Correspondences between Privacy and Nondiscrimination
Why They Should Be Studied Together

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August 7, 2018

Abstract
Privacy and nondiscrimination are related but different. We make this observation precise in two ways. First, we show that both privacy and nondiscrimination have two versions, a causal version and a statistical associative version, with each version corresponding to a competing view of the proper goal of privacy or nondiscrimination. Second, for each version, we show that a difference between the privacy edition of the version and the nondiscrimination edition of the version is related to the difference between Bayesian probabilities and frequentist probabilities. In particular, privacy admits both Bayesian and frequentist interpretations whereas nondiscrimination is limited to the frequentist interpretation. We show how the introduced correspondence allows results from one area of research to be used for the other.

1 Introduction
Privacy and nondiscrimination appear both related and yet different. The two are both norms that help to ensure that powerful entities treat people fairly by not violating social values. However, privacy is typically seen as protecting information from disclosure while nondiscrimination is seen as prohibiting certain behaviors based upon known information about protected attributes, such as gender or race.

In this paper, we explore the relationship between privacy and nondiscrimination from a technical angle. We demonstrate that key aspects of privacy and nondiscrimination mirror each other at a formal level, and make a case for the exchange of techniques between the communities independently studying the two. Further, for each norm, there exist two stances within the research community studying it: whether the norm, privacy or nondiscrimination, should be measured in terms of association (probabilistic dependence) or in terms of causation. We express, in terms of probability theory, the two stances for each of the two norms. Doing so makes precise the exact difference between each pair of stances, revealing that each pair differs over the same issue.
In more detail, for privacy, the competing stances lead to what we will call *associative privacy* and *causal privacy*. Associative privacy properties typically show up in works attempting to minimize the knowledge gained by an adversary upon observing the outcomes of a computation, that is, works attempting to provide *statistical nondisclosure* (e.g., [1]). Causal privacy instead focuses on whether some particular action leads to a large change, with differential privacy being the prime example (e.g., [2–8]). Tschantz et al. has already noted the causal nature of differential privacy [9].

As for nondiscrimination, two stances predominate in U.S. law, *disparate impact* and *disparate treatment*, with similar counterparts in the rest of the world. These are complex legal tests involving concepts that are difficult to apply to algorithms, such as intent, and numerous caveats and exceptions. However, at their cores are two standards, which we will call *associative nondiscrimination* and *causal nondiscrimination*. Associative nondiscrimination demands that members of protected classes should not disproportionately suffer adverse actions. Roughly speaking, to avoid a finding of associative discrimination, or disparate impact, a governed entity (e.g., a large employer) must ensure that the proportion of members of a protected class (e.g., women or a minority race) experiencing some adverse action (e.g., firing or not hiring) is roughly equal to the proportion of non-members experiencing it. Courts have provided statistical characterizations of disparate impact in a number of settings, such as the 80% standard in employment [10]. Causal nondiscrimination demands protected attributes do not cause people to experience some adverse action, which is the core of disparate treatment. In short, to win a case under disparate treatment, the plaintiff must show, among other things, that the defendant subjected the plaintiff to some adverse action because of the plaintiff’s status as a member of a protected class. For disparate treatment, the courts have used a common-sense approach that looks at motivations. From it, we extract a mathematical characterization that focuses on just casual processes, ignoring motivations, along the lines of Pearl’s treatment of the issue [11].

Table 1 provides a summary of the relationships between privacy and nondiscrimination. It shows that properties for each norm vary along a binary axis falling into one of two stances: an associative or causal one. We will call this axis the *dependence axis* since association and causation are two forms of *dependence* between random variables. (“Statistical independence” means a lack of association. People often speak of “causal dependence”. Sometimes the word is used ambiguously as in “the dependent variable.”) It further shows that for each nondiscrimination property there is a corresponding privacy property.

As a result of the close correspondence between both stances of privacy and nondiscrimination properties, we obtain a number of results ‘for free’, which we describe in Section 8. In particular, the Dwork–Naor impossibility result [12] translates to an impossibility of ensuring no disparate impact across arbitrary subpopulations. Also, we point out research in both areas that can be repurposed to solve the corresponding problem in the other area.

Of course, privacy and nondiscrimination are not the same. While both deal with statistical associations and causal effects, they differ in which associations
and causes are problematic. Furthermore, our simple models of nondiscrimination and privacy abstract away the nuances of these social norms, such as exceptions to the general rules we represent. However, these differences play little role in the mathematical analysis of or development of algorithms and verification techniques for the core properties capturing these norms.

There is, however, a key difference between nondiscrimination and privacy that is mathematically interesting: nondiscrimination focuses on adverse actions, whereas privacy sometimes deals with adverse actions but often deals with knowledge. That is, while privacy and nondiscrimination share the dependence axis along which their properties differ, privacy, unlike nondiscrimination, has a second axis. This axis, the endpoint axis, captures the difference between what has been called use privacy (e.g., [13]) and inferential privacy (e.g., [14]). Use privacy is similar to nondiscrimination in that it prohibits some actions from depending upon some sensitive fact. Inferential privacy is more abstract in that it refers to the knowledge of an observer requiring that the observer maintains some degree of ignorance of some sensitive fact.

The two forms of privacy are related, but not equivalent. An observer seeing an outcome that depends upon a sensitive fact may gain information about the sensitive fact, meaning that a lack of use privacy can imply a lack of inferential privacy. Furthermore, an observer who knows the sensitive fact might use it to choose outputs inappropriately, meaning that a lack of inferential privacy can imply a lack of use privacy. However, in both cases, can imply does not mean does imply. An observer not knowing the dependence between the sensitive fact and the outputs gains no information and an observer may choose to not make use of its knowledge to inappropriately select outputs.

The mathematically interesting difference between use privacy and inferential privacy is use privacy should be measured in frequentist probabilities (or, more generally, physical probabilities), whereas inferential privacy should be measured in Bayesian probabilities. Whereas the distinction between these two approaches to probability theory may appear to be merely a philosophical debate, we see here the practical distinctions between them.

|               | Associative          | Causal               |
|---------------|----------------------|----------------------|
| Nondiscrimination | Disparate impact     | Disparate treatment  |
| Privacy       | Statistical nondisclosure | Differential privacy |

Table 1: An informal summary of representative associative and causal nondiscrimination and privacy properties. For each norm (row) and stance (column), the table provides an example of a property approaching that norm from that stance. We demonstrate that the core concepts of two rows are mathematically identical to one another. We also show that columns represent a switch between causal and associative dependence. Table 4 represents each point in this grid in formal notation, exposing the structural correspondence between these properties.
In summary, we have three axes along which to navigate the space of properties: (1) the norm: privacy or nondiscrimination, (2) the notion of dependence: looking at either a change in association or in causation, and (3) for privacy, the endpoint used: either measuring the change in use (frequentist probabilities) or in knowledge (Bayesian).

We will explore and make precise these differences and relations in this paper. We first cover background and provide an overview of our results in Sections 2 and 3. We then cover related work in Section 4. In Section 5 we present a set of probabilistic definitions for formally stating causal and associative properties of systems, and prove relationships between their various forms. In Sections 6 and 7 we instantiate parameters in these definitions to obtain different notions of privacy and nondiscrimination respectively. Restating all properties in terms of this common substrate also allows us to transfer known theorems and methods about privacy to fairness, and vice versa, in Section 8.

2 Background

2.1 Probabilities, frequencies, and knowledge

Probabilities are a useful tool to characterize a number of different concepts. In this work, we distinguish between frequentist and Bayesian probabilities, or, more generally, between physical and epistemic probabilities.

Frequentist probabilities represent the frequencies of occurrence of events. They are objective in that they measure a physical property of the world. They are a useful model for representing outcomes of random coin flips or fractions of populations with a certain property. Since they are often just called frequencies, we denote frequentist probabilities using $Fr$. Frequentist conditioning restricts events to a smaller population. For example, if $O$ and $G$ are random variables representing a certain hiring outcome and gender respectively, then $Fr[O = \text{‘Hired’} \mid G = \text{‘Female’}]$ represents the frequency of women that are hired out of the overall population, that is, the number of women hired divided by the size of the population restricted to just the females. Note that in standard notation, the overall population is left implicit.

(Some authors use frequentist and frequencies in a strict sense limited to the case where the population size is infinite. They might refer to such probabilities over finite populations as empirical frequencies or physical probabilities. We note where this distinction may appear in our models of discrimination, but it can be safely ignored.)

Bayesian (or epistemic) probabilities represent how certain a reasoning agent is about propositions being true. They are subjective to the agent in that different agents may differ in the probabilities they assign. Since they are often just called credences, we denote Bayesian probabilities using $Cr$. Bayesian conditioning represents a knowledge update. For example $Cr[O = \text{‘Hired’} \mid G = \text{‘Female’}]$, represents the certainty that a person was hired given that the agent knows that he or she was female. This decision may also depend upon the agent’s back-
| Concept          | Frequentist            | Bayesian                     |
|------------------|------------------------|------------------------------|
| Probability      | Frequency              | Credence                     |
| Probability space| Population             | Background knowledge         |
| Conditioning     | Population restriction | Knowledge revision           |
| Association      | Co-occurrence          | Evidence (information leakage) |

Table 2: Comparison of frequentist and Bayesian Concepts

ground knowledge, which is considered fixed for the whole probabilistic analysis. Thus, similar to the implicit overall population for frequencies, the background knowledge is typically left implicit.

In Table 2 we contrast a few probabilistic notions and their interpretations in the frequentist and Bayesian views. In situations in this paper where results apply to both forms, we denote probabilities using \( \text{Pr} \).

2.2 Pearl’s Account of Causation

We recount Pearl’s theory of causation as background [11].

**Definition 1.** A causal model is a triple \( M = \langle U, V, F \rangle \) where \( U \) is a set of variables, called background variables (or exogenous), \( V \) is set of variables, called endogenous, and \( F \) is a set of functions \( f_1, \ldots, f_n \), called structural equations, where each \( f_i \) is a mapping that defines \( V_i \) in terms of all other variables in \( U \cup V \).

We assume, as Pearl normally does, that the equations are **recursive**, that is, there is an ordering on \( U \cup V \) such that all of \( U \) comes before all of \( V \), and that for each \( f_i \), the variables in \( U \cup V \) that it uses all come before \( V_i \). Under this assumption, given the values of the variables in \( U \), one can compute the value for any \( V_i \) in \( V \) by computing the values of each variable in that order until reaching \( V_i \). We use \( M.V_i(u) \) to denote the computed value.

An intervention \( \text{do}(X=x) \) on an endogenous variable \( X \), in a model \( M \), replaces the equation corresponding to \( X \) with \( x \), resulting in a new model \( M_{X=x} \).

**Definition 2.** A probabilistic causal model is a pair \( \langle M, B \rangle \), where \( M \) is a causal model and \( B \) is a background probability function defined over the domain of \( U \), the background variables.

\( B \) makes the the probabilities assigned to background variables explicit, whether that comes from an underlying population (frequentist) or background knowledge (Bayesian). WFor an assignment \( u \) of values to all background variables \( U \), we will write \( B(u) \) or \( \text{Pr}[U=u \mid B] \), depending upon context.
For recursive models, the probabilities over background variables can be lifted to a probability over an endogenous variable $Y$:

$$
Pr[Y = y | B] = \sum_{\{u | M.Y(u) = y\}} B(u).
$$

The probability of counterfactual statements is defined by probabilities with respect to the model $M_{X=x}$.

Pearl prefers Bayesian probabilities [11], but they can be interpreted either way. From a Bayesian viewpoint, where these probabilities represent beliefs, the lifting of probabilities is only sensible under the assumption that the agent knows the structural equations.

We modify our notation slightly make this assumption explicit by showing the structural equations as explicit conditions in our probabilities. Typically, our probabilities are of the form $Pr[\varphi | SE, B]$ where the context comprises of two parts: the background probability distribution $B$ and the structural equations $SE$. $SE$ represents structural equations that define endogenous variables. $SE$ contains exactly one structural equation for each endogenous variable of the form $X = f(U_{i1}, \ldots, U_{ik}, X_{j1}, \ldots, X_{jk})$, where $U_{i1}, \ldots, U_{ik}, X_{j1}, \ldots, X_{jk}$ are the predecessors of $X$ in the structural model.

