ON PARALLEL TRANSPORT IN QUANTUM BUNDLES
OVER ROBERTSON-WALKER SPACETIMES

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Abstract. A recently-developed theory of quantum general relativity provides a
propagator for free-falling particles in curved spacetimes. These propagators are
constructed by parallel-transporting quantum states within a quantum bundle
associated to the Poincaré frame bundle. We consider such parallel transport
in the case that the spacetime is a classical Robertson-Walker universe. An
explicit integral formula is developed which expresses the propagators for parallel
transport between any two points of such a spacetime. The integrals in this
formula are evaluated in closed form for a particular spatially-flat model.

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1. Introduction

In a recent book [1], Prugovečki describes in detail a framework for the unification of general relativity and geometro-stochastic quantum theory (cf. [2] for a review). The theoretical basis for geometro-stochastic quantization differs from the orthodox quantization procedure in that it incorporates the concept of a fundamental length in nature, which limits the accuracy with which one can measure the spacetime location of an event. This assumption resolves many of the difficulties with divergences encountered in conventional quantum field theory.

In this paper, we focus on the propagation of quantum wave functions for a massive spinless boson in a Lorentzian manifold \( \mathcal{M} \). In particular, we derive the equations governing such propagation in the case that this manifold is a classical Robertson-Walker model of our universe. The methods described could easily be applied to other models that have received attention in [3] in the context of conventional field theory in curved spacetimes.

Throughout this paper, we assume that \( \mathcal{M} \) is space-time orientable, so that it can be covered by a global coordinate system of orthonormal frames in which the timelike vector is future-pointing and the spacelike vectors form a right-handed triad. In addition, there is assumed to be a global time variable \( t \) so that we can foliate \( \mathcal{M} \) into a set of Cauchy spacelike hypersurfaces \( \sigma_t \). We adopt a metric \( g \) with the signature \((-1, 1, 1, 1)\) commonly used in relativity theory, and use Planck natural units where \( c = G = \hbar = 1 \).

In section 2 we provide a brief overview of the special-relativistic case in which a single wave function on phase space is used to describe the state of the boson. This is then extended in section 3 to the case of a globally hyperbolic spacetime through the introduction of the general quantum bundle, which is associated to the Poincaré frame bundle over that spacetime. Parallel transport on this bundle is defined by making use of the standard extension of the Levi-Civita connection used in classical general relativity, and serves as the basis for the development of the quantum-geometric propagator. In section 4, we compute the geodesics for the Robertson-Walker model and use them to develop an integral representation of the propagators for parallel transport. Finally, in section 5 we focus on a particular spatially-flat model for which these propagators turn out to have a closed-form expression.

2. Massive boson states in special relativistic phase space

Consider a spinless particle of rest mass \( m > 0 \) which moves freely in the Minkowski space \( M^4 \). The momentum representation of such a particle consists of the space \( L^2(\mathbb{V}_m^+) \) of the standard wave functions on the forward mass hyperboloid corresponding to \( m \), whose inner product is expressed using the usual invariant measure on \( \mathbb{V}_m^+ \):

\[
\langle \tilde{\varphi}_1 | \tilde{\varphi}_2 \rangle = \int_{\mathbb{V}_m^+} \tilde{\varphi}_1^*(k)\tilde{\varphi}_2(k) \, d\Omega_m(k), \quad \tilde{\varphi}_1, \tilde{\varphi}_2 \in L^2(\mathbb{V}_m^+),
\]

\[
\mathbb{V}_m^+ = \{ k \in M^4, \; |k^0| = \sqrt{k^2 + m^2} \}, \quad d\Omega_m(k) = \frac{dk}{2\sqrt{k^2 + m^2}}, \quad k = (k^0, k).
\] (2.1)

As is well-known [4], the standard interpretation of \( \tilde{\varphi} \) is that of a probability amplitude for particle 4-momentum measurements. However, a corresponding relativistically invariant probability amplitude does not exist for spacetime measurements [5,6].
Already in the 1930’s, various physicists, of whom the most prominent were Heisenberg [7] and Born [8], had conjectured that there is a fundamental length \( \ell \) in nature, such that no spacetime measurement of an event can be made with greater accuracy than \( \ell \). Measurement-theoretical arguments [9, 10] suggest that this length is in fact Planck’s length (which is 1 in the natural system of units adopted in this paper).

In the quantum-geometric framework, this hypothesis forms an integral part of the structure of the single-boson Hilbert space \( F \) which represents the state of the system. In the remainder of this section we present an overview of this theory; for details consult sections 3.7 and 3.8 of [1] and chapter 2 of [11].

To construct \( F \), we first introduce the concept of the momentum space resolution generator, denoted by \( \tilde{\eta} \). From a mathematical viewpoint, this resolution generator can be chosen to be any wave function from \( L^2(\mathbf{V}_m^+) \) satisfying the following two properties; first that \( \langle \tilde{\eta} | \tilde{\eta} \rangle = m/4\pi^3 \), and second that \( \tilde{\eta} \) is invariant under spatial rotations.

To understand the physical significance of \( \tilde{\eta} \), it is necessary to consider in detail the actual process of measuring the spacetime coordinates of a given event relative to some inertial frame. Classically, we can imagine that this process is performed with the aid of an apparatus employing pointlike particles and light signals. However, a closer examination of this procedure shows that the recoil effects produced by the emission and absorption of photons will introduce an inherent uncertainty factor. If, however, we replace the pointlike particles by extended particles whose momentum space wave function is given by \( \tilde{\eta} \), then it is possible to define this process in a physically consistent manner which takes advantage of the various methods developed in the phase space approach to quantum mechanics [12].

The introduction of quantum frames of reference induces a fundamental change in the notion of spacetime itself. Specifically, the position and momentum coordinates \( q \in \mathbb{R}^3 \) and \( p \in \mathbb{R}^3 \) of the particle which are measured at some given time will now be stochastic variables (cf. section 3.3 of [1]) whose definitions incorporate Gaussian-type distributions \( \chi_q \) and \( \chi_p \) on \( \mathbb{R}^3 \). The standard deviations of these distributions are of the order of \( \ell \) and \( \ell^{-1} \) respectively (in Planck units). Hence, it is possible to define a wave function on the phase space of the particle which provides a joint probability amplitude for measuring the particle as simultaneously having a given stochastic position and stochastic momentum, without violating the Heisenberg uncertainty principle. This will be done below. First, however, we consider what value the momentum space resolution generator should actually have.

