On the Convergence Rate of Hermite-Fejér Interpolation

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1 Introduction

For an arbitrarily given system of points

\[ \{x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}\}_{n=1}^{\infty}, \tag{1} \]

Faber [3] in 1914 showed that there exists a continuous function \( f(x) \) in \([-1, 1]\) for which the Lagrange interpolation sequence \( L_n[f] \) \((n = 1, 2, \ldots)\) is not uniformly convergent to \( f \) in \([-1, 1]\), where \( \omega_n(x) = (x - x_1^{(n)})(x - x_2^{(n)})\cdots(x - x_n^{(n)}) \)

\[ L_n[f](x) = \sum_{k=1}^{n} f(x_k^{(n)}) \ell_k^{(n)}(x), \quad \ell_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})}. \tag{2} \]

Whereas, based on the Chebyshev pointsystem

\[ x_k^{(n)} = \cos \left( \frac{2k - 1}{2n} \pi \right), \quad k = 1, 2, \ldots, n, \quad n = 1, 2, \ldots, \tag{3} \]

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Fejér [4] in 1916 proved that if $f \in C[-1, 1]$, then there is a unique polynomial $H_{2n-1}(f, x)$ of degree at most $2n - 1$ such that $\lim_{n \to \infty} \| H_{2n-1}(f) - f \|_\infty = 0$, where $H_{2n-1}(f, x)$ is determined by

$$H_{2n-1}(f, x^{(n)}_k) = f(x^{(n)}_k), \quad H'_{2n-1}(f, x^{(n)}_k) = 0, \quad k = 1, 2, \ldots, n. \quad (4)$$

This polynomial is known as the Hermite-Fejér interpolation polynomial.

It is of particular notice that the above Hermite-Fejér interpolation polynomial converges much slower compared with the corresponding Lagrange interpolation polynomial at the Chebyshev pointsystem (3) (see Fig. 1).

To get fast convergence, the following Hermite-Fejér interpolation of $f(x)$ at nodes (1) is considered [6, 7]:

$$H^*_n(f, x) = \sum_{k=1}^{n} f'(x^{(n)}_k) h^{(n)}_k(x) + \sum_{k=1}^{n} f''(x^{(n)}_k) b^{(n)}_k(x), \quad (5)$$

where $h^{(n)}_k(x) = v^{(n)}_k(x) / \ell^{(n)}_k(x)$ and $b^{(n)}_k(x) = (x - x^{(n)}_k) / \ell^{(n)}_k(x)$.

Fejér [5] and Grünwald [7] also showed that the convergence of the Hermite-Fejér interpolation of $f(x)$ also depends on the choice of the nodes. The pointsystem (1) is called normal if for all $n$

$$v^{(n)}_k(x) \geq 0, \quad k = 1, 2, \ldots, n, \quad x \in [-1, 1], \quad (6)$$

while the pointsystem (1) is called strongly normal if for all $n$

$$v^{(n)}_k(x) \geq c > 0, \quad k = 1, 2, \ldots, n, \quad x \in [-1, 1] \quad (7)$$

for some positive constant $c$. 

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**Fig. 1** $\| H_{2n-1}(f, x) - f(x) \|_\infty$, $\| L_n(f, x) - f(x) \|_\infty$ and $\| H^*_n(f, x) - f(x) \|_\infty$ at $x = -1 : 0.001 : 1$ by using Chebyshev pointsystem (3) for $f(x) = \sin(x)$, $f(x) = 1/(1+25x^2)$ and $f(x) = |x|^3$, respectively.
Fejér [5] (also see Szegö [12, pp 339]) showed that for the zeros of Jacobi polynomial \( P_n^{(\alpha, \beta)}(x) \) of degree \( n \) (\( \alpha > -1, \beta > -1 \))

\[
v_k^{(n)}(x) \geq \min\{-\alpha, -\beta\} \quad \text{for} \quad -1 < \alpha \leq 0, -1 < \beta \leq 0, \quad k = 1, 2, \ldots, n \quad \text{and} \quad x \in [-1, 1].
\]

