Existence and regularity of weak solutions to a model for coarsening in molecular beam epitaxy

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Taking into account the occurrence of a zero of the surface diffusion current and the requirement of the Ehrlich–Schwoebel effect, Siegert et al. formulated a model of Langevin type that describes the growth of pyramid-like structures on a surface under conditions of molecular beam epitaxy and that the slope of these pyramids is selected by the crystalline symmetries of the growing film. In this article, the existence and uniqueness of weak solution to an initial boundary value problem for this model is proved, in the case that the noise is neglected. The regularity of the weak solution to models, with/without slope selection, is also investigated. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction

Many processes occur mainly at surfaces of materials, such as crystal growth, catalytic reactions, production of nanostructures. Thus, surfaces are of great technological and fundamental interest. There has been increasing interest in the understanding of the kinetics of surface growth processes, for example, \cite{1, 2}. The growth of a crystalline film from a molecular or atomic beam is commonly referred to as molecular beam epitaxy (MBE), which is among the most fine methods for the growth of thin solid films and is of great importance for applied studies \cite{3}. MBE takes place in high vacuum or ultra high vacuum, for instance, some \(10^{-8}\) Pa. The most important aspect of MBE is the slow deposition rate that is typically less than 1, 000 nm per hour, so MBE allows the films to grow epitaxially. The slow deposition rates require proportionally better vacuum to achieve the same impurity levels as other deposition techniques. In turn, it is possible by using this technique to grow high-quality crystalline materials and form structures with very high precision in the vertical direction.

The mathematical theory has been developed since Langevin who proposed the equation named after him. The model by Kardar, Parisi, and Zhang \cite{4} describes well growth process such as the Eden process \cite{5}, ballistic deposition \cite{6}, and growth of various restricted solid-on-solid models \cite{7}. Thus, the Kardar, Parisi, and Zhang model has been widely accepted as a model for the growth of crystals and also has been extended to various cases. Let us mention especially the following work: in \cite{3}, the authors consider the system that has potential barriers near step edges that suppress the diffusion of adatoms to a lower terrace. This effect is now commonly called Ehrlich–Schwoebel effect. They take into account the step-flow regime and instability and proposed a continuum equation to model the growth in MBE, which is valid only at the early time as long as the slopes are much smaller than 1. However, this is too restrictive to describe the unstable three-dimensional growth of real materials in the later time regime. To take into account the occurrence of a zero in the surface diffusion current and the requirement of Ehrlich–Schwoebel effect, Seigert et al. \cite{8} thus introduced a current for a structure with a cubic symmetry so that the current is changed to the one that has a zero differing from 0 and so that the model can be also applied to the regime when the slope is much greater than 1. In this article, we shall study this model. Also, the model without slope selection will also be investigated. Because of the forming of steps, pyramid-like surfaces and so on during the crystal growth, it is more natural to assume that the initial data is in \(L^2(\Omega)\) or \(H^1(\Omega)\) than in \(H^2(\Omega)\).
To formulate the model, we need some notations. Let \( x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2 \) be a material point, \( \Omega \) is an open-bounded set with smooth boundary \( \partial \Omega \). Let \( t \) be the time variable. \( Q_t = (0, t) \times \Omega, h = h(t,x) \) is the height that is measured in a comoving frame of reference and describes the local position of the moving surface. \( \nabla_h h \) is the gradient of \( h \), and \( \Delta_x = \partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2 \) is the Laplacian. For simplicity of notations, we shall use the following notations

\[
\xi = (p, q) = \nabla_h h.
\]

Then, the equation turns out to be

\[
\frac{\partial h}{\partial t} + v \Delta_x h + \text{div}_x (J(\nabla_x h)) = 0,
\]

which is satisfied in \( Q_T \), where \( T \) is a given positive number. And the boundary and initial conditions are

\[
\frac{\partial h}{\partial n} = 0, \quad (v \nabla_x \Delta_x h + J(\nabla_x h)) \cdot n = 0, \quad \text{on } [0, T] \times \partial \Omega,
\]

\[
h(0, x) = h_0(x), \quad x \in \hat{\Omega}.
\]

Here, \( n = (n_1, n_2) \) is the unit outward normal vector to the boundary \( \partial \Omega \). We have introduced the surface diffusion current \( J = J(\nabla_x h) = J(p, q) \)

\[
J = (j_1, j_2),
\]

\[
j_1 = a \left( (p + q)f \left( (p + q)^2 \right) + (p - q)f \left( (p - q)^2 \right) \right),
\]

\[
j_2 = a \left( (p + q)f \left( (p + q)^2 \right) - (p - q)f \left( (p - q)^2 \right) \right),
\]

where \( f \) is defined by

\[
f(y) = \frac{1 - y}{(1 - y)^2 + \beta y},
\]

and \( \alpha \) is a constant of surface diffusion, \( \beta = (\ell_d)^2 \) where \( \ell_d \) is the diffusion length.

This completes the formulation of an initial boundary value problem. It is worth a remark on the nonlinearity \( f \) because there are several varieties of \( f \) that lead to different models relating to ours.

**Remark 1**

Define

\[
f(y) = \frac{1}{1 + \beta y},
\]

Then, the corresponding model is proposed by Johnson et al. [3]. However, this model does have a slope selection mechanism and is correct only for early time as long as the slopes are much smaller than 1. It is too restrictive to describe the unstable three-dimensional growth of real materials in the later time regime. Therefore, (1.7) is introduced by Siegert et al. in [8] to interpolate the two regimes. The form of the surface current \( J \) is the minimal model in the sense that the nonlinearity must be chosen such that it describes the instability and leads to slope selection. The flux still has the correct physical behavior: \( |J| \sim \sqrt{p^2 + q^2} \) for \( p^2 + q^2 \ll \frac{1}{\ell_d} \) and \( |J| \sim 1/\sqrt{p^2 + q^2} \) for \( \frac{1}{\ell_d^2} \ll p^2 + q^2 \ll 1 \). This type of fluxes gives rise to a completely different behavior than the one defined by (1.8), despite many similarities, as shown in [8]. The exact form of \( f \) does not play a role because the slope selection mechanism and the growth exponents do not depend on such details.

