Barrier Perturbation Induced "Superarrivals" and "Nonlocality" in a Time-Evolving Wave Packet

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We compute the time evolving probability of a Gaussian wave packet to be reflected from a rectangular potential barrier which is perturbed by reducing its height. A time interval is found during which this probability of reflection is enhanced ("superarrivals") compared to the unperturbed case. Such a time evolving reflection probability implies that the effect of perturbation propagates across the wave packet faster than its group velocity - a curious form of "nonlocality."

In recent years a number of interesting investigations have been reported on wave-packet dynamics in quantum-well systems [1-6]. In this paper we study the wave packet dynamics from a new perspective. The reflection/transmission probabilities for the scattering of wave packets by various obstacles are usually considered from static or unperturbed potential barriers. Generally the time-independent (asymptotic) values attained after a complete time evolution are calculated. Here we point out the striking effects that occur during the time evolution by considering dynamics of wave packet scattering from a barrier whose height is reduced to zero well before the asymptotic value of reflection probability is reached.

For an unperturbed barrier the reflection probability for an initially localized wave packet $\psi(x, t = 0)$ is calculated by considering the wave packet as a superposition of plane waves and by writing

$$|R_0|^2 = \int \left| \phi(p) \right|^2 |R(p)|^2 dp$$

(1)

where $|R(p)|^2$ is the reflection probability corresponding to the plane wave component $\exp(ipx)$ and $\phi(p)$ is Fourier transform of the initial wave packet $\psi(x, t = 0)$. Since a wave packet evolves in time, $|R_0|^2$ defined by Eq. (1) denotes the time-independent value of reflection probability pertaining to a wave packet, this value being attained in the asymptotic limit ($t_\infty$) of the time evolution. Thus $|R_0|^2$ can be expressed in the following form

$$|R_0|^2 = \int_{-\infty}^{x_1} \left| \psi(x, t_\infty) \right|^2 dx$$

(2)

where $\psi(x, t_\infty)$ is asymptotic form of the wave packet attained by evolving from

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ψ(x, t = 0) and by being scattered from a rectangular potential barrier of finite height and width. Note that x′ lies at a left edge of the initial profile of the wave packet such that \( \int_{-\infty}^{x'} |\psi(x, t = 0)|^2 \, dx \) is negligible (see Figure 1). At any instant before the constant value \(|R_0|^2\) is attained, the time evolving reflection probability in the region \(-\infty < x \leq x'\) is given by

\[
|R(t)|^2 = \int_{-\infty}^{x'} |\psi(x, t)|^2 \, dx \tag{3}
\]

Now suppose that during the time evolution of this wave packet the barrier is perturbed by reducing its height to zero within a very short but finite interval of time. Here by “short” time interval we mean that it is small compared to the time taken by the reflection probability to attain its asymptotic value \(|R_0|^2\). We compute effects of this “sudden” perturbation on \(|R(t)|^2\). The salient features of our findings are as follows:

(a) A finite time interval is found during which \(|R(t)|^2\) shows a surprising enhancement (we call this effect “superarrivals”) in the perturbed case even though the barrier height is reduced. This time interval and the amount of enhancement depend on the time over which the barrier height is made zero.

(b) The way the computed reflection probability evolves implies an incompatibility with a form of “locality condition” which is inferred from a commonly used “particle picture” which associates the mean “velocity” of a “particle” with group velocity of the corresponding wave packet. In particular, such a “picture” is phenomenologically useful in the context of design and interpretation of “single particle” interference experiments using neutrons/electrons [7,8]. The type of quantum nonlocality thus exhibited involves an action at a distance entailing a global effect on the wave packet induced by a local perturbation. The resulting action propagates across the wave packet at a finite speed which is greater than the group velocity - a distinctly nonclassical behaviour.

In order to demonstrate the above features let us begin by writing the initial wave packet (in units of \(\hbar = 1\) and \(m = 1/2\), this choice of units being convenient for numerical computation) in the form

\[
\psi(x, t = 0) = \frac{1}{\sqrt{2\pi (\sigma_0)^2}} \exp \left[ -\frac{(x - x_0)^2}{4\sigma_0^2} + ip_0x \right] \tag{4}
\]

which describes a packet of width \(\sigma_0\) centered around \(x = x_0\) with its peak moving with a group velocity \(2p_0 = \frac{ip_0}{m}\) towards a rectangular potential barrier. The point \(x_0\) is chosen such that \(\psi(x, t = 0)\) has a negligible overlap with the barrier. The expectation value of energy (\(E\)) of the wave packet is given by \(\frac{p_0^2}{2m} + \frac{1}{4}\sigma_0^2\).

