TENSOR GENERATORS ON SCHEMES AND STACKS

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Abstract. We show that an algebraic stack with affine stabilizer groups satisfies the resolution property if and only if it is a quotient of a quasiaffine scheme by the action of the general linear group, or equivalently, if there exists a vector bundle whose associated frame bundle has quasiaffine total space. This generalizes a former result of B. Totaro to non-normal and non-noetherian schemes and algebraic stacks. Also, we show that the vector bundle induces such a quotient structure if and only if it is a tensor generator in the category of quasicoherent sheaves.

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Introduction

Given a commutative ring $R$, every $R$-module $M$ admits a surjection $R^{(I)} \to M$ by a free $R$-module. Thus, on $\text{Spec}(R)$, every quasicoherent $\mathcal{O}_{\text{Spec}(R)}$-module is generated by a free one. On the other hand, if $G$ is an algebraic group, then every linear $G$-representation is generated by a family of linear representations, obtained by taking subsheaves, duals, direct sums and tensor powers of a faithful representation $V$, saying that $V$ is a tensor generator.

The goal of this article is to unify and generalize this kind of algebro-geometric characterization to general algebraic stacks that have pointwise affine stabilizer groups. The language of algebraic stacks allows us to express this uniformly. Recall, that an algebraic stack $X$ satisfies the resolution property (or has enough locally free sheaves) if every quasicoherent sheaf $\mathcal{M}$ admits a surjection $\bigoplus V_j \to \mathcal{M}$, for some family of vector bundles. This fundamental property of the category of quasicoherent sheaves was studied by various authors before and still poses a challenging problem. We refer the reader to [Tot04] for a detailed history and its connection to $K$-theory.

Here, we prove that for every quasicompact and quasiseparated algebraic stack $X$, having the resolution property and affine stabilizer groups is equivalent to the existence of a vector bundle $V$ of rank $n$ whose associated bundle of $\text{GL}_n$-frames
\( I_{\text{can}}(\mathcal{O}_{X}^{\mathbb{A}}_X, \mathcal{V}) \rightarrow X \) has quasiaffine total space, or equivalently, that \( X \) is a quotient stack \([U/\text{GL}_n]\), where \( U \) is a quasiaffine scheme acted on by some general linear group. This result was obtained by B. Totaro under the condition that the algebraic stack is noetherian and normal [Tot04, Thm. 1.1]. He also pointed out that every quotient stack of such kind has affine diagonal and hence pointwise affine stabilizer groups.

Our second aim is to show that a vector bundle induces such a global quotient stack presentation if and only if it is a tensor generator for \( \text{QCoh}(X) \) in the sense of Tannaka theory [Del90, 6.16], saying that a generating family of vector bundles can be obtained from \( \mathcal{V} \) directly by taking locally split subsheaves, duals, direct sums and tensor products.

Summarizing, we see that a vector bundle is a tensor generator if and only if its associated frame bundle has quasiaffine total space, and that the existence thereof is equivalent to the resolution property.

In order to overcome the normality assumption, we patch schemes along integral morphisms à la D. Ferrand [Fer03]. Interesting on its own, we prove that the AF-property (i.e. every finite set of points admits an affine open neighborhood) descends along integral surjections in the category of algebraic spaces. The application of approximation techniques (with their recent extension to algebraic stacks by of D. Rydh [Ryd13]) allows us to not only eliminate the noetherian hypothesis, but also to rigorously identify \( \mathcal{V} \) as a direct limit of vector bundles. For that we consider the fiber product of essentially all \( \text{GL}_n \)-torsors to get a possibly huge algebraic stack that trivializes all vector bundles simultaneously.

This article is largely based on my thesis. The recent improvements of approximating general algebraic stacks by D. Rydh made it possible to remove many technical assumptions.

The paper is organized as follows: In section 1 we define relatively generating families of finitely presented quasicoherent sheaves with respect to a morphism of algebraic stacks and prove basic permanence properties. Section 2 deals with pinching AF-schemes, and there we prove that the AF-property descends along integral surjections in the category of algebraic spaces (Theorem 2.3). From this we derive in section 3 that an algebraic stack with pointwise affine stabilizer groups must be quasiaffine when the structure sheaf is generating (Proposition 3.1). Section 4 deals with relatively generating families of locally free finite-type quasicoherent sheaves, and there we define the relative resolution property as the mere existence of the former. For the readers convenience we recast the classes of algebraic stacks where the resolution property is known to be true. Finally, in section 5 we show that a frame bundle has quasiaffine total space if and only if the corresponding vector bundle is a tensor generator and the stabilizer groups are affine (Theorem 5.3), and lastly, we prove the generalization of Totaro’s Theorem (Theorem 5.8).

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Conventions and notations. For algebraic stacks we follow the conventions in [LMB00] except that we do not require that the diagonal of an algebraic stack is separated, just quasicompact and quasiseparated as in [SP]. A vector bundle is a locally free sheaf of finite type, equivalently flat and finitely presented quasicoherent sheaf.
1. Quasicoherent generators

In this preliminary section, we define generating families of finitely presented quasicoherent sheaves, extend the definition to the relative case, and show the usual permanence properties. In forthcoming sections we restrict entirely to the case of locally free sheaves but for the sake of completeness we treat the general case here.

(1.1) Definition. A family of quasicoherent \( \mathcal{O}_X \)-modules \((\mathcal{G}_i)_{i \in I}\) is a generating family for \( X \) by abuse of notation if it is a family of finitely presented generators in the category of all quasicoherent \( \mathcal{O}_X \)-modules \( \text{QCoh}(X) \). That is, for every quasicoherent \( \mathcal{O}_X \)-module \( \mathcal{M} \) exists a surjection \( \bigoplus_{i \in I} \mathcal{G}_i^\oplus \twoheadrightarrow \mathcal{M} \).

(1.2) Remark. The existence of such a family is equivalent to the so called completeness property, saying that every quasicoherent sheaf is the direct limit of finitely presented ones, or in other words that \( \text{QCoh}(X) \) is compactly generated. This is known to hold for a vast class of algebraic stacks, including (pseudo-) noetherian and qcf stacks \[\text{Rvd13 \ 4.1}].

In case of schemes, a generating family can be given by a suitable family of ideal sheaves, as seen in the following example.

(1.3) Example ([SV04, Prop. 2.2]). Suppose that \( X \) is a noetherian scheme. Let \( \mathcal{I}_1, \ldots, \mathcal{I}_n \subset \mathcal{O}_X \) be a family of ideal sheaves such that \( (X - V(\mathcal{I}_i))_{1 \leq i \leq n} \) is an affine open covering. Then the family of all powers \( (\mathcal{I}_i^j)_{j \in \mathbb{N}, 1 \leq i \leq n} \) is generating for \( X \). If we put \( \mathcal{G} := \bigoplus_{i=1}^n \mathcal{I}_i \), then \( \text{Sym}^n(\mathcal{G})_{n \in \mathbb{N}} \) is also a generating family.

The definition of a generating family extends to the relative case analogously to as relatively ample invertible sheaves. In order to make this precise we provide a formulation in terms of an adjoint pair of functors.

(1.4) Definition. Let \( f: X \rightarrow Y \) be a quasicompact and quasiseparated morphism of algebraic stacks and let \( \mathcal{G}_i = (\mathcal{G}_i)_{i \in I} \) be a family of finitely presented quasicoherent \( \mathcal{O}_X \)-modules. We define an an adjoint pair of functors \((f_{\mathcal{G}_i}^*, f_{\mathcal{G}_i}^\bullet)\) by

\[
\begin{align*}
  f_{\mathcal{G}_i}^*: \text{QCoh}(Y)^I &\rightarrow \text{QCoh}(X), \quad (\mathcal{N}_i)_{i \in I} \mapsto \bigoplus_{i \in I} \mathcal{G}_i \otimes_{\mathcal{O}_X} f^* \mathcal{N}_i, \quad (1.4.1) \\
  f_{\mathcal{G}_i}^\bullet: \text{QCoh}(X) &\rightarrow \text{QCoh}(Y)^I, \quad \mathcal{M} \mapsto (f_\circ \text{Hom}_{\mathcal{O}_X}(\mathcal{G}_i, \mathcal{M}))_{i \in I}. \quad (1.4.2)
\end{align*}
\]

Note that \( \text{Hom}_{\mathcal{O}_X}(\mathcal{G}_i, \mathcal{M}) \) is quasicoherent because \( \mathcal{G}_i \) is of finite presentation. Using the adjunctions \((f^*, f_\circ)\) and \((\mathcal{G} \otimes, \text{Hom}_{\mathcal{O}_X}(\mathcal{G}_i, \cdot))\), \( i \in I \), it is straightforward to check that \((f_{\mathcal{G}_i}^*, f_{\mathcal{G}_i}^\bullet)\) is indeed an adjoint pair.

(1.5) Remark. For an algebraic stack \( X \) that possesses a coarse moduli space \( X_0 \), the case of singelton families that are generating with respect to the natural morphism \( \pi: X \rightarrow X_0 \) was studied in \[\text{OS03 \ Section \ 5}].

We present three equivalent ways of constructing relative resolutions.

