LIOUVILLE AND CARATHÉODORY COVERINGS IN RIEMANNIAN AND COMPLEX GEOMETRY

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To S. G. Krein with admiration and love.

INTRODUCTION

A Riemannian manifold resp. a complex space $X$ is called Liouville if it carries no nonconstant bounded harmonic resp. holomorphic functions. It is called Carathéodory, or Carathéodory hyperbolic, if bounded harmonic resp. holomorphic functions separate the points of $X$. The problems which we discuss in this paper arise from the following question:

When a Galois covering $X$ with Galois group $G$ over a Liouville base $Y$ is Liouville or, at least, is not Carathéodory hyperbolic?

An infinite abelian$^1$ covering of a Liouville base $Y$ need not be Liouville even for an open Riemann surface $Y$. In [LySu] a $\mathbb{Z}^{\infty}$-covering of this kind was constructed. Moreover, there is a non-Liouville $\mathbb{Z}$-covering of a Liouville complex surface [Li2] (see Remark 1.9.1). Thus, to ensure the Liouville property of $X$ one must subject $Y$ to a stronger condition.

By this reason, we require $Y$ to be compact or, more generally, to carry no nonconstant bounded subharmonic resp. plurisubharmonic functions. Then, to some extent, the coverings over Riemannian and complex spaces behave similarly. Roughly speaking, $X$ is Liouville if $G$ is small enough, say nilpotent [LySu, Li2]; and a solvable cocompact covering can be even Carathéodory hyperbolic (see Theorem 1.6 and §3). But in the intermediate class of polycyclic coverings this similarity fails: such a covering over a compact Riemannian resp. Kähler base $Y$ is Liouville [Ka1], while there is a non-Kähler compact complex surface with non-Liouville polycyclic universal covering [Li2] (see Theorem 1.1 and §4).

We start in §1 with a brief survey of some known results, sketching a few proofs, and proceed with certain new observations. In particular, combining a theorem of Varopoulos [VSCC] with a theorem in [LySu], we establish that a $G$-covering over a compact Riemannian resp. Kähler manifold is Liouville if $G$ is an extension of an almost nilpotent group by $\mathbb{Z}$ or by $\mathbb{Z}^2$ (see Theorems 1.4, 1.6, and Corollary 1.8).

The authors thank the Max-Planck-Institut für Mathematik in Bonn for hospitality. The research of the first author was partially supported by the Fund of the Israel Science Foundation.

$^1$A Galois covering $X \to Y$ with Galois group $G$ is referred to as a $G$-covering. If $G$ is abelian (resp. nilpotent, solvable, polycyclic, etc.) the covering is called abelian (resp. nilpotent, solvable, polycyclic, etc.). It is said to be cocompact if its base $Y$ is compact.
We discuss Liouville-type properties not only for $G$-coverings $X \to Y$ but also for general $G$-spaces $X$, Riemannian or complex. We show, in particular, that $X$ must be Liouville whenever the induced diagonal $G$-action in $X \times X$ has a dense orbit (Proposition 1.10(b)).

Further, we consider the period subgroup of $G$ consisting of all $g \in G$ that do not affect the bounded harmonic resp. holomorphic functions on $X$. If the given $G$-action $T$ on $X$ is cocompact, or $T$ is ultra-Liouville (see §1.5 below for the definition) and $G$ is amenable, then the period subgroup contains any central element of $G$ and, moreover, any element with finite conjugacy class [Li2] (see Corollary 1.14). In §2 we extend the latter result to the FC-hypercentral elements and establish Liouville property of FC-(hyper)nilpotent coverings (see Definitions 2.1 for terminology).

§3 and §4 are devoted to certain examples of non-Liouville and, especially, Carathéodory hyperbolic cocompact coverings with relatively small Galois groups. In §3 for any compact Riemann surface $Y$ of genus $g \geq 2$ we produce a Carathéodory hyperbolic two-step solvable covering $X$ over $Y$. This is based on a construction due to Lyons and Sullivan [LySu].

In §4 we consider the universal covering $X \to \mathcal{I}$ over an Inoue surface $\mathcal{I}$ [In], which is a non-Kähler compact complex surface with a polycyclic fundamental group $G = \pi_1(\mathcal{I})$. In fact, $X \cong \mathbb{H} \times \mathbb{C}$ ($\mathbb{H}$ is the upper halfplane); it is neither Liouville nor Carathéodory. We show that $X$ admits bounded holomorphic functions which are nonconstant on the orbits of suitable infinite conjugacy classes in $G$. Notice that, by Corollary 1.14, this is impossible for a finite conjugacy class.

Throughout the paper, all manifolds and complex spaces are assumed to be connected.

§1. Liouville-type properties of coverings and $G$-spaces: A survey

This brief survey is neither complete nor chronological; it contains only some selected results on Liouville-type properties. We do not address the case of harmonic functions on a discrete group with a probability measure, which is closely related to our topic (see e. g. [Av, Fu, Mar, KaVe, Ka1, Ka3, VSCC, Wo]).

Coverings over a compact base

1.1. Theorem [LySu, Ka1]. Let $X \to Y$ be a Galois covering with Galois group $G$ over a Riemannian manifold $Y$. Suppose that the base $Y$ is compact; then

a) $X$ is Liouville whenever $G$ is polycyclic\(^2\) or of subexponential growth\(^3\) [Ka1];

b) if $G$ is nilpotent, then $X$ carries no nonconstant positive harmonic functions\(^4\) [LySu].

Furthermore, for any base $Y$ the group $G$ must be amenable\(^5\) if $X$ is Liouville [LySu].

1.2. Remarks. 1. Every compact Riemann surface $R$ of genus $g \geq 2$ admits a non-Liouville solvable covering [LySu] (see also §3 for a stronger example of such kind). Thus, Theorem 1.1(a), in general, does not hold for nonpolycyclic solvable cocompact coverings.

\(^{2}\)i. e. $G$ admits a finite normal series with cyclic quotients, or, equivalently, $G$ is solvable and all its subgroups are finitely generated (see e. g. [Ha,Se]; in [Ha] polycyclic groups were called supersolvable).

\(^{3}\)Any finitely generated solvable nonpolycyclic group is of exponential growth (see [Mi]).

\(^{4}\)See also [Gui, Mar, LiPi].

\(^{5}\)According to von Neumann, $G$ is called amenable if the Banach space $L^\infty(G)$ of all bounded functions on $G$ admits a $G$-(right)invariant mean; see, for instance, [Gre].
2. Let \( X \xrightarrow{G} Y \xrightarrow{K} Z \) be the tower of Galois coverings corresponding to a group extension \( 1 \rightarrow G \rightarrow \widetilde{G} \rightarrow K \rightarrow 1 \). If \( K \) is finite then \( Y = X/G \) is compact. Hence, the statements (a), (b) of Theorem 1.1 hold true for any finite extension \( \widetilde{G} \) of \( G \) if \( G \) is as in these statements.

3. One says that a group \( G \) is almost nilpotent\(^6\) (resp. almost solvable, almost polycyclic, etc.) if it contains a nilpotent (resp. solvable, polycyclic, etc.) subgroup of finite index. Such a subgroup may clearly be assumed being normal. Thus, by Theorem 1.1(a), (b), an almost polycyclic resp. an almost nilpotent covering over a compact Riemannian manifold is Liouville resp. carries no nonconstant positive harmonic function.

4. A holomorphic function on a Kähler manifold is harmonic with respect to the Laplace-Beltrami operator related to the Kähler metric. Hence, Theorem 1.1(a) holds true for holomorphic functions on coverings of compact Kähler manifolds. It holds also true for compact semi-Kählerian manifolds \([Ka2]\). The latter class of Hermitian manifolds may actually be characterized by the harmonicity of all holomorphic germs \([Ga]\).

5. Theorem 1.1(a) does not hold, in general, in the case of holomorphic functions on polycyclic coverings of non-Kähler compact complex manifolds. For instance, consider the universal covering \( X \) of the Inoue surface \( I \) \([In, Li2]\); see also §4). The Galois group \( G = \pi_1(I) \) contains a normal subgroup \( G_0 \cong \mathbb{Z}^3 \) with \( G/G_0 \cong \mathbb{Z} \). Hence, \( G \) is a semidirect product \( \mathbb{Z}^3 \rtimes \mathbb{Z} \) and so it is a metabelian (i. e. a two-step solvable) polycyclic group. Furthermore, \( X \cong \mathbb{H} \times \mathbb{C} \) (\( \mathbb{H} \) is the upper halfplane) is not Liouville. However, nilpotent coverings of compact complex spaces are Liouville \([Li2]\); see Theorem 1.6 below).

6. The last assertion of Theorem 1.1 has no direct analog in complex geometry. For instance, set \( U = \{(x, y) \in \mathbb{C}^2 \mid \text{either } |x| < 1 \text{ and } y \neq 2 \text{ or } x \neq 2 \text{ and } |y| < 1 \} \). Then \( \pi_1(U) \) is a free group of rank 2 (i. e. nonamenable), and the universal covering of \( U \) is Liouville. Moreover, for any finitely presented group \( G \) there is a Stein manifold \( Y \) with \( \pi_1(Y) \cong G \) and Liouville universal covering \( X \) \([V. Lin, unpublished; cf. also [Li1, §7.2]).\)

1.3. Definition. A Riemannian manifold \( Y \) is called transient if it carries a nonconstant bounded subharmonic function, or, equivalently, if it possesses positive Green function. Nontransient manifolds are called recurrent; this property is equivalent to the recurrence of the random motion on \( Y \) (see e. g. [SNWC, Gri, LySu]). In \([Li2]\) the following terminology was suggested: a Riemannian manifold resp. a complex space \( Y \) is called ultra-Liouville if any bounded continuous subharmonic resp. plurisubharmonic function on \( Y \) is constant. Ultra-Liouville Riemannian manifolds are recurrent, and vice versa.

