Do zeta functions for intermittent maps have branch points?

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Abstract - We present numerical evidence that the dynamical zeta function $\zeta(z)$ and the Fredholm determinant $F(z)$ of intermittent maps with a neutral fix point have branch point singularities at $z = 1$. We consider the power series expansion of $1/\zeta(z)$ and $F(z)$ around $z = 0$ with the fix point pruned. This power series is computed up to order 20, requiring $\sim 10^5$ periodic orbits. We also discuss the relation between correlation decay and the nature of the branch point. We conclude by demonstrating how zeros of zeta functions with thermodynamic weights that are close to the branch point can be efficiently computed by a resummed cycle expansion. The idea is quite similar to that of Pade approximants, but the ansatz is a generalized series expansion around the branch point instead of a rational function.

In this note we will present numerical evidence that dynamical zeta functions and Fredholm determinants exhibit branch point singularities for intermittent maps. We will consider a family of one dimensional maps $x \mapsto f(x)$ on the unit interval, with

$$f(x) = \begin{cases} x + 2^s x^{1+s} & 0 \leq x < 1/2 \\ 2x - 1 & 1/2 \leq x \leq 1 \end{cases},$$

(1)

where $s > 0$. For $s = 0$ the map would just be the binary shift map, which is uniformly hyperbolic, but for $s > 0$ it is intermittent; the fix point $x = 0$ is neutrally stable: $f''(0) = 1$. The map admits a binary coding, we associate the letter 0 with the left leg, and 1 with the right leg. The neutral fix point now corresponds to the periodic orbit 0.
The dynamical zeta function is defined by

\[ \frac{1}{\zeta(z)} = \prod_p \left( 1 - \frac{z^{n_p}}{|\Lambda_p|} \right). \tag{2} \]

The product in (2) runs over all primitive periodic orbits \( p \), having period \( n_p \) and stability \( \Lambda_p = \frac{d^{n_p}}{dx^{n_p}}|_{x=x_p} \) with \( x_p \) being any point along \( p \).

In our considerations we will prune the neutral fix point

\[ \frac{1}{\tilde{\zeta}(z)} = (1 - z)^{1/\tilde{\zeta}(z)}, \tag{3} \]

where

\[ \frac{1}{\tilde{\zeta}(z)} = \prod_{p \neq 0} \left( 1 - \frac{z^{n_p}}{|\Lambda_p|} \right). \tag{4} \]

We will also consider the Fredholm determinant

\[ \tilde{F}(z) = \prod_{p \neq 0} \prod_{m=0}^{\infty} \left( 1 - \frac{z^{n_p}}{|\Lambda_p|^{\Lambda_m}} \right). \tag{5} \]

In the previous case, it is not essential to prune the neutral periodic orbit, but when considering the Fredholm determinant it is essential, the extra factor \((1 - z)^\infty\) would of course be devastating for our investigations.

We will consider the series expansion of the functions \( 1/\tilde{\zeta}(z) \) and \( \tilde{F}(z) \)

\[ 1/\tilde{\zeta}(z) = 1 - \sum_{i=1}^{\infty} a_i z^i, \tag{6} \]

\[ \tilde{F}(z) = 1 - \sum_{i=1}^{\infty} b_i z^i. \tag{7} \]

The nature of the leading singularity will be reflected in the asymptotic behaviour of the coefficients of these power series. To get an idea what this asymptotic behaviour may be we consider the fundamental part of a cycle expansion of \( 1/\tilde{\zeta}(z) \) \[1, 2\]

\[ 1/\tilde{\zeta}(z) \approx 1 - \sum_{n=0}^{\infty} \frac{z^{n+1}}{\prod_{m=1}^{n+1}}. \tag{8} \]

