The Post correspondence problem in groups

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Abstract. We generalize the classical Post correspondence problem ($\text{PCP}_n$) and its non-homogeneous variation ($\text{NPCP}_n$) to non-commutative groups and study the computational complexity of these new problems. We observe that $\text{PCP}_n$ is closely related to the equalizer problem in groups, while $\text{NPCP}_n$ is connected to the double twisted conjugacy problem for endomorphisms. Furthermore, it is shown that one of the strongest forms of the word problem in a group $G$ (we call it the hereditary word problem) can be reduced to $\text{NPCP}_n$ in $G$ in polynomial time. The main results are that $\text{PCP}_n$ is decidable in a finitely generated nilpotent group in polynomial time, while $\text{NPCP}_n$ is undecidable in any group containing free non-abelian subgroups (though the argument is very different from the classical case of free semigroups). We show that the double endomorphism twisted conjugacy problem is undecidable in free groups of sufficiently large finite rank. We also consider the bounded $\text{PCP}$ and observe that it is in $\text{NP}$ for any group with $\text{P}$-time decidable word problem, meanwhile it is $\text{NP}$-hard in any group containing free non-abelian subgroups. In particular, the bounded $\text{PCP}$ is $\text{NP}$-complete in non-elementary hyperbolic groups and non-abelian right angle Artin groups.

1 Introduction

1.1 Motivation

In this paper, following [19], we continue our research on non-commutative discrete (combinatorial) optimization. Namely, we define the Post correspondence problem ($\text{PCP}$) for an arbitrary algebraic structure and then study this problem together with its variations for an arbitrary group $G$. The purpose of this research is threefold. Firstly, we approach $\text{PCP}$ in a very different context, facilitating a deeper understanding of the nature of $\text{PCP}$ problems in general. Secondly, we try to tackle several interesting algorithmic problems in group theory that are related to $\text{PCP}$, whose time complexity is unknown. Thirdly, we hope to unify several algebraic techniques through the framework of $\text{PCP}$ problems. We refer to [19] for the initial motivation, the set-up of the problems, and initial facts on non-commutative discrete optimization.

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1.2 The Post correspondence problem in algebra

Let \( A \) be an arbitrary algebraic structure in a language \( L \) (for example, a semi-group, a group, or a ring). The Post correspondence problem for \( A \) (abbreviated as \( \text{PCP}(A) \)) is to decide, when given two tuples of equal length \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \) of elements of \( A \), if there is a term \( t(x_1, \ldots, x_n) \) in the language \( L \) such that \( t(u_1, \ldots, u_n) = t(v_1, \ldots, v_n) \) in \( A \). In 1946 Post introduced this problem in the case of free monoids (free semigroups) and proved that it is undecidable [22]. Since then \( \text{PCP} \) took its prominent place in the theory of algorithms and theoretical computer science.

There are some interesting variations of this problem especially in the case of semigroups and groups, which we discuss in detail in Section 2. Here we mention only one, designed specifically for (semi)groups, to which we refer as the general or non-homogeneous Post correspondence problem (\( \text{NPCP} \)). In this case the terms \( t \) are just words in a fixed alphabet \( X \) (or \( X \cup X^{-1} \) in the case of groups), and the problem is to decide, when given two tuples \( u \) and \( v \) of elements in a (semi)group \( S \) as above and two extra elements \( a, b \in S \), if there is a term \( t(x_1, \ldots, x_n) \) such that \( at(u_1, \ldots, u_n) = bt(v_1, \ldots, v_n) \) in \( S \).

Above we described a decision version of \( \text{PCP} \) and \( \text{NPCP} \) in a semigroup (or a group) which is to check if there exists a term \( w \), called a solution, for a given instance of the problem. The search variation of the problem is to find a solution (if it exists) for a given instance. An even more interesting problem is to describe all the solutions for a given instance of the problem. We will have more to say about this in due course.

1.3 Algebraic meaning of PCP and NPCP in groups

Some connections between Post correspondence problems and classical questions in groups are known. We mention some of them here and refer to Sections 3 and 4 for details.

The standard (homogeneous) \( \text{PCP} \) in groups is closely related to the problem of finding the equalizer \( E(\phi, \psi) \) of two group homomorphisms \( \phi, \psi : H \to G \). This equalizer is defined as \( E(\phi, \psi) = \{ w \in H \mid \phi(w) = \psi(w) \} \). In particular, \( \text{PCP} \) in a group \( G \) is the same as deciding if the equalizer of a given pair of homomorphisms \( \phi, \psi \in \text{Hom}(H, G) \), where \( H \) is a free group of finite rank in the variety \( \text{Var}(G) \) generated by \( G \), is trivial or not (see Section 3 for details). Indeed, in this case every tuple \( u = (u_1, \ldots, u_n) \) of elements of \( G \) gives rise to a homomorphism \( \phi_u \) from a free group \( H \) with basis \( x_1, \ldots, x_n \) in the variety \( \text{Var}(G) \) such that \( \phi_u(x_1) = u_1, \ldots, \phi_u(x_n) = u_n \), and vice versa. The equalizer \( E(\phi_u, \phi_v) \) describes all solutions \( w \) for the instance \( u, v \).
It seems that the non-homogeneous Post correspondence problem $\text{NPCP}$ for groups is even more interesting than the standard $\text{PCP}$. Indeed, first of all $\text{NPCP}$ is right in the midst of the endomorphic double twisted conjugacy problem in groups, which is one of the more difficult and less studied group theoretic conjugacy-type problems. In fact, it is shown in Section 3.2 (Proposition 3.2) that the double endomorphism twisted conjugacy problem in a relatively free group in $\text{Var}(G)$ is equivalent to $\text{NPCP}(G)$, and, in general, the double endomorphism twisted conjugacy problem in $G$ $\mathcal{P}$-time reduces to $\text{NPCP}(G)$. Furthermore, we prove in Section 4 that $\text{NPCP}$ in a given group $G$ is intimately related to the word problem in $G$. Namely we show that the hereditary word problem ($\text{HWP}$) in $G$ can be reduced in polynomial time to $\text{NPCP}$ in $G$. Here $\text{HWP}$ in $G$ is to decide, when given an element $w \in G$ and a finite subset $R \subseteq G$, if $w = 1$ in the quotient $H = G/\langle R \rangle_G$ of the group $G$ by the normal subgroup $\langle R \rangle_G$ generated by $R$. Therefore, if $\text{NPCP}$ is decidable in $G$, then there is a uniform algorithm to decide the word problem in every finitely presented (relative to $G$) quotient of $G$. Further, since decidability of $\text{NPCP}$ in $G$ is inherited by all subgroups of $G$, it implies the uniform decidability of $\text{HWP}$ in every subgroup of $G$ (even every section of $G$). Thus, a decision algorithm for $\text{NPCP}$ in $G$ is very powerful and it gives a lot of information about the group $G$. Notice that finitely generated abelian and nilpotent groups have decidable $\text{HWP}$.

