Almost Ramanujan Expanders from Arbitrary Expanders via Operator Amplification

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We give an efficient algorithm that transforms any bounded degree expander graph into another that achieves almost optimal (namely, near-quadratic, \( d \leq 1/\lambda^{2+o(1)} \)) trade-off between (any desired) spectral expansion \( \lambda \) and degree \( d \). Furthermore, the algorithm is local: every vertex in the new graph can compute its new neighbors as a subset of its original neighborhood of radius \( O(\log(1/\lambda)) \). The optimal quadratic trade-off is known as the Ramanujan bound, so our construction gives almost Ramanujan expanders from arbitrary expanders.

The locality of the transformation preserves structural properties of the original graph, and thus has many consequences. Applied to Cayley graphs, our transformation shows that any expanding finite group has almost Ramanujan expanding generators. Similarly, one can obtain almost optimal explicit constructions of quantum expanders, dimension expanders, monotone expanders, etc., from existing (suboptimal) constructions of such objects. Another consequence is a "derandomized" random walk on the original (suboptimal) expander with almost optimal convergence rate. Our transformation also applies when the degree is not bounded or the expansion is not constant.

We obtain our results by a generalization of Ta-Shma’s technique in his breakthrough paper [STOC 2017], used to obtain explicit almost optimal binary codes. Specifically, our spectral amplification extends Ta-Shma’s analysis of bias amplification from scalars to matrices of arbitrary dimension in a very natural way. Curiously, while Ta-Shma’s explicit bias amplification derandomizes a well-known probabilistic argument (underlying the Gilbert–Varshamov bound), there seems to be no known probabilistic (or other existential) way of achieving our explicit operator-valued spectral amplification.

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1 Introduction

1.1 Background

Expander graphs are fundamental objects in computer science and mathematics, possessing a variety of applications in both fields [HLW06, Lub12]. Indeed, expanders (and expansion) play a central role in numerous algorithmic advances, cryptographic schemes, circuit and proof complexity lower bounds, derandomization and pseudorandom generators, error correcting codes, ... and are central to structural results in group theory, algebra, number theory, geometry, combinatorics.

A central quality measure of expansion of an infinite family of $d$-regular graphs $\{X_i\}_{i \in \mathbb{N}}$ is the second largest singular value of its normalized adjacency matrix, which we denote by $\lambda(X_i) \in [0, 1]$. We say that a family $\{X_i\}_{i \in \mathbb{N}}$ is $\lambda$-expanding, for some fixed $\lambda < 1$, if $\lambda(X_i) \leq \lambda$ for every member $X_i$ of the family. The smaller is the expansion parameter $\lambda$, the more spectrally expanding is the family. (For simplicity, we will sometimes discuss single graphs rather than families, and say that $X$ is a $(d, \lambda)$-expander if it is $d$-regular and satisfies $\lambda(X) \leq \lambda$.)

A random $d$-regular graph with $d \geq 3$ is easily shown [Pin73] to be .99-expanding with high probability, giving rise to the existence of expanding families. The quest to explicitly construct bounded degree expanders started with Margulis’ paper [Mar73], and has been an extremely active research area in the past half-century. Today, we have a large arsenal of constructions and tools to establish expansion which are quite different in nature—algebraic, analytic, combinatorial, and mixtures of these (for a short survey of this wealth see [Wig18, Sec 8.7])—and we will discuss a few of them below.

All the different constructions above yield $d$-regular $\lambda$-expanding families with some specific constants $d$ and $\lambda$. Now, a large variety of structural and algorithmic applications call for optimizing both parameters, and understanding the best trade-off between them. One example which is directly related to this paper is the study of random walks on expanders, sometimes used for randomness-efficient error-reduction of probabilistic algorithms and also in the construction of randomness extractors. The surprising expander Chernoff bound of Gillman [Gil98] informally says that a sequence of highly correlated $k$ vertices along a random walk in a $(d, \lambda)$-expander, is almost as good a sampler as a sequence of $k$ independent vertices, with respect to empirical sums. Saving randomness calls for minimizing the degree $d$ while improving the quality of the sample requires minimizing the expansion parameter $\lambda$.

However, for any choice of degree $d$, the spectral expansion $\lambda$ cannot be made arbitrarily small. The Alon–Boppana bound [Nil91] shows that $\lambda(X_i) \geq 2\sqrt{d - 1}/d - o_n(1)$. It intuitively says that the infinite $d$-regular tree is the best possible spectral expander, raising the challenge of achieving it by finite graphs. This challenge was first met by the (independent) seminal papers of [LPS88, Mar88]; they constructed optimal spectrally expanding families, dubbed Ramanujan graphs, satisfying the (Ramanujan bound) $\lambda(X_i) \leq 2\sqrt{d - 1}/d$.

The investigation of expanding families near or achieving the optimal Ramanujan bound has received much attention. However, since then, only one essentially different construc-

\[\text{1See Definition 2.2 for a formal definition.}\]
A study of almost Ramanujan expanders, in which the bound above is nearly matched, has received much attention as well. Friedman [Fri03] greatly strengthened Pinsker’s bound above [Pin73], showing that with high probability, a random $d$-regular graph $X$ satisfies $\lambda(X) \leq 2\sqrt{d - 1/d + o_n(1)}$. For explicit constructions, one successful approach was the lifting method of Bilu–Linial [BL06], which achieves $d \leq O(1/\lambda^2)$, and famously led to the (exact) Ramanujan expanders of [MSS15] mentioned above.

A different paradigm to explicitly construct almost optimal expanders, which is central for this paper, is to start with a weak expander and amplify it to a strong one via combinatorially defined graph products. This originated with the zig-zag product of [RVW00]. They showed that their basic zig-zag construction achieves an explicit family of expanders with $d \leq 1/\lambda^4$. They further derandomize the basic zig-zag product to achieve $d \leq 1/\lambda^3$. Ben-Aroya and Ta-Shma [BAT08] in their ingenious “s-wide zig-zag product”, nearly matched the optimal quadratic bound$^2$, achieving $d \leq 1/\lambda^{2+o(1)}$. Their “higher-order” version of zig-zag [BAT08] will be central in our work. Note that these results yield specific constructions but do not provide a generic technique to amplify arbitrary expanders.

While for some applications and structural results, one is free to start with any family of expanders, for many others, the initial graphs are externally given to us (as e.g. is the case for understanding the expansion of Cayley graphs of groups). So it is highly desirable to have generic algorithms for expander construction that can operate within these constraints. Moreover, seeking different constructions and analysis tools has led to surprising applications beyond those intended (e.g., the resolution of the Kadison–Singer conjecture by [MSS14] and the proof of $SL = L$ by Reingold [Rei05]). Motivated by this, we study the following question:

When can a family of weak expanders be amplified to a strong (almost-Ramanujan) one?

We show that this is always possible: any expander family can be locally and efficiently converted into an almost Ramanujan family. More precisely, starting from any family of bounded degree expanders, it is possible to obtain, for any desired target expansion $\lambda > 0$, a new family of $\lambda$-expanders close to the Ramanujan bound.

### 1.2 Main Results

Our main result for general families of expander graphs is as follows.

**Theorem 1.1** (Main I - Informal). Let $\{X_i\}_{i \in \mathbb{N}}$ be a family of $(d_0, \lambda_0)$-expanders where $\lambda_0 < 1$ is a constant. For any (target) $\lambda \in (0, 1)$ and $X_i$, we can explicitly$^3$ construct a $(d, \lambda)$-expander, $X'_i$, on the same vertex set, where $d = O(d_0/\lambda^{2+o(1)})$. Moreover, the construction is local in the sense that edges in $X'_i$ correspond to short walks in $X_i$.

The key insight is to use a result of König that says that the adjacency matrix, say $A_{X_i}$, of an arbitrary regular graph, can be written as a sum of permutation matrices. Thus

$^2$We call any such bound near-optimal or almost Ramanujan. Of course, reducing the $o(1)$ slack in the exponent is clearly of much interest.

$^3$See Definition 2.2
we have, \( A_X = \mathbb{E}_{s \sim S}[\rho_{\text{def}}(s)] \) where \( S \) is a set of permutations, and \( \rho_{\text{def}} \) is the defining representation that maps a permutation to the matrix defining it. The task is to construct a set \( S' \), with \( \|\mathbb{E}_{s \sim S'}[\rho_{\text{def}}(s)]\|_{\text{op}} \leq \lambda \). We study a general version of this question with an arbitrary group representation of a finite group (See Definition 2.3). This is directly connected to the expansion of Cayley graphs.

Cayley Graphs  
A Cayley graph \( \text{Cay}(G, S) \) on a finite group \( G \) is specified by a symmetric set of generators \( S \subseteq G \), where vertices are elements of \( G \) and \( g, h \in G \) are adjacent if and only if \( hg^{-1} \) belongs to \( S \).

While many groups admit Cayley expanders, most of these are far from the Ramanujan bound. This is true, in particular, in the case of non-Abelian finite simple groups which includes the symmetric group. Breuillard and Lubotzky [BL18] ask whether it is possible to have near-Ramanujan expanders for all families of finite simple groups. More generally,

Which (family of) groups admit expanding Cayley graphs close to the Ramanujan bound?

Our key result is that any group that admits a Cayley expander also admits one that is almost Ramanujan.

**Theorem 1.2 (Main II).** Let \( G \) be a finite group and \( S \) be such that \( \text{Cay}(G, S) \) is a \( \lambda_0 \)-expander, for some constant \( \lambda_0 \in (0, 1) \). For every \( \lambda \in (0, 1) \), there exists \( S' \) such that

- \( \text{Cay}(G, S') \) is a \( \lambda \)-expander,
- \( |S'| = O\left( |S| / \lambda^{2+\alpha(1)} \right) \), and
- \( S' \) can be computed deterministically in \( \text{poly}(|S| / \lambda) \)-time assuming an oracle for group operations.

Furthermore, if \( \text{Cay}(G, S) \) is strongly explicit\(^4\), then so is \( \text{Cay}(G, S') \).

Since expanding families of Cayley graphs are known for non-Abelian finite simple groups [BL18, Theorem 3.1], this result makes substantial progress towards the question asked therein (the \( o(1) \) term needs to be removed to resolve it completely). Moreover, these are strongly explicit (except for the Suzuki group). Thus, our result yields strongly explicit almost Ramanujan Cayley graphs for these these groups, which notably includes the symmetric group!

**Corollary 1.3 (Explicit almost Ramanujan Cayley Expanders).** For every non-Abelian finite simple\(^5\) group \( G \) and \( \lambda > 0 \), we can explicitly construct almost-Ramanujan \((d, \lambda)\)-Cayley multi-graphs on \( G \) where \( d \leq O(1 / \lambda^{2+\alpha(1)}) \).

**Remark 1.4 (Connection with Codes).** A linear \( \lambda_0 \)-balanced code over \( \mathbb{F}_2^{n_0} \) of dimension \( k \) is equivalent to a Cayley \( \lambda_0 \)-expander over \( G = \mathbb{F}_2^k \) of degree \( n_0 \). Let \( S \subseteq G \) be the rows of a generator matrix of a good \( \lambda_0 \)-balanced code (good means \( k / n_0 \) and \( \lambda_0 < 1 \) are

\(^4\)Neighbors of a vertex can be computed in time polynomial in the description length of a vertex.

\(^5\)This holds for other groups as well, as long as they have expanding generators. One non-simple example is the Cayley expanders of Rozenman, Shalev and Wigderson [RSW06].
Thus, the breakthrough construction of explicit almost optimal binary codes of Ta-Shma [TS17] close to the Gilbert–Varshamov [Gil52, Var57] bound can be viewed as a particular case of Theorem 1.2 applied to the group $G = \mathbb{F}_2^k$. Applying Theorem 1.2 above to $S$ with final expansion parameter $\lambda > 0$, we obtain a generating set $S' \subseteq G$ of a Cayley $\lambda$-expander with degree $O(k/\lambda^{2+\alpha(1)})$, or equivalently, we obtain a $\lambda$-balanced code of rate $\Theta(\lambda^{2+\alpha(1)})$.

### 1.3 Applications

We will now discuss some applications of this operator amplification technique which allows us to improve other pseudorandom objects. All the "pseudorandom" objects below are expanders (with various structural properties), and for each of these, we amplify their spectral bound to almost Ramanujan. We stress that our amplification preserves the underlying structure and therefore, produces another object with the same properties. Precise definitions of these objects will be given in Section 5.

**Quantum Expanders** Roughly speaking, a quantum expander is an operator defined by $d$ complex matrices, whose (linear) action on quantum states has a constant spectral gap. Quantum expanders were defined in [AS04, BASTS08, Has07a], and Hastings [Has07c] showed that the Ramanujan bound also applies to them. Existing explicit constructions are far from the Ramanujan bound. In [Har07], Harrow gave a generic construction using expanding Cayley graphs which is explicit if the group has a large irreducible representation and admits efficient Quantum Fourier Transform (QFT). Both these conditions are satisfied by the symmetric group $\text{Sym}_n$ using the generating family by Kassabov [Kas07] and the QFT algorithm by Beals [Bea97].

By amplifying the expansion of the generators of [Kas07], we give the first explicit family of almost Ramanujan quantum expanders.

**Corollary 1.5 (Explicit Almost Ramanujan Quantum Expanders).** For every $\lambda \in (0, 1)$, there is an explicit infinite family of (efficient) $(O(1/\lambda^{2+\alpha(1)}), \lambda)$-quantum expanders.

**Monotone Expanders** Monotone expanders are expanders, whose edge set can be decomposed into a constant number of monotone partial maps on $[n]$. Bourgain and Yehudayoff [BY13] gave the only known explicit construction of monotone expanders with constant degree. There are two natural notions of degree for a monotone expander. The usual vertex degree and the number of monotone maps. Our almost Ramanujan trade-off is with respect to the vertex degree (and the monotone degree is polynomial in the vertex degree).

**Corollary 1.6 (Almost Ramanujan Monotone Expanders).** For every $\lambda > 0$, there is an explicit family $\{X_i\}_{i \in \mathbb{N}}$ of (vertex) $d$-regular $d^{O(1)}$-monotone expanders with $d = O(1/\lambda^{2+\alpha(1)})$ and $\lambda(X_i) \leq \lambda$.

The approach is similar to that used for Theorem 1.1; we express it as a sum of permutation matrices and amplify their expansion obtaining the following result. It would be really interesting to obtain an almost Ramanujan trade-off with respect to the monotone degree.
**Dimension Expanders**  Loosely speaking, dimension expanders (over any field \( \mathbb{F} \)) are a linear algebraic extension of expanders: a collection of \( d \) linear maps on \( \mathbb{F}^n \), which significantly expands (the span of) any vector space of dimension below \( n/2 \). They were defined by Barak et al. in [BISW01]. Over complex numbers, any quantum expander is a dimension expander. More generally, Dvir and Shpilka [DS09] proved that a monotone expander directly yields a dimension expander over every field. We give spectral almost Ramanujan expanders that have the additional property of being dimension expanders. Additionally, if the starting dimension is small enough then we achieve almost doubling of the starting dimension. See Corollary 5.21 for a precise statement.

**Kazhdan Constant**  We can also amplify operators in infinite dimensional Hilbert spaces. This allows us to obtain improved (average) Kazhdan constants of groups with “Property (T)” (and ours) is to establish an “bias amplification” result of the form, for every \( \lambda > 0 \), there exists a deterministic polynomial time algorithm to construct \( W \subseteq S^1 \) of size \( |W| \leq \text{poly}(|S|, 1/\lambda) \) such that

Let \( G \) be a finitely generated group and \( S \) a finite set of generators such that the average Kazhdan constant \( \overline{\kappa}(G, S) \) is equal to \( 2 \cdot (1 - \lambda_0) \) for some constant \( \lambda_0 \in (0, 1) \). For every \( \lambda \in (0, 1) \), there is a set \( S' \subseteq G \) such that

1. \( \overline{\kappa}(G, S') \geq 2 \cdot (1 - \lambda) \), and thus, \( \overline{\kappa}(G, S') \geq 2 \cdot (1 - \lambda) \).
2. \( |S'| = O_{\lambda_0}(|S| / \lambda^{2+o_1(1)}) \), and
3. \( S' \) can be found in time \( \text{poly}(|S| / \lambda) \) assuming an oracle for group operations on \( G \).

**1.4 Techniques**

We first rephrase the technical result of our paper from the perspective of “bias amplification”. This makes it easy to compare to prior work, which we crucially build upon. Let \( f : S \rightarrow M_{d \times n}(\mathbb{C}) \) be a matrix-valued function. The quantity \( \|E_{S - S}[f(s)]\|_{\text{op}} \) is known as the bias of the operator-valued function \( f \) with respect to \( S \). The key idea in the prior works (and ours) is to establish an “bias amplification” result of the form,

Let \( S \) be a finite set and \( \lambda_0 \in (0, 1) \) be a constant. For every \( \lambda > 0 \), there exists a deterministic polynomial time algorithm to construct \( W \subseteq S^1 \) of size \( |W| \leq \text{poly}(|S|, 1/\lambda) \) such that

- **Scalar Amplification**  For every function \( f : S \rightarrow \mathbb{C} \) such that \( \|f\|_{\infty} \leq 1 \), if \( \|E_{S - S}[f(s)]\|_{\lambda_0} \leq \lambda_0 \) then we have \( \|E_{S - S}[f(s_1) \cdots f(s_t)]\|_{\lambda} \leq \lambda \).