In general, there are infinitely many possible choices for \( \tilde{\eta} \). However, in this paper we will henceforth consider only the special case

\[
\tilde{\eta}(k) = (m^3 \tilde{Z}_{\ell,m})^{-\frac{1}{2}} \exp(-\ell k^0), \quad \tilde{Z}_{\ell,m} = \frac{8\pi^4}{\ell m^2} K_2(2\ell m), \quad k = (k^0, \mathbf{k}) \in \mathbf{V}_m^+,
\]

where \( K_2 \) denotes as usual a modified Bessel function. This case corresponds to the unique resolution generator whose phase space representative displays reciprocal invariance under transposition of configuration and momentum coordinates, and is the ground state eigenvector of Born’s [13] quantum metric operator. Hence, its presence reflects the fact that stochastic spacetime localization is implemented only to the order of \( \ell \), whereas the corresponding stochastic 4-momentum spread is of the optimal order of magnitude that is in accordance with the Heisenberg uncertainty principle [12].
The above choice of \( \tilde{\eta} \) induces a mapping \( \mathcal{W}_\eta \) from \( L^2(\mathbf{V}_m^+) \) into the set of complex-valued functions defined on the relativistic phase space \( \mathcal{P}_m^+ = \{(q, p) \in M^4 \times \mathbf{V}_m^+\} \), given by
\[
\mathcal{W}_\eta : \tilde{\varphi} \mapsto \varphi, \quad \varphi(q, p) = \langle \tilde{\eta}_{q,p} | \tilde{\varphi} \rangle, \quad \tilde{\eta}_{q,p}(k) = \exp(-iq \cdot k) \tilde{\eta}(\Lambda_v^{-1}k), \quad (2.3)
\]
where \( \Lambda_v \) is the Lorentz boost corresponding to the 4-velocity \( v = p/m \). The image under \( \mathcal{W}_\eta \) of \( L^2(\mathbf{V}_m^+) \) forms a Hilbert space \( \mathbf{F} \) with inner product
\[
\langle \varphi_1 \mid \varphi_2 \rangle = \int_{\Sigma_m^+} \varphi_1^*(q, p)\varphi_2(q, p) \, d\Sigma_m(q, p), \quad \varphi_1, \varphi_2 \in \mathbf{F}. \quad (2.4)
\]
The domain of integration is the 6-dimensional hypersurface \( \Sigma_m^+ = \sigma \times \mathbf{V}_m^+ \), where \( \sigma \) is any Cauchy spacelike hypersurface in \( M^4 \). The hypersurface \( \Sigma_m^+ \) carries the relativistically covariant volume element
\[
d\Sigma_m(q, p) = 2p^i \, d\sigma_i(q) \, d\Omega_m(p), \quad (q, p) \in \Sigma_m^+. \quad (2.5)
\]
It can be shown [11,12] that the conditions imposed on the resolution generator \( \tilde{\eta} \) imply that the mapping \( \mathcal{W}_\eta \) defined in (2.3) is unitary. Consequently, for normalized \( \varphi \) the values assumed by the probability measure
\[
P_\varphi(B) = \int_B |\varphi(q, p)|^2 \, d\Sigma_m(q, p) \quad (2.6)
\]
can be interpreted as representing relativistically invariant probabilities for stochastic phase space localization.

This phase space representation has several advantages over the conventional formulation, which is usually based on wave functions defined on \( M^4 \) that satisfy the Klein-Gordon equation. In addition to providing a joint probability amplitude for stochastic position and momentum as explained above, \( \varphi \) simultaneously satisfies the Klein-Gordon equation in its \( q \)-variables and the Born-Landé equation [14,15] in its \( p \)-variables:
\[
\left( \eta^{ij} \frac{\partial^2}{q^i q^j} - m^2 \right) \varphi(q, p) = 0, \quad \left( \eta^{ij} \frac{\partial^2}{p^i p^j} - \ell^2 \right) \varphi(q, p) = 0. \quad (2.7)
\]
Moreover, on the physical side, \( \varphi \) can be used to define a positive-definite, conserved probability current, so that a physically-consistent single-particle theory becomes possible. This theory can also be extended to handle particles in external electromagnetic fields. For example, it is possible to develop a model in which the particle is coupled to a stochastic version of the usual 4-potential \( A_i \), for both the Klein-Gordon equation (section 2.10 of [11]), and the Dirac equation [16]. The time development of the wave function in such models does not give rise to transitions to negative energy states. Hence, as shown in detail in [11] and [12], inconsistencies such as the Klein paradox are avoided.

The phase space \( \mathcal{P}_m^+ \) is acted upon by the restricted Poincaré group \( ISO_0(3,1) \). Each element of \( ISO_0(3,1) \) can be represented by a pair \((b, \Lambda)\), where \( b \) is a vector in \( \mathbf{R}^4 \) and
Λ is a 4-by-4 matrix representing a Lorentz transformation. The action of \((b, \Lambda)\) on points in \(\mathcal{P}_m^+\) is given by

\[
(b, \Lambda)(q, p) = (\Lambda q + b, \Lambda p), \quad (b, \Lambda) \in ISO_0(3, 1),
\]

and has the standard composition law \((b, \Lambda)(b', \Lambda') = (b + \Lambda b', \Lambda \Lambda')\). In turn, this action induces a representation \(U\) of \(ISO_0(3, 1)\) on \(\mathbf{F}\). This representation is defined by

\[
U(b, \Lambda) : \varphi(q, p) \mapsto \varphi(\Lambda^{-1}(q - b), \Lambda^{-1}p), \quad (b, \Lambda) \in ISO_0(3, 1),
\]

and physically can be interpreted as providing the relationship between two coordinate wave functions corresponding to a particular quantum state, as measured by an arbitrary pair of inertial observers in \(M^4\) whose relative motion is determined by \(b\) and \(\Lambda\).

The resolution generator defined in (2.2) can be used to construct a special relativistic phase space propagator \(K\), which is analogous to the Feynman spacetime propagators conventionally used in quantum mechanics. To do this, we first introduce the phase space resolution generator \(\eta \in \mathbf{F}\), which is defined in terms of (2.2) and (2.3) by \(\eta = W_q \eta\). From the rotational invariance of \(\tilde{\eta}\) it follows that \(\eta\) itself satisfies a similar property, i.e. that \(U(0, \Lambda_R)\eta = \eta\) for any spatial rotation \(\Lambda_R \in SO_0(3, 1)\).