An intervention on an endogenous variable $X$ with value $x$ in the model $SE$ is denoted by $[X \rightarrow x]SE$, which replaces the equation for $X$ in $SE$ with $X=x$.

We assume that the background variables do not refer to or depend upon the structural equations. In this case, since endogenous variables are defined by structural equations when the rest of a probability expression only refers to background variables, structural equations can be substituted. In other words, the choice of structural equations is independent to all other background variables.

**Assumption 3 (SE-independence).** For each $\varphi$, consisting of only background variables, and background distribution $B$ for any sets of equations $SE$, $SE'$,

$$
Pr[\varphi | SE, B] = Pr[\varphi | SE', B]
$$

When using Bayesian probabilities, or credences, the background distribution represents background knowledge of the agent doing the reasoning, which we typically view as an adversary, and not some objective population of outcomes. In this case, Assumption 3 means that adversary does not background knowledge about the background variables that depends upon the structural equations.

We will often consider structural equation models representing the system of the following form: $SE = \{X=X, A=A, 0=s(X, A)\}$, where $X, A, 0$ are endogenous variables, and $X$ and $A$ are background. $s$ represents a system that is the subject of our privacy or nondiscrimination analysis. $0$ represents its output. $X$ represents some sensitive attribute that, intuitively, $s$ should not use to compute $0$. $X$ represents an input to the system representing $X$. $A$ are other attributes that the system intuitively may use and $A$ is the input to $s$ representing them.
Similar to Tschantz et al. [9], we distinguish between the actual attribute $X$ and an input $\bar{X}$ representing it since we want to discuss how the system’s output would change as its inputs change without considering all the other changes that would follow from changing the actual value of the sensitive attribute. For example, a causal intervention to a person’s race not only is conceptually difficult to comprehend [15], but would have a wide range of effects, such as likely resulting in a different spouse, address, and job, taking us far from the system $s$. As a result, we don’t allow intervening on the race itself variable $X$ in our model, which is enforced by Pearl’s theory disallowing interventions on background variables. On the other hand, since the endogenous variable $X$ is an input to well defined structural equations representing a system $s$, it is possible to model the effects of intervening on the values of this variable. The same reasoning applies to $A$ and $\bar{A}$.

In our work, we assume that an adversary has complete knowledge of the model. While this is a standard assumption in cryptography and privacy, interestingly, many leakage results do not apply to weaker adversaries with partial knowledge about the model. In the weaker adversary model, observing outputs can leak information about the model, which can then leak information about secrets. However, this channel for information leakage is not permitted in the case where the programs are completely known.

3 Overview

We categorize discrimination and privacy properties by the notion of dependence they use: associative or causal (see Table 3). For each combination of norm and notion of dependence, we provide a precise mathematical model of a point in the space of fairness concepts. For the privacy concepts, which have already been extensively studied by the CS research community, we sometimes find pre-existing definitions occupying one of these points. In that case, we restate the pre-existing definition in the standard form of Tschantz et al. [9] to make its associative or causal nature apparent. We also note the kind of probability employed. Statistical nondisclosure is a property about an adversary’s state of knowledge and is therefore represented as credences. On the other hand, differential privacy is a statement about the true distribution over the outcomes of a program and is therefore represented by frequencies. We also briefly consider an associative notion of privacy using frequencies and a causal notion using credences.

The nondiscrimination properties come from complex legal tests that include issues such as motivations and exceptions for unavoidable discrimination. We will not model these tests in their completeness, but rather their core conceptions of when discrimination (perhaps unavoidable and legal) occurs. Our models show that they too boil down to associative and causal variants.

Our formalizations of all of these properties can be stated as a difference in outcomes across changes in certain variables. Each of the properties is about comparing some point of comparison for two values that the a sensitive or secret
attribute \( S \) could have taken on. The property demands that the point of comparison remains roughly the same whether \( S \) took on \( s \) or \( s' \), meaning that the attribute \( S \) plays at most a minor role in the determining the value of the point of comparison.

For example, recall disparate treatment, the U.S. legal standard for when people discriminated against because of having some protected attribute. Intuitively, determining whether a woman has suffered from disparate treatment with respect to gender requires comparing two worlds: the actual world in which the women is female and a hypothetical world in which she were a male. For each world, one computes the probability of the women experiencing an adverse outcome. This calculation, done twice, is the point of comparison for disparate treatment.

For a more detailed example, recall that for a system \( s \) to have \( \epsilon \)-differential privacy the following must hold for all databases \( d \) and \( d' \) that differ by one person’s data and all outputs \( o \):

\[
F[s(d) = o] \leq e^\epsilon F[s(d') = o]
\]

(For simplicity, we only deal with discrete data in this paper, removing the need to consider sets of outputs.) The point of comparison for differential privacy is the probability of \( s \) having various outputs for various databases. We will denote this as \( F[r(s) = o] \) highlighting that the database changes from \( d \) to \( d' \) across the comparison by underlining it. In addition to the point of comparison, differential privacy is also defined by over what databases the comparison is made (those that differ by one entry), over what outputs \( o \) the comparison is made (all of them), what the comparison demands (that \( \cdot \leq e^\epsilon \cdot \) holds). However, focusing on the point of comparison highlights the differences between properties we are interested in.

To make its causal form more apparent, following Tschantz et al., we will rewrite differential privacy as

\[
F[r(s) = o \mid X \rightarrow x] \leq e^\epsilon F[r(s) = o \mid X \rightarrow x']
\]

using notation introduced in Section 2.2. \( o \) is a random variable representing the output of the system \( s \). \( X \) represents the entry that changes between the two databases. \( x \) and \( x' \) are the values by which the two databases differ, with a special value \( \perp \) denoting that the entry is missing altogether. (We are using the bounded model of differential privacy, which differs slightly from the original

| Associative  | Causal  |
|--------------|---------|
| Conditioning | Intervention |
| Independence| Causal Irrelevance |
| Association  | Influence |

Table 3: Comparison of causal and associative conceptions of dependence
definition, but not in a way material to the points we wish to make.) Recall that $SE$ is the structural equations causally relating random variables. In particular,

$$X = X$$  \hspace{1cm} (3)$$

$$A = A$$  \hspace{1cm} (4)$$

$$O = s(X, A)$$  \hspace{1cm} (5)$$

where $X$ is a background variable corresponding to the actual value of sensitive attribute (the data point that changes), $X$ is the input to the system that represents this attribute, $A$ is a background variable corresponding to the actual value of the other attributes (other data points), $A$ is the input to the system that represents these other attributes, and $O$ is the output. $\Pr[O = o \mid [X \rightarrow x \mid SE]]$ is the probability that the outcome variable $O$ takes on the value $o$ given that a causal intervention set $X$ to $x$. This causal intervention differs from standard probabilistic conditioning in that it breaks correlations, preventing confounding, similar to how randomization does so in experiments. Using this notation, the point of comparison is $\Pr[O = o \mid [X \rightarrow x \mid SE]]$.

A more extreme privacy property, a probabilistic version of noninterference [16], would demand equality:

$$\Pr[O = o \mid [X \rightarrow x \mid SE]] = \Pr[O = o \mid [X \rightarrow x' \mid SE]]$$  \hspace{1cm} (6)$$

For both differential privacy and noninterference, the point of comparison is the same, but the comparison relation differs from $\cdot \leq e^{\epsilon}$ and $\cdot = \cdot$.

Since our work does not discuss the tradeoffs between various comparison relationships, we will sometimes write just the point of comparison when discussing properties. Using this shorthand, Table 4 lists and organizes six representative properties, or really representative clusters of properties. In each case, we would have to also specify the comparison relation used and under what conditions the comparison is to be done to fully specify the property, but providing just point of comparison is sufficient to see the patterns that concern us in this work.

Table 4 reveals a similarity in both stances on the notion of dependence to use for privacy and nondiscrimination. The points of comparison are associative in the left column and causal in the right. The properties in the left column minimize or bound associations. The properties in the right column minimize or bound causal effects.

The quadrant for associative privacy is unlike the others in that we show two points of comparison belonging to this cluster of properties. The first point of comparison, which we call indirect use privacy, shows the correspondence between the four quadrants more clearly by differing from its neighbors in a minimal number of ways. We could not find this point of comparison in prior work, although it is related to Pufferfish Privacy [17], differing only by using frequencies instead of credences. To help relate our four clusters to prior work, we also show statistical nondisclosure, one of the most well known associative privacy properties. This property differs from the other quadrants' points of
|                | Associative (conditioning)                                                                 | Causal (intervening)                                                                 |
|----------------|------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
| Discrimination | Disparate impact for $\bar{x}$ in population $B$:                                           | Disparate treatment on $\bar{x}$ for individuals in $B$:                             |
|                | $\text{Fr} [0 = o \mid \bar{x} = \bar{x}, SE, B]$                                         | $\text{Fr} [0 = o \mid \bar{x} \rightarrow \bar{x} | SE, B]$                         |
| Privacy        | Indirect use privacy $\text{Fr} [0 = o \mid x = \bar{x} | SE, B]$                          | Noninterference $\text{[16]}$ and differential privacy $\text{[3]}$ with secret $x$: |
|                | Statistical nondisclosure of secret $x$ and knowledge $B$ $\text{[1]}$:                   | $\text{Fr} [0 = o \mid \bar{x} \rightarrow \bar{x} | SE, B]$                         |
|                | $\text{Cr} [x = x \mid 0 = o, SE, B]$                                                     |                                                                                      |

Table 4: Key clusters of properties. For each cluster of properties, we show just one or more of its points of comparison, the probabilities whose change in value should be minimized or bounded as the underlined term changes values. In some cases, more than one pre-existing property may arise depending upon exactly how the comparison is made, exactly what is in secondary terms (e.g., $B$), and exactly under what circumstances the comparisons are made. For example, the differences between noninterference and differential privacy include that noninterference requires exact equality for any two values of $X$ whereas differential privacy requires approximate equality for only those values of $X$ that differs in a single person’s data.
comparison by using credences and by flipping around the attributes that are measured and conditioned upon. In Section 6 we take a closer look at the differences between associative privacy definitions.

Moving across the columns of the two rows of Table 4 we see that for each stance on privacy, there exists an identical stance on nondiscrimination, and vice versa. This correspondence shows the tight relationship between privacy and nondiscrimination, and the opportunity to reuse tools from one area of research in the other.

Despite being identical at the level of abstraction shown in Table 4 privacy and nondiscrimination are, of course, not the same. As mentioned, our models of discrimination and privacy only account for the core essence of some conceptions of these complex, multifaceted, and contested concepts. Additionally, the secret or sensitive attribute \( X \) will be instantiated differently for the two norms. For example, race and gender are quintessential instantiations of \( X \) for nondiscrimination, but it is harder to argue that they should be kept secret as an instantiation of \( X \) for privacy. Furthermore, the table contains statistical nondisclosure without a corresponding property for nondiscrimination.

### 4 Related Work

While ours is the first comprehensive exploration of the correspondence between causation and association, and privacy and nondiscrimination, prior work has examined some of these connections in isolation.

**Privacy and discrimination are similar.** Implicitly making use of the similarity between privacy and nondiscrimination, Dwork et al. define a notion of fairness that requires that similar people be treated similarly, and formalize this as a Lipschitz continuity requirement [18]. They point out the relationship between this notion of fairness as continuity and differential privacy: that differential privacy can be viewed as a special case of continuity for the Hamming distance metric on databases. We analyze this relationship through a causal lens and by casting both nondiscrimination and differential privacy as restrictions on the causal use of information.

