Wess–Zumino sigma models with non-Kählerian geometry

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Abstract

Supersymmetry of the Wess–Zumino \((N = 1, D = 4)\) multiplet allows field equations that determine a larger class of geometries than the familiar Kähler manifolds, in which covariantly holomorphic vectors rather than a scalar superpotential determine the forces. Indeed, relaxing the requirement that the field equations be derivable from an action leads to complex flat geometry. The Batalin–Vilkovisky formalism is used to show that if one requires that the field equations be derivable from an action, we once again recover the restriction to Kähler geometry, with forces derived from a scalar superpotential.

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1 Introduction

One of the basic properties of supersymmetric theories is their relation to the geometry of the target manifold. The prime example of supersymmetry is the Wess–Zumino multiplet \([1]\). It was shown early on that in interacting theories of this supermultiplet the kinetic terms for the scalar fields in the action determine Kähler manifolds \([2]\). The interactions are determined by a scalar holomorphic superpotential \(W(z)\), where \(z\) represents the complex scalars of the Wess–Zumino multiplets.

Despite the fact that this multiplet has since then been presented in many courses in the subject, we discover in this paper that it still contains some surprises. Relaxing the requirement that the field equations be derivable from an action, we obtain two generalizations. First, the kinetic terms of the field equations allow a complex flat geometry, which is a larger class of geometries than the usual Kähler geometries. Secondly, the forces are derived from covariantly holomorphic vectors on the target manifold, instead of from a scalar superpotential. The ‘complex flat’ condition on the geometry can be seen as an integrability condition for the existence of these covariantly holomorphic vectors.

Admittedly, a common starting point for passage to a quantum field theory is the classical action. However, consistent sets of classical field equations that are not comprehensively derivable from an action are well-known. For example, type IIB string theory contains a self-dual 5-form field strength, whose self-duality properties are normally stated simply as a field equation. More generally, the effective field equations for the massless modes of string theories arise directly from beta function conditions. The classical action is not a primary construct in this derivation.

In this paper, we wish to distinguish between requirements on the geometry originating strictly from the supersymmetry algebra, and those that follow from the existence of an action. Examples of geometries wider than those expected from action formulations are known in the context of \(N = 2\) hypermultiplets \([3,4]\). A natural question about such examples is whether their special structure relates to the extended supersymmetry, or whether they constitute also exceptions to the expected geometry even of \(N = 1\) supersymmetry. Here, we find that, indeed, there exists a wider class of \(N = 1, D = 4\) Wess–Zumino sigma models than the standard Kählerian ones. This class is determined purely by the supersymmetry algebra. We find that one can even allow geometries that are non-Riemannian, since only the affine connection appears in the supersymmetry transformations and field equations. The allowed geometries are complex flat \([5]\). Moreover, allowed potentials can be derived from covariantly holomorphic vector functions of the scalar fields (which we shall refer to as vector potentials).

Imposing also the requirement that the field equations be derivable from an action leads one back to the standard formulation in terms of Kähler manifolds and with a scalar superpotential. We use the Batalin–Vilkovisky (BV) formalism \([6,7,8]\) to control the off-shell non-closure of the algebra. The sigma model metric makes its appearance here as the matrix that relates the non-closure functional to the variation of the action. The BV consistency conditions then require this metric to be Kählerian. Using the metric to lower the index on the (contravariant) vector potential, BV consistency then requires the curl of the result-
ing (covariant) vector potential to vanish. Hence, the standard superpotential makes its appearance by solving the vanishing curl condition.

The previously known rigid $N = 2$ examples turn out to be specializations of these complex flat geometries. Specifically, the $N = 2$ hypermultiplet geometries are hypercomplex, which bear a similar relation to the hyperkähler ones as the complex flat geometries bear to the Kähler ones. Initially, it was suspected [3, 4] that the origin of the generalized $N = 2$ hypermultiplet geometries lay in the absence of an off-shell formulation for these models. However, in this paper we see that these generalizations appear owing to the wider possibilities that exist for auxiliary field equations when they are not required to be derivable from an action.

