Development the Regularization Computing Method for Solving Boundary Value Problem to Heat Equation in the Composite Materials

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Abstract. In this paper, the heat conduction equation for composite materials posed and solved. The inverse Boundary Value Problem (BVP) described and solved. By studying and solving the direct problem for the heat equation in composite materials, we can determine the function spaces and solve the inverse problem. The Sturm-Liouville method was used to solve this problem by reduce it to the integral equation for first kind. The Boundary value problem can be formulated as an integral first kind equations. The discretization algorithm applied to reformulated the problem as linear operator problem as matrix and vectors form. Then, Lavrentiev regularization method used to find the approximation solution.

Keyword: ill-posed problem, inverse problem, Lavrentev regularization, boundary value problem, heat equation

1. Introduction
The inverse BVP for the heat equation is classified as an ill-posed problem in the sense that a “small” arbitrary variation in data can lead to “large” faults in the solution [1]. There are many methods can solve the inverse problem under the study of the heat equation. For example, the method of regularization Tikhonov A.N.[2], the method of Lavrentiev M.M. [3][4], and the method of quasi-solutions Ivanova V.K.[5], [6]. Theoretical ideas and computational implementation associated with the solution of inverse and ill-posed problems investigated by AN Tikhonov, MM Lavrent'ev, VK Ivanov are collected with some examples in [7]. In [8] the inverse problem of heat conduction was investigated using the Fourier series in eigenfunctions for an equation with a discontinuous coefficient. The Fourier transform used to derive an inverse problem to the operator equation. Then the inverse problem of the operator equation was solved by the residual method. In [9], the author solved the problem of a moving boundary according to the Cauchy data in a one-dimensional heat equation with composite materials or a multilayer region. This problem is solved using a numerical regularization method based on the method of fundamental solutions and the method of discrete Tikhonov regularization. An artificial neural network is used in [10] as an inverse tool for evaluating the thermal conductivity of a composite made of an aluminum core with an aluminum face sheet connected by an adhesive layer. The Picard method proposed to solve the inverse Cauchy problem to heat equation for composite materials [11] the Picard method applied in [12].

The main idea of this work is to reconstruct the Boundary value function of the heat equation by using the Lavrentev regularization method. The Sturm-Liouville method used to reduce the problem to the integral equation for first kind. The discretization algorithm applied to convert the integral equation into liner operator equation or system of liner algebra equations.
2. Statement of the problem

The direct problem of finding the temperature in the rod that is created from composite materials is the problem of heat conduction at any moment by using the initial and boundary values of the temperature. The BVP described [8] by the following differential equations:

\[ \frac{\partial u_1(x,t)}{\partial t} = a_1^2 \frac{\partial^2 u_1(x,t)}{\partial x^2}; \quad x \in (0, x_0); t \in (0, T], a_1 > 0, \]  
\[ \frac{\partial u_2(x,t)}{\partial t} = a_2^2 \frac{\partial^2 u_2(x,t)}{\partial x^2}; \quad x \in (x_0, 1); t \in (0, T], a_2 > 0, \]  
\[ u_1(x,0) = 0; x \in (0, x_0) \cup (x_0, 1); u_2(x,0) = 0; x \in (x_0, 1); \]
\[ u_2(1,t) = q(t); t \in [0, T], \]

where \( q(t) \in C^2(0, T) \)

\[ q(0) = q'(0) = 0, \]
\[ \frac{\partial u_1(0,t)}{\partial x} = 0; t \in [0, T], \]
\[ u_1(x_0,t) = u_2(x_0,t); t \in [0, T], \]
\[ a_1 \frac{\partial u_1(x_0,t)}{\partial x} = a_2 \frac{\partial u_2(x_0,t)}{\partial x}; t \in [0, T]. \]

The following notation is used: \( x \) – spatial coordinate, \( t \) – time coordinate, and \( a1, a2 \) – constants, called thermal diffusivity coefficients in the direct problem (1–8) we need to find the vector \( \bar{u}(x,t) \).