In more detail, mathematically, they use a metric \( d \) on individuals that captures how similar they are with respect to the classification task. They also use a second metric \( D \) over distributions of outcomes and represent classifiers as a function \( M \) from individuals to distributions over outcomes. They require that the \((D, d)\)-Lipschitz property holds: for all pairs of individuals \( x \) and \( y \), \( D(M(x), M(y)) \leq d(x, y) \). While they leave \( d \) largely abstract since it is application specific, they mostly focus on two possibilities for \( D \): statistical distance (total variation norm) and the relative \( \ell_\infty \) metric. We focus on the second here since it is more similar to the other definitions we have considered, and to differential privacy in particular (their Section 2.3). In our notation and treating it as a causal property, this would be the requirement that for all
individuals \( x \) and \( y \) and outputs \( o \),

\[
\text{Fr}[0=o \mid \{X \to x\}SE,B] \leq e^{d(x,y)} \text{Fr}[0=o \mid \{X \to y\}SE,B]
\] (7)

The point of comparison is \( \text{Fr}[0=o \mid \{X \to x\}SE,B] \), which is of the same form as differential privacy, although here, \( x \) ranges over individuals instead of databases.

The authors object to group parity as insufficient for ensuring fairness, although they do show conditions under which their condition implies group parity (their Section 3), which are somewhat similar in goal to our Theorems 9 and 17 but rather different in form. Like us, they use the connection between privacy and nondiscrimination to transport results from privacy to nondiscrimination. While they focus on algorithmic results (their Section 5) rather different from our focus on concepts and impossibility results, they do informally consider issues (their Section 3.1, Example 3, and Section 6.3) similar to ones we discuss in our Section 8.1.3.

A difference between our work and theirs is that we focus on prohibitions against using a certain attribute \( X \) whereas they focus on a requirement to use only a certain attribute implicitly defined by the metric \( d \). While their approach is principled, it differs from current antidiscrimination that places prohibitions on using protected attributes (Section 7). One could attempt to encode each approach into the other, but much of the intuition would be lost even if successful.

**Privacy and discrimination are different.** Numerous works have instead looked at how privacy and nondiscrimination are not the same. Dwork and Mulligan write that approaching problems of discrimination with the tools of privacy might just hide the discrimination [19]. Alan and Starr provide a concrete example: the nondisclosure of criminal history appears to put pressure on employees to discriminate by race as a proxy for the missing history [20]. Strahilevitz also considers how having more information can reduce the desire to discriminate [21].

**Privacy and causation.** Tschantz et al. are the first to provide a formal correspondence between causation and information flow [22], a result with implications for security, privacy, and nondiscrimination. Others had previously noted their relationship [23–25]. This paper follows on the current authors’ recent application of a similar correspondence to differential privacy [9], which we believe is the first explicit use of causal reasoning in privacy research. This viewpoint should not be confused with works that provide algorithms for causal inference while providing differential privacy, such as Kusner et al.’s [26].

While we consider differential privacy (at least when using the centralized model of data collection) to be an instance of use privacy, the study of use privacy per se appears rather young. A recent PCAST report has called for more emphasis on the appropriate use of data, instead of banning its collection [27]. Datta et al. consider use privacy and proxies [13].
Privacy and association. The difference between associative privacy and differential (casual) privacy has received much discussion, with some arguing for associative definitions (e.g., [1, 17, 28–33]; see [34] for its antecedent in security) and others for differential ones (e.g., [2, 8]).

While associative properties have a more straightforward connection to providing inferential privacy, that is, limiting the inferences of an adversary from the data released, differential privacy can also be viewed as limiting inferences. In particular, Kasiviswanathan and Smith provide theorems showing that differential privacy implies a form of inferential privacy that limits how much more an adversary can learn with an additional row in the database [6].

Others have looked at the implications of differential privacy about what the adversary can learn from the released data in total given assumptions about the adversary’s knowledge [35] or the data [36]. Others have looked at the relationship between differential privacy and mutual information [37, 38].

Discrimination and causation. The importance of causal reasoning in nondiscrimination goes back to at least Pearl, who uses it to deal with how Simpson’s paradox can make it unclear which if either of the two groups is discriminated against [11]. More recently, Hardt et al. re-examined this issue for a broad class of nondiscrimination definitions [39, 40]. Kilbertus et al. propose a way of looking at this issue using causal reasoning about “proxy” and “resolving” variables that are either prohibited or allowed for use [41]. Other causal notions of fairness have been put forward by Kusner et al. [42], Bonchi et al. [43], and Cowgill et al. [44].

Discrimination and association. Associative notions of disparate impact have been the basis for a number of mechanisms for enforcing nondiscrimination in statistical systems [45–48]. Recently a number of richer associative notions of nondiscrimination have been proposed [40, 49, 50] that take into account three variables, group membership, predicted outcome, and true outcome, in various combinations.

We believe our connection between privacy and nondiscrimination can be extended from the simple associative notions to these more complex ones, but leave it as future work.

Contrasting definitions in other ways. In this work, we focus on which distributions privacy and nondiscrimination definitions compare, that is, on what we call the point of comparison found in the definition. Mironov instead contrasts privacy definitions based upon how different definitions use different methods of doing the comparisons [51].

5 Probabilistic notions of dependence

Before turning to privacy or nondiscrimination, we consider some general probabilistic notions that apply to both norms. In particular, we look at properties that compare probability distributions, frequentist or Bayesian,
Table 5: Summary of definitions. For each information flow property, we present a version for deterministic systems (Det.) and an approximate version for randomized systems (AR). The point of comparison is the quantity computed twice, once for two different values, and compared to check whether they are equal to one another. The check is for all pairs of values $s$ and $s'$ that can go in $s$ (or $x$ and $x'$ for $x$).

across two worlds. In this section, we define a number of such properties, and prove relationships between them. In later sections, we instantiate these properties to obtain privacy and nondiscrimination notions, as well as theorems connecting these notions.

We first examine the simple case of properties for deterministic systems, and then, in Section 5.2 we examine definitions with approximate guarantees for randomized systems. We will typically state the properties as applied to our setting, modeled by $SE$, even when the properties are more generally applicable to any system of structural equations. Table 5 summarizes the properties we consider and Figure 1 summarizes the relationships between them.

5.1 The deterministic perfect case

The first property we consider is causal irrelevance.

**Definition 4** (Pearl [11]). A system $SE = \{0=s(X,A), X=X, A=A\}$ has causal irrelevance with respect to $X$ to $0$ for $B$ iff for all $x_1$, $x_2$, and $o$,

$$\Pr[0=o \mid X \rightarrow x_1, SE, B] = \Pr[0=o \mid X \rightarrow x_2, SE, B]$$

For example, consider an experiment that randomly assigns a treatment of $x_1$ or $x_2$ to a population modeled by $B$. The causal irrelevance property states that the outcomes are identical irrespective of the treatment.

For systems with control over all inputs, one approach for achieving causal irrelevance is to enforce noninterference. Here we state a simplification of Goguen and Meseguer’s definition [10].

**Definition 5.** A function $s(X,A)$ has noninterference for $X$ iff for all $x_1$, $x_2$, and $a$,

$$s(x_1, a) = s(x_2, a)$$

Noninterference yields causal irrelevance for all backgrounds $B$. 

\begin{tabular}{|l|l|c|}
\hline
Notion & Point of comparison & Det. & AR \\
\hline
Noninterference & $s(h, l)$ & 5 & 12 \\
Associative independence & $\Pr[0=o \mid X=x, SE, B]$ & 8 & 16 \\
Associative independence & $\Pr[X=x \mid O=o, SE, B]$ & 18 \\
Causal irrelevance & $\Pr[0=o \mid X \rightarrow x, SE, B]$ & 4 & 13 \\
\hline
\end{tabular}
Figure 1: Relationships between definitions. Notable assumptions made by theorems and dilution of the privacy budget are shown as subscripts. A deterministic form of the upper associative independence definition exists, but we do not consider it.
Theorem 6. Consider a system \( SE = \{0=s(X,A), X=X, A=A\} \). If \( s \) has noninterference with respect to \( X \), then for all \( B \) it has causal irrelevance with respect to \( X \).

This theorem follows directly from the following lemma.

Lemma 7. Consider a system \( SE = \{0=s(X,A), X=X, A=A\} \). If \( s \) has noninterference with respect to \( X \), then for all \( B \) and \( x \),

\[
Pr[0\rightarrow o \mid X \rightarrow x \mid SE, B] = Pr[0\rightarrow o \mid SE, B]
\]

Proof. Assume \( s(X, A) \) has noninterference with respect to \( A \). Then,

\[
\begin{align*}
Pr[0\rightarrow o \mid X \rightarrow x \mid SE, B] &= Pr[0\rightarrow o \mid 0=s(X,A), X=x, A=A, B] \\
&= Pr[0\rightarrow o \mid 0=s(X,A), X=x, A=A, B] \\
&= Pr[0\rightarrow o \mid 0=s(X,A), X=x, A=A, B] \\
&= Pr[s(X, A)=o \mid 0=s(X,A), X=x, A=A, B] \\
&= Pr[0\rightarrow o \mid 0=s(X,A), X=X, A=A, B] \\
&= Pr[0\rightarrow o \mid SE, B]
\end{align*}
\]

The second kind of property that we consider is associative independence.

Definition 8. A system \( SE = \{0=s(X,A), X=X, A=A\} \) has associative independence with respect to \( B \) iff for each \( x_1, x_2 \), and \( o \) such that \( Pr[X=x_1 \mid SE, B] > 0 \) and \( Pr[X=x_2 \mid SE, B] > 0 \),

\[
Pr[0\rightarrow o \mid X=x_1, SE, B] = Pr[0\rightarrow o \mid X=x_2, SE, B]
\]

This is identical to stating that \( 0 \perp X \mid SE, B \), that is, \( 0 \) and \( X \) are conditionally independent with respect to \( B, SE \).

Next, we show that if an input is independent of other inputs, then the causal irrelevance of that input is equivalent to the associative independence of that input.

Theorem 9. Consider a system \( SE = \{0=s(X,A), X=X, A=A\} \). If \( A \perp X \mid B \) and for all \( x \) \( Pr[X = x \mid B] > 0 \), then \( SE \) has associative independence with respect to \( X \) for \( B \) iff \( SE \) has probabilistic causal irrelevance with respect to \( X \) for \( B \).

Proof. Follows directly from Lemma 10.

Lemma 10. Consider a system \( SE = \{0=s(X,A), X=X, A=A\} \). If \( X \perp A \mid B \), and \( Pr[X = x \mid SE, B] > 0 \),

\[
Pr[0\rightarrow o \mid X=x, SE, B] = Pr[0\rightarrow o \mid X \rightarrow x \mid SE, B]
\]
Proof. Assume that $X \perp A \mid B$. Then,

\[ \Pr[A=a \mid B] = \Pr[A=a \mid B, X=x] \]  
\[ \Pr[A=a \mid SE, B] = \Pr[A=a \mid X=x, SE, B] \] 

where (8) follows from SE-independence. Thus,

\[ \Pr[0=o \mid X=x, SE, B] = \Pr[s(x,A)=o \mid X=x, SE, B] \] 
(substituting for $X,0$)
\[ = \Pr[s(x,A)=o \mid SE, B] \] 
(from (9) above)
\[ = \Pr[s(x,A)=o \mid \{X \rightarrow x\} SE, B] \] 
(SE-independence)

\[ = \Pr[0=o \mid \{X \rightarrow x\} SE, B] \]

\[ \square \]

5.2 Making things approximate

We now consider the case of systems with internal randomness. We model such systems as \( \{0=s(x,A,R), X=X, A=A\} \), where $R$ is a background variable that represents fresh randomness.

Definition 11. A random variable $R$ with distribution $B_R$ is said to be fresh for $B$ iff for all $\varphi$ that does not reference $R$ and all $r$, $\Pr[R=r \mid B_R] = \Pr[R=r \mid \varphi, B]$.

