On the regularity of weak solutions of the Boussinesq equations in Besov spaces

Dedicated to Enrique Zuazua on the occasion of his sixtieth birthday

Annamaria Barbagallo
Department of Mathematics and Applications ’’R.Caccioppoli’’
University of Naples ”Federico II”, Naples e-mail:annamaria.barbagallo@unina.it

Sadek Gala
Department of Sciences exactes, Ecole Normale Superieure de Mostaganem Box 227,
Mostaganem 27000, Algeria e-mail: sgala793@gmail.com

Maria Alessandra Ragusa
Dipartimento di Matematica e Informatica, Università di Catania, Viale Andrea Doria,
6-95125 Catania, Italy,
e-mail:maragusa@dmi.unict.it

Michel Théra
XLIM UMR-CNRS 7252 Université de Limoges and Centre for Informatics and Applied
Optimisation, Federation University Australia e-mail: michel.thera@unilim.fr

Abstract

The main issue addressed in this paper concerns an extension of a result by Z. Zhang who proved, in the context of the homogeneous Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, that, if the solution of the Boussinesq equation (1.1) below (starting with an initial data in $H^2$) is such that $(\nabla u, \nabla \theta) \in L^2 \left(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\right)$, then the solution remains smooth forever after $T$. In this contribution, we prove the same result for weak solutions just by assuming the condition on the velocity $u$ and not on the temperature $\theta$.

Mathematics Subject Classification(2000): 35Q35, 35B65, 76D05.
Key words: Boussinesq equations, Besov space, weak solution, regularity criterion.
1 Introduction

We are interested in the regularity of weak solutions of the Cauchy problem related to the Boussinesq equations in $\mathbb{R}^3$:

$$
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \pi &= \theta e_3, \\
\partial_t \theta + (u \cdot \nabla) \theta - \Delta \theta &= 0, \\
\nabla \cdot u &= 0, \\
u(x, 0) &= u_0(x), \quad \theta(x, 0) = \theta_0(x),
\end{aligned}
$$

(1.1)

where $x \in \mathbb{R}^3$ and $t \geq 0$. Here, $u : \mathbb{R}^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}^3$ is the velocity field of the flow, $\pi = \pi(x, t) \in \mathbb{R}$ is a scalar function representing the pressure, $\theta : \mathbb{R}^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}^3$ represents the temperature of the fluid and $e_3 = (0, 0, 1)^T$. Note that $u_0(x)$ and $\theta_0(x)$ are given initial velocity and initial temperature with $\nabla \cdot u_0 = 0$ in the sense of distributions.

Owing to the physical importance and the mathematical challenges, the study of (1.1) which describes the dynamics of a viscous incompressible fluid with heat exchanges, has a long history and has attracted many contributions from physicists and mathematicians [19]. Although Boussinesq equations consist in a simplification of the original 3-D incompressible flow, they share a similar vortex stretching effect. For this reason they retain most of the mathematical and physical difficulties of the 3-D incompressible flow, and therefore, these equations have been studied and applied to various fields. Examples include for instance geophysical applications, where they serve as a model, see, e.g. [21]. There are several other results on existence and blowup criteria in different kinds of spaces which have been obtained, (see [1, 3, 6, 7, 29]).

The problem of the global-in-time well-posedness of (1.1) in a three-dimensional space is highly challenging, due to the fact that the system contains the incompressible 3D Navier-Stokes equations as a special case (obtained by setting $\theta = 0$), for which the issue of global well-posedness has not been proved until now. However, the question of the regularity of weak solutions is an outstanding open problem in mathematical fluid mechanics and many interesting results have been obtained (see e.g. [4, 5, 8, 9, 11, 12, 14, 22, 23, 27, 28, 29, 30]). We are interested in the classical problem of finding sufficient conditions for weak solutions of (1.1) such that they become regular.

Realizing the dominant role played by the velocity field in the regularity issue, Ishimura and Morimoto [16] were able to derive criteria in terms of the velocity field $u$ alone. They showed that, if $u$ satisfies

$$
\nabla u \in L^1 \left(0, T; L^\infty(\mathbb{R}^3)\right),
$$

(1.2)

then the solution $(u, \theta)$ is regular on $[0, T]$. It is worthy to emphasize that there are no assumptions on the temperature $\theta$. This assumption (1.2) was weakened in [6] with the
$L^\infty$—norm replaced by norms in Besov spaces $\dot{B}_{\infty,\infty}^0$. Quite recently, Z. Zhang [31] showed that $(u, \theta)$ is a strong solution if

$$\nabla u, \nabla \theta \in L^2 \left(0, T; \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)\right),$$

where $\dot{B}_{\infty,\infty}^{-1}$ denotes the homogenous Besov space. A logarithmically improvement of Zhang’s result, controlled by its $H^3$—norm, was given by Ye [29].

The main purpose of this work is to establish an improvement of Zhang’s regularity criterion (1.3). Now, the refined regularity criterion in terms of the gradient of the velocity $\nabla u$ can be stated as follows:

**Theorem 1.1 (Main result).** Assume that $(u_0, \theta_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. Let $(u, \theta)$ be a weak solution to the Boussinesq equations on some interval $(0, T)$ with $0 < T \leq \infty$. If

$$\nabla u \in L^2 \left(0, T; \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)\right),$$

then the weak solution $(u, \theta)$ is regular in $(0, T]$, that is $(u, \theta) \in C^\infty(\mathbb{R}^3 \times (0, T])$.

