Set systems without a simplex, Helly hypergraphs and union-efficient families

Stijn Cambie* Nika Salia*

Abstract

We present equivalent formulations for concepts related to set families for which every subfamily with empty intersection has a bounded sub-collection with empty intersection. Hereby, we summarize the progress on the related questions about the maximum size of such families.

In this work we solve a boundary case of a problem of Tuza for non-trivial $q$-Helly families, by applying Karamata’s inequality and determining the minimum size of a 2-self-centered graph for which the common neighborhood of every pair of vertices contains a clique of size $q - 2$.

1 Introduction

At first, in Subsection 1.1 we introduce new and existing concepts. In Subsection 1.2 we show how these concepts are related and give a glimpse of some related problems. In Subsection 1.3, we summarize the content of the paper.

Our notation follow [18]. Thus $[n]$ denotes $n$ element set $= \{1, 2, \ldots, n\}$. A subset of the power set of $[n]$, will be called a family $\mathcal{F} \subseteq 2^n$. In the uniform case, every set of $\mathcal{F}$ is a set of size $k$, i.e. $\mathcal{F} \subseteq \binom{[n]}{k}$. Such a family is sometimes called a set system or $k$-uniform hypergraph, but in this work, we use the term family. We use the following notation for the family of complements $\mathcal{F}^c = \{A^c \mid A \in \mathcal{F}\}$ where $A^c$ denotes $[n] \setminus A$. The order and the size of a graph $G = (V, E)$ will be denoted with $|V(G)|$ and $|E(G)|$ respectively. The subgraph of $G$ induced by a set $A$ will be denoted by $G[A]$.

1.1 Introduction of the concepts

Union-efficient families

When the twin-free graph having the largest order for a given number of maximal independent sets was characterized in the work of [8], the notion of union-efficient families naturally appeared.

Definition 1. For fixed integers $n$ and $m$, we call a family $\mathcal{F} \subseteq 2^n$ union-efficient if for every subfamily $\{A_1, A_2, \ldots, A_m\} \subseteq \mathcal{F}$ for which $\cup_{i \in [m]} A_i = [n]$, there are two indices $i, j \in [m]$ for which $A_i \cup A_j = [n]$.

At first sight, the notion of union-efficient seemed to be a new concept, but by considering the complement, it is related to existing concepts. On the other hand, the terminology of union-efficient families is more in line with existing basic terminology in extremal set theory, see e.g. [18]. As such, it is also natural to consider the following variants.
Definition 2. For fixed integers \( n, m \) and \( q \), a family \( \mathcal{F} \subseteq 2^{[n]} \) is union-\( q \)-efficient if for every subfamily \( \{A_1, A_2, \ldots, A_m\} \subseteq \mathcal{F} \) for which \( \bigcup_{i \in [m]} A_i = [n] \), there is a subset \( I \subseteq [m] \) of cardinality \( q \) for which \( \bigcup_{i \in I} A_i = [n] \).

A family \( \mathcal{F} \subseteq 2^{[n]} \) is intersection-\( q \)-efficient if for every subfamily \( \{A_1, A_2, \ldots, A_m\} \subseteq \mathcal{F} \) for which \( \bigcap_{i \in [m]} A_i = \emptyset \), there is a subset \( I \subseteq [m] \) of cardinality \( q \) for which \( \bigcap_{i \in I} A_i = \emptyset \).

Note that, union-efficient denotes union-2-efficient. Trivial families play an important role in extremal set-theoretic problems. For problems about the unions/ intersections of sets, a family \( \mathcal{F} \) is called trivial if \( \bigcup_{A \in \mathcal{F}} A \neq [n] \) resp. \( \bigcap_{A \in \mathcal{F}} A \neq \emptyset \) and non-trivial if \( \bigcup_{A \in \mathcal{F}} A = [n] \) respectively \( \bigcap_{A \in \mathcal{F}} A = \emptyset \).

### Helly hypergraphs and simplices

There are multiple ways to define Helly families and simplices. We follow [33], thus at first we introduce \( q \)-linked families and then we define Helly families.

Definition 3. A family \( \mathcal{F} \) is \( q \)-linked if the intersection of any \( q \) sets in \( \mathcal{F} \) is non-empty. That is, \( \forall A_1, A_2, \ldots, A_q \in \mathcal{F}, A_1 \cap A_2 \cap \ldots \cap A_q \neq \emptyset \).

Helly’s celebrated theorem on convex sets states, a finite collection of \( n \) convex subsets of \( \mathbb{R}^d \) has a non-empty intersection if every \( d+1 \) subsets have a non-empty intersection. This inspired the following notion for families of sets.

Definition 4. A family \( \mathcal{F} \) satisfies the Helly property if every \( 2 \)-linked (pairwise intersecting) subfamily \( \mathcal{F}' \) of \( \mathcal{F} \) has non-empty intersection (\( \cap_{F \in \mathcal{F}} F \neq \emptyset \)).

A family \( \mathcal{F} \) satisfies the \( q \)-Helly property if every \( q \)-linked subfamily \( \mathcal{F}' \) of \( \mathcal{F} \) has non-empty intersection (\( \cap_{F \in \mathcal{F}} F \neq \emptyset \)).

Here we present another important concept.

Definition 5. A \( q \)-simplex is a family with \( q+1 \) sets \( \{A_1, A_2, \ldots, A_{q+1}\} \) such that their intersection is the empty set, but the intersection of any \( q \) of them is not empty.

### 1.2 Relations between notions and with other problems

The following theorem shows important connections between the concepts introduced in the previous subsection.

Theorem 6. For a family \( \mathcal{F} \subseteq 2^{[n]} \), the following statements are equivalent.

