Ultimate “SIR” in Autonomous Linear Networks with Symmetric Weight Matrices, and Its Use to Stabilize the Network - A Hopfield-like network

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Abstract

In this paper, we present and analyse two Hopfield-like nonlinear networks, in continuous-time and discrete-time respectively. The proposed network is based on an autonomous linear system with a symmetric weight matrix, which is designed to be unstable, and a nonlinear function stabilizing the whole network thanks to a manipulated state variable called “ultimate SIR”. This variable is observed to be equal to the traditional Signal-to-Interference Ratio (SIR) definition in telecommunications engineering.

The underlying linear system of the proposed continuous-time network is $\dot{x} = Bx$ where $B$ is a real symmetric matrix whose diagonal elements are fixed to a constant. The nonlinear function, on the other hand, is based on the defined system variables called “SIR”s. We also show that the “SIR”s of all the states converge to a constant value, called “system-specific Ultimate SIR”; which is equal to $r \over \lambda_{\text{max}}$ where $r$ is the diagonal element of matrix $B$ and $\lambda_{\text{max}}$ is the maximum (positive) eigenvalue of diagonally-zero matrix $(B - rI)$, where $I$ denotes the identity matrix. The same result is obtained in its discrete-time version as well.

Computer simulations for binary associative memory design problem show the effectiveness of the proposed network as compared to the traditional Hopfield Networks.

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Index Terms

Autonomous Continuous/Discrete-Time Linear Systems, Hopfield Networks, Signal to Interference Ratio (SIR).

I. INTRODUCTION

Hopfield Neural Networks [1] has been an important focus of research area in not only neural networks field but also circuit and system theory since early 1980s whose applications vary from combinatorial optimization (e.g. [2], [3] among many others) including travelling salesman problem (e.g. [4], [5] among others) to image restoration (e.g. [6]), from various control engineering optimization problems including robotics (e.g. [8] among others) to associative memory systems (e.g. [7], [11] among others), etc. For a tutorial and further references about Hopfield NN, see e.g. [10] and [9].

In this paper, we present and analyse two Hopfield-like nonlinear networks, called SALU-SIR and DSALU-SIR in continuous-time and discrete-time respectively.

The proposed networks consist of linear and nonlinear parts: An autonomous linear system with a symmetric weight matrix, which is designed to be unstable, and a nonlinear function stabilizing the whole network. The underlying linear system of the proposed continuous time-network is

\[ \dot{x} = Bx \]  

where \( \dot{x} \) shows the derivative of \( x \) with respect to time, i.e., \( \dot{x} = \frac{dx}{dt} \), and \( B \) is called system matrix or weight matrix.

We define the following system variables denoted as \( \theta_i \),

\[ \theta_i(t) = \frac{b_{ii}x_i(t)}{d_i + \sum_{j=1,j \neq i}^N b_{ij}x_j(t)} , \quad i = 1, \ldots, N \]  

where \( x_i(t) \) represents the \( i \)'th neuron and \( b_{ij} \) is the \( ij \)'th element of matrix \( B \). In this paper, we assume \( d_i = 0 \) for the sake of brevity.

We observe that the manipulated state variable \( \theta_i(t) \) of eq.(2) resembles the well-known Signal-to-Interference-Ratio (SIR) definition in telecommunications engineering, which can be found in any related textbooks: Signal-to-Interference Ratio (SIR) is an important entity in communications...
engineering which indicates the quality of a link between a transmitter and a receiver in a multi-transmitter-receiver environment (see e.g. [13], [18]). For example, let $N$ represent the number of transmitters and receivers using the same channel. Then the received SIR is given by (e.g. [13])

$$\gamma_i(t) = \frac{g_{ii}p_i(t)}{\nu_i + \sum_{j=1,j\neq i}^{N} g_{ij}p_j(t)}, \quad i = 1, \ldots, N$$

where $p_i$ is the transmission power of transmitter $i$, $g_{ij}$ is the link gain from transmitter $j$ to receiver $i$ (e.g. in case of wireless communications, $g_{ij}$ involves path loss, shadowing, etc) and $\nu_i$ is the receiver noise at receiver $i$. Typically in communication systems like cellular radio systems, every transmitter tries to optimize its power $p_i$ such that the received SIR (i.e., $\gamma_i$) in eq.(3) is kept at a target SIR value, $\gamma_i^{tgt}$.

It is well-known that the eigenvalues of the system matrix solely determine the stability of a linear dynamic networks. If any of the eigenvalues is positive, then the system is unstable. Designing the linear part of the network to have positive eigenvalue(s), we show, in this paper, that the defined “SIR” in eq.(2) for any state approaches asymptotically to a constant, called “ultimate SIR”, which is a function of the diagonal entry of the system matrix and its maximum eigenvalue. The nonlinear part of the network uses this result to stabilize the system. Finally, the proposed network is shown to exhibit features which are generally attributed to Hopfield Networks. Taking the sign of the converged states, the proposed network is applied to binary associative memory systems design.

The paper is organized as follows: The “ultimate SIR” is analysed for the underlying linear dynamic part of the network with a symmetric weight matrix in section II. Section III presents the stabilized network by the Ultimate ”SIR” to be used as a binary associative memory system. Simulation results are presented in section IV followed by the conclusions in Section V.

### II. Ultimate “SIR” Analysis

In this section, we analyse the underlying linear system of the proposed network in both continuous-time and discrete-time respectively.
A. Continuous-time Analysis

In this paper, we examine the case where weight matrix $B$ in eq.(1) is a real symmetric matrix whose diagonal elements are equal to a constant. So, the linear system in (1) can be written as follows

$$\dot{x} = (-rI + W)x$$

(4)

where

$$rI = \begin{bmatrix} r & 0 & \ldots & 0 \\ 0 & r & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & r \end{bmatrix}, \quad \text{and}$$

$$W = \begin{bmatrix} 0 & w_{12} & \ldots & w_{1N} \\ w_{21} & 0 & \ldots & w_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ w_{N1} & w_{N2} & \ldots & 0 \end{bmatrix}$$

(5)

where $w_{ij} = w_{ji}, \quad i, j = 1, 2, \ldots, N$.

In eq.(5), the design parameter $r$, which corresponds to $b_{ii}$ in the "SIR" ($\theta_i$) definition in eq.(2), is positive, $r > 0$. For the sake of brevity, we assume that the $b_i = 0, \quad i = 1, 2, \ldots, N$ in the analysis.

It’s well known that designing the weight matrix $W$ as a symmetric one yields that all eigenvalues are real (see e.g. [16]), which we assume throughout the paper for the sake of brevity of its analysis.

**Proposition 1:**

Let’s assume that the linear dynamic network of eq.(4) with a real symmetric $W$ in (5) has positive eigenvalue(s). If $r > 0$ is chosen such that it’s smaller than the maximum (positive) eigenvalue of $W$, then

1) the defined "SIR" ($\theta_i$) in eq.(2) for any state $i$ asymptotically converges to the following constant as time evolves for any initial vector $x(0)$ which is not completely perpendicular...
to the eigenvector corresponding to the largest eigenvalue of $W$. \(^1\)

$$\theta^\text{ult}_i = \theta^\text{ult} = r \frac{1}{\lambda_{\text{max}}}, \quad i = 1, 2, \ldots, N$$  \hspace{1cm} (6)

where $\lambda_{\text{max}}$ is the maximum (positive) eigenvalue of the weight matrix $W$.

2) there exists a finite time constant $t_T$ for a given small positive number $\epsilon_c > 0$ such that

$$||\theta^\text{vec}(t \geq t_T) - \theta^\text{ult,vec}|| < \epsilon_c$$  \hspace{1cm} (7)

where $\theta^\text{vec}(t) = [\theta_1(t) \ldots \theta_N(t)]^T$ and $\theta^\text{ult,vec} = \theta^\text{ult}[1 \ldots 1]^T$ and $||\cdot||$ shows a vector norm.