**Remark 1.2.** This result is expected because of the fact that the (refinement of) Beale-Kato-Majda type criterion is well known in the class $\dot{B}_{\infty,\infty}^0$ for the 3D Boussinesq equations and one may replace the vorticity by $\nabla u$ since the Riesz transforms are continuous in $\dot{B}_{\infty,\infty}^0$. Then, the temperature plays a less dominant role than the velocity field does in the regularity theory of solutions to the Boussinesq equations. Furthermore, clearly Theorem 1.1 is an improvement of Zhang’s regularity criterion (1.3).

By a weak solution, we mean that $(u, \theta, \pi)$ must satisfy (1.1) in the sense of distributions. In addition, we have the basic regularity for the weak solution

$$(u, \theta) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)),$$

for any $T > 0$. If a weak solution $(u, \theta)$ satisfies

$$(u, \theta) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)),$$

then actually $(u, \theta)$ is a strong (classical) solution. It is worth to note that for strong solutions, we can gain more regularity properties.

Throughout this paper, $C$ denotes a generic positive constant which may vary from one line to another.
2 Preliminaries

In this section we introduce the function spaces that will be used to state and prove the main result, and we collect and/or derive a number of auxiliary estimates that will be needed throughout the proof. Before introducing the homogeneous Besov and Triebel-Lizorkin spaces, we have to fix some notations. By $S$ we denote the class of rapidly decreasing functions. The dual space of $S$, i.e., the space of tempered distributions on $\mathbb{R}^3$ is denoted by $S'$. For $u \in S(\mathbb{R}^3)$, the Fourier transform of $u$ is defined by

$$\mathcal{F}u(\omega) = \hat{u}(\omega) = \int_{\mathbb{R}^3} u(x)e^{-ix \cdot \omega}dx, \quad \omega \in \mathbb{R}^3.$$ 

The homogeneous Littlewood-Paley decomposition relies upon a dyadic partition of unity. We can use for instance any $\varphi \in S(\mathbb{R}^3)$, supported in $\mathcal{C} \triangleq \{ \omega \in \mathbb{R}^3 : \frac{3}{4} \leq |\omega| \leq \frac{8}{3} \}$ such that

$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l} \omega) = 1 \quad \text{if} \quad \omega \neq 0.$$ 

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define dyadic blocks in this way:

$$\Delta_l u \triangleq \varphi(2^{-l}D)u = 2^{3l} \int_{\mathbb{R}^3} h(2^l y)u(x - y)dy, \quad \text{for each} \quad l \in \mathbb{Z},$$ 

and

$$S_l u \triangleq \sum_{k \leq l - 1} \Delta_k u.$$ 

The formal decomposition

$$u = \sum_{l \in \mathbb{Z}} \Delta_l u$$

is called the homogeneous Littlewood-Paley decomposition.

Remark 2.1. The above dyadic decomposition has nice properties of quasi-orthogonality: with our choice of $\varphi$, we have,

$$\Delta_k \Delta_l u \equiv 0 \quad \text{if} \quad |k - l| \geq 2 \quad \text{and} \quad \Delta_k (S_{k-1} u \Delta_l u) \equiv 0 \quad \text{if} \quad |k - l| \geq 5.$$ 

With the introduction of $\Delta_l$, let us recall the definition of homogeneous Besov and Triebel-Lizorkin spaces (see [26] for more details).

Definition 2.2. The homogeneous Besov space $\dot{B}^{s}_{p,q}(\mathbb{R}^3)$ is defined by

$$\dot{B}^{s}_{p,q}(\mathbb{R}^3) = \left\{ u \in S'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3) : \|u\|_{\dot{B}^{s}_{p,q}} < \infty \right\},$$
for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, where

$$
\|u\|_{\dot{B}^s_{p,q}} = \left\{ \begin{array}{ll}
\left( \sum_{j \in \mathbb{Z}} 2^{js} \|\Delta_j u\|_{L^p}^q \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty,
\sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j u\|_{L^p}, & \text{if } q = \infty,
\end{array} \right.
$$

and $\mathcal{P}(\mathbb{R}^3)$ is the set of all scalar polynomials defined on $\mathbb{R}^3$. Similarly, the homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^3)$ is a quasi-normed space equipped with the family of semi-norms $\|\cdot\|_{\dot{F}^s_{p,q}}$ which are defined by

$$
\|u\|_{\dot{F}^s_{p,q}} = \left\{ \begin{array}{ll}
\left( \sum_{j \in \mathbb{Z}} 2^{js} |\Delta_j u|^q \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty,
\sup_{j \in \mathbb{Z}} 2^{js} |\Delta_j u|_{L^p}, & \text{if } q = \infty.
\end{array} \right.
$$

Notice that there exists a universal constant $C$ such that

$$
C^{-1} \|u\|_{\dot{B}^s_{p,q}} \leq \|\nabla u\|_{\dot{B}^{s-1}_{p,q}} \leq C \|u\|_{\dot{B}^s_{p,q}}.
$$

In particular,

$$
u \in \dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3) \iff \nabla u \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3).
$$

From this observation we derive the following corollary to Theorem 1.1.

