(t) \).

2.2. Generalized hyperbolic distributions. We next describe the tails of distributions of series with the form (2.1) with \( s = \sigma + it \) that additionally satisfy the convergence property

\[
 \sum_{p} \frac{1}{p^{2\sigma}} < \infty,
\]

i.e. \( \sigma > 1/2 \). In particular, if (2.1) satisfies (2.18), then (2.1)'s probability density function \( p(x) \) satisfies

\[
 p(x) = O \left( e^{-r|x|} \right)
\]

for some \( r \geq 1 \) as \( |x| \to \infty \). This may be shown from Laurincikas [35] (Eqn. 6.17 pg. 47). We then note that the generalized hyperbolic distributions are a large family that includes many distributions that are symmetric, elliptical, and satisfy the tail constraint (2.19). If \( X(t) \) and \( X(t + \Delta) \) have such a distribution, then one can apply the general results of Blaesild & Jensen [8]

\(^4\)The conditional moments here are defined \( E\{X^m(t + \Delta)|X(t)\} = \int_{-\infty}^{\infty} x^m p_{t+\Delta|t}(x)dx \).
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((1) pg.47, Theorem 1 (b) pg. 49-50) to show that the conditional probability density of $X(t + \Delta)$ given $X(t)$ has the form

$$p_{t+\Delta|t}(x) = \frac{\alpha'}{2\pi \delta_{t+\Delta|t}} \lambda^{-1/2} \frac{K_{\lambda-1} \left( \alpha' \left( \frac{\delta_{t+\Delta|t}^2 + (x-\mu_{t+\Delta|t})^2}{R(0)h(\Delta)} \right)^{1/2} \right)}{K_{\lambda-1/2}(\alpha'\delta_{t+\Delta|t}) \left( \frac{\delta_{t+\Delta|t}^2 + (x-\mu_{t+\Delta|t})^2}{R(0)h(\Delta)} \right)^{1/2}},$$

where $K_\nu(z)$ is the $\nu$th order modified Bessel function of the second kind,

$$\alpha' = \alpha \sqrt{R(0)}, \quad \delta_{t+\Delta|t} = \sqrt{\frac{\delta^2}{R(0)} + \frac{X^2(t)}{R^2(0)}},$$

and $\lambda, \alpha,$ and $\delta$ are given parameters. Additionally from [8] (pg. 50-51) one can show that the conditional variance has the form (2.17) with

$$V(X(t)) = \frac{1}{\alpha} \sqrt{\delta^2 + \frac{X^2(t)}{R(0)}} \frac{K_{\lambda+1/2} \left( \alpha \sqrt{\delta^2 + \frac{X^2(t)}{R(0)}} \right)}{K_{\lambda-1/2} \left( \alpha \sqrt{\delta^2 + \frac{X^2(t)}{R(0)}} \right)},$$

We then note the asymptotic result [7] (DLMF 10.17.1, 10.40.2)

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z},$$

which, with (2.15), (2.17), and (2.22) shows that, for the generalized hyperbolic family,

$$\text{var} \{X(t + \Delta)|X(t)\} \sim \frac{\sqrt{R(0)}}{\alpha} \left( 1 - \left( \frac{R(\Delta)}{R(0)} \right)^2 \right)$$

as $|X(t)| \to \infty$.

We will use the properties of $P(s)$ described above as well as the properties of elliptical and generalized hyperbolic random variables to study the conditional distribution of $\log |\zeta(\sigma + i(t + \Delta))|$ when $\zeta(\sigma + it) = 0$. We will see that the conditional distribution exists for $\sigma = 1/2$ but not for $\sigma > 1/2$.

3. APPLICATION TO ZETA AND L-FUNCTIONS

We begin by using Lemma 2.1 to give the following result describing $\log |\zeta(\sigma + it)|$’s autocovariance function:

**Lemma 3.1.** If $t$ is uniformly distributed in $[T, 2T]$, then as $T \to \infty$,

$$R_{\log \zeta(\sigma, \Delta)} = E \{ \log |\zeta(\sigma + i(t + \Delta))| \log |\zeta(\sigma + it)| \}$$

$$\rightarrow \frac{1}{2} \sum_{k=1}^{\infty} \text{Re} P(k(2\sigma + i\Delta)) \frac{\alpha \sqrt{\delta^2 + \frac{X^2(t)}{R(0)}}}{k^2}$$

for $\Delta \in \mathbb{R}$.
Proof. We begin by considering $\sigma > 1$ so that we may apply the Euler product and Taylor expansion for $\log|\zeta(s)|$ to write

$$\log|\zeta(\sigma + i(t + \Delta))| \log|\zeta(\sigma + it)| = \sum_{k=1}^{\infty} \frac{\text{Re} \mathcal{P}(k(\sigma + i(t + \Delta))) \text{Re} \mathcal{P}(k(\sigma + it))}{k^2} + 2 \sum_{k=1}^{\infty} \sum_{j<k} \frac{\text{Re} \mathcal{P}(k(\sigma + i(t + \Delta))) \text{Re} \mathcal{P}(j(\sigma + it))}{kj}$$

(3.2)