In [9, 10], the current of the form

\[
J = \xi \left( 1 - |\xi|^2 \right)
\]

is used; however, it has stable zeros for all slopes with \( |\xi| = 1 \) regardless of the direction of \( \xi \). Thus, such an azimuthal symmetry is unrealistic for crystalline films. Therefore, here, \( f_i (i = 1, 2) \) are functions chosen such that \( f_1 (p^2, q^2) = f_2 (q^2, p^2) \). The simplest form that describes growth on such substrates is a current with components

\[
j_1 = p \left( 1 - p^2 - bq^2 \right),
\]

\[
j_2 = q \left( 1 - q^2 - bp^2 \right),
\]
which leads to a buildup of pyramids with selected slopes \((p_0, q_0) = (\pm 1, \pm 1)/\sqrt{1 + b}\) for \(-1 < b < 1\). This diffusion current is suitable for substrates with a quadratic symmetry.

Finally, we point out that after a coordinate transformation \(X = Ax\), where

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

we find \(J\) in (1.7) can be reduced, without loss of generality, to a simpler form

\[
j_1 = \alpha pf \left(p^2\right),
\]

\[
j_2 = \alpha qf \left(q^2\right).
\]

(1.12)

It is interesting to compare the important difference between the Cahn–Hilliard equation modeling phase ordering and the model considered here. Mathematically, Equation (1.1), with nonlinearity (1.8), (1.9), or (1.12), differs from the Cahn–Hilliard equation because of the flux term: \(J\) in this article depends on the gradient of the unknown, whereas the Cahn–Hilliard equation on the unknown only.

Numerical experiments also show the important differences between this model and the Cahn–Hilliard one. Many papers, for example, [8, 11–13] have carried out the study of the differences between the problem studied here and phase-ordering dynamics described by the Cahn–Hilliard equation [14]. These differences become apparent when the domain configurations are plotted as in Figure 1.

A domain in this context is an area of constant slope corresponding to one of the four states. The analogous case in phase-ordering dynamics is described by a four-state clock model, for example, [11, 15]. However, in that case, we shall find that domain walls do not have any particular orientation, whereas here domain walls are intersections of planes of constant slopes and therefore form straight lines. Furthermore, there are two types of domain walls: domain walls at which only one component of the slope changes such that \(f\) grows like \(\log(|x|)\) as \(|x|\) goes to \(\infty\). Suppose now that \(h = h(t, x)\) is a solution to (1.1)–(1.3).

Now, we derive the model briefly. Define the free energy by

\[
E[h] = \int_{\Omega} \left( \frac{\nu}{2} (\Delta_x h)^2 - \frac{\alpha}{2} \left( F \left( (h_{x_1} + h_{x_2})^2 \right) + F \left( (h_{x_1} - h_{x_2})^2 \right) \right) \right) dx.
\]

Here, \(F = \int^x f(y)dy\) is the primitive of \(f\) such that \(F(x)\) grows like \(-\log(|x|)\) as \(|x|\) goes to \(\infty\). Suppose now that \(h = h(t, x)\) is a solution to (1.1)–(1.3). Formal computations yield

\[
\frac{d}{dt} E[h] = \int_{\Omega} \left( -\alpha f \left( (h_{x_1} + h_{x_2})^2 \right) (h_{x_1} + h_{x_2})(h_{x_1} + h_{x_2}) r \right) dx
\]

\[
+ \int_{\Omega} \left( -\alpha f \left( (h_{x_1} - h_{x_2})^2 \right) (h_{x_1} - h_{x_2})(h_{x_1} - h_{x_2}) r + \nu \Delta_x h \Delta_x h_t \right) dx.
\]

Combining the terms containing \((h_{x_1})_t\) (or \((h_{x_2})_t\)) together, we can rewrite the right hand side of the aforementioned equality as

\[
- \int_{\Omega} \alpha f \left( (h_{x_1} + h_{x_2})^2 \right) (h_{x_1} + h_{x_2}) + f \left( (h_{x_1} - h_{x_2})^2 \right) (h_{x_1} - h_{x_2}) \left( (h_{x_1})_t \right) dx
\]

\[
- \int_{\Omega} \alpha f \left( (h_{x_1} + h_{x_2})^2 \right) (h_{x_1} + h_{x_2}) - f \left( (h_{x_1} - h_{x_2})^2 \right) (h_{x_1} - h_{x_2}) \left( (h_{x_2})_t \right) dx
\]

\[
+ \int_{\Omega} \nu \Delta_x h \Delta_x h_t dx = \int_{\Omega} \left( -J \cdot \nabla_x h_t + \nu \Delta_x h \Delta_x h_t \right) dx.
\]

(1.13)

Figure 1. Left: the configuration for the crystal growth model, from [12]; Right: the domain walls of phase ordering governed by the Cahn–Hilliard equation, from, for example, [15].
Using integration by parts and Equation (1.1), we infer from (1.13) that
\[
\frac{d}{dt} E[h] = \int_\Omega (-J - \nu \nabla_x \Delta_x h) \cdot \nabla_x h_1 \, dx \\
= \int_\Omega \text{div}_x (J + \nu \nabla_x \Delta_x h) h_1 \, dx \\
= -\int_\Omega (\text{div}_x (J + \nu \nabla_x \Delta_x h))^2 \, dx \\
\leq 0.
\] (1.14)

This implies the second law of thermodynamics is valid. The equation can be written in a gradient form with the total surface current defined by \( J_1 := J_{\text{eq}} + J(\nabla_x h) \) where \( J_{\text{eq}} := \nu \nabla_x \Delta_x h \) is called the equilibrium surface current and \( \nu \) is proportional to the surface stiffness.

**Statement of the main result.** Before the statement of our main results, we define weak solutions to problem consisting of (1.1)–(1.3).

We use the notations: \((f, g)_{\Omega_T}\) and \((f, g)_{\Omega}\) are, respectively, the \(L^2\) inner product of \(f\) and \(g\) over \(\Omega_T\) and \(\Omega\). \((f, g)_{\Omega_T}\) denotes the dual product of \(f, g\) with \(f \in L^2(0, T; X')\), \(g \in L^2(0, T; X)\), \(X\) is a Banach space, and \(X'\) is its dual. \(H^m(\Omega)\) are the standard Hilbert spaces of order \(m\). Define \(H^m_T(\Omega) := \{ f \in H^2(\Omega) \mid \frac{\partial f}{\partial n} = 0 \text{ on } \partial \Omega \}\) and its dual space is denoted by \(H^{-m}_T(\Omega)\).

