Decomposition of a System of Equations of Motion of a Spacecraft with a Considerable Asymmetry in a Rarefied Atmosphere

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Abstract. We consider a system of equations simulating an uncontrolled descent of a spacecraft in a low-density atmosphere. The spacecraft is a solid body with a shape close to a solid of revolution. An important structural feature of the spacecraft is the presence of considerable geometric and aerodynamic asymmetry in its design. It should be noted that the initial system of equations of motion for the spacecraft is essentially nonlinear. In order to enable an effective asymptotic analysis of the evolution of motion of the spacecraft, it is necessary to decompose the initial system of equations of motion of the spacecraft. The aim of this study is to decompose the system of equations of motion of the spacecraft into two subsystems: slow-motion subsystem and fast-motion subsystem. The method of integral manifolds makes it possible to lower the order of the initial system of equations. The resulting slow subsystem can be represented in the form of a single-frequency system of equations with several slow variables. An important applied result of the study is the fact that the subsequent analysis of the evolution of slow variables in a slow subsystem can be performed using known asymptotic methods.

1. Introduction

One of the most important tasks of modern flight dynamics is the study of descent of a spacecraft in the atmospheres of terrestrial planets. There are a significant number of publications on this subject [1–4]. An actual descending spacecraft (SC) have a shape close to axisymmetric. At the same time, design features of a spacecraft inevitably result in occurrence of various asymmetries such as mass-inertial and aerodynamic asymmetries. A number of studies on the subject of atmospheric descent of spacecraft show that the presence of a combination of small mass and aerodynamic asymmetries can lead to the occurrence of the principal resonance when the spacecraft moves relative to its center of mass [5]. It is known that a long-term principal resonance leads to a violation of design restrictions on the spatial angle of attack. In addition, a variety of resonant phenomena can contribute to an abnormal increase in the rotational velocity during the descent of the spacecraft. In particular, such phenomena include secondary resonance effects [6–7]. From a practical point of view, when studying the motion of an asymmetric spacecraft in the atmosphere, one should also analyze the stability of resonance phenomena. It should be noted that there are two types of resonance stability: internal and external. When analyzing the internal stability of resonances, the motion of a spacecraft in the atmosphere is considered in a small neighborhood of the resonance [8]. We study the external stability of resonances outside a small neighborhood of the resonance. In these conditions the external stability of resonances...
can lead to a significant evolution of slow system variables outside a small neighborhood of the principal resonance [9–10].

We note there is a significant amount of publications investigating resonance phenomena and their stability during the descent of a spacecraft with small asymmetry in the atmosphere. At the same time, it is of practical interest to study resonance phenomena in the motion of an asymmetric spacecraft (with a considerable asymmetry) in a low-density atmosphere. In particular, such a statement of the problem is promising given the rising international interest in the study and exploration of Mars. However, the application of known asymptotic methods to the study of the motion of a spacecraft in the atmosphere by means of the initial nonlinear system is greatly complicated by significant nonlinearity of this system.

The aim of this study is to decompose the system of equations of motion of the spacecraft into two subsystems for slow and fast-motion. The slow subsystem can be written in the form of a single-frequency system of equations with several slow variables. Therefore, the subsequent analysis of the evolutions of slow variables in the slow subsystem can be carried out by applying known asymptotic methods.

2. Initial equations of motion of an asymmetric spacecraft in a low-density atmosphere

We consider an uncontrolled descent of a spacecraft in a low-density atmosphere. The spacecraft is a solid body with a shape close to a solid of revolution. The design of the spacecraft assumes the presence of considerable mass and aerodynamic asymmetries.

By analogy with the case of motion of a spacecraft with small asymmetry in the dense layers of the atmosphere considered in [8], the system of equations of motion of a spacecraft with considerable asymmetry with respect to the center of mass in a low-density atmosphere can be represented as follows:

\[
\frac{dR}{dt} = \varepsilon \frac{M_\alpha}{I},
\]

\[
\frac{dG}{dt} = \varepsilon M_{xy} + \varepsilon \frac{C_{yy}qS}{mV} \frac{R \cos \alpha - G}{\sin \alpha},
\]

\[
\frac{d^2 \alpha}{dt^2} = \frac{(R \cos \alpha - G)(R - G \cos \alpha)}{\sin^3 \alpha} + \varepsilon \frac{M_{zn}}{I} - \varepsilon \left[ \frac{C_{\alpha} qS}{mV} \right] \frac{d \alpha}{dt},
\]

\[
\frac{d \varphi}{dt} = \frac{R \cos \alpha - G}{I_x} \sin^2 \alpha,
\]

where \( \alpha \) is the spatial angle of attack, \( \varepsilon \) is a small dimensionless parameter characterizing the magnitude of the following aerodynamic moments: \( M_{xy} = M_x \cos \alpha - M_{yz} \sin \alpha \),

