ON CONVERGENCE IN SMOOTH GRADIENT SYSTEMS WITH BRANCHING OF EQUILIBRIA

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Abstract. Our basic model is a semilinear elliptic equation with a coercive $C^1$ nonlinearity, $\Delta \psi + f(\psi) = 0$ in $\Omega$, $\psi = 0$ on $\partial \Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. Our main hypothesis ($\text{H}_\text{R}$) on the resonance branching is as follows: if a branching of equilibria occurs at any point $\psi$ with a $k$-dimensional kernel of the linearized operator $\Delta + f'(\psi)I$, the branching subset $S_k$ at $\psi$ is a locally smooth $k$-dimensional manifold.

Using the corresponding parabolic flow $u_t = \Delta u + f(u)$, with bounded initial data $u_0$, we then prove that, under the ($\text{H}_\text{R}$), the subset of equilibria $\Phi = \{\psi\}$ is evolutionary complete, i.e., any evolution orbit converges as $t \to \infty$ to a single element of $\Phi$. We also treat the non-coercive case $f(u) = |u|^{p-1}u$ for $p \in (1, \frac{N+2}{N-2})$, where the elliptic problem is known to admit at least a countable subset of solutions $\{\psi_k\}$ due to the classical Lusternik-Schnirel’man category theory. The results are extended to higher-order elliptic operators $-(-\Delta)^m u + |u|^{p-1}u$ with Dirichlet conditions on $\partial \Omega$, $1 < p < \frac{N+2m}{N-2m}$, $m \geq 2$, with the corresponding adaptation of the parabolic flows.

For $N = 1$, the first result on stabilization to a single equilibrium is due to T.I. Zelenyak (1968). We show that Zelenyak’s approach based on Lyapunov’s function analysis can be extended to general gradient systems in Hilbert spaces with a similar smooth resonance branching. We also cover the case of their asymptotically small non-autonomous perturbations. In general, the developed approach represents an alternative (and improved) method on stabilization in Hale–Raugel (1992) and other later similar techniques in gradient system theory.

Dedicated to the memory of Professor T.I. Zelenyak

This is an extended version of the paper [12], containing extra comments and updated references.

1. Introduction: the convergence problem and evolution completeness

1.1. Semilinear elliptic equations and the convergence problem. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the smooth boundary $\partial \Omega$. Our basic model is the classical semilinear coercive elliptic equation from combustion theory

$$V'(\psi) \equiv \Delta \psi + f(\psi) = 0 \text{ in } \Omega, \ \psi = 0 \text{ on } \partial \Omega,$$

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where \( f \) is a given function satisfying
\[(1.2) \quad f \in C^1(\mathbb{R}), \quad f'(u) < 0 \text{ for } |u| \gg 1.\]

This non-optimal coercivity condition is sufficient for demonstrating the approach. For instance, \( f(u) = u - u^3 \) (an analytic nonlinearity) or \( f(u) = |u|^{p-1}u - |u|^{q-1}u \) with any \( q > p > 1 \). The operator in (1.1) is potential and coercive, so the classical variational theory establishes existence of solutions as critical points of the corresponding functional in \( H^1_0(\Omega) \),
\[(1.3) \quad V(\psi) = -\frac{1}{2} \int |D\psi|^2 \, dx + \int F(\psi) \, dx,
\]
where \( F' = f \); see e.g., [20]. If \( f \) does not satisfy the coercivity condition (1.2), then, in the most well-known power case
\[(1.4) \quad f(u) = |u|^{p-1}u \text{ with } 1 < p < p_S = \frac{N+2}{N-2},\]
versions of the classical Lusternik–Schnirel’man (L-S) theory [22–24] (see e.g., [6] and [28, 29]) establish existence of at least a countable subset of critical points of the corresponding functional (1.3).

By \( \Phi = \{ \psi \} \) we denote the subset of all bounded solutions of the elliptic problem (1.1) in both coercive and non-coercive cases. The evolution completeness of the subset \( \Phi \) of solutions of (1.1) is defined by introducing the corresponding parabolic equation
\[(1.5) \quad u_t = V'(u) \equiv \Delta u + f(u) \text{ in } \Omega \times \mathbb{R}_+, \quad u = 0 \text{ on } \partial\Omega \times \mathbb{R}_+,\]
with arbitrary bounded initial data \( u_0 \). It follows from the classical parabolic theory [10], that under hypotheses (1.2) the orbit \( \{ u(\cdot, t), t > 0 \} \) is uniformly bounded. For such smooth gradient systems, one can define its \( \omega \)-limit set \( \omega(u_0) \), which is non-empty, compact, invariant and connected in the topology of \( L^2(\Omega) \) (or \( C(\Omega) \)), and
\[(1.6) \quad \omega(u_0) \subseteq \Phi;\]
see [14] and [31 pp. 483-487]. The notion of the evolution completeness of the subset \( \Phi \) of nonlinear eigenfunctions was introduced in [11] for the porous medium equation
\[(1.6) \quad u_{\tau} = \Delta(|u|^{m-1}u) + \frac{1}{m-1}u \quad (m > 1)\]
in a bounded domain. In this case, 0 is not included into \( \Phi \) (as usual, 0 is not an eigenfunction) and this determines some specific difficulties of the asymptotic analysis.

In the present problem, 0 \( \in \Phi \), so that the evolution completeness of \( \Phi \) just means that, for the parabolic flow (1.5),
\[(1.7) \quad \text{for any data } u_0, \exists \ a \ \psi \in \Phi \text{ such that } \omega(u_0) = \{ \psi \}.\]
Indeed, this is the classical convergence problem in the theory of dynamical systems. Obviously, (1.7) is true for any smooth gradient system, if \( \Phi \) consists of isolated points (a straightforward consequence of the connectedness of \( \omega(u_0) \)), i.e.,
\[(1.8) \quad \Phi \text{ is discrete } \implies \text{evolution completeness (1.7) holds.}\]

**On branching situations: two approaches.** The case, where \( \Phi \) is not discrete and consists of continuous families due to branching of equilibrium points was for a long time
a problem of concern. Existence of various branches of stationary solutions is established by classical variational techniques; see [20] and [19] for applications to semilinear elliptic problems. In some cases, the fibering method [28, 29] shows a clear picture of the connection of the number of branches with the algebraic non-monotonicity of \( f(u) \), so, in general, \( \Phi \) may contain continuous sub-families; see examples in Section 3.

The first approach to convergence in parabolic problems in the presence of branching was formulated in 1968 by T.I. Zelenyak [34], who proved (1.7) for arbitrary one-dimensional uniformly parabolic equations with smooth coefficients

\[
(1.9) \quad u_t = a(x, u, u_x)u_{xx} + b(x, u, u_x)
\]

with arbitrary nonlinear boundary conditions. Ten years later, in 1978, a similar result was proved by H. Matano [25] by Sturm’s zero set argument, which turned out to be applied to arbitrary smooth fully nonlinear uniformly parabolic equations for \( N = 1 \)

\[
(1.10) \quad u_t = F(x, u, u_x, u_{xx}).
\]

Matano’s approach is principally one-dimensional, since Sturm’s Theorem on zero sets is not available in higher dimensions, cannot be extended to parabolic PDEs in \( \mathbb{R}^N \) for \( N \geq 2 \).

