Optimal Repair/Access MDS Array Codes with
Multiple Repair Degrees

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Abstract

In the literature, most of the known high-rate \((n, k)\) MDS array codes with the optimal repair property only support a single repair degree (i.e., the number of helper nodes contacted during a repair process) \(d\), where \(k \leq d \leq n - 1\). However, in practical storage systems, the number of available nodes changes frequently. Thus, it is preferred to construct \((n, k)\) MDS array codes with multiple repair degrees and the optimal repair property for all nodes. To the best of our knowledge, only two MDS array codes have such properties in the literature, which were proposed by Ye and Barg (IEEE Trans. Inform. Theory, 63(10), 2001-2014, 2017). However, their sub-packetization levels are relatively large. In this paper, we present a generic construction method that can convert some MDS array codes with a single repair degree into the ones with multiple repair degrees and optimal repair property for a set of nodes, while the repair efficiency/degrees of the remaining nodes can be kept. As an application of the generic construction method, an explicit construction of high-rate MDS array code with multiple repair degrees and the optimal access property for all nodes is obtained over a small finite field. Especially, the sub-packetization level is much smaller than that of the two codes proposed by Ye and Barg concerning the same parameters \(n\) and \(k\).

Index Terms

Distributed storage, high-rate, MDS array codes, sub-packetization, optimal repair, repair degree.

I. INTRODUCTION

Distributed storage systems, such as those run by Hadoop, Google Colossus, Microsoft Azure, OceanStore, Total Recall, and DHash++, are widely used in not only large-scale data centers but also peer-to-peer storage settings. Currently, deployed distributed storage systems are formed of thousands of individual nodes, where the node failures are normal. Therefore, in order to ensure reliability, a certain amount of redundant data should be stored in the distributed storage system as well. Conventionally, distributed storage systems use replications to produce redundant data, such as HDFS. However, due to the large storage consumption of exact replicas, there is a trend for distributed storage systems to migrate from replications to erasure codes. Compared with the former, erasure codes can offer higher reliability at the same redundancy level and thus have been extensively deployed in distributed storage systems.

Among families of erasure codes, maximum distance separable (MDS) codes provide optimal trade-off between fault-tolerance and storage overhead. By distributing the codeword across distinct storage nodes, in the case of node failures, the missing data can be recovered from the data at some surviving nodes, which are named helper nodes. During the repair process, efficient operation of the system requires minimizing the repair bandwidth, which is defined as the amount of data downloaded to repair a failed node.

It was proved in [4] that for an \((n, k)\) MDS code with code length \(n\) and dimension \(k\), the recovery of a single failed node from \(d\) helper nodes should download at least a fraction \(\frac{d}{d - k + 1}\) of the data stored in each of the helper nodes, i.e., the repair bandwidth \(\gamma(d)\) satisfies

\[
\gamma(d) \geq \frac{d}{d - k + 1} N, \tag{1}
\]

where \(d \in [k : n]\) and \(N\) are called the repair degree and sub-packetization level, respectively. Particularly, the code is referred to as an array code if \(N > 1\). In the literature, most existing MDS codes are designed as a kind of array codes to achieve the lower bound in (1). In this paper, we also focus on MDS array codes.

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For $d \in [k : n)$, rewrite $d = k + \theta - 1$, where $2 \leq \theta \leq n - k$. For an $(n, k)$ MDS array code, if the repair bandwidth attains the lower bound in (1) when repairing a failed node by connecting $d$ helper nodes, we say that the code has the $\theta$-optimal repair property for this node. More generally, given any $m$ ($m \geq 2$) positive integers $\delta_0, \delta_1, \cdots, \delta_{m-1}$ with $2 \leq \delta_0 < \delta_1 < \cdots < \delta_{m-1} \leq r = n - k$, if a node has the $\delta_z$-optimal repair property for all $0 \leq z < m$, we say that this node has the $\delta_{(0:m)}$-optimal repair property, where $\delta_{(0:m)} = \{\delta_0, \delta_1, \cdots, \delta_{m-1}\}$. Besides the repair bandwidth, some other metrics also need to be optimized in practice. In general, during the process of repairing a failed node, a symbol downloaded from one helper node can be a linear combination of several symbols at this node, and the amount of data accessed can be more than that transmitted. When repairing a failed node by connecting $d = k + \theta - 1$ helper nodes, if the amount of data accessed from the helper nodes also meets the lower bound in (1), we say that the $(n, k)$ MDS array code has the $\theta$-optimal access property for this node. Similarly, we also say one node has the $\delta_{(0:m)}$-optimal access property if it has $\delta_z$-optimal access property for all $0 \leq z < m$. Actually, the optimal access property implies the optimal repair property, but not vice versa. In this sense, the optimal access property can be viewed as an enhanced property of the optimal repair property.

Up to now, for $k > n/2$ (i.e., the high-rate regime), some explicit constructions of MDS array codes which support a single repair degree and with the $\theta$-optimal repair property have been proposed, where $2 \leq \theta \leq r$. Among them, most constructions are limited to the case of $\theta = r$, i.e., repairing a failed node requires connecting all the $n - 1$ surviving nodes, where some of the notable works are [12–14], [15]. Only a few known explicit constructions of MDS array codes support $d < n - 1$ (i.e., $\theta < r$) [17], [20], [23]. However, they either have a large sub-packetization level (e.g., the two codes proposed in [23]) or have restrictions on the choices of the parameter $d$ or equivalently $\theta$ (e.g., the MDS array codes proposed in [17], [20]), where the two MDS array codes proposed in [23] are respectively called VBK code 1 and VBK code 2. Particularly, in this paper, the MDS array code with the optimal access property and optimal sub-packetization level proposed in [20] is called VBK code. More recently, in [11], Liu et al. proposed an explicit construction of high-rate $(n, k)$ MDS array code with the $\theta$-optimal access property for all nodes, where the sub-packetization level is $\theta/2\lceil z \rceil$, which is between that of the YB codes 1, 2 [23] and the ones proposed in [17], [20].

Constructions of high-rate MDS array codes with multiple repair degrees were first proposed by Ye and Barg [23] in 2017, where two $(n, k)$ MDS array codes with all nodes having $\delta_{(0:m)}$-optimal repair property for any subset $\delta_{(0:m)}$ of $[2 : r]$ were proposed. Both codes have sub-packetization levels $\delta^0$, where

$$\delta = \text{lcm}(\delta_0, \delta_1, \cdots, \delta_{m-1}).$$

Specifically, the parity-check matrices of the two MDS array codes are based on diagonal matrices and permutation matrices. For convenience, we refer to the one based on diagonal matrices as YB code 3 and the other one as YB code 4. To the best of our knowledge, YB codes 3 and 4 are the only two known high-rate MDS array codes with the $\delta_{(0:m)}$-optimal repair property for all nodes in the literature. However, their sub-packetization levels are relatively large.

In this paper, we aim to construct high-rate $(n, k)$ MDS array codes that have $\delta_{(0:m)}$-optimal repair property for all nodes and a lower sub-packetization level. By this motivation, we provide a generic construction method that can convert some known MDS array codes with the $\delta_0$-optimal repair property into another MDS array code, which makes a set of nodes possessing the $\delta_{(0:m)}$-optimal repair property, and simultaneously preserves the optimal repair/access property for the remaining nodes. By applying this generic construction method multiple times, an algorithm is proposed that can construct $(n, k)$ MDS array codes with the $\delta_{(0:m)}$-optimal repair property for all nodes from a class of special $(n, k)$ MDS array codes with the $\delta_0$-optimal repair property for all nodes. As application of the algorithm to VBK code in [20], we obtain an explicit high-rate $(n, k)$ MDS array code $G$ which has the $\delta_{(0:m)}$-optimal access property for all nodes. Specifically, the new code $G$ has a sub-packetization level $\delta^{[0:m]}$ for $\delta_0 = 2, 3, 4$, which is much smaller than that of YB codes 3 and 4, where $\delta$ is defined in (2). When $\delta_0 > 4$, consider the new code $G$ with $\{(4) \cup \delta_{(0:m)}\}$-optimal access property for all nodes, it not only has a smaller sub-packetization level $(\text{lcm}(4, \delta))^{[0:m]}$ than that of YB codes 3 and 4, but also can support one more repair degree than YB codes 3 and 4.

The remainder of the paper is organized as follows. Section II reviews some necessary preliminaries. Section III proposes the generic construction method and its asserted properties. Section IV gives the algorithm of this method and a new explicit construction of high-rate MDS array code which is obtained by means of this algorithm. Section V gives comparisons of key parameters among the MDS array code proposed in this paper and YB codes 3, 4. Finally, Section VI concludes this paper.
II. Preliminaries

In this section, we introduce the MDS property and optimal repair property of MDS array codes, and a series of special partitions for a given standard basis set.

A. Structure of MDS Array Codes

Let \( F_q \) be a finite field with \( q \) elements where \( q \) is a prime power. For two non-negative integers \( a \) and \( b \) with \( a < b \), define \([a : b]\) and \([a : b]\) as two ordered sets \({a, a+1, \cdots, b-1}\) and \({a, a+1, \cdots, b}\), respectively. An \((n, k)\) array code encodes a file of size \( M = kN \) into \( n \) fragments \( f_0, f_1, \cdots, f_{n-1} \), which are stored across \( n \) nodes, respectively, where \( f_i = (f_{i,0}, f_{i,1}, \cdots, f_{i,N-1})^\top \) is a column vector of length \( N \) over \( F_q \), and \( \top \) denotes the transpose operator.

In this paper, \((n, k)\) array codes are assumed to be defined in the following parity-check form:

\[
\begin{pmatrix}
A_{0,0} & A_{0,1} & \cdots & A_{0,n-1} \\
A_{1,0} & A_{1,1} & \cdots & A_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r-1,0} & A_{r-1,1} & \cdots & A_{r-1,n-1}
\end{pmatrix}
\begin{pmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{n-1}
\end{pmatrix}
= 0_{rN}
\]  

(3)

where \( r = n - k \). Throughout this paper, \( 0_N \) (resp. \( 0_{N \times M} \)) denotes the zero column of length \( N \) (resp. matrix of order \( N \times M \)), and will be abbreviated as \( 0 \) in the sequel if the length (resp. order) is clear. In (3), the \( rN \times rN \) matrix \( A \) is called the parity-check matrix of the code, which can be simplified as

\[
A = (A_{t,i})_{t \in [0:r), i \in [0:n)}
\]

to indicate the block entries. Note that for each \( t \in [0:r) \), \( \sum_{i=0}^{n-1} A_{t,i} f_i = 0 \) contains \( N \) equations, for convenience, we say that \( \sum_{i=0}^{n-1} A_{t,i} f_i = 0 \) is the \( t \)-th parity-check group (PCG), where \( A_{t,i} \) is an \( N \times N \) matrix over \( F_q \).

An \((n, k)\) code is said to have the MDS property if the original file can be reconstructed by connecting any \( k \) out of the \( n \) nodes, i.e., the data stored in any set of \( r \) nodes can be obtained by the remaining \( k \) nodes.

Lemma 1 (\([23]\)). An \((n, k)\) array code defined by (3) has the MDS property if and only if the block matrix

\[
\begin{pmatrix}
A_{0,i_0} & A_{0,i_1} & \cdots & A_{0,i_r-1} \\
A_{1,i_0} & A_{1,i_1} & \cdots & A_{1,i_r-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r-1,i_0} & A_{r-1,i_1} & \cdots & A_{r-1,i_r-1}
\end{pmatrix}
\]

of order \( rN \) is nonsingular for any \( r \)-subset \( \{i_0, i_1, \cdots, i_r-1\} \subset [0:n) \).

B. Optimal Repair Property

An \((n, k)\) MDS code with the \( \theta \)-optimal repair property is preferred, i.e., any failed node can be repaired by downloading \( \frac{N}{\theta} \) symbols from each of the \( d = k + \theta - 1 \) helper nodes. In this paper, similarly to that in \([11]\), when repairing a failed node \( i \in [0:n) \), the \( \frac{N}{\theta} \) symbols downloaded from each helper node \( j \in \mathcal{H} \) is represented by \( R_{i,j} f_j \), where \( \mathcal{H} \) denotes the indices set of the \( d \) helper nodes and the \( \frac{N}{\theta} \times N \) matrix \( R_{i,\mathcal{H}} \) of full rank is called the \( \theta \)-repair matrix of node \( i \). In addition, the code is preferred to have the \( \theta \)-optimal access property, i.e., when repairing a failed node \( i \in [0:n) \), the amount of accessed data attains the lower bound in \([1]\). Clearly, node \( i \) has the \( \theta \)-optimal access property if the \( \theta \)-repair matrix \( R_{i,\mathcal{H}} \) satisfies that each row has only one nonzero element.

Obviously, some linear independent equations should be chosen out of those \( rN \) parity-check equations in (3) to regenerating a failed node \( i \in [0:n) \). Precisely, for any \( t \in [0:r) \), we get \( \frac{N}{\theta} \) linear independent equations from the \( t \)-th PCG of (3) by
That is, there exists an \( \text{Lemma 2.} \)

\[
\begin{align*}
\text{interference by } f_j = 0,
\end{align*}
\]

where \( \mathcal{D} = [0 : n] \setminus (\mathcal{H} \cup \{i\}) \) is the index set of the \( r - \theta \) nodes which are not connected, particularly \( \mathcal{D} = \emptyset \) if \( \theta = r \).

Therefore, the optimal repair property indicates that the interference terms caused by \( f_j \) have to be cancelled by the downloaded data \( R_{i,\theta} \) from node \( j \in \mathcal{H} \), i.e.,

\[
\text{Rank} \left( \begin{pmatrix} R_{i,\theta} & S_{i,\theta}A_{0,j} & \vdots & S_{i,\theta}A_{r-1,j} \end{pmatrix} \right) = \frac{N}{\theta}
\]

for all \( j \in \mathcal{H} \) and further for all \( j \in [0 : n] \setminus \{i\} \) since \( \mathcal{H} \) is an arbitrary \( d \)-subset of \( [0 : n] \), which means that

\[
\text{Rank} \left( \begin{pmatrix} R_{i,\theta} & S_{i,\theta}A_{0,j} & \vdots & S_{i,\theta}A_{r-1,j} \end{pmatrix} \right) = \frac{N}{\theta} \quad \text{for all } j \in [0 : n] \setminus \{i\} \text{ and } t \in [0 : r).
\]

That is, there exists an \( \frac{N}{\theta} \times \frac{N}{\theta} \) matrix \( \tilde{A}_{i,j,i,\theta} \) such that

\[
S_{i,\theta}A_{t,j} = \tilde{A}_{i,j,i,\theta}R_{i,\theta} \quad \text{for } j \in [0 : n] \setminus \{i\} \text{ and } t \in [0 : r).
\]

Let \( \mathcal{D} = \{ j_0, j_1, \cdots, j_{r-\theta-1} \} \), by substituting (5) into (4), together with the data \( R_{i,\theta} \) downloaded from each helper node \( j \in \mathcal{H} \), (4) can be rewritten as

\[
\begin{pmatrix}
S_{i,\theta}A_{0,i} & \tilde{A}_{0,j_0,i,\theta} & \cdots & \tilde{A}_{0,j_{r-\theta-1},i,\theta} \\
S_{i,\theta}A_{1,i} & \tilde{A}_{1,j_0,i,\theta} & \cdots & \tilde{A}_{1,j_{r-\theta-1},i,\theta} \\
\vdots & \vdots & \ddots & \vdots \\
S_{i,\theta}A_{r-1,i} & \tilde{A}_{r-1,j_0,i,\theta} & \cdots & \tilde{A}_{r-1,j_{r-\theta-1},i,\theta}
\end{pmatrix}
\begin{pmatrix}
f_i \\
R_{i,\theta}f_{j_0} \\
\vdots \\
R_{i,\theta}f_{j_{r-\theta-1}}
\end{pmatrix}
= -\sum_{j \in \mathcal{H}}
\begin{pmatrix}
\tilde{A}_{0,j,i,\theta} \\
\tilde{A}_{1,j,i,\theta} \\
\vdots \\
\tilde{A}_{r-1,j,i,\theta}
\end{pmatrix}
\begin{pmatrix}
R_{i,\theta}f_{j_0} \\
R_{i,\theta}f_{j_1} \\
\vdots \\
R_{i,\theta}f_{j_{r-\theta-1}}
\end{pmatrix}
\]

where the term on the right hand side (RHS) of (6) is determined by the downloaded data. It is clear that there are \( N + \frac{N}{\theta}|\mathcal{D}| = N + \frac{N}{\theta}(r - \theta) = \frac{rN}{\theta} \) unknown variables with \( \frac{rN}{\theta} \) equations in (5). Then we have the following result.

**Lemma 2.** For given \( \theta \in [2 : r] \) and \( i \in [0 : n] \), if node \( i \) has the \( \theta \)-optimal repair property, then the \( \frac{rN}{\theta} \times \frac{rN}{\theta} \) coefficient matrix of (6) is nonsingular for any \( (r - \theta) \)-subset \( \{ j_0, j_1, \cdots, j_{r-\theta-1} \} \subset [0 : n] \setminus \{i\} \).

**C. Standard Basis Sets**

For any two positive integer \( s, w \geq 2 \), let \( \{e_0, e_1, \cdots, e_{s-1}\} \) be the standard basis of \( \mathbb{F}_q^s \), i.e.,

\[
e_{a} = (0, \cdots, 0, 1, 0, \cdots, 0), \quad a \in [0 : s],
\]

(7)

with only the \( a \)-th entry being nonzero.

Given \( a \in [0 : s] \), denote \( (a_{w-1}, a_{w-2}, \cdots, a_0) \) as its \( s \)-ary expansion, i.e.,

\[
a = a_{w-1}s^{w-1} + a_{w-2}s^{w-2} + \cdots + a_0,
\]

(8)

where \( a_i \) is the \( i \)-th element in the \( s \)-ary expansion of \( a \). Throughout this paper, we do not distinguish the integer \( a \) and its \( s \)-ary expansion if the context is clear.