\[ u(x,t) = \begin{cases} u_1(x,t); & x \in [0, x_0); t \in [0, T] \\ u_2(x,t); & x \in [x_0, 1); t \in [0, T] \end{cases} \]

where \( u(x,t) \in C([0,1] \times [0, T]) \cap C^{21}([0,1] \cup (x_0, 1)] \times (0, T]) \), corresponding to the Sturm-Liouville problem \( \{S_n(x)\}_{n=1}^{\infty} \),

\[ u(\bar{x}, t) = q(t) - \sum_{n=1}^{\infty} u_n S_n(\bar{x}), \bar{x} \in [0,1] \]

where function \( S_n(x) \) defined by \( S_n(x) = \begin{cases} S_n^1(x); & x \in (0, x_0), \\ S_n^2(x); & x \in (x_0, 1). \end{cases} \), there is two conditions

\[ S_n(x) = \beta_n \begin{cases} \cos \left( \frac{\lambda_n x}{a_1} \right) \sin \left( \frac{\lambda_n (1-x_0)}{a_2} \right); & x \in [0, x_0], \\ \sin \left( \frac{\lambda_n (1-x)}{a_2} \right) \cos \left( \frac{\lambda_n x_0}{a_1} \right); & x \in [x_0, 1]. \end{cases} \]
From (9), (10) and (12) can reduce problem as the following integral first kind equation.

\[
    u(\tilde{x},t) = \begin{cases}
        q(t) - \sum_{n=1}^{\infty} \frac{\alpha_n S_n(\tilde{x})}{\lambda_n} e^{-\lambda_n(x-x_0)} q'(\tau) d\tau, & \tilde{x} \in [0,1] \\
        q(t) - \sum_{n=1}^{\infty} \frac{\alpha_n S_n(\tilde{x})}{\lambda_n} e^{-\lambda_n(x-x_0)} q'(\tau) d\tau, & \tilde{x} \in [0,1]
    \end{cases}
\]

the equation (15) can writing in the form

\[
    u(\tilde{x},t) = f(t) = \int_0^t \left[ 1 - \sum_{n=1}^{\infty} \frac{\alpha_n S_n(\tilde{x})}{\lambda_n} e^{-\lambda_n(x-x_0)} \right] q'(\tau) d\tau, \tilde{x} \in [0,1]
\]

3. Discretization of the integral equation (16)
We used the algorithm in [4] which is borrowed from [13] and implanted in [14] and we assumed \( q'(\tau) = g(\tau) \)

\[
    Aq(t) = f(t) = \int_0^t P(t,\tau) g(\tau) d\tau,
\]

where \( P(t,\tau) = 1 - \sum_{n=1}^{\infty} \frac{\alpha_n S_n(\tilde{x})}{\lambda_n} e^{-\lambda_n(x-x_0)} \), from (17) the kernel \( P(t,\tau) \) corresponding integral equation, and its derivative with respect of time \( P'(t,\tau) \) belong and continues in \( C([0,1] \times [0,T]) \) we reduce the integral equation (17) to the linear algebraic equations by the following steps.

We need define an operator \( B: L_1[0,t] \rightarrow L_2[0,t] \) by the following formula

\[
    q(t) = Bg(\tau) = \frac{1}{2} \int_0^t (t-\tau) g(\tau) d\tau, \quad g(\tau), Bg(\tau) \in L_2[0,T].
\]

Then we define the operator \( C \) as following:

\[
    Cg(\tau) = ABg(\tau), \quad g(\tau), \in L_2[0,T], \quad Cg(\tau) \in L_2[0,T]
\]

from (18) and (19) we follow that \( Cg(\tau) = \int_0^t K(t,\tau) g(\tau) d\tau, \) where \( K(t,\tau) = -\int_0^t P(\tau,\tau) d\tau. \)
The finite-dimensional operator $C_n$ has been defined for computing the numerical solution for equation (17), the $C$ replaced with the operator $C_n$.

Now we need to define the following functions

$$
K_i(t) = K(\tau_i, t),
$$

(20)

$$
K_i(t); \tau_i \leq t \leq \tau_{i+1}, \ t \in [0, T], i = 0,1,...,n-1
$$

(21)

$$
K_j(t); \tau_j \leq t \leq \tau_{j+1}, \ t \leq t_{j+1}, i = 0,1,...,n-1, j = 0,1,...,m-1.
$$

(22)

$$
K_m(t, \tau) = \bar{K}_i(t).
$$

(23)

We find finite-dimensional operator $C_n \to C$, where $n, m \to \infty$

$$
C_n \ g(\tau) = \int_0^t K_m(t, \tau) g(\tau) d\tau = f(t), t \in [0, T],
$$

(24)

by implementing the algorithm in [12] we can define the vectors $g(\tau_i)$ and $f(t_j)$

$$
\bar{C}_n \ g(\tau_i) = f(t_j),
$$

(25)

where (25) equivalent the following:

$$
\bar{C}_n = \frac{1}{n} \begin{pmatrix}
K(\tau_0, t_0) & 0 & \cdots & 0 \\
K(\tau_0, t_1) & K(\tau_1, t_1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
K(\tau_0, t_{m-1}) & K(\tau_1, t_{m-1}) & \cdots & K(\tau_{m-1}, t_{m-1})
\end{pmatrix}, \ g(\tau_i) = \begin{pmatrix}
g(\tau_0) \\
g(\tau_1) \\
\vdots \\
g(\tau_{m-1})
\end{pmatrix}, \text{ and } f(t_j) = \begin{pmatrix}
f(t_0) \\
f(t_1) \\
\vdots \\
f(t_{m-1})
\end{pmatrix}
$$

the value of kernel depended on following condition $\bar{K}_i(t_j) = \begin{pmatrix}
K(\tau_i, t_j) & i \leq j \\
0 & i > j
\end{pmatrix}$