- **Operator Amplification**  For every function \( f : S \rightarrow M_{d \times n}(\mathbb{C}) \) such that \( \max_{\rho} \|f(s)\|_{\text{op}} \leq 1 \), if \( \|E_{S - S}[f(s)]\|_{\text{op}} \leq \lambda_0 \) then we have \( \|E_{S - S}[f(s_1) \cdots f(s_t)]\|_{\text{op}} \leq \lambda \).

Such a result is closely related to expansion of Cayley graphs in the following way. For a group \( G \), a set \( S \subseteq G \) is called an \( \varepsilon \)-biased set if for every non-trivial irreducible representation, \( \rho \), of \( G \), we have \( \|E_{S - S}[^{\rho}(s)]\|_{\text{op}} \leq \varepsilon \). Here, a group representation is a
homomorphism from $G$ to the general linear group of operators, $M_t(\mathbb{C})$. The notion of $\varepsilon$-biased set over $G = \mathbb{Z}_2^n$ was introduced in the pioneering work of Naor and Naor [NN90]. These small-bias sets have found numerous applications (e.g., [ABN+92, Vad12, TS17]).

A symmetric subset $S \subseteq G$ is an $\varepsilon$-biased set if and only if the $\text{Cay}(G, S)$ is an $\varepsilon$-expander. Therefore if $\text{Cay}(G, S)$ is a given $\lambda_0$-expander, we can apply Theorem 1.8 for each irreducible representation $\rho$. This yields that $\text{Cay}(G, S')$ is a $\lambda$-expander where $S' = \{s_t \cdots s_1 \mid (s_1, \cdots, s_t) \in W\}$. The degree of the new graph is $|S'| = |W|$. Note that over Abelian groups, the scalar amplification suffices as the irreducible representations of Abelian groups are 1-dimensional, i.e., scalar-valued functions called characters. However, for general finite groups, one needs the amplification result for operator-valued functions.

**A first approach** One can take $W = S^t$ with $t \approx \log_{\lambda_0}(\lambda)$. However, now the degree, $|S|^t = O(1/\lambda^{100})$, has also increased and the trade-off remains the same. Thus, we want to efficiently compute a sparse subset of $S^t$ that retains the expansion. Since we know what degree we are aiming for, we could try to take a sparse random sample $S' \subseteq S^t$ of size $d = O(1/\lambda^{2.001})$ and hope that some form of matrix concentration ensures that $\text{Cay}(G, S')$ is $\lambda'$-spectral expander with $\lambda' \approx \lambda$. Unfortunately, it is not clear how to show even the existence of a single sparse subset $S'$ that achieves the required expansion.

For example, one could use matrix Chernoff plus union bound to select a subset $S'$ such that $\|E_{s \in S'}[\rho(s)]\|_{\text{op}}$ is small for any irreducible representation $\rho$. However, this naive approach requires $|S'| \geq \Omega(\log \dim(\rho))$. For non-abelian groups, we have irreducible representations such that $\dim(\rho) = \text{poly}(|G|)$, and therefore, this cannot deduce the existence of constant-sized subsets that achieve expansion. This difficulty is also present in the proof of the Alon–Roichman theorem [AR94] and the reason why even for non-Abelian groups the only generic upper bound known uses $\Omega(\log(|G|))$ random generators to obtain an expander. Nevertheless, we will show that, when applied correctly, the equivalence between expansion and small bias distribution can be used to build almost-Ramanujan expanders.

**Prior Results on Bias Amplification**

**Scalar amplification** Much of the earlier work has focused on the case of Abelian groups, especially $G = \mathbb{Z}_2^n$, as this has connections to other objects like error-correcting codes.

Rozenman and Wigderson introduced the approach of using expanders for “scalar amplification” via an iterated application of the expander mixing lemma (see Section 6 for details). Alon (in an unpublished email6) introduced the idea of using walks on an (auxiliary) expander graph $X$, whose vertices are identified with elements of $S$. The set $W \subseteq S^t$ is chosen to be the collection of all walks of length $(t - 1)$ on $X$. This technique gives a $\lambda$-biased set $W$, of size $|W| \leq O(|S|/\lambda^{4+o(1)})$ (cf., [TS17]), which is quite good but still sub-optimal ($\lambda^{-4}$ instead of $\lambda^{-2}$).

Ta-Shma [TS17] managed to close the gap almost optimally ($\lambda^{-2-o(1)}$) using the $s$-wide replacement product to derandomize the above amplification. The $s$-wide replacement product of Ben-Aroya and Ta-Shma [BATS08] is a higher-order version of the zig-zag prod-

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6We thank the anonymous reviewer for pointing out this reference.
uct [RVW00]. Using the collection of walks on the s-wide replacement product allows for a much smaller collection $W \subseteq S^1$ with nearly optimal size. This scalar technique was later applied to the more general case of arbitrary Abelian groups by Jalan and Moshkowitz [JM21].

**Operator amplification** To extend Ta-Shma’s approach to non-Abelian groups, it is necessary to work with operator-valued functions, $f : S \rightarrow M_\ell(\mathbb{C})$, as the irreducible representations are no longer of dimension one. To the best of our knowledge, only one general result was known for general groups. Chen, Moore and Russell [CMR13] analyzed the above expander walk construction using a matrix version of the expander mixing lemma. This gives an amplification procedure for Cayley graphs of general groups, but the resulting degree $O(|S| / \lambda^{11})$ to achieve final expansion $\lambda$ is sub-optimal.

**Our Results** We view our main contribution as the identification of appropriate natural linear algebraic extensions to Ta-Shma’s amplification framework [TS17]. This gives an almost-optimal dimension independent generalization of the scalar amplification result to operator-valued functions. Our result sharpens that of Chen, Moore and Russell [CMR13] by reducing the degree from $O(|S| / \lambda^{11})$ to $O(|S| / \lambda^{2+o(1)})$.

**Theorem 1.9** (Operator Amplification (this work)). Let $S$ be a finite set and $\lambda_0 \in (0, 1)$ be a constant. For every $\lambda > 0$, there exists a deterministic polynomial time algorithm to construct $W \subseteq S^1$ of size $|W| \leq O(|S| / \lambda^{2+o(1)})$ such that for every function $f : S \rightarrow M_\ell(\mathbb{C})$ with $\|E_s f(s)\|_{op} \leq \lambda_0$ and $\max_s \|f(s)\|_{op} \leq 1$, we have $\|E_w f(w)\|_{op} \leq \lambda$.

The key extension is a simple and yet extremely useful change in the bias operator ($\Pi_f$) defined by Ta-Shma which is a central object in the analysis of both [TS17] and [JM21]. In both these cases, $f$ is scalar, and they define,

$$\Pi_f : \mathbb{C}[S] \rightarrow \mathbb{C}[S] \text{ where } \Pi_f \cdot s = f(s) \cdot s.$$

However, this approach is not readily generalizable to operators and the view we take is that if $f : S \rightarrow M_\ell(\mathbb{C})$, then, $\Pi_f$ is actually an operator on $\mathbb{C}^\ell \otimes \mathbb{C}[S]$ defined as

$$\Pi_f : \mathbb{C}^\ell \otimes \mathbb{C}[S] \rightarrow \mathbb{C}^\ell \otimes \mathbb{C}[S] \text{ where } \Pi_f (v \otimes s) = f(s) v \otimes s.$$

Clearly, in the Abelian case, we have $\ell = 1$ and this is isomorphic to the setup by Ta-Shma. This generalization is very natural and we show that not only does the older machinery gel well with this, but the proof remains intuitive with the different spaces neatly delineated.

More precisely, we first establish an operator version of the expander walk amplification, and then we derandomize it using (a suitable version of) the s-wide replacement product. Furthermore, since the result does not depend on the dimension, $\ell$, we can use it even for functions $f : S \rightarrow \mathcal{L}(\mathcal{H})$ where $\mathcal{L}(\mathcal{H})$ is the space of bounded linear operators on an arbitrary Hilbert space, $\mathcal{H}$, possibly infinite dimensional. This is useful if the underlying group is not finite but finitely generated by $S$. 

7
1.5 Discussion

The results of this paper have some curious features, which we would like to elaborate on. For most of them, we will use the following "bare bones" description of our main spectral amplification result. Namely, let $S$ be a finite set and $H$ a Hilbert space. Let $f$ be a function mapping elements of $S$ to operators on $H$ of unit norm, such that $\|\mathbb{E}_{s \in S} [f(s)]\|_{\text{op}} \leq \lambda_0$. For any $\lambda > 0$ take $t = c \log(1/\lambda)$ (for appropriate $c$). We extend $f$ from $S$ to $S^t$ by defining $f(s_1, \ldots, s_t) = f(s_1) \cdots f(s_t)$. Clearly, $\|\mathbb{E}_{r \in S^t} [f(r)]\|_{\text{op}} \leq (\|\mathbb{E}_{s \in S} [f(s)]\|_{\text{op}})^t \leq \lambda$. Our main result is an explicit construction of a (pseudorandom) subset $S' \subseteq S^t$, of size only $|S'| = O(|S|/\lambda^2 + o(1))$, with a similar guarantee, namely $\|\mathbb{E}_{s' \in S'} [f(s')]\|_{\text{op}} \leq \lambda$.

Dimension Independence Note that if the operators in $S$ are 1-dimensional, namely scalars, then the existence of a set $S'$ of this size (which is best possible even in this 1-dimensional case) follows directly from the Chernoff bound. Indeed, Ta-Shma’s construction [TS17] may be viewed as derandomizing this result, producing an explicit such $S'$.

One may try to do the same for operators in a higher dimension, say $\ell$, by appealing to the Matrix Chernoff bounds of Ahlswede–Winter [AW02] (see also Tropp [Tro15]). However, these concentration inequalities pay a factor of $\ell$ in the tail bound, resulting in a set $S'$ of size $\Omega(\log(\ell))$. As the dimension $\ell$ is arbitrary (indeed, may be infinite), such a bound is useless.

Thus, our explicit construction has no known probabilistic (or other existential) analog! What is curious is that our dimension-independent analysis follows very closely that of Ta-Shma for 1-dimension, roughly speaking, replacing scalar absolute values by the operator norm in any dimension. We feel that it would be extremely interesting to find a matrix concentration inequality for sampling product sets like $S^t$, which is dimension independent.

Algebraic vs. Combinatorial Expander Constructions Our explicit construction of the pseudorandom set $S'$ above uses expanders obtained from the $s$-wide zig-zag product of [BATS08]. This is a combinatorial construction, a refinement of the original zig-zag product construction of [RVW00]. Nonetheless, it has significant consequences to algebraic expander constructions which use group theory, namely to the expansion of Cayley graphs over the same group. We can take $S$ to be an expanding generating set of a group and $f$ to be some non-trivial irreducible representation $\rho$. Instead of using the $t$-product of elements of $S^t$ to obtain a new amplified generating set $S'$, a much sparser subset is chosen using the $s$-wide zig-zag construction. The analysis of the norm amplification discussed above yields the required expansion bound, in a way that has no dependence on the group or the representation. The flexibility in mapping element of $S$ to operators underlies the versatility of our spectral amplification. It allows us to preserve some of the structure of the expanders whose expansion are being amplified. In this case, both the starting expander and the amplified expander are Cayley graph over the same group.

It is interesting to note that this is a recurring phenomenon. In [ALW01], it was discovered that the zig-zag product may be viewed as a combinatorial generalization of the algebraic semi-direct product of groups. This connection made possible the construction of new expanding Cayley graphs in groups that are far from being simple, e.g., in
[MW04, RSW06]. It is rewarding to see again how new combinatorial constructions, sometimes inferior in certain parameters to some algebraic ones, yield new results in group theory.

**Iterated Pseudorandomness**  Another interesting aspect of our result is the following. Recall that expanders are pseudorandom objects for many purposes. One important purpose is sampling - rather than sampling \( t \) independent random elements in some set \( S \), one may sample \( t \) points along a random walk on an expander on the vertex set \( S \) and a Chernoff type bound still holds (a nontrivial result of [Gil98])—this affords significant savings in the number of random bits spent. For this result, any expander would do. What happens in this paper is an iterated use of expanders as samplers as follows. We first choose a sparse pseudorandom set of \( t \)-walks inside \( S \) using expanders walks. Then, we choose a yet sparser pseudorandom set inside it, again using walks on an additional expander. This repeated use of expanders improves the trade-off between the quality of spectral amplification and the size of the final pseudorandom set to near-optimal. The construction of this iterated selection of walks seems critical, and (as in Ta-Shma’s paper) is chosen to come from the \( s \)-wide zig-zag product of two expanders [BATS08].

**Groups and Expansion**  For us, the most surprising consequence of our results is that “weak” simple groups, especially the symmetric group,\(^7\) can have near-Ramanujan generators. The question of which groups are expanding, and just how expanding they are, is an old quest of group theory. One dichotomy is whether every finite set of generators of the group is expanding (these are “strongly expanding” groups), or if some are and some aren’t (these are “weakly expanding” groups). For the symmetric group, many finite non-expanding generating sets of constant size were long known, and Kassabov’s breakthrough construction [Kas07] designed a constant size expanding generating set. The symmetric group is then a weakly expanding group, while, e.g., simple groups of Lie type (namely, matrix groups) are believed, and in some cases known, to be strongly expanding. Nonetheless, our construction works equally well for all, and we have almost Ramanujan generators for all expanding groups.

**Closeness to the Ramanujan Bound**  As mentioned above, a family of \( d \)-regular graphs is called Ramanujan if its spectral expansion parameter \( \lambda \) is at most \( 2\sqrt{d-1}/d \). This terminology was introduced in the seminal work of Lubotzky, Phillips and Sarnak [LPS88], and it designates the optimal degree versus expansion trade-off that a family of bounded degree expanders can achieve. Several notions of closeness to the Ramanujan bound were investigated, e.g., \((2\sqrt{d-1}+\varepsilon)/d \) (with \( \varepsilon > 0 \) small or vanishing) in [Fri03, MOP20, JMO+22], \( O(1/\sqrt{d}) \) in [ACKM19], \( \text{polylog}(d)/\sqrt{d} \) in [BL06, JMO+22] and more generally \( d^a \cdot 1/d^{1/2} \).

In this work, we obtain a bound of the form \( \lambda \leq O(2^\log(d)^c/d^{1/2}) \) for some constant \( c \in (0, 1) \), which we refer to as an almost Ramanujan bound. Rephrasing in terms of the expansion parameter, we achieve expansion \( \lambda \) with degree \( O(1/\lambda^{2+c'}) \), where \( c = O(1/\log(1/\lambda)) \) for some \( c' \in (0, 1) \). We stress that the nomenclatures almost Ramanujan, near Ramanujan

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\(^7\) See also the groups in [RSW06], which are iterated wreath products of symmetric groups.
and etc, may vary depending on the author. Improving the results in this work to achieve trade-offs even closer to the Ramanujan bound (if possible) is of great interest. We suspect that new ideas may be required to substantially improve the bound to, say, expansion $\lambda$ versus degree $O(\text{polylog}(1/\lambda)/\lambda^2)$.

1.6 Outline

We start in Section 2 by summarizing basic definitions and the notation used throughout the paper. In Section 3, we generalize the simpler construction of Ta-Shma based on expander walks. Apart from serving as a nice warm-up to the more-involved construction, it will be used as a bootstrap for the more involved construction based on $s$-wide replacement product which is the subject of Section 4. Here, we prove the main amplification result (a formal version of Theorem 1.9 above) and instantiate using known constructions and those obtained from Section 3 which establishes Theorem 1.2. Section 5 discusses the permutation amplification trick and formally completes the proof of Theorem 1.1. It also discusses the other applications in more detail. Finally, Section 6 gives an operator version of the expander mixing lemma which improves the analysis of [CMR13].

2 Preliminaries

Let $X = (V, E)$ be an $n$-vertex $d$-regular undirected multigraph for some $d \geq 1$. We denote by $A_X$ the normalized adjacency matrix of $X$, i.e., the adjacency matrix divided by $d$.

Definition 2.1 ($\lambda$-spectral Expander). Let the eigenvalues of the symmetric matrix $A_X$, denoted as $\text{Spec}(A_X)$, be $\{1 = \lambda_1 \geq \cdots \geq \lambda_n\}$ and define $\lambda(X) = \max\{|\lambda_2|, |\lambda_n|\}$. We say that $X$ is a $\lambda$-spectral expander if $\lambda(X) \leq \lambda$.

We denote by $G$ a finite group (except in Section 5.5 where we only need it to be finitely generated). For a multiset $S \subseteq G$, $\text{Cay}(G, S)$ denotes a multigraph with the vertex set being $G$ and edges $\{(g, sg) \mid g \in G, \; s \in S\}$. We will assume throughout that $S = S^{-1}$ and therefore, the graph is undirected.

Definition 2.2 (Explicit graph). A family of graphs $\{X_i\}_{i \in \mathbb{N}}$ is said to be explicit if the adjacency matrix of $X_i$ can be computed deterministically in $\text{poly}(|X_i|)$-time. Moreover, it is said to be strongly explicit if the list of its neighbors of any vertex in $X_i$ can be computed $\text{poly}(\log |X_i|)$-time.

Group Representations In order to study the expansion of a Cayley graph, we will use the notion of a group representation. Weyl’s unitary trick, says that for a large family of groups (which includes all finite groups), every representation can be made unitary and thus, we can restrict to studying these.

\[\text{Additional background on representation theory of finite groups can be found in [SS96]}.\]
Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators\(^9\) on $\mathcal{H}$. We denote by $U_\mathcal{H}$ the unitary group of operators acting on $\mathcal{H}$.

