On the local time of the Half-Plane Half-Comb walk

Endre Csáki
Alfréd Rényi Institute of Mathematics, Budapest, P.O.B. 127, H-1364, Hungary. E-mail address: csaki.endre@renyi.hu

Antónia Földes
Department of Mathematics, College of Staten Island, CUNY, 2800 Victory Blvd., Staten Island, New York 10314, U.S.A. E-mail address: Antonia.Foldes@csi.cuny.edu

Abstract The Half-Plane Half-Comb walk is a random walk on the plane, when we have a square lattice on the upper half-plane and a comb structure on the lower half-plane, i.e., horizontal lines below the x-axis are removed. We study the local time of this walk.

MSC: primary 60F17, 60G50, 60J65; secondary 60F15, 60J10

Keywords: Anisotropic random walk; Strong approximation; Wiener process; Local time; Laws of the iterated logarithm;

1 Introduction and main results

The properties of a simple symmetric random walk on the square lattice $\mathbb{Z}^2$ have been extensively investigated in the literature since Dvoretzky and Erdős [9], and Erdős and Taylor [10]. For these and further results we refer to Révész [16].

Subsequent investigations concern random walks on other structures of the plane. For example, a simple random walk on the 2-dimensional comb lattice that is obtained from $\mathbb{Z}^2$ by removing all horizontal lines off the x-axis was studied by Weiss and Havlin [19], Bertacchi and Zucca [3], Bertacchi [2], Csáki et al. [5], [6].

The latter are particular cases of the so-called anisotropic random walk on the plane. The general case is given by the transition probabilities

$$P(C(N + 1) = (k + 1, j) | C(N) = (k, j)) = P(C(N + 1) = (k - 1, j) | C(N) = (k, j)) = \frac{1}{2} - p_j,$$

$$P(C(N + 1) = (k, j + 1) | C(N) = (k, j)) = P(C(N + 1) = (k, j - 1) | C(N) = (k, j)) = p_j,$$

for $(k, j) \in \mathbb{Z}^2$, $N = 0, 1, 2, \ldots$ with $0 < p_j \leq 1/2$ and $\min_{j \in \mathbb{Z}} p_j < 1/2$. See Seshadri et al. [17], Silver et al. [18], Heyde [11] and Heyde et al. [12]. The simple symmetric random walk corresponds to the case $p_j = 1/4$, $j = 0, \pm 1, \pm 2, \ldots$, while $p_0 = 1/4$, $p_j = 1/2$, $j = \pm 1, \pm 2, \ldots$ defines random walk on the comb.
In our paper [7] we combined the simple symmetric random walk with a random walk on a comb, when \( p_j = 1/4, j = 0, 1, 2, \ldots \) and \( p_j = 1/2, j = -1, -2, \ldots \), i.e., we have a square lattice on the upper half-plane, and a comb structure on the lower half-plane. We call this model Half-Plane Half-Comb (HPHC) and denote the random walk on it by \( C(N) = (C_1(N), C_2(N)), N = 0, 1, 2, \ldots \). Here, for convenient information, we first repeat the precise construction of this walk, as it was given in [7]:

On a suitable probability space consider two independent simple symmetric (one-dimensional) random walks \( S_1(\cdot), S_2(\cdot) \). We may assume that on the same probability space we have a sequence of independent geometric random variables \( \{Y_i, i = 1, 2, \ldots \} \), independent from \( S_1(\cdot), S_2(\cdot) \), with distribution
\[
P(Y_i = k) = \frac{1}{2k+1}, \quad k = 0, 1, 2, \ldots
\]
Now horizontal steps will be taken consecutively according to \( S_1(\cdot) \), and vertical steps consecutively according to \( S_2(\cdot) \) in the following way. Start from \((0,0)\), take \( Y_1 \) horizontal steps (possibly \( Y_1 = 0 \)) according to \( S_1(\cdot) \), then take 1 vertical step. If this arrives to the upper half-plane \( S_2(1) = 1 \), then take \( Y_2 \) horizontal steps. If, however, the first vertical step is in the negative direction \( S_2(1) = -1 \), then continue with another vertical step, and so on. In general, if the random walk is on the upper half-plane, \( y \geq 0 \) after a vertical step, then take a random number of horizontal steps according to the next (so far) unused \( Y_j \), independent from the previous steps. On the other hand, if the random walk is on the lower half-plane \( y < 0 \) then continue with vertical steps according to \( S_2(\cdot) \) until it reaches the \( x \)-axis, and so on.