\( M_x = C_{xy} (D_x \cos \varphi + D_y \sin \varphi) qS ) \), \( M_{yz} = \left[ \left( m_{y} + C_{xz} A \right) \cos \varphi - \left( m_{z} + C_{yz} A \right) \sin \varphi \right] qS \),

\( m_{za} = m_{za} qS + \left[ \left( m_{z} + C_{xz} A \right) \cos \varphi - \left( m_{y} + C_{yz} A \right) \sin \varphi \right] qS \), \( \bar{\alpha} \), \( \bar{\gamma} \) are aerodynamic coefficients, \( C_{xy} = C_{xy} \cos \alpha - C_{xy} \sin \alpha \), \( C_{xz} = C_{xz} \cos \alpha + C_{yz} \sin \alpha \); \( C_{yy}^{\alpha} = \frac{dC_{yy}}{da} \), \( m_{za} \) is the damping moment coefficient, \( m_{y}^f \), \( m_{z}^f \) are the coefficients of small aerodynamic moments caused by asymmetric shape of the spacecraft, set in the body-fixed coordinate system \( OXYZ \); \( m_{za} \) is the restoring aerodynamic moment coefficient, acting in the plane of the spatial angle of attack; \( \omega_x \) is the component of the angular velocity of the spacecraft in the body-fixed coordinate system \( OXYZ \); \( V \) is the speed of the center of mass of the SC, \( m \) is the mass of the spacecraft, \( \Delta y, \Delta z \) are shifts of the center of gravity of the spacecraft in the body-fixed coordinate system \( Oxyz \), the axes of which are parallel to the axes of the body-fixed coordinate system \( O, X, Y, Z, \) and the origin is superimposed

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with the reference point for aerodynamic forces and moments \( O_v \); \( R = Q_x / I, G = Q_y / I, Q_z = I_x \omega_x \),
\( Q_{sv} = I_x \omega_x \cos \alpha + I \omega_y \sin^2 \alpha \); \( I, I_x \) are the moments of inertia of the spacecraft, \( \gamma \) is the flight-path bank angle, \( \omega_y = \frac{d \gamma}{dt} \). The system (1)–(4) must be considered together with the equations of motion of the center of mass of the spacecraft [10]. This system describes the precession motion of a spacecraft with a considerable asymmetry in the atmosphere. The study of the nonlinear system of equations (1)–(4) is directly possible only using numerical methods. Therefore, various asymptotic methods are usually used to carry out an approximate analytical analysis of motion of a spacecraft. We show that the application of the method of integral manifolds also makes it possible to decompose the equations of motion of a spacecraft with considerable asymmetry with respect to the center of mass in a low-density atmosphere. In order to achieve this we present the system of equations of motion of an asymmetric spacecraft in the atmosphere (1)–(4) in the following compact form:

\[
\frac{dx}{dt} = \varepsilon X(x, \alpha, \frac{d \alpha}{dt}, \varphi), \tag{5}
\]

\[
\frac{d^2 \alpha}{dt^2} + F(x, \alpha) = \varepsilon \left[ f_1(x, \alpha, \varphi) + f_2(x, \alpha, \varphi) \frac{d \alpha}{dt} \right], \tag{6}
\]

\[
\frac{d \varphi}{dt} = \Phi(x, \alpha), \tag{7}
\]

where \( x = (q_x, q_{sv}, \omega) \) is the slow variables vector, \( \omega \) is the frequency of the spacecraft precession at \( \omega_x = 0 \), determined from the solution of the differential equation \( \frac{d \omega}{dt} = \varepsilon \frac{d \omega}{dt} \); \( q \) is the dynamic pressure, \( f_1(x, \varphi, \alpha) = \Delta M \frac{\dot{x}}{I} \); \( f_2(x, \alpha) = -c_y \frac{q_s}{m \nu} \); \( \Phi = \frac{Q_x}{l_x} \cos \alpha - \frac{Q_{sv}}{l_x \sin^2 \alpha} \cos \alpha \);

\[
F(x, \alpha) = -\varepsilon M \frac{\dot{m}}{l} \frac{(Q_x \cos \alpha - Q_{sv}) (Q_y - Q_{sv} \cos \alpha)}{l^2 \sin^3 \alpha}, \Delta M = \left[ \left( m_i \cos \gamma + C_x \Delta \gamma \right) \cos \varphi - \left( m_i \sin \gamma - C_x \Delta \gamma \right) \sin \varphi \right] q S L. \tag{8}
\]