1. \( \mathcal{F} \) does not contain an \( r \)-simplex for any \( r \geq q \)
2. \( \mathcal{F} \) is \( q \)-Helly
3. \( \mathcal{F} \) is intersection-\( q \)-efficient
4. \( \mathcal{F}^c \) is union-\( q \)-efficient

**Proof.** We prove the equivalences of (i), (ii) and (iii) in the cyclic order, after we show the equivalence of (iii) and (iv).

(i) \( \Rightarrow \) (ii) We prove the contraposition \( \neg (ii) \Rightarrow \neg (i) \). Let \( \mathcal{F}' \) be a minimal \( q \)-linked subfamily of \( \mathcal{F} \) with empty intersection. Suppose \( \mathcal{F}' \) contains \( r+1 \) sets. Then any subfamily of \( \mathcal{F}' \) with \( r \) sets would also be \( q \)-linked and thus would not have empty intersection (otherwise \( \mathcal{F}' \) was not minimal). Hence \( \mathcal{F}' \) is a \( r \)-simplex. The condition \( r \geq q \) holds since the intersection of \( q \) sets is non-empty, while the intersection of the \( r+1 \) sets is empty.
Suppose $\mathcal{F}$ is a $q$-Helly family and let $\mathcal{F}' = \{A_1, \ldots, A_r\} \subset \mathcal{F}$ be any subfamily whose intersection is the empty set. Since $\mathcal{F}$ is $q$-Helly, we know $\mathcal{F}'$ is not $q$-linked, so there are $q$ sets in $\mathcal{F}'$ with empty intersection. Since $\mathcal{F}'$ was taken arbitrarily, we know that $\mathcal{F}$ is intersection-$q$-efficient.

Proving the contraposition $\neg (i) \Rightarrow \neg (iii)$ is immediate, since an $r$-simplex with $r \geq q$ contains $r + 1$ sets with empty intersection for which no $r$ and hence no $q$ have empty intersection.

Since the intersection of $r$ sets in $\mathcal{F}$ is the empty set if and only if the union of the complements of the $r$ sets (which belong to $\mathcal{F}'$) is $[n]$, this equivalence is immediate from Definition 2.

Another equivalent form was established by Berge and Duchet [3, Thr.1].

**Theorem 7** ([3]). A family $\mathcal{F} \subseteq 2^{[n]}$ is $q$-Helly if and only if for every $A \subseteq [n]$ such that $|A| = q + 1$,

$$\bigcap_{B \in \mathcal{F} : |A \cap B| \geq q} B \neq \emptyset.$$  

Asking the question about maximal uniform families satisfying one of the presented conditions turns out to be interesting and challenging. One of the simplest non-trivial cases turns out to be equivalent to open hypergraph-Turán-problems, originating from the work of Turán [36] (see [24] for a survey). In contrast to simpler graph cases, there are plausibly many extremal hypergraphs [28] and there is no stability in general [31] for hypergraph-Turán-problems. The equivalence was observed before in [4, 33], but for completeness, we prove it here from scratch, note that it also can be seen as a corollary of Theorem 7.

**Proposition 8.** Let $\mathcal{F}$ be a subset of $\binom{[n]}{3}$, then $\mathcal{F}$ is intersection-3-efficient if and only if $\mathcal{F}$ is a 3-uniform hypergraph without a subgraph isomorphic to $K_4^{(3)}$, a four vertex 3-uniform complete hypergraph.

**Proof.** If $\mathcal{F}$ is an intersection-3-efficient family then it is $K_4^{(3)}$-free by the definition.

Let $\mathcal{F}$ be a 3-uniform $K_4^{(3)}$-free hypergraph and let $\bigcap_{i \in [m]} A_i = \emptyset$ for $\{A_1, A_2, \ldots, A_m\} \subseteq \mathcal{F}$ and an integer $m \geq 3$. If there are two sets $A_i$ and $A_j$ such that $|A_i \cap A_j| \leq 1$, then there is a set $A_k$ such that $\ell \in [m]$ and $A_k$ is disjoint from $A_i \cup A_j$ since $\bigcap_{i \in [m]} A_i = \emptyset$. Hence $\mathcal{F}$ is an intersection-3-efficient family. If for all $i, j \in [m]$ we have $|A_i \cap A_j| = 2$ then, let $A_1 = \{a_1, a_2, b_1\}$ and $A_2 = \{a_1, a_2, b_2\}$. Since the intersection $\bigcap_{i \in [m]} A_i$ is an empty set, there is a set not containing $a_1$ and there is another set not containing $a_2$. Those two sets are $\{a_2, b_1, b_2\}$ and $\{a_1, b_1, b_2\}$, since all pairwise intersections have size two. A contradiction since $A_1, A_2, \{a_2, b_1, b_2\}$ and $\{a_1, b_1, b_2\}$ is a copy of $K_4^{(3)}$.

We shortly mention that this connection was some additional motivation to work on a boundary case of a problem of Tuza. The largest $\mathcal{F} \subseteq \binom{[n]}{3}$ that are intersection-3-efficient for values $\frac{2m}{3} \geq k > 3$ turn out to be the trivial families (Theorem 16). The latter result does not give insight in the most basic hypergraph-Turán-problem. Since the largest intersection-3-efficient family $\mathcal{F} \subseteq \binom{[n]}{3}$ is non-trivial, one may hope that determining the largest non-trivial intersection-3-efficient $\mathcal{F} \subseteq \binom{[n]}{k}$ where $\frac{2n}{3} \geq k > 3$ is more interesting. That question had been posed before by Tuza [37] (see question 17) and is still widely open.

The set-up of Helly graphs has also been connected with the transversal number, see e.g. [19], and has been studied for Sperner families [5]. Also as an analog of $t$-intersecting families (for $t = 2$), bi-Helly families have been considered [39]. A related extension is the notion of a special...
simplex [17]. A special $q$-dimensional simplex is a family $\{A_1, A_2, \ldots, A_{q+1}\}$ such that there exists a set $C = \{x_1, x_2, \ldots, x_{q+1}\}$ for which $A_i \cap C = C \setminus x_i$ for every $1 \leq i \leq q + 1$ and all $A_i \setminus C$ are disjoint. In this work, we do not focus on these related versions.