Based on (8), we further define some subsets of the standard basis set \( \{e_0, e_1, \cdots, e_{s-1}\} \) as

\[
V_{i,u} = \{e_a | a_i = u, a \in [0 : s] \}, \quad 0 \leq i < w, u \in [0 : s],
\]

(9)
Example 1. When \( w = 3 \), \( s = 2 \), Table I gives \( V_{i,u} \), \( 0 \leq i < 3 \) and \( 0 \leq u < 2 \).

For easy of notation, we also denote by \( V_{i,u} \) the \( s^{w-1} \times s^w \) matrix, whose rows are formed by vectors \( e_a \) in its corresponding sets, such that \( a \) is sorted in ascending order. For example, when \( s = 2 \) and \( w = 3 \), \( V_{1,0} \) can be viewed as a \( 4 \times 8 \) matrix

\[
V_{1,0} = (e_0^T \ e_1^T \ e_4^T \ e_5^T)^T.
\]

### D. Notations

Throughout this paper, the following notations are used.

- For a matrix \( Q \), define \( Q(u,:), Q(:,v) \) as its \( u \)-th row vector, its \( v \)-th column vector and the entry in row \( u \) and column \( v \).
- For a matrix \( Q \), define \( \text{blkdiag}(Q, Q, \cdots, Q)_t \) as a block diagonal matrix with \( Q \) occurring \( t \) times.
- The symbols \( % \) denotes the modulo operation.
- For any positive \( a \), denote \( I_a \) the identity matrix of order \( a \).

## III. A GENERIC CONSTRUCTION METHOD

In this section, we propose a method that can transform an \((n,k)\) MDS array code with the \( \delta_0 \)-optimal repair property for all nodes into a new \((n,k)\) MDS array code with the \( \delta_0(0,m) \)-optimal repair property for a set of \( \rho \) \((1 \leq \rho \leq \delta_0)\) goal nodes (GNs), where these \( \rho \) GNs are required to satisfy some specific conditions and \( \delta_0(0,m) \subseteq \{2 : r\} \), while keeping the repair property of the other \( n - \rho \) remainder nodes (RNs) intact. Specifically, given an \((n,k)\) base code, let \( G \) be the set of indices of the \( \rho \) GNs which we wish to endow with the \( \delta_0(0,m) \)-optimal repair property.

### A. The Generic Construction Method

In this subsection, we propose the generic construction method, which utilizes a known \((n,k)\) MDS array code \( C_0 \) with sub-packetization level \( \alpha N \) over \( F_q \) and \( \delta_0 \)-optimal repair property as the base code, where \( \alpha \geq 1 \), \( \delta_0 \models N \). Let \( (A_i)_{r \in \{0:r\}, i \in \{0:n\}} \) be the parity-check matrix of base code \( C_0 \) while the \( \alpha N \times \alpha N \) matrices \( R_{i,\delta_2} \) and \( S_{i,\delta_2} \), \( i \in \{0 : n\} \) respectively denote the \( \delta_2 \)-repair matrix and \( \delta_2 \)-select matrix of base code \( C_0 \) if it also has \( \delta_2 \)-optimal repair property for some \( \delta_2 \) with \( z \geq 1 \). For convenience, throughout this paper, we always set \( N' = \frac{N}{\delta_0} \).

The following example shows an MDS array code that possesses the \( \delta_0 \)-optimal repair property and will be chosen as the base code throughout the examples of this paper.

**Example 2.** The \((16,10)\) MDS array code in [20] has sub-packetization level \( N = 2^6 \) and \( \delta_0 \)-optimal repair property for all nodes, and also satisfies some other specific properties which will be illustrated later, where \( \delta_0 = 2 \). It can be chosen as the base code \( C_0 \), the \( \delta_0 \)-repair matrix \( R_{i,\delta_0} \) and \( \delta_0 \)-select matrix \( S_{i,\delta_0} \) are defined by

\[
R_{i,\delta_0} = S_{i,\delta_0} = V_{\lfloor \frac{i}{2} \rfloor , i \% 2}, 0 \leq i < 16,
\]

where \( V_{j,0}, V_{j,1}, (0 \leq j < 8) \) are given in (9).

Define

\[
l_z = \begin{cases} 
\frac{\delta}{2^l}, & \text{if } 0 \leq z < m, \\
0, & \text{if } z = m,
\end{cases}
\]

where \( \delta \) is defined in (2).
The generic construction method is then carried out through the following two steps.

**Step 1: An intermediate MDS array code $C_1$ by space sharing $l_0$ instances of code $C_0$.**

Construct an intermediate MDS array code $C_1$ with sub-packetization level $l_0 \alpha N$ by space sharing $l_0$ instances of the base code $C_0$. Specifically, for each instance $a \in [0 : l_0)$, the $t$-th PCG is of the form

$$\sum_{i \in \mathcal{G}} A_{t,i} f_{t}^{(a)} + \sum_{i \in [0 : n) \setminus (\mathcal{G} \cup \mathcal{R})} A_{t,i} f_{t}^{(a)} + \sum_{j \in \mathcal{R}} A_{t,j,i} f_{t}^{(a)} = 0, \quad t \in [0 : r),$$

where $\mathcal{R}$ denotes any given $r$-subset of $[0 : n) \setminus \mathcal{G}$, $f_{t}^{(a)}$ and $f_{t}^{(a)}$ respectively denote the data stored at nodes $i$ and $j$ of an instance of the code $C_0$ for $i \in [0 : n) \setminus \mathcal{R}$, $j \in \mathcal{R}$, and $a \in [0 : l_0)$.

**Step 2: Construct code $C_2$ by appending some data of each goal node to the PCGs of $C_1$.**

Based on code $C_1$, we construct the desired storage code $C_2$ by appending the data $P_{t,i}^{(a)} (i \in \mathcal{G})$ called appended-data to the $t$-th PCG of instance $a$ of $C_1$, which leads to new parity-check equations and means that the data stored at some $r$ nodes will be modified. By convention, we assume that the data $f_{t}^{(a)}$ stored at node $j \in \mathcal{R}$ of instance $a \in [0 : l_0)$ is changed to $f_{t}^{(a)}$ and the data stored at the other nodes is unchanged. Then the $t$-th PCG of instance $a$ of new code $C_2$ is given by

$$\sum_{i \in \mathcal{G}} (A_{t,i} f_{t,i}^{(a)} + P_{t,i}^{(a)}) + \sum_{j \in [0 : n) \setminus (\mathcal{G} \cup \mathcal{R})} A_{t,j,i} f_{t}^{(a)} + \sum_{j \in \mathcal{R}} A_{t,j,i} f_{t}^{(a)} = 0, \quad a \in [0 : l_0), t \in [0 : r),$$

where

P0. The appended-data $P_{t,i}^{(a)}$ is to be designed as a linear combination of $f_{t}^{(a)}, f_{t}^{(a+1)}, \ldots, f_{t}^{(a+k-1)}$ if $l_z \leq a < l_z$ with $z \in [1 : m]$ and $P_{t,i}^{(a)} = 0$ if $l_z \leq a < l_z$ for $t \in [0 : r), i \in \mathcal{G}$.

Obviously, the new code $C_2$ maintains the MDS property of base code $C_0$.

**Theorem 1.** The new $(n, k)$ code $C_2$ maintains the MDS property of code $C_0$.

**Proof.** The new code $C_2$ possesses the MDS property if the data stored in any $r$ out of $n$ nodes can be obtained by the remaining $k$ nodes. Let $l_0, l_1, \ldots, l_{r-1}$ be the indices of those $r$ nodes. For any $a \in [0 : l_0)$, we can obtain

$$\sum_{v=0}^{r-1} A_{t,v} f_{t,v}^{(a)} + \sum_{v=0, i \in \mathcal{G}} P_{t,i,v}^{(a)} = - \sum_{v=0}^{k-1} A_{t,v} f_{t,v}^{(a)} - \sum_{v=0, j \in \mathcal{G}} P_{t,j,v}^{(a)}, \quad t \in [0 : r),$$

from (11), where $\{j_0, j_1, \ldots, j_{k-1} \} = [0 : n) \setminus \{i_0, i_1, \ldots, i_{r-1}\}$ and the two terms on RHS of (12) are determined by the data stored at the remaining $k$ nodes. We prove the MDS property by induction in the following.

i) According to P0, $P_{t,j,v}^{(a)} = 0$ for $a \in [l_0 : l_1)$, $j \in \mathcal{G}$, thus we can directly obtain $f_{t,v}^{(a)}, v \in [0 : r), a \in [l_0 : l_1)$ from (12) by means of the MDS property of the code $C_0$.

ii) Suppose that the data $f_{t,v}^{(a)}, v \in [0 : r)$, $a \in [l_0 : l_1)$ have been obtained for some $z \in [1 : m)$, then for $a \in [l_z : l_0)$, $i \in \{i_0, i_1, \ldots, i_{r-1}\} \cap \mathcal{G}$, we can compute $P_{t,i,v}^{(a)} (a \in [l_{z+1} : l_z))$ from $f_{t,v}^{(a)}(z), f_{t,v}^{(a+1)}(z), \ldots, f_{t,v}^{(a+1)}(l_z)$ according to P0. That is, the second term on left hand side (LHS) of (12) is known, then we are able to solve $f_{t,v}^{(a)}, a \in [l_{z+1} : l_z), v \in [0 : r)$ by means of the MDS property of the code $C_0$.

By i) and ii), we thus can reconstruct $f_{i_0,v}^{(a)}, f_{i_1,v}^{(a)}, \ldots, f_{i_{r-1}}^{(a)}$ for all $a \in [0 : l_0)$, i.e., the data stored at the $r$ nodes.

**B. The Precise Form of Appended-data**

In this subsection, we first introduce two sets and analyze their properties, by which we further give the precise form of the appended-data $P_{t,j}^{(a)}$.

For a given $u \in [0 : \delta_0)$, define an $\alpha N' \times \alpha N$ matrix as

$$\Phi_{u} = \text{blkdiag}(\Delta_{u}, \Delta_{u}, \ldots, \Delta_{u})_{\alpha},$$

where $\Delta_{u}$ is an $N' \times N$ matrix defined by

$$\Delta_{u} = (0_{N' \times N'}, \ldots, 0_{N' \times N'}, I_{N'}, 0_{N' \times N'}, \ldots, 0_{N' \times N'})_{u \in [0 : \delta_0)}$$

with only the $u$-th block entry being nonzero matrix.
For any column vector $f^a_i$ of length $\alpha N$, we divide it into $\delta_0$ equal parts $f^a_i[0], f^a_i[1], \ldots, f^a_i[\delta_0 - 1]$, i.e.,

$$f^a_i[u] = \Phi_{\alpha_a}(a_i)$$ for $u \in [0 : \delta_0), i \in [0 : n]$ and $a \in [0 : l_0]$, \hspace{1cm} (15)$$

where $f^a_i[u]$ is a column vector of length $\alpha N'$. 

For any two column vectors $f^a_i[u]$ and $f^b_i[v]$, we say that $f^a_i[u] \prec f^b_i[v]$ if $a < b$ or $a = b$ and $u < v$, where $i \in [0 : n]$, $a, b \in [0 : l_0]$ and $u, v \in [0 : \delta_0]$. For any $i \in G$ and $j \in [1 : m]$, define $P_{i,j}$ as an ordered set with the set elements drawing from $f^a_i[u], u \in [0 : \delta_0), a \in [0 : l_0]$ and placed in ascending order w.r.t. $\prec$, which are generated through the following Algorithm 1.

**Algorithm 1** The way to generate set $P_{i,j}$, $i \in G$ and $j \in [1 : m]$

**Output:** $P_{i,j}$, $i \in G, j \in [1 : m]$, whose elements are column vectors of length $\alpha N'$.

1: for $j = 1; j < m; j + +$ do  
2: Set $P_{i,j} = \{f^a_i[u]a \in [l_j : l_{j-1}], u \in [0 : \delta_0]\}$;  
3: end for  
4: Dividing $P_{i,1}$ into $l_1$ disjoint subsets $P^{(0)}_{i,1}, P^{(1)}_{i,1}, \ldots, P^{(l_1-1)}_{i,1}$ of equal size.  
5: for $j = 2; j < m; j + +$ do  
6: $P_{i,j} := P_{i,j} \cup \bigcup_{a=l_j}^{l_{j-1}} (P^{(a)}_{i,1} \cup \ldots \cup P^{(a)}_{i,j-1})$;  
7: Dividing $P_{i,j}$ into $l_j$ disjoint subsets $P^{(0)}_{i,j}, P^{(1)}_{i,j}, \ldots, P^{(l_j-1)}_{i,j}$ of equal size.  
8: end for

Strictly speaking, to ensure that Algorithm 1 is valid, one needs $|P_{i,j}| = (\delta_j - \delta_j-1)l_j$ for $i \in G$ and $j \in [1 : m]$, which will be shown in P3.

**Example 3.** Based on the base code in Example 2, suppose the goal is to obtain a (16, 10) MDS array code $C_2$ having \{2, 3\}-optimal repair property for the first two nodes, i.e., $\rho = 2$ and $G = [0 : 2]$. Here $\alpha = 1, m = 2, \delta_0 = 2, \delta_1 = 3$, then $l_0 = 3, l_1 = 2$ by (10). By means of Algorithm 1 the sets $P_{i,j}$ and $P^{(a)}_{i,j}, j \in [1 : 2], a \in [0 : 2]$ corresponding to GN $i \in G$ are

$$P_{i,1} = \{f^{(1)}_i[0], f^{(2)}_i[1]\}, P^{(0)}_{i,1} = \{f^{(2)}_i[0]\}, P^{(1)}_{i,1} = \{f^{(2)}_i[1]\}.$$ 

**Example 4.** Based on the base code in Example 2, suppose the goal is to obtain a (16, 10) MDS array code $C_2$ with \{2, 3, 4, 6\}-optimal repair property for the first two nodes, i.e., $\rho = 2$ and $G = [0 : 2]$. In this case, $\alpha = 1, m = 4, \delta_0 = 2, \delta_1 = 3, \delta_2 = 4, \delta_3 = 6$, then $l_0 = 6, l_1 = 4, l_2 = 3, l_3 = 2$ by (10). By means of Algorithm 1 the sets $P_{i,j}$ and $P^{(a)}_{i,j}, j \in [1 : 4], a \in [0 : 4]$ of GN $i \in G$ are given in Table II and Table III respectively.

| $j$ | 1 | 2 | 3 |
|-----|---|---|---|
| $P_{i,j}$ | \{f^{(1)}_i[0], f^{(2)}_i[1], f^{(3)}_i[0], f^{(5)}_i[1]\} | \{f^{(2)}_i[0], f^{(3)}_i[1], f^{(5)}_i[1]\} | \{f^{(2)}_i[0], f^{(5)}_i[1]\} |

| $j$ | 1 | 2 | 3 |
|-----|---|---|---|
| $P^{(a)}_{i,j}$ | \{f^{(a)}_i[0]\} | \{f^{(a)}_i[0], f^{(2)}_i[1]\} | \{f^{(a)}_i[0], f^{(5)}_i[1]\} |

| $a$ | 1 | 2 | 3 |
|-----|---|---|---|
| 0   | \{f^{(0)}_i[0]\} | \{f^{(0)}_i[0], f^{(2)}_i[1]\} | \{f^{(0)}_i[0], f^{(5)}_i[1]\} |
| 1   | \{f^{(1)}_i[1]\} | \{f^{(1)}_i[1]\} | \{f^{(1)}_i[1]\} |
| 2   | \{f^{(2)}_i[0]\} | \{f^{(2)}_i[0]\} | \{f^{(2)}_i[0]\} |
| 3   | \{f^{(3)}_i[1]\} | \{f^{(3)}_i[1]\} | \{f^{(3)}_i[1]\} |
According to Algorithm [1] we have the following properties, whose proofs are given in Appendix A.

Property 1. Given $i \in \mathcal{G}$ and $j \in [1 : m)$.

P1. $\mathcal{P}_{i,1} \cup \mathcal{P}_{i,2} \cup \cdots \cup \mathcal{P}_{i,j} = \bigcup_{a=0}^{l_j-1} (\mathcal{P}_{i,1}^{(a)} \cup \mathcal{P}_{i,2}^{(a)} \cup \cdots \cup \mathcal{P}_{i,j}^{(a)}) = \{f_i^{(a)}[u] | a \in [l_j : l_0), u \in [0 : \delta_0]\};$

P2. When $j \geq 2$, $\mathcal{P}_{i,j} \subseteq \{f_i^{(a)}[u] | a \in [l_j : l_k), u \in [0 : \delta_0]\} \cup \bigcup_{a=l_j}^{l_j-1} (\mathcal{P}_{i,1}^{(a)} \cup \mathcal{P}_{i,2}^{(a)} \cup \cdots \cup \mathcal{P}_{i,z}^{(a)})$ for all $z \in [1 : j);$

P3. $|\mathcal{P}_{i,j}| = (\delta_j - \delta_{j-1})l_j$ and $|\mathcal{P}_{i,z}^{(a)}| = \delta_j - \delta_{j-1}$ for $a \in [0 : l_j].$

Now, we present the precise form of appended-data-based on the sets $\mathcal{P}_{i,j}^{(a)}$ for $i \in \mathcal{G}, 1 \leq j < m, a \in [0 : l_j)$. For convenience of notation, we also denote $\mathcal{P}_{i,j}^{(a)}, i \in \mathcal{G}, 1 \leq j < m, a \in [0 : l_j)$ the column vector of length $(\delta_j - \delta_{j-1})\alpha N^v$, which is formed by its elements in ascending order. Then for $i \in \mathcal{G}, t \in [0 : r)$ and $a \in [0 : l_0)$, we define $\mathbf{P}_{i,t}^{(a)}$ as

\[
\mathbf{P}_{i,t}^{(a)} = \left \{ \sum_{j=1}^{m} (K_{t,i,\delta_j-\delta_0-1}, K_{t,i,\delta_j-\delta_0-1+(j-1)}, \ldots, K_{t,i,\delta_j-\delta_0-1}) \mathcal{P}_{i,j}^{(a)} \right \}, \text{ if } a \in [l_{w+1} : l_w), w \in [1 : m),
\]

otherwise,

where the $\alpha N \times \alpha N^v$ matrices $K_{t,i,a}$ for $t \in [0 : r), i \in \mathcal{G}, v \in [0 : \delta_{m-1} - \delta_0)$ are called key matrices of node $i$. According to P1, for $i \in \mathcal{G}, t \in [0 : r)$, given $z \in [1 : m)$ and $a \in [l_z : l_{z+1})$, the appended-data $\mathbf{P}_{i,t}^{(a)}$ defined by (16) is a linear combination of $f_i^{(l_z)}, f_i^{(l_{z+1})}, \ldots, f_i^{(l_{m-1})}$, i.e., P0 holds. That is, $\mathbf{P}_{i,t}^{(a)}$ is well defined for (11).

