ABSTRACT. In this paper we study the Grassmannian of submodules of a given dimension inside a finitely generated projective module $P$ for a finite dimensional algebra $\Lambda$ over an algebraically closed field. The orbit of such a submodule $C$ under the action of $\text{Aut}_\Lambda(P)$ on the Grassmannian encodes information on the degenerations of $P/C$ and has been considered by a number of authors. The goal of this article is to bound the geometry of two-dimensional orbit closures in terms of representation-theoretic data. Several examples are given to illustrate the interplay between the geometry of the projective surfaces which arise and the corresponding posets of degenerations.

1. Introduction

Let $\Lambda$ be a basic finite dimensional algebra over an algebraically closed field $k$. A fundamental problem in representation theory is the classification of finitely generated $\Lambda$-modules. An early result in this direction is the Jordan-Hölder Theorem, which groups together modules that have the same composition factors counting multiplicities. This is a very coarse classification, however, serving only as a basis for the study of finer groupings into collections of “similar” modules. Typically one starts with a rough subdivision of a given Jordan-Hölder class in terms of numerical invariants by, for instance, asking that certain of the simple composition factors hold prescribed positions (such as placement in the radical quotient of the considered modules). This leads to the study of partial orders on isomorphism classes of finitely generated $\Lambda$-modules $M$ and $N$, with $M \leq N$ signifying that $N$ results from $M$ by some form of simplification. For example, if we say $M \leq N$ if and only if $N$ is the semi-simplification of $M$, then we recover the Jordan Hölder class associated to $N$ by this partial order. There are different partial orders under consideration (see, for example, [4, 5], and also [12, Sect. 3] for a short overview).

In this paper, we will concentrate on the geometrically defined degeneration partial order. Interest in this partial order arose from work of Gabriel and Kac on the affine scheme $\text{Mod}_d(\Lambda)$ that parameterizes the left $\Lambda$-modules with fixed dimension $d$ [8, 9, 14, 15]. The reductive group $GL_d$ acts on $\text{Mod}_d(\Lambda)$ by conjugation, and the orbits under this action are in one-to-one correspondence with the isomorphism classes of $d$-dimensional $\Lambda$-modules. Suppose a left $\Lambda$-module $M$ corresponds to a point $x$ in the scheme $\text{Mod}_d(\Lambda)$. A degeneration of $M$ is any $\Lambda$-module $N$ corresponding to a point in the closure of the orbit of $x$ under $GL_d$. By setting $M \leq_{\text{deg}} N$ in this situation, we arrive at a partial order, which places increasingly simplified modules $N$ above $M$, the largest being the direct sum of the composition factors of $M$. This partial order was studied by Kraft, Riedtmann, Bongartz, Schofield, Skowronski, Zwara and many others (see for example [16, 17], [19, 20], [4, 5], [21, 22, 23]). One of the highlights is
a purely algebraic description of the degeneration order in terms of Riedtmann-Zwara exact sequences in [23].

Let \( J \) be the Jacobson radical of \( \Lambda \), and suppose \( T \) is a finitely generated semisimple \( \Lambda \)-module. In this paper we will consider \( \Lambda \)-modules \( M \) that have radical quotient \( M/JM \) isomorphic to \( T \). There is a \( \text{GL}_d \)-stable locally closed subscheme \( \text{Mod}_d^T \) of \( \text{Mod}_d(\Lambda) \) whose orbits are in bijection with the isomorphism classes of such modules \( M \).

In [3, 7, 11–13], Bongartz and the third author took an alternate geometric approach to studying degenerations \( M \leq \text{deg} N \). When \( M \) and \( N \) have the same radical quotient \( T \) this approach proceeds in the following way. Let \( P \) be a projective \( \Lambda \)-module with radical quotient \( T \). Then \( M \cong P/C \) for some \( \Lambda \)-submodule \( C \) of \( JP \). We consider \( C \) as a point in the projective scheme \( \text{Grass}_d^T \) of all submodules of \( JP \) that have codimension \( d \) in \( P \), and we let the group \( \text{Aut}_\Lambda(P) \) act canonically on \( \text{Grass}_d^T \). As above, orbits correspond bijectively to isomorphism classes of modules \( N \) with radical quotient \( T \). The relation \( M \leq \text{deg} N \) is equivalent to the existence of a point \( C' \) in the orbit closure \( \text{Aut}_\Lambda(P).C \) such that \( N \cong P/C' \). A significant advantage of this approach is that orbit closures become closed subsets of projective varieties rather than of affine ones.

Our primary goal in this paper is to bound the global geometry of \( \text{Aut}_\Lambda(P).C \) in terms of the representation theoretic data which specifies \( M = P/C \). In particular, we will study the following problem:

**Question 1.1.** Is the Euler characteristic of \( \text{Aut}_\Lambda(P).C \) bounded by a function of the dimension of \( \text{Aut}_\Lambda(P).C \), the field \( k \) and the dimension of \( C \) over \( k \)?

We answer Question 1.1 in the affirmative for simple \( T \) and orbits of dimension 2. In fact, using the classical theory of rational surfaces, we prove the following (see also Theorem 3.1):

**Theorem 1.2.** Suppose \( T \) is simple and \( \text{Aut}_\Lambda(P).C \) has dimension 2. Then there exist positive integers \( b \) and \( c \) depending only on \( k \) and \( \text{dim}_k(C) \), together with a collection of \( b \) relatively minimal smooth rational projective surfaces, such that the minimal desingularization of \( \text{Aut}_\Lambda(P).C \) can be obtained from one of the surfaces in this collection by performing at most \( c \) monoidal transformations.

**Corollary 1.3.** Under the hypotheses of Theorem 1.2, the Euler characteristic of \( \text{Aut}_\Lambda(P).C \) is bounded from above by a function of \( k \) and \( \text{dim}_k(C) \).

Without assuming that the orbit \( \text{Aut}_\Lambda(P).C \) is two-dimensional, we will in fact bound, in a similar way, the geometry of the closure of any two-dimensional affine plane contained in \( \text{Aut}_\Lambda(P).C \) (see Remark 3.3). We should note that if \( T \) is simple, the orbit \( \text{Aut}_\Lambda(P).C \) is an affine space (see [11] Prop. 2.9 and [12] Lem. 4.1).

In the final section, we give numerous examples in which we explicitly link the geometry of \( \text{Aut}_\Lambda(P).C \) to the top-stable degenerations of \( P/C \).

We now give a brief overview of existing work on the geometry of orbit closures.

In [3], Bongartz analyzed the singularities at minimal degenerations for modules over representation-finite hereditary algebras. In [2], Bender and Bongartz considered the case when \( \Lambda \) is the Kronecker algebra. They classified all minimal singularities up to smooth equivalence and showed that they are isolated Cohen-Macaulay.

In [22], Zwara proved that if \( X \) is an orbit closure in \( \text{Mod}_d(\Lambda) \) and \( y \in X \) is an element such that the orbit of \( y \) has codimension 1 in \( X \), then \( X \) is smooth at \( y \). In [25], he moreover showed that if \( \Lambda \) is the path algebra of a Dynkin quiver, then the orbit closure \( X \) is regular in codimension two.
Lemma 2.1.

2. Preliminaries: conventions and basic results

In this section, we set up our notation and summarize what we need regarding the classification of relatively minimal smooth rational projective surfaces. Let \( k \) be an algebraically closed field of arbitrary characteristic, and let \( \Lambda \) be a basic finite dimensional \( k \)-algebra with Jacobson radical \( J \). Without loss of generality, we assume that \( \Lambda = kQ/I \) for some finite quiver \( Q \) and some admissible ideal \( I \) of the path algebra \( kQ \). The quiver \( Q \) provides us with a distinguished set of primitive idempotents \( e_1, \ldots, e_n \) of \( \Lambda \), which are in bijective correspondence with the vertices of \( Q \). As is well-known, the quotient modules \( \Lambda e_i / Je_i \), \( 1 \leq i \leq n \), form a complete set of representatives for the isomorphism classes of simple \( \Lambda \)-modules.

We fix a simple \( \Lambda \)-module \( T \), corresponding to a primitive idempotent \( e = e_{i_0} \) for some \( i_0 \), together with its projective cover \( P = \Lambda e \). Let \( d \) be a positive integer with \( 1 < d < \dim_k(JP) \), and let \( d' = \dim_k(P) - d \). Denote the classical Grassmannian of \( d' \)-dimensional subspaces of the \( k \)-vector space \( JP \) by \( \mathcal{G}r(d', JP) \). We define

\[
\mathcal{G}rass^T_d = \{ C \in \mathcal{G}r(d', JP) \mid C \text{ is a } \Lambda \text{-submodule of } JP \},
\]

which is a closed subscheme of \( \mathcal{G}r(d', JP) \). We have an obvious surjection \( \phi \) from \( \mathcal{G}rass^T_d \) to the set of isomorphism classes of \( d \)-dimensional \( \Lambda \)-modules with radical quotient \( T \), where \( \phi(C) = [P/C] \). The fibers of \( \phi \) coincide with the orbits of the natural action of \( \text{Aut}_\Lambda(P) \) on \( \mathcal{G}rass^T_d \).

Suppose \( C, C' \in \mathcal{G}rass^T_d \). As we mentioned in the introduction, the partial order \( M \leq_{\text{deg}} N \) on \( \Lambda \)-modules of the form \( M = P/C \) and \( N = P/C' \), defined by requiring that \( C' \) be in the closure of the \( \text{Aut}_\Lambda(P) \)-orbit of \( C \), coincides with the degeneration order based on the \( \text{GL}_d \)-action on \( \text{Mod}^T_d \). In fact, in [7, Prop. C], it is shown that there is an inclusion-preserving bijection between the \( \text{Aut}_\Lambda(P) \)-stable subsets of \( \mathcal{G}rass^T_d \) and the \( \text{GL}_d \)-stable subsets of \( \text{Mod}^T_d \) which preserves and reflects openness, closures, connectedness, irreducibility, and types of singularities. Hence geometric results concerning orbit closures in \( \mathcal{G}rass^T_d \) can, to a large extent, be carried over to orbit closures in \( \text{Mod}^T_d \), and vice versa.

Since we assume \( T \) to be simple, [11, Prop. 2.9] shows the orbit \( \text{Aut}_\Lambda(P).C \) to be isomorphic to a full affine space \( \mathbb{A}^m_k \), where

\[
m = \dim_k \text{Hom}_\Lambda(P, JP/C) - \dim_k \text{Hom}_\Lambda(P/C, JP/C).
\]

More precisely, let \( \{ \omega_1, \ldots, \omega_m \} \) be a \( k \)-basis of \( eJe \) consisting of oriented cycles from \( e \) to \( e \), and \( \text{Stab}_{eJe}(C) \) the \( k \)-vector space consisting of the elements \( a \in eJe \) for which \( Ca \subseteq C \). Suppose \( \{ \omega_1, \ldots, \omega_m \} \) is a \( k \)-basis of \( eJe \) modulo \( \text{Stab}_{eJe}(C) \). The following is shown in Lemma 4.1 of [12]:

**Lemma 2.1.** Let \( \overline{\text{Aut}_\Lambda(P).C} \) be the closure of \( \text{Aut}_\Lambda(P).C \) in \( \mathcal{G}rass^T_d \) with the induced reduced structure. There is a morphism

\[
\Psi : \mathbb{A}^m_k \rightarrow \overline{\text{Aut}_\Lambda(P).C}
\]

which is an isomorphism from \( \mathbb{A}^m_k \) to the dense open subset \( \text{Aut}_\Lambda(P).C \) of the target. It sends the point with coordinates \( (t_1, \ldots, t_m) \) in \( \mathbb{A}^m_k \) to the element \( C \cdot (e + t_1 \omega_1 + \cdots + t_m \omega_m) \) of \( \text{Aut}_\Lambda(P).C \).

Orbit closures in both the affine and projective module schemes are always unirational, since both the general linear group and the automorphism group \( \text{Aut}_\Lambda(P) \) are connected rational, and the orbits are epimorphic images of these groups, respectively. There is no example where rationality has been found to fail, and in our present situation, rationality is actually guaranteed,
as it is in the more general case when $T$ has no simple summands of multiplicity $> 1$ (see [12, Theorem 5.1]).

**Corollary 2.2.** The orbit closure $\overline{\text{Aut}_\Lambda(P).C}$ in $\text{Grass}_d^T$ is a rational variety of dimension $m$.

Unless specified otherwise, we now assume that $m = 2$. Then $\text{Aut}_\Lambda(P).C$ is isomorphic to $\mathbb{A}^2_k$, and hence $\overline{\text{Aut}_\Lambda(P).C}$ is a rational projective surface. We use the following notation:

- $\overline{\text{Aut}_\Lambda(P).C}^\#$ denotes the normalization of $\overline{\text{Aut}_\Lambda(P).C}$, and
- $\overline{\text{Aut}_\Lambda(P).C}^\dagger$ denotes the minimal desingularization of $\overline{\text{Aut}_\Lambda(P).C}^\#$.

In general, the smooth surface $\overline{\text{Aut}_\Lambda(P).C}^\dagger$ over $k$ fails to be relatively minimal. Recall that a smooth irreducible projective surface $X$ over $k$ is said to be relatively minimal if every birational morphism from $X$ to another smooth projective surface is necessarily an isomorphism. The relatively minimal smooth rational projective surfaces are known to be, up to isomorphism, the surfaces on the following list:

- $\mathbb{P}^2_k$,
- $X_0 = \mathbb{P}^1_k \times \mathbb{P}^1_k$,
- $X_n$, $n \geq 2$ (see [10, Example V.5.8.2, Rem. V.5.8.4]). The $X_n$ are referred to as the Hirzebruch surfaces and are described in [10, Sect. V.2]. Each $X_n$ is a rational ruled projective surface defined by $X_n = \text{Proj}(\text{Sym}(E))$ for $E = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-n)$ over $\mathbb{P}^1_k$.

Here, $\text{Proj}(\text{Sym}(E))$, where $\text{Sym}(E)$ is the symmetric algebra of $E$. It follows from [10] Example V.5.7.1 and Exercise V.5.5] that each $X_n$ can be obtained from $X_0 = \mathbb{P}^1_k \times \mathbb{P}^1_k$ by a finite sequence of monoidal transformations, via the following recursion: Let $D_0 = \mathbb{P}^1_k \times \infty$ and $D'_0 = \infty \times \mathbb{P}^1_k$. For any nonnegative integer $i$, the following two steps (a) and (b) transform $X_i$ into $X_{i+1}$. Suppose that $X_i$, together with curves $D_i, D'_i \subset X_i$, has been determined.

(a) Blow up the crossing point of $D_i$ and $D'_i$. This results in a new exceptional curve $E_i$.