**Definition 1.1**

Let \( h_0 \in L^2(\Omega) \). A function \( h = h(t, x) \) with
\[
h \in L^\infty(0, T; L^2(\Omega)) \cap L^2 \left( 0, T; H^2(\Omega) \right), \ h_t \in L^2 \left( 0, T; H^2_T(\Omega) \right)
\] (1.15)
is a weak solution of Problems (1.1)–(1.3), if
\[
\langle h_t, \varphi \rangle_{\Omega_T} + \nu (\Delta_x h, \Delta_x \varphi)_{\Omega_T} + (J, \nabla_x \varphi)_{\Omega_T} = 0
\] (1.16)
holds for all \( \varphi \in L^2 \left( 0, T; H^2(\Omega) \right) \), and \( \lim_{t \to 0^+} \langle h(t), \psi \rangle_{\Omega_T} = \langle h_0, \psi \rangle_{\Omega_T} \) for all \( \psi \in L^2(\Omega) \).

Also, we need to introduce some notations that can be found, for example, in Ladyzenskaya et al. [16]. For \( k = 0, 1, \ldots \),
\[
h^{(k)}(x) \equiv \left. \frac{\partial^k h(t, x)}{\partial t^k} \right|_{t = 0},
\]
\[
A(t, x, \partial / \partial x) h := \left( \nu \Delta_x^2 h + \text{div}_x (J(\nabla_x h)) \right).
\]

We deduce from the initial condition and the equation of \( h \) that \( h^{(0)}(x) = h_0(x) \) and \( h^{(1)}(x) = A(0, x, \partial / \partial x) h(x) \); thus, for \( k > 1 \), one can have the recursion relations
\[
h^{(k+1)}(x) = \left. \frac{\partial^k h(t, x)}{\partial t^k} A(t, x, \partial / \partial x) h(t, x) \right|_{t = 0}.
\]

The \( m \)-th order compatibility conditions are
\[
\left. \frac{\partial}{\partial n} h^{(\ell)}(x) = 0 \right|_{t = 0}, \quad \nu \nabla_x \Delta_x h^{(\ell)}(x) + \frac{\partial^\ell}{\partial t^\ell} J(\nabla_x h) h \right|_{t = 0} = 0, \quad \text{for } \ell = 0, 1, \ldots, m,
\] (1.17)
and by the \( (m + 1/2) \)-th order compatibility conditions, we mean that the \( m \)-th order compatibility conditions, that is, (1.17), hold and
\[
\left. \frac{\partial}{\partial n} h^{(m+1)}(x) = 0 \right|_{t = 0}
\] (1.18)
is satisfied.

For the special case that \( m = -1 \), we understand (1.17) as no compatibility condition is necessary. Now, we are in a position to state the main results of this article.

**Theorem 1.2 (Existence)**

Suppose that the boundary of \( \Omega \) is smooth and \( h_0 \in L^2(\Omega) \). Then, there exists a unique weak solution \( h \) to Problems (1.1)–(1.3) in the sense of Definition 1.1, and the total mass \( \int_\Omega h(t, x) \, dx \) is conserved, that is, \( \int_\Omega h(t, x) \, dx = \int_\Omega h_0(x) \, dx \).

Moreover, if \( h_0 \in H^1(\Omega) \), the weak solution to Problems (1.1)–(1.3) has, which in addition to (1.15), the following regularities
\[
h \in L^\infty \left( 0, T; H^1(\Omega) \right) \cap L^2 \left( 0, T; H^2(\Omega) \right), \ h_t \in L^2 \left( 0, T; H^1(\Omega) \right)
\] (1.19)
Suppose that the boundary of $\Omega$ is smooth and $h_0 \in H^{2m}(\Omega)$ with $m \in \mathbb{N}$. Assume that if $m = 2\ell$, the $(\ell - 1)$-order (respectively, if $m = 2\ell - 1$, the $(\ell - 2 + 1/2)$-order) compatibility conditions are satisfied.

Then, the weak solution $h$ of Problems (1.1)–(1.3) satisfies

$$h \in L^{\infty} \left(0, T; H^{2m}(\Omega) \right) \cap L^2 \left(0, T; H^{2m+2}(\Omega) \right), \quad h_t \in L^2 \left(0, T; H^{2m-2}(\Omega) \right),$$

(1.20)

and

$$D^\ell_t h \in L^{\infty} \left(0, T; L^2(\Omega) \right) \cap L^2 \left(0, T; H^2(\Omega) \right), \quad if \ m = 2\ell, \ \ell \in \mathbb{N};$$

$$D^{\ell - 1}_t h \in L^{\infty} \left(0, T; H^2(\Omega) \right), \quad D^\ell_t h \in L^2 \left(0, T; L^2(\Omega) \right) \quad if \ m = 2\ell - 1, \ \ell \in \mathbb{N}. \quad (1.21)$$

Consequently, if $h_0 \in C^\infty(\bar{\Omega})$ and the $m$-th order compatibility conditions for $m = 0, 1, 2, \ldots$ are satisfied, then the solution $h$ is smooth on $Q_T$.

Now, let us recall some references related closely to our problem. In [17], an initial boundary value problem of this epitaxial model with cubic nonlinearities is studied in which the initial data is chosen in $H^1(\Omega)$, and the $H^2(\Omega)$-norm of the solution follows directly from the Clausius–Duhem inequality, the second laws of thermodynamics, but this technique does not work for our case because we assume the initial data is only in $H^1(\Omega)$. Li and Liu studied the initial boundary value problem for the MBE model with or without slope selection in [18], and the boundary conditions are chosen periodic. In both articles, they construct approximate solutions by using the Galerkin method, whereas we use a linearized problem, together with the convolution technique, to obtain a sequence of smooth approximate solutions, then establish a priori estimates for this sequence. Kohn and Otto [19] investigated the coarsening rate for the Cahn–Hilliard equation, and Kohn and Yan [20] studied the coarsening rate for an epitaxial growth model with a cubic nonlinearity. Watson and Norris [21] studied the coarsening dynamics of multiscale solutions to a dissipative singularly perturbed partial differential equation with a trigonally symmetric potential that models the evolution of a thermodynamically unstable crystalline surface.