The following two examples illustrate the whole process of our method.

Example 5. Following up from Example [2] by applying the generic construction method, we can obtain a $(16, 10)$ MDS array code $\mathbb{C}_2$ with $\{2, 3\}$-optimal repair property for the first two nodes, which is defined by the following parity-check equations:

\[
\begin{pmatrix}
A_{t,0},f_0^{(0)} + \zeta_0 V_{0,0}^T f_0^{(2)}[0] \\
A_{t,0},f_0^{(1)} + \zeta_0 V_{0,0}^T f_0^{(2)}[1] \\
A_{t,0},f_0^{(2)} \\
A_{t,1},f_1^{(0)} + \zeta_1 V_{0,1}^T f_1^{(2)}[0] \\
A_{t,1},f_1^{(1)} + \zeta_1 V_{0,1}^T f_1^{(2)}[1] \\
A_{t,1},f_1^{(2)} \\
\vdots
\end{pmatrix} + \sum_{j=2}^{15} \begin{pmatrix} A_{t,j},f_j^{(0)} \\ A_{t,j},f_j^{(1)} \\ A_{t,j},f_j^{(2)} \end{pmatrix} = \mathbf{0}, \quad t \in [0 : 6),
\]

where the $\frac{N}{2} \times \frac{N}{2}$ key matrices are

\[
K_{t,0,0} = \zeta_0^t V_{0,0}^T, K_{t,1,0} = \zeta_1^t V_{0,1}^T, \quad t \in [0 : r)
\]

for $\zeta_0 \in \mathbb{F}_q$, and the appended-data are

\[
\mathbf{P}_{t,i}^{(0)} = \zeta_0^t V_{0,0}^T f_i^{(2)}[0], \quad \mathbf{P}_{t,i}^{(1)} = \zeta_0^t V_{0,1}^T f_i^{(2)}[1] \quad \text{for } i = 0, 1.
\]

Example 6. Following up from Example [2] through the generic construction method, we can obtain a $(16, 10)$ MDS array code $\mathbb{C}_2$ with $\{2, 3, 4\}$-optimal repair property for the first two nodes, which is defined by the following parity-check equations:

\[
\begin{pmatrix}
A_{t,0},f_0^{(0)} + \zeta_0 V_{0,0}^T f_0^{(4)}[0] + \zeta_2 V_{0,0}^T f_0^{(3)}[0] + \zeta_2^2 V_{0,0}^T f_0^{(2)}[0] + \zeta_2^3 V_{0,0}^T f_0^{(2)}[1] \\
A_{t,0},f_0^{(1)} + \zeta_0 V_{0,0}^T f_0^{(4)}[1] + \zeta_2 V_{0,0}^T f_0^{(3)}[1] + \zeta_2^2 V_{0,0}^T f_0^{(2)}[1] + \zeta_2^3 V_{0,0}^T f_0^{(2)}[1] \\
A_{t,0},f_0^{(2)} + \zeta_0 V_{0,0}^T f_0^{(5)}[0] + \zeta_2 V_{0,0}^T f_0^{(5)}[1] \\
A_{t,1},f_1^{(0)} + \zeta_1 V_{0,1}^T f_1^{(4)}[0] + \zeta_2 V_{0,1}^T f_1^{(3)}[0] + \zeta_2^2 V_{0,1}^T f_1^{(2)}[0] + \zeta_2^3 V_{0,1}^T f_1^{(2)}[1] \\
A_{t,1},f_1^{(1)} + \zeta_1 V_{0,1}^T f_1^{(4)}[1] + \zeta_2 V_{0,1}^T f_1^{(3)}[1] + \zeta_2^2 V_{0,1}^T f_1^{(2)}[1] + \zeta_2^3 V_{0,1}^T f_1^{(2)}[1] \\
A_{t,1},f_1^{(2)} + \zeta_1 V_{0,1}^T f_1^{(5)}[0] + \zeta_2 V_{0,1}^T f_1^{(5)}[1] \\
A_{t,1},f_1^{(3)} + \zeta_1 V_{0,1}^T f_1^{(5)}[1] \\
A_{t,1},f_1^{(4)} + \zeta_1 V_{0,1}^T f_1^{(5)}[1] \\
A_{t,1},f_1^{(5)} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\vdots
\end{pmatrix}
\]

\[
\begin{pmatrix}
A_{t,j},f_j^{(0)} \\
A_{t,j},f_j^{(1)} \\
A_{t,j},f_j^{(2)} \\
A_{t,j},f_j^{(3)} \\
A_{t,j},f_j^{(4)} \\
A_{t,j},f_j^{(5)} \\
\vdots
\end{pmatrix}
\]

\[
\begin{pmatrix}
A_{t,j},f_j^{(0)} \\
A_{t,j},f_j^{(1)} \\
A_{t,j},f_j^{(2)} \\
A_{t,j},f_j^{(3)} \\
A_{t,j},f_j^{(4)} \\
A_{t,j},f_j^{(5)} \\
\vdots
\end{pmatrix}
\]

\[
= \mathbf{0}, \quad t \in [0 : 6),
\]

where the $\frac{N}{2} \times \frac{N}{2}$ key matrices are

\[
K_{t,0,v} = \zeta_0^t V_{0,0}^T, K_{t,1,v} = \zeta_1^t V_{0,1}^T, \quad t \in [0 : r), v \in [0 : 4),
\]

with $\zeta_0, \zeta_1, \zeta_2$ and $\zeta_3$ being four distinct elements in $\mathbb{F}_q$.

C. Repair Property

In this subsection, we show that GN $i$ possesses the $\delta_{(0; m)}$-optimal repair property and RN $j$ maintains the same optimal repair property as that of base code for all $i \in \mathcal{G}$ and $j \in [0 : n) \setminus \mathcal{G}$. Particularly, if node $i$ in the base code $\mathbb{C}_0$ has $\delta_2$-optimal
repair property for \( z \in [0 : m) \), then let the \( \frac{\alpha N}{\delta_z} \times \alpha N \) full-rank matrices \( R_{i, \delta_z} \) and \( S_{i, \delta_z} \) denote the \( \delta_z \)-repair matrix and \( \delta_z \)-select matrix, respectively.

Consider the repair of node \( i \in [0 : n) \) by connecting \( d_z = k + \delta_z - 1 \) surviving nodes where \( z \in [0 : m) \). Let the \( \frac{\lambda_0 \alpha N}{\delta_z} \times \lambda_0 \alpha N \) full-rank matrices \( R'_{i, \delta_z} \) and \( S'_{i, \delta_z} \) respectively be the \( \delta_z \)-repair matrix and \( \delta_z \)-select matrix of node \( i \) of code \( \mathbb{C}_2 \) given by

\[
R'_{i, \delta_z} = \begin{cases} 
\begin{pmatrix} 
R_{i, \delta_0} & \cdots & \cdots & \cdots & 0_{\alpha N' \times \alpha N} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0_{\alpha N' \times \alpha N} & \cdots & \cdots & \cdots & 0_{\alpha N' \times \alpha N} \\
\end{pmatrix}, & \text{if } i \in \mathcal{G}, \\
\text{blkdiag}(R_{i, \delta_z}, R_{i, \delta_z}, \ldots, R_{i, \delta_z}), & \text{otherwise},
\end{cases}
\]

\[
S'_{i, \delta_z} = \begin{cases} 
\begin{pmatrix} 
S_{i, \delta_0} & \cdots & \cdots & \cdots & 0_{\alpha N' \times \alpha N} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0_{\alpha N' \times \alpha N} & \cdots & \cdots & \cdots & 0_{\alpha N' \times \alpha N} \\
\end{pmatrix}, & \text{if } i \in \mathcal{G}, \\
\text{blkdiag}(S_{i, \delta_z}, S_{i, \delta_z}, \ldots, S_{i, \delta_z}), & \text{otherwise},
\end{cases}
\]

In other words, when \( d_z \) surviving nodes are connected, we use the equations obtained by multiplying \( S_{i, \delta_z} \) on both sides of the equations in (11) to recover the data stored at node \( i \), i.e.,

\[
S_{i, \delta_z} A_{t,i} f_i^{(a)} + \sum_{j=0,j \neq i}^{n-1} S_{i, \delta_z} A_{t,j} f_j^{(a)} + \sum_{j \in \mathcal{G}} S_{i, \delta_z} P_{t,j}^{(a)} = 0, \quad t \in [0 : r)
\]

where \( s = 0, a \in [0 : l_z) \) if \( i \in \mathcal{G} \) and \( s = z, a \in [0 : l_0) \) otherwise. Substituting (5) into the above LSEs, we then get

\[
S_{i, \delta_z} A_{t,i} f_i^{(a)} + \sum_{j=0,j \neq i}^{n-1} \tilde{A}_{t,j,i,\delta_z} R_{i, \delta_z} f_j^{(a)} + \sum_{j \in \mathcal{G}} S_{i, \delta_z} P_{t,j}^{(a)} = 0, \quad t \in [0 : r),
\]

where \( \tilde{A}_{t,j,i,\delta_z} \) is an \( \frac{\alpha N}{\delta_z} \times \frac{\alpha N}{\delta_z} \) matrix defined in (5).

First of all, we propose the repair procedure of GNs. To this end, node \( i \in \mathcal{G} \) in base code \( \mathbb{C}_0 \) is required to satisfy the following conditions.

\( C_1. \) \( S_{j, \delta_0} K_{i,v} = 0 \) for any \( i, j \in \mathcal{G} \) with \( i \neq j, t \in [0 : r) \) and \( v \in [0 : \delta_m - 1 - \delta_0) \);

\( C_2. \) For any \( i \in \mathcal{G}, z \in [1 : m) \) and \( \mathcal{D}_z = \{j_0, j_1, \ldots, j_{r-d_z-1}\} \subset [0 : n)\setminus\{i\}\), the \( r\alpha N' \times r\alpha N' \) matrix

\[
M_{i, \mathcal{D}_z} = \begin{pmatrix} 
S_{i, \delta_0} A_{0,i} & \tilde{A}_{0,j_0,i,\delta_0} & \cdots & \tilde{A}_{0,j_{r-d_z-1},i,\delta_0} & S_{i, \delta_0} K_{0,i,0} & \cdots & S_{i, \delta_0} K_{0,i,\delta_m-\delta_0-1} \\
S_{i, \delta_0} A_{1,i} & \tilde{A}_{1,j_0,i,\delta_0} & \cdots & \tilde{A}_{1,j_{r-d_z-1},i,\delta_0} & S_{i, \delta_0} K_{1,i,0} & \cdots & S_{i, \delta_0} K_{1,i,\delta_m-\delta_0-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
S_{i, \delta_0} A_{r-1,i} & \tilde{A}_{r-1,j_0,i,\delta_0} & \cdots & \tilde{A}_{r-1,j_{r-d_z-1},i,\delta_0} & S_{i, \delta_0} K_{r-1,i,0} & \cdots & S_{i, \delta_0} K_{r-1,i,\delta_m-\delta_0-1} \\
\end{pmatrix}
\]

is nonsingular over \( \mathbb{F}_q \), where \( K_{i,v} \) are key matrices of node \( i \). Particularly, we also define a \( r\alpha N' \times r\alpha N' \) matrix \( M_{i, \mathcal{D}_0} \) as in (21) with \( z = 0 \).

\( \text{The Repair Procedure of GNs:} \) Assume that GN \( i \) (\( i \in \mathcal{G} \)) fails and \( d_z = k + \delta_z - 1 \) helper nodes are connected for any given \( z \in [0 : m) \), then node \( i \) is repaired as follows:

1) Download the data \( \{R_{i, \delta_z} f_j^{(b)} | b \in [0 : l_z)\} \) from each helper node \( j \in \mathcal{H}_z \), where \( \mathcal{H}_z \) denotes the set of indices of the \( d_z \) helper nodes.

2) Choose linear system of equations (20) for \( s = 0 \) and \( a = 0, 1, \ldots, l_z - 1 \) to solve the data stored at GN \( i \), i.e.,

\[
S_{i, \delta_0} A_{t,i} f_i^{(a)} + \sum_{j=0,j \neq i}^{n-1} \tilde{A}_{t,j,i,\delta_z} R_{i, \delta_z} f_j^{(a)} + \sum_{j \in \mathcal{G}} S_{i, \delta_z} P_{t,j}^{(a)} = 0, \quad a \in [l_w+1 : l_w], t \in [0 : r)
\]
for all $w \in [z : m]$ by noting

$$\{0, 1, \cdots, l_z - 1\} = \bigcup_{w=z}^{m-1} [l_{w+1} : l_w).$$

According to (16), by first applying C1 and then substituting the downloaded data into (22), we then obtain

$$\left( \begin{array}c S_{i, \delta_0} A_{0, i} \\ S_{i, \delta_0} A_{1, i} \\ \vdots \\ S_{i, \delta_0} A_{r-1, i} \end{array} \right) f_i^{(a)} + \sum_{j \in D_z} \left( \begin{array}c \hat{A}_{0, j, i, \delta_0} \\ \hat{A}_{1, j, i, \delta_0} \\ \vdots \\ \hat{A}_{r-1, j, i, \delta_0} \end{array} \right) R_{i, \delta_0} f_j^{(a)} + \sum_{u=1}^{u} \Gamma_u P_{i, u}^{(a)} = - \sum_{u=z+1}^{w} \Gamma_u P_{i, u}^{(a)} + *, a \in [l_{w+1} : l_w) \tag{23}$$

for all $w \in [z : m]$, where $D_z = \{0 : n\} \setminus (H_z \cup \{i\})$, symbol $*$ denotes a known vector that can be determined by the downloaded data in 1), here $*$ denotes

$$- \sum_{j \in H_z} \left( \begin{array}c \hat{A}_{0, j, i, \delta_0} \\ \hat{A}_{1, j, i, \delta_0} \\ \vdots \\ \hat{A}_{r-1, j, i, \delta_0} \end{array} \right) R_{i, \delta_0} f_j^{(a)},$$

and the $\alpha \times (\delta_u - \delta_{u-1}) \alpha$ matrix

$$\Gamma_u = \left( \begin{array}c S_{i, \delta_0} K_{0, i, \delta_u-1} - \delta_0 \\ S_{i, \delta_0} K_{1, i, \delta_u-1} - \delta_0 \\ \vdots \\ S_{i, \delta_0} K_{r-1, i, \delta_u-1} - \delta_0 \end{array} \right), u \in [1 : m).$$

Let $D_z = \{j_0, j_1, \cdots, j_{r-\delta_z-1}\}$, in matrix form, we can rewritten (23) as

$$M_{i, D_z} \left( \begin{array}c f_i^{(a)} \\ R_{i, \delta_0} f_{j_0}^{(a)} \\ \vdots \\ R_{i, \delta_0} f_{j_{r-\delta_z-1}}^{(a)} \end{array} \right) = - \sum_{u=z+1}^{w} \Gamma_u P_{i, u}^{(a)} + *, a \in [l_{w+1} : l_w), \tag{24}$$

where $M_{i, D_z}$ is defined in (21).

3) Recover the data stored at GN $i$ by sequentially solving (24) when $w = z, z + 1, \cdots, m - 1$.

3-1) Compute the first term on RHS of (24) from the recovered data of GN $i$ if $w > z$.

3-2) Recover the data $f_i^{(a)}$ and $P_{i, u}^{(a)}$ for $1 \leq u \leq z$ from (24).

**Theorem 2.** GN $i \in G$ of the new code $C_2$ has the $\delta_{(0, m)}$-optimal repair/access property if RN $i$ of base code $C_0$ satisfies C1 and C2.

**Proof.** Let us consider the $\delta_z$-optimal repair property of GN $i$ for any $z \in [0 : n)$, i.e., $d_z = k + \delta_z - 1$ helper are connected. As shown in 1) of The Repair Procedure of GNs, $\gamma(d_z) = \frac{d_z \cdot l_0 \alpha N}{\delta_0} = \frac{d_z \cdot l_0 \alpha N}{\delta_0}$ since $\operatorname{Rank}(R_{i, \delta_0}) = \frac{\beta N}{\delta_0}$, $\delta_z = d_z - k + 1$ and $\frac{l_0}{\delta_0} = \frac{\alpha \beta}{\delta_z}$ from (10), which attains the lower bound in (1). Moreover, if node $i$ has the $\delta_0$-optimal access property for base code $C_0$, i.e., the repair matrix $R_{i, \delta_0}$ has only one nonzero element in each row, then by (15) GN $i$ has the $\delta_z$-optimal access property in the new code $C_2$. Thus, to prove this theorem, it suffices to show that 3) of The Repair Procedure of GNs can be executed for $w = z, z + 1, \cdots, m - 1$.

For fixed $a$, according to P3, it is easy to see that there are

$$\alpha N + (r - \delta_z) \alpha N' + \sum_{u=1}^{z} (\delta_u - \delta_{u-1}) \alpha N' = \alpha \cdot \delta_0 N' + (r - \delta_0) \alpha N' = r \alpha N'$$
unknown variables on LHS of the \( r \alpha N' \) equations in (24). Note that the coefficient matrix on LHS of (24) is nonsingular according to C2 if \( z > 0 \) and Lemma 2 if \( z = 0 \) (the \( \delta_0 \)-optimal repair property of code \( \mathcal{C}_0 \)). That is, (24) is solvable if the first term in its RHS is known, i.e., 3-2) of The Repair Procedure of GNs can be executed. Then we only need to show the following claim.

**Claim:** For any given \( w \in [z : m] \), the first term \( \sum_{u=z+1}^{w} \Gamma_u \mathcal{P}_{i,u}^{(a)} \) on RHS of (24) can be determined for all \( a \in [l_{w+1} : l_w] \).

