Regularizing Portfolio Risk Analysis: A Bayesian Approach

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Abstract

It is important for portfolio manager to estimate and analyze the recent portfolio volatility to keep portfolio’s risk within limit. Though number of financial instruments in the portfolio are very large, some times more than thousands, however daily returns considered for analysis is only for a month or even less. In this case rank of portfolio covariance matrix is less than full, hence solution is not unique. It is typically known as “large p - small n” or “ill-posed” problem. In this paper we discuss a Bayesian approach to regularize the problem. One of the additional advantages of this approach is to analyze the source of risk by estimating the probability of positive ‘conditional contribution to total risk’ (CCTR). Each source’s CCTR sum upto total volatility risk. Existing method only estimates CCTR of a source, but it does not estimate the probability of CCTR to be significantly greater (or less) than zero. This paper presents Bayesian methodology to do so. We use parallalizable and easy to use Monte Carlo (MC) approach to achieve our objective.

Keyword: Monte Carlo, Parallal Computation, Risk Analysis, Shrinkage Method, Volatility
1 Introduction

Recent Euro zone crisis remind us that the ‘risk analysis’ is always an essential part of theory of portfolio management. Markowitz(1952) [1], first proposed volatility (or standard deviation) as the risk measure. As volatility provides an idea about average loss (or gain) of portfolio; more new risk measures like ‘Value at Risk’ (VaR) and ‘Expected shortfall’ (ESF) for extreme losses are also popular. Though Basel II regulatory framework requires inclusion of VaR and ESF, till the date volatility plays an important role in finance. First, mean-variance optimization admits analytical solution to estimate the optimizing weight of a portfolio. Secondly, most of the econometric models (e.g., multi-factor risk model, stochastic volatility models etc.) have been developed to forecast volatility. Finally, volatility can be traded directly on the open market through exchange traded fund of VIX index and indirectly through derivatives. In this paper, we mainly focus on estimating and analyzing the sources of portfolio volatility.

It is essential to measure the volatility and identify the main sources of volatility. Often the number of financial instruments of well diversified mutual funds or pension funds are more than thousands. Such funds invest in foreign countries on regular basis and some times more than fifty to sixty different countries. However, portfolio managers are interested about only recent volatility which consider daily return of a month and some times even less. As the number of sectors or countries or component of a portfolio is greater than the number of days of return, the rank of the portfolio covariance matrix is less than full; such cases yield non-unique solutions. Generally it is known as “large p - small n” or “ill-posed” problem.

In a 3-series paper, Ledoit and Wolf (2003, 2004a, 2004b) [2, 3, 4] showed the use of Shrinkage estimator on actual stock market data keeping all other steps of optimization process same. By doing so they reduced the tracking error relative to benchmark index. As a result it substantially increased the realized information ratio of active portfolio managers. Golsnay and Okhrin (2007) [5] showed the improvement of portfolio selection by using the multivariate shrinkage estimator for the optimal portfolio weights. Similarly, Still etal. (2010) [6] regularized portfolio weights with the ESF as a risk measure which is related
to support vector regression, while considering L2 norm penalty prior on weight vector as regularizer. However, in this paper, we regularize the covariance matrix as our objective to analyze the portfolio risk.

Recently, Das and Dey (2010) introduced some Bayesian properties of multivariate gamma distribution for covariance matrix. In this paper we use this Bayesian approach to regularize the estimation problem. Under certain conditions, posterior distribution of portfolio covariance matrix is proper and have a closed form inverted multivariate gamma distribution. Consequently, the solution of covariance matrix is unique. The rest of the article is organized as follows. In Section 2 we discuss the posterior distribution of portfolio covariance matrix and the condition under which it is proper. In Section 3 we present the parallizable Monte Carlo algorithm to implement the posterior inference of risk. In Section 4 we demonstrate the method with an empirical data. Section 5 concludes the paper.