(b) Blow down the proper transform of $D_i$. This leads to new boundary curves $\widetilde{E}_i$ and $\widetilde{D}'_i$.

Let $D_{i+1} = \widetilde{E}_i$ and $D'_{i+1} = \widetilde{D}'_i$.

We thus obtain the following picture relating $\overline{\text{Aut}_\Lambda(P).C}$ to $\overline{\text{Aut}_\Lambda(P).C}^\#$ and $\overline{\text{Aut}_\Lambda(P).C}^\dagger$:

\[
\begin{array}{ccc}
\overline{\text{Aut}_\Lambda(P).C}^\dagger & \xrightarrow{\rho} & \overline{\text{Aut}_\Lambda(P).C}^\# \\
\text{X} & \xrightarrow{\text{Aut}_\Lambda(P).C} & \overline{\text{Aut}_\Lambda(P).C}
\end{array}
\]

Here all arrows stand for birational morphisms, and $\overline{\text{Aut}_\Lambda(P).C}^\# \to \overline{\text{Aut}_\Lambda(P).C}$ is finite. Moreover, $X$ is a relatively minimal smooth rational projective surface, and $\rho$ is a finite sequence of monoidal transformations.

3. **Orbit closures and rational surfaces**

The formulation of our theorem refers to the notation introduced in section 2. In particular, $T$ is a simple $\Lambda$-module with projective cover $P = \Lambda e$, and $C$ is a point in $\text{Grass}_d^T$ for some
fixed positive integer \(d\). Moreover, we assume that the orbit \(\text{Aut}_A(P)C\) of \(C\) in \(\text{Grass}^T_d\) is two-dimensional, and denote by \(\overline{\text{Aut}_A(P)C}\) its closure in \(\text{Grass}^T_d\), the closure being endowed with the induced reduced structure.

Recall that, by Lemma 2.1 there is an injective morphism

\[
(3.1) \quad \Psi : \mathbb{A}^2_k \to \overline{\text{Aut}_A(P)C}
\]

which maps \(\mathbb{A}^2_k\) onto the open dense subset \(\text{Aut}_A(P)C\) of \(\overline{\text{Aut}_A(P)C}\).

The following is our main result.

**Theorem 3.1.** For some natural number \(n_0\) which depends only on \(k\) and \(\dim_k(C)\), there exists a birational morphism

\[
\rho : \overline{\text{Aut}_A(P)C} \to X
\]

with the following properties: \(X\) is a relatively minimal smooth rational projective surface among \(\mathbb{P}^2_k\), \(X_n\), for \(0 \leq n \leq n_0\), \(n \neq 1\), and there is a bound \(c\), again depending solely on \(k\) and \(\dim_k(C)\), such that \(\rho\) blows down at most \(c\) irreducible curves.

For simplicity we regard \(k\) as fixed from now on, so that we will not have to address the dependence on \(k\) of various bounds. When we say that a real-valued function is bounded, we mean that it is bounded from above by some explicit function of \(\dim_k(C)\).

We prove Theorem 3.1 in several steps which are carried out in detail below. In Step 1 we bound the number of points at which the birational map \(\psi : \mathbb{P}^1_k \times \mathbb{P}^1_k - \to \overline{\text{Aut}_A(P)C}\) resulting from the morphism \(\Psi\) in (3.1) is not defined. At each point \(w\) where this rational map is undefined, we also bound the complexity of the map \(\psi\) in a neighborhood of \(w\). In Step 2 we show that for each such \(w\), the birational map \(\psi\) becomes a rational morphism near \(w\) after the blow-up of an ideal that contains a bounded power (i.e. a power with bounded exponent) of the maximal ideal of the local ring of \(w\). In Step 3 we consider the Grassmannian consisting all ideals of the local ring of \(w\) which contain a bounded power of the maximal ideal in this Grassmannian can be dominated by a blow-up of \(\mathbb{P}^1_k \times \mathbb{P}^1_k\) at any ideal in this Grassmannian can be dominated by a blow-up of \(\mathbb{P}^1_k \times \mathbb{P}^1_k\) which results from a bounded number of successive monoidal transformations. In Steps 4 and 5 we complete the proof using a theorem of Zariski about dominating proper birational morphisms between normal projective surfaces using a finite number of monoidal transformations.

**Step 1:** Write \(\mathbb{P}^1_k = \mathbb{A}^1_k \cup \{\infty\}\) and identify \(\mathbb{A}^2_k\) with \(\mathbb{A}^1_k \times \mathbb{A}^1_k\). This gives an embedding of \(\mathbb{A}^2_k\) into \(\mathbb{P}^1_k \times \mathbb{P}^1_k\). The morphism \(\Psi\) from (3.1) defines a birational map

\[
\psi : \mathbb{P}^1_k \times \mathbb{P}^1_k - \to \overline{\text{Aut}_A(P)C}.
\]

Let \(U\) be the domain of definition of \(\psi\). Then the set \(D = \mathbb{P}^1_k \times \mathbb{P}^1_k - U\) of fundamental points of \(\psi\) is a finite set of closed points, and there is an effective bound, depending only on \(\dim_k(C)\), for the cardinality of this set. Moreover, there is a projective embedding of \(\text{Grass}^T_d\) into a projective space \(\mathbb{P}^m_k\) over \(k\) satisfying the following conditions for each \(w \in D\). There are local parameters \(t_1\) and \(t_2\) at \(w\) such that the local ring of \(w\) on \(\mathbb{P}^1_k \times \mathbb{P}^1_k\) is isomorphic to the localization \(A_w\) of \(k[t_1,t_2]\) at the maximal ideal generated by \(t_1\) and \(t_2\). The restriction of \(\psi\) to the open subset \(\text{Spec}(A_w) - w\) of \(\text{Spec}(A_w)\) is a morphism to \(\mathbb{P}^m_k\) which is defined, in terms of homogeneous coordinates, by \((q_0(t_1,t_2) : \cdots : q_h(t_1,t_2))\) where the \(q_i(t_1,t_2)\) are polynomials in \(k[t_1,t_2]\) of
bounded degree and the $A_w$-ideal $I$ generated by the $g_i(t_1, t_2)$ contains a bounded power of the maximal ideal of $A_w$. (Note: We do not claim that $h$ is bounded.)

**Proof of Step 1:** Let $m = \dim_k(P)$ and fix a basis $\{b_i\}_{i=1}^m$ for $P$ over $k$. Relative to this basis, $C$ is spanned by a set of $d' = \dim_k(C)$ row vectors in $k^m$. Recall that the map $\Psi : A_k^2 = A_k^1 \times A_k^1 \to \text{Aut}_A(P) \cdot C$ of (3.1) was constructed in the following way. If $z, u \in k$ define a point $(z, u) \in A_k^2$, then

$$\Psi(z, u) = C \cdot (e + z \omega_1 + u \omega_2)$$

for some fixed elements $\omega_1$ and $\omega_2$ of $eJk$. This implies that there is a set of $d'$ row vectors $v_1, \ldots, v_{d'}$ of size $m$ whose entries are polynomials which are at most linear in the indeterminates $z$ and $u$ such that, if one specializes $z$ and $u$ to elements of $k$, the vectors $v_1, \ldots, v_{d'}$ specialize to independent vectors which span the subspace of $P$ corresponding to $\Psi(z, u)$.

It is well-known that the set $D$ of fundamental points of $\psi$ is closed in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ and has codimension 2 (see [10, Lemma V.5.1]). In other words, the morphism $\Psi$ of (3.1) has an extension which is defined off a finite set of closed points. We need to make this extension more explicit in order to bound the number of fundamental points of $\psi$.

Let us first constructively extend $\Psi$ to a large open Zariski neighborhood of the generic point of $\mathbb{P}^r \times \mathbb{P}^s$. The local ring of this generic point may be identified with the localization $R$ of the polynomial ring $k[z^{-1}, u]$ at the prime ideal $k[z^{-1}, u] \cdot z^{-1}$. We will apply the following lemma to the situation where $r = z^{-1}, s = u$, and $V$ is the matrix whose $(i, j)$ entry $v_{i,j}$ is the $j$-th component of the vector $v_i$.

**Lemma 3.2.** Suppose $k[r,s]$ is a polynomial ring in two indeterminates $r$ and $s$. Let $F = \text{Frac}(k[r,s]) = k(r,s)$, and define $R$ to be the localization of $k[r,s]$ by the prime ideal $k[r,s] \cdot r$. Suppose $V = (v_{i,j})_{1 \leq i \leq d', 1 \leq j \leq m}$ is a matrix of elements of $F$ whose entries can each be written in the form $f(r,s)/f(r,s)$ such that $f(r,s), \tilde{f}(r,s) \in k[r,s]$ have bounded degrees. Suppose furthermore that the rows of $V$ are linearly independent over $F$, so that, over $F$, they span a $d'$-dimensional subspace $W$ of $F^m$. Viewing $R^m$ as canonically embedded into $F^m$, we obtain:

i. $R^m \cap W$ is a free $R$-module direct summand of $R^m$ that has rank $d' = \dim_F W$. In particular, $R^m \cap W$ contains a basis for $W$ over $F$.

ii. There is a $d' \times d'$ matrix $Y = (y_{i,j})$ over $F$ with the following properties: The rows of $Y \cdot V$ form an $R$-basis for $R^m \cap W$. Each $y_{i,j}$ can be written as a ratio $g(r,s)/\tilde{g}(r,s)$ in which $g(r,s), \tilde{g}(r,s) \in k[r,s]$ are polynomials of bounded degrees, the bound depending only on $d' = \dim_k(C)$ and the given bound on the degrees of the numerators and denominators of the $v_{i,j}$.

**Proof.** The ring $R$ is a discrete valuation ring, and $R^m \cap W$ is a pure $R$-submodule of $R^m$ in the sense that $R^m/(R^m \cap W)$ is $R$-torsion free. Thus $R^m/(R^m \cap W)$ is free over $R$, meaning that $R^m \cap W$ is a free direct summand of $R^m$. In particular, $R^m \cap W$ is free of rank $d' = \dim_F W$ as an $R$-module.

Let $\text{ord} : F \to \mathbb{Z}$ be the discrete valuation on $F = k(r,s)$ which is associated to the prime ideal $k[r,s] \cdot r$ of $k[r,s]$. Our assumptions on the $v_{i,j}$ imply that there is an integer $l \geq 0$ such that $v_{i,j} \neq 0$ implies $|\text{ord}(v_{i,j})| \leq l$.

We prove the rest of Lemma 3.2 using induction on $d'$. If $d' = 1$, then $V$ has a single row, and $W$ is the $F$-space spanned by this row. If $t = \min_{j=1}^m \text{ord}(v_{1,j})$, then $R^m \cap W$ is the free $R$-module on the vector $r^{-t}(v_{1,1}, \ldots, v_{1,m})$. So we can take the $1 \times 1$ matrix $Y$ to be $(r^{-t})$. Since some $v_{1,j}$ is non-zero, we have $|t| \leq l$. 


Suppose now that \( d' > 1 \). Let \((i, j)\) be a pair of indices with \(1 \leq i \leq d'\) and \(1 \leq j \leq m\) such that \( q = \text{ord}(v_{i,j})\) is minimal among the orders of the non-zero entries of \( V \). As above, \(|q| \leq l\).

By multiplying \( V \) by a \( d' \times d' \) permutation matrix on the left, we may assume \( i = 1 \). The vector \( r^{-q}(v_{1,1}, \ldots, v_{1,m}) \) lies in \( R^m \cap W \) and \( r^{-q}v_{1,j} \) is a unit of \( R \). We multiply \( V \) by the diagonal matrix \( Y_0 = \text{diag}(r^{-q}, 1, \ldots, 1) \) to make the first row of \( Y_0V \) equal to \( r^{-q}(v_{1,1}, \ldots, v_{1,m}) \).

We now subtract \( v_{1,j}/(r^{-q}v_{1,j}) \) times the first row of \( Y_0V \) from the \( a^{th}\) row of \( Y_0V \) for \( 2 \leq a \leq d' \) to arrive at a matrix \( Y_1Y_0V \) which has first row \( r^{-q}(v_{1,1}, \ldots, v_{1,m}) \), zero entries in the \( j\)-th column except for the unit \( r^{-q}v_{1,j} \) such that the \( F\)-span of the rows of \( Y_1Y_0V \) equals \( W \). Let \( W' \) be the \( F\)-span of rows \( 2, \ldots, d' \) of \( Y_1Y_0V \). Note that the \( j\)-th component of every element of \( W' \) is zero. We claim that

\[
(3.2) \quad R^m \cap W = (R \cdot r^{-q}(v_{1,1}, \ldots, v_{1,m})) \oplus (R^m \cap W').
\]

It is clear that the right hand side is contained in the left hand side, since \( W \) is the \( F\)-span of all the rows of \( Y_1Y_0V \). For the opposite containment, suppose that some \( F\)-linear combination of the rows of \( Y_1Y_0V \) lies in \( R^m \). If \( \alpha \in F \) is the coefficient of the first row in this linear combination, then \( \alpha \cdot r^{-q}v_{1,j} \) is the \( j\)-th component of the linear combination, and this must be in \( R \). Since \( r^{-q}v_{1,j} \) is a unit in \( R \), this forces \( \alpha \in R \). So the multiple of the first row in the linear combination lies in \( R \cdot r^{-q}(v_{1,1}, \ldots, v_{1,m}) \), and on subtracting this off we get an element of \( R^m \cap W' \). This proves (3.2).

Because of our assumptions about the degrees of the numerators and denominators of the non-zero \( v_{i,j} \), the non-zero entries of \( Y_1Y_0 \) and of \( Y_1Y_0V \) have numerators and denominators of bounded degree. We now apply our induction hypotheses to the \((d' - 1) \times m \) matrix \( V' \) whose rows and columns are those of \( Y_1Y_0V \) when we omit the first row. This leads to a \((d' - 1) \times (d' - 1) \) matrix \( Y' \) such that the rows of \( Y'V' \) form a basis for \( R^m \cap W' \). We define \( Y_2 \) to be the \( d' \times d' \) block matrix with a one-by-one block equal to 1 in the upper left corner followed by a \((d' - 1) \times (d' - 1) \) block given by \( Y' \). Now \( Y_2Y_1Y_0V \) has first row \( r^{-q}(v_{1,1}, \ldots, v_{1,m}) \) and the remaining rows span the free \( R \)-module \( R^m \cap W' \). Because of (3.2), we may assume \( Y = Y_2Y_1Y_0 \). The degrees of the numerators and denominators of the non-zero entries of \( Y \) are bounded since this is true for the non-zero entries of \( Y_2, Y_1 \) and \( Y_0 \).

