Noncommutative integrability

C. Klimčík
Institute de Mathématiques de Luminy
163, Avenue de Luminy, 13288 Marseille, France

Abstract

I compute the cohomology of a non-commutative complex underlying the notion of the gauge field on the fuzzy sphere.
1. Noncommutative geometry is a well established mathematical discipline with a surprising and nontrivial impact on quantum field theory in general and the standard model in particular [1]. There is a subactivity in that vast subject which aims to replace the field theoretical models on the standard smooth manifolds by its counterparts defined on suitable noncommutative deformations of those manifolds [2, 3, 4]. The resulting noncommutative models usually respect all symmetries or supersymmetries of the commutative theories but they have an important advantage of possessing only a finite (though large) number of degrees of freedom. Recently such structures have emerged also in the context of the matrix model of $M$-theory [5, 6, 7].

The basic idea of the approach is as follows: One considers an Euclidean space-time which is taken to be compact for convenience. This spacetime gets equipped with a symplectic structure. A quantization of this symplectic structure gives an algebra of quantum observables which is to be taken as the definition of the non-commutative manifold. The compactness results in the finiteness of that noncommutative algebra of observables. The important feature of the formalism is that the Hamiltonian vector fields on the classical manifold survive the deformation. They are generated by the quantized Hamiltonians via the commutators. Finally, also the integration over the symplectic manifolds gets replaced in the deformed picture by the operation of taking the trace over the matrix algebra.

Having at hand the deformed notions of algebra, Hamiltonian vector fields and integration we can construct the field theoretical actions for the models involving the scalar fields on the deformed noncommutative manifold. As an example, consider a Riemann sphere as a spacetime of an Euclidean field theory.

The crucial observation is that $S^2$ is naturally a symplectic manifold; the symplectic form $\omega$ is up to a normalization just the round volume form on the sphere. Using the standard complex coordinate $z$ on the Riemann sphere, we have

$$\omega = \frac{N}{2\pi} \frac{d\bar{z} \wedge dz}{(1 + \bar{z}z)^2}, \quad (i)$$

with $N$ a real parameter$^1$. If we consider a scalar field theory, then the scalar

$^1$Note, that we have chosen a normalization which makes the form $\omega$ purely imaginary. Under quantization, hence, the Poisson bracket is replaced by a commutator without any imaginary unit factor.
field \( \phi \) is a function on the symplectic manifold or, in other words, a classical observable. The action of the massless (real) scalar field theory on \( S^2 \) is given by

\[
S = -i \int \omega R_i \phi R_i \phi,
\]

where \( R_i \) are the vector fields which generate the \( SO(3) \) rotations of \( S^2 \) and the Einstein summation convention is understood. The vector fields \( R_i \) are Hamiltonian; this means that there exists three concrete observables \( r_i \) such that

\[
\{ r_i, \phi \} = R_i \phi.
\]

Here \( \{.,.\} \) is the Poisson bracket which corresponds to the symplectic structure \( \omega \). The observables \( r_i \in \mathbb{R}^3 \) are just the coordinates of the embedding of \( S^2 \) in \( \mathbb{R}^3 \). Thus we can rewrite the action \((ii)\) as

\[
S = -i \int \omega \{ r_i, \phi \} \{ r_i, \phi \}.
\]

Suppose we quantize the symplectic structure on \( S^2 \) (probably the first who has done it was Berezin [8]). Then the algebra of observables becomes the noncommutative algebra of all square matrices with entries in \( \mathbb{C} \); the quantization of \( S^2 \) can be only performed if \( N \) is an integer, the size of the scalar field matrices \( \phi \) is then \( (N + 1) \times (N + 1) \). This algebra of matrices defines the noncommutative (or fuzzy [4]) sphere. The integration over the phase space volume form \( i\omega \) is replaced by taking a properly normalized trace \( \text{Tr} \) over the matrices and the Poisson brackets are replaced by commutators (the Hamiltonians \( r_i \) are also quantized, of course).