(1.6) Lemma. Let \( f: X \rightarrow Y \) be a quasicompact and quasiseparated morphism of algebraic stacks and let \( \mathcal{G}_i = (\mathcal{G}_i)_{i \in I} \) be a family of finitely presented quasicoherent \( \mathcal{O}_X \)-modules. Then the following properties are equivalent:

(i) Every quasicoherent \( \mathcal{O}_X \)-module \( \mathcal{M} \) admits a surjection

\[
\bigoplus_{i \in I} \mathcal{G}_i \otimes f^* \mathcal{N}_i \twoheadrightarrow \mathcal{M}, \quad (1.6.1)
\]

for some family of quasicoherent \( \mathcal{O}_Y \)-modules \((\mathcal{N}_i)_{i \in I}\).
(ii) The counit \( \varepsilon \): \( f_{\mathcal{G}_i}^* f_{\mathcal{G}_i} \Rightarrow \text{id}_{\text{QCoh}(X)} \) evaluated at any quasicoherent \( \mathcal{O}_X \)-module \( \mathcal{M} \) is a surjective map
\[
\varepsilon(\mathcal{M}): \bigoplus_{i \in I} \mathcal{G}_i \otimes_{\mathcal{O}_X} f^* f_* \text{Hom}_{\mathcal{O}_X}(\mathcal{G}_i, \mathcal{M}) \to \mathcal{M}.
\] (1.6.2)

(iii) The functor \( f_{\mathcal{G}_i}^* \) is faithful: for every non-zero morphism \( \mathcal{M}_1 \to \mathcal{M}_2 \) in \( \text{QCoh}(X) \) there exists \( i \in I \) such that the map
\[
f_* \text{Hom}(\mathcal{G}_i, \mathcal{M}_1) \to f_* \text{Hom}(\mathcal{G}_i, \mathcal{M}_2)
\]
is non-zero.

Proof. Clearly, (ii) implies (i), and the converse holds because, by adjunction, every map \( \varphi : f_{\mathcal{G}_i}^* ((N_i)_{i \in I}) \to \mathcal{M} \) factors over the counit \( \varepsilon \) by \( f_{\mathcal{G}_i}^*(\varphi^\flat) \), where \( \varphi^\flat : (N_i)_{i \in I} \to f_{\mathcal{G}_i}^*(\mathcal{M}) \) is the adjoint of \( \varphi \). The equivalence \( (ii) \Leftrightarrow (iii) \) is a formal consequence of adjunction (see [Par70, 2.3]). □

(1.7) Definition. Let \( f : X \to Y \) be a morphism of algebraic stacks. A family of quasicoherent \( \mathcal{O}_X \)-modules \( \mathcal{G}_f = (\mathcal{G}_i)_{i \in I} \) is \( f \)-generating if \( f \) is quasipartite and quasi-separated, each \( \mathcal{G}_i \) is finitely presented, and the equivalent conditions in Lemma 14 hold. If \( Y \) is affine, the definition is the same as in the absolute case (see [13]). We call the family \( \mathcal{G}_f \) universally \( f \)-generating if for every morphism of algebraic stacks \( Y' \to Y \) the family of restricted sheaves \( \mathcal{G}_f|_{X \times_Y Y'} := (\mathcal{G}_i|_{X \times_Y Y'})_{i \in I} \) is generating for the base change \( f_{Y'} : X' \to Y' \).

We begin with the usual sortes for (universally) generating families with respect to morphisms.

(1.8) Proposition. Let \( S \) be an algebraic stack, let \( f : X \to Y \) be a morphism of algebraic \( S \)-stacks, and let \( \mathcal{G}_f = (\mathcal{G}_i)_{i \in I} \) be a family of quasicoherent \( \mathcal{O}_X \)-modules.

(i) The family \( \mathcal{G}_f \) is (universally) \( f \)-generating if and only if \( \mathcal{G}_f \) is \( (\text{universally}) \) generating for some (equivalently every) 2-isomorphic morphism \( f' : X \to Y \).

(ii) The singlet family \( \mathcal{O}_X \) is universally \( f \)-generating if \( f \) is quasiaffine (for instance, if \( f \) is an affine, finite, quasi-finite finite-type separated morphism, a finite-type monomorphism, a quasipartite open immersion, or a closed immersion).

(iii) fpqc-local on the target: Let \( (S_a \to S) \) be an fpqc covering family (resp. a faithfully flat family), \( f \) is quasipartite and quasi-separated, and each \( \mathcal{G}_i \) is finitely presented. If the restricted family \( \mathcal{G}_f|_{X_{(s_a)}} \) is (universally) generating for \( f_{(s_a)} : X_{(s_a)} \to Y_{(s_a)} \) and each \( a \in A \), then \( \mathcal{G}_f \) is (universally) \( f \)-generating.

(iv) Base change: Let \( S' \to S \) be a morphism of algebraic stacks such that \( S \) has quasiaffine (resp. just quasipartite and quasi-separated) diagonal. If \( \mathcal{G}_f \) is (universally) \( f \)-generating, then the restricted family \( \mathcal{G}_f|_{X_{(s')}} \) is (universally) generating for \( f_{(s')} : X_{(s')} \to Y_{(s')} \).

(v) Composition: Let \( g : Y \to Z \) be a morphism of algebraic \( S \)-stacks, and let \( \mathcal{E}_f = (\mathcal{E}_j)_{j \in J} \) be a family of quasicoherent \( \mathcal{O}_Y \)-modules. If \( \mathcal{G}_f \) is (universally) \( f \)-generating and if \( \mathcal{E}_f \) is (universally) \( g \)-generating, then the family
\[
\mathcal{G}_f \otimes f^* \mathcal{E}_f := (\mathcal{G}_i \otimes f^* \mathcal{E}_j)_{(i,j) \in I \times J}
\]
is (universally) generating for \( g \circ f \).
(vi) Left-cancellation property: Suppose that \( g \) is quasiseparated (resp. \( \Delta_g \) quasaffine). If \( \mathcal{G}_I \) is (universally) \( f \)-generating, then \( \mathcal{G}_I \) is (universally) \( f \)-generating.

(vii) Products: Let \( f_\alpha: X_\alpha \to Y_\alpha, \alpha = 1, 2, \) be morphisms of algebraic \( S \)-stacks and denote by \( p_\alpha: X_1 \times_X X_2 \to X_\alpha \) the projections. If \( \mathcal{G}_{I_\alpha}^{(\alpha)}, \alpha = 1, 2, \) are universally \( f_\alpha \)-generating families on \( X_\alpha \), then the family
\[
\mathcal{G}_{I_1}^{(1)} \boxtimes \mathcal{G}_{I_2}^{(2)} := \left( pr_1^* \mathcal{G}_{I_1}^{(1)} \otimes pr_2^* \mathcal{G}_{I_2}^{(2)} \right)_{(i_1,i_2) \in I_1 \times I_2}
\]
is universally generating for \( f_1 \times_S f_2: X_1 \times_S X_2 \to Y_1 \times_S Y_2 \).

(viii) Reduction: If \( \mathcal{G}_I \) is (universally) \( f \)-generating, then the restricted family \( \mathcal{G}_I|_{X_{\text{red}}} \) is (universally) generating for \( f_{\text{red}}: X_{\text{red}} \to Y_{\text{red}} \).

(1.9) Remark. Let \( P \) be a property of finitely presented sheaves which is local and satisfies fpqc-descent (e.g. “locally free”). Then the permanence properties shown in Proposition [1.8] carry over to (universally) relatively generating families of finitely presented sheaves satisfying \( P \) mutatis mutandis.

Proof of [1.8] —

Proof of [\( \text{(i)} \)]: The universal case reduces to the non-universal case, which follows from Lemma [1.6](iii) because faithfulness of a functor is preserved and reflected under 2-isomorphisms.

Proof of [\( \text{(iii)} \)]: It suffices to prove the non-universal case by applying a base change. Also we may assume that \( S = Y \) by restricting the faithfully flat covering \((S_\alpha \to S)\) along \( Y \to S \). Given a faithfully flat covering \( u_\alpha: X_\alpha \to Y \), consider for each \( \alpha \) the induced 2-cartesian square
\[
\begin{array}{ccc}
X_\alpha & \xrightarrow{v_\alpha} & X \\
\downarrow f_\alpha & \square & \downarrow f \\
Y_\alpha & \xrightarrow{u_\alpha} & Y
\end{array}
\]  
(1.9.1)

Let \( i \in I \) be given. By assumption, \( f_\alpha \) is quasicompact and quasiseparated, and each \( v_\alpha^* \mathcal{G}_i \) is finitely presented. Then \( f \) is quasicompact and quasiseparated and \( \mathcal{G}_i \) is finitely presented by fpqc descent (resp. by assumption). Thus the following diagram consists of well-defined functors:

\[
\begin{array}{ccc}
\text{QCoh}(X) & \xrightarrow{v_\alpha^*} & \text{QCoh}(X_\alpha) \\
\downarrow \text{Hom}_{\mathcal{X}_X}(\mathcal{G}_i, \cdot) & & \downarrow \text{Hom}_{\mathcal{X}_X}(v_\alpha^* \mathcal{G}_i, \cdot) \\
\text{QCoh}(Y) & \xrightarrow{u_\alpha^*} & \text{QCoh}(Y_\alpha) \\
\downarrow f^* & & \downarrow f_{\alpha^*} \\
\text{QCoh}(Y) & \xrightarrow{u_\alpha^*} & \text{QCoh}(Y_\alpha)
\end{array}
\]
(1.9.2)

The upper square is 2-commutative since \( \mathcal{G}_i \) and \( v_\alpha^* \mathcal{G}_i \) are of finite presentation and \( v_\alpha^* \) commutes with the internal hom’s by flatness. The lower square is 2-commutative by flat base change [LMB00, 13.1.9]. Thus, the whole diagram is 2-commutative. The assertion follows now by a simple diagram chase: Since \( (v_\alpha) \) is a faithfully flat covering family for \( X \), the induced pullback functor \( v_\alpha: \text{QCoh}(X) \to \prod_{\alpha} \text{QCoh}(X_\alpha), \mathcal{M} \mapsto v_\alpha^*(\mathcal{M}) \) is faithful. Similarly we get a faithful functor \( u_\alpha: \text{QCoh}(Y) \to \prod_{\alpha} \text{QCoh}(Y_\alpha) \). For each \( \alpha \) let \( v_\alpha^* \mathcal{G}_i \) be the family of restricted sheaves \((v_\alpha^* \mathcal{G}_i | i \in I)\), which is generating for \( f_\alpha \) by hypothesis.
Thus, \( (f_\alpha)_{\alpha \in \mathcal{A}} \cdot \mathcal{G}_I \cdot Q\mathcal{Coh}(X_\alpha) \to Q\mathcal{Coh}(Y_\alpha)^I \) is faithful for each \( \alpha \). We conclude that the composition

\[
Q\mathcal{Coh}(X) \xrightarrow{\cup \Delta} \prod_\alpha Q\mathcal{Coh}(X_\alpha) \xrightarrow{((f_\alpha)_{\alpha \in \mathcal{A}} \cdot \mathcal{G}_I)_{\alpha \in \mathcal{A}}} \prod_\alpha Q\mathcal{Coh}(Y_\alpha)^I \cong \left( \prod_\alpha Q\mathcal{Coh}(Y_\alpha) \right)^I
\]

is faithful, \( a \) fortiori this holds for the 2-isomorphic functor

\[
Q\mathcal{Coh}(X) \xrightarrow{f_{\mathcal{G}_I}} Q\mathcal{Coh}(Y)^I \xrightarrow{(u, \alpha)^I} \left( \prod_\alpha Q\mathcal{Coh}(Y_\alpha) \right)^I
\]

By the left cancellation property for faithful functors, we conclude that \( f_{\mathcal{G}_I} \) is faithful, too. Thus \( \mathcal{G}_I \) is \( f \)-generating.

