Privacy-Preserving Asynchronous Federated Learning Algorithms for Multi-Party Vertically Collaborative Learning

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Abstract

The privacy-preserving federated learning for vertically partitioned data has shown promising results as the solution of the emerging multi-party joint modeling application, in which the data holders (such as government branches, private finance and e-business companies) collaborate throughout the learning process rather than relying on a trusted third party to hold data. However, existing federated learning algorithms for vertically partitioned data are limited to synchronous computation. To improve the efficiency when the unbalanced computation/communication resources are common among the parties in the federated learning system, it is essential to develop asynchronous training algorithms for vertically partitioned data while keeping the data privacy. In this paper, we propose an asynchronous federated SGD (AFSGD-VP) algorithm and its SVRG and SAGA variants on the vertically partitioned data. Moreover, we provide the convergence analyses of AFSGD-VP and its SVRG and SAGA variants under the condition of strong convexity. We also discuss their model privacy, data privacy, computational complexities and communication costs. To the best of our knowledge, AFSGD-VP and its SVRG and SAGA variants are the first asynchronous federated learning algorithms for vertically partitioned data. Extensive experimental results on a variety of vertically partitioned datasets not only verify the theoretical results of AFSGD-VP and its SVRG and SAGA variants, but also show that our algorithms have much higher efficiency than the corresponding synchronous algorithms.

Keywords: Vertical federated learning, stochastic gradient descent, privacy-preserving, asynchronous distributed computation

1. Introduction

Federated learning facilitates the collaborative model learning without the sharing of raw data, and increasingly attracts attentions from both tech giants and industries where privacy protection is required. Especially, in the emerging multi-party joint modeling application, the data locate at multiple (two or more) data holders and each maintains its own records of
different feature sets with common entities, which are called as vertically partitioned data (Yang et al., 2019). While an integrated dataset improves the performance of a trained learning model, organizations cannot share data due to legal restrictions or competition between participants. For example, a digital finance company, an E-commerce company, and a bank collect different information of the same person. The digital finance company has access to online consumption, loan and repayment information. The E-commerce company has access to the online shopping information. The bank has customer information like average monthly deposit, account balance. If the person submits a loan application to the digital finance company, it might want to evaluate the credit risk of approving this financial loan by comprehensively utilizing the information stored in all the three parties. Such scenarios have been popularly appearing in recent industrial applications and raise the need of efficient federated learning algorithms on the vertically partitioned data.

For the vertically partitioned data, the direct access to the data in other providers or sharing of the data are often prohibited due to the legal and commercial issues. For the legal reason, most countries worldwide have made laws in protection of data security and privacy. For example, the European Union made the General Data Protection Regulation (GDPR) (EU, 2016) to protect users’ personal privacy and data security. The recent data breach by Facebook has caused a wide range of protests (Badshah, 2018). For the commercial reason, customer data is usually a valuable business asset for corporations. For example, the real online shopping information of customers can be used to train a recommended model which could provide valuable product recommendations to customers. Thus, both of the causes require federated learning on the vertically partitioned data without the disclosure of data.

In the literature, there are many privacy-preserving federated learning algorithms for vertically partitioned data in various applications, for example, cooperative statistical analysis (Du and Atallah, 2001), linear regression (Gascón et al., 2016; Karr et al., 2009; Sanil et al., 2004; Gascón et al., 2017), association rule-mining (Vaidya and Clifton, 2002), k-means clustering (Vaidya and Clifton, 2003), logistic regression (Hardy et al., 2017; Nock et al., 2018), XGBoost (Cheng et al., 2019), random forest (Liu et al., 2019a), support vector machine (Yu et al., 2006). From the optimization standpoint, (Wan et al., 2007) proposed privacy-preservation gradient descent algorithm for vertically partitioned data. (Zhang et al., 2018) proposed a feature-distributed SVRG algorithm (FD-SVRG) for high-dimensional linear classification. However, to the best of our knowledge, existing federated learning algorithms on the vertically partitioned data are limited to synchronous computation.

Stochastic gradient descent (SGD) algorithm (Bottou, 2010) and its variants (Gu et al., 2018a; Defazio et al., 2014; Schmidt et al., 2017; Fang et al., 2018; Gu et al., 2019a; Huo et al., 2018b) have been dominant to train large-scale machine learning problems. Specifically, at each iteration SGD independently samples a sample, and uses the stochastic gradient with respect to the sampled sample to update the solution. The stochasticity makes each iteration of SGD cheap while it also causes a large variance of stochastic gradients due to random sampling. To reduce the variance of stochastic gradients, the SGD variants with different variance reduction techniques (including SVRG (Gu et al., 2018a), SAGA (Defazio et al., 2014), SAG (Schmidt et al., 2017), SARAH (Pham et al., 2019), SPIDER (Fang et al., 2018)) were proposed to speed up SGD. SVRG and SAGA are the most popular ones among them. In addition, SGD and its adaptive variants (e.g., Adagrad, RMSProp and
Adam (Goodfellow et al., 2016) have shown their successes for the training of deep neural networks.

However, it is still vacant for SGD and its various variance reduction variants to train vertically partitioned data in parallel and asynchronously while keeping data and model privacy. To the best of our knowledge, FD-SVRG Zhang et al. (2018) is the only work of privacy-preservation SGD-like methods for vertically partitioned data. However, the updating rules in FD-SVRG Zhang et al. (2018) are executed synchronously. As we know, the asynchronous computation is much more efficient than the synchronous computation, because it keeps all computational resources busy all the time (please see Figure 1). Although there have been a lot of asynchronous SGD-like algorithms proposed to solve large-scale learning problems on horizontally partitioned data (Zhao and Li, 2016; Mania et al., 2015; Huo and Huang, 2017; Leblond et al., 2017; Meng et al., 2016; Gu et al., 2016; Kungurtsev et al., 2019; Gu et al., 2019b; Gu and Huo, 2018; Gu et al., 2018b; Huo and Gu, 2018; Huo et al., 2018a), it is still a challenge for SGD-like methods to train the vertically partitioned data asynchronously while keeping data and model privacy.

![Figure 1: Asynchronous computation vs. synchronous computation.](image)

To address this challenging problem, in this paper, we propose an asynchronous federated SGD (AFSGD-VP) algorithm and its SVRG and SAGA variants for vertically partitioned data. More importantly, we provide the convergence rates of AFSGD-VP and its SVRG and SAGA variants under the condition of strong convexity for the objective function. We also discuss their model privacy, data privacy, computational complexities and communication costs. To the best of our knowledge, the proposed algorithms are the first asynchronous federated learning algorithms for vertically partitioned data. Extensive experimental results on a variety of vertically partitioned datasets not only verify the theoretical results of AFSGD-VP and its SVRG and SAGA variants, but also show that our algorithms have much higher efficiency than the corresponding synchronous algorithms. We summarize the main contributions of this paper as follows.
1. We propose asynchronous federated stochastic gradient algorithm (i.e., AFSGD-VP) and its SVRG and SAGA variants for vertically partitioned data. We provide their convergence rates under the condition of strong convexity.

2. Based on the semi-honest assumption (i.e., Assumption [7]), we prove that our AFSGD-VP and its SVRG and SAGA variants can prevent the exact and approximate inference attacks.

Notations. In order to make notations easier to follow, we give a summary of notations in the following list.

- $\hat{w}$: $w$ that inconsistently read from different workers.
- $\tilde{w}$: The snapshot of $w$ after a certain number of iterations.
- $q$: The size of workers.
- $b^\ell$: A random number generated on the $\ell$-th worker.
- $\xi(t, \ell)$: The local time counter for the global time counter $t$ on the $\ell$-th worker.
- $\xi^{-1}(u, \ell)$: The corresponding global time counter to a local time counter $u$ on the $\ell$-th worker.
- $\psi(t)$: The corresponding worker to obtain $\hat{w}_t^T x_i$.
- $\psi^{-1}(\ell, K)$: All the elements in $K$ such that $\psi(\psi^{-1}(\ell, K)) = \ell$.
- $Leaf(\cdot)$: All leaves of a tree.

2. Asynchronous Federated Learning for Vertically Partitioned Data

In this section, we first introduce the problem addressed in this paper, and then give a brief review of SGD, SVRG and SAGA. Next, we give the system structure of our asynchronous federated learning algorithms. Finally, we propose our AFSGD-VP, AFSVRG-VP and AFSAAGA-VP algorithms.

2.1 Problem Statement

In this paper, we consider the model in a linear form of $w^T x$. Given a training set $S = \{(x_i, y_i)\}_{i=1}^l$, where $x_i \in \mathbb{R}^d$ and $y_i \in \{+1, -1\}$ for binary classification or $y_i \in \mathbb{R}$ for regression. The loss function w.r.t. the sample $(x_i, y_i)$ and the model weights $w$ can be formulated as $L(w^T x_i, y_i)$. Thus, we consider to optimize the following regularized empirical risk minimization problem.

$$\min_{w \in \mathbb{R}^d} f(w) = \frac{1}{l} \sum_{i=1}^l L(w^T x_i, y_i) + g(w),$$

(1)
where $g(w)$ is a regularization term, and each $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ is considered as a smooth, possibly non-convex function in this paper. Obviously, the empirical risk minimization problem is a special case of the problem (1). In addition to the empirical risk minimization problem, problem (1) summarizes an extensive number of important regularized learning problems, such as, $\ell_2$-regularized logistic regression [Conroy and Sajda (2012)], ridge regression [Shen et al. (2013)] and least squares SVM [Suykens and Vandewalle (1999)].

As mentioned previously, in a lot of real-world machine learning applications, the input of training sample $(x, y)$ is partitioned vertically into $q$ parts, i.e., we have a partition $\{G_1, \cdots, G_q\}$ of $d$ features. Thus, we have $x = [x_{G_1}, x_{G_2}, \ldots, x_{G_q}]$, where $x_{G_\ell} \in \mathbb{R}^{d_\ell}$ is stored on the $\ell$-th worker, and $\sum_{\ell=1}^q d_\ell = d$. According to whether the label is included in a worker, we divide the workers into two types: one is active worker and the other is passive worker, where the active worker is the data provider who holds the label of a sample, and the passive worker only has the input of a sample. The active worker would be a dominating server in federated learning, while passive workers play the role of clients [Cheng et al. (2019)]. We let $D^\ell$ denote the data stored on the $\ell$-th worker. Note that the labels $y_i$ are distributed on active workers. Our goal in this paper can be presented as follows.

**Goal:** Make active workers to cooperate with passive workers to solve the regularized empirical risk minimization problem (1) on the vertically partitioned data $\{D^\ell\}_{\ell=1}^q$ in parallel and asynchronously with the SGD and its SVRG and SAGA variants, while keeping the vertically partitioned data private.

### 2.2 Brief Review of SGD, SVRG and SAGA

As mentioned before, SGD-like algorithms have been the popular algorithms for solving large-scale machine learning problems. We first give a brief review of the update framework of SGD-like algorithms which include multiple variants of variance reduction methods. Specifically, given an unbiased stochastic gradient $v$ (i.e., $\mathbb{E}v = \nabla f(w)$), the updating rule of SGD-like algorithms can be formulated as follows.

$$w \leftarrow w - \gamma v$$

(2)

where $\gamma$ is the learning rate. In the following, we present the specific forms to the unbiased stochastic gradient $v$ w.r.t. SGD, SVRG and SAGA.

**SGD:** At each iteration SGD [Bottou (2010)] independently samples a sample $(x_i, y_i)$, and uses the stochastic gradient $\nabla f_i(w)$ with respect to the sampled sample $(x_i, y_i)$ to update the solution as follows.

$$v = \nabla f_i(w)$$

(3)

**SVRG:** For SVRG [Xiao and Zhang (2014); Gu et al. (2018a)], instead of directly using the stochastic gradient $\nabla f_i(w)$, they use an unbiased stochastic gradient $v$ as follows to update the solution.

$$v = \nabla f_i(w) - \nabla f_i(\bar{w}) + \nabla f(\bar{w})$$

(4)

where $\bar{w}$ denotes snapshot of $w$ after a certain number of iterations.
**SAGA:** For SAGA [Defazio et al. (2014)], the unbiased stochastic gradient $v$ is formulated as follows.

$$v = \nabla f_i(w) - \alpha_i + \frac{1}{l} \sum_{i=1}^{l} \alpha_i$$  \hspace{1cm} (5)

where $\alpha_i$ is the latest historical gradient of $\nabla f_i(w)$, which can be updated in an online fashion.

### 2.3 System Structure of Our Algorithms

As mentioned before, AFSG-VP, AFSVRG-VP and AFSAGA-VP are privacy-preserving asynchronous federated learning algorithms on the vertically partitioned data. Figure 2 presents their system structure. Specifically, we give detailed descriptions of tree-structured communication, and data and model privacy, respectively, as follows.

#### 2.3.1 Tree-Structured Communication

To obtain $w^T x_i$, we need to accumulate the local results from different workers. Zhang et al. [Zhang et al. (2018)] proposed an efficient tree-structured communication scheme to get the global sum which is faster than the simple strategy of sending the results from all workers directly to the coordinator for sum. Take 4 workers as an example, we pair the workers so that while worker 1 adds the result from worker 2, worker 3 can add the result from worker 4 simultaneously. Finally, the results from the two pairs of workers are sent to the coordinator and we obtain the global sum (please see Figure 3a). In this paper, we use the tree-structured communication scheme to obtain $w^T x_i$. Note that, our tree-structured communication scheme works with the asynchronous pattern to obtain $w^T x_i$, that means that we do not align the iteration numbers of $w_{G_\ell}$ from different workers to compute $w^T x_i$. It is significantly different from the synchronous pattern used in [Zhang et al. (2018)] where all $w_{G_\ell}$ have one and the same iteration number.