3. Singly perturbed equations and integral manifold

In the neighborhood of the resonance corresponding to the case of \( \Phi(x, \alpha) = O(\sqrt{\varepsilon}) \), slow time \( \tau = \mu t \quad (\mu = \sqrt{\varepsilon}) \) and normalized distance to the resonance surface \( \rho(x, \alpha) = \mu^{-1} \Phi(x, \alpha) \) are introduced in the system (5)–(7). Then we obtain:

\[
\frac{dx}{d \tau} = \mu X(x, \alpha, \omega_\alpha, \varphi), \tag{8}
\]

\[
\mu \frac{d \omega_\alpha}{d \tau} = -F(x, \alpha) + \mu^2 \left[ f_1(x, \alpha, \varphi) + f_2(x, \alpha, \varphi) \omega_\alpha \right], \tag{9}
\]

\[
\mu \frac{d \alpha}{d \tau} = \omega_\alpha, \tag{10}
\]

\[
\frac{d \varphi}{d \tau} = \rho(x, \alpha) + \mu \Phi_1(x, \alpha, \omega_\alpha, \varphi), \tag{11}
\]

where \( \omega_\alpha = d \alpha / dt \). The system (8)–(9) is a singularly perturbed system. The corresponding degenerate system has the following form (\( \mu = 0 \)):

\[
\frac{dx}{d \tau} = 0, \tag{12}
\]

\[
F(x_0, \alpha_0) = 0, \tag{13}
\]
and does not contain any information about changes in the variables \( x_0, \alpha_0 \). Since it is not possible to use the system (12)–(15) to describe the change in the variables \( x_0, \alpha_0 \) in an integral manifold. For this reason, we apply the procedure for expanding the solutions of the singularly perturbed system into asymptotic series \( \alpha = \alpha_0(\tau) + \mu \alpha_1(\tau) + \ldots \). In order to decompose the system (8)-(11), we apply the method of integral manifolds. Moreover, the system (8)–(11) in the neighborhood of the root \( \alpha_0 = \psi(x_0) \) of the equation \( F(x_0, \alpha_0) = 0 \) is simplified to the standard form of the method of integral manifolds [8]:

\[
\frac{d\psi}{d\tau} = F(x, \psi),
\]

\[
\frac{d\psi}{d\tau} = A(y, \mu) \psi + Z(y, \mu).
\]

Here \( y = [x, \psi]^T \), \( A = [\Delta \omega_0, \Delta \alpha] \), \( \Delta \omega_0 = \omega_0 \), \( \Delta \alpha = \alpha - \psi \), \( Z = [Z_1, Z_2]^T \), \( Z_1 = -F(x, \psi + \Delta \alpha) + \mu^2 \left[f_1(x, \psi, \psi + \Delta \alpha) + f_2(x, \psi, \psi + \Delta \alpha) \Delta \omega_0 - \mu^2 f_2(x, \psi, \psi) \Delta \omega_0 - \mu^2 \left( \frac{\partial f_1}{\partial \psi} \right) \Delta \alpha \right] \), \( Z_2 = -\mu^2 \frac{\partial^2 F}{\partial \psi^2} \) \( \psi _1 \) \( \Delta \alpha \), \( \frac{\partial^2 X}{\partial \psi \partial \alpha} \Delta \alpha - \frac{\partial X}{\partial \psi} \Delta \alpha - \frac{\partial X}{\partial \alpha} \Delta \omega_0 \), \( A(y, \mu) = \left[ \begin{array}{cc} f_2 & -\frac{\partial F}{\partial \psi} \\ -\frac{\partial^2 X}{\partial \psi^2} & \frac{\partial X}{\partial \psi} \\ \frac{\partial^2 X}{\partial \psi \partial \alpha} & \frac{\partial X}{\partial \psi} \end{array} \right] \), where the derivatives of the functions \( F, \psi, f_1 \) can be determined subject to the following equalities: \( \omega_0 = 0 \), \( \alpha = \psi \).