We now plug into Lemma 3.2 the substitutions mentioned in the paragraph just before the statement of Lemma 3.2. In particular, \( r = z^{-1} \) and \( s = u \), and the entries of \( rV \) are polynomials in \( r \) and \( u \). We write the rows of \( V \) as \( v_1, \ldots, v_d \). Given \( Y \) as guaranteed by the lemma, we write \( Y \cdot V = (q_{i,j})_{1 \leq i \leq d', 1 \leq j \leq m} \), where \( q_{i,j} = q_{i,j}(z^{-1}, u) \) is a ratio of polynomials of bounded degrees in the indeterminates \( r = z^{-1} \) and \( u \). The fact that \( q_{i,j} \) lies in the discrete valuation ring \( R = k[z^{-1}, u][z^{-1}] \) implies that \( z^{-1} \) does not divide the denominator of any non-zero \( q_{i,j} \) when \( q_{i,j} \) is written as a quotient of coprime polynomials in \( k[z^{-1}, u] \). By construction, the rows \( q_i = (q_{i,1}, \ldots, q_{i,m}) \) of \( Y \cdot V \) span the \( R \)-module \( R^m \cap W \). We will need the following additional fact. Let \( Y \) be a least common multiple in \( k[z^{-1}, u] \) of the denominators of the entries of \( Y \), so that \( y \) is well defined up to multiplication by an element of \( k^* \). Let \( y' \) be the quotient of \( y \) by the highest power of \( z^{-1} \) which divides \( y \). Due to \( q_{i,j} \in R \), we conclude that the entries of \( y'Y \cdot V \) are polynomials in \( k[z^{-1}, u] \). Furthermore, since the entries of \( Y \) have numerators and denominators of bounded degree, the degree of \( y' \) is bounded as well.

We have already shown that \( R^m \cap W \) is a free \( R \)-module summand of \( R^m \). This implies that the image of \( R^m \cap W \) in \((R/Rz^{-1})^m \) has dimension \( d' \) over the field \( R/Rz^{-1} \cong k(u) \). Therefore, we can use the rows of \( Y \cdot V \) to define a map from a non-empty open subset of \( \text{Spec}(k[z^{-1}, u]) \) to the Grassmannian \( \text{Grass}^r_{R} \) which contains a dense open subset of the affine
line $\infty \times \mathbb{A}^1_k$ defined by setting $z^{-1}$ equal to 0. Since $Y$ is generically invertible, this map extends the morphism $\Psi : \mathbb{A}^1_k \times \mathbb{A}^1_k \to \text{Grass}^T_d$. To see at which of the points in $\infty \times \mathbb{A}^1_k$ this extension fails to be defined, let $(Y \cdot V)$ be the $d' \times m$ matrix with entries in $R/Rz^{-1} = k(u)$ which results from reducing the entries of $Y \cdot V$ modulo $Rz^{-1}$. Since the entries of $Y \cdot V$ are ratios of polynomials of bounded degree in $z^{-1}$ and $u$ and, when written in terms of coprime numerators and denominators, none of these ratios has a denominator divided by $z^{-1}$, we conclude that the entries of $(Y \cdot V)$ are ratios of polynomials in $k[u]$ of bounded degrees. The rows of $(Y \cdot V)$ are linearly independent over $k(u)$, since the image of $R^m \cap W$ in $(R/Rz^{-1})^m = k(u)^m$ has dimension $d'$ over $k(u)$. We showed above that there is a polynomial $y' \in k[z^{-1}, u]$ of bounded degree which is not divisible by $z^{-1}$ such that $y'Y \cdot V$ has all its entries in $k[z^{-1}, u]$. It follows that if $y' \in k[u]$ is the reduction of $y'$ mod $Rz^{-1}$, then $\overline{y'} \neq 0$ and the denominator of every element of $(Y \cdot V)$ divides $\overline{y'}$. Thus the rows of $(Y \cdot V)$ can be specialized to every point of $\infty \times \mathbb{A}^1_k$ which is not a zero of the polynomial $\overline{y'} \in k[u]$, and the number of such points is bounded. We need to show that these rows are independent off a bounded set of points of $\infty \times \mathbb{A}^1_k$. This is so because there is a $d' \times d'$ minor of $(Y \cdot V)$ whose determinant is not $0$ in $k(u)$. This determinant is also a ratio of polynomials in $k[u]$ of bounded degree, so the number of zeros and poles of the determinant is bounded. So we have now bounded the number of fundamental points of $\infty \times \mathbb{A}^1_k$ of the rational map

$$
\psi : \mathbb{P}^1_k \times \mathbb{P}^1_k \to -\text{Aut}_{\Lambda}(P).C \hookrightarrow \text{Grass}^T_d.
$$

One can similarly bound the number of points of $\mathbb{A}^1_k \times \infty$ which lie outside the domain of definition of $\psi$. Since there is only one other point $\infty \times \infty$ where the rational map might be undefined, this effectively bounds the total number of fundamental points of $\psi$.

We now show the remaining claims of Step 1, concerning the restriction of $\psi$ to $\text{Spec}(A_w) - \psi$. Where $A_w$ is the local ring of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at a fundamental point $w$ of $\psi$. To simplify notation, we set $A = A_w$.

Set $h = (m \choose d') - 1 = (m \choose d) - 1$. We will find that the canonical embedding

$$
\text{Grass}^T_d \hookrightarrow \mathcal{G}r(d', k^m) \to \mathbb{P} \left( \Lambda^{d'}(k^m) \right) = \mathbb{P}^h_k,
$$

where $\iota$ sends any subspace $Z \subset k^m$ of dimension $d'$ to the point in the projective space defined by the $d'^{th}$ exterior power of a basis for $Z$, satisfies the requirements spelled out in Step 1. Indeed, $\iota$ is well-known to be an injection with closed image, and the variety structure on $\mathcal{G}r(d', k^m)$ is defined so as to make $\iota$ an isomorphism from $\mathcal{G}r(d', k^m)$ onto a closed subvariety of the projective space $\mathbb{P} \left( \Lambda^{d'}(k^m) \right)$, the latter endowed with its induced reduced structure. Consequently, the restriction to $\text{Grass}^T_d$ is a closed immersion as well. We keep this immersion fixed in the sequel and regard $\psi$ as a rational map from $\mathbb{P}^1_k \times \mathbb{P}^1_k$ to $\mathbb{P}^h_k$.

Let $w$ belong to the set $D$ of fundamental points of $\psi$, and let $A = A_w$ be the local ring of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at $w$. Let $t_1, t_2$ be local parameters at $w$. Thus $A$ is the localization of $k[t_1, t_2]$ at the maximal ideal generated by $t_1$ and $t_2$. We know that $D \subset (\infty \times \mathbb{P}^1_k) \cup (\mathbb{P}^1_k \times \infty)$ consists of a bounded number of closed points. Suppose $w \in \infty \times \mathbb{P}^1_k$. Above, we constructed an extension, $\Psi : (\mathbb{P}^1_k \times \mathbb{P}^1_k - D) \to \mathbb{P}^h_k$, of the original morphism $\Psi$ on $\mathbb{A}^1_k \times \mathbb{A}^1_k$ that underlies $\psi$. In particular, this construction shows that the morphism from $\text{Spec}(A) - D$ to $\mathbb{P}^h_k$ induced by $\Psi$ has the form

$$(t_1, t_2) \to (g_0(t_1, t_2) : \cdots : g_h(t_1, t_2)) = (g_0 : \cdots : g_h),$$
where \( g_i(t_1, t_2) \in k(t_1, t_2) \) are quotients of polynomials of bounded degrees in \( t_1, t_2 \). In fact, we can bound the denominators of the \( g_i(t_1, t_2) \) by writing the entries of the matrix \( Y \) as ratios of polynomials in \( t_1 \) and \( t_2 \) and by bounding the denominators of the entries of \( Y \). Since the number of entries of \( Y \) is \( (d')^2 \), we conclude that there is a non-zero polynomial \( p = p(t_1, t_2) \in k[t_1, t_2] \) of bounded degree such that \( p(t_1, t_2)g_i(t_1, t_2) \in k[t_1, t_2] \) for all \( i \).

The pullback of \( \mathcal{O}_{\mathbb{P}^d_k}(1) \) to \( \text{Spec}(A) - w \) is a line bundle \( \mathcal{L} \) on \( \text{Spec}(A) - w \). Since \( w \) has codimension 2 in the regular scheme \( \text{Spec}(A) \), the Weil divisor class groups of \( \text{Spec}(A) - w \) and \( \text{Spec}(A) \) are the same, and these are trivial because \( A \) is a regular local ring. Hence \( \mathcal{L} \) is trivial, and we can identify \( \mathcal{L} \) with the structure sheaf \( \mathcal{O} \) of \( \text{Spec}(A) - w \). The pullbacks of the coordinate global sections of \( \mathcal{O}_{\mathbb{P}^d_k}(1) \) define elements \( q_0, \ldots, q_h \) of \( \Gamma(\text{Spec}(A) - w, \mathcal{O}) = \Gamma(\text{Spec}(A), \mathcal{O}) = A \) which generate \( \mathcal{O} \) at every point of \( \text{Spec}(A) - w \). Thus \( q_0, \ldots, q_h \) are elements of \( A \) and the \( A \)-ideal \( I \) generated by \( q_0, \ldots, q_h \) has localization \( A_P \) at every prime ideal \( P \) of \( A \) different from the maximal ideal corresponding to \( w \).

We conclude that the maps from \( \text{Spec}(A) - D \) to \( \mathbb{P}^d_k \) which are defined by

\[
(t_1, t_2) \rightarrow (q_0 : \cdots : q_h)
\]

and

\[
(t_1, t_2) \rightarrow (g_0 : \cdots : g_h)
\]

agree. We know that the \( g_i \) are ratios of polynomials in \( k[t_1, t_2] \) of bounded degree, \( pg_i \in k[t_1, t_2] \) for a non-zero \( p = p(t_1, t_2) \in k[t_1, t_2] \) of bounded degree, and that the \( q_i \) are elements of \( A \) with the property that no irreducible element of \( A \) divides all of the \( g_i \). Here \( A \) is a UFD with irreducibles equal to the irreducibles in \( k[t_1, t_2] \) which have zero constant term, and every irreducible in \( k[t_1, t_2] \) with non-zero constant term is a unit in \( A \).

By multiplying all of the \( q_i \) by a suitable unit in \( A \), we can assume that all the \( q_i \) are elements of \( k[t_1, t_2] \). We can furthermore write

\[
(g_0, \ldots, g_h) = p^{-1} \cdot (\ell_0, \ldots, \ell_h)
\]

where \( p, \ell_0, \ldots, \ell_h \) in \( k[t_1, t_2] \) have bounded degrees. We conclude that there must be a non-zero element \( H \) of \( \text{Frac}(A) = k(t_1, t_2) \) such that

\[
(3.4) \quad H \cdot (q_0, \ldots, q_h) = (g_0, \ldots, g_h) = p^{-1} \cdot (\ell_0, \ldots, \ell_h)
\]

as tuples of elements of \( \text{Frac}(A) \).

Let \( \Omega \) be the finite set of irreducible elements \( \pi \) of \( k[t_1, t_2] \) which divide either \( p \) or one of \( \ell_0, \ldots, \ell_h \). If \( \pi \) is an irreducible of \( k[t_1, t_2] \) which is not in \( \Omega \), then \( p^{-1} \ell_j = Hq_j \) has valuation zero at the discrete valuation \( \text{ord}_\pi \) of \( k[t_1, t_2] \) associated to \( \pi \in k[t_1, t_2] \). Hence

\[
(3.5) \quad -\text{ord}_\pi(H) = \text{ord}_\pi(q_0) = \cdots = \text{ord}_\pi(q_h) \quad \text{if} \quad \pi \not\in \Omega.
\]

If \( \pi \) has non-zero constant term, then it is a unit in \( A \), and we can multiply each of the \( q_i \) by \( \pi^{\text{ord}_\pi(H)} \) to be able to assume without loss of generality that \( \text{ord}_\pi(q_i) = 0 \). If \( \pi \) has zero constant term, then \( \pi \) defines an irreducible of \( A \), and we know that no irreducible element of \( A \) divides every one of \( q_0, \ldots, q_h \in A \). So for \( \pi \) with zero constant term such that \( \pi \not\in \Omega \) we conclude that \( \text{ord}_\pi(q_j) = 0 \) for some \( j \), so (3.5) shows \( \text{ord}_\pi(q_i) = 0 \) for all \( i \). We conclude that without loss of generality, we can assume that the only irreducible elements \( \pi \) of \( k[t_1, t_2] \) which occur in the factorizations of \( H \) or in one of the \( q_j \) are \( \pi \) lying in \( \Omega \).

Suppose now that \( \pi \in \Omega \). Then \( \text{ord}_\pi(p) \) and \( \text{ord}_\pi(\ell_j) \) are bounded for all \( j \). We conclude from (3.4) that \( \text{ord}_\pi(Hq_j) \) is bounded independently of \( j \), so it follows that there is a bound on \( |\text{ord}_\pi(q_j) - \text{ord}_\pi(q_i)| \) for all \( i \) and \( j \). Suppose now that \( \pi \in \Omega \) has no constant term, so that
\( \pi \) defines an irreducible in \( A \). Then as above, we conclude that \( \ord_{\pi}(q_j) \geq 0 \) with equality for at least one \( j \). It follows that we have an effective upper bound on \( \ord_{\pi}(q_i) \) for all \( i \) in this case. Suppose now that \( \pi \in \Omega \) has non-zero constant term, so that it is a unit in \( A \). Then since \( |\ord_{\pi}(q_j) - \ord_{\pi}(q_i)| \) is bounded, we can multiply all of the \( q_j \) by a suitable power of \( \pi \in A^* \) to be able to assume that \( \ord_{\pi}(q_j) \geq 0 \) for all \( j \) and that we have an upper bound on \( \ord_{\pi}(q_j) \) for all \( j \).

We conclude that without loss of generality, we may assume that all of the \( q_j \) are in \( k[t_1, t_2] \), that the only irreducibles in \( k[t_1, t_2] \) up to associates which divide any of the \( q_j \) lie in the finite set \( \Omega \), and that the powers to which these irreducibles divide the \( q_j \) are bounded. Since the degrees of the elements of \( \Omega \) are bounded, this implies that all the \( q_j \) are now polynomials of bounded degree. We know that there is no irreducible in \( k[t_1, t_2] \) which has zero constant term and which divides all of the \( q_j \). Thus no irreducible in \( A \) divides all of the \( q_j \).