We prove it by induction.

i) When \( w = z \), Claim is obvious as \( \sum_{u=z+1}^{w} \Gamma_u \mathcal{P}_{i,u}^{(a)} = 0 \).

ii) Assume that Claim holds for all \( w \in [z : v] \) where \( z \leq v < m - 1 \). Then, \( f_i^{(a)} \) and \( \mathcal{P}_{i,u}^{(a)} \) are available for all \( 1 \leq u \leq z \) and \( a \in [l_{v+1} : l_z] \), which together with P2 imply that the first term on RHS of (24) for \( w = v + 1 \) is already known. That is, Claim holds for \( w = v + 1 \) and thus for all \( z \leq w < m \) by the induction.

Then for \( w \in [z : m) \), 3) of The Repair Procedure of GNs can be executed to the end, which means that the data \( f_i^{(a)} \) and \( \mathcal{P}_{i,u}^{(a)} \) for all \( 1 \leq u \leq z \) and \( a \in [0 : l_z] \) can be recovered. Finally by P1, we already have \( f_i^{(a)}, a \in [l_z : l_0] \) from the data in set \( \bigcup_{a=0}^{z-1} (\mathcal{P}_{i,1}^{(a)} \cup \cdots \cup \mathcal{P}_{i,z}^{(a)}) \) if \( z > 0 \), i.e., all the data stored at GN \( i \) are regenerated, which finishes the proof.

Example 7. Following up from Example 6 let us consider the repair of the first GN of the \((16, 10)\) MDS array code \( \mathcal{C}_2 \) by connecting to \( d_1 = 12 \) helper nodes, i.e., we investigate the first node has 3-optimal repair property.

By using (17) and (24), the procedure of repairing the first GN is shown in Table IV. To save space, we only give unknown variables related to the first GN in (24) for \( a = 0, 1, 2, 3 \), and show how to obtain the data stored at the first GN in 3-2) of The Repair Procedure of GNs.

| \( w \) | \( a \) | The unknown variables | The eliminated variables | The solved variables |
|---|---|---|---|---|
| 1 | 3 | \( f_0^{(0)}, f_0^{(0)} [1] \) | \( f_0^{(0)} [1] \) | \( f_0^{(0)}, f_0^{(0)} [1] \) |
| 2 | 1 | \( f_0^{(0)}, f_0^{(0)} [1], f_0^{(1)} [0], f_0^{(1)} [1] \) | \( f_0^{(0)} [1], f_0^{(0)}, f_0^{(0)} [1] \) | \( f_0^{(0)}, f_0^{(0)} [1] \) |
| 3 | 0 | \( f_0^{(0)}, f_0^{(0)} [1], f_0^{(0)} [1], f_0^{(1)} [0], f_0^{(1)} [1] \) | \( f_0^{(0)} [1], f_0^{(0)} [1], f_0^{(1)} [1] \) | \( f_0^{(0)}, f_0^{(1)} [0] \) |

Next, we examine the repair property of the RNs of code \( \mathcal{C}_2 \), which is the same as that of the base code.

**The Repair Procedure of RNs:** Assume that RN \( i \in \{0 : n \} \setminus \mathcal{G} \) fails and it has \( \delta_z \)-optimal repair property for base code \( \mathcal{C}_0 \). When \( d_z = k + \delta_z - 1 \) helper nodes are connected to repair RN \( i \), let \( \mathcal{H}_z \) be the set of indices of the \( d_z \) helper nodes, its stored data is repaired as follows:

1) Download the data \( \{ R_{i,\delta_z} f_i^{(b)} | b \in [0 : l_0] \} \) from each helper node \( j \in \mathcal{H}_z \).

2) Choose the linearly system of equations (20) for \( z = s + a = 0, 1, \cdots, l_0 - 1 \) to solve the stored data at RN \( i \), i.e.,

\[
\begin{pmatrix}
S_{i,\delta_z} A_{0,i} \\
S_{i,\delta_z} A_{1,i} \\
\vdots \\
S_{i,\delta_z} A_{r-1,i}
\end{pmatrix}
+ \sum_{v=0}^{r-\delta_z-1} \begin{pmatrix}
A_{0,j_v,i,\delta_z} \\
A_{1,j_v,i,\delta_z} \\
\vdots \\
A_{r-1,j_v,i,\delta_z}
\end{pmatrix}
R_{i,\delta_z} f_{j_v}^{(a)}
+ \sum_{j \in \mathcal{G}} \begin{pmatrix}
S_{i,\delta_z} P_{0,j}^{(a)} \\
S_{i,\delta_z} P_{1,j}^{(a)} \\
\vdots \\
S_{i,\delta_z} P_{r-1,j}^{(a)}
\end{pmatrix}
= *, a \in [l_{w+1} : l_w]
\] (25)

for all \( w \in [0 : m] \), where \( \{ j_0, j_1, \cdots, j_{r-\delta_z-1} \} = \mathcal{D}_z = \{0 : n\} \setminus (\mathcal{H}_z \cup \{i\}) \) and similar to (23). \( * \) denotes a known vector that can be determined by the downloaded data. In matrix form, (25) can be described as

\[
\begin{pmatrix}
S_{i,\delta_z} A_{0,i} \\
S_{i,\delta_z} A_{1,i} \\
\vdots \\
S_{i,\delta_z} A_{r-1,i}
\end{pmatrix}
+ \begin{pmatrix}
A_{0,j_0,i,\delta_z} \\
A_{1,j_0,i,\delta_z} \\
\vdots \\
A_{r-1,j_0,i,\delta_z}
\end{pmatrix}
\begin{pmatrix}
f_i^{(a)} \\
r_{i,j_0,i,\delta_z} f_i^{(a)} \\
\vdots \\
r_{i,j_{r-\delta_z-1},i,\delta_z} f_i^{(a)}
\end{pmatrix}
+ \sum_{j \in \mathcal{G}} \begin{pmatrix}
S_{i,\delta_z} P_{0,j}^{(a)} \\
S_{i,\delta_z} P_{1,j}^{(a)} \\
\vdots \\
S_{i,\delta_z} P_{r-1,j}^{(a)}
\end{pmatrix}
= *, a \in [l_{w+1} : l_w].
\] (26)
Claim: Rank nodes.

Proof. According to The Repair Procedure of RNs, implying that we can obtain data \( R(z_i, j) \) by choosing an \( \Phi_{z_i, j} \).

Lemma 3. Given \( z \in \{0 : m\} \), \( a \in \{l_{w+1} : l_w\} \) with \( w \in \{1 : m\} \), \( t \in \{0 : r\} \), \( i \in \{0 : n\} \), and \( j \in \mathcal{G} \), the column vector \( S_{i, z, K_{t, i, z}} \) can be computed from the data in set \{ \( R_{i, z, f_j}^{(b)} \) \( b \in \{l_w : l_0\} \) \} if for base code \( \mathcal{C}_0 \), RN \( i \) has the \( \delta \)-optimal repair property and all GNs \( j \in \mathcal{G} \) satisfy

\[
C3. \text{ Rank} \begin{pmatrix} R_{i, z, f_j}^{(b)} \\ S_{i, z, K_{t, i, z}, o} \end{pmatrix} = \frac{\alpha N}{\Delta} \text{ for } t \in \{0 : r\}, u \in \{0 : \delta_0\} \text{ and } v \in \{0 : \delta_{m-1} - \delta_0\}, \text{ where } \Phi_{z, j} \text{ is the } \alpha N' \times \alpha N \text{ matrix defined by (13).}
\]

Proof. The proof is given in Appendix [B].

Theorem 3. Given \( z \in \{0 : m\} \), RN \( i \in \{0 : n\} \) \( \mathcal{G} \) of the new code \( \mathcal{C}_2 \) has the \( \delta \)-optimal repair/access property over \( \mathbb{F}_q \) if for base code \( \mathcal{C}_0 \), RN \( i \) has the \( \delta \)-optimal repair property and all GNs satisfy C3.

Proof. According to The Repair Procedure of RNs, \( \gamma(d_z) = \frac{d_z}{d_z - k - 1} \alpha N \) due to Rank \( R_{i, z, f_j}^{(b)} \) and \( \delta_z = d_z - k + 1 \), which attains the lower bound in (1). In addition, if node \( i \) of the base code \( \mathcal{C}_0 \) has the \( \delta \)-optimal access property, i.e., the repair matrix \( R_{i, \delta} \) also has only one nonzero element in each row, then by (13) RN \( i \) has the \( \delta \)-optimal access property in the new code \( \mathcal{C}_2 \). Therefore, it is sufficient to show that 3) of The Repair Procedure of RNs can be executed under C3.

According to Lemma 2 by the \( \delta \)-optimal repair property of RN \( i \) in base code \( \mathcal{C}_0 \), the coefficient matrix of the first term on LHS of (26) is nonsingular. Thus given \( w \in \{0 : m\} \) and \( a \in \{l_{w+1} : l_w\} \), the data \( f_j^{(a)} \) and \( R_{i, z, f_j^{(a)}} \), \( v \in \{0 : r - \delta_z\} \), can be repaired from (26) if the following claim holds.

Claim: Given \( w \in \{0 : m\} \), the second term on LHS of (26) for \( a \in \{l_{w+1} : l_w\} \) is known.

By P0, \( P_{e, j}^{(a)} = 0 \) for \( a \in \{l_1 : l_0\} \) and \( j \in \mathcal{G} \), i.e., Claim holds for \( w = 0 \). Suppose that Claim holds for all \( 0 \leq w \leq s < m - 1 \), implying that we can obtain data \( R_{i, z, f_j^{(a)}} \) for \( a \in \{l_{w+1} : l_w\} \) and \( v \in \{0 : r - \delta_z\} \). Therefore by Lemma 3 we are able to compute the second term on LHS of (26) from \( R_{i, z, f_j^{(b)}} \) for \( b \in \{l_{w+1} : l_0\} \) and \( j \in \mathcal{G} \), which have been either downloaded \( (j \in \mathcal{H}_z \cap \mathcal{G}) \) or repaired \( (j \in \mathcal{D}_z \cap \mathcal{G}) \). That is, Claim also holds for \( w = s + 1 \) and thus for all \( 0 \leq w < m \) by the induction. This finishes the proof.

Combining Theorems 2-3 we then have the following results.

Theorem 4. By choosing an \((n, k)\) MDS array code \( \mathcal{C}_0 \) with the \( \delta \)-optimal repair/access property for all nodes and a set \( \mathcal{G} \) of \( \rho \) nodes satisfying C1-C3, the new \((n, k)\) MDS array code \( \mathcal{C}_2 \) has the \( \delta_{(0:m)} \)-optimal repair/access property for these \( \rho \) nodes over \( \mathbb{F}_q \), and preserves the \( \delta \)-optimal repair/access property for the other nodes where \( z \in \{0 : m\} \).

Besides, we have the following lemma whose proof is given in Appendix [C]. It is very useful when we recursively apply the construction method in the next section.

Lemma 4. Assume that base code \( \mathcal{C}_0 \) has another set \( \mathcal{G}' \subset \{0 : n\} \) of \( \rho \) nodes satisfying C1-C3, whose key matrices are \( K_{t, i, v}, t \in \{0 : r\}, i \in \mathcal{G}', v \in \{0 : \delta_{m-1} - \delta_0\} \). Then, nodes in \( \mathcal{G}' \) of new code \( \mathcal{C}_2 \) still satisfy C1-C3 with key matrices of the form

\[
K_{t, i, v}' = \text{blkdiag}(K_{t, i, v}, K_{t, i, v}, \ldots, K_{t, i, v})_{l_0}, \quad i \in \mathcal{G}', t \in \{0 : r\}, v \in \{0 : \delta_{m-1} - \delta_0\}.
\]
A. Generic Algorithm for Constructing MDS Array Code with the $\delta_{(0:m)}$-optimal Repair Property for All Nodes

In this subsection, we introduce the generic algorithm based on a class of special MDS array codes, which is called transformable MDS (TMDS) array codes.

Definition 1. An $(n, k)$ MDS array code defined in the form of $C/[\delta_{(0:m)}]$ with the $\delta_0$-optimal repair property for all nodes is said to be a TMDS array code if there exists a partition $\mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_{\mu-1}$ of set $[0 : n)$ such that the nodes in $\mathcal{J}_t$ of this code satisfy C1-C2 for $\alpha = 1$, and C3 for $z = 0, \alpha = 1$.

Remark 1. If the value $\mu$ in Definition 1 is $n$, without loss of generality, we always assume $\mathcal{J}_t = \{t\}$ for $t \in [0 : \mu)$.

By means of the TMDS array code, we present a generic algorithm (Algorithm 2) that can construct an MDS array code with the $\delta_{(0:m)}$-optimal repair property for all nodes by recursively using the construction method in Section III $\mu$ times, where $\mu$ is the value given in Definition 1. In the $(s + 1)$-th round construction method, we choose code $Q_s$ as the base code and denote the resultant code as $Q_{s+1}$, where $0 \leq s < \mu$ and the key matrices of node $i \in [0 : n)$ of code $Q_s$ is defined as $K_{t,i,v}(s), t \in [0 : r), v \in [0 : \delta_{m-1} - \delta_0)$.

Algorithm 2

Input: A $(n, k)$ TMDS array code $Q_0$ with sub-packetization $N$

Output: The desired code $Q_s$ with the $\delta_{(0:m)}$-optimal repair property for all nodes, where its sub-packetization level is $l_0^sN$

1: for $s = 0; s < \mu; s + +$ do
2: \hspace{1em} Set code $Q_s$ of sub-packetization level $\alpha_sN$ as the base code, where $\alpha_s = l_0^s$
3: \hspace{1em} Designate $\mathcal{J}_s$ given in Definition 1 as the set $\mathcal{G}$ in the construction method
4: \hspace{1em} For $i \in \mathcal{G}$, setting the $\alpha_sN \times \alpha_sN'$ matrix $K_{t,i,v} = K_{t,i,v}^{(s)}$ in (16), where
\[
K_{t,i,v}^{(s)} = \text{blkdiag}(K_{t,i,v}^{(0)}, K_{t,i,v}^{(0)}, \ldots, K_{t,i,v}^{(0)}), t \in [0 : r), v \in [0 : \delta_{m-1} - \delta_0)
\]
5: \hspace{1em} Applying the construction method on code $Q_s$ to generate a new code $Q_{s+1}$ with sub-packetization level $l_0^s\alpha_sN$
6: end for

Theorem 5. By choosing an $(n, k)$ TMDS array code over $\mathbb{F}_q$ as base code, a new $(n, k)$ MDS array code generated from Algorithm 2 has the $\delta_{(0:m)}$-optimal repair property for all nodes over $\mathbb{F}_q$, where the sub-packetization level of the new $(n, k)$ MDS array code is $l_0^sN = (\frac{1}{\alpha_s})^sN$. Moreover, the new $(n, k)$ MDS array code has the $\delta_{(0:m)}$-optimal access property for all nodes if the base code has the $\delta_0$-optimal access property for all nodes.

Proof. According to Theorem 1 to obtain the desired MDS array code from Algorithm 2, it is sufficient to show that for any $s \in [0 : \mu)$, the nodes in $\mathcal{J}_s$ of code $Q_s$ satisfy C1-C3 by setting the key matrices $K_{t,i,v}^{(s)}, t \in [0 : r), i \in \mathcal{J}_s, v \in [0 : \delta_{m-1} - \delta_0)$ in (28). By recursively applying Lemma 3, we only need to verify that the $N \times N'$ matrices $K_{t,i,v}^{(0)}, t \in [0 : r), i \in \mathcal{J}_s, v \in [0 : \delta_{m-1} - \delta_0)$ are the key matrices of the nodes in $\mathcal{J}_s$ of code $Q_0$ such that they satisfy C1-C3, which is guaranteed by Definition 1.

In the following, we provide an example of Algorithm 2.

Example 8. Applying Algorithm 2 to the base code $C_0$ in Example 2, we obtain the codes $Q_1, Q_2, \ldots, Q_8$ through eight rounds of the construction method where in round $s \in [1 : 8]$, the set $\{2s - 2, 2s - 1\}$ is chosen as the set $\mathcal{G}$. Let $g_{s,i}$ of length $3^sN$ be the data stored at node $i \in [0 : 16]$ of code $Q_s$, where $s \in [1 : 8]$. For convenience, the data $g_{s,i}$ is always represented by
\[
g_{s,i} = \begin{pmatrix}
f_{i}^{(0)} \\ f_{i}^{(1)} \\ \vdots \\ f_{i}^{(3^s-1)}
\end{pmatrix}, \quad 0 \leq i < 16, \ s \in [1 : 8],
\]
where $f_{i}^{(s)}$ is a column vector of length $N$. Note that the PCGs of code $Q_1$ has been shown in Example 5, i.e., the code $C_2$ in Example 5 is the code $Q_1$. 

In what follows, we give the $t$-th PCG of the code $Q_2$, while those of $Q_3, Q_4, \ldots, Q_8$ can be obtained similarly. By means of Algorithm $7$, the sets $\mathcal{P}_{i,1}$ and $\mathcal{P}_{i,1}^{(a)}$, $a \in \{0 : 2\}$ of node $i \in G = \{2 : 4\}$ for the second round are

$$\mathcal{P}_{i,1} = \{g_{i,j}^{(2)}[0], g_{i,j}^{(2)}[1]\} = \left\{ \begin{pmatrix} f_i^{(6)}[0] \\ f_i^{(7)}[0] \\ f_i^{(8)}[0] \end{pmatrix}, \begin{pmatrix} f_i^{(6)}[1] \\ f_i^{(7)}[1] \\ f_i^{(8)}[1] \end{pmatrix} \right\},$$

$$\mathcal{P}_{i,1}^{(a)} = \{g_{i,j}^{(2)}[0]\} = \left\{ \begin{pmatrix} f_i^{(6)}[0] \\ f_i^{(7)}[0] \\ f_i^{(8)}[0] \end{pmatrix} \right\}, \mathcal{P}_{i,1}^{(a)} = \{g_{i,j}^{(2)}[1]\} = \left\{ \begin{pmatrix} f_i^{(6)}[1] \\ f_i^{(7)}[1] \\ f_i^{(8)}[1] \end{pmatrix} \right\}.$$ 

Through the generic construction method, the $t$-th PCG of the code $Q_2$ are given as

$$\begin{align*}
A_{t,0}^{(0)} &+ f_{0,0} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
A_{t,0}^{(1)} &+ f_{0,0} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
A_{t,0}^{(2)} &+ f_{0,0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
A_{t,0}^{(3)} &+ f_{0,0} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
A_{t,0}^{(4)} &+ f_{0,0} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
A_{t,0}^{(5)} &+ f_{0,0} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
A_{t,0}^{(6)} &+ f_{0,0} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
A_{t,0}^{(7)} &+ f_{0,0} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
A_{t,0}^{(8)} &+ f_{0,0} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{align*}$$

where $0 \leq t < 6$.