For (strongly) normal pointsystems, Grünwald [7] showed that for every \( f \in C^1(-1, 1) \), \( \lim_{n \to \infty} \|H_{2n-1}^*(f) - f\|_{\infty} = 0 \) if \( \{x_k^{(n)}\} \) is strongly normal satisfying (7) and \( \{f'(x_k^{(n)})\} \) satisfies

\[
|f'(x_k^{(n)})| < n^{c-\delta} \quad \text{for some given positive number} \ \delta, \quad k = 1, 2, \ldots, \quad n = 1, 2, \ldots,
\]

while \( \lim_{n \to \infty} \|H_{2n-1}^*(f) - f\|_{\infty} = 0 \) in \([-1 + \epsilon, 1 - \epsilon]\) for each fixed \( 0 < \epsilon < 1 \) if \( \{x_k^{(n)}\} \) is normal and \( \{f'(x_k^{(n)})\} \) is uniformly bounded for \( n = 1, 2, \ldots \).

Moreover, Szabados [11] showed the convergence of the Hermite-Fejér interpolation (5) at the Chebyshev pointsystem (3) satisfies

\[
\|f - H_{2n-1}^*(f)\|_{\infty} = O(1)\|f - p^*\|_{C^1[-1,1]} \quad \text{(8)}
\]

where \( p^* \) is the best approximation polynomial of \( f \) with degree at most \( 2n - 1 \) and

\[
\|f - p^*\|_{C^1[-1,1]} = \max_{0 \leq j \leq 1} \|f^{(j)} - p^{(j)}\|_{\infty}.
\]

Hermite-Fejér interpolation has plenty of use in computer geometry aided geometric design with boundary conditions including derivative information. The convergence rate under the infinity norm has been extensively studied in [5–7, 11, 14]. The efficient algorithm on the fast implementation of Hermite-Fejér interpolation at zeros of Jacobi polynomial can be found in [17].

In this paper, the following convergence rates of Hermite-Fejér interpolation \( H_{2n-1}^*(f, x) \) at Gauss-Jacobi pointsystems are considered.

- If \( f \) is analytic in \( E_\rho \) with \( |f(z)| \leq M \), then

\[
\|f(x) - H_{2n-1}^*(f, x)\|_{\infty} = \begin{cases} 
O\left(\frac{4\tau_n M [2n\rho^2 + (1 - 2n)\rho]}{\rho - 1)^2 \rho^{2n}}\right), & \gamma \leq 0, \\
O\left(\frac{n^{2+2\gamma} [2n\rho^2 + (1 - 2n)\rho]}{(\rho - 1)^2 \rho^{2n}}\right), & \gamma > 0 
\end{cases}, \quad \gamma = \max\{\alpha, \beta\}
\]

(9)

\[1\] In fact, Grünwald in [7] considered more general cases with any vector \( \{a_k^{(n)}\} \) instead of \( \{f'(x_k^{(n)})\} \).
where
\[
\tau_n = \begin{cases} 
O(n^{-1.5 - \min\{\alpha, \beta\} \log n}), & \text{if } -1 < \min\{\alpha, \beta\} \leq \gamma \leq -\frac{1}{2} \\
O(n^{-2\gamma - \min\{\alpha, \beta\} - \frac{1}{2}}), & \text{if } -1 < \min\{\alpha, \beta\} \leq -\frac{1}{2} < \gamma \leq 0 \\
O(n^{2\gamma}), & \text{if } -\frac{1}{2} < \min\{\alpha, \beta\} \leq \gamma
\end{cases}
\]
(10)

- If \( f(x) \) has an absolutely continuous \((r - 1)\)st derivative \( f^{(r - 1)} \) on \([-1, 1]\) for an integer \( r \geq 3 \), and a \( r \)th derivative \( f^{(r)} \) of bounded variation \( V_r = \text{Var}(f^{(r)}) < \infty \), then
\[
\|f(x) - H_{2n - 1}^*(f, x)\|_{\infty} = \begin{cases} 
O\left(n^{-r} \log n\right), & \gamma \leq -\frac{1}{2} \\
O\left(n^{2\gamma - r + 1}\right), & \gamma > -\frac{1}{2}
\end{cases}
\]
(11)

while if \( f(x) \) is differentiable and \( f'(x) \) is bounded on \([-1, 1]\), then
\[
\|f(x) - H_{2n - 1}^*(f, x)\|_{\infty} = \begin{cases} 
O\left(n^{-1} \log n\right), & \gamma \leq -\frac{1}{2} \\
O(n^{2\gamma}), & \gamma > -\frac{1}{2}
\end{cases}
\]