Let \( I \) be the ideal of \( A \) generated by \( q_0, \ldots, q_h \). We will show that \( A/I \) has finite bounded length, where the only simple module for the local ring \( A/I \) is its residue field \( k \). Let \( b(q_i) \) be the sum of the exponents of the irreducibles in \( A \) appearing in the factorization of \( q_i \), where \( b(q_i) = 0 \) if \( q_i \) is a unit. We will use induction on \( \beta_0 = \min(b(q_0), \ldots, b(q_h)) \), where \( \beta_0 \) is bounded. If \( \beta_0 = 0 \) then some \( q_i \) is a unit, and \( A/I = \{0\} \). Suppose now that \( \beta_0 > 0 \). After renumbering the \( q_i \), we can suppose that \( \beta_0 = b(q_0) \). Let \( \pi \in k[t_1, t_2] \) be an irreducible in \( A \) which divides \( q_0 \).

Let \( I' \) be the ideal generated by \( q_0, \pi, q_1, \ldots, q_h \). We have an exact sequence

\[
0 \to I'/I \to A/I \to A/I' \to 0
\]

Since \( \pi \) is an irreducible in \( A \) which divides \( q_0 \), there must be some \( j > 0 \) such that \( \pi \) does not divide \( q_j \). Thus we get a surjection

\[
A/(A\pi + Aq_j) \to I'/I \to 0
\]

in which \( \delta([\alpha]) = \alpha q_0/\pi \mod I \). Since \( \pi \in k[t_1, t_2] \) divides \( q_0 \), \( \pi \) has bounded degree, as does \( q_j \).

The length of \( A/(A\pi + Aq_j) \) is the local intersection number of the divisors associated to \( \pi \) and to \( q_j \), and this length is a bounded function of the degrees of \( \pi \) and \( q_j \). Since the length of \( A/I' \) is finite and bounded by induction, we conclude from (3.6) that the length of \( A/I \) is finite and bounded.

We can now filter \( A/I \) by the images of powers of the maximal ideal \( m_A \) of \( A \). By Nakayama’s Lemma, this filtration is strictly decreasing until it reaches \( \{0\} \). Thus \( I \) contains a bounded power of \( m_A \), and we are done with Step 1. \( \square \)

**Step 2:** As in Step 1, let \( w \) be an element of the finite set \( D \) of closed points of \( \mathbb{P}_k^1 \times \mathbb{P}_k^1 \) at which the rational map \( \psi: \mathbb{P}_k^1 \times \mathbb{P}_k^1 \to \mathbb{P}_k^h \) is not defined. Define \( A_w = \mathcal{O}_{\mathbb{P}_k^1 \times \mathbb{P}_k^1, w} \). By Step 1, the restriction of \( \psi \) to \( \text{Spec}(A_w) - \{w\} \) is defined by polynomials \( q_0 = q_0(t_1, t_2), \ldots, q_h = q_h(t_1, t_2) \) of bounded degree in the polynomial ring \( k[t_1, t_2] \) associated to a pair of local parameters \( t_1, t_2 \) at \( w \), where \( A_w = k[t_1, t_2]_{(t_1, t_2)} \). Let \( I_w \) be the ideal of \( A_w \) generated by the \( q_j \), so that \( I_w \) contains a bounded power of the maximal ideal of \( A_w \) by Step 1. Let \( D_w \) be the coherent sheaf of ideals on \( \mathbb{P}_k^1 \times \mathbb{P}_k^1 \) which has stalk \( \mathcal{O}_{\mathbb{P}_k^1 \times \mathbb{P}_k^1, z} \) at each point \( z \) not in \( D \), and whose stalk at \( w \) is \( I_w \). Define \( B \) to be the blow-up of \( \mathbb{P}_k^1 \times \mathbb{P}_k^1 \) at \( D_w \). Then there is a canonical projective birational morphism \( \theta: B \to \mathbb{P}_k^1 \times \mathbb{P}_k^1 \) which induces an isomorphism on the complement of \( D_w \). For \( w \in D \), the inverse image \( \theta^{-1}(w) \) is a connected one dimensional scheme. Finally, there is a morphism \( \tilde{\psi}: B \to \mathbb{P}_k^h \) which resolves the birational map \( \psi: \mathbb{P}_k^1 \times \mathbb{P}_k^1 \to \mathbb{P}_k^h \) in the sense
that $\tilde{\psi}(b) = \psi(\theta(b))$ if $\theta(b) \notin D$ and $\tilde{\psi}(B)$ is a closed subset of $\mathbb{P}^h_k$ which coincides with the orbit closure $\text{Aut}_A(P) \cdot \mathcal{C}$ when we give each of these sets the reduced induced structure.

**Figure 1.** Illustration of Step 2.

\[
\begin{array}{ccc}
\mathbb{P}_1^1 \times \mathbb{P}_1^1 & \overset{\psi}{\longrightarrow} & \text{Aut}_A(P) \cdot \mathcal{C} \\
\theta \downarrow & & \downarrow \\
\mathbb{P}_h^1 & \longrightarrow & \mathbb{P}_h^k
\end{array}
\]

**Proof of Step 2:** Let $U = (\mathbb{P}_1^1 \times \mathbb{P}_1^1) - D$, so that $\psi$ is defined on $U$ and $A_1^1 \times A_1^1$ is an open dense subset of $U$. If $z \in \psi(U)$ and $V$ is an open subset of $\mathbb{P}_k^h$ containing $z$, then $\psi^{-1}(V)$ is an open neighborhood of $z$ in $U$. Then $\psi^{-1}(V) \cap (A_1^1 \times A_1^1) \neq \emptyset$ and $V$ contains a point of the orbit $\psi(A_1^1 \times A_1^1) = \text{Aut}_A(P) \cdot \mathcal{C}$. Thus $\psi(U)$ is contained in the orbit closure $\overline{\text{Aut}_A(P) \cdot \mathcal{C}}$, so the closure $\overline{\psi(U)}$ of $\psi(U)$ equals $\overline{\text{Aut}_A(P) \cdot \mathcal{C}}$.

Let $G \subset U \times \mathbb{P}_h^k$ be the graph of $\psi$ on $U$. Here $U \times \mathbb{P}_h^k$ is an open subset of $(\mathbb{P}_1^1 \times \mathbb{P}_1^1) \times \mathbb{P}_h^k$. Then $B = \text{Proj}(\oplus_{n=0}^{\infty} F^n)$ is the closure of $G$ in $(\mathbb{P}_1^1 \times \mathbb{P}_1^1) \times \mathbb{P}_h^k$.

Since $\oplus_{n=0}^{\infty} I_w^n$ is an integral domain, $B$ is an integral surface with a proper birational morphism $\theta : B \to \mathbb{P}_1^1 \times \mathbb{P}_1^1$ that is an isomorphism away from $D$. Namely, $\theta$ is the restriction to $B$ of the first projection morphism $\pi_1 : (\mathbb{P}_1^1 \times \mathbb{P}_1^1) \times \mathbb{P}_h^k \to \mathbb{P}_1^1 \times \mathbb{P}_1^1$. The stalk $F_w$ of $F$ at each $w \in D$ is contained in the maximal ideal $m_{A_w}$ of $A_w = \mathcal{O}_{\mathbb{P}_1^1 \times \mathbb{P}_1^1}$. Thus $F_w$ contains a point of the orbit $\psi(A_1^1 \times A_1^1) = \text{Aut}_A(P) \cdot \mathcal{C}$. This is impossible because $A_w/F_w$ has finite dimension over $k$. Now $\theta^{-1}(w) = \text{Proj}(\oplus_{n=0}^{\infty}(I_w^n/m_{A_w} I_w^n))$ has at least two points. Since $\mathbb{P}_1^1 \times \mathbb{P}_1^1$ is normal, $\theta^{-1}(w)$ is connected by Zariski’s Main Theorem. It follows that $\theta^{-1}(w)$ is a connected union of finitely many possibly non-reduced curves.

We define $\tilde{\psi} : B \to \mathbb{P}_h^k$ to be the restriction to $B$ of the second projection morphism $\pi_2 : (\mathbb{P}_1^1 \times \mathbb{P}_1^1) \times \mathbb{P}_h^k \to \mathbb{P}_h^k$. Clearly $\tilde{\psi}(B) = \pi_2(B) = \pi_2(G)$ contains $\pi_2(G) = \psi(U)$. Because $\pi_2$ is projective, $\pi_2(B)$ is closed, so

\[
\overline{\psi(U)} = \text{Aut}_A(P) \cdot \mathcal{C} \subset \pi_2(B) = \tilde{\psi}(B). \tag{3.7}
\]

Suppose $z \in \mathbb{P}_h^k$ does not lie in $\overline{\text{Aut}_A(P) \cdot \mathcal{C}} = \overline{\psi(U)}$. Then there is an open neighborhood $V$ of $z$ in $\mathbb{P}_h^k$ so $V \cap \psi(U) = \emptyset$. Therefore $\pi_2^{-1}(V)$ is an open subset of $(\mathbb{P}_1^1 \times \mathbb{P}_1^1) \times \mathbb{P}_h^k$ which contains no element of the graph $G$ of $\psi$ on $U$. Hence $\pi_2^{-1}(z)$ is not in the closure $B$ of $G$, so $z \notin \pi_2(B) = \tilde{\psi}(B)$. Taking contrapositives, we have shown

\[
\tilde{\psi}(B) \subset \overline{\psi(U)} = \overline{\text{Aut}_A(P) \cdot \mathcal{C}}. \tag{3.8}
\]

Combining (3.7) and (3.8) shows $\tilde{\psi}(B) = \overline{\text{Aut}_A(P) \cdot \mathcal{C}}$ so we are done with Step 2.

**Notation for Step 3:** Fix $w \in D$ and let $t_1$ and $t_2$ be uniformizing parameters in $A = A_w = \mathcal{O}_{\mathbb{P}_1^1 \times \mathbb{P}_1^1,w}$. Thus $A = k[t_1, t_2]/(t_1, t_2)$ and $I = \mathcal{F}_w = A q_0 + \cdots + A q_h$ for some polynomials $q_0, \ldots, q_h \in k[t_1, t_2]$ of bounded degree with constant term 0. We know from Step 1 that $I$ contains a bounded power of the maximal ideal $m_A$ of $A$, so $\dim_k(A/I) = a$ is bounded. Thus
defines a closed point on the reduction \( \mathcal{G} = \mathcal{G}_a \) of the Grassmannian of all ideals of \( A \) of codimension \( a \).

Let \( \mathcal{X} = \mathbb{P}^2_{\mathcal{G}} \) be the projective plane over \( \mathcal{G} \) with structure sheaf \( \mathcal{O}_\mathcal{X} \). For all \( I \in \mathcal{G} \), let \( k(I) \) be the residue field of \( I \). By the construction of the Grassmannian, \( I \) is associated to an ideal, which we will also denote by \( I \), in \( k(I)[t_1, t_2]_{(t_1, t_2)} \). Note that \( I \) is a closed point if and only if \( k(I) = k \). The fiber \( \mathcal{X}_I \) is \( \mathbb{P}^2_{k(I)} \), which contains the affine plane \( \mathbb{A}^2_{k(I)} = \text{Spec}(k(I)[t_1, t_2]) \) with origin \( z_0 \) defined by \( t_1 = t_2 = 0 \). Let \( \mathcal{I}_I \subset \mathcal{O}_{\mathcal{X}_I} \) be the sheaf of ideals which has stalk \( \mathcal{O}_{\mathcal{X}_I,z} \) at \( z \in \mathbb{P}^2_{k(I)} - \{z_0\} \) and stalk at \( z_0 = (0, 0) \) given by the ideal \( I \) in \( k(I)[t_1, t_2]_{(t_1, t_2)} \). Define \( \mathcal{J} \subset \mathcal{O}_\mathcal{X} \) to be the sheaf of ideals which is generated by the inverse images of \( \mathcal{I}_I \) at such a morphism from \( \mathcal{X} \) to the minimal desingularization \( \mathcal{G} \). By [10, Prop. V.3.2], the rank of the Picard group of a smooth projective surface increases by 1 when one performs a monoidal transformation. Hence the rank of \( \text{Pic}(\mathcal{G}) \) is bounded because \( \mathcal{X} \) is a projective birational morphism to \( \mathcal{G} \).

\textbf{Step 3:} Let \( \mathcal{G} \) be as above. To prove Theorem [7], it will suffice to show that there is a uniform bound (which may depend on \( \mathcal{G} \)) on the number of successive monoidal transformations one must perform, in order to birationally transform \( \mathbb{P}^2_{k(I)} \) to a smooth rational surface which has a projective birational morphism to \( B_1 \), as \( I \) varies over all points of \( \mathcal{G} \).

\textbf{Proof of Step 3:} By Step 1, the set \( D \) is a finite set of closed points, and the number of points in \( D \) is bounded (always in terms of \( d' = \dim_k(C) \)). Suppose there is a uniform bound of the kind described in Step 3 for each point \( w \in D \) when we take \( I = \mathcal{F}_w \) as in the notation just prior to Step 3, so that \( I \) is an ideal of the local ring \( A = A_w \) of \( w \) on \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) which contains a bounded power of the maximal ideal. We patch together the resulting sequences of monoidal transformations as \( w \) varies over \( D \). Because the number of points in \( D \) is bounded, and the number of monoidal transformations needed for each point \( w \) of \( D \) is also bounded by the hypothesis in Step 3, we have the following conclusion. There is a bound on the number of successive monoidal transformations one must make, beginning with \( \mathbb{P}^1_k \times \mathbb{P}^1_k \), to arrive at a smooth rational surface \( S \) which has a projective birational morphism to the blow-up \( B_1 \) described in Step 2. Let \( \mu : S \to \mathbb{P}^1_k \times \mathbb{P}^1_k \) be this morphism.

There is a projective birational morphism from \( B \) to the orbit closure \( \overline{\text{Aut}_A(P).C} \). So we arrive at such a morphism from \( S \) to \( \overline{\text{Aut}_A(P).C} \). Thus there is a projective birational morphism from \( S \) to the minimal desingularization \( \overline{\text{Aut}_A(P).C} \) of the normalization \( \overline{\text{Aut}_A(P).C} \). Suppose \( \overline{\text{Aut}_A(P).C} \to X \) is a projective birational morphism to a relatively minimal smooth rational projective surface \( X \), so that \( X \) is isomorphic to \( \mathbb{P}^2_k, \mathbb{P}^1_k \times \mathbb{P}^1_k \) or to one of the surfaces \( X_n \) with \( n \geq 2 \). We obtain a projective birational morphism \( \tilde{\mu} : S \to X \) which is a composition of an unknown number of blow-downs of rational curves of self-intersection \(-1\). We obtain the diagram in Figure [2] (see also Figure [1]).