Based on the tree-structured communication scheme, we summarize the basic algorithm of computing $\sum_{\ell'=1}^{q} w_{G_{\ell'}}^T (x_i)_{G_{\ell'}}$ on the $\ell$-th active worker in Algorithm 1.
Algorithm 1 Basic algorithm of computing \( \sum_{\ell' = 1}^{q} w_{G_{\ell'}}^T (x_i)_{G_{\ell'}} \) on the \( \ell \)-th active worker

**Input:** \( w, x_i \)

{This loop asks multiple workers running in parallel.}

1: for \( \ell' = 1, \ldots, q \) do
2: Calculate \( w_{G_{\ell'}}^T (x_i)_{G_{\ell'}} \).
3: end for
4: Use tree-structured communication scheme to compute \( \xi = \sum_{\ell' = 1}^{q} w_{G_{\ell'}}^T (x_i)_{G_{\ell'}} \).

**Output:** \( \xi \).

| Workers | 1 | 2 | 3 | 4 |
|---------|---|---|---|---|
| Workers | 6 | 5 | 7 | 2 |
| Coordinator | 6 |

(a) Tree structures \( T_1 \)

| Workers | 1 | 2 | 3 | 4 |
|---------|---|---|---|---|
| Workers | 6 | 5 | 7 | 2 |
| Coordinator | 6 |

(b) Tree structures \( T_2 \)

Figure 3: Illustration of tree-structured communication with two significantly different tree structures.

### 2.3.2 Data and Model Privacy

To keep the vertically partitioned data and model privacy, we save the data \((x_i)_{G_{\ell}}\) and model weights \(w_{G_{\ell}}\) in the \(\ell\)-th worker separately and privately. We do not directly transfer the local data \((x_i)_{G_{\ell}}\) and local model weights \(w_{G_{\ell}}\) to other workers. To obtain \(w_{G_{\ell}}^T x_i\), we locally compute \(w_{G_{\ell}}^T (x_i)_{G_{\ell}}\) and only transfer \(w_{G_{\ell}}^T (x_i)_{G_{\ell}}\) to other workers for computing \(w_{G_{\ell}}^T x_i\) as shown in Algorithm 1. It is not trivial to infer the the local model coefficients \(w_{G_{\ell}}\) and \((x_i)_{G_{\ell}}\) based on the value of \(w_{G_{\ell}}^T (x_i)_{G_{\ell}}\) which is discussed in detail in Section 3.2. Thus, we achieve the data and model privacy.

Although it is not trivial to exactly infer the the local model coefficients \(w_{G_{\ell}}\) and \((x_i)_{G_{\ell}}\) based on the value of \(w_{G_{\ell}}^T (x_i)_{G_{\ell}}\), it has the risk of approximate inference attack (please refer to Definition 11). To address this issue, we propose a safer algorithm to compute \(\sum_{\ell' = 1}^{q} w_{G_{\ell'}}^T (x_i)_{G_{\ell'}}\) in Algorithm 2. Specifically, we add a random number \(b_{\ell'}\) into \(w_{G_{\ell'}}^T (x_i)_{G_{\ell'}}\), and then use the tree-structured communication scheme on a tree structure \(T_1\) to compute \(\sum_{\ell' = 1}^{q} (w_{G_{\ell'}}^T (x_i)_{G_{\ell'}} + b_{\ell'})\) which can improve the data and model security for the operation of transferring the value of \(w_{G_{\ell'}}^T (x_i)_{G_{\ell'}} + b_{\ell'}\). Finally, we need to recover the value of \(\sum_{\ell' = 1}^{q} w_{G_{\ell'}}^T (x_i)_{G_{\ell'}}\) from \(\sum_{\ell' = 1}^{q} (w_{G_{\ell'}}^T (x_i)_{G_{\ell'}} + b_{\ell'})\). In order to prevent leaking any sum of \(b_{\ell'}\) of a subtree of \(T_1\), we use a significantly different tree structure \(T_2\) on all workers (please see Definition 1 and Figure 3) to compute \(b = \sum_{\ell' = 1}^{q} b_{\ell'}\).

**Definition 1 (Two significantly different tree structures)** For two tree structures \(T_1\) and \(T_2\) on all workers \(\{1, \ldots, q\}\), they are significantly different if there does not exist a
subtree $\hat{T}_1$ of $T_1$ and a subtree $\hat{T}_2$ of $T_2$ whose sizes are larger than 1 and smaller than $T_1$ and $T_1$ respectively, such that $\text{Leaf}(\hat{T}_1) = \text{Leaf}(\hat{T}_2)$.

Algorithm 2 Safer algorithm of computing $\sum_{\ell'=1}^{q} w_{G_{\ell'}}^T(x_i)_{G_{\ell'}}$ on the $\ell$-th active worker

**Input:** $w$, $x_i$

1. for $\ell' = 1, \ldots, q$ do
2. Generate a random number $b_{\ell'}$.
3. Calculate $w_{G_{\ell'}}^T(x_i)_{G_{\ell'}} + b_{\ell'}$.
4. end for
5. Use tree-structured communication scheme based on the tree structure $T_1$ on all workers $\{1, \ldots, q\}$ to compute $\xi = \sum_{\ell'=1}^{q} \left( w_{G_{\ell'}}^T(x_i)_{G_{\ell'}} + b_{\ell'} \right)$.
6. Use tree-structured communication scheme based on the totally different tree structure $T_2$ on all workers $\{1, \ldots, q\}$ to compute $\bar{b} = \sum_{\ell'=1}^{q} b_{\ell'}$.

**Output:** $\xi - \bar{b}$.

2.4 Algorithms

In this subsection, we propose our three asynchronous federated stochastic gradient algorithms (i.e., AFSG-VP, AFSVRG-VP and AFSAGA-VP) on the vertically partitioned data.

2.4.1 AFSGD-VP

AFSGD-VP repeats the following four steps concurrently for each worker without any lock.

1. **Pick up an index:** AFSGD-VP picks up an index $i$ randomly from $\{1, \ldots, l\}$ and obtain the local instance $(x_i)_{G_{\ell}}$ from the local data $D_{\ell}$.

2. **Compute $\hat{w}^T x_i$:** AFSGD-VP uses the tree-structured communication scheme with asynchronous pattern (i.e., Algorithm 1 or 2) to obtain $\hat{w}^T x_i = \sum_{\ell'=1}^{q} (\hat{w})_{G_{\ell'}}^T(x_i)_{G_{\ell'}}$, where $\hat{w}$ denotes $w$ inconsistently read from different workers and two $(\hat{w})_{G_{\ell'}}$ may be in different local iteration stages. Note that we always have that $(w)_{G_{\ell}} = (\hat{w})_{G_{\ell}}$.

3. **Compute stochastic local gradient:** Based on $\hat{w}^T x_i$, we can compute the unbiased stochastic local gradient as $\hat{v}^\ell = \nabla_{G_{\ell}} f_i(\hat{w})$.

4. **Update:** AFSGD-VP updates the local model weights $w_{G_{\ell}}$ by $w_{G_{\ell}} \leftarrow w_{G_{\ell}} - \gamma \cdot \hat{v}^\ell$, where $\gamma$ is the learning rate.

We summarize our AFSGD-VP algorithm in Algorithm 3.

2.4.2 AFSVRG-VP

Stochastic gradients in AFSGD-VP have a large variance due to the random sampling similar to SGD algorithm [Bottou (2010)]. To handle the large variance, AFSVRG-VP uses
Algorithm 3 Asynchronous federated SGD algorithm (AFSGD-VP) for vertically partitioned data on the $\ell$-th active worker

**Input:** Local data $D^\ell$, learning rate $\gamma$.

1. Initialize $w_{G_\ell} \in \mathbb{R}^{d_\ell}$.
2. **Keep doing in parallel**
3. Pick up an index $i$ randomly from $\{1, \ldots, l\}$ and obtain the local instance $(x_i)_{G_\ell}$ from the local data $D^\ell$.
4. Compute $(w)_{G_\ell}^T (x_i)_{G_\ell}$.
5. Compute $\hat{w} = \nabla_{G_\ell} f_i(\hat{w})$ based on Algorithm 1 or 2.
6. Compute $\hat{v}^\ell = \nabla_{G_\ell} f_i(\hat{w})$.
7. Update $w_{G_\ell} \leftarrow w_{G_\ell} - \gamma \cdot \hat{v}^\ell$.
8. **End parallel loop**

**Output:** $w_{G_\ell}$

the SVRG technique [Gu et al., 2018a] to reduce the variance of the stochastic gradient, and propose a faster AFSGD-VP algorithm (i.e., AFSVRG-VP). We summarize our AFSVRG-VP algorithm in Algorithm 4. Compared to AFSGD-VP, AFSVRG-VP has the following three differences.

1. The first one is that AFSVRG-VP is to compute the full local gradient $\nabla_{G_\ell} f_i(w^s) = \frac{1}{l} \sum_{i=1}^{l} \nabla_{G_\ell} f_i(w^s)$ in the outer loop which will be used as the snapshot of full gradient, where the superscript $s$ denotes the $s$-th out loop.

2. The second one is that we compute not only $\hat{w}^T x_i$ but also $(w^s)^T x_i$ for each iteration.

3. The third one is that AFSVRG-VP computes the unbiased stochastic local gradient as $\hat{v}^\ell = \nabla_{G_\ell} f_i(\hat{w}) - \nabla_{G_\ell} f_i(w^s) + \nabla_{G_\ell} f_i(w^s)$.

2.4.3 AFSAGA-VP

As mentioned above, the stochastic gradients in SGD have a large variance due to the random sampling. To handle the large variance, AFSAGA-VP uses the SAGA technique [Defazio et al., 2014] to reduce the variance of the stochastic gradients. We summarize our AFSAGA-VP algorithm in Algorithm 3. Specifically, we maintain a table of latest historical local gradients $\alpha_\ell^i$ which is achieved by the updating rule of $\hat{\alpha}_\ell^i \leftarrow \nabla_{G_\ell} f_i(w)$ for each iteration. Based on the table of latest historical local gradients $\hat{\alpha}_\ell^i$, the unbiased stochastic local gradient in AFSAGA-VP is computed as $\hat{v}^\ell = \nabla_{G_\ell} f_i(\hat{w}) - \hat{\alpha}_i^\ell + \frac{1}{l} \sum_{i=1}^{l} \hat{\alpha}_i^\ell$.

3. Theoretical Analyses

In this section, we provide the convergence, security and complexity analyses to AFSG-VP, AFSVRG-VP and AFSAGA-VP. All the proofs can be found in the Appendix.
Algorithm 4 Asynchronous federated SVRG algorithm (AFSVRG-VP) for vertically partitioned data on the $\ell$-th active worker

**Input:** Local data $D^\ell$, learning rate $\gamma$.

1. Initialize $w^0_{G^\ell} \in \mathbb{R}^{d^\ell}$.
2. for $s = 0, 1, 2, \ldots, S - 1$ do
3. Compute the full local gradient $\nabla_{G^\ell}f(w^s) = \frac{1}{l} \sum_{i=1}^l \nabla_{G_i}f_i(w^s)$ by using tree-structured communication scheme.
4. $w^s_{G^\ell} = w^G_{G^\ell}$.
5. **Keep doing in parallel**
6. Pick up a local instance $(x_i)_{G^\ell}$ randomly from the local data $D^\ell$.
7. Compute $(w)^T_{G^\ell}(x_i)_{G^\ell}$.
8. Compute $\hat{w}^T x_i = \sum_{q=1}^q (\hat{w})^T_{G^\ell_i}(x_i)_{G^\ell_i}$ and $(w^s)^T x_i = \sum_{q=1}^q (w^s)^T_{G^\ell_i}(x_i)_{G^\ell_i}$ based on Algorithm 1 or 2.
9. Compute $\hat{v}^\ell = \nabla_{G^\ell}f_i(\hat{w}) - \nabla_{G^\ell}f_i(w^s) + \nabla_{G^\ell}f(w^s)$.
10. Update $w^s_{G^\ell} \leftarrow w^s_{G^\ell} - \gamma \cdot \hat{v}^\ell$.
11. **End parallel loop**
12. $w^{s+1}_{G^\ell} = w^s_{G^\ell}$.
13. end for
**Output:** $w^s_{G^\ell}$

Algorithm 5 Asynchronous federated SAGA algorithm (AFSAGA-VP) for vertically partitioned data on the $\ell$-th active worker

**Input:** Local data $D^\ell$, learning rate $\gamma$.

1. Initialize $w^0_{G^\ell} \in \mathbb{R}^{d^\ell}$.
2. Compute the local gradients $\alpha^\ell_i = \nabla_{G_i}f_i(w)$, $\forall i \in \{1, \ldots, n\}$ by using tree-structured communication scheme, and locally save them.
3. **Keep doing in parallel**
4. Pick up a local instance $(x_i)_{G^\ell}$ randomly from the local data $D^\ell$.
5. Compute $(w)^T_{G^\ell}(x_i)_{G^\ell}$.
6. Compute $\hat{w}^T x_i = \sum_{q=1}^q (\hat{w})^T_{G^\ell_i}(x_i)_{G^\ell_i}$, $\tilde{\alpha}^\ell = \nabla_{G^\ell}f_i(\hat{w})$, based on Algorithm 1 or 2.
7. Compute $\hat{v}^\ell = \nabla_{G^\ell}f_i(\hat{w}) - \hat{\alpha}^\ell + \frac{1}{l} \sum_{i=1}^l \tilde{\alpha}^\ell$.
8. Update $w^s_{G^\ell} \leftarrow w^s_{G^\ell} - \gamma \cdot \hat{v}^\ell$.
9. Update $\hat{\alpha}^\ell_i \leftarrow \nabla_{G^\ell}f_i(\hat{w})$.
10. **End parallel loop**
**Output:** $w^s_{G^\ell}$

3.1 Convergence Analyses

We first make several basic assumptions, then provide the results of convergence of AFSG-VP, AFSVRG-VP and AFSAGA-VP.
3.1.1 Preliminaries

In this part, we give the assumptions of strong convexity (Assumption 1), different Lipschitz smoothness (Assumption 2), and block-coordinate bounded gradients (Assumption 3), which are standard for convex analysis Defazio et al. (2014); Xiao and Zhang (2014); Zhao and Li (2016); Beck and Tetruashvili (2013); Li et al. (2017, 2016).