Any connected Zariski open subset \( Y \) of a compact complex space \( \overline{Y} \) (for instance, any quasiprojective complex variety \( Y \)) is ultra-Liouville. Indeed, by a theorem of Grauert and Remmert \([GraRe]\) (see also \([BoNa]\)) every bounded plurisubharmonic function on \( Y \) admits a plurisubharmonic extension to \( \overline{Y} \) and, hence, by the maximum principle, it is constant. Note that a smooth quasiprojective complex variety, being endowed with a Riemannian metric, may be transient; e. g., this is so for \( Y = \mathbb{C}^n, \ n \geq 2 \), with its Euclidean metric.

The following recurrence criterion of cocompact coverings was proved in \([VSCC, X.3]\).\(^7\)

\(^6\)or virtually nilpotent, or also nilpotent–by–finite.

\(^7\)See the references therein. For abelian coverings this theorem was proved in \([Gui]\) and \([LySu]\). For the classical case of Riemann surfaces see e. g. \([My, Ne, Roy, Mo, Ts]\).
1.4. Theorem. Let $X \to Y$ be a Galois covering with Galois group $G$ over a compact Riemannian manifold $Y$. Then $X$ is recurrent (or, which is equivalent, ultra-Liouville) if and only if $G$ is a Varopoulos group, that is, a finite extension of one of the groups $1$, $\mathbb{Z}$, and $\mathbb{Z}^2$.

Ultra-Liouville actions and coverings over a noncompact base

1.5. Notation and definitions. Given a Riemannian manifold resp. a complex space $X$, we denote by $I(X)$ the group of all its homotheties$^8$ Homo $(X)$ resp. the group of all its biholomorphic automorphisms Aut $(X)$. By $\mathcal{H} = \mathcal{H}(X)$ we denote the space $Harm^\infty(X)$ resp. $H^\infty(X)$ of all bounded complex valued harmonic resp. holomorphic functions on $X$.

Clearly, $I(X)$ acts in $\mathcal{H}(X)$. We say that the action of a subgroup $G \subseteq I(X)$ on $X$ is ultra-Liouville if $X$ admits no nonconstant $G$-invariant bounded continuous subharmonic resp. plurisubharmonic functions. If the quotient $Y = X/G$ exists in the same category, then the $G$-action on $X$ is ultra-Liouville if and only if $Y$ is ultra-Liouville in the sense of Definition 1.3.

Let $Z(G)$ denote the center of a group $G$. Consider the upper central series of $G$

$$1 = Z_0(G) \triangleleft Z_1(G) \triangleleft Z_2(G) \triangleleft \cdots \triangleleft Z_n(G) \triangleleft \cdots \triangleleft G;$$

here $Z_n(G)$ is the total preimage $p_{n-1}^{-1}(Z(G/Z_{n-1}(G)))$ of $Z(G/Z_{n-1}(G))$ under the natural surjection $p_{n-1} : G \to G/Z_{n-1}(G)$, $n = 1, 2, \ldots$. The upper central series is continued transfinitely in the usual way, by defining $Z_\alpha(G) = \bigcup_{\beta < \alpha} Z_\beta(G)$, when $\alpha$ is a limit ordinal.

The group $G$ is called $\omega$-nilpotent if it coincides with the union $Z_\omega(G) = \bigcup_{n \in \mathbb{N}} Z_n(G)$. $G$ is called hypernilpotent, or also hypercentral, if $G = Z_{\lim}(G)$, where $Z_{\lim}(G) = \bigcup_{\alpha} Z_\alpha(G)$ is the hypercenter of $G$ (here $\alpha$ runs over all the ordinals).

The following theorem was proved for $\omega$-nilpotent coverings of Riemannian manifolds in [LySu], and in its present form in [Li2], by different methods.

1.6. Theorem (LySu, Li2). Let $X$ be a Riemannian manifold resp. a complex space, and let $G$ be a hypernilpotent subgroup of $I(X)$. The space $X$ is Liouville whenever the $G$-action on $X$ is ultra-Liouville. In particular, if $X \to Y$ is a hypernilpotent covering over an ultra-Liouville (Riemannian or complex) base $Y$, then $X$ is Liouville.

1.7. Remark. By the maximum principle, any cocompact $G$-action$^9$ on $X$ is ultra-Liouville. Hence, for hypernilpotent coverings over a compact Riemannian manifold $Y$ the last assertion of Theorem 1.6 follows from Theorem 1.1(a) (but not vice versa!). Indeed, being a quotient of a finitely generated group $\pi_1(Y)$, the Galois group of a Galois covering $X \to Y$ is finitely generated, too. But a finitely generated hypernilpotent group is nilpotent and polycyclic (see [Ha, Se] resp. Remark 2.2.1 and references therein for the case of finitely generated $\omega$-nilpotent resp. finitely generated hypernilpotent groups).

However, unlike Theorem 1.1(a), Theorem 1.6 applies to complex spaces (Kähler or not) and also to ultra-Liouville actions, which may be neither free nor properly discontinuous nor cocompact.

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$^8$By a homothety of a Riemannian manifold $(X, d)$ we mean a transformation $g : X \to X$ such that $d(gx, gy) = Cd(x, y)$ with some constant $C = C(g)$ which does not depend on $x, y \in X$.

$^9$That is, a $G$-action such that $GT = X$ for some compact set $T \subseteq X$. 
From Theorems 1.4, 1.6 we obtain such a corollary.

1.8. **Corollary.** Let \( X \to Y \) be a Galois covering with Galois group \( G \) over a compact Riemannian resp. Kähler manifold \( Y \). If \( G \) is an extension of an almost hypernilpotent group by a Varopoulos group\(^{10}\), then \( X \) is Liouville.

1.9. **Remarks.** 1. Corollary 1.8 does not apply to general compact complex manifolds. Indeed [Li2], let \( X \to \mathcal{I} \) be the universal covering over the Inoue surface \( \mathcal{I} \) (see Remark 1.2.3 and §4). The semidirect decomposition \( G \cong \mathbb{Z}^3 \ltimes \mathbb{Z} \) provides the tower of Galois coverings \( X \to Y \to \mathcal{I} \). If \( Y \) were ultra-Liouville, then, by Theorem 1.6, the abelian covering \( X \to Y \) would be Liouville, which is wrong. Hence, due to Theorem 1.6, \( Y \) is a Liouville, but not ultra-Liouville \( \mathbb{Z} \)-covering over a compact complex surface \( \mathcal{I} \). Furthermore, the above covering \( X \to Y \) produces a tower of three \( \mathbb{Z} \)-coverings \( X = X_1 \to X_2 \to X_3 \to X_4 = Y \). Since \( X_4 = Y \) is Liouville and \( X_1 = X \) is not Liouville, at least one of \( X_i \), \( 1 \leq i \leq 3 \), is a non-Liouville \( \mathbb{Z} \)-covering over a Liouville base.

2. Theorem 1.4 does not hold for coverings over a noncompact ultra-Liouville Riemannian manifold \( Y \). Consider, for instance, the maximal abelian covering \( X \to Y \) over the punctured Riemann sphere \( Y = \mathbb{P}^1 \setminus \{3 \text{ points}\} \cong \mathbb{C} \setminus \{0, 1\} \). The Riemann surface \( X \) can be realized as an analytic curve in \( \mathbb{C}^2 \), namely, the curve with the equation \( e^x + e^y = 1 \). The covering projection \( X \to Y \cong \mathbb{C} \setminus \{0, 1\} \) is \( (x, y) \mapsto e^x \). The Galois group \( G \) of this covering is isomorphic to \( H_1(Y; \mathbb{Z}) \cong \mathbb{Z}^2 \). It is known [McKSu, LyMcK] that \( X \) is transient; hence, \( X \) is not ultra-Liouville whereas \( G \cong \mathbb{Z}^2 \) is a Varopoulos group. Note that \( X \) in this example is Liouville (see [Dem, Wa, Sh] or Theorem 1.6).

The next proposition contains some new observations concerning ultra-Liouville actions.

1.10. **Proposition.** a) Let the action of a subgroup \( G \subseteq I(X) \) on a Riemannian manifold resp. on a complex space \( X \) be ultra-Liouville. Then any \( G \)-orbit in \( X \) is a uniqueness set for the function space \( \mathcal{H} = \mathcal{H}(X) \). I.e., if \( h \in \mathcal{H} \), \( x_0 \in X \), and \( h \mid Gx_0 = 0 \), then \( h = 0 \). b) If the induced diagonal \( G \)-action \( g : (x, y) \mapsto (gx, gy) \) on \( X \times X \) is ultra-Liouville, then \( X \) is Liouville.

The proof will be done in §1.18.

1.11. **Remark.** 1. For complex spaces the following stronger form of \( (a) \) was proven in [BoNa]: a bounded holomorphic function on a complex space \( X \) with an ultra-Liouville \( G \)-action is constant whenever the set of its values on some \( G \)-orbit is finite.

2. The complement \( Gx_0 \setminus K \) of any finite subset \( K \subseteq Gx_0 \) also is a uniqueness set for \( \mathcal{H} \).