**Proof of (iv)** As the property "quasi-affine" is stable under arbitrary base change, it suffices to show that \( \mathcal{O}_X \) is \( f \)-generating. By hypothesis there is a smooth covering \( Y' \to Y \) by a scheme \( Y' \) such that the base change \( f': X':= X \times_Y Y' \to Y' \) is quasi-affine. Thus, \( \mathcal{O}_{X'} \) is \( f' \)-ample [EGA I, 5.1.2], hence \( f' \)-generating. So by (iii) we conclude that \( \mathcal{O}_X \) is \( f \)-generating.

**Proof of (v)** It suffices to treat the non-universal case by replacing \( g \circ f: X \to Y \to Z \) for a given \( Z' \to Z \) with the base change \( g' \circ f': X' \to Y' \to Z' \) and using the isomorphisms \( (\mathcal{G}_I \otimes \mathcal{O}_X f^* E_j)|_{X'} \simeq \mathcal{G}_I|_{X'} \otimes \mathcal{O}_{X'}, f'^* E_j|_{Y'} \) for all \( (i,j) \in I \times J \). By assumption \( f \) and \( g \) are quasicompact and quasiseparated, so the same holds for \( h \). Let \( (i,j) \in I \times J \) be given. Since \( \mathcal{G}_I \) and \( E_j \) are of finite presentation, so is \( \mathcal{G}_I \otimes f^* E_j \). Then we get a diagram of well-defined functors:

\[
\begin{array}{ccc}
Q\mathcal{Coh}(X) & \xrightarrow{\text{Hom}(\mathcal{G}_I, \cdot)} & Q\mathcal{Coh}(X) \\
\downarrow{\text{Hom}(\mathcal{G}_I \otimes f^* E_j, \cdot)} & & \downarrow{\text{Hom}(f^* E_j, \cdot)} \\
Q\mathcal{Coh}(X) & \xrightarrow{f_*} & Q\mathcal{Coh}(Y) \\
\downarrow{g_* f_*} & & \downarrow{g_*} \\
Q\mathcal{Coh}(Z) & & Q\mathcal{Coh}(Z)
\end{array}
\]

The upper left triangle is 2-commutative by adjunction of \( \mathcal{G}_I \otimes \cdot \) and \( \text{Hom}(\mathcal{O}_X, \mathcal{G}_I \cdot \cdot) \) in \( Q\mathcal{Coh}(X) \). The lower triangle is 2-commutative by definition. The square is 2-commutative since it corresponds by adjunction to the isomorphism \( f^* (E_j \otimes \mathcal{O}_{X'}) \cong f^* E_j \otimes \mathcal{O}_{X'} f^*(\cdot) \). Thus, the whole diagram is 2-commutative. It follows that the composition

\[
(g_{E_j})^I \circ f_{\mathcal{G}_I} \cdot Q\mathcal{Coh}(X) \xrightarrow{f_{\mathcal{G}_I}} Q\mathcal{Coh}(Y)^I \xrightarrow{(g_{E_j})^I} (Q\mathcal{Coh}(Z))^I \cong (Q\mathcal{Coh}(Z)^I)^J
\]

is 2-isomorphic to the functor

\[
((g \circ f)_{\mathcal{G}_I \otimes f^* E_j})_*: Q\mathcal{Coh}(X) \to Q\mathcal{Coh}(Z)^I \times J.
\]

By hypothesis, \( f_{\mathcal{G}_I} \) and \( E_j \) are faithful. Then the constant functor \( (g_{E_j})^I \) is faithful, so too is the composition \( (g_{E_j})^I \circ f_{\mathcal{G}_I} \simeq ((g \circ f)_{\mathcal{G}_I \otimes f^* E_j})_* \) as required.

**Proof of (vi)** Let us first prove the non-universal case. By assumption, \( \mathcal{G}_I \) is a family of finitely presented \( \mathcal{O}_X \)-modules. Since \( g \circ f \) is quasicompact and quasiseparated, it follows that \( f \) is quasicompact and quasiseparated since \( \Delta_B \) is by assumption. Consider now diagram (1.9.3) with the singleton family \( \mathcal{E} = \mathcal{O}_Y \). Here, the lower triangle is well-defined if we extend the lower right corner by the inclusion \( i: Q\mathcal{Coh}(Z) \to \text{Mod}(Z) \), which is a faithful (and full) functor. As above, a diagram chase shows us that

\[
(g_*)^I \circ f_{\mathcal{G}_I} \cdot Q\mathcal{Coh}(X) \to Q\mathcal{Coh}(Y)^I \to \text{Mod}(Z)^I
\]
is 2-isomorphic to
\[ i^! \circ (g \circ f)_{G_1 \times} : \text{QCoh}(X) \to \text{QCoh}(Z)^I \to \text{Mod}(Z)^I \]

By hypothesis \((g \circ f)_{G_1 \times}\) is faithful. Also, the constant functor \(i^!\) is faithful. Thus, \((g_i)^! \circ f^i_{G_1 \times}, a \text{ a fortiori} the factor } f^i_{G_1 \times} \text{ is faithful as asserted.}

For the universal case we use the standard argument that \(f\) factors up to 2-isomorphism as the composition of the upper horizontal morphisms of the following two 2-cartesian squares:

\[
\begin{array}{ccc}
X & \xrightarrow{\Gamma_f} & X \times_Z Y \\
\downarrow f & \square & \downarrow f \times 1 \\
Y & \xrightarrow{\Delta_S} & Y \times_Z Y
\end{array}
\quad (1.9.4)
\]

\[
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{q} & Y \\
\downarrow p & \square & \downarrow g \\
X & \xrightarrow{g \circ f} & Z
\end{array}
\]

By hypothesis \(\Delta_S\) and hence \(\Gamma_f\) are quasiaffine. Thus, \(\mathcal{O}_X\) is universally \(\Gamma_f\)-generating by [ii]. Since \(G_1\) is universally \(g \circ f\)-generating by assumption, \(p^! G_1\) is universally \(q\)-generating. So by applying [v] to \(f = q \circ \Gamma_f\) we conclude that \(\{\mathcal{O}_X\} \otimes \Gamma_f^! (p^! G_1)\) is universally \(f\)-generating. But in light of the identity \(p \circ \Gamma_f = \text{id}_X\) this means that \(G_1\) is universally \(f\)-generating.

**Proof of [iv]** Clearly, universal relatively generating families are stable under base change. Hence, it suffices to treat the non-universal case. Choose a smooth covering family \(S'_\alpha \to S'\) of affine scheme \(S'_\alpha\). Then each composition \(S'_\alpha \to S' \to S\) is a quasiaffine morphism because \(\Delta_{S'/Z}\) is quasiaffine. Then the base change \(Y_{S'_\alpha} \to Y\) and \(X_{(S'_\alpha)} \to X\) are quasiaffine, too, so that \(\mathcal{O}_{Y_{S'_\alpha}}\) and \(\mathcal{O}_{X_{(S'_\alpha)}}\) are relatively generating by [ii]. It follows that the family of restricted sheaves \(G_1|_{X_{(S'_\alpha)}}\) is generating for the composition \(X_{S'_\alpha} \to X \to Y\) by [vii] a fortiori for the 2-isomorphic morphism \(X_{(S'_\alpha)} \to Y_{S'_\alpha}\) \(\to Y\) and hence for \(X_{(S'_\alpha)} \to Y_{S'_\alpha}\) by [vi] because \(Y_{S'_\alpha}\) \(\to Y\) has quasiaffine diagonal. Then [iii] implies that \(G_1|_{S'}\) is \(f_{S'}\)-generating.

**Proof of [vii]** The product morphism \(f_1 \times_S f_2\) is the decomposition of the upper horizontal morphisms of the following 2-cartesian squares, where \(p_\alpha, q_\alpha, r_\alpha\) denote the projections on the \(\alpha\)-th factor:

\[
\begin{array}{ccc}
X_1 \times_S X_2 & \xrightarrow{(f_1, \text{id}_{X_2})} & Y_1 \times_S X_2 \\
\downarrow p_1 & \square & \downarrow q_1 \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
\quad (1.9.5)
\]

\[
\begin{array}{ccc}
Y_1 \times_S X_2 & \xrightarrow{(\text{id}_{X_1}, f_2)} & Y_1 \times_S Y_2 \\
\downarrow q_2 & \square & \downarrow r_2 \\
X_2 & \xrightarrow{f_2} & Y_2
\end{array}
\]

Then the family \(p_1^* G_{i_1}^{(1)}\) is universally \((f_1, \text{id}_{X_2})\)-generating, and the family \(q_2^* p_1^* G_{i_2}^{(2)}\) is universally \((\text{id}_{X_1}, f_2)\)-generating. By using property [v] it follows that \(p_1^* G_{i_1}^{(1)} \otimes (f_1, \text{id}_{X_2})^* (q_2^* p_1^* G_{i_2}^{(2)})\) is universally \(f_1 \times_S f_2\)-generating. But due to \(q_2 \circ (f_1, \text{id}_{X_2}) \simeq p_2\) we can identify the right factor of the latter tensor product with the family \(p_2^* G_{i_2}^{(2)}\). This proves the assertion.