Finally, we observe that there are more notions that are similar in flavor. A family $F$ of sets has the $(p, q)$ property if among any $p$ sets of $F$ some $q$ have a nonempty intersection. This property was invented by Hadwiger and Debrunner [20], where they extend the result of Helly on convex sets in $\mathbb{R}^k$. While the definition is stated for set families, it did not get attention in this more general framework. A few exceptions are $p = q$ and $q = 2$, stated in a different way. A family satisfies the $(p, p)$ property precisely if it is $p$-intersecting. When $q = 2$, a family $F$ of sets has the $(p, 2)$ property if $F$ contains no $p$ disjoint sets. Hence Kleitman [27] studied this case already. More general, one can ask about the maximum size of a family $F \subseteq 2^{[n]}$ which has the $(p, q)$ property for $p > q > 2$.

**Question 9.** Given $p > q > 2$. What is the maximum size of a family $F \subseteq 2^{[n]}$ which has the $(p, q)$ property.

### 1.3 Overview of content

In Section 2, we summarize the progress on maximum Helly families and families without simplices. We end with a problem of Tuza [37] about the maximum size of a non-trivial uniform $q$-Helly family $F \subseteq \binom{[n]}{q}$, which we solve in a boundary case $n = k \frac{4}{q-1}$. Equivalently, we determine the largest non-trivial union-$q$-efficient family $F \subseteq \binom{[n]}{q}$ where $n = qk$. In Section 3 we solve the case $k = 2$ separately. In Section 4 this is done for $k \geq 3$. Now we present the main result of this work.

**Theorem 10.** Let $n = qk$ where $q, k \geq 2$, $(q, k) \neq (2, 2)$ and let $F \subseteq \binom{[n]}{q}$ be a non-trivial union-$q$-efficient family. Then $|F| \leq \binom{n-1}{q-1} + q$. Furthermore, the extremal family is unique up to isomorphism.

The proof uses some results on self-centered (graphs with the radius equal to the diameter) graphs for which the common neighborhood of any 2 vertices contains a $K_{q-2}$. For $q = 3$, this shows that adding the triangle-property as a condition to the result of Buckley [7], implies that the lower bound on the size of a 2-self-centered graph (graph with $\text{rad}(G) = \text{diam}(G) = 2$) goes up from $2n - 5$ to $2n - 3$. Finally, we give some concluding remarks in Section 6.

### 2 Overview of results on maximum Helly families and families without simplices

In this section, we summarize some important theorems connected to the previously presented concepts.

#### 2.1 Non-uniform families of maximal size

Milner, as mentioned in [14], proved the following theorem.

**Theorem 11** (Milner). Let $F \subseteq 2^{[n]}$ be a family without 3-simplex. Then $|F| \leq 2^{n-1} + n$.

The family $F = \{A \mid 1 \in A \subseteq [n]\} \cup \binom{[n]}{2}$ shows sharpness of this theorem. Note that by Theorem 6 we have the family $F = 2^{[n-1]} \cup \binom{[n]}{2}$ is an extremal union-efficient family. Bollobás and Duchet [4, Cor. 3], with the uniqueness statement proven in [5, Thr. 2], and Mulder [33, Thr. 2] generalized the above theorem for $q$-Helly families.
Theorem 12 ([4],[33]). Let $\mathcal{F} \subseteq 2^{[n]}$ be a $q$-Helly family. Then $|\mathcal{F}| \leq 2^{n-1} + \binom{n-1}{\geq n-q}$. Furthermore, equality holds if and only if for some $i \in [n] \mathcal{F}$ equals $\{A \mid i \in A \subseteq [n]\} \cup \binom{[n]}{\leq q-1}$.

It took 25 years to prove that the same bound holds for families without a $q$-simplex. Keevash and Mubayi [25] proved the following theorem.

Theorem 13 ([25]). Let $\mathcal{F} \subseteq 2^{[n]}$ be a family without $q$-simplex. Then $|\mathcal{F}| \leq 2^{n-1} + \binom{n-1}{\geq n-q}$.

2.2 Maximum uniform families without $r$-simplices are typically trivial

In 1974, inspired by a problem of Erdős [14] and the Erdős-Ko-Rado theorem [15], Chvátal [10] conjectured that maximum uniform families without $r$-simplices are typically trivial. This conjecture is known as the Erdős-Chvátal Simplex Conjecture.

Conjecture 14 ([10]). Let $k > r \geq 1$ and $n \geq \frac{k+1}{r}k$. A family $\mathcal{F} \subseteq \binom{[n]}{k}$ without a $r$-simplex contains at most $\binom{n-1}{k-1}$ sets.

The theorem of Erdős-Ko-Rado [15] can be formulated in this form.

Theorem 15 ([15]). A family $\mathcal{F} \subseteq \binom{[n]}{k}$ without a 1-simplex contains at most $\binom{n-1}{k-1}$ sets.

If one forbids all simplices of size at least $q$, instead of only the simplices of size $q$, the problem is easier. In [33, Thr. 1], Mulder proved the upper bound for $q$-Helly families.

Theorem 16 ([33]). Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a $q$-Helly family, where $k > q$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and equality is attained only if $\mathcal{F}$ is a trivial family.

Chvátal [10] proved the $k = r + 1$ case of Conjecture 14. The case $r = 2$, which was the initial problem of Erdős, was proven only 30 years later by Mubayi and Verstraëte [32]. Here they considered hypergraphs without a non-trivial intersecting sub(hyper)graph of size $q+1$. Here it is interesting to note that in that set-up [32, Thr.3], for $k = 3$, if the size of the non-trivial intersecting family has a large size $q+1 \geq 11$ the star is not extremal. Liu [29] proved that the star is still extremal if $k > 3$ and $n$ is sufficiently large.

