MOMENTS OF THE GAUSSIAN $\beta$ ENSEMBLES AND THE LARGE-$N$ EXPANSION OF THE DENSITIES

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Abstract. The loop equation formalism is used to compute the $1/N$ expansion of the resolvent for the Gaussian $\beta$ ensemble up to and including the term at $O(N^{-6})$. This allows the moments of the eigenvalue density to be computed up to and including the 12-th power and the smoothed density to be expanded up to and including the term at $O(N^{-6})$. The latter contain non-integrable singularities at the endpoints of the support — we show how to nonetheless make sense of the average of a sufficiently smooth linear statistic. At the special couplings $\beta = 1, 2$ and $4$ there are characterisations of both the resolvent and the moments which allows for the corresponding expansions to be extended, in some recursive form at least, to arbitrary order. In this regard we give fifth order linear differential equations for the density and resolvent at $\beta = 1$ and $4$, which complements the known third order linear differential equations for these quantities at $\beta = 2$.

1. Introduction

One of the most active topics in random matrix theory at present is the study of the $\beta$-ensembles. Motivations come from varying viewpoints, including universality, integrability, asymptotics and applications to matrix models and field theories. In applications to matrix models and field theories the average

$$\langle \operatorname{Tr} G^{2k} \rangle_{G \in \text{GUE}^*}, \quad k \in \mathbb{Z}_{\geq 0},$$

features prominently. Here $\text{GUE}^*$ ($\beta = 2$) denotes the set of $N \times N$ GUE matrices each multiplied by $1/\sqrt{2N}$, where this scaling is chosen so that the leading order support of the eigenvalues is $(-1, 1)$. The large $N$ asymptotics of (1.1) is one of the earliest examples of a topological expansion in the theory of matrix integrals [9, 33, 12, 23]. This expansion counts maps on surfaces of definite genus, or equivalently the number $c(g; k)$ of pairings of vertices of a regular $2k$-gon which corresponds to a surface of genus $g$. Thus purely combinatorial reasoning gives

$$\frac{1}{N} \langle \operatorname{Tr} G^{2k} \rangle_{\text{GUE}^*} = \sum_{g=0}^{[k/2]} \frac{c(g; k)}{N^{2g}}.$$  

It is furthermore the case that the average $\langle \operatorname{Tr} G^{2k} \rangle$ with respect to $\text{GOE}^*$ ($\beta = 1$) or $\text{GSE}^*$ ($\beta = 4$) (see [13] below for the precise definitions) also admits a combinatorial interpretation, with the corresponding coefficients again being related to certain maps [33].

A primary concern of the present work is the analysis and generation of the large $N$ expansion of the average $\langle \operatorname{Tr} G^{2k} \rangle$, where the matrices $G$ belong to the Gaussian $\beta$-ensemble $\text{G}\beta\text{E}^*$. For general $\beta > 0$ such matrices can be defined as real symmetric matrices with independent entries (see e.g. [15]), but since $\operatorname{Tr} G^{2k} = \sum_{m=1}^{N} \lambda_{2m}^{2k}$, for present purposes it suffices to specify the eigenvalue probability density function. This is proportional to

$$\prod_{i=1}^{N} e^{-\beta N \lambda_i^2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta,$$

As in the notation $\text{GUE}^*$, which corresponds to the case $\beta = 2$, the asterisk in $\text{G}\beta\text{E}^*$ denotes the particular scaling of the eigenvalues that gives the leading order eigenvalue density supported on $(-1, 1)$. There has been recent interest in this $\beta$ generalized moment from the viewpoint of topological expansions of matrix models [8, 6, 4, 32].

More general than the calculation of the moments is the problem of computing the asymptotic smoothed signed densities. Consider a linear statistic of the eigenvalue, which refers to a function of the form $A =
The existence of the large $N$ expansion (1.4) can be found in the recent work [7]. Our determination of \( \tilde{\rho}_{(1)}(\lambda; \beta) \) is consistent with the expansion (1.5) only involving even inverse powers of \( \rho_{(1)}(\lambda; \beta) \) is given in terms of the eigenvalue density according to

\[
\mu_{N, \beta}[A] = \int_{-\infty}^{\infty} a(\lambda) \rho_{(1)}^N(\lambda; \beta) \, d\lambda.
\]

Performing a large $N$ expansion analogous to \([12]\) allows the asymptotic smoothed signed densities \( \{\tilde{\rho}_{(1), g}(\lambda; \beta)\}_{g=0,1,...} \) to be defined according to

\[
\frac{1}{N^{\mu_{N, \beta}}[A]} = \sum_{g=0}^{\infty} \frac{1}{N^g} \int_{-\infty}^{\infty} a(\lambda) \tilde{\rho}_{(1), g}(\lambda; \beta) \, d\lambda.
\]

In the case of $\beta = 2$ \([13]\) is known rigorously from the work of Ercolani and McLaughlin \([14]\) and from Haagerup and Thorbjørnsen \([21]\), and in fact all the terms with $g$ odd vanish. For general $\beta$ a rigorous demonstration of (1.5) can be found in the recent work \([7]\). Our determination of \( \{\tilde{\rho}_{(1), g}(\lambda; \beta)\} \) assumes the existence of the large $N$ expansion (1.5).

The best known result relating to (1.5), which requires certain technical assumptions on $a(\lambda)$, is the limit theorem (see e.g. \([35]\))

\[
\lim_{N \to \infty} \frac{1}{N} \mu_{N}[A] = \int_{-\infty}^{\infty} a(\lambda) \tilde{\rho}_{(1), 0}(\lambda, \beta) \, d\lambda,
\]

where, with $\chi_{\lambda \in J} = 1$ for $\lambda \in J$ and $\chi_{\lambda \in J} = 0$ otherwise,

\[
\tilde{\rho}_{(1), 0}(\lambda, \beta) = \frac{2}{\pi} \sqrt{\lambda^2 - \lambda^2 \chi_{\lambda \in (-1, 1)}},
\]

this being the celebrated Wigner semi-circle law. Note that \( \tilde{\rho}_{(1), 0}(\lambda, \beta) \) is not dependent on $\beta$.

Let us illustrate our method of determination of \( \tilde{\rho}_{(1), g}(\lambda; \beta) \) by deriving (1.7). In fact we will make use of one of the standard approaches (see e.g. \([35]\)) which proceeds by considering the particular linear statistic

\[
a(\lambda) = (z - \lambda)^{-1}.
\]

For the Gaussian $\beta$-ensemble, with

\[
R(z) \equiv \lim_{N \to \infty} \frac{1}{N} \mu_{N} \left[ \sum_{j=1}^{N} \frac{1}{z - \lambda_j} \right],
\]

it is possible to deduce that \([24]\)

\[
R(z)^2 - 4zR(z) + 4 = 0,
\]

which gives

\[
R(z) = 2 \left(z - \sqrt{z^2 - 1}\right).
\]

But from the definition of $R(z)$ and \( \tilde{\rho}_{(1), 0}(x; \beta) \)

\[
\tilde{\rho}_{(1), 0}(x; \beta) = \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left( R(x - i\epsilon) - R(x + i\epsilon) \right),
\]

and (1.7) follows.

In addition to knowledge of the explicit form of \( \tilde{\rho}_{(1), 0}(x; \beta) \) as given by (1.7), it is furthermore well known that \([24, 16]\)

\[
\tilde{\rho}_{(1), 1}(\lambda; \beta) = \left( \frac{1}{\beta} - \frac{1}{2} \right) \left\{ \frac{1}{2} \left[ \delta(\lambda - 1) + \delta(\lambda + 1) \right] - \frac{1}{\pi \sqrt{1 - \lambda^2}} \right\}.
\]

Note that the dependence on $\beta$ is a linear polynomial in $1/\beta$ which vanishes at $\beta = 2$, and that this latter feature is consistent with the expansion (1.5) only involving even inverse powers of $N$ for $\beta = 2$. The result
(1.13) follows from the $1/N$ term in the expansion

$$\frac{1}{N} \mu_N \left[ \sum_{j=1}^{N} \frac{1}{z - \lambda_j} \right] = R(z) + \frac{1}{N} \left( \frac{1}{\beta} - \frac{1}{2} \right) \left[ \frac{1}{\sqrt{z^2 - 1}} - \frac{z}{z^2 - 1} \right] + O(N^{-2}),$$

and application of (1.12). It might then seem that our task is to extend the expansion (1.14) to higher order. While this is essentially the case, there are some complications. For example, the next term in the expansion (1.14) is [10, 30]

$$\frac{1}{N^2} \left\{ \frac{1}{2} \left( 1 - \frac{2}{\beta} \right)^2 \left[ \frac{z^2 + \frac{1}{2}}{(z^2 - 1)^{5/2}} - \frac{z}{(z^2 - 1)^2} \right] + \frac{1}{16} \frac{1}{(z^2 - 1)^{5/2}} \right\}.$$  

Application of (1.12) then gives that $\rho_{(1),2}(\lambda; \beta)$ contains a term proportional to $(1 - \lambda^2)^{-5/2}$ and thus is not integrable at $\lambda = \pm1$. One of our tasks then is to give meaning to the terms in (1.15) in the light of such singularities.

In the existing literature one can find the expansion (1.14) extended up to and including the term $O(N^{-4})$ in [10], and up to and including the term $O(N^{-6})$ in [30]. However, examination of the two sets of results show that they disagree in the term $O(N^{-4})$. Thus we have no option but to go back to scratch and to derive the expansion (1.14) and its extension to higher orders for ourselves. As in [10, 30] we use the loop equation method, the details of which are given in [12].

A corollary of knowledge of the expansion (1.14) (extended to higher orders) is knowledge of the expansion of (1.3) in the case of $a(\lambda) = \lambda^{2p}$, $p \in \mathbb{Z}_{\geq 0}$, corresponding to the moments of the density. Thus

$$\mu_{N,\beta} \left[ \sum_{j=1}^{N} \frac{1}{z - \lambda_j} \right] = \sum_{p=0}^{\infty} \frac{1}{z^{1 + 2p}} \mu_{N,\beta} \left[ \sum_{j=1}^{N} \lambda_j^{2p} \right] = \sum_{p=0}^{\infty} \frac{1}{z^{1 + 2p}} \langle \text{Tr} G^{2p} \rangle_{G_{\beta}^{\mathbb{E}^*}}.$$  

This should be interpreted as an asymptotic expansion as the sum need not converge. Moreover, the moments have the crucial property that their large $N$ expansion terminates,

$$\frac{1}{N} \langle \text{Tr} G^{2p} \rangle_{G_{\beta}^{\mathbb{E}^*}} = \sum_{g=0}^{p} \frac{1}{N^g} \int_{-\infty}^{\infty} \lambda^2 \hat{\rho}_{(1),g}(\lambda; \beta) d\lambda.$$  

This will play a crucial role in us giving meaning to the expansion (1.5) for general $a(\lambda)$, in the light of the non-integrable singularities of the $\hat{\rho}_{(1),g}(\lambda; \beta)$ identified below (1.15).

One possible approach to study the expansion (1.5) is to compute the large $N$ expansion of the $\rho_{(1)}(\lambda; \beta)$ itself, rather than (1.17). Actually this is rather complicated as there are three distinct scaling regimes: $-1 < \lambda < 1$, $\lambda \approx \pm 1 + X/2N^{2/3}$, and $|\lambda| > 1$ which correspond to the bulk, soft edge and exponentially small portion of $\rho_{(1)}(\lambda; \beta)$ respectively. Moreover in the bulk regime there are both oscillatory and non-oscillatory terms. In fact the explicit carrying out of this expansion (17) up to including the term at order $N^{-2}$ shows that $\hat{\rho}_{(1),g}(\lambda; \beta)$ for $g = 0, 1, 2$ is the same as that obtained by expanding $\rho_{(1)}(\lambda; \beta)$ in the region $-1 < \lambda < 1$ and ignoring the oscillatory terms. For $\beta = 2$ this expansion can be generated to higher order using the linear differential equation satisfied by $\rho_{(1)}$, see (1.10) below, and the same property is observed.

We commented that the original motivation for considering the expansion (1.17) came from its relevance to matrix models and their topological interpretation. Further interest was then identified with respect to the underlying asymptotic smoothed signed densities, and in particular to the analytic challenge issued by their in general non-integrable singularities at $x = \pm 1$. As suggested by the final sentence of the above paragraph, integrability provides yet another motivation. It turns out that orthogonal polynomial expressions for the density $\rho_{(1)}^N(\lambda; \beta)$ in the cases $\beta = 1, 2$ and $4$ (see e.g. [15] Ch. 5&6)), can be used to determine linear differential equations for the density and its Stieltjes transform or equivalently the resolvent. In the case $\beta = 2$ these differential equations, which are both third order with the same homogeneous part, are known from earlier work [18, 21].

We begin in §2 by detailing the loop equation formalism as it applies to the Gaussian $\beta$ ensemble, and we give the explicit form of the first three terms in the large $N$ expansion. This is supplemented in §3 by the specification of the next three terms in this expansion. This knowledge is used to compute the asymptotic smoothed signed densities and moments up the corresponding order, and we furthermore address the problem
imposed by the singularities of the former in the computation of the integrals in (1.5). In §4 we first make note of known characterisations of the moments for $\beta = 1, 2$ and 4 to general order, as these provide checks on our results for general $\beta$. We consider the problem of determining the resolvent at these couplings for general order, and this leads us to the consideration of linear differential equations. In addition to giving a self contained derivation of the known third order linear differential equations for the density and resolvent at $\beta = 2$, we derive fifth order linear differential equations for these quantities in the cases $\beta = 1$ and 4.

2. Loop equations and the large $N$ expansion of the resolvent for the Gaussian $\beta$-ensemble

The Gaussian $\beta$-ensemble eigenvalue PDF (1.3) is the special case $\kappa = \beta/2$, $V(\lambda) = \frac{1}{2}\lambda^2$ of the eigenvalue PDF proportional to

$$\prod_{l=1}^{N} e^{-N\kappa V(\lambda_l)/g} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^{2\kappa}.$$  

We will assume henceforth that $g, N, \kappa > 0$. Note that this is a slight variant on the average given in (1.3) where we have introduced an extra coupling constant factor $4g$. This new average will be denoted as $G\beta E^*(g)$. Much of the theory that follows is applicable to general weights which are parameterised in the form

$$V = g_0 + \sum_{k=1}^{K} g_k \lambda^k.$$  

By definition the smoothed eigenvalue density is computed from knowledge of the mean (1.4) of a sufficiently general linear statistic $A$, with a particularly convenient choice being that corresponding to (1.8). Thus one wants to compute the so-called resolvent

$$W_1(x) := \left\langle \sum_{j=1}^{N} \frac{1}{x - \lambda_j} \right\rangle_{G\beta E^*(g)} \sim \sum_{k=0}^{\infty} m_k^{*} x^{k+1}, \quad m_k^{*} = \int_{-\infty}^{\infty} \lambda^k \rho_{(1)}(\lambda) d\lambda.$$  

As already noted, interest in this quantity also stems from its relationship (1.19), (1.17) to the moments of the eigenvalue density.

The large $N$ expansion of the resolvent is in fact a classical problem in random matrix theory and the theory of matrix models. Its solution involves a recursive set of equations, known as either the Pastur [36] or loop equations [3] (we will use the latter terminology) in the mathematical literature, or as Virasoro constraints, Schwinger-Dyson equations or Ward identities in the physics literature.