For the purpose of computing \(|R(t)|^2\) given by Eq. (3) the time dependent Schrodinger equation is solved by using the numerical methods as developed by Goldberg, Schey and Schwartz [9]. In this treatment the parameters are chosen in a way ensuring the spreading of packet to be negligible during the scattering process so that it doesn’t mask the effects of interest. Here we choose \(x_0 = 1.2\),
σ₀ = 0.05/√2 and p₀ = 50π. The barrier is centered around x_c = 1.5 with width = 0.064. Height of the barrier (V) before perturbation is chosen to be V = 2E. This choice satisfies the following criteria:

1. V is such that the reflection probability is very close to 1 since we are interested only in the reflection probability.
2. At the same time V is chosen not to be too large. This is in order to ensure that reduction of the barrier height need not be too fast.

|R(t)|² is computed according to Eq. (3) by taking x' = x₀ - 3σ₀/√2. The computed evolution of |R(t)|² corresponds to the building up of reflected particles with time. More precisely, it means that a detector located within the region \(-∞ < x < x'\) measures |R(t)|² by registering the reflected particles arriving in that region up to various instants.

First, we compute |R(t)|² for the wave packet scattered from a static barrier V = 2E. The relevant curve is shown in Figure 2 which tends towards a time-independent value which is the stationary state reflection probability |R₀|² given by Eq (2); this is numerically verified to be equivalent to the expression for |R₀|² given by Eq. (1). We then proceed to study the consequences of reducing the barrier height from V = 2E to V = 0. The time evolution of |R(t)|² in the perturbed cases is found to show a number of interesting features.

In all the cases we study, the potential V goes to zero linearly within a switching off time ε around t = t_p chosen to be 8 × 10⁻⁴ (note that numbers denoting the various instants are in terms of time steps; for example, t = 8 × 10⁻⁴ corresponds to 400 time steps). Here ε ≪ t₀, t₀ being the time required for |R(t)|² to attain the asymptotic value |R₀|². This short time span ε over which the perturbation takes place is thus given by \([t_p - \frac{p}{2}, t_p + \frac{p}{2}]\). Profile of the wave packet at t₀ = 8 × 10⁻⁴ is shown in Figure 3. Note that at that instant the overlap of the wave packet with the barrier is significant.

Figure 4 shows the evolution of |R(t)|² for various values of ε. Different ε correspond to different N where N is the number of time steps involved in computing the reduction of V from 2E to zero, \(\frac{p}{2}\) being the magnitude of each time step. Varying ε signifies changing the time span over which the barrier height goes to zero which in turn means different rates of reduction. We now compare the computed |R(t)|² for the particular case N=2 (denoted by |R_p(t)|²) with that calculated for a static barrier (denoted by |R_s(t)|²). This comparison is shown in Figure 5 which reveals that

\[
|R_p(t)|^2 = |R_s(t)|^2 \quad t \leq t_d
\]

\[
|R_p(t)|^2 > |R_s(t)|^2 \quad t_d < t \leq t_c
\]

\[
|R_p(t)|^2 < |R_s(t)|^2 \quad t > t_c
\]

where t_p is the instant around which the perturbation takes place, t_c is the instant when the two curves cross each other, and t_d is the time from which...
the curve corresponding to the perturbed case starts deviating from that in the unperturbed case. Here $t_c > t_d > t_p$.

Let us now focus on a striking feature embodied in the inequality (6). As the barrier height is made zero, one does not expect at any time an increase in the reflected particle flux compared to that in the unperturbed case. Nevertheless, the inequality (6) shows that there is a finite time interval $\Delta t = [t_d, t_c]$ during which the probability of finding a reflected particle is more (“superarrivals”) in the perturbed case than when the barrier is left unperturbed (see Figure 5). A detector placed in the region $x < x'$ would therefore register more counts during this time interval $\Delta t$ even though the barrier height had been reduced to zero prior to that. It has been checked that this effect of “superarrivals” occur for other values of $N$ (or, $\epsilon$) as well; see Figure 4.