**Assumption 1 (Strong convexity)** The differentiable function $f_i$ ($\forall i \in \{1, \cdots, l\}$ in the problem (1) is strongly convex with parameter $\mu > 0$, which means that $\forall w$ and $\forall w'$, we have

$$f_i(w) \geq f_i(w') + \langle \nabla f_i(w'), w - w' \rangle + \frac{\mu}{2} \|w - w'\|^2 \tag{6}$$

**Assumption 2 (Lipschitz smoothness)** The function $f_i$ ($\forall i \in \{1, \cdots, l\}$ in the problem (1) is Lipschitz smooth with constant $L$, which means that, $\forall w$ and $\forall w'$, we have:

$$\|\nabla f_i(w) - \nabla f_i(w')\| \leq L \|w - w'\| \tag{7}$$

The function $f_i$ ($\forall i \in \{1, \cdots, l\}$ in the problem (1) is block-coordinate Lipschitz smooth w.r.t. the $\ell$-th block $G_\ell$ with constant $L_\ell$, such that, $\forall w$, and $\forall \ell \in \{1, \cdots, q\}$, we have:

$$\|\nabla_{G_\ell} f_i(w + U_\ell \Delta_\ell) - \nabla_{G_\ell} f_i(w)\| \leq L_\ell \|\Delta_\ell\| \tag{8}$$

where $\Delta_\ell \in \mathbb{R}^{d_\ell}$, $U_\ell \in \mathbb{R}^{d_x d_\ell}$ and $[U_1, U_2, \ldots, U_q] = I_d$.

According to the definition of block-coordinate Lipschitz smooth constant $L_\ell$ in Assumption 2, we define $L_{\text{max}} = \max_{\ell=1,\ldots,q} L_\ell$. Furthermore, we have $L \leq qL_{\text{max}}$ which is proved in Lemma 2 of Nesterov (2012).

**Assumption 3 (Block-coordinate bounded gradients)** For smooth function $f_i(x)$ ($\forall i \in \{1, \ldots, l\}$) in (1), the block-coordinate gradient $\nabla_{G_\ell} f_i(w)$ is called bounded if there exists a parameter $G$ such that $\|\nabla_{G_\ell} f_i(w)\|^2 \leq G$, $\forall i \in \{1, \ldots, l\}$ and $\forall \ell \in \{1, \ldots, q\}$.

3.1.2 Difficulties

In this part, we discuss the difficulties of globally labeling the iterates, global updating rules and the relationship between $w_t$ and $\hat{w}_t$.

**Globally labeling the iterates:** As shown in Algorithms 3 and 4, we do not globally label the iterates from different workers. Although it is fine for the implementation, how we choose to define the iteration counter $t$ to label an iterate $w_t$ matters in the analysis. More specifically, the global time counter plays a fundamental role in the convergence rate analyses of AFSG-VP, AFSVRG-VP and AFSAGA-VP. To address this issue, we propose the strategy of “after communication” labeling Leblond et al. (2017), in which we update our iterate counter as one worker finishes computing $\hat{w}_t^T x_i$. This means that $\hat{w}_t$ (or $\hat{w}_t^*$) is the $(t + 1)$-th fully completed the computation of $\hat{w}_t^T x_i$. The strategy of “after communication” labeling guarantees both that the $i_t$ are uniformly distributed and that $i_t$ and $\hat{w}_t$ are independent.

We define a minimum set of successive iterations of fully visiting all coordinates from the time counter $t$ as $K(t)$ in Definition 11.
Definition 2 (Set \( K(t) \)) Let \( \overline{K}(t) = \{ \{ t, t+1, \ldots, t+\sigma \} : \psi(\{ t, t+1, \ldots, t+\sigma \}) = \{1, \ldots, q\} \} \). The minimum set of successive iterations of fully visiting all coordinates from the time counter \( t \) is defined as \( K(t) = \arg\min_{K'(t) \in \overline{K}(t)} |K'(t)| \).

Let \( \psi^{-1}(\ell, K) \) denote all the elements in \( K \) such that \( \psi(\psi^{-1}(\ell, K)) = \ell \). We assume that there exists an upper bound \( \eta_1 \) to the size of \( \psi^{-1}(\ell, K(t)) \) (Assumption 4).

Assumption 4 (Bounded size of \( \psi^{-1}(\ell, K(t)) \)) \( \forall t \), and \( \forall \ell \in \{1, \ldots, q\} \), the sizes of all \( \psi^{-1}(\ell, K(t)) \) are upper bounded by \( \eta_1 \), i.e., \( |\psi^{-1}(\ell, K(t))| \leq \eta_1 \).

Based on the definition of \( K(t) \), we define the epoch number of fully visiting all coordinates for the global \( t \)-th iteration as \( \nu(t) \), and the start start time counter in one epoch as \( \psi(t) \) in Definition 3. Our convergence rate analyses are build on the epoch number \( \nu(t) \).

Definition 3 (Epoch number \( \nu(t) \) and start time counter \( \psi(t) \)) Let \( P(t) \) is a partition of \( \{0, 1, \ldots, t-\sigma'\} \), where \( \sigma' \geq 0 \). For any \( \kappa \in P(t) \) we have that, there exists \( t' \leq t \) such that \( K(t') = \kappa \), and there exists \( \kappa_1 \in P(t) \) such that \( K(0) = \kappa_1 \). The epoch number \( \nu(t) \) is defined as the maximum cardinality of \( P(t) \). Given a global time counter \( u \leq t \), if there exists \( \kappa \in P(t) \) such that \( u \in \kappa \), we define the start time counter \( \psi(t) \) as the minimum element of \( \kappa \), otherwise \( \psi(t) = t - \sigma' + 1 \).

Global updating rule: The updating rules (such as \( w_{\hat{G}_i} \leftarrow w_{\hat{G}_i} - \gamma \cdot \hat{v}^t \)) in Algorithms 3, 4 and 5 are updating rules locally working on a certain worker. To provide the convergence rate analyses of AFSG-VP, AFSVRG-VP and AFSG-VP-VP, we need provide the global updating rules of AFSG-VP, AFSVRG-VP and AFSG-VP-VP. Due to the commutativity of the add operations used in \( w_{\hat{G}_i} \leftarrow w_{\hat{G}_i} - \gamma \cdot \hat{v}^t \), the order in which these updates are finished in the corresponding worker is irrelevant. Hence, we provide the global updating rules of AFSG-VP, AFSVRG-VP and AFSG-VP-VP as follows.

\[
 w_{t+1} = w_t - \gamma U_{\psi(t)} \hat{v}_{\psi(t)}^t 
\]  

(9)

Note that the global updating rule (9) which defines the relation of two adjacent iterates, does not conflict with the rule of globally labeling the iterates due to the commutativity of the add operations.

Relationship between \( w_t \) and \( \hat{w}_t \): As mentioned before, AFSG-VP, AFSVRG-VP and AFSG-VP-VP use the tree-structured communication scheme with asynchronous pattern to obtain \( \hat{w}_T x_i = \sum_{i'=1}^q (\hat{w})_{G_{i'}} (x_i)_{G_{i'}} \), where \( \hat{w} \) denotes \( w \) inconsistently read from different workers. Thus, the vector \( (\hat{w})_{G_{i'}} \) for \( i' \neq i \) may be inconsistent to the vector \( (w)_i G_{i'} \), which means that some blocks of \( \hat{w}_i \) are same with the ones in \( w_i \) (e.g., \( (w)_{G_{i'}} = (\hat{w})_{G_{i'}} \)), but others are different to the ones in \( w_i \). To address the challenge, we assume an upper bound to the delay of updating. Specifically, we define a set \( D(t) \) of iterations, such that:

\[
 \hat{w}_t - w_t = \gamma \sum_{u \in D(t)} U_{\psi(u)} \hat{v}_{\psi(u)}^u 
\]  

(10)

where \( \forall u \in D(t) \), we have \( u < t \). It is reasonable to assume that there exists an upper bound \( \tau \) such that \( \tau \geq t - \min\{t' | t' \in D(t)\} \) \( \psi(t) \) (i.e., Assumption 5).
Assumption 5 (Bounded overlap) There exists an upper bound $\tau$ such that $\tau \geq t - \min\{u|u \in D(t)\}$ for all iterations $t$ in AFSG-VP, AFSVRG-VP and AFSAGA-VP.

In addition, we assume that there exist an upper bound $\eta_2$ to the size of $\psi^{-1}(\ell, D(t))$ (Assumption 6).

Assumption 6 (Bounded size of $\psi^{-1}(\ell, D(t))$) $\forall t$, and $\forall \ell \in \{1, \ldots, q\}$, the sizes of all $\psi(\psi^{-1}(\ell, D(t)))$ are upper bounded by $\eta_2$, i.e., $|\psi(\psi^{-1}(\ell, K))| \leq \eta_2$.

3.1.3 AFSGD-VP

We provide the convergence result of AFSGD-VP in Theorem 4.

Theorem 4 Under Assumptions 1-6, to achieve the accuracy $\epsilon$ of (1) for AFSGD-VP, i.e., $\mathbb{E}f(w_t) - f(w^*) \leq \epsilon$, we set

$$
\gamma = \frac{-L_{\text{max}} + \sqrt{L_{\text{max}}^2 + 2\mu(L^2q\eta_1^2 + \eta_2L^2\tau)}}{2L^2(q\eta_1^2 + \eta_2\tau)}
$$

(11)

and the epoch number $v(t)$ should satisfy the following condition.

$$
\gamma = \frac{2L^2(q\eta_1^2 + \eta_2\tau)}{\mu - L_{\text{max}} + \sqrt{L_{\text{max}}^2 + 2\mu(L^2q\eta_1^2 + \eta_2L^2\tau)}} \cdot \log \left( \frac{2(f(w_0) - f(w^*))}{\epsilon} \right)
$$

(12)

Remark 5 Theorem 6 shows that, the convergence rate of AFSGD-VP is $O\left(\frac{1}{\sqrt{\epsilon}} \log \left(\frac{1}{\epsilon}\right)\right)$ to reach the accuracy $\epsilon$. The theorem shows that if we try to obtain a more accurate solution with a smaller stepsize, the convergence rate slows down.

3.1.4 AFSVRG-VP

We provide the convergence result of AFSVRG-VP in Theorem 6.

Theorem 6 Under Assumptions 1-6, to achieve the accuracy $\epsilon$ of (1) for AFSVRG-VP, i.e., $\mathbb{E}f(w_t) - f(w^*) \leq \epsilon$, let $C = (\eta_1\gamma L^2q + L_{\text{max}}) \frac{\gamma^2}{2}$ and $\rho = \frac{\gamma^2}{2} - \frac{16L^2q\eta_1C}{\mu}$, we choose $\gamma$ such that

$$
\rho < 0
$$

(13)

$$
\frac{8L^2q\eta_1C}{\rho \mu} \leq 0.5
$$

(14)

$$
\gamma^3 \left( \left( \frac{1}{2} + \frac{2C}{\gamma} \right) \eta_2\tau + \frac{4C}{\gamma} \eta_1^2 \right) \frac{\eta_1qL^2G}{\rho} \leq \frac{\epsilon}{8}
$$

(15)

the inner epoch number should satisfy $v(t) \geq \frac{\log 0.25}{\log(1 - \rho)}$, and the outer loop number should satisfy $S \geq \frac{\log \frac{2(f(w_0) - f(w^*))}{\log(1 - \rho)}}{\log \frac{4}{3}}$.

Remark 7 Theorem 6 shows that, the convergence rate of AFSVRG-VP is $O(\log \left(\frac{1}{\epsilon}\right))$ to reach the accuracy $\epsilon$. 
3.1.5 AFSAGA-VP

We provide the convergence result of AFSAGA-VP in Theorem 8.

**Theorem 8** Under Assumptions 1-6, to achieve the accuracy \( \epsilon \) of (1) for AFSAGA-VP, i.e., \( \mathbb{E}f(w_t) - f(w^*) \leq \epsilon \), let

\[
c_0 = (\frac{\eta_2^2}{4} + 3(\gamma q \eta_1^2 + L_{\text{max}})(\eta_1 + 2\eta_2))\tau + (\gamma L^2 q \eta_1^2 + 8L_{\text{max}})\eta_1 \eta_1) \gamma^4 L^2 \eta_1 q G,
\]

\[
c_1 = (\gamma L^2 q \eta_1^2 + L_{\text{max}}) \gamma^2 \eta_1 q 2L^2, c_2 = 4 (\gamma L^2 q \eta_1^2 + L_{\text{max}}) \frac{L^2 q}{4} \gamma^2, \text{ and let } \rho \in (1 - \frac{1}{4}, 1), \text{ we choose } \gamma \text{ such that}
\]

\[
\frac{4c_0}{\gamma \mu (1 - \rho) \left( \frac{\gamma^2}{4} - 2c_1 - c_2 \right)} \leq \frac{\epsilon}{2}
\]

\[
0 < 1 - \frac{\gamma \mu}{4} < 1
\]

\[
- \frac{\gamma^2}{4} + 2c_1 + c_2 \left( 1 + \frac{1}{1 - \frac{1}{4} \rho} \right) \leq 0
\]

\[
- \frac{\gamma^2}{4} + c_2 + c_1 \left( 2 + \frac{1}{1 - \frac{1}{4} \rho} \right) \leq 0
\]

the epoch number should satisfy \( \nu(t) \geq \frac{\log \frac{2(2\mu - 1 + \frac{\mu}{4}) (f(w_0) - f(w^*))}{\epsilon (\rho - 1 + \frac{\mu}{4}) \frac{2q^4}{4} - 2c_1 - c_2}} {\log \frac{1}{\rho}} \).