3. It follows from Proposition 1.10(b) that a complex space \( X \) is Liouville if the action of the group \( I(X) \) on \( X \) is almost doubly transitive, meaning that the induced diagonal \( G \)-action on \( X \times X \) possesses a dense orbit. This simple observation yields yet another proof of the classical Liouville Theorem. Indeed, the affine transformation group \( \text{Aff}(\mathbb{C}) = \text{Aut}(\mathbb{C}) \) is doubly transitive on \( \mathbb{C} \); hence, the diagonal action on \( \mathbb{C} \times \mathbb{C} \) is transitive outside of the diagonal \( \Delta \), and \( (\mathbb{C} \times \mathbb{C}) \setminus \Delta \) is a dense orbit in \( \mathbb{C} \times \mathbb{C} \).

\(^{10}\)see Theorem 1.4.
1.12. Definition. Given a $G$-space $X$, the corresponding $G$-action in the vector space $\mathbb{C}^X$ of all complex valued functions on $X$ is denoted by $f \mapsto f^g$, $f^g(x) = f(gx)$. We say that an element $g \in G$ is a period of a function $f \in \mathbb{C}^X$, or $f$-period, if $f$ is $g$-invariant, i.e. $f(gx) = f(x)$ for all $x \in X$. For a function $f \in \mathbb{C}^X$ the set of all its periods form a subgroup in $G$, which is denoted by $G_f$. It is a stationary subgroup of $f$ with respect to the $G$-action on $\mathbb{C}^X$. For a subspace $\mathcal{F} \subseteq \mathbb{C}^X$ denote by $G_{\mathcal{F}}$ the intersection of all the subgroups $G_f$, $f \in \mathcal{F}$. We call $G_{\mathcal{F}}$ the subgroup of $\mathcal{F}$-periods, or simply the period subgroup. It is easily seen that $G_{\mathcal{F}}$ is a normal subgroup of $G$ if $\mathcal{F}$ is $G$-invariant. In particular, for any subgroup $G \subseteq I(X)$ the $H$-period subgroup $G_H$ is normal in $G$, where, as before, $H = H(X)$ denotes the space of bounded harmonic resp. holomorphic functions on a Riemannian manifold resp. on a complex space $X$.

For a subgroup $G \subseteq I(X)$ and an element $s \in I(X)$ we denote by $[s, G]$ the subgroup of $I(X)$ generated by all the commutators $[s, g] = sgs^{-1}g^{-1}$, $g \in G$.

The following theorem provides certain information on the period subgroup $I(X)_f$ of any bounded harmonic resp. holomorphic function $f$ on $X$.

1.13. Theorem [Li2, Thms. 2.10, 3.9]. Let, as before, $X$ be a Riemannian manifold resp. a complex space, and let $G$ be a subgroup of the group $I(X) = \text{Homo}(X)$ resp. $I(X) = \text{Aut}(X)$. Suppose that one of the following two conditions is fulfilled:

* $G$ is amenable and its action on $X$ is ultra-Liouville;
* the action of $G$ on $X$ is cocompact.

Let $f$ be a bounded harmonic resp. holomorphic function on $X$. Suppose that an element $s \in I(X)$ satisfies the condition $[s, G] \subseteq I(X)_f$. Then $s \in I(X)_f$. In other words, if $f$ is invariant under all the commutators $[s, g] = sgs^{-1}g^{-1}$, $g \in G$, then $f$ is also $s$-invariant.

In Corollary 1.14 and Remarks 1.15 below we use the same notation as in Theorem 1.13; see also Definitions 1.5, 1.12.

1.14. Corollary [Li2, Lemma 3.3 and Thms. 3.4, 3.9]. Either of the above conditions (*) implies the following statements:

a) The center $Z(G)$ of $G$ is contained in the $H$-period subgroup $G_H$. Moreover, the hypercenter $Z_{\text{lim}}(G) = \bigcup_{\alpha} Z_{\alpha}(G)$ of $G$ is contained in $G_H$, as well. Hence, $X$ is Liouville whenever $G$ is hypernilpotent or almost hypernilpotent\(^{11}\).

b) If the center $Z(G)$ of $G$ is nontrivial, then the space $X$ is not Carathéodory hyperbolic. In particular, a Galois covering $X$ over a quasiprojective variety cannot be Carathéodory hyperbolic whenever its Galois group is an amenable group with nontrivial center.

c) If an element $s \in G$ is central in a finite index subgroup $S \subseteq G$ (or, more generally, is contained in the centralizer of such a subgroup in $G$), then $s$ is an $H$-period: $s \in G_H$.

1.15. Remarks. 1. The statement $Z(G) \subseteq G_H$ of Corollary 1.14(a) follows immediately from Theorem 1.13. Further, to prove the inclusion $Z_{\text{lim}}(G) \subseteq G_H$ one shows, by transfinite

\(^{11}\)Note that the condition b) in §3.1 of [Li2] must sound as follows: $[H, H_\alpha] \subseteq \bigcup_{\beta < \alpha} H_\beta$ for each $\alpha \in A$.\n
induction, that the members $Z_\alpha(G)$ of the transfinite upper central series of $G$ are contained in $G_H$. In turn, Corollary 1.14(a) gives a proof of Theorem 1.6.

2. We shall see in §3 that Corollary 1.14(b) might be wrong, even for solvable coverings of compact Riemann surfaces, if one omits the condition that the center is nontrivial.

3. Corollary 1.14(c) shows that if the conjugacy class $s^G = \{g^{-1}sg \mid g \in G\}$ of an element $s \in G$ is finite, then any function $h \in \mathcal{H}$ is constant on the $s^G$-orbit $s^Gx$ of any point $x \in X$. An element with the finite conjugacy class is called an FC-element; in §2 we study some generalizations and applications of this property.

On the other hand, in general, holomorphic functions need not be constant on the orbits of infinite conjugacy classes, even for a free cocompact holomorphic $G$-action of a (nonnilpotent) polycyclic group $G$ (see §4, especially Proposition 4.4(b) and Remark 4.7).

4. For some applications of Corollary 1.14(b) see [DetZa, DetOrZa].

Some proofs
Following the scheme suggested in [Li2], we sketch here the proofs of Proposition 1.10 and Theorem 1.13.

We denote by $\beta G$ the Stone-Čech compactification of a discrete topological space $G$, or, which is the same, the Gel’fand spectrum of the Banach algebra $L^\infty(G)$ of all bounded complex valued functions on $G$. Recall that the space $\beta G$ is compact and Hausdorff, and $L^\infty(G) \cong C(\beta G)$. For $f \in L^\infty(G)$ denote by $\hat{f}$ the unique continuous extension of $f$ to $\beta G$, and by $M(f) \subseteq \beta G$ the peak point set of the function $\hat{f}$:

$$M(f) = \left\{ \xi \in \beta G \mid |\hat{f}(\xi)| = \|\hat{f}\|_{C(\beta G)} \right\}.$$ 

The right action of a discrete group $G$ onto itself extends to the right $G$-action on $\beta G$.

Let $X$ be a Riemannian manifold resp. a complex space, and let $G$ be a subgroup of the group $I(X)$ (see 1.5). For any function $h \in \mathcal{H} = \mathcal{H}(X)$ we set $\|h\|_X = \sup_{x \in X} |h(x)|$. Let $\mathcal{K} = \mathcal{K}(X)$ denote the convex cone of all nonnegative bounded continuous subharmonic resp. plurisubharmonic functions on $X$.

1.16. Proposition [Li2]. Let $X, G$, and $\mathcal{H}$ be as above. Assume that

(i) the $G$-action on $X$ is ultra-Liouville, i.e. the cone $\mathcal{K}$ contains no nonconstant $G$-invariant function.

Let $h \in \mathcal{H}$. Set $h_x(g) = h(gx)$ and $\varphi_h(x) = \|\hat{h}_x\|_{C(\beta G)}$. Then

a) $\varphi_h = \text{const}$ and b) $\|\hat{h}_x\|_{C(\beta G)} = \varphi_h = \|h\|_X$;

c) the peak point set $M(\hat{h}_x) \subseteq \beta G$ of the function $\hat{h}_x$ does not depend on $x \in X$; moreover, the subset $M(h) := M(\hat{h}_x) \subseteq \beta G$ is $G$-invariant;

d) for any $G$-invariant regular probability Borel measure $\mu$ on $\beta G$ the $L^2(\mu)$-class $[\hat{h}_x]$ of the function $\hat{h}_x$ does not depend on $x \in X$. 

If, in addition, the group $G$ is amenable, then

e) $\beta G$ carries a $G$-invariant probability measure $\mu$ supported in $M(h)$;

f) $h = 0$ whenever $\left[ \hat{h}_x \right] = 0$ in $L^2(\mu)$ for a measure $\mu$ as in (e).