**Corollary 2.3.** Suppose that $(u, \theta)$ is a weak solution of the Boussinesq equations on $(0, T)$. If

$$
u \in L^2(0, T; \dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3),
$$

then the weak solution $(u, \theta)$ is regular in $(0, T]$.

Next, we introduce the following Bernstein lemma due to [10].

**Lemma 2.4 (Bernstein).** For all $k \in \mathbb{N}$, $j \in \mathbb{Z}$, and $1 \leq p, q \leq \infty$, we have for all $f \in \mathcal{S}(\mathbb{R}^3)$:

(i)

$$
\sup_{|\alpha|=k} \|\nabla^\alpha \Delta_j f\|_{L^q} \leq C_1 2^{j(k+3j)(\frac{1}{p} - \frac{1}{q})} \|\Delta_j f\|_{L^p}
$$

(ii)

$$
\|\Delta_j f\|_{L^p} \leq C_2 2^{-j} \sup_{|\alpha|=k} \|\nabla^\alpha \Delta_j f\|_{L^p},
$$

where $C_1$, $C_2$ are positive constants independent of $f$ and $j$. 

5
The proof of the main result needs a logarithmic Sobolev inequality in terms of Besov space. It will play an important role in the proof of Theorem 1.1. The following is a well-known embedding result, (cf. [26], pp. 244):

\[
L^\infty(\mathbb{R}^3) \hookrightarrow BMO(\mathbb{R}^3) = F^0_{\infty,2} \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^3),
\]

where \(BMO(\mathbb{R}^3)\) stands for the Bounded Mean Oscillations space [26].

We state and prove the following lemma.

**Lemma 2.5.** Suppose that \(\nabla f \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\) and \(f \in H^s(\mathbb{R}^3)\) for all \(s > \frac{3}{2}\). Then, there exists a constant \(C > 0\) such that

\[
\|f\|_{L^\infty} \leq C \left[ 1 + \|\nabla f\|_{\dot{B}^{-1}_{\infty,\infty}} \left( \ln^+ \|f\|_{H^s} \right)^{\frac{1}{2}} \right],
\]

holds, where \(H^s\) denotes the standard Sobolev space and

\[
\ln^+ x = \begin{cases} 
\ln x, & \text{if } x > e, \\
1, & \text{if } 0 < x \leq e.
\end{cases}
\]

**Proof.** The proof is an easy modification of the one in [17]. Owing the Littlewood–Paley decomposition, we can rewrite

\[
f = \sum_{j \in \mathbb{Z}} \Delta_j f = \sum_{j=-\infty}^{-N-1} \Delta_j f + \sum_{j=-N}^{N} \Delta_j f + \sum_{j=N+1}^{+\infty} \Delta_j f,
\]

where \(N\) is a positive integer to be determined later. Bernstein’s lemma and Young’s inequality give rise to

\[
\|f\|_{L^\infty} \leq \sum_{j=-\infty}^{-N-1} \|\Delta_j f\|_{L^\infty} + \sum_{j=-N}^{N} \|\Delta_j f\|_{L^\infty} + \sum_{j=N+1}^{+\infty} \|\Delta_j f\|_{L^\infty}
\]

\[
\leq C \sum_{j=-N}^{N} 2^{\frac{3j}{2}} \|\Delta_j f\|_{L^2} + CN \|f\|_{B^0_{\infty,\infty}} + C \sum_{j>N} 2^{(-s+\frac{3}{2})j} \|\Delta_j f\|_{L^2} 2^{js}
\]

\[
\leq C \left( 2^{-\frac{3N}{2}} \|f\|_{L^2} + N \|\nabla f\|_{B^{-1}_{\infty,\infty}} + \sum_{j>N} 2^{(-s+\frac{3}{2})j} \|f\|_{\dot{B}^s_{2,\infty}} \right)
\]

\[
(2.7)
\]

where we have used the fact that \(s > \frac{3}{2}\) and the Besov embedding \(H^s \hookrightarrow \dot{B}^s_{2,\infty}\).

Setting \(\alpha = \min \left( s - \frac{3}{2}, \frac{3}{2} \right) \), we derive

\[
\|f\|_{L^\infty} \leq C \left( 2^{-\alpha N} \|f\|_{H^s} + N \|\nabla f\|_{B^{-1}_{\infty,\infty}} \right). \tag{2.8}
\]

Now choose \(N\) such that \(2^{-\alpha N} \|f\|_{H^s} \leq 1\). Thus we get

\[
N \geq \frac{\log \|f\|_{H^s}}{\alpha \log 2}.
\]
Next, the following lemma is needed.