**Proof of [viii]** The morphisms \(f\) and \(f_{\text{red}}\) fit in a 2-commutative square, where the horizontal morphisms are closed immersions:

\[
\begin{array}{ccc}
X_{\text{red}} & \xrightarrow{u} & X \\
\downarrow f_{\text{red}} & & \downarrow f \\
Y_{\text{red}} & \xrightarrow{u} & Y
\end{array}
\quad (1.9.6)
\]
Since $O_{X, \red}$ is universally $v$-generating and $u$ has quasi-affine diagonal, the assertion is a consequence of (1.1) applied to $f \circ v$ and (1.1) applied to $u \circ f_{\red}$.

(1.10) Corollary. Let $f : X \to Y$ be a quasicompact and quasiseparated morphism of algebraic stacks. If $Y$ has quasi-affine diagonal, then every $f$-generating family is universally $f$-generating.

(1.11) Remark. For families of quasicoherent sheaves on algebraic stacks without quasi-affine diagonal the properties "universally generating" and "generating" do not coincide. For a quasiseparated morphism $f : X \to Y$, the structure sheaf $O_X$ is generating for $\Delta_f : X \to X \times_Y X$, by applying Proposition [LS][iv] to the factorization $id_X \circ \Delta_f$. However, $O_X$ is not necessarily universally $\Delta_f$-generating. To give a counterexample, let $A \to Spec \, k$ be an abelian scheme of positive dimension. Then the trivial torsor $p : Spec \, k \to BA$ induces a 2-cartesian square

$$
\begin{array}{ccc}
A & \longrightarrow & BA \\
\downarrow \pi & & \downarrow \Delta \\
Spec \, k & \longleftarrow & BA \times_k BA
\end{array}
$$

Although $O_{BA}$ is $\Delta$-generating, $O_A$ is not $\pi$-generating (equivalently $\pi$-ample) since $A$ is not quasi-affine. Hence, $O_{BA}$ is not universally $\Delta$-generating.

As expected, the property "universally generating" can be tested over affines:

(1.12) Proposition. Let $f : X \to Y$ be a morphism of algebraic stacks and let $G_i = (G_i)_{i \in I}$ be a family of quasicoherent $O_X$-modules. Then the following properties are equivalent:

(i) $G_i$ is universally $f$-generating.

(ii) For every morphism $Spec \, A \to Y$, the family of restricted sheaves $G_i|_{X_A}$ is generating for $X_A$.

(iii) There exists a fpqc covering family $(Y_\alpha \to Y)$ of algebraic stacks $Y_\alpha$ with quasi-affine diagonal such that each restricted family $G_i|_{(Y_\alpha)}$ is generating for $X_{(Y_\alpha)} \to Y_\alpha$.

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. For (iii) $\Rightarrow$ (i) note that for each $\alpha$, the restriction $G_i|_{(Y_\alpha)}$ is universally generating for $X_{(Y_\alpha)} \to Y_\alpha$ by Proposition [LS][iv] using that $Y_\alpha$ has quasi-affine diagonal. Therefore $G_i$ is universally $f$-generating by fpqc descent (Proposition [LS][iii]).

The following establishes descent of the completeness property along finite, flat, finitely presented surjections. It seems to be known before only for étale maps [Tho87], and was independently proven by D. Rydh.

(1.13) Proposition. Let $f : X \to Y$ be a finite, faithfully flat and finitely presented morphism, and let $g : Y \to Z$ be a quasicompact and quasiseparated morphism of algebraic stacks. If $G_i$ is a (universally) $g \circ f$-generating family of $O_X$-modules, then the family of $O_Y$-modules $f_*G_i = (f_*G_i)_{i \in I}$ is (universally) $g$-generating.

Proof. It suffices to treat the non-universal case by applying an appropriate base change. So let us assume that $G_i$ is a $g \circ f$-generating family of $O_Y$-modules. We invoke now Grothendieck duality for finite morphisms. Recall that $f_*$ preserves finitely presented sheaves because $f$ is finite and locally free, and that $f_*$ has a right adjoint $f^!$ defined by $f_*f^!(\cdot) = Hom_{O_Y}(f_*(\cdot), \cdot)$. Then the adjunction formula $f_*Hom_{O_X}(\cdot, f^!(\cdot)) = Hom_{O_Y}(f_*(\cdot), \cdot)$ implies that for each $i \in I$ holds $g_* \circ Hom_{O_Y}(f_*(G_i), \cdot) = g_* \circ Hom_{O_X}(G_i, \cdot) \circ f^!$ as isomorphism of functors...
Qcoh(Y) → Qcoh(Z). Using Proposition 1.6(iii) one can see that f_* maps g ∘ f-generating families to g-generating families if f^* is faithful. The latter is equivalent to the property that the counit f_* f^*(M) → M is surjective for every quasicoherent O_Y-module M. By applying Hom_{O_Y}(−, M) to the canonical map ϕ_f : O_Y → f_* O_X, we see that this happens precisely if ϕ_f is an fpf locally split monomorphism of quasicoherent O_Y-modules. The latter is true by faithfully flatness of f because ϕ_f is a map of O_Y-algebras.

We do not know a general descent method for non-finite flat affine coverings. The main obstacle is that the pushforward of a finitely presented quasicoherent sheaf is no longer finitely presented. The following technical lemma is a reminiscence of this approach and will be helpful to construct generating families on low dimensional stacks.

(1.14) Lemma. Let f : Y → X be an affine and faithfully flat morphism of algebraic stacks such that Y is quasiaffine. Then every quasicoherent O_X-module M admits a surjection lim M α → M, for some family of quasicoherent O_X-submodules N α ⊂ f_* O_Y^⊗ n α, n α ∈ N.

Proof. Since f is faithfully flat, the unit δ : M → f_* f^* M is injective, so that we may identify M with a subsheaf of f_* f^* M. As Y is quasiaffine, there exists a surjection ϕ : O_Y(1) → f^* M. Using that f is affine, it follows that ψ = f_*(ϕ) : f_* O_Y(1) → f_* f^* M is surjective. By identifying f_* O_Y(1) as a direct limit lim M α → f_* O_Y^⊗ n α, the preimages N α := ψ_−1 f^* M ⊂ O_Y^⊗ n α (where ψ_α : O_Y^⊗ n α → f_* f^* M and lim M α → f^* M) form a direct system whose limit surjects on M.

(1.15) Remark. In case of the classifying stack BG of an algebraic group scheme G one recovers the fact that every linear representation is contained in a finite direct sum of the left regular representation, by using the correspondence between quasicoherent OBG-modules and linear G-representations.

(1.16) Corollary. On a reduced quasicompact algebraic stack with affine diagonal every quasicoherent sheaf is a quotient of a torsionfree quasicoherent sheaf.

2. Pinching schemes

Recall that every quasicompact algebraic space X is finitely parametrized by a scheme, saying that it admits a finite and finitely presented surjection f : Z → X from a scheme Z (see [LMB00, 16.6] for the noetherian case and [Rvd13, Thm. B] for the general case). In this section we show that if every fiber of f is contained in an affine open subset, then X is representable by a scheme.

(2.1) Definition. An algebraic space X is an AF-scheme (or satisfies the Kleiman-Chevalley property) if the following condition is satisfied:

(AF) Every finite set of points x_1, . . . , x_n ∈ |X| is contained in a Zariski open neighborhood that is representable by an affine scheme.

(2.2) Remark. —

(i) Every AF-scheme is separated.

(ii) Every AF-scheme with finitely many points is affine.

(iii) If f : X → Y is a strongly representable morphism of algebraic spaces such that X admits a relatively ample invertible sheaf and Y is an AF-scheme, then X is an AF-scheme. In particular, this holds if f is affine or quasiaffine.
(2.3) Theorem. Let $Z \to X$ be an integral surjective morphism of algebraic spaces. If $Z$ is a quasicompact and quasiseparated scheme that admits an ample invertible sheaf, then $X$ is representable by a quasicompact and separated AF-scheme. If $X$ is noetherian and normal, then $X$ admits an ample invertible sheaf.

(2.4) Remark. The result is well known if $X$ is a noetherian normal scheme using the norm map [EGA II, 6.6.2], or if $f$ is flat, finite and finitely presented, or if $f$ is quotient map $Z \to Z/G$ of a geometric quotient by a finite group. If $Z$ is affine, then $X$ is affine by Chevalley’s Theorem for affines ([Ryd13, 8.1], or [Knu71] in the noetherian case).

Proof of Theorem 2.3. Let us call $X$ finitely parametrized if there exists such a surjection $Z \to X$. We frequently use that this property ascends along finite maps $X' \to X$.

Note that $X$ is quasicompact since $Z$ is quasicompact and $f$ is surjective. $X$ is separated because $Z$ is separated and $p : Z \to X$ universally closed.

Step 1. Reduction to the case that $f$ is finite and finitely presented: $Z$ is the filtered projective limit $\varprojlim \lambda Z_\lambda$ of integral and finitely presented (hence finite) algebraic $X$-spaces $Z_\lambda$ with affine bonding maps $Z_\lambda \to Z_\mu$ [Ryd13, Thm. D]. Then for sufficiently large $\lambda$, each $Z_\lambda$ is a quasicompact and separated scheme that carries an ample invertible sheaf $\mathcal{L}_\lambda$ ([TT90, C.8] if $X$ is an affine scheme but the proof applies also in the general case that $X$ is an algebraic space). So we may assume that $f$ is finite and finitely presented by replacing $f$ with $Z_\lambda \to X$.