1.4 Results

In Section 4 we show that $\text{NPCP}$ is undecidable in every non-abelian free group, as well as in every group containing free non-abelian subgroups. In particular, $\text{NPCP}$ is undecidable in the following groups: non-elementary hyperbolic, non-abelian right angled Artin, braid groups $B_n$ ($n \geq 3$), non-solvable defined by a single relator (thus all one-relator groups with more than two generators), etc. A similar argument shows that the bounded Post correspondence problem in all groups mentioned above is $\text{NP}$-complete. Here in the bounded version of $\text{NPCP}$ one is looking only for solutions (the words $t(x_1, \ldots, x_n)$) whose length is bounded by a given number. We emphasize that the argument used to prove the undecidability results here has nothing to do with the original argument for undecidability of $\text{PCP}$ or $\text{NPCP}$ in free non-commutative semigroups, even though all the groups mentioned above contain such semigroups. In fact, it is still unclear if the $\text{PCP}$ in a free semigroup can be reduced to $\text{PCP}$ in a free non-abelian group. Furthermore, it is still one of the most intriguing open problems whether $\text{PCP}$ in a free non-abelian group is decidable or not.

As a corollary of the undecidability of $\text{NPCP}$ in free non-abelian groups we show that the double endomorphism twisted conjugacy problem in free groups $F_n$
of rank $n \geq 28$ is undecidable. Whether the double endomorphism twisted conjugacy problem is decidable or not in free non-abelian groups of smaller rank remains to be seen.

We also show that free solvable groups $S_{m,n}$ of class $m \geq 3$ and sufficiently high rank $n$ have undecidable double endomorphism twisted conjugacy problem, as well as $\text{NPCP}$. This result is based on examples of finitely presented solvable groups with undecidable word problem constructed by Kharlampovich in [16].

In the opposite direction we show in Section 5 that $\text{PCP}$ is decidable in polynomial time in every finitely generated nilpotent group $G$. This is the best known positive result up to date on $\text{PCP}$ in groups.

2 Post correspondence problems

2.1 The classical Post correspondence problem

Let $A$ be a finite alphabet with $|A| \geq 2$. Denote by $A^*$ the free monoid with basis $A$ viewed as the set of all words in $A$ with concatenation as the multiplication. Let $X$ be an infinite countable set of variables and $X^*$ the corresponding free monoid.

The classical Post correspondence problem ($\text{PCP}$) in $A^*$. Given a finite set of pairs $(g_1, h_1), \ldots, (g_n, h_n)$ of elements of $A^*$, determine if there is a non-empty word $w(x_1, \ldots, x_n) \in X^*$ such that $w(g_1, \ldots, g_n) = w(h_1, \ldots, h_n)$ in $A^*$.

Post showed in [22] that this problem is undecidable (see [26] for a simpler proof).

Nowadays there are several variations of $\text{PCP}$ in $A^*$; the following restricted version is the most typical.

$\text{PCP}_n$ in $A^*$. Let $n$ be a fixed positive integer. Given a finite sequence of pairs $(g_1, h_1), \ldots, (g_m, h_m)$ of $G$, where $m \leq n$, determine if there is a non-empty word $w(x_1, \ldots, x_m) \in X^*$ such that $w(g_1, \ldots, g_m) = w(h_1, \ldots, h_m)$ in $A^*$.

Breaking $\text{PCP}$ into the collection of restricted problems $\text{PCP}_n$ makes the boundary between decidable and undecidable more clear: $\text{PCP}_n$ in $A^*$ is decidable for $n \leq 3$, and undecidable for $n \geq 7$, see [6, 12, 17].

Another version of interest is the non-homogeneous $\text{NPCP}$ in the free monoid $A^*$, in which case an input to $\text{NPCP}$ contains a finite sequence of pairs $(g_1, h_1), \ldots, (g_n, h_n)$ as above and also four elements $a, b, c, d \in A^*$, while the task is to find a word $w(x_1, \ldots, x_n) \in X^*$ such that

$$aw(g_1, \ldots, g_n)b = cw(h_1, \ldots, h_n)d$$

in $A^*$. This problem is also undecidable in $A^*$. 
There are marked variations of the \( \text{PCP}_n \) in \( A^* \), in which case for each pair \( (g_i, h_i) \) in the instance the initial letters in \( g_i \) and \( h_i \) are not equal. These problems are known to be decidable [13]. We refer to [11] for some recent developments on the Post correspondence problem in free semigroups.

Finishing our short survey of known results we would like to mention that \( \text{PCP} \) is undecidable in a free non-abelian semigroup as well (the same argument as for free monoids). Hence, the semigroup version of \( \text{PCP} \) is also undecidable in semigroups containing free non-abelian subsemigroups, in particular, in groups containing free non-abelian subgroups, or solvable not virtually nilpotent groups (they contain free non-abelian subsemigroups).

In what follows we focus only on the group theoretic versions of the Post corresponding problems \( \text{PCP} \) and \( \text{NPCP} \) in groups, which is different from the original semigroup version since one has to take inversion of elements into account.

### 2.2 The Post correspondence problem in groups

Throughout the paper we use the following notation: \( G \) is an arbitrary fixed group generated by a finite set \( A \), and \( F(X) \) the free group with basis \( X = \{x_1, \ldots, x_n\} \). We view elements of \( F(X) \) as reduced words in the set \( X \cup X^{-1} \). Sometimes we denote \( F(X) \) as \( F(x_1, \ldots, x_n) \), or simply as \( F_n \).

As we mentioned earlier, the group theoretic version of the Post correspondence problem involves terms (words) with inversion.