Figure 4 also reveals that the probability of “superarrivals” (i.e., the enhancement of reflection probability) depends on $N$ (or, $\epsilon$). The maximum enhancement takes place for $N = 2$ and the amount of enhancement decreases with increasing $N$. But interestingly there is no appreciable change in the magnitude of the time interval $\Delta t$ over which this enhancement occurs. In order to have a quantitative measure of “superarrivals” we define the parameter $\eta$ given by

$$\eta = \frac{I_p - I_s}{I_s}$$

where the quantities $I_p$ and $I_s$ are defined with respect to $\Delta t$ during which “superarrivals” occur

$$I_p = \int_{\Delta t} |R_p(t)|^2 \, dt \quad (9)$$

$$I_s = \int_{\Delta t} |R_s(t)|^2 \, dt \quad (10)$$

The relevant numerical results are displayed in Table 1 and variation of $\eta$ with $N$ (or, $\epsilon$) for different cases of perturbation are shown in Figure 6. These results are summarised below:

(a) There exists a finite time interval $\Delta t$ during which an increase in the reflection probability (“superarrivals”) occurs for the perturbed cases relative to the unperturbed situation.

(b) The time interval $\Delta t$ over which this probability enhancement takes place is not sensitive to the span of the switching off time $\epsilon$ within which the barrier height is reduced to zero.

(c) Magnitude of this probability enhancement falls off linearly with increasing $\epsilon$.

We also note that both $\Delta t$ and “superarrivals” given by $\eta$ depend on the instant $t_p$ around which the barrier is switched off. Hence choice of this instant for demonstrating the above effects needs to be appropriate. From the profiles of wave packet corresponding to different times of perturbation $t_p$ (as shown in
Figure 6) it is seen that “superarrivals” is appreciable in cases where the wave packet has some significant overlap with the barrier during its switching off. What optimal condition determining the choice of $t_p$ maximises “superarrivals” needs to be clarified. In particular, there could be choice(s) of $t_p$ for which “superarrivals” is more than what is obtained in the present work. Further, it should be interesting to investigate whether any lower bound of $t_p$ exists prior to which switching off the barrier does not give rise to “superarrivals”.

Quantum Nonlocality. - We now proceed to discuss in what sense the computed time development of $|R(t)|^2$ in the perturbed cases mentioned above entail an incompatibility with a certain form of locality condition. This locality condition is formulated in terms of a “particle picture” which is commonly used in interpreting wave packet behaviour. In particular, such a “picture” is phenomenologically motivated, being used in analyzing the results of neutron/electron interferometric experiments [7,8] performed in the region of so-called “self interference” where only one “particle” is required to be present inside the device at a time. In order to ensure this condition it is necessary to associate a mean “velocity” with an individual neutron/electron so that the time it stays inside the interferometer can be calculated and hence suitably adjusted by varying the relevant parameters (such as the rate of emission from a source).

A crucial point is that this mean “velocity” is assumed to be the group velocity ($v_g$) of the wave packet associated with the particle [10]. The average time of transit $\Delta T$ of a neutron/electron inside the device is then estimated by using the relation $\Delta T = L/v_g$ where $L$ is the distance travelled within the device. Such a relation characterizes the type of “particle picture” which constitutes a crucial ingredient in designing and interpreting the neutron/electron self interference experiments. This “particle picture”, if applied in the specific context of the example discussed in the present paper, leads to the following Propositions.

**Proposition 1:** If a particle is detected at time $t$, it is inferred to be reflected from the barrier at an earlier instant $t - \frac{D}{v_g}$ where $D$ is the separation between the detector and the barrier.

**Proposition 2:** Locally perturbing a barrier has no effect on those particles already reflected from the barrier.

Note that Proposition 2 is some form of locality condition which assumes that detection probability of the particles reflected from the barrier prior to its perturbation does not bear any signature of the perturbation. On the basis of the above Propositions we now proceed to derive the following constraint condition.

In our computed cases the relevant perturbation (i.e., reduction of the barrier height) commences from the instant $t_p - \frac{\epsilon}{2}$. From Proposition 1 it follows that particles reflected from the barrier until this particular instant are registered at a detector (placed at a distance $D$) up to an instant $\tau$ which is given by $\tau = (t_p - \frac{\epsilon}{2}) + \frac{D}{v_g}$. The locality condition (Proposition 2) therefore requires that the measured particle statistics at this detector should not be affected by perturbation of the barrier until the instant $\tau$. This means that the time evolving reflection probability $|R_p(t)|^2$ in the perturbed cases is permitted by this locality condition to deviate from the reflection probability $|R_s(t)|^2$ in the static case only after the instant $\tau$. In other words, if $|R_p(t)|^2$ and $|R_s(t)|^2$
are found to differ from the instant $t_d$, the locality condition in the form of Proposition 2 requires

$$t_d > \tau$$

(11)

However, the results of our computations of $|R_p(t)|^2$ and $|R_s(t)|^2$ indicate a rather strong violation of the inequality (10). Choosing $D = 0.375, v = 2p_0 = 100\pi$ and $t_p = 8 \times 10^{-4}$, the results of our calculation are displayed in Table 2.