We now define the quantum propagator as follows. Let \((q', p')\) and \((q'', p'')\) be any two points in \(\mathcal{P}_m^+\). Then the propagator between these points is given in terms of the inner product (2.4) on \(\mathbf{F}\) by

\[
K(q'', p''; q', p') = \langle \eta_{q'', p''} | \eta_{q', p'} \rangle, \quad \eta_{q'', p''} = U(q'', \Lambda_{q''})\eta, \quad \eta_{q', p'} = U(q', \Lambda_{q'})\eta.
\]

Using (2.10) it can then be shown that \(K\) satisfies all the properties conventionally required of propagators. In particular, if \((q', p')\) and \((q'', p'')\) are arbitrary points in \(\mathcal{P}_m^+\), then we have

\[
K(q'', p''; q', p') = K^*(q', p'; q'', p''),
\]

and

\[
K(q'', p''; q', p') = \int_{\Sigma_m^+} K(q'', p''; q''', p''') K(q''', p'''; q', p') d\Sigma_m(q''', p'''),
\]

where, as before, \(\Sigma_m^+\) is any allowable hypersurface in \(\mathcal{P}_m^+\). From (2.12) it follows that \(K\) is a reproducing kernel for \(\mathbf{F}\), so that the probability amplitudes corresponding to arbitrary phase points in \(\mathcal{P}_m^+\) are related by

\[
\varphi(q'', p'') = \int_{\Sigma_m^+} K(q'', p''; q', p') \varphi(q', p') d\Sigma_m(q', p'), \quad \varphi \in \mathbf{F}.
\]

Although the quantum propagator enjoys all the usual properties of conventional propagators, it has the advantage of being an analytic function as opposed to a distribution. Indeed, for the specific choice of momentum resolution generator given in (2.2), it turns out that the propagator (2.10) can be evaluated in the closed form

\[
K(q'', p''; q', p') = \frac{2\pi}{m^2 Z_{\ell,m}} \frac{K_1\left(m f_{\ell,m}(q, p)\right)}{f_{\ell,m}(q, p)}, \quad q = q'' - q', \quad p = p'' + p',
\]

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where $K_1$ again denotes a modified Bessel function, and $f_{\ell,m}$ is given by the Poincaré invariant expression

$$f_{\ell,m}(q,p) = \sqrt{q \cdot q - \frac{2i\ell}{m} q \cdot p - \frac{\ell^2}{m^2} p \cdot p}, \quad q, p \in M^4. \tag{2.15}$$

This result enables us to manipulate propagators algebraically, without having to deal with any of the difficulties which arise from attempting to multiply distributions. In particular, it will be shown in the next section that the special relativistic propagator can be used to construct a quantum-geometric propagator for curved spacetimes.

3. Quantum propagators in globally hyperbolic spacetimes

In section 2 we constructed a model of quantum mechanics in special relativistic phase space. The system in consideration was represented by a single wave function $\varphi$ defined on the phase space $P_m^+$. In a general curved spacetime $M$ this is no longer possible in a relativistically consistent manner. Instead, it is necessary to construct a representation based on various principal and associated bundles over $M$ (cf. sections 2.3-2.6 of [1]).

Within this context, the set of all possible local states of single-boson quantum systems will be represented by a bundle $E$ associated to the Poincaré frame bundle $PM$, with its typical fibre the Hilbert space $F$ defined in section 2, and with gauge group $ISO_0(3,1)$ acting on $F$ in accordance with (2.9).

This associated bundle is most easily constructed through the use of a $G$-product, in terms of which it can be written as $E = PM \times_G F$. This construction implicitly defines a projection map $\pi : E \to M$ which maps each local quantum state in $E$ into its corresponding base point in $M$. For any $x \in M$, the fibre $\pi^{-1}(x)$ above $x$ is associated to the typical fibre $F$ by the soldering map $[1,17,18]$

$$\sigma^u_x : \pi^{-1}(x) \to F, \quad \Psi_x \mapsto \Psi^u_x, \quad u = (a, e_i) \in \Pi^{-1}(x), \tag{3.1}$$

where $\Pi$ is the canonical projection map from $PM$ to $M$ which identifies the base point in $M$ of any Poincaré frame from $PM$. As usual, the local Poincaré frame $u$ has been decomposed into a displacement vector $a \in T_xM$ and a local Lorentz frame $(e_0, e_1, e_2, e_3) \in (T_xM)^4$.

The physical interpretation of these maps is as follows. The state of a particular system corresponds to a section $\Psi$ in $E$, whose value is given by the quantum-geometric propagator described below. Above each point $x$ in $M$ is a local quantum state vector $\Psi_x$ which represents the state of the system at $x$. The coordinate wave function $\Psi^u_x : P_m^+ \to \mathbb{C}$ of this state as measured by an observer at $x$ is determined by applying to $\Psi_x$ the soldering map defined in (3.1) with $u = (0, e_i)$, where $(e_i)$ is the local Lorentz frame relative to which the measurements are performed.

Since the quantum bundle $E$ is associated to $PM$, we can use the standard extension [19,20] to $PM$ of the Levi-Civita connection on $LM$ to construct a connection on $E$. The connection on $PM$ can be specified by means of the usual 1-forms for infinitesimal Poincaré
transf orms, whose values in terms of the chosen Poincaré gauge provided by a section \(s\) of \(PM\) are given by \([18]\)

\[
\omega_{ij}^s(X) = \frac{1}{2} X^k \left[ g(e_i, [e_j, e_k]) + g(e_j, [e_k, e_i]) - g(e_k, [e_i, e_j]) \right],
\]

\[
\theta^i_s(X) = X^i + e^i(\nabla a), \quad X = X^i e_i \in TM.
\]

By standard results of differential geometry \([19]\), this connection induces a corresponding connection on \(E\), which we refer to as the quantum connection. Below, we examine this connection from the viewpoint of coordinate wave functions.

To do this, we first choose a Poincaré gauge \(s\). For any smooth curve \(\gamma\) joining the points \(x'\) and \(x''\), the connection on \(PM\) gives rise to an operator \(\tau_\gamma(x'', x') : \Pi^{-1}(x') \to \Pi^{-1}(x'')\) for parallel transport. In turn, this defines the transformation \(g^s_\gamma(x'', x') \in ISO_0(3,1)\) given by

\[
\tau_\gamma(x'', x')s(x') = s(x'')g^s_\gamma(x'', x'),
\]

where the term on the right side incorporates the (right) group action of \(ISO_0(3,1)\) on the fibre \(\Pi^{-1}(x'') \subseteq PM\). In view of the fact (cf. chapter 8 of \([21]\)) that this action satisfies the compatibility condition \(\tau_\gamma(x'', x')(u'g) = (\tau_\gamma(x'', x')u')g\) for any \(u' \in \Pi^{-1}(x')\) and any \(g \in ISO_0(3,1)\), the transformations defined above satisfy an important composition law. Namely, if \(\gamma\) is smoothly extended to any third point \(x''' \in M\), then we have the relation

\[
g^s_\gamma(x''', x') = g^s_\gamma(x''', x'')g^s_\gamma(x'', x').
\]