In paper [7] we investigated the almost sure limit properties of this walk by using strong approximation methods. Our first result was a strong approximation of both components of the random walk \( C(\cdot) \) by certain time-changed Wiener processes (Brownian motions) with rates of convergence. Before stating it, we need some definitions. Assume that we have two independent standard Wiener processes \( W_1(t), W_2(t), t \geq 0 \), and consider
\[
\alpha_2(t) := \int_0^t I\{W_2(s) \geq 0\} \, ds,
\]
i.e., the time spent by \( W_2(\cdot) \) on the non-negative side during the interval \([0,t]\). The process \( \gamma_2(t) := \alpha_2(t) + t \) is strictly increasing, hence we can define its inverse: \( \beta_2(t) := (\gamma_2(t))^{-1} \). Observe that the processes \( \alpha_2(t), \beta_2(t) \) and \( \gamma_2(t) \) are defined in terms of \( W_2(t) \), so they are independent from \( W_1(t) \). It can be seen moreover that \( 0 \leq \alpha_2(t) \leq t \), and \( t/2 \leq \beta_2(t) \leq t \).

**Theorem A** On an appropriate probability space for the HPHC random walk
\( \{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots \} \) one can construct two independent standard Wiener processes \( \{W_1(t); t \geq 0\}, \{W_2(t); t \geq 0\} \) such that, as \( N \to \infty \), we have with any \( \varepsilon > 0 \)
\[
|C_1(N) - W_1(N - \beta_2(N))| + |C_2(N) - W_2(\beta_2(N))| = O(N^{3/8+\varepsilon}) \quad \text{a.s.}
\]
Our second result in paper [7] was the following LIL.
Theorem B We have

\[ \limsup_{N \to \infty} \frac{C_1(N)}{\sqrt{N \log \log N}} = \limsup_{N \to \infty} \frac{C_2(N)}{\sqrt{N \log \log N}} = 1 \text{ a.s.} \]

Furthermore

\[ \liminf_{N \to \infty} \frac{C_1(N)}{\sqrt{N \log \log N}} = -1 \text{ a.s.,} \quad \liminf_{N \to \infty} \frac{C_2(N)}{\sqrt{N \log \log N}} = -\sqrt{2} \text{ a.s.} \]

Moreover we gave an explicit formula for the \(N\)-step return probability of the walk, which however was too complicated to conclude the asymptotic limit. The aim of the present paper is to study the local time of this walk. Based on the just mentioned formula and a beautiful result of Sparre Andersen, we first get the asymptotic limit of this return probability, and then use it for getting local time results.

2 Preliminaries

Let \(X_1, X_2, \ldots\) be i.i.d. random variables with \(P(X_1 = \pm 1) = 1/2\), and define \(S(0) = 0, S(i) = \sum_{j=1}^{i} X_j\). Then \(\{S(n), n = 0, 1, \ldots\}\) is a simple symmetric random walk on the line with local time

\[ \xi(x, n) = \# \{ j : 0 \leq j \leq n, S(j) = x \}, \quad x \in \mathbb{Z}, \]

and put

\[ A(n) = \sum_{j=0}^{\infty} \xi(j, n - 1), \quad n = 1, 2, \ldots \]

\[ P(2n, r) = P(A(2n) = r, S(2n) = 0), \quad r = 1, 2, \ldots, 2n. \] (2.1)

