Wei-Norman equations for classical groups via cominuscule induction

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Abstract

We show how to reduce the nonlinear Wei-Norman equations, expressing the solution of a linear system of non-autonomous equations on a Lie algebra, to a hierarchy of matrix Riccati equations using the cominuscule induction. The construction works for all reductive Lie algebras with no simple factors of type $G_2$, $F_4$ or $E_8$. A corresponding hierarchy of nonlinear, albeit no longer Riccati equations, is given for these exceptional cases.

Keywords: Lie equations, linear non-autonomous system, Wei-Norman equations, Riccati equations, parabolic subgroups

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1. Introduction

The Wei-Norman method [1] was developed by its authors to reduce a systems of linear differential equations with variable coefficients to a nonlinear one. At first sight advantages of such a reduction seem questionable, but in fact, in many applications it provides an useful method of analyzing linear non-autonomous systems, alternative to the commonly used in physics expansion in a series of time-ordered multiple integrals [1]. Another advantage can follow from the fact that the nonlinear system can be integrated in steps, each involving smaller number of variables than the original linear system. In [2] we were able to show that in the unitary case, i.e. when the original linear system of equations describes an evolution within the unitary group, the Wei-Norman equations can be reduced to a system of nonlinear matrix Riccati equations. In [3] we generalized this result to all classical Lie groups. We also observed that the resulting equations are in general not of the matrix Riccati form for the non-classical simple Lie groups showing this fact explicitly for the $G_2$ group, but leaving other non-classical cases inconclusive. In the present paper we give a unified approach to the problem based on the cominuscule induction, which works for all complex reductive Lie groups having no simple factors of type $G_2$, $F_4$ or $E_8$. Moreover for the cases of simple Lie groups for which the cominuscule induction is not applicable, we show that employing the contact grading of the corresponding Lie algebra gives the Wei-Norman equations in form of coupled first-order equations of at most fourth degree (compared to two in the cominuscule case). Since the contact grading exists for all simple Lie algebras our results exhibit the structure of the Wei-Norman equations for all reductive complex Lie groups.

Let $G$ be a complex reductive Lie group, and $\mathfrak{g}$ its Lie algebra. Denote by $R_\xi : G \to G$ the map $h \mapsto h\xi$. Given a continuous map $X : I \to \mathfrak{g}$ from an interval $I \subset \mathbb{R}$ containing 0, the development of $X$ in $G$ is a solution $x : I \to G$ of the ODE

$$\frac{dx}{dt} = R_{\gamma}(t)X,$$

$x(0) = e_G$.

Given a vector space $E$, a vector Riccati equation for a function $\xi : I \to E$ is an ODE of the form

$$\frac{d\xi(t)}{dt} = \gamma(t) + \alpha(t)(\xi) + \beta(t)(\xi,\xi)$$

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where \( \gamma : I \to E, \alpha : I \to E^* \otimes E, \beta : I \to S^2 E^* \otimes E \) are given functions corresponding to terms of degree, respectively, zero, one and two in \( \xi \).

We shall prove the following.

**Theorem 1.** Assume no simple factor of \( G \) is of type \( G_2, F_4 \) or \( E_6 \). Then computing the development of \( X \) in \( G \) can be, locally on \( I \), reduced to solving a system of finitely many vector Riccati equations on functions \( \xi_i : I \to \mathfrak{g} \) of the form

\[
\frac{d \xi_i(t)}{dt} = \gamma_i(t) + \alpha_i(t) \xi_i + \beta_i(t) \xi_i \xi_i, \quad \xi_i(0) = 0
\]

for \( i = 1, \ldots, r \), where the coefficients \( \gamma_i(t), \alpha_i(t), \beta_i(t) \) can be expressed explicitly in terms of \( \xi_j(t), \ j < i \), and the structure constants of \( \mathfrak{g} \) in a Chevalley basis. The solution is then given in the Wei-Norman form \( x(t) = e^{\xi_1(t)} \cdots e^{\xi_r(t)} \).

Furthermore \( r \leq 2 \text{rank}\ G^{ss} + \text{rank}\ G \), where \( G^{ss} \) is the semisimple part of \( G \).