Step 2. Normal case: $X$ is a geometric quotient of a noetherian normal scheme $X'$ by a finite group $G$ [LMB06, 16.6.2]. It follows that $X'$ is finitely parametrized: Since $X' \to X$ is finite, the pullback $pr_1 : Z' := Z \times_X X' \to Z$ is finite, so that $Z'$ admits an ample line bundle $\mathcal{L}'$ by hypothesis on $Z$. Since $pr_2 : Z' \to X'$ is finite and surjective, $N_{Z'/X'}(\mathcal{L}')$ is an ample $\mathcal{O}_{X'}$-module [EGA II, 6.6.2], showing that $X'$ is an AF-scheme [EGA II, 4.5.4]. Then it is well-known that the geometric quotient $X = X'/G$ is representable by an AF-scheme (see [Ryd07, 4.8] for the non-noetherian case).

Step 3. Final step: By approximating $X$ and $p$, we may assume that $X$ is of finite type over $Z$ (the reduction step in the proof of Chevalley’s Theorem [Ryd13, 8.1] applies literally). In particular, $X$ is noetherian and Nagata. If $X_{red}$ is an AF-scheme, then $X$ is an AF-scheme as a consequence of Chevalley’s Theorem. Therefore we may assume that $X$ is reduced since $X_{red}$ is finitely parametrized. The normalization $f : X' \to X$ is finite since $X$ is Nagata, hence $X'$ is noetherian and finitely parametrized. By step 2 we know that $X'$ is representable by an AF-scheme. Let $i : Y = \text{Supp} \text{ Ann coker}(\mathcal{O}_X \to f_* \mathcal{O}_{X'}) \subset X$ be the conductor subspace, set $i' : Y' = Y \times_X X' \subset X'$ and $g := f|_{Y'}$. Then $X$ is the pushout of $i'$ and $g$ in the category of algebraic spaces because $f$ has schematically dense image (see Lemma 2.25 below). Since $Y \subset X$ is a proper subspace that is finitely parametrized, by noetherian induction we may assume that $Y$, and hence $Y'$, is an AF-scheme. Thus, the pushout $X_0 := X' \sqcup_{f^{-1}(Y)} Y$ exists already in the category of ringed spaces and is an AF-scheme since $X'$ and $Y$ are AF-schemes by Ferrand [Fer03, 5.4]. The quotient map $f_0 : X' \to X_0$ is finite, surjective and has schematically dense image. We claim that $X = X_0$. For that it suffices to show that $X$ is a scheme, which is a Zariski local property. By the universal property there exists a map of algebraic spaces $h : X \to X_0$ such that $h \circ f = f_0$. By taking a Zariski covering of affine open subschemes of $X_0$, we may assume that $X_0$ is affine. Then
X is affine using that \( f_0 \) is affine. Consequently, \( X \) is affine by Chevalley’s Theorem since \( f \) is finite and surjective, proving the assertion.

The following preparatory lemma is folklore but stated for lack of reference.

**Lemma.** Given a morphism of algebraic spaces \( f : X' \to X \), the closed immersion \( i : Y \hookrightarrow X \) defined by the conductor ideal \( \operatorname{Ann}_{\mathcal{O}_X} \ker(\mathcal{O}_X \to f_*\mathcal{O}_{X'}) \) and the preimage \( Y' := f^{-1}(Y) \) with closed immersion \( i' : Y' \hookrightarrow X' \) and restriction \( g = f|_{Y'} : Y' \to Y \) give rise to a cartesian square:

\[
\begin{array}{ccc}
Y' & \xrightarrow{i'} & X' \\
\downarrow{g} & & \downarrow{f} \\
Y & \xrightarrow{i} & X
\end{array}
\]

\[(2.5.1)\]

If \( f \) is finite with schematically dense image (i.e. \( \mathcal{O}_X \to f_*\mathcal{O}_{X'} \) is injective), then the square is cocartesian in the category of algebraic spaces.

**Proof.** In case that \( X \) and hence \( X', Y, Y' \) are affine schemes, the conductor square is cocartesian in the category of algebraic spaces ([Fer03, §1.1] and [BC10, Proof of Thm. 2.2.2]). It follows that for every étale covering \( u : U \to X \) with \( U \) affine, one recovers \( U \) as the pushout of \( g_U : Y'_U \to Y_U \) and \( i'_U : Y'_U \to X'_U \).

In order to see that the square is cocartesian, let \( h : X' \to T \), \( j : Y \to T \) be given morphisms satisfying \( hi' = jg \). We have to construct a unique map \( t : X \to T \) with \( tf = h \) and \( ti = j \). Suppose there are two maps \( t_1, t_2 : X \to T \) satisfying this condition. Then \( t_1u = t_2u \) by uniqueness of the former case, hence \( t_1 = t_2 \) since \( \operatorname{Hom}(\cdot, T) \) is a separated presheaf. This shows uniqueness. Regarding the existence, observe that \( X'_U \to T \) and \( Y'_U \to T \) factor over a unique map \( t' : U \to T \). It gives rise to two morphisms \( t' \circ \operatorname{pr}_\alpha : U \times_X U \to T, \alpha = 1, 2 \), and both satisfy the compatibility condition after restricting \[(2.5.1)\] along the étale covering \( U \times_X U \to X \). So by uniqueness, we infer \( t' \circ \operatorname{pr}_1 = t' \circ \operatorname{pr}_2 \). Since \( \operatorname{Hom}(\cdot, T) \) is a sheaf, there is a map \( t : X \to T \) with \( tu = t' \). The condition \( tf = h \) (resp. \( ti = j \)) is local over \( X' \) (resp. \( Y \)), hence follows by restricting \[(2.5.1)\] along \( u \).

Recall that for a given map \( f : Y \to X \) of topological spaces the saturation \( V^s \subset V \) of a subset \( V \subset Y \), is defined as \( \{ y \in Y : f^{-1}(f(y)) \subset V \} \). From \( V^s = Y - f^{-1}(f(Y - V)) \) we see that \( V^s \) is the preimage of an open subset if \( f \) is closed.

**Corollary.** Let \( f : Y \to X \) be an integral surjective morphism of algebraic spaces. A set of points \( P \subset |X| \) is contained in an affine open subspace if and only if \( f^{-1}(P) \subset |Y| \) is contained in an affine open subspace.

**Proof.** The condition is clearly necessary since \( f \) is affine. Conversely, suppose that \( V \subset Y \) is an affine Zariski open neighborhood of \( f^{-1}(P) \). Using that \( f \) is closed, there is an open algebraic subspace \( U \subset X \) such that \( f^{-1}(U) = V^s \subset V \) is a quasiaffine subscheme. Then \( U \) is representable by an AF-scheme by Theorem 2.3. From \( f^{-1}(P) \subset V^s \) we conclude \( P \subset U \).

**Corollary.** Let \( f : Y \to X \) be an integral surjective morphism of algebraic spaces with finite topological fibers. Then every fiber of \( f \) is contained in an affine open neighborhood if and only if \( X \) is a scheme.

As D. Rydh pointed out, if \( f \) has universally finite topological fibers, then the integrality condition on \( f \) is automatic under weaker assumptions.
(2.8) Corollary. Let $f: Y \to X$ be a separated, surjective, universally closed morphism of algebraic spaces with universally finite topological fibers. If every topological fiber of $f$ is contained in an affine open subspace of $Y$, then $f$ is an integral morphism of schemes.

Proof. It suffices to show that $f$ has affine fibers because then $f$ is integral by [Ryd13 Thm. 8.5], and $Y$ is a scheme by Corollary 2.7. For that let $F := \text{Spec } k \times_X Y$ be the fiber over a given point $x: \text{Spec } k \to X$. The morphism $x$ is quasiaffine because $X$ has quasiaffine diagonal. Thus, $\text{pr}_2: F \to Y$ is quasiaffine as well. Since the image $\text{pr}_2(F)$ is contained in some affine open subspace $V \subseteq Y$, the inverse image $F = \text{pr}_2^{-1}(V)$ is a quasiaffine scheme. In particular, it satisfies the AF-property, and we conclude that it is an affine scheme. □

(2.9) Corollary. Let $f: Y \to X$ be a morphism of algebraic spaces that is separated, surjective, universally closed and has finite topological fibers. Then $X$ is an AF-scheme if and only if $Y$ is.

3. Global generation of sheaves and quasiaffiness

In this section we show that for a quasicompact and quasiseparated algebraic stack $X$ with affine stabilizer groups, the condition that every quasicoherent sheaf is globally generated implies that $X$ is a quasiaffine scheme. This is well-known if $X$ is a separated scheme ([EGA II 5.1.2]).

(3.1) Proposition. A quasicompact and quasiseparated morphism of algebraic stacks $f: X \to Y$ is quasiaffine if and only if the following conditions are satisfied:

(i) $\mathcal{O}_X$ is universally $f$-generating.

(ii) $f$ has affine relative stabilizer groups at geometric points, i.e. the geometric fibers of the relative inertia $I_f \to X$ are affine (equiv. quasiaffine). This holds for instance, if $f$ has quasiaffine diagonal (e.g. if $\Delta_f$ is quasifinite).

Proof. The conditions are necessary by Proposition 1.8(ii) so let us verify the sufficiency. Both assumptions (i) and (ii) are stable under base change, and the assertion is local over $Y$. Therefore, we may assume that $Y = \text{Spec } A$ is affine and that $X$ is quasicompact and quasiseparated, by replacing $Y$ with an appropriate smooth covering.