Several auxiliary quantities are required. Thus we introduce the correlators

$$W_n(x_1, \ldots, x_n) := \left\langle \sum_{\lambda_{i_1} = 1}^{N} \frac{1}{x_1 - \lambda_{i_1}} \cdots \sum_{\lambda_{i_n} = 1}^{N} \frac{1}{x_n - \lambda_{i_n}} \right\rangle_c,$$

$$P_1(x) := \left\langle \sum_{j=1}^{N} \frac{V'(x) - V'(\lambda_j)}{x - \lambda_j} \right\rangle,$$

$$P_{n+1}(x; x_1, \ldots, x_n) := \left\langle \sum_{i=1}^{N} \frac{V'(x) - V'(\lambda_i)}{x - \lambda_i} \times \sum_{i_1=1}^{N} \frac{1}{x_1 - \lambda_{i_1}} \cdots \sum_{i_n=1}^{N} \frac{1}{x_n - \lambda_{i_n}} \right\rangle_c, \quad n \geq 1,$$

where the notation $\langle \cdot \rangle_c$ denotes the fully truncated (connected) average (for this latter notation see e.g. [15] Equation (5.3))). With this notation, and furthermore $I = \{x_1, \ldots, x_{n-1}\}$, the general loop equation reads ([5] Equation (2.19)] with $a \to \infty$ and $d/dx_i$ corrected to read $-d/dx_i$, [10] Equation (2.25)] and references
therein)

\begin{equation}
\sum_{J \leq l} \kappa W_{J+1}(x, J) \kappa W_{n-J}(x, I \setminus J) + \kappa W_{n+1}(x, I) + (\kappa - 1) \frac{\partial W_n(x, I)}{\partial x}
= \frac{\kappa N}{g} [V'(x)W_n(x, I) - P_n(x; I)] - \sum_{x_i \in I} \frac{\partial}{\partial x_i} \left[ \frac{W_{n-1}(x, I \setminus \{x_i\}) - W_{n-1}(I)}{x - x_i} \right].
\end{equation}

In particular, for \( n = 1 \) (Equations (2.13) and (2.17) \([5]\)) reads

\begin{equation}
\kappa W_2(x, x) + \kappa W_1(x)^2 + (\kappa - 1) W_1(x) = \frac{\kappa N}{g} [V'(x)W_1(x) - P_1(x)],
\end{equation}

and for \( n = 2 \) (Equation (2.19) \([5]\)) it reads

\begin{equation}
\kappa W_3(x, x_1, x_1) + 2 \kappa W_1(x) W_2(x, x_1) + (\kappa - 1) \frac{\partial}{\partial x} W_2(x, x_1)
= \frac{\kappa N}{g} [V'(x)W_2(x, x_1) - P_2(x, x_1)] - \frac{\partial}{\partial x_1} \left[ \frac{W_1(x) - W_1(x_1)}{x - x_1} \right],
\end{equation}

and for \( n = 3 \) it reads

\begin{equation}
\kappa W_4(x, x_1, x_1, x_2) + 2 \kappa W_1(x) W_3(x, x_1, x_2) + 2 \kappa W_2(x, x_1) W_2(x, x_2)
+ (\kappa - 1) \frac{\partial}{\partial x} W_3(x, x_1, x_2)
= \frac{\kappa N}{g} [V'(x)W_3(x, x_1, x_2) - P_3(x, x_1, x_2)]
- \frac{\partial}{\partial x_1} \left[ \frac{W_2(x, x_1) - W_2(x_1, x_2)}{x_1 - x_1} \right]
- \frac{\partial}{\partial x_2} \left[ \frac{W_2(x, x_1) - W_2(x_1, x_2)}{x_2 - x_2} \right],
\end{equation}

and so forth.

Our quantity of interest, \( W_1(x) \), thus couples with the auxiliary quantities \( (2.4) \) for general \( n \), and thus in this sense the loop equations are not closed. The utility of the loop equations reveals itself by hypothesizing that for large \( N \), \( W_n \) has leading term proportional to \( N^{2-n} \), with higher order terms a power series in inverse powers of \( N \). Thus one writes (Equation (2.26) \([5]\))

\begin{equation}
\kappa^{n-1} \left( \frac{g}{N} \right)^{2-n} W_n = \sum_{l=0}^{\infty} \left( \frac{N \sqrt{\kappa}}{g} \right)^{-l} W_n^l,
\end{equation}

where each \( W_n^l \) is independent of \( N \), and also (see §2.6.2 \([5]\))

\begin{equation}
\kappa^{n-1} \left( \frac{g}{N} \right)^{2-n} P_n = \sum_{l=0}^{\infty} \left( \frac{N \sqrt{\kappa}}{g} \right)^{-l} P_n^l,
\end{equation}

where each \( P_n^l \) is similarly independent of \( N \). The partition function has the expansion (see Equation (2.25) of \([5]\))

\begin{equation}
F = \sum_{l \geq 0} \left( \frac{\sqrt{\kappa} N}{g} \right)^{2-l} F^l.
\end{equation}

The large \( N \) expansion differs from a genus or semi-classical expansion, wherein the latter would employ a development in powers of both the expansion parameters

\begin{equation}
\nu = \frac{N \sqrt{\kappa}}{g}, \quad \hbar = \frac{g}{N} (1 - \kappa^{-1}).
\end{equation}

Substituting (2.9) and (2.10) and equating like powers of \( N \) one sees that a quasi-triangular system of equations result, which can be solved recursively. Moreover, one sees too that the dependence on \( \kappa \) is each
\(W_n^l\) is a polynomial in \((\sqrt{\kappa} - 1/\sqrt{\kappa})\) of degree \(l\),

\[
(2.13) \quad W_n^l = \left(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}\right)^l W_n^0 + \left(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}\right)^{l-2} W_n^{1,l-2} + \cdots + \left(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}\right)^{l-2[l/2]} W_n^{[l/2],l-2[l/2]},
\]

where \(\{W_n^{g,l-2g}\}_{g=0}^{[l/2]}\) are independent of \(\kappa\) and \(V\).

We now specialize to the Gaussian potential \(V(x) = \frac{1}{2}x^2\) as corresponds to (1.3). Since then

\[
(2.14) \quad \frac{V'(x) - V'(\lambda)}{x - \lambda} = 1,
\]

the definition (2.4) gives

\[
(2.15) \quad P_1 = \left(\sum_{j=1}^{N} 1\right) = N(1) = N = \frac{N}{g} P_1^0,
\]

and thus

\[
(2.16) \quad P_l = \begin{cases} g, & l = 0, \\ 0, & \text{otherwise}. \end{cases}
\]

Also, using (2.14) in the definition (2.4) we have

\[
P_{n+1}(x; x_1, \ldots, x_n) = \left(\sum_{i=1}^{N} \sum_{i_1=1}^{N} \frac{1}{x_1 - \lambda_{i_1}} \cdots \sum_{i_n=1}^{N} \frac{1}{x_n - \lambda_{i_n}}\right)_c.
\]

In fact this vanishes identically for \(n > 1\).

**Theorem 1.** For each \(n = 1, 2, \ldots\) we have

\[
(2.17) \quad P_{n+1}(x; x_1, \ldots, x_n) = 0.
\]

**Proof.** This is a special case of the more general result

\[
(2.18) \quad \langle 1 \cdot A_1 \cdots A_n \rangle_c = 0,
\]

valid for \(n \geq 1\). We can establish (2.18) by induction. The base case \(n = 1\) when we have

\[
\langle 1 \cdot A_1 \rangle_c = \langle 1 \cdot A_1 \rangle - \langle 1 \rangle \langle A_1 \rangle = 0,
\]

as required. We now assume (2.18) is valid for \(n = 1, \ldots, m\), with our remaining task being to show that it is true for \(n = m + 1\). For this purpose, let \(A_0 := 1\) and \(A_I := A_{i_1} \cdots A_{i_n}, I = \{i_1, \ldots, i_k\}\). Then from the definition of a connected correlator we have

\[
(2.19) \quad \langle 1 \cdot A_1 \cdots A_{m+1} \rangle = \langle 1 \cdot A_1 \cdots A_{m+1} \rangle_c + \sum_{k=2}^{m+2} \sum_{I_1 \cup \cdots \cup I_k = \{0, \ldots, m+1\}} \prod_{j=1}^{k} \langle A_{I_j} \rangle_c.
\]

By the induction hypothesis, if \(0 \in I_j\) and \(|I_j| > 1\) we have \(\langle A_{I_j} \rangle_c = 0\) (i.e. the 0 index must be in a subset of its own). Thus

\[
\sum_{k=2}^{m+2} \sum_{I_1 \cup \cdots \cup I_k = \{0, \ldots, m+1\}} \prod_{j=1}^{k} \langle A_{I_j} \rangle_c = \sum_{k=1}^{m+1} \sum_{I_1 \cup \cdots \cup I_k = \{1, \ldots, m+1\}} \prod_{j=1}^{k} \langle A_{I_j} \rangle_c = \langle A_1 \cdots A_{m+1} \rangle.
\]

Substituting this back in (2.19) implies (2.18). \(\square\)
The loop equations, for the Gaussian potential and resolved into the large $N$ expansion, are then solved in the following hierarchy, whose initial parts we give in 5 steps. -

**Step 1 -** Order $N^2$ terms of the $n = 1$ loop equations:

$$(W_1^0(x))^2 - xW_1^0(x) + g = 0,$$

with the solution

$$W_1^0(x) = \frac{1}{2} \left( x - \sqrt{x^2 - 4g} \right),$$

where the negative sign is chosen so that $W_1^0 \sim_{x \to \infty} O(x^{-1})$. Thus

$$W_1^0 = \frac{1}{2} \sum_{k \geq 0} \frac{(4g)^{k+1}}{x^{2k+1}} \frac{\Gamma(k + \frac{1}{2})}{(k + 1)! \Gamma(\frac{1}{2})},$$

which yield the moments of the Wigner semi-circle law

$$\tilde{\rho}_{(1),0}(x; \beta) = \frac{1}{2\pi} \sqrt{4g - x^2} \quad \text{on} \ (-2\sqrt{g}, 2\sqrt{g}).$$

**Step 2 -** Order $N^1$ terms of the $n = 1$ loop equations:

$$(2W_1^0(x) - x)W_1^1(x) + \left( \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) \frac{\partial}{\partial x} W_1^0(x) = 0,$$

with the solution

$$W_1^1(x) = \frac{1}{2} \left( \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) \left[ \frac{1}{\sqrt{x^2 - 4g}} - \frac{x}{x^2 - 4g} \right].$$

Its large $x$ expansion is

$$W_1^1 = \frac{1}{2} \left( \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) \sum_{k \geq 0} \frac{(4g)^k}{x^{2k+1}} \left[ \frac{\Gamma(k + 1)}{k!} - 1 \right] \sim - \left( \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) g x^{-3} + \cdots.$$ 

**Step 3 -** Order $N^1$ terms of $n = 2$ loop equation and order $N^0$ terms of the $n = 1$ loop equations:

$$(2W_1^0(x) - x)W_2^0(x, x_1) + \frac{\partial}{\partial x_1} \frac{W_1^0(x) - W_1^0(x_1)}{x - x_1} = 0,$$

with its solution

$$W_2^0(x, x_1) = \frac{1}{2} (x - x_1)^{-2} \left[ \frac{\sqrt{x_1^2 - 4g}}{x^2 - 4g} - 1 \right] + \frac{1}{2} (x - x_1)^{-1} \frac{x_1}{\sqrt{x_1^2 - 4g} \sqrt{x_2 - 4g}},$$

followed by

$$(2W_1^0(x) - x)W_2^2(x, x) + W_2^0(x, x) + (W_1^0) + \left( \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) \frac{\partial}{\partial x} W_1^1(x) = 0,$$

and its solution

$$W_2^0(x) = \left( \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) \left[ -\frac{x}{(x^2 - 4g)^2} + \frac{x^2 + g}{(x^2 - 4g)^{5/2}} \right] + \frac{g}{(x^2 - 4g)^{5/2}}.$$

This has the large $x$ expansion

$$W_2^0(x) \sim_{x \to \infty} \left( \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right)^2 \left( \frac{3g}{x^3} + \cdots \right) + \frac{g}{x^3} + \cdots.$$ 

**Step 4 -** Order $N^0$ terms of $n = 2$ loop equation and order $N^{-1}$ terms of the $n = 1$ loop equations:

$$(2W_1^0(x) - x)W_2^2(x, x_1) + 2W_1^1(x)W_2^0(x, x_1)$$

$$+ \left( \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) \frac{\partial}{\partial x} W_2^0(x, x_1) + \frac{\partial}{\partial x_1} \frac{W_1^0(x) - W_1^0(x_1)}{x - x_1} = 0,$$
and

$$(2W_1^0(x) - x)W_1^3(x) + W_1^2(x, x) + 2W_1^1(x)W_1^2(x) + \left(\sqrt{\kappa - \frac{1}{\sqrt{\kappa}}} \right) \frac{\partial}{\partial x} W_1^2(x) = 0.$$ 

**Step 5** - Order $N^0$ terms of $n = 3$ loop equation, order $N^{-1}$ terms of the $n = 2$ loop equations and order $N^{-2}$ terms of the $n = 1$ loop equations:

$$(2W_1^0(x) - x)W_3^0(x, x_1, x_2) + 2W_2^0(x, x_1)W_2^0(x, x_2) + \frac{\partial}{\partial x_1} \left[ W_2^0(x, x_2) - W_2^0(x_1, x_2) \right] + \frac{\partial}{\partial x_2} \left[ W_2^0(x, x_1) - W_2^0(x_1, x_2) \right] = 0,$$

and

$$(2W_1^0(x) - x)W_2^3(x, x_1) + W_3^0(x, x_1) + 2W_1^1(x)W_2^0(x, x_1) + \left(\sqrt{\kappa - \frac{1}{\sqrt{\kappa}}} \right) \frac{\partial}{\partial x} W_2^2(x, x_1) + \frac{\partial}{\partial x_1} \left( W_2^2(x) - W_1^2(x_1) \right) = 0,$$

followed by

$$(2W_1^0(x) - x)W_4^1(x) + W_2^3(x, x) + (W_2^2(x))^2 + 2W_1^1(x)W_3^0(x) + \left(\sqrt{\kappa - \frac{1}{\sqrt{\kappa}}} \right) \frac{\partial}{\partial x} W_3^1(x) = 0.$$ 

In §3 we supplement our computation of $W_1^0, W_1^1, W_2^1$ by specifying $W_1^{(j)}$ for $j$ up to 6, and we furthermore use this to compute the asymptotic smoothed densities and the moments.