2 Posterior Distribution of Covariance Matrix

Suppose $S$ is a real symmetric matrix of order $p$ with $\frac{p(p+1)}{2}$ variables $s_{ij}$ and $\Sigma = ((\sigma_{ij}))$ is corresponding dispersion matrix, such that for diagonal matrix $D$ with diagonal elements 1 and $-1$, $(D\Sigma D)^{-1}$ has non-positive off diagonal elements. Hence due to Bapat’s condition, Bapat (1989)\[8\], $S$ with characteristic function as

$$\psi_S(t) = E[i \exp\{tr(TS)\}] = |I_p - i\beta\Sigma T|^{-\alpha},$$

having the density function as

$$f(S) = \frac{|\Sigma|^\alpha}{\Gamma_p(\alpha)^{\beta^p}} \exp\left\{-\frac{1}{\beta} tr\Sigma^{-1} S\right\} |S|^{\alpha-\frac{1}{2}(p+1)}, \quad S > 0$$

has an infinitely divisible multivariate gamma distribution with parameters $\alpha \geq \frac{p-1}{2}$, $\beta \geq 0$ and $\Sigma$ a positive definite matrix. We denote that as $S \sim MG_p(\alpha, \beta, \Sigma)$. Note that if $0 \leq \alpha \leq \frac{p-1}{2}$, $S$ has a degenerate distribution. If we choose $\alpha = \frac{n-1}{2}$ and $\beta = 2$ then $S$ follows a Wishart distribution, i.e., $S \sim Wishart(n - 1, \Sigma)$ (see Anderson, (1984) \[9\], pp 252). Therefore if $p \geq n$, then $S$ is less than full rank and the sampling distribution of $S$ is degenerate. That is no valid statistical inference can be implemented for such cases.
Das and Dey (2010) [7] showed that if $\Sigma$ has prior as inverted multivariate gamma distribution, i.e., if $\Sigma \sim \text{InvMG}_p(a, \beta, \Psi)$ then the posterior distribution of $\Sigma$ is

$$\Sigma|S \sim \text{InvMG}_p(\alpha + a, \beta, S + \Psi).$$

Note that as long as $(\alpha + a) \geq \frac{p-1}{2}$, the posterior distribution is proper. Suppose $n \leq p$, i.e., $\alpha \leq \frac{p-1}{2}$, where $\alpha = \frac{n-1}{2}$; then the sampling distribution of $S$ is degenerate. However, if we choose the prior degrees of freedom parameter $a$ to be such that

$$\alpha + a \geq \left(\frac{p-1}{2}\right),$$

where $a \geq \frac{p-n}{2}$ then the posterior distribution of $\Sigma$ is proper.

If we choose $a = \frac{n_0}{2}$ and $\beta = 2$ then the prior of $\Sigma$ is inverse Wishart distribution,

$$\Sigma \sim \text{InvWishart}(n_0, \Psi)$$

(1)

and posterior distribution of $\Sigma$ is

$$\Sigma|S \sim \text{InvWishart}(n_0 + n - 1, \Psi + S),$$

(2)

for detail see Anderson (1984) [9]. If $n < p$, then we choose the prior degrees of freedom as

$$n_0 = (p - n) + 1.5.$$  

(3)

This ensures posterior distribution to be proper. Posterior mode of $\Sigma$ is

$$M(\Sigma|S) = \frac{\Psi + S}{n_0 + n - p - 2}$$

$$= \frac{n_0 + p + 1}{n_0 + n + p} \cdot \frac{\Psi}{n_0 + p + 1} + \frac{n - 1}{n_0 + n + p} \cdot \frac{S}{n - 1}$$

$$= \omega \frac{\Psi}{n_0 + p + 1} + (1 - \omega) \frac{S}{n - 1},$$

where $\omega = \frac{n_0 + p + 1}{n_0 + n + p}$. Clearly posterior mode of $\Sigma$ is a shrinkage estimator which is a weighted average of prior distribution’s mode and sample covariance estimator. Das and Dey (2010) [7] showed posterior mode is Bayes estimator under a Kullback-Leibler type loss function.
Therefore under properly chosen prior degrees of freedom (\(\alpha\) or \(n_0\)) and positive definite \(\Psi\), posterior mode of \(\Sigma\) is Bayes shrinkage estimator which regularize the solution. Also posterior distribution of \(\Sigma\) is proper, which will help us to regularize the risk analysis in the next section.