In section 2 we show how the closure of the supersymmetry algebra on-shell restricts the allowed field equations. The result, as shown in section 3, is that the target-space manifold must have a complex flat geometry. In section 4 we show all of this again in superspace. In section 5 we show how one recovers the usual Kähler geometry with a superpotential once one requires the field equations to be derivable from an action. We use the BV formalism to give a general derivation that does not depend on any assumption of an off-shell supersymmetric or superspace action. In the concluding section 6 we discuss possible extensions of these results to supergravity couplings.

## 2 Closure conditions constrain the geometry

In order to make as few restrictive assumptions as possible, we will work with explicit component physical fields only, i.e. without auxiliary fields. We will investigate how the $N = 1$, $D = 4$ supersymmetry algebra can be realized on chiral multiplets, with closure following solely from dynamical constraints on the physical fields. Specifically, for 2 supersymmetry transformations with rigid parameters $\epsilon_1$ and $\epsilon_2$, we require the commutator

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \bar{\epsilon}_2 \gamma^\mu \epsilon_1 \partial_\mu = (\bar{\epsilon}_{2L} \gamma^\mu \epsilon_{1R} + \bar{\epsilon}_{2R} \gamma^\mu \epsilon_{1L}) \partial_\mu ,$$

(2.1)

to be realized on all fields, up to dynamical equations of motion.

By (left-handed) chiral multiplets we mean the following. These are multiplets of complex scalars $z^a$, where $a$ is the label of a particular multiplet, together with their spinor superpartners $\zeta^a$. The left-handed spinor $\zeta^a_L$ is defined as normal to satisfy

$$\zeta^a_L = \frac{1}{2}(1 + \gamma_5)\zeta^a_L ,$$

(2.2)

where $(\gamma_5)^2 = 1$. The right-handed spinors are charge conjugates of the corresponding left-handed ones,

$$\zeta^a_R = (\zeta^a_L)^* = \frac{1}{2}(1 - \gamma_5)\zeta^a_R .$$

(2.3)

Without further ado, we assume that the target manifold of the $z^a$ (with complex conjugates $\bar{z}^a$) is a complex manifold, with complex structure

$$J^b_a = i \delta^b_a \quad , \quad J^b_{\bar{a}} = -i \delta^b_{\bar{a}} \quad , \quad J^b_a = J^b_{\bar{a}} = 0 .$$

(2.4)
The definition of a chiral multiplet is now that \( z^a \) transforms only with a (left-handed) chiral supersymmetry parameter:

\[
\delta(\epsilon) z^a = \bar{\epsilon}_L \zeta^a_L, \quad \text{where} \quad \bar{\epsilon}_L = \frac{1}{2} \bar{\epsilon}_L (1 + \gamma_5). \tag{2.5}
\]

Note that the Lorentz conjugate \( \bar{\epsilon}_L \) used here may also be given as \( \epsilon^+_R \gamma^0 \). Accordingly, \( \zeta^a_L \) has the same holomorphic index type as \( \zeta^a_L \).

Now we look for the most general transformation of the fermions. This can be parametrized as

\[
\delta(\epsilon) \zeta^a_L = \gamma^\mu \epsilon^a_R X^a_{\mu} + \gamma^\mu \nu \epsilon^a_L T^a_{\mu \nu} + h^a \epsilon^a_L. \tag{2.6}
\]

Imposing the supersymmetry algebra (2.1) on the scalars leads to

\[
X^a_{\mu} = \partial_{\mu} z^a, \quad T^a_{\mu \nu} = 0, \tag{2.7}
\]

while \( h^a \) remains an arbitrary complex function of the scalars and fermions. Thus we obtain

\[
\delta(\epsilon) \zeta^a_L = \partial z^a \epsilon^a_R + h^a \epsilon^a_L. \tag{2.8}
\]

