The harmonic analysis represents a fundamental tool for the study of partial differential equations. Beside providing explicit expression for the solutions, it appears often as a second step of the investigation, once theorems of existence and uniqueness are established by other methods. The aim is then to provide a more precise insight to the properties of the solutions, by taking care simultaneously of the values of the function in the space domain, as well as of the frequency components. This proceeding is sometimes named micro-local analysis, synonym of time-frequency analysis or phase-space analysis.

Ideally, one would like to know exactly the frequencies occurring at a certain point for the solution. This is however out of reach, in view of the uncertainty principle of Heisenberg. So instead, we fix a partition of the phase-space into sufficiently large subsets, split consequently the function into wave packets, and establish which wave packets are present, or dominant, in the expression.

Such micro-local decomposition can be done in different ways, the choice depending on the equation and on the problem under consideration. The aim is to obtain a sparse representation of the resolvent, or propagator. Namely, fixing attention on the Cauchy problem we want that the wave packets of the initial datum are moved, at any fixed time $t \neq 0$, in a well determined way, so that only a controlled number of overlappings is allowed. Sparsity is extremely important in the numerical applications, by suggesting a natural proceeding of approximation.

In the present paper we choose as micro-local decomposition the Gabor decomposition, corresponding geometrically to a uniform partition of the phase-space into boxes, each wave packet occupying a box, essentially. Following [15], we shall apply the Gabor decomposition to a class of evolution equations. We shall fix here attention on parabolic equations, performing some numerical experiments.
We begin by recalling the definition of Gabor frame, addressing to the next Section 2 for details and notation.

Fix a function \( g \in L^2(\mathbb{R}^d) \) and consider the time-frequency shifts
\[
\pi(\lambda)g = e^{2\pi inx}g(x-m), \quad \lambda = (m,n) \in \Lambda,
\]
for some lattice \( \Lambda \subset \mathbb{R}^{2d} \). The set of functions \( \{\pi(\lambda)g\}_{\lambda \in \Lambda} \) is called Gabor system. If moreover there exist \( A, B > 0 \) such that
\[
A\|f\|^2_{L^2} \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|^2_{L^2}
\]
for every \( f \in L^2(\mathbb{R}^d) \), we say that \( \{\pi(\lambda)g\}_{\lambda \in \Lambda} \) is a Gabor frame; see e.g. [20, 30, 51, 23].

Gabor frames have found important applications in signal processing and, more generally, to several problems in Numerical Analysis, see e.g. [6, 46], and the references therein. More recently, the decomposition by means Gabor frames was applied to the analysis of certain partial differential equations, in particular the constant coefficient Schrödinger, wave and Klein-Gordon equations [1, 2, 9, 19, 31, 32, 33, 36, 52, 53, 54, 55].

We also refer to the survey [43] and the monograph [56]. The analysis of variable coefficients Schrödinger-type operators was carried out in [7, 8, 11, 47] for smooth symbols and in [16, 17] in the analytic category; see also [37].

The fact of the matter is that, together with the decomposition of functions, say by a Gabor frame, there is a corresponding decomposition of operators; namely a linear operator \( T \) can be regarded as the infinite matrix
\[
\langle T\pi(\mu)g, \pi(\lambda)g \rangle, \quad \lambda, \mu \in \Lambda.
\]

The more this matrix is sparse, the more this representation is useful, both for theoretical and numerical purposes.

In the applications of evolution equations, \( T \) will be the propagator of some well-posed Cauchy problem, and will belong to some class of pseudodifferential operators (PSDO), or Fourier integral operators (FIO).

In [15] we have shown that Gabor frames may work as appropriate tool for theoretical and numerical analysis of the Cauchy problem for a large class of partial differential equations, including hyperbolic, weakly hyperbolic and parabolic equations with constant coefficients.

By fixing for a moment attention on the hyperbolic case, Gabor’s approach may certainly look striking, since for the corresponding solutions the analysis is limited, in the most part of the literature, to the precise location of singularities in the space variables, the treatment of the frequency components being somewhat rough. Namely, in [3, 4, 29] and many others, the wave packets (the Hörmander’s wave-front set) are concentrated in a neighborhood, as small as we want, of each point \( x_0 \) in the space variables, geometrically multiplied by a conic neighborhood of \( \xi_0 \) in the frequency space, providing as a whole an infinite large domain.