We should also mention that different modeling approaches are used to describe the interplay of surface diffusion and Schwoebel barriers in MBE. These lead to related fourth-order models without apparent energy structure, the analysis of which is more delicate. Relevant references can be found, for example, [22] by Blömker et al., [23] by Winkler. In [22], it is studied the regularity and blow-up criteria for a surface growth model, which seems to have similar properties to the three-dimensional Navier–Stokes; in [23], Winkler proves the existence of global solutions to a fourth-order parabolic equation in higher dimensions that models epitaxial thin-film growth.

We give now a remark on the choice of the initial data in this article.

**Remark 2**

The assumption that initial data $h_0$ is in $H^1$ is more natural than the one that $h_0$ is in $H^2$. The reason is that $h$ is piecewise affine in the case that the surfaces are high symmetric, such as pyramid-like ones. One evidence can be also seen from a typical scanning tunneling microscope picture, Figure 2, which shows clearly that the surface are not smooth. Correspondingly, a good mathematical model should consider this feature.

If the initial data is in $H^1$, the problem on the regularity (i.e., to prove the solution is in $L^\infty(0, T; H^1)$) of weak solutions to Problems (1.1)–(1.3) in which the nonlinearity is cubic, like (1.9), also (1.10)–(1.11), is still open and may be interesting. To solve such a problem, I surmise we need to invent an inequality of the Brezis–Gallouet type [25].

![Figure 2](image-url)
The organization of the remaining parts of this article is as follows. The existence and uniqueness of weak solutions is studied in Section 2 by constructing smooth approximate solutions and using a priori estimates. Section 3 consists of two parts, each of which is concerned, respectively, with the regularities of weak solutions to models with and without slope selection.

2. Existence of weak solutions

2.1. Existence for the approximate problem

To prove the existence Theorem 1.2, we construct smooth approximate solutions in a similar way as carried out in [26], provided the initial data is smooth. Then, we establish uniform a priori estimates of these solutions by which we conclude compactness. Before formulating an approximate problem, we introduce the modifier \( \chi = \chi(t,x) \) such that \( \chi \in C^2_0(Q_T) \) satisfies \( \int_{\mathbb{R}^3} \chi(t,x) \, dx = 1 \). For \( \varepsilon > 0 \), we set

\[
\chi_\varepsilon(t,x) := \frac{1}{\varepsilon^2} \chi \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right),
\]

and for any function \( f \in L^\infty(Q_T) \), we define

\[
\tilde{f}(t,x) = (\chi_\varepsilon * f)(t,x) = \int_{\mathbb{R}^3} \chi_\varepsilon(t-s,x-y)f(s,y) \, dy.
\]

We choose a smooth sequence \( h_0^n \) such that

\[
\|h_0^n - h_0\|_X \to 0
\]
as \( \varepsilon \to 0 \). Here, \( X = L^2(\Omega) \) or \( X = H^1(\Omega) \).

Then, the smoothed problem turns out to be

\[
\frac{\partial h}{\partial t} + \nu \Delta_2^2 h + \text{div} \left( (\nabla_2 h) \right) = 0,
\]

with the following boundary and initial conditions

\[
\frac{\partial h}{\partial n} = 0, \quad (\nu \nabla_2 \Delta_2 h + \text{div} (\nabla_2 h)) \cdot n = 0, \quad \text{on } [0,T] \times \partial \Omega,
\]

\[
h(0,x) = h_0^n(x), \quad x \in \bar{\Omega}.
\]

Note that Equation (2.2) is a linear fourth-order parabolic equation with a smooth known term. By the existence theorem for higher order parabolic equations in the book by Ladyzenskaya et al. [16] or the book by Eidelman [27], we have

**Theorem 2.1 (Existence of smooth approximate solutions)**

Suppose that the assumptions of Theorem 1.2 are satisfied and \( \varepsilon \) is a given positive constant. Let \( \hat{h} \in L^2(0,T;H^2(\Omega)) \).

Then, for any \( T > 0 \), there exists a unique smooth solution \( h \) of Problems (2.2)–(2.4), which satisfies that the total mass \( \int_{\Omega} h(t,x) \, dx \) is conserved.

The solution \( h \) constructed in Theorem 2.1 depends on the small parameter \( \varepsilon \). To prove the existence of weak solutions to the original Problems (1.1)–(2.4), we need to establish some a priori estimates that are independent of \( \hat{h} \) and \( \varepsilon \) and thus guarantee the passage to limit of \( h_\varepsilon \) as \( \varepsilon \to 0 \).

2.2. A priori estimates

Assume that there exists a classical solution \( h_\varepsilon \) to Problems (2.2)–(2.4) with smooth initial data \( h_0^n \) and \( \hat{h} \in L^2(0,T;H^2(\Omega)) \) satisfying \( \|\hat{h}\|_{L^2(0,T;H^2(\Omega))} \leq \hat{C} \). In what follows, \( C \) denotes a constant that is independent of \( \varepsilon \) and \( \hat{h} \). The \( L^2(\Omega) \)-norm of \( f \) is denoted by \( \|f\| \). We shall derive a priori estimates for this solution. To begin with, we first state the following lemma on the nonlinearity \( f \).