**The Post correspondence problem (PCP) in a group** \( G \). Given a finite set of pairs \( (g_1, h_1), \ldots, (g_n, h_n) \) of elements of a group \( G \), determine if there is a word \( w(x_1, \ldots, x_n) \in F(x_1, \ldots, x_n) \), which is not an identity of \( G \), such that

\[
w(g_1, \ldots, g_n) = w(h_1, \ldots, h_n)
\]

in \( G \).

Several comments are in order here. Recall that an identity on \( G \) is a word \( w(x_1, \ldots, x_n) \) such that \( w(g_1, \ldots, g_n) = 1 \) in \( G \) for any \( g_1, \ldots, g_n \in G \). If the group \( G \) does not have non-trivial identities, then the requirement that \( w \) is not an identity becomes the same as in the original Post formulation that \( w \) is non-empty. Meanwhile, any non-trivial identity \( w(x_1, \ldots, x_n) \) in \( G \) gives a solution to any instance of \( \text{PCP} \) in \( G \), which is not very interesting. Sometimes we refer to words \( w \) which are identities in \( G \) as trivial solutions of \( \text{PCP} \) in \( G \), while the solutions which are not identities in \( G \) are termed non-trivial. In this regard \( \text{PCP}(G) \) asks for a non-trivial solution to \( \text{PCP} \) in \( G \).

In the sequel, unless stated otherwise, by \( \text{PCP} \) for a group \( G \) we always mean the group theoretic version of \( \text{PCP} \) stated above, not the semigroup version.
By definition, \textit{PCP}(G) depends on the given generating set of G; however, it is easy to see that \textit{PCP}(G) for different finite generating sets are polynomial time equivalent to each other, i.e., each one reduces to the other in polynomial time. Since in all our considerations the generating sets are finite, we omit them from notation and write \textit{PCP}(G).

Similar to the classical case one can define the restricted version \textit{PCP}_n for a group G, in which case the number of pairs in each instance of \textit{PCP}_n is bounded by \(n\), and the non-homogeneous version \textit{NPCP} (or \textit{NPCP}_n), where there are some constants involved. Since the non-homogeneous version is of crucial interest for us, we state it precisely.

The non-homogeneous Post correspondence problem (NPCP) in a group G. Given a finite sequence of pairs \((g_1, h_1), \ldots, (g_n, h_n)\) and two pairs \((a_1, b_1)\) and \((a_2, b_2)\) of elements of G (called the \textit{constants} of the instance), determine if there is a word \(w(x_1, \ldots, x_n) \in F(x_1, \ldots, x_n)\) such that

\[ a_1 w(g_1, \ldots, g_n) b_1 = a_2 w(h_1, \ldots, h_n) b_2 \]

in G.

Note that the requirement on the word \(w\) is different for \textit{PCP} and \textit{NPCP}; in the former case \(w\) must not be an identity of the group G, while in the latter case \(w\) can be arbitrary. Given this distinction, it is not immediately clear whether solvability of \textit{NPCP} in a given group implies solvability of \textit{PCP}. Hence we prefer to use the term \textit{non-homogeneous} \textit{PCP} over \textit{general} \textit{PCP} in groups.

Two lemmas are due here.

**Lemma 2.1.** For any group G, \textit{NPCP}(G) is linear time equivalent to the restriction of \textit{NPCP}(G) where the constants \(b_1, b_2, a_2\) are all equal to 1.

**Proof.** Indeed, in the notation above, notice that

\[ a_1 w(g_1, \ldots, g_n) b_1 = a_2 w(h_1, \ldots, h_n) b_2 \]

in G if and only if

\[ a_2^{-1} a_1 w(g_1, \ldots, g_n) b_1 b_2^{-1} = w(h_1, \ldots, h_n), \]

so \textit{NPCP} in G is equivalent to \textit{NPCP} with \(a_2 = 1, b_2 = 1\). Moreover,

\[ a w(g_1, \ldots, g_n) b = w(h_1, \ldots, h_n) \]

in G if and only if

\[ a b b^{-1} w(g_1, \ldots, g_n) b = w(h_1, \ldots, h_n), \]
i.e.,

\[ ab \omega(g_1^b, \ldots, g_n^b) = w(h_1, \ldots, h_n). \]

Hence \( \text{NPCP}(G) \) is linear time equivalent to \( \text{NPCP}(G) \) with \( b_1 = a_2 = b_2 = 1 \), as claimed.

From now on we often assume that in \( \text{NPCP} \) each instance has the three constants \( b_1, b_2, a_2 \) all equal to 1, in which case we denote \( a_1 \) by \( a \) and term it the constant of the instance.

**Lemma 2.2.** For any group \( G \) and for any instance \( (g_1, h_1), \ldots, (g_n, h_n), a \) of \( \text{NPCP}(G) \), all solutions \( w \) to this instance can be described as \( w = w_0u \), where \( w_0 \) is a particular fixed solution to this instance and \( u \) is an arbitrary (perhaps, trivial) solution to \( \text{PCP}(G) \) for the instance \( (g_1, h_1), \ldots, (g_n, h_n) \).

**Proof.** Suppose \( w_0 \) is a particular fixed solution to \( \text{NPCP}(G) \) for the instance \( (g_1, h_1), \ldots, (g_n, h_n), a \), so \( aw_0(g_1, \ldots, g_n) = w_0(h_1, \ldots, h_n) \). If \( w \) is an arbitrary solution to the same instance in \( G \), then

\[ aw(g_1, \ldots, g_n) = w(h_1, \ldots, h_n), \]

so

\[ w^{-1}_0(g_1, \ldots, g_n)w(g_1, \ldots, g_n) = w^{-1}_0(h_1, \ldots, h_n)w(h_1, \ldots, h_n), \]

hence \( u = w^{-1}_0w \) solves \( \text{PCP}(G) \) for the instance \( (g_1, h_1), \ldots, (g_n, h_n) \). Therefore, \( w = w_0u \) as claimed.

Lemma 2.2 shows that to get all solutions of \( \text{NPCP} \) in \( G \) for a given instance one needs only to find a particular solution of \( \text{NPCP}(G) \) and all solutions of \( \text{PCP}(G) \) for the same instance.