Comparing these results with
\[
f(x) - H_{2n - 1}(f, x) = \begin{cases} 
O\left(n^{-1} \log n\right), & \gamma \leq -\frac{1}{2} \\
O(n^{2\gamma}), & \gamma > -\frac{1}{2}
\end{cases}, \quad (\text{Vértesi [14]})
\]

which is sharp and attainable (see Fig. 2), we see that \( H_{2n - 1}^*(f, x) \) converges much faster than \( H_{2n - 1}(f, x) \) for analytic functions or functions of higher regularities (see Fig. 1). Particularly, \( H_{2n - 1}(f, x) \) diverges at Gauss-Jacobi pointsystems with \( \gamma \geq 0 \), whereas, \( H_{2n - 1}^*(f, x) \) converges for functions analytic in the Bernstein ellipse or of finite limited regularity.

Fig. 2 \( \|H_{2n - 1}(f, x) - f(x)\|_{\infty} \) at \( x = -1 : 0.001 : 1 \) by using Gauss-Jacobi pointsystem for \( f(x) = |x| \) with different \( \alpha \) and \( \beta \), respectively.
For simplicity, in the following we abbreviate \( x_k^{(n)} \) as \( x_k \), \( \ell_k^{(n)}(x) \) as \( \ell_k(x) \), \( h_k^{(n)}(x) \) as \( h_k(x) \), and \( b_k^{(n)}(x) \) as \( b_k(x) \). \( A \sim B \) denotes there exist two positive constants \( c_1 \) and \( c_2 \) such that \( c_1 \leq |A|/|B| \leq c_2 \).

## 2 Main Results

Suppose \( f(x) \) satisfies a Dini-Lipschitz condition on \([-1, 1]\), then it has the following absolutely and uniformly convergent Chebyshev series expansion

\[
f(x) = \sum_{j=0}^{\infty} c_j T_j(x), \quad c_j = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_j(x)}{\sqrt{1-x^2}} dx, \quad j = 0, 1, \ldots.
\] (12)

where the prime denotes summation whose first term is halved, \( T_j(x) = \cos(j \cos^{-1} x) \) denotes the Chebyshev polynomial of degree \( j \).

**Lemma 1**

(i) (Bernstein [2]) If \( f \) is analytic with \( |f(z)| \leq M \) in the region bounded by the ellipse \( E_\rho \) with foci \( \pm 1 \) and major and minor semiaxis lengths summing to \( \rho > 1 \), then for each \( j \geq 0 \),

\[
|c_j| \leq \frac{2M}{\rho^j}.
\] (13)

(ii) (Trefethen [13]) For an integer \( r \geq 1 \), if \( f(x) \) has an absolutely continuous \((r-1)\)st derivative \( f^{(r-1)} \) on \([-1, 1]\) and a \( r \)th derivative \( f^{(r)} \) of bounded variation \( V_r = \text{Var}(f^{(r)}) < \infty \), then for each \( j \geq r + 1 \),

\[
|c_j| \leq \frac{2V_r}{\pi j(j-1) \cdots (j-r)}.
\] (14)

Suppose \(-1 < x_n < x_{n-1} < \cdots < x_1 < 1\) in decreasing order are the roots of \( P_n^{(\alpha,\beta)}(x) \) (\( \alpha, \beta > -1 \)), and \( \{w_j\}_{j=1}^{n} \) are the corresponding weights in the Gauss-Jacobi quadrature.

**Lemma 2** For \( j = 1, 2, \ldots, n \), it follows

\[
(x-x_j)\ell_j(x) = \sigma_n(-1)^j \sqrt{1-x_j^2} w_j \frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} P_n^{(\alpha,\beta)}(x),
\] (15)

where \( \sigma_n = +1 \) for even \( n \) and \( \sigma_n = -1 \) for odd \( n \).
Proof Let \( z_n = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} [P_n^{(\alpha,\beta)}(x)]^2 \, dx \) and \( K_n \) the leading coefficient of \( P_n^{(\alpha,\beta)}(x) \). From Abramowitz and Stegun [1], we have

\[
 z_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}, \quad K_n = \frac{1}{2^n n!\Gamma(n+\alpha+\beta+1)}.
\]

Furthermore, by Szegö [12, (15.3.1)] (also see Wang et al. [15]), we obtain

\[
 (x-x_j)\ell_j(x) = \frac{1}{\omega_n'(x_j)\omega_n(x)} = \sigma_n(-1)^j \sqrt{\frac{K_n^22n(1-x_j^2)w_j}{2n(2n+\alpha+\beta+1)z_n}} \omega_n(x)^2n(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x),
\]

which implies the desired result (15).