**Remark 9** Theorem 6 shows that, the convergence rate of AFSAGA-VP is \( O(\log(\frac{1}{\epsilon})) \) to reach the accuracy \( \epsilon \).

3.2 Security Analysis

We discuss the data and model security (in other words, prevent local data and model on one worker leaked to or inferred by other workers) of AFSG-VP, AFSVRG-VP and AFSAGA-VP under the semi-honest assumption. Note that the semi-honest assumption (i.e., Assumption 7) is commonly used in previous works [Wan et al. (2007); Hardy et al. (2017); Cheng et al. (2019)].

**Assumption 7 (Semi-honest security)** All workers will follow the algorithm to perform the correct computations. However, they may retain records of the intermediate computation results which they may use later to infer the other work’s data and model.

Before discussing the data and model privacy in detail, we first introduce the concepts of exact and approximate inference attacks in Definitions 10 and 11.

**Definition 10 (Exact inference attack)** An exact inference attack on the \( \ell \)-th worker is to exactly infer some feature group \( \mathcal{G} \) of one sample \( x \) or model \( w \) which belongs from other workers without directly accessing it.

**Definition 11 (\( \epsilon \)-approximate inference attack)** An \( \epsilon \)-approximate inference attack on the \( \ell \)-th worker is to infer some feature group \( \mathcal{G} \) of one sample \( x \) (model \( w \)) as \( \hat{x}_\mathcal{G} \) (\( \hat{w}_\mathcal{G} \)) with the accuracy of \( \epsilon \) (i.e., \( \| \hat{x}_\mathcal{G} - x_\mathcal{G} \|_\infty \leq \epsilon \) or \( \| \hat{w}_\mathcal{G} - w_\mathcal{G} \|_\infty \leq \epsilon \)) which belongs from other workers without directly accessing it.
Security Analysis based on Algorithm 1

Firstly, we show that AFSG-VP, AFSVRG-VP, and AFSAGA-VP based on Algorithm 1 can prevent the exact inference attack, however has the risk of approximate inference attack.

Specifically, in order to infer the information of \((w_t)_{\mathcal{G}_t}\) on the \(\ell'\)-th worker where \(\ell' \neq \ell\), we only have a sequence of linear system of \(o_t = (w_t)_{\mathcal{G}_t}^T(x_i)_{\mathcal{G}_t}\) with a sequence of trials of \((x_i)_{\mathcal{G}_t}\) and \(o_t\) while only \(o_t\) are known. Thus, it is impossible to infer the exact information of \((w_t)_{\mathcal{G}_t}\) from the linear system of \(o_t = (w_t)_{\mathcal{G}_t}^T(x_i)_{\mathcal{G}_t}\) even the size of feature group \(\mathcal{G}_t\) is one. Similarly, we cannot infer the exact information of \((x_i)_{\mathcal{G}_t}\).

However, it has the potential to approximately infer \((w_t)_{\mathcal{G}_t}\) from the linear system of \(o_t = (w_t)_{\mathcal{G}_t}^T(x_i)_{\mathcal{G}_t}\) if the size of feature group \(\mathcal{G}_t\) is one. Specifically, if we know the region of \((x_i)_{\mathcal{G}_t}\) as \(\mathcal{I}\), we can have that \(o_t/w_{\mathcal{G}_t}\) which can infer \(w_{\mathcal{G}_t}\) approximately. Further, we can infer \((x_i)_{\mathcal{G}_t}\) approximately. We say that Algorithm 1 has the risk of approximate inference attack.

Security Analysis based on Algorithm 2

Next, we show that AFSG-VP, AFSVRG-VP, and AFSAGA-VP based on Algorithm 2 can prevent the approximate inference attack.

As discussed above, the key to preventing the approximate inference attack is to mask the value of \(o_t\). As described in lines 2-5 of Algorithm 2, we add an extra random variable \(b^{\ell'}\) into \(w^{T}_{\mathcal{G}_t}(x_i)_{\mathcal{G}_t}\), and transfer the value of \(w^{T}_{\mathcal{G}_t}(x_i)_{\mathcal{G}_t} + b^{\ell'}\) to another worker. This operation makes the received part cannot directly get the value of \(o_t\). Finally, the \(\ell\)-th active worker gets the global sum \(\sum_{t=1}^{q}(w^{T}_{\mathcal{G}_t}(x_i)_{\mathcal{G}_t} + b^{\ell'})\) by using a tree-structured communication scheme based on the tree structure \(T_1\). Thus, the lines 2-5 of Algorithm 2 keeps data privacy.

Line 6 of Algorithm 2 is trying to get \(w^T x\) by removing \(\bar{b} = \sum_{t=1}^{q} b^{\ell'}\) from the sum \(\sum_{t=1}^{q}(w^{T}_{\mathcal{G}_t}(x_i)_{\mathcal{G}_t} + b^{\ell'})\). To prove that Algorithm 2 can reduce the risk of approximate inference attack, we only need to prove that the calculation of \(\bar{b} = \sum_{t=1}^{q} b^{\ell'}\) in line 6 of Algorithm 2 does not disclose the value of \(b^{\ell'}\) or the sum of \(b^{\ell'}\) on a node of tree \(T_1\) (please see Lemma 12, the proof is provided in the Appendix).

Lemma 12 Using a tree structure \(T_2\) on all workers which is significantly different to the tree \(T_1\) to compute \(\bar{b} = \sum_{t=1}^{q} b^{\ell'}\), there is no risk to disclose the value of \(b^{\ell'}\), or the sum of \(b^{\ell'}\) on all nodes of a subtree of \(T_1\) whose sizes are larger than 1 and smaller than \(T_1\).

3.3 Complexity Analysis

We give the computational complexities and communication costs of AFSG-VP, AFSVRG-VP, and AFSAGA-VP as follows.

The computational complexity for one iteration of AFSGD-VP is \(O(d + q)\). Thus, the total computational complexity of AFSGD-VP is \(O((d + q)t)\), where \(t\) denotes the iteration number. Further, the communication cost for one iteration of AFSGD-VP is \(O(q)\), and the total communication cost is \(O(qt)\).

For AFSVRG-VP, the computational complexity and communication cost of line 3 are \(O((d + q)t)\) and \(O(ql)\) respectively. Assume that the inner loop number of AFSVRG-VP is \(t\). Thus, the total computational complexity of AFSVRG-VP is \(O((d + q)(l + t)s)\), and the communication cost is \(O(q(l + t)s)\).
For AFSAGA-VP, the computational complexity and communication cost of line 2 are $O((d+q)l)$ and $O(ql)$ respectively. Assume that the loop number of AFSAGA-VP is $t$. Thus, the total computational complexity of AFSAGA-VP is $O((d+q)(l+t))$, and the communication cost is $O(q(l+t))$.

4. Experimental Results

In this section, we first present the experimental setup, and then provide the experimental results and discussions.

4.1 Experimental Setup

4.1.1 Design of Experiments

In the experiments, we not only verify the theoretical results of AFSG-VP, AFSVRG-VP and AFSAGA-VP, but also show that our algorithms have much better efficiency than the corresponding synchronous algorithms (i.e., FSG-VP, FSVRG-VP and FSAGA-VP). We compare our asynchronous vertical SGD, SVRG and SAGA algorithms (i.e., AFSG-VP, AFSVRG-VP and AFSAGA-VP) with synchronous version of vertical SGD, SVRG and SAGA (denoted as FSG-VP, FSVRG-VP and FSAGA-VP respectively) on classification and regression tasks, where FSVRG-VP is almost same to FD-SVRG [Zhang et al. (2018)]. For the classification tasks, we consider the $\ell_2$-norm regularized logistic regression model as follows:

$$\min_w f(w) = \frac{1}{l} \sum_{i=1}^{l} \log(1 + e^{-y_i w^T x_i}) + \frac{\lambda}{2} \|w\|^2$$  \hspace{1cm} (20)

For the regression tasks, we use the ridge linear regression method with $\ell_2$-norm regularization as follows:

$$\min_{w,b} f(w,b) = \frac{1}{l} \sum_{i=1}^{l} (w^T x_i + b - y_i)^2 + \frac{\lambda}{2} (\|w\|^2 + b^2)$$  \hspace{1cm} (21)

4.1.2 Experiment Settings

We run all the experiments on a cluster with 32 nodes of 20-core Intel Xeon E5-2660 2.60 GHz (Haswell). The nodes are connected with 56 Gb FDR. We use OpenMPI [Graham et al. (2005)] v3.1.1 with multi-thread support for communication between worker processes and Armadillo [Sanderson and Curtin (2016)] v9.700.3 for efficient matrix computation. Each worker is placed on a different machine node. For the $\ell_2$ regularization term, we set the coefficient $\lambda = 1e^{-4}$ for all experiments. We also choose the best learning rate $\in \{5e^{-1}, 1e^{-1}, 5e^{-2}, 1e^{-2}, ...\}$ for each algorithm on different learning tasks. There is a synthetic straggler node which may be 40% to 300% slower than the fastest worker node to simulate the real application scenario. In practice, it is normal that different parties in a federated learning system will possess different computation and communication power and resources.
4.1.3 Implementation Details

Our asynchronous algorithms are implemented under the decentralized framework, where a worker owns its own part of data and model parameters. There is no master node for aggregating data/features/gradients which may lead to undesired user information disclosure. Instead, we utilize a coordinator as in Figure 2 to collect the product computed from local data and parameters from other workers. Each worker node can independently call the coordinator to enable the asynchronous model update. The aggregation of local product is performed in a demand-based manner, which means that only when a worker node needs to update its local parameter will it request the coordinator to pull the local product from other worker nodes. Different from horizontal federated learning Yang et al. (2019); Liu et al. (2019b); So et al. (2019), it will be much harder for an attacker to restore the information of the user data in a worker node using the local product than the gradient.

Specifically, in our asynchronous algorithms, each worker node performs computation rather independently. The main thread of a worker process performs the major workload of gradient computation and model update operation. Another listener thread keeps listening for the request and sends back the local product to the requesting source. The computation diagram can be summarized as follows for a worker:

1. Randomly select an index of the data.
2. Call the coordinator to broadcast the index to the listeners of other workers.
3. Reduce the sum of the local product back from the listeners.
4. Perform gradient computation and model parameters update.

Note that the local product is computed based on a worker’s current parameters. Overall speaking, however, some workers may have updated their parameters more times than other workers. Different from common asynchronous horizontal algorithms Meng et al. (2016); Gu et al. (2016), although the worker processes run asynchronously, all the parameters a worker uses to compute gradient is most up-to-date. The broadcast and reduce operation are also realized in a tree-structured scheme to reduce communication costs.

4.1.4 Datasets

Table 1: The datasets used in the experiments.

| Classification Tasks | Regression Tasks |
|----------------------|------------------|
| UCICreditCard        | GiveMeSomeCredit |
| news20               | rcrv1            |
| url                  | webspam          |
| E2006-tfidf          | YearPredictionMSD|

| #Train | #Test | #Feature |
|--------|-------|----------|
| 24,000 | 6,000 | 90       |
| 96,257 | 24,012| 92       |

To fully demonstrate the scalability of our asynchronous vertical federated learning algorithms, we conduct experiments on eight datasets as summarized in Table 1 for binary...
classification and regression tasks. Two real and relatively small financial datasets, UCI-CreditCard and GiveMeSomeCredit are from the Kaggle website. The other six datasets are from the LIBSVM website Chang and Lin (2011). We split news20, url and webspam datasets into training data and testing data randomly with a ratio of 4:1. We also use rcv1’s testing data for training and training data for testing as there are more instances in the testing data.

4.2 Results and Discussions

Figure 4: Convergence of different algorithms for classification task (left two) and regression task (right two).

Figure 5: Convergence of different algorithms for binary classification task on more large-scale datasets.

Figure 6: Scalability (url dataset for classification task).

1. https://www.kaggle.com/datasets
2. https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/
4.2.1 Classification Tasks

We first compared our asynchronous federated learning algorithm with synchronous version on financial datasets to demonstrate the ability to address real application. In asynchronous algorithms, each worker saves its local parameters every fixed interval for testing. In the synchronous setting, each worker saves the parameters every fixed number of iterations as all the workers run at the same pace. We follow this scheme for the other experiments.

The original total numbers of features of UCICreditCard and GiveMeSomeCredit dataset are 23 and 10 respectively. We apply one-hot encoding for categorical features and standardize other features column-wisely. The numbers of features become 90 and 92 respectively after the simple data preprocessing.

Four worker nodes are used in this part of the experiment. As shown by the left two figures in Figure 4, our asynchronous vertical algorithms consistently surpass their synchronous counterparts. The y-axis function sub-optimality represents the error of objective function to the global optimal. The shape of the convergence curve is firstly determined by the optimization method we choose, i.e., SGD, SVRG and SAGA. The error precision of SGD is usually higher than SVRG, while that of SAGA is similar to SVRG. Then the convergence speed is mostly influenced by the computation and communication complexity. In asynchronous settings there is no inefficient idle time to wait for other workers, so the update frequency is much higher, which results in faster convergence speed of our asynchronous algorithm with regard to wall clock time.