Define

\[ G_n = \# \{ j : 0 \leq j < n, S(j) \geq 0 \}. \]

Then we can rephrase the definition of \(P(2n, r)\) as follows:

\[ P(2n, r) = P(G_{2n} = r, S(2n) = 0). \] (2.2)

We proved in [7], that

\[ P(C(2N) = (0, 0)) = \binom{2N}{N} \frac{1}{4^{2N}} + \sum_{n=1}^{N} \sum_{r=1}^{2n} P(2n, r) \binom{2N - 2n}{N - n} \frac{1}{2^{N-2n}} \binom{2n - 2n + r}{r} \frac{1}{2^{N-2n+r}}, \] (2.3)

where it was shown that

\[ P(2n, 2r - 1) = P(2n, 2r), \] (2.4)
and we concluded the following complicated formula for $P(2n, 2r)$ (see Lemma 5.2 in [7]):

$$P(2n, 2r) = \frac{1}{2^{2n}} \sum_{j=1}^{r} \frac{1}{2j - 1} \binom{2j - 1}{j} \frac{1}{2n + 1 - 2j} \binom{2n + 1 - 2j}{n + 1 - j}.$$  

However, in order to proceed, we need a closed form for $P(2n, 2r)$.

Sparre Andersen [1] proved some elegant results about the fluctuation of the sums of random variables. We only quote the case of simple symmetric random walk of his much more general results. In his formula (5.12) he defines

$$K_n = \#\{j : 0 < j \leq n, S(j) > 0\},$$

and gives the probability of

$$P(K_{2n-1} = 2r, S(2n) = 0) = P(K_{2n-1} = 2r + 1, S(2n) = 0)$$

$$= \frac{1}{2} c_{2n} \frac{1}{n+1} \left( 1 + \frac{n - 2r}{n} \left( -\frac{1}{2} \right) \left( \frac{-1}{n-r} \right) \left( \frac{1}{n} \right) \right), \quad r = 0, \ldots, n - 1,$$

where

$$c_{2n} := P(S(2n) = 0) = (-1)^n \left( \frac{-1}{n} \right) = \frac{1}{2^{2n}} \binom{2n}{n}.$$  

However $P(2n, 2r) = P(G_{2n} = 2r, S(2n) = 0)$, given in (2.2) is slightly different from the above one. We will show the following

Lemma 2.1

$$P(2n, 2r) = P(G_{2n} = 2r, S(2n) = 0) = \frac{1}{2^{2n+1}} \binom{2n}{n} \frac{1}{n+1} \left( 1 + \frac{n - 2r}{n} \left( \frac{2r}{n} \right) \binom{2n-2r}{n-r} \right).$$  

(2.5)

Proof: Recall the definition of $K_n$ and $G_n$ and let

$$M_n = \#\{j : 0 < j \leq n, S(j) \leq 0\}.$$  

Then observe that

$$P(M_{2n} = k, S(2n) = 0) = P(G_{2n} = k, S(2n) = 0).$$  

Moreover, the following two events are the same:

$$\{M_{2n} = r, S(2n) = 0\} = \{K_{2n-1} = 2n - r, S(2n) = 0\}.  \quad (2.7)$$
\[ P(2n, 2r) = P(G_{2n} = 2r, S(2n) = 0) = P(M_{2n} = 2r, S(2n) = 0) = P(K_{2n-1} = 2n - 2r, S(2n) = 0), \]

which immediately implies our lemma. □

Recall now the definition of the sequence of i.i.d. geometric random variables given in the introduction

\[ P(Y_i = k) = 2^{-(k+1)} \quad i = 1, 2, \ldots, \quad k = 0, 1, 2, \ldots, \]

and let

\[ U = U_K = \sum_{i=1}^{K} Y_i. \]

Then \( U_K \) is negative binomial with \( E(U_K) = K \), \( Var(U_K) = 2K \) and

\[ P(U_K = r) = \binom{K - 1 + r}{r} \frac{1}{2^{K+r}}, \quad r = 0, 1, 2, \ldots \]

We will need the following two well-known identities about the negative binomial distribution:

\[ \sum_{r=0}^{a} \binom{a + r}{r} \frac{1}{2^{a+r}} = 1 \]

\[ \sum_{r=0}^{\infty} \binom{a + r}{r} \frac{1}{2^{a+r}} = 2 \]

See the first one, e.g., in Pitman [15] (page 220), while the second one is equivalent with \( \sum_{r=0}^{\infty} P(U_K = r) = 1 \).