**Definition 2.3** (Group Representation). For a group $G$, a representation is a pair $(\rho, \mathcal{H})$ where $\rho : G \rightarrow U_\mathcal{H}$ is a group homomorphism, i.e., for every $g_1, g_2 \in G$, we have $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$. A representation is irreducible if the only subspaces of $\mathcal{H}$ that are invariant under the action of $\rho(G)$ are the empty space, $\{0\}$, and the entire space, $\mathcal{H}$.

Every group has two special representations, which are,

1. (Trivial representation) – $(\rho, \mathbb{C})$ where for every $g$, $\rho(g) = 1$.

2. ((left) Regular representation) – $(\rho_{\text{reg}}, \mathcal{V}_{\text{reg}})$ where, $\mathcal{V}_{\text{reg}} = \mathbb{C}[G]$ is a vector space with the elements of $G$ being an orthonormal basis, and $\rho_{\text{reg}}(g) : h \mapsto g \cdot h$.

**Fact 2.4.** Let $G$ be a finite group and let $\mathcal{V}_{\text{reg}}$ be the regular representation over $\mathbb{C}$. We have

$$\mathcal{V}_{\text{reg}} \cong \bigoplus_{(\rho,\mathcal{V}_\rho) \in \text{Irrep}(G)} \text{dim}(\rho) \cdot \mathcal{V}_\rho,$$

where $\text{Irrep}(G)$ denotes the set of irreducible representations of $G$.

**Expanders and Biased Distributions** It follows from definitions that the normalized adjacency matrix of $\text{Cay}(G, S)$ is given by $A = \mathbb{E}_{s \sim S} [\rho_{\text{reg}}(s)]$. Moreover, the copy of the trivial representation is the space spanned by the all-ones vector. **Fact 2.4** implies that this can be block diagonalized and therefore,

$$\text{Spec}(A) = \bigcup_{\rho \in \text{Irrep}(G)} \text{Spec} \left( \mathbb{E}_{s \sim S} [\rho(s)] \right),$$

and thus,

$$\lambda(\text{Cay}(G, S)) = \max_{\rho \text{ is non-trivial}} \sup_{h \in \mathcal{H}} \left\| \mathbb{E}_{s \sim S} [\rho(s)] \right\|_{\text{op}}.$$

For any bounded linear operator, $T : \mathcal{H} \rightarrow \mathcal{H}'$, between Hilbert spaces, we have

$$\|T\|_{\text{op}} = \sup_{v \in \mathcal{H} : \|v\| = 1} \|Tv\| = \sup_{v \in \mathcal{H}, \|v\| = 1, w \in \mathcal{H}' : \|w\| = 1} |\langle Tv, w \rangle|,$$

where $\|v\| = \sqrt{\langle v, v \rangle}_\mathcal{H}$ and $\|w\| = \sqrt{\langle w, w \rangle}_{\mathcal{H}'}$. Given this equivalence, we will find it convenient to work with the operator norm version referred to as bias in the literature [CMR13].

**Definition 2.5** (Biased Set on $G$). Let $\epsilon \in (0, 1)$. We say that a multiset $S$ of elements of a group $G$ is $\epsilon$-biased if for every non-trivial irreducible representation $\rho$, we have $\|\mathbb{E}_{s \sim S} [\rho(s)]\|_{\text{op}} \leq \epsilon$. We sometimes use the shorthand bias($S$) $\leq \epsilon$, where bias($S$) $= \lambda(\text{Cay}(G, S))$.

Irreducible representations of Abelian groups, called characters, have dimension 1. Thus, this definition coincides with the usual one of $\epsilon$-biased distribution fooling non-trivial characters [NN90, AGHP92]. These pseudorandom distributions were introduced in the pioneering work of Naor and Naor where several applications to derandomization were given [NN90].

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\(^9\)For most applications, one can think of $\mathcal{H} = \mathbb{C}^n$ for some $n$, and $\mathcal{L}(\mathcal{H}) = M_n(\mathbb{C})$, the space of $n \times n$ complex matrices. However, we will need the generality in Section 5.5.
3 Operator Bias Reduction via Expander Walks

In this section, we establish a new operator analog of the expander walk–based bias amplification procedure for scalars. An analysis of this scalar amplification was given by Ta-Shma in [TS17]. We prove an operator analog that can amplify from any bias (Theorem 3.6) which implies the main result below (Theorem 3.1), that amplifies from constant bias.

**Theorem 3.1** (Operator Amplification via Expander Walks). Let $X$ be a $\lambda(X)$-spectral expander, and let $W_t$ be the collection of walks obtained from walks of length $t$ on $X$. Then for any operator valued function $f$ such that $\|E_{x \in V_X} f(x)\|_{op} \leq \lambda_0$ and $\max_{x \in V_X} \|f(x)\|_{op} \leq 1$, we have

$$\left\| \mathbb{E}_{(x_0, \ldots, x_t) \in W_t} [f(x_t) \cdots f(x_0)] \right\|_{op} \leq (2\lambda(X) + \lambda_0)^{1/2} \lambda_0$$

---

**Notation**

Since we deal with various vector spaces and graphs, we will find it useful to establish some convenient notation. While we recall these in the relevant section, the following is a summary for ready reference.

- The main multigraphs we study will be $X$ and $Y$ with vertices $V_X$, $V_Y$ and normalized adjacency operators $A_X, A_Y$.
- We denote vertices of $X, Y$ by $x, y$ and an ordered tuple of vertices by $\vec{x} = (x_0, \ldots, x_t)$.
- We use $u, v, w$ to denote arbitrary vectors in $\mathcal{H}$ and $x, y$ for basis vectors of $\mathbb{C}[V_X], \mathbb{C}[V_Y]$ where $\mathbb{C}[V_X]$ is the complex vector space with the elements of $V_X$ being a orthonormal basis.
- The tensored vector spaces have an induced inner product. For $\mathcal{X}_\mathcal{H} := \mathcal{H} \otimes \mathbb{C}[V_X]$, it is $\langle v \otimes x, w \otimes x' \rangle = (v, w)_\mathcal{H} \langle x, x' \rangle$. Similarly, we have one on $\mathcal{X}_\mathcal{Y} := \mathcal{H} \otimes \mathbb{C}[V_Y]$.
- Orthogonal decomposition: $\mathcal{X}_\mathcal{H} = \mathcal{X}^\|_\mathcal{H} \oplus \mathcal{X}^\bot_\mathcal{H}$ where $\mathcal{X}^\|_\mathcal{H} := \text{span}\{v \otimes \vec{1} \mid v \in \mathcal{H}\}$. Here, $\vec{1}$ denotes the un-normalized all-ones vector. Similarly, $\mathcal{X}_\mathcal{Y} = \mathcal{X}^\|_\mathcal{H} \oplus \mathcal{X}^\bot_\mathcal{H}$, where $\mathcal{X}^\|_\mathcal{H} := \text{span}\{z \otimes \vec{1} \mid z \in \mathcal{X}_\mathcal{H}\}$.
- The operator $\hat{A}$ denotes the extension of operator $A$ to a tensor product of spaces where it acts as identity on the other spaces. For example, $A_X$ acts on $\mathbb{C}[V_X]$ and its extension to $\mathcal{X}_\mathcal{H}$ is $A_X = 1_\mathcal{H} \otimes A_X \otimes 1_Y$. However, if we were working on $\mathcal{X}_\mathcal{Y}$, it would be $\hat{A}_X = 1_\mathcal{H} \otimes A_X \otimes 1_Y$ instead\(^\text{10}\).
- Given an operator valued function $f : V_X \to \mathcal{L}(\mathcal{H})$, the generalized bias operator is defined on the basis as\(^\text{11}\),

$$\Pi_f : \mathcal{X}_\mathcal{H} \to \mathcal{X}_\mathcal{H}, \ v \otimes x \mapsto f(x)v \otimes x.$$
We remark that a precursor of these techniques, in the simpler setting of Abelian groups appears in the pioneering work of Naor and Naor introducing \( \epsilon \)-biased distributions over the group \( \mathbb{Z}_2^m \) using expanders [NN90].

This simpler amplification of Theorem 3.1 will be crucially used to bootstrap the almost-optimal amplification. Moreover, it yields a construction of expanding Cayley graphs close to any desired size, which will be required later.

This bias reduction procedure uses walks on an auxiliary expander graph. Here, we only use its expansion property (as opposed to later when we rely on its structure for the \( s \)-wide construction). With this it is already possible to obtain \( 1/\lambda^{4+o(1)} \) dependence on the final degree of an \( \lambda \)-expander.

**Theorem 3.2.** Let \( S \subseteq G \) such that \( \lambda(\text{Cay}(G, S)) = \lambda_0 < 1 \). For every \( \lambda \in (0, 1) \) and constant \( \beta \in (0, 1) \), we can find \( S' \subseteq G \) in time \( \text{poly}(|S|, 1/\lambda_0, 1/\lambda) \) such that \( \lambda(\text{Cay}(G, S')) \leq \lambda \) and \( |S'| = O_{\lambda_0}(|S|/\lambda^{4+\beta}) \).

**Notation Recall**  Let \( S \) be any finite set and let \( X \) be a graph on the vertex set \( V_X = S \) with \( A_X \) being its normalized adjacency matrix. Let \( \mathcal{H} \) be a complex Hilbert space and \( \mathcal{L}(\mathcal{H}) \) be the (bounded) operators on \( \mathcal{H} \); an important example will be \( \mathcal{L}(\mathcal{H}) = M_d(\mathbb{C}) \). For any operator valued function, \( f : S \rightarrow \mathcal{L}(\mathcal{H}) \), we define the generalized bias operator as

\[
\Pi_f : \mathcal{H} \otimes \mathbb{C}[V_X] \mapsto \mathcal{H} \otimes \mathbb{C}[V_X], \quad \Pi_f(v \otimes x) = f(x) v \otimes x.
\]

In the scalar case, since \( \mathcal{H} = \mathbb{C} \), earlier works [TS17, JM21] used the implicit identification \( \mathbb{C} \otimes \mathbb{C}[V_X] \cong \mathbb{C}[V_X] \) and defined \( \Pi_f \) as a diagonal matrix. This identification is no longer suitable when \( f \) is operator valued in dimension \( > 1 \). However, a simple yet crucial observation is that merely decoupling the spaces allows us to collect the terms as we proceed along the walk.

### 3.1 Operator Norm Decay

**Lemma 3.3.** Let \( \mathcal{W}_t \subseteq V_X^{t+1} \) be the collection of all length \( t \) walks on the graph \( X \) and we define \( \hat{A}_X = I_{\mathcal{H}} \otimes A_X \). Then, we have

\[
\left\| \mathbb{E}_{(x_0, \ldots, x_t) \in \mathcal{W}_t} [f(x_t) \cdots f(x_0)] \right\|_{\text{op}} \leq \left\| \Pi_f (\hat{A}_X \Pi_f)^t \right\|_{\text{op}} \leq \left\| (\hat{A}_X \Pi_f)^2 \right\|_{\text{op}}^{t/2}.
\]

**Proof:**

\[
\Pi_f (\hat{A}_X \Pi_f)^t \mathbb{E}_{x \in V_X} [v \otimes x] = \mathbb{E}_{(x_1, \ldots, x_t) \in \mathcal{W}_t} [f(x_t) \cdots f(x_0)] v \otimes x_t.
\]

This can be shown easily via induction on \( t \), and we refer to Lemma 4.7 for a formal proof of a more general statement. We use projection and lifting maps to move between the spaces \( \mathcal{X}_\mathcal{H} \) and \( \mathcal{H} \). Define \( P_\mathcal{H} : \mathcal{X}_\mathcal{H} \rightarrow \mathcal{H} \) and \( L_\mathcal{H} : \mathcal{H} \rightarrow \mathcal{X}_\mathcal{H} \), as,

\[
P_\mathcal{H}(v \otimes x) = w, \quad L_\mathcal{H}(w) = \mathbb{E}_{x \in V_X} [v \otimes x] = \frac{1}{|V_X|} v \otimes \bar{1}.
\]
From the definition, \(\|L_H(v)\| = \|v\| \|\overline{f}\|_{a_X} = \|v\|_{\sqrt{\mathcal{V}_X}}\) and thus, \(\|L_H\|_{\text{op}} = 1/\sqrt{|\mathcal{V}_X|}\). We can use Cauchy-Schwarz to get that \(\|P_H\|_{\text{op}} = \sqrt{|\mathcal{V}_X|}\). Now, we put this together to obtain a simple expression on the quantity we need to bound

\[
\left\| \mathbb{E}_{(x_0, \ldots, x_t) \in W_t} [f(x_t) \cdots f(x_0)] \right\|_{\text{op}} = \sup_{\|v\| = 1} \left\| \mathbb{E}_{(x_0, \ldots, x_t) \in W_t} [f(x_t) \cdots f(x_0)] v \otimes x_t \right\|_2
\]

\[
= \sup_{\|v\| = 1} \left\| P_H \left( \mathbb{E}_{(x_0, \ldots, x_t) \in W_t} [f(x_t) \cdots f(x_0)] v \otimes x_t \right) \right\|_2
\]

\[
= \sup_{\|v\| = 1} \left\| P_H \Pi_f (A_X \Pi_t)^t \mathbb{E}_{x \in V_X} [v \otimes x] \right\|_2
\]

\[
= \sup_{\|v\| = 1} \left\| P_H \Pi_f (A_X \Pi_t)^t L_H v \right\|_2
\]

\[
\leq \Pi_f (A_X \Pi_t)^t \|P_H\|_{\text{op}} \|L_H\|_{\text{op}}
\]

\[
\leq \Pi_f (A_X \Pi_t)^t \|\Pi\|_{\text{op}}.
\]

The last inequality follows from submultiplicativity of the operator norm and the observation that \(\|\Pi\|_{\text{op}} = \|f\|_{\infty} \leq 1\). 

Now that we have reduced the problem to studying the operator norm, we will study how the norm decays as we take walks. We use the decomposition, \(\mathcal{X}_H = \mathcal{X}_H^\| \oplus \mathcal{X}_H^\perp\) where \(\mathcal{X}_H^\| := \text{span}\{v \otimes \overline{1} \mid v \in \mathcal{H}\}\). The decay comes from two sources. For \(z \in \mathcal{X}_H^\perp\), we get a decay by \(\lambda(X)\) by the definition of \(X\) being an expander. 

**Claim 3.4.** For \(z \in \mathcal{X}_H^\|\), we have

\[
\left\| (\Pi_f z) \right\|_2 \leq \mathbb{E}_{x \in V_X} [f(x)] \cdot \|z\|_2.
\]

**Proof:** From definition of \(\mathcal{X}_H^\|\), we can assume that \(z = u \otimes \overline{1}\). Computing we have,

\[
\left\| (\Pi_f (u \otimes \overline{1})) \right\|_2 = \sup_{w \in \mathcal{H}} \frac{\left\langle w \otimes \overline{1}, \Pi_f (u \otimes \overline{1}) \right\rangle}{\|w\|_{a_X} = 1}
\]

\[
= \sup_{w \in \mathcal{H}} \frac{\left\langle w \otimes \overline{1}, \Pi_f (u \otimes \sum_{x \in V_X} x) \right\rangle}{\|w\|_{a_X} = 1}
\]

\[
= \sup_{w \in \mathcal{H}} \frac{\left\langle w \otimes \overline{1}, \sum_{x \in V_X} (f(x) u \otimes x) \right\rangle}{\|w\|_{a_X} = 1}
\]

\[
= \sup_{w \in \mathcal{H}} \frac{\left\langle w, f(x) u \right\rangle}{\|w\|_{a_X} = 1} \cdot \left\langle \overline{1}, x \right\rangle,
\]
To analyze \( \| \hat{A}_X \Pi_f \hat{A}_j \|_\text{op} \), let \( z \in \mathcal{X}_H \) be a unit vector which is decomposed as \( z = z^+ + z^- \).

We have,

\[
\| \hat{A}_X \Pi_f \hat{A}_j \|_2 \leq \| \hat{A}_X \Pi_f \hat{A}_j \Pi_f \hat{A}_j \|_2 \leq \hat{A}_X \| (\Pi_f z^+) + (\Pi_f z^-) \|_2 = \| (\Pi_f z^+) \|_2 \leq \lambda_0. \quad \text{ (By Claim 3.4)}
\]

Thus, \( \| \hat{A}_j \Pi_f \hat{A}_j \|_\text{op} \leq \lambda_0 \). Recall that \( \| \Pi_f \|_\text{op} \leq 1 \) since \( \max_x \| f(x) \|_\text{op} \leq 1 \), and we also have \( \| E \|_\text{op} \cdot \| A_j \|_\text{op} \leq 1 \). Then,

\[
\| \hat{A}_X \Pi_f \hat{A}_j \|_\text{op} \leq (1 - \lambda)^2 \lambda_0 + 2 \lambda(1 - \lambda) + \lambda^2 = (1 - \lambda)^2 \lambda_0 + 1 - (1 - \lambda)^2,
\]

concluding the proof. □

\[\text{This is the source of loss of a factor of 2 in the exponent (which leads to degree O(|S|/\lambda^{4+o(1)}) rather than the desired degree of O(|S|/\lambda^{2+o(1)}) we will later achieve. Note that the same loss occurs in the original zig-zag analysis of [RVW00], which was later remedied by the s-wise zig-zag of [BATS08].}\]
Theorem 3.1 trivializes if $2\lambda(X) + \lambda_0 \geq 1$. Nonetheless, we now show that amplification does occur under much weaker conditions, namely, whenever $\|E_{x \in V_X}[f(x)]\|_{op} < 1$ and the auxiliary graph $X$ has expansion $\lambda(X) < 1$. This regime of bias amplification was instrumental in the breakthrough SL = L result of Reingold [Rei05].

