BESSEL FUNCTIONS AND THE WAVE EQUATION

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Abstract. We solve the Cauchy problem for the n-dimensional wave equation using elementary properties of the Bessel functions.

With $\nabla^2 = D^2_{x_1x_1} + \cdots + D^2_{x_nx_n}$ the Laplacian in $\mathbb{R}^n$, where

$$D^2_{x_kx_k} = \frac{\partial^2}{\partial x_k^2}, \quad 1 \leq k \leq n,$$

and $D_t$ and $D_{tt}$ indicating the first and second order derivatives with respect to the variable $t \in \mathbb{R}$, respectively, the wave equation in the upper half-space $\mathbb{R}^n_+$ is given by

$$D^2_{tt}u(x,t) = \nabla^2 u(x,t), \quad x \in \mathbb{R}^n, t > 0,$$

and the Cauchy problem for this equation consists of finding $u(x,t)$ that satisfies (1) subject to the initial conditions

$$u(x,0) = \varphi(x) \quad \text{and} \quad D_t u(x,0) = \psi(x), \quad x \in \mathbb{R}^n,$$

where for simplicity we shall take $\varphi$ and $\psi$ in $\mathcal{S}(\mathbb{R}^n)$.

Applying the Fourier transform to (1) in the space variables, considering $t$ as a parameter, it readily follows that $\hat{\nabla^2} u(\xi,t) = -|\xi|^2 \hat{u}(\xi,t)$, and so $\hat{u}$ satisfies

$$D^2_{tt} \hat{u}(\xi,t) + |\xi|^2 \hat{u}(\xi,t) = 0, \quad \xi \in \mathbb{R}^n, t > 0,$$

subject to

$$\hat{u}(\xi,0) = \hat{\varphi}(\xi) \quad \text{and} \quad D_t \hat{u}(\xi,0) = \hat{\psi}(\xi), \quad \xi \in \mathbb{R}^n.$$

For each fixed $\xi \in \mathbb{R}^n$ this resulting ordinary differential equation in $t$ is the simple harmonic oscillator equation with constant angular frequency $|\xi|$, and so

$$\hat{u}(\xi,t) = \hat{\varphi}(\xi) \cos(t|\xi|) + \hat{\psi}(\xi) \frac{\sin(t|\xi|)}{|\xi|}, \quad \xi \in \mathbb{R}^n, t > 0.$$

Hence, the Fourier inversion formula gives for $(x,t) \in \mathbb{R}^{n+1}_+$,

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \cos(t|\xi|) e^{i\xi \cdot x} d\xi $$

$$+ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\psi}(\xi) \frac{\sin(t|\xi|)}{|\xi|} e^{i\xi \cdot x} d\xi.$$

Since the first integral in (2) can be obtained from the second by differentiating with respect to $t$, we will concentrate on the latter. The idea is to
interpret $\frac{\sin(|\xi|t)}{|\xi|}$ as the Fourier transform of a tempered distribution, and the key ingredient for this are the following representation formulas established in [1].

**Representation Formulas.** Assume that $n$ is an odd integer greater than or equal to 3. Then, with $\sigma$ the element of surface area on $\partial B(0,R)$,

$$
\frac{\sin(R|\xi|)}{|\xi|} = c_n \left( \frac{1}{R} \frac{\partial}{\partial R} \right)^{(n-3)/2} \left( \frac{1}{\omega_n R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) \right),
$$

where $R > 0$, $\omega_n$ is the surface measure of the unit ball in $\mathbb{R}^n$, and $c_n^{-1} = (n-2)(n-4) \cdots 1$.

On the other hand, if $n$ is an even integer greater than or equal to 2,

$$
\frac{\sin(R|\xi|)}{|\xi|} = d_n \left( \frac{1}{R} \frac{\partial}{\partial R} \right)^{(n-2)/2} \left( \frac{1}{\nu_n R} \int_{B(0,R)} \frac{1}{\sqrt{R^2 - |x|^2}} e^{-ix \cdot \xi} d\sigma(x) \right),
$$

where $R > 0$, $d_n^{-1} = n(n-2)(n-4) \cdots 2$, and $\nu_n$ is the volume of the unit ball in $\mathbb{R}^n$.

