A Refutation of Bell’s Theorem∗

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Bell’s Theorem was developed on the basis of considerations involving a linear combination of spin correlation functions, each of which has a distinct pair of arguments. The simultaneous presence of these different pairs of arguments in the same equation can be understood in two radically different ways: either as ‘strongly objective,’ that is, all correlation functions pertain to the same set of particle pairs, or as ‘weakly objective,’ that is, each correlation function pertains to a different set of particle pairs.

It is demonstrated that once this meaning is determined, no discrepancy appears between local realistic theories and quantum mechanics: the discrepancy in Bell’s Theorem is due only to a meaningless comparison between a local realistic inequality written within the strongly objective interpretation (thus relevant to a single set of particle pairs) and a quantum mechanical prediction derived from a weakly objective interpretation (thus relevant to several different sets of particle pairs).

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1. INTRODUCTION

Bell’s Theorem[1] exhibits a peculiar discrepancy between any local realistic theory and Quantum Mechanics, which leads to empirically distinguishable alternatives. The quandary is that neither local realistic conceptions nor Quantum Mechanics are easy to abandon. Indeed, classical physics and common sense are usually based upon the former, while the latter is rightly presented as the most successful theory of all times. Several experiments have been done, all but a few[2] show violations of Bell inequalities. Yet, the ideas brought forth by Bell’s Theorem are so disconcerting that there is still incredulity, not to mention antipathy, evoked by the verdict. The purpose of this article is to provide a refutation of this theorem, within a strictly quantum theoretical framework, without the use of outside assumptions.

Although experiments showing violations of Bell’s inequalities are getting increasingly accurate and loophole-free[3], it must be stressed that they, no matter how accurate and close to the ideal, can prove no more than the validity of quantum mechanics, and not the validity of the theorem. Herein, it will be assumed that all tests conducted so far prove conclusively the validity of Quantum Mechanics. In other words, the purpose of this article is not to criticise the numerous experiments, or quantum mechanics for that matter, but Bell’s Theorem itself.

2. THE EPRB GEDANKEN EXPERIMENT

2.1. Spin observables and the singlet state

Bell’s theorem is usually based on a didactic reformulation of the EPR (Einstein, Podolsky and Rosen[4]) gedanken experiment, due to D. Bohm[5]. In this EPRB gedanken experiment, a pair of spin-$\frac{\hbar}{2}$ particles with total spin zero is produced such that each particle moves away from the source in opposite directions along the y-axis. Two Stern-Gerlach devices are placed at opposite points (left and right) on the y-axis, and are oriented respectively along the directions $\mathbf{u}$ and $\mathbf{v}$.

The spin observable associated with a measurement given by a Stern-Gerlach device oriented along the unit vector $\mathbf{u}$ is $\sigma \cdot \mathbf{u}$ (to simplify notation, $\hbar/2$ is set to 1 throughout); where the components of $\sigma$ are then the Pauli matrices $\sigma_x$, $\sigma_y$, and $\sigma_z$. Let $H_L$ and $H_R$ be the Hilbert spaces associated with each Stern-Gerlach device respectively. The Hilbert space $H$ associated with the entire EPRB system is the direct product of the Hilbert spaces associated with each Stern-Gerlach device:

$$H \equiv H_L \otimes H_R.$$  \hspace{1cm} (1)
The spin observables relevant to $\mathcal{H}_L$ and $\mathcal{H}_R$ have their respective counterpart in this new product space $\mathcal{H}$ as

$$\sigma_L \cdot u \equiv \sigma \cdot u \otimes \mathbb{I}_R,$$

$$\sigma_R \cdot v \equiv \mathbb{I}_L \otimes \sigma \cdot v,$$

where $\mathbb{I}_L$ and $\mathbb{I}_R$ are the identity operators of $\mathcal{H}_L$ and $\mathcal{H}_R$. Contrary to the observables $\sigma \cdot u$ and $\sigma \cdot v$ which are mutually non commuting when $u \neq v$, these new observables $\sigma_L \cdot u$ and $\sigma_R \cdot v$ do commute, reflecting the fact that the Stern-Gerlach devices are arbitrarily far from each other, and are thus measuring distinct subsystems. The product of these two observables

$$(\sigma_L \cdot u)(\sigma_R \cdot v) = \sigma \cdot u \otimes \sigma \cdot v$$

is therefore also an observable and can be understood as a spin correlation observable corresponding to the joint spin measurement of both Stern-Gerlach devices.