**Theorem 3.6 (Operator Amplification via Expander Walks (strengthening of Theorem 3.1)).**

Let $X$ be a $\lambda(X)$-spectral expander and let $W_t$ be the collection of walks obtained from walks of length $t$ on $X$. Then for any operator valued function $f$ such that $\|E_{x \in V_X}[f(x)]\|_{op} \leq \lambda_0$ and $\max_{x \in V_X} \|f(x)\|_{op} \leq 1$, we have

$$\left\| \mathbb{E}_{(x_0, \ldots, x_t) \in W_t} [f(x_0) \cdots f(x_t)] \right\|_{op} \leq \left[1 - (1 - \lambda(X))(1 - \lambda_0) \right]^{t/2}.$$ 

**Proof:** Follows by combining Lemma 3.3 and Lemma 3.5.

This establishes that expander walks can be used to derandomize powers of an operator, itself given by an average of bounded operators, in the general case. In this derandomization, we still have an exponential norm decay, but we only “pay additional randomness” proportional to the degree of the auxiliary expander regardless of the number of operators.

### 3.2 Cayley Graphs and the Construction of Amplified Biased Sets

In this section, we will prove the following,

**Theorem 3.2.** Let $S \subseteq G$ such that $\lambda(Cay(G, S)) = \lambda_0 < 1$. For every $\lambda \in (0, 1)$ and constant $\beta \in (0, 1)$, we can find $S' \subseteq G$ in time $\text{poly}(|S|, 1/\lambda_0, 1/\lambda)$ such that $\lambda(Cay(G, S')) \leq \lambda$ and $|S'| = O_{\lambda_0}(|S|/\lambda^{4+\beta})$.

Towards this, we first formalize the connection between bias of a special subset of a group and the operator norm of a certain operator. The subset is obtained by taking random walks over an expander graph as mentioned above. We then proceed to bound this operator norm. Finally, we instantiate our construction with an explicit expander graph due to [Alo21].

The particular case of $S \subseteq G$ (for some group $G$) and the function $f$ being a representation $\rho$ on $H$ leads to the amplification of biased sets. We will construct a new multiset $S' \subseteq G$ such if $\|E_{s \in S}[\rho(s)]\|_{op} \leq \lambda_0$, then we have $\|E_{s \sim S'}[\rho(s)]\|_{op} \leq \lambda \ll \lambda_0$. Note here that the construction of $S'$ is agnostic to $\rho$, and thus we can reduce the bias of all irreducible representations simultaneously! Assume that we have a graph $X$ on the vertex set $S$. For $s \in S$, we have $f(s) = \rho(s)$ in this case. Let

$$S' = \{ s_t s_{t-1} \cdots s_0 \mid (s_0, s_1, \ldots, s_t) \in W_t \},$$

which will be our new amplified biased set. Using the homomorphism property of $\rho$, we have the following simplification

$$\mathbb{E}_{w=(s_0, \ldots, s_t) \in W_t} [f(s_t) \cdots f(s_0)] = \mathbb{E}_{(s_0, \ldots, s_t) \in W_t} [\rho(s_t) \cdots \rho(s_0)] = \mathbb{E}_{s' \in S'} [\rho(s')] , \quad (2)$$

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We now establish the key amplification lemma.

**Lemma 3.8.** Let \( S \subseteq G \) such that bias\((S) = \lambda_0 < 1 \) and let \( \epsilon_0 \) be a constant such that \((1 + 2\epsilon_0)\lambda_0 < 1 \). Then, for any \( \lambda > 0 \), we can explicitly compute \( S' \) such that bias\((S') \leq \lambda \) and \( |S'| = O_{\epsilon_0, \lambda_0} \left( \frac{|S|}{\lambda^{4 + 4 \log \lambda}} \right) \) where \( \delta(\lambda_0, \epsilon_0) = \frac{\log(3 + 6\epsilon_0) + \log(1/\epsilon_0)}{\log(1/\lambda_0)} \).

**Proof:** Pick a constant \( \epsilon_0 \) such that \( \lambda_1 := (1 + 2\epsilon_0)\lambda_0 < 1 \) and use Theorem 3.7 to obtain an explicit \((m|S|, d, \epsilon_0\lambda_0)\)-graph \( X \) where \( d \leq \frac{9}{\epsilon_0\lambda_0} \). Let \( S_1 \) be the multiset consisting of \( m \) copies of \( S \). The bias remains the same and now, \( |V(X)| = |S_1| \). We construct \( S' \) by multiplying elements of t-length walks on \( X \) where \( t = \lceil 2(1 + \log \lambda_0(\lambda)) \rceil \). The size of \( S' \) is

\[
|S'| = (m|S|) \cdot d^t = m \cdot |S| \cdot \left( \frac{3}{\epsilon_0\lambda_0} \right)^{4 + 4 \log \lambda_1 / \lambda_0}
\]

\[
= m \left( \frac{3}{\epsilon_0\lambda_0} \right)^4 |S| \cdot \lambda \cdot \left( 1 + \frac{\log(3 + 6\epsilon_0)}{\log(1/\lambda_0)} \right)
\]

\[
\leq O_{\epsilon_0, \lambda_0} (|S|) \cdot \lambda.
\]

Let \( \rho \) be any irreducible representation of \( G \). From Eq. (3) and Theorem 3.1, we get,

\[
\left\| \mathbb{E}_{s_0 \cdots s_t \in S'} [\rho(s_t \cdots s_0)] \right\|_{op} \leq (2\lambda(X) + \text{bias}(S))^{1/2 - 1} \leq (\lambda_1)^{1/2 - 1} \leq \lambda.
\]

Using the amplification above, we now derive our first simplified explicit construction.

**Theorem 3.2.** Let \( S \subseteq G \) such that \( \lambda(\text{Cay}(G, S)) = \lambda_0 < 1 \). For every \( \lambda \in (0, 1) \) and constant \( \beta \in (0, 1) \), we can find \( S' \subseteq G \) in time \( \text{poly}(|S|, 1/\lambda_0, 1/\lambda) \) such that \( \lambda(\text{Cay}(G, S')) \leq \lambda \) and \( |S'| = O_{\lambda_0} (|S|/\lambda^{4 + \beta}) \).
We cannot use existing results like the recent work of Alon [Alo21] or the construction in [TS17]. This is because Alon’s construction does not have a Cayley graph structure (which our proof utilizes). On the other hand, the construction in [TS17] is a Cayley graph based on [LPS88], but it only guarantees a graph of size $O(n)$ rather than $n + o(n)$.

Recall that $\text{SL}_2(p)$ is the group of $2 \times 2$ invertible matrices over $\mathbb{F}_p$ with determinant 1. We obtain a base generating set for $\text{SL}_2(p)$ via the following result.

**Theorem 3.9** ([Lub11]). There exists an absolute constant $\lambda_0 < 1$ such that for every $p > 17$, there exists an explicit generating set $S$ (of constant size independent of $p$) for $\text{SL}_2(p)$, such that $\lambda(\text{Cay}(\text{SL}_2(p), S)) \leq \lambda_0$.

**Theorem 3.10** ([Che10]). For every $n \geq 2^{3 \cdot 2^{15}}$, there exists a prime in $[n, n + 4n^{2/3}]$.

**Corollary 3.11.** For any $n > 2^{9 \cdot 2^{15}}$, $\lambda > 0$, there is a deterministic polynomial time algorithm to construct an $(n', d, \lambda)$-graph $\text{Cay}(\text{SL}_2(p), S)$, where $n' = n + O(n^{8/9})$ and $d = O(\lambda^{-4.1})$.

**Proof:** Find a prime $p \in [n^{1/3} + 1, n^{1/3} + O(n^{2/9})]$, which exists due to Theorem 3.10, via brute-force search. Since, $\text{SL}_2(p)$ is a group of order $(p^2 - 1)p$, we have $n \leq |\text{SL}_2(p)| \leq n + O(n^{8/9})$. We use the constant-sized generating set $S$ from Theorem 3.9 and amplify using Theorem 3.2.

### 4 Operator Bias Reduction via the s-wide Replacement Walk

We have seen in Section 3 that bias reduction via random walks on an expander $X$ is suboptimal (by a factor of 2 in the exponent). We will derandomize this random walk construction to achieve an almost optimal bias reduction. The idea is to introduce a new graph $Y$ which has a much smaller degree, and to “simulate” a random walk on $X$ via a walk on $Y$. This is realized by a higher-order version of the zig-zag product [RVW00] called the $s$-wide replacement product defined by Ben-Aroya and Ta-Shma [BATS08] (see Definition 4.5).

This section establishes our key technical result which states that given any initial operator valued function of constant bias $< 1$, we amplify the bias in an almost optimal way. This generalizes the analysis of Ta-Shma [TS17] from scalar valued functions to operator valued functions.
Theorem 4.1 (Operator Generalization of Theorem 24 [TS17]). Fix integers \( t \geq s \geq 1 \). Let \( X \) be any \( d_1 \)-regular graph (with \( d_1 \) a power of 2), and \( Y \) be any \( d_2 \)-regular Cayley graph on \( \mathbb{F}_2^{s \log d_1} \). Let \( sW_t \) be the collection of length \( t \) walks on the \( s \)-wide replacement product of \( X \) and \( Y \). Let \( \mathcal{H} \) be a Hilbert space. For any operator valued function \( f : V_X \rightarrow \mathcal{L}(\mathcal{H}) \), satisfying \( \max_{x \in V_X} \| f(x) \|_{\text{op}} \leq 1 \) and \( \| \mathbb{E}_{x \in V_X} [f(x)] \|_{\text{op}} := \lambda_0 \leq \lambda(Y)^2 - 2\lambda(X) \), we have

\[
\left\| \frac{1}{|sW_t|} \sum_{(x_0, \ldots, x_t) \in sW_t} f(x_0) \cdots f(x_t) \right\|_{\text{op}} \leq (\lambda(Y)^s + s \cdot \lambda(Y)^{s-1} + s^2 \cdot \lambda(Y)^{s-3})^{t/s} \leq O_s (\lambda(Y))^{(1-o_s(1))t}.
\]

Furthermore, the size of the collection is \( |sW_t| = |X| \cdot d_1^s \cdot d_2^t \).

Remark 4.2. Note that there is an inherent trade-off between the spectral bound amplification (on the operator norm), and the degree bound (on the number of walks), which causes the suboptimality in how close this technique lets us approach the Ramanujan bound. As in [TS17], the \( o_s(1) \) term we obtain from the bound above is \( (1/\log(1/\lambda))^c \) for some \( c > 0 \) (see Theorem 4.19 for the precise computation).

Section Outline. In Section 4.1, we recall the \( s \)-wide replacement product and describe random walks on it. Then, in Section 4.2, we formalize the distributions we work with and reprove the result that if \( Y \) is a Cayley graph over any product group of appropriate size, then it enables the transfer of pseudorandomness from \( Y \) to \( X \). The key generalization to operator valued functions is established in Lemma 4.7 which is identical in spirit to Eq. (1). In Section 4.3, we then finish the amplification analysis in a manner similar to [TS17]. In Section 4.4, we provide details about instantiating the setup by explicitly constructing the graphs we need.

4.1 The \( s \)-wide Replacement Product and its Walks

Let \( X \) be a \( d_1 \)-regular graph. For each \( x \in V_X \) and \( j \in [d_1] \), let \( x[j] \) be the \( j \)-th neighbor of \( x \).

Definition 4.3 (Locally Invertible Rotation Map). \( X \) admits a locally invertible rotation map if there exists a bijection \( \varphi : [d_1] \rightarrow [d_1] \) such that for every \( (x, j) \in V_X \times [d_1] \),

\[
\text{if } x' = x[j], \text{ then, } x'[\varphi(j)] = x.
\]

Example 4.4 (Cayley Graphs are Locally Invertible). Let \( G \) be a group and \( A \subseteq G \) where the set \( A \) is closed under inversion. Label the neighbors of vertices in \( \text{Cay}(G, A) \), by elements of \( A \) such that \( g[a] = a \cdot g \). Then, \( \text{Cay}(G, A) \) is locally invertible as the map \( \varphi : A \rightarrow A \) defined as \( \varphi(a) = a^{-1} \) clearly satisfies the criteria,

\[
\text{if } g' = g[a] = a \cdot g, \text{ then, } g'[\varphi(a)] = a^{-1} \cdot g' = g,
\]

for every \( g \in G, a \in A \).

We now define the \( s \)-wide replacement product which is a generalization of standard replacement product of graphs which can be seen as the special case when \( s = 1 \).

Definition 4.5 (s-wide Replacement Product). Suppose we are given the following:
- A $d_1$-regular graph $X$ with a bijection $\varphi : [d_1] \to [d_1]$ which defines a locally invertible rotation map.

- A $d_2$-regular graph $Y$ on the vertex set $[d_1]^s$.

And we define:

- For $i \in \{0, 1, \ldots, s - 1\}$, we define $\text{Rot}_i : V_X \times V_Y \to V_X \times V_Y$ such that,

  $\text{Rot}_i((x, (a_0, \ldots, a_{s-1}))) := (x[a_i], (a_0, \ldots, a_{i-1}, \varphi(a_i), a_{i+1}, \ldots, a_{s-1}))$,

  for every $x \in V_X$ and $(a_0, \ldots, a_{s-1}) \in V_Y = [d_1]^s$. (Note that the $Y$ component of the rotation map depends only on a vertex’s $Y$ component, not its $X$ component.)

- Denote by $X_i$, the operator on $\mathbb{C}[V_X \times V_Y]$ which acts on the natural basis via the permutation $\text{Rot}_i$, and let $A_Y$ be the normalized random walk operator of $Y$.

Then $t$ steps of the $s$-wide replacement product are given by the operator

$$X_{t-1 \mod s} \hat{A}_Y \cdots X_1 \mod s \hat{A}_Y X_0 \mod s \hat{A}_Y.$$

Observe that a walk on the $s$-wide replacement product yields a walk on the outer graph $X$ by recording the $X$ component after each step of the walk. Since a walk is completely determined by its intra-cloud steps, the number of $t$-step walks on the $s$-wide replacement product is,

$$|V_X| \cdot |V_Y| \cdot d_1^2 = n \cdot d_1^s \cdot d_2^1 \ll n \cdot d_1^1,$$

which therefore gives us a very sparse subset of all $t$-walks on $X$. Thus the $s$-wide replacement product will be used to simulate random walks on $X$ while requiring a reduced amount of randomness (as we shall see this simulation is only possible under special conditions, namely, when we are uniformly distributed on each cloud).

### 4.2 The Collection of Derandomized Walks

We now describe the distribution obtained by the walks on the $s$-wide replacement product using the language of operators.

**Definition 4.6 (Operators and Distributions).** Given a tuple of random walk operators$^{13}$ $B = (B_0, \cdots, B_{t-1})$ on $\mathbb{C}[V_X] \otimes \mathbb{C}[V_Y]$ and a starting vertex $x_0 \in V_X$, we can define a distribution induced by the walk using these operators. More precisely, $\mathcal{D}(B, x_0)$ is the distribution on $(V_X \times V_Y)^{t+1}$ such that for every $1 \leq \ell \leq t$,

$$\left(B_{t-1} \cdots B_0\right) \left(x_0 \otimes \frac{1}{|V_Y|}\right) = E_{(\hat{x}, \hat{y}) \sim \mathcal{D}(B, x_0)} x_\ell \otimes y_\ell. \quad (4)$$

We typically suppress $x_0$ as it will not matter and denote $\mathcal{D}_X(B)$, and $\mathcal{D}_Y(B)$ to specify the projections to $V_X$, and $V_Y$ respectively.

$^{13}$Markov chain operators on $V_X \times V_Y$. 

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The next lemma is a generalization of Eq. (1) which we need for the s-wide replacement walk. This can also be specialized to prove Eq. (1) by letting \( Y \) be a graph with one vertex (and thus \( \mathcal{X}_H \cong \mathcal{X}_Y \)). Recall that \( f_t \) \((v \otimes x \otimes y) = f(x) v \otimes x \otimes y \).

**Lemma 4.7 (Operator Generalization).** For any tuple of random walk operators \( \mathbf{B} \), any operator valued \( f \), and any \( v \in \mathcal{H} \), \( x_0 \in \mathcal{V}_X \), we have

\[
\left( \mathbf{B}_t \hat{\mathcal{F}} \cdots \mathbf{B}_0 \hat{\mathcal{F}} \right) (\nu \otimes x_0 \otimes \frac{1}{|\mathcal{V}_Y|} \hat{1}) = \frac{1}{|\mathcal{V}_Y|} \mathcal{E} \left[ f(x_{t-1}) \cdots f(x_0) \nu \otimes x_t \otimes y_t \right].
\]

**Proof:** We prove the computation via induction on \( t \). The base case is when \( t = 1 \)

\[
\left( \mathbf{B}_0 \hat{\mathcal{F}} \right) (\nu \otimes x_0 \otimes \frac{1}{|\mathcal{V}_Y|} \hat{1}) = \mathbf{B}_0 \left( f(x_0) \nu \otimes x_0 \otimes \frac{1}{|\mathcal{V}_Y|} \hat{1} \right) = \mathcal{E}_{(\tilde{x}, \tilde{y}) \sim \mathcal{D}(\mathbf{B})} [f(x_0) \nu \otimes x_1 \otimes y_1] \quad \text{(Using Eq. (4) for } \ell = 1)\]

Let \( y_0 = \frac{1}{|\mathcal{V}_Y|} \hat{1} \) and assume the statement holds for \( t - 1 \). Then,

\[
\left( \mathbf{B}_{t-1} \hat{\mathcal{F}} \cdots \mathbf{B}_0 \hat{\mathcal{F}} \right) (\nu \otimes x_0 \otimes y_0) = \mathbf{B}_{t-1} \hat{\mathcal{F}} \cdots \mathbf{B}_0 \hat{\mathcal{F}} (\nu \otimes x_0 \otimes y_0) = \mathbf{B}_{t-1} \hat{\mathcal{F}} \mathcal{E}_{(\tilde{x}, \tilde{y}) \sim \mathcal{D}(\mathbf{B})} [f(x_{t-2}) \cdots f(x_0) \nu \otimes x_{t-1} \otimes y_{t-1}] = \mathbf{B}_{t-1} \mathcal{E}_{(\tilde{x}, \tilde{y}) \sim \mathcal{D}(\mathbf{B})} [f(x_{t-1}) \cdots f(x_0) \nu \otimes x_t \otimes y_t].
\]

The second equality uses the inductive hypothesis, and the last two equalities use Eq. (4) for \( \ell = t - 1 \) and \( \ell = t \) respectively. \( \blacksquare \)

Using **Definition 4.6**, we further define the operators for the distributions we wish to study.