**Lemma A** Berry-Esseen bound: [14] (page 150) Let \( X_1, \ldots, X_n \) be i.i.d. random variables. Let

\[ E(X_1) = 0, \quad Var(X_1) = \sigma^2 > 0, \quad E(|X|^3) < \infty, \quad \rho = E(|X|^3)/\sigma^3. \]

Then with some constant \( A > 0 \) we have

\[ \sup_x \left| P \left( \sigma^{-1} n^{-1/2} \sum_{j=1}^{n} X_j < x \right) - \Phi(x) \right| \leq A \rho n^{-1/2}, \]

where \( \Phi(\cdot) \) is the standard normal distribution function.
3 Asymptotic return probability

We want to determine the asymptotic probability that the HPHC random walk returns to the starting point in $2N$ steps.

**Theorem 3.1** For the asymptotic return probability of the HPHC walk, starting at $(0,0)$, we have

$$P(C(2N) = (0,0)) \sim \frac{2}{\pi N}, \quad \text{as} \quad N \to \infty.$$

**Proof:** Recall the definition of $U_K$ in (2.10). In what follows let $U = U_{2N-2n+1}$. Introduce the notation

$$Q(r, n) := \binom{2r}{r} \binom{2n-2r}{n-r} \binom{2n}{n}^{-1}.$$

Combining formulas (2.3), (2.4) and (2.6) we have that

$$P(C(2N) = (0,0)) = \binom{2N}{N} \frac{1}{4^{2N}}$$

$$+ \sum_{n=1}^{N} \binom{2N - 2n}{N - n} \frac{1}{2^{2N-2n}} \sum_{r=1}^{2n} P(2n, r) \binom{2N - 2n + r}{r} \frac{2}{2^{2N-2n+r+1}}$$

$$= \binom{2N}{N} \frac{1}{4^{2N}} + \sum_{n=1}^{N} \binom{2N - 2n}{N - n} \frac{1}{2^{2N-2n}} \sum_{r=1}^{n} 2P(2n, 2r)P(2r - 1 \leq U \leq 2r)$$

$$= \binom{2N}{N} \frac{1}{4^{2N}} + \sum_{n=1}^{N} \binom{2N - 2n}{N - n} \frac{1}{2^{2N-n+1}} \binom{2n}{n} \sum_{r=1}^{n} \left(1 + \frac{2r - n}{n} Q(r, n)\right) P(2r - 1 \leq U \leq 2r).$$

The first term in (3.1) is negligible, since

$$\binom{2N}{N} \frac{1}{4^{2N}} = O\left(\frac{1}{4^{N}}\right).$$

Thus

$$P(C(2N) = (0,0))$$

$$\sim \sum_{n=1}^{N} \binom{2N - 2n}{N - n} \frac{1}{2^{2N}} \binom{2n}{n} \frac{1}{n} P(U \leq 2n)$$

$$+ \sum_{n=1}^{N} \binom{2N - 2n}{N - n} \frac{1}{2^{2N}} \binom{2n}{n} \frac{1}{n} \sum_{r=1}^{n} \frac{2r - n}{n} Q(r, n) P(2r - 1 \leq U \leq 2r) = I + II.$$
Observe that
\[
\binom{2N - 2n}{N - n} \frac{1}{2^{2N}} \binom{2n}{n} = c_{2N-2n}c_{2n} \sim \frac{1}{\pi \sqrt{n}} \frac{1}{\sqrt{N - n}}, \quad \text{when } n \to \infty \text{ and } N - n \to \infty. \tag{3.2}
\]
Moreover, if only \( n \to \infty \) but \( N - n \) might be small, then
\[
c_{2N-2n}c_{2n} \leq \frac{c}{\sqrt{n}}. \tag{3.3}
\]
Here and in what follows \( c \) is a positive constant whose value can change from line to line. It is clear that for \( 1 \leq r \leq n \)
\[
-1 \leq \frac{2r - n}{n} \leq 1.
\]

