Research Article

Lag Synchronization of Coupled Multidelay Systems

Luo Qun, Peng Hai-Peng, Xu Ling-Yu, and Yang Yi Xian

Information Security Center, Beijing University of Posts and Telecommunications, P.O. Box 145, Beijing 100876, China

Correspondence should be addressed to Peng Hai-Peng, penghaipeng2003@yahoo.com.cn

Received 23 February 2012; Revised 15 April 2012; Accepted 20 May 2012

Academic Editor: Mohamed A. Zohdy

Copyright © 2012 Luo Qun et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Chaos synchronization is an active topic, and its possible applications have been studied extensively. In this paper we present an improved method for lag synchronization of chaotic systems with coupled multidelay. The Lyapunov theory is used to consider the sufficient condition for synchronization. The specific examples will demonstrate and verify the effectiveness of the proposed approach.

1. Introduction

Since synchronization of chaotic systems was first realized by Fujisaka and Yamada [1] and Pecora and Carroll [2], chaos synchronization has received increasing interest and has become an active research topic. Currently its possible applications in various fields are in great interest, for example, applications to control theory [3], telecommunications [4–7], biology [8, 9], lasers [10, 11], secure communications [12], and so on.

Roughly speaking, chaotic communication schemes rely on the synchronization technique: the information signal is mixed at the master, and a driving signal is then generated and is sent to the slave; as a result, their chaotic trajectories remain in step with each other during temporal evolution. Besides identical synchronization [13], several new types of chaos synchronization of coupled oscillators have occurred, that is, generalized synchronization [12], phase synchronization [14], lag synchronization [15], anticipation synchronization [16], and projective synchronization [17].

Lag synchronization can be realized when the strength of the coupling between phase-synchronized oscillators is increased. There, the driving signal is constituted by the sum of multiple nonlinear transformations of delayed state variable [18]. Master and slave’s formulas are in the form of single delay [19, 20] and multidelay [21–23]. From the application point
of view, this new multidelay synchronization, different from conventional synchronization without lag, offers a significant advantage in terms of security of communication. Since the constructed state variable of the master system with lag becomes more complex than that of the conventional system, multilag systems achieve high security. Intruders cannot reconstruct the attractors of driving signal by using conventional reconstruction methods [24, 25] so as not to decipher the transferred message.

In the present paper, we proposed a systematic and rigorous scheme for lag synchronization of coupled multidelay systems based on the Lyapunov stability theory. Furthermore, the zero solution of lag synchronization differential equation is globally asymptotically stable. The effectiveness of the proposed scheme is confirmed by the numerical simulation of specific example.

2. The Schemes of Lag Synchronization

2.1. The Proposed Lag Synchronization Model

Lag synchronization was first investigated by Rosenblum [15], and it can be considered that the state variable of the slave is delayed by the positive time lag $\tau_d$ in comparison with that of the master while their amplitudes follow each other, that is, $\lim_{t \to \infty} |y(t) - x(t - \tau_d)| = 0$.

We consider the following model of lag synchronization.

Master:

$$\frac{dx(t)}{dt} = -ax(t) + \sum_{i=1}^{P} m_i f_i[x(t - \alpha_i)].$$  \hspace{1cm} (2.1)

Driving signal:

$$DS(t) = \sum_{i=1}^{Q} k_i g_i[x(t - \beta_i)] + Wx(t - \tau_d).$$ \hspace{1cm} (2.2)

Slave:

$$\frac{dy(t)}{dt} = -ay(t) + \sum_{i=1}^{R} n_i h_i[y(t - \gamma_i)] + DS(t) - Wy(t),$$ \hspace{1cm} (2.3)

where coefficients $a, m_i, k_i, n_i, \alpha_i, \beta_i, \gamma_i, W \in \mathbb{R}$, and $P, Q, R$ are positive integers. State variables $x, y \in \mathbb{R}$, and $f_i(\cdot), g_i(\cdot), h_i(\cdot) \in \mathbb{R} \to \mathbb{R}$ are three continuous nonlinear functions. The driving signal $DS(t)$ in (2.2) is constituted by the sum of multiple nonlinear transformations of delayed state variable; $\sum_{i=1}^{Q} k_i g_i[x(t - \beta_i)]$ is added with $Wx(t - \tau_d)$. The polynomial $-Wy(t)$ is added to the right side of $\frac{dy(t)}{dt} = -ay(t) + \sum_{i=1}^{R} n_i h_i[y(t - \gamma_i)] + DS(t)$, forming the slave equation which is shown as (2.3).