Putting together, we can consider along with \((iv)\) a noncommutative action

\[
S = -\frac{1}{N + 1} \text{Tr}([r_i, \phi], [r_i, \phi]).
\]

The action \((v)\) has a few nonstandard properties. First of all, the space of all ”fields” (=matrices) is finite dimensional and the product of fields is noncommutative. The latter property may seem awkward but in all stages of analysis we shall never encounter a problem which this noncommutativity might create. The former property, however, is highly desirable, since all divergences of the usual field theories are automatically eliminated. We may interpret \((v)\) as the regularized version of \((iv)\); the fact that \((v)\) goes to \((iv)\) in
the limit $N \to \infty$ is just the statement that classical mechanics is the limit of the quantum one for the value of the Planck constant $1/N$ approaching zero. Remarkably, unlike lattice regularizations, $(v)$ preserves the $SO(3)$ isometry of the sphere (="spacetime"). Indeed, under the variation $\delta \phi = [r_i, \phi]$ the action $(v)$ remains invariant.

2. A question of obvious interest consists in enlarging the above-mentioned framework of constructing the field theories on noncommutative manifolds also to the case of nonscalar fields. In practice, one is interested in spinor and 1-form fields (gauge potentials). While the quantization gives automatically scalars the rest of the story is not evident because in the literature on quantization one did not consider the question of quantizing the vector bundles as the modulus of algebra. The idea adopted in [2] for quantizing spinors is simple: one enlarges the algebra appropriately to include the spinor fields with the scalars. The resulting enlarged algebra is known as the algebra of superfields; by the way, one gains in this way a possibility of constructing noncommutative supersymmetric field theories along the same lines as above. The issue of the gauge fields turned out to be more complicated than the story of spinors . It was necessary to construct the deformation of the whole de Rham complex for being able to define the notion of the gauge field in the noncommutative case [9].

Actually, one needs more than the deformation of the algebraic structure of the de Rham complex. Indeed, the notion of the exterior derivative $d$ has to survive the quantization. Since $d$ has to be a derivative, one has to find Hamiltonian vector fields to express $d$. It turned out that this can be done by paying the price of injecting the standard classical de Rham complex to a larger complex which can be deformed with all its relevant structures. This new complex was referred to as the Hamiltonian de Rham complex in [9], reflecting the fact that the exterior derivative could be expressed in terms of the Hamiltonian vector fields and thus quantized. There remains a mathematical question which was only touched upon in [9] but which is quite important in order to have a feeling of general consistency of the deformed picture. The question reads: What is the cohomology of the deformed complex? A satisfactory answer must be that it is the same as the cohomology

\[\text{This question can well become physical in the context of the so-called world-sheet } T\text{-duality.}\]
of the undeformed complex. The reason for this is simple: the quantization should influence only the short distance properties of the manifold but not its topology; obviously, the cohomology of the de Rham complex reflects the topology of the underlying manifold. In this note, we give a so far missing proof that the deformation of the sphere does not change the cohomological content of the Hamiltonian de Rham complex.

3. We should first review what are the nondeformed and deformed Hamiltonian complexes over the sphere following [9], then we shall actually compute their cohomologies. We shall not review the way how the standard de Rham complex is injected into the Hamiltonian one. The interested reader may find it again in [9].