**Uniform Distribution** Let us first capture, using this notation, the uniform distribution on walks on \( X \) starting from \( x_0 \in \mathcal{V}_X \). We define \( B_i \) where for each \( i \), \( B_i = A_X \otimes I_Y \) for every \( i \). Then, for any \( \ell \), \( (A_X \otimes I_Y)^\ell = A_X^\ell \otimes I_Y \). Therefore, we obtain that \( \mathcal{D}(B_i) \) is the t-step random walk distribution on \( X \) i.e., \( x_i \sim A_X^ix_0 \).

**The s-wide Distribution** This is the distribution obtained by the s-wide walks as described in the earlier section. For \( 0 \leq a \leq b < s \), we define

\[
\mathbf{B}[a, b] = (X_a \hat{A}_Y, X_{a+1} \hat{A}_Y, \cdots, X_b \hat{A}_Y).
\]

We can view this random walk as occurring in two steps. The first being picking an initial vertex \( y_0 \in Y \) and then, picking the sequence of neighbors according to which we will perform the walk in \( Y \). To formalize this, let \( \hat{A}_Y = (1/d_2) \sum_{j=1}^{d_2} P_j \) where \( P_j \) are permutation
matrices and let \( J = (j_0, \cdots, j_{b-a}) \in [d_2]^{b-a+1} \). For a fixed sequence of permutations \( J \), the conditional distribution (conditioned on \( J \)) is defined by,

\[
B[a, b, J] = (X_a \, \hat{P}_{j_0}, X_{a+1} \, \hat{P}_{j_1}, \cdots, X_b \, \hat{P}_{j_{b-a}})
\]

We would like these two distributions, i.e., the uniform distribution and the

\[
\text{distribution , to be the same and a graph } Y \text{ is said to be } \textit{compatible} \text{ with respect to } (X, \varphi), \text{ if for any fixed sequence, } J, \text{ of a walk of length } \ell \leq s, \text{ the distribution obtained on } X \text{ via the uniform sampling of } y_0, \text{ is the same as the usual } \ell \text{-length walk on } X \text{ from any fixed initial vertex, } x_0. \text{ Thus, the randomness of sampling a vertex from } Y \text{ is effectively } \textit{transferred} \text{ to a random walk on } X.
\]

\textbf{Definition 4.8 (Compatible).} A graph \( Y \) is \textit{compatible} with respect to \((X, \varphi)\) if for every \( 0 \leq a \leq b < s, \, J \in [d_2]^{b-a+1} \) and \( x_0 \in V_X \), we have\(^{14}\)

\[
D_X(B[a, b, J], x_0) = D_X(B_{U_1}, x_0) = A_X^{b-a+1} x_0.
\]

This \textit{compatible} property is the same as the 0-pseudorandom property in [TS17]. We rename it as it is more of a structural compatibility property than a pseudorandomness one. We now prove, for the sake of completeness, that Cayley graphs are compatible with every locally invertible graph.

\textbf{Lemma 4.9 ([TS17, Lemma 29]).} Let \( Y = \text{Cay}(G^s, T) \) where \(|G| = d_1\). Then, \( Y \) is compatible with respect to any \( X, \varphi \).

\textbf{Proof:} Let \( x_0 \in V_X \) be arbitrary and let \( y_0 = (r_1, \cdots, r_s) \sim G^s \) be sampled uniformly. Since \( Y \) is a Cayley graph, the sequence of permutations, \( J \), is equivalent to a sequence of generators \((t^1, \cdots, t^s) \in T^s \), and the permutation is group multiplication, \( y \mapsto t^k \cdot y \).

For each \( 1 \leq i \leq s \), let \( t^i \cdot y_{i-1} = (r_i^1, \cdots, r_i^s) \). The walk evolves as follows,

\[
\begin{align*}
  x_i &= X_i(x_{i-1}, t^i \cdot y_{i-1}) = x_{i-1}[r_i^1] \\
  y_i &= \varphi(t^i \cdot y_{i-1}) = (r_i^1, \cdots, r_i^s, \varphi(r_i^1), r_i^{s+1}, \cdots, r_i^{s+k})
\end{align*}
\]

Compatibility requires that \( x_i \sim A_X^{i} x_0 \), which is inductively implied if for every \( i, r_i^k \) is uniform over \( G \) and independent of \( r_j^k \) for \( j \neq i \).

This claim is true initially as \( y_0 \) is uniform over \( G^s \). Assume it is true for \( y_{i-1} \). Since, \( t^i, \varphi \) are fixed permutations, they do not affect the uniformity of the distribution of \( r_k^i \) for any \( k \). Since, \( \varphi \) acts only on the \( i \text{th} \) component, the independence of \( r_k^i \) and \( r_j^k \), guaranteed by inductive claim, implies the independence of \( r_k^i \) and \( \varphi(r_i^1) \).

For a fixed \( x_0 \) and \( J \), we say that \( y_0 \) gives rise to a sequence \( \tilde{z} = (z_1, \cdots, z_t) \) if \( z_i = x_i \) where \( x_i \) is as defined in the above proof.

\textbf{Observation 4.10.} Fix \( x_0 \in V_X \) and a sequence \( J \). For any \( \tilde{z} = (z_1, \cdots, z_k) \), the number of \( y_0 \) that gives rise to \( \tilde{z} \) is \( d_2^{a-k} \).

\textbf{Proof:} From the proof of \textbf{Lemma 4.9}, we see that enforcing \( z_i = x_i \) for each \( i \) starting from \( i = 1 \), forces exactly \( r_1 \) to be fixed. Thus, the remaining \( (r_{k+1}, \cdots, r_s) \) are free. \( \blacksquare \)

\(^{14}\)It is important to note that \( D_Y(B[a, b, J]) \neq D_Y(B_{U_1}) \).
4.3 The $s$-wide Operator Norm Decay

We are now ready to establish the key technical lemma in the analysis of the $s$-wide replacement which is an operator-valued generalization of the scalar version of present in [TS17].

**Lemma 4.11 (Simulation Lemma (generalization of Lemma 26 from [TS17])).** Let $0 \leq s_1 \leq s_2 < s$. For every pair of vectors $z, z' \in \mathcal{X}_H$, we have,

$$\left\langle \left( \sum_{i=s_1}^{s_2} (\hat{X}_i \hat{A}_Y \Pi_t) \left( \frac{1}{|V_Y|} \hat{1} \right), z' \otimes \hat{1} \right) \right\rangle = \left\langle \left( \hat{A}_X \Pi_t \right)^{s_2-s_1+1} z, z' \right\rangle.$$  

**Proof:** Let $z = \sum_x v_x \otimes x$ and $z' = \sum_x w_x \otimes x$. Since the expression is bilinear, it suffices to prove the equation for $v \otimes x_0$, $w \otimes x'$ for an arbitrary pair $(x_0, x')$. Let $t = s_2 - s_1 + 1$.

$$\left\langle \left( \sum_{i=s_1}^{s_2} (\hat{X}_i \hat{A}_Y \Pi_t) \right), v \otimes x_0 \otimes \frac{1}{|V_Y|} \hat{1} \right\rangle = \mathbb{E}_{(\tilde{x}, \tilde{y}) \sim \mathcal{D}(B[s_1, s_2], J)} \left[ \sum_{x \in \mathcal{X}_t} f(x_{t-1}) \cdots f(x_0) v \otimes x_t \otimes y_t \right]$$

$$= \sum_{x \in \mathcal{X}_t} \mathbb{E}_{y_0 \sim V_Y} \left[ f(\tilde{x}) v \otimes x_t \otimes y_t \right] \mathbb{I}[y_0 \text{ gives rise to } \tilde{x}],$$

$$= \frac{d^s}{d^t} \sum_{x \in \mathcal{X}_t} f(\tilde{x}) v \otimes x_t \otimes y_t,$$

where $f(\tilde{x}) = f(x_{t-1}) \cdots f(x_0)$. The last equality uses Observation 4.10. Therefore, the conditioning on $y$ does not change the distribution $\mathcal{D}_X$ and when we take inner products, we obtain,

$$\left\langle \left( \sum_{i=s_1}^{s_2} (\hat{X}_i \hat{A}_Y \Pi_t) \right), v \otimes x_0 \otimes \frac{1}{|V_Y|} \hat{1} \right\rangle = \frac{d^s}{d^t} \sum_{x \in \mathcal{X}_t} \langle x_t, x' \rangle \left\langle f(x_{t-1}) \cdots f(x_0) v, w \right\rangle$$

$$= \mathbb{E}_{\tilde{x} \sim \mathcal{D}_X(B[s_1, s_2], J)} \left[ \langle x_t, x' \rangle \left\langle f(x_{t-1}) \cdots f(x_0) v, w \right\rangle \right].$$

We now use\(^{15}\) Lemma 4.7 for $B_{\mathcal{U}}$ and take inner product to get,

$$\left\langle \left( \hat{A}_X \Pi_t \right)^{s_2-s_1+1} (v \otimes x_0), w \otimes x' \right\rangle = \mathbb{E}_{\tilde{x} \sim \mathcal{D}_X(B_{\mathcal{U}})} \left[ \langle x_t, x' \rangle \langle f(x_{t-1}) \cdots f(x_0) v, w \rangle \right].$$

From Lemma 4.9, we know that $Y$ is compatible and thus, $\mathcal{D}_X(B[s_1, s_2], J) = \mathcal{D}_X(B_{\mathcal{U}})$. Thus, the right hand side (and therefore left hand sides) of these two equations above are equal.  

\(^{15}\)As we only want to work with the space $\mathcal{X}_H$ here, we can assume in the application of the lemma that $|V_Y| = 1$. Else, one could directly apply Eq. (1) and use the observation that $\mathcal{D}_X(B_{\mathcal{U}})$ is the same as the random walk distribution on $X$.  

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The s-step Decay. Just like the amplification in Section 3 was analyzed by studying the norm decay obtained in every two steps (cf., Lemma 3.8), this amplification via the s-wide walks will be analyzed by bounding the norm decay for steps of length s using Lemma 4.11 similarly to [BATS08, TS17]. We will use the shorthand \( L_i := x_i f_i \lambda Y \).

The goal is to bound \( \|L_{s-1} \cdots L_0\|_{\text{op}} \) which controls the bias of the set obtained by s-long, s-wide walks (cf., proof of Eq. (2)). Equivalently, we will bound \( \langle (\prod_{i=1}^s L_i) v_0, w_s \rangle \) for any unit vectors\(^{16}\) \( v_0, w_s \in \mathcal{X}^{\mathcal{Y}_{H}} \). We will use the orthogonal decomposition,

\[
\mathcal{X} \mathcal{Y}_H := \mathcal{X}_H \otimes \mathbb{C}[\mathcal{V}_Y] = \mathcal{X}_H^{\|} \oplus \mathcal{X}_H^{\perp}
\]

where \( \mathcal{X}_H^{\|} := \text{span}\{ z \otimes \bar{1} \mid z \in \mathcal{X}_H \} \).

For \( i \geq 1 \), we inductively define the vectors \( v_i, w_i, z_i \) and bound their norms\(^{17}\),

\[
\|v_i\| \leq \lambda(Y)^i, \quad \|z_i\| \leq \lambda(Y)^{i-1}, \quad \|w_i\| \leq \lambda(Y)^i.
\]

The following lemma follows readily from a calculation and we omit its proof.

**Lemma 4.12.** For any \( v_0, w_s \) and \( 0 \leq r \leq s-2 \) we have,

\[
L_{s-1} \cdots L_0 v_0 = v_s + \sum_{i=0}^{s-1} L_{s-1} \cdots L_i v_i \]

\[
L_{s-1}^* w_s = w_{s-1} + z_{s-1}^i
\]

\[
L_r^* \cdots L_{s-1}^* w_s = w_r + z_r^i + \sum_{i=r+1}^{s-1} L_r^* \cdots L_{i-1}^* z_i^i
\]

**Theorem 4.13** (Operator Generalization of Theorem 24 [TS17]). Let \( X \) be any \( d_1 \)-regular graph and \( Y \) be a Cayley graph on \( \mathbb{F}_2^{d_1 \log d_1} \). Let \( W_t \) be the collection of t-length s-wide walks, on the s-wide replacement product on \( X \) and \( Y \). For any operator valued function \( f \) on \( \mathcal{V}_X \), such that \( \max_{x \in \mathcal{V}_X} \|f(x)\|_{\text{op}} \leq 1 \) and \( \|\mathbb{E}_{x \in \mathcal{V}_X} f(x)\|_{\text{op}} := \lambda_0 \leq \lambda(Y)^2 - 2\lambda(X) \),

\[
\left\| \mathbb{E}_{(x_0, \ldots, x_t) \in W_t} [f(x_t) \cdots f(x_0)] \right\|_{\text{op}} \leq (\lambda(Y)^s + s \cdot \lambda(Y)^{s-1} + s^2 \cdot \lambda(Y)^{s-3})^{t/s}.
\]

**Proof:** Using Lemma 4.7, we can repeat the proof of Eq. (2) to see that,

\[
\left\| \mathbb{E}_{(x_0, \ldots, x_t) \in W_t} [f(x_t) \cdots f(x_0)] \right\|_{\text{op}} \leq \|L_t \cdots L_0\|_{\text{op}} \leq \|L_{s-1} \cdots L_0\|_{\text{op}}^{t/s}.
\]

\[
\langle L_{s-1} \cdots L_0 v_0, w_s \rangle = \langle v_s, w_0 \rangle + \sum_{r=0}^{s-1} \langle L_{s-1} \cdots L_r v_r, w_s \rangle
\]

\[
= \langle v_s, w_s \rangle + \sum_{r=0}^{s-1} \left( \langle v_r, L_r^* \cdots L_{s-1}^* w_s \rangle \right)
\]

\(^{16}\)Here we deviate from our notation and use \( v, w \) for vectors in \( \mathcal{X}^{\mathcal{Y}_H} \).

\(^{17}\)By definition \( \|v_i\| \leq \|\lambda Y v_{i-1} \| \leq \lambda(Y) \|v_{i-1}\| \). The computation is similar for \( w \) and \( z \).
\[
\begin{align*}
\langle v_s, w_s \rangle & = \langle v_s, w_s \rangle + \sum_{i=0}^{s-1} \langle v_r^i, w_r^i + z_r^i \rangle + \sum_{r=0}^{s-2} \sum_{i=r+1}^{s-1} \langle v_r^i, L_r^i \cdots L_{i-1}^i z_r^i \rangle \\
& = \langle v_s, w_s \rangle + \sum_{i=0}^{s-1} \langle v_r^i, z_r^i \rangle + \sum_{r=0}^{s-2} \sum_{i=r+1}^{s-1} \langle v_r^i, L_r^i \cdots L_{i-1}^i z_r^i \rangle.
\end{align*}
\]

The last step uses \(\langle v_r^i, w_r^i \rangle = \langle \hat{A}_Y v_r^i, z_r^i \rangle = 0\). Using Eq. (6), we get \(\langle v_r^i, z_r^i \rangle \leq \lambda(Y)^s-1\). To bound the last term, we finally use Lemma 4.11. Let \(v_r^i = v_r' \otimes \hat{I}\), and \(z_r^i = z_r' \otimes \frac{1}{v_r'} \hat{I}\). Then,

\[
\begin{align*}
\langle v_r^i, L_r^i \cdots L_{i-1}^i z_r^i \rangle &= \langle v_r', (\hat{A}_X \Pi_l)^{i-r} z_r' \rangle \\
& \leq \| (\hat{A}_X \Pi_l)^{i-r} \|_{\text{op}} \| z_r' \| \| v_r' \| \\
& \leq \lambda(Y)^2 l^{-1} \lambda(Y)^{r-s+1} \lambda(Y)^{s-1} \lambda(Y)^{s-3},
\end{align*}
\]

where the penultimate inequality uses Theorem 3.1 and plugs in the assumption that \(2 \lambda(X)+\|B_{x \in \mathcal{V}_X} f(x) \|_{\text{op}} \leq \lambda(Y)^2\). Substituting this back in our expression above gives us the result.

\[
\square
\]

### 4.4 Instantiating the \(s\)-wide Replacement Product

The goal of this section is to prove the following result that implies Theorem 1.2.