**Lemma 2.6.** Let \( g, h \in H^1(\mathbb{R}^3) \) and \( f \in \text{BMO}(\mathbb{R}^3) \). Then we have

\[
\int_{\mathbb{R}^3} f \cdot \nabla (gh) \, dx \leq C \| f \|_{\text{BMO}} \left( \| \nabla g \|_{L^2} \| h \|_{L^2} + \| g \|_{L^2} \| \nabla h \|_{L^2} \right).
\]

**Proof.** The proof of the above lemma requires some paradifferential calculus. We have to recall here that paradifferential calculus enables to define a generalized product between distributions. It is continuous in many functional spaces where the usual product does not make sense (see the pioneering work of J.-M. Bony in [2]). The paraproduct between \( f \) and \( g \) is defined by

\[
T_f g \triangleq \sum_{j \in \mathbb{Z}} S_{j-1} f \Delta_j g.
\]

We thus have the following formal decomposition (modulo a polynomial):

\[
f g = T_f g + T_g f + R(f, g),
\]

with

\[
R(f, g) = \sum_{|j-k| \leq 1} \Delta_j f \Delta_k g.
\]

Coming back to the proof of Lemma 2.6, we split \( \int_{\mathbb{R}^3} f \cdot \nabla (gh) \, dx \) into

\[
\int_{\mathbb{R}^3} f \cdot \nabla (gh) \, dx = \int_{\mathbb{R}^3} f \cdot \nabla (T_g h) \, dx + \int_{\mathbb{R}^3} f \cdot \nabla (g T_h) \, dx + \int_{\mathbb{R}^3} f \cdot \nabla R(g, h) \, dx = I_1 + I_2 + I_3.
\]

Since we know that \( \text{BMO} = F^0_{\infty, 2} \) (see pp. 243–244 of [26]), the duality between \( F^0_{\infty, 2} \) and \( F^0_{1, 2} \) guarantees that

\[
I_1 = \int_{\mathbb{R}^3} f \cdot (T_g h) \, dx + \int_{\mathbb{R}^3} f \cdot (T_h \nabla g) \, dx \\
\leq \| f \|_{\text{BMO}} (\| T_g h \|_{F^0_{1, 2}} + \| T_h \nabla g \|_{F^0_{1, 2}}) \\
= \| f \|_{\text{BMO}} (I_{11} + I_{12}).
\]

In view of the boundedness of the Hardy-Littlewood maximal operator \( \mathcal{M} \) in \( L^p \) spaces \((1 < p < \infty)\) (c.f. Stein [25, Chap II, Theorem 1]), we can estimate the term \( I_{11} \) as follows:

\[
I_{11} \approx \left\| \left( \sum_{j \in \mathbb{Z}} |S_{j-1}(\nabla g)|^2 |\Delta_j h|^2 \right)^{\frac{1}{2}} \right\|_{L^1} \leq C \left\| \mathcal{M}(\nabla g) \left( \sum_{j \in \mathbb{Z}} |\Delta_j h|^2 \right)^{\frac{1}{2}} \right\|_{L^1} \\
\leq C \| \mathcal{M}(\nabla g) \|_{L^2} \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_j h|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \leq C \| \nabla g \|_{L^2} \| h \|_{L^2}.
\]
Repeating the same arguments, we also have for $I_{12}$

$$I_{12} \approx \left\| \left( \sum_{j \in \mathbb{Z}} |S_{-1}(g)|^2 |\Delta_j(\nabla h)|^2 \right)^{\frac{1}{2}} \right\|_{L^1} \leq C \left\| M(g) \left( \sum_{j \in \mathbb{Z}} |\Delta_j(\nabla h)|^2 \right)^{\frac{1}{2}} \right\|_{L^1}$$

$$\leq C \| g \|_{L^2} \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_j(\nabla h)|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \leq C g \| \nabla h \|_{L^2}.$$  

Collecting these estimates, we obtain

$$I_1 \leq C \| f \|_{BMO} (\| \nabla g \|_{L^2} \| h \|_{L^2} + \| g \|_{L^2} \| \nabla h \|_{L^2}).$$

As a result, estimating $I_2$ following the same arguments, we obtain

$$I_2 \leq C \| f \|_{BMO} (\| \nabla g \|_{L^2} \| h \|_{L^2} + \| g \|_{L^2} \| \nabla h \|_{L^2}).$$

For the third term $I_3$, using the embedding relation $\dot{B}^{0}_{1,1} \subset F^{0}_{1,2}$ and in view of Bernstein’s lemma, we can deduce that

$$I_3 \leq \| f \|_{BMO} \| \nabla R(g,h) \|_{\dot{B}^{0}_{1,1}} \leq C \| f \|_{BMO} \| R(g,h) \|_{\dot{B}^{0}_{1,1}}$$

$$\leq C \| f \|_{BMO} \sum_{j \in \mathbb{Z}} 2^j \| \Delta_j g \cdot \Delta_j h \|_{L^1}$$

$$\leq C \| f \|_{BMO} \sum_{j \in \mathbb{Z}} 2^j \| \Delta_j g \|_{L^2} \| \Delta_j h \|_{L^2}$$

$$\leq C \| f \|_{BMO} \| \nabla g \|_{L^2} \| h \|_{L^2}.$$  

so that the proof of Lemma 2.6 is achieved. \(\square\)

We often use the following well-known lemma.