We next note that

$$\sum_{k=1}^{\infty} \frac{|\text{Re} \mathcal{P}(k(\sigma + i(t + \Delta))) \text{Re} \mathcal{P}(k(\sigma + it))|}{k^2} \leq \sum_{k=1}^{\infty} \frac{\mathcal{P}^2(k\sigma)}{k^2} < \infty \quad (3.3)$$

and

$$2 \sum_{k=1}^{\infty} \sum_{j<k} \frac{|\text{Re} \mathcal{P}(k(\sigma + i(t + \Delta))) \text{Re} \mathcal{P}(j(\sigma + it))|}{kj} \leq 2 \mathcal{P}(\sigma) \sum_{k=1}^{\infty} \frac{\mathcal{P}(k\sigma)}{j} < 2 \mathcal{P}(\sigma) \sum_{k=1}^{\infty} \mathcal{P}(k\sigma) < \infty. \quad (3.4)$$

These upper bounds may be easily shown to be finite from the asymptotic decay rates implied by (2.1) and the ratio test. The finite bounds (3.3) and (3.4) show that (3.2)’s series satisfy the condition (2.7). We may therefore apply the expectation to (3.2) term-by-term with $a = T$, $b = 2T$, and $T \to \infty$. We do so and apply Lemma 2.1 to show that the expected value of (3.2)’s second term vanishes while the expected value of its first term is given by (3.1). \[ \Box \]

We note that, similarly to Lemma 2.1, Lemma 3.1’s result (3.1) is convergent using the prime sum (2.1) for all $\sigma > 1/2$, and it extends to a still larger domain in $\sigma > 0$ using the analytic continuation (2.2). We will next apply the above results to study the statistical behavior of $\log|\zeta(s)|$ both on and off the critical line.

3.1. On the critical line. We first show that, for $\sigma = 1/2$, Lemma 3.1’s result (3.1) reproduces the logarithmic correlations described by Bourgade [13] as well as Fyodorov, Hiary, & Keating [23] [22]. Showing this will use several principles that we will also apply in the proof of Theorem 1.1 below. We first note that, since $\mathcal{P}(s) = O(1)$ for all $\text{Re}(s) > 1$, (3.1) with $\sigma = 1/2$ satisfies

$$R_{\log \zeta}(1/2, \Delta) = \frac{1}{2} \text{Re} \mathcal{P}(1 + i\Delta) + O(1),$$
which we can expand using (2.2) to write

\( R_{\log \zeta} (1/2, \Delta) = \frac{1}{2} \log |\zeta(1 + i\Delta)| + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \log |\zeta(n (\sigma + i\Delta))| + O(1). \)

We next make the general note that, for \( \sigma \geq 1/2 \) and \( \tau \in \mathbb{R} \),

\( \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \log |\zeta(n (\sigma + i\tau))| = O(1) \)

because \( \zeta(s) \) has no poles with \( \text{Re}(s) > 1 \) and the summands in (3.6) are \( O\left(\frac{1}{n^{2\sigma}}\right) \) as \( n \to \infty \). Hence

\( R_{\log \zeta} (1/2, \Delta) = \frac{1}{2} \log |\zeta(1 + i\Delta)| + O(1). \)

We then note that \( \zeta(s) \) has the following Laurent series expansion around its simple pole at \( s = 1 \):

\( \zeta(s) = \frac{1}{s - 1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k (s - 1)^k, \)

where the \( \gamma_k \) are the Stieltjes constants. It is clear from (3.8) that \( \zeta(1 + i\Delta) \approx \frac{1}{\Delta} \) for small \( |\Delta| \) and hence, by (3.7),

\( R_{\log \zeta} (1/2, \Delta) \approx -\frac{1}{2} \log |\Delta|, \)

reproducing logarithmic correlations over short distances.

We next will directly use Selberg’s limit theorem (1.2) as well as Lemma 3.1 to prove Theorem 1.1.

**Proof.** Let \( t \) be uniformly distributed in \([T, 2T]\) with \( \Delta \in [T - t, 2T - t] \). By (1.2), \( \log |\zeta(1/2 + it)| \) and \( \log |\zeta(1/2 + i(t + \Delta))| \) are normally distributed for large \( T \). Thus the conditional mean has the limiting form (2.16), which we combine with (3.1) to write

\( E \{ \log |\zeta(1/2 + i(t + \Delta))|| \log |\zeta(1/2 + it)| \} \)

\( \sim \frac{R_{\log \zeta}(1/2, \Delta)}{R_{\log \zeta}(1/2, 0)} \log |\zeta(1/2 + it)| = \frac{\sum_{k=1}^{\infty} \frac{\text{Re}(P(k(1+i\Delta)))}{k^2}}{\sum_{k=1}^{\infty} \frac{P(k)}{k^2}} \log |\zeta(1/2 + it)|. \)

We then apply (2.2) and use (3.6) and (3.8) to note that

\( \sum_{k=1}^{\infty} \frac{P(k)}{k^2} = P(1) + O(1) = \log \zeta(1) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \log \zeta(n) + O(1) \)