The purpose of this note is to establish (3) and (4) using elementary properties of Bessel functions. $J_\nu(x)$, the Bessel function of order $\nu$, is defined as the solution of the second order linear equation

$$
x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0.
$$

Several basic properties of the Bessel functions follow readily from their power series expression [2]. They include the recurrence formula

$$
\frac{d}{dx} (x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x),
$$

the integral representation of Poisson type

$$
J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_{-1}^1 (1-s^2)^{\nu-1/2} e^{ixs} ds,
$$

and the identity

$$
J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi x^{1/2}}} \sin(x),
$$

for $x > 0$.

We will consider the odd dimensional case first. The dimensional constant $c_n$ may vary from appearance to appearance until it is finally determined at the end of the proof. To begin recall that for $n \geq 3$, as established in (18) in [1],

$$
\frac{1}{\omega_{n-1} R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) = R^{n-2} \int_{-1}^1 e^{iR|\xi|s} (1-s^2)^{(n-3)/2} ds,
$$
which combined with (6) above with $\nu - 1/2 = (n - 3)/2$ there, i.e., $\nu = (n - 2)/2$, gives

$$\frac{1}{\omega_n R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) = c_n R^{n-2} J_{(n-2)/2}(R|\xi|) / (R|\xi|)^{(n-2)/2}$$

$$= c_n \frac{1}{|\xi|^{n-2}} (R|\xi|)^{(n-2)/2} J_{(n-2)/2}(R|\xi|).$$

Now, by (5) we obtain that

$$\frac{\partial}{\partial R} \left( \frac{1}{\omega_n R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) \right) = c_n \frac{1}{|\xi|^{n-2}} |(R|\xi|)^{(n-2)/2} J_{(n-4)/2}(R|\xi|),$$

or

$$\frac{1}{R} \frac{\partial}{\partial R} \left( \frac{1}{\omega_n R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) \right) = c_n \frac{1}{|\xi|^{n-4}} (R|\xi|)^{(n-4)/2} J_{(n-4)/2}(R|\xi|).$$

Thus, applying the above reasoning $(n - 3)/2$ times, (7) gives

$$\left( \frac{1}{R} \frac{\partial}{\partial R} \right)^{(n-3)/2} \left( \frac{1}{\omega_n R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) \right)$$

$$= c_n \frac{1}{|\xi|} (R|\xi|)^{1/2} J_{1/2}(R|\xi|)$$

$$= c_n \frac{1}{|\xi|} (R|\xi|)^{1/2} \sin(R|\xi|)$$

$$= c_n \frac{\sin(R|\xi|)}{|\xi|}.$$ 

The value of $c_n$ is readily obtained as in [1], and (3) has been established.

To consider the case $n$ even, one generally proceeds at this point by a reasoning akin to Hadamard’s method of descent, i.e., the desired result for the wave equation in even dimension $n$ is derived from the result in odd dimension $n + 1$, as is done for instance in [1] for the representation formulas. On the other hand, Bessel functions provide the desired result for the wave equation in even dimensions directly, by a method akin to ascent: the result for the wave equation for dimension $n = 2$ is obtained explicitly, and for even dimension $n + 2$ is obtained from the result in even dimension $n$.

We will first prove a preliminary result. The dimensional constant $d_n$ may vary from appearance to appearance until it is finally determined at the end of the proof.

**Lemma.** The following three statements hold.

(8)  $$\int_0^\infty \sin(R\rho) J_0(t \rho) \, d\rho = \frac{1}{\sqrt{R^2 - t^2}} H(R - t), \quad R, t > 0,$$

where $H$ denotes the Heavyside function.

Furthermore, for $\nu \geq 1$,

(9)  $$\left( \frac{1}{R} \frac{\partial}{\partial R} \right) \left( \int_0^\infty \sin(R\rho) \rho^{\nu-1} J_{\nu-1}(t \rho) \, d\rho \right) = \frac{1}{t} \int_0^\infty \sin(R\rho) \rho^\nu J_\nu(t \rho) \, d\rho.$$
and, consequently, for 1 ≤ j ≤ ν,

\[ \left( \frac{1}{R} \frac{\partial}{\partial R} \right)^j \left( \int_0^\infty \sin(R\rho) \rho^{\nu-j} J_{\nu-j}(t \rho) \, d\rho \right) = \frac{1}{t^j} \int_0^\infty \sin(R\rho) \rho^\nu J_{\nu}(t \rho) \, d\rho. \]

Proof. (8) is Formula (6) in \[2\], page 405.