The dependence of the stabilities \( \Lambda_{n+1} \) on \( n \) can be estimated by replacing the difference equation \( x_{n+1} = x_n + 2^s x_n^{1+s} \) by the differential equation\[3\]

\[ \frac{dx_n}{dn} = 2^s x_n^{1+s}, \tag{9} \]

having solution

\[ x_n = [x_0^{-s} - s2^s n]^{-1/s}. \tag{10} \]
First, because $x_n \sim 1$ we get a relation between $x_0$ and $n$

$$x_0 \sim n^{-1/s}, \quad (11)$$

telling us how deep into the intermittent region the orbit $0^n1$ penetrate. We are only interested in the dependence on $n$ and have discarded other information. We can now obtain the stabilities by differentiation

$$\Lambda_{\text{lin}} = 2 \frac{dx_n}{dx_0} \approx 2s_{n+1} x_0^{-(s+1)} n^{(s+1)/s}. \quad (12)$$

This suggest that $\hat{Z}(z)$ contains a singularity of the type

$$(1 - z)^{1/s} \quad 1/s \notin N \quad (1 - z)^{1/s} \log(1 - z) \quad 1/s \in N^+, \quad (13)$$
as can be realized through the Tauberian theorems for power series.

A somewhat more refined, although closely related estimate is obtained by the piece-wise affine approximation of ref. [6], leading to the same predictions. In [6] there are also much stronger arguments that $z = 1$ is a branch point but it is still not proven that $1/\zeta(z)$ can be analytically continued outside the unit disk. That this can actually be done is suggested by the numerical data we are now going do discuss.

In order to numerically determine the coefficients $\{a_i\}$ we simply expand the product (4) using all periodic orbits up to period 20. This set contains 111011 periodic orbits which means that we cannot go very much further within reasonable computer time. When expanding the Fredholm determinant (5) we use Euler’s formula to expand the inner product:

$$\tilde{F}(z) = \prod_{p \neq 0} \prod_{m=0}^{\infty} (1 - \frac{z^{n_p}}{|\Lambda_p|^m}) \quad (14)$$

$$= \prod_{p \neq 0} \left( \sum_{j=0}^{\infty} (-1)^j \frac{\Lambda^{-j(j-1)/2}}{|\Lambda_p|^j \prod_{k=1}^{j} (1 - \Lambda^{-k})} z^{j\cdot n_p} \right).$$

Periodic orbits are determined by a Newton-Raphson procedure. To this end we look for fix points of some iterate of the inverted map, choosing branch according to the symbol code.

The coefficients for three parameter values are plotted in figs. 1-3 together with the expected slope according to eqs. (8) and (12). Our set of data is consistent with the expected leading singularity (13) and indicate no other singularity close to the unit circle; this would have induced oscillations superposed on the sequence of coefficients.

For the binary shift map ($s = 0$) we would have $a_i = 1/2^i$. For a slightly higher value of the parameter $s = 0.1$ (fig. 1) the coefficients for small $i$ conform with this behaviour but eventually they bend off, approaching the expected slope.

One may also observe that whereas eq. (6) seems to provide the correct order of the singularity it does not predict the correct size (prefactor) of it. The curvature corrections of ref. [1] exhibit the same type of singularity as does the fundamental part.
Why is this interesting? It is known that the spectra of zeta functions and Fredholm determinants are, at least in some cases, related to the (typical) decay of correlations. For exponentially mixing system the mixing rate is given by the size of the gap between the leading and next-to-leading zeros of $F(z)$ \[4\]. We believe that a similar coupling can be made system without a gap. Let us take the (formal) trace of the transfer operator
\[
\text{tr}L^n = \int_\epsilon^1 \delta(x - f^n(x))dx = \sum_{p \neq 0} n_p \sum_{r=1}^\infty \frac{\delta_{n,rn_p}}{|\text{det}(1 - \Lambda_p^r)|} = \frac{1}{2\pi i} \int_{|z|<1} z^{-n} \tilde{F}'(z) \frac{dz}{F(z)}
\] (15)
where $\epsilon$ is a small number. The trace is formal because it makes no explicit reference to eigenvalues of an operator, it is just the trace over the integral kernel of the operator. We claim that this trace serves as an archetype correlation function. If there is a gap, residue calculus tells us that the trace will approach unity exponentially fast and the rate is provided by the size of the gap. But could this really work for the intermittent case where the leading zero is connected by a branch cut running along the line $\text{Im}(z) = 0 \Re(z) > 1$. Let us assume that $\tilde{F}(z)$ is holomorphic and zero-free, except along the cut, in the disk $1 < |z| < C$ where $C > 1$. The value of the (15) for large $n$ above will be governed by the vicinity of $z = 1$ and can be evaluated asymptotically. Inserting $\hat{F}$ with the suggested branch point singularity into (15) we get the following asymptotic behaviour for the trace
\[
\text{tr}L^n \sim \begin{cases} 
1 + C/n^{1/s-1} & 0 < s < 1 \\
1 + C/\log n & s = 1 \\
1/s & s < 1
\end{cases}
\] (16)
For $0 < s < 1$ this suggests that typical correlations should decay as $\sim 1/n^{1/s-1}$ which indeed agrees with the rigorous results \[3\]. The failure of the trace to approach unity for $s > 1$ reflect the fact that the invariant density is not normalizable anymore.