\( = \log \zeta(1) + O(1) = \log \left( \lim_{z \to 1^+} \frac{1}{z - 1} \right) + O(1) = \lim_{z \to 0^+} \log \left( \frac{1}{z} \right) + O(1). \)
We additionally note that when \( \zeta(1/2 + it) = 0 \)
\[
\log |\zeta(1/2 + it)| = \lim_{z \to 0^+} \log (z).
\]
Substituting the two limits (3.11)-(3.12) into (3.10) and cancelling gives the conditional expectation
\[
E \{ \log |\zeta(1/2 + i(t + \Delta))| | \zeta(1/2 + it) = 0 \} \sim - \sum_{k=1}^{\infty} \frac{\text{Re} \mathcal{P}(k(1+i\Delta))}{k^2}.
\]
We next consider the conditional variance which, by (1.2), has the limiting form (2.17) where \( V(.) = 1 \) and \( h(\Delta) \) has the form (2.15). We also note from Selberg’s central limit theorem that the leading factor \( R(0) \) in (2.15) may be replaced with \( \sim \frac{1}{2} \log \log T \). We apply these facts to write
\[
\text{var} \{ \log |\zeta(1/2 + i(t + \Delta))| | \log |\zeta(1/2 + it)| \}
\sim \left( \frac{1}{2} \log \log T \right) \left( 1 - \left( \frac{R_{\log}(1/2, \Delta)}{R_{\log}(1/2, 0)} \right)^2 \right)
\]
\[
= \left( \frac{1}{2} \log \log T \right) \left( 1 - \left( \sum_{k=1}^{\infty} \frac{\text{Re} \mathcal{P}(k(1+i\Delta))}{k^2} \right)^2 \right).
\] (3.14)
We then note from (2.2) that \( \mathcal{P}(1+i\Delta) \) and hence the numerator in (3.14)’s second term,
\[
\left( \sum_{k=1}^{\infty} \frac{\text{Re} \mathcal{P}(k(1+i\Delta))}{k^2} \right)^2,
\]
has no singularities, i.e., only takes finite values with \( \Delta \neq 0 \) while the denominator is \( (\mathcal{P}(1) + O(1))^2 \), which, by (3.11), diverges. We thus conclude that
\[
\text{var} \{ \log |\zeta(1/2 + i(t + \Delta))| | \log |\zeta(1/2 + it)| \} \sim \left( \frac{1}{2} \log \log T \right) \left( 1 + o(1) \right).
\]
This completes the proof. \( \square \)

3.2. Off the critical line. By (2.13) the unconditional variance of \( \text{Re} \mathcal{P}(\sigma + it) \) for \( \sigma > 1/2 \) is given by
\[
\frac{\mathcal{P}(2\sigma)}{2} = \frac{1}{2} \sum_p \frac{1}{p^{2\sigma}} = O(1).
\] (3.16)
Since the variance (3.16) is convergent, in contrast to when \( \sigma = 1/2 \), a central limit theorem does not hold for \( \text{Re} \mathcal{P}(\sigma + it) \) with \( \sigma > 1/2 \). Relatedly, we can show the following result describing the nonexistence of \( \log |\zeta(\sigma + i(t + \Delta))| \)’s conditional distribution when \( \zeta(\sigma + it) = 0 \) with \( \sigma > 1/2 \):
Theorem 3.1. Suppose $t$ is uniformly distributed in $[T, 2T]$ with $T \to \infty$ and let $p_{t+\Delta \mid t}^\ast (x)$ denote the conditional probability density function of $\log |\zeta (\sigma + i(t + \Delta))|$ given $\zeta (\sigma + it)$. Then

\begin{equation}
E \{ \log |\zeta (\sigma + i(t + \Delta))| \mid \zeta (\sigma + it) = 0 \} = \begin{cases}
-\infty ; & \Re \mathcal{P} (2\sigma + i\Delta) > 0 \\
+\infty ; & \Re \mathcal{P} (2\sigma + i\Delta) < 0 \\
O(1) ; & \Re \mathcal{P} (2\sigma + i\Delta) = 0
\end{cases}
\end{equation}

and, if $\sigma > 1/2$ and $\zeta (\sigma + it) = 0$, then

\begin{equation}
p_{t+\Delta \mid t}^\ast (x) = 0
\end{equation}

for all $|x| < \infty$ and all $\Delta \in \mathbb{R}$ such that $0 < |\Re \mathcal{P} (2\sigma + i\Delta) / \mathcal{P} (2\sigma)| < 1$.

Proof. We first apply (2.2) and (3.6) to write

\begin{equation}
\Re \mathcal{P} (\sigma + i\tau) = \log |\zeta (\sigma + i\tau)| + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \log |\zeta (n\sigma + n\tau)| = \log |\zeta (\sigma + i\tau)| + O(1).
\end{equation}

We next apply (2.16) with (2.13) to give the following result for the conditional expectation of $\Re \mathcal{P} (\sigma + i(t + \Delta))$ given $\Re \mathcal{P} (\sigma + it)$:

\begin{equation}
E \{ \Re \mathcal{P} (\sigma + i(t + \Delta)) \mid \Re \mathcal{P} (\sigma + it) \} = \frac{\Re \mathcal{P} (2\sigma + i\Delta)}{\mathcal{P} (2\sigma)} \Re \mathcal{P} (\sigma + it).
\end{equation}