In terms of the transformation defined in (3.3), the coordinate wave function of any local quantum state in \(\pi^{-1}(x') \subset E\) which is parallel transported from \(x'\) to \(x''\) along \(\gamma\) is then given by

\[
\Psi^s_{\gamma,x''}(q,p) = U(g^s_\gamma(x'', x')) \Psi^s_{\gamma,x'}(q,p), \quad (q,p) \in P_m^+,
\]

where \(U\) is given by (2.9). If we now denote the translation and Lorentz transformation parts of \(g^s_\gamma(x'', x')\) in (3.3) as follows,

\[
g^s_\gamma(x'', x') = (b^s_\gamma(x'', x'), \Lambda^s_\gamma(x'', x')),
\]

then, according to (2.9), (3.5) can be written in the form

\[
\Psi^s_{\gamma,x''}(q,p) = \Psi^s_{\gamma,x'} \left( \Lambda^s_\gamma(x'', x')^{-1}(q - b^s_\gamma(x'', x')), \Lambda^s_\gamma(x'', x')^{-1}p \right).
\]

The connection constructed on \(E\) enables us to generalize the special relativistic phase space propagators defined in (2.10) to curved spacetimes. Specifically, for any Poincaré gauge \(s\), and any two points \(x'\) and \(x''\) joined by a smooth curve \(\gamma\), we define the propagator for parallel transport from \(x'\) to \(x''\) along \(\gamma\) in the gauge \(s\) to be a function \(K^s_\gamma(x'', \zeta''; x', \zeta')\) from \(P_m^+ \times P_m^+ \to C\) given by \([1,17,18]\)

\[
K^s_\gamma(x'', \zeta''; x', \zeta') = \langle \eta_{\zeta''} \mid U(g^s_\gamma(x'', x')) \eta_{\zeta'} \rangle.
\]
In the above equation, we have introduced the composite phase variable \( \zeta = (q, p) \), so that the integration measure \( d\Sigma_m(q, p) \) on \( F \) can be written simply as \( d\Sigma_m(\zeta) \). This notation will be employed throughout the rest of this section.

In the case where the base manifold \( M \) is flat, this propagator is independent of the choice of \( \gamma \). It is then not difficult to show \([17]\) that with an appropriate choice of section \( s \), the propagator for parallel transport assumes the same role as the quantum propagator considered in section 2. Namely, if we take for \( s \) the gauge in which \( a \) is identically \( 0 \) and \( e_i \) is parallel to the positive \( x^i \)-axis, then the propagators defined in (2.10) and (3.8) are related by

\[
K^s_\gamma(x'', \zeta'' ; x', \zeta') = K(q'' + x'', p'' ; q' + x', p'), \quad \zeta' = (q', p'), \quad \zeta'' = (q'', p'').
\]

With curved spacetimes the situation is more complicated. In chapter 4 of \([1]\), Prugovečki proposes the idea of a quantum-geometric propagator, denoted by \( K \). This propagator is constructed in a similar manner as the conventional Feynman propagator, except that the space of paths which is integrated over is formed from arcs of geodesics in \( M \) instead of the usual straight line segments. Physically, \( K \) is to be interpreted as representing the evolution of a single-particle system against a geometro-dynamic spacetime background (cf. section 4.7 of \([1]\) for details), so that, in agreement with (2.13), the quantum-geometric wave function

\[
\Psi^s(x'') (\zeta'') = \int_{\Sigma_m} K^s(x'', \zeta'' ; x', \zeta') \Psi^s(x') (\zeta') d\Sigma_m(\zeta'),
\]

represents the relative probability amplitude for the detection at \( x'' \) of the stochastic phase space value \( \zeta'' \) (relative to the gauge \( s \)), for a system prepared at \( x' \) in the local state \( \Psi_{x'} \).

To construct this propagator, we first foliate the given spacetime into a family of Cauchy hypersurfaces \( \sigma_t \) indexed by the global time parameter \( t \). Choosing two hypersurfaces \( \sigma_{t'} \) and \( \sigma_{t''} \) with \( t'' > t' \) and some positive integer \( N \), we insert the \( N - 1 \) intermediate hypersurfaces \( \sigma_{t_n} \), where \( t_n = \frac{1}{N}[(N - n)t' + nt''] \) for \( n = 1, 2, \ldots, N - 1 \). For any two points \( x' \) and \( x'' \) lying on the surfaces \( \sigma_{t'} \) and \( \sigma_{t''} \) respectively, and for any two phase space points \( \zeta' \) and \( \zeta'' \) (relative to the gauge \( s \)), the quantum-geometric propagator \( K^s(x'', \zeta'' ; x', \zeta') \) is defined to be the limit \([1,17,18]\)

\[
K^s(x'', \zeta'' ; x', \zeta') = \lim_{N \to \infty} \prod_{n=1}^{N} K^s_\gamma(x_n, x_{n-1}) (x_n, \hat{\zeta}_n ; x_{n-1}, \hat{\zeta}_{n-1}) d\Sigma(x_n, p_n).
\]

Here, \( \gamma(x_n, x_{n-1}) \) is the geodesic joining \( x_{n-1} \in \sigma_{t_{n-1}} \) to \( x_n \in \sigma_{t_n} \), the measure on the hypersurfaces \( \Sigma = \sigma \times V^+_m \) is given by

\[
d\Sigma(x_n, p_n) = 2(p_n)^{\frac{1}{2}} d\sigma(x_n) d\Omega(p_n),
\]

and the intermediate phase points are chosen so that they correspond with the point of contact between the tangent space and the base manifold:

\[
\hat{\zeta}_n = (-a(x_n), p_n), \quad a^i(x_n) e_i = a(x_n), \quad n = 1, 2, \ldots, N - 1.
\]
The points \( x_0 \) and \( x_N \) correspond to the initial and final points between which propagation takes place, i.e. we have \( x_0 = x', \ x_N = x'' \), \( \zeta_0 = \zeta' \), and \( \zeta_N = \zeta'' \). Note also that the product in (3.11) does not include an integration over \( \sigma' \), so that only the intermediate hypersurfaces are integrated over.

In the case of Minkowski space, and with the appropriate choice of gauge, it can be shown [1,17,18] using (2.12) and (3.9) that this quantity reduces to the propagator for parallel transport given in (3.8). However, very little is known in the case where \( \zeta' \) is given in (2.2), it follows from (2.10) that (3.8) can be written as

\[
K^s_{\gamma}(x'', \zeta''; x', \zeta') = \langle U(q', \Lambda_{\nu'}) \eta \mid U(b + \Lambda q', \Lambda_{\nu'}) \eta \rangle,
\]

where \((b, \Lambda) = g^s_{\gamma}(x'', x')\). In order to evaluate this expression, we use the fact [17] that we can write the product of Lorentz transformations in the inner product as \( \Lambda \Lambda' = \Lambda_{\nu'} \Lambda_R \), where \( \Lambda_R \) is a spatial rotation. Since \( \eta \) is rotationally invariant (recall section 2), it follows from (2.10) and (2.14) that

\[
K^s_{\gamma}(x'', \zeta''; x', \zeta') = \frac{2\pi}{m^2 \tilde{z}_{\ell,m}} \frac{K_1(m f_{\ell,m}(q,p))}{f_{\ell,m}(q,p)}, \quad p = p'' + \Lambda p', \quad q = q'' - \Lambda q' - b,
\]

where \( f_{\ell,m} \) is given by (2.15). The calculation of the propagator for parallel transport thus reduces to the problem of determining \( b \) and \( \Lambda \).