By [10] Prop. V.3.2], the rank of the Picard group of a smooth projective surface increases by 1 when one performs a monoidal transformation. Hence the rank of \( \text{Pic}(S) \) is bounded because \( \mu : S \to \mathbb{P}^1_k \times \mathbb{P}^1_k \) involves a bounded number of blow-downs. This implies there is a bound on the number of blow-downs needed to factor \( \tilde{\mu} : S \to X \).
We define an irreducible effective smooth curve $L$ on $X$ in the following way. If $X = \mathbb{P}^2$ let $L$ be any projective line on $X$, so that $L$ has self-intersection $L \cdot L = 1$. If $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$, let $L = \mathbb{P}^1_k \times \{x\}$ for a point $x \in \mathbb{P}^1_k$, so that $L \cdot L = 0$. Finally, if $X = X_n$ for some $n \geq 2$, there is by [10, Example V.2.11.3] a smooth curve $L$ on $X_n$ such that $L \cdot L = -n$.

By [10, Prop. 3.2], if $S'' \rightarrow S'$ is a monoidal transformation of smooth projective surfaces, and $L'$ is a smooth curve on $S'$ with proper transform $L''$ on $S''$, then either $L'' \cdot L'' = L' \cdot L'$ or $L'' \cdot L'' = L' \cdot L' + 1$. Let $L_S$ be the proper transform of $L$ under $\mu : S \rightarrow X$, and suppose $\hat{\mu}$ can be factored into $\hat{c}$ blow-downs, where we have bounded $\hat{c}$. Then $(L_S) \cdot (L_S) \leq L \cdot L + \hat{c}$.

We claim that it is impossible that $X = X_n$ for an $n > \hat{c} + 1$. For otherwise, $(L_S) \cdot (L_S) \leq L \cdot L + \hat{c} = n + \hat{c} < -1$. If a curve is sent to a point by a blow-down, then its self-intersection must be $-1$, and if it is not sent to a point then its self-intersection cannot increase. Hence we would have that the image $L_0$ of $L_S$ under $\mu : S \rightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$ would be an effective curve of self-intersection $L_0 \cdot L_0 < -1$. There are no such curves, since the self-intersection pairing is non-negative on the cone of effective curves in $\mathbb{P}^1_k \times \mathbb{P}^1_k$. This shows that the $n$ for which $X$ could be isomorphic to $X_n$ are bounded. Since we have also bounded the number of blow-downs needed to factor the morphism $\hat{\mu} : S \rightarrow X$, this completes Step 3.

**Step 4**: Let $\xi : \mathcal{B} \rightarrow \mathcal{X} = \mathbb{P}^2_G$ be as in Step 3. Let $C$ be an irreducible component of $\mathcal{G}$, and let $\mathcal{B}_C$ be the pullback of $\mathcal{G}$ to $C$. Define $s : C \rightarrow \mathcal{X}_C = \mathbb{P}^2_C$ to be the section of $\mathcal{X}_C = \mathbb{P}^2_C \rightarrow C$ defined by the origin on $\mathbb{A}^2_C \subset \mathcal{X}_C$. Define $\mathcal{B}_C^z$ to be the normalization of $\mathcal{B}_C$. Then $\mathcal{B}_C^z \rightarrow \mathcal{B}_C$ is finite and an isomorphism off of $\xi^{-1}(s(C))$, where $s(C)$ is the image of the section $s : C \rightarrow \mathcal{X}_C$. The fiber $(\mathcal{B}_C^z)_{\eta_C}$ of $\mathcal{B}_C^z$ over the generic point $\eta_C$ of $C$ is a normal projective surface over the residue field $k(\eta_C)$.

**Proof of Step 4**: The local rings of $\mathcal{B}_C^z$ are integrally closed in their fraction fields. For points $z$ on $(\mathcal{B}_C^z)_{\eta_C}$, the local ring of $z$ on $\mathcal{B}_C^z$ is equal to the local ring of $z$ on $(\mathcal{B}_C^z)_{\eta_C}$. Thus $(\mathcal{B}_C^z)_{\eta_C}$ is a normal projective surface, and the rest of Step 4 is clear.

**Step 5**: In this step we show that there is a uniform bound as specified in Step 3, which will complete the proof of Theorem 3.7. The plan is to apply a Theorem of Zariski to the general fiber $(\mathcal{B}_C^z)_{\eta_C}$ of $\mathcal{B}_C^z$. This produces a bound of the kind required in Step 3 when the point $I$ of
$\mathcal{G}$ is the generic point of the irreducible component $\mathcal{C}$ of $\mathcal{G}$. We then show how to extend the monoidal transformations involved over an open dense subset of $\mathcal{C}$, to be able to handle all $I$ in such a subset. Since there are a bounded number of irreducible components $\mathcal{C}$ of $\mathcal{G}$, this provides us with a bound of the required kind for all $I$ in an open dense subset of $\mathcal{G}$. Finally, we apply Noetherian induction to deal with all $I$ in $\mathcal{G}$.

**Proof of Step 5:** To begin the proof, we recall a Theorem of Zariski concerning birational projective morphisms $f : Z' \to Z$ between projective normal surfaces over a field $F$. Such morphisms are called modifications by Artin in [1].

A normalized monoidal transformation of $Z$ is a morphism $Z_1 \to Z$ in which $Z_1$ is the normalization of the monoidal transformation of $Z$ at the maximal ideal of a closed point. Zariski proves that every modification $f : Z' \to Z$ is dominated by a morphism $g : Z'' \to Z$ which is formed as the composition of a finite series of normalized monoidal transformations. Here domination means that there is a birational morphism $h : Z'' \to Z'$ compatible with $f : Z' \to Z$ and $g : Z'' \to Z$.

With the notations of Step 4 we now apply Zariski’s Theorem to the morphism

$$f : Z' = (\mathcal{B}_C^2)_{\eta_C} \to Z = \mathbb{P}_k^2$$

with $F = k(\eta_C)$. We obtain a birational morphism

$$g : Z'' \to Z$$

of normal projective surfaces over $k(\eta_C)$ which is the composition of a finite number of normalized monoidal transformations together with a birational morphism $h : Z'' \to Z'$ which is compatible with $f$ and $g$. Since $Z = \mathbb{P}_k^2$ is regular, and monoidal transformations preserve regularity, we never have to normalize any of the results of the monoidal transformations used in constructing $g : Z'' \to Z$.

Define $Z = \mathbb{P}_C^2$, so that the generic fiber of $Z$ over $C$ is $Z$. Take a closed point $Q$ on $Z$, and let $\overline{Q}$ be the Zariski closure of $Q$ in $Z$. The scheme $\overline{Q}$ need not be reduced, since $Q$ might have residue field $k(Q)$ which is a finite inseparable extension of $k(\eta_C)$. But if we let $\overline{Q}^{red}$ be the reduction of $\overline{Q}$, then $\overline{Q}^{red}$ is finite over $C$ whose fibers over each point $I$ of $C$ are a disjoint union of reduced points which are closed in the fiber over $I$. The number of such points in each fiber is bounded by the relative degree $[k(Q) : k(\eta_C)]$.

Let $Z_1$ be the blow-up of $Z$ at the sheaf of ideals which defines $\overline{Q}^{red}$. Since $Q$ is a closed point on the general fiber $Z$ of $\overline{Q}^{red}$, we conclude that $Z_1$ has as general fiber the blow-up $Z_1$ of $Z$ at $Q$. Furthermore, each fiber of $Z_1$ over $C$ is the blow-up of the fiber of $Z$ at the disjoint union of a finite number of reduced closed points, and the number of such points is uniformly bounded by $[k(Q) : k(\eta_C)]$. We conclude that each fiber of $Z_1$ over $C$ results from a number of monoidal transformations of the corresponding fiber of $Z$ over $C$, with the number of transformations involved being bounded by $[k(Q) : k(\eta_C)]$.

We apply this process to each of the monoidal transformations involved in producing the morphism $g : Z'' \to Z$. We conclude that $g$ is the general fiber over $C$ of a projective birational morphism $\tilde{g} : Z'' \to Z = \mathbb{P}_C^2$ which has the following property. For each point $I$ of $C$, the fiber $\tilde{g}_I : Z''_I \to Z_I = (\mathbb{P}_C^2)_I = \mathbb{P}_{k(I)}^2$ of $\tilde{g}$ at $I$ is an isomorphism on the complement of the origin of $\mathbb{A}^2_{k(I)} \subset \mathbb{P}^2_{k(I)}$, and $\tilde{g}_I$ is the composition of a bounded number of monoidal transformations.

Recall now that we have a proper birational morphism

$$h : Z'' \to Z' = (\mathcal{B}_C^2)_{\eta_C}$$
of normal surfaces over \( k(\eta_C) \) which gives a commutative diagram

\[
\begin{array}{ccc}
Z'' & \xrightarrow{h} & Z' = (\mathfrak{B}_C^\sharp)_{\eta_C} \\
\downarrow g & & \downarrow f \\
Z = \mathbb{P}_C^2 & \xrightarrow{\xi} & \mathbb{P}_C^2
\end{array}
\]

Here \( h \) defines an isomorphism between dense open subsets of \( Z'' \) and \( Z' \).

Let \( \tau : \mathfrak{B}_C \to \mathfrak{B}_C \) be the (finite) normalization morphism. We will use a subscript \( M \) to denote the pullback of a scheme or a morphism between schemes over a subscheme \( M \) of \( \mathfrak{B} \).

Then \( f_{\eta_C} : (\mathfrak{B}_C^\sharp)_{\eta_C} \to \mathbb{P}_C^2 \) is the composition \( \xi_{\eta_C} \circ \tau_{\eta_C} \) where \( \xi : \mathfrak{B} \to \mathbb{P}_C^2 \) is as in the notation stated just prior to Step 3.

By considering the denominators of the coefficients in \( k(\eta_C) \) of homogeneous polynomials defining \( h \), we see that there is a dense open affine subset \( \text{Spec}(\mathcal{R}) \) of \( C \) such that \( h \) extends to a morphism \( \tilde{h}_R : Z''_R \to \mathfrak{B}_C^\sharp_R \) where \( Z''_R \) is the pullback to \( \text{Spec}(\mathcal{R}) \) of the scheme \( Z'' \) constructed above, and \( \mathfrak{B}_C^\sharp_R \) is the pullback of \( \mathfrak{B}^\sharp \) to \( \text{Spec}(\mathcal{R}) \). We can further shrink \( \text{Spec}(\mathcal{R}) \) so that \( \tilde{h}_R \) is compatible with the restriction \( \tilde{g}_R : Z''_R \to Z_1 \to \mathbb{P}_C^2 \) over \( \text{Spec}(\mathcal{R}) \) and with the restriction \( (\xi \circ \tau)_R : \mathfrak{B}_C^\sharp_R \to \mathbb{P}_C^2_R \) of \( \xi \circ \tau \) to \( \text{Spec}(\mathcal{R}) \).

We conclude that for all points \( I \in \text{Spec}(\mathcal{R}) \), we have a birational morphism

\[
\tilde{h}_I : Z''_I \to (\mathfrak{B}_C^\sharp)_I
\]

compatible with the bounded composition of monoidal transformations

\[
\tilde{g}_I : Z''_I \to \mathbb{P}_{k(I)}^2 = (\mathbb{P}_C^2)_I
\]

and the birational morphism

\[
(\xi \circ \tau)_I : (\mathfrak{B}_C^\sharp)_I \to \mathbb{P}_{k(I)}^2 = (\mathbb{P}_C^2)_I
\]

which is the composition of \( \tau_I : (\mathfrak{B}_C^\sharp)_I \to \mathfrak{B}_I \) with the blow-up morphism \( \xi_I : \mathfrak{B}_I \to \mathbb{P}_{k(I)}^2 = (\mathbb{P}_C^2)_I \). Since \( \tau_I \) is an isomorphism on the complement of the one-dimensional fiber of \( \mathfrak{B}_I \) over the origin \( k^2_{k(I)} \subset \mathbb{P}_{k(I)}^2 \), we know that \( \tau_I \) is birational. Thus the composition of \( \tilde{h}_I \) with \( \tau_I \) is a birational morphism

\[
\zeta_I : Z''_I \to \mathfrak{B}_I
\]

compatible with \( \tilde{g}_I \) and \( \xi_I \). Since \( \tilde{h}_I \) is the composition of a bounded number of monoidal transformations, we have now produced the bound required in Step 3 for all \( I \) in the open dense subset \( \text{Spec}(\mathcal{R}) \) of \( C \).

We now do the above construction for the bounded number of irreducible components \( C \) of \( \mathcal{G} \). This produces a bound of the kind needed in Step 3 for all \( I \) in an open dense subset of \( \mathcal{G} \). The complement of these \( I \) is a closed subset of dimension strictly smaller than the dimension of \( \mathcal{G} \). Continuing by Noetherian induction on these subsets, we use the fact that \( \text{dim}(\mathcal{G}) \) is bounded.
to conclude that we have a bound of the required kind which applies to all points $I$ of $\mathcal{G}$. In view of Step 3, this completes the proof of Theorem 3.1.

Remark 3.3. The proof of Theorem 3.1 can be easily modified to show the following generalization:

Suppose $T$ is simple as before, but the orbit $\text{Aut}_A(P).C$ of $C$ in $\text{Grass}^T_d$ is $m$-dimensional for some $m \geq 2$. Let $Z$ be any two-dimensional affine plane contained in $\text{Aut}_A(P).C$. Let $\overline{Z}$ be the minimal desingularization of the normalization of the closure $\overline{Z}$ of $Z$ in $\text{Grass}^T_d$.

There is a bound $n_0$ which depends only on $k$ and $\text{dim}_k(C)$ such that there is a birational morphism from $\overline{Z}$ to a relatively minimal smooth rational projective surface which is either $\mathbb{P}^2_k$ or $X_n$ for some integer $0 \leq n \leq n_0$ with $n \neq 1$. There is furthermore a bound depending on $k$ and $\text{dim}_k(C)$ for the number of irreducible curves which are blown down to points by this morphism.

4. Bounding Euler characteristics of orbit closures

In this section we assume the hypotheses and notations of Theorem 3.1. To prove Corollary 1.3 we must show that the Euler characteristic $\chi(\text{Aut}_A(P).C)$ is bounded above by a function which depends only on $k$ and $\text{dim}_k(C)$. Here we define the Euler characteristic $\chi(V)$ of an arbitrary separated open subscheme $V$ of a closed subscheme $\overline{V}$ of a projective space over $k$ to be the étale Euler characteristic with compact support

$$\chi(V) = \sum_i (-1)^i \cdot \dim_{\mathbb{Q}_\ell} H^i_c(V, \mathbb{Q}_\ell)$$

for any prime $\ell$ different from the characteristic of $k$ (see [18, p. 93]). By [18, Remark 1.30], if $Z$ is a closed subscheme of $V$, there is a long exact sequence

$$\cdots \rightarrow H^i_c(V - Z, \mathbb{Q}_\ell) \rightarrow H^i_c(V, \mathbb{Q}_\ell) \rightarrow H^i_c(Z, \mathbb{Q}_\ell) \rightarrow \cdots$$

This gives an equality of Euler characteristics

$$(4.1) \quad \chi(V) = \chi(Z) + \chi(V - Z).$$

If $T$ is a constructible sheaf of $\mathbb{Q}_\ell$-vector spaces on $V$ for some prime $\ell$ different from the characteristic of $k$, we will denote the compactly supported Euler characteristic of $T$ by

$$\chi(V, T) = \sum_i (-1)^i \cdot \dim_{\mathbb{Q}_\ell} H^i_c(V, T).$$

Thus $\chi(V) = \chi(V, \mathbb{Q}_\ell)$.