Sketch of the proof. Note that the space $\mathcal{H}$ and the convex cone $\mathcal{K}$ satisfy the following two conditions (ii), (iii):

(ii) $\mathcal{H}$ contains all the constant functions, and for any $\| \cdot \|_X$-bounded subset $\mathcal{F} \subset \mathcal{H}$ the function $k^2_{\mathcal{F}}, \ k^2_{\mathcal{F}}(x) = \sup_{f \in \mathcal{F}} |f(x)|^2$, belongs to the cone $\mathcal{K}$;

(iii) for any closed ball $B$ in the space $BC(X)$ of all complex valued bounded continuous functions on $X$ the sets $\mathcal{H} \cap B$ and $\mathcal{K} \cap B$ are closed in the compact open topology.
Thus, by (ii), $\varphi^2_p \in K$. Since the function $\varphi^2_p$ is $G$-invariant, (a) follows from (i). Clearly,

$$\varphi_p \equiv \varphi_p(x) = \|\hat{h}_x\|_{C(\beta G)} = \|h_x\|_{L^\infty(G)} = \sup_{g \in G} |h(gx)| \leq \sup_{y \in \chi} |h(y)| = \|h\|_\chi.$$ 

If the latter inequality were strict, then for some $x_0 \in X$ we would have

$$\varphi_p < |h(x_0)| \leq \sup_{g \in G} |h^g(x_0)| = \sup_{g \in G} |h(gx_0)|$$

$$= \sup_{g \in G} |h(x_0)(g)| = \|\hat{h}_{x_0}\|_{C(\beta G)} = \varphi_p(x_0) = \varphi_p,$$

which is impossible; this proves (b).

Given $x_0 \in X$ and a point $\xi_0$ in the peak point set $M(\hat{h}_{x_0})$, consider the function $h^{\xi_0}(x) = \hat{h}_x(\xi_0)$. It follows from (iii) that this function is in $H$, and the function $|h^{\xi_0}(x)|$ attains its maximal value $\|h\|_\chi$ at the point $x = x_0$. The maximum principle

(iv) $h = \text{const}$ whenever $h \in H$ and $|h(x_0)| = \|h\|_\chi$ for some point $x_0 \in X$

implies that $h^{\xi_0} = \text{const}$. Hence,

$$\left|\hat{h}_x(\xi_0)\right| = |h^{\xi_0}(x)| \equiv |h^{\xi_0}(x_0)| = \|h\|_\chi = \left|\hat{h}_{x_0}(\xi_0)\right|.$$

This shows that $\xi_0 \in M(\hat{h}_x)$ for any $x \in X$, which proves the first assertion of (c). The constant function $h^{\xi_0}$ is certainly $G$-invariant, and hence $h^{\xi_0}g = h^{\xi_0}$ for any $g \in G$. This yields $\left|\hat{h}_{x_0}(\xi_0g)\right| = |h^{\xi_0}g(x_0)| = |h^{\xi_0}(x_0)| = \|h\|_\chi$ and $\xi_0g \in M(\hat{h}_{x_0}) = M(h)$, which proves the second assertion of (c).

Given a $G$-invariant regular probability Borel measure $\mu$ on $\beta G$, define the function

$$\Phi^2: X \to \mathbb{R}, \quad \Phi^2(x) = \left\|\left[\hat{h}_x\right]_{L^2(\mu)}^2 = \int_{\beta G} \left|\hat{h}_x(\xi)\right|^2 d\mu(\xi).$$

This function is $G$-invariant, and it follows from (ii), (iii) that $\Phi^2 \in K$; by (i), $\Phi^2 = \text{const}$. Fix a point $x_0 \in X$, and consider the mapping $X \ni x \mapsto F(x) = \left[\hat{h}_x\right] \in L^2(\mu)$ and the inner product $\psi(x) = \langle F(x), F(x_0) \rangle$. It follows from (iii) that $\psi \in H$. Clearly,

$$|\psi(x)| \leq \|F(x)\|_{L^2(\mu)} \|F(x_0)\|_{L^2(\mu)} = \Phi(x)\Phi(x_0) \equiv \Phi^2(x_0)$$

and $|\psi(x_0)| = \Phi^2(x_0)$; hence, by maximum principle (iv), $\langle F(x), F(x_0) \rangle$ is const. Set $a = F(x_0)$ and $b = F(x)$; then we have $\langle b, a \rangle = \|a\|^2$ and $\|b\| = \|a\|$. Since the norm in the Hilbert space $L^2(\mu)$ is strictly convex, this implies $b = a$, that is, $F = \text{const}$. This proves (d).

The statement (e) follows from (c) and the Fixed Point Theorem for amenable groups [Gre, Thm. 3.3.5] applied to the natural $G$-action on the convex compact set of all probability measures supported in the $G$-invariant set $M(h)$.

Finally, (f) follows from (e). Indeed, the function $\hat{h}_x$ is continuous on $C(\beta G)$, and hence $\left[\hat{h}_x\right] = 0$ implies $\hat{h}_x \mid \text{supp} \mu = 0$. Since $\text{supp} \mu \subseteq M(h) = M(\hat{h}_x)$, it follows that $\hat{h}_x = 0$ for any $x \in X$. Thus, $h = 0$. \qed
1.17. Remark. Let $X$ be a topological space endowed with a $G$-action preserving a subspace $\mathcal{H} \subset BC(X)$ and a convex cone $\mathcal{K} \subset BC_{\mathbb{R}}(X)$. Suppose that the conditions $(i) – (iv)$ introduced above are fulfilled. All the assertions of Proposition 1.16 hold true in this more general setting. This yields analogs of Theorems 1.6 and 1.13 for certain equivariant second order elliptic operators on smooth manifolds and for harmonic functions on discrete groups [Li2, §2.12–2.15]. Moreover, there is a version of Proposition 1.16 which applies to the case when, instead of a $G$-action on $X$, one deals with $G$-actions in the space $\mathcal{H}$ and in the cone $\mathcal{K}$. This leads to an analog of Theorem 1.6 for suitable complex Lie group actions on complex spaces (see [Li2, §2 and Thm. 2.17]).

1.18. Proof of Proposition 1.10. $(a)$ is an immediate consequence of Proposition 1.16$(a, b)$. To prove $(b)$, fix a function $h \in \mathcal{H}(X)$ and define the function $\tilde{h} \in \mathcal{H}(X \times X)$ by $\tilde{h}(x, y) = h(x) - h(y)$. Clearly, $\tilde{h}$ vanishes on the diagonal $\Delta \subset X \times X$, which is invariant under the diagonal $G$-action in $X \times X$; hence, $\tilde{h}$ vanishes on the orbit of any point $(x, x) \in \Delta$. Since the diagonal action on $X \times X$ is assumed to be ultra-Liouville, the statement $(a)$ implies $\tilde{h} = 0$. Thus, $h = \text{const}$ and $X$ is Liouville. \hfill $\square$

1.19. Proof of Theorem 1.13 for amenable $G$ and an element $s \in G$. Actually, as in the proof of Proposition 1.16, the only essential assumptions about the space $\mathcal{H}$ and the convex cone $\mathcal{K}$ are those $(i)$-$(iv)$ above. We deal with a function $f \in \mathcal{H}$ and an element $s \in G$ such that $f^{[s, g]} = f$ for all $g \in G$, where $f^{[s, g]}(x) = f([s, g]x) = f(sgs^{-1}g^{-1}x)$. Thus, $f_{sx}(g) = f(gs) = f(gsx) = f^{s}(gsx) = (f^{s})_{x}(g)$ for all $g \in G$ and all $x \in X$, and hence

$$\widehat{f_{sx}} = (\widehat{f^{s}})_{x} \quad \text{for all } x \in X. \quad (*)$$

Set $h = f^{s} - f \in \mathcal{H}$. We must show that $h = 0$. Let $\mu$ be a measure as in Proposition 1.16$(e)$. Since it is $G$-invariant, Proposition 1.16$(d)$ implies that the $L^{2}(\mu)$-class $[\hat{f}_{x}]$ does not depend on $x \in X$. In particular, $[\hat{f_{sx}}] = [\hat{f}_{x}]$ and

$$[\hat{f_{sx}} - \hat{f}_{x}] = [\hat{f_{sx}} - \hat{f}_{x}] = [\hat{f_{sx}}] - [\hat{f}_{x}] = 0.$$

Combined with $(*)$ this leads to $[(f^{s})_{x} - f_{x}] = 0$. Therefore, $[\hat{h}_{x}] = [(f^{s} - f)_{x}] = 0$. Proposition 1.16$(f)$ implies $h = 0$. \hfill $\square$

In the case when the element $s$ of $I(X)$ is not in $G$ and there is no amenable subgroup in $I(X)$ containing both $s$ and $G$, the above argument does not work. To treat this case one should deal with actions of the amenable group $G \times \mathbb{Z}$ in suitable function spaces (see [Li2] for details; see also Remark 1.17 above).

The proof of Theorem 1.13 for cocompact actions is based on the compactness principle and a version of the Harnack inequality. This approach goes back to Dynkin, Malyutov [DyMal] and Margulis [Mar] who considered bounded resp. positive harmonic functions on nilpotent groups (see [Li2] for details).
§ 2. Upper FC-series and Liouville-type properties

The concept of the upper FC-series [Hai] (see also Definition 2.1 below) allows us to generalize Theorem 1.13 and Corollary 1.14. Namely, let $X \to Y$ be a Galois covering with Galois group $G$ over an ultra-Liouville base $Y$. We show that the period subgroup $G_H$ contains all the members of the upper FC-series of $G$ and, hence, their union, too (see Corollary 2.5). This leads also to a generalization of Theorem 1.6 and Corollary 1.8 on the Liouville property of coverings (see Corollary 2.6).

2.1. Definitions. 1. FC-groups and FC-series. A group $G$ is called FC-group [Ba, Ku, To] if the conjugacy class of each element $g \in G$ is finite. Almost abelian groups or groups with finite commutator subgroups are so [Neu]. Both of the latter classes contain the proper subclass of groups $G$ with finite quotients $G/Z(G)$ by the center [Neu; Er; To, Thm. 1.1]. For any FC-group $G$ the quotient $G/Z(G)$ is a periodic group (see [Ba; To, Thm. 1.4]).