**Proof:**

Defining the following matrix series equation

$$e^{(-rI+W)t} = \sum_{k=0}^{\infty} \frac{(-rI + W)^k t^k}{k!}$$  \hspace{1cm} (8)

$$= I + \frac{(-rI + W)^1 t^1}{1!} + \frac{(-rI + W)^2 t^2}{2!} + \ldots + \frac{(-rI + W)^k t^k}{k!} + \ldots$$  \hspace{1cm} (9)

where (!) shows the factorial, and $I$ represents the identity matrix of right dimension, it’s well known that the solution of the autonomous dynamic linear system in eq.(4) is (see e.g. [17]):

$$x(t) = e^{(-rI+W)t}x(0)$$  \hspace{1cm} (10)

where $x(0)$ shows the initial state vector at time zero. So, the state transition matrix of the linear system of eq.(4) is $e^{(-rI+W)t}$ in eq. (8) (see e.g. [17]).

Let us first examine the powers of the matrix ($-rI + W$) in (8) in terms of matrix $rI$ and the eigenvectors of matrix $W$. First let’s remind some spectral features of the symmetric real square matrices that we use in the proof later on:

It’s well known that any symmetric real square matrix can be decomposed into

\(^1\) It’s easy to check in advance if the initial vector $x(0)$ is completely perpendicular to the eigenvector of the maximum (positive) eigenvalue of $W$ or not. If this is the case, then this can easily be overcome by introducing a small random number to $x(0)$ so that it’s not completely perpendicular to the mentioned eigenvector.
\[ W = \sum_{i=1}^{N} \lambda_i v_i v_i^T \]  \hspace{1cm} (11)

where \( \{\lambda_i\}_{i=1}^{N} \) and \( \{v_i\}_{i=1}^{N} \) show the (real) eigenvalues and the corresponding orthonormal eigenvectors (see e.g. [16]), i.e.,

\[ v_i^T v_j = \begin{cases} 1 & \text{if and only if } i = j \\ 0 & \text{otherwise,} \end{cases} \]  \hspace{1cm} (12)

where \( i, j = 1, 2, \ldots, N \). Defining the outer-product matrices of the eigenvectors \( \{\lambda_i\}_{i=1}^{N} \) as \( V_j = v_i v_i^T \) in eq. 11 gives

\[ W = \sum_{i=1}^{N} \lambda_i V_i \]  \hspace{1cm} (13)

The matrix \((-rI + W)\) can be written using eq.(11) as follows

\[ -rI + W = -rI + \sum_{i=1}^{N} \beta_i(1) V_i \]  \hspace{1cm} (14)

where \( r \) is the diagonal element of matrix \( rI \), and where \( \beta_i(1) \) is equal to

\[ \beta_i(1) = \lambda_i \]  \hspace{1cm} (15)

The matrix \((-rI + W)^2\) can be written using eq.(11)-(13) as

\[ (-rI + W)^2 = r^2I + \sum_{i=1}^{N} \beta_i(2) V_i \]  \hspace{1cm} (16)

where \( \beta_i(2) \) is equal to

\[ \beta_i(2) = -r\lambda_i + (-r + \lambda_i)\beta(1) \]  \hspace{1cm} (17)

Similarly, the matrix \((-rI + W)^3\) is obtained as

\[ (-rI + W)^3 = -r^3I + \sum_{i=1}^{N} \beta_i(3) V_i \]  \hspace{1cm} (18)

where \( \beta_i(3) \) is
\[ \beta_i(3) = r^2 \lambda_i + (-r + \lambda_i) \beta(2) \quad (19) \]

For \( k = 4 \),

\[ (-r \mathbf{I} + \mathbf{W})^4 = r^4 \mathbf{I} + \sum_{i=1}^{N} \beta_i(4) \mathbf{V}_i \quad (20) \]

where \( \beta_i(4) \) is

\[ \beta_i(4) = -r^3 \lambda_i + (-r + \lambda_i) \beta(3) \quad (21) \]

So, when we continue, we observe that the \( k \)'th power of the matrix \((-r \mathbf{I} + \mathbf{W})\) is obtained as

\[ (-r \mathbf{I} + \mathbf{W})^k = (-r)^k \mathbf{I} + \sum_{i=1}^{N} \beta_i(k) \mathbf{V}_i \quad (22) \]

where \( \beta_i(k) \) for \( k \geq 2 \) is equal to

\[ \beta_i(k) = (-r)^{k-1} \lambda_i + (-r + \lambda_i) \beta_i(k-1) \quad k = 2, 3, 4, \ldots \quad (23) \]

In what follows, we summarize the findings about the auxiliary variable \( \beta_i(k) \) from eq.(15) and (23):

\[ \beta_i(1) = \lambda_i \quad (24) \]
\[ \beta_i(2) = -r \lambda_i + (\lambda_i - r) \lambda_i \quad (25) \]
\[ \beta_i(3) = r^2 \lambda_i - r \lambda_i (\lambda_i - r) + (\lambda_i - r)^2 \lambda_i \quad (26) \]
\[ \vdots \]
\[ \beta_i(k) = \sum_{m=1}^{k} \lambda_i (-r)^{k-m} (\lambda_i - r)^{m-1} \quad (27) \]
\[ = \lambda_i \left( (-r)^{k-1} + (-r)^{k-2} (\lambda_i - r)^1 \right. \]
\[ + (-r)^{k-3} (\lambda_i - r)^2 + \ldots - r (\lambda_i - r)^{k-2} \]
\[ + (\lambda_i - r)^{k-1} \right) \quad (28) \]
Defining $\zeta_i = \lambda_i / r$, we write the equation eq.(29) as follows

$$\beta_i(k) = r^k \zeta_i (-1)^{k-1} (1 - (\zeta_i - 1) + (\zeta_i - 1)^2 - \ldots$$

$$-(-1)^{k-1}(\zeta_i - 1)^{k-2}$$

$$+(\zeta_i - 1)^{k-1})$$

(30)

Defining the sum $S(k)$

$$S(k) = (-1)^{k-1} (1 - (\zeta_i - 1) + (\zeta_i - 1)^2 - \ldots$$

$$-(-1)^{k-1}(\zeta_i - 1)^{k-2}$$

$$+(\zeta_i - 1)^{k-1})$$

(31)

the eq.(30) is written as

$$\beta_i(k) = r^k \zeta_i S(k)$$

(32)

Summing $S(k)$ with $(\zeta_i - 1)S(k)$ yields $S(k)$ as follows

$$S(k) = \frac{(-1)^{k-1} + (\zeta_i - 1)^k}{\zeta_i}$$

(33)

From eq.(32) and (33)

$$\beta_i(k) = r^k \left((-1)^{k-1} + (\zeta_i - 1)^k\right)$$

(34)

From eq.(8), (22) and (34), the state transition matrix is

$$e^{(-rI+W)t} = \sum_{k=0}^{\infty} \frac{(-rI+W)^k t^k}{k!}$$

(35)

$$= \sum_{k=0}^{\infty} \frac{(-rt)^k I}{k!} + \sum_{k=0}^{\infty} \sum_{i=1}^{N} \frac{\beta_i(k) t^k}{k!} V_i$$

(36)

The first phrase of eq.(36) is equal to the exponential matrix series of $-rtI$, i.e.,

$$\sum_{k=0}^{\infty} \frac{(-rt)^k I}{k!} = e^{-rtI}$$

(37)
The second phrase in eq.(36), where the sums over \( i \) and \( k \) are interchangable, is obtained using eq.(34) as

\[
\sum_{i=1}^{N} \sum_{k=0}^{\infty} \beta_i(k) \frac{t^k}{k!} V_i = \left( \sum_{i=1}^{N} \sum_{k=0}^{\infty} -\frac{(-rt)^k}{k!} + \sum_{i=1}^{N} \sum_{k=0}^{\infty} \frac{(r(\zeta_i - 1)t)^k}{k!} \right) V_i \tag{38}
\]

\[
= \left( \sum_{i=1}^{N} -e^{-rt} + \sum_{i=1}^{N} e^{r(\zeta_i - 1)t} \right) V_i \tag{39}
\]