Previous experiments show that our asynchronous federated learning algorithms could address real financial problems more efficiently. In this part we will use large-scale benchmark datasets, i.e. large number of data instances and high-dimensional features, for further validations. In our experiments, 8 worker nodes are used for experiments on new20 and rcv1 datasets; 16 worker nodes are used for experiments on url and webspam datasets. The results are visualized in Figure 5. As the total computation budget grows, the speedup of the asynchronous algorithm becomes more obvious. So it will be be much more efficient when put into large-scale practical use. Our asynchronous SGD, SVRG and SAGA still surpass their synchronous counterparts in the experiments on all the four datasets.

4.2.2 Regression Tasks

To further illustrate the advantages of asynchronous algorithms can scale to various tasks, we also conduct experiments on regression problems as shown by the right two figures in Figure 4. Both the E20060-tfidf with a smaller number of data instances but a larger number of features, and the YearPredictionMSD with larger number of instances but a smaller number of features are tested. 4 worker nodes are used in this experiment and similar conclusions as previous can be reached.

4.2.3 Asynchronous Efficiency

The speedup results of asynchronous algorithms compared with synchronous ones are summarized in Table 2. The speedup is computed based on the time when the algorithm reaches a certain precision of optimality (1e^{-4} for SVRG and SAGA; 1e^{-2.5} or 1e^{-1.5} for SGD based on different datasets).
To further analyze the efficiency of our asynchronous algorithms, we quantify the composition of the time consumption of asynchronous and synchronous algorithms as in Figure 7. The execution time and update frequency are scaled by those of the straggler in the synchronous algorithm. The computation time of stragglers is much higher than non-stragglers, which leads to a large amount of synchronization time for non-stragglers in synchronous algorithms. While in our asynchronous algorithms, non-stragglers pull the update-to-date product information from stragglers without waiting the straggler to finish its current iteration. As a result, the synchronization time is eliminated. Although the communication cost increases because each worker needs to independently aggregate product from other workers, we can achieve a large gain in terms of the update frequency.

4.2.4 Scalability

The scalability in terms of number of workers is shown in Figure 6. Synchronous algorithms cannot address the problem of straggler and behaves poorly. Using synchronization barrier keeps non-stragglers inefficiently waiting for the straggler. Our asynchronous algorithms behave like ideal in the beginning as they can address the straggler problem well, and deviate from ideal when the number of workers continues to grow because the communication overheads will limit the speedup.
5. Conclusion

In this paper, we proposed an asynchronous federated SGD (AFSGD-VP) algorithm and its SVRG and SAGA variants for vertically partitioned data. To the best of our knowledge, these algorithms are the first asynchronous federated learning algorithms for vertically partitioned data. Importantly, we provided the convergence rates of AFSGD-VP and its SVRG and SAGA variants under the condition of strong convexity for the objective function. We also proved the model privacy and data privacy. Extensive experimental results on a variety of vertically partitioned datasets not only verify the theoretical results of AFSGD-VP and its SVRG and SAGA variants, but also show that our algorithms have much better efficiency than the corresponding synchronous algorithms.

Appendix

Appendix A: Proof of Theorem 4

Before proving Theorem 4, we first provide several basic inequalities in Lemma 13.

**Lemma 13** For AFSGD-VP, under Assumptions 3 and 4, we have that

\[ \sum_{u \in K(t)} \| \nabla g_{\psi(u)} f(w_u) \|^2 \geq \frac{1}{2} \sum_{u \in K(t)} \| \nabla g_{\psi(u)} f(w_t) \|^2 - \eta_1 \gamma^2 L^2 \sum_{u \in K(t)} \sum_{v \in \{t, \ldots, u\}} \| \hat{\psi}_v \|^2 \]

(22)

**Proof** For any \( u \in K(t) \), we have that

\[ \| \nabla g_{\psi(u)} f(w_t) \|^2 \leq (\| \nabla g_{\psi(u)} f(w_t) - \nabla g_{\psi(u)} f(w_u) \|^2 + \| \nabla g_{\psi(u)} f(w_u) \|^2)^{1/2} \]

(23)

\[ \leq 2 \| \nabla g_{\psi(u)} f(w_t) - \nabla g_{\psi(u)} f(w_u) \|^2 + 2 \| \nabla g_{\psi(u)} f(w_u) \|^2 \]

\[ \leq 2 \| \nabla f(w_t) - \nabla f(w_u) \|^2 + 2 \| \nabla g_{\psi(u)} f(w_u) \|^2 \]

\[ \leq 2L^2 \gamma^2 \sum_{v \in \{t, \ldots, u\}} U_{\psi(v)} \hat{\psi}_v \|^2 + 2 \| \nabla g_{\psi(u)} f(w_u) \|^2 \]

where the inequality (a) uses Assumption 2, the last inequality uses Assumption 4. According to (23), we have that

\[ \| \nabla g_{\psi(u)} f(w_u) \|^2 \geq \frac{1}{2} \| \nabla g_{\psi(u)} f(w_t) \|^2 - \eta_1 L^2 \gamma^2 \sum_{v \in \{t, \ldots, u\}} \| \hat{\psi}_v \|^2 \]

(24)

Summing (24) over all \( u \in K(t) \), we obtain the conclusion. This completes the proof. ■

Based on the basic inequalities in Lemma 13, we provide the proof of Theorem 6 in the following.
Proof For $u \in K(t)$, we have that

$$\mathbb{E} f(w_{u+1}) \leq \mathbb{E} f(w_u) + \langle \nabla f(w_u), w_{u+1} - w_u \rangle + \frac{L_{\psi}(u)}{2} \|w_{u+1} - w_u\|^2$$

(25)

$$= \mathbb{E} f(w_u) - \gamma \langle \nabla f(w_u), \hat{\psi}(u) \rangle + \frac{L_{\psi}(u)}{2} \|\hat{\psi}(u)\|^2$$

$$= \mathbb{E} \left( f(w_u) + \frac{L_{\psi}(u)}{2} \|\hat{\psi}(u)\|^2 - \gamma \langle \nabla f(w_u), \hat{\psi}(u) \rangle - \nabla g_{\psi(u)} f_i(w_u) + \nabla g_{\psi(u)} f_i(w_u) \right)$$

$$= \mathbb{E} f(w_u) - \gamma \mathbb{E} \langle \nabla f(w_u), \nabla g_{\psi(u)} f(w_u) \rangle$$

$$+ \gamma \mathbb{E} \langle \nabla f(w_u), \nabla g_{\psi(u)} f_i(w_u) \rangle - \nabla g_{\psi(u)} f_i(w_u) + \frac{L_{\psi}(u)}{2} \mathbb{E} \|\hat{\psi}(u)\|^2$$

(26)

$$\leq \mathbb{E} f(w_u) - \frac{\gamma}{2} \mathbb{E} \|\nabla g_{\psi(u)} f_i(w_u) - \nabla g_{\psi(u)} f_i(w_u)\|^2 + \frac{L_{\psi}(u)}{2} \mathbb{E} \|\hat{\psi}(u)\|^2$$

$$\leq \mathbb{E} f(w_u) - \frac{\gamma}{2} \mathbb{E} \|\nabla g_{\psi(u)} f_i(w_u)\|^2 + \frac{L_{\psi}(u)}{2} \mathbb{E} \|\hat{\psi}(u)\|^2$$

$$\leq \mathbb{E} f(w_u) - \frac{\gamma}{2} \mathbb{E} \|\nabla g_{\psi(u)} f_i(w_u)\|^2 + \frac{L_{\psi}(u)}{2} \mathbb{E} \|\hat{\psi}(u)\|^2$$

where the inequalities (a) and (c) use Assumption 2, the inequality (b) uses the fact of

$$\langle a, b \rangle \leq \frac{1}{2} (\|a\|^2 + \|b\|^2),$$

the inequality (d) uses (10).

Summing (25) over all $u \in K(t)$, we obtain

$$\mathbb{E} f(w_{t+1}) - \mathbb{E} f(w_t) \leq -\frac{\gamma}{2} \sum_{u \in K(t)} \mathbb{E} \|\nabla g_{\psi(u)} f_i(w_u)\|^2 + \frac{L_{\max}(t)^2}{2} \sum_{u \in K(t)} \mathbb{E} \|\hat{\psi}(u)\|^2$$

$$\leq -\frac{\gamma}{2} \left( \frac{1}{2} \sum_{u \in K(t)} \|\nabla g_{\psi(u)} f_i(w_u)\|^2 - \eta_1 \gamma^2 L^2 \sum_{u \in K(t)} \sum_{v \in \{t, \ldots, u\}} \|\hat{\psi}(v)\|^2 \right)$$

$$+ \frac{\eta_2 L^2 \gamma^2}{2} \sum_{u \in K(t)} \sum_{u' \in D(u)} \mathbb{E} \|\hat{\psi}(u')\|^2 + \frac{L_{\max}(t)^2}{2} \sum_{u \in K(t)} \mathbb{E} \|\hat{\psi}(u)\|^2$$

$$= -\frac{\gamma}{4} \sum_{u \in K(t)} \|\nabla g_{\psi(u)} f_i(w_t)\|^2 + \frac{L_{\max}(t)^2}{2} \sum_{u \in K(t)} \mathbb{E} \|\hat{\psi}(u)\|^2$$

$$+ \frac{\eta_1 \gamma^3 L^2}{2} \sum_{u \in K(t)} \sum_{v \in \{t, \ldots, u\}} \|\hat{\psi}(v)\|^2 + \frac{\eta_2 L^2 \gamma^2}{2} \sum_{u \in K(t)} \sum_{u' \in D(u)} \mathbb{E} \|\hat{\psi}(u')\|^2$$

$$\leq -\frac{\gamma}{4} \|\nabla f(w_t)\|^2 + \frac{\eta_1 \gamma^3 L^2}{2} \sum_{u \in K(t)} \sum_{v \in \{t, \ldots, u\}} \|\hat{\psi}(v)\|^2$$
Before proving Theorem 6, we first provide Lemma 14 to provides an upper bound to $E \| \tilde{v}_u \|^2$.

Appendix B: Proof of Theorem 6

Because $\log \frac{\gamma L^2}{b}$, we have that

Let $\gamma \cdot \gamma L^2(q \eta_1)^2 + \eta_2 \eta_2 q \eta_1 \tau + L_{\text{max}} \gamma^2 q \eta_1 \leq 2$, we have that $\gamma \leq \frac{-L_{\text{max}} + \sqrt{L_{\text{max}}^2 + 2b \mu (L^2(q \eta_1)^2 + \eta_2 \eta_2 q \eta_1 \tau)}}{2b \mu L^2(q \eta_1)^2 + \eta_2 \eta_2 q \eta_1 \tau}$.

Let $1 - \frac{\gamma \mu}{2} \cdot (f(w_0) - f(w^*)) \leq \frac{\epsilon}{2}$, we have that

Because $\log \left( \frac{1}{\epsilon} \right) \geq 1 - \rho$ for $0 < \rho \leq 1$, we have that

This completes the proof.

Appendix B: Proof of Theorem 6

Before proving Theorem 6, we first provide Lemma 14 to provides an upper bound to $E \| \tilde{v}_u \|^2$. 

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Lemma 14  For AFSVRG-VP, under Assumptions 1 and 2, let \( u \in K(t) \), we have that

\[
E \left\| \hat{v}_u^\ell \right\|^2 \leq \frac{16L^2}{\mu} E(f(w_t^*) - f(w^*)) + \frac{8L^2}{\mu} E(f(w^*) - f(w^*)) + 4L^2\gamma^2 \eta_1 \sum_{v \in \{t, \ldots, u\}} E \left\| \hat{g}_{v}(v') \right\|^2 + 2\eta_2 L^2 \gamma^2 E \sum_{w' \in D(u)} \left\| \hat{g}_{u'}(w') \right\|^2
\]

Proof Define \( v_u^\ell = \nabla g_{u} f_t(w_u^*) - \nabla g_{u} f_t(w^*) + \nabla g_{u} f_t(w^*) \). We have that \( E \left\| \hat{v}_u^\ell \right\|^2 = E \left\| \hat{v}_u^\ell - v_u^\ell + v_u^\ell \right\|^2 \leq 2E \left\| \hat{v}_u^\ell - v_u^\ell \right\|^2 + 2E \left\| v_u^\ell \right\|^2 \). Firstly, we give the upper bound to \( E \left\| v_u^\ell \right\|^2 \) as follows.

\[
E \left\| v_u^\ell \right\|^2 \leq \frac{8L^2}{\mu} E(f(w_t^*) - f(w^*)) + 4L^2\gamma^2 \eta_1 \sum_{v \in \{t, \ldots, u\}} E \left\| \nabla g_{v}(v') \right\|^2 + 4L^2 \gamma^2 \sum_{v \in \{t, \ldots, u\}} E \left\| \hat{g}_{v}(v) \right\|^2 + 2\eta_2 L^2 \gamma^2 E \sum_{w' \in D(u)} \left\| \hat{g}_{u'}(w') \right\|^2
\]

where the inequality (a) uses \( E \left\| x - Ex \right\|^2 \leq E \left\| x \right\|^2 \), the inequality (b) uses Assumption 2, the equality (c) uses Eq. (10), and the inequality (d) uses Assumption 1.

Next, we give the upper bound to \( E \left\| \hat{v}_u^\ell - v_u^\ell \right\|^2 \) as follows.

\[
E \left\| \hat{v}_u^\ell - v_u^\ell \right\|^2 = E \left\| \nabla g_{u} f_t(w_u^*) - \nabla g_{u} f_t(w_u^*) \right\|^2 \leq L^2 E \left\| \hat{w}_u^\ell - w_u^\ell \right\|^2
\]

Combining (32) and (33), we have that

\[
E \left\| \hat{v}_u^\ell \right\|^2 \leq 2E \left\| \hat{v}_u^\ell - v_u^\ell \right\|^2 + 2E \left\| v_u^\ell \right\|^2 \]

(34)
\[
\leq \frac{16L^2}{\mu} \mathbb{E}(f(w^s_t) - f(w^s)) + \frac{8L^2}{\mu} \mathbb{E}(f(w^s) - f(w^s)) \\
+ 4L^2\gamma^2\eta_t \sum_{v \in \{t, \ldots, u\}} \mathbb{E} \left\| \tilde{v}^{(v)} \right\|^2 + 2\eta_2 L^2\gamma^2 \mathbb{E} \sum_{u' \in D(u)} \left\| \tilde{v}^{(u')} \right\|^2
\]

This completes the proof.

