Differential Geometry

Extension of Reilly formula with applications to eigenvalue estimates for drifting Laplacians

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Extension de la formule de Reilly avec applications aux estimées de valeurs propres pour les laplaciens avec dérive

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1. Introduction

Among the important formulae in differential geometry, Reilly formula [5] is an important tool used to give a lower bound of eigenvalues of Laplacian operator on a Riemannian manifold with smooth boundary. Motivated by important work of G. Perelman [19] and the optimal transport theory [22], we study an extension of Reilly formula for drifting Laplacian operator associated with weighted measure and Bakry–Emery–Ricci tensor on a compact Riemannian manifold with smooth boundary. Then we give applications of this formula to the eigenvalue estimates of the drifting Laplacian on manifolds with boundary. The important motivation for such a study is its close connection with fundamental gaps of the classical Laplacian operator on manifolds [16].

Let \( (M, g) \) be a compact \( n \)-dimensional Riemannian manifold with boundary. Let \( L = \Delta \) be the Laplacian operator on the compact Riemannian manifold \( (M, g) \). Given \( h \) a smooth function on \( M \). We consider the elliptic operator with drifting \( L_h = \Delta - \nabla h \nabla \) associated with the weighted volume form \( dm = e^{-h} \, dv_g \). We also call \( L_h \) the \( h \)-Laplacian on \( M \). Assume that

\[-L_h u = \lambda u,\]

\( \lambda \) is an eigenvalue of \( L_h \).
with the Dirichlet or Neumann boundary condition. We shall always assume that \( \lambda > 0 \) and \( \int u^2 \, dm = 1 \). Then \( \lambda = \int |\nabla u|^2 \, dm \).

With the help of the Bochner formula for a smooth function \( f \) (see [19-14,17,18,23,6] and [21]), \( \frac{1}{2} L |\nabla f|^2 = |D^2 f|^2 + (\nabla f, \nabla L f) + \Ric(\nabla f, \nabla f) \), we can show the following Bochner formula for Bakry–Emery–Ricci tensor (see [2,3,15]):

\[
\frac{1}{2} L_h |\nabla f|^2 = |D^2 f|^2 + (\nabla f, \nabla L_h f) + (\Ric + D^2 h)(\nabla f, \nabla f).
\]  

(2)

We remark that the tensor \( \Ric^h := \Ric + D^2 h \) is called Bakry–Emery–Ricci tensor which arises naturally from the study of Ricci solitons [7].

Then we have \( \frac{1}{2} L_h |\nabla u|^2 = |D^2 u|^2 - \lambda |\nabla u|^2 + (\Ric + D^2 h)(\nabla u, \nabla u) \). Recall that the second fundamental form of \( \partial M \) is defined by \( I(X,Y) = g(\nabla_X v, Y) \), where \( v \) is the outer unit normal vector to \( \partial M \). And \( H = \text{tr} I \) is the mean curvature. We shall denote by \( h_v \) the normal derivative of \( h \) on \( \partial M \) or on the hypersurface \( P \).

Using the integration by part on \( P \), we have the following extension of Reilly formula:

**Theorem 1.** We have the following extension of Reilly formula:

\[
\int_M \left( |L_h f|^2 - |D^2 f|^2 \right) \, dm = \int_M \Ric^h(\nabla f, \nabla f) \, dm + \int_{\partial M} (H f_v - \nabla h \nabla f + \Delta_{\partial} f) f_v \, dm
\]

\[
+ \int_{\partial M} \left( \langle \nabla_{\partial} f, \nabla_{\partial} f \rangle - \langle \nabla_{\partial} f, \nabla_{\partial} f_v \rangle \right) \, dm.
\]

(3)

Here and below, the symbol \( \nabla_{\partial} \) means covariant derivative taken with respect to the induced metric on \( \partial M \).

We shall apply the above result to study the eigenvalue estimate for drifting Laplacian operators on \( M \). We impose either Dirichlet boundary condition \( u = 0 \) on \( \partial M \) or the Neumann boundary condition \( \frac{\partial u}{\partial n} = 0 \). The corresponding first nontrivial eigenvalue of the \( h \)-Laplacian is denoted by \( \lambda_{D} \) or \( \lambda_{N} \) respectively. In below, for notation simplicity, we shall denote by \( \lambda \) for \( \lambda_{D} \) or \( \lambda_{N} \) when it is clear in the context.

**Theorem 2.** Assume that

\[
\Ric + D^2 h \geq \left( \frac{|Dh|^2}{nz^2} + A \right) g,
\]

(4)

for some \( A > 0 \) and \( z > 0 \).