The left hand side represents the true frequency distribution of a random variable $R$ according to $B_R$. The freshness condition can be instantiated for both frequentist and Bayesian probabilities. For frequentist probabilities, this condition can be interpreted as $R$ being uncorrelated with any other background variables. For Bayesian probabilities this condition can be interpreted as the agent knowing nothing about $R$ apart from its natural frequentist distribution $B_R$. Since $R$ has the same distribution under $B_R$ and $B$ when it is fresh, we will not mention $B_R$ when $R$ is fresh for $B$.

Also, instead of requiring equalities, we allow the probabilities above to be approximately equal, arriving at a version of differential privacy for functions.

Definition 12. A function $s(X,A,R)$ has $\epsilon$-noninterference for $X$ given a distribution $B$ over $R$ if for all $x_1, x_2$, and $a$,

\[ \Pr[s(x_1,a,R)=o \mid B] \leq e^\epsilon \Pr[s(x_2,a,R)=o \mid B] \]

Note that the only role $B$ plays in Definition 12 is assigning a probability distribution to the randomization within $s$ provided by $R$.

Definition 13. For a system $SE = \{0=s(x,A,R), X=X, A=A\}$, $X$ has $\epsilon$-probabilistic causal irrelevance for $O$ with respect to $B$ if for all $x_1, x_2$, and $o$,

\[ \Pr[0=o \mid \{X \rightarrow x_1\} SE, B] \leq e^\epsilon \Pr[0=o \mid \{X \rightarrow x_2\} SE, B] \]
Theorem 14. If $s(X, A, R)$ has $\epsilon$-noninterference, then for each $B$ such that $R$ is fresh, $X$ has $\epsilon$-probabilistic causal irrelevance for $O$ with respect to $B$, given $SE = \{0=s(X, A, R), X=X, A=A\}$.

Proof. Assume $s$ has $\epsilon$-noninterference. Therefore,
\[
\Pr[0=0 | X\rightarrow x_1]SE, B] \\
= \sum_a \Pr[0=0 | A=a, X\rightarrow x_1]SE, B] \Pr[A=a | X\rightarrow x_1]SE, B] \\
= \sum_a \Pr[0=0 | A=a, X\rightarrow x_1]SE, B] \Pr[A=a | B] \quad \text{(exogeneity)} \\
= \sum_a \Pr[s(x_1, a, R)=0 | A=a, X\rightarrow x_1]SE, B] \Pr[A=a | B] \\
= \sum_a \Pr[s(x_1, a, R)=0 | A=a, B] \Pr[A=a | B] \quad \text{(SE-independence)} \\
= \sum_a \Pr[s(x_1, a, R)=0 | B] \Pr[A=a | B] \quad \text{(R fresh)} \\
\leq \sum_a \epsilon \Pr[s(x_2, a, R)=0 | B] \Pr[A=a | B] \quad \text{(noninterference of $R$ and $B$)} \\
= \epsilon \sum_a \Pr[s(x_2, a, R)=0 | A=a, B] \Pr[A=a | B] \quad \text{(R fresh)} \\
= \epsilon \sum_a \Pr[s(x_2, a, R)=0 | A=a, X\rightarrow x_1]SE, B] \Pr[A=a | B] \\
= \epsilon \sum_a \Pr[s(x_2, a, R)=0 | A=a, X\rightarrow x_2]SE, B] \Pr[A=a | B] \\
= \epsilon \Pr[0=0 | X\rightarrow x_2]SE, B] \]

Corollary 15. If $s(X, A, R)$ has $\epsilon$-noninterference, then for all $B$ such that $R$ is fresh, for all $x$ and $o$,
\[
\Pr[0=0 | SE, B] \leq \epsilon \Pr[0=0 | X\rightarrow x]SE, B], \text{ and} \\
\Pr[0=0 | X\rightarrow x]SE, B] \leq \epsilon \Pr[0=0 | SE, B]
\]
Proof.

\[ \Pr[O=o \mid SE, B] = \sum_{x'} \Pr[O=o \mid X=x', SE, B] \Pr[X=x' \mid SE, B] \]

\[ = \sum_{x'} \Pr[O=o \mid X=x', [X \to x']SE, B] \Pr[X=x' \mid [X \to x']SE, B] \]

(substituting for \( X \))

\[ \leq e^\epsilon \sum_{x} \Pr[O=o \mid X=x', [X \to x]SE, B] \Pr[X=x' \mid SE, B] \quad \text{(Theorem 14)} \]

\[ = e^\epsilon \Pr[O=o \mid X=x]SE, B] \]

The other direction follows similarly.

Definition 16. A system \( SE = \{0=s(x, a, R), x=X, a=A\} \) has \( \epsilon \)-probabilistic associative independence on \( 0 \) with respect to \( X \) for \( B \) iff for each \( x_1, x_2 \) and \( o \) such that \( \Pr[X=x_1 \mid SE, B] > 0 \) and \( \Pr[X=x_2 \mid SE, B] > 0 \),

\[ \Pr[O=o \mid X=x_1, SE, B] \leq e^\epsilon \Pr[0=o \mid X=x_2, SE, B] \]

\( \epsilon \)-probabilistic associative independence coincides with causal irrelevance when the sensitive input is independent of other inputs.

Theorem 17. Consider a system \( SE = \{0=s(x, a, R), x=X, a=A\} \). If \( X \perp A \mid B \), and \( R \) is fresh, then, for \( SE \), \( X \) has \( \epsilon \)-probabilistic associative independence for \( O \) with respect to \( B \) iff \( X \) has \( \epsilon \)-probabilistic causal irrelevance for \( O \) with respect to \( B \).

Proof. This follows directly from Lemma 10.

Finally, we show that the two forms of associative independence considered in Table 5 are not that different. Approximate associative notions of dependence may relax independence in a number of different ways. The form of approximate associative independence in Definition 16 is employed by the Pufferfish privacy formalism [17, 29]. Statistical nondisclosure is a different form that for Bayesian probabilities compares posterior beliefs to prior beliefs [1]. An approximate form dropping the requirement of Bayesian probabilities follows:

Definition 18. A system \( SE = \{0=s(x, a, R), x=X, a=A\} \) has \( \epsilon \)-probabilistic associative independence on \( X \) with respect to \( O \) for \( B \) iff for each \( x_1, x_2 \), and \( o \) such that \( \Pr[O=o \mid SE, B] > 0 \),

\[ \Pr[X=x \mid O=o, SE, B] \leq e^\epsilon \Pr[X=x \mid SE, B] \quad \text{and} \quad (10) \]

\[ \Pr[X=x \mid SE, B] \leq e^\epsilon \Pr[X=x \mid O=o, SE, B] \quad \text{(11)} \]
We show here that the two formulations are very closely related. Since the structural equations $SE$, and background $B$ don’t change throughout this section, we elide them from the statements in the rest of this section.

**Theorem 19.** If for all $o, x_1$, and $x_2$, such that $\Pr[X=x_1 \mid SE, B] > 0$ and $\Pr[X=x_2 \mid SE, B] > 0$,

$$\Pr(O=o \mid X=x_1, SE, B) \leq e^\epsilon \Pr(O=o \mid X=x_2, SE, B)$$

then, for all $o$ and $x$ such that $\Pr(O=o \mid SE, B] > 0$,

$$\Pr[X=x \mid O=o, SE, B] \leq e^\epsilon \Pr[X=x \mid SE, B]$$

and

$$\Pr[X=x \mid SE, B] \leq e^\epsilon \Pr[X=x \mid O=o, SE, B]$$

**Proof.** Assume for all $o, x_1$, and $x_2$ such that $\Pr[X=x_1 \mid SE, B] > 0$ and $\Pr[X=x_2 \mid SE, B] > 0$,

$$\Pr(O=o \mid X=x_1, SE, B) \leq e^\epsilon \Pr(O=o \mid X=x_2, SE, B)$$

This implies that, for all $o, x_1$, and $x_2$ such that $\Pr[X=x_1 \mid SE, B] > 0$ and $\Pr[X=x_2 \mid SE, B] > 0$,

$$\Pr(O=o \mid X=x_1, SE, B) \Pr[X=x_2 \mid SE, B] \Pr[X=x_2 \mid SE, B] \leq e^\epsilon \Pr(O=o \mid X=x_2, SE, B) \Pr[X=x_2 \mid SE, B]$$

Since (12) holds for all $x_2$ such that $\Pr[X=x_2 \mid SE, B] > 0$, the following must also hold, for all $o$ and $x_1$ such that $\Pr[X=x_1 \mid SE, B] > 0$:

$$\sum_{x_2 \in \mathcal{X}^\prime} \Pr(O=o \mid X=x_1, SE, B) \Pr[X=x_2 \mid SE, B] \leq e^\epsilon \sum_{x_2 \in \mathcal{X}^\prime} \Pr(O=o \mid X=x_2, SE, B) \Pr[X=x_2 \mid SE, B]$$

$$\Pr(O=o \mid X=x_1, SE, B) \sum_{x_2 \in \mathcal{X}^\prime} \Pr[X=x_2 \mid SE, B] \leq e^\epsilon \sum_{x_2 \in \mathcal{X}} \Pr(O=o \mid X=x_2, SE, B) \Pr[X=x_2 \mid SE, B]$$

where $\mathcal{X}_2$ is the range of $X$ and $\mathcal{X}^\prime$ is the support of $X$ given $SE$ and $B$: $\{x \in \mathcal{X} \mid \Pr[X=x \mid SE, B] > 0\}$. From this it follows that for all $o$ and $x_1$ such that $\Pr(O=o \mid SE, B] > 0$ and $\Pr[X=x_1 \mid SE, B] > 0$:

$$\frac{\Pr(O=o \mid X=x_1, SE, B) \Pr[X=x_1 \mid SE, B]}{\Pr(O=o \mid SE, B]} \leq e^\epsilon \Pr[X=x_1 \mid SE, B]$$

$$\Pr[X = x_1 \mid O=o, SE, B] \leq e^\epsilon \Pr[X=x_1 \mid SE, B]$$

In the case where $\Pr[X=x_1 \mid SE, B] = 0$, the final inequality also holds since both sides are 0.
Similarly, we can show that
\[
\Pr[X = x_1 | SE, B] \leq e^\epsilon \Pr[X = x_1 | O = o, SE, B]
\]

**Theorem 20.** If for all \(o\) and \(x\) such that \(\Pr[O = o | SE, B] > 0\),
\[
\Pr[X = x | O = o, SE, B] \leq e^\epsilon \Pr[X = x | SE, B]
\]
and
\[
\Pr[X = x | SE, B] \leq e^\epsilon \Pr[X = x | O = o, SE, B]
\]
then, for all \(o\), \(x_1\), and \(x_2\), such that \(\Pr[X = x_1 | SE, B] > 0\) and \(\Pr[X = x_2 | SE, B] > 0\),
\[
\Pr[O = o | X = x_1, SE, B] \leq e^{2\epsilon} \Pr[O = o | X = x_2, SE, B]
\]

**Proof.** Assume for all \(o\) and \(x\) such that \(\Pr[O = o | SE, B] > 0\),
\[
\Pr[X = x | O = o, SE, B] \leq e^\epsilon \Pr[X = x | SE, B]
\]
From this and Bayes’s Rule, it follows that for all \(o\) and \(x\) such that \(\Pr[O = o | SE, B] > 0\) and \(\Pr[X = x | SE, B] > 0\),
\[
\frac{\Pr[O = o | X = x, SE, B] \Pr[X = x | SE, B]}{\Pr[O = o | SE, B]} \leq e^\epsilon \Pr[X = x | SE, B]
\]
\[
\Pr[O = o | X = x, SE, B] \leq e^\epsilon \Pr[O = o | SE, B]
\]
The last line also holds in the case where \(\Pr[O = o | SE, B] > 0\) since both sides will be zero.
Similarly, we can show that for all \(o\) and \(x\) such that \(\Pr[X = x | SE, B] > 0\),
\[
\Pr[O = o | SE, B] \leq e^\epsilon \Pr[O = o | X = x, SE, B]
\]
Putting the two together, we get for all \(o\), \(x_1\), and \(x_2\) such that \(\Pr[X = x_1 | SE, B] > 0\) and \(\Pr[X = x_2 | SE, B] > 0\),
\[
\Pr[O = o | X = x_1, SE, B] \leq e^\epsilon \Pr[O = o | SE, B] \leq e^{2\epsilon} \Pr[O = o | X = x_2, SE, B]
\]

## 6 Privacy

We now instantiate the definitions above to obtain various definitions of privacy reported in prior work. We will organize this section around the two types of privacy definitions found in Table 3. For each type, we will first discuss its general nature and then relate it to the most well known formal notion of privacy, differential privacy.
6.1 Direct Use Privacy: Frequentist Causal Irrelevance

The first kind of privacy requirements we examine is direct use privacy, which requires that some data be not used to produce some output. One way of formalizing non-use is Pearl’s notion of causal irrelevance, which holds if the outcomes over a population represented by \( B \) are identical when the sensitive input is intervened on. Since in this section we will be using more than one type of background distribution \( B \), we will denote ones modeling populations as \( B_{P} \). This view is the basis for Tschantz et al.’s prior work on inferring information use by detecting causation [22]. Their work started with noninterference, the classic security property formalizing information flow [16], which is mathematically equivalent to direct use privacy despite the difference in application. It showed that noninterference is equivalent to Pearl’s notion of causation. That is, absolute direct use privacy holds iff for all \( o, x_1, x_2, \) and \( B_{P} \),

\[
\text{Fr}[0 \rightarrow o \mid X \rightarrow x_1]SE, B_{P} = \text{Fr}[0 \rightarrow o \mid X \rightarrow x_2]SE, B_{P}
\]

(13)

where \( SE = \{0 = s(X, A), X = X, A = A\} \) as usual. Differential privacy can be understood as a relaxation of this property, which we’ll cover in detail below.