Conjecture 14 was proven to be true provided that $n$ is sufficiently large in terms of $k, r$ by Frankl and Furedi [17] and in terms of only $r$ by Keller and Lifshitz [26]. Currier [11] proved it when $n \geq 2k - r + 2$. For more insights on the history of Conjecture 14 and related problems, we refer the reader to [30, Sec. 6.4].

2.3 Maximum non-trivial Helly families

Since the extremal families are the trivial ones (Theorem 16), it is natural to wonder what happens with non-trivial families.

Question 17 ([37]). What is the maximum possible size of a non-trivial $k$-uniform $q$-Helly family $\mathcal{F} \subseteq \binom{[n]}{k}$?

Tuza [38, Thr. 1.5] solved this question for $q = 2$ provided that $n >> k$.

Theorem 18 ([38]). For $n$ sufficiently large in terms of $k$, a non-trivial Helly family $\mathcal{F} \subseteq \binom{[n]}{k}$ satisfies

$$|\mathcal{F}| \leq \binom{n-k-1}{k-1} + \binom{n-2}{k-2} + 1.$$ 

Furthermore, the extremal family is unique (up to isomorphism).
3 Maximum size graphs for which every spanning subgraph has a perfect matching

In this section, we prove the $k = 2$ case of Theorem 10 (the $k \geq 3$ case will be handled with a different strategy in Section 4. This case can be stated completely with basic terminology in graph theory: spanning subgraphs and perfect matchings. We first prove the case where the graph is connected.

Theorem 19. Let $n = 2q$ and $G$ be a connected graph of order $n$ for which every spanning (not necessarily connected) subgraph $H$ has a perfect matching. Then the maximum size of $G$ equals $1$ or $4$ if $q \in \{1, 2\}$, or $\left(\frac{q+1}{2}\right)$ if $q \geq 3$. Furthermore, the extremal graph is unique.

Proof. For $q = 1$ we are trivially done. For $q = 2$ the graph does not contain a vertex adjacent to the rest of the vertices, therefore the maximum degree is two and we have at most four edges. Note that the bound is tight since a cycle of length four has the desired properties.

Let us assume $q \geq 3$. Let $M = \{u_i v_j\}_{1 \leq i, j \leq q}$ be a perfect matching of $G$ (which exists by choosing $H = G$). We claim that for every edge of $M$, one of the two vertices is a leaf.

Claim 20. For every edge $u_i v_j \in M$, one of its end-vertices $u_i, v_j$ is a leaf.

Proof. Without loss of generality, we may assume $i = 1$. Since $G$ is a connected graph, $u_1$ or $v_1$ is adjacent to a vertex of $G$ different from $u_1$ and $v_1$. Without loss of generality, we may assume that $u_1$ is adjacent to $u_2$. Now we will prove that $v_1$ is a leaf. The vertex $v_1$ is not adjacent to vertex $u_2$. Since otherwise the edge set $\{u_2 v_1, u_2 v_2, u_2 u_1\} \cup \{u_i v_i\}_{3 \leq i \leq q}$ spans a graph $G$ without a perfect matching. The vertex $v_1$ is not adjacent to any vertex with an index greater than two. Since otherwise, let us assume without loss of generality that $v_1$ is adjacent with $v_3$, then $v_1 v_3 \in E(G)$, then $\{u_2 v_2, u_2 u_1, v_1 v_3, v_3 u_3\} \cup \{u_i v_i\}_{4 \leq i \leq q}$ spans graph $G$ without a perfect matching. Finally, if $v_1$ is adjacent to the vertex $v_2$ then by the previous argument none of the vertices $v_1, v_2, u_1, u_2$ vertices is adjacent to a vertex with index larger than 2, a contradiction since $G$ is a connected graph with more than four vertices. ◊

Since the graph has $q$ leaves, we note that $G$ is a subgraph of a clique $K_q$ with a pendant vertex for each vertex of $K_q$. The latter graph has size $\left(\frac{q+1}{2}\right)$ and every spanning subgraph contains a perfect matching.

As a corollary, we derive Theorem 10 for $k = 2$.

Corollary 21. Let $n = 2q$ and $\mathcal{F} \subseteq \binom{[n]}{2}$ be a non-trivial union-$q$-efficient family. Then $|\mathcal{F}| \leq \left(\frac{q+1}{2}\right)$ whenever $q \geq 3$.

Proof. Note that a family $\mathcal{F} \subseteq \binom{[n]}{2}$ corresponds with the edge-set of a graph $G$. Being union-$q$-efficient implies here that any spanning graph contains a perfect matching. Thus if $G$ is connected we are done by Theorem 19. If $G$ is not connected then we are done by induction since the following inequalities hold $\left(\frac{q+1}{2}\right)^2 > \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right), \left(\frac{q+2}{2}\right)^2 > \left(\frac{q+1}{2}\right) + 1, \left(\frac{q+3}{2}\right)^2 > \left(\frac{q+1}{2}\right) + 4, \left(\frac{q+1}{2}\right)^2 > 4 + 4$ and $\left(\frac{3q+1}{2}\right)^2 > 4 + 1$. ◊

Remark that the maximum size of a family $\mathcal{F} \subseteq \binom{n}{2}$ without $q$-simplex is different for $n = 2q \geq 10$. Let $F^*$ be the graph $K_{n-4}$ with four additional vertices connected to the same vertex of the $K_{n-4}$. This graph has $\binom{n-4}{2} + 4 > \left(\frac{q+2}{2}\right)$ edges, while there are no $q + 1$ edges spanning all vertices of the graph and thus in $\mathcal{F}$ there is no $q$-simplex. This indicates the clear difference between forbidding a $q$-simplex and forbidding all $r$-simplices with $r \geq q$. 