The product space $\mathcal{H}$ is spanned by the product basis formed by the four eigenvectors $\{|+\rangle, |+\rangle, |--\rangle, |--\rangle\}$ associated with the spin correlation observable $(\sigma_L \cdot u)(\sigma_R \cdot v)$ where $u$ is a unitary vector. In an EPRB gedanken experiment, the source produces particle pairs with zero total spin, represented by the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[ |+\rangle - |+\rangle \right].$$

This singlet state has the important property of being invariant under rotation, which permits one to ignore the explicit form of $u$ in expressing the $\mathcal{H}$ basis (see, for instance, Ref. [6]).

### 2.2. Statistical properties of the singlet state

As it is, nothing certain can be said either about a single spin measurement, or about a single spin correlation measurement, performed on a system represented by the singlet state. According to the Born interpretation of the state vector, only probabilistic predictions—such as expectation values relevant to numerous measurements in the measurement, performed on a system represented by the singlet state. According to the Born interpretation of the $\langle \psi | A | \psi \rangle$ relevant to numerous measurements in the $\mathcal{H}$ space spanned by the product basis formed by the four eigenvectors $\{|+\rangle, |+\rangle, |--\rangle, |--\rangle\}$ associated with the spin correlation observable $(\sigma_L \cdot u)(\sigma_R \cdot v)$ where $u$ is a unitary vector. In an EPRB gedanken experiment, the source produces particle pairs with zero total spin, represented by the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[ |+\rangle - |+\rangle \right].$$

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#### 2.3. Perfect correlations and hidden-variables

When $u = v$, the expectation value of the spin correlation observable $\langle \psi | A | \psi \rangle$ is equal to $-1$, meaning that if both Stern-Gerlach devices are oriented along the same direction, then with certainty the outcomes will be found to be opposite. Since the Stern-Gerlach devices are arbitrarily far from each other, a perfect correlation can be understood from a realistic point of view as invested in the particles at their inception. This, however, would mean that the singlet state is incomplete, and therefore, that it should be possible to give a more precise specification using additional “hidden-variables”. On the other hand, if a more complete description is impossible, then this perfect correlation seems rather mysterious, since a measurement performed on one of the subsystems seems to be capable of influencing the measurement on the other subsystem, whatever the distance between them.

In order to facilitate a choice between incompleteness and non-locality of Quantum Mechanics, Bell’s idea was to specify mathematical requirements for a generic local hidden-variables theory, and then to compare its predictions with those from quantum mechanics and the results of experiments.
In a local realistic hidden-variables model, a single particle pair is thus supposed to be entirely characterised by means of a set of hidden-variables, which are symbolically represented by a parameter $\lambda$, so that the measurement result on the left along $u$ can be written as $A(u, \lambda)$, and the result on the right along $v$ as $B(v, \lambda)$. Although the hidden-variables model is supposed to be fully deterministic, it must also be capable of reproducing the stochastic nature of the EPRB gedanken experiment expressed in Eqs. (6a) and (6b). For that purpose, the complete state specification $\lambda_i$ of any particle pair with label $i$ must be a random variable: its complete state $\lambda_i$ is supposed to be drawn randomly according to a probability distribution $\rho$ (see Refs. [2] and [3]), meaning that the probability of having $\lambda_i$ equal to a particular $\lambda$ is $\rho(\lambda)$.

Consider a set of $N$ particle pairs $\{i = 1, \ldots, N\}$, the mean value of joint spin measurements for this set is:

$$M^\rho(u, v) = \frac{1}{N} \sum_{i=1}^{N} A(u, \lambda_i)B(v, \lambda_i).$$

(7)

The probability distribution $\rho$ is supposed to assure the equality between this mean value and the expected value, Eq. (6b), given by Quantum Mechanics when $N$ goes to infinity.