### 3. Expansions of the resolvent, moments and smoothed densities for general $\beta$

#### 3.1. Resolvent expansion.** We record here the results of the large $N$ expansion of the resolvent as computed using the loop equations given in §2 and make a number of observations on this data. We recall from (2.9) that this expansion has the form

$$(3.1) \quad \frac{g}{N} \tilde{W}_1 = W_1^0 + \left( \frac{g}{\sqrt{\kappa}N} \right) W_1^1 + \left( \frac{g}{\sqrt{\kappa}N} \right)^2 W_1^2 + \left( \frac{g}{\sqrt{\kappa}N} \right)^3 W_1^3 + \ldots ,$$

which as noted in the Introduction is known to be rigorously valid for the $G\beta\mathbb{E}^*(g)$ ensembles. Relative to (2.4), on the LHS we have written $\tilde{W}_1$ in place of $W_1$, so we can distinguish the expanded form from the definition (2.3). We will utilise the following abbreviations for the variable characterising the $\beta$-deformation or deviation from the hermitian case, and the single-cut spectral curve

$$(3.2) \quad h := \sqrt{\kappa - \frac{1}{\sqrt{\kappa}}}, \quad y(x) := \sqrt{x^2 - 4g}.$$ 

In the compact two-term form the first six coefficients are

$$(3.3) \quad W_1^0 = \frac{1}{2} \left[ x - y \right],$$

$$(3.4) \quad W_1^1 = h \frac{1}{2} \left[ \frac{1}{y} - \frac{x}{y^2} \right],$$

$$(3.5) \quad W_1^2 = h^2 \left[ -\frac{x}{y^4} + \frac{x^2 + g}{y^3} \right] + \frac{g}{y^5},$$

$$(3.6) \quad W_1^3 = h^3 \left[ \frac{x^2 + g}{y^7} - \frac{x^3 + 2gx}{y^8} \right] + h^2 \left[ \frac{x^2 + 6g}{y^7} - \frac{x^3 + 30gx}{y^8} \right].$$
\begin{align}
W_1^4 &= h^4 \left[ \frac{37x^3 + 92gx}{y^{10}} + \frac{37x^4 + 123gx^2 + 21g^2}{y^{11}} \right] \\
&\quad + h^2 \left[ \frac{23x^3 + 180gx}{2y^{10}} + \frac{23x^4 + 454gx^2 + 176g^2}{2y^{11}} \right] + \frac{21g(x^2 + g)}{y^{11}},
\end{align}

\begin{align}
W_1^5 &= h^5 \left[ \frac{353x^4 + 1527gx^2 + 399g^2}{y^{13}} - \frac{353x^5 + 1766gx^3 + 848g^2x}{y^{14}} \right] \\
&\quad + h^3 \left[ \frac{445x^4 + 4332gx^2 + 1512g^2}{2y^{13}} - \frac{445x^5 + 7714gx^3 + 7440g^2x}{2y^{14}} \right] \\
&\quad + h \left[ \frac{21(x^4 + 20gx^2 + 14g^2)}{2y^{13}} - \frac{3(7x^5 + 628gx^3 + 1200g^2x)}{2y^{14}} \right],
\end{align}

\begin{align}
W_1^6 &= h^6 \left[ \frac{-4081x^5 + 26392gx^3 + 18976gx^2 + 1978g^2}{y^{16}} + \frac{4081x^6 + 28625gx^4 + 26832g^2x^2 + 1738g^3}{y^{17}} \right] \\
&\quad + h^4 \left[ \frac{-8567x^5 + 101288gx^3 + 93600gx^2}{2y^{16}} + \frac{8567x^6 + 147556gx^4 + 243180g^2x^2 + 31236g^3}{2y^{17}} \right] \\
&\quad + h^2 \left[ \frac{-618x^5 + 13104gx^3 + 18000gx^2}{y^{16}} + \frac{618x^6 + 32043gx^4 + 91299gx^2 + 16834g^3}{y^{17}} \right] \\
&\quad + \frac{11g(135x^4 + 558gx^2 + 158g^2)}{y^{17}},
\end{align}

Remark 3.1. Essentially the same number of coefficients were reported in Eq. (3.3) of [30], which agree with our results after correcting for the typographical errors in p\textsubscript{1.5}. Partial results have also been given in Eq. (2.60) of [10] up to \( W_1^4 \) but their result for \( W_{1,2}(p) \) differs from the coefficient of \( h^2 \) in (3.7). Our results, specialised to \( \kappa = 1 \), are consistent with the recurrence system (2.22), (2.23) given in [21] and the expansion, Eq. (3.60), of [31].

Inspection of (3.3) to (3.9) suggest the following analytic form for the \( W_1^4 \).

**Conjecture 1.** Let \( y = y(x) \) be given by (3.2). For \( l \geq 2 \) even we have

\begin{equation}
W_1^l(x) = h^l \left[ \frac{P_l(x)}{y^{l-2}} + \frac{P_{l+1}(x)}{y^{l-1}} \right] + h^{l-2} \left[ \frac{P_l(x)}{y^{l-2}} + \frac{P_{l+1}(x)}{y^{l-1}} \right] + \ldots + h^2 \left[ \frac{P_l(x)}{y^{l-2}} + \frac{P_{l+1}(x)}{y^{l-1}} \right] + \frac{P_l(x)}{y^{l-1}},
\end{equation}

where \( \deg_{x} P_j^{l} = l - 1 \) for \( j = 1, 3, \ldots, l - 1 \), \( \deg_{x} P_j^{l} = l \) for \( j = 2, 4, \ldots, l \) and \( \deg_{x} P_{l+1}^{l} = l - 2 \). For \( l \geq 1 \) odd we have

\begin{equation}
W_1^l(x) = h^l \left[ \frac{P_l(x)}{y^{l-2}} + \frac{P_{l+1}(x)}{y^{l-1}} \right] + h^{l-2} \left[ \frac{P_l(x)}{y^{l-2}} + \frac{P_{l+1}(x)}{y^{l-1}} \right] + \ldots + h \left[ \frac{P_l(x)}{y^{l-2}} + \frac{P_{l+1}(x)}{y^{l-1}} \right],
\end{equation}

where the polynomial numerators have \( \deg_{x} P_j^{l} = l - 1 \) for \( j = 1, 3, \ldots, l \) and \( \deg_{x} P_j^{l} = l \) for \( j = 2, 4, \ldots, l + 1 \). The polynomial numerators \( P_j^{l} \) are either even or odd with respect to \( x \to -x \) according as the degree is even or odd respectively. Furthermore, the leading term in the \( x \to \infty \) expansion of \( W_1^l(x) \) is of order \( x^{-2l-1} \) for all \( l \geq 0 \).

3.2. Expansion of the smoothed density. Having the resolvent at hand, in the form of a development in descending powers of \( N \), we come the extract meaning to the density via the inversion Sokhotski-Plemelj formula

\begin{equation}
\tilde{\rho}_{(1)}(x) = \frac{1}{2\pi i} \left[ \tilde{W}_1(x - i\epsilon) - \tilde{W}_1(x + i\epsilon) \right]_{x \in (-2\sqrt{\gamma}, 2\sqrt{\gamma})},
\end{equation}

However by using the large \( N \) expansion for \( W_1(x) \) this formulae does not yield the true density, as we have indicated by our notation, but rather the smoothed density \( \tilde{\rho}_{(1)}(x) \). The smoothed density does not possess any of the oscillatory contributions of the true density, the leading order contributions of which have been
found in a number of studies (see e.g. [17, 16, 11]), but rather the remnant of these when integrated against classes of test functions (usually continuously differentiable of all orders and bounded functions $C_c^\infty$). The smoothed density is in fact a distribution with respect to such a class of functions.

To facilitate the extraction of the smoothed density $\tilde{\rho}_1(x)$ it is necessary to express the coefficients of (3.1) in a partial-fraction form with terms containing factors of $y^{-\sigma}$, $xy^{-\sigma}$ where $\sigma \in \mathbb{Z}, \mathbb{Z} + \frac{1}{2}$. Noting the above formula and (3.3) we have a similar large-$N$ expansion (although defined slightly differently from (1.3), in which case there is no variable $g$

$$\frac{g}{N}\tilde{\rho}_1 = \tilde{\rho}_{1,0} + \left(\frac{g}{\sqrt{\pi N}}\right)\tilde{\rho}_{1,1} + \left(\frac{g}{\sqrt{\pi N}}\right)^2\tilde{\rho}_{1,2} + \left(\frac{g}{\sqrt{\pi N}}\right)^3\tilde{\rho}_{1,3} + \cdots . \tag{3.13}$$

In addition to the indicator or step-function $\chi_{x \in (-2\sqrt{g}, 2\sqrt{g})}$ let us define the Dirac delta distributions

$$\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(l)} = \delta^{(l)}(x - 2\sqrt{g}) + (-1)^l\delta^{(l)}(x + 2\sqrt{g}), \quad l = 1, 2, \ldots , \tag{3.14}$$

where we note that the first of these

$$\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(1)} = \frac{d}{dx}\chi_{x \in (-2\sqrt{g}, 2\sqrt{g})}. \tag{3.15}$$

Using a partial fraction expansion of the coefficients in (3.13) along with (3.3)-(3.9) we have

$$\tilde{\rho}_{1,0}(x) = \frac{1}{2\pi}\sqrt{4g - x^2} \chi_{x \in (-2\sqrt{g}, 2\sqrt{g})}, \tag{3.16}$$

$$\tilde{\rho}_{1,1}(x) = h \left\{ \frac{1}{2\pi}(4g - x^2)^{-1/2} \chi_{x \in (-2\sqrt{g}, 2\sqrt{g})} - \frac{1}{4}\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(0)} \right\}, \tag{3.17}$$

$$\tilde{\rho}_{1,2}(x) = h^2 \left\{ \frac{1}{\pi}(x^2 + g)(4g - x^2)^{-5/2} \chi_{x \in (-2\sqrt{g}, 2\sqrt{g})} + \frac{1}{8\sqrt{g}}\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(1)} \right\} + \frac{1}{4g}(4g - x^2)^{-5/2} \chi_{x \in (-2\sqrt{g}, 2\sqrt{g})}, \tag{3.18}$$

$$\tilde{\rho}_{1,3}(x) = h^3 \left\{ -\frac{5}{\pi}(x^2 + g)(4g - x^2)^{-7/2} \chi_{x \in (-2\sqrt{g}, 2\sqrt{g})} \right. \right.$$

$$\left. - \frac{5}{512g^{1/2}}\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(1)} - \frac{256g}{256g^{1/2}}\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(2)} - \frac{5}{128g^{1/2}}\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(3)} \right\} + h \left\{ \frac{1}{2\pi}(x^2 + 6g)(4g - x^2)^{-7/2} \chi_{x \in (-2\sqrt{g}, 2\sqrt{g})} \right. \right.$$

$$\left. + \frac{13}{1024g^{3/2}}\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(1)} + \frac{13}{512g^{3/2}}\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(2)} + \frac{17}{768g^{3/2}}\epsilon_{x \in (-2\sqrt{g}, 2\sqrt{g})}^{(3)} \right\}. \tag{3.19}$$

We present $\tilde{\rho}_{1,4}(x), \tilde{\rho}_{1,5}(x)$ and $\tilde{\rho}_{1,6}(x)$ in the Appendix.

Some comments are in order regarding the meaning of these results in relation to their use in (1.5). The correct meaning of the integral in (1.5) is the Hadamard regularised form, or the partie finie [22], which was shown by Riesz [37] to be the meromorphic continuation of a finite integral. Here we indicate this with the relevant example for the power law singularities of $\tilde{\rho}_{1,0}$ at $x = \pm 1$,

$$I(\alpha) := \int_{-1}^{1} dx f(x)(1 - x^2)^{-\alpha}, \tag{3.20}$$

where $f(x)$ is continuously differentiable up to order $p + 1$ for $x \in (0, 1)$, and $\alpha \in \mathbb{R} > 0$. Changing variables $y = x^2$ and defining $F(y) = \frac{1}{2}(f(\sqrt{y}) + f(-\sqrt{y})$ we subtract off the first $p + 1$ terms of the Taylor expansion
of $F$ in the integrand giving

\[ I(\alpha) = \int_0^1 dy \ y^{-1/2}(1-y)^{-\alpha} \left[ F(y) - F(1) - (y-1)F'(1) - \cdots - \frac{1}{p!}(y-1)^p F^{(p)}(1) \right] \\
+ F(1) \int_0^1 dy \ y^{-1/2}(1-y)^{-\alpha} + \cdots + \frac{(-1)^p}{p!} F^{(p)}(1) \int_0^1 dy \ y^{-1/2}(1-y)^{p-\alpha}, \]

where the first integral is clearly an ordinary integral if $p-\alpha > -2$ and the latter integrals are to be Hadamard regularised. These latter integrals are examples of Euler $\beta$ integrals and can be evaluated according to

\[ \int_0^1 dy \ y^{-1/2}(1-y)^{q-\alpha} = \frac{\Gamma(1/2)\Gamma(q-\alpha+1)}{\Gamma(q-\alpha+3/2)}, \]

where the meromorphic continuation is with respect to $\alpha$ and through the explicit form of the Gamma function.

It is furthermore the case that for the singularities of $\tilde{\rho}(1), g, \alpha$ in (3.20) is a positive half-integer $\alpha = n + \frac{1}{2}$. Then the denominator of the right-hand side of (3.22) is $\Gamma(p-n+1)$. Also, taking $p = n-1$ leaves us with a convergent integral in the first line of (3.21). But for $0 \leq p \leq n-1$ the argument of $\Gamma(p-n+1)$ is a negative integer and thus the finite-part is actually zero (the numerator Gamma functions have half-integer arguments). Thus in the sense of Hadamard regularisation we have $(p = n-1)$

\[ I(n + \frac{1}{2}) = \int_0^1 dy \ y^{-1/2}(1-y)^{-\alpha} \left[ F(y) - F(1) - (y-1)F'(1) - \cdots - \frac{1}{p!}(y-1)^p F^{(p)}(1) \right]. \]

An alternative understanding of (3.23) is possible. First we note that the final statement in Conjecture 1 is equivalent to the moment identity

\[ \int_{-\infty}^\infty dx \ x^{2\sigma} \tilde{\rho}(1),l(x) = 0, \quad 0 \leq \sigma \leq l-1, \quad l \geq 1. \]

It can be shown from the explicit forms of (3.17)-(3.19) and (6.1)-(6.3) that (3.24) is satisfied for $0 \leq \sigma \leq l-1$ and $1 \leq l \leq 6$. The mechanism of how this occurs is that there is mutual cancellation amongst the terms with delta function derivatives for low moment orders $0 \leq \sigma \leq l-2$, whilst the cancellation at $\sigma = l-1$ is between the first non-zero integral and the delta-function terms. Use of (3.24) shows that subtracting the first $l-1$ terms of the power terms in the variable $1-x^2$ of an averaged function $f(x)$ leaves the average unchanged and moreover transforms the divergent integral to a convergent one.