As usual in discrete optimization there are several other standard variations of \( \text{PCP} \) problems: bounded, search, and optimal. We mention them briefly now and refer to [19] for a thorough discussion of these types of problems in groups. The bounded version of \( \text{PCP} \) (or \( \text{NPCP} \)) requires that the word \( w \) in question should be of length bounded from above by a given number \( M \). We denote these versions by \( \text{BPCP}(G) \) or \( \text{BNPCP}(G) \). The search variation of \( \text{PCP} \) (or \( \text{NPCP} \)) is to find a word \( w \) that gives a non-trivial solution to a given instance of the problem (if such a solution exists). The optimization version of \( \text{PCP} \) (or \( \text{NPCP} \)) is a variation of the search problem, in which the solution is required to satisfy some “optimal” conditions. In our case, if not stated otherwise, the optimal condition is to find a shortest possible word \( w \) which is a solution to the given instance of the problem.
3 Connections to group theory

3.1 PCP and the equalizer problem

As above, let $G$ be a fixed arbitrary group with a finite generating set $A$, and let $F_n = F(x_1, \ldots, x_n)$ be the free group with basis $X = \{x_1, \ldots, x_n\}$.

An $n$-tuple of elements $g = (g_1, \ldots, g_n) \in G^n$ gives a homomorphism

$$\phi_g : F_n \to G, \quad \phi_g(x_1) = g_1, \ldots, \phi_g(x_n) = g_n,$$

and vice versa, every homomorphism $F_n \to G$ gives a tuple as above. In this sense each instance $(u_1, v_1), \ldots, (u_n, v_n)$ of PCP($G$) can be uniquely described by a pair of homomorphisms

$$\phi_u, \phi_v : F_n \to G,$$

where $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n)$. In this case we refer to such a pair of homomorphisms as an instance of PCP in $G$.

Now, given groups $H, G$, and two homomorphism $\phi, \psi \in \text{Hom}(H, G)$, one can define the equalizer $E(\phi, \psi)$ of $\phi, \psi$ as

$$E(\phi, \psi) = \{w \in H \mid w^\phi = w^\psi\}, \quad (3.1)$$

which is obviously a subgroup of $H$. If $G$ does not have non-trivial identities, then all non-trivial words from $E(\phi, \psi)$ give all solutions to PCP in $G$ for a given instance $\phi, \psi \in \text{Hom}(F_n, G)$. However, if $G$ has non-trivial identities, then some words from $E(\phi, \psi)$ are identities which are not solutions to PCP($G$). To accommodate all the cases at once we suggest replacing the free group $F_n$ above by the free group $F_{G,n}$ in the variety Var($G$) of rank $n$ with basis $\{x_1, \ldots, x_n\}$. Then, similar to the above, every tuple $u \in G^n$ gives rise to a homomorphism

$$\phi_u : F_{G,n} \to G, \quad \phi_u(x_1) = u_1, \ldots, \phi_u(x_n) = u_n,$$

and non-trivial elements of the equalizer $E(\phi_u, \phi_v)$ describe all solutions of PCP($G$) for the instance $u, v \in G^n$. This connects PCP$_n$ in $G$ with the equalizers of homomorphisms from $\text{Hom}(F_{G,n}, G)$.

There are two general algorithmic problems in groups concerning equalizers.

The triviality of the equalizer problem (TEP($H, G$)) for groups $H$ and $G$. Given two homomorphisms $\phi, \psi \in \text{Hom}(H, G)$, decide if the subgroup $E(\phi, \psi)$ in $H$ is trivial or not.

The equalizer problem (EP($H, G$)) for groups $H$ and $G$. Given two homomorphisms $\phi, \psi \in \text{Hom}(H, G)$, find the equalizer $E(\phi, \psi)$. In particular, if EP($H, G$) is finitely generated, then find a finite generating set of $E(\phi, \psi)$. 

The formulation above needs some explanation on how we mean “to find” a subgroup in a group. If the subgroup is finitely generated, then “to find” usually means to list a finite set of generators. It might happen that the subgroup is not finitely generated, but allows a finite set of generators as a normal subgroup, or as a module under some action. In this case to solve $\text{EP}(H, G)$ one has to list a finite set of these generators of $\text{EP}(H, G)$. In this paper we consider equalizers of homomorphisms of finitely generated nilpotent groups, so in this case they are finitely generated and the problem of describing equalizers becomes well-stated.

Equalizers $E(\phi, \psi)$ were studied before, but mostly in the case when $H = G$ and $\phi, \psi$ are automorphisms of $G$. There are few results on equalizers of endomorphisms in groups. Goldstein and Turner have proved in [10] that the equalizer of two endomorphisms of $F_n$ is a finitely generated subgroup in the case where one of the two maps is injective. However, it is not known whether there is an algorithm to decide if the equalizer of two endomorphisms in a free group $F_n$ is trivial or not. Ciobanu, Martino and Ventura showed that generically equalizers of endomorphisms in free groups are trivial [5], so on most inputs in a free non-abelian group $F$, $\text{PCP}(F)$ does not have a solution, in this sense $\text{PCP}(F)$ is generically decidable.

We summarize the discussion above in the following easy lemma.

**Lemma 3.1.** Let $G$ be a group. Then the following holds for any natural $n > 0$:

1. $\text{PCP}_n(G)$ is equivalent (being just a reformulation) to $\text{TEP}$ for homomorphisms from $\text{Hom}(F_{G,n}, G)$.

2. Finding all solutions for a given instance of $\text{PCP}_n(G)$ is equivalent (being just a reformulation) to $\text{EP}(F_{G,n}, G)$ for the same instance.

### 3.2 NPCP and the double twisted conjugacy

Let $\phi, \psi$ be two fixed automorphisms of a group $G$. Two elements $u, v \in G$ are termed $(\phi, \psi)$-double-twisted conjugate if there is an element $w \in G$ such that $uw^\phi = w^\psi v$. In particular, if $\psi = 1$, then $u$ and $v$ are called $\phi$-twisted conjugate, while in the case $\phi = \psi = 1$, $u$ and $v$ are just conjugates of each other in the usual sense. The twisted (or double twisted) conjugacy problem in $G$ is to decide whether or not two given elements $u, v \in G$ are twisted (double twisted) conjugate in $G$ for a fixed pair of automorphisms $\phi, \psi \in \text{Aut}(G)$. Observe that, since $\psi$ has an inverse, the $(\phi, \psi)$-double-twisted conjugacy problem reduces to the $\phi \psi^{-1}$-twisted conjugacy problem, so in the case of automorphisms it is sufficient to consider only the twisted conjugacy problem. This problem is much studied in groups, we refer to [1, 2, 7–9, 23, 24, 27] for some recent results.
Much stronger versions of the problems above appear when one replaces automorphisms by arbitrary endomorphisms $\phi, \psi \in \text{End}(G)$. Not much is known about the double twisted conjugacy problem in groups with respect to endomorphisms.