Another direction of the development of the stability theory of gradient systems is associated with the Lojasiewicz–Simon inequality and technique. It is well-known that, for gradient dynamical systems in a real Banach space \( E \)

\[
(1.11) \quad u_t = V'(u) \quad \text{for} \quad t > 0, \quad u(0) = u_0 \in E,
\]

where \( V \) is a \( C^2 \) functional, with the \( C^1 \) Frechet derivative \( V' : E \to E' \), \( E' \) being the dual space, the Lojasiewicz–Simon inequality is an effective tool of studying of the stabilization phenomena. In particular, it completely settles the case of analytic nonlinearities; see references in [5]. Given an equilibrium point \( \psi \), with \( V'(\psi) = 0 \), this classical inequality has the form

\[
(1.12) \quad |V(v) - V(\psi)|^{1-\theta} \leq c\|V'(v)\|_{E'}
\]

in a neighbourhood of \( \psi \), where \( c > 0 \) is a constant and \( \theta \in (0, \frac{1}{2}] \) is called the Lojasiewicz exponent; see basic references and historical comments in [3]. If (1.12) with a fixed exponent \( \theta \) holds for the critical manifold [5, Thm. 3.10], then, under natural assumptions on the spectral properties of linearized operators (all those hold for our elliptic case), it is possible to establish:

(i) that the \( \omega \)-limit set \( \omega(u_0) \) of any bounded orbit consists of a unique equilibrium, i.e., (1.7) holds, and

(ii) the rate of convergence to \( \psi \) as \( t \to \infty \).

For instance, in the case of finite regularity, the best possible constant \( \theta = \frac{1}{2} \) guarantees the exponential convergence, [16, Thm. 1.1],

\[
(1.13) \quad \|u(t) - \psi\|_E \leq c e^{-\delta t} \quad \text{as} \quad t \to \infty \quad (\delta > 0).
\]
In general, roughly speaking (see examples in [5])

\[(1.14) \quad \theta \in (0, \frac{1}{2}) \implies \|u(t) - \psi\|_E \leq c t^{-\frac{\theta}{1-2\theta}} \text{ as } t \to \infty.\]

Concerning applications of the Lojasiewicz–Simon approach to parabolic equations like (1.5), it does not assume any restrictions of the dimension \(N\), though, in presence of branching, checking the validity of the corresponding inequality on the singular subset becomes an extremely difficult problem; see examples in [5]. On the other hand, in Section 5 we show that the Lojasiewicz–Simon inequality (1.12) does not apply to general \(C^\infty\) dynamical systems, i.e., the exponent \(\theta \in (0, \frac{1}{2}]\) does not always exist. Actually, this means that this kind of analysis needs other non-rational moduli of continuity in the inequality (1.12).

**Counterexamples and the resonance branching hypothesis.** It is well-known for a long time (see [27]) that, for general non-analytic gradient systems on the plane \(\mathbb{R}^2\), the convergence result (1.7) is not true. The idea of such a construction has been extended to prove existence of non-convergent orbits for special parabolic equations like (1.5) with \(C^\infty\) function \(f = f(u, x)\); see [30] and earlier references therein. The principal part of the construction in [27, 30] uses the fact that the dimension of manifold of equilibrium points is less than the dimension of the corresponding kernel of the linearized operator meaning a certain “defect” of dimensions.

It is well-known that there are special (often similar to ours) cases of branching of equilibria, where Hale–Raugel’s approach applies to guarantee convergence of the orbits of gradient systems; see [15], where a survey and further references are given. This approach essentially relies on spectral properties of the linearized operator (main Hale–Raugel’s hypothesis is that 0 has multiplicity at most one, or \(k > 1\) under special hypothesis) and uses properties of stable, unstable and center manifolds, that are difficult to justify for some less smooth equations. We also refer to more recent paper [4], where a similar approach to stabilization is used and other references can be found.

Our method is different and uses Zelenyak’s ideas (1968) from one dimension that are mainly connected with Lyapunov functions only. It is important that Zelenyak’s approach yields the exponential rate of convergence to a single equilibrium and can be extended to non-autonomous perturbations of gradient systems, for which the classic stable-centre manifold theory fails.

Let us present the main “resonance” hypothesis, under which non-convergent orbits do not exist. We formulate it in a general form, where, for the PDEs (1.5), by the Frechet derivative we mean \(V''(\psi) = \Delta + f'(\psi)I\) assumed to be a Fredholm operator of index zero at any equilibrium point from \(\Phi\).

**Hypothesis \((H_R)\).** If branching occurs at any equilibrium \(\psi\) and \(\dim \ker V''(\psi) = k \geq 1\), then the corresponding branching subset \(S_k\) of equilibrium points is a locally smooth \(k\)-dimensional manifold (a relatively open surface).

As a first comment, we note that, obviously, \((H_R)\) always holds in the linear case with \(f(\psi) = \lambda \psi\), where branching of equilibria occurs for \(\lambda \in \sigma(\Delta)\). Then the subsets coincide,

\[(1.15) \quad S_k = \ker V''(\psi) \quad (V''(\psi) = \Delta + \lambda I).\]
Indeed in the linear case the convergence of all the orbits follows from the completeness and closure of the eigenfunctions of $\Delta$ as a straightforward consequence of general theory of self-adjoint operators, [3], and we do not treat this case here.

The condition in $(H_R)$ cannot be improved in the sense that, in a non-resonance branching, where the smooth stationary manifold $S_k$ has dimension $k < \dim \ker V''(\psi)$, this makes it possible to apply the idea of construction of non-converging orbits from [27, 30]. According to our proof, any defect of dimensions destroys the approach. Concerning the resonance branching condition, we refer to the classical branching theory for nonlinear equations in Banach spaces [33, Ch. 7,8], [2, Ch. 6], [7, § 30.1] for the conditions, which guarantee necessary branching in variational setting, and also to [19], where applications to elliptic problems of interest are discussed in detail. Notice that, for (1.1), the branching parameter is not available explicitly in the equation and such branching cases are less studied in the literature. We will present simple examples showing that in standard nonlinear elliptic problems, the resonance branching with $k = N - 1$ actually exists in dimensions $N \geq 2$.

1.2. Main result, plan and extensions. We now state the main result of the paper, which, as we have mentioned, are established by using key Zelenyak’s ideas from [34].

**Theorem 1.1.** Let $f$ be a $C^1$ function. Let $\Phi = \{\psi\}$ be the subset of all bounded solutions of the elliptic problem (1.1) and let $(H_R)$ hold. Let, given bounded initial data $u_0$, $u(x,t)$ be the unique classical solution of (1.5). Then:

(i) in the coercive case (1.2), there exists a unique $\psi \in \Phi$ such that $\omega(u_0) = \{\psi\}$;

(ii) in the non-coercive case (1.4), the same is true for any uniformly bounded orbit;

(iii) in both cases, the convergence to $\psi$ is exponential.

The hypothesis of boundedness of the orbit in (ii) is essential since (1.5) admits solutions, which blow up in finite time and hence exhibit entirely different asymptotic patterns (specific asymptotic techniques are explained in [13]). As we have mentioned, for analytic nonlinearities $f(u)$ (for (1.1) this means $p = 3, 5, ...$), the evolution completeness conclusion follows from the classical Lojasiewicz–Simon inequality; see references in [3, 9], where versions of such inequalities were proposed for some non-analytic settings. The result (iii) suggests that in the resonance branching, the Lojasiewicz–Simon inequality with $\theta = \frac{1}{2}$ holds in a neighbourhood of $S_k$, though we do not study this.

The layout of the paper is as follows. Theorem 1.1 is proved in Section 2. In Section 3 we present some examples showing the possibility of resonance branching. We show in Section 5 that, for some $C^\infty$ nonlinearities, the Lojasiewicz–Simon inequality fails for any arbitrarily small $\theta > 0$, so the stabilization technique needs other non-rational moduli of continuity in (1.12), which are proposed. It follows from (iii) in Theorem 1.1 that such essentially non-analytic counterexamples correspond to isolated equilibria.