So, let’s put the eigenvalues into two sets: Let those eigenvalues which are smaller that \( r \), belong to set \( T = \{\lambda_{jt}\}_{jt=1}^{N_t} \) where \( N_t \) is the length of the set; and let those eigenvalues which are larger than \( r \) belong to set \( S = \{\lambda_{js}\}_{js=1}^{N_s} \) where \( N_s \) is the length of the set. Using eqs. (37)-(39) and the definition \( \lambda_i = \zeta_i r \), we write the state transition matrix in eq.(35) as follows

\[
e^{(-rI+W)t} = M_{tp}(t) + M_{sp}(t) \tag{40}
\]

where

\[
M_{tp}(t) = e^{-rtI} - \sum_{i=1}^{N} e^{-rt} V_i + \sum_{j_t \in T} e^{(\lambda_{jt} - r)t} V_{j_t} \tag{41}
\]

and

\[
M_{sp}(t) = \sum_{j_s \in S} e^{(\lambda_{js} - r)t} V_{j_s} \tag{42}
\]

We call the matrices \( M_{tp}(t) \) and \( M_{sp}(t) \) in (41) and (42) as transient phase part and steady phase part, respectively, of the state transition matrix.

In eq.(41), because \( r > 0 \) and \( \lambda_{jt} - r < 0 \) are finite numbers, the matrix \( M_{tp}(t) \) exponentially vanishes (approaches to zero matrix) and there exists a finite time constant \( t_{T_1} \) for a given small positive number \( \epsilon_1 > 0 \) such that

\[
||M_{tp}(t \geq t_{T_1})|| < \epsilon_1 \tag{43}
\]
where $|| \cdot ||$ shows a matrix norm. From eq. (40) and (43), the $M_{tp}(t)$ affects only the transitory phase, and what shapes the steady state behavior is merely $M_{sp}$. So, let’s examine the steady phase solution in the following:

The steady phase solution, denoted as $x_{sp}(t)$, is obtained from eq. (10) and (40) as

$$x_{sp}(t) = M_{sp}(t)x(0) = \sum_{js \in S} e^{(\lambda_{js} - r)t}V_{js}x(0)$$

(44)

Let’s define the interference vector $J_{sp}(t)$ as

$$J_{sp}(t) = Wx_{sp}(t)$$

(45)

Using eq.(11) in (45) and the orthonormal features in (13) yields

$$J_{sp}(t) = \sum_{js \in S} \lambda_{js} e^{(\lambda_{js} - r)t}V_{js}x(0)$$

(46)

Defining $V_{js}x(0) = u_{js}$, we rewrite the eq.(44) and (46), respectively, as

$$x_{sp}(t) = M_{sp}(t)x(0) = \sum_{js \in S} e^{(-r + \lambda_{js})t}u_{js}$$

(47)

and

$$J_{sp}(t) = \sum_{js \in S} \lambda_{js} e^{(-r + \lambda_{js})t}u_{js}$$

(48)

In eq.(47) and (48), we assume that the $u_{js} = V_{js}x(0)$ corresponding to the eigenvector of the largest eigenvalue is different than zero vector. This means that we assume in the analysis here that $x(0)$ is not completely perpendicular to the mentioned eigenvector. This is something easy to check in advance. If it is the case, then this can easily be overcome by introducing a small random number to $x(0)$ so that it’s not completely perpendicular to the mentioned eigenvector.

Taking eq.(43) into account and dividing the vector $x_{sp}(t)$ of eq.(47) to $J(t)$ of eq.(48) elementwise and comparing the outcome with the “SIR” $(\theta_i(t))$ definition in eq.(2) where $b_{ii} = r$ results in
\[
\frac{x_{sp}^{i}(t)}{J_{sp,i}(t)} = \frac{1}{r} \theta_i(t), \quad t \to \infty, \quad i = 1, \ldots, N \tag{49}
\]
\[
= \frac{\sum_{j_s \in S} e^{(-r+\lambda_{j_s})t} u_{j_s}(i)}{\sum_{j_s \in S} \lambda_{j_s} e^{(-r+\lambda_{j_s})t} u_{j_s}(i)} \tag{50}
\]
where \(u_{j_s}(i)\) is the \(i\)'th element of the vector \(u_{j_s}\). From the analysis above, we observe that

1) If there is only one positive eigenvalue and it’s a multiple one, denoted as \(\lambda_c\) (i.e. \(\lambda_{j_s} = \lambda_c, \quad j_s \in S\)), then it’s seen from (50) that

\[
\theta_i(t) \to \frac{r}{\lambda_c}, \quad i = 1, \ldots, N, \quad t \geq t_{T_1} \tag{51}
\]

2) Similarly, if there is only one positive eigenvalue and it’s a single one, shown as \(\lambda_c\), (i.e., \(S = \{\lambda_c\}\)), then eq.(51) holds.

3) If there are more than two different (positive) eigenvalues and the largest positive eigenvalue is single (not multiple), then we observe from (50) that the largest (positive) eigenvalue dominates the dynamics of eq.(50) as time evolves because of the fact that a relatively small increase in the power of the exponential causes exponential increase as time evolves. This can be seen as follows:

Let’s show the two largest (positive) eigenvalues as \(\lambda_{max}\) and \(\lambda_j\) respectively and the difference between them as \(\Delta \lambda\). So, \(\lambda_{max} = \lambda_j + \Delta \lambda\). We define the following ratio

\[
K(t) = \frac{e^{(\lambda_j-r)t}}{e^{(-r+\lambda_{max})t}} \tag{52}
\]
\[
= \frac{1}{e^{(\Delta \lambda)t}}, \quad \Delta \lambda > 0 \tag{53}
\]

In eq.(53), because \(\Delta \lambda > 0\) is a finite number, the \(K(t)\) exponentially vanishes (approaches to zero), as will be depicted in Fig. 1, and there exists a finite time constant \(t_{T_2}\) for a given small positive number \(\epsilon_2 > 0\) such that

\[
|K(t \geq t_{T_2})| < \epsilon_2 \tag{54}
\]

Similarly, for the denominator terms, we define the following ratio
\[
\frac{\lambda_j e^{(\lambda_j - r)t}}{\lambda_{\text{max}} e^{(-r + \lambda_{\text{max}})t}} = K(t)(1 - \frac{\Delta \lambda}{\lambda_j + \Delta \lambda}) < K(t)
\]
\[
\Delta \lambda, \lambda_j > 0
\]

The ratio between the exponentials of the highest two positive eigenvalues with time

![Graph showing the ratio of exponentials]

Fig. 1. The figure shows the ratio \(K\) in eq.(52) for some different \(\Delta \lambda\) values.

So, the \(K(t)\) in eq.(52) is obtained as a multiplier in the ratio of the two greatest terms of the sum in both nominator and denominator. We plot the ratio \(K(t)\) in Fig. 1 for some different \(\Delta \lambda\) values. The Figure 1 shows that the term related to the \(\lambda_{\text{max}}\) dominates the sum of the nominator and that of the denominator respectively. This observation implies from eq.(50) that

\[
\frac{x_{sp}^i(t)}{J_{sp,i}(t)} = \frac{\sum_{js \in S} e^{(-r + \lambda_{js})t} u_{js}(i)}{\sum_{js \in S} \lambda_{js} e^{(-r + \lambda_{js})t} u_{js}(i)}
\]

\[
\rightarrow \frac{e^{(-r + \lambda_{\text{max}})t}}{\lambda_{\text{max}} e^{(-r + \lambda_{\text{max}})t}} = \frac{1}{\lambda_{\text{max}}}
\]

4) If the largest positive eigenvalue in case 3 above is a multiple eigenvale, then, similarly, the corresponding terms in the sum of the nominator and that of the denominator become dominant, which implies from eq.(50) that \(\frac{x_{sp}^i(t)}{J_i(t)}\) exponentially converges to \(\frac{1}{\lambda_{\text{max}}}\) as time evolves.
Using the observations 1 to 4, and the "SIR" definition in eq.(2), we conclude from eq.(49), (56) and (57) that for any initial vector $x(0)$ which is not completely perpendicular to the eigenvector corresponding to the largest eigenvalue of $W$,

$$\theta_i(t \to \infty) = \frac{r x_{i}^{sp}(t)}{J_{sp,i}(t)} \to \frac{r}{\lambda_{max}} \quad i = 1, \ldots, N,$$

(58)

which completes the first part of the proof. Furthermore, from eq. (43), (54) and (58) we conclude that there exists a finite time constant $t_T$ for a given small positive number $\epsilon_c > 0$ such that

$$|\theta^{vec}(t \geq t_T) - \theta^{ult,vec}| \leq \epsilon_c$$

(59)

where $\theta^{vec}(t) = [\theta_1(t) \ldots \theta_N(t)]^T$ and $\theta^{ult,vec} = \theta^{ult}[1 \ldots 1]^T$, which completes the proof.

