s-th power of Fibonacci number of the form

\[ 2^a + 3^b + 5^c \]

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Abstract: In this paper, we solve the Diophantine equation \( F_n^s = 2^a + 3^b + 5^c \), where \( a, b, c \) and \( s \) are positive integers with \( 1 \leq \max\{a, b\} \leq c \).

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1 Introduction

Let \((F_n)_{m \geq 0}\) be the Fibonacci sequence given by the relation \( F_n = F_{n-1} + F_{n-2} \) with \( F_0 = 0, F_1 = 1 \) for all \( n \geq 2 \). It has many amazing combinatorial identities (see [7]). Put \( \alpha = (1 + \sqrt{5})/2 \) and \( \beta = (1 - \sqrt{5})/2 \). Then the well-known Binet formula

\[ F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \]

holds for \( n \geq 0 \).

The problem of finding the different types of numbers among the terms of a linear recurrence has a long history. One of the popular results by Bageaud, Mignotte and Siksek [2] is that the integers 0, 1, 8, 144 among the Fibonacci numbers and the integers 1, 4 among the Lucas
numbers (an associated sequence of Fibonacci) can be written in the form $y^t$ where $t > 1$. Szalay and Luca [4] showed that there are only finitely many quadruples $(n, a, b, p)$ such that $F_n = p^a \pm p^b + 1$ where $p$ is a prime number. Marques and Togbé [5] determined the Fibonacci numbers and the Lucas numbers of the form $2^a + 3^b + 5^c$ under $1 \leq \max \{a, b\} \leq c$. Bertők, Hajdu, Pink and Rábai [1] removed this condition. Namely, they gave full solutions of the equation

$$U_n = 2^a + 3^b + 5^c,$$

where $U_n$ is the $n$-th Fibonacci, Lucas, Pell or Pell–Lucas number. We refer to the paper of Shorey and Stewart [9] for pure powers in recurrence sequences and some related Diophantine equations.

In this work, we generalize the problem of Marques and Togbé. We solve the Diophantine equation

$$F_n^s = 2^a + 3^b + 5^c$$

for $s \geq 1$ integer and $1 \leq \max \{a, b\} \leq c$. Our result is following.

**Theorem 1.1.** The solution of the equation (2) is $(n, s, a, b, c) = (3, 5, 2, 1, 2)$.

### 2 Auxiliary results

Before going further, we present several lemmas. The following lemma was given by Matveev [6].

**Lemma 2.1.** Let $K$ be a number field of degree $D$ over $\mathbb{Q}$, $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers of $K$, and $b_1, b_2, \ldots, b_t$ be rational integers. Put

$$B \geq \max \{|b_1|, |b_2|, \ldots, |b_t|\},$$

and

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be real numbers such that

$$A_i \geq \max \{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \quad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp \left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 \times (1 + \log D) (1 + \log B) A_1 \ldots A_t\right)$$

As usual, in the above lemma, the logarithmic height of the algebraic number $\eta$ is defined as

$$h(\eta) = \frac{1}{d} \left(\log a_0 + \sum_{i=1}^{d} \left(\max \{|\eta^{(i)}|, 1\}\right)\right)$$

with $d$ being the degree of $\eta$ over $\mathbb{Q}$ and $(\eta^{(i)})_{1 \leq i \leq d}$ being the conjugates of $\eta$ over $\mathbb{Q}$. Application of the Matveev theorem gives the large upper bound. In order to reduce this bound, we use the following lemma.
Lemma 2.2. Suppose that $M$ is a positive integer. Let $p/q$ be a convergent of the continued fraction expansion of the irrational number $\gamma$ such that $q > 6M$ and $\epsilon = \| \mu q \| - M \| \gamma q \|$, where $\mu$ is a real number and $\| \cdot \|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution to the inequality

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers $m$ and $n$ with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m < M.$$

The following lemma is in the paper [3] (the case $k = 2$).

Lemma 2.3. For every positive integer $n \geq 2$, we have

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1},$$

where $\alpha$ is the dominant root of the characteristic equation $x^2 - x - 1 = 0$.

Lemma 2.4. There is no solution of the equation

$$2^s = 2^a + 3^b + 5^c$$

for $1 \leq \max\{a, b\} \leq c$, $c \geq 6$ and $s$ being positive integers.