As shown in [18], the 1-forms given in (3.2) can be used to compute an explicit formula for \( g^s_{\gamma}(x'', x') \). We shall now present that derivation in a form best suited to the computations in the subsequent two sections.

Suppose that \( x' \) and \( x'' \) in \( M \) are connected by a smooth curve \( \gamma \) with \( \gamma(t') = x' \) and \( \gamma(t'') = x'' \). Choose a positive integer \( n \) and divide \( \gamma \) into \( n \) pieces by defining the points

\[
x_k = \gamma(t_k), \quad t_k = \frac{(n - k)t' + kt''}{n}, \quad k = 0, 1, 2, \ldots, n,
\]

so that in particular \( x_0 = x' \) and \( x_n = x'' \). By induction on (3.4) we have

\[
g^s_{\gamma}(x'', x') = \prod_{k=n-1}^{0} g^s_{\gamma}(x_{k+1}, x_k).
\]

Applying the composition law for \( ISO_0(3,1) \) to the right side and using (3.6) we see that for any \( n = 1, 2, 3, \ldots \), the total translation and total Lorentz transformation from \( x' \) to \( x'' \) are respectively given by

\[
b = \sum_{k=0}^{n-1} \prod_{l=n-1}^{k+1} \Lambda^s_\gamma(x_{l+1}, x_l) b^s_\gamma(x_{k+1}, x_k), \quad \Lambda = \prod_{k=n-1}^{0} \Lambda^s_\gamma(x_{k+1}, x_k).
\]

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In order to compute these expressions, we use the exponential map from $iso(3, 1)$ to $ISO_0(3, 1)$. Specifically, let $\Delta t = (t'' - t')/n$ be the increment in the parameter $t$ corresponding to (3.16). Then to the first order in $\Delta t$ we have

$$b^s(x_{k+1}, x_k) \approx \Delta t P^s(X_k), \quad \Lambda^s(x_{k+1}, x_k) \approx \exp(\Delta t M^s(X_k)),$$

(3.19)

where $X_k = \gamma'(t_k)$ is the tangent vector to $\gamma$ at $t = t_k$, and the infinitesimal translation and Lorentz transformation are defined in terms of the standard basis elements $P_i$ and $M^{ij}$ of $iso(3, 1)$, and the 1-forms given in (3.2), by

$$P^s(X) = -\theta^s_i(X) P_i, \quad M^s(X) = -\frac{1}{2} \omega^{s}_{ij}(X) M^{ij}, \quad X \in TM.$$

(3.20)

Substituting (3.19) into (3.18) gives the approximations

$$b \approx \Delta t \sum_{k=0}^{n-1} \left[ \prod_{l=n-1}^{k+1} \exp(\Delta t M^s(X_l)) \right] P^s(X_k), \quad \Lambda \approx \prod_{k=n-1}^{0} \exp(\Delta t M^s(X_k)).$$

(3.21)

In the limit as $n \to \infty$ the approximations will become exact, and we have as our final answer

$$b = \int_{t'}^{t''} \left[ T \exp \left( \int_{t'}^t M^s(\gamma'(s)) ds \right) P^s(\gamma'(t)) \right] dt,$$

(3.22)

$$\Lambda = T \exp \left( \int_{t'}^{t''} M^s(\gamma'(t)) dt \right).$$

(3.23)

4. Quantum parallel transport in Robertson-Walker spacetimes

As shown in section 3, the propagators for parallel transport are completely determined by the quantities $b$ and $\Lambda$ given in (3.22) and (3.23). An example of the application of these formulae is provided by the classical Robertson-Walker model, whose metric can be expressed [22] in the form

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2,$$

(4.1)

where $a(t)$ is a smooth positive function of $t$. The parameter $k$ can be either $-1$, $0$, or $1$, corresponding respectively to spatially hyperbolic, spatially flat, and spatially elliptic universes. In this section, we compute the geodesics in the Robertson-Walker model (employing a method suggested by Prugovečki [23]), and use them to arrive at integral representations for $b$ and $\Lambda$.

To begin, it is convenient to define a new variable $R$ by

$$R = \begin{cases} 
\arcsinh r, & \text{if } k = -1; \\
r, & \text{if } k = 0; \\
\arcsin r, & \text{if } k = 1,
\end{cases}$$

(4.2)
which reduces the metric to the form

\[ ds^2 = -dt^2 + a^2(t) [dR^2 + r^2 d\Omega^2]. \]  

(4.3)

For the calculation of the geodesics, we concentrate on curves whose spatial projections follow paths passing through the origin. Clearly, any geodesic can be obtained from one of this form by a suitable shift of the origin. Along such lines \( \theta \) and \( \phi \) are constant, and so the distance along a curve \( R(t), t' \leq t \leq t'' \) can be expressed as

\[ l = \int_{t'}^{t''} \mathcal{L}(R, \dot{R}, t) \, dt, \quad \mathcal{L}(R, \dot{R}, t) = \sqrt{\pm (1 - a^2(t) \dot{R}^2)}, \]  

(4.4)

where the plus sign is taken for timelike geodesics and the minus sign for spacelike geodesics.

To minimize or maximize \( l \) as a functional of the path we use the Euler-Lagrange equation, which in the case of timelike geodesics takes the form [24]

\[ 0 = \frac{\partial \mathcal{L}}{\partial R} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{R}} = \frac{d}{dt} \left( \frac{a^2(t) \dot{R}}{\sqrt{1 - a^2(t) \dot{R}^2}} \right). \]  

(4.5)

Integrating with respect to \( t \) gives

\[ R(t) = C \int \frac{dt}{a(t) \sqrt{C^2 + a^2(t)}}, \]  

(4.6)

where \( C \geq 0 \) is a constant of integration. A similar computation for the spacelike case yields

\[ R(t) = C \int \frac{dt}{a(t) \sqrt{C^2 - a^2(t)}}. \]  

(4.7)

We now turn to the problem of calculating \( b \) and \( \Lambda \). For simplicity, we adopt a Poincaré gauge in which the displacement vector \( \mathbf{a} \) is identically \( \mathbf{0} \). From (4.3) it follows that a natural choice for the basis vectors is given by

\[ \mathbf{e}_0 = \partial_t, \quad \mathbf{e}_1 = \frac{1}{a(t)} \partial_R, \quad \mathbf{e}_2 = \frac{1}{r a(t)} \partial_\theta, \quad \mathbf{e}_3 = \frac{1}{r a(t) \sin \theta} \partial_\phi. \]  