Definition: System-Specific Ultimate SIR value: In proposition 1, we showed that the SIR in (2) for every state in the autonomous linear dynamic networks in eq.(4) converges to a constant value as time goes to infinity. We call this converged constant value as "system specific ultimate SIR" and denote as $\theta^{ult}$:

$$\theta^{ult} = \frac{r}{\lambda_{max}}$$

(60)

where $r > 0$ is the design parameter and $\lambda_{max}$ is the maximum (positive) eigenvalue of matrix $W$.

B. Discrete-time Analysis

In this subsection, we analyse the defined “SIR” for the the underlying linear part of the proposed discrete-time autonomous network which is obtained by discretizing the continuous-time system of eq.(4) by using well-known Euler method:

$$x(k + 1) = (I + \alpha(-r I + W))x(k)$$

(61)

where $I$ is the identity matrix, $r$ is a positive real number, $-r I$ and $W$ is as eq.(5), $x(k)$ shows the state vector at step $k$, and $\alpha > 0$ is the step size.
**Proposition 2:**

In the autonomous discrete-time linear network of eq.(61), let’s assume that the spectral radius of the system matrix \((I + \alpha(-rI + W))\) is larger than 1. (This assumption is equal to the assumption that \(W\) has positive eigenvalue(s) and \(r > 0\) is chosen such that \(\lambda_{max} > r\), where \(\lambda_{max}\) is the maximum (positive) eigenvalue of \(W\). If \(\alpha\) is chosen such that \(0 < \alpha r < 1\), then

1) the defined "SIR" \(\theta_i(k)\) in eq.(2) for any state \(i\) asymptotically converges to the “ultimate SIR” constant in (6) as time step evolves for any initial vector \(x(0)\) which is not completely perpendicular to the eigenvector corresponding to the largest eigenvalue of \(W\).

2) there exists a finite step number \(k_T\) for a given small positive number \(\epsilon_d > 0\) such that

\[
||\theta^{vec}(k \geq k_T) - \theta^{ult,vec}|| < \epsilon_d
\]

where \(\theta^{vec}(k) = [\theta_1(k) \ldots \theta_N(k)]^T\) and \(\theta^{ult,vec} = \theta^{ult}[1 \ldots 1]^T\).

**Proof:**

From eq. (61), it’s obtained

\[
x(k) = \left(I + \alpha(-rI + W)\right)^k x(0)
\]

where \(x(0)\) shows the initial state vector at step zero. Let us first examine the powers of the matrix \((I + \alpha(-rI + W))\) in (63) in terms of matrix \(rI\) and the eigenvectors of matrix \(W\): It’s well known that any symmetric real square matrix can be decomposed into

\[
W = \sum_{i=1}^{N} \lambda_i v_i v_i^T = \sum_{i=1}^{N} \lambda_i V_i
\]

where \(\{\lambda_i\}_{i=1}^{N}\) and \(\{v_i\}_{i=1}^{N}\) show the (real) eigenvalues and the corresponding eigenvectors and the eigenvectors \(\{v_i\}_{i=1}^{N}\) are orthonormal (see e.g. [16]), i.e.,

\[
v_i^T v_j = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j,
\end{cases}
where \(i, j = 1, 2, \ldots, N\)

Let’s define the outer-product matrices of the eigenvectors \(\{\lambda_i\}_{i=1}^{N}\) as \(V_j = v_i v_i^T\), \(i = 1, 2, \ldots, N\); and, furthermore, the matrix \(M\) as
\[ M = I + \alpha(-rI + W) \]  

which is obtained using eq.(64) as

\[ M = (1 - \alpha r)I + \sum_{i=1}^{N} \beta_i(1)V_i \]  

where \( r > 0, \alpha > 0, \) and where \( \beta_i(1) \) is equal to

\[ \beta_i(1) = \alpha \lambda_i. \]  

The matrix \( M^2 \) can be written as

\[ M^2 = (1 - \alpha r)^2I + \sum_{i=1}^{N} \beta_i(2)V_i \]  

where \( \beta_i(2) \) is equal to

\[ \beta_i(2) = \alpha(1 - \alpha r)\lambda_i + (1 - \alpha r + \alpha \lambda_i)\beta_i(1) \]  

Similarly, the matrix \( M^3 \) can be written as

\[ M^3 = (1 - \alpha r)^3I + \sum_{i=1}^{N} \beta_i(3)V_i \]  

where \( \beta_i(3) \) is equal to

\[ \beta_i(3) = \alpha(1 - \alpha r)^2\lambda_i + (1 - \alpha r + \alpha \lambda_i)\beta_i(2) \]  

So, \( M^4 \) can be written as

\[ M^4 = (1 - \alpha r)^4I + \sum_{i=1}^{N} \beta_i(4)V_i \]  

where \( \beta_i(4) \) is equal to

\[ \beta_i(4) = \alpha(1 - \alpha r)^3\lambda_i + (1 - \alpha r + \alpha \lambda_i)\beta_i(3) \]  

So, at step \( k \), the matrix \((M)^k\) is obtained as
\[ M^k = (1 - \alpha r)^k I + \sum_{i=1}^{N} \beta_i(k) V_i \]  

where \( \beta_i(k) \) is equal to

\[ \beta_i(k) = \alpha(1 - \alpha)^{k-1} \lambda_i + (1 + \alpha(\lambda_i - r)) \beta_i(k - 1) \]  

Using eq.(68) and (76), the \( \beta_i(k) \) is obtained as

\[ \begin{align*}
\beta_i(1) &= \alpha \lambda_i \\
\beta_i(2) &= \alpha \lambda_i \left((1 - \alpha r) + (1 + \alpha(\lambda_i - r))\right) \\
\beta_i(3) &= \alpha \lambda_i \left((1 - \alpha r)^2 + (1 - \alpha r)(1 + \alpha(\lambda_i - r)) + (1 + \alpha(\lambda_i - r))^2\right) \\
&\vdots \\
\beta_i(k) &= \alpha \lambda_i \sum_{m=1}^{k} (1 - \alpha r)^{k-m}(1 + \alpha(\lambda_i - r))^{m-1}
\end{align*} \]

Defining \( \lambda_i = \zeta_i(1 - \alpha r) \), we obtain

\[ (1 - \alpha r)^{k-m}(1 + \alpha(\lambda_i - r))^{m-1} = (1 - \alpha r)^{k-1}(1 + \alpha \zeta_i)^{m-1} \]

Writing eq.(81) in eq.(80) gives

\[ \beta_i(k) = \alpha \zeta_i(1 - \alpha r)^k S(k) \]  

where \( S(k) \) is

\[ S(k) = \sum_{m=1}^{k} (1 + \alpha \zeta_i)^{m-1} \]

Summing \(-S(k)\) with \((1 + \alpha \zeta_i)S(k)\) yields

\[ S(k) = \frac{(1 + \alpha \zeta_i)^k - 1}{\alpha \zeta_i} \]