Consider the algebra of functions on the complex $C^{2,1}$ superplane, i.e. algebra generated by bosonic variables $\tilde{\chi}^i, \chi^i, i = 1, 2$ and by fermionic ones $\tilde{a}, a$. The algebra is equipped with the graded involution
\[
(\chi^i)^\dagger = \tilde{\chi}^i, \quad (\tilde{\chi}^i)^\dagger = \chi^i, \quad a^\dagger = \tilde{a}, \quad \tilde{a}^\dagger = -a
\]
and with the super-Poisson bracket
\[
\{f, g\} = \partial_{\chi^i} f \partial_{\chi^i} g - \partial_{\tilde{\chi}^i} f \partial_{\tilde{\chi}^i} g + (-1)^{f+1}[\partial_{\tilde{a}} f \partial_{a} g + \partial_{a} f \partial_{\tilde{a}} g].
\]
Here and in what follows, the Einstein summation convention applies. We can now apply the (super)symplectic reduction with respect to a moment map $\tilde{\chi}^i \chi^i + \tilde{a}a$. The result is a smaller algebra $\mathcal{A}$, that by definition consists of all functions $f$ with the property
\[
\{f, \tilde{\chi}^i \chi^i + \tilde{a}a\} = 0.
\]
Moreover, two functions obeying (55) are considered to be equivalent if they differ just by a product of $(\tilde{\chi}^i \chi^i + \tilde{a}a - 1)$ with some other such function. The algebra $\mathcal{A}$ has a subalgebra $\mathcal{A}_e$ which consists of all even elements of $\mathcal{A}$; that means that the odd generators $a, \tilde{a}$ appears only in the combination $\tilde{a}a$. We identify $\mathcal{A}_e$ with the space of the (complex) Hamiltonian 0-forms $\Omega_0$ and also with the space of the (complex) Hamiltonian 2-forms $\Omega_2$. The space of the (complex) Hamiltonian 1-forms is defined as
\[
\Omega_1 \equiv \mathcal{A}_a \oplus \mathcal{A}_a \oplus \mathcal{A}_a \oplus \mathcal{A}_a,
\]
where the space \( \mathcal{A}_a (\mathcal{A}_{\bar{a}}) \) consists of all odd elements of \( \mathcal{A} \) not depending on \( \bar{a}(a) \).

In order to define the exterior derivative \( d \) we introduce the following Hamiltonian vector fields \( T_i, \bar{T}_i \):

\[
T_i = \chi^i \partial_a - a \partial \chi^i, \quad \bar{T}_i = \bar{a} \partial \chi^i + \chi^i \partial \bar{a}.
\]

Of course, they annihilate the moment map \( (\bar{\chi}^i \chi^i + \bar{a}a) \), otherwise they would not be well defined differential operators acting on \( \mathcal{A} \). Their Hamiltonians are

\[
t_i = \bar{\chi}^i a, \quad \bar{t}_i = \chi^i \bar{a}.
\]

The multiplication in \( \Omega \) is entailed by one in \( \mathcal{A} \), the only non-obvious thing is to define the product of 1-forms. Here it is

\[
(A_1, A_2, \bar{A}_1, \bar{A}_2)(B_1, B_2, \bar{B}_1, \bar{B}_2) \equiv A_1 \bar{B}_1 + A_2 \bar{B}_2 + \bar{A}_1 B_1 + \bar{A}_2 B_2.
\]

Of course, the r.h.s. is viewed as an element of \( \Omega_2 \). The product of a 1-form and a 2-form is set to zero by definition. Now the coboundary operator \( d \) is given by

\[
df \equiv (T_1 f, T_2 f, \bar{T}_1 f, \bar{T}_2 f), f \in \Omega_0; \\
d(A_1, A_2, \bar{A}_1, \bar{A}_2) \equiv T_1 \bar{A}_1 + T_2 \bar{A}_2 + \bar{T}_1 A_1 + \bar{T}_2 A_2, \quad (A_1, A_2, \bar{A}_1, \bar{A}_2) \in \Omega_1; \\
dh = 0, \quad h \in \Omega_2.
\]

It maps \( \Omega_i \) to \( \Omega_{i+1} \) and it satisfies

\[
d^2 = 0, \quad d(AB) = (dA)B + (-1)^{\bar{A}} A(dB).
\]