**Lemma 2.1**

There hold, for all \( y \geq 0 \), that

\[
|yf(y)| \leq C, \quad (2.5)
\]

\[
|y^{m+1} f^{(m)}(y)| \leq C, \quad m \in \mathbb{N}. \quad (2.6)
\]

Here, \( f^{(m)} \) is the \( m \)-th order derivative of \( f \).
Proof
One needs to investigate the behavior of the function

\[-yf(y) = \frac{y^2 - y}{(1 - y)^2 + \beta y}\]

for all \(y \geq 0\). It is easy to see that \(-yf(y) \to 1\) as \(y \to \infty\), hence \(|-yf(y)| \leq C\) for \(y \geq M\), where \(M\) is a suitably large constant. We shall prove that \(-yf(y)\) is also bounded on the interval \([0, M]\). To this end, we need to prove that the denominator \(g(y) = (1 - y)^2 + \beta y\) is greater than a positive constant. Note that \(g(y)\) is nonnegative for \(y \geq 0\). Thus, \(0\) is the only possible minimum of \(g(y)\). However, \(g(y) = 0\) implies that \(1 - y = 0\) and \(y = 0\), which cannot be satisfied simultaneously. Therefore, the minimum of \(g\) must be positive, and because \(M\) is finite, we infer from the continuity of \(g(y)\) that

\[\min_{0 \leq y \leq M} g(y) = C_1 > 0.\]  

(2.7)

We then obtain, for all \(y \in [0, M]\), that

\[|yf(y)| \leq \frac{M^2 + M}{C} \leq C.\]

Next, we consider the behavior of derivatives of \(f\). Let \(m \leq 1\) be an integer. Rewrite

\[f^{(m)}(y) = ((1 - y)g(Y(y)))^{(m)}\]

where \(g(Y) = Y^{-1}\) and \(Y = (1 - y)^2 + \beta y\). Invoking the product rule

\[f^{(m)}_0 = (f_1 \cdot f_2)^{(m)} = \sum_{k=0}^{m} C_m^k f_1^{(k)} f_2^{(m-k)},\]

where \(C_m^k\) denotes the number of \(k\)-combinations of an \(m\)-element set, we have

\[f^{(m)}(y) = C_m^0 (1 - y)g(Y(y))^{(m)} + C_m^1 (1 - y)^{y}g(Y(y))^{(m-1)}\]

\[= (1 - y)g(Y(y))^{(m)} - mg(Y(y))^{(m-1)}.\]  

(2.8)

Making use of the Faà di Bruno formula, that is,

\[\frac{d^m}{dx^m} g(Y(y)) = \sum_{1 l_1 + 2 l_2 + \ldots + m l_m = m} \frac{m!}{l_1! l_2! \ldots l_m!} g^{(l_1 + l_2 + \ldots + l_m)}(Y(y)) \prod_{j=1}^{m} \left(\frac{y^{(j)}(y)}{j!}\right)^{l_j}\]  

(2.9)

and recalling \((g(Y))^{(m)} = (Y^{-1})^{(m)} = (-1)^m m! Y^{-m-1}\), \(Y'(y) = -2(1 - y) + \beta\), \(Y''(y) = 2\), and \(Y^{(j)}(y) = 0\) for any \(j > 2\), we can reduce (2.9) to

\[\frac{d^m}{dx^m} g(Y(y)) = \sum_{1 l_1 + 2 l_2 = m} (-1)^{l_1 + l_2} \frac{m!(l_1 + l_2)!}{l_1! l_2!} Y^{-(l_1 + l_2 + 1)}(y) \prod_{j=1}^{2} \left(\frac{y^{(j)}(y)}{j!}\right)^{l_j}.\]  

(2.10)

For \(y \in [0, M]\), \(f^{(m)}(y)\) is smooth by (2.7). One thus needs only to investigate the behavior for large \(y \in [M, \infty)\), and it is enough to calculate the highest exponent. From (2.10), it follows that the highest exponent is less than or equal to

\[-2(l_1 + l_2 + 1) + l_1 + 0 l_2 = -l_1 - 2l_2 - 2 = -m - 2.\]

Therefore, invoking (2.8), we assert that there exists a constant \(\gamma\) such that

\[f^{(m)}(y) \sim \gamma y^{-m-1} \text{ as } y \to \infty,\]  

(2.11)

where for two functions in \(y A, B\), the expression \(A \sim B\) means that \(A\) is asymptotically equivalent to \(B\) as \(y \to \infty\), that is, \(\lim_{y \to \infty} A/B = 1\) if \(B \neq 0\) for large \(y\). Hence, this implies (2.6). Thus, the proof of this lemma is complete.

\[\square\]
From now on, we are going to derive a priori estimates. The first is

**Lemma 2.2**

There hold for all $t \in [0, T]$\(^2\)
\[
\int_{\Omega} h(t,x) dx = \int_{\Omega} h_0(x) dx, \quad (2.12)
\]
\[
\|h(t)\|^2 + \int_0^t \| \Delta_x h(\tau) \|^2 d\tau \leq C. \quad (2.13)
\]

**Proof**

Integrating (2.2) with respect to $x$ yields
\[
\frac{d}{dt} \int_{\Omega} h(t,x) dx = 0, \quad (2.14)
\]
which implies (2.12).

Multiplying (2.2) by $h$ and integrating the resulting equation with respect to $x$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|h\|^2 + v \|\Delta_x h\|^2 + \int_{\Omega} J \left( \nabla_x h \right) \cdot \nabla_x h dx = 0. \quad (2.15)
\]
Applying Lemma 2.1, we obtain
\[
\int_{\Omega} J \left( \nabla_x h \right) \cdot \nabla_x h dx \leq C \|\nabla_x h\|. \quad (2.16)
\]
Using the Poincaré inequality of the form $\|\nabla_x h\| \leq C \|D^2 h\| + C (\int_{\Omega} h dx)$, applying the elliptic estimates $\|D^2 h\| \leq C \|\Delta_x h\|$, and from (2.12), it then follows that
\[
\|\nabla_x h\| \leq C \|\Delta_x h\| + C.
\]
Therefore, (2.15) becomes
\[
\frac{1}{2} \frac{d}{dt} \|h\|^2 + v \|\Delta_x h\|^2 \leq \frac{v}{2} \|\Delta_x h\|^2 + C. \quad (2.17)
\]
From this, estimate (2.13) follows. And the proof of this lemma is complete. \(\square\)

**Lemma 2.3**

There holds
\[
\|h(t)\|_{H_{\Omega}^1}^2 + \int_0^t \|\nabla_x \Delta_x h(\tau)\|^2 d\tau \leq C. \quad (2.18)
\]