2.2. Proof for the Lag Synchronization Model

The desired synchronization manifold is expressed by the following relation $y(t) \to x(t - \tau_d)$ as $t \to \infty$, where $\tau_d$ is a lag time.
Mathematical Problems in Engineering

We choose suitable DS$(t)$ to satisfy $e(t) = y(t) - x(t - \tau_d) \to 0$ as $t \to \infty$.

Assumption 2.1. $Q = P + R - I, I < \min\{P, Q, R\}$ where $I$ is integer. When $i = 1, \ldots, I, f_i \equiv g_i \equiv h_i$,

\[
\gamma_i = \alpha_i, \quad \beta_i = \alpha_i + \tau_d, \quad k_i = m_i - n_i.
\] \hspace{1cm} (2.4)

Assumption 2.2. When $j = 1, \ldots, (P - I), g_{i+j} \equiv f_{i+j}$,

\[
k_{i+j} = m_{i+j}, \quad \beta_{i+j} = \alpha_{i+j} + \tau_d.
\] \hspace{1cm} (2.5)

When $j = 1, \ldots, (R - I), g_{p+j} \equiv h_{i+j}$,

\[
k_{p+j} = -n_{i+j}, \quad \beta_{p+j} = \gamma_{i+j} + \tau_d.
\] \hspace{1cm} (2.6)

Assumption 2.3. Nonlinearity $f_i$ $(i = 1, \ldots, I)$ satisfies Lipshitz condition; that is, there exists a positive constant $L$ for all time variables $a$ and $b$, such that $|f_i(a+b) - f_i(a)| \leq L|b|$ $(i = 1, \ldots, I)$.

Here we give the sufficient condition for system synchronization.

**Theorem 2.4.** If the system (2.1), (2.2), and (2.3) satisfies Assumptions 2.1, 2.2, and 2.3 and if

\[
-\alpha - W + \frac{1}{2} I + \frac{1}{2} \sum_{i=1}^{I} n_i^2 L^2 < 0,
\] \hspace{1cm} (2.7)

then $\lim_{t \to \infty} [y(t) - x(t - \tau_d)] = 0$.

**Proof.** The dynamics of synchronization error is

\[
\frac{de(t)}{dt} = \frac{dy(t)}{dt} - \frac{dx(t - \tau_d)}{dt}
\]

\[
= -\alpha e(t) + \sum_{i=1}^{I} n_i h_i [y(t - \gamma_i)] + \sum_{i=1}^{Q} k_i g_i [x(t - \beta_i)] + W x(t - \tau_d) - W y(t)
\]

\[
- \sum_{i=1}^{P} m_i f_i [x(t - \alpha_i - \tau_d)].
\] \hspace{1cm} (2.8)

By applying Assumption 2.1, (2.8) can be rewritten as

\[
\frac{de(t)}{dt} = -\alpha e(t) - W e(t) + \sum_{i=1}^{I} n_i f_i [y(t - \gamma_i)] - \sum_{i=1}^{I} (m_i - k_i) f_i [x(t - \alpha_i - \tau_d)]
\]

\[
+ \sum_{i=I+1}^{R} n_i h_i [y(t - \gamma_i)] + \sum_{i=I+1}^{Q} k_i g_i [x(t - \beta_i)] - \sum_{i=I+1}^{P} m_i f_i [x(t - \alpha_i - \tau_d)].
\] \hspace{1cm} (2.9)
By applying Assumption 2.2, if \( Q-I = (P-I)+(R-I) \) and \( y(t-\gamma_i) = x(t-\gamma_i-\tau_d)+e(t-\gamma_i) \), we have

\[
\sum_{i=1}^{R} n_i h_i[y(t-\gamma_i)] + \sum_{i=1}^{Q} k_i g_i[x(t-\beta_i)] - \sum_{i=1}^{P} m_i f_i[x(t-\alpha_i-\tau_d)]
\]

\[
= \sum_{j=1}^{R-1} n_i h_{i,j}[x(t-\gamma_{i,j}-\tau_d) + e(t-\gamma_{i,j})] + \sum_{j=1}^{P-1} k_{i,j} g_{i,j}[x(t-\beta_{i,j})] 
\]

\[
+ \sum_{j=1}^{Q-1} k_{p,j} g_{p,j}[x(t-\beta_{p,j})] - \sum_{i=1}^{P-1} m_i f_{i,j}[x(t-\alpha_{i,j}-\tau_d)] = 0, \tag{2.10}
\]

where \( e(t-\gamma_{i,j}) = 0 \) as well as synchronization established, in fact, \( e(t-\gamma_{i,j}) \) reduces during establishing the synchronization regime.