3. THE ‘CHSH’ FUNCTION

In order to establish Bell’s Theorem, a linear combination of correlation functions $c(a, b)$ with different arguments is considered, once when these correlation functions are expectation values $E^\psi(u, v)$ given by Quantum Mechanics; i.e., Eq. (6); and once when they are mean values $M^\rho(u, v)$ given by local hidden-variables theories, Eq. (7); then the results are to be compared. A well known choice of such a linear combination is the CHSH (Clauser, Horne, Shimony and Holt) function, written with four pairs of arguments:

$$S \equiv |c(a, b) - c(a, b') + c(a', b) + c(a', b')|.$$  

(8)

Bell’s Theorem consists in showing that the quantum prediction for the CHSH function violates the maximum possible value given by any local realistic hidden-variables theory. Thus, no such theory will ever be capable of explaining or reproducing these quantum results. Herein, this claim is refuted by demanding the rules of Quantum Mechanics be consistently and meaningfully applied.

To begin, the exact meaning of the simultaneous presence of different arguments in a CHSH function must be clarified. Basically, there are two possible interpretations, the strongly objective interpretation and the weakly objective interpretation:

**Strongly Objective Interpretation** implies that all correlation functions are relevant to the same set of $N$ particle pairs, that is, all four pairs of directions are considered simultaneously relevant to each particle pair. As such they cannot be relevant to actual experiments but rather with what result would have been obtained if measured on the same set of $N$ particle pairs along different directions.

**Weakly Objective Interpretation** implies that each correlation function is actually to be measured on distinct sets of $N$ particle pairs. Each set of $N$ particle pairs pertains to only one pair of arguments, that is, for each pair only one joint spin measurement is executed.

The CHSH function was developed specifically for experimental convenience. Many experiments have been done (the most famous being Aspect’s) obviously invoking the natural interpretation, namely the weakly objective one. Nevertheless, the strongly objective interpretation must also be considered, since it remains a possible interpretation a priori, and since the choice between strong and weak objectivity is not made at all explicit in many papers, including Bell’s.

It must be stressed, moreover, that these interpretations are radically different, not only epistemologically, but also physically. Indeed, the strongly objective interpretation pertains to a single set of $N$ particle pairs characterised by the corresponding set of parameters $\{\lambda_i : i = 1, \ldots, N\}$; whereas the weakly objective interpretation pertains to no less than 4 sets of $N$ particle pairs. The fact is that a finite set of $N$ particle pairs characterised by $\{\lambda_i\}$ can’t be identically reproduced, either theoretically (for each complete state $\lambda_i$ of any particle pair $i$ is a random variable, as defined in Section 2.3), or empirically (for the experimenter has no control over the complete state of a particle pair in a singlet state). Of course, when $N$ goes to infinity, these four sets of $N$ particle pairs necessarily converge to the same ideal set described by the probability distribution $\rho$. However, as soon as real experiments are concerned, then $N \neq \infty$ and these four sets are necessarily four different sets of particle pairs (see Ref. [3], page 348) respectively characterised...
by four different sets of hidden-variables parameters \( \{\lambda_{1,i}\}, \{\lambda_{2,i}\}, \{\lambda_{3,i}\} \) and \( \{\lambda_{4,i}\} \) (for an alternate approach, see \[15\]). The difference between each interpretation can therefore be embodied in the number of degrees of freedom of the whole system. Let \( f \) be the degrees of freedom of a single particle pair. In the strongly objective interpretation the degrees of freedom of the whole CHSH system is then \( Nf \), whereas in the weakly objective interpretation it is 4 times as large, that is, \( 4Nf \).

Thus, before initiating Bell’s analysis, one has to choose explicitly one interpretation and stick to it. Unfortunately, this is not what has been done. It will be shown here that the discrepancy exhibited by Bell’s Theorem is due to a meaningless comparison between strongly objective and weakly objective results, which means comparing the numerical value of the CHSH function for two systems, one having \( Nf \) degrees of freedom, the other \( 4Nf \).