### 3 Bayesian Inference for Analyzing Risk

Once portfolio risk is being calculated, next step is to evaluate the sources of risk and how they interrelate. There could be many different sources of risk, like individual stocks, sectors, asset classes, industries, currencies or style factors. Sources of risk could be anything, so we would keep it generic. We consider an investment period and suppose \(r_j\) denote return to source \(j\) for the same period, where \(j = 1, 2, ..., p\). The portfolio return over the period is

\[
R_p = \sum_{j=1}^{p} \omega_j r_j,
\]

where \(\omega_j\) is the portfolio exposure to the source \(j\), i.e., portfolio weight, such that \(\omega_j \geq 0\) and \(\sum_{j=1}^{p} \omega_j = 1\), see Rupert (2004) \[\text{[10]}\]. Portfolio manager determines the size of \(\omega_j\) at the beginning of the investment period. Portfolio volatility is defined as

\[
\sigma_p = \sqrt{\omega^T \Sigma \omega},
\]

where \(\omega^T = \{\omega_1, \omega_2, ..., \omega_p\}\). Clearly, weights \((\omega_j)\) are the main regulators of the portfolio’s total volatility. So it is important for a manager to quantify, how sensitive the portfolio volatility is with respect to a small change in \(\omega\). This can be achieved by differentiating the volatility with respect to weight, i.e.,

\[
\frac{\partial (\sigma_p)}{\partial \omega} = \frac{1}{\sigma_p} \Sigma \omega = \varrho,
\]

where \(\varrho = \{\varrho_1, \varrho_2, ..., \varrho_p\}\), is known as ‘marginal contribution to total risk’ (MCTR). Note that MCTR of source \(i\) is

\[
\varrho_i = \frac{1}{\sigma_p} \sum_{j=1}^{p} \sigma_{ij} \omega_j.
\]
The CCTR is the amount that a source contribute to total portfolio volatility. In other words, as \( \zeta_j = \omega_j \varrho_j \) is the CCTR of source \( j \) then

\[
\sigma_p = \sum_{j=1}^{p} \zeta_j = \sum_{j=1}^{p} \omega_j \varrho_j.
\]

Therefore total volatility can be viewed as weighted average of MCTR.

Now in order to estimate the MCTR and CCTR, estimation of \( \Sigma \) is good enough, because portfolio weight is predetermined by manager. However, a manager is more interested about estimating the \( P(\varrho_j < 0) \) or \( P(\zeta_j < 0) \). Because, negative \( \varrho_j \) or \( \zeta_j \) means that source \( j \) is reducing the total risk. In section [2], we presented that the posterior distribution of \( \Sigma \) follows \( \text{InvWishart}(n_0 + n - 1, S + \Psi) \). Now in order to estimate the \( P(\varrho_j > 0) \) we present Monte Carlo (MC) method as follows:

- **Step 1:** For iteration \( i \), generate a sample \( \Sigma^{(i)} \) from \( \text{InvWishart}(n_0 + n - 1, S + \Psi) \)
- **Step 2:** Compute \( \sigma_p^{(i)} = \sqrt{\omega^T \Sigma^{(i)} \omega} \)
- **Step 3:** Compute \( \varrho^{(i)} = \frac{1}{\sigma_p^{(i)}} \Sigma^{(i)} \omega \)
- **Step 4:** Compute \( \zeta^{(i)}_j = \varrho^{(i)}_j \omega_j \) for all \( j = 1, 2, ..., p \).
- **Step 5:** Set \( i = i + 1 \) and go to Step 1.

Note that all the steps of iteration \( i \), does not depend on previous step \( (i - 1) \). So if two parallel processors are available then iteration \( i \) and \( (i - 1) \) can be implemented parallely at the same time. In fact one can consider the algorithm as ‘embarassingly parallel’ algorithm. Implementing the algorithm in parallel might not be required if \( p \) is small, such as if we consider four or five different asset classes as the risk sources. But if \( p \) is very large, like for any mutual fund portfolio, number of stocks might be more than thousands. For such large \( p \), generating \( \Sigma^{(i)} \) will be slow and consequent calculation of all other steps will be slow as well. In such cases, parallalization of the algorithm is required to improve the time efficiency.