There remains to clarify the issues of reality and cohomology. An involution \( \dagger \) is defined as follows

\[
f^\dagger = f^\dagger, \quad f \in \Omega_0, \quad h^\dagger = -h^\dagger, \quad h \in \Omega_2; \\
(A_i, \bar{A}_i)^\dagger = (\bar{A}_i^\dagger, -A_i^\dagger), \quad (A_i, \bar{A}_i) \in \Omega_1.
\]

The involution \( \dagger \) \((\dagger^2 = 1)\) preserves the linear combinations with real coefficients and the multiplication, and commutes with the coboundary operator \( d \):

\[
(af + bg)^\dagger = af^\dagger + bg^\dagger, \quad a, b \in \mathbb{R}, \quad f, g \in \Omega;
\]
\[(fg)^\dagger = f^\dagger g^\dagger, \quad f, g \in \Omega; \tag{15}\]
\[(df)^\dagger = df^\dagger, \quad f \in \Omega. \tag{16}\]

The real forms under the involution \(\dagger\) form the real Hamiltonian complex. Its cohomology contains only two nontrivial classes: a 0-form 1 and a 2-form \(i\bar{a}a\).

Now we are ready to quantize the infinitely dimensional algebra \(A\) with the goal of obtaining its (noncommutative) finite dimensional deformation. We start with the well-known quantization of the complex plane \(C^{2,1}\). The generators \(\bar{\chi}^i, \chi^i, \bar{a}, a\) become creation and annihilation operators on the Fock space whose commutation relations are given by the standard replacement

\[\{.,.\} \to \frac{1}{\hbar}[.,.]. \tag{17}\]

Here \(\hbar\) is a real parameter (we have absorbed the imaginary unit into the definition of the Poisson bracket) referred to as the "Planck constant". Explicitely

\[\chi^i, \bar{\chi}^j = \hbar \delta^{ij}, \quad [a, \bar{a}] = \hbar \tag{18}\]

and all remaining graded commutators vanish. The Fock space is built up as usual, applying the creation operators \(\bar{\chi}^i, \bar{a}\) on the vacuum \(|0\rangle\), which is in turn annihilated by the annihilation operators \(\chi, a\). The scalar product on the Fock space is fixed by the requirement that the barred generators are adjoint of the unbarred ones. We hope that using the same symbols for the classical and quantum generators will not confuse the reader; it should be fairly obvious from the context which usage we have in mind.

Now we perform the quantum symplectic reduction with the self-adjoint moment map \((\bar{\chi}^i\chi^i + \bar{a}a)\). First we restrict the Hilbert space only to the vectors \(\psi\) satisfying the constraint

\[(\bar{\chi}^i\chi^i + \bar{a}a - 1)\psi = 0. \tag{19}\]

Hence operators \(\hat{f}\) acting on this restricted space have to fulfil

\[[\hat{f}, \bar{\chi}^i\chi^i + \bar{a}a] = 0 \tag{20}\]

and they are to form our deformed version of \(A\).

The spectrum of the operator \((\bar{\chi}^i\chi^i + \bar{a}a - 1)\) in the Fock space is given by a sequence \(mh - 1\), where \(m\)'s are integers. In order to fulfil (19) for a
non-vanishing $\psi$, we observe that the inverse Planck constant $1/h$ must be an integer $N$. The constraint (19) then selects only $\psi$’s living in the eigenspace $H_N$ of the operator $(\bar{\chi}^i \chi^i + \bar{a}a - 1)$ with the eigenvalue 0. This subspace of the Fock space has the dimension $2N + 1$ and the algebra $A_N$ of operators $\hat{f}$ acting on it is $(2N + 1)^2$-dimensional.

When $N \to \infty$ (the dimension $(2N + 1)^2$ then also diverges) we have the Planck constant approaching 0 and, hence, the algebras $A_N$ tend to the classical limit $A$.