(4.8)

Their Lie brackets are then computed to be

\[ [\mathbf{e}_0, \mathbf{e}_1] = -\frac{a'(t)}{a(t)} \mathbf{e}_1, \quad [\mathbf{e}_0, \mathbf{e}_2] = -\frac{a'(t)}{a(t)} \mathbf{e}_2, \quad [\mathbf{e}_0, \mathbf{e}_3] = -\frac{a'(t)}{a(t)} \mathbf{e}_3, \]

\[ [\mathbf{e}_1, \mathbf{e}_2] = -\frac{1}{r a(t) \frac{dr}{dR}} \mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = -\frac{1}{r a(t) \frac{dr}{dR}} \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\frac{\cot \theta}{r a(t)} \mathbf{e}_3. \]  

(4.9)

From (3.2), the nonzero coefficients corresponding to Lorentz boosts are found to be

\[ \omega^{a}_{01}(\mathbf{e}_1) = \omega^{a}_{02}(\mathbf{e}_2) = \omega^{a}_{03}(\mathbf{e}_3) = \frac{a'(t)}{a(t)}, \]  

(4.10)
while for spatial rotations they are
\[
\omega^s_{12}(e_2) = \omega^s_{13}(e_3) = \frac{1}{r a(t)} \frac{dr}{dR}, \quad \omega^s_{23}(e_3) = \frac{\cot \theta}{r a(t)}.
\]

(4.11)

Since the displacement vector \(a\) is assumed to be identically 0, it follows from (3.2) that the 1-forms for translation will simply be given by \( \theta_i(X) = X_i \).

For the computations of \(b\) and \(\Lambda\) we will take \(\gamma\) to be one of the special geodesics considered in (4.4). Because of the similarity of (4.6) and (4.7), it is convenient in the following calculations to consider the cases of timelike and spacelike geodesics together.

Following the example of (4.4), we will adopt the convention that whenever a \(\pm\) sign occurs the plus refers to the timelike case and the minus to the spacelike case.

Let the initial and final points of \(\gamma\) be \(x'\) and \(x''\), corresponding to the global time values \(t'\) and \(t''\) respectively. From (4.6) and (4.7) the tangent vector is seen to be
\[
\gamma'(t) = e_0 + \frac{C}{c(t)} e_1, \quad c(t) = \sqrt{C^2 + a^2(t)}.
\]

(4.12)

It then follows from (3.20) and (4.10)-(4.12) that the infinitesimal translation and Lorentz transformation elements of \(iso(3,1)\) along \(\gamma\) are respectively
\[
P^s(\gamma'(t)) = -P_0 - \frac{C}{c(t)} P_1, \quad M^s(\gamma'(t)) = -\frac{Ca'(t)}{a(t)c(t)} M^{01}.
\]

(4.13)

Clearly the quantities \(M^s(\gamma'(t))\) commute for all values of \(t\), and so (3.22) and (3.23) become
\[
b = -\int_{t'}^{t''} \left[ \exp \left( -\int_{s}^{t''} \frac{Ca'(s)}{a(s)c(s)} M^{01} ds \right) \left( 1, \frac{C}{c(t)}, 0, 0 \right) \right] dt, \quad (4.14)
\]

\[
\Lambda = \exp \left( -\int_{t'}^{t''} \frac{Ca'(t)}{a(t)c(t)} M^{01} dt \right). \quad (4.15)
\]

The case \(a(t) = 1\) corresponds to Minkowski space. Here, (4.14) and (4.15) reduce to the equations
\[
b = (t' - t'') \left[ P_0 + \frac{C}{\sqrt{C^2 + 1}} P_1 \right], \quad \Lambda = 1. \quad (4.16)
\]

In this case it follows from (3.7) that the coordinate wave functions at the endpoints are related by
\[
\Psi^s_{\gamma,x''}(q,p) = \Psi^s_{x'}(q + (t'' - t') \left[ P_0 + \frac{C}{\sqrt{C^2 + 1}} P_1 \right], p), \quad (4.17)
\]

confirming the fact that two observers in flat space at rest relative to one another and with coincident axes will measure the same wave function for the system, modulo a spacetime translation.
5. Explicit propagators for a spatially-flat Robertson-Walker model

The integral representations for $b$ and $\Lambda$ which were derived in section 4 generally cannot be evaluated in closed form. Indeed, even in the spatially-flat case corresponding to $R = r$, the determining equations for $C$ which are obtained from (4.6) or (4.7) cannot be solved explicitly, except in special cases. In this section we consider one such case, the rapidly-expanding universe corresponding to $a(t) = t$.

We begin by choosing coordinates $(t, x^1, x^2, x^3)$ for $M$, in terms of which the metric assumes the form $ds^2 = -dt^2 + t^2((dx^1)^2 + (dx^2)^2 + (dx^3)^2)$, and we take as our Poincaré gauge a section $s$ of $PM$ in which the displacement vector $a$ is identically $0$ and the orthonormal frame is given by

$$e_0 = \frac{\partial}{\partial t}, \quad e_1 = \frac{1}{t} \frac{\partial}{\partial x^1}, \quad e_2 = \frac{1}{t} \frac{\partial}{\partial x^2}, \quad e_3 = \frac{1}{t} \frac{\partial}{\partial x^3}. \tag{5.1}$$

The initial and final points of the geodesic are taken to be $x' = (t', \mathbf{x}')$ and $x'' = (t'', \mathbf{x}'')$, with $t'' > t'$. We decompose the spatial separation of the two points as $\mathbf{x}'' - \mathbf{x}' = \rho \mathbf{n}$, where $\rho \geq 0$ and $\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector. For future reference we define the quantities $\rho_{\text{null}}$ and $\rho_{\text{sl}}$ by

$$\rho_{\text{null}} = \ln \frac{t''}{t'}, \quad \rho_{\text{sl}} = \ln \left( \frac{t'' + \sqrt{(t'')^2 - (t')^2}}{t'} \right). \tag{5.2}$$

Consider first the case where the geodesic joining $x'$ to $x''$ is timelike. We begin by calculating the constant of integration in (4.6), which for definitiveness will be denoted as $C_{t1}$. For this purpose it is convenient to define the new variable $u = \sqrt{t^2 + C_{t1}^2}$, with $u'$ and $u''$ defined accordingly. Using (4.6) with $a(t) = t$ we get

$$\rho = \int_{t'}^{t''} \frac{C_{t1}}{t \sqrt{C_{t1}^2 + t^2}} dt = \frac{1}{2} \ln \left( \frac{(u' + C_{t1})(u'' - C_{t1})}{(u' - C_{t1})(u'' + C_{t1})} \right). \tag{5.3}$$