The results are true for smooth higher-order equations. E.g., Theorem 1.1 is valid for a general $2m$th-order parabolic flow for any $m \geq 1$

\begin{equation}
(1.16) \quad u_t = (-1)^{m+1} \Delta^m u + f(u) \quad \text{in } \Omega \times \mathbb{R}_+,
\end{equation}
with \( m \) zero Dirichlet boundary conditions. The critical Sobolev exponent for (1.4) is
\[
\frac{N + 2m}{N - 2m} \quad (N > 2m).
\]
We can also consider quasilinear fourth-order uniformly parabolic equations like
\[
(1.18) \quad u_t = -\Delta g(\Delta u) + f(u), \quad g'(s) \geq c_0 > 0,
\]
or with others potential operators in various metrics. In Section 6 we extend the approach to smooth gradient systems in Hilbert spaces. In Section 7 we treat the case of asymptotically small non-autonomous perturbations of gradient systems. We show that the main result of convergence (1.7) of bounded orbits remains true for perturbed parabolic equations (1.1) such as
\[
(1.19) \quad u_t = \Delta u + f(u) + h(t)g(u), \quad \text{where } h(t) \to 0 \text{ as } t \to \infty,
\]
and \( g(u) \) is a given smooth function, and also applies for similar perturbations of PDEs (1.16) and (1.18). We then impose a certain restriction on the rate of decay of \( h(t) \) as \( t \to \infty \), but, anyway, we can treat non-exponentially small perturbations, e.g., \( h(t) = O(\frac{1}{t^3}) \).

2. Proof of Theorem 1.1

Thus we consider the uniformly elliptic equation (1.1) and the corresponding uniformly parabolic flow (1.5). Notice that the subset of all stationary solutions \( \Phi = \{ \psi \} \) includes the trivial one \( \psi = 0 \). Obviously, 0 is exponentially stable for (1.5) with nonlinearity (1.4) and hence must be taken into account in the evolution completeness analysis.

Let us mention again that according to the potential structure of the equation (1.1) and to (H_r), we are going to use the following properties:
(i) \( S_k \) is open, so any \( \psi \in S_k \) has a neighbourhood relatively open in \( S_k \), and
(ii) \( V(\psi) = B = \text{const.} \), for all \( \psi \in S_k \).

2.1. Proof Theorem 1.1. Thus we consider a smooth flow (1.5) in, say, \( H^2_0(\Omega) \) and study the \( \omega \)-limit set \( \omega(u_0) \) of a uniformly bounded orbit \( \{ u(t), t > 0 \} \). Take an arbitrary equilibrium \( \psi \) from the \( \omega \)-limit set \( \omega(u_0) \), which is non-empty, compact and connected for such smooth gradient systems, [14]. So that there exists a monotone sequence \( \{ t_k \} \to \infty \) such that
\[
(2.1) \quad u(t_k) \to \psi \quad \text{as } k \to \infty.
\]
First of all, if \( 0 \notin \sigma(V''(\psi)) \), where \( V''(\psi) = \Delta + f'(\psi)I \), then \( \psi \) is isolated [33] [20] and the convergence result (1.7) follows for given \( u_0 \). Second, assume that, under hypothesis (H_r), \( 0 \in \sigma(V''(\psi)) \) and \( \psi \) is not isolated and belongs to a smooth manifold \( S_k \) branching at \( \psi \), where \( k \) coincides with the dimension of \( \ker V''(\psi) \).

Our further analysis uses the structure of Zelenyak’s proof in the case \( N = 1 \), [34] and consists of seven steps.

Step 1: optimal approximation of \( u(t) \) on \( S_k \). Let \( S_k = \{ \psi_\mu \} \), where \( \mu \in \mathbb{R}^k \) denotes local coordinates on \( S_k \) such that \( \psi = \psi_0 \). Given a point \( u(t) \) for some \( t \gg 1 \), when
\( u(t) \approx \psi \), we define the optimal approximation \( \psi_{\mu(t)} \) of \( u(t) \) on \( S_k \) by minimizing the distance

\[
(2.2) \quad h(\mu) = \|u(t) - \psi_{\mu}\|
\]

where \( \| \cdot \| \) denotes the \( L^2(\Omega) \)-norm induced by the scalar product \( \langle \cdot, \cdot \rangle \). Then \( \inf_{\mu} h(\mu) \) is attained at some \( \mu = \mu(t) \approx 0 \), and, by construction, the orthogonality condition holds,

\[
(2.3) \quad u(t) - \psi_{\mu(t)} \perp \ker V''(\psi_{\mu(t)}).
\]

In the further evolution analysis we deal with equilibria \( \psi_\mu \) in \( S_k \) for \( \mu \approx 0 \). It suffices to prove that there exists the limit

\[
\mu(t) \to 0 \quad \text{as} \quad t \to \infty.
\]

**Step 2: a priori bound for the linearized stationary equation.** We linearize the stationary equation (1.1) at \( \psi = \psi_{\mu(t)} \) and consider the corresponding inhomogeneous problem

\[
(2.4) \quad V''(\psi_{\mu(t)}) w = g,
\]

where \( g \in L^2 \) is a given function. Then, in view of the orthogonality to the kernel, (2.3), by the standard theory of elliptic self-adjoint operators (see e.g. [2, 3]), there exists a constant \( C_1 > 0 \) such that

\[
(2.5) \quad \|w\|_{H^2} \equiv \|u(t) - \psi_{\mu(t)}\|_{H^2} \leq C_1 \|g\|.
\]

**Step 3: spectral gap.** By hypothesis (HR), \( 0 \in \sigma(V''(\psi_\mu)) \) for all \( \mu \approx 0 \). Moreover, the resonance condition also assumes that the spectral gap

\[
(2.6) \quad \Lambda(\mu) = \inf\{|\lambda| : \lambda \in \sigma(A'(\psi_\mu)), \lambda \neq 0\}
\]

is uniformly bounded away from zero, i.e., there exists a constant \( c_2 > 0 \) such that

\[
(2.7) \quad \Lambda(\mu) \geq c_2 > 0 \quad \text{for all} \quad \mu \approx 0.
\]

This follows from the fact that, by (HR), the kernel \( \ker A'(\psi_\mu) \) changes continuously with \( \mu \approx 0 \) as the tangent space of the changing continuously points \( \psi_\mu \) on the given smooth manifold \( S_k \).

**Step 4: estimate for the linearized parabolic equation.** We now set \( w(t) = u(t) - \psi_{\mu(t)} \). Consider next the parabolic equation (1.5), linearize the right-hand side at \( u = w_{\mu(t)} \) and write down it as the inhomogeneous elliptic equation (2.4). Then, by linearization, we obtain an extra quadratic term \( g_1(u) = \frac{1}{2} f''(\xi_\mu)(u - \psi_{\mu(t)})^2 \), so that, in (2.4),

\[
(2.8) \quad g = u_t + g_1(u), \quad \text{with} \quad |g_1(u)| \leq C_3 \|u(t) - \psi_{\mu(t)}\|_{H^2}^2.
\]

By convergence in such a smooth parabolic equation, we have that the smallness of \( w(x,t) = u(x,t) - w_{\mu(t)}(x) \) implies the smallness of the derivatives, so that we may assume that \( C_3 \) is uniformly bounded on any intervals \( [t_k, t_k + T] \) as \( k \to \infty \).
Combining the elliptic estimate (2.5) and the parabolic one (2.8) yields the following bound:

\[(2.9) \quad \|u(t) - \psi_{\mu(t)}\|_{H^2}^2 \leq C_4\|u_t\|^2,\]

for all \(t \gg 1\) such that the orthogonality (2.3) holds.