We showed in the first paragraph of the proof of Step 3 of the proof of Theorem 3.1 the following fact. Once Step 3 was completed, there is a smooth rational surface $S$ together with morphisms $\mu : S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1_k$ and $\nu : S \rightarrow \overline{\text{Aut}_A(P).C}$ with the following properties. The morphism $\mu$ is a composition of a bounded number of monoidal transformations, and it is an isomorphism over $A^2_k = A^1_k \times A^1_k$. The morphism $\nu$ is proper and birational. On $\mu^{-1}(A^2_k)$, $\nu$ is
equal to $\Psi \circ \mu$, where $\Psi : A_k^2 \to \text{Aut}_A(P.C)$ is the isomorphism in (3.1).

(4.2)\[ \begin{array}{c} \mathbb{P}_k^1 \times \mathbb{P}_k^1 \\ \downarrow \mu \\ A_k^2 \\ \downarrow \Psi \\ \text{Aut}_A(P.C) \end{array} \]

We write $\mathbb{P}_k^1 \times \mathbb{P}_k^1 - A_k^2 = \Delta$ as the union of the two rational curves $\infty \times \mathbb{P}_k^1$ and $\mathbb{P}_k^1 \times \infty$ crossing transversely at $\infty \times \infty$. By induction on the number $n$ of monoidal transformations involved in factoring $\mu : S \to \mathbb{P}_k^1 \times \mathbb{P}_k^1$, we see that $\mu^{-1}(\Delta)$ is reduced and a tree of $n + 2$ smooth $\mathbb{P}_k^1$-curves crossing transversely, with each point lying on at most two irreducible components. The normalization $\mu^{-1}(\Delta)^\#$ is thus the disjoint union of $n + 2$ copies of $\mathbb{P}_k^1$. Furthermore, the canonical morphism $\mu^{-1}(\Delta)^\# \to \mu^{-1}(\Delta)$ is an isomorphism off the $r$ points of $\mu^{-1}(\Delta)$ which lie on two distinct irreducible components. Since $\mu^{-1}(\Delta)$ is a tree of $\mathbb{P}_k^1$-curves, we see by induction on the number $n + 2$ of irreducible components of this tree that $r = n + 1$. We therefore have from (4.1) the Euler characteristic identities

$$\chi(S) = \chi(\mu^{-1}(A_k^2)) + \chi(\mu^{-1}(\Delta)) = 1 + \chi(\mu^{-1}(\Delta))$$

and

$$2n + 4 = (n + 2) \cdot \chi(\mathbb{P}_k^1) = \chi(\mu^{-1}(\Delta)^\#) = \chi(\mu^{-1}(\Delta)) + n + 1.$$ 

Thus \[ \chi(S) = n + 4. \]

In a similar way, we can write the orbit closure $\overline{\text{Aut}_A(P.C)}$ as the disjoint union of $\Psi(A_k^2 \cong A_k^2$ with the closed subset $\nu(\mu^{-1}(\Delta))$. Here $\nu(\mu^{-1}(\Delta))$ is the connected union of some number $t$ of distinct rational curves which may be singular, and $t \leq n + 2$. By the same reasoning above, we find that

(4.3) \[ \chi(\text{Aut}_A(P.C)) = \chi(A_k^2) + \chi(\nu(\mu^{-1}(\Delta))) = 1 + \chi(\nu(\mu^{-1}(\Delta))). \]

To bound $\chi(\nu(\mu^{-1}(\Delta)))$ we use the fact that the normalization $\nu(\mu^{-1}(\Delta))^\#$ is the disjoint union of $t$ copies of $\mathbb{P}_k^1$. Therefore

(4.4) \[ \chi(\nu(\mu^{-1}(\Delta))^\#) = 2t. \]

Let $\sigma : \nu(\mu^{-1}(\Delta))^\# \to \nu(\mu^{-1}(\Delta))$ be the canonical morphism. We thus have an exact sequence of étale sheaves on $\nu(\mu^{-1}(\Delta))$ given by

\[ 0 \to \mathbb{Q}_\ell \to \sigma_* \mathbb{Q}_\ell \to \mathcal{T} \to 0 \]

in which $\mathcal{T}$ is supported on the finitely many singular points of $\nu(\mu^{-1}(\Delta))$. This gives the following equality of compactly supported Euler characteristics on $\nu(\mu^{-1}(\Delta))$: \[ \chi(\nu(\mu^{-1}(\Delta))) = \chi(\nu(\mu^{-1}(\Delta)), \mathbb{Q}_\ell) \]

(4.5) \[ = \chi(\nu(\mu^{-1}(\Delta)), \sigma_* \mathbb{Q}_\ell) - \chi(\nu(\mu^{-1}(\Delta)), \mathcal{T}). \]
Since \( \sigma \) is finite, the higher derived functors \( R^q\sigma_* \) vanish on \( \mathbb{Q}_\ell \) for \( q > 0 \). Therefore the spectral sequence
\[
H^p(\nu(\mu^{-1}(\Delta)), R^q\sigma_* \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\nu(\mu^{-1}(\Delta))^\#_* \mathbb{Q}_\ell)
\]
degenerates to give
\[
(4.6) \quad \chi(\nu(\mu^{-1}(\Delta)), \sigma_* \mathbb{Q}_\ell) = \chi(\nu(\mu^{-1}(\Delta))^\#_* \mathbb{Q}_\ell) = \chi(\nu(\mu^{-1}(\Delta))^\#) = 2t.
\]

Since the higher cohomology groups of \( T \) vanish, we have
\[
(4.7) \quad \chi(\nu(\mu^{-1}(\Delta)), T) = \dim_{\mathbb{Q}_\ell} H^0_c(\nu(\mu^{-1}(\Delta)), T).
\]
Putting (4.6) and (4.7) into (4.5) gives
\[
(4.8) \quad \chi(\nu(\mu^{-1}(\Delta))) = 2t - \dim_{\mathbb{Q}_\ell} H^0_c(\nu(\mu^{-1}(\Delta)), T) \leq 2t \leq 2n + 4.
\]
This and (4.3) complete the proof of Corollary 1.3 (With a little more work, one can in fact show \( \chi(\nu(\mu^{-1}(\Delta))) \leq t + 1 \leq n + 3 \).)

5. Examples

In this section, we provide some examples of closures of orbits of dimension 2.

Example 5.1. In our first two examples, the orbit closures are isomorphic to \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) and \( \mathbb{P}^2_k \), respectively.

(a) Let \( \Lambda = kQ/I \), where
\[
Q = \begin{array}{c}
\omega_1 \\
\omega_2 \\
1 \\
\end{array}
\quad \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
2 \\
\end{array}
\quad \begin{array}{c}
3 \\
\end{array}
\]
and
\[
I = \langle \omega_1^2, \omega_1\omega_2, \omega_2\omega_1, \omega_2^2, \alpha_1\omega_2, \alpha_2\omega_1 \rangle .
\]
Then \( P = \Lambda e_1 \) may be visualized by the diagram
\[
P = \begin{array}{c}
1 \\
\alpha_1 \\
2 \\
\end{array}
\quad \begin{array}{c}
\omega_1 \\
\omega_2 \\
\end{array}
\quad \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
2 \\
\end{array}
\quad \begin{array}{c}
3 \\
\end{array}
\]
The action of \( \text{Aut}_\Lambda(P) \) on any submodule \( C \) of \( P \) is given by right multiplication by \( e_1 + t_1\omega_1 + t_2\omega_2 \) for \( t_1, t_2 \in k \). For \( C = \Lambda\alpha_1 + \Lambda\alpha_2 = k\alpha_1 + k\alpha_2 \), the orbit \( \text{Aut}_\Lambda(P)C \) therefore consists of the points
\[
(5.1) \quad C(e_1 + t_1\omega_1 + t_2\omega_2) = k(\alpha_1 + t_1\alpha_1\omega_1) + k(\alpha_2 + t_2\alpha_2\omega_2) .
\]
Introducing projective coordinates \( (u_0 : u_1) \times (v_0 : v_1) \) for \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) and letting \( t_1 = \frac{u_1}{u_0} \) and \( t_2 = \frac{v_1}{v_0} \), we find the family (5.11) of modules over \( \mathbb{A}^2_k = \mathbb{A}^1_k \times \mathbb{A}^1_k \) to correspond to the affine patch, where \( u_0 \neq 0 \) and \( v_0 \neq 0 \), of the family of modules
\[
\tilde{C} = k[u_0, u_1, v_0, v_1] (u_0\alpha_1 + u_1\alpha_1\omega_1) + k[u_0, u_1, v_0, v_1] (v_0\alpha_2 + v_1\alpha_2\omega_2)
\]
over $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. Sending $(u_0 : u_1) \times (v_0 : v_1) \in \mathbb{P}_k^1 \times \mathbb{P}_k^1$ to

$$
\Lambda(u_0 \alpha_1 + u_1 \alpha_1 \omega_1) + \Lambda(v_0 \alpha_2 + v_1 \alpha_2 \omega_2)
$$

in $\text{Grass}^{S_i}_0$, we obtain a well-defined morphism of schemes

$$
\mathbb{P}_k^1 \times \mathbb{P}_k^1 \to \text{Grass}^{S_i}_0.
$$

It follows that $\text{Aut}_\Lambda(P).C \cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$. The boundary $\overline{\text{Aut}_\Lambda(P).C} - \text{Aut}_\Lambda(P).C$ consists of precisely three $\text{Aut}_\Lambda(P)$-orbits (two of them 1-dimensional), meaning that $M = P/C$ has precisely three top-stable degenerations, up to isomorphism.

(b) Let $\Lambda = kQ/I$, where

$$
Q = \begin{array}{c}
\omega_1 \\
\omega_2 \\
1
\end{array}
\xrightarrow{\alpha} 2
\quad \text{and} \quad
I = \{\omega_i \omega_j \mid i, j \in \{1, 2\}\}.
$$

Let $P = \Lambda e_1$ and $C = \Lambda \alpha = k \alpha$. Sending $(z_0 : z_1 : z_2) \in \mathbb{P}_k^2$ to $\Lambda(z_0 \alpha + z_1 \alpha \omega_1 + z_2 \alpha \omega_2)$ in $\text{Grass}^{S_i}_0$, we obtain a well-defined morphism of schemes

$$
\mathbb{P}_k^2 \to \text{Grass}^{S_i}_0
$$

with the affine patch $z_0 \neq 0$ corresponding to $\text{Aut}_\Lambda(P).C$. We conclude that $\overline{\text{Aut}_\Lambda(P).C} \cong \mathbb{P}_k^2$. The boundary $\overline{\text{Aut}_\Lambda(P).C} - \text{Aut}_\Lambda(P).C$ consists of infinitely many 0-dimensional $\text{Aut}_\Lambda(P).C$-orbits corresponding to a $\mathbb{P}_k^1$-family of isomorphism classes of top-stable degenerations of $M = P/C$. Namely, this family is given by $(P/C(\alpha^{(a:b)}))$, where $C(\alpha^{(a:b)}) = \Lambda(\alpha \omega_1 + b \omega_2)$.

**Example 5.2.** In our next example, the orbit closure $\overline{\text{Aut}_\Lambda(P).C}$ is isomorphic to the Hirzebruch surface $X_2$. Let $\Lambda = kQ/I$, where

$$
Q = \begin{array}{c}
\omega \\
1
\end{array}
\xrightarrow{\alpha} 2
\quad \text{and} \quad
I = \{\omega^3, \beta \omega^2\}.
$$

Moreover, we let $P = \Lambda e_1$ and

$$
C = \Lambda(\alpha + \beta) + \Lambda \alpha \omega + \Lambda \gamma \omega = k(\alpha + \beta) + k \alpha \omega + k \gamma \omega \in \text{Grass}^{S_i}_8.
$$

The module $M = P/C$ is displayed at the top of Figure 3. Again, $\text{Aut}_\Lambda(P).C$ consists of the points

$$
C(e_1 + t_1 \omega + t_2 \omega^2) = k(\alpha + \beta + t_1 \alpha \omega + t_1 \beta \omega + t_2 \alpha \omega^2) + k(\alpha \omega + t_1 \alpha \omega^2) + k(\gamma \omega + t_1 \gamma \omega^2),
$$

for $(t_1, t_2) \in k^2_2$.