For any group $G$ the union $\text{FC}(G)$ of all finite conjugacy classes is a normal subgroup of $G$. Clearly, $\text{FC}(G)$ is an FC-group; it is called the FC-center of $G$ [To]. Set $\text{FC}_1(G) = \text{FC}(G)$ and for any $n \geq 1$ denote by $\text{FC}_{n+1}(G)$ the total preimage of $\text{FC}(G/\text{FC}_n(G))$ under the natural surjection $G \to G/\text{FC}_n(G)$. We obtain the upper FC-series of $G$ [Hai]:

$$1 \triangleleft \text{FC}_1(G) \triangleleft \text{FC}_2(G) \triangleleft \cdots \triangleleft \text{FC}_n(G) \triangleleft \cdots \triangleleft G.$$ 

Clearly, $\text{FC}_n(G)$ is a normal subgroup of $G$; in fact, it is strictly characteristic $^{12}$ [Hai]. The upper FC-series extends transfinitely in the usual way [Du], by defining $\text{FC}_\alpha(G) = \bigcup_{\beta < \alpha} \text{FC}_\beta(G)$ for each limit ordinal $\alpha$. We set

$$\text{FC}_\omega(G) = \bigcup_{n \in \mathbb{N}} \text{FC}_n(G) \quad \text{and} \quad \text{FC}_{\lim}(G) = \bigcup_{\alpha} \text{FC}_\alpha(G),$$

where $\alpha$ runs over all the ordinals. The normal subgroup $\text{FC}_{\lim}(G) \triangleleft G$ is called the FC-hypercenter of $G$; we say that the elements of $\text{FC}_{\lim}(G) \triangleleft G$ are FC-hypercentral in $G$.

2. FC-nilpotent and FC-hypernilpotent groups [Hai, Du, Rob]. If $G = \text{FC}_n(G)$ for some $n \in \mathbb{N}$ and $G \neq \text{FC}_{n-1}(G)$, then $G$ is called FC-nilpotent of class $n$, or simply FC-nilpotent. We say that the group $G$ is FC-\(\omega\)-nilpotent resp. FC-hypernilpotent if $G = \text{FC}_\omega(G)$ resp. $G = \text{FC}_{\lim}(G)$. Clearly, an FC-\(\omega\)-nilpotent group is locally FC-nilpotent, meaning that any finitely generated subgroup of $G$ is FC-nilpotent.

2.2. Remarks. 1. A nilpotent (resp. \(\omega\)-nilpotent, hypernilpotent, locally nilpotent) group is FC-nilpotent (resp. FC-\(\omega\)-nilpotent, FC-hypernilpotent, locally FC-nilpotent). The properties of FC-nilpotence, FC-\(\omega\)-nilpotence, and FC-hypernilpotence are inherited by the subgroups, the quotient groups, and the finite extensions. So, a finite extension of a nilpotent group is FC-nilpotent. Vice versa, a finitely generated FC-nilpotent group of class $n$ is a finite extension of a nilpotent group of class at most $n$ (see [DuMcL, Thm. 2]). A finitely generated FC-hypernilpotent group is almost nilpotent and almost polycyclic [McL; Rob, Vol. 1, p. 133]. Thus, an FC-hypernilpotent group is locally FC-nilpotent.

$^{12}$A subgroup $H \subseteq G$ is called strictly characteristic if $\phi(H) \subseteq H$ for any epimorphism $\phi: G \to G$. 
2. Every locally FC-nilpotent, and so, every FC-hypernilpotent, group $G$ is amenable. Indeed, $G$ is the union of the direct system of its finitely generated FC-nilpotent subgroups. Therefore, by Theorem 1.2.7 in [Gre], the statement follows once we know that any finitely generated FC-nilpotent group is amenable. The latter holds since a finitely generated FC-group is almost nilpotent [DuMcL, Thm. 2] (see Remark 2.2.1 above).

3. The following examples show that, in general, the FC-(hyper)center of a finitely generated group need not be finitely generated, and a countable FC-group which is not finitely generated may be neither almost solvable nor almost hypernilpotent.

2.3. Examples. 1 (see [PHa] and also [Rob, Vol. 1, Thm. 5.36]). Let $G = \langle a, b \mid [b_i, b_j] = 1, [b_i, b_j] = [b_{i+k}, b_{j+k}], i, j, k \in \mathbb{Z} \rangle$, where $b_i = a^{-i}b^i$. Then the center $Z(G)$ coincides with the subgroup $H \subset G$ generated by the elements $d_r = [b_0, b_r]$, $r \in \mathbb{N}$. It is a free abelian group of infinite rank. Furthermore, $Z(G/H) = 1$, and hence $Z_{\lim}(G) = Z(G) = H$. It is easily seen that actually $FC_{\lim}(G) = FC(G) = Z(G) = H$.

2 (see [Ku, §38]). Let $P = \prod_{n=2k+1, k \geq 2} A_n$ and $H = \bigoplus_{n=2k+1, k \geq 2} A_n$ be, respectively, the direct product and the direct sum of the alternating groups $A_n \subset S_n$ of all odd degrees $n \geq 5$, where $S_n$ stands for the symmetric group of degree $n$. It is known that for any odd $n$ the group $A_n$ is generated by the cyclic permutations $a_n = (1, ..., n)$ and $b_n = (1, 2, 3)$. Let $G \subset P$ be the subgroup generated by the elements $a = (a_5, ..., a_{2k+1}, ...) \in P$ and $b = (b_5, ..., b_{2k+1}, ...) \in P$. Then $G \supset H$. It is easily seen that the center $Z(G)$ is trivial, whereas the FC-center $FC(G)$ coincides with the subgroup $H$. Moreover, $FC(G/H) = 1$, and hence $FC_{\lim}(G) = FC(G) = H$. The FC-group $H = FC(G)$ is neither almost solvable nor almost hypernilpotent. Indeed, for any normal subgroup $N$ of finite index in $H$ the intersection $N_n = A_n \cap N \neq 1$ if $n$ is sufficiently large. Clearly, $A_n = N_n \subset N$ for such $n$ (since $A_n$ is simple), and thus $N$ is neither solvable nor hypernilpotent.

The following lemma is an easy consequence of Theorem 1.13. We use the same notation as in Theorem 1.13; see also Definitions 1.5, 1.12.

2.4. Lemma. Suppose that one of the two conditions of Theorem 1.13 is fulfilled, i.e. either

* $G$ is amenable and its action on $X$ is ultra-Liouville, or

* the action of $G$ on $X$ is cocompact.

Let $N \ll G$ be a normal subgroup of $G$ contained in the $H$-period subgroup $G_H$. Let $\bar{s}$ denote the image of an element $s \in G$ in the quotient group $\overline{G} = G/N$. Suppose that the conjugacy class $\bar{s}^{-1} \bar{g} \bar{s} = \{ \bar{g}^{-1} \bar{g} \bar{s} \mid \bar{g} \in \overline{G} \}$ of $\bar{s}$ in $\overline{G}$ is finite. Then $s \in G_H$.  \[13\] i.e. the restricted direct product.


Proof. By our assumption, the centralizer $C$ of the element $s$ is of finite index in $G$. The total preimage $C$ of $\overline{C}$ in $G$ is a subgroup of finite index. Hence (see [Li2, Lemma 3.3]) $C$ satisfies the same condition (* ) as $G$. Furthermore, $C$ contains both $s$ and $N$. Since $s$ is central in $C$, we have $[s, C] \subseteq N \subseteq G_H$. Now Theorem 1.13 shows that $s \in C_H \subseteq G_H$. □

2.5. Corollary 14. Suppose that one of the conditions (∗) of Lemma 2.4 is fulfilled. Then $FC_{\lim}(G) \subseteq G_H$, i.e. any FC-hypercentral element of $G$ is an $H$-period.

Proof. Starting with the unit subgroup $1 \subseteq G_H$, we proceed by transfinite induction. Suppose that $FC_{\alpha}(G) \subseteq G_H$. Set $N = FC_{\alpha}(G) < G$. By Lemma 2.5, for any element $s \in FC_{\alpha+1}(G)$ we have $s \in G_H$, and thus $FC_{\alpha+1}(G) \subseteq G_H$. Furthermore, if $\alpha$ is a limit ordinal and $FC_{\beta}(G) \subseteq G_H$ for all $\beta < \alpha$, then $FC_{\alpha}(G) = \bigcup_{\beta<\alpha} FC_{\beta}(G) \subseteq G_H$. By induction, it follows that $FC_{\lim}(G) = \bigcup_{\alpha} FC_{\alpha}(G) \subseteq G_H$.

2.6. Corollary 15. Let $X \to Y$ be a Galois covering with Galois group $G$ over a compact Riemannian resp. Kähler manifold $Y$. If $G$ is an extension of an FC-hypernilpotent group by a Varopoulos group, then $X$ is Liouville.

2.7. Remark. It follows from Corollary 2.5 that either of the conditions (∗) of Lemma 2.4 implies the following property of the period subgroup $G_H < G$: the FC-center $FC(G/G_H)$ of the quotient $G/G_H$ is trivial, that is, each nontrivial (i.e. $\neq \{1\}$) conjugacy class in $G/G_H$ is infinite. Clearly, the subgroup $FC_{\lim}(G) < G_H$ has the same property. It would be interesting to find an example (if it does exist) in which $FC_{\lim}(G) \neq G_H$.