B. An $(n, k)$ MDS array code $G$ by Applying Algorithm 2 to VBK code in [20]

In this subsection, we generate an MDS array code $G$ by applying Algorithm 2 to the $(n, k)$ VBK code which has the $\delta_0$-optimal access property for all nodes and sub-packetization level $N = \delta_0^\tau$, where $\tau = \left\lfloor \frac{\log q}{\delta_0^0} \right\rfloor$, $\delta_0 \in \{2, 3, 4\}$ and $r = n - k > \delta_0$.

In what follows, we first visit the definition of the VBK code.

Let $\epsilon \not\in \{0, 1\}$ be an element in the field $\mathbb{F}_q$. For $x \in \{0 : \tau\}$ and $\delta_0 = 2, 3, 4$, respectively define $\delta_0 \times \delta_0$ matrix $\Theta_x$ as

$$\Theta_x = \begin{cases} 
\begin{pmatrix}
\vartheta_{0,x} & \epsilon \vartheta_{1,x} \\
\vartheta_{1,x} & \vartheta_{0,x}
\end{pmatrix}, & \text{if } \delta_0 = 2, \\
\begin{pmatrix}
\vartheta_{0,x} & \epsilon \vartheta_{1,x} & \epsilon \vartheta_{2,x} \\
\vartheta_{1,x} & \vartheta_{0,x} & \epsilon \vartheta_{3,x} \\
\vartheta_{2,x} & \vartheta_{3,x} & \vartheta_{0,x}
\end{pmatrix}, & \text{if } \delta_0 = 3, \\
\begin{pmatrix}
\vartheta_{0,x} & \epsilon \vartheta_{1,x} & \epsilon \vartheta_{2,x} & \epsilon \vartheta_{3,x} \\
\vartheta_{1,x} & \vartheta_{0,x} & \epsilon \vartheta_{3,x} & \epsilon \vartheta_{2,x} \\
\vartheta_{2,x} & \vartheta_{3,x} & \vartheta_{0,x} & \epsilon \vartheta_{1,x} \\
\vartheta_{3,x} & \vartheta_{2,x} & \vartheta_{1,x} & \vartheta_{0,x}
\end{pmatrix}, & \text{if } \delta_0 = 4,
\end{cases}$$

where $\{\vartheta_{i,x}, \epsilon \vartheta_{i,x}, \vartheta_{0,x} | i \in \{1, 2, 3\}, x \in \{0 : \tau\}\}$ is a collection of distinct elements in $\mathbb{F}_q$ with

$$q \geq \begin{cases} 
6 \left\lceil \frac{\tau}{2} \right\rceil + 2, & \text{if } \delta_0 = 2, \\
18 \left\lceil \frac{\tau}{3} \right\rceil + 2, & \text{if } \delta_0 = 3, 4.
\end{cases}$$

(29)

Let $s$ be any given positive integer, for any $0 \leq x < s$, $u \in \{0 : \delta_0\}$ and given $a = (a_{s-1}, a_{s-2}, \ldots, a_0) \in \{0 : \delta_0^0\}$, define

$$\pi_x(a, x, u) = (a_{s-1}, \ldots, a_{x+1}, u, a_{x-1}, \ldots, a_0),$$

(30)

i.e., replace the $x$-digit $a_x$ of the vector $a = (a_{s-1}, \ldots, a_{x+1}, u, a_{x-1}, \ldots, a_0)$ by $u$. For $x \in \{0 : \tau\}$, $y \in \{0 : \delta_0\}$, and $t = 0, 1, \ldots, r - 1$, define an $N \times N$ matrix $A_{t,\delta_0+y}$ as

$$A_{t,\delta_0+y} = \sum_{a=0}^{N-1} \lambda_{\delta_0, x+y, a}^t e_a^T e_a + \sum_{a=0}^{N-1} \sum_{a \neq y, a \neq 0} \varepsilon_{u,y} \lambda_{\delta_0, x+y, a}^t e_a^T e_{\pi_x(a, x, u)},$$

(31)
where \( \{e_a | a \in [0 : N] \} \) is the standard basis of \( \mathbb{F}_q^N \) defined in (7),
\[
\lambda_{\delta_0 x+y,v} = \Theta_x(v,y) \quad \text{for } 0 \leq x < \tau, 0 \leq v, y < \delta_0
\]
and
\[
\varepsilon_{u,y} = \begin{cases} 
\varepsilon, & \text{if } u < y, \\
1, & \text{if } u > y.
\end{cases}
\] (32)

Then, the \((n,k)\) VBK code is defined by (3) with parity-check matrices \((A_{t,i})_{t \in [0:r], i \in [0:n]}\) given in (31), \(\delta_0\)-repair matrices \(R_{t,\delta_0}\) and \(\delta_0\)-select matrices \(S_{t,\delta_0}\) given as
\[
R_{t,\delta_0} = S_{t,\delta_0} = V_{\frac{1}{\delta_0}}(1, i, \%_{\delta_0}, i \in [0 : n]).
\] (33)

In the sequel, we show that the VBK code is a TMDS array code with the sets \(J_0, J_1, \cdots, J_{\mu-1}\) in Definition 1 being
\[
J_s = \begin{cases} 
\{ s\delta_0, s\delta_0 + 1, \cdots, s\delta_0 + \delta_0 - 1 \}, & \text{if } 0 \leq s < \mu - 1, \\
\{ s\delta_0, s\delta_0 + 1, \cdots, n - 1 \}, & \text{if } s = \mu - 1,
\end{cases}
\] (34)

where \(\mu = \tau\) and the matrix \(K_{t,i,v}^{(0)}\) in (28) is set as
\[
K_{t,i,v}^{(0)} \triangleq \zeta_i R_t^T, \quad t \in [0 : r], i \in [0 : n], v \in [0 : \delta_{m-1} - \delta_0),
\] (35)

with \(\zeta_0, \zeta_1, \cdots, \zeta_{\delta_{m-1} - \delta_0 - 1}\) being \(\delta_{m-1} - \delta_0\) distinct elements in \(\mathbb{F}_q\{\lambda_{i,u} | i \in [0 : n], u \in [0 : \delta_0]\} \).

In what follows, we check that for any \(x \in [0 : \mu]\), the nodes is set \(J_x\) of VBK code satisfy C1-C2 for \(\alpha = 1\), and C3 for \(\alpha = 1, z = 0\). For convenience, let \(\epsilon_0, \epsilon_1, \cdots, \epsilon_{N'-1}\) be the standard basis of \(\mathbb{F}_q^{N'}\) defined in (7) from now on, where \(N' = \frac{N}{\delta_0} = \delta_0^{r-1}\).

First of all, we verify that for any \(x \in [0 : \mu]\), the nodes in set \(J_x\) of VBK code satisfy C1 for \(\alpha = 1\) and C3 for \(\alpha = 1, z = 0\) with the help of Lemma 5 whose proof is given in Appendix D.

**Lemma 5.** For any \(0 \leq \bar{x} \neq x < \tau\) and \(0 \leq u, v, h < \delta_0\),
(i) \(V_{\bar{x},u} V_{\bar{x},u}^T = I_{N'}\) and \(V_{\bar{x},u} V_{\bar{x},u}^T = 0\) if \(u \neq v\); and
(ii) \(V_{\bar{x},u} (V_{\bar{x},u}^T \Delta_h) = T_{\bar{x},\bar{x},u,v} V_{\bar{x},u}\) for some \(N' \times N'\) matrices \(T_{\bar{x},\bar{x},u,v}\) with
\[
T_{\bar{x},\bar{x},u,v}(a,\cdot) = \begin{cases} 
\epsilon(h,a_r-1,\cdots,a_{\bar{x}-1,a_{\bar{x}}}a_{\bar{x}-1},a_0), & \text{if } 0 \leq x < \tau, a_{\bar{x}-1} = v, \\
\epsilon(h,a_r-1,\cdots,a_{\bar{x},a_{\bar{x}}-1,1\cdots,a_0}, & \text{if } 0 \leq x < \tau, a_{\bar{x}} = v, \\
0, & \text{otherwise},
\end{cases}
\] (36)

where \(a = (a_{r-2},a_{r-3},\cdots,a_0) \in [0 : N']\) and \(\Delta_h\) is the \(N' \times N'\) matrix defined in (14).

**Theorem 6.** By setting the \(N \times N'\) key matrix \(K_{t,i,v}^{(0)}\) of node \(i\) of VBK code as in (35), the nodes with indices in \(J_x\) \((x \in [0 : \mu])\) of VBK code satisfy C1 for \(\alpha = 1\) and C3 for \(z = 0, \alpha = 1\).

**Proof.** For any \(i, j \in [0 : n]\), let \(u = i \% \delta_0\) and \(u' = j \% \delta_0\).

Firstly, consider \(i, j \in J_x\) with \(i \neq j\) for \(x \in [0 : \mu]\). According to (34), we have \(\frac{\bar{x}}{\delta_0} = \frac{(\bar{x}' + 1)}{\delta_0}\) = \(x\neq u'\). Thus by (i) of Lemma 5 and (33),
\[
S_{i,\delta_0} K_{t,i,u}^{(0)} = S_{i,\delta_0} \cdot \zeta_i R_t^T \cdot S_{i,\delta_0} = \zeta_i V_{\bar{x},u} V_{\bar{x},u}^T = 0, t \in [0 : r], v \in [0 : \delta_{m-1} - \delta_0),
\]
which means that the nodes with indices in \(J_x\) of VBK code satisfy C1.

Next we show that any node \(j \in J_x\) satisfy C3 for \(z = 0, \alpha = 1\). For any \(i \in J_{\bar{x}}\) with \(0 \leq \bar{x} \neq x < \mu = \tau\), then
\[
\text{Rank} \left( \begin{pmatrix} R_{t,\delta_0} & S_{i,\delta_0}(K_{t,i,v}^{(0)}) \end{pmatrix} \right) = \text{Rank} \left( \begin{pmatrix} V_{\bar{x},u} & \zeta_i V_{\bar{x},u} (V_{\bar{x},u}^T \Delta_h) \end{pmatrix} \right) = \text{Rank} \left( \begin{pmatrix} V_{\bar{x},u} & \zeta_i T_{\bar{x},\bar{x},u,v} V_{\bar{x},u} \end{pmatrix} \right) = \text{Rank}(V_{\bar{x},u}) = \frac{N}{\delta_0}.
\]
for \( h \in [0 : \delta_0), t \in [0 : r) \) and \( v \in [0 : \delta_{m-1} - \delta_0) \), where the first equality holds due to (13), (33) and (35), and the second equality follows from (ii) of Lemma 5. Then, the nodes with indices in set \( \mathcal{J}_x \) satisfy C3 for \( z = 0, \alpha = 1 \). \( \square \)

Next, we show that for any \( s \in [0 : \mu) \), the nodes with indices in set \( \mathcal{J}_s \) of VBK code satisfy C2 for \( \alpha = 1 \). That is, we need to verify that the matrix \( M_{1, D_s} \) defined in (21) with \( \alpha = 1 \) is nonsingular for any given \( i \in [0 : n) \), \( z \in [1 : m) \) and \( D_s = \{ j_0, j_1, \cdots, j_r-\delta_s-1 \} \subset [0 : n) \{i\} \). According to the definition of matrix \( M_{1, D_s} \), the verification of its nonsingularity requires to determine the form of \( S_{t,0} A_{t,i} \) and \( \tilde{A}_{t,j,i,0} \) for \( 0 \leq i \neq j < n \), which will be ensured by Lemma 6. In addition, Lemmas 7 and 8 are also critical to proving the invertibility of the matrix \( M_{1, D_s} \). Lemmas 7 and 8 are proved similar to the proof of MDS property of VBK code in (20), thus we omit it here. Whereas, the proofs of Lemmas 6 and 8 are given in Appendix D

**Lemma 6.** For any \( i = \delta_0 x + \tilde{y} \) and \( j = \delta_0 x + y \in [0 : n) \{i\} \), where \( 0 \leq x, \tilde{x} < \tau \) and \( 0 \leq y, \tilde{y} < \delta_0 \),

(i) \( S_{t,0} A_{t,i} = \lambda_{t,\tilde{y}} V_{\tilde{x},\tilde{y}} + \sum_{u=0}^{\delta_0-1} \varepsilon_{u,\tilde{y}} \lambda_{t,u} V_{\tilde{x},u} \),

(ii) The matrix \( \tilde{A}_{t,j,i,0} \) is of the form

\[
\tilde{A}_{t,j,i,0} = \begin{cases} 
\sum_{a=0}^{N'-1} \lambda_{t,a}^T \tau_a + \sum_{a=0}^{N'-1} \sum_{a=0}^{\delta_0-1} \delta_0 \varepsilon_{u} \lambda_{t,u} \tau_{a} \varepsilon_{\tau_{a-1}(a,u)} , & \text{if } x < \tilde{x}, \\
\lambda_{t,\tilde{y}}^T N, & \text{if } x = \tilde{x}, \\
\sum_{a=0}^{N'-1} \lambda_{t,a}^T \tau_a + \sum_{a=0}^{N'-1} \sum_{a=0}^{\delta_0-1} \delta_0 \varepsilon_{u} \lambda_{t,u} \tau_{a} \varepsilon_{\tau_{a-1}(a,u-1,u)} , & \text{if } x > \tilde{x}, 
\end{cases}
\]

where \( A_{t,i} \) and \( S_{t,0} \) are defined in (31) and (33), respectively.

**Lemma 7.** For any given \( i \in [0 : n) \) and \( \{ j_0, j_1, \cdots, j_{s-1} \} \subset [0 : n) \{i\} \) with \( 1 \leq s < r \), when \( \delta_0 = 2, 3, 4 \), the block matrix

\[
\begin{pmatrix}
\tilde{A}_{0,j_0,i,0} & \tilde{A}_{0,j_1,i,0} & \cdots & \tilde{A}_{0,j_{s-1},i,0} \\
\tilde{A}_{1,j_0,i,0} & \tilde{A}_{1,j_1,i,0} & \cdots & \tilde{A}_{1,j_{s-1},i,0} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}_{s-1,j_0,i,0} & \tilde{A}_{s-1,j_1,i,0} & \cdots & \tilde{A}_{s-1,j_{s-1},i,0}
\end{pmatrix}
\]

of order \( sN' \) is nonsingular over \( \mathbb{F}_q \), where the \( N' \times N' \) matrix \( \tilde{A}_{t,j,i,0} \) is given by (37).

**Lemma 8.** Let \( \beta_0, \beta_1, \cdots, \beta_{r-1} \) be the elements in \( \mathbb{F}_q \). For any given \( i = \delta_0 x + \tilde{y} \) and any \( 0 \leq s \leq p < r \), define a \( (r-s)N' \times (r-s)N' \) matrix \( H_{i,p,s} \) as

\[
H_{i,p,s} = \begin{pmatrix}
I_{N'} & I_{N'} & \cdots & I_{N'} \\
\beta_1 I_{N'} & \beta_2 I_{N'} & \cdots & \beta_{p-1} I_{N'} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_s^{-s-1} I_{N'} & \beta_s^{-s-1} I_{N'} & \cdots & \beta_s^{-s-1} I_{N'} \\
\beta_{s+1} I_{N'} & \beta_{s+1} I_{N'} & \cdots & \beta_{s+1} I_{N'} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{p-1}^{-s-1} I_{N'} & \beta_{p-1}^{-s-1} I_{N'} & \cdots & \beta_{p-1}^{-s-1} I_{N'} \\
\tilde{A}_{r-s,j_0,i,0} & \tilde{A}_{r-s,j_1,i,0} & \cdots & \tilde{A}_{r-s,j_{r-p-1},i,0}
\end{pmatrix}
\]

where \( j_0, j_1, \cdots, j_{r-p-1} \in [0 : n) \{i\} \). Then for any \( 0 \leq s < p \), \( |H_{i,p,s}| \neq 0 \) if

(i) \( |H_{i,p,s}| \neq 0 \); 
(ii) \( \beta_u \neq \beta_v \) and \( \beta_u \neq \lambda_{j,u} \) for any \( 0 \leq u \neq v < p \) and \( j \in [0 : n) \{i\} \) with \( \frac{1}{\beta_u} = \frac{1}{\beta_v} \); 
(iii) \( \beta_u \neq \lambda_{j,u} \) for any \( 0 \leq u < p \), \( 0 \leq v < \delta_0 \) and \( j \in [0 : n) \{i\} \) with \( \frac{1}{\beta_u} = \frac{1}{\beta_v} \).

**Theorem 7.** By setting the \( N \times N' \) key matrix \( K^{(0)}_{i,t,i,u} \) of node \( i \) of VBK code to be the form in (35), the matrix \( M_{1, D_s} \) in (21) is nonsingular over \( \mathbb{F}_q \) for any given \( i = \delta_0 x + \tilde{y} \in [0 : n) \) and \( D_s = \{ j_0, j_1, \cdots, j_{r-\delta_s-1} \} \subset [0 : n) \{i\} \) with \( z \in [1 : m) \), where \( q \) is determined in (20).

**Proof.** For simplicity, let

\[
\beta_u = \begin{cases}
\lambda_{i,u}, & \text{if } u \in [0 : \delta_0), \\
\zeta_{u-\delta_0}, & \text{if } u \in [\delta_0 : \delta_{m-1}),
\end{cases}
\]

where it is noting from the definitions of \( \lambda_{i,u}, \zeta_{u-\delta_0} \) that
(i) \( \beta_u \neq \beta_v, \beta_u \neq \lambda_j, a \) and \( \beta_u \neq \lambda_j', \bar{g} \) for any \( 0 \leq u \neq v \leq \delta_z, a \in [0 : \delta_0), j \in [0 : n) \) with \( \left[ \frac{u}{\delta_0} \right] \neq \bar{x} \) and \( j' \in [0 : n) \setminus \{i\} \) with \( \left[ \frac{x_i}{\delta_0} \right] = \bar{x} \).