6
4 Largest non-trivial union-\(q\)-efficient families

In this section, we prove Theorem 10 which we restate for the convenience of the reader.

**Theorem.** Let \( n = qk \) where \( k \geq 3 \) and let \( \mathcal{F} \subseteq \binom{[n]}{k} \) be a non-trivial union-\(q\)-efficient family. Then \( |\mathcal{F}| \leq \left( \binom{n-1}{k-1} + q \right) \). Furthermore, the extremal family is unique up to isomorphism.

**Proof.** The case \( q = 1 \) trivially holds. The case \( q = 2 \) holds by an easier version of the proof for \( q \geq 3 \). Thus we assume \( q \geq 3 \). Let \( \mathcal{F} \subseteq \binom{[n]}{k} \) be a non-trivial union-\(q\)-efficient family as in the statement. For every \( j \in [n] \), let \( \mathcal{F}(j) = \{ A \in \mathcal{F} \mid j \in A \} \) be the family of sets in \( \mathcal{F} \) containing \( j \) and \( X(j) = \bigcup \{ A \in \mathcal{F} \mid j \in A \} \subseteq [n] \) be the set of elements which are covered by \( \mathcal{F}(j) \). An important observation is the following.

**Claim 22.** There is no index set \( J \) with \( J \subseteq [n] \) and \( |J| = q - 1 \) such that \( \bigcup_{j \in J} X(j) = [n] \)

**Proof.** Suppose by way of contradiction that there is index set \( J \) with \( J \subseteq [n] \) and \( |J| = q - 1 \) such that \( \bigcup_{j \in J} X(j) = [n] \). Then since \( \mathcal{F} \) is union-\(q\)-efficient, there must be \( q \) sets such that their union is \([n]\). Thus these sets must be disjoint since \( n = qk \), but this is impossible since there will be at least two sets sharing an element from \( J \) by the pigeonhole principle, a contradiction. \( \diamond \)

Next, we prove an upper bound for the size of \( \mathcal{F}(j) \).

**Claim 23.** For every \( j \in [n] \) we have
\[
|\mathcal{F}(j)| \leq \left( \frac{|X(j)| - 2}{k-1} \right) + 1.
\]

**Proof.** The family \( \mathcal{F} \) is union-\(q\)-efficient and non-trivial, thus there are \( q \) sets \( A_1, A_2, \ldots, A_q \in \mathcal{F} \) such that \( \bigcup_{i \in [q]} A_i = [n] \). Thus since \( n = qk \) and \( \mathcal{F} \) is \( k \)-uniform the sets \( A_i, i \in [q] \) are disjoint. Without loss of generality, we may assume that \( j \in A_1 \). By Claim 22 it is easy to note that \( A_2, A_3, \ldots, A_q \subseteq X(j) \). Thus each \( A_i, 1 < i \leq q \), contains a unique element of \([n]\) which is not an element of \( X(j) \). The family \( \mathcal{F}(j) \setminus \{ A_1 \} \) does not covers all elements of \( A_1 \), since otherwise the family \( \mathcal{F}(j) \setminus \{ A_1 \} \cup (\bigcup_{i=2}^{q} \{ A_i \}) \) covers \([n]\), but there are no \( q \) sets covering \([n]\) contradicting to the condition that \( \mathcal{F} \) is union-\(q\)-efficient. Hence \( \mathcal{F}(j) \setminus A_1 \) covers at most \(|X(j)| - 1\) elements of \( X(j) \), thus we have the desired inequality. \( \diamond \)

Let \( G \) be the graph with vertex set \([n]\) for which \( i, j \in E(G) \) if and only if there is no \( A \in \mathcal{F} \) for which \( \{i, j\} \subseteq A \), i.e. \( G \) is a complement of the \( 2 \)-shadow of \( \mathcal{F} \). This graph satisfies the following properties.

**Claim 24.** For every two vertices \( u, v \) in \( G \), their common neighborhood \( G[N(u) \cap N(v)] \) contains a clique on \( q - 2 \) vertices. The minimum degree of \( G \) is at least \( q - 1 \) and the maximum degree is bounded by \((q - 1)k\).

**Proof.** For every \( j \in [n] \) we have \(|X(j)| \leq n - (q - 1)\) by Claim 22. Thus the degree of vertex \( j \) in \( G \) is at least \( q - 1 \) i.e. \( \delta(G) \geq q - 1 \).

By Claim 22, for every \( j, j' \in [n] \) there is \( i_1 \notin X(j) \cup X(j') \), i.e., \( j \) and \( j' \) have a common neighbor in \( G \). One can repeat this for \( J_{\ell} = \{ j, j', i_1, \ldots, i_{\ell} \} \) by taking an \( i_{\ell+1} \notin \bigcup_{\gamma \in J_{\ell}} X(\gamma) \) whenever \( \ell \leq q - 3 \). Thus \( G[\{i_1, \ldots, i_{q-2}\}] \) is a clique and all vertices \( \{i_1, \ldots, i_{q-2}\} \) are adjacent to both \( j \) and \( j' \).