§3. Solvable Carathéodory hyperbolic coverings of a compact Riemann surface

It was shown in [LySu] that any compact Riemann surface $Z$ of genus $g \geq 2$ admits a non-Liouville Galois covering $X \to Z$ with a metabelian (i.e. two-step solvable) Galois group. Modifying the construction of Lyons and Sullivan, we prove the following theorem.

3.1. Theorem. Each compact Riemann surface $Z$ of genus $g \geq 2$ admits a Carathéodory hyperbolic metabelian covering $X \to Z$.

Proof. Let $\Gamma = \pi_1(Z)$ and let $p_1 : Y \to Z$ be the maximal abelian covering over $Z$ (i.e., the covering corresponding to the commutator subgroup $\Gamma' = [\Gamma, \Gamma]$); its Galois group $G = \Gamma/\Gamma' \cong H_1(Z, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Since $\text{rk } G = 2g \geq 4$, a theorem of A. Mori [Mo] implies that for any point $y \in Y$ there exists a unique positive Green function, say $g_y$, with pole at $y$ (see also [Ts, Theorem X.46]).

Let $D \subset Y \setminus \{y\}$ be a simply connected domain. Then there is a conjugate harmonic function $g_y^* \in D$, which is defined uniquely up to an additive real constant. Therefore, the differential $\omega_y = df_y$, where $f_y = g_y + ig_y^*$, is a well-defined holomorphic 1-form on $Y \setminus \{y\}$. Its real part $\text{Re } \omega_y = dg_y$ is an exact 1-form on $Y \setminus \{y\}$. Hence, the real part of each period $\int_{\gamma} \omega_y$ of $\omega_y$, where $\gamma \in H_1(Y \setminus \{y\}; \mathbb{Z})$, is zero. Thus, $\omega$ defines a homomorphism $H_1(Y \setminus \{y\}; \mathbb{Z}) \to i\mathbb{R}$.

14cf. Corollary 1.14(a).
15cf. Theorem 1.4 and Corollary 1.8.
Fix a point \( z_0 \in Y \setminus \{y\} \). For any particular choice of \( g_y^* \) consider the function
\[
\varphi_y(z) = \exp \left( -2\pi f_y(z) \right) = \exp \left( -2\pi \left( f_y(z_0) + \int_{z_0}^z \omega_y \right) \right).
\]
This is a multi-valued holomorphic function on \( Y \) with values in the unit disk \( \mathbb{D} \). For a given \( y \in Y \) any two such functions coincide up to a constant factor \( \lambda \in \mathbb{T} \), where \( \mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \). For each \( y \in Y \) choose, once forever, one of the functions \( \varphi_y \).

Any two values of \( \varphi_y \) differ by a factor of the form \( \exp \left( -2\pi \int_{\gamma} \omega_y \right) \in \mathbb{T} \), where \( \gamma \in H_1(Y; \mathbb{Z}) \). More precisely, we have a well-defined character
\[
\alpha_y : H_1(Y \setminus \{y\}; \mathbb{Z}) \ni \gamma \mapsto \exp \left( -2\pi \int_{\gamma} \omega_y \right) \in \mathbb{T}.
\]

Actually, it yields a character
\[
\alpha_y : H_1(Y; \mathbb{Z}) \to \mathbb{T}.
\]
Indeed, consider the exact sequence
\[
0 \to \mathbb{Z} \to H_1(Y \setminus \{y\}; \mathbb{Z}) \to H_1(Y; \mathbb{Z}) \to 0,
\]
where the subgroup \( \mathbb{Z} \subset H_1(Y \setminus \{y\}; \mathbb{Z}) \) is generated by a small circle \( \sigma_y \) in \( Y \) centered at \( y \). In a small disk \( \delta_y \) around \( y \) we have \( g_y(z) = -\frac{1}{2\pi} \log|z-y| + h_y(z) \), where \( h_y \) is a single-valued harmonic function in \( \delta_y \); hence, \( f_y(z) = -\frac{1}{2\pi} \log(z-y) + \tilde{f}_y(z) \), where \( \tilde{f}_y \) is a single-valued holomorphic function in \( \delta_y \). It follows that
\[
2\pi \int_{\sigma_y} \omega_y = 2\pi \int_{\sigma_y} \left( -\frac{1}{2\pi} d \log(z-y) + d \tilde{f}_y \right) \in 2\pi i \mathbb{Z}.
\]

Thereby, \( \exp \left( -2\pi \int_{\sigma_y} \omega_y \right) = 1 \), the restriction of the homomorphism \( \alpha_y \) to the kernel subgroup \( \mathbb{Z} \) in the above exact sequence is trivial, and \( \alpha_y \) can be pushed down to the quotient group.

The set of values of the function \( \varphi_y \) at a point \( z \in Y \setminus \{y\} \) coincides with a coset of the subgroup Image(\( \alpha_y \)) in the multiplicative group \( \mathbb{C}^* \), whereas all its values at the point \( y \) are zero.

Let \( \rho : \pi_1(Y) \to H_1(Y; \mathbb{Z}) \cong \pi_1(Y)/\pi'_1(Y) \) be the canonical surjection. Set \( \tilde{\alpha}_y = \alpha_y \circ \rho \).

The covering \( X_y \to Y \) over \( Y \) corresponding to the subgroup Ker\( \tilde{\alpha}_y \triangleleft \pi_1(Y) \) is the minimal one such that the function \( \varphi_y \) becomes single-valued when lifted to \( X_y \). Set
\[
K = \bigcap_{y \in Y} \text{Ker} \alpha_y \subset H_1(Y; \mathbb{Z}),
\]
and \( \tilde{K} = \rho^{-1}(K) \subset \pi_1(Y) \). Let \( p_2 : X \to Y \) be the abelian covering over \( Y \) associated with the subgroup \( \tilde{K} \triangleleft \pi_1(Y) \). First we show that \( p = p_1 \circ p_2 : X \xrightarrow{p_2} Y \xrightarrow{p_1} Z \) is a Galois (and hence a metabelian) covering.
Let $\Gamma'' = [\Gamma', \Gamma']$. The group $G = \Gamma/\Gamma' \cong H_1(Z, \mathbb{Z})$ acts isometrically on $Y$; thus, $g_y \circ \hat{\gamma} = g_{\hat{\gamma}(y)}$ and $\gamma^*(\omega_{y}) = \omega_{\hat{\gamma}(y)}$ for any $\gamma \in \Gamma/\Gamma'$. Hence the subgroup $K \subset H_1(Y; \mathbb{Z}) \cong \Gamma'/\Gamma''$ is invariant with respect to the induced action of $\Gamma/\Gamma'$ in homology $H_1(Y; \mathbb{Z})$; denote this action as $\mu$. It is easily seen that $\mu$ coincides with the “adjoint” representation $\Gamma/\Gamma' \ni \gamma \mapsto T_\gamma \in \text{Aut}(\Gamma'/\Gamma'')$, $T_\gamma(v) = \gamma^{-1} v \gamma$, $v \in \Gamma'/\Gamma''$. It follows that $p_*(\pi_1(X)) = (p_1)_*(\overline{K}) \subset \pi_1(Z)$ is a normal subgroup of $\pi_1(Z)$, and hence $p: X \to Z$ is a metabelian Galois covering.

Clearly, $p_2: X \to Y$ is the minimal covering over $Y$ such that all the functions $\{\varphi_y\}_{y \in Y}$ become single-valued when lifted to $X$. Let $E = \{\hat{\varphi}_y\}_{y \in Y} \subset H^\infty(X)$ be the collection of all the lifted functions. We will show that $E$ separates the points of $X$.

Denote by $F_y = p^{-1}(y) \subset X$ the fiber of $p$ over $y \in Y$. For any two distinct points $y, y' \in Y$ the function $\hat{\varphi}_y$ vanishes identically on $F_y$ and does not vanish at the points of $F_y'$. Therefore, $E$ separates the fibers $\{F_y\}$.

Thus, it is sufficient to show that $E$ separates the points of each fiber $F_y$. It is easily seen that for $y' \neq y$ the function $\hat{\varphi}_y$ separates the points of $F_y'$ if and only if $\text{Ker} \alpha_y = K$. If the latter equality holds for a certain pair of distinct points $y_1, y_2 \in Y$, then the points of each fiber $F_y$, $y \in Y$, are separated by at least one of the functions $\hat{\varphi}_{y_1}, \hat{\varphi}_{y_2}$. Hence, the theorem follows from the next claim.

**Claim 1.** There exists a countable union $C = \bigcup_{n \in \mathbb{N}} C_n \subset Y$ of real analytic curves $C_n$ in $Y$ such that $\text{Ker} \alpha_y = K$ for each point $y \in Y \setminus C$.

The proof is based on the following statement$^{16}$:

**Claim 2.** The function of two complex variables $g(y, y') = g_y(y')$ is harmonic on the complement $(Y \times Y) \setminus \Delta$, where $\Delta \subset Y \times Y$ is the diagonal.

**Proof of Claim 2.** By the symmetry property of Green function [Ts, Theorem I.16], we have $g_y(y') = g_{y'}(y)$ for any $y \neq y'$, $y, y' \in Y$. Hence, $g(y, y')$ is a harmonic function in each argument on $(Y \times Y) \setminus \Delta$. It is sufficient to show that it is harmonic as a function of two complex variables in each bidisk $\delta \times \delta' \subset C (Y \times Y) \setminus \Delta$, where $\delta$, $\delta'$ are two small disks in $Y$. Being harmonic in each variable, the function $g(y, y')$ in the bidisk $\delta \times \delta'$ satisfies the Laplace equation $\Delta_y g(y, y') = \Delta_{y'} g(y, y') + \Delta_{y''} g(y, y') = 0$, where $\Delta_y$, $\Delta_{y'}$ are the usual Laplacians. Therefore, $g(y, y')$ is harmonic in $\delta \times \delta'$ as soon as it is continuous there.