**Theorem 4.14** (Almost Ramanujan Expanders I). Let \(\text{Cay}(G, S)\) be \(\lambda_0\)-expander with constant \(\lambda_0 \in (0, 1)\). For every function \(\beta(\lambda) > 0\), and for any \(\lambda > 0\), sufficiently small such that

\[
\frac{32}{\beta(\lambda)} \leq \left( \frac{\log(1/\lambda)}{4 \log \log(1/\lambda)} \right)^{1/3},
\]

there exists a deterministic polynomial time algorithm to construct \(S'\) such that \(\text{Cay}(G, S')\) is a \(\lambda\)-expander with degree \(|S'| = O_{\lambda_0}(|S|/\lambda^{2+\beta})\).

Furthermore, each element in \(S'\) is the product of \(O(\log(1/\lambda))\) elements of \(S\).

**Overview**

We will explicitly construct the graphs \(X\) and \(Y\) as needed for the \(s\)-wide product. Once we obtain the graphs, we identify the vertices of \(X\), i.e., \(V_X\) with the initial generating set \(S\), or perhaps a slightly modified set \(S'\), obtained by duplicating and adding identities. The final set is obtained by multiplying elements along each \((t-1)\)-length walk on the \(s\)-wide replacement product of \(X\) and \(Y\). The way we choose parameters and objects for it borrows heavily from Ta-Shma’s arguments in [TS17]. The analysis follows an analogous structure of [JQST20] for binary codes, which in turn builds on the original analysis of Ta-Shma [TS17]. We will also use the following result from that work,

**Lemma 4.15** (Based on Lemma 6 [TS17]). For every \(m \in \mathbb{N}^+\) and \(d = 2^{2k} \leq 2^m\), there exists a fully explicit set \(A \subseteq \mathbb{Z}_2^m\) such that the graph \(\text{Cay}(\mathbb{Z}_2^m, A)\) is a \((2^m, d, \lambda = \frac{m}{\sqrt{d}})\)-expander graph.
The construction  Given as input \( n := |S|, \lambda \) and a slowly growing function \( \beta(\lambda) \), we construct the graphs \( X, Y \) as described below with the parameters \( (s, d_1, d_2, \lambda_1, \lambda_2) \) which are all functions of \( \lambda \) and \( \beta(\lambda) \). These are summarized in a table below for reference\(^{18}\). Recall that a \((n, d, \lambda)\)-graph has \( n \) vertices, is \( d \)-regular, and has the second largest singular value of its normalized adjacency matrix at most \( \lambda \).

- Outer Graph — The outer graph \( X \) will be an \((n', d_1, \lambda_1)\)-graph which is a Cayley graph on \( \text{SL}_2(p) \) constructed using Corollary 3.11 with \((n, \lambda_1)\) as input. By Example 4.4, we obtain a locally invertible graph on \( n' \approx n \). The condition on the size is satisfied as \( n = 2|S|d_2^5 \geq d_2^5 \geq 2^{17} \) by the assumption that \( s \geq 2^{10} \). Moreover, the degree is

\[
\frac{c}{\lambda_1^4 + 1} \leq \frac{cd_2^5}{b_2^2} \leq d_2^5.
\]

We increase its degree to \( d_2^5 \) by taking multiple copies of the generating set which does not change bias\(^{19}\). Thus, we obtain a \((n', d_1, \lambda_1)\)-graph where \( n' = n + O(n^{8/9}) \).

- Inner Graph — The inner graph \( Y \) will be a \((d_1^5, d_2, \lambda_2)\)-graph which is a Cayley graph on \( \mathbb{Z}_m^2 \) and therefore by Lemma 4.4, it is compatible. For this, we use the construction of Alon et al. [AGHP92] and the analysis of Ta-Shma (Lemma 4.15).

We summarize the construction and the choice of parameters \( n', d_1, d_2, \lambda_1, \lambda_2 \) and \( s \), which are chosen as follows for a fixed \( \beta(\lambda) \) here -

| Parameter | Value |
|-----------|-------|
| \( s \) | is the smallest power of 2 such that \( \frac{32}{\beta} \leq s \leq \left( \frac{\log(1/\lambda)}{4 \log \log(1/\lambda)} \right)^{1/3} \) |
| Every other parameter | is a function of \( s \). |
| \( Y : (n_2, d_2, \lambda_2), \) \( n_2 = d_2^{5s}, \ d_2 = s^{4s}, \ \lambda_2 \leq \frac{b_2}{\sqrt{d_2}}, \ b_2 = 5s \log d_2 \) |
| \( X : (n', d_1, \lambda_1), \) \( n' \approx n = O(|S| d_2^5), \ d_1 = d_2^5, \ \lambda_1 = \frac{\lambda_2^2}{10} \) |
| \( t : \) smallest integer such that \( (\lambda_2)^{(1-5\alpha)(1-\alpha)(t-1)} \leq \lambda_1 \) ; where \( \alpha = 1/s \) |

**Note:** We assume that \( s \geq 2^{10} \) since otherwise \( \lambda \) is a constant and we can use Theorem 3.2. Now, we mention the central claim that we need from our choice of parameters.

**Claim 4.16.** The selection of the parameters above implies the following bounds on \( t \),

(i) \( t - 1 \geq 2s^2 \)

(ii) \( (d_2)^{(t-1)} \leq \lambda^{-2(1+10\alpha)} \),

**Proof:** Proof of (i) : Using \( d_2 = s^{4s} \) and the upper bound on \( s \), we have

\[
\left( \frac{1}{\lambda_2} \right)^{(1-5\alpha)(1-\alpha)2s^2} \leq \left( \frac{1}{\lambda_2} \right)^{2s^2} \leq \left( \frac{d_2}{b_2^2} \right)^{s^2} = (d_2)^{s^2} = s^{4s^3}
\]

\(^{18}\)The choice of parameters is similar but not identical to Ta-Shma’s choice.

\(^{19}\)This is wasteful, but we do it to ensure that \( V(Y) = d_1^5 \) and that \( d_1^5 \) is a power of 2.
\(= 2^{4s^3 \log_2(s)} \leq 2^{\log_2(1/\lambda)} = \frac{1}{\lambda}.\)

Hence, \(\lambda \cdot (1-5\alpha)(1-\alpha)s/\alpha \geq \lambda\) and thus \(t - 1\) must be at least \(2s^2\). Also observe that,

\[
\lambda^{(1-5\alpha)(1-\alpha)^2}(t-1) = \lambda^{(1-5\alpha)(1-\alpha)(t-2)} \left( \frac{(1-\alpha)}{1/(t-1)} \right) \\
\geq \lambda^{(1-5\alpha)(1-\alpha)(t-2)} \quad \text{(t - 1 \geq s = 1/\alpha)} \\
\geq \lambda \quad \text{(From the choice of minimal t)}
\]  

(7)  

(8)  

(9)

Since \(b_2 = 5s \log_2(d_2) = 20s^2 \log_2(s) \leq s^4\) (recall that \(s = 1/\alpha \geq 2^{10}\)),

\[
d_2^{1-2\alpha} = \frac{d_2}{d_2^{2\alpha}} = \frac{d_2}{s^5} \leq \frac{d_2}{b_2} = \frac{1}{\lambda^2}.
\]

We obtain claim (ii) by the following computation,

\[
(d_2)^{(t-1)} \leq \lambda^{(t-1)} \frac{1}{2^{2\alpha}} \\
\leq \lambda^{(1-2\alpha)(1-5\alpha)(t-1)/2} \\
\leq \lambda^{-(1+10\alpha)}.
\]  

(Using Eq. (9))

\[\Box\]

**Lemma 4.17.** The number of walks of length \(t - 1\) on the \(s\)-wide replacement product of \(X\) and \(Y\) is \(O(|S|/\lambda^{2+\beta})\).

**Proof:** Since each step of the walk has \(d_2\) options, the number of walks is

\[
|V(X)||V(Y)| \cdot d_2^{(t-1)} = n' \cdot d_2^{(t-1)+5s} = \Theta \left( |S| \cdot d_2^{(t-1)+5s+5} \right) \\
= O \left( |S| \cdot d_2^{(1+5\alpha)(t-1)} \right).
\]

which from **Claim 4.16** (ii), implies a size of

\[
O \left( |S| \cdot d_2^{(1+5\alpha)(t-1)} \right) = O \left( \frac{|S|}{\lambda^{2+10\alpha}} \right) = O \left( \frac{|S|}{\lambda^{2+32\alpha}} \right) = O \left( \frac{|S|}{\lambda^{2+\beta}} \right).
\]  

\[\Box\]

Before we prove the main result, we need the following simple observation that will be used to construct a modified \((\epsilon + o(1))\)-biased set starting from an \(\epsilon\)-biased set, \(S\). This is needed because the graph obtained from **Corollary 3.11** does not have size exactly \(|S|\) but is only guaranteed to be at most \((1 + o(1))|S|\).

**Lemma 4.18.** Let \(S\) be an \(\epsilon\)-biased set of a group \(G\). And let \(S'\) be obtained by adding \(\Theta(|S|)\) many identity elements. Then, \(S'\) is an \((\epsilon + \Theta)\)-biased set.
Proof: Denote by $e$ the identity element of $G$. Let $\rho$ be any non-trivial irreducible representation of a group $G$. From the computation we have,

$$\|\mathbb{E}_{s \in S'} \rho(s)\|_{\text{op}} = \frac{1}{1 + \theta} \left( \|\mathbb{E}_{s \in S} \rho(s) + \theta \cdot \mathbb{E}_{s \in S \setminus S'} \rho(1)\|_{\text{op}} \right)$$

\[
\leq \|\mathbb{E}_{s \in S} \rho(s)\|_{\text{op}} + \theta
\]

\[
Theorem 4.19 \text{ (Almost Ramanujan Expanders I).} \text{ Let } \text{Cay}(G, S) \text{ be } \lambda_0\text{-expander with constant } \lambda_0 \in (0, 1). \text{ For every function } \beta(\lambda) > 0, \text{ and for any } \lambda > 0, \text{ sufficiently small such that}

\[
\frac{32}{\beta(\lambda)} \leq \left( \frac{\log(1/\lambda)}{4 \log \log(1/\lambda)} \right)^{1/3},
\]

there exists a deterministic polynomial time algorithm to construct $S'$ such that $\text{Cay}(G, S')$ is a $\lambda$-expander with degree $|S'| = \tilde{O}_{\lambda_0}(|S|/\lambda^{2+\beta})$.

Furthermore, each element in $S'$ is the product of $O(\log(1/\lambda))$ elements of $S$.

Proof: We can assume that $s \geq 2^{10}$ since otherwise $\lambda$ is a constant and we can just use Theorem 3.2.

Initial Boost We first boost the expansion from $\lambda_0$ to $1/d_2 \leq \lambda_2^2/3$. Using Theorem 3.2 (with its parameter $\beta$ equal to 1), we can find a new set of generators, $S_1$, such that $\text{Cay}(G, S_1)$ is $1/d_2$-spectral expander and $|S_1| = O(|S| d_5^5)$. Moreover, we also know that each element in $S_1$ is a multiple of at most $\log((d_2/\lambda)^2)$ elements in $S$. We add multiple copies of the entire set to make the size $|S| d_5^2$.

The s-wide walk Obtain an $(n', d_1, \lambda_1)$ Cayley graph $X$ from Corollary 3.11 as explained before. We add $n' - n = O(n^8/9)$ copies of the identity to $S_1$ to obtain $S_2$. By Lemma 4.18 and the assumption that $s \geq 2^{10}$, $S_2$ is a $\lambda_2^2/3 + O(n^{-1/6}) \leq 2\lambda_2^2/3$-biased set. We denote by $S'$ the final set of generators obtained by $t$ steps of the $s$-wide replacement product of $X$ and $Y$. By definition, each element in $S'$ is a product of $t$ elements in $S_2$ which has the same elements as $S_1$. Thus, each element in $S'$ is a product of at most

$$O(t \log(d_2)) \leq O((1 + 10\alpha) \log(1/\lambda)) \quad \text{(Using Claim 4.16 [ii])}
\leq O(\log(1/\lambda)) \quad \text{(By the assumption that } \alpha \leq 1/128)$$

elements of $S$. The only thing that remains is to prove expansion of $\text{Cay}(G, S')$. We pick any irreducible representation $\rho$ and apply Theorem 4.13 to the function $\rho$ on $S_2 \leftrightarrow V(X)$. The condition that $2\lambda(X) + \|\mathbb{E}_{g \in S_2} [\rho(g)]\|_{\text{op}} \leq \lambda(Y)^2$ translates to $\lambda_1 \leq \lambda_2^2/6$ which is satisfied by our choice of $\lambda_1$. Thus, the final expansion is given by,

$$\|\mathbb{E}_{g \in S'} [\rho(g)]\|_{\text{op}} := (\lambda_2^2 + s \cdot \lambda_2^{2-1} + s^2 \cdot \lambda_2^{3-3})^{(1-t)/s}$$

\[
\leq (3s^2 \lambda_2^{3-3})^{(1-t)/s-1}
\]
\[
\leq (\lambda_2^{s-1})^{(t-1-s)/s} \\
\leq \lambda_2^{(1-5/(s(1-s/(t-1))})(t-1)} \\
\leq \lambda_2^{(1-5\alpha)(1-\alpha)(t-1)} \\
= \lambda_2^{(1-5\alpha)(1-\alpha)(t-1)} \leq \lambda, \\
\]
(Using Claim 4.16 [ii])
(From the choice of \(t\)).

5 Some Applications

Our operator amplification leads to almost optimal explicit constructions of many pseudorandom objects (from existing suboptimal ones): transforming arbitrary regular expander graphs into almost-Ramanujan expanders (Section 5.2), quantum expanders (Section 5.3), monotone expanders (Section 5.4), to generating sets with improved (average) Kazhdan constants (Section 5.5) and to dimension expanders (Section 5.6). These pseudorandom objects embody various notions of expansion.

**Permutation Amplification**  The key to these applications is observing that the adjacency matrix of an arbitrary graph and that of a monotone expander can be written as a sum of permutation matrices which can be interpreted as \(P_\sigma = \rho_{\text{def}}(\sigma)\) for the defining (or natural) representation \(\rho_{\text{def}}\). We plug in the collection of these permutations \(\{\sigma\}\) in our amplification machinery to obtain almost optimal spectral expanders and monotone expanders.

**Almost Ramanujan Expanders for the Symmetric Group**  Constructing constant size expanding generating sets for the symmetric group was quite challenging (even non-explicitly). In a breakthrough work [Kas07], Kassabov provided the first family of such expanding generators which was also explicit. However, this family was not close to the Ramanujan bound and no such generating set was known. Theorem 1.2 lets us amplify Kassabov’s generating set to one close to optimum bound showing that the symmetric group has explicit almost Ramanujan Cayley expanders. The same obviously holds for every expanding group.

**Quantum Expanders**  A quantum expander is a generalization of an expander graph having many applications in quantum information theory [AS04, BASTS08, Has07b, Has07a, HH09, AHL14]. Harrow [Har07] showed that Cayley graphs can be used to construct quantum expanders inheriting the expansion of the starting Cayley graph. However, the construction is only explicit if the group admits an efficient quantum Fourier transform (QFT). Since we can now obtain almost Ramanujan Cayley graphs for the symmetric group which has a known efficient QFT [Bea97], we obtain the first explicit almost Ramanujan quantum expanders.

**Improving the Kazhdan Constant**  The Kazhdan constant \(K(G, S)\) of a finitely generated group \(G\), with respect to a generating set \(S\), is a quantitative version of Property (T) which
has been used to construct explicit expanders (e.g., Margulis [Mar88]). We show that this can be amplified by considering a slightly different version called the average Kazhdan constant which directly relates to the bias of the set $S$. This is interesting as typically the bound on the Kazhdan constant is used to construct expanders but here we construct expanding generating sets to improve the constant! The improved constants and the generating sets have algorithmic implications and we mention two of them.

- **Dimension expanders** - Lubotzky and Zelmanov [LZ08] showed that the image of a generating set of a group under an irreducible representation gives a dimension expander and its expansion is controlled by its Kazhdan constant.

- **Product replacement algorithm** - uses random walks on $k$-tuples of groups elements. Lubotzky and Pak [LP00] showed that the mixing time of the algorithm relates to the Kazhdan constant of certain lattice groups like $SL_n(\mathbb{Z})$, assuming Property (T). This crucial assumption was proven in complete generality recently by Kaluba, Kielak and Nowak [KKN21]. In particular, we have a mixing time bound of $\frac{4 \log |G|}{\lambda(G,S)^2}$.

Using our amplified generating set (Corollary 5.15), we can improve both these results.

### 5.1 Permutation Amplification

The *defining representation* - $(\rho_{\text{def}}(\sigma), \mathbb{C}^n)$ for $\text{Sym}_n$ is defined as the representation that maps a permutation to the matrix defining it. More formally, $\rho_{\text{def}}(\sigma)e_i = e_{\sigma(i)}$ for every unit basis vector $e_i$ of $\mathbb{C}^n$. It is a fact that $\mathcal{V}_{\text{def}} = \mathcal{V}_{\text{triv}} \oplus \mathcal{V}_{\text{standard}}$ where $\mathcal{V}_{\text{standard}}$ is an irreducible non-trivial representation. Note that if we are given a set $\{P_1, \cdots, P_r\}$ of permutation matrices acting on $\mathbb{C}^n$, we can identify a set $S = \{\sigma_1, \cdots, \sigma_r\} \subseteq \text{Sym}_n$ such that $\rho_{\text{def}}(\sigma_i) = P_i$.