**Proof**

Integrating, with respect to $x$ over $\Omega$, Equation (2.2) multiplied by $-\Delta_x h$ and using integration by parts yield
\[
\frac{1}{2} \frac{d}{dt} \|\nabla_x h\|^2 + v \|\nabla_x \Delta_x h\|^2 - \int_{\Omega} J \left( \nabla_x h \right) \cdot \nabla_x \Delta_x h dx = 0. \quad (2.19)
\]
Making use of Lemma 2.1 again, we can easily prove that
\[
\left| \int_{\Omega} J \left( \nabla_x h \right) \cdot \nabla_x \Delta_x h dx \right| \leq C + \frac{v}{2} \|\nabla_x \Delta_x h\|^2. \quad (2.20)
\]
Thus, integrating (2.19) with respect to $t$, one obtains
\[
\frac{1}{2} \|\nabla_x h\|^2 + v \int_0^t \|\nabla_x \Delta_x h\|^2 d\tau \leq C + \frac{v}{2} \int_0^t \|\nabla_x \Delta_x h\|^2 d\tau, \quad (2.21)
\]
which implies the estimate (2.18). The proof of this lemma is complete. \(\square\)
2.3. Existence of solutions to the phase field model

In this section, we shall make use of the \textit{a priori} estimates established in the previous subsection to study the convergence of the solutions \( h^m \) of the approximate problem for \( m \to \infty \), thereby proving Theorem 1.2. In our investigation, we need the following well-known results, for instance, Lions [28] and Evans [29]:

We shall make use of the following lemma that is of Aubin–Lions type.

\textit{Lemma 2.4}

Let \( B_0, B_1 \) be Banach spaces which satisfy that \( B_0, B_1 \) are reflexive and that

\[ B_0 \subset \subset B \subset B_1. \]

Here, by \( \subset \subset \), we denote the compact imbedding. Define

\[ W = \left\{ f \mid f \in L^\infty(0, T; B_0), \quad \frac{df}{dt} \in L'(0, T; B_1) \right\} \]

with \( T \) being a given positive number and \( 1 < r < \infty \).

Then, the embedding of \( W \) in \( C([0, T]; B) \) is compact.

To deal with the nonlinear terms, we also need

\textit{Lemma 2.5}

Let \( \Gamma \) be an open set in \( \mathbb{R}^m \). Suppose functions \( f_n, f \) are in \( L^q(\Gamma) \) for any given \( 1 < q < \infty \), which satisfy

\[ \| f_n \|_{L^q(\Gamma)} \leq C, \quad f_n \to f \text{ almost everywhere in } \Gamma. \]

Then, \( f_n \) converges to \( f \) weakly in \( L^q(\Gamma) \).

We now turn to prove the existence of weak solutions for the initial function \( h_0 \) that is assumed in \( L^2(\Omega) \). The existence for the initial data in \( H^1(\Omega) \) is easy by recalling Lemma 2.3. Using Problems (2.2)–(2.4), we can construct smooth approximate solutions as follows: Let \( h^1 \) be a given function. By solving Problems (2.2)–(2.4) with \( h = h^1 \), one obtains \( h^2 \). Suppose we have obtained \( h^m \) for some \( m \in \mathbb{N} \), then set \( \varepsilon = 1/m \) and \( h = h^m \). Thus, we can define \( h^{m+1} \) successively. Therefore, we have a sequence of approximate solutions \( h^m \). Because equation (1.1) is nonlinear, we need some results about strong and pointwise convergence.

Define \( f = h^1 \) and \( r = 2 \). Set

\[ B_0 = H^2_N(\Omega), \quad B = H^1(\Omega), \quad B_1 = H^{-2}_N(\Omega), \]

it follows from Lemma 2.2 that

\[ \| h \|_{L^2(0, T; B_0)} \leq C, \quad \| h \|_{L^2(0, T; B_1)} \leq C. \]

Applying Lemma 2.4, we then conclude that \( \{ h^m \} \) is compact in \( C([0, T]; B) \), namely, \( C([0, T]; H^1(\Omega)) \), so \( \{ h^m \}, \) for \( i = 1, 2 \), is compact in \( C([0, T]; L^2(\Omega)) \). Therefore, there exists a subsequence, still denote it by \( \{ h^m \} \), such that

\[ \| h^m - h \|_{C([0, T]; H^1(\Omega))} \to 0, \quad \| h^m - h \|_{C([0, T]; L^2(\Omega))} \to 0 \]

as \( m \to \infty \). Moreover, we can select furthermore a subsequence such that \( h^m_k \) converges to \( h_k \) almost everywhere. Setting \( k = \frac{1}{m} \),

\[ S^k = \nabla h^m_k \]

By the properties of convolution and (2.22), we have

\[ \| \chi_k \ast S^k - S \|_{L^2(Q_T)} \leq \| \chi_k \ast (S^k - S) \|_{L^2(Q_T)} + \| (S - \chi_k \ast S) \|_{L^2(Q_T)} \]

\[ \leq \| (S - \chi_k \ast S) \|_{L^2(Q_T)} + \| S^k - S \|_{L^2(Q_T)} \to 0, \]

for \( k \to 0 \), whence we can select a subsequence, still denote by \( \chi_k \ast S^k \) converges to \( S \) almost everywhere.

Consequently, we assert that

\[ J(\nabla h^m) \text{ converges to } J(\nabla h) \]

almost everywhere as \( m \to \infty \). Remembering that \( \| J(\nabla h^m) \|_{L^2(Q_T)} \leq C \), using Lemma 2.5 we assert that

\[ J(\nabla h^m) \to J(\nabla h) \]

in \( L^2(Q_T) \) as \( m \to \infty \).

For the linear terms, by weak compactness, one can easily obtain

\[ \langle h^m_t, \phi \rangle \to \langle h_t, \phi \rangle, \quad (\Delta x h^m, \Delta x \psi \to \Delta x h, \Delta x \psi \rangle \]

as \( m \to \infty \), for all \( \psi \in L^2(0, T; H^2_N(\Omega)) \).
Taking the inner product of (2.2) and \( \psi \), we arrive at
\[
0 = \left( \frac{\partial h^n}{\partial t}, \psi \right) + v(\Delta_x h^n, \Delta_x \psi) - \left( J(\nabla_x \dot{h}^m), \nabla_x \psi \right) \\
- \left( \frac{\partial h^m}{\partial t}, \psi \right) + v(\Delta_x h, \Delta_x \psi) - \left( J(\nabla_x h), \nabla_x \psi \right).
\]

Thus, (1.16) is proved. From (2.12), (2.22), and the choice of the smooth initial data \( h_0 \) (let \( \varepsilon = 1/m \)), we have
\[
\int_\Omega h^n(t,x)dx = \int_\Omega h_0^n(x)dx \to \int_\Omega h_0(x)dx,
\]
and the left hand side converges to \( \int_\Omega h(t,x)dx \); thus, the mass is conserved for weak solution. And the existence of weak solutions is complete.