In (1–8) we consider the inverse boundary value problem for the heat equation, we need to find $q_0(t)$ where the $q_0(t) \in H^2[0, T]$

$$
q_0(t) = Bg_0(\tau),
$$

(26)

it is necessary defined the $g_0(\tau) \in M_r$,

$$
M_r = \{g(\tau): g(\tau) \in L_2[0, T], \|g\| \leq r\}
$$

(27)

Suppose we know a function $f_0(t) \in L_2[0, T]$, which is a solution to the direct problem.

$$
f_0(t) = u(\tilde{x}, t),
$$

(28)

where $\tilde{x} \in [0, 1], t \in [0, T]$, additionally assume we do not know the exact value of the function $f_0(t)$ instated of the $f_0(t) \in L_2[0, T]$ and $\delta > 0$ are given such that

$$
\left\|f_\delta(t) - f_0(t)\right\|^2 \leq \delta^2
$$

(29)

By using the given data of the problem $f_\delta(t), \delta$, and $M_r$, it is required to determine the approximate value $q_\delta(t)$, and get an error estimate $\left\|q_\delta(t) - q_0(t)\right\|$. 

4. Lavrentiev method

The operator $\bar{C}_n$ in (25) is non-injective operator, because the matrix of this operator is triangular matrix and $N(\bar{C}_n) \neq \{0\}$. In this case the Tikhonov’s regularization method is not suitable because
\[ N(\overline{C}_n) \neq \{0\} \text{ and } f_\delta \perp N(\overline{C}_n) = 0. \] Therefore, the Lavrenteva method is suggested for solving the problem (25).

This method is described in [13] and borrowed from [3]. It is based on the substitution of operator equation (25) with a family of operator equation of the second kind depending on the parameter \( \alpha > 0 \) as shown in (30).

\[ Aq + \alpha Qq = f \] (30)

The equation (30) ill-posed in sense of the operator \( A^{-1} \) exists and \( \|A^{-1}\| = \infty \). By using the polar decomposition \( \overline{A} = \sqrt{A^*A}, \overline{B} = \sqrt{BB^*} \) and \( A = Q\overline{A} \), where \( Q^{-1} \) is a unitary operator.

The Lavrentevia method used the regularizing family of operators \( \{R_\alpha : 0 < \alpha \leq \alpha_0\} \), \( R_\alpha : H \rightarrow H \) defined by the following formula

\[ R_\alpha = \overline{B}(\overline{C} + \alpha E)^{-1}Q^*, 0 < \alpha \leq \alpha_0, \] (31)

where \( \overline{C} = \overline{A}\overline{B} \), but we have a finite-dimensional operator \( \overline{C}_n \) instead of \( \overline{C} \).

The regularizing family of finite-dimensional operators \( \{R_{\alpha} : 0 < \alpha \leq \alpha_0\} \), \( R_{\alpha} : H \rightarrow H \) defined by the formula

\[ R_{\alpha} = \overline{B}_n(\overline{C}_n + \alpha E)^{-1}Q_{\alpha}^*, 0 < \alpha \leq \alpha_0, \] \[ \text{the regularized solution } q^\alpha_\delta \text{ of the inverse problem is defined by the formula} \]

\[ q^\alpha_\delta = R_{\alpha}f_\delta \] (33)

We consider the variational problem in order to give the definition of finite-dimensional approximation in the method of M.M. Lavrentevia.

\[ \inf \{ \overline{C}_n g + \alpha Q_n\overline{B}_n g - f_\delta : g \in M_r \}, \] (34)

by using (34) we can select the best value of regularization parameter \( \alpha \)

5. Conclusion

This work deals with BVP for the heat equation in composed materials. This problem is an ill-posed problem. The Sturm-Liouville method used to represent the partial differential equation as an integral equation of the first kind. By using the discretization method, the integral equation converted linear operator equation first kind. Lavrenteva regularization method was an efficient method to obtain the approximation solution.

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