Checking the supersymmetry algebra on the spinors, we find that closure (allowing only constraints that are dynamical on the physical fields \( z \) and \( \zeta \)) requires that \( \delta(\epsilon) h \) does not contain \( \epsilon^a_L \):

\[
\delta h^a = \bar{\epsilon}_R \lambda^a_R, \tag{2.9}
\]

where \( \lambda^a_R \) is a so-far undetermined right handed spinor. The resulting algebra for the fermions is

\[
[\delta(\epsilon_1), \delta(\epsilon_2)] \zeta^a_L = \left( \partial_{\mu} \zeta^a_L - \frac{1}{2} \gamma_{\mu} I^a_R \right) \bar{\epsilon}_2 \gamma^\mu \epsilon^a_1, \tag{2.10}
\]

where

\[
I^a_R = \partial \zeta^a_L - \lambda^a_R \tag{2.11}
\]

is the non-closure functional.

We require that \( h^a \) be given in terms of the physical fields. (Normally \( h^a \) would be the on-shell value of the auxiliary field as determined by an action, but we are not making such assumptions here). We assume only that the \( h^a \) are scalars with respect to the Lorentz group. We may expand in powers of fermions; dimensional arguments restrict this to terms at most quadratic in the fermions:

\[
h^a = W^a(z, \bar{z}) + \Gamma^a_{bc}(z, \bar{z}) \bar{\zeta}^b_R \zeta^c_L + \frac{1}{2} \Gamma^a_{bc}(z, \bar{z}) \bar{\zeta}^b_R \zeta^c_L, \tag{2.12}
\]

where \( W^a, \Gamma^a_{bc} \) and \( \Gamma^a_{bc} \) are so-far undetermined functions of the scalars \( (z^a, \bar{z}^a) \). The conditions for the transformation of \( h^a \) not to contain \( \epsilon^a_L \) are

\[
\nabla_b W^a \equiv \partial_b W^a + \Gamma^a_{bc} W^c = 0, \tag{2.13}
\]

\[
\Gamma^a_{bc} = 0, \tag{2.14}
\]

\[
R_{abc}^d \equiv 2 \partial_{[a} \Gamma_{bc]}^d + 2 \Gamma_{[a}^e \Gamma_{bc]}^c \epsilon^e = 0. \tag{2.15}
\]

\(^1\text{See this approach already in [9, section 3.1].}\)
Thus, the final expression for $h^a$ is

$$h^a = W^a(z, \bar{z}) + \frac{1}{2} \Gamma_{bc}^a(z, \bar{z}) \bar{\zeta}_L^b \bar{\zeta}_L^c, \quad (2.16)$$

and the non-closure functional $(2.11)$ is

$$I^a_R = \nabla \zeta_L^a - \zeta_R^b \partial_b W^a - \frac{1}{2} \zeta_R^d \bar{\zeta}_L^b R_{dbc}^a, \quad (2.17)$$

where

$$\nabla \mu \zeta_L^a \equiv \partial_{\mu} \zeta_L^a + \zeta_L^b \Gamma_{bc}^a \partial_{\mu} z^c, \quad R_{dbc}^a \equiv \partial_{\mu} \Gamma_{bc}^a. \quad (2.18)$$

Setting $I^a_R = 0$ gives the fermionic field equation. The bosonic field equation follows from a supersymmetry variation of the fermionic one. This is

$$\Box^a = -\bar{W}^b \partial_b W^a + \frac{1}{2} \zeta_R^d \bar{\zeta}_L^b \nabla_b \partial_c W^a - \frac{1}{2} \zeta_R^d \bar{\zeta}_L^b \nabla_c R_{dbc}^a - \zeta_R^d \bar{\zeta}_L^b \partial_{\mu} \zeta_L^c R_{dbc}^a = 0. \quad (2.19)$$