The next statement (which follows from the discussion above) relates the double-twisted conjugacy problem for endomorphisms to the non-homogeneous Post correspondence problem.

**Proposition 3.2.** Let $G$ be a group generated by a finite set $A = \{a_1, \ldots, a_n\}$. Then the following holds:

1. The double-twisted conjugacy problem for endomorphisms in $G$ is linear time reducible to $\text{NPCP}_n(G)$.
2. If $G$ is relatively free with basis $A$, then the double-twisted conjugacy problem for endomorphisms in $G$ is linear time equivalent to $\text{NPCP}_n(G)$.

### 4 The hereditary word problem and NPCP

It is easy to see that decidability of $\text{PCP}_n$ or $\text{NPCP}_n$ in a group $G$ has some implications for the word problem in $G$. Indeed, an element $g$ is equal to 1 in $G$ if and only if $\text{NPCP}_1$ is solvable in $G$ for the instance consisting of a single pair $(1, 1)$ and the constant $g$. Similarly, if $G$ is torsion-free, then $g = 1$ in $G$ if and only if $\text{PCP}$ is solvable in $G$ for the instance pair $(g, 1)$. In this section we show that the whole lot of word problems in the quotients of $G$ is reducible to $\text{NPCP}$ in $G$.

Let $G$ be a group generated by a finite set $A$. For a subset $R \subseteq G$ by $\langle R \rangle_G$ we denote the normal closure of $R$ in $G$.

**The hereditary word problem (HWP($G$)) in $G$.** Given a finite set $R$ of words in the alphabet $A \cup A^{-1}$, decide whether or not $w$ is trivial in the quotient $G/\langle R \rangle_G$.

Note that this problem can also be stated as the uniform membership problem to normal finitely generated subgroups of $G$. Observe also that $\text{HWP}(G)$ requires a uniform algorithm for the word problems in the quotients $G/\langle R \rangle_G$.

It seems that groups with decidable $\text{HWP}$ are rare. Notice that the hereditary word problem is decidable in finitely generated abelian or nilpotent groups.

**Proposition 4.1.** Let $G$ be a finitely generated group. Then the hereditary word problem in $G$ $\text{P}$-time reduces to $\text{NPCP}(G)$.

**Proof.** Let $A$ be a finite generating set of $G$. Suppose $R$ is a finite set of elements of $G$, represented by words in $A \cup A^{-1}$. Denote $H = G/\langle R \rangle_G$. 


Put

\[ D_R = \{(a, a^{-1}) \mid a \in A\} \cup \{(a^{-1}, a) \mid a \in A\} \]
\[ \cup \{(r, 1) \mid r \in R\} \cup \{(r^{-1}, 1) \mid r \in R\}. \]

**Claim 1.** Let \( w \) be a word \( w \in (A \cup A^{-1})^* \). Then \( w =_H 1 \) if and only if there is a finite sequence of pairs \((u_1, v_1), \ldots, (u_n, v_n) \in D_R\) such that

\[ v_n (\cdots (v_2 (v_1 w_1 u_2) u_2) \cdots) u_n =_G 1. \]  
\[ (4.1) \]

Indeed, if (4.1) holds, then

\[ w =_G v_{n-1}^{-1} \cdots v_1^{-1} (v_n^{-1} u_n^{-1}) u_{n-1}^{-1} \cdots u_1^{-1} =_H 1 \]

since, for every pair \((u, v) \in D_R\), one has \( uv = 1 \) in \( H \).

To show the converse, suppose \( w =_H 1 \), i.e., \( w \in \langle R \rangle_G \). In this case

\[ w =_G w_1 r_1 w_2 \cdots w_m r_m w_{m+1} \]  
\[ (4.2) \]

with \( r_i \in R^{\pm 1}, w_i \in A^* \) and \( w_1 w_2 \cdots w_{m+1} =_G 1 \). Rewriting (4.2) one gets

\[ r_1^{-1} \cdot w_1^{-1} \cdot w \cdot w_1 \cdot 1 =_G w_2 r_2 w_3 \cdots w_m r_m w_{m+1} w_1. \]  
\[ (4.3) \]

Notice that the product on the left is in the form required in (4.1), and the product on the right is in the form required in (4.2). Now the result follows by induction on \( m \). This proves the claim.

**Claim 2.** Let \( R \subseteq (A \cup A^{-1})^* \) be a finite set and \( w \in (A \cup A^{-1})^* \). Then \( \text{NPCP}(G) \) has a solution for the instance \( \hat{D}_R = \{(u, v^{-1}) \mid (u, v) \in D_R\} \) with constant \( w \) if and only if \( w = 1 \) in \( H \).

Indeed, a sequence

\[ (u_1, v_1^{-1}), \ldots, (u_M, v_M^{-1}) \in \hat{D}_R \]  
\[ (4.4) \]

gives a solution to \( \text{NPCP}(G) \) for the instance \( \hat{D}_R \) with constant \( w \) if and only if

\[ wu_1 u_2 \cdots u_M =_G v_1^{-1} v_2^{-1} \cdots v_M^{-1} \iff v_M (\cdots (v_2 (v_1 w_1 u_2) u_2) \cdots) u_M =_G 1, \]

which, by the claim above, is equivalent to \( w =_H 1 \).