The Hilbert space $H_N$ is naturally graded. The even subspace $H_{eN}$ is created from the Fock vacuum by applying only the bosonic creation operators:

$$(\bar{\chi}^1)^{n_1}(\bar{\chi}^2)^{n_2}|0\rangle, \quad n_1 + n_2 = N,$$

while the odd one $H_{oN}$ by applying both bosonic and fermionic creation operators:

$$(\bar{\chi}^1)^{n_1}(\bar{\chi}^2)^{n_2}\bar{a}|0\rangle, \quad n_1 + n_2 = N - 1. \quad (22)$$

Correspondingly, the algebra of operators $A_N$ on $H_N$ consists of an even part $A_{eN}$ (operators respecting the grading) and an odd part (operators reversing the grading). The odd part can be itself written as a direct sum $A_{aN} \oplus A_{\bar{a}N}$. The two components in the sum are distinguished by their images: $A_{aN}H_N = H_{eN}$ while $A_{\bar{a}N}H_N = H_{oN}$. $A_{aN}$ is spaned by operators

$$(\bar{\chi}^1)^{n_1}(\bar{\chi}^2)^{n_2}(\chi^1)^{m_1}(\chi^2)^{m_2}a, \quad n_1 + n_2 = m_1 + m_2 + 1 = N, \quad (23)$$

$A_{\bar{a}N}$ by

$$(\bar{\chi}^1)^{n_1}(\bar{\chi}^2)^{n_2}\bar{a}(\chi^1)^{m_1}(\chi^2)^{m_2}, \quad n_1 + n_2 + 1 = m_1 + m_2 = N \quad (24)$$

and $\mathcal{A}_{eN}$ by

$$(\bar{\chi}^1)^{n_1}(\bar{\chi}^2)^{n_2}(\chi^1)^{m_1}(\chi^2)^{m_2}(\bar{a}a)^k, \quad n_1 + n_2 = m_1 + m_2 = N - k. \quad (25)$$

Here the graded involution $\hat{\dagger}$ in the noncommutative algebra $A_N$ is defined exactly as in (1).

Define a non-commutative Hamiltonian de Rham complex $\Omega_N$ of the fuzzy sphere $S^2$ as the graded associative algebra with unit

$$\Omega_N = \Omega_{0N} \oplus \Omega_{1N} \oplus \Omega_{2N},$$

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where
\[ \Omega_{0N} = \Omega_{2N} = \mathcal{A}_N \] (27)
and
\[ \Omega_{1N} = \mathcal{A}_{aN} \oplus \mathcal{A}_{aN} \oplus \mathcal{A}_{aN} \oplus \mathcal{A}_{aN}. \] (28)

The multiplication in \( \Omega_N \) with the standard properties with respect to the grading is entailed by one in \( \mathcal{A}_N \). The product of 1-forms is given by the same formula as in the graded commutative case
\[
(A_1, A_2, \bar{A}_1, \bar{A}_2)(B_1, B_2, \bar{B}_1, \bar{B}_2) \equiv A_1 B_1 + A_2 B_2 + \bar{A}_1 B_1 + \bar{A}_2 B_2. \] (29)

Of course, by definition, the r.h.s. is viewed as an element of \( \Omega_{2N} \). Here we note an important difference with the graded commutative case: the product \( AA \) of a 1-form \( A \) with itself automatically vanishes in the commutative case but may be a non-vanishing element of \( \Omega_{2N} \) in the deformed picture. The product of a 1-form and a 2-form is again set to zero by definition. Now the coboundary operator \( d \) is given by
\[
df \equiv (T_1 f, T_2 f, \bar{T}_1 f, \bar{T}_2 f), \quad f \in \Omega_{0N}; \] (30)
\[
d(A_1, A_2, \bar{A}_1, \bar{A}_2) \equiv T_1 \bar{A}_1 + T_2 \bar{A}_2 + \bar{T}_1 A_1 + \bar{T}_2 A_2, \quad (A_1, A_2, \bar{A}_1, \bar{A}_2) \in \Omega_{1N}; \] (31)
\[
dh = 0, \quad h \in \Omega_{2N}, \] (32)
where the action of \( T_i, \bar{T}_i \) is given by the noncommutative version of (8):
\[
T_i X \equiv N(t_i X - (-1)^X t_i), \quad \bar{T}_i X \equiv N(\bar{t}_i X - (-1)^\bar{X} \bar{t}_i), \quad X \in \mathcal{A}_N, \] (33)
where
\[
t_i = \chi^i a, \quad \bar{t}_i = \chi^i \bar{a}. \] (34)