From eq.(82), (83) and (84), we obtain
\[
\beta_i(k) = (1 - \alpha r)^k (1 + \alpha \zeta_i)^k - (1 - \alpha r)^k
\]  \hspace{1cm} (85)

Using the definition \(\zeta_i = \lambda_i/(1 - \alpha r)\) in eq.(85) gives

\[
\beta_i(k) = (1 + \alpha (\lambda_i - r))^k - (1 - \alpha r)^k
\]  \hspace{1cm} (86)

From eq.(75) and eq.(86),

\[
M^k = (1 - \alpha r)^k I + \sum_{i=1}^{N} (1 + \alpha (\lambda_i - r))^k V_i - \sum_{i=1}^{N} (1 - \alpha r)^k V_i
\]  \hspace{1cm} (87)

Let’s put the \(N\) eigenvalues of matrix \(W\) into two groups as follows: Let those eigenvalues which are smaller than \(r\), belong to set \(T = \{\lambda_{j_t}\}_{j_t=1}^{N_t}\) where \(N_t\) is the length of the set; and let those eigenvalues which are larger than \(r\) belong to set \(S = \{\lambda_{j_s}\}_{j_s=1}^{N_s}\) where \(N_s\) is the length of the set. We write the matrix \(M^k\) in eq.(87) using this eigenvalue grouping

\[
M^k = M_{tp}(k) + M_{sp}(k)
\]  \hspace{1cm} (88)

where

\[
M_{tp}(k) = (1 - \alpha r)^k I - \sum_{i=1}^{N_t} (1 - \alpha r)^k V_i + \sum_{j_t \in T} (1 + \alpha (\lambda_{j_t} - r))^k V_{j_t}
\]  \hspace{1cm} (89)

and

\[
M_{sp}(k) = \sum_{j_s \in S} (1 + \alpha (\lambda_{j_s} - r))^k V_{j_s}
\]  \hspace{1cm} (90)

We call the matrices \(M_{tp}(k)\) and \(M_{sp}(k)\) in (89) and (90) as transitory phase part and steady phase part, respectively, of the matrix \(M^k\).

It’s observed from eq.(89) that the \(M_{tp}(k)\) converges to zero matrix as time step number evolves because relatively small step number \(\alpha > 0\) is chosen such that \((1 - \alpha r) < 1\) and \(1 + \alpha (\lambda_{j_t} - r) < 1\). Therefore, there exists a finite time step number \(k_{T_1}\) for a given small positive number \(\epsilon_3 > 0\) such that
Thus, (from eq.(66) and (88)), what shapes the steady state behavior of the system in eq.(63) is merely the $M_{sp}(k)$ in eq.(90). So, the steady phase solution is obtained from eqs.(63), (66) and (88)-(90) using the above observations as follows

\[ x_{sp}(k) = M_{sp}(k)x(0) \]

\[ = \sum_{j_x \in S} (1 + \alpha(\lambda_{j_x} - r))^k V_{j_x}x(0) \]

Let’s define the interference vector, $J_{sp}(k)$ as

\[ J_{sp}(k) = Wx_{sp}(k) \]

Using eq.(64) in (94) and the orthonormal features in (65) yields

\[ J_{sp}(k) = \sum_{j_x \in S} \lambda_{j_x}(1 + \alpha(\lambda_{j_x} - r))^k V_{j_x}x(0) \]

First defining $V_jx(0) = u_j$, and $\xi = \frac{\alpha}{1-\alpha r}$, then dividing vector $x_{sp}(k)$ of eq.(93) to $J_{sp}(k)$ of eq.(95) elementwise and comparing the outcome with the ”SIR” definition in eq.(2) results in

\[ \frac{x_{sp,i}(k)}{J_{sp,i}(k)} = \frac{1}{r} \theta_i(k), \quad i = 1, \ldots, N \]

\[ \theta_i(k) = \frac{r}{\lambda_b}, \quad k \to \infty, \quad i = 1, \ldots, N \]
2) Similarly, if there is only one positive eigenvalue which is larger than \( r \) and it’s a single one, shown as \( \lambda_b \), then eq.(98) holds.

3) If there are more than two different (positive) eigenvalues and the largest positive eigenvalue is single (not multiple), then we see from (96) that the term related to the largest (positive) eigenvalue dominates the sum of the nominator. Same observation is valid for the sum of the denominator. This is because a relatively small increase in \( \lambda_j \) causes exponential increase as time step evolves, which is shown in the following: Let’s show the two largest (positive) eigenvalues as \( \lambda_{\text{max}} \) and \( \lambda_j \) respectively and the difference between them as \( \Delta \lambda \). So, \( \lambda_{\text{max}} = \lambda_j + \Delta \lambda \). Let’s define the following ratio between the terms related to the two highest eigenvalues in the nominator

\[
K_n(k) = \frac{(1 + \xi \lambda_j)^k}{(1 + \xi (\lambda_j + \Delta \lambda))^k}
\]  (99)

where

\[
\xi = \frac{\alpha}{1 - \alpha r}.
\]  (100)

In eq.(99), because \( \Delta \lambda > 0 \), there exists a finite time step number \( k_{T_2} \) for a given small positive number \( \epsilon_3 > 0 \) such that

\[
|K_n(k)| < \epsilon_3
\]  (101)

Similarly, let’s define the ratio between the terms related to the two highest eigenvalues in the denominator as

\[
K_d(k) = \frac{\lambda_j(1 + \xi \lambda_j)^k}{(\lambda_j + \Delta \lambda)(1 + \xi (\lambda_j + \Delta \lambda))^k}
\]  (102)

From eq.(99) and (102), because \( \Delta \lambda > 0 \) and \( \frac{\lambda_j}{\lambda_j + \Delta \lambda} < 1 \),

\[
K_d(k) < K_n(k).
\]  (103)

and there exists a finite time step number \( k_{T_2} \) for a given small positive number \( \epsilon_4 > 0 \) such that
\[ |K_n(k \geq k_{T_2})| < \epsilon_4. \] (104)

The ratio between the exponentials of the highest two positive eigenvalues with time

![Graph](image-url)

Fig. 2. The figure shows the ratio \( K_n \) in eq.(99) for some different \( \Delta \lambda \) values (\( \lambda = 5, \xi = 0.11 \)).

We plot the ratio \( K_n(k) \) in Fig. 2 for some different \( \Delta \lambda \) values and for a typical \( \xi \) value. The Figure 2 and eq.(103) implies that the terms related to the \( \lambda_{\text{max}} \) dominate the sum of the nominator and that of the denominator respectively. So, from eq.(97) and (100),

\[
\frac{x_{sp,i}(k)}{J_{sp,i}(k)} = \frac{\sum_{j_s \in S} (1 + \xi \lambda_{j_s})^k u_{j_s,i}}{\sum_{j_s \in S} \lambda_{j_s} (1 + \xi \lambda_{j_s})^k u_{j_s,i}} \rightarrow \frac{(1 + \xi \lambda_{\text{max}})^k}{\lambda_{\text{max}} (1 + \xi \lambda_{\text{max}})^k} = \frac{1}{\lambda_{\text{max}}},
\] (105)

4) If the largest positive eigenvalue in case 3 above is a multiple eigenvalue, then, similarly, the corresponding terms in the sum of the nominator and that of the denominator become dominant, which implies from eq.(97), (99) and (102) that \( \frac{x_{sp,i}(k)}{J_{sp,i}(k)} \) converges to \( \frac{1}{\lambda_{\text{max}}} \) as step number increases.