Since the function $g(y, 0)$ is continuous in the closed disk $\overline{\delta}$, the family $g_y = g_y(y')$ of positive harmonic functions in $\delta'$ is equicontinuous in every smaller closed disk (this easily follows by the Harnack inequality). This implies that $g = g(y, y')$ is a continuous function in $\delta \times \delta'$, which completes the proof.

**Proof of Claim 1.** It is sufficient to check our statement locally. Fix a small disk $\delta \subset Y$. We will show that $\text{Ker} \alpha_y = K$ for all $y \in \delta$ outside of a countable union $C_\delta \subset \delta$ of closed real analytic curves in $\delta$.

It follows from Claim 2 that in each local chart $\Omega$ in $Y$ the coefficients of the holomorphic 1-form $\omega_y$ are real analytic functions of $y \in Y \setminus \Omega$.

---

$^{16}$It should be well known; for the sake of completeness we give a simple proof.
Let a sequence \( \{\gamma_n\}_{n \in \mathbb{N}} \) of 1-cycles in \( Y \) be a free basis of the homology group

\[
H_1(Y; \mathbb{Z}) \cong \mathbb{Z}^\infty = \bigoplus_{1}^{\infty} \mathbb{Z}.
\]

We may assume that they do not meet the closed disk \( \overline{\delta} \). The periods \( c_{n}(y) = \int_{\gamma_n} \omega_y, \ n \in \mathbb{N}, \) are (imaginary-valued) real analytic functions of \( y \in \delta \). For \( \gamma = \sum_{j=1}^{n} a_j \gamma_j \in H_1(Y; \mathbb{Z}) \) we have

\[
\langle \gamma, \omega_y \rangle = \int_{\gamma} \omega_y = \sum_{j=1}^{n} a_j \int_{\gamma_j} \omega_y = \sum_{j=1}^{n} a_j c_{j}(y) = \langle \alpha, c(y) \rangle,
\]

where \( \alpha = (a_1, \ldots, a_n, 0, \ldots) \) and \( c(y) = (c_{j}(y))_{j=1}^{\infty} \). By the definition of the character \( \alpha_y: H_1(Y; \mathbb{Z}) \to \mathbb{T} \), we have

\[
\text{Ker} \ \alpha_y = \{ \gamma \in H_1(Y; \mathbb{Z}) \mid \langle \gamma, i\omega_y \rangle = \langle \alpha, ic(y) \rangle \in \mathbb{Z} \}.
\]

Set \( L = \{ \alpha \in \mathbb{Z}^\infty \mid \langle \alpha, ic(y) \rangle \in \mathbb{Z} \text{ for all } y \in \delta \} \). For each \( \alpha \in \mathbb{Z}^\infty \setminus L \) and for each \( k \in \mathbb{Z} \) consider the real analytic curve \( C_{\alpha, k} = \{ y \in \delta \mid \langle \alpha, ic(y) \rangle = k \} \). Put

\[
C_\delta = \bigcup_{\alpha \in \mathbb{Z}^\infty \setminus L; \ k \in \mathbb{Z}} C_{\alpha, k}.
\]

It is easily seen that for \( y \in \delta \setminus C_\delta \) the subgroup \( \text{Ker} \ \alpha_y \subset H_1(Y; \mathbb{Z}) \) does not depend on \( y \) and coincides with \( L \). Furthermore, for any \( y \in C_\delta \) we have \( \text{Ker} \ \alpha_y \supset L \), and hence \( L = K \). This proves Claim 1 and completes the proof of Theorem 3.1. \( \square \)

§4. \( H^\infty \)-hulls in a solvable covering of Inoue surface

In this section we study in more details the universal covering \( \pi: X \to I \) over one of the Inoue surfaces \( I \) [In]. We start with a description of the Inoue surface.

Let \( A \in \text{SL}(3; \mathbb{Z}) \) be a matrix with one real eigenvalue \( \alpha > 1 \) and two complex conjugate eigenvalues \( \beta, \overline{\beta} \in \mathbb{C} \setminus \mathbb{R} \) (certainly \( |\beta| < 1 \)). Let \( a = (a_1, a_2, a_3) \) resp. \( b = (b_1, b_2, b_3) \) be a real resp. a complex eigenvector of \( A \) corresponding to the eigenvalue \( \alpha \) resp. \( \beta \).

Set \( \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im} \ z > 0 \} \) (the upper halfplane) and \( X = \mathbb{H} \times \mathbb{C} \). Consider the subgroup \( G \subset \text{Aut} \ X \) generated by the following four automorphisms \( g_j \):

\[
g_0(z, w) = (\alpha z, \beta w), \quad g_j(z, w) = (z + a_j, w + b_j), \quad 1 \leq j \leq 3, \quad (z, w) \in X = \mathbb{H} \times \mathbb{C}.
\]

The action of the group \( G \) on \( X \) is free, properly discontinuous, and cocompact. The smooth compact complex surface \( I = X/G \) is one of the Inoue surfaces [In].

The subgroup \( G_0 \subset G \) generated by \( g_1, g_2, g_3 \) is isomorphic to \( \mathbb{Z}^3 \); this subgroup is normal in \( G \), and the quotient group \( G/G_0 \) is isomorphic to \( \mathbb{Z} \). Thus, we have the exact sequence

\[
0 \longrightarrow \mathbb{Z}^3 \longrightarrow G \xrightarrow{\tau} \mathbb{Z} \longrightarrow 0,
\]  (1)
and the corresponding tower of abelian coverings $X \xrightarrow{\varphi_3} Y \xrightarrow{\varphi_2} \mathcal{I}$, where $Y = X/G_0$. In particular, $G$ is a metabelian (i.e., two-step solvable) polycyclic group, and $X \to \mathcal{I}$ is a polycyclic covering with the Galois group $G$.

To establish certain analytic properties of the covering $X \to \mathcal{I}$, we need the following simple observations.

First, note that the sequence (1) splits; a splitting $\rho: \mathbb{Z} \to G$ ($\tau \circ \rho = \text{id}_\mathbb{Z}$) may be defined by $\mathbb{Z} \ni m \mapsto g_m \in G$. Therefore, $G$ is a semidirect product $\mathbb{Z}^3 \rtimes \mathbb{Z}$, and any element $g \in G$ admits a unique representation of the form

$$g = g_0 \tilde{g} = g_0^m g_1^r g_2^b g_3^c, \quad \text{where} \quad m = \tau(g) \in \mathbb{Z}, \quad r, b, c \in \mathbb{Z}, \quad \tilde{g} = g_1^r g_2^b g_3^c \in G_0.$$

Using this normal form, for any $d \in \mathbb{Z}$ we can write

$$g^{-1} g_0^d g = (g_0^m \tilde{g})^{-1} \cdot g_0^d \cdot (g_0^m \tilde{g}) = \tilde{g}^{-1} g_0^{-m} \cdot g_0^d \cdot g_0^m \tilde{g} = \tilde{g}^{-1} g_0^{-d} \tilde{g} = g_3^{-r} g_2^{-b} g_1^{-c} \cdot g_0^d \cdot g_1^r g_2^b g_3^c.$$

**4.1. Lemma.** The conjugacy class $s^G$ of the element $s = g_0^d$ consists of all the transformations of the form

$$(z, w) \mapsto (\alpha^d z + (\alpha^d - 1)(r_1 a_1 + r_2 a_2 + r_3 a_3), \beta^d w + (\beta^d - 1)(r_1 b_1 + r_2 b_2 + r_3 b_3)),$$

where $r_1, r_2$, and $r_3$ run over $\mathbb{Z}$.

**Proof.** Since the elements $g_j$, $j = 1, 2, 3$, commute, the lemma follows from (2) and the formula

$$g_j^{-r} g_0^d g_j^r (z, w) = (\alpha^d z + (\alpha^d - 1)r a_j, \beta^d w + (\beta^d - 1)r b_j).$$

**4.2. Lemma.** a) The real eigenvalue $\alpha$ of the matrix $A$ is a nonquadratic irrationality.

b) The coordinates $a_1, a_2, a_3$ of the corresponding eigenvector $\mathbf{a}$ are linearly independent over $\mathbb{Q}$.

c) For any subgroup $L \subseteq \mathbb{Z}^3$ of rank $rk L \geq 2$ and for any finite subset $S \subseteq L$ we have

$$\inf \{ |r_1 a_1 + r_2 a_2 + r_3 a_3| \mid r = (r_1, r_2, r_3) \in L \setminus S \} = 0.$$

**Proof.** a) The characteristic polynomial $P(z) = z^3 + pz^2 + qz + 1$, $p, q \in \mathbb{Z}$, of the unimodular matrix $A$ has no rational root except, possibly, of $\pm 1$. Since $\alpha, \beta \neq \pm 1$, the polynomial $P$ is irreducible over $\mathbb{Q}$, which proves (a).