Finally, every \( j \in [n] \) belongs to at least one \( k \)-set in \( \mathcal{F} \) since \( \mathcal{F} \) is non-trivial and thus \( \Delta(G) \leq n - k = (q - 1)k \). \( \diamond \)
Let \( \{d_1, d_2, \ldots, d_n\} \) be the degree sequence of \( G \). As Claim 24 implies that \( G \) satisfies \( \text{diam}(G) = \text{rad}(G) = 2 \) and the property that every common neighborhood of 2 vertices contains a \( K_{q-2} \), by Theorem 30 (for \( q = 3 \)) and Theorem 25 (when \( q > 3 \)) and the handshaking lemma, we know that \( \sum_{i=1}^{n} d_i \geq (q-1)(2n-q) \). If this inequality is strict, decrease some of the values with the constraint that all their values are still at least \( q - 1 \) in such a way that the sum is \( (q-1)(2n-q) \). Let the resulting sequence be \( \{x_1, x_2, \ldots, x_n\} \). Note that the latter sequence is majorized by \( \{q-1, \ldots, q-1, (q-1)k, \ldots, (q-1)k\} \). That is, the sum of the largest \( i \) elements in the latter sequence is at least the sum of the largest \( i \) elements in the sequence \( \{x_1, x_2, \ldots, x_n\} \), for every \( 1 \leq i \leq n \), with equality if \( i = n \). Let \( f : \mathbb{R} \to \mathbb{R} : x \mapsto (n-x-2)k + 1 \). Restricted to the interval \( [q-1, (q-1)k] \), this is a strictly convex function. By Karamata’s inequality [23] and the fact that \( f \) is decreasing, we have

\[
\sum f(d_i) \leq \sum f(x_i) \leq (n-q)f(q-1) + qf((q-1)k).
\]

By Claim 23, where \( |X(j)| = n - d_j \) and double-counting (each set is counted \( k \) times), we conclude that

\[
|F| = \sum_{j \in [n]} \frac{|F(j)|}{k} \leq \frac{(n-q)f(q-1) + qf((q-1)k)}{k} = \frac{n + (n-q)(n-q-1)}{k} = q + \binom{n-q}{k}.
\]

When equality is attained, there are \( q \) elements (without loss of generality \( n - q + 1 \) till \( n \)) belonging to a unique \( k \)-set (since \( d_i = (q-1)k \) and there are at most \( \binom{n-q}{k} \) other \( k \)-sets which do not contain any of these \( q \) elements, so all of these \( k \)-sets need to be contained in \( F \). The first \( q \) sets have to be different and if they are not disjoint, there is an element \( j \in [n] \) for which \( |X(j)| \geq n - k + 2 \), which is a contradiction. Hence equality does occur if and only if there are \( q \) elements belonging to a unique (disjoint) \( k \)-set, and all \( k \)-sets of the remaining \( n-q \) elements belong to \( F \). Noting that this family is union-\( q \)-efficient is immediate since a union of sets from the family can only be equal to \([n]\) if the \( q \) disjoint sets all belong to the family. An example of a maximum family \( F \) and the corresponding graph \( G \) has been given in Figure 1 for \( k = 4, q = 3 \). Here every 4-set within the light grey box belongs to \( F \).

![Figure 1: Sketch of a maximum union-3-efficient family \( \binom{12}{4} \) and the associated graph \( G \) with \( \text{diam}(G) = \text{rad}(G) = 2 \) and the triangle property](image)

We remark that as was the case with \( k = 2 \), the largest non-trivial \( q \)-simplex-free families can have a larger size than the largest non-trivial \( q \)-Helly family. E.g., when \( n = qk \), let \( F^c = \{A \in \binom{[n]}{k} \mid |A \cap [q+2]| \leq 1\} \). It has size \( \binom{n-q}{k} + q\binom{n-q-2}{k-1} - \binom{n-q-2}{k-2} \) and is \( q \)-simplex-free.

### 5 Minimum size of 2-self-centered graphs

Estimating the size of graphs (determining the minimum and the maximum) with some given parameters (mostly order and one other parameter) is a fundamental question in extremal
combinatorics. For example, finding the minimum/maximum size of certain critical graphs with given order and diameter 2 is challenging. In [2, 22, 9] authors proved that the minimum size of a vertex-diameter-2-critical graph (a graph with diameter 2 for which the diameter increases by deleting any of its vertices) is roughly $\frac{2}{5}n$.

In [7] it was proven that a self-centered graph (a graph for which diameter and radius are equal, initially called equi-eccentric graph) with a diameter equal to 2, has a size of at least $2n - 5$. The 2-self-centered graphs with size equal to $2n - 5$ have been characterized in [1]. By observing that a graph with diameter 2 has radius 2 if and only if the maximum degree satisfies $\Delta < n - 1$, the bound of $2n - 5$ edges had been derived before by Erdős and Rényi [12].

A related question was solved in [6], where the non-adjacent vertices have a minimum number of common neighbors. We consider a similar question, where adjacent vertices have at least one common neighbor, i.e. the graph $G$ has the triangle-property: every edge of $G$ is contained in a triangle. This property has been studied before e.g. in [35] for 4-regular graphs. Since every graph with the triangle-property is the union of some triangles, for a connected graph with the triangle-property, the size $m$ satisfies $m \geq \frac{n}{2}(n - 1)$. If $G$ is 2-self-centered and has the triangle-property, we prove that its size is at least $2n - 3$. We first prove such a result for when any 2 vertices in their neighborhood share a $K_{q-2}$ for $q \geq 4$.

5.1 The minimum size of a graph $G$ with a $K_{q-2}$ in the common neighborhood of every 2 vertices

In this subsection, we determine the minimum size of a 2-self-centered graph $G$ with a copy of $K_{q-2}$ in the common neighborhood of every pair of vertices. More precisely, we prove the following theorem.

**Theorem 25.** Let $q \geq 4$ and $n \geq 3q$ be integers. Let $G$ be a graph for which $\text{rad}(G) = 2$ and such that for every $u, v \in G$, $G[N(u) \cap N(v)]$ contains a $K_{q-2}$. Then the number of edges of $G$ is at least $(q - 1)n - \left(\frac{q}{2}\right)$.

The lower bound in Theorem 25 is sharp. Equality is attained by an $n$ vertex graph $G$, such that the vertex set of $G$ is partitioned into $q + 1$ non-empty sets $A_0 = \{q\}, A_1, \ldots, A_q$, where $G[A_0]$ is isomorphic to $K_q$, $A_i$ for $i \in [q]$ are independent sets and for each $i \in [q]$ every vertex of $A_i$ is adjacent to all vertices from $A_0 \setminus \{i\}$.