Using the observations 1 to 4, eq.(91), the "SIR" definition in eq.(2), eq.(96) and (97), we conclude that

\[
\theta_i(k \to \infty) = \frac{r x_{sp,i}(k)}{J_{sp,i}(k)} \rightarrow \frac{r}{\lambda_{\text{max}}},
\] (106)
where \( i = 1, \ldots, N \), and \( \lambda_{\max} \) is the largest (positive) eigenvalue of the matrix \( W \), which completes the first part of the proof. Furthermore, from eq. (91), (104) and (106), we conclude that there exists a finite time constant \( k_T \) for a given small positive number \( \epsilon_d > 0 \) such that

\[
||\theta^{vec}(k \geq k_T) - \theta^{ult,vec}|| < \epsilon_d
\]

where \( \theta^{vec}(k) = [\theta_1(k) \ldots \theta_N(k)]^T \) and \( \theta^{ult,vec} = \theta^{ult}[1 \ldots 1]^T \), which completes the proof.

III. STABILIZED SIR SYSTEM

Do the results of the ultimate SIR analysis in Section II above have any practical meanings? Our answer is yes. In this section, we propose two Hopfield-like networks in continuous and discrete-time domain respectively where the “system-specific ultimate SIR” is used to stabilize the system.

A. Continuous-time SALU-”SIR”

Defining the following \( g(a) \) function,

\[
g(a) = \begin{cases} 
1 & \text{if } |a| \geq \epsilon, \\
0 & \text{otherwise}
\end{cases}
\]

we propose the following dynamic network,

\[
\dot{x} = (-rI + W)xg(||\theta^{vec}(t) - \theta^{ult,vec}||)
\]

\[
y = \text{sign}(x)
\]

where \( \theta^{vec}(t) = [\theta_1(t) \ldots \theta_N(t)]^T \) and \( \theta^{ult,vec} = \theta^{ult}[1 \ldots 1]^T \), and \( y \) is the output of the network.

We call the network in eqs.(108)-(110) as Stabilized Autonomous Linear Networks by Ultimate “SIR” (SAL-U”SIR”).

Proposition 3:
The proposed dynamic network of eqs. (109)-(110) with the weight matrix $(-rI + W)$ as defined in eq. (5) is stable for any initial condition $x(0)$.

**Proof:**

If all the eigenvalues of the symmetric matrix $(-rI + W)$ as defined in eq. (5) are negative, then it’s well known from linear systems theory that the states go to zero exponentially for any initial vector $x(0)$ (see e.g. [17]).

Otherwise, if there exists positive eigenvalue(s) of $(-rI + W)$, which is the case in our design, then the proposition 1 above proves for the underlying linear system of (109) that i) the defined SIR in eq.(2) for any state $i (x_i(t), 1, 2, \ldots, N,)$ converges, as time evolves, to the constant system-specific ultimate SIR value in eq.(6) for any initial vector $x(0)$, and ii) there exists a finite time constant $t_T$ for a given small positive number $\epsilon > 0$ such that $||\theta^{vec}(t \geq t_T) - \theta^{ult,vec}|| < \epsilon$. So, the function $g(\cdot)$ stabilizes the system within the $t_T$ seconds, i.e., once $||\theta^{vec}(t) - \theta^{ult,vec}|| < \epsilon$ is met. So, the system is stable.

From the analysis above for symmetric $W$ and positive $rI$, we observe that

1) The SAL-U"SIR" does not show oscillatory behaviour because all eigenvalues are real (i.e., no imaginary part), and at least one eigenvalue is positive, which is assured by choosing the matrix $rI$ accordingly.

2) The transition phase of the "unstable" linear network is shaped by the initial state vector $x(0)$. 

Fig. 3. The figure depicts a sketch of the proposed network for numerical implementation purposes, omitting the considerations on the circuit, called Stabilized Autonomous Linear Networks with Ultimate “SIR” (SAL-U"SIR").
and the phase space characteristics formed by the eigenvectors. The network is stabilized by the function \( g(\cdot) \) once the network has passed the transition phase. The output of the network then is formed taking the sign of the converged states. Whether the state converges to a plus or minus value is dictated by the phase space shaped by the eigenvectors of \( W \) and the initial vector \( x(0) \).

From observation 1 and 2 above, we see that choosing the positive matrix \( rI \) such that the matrix \( (-rI + W) \) has positive eigenvalues makes the proposed SAL-U"SIR" exhibit similar features as the Hopfield Network does. The computer simulations in section IV shows the performance of the proposed network as compared to the Hopfield Network in some simple associative memory systems examples.

As far as the possible circuital implementations of the proposed network is concerned, a further research is needed on especially how the nonlinear function \( g(\cdot) \) could be implemented on the circuit. In this paper, we mainly focus on the analysis part, and the circuital implementation part is left as a future research topic. However, in order to give an insight, in what follows, we propose the following simplified network for numerical implementation purposes:

It’s well known that a linear dynamic network like (4) can be implemented by RC (Resistance-Capacitance) circuits. Corresponding RC dynamic network is given as follows:

\[
\begin{bmatrix}
C\dot{V}_{C_1} \\
C\dot{V}_{C_2} \\
\vdots \\
C\dot{V}_{C_N}
\end{bmatrix}
= \begin{bmatrix}
R_{11} & R_{12} & \ldots & R_{1N} \\
R_{21} & 0 & \ldots & R_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
R_{N1} & R_{N2} & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
V_{C_1} \\
V_{C_2} \\
\vdots \\
V_{C_N}
\end{bmatrix}
\tag{111}
\]

where \( C \) represents the capacitance, \( V_{C_i} \) shows the voltage of the capacitance \( C_i \), which is the state \( i \) of the network and \( R_{ij} \) is the resistance. From eq.(1), (5) and (111),

\[
R_{ij} = \frac{1}{C} w_{ij}, \quad i, j = 1, 2, \ldots, N \quad \text{and} \quad i \neq j, \tag{112}
\]

\[
R_{ii} = r \tag{113}
\]

So, we sketch a simplified numerical implementation of the proposed SAL-U"SIR" in Fig. 3, omitting the considerations on circuits, where the function \( g(\cdot) \) is represented by a switch ("ultimate SIR checking").
B. Discrete-time SALU-“SIR”

The proposed autonomous network in discrete-time, called DSAL-USIR (Discrete Stabilized Autonomous Linear networks by Ultimate “SIR”) is given as follows

\[
x(k+1) = (I + \alpha(-rI + W))x(k)g(||\theta^{vec}(t) - \theta^{ult,vec}||) \tag{114}
\]

\[
y(k) = \text{sign}(x(k)) \tag{115}
\]

where \(W\) is defined by eq.(5), the function \(g(\cdot)\) is defined by eq.(108), \(\alpha > 0\) is step size, \(I\) is identity matrix and \(r > 0\) as in eq.(61), \(\theta^{vec}(t) = [\theta_1(t) \ldots \theta_N(t)]^T\) and \(\theta^{ult,vec} = \theta^{ult}[1 \ldots 1]^T\), and \(y(k)\) is the output of the network.

**Proposition 4:**

The proposed discrete-time networks, DSAL-U”SIR”, in eq.(114) and (115) is stable for any initial vector \(x(0)\).

**Proof:**

If the spectral radius of the system matrix \((I + \alpha(-rI + W))\) is smaller than 1, then it’s well known from the discrete-time linear systems theory that the states go to zero exponentially for any initial vector \(x(0)\) (see e.g. [17]).

If, on the other hand, the spectral radius is larger than 1, which is the case in our design, then the proposition 2 above proves for the underlying linear system (61) that i) the defined “SIR” \(\theta_i(k)\) in eq.(2) for any state \(i\) asymptotically converges to the “system-specific ultimate SIR” constant in (6) as time step evolves for any initial vector \(x(0)\) ii) there exists a finite step number \(k_T\) for a given small positive number \(\epsilon_d > 0\) such that \(||\theta^{vec}(k \geq k_T) - \theta^{ult,vec}|| < \epsilon_d\). So, the function \(g(\cdot)\) stabilizes the system within the \(k_T\) steps, i.e., once \(||\theta^{vec}(t) - \theta^{ult,vec}|| < \epsilon_d\) is met. So, the system is stable.