Lemma 3 For \( j = 1, 2, \ldots, n \), it follows

\[
 (1-x_j^2)w_j = O\left(n^{-1}\right). \tag{16}
\]

Proof From \( w_j = O\left(2^{\alpha+\beta+1}\pi\left(\sin\frac{\theta_j}{2}\right)^{2^{\alpha+1}}\cos\frac{\theta_j}{2}\right)^{2^{\alpha+1}}\) Szegö [12, (15.3.10)], we see for \( x_j = \cos\theta_j \) that \( (1-x_j^2)w_j = O\left(2^{\alpha+\beta+3}\pi\left(\sin\frac{\theta_j}{2}\right)^{2^{\alpha+1}}\cos\frac{\theta_j}{2}\right)^{2^{\alpha+3}}\), which derives the desired result.

Lemma 4 ([10, 16]) For \( t \in [-1, 1] \), let \( x_m \) be the root of the Jacobi polynomial \( P_n^{(\alpha,\beta)} \) which is closest to \( t \). Then for \( k = 1, 2, \ldots, n \), we have

\[
 \ell_k(t) = \begin{cases} 
 O\left(|k-m|^{-1} + |k-m|^\gamma^{-\frac{1}{2}}\right), & k \neq m, \\
 O(1), & k = m, \quad \gamma = \max\{\alpha, \beta\}. 
\end{cases} \tag{17}
\]

Lemma 5 (Szegö [12, Theorem 8.1.2]) Let \( \alpha, \beta \) be real but not necessarily greater than \(-1\) and \( x_k = \cos\theta_k \). Then for each fixed \( k \), it follows

\[
 \lim_{n \to \infty} n\theta_k = j_k, \tag{18}
\]

where \( j_k \) is the \( k \)th positive zero of Bessel function \( J_\alpha \).

Lemma 6 For \( k = 1, 2, \ldots, n \), it follows

\[
 v_k(x) = 1 - (x-x_k)\frac{\omega_n''(x_k)}{\omega_n'(x_k)} = O(n^2). \tag{19}
\]
Proof Note that \( P_n^{(\alpha, \beta)}(x) \) satisfies the second order linear homogeneous Sturm-Liouville differential equation [12, (4.2.1)]
\[
(1 - x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta + 1)y = 0.
\]
By \( \omega_n(x) = \frac{P_n^{(\alpha, \beta)}(x)}{K_n} \), we get
\[
\frac{\omega_n''(x_j)}{\omega_n'(x_j)} = -\frac{\beta - \alpha - (\alpha + \beta + 2)x_j}{1 - x_j^2} \quad ([12, (14.5.1)]).
\]
In addition, by Lemma 5 with \( x_j = \cos \theta_j \), we see that \( \theta_1 \sim \frac{1}{n} \). Similarly, by \( P_n^{(\alpha, \beta)}(-x) = (-1)^nP_n^{(\beta, \alpha)}(x) \) we have \( \theta_n \sim \frac{1}{n} \). These together yield
\[
\frac{1}{1 - x_1^2} = O(n^2), \quad \frac{1}{1 - x_n^2} = O(n^2), \quad \frac{1}{1 - x_j^2} \leq \max \left( \frac{1}{1 - x_1^2}, \frac{1}{1 - x_n^2} \right) = O(n^2)
\]
and then by (20) it deduces the desired result. \( \square \)

Theorem 1 Suppose \( \{x_j\}_{j=1}^n \) are the roots of \( P_n^{(\alpha, \beta)}(x) \) with \( \alpha, \beta > -1 \), then the Hermite-Fejér interpolation (5) for \( f \) analytic in \( E_\rho \) with \( |f(z)| \leq M \) at \( \{x_j\}_{j=1}^n \) has the convergence rate (9).