Since every vertex belongs to a clique $K_q$, we have $\delta(G) \geq q - 1$. We prove the cases $\delta(G) = q - 1$ and $\delta \geq q$ separately in the following two propositions.

First, we prove the statement in a more general form for $q \geq 4$ in the case that the minimum degree is exactly $q - 1$.

**Proposition 26.** Let $q \geq 4$ and $n \geq 2q$ be integers. Let $G$ be a graph with $\text{rad}(G) = 2$, $\delta(G) = q - 1$ such that for every $u, v \in G$, $|N(u) \cap N(v)| \geq q - 2$. Then the number of edges in $G$ is at least $(q - 1)n - \left(\frac{q}{2}\right)$.

**Proof.** Let $v$ be a vertex of minimum degree of $G$, $\text{deg}(v) = \delta(G) = q - 1$ and let the neighborhood of $v$ be $N(v) = \{v_i : i \in [q - 1]\}$. Since for every $i \in [q - 1]$, $v_i$ and $v$ share $q - 2$ common neighbors $G[N[v]]$ is isomorphic to $K_q$. Since $\text{rad}(G) = 2$, for every vertex $v_i$, $i \in [q - 1]$, there is a vertex $s_i \in V \setminus N[v]$ not adjacent to $v_i$. Since each vertex has at least $q - 2$ neighbors in $N(v)$, vertices $s_i$ are distinct. Let us denote

$$R = \{w_1w_2 \in E(G) : \text{deg}(w_1) = q - 1, |N(w_1) \cap N(v)| = q - 2, w_1, w_2 \notin N[v]\}.$$

Let $w_1w_2$ be an edge from $R$. We may assume $N(w_1) \cap N(v) = \{v_i : i \in [q - 2]\}$. Even more, since the common neighborhood of $w_1$, $w_2$ contains $q - 2$ vertices, $w_2$ is adjacent with $v_1, \ldots, v_{q-2}$. 


The vertex \( w_2 \) is also incident with \( \{ s_i : i \in [q-2] \} \), since \( w_1 \) and \( s_i \ (i \in [q-2]) \) have at least \( q-2 \) vertices in the common neighborhood. Note that if \( N(w_2) \cap N(v) = \{ v_i : i \in [q-2] \} \), then \( \deg(s_i) \geq q \) for \( 1 \leq i \leq q-2 \), since \( w_2 \) and \( s_i \) share at least \( q-2 \) common neighbors.

Now we are ready to lower bound the number of edges. There are \( \binom{q}{2} \) edges in \( N[v] \). For each vertex \( u \in V \setminus N[v] \), there are at least \( q-2 \) edges from \( u \) to \( N(v) \), let \( V_1 \subseteq V \setminus N[v] \) be vertices with exactly \( q-2 \) neighbors in \( N(v) \). For each vertex \( u \in V_1 \) either \( \deg(u) = q-1 \) (and it is incident with exactly one edge in \( R \)) or \( u \) is incident with at least \( q \) edges from \( E(G) \setminus R \), since if it is incident to at least one edge from \( R \), then it is adjacent to \( q-2 \) vertices from \( \bigcup_{i \in [q-1]} \{ s_i \} \) none of which has degree \( q-1 \). Note that \( q-2 \geq 2 \). Thus for each vertex in \( V \setminus N[v] \) there are either \( q-1 \) edges to \( N(v) \), or there are \( q-2 \) edges to \( N(v) \) and exactly one edge from \( R \), or there are \( q-2 \) edges to \( N(v) \) and at least two edges in \( E(G[V \setminus N[v]]) \setminus R \). If we associate these edges with the vertex, except the edges in \( E(G[V \setminus N[v]]) \setminus R \) which are taken with a weight of a half, then every edge is counted (with weight) at most once and every vertex in \( V \setminus N[v] \) is associated with a total weight of edges of at least \( q-1 \). Hence there are at least \( \binom{q}{2} + (n-q)(q-1) \) edges. An example for which equality is attained is shown in Figure 2, where the edges in \( R \) are presented in red.

\[ \text{Figure 2: Extremal graph for Proposition 26 with } n=8 \text{ and } q=4 \]

**Remark 27.** The condition \( \text{rad}(G) = 2 \) is necessary here. Without that condition, one can take \( \frac{n-(q-2)}{2} \) copies of \( K_q \) which pairwise intersect in a fixed copy of \( K_{q-2} \). The latter construction has a smaller size.

Next, we prove the case where \( \delta \geq q \). Here the statement is true without the constraint \( \text{rad}(G) = 2 \).

**Proposition 28.** Let \( q \geq 3 \) and \( n \geq 3q - 4 \) be integers. Let \( G \) be a graph such that for every \( u, v \in G \), \( G[N(u) \cap N(v)] \) contains a \( K_{q-2} \) and \( \delta(G) \geq q \). Then the number of edges in \( G \) is at least \( (q-1)n - \binom{q}{2} \).

*Proof.* If \( \delta(G) \geq 2(q-1) \), then by the handshaking lemma we have \( |E(G)| \geq (q-1)n \) and we are done. So assume \( 2q - 3 \geq \delta(G) \geq q \) and let \( a = \delta(G) - (q-1) \). Let \( v \) be a vertex with \( \deg(v) = \delta(G) \). Since for every \( u \in N[v] \), \( uv \) is in a \( K_q \), \( \delta(G[N[v]]) \geq q-1 \). Take a \( K_q \)
containing \( v \) and let \( A \) be the set with the \( a \) other vertices of \( N[v] \). The number of edges in 
\( G[N[v]] \) containing at least one vertex in \( A \) is equal to 
\[
\sum_{u \in A} \deg_{G[N[v]]}(u) - |E[A]| \geq (q - 1)a - \left( \frac{a}{2} \right).
\]
Hence the number of edges in \( G[N[v]] \) is at least \( \binom{q}{2} + (q - 1)a - \left( \frac{a}{2} \right) \). Every vertex \( z \in V \setminus N[v] \) has at least \( q - 2 \) neighbors in \( N(v) \) and at least \( a + 1 \) additional neighbors. This implies that 
\[
|E(G)| \geq \binom{q}{2} + (q - 1)a - \left( \frac{a}{2} \right) + (n - a - q) \left( q - 1 + \frac{a - 1}{2} \right)
\]
\[
= n(q - 1) - \binom{q}{2} + (n - 2a - q) \frac{a - 1}{2}
\]
\[
\geq n(q - 1) - \binom{q}{2}.
\]