Based on the basic inequalities in Lemma 13, we provide the proof of Theorem 6 in the following.

**Proof** Similar to (25), for \( u \in K(t) \) at \( s \)-th outer loop, we have that

\[
\mathbb{E} f(w^s_{u+1})
\]

\[
\leq \mathbb{E} \left( f(w^s_u) + \langle \nabla f(w^s_u), w^s_{u+1} - w^s_u \rangle + \frac{L\psi(u)}{2} \|w^s_{u+1} - w^s_u\|^2 \right)
\]

\[
= \mathbb{E} \left( f(w^s_u) - \gamma \langle \nabla f(w^s_u), \tilde{v}^{(u)} \rangle + \frac{L\psi(u)}{2} \|\tilde{v}^{(u)}\|^2 \right)
\]

\[
= \mathbb{E} f(w^s_u) - \gamma \mathbb{E} \langle \nabla f(w^s_u), \nabla \psi(u) f_i(u) (\tilde{w}^s_u) \rangle + \frac{L\psi(u)}{2} \mathbb{E} \|\tilde{v}^{(u)}\|^2
\]

where the inequalities (a) use Assumption 2, the equality (b) uses the fact that the stochastic local gradient \( \tilde{v}^u \) is unbiased, the inequality (c) follows the proof in (25).

Summing (35) over all \( u \in K(t) \), we obtain

\[
\mathbb{E} f(w^s_{t+K(t)}) - \mathbb{E} f(w^s_t)
\]

\[
\leq -\frac{\gamma}{2} \sum_{u \in K(t)} \mathbb{E} \|\nabla \psi(u) f(w^s_u)\|^2 + \frac{L_{\text{max}} \gamma^2}{2} \sum_{u \in K(t)} \mathbb{E} \|\tilde{v}^{(u)}\|^2
\]

\[
+ \frac{\eta_2 L^2 \gamma^2}{2} \sum_{u \in K(t)} \sum_{u' \in D(u)} \mathbb{E} \|\tilde{v}^{(u')}\|^2
\]

\[
\leq -\frac{\gamma}{2} \left( \frac{1}{2} \sum_{u \in K(t)} \|\nabla \psi(u) f(w^s_u)\|^2 - \eta_1 \gamma^2 L^2 \sum_{u \in K(t)} \sum_{v \in \{t, \ldots, u\}} \|\tilde{v}^{(v)}\|^2 \right)
\]

\[
+ \frac{\eta_2 L^2 \gamma^2}{2} \sum_{u \in K(t)} \sum_{u' \in D(u)} \mathbb{E} \|\tilde{v}^{(u')}\|^2 + \frac{L_{\text{max}} \gamma^2}{2} \sum_{u \in K(t)} \mathbb{E} \|\tilde{v}^{(u)}\|^2
\]

\[
= -\frac{\gamma}{4} \sum_{u \in K(t)} \|\nabla \psi(u) f(w^s_u)\|^2 + \frac{L_{\text{max}} \gamma^2}{2} \sum_{u \in K(t)} \mathbb{E} \|\tilde{v}^{(u)}\|^2
\]
Thus, to achieve the accuracy $\epsilon$ of (1) for AFSVRG-VP, i.e., $\mathbb{E}f(w_S) - f(w^*) \leq \epsilon$, we can carefully choose $\gamma$ such that

$$\frac{8L^2\eta_1 q C}{\rho \mu} \leq 0.5$$



(37)
\[ \gamma^3 \left( \left( \frac{1}{2} + \frac{2C}{\gamma} \right) \eta_2 \tau + 4 \frac{C}{\gamma} \eta_1 q \right) \frac{\eta q L^2 G}{\rho} \leq \frac{\epsilon}{8} \] (40)

and let \( (1 - \rho)^{\nu(t)} \leq 0.25 \), i.e., \( \nu(t) \geq \frac{\log 0.25}{\log(1 - \rho)} \), we have that

\[ e^{s+1} \leq 0.75 e^s + \frac{\epsilon}{8} \] (41)

Recursively apply (69), we have that

\[ e^S \leq (0.75)^S e^0 + \frac{\epsilon}{2} \] (42)

Finally, the outer loop number \( S \) should satisfy the condition of \( S \geq \frac{\log \frac{2e^0}{\log 4}}{\log 3} \). This completes the proof.

Appendix C: Proof of Theorem 8
Before proving Theorem 8, we first provide Lemma 17 to provides an upper bound to \( \mathbb{E} \| \hat{e}_u^\ell \|^2 \).

**Lemma 15** For AFSAGA-VP, we have that

\[ \mathbb{E} \left\| \alpha_{i_u}^{u,\ell} - \nabla G_i f_{i_u}(w^*) \right\|^2 \leq \frac{L^2}{l} \sum_{u' = 1}^{\xi(u,\ell) - 1} \left( 1 - \frac{1}{l} \right)^{\xi(u,\ell) - u' - 1} \sigma(w_{\xi^{-1}(u',\ell)}) + L^2 \left( 1 - \frac{1}{l} \right)^{\xi(u,\ell)} \sigma(w_0) \] (43)

\[ \mathbb{E} \left\| \alpha_{i_u}^{u,\ell} - \bar{\alpha}_{i_u}^{u,\ell} \right\|^2 \leq \frac{\eta_0 L^2}{l} \sum_{u' = 1}^{\xi(u,\ell) - 1} \sum_{\bar{u} \in D(\xi^{-1}(u',\ell))} \left( 1 - \frac{1}{l} \right)^{\xi(u,\ell) - u' - 1} \mathbb{E} \left\| \hat{v}_{\bar{u}}^{i_u} \right\|^2 \]

where \( \sigma(w_u) = \mathbb{E} \| w_u - w^* \|^2 \).

**Proof** Firstly, we have that

\[ \mathbb{E} \left\| \alpha_{i_u}^{u,\ell} - \nabla G_i f_{i_u}(w^*) \right\|^2 \] (44)

\[ = \frac{1}{l} \sum_{i = 1}^l \mathbb{E} \left\| \alpha_{i_u}^{u,\ell} - \nabla G_i f_i(w^*) \right\|^2 \]

\[ = \frac{1}{l} \sum_{i = 1}^l \sum_{u' = 0}^{\xi(u,\ell) - 1} \mathbb{E} \sum_{u'' = u'}^{u'' = u} \| \nabla G_i f_i(w_{\xi^{-1}(u',\ell)}) - \nabla G_i f_i(w^*) \|^2 \]

\[ = \frac{1}{l} \sum_{u' = 0}^{\xi(u,\ell) - 1} \sum_{i = 1}^l \mathbb{E} \| \nabla G_i f_i(w_{\xi^{-1}(u',\ell)}) - \nabla G_i f_i(w^*) \|^2 \]
where $u^\gamma_i$ denote the last iterate to update the $\alpha^u_i$. We consider the two cases $u' > 0$ and $u' = 0$ as following.

For $u' > 0$, we have that

$$\mathbb{E} \left( 1_{\{u^\gamma_i = u'\}} \left\| \nabla g_i(f_i(w_{\xi^{-1}(u',\ell)})) - \nabla g_i f_i(w^*) \right\|^2 \right) \quad (45)$$

\[ \leq \mathbb{E} \left( 1_{\{i_u' = 1\}} 1_{\{i_{u'} \neq i, \forall v \text{ s.t. } u' + 1 \leq v \leq \xi(u, \ell) - 1\}} \left\| \nabla g_i f_i(w_{\xi^{-1}(u',\ell)}) - \nabla g_i f_i(w^*) \right\|^2 \right) \quad (a) \]

\[ \leq P\{i_{u'} = i\} P\{i_v \neq i, \forall v \text{ s.t. } u' + 1 \leq v \leq \xi(u, \ell) - 1\} \mathbb{E} \left\| \nabla g_i f_i(w_{\xi^{-1}(u',\ell)}) - \nabla g_i f_i(w^*) \right\|^2 \quad (b) \]

\[ \leq \frac{1}{l} \left(1 - \frac{1}{l}\right)^{\xi(u,\ell) - u' - 1} \mathbb{E} \left\| \nabla g_i f_i(w_{\xi^{-1}(u',\ell)}) - \nabla g_i f_i(w^*) \right\|^2 \quad (c) \]

where the inequality (a) uses the fact $i_u'$ and $i_v$ are independent for $v \neq u'$, the inequality (b) uses the fact that $P\{i_u = i\} = \frac{1}{l}$ and $P\{i_v \neq i\} = 1 - \frac{1}{l}$.

For $u' = 0$, we have that

$$\mathbb{E} \left( 1_{\{u^\gamma_i = 0\}} \left\| \nabla g_i f_i(w_0) - \nabla g_i f_i(w^*) \right\|^2 \right) \quad (46)$$

\[ \leq \mathbb{E} \left( 1_{\{i_v \neq i, \forall v \text{ s.t. } 0 \leq v \leq \xi(u, \ell) - 1\}} \left\| \nabla g_i f_i(w_0) - \nabla g_i f_i(w^*) \right\|^2 \right) \]

\[ \leq P\{i_v \neq i, \forall v \text{ s.t. } 0 \leq v \leq \xi(u, \ell) - 1\} \mathbb{E} \left\| \nabla g_i f_i(w_0) - \nabla g_i f_i(w^*) \right\|^2 \]

\[ \leq \left(1 - \frac{1}{l}\right)^{\xi(u,\ell)} \mathbb{E} \left\| \nabla g_i f_i(w_0) - \nabla g_i f_i(w^*) \right\|^2 \]

Substituting (45) and (46) into (44), we have that

$$\mathbb{E} \left\| \alpha_{i_u}^{u,\ell} - \nabla g_i f_i(u^\gamma_i(w^*)) \right\|^2 \quad (47)$$

\[ = \frac{1}{l} \sum_{u' = 0}^{\xi(u,\ell) - 1} \sum_{i=1}^{l} \mathbb{E} 1_{\{u^\gamma_i = u'\}} \left\| \nabla g_i f_i(w_{\xi^{-1}(u',\ell)}) - \nabla g_i f_i(w^*) \right\|^2 \]

\[ \leq \frac{1}{l} \sum_{u' = 0}^{\xi(u,\ell) - 1} \sum_{i=1}^{l} \left(1 - \frac{1}{l}\right)^{\xi(u,\ell) - u' - 1} \mathbb{E} \left\| \nabla g_i f_i(w_0) - \nabla g_i f_i(w^*) \right\|^2 \quad (a) \]

\[ + \frac{1}{l} \sum_{i=1}^{l} \left(1 - \frac{1}{l}\right)^{\xi(u,\ell) - 1} \mathbb{E} \left\| \nabla g_i f_i(w_0) - \nabla g_i f_i(w^*) \right\|^2 \]

\[ = \frac{1}{l} \sum_{u' = 0}^{\xi(u,\ell) - 1} \left(1 - \frac{1}{l}\right)^{\xi(u,\ell) - u' - 1} \mathbb{E} \left\| \nabla g_i f_i(w_{\xi^{-1}(u',\ell)}) - \nabla g_i f_i(w^*) \right\|^2 \]

\[ + \frac{1}{l} \sum_{i=1}^{l} \left(1 - \frac{1}{l}\right)^{\xi(u,\ell) - 1} \mathbb{E} \left\| \nabla g_i f_i(w_0) - \nabla g_i f_i(w^*) \right\|^2 \]

\[ = \frac{1}{l} \sum_{u' = 0}^{\xi(u,\ell) - 1} \left(1 - \frac{1}{l}\right)^{\xi(u,\ell) - u' - 1} \mathbb{E} \left\| \nabla g_i f_i(w_{\xi^{-1}(u',\ell)}) - \nabla g_i f_i(w^*) \right\|^2 \]

\[ + \left(1 - \frac{1}{l}\right)^{\xi(u,\ell) - 1} \mathbb{E} \left\| \nabla g_i f_i(w_0) - \nabla g_i f_i(w^*) \right\|^2 \]
Given a global time counter $u$, we let $\{\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{v(u)-1}\}$ be the all start time counters for the global time counters from 0 to $u$. Thus, for AFSAGA-VP, we have that

$$
\mathbb{E} \left\| v_\ell \right\|^2 
\leq 4 \frac{L^2 \eta_1}{l} \sum_{k'=1}^{v(u)} \left( 1 - \frac{1}{l} \right)^{v(u) - k'} \sigma(w_{\overline{u}_{k'}}) + 8 L^2 \gamma^2 \eta_1^2 g G + 2 L^2 \left( 1 - \frac{1}{l} \right)^{v(u)} \sigma(w_0) + 4 L^2 \sigma(w_{\psi(u)})
$$

where the inequality (a) uses \[45\] and \[46\], the inequality (b) uses Assumption \[1\].