Proof. By (4) together with the facts that $2 < \sqrt{5}$ and $3 < 5^{0.7}$, we get

$$|1 - 2^s5^{-c}| < \frac{2}{(1.6)^c}. \quad (5)$$

We take $\alpha_1 := 2$, $\alpha_2 := 5$, $b_1 := s$, $b_2 := c$. For this choice $D = 1$, $t = 2$, $B = s$ and $A_1 = 0.7 > \log 2$, $A_2 = 1.61 > \log 5$. The Lemma 2.1 yields that

$$\exp(C \cdot (1 + \log s)) < |1 - 2^s5^{-c}| < \frac{2}{(1.6)^c}, \quad (6)$$

where $C := 1.4 \cdot 305 \cdot 2^{4.5} \cdot 0.7 \cdot 1.61$. Since $2^s < 5^{c+1}$, we have that $0.4s < c+1$. So, the inequality

$$s < 2.5 \cdot 10^{11}$$

is obtained. Let $z := |s \log 2 - c \log 5|$. Note that (5) can be written as

$$|1 - e^z| < \frac{3.2}{(1.2)^s},$$

since $0.4s < c + 1$ holds. Since $1 < 2^s5^{-c}$, then $z > 0$ holds. We obtain that

$$0 < |s \log 2 - c \log 5| < |1 - e^z| < \frac{3.2}{(1.2)^s}.$$
Dividing both sides by $\log 5$ yields that
\[
\left| s \frac{\log 2}{\log 5} - c \right| < \frac{2}{(1.2)^s}.
\]

Let $\gamma := \frac{\log 2}{\log 5}$ and $[a_0, a_1, a_2, \ldots] = [0, 2, 3, 9, 2, \ldots]$ be the continued fraction of $\gamma$, and let $p_k/q_k$ be its $k$-th convergent. Mathematica reveals that $q_{23} < 2.5 \cdot 10^{11} < q_{24}$.

$a_M := \max \{a_i; i = 0, \ldots, 24\} = a_{23} = 42$. By the properties of continued fractions, we obtain
\[
\frac{1}{(a_M + 2) s} < \left| s \frac{\log 2}{\log 5} - c \right| < \frac{2}{(1.2)^s}
\]
which yields that $s \leq 45$ as $a_M = 42$. Since $5^c < 2^s$, then we deduce that $c \leq 19$. A quick inspection using Mathematica reveals that there is no solution of the equation (4) with $1 \leq \max \{a, b\} \leq c$ and $6 \leq c \leq 19$.

3 Proof of Theorem 1.1

Firstly, assume that $1 \leq c \leq 5$. Then the solution of the equation (2) is given in Theorem 1.1. From now on, suppose that $c \geq 6$. Lemma 2.3 gives that $\alpha_s^{(n-2)} < F_n^s < 3 \cdot 5^c < 5^{1.1c}$. So, we have the fact $s < c$. Since $a, b, c \geq 1$, then $n \geq 3$ holds. If $n = 3$, then we can rewrite formula (2) as
\[
2^s = 2^a + 3^b + 5^c.
\]
which is investigated in Lemma 2.4. If $n = 4$ and $n = 5$ hold, then we arrive at a contradiction since the left-hand side of the equation $F_n^s = 2^a + 3^b + 5^c$ is odd, while right-hand side is even. Therefore, we suppose that $n \geq 6$.

Using formula (1), we rewrite the equation (2) as
\[
F_n^s - 5^c = 2^a + 3^b. \tag{7}
\]
Since $\max \{a, b\} \geq 1$, then the right-hand side of above equation is positive. Dividing both sides of the equation (7) by $5^c$, we obtain
\[
\left| F_n^s 5^{-c} - 1 \right| < \frac{2}{5^{0.3c}} \tag{8}
\]
where we use $2 < 3 < 5^{0.7}$.

In the application of theorem of Matveev, we take $\alpha_1 = F_n, \alpha_2 = 5, b_1 = s, b_2 = c$. We also take
\[
\Lambda := F_n^s 5^{-c} - 1.
\]
Since we assume $n \geq 6$, then it is obvious that $\Lambda \neq 0$. We can take the degree $D = 1$. Then $A_1 = \log F_n$ and $A_2 = 1.61 > \log 5$ follow. As $s < c$, then we get $B = c$ together with $t = 2$.
After applying the inequality (3) to get lower bound for the form \( \Lambda \), then we have
\[
e^{-C_{2,1}(1+\log c)\times 1.61\times \log F_n} < \frac{2}{5^{0.3c}},
\]
where \( C_{2,1} = 1.4 \times 30^5 \times 2^{4.5} \). Hence, we obtain that
\[
\frac{c}{\log c} < 2.5 \times 10^9 (n - 1),
\]
where we used the fact \( 1 + \log c < 2 \log c \). It is easy to prove that \( \frac{x}{\log x} < A \) yields \( x < 2A \log A \).
After rewriting the formula (9), we obtain
\[
c < 7.3 \times 10^{10} (n - 1) \log (n - 1)
\]
by the inequality 21.64 + \( \log (n - 1) < 14.6 \times \log (n - 1) \).