We will show that term II is negligible compared to term I, so we use the above fact to give the following upper bound for II:
\[
|II| \leq II^* := \sum_{n=1}^{N} \frac{1}{n} \binom{2n}{n} \left( \frac{2N - 2n}{N - n} \right) \frac{1}{2^{2N}} \sum_{r=1}^{n} Q(r, n) P(2r - 1 \leq U \leq 2r).
\]
First we deal with the term I, dividing the sum for \( n \) into 5 parts:
\[
(i) \quad 1 \leq n < \frac{N}{4} \\
(ii) \quad \frac{N}{4} \leq n < \frac{N}{2} - N^{1/2 + \alpha} \\
(iii) \quad \frac{N}{2} - N^{1/2 + \alpha} \leq n < \frac{N}{2} + N^{1/2 + \alpha} \\
(iv) \quad \frac{N}{2} + N^{1/2 + \alpha} \leq n < N - N^{1/2 - \alpha} \\
(v) \quad N - N^{1/2 - \alpha} \leq n \leq N,
\]
with some \( 0 < \alpha < 1/2 \). Observe that
\[
I = \sum_{n=1}^{N} \frac{1}{n} c_{2N-2n}c_{2n} P(U \leq 2n). \tag{3.4}
\]
Let us start with (i). In this case we can use the estimation
\[
P(U \leq 2n) = \sum_{r=0}^{2n} \binom{2N - 2n + r}{r} \frac{1}{2^{2N-2n+r+1}} \leq 2n \binom{2N}{2n} \frac{1}{2^{2N}}, \tag{3.5}
\]
since the largest term in the previous sum corresponds to \( r = 2n \). Thus

\[
\sum_{(i)} \leq \sum_{1 \leq n < N/4} c_{2N-2n} c_{2n} \frac{1}{n} \frac{2n}{2N} \frac{1}{2^{2N}} \leq c \frac{1}{2^{2N}} \sum_{1 \leq n < N/4} \left( \frac{2N}{2n} \right) \leq c \frac{1}{2^{2N}} \frac{N}{4} \left( \frac{2N}{N/2} \right) \leq c \sqrt{N} \left( \frac{4}{3 \sqrt{3}} \right)^N
\]  

with some constant \( c \), by observing that the first two factors in our sum is the product of two probabilities. We used Stirling formula to get the last inequality.

In case (ii) we use normal approximation for negative binomial distribution, with Berry-Esseen bound as in (2.14) to get that for \( n \) belonging to the set (ii)

\[
\mathbb{P}(U \leq 2n) = \Phi \left( \frac{4n - 2N - 1}{\sqrt{2(2N - 2n + 1)}} \right) + O \left( \frac{1}{\sqrt{N - n}} \right) \leq \Phi(-2N^\alpha) + \frac{c}{\sqrt{N}} \leq \frac{c}{\sqrt{N}},
\]

being the normal term exponentially small. Moreover, using (3.2)

\[
\sum_{(ii)} \leq c \sum_{N/4 < n \leq N/2 - N^{1/2 + \alpha}} \frac{1}{n} \frac{1}{\sqrt{n(N - n)}} \leq \frac{c}{N^{3/2}}.
\]

Considering now term (iii), we can overestimate \( \mathbb{P}(U \leq 2n) \) by 1, and obtain, using (3.2) again,

\[
\sum_{(iii)} \sim \frac{1}{\pi} \sum_{N/2 - N^{1/2 + \alpha} \leq n < N/2 + N^{1/2 + \alpha}} \frac{1}{n^{3/2}(N - n)^{1/2}} \leq \frac{c}{N^{3/2 - \alpha}}.
\]