4. THE STRONGLY OBJECTIVE INTERPRETATION: COUNTERFACTUAL PROPERTIES OF N PARTICLE PAIRS

4.1. A Local realistic inequality within the strongly objective interpretation

The local realistic formulation of the CHSH function within strong objectivity is written

\[
S_{\text{strong}}^\rho = \left| M^\rho(a, b) - M^\rho(a, b') + M^\rho(a', b) + M^\rho(a', b') \right|, \tag{9}
\]

or explicitly (using Eq. \[7\])

\[
S_{\text{strong}}^\rho = \left| \frac{1}{N} \sum_{i=1}^{N} A(a, \lambda_i)B(b, \lambda_i) - A(a, \lambda_i)Bb', \lambda_i \right. \\
+ \left. A(a', \lambda_i)B(b, \lambda_i) + A(a', \lambda_i)B(b', \lambda_i) \right|, \tag{10}
\]

which after factorisation becomes

\[
S_{\text{strong}}^\rho = \left| \frac{1}{N} \sum_{i=1}^{N} A(a, \lambda_i) \left[B(b, \lambda_i) - B(b', \lambda_i)\right] \\
+ A(a', \lambda_i) \left[B(b, \lambda_i) + B(b', \lambda_i)\right] \right|, \tag{11}
\]

where each term can have two values in the summation \[3, 8\]

\[
A(a, \lambda_i) \left[B(b, \lambda_i) - B(b', \lambda_i)\right] \\
+ A(a', \lambda_i) \left[B(b, \lambda_i) + B(b', \lambda_i)\right] = \pm 2, \tag{12}
\]

so that the most restrictive local realistic inequality within the strongly objective interpretation is:

\[
S_{\text{strong}}^\rho \leq 2. \tag{13}
\]

This is the well known generalised formulation of Bell’s inequality due to CHSH \[11\]. It must be stressed once more, however, that this inequality has been established only within the strongly objective interpretation, which means that each expectation value is relevant to the same set of \( N \) particle pairs. Hence, this result cannot be compared directly with results from real experimental tests, where in fact mean values from four distinct sets of \( N \) particle pairs are measured. The question whether the same inequality can be applied to real experiments will be discussed in Section 5.2 (weak objectivity).

4.2. The Quantum mechanical prediction within the strongly objective interpretation

The quantum prediction for the CHSH function within the strongly objective interpretation is written

\[
S_{\text{strong}}^\psi = |E^\psi(a, b) - E^\psi(a, b') + E^\psi(a', b) + E^\psi(a', b')|. \tag{14}
\]
This equation is usually directly evaluated by replacing each expectation value by the scalar product result of Eq. (19). This, unfortunately, is all too hasty.

Indeed, in order to understand better the quantum mechanical meaning of Eq. (14), it is advantageous to take a step backward using Eq. (10)

\[ S^\psi_{\text{strong}} = \left| \langle \psi | (\sigma_L \cdot a)(\sigma_R \cdot b) | \psi \rangle - \langle \psi | (\sigma_L \cdot a')(\sigma_R \cdot b') | \psi \rangle \right| 
+ \left| \langle \psi | (\sigma_L \cdot a')(\sigma_R \cdot b) | \psi \rangle + \langle \psi | (\sigma_L \cdot a')(\sigma_R \cdot b') | \psi \rangle \right| ,
\] (15)

or again

\[ S^\psi_{\text{strong}} = \left| \langle \psi | (\sigma_L \cdot a)(\sigma_R \cdot b) - (\sigma_L \cdot a)(\sigma_R \cdot b') + (\sigma_L \cdot a')(\sigma_R \cdot b) + (\sigma_L \cdot a')(\sigma_R \cdot b') | \psi \rangle \right|. \]