Assume that \( n \in [6, 233] \). Label \( z := s \log F_n - c \log 5 \). Hence, by the equation (8)
\[
0 < z < e^z - 1 < \frac{2}{5^{0.3c}}
\]
follows. Dividing both sides by \( \log 5 \), we obtain
\[
0 < s\gamma - c < 1.25 \times 5^{-0.3c}
\]
where \( \gamma := \frac{\log F_n}{\log 5} \). Let \([a_0, a_1, a_2, \ldots] \) be the continued fraction of \( \gamma \), and let \( p_k/q_k \) be its \( k \)-th convergent. We have
\[
s < c < 9.23 \times 10^{13}
\]
by the inequality (11). A quick inspection using \textit{Mathematica} reveals that \( q_{40} > M \). Moreover, \( a_M := \max \{a_i, i = 0, 1, \ldots, 40\} = 3996 \). From the properties of continued fractions, we get that
\[
|s\gamma - c| > \frac{1}{(a_M + 2) s}.
\]
Comparing the estimates (12) and (13) we get
\[
\frac{1}{3998s} < 1.25 \times 5^{-0.3c} \Rightarrow 5^{0.3s} < 5^{0.3c} < 4997.5s,
\]
which yields that \( s \leq 23 \). Hence, \( c \leq 1581 \) follows. In order to decrease the upper bound for \( c \), we use that \( \nu_5 (F_n^s - 2^a - 3^b) = c \). Thus, \textit{Mathematica} returns \( \nu_5 (F_n^s - 2^a - 3^b) = c \leq 12 \) for \( 1 \leq s \leq 23 \), \( 6 \leq n \leq 233 \) and \( c > \max \{a, b\} \geq 1 \). Therefore, \( c \leq 12 \) gives that \( n \leq 44 \). Then the solutions of the equation (2) are given Theorem 1.1.

From now on, assume that \( n > 233 \). In order to find the upper bound for \( c \), we use the key argument in the paper [8]. Let \( x := \frac{s}{\alpha^{2n}} \). From the above inequality (11), it follows that
\[
x < \frac{7.3 \times 10^{10} (n - 1) \log (n - 1)}{\alpha^{2n}} < \frac{2}{\alpha^n},
\]
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where it holds for $n > 233$. We now write

$$F_n^s = \frac{\alpha^{ns}}{5^s} \left(1 - \frac{(-1)^n}{\alpha^{2n}}\right)^s. \quad (14)$$

In the paper of Luca and Oyono [8], it was proven that

$$\left| \left(1 - \frac{(-1)^n}{\alpha^{2n}}\right)^s - 1 \right| < \frac{2}{\alpha^n}. \quad (15)$$

Let $\Lambda_2 := 5^{c+\frac{7}{2} \alpha^{-ns}} - 1$. From the formulas (2) and (14) together with the inequality (15), we have

$$|\Lambda_2| < \frac{2}{\alpha^n} + \frac{(2^a + 3^b)\, 5^{\frac{7}{2}}}{\alpha^{ns}}. \quad (16)$$

For the inequality (16) the facts that $2^a + 3^b < 2 \times 5^{0.7c}$ and $n > 233$ yield that

$$|\Lambda_2| < 0.8.$$ 

The last inequality gives that $\frac{5^{\frac{7}{2}}}{\alpha^{ns}} < \frac{2}{5^c}$. The inequality (16) yields

$$|\Lambda_2| < \frac{2}{\alpha^c} + \frac{2}{\alpha^n} = \frac{4}{\alpha^l},$$

where $l = \min \{n, c\}$. We use again the theorem of Matveev. We take $k = 2$, $\alpha_1 := \alpha$, $\alpha_2 := 5$, $b_1 := ns$, $b_2 := c + \frac{s}{2}$. As in the previous application of Matveev’s result, we can take $D := 2$, $A_1 := 0.5$, $A_2 := 1.61$. Note that $\alpha^c < 5^c < \alpha^{s(n-1)}$ gives $c < s < ns$. So, we take $B := ns$. We thus get that

$$\exp\left(-C_{2,2} \left(1 + \log ns\right) \times 0.5 \times 1.61\right) < \frac{4}{\alpha^l},$$

where $C_{2,2} = 1.4 \times 30^5 \times 2^{4.5} \times 4 \left(1 + \log 2\right)$. This leads to