1. In the Dirichlet case, if the modified mean curvature \( H - h_v \) of \( \partial M \) is non-negative, then \( \lambda_{D} \geq \frac{n(n+1)A}{(n+1)(1-\beta)} \).

2. In the Neumann case, if \( \partial M \) is convex, that is, the second fundamental form (defined by \( I(X,Y) = g(\nabla_X v, Y) \)) is non-negative, then \( \lambda_{N} \geq \frac{n(n+1)A}{(n+1)(1-\beta)} \).

Recall that, by definition, a minimal \( h \)-hypersurface \( P \) in \( M \) is a hypersurface \( P \) with \( H = h_v = 0 \), where \( v \) is the unit normal vector which defines the second fundamental form of \( P \) in \( M \). We denote by \( \Delta_P \) the Laplacian operator of the induced metric on \( P \). Then we can prove the following result, which generalizes a result of Choi and Wang [4]:

**Theorem 3.** Let \((M^n, g)\) be a closed orientable manifold with \( \Ric^h \geq (n-1)K \). Let \( h \) be a smooth function on \( M \). Let \( P \subset M \) be an embedded minimal \( h \)-hypersurface dividing \( M \) into two submanifolds \( M_1 \) and \( M_2 \) (i.e., \( H = h_v \), this equality being independent on the orientation of the unit normal \( v \)). Then for the drifting Laplacian \( \Delta^h_p := \Delta_P - \nabla h \nabla_P \), \( \lambda_1(\Delta^h_p) \geq \frac{2(n-1)K}{n} \).

This paper is organized as follows. In Section 2 we prove Theorem 1, and Theorem 2 is proved in Section 3. Theorem 3 is proved in Section 4.

2. **Proof of Theorem 1**

We now prove Theorem 1.

**Proof.** We shall integrate the formula (2). Choose a set of local orthonormal frame fields \( \{e_i\} \) such that \( e_v = v \) on the boundary \( \partial M \). Note that \( \frac{1}{2} \int_M L_h |\nabla f|^2 \, dm = \int_M f_i f_j v_i v_j \, dm \), and \( \int_M (\nabla f, \nabla L_h f) \, dm = \int_M L_h f_i v_j v_i \, dm - \int_M |L_h f|^2 \, dm \), where, for the sake of simplicity, we still denote by \( dm \) the measure induced on \( \partial M \).

We shall use the classical notations that \( f_1 = df(e_v) \) and \( f_1 = D^2 f(e_v, e_v) \), etc. Then we have \( \int_M (|L_h f|^2 - |D^2 f|^2) \, dm = \int_M \Ric^h(\nabla f, \nabla f) \, dm + \int_{\partial M} (f_n L_h f - f_i f_in) \, dm \). Recall that \( L_h f = \Delta f - \nabla h \nabla f \). Then we have \( f_n L_h f - f_i f_in = -f_n \nabla h \cdot \nabla f + \sum_{j=1}^{m} (f_j f_n - f_j f_n) \).
Now
\[
\sum_{j<n} f_{jj} = \sum_{j<n} (e_j(e_j f) - (\nabla e_j e_j f)) = \sum_{j<n} ((\nabla_e e_j f) - (\nabla e_j e_j f) + \Delta_\lambda f)
\]
\[= Hf_n + \Delta_\lambda f.
\]

For \(j < n\),
\[
f_{jn} = f_{nj} = e_j(e_n f) - (\nabla e_j e_n f)
\]
\[= e_j(f_n) - \sum_{k<n} \delta_{jk} f_k.
\]

Then we have
\[
\sum_{j<n} f_{j} f_{jn} = (\nabla_\delta f, \nabla_\delta f_n) - \delta_{jk} f_j f_k.
\]

Putting all these together we have
\[
\int_M (|L_h f|^2 - |D^2 f|^2) \, dm = \int_M \text{Ric}^h(\nabla f, \nabla f) \, dm + \int_{\partial M} (Hf_n - \nabla h \nabla f + \Delta_\lambda f) f_n \, dm
\]
\[+ \int_{\partial M} (\|\nabla_\delta f, \nabla_\delta f) - (\nabla_\delta f, \nabla_\delta f_n) \| \, dm.
\]

The result follows. \(\Box\)

3. Proof of Theorem 2

The idea in the proof of Theorem 2 is similar to the one used by Reilly in [20] (see also [8]). We use the extension of Reilly formula to prove Theorem 2 below.