Similarly, we have that

$$
\mathbb{E} \left\| \alpha_{u,\ell}^\prime - \check{\alpha}_{u,\ell}^\prime \right\|^2 = \frac{1}{l} \sum_{i=1}^l \mathbb{E} \left\| \alpha_{i,\ell}^\prime - \check{\alpha}_{i,\ell}^\prime \right\|^2
$$

$$
= \frac{1}{l} \sum_{u'=0}^{l} \sum_{u'=1}^{l} \mathbb{E} 1_{\{u'=u\}} \left\| \alpha_{i,\ell}^u - \check{\alpha}_{i,\ell}^u \right\|^2
$$

$$
\leq \frac{1}{l} \sum_{u'=1}^{l} \sum_{u'=1}^{l} \left( \frac{1}{l} \sum_{i=1}^l \mathbb{E} \left\| \alpha_{i,\ell}^u - \check{\alpha}_{i,\ell}^u \right\|^2 - \xi(\ell) \right)
$$

$$
\leq \frac{1}{l} \sum_{u'=1}^{l} \sum_{u'=1}^{l} \left( \frac{1}{l} \sum_{i=1}^l \mathbb{E} \left\| \alpha_{i,\ell}^u - \check{\alpha}_{i,\ell}^u \right\|^2 - \xi(\ell) \right)
$$

Thus, this completes the proof.

**Lemma 16**: Given a global time counter $u$, we let $\{\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{v(u)-1}\}$ be the all start time counters for the global time counters from 0 to $u$. Thus, for AFSAGA-VP, we have that

$$
\mathbb{E} \left\| v_\ell \right\|^2 
\leq 4 \frac{L^2 \eta_1}{l} \sum_{k'=1}^{v(u)} \left( 1 - \frac{1}{l} \right)^{v(u) - k'} \sigma(w_{\overline{u}_{k'}}) + 8 L^2 \gamma^2 \eta_1^2 g G + 2 L^2 \left( 1 - \frac{1}{l} \right)^{v(u)} \sigma(w_0) + 4 L^2 \sigma(w_{\psi(u)})
$$
where \( \sigma(w_u) = \mathbb{E}\|w_u - w^*\|^2 \).

**Proof** We have that

\[
\begin{align*}
\mathbb{E}\left\| v_{i,u}^{\ell} \right\|^2 \\
= & \mathbb{E}\left\| \nabla_{g_i} f_{i_u}(w_u) - \alpha_{i,u}^{u,\ell} + \frac{1}{l} \sum_{i=1}^l \alpha_{i,u}^{u,\ell} \right\|^2 \\
= & \mathbb{E}\left\| \nabla_{g_i} f_{i_u}(w_u) - \nabla_{g_i} f_{i_u}(w^*) - \alpha_{i,u}^{u,\ell} + \nabla_{g_i} f_{i_u}(w^*) \right. \\
& \left. + \frac{1}{l} \sum_{i=1}^l \alpha_{i,u}^{u,\ell} - \nabla_{g_i} f(w^*) + \nabla_{g_i} f(w^*) \right\|^2 \\
\overset{(a)}{\leq} & 2\mathbb{E}\left\| \nabla_{g_i} f_{i_u}(w^*) - \alpha_{i,u}^{u,\ell} + \frac{1}{l} \sum_{i=1}^l \alpha_{i,u}^{u,\ell} - \nabla_{g_i} f(w^*) \right\|^2 \\
& + 2\mathbb{E}\left\| \nabla_{g_i} f_{i_u}(w_u) - \nabla_{g_i} f_{i_u}(w^*) \right\|^2 \\
\overset{(b)}{\leq} & 2\mathbb{E}\left\| \alpha_{i,u}^{u,\ell} - \nabla_{g_i} f_{i_u}(w^*) \right\|^2 + 2\mathbb{E}\left\| \nabla_{g_i} f_{i_u}(w_u) - \nabla_{g_i} f_{i_u}(w^*) \right\|^2 \\
\overset{(c)}{\leq} & 2 \frac{L^2}{l} \sum_{u' = 1}^{\xi(u,\ell) - 1} \left( 1 - \frac{1}{l} \right) \xi(u,\ell) - u' - 1 \mathbb{E}\left\| w_{\xi^{-1}(u',\ell)} - w^* \right\|^2 \\
& + 2L^2 \left( 1 - \frac{1}{l} \right) \xi(u,\ell) \sigma(w_0) + 2L^2 \mathbb{E}\left\| w_u - w^* \right\|^2 \\
= & 2 \frac{L^2}{l} \sum_{u' = 1}^{\xi(u,\ell) - 1} \left( 1 - \frac{1}{l} \right) \xi(u,\ell) - u' - 1 .
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}\left\| w_{\xi^{-1}(u',\ell)} - w_{\varphi(\xi^{-1}(u',\ell))} + w_{\varphi(\xi^{-1}(u',\ell))} - w^* \right\|^2 \\
& + 2L^2 \left( 1 - \frac{1}{l} \right) \xi(u,\ell) \sigma(w_0) + 2L^2 \mathbb{E}\left\| w_u - w^* \right\|^2 \\
\overset{(d)}{\leq} & 2 \frac{L^2}{l} \sum_{u' = 1}^{\xi(u,\ell) - 1} \left( 1 - \frac{1}{l} \right) \xi(u,\ell) - u' - 1 .
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}\left( 2\| w_{\xi^{-1}(u',\ell)} - w_{\varphi(\xi^{-1}(u',\ell))} \|^2 + 2\| w_{\varphi(\xi^{-1}(u',\ell))} - w^* \|^2 \right) \\
& + 2L^2 \left( 1 - \frac{1}{l} \right) \xi(u,\ell) \sigma(w_0) + 4L^2 \mathbb{E}\left\| w_{\varphi(u)} - w^* \right\|^2 + 4L^2 \gamma^2 \mathbb{E}\left\| \sum_{v \in \{\varphi(u),...\}} U_{\psi(v)} \tilde{\psi}_v \right\|^2 \\
\overset{(e)}{\leq} & 2 \frac{L^2}{l} \sum_{u' = 1}^{\xi(u,\ell) - 1} \left( 1 - \frac{1}{l} \right) \xi(u,\ell) - u' - 1 \mathbb{E}\left( 2\eta_1 \gamma^2 \cdot \sum_{v \in \{\varphi(\xi^{-1}(u',\ell))...\}} \| \tilde{\psi}_v \|^2 \right) \\
& + 2\| w_{\varphi(\xi^{-1}(u',\ell))} - w^* \|^2 \\
\end{align*}
\]

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\[+ 2L^2 \left(1 - \frac{1}{t}\right)^{v(u)} \sigma(w_0) + 4L^2 \mathbb{E} \|w_{\varphi(u)} - w^*\|^2 + 4L^2 \gamma^2 \eta_{1} \sum_{v \in \{\varphi(u), \ldots, u\}} \mathbb{E} \|\hat{v}_v^{\psi(v)}\|^{2}\]

\[\leq 2L^2 \sum_{u' = 1}^{\xi(u, \ell) - 1} \left(1 - \frac{1}{t}\right) \xi(u, \ell) - u' - 1 \cdot \mathbb{E} \left(2\eta_1^2 \gamma^2 qG + 2\|w_{\varphi(\xi^{-1}(u', \ell))} - w^*\|^2\right)\]

\[+ 2L^2 \left(1 - \frac{1}{t}\right)^{v(u)} \sigma(w_0) + 4L^2 \mathbb{E} \|w_{\varphi(u)} - w^*\|^2 + 4L^2 \gamma^2 \eta_{1}^2 qG\]

\[\leq 4 \frac{L}{l} \sum_{u' = 1}^{\xi(u, \ell) - 1} \left(1 - \frac{1}{t}\right) \xi(u, \ell) - u' - 1 \mathbb{E} \|w_{\varphi(\xi^{-1}(u', \ell))} - w^*\|^2\]

\[+ 2L^2 \left(1 - \frac{1}{t}\right)^{v(u)} \sigma(w_0) + 4L^2 \mathbb{E} \|w_{\varphi(u)} - w^*\|^2 + 8L^2 \gamma^2 \eta_{1}^2 qG\]

where the inequalities (a) and (d) uses \(\|\sum_{i=1}^{n} a_i\|^2 \leq n \sum_{i=1}^{n} \|a_i\|^2\), the inequality (b) follows from \(\mathbb{E} \|x - \mathbb{E} x\|^2 \leq \mathbb{E} \|x\|^2\), the inequality (c) uses Lemma 15, the inequality (e) uses Assumption 3, the inequality (f) uses Assumption 3, and the inequality (g) uses the fact \(\sum_{u' = 1}^{\xi(u, \ell) - 1} \left(1 - \frac{1}{t}\right) \xi(u, \ell) - u' - 1 \leq l\). This completes the proof.

**Lemma 17** For AFSVRG-VP, under Assumptions 3 and 4, let \(u \in K(t)\), we have that

\[\mathbb{E} \left\|\hat{v}_u^{\ell}\right\|^2 \leq 6\eta_{1} L^2 \gamma^2 \sum_{u' \in D(u)} \mathbb{E} \left\|\hat{v}_{u'}^{\psi(u')}\right\|^2 + \frac{12\eta_{2} L^2 \gamma^2 \xi(u, \ell) - 1}{l} \sum_{u' = 1}^{\xi(u, \ell)} \left(1 - \frac{1}{t}\right) \xi(u, \ell) - u' - 1 \mathbb{E} \left\|\hat{v}_{u}^{\psi(u)}\right\|^2 + 2\mathbb{E} \left\|v_u^{\ell}\right\|^2.\]  

**Proof** Define \(v_u^{\ell} = \nabla_{g_\ell} f_{i}(w_u) - \alpha_\ell + \frac{1}{l} \sum_{i=1}^{l} \alpha_i^{u, \ell}\). We have that \(\mathbb{E} \left\|\hat{v}_u^{\ell}\right\|^2 = \mathbb{E} \left\|\hat{v}_u^{\ell} - v_u^{\ell} + v_u^{\ell}\right\|^2 \leq 2\mathbb{E} \left\|\hat{v}_u^{\ell} - v_u^{\ell}\right\|^2 + 2\mathbb{E} \left\|v_u^{\ell}\right\|^2.

Next, we give the upper bound to \(\mathbb{E} \left\|\hat{v}_u^{\ell} - v_u^{\ell}\right\|^2\) as follows. Next, we have that

\[\mathbb{E} \left\|\hat{v}_u^{\ell} - v_u^{\ell}\right\|^2 \leq 3\mathbb{E} \left\|\nabla_{g_\ell} f_{i}(\tilde{w}_u) - \nabla_{g_\ell} f_{i}(w_u)\right\|^2 + 3\mathbb{E} \left\|\alpha_u^{u, \ell} - \hat{\alpha}_u^{u, \ell}\right\|^2 + 3\mathbb{E} \left\|\frac{1}{l} \sum_{i=1}^{l} \alpha_i^{u, \ell} - \frac{1}{l} \sum_{i=1}^{l} \hat{\alpha}_i^{u, \ell}\right\|^2.\]
where the inequality (a) uses $\|\sum_{i=1}^n a_i\|^2 \leq n \sum_{i=1}^n \|a_i\|^2$. We will give the upper bounds for the expectations of $Q_1$, $Q_2$ and $Q_3$ respectively.

\[
\mathbb{E}Q_1 = \mathbb{E}\|\nabla_{G_{i_u}} f_{i_u}(\tilde{w}_u) - \nabla_{G_{i_u}} f_{i_u}(w_u)\|^2 \\
\leq L^2 \mathbb{E}\|\tilde{w}_u - w_u\|^2 = L^2 \gamma^2 \mathbb{E}\left\| \sum_{u' \in D(u)} U_{\psi(u')} \hat{v}_{u'} \right\|^2 \\
\leq \eta_1 L^2 \gamma^2 \sum_{u' \in D(u)} \mathbb{E}\left\| \hat{v}_{u'} \right\|^2
\]

(53)

where the first inequality uses Assumption 2, the second inequality uses (54).

\[
\mathbb{E}Q_2 = \mathbb{E}\left\| \alpha_{i'u}^u - \tilde{\alpha}_{i'u} \right\|^2 \\
\leq \frac{\eta_2 L^2 \gamma^2 \xi(u,\ell)}{l} \sum_{u'=1}^{\xi(u,\ell) - 1} \sum_{\tilde{u} \in D(\xi^{-1}(u,\ell))} \left( 1 - \frac{1}{l} \right) \mathbb{E}\left\| \hat{v}_{\tilde{u}} \right\|^2
\]

(54)

where the inequality uses Lemma 15.

\[
\mathbb{E}Q_3 = \mathbb{E}\left\| \frac{1}{l} \sum_{i=1}^l \alpha_{i'u}^u - \frac{1}{l} \sum_{i=1}^l \tilde{\alpha}_{i'u} \right\|^2 \\
\leq \frac{1}{l} \sum_{i=1}^l \mathbb{E}\left\| \alpha_{i'u}^u - \tilde{\alpha}_{i'u} \right\|^2 \\
\leq \frac{\eta_2 L^2 \gamma^2 \xi(u,\ell)}{l} \sum_{u'=1}^{\xi(u,\ell) - 1} \sum_{\tilde{u} \in D(\xi^{-1}(u,\ell))} \left( 1 - \frac{1}{l} \right) \mathbb{E}\left\| \hat{v}_{\tilde{u}} \right\|^2
\]

(55)

where the first inequality uses $\|\sum_{i=1}^n a_i\|^2 \leq n \sum_{i=1}^n \|a_i\|^2$, the second inequality uses (54).

\[
\mathbb{E}\left\| \hat{v}_{\tilde{u}} \right\|^2 \leq 2 \mathbb{E}\left\| \hat{v}_{u} - (v^{t+1})_f \right\|^2 + 2 \mathbb{E}\left\| v_u^f \right\|^2 \\
\leq 6\mathbb{E}Q_1 + 6\mathbb{E}Q_2 + 6\mathbb{E}Q_3 + 2 \mathbb{E}\left\| v_u^f \right\|^2 \\
\leq 6\eta_1 L^2 \gamma^2 \sum_{u' \in D(u)} \mathbb{E}\left\| \hat{v}_{u'} \right\|^2 + 2 \mathbb{E}\left\| v_u^f \right\|^2 \\
+ \frac{12\eta_2 L^2 \gamma^2 \xi(u,\ell)}{l} \sum_{u'=1}^{\xi(u,\ell) - 1} \sum_{\tilde{u} \in D(\xi^{-1}(u',\ell))} \left( 1 - \frac{1}{l} \right) \mathbb{E}\left\| \hat{v}_{\tilde{u}} \right\|^2
\]

(56)

where the second inequality uses Lemma 15. This completes the proof. 

\[\square\]
Based on the basic inequalities in Lemma 13, we provide the proof of Theorem 6 in the following.