Once the samples are generated, required MC statistics can be estimated easily, like

\[
P(\zeta_j > 0) = \frac{1}{N} \sum_{i=1}^{N} I(\zeta^{(i)}_j > 0),
\]
where $N$ is the simulation size, $I(A) = 1$ if $A$ is true and $I(A) = 0$ if $A$ is false. In the next section, we present an empirical study for a portfolio with five asset classes.

4 Empirical Study

In this section we demonstrate the methodology with empirical data. We consider a portfolio with five asset classes, viz. (i) Hybrid Bond, (ii) Emerging Market, (iii) Commodity, (iv) Bond and (v) Stocks. We consider two time periods of study. First time period is consisted of three months of monthly return data (May, June and July of 2008) which is presented in table 1. Second time period (for August, September and October of 2008) is presented in table 2. As second month of period 2 is September 2008, when event like ‘Bankruptcy of Lehman Brothers’ took place, the total volatility during this period is very high. However, for a portfolio manager it is important to identify the different sources of risk contribution from highest to least.

We consider same portfolio weight (see table 3) during both periods for fair comparison. Here $n = 3$ and $p = 5$, which implies that regular sample covariance matrix estimator is not stable, as it provides non-unique solution for both periods. So we use Bayesian approach as discussed in section 2. We select the degrees of freedom for Wishart prior to be $n_0 = 3.5$ and choose $\Psi = \lambda I$, where $\lambda = 0.001$ and $I$ is the identity matrix, as described in equation (1), (2) and (3). We considered simulation size as $N = 10000$.

We present posterior estimates of total volatility in table 4 and posterior density in figure 1. Monthly level total portfolio volatility is estimated by posterior mean and it goes up from 5.18% to 15.21% during first to second period. Posterior standard deviation indicates the variability of the estimate and that also goes up during the same period, because portfolio density becomes more positively skewed during the period 2 as presented in figure 1.

We present the posterior density of CCTR ($\zeta$) of four asset classes, viz. (i) Hybrid bond, (ii) Emerging Market, (iii) Commodity and (iv) Stock in figure 2. Clearly we can say from figure 2 that posterior density of CCTR of these four asset classes shifted towards right and become more positively skewed during the second period. It means that these
four asset classes have contributed heavily to total volatility risk during second period. Posterior estimates of all five asset classes are presented in table 6. Except ‘Bond’, for four of the other asset classes, the posterior mean of CCTR went up significantly during the second period. Also the Bayesian 95% CI of these four asset classes do not include zero and all four intervals are on the positive side of the real line. Therefore we can conclude these four assets classes contributed statistically significantly large amount towards total volatility risk of the portfolio. One point to be noted here is that among these four asset classes, the CCTR of ‘Hybrid Bond’ (0.71%) is least compared to the CCTR of ‘Emerging Mkt’ (1.5%), ‘Commodity’ (3.61%) and ‘Stock’ (8.83%).

However, ‘Bond’ is the only asset class whose posterior density, as presented in figure 3, did not shift towards right. Rather it became more concentrated around zero. In fact Bayesian estimate of CCTR of Bond has reduced to 0.55% from 0.90%, during first to second period. Also we can see from table 6 that Bayesian 95% CI for Bond become more concentrated around zero during the second period, compared to the first period.

Finally, using the formula as presented in equation (4), we calculated the probability of positive CCTR for both periods and presented the result in table 5. Probability of positive CCTR for all the asset classes, except Bond, goes up to 0.99 during period 2. During the same period, the probability of positive CCTR for bond is 0.756, which is much less than other classes. This showed us that during period 2, which is known as crisis period, bond was the only asset class whose volatility risk was much less than the other asset classes of the portfolio. This analysis clearly align with the intuitive understanding of financial domain. The advantage of this analysis is that it can measure such intuitive understanding in terms of Bayesian probability.