Exponentiating and squaring this equation gives the pair of relations

$$(u' + C_{t1})(u'' - C_{t1}) = e^\rho t' t'', \quad (u' - C_{t1})(u'' + C_{t1}) = e^{-\rho} t' t'' \tag{5.4}$$

which when added together lead to the result

$$C_{t1} = \frac{e^{-\rho/2}(e^{2\rho} - 1)t' t''}{2 \sqrt{(t'')^2 - (e^\rho t'')^2}}, \quad 0 \leq \rho < \rho_{\text{null}}. \tag{5.5}$$

We next proceed to the calculation of $\Lambda$. To simplify this computation, it is convenient to first consider the case in which $x^2$ and $x^3$ are constant along $\gamma$, so that $\mathbf{n} = (1, 0, 0)$. In this case the Lorentz transformation is found from (4.15) to be

$$\Lambda = \exp \left( - \int_{t'}^{t''} \frac{C_{t1}}{t \sqrt{C_{t1}^2 + t^2}} M^{01} dt \right), \quad \mathbf{n} = (1, 0, 0). \tag{5.6}$$
A comparison of this formula with (5.3) shows that $\Lambda$ is the standard Lorentz boost in the $x^1$ direction, with $\cosh \rho$ in its $(0,0)$ and $(1,1)$ entries and $-\sinh \rho$ in its $(0,1)$ and $(1,0)$ entries. This result can then be extended to arbitrary values of $n$ by noting that since the spacetime in question is spatially flat, it is only necessary to conjugate $\Lambda$ with an appropriate spatial rotation matrix which maps $(1,0,0)$ into $n$. In this manner the Lorentz transformation for a general geodesic $\gamma$ is found to be

$$
\Lambda = \begin{bmatrix}
\cosh \rho & -n_1 \sinh \rho & -n_2 \sinh \rho & -n_3 \sinh \rho \\
-n_1 \sinh \rho & n_1^2 (\cosh \rho - 1) + 1 & n_1 n_2 (\cosh \rho - 1) & n_1 n_3 (\cosh \rho - 1) \\
-n_2 \sinh \rho & n_2 n_1 (\cosh \rho - 1) & n_2^2 (\cosh \rho - 1) + 1 & n_2 n_3 (\cosh \rho - 1) \\
-n_3 \sinh \rho & n_3 n_1 (\cosh \rho - 1) & n_3 n_2 (\cosh \rho - 1) & n_3^2 (\cosh \rho - 1) + 1
\end{bmatrix}. \quad (5.7)
$$

To calculate the translation $b$ provided by (4.14), note that due to the spatial flatness of $\mathbf{M}$ the spatial part of $b$ will be parallel to the direction of propagation. Thus, if we denote the temporal and spatial parts of $b$ by $b_T$ and $b_S$, then $b$ can be written in the form $b = (b_T, b_S n)$. In order to compute these quantities, we again consider first the case of propagation along the $x^1$-axis, for which $n = (1,0,0)$, and obtain from (4.14) the formula

$$
b = -\int_t^{t''} \left[ \exp \left( -\int_t^{t''} \frac{C_{t1}}{s \sqrt{C_{t1}^2 + s^2}} M^{01} ds \right) \left( 1, \frac{C_{t1}}{\sqrt{C_{t1}^2 + t''^2}}, 0, 0 \right) \right] dt. \quad (5.8)
$$

Using (5.3) to evaluate the matrix integral inside the exponential, we get by equating the first two components of the left and right sides of (5.8) the equations

$$
b_T = -\int_{t'}^{t''} \left[ \cosh (\ln v) - \frac{C_{t1} \sinh (\ln v)}{u} \right] dt, \quad b_S = -\int_{t'}^{t''} \left[ \frac{C_{t1} \cosh (\ln v)}{u} - \sinh (\ln v) \right] dt, \quad v = \frac{\sqrt{u + C_{t1} \sqrt{u''^2 - C_{t1}^2}}}{\sqrt{u - C_{t1} \sqrt{u''^2 + C_{t1}^2}}}.
$$

These integrals are most easily computed by expressing them in a form in which the limits are $u'$ and $u''$. In the case of $b_T$, this change of variables produces an elementary integral,

$$
b_T = \frac{1}{2} \int_{u'}^{u''} \frac{C_{t1} (v^2 - 1) - u (v^2 + 1)}{v \sqrt{u^2 - C_{t1}^2}} \, du = -\frac{u''}{t''} \int_{u'}^{u''} \, du = \frac{u'' (u' - u'')}{{t''}}, \quad (5.10)
$$

while for $b_S$ a similar calculation gives

$$
b_S = \frac{1}{2} \int_{u'}^{u''} \frac{u (v^2 - 1) - C_{t1} (v^2 + 1)}{v \sqrt{u^2 - C_{t1}^2}} \, du = -\frac{C_{t1}}{t''} \int_{u'}^{u''} \, du = \frac{C_{t1} (u' - u'')}{{t''}}. \quad (5.11)
$$

Substituting $u = \sqrt{t''^2 + C_{t1}^2}$ into (5.10) and (5.11) and using (5.5) to eliminate $C_{t1}$ gives a pair of expressions involving $e^\rho$. These can be simplified by converting the exponentials into hyperbolic functions, and the final result is then found to be

$$
b_T = t' \cosh \rho - t'', \quad b_S = -t' \sinh \rho. \quad (5.12)$$
Using the spatial flatness of $M$, it is not difficult to see that these formulae in fact hold for all values of the direction vector $n$.

The computations performed above can be repeated for the spacelike case, with similar results. However, one important difference arises. To see this, consider first the determination of the constant of integration in (4.7), denoted this time by $C_{sl}$. Substitution of $a(t) = t$ into (4.7) gives

$$\rho = \int_{t'}^{t''} \frac{C_{sl}}{t \sqrt{C_{sl}^2 - t^2}} dt.$$  \hspace{1cm} (5.13)

A comparison of this equation with (5.3) immediately gives the solution $C_{sl} = iC_{tl}$, with $C_{tl}$ given by (5.5). Note that $C_{sl}$ is actually real since in the case of spacelike geodesics we have $\rho > \rho_{null}$ and hence $t'' - e^\rho t' < 0$. However, it follows also from (5.13) that the range of $C_{sl}$ will be restricted to the interval $[t'', \infty)$. The value $C_{sl} = t''$ corresponds to the greatest possible spatial separation of the points $x'$ and $x''$, which is achieved by a geodesic with points given by $\{(t, x' + r_{max}(t)n) | t' \leq t \leq t''\}$, where

$$r_{max}(t) = \int_{t'}^{t''} \frac{t''}{s \sqrt{(t'')^2 - s^2}} ds = \frac{1}{2} \ln \left[ \frac{(t'' + \sqrt{(t'')^2 - (t')^2})(t'' - \sqrt{(t'')^2 - t^2})}{(t'' - \sqrt{(t'')^2 - (t')^2})(t'' + \sqrt{(t'')^2 - t^2})} \right].$$  \hspace{1cm} (5.14)

Substitution of $t = t''$ into this equation and comparison with the second equation in (5.2) reveals that such a geodesic will achieve a spatial separation of magnitude $\rho_{sl}$ between $x'$ and $x''$. Consequently, geodesics connecting two points with a spatial separation greater that $\rho_{sl}$ are formed by smoothly extending the geodesic given by (5.14) into a straight line with spatial projection parallel to $n$, lying in the Cauchy hypersurface $\Sigma_{\nu'} = \{(t'', x) | x \in \mathbb{R}^3\}$. This extension is easily seen to satisfy the geodesic equation at every point.