Note, however, that the four spin correlation observables in this equation are non commuting observables (this can be shown by calculating the commutator of \((\sigma_L \cdot u)(\sigma_R \cdot v)\) and \((\sigma_L \cdot u')(\sigma_R \cdot v')\) with \(v \neq v'\), so that the meaning of their combination must be questioned. The problem is that it is actually impossible to find an eigenvector for this combination of observables. Indeed, this linear combination of observables is, after factorisation, an operator of the form \(A \otimes B + \hat{C} \otimes \hat{D}\) with \([A, \hat{C}] \neq 0\) and \([\hat{B}, \hat{D}] \neq 0\). In the Hilbert space \(H\), an hypothetical eigenvector \(|\phi \rangle \otimes |\chi\rangle\) (with \(\alpha\) being its eigenvalue) of this operator should satisfy

\[ [\hat{A} \otimes \hat{B} + \hat{C} \otimes \hat{D}] |\phi \rangle \otimes |\chi\rangle = \alpha (|\phi \rangle \otimes |\chi\rangle) , \]

that is,

\[ \hat{A}|\phi \rangle \otimes \hat{B}|\chi\rangle + \hat{C}|\phi \rangle \otimes \hat{D}|\chi\rangle = \alpha (|\phi \rangle \otimes |\chi\rangle) . \]

This equation can have solutions only if its left hand side can be factored; that is, either \(\hat{A}|\phi \rangle\) and \(\hat{C}|\phi \rangle\), or \(\hat{B}|\chi\rangle\) and \(\hat{D}|\chi\rangle\) must be colinear vectors. This, however, can never happen because both \([A, \hat{C}] \neq 0\) and \([\hat{B}, \hat{D}] \neq 0\). Hence, Eq. (17) has no solution, and the linear combination of observables in Eq. (10) has no eigenvector: it is not an observable, and thus it can’t be given physical meaning. Therefore, \(S^\psi_{\text{strong}}\) is meaningless and is not a proper equation to use in order to make physical predictions.

Of course, this does not imply that Quantum Mechanics cannot provide any meaning at all for the CHSH function; it implies only that this meaning cannot be strongly objective. Indeed, according to Von Neumann \(14\), any linear combination of expectation values of different observables \(\hat{R}, \hat{S}, \ldots\) is meaningful in Quantum Mechanics:

\[ \langle \hat{R} + \hat{S} + \ldots \rangle_\phi = \langle \hat{R} \rangle_\phi + \langle \hat{S} \rangle_\phi + \ldots \]

even if \(\hat{R}, \hat{S}, \ldots\) are non commuting observables. The explanation is that Quantum Mechanics is only a weakly objective theory \(12, 14\), and that expectation values given by Quantum Mechanics are also weakly objective statements, that is to say, statements relevant to observations. Hence, when \(\hat{R}, \hat{S}, \ldots\) are non commuting observables, the expectation values cannot be simultaneously relevant to the same set of \(N\) systems: each expectation value is necessarily relevant to a distinct set of \(N\) systems (all systems being represented by the quantum state \(|\phi\rangle\)). Likewise, the only possible meaning of Eq. (15) is therefore weakly objective, not strongly objective as desired.

Since these expectation values are known with certainty, it is tempting to consider them as counterfactual entities. However, counterfactuality requires at least measurement compatibility, that is, commuting observables. The certainty of a contextual prediction is not sufficient to make it a counterfactual prediction; in other words, weakly objective results known with certainty are not strongly objective results. Incidentally, this is also true in the case of perfect correlations, so that as a general rule, one may not manipulate weakly objective results as if they were strongly objective.

The local realistic inequality \(S^\psi_{\text{strong}}\) cannot be compared with any strongly objective prediction given by Quantum Mechanics, so that Bell’s Theorem cannot be verified with a strongly objective interpretation given to the CHSH function, simply because Quantum Mechanics is not a strongly objective theory.

This restriction is the first part of a refutation of Bell’s theorem, though maybe not conclusive, since the strength of Bell’s Theorem is mainly its amenability to experimental test. Still, this was necessary, for now that a strongly objective interpretation is precluded, there is no choice but to rely on the weakly objective interpretation in order to compare hidden-variables theories and Quantum Mechanics.

In the next section, a simple method will be provided in order to obtain a unique and meaningful quantum prediction for the CHSH function within weak objectivity.
5. THE WEAKLY OBJECTIVE INTERPRETATION : CONTEXTUAL MEASUREMENTS ON 4 DISTINCT SETS OF N PARTICLE PAIRS

5.1. A Quantum mechanical prediction within the weakly objective interpretation

It was shown in Section 3 that strong objectivity and weak objectivity pertain to different physical systems. This difference should therefore appear in the relevant equations. Indeed, the correlation expressed in Eq. (5) is relevant to spin measurements performed on particles that once constituted a single parent particle. Yet, two particles issued from two distinct parents never have interacted with each other, so that spin measurements performed on such particle pairs can not be correlated. Hence, if left and right spin measurements are performed on two distinct sets of $N$ particle pairs, instead of the same set, there should be no correlation, and this property should appear in a generalised spin correlation function (i.e. generalised to the case of spin measurements performed on different sets of particle pairs).