By replacing \( K_{t,i,v} \) with \( K_{t,i,0}^{(0)} \) in (21) with \( K_{t,i,v}^{(0)} \), we first calculate

\[
\begin{pmatrix}
S_{t,0} A_{t,i} & A_{t,j,0,i,\delta_0} & \cdots & A_{t,j,-r-1,i,\delta_0}
\end{pmatrix}
\begin{pmatrix}
S_{t,0} K_{t,i,0}^{(0)} & \cdots & S_{t,0} K_{t,1,0}^{(0)}
\end{pmatrix}
\]

\[
= \left( \lambda_x^{(0)} V_{x,0} + \sum_{u=0, u \neq \bar{x}}^{\delta_0-1} \varepsilon_{u,\bar{g}} A_{t,i,u,0} \right) \cdots \left( \lambda_{x,-r-1}^{(0)} V_{x,-r-1,0} \right) B_{VBK}
\]

\[
= \left( \beta_0^{(0)} I_{N'} \cdots \beta_{\delta_0-1}^{(0)} I_{N'} \right) B_{VBK}
\]

where the first equality follows from Lemma 6 and (33), and the second equality follows from (39) and the \( rN' \times rN' \) matrix

\[
B_{VBK} \triangleq \begin{pmatrix}
w_0 V_{x,0} \\
\vdots \\
w_{\delta_0-1} V_{x,\delta_0-1}
\end{pmatrix}
\]

with \( w_u = \begin{cases} \varepsilon_{u,\bar{g}}, & \text{if } u \neq \bar{g}, \\ 1, & \text{otherwise} \end{cases} \).

It is easy to see that the block matrix \( B_{VBK} \) is nonsingular.

**Case 1.** If \( \delta_z = r \), i.e., \( D_z = \emptyset \), then by (21) and (40), we have that the matrix \( M_{i,D_z} \) is of the form

\[
M_{i,D_z} = \begin{pmatrix}
I_{N'} & I_{N'} & \cdots & I_{N'} \\
\beta_0 I_{N'} & \beta_1 I_{N'} & \cdots & \beta_{r-1} I_{N'} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{\delta_0-1}^{(r-1)} & \beta_{\delta_0-1}^{(r-2)} & \cdots & \beta_{\delta_0-1}^{(0)} I_{N'}
\end{pmatrix} B_{VBK},
\]

(41)

Note that the first block Vandermonde matrix on \( \text{RHS of (41)} \) is nonsingular according to (i), so is the matrix \( M_{i,D_z} \).

**Case 2.** If \( \delta_z < r \), we have that the matrix \( M_{i,D_z} \) is of the form

\[
M_{i,D_z} = \begin{pmatrix}
I_{N'} & I_{N'} & \cdots & I_{N'} \\
\beta_0 I_{N'} & \beta_1 I_{N'} & \cdots & \beta_{r-1} I_{N'} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{\delta_0-1}^{(r-1)} & \beta_{\delta_0-1}^{(r-2)} & \cdots & \beta_{\delta_0-1}^{(0)} I_{N'}
\end{pmatrix} B_{VBK}
\]

according to (21) and (40). Then, the matrix \( M_{i,D_z} \) is nonsingular if the block matrix \( A_{VBK} \) is nonsingular. By switching some block columns of \( A_{VBK} \), we get \( |A_{VBK}| \neq 0 \) if and only if \( |H_{i,\delta_z,0}| \neq 0 \), where \( H_{i,\delta_z,0} \) is defined in (38). By (i) and Lemma 8 \( |H_{i,\delta_z,0}| \neq 0 \) if \( |H_{i,\delta_z,\delta_z}| \neq 0 \), where the invertibility of \( H_{i,\delta_z,\delta_z} \) follows from Lemma 7.

Collecting the above two cases, we can conclude that the matrix \( M_{i,D_z} \) with \( \alpha = 1 \) is nonsingular over \( \mathbb{F}_q \) for any \( i \in [0 : n] \) and \( D_z \in [0 : n) \setminus \{i\} \) with \( z \in [1 : m] \). This finishes the proof.

By combining Theorems 6 and 7 and the \( \delta_0 \)-optimal repair property of VBK code, we have the following theorem.

**Theorem 8.** The \((n, k)\) VBK code is a TMDS array code over \( \mathbb{F}_q \) with \( q \) in (29), and

- The sets \( J_0, J_1, \ldots, J_{\mu-1} \) in Definition 7 are given by (34), where \( \mu = \left[ \frac{\tau}{\delta_0} \right] \).
- For \( i \in [0 : n), t \in [0 : r) \) and \( u \in [0 : \delta_0-1 - \delta_0) \), the \( N \times N' \) key matrix \( K_{t,i,v}^{(0)} \) in (28) is \( \zeta_{u,\delta_0}^{(T)} R_{i,\delta_0}^{\top} \), where \( \zeta_{0,1}, \ldots, \zeta_{\delta_0-1 - \delta_0-1} \) are all distinct elements in \( \mathbb{F}_q \setminus \{\lambda_{t,i}, i \in [0 : n), u \in [0 : \delta_0) \} \).

The following theorem immediately follows from Theorems 3 and 8.

**Theorem 9.** By choosing VBK code as base code in Algorithm 2 an \((n, k)\) MDS array code \( \mathcal{C} \) with the \( \delta_{(0,m)} \)-optimal access property for all nodes over \( \mathbb{F}_q \) can be obtained, where \( q \geq \lfloor \frac{n+1}{2} \rfloor + 2 \) if \( \delta_0 = 2 \) and \( q \geq \lceil \frac{n+1}{2} \rceil + 2 \) if \( \delta_0 = 3, 4 \). Especially, the sub-packetization level of the MDS array code \( \mathcal{C} \) is \( \delta \left[ \frac{1}{\delta_0} \right] \) with \( \delta = \text{lcm}(\delta_0, \delta_1, \ldots, \delta_{m-1}) \).
Remark 2. Note that the generic construction method and Algorithm 2 have wide potential applications. For example, it can be verified that both the YB codes 1 and 2 in [23] are TMDS array codes and can be chosen as the base code. However, the resultant codes are not as good as the code \(G\) particularly in terms of the sub-packetization level, because the sub-packetization levels of YB codes 1 and 2 in [23] are much larger than that of the VBK code in [20], i.e., the base code of \(G\). Therefore, we do not present the two resultant codes in this paper.

V. COMPARISONS

In this section, we give comparisons of some key parameters among the proposed MDS array code \(G\) and some existing notable MDS codes with \(\delta_{[0,m]}\)-optimal repair property for all nodes, where \(\delta_{[0,m]} = \{\delta_0, \delta_1, \cdots, \delta_{m-1}\}\).

Table VI compares the details of these codes, while Table VII-VIII compare the new MDS array code \(G\), YB codes 3 and 4 in terms of the sub-packetization level, the smallest possible size of field with characteristic two, and the storage capacity \((N \log q)\) for \(\delta_0 = 2, 3\) and 4, respectively. From these tables, we see that the proposed MDS array code \(G\) has the following advantages:

- The new MDS array code \(G\) has the \(\delta_{[0,m]}\)-optimal access property for all nodes.
- Compared with YB code 3, the new \((n, k)\) MDS array code \(G\) has a smaller finite field size under the same parameters \(n, k\) and set \(\delta_{[0,m]}\), but do not possess the optimal update property.
- In contrast to YB codes 3 and 4 with the same \(\delta\) and \(n\), the sub-packetization level of code \(G\) is much smaller than that of YB codes 3 and 4. More precisely,
  1. For \(\delta_0 = 2, 3, 4\), the sub-packetization level of code \(G\) is decreased by a factor of \(\delta^{\lceil n / \delta \rceil}\) in contrast to YB codes 3 and 4;
  2. For \(\delta_0 > 4\), consider the new code \(G\) with the \((\{4\} \cup \delta_{[0,m]}\)\)-optimal access property for all nodes, its sub-packetization level \(N = (\text{lcm}(4, \delta))^\frac{1}{\eta}\) is decreased by a factor of \(\eta\) in contrast to YB codes 3 and 4, where

\[
\eta = \begin{cases} 
\delta^{\lceil n / \delta \rceil}, & \text{if } 4 \mid \delta \\
\frac{\delta^{\lceil n / \delta \rceil}}{4^\frac{1}{\eta}}, & \text{if } 2 \mid \delta \text{ and } 4 \nmid \delta \\
\frac{2^{\lceil n / \delta \rceil}}{\frac{4^\frac{1}{\eta}}, & \text{otherwise}
\end{cases}
\]

Moreover, it supports one more repair degree than YB codes 3 and 4 in this case.

- Compared with YB codes 3 and 4, the field size of new code \(G\) is smaller than that of YB code 3 but at most 6 times larger than that of YB code 4. Since the sub-packetization level of code \(G\) is decreased logarithmically with the code length \(n\) and value \(\delta_0\), thus the total storage \((N \log q \text{ bits})\) at each node of our new code \(G\) is much smaller than those of YB codes 3 and 4 under the same condition that all of them are constructed over the smallest possible finite field \(\mathbb{F}_q\), as shown in Tables VI-VIII.

### TABLE V
A COMPARISON OF SOME KEY PARAMETERS AMONG THE \((n, k)\) MDS ARRAY CODES \(G\), YB CODES 3 AND 4.

| Sub-packetization level \(N\) | Field size | Remark | Reference |
|-----------------------------|------------|--------|-----------|
| New MDS array code \(G\) \(\delta^{\lceil n / \delta \rceil}\), if \(\delta_0 \in [2 : 5]\); \(\frac{\lcm(4, \delta)^{\frac{1}{\eta}}}{\delta}\), if \(\delta_0 \geq 5\) | \(q \geq (\delta_0 + 2)\) if \(\delta_0 = 2\); \(18(\delta_0 + 2)\) if \(\delta_0 = 3, \delta_0 \geq 5\) | Optimal access | Theorem 9 |
| YB code 3 \(\delta^n\) | \(q \geq \delta n\) | Optimal update | [20] |
| YB code 4 \(\delta^n\) | \(q \geq n + 1\) | Optimal access | [20] |

VI. CONCLUSION

In this paper, we provided a generic construction method and further proposed an algorithm that can transform an existing TMDS array code with \(\delta_0\)-optimal repair property for all nodes into a new MDS array code with all nodes having \(\delta_{[0,m]}\)-optimal repair property, where \(1 < \delta_0 < \delta_1 < \cdots < \delta_{m-1} \leq r\). A new explicit construction of high-rate MDS array code \(G\) is obtained.
by directly applying the algorithm to VBK code, where each node of the new code G has the $\delta_{(0,m)}$-optimal access property. The comparisons show that the new code G outperforms existing MDS array codes (i.e., YB codes 3 and 4) in terms of the field size and/or the sub-packetization level under the same parameters $n, k$ and subset $\delta_{(0,m)}$ of $[2:r]$. Extending our generic construction method and specific algorithm to any MDS array codes with $\delta_0$-optimal repair property for all nodes is part of our ongoing work.

### TABLE VI
A comparison of some parameters among the (24, 20) MDS array codes G, YB codes 3 and 4 for $\delta_0 = 2$, where we set the finite field size as the power of 2

| Set of $\delta_{(0,m)}$ | Sub-packetization level $N$ | The finite field size $q$ | Storage capacity $(N \log q)$ | Storage capacity of YB code 4 |
|-------------------------|----------------------------|---------------------------|-------------------------------|-----------------------------|
| New code G              | $6^{12}$                   | $2^7$                     | $7 \times 6^{12}$             | $6.43 \times 10^{-17}$      |
| YB code 3               | $6^{24}$                   | $2^8$                     | $8 \times 6^{24}$             | $1.6$                       |
| YB code 4               | $6^{24}$                   | $2^5$                     | $5 \times 6^{24}$             | $1$                         |
| New code G              | $4^{12}$                   | $2^7$                     | $7 \times 4^{12}$             | $8.34 \times 10^{-8}$       |
| YB code 3               | $4^{24}$                   | $2^7$                     | $7 \times 4^{24}$             | $1.4$                       |
| YB code 4               | $4^{24}$                   | $2^5$                     | $5 \times 4^{24}$             | $1$                         |
| New code G              | $12^{12}$                  | $2^7$                     | $7 \times 12^{12}$            | $1.57 \times 10^{-15}$      |
| YB code 3               | $12^{24}$                  | $2^9$                     | $9 \times 12^{24}$            | $1.8$                       |
| YB code 4               | $12^{24}$                  | $2^5$                     | $5 \times 12^{24}$            | $1$                         |

### TABLE VII
A comparison of some parameters among the (24, 19) MDS array codes G, YB codes 3 and 4 for $\delta_0 = 3$, where we set the finite field size as the power of 2

| Set of $\delta_{(0,m)}$ | Sub-packetization level $N$ | The finite field size $q$ | Storage capacity $(N \log q)$ | Storage capacity of YB code 4 |
|-------------------------|----------------------------|---------------------------|-------------------------------|-----------------------------|
| New code G              | $12^6$                     | $2^8$                     | $7 \times 12^6$               | $7.57 \times 10^{-18}$      |
| YB code 3               | $12^{24}$                  | $2^9$                     | $9 \times 12^{24}$            | $1.8$                       |
| YB code 4               | $12^{24}$                  | $2^5$                     | $5 \times 12^{24}$            | $1$                         |
| New code G              | $15^6$                     | $2^8$                     | $8 \times 15^6$               | $2.44 \times 10^{-19}$      |
| YB code 3               | $15^{24}$                  | $2^9$                     | $9 \times 15^{24}$            | $1.8$                       |
| YB code 4               | $15^{24}$                  | $2^5$                     | $5 \times 15^{24}$            | $1$                         |
| New code G              | $60^6$                     | $2^8$                     | $8 \times 60^6$               | $5.67 \times 10^{-26}$      |
| YB code 3               | $60^{24}$                  | $2^{11}$                  | $11 \times 60^{24}$           | $2.2$                       |
| YB code 4               | $60^{24}$                  | $2^5$                     | $5 \times 60^{24}$            | $1$                         |

### TABLE VIII
A comparison of some parameters among the (24, 18) MDS array codes G, YB codes 3 and 4 for $\delta_0 = 4$, where we set the finite field size as the power of 2

| Set of $\delta_{(0,m)}$ | Sub-packetization level $N$ | The finite field size $q$ | Storage capacity $(N \log q)$ | Storage capacity of YB code 4 |
|-------------------------|----------------------------|---------------------------|-------------------------------|-----------------------------|
| New code G              | $20^6$                     | $2^7$                     | $7 \times 20^6$               | $5.34 \times 10^{-24}$      |
| YB code 3               | $20^{24}$                  | $2^9$                     | $9 \times 20^{24}$            | $1.8$                       |
| YB code 4               | $20^{24}$                  | $2^5$                     | $5 \times 20^{24}$            | $1$                         |
| New code G              | $12^6$                     | $2^7$                     | $7 \times 12^6$               | $5.26 \times 10^{-26}$      |
| YB code 3               | $12^{24}$                  | $2^9$                     | $9 \times 12^{24}$            | $1.8$                       |
| YB code 4               | $12^{24}$                  | $2^5$                     | $5 \times 12^{24}$            | $1$                         |
| New code G              | $60^6$                     | $2^7$                     | $7 \times 60^6$               | $1.38 \times 10^{-24}$      |
| YB code 3               | $60^{24}$                  | $2^{11}$                  | $11 \times 60^{24}$           | $2.2$                       |
| YB code 4               | $60^{24}$                  | $2^5$                     | $5 \times 60^{24}$            | $1$                         |
APPENDIX A

PROOF OF PROPERTY 1

All the three properties rely on a fact from Lines 2 and 6 of Algorithm 1 that

\[ P_{i,j} = \bigcup_{a=0}^{l_j-1} P_{i,j}^{(a)} = \{ f_1^{(a)}[u] \mid a \in \{ l_j : l_{j-1} \}, u \in [0 : \delta_0) \} \cup \bigcup_{a=l_j}^{l_{j-1} - 1} (P_{i,1}^{(a)} \cup \cdots \cup P_{i,j-1}^{(a)}), j \in [1 : m]. \]  

(42)

Firstly, we prove P1 by induction.

i) If \( j = 1 \), then P1 is a direct consequence of Lines 2 and 4 of Algorithm 1.

ii) Suppose that P1 holds for \( j = w \), where \( 1 \leq w < m - 1 \), i.e.,

\[ P_{i,1} \cup P_{i,2} \cdots \cup P_{i,w} = \bigcup_{a=0}^{l_{w+1}-1} (P_{i,1}^{(a)} \cup P_{i,2}^{(a)} \cup \cdots \cup P_{i,w}^{(a)}) = \{ f_1^{(a)}[u] \mid a \in [l_{w+1} : l_w), u \in [0 : \delta_0) \}. \]  

(43)

Then, it follows from (42) and (43) that

\[ P_{i,1} \cup P_{i,2} \cdots \cup P_{i,w+1} = \bigcup_{a=0}^{l_{w+1}-1} (P_{i,1}^{(a)} \cup P_{i,2}^{(a)} \cup \cdots \cup P_{i,w+1}^{(a)}) \]

\[ = (\bigcup_{a=0}^{l_{w+1}-1} P_{i,1}^{(a)}) \cup (\bigcup_{a=0}^{l_{w+1}-1} P_{i,2}^{(a)}) \cup \cdots \cup (\bigcup_{a=0}^{l_{w+1}-1} P_{i,w+1}^{(a)}) \]

\[ = \{ f_1^{(a)}[u] \mid a \in [l_{w+1} : l_w), u \in [0 : \delta_0) \} \bigcup \bigcup_{a=0}^{l_{w+1}-1} (P_{i,1}^{(a)} \cup P_{i,2}^{(a)} \cup \cdots \cup P_{i,w+1}^{(a)}) \]

\[ = \{ f_1^{(a)}[u] \mid a \in [l_{w+1} : l_0), u \in [0 : \delta_0) \}, \]

i.e., P1 also holds for \( j = w + 1 \).