There are indications that the identification between the behaviour of the formal trace and the typical correlation functions is possible also for the Sinai billiard which seems to exhibit the decay law $C(t) \sim 1/t$ \[5, 7\].

Our findings are also interesting from a more practical point of view. Consider the problem of computing the topological pressure which amounts to compute the leading zero of
\[
1/\tilde{\zeta}(z, \beta) = \prod_{p \neq 0} (1 - z^{n_p} \frac{1}{|\Lambda_p|^\beta}) .
\] (17)
If $\beta$ is close to, but less than unity this leading zero will be close to $z = 1$ and it will be extremely inefficient to compute it from the truncated power series in $z$. It is natural to try to expand $1/\tilde{\zeta}(z, \beta)$ in a generalized power series around $z = 1$. If the leading singularity is of the form $(1 - z)^\alpha$ the simplest possible expansion would be
\[
1/\tilde{\zeta}(z, \beta) = \sum_{i=0}^\infty c_i (1 - z)^i + (1 - z)^\alpha \sum_{i=0}^\infty d_i (1 - z)^i .
\] (18)
According to our previous findings we expect that \( \alpha = \beta \frac{(s + 1)}{s} - 1 \). Suppose now that we replace these infinite sums by finite sums of increasing degrees, \( n_c \) and \( n_d \), and require that

\[
\sum_{i=0}^{n_c} c_i (1 - z)^i + (1 - z)^\alpha \sum_{i=0}^{n_d} d_i (1 - z)^i = \frac{1}{\tilde{\zeta}(z, \beta)} + O(z^{n+1}) .
\] (19)

If \( n_c + n_d = n + 1 \) we just get a linear system of equation to solve in order to determine the coefficients \( c_i \) and \( d_i \) from those of the series expansion around \( z = 0 \). This strategy is quite similar to that of Padé approximants. It also natural to require that \( |n_d + \alpha - n_c| < 1 \). So far we have assumed that \( \alpha \) is an non integer. The case with integer \( \alpha \) can be worked out in close analogy.

To test the idea we choose (arbitrarily) the parameters \( s = 0.7 \) and \( \beta = 0.9 \). In fig 4 we plot the leading zero versus truncation length \( n \) determined from expansion (19) and the expansion around \( z = 0 \). The improvement is obvious. However, we do not claim that the simple expansion (18) is entirely able to capture the complicate analytic structure around \( z = 1 \).

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Figure 1: Expansion coefficients for the functions $\hat{Z}(z)$ and $\hat{F}(z)$ for the parameter values $s = 0.1$.

Figure 2: Same as fig 1 but with $s = 1$. 


Figure 3: Same as fig 1 but with $s = 2$.

Figure 4: Leading zero for $s = 0.7$ and $\beta = 0.9$ versus truncation length determined from the generalized series expansion around $z = 1$ and the power series around $z = 0$