**3.3. Moments.** For purposes of comparison with earlier works in this section we will specialise to the ensemble $G\beta E(N)$ (recall text below (2.1)) with the PDF

\[ \prod_{l=1}^N e^{-\frac{\kappa}{2}\lambda_l^2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^{2\kappa}. \]

Thus we have $m_{2p}(N, \kappa) := \langle \text{Tr} \ G^{2p} \rangle_{G\beta E}$ in comparison to our earlier definition (2.3)

\[ m_{2l}^2(N, \kappa) = \left( \frac{q}{N} \right)^l m_{2l}(N, \kappa), \quad l \in \mathbb{Z}_{\geq 0}. \]
Moments of the density can be readily computed from the resolvent coefficients (3.4)-(3.9). Thus we find

\begin{align}
(3.27) & \quad m_0 = N, \\
(3.28) & \quad m_2 = N^2 + N (-1 + \kappa^{-1}), \\
(3.29) & \quad m_4 = 2N^3 + 5N^2 (-1 + \kappa^{-1}) + N (3 - 5\kappa^{-1} + 3\kappa^{-2}), \\
(3.30) & \quad m_6 = 5N^4 + 22N^3 (-1 + \kappa^{-1}) + N^2 (32 - 54\kappa^{-1} + 32\kappa^{-2}) \\
& \quad \quad + N (-15 + 32\kappa^{-1} - 32\kappa^{-2} + 15\kappa^{-3}), \\
(3.31) & \quad m_8 = 14N^5 + 93N^4 (-1 + \kappa^{-1}) + N^3 (234 - 398\kappa^{-1} + 234\kappa^{-2}) \\
& \quad \quad + N^2 (-260 + 565\kappa^{-1} - 565\kappa^{-2} + 260\kappa^{-3}) \\
& \quad \quad + N (105 - 260\kappa^{-1} + 331\kappa^{-2} - 260\kappa^{-3} + 105\kappa^{-4}), \\
(3.32) & \quad m_{10} = 42N^6 + 386N^5 (-1 + \kappa^{-1}) + 10N^4 (145 - 248\kappa^{-1} + 145\kappa^{-2}) \\
& \quad \quad + 550N^3 (-5 + 11\kappa^{-1} - 11\kappa^{-2} + 5\kappa^{-3}) \\
& \quad \quad + N^2 (2589 - 6545\kappa^{-1} + 8395\kappa^{-2} - 6545\kappa^{-3} + 2589\kappa^{-4}) \\
& \quad \quad + N (-945 + 2589\kappa^{-1} - 3795\kappa^{-2} + 3795\kappa^{-3} - 2589\kappa^{-4} + 945\kappa^{-5}), \\
(3.33) & \quad m_{12} = 132N^7 + 1586N^6 (-1 + \kappa^{-1}) + N^5 (8178 - 14046\kappa^{-1} + 8178\kappa^{-2}) \\
& \quad \quad + N^4 (-22950 + 50945\kappa^{-1} - 50945\kappa^{-2} + 22950\kappa^{-3}) \\
& \quad \quad + 4N^3 (9125 - 23403\kappa^{-1} + 30173\kappa^{-2} - 23403\kappa^{-3} + 9125\kappa^{-4}) \\
& \quad \quad + N^2 (-30669 + 85796\kappa^{-1} - 127221\kappa^{-2} + 127221\kappa^{-3} - 85796\kappa^{-4} + 30669\kappa^{-5}) \\
& \quad \quad + 3N (3465 - 10223\kappa^{-1} + 16432\kappa^{-2} - 18853\kappa^{-3} + 16432\kappa^{-4} - 10223\kappa^{-5} + 3465\kappa^{-6}).
\end{align}

**Remark 3.2.** Results for moments up to $m_6$ were given in [13], see pg. 9 of that work, and up to $m_8$ were also given by Eq.(24) in [30], both sets of which coincide with our calculations.

In the study of Dimitriu and Edelman [13] structural properties for the moments were established using Jack polynomial theory, and in particular we have the following result.

**Theorem 2** (Thm 2.8 of [13]). The general moment $m_{2l}(N, \kappa)$, $l \geq 0$ is a polynomial of degree $l+1$ in $N$ and has a vanishing tail coefficient, i.e. is proportional to $N$. The coefficients with respect to $N$ are polynomials in $\kappa^{-1}$ with degree increasing linearly by unity from the leading term whose degree is zero. These coefficients have a numerator which are a palindromic polynomial in $\kappa$ if of even degree or an anti-palindromic polynomial if of odd degree. In the latter case the numerator has a factor of $\kappa - 1$. This property can be expressed by the duality relation which

\begin{align}
(3.34) & \quad m_{2l}(N, \kappa) = (-1)^{l+1} \kappa^{-l-1} m_{2l}(-\kappa N, \kappa^{-1}), \quad \forall l \geq 0, \ \kappa > 0.
\end{align}

**Remark 3.3.** It is immediate that (3.27) - (3.33) satisfy (3.34).

In addition to the resolvent $W_1(x) = W_1(x, N, \kappa)$, let us introduce the exponential generating functional

\begin{align}
(3.35) & \quad u(t, N, \kappa) \equiv \sum_{p=0}^{\infty} \frac{t^{2p}}{(2p)!} \left\langle \text{Tr} G^{2p} \right\rangle_{G;\beta E} = \left\langle \sum_{j=1}^{N} e^{\epsilon \lambda_j} \right\rangle_{G;\beta E} = \left\langle \text{Tr} e^{tG} \right\rangle_{G;\beta E}.
\end{align}

The formal relation with the resolvent, as defined by (2.11) and (2.13), is

\begin{align}
(3.36) & \quad \int_{0}^{\infty} dt \ e^{-xt} u(t, N) = \sqrt{\frac{\beta}{N}} W_1(\sqrt{\frac{\beta}{N}} x, N),
\end{align}

but in line with our earlier remarks the existence of the integral needs to be examined. Other generating functions have been employed, including a “sub”-exponential type defined by

\begin{align}
(3.37) & \quad \phi(s, N) \equiv \sum_{p=0}^{\infty} \frac{s^{2p}}{(2p - 1)!!} \left\langle \text{Tr} G^{2p} \right\rangle_{G;\beta E}.
\end{align}
It will be observed that (3.37) is a convergent sum, and both (3.35) and (3.37) can be seen as Borel resum-
mations of the divergent expansion for the resolvent.

From (3.34) we can immediately deduce the consequences for the generating functions themselves.

**Corollary 1.** The generating functions satisfy the following duality relations for \( \kappa, N > 0 \)

\[
\begin{align*}
W_1(x, N, \kappa) &= -\kappa^{-1}W_1(x, -\kappa N, \kappa^{-1}), \\
u(t, N, \kappa) &= -\kappa^{-1}u(\kappa^{-1/2}it, -\kappa N, \kappa^{-1}), \\
\phi(s, N, \kappa) &= -\kappa^{-1}\phi(\kappa^{-1/2}is, -\kappa N, \kappa^{-1}).
\end{align*}
\]

**Remark 3.4.** One can verify that (3.1) together with (3.10) and (3.11) satisfies (3.38) for general \( x, N, \kappa \).

To the foregoing result on the low order moments we can add some explicit detail concerning the coefficients of the general 2\( l \)-th moment. It is a classical result that the leading coefficient with respect to \( N \) is the \( l \)-th Catalan number, \( C_l \equiv \frac{(2l)!}{(l+1)!} \), which expresses the appearance of the Wigner semi-circle law in the bulk scaling of \( \frac{1}{N}W_1(x, N) \) in the \( N \to \infty \) limit. We can give simple explicit formulae for the sub-leading coefficients as well, for general values of \( l \).

**Theorem 3.** The moment \( m_{2l} \) has the further properties -
The coefficient of \( \kappa^{-1/2}N^l \) in \( m_{2l} \) is (which is the integer sequence \( \text{A000346} \))

\[
2^{2l-1} \left[ -1 + \frac{\Gamma\left(l + \frac{1}{2}\right)}{\sqrt\pi \Gamma(l+1)} \right] h, \quad l \geq 1.
\]

The coefficient of \( \kappa^{-1}N^{l-1} \) in \( m_{2l} \) is

\[
\frac{1}{3} 4^{l-1} l \left[ -3 + (5l + 1) \frac{\Gamma\left(l + \frac{1}{2}\right)}{\sqrt\pi \Gamma(l+1)} \right] h^2 + \frac{1}{3} 4^{l-1} l(l-1) \frac{\Gamma\left(l + \frac{1}{2}\right)}{\sqrt\pi \Gamma(l+1)}, \quad l \geq 2.
\]

The coefficient of \( \kappa^{-3/2}N^{l-2} \) in \( m_{2l} \) is

\[
\frac{5}{3} 4^{l-3} l^2(l-1) \left[ -3 + \frac{8\Gamma\left(l + \frac{1}{2}\right)}{\sqrt\pi \Gamma(l+1)} \right] h^3 + \frac{1}{3} 2^{2l-7} l(l-1) \left[ 28 - 17l + \frac{16(l-1)\Gamma\left(l + \frac{1}{2}\right)}{\sqrt\pi \Gamma(l+1)} \right] h, \quad l \geq 3.
\]

The coefficient of \( \kappa^{-2}N^{l-3} \) in \( m_{2l} \) is

\[
2^{2l-7} l(l-1)(l-2) \left[ \frac{1}{3} (8 - 15l) + \frac{4(1105l^2 - 193l - 42)\Gamma\left(l + \frac{1}{2}\right)}{945\sqrt\pi \Gamma(l+1)} \right] h^4 \\
+ 4^{l-4} l(l-1)(l-2) \left[ \frac{1}{3} (28 - 17l) + \frac{16(590l^2 - 1259l - 84)\Gamma\left(l + \frac{1}{2}\right)}{945\sqrt\pi \Gamma(l+1)} \right] h^2 \\
+ 2^{2l-5} l(l-1)(l-2)(l-3) \frac{(5l - 2)\Gamma\left(l + \frac{1}{2}\right)}{45\sqrt\pi \Gamma(l+1)}, \quad l \geq 4.
\]

The coefficient of \( \kappa^{-5/2}N^{l-4} \) in \( m_{2l} \) is

\[
2^{2l-13} l^2(l-1)(l-2)(l-3) \left[ \frac{1}{3} (99 - 113l) + \frac{128(1105l^2 - 1243)\Gamma\left(l + \frac{1}{2}\right)}{945\sqrt\pi \Gamma(l+1)} \right] h^5 \\
+ 4^{l-7} l(l-1)(l-2)(l-3) \left[ \frac{1}{45} (5677l^2 - 17271l + 4952) + \frac{(302080l^2 - 698368l + 10752)\Gamma\left(l + \frac{1}{2}\right)}{945\sqrt\pi \Gamma(l+1)} \right] h^3 \\
\times \left[ \frac{1}{15} (l-1)(239l - 886) + \frac{128(l-3)(5l - 2)\Gamma\left(l + \frac{1}{2}\right)}{45\sqrt\pi \Gamma(l+1)} \right] h, \quad l \geq 5.
\]
The coefficient of $\kappa^{-3}N^{l-5}$ in $m_{2\ell}$ is

\begin{equation}
4^{l-7}l(l-1)(l-2)(l-3)(l-4) \times \left[ -\frac{1}{15}(565l^2 - 1295l + 512) + \frac{128(82825l^3 - 135690l^2 + 8081l + 1716)\Gamma(l+\frac{1}{2})}{405405\sqrt{\pi}\Gamma(l+1)} \right] h^6 \\
+ 2^{2l-15}l(l-1)(l-2)(l-3)(l-4) \times \left[ -\frac{1}{45}(5677l^2 - 19991l + 9432) + \frac{256(5929l^3 - 23320l^2 + 12861l + 312)\Gamma(l+\frac{1}{2})}{12285\sqrt{\pi}\Gamma(l+1)} \right] h^4 \\
+ 4^{l-7}l(l-1)(l-2)(l-3)(l-4) \times \left[ -\frac{1}{15}(l-1)(239l - 886) + \frac{128(93427l^3 - 549765l^2 + 623360l + 9438)\Gamma(l+\frac{1}{2})}{405405\sqrt{\pi}\Gamma(l+1)} \right] h^2 \\
+ 2^{2l-7}l(l-1)(l-2)(l-3)(l-4)(l-5) \left( \frac{35l^2 - 77l + 12}{2835\sqrt{\pi}\Gamma(l+1)} \right), \quad l \geq 6.
\end{equation}

\textbf{Proof.} The first two equalities follow from Theorems 12 and 18, and the previous theorem - see the formulae (3.32) and (4.72). All of the relations (3.41) - (3.46) can be established by computing the general term in the large $x$ expansion (which are convergent expansions) of $W_{\beta}^l(x)$ for $l = 1, \ldots, 6$ respectively, as given by (3.4) - (3.9). \hfill \Box

We conclude with an observation on the location of the zeros of the $N$ coefficients that is satisfied by all the cases that are accessible to us.

**Conjecture 2.** In addition to the palindromic/anti-palindromic property the numerator of the coefficients with respect to $N$ have simple zeros all lying on the unit circle, $|\kappa| = 1$, and thus form complex conjugate pairs.

\section{The special cases $\beta = 1, 2$ and 4}

For simplicity in the final results for the moments and the exponential-type generating functions\footnote{In contrast to our choice of the eigenvalue PDF in \cite{2} given by (2.1) and the potential $V(\lambda)$.} we will employ averages with respect to the $G\beta E$ ensemble with the PDF \begin{equation} (2.25). \end{equation} Thus we have $m_{2\ell}(N, \kappa) := \langle \mathrm{Tr} G^{2\ell} \rangle_{G\beta E}$ in comparison to our earlier definition (2.3). There are special orthogonal structures for $\beta = 1, 2$ and 4 which enables special characterisations of the moments and the resolvent not available for general $\beta$. This structure rests on the semi-classical character of the underlying orthogonal polynomial system. For such a system the reproducing kernel is defined as

\begin{equation}
K_N(x, x) = \sqrt{N}e^{-\frac{1}{2}\kappa^2} \left[ p_N(x)p_{N-1}(x) - p_N(x)p'_{N-1}(x) \right],
\end{equation}

and the orthogonal polynomials $\{p_n(x)\}_{n=0}^\infty$ normalised with respect to $e^{-\frac{1}{2}\kappa^2}$ are given by

\begin{equation}
p_n(x) = \frac{1}{\sqrt{\sqrt{2\pi}2^n n!}} \frac{1}{H_n \left( \frac{x}{\sqrt{2}} \right)} = \frac{1}{\sqrt{\sqrt{2\pi}n!}} H_n(x),
\end{equation}

where $H_n(x)$, $He_n(x)$ are the standard Hermite polynomials, see §18.3 of [34]. The density is normalised so that

\begin{equation}
\int_{-\infty}^{\infty} dx \, \rho_{(1)}(x) = N.
\end{equation}

The key relations we require are the generic three-term recurrence relation, which in our context is

\begin{equation}
xp_n(x) = \sqrt{n + 1}p_{n+1}(x) + \sqrt{n}p_{n-1}(x),
\end{equation}

the semi-classical property of the derivative

\begin{equation}
p'_n(x) := \frac{d}{dx} p_n(x) = \sqrt{n}p_{n-1}(x),
\end{equation}

and
and as a consequence the eigenvalue or second-order differential equation

\[ p'' - xp' + np_n = 0. \]

4.1. \( \kappa = 1 \) GUE Moments. The density, as the one-point correlation function, has the classical evaluation \[ (4.8) \] of a determinant of the reproducing kernel

\[ \rho(1)(x) = \sqrt{\frac{N}{g}} K_N(x, x)|_{x \to \sqrt{g} x}, \]

where the kernel is given in \[ (4.1) \].

A third order ordinary differential equation was found for the density and resolvent directly, in the works of Götze and Tikhomirov \[ (18) \] and Haagerup and Thorbjørnsen \[ (21) \], however we will give an independent proof of this fact from first principles.