First, we show that $X$ is representable. For that, it suffices to show that $f$ has representable geometric fibers. Therefore, we may assume that $Y$ is the spectrum of an algebraically closed field, by applying base change with a given point. Now, we have to show that for every point $x$: $\text{Spec } k \to X$, the stabilizer group $G_x$ is trivial, which is an affine (algebraic) group scheme by assumption on $I_f$.

Let $\xi \in |X|$ be the point induced by $x$. Then $x$ factors over the residual gerbe $G_\xi$ by an epimorphism $F$: $\text{Spec } k \to G_\xi$ followed by a monomorphism $G_\xi \hookrightarrow X$ [LMB00 11.1]. $G_\xi$ is an algebraic stack of finite type over the residue field $k(\xi)$, which is the sheafification of $G_\xi$, and the monomorphism $G_\xi \hookrightarrow X$ is quasiaffine [Ryd11 Theorem B.2]. It follows that there exists a finite field extension $k(\xi) \subset L$ such that $G_\xi \otimes_{k(\xi)} L \simeq B G_{x'}$, where $G_{x'} \to \text{Spec } L$ is the stabilizer group at the induced representative $x': \text{Spec } L \to \text{Spec } k(\xi) \hookrightarrow X$ of $\xi$. The upshot is that the composition $B G_{x'} \to G_\xi \hookrightarrow X$ is a quasiaffine map, so that $\mathcal{O}_{B G_{x'}}$ is relatively generating. Since $\mathcal{O}_X$ is generating for $X$, we conclude that $\mathcal{O}_{B G_{x'}}$ is an absolute generator for $B G_{x'}$ by Proposition 1.8(v). But then $G_{x'} \to \text{Spec } L$ is the trivial algebraic $L$-group scheme because every linear representation is generated by the trivial representation. As $G_x$ and $G_{x'}$ are isomorphic over some common field extension, we infer that $G_x$ is trivial by fpqc-descent.
Therefore, \( X \) is representable by an algebraic space. Take a finite, finitely presented and surjective morphism \( p: Z \to X \) for some scheme \( Z \) [Ryd13, Thm B]. Since \( p \) is quasiaffine, \( \mathcal{O}_Z = p^* \mathcal{O}_X \) is generating for \( Z \). If \( Z \) is quasiaffine, then \( X \) must be a scheme by Theorem 2.3.

This reduces to the final case that \( X \) is a scheme. Since \( p \) is quasiaffine, \( \mathcal{O}_Z = p^* \mathcal{O}_X \) is generating for \( Z \). If \( Z \) is quasiaffine, then \( X \) must be a scheme by Theorem 2.3. This reduces to the final case that \( X \) is a scheme. Since \( \mathcal{O}_X \) is generating, every quasicoherent ideal sheaf is a quotient of a free \( \mathcal{O}_X \)-module. This shows that the open subsets \( X_f \) define a base of the Zariski-topology, where \( f \) runs over the set of global sections of \( \mathcal{O}_X \). Since \( X \) is covered by affine open subschemes, there exists a subbase of affines \( X_f, f \in \Gamma_0 \subset \Gamma(X, \mathcal{O}_X) \), so that the affine hulls \( p_f: X_f \to \text{Spec} \Gamma(X_f, \mathcal{O}_{X_f}) \) are isomorphisms. Consequently, they glue to a quasicompact open immersion \( X \to \text{Spec} \Gamma(X, \mathcal{O}_X) \) using the canonical identification \( \Gamma(X_f, \mathcal{O}_{X_f}) \simeq \Gamma(X, \mathcal{O}_X)_f \). This proves the assertion. \( \square \)

(3.2) Remark. In case that \( f \) is strongly representable (or representable and finitely presented) and \( Y \) has quasiaffine diagonal, Proposition 3.1 was proven in [JA10, 5.1] by different methods. If \( X \) is noetherian and normal it can be deduced from the proof of Totaro’s Theorem [Tot04, 1.1].

(3.3) Corollary. A morphism of algebraic stacks \( f: X \to Y \) has quasiaffine diagonal \( \Delta_f \) if and only if \( \mathcal{O}_X \) is universally \( \Delta_f \)-generating.

(3.4) Corollary. A morphism of algebraic stacks \( f: X \to Y \) is quasiaffine if and only if \( \mathcal{O}_X \) is universally generating for \( f \) and \( \Delta_f \).

4. The resolution property

In this section we define the resolution property of a morphism in terms of locally free generating sheaves and recast the example classes where it is known to hold. From now on we implicitly assume that every vector bundle has constant rank.

(4.1) Definition. An algebraic stack \( X \) has the resolution property if \( X \) is quasicompact and quasiseparated and if there exists a generating family of locally free \( \mathcal{O}_X \)-modules. We say that a morphism \( f: X \to Y \) of algebraic stacks has the resolution property, or that \( X \) has the resolution property over \( Y \) (relative to \( f \)), if \( f \) is quasicompact and quasiseparated and if there exists a universally \( f \)-generating family of locally free \( \mathcal{O}_X \)-modules (see Definition 1.7).

(4.2) Remark. For a noetherian algebraic stack this definition is equivalent to Totaro’s [Tot04], saying that \( X \) has the resolution property if and only if every coherent sheaf is a quotient of a coherent locally free sheaf, by taking the family of all vector bundles (up to isomorphism) because \( X \) has the completeness property (cf. Remark 1.2).

Let us give the usual sorites for this class of morphisms.

(4.3) Proposition. —

(i) Every affine, finite, quasi-finite finite-type separated morphism, finite-type monomorphism, quasicompact immersions, or more generally quasiaffine morphism has the resolution property.

(ii) Let \( Y' \to Y \) be a morphism. If a morphism \( f: X \to Y \) has the resolution property, then so has the base change \( f': X' \to Y' \).

(iii) Let \( f: X \to Y \) be morphism and let \( Y' \to Y \) be a fpqc morphism. If the base change \( f': X' \to Y' \) has the resolution property given by a family of locally free \( \mathcal{O}_{X'} \)-modules \( \mathcal{G}'_i = (G'_i)_{i \in I} \) endowed with a descent datum
relative to \( X' \to X \) (i.e. isomorphisms \( \sigma_i : \pr_1^* \mathcal{G}_i \cong \pr_2^* \mathcal{G}_i \) for each \( i \in I \), where \( \pr_\alpha : X' \times_X X' \to X \), that satisfy the cocycle condition over \( X' \times_X X' \times_X X' \)), then \( f \) has the resolution property and there is a universally \( f \)-generating family \( \mathcal{G}_I = (\mathcal{G}_i)_{i \in I} \) such that \( \mathcal{G}_I |_{X'} \cong \mathcal{G}_i \) for each \( i \in I \).

(iv) If two morphisms \( f : X_\alpha \to Y_\alpha \), \( \alpha = 1, 2 \), over an algebraic stack \( S \), have the resolution property, then so has \( f \times_S g : X_1 \times_S X_2 \to Y_1 \times_S Y_2 \).

(v) If \( f : X \to Y \) and \( g : Y \to Z \) have the resolution property, then so has \( g \circ f \).

(vi) Suppose that \( \Delta_g \) is quasiaffine. If \( g \circ f \) has the resolution property, then so has \( f \).

(vii) Suppose that \( f \) is finite, faithfully flat and finitely presented. If \( g \circ f \) has the resolution property, then so has \( g \).

(viii) If \( f : X \to Y \) has the resolution property, then so has \( f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}} \).

Proof. The property “locally free and finitely presented” of quasicoherent sheaves is stable under taking pullbacks or tensor products and satisfies descent with respect to fpqc covers. Thus Proposition 4.3 holds mutatis mutandis for generating and universally generating families of locally free finitely presented quasicoherent sheaves. From this one easily deduces properties (i)-(vi) and (viii). Finally, property (vii) is a consequence of Proposition 1.13.

(4.4) Lemma (Finite fppf groupoids). Let \( R \equiv U \) be a finite, faithfully flat, finitely presented groupoid of algebraic \( S \)-spaces. If \( U \) (and hence \( R \)) satisfies the resolution property over \( S \), then so does the quotient stack \( X = [R \equiv U] \).

Proof. The quotient map \( q : U \to X \) is finite, finitely presented and faithfully flat. Thus, Proposition 1.13 (vii) applies.

(4.5) Corollary. Let \( G \to S \) be a flat, finite and finitely presented (equiv. finite, locally free) group algebraic space over an algebraic space \( S \) that satisfies the resolution property. Then the classifying stack \( BG \) has the resolution property.

Proof. The trivial \( G \)-torsor \( S \to BG \) is finite, finitely presented and faithfully flat.

(4.6) Remark. This result is well-known if \( G \to S \) is étale [Tho87, 2.14].

(4.7) Corollary. Let \( X \) be a regular algebraic stack that admits a finite, finitely presented surjection \( f : Y \to X \) such that \( Y \) is Cohen-Macaulay and satisfies the resolution property. Then \( X \) has the resolution property.

Proof. The regularity properties of \( Y \) and \( X \) imply that \( f \) is flat.

(4.8) Lemma (Stacks with regular noetherian covers of dimension \( \leq 1 \)). Let \( X \) be an algebraic stack that admits an affine faithfully flat map \( f : Y \to X \) with \( Y \) a noetherian regular scheme of dimension \( \leq 1 \). Then \( X \) has the resolution property.