We stress that the method is intrinsic to the Lie algebra \( \mathfrak{g} \), and does not rely on any particular representation (i.e. does not proceed by reduction to the classical result for matrix equations). Our approach uses the existence of so-called cominuscule parabolic subgroups, which appear precisely in those reductive groups that satisfy the hypothesis of the Theorem.

Let us briefly recall the notion of a general parabolic subgroup of a reductive complex Lie group \( G \): it is a proper closed subgroup \( P \subset G \) such that the homogeneous space \( G/P \) is a compact complex manifold (in fact, a projective algebraic variety). The group \( P \) admits a Levi decomposition \( P = L \rtimes U \) where \( U \) is the unipotent radical of \( P \), while \( L \) is the Levi factor of \( P \). On the level of Lie algebras, one may associate with \( P \) a grading

\[
\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k,
\]

for certain \( k > 0 \), such that \( \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k \) is the Lie algebra of \( P \), and \( \mathfrak{g}_0 \) is the Lie algebra of \( L \) (it is understood that \( \mathfrak{g}_k = 0 \) whenever \( |k| > k \)). In fact, the grading as above determines the subgroup \( P \) up to isogeny, and a maximal compatible \( P \) may be chosen by requiring that \( G/P \) is connected and simply-connected.

Associated with \( P \) is another parabolic subgroup \( P' \), called the **opposite parabolic**, with the property that \( P \cap P' = L \) and \( P' \cdot P \) is dense in \( G \). Writing \( P' = U' \rtimes L \) for the Levi decomposition of the opposite parabolic, one has that the map

\[
U' \times L \times U \to G
\]

induced by the group operation is an embedding onto a dense open subset. The grading corresponding to \( P' \) is simply the opposite of the grading associated with \( P \), so that in particular \( \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \) is the Lie algebra of \( U' \), and the above open embedding gives

\[
\begin{array}{c}
\text{Lie } U' \\
\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \\
\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \\
\text{Lie } L \\
\text{Lie } U
\end{array}
\]

on the infinitesimal level.

2. **Proof of the main Theorem**

A parabolic subgroup of a reductive complex Lie group is called **cominuscule** if it satisfies the equivalent statements in the following:

**Proposition 1** (cf. [4, 5]). Let \( P \subset G \) be a parabolic subgroup. The following are equivalent:

1. \( G/P \) is a Hermitian symmetric space of compact type,
2. the unipotent radical of \( P \) is abelian,
3. there is a grading \( \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) such that \( \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \).

The proof of the Theorem 1 relies on **cominuscule induction** (see e.g. [6] for a geometric application). The following will be proven later.

**Proposition 2.** Let \( \mathcal{P} \) be a property of complex reductive Lie groups such that:
1. \( \mathcal{P} \) holds for all tori,
2. if \( \mathcal{P} \) holds for the Levi factor of some cominuscule parabolic subgroup in \( G \), then it holds for \( G \).

Then \( \mathcal{P} \) holds for all reductive complex Lie groups with no simple factors of type \( G_2, F_4 \) or \( E_8 \).

**Proof of Theorem 1.** Given a complex reductive \( G \) we let \( \mathcal{P}(G) \) stand for the conclusion of the Theorem. Clearly, computing the development in a torus reduces to integrating and exponentiating a function, hence to a degenerate vector Riccati equation, so that \( \mathcal{P} \) holds for all tori.

Let now \( P \subset G \) be a cominuscule parabolic subgroup. We use the grading

\[ \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \]

such that \( \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) (cf. Prop. 1). One then has that \( \mathfrak{g}_0 \) is the Lie algebra of the Levi factor \( L \) of \( P \), while \( \mathfrak{g}_1 \) is the Lie algebra of its unipotent radical \( U \). The opposite parabolic \( P' \) has the property that \( P' \cap P = L \) and \( L \) is the Levi factor of \( P' \). One then has \( \mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \) and \( \mathfrak{g}_{-1} \) is the Lie algebra of the unipotent radical \( U' \) of \( P' \). The group operation induces an embedding

\[ U' \times L \times U \hookrightarrow G \]