**Theorem 4** (Lemma 2.1 of \[ (13) \], Prop. 2.2 and Lemma 4.1 of \[ (21) \]). The resolvent \( W_1(x) \) satisfies the third order, inhomogeneous ordinary differential equation

\[ \frac{g^2}{N^2} W''_1 + (4g - x^2) W'_1 + x W_1 = 2N, \quad x \notin \mathbb{R}, \]

subject to the boundary conditions, for fixed \( g, N \)

\[ W_1(x) \xrightarrow{x \to \infty} \frac{m_0}{x} + \frac{g m_2}{N x^3} + \frac{g^2 m_4}{N^2 x^5} + \frac{g^3 m_6}{N^3 x^7} + \ldots, \]

and the moments are given by \[ (4.31) \]. Furthermore, the density satisfies the homogeneous part of \[ (4.8) \].

Proof. We will establish \[ (4.8) \] in a few steps, initially establishing that the density satisfies the homogeneous form of \[ (4.8) \]. However we will work with the version where the independent variable is not scaled for the bulk scaling regime purely for convenience and entailing no loss of generality, which is just the version with \( g \to N \)

\[ \rho''(1) + (4N - x^2) \rho'(1) + x \rho(1) = 0. \]

Thus \( \rho(1) = K_N(x, x) \) and using \[ (4.1) \] we compute the first three derivatives with respect to \( x \)

\[ \frac{1}{\sqrt{N}} e^{\frac{i}{2} x^2} K_N = p_N p_{N-1} - p_N p'_N, \]

\[ \frac{1}{\sqrt{N}} e^{\frac{i}{2} x^2} K'_N = -p_N p_{N-1}, \]

\[ \frac{1}{\sqrt{N}} e^{\frac{i}{2} x^2} K''_N = x p_N p_{N-1} - p'_N p_{N-1} - p_N p'_N, \]

\[ \frac{1}{\sqrt{N}} e^{\frac{i}{2} x^2} K'''_N = (4N - x^2) p_N p_{N-1} - x p'_{N} p_{N-1} + x p_N p'_{N-1}, \]

where we have repeatedly used \[ (4.6) \]. Furthermore, of the four bilinear products \( p_N p_{N-1}, p'_N p_{N-1}, p_N p'_N, p'_N p'_{N-1} \), only three are independent as one can deduce

\[ p'_N p'_{N-1} = -N p_N p_{N-1} + x p'_N p_{N-1}, \]

from \[ (4.5) \] and \[ (4.4) \]. Using the first three relations one can invert these for \( p_N p_{N-1}, p'_N p_{N-1}, p_N p'_N \) as the determinant of the transformation is non-vanishing. Thus we have

\[ e^{-\frac{i}{2} x^2} p_{N-1} = -\frac{1}{\sqrt{N}} K'_N, \]

\[ e^{-\frac{i}{2} x^2} p'_{N-1} = \frac{1}{2\sqrt{N}} [K_N - x K'_N - K''_N], \]

\[ e^{-\frac{i}{2} x^2} p_N p'_{N-1} = -\frac{1}{2\sqrt{N}} [K_N + x K'_N + K''_N]. \]
Substituting these into the fourth relation gives (4.10). The inhomogeneous relation now follows from the sequence of steps

\[ 0 = \int_{-\infty}^{\infty} dx \frac{1}{z-x} \left[ \rho''(1) + (4N-x^2)\rho'(1) + x\rho(1) \right] = \int_{-\infty}^{\infty} dx \frac{\rho''(1)}{z-x} \]

\[ + (4N-z^2) \int_{-\infty}^{\infty} dx \frac{\rho'(1)}{z-x} + \int_{-\infty}^{\infty} dx (z+x)\rho(1) + z \int_{-\infty}^{\infty} dx \frac{\rho(1)}{z-x} - \int_{-\infty}^{\infty} dx \rho(1). \]

Now we integrate by parts the first three terms using \( \partial_x(z-x)^{-1} = -\partial_x(z-x)^{-1} \) and assuming \( z \notin \mathbb{R} \), \( \rho(1)(x) \), \( x\rho(1)(x) \), \( \rho'(1)(x) \) and \( \rho''(1)(x) \) all vanish sufficiently rapidly as \( x \to \pm\infty \). This is justified because our solution to (4.10) is the single one, out of the three possible, that possesses exponential decay at the boundaries. This also justifies our interchange of derivative and integral as the integrals are uniformly and absolutely convergent. We find that the only boundary terms remaining on the right-hand side are two copies of the normalisation integral, and thus (4.8) follows once the bulk scaling \( x \mapsto \sqrt{\frac{N}{g}} x \) is re-instated. □

**Remark 4.1.** As a consistency check we observe that the \( 1/N \) expansion of the resolvent (3.1) along with coefficients (3.3)–(3.9), under the specialisation \( \kappa \to 1 \), identically satisfies (4.8) up to the error term of \( O(N^{-8}) \).

As one can see the third-derivative term in (4.8) can be interpreted as a correction term in the large \( N \) expansion, so this relation can serve to generate successive terms in such an expansion. An explicit large \( N \) expansion for the resolvent was found in Prop. 4.5 of [21], along with a recurrence for the coefficients. Let

\[ \frac{g}{N} W_1(x) = \eta_0(x) + \frac{\eta_1(x)}{N^2} + \ldots + \frac{\eta_k(x)}{N^{2k}} + O(N^{-2k-2}), \quad k \in \mathbb{N}, \]

where

\[ \eta_0(x) = \frac{1}{4} \left[ x - \sqrt{x^2 - 4g} \right], \]

and

\[ \eta_j(x) = \sum_{r=2j}^{3j-1} C_{j,r}(x^2 - 4g)^{-r-1/2}, \quad j \in \mathbb{N}. \]

Then for \( 2j+2 \leq r \leq 3j+2 \) we have

\[ C_{j+1,r} = g^2 \frac{(2r-3)(2r-1)}{r+1} \left[ (r-1)C_{j,r-2} + g(4r-10)C_{j,r-3} \right], \]

and \( C_{j,2j-1} = C_{j,3j} = 0 \).

In Haagerup and Thorbjørnsen [20] and Ledoux’s works [27, 26] a linear ordinary differential equation was derived for the exponential generating function (3.3a). In Haagerup and Thorbjørnsen’s [20] notation we have \( c(p,N) = m_{2p}(N,1) \) whilst in Ledoux’s notation \( a_p^N \equiv m_{2p}(N,1) \). We give an alternative proof of this result using the result of Theorem 4.

**Theorem 5** (Eqs. (0.2, 2.16) of [20], Eq. (18) of [27]). The GUE exponential generating function \( u(t, N) \) satisfies the differential equation

\[ tu'' + 3u' - t(t^2 + 4N)u = 0, \]

subject to the boundary conditions

\[ u(t, N) \sim N + \frac{N^2}{2!} t^2 + \ldots. \]

The solution defined above is

\[ u(t, N) = N e^{-t^2/2} F_1 \left( 1 + N, 2; t^2 \right), \]

where \( F_1(1 + N, 2; t^2) \) is the regular confluent hypergeometric function [34].
Proof. We start with the Laplace transform (3.36) and compute the difference of the left-hand side and right-hand sides of (4.8), yielding

\begin{equation}
0 = \sqrt{\frac{N}{g}} \int_0^\infty dt \ e^{-\sqrt{\frac{N}{g}} xt} \left[ -\sqrt{\frac{N}{g}} t^3 - (4g - x^2)\sqrt{\frac{N}{g}} t + x \right] - 2N.
\end{equation}

Now we employ the identity \(-\sqrt{\frac{N}{g}} xe^{-\sqrt{\frac{N}{g}} xt} = e^{-\sqrt{\frac{N}{g}} xt}\) and for higher orders, and integrate by parts which gives us

\begin{equation}
0 = \int_0^\infty dt \ e^{-\sqrt{\frac{N}{g}} xt} \left[ (tu)^\prime\prime + u\prime - t^3 u - 4Ntu \right] + \left[ -\sqrt{\frac{N}{g}} tu - 2u\prime \right] e^{-\sqrt{\frac{N}{g}} xt} \right]_0^\infty - 2N.
\end{equation}

Assuming the upper limit vanishes for each of the three terms, in some sector \(\arg(t) < \pi\), then the evaluations \(u(0) = N, u\prime(0) = 0\) lead to the cancellation of the inhomogeneous terms. Thus we have (4.24). In fact (4.24), after removing the factor \(e^{-t^2/2}\), is one of the standard forms of the confluent hypergeometric differential equation, see §13.2 of [31], and only the regular part is admissible because the other solution, \(U(N + 1, 2, t^2) \sim t^{-2}/N!\) as \(t \to 0\) and has \(\log(t)\) terms. □

Remark 4.2. As \(t \to +\infty\) with \(\arg(t^2) \leq \frac{\pi}{2} - \delta\) and \(\delta > 0\)

\begin{equation}
u(t) \sim Ne^{-t^2/2} \cdot e^{t^2} \frac{t^{N-1}}{(N-1)!} = \frac{t^{N-1}}{(N-1)!} e^{t^2/2},
\end{equation}

which implies that the Laplace transform of \(\nu(t)\) does not exist, unless the integral is taken along a ray such that \(\text{Re}(t^2) < 0\).

A direct consequence of the Theorems 4 or 5 is that the moments satisfy a linear recurrence relation. We can give an independent derivation of such a recurrence relation.

Theorem 6 ([23, Theorem 4.1 of [20], Theorem 1 of [27]). The moments of the GUE satisfy the linear difference equation

\begin{equation}(p + 1)m_{2p} = (4p - 2)Nm_{2p-2} + (p - 1)(2p - 1)(2p - 3)m_{2p-4},\end{equation}

subject to the initial conditions \(m_0 = N, m_2 = N^2\).

Proof. Taking the homogeneous form of (4.8) for the density and integrating it against the monomial \(x^{2p-1}\) we deduce using integration by parts

\begin{align*}
0 &= \int_{-\infty}^\infty dx \ x^{2p-1} \left[ x\rho_{(1)} + (4g - x^2)\rho_{(1)}\prime + \frac{g^2}{N^2} \rho_{(1)}\prime\prime \right] \\
&= m_{2p}^* + 4g \left\{ \left[ x^{2p-1} \rho_{(1)} \right]_{-\infty}^\infty - (2p - 1) \int_{-\infty}^\infty dx \ x^{2p-2} \rho_{(1)} \right\} \\
&- \left\{ \left[ x^{2p+1} \rho_{(1)} \right]_{-\infty}^\infty - (2p + 1) \int_{-\infty}^\infty dx \ x^{2p} \rho_{(1)} \right\} + \frac{g^2}{N^2} \left\{ \left[ x^{2p-1} \rho_{(1)}\prime \right]_{-\infty}^\infty - (2p - 1) \int_{-\infty}^\infty dx \ x^{2p-2} \rho_{(1)}\prime \right\}.
\end{align*}

Further use of integration by parts shows

\begin{align*}
0 &= m_{2p}^* - 4g(2p - 1)m_{2p-2} + (2p + 1)m_{2p} - \frac{g^2}{N^2} (2p - 1)(2p - 2)(2p - 3)m_{2p-4} \\
&+ \left[ 4g x^{2p-1} \rho_{(1)} - x^{2p+1} \rho_{(1)} \right] \\
&+ \frac{g^2}{N^2} \left( x^{2p-1} \rho_{(1)}\prime - (2p - 1)x^{2p-2} \rho_{(1)}\prime + (2p - 1)(2p - 2)x^{2p-3} \rho_{(1)} \right) \right]_{-\infty}^\infty.
\end{align*}

Now we require our density to satisfy \(x^{2p+1} \rho_{(1)}, x^{2p-2} \rho_{(1)}, x^{2p-1} \rho_{(1)}\prime \to 0\) as \(x \to \pm\infty\) for all \(p \in \mathbb{N}\), which our solution indeed satisfies. Adjusting for (3.20) we have (4.30). □
Using (4.30) one can efficiently generate the low-order moments and we record the first seven for checking purposes

\[
m_0 = N, \\
m_2 = N^2, \\
m_4 = N(2N^2 + 1), \\
m_6 = 5N^2(N^2 + 2), \\
m_8 = 7N(2N^4 + 10N^2 + 3), \\
m_{10} = 21N^2(2N^4 + 20N^2 + 23), \\
m_{12} = 33N(4N^6 + 70N^4 + 196N^2 + 45).
\]

(4.31)

Ledoux [27] has utilised this recurrence relation to derive the large \(N\) behaviour of the moments up to the first non-zero correction. It is easy to extend this method to obtain more corrections.

**Theorem 7.** As \(N \to \infty\) for fixed \(p\), \(m_{2p}(N,1) = O(N^{p+1})\). Furthermore

\[
m_{2p}(N,1) = \frac{1}{C_p N^{p+1}} \left[ 1 + \frac{1}{12} (p+1)p(p-1)N^{-2} + \frac{1}{1440} (p+1)p(p-1)(p-2)(5p-2)N^{-4} \\
+ \frac{1}{362880} (p+1)p(p-1)(p-2)(p-3)(5p-5)(35p^2 - 77p + 12)N^{-6} + O(N^{-8}) \right].
\]

(4.32)

**Proof.** We proceed by peeling off successive terms in the large \(N\) development of

\[
m_{2p} = C_p N^{p+1} \left[ 1 + a_p N^{-2} + b_p N^{-4} + c_p N^{-6} + \ldots \right],
\]

the odd orders are not present as one can see from the substitution \(m_{2p} = \frac{(2p)!}{p!(p+1)!} N^{p+1} X_p\) which yields the difference equation \(X_p = X_{p-1} + \frac{p(p-1)}{4N} X_{p-2}\). At the \(N^{-1}\) order we find the first order inhomogeneous difference equation \(p - p^2 - 4a_{p-1} + 4a_p = 0\) which is solved with \(a_1 = 0\) to give \(a_p = \frac{1}{p!(p^3 - p)}\). Using this solution we find at the \(N^{-3}\) order another first order inhomogeneous difference equation \(p(p-1)^2(p-2)(p-3)(p-4)N^{-6} + O(N^{-8})\).

Remark 4.3. This agrees with Theorem 3 in the case \(h = 0, \kappa = 1\).

An explicit formula for the moments is given in Mehta [28], see §6.5.6, and also by Mezzadri and Simm 2011 [29], see Theorem 2.9, Equations (31) and (32). In their notations our moment can be written \(m_{2p}(N,1) \equiv C(p, N)\) and \(m_{2p}(N,1) \equiv 2pM_{C_p}^{(N)}(2p, N)\).

**Theorem 8** (28, 29). For all \(N > 0\), \(p \in \mathbb{Z} \geq 2p\) the GUE moments are given by Mehta’s evaluation

\[
m_{2p}(N,1) = \frac{(2p)!}{2^p p!} N_2 F_1(-p, 1 - N; 2; 2) = \frac{(2p)!}{2^p p!} \sum_{j=0}^{p} \binom{p}{j} \left( \frac{N}{j+1} \right)^{2j},
\]

(4.33)

or by Mezzadri and Simm’s

\[
m_{2p}(N,1) = \begin{cases} \\
\frac{2^{N+p} \Gamma \left( \frac{N}{2} + 1 \right)}{\pi^{1/2} (2p+1) \Gamma(N)} \sum_{j=0}^{\min\left(\frac{N}{2}-1, p\right)} \binom{p}{j} \left( \frac{N}{j+1} \right)^{p+1} \left( \frac{N}{2} - j \right)_{p+\frac{1}{2}}, & \text{N even,} \\
\frac{2^{N+p} \Gamma \left( \frac{N+1}{2} \right)^2}{\pi^{1/2} (2p+1) \Gamma(N)} \sum_{j=0}^{\min\left(\frac{N}{2}+1, p\right)} \binom{p}{j} \left( \frac{N}{2} - j \right) \left( \frac{N}{2} + 1 - j \right)_{p+\frac{1}{2}}, & \text{N odd.}
\end{cases}
\]

(4.34)
Thus we first recast (4.36) in terms of the orthonormal polynomials \{p_n\}.