Proof. Since \( Y \) is quasicompact there is a finite family Zariski covering of affine open subschemes \( Y_i \subset Y \). Then the open immersions \( Y_i \to Y \) are affine because \( \dim Y \leq 1 \). So by composing \( f \) with the affine map \( \prod Y_i \to Y \) we may assume that \( Y \) is affine. Let \( \mathcal{M} \) be a given coherent \( \mathcal{O}_X \)-module. Then \( \mathcal{M} \) admits a surjection \( \varphi : \mathcal{N} \to \mathcal{M} \) by a coherent subsheaf \( \mathcal{N} \subset f_* \mathcal{O}_Y^n \) for some \( n \geq 1 \) by Lemma 1.14. Then \( f^* \mathcal{N} \) is a subsheaf of the quasicoherent \( \mathcal{O}_Y \)-module \( f^* f_* \mathcal{O}_Y^n \) which is flat because \( f \) is a flat and affine map and \( \mathcal{O}_Y^n \) is a flat \( \mathcal{O}_X \)-module. Thus, \( f^* \mathcal{N} \) is
locally torsion-free as \( Y \) is locally integral. So by the regularity assumption for \( Y \), we infer that \( f^* \mathcal{N} \) is locally free. This shows that \( \mathcal{N} \) is a vector bundle by flat descent. \( \square \)

(4.9) Example (Schemes). Given a noetherian scheme \( X \), the resolution property is known to hold in the following cases:

(i) \( X \) is divisorial. That is, every point \( x \in X \) admits an affine open neighborhood that is the non-vanishing locus of a global section \( s \in \Gamma(X, \mathcal{L}) \) for some invertible sheaf \( \mathcal{L} \) (Bor63, Bor67). This is true, if \( X \) is quasifinite, or quasiprojective over a noetherian ring [EGA II, 5.3.2] (including all algebraic curves and all separated algebraic surfaces with finitely many isolated singularities that are contained in an affine open [Kle66, Cor. 4, p.328]). This also holds if \( X \) is normal and \( \mathbb{Q} \)-factorial with affine diagonal ([BS03, 1.3], and the case of separated, regular noetherian schemes is due to Kleiman and independently Illusie [SGA 6, II.2.2.7]).

(ii) \( X \) is separated and of finite type over a Dedekind ring and \( \dim(X) \leq 2 \) (Gro12, 5.2, and for normal separated algebraic surfaces [SV04, 2.1]). In dimension \( \geq 2 \) there exist normal, separated algebraic schemes that have no non-trivial invertible sheaves, and hence are not divisorial (see [Sch99] for algebraic surfaces).

(4.10) Example (Classifying stacks of algebraic group schemes). Given an affine, flat and finitely presented group scheme \( \pi: G \to S \) over a noetherian and separated scheme, Thomason [Tho87] verified the absolute resolution property for \( BG \) in the following cases:

(i) \( S \) is regular with \( \dim(S) \leq 1 \).

(ii) \( S \) is a regular with \( \dim(S) = 2 \) and \( \pi_* \mathcal{O}_G \) is a locally projective \( \mathcal{O}_S \)-module; for instance, if \( G \to S \) is smooth with connected fibers.

(iii) \( S \) satisfies the resolution property, \( G \to S \) is reductive and either \( G \) is semisimple, or \( S \) is normal, or the radical and coradical of \( G \) are isotrivial (i.e. diagonalizable on a finite étale cover of \( S \)).

(4.11) Example. By Example 4.10.(i) the classifying stack \( B \text{GL}_n, \mathbb{Z} \) has the resolution property. More directly, as quasicoherent sheaves on \( B \text{GL}_n \) correspond to \( \mathbb{Z}[\text{GL}_n] \)-comodules, the representation theory of \( \text{GL}_n, \mathbb{Z} \) implies that every quasicoherent sheaf on \( B \text{GL}_n, \mathbb{Z} \) can be generated by subsheaves \( \mathcal{G}_i \subset \mathcal{P}(V, V^\vee) \), where \( V \) denotes the locally free sheaf of rank \( n \) that corresponds to the standard representation, and \( P \) runs over all polynomials \( P \in \mathbb{N}[t, s] \) (it suffices to generate all locally free sheaves on \( B \text{GL}_n, \mathbb{Z} \) by Lemma 1.14 and then the proof of [Wat79, Theorem 3.5] carries over from a base field to a Dedekind ring as because locally free sheaves are non-equivariantly free).

(4.12) Remark. By Totaro’s Theorem and its generalization to arbitrary algebraic stacks (Theorem 5.3 below) we know that a quasicompact and quasiseparated algebraic stack with affine stabilizer groups that satisfies the resolution property must have quasifinite diagonal. So every algebraic with quasifinite and non-affine diagonal has not the resolution property. As an example, glue two copies of \( \mathbb{A}^1_k \) at the complement of the origin to get a scheme with quasifinite and non-affine diagonal. Similarly, take the quasifinite group scheme \( G \) obtained from \( \mathbb{Z}/2\mathbb{Z} \to \mathbb{A}^1_k \) by removing the origin in the identity component, then the classifying stack \( BG \) has quasifinite but not affine diagonal.
There is an example of an algebraic stack with affine diagonal that does not have the resolution property. It is the \(G_m\)-gerbe over a complex algebraic surface \(Y\) corresponding to a non-torsion element of the cohomological Brauer group \(H^2_{et}(Y, G_m)\).

We do not know if every algebraic stack with quasifinite and affine diagonal has the resolution property, even in case of normal, separated algebraic schemes of an algebraically closed field (like toric threefolds, see [Pav09]).

Étale locally, every algebraic stack with quasifinite and locally separated diagonal has the resolution property [Ryd13, Cor. 2.7]

\((4.13)\) **Remark.** If an algebraic stack \(X\) is fibered over an algebraic stack \(Y\) by means of a morphism \(f: X \to Y\), then the question whether the resolution property holds or not, can be broken down to the relative resolution property of \(f\) and the resolution property of the base \(Y\). For example, from this point of view one can tackle the *equivariant resolution property* of an algebraic space \(Y\), acted on by an affine, flat and finitely presented group scheme \(G\). It says that every quasicoherent \(\mathcal{O}_Y\)-G-comodule is a quotient of direct sum of locally free and finitely presented \(\mathcal{O}_Y\)-comodules. Now, quasicoherent \(\mathcal{O}_Y\)-G-comodules correspond to quasicoherent sheaves on the quotient stack \(X := [Y/G]\). On the one hand, the affine and faithfully flat quotient map \(Y \to [X/G]\) is a \(G\)-torsor, and we get a \(G\)-fibration of \(Y\) over the base \(X\). On the other hand, the classifying morphism \(X \to BG\) imposes on \(X\) an \(Y\)-fibration over \(BG\).

\((4.14)\) **Proposition.** Let \(S\) be an algebraic space and let \(G \to S\) be an affine, flat and finitely presented algebraic group space that acts on an algebraic \(S\)-space \(Y\). Then the following conditions are equivalent:

- (i) The classifying map \([X/G] \to BG\) has the resolution property,
- (ii) \(X\) has a family of \(G\)-linearized locally free \(\mathcal{O}_X\)-modules of finite type that is universally generating for \(X\) over \(S\).

Moreover, if \(BG \to S\) has the resolution property, then the conditions are equivalent to:

- (i) \([Y/G]\) has the resolution property over \(S\),

**Proof.** By definition of the classifying stack \(BG\) there exists a 2-cartesian square of \(S\)-stacks, where the vertical arrows are fpqc-coverings:

\[
\begin{array}{cccc}
Y & \longrightarrow & S \\
\downarrow & & \downarrow \\
[Y/G] & \longrightarrow & BG
\end{array}
\]

\((4.14.1)\)

Hence, a generating family of quasicoherent sheaves for \([Y/G] \to BG\) restricts to a quasicoherent family on \(Y \to S\) with descent datum, which is equivalent to giving a \(G\)-linearization. Conversely, every \(G\)-linearized family of quasicoherent sheaves for \(Y \to S\) descends to a family of quasicoherent sheaves on \([Y/S]\). During this restriction and descent process the property of being a relative generating family of finitely presented locally free sheaves is preserved.

\((4.15)\) **Remark.** An algebraic stack \(X\) is a *global quotient stack* if it is 2-isomorphic to an algebraic stack \([Y/GL_n]\), where \(Y\) is an algebraic space acted on by \(GL_n\) for some integer \(n \geq 0\). It seems to be unknown whether the resolution property descends along the quotient map \(q: Y \to [Y/GL_n]\), in general. Though, if \(Y\) is normal scheme that admits an ample family of invertible sheaves \(\{L_i\}_{i=1}^n\), then
the powers $\{L^m\}$ admit a $\text{GL}_n$-linearization for $m$ sufficiently large by Sumihiros Theorem \cite{sum0}, and hence descent to invertible sheaves, whose duals form a generating family for $[Y/G] \to BG$. Consequently, a large class of global quotient stacks $[Y/G]$ satisfy the resolution property.

5. Tensor Generators and Totaro’s Theorem

In this section we define the property of a vector bundle to be a tensor generator and show that this property is equivalent to the property that its associated frame bundle has quasiaffine total space when the stabilizer groups are affine. Moreover, we prove the generalization of Totaro’s Theorem \cite{totaro} to the general relative case.

(5.1) Definition. Given a morphism of algebraic stacks $f: X \to Y$, and a vector bundle $V$ on $X$ of constant rank $n$ with associated frame bundle $p: \underline{\text{Isom}}_X(O^n_X, V) = F(V) \to X$, we shall say that $V$ is a tensor generator for $X$ over $Y$ (or just $f$-tensor generating), if the the family of all $p$-locally split subsheaves $G \subset P(V, V^\vee)$ with polynomials $P \in \mathbb{N}[t, s]$, is a universally $f$-generating family of quasicoherent sheaves. Here, by $p$-locally split, we mean that the restriction $G|_{F(V)} \subset P(V, V^\vee)|_{F(V)}$ admits a left-inverse.

(5.2) Remark. Suppose that $Y = \text{Spec } k$ for a field $k$, and that $X = BG$ where $G$ denotes an algebraic group scheme. Then the definition above coincides with the one in Tannaka theory \cite[6.16]{deligne} if we identify quasicoherent $O_{BG}$-modules with $k$-linear $G$-representations.