**Step 5: local exponential convergence of the Lyapunov function.** The Lyapunov function \(V(u(t))\) is strictly monotone increasing on evolution orbits,

\[(2.10) \quad \frac{d}{dt} V(u(t)) = \int_{\Omega} (u_t)^2 > 0 \quad (u_t \neq 0),\]

so that there exists the finite limit

\[(2.11) \quad V(u(t)) \to B^+ \quad \text{as} \quad t \to \infty,\]

and then \(V \equiv B\) on \(S_k\). Fixing the parameter of the optimal approximation \(\mu = \mu(t)\), by using the standard expansion at \(\psi_{\mu}\), one can see that

\[(2.12) \quad |V(u) - V(\psi_{\mu})| \leq C_5\|u - \psi_{\mu}\|_{H^1} \leq C_5\|u - \psi_{\mu}\|_{H^2}.\]

Therefore from (2.10), by (2.9) and (2.12), we have that

\[(2.13) \quad \frac{d}{dt} [B - V(u(t))] = - \int_{\Omega} (u_t)^2 \leq - C_6\|u - \psi_{\mu(t)}\|_{H^2}^2 \leq - C_7[B - V(u(t))].\]

Integrating this inequality yields the local exponential convergence of \(V(u(t))\) to \(B\) on any arbitrarily large bounded intervals \(t \in [t_k, t_k + T]\) as \(k \to \infty\),

\[(2.14) \quad B - V(u(t)) \leq [B - V(u(t_k))] e^{-C_7(t - t_k)}.\]

**Step 6: local exponential convergence of \(V(u(t))\) implies exponential convergence of \(u(t)\) to equilibrium.** Here we use Lemma 4 in [33], actually establishing a weighted Gronwall’s-type inequality by using a discrete partition technique. Later on, finite partitions have been widely used for more general Gronwall’s weighted inequalities; see Henry’s famous book [18, Ch. 7].

**Lemma 2.1.** Let, for all \(t > 0\),

\[(2.15) \quad \int_t^\infty \int_{\Omega} (u_t)^2 \leq C_8 e^{-t}.\]

Then there exists a constant \(C_0 > 0\) such that

\[(2.16) \quad \|u(t) - u(\tau)\| \leq C_0 e^{-\frac{t}{2}} \quad \text{for all} \quad 0 < t \leq \tau < \infty.\]

**Proof.** Firstly, if \(|t - \tau| \leq 1\) we apply the Hölder inequality to get that

\[\|u(t) - u(\tau)\|^2 = \int_{\Omega} (\int_t^\tau u_t)^2 \leq (\tau - t) \int_{\Omega} (\int_t^\tau (u_t)^2) \leq C_8 e^{-t}.\]
If $|t - \tau| > 1$, then we perform a uniform partition of the interval $[t, \tau]$ into $K$ parts of length 1 with a reminder, and apply the Hölder inequality in each subinterval to get

$$
\|u(t) - u(\tau)\|_2^2 \leq \sum_{i=0}^{K-1} \int_{t+i}^{t+i+1} \int_\Omega (u_t)^2 + \int_\Omega (u_t)^2 
\leq C_8 \sum_{i=0}^{K-1} e^{-(t+i)} + C_8 e^{-(t+i+K)} \leq C_9 e^{-t}.
$$

\[\square\]

**Step 7: exponential estimate implies convergence to $\psi$.** We continue to describe the evolution of $u(t)$ on a large finite interval $[t_k, t_k + T]$ with $k \gg 1$, on which, by assumption, the orbit enjoys all the estimates following from the optimal approximation on the stationary subset $S_k$. So for any $t \in [t_k, t_k + T]$, there holds:

(a) By (2.14) and (2.10),

$$
\int_t^\infty \int_\Omega \int_\Omega (u_t)^2 = B - V(u(t)) \text{ is exponentially small;}
$$

(b) By Lemma 2.1 $\|u(t_k) - u(t)\|$ is exponentially small;

(c) Therefore, $u(t) \approx u(t_k) \approx \psi$ for all $t \geq t_k$, completing the proof in the case (i).

The case (ii) is the same once we have fixed a uniformly bounded orbit.

(iii) See (b) above. \[\square\]

### 3. Discussion: resonance branching and exponential convergence

**3.1. Example: non-isolated equilibria and resonance branching.** We take the equation with the cubic analytic nonlinearity in the unit ball in $\mathbb{R}^N$

$$
\Delta \psi + \psi^3 = 0 \quad \text{in} \quad B_1 = \{ |x| < 1 \} \subset \mathbb{R}^N, \quad \psi|_{\partial B_1} = 0.
$$

Then $p = 3$ is in the subcritical Sobolev range if $3 < \frac{N+2}{N-2}$, i.e., we need $N < 4$. Taking other $p < p_S$ provides us with similar examples for any $N \geq 2$.

Firstly, we consider (3.1) in the half of the ball, $\Omega_+ = B_1 \cap \{ x_1 > 0 \}$. There exists a unique strictly positive classical solution $\psi_+ \in H_0^1(\Omega_+)$ of (3.1) in $\Omega_+$ constructed by the standard variational technique, \[2, 29\].

Next, since equation (3.1) is invariant under the reflection $x_1 \mapsto -x_1$, setting $\psi = \psi_+$ in $\Omega_+$ and $\psi = -\psi_+$ in $\Omega_-$ yields a non-radial $H_0^1$-solution of (3.1) in $B_1$. This extended function is a weak solution of (3.1) in the sense that it satisfies the corresponding integral identity and hence is a classical solutions by the theory of uniformly elliptic equations. Any smooth invariant orthogonal transformation in $\mathbb{R}^N$ leaving the Laplacian and the Dirichlet boundary condition invariant produces continuous families of solutions consisting of non-isolated points. By the classical branching theory \[20\], in this case, 0 belongs the spectrum of the corresponding linearized operator $V''(\psi) = \Delta + 3\psi^2 I$.
In particular, we fix \( N - 1 \) angels of rotations \( \{ \theta_1, \ldots, \theta_{N-1} \} \) of the polar coordinate system in \( \mathbb{R}^N \) to get at least \( N - 1 \) linearly independent elements of the kernel of \( \Delta + 3\psi^2 I \) given by

\[
\varphi_k = \frac{d}{d\theta_k} \psi, \quad k = 1, \ldots, N - 1.
\]

Bearing in mind these rotations, the stationary subset \( S_{N-1} \) is expected to be precisely \( (N-1) \)-dimensional, and this \( k = N - 1 \) coincides with the kernel dimension. Unfortunately, we cannot prove the exact equality, and present later on another non-local model, for which Hypothesis \((H_R)\) is guaranteed. For the problem on the plane \((N = 2)\), the corresponding eigenfunction of \( V''(\psi) \) with \( \lambda = 0 \) is given by rotation by the single angle \( \theta = \theta_1 \), so

\[
(3.2) \quad \varphi_1 = \frac{d}{d\theta} \psi,
\]

and the kernel seems to be one-dimensional (we do not prove this either).

Using similar reflections, one can construct in the unit ball \( B_1 \subset \mathbb{R}^N \) the nonlinear eigenfunctions \( \psi_l(x) \) satisfying \((3.1)\) with arbitrarily number \( l \geq 1 \) of connected positivity and negativity components. By orthogonal invariant transformation, each one generates continuous families of other stationary solutions.

3.2. On exponential convergence. We first illustrate the reason for the exponential convergence by using the above example in \( \mathbb{R}^2 \), where the kernel is one-dimensional with the basis function \((3.2)\).