This proves Claim 2 together with the proposition.

\[ \square \]

**Corollary 4.2.** Let \( F \) be a free non-abelian group of finite rank. Then \( \text{NPCP}(F) \) is undecidable.

**Proof.** It is known [18] that for any natural number \( n \geq 2 \) there are finitely presented groups with \( n \) generators and undecidable word problem. Therefore, \( \text{HWP}(F) \) is undecidable. By Proposition 4.1 \( \text{NPCP}(F) \) is also undecidable.  
\[ \square \]
For a finite group presentation \( P = \langle a_1, \ldots, a_k \mid r_1, \ldots, r_\ell \rangle \) denote by
\[
N(P) = k + \ell
\]
the total sum of the number of generators and relators in \( P \). Let \( N \) be the least number \( N(P) \) among all finite presentations \( P \) with undecidable word problem. In [3] Borisov constructed a finitely presented group with four generators and twelve relations which has undecidable word problem. Further, by Miller’s refinement of Higman–Neumann–Neumann Theorem (see [18, Corollary 3.7]), this group embeds in a two-generated group with twelve relations, so \( N \leq 2 + 12 = 14 \).

**Corollary 4.3.** Let \( F_n \) be a free group of rank \( n \geq 28 \). Then the endomorphism double twisted conjugacy problem in \( F_n \) (as well as \( \text{NPCP}_n(F_n) \)) is undecidable.

**Proof.** Let \( P_0 = \langle a_1, a_2 \mid r_1, \ldots, r_{12} \rangle \) be the above presentation with undecidable word problem and let \( F_n = \langle a_1, \ldots, a_n \rangle \) be a free group of rank \( n \geq 28 \). Claim 2 in the proof of Proposition 4.1 shows that the word problem in the group \( H \) defined by the presentation \( P_0 \) is polynomial time reducible to \( \text{NPCP}_n(F_n) \), hence the latter is undecidable. Now part (2) of Proposition 3.2 shows that the endomorphism double twisted conjugacy problem in \( F_n \) is also undecidable, as claimed. \( \square \)

Note that the automorphism twisted conjugacy problem is decidable in free groups [1] (see also [4, 21]). Together with Corollary 4.3, this gives:

**Corollary 4.4.** Free groups of rank at least 28 have decidable twisted conjugacy problem but undecidable endomorphism double twisted conjugacy problem.

**Remark 4.5.** Note that for a given group, decidability of the endomorphism double twisted conjugacy problem implies decidability of the twisted conjugacy problem, which in turn implies decidability of the conjugacy problem. It was shown in [2] that the converse to the latter implication is false in general. The above Corollary 4.4 answers E. Ventura’s question of whether or not the converse to the former implication is true.

Similar results hold for free solvable groups. Let \( N_{\text{sol}} \) be the least number \( N(P) \) among all finite presentations \( P \) which define a solvable group with undecidable word problem. In [16] Kharlampovich constructed a finitely presented solvable group with undecidable word problem, so such a number \( N_{\text{sol}} \) exists.

**Corollary 4.6.** Let \( S_{m,n} \) be a free solvable non-abelian group of class \( m \geq 3 \) and rank \( n \geq N_{\text{sol}} \). Then the endomorphism double twisted conjugacy problem in \( S_{m,n} \) (as well as \( \text{NPCP}_n(S_{m,n}) \)) is undecidable.

**Proof.** Similar to the argument in Corollary 4.3. \( \square \)
Observe that it immediately follows from definitions that decidability of \( \text{PCP} \) or \( \text{NPCP} \) in a finitely generated group is inherited by all finitely generated subgroup of \( G \). Therefore, the results above give a host of groups with undecidable \( \text{NPCP} \) (as well as \( \text{NPCP}_n \)).

**Corollary 4.7.** If a group \( G \) contains a free non-abelian subgroup \( F_2 \), then the problem \( \text{NPCP}(G) \) is undecidable.

Therefore \( \text{NPCP} \) is undecidable, for example, in non-elementary hyperbolic groups, non-abelian right angled Artin groups, groups with non-trivial splittings into free products with amalgamation or HNN extensions, braid groups \( B_n \), non-virtually solvable linear groups, etc.

Another corollary of the results above concerns with complexity of the bounded \( \text{NPCP} \) in groups.

**Corollary 4.8.** Let \( F \) be a non-abelian free group of finite rank. Then the bounded \( \text{NPCP}(F) \) is \( \text{NP} \)-complete.

**Proof.** Let \( F = F(A) \) be a free non-abelian group with finite basis \( A \). It is showed in [25, Corollary 1.1] that there exists a finitely presented group \( H = \langle B \mid R \rangle \) with \( \text{NP} \)-complete word problem and polynomial Dehn function \( \delta_H(n) \). Passing to a subgroup of \( F(A) \), we may assume that \( A = B \). One can see that in the case of a free group \( G = F(A) \), \( M \) in (4.4) is bounded by a polynomial (in fact, linear) function of \( |w| \) and the number \( m \) of relators in (4.2) (see [20, Lemma 1] for details). Note that there exists \( m \) as above bounded by \( \delta_H(|w|) \), so \( M \) is bounded by some polynomial \( q(|w|) \). Therefore, the map \( w \rightarrow (w, D, M = q(|w|)) \) is a \( \text{P} \)-time reduction of the word problem in \( H \) to the bounded \( \text{NPCP}(F(A)) \). It follows that the latter is \( \text{NP} \)-hard and therefore \( \text{NP} \)-complete (since the word problem in \( F(A) \) is \( \text{P} \)-time decidable).

**Corollary 4.9.** If a group \( G \) contains a free non-abelian subgroup \( F_2 \), then the bounded \( \text{NPCP}(G) \) is \( \text{NP} \)-hard.

## 5 PCP in nilpotent groups

In this section we study the complexity of Post correspondence problems in nilpotent groups.

**Proposition 5.1.** There is a polynomial time algorithm that given finite presentations of groups \( A, B \) in the class of abelian groups and a homomorphism \( \phi : A \rightarrow B \) computes a finite set of generators of the kernel of \( \phi \).
Proof. Results of [14] provide a polynomial time algorithm to bring an integer matrix to its canonical diagonal (Smith) normal form. Since computing the canonical presentation of a finitely presented abelian group reduces by a standard argument to finding Smith form of an integer matrix (determined by relators in a given presentation), we may find in polynomial time the canonical presentation of $B$, i.e., a direct decomposition $B = \mathbb{Z}^l \times K$, where $K$ is a finite abelian group. Once $B$ is in its canonical form, computing the kernel of $\phi$ reduces to solving a system of linear equations in $\mathbb{Z}^l$ and $K$, which can be done in polynomial time by the same results [14].