Comparing the values of $\tau$ with those of $t_d$ as shown in Table 2 we find that $t_d < \tau$ for all values of $\epsilon$. A clear violation of the locality condition (10) is thus demonstrated. This form of quantum nonlocality is quite distinct from the usual nonlocality [10] which is inferred from many particle entangled states. We now elaborate a bit on the significance of this new form of quantum nonlocality.

What our computed results show is that a local change in potential (in our specific case, a reduction of the barrier height) affects a wave packet globally, the global effect being manifested through time evolution of the packet. The action due to local perturbation propagates across the wave packet at a finite speed, say, $v_e$ affecting the time evolving reflection probability which can be measured at different points. Thus a distant observer who records the growth of reflection probability becomes aware of perturbation of the barrier (occurring around an instant $t_p$) from the instant $t_d$ when the time varying reflection probability starts deviating from that measured in the unperturbed case. Then $v_e$ is given by

$$v_e = \frac{D}{t_d - (t_p - \frac{\epsilon}{2})}$$

(12)

From Eq. (12) it follows that the violation of locality condition (10) implies

$$v_e > v_g$$

(13)

i.e., the effect caused by reducing the barrier height travels across the wave packet at a speed exceeding the packet’s group velocity. This is an intrinsically nonclassical “action at a distance” which is manifested even when spreading of the wave packet is ensured to be negligibly small.

In general, $v_e$ depends on $D, t_p, t_d,$ and $\epsilon$. However, $t_d$ is essentially determined by $\epsilon$ for a given $t_p$. Hence for fixed values of $D$ and $t_p$, $v_e$ depends only on $\epsilon$. For a particular choice of $D = 0.343$ and $t_p = 8 \times 10^{-4}$, Table 3 indicates the way calculated values of $v_e$ vary with $\epsilon$; the corresponding variation of the ratio of $v_e$ with $v_g$ is also shown in Table 3. Of course, more detailed studies are called for in order to have a precise quantitative idea about the dependence of $v_e$ on the relevant parameters.

An important point to note from Table 3 is that $v_e$ can exceed $v_g$ substantially. It should therefore be interesting to investigate whether any bound exists on the ratio of $v_e$ with $v_g$. It may also be worthwhile to compute our example for various cases by varying mass and width of the wave packet to see what happens to this ratio in the various limiting situations such as for large mass (classical limit) and for broad wave packets (plane wave limit).

To sum up, our work serves to reveal that much interesting and counterintuitive physics is concealed within the time evolution of reflection/transmission
probability for a wave packet scattered from a perturbed barrier. This has so far remained unexplored because attention is usually focused only on the final time independent values of reflection/transmission probability. In particular, the effects uncovered in this paper involve an intriguing interplay between “particle” and “wave” aspects of a wave packet, its conceptual ramifications warranting further probing. Such effects could also be amenable to experimental verification using the available neutron/electron “single particle” experimental arrangements [7,8].

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[1] B. M. Garraway and K-A. Souminen, Rep. Prog. Phys. 58, 365 (1995); W. S. Warren, H. Rabitz, and M. Dahleh, Science 259, 1581 (1993); M. J. J. Vrakking, D. M. Villeneuve, and Albert Stolow, Phys. Rev. A 54, 37 (1996).
[2] I. Sh. Averbukh, M. J. J. Vrakking, D.M. Villeneuve, and Albert Stolow, Phys. Rev. Lett. 77, 3518 (1996).
[3] R. Bluhm, V.A. Kostelecky, and J.A. Porter, Am. J. Phys. 64, 944 (1996).
[4] M.V. Berry, J. Phys. A 29, 6617 (1996); M.V. Berry and S. Klein, J. Mod. Opt. 43, 2139 (1996).
[5] D.L. Aronstein and C.R. Stroud, Phys. Rev. A 55, 4526 (1997).
[6] A. Venugopalan and G.S. Agarwal, Phys. Rev. A 59, 1413 (1999).
[7] A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki and H. Ezawa, Am. J. Phys., 57 (2), 117 (1989); S. A. Werner in Fundamental Problems in Quantum Theory, eds. D. M. Greenberger and A. Zeilinger (N. Y. Academy of Science, 1995), p. 241.
[8] H. Rauch, in Fundamental Problems in Quantum Theory, eds. D. M. Greenberger and A. Zeilinger (N. Y. Academy of Science, 1995), p. 263.
[9] A. Goldberg, H. M. Schey and J. L. Schwartz, Am. J. Phys. 35, 177 (1967).
[10] D. M. Greenberger, Review of Modern Physics, 55, 875 (1983).
[11] J. S. Bell, Physics, 1, 195 (1964). For a recent overview see, for example, D. Home, Conceptual Foundations of Quantum Physics - An Overview from Modern Perspectives (Plenum Press, New York, 1997), Chapter 4 and references there in.
Table Captions