\( d \) maps \( \Omega_{iN} \) to \( \Omega_{i+1,N} \) and it satisfies
\[
d^2 = 0, \quad d(AB) = (dA)B + (-1)^A A(dB). \] (35)

Using the graded involution \( \dagger \), we define the standard involution \( \dagger (\dagger^2 = 1) \) on the noncommutative complex \( \Omega_N \):
\[
f^\dagger = f^\dagger, \quad f \in \Omega_{0N}, \quad g^\dagger = -g^\dagger, \quad g \in \Omega_{2N}; \] (36)
\[
(A_i, \bar{A}_i)^\dagger = (\bar{A}_i^\dagger, -A_i^\dagger), \quad (A_i, \bar{A}_i) \in \Omega_{1N}. \] (37)
The coboundary map $d$ is compatible with the involution, however, due to noncommutativity, it is no longer true that the product of two real elements of $\Omega_N$ gives a real element. Thus we cannot define the real noncommutative Hamiltonian de Rham complex. For field theoretical applications this is not a drawback, nevertheless, because for the formulation of the field theories we shall not need the structure of the real subcomplex, but only the involution on the complex Hamiltonian de Rham complex.

5. We are now ready to compute the cohomology of the noncommutative complex $\Omega_N$.

**Theorem:**

i) Let $f \in \Omega_{0N}, \ df = 0$. Then $f$ is the unit element of $\Omega_{0N}$ (unit matrix acting on $H_N$) multiplied by some number.

ii) Let $A \equiv (A_1, A_2, A_1, A_2) \in \Omega_{1N}, \ dA = 0, \ A = A^\dagger$. Then $A = dg$ for some $g \in \Omega_{0N}, \ g = g^\dagger$.

iii) Let $F \in \Omega_{2N}$ (i.e. $dF$ automatically vanishes), $F = F^\dagger$. Then $F$ can be written as $F = p\text{Id} + dB$, where $B \in \Omega_{1N}, \ B = B^\dagger$ is some 1-form, $\text{Id}$ is the unit element in $A_{eN}$ and $p$ is an imaginary number. $\text{Id} \in \Omega_{2N}$ itself cannot be written as a coboundary of some 1-form.

Thus the theorem implies that $\text{Id}$ is the only nontrivial cohomology class in $H^2(\Omega_N)$ and $H^0(\Omega_N)$, and $H^1(\Omega_N)$ vanishes.

**Proof:**

i) One notices that the Hamiltonians $t_i, \bar{t}_i$ of the vector fields $T_i, \bar{T}_i$ generates the whole algebra $A_N$ and therefore also its subalgebra $A_{eN} = \Omega_{0N}$.

According to (33), the $T_i, \bar{T}_i$ act on an element $f \in \Omega_{0N}$ as commutators $N[t_i, f], N[\bar{t}_i, f]$, respectively. Thus vanishing of the commutators means that $f$ commutes with all matrices in $A_N$. Hence $f$ is a multiple of the unit matrix $\text{Id}$.

ii) We want to show that $A_i = [t_i, \Phi]; \bar{A}_i = [\bar{t}_i, \Phi]$ for some hermitian matrix $\Phi = \Phi^\dagger; \Phi \in A_{0N}$. The first step is to prove the following