Proof Since the Chebyshev series expansion of \( f(x) \) is uniformly convergent under the assumptions, and the error of Hermite-Fejér interpolation (5) on Chebyshev polynomials satisfies \( |E(T_j, x)| = |T_j(x) - H_{2n-1}^+(T_j, x)| = 0 \) for \( j = 0, 1, \ldots, 2n - 1 \), then it yields
\[
|E(f, x)| = |f(x) - H_{2n-1}^+(f, x)| = \sum_{j=0}^{\infty} c_j E(T_j, x) \leq \sum_{j=2n}^{\infty} |c_j| |E(T_j, x)|.
\]
(21)
Furthermore, \( |E(T_j, x)| = |T_j(x) - \sum_{i=1}^n T_j(x_i)h_i(x) - \sum_{i=1}^n T'_j(x_i)b_i(x)| \). In the following, we will focus on estimates of \( |E(T_j, x)| \) for \( j \geq 2n \).

In the case \( \gamma \leq 0 \): Notice that the pointsystem is normal which implies \( h_i(x) \geq 0 \) for all \( i = 1, 2, \ldots, n \) and for all \( x \in [-1, 1] \),
\[
1 \equiv \sum_{i=1}^n h_i(x) = \sum_{i=1}^n v_i(x) \ell_i^2(x).
\]
Then we have
\[
|\sum_{i=1}^n T_j(x_i)h_i(x)| \leq \sum_{i=1}^n h_i(x) = 1, \quad j = 0, 1, \ldots
\]
(22)
Additionally, by Lemma 2, it obtains for \( j = 2n, 2n + 1, \ldots \) that

\[
| \sum_{i=1}^{n} T'_j(x_i)b_i(x) | \\
= j | \sum_{i=1}^{n} U_{j-1}(x_i)(x - x_i)\ell_i^2(x) | \\
= \frac{2(\alpha+\beta+1)/2}{\sqrt{n!\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+2)\Gamma(n+\beta+2)}} | P^{(\alpha,\beta)}_n(x) \sum_{i=1}^{n} U_{j-1}(x_i) \sqrt{(1 - x_i^2)\ell_i(x)} | \\
= \frac{2(\alpha+\beta+1)/2}{\sqrt{n!\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+2)\Gamma(n+\beta+2)}} | P^{(\alpha,\beta)}_n(x) \sum_{i=1}^{n} \sin((j-1) \arccos(x_i))\sqrt{\ell_i(x)} | \\
= j O \left( \frac{| P^{(\alpha,\beta)}_n(x) \sum_{i=1}^{n} \ell_i(x) | \| w_i \|_1}{\Lambda_n} \right)
\]

\((U_{j-1} \text{ is the second kind of Chebyshev polynomial of degree } j - 1)\) since 

\[
\sqrt{\frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+2)\Gamma(n+\beta+2)}} \text{ is uniformly bounded in } n \text{ for } \alpha, \beta > -1 \text{ due to }
\]

\[
\frac{(n+1)!\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+2)\Gamma(n+\beta+2)} = \left( 1 - \frac{\alpha\beta}{(n+1)^2 + (\alpha + \beta)(n+1) + \alpha\beta} \right) \\
\times \frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}
\]

which implies \( \sqrt{\frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+2)\Gamma(n+\beta+2)}} \) is uniformly bounded in \( n \) and then 

\[
\sqrt{\frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+2)\Gamma(n+\beta+2)}} \text{ is uniformly bounded. Here } \Lambda_n = \max_{x \in [-1, 1]} \sum_{i=1}^{n} | \ell_i(x) | \text{ is the Lebesgue constant.}
\]

Then from

\[
P^{(\alpha,\beta)}_n(x) = \begin{cases} 
    O(n^{-1/2}), & \text{if } \max\{\alpha, \beta\} \leq -\frac{1}{2}, \\
    O(n^{\max\{\alpha, \beta\}}), & \text{if } \max\{\alpha, \beta\} > -\frac{1}{2}
\end{cases}
\]

\[
w_i = \begin{cases} 
    O(n^{-2-2\min\{\alpha, \beta\}}), & \text{if } \min\{\alpha, \beta\} \leq -\frac{1}{2}, \\
    O(n^{-1}), & \text{if } \min\{\alpha, \beta\} > -\frac{1}{2}
\end{cases}
\]

(see Szegö [12, pp 168, 354]) and

\[
\Lambda_n = \begin{cases} 
    O(\log n), & \text{if } \max\{\alpha, \beta\} \leq -\frac{1}{2}, \\
    O(n^{\max\{\alpha, \beta\}+\frac{1}{2}}), & \text{if } \max\{\alpha, \beta\} > -\frac{1}{2}
\end{cases} \quad ([12, \text{pp 338})],
\]

we have

\[
| \sum_{i=1}^{n} T'_j(x_i)b_i(x) | = j \tau_n. \quad (23)
\]
Then by (22) and (23), we find \( |E(T_j, x)| \leq 2 + j \tau_n < 2 j \tau_n \) for \( j \geq 2n \), and consequently

\[
|E(f, x)| = |f(x) - H^*_{2n-1}(f, x)| \leq \sum_{j=2n}^{\infty} |c_j||E(T_j, x)| = 2 \tau_n \sum_{j=2n}^{\infty} j|c_j|,
\]

which, directly following [18], leads to the desired result.