Skipping term (iv) to finish estimating the negligible terms, it is easy to see that

\[
\sum_{(v)} \leq \sum_{N - N^{1/2 - \alpha} \leq n < N} \frac{1}{n^{3/2}} \leq c \frac{N^{1/2 - \alpha}}{N^{3/2}} = \frac{c}{N^{1+\alpha}}.
\]

using again only that \( \mathbb{P}(U \leq 2n) \leq 1 \) and (3.3).

Now we want to show that part (iv) in sum I will give the order of magnitude claimed in the theorem. It is easy to see by normal approximation again that for \( n \in (iv) \) we obtain

\[
\Phi(cN^\alpha) \leq \mathbb{P}(U \leq 2n) \leq 1,
\]

to conclude that for \( n \in (iv) \)

\[
\mathbb{P}(U \leq 2n) = 1 - o(1), \quad \text{as} \quad N \to \infty.
\]
So we need the asymptotic value of

\[ \sum_{(iv)} \sim \frac{1}{\pi} \sum_{N/2 + N^{1/2 + \alpha} \leq n < N - N^{1/2 - \alpha}} \frac{1}{n^{3/2}(N - n)^{1/2}}. \] (3.7)

By showing that

\[ \sum_{N/2 \leq n \leq N/2 + N^{1/2 + \alpha}} \frac{1}{n^{3/2}(N - n)^{1/2}} \leq c \frac{N^{1/2 + \alpha}}{N^2} = \frac{c}{N^{3/2 - \alpha}} \]

and

\[ \sum_{N - N^{1/2 - \alpha} \leq n \leq N} \frac{1}{n^{3/2}(N - n)^{1/2}} \leq c \frac{N^{1/2 - \alpha}}{N^{3/2}} = \frac{c}{N^{1+\alpha}}, \]

we can extend the interval of summation in (3.7) without changing the limit of the sum as follows:

\[ I \sim \frac{1}{\pi N} \sum_{N/2 < n < N} \left( \frac{n}{N} \right)^{3/2} \left( 1 - \frac{n}{N} \right)^{1/2} \frac{1}{n} \sim \frac{1}{\pi N} \int_{1/2}^{1} \frac{dv}{v^{3/2}(1 - v)^{1/2}} = \frac{2}{\pi N}. \]

Concerning the term \( II^* \), it is clear that \( Q(r, n) \) being a probability, the four negligible terms which we investigated as terms of \( I \) are also negligible compared to the main term. The only problem is to estimate the sum \( II^* \) for \( n \in (iv) \). This however is a delicate calculation. We split the sum for \( r \) into 3 parts:

1. \( 0 \leq r \leq n/4 \),
2. \( n/4 < r \leq n - n^\beta \),
3. \( n - n^\beta < r \leq n \) with some \( 0 < \beta < 1 \).

For (1) we use that \( Q(r, n) \) is a probability, obtaining just as in (ii) in (3.5) that

\[ \sum_{r \leq n/4} Q(r, n)P(2r - 1 \leq U \leq 2r) \leq cP(U \leq N/2) < \frac{N}{2} \left( \frac{2N}{N/2} \right) \frac{1}{2^{2N}}, \]