**Corollary 5.1 (Permutation Amplification).** Let $P = \{P_1, \cdots, P_r\}$ be a collection of permutation matrices such that $\lambda(\mathbb{E}_{n-1}[P_1]) \leq \lambda_0$. Then, for any $\lambda \in (0,1)$, we can explicitly construct a collection $P'$ such that

1. $\lambda(\mathbb{E}_{M-P}[M]) \leq \lambda$,
2. $|P'| \leq O\left(|P|/\lambda^{2+\omega(1)}\right)$ and
3. each $P'_i \in P'$ is a product of at most $O_{\lambda_0}(\log(1/\lambda))$ many matrices from $P$.

**Proof:** Let $P_i = \sigma_i$. Applying *Theorem 4.13* to the set $S = \{\sigma_i\}$ we get a larger set of permutations, $S'$ of the form $\sigma' = \sigma_{i_1} \cdots \sigma_{i_k}$ where $k = O_{\lambda_0}(\log(1/\lambda))$. By the decomposition of the defining representation, we have that

$$\text{Spec}\left(\mathbb{E}_{M-P'}[M]\right) = \text{Spec}\left(\mathbb{E}_{\sigma'\sim S'}[\rho_{\text{def}}(\sigma')]\right)$$

$$= \{1\} \cup \text{Spec}\left(\mathbb{E}_{\sigma'\sim S'}[\rho_{\text{standard}}(\sigma')]\right).$$

---

$^{20}$[KKN21] prove that $\text{Aut}(F_n)$, the automorphism group of the free group generated by $n$ elements, has Property (T). This implies Property (T) for quotients of $\text{Aut}(F_n)$, which includes $SL_n(\mathbb{Z})$. 

30
where the 1 corresponds to the eigenvalue from the trivial representation. Since the operator amplification reduces the bias of every non-trivial irreducible representation, it also does so for $\nu_{\text{standard}}$. ■

5.2 Arbitrary Expanders via Permutation Amplification

We can make any family of bounded degree expander graphs into an almost Ramanujan family while preserving their adjacency structure. First, we recall König’s theorem that says that the adjacency matrix of a $d$-regular graph can be expressed in terms of permutation matrices.

**Theorem 5.2 (König).** Let $A_X$ be normalized adjacency matrix of a $d$-regular $n$-vertex simple graph $X$. Then, there exists $d$ permutation matrices $P_1, \ldots, P_d \in \mathbb{R}^{n \times n}$ such that

$$A_X = \frac{1}{d} \sum_{j=1}^{d} P_j.$$

It is also efficient to find permutation matrices as above.

**Claim 5.3.** The permutations in Theorem 5.2 can be found in time $\text{poly}(n)$.

**Proof:** We view $A_X$ as encoding the adjacency relation of a bipartite graph with vertex bipartition $(A = V(X), B = V(X))$. This bipartite graph is $d$-regular so it has at least one perfect matching $M$, which can be found in $\text{poly}(n)$ time. We remove this matching $M$ obtaining a $(d-1)$-regular graph and we repeat till the resulting graph is empty. ■

Our general transformation into an almost Ramanujan bound follows by using Claim 5.3 to obtain an initial set of permutation matrices which are amplified using Corollary 5.1.

**Theorem 5.4 (Main I (Formal version of Theorem 1.1)).** Let $\{X_i\}_{i \in \mathbb{N}}$ be a family of $d_0$-regular $\lambda_0$-expanders with constant $\lambda_0 < 1$. For any $\lambda \in (0,1)$ and any expander $X_i$, we can deterministically compute a $d$-regular $\lambda$-expander $X'_i$ with $d = O_{\lambda_0}(d_0/\lambda^{2+o(1)})$ in time $\text{poly}(|V(X_i)|)$. Moreover, the construction is local in the sense that edges in $X'_i$ correspond to short walks in $X_i$. More precisely, if the adjacency matrix of $X_i$ is

$$A_{X_i} = \frac{1}{d_0} \sum_{j=1}^{d_0} P_j,$$

where $P_1, \ldots, P_{d_0}$ are permutation matrices, then the adjacency matrix of $X'_i$ is

$$A_{X'_i} = \frac{1}{d} \sum_{j=1}^{d} P'_j,$$

where each $P'_j$ is the product of at most $k = O_{\lambda_0}(\log(1/\lambda))$ permutation matrices among $P_1, \ldots, P_{d_0}$. 31
5.3 Explicit Almost Ramanujan Quantum Expanders

Quantum expanders were defined in [AS04, BASTS08, Has07a] and have found many applications in quantum information theory. For instance, they can be used in the construction of designs and gates sets [HH09], in quantum statistical zero knowledge (QSZK) [BASTS08], in detecting EPR pairs [AHL+14] and in the study of entanglement [Has07b].

Roughly speaking, a quantum expander is a generalization of an expander graph (see Definition 5.5 for precise details). While a usual degree-\(d\) expander graph \(X = (V, E)\) is given by \(d\) permutation matrices acting on a vector space \(C[V]\), a quantum expander is given by \(d\) (suitable) linear operators acting on quantum states (i.e., PSD matrices of trace 1). The normalized adjacency matrix of a \(\lambda\)-expander shrinks the \(\ell_2\)-norm of vectors orthogonal the all ones function by a factor of \(\lambda\). Similarly, a quantum expander shrinks the Frobenius norm of PSD matrices orthogonal \(21\) to the identity matrix (the quantum analogue of the all ones function) by a factor of \(\lambda\) (the quantum expansion parameter).

Definition 5.5 (Quantum Expander [AHL+14]). The (super) operator \(\Phi : C^{N \times N} \to C^{N \times N}\) is an \((N, d, \lambda)\) quantum expander if

- “Degree” – The operator \(\Phi\) can be expressed as a sum of \(d\) linear operators as follows, \(\Phi(\rho) = \sum_{i=1}^{d} B_i \rho B_i^\dagger\) where \(\sum_{i=1}^{d} B_i^\dagger B_i = I_N\).
- “Expansion” – The second largest eigenvalue \(23\) of \(\Phi\) as a linear map is \(\leq \lambda\).

In [Has07c], Hastings showed that the Ramanujan bound also applies to quantum expanders and that \(d\) random unitaries get arbitrarily close to the bound. However, such a construction cannot be efficiently implemented and thus used in applications like [AHL+14] which rely on existing explicit constructions (e.g., [BASTS08, Har07]) that are far from the Ramanujan bound and thus give sub-optimal results.

Harrow [Har07] proved that one can construct a quantum expander using an expander Cayley graph over a group for which efficient Quantum Fourier Transform (QFT) is known [Bea97].

Theorem 5.6 (Harrow [Har07]). Let \(G\) be a group and \(S \subseteq G\) be a multiset such that \(\text{Cay}(G, S)\) is a \(\lambda\)-spectral expander. Let \(V^\mu\) be an irreducible representation of \(G\) of dimension \(N\). Then, there exists an \((N, |S|, \lambda)\)-quantum expander. Furthermore, if the group \(G\) admits an efficient QFT and \(\log N = \Omega(\log |G|)\), then the quantum expander is explicit.

Until now, we did not have almost-Ramanujan expanders over such a group. Since the symmetric group admits such a QFT algorithm, we deduce the existence of explicit families of almost Ramanujan quantum expanders by applying our amplification to the Cayley graphs over the symmetric group due to Kassabov [Kas07].

Corollary 5.7 (Explicit Almost Ramanujan Quantum Expanders). For every \(\lambda \in (0, 1)\), there is an explicit infinite family of (efficient) \(\mathcal{O}(1/\lambda^{2+o_1(1)}), \lambda\)-quantum expanders.

\(21\)With respect to the Hilbert–Schmidt inner product.
\(22\)A useful special case is when each \(B_i\) is a (normalized) unitary.
\(23\)If \(\rho\) satisfies \(\text{Tr}(\rho) = 0\), then \(\|\Phi(\rho)\|_2 \leq \lambda \|\rho\|_2\), where \(\|\rho\|_2 := \sqrt{\text{Tr}(\rho^\dagger \rho)}\).
5.4 Explicit Almost Ramanujan Monotone Expander

We now show how to obtain almost Ramanujan monotone expanders starting from the explicit construction in Bourgain and Yehudayoff [BY13]. First, we recall the definition of a monotone graph. All graphs we consider are undirected.

**Definition 5.8 (Monotone partial map).** A partial map \( f : [n] \to [n] \) is monotone if for every pair \( \{x, y\} \) for which \( f \) is defined, if \( x < y \), we have \( f(x) < f(y) \).

**Definition 5.9 (Monotone Graph).** A bipartite graph \( X = ([n]_A \sqcup [n]_B, E) \) is a d-monotone graph if there are \( d \) monotone partial maps \( f_1, \ldots, f_d : [n] \to [n] \), such that the edges set \( E \) is the following disjoint union,

\[
E = \bigsqcup_{i=1}^{d} \{(v_A, f_i(v_B)) \mid v \in \text{Domain}(f_i)\}.
\]

We observe that there are two notions of degree of a monotone graph: the usual vertex degree and the number of monotone functions. Clearly, if a graph is \( d \)-monotone, all vertex degrees are at most \( d \). The converse is not necessarily true; for example, the complete bipartite graph on \( 2 \) vertices on each side, \( K_{2,2} \), has vertex degree \( 2 \), but the graph is not \( 2 \)-monotone. We stress that our almost Ramanujan bound is with respect to the usual notion of vertex degree (and keeps the number of monotone maps polynomial in the vertex degree).

**Definition 5.10 (Monotone Vertex Expander).** We say that \( X = (A = [n]_A \sqcup B = [n]_B, E) \) is a \( d \)-monotone expander if it is a \( d \)-monotone graph and there exists \( \delta > 0 \) such that for all \( A' \subseteq A \) with \( |A'| \leq n/2 \), we have \( |\partial(A')| \geq (1 + \delta)|A'| \), where \( \partial(A') \) is the set of vertices in \( B \) adjacent to \( A' \).

**Theorem 5.11 (Bourgain and Yehudayoff [BY13]).** There is an explicit family \( \{X_n\}_{n \in \mathbb{N}} \) of \( d \)-monotone vertex expanders with \( d = \Theta(1) \).

We will work with a spectral definition of monotone expander. For a bipartite graph \( X \), we define its biadjacency matrix, \( B_X \) such that the adjacency matrix \( A_X = \begin{pmatrix} 0 & B_X \\ B_X^T & 0 \end{pmatrix} \). Precisely, for a monotone graph \( X = ([n]_A \sqcup [n]_B, E) \), we have \( (B_X)_{ij} = 1 \text{ if } (i_A, j_B) \in E \). Note that if \( X \) is \( d \)-regular, then \( B_X \) is \( d \)-regular. We will define the graph \( X \) via \( B_X \) throughout.

**Definition 5.12 (Spectral Monotone Expander).** We say that a \( d \)-monotone graph, \( X \), is a \( \lambda \)-spectral monotone expander if \( \lambda(X) = \max\{||\lambda_2(B_X)||, ||\lambda_n(B_X)||\} < \lambda \).

It is well-known that starting from a monotone expander (not necessarily a vertex regular graph), we can add monotone partial functions to obtain a monotone graph of regular (vertex) degree that is still expanding. We use this to establish the following.

**Corollary 5.13.** There is explicit family \( \{X_n\}_{n \in \mathbb{N}} \) of \( d_0 \)-regular \( 2d_0 \)-monotone expanders with \( \lambda(X_n) \leq \lambda_0 < 1 \) and \( d_0 = \Theta(1) \). Furthermore, the unnormalized adjacency matrix of \( X_n \) can be written as a sum of \( d_0 \) permutation matrices each corresponding to two monotone maps.
Proof Sketch: Let \( \{X'_n\}_{n \in \mathbb{N}} \) be the family in Theorem 5.11. Let \( X = X'_n \) be a fixed \( d_0 \)-regular graph that is also \( d_0 \)-monotone expander with the maps \( \{f_i\} \).

For each monotone function \( f_i \), we define its "complement", \( \overline{f_i} \), as the (unique) monotone partial function \( \overline{f_i} \) such that \( f_i \cup \overline{f_i} \) is a total function. Let \( Y \) be the \( 2d_0 \)-monotone graph corresponding to the maps \( \{f_i, \overline{f_i}\} \). Then, we have

\[
B_Y = \sum_{i=1}^{d_0} P_i,
\]

where \( P_i = M_{f_i} + M_{\overline{f_i}} \) and \( (M_{f_i})_{x,y} = \chi f_i(x) = y \).

Each matrix \( P_i \) is a permutation matrix as \( f_i \cup \overline{f_i} \) is a total function. Adding more maps preserves the constant vertex expansion parameter which (together with having constant vertex degree) implies constant spectral expansion bounded away from 1 (see [Vad12, Theorem 4.9]). Thus, \( \{Y_n\}_{n \in \mathbb{N}} \) is the required family. \( \square \)

In the amplification process, we will be multiplying permutation matrices rather than just composing monotone maps since the latter operation can result in a map with empty domain. We now establish the derandomized spectral amplification of monotone expanders.

**Corollary 5.14** (Almost Ramanujan Monotone Expanders). For every \( \lambda > 0 \), there is an explicit family \( \{X_i\}_{i \in \mathbb{N}} \) of (vertex) \( d \)-regular \( d^{O(1)} \)-monotone expanders with \( d = O(1/\lambda^{2+\alpha(1)}) \) and \( \lambda(X_i) \leq \lambda \).

**Proof:** Let \( \{X'_n\}_{n \in \mathbb{N}} \) be the family in Corollary 5.13. Fix \( X = X'_n \) and let \( P_1, \ldots, P_{d_0} \in \mathbb{R}^{n \times n} \) be the permutation matrices guaranteed by Corollary 5.13, where each \( P_i = M_{f_i} + M_{\overline{f}_i} \).

Use Corollary 5.1 to obtain a collection \( P' \) of size \( |P'| = d := O(1/\lambda^{2+\beta}) \) such that,

\[
P' = \{ \sigma \mid \sigma = P_{i_1} \cdots P_{i_k} \text{ for some } i_1, \ldots, i_k \in [d_0] \}.
\]

Our final bipartite monotone graph will be \( g \) given by \( B_Y = \sum_{\sigma \in P'} \sigma \). To compute its monotone degree, we observe that,

\[
P_{i_1} \cdots P_{i_k} = \sum_{g_{i_1} \in \{f_{i_1}, \overline{f}_{i_1}\}} \cdots \sum_{g_{i_k} \in \{f_{i_k}, \overline{f}_{i_k}\}} M_{g_{i_1}} \cdots M_{g_{i_k}}
\]

\[
= \sum_{g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_k}} M_{g_{i_1} \circ g_{i_2} \circ g_{i_3} \circ \cdots \circ g_{i_k}},
\]

where \( g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_k} \) is the composed map which is monotone (possibly with an empty domain). This means that we can have at most \( 2^k \) monotone maps (and at least one since \( P_{i_1} \cdots P_{i_k} \neq 0 \)). Therefore, the total number of maps is at most \( d \cdot 2^k = d^{O(1)} \) as \( k = O_{\lambda_0}(\log(1/\lambda)) \).

\section{5.5 Amplifying the Average Kazhdan Constant}

The *Kazhdan constant* is a notion of "spectral gap" (and so it is related to bias) for discrete groups which predates, and was central to the study of expansion in finite groups and
graphs. In particular, we can work with finitely generated groups that can have infinitely many irreducible representations on more general Hilbert spaces, possibly of infinite dimension. Nonetheless, we can still apply our operator version of Ta-Shma’s amplification procedure as it is independent of dimension and works for any unitary representation ρ. Therefore, we amplify the average Kazhdan constant which also amplifies the Kazhdan constant. We now define these two parameters formally.

Let $G$ be a group generated by a finite set $S$ of generators. The Kazhdan constant of $G$ with respect to generators $S$ is defined as

$$K(G, S) := \inf \{ K(G, S, \rho) | (\rho, H) \text{ irreducible and non-trivial unitary representation} \},$$

where $K(G, S, \rho) := \inf_{v \in H: \|v\|_2 = 1} \max_{g \in S} \|\rho(g) v - v\|_2^2$.

Analogously, an average version of the Kazhdan constant, as in the work of Pak and Zuk [PZ01], can be defined as

$$\overline{K}(G, S) := \inf \{ \overline{K}(G, S, \rho) | (\rho, H) \text{ irreducible and non-trivial unitary representation} \}.$$

\[
\overline{K}(G, S, \rho) := \inf_{v \in H: \|v\|_2 = 1} \frac{1}{|S|} \sum_{g \in S} \|\rho(g) v - v\|_2^2
\]

\[
= \inf_{v \in H: \|v\|_2 = 1} \frac{1}{|S|} \sum_{g \in S} 2 - 2 \langle \rho(g) v, v \rangle
\]

\[
= \inf_{v \in H: \|v\|_2 = 1} 2 - 2 \left( \mathbb{E}_{g \sim S} [\rho(g)] v, v \right)
\]

\[
= 2 \left( 1 - \left\| \mathbb{E}_{g \sim S} [\rho(g)] \right\|_{op} \right).
\]

**Theorem 1.2** gives an improved generating set in this more general setting.

**Corollary 5.15 (Amplifying Average Kazhdan Constant).** Let $G$ be a finitely generated group and $S$ a finite set of generators such that the average Kazhdan constant $\overline{K}(G, S)$ is equal to $2 \cdot (1 - \lambda_0)$ for some constant $\lambda_0 \in (0, 1)$. For every $\lambda \in (0, 1)$, there is a set $S' \subseteq G$ such that

1. $\overline{K}(G, S') \geq 2 \cdot (1 - \lambda)$, and thus, $K(G, S') \geq 2 \cdot (1 - \lambda)$.
2. $|S'| = O_{\lambda_0}(|S| / \lambda^{2 + o(1)})$, and
3. $S'$ can be found in time $\text{poly}(|S| / \lambda)$ assuming an oracle for group operations on $G$.