Let us first consider the geodesics for which $\rho_{null} < \rho \leq \rho_{sl}$. In this case the translation and Lorentz transformation for geodesics whose spatial projection is parallel to the $x^1$ axis are found from (4.14) and (4.15) to be

$$b = -\int_{t'}^{t''} \left[ \exp \left( -\int_{t'}^{t''} \frac{C_{sl}}{s \sqrt{C_{sl}^2 - s^2}} M_{01}^1 ds \right) \left( 1, \frac{C_{sl}}{\sqrt{C_{sl}^2 - t^2}}, 0, 0 \right) \right] dt,$$  \hspace{1cm} (5.15)

$$\Lambda = \exp \left( -\int_{t'}^{t''} \frac{C_{sl}}{t \sqrt{C_{sl}^2 - t^2}} M_{01}^1 dt \right).$$  \hspace{1cm} (5.16)

Substitution of $iC_{tl}$ for $C_{sl}$ in the above expressions and comparison with (5.6) and (5.8) shows that the formulae for $b$ and $\Lambda$ in the spacelike case formally reduce to those for the timelike case.

For geodesics $\gamma$ corresponding to $\rho \geq \rho_{sl}$ we define an intermediate point $\bar{x} = (\bar{t}, \bar{x})$ on $\gamma$ by $\bar{t} = t''$ and $\bar{x} = x' + \rho_{sl}n$, so that as $x$ varies from $x'$ to $\bar{x}$, $\gamma$ follows the geodesic given by (5.14), while as $x$ varies from $\bar{x}$ to $x''$, $\gamma$ follows a straight line along the hypersurface $\Sigma_{\nu'}$. It then follows from the composition law for $ISO_0(3,1)$ that the translation and Lorentz transformation from $x'$ to $x''$ are given by

$$\Lambda = \Lambda^\gamma_{\nu'}(x'', \bar{x}) \Lambda^\gamma_{\nu'}(\bar{x}, x'), \quad b = b^\gamma_{\nu'}(x'', \bar{x}) + \Lambda^\gamma_{\nu'}(x'', \bar{x}) b^\gamma_{\nu'}(\bar{x}, x').$$  \hspace{1cm} (5.17)
The quantity $\Lambda_{\gamma}^s(x'', \bar{x})$ can be obtained from (3.23) by considering $t$ to represent spatial instead of temporal distance, and integrating from $\rho_{sl}$ to $\rho$ along the tangent vector $t'' \mathbf{e}_1$. An easy computation then shows that $\Lambda_{\gamma}^s(x'', \bar{x})$ is a matrix of the form (5.7) with $\cosh \rho$ and $\sinh \rho$ replaced by $\cosh(\rho - \rho_{sl})$ and $\sinh(\rho - \rho_{sl})$ respectively. Consequently, it follows from the first equation in (5.17) that the total Lorentz transformation from $x'$ to $x''$ is given as before by (5.7).

To calculate the total translation, we decompose $b_T^s(\bar{x}, x'')$ and $b_S^s(x'', \bar{x})$ into temporal and spatial parts in the earlier-mentioned manner. In terms of this decomposition, the second equation in (5.17) can be expressed in the matrix form

$$
\begin{bmatrix}
  b_T \\
  b_S
\end{bmatrix} = \begin{bmatrix}
  b_T(x'', \bar{x}) \\
  b_S(x'', \bar{x})
\end{bmatrix} + \begin{bmatrix}
  \cosh(\rho - \rho_{sl}) & -\sinh(\rho - \rho_{sl}) \\
  -\sinh(\rho - \rho_{sl}) & \cosh(\rho - \rho_{sl})
\end{bmatrix} \begin{bmatrix}
  b_T(\bar{x}, x') \\
  b_S(\bar{x}, x')
\end{bmatrix}.
$$

To calculate $b_T(\bar{x}, x')$ and $b_S(\bar{x}, x')$, first note that from (5.2) we have

$$
cosh \rho_{sl} = \frac{t''}{t'}, \quad \sinh \rho_{sl} = \frac{\sqrt{(t'')^2 - (t')^2}}{t'}.
$$

Substituting $\rho = \rho_{sl}$ into (5.12) and applying the first formula above then gives

$$
b_T(\bar{x}, x') = 0, \quad b_S(\bar{x}, x') = -t' \sinh \rho_{sl}.
$$

Using (3.22), the quantities $b_T(x'', \bar{x})$ and $b_S(x'', \bar{x})$ can be evaluated by a calculation similar to the derivation of $\Lambda_{\gamma}^s(x'', \bar{x})$ as

$$
b_T(x'', \bar{x}) = t''[\cosh(\rho - \rho_{sl}) - 1], \quad b_S(x'', \bar{x}) = -t'' \sinh(\rho - \rho_{sl}),
$$

and so we get by substituting (5.20) and (5.21) into (5.18) that for $\rho \geq \rho_{null}$ the spacetime translation is given by

$$
b_T = t''[\cosh(\rho - \rho_{sl}) - 1] + t' \sinh(\rho - \rho_{sl}) \sinh \rho_{sl},
\quad
b_S = -t'' \sinh(\rho - \rho_{sl}) - t' \cosh(\rho - \rho_{sl}) \sinh \rho_{sl}.
$$

Expanding out the hyperbolic functions in these expressions and then using (5.19), we find that once again they reduce formally to (5.12). As before, this result is easily seen to hold for all choices of the direction vector $\mathbf{n}$.

So far, we have not made any mention of the case where the geodesic joining $x'$ to $x''$ is null. Although this case could be handled directly using (3.22) and (3.23), it is more straightforward to note that since $b$ and $\Lambda$ are continuous functionals of the path $\gamma$, these quantities will be given by the formulæ already derived.

In conclusion, the expressions in (5.7) and (5.12) provide, through (2.15) and (3.15), an explicit form of the propagator for parallel transport between any two points $x'$ and $x''$ in $\mathbf{M}$ for which $t'' > t'$. In the concluding paragraph of [18], Prugovecki speculates that the quantum-geometric propagators associated with rapidly expanding or contracting universes may lead to violations of strict Einstein causality. The calculations presented above can serve as a basis for the investigation of such phenomena.
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