onto a dense open subset. The exponential map gives Lie group isomorphisms \( \mathfrak{g}_{-1} \cong U' \), \( \mathfrak{g}_1 \cong U \), where the Lie group structure on \( \mathfrak{g}_{\pm 1} \) is induced by the vector space structure. Observe that \( \text{ad}_X \) is a derivation of degree \( \pm 1 \) for \( X \in \mathfrak{g}_{\pm 1} \), while \( \mathfrak{g} \) is graded in degrees \( -1, 0, 1 \), so that \( \text{ad}_X^3 = 0 \). It follows that

\[ \text{Ad}_{\exp} X = \sum k! \text{ad}_X^k = \text{id} + \text{ad}_X + \frac{1}{2} \text{ad}_X^2 \]

for \( X \in \mathfrak{g}_{\pm 1} \). We shall use the exponential to view elements of \( \mathfrak{g}_{\pm 1} \) as elements of \( G \), so that

\[ \text{Ad}_X Y = Y + [X, Y] + \frac{1}{2} [X, [X, Y]] \]

for \( X \in \mathfrak{g}_{\pm 1}, Y \in \mathfrak{g} \).

Consider now the development equation

\[ \frac{dx}{dt} = R_x X, \quad x(0) = e_G. \]

Possibly shrinking the domain interval \( I \), we can assume \( x(t) \in U'LU \), so that the equation becomes equivalent to

\[ R_{\text{exp} x}^{-1} \frac{d}{dt} W y V = Z' + Y + Z, \quad W(0) = 0, \quad y(0) = e_L, \quad V(0) = 0 \]

where

\[ X = Z' + Y + Z, \quad x = W y V \]

for

\[ Z', W : I \rightarrow \mathfrak{g}_{-1}, \quad Z, V : I \rightarrow \mathfrak{g}_1 \]

and

\[ Y : I \rightarrow \mathfrak{g}_0, \quad y : I \rightarrow L. \]

We compute:

\[ R_{\text{exp} x}^{-1} \frac{d}{dt} W y V = \frac{dW}{dt} + \text{Ad}_W R_{\gamma x}^{-1} \frac{dy}{dt} + \text{Ad}_W \text{Ad}_y \frac{dV}{dt}. \]

Comparing with \( Z' + Y + Z \) and decomposing according to the grading, we then have

\[ \frac{dW}{dt} + [W, R_{\gamma x}^{-1} \frac{dy}{dt}] + \frac{1}{2} [W, [W, \text{Ad}_y \frac{dV}{dt}]] = Z' \]

\[ R_{\gamma x}^{-1} \frac{dy}{dt} + [W, \text{Ad}_y \frac{dV}{dt}] = Y \]

\[ \text{Ad}_y \frac{dV}{dt} = Z. \]
Elimination yields the equivalent system
\[
\frac{dW}{dt} + [W, Y] - \frac{1}{2}[W, [W, Z]] = Z' \\
R_y \frac{dy}{dt} + [W, Z] = Y \\
\text{Ad}_y \frac{dV}{dt} = Z.
\]

It follows that solving the development equation is equivalent to the following sequence:

1. solving the Riccati equation
   \[
   \frac{dW}{dt} = Z' - [W, Y] + \frac{1}{2}[W, [W, Z]]
   \]
on $W : I \to g^{-1}$,
2. solving the development equation
   \[
   \frac{dy}{dt} = R_y (Y - [W, Z])
   \]
on $y : I \to L$,
3. integrating
   \[
   V = \int \text{Ad}_{e^{-1}} Z dt.
   \]

Clearly, steps (2) and (3) can be considered as degenerate Riccati equations themselves. Denoting the conclusion of the Theorem with $P(G)$, we have just proven that $P$ satisfies the hypotheses of Prop. 2. Hence it holds for all reductive complex Lie groups whose simple factors admit cominuscule parabolic subgroups. \(\square\)

### 3. Proof of Parabolic Induction

It remains to prove Proposition 2. We recall the description of parabolic subgroups in terms of root data:

**Proposition 3** (cf. [5]). Let $G \supset B \supset T$ be a complex reductive Lie group, a Borel subgroup, and a maximal torus. Denote by $\Phi$ the root system of $g$ with respect to $T$, and by $\Phi^+$ the subset of positive roots determined by $B$ (i.e. $B$ is generated by $T$ and by root subgroups corresponding to positive roots). Let $\Delta \subset \Phi^+$ be the simple roots. Define for each subset $\Sigma \subset \Delta$ the function $ht_{\Sigma} : \Phi \to \mathbb{Z}$ sending a root $\alpha \in \Phi$ to the sum of the coefficients of elements of $\Sigma$ in the simple root decomposition of $\alpha$. Let $P_{\Sigma} \subset G$ be the subgroup generated by $B$ and root subgroups $G_{-\alpha}$ with $ht_{\Sigma}(\alpha) = 0$. Then:

1. $P_{\Sigma}$ is parabolic,
2. $\Sigma \mapsto P_{\Sigma}$ gives a one-to-one correspondence between nonempty subsets of $\Delta$ and conjugacy classes of parabolic subgroups in $G$, 
3. the Dynkin diagram of the Levi factor $L_{\Sigma}$ of $P_{\Sigma}$ is obtained by removing nodes corresponding to $\Sigma \subset \Delta$ from the Dynkin diagram of $G$.

Parabolic subgroups of the form $P_{\Sigma}$ are called standard; every parabolic subgroup is standard for some choice of a maximal torus and positive roots. In particular, cominuscule parabolic subgroups are characterized as follows:

**Lemma 1** (cf. [4, 5]). Let $P_{\Sigma}$ be a standard parabolic for $\Sigma \subset \Delta$. The following are equivalent:

1. $P_{\Sigma}$ is cominuscule,
2. $ht_{\Sigma}(\Phi_+) = \{0, 1\}$. 

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4
3. marking the nodes corresponding to $\Sigma$ in the Dynkin diagram $\Gamma$ of $G$, each connected component of $\Gamma$ contains either no marked nodes, or a single marked cominuscule node, where the cominuscule nodes for connected diagrams are as follows (in Bourbaki labelling):

- $A_n : \alpha_i, 1 \leq i \leq n$
- $B_n : \alpha_1$
- $C_n : \alpha_n$
- $D_n : \alpha_1, \alpha_{n-1}, \alpha_n$
- $E_6 : \alpha_1, \alpha_6$
- $E_7 : \alpha_7$

As a consequence, we have:

**Lemma 2.** Assume no simple factor of $G$ is of type $G_2$, $F_4$, $E_8$. Then $G$ admits a cominuscule parabolic subgroup, and furthermore for each cominuscule parabolic subgroup $P \subset G$, the Levi factor $L$ of $P$ has no simple factor of type $G_2$, $F_4$, $E_8$.

**Proof.** By inspection of Dynkin diagrams.

**Proof of Proposition 2.** We shall proceed by induction on the semisimple rank. Note that the Proposition holds tautologically when restricted to tori. Suppose the Proposition holds when restricted to reductive complex Lie groups of semisimple rank less than $n$. Let $P$ be a property of complex reductive Lie groups satisfying the hypotheses of the Proposition, and let $G$ be a reductive complex Lie group of semisimple rank $n$, and with no simple factors of type $G_2$, $F_4$ or $E_8$. By Lemma 2, there is a cominuscule parabolic subgroup $P \subset G$, and $P$ holds for the Levi factor $L$ of $P$ by the inductive hypothesis ($\dim L < n$). Hence $P$ holds for $G$ by hypothesis (2) of the Proposition.

4. Remark on the remaining cases of simple groups

As the reader of our proof will notice, it is not difficult to give a weaker general result for all reductive groups. The reduction to Riccati equations relied on the single fact that $\text{ad}_X g = 0$ for $X \in g_{\pm 1}$ in the cominuscule grading; the class of groups under consideration was precisely those whose Lie algebras admit such gradings. It turns out that the next best thing happens in the general case: every simple Lie algebra of rank at least two admits a contact grading, i.e. one of the form

$$g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$$

with $\dim g_{\pm 2} = 1$ (cf. [5]). We then have $\text{ad}_X^5 g = 0$ for $X \in g_{\pm 1}$ and $\text{ad}_X^4 g = 0$ for $X \in g_{\pm 2}$, whence following the lines of the proof of Theorem 1 for such a grading one is left with a system of first-order ODEs of degree at most four (compared to two in the cominuscule case).

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