Secondly, we prove P2. When \( z = j - 1 \), P2 is obvious from (42). For \( z \leq j-2 \), by applying (42) \( j-z \) times we have

\[ P_{i,j} = \bigcup_{a=0}^{l_j-1} (P_{i,1}^{(a)} \cup P_{i,2}^{(a)} \cup \cdots \cup P_{i,j-1}^{(a)}) \]

\[ \subseteq \bigcup_{a=0}^{l_j-1} (P_{i,1}^{(a)} \cup P_{i,2}^{(a)} \cup \cdots \cup P_{i,j-1}^{(a)}) \cup \bigcup_{a=l_j}^{l_{j-1} - 1} (P_{i,1}^{(a)} \cup P_{i,2}^{(a)} \cup \cdots \cup P_{i,j-2}^{(a)}) \]

\[ = \bigcup_{a=0}^{l_j-1} (P_{i,1}^{(a)} \cup P_{i,2}^{(a)} \cup \cdots \cup P_{i,j-2}^{(a)}) \]

\[ \vdots \]

\[ \subseteq \bigcup_{a=0}^{l_{j+1}-1} (P_{i,1}^{(a)} \cup P_{i,2}^{(a)} \cup \cdots \cup P_{i,j-z}^{(a)}) \]

\[ = \bigcup_{a=0}^{l_{j+1}-1} (P_{i,1}^{(a)} \cup P_{i,2}^{(a)} \cup \cdots \cup P_{i,j-z}^{(a)}). \]

Thirdly, we prove P3 for \( 1 \leq j < m \) by induction.

i) If \( j = 1 \), then P3 follows from Lines 2 and 4 of Algorithm 1.

ii) Suppose that P3 holds for all \( j = 1, 2, \cdots, w \) where \( 1 \leq w < m - 1 \), i.e., \( |P_{i,j}| = (\delta_j - \delta_{j-1})l_j \) and \( |P_{i,j}^{(a)}| = \delta_j - \delta_{j-1} \) for \( a \in [0 : l_j) \) and \( j = 1, 2, \cdots, w \). We next prove that P3 holds for \( j = w + 1 \).
Given \( i \in \mathcal{G} \) and \( w \in [1 : m] \), by P1 and the hypothesis, we have

\[
\delta_0(l_0 - l_w) = |\{f_i^{(a)}[u] | a \in [l_w : l_0), u \in [0 : \delta_0)\}| \\
= | \bigcup_{a=0}^{l_w-1} (P_{t,i}^{(a)} \cup P_{t',i}^{(a)} \cup \cdots \cup P_{t',i}^{(a)}) | \\
\leq \sum_{a=0}^{l_w-1} (|P_{t,i}^{(a)}| + |P_{t',i}^{(a)}| + \cdots + |P_{t',i}^{(a)}) \\
= l_w(\delta_1 - \delta_0) + (\delta_2 - \delta_1) + \cdots + (\delta_w - \delta_{w-1}) \\
= l_w(\delta_w - \delta_0) \\
= \delta_0(l_0 - l_w),
\]

which implies

\[
| \bigcup_{a=0}^{l_w-1} (P_{t,i}^{(a)} \cup P_{t',i}^{(a)} \cup \cdots \cup P_{t',i}^{(a)}) | = \sum_{a=0}^{l_w-1} (|P_{t,i}^{(a)}| + |P_{t',i}^{(a)}| + \cdots + |P_{t',i}^{(a)}) 
\]

i.e.,

\[
P_{t,i}^{(a)} \cap P_{t,i'}^{(a')} = \emptyset \text{ for } s, s' \in [1 : w] \text{ and } a, a' \in [0 : l_w) \text{ with } (s, a) \neq (s', a').
\]

Then, by (10), (42) and (45), similarly to (44) we obtain

\[
|P_{t,w+1}^{(a)}| = |\{f_i^{(a)}[u] | u \in [0 : \delta_0), a \in [l_w+1 : l_w)\}| + (l_w - l_{w+1})(\delta_w - \delta_0) \\
= \delta_0(l_w - l_{w+1}) + (l_w - l_{w+1})(\delta_w - \delta_0) \\
= \delta_w(l_w - l_{w+1}) \\
= (\delta_{w+1} - \delta_w)l_{w+1},
\]

which together with Line 7 of Algorithm 1 shows P3 holds for \( j = w + 1 \). This finishes the proof.

**APPENDIX B**

**PROOF OF LEMMA 3**

Given \( a \in [l_{w+1} : l_w) \) with \( 1 \leq w < m \), according to P3, we set

\[
P_{j,s}^{(a)} = \left\{ f_j^{(b_{j-1} - \delta_0)}[u_{s_{j-1} - \delta_0}], \ldots, f_j^{(b_s - \delta_0 - 1)}[u_{s_{j-1} - \delta_0}] \right\} \text{ for all } s \in [1 : w], j \in \mathcal{G},
\]

which together with (16) gives

\[
S_{t,i}^{(a)} P_{t,i}^{(a)} = \sum_{v=0}^{\delta_w - \delta_0 - 1} S_{t,i}^{(a)} K_{t,j,v} f_j^{(b_v)}[u_v] = \sum_{v=0}^{\delta_w - \delta_0 - 1} S_{t,i}^{(a)} K_{t,j,v} \Phi_{\alpha,uv} f_j^{(b_v)}
\]

for \( i \in [0 : n] \setminus \mathcal{G}, j \in \mathcal{G}, t \in [0 : r) \) and \( z \in [0 : m] \), where \( \{(b_p, u_p) | p \in [0 : \delta_w - \delta_0)\} \subset [0 : l_0) \times [0 : \delta_0) \) and the second equality holds because of (15). Moreover, it follows from P1 and Lines 2, 7 of Algorithm 1 that

\[
P_{j,s}^{(a)} \subseteq \left\{ f_j^{(b)}[u] | b \in [l_w : l_0), u \in [0 : \delta_0) \right\}, s \in [1 : w],
\]

which implies that \( b_p \in [l_w : l_0) \) for any \( p \in [0 : \delta_w - \delta_0) \).

Then, for any \( a \in [l_{w+1} : l_w) \) with \( w \in [1 : m] \), any \( z \in [0 : m] \), \( t \in [0 : r) \), \( i \in [0 : n] \) \( \mathcal{G} \) and \( j \in \mathcal{G} \), according to (46), we are able to compute the vector \( S_{t,i}^{(a)} P_{t,i}^{(a)} \) from the data in set \( \{ R_{t,i} f_j^{(b)} | b \in [l_w : l_0) \} \) based on C3.

**APPENDIX C**

**PROOF OF LEMMA 4**

Our task is to prove that the matrices \( K_{t,i,v}^{(a)} \) \( t \in [0 : r), i \in \mathcal{G}, v \in [0 : \delta_{m-1} - \delta_0) \) given in (27) are the key matrices such that the nodes in set \( \mathcal{G}' \) of new code \( \mathcal{C}_2 \) satisfy C1-C3.
Firstly, we verify that the nodes in set \( G' \) of new code \( C_2 \) satisfy C1. For any \( t \in [0 : r) \), \( v \in [0 : \delta_{m-1} - \delta_0) \) and \( i, j \in G' \) with \( i \neq j \), by (19) and (27), we then have
\[
S_{t,\delta_0} K_{t,j,v} = \text{blkdiag}(S_{t,\delta_0} K_{l,j}, \ldots, S_{t,\delta_0} K_{l,j,v})_{l_0}.
\]
Recall that the nodes in set \( G' \) of base code \( C_0 \) satisfy C1, thus the nodes in set \( G' \) of new code \( C_2 \) also satisfy C1.

Secondly, we check that the nodes in set \( G' \) of new code \( C_2 \) satisfy C2. Given a node \( j \in [0 : n) \) of code \( C_2 \), let \( A'_{t,j} \) \((t \in [0 : r))\) be its parity-check matrix and \( g_j = ((f_j^{(0)})^\top, (f_j^{(1)})^\top, \ldots, (f_j^{(l_0-1)})^\top)^\top \) be the data stored at node \( j \). Then, for any \( j \in G' \), by (11),
\[
A'_{t,j} g_j = \begin{cases} 
\text{blkdiag}(A_{t,j}, A_{t,j}, \ldots, A_{t,j})_{l_0} g_j + \begin{pmatrix} P_{t,j}^{(0)} \\ P_{t,j}^{(1)} \\ \vdots \\ P_{t,j}^{(l_0-1)} \end{pmatrix}, & \text{if } j \in G, \\
\text{blkdiag}(A_{t,j}, A_{t,j}, \ldots, A_{t,j})_{l_0} g_j, & \text{otherwise.}
\end{cases}
\]
Thus, for any \( i \in G' \) and \( j \in [0 : n) \setminus \{i\} \), by (5), (13) and (19), we have
\[
S'_{t,\delta_0} A'_{t,i} = \text{blkdiag}(S_{t,\delta_0} A_{t,i}, S_{t,\delta_0} A_{t,i}, \ldots, S_{t,\delta_0} A_{t,i})_{l_0}
\]
and
\[
\tilde{A}'_{t,i,i,\delta_0} = \begin{pmatrix} \tilde{A}_{t,i,i,\delta_0} \\ \vdots \\ \tilde{A}_{t,i,i,\delta_0} \end{pmatrix}, \text{ if } j \in G, \quad \text{otherwise},
\]
where the case of \( j \in G \) follows from P0 and Lemma 3. \( \tilde{A}'_{t,i,i,\delta_0} \) is the matrix defined in (5), and symbol \( \# \) denotes some matrices which we do not care about the exact expression.

Let \( M'_{t,D_z} \) be the matrix defined in (21), where one should note that the symbols \( A, K, S \) in \( 21 \) are replaced by \( A', K', S' \), respectively. According to (47) and (48), by exchanging some block rows and block columns of matrix \( M'_{t,D_z} \), we obtain
\[
\text{Rank}(M'_{t,D_z}) = \text{Rank}
\begin{pmatrix} 
M_{i,D_z} \\ M_{i,D_z} \\ \vdots \\ M_{i,D_z} 
\end{pmatrix},
\]
which finishes the proof of C2, together with the fact that the nodes in \( G' \) of base code satisfy C2 as well.

Finally, we show that the nodes in \( G' \) of new code \( C_2 \) satisfy C3. Note that the sub-packetization level of new code \( C_2 \) is \( l_0 \alpha N \), let \( \alpha' = l_0 \alpha \). Then by (13),
\[
\Phi_{\alpha',u} = \text{blkdiag}(\Delta_u, \Delta_u, \ldots, \Delta_u)_{l_0} = \text{blkdiag}(\Phi_{\alpha,u}, \Phi_{\alpha,u}, \ldots, \Phi_{\alpha,u})_{l_0}
\]
due to \( \alpha' = l_0 \alpha \). Thus by (19), (27) and (49), for any \( j \in G' \), \( i \in [0 : n) \setminus \{j\} \), \( v \in [0 : \delta_{m-1} - \delta_0) \) and \( u \in [0 : \delta_0) \), we get
\[
S'_{t,\delta_0} K'_{t,j,v} \Phi_{\alpha,u} = \begin{cases} 
\begin{pmatrix} 
S_{t,\delta_0} K_{l,j,t} \Phi_{\alpha,u} \\ \vdots \\ S_{t,\delta_0} K_{l,j,t} \Phi_{\alpha,u} \end{pmatrix}, & \text{if } i \in G, \\
\text{blkdiag}(S_{t,\delta_0} K_{l,j,t} \Phi_{\alpha,u}, S_{t,\delta_0} K_{l,j,t} \Phi_{\alpha,u}, \ldots, S_{t,\delta_0} K_{l,j,t} \Phi_{\alpha,u})_{l_0}, & \text{otherwise},
\end{cases}
\]
where one should note that (50) holds for \( i \notin G \) if and only if the node \( i \) has \( \delta_2 \)-optimal repair property in new code \( C_2 \).
Combining (18), (19) and (50), we obtain
\[
\text{Rank} \left( \begin{pmatrix} R_{i,\delta z} \\ S_{i,\delta z} \end{pmatrix} \right) = l_z \cdot \text{Rank} \left( \begin{pmatrix} R_{i,\delta z} \\ S_{i,\delta z} \end{pmatrix} \right) = l_z \cdot \frac{\alpha N}{\delta z} = \frac{l_0 \alpha N}{\delta z} \quad \text{if } i \in G
\]
and
\[
\text{Rank} \left( \begin{pmatrix} A_{i,\delta z} \\ B_{i,\delta z} \end{pmatrix} \right) = l_0 \cdot \text{Rank} \left( \begin{pmatrix} A_{i,\delta z} \\ B_{i,\delta z} \end{pmatrix} \right) = \frac{l_0 \alpha N}{\delta z} \quad \text{if } i \in [0 : n] \setminus (G \cup G')
\]
since the nodes in $G'$ of base code $G_0$ satisfy C3, where we make use of the fact $\frac{l_0}{\delta_0} = \frac{l_0}{\delta_z}$ from (10). That is, the nodes in $G'$ of new code $G_2$ satisfy C3.

APPENDIX D

PROOFS OF LEMMAS 5, 6 AND 8

Before proving those three lemmas, let us introduce some necessary notations. Note that $N' = \frac{N}{\delta_0} = \delta_0^{\tau-1}$. For a given $a = (a_{\tau-2}, a_{\tau-3}, \ldots, a_0) \in [0 : N')$, define
\[
\varphi(a, x, u) = (a_{\tau-2}, \ldots, a_x, u, a_{x-1}, \ldots, a_0)
\]
for any $x \in [0 : \tau)$, $u \in [0 : \delta_0)$, i.e., insert the value $u$ between the $x$-digit and $(x-1)$-digit of the vector $a$ if $x < \tau - 1$, and insert the value $u$ before the $(\tau-1)$-digit if $x = \tau - 1$. Then by (9), we easily get
\[
V_{x,u}(a, \cdot) = e_{\varphi(a,x,u)}, \quad a \in [0 : N').
\]  
(52)

By (7), (50) and (51), we have the following simple facts.

**Fact.** 1) For $0 \leq a = (a_{\tau-1}, a_{\tau-2}, \ldots, a_0), b = (b_{\tau-1}, b_{\tau-2}, \ldots, b_0) < N$,
\[
e_{a}e_{b}^T = \begin{cases} 1, & \text{if } a = b, \\
0, & \text{otherwise}. \end{cases}
\]  
(53)

2) For $a = (a_{\tau-2}, a_{\tau-3}, \ldots, a_0) \in [0 : N')$, $0 \leq x, x < \tau$ and $0 \leq u, v < \delta_0$,
\[
(\varphi(a, \bar{x}, u))_x = \begin{cases} a_x, & \text{if } x < \bar{x}, \\
u, & \text{if } x = \bar{x}, \\
a_{x-1}, & \text{if } x > \bar{x}, \end{cases}
\]  
(54)

and
\[
\pi_x(\varphi(a, \bar{x}, u), v) = \begin{cases} \varphi(\pi_{\tau-1}(a, x, v), \bar{x}, u), & \text{if } x < \bar{x}, \\
\varphi(a, \bar{x}, v), & \text{if } x = \bar{x}, \\
\varphi(\pi_{\tau-1}(a, x - 1, v), \bar{x}, u), & \text{if } x > \bar{x}. \end{cases}
\]  
(55)

**Proof of Lemma 5**

Clearly, (i) is true because of (9) and (53).

Next, we prove (ii) for $0 \leq u, v, h < \delta_0$ and $0 \leq x \neq \bar{x} < \tau$. Given $a = (a_{\tau-2}, a_{\tau-3}, \ldots, a_0) \in [0 : N')$, on one hand,
\[
V_{x,u}^T(a, \cdot) \cdot (V_{x,u}^T \Delta h) = e_{\varphi(a,x,u)} \cdot e_{\varphi(0,\bar{x},v)}^T e_{\varphi(1,\bar{x},v)}^T \cdots e_{\varphi(N'-1,\bar{x},v)}^T \begin{pmatrix} e_{h,N'} \\
e_{h,N'+1} \\
\vdots \\
e_{h,N'+N'-1} \end{pmatrix}
\]

\[
= \sum_{b=0}^{N'-1} e_{\varphi(a,x,u)} e_{\varphi(b,\bar{x},v)} e_{h,N'+b}
\]

\[
= \sum_{b=0}^{N'-1} (e_{\varphi(a,x,u)} e_{\varphi(b,\bar{x},v)}) e_{h,N'+b}
\]

\[
= e_{h,N'+b}
\]  
(56)
where the first identity follows from (14) and (52), and the fourth identity comes from (53).

For \( x \neq \bar{x} \), by (51), we have \( \varphi(a, x, u) = \varphi(b, \bar{x}, v) \) if and only if

\[
\bar{b} = \begin{cases} 
    (a_{r-2}, \ldots, a_{\bar{x}+1}, a_{\bar{x}-1}, \ldots, a_0), & \text{if } 0 \leq x < \bar{x} < \tau, a_{\bar{x}-1} = v, \\
    (a_{r-2}, \ldots, a_{\bar{x}}, u, a_{\bar{x}-1}, \ldots, a_0), & \text{if } 0 \leq \bar{x} < \tau, a_{\bar{x}} = v.
\end{cases}
\]

(57)

On the other hand, applying (56), (52) and (57), we get

\[
T_{x, \bar{x}, v, h}(a, :), V_{x, u} = \begin{cases} 
    e_{\bar{a}} V_{x, u}, & \text{if } 0 \leq x < \bar{x} < \tau, a_{\bar{x}-1} = v \text{ or } 0 \leq \bar{x} < \tau, a_{\bar{x}} = v \\
    0, & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
    e_{\varphi(a, x, u)}, & \text{if } 0 \leq x < \bar{x} < \tau, a_{\bar{x}-1} = v \text{ or } 0 \leq \bar{x} < \tau, a_{\bar{x}} = v \\
    0, & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
    e_{b}, & \text{if } \varphi(b, \bar{x}, v) = \varphi(a, x, u), \\
    0, & \text{otherwise,}
\end{cases}
\]

(58)

where

\[
\bar{a} = \begin{cases} 
    (h, a_{r-2}, \ldots, a_{\bar{x}-1}, a_{\bar{x}}, a_{\bar{x}+1}, \ldots, a_0), & \text{if } 0 \leq x < \bar{x} < \tau, a_{\bar{x}-1} = v \\
    (h, a_{r-2}, \ldots, a_{\bar{x}}, a_{\bar{x}+1}, \ldots, a_0), & \text{if } 0 \leq \bar{x} < \tau, a_{\bar{x}} = v
\end{cases}
\]

Collecting (56) and (58), we complete the proof.