Under the conditions of the theorem on the existence of an integral manifold [11], the system (12)–(15) has a unique manifold \( z = H(y, \mu) \) of \( n+1 \) parameters. Motion in respect for this manifold for each \( \mu \in (0, \mu_0) \) at \( y \in R^{n+1} \) is described by an equation of the following form:

\[
\frac{d\psi}{d\tau} = Y(y, H(y, \mu), \mu).
\]

The integral manifold \( z = H(y, \mu) \) expands into an asymptotic series in \( \mu \). By determining the desired terms of this series \( H_i(y, \mu), i = 1, 2, \ldots \) up to the terms \( o(\mu^2) \) it can be shown that the integral manifold is written in the following form:

\[
\omega_0 = H_1(x, \psi, \mu),
\]

\[
\alpha = \psi + H_2(x, \psi, \mu).
\]

Here \( H_1 = \mu^2 \frac{\partial \psi}{\partial x} X(x, \psi, \psi) + \frac{\partial F}{\partial \psi} \), \( H_2 = \mu^2 f_1(x, \psi, \psi) \left( \frac{\partial F}{\partial \psi} \right)^{-1} \).

4. Conditions for the stability of solutions in the integral manifold

Decomposition of the initial system by the method of integral manifolds involves the following two principal conditions [8]:

**Condition 1.** The equation \( F(x_0, \alpha_0) = 0 \) has an isolated root \( \alpha_0 = \psi(x_0) \). This root must satisfy the condition \( \frac{\partial F}{\partial \alpha}(x_0) > C_1 > 0 \) where \( C_1 \) is a constant.
Let us consider the fulfillment of condition 1 in more detail. In the first place, we transform the expression $F(x, \alpha) = -\frac{M_{zn}}{I} + \frac{(Q_x \cos \alpha - Q_z)(Q_z - Q_x \cos \alpha)}{I^2 \sin^3 \alpha}$. Here the kinetic moments $Q_x, Q_{xy}$ are respectively:

$$Q_x = I_x \omega_x,$$

$$Q_{xy} = I_x \omega_x \cos \alpha + I \omega_y \sin^2 \alpha.$$  \hspace{1cm} (20)

Given the expressions (20)–(21), the function $F(x, \alpha)$ takes the following form:

$$F(x, \alpha) = -\frac{M_{zn}}{I} + \omega_y \sin \alpha (I_x \omega_x - \omega_y \cos \alpha).$$  \hspace{1cm} (22)

It can be shown, that with $\omega_y = \frac{I_x \omega_x}{2 \cos \alpha} \pm \sqrt{\left(\frac{0.25 I_x^2 \omega_x^2 + \omega^2}{\cos \alpha}\right)}$ and $\omega^2 = -M_{zn} I^{-1} \cot \alpha$, we obtain $F(x, \alpha) = 0$. Indeed, the expression $\omega_y = \frac{I_x \omega_x}{2 \cos \alpha} \pm \sqrt{\left(\frac{0.25 I_x^2 \omega_x^2 + \omega^2}{\cos \alpha}\right)}$ is obtained by finding $Q_{xy}$ from the equation $F(x, \alpha) = -\frac{M_{zn}}{I} - \frac{(Q_x \cos \alpha - Q_z)(Q_z - Q_x \cos \alpha)}{I^2 \sin^3 \alpha} = 0$ with the subsequent use of expressions (20)–(21). Upon equating function (22) to zero, we obtain the equation

$$-\frac{M_{zn}}{I} + \omega_y \sin \alpha_0 (I_x \omega_x - \omega_y \cos \alpha_0) = 0.$$  \hspace{1cm} (23)

We find the root $\alpha_0 = \psi(x_0)$ from the numerical solution of the nonlinear algebraic equation (23). Note that the desired values of the angle of attack $\alpha_0 = \psi(x_0)$ must be within the considered interval $0 < \alpha_0 < 0.5\pi$. In this case, the spatial angle of attack is equal to the value of this angle in the manifold $\alpha = \psi$. However, this root $\alpha_0 = \psi(x_0)$ must satisfy also the condition $\frac{\partial F}{\partial \alpha}(x_0, \alpha_0) > 0$.