**Differential Privacy.** Differential privacy is a relaxed form of direct use privacy. It can be expressed as either \( \epsilon \)-noninterference or \( \epsilon \)-causal irrelevance for database rows. For the setting of database privacy, where a function operates on a database comprising a set of rows, differential privacy requires that the distribution over outcomes is not significantly affected by the value of a single row in the database.

**Definition 21.** A system \( s(d, R) \), operating over a data set \( d \) has \( \epsilon \)-differential privacy iff for all rows \( i \), databases \( d \), and row values \( x, x' \), and outputs \( o \)

\[
\text{Fr}[s(d_{-i}x, R) = o] \leq e^{\epsilon} \text{Fr}[s(d_{-i}x', R)]
\]

In the definition above, for a database \( d \), \( d_{-i} \) refers to the remainder of the database with the row indexed by \( i \) left out, and \( d_{-i}x \) refers to the database \( d \) with the value \( x \) in row \( i \). We allow a special value \( \perp \) for \( x \) or \( x' \) where \( d_{-i} \perp \) denotes the database without replacing the \( i \)th entry. (This is the so-call *bounded* formulation of differential privacy.) In the notation we used before, \( d_{-i} \) can be thought of as the value of \( A \) and \( x \) as the value of \( X \) with the split of the data points into \( A \) and \( X \) varying with the value of \( i \). \( R \), as before, represents the randomness used by \( s \). That is, differential privacy is \( \epsilon \)-noninterference with respect to each row in the database. It means that intervening on any row does not affect the distribution over outcomes significantly.

**Proposition 22.** A function \( s(D, R) \), where \( D = (D_1, \ldots, D_k) \), has \( \epsilon \)-differential privacy iff for each \( i, s(D, R) \) has \( \epsilon \)-noninterference with respect to \( D_i \).
This equivalence follows directly from the definitions. Moreover, we can express differential privacy as causal irrelevance:

**Proposition 23.** Consider a system defined by the structural equations $SE = \{O = s(D, R), A = A, D_1 = D_1, \ldots, D_k = D_k\}$. The function $s$ has $\epsilon$-differential privacy iff for all $i$ and all $x_1$ and $x_2$, and for all $B_P$,

$$Fr[s(D, R) | [D_i \rightarrow x_1]SE, B_P] = e^\epsilon Fr[s(D, R) | [D_i \rightarrow x_2]SE, B_P]$$

where we use $D$ as short for $\langle D_1, \ldots, D_k \rangle$.

This equivalence is the corollary of Theorem 14. Tschantz et al. provide details [9].

### 6.2 Associative Inferential Privacy: Bayesian Associative Independence

Associative inferential privacy tries to capture the change in the beliefs of an adversary after observing the outcome of a program. This notion is well captured by measuring the difference in the Bayesian distributions representing the adversary’s beliefs with and without conditioning on the outcome. That is, if $\varphi$ is the proposition of interest that the adversary is attempting to learn about, we compare $Cr[\varphi | O = o, B_K]$ to $Cr[\varphi | B_K]$ where $B_K$ is a background distribution representing the adversary’s background knowledge about background variables. Given that the definition we consider in this section does not use causal interventions, $B_K$ can more generally refer to any background knowledge. In the absolute case, we require that

$$Cr[\varphi | O = o, B_K] = Cr[\varphi | B_K]$$

This concept goes back to at least Dalenius in 1977 as statistical nondisclosure [1]. The definition looks more familiar if we consider the special case where $\varphi$ is $X = x$:

$$Cr[X = x | O = o, B_K] = Cr[X = x | B_K]$$

This requirement is closely related to a similar special case Pufferfish privacy [17, 29]:

$$Cr[0 = o | X = x_1, SE, B_K] = Cr[0 = o | X = x_2, SE, B_K]$$

Section 5.2 shows approximate forms of these requirements that, similar to differential privacy, allows them to differ by factors of $e^\epsilon$. The section also shows that they are nearly equivalent, with the relaxation of (16) implying the relaxation of (15) and the relaxation of (15) implying the relaxation of (16) with the privacy budget diluted from $\epsilon$ to $2\epsilon$.

Other approaches to measuring an associative difference include measuring a notion of accuracy [52] or a difference in beliefs over the runs of a system [54].
Such associative inferential requirements are hard to meet. Typically, we would not know the background knowledge of the adversary. In the worst case, we might need to consider any background knowledge as possible. Dwork and Naor have shown that it is impossible, in general, to both satisfy this requirement for all background knowledge sets \( B_K \) and release an output \( O \) providing utility [12]. In Section 8.1, we review and re-present Dwork and Naor’s proof.

**Differential Privacy implies Associative Inferential Privacy for Independent Data Points.** One way to get around the aforementioned impossibility result is to attain the property for a restricted set of background knowledge. If we require that the data points be independent under the adversary’s background knowledge, then differential privacy implies associative inferential privacy. We can now use the results proved in Section 5.2 for differential privacy.

The implication states that for background knowledge such that each row is independent, conditioning on the value of a row does not change the beliefs about the outcomes. The requirement for independence to obtain such an epistemic guarantee has been pointed out in prior work [17, 29], and follows directly from Theorem 14 and Theorem 17.

**Proposition 24.** Consider a system \( SE = \{D \rightarrow s(D, R), D = D, A = A\} \) and \( D = \langle D_1, \cdots, D_k \rangle \), and \( D = \langle D_1, \cdots, D_k \rangle \). If \( D_i \perp D_{-i}|B_K \) for all \( i \), and \( R \) is fresh, then for all \( i, d_i, d'_i, \) and \( o \),

\[
Cr[O = o | D_i = d_i, SE, B_K] \leq e^\epsilon Cr[O = o | D_i = d'_i, SE, B_K]
\]

if \( s \) is \( \epsilon \)-differentially private.

Another reasonable restriction on background knowledge is to tie the background knowledge to an objective population frequency distribution: the distribution that captures the associations known about a population or set of populations. This approach has its own challenges that we discuss below.

### 6.3 Indirect Use Privacy: Frequentist Associative Independence

Above, we considered a frequentist causal property and a Bayesian associative property. Here, we consider the possibility of a frequentist associative notion of privacy. Let *indirect use privacy* be the requirement that the output is not statistically associated with the sensitive attribute \( X \). For a population \( B_P \), absolute indirect use privacy holds iff for all \( o, x_1, \) and \( x_2, \)

\[
Fr[O = o | X = x_1, SE, B_P] = Fr[O = o | X = x_2, SE, B_P]
\]

where \( SE = \{0 = s, A, X = X, A = A\} \) as usual. This property can be understood as a more objective version of associative inferential privacy. That notion of
privacy is subjective in the sense that it depends upon the state of knowledge of an adversary. This definition replaces that background knowledge $B_K$ with the actual population $B_P$ as it switches from credences to frequencies, making it more objective by instead referring to the state of the outside world. Furthermore, this requirement will imply associative inferential privacy for adversaries whose background knowledge is limited to knowing the population.

Indirect use privacy prevents the release of any output that is associated with the sensitive attribute, even if that release is not caused by the sensitive attribute. For example, suppose the sensitive attribute $X$ is whether someone has cancer. Indirect use privacy would prevent the release of whether someone smokes, which is typically public knowledge, as the output $O$ since it is associated with getting cancer.

Generally, the population $B_P$ might not be completely known to whoever designs the system. Guaranteeing indirect use privacy, in this case, requires meeting the definition for all populations $B_P$ that the designer believes to be possible. In the extreme case, this would require protecting the information for all populations $B_P$, leading to the same impossibility result as for associative inferential privacy.

Perhaps due to how strong this requirement is, we do not know of any work that attempts to achieve this requirement for a comprehensive class of populations $B_P$. However, some prior works can be viewed as approximations of this requirement. We will discuss two such approximations, one for privacy in general and one related to differential privacy in particular.

**Proxies.** Prior work has prohibited the direct use of proxies, which are variables statistically associated with the sensitive attribute $X$ (e.g., [54]), for example ZIP Codes and race. Directly using a variable that is associated with $X$ is required to violate indirect use privacy assuming that the system cannot predict the value of $X$ a priori. However, proxies may be used as inputs without introducing associations between the outcomes and sensitive attributes, such as the use of ZIP Code to target geographic locations that may have heterogeneous racial demographics. Identifying the use of proxies in a system exposes points at which normative judgments can be made about the acceptable use of proxies.

**Inferential Guarantees from Differential Privacy.** Ghosh and Kleinberg reason about what an adversary could learn from a differentially private data release when the adversary’s knowledge is characterized by the correlations in the underlying population [14]. In our notation, this corresponds to assuming that $B_K$ is determined by $B_P$. They show that the amount of associative inferential privacy lost is bounded when the correlations in $B_P$ are bounded in particular ways.

**Definitions Going Beyond Differential Privacy.** Recall that Proposition 24 showing that differential privacy implies a form of associative inferential privacy under the pre-condition that the data points are independent...
under the adversary’s background knowledge $B_K$. That pre-condition is not optional: numerous works have noted that the output of a differentially private function can allow an adversary to draw an inference about a single data point due to output still being strongly associated (more so than by a factor of $e^\epsilon$) with the data point (e.g., [17, 28–33]). Some of these authors have responded by proposing definitions that are similar to differential privacy but requiring that a larger set of databases are treated as neighboring [17, 29, 31–33]. This means that the algorithm will have to produce nearly identical (within a factor of $e^\epsilon$) distributions over outputs for more pairs of databases. Intuitively, these additional pairs of databases are ones that differ not in the data point $D_i$ currently under consideration, but rather other data points $D_j$ that are closely associated with $D_i$. By requiring the output to not depend much upon these associates, such definitions are similar to proxy prohibitions where the correlated data points $D_j$ are treated as the prohibited proxies. By prohibiting their use, these definitions attempt to avoid a strong association between the output and a single data point. Between this motivation and being implemented as a restriction over outputs, these definitions are similar to indirect use privacy in the same sense as proxy prohibitions are similar.

### 6.4 Causal Inferential Privacy

We have seen Bayesian associative notions of privacy and frequentist causal ones. This raises the question of what Bayesian causal ones exist. Intuitively, such causal epistemic properties require that intervening on the sensitive attribute does not change significantly an observer’s beliefs about any proposition $\varphi$ about the background.