Now,

\[ \frac{\partial}{\partial R} \left( \int_0^\infty \sin(R\rho) \rho^{\nu-1} J_{\nu-1}(t \rho) \, d\rho \right) = - \int_0^\infty \cos(R\rho) \rho^\nu J_{\nu-1}(t \rho) \, d\rho \]

which, by (1), equals

\[ \frac{-1}{t^{\nu+1}} \int_0^\infty \cos(R\rho) \frac{\partial}{\partial \rho} ((t \rho)^\nu J_{\nu}(t \rho)) \, d\rho = \frac{R}{t} \int_0^\infty \sin(R\rho) \rho^\nu J_{\nu}(t \rho) \, d\rho, \]

which proves (9).

(10) follows by repeated applications of (9), and we have finished. \(\Box\)

Finally, recall that the Fourier transform of a radial function \(f\) on \(\mathbb{R}^n\) is given by the expression \[2\],

\[ \hat{f}(\xi) = d_n \frac{1}{|\xi|^{(n-2)/2}} \int_0^\infty \rho^{n/2} f(\rho) J_{(n-2)/2}(|\xi| \rho) \, d\rho. \]

In particular, we have

\[ \int_{\mathbb{R}^n} \frac{\sin(R|\xi|)}{|\xi|} e^{-ix \cdot \xi} \, d\xi = d_n \frac{1}{|x|^{(n-2)/2}} \int_0^\infty \rho^{n/2} \frac{\sin(R\rho)}{\rho} J_{(n-2)/2}(|x| \rho) \, d\rho. \]

Let now \(n = 2k\) be an even integer. Then by (11),

\[ \int_{\mathbb{R}^n} \frac{\sin(R|\xi|)}{|\xi|} e^{-ix \cdot \xi} \, d\xi = d_n \frac{1}{|x|^{(k-1)}} \int_0^\infty \sin(R\rho) \rho^{k-1} J_{k-1}(|x| \rho) \, d\rho, \]

and, therefore, (10) with \(\nu = j = k - 1\) there yields

\[ \int_{\mathbb{R}^n} \frac{\sin(R|\xi|)}{|\xi|} e^{-ix \cdot \xi} \, d\xi = d_n \frac{1}{|x|^{(k-1)}} \int_0^\infty \sin(R\rho) \rho^{(k-1)} J_{k-1}(|x| \rho) \, d\rho \]

\[ = d_n \left( \frac{1}{R} \frac{\partial}{\partial R} \right)^{(k-1)} \left( \int_0^\infty \sin(R\rho) J_0(|x| \rho) \, d\rho \right) \]

\[ = d_n \left( \frac{1}{R} \frac{\partial}{\partial R} \right)^{(k-1)} \left( \frac{1}{\sqrt{R^2 - |x|^2}} H(R - |x|) \right). \]

Thus by the Fourier inversion formula,

\[ \frac{\sin(R|\xi|)}{|\xi|} = d_n \left( \frac{1}{R} \frac{\partial}{\partial R} \right)^{(n-2)/2} \left( \int_{\mathbb{R}^n} \frac{1}{\sqrt{R^2 - |x|^2}} H(R - |x|) e^{-ix \cdot \xi} \, dx \right) \]

\[ = d_n \left( \frac{1}{R} \frac{\partial}{\partial R} \right)^{(n-2)/2} \left( \frac{1}{v_n} \int_{B(0,R)} \frac{1}{\sqrt{R^2 - |x|^2}} e^{-ix \cdot \xi} \, dx \right), \]
The constant $d_n$ is readily determined as in [1], and we have finished.

References

[1] A. Torchinsky, The Fourier transform and the wave equation, Amer. Math. Monthly 118 (2011) no.7, 599-609.
[2] G. N. Watson, A treatise on the theory of Bessel functions. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.