So

\[ \sum_{n \in (iv)} \frac{1}{n} \binom{2n}{N} \left( \frac{2N - 2n}{N - n} \right) \frac{1}{2^{2N}} \sum_{r \in (1)} Q(r, n)P(2r - 1 < U \leq 2r) \]

\[ \leq \frac{cN}{n} \binom{2N}{N/2} \frac{1}{2^{2N}} \leq c \left( \frac{2N}{N/2} \right) \frac{1}{2^{2N}} \leq c \left( \frac{4}{3\sqrt{3}} \right)^N, \]

where the last inequality is coming from Stirling formula as in (3.6).
In case (2), using Stirling formula, we have

\[
\sum_{r \in (2)} Q(r, n)P(2r - 1 \leq U \leq 2r) \leq \sum_{r \in (2)} \frac{c\sqrt{n}}{\sqrt{r\sqrt{n - r}}}P(2r - 1 \leq U \leq 2r) \\
\leq \frac{c}{n^{3/2}} \sum_{r \in (2)} P(2r - 1 \leq U \leq 2r) \leq \frac{c}{n^{3/2}} \leq \frac{c}{N^{3/2}},
\]

where the last inequality holds as \( n \in (iv) \). Consequently, similarly to (iv) in calculating I, we have \( \frac{c}{N^{3/2}} \) times the sum in (3.7) implying that

\[
\sum_{n \in (iv)} \sum_{r \in (2)} \leq \frac{c}{N^{1+3/2}}.
\]

For the case \( r \in (3) \) we have

\[
\sum_{r \in (3)} Q(r, n)P(2r - 1 \leq U \leq 2r) \leq cP(2n - 2n^{\frac{\beta}{2}} - 1 \leq U \leq 2n),
\]

which, using normal approximation, can be seen to tend to zero, as \( N \to \infty \), so the term \( \sum_{n \in (iv)} \sum_{r \in (3)} \) is negligible compared to (3.7), the main term in I.

This completes the proof of Theorem 3.1. \( \square \)

4 Laws of the iterated logarithm for the local time

Define the local time of the random walk on the HPHC lattice as

\[
\Xi((k, j), n) = \sum_{r=0}^{N} I\{C(r) = (k, j)\}, (k, j) \in \mathbb{Z}^2
\]

From Theorem 3.1 we can calculate the truncated Green function \( g(.) \) :

\[
g(N) = \sum_{k=0}^{[N/2]} P(C(2k) = 0) \sim \frac{2}{\pi} \log N \quad \text{as } n \to \infty.
\]

Our random walk being Harris recurrent, we can infer (e.g. Chen [1]) that

\[
\lim_{N \to \infty} \frac{\Xi((k_1, j_1), N)}{\Xi((k_2, j_2), N)} = \frac{\mu(k_1, j_1)}{\mu(k_2, j_2)} \quad \text{a.s.,}
\]

where \( \mu(\cdot) \) is an invariant measure. Here the invariant measure is defined as

\[
\mu(A) = \sum_{(k, j) \in \mathbb{Z}^2} \mu(k, j)P(C(N + 1) \in A|C(N) = (k, j)).
\]
For \((k, j) \in \mathbb{Z}^2\), in our case we have

\[
\mu(k, j) = \mu(k + 1, j) \left( \frac{1}{2} - p_j \right) + \mu(k - 1, j) \left( \frac{1}{2} - p_j \right) + \mu(k, j + 1) + \mu(k, j - 1),
\]

where

\[
p_j = \frac{1}{4} \quad \text{if} \quad j \geq 0 \quad \text{and} \quad p_j = \frac{1}{2} \quad \text{if} \quad j < 0.
\]

It is easy to see that

\[
\mu(k, j) = \frac{1}{p_j}, \quad (k, j) \in \mathbb{Z}^2
\]

is one possible invariant measure. So from Theorem 17.3.2 of Meyn and Tweedie [13] we get the following result.

**Corollary 4.1** For all integers \(k_1, k_2\) we have

\[
\lim_{N \to \infty} \frac{\Xi((k_1, j_1), N)}{\Xi((k_2, j_2), N)} = 1 \quad \text{a.s. if} \quad j_1 \geq 0, j_2 \geq 0 \quad \text{or} \quad j_1 < 0, j_2 < 0,
\]

and

\[
\lim_{N \to \infty} \frac{\Xi((k_1, j_1), N)}{\Xi((k_2, j_2), N)} = 2 \quad \text{a.s. if} \quad j_1 \geq 0, j_2 < 0.
\]

Using \(g(N)\) again, we get from Darling and Kac [8] the following result.