A trivial way of getting such a requirement is to replace the frequencies in the causal notion of privacy found in Section 6.1 (13) with Bayesian probabilities. Given Pearl’s preference for using Bayesian probabilities [11], such a replace would bring our causal notion into closer correspondence to his model of causation.

However, doing such a wholesale replace ignores the distinction between probabilities measuring properties of a population $B_P$ and properties of the adversary’s background knowledge $B_K$. A more nuanced approach would use both forms of probabilities, combining frequencies, credences, causal interventions. We leave this effort to future work.

**Semantic (Differential) Privacy.** Kasiviswanathan and Smith provide a “semantic” version of differential privacy, which they call *semantic privacy* [6]. It requires that the probability that the adversary assigns to the all input data points does not change much whether an individual $i$ submits data or not. While they did not express their definition in terms of causation, we conjecture that this definition could be expressed in such terms, using a combination of frequencies and credences. Indeed, while they do not formally distinguish between frequencies and credences, their notation suggests such a distinction as they classify some probabilities as coming from the algorithm (their Pr)
and others from the adversary’s beliefs (their \( b \)). Kasiviswanathan and Smith prove that differential privacy and semantic (differential) privacy are closely related [6, Thm. 2.2], and such a result should also apply to our causal view.

7 Nondiscrimination

Similar properties to the ones discussed above appear as nondiscrimination properties. For example, group parity, a nondiscrimination property that the frequency of individuals hired from a protected group be similar to the frequency of individuals hired from the rest of the population, is an associative notion. Furthermore, the use of a protected attribute such as race or gender can be viewed as a causal property, and is an important part of the concept of disparate treatment.

Much as differential privacy served as a point of reference for understanding the forms of privacy properties, we use the U.S. legal notions of disparate treatment and disparate impact here. However, whereas, differential privacy fit squarely under one of our forms of privacy, the situation is more complex here due to the complexity of the law.

7.1 Direct Nondiscrimination: Frequentist Causal Irrelevance

The first kind of nondiscrimination requirements we examine is direct nondiscrimination, which requires that some protected attribute not be used to produce some output. As with direct use privacy, we formalize non-use as Pearl’s notion of causal irrelevance. That is, absolute direct nondiscrimination holds iff for all \( o, x_1, x_2, \) and \( B_P \),

\[
Fr[0=o \mid [X\rightarrow x_1]SE, B_P] = Fr[0=o \mid [X\rightarrow x_2]SE, B_P]
\]

where \( SE = \{0=s(X,A), X=X, A=A\} \) as usual. This property is identical absolute direct use privacy. However, for privacy, the variable \( X \) would intuitively be some attribute kept private, perhaps a medical diagnosis, whereas for nondiscrimination, the variable \( X \) would intuitively be some protected attribute. For example, \( X \) could be gender or race, which are typically publicly known.

Disparate Treatment. The above formulation is related to the legal notion of disparate impact. In U.S. law, Section 703 of Title VII of the Civil Rights Act of 1964 (codified as 42 U.S.C. §2000e-2) contains

(a) It shall be an unlawful employment practice for an employer –

(1) to fail or refuse to hire or to discharge any individual, or otherwise to discriminate against any individual with respect to his compensation, terms, conditions, or privileges of
employment, because of such individual’s race, color, religion, sex, or national origin; or […]

Disparate treatment is one way in which this provision could be violated. The other is *disparate impact*, which we turn to next. The U.S. Supreme Court has described *disparate treatment* as follows [Footnote 15]:

“Disparate treatment” such as is alleged in the present case is the most easily understood type of discrimination. The employer simply treats some people less favorably than others because of their race, color, religion, sex, or national origin. Proof of discriminatory motive is critical, although it can in some situations be inferred from the mere fact of differences in treatment.

This definition leaves some ambiguity about whether unintentional but causal discrimination counts as disparate treatment. For example, suppose an employer harbors a subconscious basis against women that causes him to not hire them despite being unaware of it. Arguably, sex would be the cause of less favorable treatment, but without intent.

We suspect that such ambiguity has not posed problems in legal cases for two reasons. First, proving, outside of a laboratory setting, the existence of such a subconscious bias and its role in decision making is so difficult that such cases are unlikely to arise due to a lack of evidence. Second, these difficulties do not apply to showing disparate impact, providing plaintiffs with an easier alternative course of action in such cases, which means that few would feel tempted to push the envelope on what counts as disparate treatment.

Turning to automated systems, this ambiguity may need to be addressed. Some have argued that machines can, in a sense, have intent (e.g., [56]), but, arguably, they cannot. With this in mind, perhaps, the notion of disparate treatment is best limited to humans. However, we are interested in the case where a system uses protected information about a person to select an outcome, that is, systems in which the protected output causes the outcome. Such cases are violations of direct nondiscrimination and would be illegal under the quotes above given a causal interpretation of the word *because* found in them both. While this interpretation loses some of the nuances of intent, we believe it maintains the essence of the stance that gives raise to it: no one should be treated poorly as an effect their protected attributes.

Under this view, at a high level, showing disparate treatment requires establishing direct discrimination. Experimental methods such as randomized controlled experiments and situation testing over populations can establish such causation.

In practice, disparate treatment cases look little like scientific studies establishing causation for two reasons. First, running such experiments over actual workplaces is typically impossible except in limited circumstances (see, e.g., [57]). Second, winning a disparate treatment case is more complex than just showing direct discrimination. For one, nondiscrimination laws only cover certain entities making certain types of decisions. Further complicating
matters, exceptions exist. For example, a gym will typically be able to defend hiring only females to attend its women’s locker room as a *bona fide occupational qualification*, which permits an exception to the prohibitions against employment discrimination. Furthermore, showing direct discrimination yields little reward in *mixed-motives* cases. For example, the defendant may show that even if it had not discriminated against the plaintiff, the plaintiff would still not have been hired for some other legal reason, such as lacking a required quantification. In such cases, under U.S. law, the defendant does not receive damages (42 U.S.C. §2000e-5(g)(2)(B)).

For these reasons, legal formulations and case law surrounding disparate treatment centers around discrimination against individuals [58, Chapter 10.1], where a plaintiff claims that a protected attribute was a sole reason or motivating factor for some adverse action against them and had it not been for the protected attribute the adverse action would not have happened. (Meanwhile, the defendant typically claims that this is not the case and offers other lawful reasons that supposedly motivated the adverse action.) Showing that the adverse action would not have happened to a particular plaintiff (not just a claim about a population of plaintiffs) means showing not just causation but the more complex property called *actual causation*. (What we are calling causation is more precisely known as *type causation* when contrasted with actual causation.) Actual causation is formally identified using analytical methods (e.g., [59]). Practically, this relies on indirect ways of establishing causation such as verbal claims or inconsistencies in applying rules [60].

For these reasons, we only claim that the core essence of disparate treatment is reduced to a question of causation.

### 7.2 Indirect Discrimination: Frequentist Associative Independence

Another form of discrimination happens when a policy or system has disproportionate impacts upon one group by lacking group parity. We call this *indirect discrimination*. It is defined in a similar manner as *indirect use privacy*: the output is not statistically associated with the sensitive attribute $X$. Absolute indirect nondiscrimination holds iff there is frequentist associative independence: for all $o$, $x_1$, $x_2$, and $B_P$,

$$\text{Fr}[0=o \mid x=x_1, SE, B_P] = \text{Fr}[0=o \mid x=x_2, SE, B_P]$$

where $SE = \{0=s(X, A), X=X, A=A\}$ as usual. As before, $X$ is intuitively a protected attribute, such as gender or race, instead of a private attribute.

Preventing indirect discrimination (ensuring group parity) is the basis for a number technical approaches for nondiscrimination in machine learning [47, 48, 61, 62].

**Disparate Impact.** Whereas the disparate treatment form of illegal discrimination looked at employment practices that directly treat protected
groups differently from others, disparate impact is a form of illegal discrimination that can arise when even facially neutral policies lead to indirect discrimination. In employment law, the 80 percent rule \cite{10} is used to test whether an employer’s selection system has a disproportionate impact on protected groups by measuring the ratio of positive outcomes across groups. If the ratio is less than 0.8, then the employer may be called upon to explain this disparity.

**Proposition 25.** The 80 percent rule is satisfied by a system iff it has $\epsilon$-frequentist associative independence for protected groups, where $e^{-\epsilon} = 0.8$.

Failing the 80 percent rule does not, in itself, mean that the employer will be found liable for disparate impact. As with disparate treatment, only certain entities are covered and exceptions exist. For example, the employer will not be held liable if the practice with a disparate impact is a *business necessity*. Thus, similarly, we do not claim that all of disparate impact can be reduced to a statistical test, but its core concept can be.

8 Results for Free

Having recognized the relationship between privacy and nondiscrimination, we get theorems and methods from one for the other at no additional cost. This motivates the case for the exchange of techniques between the communities studying the two independently. In particular, the Dwork–Naor impossibility result for statistical disclosure \cite{12} translates to an impossibility result for fairness that states that there will always exist a subpopulation for which a system violates group parity. Further, we identify techniques in probabilistic program analysis and data privacy geared towards minimizing quantitative information flow that can be used to enforce nondiscrimination. Finally, we mention ideas for generalizing the definitions found in this work that can apply to either value.

8.1 Dwork–Naor’s Impossibility Result

Dwork and Naor showed the impossibility of statistical nondisclosure (associative inferential privacy) when releasing useful outputs \cite{12}. That theorem will carry over for discrimination.

They write \cite{12} p. 1:

The intuition behind the proof of impossibility is captured by the following parable. Suppose one’s exact height were considered a sensitive piece of information, and that revealing the exact height of an individual were a privacy breach. Assume that the database yields the average heights of women of different nationalities. An adversary who has access to the statistical database and the auxiliary information “Terry Gross is two inches shorter than the
average Lithuanian woman” learns Terry Gross’ height, while anyone learning only the auxiliary information, without access to the average heights, learns relatively little.

Generalizing this parable to sensitive conditions other than Terry Gross’ height and to computations other than the average, requires characterizing the sensitive conditions, computations, and adversaries for which such reasoning holds. Dwork and Naor do so for a fairly general characterization. We will do so for a narrower but simpler characterization, which allows making the points we wish to make more straightforward.

To understand our characterization, first note that the reasoning in the parable requires that the sensitive condition is not already known to the adversary. So a straightforward way of generalizing it would be

For all informative computations $s$, outputs $o$, and sensitive conditions $\varphi$, there exists some background condition $B$ such that $\varphi$ is not known from $B$ and $SE$, but $\varphi$ will become known upon seeing that the output is $o$.

The auxiliary information that “Terry Gross is two inches shorter than the average Lithuanian woman” can be generalized to $O=o$ implies $\varphi$ (or $\neg\varphi$ if $\varphi$ is actually false). However, this does not hold. The reason is that such auxiliary knowledge along with $SE$ (which is assumed to be known by all) can alone imply that $\varphi$ holds. One reason this is possible is that $\varphi$ can just follow from $SE$. Alternatively, $O=o$ might follow from $SE$. More complicatedly, $\varphi$ might be known to hold for every output other than $o$ from $SE$. In this case, the auxiliary information covers the only case where $\varphi$ was not known to hold, implying that $\varphi$ holds without needing to see the output. (Appendix A provides such a case.)

Rather than consider the possibility that some more complex auxiliary information could deal with this case, we instead prove a more restricted theorem. Intuitively, it is

For all informative computations $s$, outputs $o$, and sensitive conditions $\varphi$ whose truth or falsehood is not simply known from any of the outputs, there exists some auxiliary information $B$ such that $\varphi$ is not known from $B$ and $SE$, but $\varphi$ will become known upon seeing that the output is $o$.

Since our focus is showing how privacy and nondiscrimination are related, rather prove this result for just an epistemic notion of probability, we provide a generic proof working for frequentist or Bayesian probabilities.

To do so, we generalize auxiliary information to be any background condition. We also introduce the terms closed and open to generalize known and unknown. Section S.1.1 provides the generic results while Sections S.1.2 and S.1.3 discuss how to interpret these results for privacy and nondiscrimination.