**Lemma 2.7.** Let $1 \leq q, r < \infty$ and $m \leq k$. Suppose that $\theta$ and $j$ satisfy $m \leq j \leq k$, $0 \leq \theta \leq 1$ and define $p \in [1, +\infty]$ by

$$\frac{1}{p} = \frac{j}{3} + \theta \left( \frac{1}{r} - \frac{m}{3} \right) + (1 - \theta) \left( \frac{1}{q} - \frac{k}{3} \right).$$

Then, the inequality

$$\| \nabla^j f \|_{L^p} \leq C \| \nabla^m f \|_{L^q}^{1-\theta} \| \nabla^k f \|_{L^r}^\theta$$

holds with some constant $C > 0$.  

For $f \in W^{m,\theta}(\mathbb{R}^3) \cap W^{k,\gamma}(\mathbb{R}^3)$,
3 Proof of Theorem 1.1

Now we are ready to prove our main result of this section.

Proof

First, note that a weak solution \((u, \theta)\) to (1.1) has at least one global weak solution
\[
(u, \theta) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)),
\]
which satisfies the following energy inequality
\[
\frac{1}{2}(\|u(\cdot, t)\|_2^2 + \|\theta(\cdot, t)\|_2^2) + \int_0^t (\|\nabla u(\cdot, \tau)\|_2^2 + \|\nabla \theta(\cdot, \tau)\|_2^2) d\tau \\
\leq \frac{1}{2}(\|u_0\|_2^2 + \|\theta_0\|_2^2),
\]
for almost every \(t \geq 0\).

In order to prove that \((u, \theta) \in C^\infty(\mathbb{R}^3 \times (0, T])\), as it is well known, it suffices to show that the weak solution \((u, \theta)\) is also a strong solution on \((0, T]\), which means that:
\[
(u, \theta) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)).
\]

Owing to (1.4), we know that for any small constant \(\epsilon > 0\), there exists \(T_0(T_0(\epsilon)) < T\) such that
\[
\int_{T_0}^T \|\nabla u(\cdot, \tau)\|_{B_{\infty, \infty}^1}^2 d\tau \leq \epsilon.
\]

To do so, we shall work on the local strong solution with the initial datum \((u_0, \theta_0)\) on its maximal existence time interval \((0, T_0)\). Then, we have only to show that
\[
\sup_{0 \leq t < T_0} (\|\nabla u(\cdot, t)\|_2^2 + \|\nabla \theta(\cdot, t)\|_2^2) + \int_0^{T_0} (\|\nabla u(\cdot, \tau)\|_2^2 + \|\nabla \theta(\cdot, \tau)\|_2^2) d\tau \leq C < \infty,
\]
here and in what follows \(C\) denotes various positive constants which are independent from \(T_0\).

Take the operator \(\nabla\) in equations (1.1) and (1.1)2, respectively, and the scalar product of them \(\nabla u\) and \(\nabla \theta\), respectively and add them together, to obtain
\[
\frac{1}{2} \frac{d}{dt} (\|\nabla u\|_2^2 + \|\nabla \theta\|_2^2) + \|\Delta u\|_2^2 + \|\Delta \theta\|_2^2 \\
= -\int_{\mathbb{R}^3} \theta e_3 \cdot \Delta u dx - \sum_{i=1}^3 \int_{\mathbb{R}^3} (\partial_i u \cdot \nabla) u \partial_i u dx - \sum_{i=1}^3 \int_{\mathbb{R}^3} (\partial_i u \cdot \nabla) \theta \partial_i \theta dx
\]
\[
(3.12) \quad = I_1 + I_2 + I_3.
\]
In the following, we estimate each term at the right-hand side of (3.12) separately below.

To bound $I_1$, we integrate by parts and apply Hölder’s inequality to obtain

$$|I_1| \leq C \| \nabla u \|_{L^2} \| \nabla \theta \|_{L^2} \leq C (\| u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2).$$

In order to deal with the terms $I_2$ and $I_3$, we need the following elegant Machihara-Ozawa inequality [18] (see also Meyer [20])

$$\| \nabla u \|_{L^4}^2 \leq C \| u \|_{\dot{B}^0_{\infty, \infty}} \| \Delta u \|_{L^2}. \quad (3.13)$$

We now bound $I_2$. By (3.13) and Young’s inequality

$$|I_2| \leq C \| \nabla u \|_{L^2} \| \nabla u \|_{L^4} \leq C \| \nabla u \|_{L^2} \| u \|_{\dot{B}^0_{\infty, \infty}} \| \Delta u \|_{L^2} \leq C \| \nabla u \|_{L^2} \| u \|_{BMO} \| \Delta u \|_{L^2} \leq \frac{1}{2} \| \Delta u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| u \|_{BMO}^2.$$

By integration by parts, we can rewrite and estimate $I_3$ as follows

$$|I_3| = \left| \sum_{i=1}^{3} \int_{\mathbb{R}^3} (\partial_i u \cdot \nabla) \theta \cdot \partial_i \theta \, dx \right| = \left| \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} \partial_i (\partial_i \theta \cdot \delta_k \partial_j \theta) u_j \, dx \right| \leq C \| u \|_{BMO} \| \nabla \theta \|_{L^2} \| \Delta \theta \|_{L^2} \leq \frac{1}{6} \| \Delta \theta \|_{L^2}^2 + C \| u \|_{BMO}^2 \| \nabla \theta \|_{L^2}^2.$$