From (2.10) and (2.9) we get

\[
\frac{de(t)}{dt} = (-\alpha - W)e(t) + \sum_{i=1}^{I} n_i [f_i(x(t-\alpha_i-\tau_d) + e(t-\alpha_i)) - f_i(x(t-\alpha_i-\tau_d))]. \tag{2.11}
\]

Define a Lyapunov function \( [26] \) as

\[
V = \frac{1}{2} e^2(t) + \frac{1}{2} \sum_{i=1}^{I} \int_{t-\alpha_i}^{t} e^2(s)ds. \tag{2.12}
\]

Then, we obtain

\[
\frac{dV}{dt} = e(t) \frac{de(t)}{dt} + \frac{1}{2} \sum_{i=1}^{I} e^2(t) - \frac{1}{2} \sum_{i=1}^{I} e^2(t-\alpha_i). \tag{2.13}
\]

Here, we have

\[
\frac{dV}{dt} = (-\alpha - W)e^2(t) + \sum_{i=1}^{I} n_i e(t) [f_i(x(t-\alpha_i-\tau_d) + e(t-\alpha_i))
\]

\[
- \sum_{i=1}^{I} (m_i - k_i)e(t) f_i(x(t-\alpha_i-\tau_d)) + \frac{1}{2} \sum_{i=1}^{I} e^2(t) - \frac{1}{2} \sum_{i=1}^{I} e^2(t-\alpha_i). \tag{2.14}
\]

According to Assumption 2.1, we have

\[
\frac{dV}{dt} = (-\alpha - W)e^2(t) + \sum_{i=1}^{I} n_i e(t) [f_i(x(t-\alpha_i-\tau_d) + e(t-\alpha_i)) - f_i(x(t-\alpha_i-\tau_d))]
\]

\[
+ \frac{1}{2} \sum_{i=1}^{I} e^2(t) - \frac{1}{2} \sum_{i=1}^{I} e^2(t-\alpha_i). \tag{2.15}
\]
In our model, \( (1/2) \sum_{i=1}^{l} e^2(t) \) can be rewritten as \( (1/2) I e^2(t) \). By Assumption 2.3, we have

\[
\frac{dV}{dt} \leq \left(-\alpha - W + \frac{1}{2} I\right) e^2(t) + \frac{l}{2} \sum_{i=1}^{l} n_i^2 L^2 e^2(t) + \frac{l}{2} \sum_{i=1}^{l} e^2(t - \alpha_i) - \frac{l}{2} \sum_{i=1}^{l} e^2(t - \alpha_i). \tag{2.16}
\]

According to \( 2xy \leq x^2 + y^2 \), where \( x, y \in \mathbb{R} \), we get

\[
\frac{dV}{dt} \leq \left(-\alpha - W + \frac{1}{2} I\right) e^2(t) + \frac{l}{2} \sum_{i=1}^{l} n_i^2 L^2 e^2(t) + \frac{l}{2} \sum_{i=1}^{l} e^2(t - \alpha_i) - \frac{l}{2} \sum_{i=1}^{l} e^2(t - \alpha_i). \tag{2.17}
\]

Finally, we obtain

\[
\frac{dV}{dt} \leq \left(-\alpha - W + \frac{1}{2} I + \frac{l}{2} \sum_{i=1}^{l} n_i^2 L^2\right) e^2(t). \tag{2.18}
\]

The proof is completed.

\[\square\]

Note 1. The advantages of our lag synchronization model are as follows.

1. The nonlinear function \( f(\cdot) \) satisfies \( |f(a + b) - f(a)| \leq L|b| \), so the zero solution of lag synchronization error system is globally asymptotically stable. The condition for synchronization is easy to be realized.

2. We can choose nonlinear function in many ways, and \( f_i, g_i, h_i \) vary as \( i \) changes. Moreover, the format of function can be different even if \( i \) is the same value.

3. In order to enhance the complexity of the system, \( P, Q, R \) can be different positive integers, and the number of multiple time delays can be chosen as many values.

### 3. Numerical Simulations

The following example will demonstrate synchronization between systems with multidelay. Functions of systems are chosen from the set of \( \{ \sin u, u/(1 + u^5), u/(1 + u^{10}) \} \). Let us consider synchronization model with the master’s and slave’s equations defined as.