In the case \( \gamma > 0 \): From

\[
|E(T_j, x)| = |T_j(x) - \sum_{i=1}^{n} T_j(x_i) h_i(x) - \sum_{i=1}^{n} T_j'(x_i) b_i(x)|,
\]

by Lemmas 3 and 6 we obtain

\[
\sum_{i=1}^{n} |v_i(x)| \ell_i^2(x) = O \left( n^{2} \int_{1}^{n} t^{2\gamma-1} dt \right) = O(n^{2+2\gamma}),
\]

and

\[
T_j(x) - \sum_{i=1}^{n} T_j(x_i) h_i(x) = T_j(x) - \sum_{i=1}^{n} T_j(x_i) v_i(x) \ell_i^2(x) = O \left( n^{2+2\gamma} \right).
\]

These together with

\[
| \sum_{i=1}^{n} T_j'(x_i) b_i(x) | = \frac{j^{\frac{1}{2(\alpha+\beta+1)}} \sqrt{n! \Gamma(n+\alpha+\beta+1)}}{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)} |P_n^{(\alpha, \beta)}(x) \sum_{i=1}^{n} \sin((j-1) \arccos(x_i)) \sqrt{w_i} \ell_i(x)| = j \tau_n
\]

and then \( |E(T_j, x)| = O \left( j^{2+2\gamma} \right) \) for \( j \geq 2n \), similar to the above proof in the case of \( \gamma \leq 0 \), implies the desired result.

From the definition of \( \tau_n \), we see that when \( \alpha = \beta = -\frac{1}{2} \) the convergence order on \( n \) is the lowest. In addition, if \( f \) is of limited regularity, we have

**Lemma 7 (Vértesi [14])** Suppose \( \{x_j\}_{j=1}^{n} \) are the roots of \( P_n^{(\alpha, \beta)}(x) \), for every continuous function \( f(x) \) we have

\[
|H_{2n-1}(f, x) - f(x)| = O(1) \sum_{j=1}^{n} \left[ w(f; \frac{j^2 \sqrt{1 - x^2}}{n}) + w(f; \frac{j^2 |x|}{n^2}) \right] j^{2\tilde{\gamma}-1},
\]

where \( w(f; t) = w(t) \) is the modulus of continuity of \( f(x) \), and \( \tilde{\gamma} = \max(\alpha, \beta, -\frac{1}{2}) \).

**Theorem 2** Suppose \( \{x_j\}_{j=1}^{n} \) are the roots of \( P_n^{(\alpha, \beta)}(x) (\alpha, \beta > -1) \), and \( f(x) \) has an absolutely continuous \((r-1)st\) derivative \( f^{(r-1)} \) on \([-1, 1]\) for some \( r \geq 3 \),
and a \( r \)th derivative \( f^{(r)} \) of bounded variation \( V_r < \infty \), then the Hermite-Fejér interpolation (5) at \( \{x_j\}_{j=1}^n \) has the convergence rate (11).

**Proof** Consider the special functional \( L(g) = E_n(g, x) \), where \( E_n(g, x) \) is defined for \( \forall g \in C^1([-1, 1]) \) by

\[
E_n(g, x) = g(x) - \sum_{j=1}^n g(x_j)v_j(x)\ell^2_j(x) - \sum_{j=1}^n g'(x_j)(x - x_j)\ell^2_j(x). \tag{25}
\]