We then note from (3.19) that when $\zeta (\sigma + it) = 0$ with $\sigma > 1/2$

\begin{equation}
\Re \mathcal{P} (\sigma + it) = \lim_{z \to 0^+} \log(z) + O(1),
\end{equation}

which, with (3.20) and (3.16), shows that

\begin{equation}
E \{ \Re \mathcal{P} (\sigma + i(t + \Delta)) \mid \zeta (\sigma + it) = 0 \} = \begin{cases}
-\infty ; & \Re \mathcal{P} (2\sigma + i\Delta) > 0 \\
+\infty ; & \Re \mathcal{P} (2\sigma + i\Delta) < 0 \\
O(1) ; & \Re \mathcal{P} (2\sigma + i\Delta) = 0
\end{cases}
\end{equation}

We additionally note from (3.19) that, for an arbitrary event $Y$,

\begin{equation}
E \{ \Re \mathcal{P} (\sigma + i\tau) \mid Y \} = E \{ \log |\zeta (\sigma + i\tau)| \mid Y \} + O(1).
\end{equation}

From the results (3.22) and (3.23) we conclude (3.17).

We then let $p_{t+\Delta \mid t} (x)$ denote the conditional probability density function of $\Re \mathcal{P} (\sigma + i(t + \Delta))$ given $\Re \mathcal{P} (\sigma + it)$ and note that it has the elliptical form (2.14) with $\mu_{t+\Delta \mid t}$ given by (3.20) and

\begin{equation}
h(\Delta) = \frac{\mathcal{P}(2\sigma)}{2} \left( 1 - \left( \frac{\Re \mathcal{P} (2\sigma + i\Delta)}{\mathcal{P}(2\sigma)} \right)^2 \right).
\end{equation}
Due to the requirement that \( \int_{-\infty}^{\infty} p_{t+\Delta t}(x) dx \) converges, \( g(u) = o(u^{-1/2}) \). Hence, from (2.14) and (3.20), for nonzero (3.24) we have

\[
(3.25) \quad p_{t+\Delta t}(x) = o \left( \left| x - \frac{\text{Re} \mathcal{P}(2\sigma + i\Delta)}{\mathcal{P}(2\sigma)} \text{Re} \mathcal{P}(\sigma + it) \right|^{-1} \right).
\]

(3.21)-(3.22) and (3.24)-(3.25) show that \( p_{t+\Delta t}(x) = 0 \) if \( \zeta(\sigma + it) = 0 \) for all \( x < \infty \) and \( \Delta \) such that \( 0 < |\text{Re} \mathcal{P}(2\sigma + i\Delta)/\mathcal{P}(2\sigma)| < 1 \). Then from (3.19) the result (3.18) follows. \( \square \)

The proof of Theorem 3.1 shows that, for \( \sigma > 1/2 \) and most values of \( \Delta \), the conditional moments of \( \log |\zeta(\sigma + it + \Delta)| \) are infinite and its conditional probability density function is identically zero when \( \zeta(\sigma + it) = 0 \). A probability density function cannot be zero everywhere on the real line. We thus conclude from this result that, for most values of \( \Delta \), the conditional distribution of \( \log |\zeta(\sigma + it + \Delta)| \) does not exist when \( \zeta(\sigma + it) = 0 \).

Distances \( \Delta \) where \( \text{Re} \mathcal{P}(2\sigma + i\Delta) = 0 \), i.e., where \( \log |\zeta(\sigma + it + \Delta)| \) and \( \text{Re} \mathcal{P}(\sigma + it) \) are uncorrelated (see Appendix B) are a possible exception to the outcome described above. At these distances \( \log |\zeta(\sigma + it + \Delta)| \)'s conditional expectation can exist, but the status of its higher order moments and density is unclear and depends on the specific, non-normal, unconditional distribution of \( \text{Re} \mathcal{P}(\sigma + it) \).

We recall from (3.16) that (2.18)-(2.19) is satisfied by \( \text{Re} \mathcal{P}(\sigma + it) \) with \( \sigma > 1/2 \), which makes it likely that \( \text{Re} \mathcal{P}(\sigma + it) \) and \( \text{Re} \mathcal{P}(\sigma + it + \Delta) \) have a generalized hyperbolic distribution. If this is true, then we can easily prove that \( \log |\zeta(\sigma + it + \Delta)| \)'s conditional density \( p_{t+\Delta t}(x) \) is also identically zero at distances \( \Delta \) where \( \text{Re} \mathcal{P}(2\sigma + i\Delta) = 0 \).

**Corollary 3.1.** If \( \text{Re} \mathcal{P}(\sigma + it) \) and \( \text{Re} \mathcal{P}(\sigma + it + \Delta) \) have a generalized hyperbolic distribution with \( \sigma > 1/2 \) and \( t \) uniformly distributed in \( [T, 2T] \) with \( T \to \infty \), then, if \( \text{Re} \mathcal{P}(2\sigma + i\Delta) \) is 0 and \( \zeta(\sigma + it) = 0 \),

\[
(3.26) \quad p_{t+\Delta t}^*(x) = 0
\]

for all \( |x| < \infty \).