**Master:**

\[
\frac{dx(t)}{dt} = -ax(t) + m_1 \sin[x(t - \alpha_1)] + m_2 \sin[x(t - \alpha_2)] + m_3 \sin[x - (t - \alpha_3)] + m_4 \frac{x(t - \alpha_4)}{1 + x^5(t - \alpha_4)} + m_5 \frac{x(t - \alpha_5)}{\alpha + x^{10}(t - \alpha_5)}. \tag{3.1}
\]
Slave:

\[
\frac{dy(t)}{dt} = -ax(t) + n_1 \sin[y(t - \gamma_1)] + n_2 \sin[y(t - \gamma_2)] + n_3 \sin[y(t - \gamma_3)] \\
+ n_4 \sin[y(t - \gamma_4)] + DS(t) - Wy(t).
\] (3.2)

Therefore the equation for driving signal is chosen as.

Driving signal:

\[
DS(t) = k_1 \sin[x(t - \beta_1)] + k_2 \sin[x(t - \beta_2)] + k_3 \sin[x(t - \beta_3)] + k_4 \frac{x(t - \beta_4)}{1 + x^8(t - \beta_4)} \\
+ k_5 \frac{x(t - \beta_5)}{1 + x^{10}(t - \beta_5)} + k_6 \sin[x(t - \beta_6)] + k_7 \sin[x(t - \beta_7)] + Wx(t - \tau_d),
\] (3.3)

where \( P = 5, Q = 7, R = 4, I = 2 \) satisfy \( Q = P + R - I \).

According to (2.4)–(2.6), the relation of the delays and parameters is expressed as

\[
m_1 - k_1 = n_1, m_2 - k_2 = n_2, k_3 = m_3, k_4 = m_4, k_5 = m_5, k_6 = -n_5, k_7 = -n_4, \\
\beta_1 = \alpha_1 + \tau_d, \beta_2 = \alpha_2 + \tau_d, \beta_3 = \alpha_3 + \tau_d, \beta_4 = \alpha_4 + \tau_d, \beta_5 = \alpha_5 + \tau_d, \\
\gamma_1 = 3.4, \gamma_2 = 4.5, \gamma_3 = 2.0, \gamma_4 = 7.3, \\
\beta_6 = 6.4, \beta_7 = 7.5, \beta_8 = 9.5, \beta_9 = 8.3, \beta_{10} = 5.9, \beta_{11} = 5.0, \beta_{12} = 10.3.
\]

In Figure 1, the portrait of \( x(t - \tau_d) \) versus \( y(t) \) illustrates that the lag synchronization of coupled partly nonidentical systems is established. However their trajectories do not remain in step with each other during a short part of evolution, because they are not in synchronization as \( t < \tau_d \).

It is clear to observe from Figure 2 that synchronization error \( e(t) \) leaps at a sudden as \( \tau_d = 3.0 \) and vanishes eventually in a short time. Then \( e(t) \) stays at zero.

As shown in Figure 3, the slave’s state variable is retarded with the time length of \( \tau_d = 3.0 \) in comparison with master’s. The desired lag synchronization is realized.

4. Conclusions

In this paper, we have presented a lag synchronization model as well as researched on it. Based on Lyapunov theory, the sufficient conditions of the synchronization model are given. Simulation results of the lag synchronization model are provided to illustrate the effectiveness and feasibility of the proposed method.
Figure 1: Portrait of $x(t - 3.0)$ versus $y(t)$.

Figure 2: Synchronization error $e(t) = y(t) - x(t - 3.0)$.

Figure 3: Time series of state variables $x(t)$ and $y(t)$. 
Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grants nos. 61170269, 61121061), the Specialized Research Fund for the Doctoral Program of Higher Education (Grant no. 20100005110002) and the Fundamental Research Funds for the Central Universities (Grant no. BUPT2011RC0211).