8.1.1 Generic Results

We start by making the concepts mentioned above precise.
We again use $B$ for background information, but since we do not use causal interventions for these results, $B$ is not limited background variables. Let $\varphi$ be closed for $B$ iff $B$ is consistent and $\Pr[\varphi \mid B]$ is 0 or 1. Since we are working with discrete probabilities, this is the same as $B$ either proving or disproving $\varphi$.

If $\varphi$ is closed for $B$, then $\neg\varphi$ is closed for $B$.

Let $\varphi$ be open for $B$ iff $B$ is consistent and $0 < \Pr[\varphi \mid B] < 1$. If $\varphi$ is open for $B$, then $\neg\varphi$ is open for $B$. If $\varphi$ is open for $B$, then $\{\varphi, B\}$ is consistent since $\Pr[\varphi \mid SE] > 0$.

For consistent $B$, $\varphi$ is either open for $B$ or closed for $B$, but not both.

We say that a system is uninformative if there exists an output $o$ such that $\Pr[0=o \mid SE] = 1$. In this case, there is no point to observing its output since its output was already known from $SE$. A system might be uninformative because it is a constant function or because only a single input pair is possible: $\Pr[X=x \land A=a \mid SE] = 1$ for some $x$ and $a$. We say the system is informative if it is not uninformative: for all $o$, $\Pr[0=o \mid SE] < 1$.

We say that a system $s$ can trivially close $\varphi$ given $SE$ iff $\varphi$ is open for $SE$, but $s$ can produce an output $o$ that shows that either $\varphi$ or $\neg\varphi$ holds. That is, for some $o$, $\varphi$ is open for $SE$ but closed for $\{0=o, SE\}$. Even without considering the background information of an adversary, such a system closes $\varphi$ for at least one output.

**Theorem 26.** Consider $SE = \{0=s(X,A), X=X, A=A\}$. For all conditions $\varphi$, one of the following is true: $\varphi$ is closed for $SE$, $s$ is uninformative, $s$ can trivially close $\varphi$ given $SE$, or for all $o$, $\varphi$ is open for $\{0\neq o \lor \varphi, SE\}$ but closed for $\{0=o, 0\neq o \lor \varphi, SE\}$.

This result follows from a series of lemmas, starting with one about open propositions.

**Lemma 27.** If $\varphi$ is open for $\{\psi, B\}$, then $\varphi$ is open for $\{\varphi \lor \psi, B\}$.

**Proof.** We must show that, when the above condition holds,

\begin{align*}
0 &< \Pr[\varphi \mid \varphi \lor \psi, B] < 1 \\
0 &< \frac{\Pr[\varphi \lor \psi \mid \varphi, B] \ast \Pr[\varphi \mid B]}{\Pr[\varphi \lor \psi \mid B]} < 1 \quad (17) \\
0 &< \frac{1 \ast \Pr[\varphi \mid B]}{\Pr[\varphi \mid B] + \Pr[\psi \mid B] - \Pr[\varphi \land \psi \mid B]} < 1 \quad (18) \\
0 &< \frac{\Pr[\varphi \mid B] + \Pr[\psi \mid B] - \Pr[\psi \mid B] \ast \Pr[\varphi \mid \psi, B]}{\Pr[\varphi \mid B]} < 1 \quad (19)
\end{align*}

Since $\varphi$ is open for $\{\psi, B\}$, $0 < \Pr[\varphi \mid \psi, B] < 1$. By Bayes rule,

\begin{align*}
0 &< \Pr[\varphi \mid \psi, B] \quad (20) \\
0 &< \Pr[\varphi \mid B] \ast \frac{\Pr[\psi \mid \varphi, B]}{\Pr[\psi \mid B]} \quad (21) \\
0 &< \Pr[\varphi \mid B] \ast \frac{\Pr[\psi \mid \varphi, B]}{\Pr[\psi \mid B]} \quad (22)
\end{align*}

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which implies that $0 < \Pr[\varphi | B]$. Thus,

$$0 < \frac{\Pr[\varphi | B]}{\Pr[\varphi | B] + \Pr[\psi | B] - \Pr[\varphi | \psi, B] * \Pr[\varphi | \psi, B]}$$ (23)

Since $0 < \Pr[\varphi | \psi, B] < 1$,

$$\Pr[\psi | B] * \Pr[\varphi | \psi, B] < \Pr[\psi | B]$$ (24)

Thus,

$$0 < \Pr[\psi | B] - \Pr[\psi | B] * \Pr[\varphi | \psi, B]$$ (25)

and

$$\Pr[\varphi | B] < \Pr[\psi | B] - \Pr[\psi | B] * \Pr[\varphi | \psi, B] + \Pr[\varphi | B]$$ (26)

and

$$\frac{\Pr[\varphi | B]}{\Pr[\varphi | B] + \Pr[\psi | B] - \Pr[\psi | B] * \Pr[\varphi | \psi, B]} < 1$$ (27)

Together (23) and (27) show that (20) holds. Thus, $\varphi$ is open for $\{\varphi \lor \psi, B\}$.

We next prove a theorem about systems $s$ that cannot trivially close $\varphi$.

**Lemma 28.** Consider $SE = \{0=\text{s}(x, a), x=x, a=a\}$. For all systems $s$ that are informative, if for all $o$, $\varphi$ is open for $\{0=o, SE\}$, then for all $o$, $\varphi$ is open for $\{0\neq o, SE\}$.

**Proof.** We must show that, when the above conditions hold, for all $o$,

$$0 < \Pr[\varphi | 0\neq o, SE] < 1$$ (28)

$$0 < \frac{\Pr[\varphi \land 0\neq o | \varphi] \land 0=\varphi | SE]}{\Pr[0\neq o | SE]} < 1$$ (29)

$$0 < \frac{\Pr[\varphi \land \bigvee_{o' \in O \text{ s.t. } o' \neq o} 0=\varphi | SE]}{\Pr[\bigvee_{o' \in O \text{ s.t. } o' \neq o} 0=\varphi | SE]} < 1$$ (30)

$$0 < \frac{\sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[\varphi \land 0=\varphi | SE]}{\sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[0=\varphi | SE]} < 1$$ (31)

Since $\varphi$ is open, for all $o$, $\Pr[\varphi | O=\varphi, SE] < 1$. Thus,

$$\sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[\varphi \land 0=\varphi | SE] = \sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[\varphi | 0=\varphi, SE] * \Pr[0=\varphi | SE]$$ (32)

$$< \sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[0=\varphi | SE]$$ (33)

which implies that

$$\frac{\sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[\varphi \land 0=\varphi | SE]}{\sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[0=\varphi | SE]} < 1$$ (34)
Since $s$ is informative, for all outputs $o'$, $\Pr[0=o' \mid SE] < 1$. Thus, there must exist at least two outputs $o_1$ and $o_2$ such that $0 < \Pr[0=o_i \mid SE]$ for $i \in \{1, 2\}$. Since, for all $o$, $\varphi$ is open for $\{0=o, SE\}$, for $o_i \in \{o_1, o_2\}$, $0 < \Pr[\varphi \mid 0=o_i, SE]$. Since $0 < \Pr[\varphi \mid 0=o_i, SE]$ and $0 < \Pr[0=o_i \mid SE]$,

$$0 < \Pr[\varphi \mid 0=o_1, SE] \ast \Pr[0=o_1 \mid SE] = \Pr[\varphi \land 0=o_1 \mid SE] \quad (35)$$

This implies that

$$0 < \sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[\varphi \land 0=o' \mid SE] \quad (36)$$

since at most of one $o_1$ and $o_2$ can be $o$ and for all $o'$, $0 \leq \Pr[\varphi \land 0=o' \mid SE]$. Thus,

$$0 < \frac{\sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[\varphi \land 0=o' \mid SE]}{\sum_{o' \in O \text{ s.t. } o' \neq o} \Pr[0=o' \mid SE]} \quad (37)$$

The inequalities (34) and (37) together imply that (31) holds. Thus, $\varphi$ is open for $\{0 \neq o, SE\}$. \hfill \Box

Theorem 26 follows directly from the next lemma.

**Lemma 29.** Consider $SE = \{0=s(x, A), x=X, A=A\}$ such that $s$ is informative. For all conditions $\varphi$ such that $\varphi$ is open for $SE$ and $s$ cannot trivially close $\varphi$ given $SE$, for all outputs $o$, $\varphi$ is open for $\{0 \neq o \lor \varphi, SE\}$ but closed for $\{0=o, 0 \neq o \lor \varphi, SE\}$.

**Proof.** Since $\varphi$ is open for $SE$ and $s$ cannot trivially close $\varphi$ given $SE$, $\varphi$ is open for $\{0=\ast, SE\}$ for all $\ast$ including $o$. Thus, since $s$ is informative, Lemma 28 applies and $\varphi$ is open for $\{0 \neq \ast, SE\}$ for all $\ast$ including $o$. Since $\varphi$ is open for $\{0 \neq o, SE\}$, Lemma 27 applies and $\varphi$ is open for $\{\varphi \lor 0 \neq o, SE\}$.

On the other hand,

$$\Pr[\varphi \mid \varphi \lor 0 \neq o, 0=o, SE] = \Pr[\varphi \mid \varphi, 0=o, SE] = 1 \quad (38)$$

Thus, $\varphi$ is closed for $\{0=o, 0 \neq o \lor \varphi, SE\}$. \hfill \Box

We can instantiate Theorem 26 for either privacy or nondiscrimination, as we do so below.

### 8.1.2 Privacy Interpretation

For privacy, Theorem 26 corresponds to saying that there always exist an adversary who will prevent inferential privacy of the statistical nondisclosure form.

More precisely, for privacy, let *resolved* from $B$ mean known to be true or known to be false from just $B$. We say that someone *learns* a proposition $\varphi$ from an observation if $\varphi$ is unresolved from that entity’s background knowledge.
but is resolved from the entity’s background knowledge with observation added to it. We say that \( s \) can trivially resolve \( \varphi \) given \( SE \) if \( \varphi \) is unresolved for \( SE \) but is for the combination of an output of \( s \) and \( SE \).

We can restate Theorem 26 as

**Corollary 30.** Consider \( SE = \{0=s(X,A), X=X, A=A\} \). For all propositions \( \varphi \), one of the following is true: \( \varphi \) is resolved from \( SE \), \( s \) is uninformative, \( s \) can trivially resolve \( \varphi \) given \( SE \), or for all \( o \), \( \varphi \) is unresolved for \( \{0\neq o \lor \varphi, SE\} \) but resolved for \( \{0=o, 0\neq o \lor \varphi, SE\} \).

Corollary 30 implies the following:

**Corollary 31.** Consider \( SE = \{0=s(X,A), X=X, A=A\} \). For all propositions \( \varphi \), if \( \varphi \) is unresolved from \( SE \) and \( s \) is informative, then there exists an adversary that experiences a statistical disclosure for \( \varphi \) upon seeing some output of \( s \).

*Proof.* Recall that a statistical disclosure happens for an adversary with background knowledge \( B \) if \( Cr[\varphi \mid O=o, B] \neq Cr[\varphi \mid B] \), which will happen if \( \varphi \) goes from unresolved to resolved.

Since \( \varphi \) is unresolved from \( SE \) and \( s \) is informative, by Corollary 30 either \( s \) can trivially resolve \( \varphi \) given \( SE \) or for all \( o \), \( \varphi \) is unresolved for \( \{0\neq o \lor \varphi, SE\} \) but resolved for \( \{0=o, 0\neq o \lor \varphi, SE\} \). In the first case, \( \varphi \) goes from unresolved for \( SE \) to resolved for \( \{O=o, SE\} \) for some output \( o \). In the second case, \( \varphi \) goes from unresolved for \( \{0\neq o \lor \varphi, SE\} \) but to resolved for \( \{0=o, 0\neq o \lor \varphi, SE\} \) for any output \( o \). Either way, there is a statistical disclosure for some output \( o \).