We represent the function $F(x, \alpha)$ taking into account the order of the terms included in this function:

$$F(x, \alpha) = \omega_y \sin \alpha (I_x \omega_x - \omega_y \cos \alpha) - \frac{M_{zn}}{I}.$$  \hspace{1cm} (24)

Upon excluding a small quantity $\frac{M_{zn}}{I}$ from the expression (24), we obtain:

$$F(x, \alpha) = \omega_y \sin \alpha (I_x \omega_x - \omega_y \cos \alpha).$$  \hspace{1cm} (25)

Upon differentiating the function (25) with respect to the angle of attack, we obtain the condition $\frac{\partial F}{\partial \alpha}(x_0, \alpha_0) > 0$ in the following form:

$$\frac{\partial F}{\partial \alpha}(x_0, \alpha_0) = \omega_y^2 + I_x \omega_{x0} \omega_y \cos \alpha_0 - 2 \omega_y \cos^2 \alpha_0 > 0.$$  \hspace{1cm} (26)

From the solution of the inequality (26), taking into account the expression

$$\omega_y = \frac{I_x \omega_x}{2 \cos \alpha} \pm \sqrt{\left(\frac{0.25 I_x^2 \omega_x^2 + \omega^2}{\cos \alpha}\right)} \text{ at } 0 < \alpha_0 < 0.25 \pi,$$  \hspace{1cm} (27)

we obtain:

$$|\omega_{x0}| > 2 I_x^{-1} |\omega| \cot \alpha_0.$$  \hspace{1cm} (28)

Note that at $0.25 \pi < \alpha_0 < 0.5\pi$ the inequality (27) is fulfilled for arbitrary values $\omega_{x0} > 0, \omega > 0$ or $\omega_{x0} < 0, \omega < 0$. Therefore, the desired isolated root $\alpha_0 = \psi(x_0)$ is found from the solution of equation (23). In this case, the root $\alpha_0 = \psi(x_0)$ must satisfy the condition (27).
Condition 2. The eigenvalues of the matrix $A(y, \mu)$ calculated with the consideration of the system (8)-(11) must have negative real parts. Let us consider the fulfillment of this condition. The characteristic equation has the following form:

$$
\begin{vmatrix}
 f_2 - \lambda & -\frac{\partial F}{\partial y} + \frac{\partial f_1}{\partial x} \\
 -\frac{\partial X}{\partial y} & \frac{\partial X}{\partial x} - \lambda
\end{vmatrix} = 0.
$$

(28)

Upon solving the equation (28), we find the values of the matrix $A(y, \mu)$

$$
\lambda_{1,2}(y, \mu) = -0.5 \delta \pm 0.5 \left( \delta_1^2 - 4 \delta_2 \right)^{0.5},
$$

(29)

where $\delta_1 = f_2 + \frac{\partial X}{\partial y} \frac{\partial \varphi}{\partial x}, \delta_2 = \left( \frac{\partial F}{\partial y} - \frac{\partial f_1}{\partial x} \right) \left( 1 - \frac{\partial X}{\partial y} \frac{\partial \varphi}{\partial x} \right) f_2 - \frac{\partial X}{\partial y} \frac{\partial \varphi}{\partial x} \frac{\partial X}{\partial x} - \lambda$.

With the appropriate choice of constants $c_1$ $(\frac{\partial F(x_0)}{\partial \alpha} > c_1 > 0)$, the discriminant of the characteristic quadratic equation (28) has a negative sign. In this case, the matrix $A(y, \mu)$ has complex conjugate eigenvalues $\lambda_{1,2}$. Let their real part satisfy the condition

$$
\frac{\mu^2}{2} \left( f_2 - \frac{\partial X}{\partial y} \frac{\partial \varphi}{\partial x} \right) < -2\gamma < 0,
$$

(30)

where $\gamma = O\left(\mu^2\right)$ is a positive constant.

The negative condition for the real parts of the eigenvalues of the matrix $A(y, \mu)$ for the system (8)–(11) is reduced to the existence of a dissipative moment acting in the plane of the spatial angle of attack $\alpha$:

$$
f_2(x, \alpha_0) = -c_\alpha \frac{\rho SV}{2m} < -\frac{4\gamma}{\mu^2} < 0.
$$

(31)

The condition (31) is ensured by the corresponding damping characteristics of the spacecraft.