As far as the design of weight matrix \(W\) and \(r\) is concerned, we propose to use the following method which is based on the well known Hebb-learning rule [15].
C. Outer products based network design

Let’s assume that $L$ desired prototype vectors, $\{d_s\}_{s=1}^L$, are given from $(-1, +1)^N$. The proposed method is based on well-known Hebb-learning [15] as follows:

Step 1: Calculate the sum of outer products of the prototype vectors (Hebb Rule, [15])

$$Q = \sum_{s=1}^{L} d_s d_s^T$$  \hspace{1cm} (116)

Step 2: Determine the diagonal matrix $rI$ and $W$ as follows:

$$r = q_{ii} + \rho$$  \hspace{1cm} (117)

where $\rho$ is a real number and

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \text{ } i, j = 1, \ldots, N \\ q_{ij} & \text{if } i \neq j \end{cases}$$  \hspace{1cm} (118)

where $q_{ij}$ shows the entries of matrix $Q$, $N$ is the dimension of the vector $x(t)$ and $L$ is the number of the prototype vectors ($N > L > 0$). From (116), $q_{ii} = L$ in eq.(117) since $\{d_s\}$ is from $(-1, +1)^N$.

If the desired prototype vectors are orthogonal, then it can be shown, using the steps of the proofs of prepositions 1 and 3 for continuous and discrete-time respectively, that the “system specific ultimate SIR” be $\theta^{ult} = \frac{r}{N-L}$.

IV. SIMULATION RESULTS

We take the same examples as in [14] for comparison reasons and for the sake of brevity.

In this section, we apply the the proposed networks SAL-U”SIR” and DSAL-U”SIR”, in continuous and discrete-time respectively, to associate memory systems design, and present their simulation results as compared to those of corresponding Hopfield Networks. The weight matrices of the proposed networks and the Hopfield Networks are designed by the outer-products (Hebb learning [15]) learning rule described in Section III-C.
A. Continuous-time examples

In this section, we present two examples, one with 8 neurons and one with 16 neurons, in Example 1 and 2 respectively.

The proposed DSALU-SIR network is given by eqs. (109) and (110). The Hopfield Network [1], used as the reference network, is given by

\[
\dot{x} = -rx + Wf(x(t)) + b \tag{119}
\]

\[
y(t) = f(x(t)) \tag{120}
\]

where \( W \) is the weight matrix and \( x(t) \) is the state at time \( t \), \( b = 0 \), \( f(x) = [f(x_1)f(x_2)\ldots f(x_N)]^T \), the \( f(\cdot) \) is a sigmoid function, i.e., \( f(x_i) = 1 - \frac{1}{1+\exp(-\sigma x_i)} \), where \( \sigma > 0 \).

Example 1:

This example is taken from example 1 in [14]. In the design, \( \sigma = 2 \) and \( \rho \) is chosen as \(-2\), and \( r = 1 \). The desired prototype vectors are given in the rows of matrix \( D \) as follows,

\[
D = \begin{bmatrix}
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix} \tag{121}
\]

The weight matrix, using the design rule in Section III-C, is obtained as

\[
W = \begin{bmatrix}
0 & 1 & 1 & -1 & 1 & -1 & -1 & -3 \\
1 & 0 & -1 & 1 & -1 & 1 & -3 & -1 \\
1 & -1 & 0 & 1 & -1 & -3 & 1 & -1 \\
-1 & 1 & 1 & 0 & -3 & -1 & -1 & 1 \\
1 & -1 & -1 & -3 & 0 & 1 & 1 & -1 \\
-1 & 1 & -3 & -1 & 1 & 0 & -1 & 1 \\
-1 & -3 & 1 & -1 & 1 & -1 & 0 & 1 \\
-3 & -1 & -1 & 1 & -1 & 1 & 1 & 0
\end{bmatrix}, \tag{123}
\]
The Figure 4 shows the percentages of correctly recovered desired patterns for all possible initial conditions $x(t = 0) \in (-1, +1)^N$, in the proposed SALU-”SIR” as compared to traditional Hopfield network.

Let $m_d$ show the number of prototype vectors and $C(N, K)$, (such that $N \geq K \geq 0$), represent the combination $N, K$, which is equal to $C(N, K) = \frac{N!}{(N-K)K!}$, where ! shows factorial. In our simulation, the prototype vectors are from $(-1, 1)^N$ as seen above. For initial conditions, we alter the sign of $K$ states where $K=0, 1, 2, 3$ and 4, which means the initial condition is within $K$-Hamming distance from the corresponding prototype vector. So, the total number of different possible combinations for the initial conditions for this example is 24, 84 and 168 for 1, 2 and 3-Hamming distance cases respectively, which could be calculated by $m_d \times C(8, K)$, where $m_d = 3$ and $K = 1, 2$ and 3.

As seen from Figure 4, the performance of the proposed network SALU”SIR” is the same as that of the continuous Hopfield Network for 1-Hamming distance case (%100 for both networks) and is slightly higher than that of the Hopfield Network for 2 distance case.

**Example 2:**

This example is taken from example 2 in [14]. The desired prototype vectors as well as the
obtained weight matrix $W$ are shown in Appendix I. The other network parameters are chosen as in example 1: $\sigma = 2$ and $\rho = -2$.

The Figure 5 shows the percentages of correctly recovered desired patterns for all possible initial conditions $x(t = 0) \in (-1, +1)^{16}$, in the proposed SALU”SIR” as compared to the traditional Hopfield network.

![Percentage of correctly recovered prototype vectors, N=16](image)

Fig. 5. The figure shows percentage of correctly recovered desired patterns for all possible initial conditions in example 2 for the proposed SALU”SIR” as compared to traditional Hopfield network with 16 neurons.

The total number of different possible combinations for the initial conditions for this example is 64, 480 and 2240 and 7280 for 1, 2, 3 and 4-Hamming distance cases respectively, which could be calculated by $m_d \times C(16, K)$, where $m_d = 4$ and $K = 1, 2, 3$ and 4.

As seen from Figure 5 the performance of the proposed network SALU”SIR” is the same as that of Hopfield Network for 1, 2 and 3-Hamming distance cases (%100 for both networks), and gives comparable performance with the Hopfield Network for 4-Hamming distance case.

B. Discrete-time examples

In this section, we present two examples, one with 8 neurons (Example 3) and one with 16 neurons (Example 4). The traditional discrete Hopfield network [1], shown in the following, is used as a reference network:
\[ x^{k+1} = \text{sign}(Wx^k) \] (124)

where \( W \) is the weight matrix and \( x^k \) is the state at time \( k \), and at most one state is updated at a step.

In the simulations in this subsection, we also examine the following version of the DSAL-U"SIR" for comparison reasons:

\[
x(k+1) = (\rho I + W)x(k)g(|\theta^{vec}(t) - \theta^{alt,vec}|)
\] (125)

\[
y(k) = \text{sign}(x(k))
\] (126)

where \( I \) is the identity matrix, \( 1 > \rho > 0 \) and \( W \) is defined in eq.(5), and \( y(k) \) is the output of the network. It can be shown that the above network is stable using the steps in DSAL-U"SIR" in previous section. Here, we omit the proof for the sake of brevity and present only the results for comparison reasons.

Let’s denote the original network in eqs.(114) - (115) as DSAL-U"SIR"1, and let’s call the network in eq.(125)-(126) as DSAL-U"SIR"2.

**Example 3:**

The desired prototype vectors are given in the rows of the following matrix

\[
D = \begin{bmatrix}
1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1
\end{bmatrix}
\] (127)

The weight matrices \( \rho I \) and \( W \), and the threshold vector \( b \) are obtained as follows by using the outer-products-based design mentioned above and \( \rho \) is chosen as -1 and for the DSALU-U"SIR"2 network, \( \rho = 0.5 \).
\[ rI = 2I, \quad (128) \]

\[
W = \begin{bmatrix}
0 & 2 & 0 & 0 & 0 & 0 & -2 & -2 \\
2 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\
0 & 0 & 0 & 2 & -2 & -2 & 0 & 0 \\
0 & 0 & 2 & 0 & -2 & -2 & 0 & 0 \\
0 & 0 & -2 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -2 & -2 & 2 & 0 & 0 & 0 \\
-2 & -2 & 0 & 0 & 0 & 0 & 0 & 2 \\
-2 & -2 & 0 & 0 & 0 & 0 & 2 & 0
\end{bmatrix}, \quad (129) 
\]

\[ d = 0 \quad (130) \]

The Figure 6 shows the percentages of correctly recovered desired patterns for all possible initial conditions \( x(k = 0) \in (-1, +1)^N \), in the proposed DSALU-"SIR"1 and DSALU-"SIR"2 as compared to the traditional discrete Hopfield network in (124).