We will exhibit an isomorphism from

$$
X_2 = \{(z_0 : z_1) \times (y_0 : y_1 : y_2 : y_3) \in \mathbb{P}_k^1 \times \mathbb{P}_k^3 \mid z_0 y_1 = z_1 y_0 \text{ and } z_0 y_2 = z_1 y_1\}.
$$
Figure 3. The hierarchy of top-stable degenerations in Example 5.2.

\[ (1 : 1) \in \mathbb{P}_k^1 \]

\[ (a : b) \in \mathbb{P}_k^1 \]
\[ a, b \in k^* \]

\[ \text{to } \text{Aut}_\Lambda(P).C, \text{ which sends a point } (z_0 : z_1) \times (y_0 : y_1 : y_2 : y_3) \text{ of } X_2 \text{ with } z_0 \neq 0 \text{ and } y_0 \neq 0 \text{ to the point} \]
\[ C \left( e_1 + \frac{z_1}{z_0} \omega + \frac{y_3}{y_0} \omega^2 \right) \in \text{Aut}_\Lambda(P).C. \]
In other words, we will check that the Zariski-continuous extension $\varphi : X_2 \to \overline{\text{Aut}_\Lambda(P)} \cdot C$ of this assignment is indeed an isomorphism onto $\overline{\text{Aut}_\Lambda(P)} \cdot C$. For this purpose, we consider the following affine open cover $(U_i)_{1 \leq i \leq 4}$ of $X_2$: Namely, $U_1$ consists of the points with $z_0 = y_0 = 1$, that is, the following copy

$$U_1 = \{(1 : z_1) \times (1 : z_1 : z_1^2 : y_3) \mid z_1, y_3 \in k\};$$

of affine 2-space; $U_2$ consists of the points with $z_0 = y_3 = 1$, that is,

$$U_2 = \{(1 : z_1) \times (y_0 : z_1, y_0 : z_1^2 y_0 : 1) \mid z_1, y_0 \in k\};$$

$U_3$ consists of the points with $z_1 = y_2 = 1$, that is,

$$U_3 = \{(z_0 : 1) \times (z_0^2 : z_0 : 1 : y_3) \mid z_0, y_3 \in k\};$$

and $U_4$ consists of the points with $z_1 = y_3 = 1$, that is,

$$U_4 = \{(z_0 : 1) \times (z_0^2 y_2 : z_0 y_2 : y_2 : 1) \mid z_0, y_2 \in k\}.$$

In describing the restrictions $\varphi_i$ of $\varphi$ to the affine charts $U_i$, we use the following convention: If $\tau : \mathbb{A}_k^n - \{a_1, \ldots, a_r\} \to \overline{\text{Aut}_\Lambda(P)} \cdot C$ is a smooth curve and $\tau$ its unique extension to a curve defined on $\mathbb{P}^1$, we denote $\tau(a_j)$ by $\lim_{t \to a_j} \tau(t)$. The morphism $\varphi_2$ sends a point $(1 : z_1) \times (1 : z_1 : z_1^2 : y_3)$ of $U_1$ to $C(e_1 + z_1 \omega + y_3 \omega^2)$, and thus induces an isomorphism from $U_1$ onto $\overline{\text{Aut}_\Lambda(P)} \cdot C$. The restriction $\varphi_2$ sends a point $(1 : z_1) \times (y_0 : z_1 y_0 : z_1^2 y_0 : 1)$ of $U_2$ to $C(e_1 + z_1 \omega + \frac{1}{y_0} \omega^2)$ if $y_0 \neq 0$, and to

$$\lim_{t \to 0} C \left( e_1 + z_1 \omega + \frac{1}{t} \omega^2 \right)$$

if $y_0 = 0$. For $t \neq 0$ a short calculation shows

$$C \left( e_1 + z_1 \omega + \frac{1}{t} \omega^2 \right) = \Lambda \left( t \alpha + t \beta + t z_1 \alpha \omega + t z_1 \beta \omega + \alpha \omega^2 \right) + \Lambda \left( \alpha \omega + z_1 \alpha \omega^2 \right) + \Lambda \left( \gamma \omega + z_1 \gamma \omega^2 \right).$$

Thus

$$\lim_{t \to 0} C \left( e_1 + z_1 \omega + \frac{1}{t} \omega^2 \right) = \Lambda \alpha \omega^2 + \Lambda \alpha \omega + \Lambda \left( \gamma \omega + z_1 \gamma \omega^2 \right).$$

As $z_1$ traces $k$, the latter points trace the orbit $\overline{\text{Aut}_\Lambda(P)} \cdot E$, where $E = \Lambda \alpha \omega^2 + \Lambda \alpha \omega + \Lambda \gamma \omega$. This is the only 1-dimensional orbit in $\overline{\text{Aut}_\Lambda(P)} \cdot C$; the corresponding degeneration $P/E$ of $M = P/C$ is depicted in the left-hand position of the second row of Figure 3.

The restriction $\varphi_3$ sends any point $(z_0 : 1) \times (z_0^2 : z_0 : 1 : y_3)$ of $U_3$ to $C(e_1 + \frac{1}{z_0} \omega + \frac{y_3}{z_0^2} \omega^2)$ if $z_0 \neq 0$ and to

$$\lim_{t \to 0} C \left( e_1 + \frac{1}{t} \omega + \frac{y_3}{t^2} \omega^2 \right)$$

if $z_0 = 0$. For $t \neq 0$ a calculation similar to the one above shows

$$\lim_{t \to 0} C \left( e_1 + \frac{1}{t} \omega + \frac{y_3}{t^2} \omega^2 \right) = \Lambda \left( (1 - y_3) \alpha \omega + \beta \omega \right) + \Lambda \alpha \omega^2 + \Lambda \gamma \omega^2.$$

When $y_3 = 1$ this gives the point $\Lambda \beta \omega + \Lambda \alpha \omega^2 + \Lambda \gamma \omega^2$ which constitutes the 0-dimensional orbit corresponding to the degeneration that appears on the far right in the second row of Figure 3. As $1 - y_3$ varies over $k^*$ (i.e. $y_3$ varies over $k - \{1\}$) we have a $k^*$-family of degeneration of $M = P/C$ as shown in the central position of Figure 3.
Finally, we consider the restriction \( \varphi_4 \) of \( \varphi \) to points \( Q = (z_0 : 1) \times (z_0^2 y_2 : z_0 y_2 : y_2 : 1) \in U_4 \). If \( z_0 \neq 0 \) and \( y_2 \neq 0 \), then \( \varphi_4(Q) = C \left( e_1 + \frac{1}{z_0} \omega + \frac{1}{z_0^2 y_2} \omega^2 \right) \). For \( z_0 = 0 \) and \( y_2 \neq 0 \), we find as above that

\[
\varphi_4(Q) = \lim_{t \to 0} C \left( e_1 + \frac{1}{t} \omega + \frac{1}{t^2 y_2} \omega^2 \right) = \Lambda \left( 1 - \frac{1}{y_2} \right) \alpha \omega + \beta \omega + \Lambda \alpha \omega^2 + \Lambda \gamma \omega^2,
\]

yielding points already encountered in \( \text{Im}(\varphi_3) \).

If \( z_0 \neq 0 \) and \( y_2 = 0 \), one finds in a similar way that

\[
\varphi_4(Q) = \lim_{t \to 0} \left( \lim_{s \to 0} C \left( e_1 + \frac{1}{s} \omega + \frac{1}{t^2 y_2} \omega^2 \right) \right) = \Lambda \alpha \omega + \Lambda \alpha \omega^2 + \Lambda \gamma \omega^2.
\]

This value of \( \varphi_4 \) gives rise to the maximal top-stable degeneration that appears in the third row of Figure 3.

One checks that the maps \( \varphi_i \) coincide on the overlaps of their domains, that each yields an isomorphism from \( U_i \) to an open subvariety of \( \overline{\text{Aut}_\Lambda(P).C} \), and that their images cover \( \overline{\text{Aut}_\Lambda(P).C} \).

**Example 5.3.** In this example, we retain the same algebra \( \Lambda = kQ/I \) introduced in Example 5.2 as well as the projective module \( P \). But we use a different point \( C \) in \( \text{Grass}^8_{S^1} \). We will show that in this case the minimal desingularization \( \overline{\text{Aut}_\Lambda(P).C} \) is isomorphic to a blow-up \( B \) of \( X_2 \) at a closed point such that both \( \mathbb{P}^2_k \) and \( X_2 \) are relatively minimal models. Moreover, we will show that the orbit closure \( \overline{\text{Aut}_\Lambda(P).C} \) is obtained from \( B \) by blowing down a curve of self-intersection \( -2 \), resulting in a singular projective surface. In other words, we have the following diagram of birational morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\rho_1} & \overline{\text{Aut}_\Lambda(P).C} \\
\downarrow \rho_2 & & \downarrow \rho_3 \\
\mathbb{P}^2_k & \xleftarrow{f} & X_2
\end{array}
\]

where each of \( \rho_1, \rho_2 \) and \( \rho_3 \) results from blowing down a curve of self-intersection \( -1 \) and the morphism \( \overline{\text{Aut}_\Lambda(P).C} \xrightarrow{f} \overline{\text{Aut}_\Lambda(P).C} \) results from blowing down a curve of self-intersection \( -2 \).

Let \( \Lambda = kQ/I \) be as in Example 5.2 let \( P = \Lambda e_1 \), and define

\[
C = \Lambda (\alpha + \beta) + \Lambda \alpha \omega + \Lambda \gamma = k(\alpha + \beta) + k\alpha \omega + k\gamma \in \text{Grass}^8_{S^1}.
\]
The module $M = P/C$ is displayed at the top of the Figure. Again, $\text{Aut}_\Lambda(P).C$ consists of the points

$$C(e_1 + t_1\omega + t_2\omega^2) = k(\alpha + \beta + t_1\alpha\omega + t_1\beta\omega + t_2\alpha\omega^2) + k(\alpha\omega + t_1\alpha\omega^2) + k(\gamma + t_1\gamma\omega + t_2\gamma\omega^2),$$

for $(t_1, t_2) \in \mathbb{A}^2$.

We consider the blow-up $B$ of

$$X_2 = \{(z_0 : z_1) \times (y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^1 \times \mathbb{P}^3 \mid z_0y_1 = z_1y_0 \text{ and } z_0y_2 = z_1y_1\}$$
at the closed point $(z_0 : z_1) \times (y_0 : y_1 : y_2 : y_3) = (0 : 1) \times (0 : 1 : 0)$. In other words,

$$B = \{(z_0 : z_1) \times (y_0 : y_1 : y_2 : y_3) \times (x_0 : x_1) \in \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^1 \mid z_0y_1 = z_1y_0, z_0y_2 = z_1y_1 \text{ and } z_0x_1 = y_3x_0\}.$$

We consider the following affine cover of $B$, consisting of 5 affine 2-spaces:

$$\tilde{U}_1 = \{(1 : z_1) \times (1 : z_1 : z_1^2 : y_3) \times (1 : y_3) \mid z_1, y_3 \in k\};$$

$$\tilde{U}_2 = \{(1 : z_1) \times (y_0 : z_1y_0 : z_1^2y_0 : 1) \times (1 : 1) \mid z_1, y_0 \in k\};$$

$$\tilde{U}_{31} = \{(z_0 : 1) \times (z_0 : z_0 : 1 : z_0) \times (1 : 1) \mid z_0, \xi \in k\};$$

$$\tilde{U}_{32} = \{((\zeta y_3 : 1) \times (\xi y_3^2 : \xi y_3 : 1) \times (\zeta : 1) \mid \zeta, y_3 \in k\};$$

$$\tilde{U}_4 = \{(z_0 : 1) \times (z_0 : z_0y_2 : z_0y_2 : 1) \times (1 : 1) \mid z_0, y_2 \in k\}.$$

We now define a morphism $\varphi : B \to \text{Aut}_\Lambda(P).C$ by defining the restrictions of $\varphi$ to these 5 affine patches as follows.

The restriction $\varphi_1$ of $\varphi$ to $\tilde{U}_1$ sends a point $(1 : z_1) \times (1 : z_1 : z_1^2 : y_3) \times (1 : y_3)$ of $\tilde{U}_1$ to $C(e_1 + z_1\omega + y_3\omega^2)$, and thus induces an isomorphism from $\tilde{U}_1$ onto $\text{Aut}_\Lambda(P).C$.

The restriction $\varphi_2$ sends a point $(1 : z_1) \times (y_0 : z_1y_0 : z_1^2y_0 : 1) \times (1 : 1)$ of $\tilde{U}_2$ to $C\left(e_1 + z_1\omega + \frac{1}{y_0}\omega^2\right)$ if $y_0 \neq 0$, and to

$$\lim_{t \to 0} C\left(e_1 + z_1\omega + \frac{1}{t}\omega^2\right) = \Lambda\omega^2 + \Lambda\alpha\omega + \Lambda\gamma\omega^2$$

if $y_0 = 0$. The corresponding degeneration of $M = P/C$ is depicted in the left-hand position of the second row of Figure.

The restriction $\varphi_{31}$ sends any point $(z_0 : 1) \times (z_0^2 : z_0 : 1 : z_0) \times (1 : \xi) \in \tilde{U}_{31}$ to $C\left(e_1 + \frac{1}{z_0}\omega + \xi\omega^2\right)$ if $z_0 \neq 0$ and to

$$\lim_{t \to 0} C\left(e_1 + \frac{1}{t}\omega + \xi\omega^2\right) = \Lambda(\alpha\omega + \beta\omega) + \Lambda\alpha\omega^2 + \Lambda(\gamma\omega + \xi\gamma\omega^2)$$

if $z_0 = 0$. As $\xi$ traces $k$, the latter points trace the orbit $\text{Aut}_\Lambda(P).D$, where $D = \Lambda(\alpha\omega + \beta\omega) + \Lambda\alpha\omega^2 + \Lambda\gamma\omega$. This is the only 1-dimensional orbit in $\text{Aut}_\Lambda(P).C$; the corresponding degeneration $P/D$ of $M = P/C$ is depicted in the second-to-left position of the second row of Figure.

Next, we consider the restriction $\varphi_{32}$ of $\varphi$ to points $R = (\zeta y_3 : 1) \times (\zeta^2 y_3^2 : \zeta y_3 : 1 : y_3) \times (\zeta : 1) \in \tilde{U}_{32}$. If $\zeta \neq 0$ and $y_3 \neq 0$, then $\varphi_{32}(R) = C(e_1 + \frac{1}{\zeta y_3}\omega + \frac{1}{\zeta^2 y_3^2}\omega^2)$. For $\zeta = 0$ and $y_3 \neq 0$, we get that $\varphi_{32}(R)$ is equal to the limit

$$\lim_{t \to 0} C\left(e_1 + \frac{1}{t y_3}\omega + \frac{1}{t^2 y_3^2}\omega^2\right) = \Lambda(\beta\omega + (1 - y_3)\alpha\omega) + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2$.}
Figure 4. The hierarchy of top-stable degenerations in Example 5.3.

When $y_3 = 1$ this gives the point $A_{\beta \omega} + A_{\alpha \omega^2} + A_{\gamma \omega^2}$ which constitutes the 0-dimensional orbit corresponding to the degeneration that appears on the far right in the second row of Figure 4.
As $1 - y_3$ varies over $k^*$ (i.e. $y_3$ varies over $k - \{1\}$) we obtain a $k^*$-family of degenerations of $M = P/C$ as shown in the second-to-right position of the second row of Figure 4.

If $\zeta \neq 0$ and $y_3 = 0$, one finds in a similar way that

$$\varphi_{32}(R) = \lim_{t \to 0} C \left( e_1 + \frac{1}{\zeta t} \omega + \frac{1}{\zeta^2 s} \omega^2 \right) = \Lambda(\alpha \omega + \beta \omega) + \Lambda \alpha \omega^2 + \Lambda \left( \gamma \omega + \frac{1}{\zeta} \gamma \omega^2 \right).$$

The resulting points in the boundary of $\text{Aut}_\Lambda(P).C$, as $\zeta$ ranges over $k^*$, retrace the points in the orbit $\text{Aut}_\Lambda(P).D$, where $D \in \text{Im}(\varphi_{31})$ is specified as above.

If $\zeta = y_3 = 0$, on the other hand, then

$$\varphi_{32}(R) = \lim_{t \to 0} \left( \lim_{s \to 0} C \left( e_1 + \frac{1}{t} \omega + \frac{1}{t^2 s} \omega^2 \right) \right) = \Lambda(\alpha \omega + \beta \omega) + \Lambda \alpha \omega^2 + \Lambda \gamma \omega^2.$$

This value of $\varphi_{32}$ gives rise to the maximal top-stable degeneration that appears in the third row of Figure 4.