$$l < \frac{C_{2,2} \left(\log ns\right) \times 1.61}{\log \alpha}.$$ 

If $l = n$, then the last inequalities

$$n < \frac{C_{2,2} \left(\log ns\right) \times 1.61}{\log \alpha}$$

$$< \frac{C_{2,2} \left(\log n \left(7.3 \times 10^{10} \left(n - 1\right) \log \left(n - 1\right)\right)\right) \times 1.61}{\log \alpha}$$

$$< \frac{C_{2,2} \left(\log \left(7 \times 10^{10} n^3\right)\right) \times 1.61}{\log \alpha}$$

give that $n < 1.92 \times 10^{12}$. By the inequality (11), we get

$$c < 7.3 \times 10^{10} \left(n - 1\right) \log \left(n - 1\right) < 4 \times 10^{24}.$$ 

If $l = c$, then we have that

$$c < \frac{C_{2,2} \left(\log ns\right) \times 1.61}{\log \alpha} < \frac{C_{2,2} \left(\log 6.8c\right) \times 1.61}{\log \alpha}$$
yields \( c < 6 \times 10^{11} \), where we used the fact
\[
\alpha^{ns} < 5^{1.2c} \alpha^{2s} < \alpha^{4.8c} \alpha^{2s} = \alpha^{6.8c}.
\]
At any rate, we get
\[
c < 4 \times 10^{24}.
\]
Next we take \( \Gamma := (\frac{s}{2} + c) \log 5 - ns \log \alpha \). Observe that \( \Lambda_2 = e^\Gamma - 1 \). Since \( |\Lambda_2| < 0.8 \), then we have \( |e^\Gamma - 1| < 0.8 \), which yields that \( e^{|\Gamma|} < 2 \). Hence,
\[
|\Gamma| \leq e^{|\Gamma|} |e^\Gamma - 1| < 2 |\Lambda_2| < \frac{2}{\alpha^c} + \frac{2}{\alpha^n}.
\]
This leads to
\[
\left| \frac{\log \alpha}{\log 5} - \frac{c + s}{ns} \right| = \left| \frac{\log \alpha}{\log 5} - \frac{2c + s}{2ns} \right| < \frac{1}{ns \log 5} \left( \frac{2}{\alpha^c} + \frac{2}{\alpha^n} \right) < \frac{1}{374s} \left( \frac{2}{\alpha^c} + \frac{2}{\alpha^n} \right) (17)
\]
since \( n > 233 \). Assume that \( c \geq 30 \). In this case, note that \( \alpha^n > 160c^2 \) (as \( n < ns < 6.8c \)) and \( \alpha^c > 160c^2 \). Hence, we get that by the inequality (17) by the fact \( \alpha^{ns} < \alpha^{6.8c} \),
\[
\left| \frac{\log \alpha}{\log 5} - \frac{2c + s}{2ns} \right| < \frac{1}{14960s^2} < \frac{1}{14960c^2} < \frac{6.8^2}{14960 (ns)^2} < \frac{1}{80 (2ns)^2}. (18)
\]
By a criterion of Legendre, the rational number \( \frac{2c + s}{2ns} \) converts to \( \gamma := \frac{\log \alpha}{\log 5} \).

Let \([a_0, a_1, a_2, \ldots] = [0, 3, 2, 1, \ldots] \) be the continued fraction of \( \gamma \), and \( p_k/q_k \) be its \( k \)-th convergent. Assume that \( \frac{2c + s}{2ns} = \frac{p_t}{q_t} \) for some \( t \). We have \( q_{49} > 4 \times 10^{24} \). Thus, \( t \in \{0, 1, \ldots, 49\} \). Furthermore, \( a_k \leq 59 \), for \( k = 0, 1, \ldots, 49 \). From the well-known properties of continued fractions, we get that
\[
\left| \gamma - \frac{2c + s}{2ns} \right| = \left| \gamma - \frac{p_t}{q_t} \right| > \frac{1}{(at + 2) q_t^2} > \frac{1}{(at + 2) (2ns)^2} > \frac{1}{4 \times 61 \times 6.8^2 \times c^2}. (19)
\]
After combining the inequalities (18) and (19), then
\[
\frac{1}{4 \times 61 \times 6.8^2 \times c^2} < \frac{1}{14960c^2s}
\]
gives \( s < 1 \). But, this is not possible.

Therefore, \( c \) is at most 29. By the inequality \( ns < 6.8c \), we obtain that \( ns \leq 197 \), which is false as \( n > 233 \).

Hence, the proof theorem is completed.

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