For the proof of that Theorem we will establish the result for the unscaled system (4.2).

(4.36) \[ \phi(s, N) = \frac{1}{2^{s^2}} \left[ \left( 1 + \frac{s^2}{1 - s^2} \right)^N - 1 \right], \]

which follows directly from the definition (3.37) and the evaluation (4.33). The latter explicit formula agrees with the specialisation of \( \kappa = 1 \) in the cases \( 0 \leq p \leq 6 \) of (3.27)-(3.33).

4.2. \( \kappa = 1/2 \) GOE Moments. We revise the well-known explicit result for the density of eigenvalues in the orthogonal Gaussian ensemble, as given in §4, p. 158, 9 of [2] (after correcting for the typographical error), and adapted to our slightly differing conventions. From this work we deduce

(4.36) \[ \rho_{(1)}(x, N) = \sqrt{\frac{N}{g}} \left\{ K_N(x, x) + \frac{e^{-\frac{4}{g^2} x^2} H_{N-1}(2^{-1/2} x)}{\sqrt{2} N^{1/2}} \int_{-\infty}^{\infty} dt \frac{\sqrt{\pi}}{2 N} e^{-\frac{4}{g^2} x^2} H_{N-1}(2^{-1/2} x) \right\} \]

where \( K_N(\cdot, \cdot) \) is given by the kernel (1.1).

The orthogonal analogue of (4.36) is a fifth order ordinary differential equation for the resolvent.

Theorem 9. The resolvent \( W_{1}(x) \) for the GOE satisfies the fifth order, linear inhomogeneous ordinary differential equation

(4.37) \[ -\frac{2}{N^4} W_1^{(V)} + \frac{6}{N^2} W_1^{(V)} + 6 \frac{g^2}{N^2} x W_1'' - 6 \frac{g^2}{N^2} x W_1'' + \left[ x^4 - (4N - 2) \frac{g}{N} x^2 + (-16N^2 + 16N + 2) \frac{g^2}{N^2} \right] W_1 + x \left[ x^2 - (4N - 2) \frac{g}{N} \right] W_1 = 2N(x^2 - 4g) + 10g, \]

subject to the boundary conditions (4.9), for fixed \( g, N \), and the moments are given by (4.51).

Proof. Our proof will be a natural extension of the methods adopted in the proof of Theorem 4. As in the proof of that Theorem we will establish the result for the unscaled system \( (g \mapsto N) \) to simplify matters.

Thus we first recast (4.36) in terms of the orthonormal polynomials \( \{p_n\}_{n=0}^\infty \)

(4.38) \[ \frac{1}{\sqrt{N}} \rho_{(1)} = e^{-\frac{4}{g^2} x^2} \left[ p_N p_{N-1} - p_N \right] + e^{-\frac{4}{g^2} x^2} \left[ \frac{1}{g} q_N + A_N \right] p_{N-1}, \]

where the constant \( A_n \) is defined as

(4.39) \[ A_n = \frac{1}{4} \chi_{n \in 2^Z} D_n + \frac{1}{\sqrt{n}} \chi_{n \in 2^{Z+1}} D^{-1}_{n-1}, \quad D_n = \pi^{1/4} 2^{3/4} \sqrt{\frac{(n-1)!}{n!}}. \]

The new variable \( q_n \) is defined as

(4.40) \[ q_n(x) := \int_{-\infty}^{x} dt \ e^{-\frac{4}{g^2} t} p_n(t), \]

and clearly \( q_n = e^{-\frac{4}{g^2} x^2} p_n(x) \). The basis of bilinear products is now five dimensional with basis

\[ \{p_N p_{N-1}, p_N \rho_{N-1}, p_N \rho_{N-1}, \frac{1}{g} q_N + A_N \rho_{N-1}, \frac{1}{g} q_N + A_N \rho_{N-1}\}. \]
Using the relation for \( q_n' \) and (4.6) we compute the first four derivatives of \( \rho_{(1)} \) which we write in matrix form

\[
(4.41) \quad \frac{1}{\sqrt{N}} \left( \begin{array}{c}
\rho_{(1)} \\
\rho_{(1)}' \\
\rho_{(1)}'' \\
\rho_{(1)}'''(V)
\end{array} \right) =
\left( \begin{array}{cccc}
0 & 1 & -1 & 0 \\
-\frac{1}{2} & 0 & 0 & 1 \\
\frac{1}{4} - \frac{1}{2}x^2 + \frac{1}{2}(N + \frac{3}{2})x & \frac{1}{4}x & \frac{1}{4}x & -\frac{1}{4}x \\
\frac{1}{16}x^4 - \frac{1}{4}(N + \frac{3}{2})x + \frac{1}{4}x & 1 & \frac{1}{16}x^4 - \frac{1}{2}(N + \frac{3}{2})x^2 + x^2 - N + \frac{1}{2}
\end{array} \right) \left( \begin{array}{c}
e^{-\frac{1}{2}x^2} p_{NP_{N-1}} \\
e^{-\frac{1}{2}x^2} p_{NP_{N-1}}' \\
e^{-\frac{1}{2}x^2} [\frac{1}{2}q_{N} + A_N]p_{NP_{N-1}} \\
e^{-\frac{1}{2}x^2} [\frac{1}{2}q_{N} + A_N]p'_{N-1}
\end{array} \right) 
\]

This is invertible for \( N > 1 \) as the determinant is \(-\frac{\varpi}{2}(N - 1)\). The fifth derivative is

\[
(4.42) \quad \frac{1}{\sqrt{N}} \rho_{(1)}'''(V) = \left[ -\frac{1}{32}x^4 + \frac{1}{4}(N + \frac{7}{4})x^3 - \frac{1}{2}N^2 - \frac{11}{4}N + 1 \right] e^{-\frac{1}{2}x^2} p_{NP_{N-1}} \\
+ \left[ -\frac{1}{64}x^3 + \frac{1}{4}(N + \frac{5}{2})x^2 + \frac{1}{2}(-N^2 + 3N + \frac{1}{4})x \right] e^{-\frac{1}{2}x^2} p_{NP_{N-1}}' \\
+ \left[ \frac{1}{64}x^3 + \frac{1}{4}(N + \frac{5}{2})x^2 + \frac{1}{2}(N^2 - 3N - \frac{1}{4})x \right] e^{-\frac{1}{2}x^2} [\frac{1}{2}q_{N} + A_N]p_{NP_{N-1}} \\
+ \left[ \frac{1}{16}x^4 + \frac{1}{4}(N + \frac{3}{2})x^3 + \frac{1}{4}(-N^2 - 3N - \frac{1}{4})x \right] e^{-\frac{1}{2}x^2} [\frac{1}{2}q_{N} + A_N]p'_{N-1}
\]

Substituting the solution for the bilinear products into this expression we get

\[
(4.43) \quad -4\rho_{(1)}'''(V) + 5(x^2 - 4N + 2)\rho_{(1)}'' + 6x\rho_{(1)}' + [-(x^4 + (8N - 4)x^2 - 16N^2 + 16N + 2)\rho_{(1)} + x(x^2 - 4N + 2)\rho_{(1)} = 0.
\]

Restoring the bulk scaling \( x \mapsto \sqrt{\frac{1}{g}}x \) we have the homogeneous form of (4.3.1). To find the differential equation for \( W_1 \) we repeat the methods employed in the proof of Theorem [4] except that there are more terms to treat. Integrating the homogeneous form of the differential equation against \((z - x)^{-1}\) on \(\mathbb{R}\), and performing the substitutions for the \(x\)-dependent coefficients we arrive at

\[
(4.44) \quad 0 = -4\frac{g^4}{N^4} \int_{-\infty}^{\infty} dx \left\{ \rho_{(1)}'''(V)\left(\frac{1}{z-x}\right) + 5\left[ z^2 - (4N - 2)\frac{g}{N}\right] \int_{-\infty}^{\infty} dx \frac{\rho_{(1)}''}{\left(\frac{1}{z-x}\right) - \frac{g^2}{2N^2} \int_{-\infty}^{\infty} dx \frac{\rho_{(1)}}{\left(\frac{1}{z-x}\right) - \frac{5}{3}z \int_{-\infty}^{\infty} dx \frac{\rho_{(1)}}{\left(\frac{1}{z-x}\right) - \frac{g^2}{2N^2} \int_{-\infty}^{\infty} dx \frac{\rho_{(1)}}{(x + z)} - \frac{6}{N^2} \int_{-\infty}^{\infty} dx \rho_{(1)}'' + \frac{6^2}{N^2} \int_{-\infty}^{\infty} dx \rho_{(1)}''' + \int_{-\infty}^{\infty} dx [-2z^2 + 12(2N - 6)\frac{g}{N}m_0 - 4\frac{g^2}{N^2}m_2].\right]\right\}
\]

Making similar observations on \( \rho_{(1)} \) concerning its decay as \( x \to \pm \infty \) as we did in the proof of Theorem [4] we can conclude \( \int dx (z-x)^{-1}\partial_n^2 \rho_{(1)}(x) = \partial_n^2 W_1(z) \) for \( 0 \leq n \leq 5 \). Of the four final terms of the above expression only a few are non-zero and these contribute \( -(2z^2 + (12N - 6)\frac{g}{N})m_0 - 4\frac{g^2}{N^2}m_2 \). From the knowledge of the first two moments we deduce the inhomogeneous term and arrive at (4.3.7).
Remark 4.5. As a check we can verify that the $1/N$ expansion of the resolvent \( (3.1) \) along with coefficients \( (3.3)-(3.9) \), under the specialisation $\kappa \to 1/2$, identically satisfies \( (4.37) \) up to the error term of $O(N^{-7})$.

In \cite{27} a linear ordinary differential equation was derived for the exponential generating function, however we can easily re-derive this from the preceding theorem. Here Ledoux’s definitions imply $b_p^N = m_{2p}(N, 1/2)$.

**Theorem 10** \textbf{\cite{27}. Equation (27).} \textit{The GOE exponential generating function} \( u(t, N) \text{ satisfies the fourth order linear ordinary differential equation} \)

(4.45) \[ tu^{(IV)} + 5u'' - t(5t^2 + 8N + 20N - 10)u'' - (36t^2 + 20N - 10)u(f) \]
\[ + t \left[ 4t^4 + (20N - 10)t^2 + 16N^2 - 16N - 44 \right] u = 0, \]

\textit{or equivalently if we define} \( U \equiv u'' - (4t^2 + 4N - 2)u \text{ then} U(t) \text{ satisfies} \)

(4.46) \[ tU'' + 5U' - t(t^2 + 4N - 2)U = 0. \]

\textit{The solutions are subject to the boundary conditions} \( (4.47) \)

(4.47) \[ u(t) \sim N + \frac{N(N + 1)}{2!} t^2 + \frac{N(2N^2 + 5N + 5)}{4!} t^4 + \frac{N(5N^3 + 22N^2 + 52N + 41)}{6!} t^6 \ldots. \]

\textit{Proof.} We utilise the same method as given in the proof of Theorem \textbf{5}. In our intermediate step we find

(4.48) \[ 0 = \frac{g}{N} \int_0^\infty dt e^{-\sqrt{\frac{N}{g}}xt} \left\{ \partial_t^4 (tu) + \partial_t^3 u - \partial_t^2 [5t^3 u + (8N - 4)tu] - \partial_t [6t^2 u + (4N - 2)u] \right. \]
\[ + \left. [4t^4 + 5(4N - 2)t^3 - (-16N^2 + 16N + 2)t]u \right\} \]
\[ + \frac{g}{N} \left[ e^{-\sqrt{\frac{N}{g}}xt} (-\partial_t^3 (tu) - \partial_t^2 u + \partial_t (5t^3 u + (8N - 4)tu) + (6t^2 + 4N - 2)u) \right. \]
\[ + \partial_t e^{-\sqrt{\frac{N}{g}}xt} (\partial_t^2 (tu) + \partial_t u - (5t^3 u + (8N - 4)tu) + \partial_t^2 e^{-\sqrt{\frac{N}{g}}xt} (-\partial_t (tu) - u) + \partial_t^3 e^{-\sqrt{\frac{N}{g}}xt} (tu) \right] \]
\[ - 2N(x^2 - 4g) - 10g. \]

Again assuming the upper terminal contribution of the boundary term vanishes we compute the lower terminal to be $\frac{g}{N} \left( 4u''(0) - (12N - 6)u(0) + 2 \frac{N}{g} x^2 u(0) \right)$. Using the data for $m_0, m_2$ this cancels the inhomogeneous terms from the original differential equation and we have \( (4.46) \). \( \square \)

Ledoux has also shown that a linear recurrence relation for the moments follows from the above result, which can also be derived directly from Theorem \textbf{9}.