![Percentage of correctly recovered prototype vectors, N=8](image)

Fig. 6. The figure shows percentage of correctly recovered desired patterns for all possible initial conditions in example 3 for the proposed DSALU-"SIR"1 and DSALU-"SIR"2 as compared to traditional discrete Hopfield network with 8 neurons.

As seen from Figure 6, the performances of the DSALU-"SIR"1 and 2 are the same as that
of the discrete-time Hopfield Network for 1-Hamming distance case (%100 for both networks) and are comparable for 2 and 3-Hamming distance cases respectively.

**Example 4:**

The desired prototype vectors as well as the obtained weight matrices are given in in Appendix II (eq.(134)).

For matrix $r\mathbf{I}$, $\rho$ is chosen as -2. The other network parameters are chosen as in example 3.

The Figure 7 shows the percentages of correctly recovered desired patterns for all possible initial conditions $\mathbf{x}(k = 0) \in (-1, +1)^{16}$, in the proposed DSALU"SIR"1 and 2 as compared to the traditional discrete Hopfield network.

![Percentage of correctly recovered prototype vectors, N=16](image)

Fig. 7. The figure shows percentage of correctly recovered desired patterns for all possible initial conditions in example 4 for the proposed DSALU"SIR"1 and DSALU"SIR"2 as compared to the traditional discrete Hopfield network with 16 neurons.

The total number of different possible combinations for the initial conditions for this example is 64, 480 and 2240 and 7280 for 1, 2, 3 and 4-Hamming distance cases respectively.

As seen from Figure 7 the performance of the proposed networks DSALU"SIR"1 and 2 are the same as that of Hopfield Network for 1 and 2-Hamming distance cases (%100 for both networks), and are comparable for 3, 4 and 5-Hamming distance cases respectively.
V. CONCLUSIONS

In this paper, we present and analyse two Hopfield-like nonlinear networks, in continuous-time and discrete-time respectively. The proposed network is based on an autonomous linear system with a symmetric weight matrix, which is designed to be unstable, and a nonlinear function stabilizing the whole network thanks to a manipulated state variable. This variable is observed to be equal to the traditional Signal-to-Interference Ratio (SIR) definition in telecommunications engineering.

The underlying linear system of the proposed continuous-time network is \( \dot{x} = Bx \) where \( B \) is a real symmetric matrix whose diagonal elements are fixed to a constant. The nonlinear function, on the other hand, is based on the defined system variables called “SIR”s. We also show that the “SIR”s of all the states converge to a constant value, called “system-specific Ultimate SIR”; which is equal to \( \frac{r}{\lambda_{max}} \) where \( r \) is the diagonal element of matrix \( B \) and \( \lambda_{max} \) is the maximum (positive) eigenvalue of diagonally-zero matrix \( (B - rI) \), where \( I \) denotes the identity matrix. The same result is obtained in its discrete-time version as well.

Computer simulations for binary associative memory design problem show the effectiveness of the proposed network as compared to the traditional Hopfield Networks.

ACKNOWLEDGMENTS

This work was supported in part by Academy of Finland and Research Foundation (Tukisäätiö) of Helsinki University of Technology, Finland.

The author would also like to thank the anonymous four reviewers for their valuable comments which helped in improving the structure and the content of the paper.

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APPENDIX I

In Example 2, the matrix which has the desired prototype vectors as its rows is

\[
D = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix}
\]  

(131)

In Example 2, the weight matrices $rI$ and $W$, which are obtained by the outer products based design as explained in Section III-C, are as follows:
\[ rI = 2I \]  
\[
\begin{bmatrix}
0 & 2 & 2 & 0 & 2 & 0 & 0 & -2 & 2 & 0 & 0 & -2 & 0 & -2 & -2 & -4 \\
2 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 2 & -2 & 0 & -2 & 0 & -4 & -2 \\
2 & 0 & 0 & 2 & 0 & -2 & 2 & 0 & 0 & -2 & 2 & 0 & -2 & -4 & 0 & -2 \\
0 & 2 & 2 & 0 & -2 & 0 & 0 & 2 & -2 & 0 & 0 & 2 & -4 & -2 & -2 & 0 \\
2 & 0 & 0 & -2 & 0 & 2 & 2 & 0 & 0 & -2 & -2 & -4 & 2 & 0 & 0 & -2 \\
0 & 2 & -2 & 0 & 2 & 0 & 0 & 2 & -2 & 0 & -4 & -2 & 0 & 2 & -2 & 0 \\
0 & -2 & 2 & 0 & 2 & 0 & 0 & 2 & -2 & -4 & 0 & -2 & 0 & -2 & 2 & 0 \\
-2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & -4 & -2 & -2 & 0 & -2 & 0 & 0 & 2 \\
2 & 0 & 0 & -2 & 0 & -2 & -2 & -4 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & -2 \\
0 & 2 & -2 & 0 & -2 & 0 & -4 & -2 & 2 & 0 & 0 & 2 & 0 & 2 & -2 & 0 \\
0 & -2 & 2 & 0 & -2 & -4 & 0 & -2 & 2 & 0 & 0 & 2 & 0 & -2 & 2 & 0 \\
-2 & 0 & 0 & 2 & -4 & -2 & -2 & 0 & 0 & 2 & 2 & 0 & -2 & 0 & 0 & 2 \\
0 & -2 & -2 & -4 & 2 & 0 & 0 & -2 & 2 & 0 & 0 & -2 & 0 & 2 & 2 & 0 \\
-2 & 0 & -4 & -2 & 0 & 2 & -2 & 0 & 0 & 2 & -2 & 0 & 2 & 0 & 0 & 2 \\
-2 & -4 & 0 & -2 & 0 & -2 & 2 & 0 & 0 & -2 & 2 & 0 & 2 & 0 & 0 & 2 \\
-4 & -2 & -2 & 0 & -2 & 0 & 0 & 2 & -2 & 0 & 0 & 2 & 0 & 2 & 2 & 0
\end{bmatrix}
\]  
\[ W = \]  
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  

**APPENDIX II**

In Example 4, the matrix which has the desired prototype vectors as its rows is

\[
D = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix}
\]  

In Example 4, the weight matrices \( rI \) and \( W \) obtained are as follows:
\[
\begin{bmatrix}
0 & 3 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -3 & -3 \\
3 & 0 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -3 & -3 \\
1 & 1 & 0 & 3 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -3 & -3 & -1 & -1 \\
1 & 1 & 3 & 0 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -3 & -3 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 3 & 1 & 1 & -1 & -1 & -3 & -3 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 3 & 0 & 1 & 1 & -1 & -1 & -3 & -3 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 0 & 3 & -3 & -3 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 3 & 0 & -3 & -3 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -3 & -3 & 3 & 0 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -3 & -3 & 3 & 0 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -3 & -3 & -1 & -1 & 1 & 1 & 0 & 3 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -3 & -3 & -1 & -1 & 1 & 1 & 3 & 0 & -1 & -1 & 1 & 1 \\
-1 & -1 & -3 & -3 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 3 & 1 & 1 \\
-1 & -1 & -3 & -3 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 3 & 0 & 1 & 1 \\
-3 & -3 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 0 & 3 & 0 \\
-3 & -3 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 3 & 0
\end{bmatrix}
\]

\[rI = 3I,\]

\[W = \begin{bmatrix}
\end{bmatrix},\]