Proof. Let \(L_h u + \lambda u = 0\). We shall integrate the extension of Reilly formula (3).

Note that \((a + b)^2 \geq \frac{a^2}{2 + t} - \frac{b^2}{2}\) for any \(z > 0\). So, we have \((\Delta u)^2 = (\lambda u + g(\nabla h, \nabla u))^2 \geq \frac{\lambda^2 z^2}{2 + t} - \frac{g(\nabla h, \nabla u)^2}{2}\). Then we have
\[
\int_M (|L_h u|^2 - |D^2 u|^2) \leq \int_M \left(\lambda^2 u^2 - \frac{1}{n} (\Delta u)^2\right) \, dm \leq \int_M \left(\frac{\lambda^2 u^2 (n + 1)}{n(n + 1)} + \frac{g(\nabla h, \nabla u)^2}{nz}\right) \, dm. \tag{5}
\]

Note that for either Dirichlet or Neumann cases, we have
\[
\int_{\partial M} (Hu_v - g(\nabla h, \nabla u) + \Delta_\delta u) u_v \, dm + \int_{\partial M} (\|\nabla_\delta u, \nabla_\delta u\) - (\nabla_\delta u, \nabla_\delta u_n) \| \, dm
\]
\[= \int_{\partial M} (Hu_v^2 - h_v u_v^2) \, dm + \int_{\partial M} (\nabla_\delta u, \nabla_\delta u) \, dm \geq 0.
\]

In the last inequality we have used our assumption on the geometry of \(\partial M\).

Then by our assumption (4) we have
\[
\int_M \text{Ric}^h(\nabla u, \nabla u) \, dm \geq \int_M \left(\frac{|Dh|^2}{nz} + A\right)|\nabla u|^2 \, dm. \tag{6}
\]

Putting (5) and (6) together we have
\[
\int_M \frac{|Dh|^2}{nz} |\nabla u|^2 \, dm + A \lambda \leq \frac{\lambda^2 (n+1)}{n(n+1)} + \int_M \frac{|Dh|^2 |\nabla u|^2}{nz} \, dm,
\]
and noting \(\lambda \neq 0\), we have \(\lambda \geq \frac{n+1}{n(n+1)} A\). The result is proved. \(\Box\)
4. Proof of Theorem 3

Suppose $\Delta^h f + \lambda f = 0$. Substituting possibly $-\nu$ to $\nu$, there exists a choice of the orientation of the unit normal vector $\nu$ such that $\int_{\partial M_1} \langle \nabla \nu, \nabla u \rangle \, d\nu \geq 0$. Fixing this choice of the orientation of $\nu$ between the two open submanifolds $M_1$ and $M_2$, we decide to call $M_1$ the one which admits $\nu$ as the unit outer normal vector.

Define $f$ on $M_1$ such that $L_h f = 0$, on $M$ with the boundary condition $f = u$ on $\partial M_1$. By Theorem 1 we have $0 \geq \int_{\partial M_1} (-|D^2 f|^2) \, dM = \int_{M_1} \text{Ric}^h(\nabla f, \nabla f) \, dM + \int_{\partial M_1} (H_f - \nu \nabla f + \Delta_p u) f_n \, dM + \int_{\partial M_1} (-\langle \nabla_p f, \nabla f \rangle) \, dM$. Note that

$$
\int_{\partial M_1} (H_f - \nabla_h \nabla f + \Delta_p u) f_n \, dM = \int_{\partial M_1} ((H - h_n) f_n - \nabla_p h \nabla f + \Delta_p u) f_n \, dM
$$

$$
= -\int_{\partial M_1} ((\nabla \nu h - H) \nabla f + \lambda u) f_n \, dM = -\lambda \int_{\partial M_1} u f_n \, dM,
$$

and

$$
\int_{\partial M_1} (-\langle \nabla_p f, \nabla f \rangle) \, dM = \int_{\partial M_1} (\Delta^h f) f_n \, dM = -\lambda \int_{\partial M_1} u f_n \, dM.
$$

Compute

$$
2 \int_{\partial M_1} u f_n \, dM = \int_{\partial M_1} (f^2) \, dM = \int_{M_1} L_h(f^2) \, dM = 2 \int_{M_1} |\nabla f|^2 \, dM.
$$

Using our assumption we have $0 \geq ((n-1)K - 2\lambda) \int_{M_1} |\nabla f|^2 \, dM$. Since $\int_{M_1} |\nabla f|^2 \, dM > 0$, we get $\lambda \geq \frac{(n-1)K}{2}$.

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