This can be easily done within a quantum theoretical framework by means of a distinct EPRB space for each set of $N$ particle pairs. Let $H_j$ be the EPRB Hilbert space associated with the $j$th set of particle pairs. In this Hilbert space, the EPRB gedanken experiment is represented by the singlet state $|\psi_j\rangle$ (see Section 2),

$$|\psi_j\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle_j - |-\rangle_j\right].$$

(20)

The whole CHSH experiment with the four sets of particle pairs can be expressed then in terms of a new direct product space $H_{1234} \equiv H_1 \otimes H_2 \otimes H_3 \otimes H_4$ in which the state vector is

$$|\psi_{1234}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \otimes |\psi_4\rangle.$$  

(21)

The counterparts of observables in $H_{1234}$ are obtained as in Section 2.1. For instance, the observables pertaining to the right Stern-Gerlach device for the 1st, 2nd, 3rd and 4th set of particle pairs are respectively

$$\sigma_{1,R} \cdot u \equiv (\sigma_R \cdot u) \otimes I_2 \otimes I_3 \otimes I_4,$$

(22a)

$$\sigma_{2,R} \cdot u \equiv I_1 \otimes (\sigma_R \cdot u) \otimes I_3 \otimes I_4,$$

(22b)

$$\sigma_{3,R} \cdot u \equiv I_1 \otimes I_2 \otimes (\sigma_R \cdot u) \otimes I_4,$$

(22c)

$$\sigma_{4,R} \cdot u \equiv I_1 \otimes I_2 \otimes I_3 \otimes (\sigma_R \cdot u),$$

(22d)

where $I_j$ is the identity operator of the EPRB space $H_j$. Hence, the expectation value of the product of two spin observables, the first belonging to the $k$th set and the second to the $l$th set, is

$$E_{kl}^{\psi}(u, v) \equiv \langle \psi_{1234}\rangle |(\sigma_{k,L} \cdot u)(\sigma_{l,R} \cdot v)|\psi_{1234}\rangle.$$  

(23)

and this is the generalised expectation value of spin correlation observables that was sought. The expectation value for measurements performed on the same set ($k = l$) of particle pairs is already known, Eq. (4), and $E_{kk}^{\psi}(u, v)$ should provide the same result. Indeed, using Eqs. (22) and (22) leads to

$$E_{kk}^{\psi}(u, v) = \langle \psi_k\rangle |(\sigma_L \cdot u)(\sigma_R \cdot v)|\psi_k\rangle$$

$$= -u \cdot v,$$

(24)

but when $k \neq l$, the result is quite different:

$$E_{kl}^{\psi, k \neq l}(u, v) = \langle \psi_k\rangle |(\sigma_L \cdot u)(\sigma_R \cdot v)|\psi_k\rangle$$

$$= \langle \psi_k\rangle |(\sigma \cdot u \otimes I_R)|\psi_k\rangle$$

$$= \langle \psi_k\rangle |(\sigma \otimes I_L)|\psi_k\rangle$$

$$= 0,$$

(25)

in accord with Eq. (3). There are indeed no correlations between two sets of particle pairs, as stipulated in the beginning of this section.