Combining the estimates for $I_1$, $I_2$ and $I_3$, we find

$$\frac{d}{dt} \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) + \| \Delta u \|_{L^2}^2 + \| \Delta \theta \|_{L^2}^2 \leq C (1 + \| u \|_{BMO}^2) (\| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2).$$

Using the Gronwall inequality on the time interval $[T_0, t]$, one has the following inequality

$$\| \nabla u(\cdot, t) \|_{L^2}^2 + \| \nabla \theta(\cdot, t) \|_{L^2}^2 \leq \| \nabla u(\cdot, T_0) \|_{L^2}^2 + \| \nabla \theta(\cdot, T_0) \|_{L^2}^2 \leq C \int_{T_0}^{t} \| u(\cdot, \tau) \|_{BMO}^2 \, d\tau.$$

Let us denote for any $t \in [T_0, T)$

$$F(t) \triangleq \max_{T_0 \leq \tau \leq t} \left( \| u(\cdot, \tau) \|_{H^2}^2 + \| \theta(\cdot, \tau) \|_{H^2}^2 \right). \quad (3.14)$$
It should be noted that the function $F(t)$ is nondecreasing. Using (2.6), we obtain

$$\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\nabla \theta(\cdot, t)\|_{L^2}^2 + \int_{T_0}^t (\|\Delta u(\cdot, \tau)\|_{L^2}^2 + \|\Delta \theta(\cdot, \tau)\|_{L^2}^2) d\tau \leq C(T_0) \exp \left( C \int_{T_0}^t (1 + \|u(\cdot, \tau)\|_{L_{\infty, \infty}}^2 \log(\|u(\cdot, \tau)\|_{H^0}^2 + \|\theta(\cdot, \tau)\|_{H_0}^2)) d\tau \right)$$

$$\leq C(T_0) \exp \left( C \int_{T_0}^t \|\nabla u(\cdot, \tau)\|_{L^2}^2 \log(\|u(\cdot, \tau)\|_{H^0}^2 + \|\theta(\cdot, \tau)\|_{H_0}^2) d\tau \right)$$

$$\leq C(T_0) \exp \left( C \int_{T_0}^t \|\nabla u(\cdot, \tau)\|_{L^2}^2 d\tau \sup_{T_0 \leq \tau \leq t} \log(\|u(\cdot, \tau)\|_{H^0}^2 + \|\theta(\cdot, \tau)\|_{H_0}^2) \right)$$

$$\leq C(T_0) \exp(C \epsilon \log F(t))$$

$$\leq C(T_0) [F(t)]^{C\epsilon},$$

where

$$C(T_0) = C \left( \|\nabla u(\cdot, T_0)\|_{L^2}^2 + \|\nabla \theta(\cdot, T_0)\|_{L^2}^2 \right).$$

Next, applying $\Delta$ to the equations (1.1)$_1$, (1.1)$_2$, taking the $L^2$ inner product of the obtained equations with $-\Delta u$ and $-\Delta \theta$, respectively, adding them up and using the incompressible conditions $\nabla \cdot u = 0$, we arrive at

$$\frac{1}{2} \frac{d}{dt} (\|\Delta u\|_{L^2}^2 + \|\Delta \theta\|_{L^2}^2) + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 \theta\|_{L^2}^2$$

$$= \int_{\mathbb{R}^3} \Delta(\theta e_3) \cdot \Delta u dx - \int_{\mathbb{R}^3} \Delta(u \cdot \nabla u) \cdot \Delta u dx - \int_{\mathbb{R}^3} \Delta(u \cdot \nabla \theta) \cdot \Delta \theta dx$$

$$\leq \left| \int_{\mathbb{R}^3} \Delta(\theta e_3) \cdot \Delta u dx \right| + \left| \int_{\mathbb{R}^3} (\Delta u \cdot \nabla u) \cdot \Delta u dx \right| + 2 \sum_{i=1}^3 \left| \int_{\mathbb{R}^3} (\partial_i u \cdot \nabla \partial_i u) \cdot \Delta u dx \right|$$

$$+ \left| \int_{\mathbb{R}^3} (\Delta u \cdot \nabla \theta) \cdot \Delta \theta dx \right| + 2 \sum_{i=1}^3 \left| \int_{\mathbb{R}^3} (\partial_i u \cdot \nabla \partial_i \theta) \cdot \Delta \theta dx \right|$$

$$\leq \sum_{k=1}^5 A_k.$$  

Now we will estimate the terms on the right-hand side of (3.15) one by one as follows. Let us begin with estimating the term $A_4$.