(5.3) Theorem. Let $f: X \to Y$ be a quasicompact morphism of algebraic stacks. Given a vector bundle $V$ with associated frame bundle $p: F(V) \to X$, the following properties are equivalent:

(i) (a) $V$ is a tensor generator for $f: X \to Y$.

(b) The relative inertia stack $I_f \to Y$ has affine fibers.

(ii) $f \circ p: F(V) \to Y$ is quasiaffine, or equivalently, the classifying morphism $c_V: X \to B\text{GL}_n_Y$ is quasiaffine.

Proof. First, suppose that $V$ is a tensor generator for $f$, and that $I_f \to Y$ has affine fibers. In order to show that $F(V) \to Y$ is quasiaffine, we may assume that $Y$ is affine by smooth descent, and that $Y = \text{Spec } \mathbb{Z}$. Moreover, we may assume that $X$ has affine stabilizer groups at geometric points. Then the associated generating family of $p$-locally split vector bundles $(G_i \subset P_i(V, V^\vee))_{i \in I}$, when restricted to $F(V)$, becomes a family of globally generated sheaves because each $P_i(V, V^\vee)|_{F(V)}$ is trivial. Thus, $O_{F(V)}$ is universally generating. Using that $X$ has affine stabilizer groups, it follows that the same holds for $F(V)$ since $p$ is affine, and we know from Proposition \ref{prop:affine_stabilizers} that $F(V)$ is representable by a quasiaffine scheme.

To prove the reverse implication, let us assume that $F(V) \to Y$ is quasiaffine. Then the classifying map $c_V: X \to B\text{GL}_n_Y$ is quasiaffine, hence the tensor functor $c_V^*: \text{Qcoh}(B\text{GL}_n_Y) \to \text{Qcoh}(X)$ preserves $Y$-relative generating families of quasicoherent sheaves. Moreover, it maps the standard representation $\mathcal{E} \otimes 1$ of $\text{GL}_n_Y = \text{GL}_n \times Y$ to $V$. Therefore, it suffices to show that $\mathcal{E} \otimes 1$ is a tensor generator over $Y$. But by Example \ref{example:inertia}, $\mathcal{E}$ is a tensor generator for $B\text{GL}_n, \mathbb{Z} \to \text{Spec } \mathbb{Z}$, so the base change $\mathcal{E} \otimes 1$ is a tensor generator for $B\text{GL}_n_Y \to Y$. \hfill $\square$

If the structure group of the vector bundle $V$ is linearly reductive, then a generating family can be deduced from $V$ without taking locally split subsheaves.
(5.4) Proposition. With the preceding notations, suppose that $V$ is a tensor generator of rank $n$. Suppose that the associated $\text{GL}_n$-frame bundle $p: F(V) \to X$ is induced by a $G$-torsor $\pi: P \to X$, for some subgroup scheme $G \subset \text{GL}_{n,Y}$ with quasiaffine quotient $\text{GL}_n/G \to Y$. If $G \to Y$ is linearly reductive (for example if $V = \bigoplus_{j=1}^n L_j$ is a direct sum of line bundles with the diagonal embedding $G = \prod_j \mathbb{G}_m \to \text{GL}_{n,Y}$, or if $G = \text{GL}_{n,Y}$ and $Y$ is of characteristic 0), then $(P(V, V^\vee))_{p \in \mathbb{N}[t,s]}$ is a countable universally generating family for $X \to Y$.

Proof. First note that the classifying morphism $c_{\pi}: X \to BG$ is quasiaffine.

To see this, observe that the quasiaffine morphism $c_{\pi}: V \to BG$ factors as a composition $X \to BG \to B\text{GL}_n$. By left cancellation $c_{\pi}: X \to BG$ must be quasiaffine too, because the diagonal $\Delta: BG \to BG \times_B \text{GL}_{n,Y}$ is quasiaffine by assumption on the quotient $\text{GL}_{n,Y} \to Y$. The upshot is that $c_{\pi}^*: \text{QCoh}(BG) \to \text{QCoh}(X)$ preserves generating families, which reduces the statement to the case $X = BG$ for some linearly reductive group scheme $G \to Y$. Since the assertion is local over $Y$, we may assume that $Y$ is affine. Now, every $p$-locally split subsheaf $G \subset P(V, V^\vee)$ is globally split because $Y \to BG$ is cohomologically affine using that $G \to Y$ is linearly reductive.

□

(5.5) Corollary. Let $X$ be an algebraic stack with affine stabilizer groups, and let $(L_j)_{j=1}^n$ be a family of invertible sheaves. Then the following properties are equivalent:

(i) $\bigoplus_{j=1}^n L_j$ is a tensor generator.

(ii) The family $(L_j^i)_{1 \leq j \leq n, i \in \mathbb{Z}}$ is generating.

(iii) The associated $\prod_j \mathbb{G}_m$-torsor $P \to X$ has quasiaffine total space $P$.

(5.6) Remark. The corollary generalizes [Hau02, 1.1] to algebraic stacks that are not algebraic varieties.

(5.7) Proposition. Let $f: X \to Y$ be a morphism of algebraic stacks such that the inertia $I_f \to X$ has affine fibers. Let $V$ be a vector bundle on $X$. Then $V$ is a tensor generator for $f$ if and only if $V|_{X_{\text{red}}}$ is a tensor generator for $f_{\text{red}}: X_{\text{red}} \to Y_{\text{red}}$.

Proof. The total space of the frame bundle $F(V)$ is quasiaffine over $Y$ if and only if $F(V|_{X_{\text{red}}})$ is quasiaffine over $Y_{\text{red}}$. Hence, the result is a direct consequence of Theorem 5.3. □

Let us finally prove the generalization of Totaro’s theorem to the general relative case.

(5.8) Theorem. Let $f: X \to Y$ be a morphism of algebraic stacks with $Y$ quasi-compact. Then the following conditions are equivalent:

(i) (a) $f$ has the resolution property, and

(b) the relative inertia stack $I_f \to X$ has affine fibers (for instance, if $\Delta_f$ is quasiaffine).

(ii) For sufficiently large $n \geq 0$ the morphism $f$ admits a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{g} & B\text{GL}_{n,Y} \\
& \searrow & \downarrow \pi \\
& & Y
\end{array}
$$

(5.8.1)
where \( g \) is quasiaffine, which is the classifying morphism of the frame bundle for some vector bundle \( V \) of rank \( n \), and \( q \) is the structure morphism. In particular, the diagonal \( \Delta_f \) is affine.

**Proof.** Let \( f : X \to Y \) be a morphism of algebraic stacks. Suppose that \( f \) factors by a quasiaffine morphism \( X \to B\text{GL}_n,Y \) followed by the projection \( B\text{GL}_n,Y \to Y \). Then both morphisms have the resolution property, so has the composition \( f \). Moreover, both morphisms have affine diagonal, so \( f \) has too, and we conclude that the inertia \( I_f = X \times_{X \times_Y X} X \to X \) is affine.

Conversely, suppose that \( f \) has resolution property and that \( I_f \to X \) has affine fibers at geometric points. Let \( (V_i)_{i \in I} \) be an \( f \)-generating family of vector bundles on \( X \), for instance the family of all vector bundles on \( X \) up to isomorphism. We have to show that there exists a quasiaffine morphism \( X \to B\text{GL}_n,Y \). For every finite subset \( J \subset I \) the \( X \)-fiber product \( p_J : F_J := \left( \prod_{i \in J} F(V_i) \right) \to X \) is an affine morphism. Choose an inverse system \( (F_J \to F_K) \) for the family \( (F_J \to X) \). Then the inverse limit of \( X \)-stacks \( F := \lim_{J} F_J \) is an algebraic stack over \( X \) because the bonding maps \( F_J \to F_K \) are affine. The projection \( p : F \to X \) is an affine morphism and has the property, that \( p^*(V_i) \) is trivial for every \( i \in I \). So \( O_F \) is universally \( f \circ p \)-generating. Since \( I_f \to X \) has affine fibers, the inertia \( I_f \circ p \to F \) also has affine fibers because \( p \) is affine. Hence, \( p \) must be quasiaffine by Proposition 3.1. But then for \( J \subset I \) sufficiently large each \( p_J : F_J \to Y \) must be already quasiaffine since \( Y \) is quasicompact [Ryd13, Thm. C]. The morphism \( p_J \) is a torsor for the relative product group \( G := \left( \prod_{i \in J} \text{GL}_{n_i,Y} \right) \), where \( n_i = \text{rank} V_i \), and the classifying morphism \( X \to B\text{GL}_n,Z \) is quasiaffine because \( p_J \) is.

To finish the proof it suffices to construct a quasiaffine morphism \( B\text{G} \to B\text{GL}_n,Z \). The diagonal embedding \( G \hookrightarrow \text{GL}_n,Z, n = \sum_{i \in J} \text{rank} V_i \), induces a morphism of torsors and therefore a morphism \( B\text{G} \to B\text{GL}_n,Z \), which is affine by smooth descent because the base change along the natural map \( \text{Spec}(Z) \to B\text{GL}_n,Z \) is the affine Stiefel scheme \( \text{GL}_n,Z / G \). 

In the absolute case (\( Y \) is affine) the result reads as follows.

**5.9 Corollary.** Let \( X \) be a quasicompact and quasiseparated algebraic stack. Then the following conditions are equivalent:

(i) \( X \) has the resolution property and affine stabilizer groups at geometric points.

(ii) \( X = [U / \text{GL}_n] \) for some quasiaffine scheme \( U \) acted on by \( \text{GL}_n \), \( n \geq 0 \).

**5.10 Remark.** This result was proven by B. Totaro [Tot04, Thm. 1.1] for normal noetherian stacks.

**5.11 Corollary.** Let \( f \) be a quasicompact and quasiseparated morphism of algebraic stacks. If \( f \) has the resolution property and the relative inertia \( I_f \to X \) has affine fibers, then the diagonal \( \Delta_f : X \to X \times_Y X \) is affine.

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