The target-space covariant derivative $\nabla_b$ was already introduced in $(2.13)$ when acting on a target space vector; on more general tensors like $R_{dbc}^a$ it acts according to the standard covariant rule. From this we already note that $\Gamma$ acts as a connection. We shall return to the details of this geometry in section 3. The covariant d’Alembertian $\Box$ is defined using a pulled back covariant derivative $\nabla_{\mu}$:

$$\Box^a = \nabla_{\mu} \partial_{\mu} z^a + \partial_{\mu} \partial_{\mu} z^a + \Gamma_{bc}^a (\partial_{\mu} z^b) (\partial_{\mu} z^c). \quad (2.20)$$

Then one checks that the supersymmetry transformation of the bosonic field equation leads to no new conditions, i.e. it leads only to equations involving derivatives of the original one. This will become more clear using superspace notation in the next section.

The interesting result here is that we find the requirements $(2.13 - 2.15)$. These lead us to the concept of ‘complex flat’ geometry, as we will show in the next section.

### 3 Complex flat geometry

In the previous section we found a complex geometry, with complex structures defined by $(2.4)$. It has a torsionless affine connection with purely holomorphic indices $\Gamma_{be}^a$ (together with its complex conjugate $\bar{\Gamma}_{be}^a$). The torsionless condition is the symmetry of the connection as is manifest from its introduction in $(2.12)$.

The purely holomorphic character of the connection can be interpreted as the condition for the complex structure $(2.4)$ to be covariantly constant. This promotes the scalar-field target space from an almost complex to a complex manifold. Explicitly, the vanishing of the mixed components implies

$$\nabla_{\bar{c}} J^b_a = -\Gamma_{\bar{c}a}^d J^b_d + \Gamma_{\bar{c}d}^b J^d_a = -2i \Gamma_{\bar{c}a}^b = 0,$$
$$\nabla_{\bar{c}} \bar{J}^b_a = -\Gamma_{\bar{c}a}^d \bar{J}^b_d + \Gamma_{\bar{c}d}^b \bar{J}^d_a = 2i \Gamma_{\bar{c}a}^b = 0. \quad (3.1)$$
The remaining restriction (2.15) that we found on the geometry, now interpreted as the vanishing of the purely holomorphic components of the curvature, defines [5, Definition 6.1] a complex flat manifold.

Aside from the geometry, our results include a covariantly holomorphic vector potential \( W^a \), see (2.13). This generalizes the gradient of the scalar superpotential that appears in the standard sigma models, where one assumes the existence of an action. The integrability condition for such a covariantly holomorphic vector to exist is precisely the complex flat condition \( R^{abcd} = 0 \).

Examples of such geometries are the hypercomplex geometries that were found in the context of \( N = 2 \) supersymmetric field equations without an action, for a system of hypermultiplets. This follows from Proposition 7.1 in [5], which states that a hypercomplex manifold is necessarily complex flat with respect of any one of its complex structures. Therefore the explicit examples based on group manifolds given in [10,4] are therefore also explicit examples of the complex flat manifolds considered here.

### 4 Covariant superspace reformulation

We reformulate our results now in two respects. The transformations of the fermions can be written in a way that is known from the standard supersymmetric sigma models, and we can also introduce a covariant superspace formulation. The former is clear from combining the results (2.8) and (2.16), using a Fierz transformation to give

\[
\delta(\epsilon) \zeta_L^a = \partial z^a \epsilon_R + W^a \epsilon_L + \frac{1}{2} \zeta_L^b \zeta_L^c \epsilon_L \Gamma_{bc}^a \\
= \partial z^a \epsilon_R + W^a \epsilon_L - \zeta_L^b \Gamma_{bc} \delta(\epsilon) z^c.
\]

We shall soon see how this can be interpreted as a covariant superspace transformation.