By the Peano kernel theorem for \( n \geq r \) (see Peano [9] or Kowalewski [8]), \( E_n(f, x) \) can be represented as

\[
E_n(f, x) = \int_{-1}^1 f^{(r)}(t)K_r(t)dt \tag{26}
\]

with \( K_r(t) = \frac{1}{(r-1)!}L\left((x-t)^{r-1}_+\right) \) for \( r = 3, 4, \ldots \), that is

\[
K_r(t) = \frac{1}{(r-1)!}(x-t)^{r-1}_+ - \frac{1}{(r-1)!}\sum_{j=1}^n (x_j-t)^{r-1}_+ v_j(x)\ell^2_j(x)
\]

\[- \frac{1}{(r-1)!}\sum_{j=1}^n (x_j-t)^{r-2}_+ (x-x_j)\ell^2_j(x),
\]

where

\[
(x-t)^{k-1}_+ = \begin{cases} (x-t)^{k-1}, & x \geq t; \quad (k \geq 2), \\ 0, & x < t, \end{cases}
\]

\[
(x-t)^0_+ = \begin{cases} 1, & x \geq t; \\ 0, & x < t. \end{cases} \quad (k = 1).
\]

Moreover, noting that

\[
\frac{1}{(k-2)!}(x-u)^{k-2}_+ = \int_u^1 \frac{1}{(k-3)!}(x-t)^{k-3}_+dt, \quad k = 3, 4, \ldots ,
\]

we get the following identity

\[
K_{s-1}(u) = \int_u^1 K_{s-2}(t)dt, \quad s = 4, 5, \ldots ,
\]

where \( K_2(t) \) is defined by

\[
K_2(t) = (x-t)^1_+ - \sum_{j=1}^n (x_j-t)^1_+ v_j(x)\ell^2_j(x) - \sum_{j=1}^n (x_j-t)^0_+ (x-x_j)\ell^2_j(x).
\]
In addition, it can be easily verified that \( K_s(-1) = K_s(1) = 0 \) for \( s = 2, 3, \ldots \).

Since \( f^{(r)} \) is of bounded variation, directly applying the similar skills of Theorem 2 and Lemma 4 in [16], we get

\[
\|E_n(f, x)\|_\infty \leq V_r \|K_{r+1}\|_\infty, \tag{27}
\]

and

\[
\|K_{s+1}\|_\infty \leq \frac{\pi}{2n-s} \sup_{-1 \leq t \leq 1} |K_s(t)|, \quad \text{for } s = 2, 3, \ldots, \tag{28}
\]

respectively. Then from (27) and (28), we can obtain that

\[
\|E_n(f, x)\|_\infty \leq \frac{\pi^{r-1}V_r}{(2n-2)(2n-3) \ldots (2n-r)} \|K_2\|_\infty. \tag{29}
\]

In addition, by Lemma 7, we have

\[
\|(x - t)^\perp_+ - \sum_{j=1}^{n} (x_j - t)^\perp_+ v_j(x)\ell^2_j(x)\|_\infty = \begin{cases} O\left(\frac{\log n}{n}\right), & \gamma \leq -\frac{1}{2} \\ O\left(n^{2\gamma}\right), & \gamma > -\frac{1}{2} \end{cases} \tag{30}
\]

while by Lemmas 2–3, we get

\[
|\sum_{j=1}^{n} (x_j - t)^0_+ (x - x_j)\ell^2_j(x)| \leq \sum_{j=1}^{n} |(x - x_j)\ell^2_j(x)| = \begin{cases} O\left(\frac{\log n}{n}\right), & \gamma \leq -\frac{1}{2} \\ O\left(n^{2\gamma}\right), & \gamma > -\frac{1}{2} \end{cases}. \tag{31}
\]

Together (30) and (31), we can obtain the desired results by using

\[
K_2(t) = \begin{cases} O\left(\frac{\log n}{n}\right), & \gamma \leq -\frac{1}{2} \\ O\left(n^{2\gamma}\right), & \gamma > -\frac{1}{2} \end{cases}.
\]

Finally, We use a function of analytic \( f(x) = \frac{1}{1+25x^2} \) and a function of limited regularity \( f(x) = |x|^5 \) to show that the convergence rate of \( \|f(x) - H^*_2n^{-1}(f, x)\|_\infty \) is dependent on \( \alpha \) and \( \beta \) in Fig. 3.
$f(x) = 1/(1+25x^2)$

$\|H_{2n-1}^s(f, x) - f(x)\|_{\infty}$ at $x = -1 : 0.001 : 1$ by using Gauss-Jacobi pointsystem for $f(x) = 1/(1+25x^2)$ and $f(x) = |x|^5$ with different $\alpha$ and $\beta$, respectively

**Fig. 3**

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