Now, contrary to what was done in Section 4.2, it is possible to proceed here in full accord with the quantum mechanical postulates, because the spin correlation observables, Eqs. (22), are mutually commuting, so that a linear combination of these commuting observables is an observable as well. The CHSH experiment can therefore be described by a new observable

$$\hat{S}_{\text{weak}} \equiv (\sigma_{1,L} \cdot a)(\sigma_{1,R} \cdot b) - (\sigma_{2,L} \cdot a)(\sigma_{2,R} \cdot b')$$

$$+ (\sigma_{3,L} \cdot a')(\sigma_{3,R} \cdot b) + (\sigma_{4,L} \cdot a')(\sigma_{4,R} \cdot b'),$$

(26)
and the quantum prediction for the CHSH function within a weakly objective interpretation is therefore obtained by calculating the expectation value of the observable $S_{\text{weak}}$ when the system is in the quantum state $|\psi_{1234}\rangle$:

$$S_{\text{weak}}^\psi = \langle \psi_{1234} | S_{\text{weak}} | \psi_{1234} \rangle,$$

which using Eqs. (22) and (23) is

$$S_{\text{weak}}^\psi = \left| \langle \psi_1 | (\sigma_L \cdot a)(\sigma_R \cdot b) | \psi_1 \rangle - \langle \psi_2 | (\sigma_L \cdot a') (\sigma_R \cdot b) | \psi_2 \rangle \right. + \langle \psi_3 | (\sigma_L \cdot a)(\sigma_R \cdot b') | \psi_3 \rangle + \langle \psi_4 | (\sigma_L \cdot a') (\sigma_R \cdot b') | \psi_4 \rangle \right|,$$

that is, using Eq. (24),

$$S_{\text{weak}}^\psi = \left| E_{11}^\psi (a, b) - E_{22}^\psi (a, b') + E_{33}^\psi (a', b) + E_{44}^\psi (a', b') \right|.$$

This equation is not ambiguous (as was Eq. (15)): it is a linear combination of expectation values, each relevant to a distinct set of $N$ particle pairs. This equation is therefore weakly objective, as requested.

Finally, using Eq. (24), yields

$$S_{\text{weak}}^\psi = |a \cdot b - a \cdot b' + a' \cdot b + a' \cdot b'|,$$

with a well known maximum equal to

$$\max(S_{\text{weak}}^\psi) = 2\sqrt{2}.$$

This numerical result is indeed the one given in the literature, the only difference here being the fact that the meaning of this result is unambiguously weakly objective. Quantum Mechanics, which is a weakly objective theory [12], provides a clear answer to the CHSH function understood as a weakly objective question.

### 5.2. A Local realistic inequality within the weakly objective interpretation

The last step consists in comparing the quantum prediction $S_{\text{weak}}^\psi$ with its local realistic counterpart $S_{\text{weak}}^\rho$. As was stressed in Section 3, the $j$th set of particle pairs must be characterised by a distinct set of hidden-variables parameters $\{\lambda_{j,i} : i = 1, \ldots, N\}$. Hence, to the generalised expectation value of the spin correlation observable Eq. (23) corresponds the generalised mean value of joint spin measurements:

$$M_{kl}^\rho (u, v) \equiv \frac{1}{N} \sum_{i=1}^{N} A(u, \lambda_{k,i}) B(v, \lambda_{l,i}),$$

which is a priori capable of reproducing not only the $k = l$ prediction, Eq. (24), but also the $k \neq l$ prediction, Eq. (25). The local realistic CHSH function with a weakly objective interpretation is therefore

$$S_{\text{weak}}^\rho = \left| M_{11}^\rho (a, b) - M_{22}^\rho (a, b') + M_{33}^\rho (a', b) + M_{44}^\rho (a', b') \right|,$$

and that is explicitly

$$S_{\text{weak}}^\rho = \left| \frac{1}{N} \sum_{i=1}^{N} \left[ A(a, \lambda_{1,i}) B(b, \lambda_{1,i}) - A(a, \lambda_{2,i}) B(b', \lambda_{2,i}) + A(a', \lambda_{3,i}) B(b, \lambda_{3,i}) + A(a', \lambda_{4,i}) B(b', \lambda_{4,i}) \right] \right|.$$