Orthogonality conditions for branching. The branching occurs in \((3.1)\) from solution \( \psi_0 = \psi \), so by the classical branching theory \([33]\), we look for a smooth curve of solutions in the form

\[
(3.3) \quad \psi_\mu = \psi_0 + \mu \psi_1 + \mu^2 \psi_2 + \mu^3 \psi_3 + \mu^4 \psi_4 + \ldots,
\]

where, up to scaling, \( \mu \) can be attributed to the angle of rotation. Substituting \((3.3)\) into \((3.1)\) and equating similar terms yields the system for expansion coefficients

\[
(3.4) \quad \begin{cases}
\Delta \psi_0 + \psi_0^3 = 0, \\
\Delta \psi_1 + 3\psi_0^2 \psi_1 = 0, \\
\Delta \psi_2 + 3\psi_0^2 \psi_2 = -3\psi_0 \psi_1^2, \\
\Delta \psi_3 + 3\psi_0^2 \psi_3 = -\psi_1^3 - 6\psi_0 \psi_1 \psi_2, \\
\Delta \psi_4 + 3\psi_0^2 \psi_4 = -3\psi_0 \psi_2^2 - 3\psi_1^2 \psi_2 - 6\psi_0 \psi_1 \psi_3, \\
\ldots \quad \ldots \quad \ldots
\end{cases}
\]

The second equations says that \( \psi_1 \) is from the non-empty kernel and hence is given by \((3.2)\). The third equation then yields the first orthogonality condition

\[
(3.5) \quad \int_\Omega \psi_0 \psi_1^3 = 0,
\]

which is necessary for branching to occur. Determining \( \psi_2 \) up to a constant \( \beta_2 \),

\[
\psi_2 = \bar{\psi}_2 + \beta_2 \psi_1,
\]

...
where $\bar{\psi}_2$ is a solution of the inhomogeneous equation, and substituting into the fourth one, for its solvability we have to have that

$$\int_{\Omega} \psi_1^4 + 6 \int_{\Omega} \psi_0 \psi_1^2 \bar{\psi}_2 + 6 \beta_2 \int_{\Omega} \psi_0 \psi_1^3 = 0.$$ 

Since the last term containing $\beta_2$ vanishes by (3.5), we arrive at the second orthogonality condition

(3.6) $$\int_{\Omega} \psi_1^4 + 6 \int_{\Omega} \psi_0 \psi_1^2 \bar{\psi}_2 = 0.$$ 

Next choosing similarly

$$\psi_3 = \bar{\psi}_3 + \beta_3 \psi_1,$$

we obtain from the fifth equation that

$$2 \int_{\Omega} \psi_0 \psi_1^2 \bar{\psi}_3 + \int_{\Omega} \psi_0 \psi_1 \bar{\psi}_2^2 + \int_{\Omega} \psi_1^3 \bar{\psi}_2 + \frac{2}{3} \beta_2 \int_{\Omega} \psi_1^4 = 0.$$ 

This determines $\beta_2$ and so on. Thus, in general, this kind of branching demands more than one orthogonality condition (and, of course, existence of a non-trivial kernel).

**On a centre manifold link.** Let us now discuss how these branching conditions affect the evolution properties of the corresponding parabolic equation

(3.7) $$u_t = \Delta u + u^3 \text{ in } B_1 \times \mathbb{R}_+.$$ 

In this analytic case, there exists a local centre $C^\infty$ manifold $W_{loc}^c(\psi_0)$; see general theory in [20], [21], § 9], [32] for applications to reaction-diffusion equations and [15], where the convergence problem is studied.

Since $W_{loc}(\psi_0)$ is tangent to the corresponding centre subspace $E^c(0) = \text{Span}\{\psi_1\}$, looking for the centre manifold behaviour with solutions of the form (see references and accurate estimates for the non-analytic flow in the next section)

$$u(x,t) = \psi_0(x) + a_1(t)\phi_1(x) + ... \text{ for } t \gg 1,$$

and projecting the PDE (3.7) on $E^c(0)$, we obtain a perturbed ODE

(3.8) $$a_1' = \gamma a_1^2 + ...,$$

where, as one can see, the constant $\gamma$ vanishes,

(3.9) $$\gamma = \int_{\Omega} \psi_0 \psi_1^3 = 0,$$

by the first orthogonality branching condition (3.5). Therefore the higher-order terms should be taken into account, but these seem also do not help to detect a suitable non-trivial evolution on the centre manifold.

Indeed one can see that this one-dimensional branching manifold $S_1$ is a centre manifold for the parabolic problem. Under the above assumptions, denoting by $\{\phi_k\}$ the complete subset of eigenfunctions of $V''(\psi_0) = \Delta + 3\psi_0^2 I$, we may assume the following expansion (cf. (3.3))

$$S_1 = \{\psi_\mu = \psi_0 + a_1(\mu)\phi_1 + a_2(\mu)\phi_2 + ...\}.$$
where \(a_k(0) = 0\) for any \(k\). Therefore, the evolution on this centre manifold with

\[ u(x, t) = \psi_0(x) + \sum_{(k)} a_k(t)\phi_k(x) \]

is governed by the trivial dynamical system

\[ a'_k = 0, \quad k = 1, 2, \ldots, \]

which we have observed in (3.8), (3.9). Notice that a centre manifold consisting of pure equilibria is not an exceptional situation in parabolic problems; cf. [8], where such an invariant exponentially stable centre manifold has been detected in the free-boundary Mullins–Sekerka model (a kind of Hele–Shaw flow).

Thus in this case, as well as and in other resonance cases, there exists a local centre manifold for the parabolic problem precisely coinciding with the stationary subset \(S_k\), so no evolution of orbits on \(W_{loc}^c(\psi_0)\) can be observed. Then the exponential convergence in the Theorem 1.1 (iii) can be associated with the fact that, in the orthogonal complement of the kernel, the behaviour is purely exponential and corresponds to the evolution on the orthogonal stable manifold constructed at a different equilibrium \(\psi = \psi_{\mu(t)} \approx \psi_0\); cf. Step 1 of the proof in Section 2. In the invariant manifold theory, this is usually expressed by the fact that, for \(C^2\) nonlinearities and good linearized sectorial operators with discrete spectrum and finite-dimensional unstable and centre subspaces, etc., the centre manifold is exponentially stable; see e.g. [21, Prop. 9.2.3] and [32].

4. An explicit example with non-local nonlinearity

Let \(\Omega\) be a bounded smooth domain in \(\mathbb{R}^N\) and let \(\{\lambda_k\} \) and \(\{\psi_k\} \) be the eigenvalues of \(\Delta\), where each one \(\lambda_k\) repeated as many times as its multiplicity \(\kappa_k \geq 1\), and the corresponding complete, closed subset of eigenfunctions that are orthonormal in \(L^2(\Omega)\). The case of the unit ball, \(\Omega = B_1\), is classical. Here, for \(\Delta\), all the eigenvalues, eigenfunctions, multiplicities, etc., are described by the Laplace-Beltrami operator \(\Delta_\sigma\) on the unit sphere \(S^{N-1} = \partial B_1\) in \(\mathbb{R}^N\), which in the polar coordinates \((r, \sigma)\) is given by

\[
\Delta_\sigma = \Delta_r + \frac{1}{r} \Delta_\sigma,
\]

\(\Delta_\sigma\) is a regular operator with the discrete spectrum in \(L^2(S^{N-1})\) (again each one repeated as many times as its multiplicity)

\[
\sigma(-\Delta_\sigma) = \{\nu_k = k(k + N - 2), \ k \geq 0\}
\]

and an orthonormal, complete, closed subset \(\{V_k(\sigma)\}\) of eigenfunctions, which are homogeneous harmonic \(k\)-th order polynomials restricted to \(S^{N-1}\).

Fix an \(l \geq 2\) and consider the PDE with a non-local cubic nonlinear term

\[
u_t = \Delta u + \lambda_t u - \left( \int_{\Omega} u^2 \right) u
\]

with the same Dirichlet boundary conditions and bounded initial data \(u_0\). Such non-local parabolic models can provide us with further examples of delicate asymptotics obtained
via explicit computations; see [29]. Firstly, it is easy to solve the stationary equation

\[ \Delta \psi + \lambda_l \psi - \left( \int_\Omega \psi^2 \right) \psi = 0. \]

Namely, there exist non-trivial equilibria of the form

\[ \psi(x) = \pm \sqrt{\lambda_l - \lambda_j} \psi_j(x) \quad \text{provided that } \lambda_l > \lambda_j, \]

and other obviously constructed linear combinations of such functions corresponding to the same eigenvalues.