The superspace formulation of our results uses standard chiral superfields \( \phi^a \), i.e. \( \bar{D}_\alpha \phi^a = 0 \). The dynamics is defined by imposing an extra condition that is the superspace version of (2.16) together with its transformed equations (2.17) and (2.19):

\[
\nabla_\alpha D_\alpha \phi^a = 4iW^a,
\]

where \( \nabla_\alpha \) is defined similarly to \( \nabla_\mu \), i.e.

\[
\nabla_\alpha \lambda_\beta^a = D_\alpha \lambda_\beta^a + D_\alpha \phi^b \Gamma_{bc}^a \lambda_\beta^c.
\]

Equation (4.3) is now recognizable as the component version of the covariantized superspace supersymmetry transformation generated by \( (\nabla_\alpha, \nabla_\alpha) \).

The superspace equation of motion should be covariantly antichiral in order for the equations following from it by differentiation to involve only the expected dynamical equations for the physical fermions and scalars. In particular, we want to avoid the appearance of algebraic constraints on the physical fields.

In order for the left-hand side to be covariantly antichiral (i.e. to be annihilated by \( \nabla_\beta \)), we require \( \{\nabla_\alpha, \nabla_\beta\} = 0 \). This in turn requires (2.15). For the right-hand side this is accomplished by having \( W^a \) be covariantly holomorphic, i.e. (2.13). Thus, the superspace formulation succinctly summarizes all of the geometrical conclusions arrived at in section 2.
5 Requiring an action recovers the standard geometry

We will now see how requiring that equations of motion be derivable from an action leads one back to Kähler geometry and forces derived from a scalar superpotential. The above complex flat geometry, as far as one can see, is free from the inconsistencies that may accompany equations that are not derivable from an action. In particular, requiring the geometry to be complex flat and the vector potential to be covariantly holomorphic precludes the appearance of hidden constraints in the equations of motion. It is clear that these equations cannot be derived from a standard action, however, because a sigma-model action requires a target space metric, which we have not even defined. In short, the above geometries are purely affine.

Clearly it is of interest to see what further restrictions on the geometry arise when one requires the existence of an action. Instead of following the textbook approach that starts from a superspace formulation [2], we will try to make as few assumptions as possible, and will derive the Kähler geometry from the BV formalism without specifying a particular classical action.

The Batalin–Vilkovisky (BV) or ‘antifield’ formalism [6,7,8] is a convenient bookkeeping device to keep track of the elementary relations of gauge theories. Here we consider only rigid supersymmetric theories, but we can nonetheless use a formalism with ghosts (and their antifields) that are taken to be constants rather than spacetime-dependent fields. We introduce ghosts $c^\mu$ for translations and $c^L$ and $c^R$ for supersymmetry.

Briefly, the BV formalism introduces an antifield $\Phi^*_A$ for every field $\Phi^A$, including the ghosts, and the main task is to ensure the validity of the master equation for the ‘extended’ action $S_{BV}(\Phi, \Phi^*)$:

$$\left( S_{BV}, S_{BV} \right) = 2 S_{BV} \frac{\delta}{\delta \Phi^*_A} \frac{\delta}{\delta \Phi^*_A} S_{BV} = 0 . \quad (5.1)$$

Expanding in terms of antifield number (see table below), this extended action begins with a classical action $S_0(\Phi)$, which we do not specify. Suppressing integrals $\int d^4x$, and using the supersymmetry algebra and physical field transformations and that we already computed, we can write the BV action as

$$S_{BV} = S_0(z, \bar{z}, \zeta_L, \zeta_R) + S_1 + S_2 ,$$

$$S_1 = z^*_a c^\mu \partial_\mu z^a + \zeta^*_L (\partial_\mu \zeta^a_L) c^\mu + z^*_a \zeta^a_L \gamma^\mu c^L + \zeta^*_L [\bar{\zeta} z^a c_R + h^a(z, \bar{z}, \zeta_L) c_L] + \text{h.c.} ,$$

$$S_2 = c^*_\mu \bar{\zeta} L \gamma^\mu c_R + \frac{1}{2} \zeta_L \gamma^\mu \bar{\zeta} R \gamma^\mu c_R g^{ab}(z, \bar{z}) . \quad (5.2)$$