In what follows, we give the proofs of Lemma 6 and 8, in which we always let \( i = \delta_0 \bar{x} + \bar{y} \) and \( j = \delta_0 x + y \), where \( 0 \leq \bar{x}, x < \tau \) and \( 0 \leq \bar{y}, y < \delta_0 \).

Proof of Lemma 6

For any given \( a = (a_{r-2}, a_{r-3}, \ldots, a_0) \in \{0 : N^\prime\} \), according to (31), (32) and (52), we have

\[
V_{\bar{x}, \bar{y}}(a, :) A_{t, j} = e_{\varphi(a, \bar{x}, \bar{y})} \cdot A_{t, j}
\]

\[
= e_{\varphi(a, \bar{x}, \bar{y})} \left( \sum_{b=0}^{N^\prime-1} \lambda_{j, b} e_b e_b^\top + \sum_{b=0}^{N^\prime-1} \epsilon_{u, y} \lambda_{j, \bar{u}} e_b^\top e_{\pi(b, x, u)} \right)
\]

\[
= \begin{cases} 
    \lambda_{i, \bar{y}} e_{\varphi(a, \bar{x}, \bar{y})} + \sum_{u=0}^{\delta_0-1} \epsilon_{u, y} \lambda_{j, \bar{u}} e_{\pi(u, \varphi(a, \bar{x}, \bar{y}))}, & \text{if } (\varphi(a, \bar{x}, \bar{y})) = y, \\
    \lambda_{j, \bar{y}} e_{\varphi(a, \bar{x}, \bar{y})}, & \text{otherwise,}
\end{cases}
\]

(59)

due to (53).

Particularly, when \( x = \bar{x} \), (59) becomes

\[
V_{\bar{x}, \bar{y}}(a, :) A_{t, j} = \begin{cases} 
    \lambda_{i, \bar{y}} V_{\bar{x}, \bar{y}}(a, :) + \sum_{u=0}^{\delta_0-1} \epsilon_{u, y} \lambda_{i, \bar{u}} V_{\bar{x}, u}(a, :), & \text{if } i = j, \\
    \lambda_{j, \bar{y}} V_{\bar{x}, \bar{y}}(a, :) , & \text{if } i \neq j \text{ and } x = \bar{x}
\end{cases}
\]

since \( (\varphi(a, \bar{x}, \bar{y})) = \bar{y} \) and \( e_{\pi(u, \varphi(a, \bar{x}, \bar{y}))} = e_{\varphi(a, \bar{x}, u)} \) according to (52), (54) and (55). That is,

\[
V_{\bar{x}, \bar{y}} A_{t, j} = \begin{cases} 
    \lambda_{i, \bar{y}} V_{\bar{x}, \bar{y}} + \sum_{u=0}^{\delta_0-1} \epsilon_{u, y} \lambda_{i, \bar{u}} V_{\bar{x}, u}, & \text{if } i = j, \\
    \lambda_{j, \bar{y}} V_{\bar{x}, \bar{y}} , & \text{if } i \neq j \text{ and } x = \bar{x}
\end{cases}
\]

which together with (33) implies

\[
S_{i, \delta_0} A_{t, i} = \lambda_{i, \bar{y}} V_{\bar{x}, \bar{y}} + \sum_{u=0}^{\delta_0-1} \epsilon_{u, y} \lambda_{i, \bar{u}} V_{\bar{x}, u}
\]

and

\[
S_{i, \delta_0} A_{t, j} = \lambda_{j, \bar{y}} R_{i, \delta_0} \text{ for } 0 \leq i < j < n \text{ with } x = \bar{x},
\]
i.e., (i) is true and (ii) holds for \( 0 \leq i \neq j < n \) with \( x = \bar{x} \).
Next we prove this lemma for \( x \neq \hat{x} \). Herein we only check the case of \( x < \hat{x} \) since the case of \( x > \hat{x} \) can be proved in a similar manner. In this case, i.e., \((\varphi(a, \hat{x}, \hat{y})), x = a_x \) by (54), then (59) turns into

\[
V_{\hat{x}, \hat{y}}(a_x, \cdot) A_{t, j} = \begin{cases} 
\sum_{u=0, u \neq y}^{\delta_0-1} \epsilon_{u, y} \lambda_{j, u}^T \epsilon_{a_x}^T \varphi(\pi_{r-1}(a, x, u), \hat{x}, \hat{y}), & \text{if } a_x = y, \\
\lambda_{j, a_x}^T \epsilon_{a_x}, & \text{otherwise,}
\end{cases}
\]

\[
= \begin{cases} 
\sum_{u=0, u \neq y}^{\delta_0-1} \epsilon_{u, y} \lambda_{j, u}^T \epsilon_{\pi_{r-1}(a, x, u)}^T V_{\hat{x}, \hat{y}}(\pi_{r-1}(a, x, u), \cdot), & \text{if } a_x = y, \\
\lambda_{j, a_x}^T \epsilon_{a_x} \cdot V_{\hat{x}, \hat{y}}, & \text{otherwise,}
\end{cases}
\]

\[
= \begin{cases} 
\epsilon_a \sum_{b=0}^{N-1} \lambda_{j, b_x}^T \epsilon_{b} + \sum_{b=0}^{N-1} \sum_{b=0, b_x = y = 0, u \neq y}^{\delta_0-1} \epsilon_{u, y} \lambda_{j, u}^T \epsilon_{b} \epsilon_{\pi_{r-1}(b, x, u)}^T V_{\hat{x}, \hat{y}}, & \text{if } a_x = y, \\
\lambda_{j, a_x}^T \epsilon_{a_x} \cdot V_{\hat{x}, \hat{y}}, & \text{otherwise,}
\end{cases}
\]

\[
= \epsilon_a A_{t, j, i, \delta_0} V_{\hat{x}, \hat{y}} = A_{t, j, i, \delta_0} (a_x, \cdot) \cdot V_{\hat{x}, \hat{y}}
\]

(60)

where the first equality follows from (55), the second equality comes from (52), and the fourth equality can be derived similarly to the third equality in (59). Applying (33) and (60), we have \( S_{i, \delta_0} A_{t, j} = A_{t, j, i, \delta_0} R_{t, \delta_0} \), which finishes the proof.

**Proof of Lemma 8**

Hereafter we only check the case of \( 0 < s < p - 1 \) since the case of \( s = p - 1 \) can be verified similarly. For \( i, j \in [0 : n) \) with \( i \neq j \), define

\[
\Upsilon_{j, i} = \begin{cases} 
\sum_{a=0}^{N-1} \lambda_{j, a_x}^T \epsilon_a, & \text{if } x < \hat{x}, \\
\lambda_{j, \hat{y}}^T \epsilon_{N'}, & \text{if } x = \hat{x}, \\
\sum_{a=0}^{N-1} \lambda_{j, a_x-1}^T \epsilon_a, & \text{if } x > \hat{x},
\end{cases}
\]

(61)

which together with (37) implies

\[
A_{t, j, i, \delta_0} \Upsilon_{j, i} = \begin{cases} 
\sum_{a=0}^{N-1} \lambda_{j, a_x}^T \epsilon_a + \sum_{a=0, a_x = y = 0, u \neq y}^{\delta_0-1} \epsilon_{u, y} \lambda_{j, u}^T \epsilon_{\pi_{r-1}(a, x, u)}^T \sum_{b=0}^{N-1} \lambda_{j, b_x}^T \epsilon_b, & \text{if } x < \hat{x}, \\
\lambda_{j, \hat{y}}^T \epsilon_{N'}, & \text{if } x = \hat{x}, \\
\sum_{a=0}^{N-1} \lambda_{j, a_x}^T \epsilon_a + \sum_{a=0, a_x = y = 0, u \neq y}^{\delta_0-1} \epsilon_{u, y} \lambda_{j, u}^T \epsilon_{\pi_{r-1}(a, x, u)}^T \sum_{b=0}^{N-1} \lambda_{j, b_x}^T \epsilon_b, & \text{if } x > \hat{x},
\end{cases}
\]

(62)
for \(0 \leq i \neq j < n\) and \(0 \leq t < r - 1\). Let us define a block lower triangular matrix \(\Psi_s\) of order \((r - s)N'\) as

\[
\Psi_s = \begin{pmatrix}
I_{N'} & \beta_1 I_{N'} & -I_{N'} & \cdots \\
\beta_2 I_{N'} & -I_{N'} & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{r-s} I_{N'} & \cdots & \cdots & -I_{N'}
\end{pmatrix}_{(r-s) \times (r-s)}.
\]

Then by (62), multiplying \(H_{i,s}^p\) on the left by \(\Psi_s\) we obtain

\[
\Psi_s H_{i,p,s} = \begin{pmatrix}
I_{N'} & W_1 & W_2 & W_3 & W_4
\end{pmatrix}_{(r-s)N' \times N'},
\]

where

\[
W_1 = (I_{N'}, \ldots, I_{N'}),
\]

\[
W_2 = \begin{pmatrix}
\tilde{A}_{0,j_0,i} & \ldots & \tilde{A}_{0,j_{r-p-1},i}
\end{pmatrix},
\]

\[
W_3 = \begin{pmatrix}
I_{N'} & \ldots & I_{N'} \\
\beta_{s+1} I_{N'} & \ldots & \beta_{r-s-1} I_{N'} \\
\vdots & \vdots & \vdots \\
\beta_{r-s} I_{N'} & \ldots & \beta_{s-1} I_{N'}
\end{pmatrix},
\]

\[
W_4 = \begin{pmatrix}
\tilde{A}_{0,j_0,i} & \ldots & \tilde{A}_{0,j_{r-p-1},i}
\end{pmatrix},
\]

and

\[
Q_{s,0} = \begin{pmatrix}
(\beta_s - \beta_{s+1}) I_{N'} \\
\vdots \\
(\beta_s - \beta_{p-1}) I_{N'}
\end{pmatrix},
\]

\[
Q_{s,1} = \begin{pmatrix}
\beta_s I_{N'} - \Upsilon_{j_0,i} \\
\vdots \\
\beta_s I_{N'} - \Upsilon_{j_{r-p-1},i}
\end{pmatrix}.
\]

Then we obtain

\[
|\Psi_s||H_{i,p,s}| = |I_{N'}| \left| \begin{pmatrix}
W_3 & Q_{s,0} \\
W_4 & Q_{s,1}
\end{pmatrix} \right| = \left| \begin{pmatrix}
I_{N'} & W_3 \\
W_4 & Q_{s,1}
\end{pmatrix} \right| Q_{s,0} = |\Psi_s||H_{i,p,s}||Q_{s,0}| / |Q_{s,1}|,
\]

where the last equality follows from the definition of matrix \(H_{i,p,s}\) in (38). It is clear that \(Q_{s,0} \neq 0\) since \(\beta_u \neq \beta_v\) for any \(0 \leq u \neq v < p\). Additionally, for any \(j \in [0 : n] \setminus \{i\}\), by (61) we have

\[
\beta_u I_{N'} - \Upsilon_{j,i} = \begin{cases} 
\sum_{a=0}^{N'-1} (\beta_u - \lambda_{j,a}) \tilde{e}_a, & \text{if } x < \tilde{x}, \\
(\beta_u - \lambda_{j,0}) I_{N'}, & \text{if } x = \tilde{x}, \\
\sum_{a=0}^{N'-1} (\beta_u - \lambda_{j,a-1}) \tilde{e}_a, & \text{if } x > \tilde{x},
\end{cases}
\]

which is nonsingular according to conditions (ii) and (iii) in this lemma. Then, we arrive at the desired conclusion.

REFERENCES

[1] M. Blaum, P.G. Farrell, and H. van Tilborg, “Array codes,” Handbook of Coding Theory, V. Pless and W. C. Huffman, Eds. Elsevier Science, 1998, vol. II, ch. 22, pp. 1855-1909.
[2] D. Borthakur, “HDFS Architecture Guide,” in Hadoop Apache Project, 2008. [Online]. Available: http://hadoop.apache.org/common/docs/current/hdfs-design.pdf
[3] R. Bhagwan, K. Tati, Y.-C. Cheng, S. Savage, and G.M. Voelker, “Total recall: System support for automated availability management,” in Proc. 1st Symposium on Networked Systems Design and Implementation (NSDI), San Francisco, CA, Mar. 2004.
[4] A.G. Dimakis, P. Godfrey, Y. Wu, M. Wainwright, and K. Ramchandran, “Network coding for distributed storage systems,” IEEE Trans. Inform. Theory, vol. 56, no. 9, pp. 4539-4551, Sep. 2010.
[5] F. Dabek, J. Li, E. Sit, J. Robertson, M. Kaashoek, and R. Morris, “Designing a DHT for low latency and high throughput,” in Proc. 1st Symposium on Networked Systems Design and Implementation (NSDI), San Francisco, CA, Mar. 2004.
[6] C. Huang, H. Simitci, Y. Xu, A. Ogus, B. Calder, P. Gopalan, J. Li, and S. Yekhanin, “Erasure coding in Windows Azure storage,” in Proc. 2012 USENIX Annual Technical Conference, Boston, MA, pp. 1-12, Jun. 2012.

[7] J. Li and X.H. Tang, “Optimal exact repair strategy for the parity nodes of the \((k+2, k)\) Zigzag code,” IEEE Trans. Inform. Theory, vol. 62, no. 9, pp. 4848-4856, Sep. 2016.

[8] J. Li, X.H. Tang, and W. Xiang, “A New Construction of \((k+2, k)\) Minimal Storage Regenerating Code Over \(F_3\) with Optimal Access Property for All Nodes,” IEEE Communications Letters, vol. 20, no. 7, pp. 1289-1292, Jul. 2016.

[9] J. Li, X.H. Tang, and U. Parampalli, “A framework of constructions of minimal storage regenerating codes with the optimal access/update property,” IEEE Trans. Inform. Theory, vol. 61, no. 4, pp. 1920-1932, Apr. 2015.

[10] J. Li, X.H. Tang, and C. Tian, “A Generic Transformation for Optimal Repair Bandwidth and Rebuilding Access in MDS codes,” Proc. IEEE Int. Symp. Inform. Theory, Aachen, Germany, pp. 1623-1627, Jun. 2017.

[11] Y. Liu, J. Li, and X.H. Tang, “A Generic Transformation to Generate MDS Codes with \(\delta\)-Optimal Access Property,” arxiv preprint arXiv:2107.07733, 2021.

[12] N. Raviv, S. Natalia, and E. Tuvi, “Constructions of high-rate minimum storage regenerating codes over small fields,” IEEE Trans. Inform. Theory, vol. 63, no. 4, pp. 2015-2038, Apr. 2017.

[13] I. Reed and G. Solomon, “Polynomial codes over certain finite fields,” J. Soc. Ind. Appl. Math., vol. 8, no. 2, pp. 300-304, Jun. 1960.

[14] S. Rhea, C. Wells, P. Eaton, D. Geels, B. Zhao, H. Weatherspoon, and J. Kubiatowicz, “Maintenance-free global data storage,” IEEE Internet Comput., vol. 5, no. 5, pp. 40-49, Sep.-Oct. 2001.

[15] B. Sasidharan, G.K. Agarwal, and P.V. Kumar, “A high-rate MSR code with polynomial sub-packetization level,” Proc. IEEE Int. Symp. Inform. Theory, Hong Kong, China, pp. 2051-2055, Jun. 2015.

[16] B. Sasidharan, V. Myna, and P.V. Kumar, “An explicit, coupled-layer construction of a high-rate MSR code with low sub-packetization level, small field size and all-node repair,” arxiv preprint arXiv:1607.07335.

[17] B. Sasidharan, V. Myna, and P.V. Kumar, “An explicit, coupled-layer construction of a high-rate MSR code with low sub-packetization level, small field size and \(d < (n-1)\),” Proc. IEEE Int. Symp. Inform. Theory, Aachen, Germany, pp. 2048-2052, Jun. 2017.

[18] X.H. Tang, B. Yang, J. Li, and H.D.L. Hollmann, “A new repair strategy for the hadamard minimum storage regenerating codes for distributed storage systems,” IEEE Trans. Inform. Theory, vol. 61, no. 10, pp. 5271-5279, Oct. 2015.

[19] T. Tamo, Z. Wang, and J. Bruck, “Zigzag codes: MDS array codes with optimal rebuilding,” IEEE Trans. Inform. Theory, vol. 59, no. 3, pp. 1597-1616, Mar. 2013.

[20] M. Vajha, B.S. Babu, and P.V. Kumar, “Explicit MSR Codes with Optimal Access, Optimal Sub-packetization and Small Field Size for \(d = k + 1, k + 2, k + 3\),” arxiv preprint arXiv:1804.00598, 2018.

[21] Z. Wang, T. Tamo, and J. Bruck, “On codes for optimal rebuilding access,” in Proc. 49th Annu. Allerton Conf. Commun., Control, Comput., Monticello, IL, pp. 1374-1381, Sep. 2011.

[22] Z. Wang, T. Tamo, and J. Bruck, “Explicit minimum storage regenerating codes,” IEEE Trans. Inform. Theory, vol. 62, no. 8, pp. 4466-4480, Aug. 2016.

[23] M. Ye and A. Barg, “Explicit constructions of high-rate MDS array codes with optimal repair bandwidth,” IEEE Trans. Inform. Theory, vol. 63, no. 4, pp. 2001-2014, Apr. 2017.

[24] M. Ye and A. Barg, “Explicit constructions of optimal-access MDS codes with nearly optimal sub-packetization,” IEEE Trans. Inform. Theory, vol. 63, no. 10, pp. 6307-6317, Oct. 2017.