**Proof.** If generalized hyperbolic, then \( \text{Re} \mathcal{P}(\sigma + it + \Delta) \)'s conditional density has the form (2.20) and, by (3.20), \( \mu_{t+\Delta t} = 0 \) for \( \text{Re} \mathcal{P}(2\sigma + i\Delta) = 0 \). Additionally from (2.21) it is clear that as \( |\text{Re} \mathcal{P}(\sigma + it)| \to \infty \),

\[
(3.27) \quad \delta_{t+\Delta t} \sim \frac{|\text{Re} \mathcal{P}(\sigma + it)|}{\mathcal{P}(0)}.
\]

Applying \( \mu_{t+\Delta t} = 0 \) and (3.27) along with (2.23) in (2.20) and simplifying shows that, for all \( |x| < \infty \)

\[
(3.28) \quad p_{t+\Delta t}(x) = O \left( |\text{Re} \mathcal{P}(\sigma + it)|^{-1/2} \right),
\]

which, with (3.21) and (3.19), proves (3.26). \( \square \)
We may also easily show that, under a generalized hyperbolic distribution assumption for $\text{Re} \mathcal{P}(\sigma + it)$ and $\text{Re} \mathcal{P}(\sigma + i(t + \Delta))$, the conditional variance of $\log |\zeta(\sigma + i(t + \Delta))|$ is infinite if $\zeta(\sigma + it) = 0$ at distances $\Delta$ where $\text{Re} \mathcal{P}(2\sigma + i\Delta) = 0$ and indeed at any distances where the autocorrelation is not $\pm 1$.

**Proposition 3.1.** If $\text{Re} \mathcal{P}(\sigma + it)$ and $\text{Re} \mathcal{P}(\sigma + i(t + \Delta))$ have a generalized hyperbolic distribution with $\sigma > 1/2$ and $t$ uniformly distributed in $[T, 2T]$ with $T \to \infty$, then

\begin{equation}
\text{var} \{ \log |\zeta(\sigma + i(t + \Delta))| \} | \zeta(\sigma + it) = 0 \} = \infty
\end{equation}

for all $\Delta \in \mathbb{R}$ such that $|\text{Re} \mathcal{P}(2\sigma + i\Delta) / \mathcal{P}(2\sigma)| < 1$.

**Proof.** We apply (2.24) with (2.13) and (3.16) to note that

\begin{equation}
\text{var} \{ \text{Re} \mathcal{P}(\sigma + i(t + \Delta)) | \text{Re} \mathcal{P}(\sigma + it) \} \sim |\text{Re} \mathcal{P}(\sigma + it)| \frac{1}{\alpha} \sqrt{\frac{\mathcal{P}(2\sigma)}{2}} \left( 1 - \left( \frac{\text{Re} \mathcal{P}(2\sigma + i\Delta)}{\mathcal{P}(2\sigma)} \right)^2 \right)
\end{equation}

as $|\text{Re} \mathcal{P}(\sigma + it)| \to \infty$. The result (3.30) along with (3.16) and (3.21) shows that

\begin{equation}
\text{var} \{ \text{Re} \mathcal{P}(\sigma + i(t + \Delta)) | \zeta(\sigma + it) = 0 \} = \infty
\end{equation}

for all $\Delta \in \mathbb{R}$ such that $|\text{Re} \mathcal{P}(2\sigma + i\Delta) / \mathcal{P}(2\sigma)| < 1$. We next denote (3.19)’s $O(1)$ term as $B(\sigma, \tau)$ and note that, for an arbitrary event $Y$,

\begin{equation}
\begin{aligned}
\text{var} \{ \text{Re} \mathcal{P}(\sigma + it) | Y \} &= \text{var} \{ \log |\zeta(\sigma + it)| | Y \} + \text{var} \{ B(\sigma, \tau) | Y \} \\
&\quad + 2 \alpha \text{cov} \{ \log |\zeta(\sigma + it)|, B(\sigma, \tau) | Y \}.
\end{aligned}
\end{equation}

We then apply the Cauchy-Schwartz inequality to (3.32)’s last term to write

\begin{equation}
\begin{aligned}
\text{var} \{ \text{Re} \mathcal{P}(\sigma + it) | Y \} &= \text{var} \{ \log |\zeta(\sigma + it)| | Y \} + \text{var} \{ B(\sigma, \tau) | Y \} \\
&\quad + 2 \alpha \sqrt{\text{var} \{ \log |\zeta(\sigma + it)| | Y \} \text{var} \{ B(\sigma, \tau) | Y \}},
\end{aligned}
\end{equation}

where $|\alpha| \leq 1$. Since $B(\sigma, \tau) = O(1)$ it is then clear that

\begin{equation}
\text{var} \{ \text{Re} \mathcal{P}(\sigma + it) | Y \} = \infty \implies \text{var} \{ \log |\zeta(\sigma + it)| | Y \} = \infty.
\end{equation}

The results (3.31) and (3.34) complete the proof of (2.29). \qed

### 3.3. Extension to $L$-functions

$L$-functions $L(s, \chi)$ are an important generalization of $\zeta(s)$ that are useful for studying the distribution of primes in arithmetic progressions and related topics. Like $\zeta(s)$, these functions have a Dirichlet series and Euler product representation

\begin{equation}
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1},
\end{equation}
where \( \chi \) is a given Dirichlet character modulo \( M \), which has the definition

\[
\chi(nm) = \chi(n)\chi(m),
\]

\[
\chi(n) = \begin{cases} 
0; & \gcd(n, M) > 1 \\
\neq 0; & \gcd(n, M) = 1 
\end{cases}
\]