Next, we are going to study the

**Stability and Uniqueness.** Let \( h_1, \ h_2 \) be two weak solutions corresponding to initial data \( h_0^1 \) and \( h_0^2 \), respectively. Define \( u = h_1 - h_2 \). We write
\[
\nabla_x h = (p_i, q_i), \ (i = 1, 2), \ \nabla_x u = (p, q), \ J_i = J(\nabla_x h_i).
\]
Then, by the estimates in Lemma 2.1, we have \( |J_1 - J_2| \leq C|\nabla_x h_1 - \nabla_x h_2| = C|\nabla_x u| \), hence
\[
0 = \frac{1}{2} \frac{d}{dt} \| u \|^2 + v \| \Delta_x u \|^2 + \int_\Omega (J_1 - J_2) \cdot \nabla_x u dx \\
\geq \frac{1}{2} \frac{d}{dt} \| u \|^2 + v \| \Delta_x u \|^2 - C \| \nabla_x u \|^2.
\]

Making use of the Nirenberg inequality of the following form
\[
\| \nabla_x u \| \leq C \| \Delta_x u \|^\frac{1}{2} \| u \|^\frac{1}{2} + C' \| u \|
\]
and the Young inequality, from (2.28), one obtains
\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + v \| \Delta_x u \|^2 \leq C \left( \| \Delta_x u \| \| u \| + \| u \|^2 \right) \\
\leq \frac{v}{2} \| \Delta_x u \|^2 + C \| u \|^2.
\]

Now, using the Gronwall inequality, we obtain
\[
\| u(t) \|^2 \leq \| u(0) \|^2 e^{Ct}.
\]
Here, \( u(0) = h_0^1 - h_0^2 \). Thus, the solution depends continuously on the initial data.
Consequently, if \( h_0^1 = h_0^2 \), that is \( \| u(t) \|^2 = 0 \) which implies \( \| u(t) \|^2 = 0 \), so the weak solution is unique.
Therefore, the proof of Theorem 1.3 is complete.

### 3. Regularity of weak solutions

We shall investigate the regularity of weak solutions to both models with and without slope selection, whereas for the latter model, we can only carry out such study in one space dimension.

#### 3.1. The model with slope selection

Suppose now that \( h_0 \in H^{2m}(\Omega) \) with \( m \in \mathbb{N} \) and there exists a unique solution \( h \) to Problems (1.1)–(1.3). In this section, we shall investigate the regularities of this solution. We first consider the case that \( m = 1 \).

**Lemma 3.1**
There hold for \( h_0 \in H^2(\Omega) \) that
\[
\| \dot{h}(t) \|^2_{H^1(\Omega)} + \| h(t) \|^2_{L^2(0,T; L^2(\Omega))} \leq C, \tag{3.1}
\]
\[
\| \dot{h} \|^2_{L^2(0,T; H^1(\Omega))} \leq C. \tag{3.2}
\]
Proof

Multiplying Equation (1.1) by \( h_t \) and integrating the resulting equation with respect to \( x \) yield

\[
0 = \|h_t\|^2 + \frac{\nu}{2} \frac{d}{dt} \|\Delta_x h\|^2 + \int_\Omega J \cdot \nabla_x h_t \, dx
\]

(3.3)

Thus, one has

\[
\int_0^T \|h_t\|^2 \, dt \leq C.
\]

Recalling the definitions of \( E[h] \) and \( f \), we arrive at (3.1).

From Equation (1.1), making use of the estimates in Lemmas 2.1 and (3.1), we obtain

\[
\int_0^T \|\Delta^2_x h(t)\|^2 \, dt \leq C \int_0^T \|h_t(t)\|^2 \, dt + C \int_0^T \|D^2 h\|^2 \, dt,
\]

\[
\leq C.
\]

Thus, (3.2) is proved. And the proof of this lemma is complete.

To obtain the a priori estimates for higher order derivatives, we differentiate the equation with respect to \( t \) to obtain

\[
\frac{\partial h_t}{\partial t} + \nu \Delta^2_x h_t + \text{div}_x ((J \nabla_x h)_t) = 0.
\]

(3.4)

Such computations are formal. However, by using the technique of difference quotient, one can justify easily. In a similar way for deriving the estimates for \( h \), we arrive at

Lemma 3.2

Suppose that \( h_0 \in H^4(\Omega) \), that is, \( m = 2 \). There holds for any \( t \in [0, T] \) that

\[
\|h_t\|^2 + \|h_t\|_{H^4(\Omega)}^2 + \|h_t\|_{L^2(0,T;H^4(\Omega))}^2 \leq C.
\]

(3.5)

Proof

Multiplying Equation (3.4) by \( h_t \) and integrating the resulting equation with respect to \( x \) yield

\[
0 = \frac{1}{2} \frac{d}{dt} \|h_t\|^2 + \nu \|\Delta_x h_t\|^2 + \int_\Omega J_t \cdot \nabla_x h_t \, dx
\]

(3.6)

\[
\leq \frac{1}{2} \frac{d}{dt} \|h_t\|^2 + \nu \|\Delta_x h_t\|^2 - C \|\nabla_x h_t\|^2.
\]

With the help of the Nirenberg inequality (2.29), one obtains from (3.6) and the Young inequality that

\[
\frac{1}{2} \frac{d}{dt} \|h_t\|^2 + \nu \|\Delta_x h_t\|^2 \leq C \|\Delta_x h_t\| \|h_t\| + C \|h_t\|^2
\]

(3.7)

\[
\leq \frac{\nu}{2} \|\Delta_x h_t\|^2 + C \|h_t\|^2.
\]

From which, by the Gronwall inequality, it follows that

\[
\|h_t\|^2 + \nu \int_0^T \|\Delta_x h_t\|^2 \, dt \leq C.
\]