The last term in $S_2$ involves a new quantity $g^{\bar{a}b}(z, \bar{z}^*)$, which in non-BV language determines how the non-closure functional $I^a$ is proportional to the field equations:

$$I^a_R = -g^{ab} \delta S_0 \delta \zeta^b_R . \quad (5.3)$$

\[2\] The BV formalism was also used to determine possibilities in supersymmetry and supergravity in [11,12].

\[3\] For more details on the use of rigid symmetries in the BV formalism, see [13].
In the check of the master equation this occurs in the term proportional to $\bar{c}_L\gamma^\mu c_R\bar{c}_L^*\gamma^\mu$. It is this quantity $g^{ab}$ that will become (the inverse of) the metric.

To see why the action (5.2) stops at antifield number 2, we have to characterize all the fields and antifields in a 3-fold way, listing dimension, ghost number and antifield number for each field and antifield of the theory. In order for the antibracket operation $(\cdot, \cdot)$ to be dimensionality consistent, the dimensions of any field and its antifield should add up to the same number for all pairs. We choose this to be 1. Note that, owing to ghost number conservation, there is an arbitrariness in assigning dimensions. We choose the assignments in table I which have the advantage that no fields are of negative dimension. The action has dimension 2. It has no definite antifield number, and the expansion in (5.2) is according to antifield number. The dimensional arguments can be used to see that $g^{ab}$ can only be a function of the scalars.

One can now easily establish, owing to the absence of objects with negative dimension, that if there were further terms in the BV action, they would have to involve the $c^\mu$ ghosts. We do not expect such terms because the algebra of translations is well behaved, but it can also easily be checked that even if such terms were to occur, they would not spoil the arguments below.

We now consider the terms in $(S, S)$ that are quadratic in the $\zeta^*$ antifield and are cubic in the supersymmetry ghosts. The terms quadratic in $c_L$ and linear in $c_R$ are

$$(S, S) = \tilde{\zeta}_*^a \gamma^\mu \zeta^*_{Ld} \gamma^\mu c_R \partial_d g^{ab}(z, \bar{z}) \tilde{c}_L g^{ab} + \zeta^*_{La} c_L \left[ h^a(z, \bar{z}, \zeta_L) \frac{\delta}{\delta \zeta^a_{Ld}} \gamma^\mu c_R g^{db}(z, \bar{z}) \right].$$

The vanishing of this expression, using (2.16), leads after a few Fierz transformations to

$$(S, S) = -\frac{1}{8} \zeta^*_{La} \gamma^\mu \zeta^*_{Rd} \gamma^\mu c_R \partial_d g^{ab}(z, \bar{z}) \Gamma_{dc}^a z(z, \bar{z}) g^{db}(z, \bar{z}).$$

Table 1: Dimensions, ghost and antifield numbers of all independent fields and their antifields, the action and the auxiliary fields

| field | dim | gh | afn | antifield | dim | gh | afn |
|-------|-----|----|-----|-----------|-----|----|-----|
| $z^a$ | 0   | 0  | 0   | $z^a$     | 1   | -1 | 1   |
| $\zeta^a$ | $\frac{1}{2}$ | 0 | 0 | $\zeta^*$ | $\frac{1}{2}$ | -1 | 1 |
| $c^\mu$ | 0   | 1  | 0   | $c^*_\mu$ | 1   | -2 | 2   |
| $c$   | $\frac{1}{2}$ | 1 | 0   | $c^*$    | $\frac{1}{2}$ | -2 | 2 |
| $h^a$ | 1   | 0  | 0   |           |     |    |     |
| $S$   | 2   | 0  | 0   |           |     |    |     |
This proves that the $g^{ab}$ is covariantly constant for the affine connection that we intro-
duced. We may now define the inverse of this $g^{ab}$ as $g_{ab}$:

\[ g_{ab} g^{bc} = \delta_a^c \, . \]  