4.1. **Result on convergence.** Secondly, studying the parabolic equation (4.3) and looking for solutions in the form of the eigenfunction expansion

\[ u(x, t) = \sum_{(k)} a_k(t) \psi_k(x), \]

we obtain the following dynamical system for the expansion coefficients:

\[ a_k'(t) = (\lambda_l - \lambda_k - |a|^2) a_k, \quad k = 1, 2, \ldots, \quad \text{where } |a|^2 = \sum a_k^2. \]

Integrating yields \( a_k(t) = a_k(0) e^{(\lambda_l - \lambda_k)t} e^{-\int_0^t |a|^2(s) ds}. \) Calculating the sum \( |a|^2 \), we derive

\[ |a|^2 = \Psi(t) e^{-\int_0^t |a|^2(s) ds}, \quad \text{with } \Psi(t) = \sum e^{2(\lambda_l - \lambda_k)t} a_k^2(0). \]

Setting \( Z = e^{-\int_0^t |a|^2(s) ds} \) yields a simple ODE, \( Z' = -2 \Psi(t) Z^2 \), so, on integration,

\[ |a|^2(t) = \frac{a_l^2(0) + \sum_{\lambda_m \neq \lambda_l} e^{(\lambda_l - \lambda_m)t} a_m^2(0)}{1 + 2a_l^2(0) + \sum_{\lambda_m \neq \lambda_l} e^{(\lambda_l - \lambda_m)t} a_m^2(0)}. \]

where for simplicity \( a_l^2(0) \) denotes the sum \( \sum_{\lambda_k=\lambda_l} a_k^2(0) \). Finally we arrive at the following expressions for the expansion coefficients:

\[ a_k(t) = a_k(0) \frac{e^{(\lambda_l - \lambda_k)t}}{\sqrt{1 + 2a_l^2(0) + \sum_{\lambda_m \neq \lambda_l} e^{(\lambda_l - \lambda_m)t} a_m^2(0)}}. \]

This reveals three different cases of solutions with exponential and algebraic rate of convergence to equilibria, depending on the dominant term in the long square root in the denominator. Namely, we take initial function

\[ u_0(x) = \sum a_k(0) \psi_k(x) \]

such that there exists \( j > 1 \), for which

\[ a_j(0) = \ldots = a_{j-1}(0) = 0 \quad \text{and} \quad a_j(0) \neq 0. \]

It is easy to derive from (4.6) the following result.
Proposition 4.1. Let \((4.7)\) hold. Then, as \(t \to \infty\),

\[
(4.8) \quad (i) \quad u(x,t) = O(e^{-(\lambda_j - \lambda_l)t}) \to 0, \quad \text{if} \quad \lambda_j > \lambda_l;
\]

\[
(4.9) \quad (ii) \quad u(x,t) = O\left(\frac{1}{\sqrt{t}}\right) \to 0, \quad \text{if} \quad \lambda_j = \lambda_l; \quad \text{and}
\]

\[
(4.10) \quad (iii) \quad u(x,t) = \psi(x) + O\left(t e^{-2(\lambda_l - \lambda_j)t}\right), \quad \text{if} \quad \lambda_j < \lambda_l,
\]

where \(\psi \neq 0\) is an equilibrium.

It is important that the case (iii), where branching of equilibria is available by \((4.5)\), precisely demonstrates that the convergence to non-trivial stationary solutions is exponential; cf. Theorem 1.1(iii). It is worth mentioning that the rate of convergence in \((4.10)\) is not purely exponential and, in general, contains a lower-order algebraic factor \(t\) that is induced by the \(O(t)\)-term in the square root in \((4.6)\). Different \(ln t\)-perturbations can occur in problems with usual (local) nonlinearities; see the Remark in Section 5.

The only case in Proposition 4.1, where the convergence is not exponentially fast, is (ii), in which, by \((4.5)\), the only equilibrium is trivial, \(\psi = 0\), and is isolated.

4.2. Hypothesis \((H_R)\) is valid. We now present a rigorous evidence that Hypothesis \((H_R)\) is valid for such operators. Consider the elliptic equation \((4.4)\). Fix a \(j > l\), let \(\kappa_j = 1 + k \geq 2\) be the multiplicity of \(\lambda_j\) and let \(\{\psi_{j,1}, ..., \psi_{j,k+1}\}\) be the orthonormal subset of eigenfunctions of \(\Delta\) corresponding to \(\lambda_j\). We fix an equilibrium

\[
(4.11) \quad \hat{\psi}(x) = \sum_{i=1}^{k+1} \hat{c}_i \psi_{j,i}(x),
\]

where, on substitution into \((4.4)\), the coefficients satisfy (cf. \((4.5)\))

\[
(4.12) \quad \sum_{i=1}^{k+1} \hat{c}_i^2 = \lambda_l - \lambda_j.
\]

Stationary subset \(S_k\). Obviously the subset \(S_k\) containing given equilibrium \((4.11)\) is described by

\[
(4.13) \quad \psi = \hat{\psi}(x) = \sum_{i=1}^{k+1} c_i \psi_{j,i}, \quad \text{where} \quad \sum_{i=1}^{k+1} c_i^2 = \lambda_l - \lambda_j.
\]

Therefore

\[
(4.14) \quad \dim S_k = k = \kappa_j - 1 \geq 1.
\]

Kernel of \(V''(\hat{\psi})\). It follows that

\[
(4.15) \quad V''(\hat{\psi})v = \Delta v + \lambda_l v - 2\hat{\psi}\left(\int \hat{\psi} v\right) - v\left(\int \hat{\psi}^2\right).
\]

Substituting \((4.11)\) yields the following equation for the kernel:

\[
(4.16) \quad V''(\hat{\psi})v \equiv \Delta v + \lambda_j v - 2\hat{\psi}(\int \hat{\psi} v) = 0.
\]

Finally taking \(v\) in the form

\[
v = \sum_{i=1}^{k+1} b_i \psi_{j,i}
\]
and substituting into (4.16) gives a single condition on \( k+1 \) expansion coefficients
\[
\sum_{i=1}^{k+1} b_i \hat{c}_i = 0,
\]
so that the kernel is precisely \( k \)-dimensional.

By (4.14) this completes the analysis and proves that \((H_R)\) is always valid for such non-local elliptic operators. Without any changes this example extends to such operators of arbitrary order
\[
V'(\psi) = -(-\Delta)^m \psi + \lambda_l \psi - \left( \int_{\Omega} \psi^2 \right) \psi \quad \text{with any } m \geq 1.
\]

5. **Rate of convergence can be arbitrarily slow**: \( \theta = 0 \)

Here we present some estimates showing that the rate of convergence in parabolic problems with specially designed nonlinearities can be arbitrarily slow.