5. Decomposition of the singularly perturbed system into two subsystems

According to the well-known method for singularly perturbed systems, we intend to look for a solution to the system (15)–(16) in the following form

$$
y = \bar{y} + y^*, \ z = \bar{z} + z^*.
$$

(32)

Here $\bar{y} = y_0(t) + \mu y_1(t) + O(\mu^2) + \ldots, \ \bar{z} = z_0(t) + \mu z_1(t) + O(\mu^2) + \ldots$ are slow-motion series, $y^* = y_0^*(t) + \mu y_1^*(t) + O(\mu^2) + \ldots, \ z^* = z_0^*(t) + \mu z_1^*(t) + O(\mu^2) + \ldots$ are fast-motion series.

We substitute the expressions (32) in the system of equations (15)–(16).

$$
\mu \frac{d\bar{y}^*}{d\tau} = \mu Y\left(\bar{y} + y^*, \bar{z} + z^*, \mu\right),
$$

(33)

$$
\mu \frac{d\bar{z}^*}{d\tau} = A \left(\bar{y} + y^*, \bar{z} + z^*, \mu\right).
$$

(34)

We write the functions contained in the right-hand sides of the equations (33)–(34) in the form of sums of slow and fast components

$$
Y = \bar{Y} + Y^*, \ Z = \bar{Z} + Z^*, \ A = \bar{A} + A^*.
$$

(35)

Here $\bar{Y} = \bar{Y}(\bar{y}, \bar{z}, \mu), \ \bar{Z} = \bar{Z}(\bar{y}, \bar{z}, \mu)$ are slow-motion vectors,

$$
Y^* = Y\left(\gamma(\mu t) + y^*, \bar{z}(\mu t) + z^*, \mu\right) - Y(\bar{y}(\mu t), \bar{z}(\mu t), \mu), \ Z^* = Z\left(\gamma(\mu t) + y^*, \bar{z}(\mu t) + z^*, \mu\right) - Z(\bar{y}(\mu t), \bar{z}(\mu t), \mu).
$$
are fast-motion vectors, \( \overline{A} = \overline{A}(\overline{y}, \overline{z}, \mu) \) is the slow-motion matrix, 
\( A^* = \overline{A}\left(\overline{y}(\mu t) + y^*, \overline{z}(\mu t) + z^*, \mu\right) - \overline{A}(\overline{y}(\mu t), \overline{z}(\mu t), \mu) \) is the fast-motion matrix.

We substitute the expressions (32) and (35) into the system of equations (15)–(16). Upon equating terms depending on \( \tau \) and \( t \) separately from each other, we obtain two subsystems: slow-motion subsystem and fast-motion subsystem.

The slow-motion subsystem is written as follows:
\[
\frac{d\overline{y}}{d\tau} = \overline{A} \overline{y},
\]
(36)
\[
\mu \frac{d\overline{z}}{d\tau} = \overline{A} \overline{z} + \overline{Z}.
\]
(37)

The fast-motion subsystem is written as follows:
\[
\frac{dy^*}{dt} = y^*,
\]
(38)
\[
\mu \frac{dz^*}{dt} = (\overline{A} + A^*)z^* + A^* z + Z^*.
\]
(39)

First, we consider the fast-motion subsystem (39). We expand the right-hand side of equation (39) in a Maclaurin series for a small quantity \( y^* \)
\[
\frac{dz^*}{dt} = \overline{A} z^* + \mu \frac{\partial A^*(0)}{\partial y^*}(\overline{z} + z^*)y^* + \overline{Z} + Z^*,
\]
(40)
where \( Z^* = \mu \frac{1}{2} \frac{\partial^2 A^*(0)}{\partial y^*^2} (y^*)^2 z + \mu \frac{1}{2} \frac{\partial^2 Z(0)}{\partial y^*^2} (y^*)^2 + \ldots \). Note that upon obtaining (40) the matrix \( A \) also expands into a Maclaurin series for \( y^* \). As a result, at \( \mu = 0 \) the matrix \( \overline{A} = \begin{bmatrix} 0 & -\frac{\partial F}{\partial \alpha} \\ 1 & 0 \end{bmatrix} \). Here the following equalities are satisfied: \( \omega_{\alpha} = 0 \), \( \alpha = \psi \). Further in equations (40) it is assumed that \( z = \overline{z} + z^* = \overline{z} \). By analogy with [8], we represent the system of equations (40) as a second-order equation. As a result, we obtain
\[
\frac{d^2 \alpha^*}{dt^2} + f_3 \frac{d \alpha^*}{dt} + f_4 \alpha^* = 0,
\]
(41)
where the functions \( f_3 = -f_2 \), \( f_4 = \frac{\partial F}{\partial \alpha} \) are positive. It should be noted that the equation (41) describes a steady decrease in the fast component of the angle of attack \( \alpha^* \) from a small initial value to zero.