(5.6)

Then we have

\[ \partial_c g_{ab} - \Gamma^{\,d}_{ca} g_{db} = 0 \, . \]  

(5.7)

The symmetric part in $(ca)$ of this equation determines that $\Gamma$ is the Levi-Civita connection
of $g$:

\[ \Gamma^{\,c}_{ab} = g^{cd} \partial_{(a} g_{b)d} \, . \]  

(5.8)

The antisymmetric part says that $g$ is a Kähler metric:

\[ \partial_c (g_{ab}) = 0 \, . \]  

(5.9)

We derived (5.3) as a consequence of the BV equations. We can write this now as
(introducing spinor indices $\alpha$ now to be explicit)

\[ g_{\dot{a} \dot{b}} I_{\dot{R} \alpha}^{a} = \delta S_0 \partial \bar{\zeta}^{\alpha} \bar{\beta} \bar{c} \bar{R}^{I} \dot{a} \dot{R} \beta \, . \]  

(5.10)

We have not committed ourselves so far to any particular expression for the classical action $S_0$. But if any $S_0$ should exist, we obtain here an integrability condition

\[ g_{\dot{a} \dot{b}} \frac{\delta}{\partial \bar{\zeta}^{\alpha}} I_{\dot{R} \alpha}^{a} + (\bar{b} \alpha \leftrightarrow \bar{c} \beta) = 0 \, . \]  

(5.11)

Using the explicit expression (2.17) and working to lowest order in the fermions, gives, with $C_{\alpha \beta}$ the (antisymmetric) charge conjugation matrix,

\[ g_{\dot{a} \dot{b}} \partial_{\bar{c}} W^{a} C_{\alpha \beta} + (\bar{b} \alpha \leftrightarrow \bar{c} \beta) = 0 \quad \rightarrow \quad g_{\dot{a} \dot{b}} \partial_{\bar{c}} W^{a} = 0 \, . \]  

(5.12)

As we know meanwhile that $g$ is covariantly constant, we derive from this that

\[ \partial_{\bar{c}} W_{\dot{b}] = 0} \, , \quad \text{with} \quad W_{\dot{b}} = g_{\dot{a} \dot{b}} W^{a} \, . \]  

(5.13)

Thus, $W_{\dot{a}}$ is (locally) derivable from a ‘prepotential’ $W$:

\[ W_{\dot{a}} = \partial_{\dot{a}} \bar{W} \, . \]  

(5.14)

The condition (2.13) then implies that $\bar{W}$ is antiholomorphic. Thus we have recovered all
the usual features of actions for the Wess-Zumino chiral multiplet.
6 Conclusion and discussion

We have found that imposing the supersymmetry algebra on the fields of the WZ chiral multiplet leads to ‘complex flat geometry’ rather than Kähler geometry, \(i.e.\) to connections that have only holomorphic indices (together with their complex conjugates) and that satisfy

\[
R_{abc}^d \equiv 2\partial_{[a} \Gamma_{b|c]}^d + 2\Gamma_{e[a}^d \Gamma_{b|c]}^e = 0. \tag{6.1}
\]

The potential terms depend on arbitrary covariantly holomorphic vectors \(W^a(z, \bar{z})\):

\[
\nabla_b W^a(z, \bar{z}) = 0. \tag{6.2}
\]

If we impose the further requirement that an action invariant under these transformations exists, then we reobtain the usual extra condition that the connection is the Levi-Civita connection of a Kähler metric, and that the potential is derivable from a scalar holomorphic prepotential.