b) Assume, on the contrary, that $a_1, a_2, a_3$ are linearly dependent over $\mathbb{Q}$. Let $A = (a_{ij})_{i,j=1}^3$, where $a_{ij} \in \mathbb{Z}$. Then we have the following system

$$r_1 a_1 + r_2 a_2 + r_3 a_3 = 0$$

$$(a_{11} - \alpha) a_1 + a_{12} a_2 + a_{13} a_3 = 0$$

$$a_{21} a_1 + (a_{22} - \alpha) a_2 + a_{23} a_3 = 0$$

$$a_{31} a_1 + a_{32} a_2 + (a_{33} - \alpha) a_3 = 0.$$
with some \((r_1, r_2, r_3) \in \mathbb{Z}^3 \setminus \{0\}\). Since \((a_1, a_2, a_3) \neq 0\), we obtain the following three equations (each of degree at most 2) for \(\alpha\):

\[
\det \begin{pmatrix} r_1 & r_2 & r_3 \\ a_{21} & a_{22} - \alpha & a_{23} \\ a_{31} & a_{32} & a_{33} - \alpha \end{pmatrix} = 0, \quad \det \begin{pmatrix} a_{11} - \alpha & a_{12} & a_{13} \\ r_1 & r_2 & r_3 \\ a_{31} & a_{32} & a_{33} - \alpha \end{pmatrix} = 0, \\
\det \begin{pmatrix} a_{11} - \alpha & a_{12} & a_{13} \\ a_{21} & a_{22} - \alpha & a_{23} \\ r_1 & r_2 & r_3 \end{pmatrix} = 0.
\]

At least one of these equations must certainly be of degree 2 (for \((r_1, r_2, r_3) \neq 0\)), which contradicts (a).

c) By (b), the homomorphism \(\chi: L \ni r = (r_1, r_2, r_3) \mapsto r_1a_1 + r_2a_2 + r_3a_3 \in \mathbb{R}\) is injective. Hence, \(M = \chi(L) \subset \mathbb{R}\) is a free abelian subgroup of rank \(\text{rk } M = \text{rk } L \geq 2\). The closure \(\overline{M}\) of \(M\) in \(\mathbb{R}\) coincides with \(\mathbb{R}\) (for otherwise, \(\overline{M} \cong \mathbb{Z}\) and hence \(M \cong \mathbb{Z}\), which contradicts the property \(\text{rk } M \geq 2\)). This implies (c). \(\square\)

4.3. Definition. The \(H^\infty(X)\)-hull \(\widehat{Y}\) of a set \(Y \subseteq X\) in a complex space \(X\) is defined as follows:

\[
\widehat{Y} = H^\infty - \text{hull}_X(Y) = \left\{ x \in X \mid |f(x)| \leq \sup_{y \in \overline{Y}} |f(y)| \quad \text{for all } f \in H^\infty(X) \right\}.
\]

4.4. Proposition (cf. Corollary 1.14(c)). Let, as in Lemma 4.1, \(s = g_0^d\). Suppose that \(\alpha^d > 2\). Then:

a) The \(s^G\)-orbit \(s^G(x_0) = \{g^{-1}sgx_0 \mid g \in G\}\) of the point \(x_0 = (i, 0) \in X = \mathbb{H} \times \mathbb{C}\) consists of all the points \(x = (z, w) \in X = \mathbb{H} \times \mathbb{C}\) of the form

\[
(z, w) = ((\alpha^d - 1)(r_1a_1 + r_2a_2 + r_3a_3) + i\alpha^d, (\beta^d - 1)(r_1b_1 + r_2b_2 + r_3b_3)),
\]

where \(r_1, r_2, r_3 \in \mathbb{Z}\).

b) The bounded holomorphic function \(F(z, w) = 2(z + i)^{-1}\) on \(X = \mathbb{H} \times \mathbb{C}\) satisfies the inequality

\[
|F(x_0)| = 1 > \frac{2}{3} > \sup_{x \in s^G(x_0)} |F(x)|.
\]

In particular, the \(H^\infty(X)\)-hull \(\overline{s^G(x_0)}\) of the \(s^G\)-orbit \(s^G(x_0)\) does not contain the point \(x_0\) itself, and for any mean \(m\) on \(L^\infty(s^G(x_0))\) we have \(F(x_0) \neq m(F \mid s^G(x_0))\).

Proof. (a) follows immediately from Lemma 4.1. In view of the assumption \(\alpha^d > 2\), (a) implies that

\[
\sup_{x \in s^G(x_0)} |F(x)| = 2 \sup_{r_1, r_2, r_3 \in \mathbb{Z}} \left| (\alpha^d - 1)(r_1a_1 + r_2a_2 + r_3a_3) + i(\alpha^d + 1) \right|^{-1}
\]

\[
= 2 \left[ \inf_{r_1, r_2, r_3 \in \mathbb{Z}} \left| (\alpha^d - 1)(r_1a_1 + r_2a_2 + r_3a_3) + i(\alpha^d + 1) \right|^{-1} \right] \leq \frac{2}{\alpha^d + 1} < \frac{2}{3},
\]

which proves (b). \(\square\)
The $H^\infty(X)$-hull $\hat{Y}$ of a subset $Y \subseteq X$ may be found as follows: $\hat{Y} = \overline{\text{pr}_H Y \times \mathbb{C}}$, where \text{pr}_H: $X = \mathbb{H} \times \mathbb{C} \to \mathbb{H}$ is the natural projection and $\overline{\text{pr}_H Y}$ is the $H^\infty(\mathbb{H})$-hull of the subset $\text{pr}_H Y \subseteq \mathbb{H}$. In view of Proposition 4.4(b), we would like to pose the following question.

4.5. Question. For which subsets $\Gamma \subseteq G \setminus \{1\}$

(*) any point $x \in X$ is contained in the $H^\infty(X)$-hull $\hat{\Gamma(x)}$ of its $\Gamma$-orbit $\Gamma(x)$?

Proposition 4.6 below provides examples of subsets $\Gamma \subseteq G \setminus \{1\}$ with the property (*).

Recall that the elements $g_1, g_2, g_3$ form a free basis of the normal subgroup $G_0 \cong \mathbb{Z}^3$ in $G$. Any subgroup $H \subseteq G_0$ is a free abelian group of rank $rk \ H \leq 3$.

4.6. Proposition. Let $H \subseteq G_0$ be a subgroup of rank $rk \ H \geq 2$, and let $\Gamma \subseteq H$ be the complement of a finite subset$^{17}$ $S \subset H$. Then $x \in \hat{\Gamma(x)}$ for any $x \in X$. In particular, $x \in G(x) \setminus \{x\}$ for any $x \in X$.

Proof. By Liouville Theorem, any function $f \in H^\infty(X) = H^\infty(\mathbb{H} \times \mathbb{C})$ is of the form

$$f = \tilde{f} \circ \text{pr}_H,$$  \hspace{1cm} (3)

Hence, for any point $x = (z, w) \in \mathbb{H} \times \mathbb{C}$ and any element $h = g_1^r g_2^r g_3^r \in H$, we have

$$hx = (z + r_1 a_1 + r_2 a_2 + r_3 a_3, w + r_1 b_1 + r_2 b_2 + r_3 b_3)$$  \hspace{1cm} (4)

and

$$f(hx) = \tilde{f}(z + r_1 a_1 + r_2 a_2 + r_3 a_3).$$  \hspace{1cm} (5)

When $h$ runs over $H$ (resp. over the complement $\Gamma = H \setminus S$), the corresponding vector $r = (r_1, r_2, r_3) \in \mathbb{Z}^3$ in (4) and (5) runs over a sublattice $\widetilde{H} \subseteq \mathbb{Z}^3$ isomorphic to $H$ (resp. over the complement $\widetilde{\Gamma} = \widetilde{H} \setminus \widetilde{S}$ of a finite subset $\widetilde{S} \subset \widetilde{H}$); in particular, $rk \ \widetilde{H} = rk \ H \geq 2$. Since $\tilde{f}$ is a continuous function, it follows from (3), (5) and Lemma 4.2(c) that

$$f(x) = \tilde{f}(z) \in \overline{\tilde{f}(\Gamma(x))}$$

(the closure in $\mathbb{C}$). Therefore, $|f(x)| \leq \sup_{y \in \Gamma(x)} |f(y)|$, and hence $x \in \hat{\Gamma(x)}$. $\square$

4.7. Remark. Despite Proposition 4.4(b), the following fact holds$^{18}$:

For any integer $d \neq 0$ and for any $x = (z, w) \in X = \mathbb{H} \times \mathbb{C}$, the $s^G$-orbit $s^G(x)$ is a uniqueness set for bounded holomorphic functions on $X$.

That is, $f = 0$ whenever $f \in H^\infty(X)$ and $f \mid s^G(x) = 0$. Indeed, any $f \in H^\infty(X)$ is of the form (3); hence, $f \mid s^G(x) = 0$ implies $\tilde{f} \mid \text{pr}_H [s^G(x)] = 0$. However, it follows from Lemmas 4.1 and 4.2(c) that the point $\alpha^d z \in \mathbb{H}$ is a limit point of the set $\text{pr}_H [s^G(x)] = \{\alpha^d z + (\alpha^d - 1)(r_1 a_1 + r_2 a_2 + r_3 a_3) \mid r_1, r_2, r_3 \in \mathbb{Z}\} \subseteq \mathbb{H}$. Thus, $\tilde{f} = 0$, and so $f = 0$.

---

$^{17}$The statement of the proposition is trivial if $1 \in \Gamma$; however, it is meaningful whenever $1 \in S$.

$^{18}$cf. Proposition 1.10(a).
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