This expression is to be compared with the one pertaining to the strongly objective interpretation, Eq. (14), which contained terms that could be factored. Here, since each term is different from the others, no factorisation is possible; i.e., there is no way to derive a Bell inequality. This is not the first time this fact has been noticed (see A. Bohm pp. 351, 352 [5]), unfortunately, no conclusion was drawn then. Yet, this fact cannot be ignored, for it has been shown in Section 4.2 that Bell’s Theorem cannot be demonstrated within a strongly objective interpretation.
Here, the only local realistic inequality that can be derived is obtained by considering (as was done with Eq. (12) the possible numerical values of each term of the summation in Eq. (34), that is,

\[ A(a, \lambda_1, i)B(b, \lambda_1, i) - A(a, \lambda_2, i)B(b, \lambda_2, i) + A(a', \lambda_3, i)B(b, \lambda_3, i) + A(a', \lambda_4, i)B(b', \lambda_4, i) = +4, +2, 0, -2, -4, \]

for which the extrema are +4 and -4, so that the narrowest local realistic inequality that can be derived from Eq. (34) is nothing but

\[ S_{\text{weak}} \leq 4. \]  

(36)

This most restrictive local realistic inequality (which can also be found in Accardi [18]) is not incompatible with the quantum mechanical prediction, as the maximum of \( S_{\psi} \) is \( 2\sqrt{2} \). This shows that experiments intended to test Bell’s Theorem were unfortunately not testing the strongly objective inequality (a Bell inequality, Eq. (13), but this weakly objective one, Eq. (36), since all experimental tests necessarily are executed in a weakly objective way, due to the irreducible incompatibility between spin measurements. As was stressed by Sica [19] and Accardi [18], a local realistic inequality is nothing but an arithmetic identity, and inequality (36) is definitely too lax to be violated by experimental tests.

6. CONCLUSION

It was shown that Bell’s Theorem cannot be derived, either within a strongly objective interpretation of the CHSH function, because Quantum Mechanics gives no strongly objective results for the CHSH function (see Section 4.2), or within a weakly objective interpretation, because the only derivable local realistic inequality is never violated, either by Quantum Mechanics or by experiments (see Section 5.2). It was demonstrated that the discrepancy in Bell’s Theorem is due only to a meaningless comparison between \( S_{\rho}^{\text{strong}} \leq 2 \) and \( S_{\psi}^{\text{weak}} = 2\sqrt{2} \), where the former is relevant to a system with \( Nf \) degrees of freedom, whereas the latter to one with \( 4Nf \) (see Section 3). The only meaningful comparison is between the weakly objective local realistic inequality \( S_{\rho}^{\text{weak}} \leq 4 \) and the weakly objective quantum prediction \( S_{\psi}^{\text{weak}} = 2\sqrt{2} \), but these results are not incompatible. Bell’s Theorem, therefore, is refuted.

[1] J. S. Bell, Physics 1, 195 (1964).
[2] F. Selleri, Le grand débat de la mécanique quantique (Champs Flammarion, Paris, 1986).
[3] A. Aspect, Nature 398, 189 (1999).
[4] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[5] D. Bohm, Phys. Rev. 85, 166 (1952).
[6] D. Greenberger, M. Horne, A. Shimony and A. Zeilinger, Am. J. Phys. 58, 1131 (1990).
[7] C. Cohen-Tannoudji, B.Diu, and F. Lalouë, Mécanique Quantique, vol. I (Hermann, Paris, 1973).
[8] A. Bohm, Quantum Mechanics, Foundations and applications (Springer-Verlag, New York, 1979).
[9] J. S. Bell, in Proceedings of the international School of physics ‘Enrico Fermi’, course II: Foundations of quantum mechanics (Academic, New York, 1971), p. 171.
[10] J. S. Bell, Epistemological Letters, p. 2 (July, 1975).
[11] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[12] B. d’Espagnat, Veiled Reality: An Analysis of Present Day Quantum Mechanical Concepts, (Addison-Wesley, 1995).
[13] B. d’Espagnat, http://arXiv/quant-ph/0802046.
[14] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).
[15] A. Khrennikov, http://arXiv/quant-ph/0006017.
[16] J. von Neumann, Mathematical Foundations of Quantum Mechanics (Princeton University Press, 1955).
[17] B. d’Espagnat, Conceptual foundations of Quantum Mechanics, (W.A. Benjamin, Massachusetts, 1976).
[18] L. Accardi, http://arXiv/quant-ph/0007005.
[19] L. Sica, Opt. Commun., 170, 55 (1999).