(3.36)

\[
\chi(n + M) = \chi(n)
\]

If \( \chi \) is a principal character, \( \chi_0 \), meaning it has the simple definition

\[
\chi_0(n) = \begin{cases} 
0; & \gcd(n, M) > 1 \\
1; & \gcd(n, M) = 1 
\end{cases}
\]

(3.37)

then, similarly to \( \zeta(s) \), \( L(s, \chi_0) \) may be extended by analytic continuation to all \( s \) in \( \mathbb{C} \) except \( s = 1 \) where there is a simple pole. If \( \chi \) is not principal, then \( L(s, \chi) \) is an entire function. Also like \( \zeta(s) \), it is hypothesized (the so-called generalized Riemann hypothesis) that all the the nontrivial zeros of any \( L(s, \chi) \) reside on the line \( \text{Re}(s) = 1/2 \).

We may generalize (2.1)-(2.2) for the consideration of L-functions \( L(s, \chi) \). We generalize (2.1) by defining the prime L-function with the series

\[
P_\chi(s) = \sum_p \frac{\chi(p)}{p^s}
\]

(3.38)

where \( \chi(p) = e^{-itp} \) for primes \( p \) such that \( p \nmid M \) and \( \chi(p) = 0 \) for primes \( p \) such that \( p|M \). We next note that one may take the logarithm of \( L(s, \chi) \)’s Euler product, Taylor expand, and apply Möbius inversion to give a generalized version of (2.2) relating L-functions \( L(s, \chi) \) and their corresponding series \( P_\chi(s) \) for \( \text{Re}(s) > 0 \):

\[
P_\chi(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log L(ns, \chi^n)
\]

(3.39)

By the same reasoning as that described in Section 2, the summands of (3.38)’s \( \text{Re}P_\chi(\sigma + it) \) are statistically independent under \( t \) uniformly distributed in \([a, b]\) with \( b - a \to \infty \). It is also clear from the same reasoning that \( \text{Re}P_\chi(\sigma + it) \) and \( \text{Re}P_\chi(\sigma + i(t + \Delta)) \) have an elliptical joint distribution. Additionally, for \( \sigma > 1/2 \), \( \text{Re}P_\chi(\sigma + it) \)’s probability density function satisfies (2.19), which is evidence that \( \text{Re}P_\chi(\sigma + it) \) and \( \text{Re}P_\chi(\sigma + i(t + \Delta)) \) with \( \sigma > 1/2 \) have a generalized hyperbolic distribution.

Essentially equivalent reasoning to that in the proof of Lemma 2.1 shows that, for \( t \) uniformly distributed in \([T, 2T]\) with \( T \to \infty \)

\[
E \{ \text{Re}P_\chi(k(\sigma + i(t + \Delta))) \text{Re}P_{\chi'}(j(\sigma + it)) \}
\]

\[
\to \begin{cases} 
\frac{1}{2} \text{Re}P_{\chi_0}(k(2\sigma + i\Delta)); & k = j \\
0; & k \neq j 
\end{cases}
\]

(3.40)
Note the dependence on the principal character $\chi_0$. The result (3.40) gives $\text{Re} \mathcal{P}_\chi(\sigma + it)$’s autocovariance function:

$$R_{\mathcal{P}_\chi}(\sigma, \Delta) = E \{\text{Re} \mathcal{P}_\chi(\sigma + i(t + \Delta)) \text{Re} \mathcal{P}_\chi(\sigma + it)\} \rightarrow \frac{1}{2} \text{Re} \mathcal{P}_{\chi_0}(2\sigma + i\Delta)$$

(3.41)

We can then apply (3.40) similarly to Lemma 2.1 in the proof of Lemma 3.1 to give $\log L(\sigma + it, \chi)$’s autocovariance function:

$$R_{\log L_\chi}(\sigma, \Delta) = E \{\log L(\sigma + i(t + \Delta), \chi) \log L(\sigma + it, \chi)\} \rightarrow \frac{1}{2} \sum_{k=1}^{\infty} \frac{\text{Re} \mathcal{P}_{\chi_0}(k(2\sigma + i\Delta))}{k^2}$$

(3.42)

We next make the important note that Selberg’s central limit theorem (1.2) also applies for $L$-functions on the critical line [31]. $\log |L(1/2 + it, \chi)|$ and $\log |L(1/2 + i(t + i\Delta), \chi)|$ are therefore normally distributed under $t$ uniformly distributed in $[T, 2T]$ with $T$ large and $\Delta \in [T - t, 2T - t]$. We hence can apply the conditional mean formula (2.16) with (3.42) in the same way as we use (2.16) and (3.1) in the proof of Theorem 1.1, noting from (3.38), (3.39), and an analogous result to (3.6) that

$$\sum_{k=1}^{\infty} \frac{\mathcal{P}_{\chi_0}(k)}{k^2} = \mathcal{P}_{\chi_0}(1) + O(1) = \log L(1, \chi_0) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \log L(n, \chi_0) + O(1)$$