**Lemma:** Any real 1-form $(A_i, \bar{A}_i)$ can be written in terms of two hermitian matrices $\Phi_1, \Phi_2$ which have zeros on their diagonals and two supertraceless diagonal hermitian matrices $\Delta, \bar{\Delta}$ as follows

$$A_1 = [t_1, \Phi_1 + \Delta + i\Delta]; \ A_2 = [t_2, \Phi_2 + \Delta + i\bar{\Delta}];$$
\[ \bar{A}_1 = [\bar{t}_1, \Phi_1 + \Delta - i\tilde{\Delta}]; \bar{A}_2 = [\bar{t}_2, \Phi_2 + \Delta - i\tilde{\Delta}]. \] (38)

It easy to prove lemma by giving an explicit formula how to find \( \Phi_i \) in terms of \( A_i \). Here it is in terms of the matrix elements

\[ (\Phi_1)_{ij} = \frac{1}{i-j}(\{t_1, A_1\} + \{\bar{t}_1, \bar{A}_1\})_{ij}, (\Phi_2)_{ij} = \frac{1}{j-i}(\{t_2, \bar{A}_2\} + \{\bar{t}_2, A_2\})_{ij}; i \neq j. \] (39)

The formula giving the \( \Delta \)-s in terms of \( A_i \) is somewhat cumbersome and we invite the interested reader to work it out as an exercise.

The proof of the part ii) of the theorem then finishes by noting that the condition \( dA = \{t_1, A_1\} + \{\bar{t}_1, A_1\} + \{t_2, \bar{A}_2\} + \{\bar{t}_2, A_2\} = 0 \) clearly entails \( \Phi_1 = \Phi_2 \) and one can also show that it gives \( \tilde{\Delta} = 0 \).

iii) We have to show that every supertraceless antihermitian matrix \( \Psi = -\Psi^\dagger \) can be written as \( \Psi = dA = \{t_1, A_1\} + \{\bar{t}_1, A_1\} + \{t_2, \bar{A}_2\} + \{\bar{t}_2, A_2\} \) for a certain real 1-form \( A = A^\dagger \). First of all we note that \( dA \) is always supertraceless; this explains why \( \Phi \) has to be supertraceless. Then we have to find for each supertraceless antihermitian \( \Phi \) a set of two hermitian matrices \( \Phi_1, \Phi_2 \) which have zeros on their diagonals and two supertraceless diagonal hermitian matrices \( \Delta, \tilde{\Delta} \). Remind that according to ii) those data encode unambiguously a real one-form \( A \). One finds \( \Delta = 0 \); to find \( \tilde{\Delta} \) is easy but the explicit formula is somewhat cumbersome and I do not list it here. There is slightly more work needed to identify \( \Phi_1 \) and \( \Phi_2 \), on the other hand the explicit formulas for the matrix elements of \( \Phi_1 \) and \( \Phi_2 \) are much nicer than those for \( \Delta \)-s. Here there are:

\[ (\Phi_1)_{ij} = \frac{1}{i-j}(\Psi)_{ij}; (\Phi_2)_{ij} = \frac{1}{j-i}(\Psi)_{ij}. \] (40)

Of course this solution is not unique for the ”gauge” transformed form \( A + d\Phi \) gives also a solution.

The only nontrivial cohomology class in \( H^2(\Omega_N) \) is therefore supertraceful and can be chosen to be an imaginary multiple of \( Id \). The theorem is proved.

6. We conclude by interpreting the result and sketching its possible application. The fact that a closed one-form can be written as an exterior derivative of a zero-form is often referred to by saying that the one-form is integrable. We have shown here that this integrability is not touched upon by the noncommutative deformation of the complex. This result was by no
means evident and one had to use different technical tools than the infinitesimal calculus in order to reveal the cohomological content of the deformed complex. An interesting application of the result may reside in the world of the string-theoretical target space duality. The sigma-models formulated on the noncommutative world-sheet become dualizable in the similar way than their commutative counterparts. The details of the story are currently in preparation.

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