**Proof** Similar to (36), we have that

\[
\mathbb{E} f(w_{t+1} | K(t)) - \mathbb{E} f(w_t) \leq \frac{\gamma}{4} \| \nabla f(w_t) \|^2 + \frac{\eta_2 \gamma^2 L^2}{2} \sum_{u \in K(t)} \sum_{u' \in D(u)} \mathbb{E} \| \hat{v}_u^{(u')} \|^2 \\
+ \left( \frac{\eta_1 \gamma^3 L^2 q \eta_1}{2} + \frac{L_{\text{max}} \gamma^2}{2} \right) \sum_{u \in K(t)} \mathbb{E} \| \hat{v}_u^{(u')} \|^2 \\
\leq \frac{\gamma}{4} \| \nabla f(w_t) \|^2 + \frac{\eta_2 \gamma^2 L^2}{2} \sum_{u \in K(t)} \sum_{u' \in D(u)} \mathbb{E} \| \hat{v}_u^{(u')} \|^2 \\
+ \left( \frac{\eta_1 \gamma^3 L^2 q \eta_1}{2} + \frac{L_{\text{max}} \gamma^2}{2} \right) \sum_{u \in K(t)} \left( 6 \eta_1 L^2 \gamma^2 \sum_{u' \in D(u)} \mathbb{E} \| \hat{v}_u^{(u')} \|^2 \right)
\]

\[
\leq -\frac{\gamma}{4} \| \nabla f(w_t) \|^2 + \left( \gamma L^2 q \eta_1^2 + L_{\text{max}} \right) \gamma^2 \sum_{u \in K(t)} \mathbb{E} \| v_u^{(u)} \|^2 \\
+ \left( \frac{\eta_2}{2} + 3 \gamma (q \eta_1^2 + L_{\text{max}}) (\eta_1 + 2 \eta_2) \right) \gamma^2 L^2 \eta_1 q \tau G
\]

\[
\leq \frac{\gamma}{4} \| \nabla f(w_t) \|^2 + \left( \gamma L^2 q \eta_1^2 + L_{\text{max}} \right) \gamma^2 \sum_{u \in K(t)} \mathbb{E} \| v_u^{(u)} \|^2 \\
+ \left( \frac{\eta_2}{2} + 3 \gamma (q \eta_1^2 + L_{\text{max}}) (\eta_1 + 2 \eta_2) \right) \gamma^2 L^2 \eta_1 q \tau G
\]

\[
\leq \frac{\gamma}{4} \| \nabla f(w_t) \|^2 + \left( \gamma L^2 q \eta_1^2 + L_{\text{max}} \right) \gamma^2 \sum_{u \in K(t)} \mathbb{E} \| v_u^{(u)} \|^2 \\
+ \left( \frac{\eta_2}{2} + 3 \gamma (q \eta_1^2 + L_{\text{max}}) (\eta_1 + 2 \eta_2) \right) \gamma^2 L^2 \eta_1 q \tau G
\]

\[
\leq -\frac{\gamma}{4} \| \nabla f(w_t) \|^2 + \left( \gamma L^2 q \eta_1^2 + L_{\text{max}} \right) \gamma^2 \sum_{u \in K(t)} \mathbb{E} \| v_u^{(u)} \|^2 \\
+ \left( \frac{\eta_2}{2} + 3 \gamma (q \eta_1^2 + L_{\text{max}}) (\eta_1 + 2 \eta_2) \right) \gamma^2 L^2 \eta_1 q \tau G
\]

\[
\leq -\frac{\gamma}{4} \| \nabla f(w_t) \|^2 + \left( \gamma L^2 q \eta_1^2 + L_{\text{max}} \right) \gamma^2 \sum_{u \in K(t)} \mathbb{E} \| v_u^{(u)} \|^2 \\
+ \left( \frac{\eta_2}{2} + 3 \gamma (q \eta_1^2 + L_{\text{max}}) (\eta_1 + 2 \eta_2) \right) \gamma^2 L^2 \eta_1 q \tau G
\]
where the inequalities (a) use (36), the equality (b) uses Lemma 17, the inequality (c) uses Assumption 3, the inequality (d) uses Lemma 16, the inequality (e) uses Assumption 1.

According to (57), we have that

\[ e(w_{t+|K(t)|}) \leq \left(1 - \frac{\gamma \mu}{4}\right) e(w_t) + c_1 \left(1 - \frac{1}{l}\right) \sum_{k'=1}^{v(t)} \left(1 - \frac{1}{l}\right)^{v(t)-k'} \sigma(w_{\pi_{k'}}) \]

\[ + c_0 + c_2 \sum_{k'=1}^{v(t)} \left(1 - \frac{1}{l}\right)^{v(t)-k'} \sigma(w_{\pi_{k'}}) - \frac{\gamma \mu^2}{4} \sigma(w_t) \]

\[ = \left(1 - \frac{\gamma \mu}{4}\right) e(w_t) + \left(-\frac{\gamma \mu^2}{4} + 2c_1 + c_2\right) \sigma(w_t) \]

\[ + c_1 \left(1 - \frac{1}{l}\right)^{v(t)} \sigma(w_0) + c_2 \sum_{k'=1}^{v(t)-1} \left(1 - \frac{1}{l}\right)^{v(t)-k'} \sigma(w_{\pi_{k'}}) + c_0 \]

where \( c_0 = \left(\frac{\eta_1}{2} + 3(\gamma q \eta_1^2 + L_{\text{max}})(\eta_1 + 2\eta_2)\right)\tau + (\gamma L^2 q \eta_1^2 + 8L_{\text{max}}) \eta_1 q \eta_1) \gamma^4 L^2 \eta_1 q G, c_1 = (\gamma L^2 q \eta_1^2 + L_{\text{max}}) \gamma^2 \eta_1 q 2L^2, c_2 = 4(\gamma^4 L^2 \eta_1^2 + L_{\text{max}}) \frac{L^2 q \eta_1^2}{l} \gamma^2 \), \( \{\pi_0, \pi_1, \ldots, \pi_{v(u)-1}\} \) are the all start time counters for the global time counters from 0 to \( u \).

We define the Lyapunov function as \( L_t = \sum_{k=0}^{v(t)} \rho^{v(t)-k} e(w_{\pi_k}) \) where \( \rho \in (1 - \frac{1}{l}, 1) \), we have that

\[ L_{t+|K(t)|} \leq \rho^{v(t)+1} e(w_0) + \sum_{k=0}^{v(t)} \rho^{v(t)-k} e(w_{\pi_{k+1}}) \]

\[ \leq \rho^{v(t)+1} e(w_0) + \sum_{k=0}^{v(t)} \rho^{v(t)-k} \left[ \left(1 - \frac{\gamma \mu}{4}\right) e(w_0) \right. \]

\[ + \left( -\frac{\gamma \mu^2}{4} + 2c_1 + c_2\right) \sigma(w_{\pi_k}) + c_1 \left(1 - \frac{1}{l}\right)^k \sigma(w_0) \]

\[ + c_2 \sum_{k'=1}^{k-1} \left(1 - \frac{1}{l}\right)^{k-k'} \sigma(w_{\pi_{k'}}) + c_0 \]

\[ = \rho^{v(t)+1} e(w_0) + \left(1 - \frac{\gamma \mu}{4}\right) L_t + \sum_{k=0}^{v(t)} \rho^{v(t)-k} \]

\[ \left[ \left( -\frac{\gamma \mu^2}{4} + 2c_1 + c_2\right) \sigma(w_{\pi_k}) + c_1 \left(1 - \frac{1}{l}\right)^k \sigma(w_0) \right] \]

34
Based on (61), we can carefully choose \( \gamma \) such that the terms related to \( \sigma(w_{\pi_k}) \) \( (k = 0, \ldots, v(t) - 1) \) are negative, because the signs related to the lowest orders of \( \sigma(w_{\pi_k}) \) \( (k = 0, \ldots, v(t) - 1) \) are negative. In the following, we give the detailed analysis of choosing \( \gamma \) such that the terms related to \( \sigma(w_{\pi_k}) \) \( (k = 0, \ldots, v(t) - 1) \) are negative. We first consider \( k = 0 \). Assume that \( C(\sigma(w_0)) \) is the coefficient term of \( \sigma(w_0) \) in (60), we have that

\[
C(\sigma(w_0)) = \rho^{v(t)} \left( -\gamma \mu^2 \frac{1}{4} + 2c_1 + c_2 \right) + c_1 \sum_{k=0}^{v(t)} \rho^{v(t)-k} \left( 1 - \frac{1}{\rho} \right)^k
\]

Based on (61), we can carefully choose \( \gamma \) such that

\[
-2\mu^2 \frac{1}{4} + 2c_1 + c_2 \left( 2 + \frac{1}{1 - \frac{1}{\rho}} \right) \leq 0.
\]

Assume that \( C(\sigma(w_{\pi_k})) \) is the coefficient term of \( \sigma(w_{\pi_k}) \) \( (k = 1, \ldots, v(t) - 1) \) in the big square brackets of (60), we have that

\[
C(\sigma(w_{\pi_k})) = \rho^{v(t)-k} \left( -\gamma \mu^2 \frac{1}{4} + 2c_1 + c_2 \right) + c_2 \sum_{v=k+1}^{v(t)-1} \left( 1 - \frac{1}{\rho} \right)^{v-k} \rho^{v(t)-v}
\]
\[
\leq \rho^{v(t)-k} \left( -\frac{\gamma \mu^2}{4} + 2c_1 + c_2 \left( 1 + \frac{1}{1 - \frac{1-\gamma \mu}{\rho}} \right) \right)
\]

Based on (62), we can carefully choose \( \gamma \) such that 
\[-\frac{2\mu^2}{4} + 2c_1 + c_2 \left( 1 + \frac{1}{1 - \frac{1-\gamma \mu}{\rho}} \right) \leq 0.\]

Thus, based on (59), we have that
\[
\left( \frac{\gamma \mu^2}{4} - 2c_1 - c_2 \right) \frac{2}{L} e(w_{\pi_k})
\leq \left( \frac{\gamma \mu^2}{4} - 2c_1 - c_2 \right) \frac{2}{L} e(w_{\pi_k}) + \mathcal{L}_{t+K(t)}
\leq \rho^{v(t)+1} e(w_0) + \left( 1 - \frac{\gamma \mu}{4} \right) \mathcal{L}_t + \frac{c_0}{1 - \rho}
\leq \left( 1 - \frac{\gamma \mu}{4} \right)^{v(t)+1} \mathcal{L}_0 + \rho^{v(t)+1} e(w_0) \sum_{k=0}^{v(t)+1} \left( 1 - \frac{\gamma \mu}{4} \right)^{k}
= \left( 1 - \frac{\gamma \mu}{4} \right)^{v(t)+1} e(w_0) + \rho^{v(t)+1} e(w_0) \frac{1}{1 - \frac{1-\gamma \mu}{\rho}} + \frac{c_0}{1 - \rho} \frac{4}{\gamma \mu}
\leq \frac{2\rho - 1 + \frac{\gamma \mu}{4}}{\rho - 1 + \frac{\gamma \mu}{4}} \rho^{v(t)+1} e(w_0) + \frac{c_0}{1 - \rho} \frac{4}{\gamma \mu}
\]

where the inequality (a) follows from (59), the inequality (b) holds by using the inequality (59) recursively, the inequality (c) uses the fact that \( 1 - \frac{\gamma \mu}{4} < \rho. \)

According to (63), we have that
\[
e(w_{\pi_k}) \leq \frac{2\rho - 1 + \frac{\gamma \mu}{4}}{\rho - 1 + \frac{\gamma \mu}{4}} \rho^{v(t)+1} e(w_0)
+ \frac{4c_0}{\gamma \mu(1 - \rho) \left( \frac{2\mu^2}{4} - 2c_1 - c_2 \right)}
\]

Thus, to achieve the accuracy \( \epsilon \) of (1) for AFSAGA-VP, i.e., \( E[w_{\pi_k}] - f(w^*) \leq \epsilon \), we can carefully choose \( \gamma \) such that
\[
-\frac{\gamma \mu^2}{4} + 2c_1 + c_2 \left( 1 + \frac{1}{1 - \frac{1-\gamma \mu}{\rho}} \right) \leq 0.
\]

\[\text{36}\]
\[-\frac{\gamma \mu^2}{4} + c_2 + c_1 \left( 2 + \frac{1}{1 - \frac{1}{\rho}} \right) \leq 0 \]  

(68)

and let 

\[ \frac{2^\rho - 1 + \frac{\gamma \mu}{2}}{(\rho - 1 + \frac{\gamma \mu}{4}) \left( \frac{\mu^2}{4} - 2c_1 - c_2 \right)} \rho^{v(t)+1} e(w_0) \leq \frac{\epsilon}{2}, \]

we have that

\[ v(t) \geq \frac{\log \frac{2(2^\rho - 1 + \frac{\gamma \mu}{4}) e(w_0)}{\epsilon (\rho - 1 + \frac{\gamma \mu}{4}) \left( \frac{\mu^2}{4} - 2c_1 - c_2 \right)}}{\log \frac{1}{\rho}} \]

(69)

This completes the proof. 

\[ \Box \]

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