(3.43)

$$= \log L(1, \chi_0) + O(1) = \log \left( \lim_{z \to 1^+} \frac{1}{z - 1} \right) + O(1) = \lim_{z \to 0^+} \log \left( \frac{1}{z} \right) + O(1),$$

since all principal $L$-functions have a simple pole at $s = 1$. The divergent quantity (3.43) cancels with $\log |L(1/2 + it, \chi)|$ when $L(1/2 + it, \chi) = 0$, which produces a finite result for $\log |L(1/2 + i(t + \Delta), \chi)|$’s conditional mean. Lastly we can apply Selberg’s central limit theorem (1.2) for $L$-functions along with (3.42) in (2.15) and (2.17) to show that the conditional variance is $\sim \frac{1}{2} \log \log T$. These results for the conditional mean and variance show that

$$\log |L\left(\frac{1}{2} + i(t + \Delta), \chi\right)| \bigg| L\left(\frac{1}{2} + it, \chi\right) = 0$$

(3.44)

$$d \to \mathcal{N} \left( -\sum_{k=1}^{\infty} \frac{\text{Re} \mathcal{P}_{\chi_0}(k(1 + i\Delta))}{k^2}, \frac{1}{2} \log \log T \right)$$

for $t$ uniformly distributed in $[T, 2T]$ with $\Delta \in [T - t, 2T - t]$ and $T \to \infty$. This gives an analog of Theorem 1.1 for general $L$-functions.

Off the critical line we apply the same reasoning as in Theorem 3.1’s proof, using (2.16) with (3.41) and noting from (3.39) that, if $L(\sigma + it, \chi) = 0$, then

$$\text{Re} \mathcal{P}_\chi(\sigma + it) = \log |L(\sigma + it, \chi)| + O(1) = \lim_{z \to 0^+} \log(z) + O(1).$$
This shows that, for $\sigma > 1/2$, $t$ uniformly distributed in $[T, 2T]$ with $T \to \infty$, and for all $\Delta \in \mathbb{R}$ such that $0 < |\text{Re}\mathcal{P}_{\chi}(2\sigma + i\Delta )/\mathcal{P}_{\chi}(2\sigma )| < 1$:

$$E\{\log |L(\sigma + it, \chi)| |L(\sigma + it, \chi) = 0\} = \pm \infty$$

(3.45)

and $\log |L(\sigma + it + \Delta, \chi)|$'s conditional probability density function given the value of $L(\sigma + it, \chi)$, denoted $p_{\chi,t+\Delta|t}(x)$, satisfies

$$p_{\chi,t+\Delta|t}(x) = 0$$

(3.46)

if $L(\sigma + it, \chi) = 0$ for all $|x| < \infty$. We conclude that, for $\sigma > 1/2$ and most values of $\Delta$, the conditional distribution of $\log |L(\sigma + it + \Delta, \chi)|$ does not exist when $L(\sigma + it, \chi) = 0$. This gives an analog of Theorem 3.1 for L-functions off the critical line.

We may then apply similar reasoning to that in the proofs of Corollary 3.1 and Proposition 3.1 to show that, under a generalized hyperbolic distribution assumption for $\text{Re}\mathcal{P}_{\chi}(\sigma + it)$ and $\text{Re}\mathcal{P}_{\chi}(\sigma + i(t + \Delta))$, the conditional density $p_{\chi,t+\Delta|t}(x)$ is also identically zero if $L(\sigma + it, \chi) = 0$ at distances $\Delta$ where $\text{Re}\mathcal{P}_{\chi}(2\sigma + i\Delta ) = 0$ and additionally

$$\text{var}\{\log |L(\sigma + it + \Delta, \chi)| |L(\sigma + it, \chi) = 0\} = \infty$$

(3.47)

for all $\Delta \in \mathbb{R}$ such that $|\text{Re}\mathcal{P}_{\chi}(2\sigma + i\Delta )/\mathcal{P}_{\chi}(2\sigma )| < 1$. These results show that, similarly to the zeta function, the conditional distribution of an L-function’s magnitude likely does not exist for any distances above or below nontrivial zeros off the critical line.

4. Conclusion

The pole at $\zeta(1)$ is critically important to why $\log |\zeta(1/2 + i(t + \Delta ))|$’s conditional distribution exists when $\zeta(1/2 + it) = 0$. Firstly, it provides an important cancellation in the conditional expectation formula that produces a convergent result around zeta zeros. Secondly, the pole-induced divergence of $\log |\zeta(1/2 + it)|$’s variance, i.e., the fact that $\text{var}\{\log |\zeta(1/2 + it)|\} \sim \frac{1}{2} \log \log T$ for $t \in [T, 2T]$ produces the central limit theorem and normal distribution on the critical line. The normal distribution implies that $\log |\zeta(1/2 + i(t + \Delta ))|$'s conditional variance and higher order conditional moments do not depend on the value of $\zeta(1/2 + it)$.

In contrast, with $\sigma > 1/2$, there is no corresponding pole at $\zeta(2\sigma )$, so there is no stabilizing cancellation in $\log |\zeta(\sigma + it(t + \Delta ))|$’s conditional expectation formula and no central limit theorem providing a normal distribution. As a result the conditional moments and distribution of $\log |\zeta(\sigma + it(t + \Delta ))|$ do not exist if $\zeta(\sigma + it) = 0$ except possibly for specific distances $\Delta$ where the correlation between $\text{Re}\mathcal{P}(\sigma + it)$ and $\text{Re}\mathcal{P}(\sigma + i(t + \Delta ))$ vanishes. Further research is needed on these distances, however, we have shown that under relatively mild assumptions, the conditional distribution does not exist for these distances as well. All of this reasoning generalizes to L-functions.