**Table 1:** Magnitudes of the time interval ($\Delta t$) over which “superarrivals” occur and the measure ($\eta$) of “superarrivals”. Note that $\Delta t$ is *almost independent* of the span of perturbation $\epsilon$ whereas $\eta$ has an explicit *dependence* on $\epsilon$.

**Table 2:** *Violation* of the locality condition $t_d > \tau$.

**Table 3:** Velocity $v_e$ with which the effect of perturbation propagates across the wave packet is substantially *higher* than the group velocity $v_g$. Note that $v_e/v_g$ is *independent* of the span of perturbation.

Figure Captions

**Figure 1:** Profile of the wave packet at $t=0$.

**Figure 2:** The time evolution of reflection probability in the unperturbed situation. Note that the curve gradually tends towards its asymptotic (time-independent) value.

**Figure 3:** Profile of the wave packet at $t = t_p$. Overlap of the wave packet with the barrier is crucial for the effect of “superarrivals”.

**Figure 4:** The time evolution of $|R_p(t)|^2$ for various magnitudes of $N(\text{or} \epsilon)$.

**Figure 5:** A comparison between $|R_x(t)|^2$ and $|R_p(t)|^2$ for $N = 2$.

**Figure 6:** Profiles of the wave packet at different times of perturbation.
### TABLE 1

| N | $\epsilon \times 10^{-3}$ | $t_d \times 10^{-3}$ | $t_c \times 10^{-3}$ | $\Delta t = t_c - t_d$ | $\eta$ |
|---|-----------------|-----------------|-----------------|------------------|------|
| 2  | 0.004           | 1.122           | 1.832           | 0.71             | 0.50 |
| 10 | 0.02            | 1.114           | 1.828           | 0.714            | 0.46 |
| 30 | 0.06            | 1.094           | 1.814           | 0.72             | 0.37 |
| 50 | 0.1             | 1.072           | 1.792           | 0.72             | 0.28 |

### TABLE 2

| N | $\epsilon \times 10^{-3}$ | $t_d \times 10^{-3}$ | $\tau \times 10^{-3}$ | locality condition |
|---|-----------------|-----------------|-----------------|------------------|
| 2  | 0.004           | 1.122           | 1.890           | violated         |
| 10 | 0.02            | 1.114           | 1.882           | do               |
| 30 | 0.06            | 1.094           | 1.862           | do               |
| 50 | 0.1             | 1.072           | 1.842           | do               |

### TABLE 3

| N | $\epsilon \times 10^{-3}$ | $t_d \times 10^{-3}$ | $\tau \times 10^{-3}$ | $v_c$ | $v_0 = 2p_0$ | $\frac{v_c}{2p_0}$ |
|---|-----------------|-----------------|-----------------|------|------------|------------------|
| 2  | 0.004           | 1.122           | 1.890           | 337.15$\pi$ | 100$\pi$ | 3.37 |
| 10 | 0.02            | 1.114           | 1.882           | 337.15$\pi$ | do       | 3.37 |
| 30 | 0.06            | 1.094           | 1.862           | 337.15$\pi$ | do       | 3.37 |
| 50 | 0.1             | 1.072           | 1.842           | 339.24$\pi$ | do       | 3.39 |
Figure 2
Figure 4

- Static case
- $N=2$
- $N=30$
- $N=50$
Figure 5

$|R_s(t)|^2$

$|R_p(t)|^2$

$|R(t)|^2$

Time $x 10^3$

0 0.5 1 1.5 2 2.5 3

0 0.2 0.4 0.6 0.8 1
Figure 6