5.1. **The original semilinear parabolic equation.** Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \) such that \( \lambda_1 = -1 \) is the first simple eigenvalue of \( \Delta \) in \( L^2(\Omega) \) with domain \( H^2_0(\Omega) \) and the normalized eigenfunction \( \phi_1 > 0 \) in \( \Omega \). Consider the following parabolic equation:
\[
(5.1) \quad u_t = V'(u) \equiv \Delta u + u - e^{-1/u^2} \quad \text{in } \Omega \times \mathbb{R}_+, \quad u = 0 \quad \text{on } \partial \Omega \times \mathbb{R}_+,
\]
where we set \( f(0) = 0 \) by continuity. Here \( e^{-1/u^2} \) is the standard \( C^\infty \) function, which is not analytic at \( u = 0 \). In view of the perfect spectral properties of \( \Delta \), we have that there exists a local invariant, \( C^\infty \), one-dimensional centre manifold \( W^c_{\text{loc}}(0) \) of operator \( V'(u) \), which is tangent to the eigenspace \( E^c(0) = \text{Span}\{\phi_1\} \) of \( \Delta + I \); see [21, Thm. 9.2.2] and [32]. Looking for the corresponding center manifold behaviour, we decompose the solution in the form
\[
(5.2) \quad u(x,t) = a_1(t) \phi_1(x) + u_2(x,t) \quad \text{for } t \gg 1,
\]
where \( u_2(x,t) = \sum_{k \geq 2} a_k(t) \phi_k(x) \perp E^c(0) \), \( u_2(\cdot,t) = o(a_1(t)) \) as \( t \to \infty \) and we assume that \( a_1(t) > 0 \) for \( t \gg 1 \), e.g., we take positive solutions. Projecting the PDE onto \( E^c(0) \) yields
\[
(5.3) \quad a_1' = -\langle e^{-1/u^2}, \phi_1 \rangle = -\langle e^{-1/(a_1^2 \phi_1^2 + \ldots)}, \phi_1 \rangle
\]
for \( t \gg 1 \), where \( \langle \cdot, \cdot \rangle \) denoted the inner product in \( L^2(\Omega) \). Denoting \( \rho_1 = \max \phi_1(x) > 0 \) and \( \rho_2 = \int \phi_1 > 0 \) yields the ordinary differential inequality
\[
(5.4) \quad a_1' \geq -\rho_2 e^{-1/4 \rho_1^2 a_1^2},
\]
so that \( a_1(t) \geq \bar{a}_1(t) \), where \( \bar{a}_1 \) solves the corresponding ODE with the equality sign in (5.4). Finally, we obtain the following estimate:
\[
(5.5) \quad a_1(t) \geq \frac{1}{3 \rho_1 \sqrt{4t}} \quad \text{for } t \gg 1
\]
for the stabilization on the centre manifold. Obviously, this corresponds to \( \theta = 0 \) in (1.14), so that the rational algebraic modulus \( | \cdot |^{1-\theta} \) in (1.12) cannot be applied to such
$C^\infty$ nonlinearities. One can introduce an appropriate modulus, for which the rate (5.5) is acceptable. For instance, for a slightly modified function

$$V(u) = \frac{1}{2} e^{-1/u^2} \implies V'(u) = \frac{1}{u^3} e^{-1/u^2},$$

we have the generalized gradient inequality inequality at $\psi = 0$ for $u \approx 0$ (actually, it is equality)

$$\tilde{\omega}(V(u)) \leq |V'(u)| \text{ with modulus } \tilde{\omega}(s) = 2|s| |\ln(2s)|^{3/2},$$

which is “almost” linear, strictly concave function as $s \to 0$.

It is easy to present other $C^\infty$, non-analytic nonlinearities generating arbitrarily slow rate of convergence. Recall that, according to the results in Section 3, any such slow convergence corresponds to isolated equilibria.

**Remark: a cubic nonlinearity.** In the presence of non-trivial kernels, a full asymptotic expansion of solutions can be a difficult problem even in the analytic cubic case

(5.6) \[ u_t = \Delta u + u - u^3. \]

It was shown in [1] that, under the same kernel assumption

(5.7) \[ \dim \ker (\Delta + I) = 1, \]

there exist solutions with the following logarithmically perturbed decay as $t \to \infty$:

$$u(x, t) = \sum_{k=0}^{n} t^{-\frac{1}{2} - k} \sum_{j=0}^{k} \varphi_{kj}(x)(\ln t)^j + O(t^{-\frac{3}{2} - n}),$$

for some integer $n > 0$, where $\varphi_{kj}$ are solutions of well-posed elliptic problems. This corresponds to a special case of centre manifold behaviour, where the leading term of convergence to $\psi = 0$ is of order $O(t^{-\frac{1}{2}})$ that does not contain the $\ln t$-factor. It is easy to see why in this case branching of equilibria from zero is not possible. Indeed, if this occurs, then in view of (5.7), by branching theory [33], the equilibrium branch should have the representation $\psi = \mu \varphi_0 + ..., $ where $\mu$ is the branching parameter and $\varphi_0$ is the eigenfunction of $\Delta$ with $\lambda = -1$. Substituting this into the stationary equation $\Delta \psi + \psi - \psi^3 = 0$ and multiplying by $\varphi_0$ yields $\int_{\Omega} \varphi_0^4 = 0$, a contradiction, so $\psi = 0$ is the isolated equilibrium.

5.2. **A non-local semilinear parabolic equation.** As usual the computations are simplified for semilinear non-local parabolic flows like

(5.8) \[ u_t = \Delta u + \mu u - g'( \int_{\Omega} u^2) u, \]

where $g'(s)$ is a given $C^\infty$, non-analytic function. The potential here is

(5.9) \[ V(u(t)) = -\frac{1}{2} \int |Du(t)|^2 + \frac{1}{2} \int u^2 - g( \int u^2). \]

Assuming again that $\mu = \lambda_1$ is the first eigenvalue of $\Delta$ in $L^2(\Omega)$, we obtain that the centre manifold behaviour (5.2) is described by the ODE

(5.10) \[ a_1' = -g'(a_1^2 + ...) \text{ for } t \gg 1 \]
Choosing non-analytic $C^\infty$ functions $g$, e.g.,
\[ g(s) = s^{3/2}e^{-1/s} \quad \text{for } s > 0, \]
integrating (5.10) asymptotically yields a non-algebraic decay
\[ a_1(t) \approx \frac{1}{\sqrt{\ln 2}} \quad \text{as } t \to \infty. \]
Notice that, besides slow decay behaviour, the present “less nonlinear” model (5.8) is suitable for revealing refined evolution properties of orbits describing stabilization phenomena.

6. On applications to smooth gradient systems in Hilbert spaces

Such an extension is straightforward, since, in the proof of Theorem 1.1 in Section 2, we have minimally used the specific properties of the second-order elliptic and parabolic equations under consideration.

Denoting by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the induced norm in a separable Hilbert space $H$, we consider a smooth gradient flow in $H$,
\[ u_t = V'(u) \quad \text{for } t > 0, \quad (6.1) \]
where the operator $V'$ with a dense, compactly embedded domain $H^2 = D(V') \subset H$ is the Frechet derivative of a $C^2$-functional $V : H^2 \to \mathbb{R}$. We assume that $V''(u) : H^2 \to H$ is a Fredholm operator of index zero admitting a suitable self-adjoint extension with discrete spectrum, compact resolvent and a subset of eigenfunctions, which is complete and closed in $H$. We impose the necessary condition of coercivity of $V'$ to guarantee existence of global orbits \{u(t)\} for arbitrary initial data $u_0 \in H^2$, which are sufficiently regular to satisfy the gradient identity (cf. (2.10))
\[ \frac{d}{dt} V(u(t)) = \|u_t\|^2 \geq 0. \quad (6.2) \]

We impose the same main Hypothesis ($H_R$) and will next check, using the scheme of the proof from Section 2, which conditions we need to guarantee the result (i) and, hence, (iii) in Theorem 1.1.

**Steps 1–3.** The construction remains the same and (2.2), (2.5) and (2.7) are guaranteed by assumed good spectral properties of the linearized self-adjoint operator $V''(\psi_\mu)$ for any $\psi_\mu \in S_k$ uniformly in $\mu \approx 0$.