**Corollary 4.2**

\[
\lim_{N \to \infty} \mathbb{P} \left( \frac{\Xi(0, 0), N)}{g(N)} \geq x \right) = \lim_{N \to \infty} \mathbb{P} \left( \frac{\pi \Xi(0, 0), N)}{2 \log N} \geq x \right) = e^{-x}.
\]

As to the law of the iterated logarithm, we get from Theorem 2.4 of Chen [4] that it reads as follows.

**Corollary 4.3**

\[
\limsup_{N \to \infty} \frac{\Xi((0, 0), N)}{\log N \log \log \log N} = \frac{2}{\pi} \quad \text{a.s.}
\]

**References**

[1] Andersen, E. Sparre (1953). On the fluctuations of sums of random variables. *Math. Scand.* 1 263-285.

[2] Bertacchi, D. (2006). Asymptotic behaviour of the simple random walk on the 2-dimensional. *Electron J. Probab.* 11 1184–1203.
[3] Bertacchi, D. and Zucca, F. (2003). Uniform asymptotic estimates of transition probabilities on combs. *J. Aust. Math. Soc.* 75 325–353.

[4] Chen, X. (1999). How often does a Harris recurrent Markov chain recur? *Ann. Probab.* 27 1324–1346.

[5] Csáki, E., Csörgő, M., Földes, A. and Révész, P. (2009). Strong limit theorems for a simple random walk on the 2-dimensional comb. *Electr. J. Probab.* 14 2371–2390.

[6] Csáki, E., Csörgő, M., Földes, A. and Révész, P. (2011). On the local time of random walk on the 2-dimensional comb. *Stoch. Process. Appl.* 121 1290–1314.

[7] Csáki, E., Csörgő, M., Földes, A. and Révész, P. (2012). Random walk on the half-plane half-comb structure. *Annales Mathematicae et Informaticae* 39 29-39.

[8] Darling, D.A. and Kac, M. (1957). On occupation times for Markoff processes. *Trans. Amer. Math. Soc.* 84 444–458.

[9] Dvoretzky, A. and Erdős, P. (1951). Some problems on random walk in space. In: *Proc. Second Berkeley Symposium*, pp. 353–367.

[10] Erdős, P. and Taylor, S.J. (1960). Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar.* 11 137–162.

[11] Heyde, C.C. (1982). On the asymptotic behaviour of random walks on an anisotropic lattice. *J. Statist. Physics* 27 721–730.

[12] Heyde, C.C., Westcott, M. and Williams, R.J. (1982). The asymptotic behavior of a random walk on a dual-medium lattice. *J. Statist. Physics* 28 375–80.

[13] Meyn, S.P. and Tweedie, R.L. (1993). *Markov Chains and Stochastic Stability*, Springer, London.

[14] Petrov, V.V. (1995). *Limit Theorems of Probabilty Theory. Sequences of Independent Random Variables*, Oxford Stud. Probab. 4, Clarendon Press, Oxford.

[15] Pitman, J. (1993). *Probability*, Springer Text in Statistics, Springer, New York.

[16] Révész, P. (2013). *Random walk in Random and Non-Random Environments*, 3rd ed. World Scientific, Singapore.

[17] Seshadri, V., Lindenberg, K. and Schuler, K.E. (1979). Random walks on periodic and random lattices II. Random walk properties via generating function techniques. *J. Statist. Physics* 21 517–548.
[18] Silver, H., Shuler, K.E. and Lindenberg, K. (1977). Two-dimensional anisotropic random walks. In: Statistical Mechanics and Statistical Methods in Theory and Applications Proc. Sympos., Univ. Rochester, Rochester, N.Y., 1976, Plenum, New York, pp. 463–505.

[19] Weiss, G.H. and Havlin, S. (1986). Some properties of a random walk on a comb structure. *Physica A* **134** 474–482.