5 Conclusion

In this paper we discussed about Bayesian approach to regularize the ‘ill-posed’ covariance estimation problem. The method also analyze the source of risk by estimating the probability of positive CCTR. As CCTR sums up to total volatility, it provides each source’s contribution to total volatility. Regular method only estimates CCTR of a source, but it
does not estimate the probability of CCTR to be significantly greater (or less) than zero. This paper discussed the methodology to do so. We presented parallelizable and easy to implement Monte Carlo method to achieve that. We further presented an empirical study which showed that during ‘Lehman’s Bankruptcy’ crisis of 2008, a portfolio consists five asset classes experienced large volatility risk due to significant increase in the contribution of Stock and Commodity. During the same period ‘Bond’ was the asset class which contributed least to volatility risk. Such analysis can easily be implemented under the Bayesian probability framework.

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Table 1: Percentage return of five asset classes during the May-July, 2008

|       | Hybrid Bond | Emerging Mkt | Commodity | Bond | Stock |
|-------|-------------|--------------|-----------|------|-------|
| May 2008 | 0.13        | 3.53         | 2.56      | -0.94| 2.09  |
| Jun 2008 | -1.88       | -11.11       | 8.54      | 0.10 | -8.57 |
| Jul 2008 | -0.82       | -3.03        | -12.77    | 0.00 | -0.74 |

Table 2: Percentage return of five asset classes during the Aug-Oct, 2008

|       | Hybrid Bond | Emerging Mkt | Commodity | Bond | Stock |
|-------|-------------|--------------|-----------|------|-------|
| Aug 2008 | 0.41        | -8.46        | -7.71     | 0.84 | 1.54  |
| Sep 2008 | -6.03       | -18.83       | -12.38    | -1.15| -9.72 |
| Oct 2008 | -13.27      | -37.43       | -25.01    | -0.53| -26.98|

Table 3: Portfolio weight of five asset classes

|       | Hybrid Bond | Emerging Mkt | Commodity | Bond | Stock |
|-------|-------------|--------------|-----------|------|-------|
| Weight | 5%          | 5%           | 20%       | 40%  | 30%   |

Table 4: Posterior Statistics of Total Portfolio Volatility

| Asset Class | Posterior Statistics | Period 1          | Period 2          |
|-------------|-----------------------|-------------------|-------------------|
| Portfolio   | Mean (SD)             | 5.18 (15.50)      | 15.21 (44.88)     |
|             | 95% CI                | (1.05 , 22.67)    | (2.96 , 66.68)    |

Table 5: Table provide the Prob(ζ > 0) for all five asset class over two periods

| Asset Class | Period 1 | Period 2 |
|-------------|----------|----------|
| Hybrid Bond | 0.6027   | 0.9970   |
| Emerging Mkt| 0.7200   | 0.9992   |
| Commodity   | 0.7775   | 0.9966   |
| Bond        | 0.6968   | 0.7560   |
| Stock       | 0.7312   | 0.9997   |
Table 6: Posterior Statistics of CCTR five asset classes

| Asset Class | Posterior Statistics | Period 1       | Period 2       |
|-------------|----------------------|----------------|----------------|
| Hybrid Bond | Mean (SD)            | 0.05 (1.05)    | 0.71 (2.17)    |
|             | 95% CI               | (-0.70, 0.94)  | (0.12, 3.17)   |
| Emerging Mkt| Mean (SD)            | 0.29 (1.89)    | 1.50 (4.49)    |
|             | 95% CI               | (-1.18, 2.82)  | (0.28, 6.61)   |
| Commodity   | Mean (SD)            | 2.42 (12.55)   | 3.62 (10.83)   |
|             | 95% CI               | (-5.40, 16.71) | (0.61, 15.97)  |
| Bond        | Mean (SD)            | 0.90 (7.09)    | 0.55 (2.96)    |
|             | 95% CI               | (-4.29, 8.20)  | (-1.47, 4.01)  |
| Stock       | Mean (SD)            | 1.52 (8.23)    | 8.83 (26.18)   |
|             | 95% CI               | (-5.30, 12.82) | (1.67, 38.98)  |

Figure 1: plot showing posterior density of volatility of two periods.
Figure 2: Plot showing posterior density of CCTR for four asset classes - (i) Hybrid Bond, (ii) Emerging Mkt, (iii) Commodity and (iv) Stocks.
Figure 3: Plot showing posterior density of CCTR of Bond.