**Remark 5.16.** Note that the above amplification for $\overline{K}$ immediately implies the same amplification for $K$ (since the maximum is at least the average, $\overline{K}(G, S) \leq K(G, S)$). Moreover, we remark that the above amplification can also similarly improve the constant of Lubotzky’s property (τ) (the latter being a weaker version of property (T)), so it is more general and applies to expansion in many more discrete groups [RL10].

In Section 5.6, we will apply this corollary to a specific family of representations which will give a simple improvement to the bounds on the dimension expander constructed in [LZ08].
5.6 Explicit Almost Ramanujan Dimension Expanders

Dimension expanders were defined in [BISW01] motivated by applications in theoretical computer science. A conjectured construction based on irreducible representations was suggested by Wigderson to hold over every field. The conjecture was subsequently established by Lubotzky and Zelmanov [LZ08] for fields of characteristic zero. We now define dimension expanders, explain the [LZ08] proof, and our amplification in this setting.

**Definition 5.17 ((ε, γ) Dimension Expander).** Let $\mathbb{F}$ be a field, $d \in \mathbb{N}$, $\epsilon > 0$, $V$ be a vector space of dimension $n$ and $T_1, \ldots, T_d: V \rightarrow V$ be linear transformations. We say that $(V, \{T_i\}_{i \in [d]})$ is an $(\epsilon, \gamma)$-dimension expander if for every subspace $W \subseteq V$ of dimension at most $\gamma n$, we have $\dim(W + \sum_{i=1}^d T_i(W)) \geq (1 + \epsilon) \cdot \dim(W)$.

**Remark 5.18.** Observe that if the maps $T_i$ are restricted to being permutation matrices, and the expansion condition is restricted only to subspaces $W$ generated by elementary basis vectors, then one obtains the usual definition of vertex expansion of graphs. Thus dimension expanders may be viewed as a linear-algebraic extension of expander graphs.

For an irreducible unitary representation $\rho$, there exists an associated representation $\text{adj}_\rho$. The construction in [LZ08] relates dimension expansion with the Kazhdan constant. Their result gives a dimension expander, which gives expansion for all subspaces $W$, such that $\dim(W) \leq n/2$, but their expansion guarantee is significantly stronger when $\dim(W)$ is smaller. To obtain this, we first state a simple improvement to a computation in [LZ08].

**Claim 5.19.** Let $W, W' \subseteq \mathbb{C}^d$ be two vector spaces. Let $P, P'$ be orthogonal projectors onto $W, W'$, respectively. Then,

$$\text{Re}(\text{Tr}(PP')) = \text{Tr}(PP') \geq \dim(W \cap W').$$

**Proof:** Let $U_0 = W \cap W'$, $U_1 = W \cap U_0^\perp$ and $U_2 = W' \cap U_0^\perp$, with orthonormal bases $\{u_1, \ldots, u_k\}, \{a_1, \ldots, a_\ell\}$ and $\{b_1, \ldots, b_m\}$, respectively. We can write $P$ and $P'$ as

$$P = \sum_{i=1}^k u_i u_i^T + \sum_{i=1}^\ell a_i a_i^T \quad \text{and} \quad P' = \sum_{i=1}^k u_i u_i^T + \sum_{i=1}^m b_i b_i^T.$$

Using linearity and orthogonality, we obtain

$$\text{Tr}(PP') = \sum_{i=1}^k \|u_i\|^2 \text{Tr}(u_i u_i^T) + \sum_{i=1}^\ell \sum_{j=1}^m \langle a_i, b_j \rangle \text{Tr}(a_i b_j^T)$$

$$= k + \sum_{i=1}^\ell \sum_{j=1}^m \langle a_i, b_j \rangle^2 \geq \dim(U_0),$$

**Remark 24.** Let $sl_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \text{tr}(A) = 0\}$. Equip the space with the Frobenius inner product defined as $\langle A, B \rangle = \text{tr}(A^T B)$ where $A^T$ is the conjugate transpose. For any finite-dimensional unitary representation $\rho: G \rightarrow U_n$, we have an adjoint representation $(\text{adj}_\rho, sl_n)$ where the action is by conjugation $\text{adj}_\rho(g) \cdot A = \rho(g) A \rho(g)^{-1}$. Since conjugation by unitary matrices preserves the trace, $sl_n$ is closed under the representation. Moreover, it is unitary as

$$\langle \text{adj}_\rho(g) A, \text{adj}_\rho(g) B \rangle = \text{tr} \left( \rho(g) A^\dagger \rho(g)^\dagger \rho(g) B \rho(g)^{-1} \right) = \langle A, B \rangle.$$
here in the last step we used that \( \|u_i\|^2 = \text{Tr}(u_i u_i^T) = 1 \). □

The above claim is a variant of the one used in [LZ08] to prove their main result. By plugging in Claim 5.19 in their proof we obtain,

**Proposition 5.20** (Adapted from [LZ08] using Claim 5.19). Let \( \rho : G \to U_{\mathbb{C}^n} \) be a unitary irreducible representation. Then \((\mathbb{C}^n, \{\rho(g)\}_{g \in S})\) is \((1 - \lambda - o_n(1), 1/2 - O(\lambda))-\)dimension expander, where \(2(1 - \lambda) := K(G, S, \text{adj}_\rho)\).

**Corollary 5.21.** Let \( \lambda > 0 \) be any fixed constant. Then, there exists an explicit infinite family of \((1 - \lambda - o_n(1), 1/2 - O(\lambda))-\)dimension expanders.

**Proof:** Pick a family of groups \( \{G_n\}_n \) such that each \( G_n \) satisfies the condition of Corollary 5.15; for example, one can take any non-abelian finite simple group. By definition, for any such \( G \), we have \( K(G, S, \text{adj}_\rho) \geq K(G, S) \) and therefore we obtain a set \( S' \) such that \( K(G, S', \text{adj}_\rho) \geq 2(1 - \lambda) \) for the given \( \lambda \). We can now apply Proposition 5.20. □

**Remark 5.22.** Forbes and Guruswami [FG15] point out that the quantum expander construction of Harrow [Har07] also yields a dimension expander (with a similar construction of the dimension expanders from [LZ08]). As mentioned earlier, monotone expanders are dimension expanders over any field [DS09, DW10]. Moreover, the Bourgain and Yehudayoff [BY13] construction of monotone expanders with constant generating set yields such dimension expanders with constant generating set!

### 5.7 Diameter of Finite Groups

The study of the diameter of Cayley graphs can take many forms, e.g., it can be with respect to every generating set (as in the celebrated Babai–Seress conjecture [BS88]) or with respect to some constant size generating set as in [BKL89]. Here, we explore the latter case.

First, recall that any \( n \)-vertex degree-\( d \) graph has diameter at least \( \log_{d-1}(n) \). On the other hand, it is well-known that expansion directly implies diameter at most \( C \cdot \log_{d-1}(n) \) for some constant \( C \geq 1 \) (depending on the expansion). The best upper bound a spectral proof can provide is 2 due to the Alon–Bopanna bound for spectral expansion. Using our amplification to almost-optimal spectral expansion, deduce that any expanding group \( G \) has a constant degree-\( d \) Cayley expander of diameter \( 2 + o_d(1) \).

More precisely, we have the following.

**Lemma 5.23.** Suppose \( \{\text{Cay}(G_i, S_i)\}_{i \in \mathbb{N}} \) is a family of bounded degree Cayley expanders. Then, there exists a family \( \{\text{Cay}(G_i, S'_i)\}_{i \in \mathbb{N}} \) of constant degree-\( d \) Cayley expanders with diameter at most \((2 + o_d(1)) \cdot \log_{d-1}(G_i)\).

**Proof:** We apply Theorem 1.2 to the family \( \{\text{Cay}(G_i, S_i)\}_{i \in \mathbb{N}} \) obtaining a new family of \( \{\text{Cay}(G_i, S'_i)\}_{i \in \mathbb{N}} \) of \((d, \lambda))-\)expanders with \( d = 1/\lambda^{2+\beta} \) for some sufficiently small constants \( \lambda, \beta > 0 \). Let \( A_i \) be the normalized adjacency matrix of \( \text{Cay}(G_i, S'_i) \) and \( n_i = |G_i| \). Let \( e_g \) be the indicator vector of some fixed \( g \in G_i \). Note that

\[
\| (A_i - J/n_i) e_g \|_2 \leq \lambda^i = d^{-i/(2+\beta)} < 1/|G_i|,
\]
for \( t = (2 + 2\beta) \cdot \log_d(|G_i|) = (2 + o_d, \beta(1)) \cdot \log_d -1(|G_i|). \)

This implies that \( \Lambda_i e_g \) is supported on all elements of \( G_i \), and thus the diameter of \( G_i \) is at most \( t \).

## 6 Operator Expander Mixing Lemma

In Section 3, we showed an operator amplification based on walks on an auxiliary expander. The same construction can be analyzed differently by using the expander mixing lemma (EML) iteratively.

For the scalar case, this is a classic result due to Rozenman–Wigderson (see also [Bog12]). An operator version of this approach was proved by Chen, Moore, and Russell [CMR13] but they obtain a much larger degree, \( \lambda^{-11} \), rather than the \( \lambda^{-4} \) similar to the expander walk approach (Theorem 3.2). Our proof obtains the bound of \( \lambda^{-4} \).

### Theorem 6.1 (Iterated Operator EML)

Let \( S \subseteq G \). Suppose \( \lambda(\text{Cay}(G, S)) = \lambda_0 < 1 \), where \( \lambda_0 \in (0, 1) \). For every \( \lambda \in (0, 1) \), we can find \( S' \subseteq G \) such that,

1. \( \lambda(\text{Cay}(G, S')) \leq \lambda \) and \( |S'| = O(\lambda_0^4 \lambda^{-1} + o(1)) \), and
2. the running time is \( \text{poly}(|S|, 1/\lambda_0, 1/\lambda) \).

We now show an operator version of the expander mixing lemma for completeness. As we mentioned above, a similar result was first derived in [CMR13]. While a simple generalization of EML, it is of the same nature of the generalizations of this paper and is of independent interest.

### Lemma 6.2 (Matrix EML [CMR13])

Let \( X = (V, E) \) be a \( \lambda(X) \)-spectral expander and let \( f : V \rightarrow \mathcal{L}(\mathcal{H}) \). Then,

\[
\| \mathbb{E}_{(x', x) \in E} [f(x') \cdot f(x)] - \left( \mathbb{E}_{x \in V_X} [f(x)] \right)^2 \|_{op} \leq \lambda(X) \cdot \max_{x \in V_X} \| f(x) \|_{op}^2.
\]

We start with a simple claim describing an operator form the process of sampling according to the edges of an expander and sampling according to pairs of vertices. Recall the following maps from Section 3, \( \text{P}_H : \mathcal{X}_H \rightarrow \mathcal{H} \) and \( \text{L}_H : \mathcal{H} \rightarrow \mathcal{X}_H \),

\[
\text{P}_H(w \otimes x) = w, \quad \text{L}_H(v) = \mathbb{E}_{x \in V_X} [v \otimes x].
\]

We will need again that \( \| \text{P}_H \|_{op} \| \text{L}_H \|_{op} = 1 \).

### Claim 6.3

Let \( A_X \) be the normalized adjacency matrix of a \( d \)-regular graph \( X \) and let \( J_X \) be the normalized \( |V_X| \times |V_X| \) all-ones matrix.

\[
\mathbb{E}_{(x, x') \in E} [f(x') \cdot f(x)] = \text{P}_H \Pi_z \hat{A}_X \Pi_z \text{L}_H.
\]

\[
\mathbb{E}_{x, x' \in V} [f(x') \cdot f(x)] = \text{P}_H \Pi_z J_X \Pi_z \text{L}_H.
\]
Proof: The proof is identical for both so we prove just the first one. For any \( w \in \mathcal{H} \), we have

\[
P_{\mathcal{H}} \Pi_z \tilde{A}_X \Pi_L \mathcal{H} w = \frac{1}{|V_X|} P_{\mathcal{H}} \Pi_z \tilde{A}_X \Pi_z \left( \sum_{x \in V_X} x \otimes w \right)
\]

\[
= \frac{1}{|V_X|} P_{\mathcal{H}} \Pi_z \tilde{A}_X \left( \sum_{x \in V_X} x \otimes f(x) w \right).
\]

\[
= \frac{1}{d|V_X|} P_{\mathcal{H}} \Pi_z \left( \sum_{x \in V_X} \sum_{x' \sim x} x' \otimes f(x') w \right).
\]

\[
= \frac{1}{|E|} P_{\mathcal{H}} \left( \sum_{x \sim x'} f(x') f(x) w \right) = \mathbb{E}_{(x', x) \in E} [f(x') \cdot f(x)] w.
\]

as claimed.

We now prove the operator mixing lemma above.

Proof: [Proof of Lemma 6.2] By Claim 6.3, it is enough to bound the operator norm

\[
\| P_{\mathcal{H}} \Pi_z \left( \tilde{A}_X - \tilde{J}_X \right) \Pi_L \mathcal{H} \|_{op} \leq \| P_{\mathcal{H}} \|_{op} \| \Pi_z \|_{op}^2 \left\| \left( \tilde{A}_X - \tilde{J}_X \right) \right\|_{op} \| \Pi_L \mathcal{H} \|_{op}
\]

\[
\leq \lambda(X) \cdot \| \Pi_z \|_{op}^2 = \lambda(X) \cdot \max_{x \in V_X} \| f(x) \|_{op}^2,
\]

concluding the proof.

Corollary 6.4 (Non-abelian EML). Let \( X = (V, E) \) be a \( \lambda(X) \)-spectral expander, \( \rho : G \to U_{\mathcal{H}} \) be an unitary representation and \( (g_v)_{v \in V} \in G^V \). Then

\[
\left\| \mathbb{E}_{(u, v) \in E} \left[ \rho(g_u) \cdot \rho(g_v) \right] - \left( \mathbb{E}_{u \in V} [\rho(g_u)] \right)^2 \right\|_{op} \leq \lambda(X).
\]

Proof: Follows immediately from Lemma 6.2 and the fact that unitary operators have operator norm bounded by 1.

We now prove the main result of this section. This iterated amplification also appears in the derandomized squaring of Rozenman and Vadhan [RV05] used to give an alternative proof of the SL = L result of Reingold [Rei04].

Proof: [Proof of Theorem 6.1] We amplify the expansion in two phases. The first phase amplifies the initial expansion of \( S \) from \( \lambda_0 \) to a constant expansion \( \lambda_0' = 1/4 \). This phase increases the size of the generator set by a constant factor.

(First Phase) Let \( \varepsilon_0, \gamma_0 \) be constants such that

\[
\varepsilon_0 = \lambda_0 (1 - \lambda_0) / 2, \quad 0 < \gamma_0 \leq (1 - \lambda_0) / 2 < 1
\]

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Let \( X_0 = (V_0, E_0) \) be an explicit expander via Theorem 3.7, with \( \lambda(X_0) \leq \varepsilon_0 \), degree \( O(1/\varepsilon_0^2) \) and with the number of vertices \(|V_0| = m|S|\) with \( m = O(1) \). Replicate each element of \( S \) \( m \) times and still call the resulting multiset \( S \) (observe that expansion remains \( \lambda_0 \)). For every edge \((u, v) \in E_0\), add \( g_u g_v \) to \( S_0 \). By Corollary 6.4,

\[
\lambda(G, S_0) \leq \lambda_0^2 + \varepsilon_0 \leq \lambda_0 (1 - \gamma_0), \quad |S_0| = 9|S|/\varepsilon_0^2 = O(|S|)
\]

Repeat this procedure \( \log_{1-\gamma_0} 1/4\lambda_0 \) times which ensures that the expansion is \( \lambda_0'' = 1/4 \). Let \( S_0 \) be this final set.

(Second Phase) We will amplify the bias inductively using a stronger (i.e., more expanding) auxiliary expander graph \( X_i \) at each step. As mentioned, this inductive amplification is similar to the derandomized squaring of Rozenman and Vadhan [RV05]. We start with \( S_0 \) and expansion \( \lambda_0'' = 2^{-2} \) as in the first phase. At each step assume that you have a set \( S_{i-1} \) with expansion \( \lambda_{i-1} \). Use Theorem 3.7, to construct \( X_{i-1} \) to have expansion \( \lambda_{i-1}^2 \) and degree at most \( 9/\lambda_{i-1}^4 \). Then, \( S_i \) is obtained via edges of \( X_i \) as before and we have \( \lambda_i \leq 2\lambda_{i-1}^2 \). It is easy to check that the recurrence yields \( \lambda_i \leq 2^{-2i} \) for \( i \geq 1 \). Assume for convenience that \( \log \lambda = -2^i \). Clearly, then we need to iterate this \( r \) times. In each iteration, the size grows by a factor of the degree which is \( 9/\lambda_{i-1}^4 \) and thus the final size of \( S' \) can be bounded as,

\[
|S'| = |S_0| \prod_{i=0}^{r-1} \frac{9}{\lambda_i^4} \leq |S_0| \cdot 9^{r+4} = \frac{|S_0|}{\lambda^4 \cdot \log^9 \lambda} \cdot |S_0| \cdot \frac{1}{\log \lambda} \cdot \lambda_0 \cdot \left(\frac{|S|}{\lambda^{4+o(1)}}\right) \cdot O(1).
\]

concluding the proof.

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