Clearly, a major question about the physical acceptability of the wider class of complex-flat sigma models that we have discussed here concerns their possible quantization. Without an action, standard path-integral approaches do not apply. But there are other ways to quantize using the equations of motion directly, \(e.g.\) Schwinger-Dyson quantization. The effective field equations following from string theory also arise directly from beta function conditions, without the classical action playing a direct rôle. Admittedly, Zamolodchikov’s \(C\)-theorem does give the central charge functional a rôle as an effective action, so the conformal field theory implications of the target space geometries that we have studied would have to be carefully re-examined.

A related point involves the use of dimensional considerations. In our BV examination of the consequences of assuming that an action exists, we restricted the allowed terms to those that are dimensionally consistent with standard second-order scalar kinetic terms. After quantization, these restrictions could be relaxed, potentially allowing the BV discussion to reach a more general conclusion than the standard Kähler geometry with a scalar superpotential that we reobtained for theories derivable from an action.

To summarize the allowed manifolds without and with the assumption that there exists a preserved metric, we have the situation as shown in Table 2. A general curvature tensor with non-zero components \(R_{abc}^d\) (and complex conjugates), allows a general holonomy \(G \ell(n, \mathbb{C})\). When one requires an action, the holonomy group becomes a subgroup of \(\text{SO}(2n)\). The intersection of \(G \ell(n, \mathbb{C})\) with \(\text{SO}(2n)\) is \(U(n)\).

We may also think about how this framework can be embedded into local supersymmetry, \(i.e.\) in the presence of supergravity. This can be approached using superconformal tensor calculus. One first imposes conformal symmetry on the geometry and includes at the same time an extra chiral multiplet, called the compensating multiplet. The conformal symmetry requires the existence of a closed homothetic Killing vector on this extended scalar space. The superconformal symmetry is then gauged using the Weyl multiplet. The latter involves the graviton, the gravitino and a \(U(1)\) gauge field. Indeed, the superconformal group contains such a \(U(1)\), which is the R-symmetry group. Then the superfluous gauge symmetries are
Table 2: Manifolds for chiral multiplet couplings in $D = 4$, $N = 1$ supersymmetry. The corresponding holonomy groups are as indicated, assuming positive definite kinetic energies for theories derivable from an action.

| no preserved metric | with a preserved metric |
|---------------------|-------------------------|
| complex flat $G \ell(n, \mathbb{C})$ | Kähler $U(n)$ |

removed by gauge fixing. This removes one complex scalar (corresponding to dilatation and U(1) gauge fixings) and one fermion (corresponding to the special supersymmetry present in the superconformal group). These steps are similar to what has been done for hypermultiplets in [4] and in a forthcoming paper [13].

In general, supergravity coupling leads to a nontrivial factor in the holonomy group corresponding to the R-symmetry. In the superconformal framework this comes from the U(1) mentioned above. A U(1) holonomy component can also be non-trivial for purely flat space theories without supergravity coupling, but the couplings in the fermionic sector of the theory would then be different from those arising from supergravity coupling. One thing that is known to change in the presence of supergravity coupling is an integrality condition on the scalar target manifold. When an action exists, the Kähler gauge transformation needs to be accompanied by a U(1) super-Weyl transformation upon coupling to supergravity. For theories with an action, the presence of this U(1) factor, taken together with global requirements, leads to a restriction from Kähler manifolds to Hodge-Kähler manifolds [15]. Such manifolds have integral periods for the Kähler form. The analogous structure without the assumption that an action exists remains to be clarified.

We find it amusing that the fundamental multiplet of supersymmetry has reserved some surprises for us, even 29 years after its introduction.

Acknowledgments and Dedication.

We would like to acknowledge stimulating discussions on this work with Ian Kogan, only two weeks before his tragic death. Ian was a wonderful person and an ever-enthusiastic colleague and we shall deeply miss him. This article is dedicated to his memory.

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