**Step 4.** Estimate (2.9) is valid for sufficiently smooth operator $V'$ in $H^2$.

**Step 5.** Here we need estimate (2.12), which is indeed Lagrange’s formula for the smooth functional $V$ in $H^2$.

**Step 6.** Lemma 2.1 remains valid in the topology of $H$.

**Step 7** remains unchanged.

Hence Theorem 1.1 (i), (iii) is true for arbitrary smooth gradient flows under the presence of resonance branching. It follows that the convergence (evolution completeness) result holds for the 2mth-order PDEs (1.16) and for other smooth gradient parabolic flows governed by higher-order operators.
7. Non-autonomous perturbations of gradient systems

More modifications of the approach are necessary to prove the convergence result in Theorem 1.1 (i) for non-autonomous perturbations of (6.1). The main features of our analysis are illustrated by the following example:

(7.1) \( u_t = V'(u) + h(t)W'(u) \) for \( t > 0 \),

where \( W(u) : H^2 \to \mathbb{R} \) is a \( C^1 \)-functional. Without loss of generality and for simplification of future manipulations, we assume that the decay rate of perturbation \( h \in C^2 \) satisfies

(7.2) \( h(t) \to 0 \) as \( t \to \infty \), and \( h(t) > 0, h'(t) < 0 \) for \( t \geq 0 \),

so that, in particular, \( h' \in L^1(\mathbb{R}_+) \). Otherwise, for non-monotone and changing sign rates of perturbations, we can use estimates from above and below with functions \( h_{\pm}(t) \) satisfying necessary assumptions.

Actually, we will need a more restrictive condition on the decay rate,

(7.3) \( \sqrt{h} \in L^1(\mathbb{R}_+) \).

It is important to deal with non-exponentially small perturbations such as

(7.4) \( h(t) = \frac{1}{(1+t)\alpha} \) with any \( \alpha > 2 \),

for which (7.3) holds. For the future purpose of integration of ordinary differential inequalities, we characterize this class of such slow decaying functions as follows: for any constant \( \beta > 0 \),

(7.5) \( \int_0^t h(s)e^{\beta s} \, ds = \frac{1}{\beta} h(t)e^{\beta t}(1 + o(1)) \) as \( t \to \infty \),

or, equivalently, by L’Hospital’s rule,

(7.6) \( \frac{h'(t)}{h(t)} \to 0^- \) as \( t \to \infty \).

Since (7.1) is not a gradient system in the sense that a monotone Lyapunov function, in general, does not exist, we need first to prove that the crucial characterization (1.6) of the \( \omega \)-limit set remains valid, so we should begin with

**Step 0:** \( \omega(u_0) \subseteq \Phi \). Multiplying (7.1) by \( u_t \) in \( H \), instead of (6.2), we obtain the identity

(7.7) \( \frac{d}{dt} V(u(t)) + \frac{d}{dt} (h(t)W(u)) - h'(t)W(u) = \|u_t\|^2 \geq 0 \).

Integrating over \( (t, \infty) \) and using that \( h' \in L^1(\mathbb{R}_+) \) yields the convergence

(7.8) \( \int_t^\infty \|u_t(s)\|^2 \, ds < \infty \).

Hence, given a sequence \( \{t_k\} \to \infty \) such that \( u(t_k) \to \psi \in \omega(u_0) \), we obtain by the Hölder inequality that, for arbitrarily large fixed \( t > 0 \),

(7.9) \( \|u(t_k + t) - u(t_k)\|^2 \leq t \int_{t_k}^\infty \|u_t\|^2 \, ds \to 0 \) as \( k \to \infty \).
Passing to the limit as \( t = t_k + t \to \infty \) in (7.1), we then conclude that \( u(t_k + t) \) converges in \( H \) to a solution \( u(t) \) of the limit autonomous equation (6.1), which is independent of \( t \), so it is an equilibrium \( \psi \in \Phi \).

We now return to seven steps of the proof in Section 2.

**Steps 1-3** are unchanged.

**Step 4.** In view of the extra term in (7.1), instead of (2.8), we will have

\[
g = u_t + g_1(u) + h(t)W'(u).
\]

Therefore, by (7.2), instead of (2.9), we obtain

\[
\|u(t) - \psi_{\mu(t)}\|_{H^2} \leq C_4(\|u_t\| + h(t)).
\]

**Step 5.** It follows from (7.7) that the convergence (2.11) takes place but not necessarily from below, which is not important. Then, instead of (2.13), we have

\[
\frac{d}{dt}[B - V(u(t))] = -\|u_t\|^2 + \langle h(t)W'(u), u_t \rangle \leq -C_7[B - V(u(t))] + C_7h(t).
\]

Integrating this inequality by using the slow decay hypotheses (7.5), (7.6), we conclude that, instead of the exponential estimate (2.14), on large bounded intervals \( t \in [t_k, t_k + T] \) as \( k \to \infty \), with \( T \gg 1 \) and \( t \gg 1 \),

\[
B - V(u(t)) \leq 2h(t_k + t).
\]

**Step 6** needs a major revision, since Zelenyak’s Lemma 2.1 applies only to exponential decay estimates.

**Lemma 7.1.** Assume (7.2) and (7.3) hold and, for all \( t > 0 \),

\[
\int_t^\infty \|u_t(s)\|^2 \, ds \leq h(t).
\]

Then

\[
\|u(t) - u(\tau)\| \leq \int_{t-1}^\infty \sqrt{h(s)} \, ds \quad \text{for all } 0 < t \leq \tau \leq \infty.
\]

**Proof.** The result is obvious if \( |t - \tau| \leq 1 \), where, by the Hölder inequality,

\[
\|u(t) - u(\tau)\| \leq \sqrt{\tau - t} \int_t^\tau \|u_t(s)\|^2 \, ds \leq \sqrt{\tau - t} \sqrt{h(t)}.
\]

For large intervals \( \tau - t > 1 \), we perform a partition with a sequence of time-steps \( \{\Delta_i, i = 0, 1, ..., K\} \), so that, instead of (2.17), we obtain

\[
\|u(t) - u(\tau)\| \leq \sum_{i=0}^K \sqrt{\Delta_i} \int_{t_i}^{t_{i+1}} \|u_t(s)\|^2 \, ds = \sum_{i=0}^K \sqrt{\Delta_i} \sqrt{h(t_i)},
\]

where \( t_{i+1} = t_i + \Delta_i \). Setting \( \Delta_i = 1 \) for \( i = 0, 1, ..., K - 1 \) and \( \Delta_K \leq 1 \), by the integral test of convergence of series, we obtain (7.15). \( \square \)
Step 7 remains the same, where we replace “exponentially small” in (a) and (b) by “$O(\sqrt{h(t)})$-small”.

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Comment. Almost two years after publishing the present paper in

Sbornik: Math., 198:6 (2007), 817–838; see [12],
in January 2009, the first author found that similar Zelenyak’s ideas and techniques were applied in [17, § 5] to convergence in 1D wave equations. Actually, Zelenyak’s Lemma[21] were introduced in [17] earlier, in Section 2, and was a key ingredient of the proof of Theorem 1.1 on exponential convergence in a Hilbert space setting (similar to our Theorem 1.1 in an analogous Hilbert space framework explained in Section [6]). It is interesting and truly remarkable that such a growing interest to Zelenyak’s fundamental idea on proving exponential convergence from the 1960s [34], in the 21st century, occurred approximately in the same year, about 2007, according to the publication dates.

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1It is worth mentioning that this research was essentially completed in 2005, but publishing the paper [12] took quite a while, since the first its submission to the J. Differ. Equat., after a rather long time, was rejected with the critics from a Referee saying, loosely speaking, that almost all the obtained results directly follow from Hale and Raugel’s ones [15] of 1992, and that the exponential convergence in Theorem [11] (iii) (not obtained elsewhere at that time) can be also somehow easily proved. The latter is wrong, since the exponential convergence cannot in principle derived from any kind of invariant manifold theory (in particular, as we have mentioned, this has nothing to do with the classic exponential stability of centre manifolds). This exponential convergence is one of the main achievements of Zelenyak’s approach, which he developed for 1D second-order parabolic equations [21] that we here extend to more general higher-order parabolic flows in $\mathbb{R}^N$ and to some autonomous or perturbed ODEs in Hilbert spaces. Note that the exponential convergence in [17, Theorem 1.1(iii)] (a Hilbert space framework) is also proved by Zelenyak’s Lemma and his related techniques.
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