Thus, it is impossible to have a system that both produces interesting outputs and always prevents statistical disclosures. Dwork and Naor’s impossibility result [12], which is even stronger, can be viewed as a justification for focusing on the weaker property of differential privacy.

### 8.1.3 Nondiscrimination Interpretation

For discrimination, the generic result corresponds to saying that a lack of group parity (disparate impact) will always exist for some subpopulation.

In more detail, for nondiscrimination, we say that a population lacks all diversity of an attribute \( \varphi \) if either everyone in that population has \( \varphi \) or everyone in the population lacks \( \varphi \). Otherwise, we say that the population has some diversity of \( \varphi \). Given a population \( B_P \), let its \( \psi \) subpopulation be the subpopulation of \( B_P \) identified by keeping only those members of \( B_P \) who have the attribute \( \psi \). We say that a population loses diversity for \( \varphi \) in its \( \psi \) subpopulation if the population has some diversity of \( \varphi \) but its \( \psi \) subpopulation lacks all diversity of \( \varphi \). We say that \( s \) can trivially remove diversity for \( \varphi \) from \( SE \) if for some output \( o \) of \( s \), \( SE \) loses diversity for \( \varphi \) in its \( O=o \) subpopulation.

We can restate Theorem 26 as

**Corollary 32.** Consider \( SE = \{0=s(X,A), X=X, A=A\} \). For all attributes \( \varphi \), one of the following is true: \( \varphi \) lacks all diversity for \( SE \), \( s \) is uninformative, \( s \) can...
trivially remove diversity for $\varphi$ from $SE$, or for all outputs $o$, the $\{0\neq o \lor \varphi, SE\}$ subpopulation of $SE$ loses diversity of $\varphi$ in its $0= o$ subpopulation.

An example will make the consequences of this result more clear. Consider a system $s$ that decided who to hire with the output $o$ meaning hired and consider the attribute of being male $\varphi$. Suppose that the system does not simply hire or not hire everyone, meaning that it is informative. Further suppose that it hires some but not all males, meaning it does not trivially remove diversity for $\varphi$. Then, there exists some subpopulation of all applicants such that that subpopulation contains both males and females, but only the males are hired from it. In particular, the subpopulation of men and non-hired women will go from having diversity to lacking it when focusing on just the hired ones. The consequence of this is that while we can demand that the system hires both males and females, we cannot expect this to hold for all subpopulations.

Given the contrived nature of the subpopulation of men and non-hired women, this may seem obvious and uninteresting. However, three points are worth bearing in mind.

First, this result shows that the absence of disparate impact for all subpopulations cannot be met for interesting systems $s$. Some bounds on the subpopulations considered must be placed on demands for group parity, and future work can search for reasonable bounds instead of trying to provide mechanisms providing universal nondiscrimination. This is similar in spirit to Dwork and Naor’s result in that the adversary was not particularly realistic there either, but the result still forced researchers to give up the search for mechanisms ensuring statistical nondisclosure in all cases.

Second, if any non-empty subpopulation of the $\{0\neq o \lor \varphi, SE\}$ subpopulation can be identified using some other attribute $\chi$, then it may seem that the attribute $\chi$ is causing some form of intersectional discrimination against those identified by $\varphi \land \chi$ for attribute $\psi$ of getting hired ($0= o$). For example, suppose that all the applicants who are black falls into the set of men and non-hired women, that is, no black women were hired. It is possible that some form of intersectional discrimination is at play, that is, discrimination not against women per se, nor black people per se, but rather against black women.

It is also possible that outcome came about by noise. The possibility of noise looks more likely as the third attribute $\chi$ gets more contrived looking. For example, if instead of it being a simple racial attribute, consider $\chi$ identifying applicants who are from Spain and over the age of 40. Is there intersectional discrimination against older Spanish women? Or is it just a fluke?

While we do not do so here, we conjecture that modeling the input space $A$ more explicitly as a rich and diverse set of attributes would allow us to show that one can always identify some such $\chi$ that is simple.

Third, returning to the goal of this paper, our real aim was to show that one can produce general results that apply to both privacy and nondiscrimination and then specialize them for each. Corollaries 30 and 32 meets this goal.
8.2 Probabilistic Program Analysis

There exists a significant body of work in the formal security literature on achieving formal bounds on probabilistic properties, aimed at verifying probabilistic security properties of programs. Techniques used to estimate the probability of outcomes include abstract interpretation [63], volume computation [64], and sampling [65]. While this body of work has largely focused on security properties, as the properties are equivalent at a mathematical level, techniques for the analysis of probabilistic programs can be directly used for fairness properties.

In one example, Albarghouti et al. [66] use a volume computation technique to either verify that the program satisfies a probabilistic fairness property or provides a counterexample that demonstrates the program is biased. They focus on disparate impact as a notion of discrimination:

\[
\Pr[O = o \mid X = x_1] \leq k \cdot \Pr[O = o \mid X = x_2]
\]

Their approach requires the specification of two programs: popModel and dec. The program dec is the program that decides the outcomes O for individuals, and popModel generates random individuals from a population which are fed to dec. The problem of estimating disparate impact, then reduces to estimating the quantities \( \Pr[O = o \land X = x_i] \) and \( \Pr[X = x_i] \) for \( i \in \{1, 2\} \). All of these quantities can be estimated from the outcome of the composed program \( \text{dec} \circ \text{popModel} \) using methods from security.

As another example of the connection between privacy and nondiscrimination, we note that probabilistic couplings form the core of approaches to ensuring both. On the one hand, a line of work has used probabilistic couplings to verify that an algorithm provides differential privacy [67–70]. On the other hand, Friedler et al. have presented a series of probabilistic notions of (un)fairness [71]. Their definitions structural bias (their Def. 3.5), direct discrimination (their Def. 3.6), and nondiscrimination (their Def. 3.7) each have couplings (their Def. 2.5) at their cores.

8.3 Statistical Scrubbing of Inputs

A separate body of work [72–75] addresses inference attacks in data sets by mapping each row \( D \) to a row \( D' \) such that \( Y \) is independent to sensitive attributes with respect to some fixed background knowledge \( B \). The key idea is that if the inputs \( D \) are independent to sensitive attributes \( X, D' \perp X \mid B \), then for any function \( f, f(D') \perp X \mid B \). We term such mappings from \( D \) to \( D' \) statistical scrubbing.

Independently, a body of work [47, 48] in the machine learning literature attempts to solve the same problem of statistical scrubbing in a data set to achieve independence with respect to sensitive attributes for providing group parity.
8.4 Generalized Distance Metrics for Differential Privacy

One idea that appears in both nondiscrimination and privacy literature is the generalization of differential privacy from indistinguishability over neighboring databases to indistinguishability over nearby inputs as measured by some distance metric. In [18], Dwork et al. propose a notion of fairness that states that similar people should be treated similarly. Formally the definition requires that the distance in distribution over outputs be bounded by the distance between input, a classical Lipschitz continuity requirement. In [76], Chatzikokolakis et al. propose a similar generalization for different privacy notions that and apply the definition in the contexts of smart meter privacy and geo-location privacy.

9 Discussion

We have explored the relationship between privacy and nondiscrimination at a mathematical level. We have shown that both use privacy and nondiscrimination definitions come in associative and causal flavors. We have shown, at a mathematical level, that the basic form of the two associative definitions are identical and that the basic form of the two causal definitions are identical. This similarity has allowed us to show what the relationships are between both associative definitions and both causal definitions at once. It has also allowed us to re-use a proof about privacy to say something about nondiscrimination. We believe this work shows that, to avoid duplication of effort, computer science research on privacy and nondiscrimination should not be siloed into separate communities.

This is not to say that privacy and nondiscrimination are or should be collapsed into one problem. The definitions, despite having the same formal structures, differ in their interpretations and what the variables model. Neither can do the job of the other; attention must be paid to both.

Furthermore, privacy and nondiscrimination can interact both positively and negatively. For example, not disclosing one's race on an employment application may help prevent discrimination in some settings. Furthermore, nondiscrimination can lessen some privacy concerns. For example, banning health insurers from discriminating based on pre-existing conditions may lessen the degree of privacy patients with expensive conditions seek. On the other hand, privacy can hide discrimination. Our work may be a starting point for exploring these interactions.

Lastly, in all cases, the mathematical characterizations we use are simplifications of the the social norms and laws they characterize. For one, both privacy and nondiscrimination norms contain exceptions. An example for privacy is that patient–physician confidentiality may be set aside by court orders. As for nondiscrimination, disparate impact is allowed where a “business necessity” exists. Furthermore, nondiscrimination law contains many features that we do not capture, such as intent. In particular, we replaced the vague
concept of *intent* with the more precise concept of *cause*. Exploring what intent might look like for automated systems could provide more nuanced models of disparate treatment. Nevertheless, we believe our mathematical characterizations capture the essence of what each norm or law is attempting to achieve.

**Acknowledgements.** We gratefully acknowledge funding support from the National Science Foundation (Grants 1514509, 1704845, and 1704985). The opinions in this paper are those of the authors and do not necessarily reflect the opinions of any funding sponsor or the United States Government.

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A Example System

Here we present an example system showing that $\varphi$ could be closed for $\{O\neq o \lor \varphi, SE\}$ despite not being closed for $SE$. This motivates the condition that the system $s$ cannot trivially close $\varphi$ given $SE$ in Lemma 29. In particular, note that example below does trivially close $\varphi$ given $SE$ when the output is nonpositive.

As before, let $SE$ be $0=s(X, A)$, $x=X$, and $a=A$, and let it hold. Consider the program $s(x, a) = \text{pos}(x, a)$ which returns whether $x$ is positive, with positive meaning yes it is and nonpositive meaning no it is not. Since $\text{pos}$ ignores its second input, we will drop it. Suppose that the range $\mathcal{X}$ of $X$ is $\{0, 1, 2\}$.

Consider the sensitive condition $\varphi$ that is whether $X$ is even, that is, $\text{even}(X)$. Seeing the output $0$ only sometimes reveals whether $\text{even}(X)$ holds. If the output is $\text{pos}(X) = \text{nonpositive}$, then the input must have been $0$, and $\text{even}(X)$ must hold. If the output is $\text{pos}(X) = \text{positive}$, then the input could have been $1$ or $2$, and $\text{even}(X)$ may or may not hold. The following diagram summarizes this state of affairs:
even($X$)  $X$  $X = X$  $O = \text{pos}(X)$

| 0 | 0 |
|---|---|
| true | nonpositive |
| false | positive |
| 1 | 1 |
| 2 | 2 |

To understand how this relates to Lemma 29, let us focus on when the output $O$ takes on the value $o = \text{positive}$. Note that $\varphi$ is open for \{$0=\text{positive}, SE\}$.

Consider the background context $0\neq\text{positive} \lor \text{even}(X)$, that is, that $0=\text{positive}$ implies even($X$). Given this context, seeing the output $\text{positive}$ would imply even($X$). Thus, given this context, one can learn the sensitive condition for either output of $s$.

However, one would not even need to see the output to learn the sensitive condition given this context. Consider an analysis by cases. If $0\neq\text{positive}$ holds, then $X$ must be 0 and even($X$) holds. If even($X$) holds, then even($X$) must hold. So, either way even($X$) must hold.

The issue is that $0$ not having the value $\text{positive}$ implied too much: it implies that the output must be $\text{nonpositive}$, which implies $\varphi$ holds. The following table, which shows the values of $X$ that lead to each value for $\varphi$ and $0$, illustrates this issue:

| even($X$) |
|-----------|
| $\text{pos}(X)$ | true | false |
| nonpositive | 0 | false |
| positive | 2 | 1 |

Note that no value is both odd and nonpositive, meaning that $s$ trivially closes $\varphi$ for the nonpositive output. This means that set of value of $X$ that makes $0\neq\text{positive} \lor \varphi$ true, which is equivalent to $0=\text{nonpositive} \lor \varphi$, only has a value of $X$ for when $\varphi$ is true. Thus, it implies the truth of $\varphi$ without needing to see the value of $0$. 

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