**Theorem 11** \textbf{\cite{27}. Theorem 2.} \textit{The GOE moments} \( m_p(N, 1/2) \text{ satisfy the fourth order, linear difference equation} \)

(4.49) \[ (p + 1)m_{2p} = (4p - 1)(2N - 1)m_{2p-2} + (2p - 1)(10p^2 - 9p - 8N^2 + 8N)m_{2p-4} \]
\[ - 5(2p - 3)(2p - 4)(2p - 5)(2N - 1)m_{2p-6} - 2(2p - 3)(2p - 4)(2p - 5)(2p - 6)(2p - 7)m_{2p-8}, \]

\textit{subject to the initial values} \( m_0 = N, \ m_2 = N(N + 1) \text{ from} p \geq 2. \)
Proof. As in the proof of Theorem 6 we integrate $x^{2p-3}$ against the homogeneous form of (4.37) for $\rho_{(1)}$, and after integrating by parts we find

\begin{equation}
(4.50) \quad 0 = \left( \frac{g}{N} \right)^p \left\{ (2p+2)m_{2p} - (4N-2)(4p-1)m_{2p-2} - (2p-3)[-16N^2 + 16N + 2 + 6(2p-2) + 5(2p-1)(2p-2)]m_{2p-4} + 5(4N-2)(2p-3)(2p-4)(2p-5)m_{2p-6} + 4(2p-3)(2p-4)(2p-5)(2p-6)(2p-7)m_{2p-8} \right\}
\end{equation}

\begin{equation}
\begin{split}
+ & \left[ -x^{2p+1}\rho_{(1)} + (8N-4)\frac{g}{N}x^{2p-1}\rho_{(1)} + (-16N^2 + 16N + 2)\frac{g^2}{N^3}x^{2p-3}\rho_{(1)} \\
& \quad - 5\frac{g^2}{N^2} \rho_{(1)}' + (2p-1)x^{2p-2}\rho_{(1)}' + (2p-1)(2p-2)x^{2p-3}\rho_{(1)}' \right]
\end{split}
\end{equation}

Clearly $x^{2p+1}\rho_{(1)}$, $x^{2p-2}\rho_{(1)}'$, $x^{2p-1}\rho_{(1)}''$, $x^{2p-4}\rho_{(1)}'''(IV)$ all vanish exponentially fast as $x \to \pm\infty$ for all $p \geq 4$ and we are justified in neglecting the boundary terms. Eq. (4.49) then follows. \qed

This recurrence relation is an efficient way to generate low order moments, of which we list the first seven for checking purposes

\begin{align*}
m_0 &= N, \\
m_2 &= N^2 + N, \\
m_4 &= 2N^3 + 5N^2 + 5N, \\
m_6 &= 5N^4 + 22N^3 + 52N^2 + 41N, \\
m_8 &= 14N^5 + 93N^4 + 374N^3 + 690N^2 + 509N, \\
m_{10} &= 42N^6 + 386N^5 + 2290N^4 + 7150N^3 + 12143N^2 + 8229N, \\
m_{12} &= 132N^7 + 1586N^6 + 12798N^5 + 58760N^4 + 167148N^3 + 258479N^2 + 166377N.
\end{align*}

Again the recurrence relation enables one to compute the large $N$ corrections to the moments and Ledoux has given the first correction beyond the leading order in [27]. We require more terms beyond the first correction, and these can be easily found using the recurrence relation.
Theorem 12. As $N \to \infty$ for fixed $p$ then $m_{2p} = O(N^{p+1})$ and the sub-leading coefficients are given by

$$
\frac{m_{2p}(N, \frac{1}{2})}{N^{p+1}} = C_p + 2^{2p-1} \left[ 1 - \frac{\Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-1} + \frac{1}{3} \frac{4^{p-1} p}{\Gamma(p + 1)} \left[ -3 + \frac{(7p - 1) \Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-2} + \frac{1}{3} 4^{p-2} p(p - 1) \left[ 8p - 7 - \frac{(14p - 4) \Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-3} + \frac{1}{45} 2^{2p-5} p(p-1)(p-2) \left[ -15(8p - 9) + \frac{(185p^2 - 317p + 6) \Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-4} + \frac{1}{45} 2^{2p-9} p(p-1)(p-2)(p-3)(p-4) \left[ -63(320p^2 - 1168p + 675) + \frac{4(6209p^3 - 29106p^2 + 26605p - 60) \Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-5} + O(N^{-6}).
$$

Proof. This is derived using the same methods as given in the proof of Theorem 7. There are three practical differences with the GUE case - the odd orders are present in addition the even ones, that the inhomogeneous difference equations are now of second order and the inhomogeneous terms involve Gamma functions. □

Remark 4.6. This agrees with Theorem 5 in the case $\kappa = 1/2$.

In Theorem 4.2 of [19] Goulden and Jackson derived an explicit formula for the GOE moments. In addition another formula for these moments has been deduced by Mezzadri and Simm 2011 [29], see Equation (34). Their notation is related to ours by $m_{2p}(N, 1/2) \equiv 2^p M_G^{(1)}(2p, N)$.

Theorem 13 [19, 29]. For all $p, N$ the GOE moments are

$$
m_{2p}(N, 1/2) = m_{2p}(N - 1, 1) + p! \sum_{i=0}^{p} 2^{2p-i} \sum_{j=0}^{p-i} \binom{p - \frac{1}{2}}{i} \binom{p - \frac{1}{2} - j}{j} \frac{N - i - j}{(N - j)^{2p + \frac{1}{2}}} + \phi_p(N),
$$

For $N$ even the GOE moments were given by Mezzadri and Simm as

$$
m_{2p}(N, 1/2) = m_{2p}(N - 1, 1)
$$

$$
= -2^p \sum_{j=1}^{\min\left(\frac{N}{2} - 1, p\right)} \sum_{i=0}^{\min\left(\frac{N}{2} - 1, p - j\right)} \binom{p}{i} \binom{p}{i + j} \frac{N - i - j}{(N - j)^{2p + \frac{1}{2}}} + \phi_p(N),
$$

where

$$
\phi_p(N) = \begin{cases} 
\frac{(2p)! \frac{N}{2}^{p - \frac{N}{2} - 1}}{\Gamma\left(\frac{N}{2}\right)} \sum_{j=0}^{\frac{N}{2} - 1} \sum_{i=0}^{\frac{N}{2} - 1} \binom{N - 1}{2j} \binom{(-1)^j 2^{-j - 2i}}{(2j + 2i + 1)! (p - \frac{N}{2} - j)!} & N \leq 2p, \\
+ \frac{(2p)! \frac{N}{2}^{p - \frac{N}{2} - 1}}{\Gamma\left(\frac{N}{2}\right)} \sum_{j=0}^{\frac{N}{2} - 1} \sum_{i=0}^{j} \frac{(N - i - 1)!}{(j - i)! (p - j)! 2^{p - 2j}} \binom{N - 1}{N - 2i - 1} & N > 2p,
\end{cases}
$$

Remark 4.7. For $0 \leq p \leq 6$ this agrees with Eqs. (3.32) - (3.33) in the case $\kappa = 1/2$. 

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4.3. \( \kappa = 2 \) GSE Moments. We recount the well-known explicit result for the density of eigenvalues in the symplectic Gaussian ensemble, as say given in §4, p. 159 of [2], but adapted to our slightly differing conventions. From this work we deduce

\begin{equation}
\rho_{(1)}(x, N) = \frac{1}{2} \sqrt{\frac{N}{g}} \left\{ \sqrt{2} K_{2N}(\sqrt{2x}, \sqrt{2x}) + \frac{e^{-\frac{x^2}{2}} H_{2N}(x)}{\sqrt{\pi} 2^{2N} (2N - 1)!} \int_{-\infty}^{x} dt \ e^{-\frac{t^2}{2}} H_{2N-1}(t) \right\}_{x = \sqrt{\frac{2}{N} x}},
\end{equation}

where \( K_N(\cdot, \cdot) \) is given by the kernel \((4.11)\).

All the results we give in this subsection for the GSE case can be expressed by the duality relations with the GOE. In \([43, 33]\) the general expression for all \( \kappa \) was given and also details of the implications for the generating functions in Corollary \( \mathbb{1} \) so we will refrain from repeating all of that here.

**Theorem 14** \([33, \text{Theorem 6 of } 27]\). For all \( p \in \mathbb{Z} \) and \( N \geq 1 \) the moments satisfy the duality relation

\begin{equation}
m_{2p}(N, 2) = (-)^{p+1} 2^{-p-1} m_{2p}(-2N, 1/2),
\end{equation}

as implied by \([33, 34]\) with \( \kappa = 2 \), together with the corresponding formula of Corollary \( \mathbb{1} \).

However, it is of independent interest to derive the results from first principles, and we will proceed in this manner.

The symplectic analogue of \((4.38)\) and \((4.37)\) is a fifth order ordinary linear differential equation for the density and an inhomogeneous version for the resolvent.

**Theorem 15.** The resolvent \( W_1(x) \) satisfies the fifth order, linear inhomogeneous ordinary differential equation

\begin{equation}
- \frac{1}{4} \frac{g^4}{N^4} W_1^{(V)} + 5 \left[ \frac{1}{4} x^2 - \frac{g}{N} (N + \frac{1}{2}) \right] \frac{g^2}{N^2} W_1''' + \frac{3}{4} \frac{g^2}{N^2} x W_1''
\end{equation}

\begin{equation}
+ \left[ -x^4 + (8N + 2) \frac{g^2}{N} x^2 + (-16N^2 - 8N + \frac{1}{2}) \frac{g^2}{N^2} \right] W_1' + x \left[ x^2 - (4N + 1) \frac{g}{N} \right] W_1 = 2N(x^2 - 4g) - 5g
\end{equation}

subject to the boundary conditions \((4.9)\), for fixed \( g, N \), with the moments given by \((4.71)\).

**Proof.** As in Theorems \( \mathbb{4} \) and \( \mathbb{9} \) we will derive the result for the unscaled independent variable and make the scaling at the conclusion of the derivation. We take as our starting point a simplified variant of \((4.56)\)

\begin{equation}
B_N \rho_{(1)}(x) = e^{-x^2} \left[ H_{2N}(x) H_{2N-1}(x) - H_{2N}(x) H_{2N-1}(x) \right] + e^{-\frac{x^2}{2}} H_{2N}(x) Q_{2N-1}(x),
\end{equation}

where \( B_N = \pi^{1/2} 2^{2N+1} (2N - 1)! \) and the new variable is defined as

\begin{equation}
Q_n(x) := \int_{-\infty}^{x} dt \ e^{-\frac{t^2}{2}} H_n(t).
\end{equation}

We will work with the Hermite polynomials instead of the \( p_n \) to avoid unnecessary factors of two appearing in the workings, and the identities we require that correspond to the ones employed for the \( p_n \) are

\[
H_{n+1} = 2x H_n - 2n H_{n-1},
\]

\[
H_n' = 2n H_{n-1},
\]

\[
H_n'' - 2x H_n' + 2n H_n = 0,
\]

\[
Q_n' = e^{-\frac{x^2}{2}} H_n,
\]

\[
H_n' H_{n-1} + 2x H_n H_{n-1} - 2x H_n' H_{n-1} = 0.
\]

Again we successively differentiate the density and employ the above identities to reduce the expressions to linear combinations of the independent bilinear products

\begin{equation}
\{ H_{2N} H_{2N-1}, H_{2N} H_{2N-1}, H_{2N} H_{2N-1}, H_{2N} Q_{2N-1}, H_{2N} Q_{2N-1} \}.
\end{equation}
The result for the first four derivatives is

(4.62) \[ B_N \begin{pmatrix} \rho'(1) \\ \rho''(1) \\ \rho'''(1) \\ \rho''''(1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & -x & 1 \\ x & 0 & -1 & x^2 - 4N - 1 & 0 \\ x^3 - (4N - 7)x & -4 & -x^2 + 4N - 1 & x^4 - (8N + 6)x^2 + 16N^2 + 8N + 3 & 4x \end{pmatrix} \begin{pmatrix} e^{-x^2} H_{2N} H_{2N-1} \\ e^{-x^2} H_{2N}^2 H_{2N-1} \\ e^{-x^2} H_{2N} H_{2N-1}^2 \\ e^{-x^2} H_{2N} Q_{2N-1} \\ e^{-x^2} H_{2N} Q_{2N-1} \end{pmatrix}. \]

In this case the determinant of the transformation is \(-36(2N + 1)\). The fifth order derivative is computed to be

(4.63) \[ B_N \rho''''(1) = \left[ -x^4 + (8N + 19)x^2 - 16N^2 + 52N + 8 \right] e^{-x^2} H_{2N} H_{2N-1} \\
+ \left[ -x^3 + (4N + 1)x \right] e^{-x^2} H_{2N}^2 H_{2N-1} + \left[ x^3 + (-4N + 5)x \right] e^{-x^2} H_{2N} H_{2N-1}^2 \\
+ \left[ -x^5 + (8N + 10)x^3 - (16N^2 + 40N + 15)x \right] e^{-x^2} H_{2N} Q_{2N-1} \\
+ \left[ x^4 - (8N + 2)x^2 + 16N^2 + 8N + 7 \right] e^{-x^2} H_{2N} Q_{2N-1}. \]

and after substituting for the basis products we find the homogeneous fifth order ordinary differential equation

(4.64) \[ -\frac{1}{4} \rho''''_{(1)} + 5 \left[ \frac{1}{4} x^2 - (N + \frac{1}{4}) \right] \rho''_{(1)} - \frac{3}{2} x \rho'_{(1)} \\
+ \left[ -x^4 + (8N + 2)x^2 - 16N^2 - 8N + \frac{1}{2} \right] \rho_{(1)} + x \left[ x^2 - (4N + 1) \right] \rho_{(1)} = 0. \]

Restoring the bulk scaling \( x \mapsto \sqrt{\frac{N}{g}} x \) we have the homogeneous form of (4.58). To find the differential equation for \( W_1 \) we repeat the methods employed in the proof of Theorems 4 and 9. Integrating the homogeneous form of the differential equation against \((z - x)^{-1}\) on \( \mathbb{R} \), and performing the subtraction for the \( x \)-dependent coefficients we arrive at

(4.65) \[ 0 = -\frac{1}{4} g^2 N^4 \int_{-\infty}^{\infty} dx \frac{\rho''''_{(1)}}{(z - x)} + 5 \left[ \frac{1}{4} z^2 - (N + \frac{1}{4}) \right] \frac{g^2}{N^2} \int_{-\infty}^{\infty} dx \frac{\rho''_{(1)}}{(z - x)} \\
- \frac{3}{4} \frac{g^2}{N^2} z \int_{-\infty}^{\infty} dx \frac{\rho''_{(1)}}{(z - x)} + \left[ -z^4 + (8N + 2) \frac{g}{N} z^2 + (-16N^2 - 8N + \frac{1}{2}) \frac{g^2}{N^2} \right] \int_{-\infty}^{\infty} dx \frac{\rho_{(1)}}{(z - x)} \\
+ z \left[ z^2 - (4N + 1) \frac{g}{N} \right] \int_{-\infty}^{\infty} dx \frac{\rho_{(1)}}{(z - x)} \\
- \frac{5}{4} \frac{g^2}{N^2} \int_{-\infty}^{\infty} dx (x + z) \rho''_{(1)} - \frac{3}{2} \frac{g^2}{N^2} \int_{-\infty}^{\infty} dx \rho'_{(1)} + \int_{-\infty}^{\infty} dx \left[ z^3 + z^2 x + z x^2 + x^3 - (8N + 2) \frac{g}{N} (x + z) \right] \rho_{(1)} \\
+ \int_{-\infty}^{\infty} dx \left[ -z^2 - z x - x^2 + (4N + 1) \frac{g}{N} \right] \rho_{(1)}. \]

Making similar observations on \( \rho_{(1)} \) concerning its decay as \( x \rightarrow \pm \infty \) as we did in the proof of Theorems 4 and 9 we can use \( \int dx (z - x)^{-1} \rho''_{(1)}(x) = \partial_x W_1(z) \) for \( 0 \leq n \leq 5 \). Only a few of the remaining terms are
non-zero and these contribute \([-2z^2 + (12N + 3)\frac{N}{2}]m_0 - 4\frac{N}{2}m_2\). Using \(m_0 = N\) and \(m_2 = N(N - \frac{1}{2})\) we deduce the inhomogeneous term and arrive at (4.65).

**Remark 4.8.** As a check we can substitute the \(W_1\) duality formula (3.38) into GSE ordinary differential equation (4.55) and easily recover its GOE equivalent (1.37). Furthermore we can verify that the 1\(/N\) expansion of the resolvent (5.11) along with coefficients (6.8)- (6.13), under the specialisation \(\kappa \to 2\), identically satisfies (4.55) up to the error term of \(O(N^{-7})\).

The result for the linear ordinary differential equation for the exponential generating function in the GOE case Theorem 10 has a symplectic analogue. This was also given by Ledoux [27] but can be directly deduced from the previous result. In the symplectic case Ledoux has defined the moments \(c_{2p}^{N} = 2^{p}m_{2p}(N, 2)\).