Using Lemma 2.7 with $p = q = r = j = 2$, $k = 3$ and $m = 1$, $A_1$ can be bounded
above as follows:

\[ A_1 \leq C \| \Delta u \|_{L^2} \| \Delta \theta \|_{L^2} \]
\[ \leq C \| \nabla u \|_{L^2}^2 \| \nabla^3 u \|_{L^2}^{\frac{3}{2}} \| \nabla \theta \|_{L^2}^{\frac{3}{2}} \| \nabla^3 \theta \|_{L^2}^{\frac{3}{2}} \]
\[ = \left( \| \nabla^3 u \|_{L^2}^2 \right)^{\frac{1}{4}} \left( \| \nabla^3 \theta \|_{L^2}^2 \right)^{\frac{1}{4}} (C \| \nabla u \|_{L^2} \| \nabla \theta \|_{L^2})^{\frac{1}{2}} \]
\[ \leq \frac{1}{16} \| \nabla^3 u \|_{L^2}^2 + \frac{1}{16} \| \nabla^3 \theta \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| \nabla \theta \|_{L^2} \]
\[ \leq \frac{1}{16} \| \nabla^3 u \|_{L^2}^2 + \frac{1}{16} \| \nabla^3 \theta \|_{L^2}^2 + C (\| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2) . \]

Let us now recall Gagliardo–Nirenberg’s inequality

\[ \| \Delta f \|_{L^4} \leq C \| \nabla f \|_{L^2}^{\frac{1}{4}} \| \nabla^3 f \|_{L^2}^{\frac{7}{8}} . \]

Thus, we obtain

\[ A_2, A_3 \leq C \| \nabla u \|_{L^2} \| \nabla \Delta u \|_{L^4} \]
\[ \leq C \| \nabla u \|_{L^2} \| \nabla \|_{L^2}^{\frac{3}{2}} \| \nabla^3 u \|_{L^2}^{\frac{3}{2}} \]
\[ = C \| \nabla u \|_{L^2}^{\frac{3}{2}} \| \nabla^3 u \|_{L^2} \]  
\[ = (C \| \nabla u \|_{L^2}^{10})^{\frac{1}{2} \cdot \frac{7}{8}} \left( \| \nabla^3 u \|_{L^2}^2 \right)^{\frac{7}{8}} \leq \frac{1}{16} \| \nabla^3 u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^{10} . \]

Similarly to the estimate of \( A_1 \), the terms \( A_4 \) and \( A_5 \) can be bounded above as

\[ A_4, A_5 \leq C \| \nabla \theta \|_{L^2} \| \Delta \theta \|_{L^4} \| \Delta u \|_{L^4} \]
\[ \leq C \| \nabla \theta \|_{L^2} \left( \| \Delta u \|_{L^4}^{2} + \| \Delta \theta \|_{L^4}^{2} \right) \]
\[ \leq C \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^3 u \|_{L^2}^{\frac{3}{2}} + C \| \nabla \theta \|_{L^2}^{\frac{3}{2}} \| \nabla^3 \theta \|_{L^2}^{\frac{3}{2}} \]
\[ \leq \frac{1}{4} \| \nabla^3 u \|_{L^2}^2 + C \| \nabla \theta \|_{L^2}^{10} \| \nabla u \|_{L^2}^{2} + \frac{1}{2} \| \nabla^3 \theta \|_{L^2}^2 + C \| \nabla \theta \|_{L^2}^{10} \]
\[ \leq \frac{1}{16} \| \nabla^3 u \|_{L^2}^2 + \frac{1}{4} \| \nabla^3 \theta \|_{L^2}^2 + C \| \nabla \theta \|_{L^2}^{10} \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) . \]

Summarizing all the estimates and absorbing the dissipative term, we can derive

\[ \frac{d}{dt} (\| \Delta u \|_{L^2}^2 + \| \Delta \theta \|_{L^2}^2) \leq C \| \nabla u \|_{L^2}^{10} + C \| \nabla \theta \|_{L^2}^{10} \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) \]
\[ \leq C \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) \]
\[ \leq C \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) \]
\[ \leq C \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) \]
\[ \leq C(T_0) [F(t)]^{5C} \].
Integrating the above estimate over interval \((T_0, t)\) and observing that \(F(t)\) is a monotonically increasing function, we thus have

\[
\|\Delta u(\cdot, t)\|_{L^2}^2 + \|\Delta \theta(\cdot, t)\|_{L^2}^2 \\
\leq \|\Delta u(\cdot, T_0)\|_{L^2}^2 + \|\Delta \theta(\cdot, T_0)\|_{L^2}^2 + C(T_0) \int_{T_0}^{t} [F(\tau)]^{5C \epsilon} d\tau.
\]

By using (3.14), it follows that

\[
F(t) \leq \|u(\cdot, T_0)\|_{H^2}^2 + \|\theta(\cdot, T_0)\|_{H^2}^2 + C \int_{T_0}^{t} [F(\tau)]^{5C \epsilon} d\tau \\
\leq \|u(\cdot, T_0)\|_{H^2}^2 + \|\theta(\cdot, T_0)\|_{H^2}^2 + C(T_0)(t - T_0) [F(t)]^{5C \epsilon}.
\]

Choosing \(\epsilon\) such that \(5C \epsilon < 1\), the above inequality yields for any \(t \in [T_0, T)\)

\[
F(t) \leq C < \infty,
\]

which implies that \((u, \theta) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3))\). This completes the proof of Theorem 1.1.

**Remark 3.1.** Comparing our result with [30], we have simplified the proof of Theorem 1.1 in [30], in fact we only need \(H^2\) a priori estimates of solutions.

4 Acknowledgment.

Part of the work was carried out while the third author was long-term visitor at University of Catania. The hospitality of Catania University is graciously acknowledged. This research is partially supported by Piano della Ricerca 2016-2018 - Linea di intervento 2: ”Metodi variazionali ed equazioni differenziali”. M.A. Ragusa wish to thank the support of “RUDN University Program 5-100”. The authors wish to express their thanks to the referees for their very careful reading of the paper, giving valuable comments and helpful suggestions.