**Remark 29.** The bound in Proposition 28 does not hold if one relaxes the condition \( G[N(u) \cap N(v)] \) contains a \( K_{q-2} \) to the condition \( |N(u) \cap N(v)| \geq q - 2 \), as was the case with Proposition 26. Since one can take a graph \( (K_{q-2} \setminus M) + \left( \frac{n-(q-2)}{3} K_3 \right) \) which is the graph that contains a clique of size \( q - 2 \) minus a maximal matching \( K_{q-2} \setminus M \) and \( \frac{n-(q-2)}{3} \) disjoint copies of \( K_3 \), such that every vertex of each \( K_3 \) is adjacent to the \( q - 2 \) vertices of the clique minus a matching. This construction has fewer edges than in Proposition 28 for every \( q \geq 6 \) and for every \( n \geq q + 1 \) congruent to \( q - 2 \) modulo 3.

**5.2 The minimum size of a 2-self-centered graph \( G \) with the triangle-property**

**Theorem 30.** Let \( G \) be an \( n \)-vertex graph satisfying the triangle-property and \( \text{diam}(G) = \text{rad}(G) = 2 \). Then the number of edges of \( G \) is at least \( 2n - 3 \).

**Proof.** First suppose that \( G \) has a vertex \( v \) of degree \( n - 2 \) i.e., \( N[v] = V \setminus \{u\} \) for some vertex \( u \) distinct from \( v \). Let \( c_v \) be the number of components in \( G[N(v)] \). Since \( G \) has diameter 2 and has the triangle-property, \( u \) has at least two neighbors in every component and thus \( |E(G)| \geq (n - 2) + (n - 2 - c_v) + 2c_v \geq 2n - 3 \).

Now assume that \( G \) has the minimum number of vertices for which the statement of Theorem 30 does not hold. Thus \( G \) has no vertex of degree \( n - 1 \). We first observe that the minimum degree of \( G \) is at least 3.

**Claim 31.** We have \( \delta(G) \geq 3 \)

**Proof.** Since \( G \) has the triangle-property, \( \delta(G) \geq 2 \). Suppose by way of contradiction that there is a vertex \( u \) in \( G \) of degree 2, and let \( N(u) = \{a, b\} \). By the triangle-property \( ab \) is an edge of \( G \). If the edge \( ab \) belongs to a triangle different from \( abu \), then \( G \setminus u \) is a smaller graph with a diameter and radius equal to 2 which has the triangle-property, \( n - 1 \) vertices and less than \( 2(n - 1) - 3 \) edges, contradicting the minimality of \( G \).

If \( N(a) \cap N(b) = \{u\} \), then let \( A = N(a) \setminus \{u, b\} \) and \( B = N(b) \setminus \{u, a\} \). Let \( G[A] \) and \( G[B] \) have \( c_a \) and \( c_b \) components respectively. Note here that \( \delta(G[a]), \delta(G[b]) \geq 1 \) since \( G \) has the triangle-property and thus every edge between \( a \) and \( A \) belongs to a triangle with an edge in \( A \). Since \( \text{diam}(G) = 2 \), there is an edge from every component of \( A \) to every component of \( B \),

---

\(^{1}\) this is the graph join of \( K_{q-2} \) and \( \frac{n-(q-2)}{3} K_3 \)
even more, since every edge is in a triangle there are at least 2 edges between each such pair of components. Note that since Hence the size of \( G \) is at least

\[
3 + |A| + |B| + (|A| - c_a) + (|B| - c_b) + 2c_ac_b = 2n - 3 + (2c_ac_b - c_a - c_b) \geq 2n - 3. \]

Since \( G \) has a diameter equal to two and has a triangle-property for every two vertices of \( G \) there is a vertex in the common neighborhood. Thus by Proposition 28 for \( q = 3 \) and \( n \geq 5 \) we are done since \( \delta(G) = 3 \). For \( n \leq 5 \) there are no graphs satisfying conditions of Theorem 30.

6 Conclusion

In this paper, we determined the largest non-trivial family \( \mathcal{F} \subseteq \left( \binom{n}{k} \right) \) which is union-\( q \)-efficient for \( n = qk \). Due to its similarities with the Erdős matching conjecture [13], one may wonder about the largest non-trivial family \( \mathcal{F} \subseteq \left( \binom{n}{k} \right) \) without \( q \) pairwise disjoint sets in the regime where the trivial family \( \left( \binom{kq-1}{k} \right) \) (see [16]) is extremal. The question of Tuza (Question 17), asking for the largest non-trivial family \( \mathcal{F} \subseteq \left( \binom{n}{k} \right) \) which is union-\( q \)-efficient, is still widely open when \( k + q \leq n < qk \). Here the case \( n = q + k \) is equivalent with a hypergraph-Turan problem (Proposition 8). The same question for non-trivial families \( \mathcal{F} \subseteq \left( \binom{n}{k} \right) \) without \( q \)-simplex is equally natural and interesting.

The largest non-trivial \( d \)-wise intersecting families have been determined by O’Neill and Verstraëte [34], proving a conjecture of Hilton and Milner [21]. Analogous to some other results about the equivalence of the largest families, one may expect that these constructions are also the largest non-trivial families which do not contain a \((d - 1)\)-simplex?

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