**Theorem 16.** The GSE exponential generating function \(u(t, N)\) satisfies the fourth order, linear differential equation

\[
(4.66) \quad tu^{(IV)} + 5u''' - t(\frac{4}{9}t^2 + 8N + 2)u'' - (9t^2 + 20N + 5)u' + t[\frac{1}{2}t^4 + 5(N + \frac{1}{2})t^2 + 16N^2 + 8N - 11]u = 0,
\]

subject to the boundary conditions

\[
(4.67) \quad u(t, N) \sim N + \frac{1}{2!}N(N - \frac{1}{2})t^2 + \frac{1}{4!}N(2N^2 - \frac{5}{2}N + \frac{5}{2})t^4 + \frac{1}{6!}N(5N^3 - 11N^2 + 13N - \frac{41}{6})t^6 \cdots .
\]

**Proof.** The method is the same as in the case of the GUE and GOE cases so we confine ourselves to recording the intermediate step

\[
(4.68) \quad 0 = \frac{g}{N} \int_{0}^{\infty} dt \left\{ \partial_t^4(tu) + \partial_t^3u - \partial_t^2\left[\frac{4}{9}t^2u + (8N + 2)tu\right] - \partial_t[\frac{4}{9}t^2u + (4N + 1)u] \right. \\
+ \left. [\frac{t^4}{2} + 5(N + \frac{1}{2})t^2 - (16N^2 - 8N + \frac{1}{4})t]u \right\} \\
+ \frac{g}{N} \left[ e^{-\sqrt{\frac{2}{N}}xt} (-\partial_t^4(tu) - \partial_t^3u + \partial_t((-\frac{4}{9}t^2 + 8N + 2)tu) + \frac{4}{9}t^2 + 4N + 1)u) \right. \\
+ \partial_t e^{-\sqrt{\frac{2}{N}}xt} (\partial_t^3(tu) + \partial_t u - (\frac{5}{4}t^2 + 8N + 2)tu) + \partial_t^2 e^{-\sqrt{\frac{2}{N}}xt} (-\partial_t(tu) - u) + \partial_t^3 e^{-\sqrt{\frac{2}{N}}xt}(tu) \right\}_{0}^{\infty} \\
- 2N(x^2 - 4g) + 5g.
\]

Again assuming the upper terminal contribution of the boundary term vanish we compute the lower terminal to be \(\frac{g}{N} \left( 4u''(0) - (12N + 3)u(0) + 2\frac{N}{2}x^2u(0) \right)\). Using the data for \(m_0, m_2\) this cancels the inhomogeneous terms from the original differential equation and we have (4.66). \(\square\)

**Remark 4.9.** Employing the duality formula for \(u\), (3.39), and the implied mapping of the independent variable into the preceding symplectic formula (4.66) we recover (4.45).

Ledoux has shown that a linear recurrence relation for the moments follows from the above result. This also follows directly from the ordinary differential equation for \(W_1\) as given in Theorem 10.

**Theorem 17 [27].** The GSE moments \(m_{2p}(N, 2)\) satisfy the fourth order, linear difference equation

\[
(4.69) \quad (p + 1)m_{2p} = \frac{1}{2}(4p - 1)(4N + 1)m_{2p-2} + \frac{1}{4}(2p - 3)(10p^2 - 9p - 32N^2 - 16N)m_{2p-4} \\
- \frac{5}{8}(2p - 3)(2p - 4)(2p - 5)(4N + 1)m_{2p-6} - \frac{1}{8}(2p - 3)(2p - 4)(2p - 5)(2p - 6)(2p - 7)m_{2p-8},
\]

subject to the initial values \(m_0 = N, m_2 = N(N - \frac{1}{2})\) for \(p \geq 2\).
Proof. As in the proof of Theorems 6, 11 we integrate $x^{2p-3}$ against the homogeneous form of (4.58) for $\rho_{(1)}$, and after integrating by parts we find

\[
(4.70) \quad 0 = \left(\frac{g}{N}\right)^P \left\{ (2p+2)m_{2p} - (4N+1)(4p-1)m_{2p-2} + (2p-3)[16N^2 + 8N - \frac{1}{2} - \frac{3}{2}(2p-2) - \frac{5}{2}(2p-1)(2p-2)]m_{2p-4} + \frac{5}{4}(4N+1)(2p-3)(2p-4)(2p-5)m_{2p-6} + \frac{1}{4}(2p-3)(2p-4)(2p-5)(2p-6)(2p-7)m_{2p-8} \right\} + \left[ -x^{2p+1}\rho_{(1)} + (8N+2)\frac{g}{N}x^{2p-1}\rho_{(1)} + (-16N^2 - 8N + \frac{1}{2})\frac{g^2}{N^2}x^{2p-3}\rho_{(1)} - \frac{3}{2}\frac{g^2}{N^2} \left( x^{2p-2}\rho_{(1)}' - (2p-2)x^{2p-3}\rho_{(1)} \right) + \frac{5}{4}\frac{g^2}{N^2} \left( x^{2p-1}\rho_{(1)}'' - (2p-1)x^{2p-2}\rho_{(1)}' + (2p-1)(2p-2)x^{2p-3}\rho_{(1)} \right) - \frac{5}{4}(4N+1)\frac{g^3}{N^3} \left( x^{2p-3}\rho_{(1)}'' - (2p-3)x^{2p-4}\rho_{(1)}' + (2p-3)(2p-4)x^{2p-5}\rho_{(1)} \right) - \frac{1}{4}\frac{g^4}{N^4} \left( x^{2p-4}\rho_{(1)}''' - (2p-3)x^{2p-5}\rho_{(1)}'' + (2p-3)(2p-4)x^{2p-6}\rho_{(1)}' \right) - (2p-3)(2p-4)(2p-5)x^{2p-6}\rho_{(1)}' + (2p-3)(2p-4)(2p-5)(2p-6)x^{2p-7}\rho_{(1)} \right\}\left[ -\infty \right].
\]

Again $x^{2p+1}\rho_{(1)}$, $x^{2p-2}\rho_{(1)}'$, $x^{2p-1}\rho_{(1)}''$, $x^{2p-3}\rho_{(1)}'''$ all vanish exponentially fast as $x \to \pm \infty$ for all $p \geq 4$ and we are justified in neglecting the boundary terms. Eq. (4.69) then follows. \[\Box\]

**Remark 4.10.** Employing the duality formula (4.57) into the GSE recurrence (4.69) we recover the GOE analog (4.49).

Initial data on the GSE moments are efficiently computed with this recurrence and we give the first seven cases

\[
\begin{align*}
m_0 &= N, \\
2m_2 &= 2N^2 - N, \\
4m_4 &= 8N^3 - 10N^2 + 5N, \\
8m_6 &= 40N^4 - 88N^3 + 104N^2 - 41N, \\
16m_8 &= 224N^5 - 744N^4 + 1496N^3 - 1380N^2 + 509N, \\
32m_{10} &= 1344N^6 - 6176N^5 + 18320N^4 - 28600N^3 + 24286N^2 - 8229N, \\
64m_{12} &= 8448N^7 - 50752N^6 + 204768N^5 - 470080N^4 + 668592N^3 - 516958N^2 + 166377N.
\end{align*}
\]

In addition we can make statements about the leading terms of the general $2p$-th moment, in the sense of large $N$ using the recurrence relation.
Theorem 18. As $N \to \infty$ with $p$ fixed then $m_{2p} = O(N^{p+1})$ and the leading coefficients are given by

\begin{align*}
(4.72) \quad \frac{m_{2p}}{N^{p+1}} &= C_p + 4^{p-1} \left[ -1 + \frac{\Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-1} \\
&\quad + \frac{1}{3} 4^{p-2} p \left[ -3 + \frac{(7p - 1) \Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-2} \\
&\quad + \frac{1}{3} 2^{p-7} p(p - 1) \left[ -8p + 7 + \frac{2(7p - 2) \Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-3} \\
&\quad + \frac{1}{45} 2^{2p-9} p(p - 1)(p - 2) \left[ -15(8p - 9) + \frac{(185p^2 - 317p + 6) \Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-4} \\
&\quad + \frac{1}{45} 2^{2p-13} p(p - 1)(p - 2)(p - 3) \left[ -320p^2 + 1008p - 487 + \frac{4(185p^2 - 387p + 28) \Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-5} \\
&\quad + \frac{1}{2835} 2^{2p-15} p(p - 1)(p - 2)(p - 3)(p - 4) \left[ -63(320p^2 - 1168p + 675) + \frac{4(6209p^3 - 29106p^2 + 26605p - 60) \Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right] N^{-6} + O(N^{-7}).
\end{align*}

Proof. This result can be shown using the methods of Theorem 7 with the additional features noted in proof of Theorem 12. \hfill \Box

Remark 4.11. This agrees with Theorem 3 in the case $k = 2$.

Again one has an explicit formula for the moments, given by Mezzadri and Simm 2011 [29], Equation (33) where we note $m_{2p}(N, 2) \equiv 2^p M^{(1)}_{p \chi}(2p, N)$.

Theorem 19 ([29]). For all $N > 0$ the GSE moments are given by

\begin{align*}
(4.73) \quad m_{2p}(N, 2) &= 2^{-p-1} m_{2p}(2N, 1) \\
&\quad - \frac{\Gamma(N + 1) \Gamma(N)}{\pi^{1/2} 4^{1-N} \Gamma(2N)} \sum_{j=1} \sum_{i=0} \binom{p}{i} \binom{p}{i+j} (N - i - j + 1)^{p-1}.
\end{align*}

Remark 4.12. For $0 \leq p \leq 6$ this agrees with Eqs. 32-33 in the case $k = 2$.

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6. Appendix

In this Appendix we record the values of $\hat{\rho}_{(1,4)}(x)$, $\hat{\rho}_{(1,5)}(x)$ and $\hat{\rho}_{(1,6)}(x)$ computed as for $\hat{\rho}_{(1,0)}(x)$, \ldots, $\hat{\rho}_{(1,3)}(x)$ in (3.16)-(3.19). We find

\begin{align*}
(6.1) \quad \hat{\rho}_{(1,4)}(x) &= \frac{1}{\pi} \left\{ \frac{1}{2048g^5/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} + \frac{1}{1024g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} + \frac{1}{96g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} - \frac{15}{768g^1/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} \right\} \\
&\quad + \frac{1}{27} \left\{ \frac{1}{2048g^5/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} + \frac{39}{4096g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} - \frac{7}{384g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} - \frac{17}{1536g^1/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} \right\} \\
&\quad - \frac{1}{17} \left\{ \frac{1}{2048g^5/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} + \frac{39}{4096g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} - \frac{7}{384g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} - \frac{17}{1536g^1/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} \right\} \\
&\quad - \frac{1}{17} \left\{ \frac{1}{2048g^5/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} + \frac{39}{4096g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} - \frac{7}{384g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} - \frac{17}{1536g^1/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} \right\} \\
&\quad - \frac{1}{17} \left\{ \frac{1}{2048g^5/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} + \frac{39}{4096g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} - \frac{7}{384g^3/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} - \frac{17}{1536g^1/2} \chi_{x \in (-\sqrt{2}, \sqrt{2})} \right\}.
\end{align*}
\[ \tilde{\rho}_{(1),A}(x) = h^5 \left\{ \frac{1}{\pi} (353x^4 + 1527gx^2 + 399g^2)(4g - x^2)^{-13/2} \chi_{x \in (-2,\sqrt{2},2,\sqrt{2})} + \frac{425}{524288g^{7/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{262144g^3}{2}(-2,\sqrt{2},2,\sqrt{2}) + \frac{159}{49152g^{5/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{847}{196080g^4(-2,\sqrt{2},2,\sqrt{2})} + \frac{705}{491520g^{3/2}(-2,\sqrt{2},2,\sqrt{2})} - \frac{1695}{737280g}(-2,\sqrt{2},2,\sqrt{2}) \right\} \]

\[ + h^3 \left\{ \frac{1}{2\pi} (445x^4 + 4332gx^2 + 1512g^2)(4g - x^2)^{-13/2} \chi_{x \in (-2,\sqrt{2},2,\sqrt{2})} + \frac{3019}{1048576g^{7/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{3019}{524288g^3(-2,\sqrt{2},2,\sqrt{2})} + \frac{157}{98304g^{5/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{1763}{5837}(-2,\sqrt{2},2,\sqrt{2}) - \frac{5837}{1474560g}(-2,\sqrt{2},2,\sqrt{2}) \right\} \]

\[ + h \left\{ \frac{1}{2\pi} (21x^4 + 420gx^2 + 294g^2)(4g - x^2)^{-13/2} \chi_{x \in (-2,\sqrt{2},2,\sqrt{2})} - \frac{1533}{524288g^{7/2}(-2,\sqrt{2},2,\sqrt{2})} - \frac{1533}{262144g^4(-2,\sqrt{2},2,\sqrt{2})} - \frac{327}{49152g^{5/2}(-2,\sqrt{2},2,\sqrt{2})} - \frac{1083}{196080g^2(-2,\sqrt{2},2,\sqrt{2})} - \frac{1533}{491520g^{3/2}(-2,\sqrt{2},2,\sqrt{2})} - \frac{717}{737280g}(-2,\sqrt{2},2,\sqrt{2}) \right\} \]

\[ \tilde{\rho}_{(1),B}(x) = h^5 \left\{ \frac{1}{\pi} (4081x^6 + 28625gx^4 + 26832g^2x^2 + 1738g^3)(4g - x^2)^{-17/2} \chi_{x \in (-2,\sqrt{2},2,\sqrt{2})} + \frac{161}{2097152g^{9/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{161}{1048576g^5(-2,\sqrt{2},2,\sqrt{2})} + \frac{1197}{1572864g^{7/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{259}{196080g^6(-2,\sqrt{2},2,\sqrt{2})} + \frac{259}{196080g^6(-2,\sqrt{2},2,\sqrt{2})} + \frac{1849}{983040g^{7/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{6075}{2949120g^{3/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{11865}{10321920g}(-2,\sqrt{2},2,\sqrt{2}) \right\} \]

\[ + h^4 \left\{ \frac{1}{2\pi} (8567x^6 + 147556gx^4 + 243180g^2x^2 + 31236g^3)(4g - x^2)^{-17/2} \chi_{x \in (-2,\sqrt{2},2,\sqrt{2})} + \frac{7987}{4194304g^{9/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{7987}{2097152g^5(-2,\sqrt{2},2,\sqrt{2})} + \frac{27543}{3145728g^{7/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{4889}{393216g^6(-2,\sqrt{2},2,\sqrt{2})} + \frac{4889}{393216g^6(-2,\sqrt{2},2,\sqrt{2})} + \frac{20683}{166080g^{7/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{34305}{5898240g^{3/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{39739}{2064340g^{5/2}(-2,\sqrt{2},2,\sqrt{2})} \right\} \]

\[ + h^2 \left\{ \frac{1}{\pi} (618x^6 + 32043gx^4 + 91299g^2x^2 + 16834g^3)(4g - x^2)^{-17/2} \chi_{x \in (-2,\sqrt{2},2,\sqrt{2})} + \frac{10731}{2097152g^{9/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{10731}{1048576g^5(-2,\sqrt{2},2,\sqrt{2})} + \frac{17679}{1572864g^{7/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{1737}{196080g^6(-2,\sqrt{2},2,\sqrt{2})} + \frac{5007}{983040g^{7/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{6033}{2949120g^{3/2}(-2,\sqrt{2},2,\sqrt{2})} + \frac{5019}{10321920g^{5/2}(-2,\sqrt{2},2,\sqrt{2})} \right\} \]

\[ + h \left\{ \frac{1}{\pi} (1485x^4 + 6138g^2x^2 + 1738g^3)(4g - x^2)^{-17/2} \chi_{x \in (-2,\sqrt{2},2,\sqrt{2})} \right\} \].

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