(3.8)

Furthermore, one can obtain from Equation (1.1) and Lemma 2.1 that

\[
\|\Delta^2_x h\|^2 \leq C \|h_t\|^2 + C \|\Delta^2_x h\|^2
\]

(3.9)

which implies that \( h \in L^\infty(0,T;H^4(\Omega)) \). The proof of this lemma is thus complete.
For the higher order derivatives with both \( x \) and \( t \), we have

**Lemma 3.3**

Let \( h_0 \in H^{2m}(\Omega) \). There hold for any \( t \in [0, T] \) that

\[
\| h(t) \|_{H^{2m}(\Omega)}^2 + \| h \|_{L^2(0,T;H^{2m+2}(\Omega))}^2 + \| h_t \|_{L^2(0,T;H^{\ell-2}(\Omega))}^2 \leq C, \tag{3.10}
\]

and

\[
\| D^\ell h \|_{L^2(0,T;H^{\ell}(\Omega))}^2 + \| D^\ell h \|_{L^2(0,T;H^{\ell}(\Omega))}^2 \leq C, \quad \text{if } m = 2\ell, \tag{3.11}
\]

\[
\| D^\ell h \|_{L^2(0,T;H^{\ell}(\Omega))}^2 + \| D^\ell h \|_{L^2(0,T;H^{\ell}(\Omega))}^2 \leq C, \quad \text{if } m = 2\ell - 1. \tag{3.12}
\]

**Proof**

We employ the mathematical induction. From Lemma 3.1, it is easy to see that (3.10) and (3.12) are true for \( m = 1 \) that implies \( \ell = 1 \) too. By Lemma 3.2, estimate (3.11) holds when \( m = 2 \). Assume that (3.10) is true for any \( k \leq m \in \mathbb{N} \) and (3.11) and (3.12) are true, respectively, for even and odd \( m \). Next, we shall prove that they are true for \( k \leq m + 1 \) when \( h_0 \in H^{2(m+1)}(\Omega) \).

For the case that \( m + 1 \) is even (resp. odd), differentiating \( m + 1/2 \) (resp. \( m/2 \)) times Equation (1.1) with respect to \( t \), letting \( v = D_m^{m+1/2} h \) (resp. \( v = D_m^{m/2} h \)), repeating the argument of Lemma 3.2 (resp. Lemma 3.1) for this function \( v \), and using the estimates in Lemma 2.1, we then conclude that (3.10) holds for \( m + 1 \); moreover, (3.11) (resp. (3.12)) is true for \( \ell = m + 1/2 \) (resp. \( \ell = m/2 \)). Thus, the proof of this lemma is complete. \( \square \)

### 3.2. The one-dimensional cubic model without slope selection

In this subsection, we shall study the one-dimensional problem with a cubic current, that is, (1.9), or (1.10)–(1.11), which is studied in [9, 10, 21]. However, the original two-dimensional problem is still open. The problem is

\[
h_t + \nu h_{xxxx} + (J(h_x))_x = 0 \quad \text{in } Q_T, \tag{3.13}
\]

\[
h_x = 0, \quad h_{xx} = 0 \quad \text{on } [0, T] \times \partial \Omega, \tag{3.14}
\]

\[
h|_{t=0} = h_0. \tag{3.15}
\]

Here, \( \Omega = (a, b) \subset \mathbb{R}, \ a, b \in \mathbb{R} \) and \( J = \alpha h_x(1 - h_x^2) \).

Assume that \( h_0 \in L^2(\Omega) \). Multiplying (3.13) by \( h \) and integrating it with respect to \( x \) give

\[
0 = \frac{1}{2} \frac{d}{dt} \| h \|_2^2 + \nu \| h_{xx} \|_2^2 - \int_\Omega J(h_x)h_x dx
\]

\[
= \frac{1}{2} \frac{d}{dt} \| h \|_2^2 + \nu \| h_{xx} \|_2^2 + \frac{1}{2} \| h_x \|_4^4 - \| h_x \|_2^2. \tag{3.16}
\]

By the Young inequality \( \alpha^2 \leq \frac{1}{2} \alpha^4 + \frac{1}{2} \), we infer from (3.16) that

\[
\frac{1}{2} \frac{d}{dt} \| h \|_2^2 + \nu \| h_{xx} \|_2^2 + \frac{1}{2} \int_\Omega |h_x|_4^4 dx \leq \frac{1}{2} \int_\Omega |h_x|_4^4 dx + \frac{1}{2}, \tag{3.17}
\]

which gives

**Lemma 3.4**

We have

\[
\| h \|_2^2 + \int_0^T \left( \| h(t) \|_{H^2(\Omega)}^2 + \frac{1}{2} \int_\Omega |h_x|_4^4 dx \right) dt \leq C. \tag{3.18}
\]

On the basis of this lemma, we can define weak solutions and prove the existence and uniqueness in a similar way as in Section 2 for \( h_0 \in L^2(\Omega) \). Suppose that \( h_0 \in H^1(\Omega) \), is the weak solution regular, that is, \( h \in L^2(0,T;H^1(\Omega)) \)? To answer this question, we need more estimates. From Equation (3.13), we obtain

\[
0 = \frac{1}{2} \frac{d}{dt} \| h_x \|_2^2 + \nu \| h_{xxx} \|_2^2 - \int_\Omega J(h_x)h_{xx} dx, \tag{3.19}
\]

\[
= \frac{1}{2} \frac{d}{dt} \| h_x \|_2^2 + \nu \| h_{xxx} \|_2^2 + \frac{3}{2} \int_\Omega |h_x|^2 |h_{xx}|^2 dx - \| h_{xx} \|_2^2. \tag{3.20}
\]

Therefore,

\[
\| h_x \|_2^2 + \int_0^T \left( \nu \| h_{xxx} \|_2^2 + \frac{3}{2} \int_\Omega |h_x|^2 |h_{xx}|^2 dx \right) dt \leq C.
\]

So, the weak solution is more regular if \( h_0 \in H^1(\Omega) \).

However, for the two-dimensional problem, we cannot obtain (3.19) so that we obtain the good term \( \int_\Omega 3|h_x|^2 |h_{xx}|^2 dx \) in (3.20).
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