Corollary 5.2. There is a polynomial time algorithm that, given finite presentations of groups $A$, $B$ in the class of abelian groups, and $\phi, \psi \in \text{Hom}(A, B)$, computes a finite set of generators of the equalizer $E(\phi, \psi)$.

Proof. Observe that the map $\xi : A \to B$ defined by $\xi(g) = \phi(g)\psi(g)^{-1}$ is a homomorphism from $A$ to $B$ and $E(\phi, \psi) = \ker \xi$. Now the result follows from Proposition 5.1.

One can slightly strengthen the corollaries above.

Corollary 5.3. Let $c$ be a fixed positive integer.

1. There is a polynomial time algorithm that, given a finite presentation of a group $A$, and a finite presentation of a group $B$ in the class of abelian groups, and a homomorphism $\phi \in \text{Hom}(A, B)$, computes a finite set of generators of the kernel $\ker \phi$ modulo the commutator $[A, A]$.

2. There is a polynomial time algorithm that, given a finite presentation of a group $A$, and a finite presentation of a group $B$ in the class of abelian groups, and homomorphisms $\phi, \psi \in \text{Hom}(A, B)$, computes a finite set of generators of the equalizer $E(\phi, \psi)$ modulo the commutator $[A, A]$.

Proof. Follows immediately from Proposition 5.1 and Corollary 5.2.

By $\gamma_c(G)$ we denote the $c$-th term of the lower central series of $G$. Recall that the iterated commutator of elements $g_1, \ldots, g_c$ is

$$[g_1, g_2, \ldots, g_c] = [\ldots[[g_1, g_2], g_3], \ldots].$$

The following lemma is well known (for example, see [15, Lemma 17.2.1]).

Lemma 5.4. Let $G$ be a group generated by elements $x_1, \ldots, x_n \in G$. Then $\gamma_c(G)$ is generated as a subgroup by $\gamma_{c+1}(G)$ and iterated commutators $[x_{i_1}, \ldots, x_{i_c}]$. 
Lemma 5.5. Let $c_0$ be a fixed positive integer. There is a polynomial time algorithm that, given a finite group presentation of a group $G$ in the class of nilpotent groups of class $\leq c_0$, finds subgroup generators of $[G, G]$.

Proof. Follows from Lemma 5.4 by an inductive construction since there are at most $n^{c_0+1}$ iterated commutators $[x_{i_1}, \ldots, x_{i_c}]$, $c \leq c_0$, in a group generated by $n \geq 2$ elements $x_1, \ldots, x_n$ (the case $n = 1$ is obvious). \qed

Theorem 5.6. Let $c_0$ be a fixed positive integer. Then there is a polynomial time algorithm that, given positive integers $c_H, c_G \leq c_0$, finite presentations of groups $H, G$ in the classes of nilpotent groups of class $c_H$ and $c_G$, respectively, and homomorphisms $\phi, \psi \in \text{Hom}(H, G)$, computes a generating set of the equalizer $E(H, \phi, \psi)$ as a subgroup of $H$.

Proof. Let $Y$ and $Z$ be finite generating sets of $H$ and $G$, respectively. We use induction on the nilpotency class $c = c_G$ of $G$. If $c = 1$, then $G$ is abelian and the result follows from Corollary 5.3 (2).

Suppose now that $c > 1$ and we are given $\phi, \psi \in \text{Hom}(H, G)$. Consider the quotient group $G = G/\gamma_c(G)$, which is a nilpotent group of class $c - 1$. The homomorphisms $\phi, \psi$ induce some homomorphisms $\phi', \psi' \in \text{Hom}(H, G)$. Observe that the size of $\phi'(y), \psi'(y), y \in Y$ (as words in $Z$) is the same as that of $\phi, \psi$. Also observe that $\bar{G}$ is described in the class of nilpotent groups of class $c - 1$ by the same presentation that describes $G$ in the class of nilpotent groups of class $c$. By induction we can compute in polynomial time a finite generating set, say $L = \{h_1, \ldots, h_k\}$, of $E' = E(H, \phi', \psi')$ as a subgroup of $H$. By construction, for $g \in E'$ one has $\phi'(g) = \psi'(g) \mod \gamma_c(G)$, hence the map $\xi(g) = \phi(g)\psi(g)^{-1}$ defines a homomorphism $\xi : E' \to \gamma_c(G)$. Obviously, $E(\phi, \psi) = \ker \xi$. Further, note that the size of $L$ is polynomial in terms of the size of the input, and the size of a generating set for $\gamma_c(G)$ is polynomial (of degree that depends on $c$) in terms of the size of a generating set for $G$ by Lemma 5.4. Now the result follows from Corollary 5.3 (1), since $\gamma_c(G)$ is abelian, and Lemma 5.5 \qed

Theorem 5.7. Let $c$ be a fixed positive integer.

1. Let $G$ be a finitely generated nilpotent group of class $c$. Then, for any homomorphisms $\phi, \psi \in \text{Hom}(F_n, G)$, the subgroup $E(\phi, \psi) \leq F_n$ contains the subgroup $\gamma_{c+1}(F_n)$ and is finitely generated modulo $\gamma_{c+1}(F_n)$.

2. There is a polynomial time algorithm that, given a positive integer $n$, a presentation of a group $G$ in the class of nilpotent groups of class $c$, and homomorphisms $\phi, \psi \in \text{Hom}(F_n, G)$, computes a finite set of generators of $E(\phi, \psi)$ in $F_n$ modulo the subgroup $\gamma_{c+1}(F_n)$. 
Proof. Let $F_n = F_n(X)$ with $X = \{x_1, \ldots, x_n\}$. Fix $\phi, \psi \in \text{Hom}(F_n, G)$. Since the group $G$ is nilpotent of class $c$, one has $\gamma_{c+1}(G) = 1$, so $E(\phi, \psi) \succeq \gamma_{c+1}(F_n)$. The quotient $N_{n,c} = F_n/\gamma_{c+1}(F_n)$ is the finitely generated free nilpotent group of rank $n$ and class $c$, hence every subgroup, in particular the image $\bar{E}$ of $E(\phi, \psi)$, is finitely generated. It follows that the group $E(\phi, \psi)$ is finitely generated modulo $\gamma_{c+1}(F_n)$. This proves (1). Notice that the argument above allows one to reduce everything to the case of nilpotent groups, i.e., to consider the induced homomorphisms $\tilde{\phi}, \tilde{\psi} \in \text{Hom}(N_{n,c}, G)$, instead of $\phi, \psi$, and the subgroup $\bar{E}$ instead of $E(\phi, \psi)$. Now the result follows from Theorem 5.6.