Finally, we consider the restriction $\varphi_4$ of $\varphi$ to points $Q = (z_0 : 1) \times \left( \frac{2}{z_0} y_2 : z_0 y_2 : 1 \right) \times (z_0 : 1) \in \tilde{U}_4$. If $z_0 \neq 0$ and $y_2 \neq 0$, then $\varphi_4(Q) = C \left( e_1 + \frac{1}{z_0} \omega + \frac{1}{z_0 y_2} \omega^2 \right)$. For $z_0 = 0$ and $y_2 \neq 0$, we get that $\varphi_4(Q)$ is equal to the limit

$$\lim_{t \to 0} C \left( e_1 + \frac{1}{t} \omega + \frac{1}{t^2 y_2} \omega^2 \right) = \Lambda \left( \beta \omega + \left( 1 - \frac{1}{y_2} \right) \alpha \omega \right) + \Lambda \alpha \omega^2 + \Lambda \gamma \omega^2.$$

Thus, we obtain points already encountered in $\text{Im}(\varphi_{32})$.

If $z_0 \neq 0$ and $y_2 = 0$, one finds in a similar way that

$$\varphi_4(Q) = \lim_{t \to 0} C \left( e_1 + \frac{1}{z_0} \omega + \frac{1}{z_0 t} \omega^2 \right) = \Lambda \alpha \omega + \Lambda \alpha \omega^2 + \Lambda \gamma \omega^2,$$

yielding the degeneration already encountered in $\text{Im}(\varphi_2)$. If $z_0 = y_2 = 0$, then we again obtain that

$$\varphi_4(Q) = \lim_{t \to 0} \left( \lim_{s \to 0} C \left( e_1 + \frac{1}{t} \omega + \frac{1}{t^2 s} \omega^2 \right) \right) = \Lambda \alpha \omega + \Lambda \alpha \omega^2 + \Lambda \gamma \omega^2.$$

One checks that the maps $\varphi_i$, for $i \in \{1, 2, 31, 32, 4\}$ coincide on the overlaps of their domains, and that their images cover $\tilde{\text{Aut}}_\Lambda(P).C$. Moreover, for $i \in \{1, 31, 32\}$, $\varphi_i$ yields an isomorphism from $U_i$ to an open subvariety of $\tilde{\text{Aut}}_\Lambda(P).C$. On $U_2$, however, $\varphi_2$ sends the curve $y_0 = 0$, corresponding to the points $(1 : z_1) \times (0 : 0 : 0 : 1)$ for $z_1 \in k$, to the single point $\Lambda \alpha \omega^2 + \Lambda \alpha \omega + \Lambda \gamma \omega^2$. Similarly, on $U_4$, $\varphi_4$ sends the curve $y_2 = 0$, corresponding to the points $(z_0 : 1) \times (0 : 0 : 0 : 1)$ for $z_0 \in k$, again to the same point $\Lambda \alpha \omega^2 + \Lambda \alpha \omega + \Lambda \gamma \omega^2$. The curves $y_0 = 0$ on $U_2$, respectively, $y_2 = 0$ on $U_4$, define the same projective curve on the blow-up $B$. We need to blow down this curve on $B$ to a point. Our explicit description of the maps $\varphi_i$ shows that we do not need to blow down any further curves on $B$, but that this blow-down is isomorphic to the orbit closure $\tilde{\text{Aut}}_\Lambda(P).C$. We also see that the point on $X_2$ we blew up to construct $B$ does not lie on the curve we need to blow down. Computing the self-intersections of all the curves on $B$ corresponding to boundary points of $\tilde{\text{Aut}}_\Lambda(P).C$, it follows that this curve has self-intersection $-2$. Therefore, $\tilde{\text{Aut}}_\Lambda(P).C$ is a singular projective surface.

The blow-up $B'$ of $X_1 = \{(z_0 : z_1) \times (u_0 : u_1 : u_2) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \mid z_0 u_1 = z_1 u_0\}$ at the closed point $(z_0 : z_1) \times (u_0 : u_1 : u_2) = (0 : 1) \times (0 : 0 : 1)$ can be described as

$$B' = \{(z_0 : z_1) \times (u_0 : u_1 : u_2) \times (t_0 : t_1) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \times \mathbb{P}_k^1 \mid z_0 u_1 = z_1 u_0 \text{ and } z_0 t_1 = u_1 t_0\}. $$
Using an appropriate affine open cover, consisting of 5 affine 2-spaces, it is straightforward to check that $B$ and $B'$ are isomorphic as schemes, giving the diagram \((\ref{fig:example-5.4-digraph})\).

**Example 5.4.** In our last example, $\text{Aut}_\Lambda(P).C$ is a smooth rational projective surface that fails to be relatively minimal. More precisely, $\text{Aut}_\Lambda(P).C$ is isomorphic to the blow-up $B_0$ of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at $\infty \times \infty$ such that both $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$ are relatively minimal models.

Let $\Lambda = kQ/I$, where

$$Q = \begin{pmatrix} \gamma & \alpha \\ \delta & \beta \end{pmatrix}$$

and

$$I = \langle \omega_1^2, \omega_1 \omega_2, \omega_2 \omega_1, \omega_2^2, \alpha \omega_2, \beta \omega_1, \gamma \omega_2, \delta \omega_1 \rangle.$$

As before, let $P = \Lambda e_1$ and

$$C = \Lambda \alpha + \Lambda \beta + \Lambda \gamma + \Lambda \delta + \Lambda (\alpha \omega_1 + \beta \omega_2) = k \alpha + k \beta + k \gamma + k \delta + k (\alpha \omega_1 + \beta \omega_2) \in \text{Grass}_6^2.$$

The module $M = P/C$ is displayed at the top of Figure \ref{fig:example-5.4-digraph}. Again, $\text{Aut}_\Lambda(P).C$ consists of the points

$$C(e_1 + t_1 \omega_1 + t_2 \omega_2) = k(\alpha + \beta + t_1 \alpha \omega_1) + k(\beta + t_2 \beta \omega_2) + k(\gamma + t_1 \gamma \omega_1) + k(\delta + t_2 \delta \omega_2)$$

for $(t_1, t_2) \in k^2$.

Consider the blow-up $B_0$ of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at the closed point $(z_0 : z_1) \times (y_0 : y_1) = (0 : 1) \times (0 : 1)$. More precisely,

$$B_0 = \{(z_0 : z_1) \times (y_0 : y_1) \times (x_0 : x_1) \in \mathbb{P}^1_k \times \mathbb{P}^1_k \times \mathbb{P}^1_k | z_0 x_1 = y_0 x_0 \}.$$

Let $O_1, O_2, O_3, O_4$ and $O_5$ be the affine patches where $z_0 y_0, z_0 y_1, z_1 y_0, z_1 y_1 x_0$ and $z_1 y_1 x_1$, respectively, are not 0. Using these affine patches, one uses calculations similar to those of Example \ref{example:5.3} to produce an isomorphism $\varphi : B_0 \rightarrow \text{Aut}_\Lambda(P).C$ with the following properties.

A point $(z_0 : z_1) \times (y_0 : y_1) \times (x_0 : x_1)$ of $B_0$ with $z_0 \neq 0$ and $y_0 \neq 0$ is sent to the point $C(e_1 + \frac{z_1 \omega_1}{y_0} + \frac{y_1 \omega_2}{y_0}) \in \text{Aut}_\Lambda(P).C$.

The restriction of $\varphi$ to $O_2$ sends a point $(1 : z_1) \times (y_0 : 1) \times (1 : y_0)$ to $C\left(e_1 + z_1 \omega_1 + \frac{1}{y_0} \omega_2\right)$ if $y_0 \neq 0$, and to $\Lambda \alpha + \Lambda \omega_1 + \Lambda \beta \omega_2 + \Lambda (\gamma + z_1 \gamma \omega_1) + \Lambda \delta \omega_2$ if $y_0 = 0$. As $z_1$ traces $k$, the latter points trace the orbit $\text{Aut}_\Lambda(P).E_1$, where $E_1 = \Lambda \alpha + \Lambda \gamma + \Lambda \omega_1 + \Lambda \beta \omega_2 + \Lambda \delta \omega_2$. The corresponding degeneration $P/E_1$ of $M = P/C$ is depicted in the left position of the second row of Figure \ref{fig:example-5.4-digraph}.

Analogously, the restriction of $\varphi$ to $O_3$ sends a point $(z_0 : 1) \times (1 : y_1) \times (z_0 : 1)$ of $O_3$ to $C\left(e_1 + \frac{1}{z_0} \omega_1 + y_1 \omega_2\right)$ if $z_0 \neq 0$ and to $\Lambda \alpha \omega_1 + \Lambda \beta + \Lambda \beta \omega_2 + \Lambda (\delta + y_1 \delta \omega_2)$ if $z_0 = 0$. As $y_1$ traces $k$, the latter points trace the orbit $\text{Aut}_\Lambda(P).E_2$, where $E_2 = \Lambda \alpha \omega_1 + \Lambda \beta + \Lambda \beta \omega_2 + \Lambda \gamma \omega_1 + \Lambda \delta$. The corresponding degeneration $P/E_2$ of $M = P/C$ is depicted in the right position of the second row of Figure \ref{fig:example-5.4-digraph}.

The restriction of $\varphi$ to $O_4$ has the following image on $R = (z_0 : 1) \times (\mu z_0 : 1) \times (1 : \mu)$. If $z_0 \neq 0$ and $\mu \neq 0$, then $\varphi_4(R) = C\left(e_1 + \frac{1}{z_0} \omega_1 + \frac{1}{\mu z_0 \omega_2}\right)$. If $z_0 = 0$ and $\mu \neq 0$, $\varphi_4(R) = \Lambda (\alpha + \mu \beta) + \Lambda \alpha \omega_1 + \Lambda \beta \omega_2 + \Lambda \gamma \omega_1 + \Lambda \delta \omega_2$. As $\mu$ ranges over $k^*$, we obtain the degenerations
Figure 5. The hierarchy of top-stable degenerations in Example 5.4.

\[
\begin{array}{c}
\omega_1 \quad 1 \\
1 \\
\gamma \quad \alpha \quad 2 \\
\omega_2 \quad \beta \\
1 \\
3 2 3 \\
\end{array}
\]

\[
\begin{array}{c}
\omega_1 \quad 1 \\
1 \\
\gamma \quad \alpha \quad 3 \\
\omega_2 \quad \beta \\
1 \\
3 2 3 \\
\end{array}
\]

\[
\begin{array}{c}
\omega_1 \quad 1 \\
1 \\
\gamma \quad \alpha \quad 3 \\
\omega_2 \quad \beta \\
1 \\
3 2 3 \\
\end{array}
\]

\[
\begin{array}{c}
\omega_1 \quad 1 \\
1 \\
\gamma \quad \alpha \quad 3 \\
\omega_2 \quad \beta \\
1 \\
3 2 3 \\
\end{array}
\]

in the third row of Figure 5 corresponding to the points \((1 : \mu) \in \mathbb{P}^1_k\) for \(\mu \in k^*\). If \(z_0 \neq 0\) and \(\mu = 0\), \(\varphi_4(R) = \Lambda \alpha + \Lambda \alpha \omega_1 + \Lambda \beta \omega_2 + \Lambda \left(\gamma + \frac{1}{z_0} \gamma \omega_1\right) + \Lambda \delta \omega_2\); these are degenerations already encountered in the image of \(\varphi_2\). If \(z_0 = 0 = \mu\), then \(\varphi_4(R) = \Lambda \alpha + \Lambda \alpha \omega_1 + \Lambda \beta \omega_2 + \Lambda \gamma \omega_1 + \Lambda \delta \omega_2\). This is the 0-dimensional orbit corresponding to the degeneration in the left position of the fourth row of Figure 5.

The restriction of \(\varphi\) to points \(Q = (\nu y_0 : 1) \times (y_0 : 1) \times (\nu : 1) \in O_5\) yields the following additional degenerations. For \(\nu \neq 0\) and \(y_0 = 0\), \(\varphi_5(Q) = \Lambda(\nu \alpha + \beta) + \Lambda \alpha \omega_1 + \Lambda \beta \omega_2 + \Lambda \gamma \omega_1 + \Lambda \delta \omega_2\). As \(\nu\) ranges over \(k^*\), we obtain the degenerations in the third row of Figure 5 corresponding to the points \((\nu : 1) \in \mathbb{P}^1_k\) for \(\mu \in k^*\). On the other hand, if \(\nu = 0 = y_0\), then \(\varphi_5(Q) = \Lambda \alpha \omega_1 + \Lambda \beta + \Lambda \beta \omega_2 + \Lambda \gamma \omega_1 + \Lambda \delta \omega_2\). This is the 0-dimensional orbit corresponding to the degeneration in the right position of the fourth row of Figure 5.
The blow-up $B'_0$ of $X_1 = \{(z_0 : z_1) \times (u_0 : u_1 : u_2) \in \mathbb{P}^1_k \times \mathbb{P}^2_k \mid z_0 u_1 = z_1 u_0\}$ at the closed point $(z_0, u_2) = (0, 0)$ can be described as

$$B'_0 = \{(z_0 : z_1) \times (u_0 : u_1 : u_2) \times (t_0 : t_1) \in \mathbb{P}^1_k \times \mathbb{P}^2_k \times \mathbb{P}^1_k \mid z_0 u_1 = z_1 u_0 \text{ and } z_0 t_1 = u_2 t_0\}.$$ 

Using an appropriate affine open cover of $B'_0$, consisting of 5 affine 2-spaces, we see that $B_0$ and $B'_0$ are isomorphic as schemes.

**Concluding Questions 5.5.** Let $T$ be a simple module, $P$ its projective cover, and $C \subseteq JP$ such that $\dim \text{Aut}_A(P).C = 2$.

1. Is there a uniform upper bound (not depending on $k$ and $\dim_k C$) on the positive integers $n$ with the property that $\overline{\text{Aut}_A(P).C}$ has a relatively minimal model among $\mathbb{P}^2_k, X_0, X_2, \ldots, X_n$?

2. In Example 5.3, the boundary of $\overline{\text{Aut}_A(P).C}$ has three irreducible components; in all other examples, the number of irreducible components is 1 or 2. Can more than three components be realized in $\text{Aut}_A(P).C - \text{Aut}_A(P).C$ under our side conditions on $T$ and $C$?

3. How does the geometric structure of the surface $\overline{\text{Aut}_A(P).C}$ (resp. of $\overline{\text{Aut}_A(P).C}$) pertain to degeneration-theoretic information about the module $P/C$?

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