Figure 1 shows the change in the fast component of the angle of attack as a function of time of the spacecraft motion. This result was obtained by numerically solving the equation (41). It follows from Figure 1 that the influence of aerodynamic damping provides oscillatory movements \( \alpha^* \) relative to zero with monotonically decreasing amplitude. In this case, the amplitude of these oscillations decreases from a small initial value of 0.2 rad to zero within less than 300 seconds of motion.

6. Low-frequency system of equations of motion of an asymmetric spacecraft in the atmosphere

In the following, we will focus on the consideration of the slow-motion subsystem (36)–(37). The slow component of the angle of attack \( \overline{\alpha} \) is denoted \( \alpha \) below. The motion with respect to the integral manifold is described by differential equations (1), (2), (4). Where \( \omega_{\alpha} \) and \( \alpha \) are determined by the expressions (18)–(19). We transform the system of equations describing the motion with respect to the
integral manifold into a form convenient for the subsequent application of asymptotic methods.

Figure 1. Changes in the fast component of the angle of attack with aerodynamic damping.

First of all, we obtain the differential equation for the variable $\alpha$ when the system (1)–(4) is in motion with respect to the integral manifold. After that, by differentiating the expression for $\alpha$ (19), we obtain

$$\mu \frac{d\alpha}{d\tau} = \mu \frac{d\psi}{d\tau} + \frac{2}{\mu} \frac{d}{d\tau} f_1 \left( \frac{\partial F}{\partial \psi} \right)^{-1} + O\left( \mu^3 \right),$$

(42)

where the derivative $d\psi / d\tau$ is determined from the derivative of the function $F(x, \psi)$ and has the form

$$\frac{d\psi}{d\tau} = -\mu \frac{\partial F}{\partial x} \left[ \frac{\partial F}{\partial \psi} \right]^{-1} \chi(x, \phi, \psi).$$

Moreover, in equation (42), we can take into account that $\psi \approx \alpha$. When solving equation (42), the initial conditions for the angle $\alpha$ must belong to the integral manifold (18)–(19). As a result, we obtain the equation for the spatial angle of attack:

$$\frac{d\alpha}{dt} \left[ \omega_x^2 + \left( I_x \omega_x - \omega_y \cos \alpha \right) \left( I_x \omega_x - 2 \omega_x \cos \alpha \right) \right] =$$

$$= \mu \frac{M_{\psi n}}{l q} \frac{d\phi}{dt} + \mu \frac{M_{\psi x}}{l} \frac{d\phi}{dt} + \mu \left( I_x \omega_x - 2 \omega_x \cos \alpha \right) \frac{M_{\psi x}}{l} -$$

$$- \mu \frac{M_{\psi x}}{l} \omega_y \sin \alpha - \mu \frac{C_{yy} q S}{mV} \left( I_x \omega_x - \omega_y \cos \alpha \right) \left( I_x \omega_x - 2 \omega_x \cos \alpha \right).$$

(43)

where $M_{\psi n} = -C_{x1} A_{z n} q S + \left( m^f \cos \phi_n - m^f \sin \phi_n \right)$.

From the equation (1), taking into account the expression (20), we obtain:

$$I_x \frac{d\omega_x}{dt} = \mu M_{\psi n},$$

(44)

From the equation (4), taking into account the expressions (20)–(21), we obtain:

$$\frac{d\phi}{dt} = \omega_x - \omega_y \cos \alpha,$$

(45)

In the asymptotic analysis of the system of equations (43)–(45), the angular velocity $\omega_y$ is calculated directly from the nonlinear equation $F(x, \alpha) = 0$ and takes the form:
where the signs "+" and "-" correspond to the "direct" and "reverse" precessions of the spacecraft [6].