We have thus shown that, although zeta and L-function magnitudes have well-understood vertical statistical structure around zeros on the critical
line, they cannot have vertical statistical structure at most distances around nontrivial zeros off the critical line. This provides a novel, probabilistic explanation for why the Riemann hypothesis is likely to be true. Our proofs are relatively simple, involving the statistical properties of the log-Euler product and related prime sums, the conditional distribution structure of elliptical random variables, and the elementary pole structure of zeta and L-functions. Further research is needed to understand if these results could serve as a starting point for a proof of the Riemann hypothesis.

**Appendix A. On statistical independence**

The characteristic function of a random variable $X$ is defined

\[ \varphi_X(\lambda) = E\{e^{i\lambda X}\}. \]

We suppose $t$ is uniformly distributed in $[a,b]$ with $b-a \to \infty$ and consider the following sum over some set of primes $p$:

\[ X(t) = \sum_p a_p e^{-i(\alpha_p t \log p + \phi_p)}, \]

where $a_p, \phi_p \in \mathbb{R}$ and $\alpha_p \in \mathbb{Q}$. We substitute (A.2)'s real part into (A.1) to write

\[ \varphi_X(t)(\lambda) = E\left\{ \prod_p \exp \left( i\lambda a_p \cos (\alpha_p t \log p + \phi_p) \right) \right\}. \]

We then expand (A.3) using the Bessel function identity

\[ e^{ix \cos y} = \sum_{n=-\infty}^{\infty} i^n J_n(x) e^{iny} \]

where $J_n(.)$ is the $n$th-order Bessel function of the first kind. This gives

\[ \varphi_X(t)(\lambda) = E\left\{ \sum_{n_1,n_2,\ldots} (i^{n_1+n_2+\ldots} J_{n_1}(\lambda a_{p_1}) J_{n_2}(\lambda a_{p_2}) \ldots \times e^{i(n_1 \phi_{p_1}+n_2 \phi_{p_2}+\ldots)} e^{it(n_1 \alpha_{p_1} \log p_1+n_2 \alpha_{p_2} \log p_2+\ldots)}) \right\}. \]

The exponential terms on (A.5)'s far right-hand side are unit circle rotations with $t$. Therefore taking the expected value will cause all terms to vanish except those for which

\[ n_1 \alpha_{p_1} \log p_1 + n_2 \alpha_{p_2} \log p_2 + n_3 \alpha_{p_3} \log p_3 + \ldots = 0. \]

However, by unique-prime-factorization, the log $p$'s are linearly independent over the rational numbers. Therefore the only solution to (A.6) is given by

\[ n_1 = n_2 = n_3 = \ldots = 0. \]
This simplifies (A.5) to give
\[
\varphi_X(t)(\lambda) = \prod_p J_0(\lambda a_p).
\]
We next note from (A.1) and (A.11) that the characteristic function for a single summand in (A.2)'s real part, \(a_p \cos(\alpha_p t \log p + \phi_p)\), is given by
\[
\varphi_p(\lambda) = J_0(\lambda a_p).
\]
Therefore, by (A.8) and (A.9),
\[
\varphi_X(t)(\lambda) = \prod_p \varphi_p(\lambda).
\]
This shows that (A.2)'s summands are independent. Essentially equivalent reasoning using the identity
\[
e^{ix \sin \phi} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\phi}
\]
gives equivalent results for (A.2)'s imaginary part. This completes the proof sketch. For further details and complete proof, apply results in [35] (pg. 36, Cor. 5.2 pg. 38, Eqn. 6.4 pg. 41, Cor. 6.7 pg. 43, Cor. 6.8 pg. 44).

**Appendix B. Uncorrelated shifts**

We consider \(\sigma > 1\) and apply the Euler product to write
\[
\log |\zeta(\sigma + it + \Delta)| \Re P(\sigma + it) = \sum_{k=1}^{\infty} \frac{\Re P(k(\sigma + it + \Delta)) \Re P(\sigma + it)}{k}
\]
We then note that the sum of the absolute values of (B.1)'s summands has the upper bound
\[
P(\sigma) \sum_{k=1}^{\infty} \frac{P(k\sigma)}{k} < \infty,
\]
which shows that (B.1) satisfies (2.7). We may therefore apply the expectation to (B.1) term-by-term. We do so under \(a = T, b = 2T,\) and \(T \to \infty\) and apply Lemma 2.1 to show that the expected value of each term in (B.1) vanishes except the term corresponding to \(k = 1\). By (2.13), the expected value of the \(k = 1\) term gives
\[
E \{ \log |\zeta(\sigma + it + \Delta)| \Re P(\sigma + it) \} \to \frac{1}{2} \Re P(2\sigma + i\Delta),
\]
which, as was noted after the proof of Lemma 2.1, has a domain extending to \(\sigma > 0\). Therefore, if \(\Re P(2\sigma + i\Delta) = 0\), then \(\log |\zeta(\sigma + it + \Delta)|\) and \(\Re P(\sigma + it)\) are uncorrelated.
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