References

[1] H. Abidi and T. Hmidi. On the global well-posedness for Boussinesq system. J. Differential Equations, 233(1):199220, 2007.

[2] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. Ecole Norm. Sup. 14 (1981), 209-246.
[3] J. R. Cannon and E. DiBenedetto. The initial value problem for the Boussinesq equations with data in Lp. In Approximation methods for Navier-Stokes problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979), volume 771 of Lecture Notes in Math., pages 129-144. Springer, Berlin, 1980.

[4] D. Chae and H.-S. Nam, Local existence and blow-up criterion for the Boussinesq equations, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), 935-946.

[5] D. Chae, S.-K. Kim and H.-S. Nam, Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations, Nagoya Math. J. 155 (1999), 55-80.

[6] R. Danchin and M. Paicu. Existence and uniqueness results for the Boussinesq system with data in Lorentz spaces. Phys. D, 237(10-12):1444-1460, 2008.

[7] R. Danchin and M. Paicu. Les theore`mes de Leray et de Fujita-Kato pour le syst`eme de Boussinesq partiellement visqueux. Bull. Soc. Math. France, 136(2):261-309, 2008.

[8] J. Fan and T. Ozawa, Regularity criteria for the 3D density-dependent Boussinesq equations, Nonlinearity 22 (2009), 553-568.

[9] J. Fan and Y. Zhou, A note on regularity criterion for the 3D Boussinesq system with partial viscosity, Appl. Math. Lett. 22 (2009), 802-805.

[10] J.-Y. Chemin, Perfect Incompressible Fluids, Oxford University Press, New York, 1998.

[11] S. Gala, Z. Guo and M.A. Ragusa, A remark on the regularity criterion of Boussinesq equations with zero heat conductivity, Appl. Math. Lett. 27 (2014), 70-73.

[12] Z. Guo and S. Gala, Regularity criterion of the Newton-Boussinesq equations in $R^3$, Commun. Pure Appl. Anal. 11 (2012), 443-451.

[13] S. Gala, On the regularity criterion of strong solutions to the 3D Boussinesq equations, Appl. Anal. 90 (2011), 1829-1835.

[14] S. Gala and M. A. Ragusa, Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices, Appl. Anal. 95 (2016), 1271-1279.

[15] G.P. Galdi, An introduction to the Navier-Stokes initial-boundary value problem, In : Fundamental Directions in Mathematical Fluid Mechanics, Adv. Math. Fluid Mech. Basel-Birkhauser, 2000, 1-70.
[16] N. Ishimura and H. Morimoto, Remarks on the blow-up criterion for 3D Boussinesq equations, Math. Models Methods Appl. Sci. 9 (1999), 1323-1332.

[17] H. Kozono, T. Ogawa and Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations, Math. Z. 242 (2002), 251-278.

[18] S. Machihara and T. Ozawa, Interpolation inequalities in Besov spaces, Proc. Amer. Math. Soc. 131 (2002), 1553-1556.

[19] A. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean, Courant Lecture Notes in Mathematics Vol. 9, AMS/CIMS, Providence, RI, 2003.

[20] Y. Meyer, Oscillating Patterns in Some Nonlinear Evolution Equations, in: M. Cannone, T. Miyakawa (Eds.), Mathematical Foundation of Turbulent Viscous Flows, Lecture Notes in Mathematics, vol. 1871, Springer-Verlag, 2006, pp. 101-187.

[21] Geophysical Fluids Dynamics. Springer Verlag, New York, 1979.

[22] Y. Qin, X. Yang, Y. Wang, X. Liu, Blow-up criteria of smooth solutions to the 3D Boussinesq equations, Math. Methods Appl. Sci. 35 (2012), 278-285.

[23] H. Qiu, Y. Du and Z. Yao, A blow-up criterion for 3D Boussinesq equations in Besov spaces, Nonlinear Anal. 73 (2010), 806-815.

[24] J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal. 9 (1962), 187-195.

[25] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, 1970.

[26] H. Triebel, Theory of Function Spaces, Birkhäuser Verlag, Basel, Boston, 1983.

[27] Z. Xiang, The regularity criterion of the weak solution to the 3D viscous Boussinesq equations in Besov spaces, Math. Methods Appl. Sci. 34 (2011), 360-372.

[28] F. Xu, Q. Zhang and X. Zheng, Regularity criteria of the 3D Boussinesq equations in the Morrey-Campanato space, Acta Appl. Math. 121 (2012), 231-240.

[29] Z. Ye, Blow-up criterion of smooth solutions for the Boussinesq equations, Nonlinear Anal. 110 (2014), 97-103.

[30] Z. Ye, A logarithmically improved regularity criterion of smooth solutions for the 3D Boussinesq equations, Osaka J. Math. 53 (2016), 417-423.
[31] Z. Zhang, Some regularity criteria for the 3D Boussinesq equations in the Class $L^2(0, T; B^{-1}_{\infty, \infty}(\mathbb{R}^3))$, ISRN Appl. Math. (2014), Art. ID 564758, 4 pp.