Note that system (43)-(45) differs in its form from the low-frequency system [8] describing the motion of spacecraft with a small asymmetry in the atmosphere only in the absence of the term \(M \frac{\dot{\alpha}}{\cos \alpha} / I\) on the left side of the approximate equation for the spatial angle of attack. The low-frequency system of equations (43)–(45) is solved numerically together with three equations describing the motion of the center of mass of the spacecraft in the atmosphere, namely, the equation for speed, the equation for height, and the equation for the flight-path angle [10]. This system describes the motion of the spacecraft in the following interval of variation of the spatial angle of attack: \(\alpha \in [0, 0.5\pi]\).

Figure 2 shows the change in the slow component of the angle of attack from the time of the spacecraft motion. Figure 2 was constructed as a result of numerical integration of the equation (43).

\[
\omega_{\gamma,\pm} = \frac{d\gamma}{dt} = \frac{\tilde{T}_x \dot{\alpha}_x}{2 \cos \alpha} \pm \frac{1}{\cos \alpha} \left[ \frac{\tilde{T}_x^2 \dot{\alpha}_x^2}{4} - \frac{M \frac{\dot{\gamma}}{\cos \alpha}}{I} \right]^{1/2} + O(\varepsilon),
\] (46)

Figure 2. Change in the slow component of the angle of attack, taking into account the effect of asymmetry.

From the comparison of Figures 1–2 it follows that the magnitude of the slow component is significantly larger than the amplitude of the oscillations of the fast component of the angle of attack. This result corresponds to the earlier assumption about the values of these components.

7. Conclusion
The initial nonlinear system of equations of motion of a spacecraft with considerable asymmetry in a low-density atmosphere is represented in a singularly perturbed form. We consider the problem of decomposition of the initial system of equations into two subsystems: slow-motion subsystem and fast-motion subsystem. The method of integral manifolds has been used to decrease the order of the initial system. The resulting slow-motion subsystem is called low-frequency system of equations. This approximate system makes it possible to use known asymptotic methods to study the evolution of slow variables in the problem of descent of an asymmetric spacecraft in a low-density atmosphere. In particular, this provides for application of the averaging method [12] to the problem being considered.

8. References
[1] Barinova E V and Timbai I A 2015 Studying transient motion modes with respect to the landingmodule angle of attack with the restoring moment triharmonic characteristic on entering the atmosphere Cosm. Res. 53(3) 246-255.
[2] Zabolotnov Yu M 2013 Statistical analysis of attitude motion of a light capsule entering the atmosphere *Cosm. Res.* **51**(3) 213-224.

[3] Desai P N, Prince J L, Queen E M, Schoenenberger M, Cruz J R and Grover M R 2011 Entry, descent, and landing performance of the Mars Phoenix lander *Journal of SC and Rockets* **48**(5) 798-808.

[4] Way D W, Davis J L and Shidner J D 2013 Assessment of the Mars Science Laboratory entry, descent, and landing simulation *Adv. in the Astron. Sci.* **148** 563-581.

[5] Lyubimov V V 2018 Induced resonant torques during the descent of a small asymmetric spacecraft in the atmosphere *Journal of Physics: Conf. Series* **1096**(1) 1-10.

[6] Lyubimov V V 2014 Asymptotic analysis of the secondary resonance effects in the rotation of a spacecraft with small asymmetry in the atmosphere *Russ.aeronaut.* **57** 245-252.

[7] Zabolotnov Yu M and Lyubimov V V 1998 Secondary resonance effect in the motion of a SC in the atmosphere *Cosm. Res.* **36**(2) 194-201.

[8] Zabolotnov Yu M 1999 A method of studying the resonance motion of a single nonlinear oscillation system *Mechanics of Solids* **1** 33-45.

[9] Lyubimov V V 2010 External stability of resonances in the movement of an asymmetric rigid body with a strong magnet in the geomagnetic field *Mech. Solids* **45** 10-21.

[10] Lyubimov V V and Lashin V S 2017 External stability of a resonance during the descent of a spacecraft with a small variable asymmetry in the martian atmosphere *Advanc. Space Resear*. **59** 1607-1613.

[11] Shchepakina E A, Sobolev V A and Mortell M P 2014 Singular Perturbations. Introduction to System Order Reduction Methods with Applications *Lecture Notes in Mathematics* (Cham: Springer International Publishing AG) p 212.

[12] Sanders J A, Verhulst F and Murdock J 2007 Averaging methods in nonlinear dynamical systems *Applied Mathematical Sciences* (New York: Springer-Verlag) p 434.