References

[1] H. Fujisaka and T. Yamada, “Stability theory of synchronized motion in coupled-oscillator systems,” Progress of Theoretical Physics, vol. 69, no. 1, pp. 32–47, 1983.
[2] L. M. Pecora and T. L. Carroll, “Synchronization in chaotic systems,” Physical Review Letters, vol. 64, no. 8, pp. 821–824, 1990.
[3] K. Pyragas, “Continuous control of chaos by self-controlling feedback,” Physics Letters A, vol. 170, no. 6, pp. 421–428, 1992.
[4] K. M. Cuomo and A. V. Oppenheim, “Circuit implementation of synchronized chaos with applications to communications,” Physics Review Letters, vol. 71, no. 1, pp. 65–68, 1993.
[5] C. W. Wu and L. O. Chua, “A unified framework for synchronization and control of dynamical systems,” International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, vol. 4, no. 4, pp. 979–998, 1994.
[6] R. Brown, N. F. Rulkov, and E. R. Tracy, “Modeling and synchronizing chaotic systems from time-series data,” Physical Review E, vol. 49, no. 5, pp. 3784–3800, 1994.
[7] L. Kocarev and U. Parlitz, “General approach for chaotic synchronization with applications to communication,” Physical Review Letters, vol. 74, no. 25, pp. 5028–5031, 1995.
[8] S. K. Han, C. Kurrer, and Y. Kuramoto, “Dephasing and bursting in coupled neural oscillators,” Physical Review Letters, vol. 75, no. 17, pp. 3190–3193, 1995.
[9] R. C. Elson, A. I. Selverston, R. Huerta, N. F. Rulkov, M. I. Rabinovich, and H. D. I. Abarbanel, “Synchronous behavior of two coupled biological neurons,” Physical Review Letters, vol. 81, no. 25, pp. 5692–5695, 1998.
[10] I. Fabiny, P. Colet, R. Roy, and D. Lenstra, “Coherence and phase dynamics of spatially coupled solid-state lasers,” Physical Review A, vol. 47, no. 5, pp. 4287–4296, 1993.
[11] W. Yu, J. Cao, K.-W. Wong, and J. Lu, “New communication schemes based on adaptive synchronization,” Chaos, vol. 17, no. 3, p. 033114, 13, 2007.
[12] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, “Generalized synchronization of chaos in directionally coupled chaotic systems,” Physical Review E, vol. 51, no. 2, pp. 980–994, 1995.
[13] D. Huang and R. Guo, “Identifying parameter by identical synchronization between different systems,” Chaos, vol. 14, no. 1, pp. 152–159, 2004.
[14] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, “Phase synchronization of chaotic oscillators,” Physical Review Letters, vol. 76, no. 11, pp. 1804–1807, 1996.
[15] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, “From phase to lag synchronization in coupled chaotic oscillators,” Physical Review Letters, vol. 78, no. 22, pp. 4193–4196, 1997.
[16] H. U. Voss, “Anticipating chaotic synchronization,” Physical Review E, vol. 61, no. 5 A, pp. 5115–5119, 2000.
[17] R. Mainieri and J. Rehacek, “Projective synchronization in three-dimensional chaotic systems,” Physical Review Letters, vol. 82, no. 15, pp. 3042–3045, 1999.
[18] K. Pyragas, “Synchronization of coupled timedelay systems: analytical estimations,” Physical Review E, vol. 58, pp. 3067–3071, 1998.
[19] E. M. Shahverdiev, S. Sivaprakasam, and K. A. Shore, “Lag synchronization in time-delayed systems,” Physics Letters, Section A, vol. 292, no. 6, pp. 320–324, 2002.
[20] E. M. Shahverdiev, S. Sivaprakasam, and K. A. Shore, “Lag times and parameter mismatches in synchronization of unidirectionally coupled chaotic external cavity semiconductor lasers,” Physical Review E, vol. 66, no. 3, Article ID 037202, pp. 037202/1–037202/4, 2002.
[21] E. M. Shahverdiev, R. A. Nuriev, R. H. Hashimov, and K. A. Shore, “Chaos synchronization between the Mackey-Glass systems with multiple time delays,” Chaos, Solitons and Fractals, vol. 29, no. 4, pp. 854–861, 2006.
[22] C. W. Wu and L. O. Chua, “A simple way to synchronize chaotic systems with applications to secure communication systems,” *International Journal of Bifurcation and Chaos*, vol. 3, no. 6, pp. 1619–1627, 1993.

[23] T. M. Hoang and M. Nakagawa, “Enhancing security for chaos-based communication system with change in synchronization manifolds’ delay and in encoder’s parameters,” *Journal of the Physical Society of Japan*, vol. 75, no. 6, Article ID 064801, 2006.

[24] T. M. Hoang, D. T. Minh, and M. Nakagawa, “Chaos synchronization of multi-delay feedback systems with multi-delay driving signal,” *Journal of the Physical Society of Japan*, vol. 74, no. 8, pp. 2374–2378, 2005.

[25] T. M. Hoang and M. Nakagawa, “Synchronization of coupled nonidentical multidelay feedback systems,” *Physics Letters, Section A*, vol. 363, no. 3, pp. 218–224, 2007.

[26] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, 3rd edition, 2002.