\textbf{Theorem 5.8.} Let $G$ be a finitely generated nilpotent group. Then $\text{PCP}_n(G) \in \text{P}$ for every $n \in \mathbb{N}$.

Proof. Indeed, by Theorem 5.7, one can compute in $\text{P}$-time a finite set of elements $h_1, \ldots, h_m \in F_n$ such that $E(\phi, \psi) = \langle h_1, \ldots, h_m, \gamma_{c+1}(F_n) \rangle$. Now the instance of $\text{PCP}_n$ defined by $(\phi, \psi)$ has a non-trivial solution in $G$ if and only if there is $i$ such that $\phi(h_i) \neq 1$ in $G$. Indeed, in this case $\phi(h_i) = \psi(h_i) \neq 1$ in $G$. Otherwise, $\phi(E(\phi, \psi)) = 1$ in $G$ so there is no non-trivial solution in $G$ to the instance of $\text{PCP}_n$ determined by $\phi$ and $\psi$. This proves the theorem.

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\textbf{Bibliography}

[1] O. Bogopolski, A. Martino, O. Maslakova and E. Ventura, The conjugacy problem is solvable in free-by-cyclic groups, \textit{Bull. Lond. Math. Soc.} \textbf{38} (2006), 787–794.

[2] O. Bogopolski, A. Martino and E. Ventura, Orbit decidability and the conjugacy problem for some extensions of groups, \textit{Trans. Amer. Math. Soc.} \textbf{362} (2010), 2003–2036.

[3] V. Borisov, Simple examples of groups with unsolvable word problem, \textit{Math. Notes} \textbf{6} (1969), 768–775.

[4] M. Bridson and D. Groves, The quadratic isoperimetric inequality for mapping tori of free group automorphisms, \textit{Mem. Amer. Math. Soc.} \textbf{955} (2010).

[5] L. Ciobanu, A. Martino and E. Ventura, The generic Hanna Neumann conjecture and Post correspondence problem, preprint (2008), http://www.epsem.upc.edu/~ventura/ventura/engl/abs-e.htm#(31).

[6] A. Ehrenfeucht, J. Karhumäki and G. Rozenberg, The (generalized) Post correspondence problem with lists consisting of two words is decidable, \textit{Theoret. Comput. Sci.} \textbf{21} (1982), 119–144.
[7] A. Fel’shtyn, Reidemeister number of any automorphism of a Gromov hyperbolic group is infinite, preprint (2001), http://arxiv.org/pdf/math/0101010v1.pdf.

[8] A. Fel’shtyn, Y. Leonov and E. Troitsky, Twisted conjugacy classes in saturated weakly branch groups, Geom. Dedicata 134 (2008), 61–73.

[9] A. Fel’shtyn and E. Troitsky, Twisted conjugacy separable groups, preprint (2012), http://arxiv.org/abs/math/0606764.

[10] R. Z. Goldstein and E. C. Turner, Fixed subgroups of homomorphisms of free groups, Bull. Lond. Math. Soc. 18 (1986), 468–470.

[11] V. Halava and T. Harju, Some new results on Post correspondence problem and its modifications, Bull. Eur. Assoc. Theor. Comput. Sci. EATCS 73 (2001), 131–141.

[12] V. Halava, T. Harju and M. Hirvensalo, Binary (generalized) Post correspondence problem, Theoret. Comput. Sci. 276 (2002), 183–204.

[13] V. Halava, M. Hirvensalo and R. de Wolf, Decidability and undecidability of marked PCP, in: Proceedings of the 16th Annual Conference on Theoretical Aspects of Computer Science – STACS’99 (Trier 1999), Springer-Verlag, Berlin (1999), 207–216.

[14] R. Kannan and R. Bachem, Polynomial time algorithms for computing Smith and Hermite normal forms of an integer matrix, SIAM J. Comput. 8 (1979), 499–507.

[15] M. I. Kargapolov and J. I. Merzlyakov, Fundamentals of Theory of Groups, Grad. Texts in Math. 62, Springer-Verlag, New York, 1979.

[16] O. Kharlampovich, A finitely presented solvable group with unsolvable word problem, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), 852–873.

[17] Y. Matiyasevich and G. Senizergues, Decision problems for semi-Thue systems with a few rules, Theoret. Comput. Sci. 330 (2005), 145–169.

[18] C. F. Miller III, Decision problems for groups – survey and reflections, in: Algorithms and Classification in Combinatorial Group Theory (Berkeley 1989), Springer-Verlag, New York (1992), 1–59.

[19] A. Myasnikov, A. Nikolaev and A. Ushakov, Knapsack problems in groups, Math. Comp., to appear.

[20] A. Ol’shanskii and M. Sapir, Length and area functions on groups and quasi-isometric Higman embeddings, Internat. J. Algebra Comput. 11 (2001), 137–170.

[21] A. Ol’shanskii and M. Sapir, Groups with small Dehn functions and bipartite chord diagrams, Geom. Funct. Anal. 16 (2006), 1324–1376.

[22] E. L. Post, A variant of a recursively unsolvable problem, Bull. Amer. Math. Soc. 52 (1946), 264–268.

[23] V. Romankov, The twisted conjugacy problem for endomorphisms of polycyclic groups, J. Group Theory 13 (2010), 355–364.
[24] V. Romankov, Twisted conjugacy classes in nilpotent groups, *J. Pure Appl. Algebra* **215** (2011), 664–671.

[25] M. V. Sapir, J.-C. Birget and E. Rips, Isoperimetric and isodiametric functions of groups, *Ann. of Math. (2)* **156** (2002), 345–466.

[26] M. Sipser, *Introduction to the Theory of Computation*, 2nd ed., Thompson, Boston, 2006.

[27] E. Ventura and V. Romankov, The twisted conjugacy problem for endomorphisms of metabelian groups, *Algebra Logic* **48** (2009), 89–98.

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