So, the Gabor’s approach and Hörmander’s approach are both compatible with the uncertainty principle of Heisenberg. The information given on the solutions of the hyperbolic equations are however quite different. By the Gabor analysis, in fact, we cannot identify any more where singularities exactly are, on the other hand the information on the frequency components is much more precise.
As disadvantage of the Gabor analysis, we also observe that Gabor frames do not work as soon as the hyperbolic operator is allowed to have non-constant coefficients. A simple example is given by the transport equation

$$\partial_t u - \sum_{j=1}^{d} a_j(x) \partial_{x_j} u = 0, \quad u(0, x) = u_0(x),$$

whose solution at a fixed time $t \neq 0$ is expressed by a change of variables in $u_0(x)$. A nonlinear change of variable is well-behaved with respect to Hörmander’s wave front set [29, Theorem 8.2.4, Vol. I], whereas its representation with respect to Gabor frames is not sparse, cf. [12, 14].

As advantage of the Gabor decomposition, apart from detecting the frequency components, we emphasize that the same procedure works also for weakly hyperbolic equations and parabolic equations, whose numerical analysis is usually performed in a different way. Besides, for all these equations, we have exponentially sparse representation of the propagator $T$:

$$|\langle T \pi(\mu) g, \pi(\lambda) g \rangle| \lesssim \exp(-\epsilon |\lambda - \mu|^{1/s}),$$

for every $\lambda, \mu$ in the lattice $\Lambda$, and for some positive constants $s, \epsilon$.

The contents of the next sections is the following. In Section 2 we recall some results on Gelfand-Shilov spaces, cf. [22, 38], and time-frequency representations, cf. [5, 18, 23, 27, 46, 49]. Section 3 is devoted to the almost-diagonalization (sparsity) of pseudodifferential operators. Basic references here are [15, 26], see also [24, 25, 42]. Section 4 concerns applications to evolution equations. The numerical experiments, which are new with respect to [15], are given in 4.2, 4.3.

2. Preliminaries

2.1. Notations. We denote the Schwartz class by $S(\mathbb{R}^d)$ and the space of tempered distributions by $S'(\mathbb{R}^d)$. We use the brackets $\langle f, g \rangle$ to denote the extension to $S'(\mathbb{R}^d) \times S(\mathbb{R}^d)$ of the inner product $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$ on $L^2(\mathbb{R}^d)$.

We denote the Euclidean norm of $x \in \mathbb{R}^d$ by $|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$, and $\langle x \rangle = (1 + |x|^2)^{1/2}$. We set $xy = x \cdot y$ for the scalar product on $\mathbb{R}^d$, for $x, y \in \mathbb{R}^d$.

The Fourier transform is normalized to be $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(t) e^{-2\pi i t \xi} dt$. We define the translation and modulation operators, $T$ and $M$, by

$$T_x f(\cdot) = f(\cdot - x) \quad \text{and} \quad M_x f(\cdot) = e^{2\pi i x \cdot \cdot} f(\cdot), \quad x \in \mathbb{R}^d.$$  

For $z = (x, \xi)$ we shall also write

$$\pi(z) f = M_\xi T_x f.$$

We shall use the notation $A \lesssim B$ to express the inequality $A \leq cB$ for a suitable constant $c > 0$, and $A \asymp B$ for the equivalence $c^{-1}B \leq A \leq cB$. 

2.2. Gelfand-Shilov Spaces. Gelfand-Shilov spaces can be considered a refinement of the Schwartz class, and they turn out to be useful when a more quantitative information about regularity and decay is required. Let us recall their definition and main properties; see [22, 38] for more details and proofs.

**Definition 2.1.** Let there be given $s, r > 0$. The Gelfand-Shilov type space $S^s_r(\mathbb{R}^d)$ is defined as all functions $f \in \mathcal{S}(\mathbb{R}^d)$ such that

$$|x^\alpha \partial^\beta f(x)| \lesssim A^{\alpha |\alpha|} B^{\beta |\beta|} (\alpha!)^r (\beta!)^s, \quad \alpha, \beta \in \mathbb{N}^d,$$

for some $A, B > 0$.

We observe that the space $S^s_r(\mathbb{R}^d)$ is nontrivial if and only if $r + s \geq 1$. So the smallest nontrivial space with $r = s$ is provided by $S^{1/2}_{1/2}(\mathbb{R}^d)$. Every function of the type $P(x)e^{-a|x|^2}$, with $a > 0$ and $P(x)$ polynomial on $\mathbb{R}^d$, is in the class $S^{1/2}_{1/2}(\mathbb{R}^d)$. We observe the trivial inclusions $S^1_{r_1}(\mathbb{R}^d) \subset S^1_{r_2}(\mathbb{R}^d)$ for $s_1 \leq s_2$ and $r_1 \leq r_2$.

The Fourier transform maps $S^s_r(\mathbb{R}^d) \to S^r_s(\mathbb{R}^d)$. Therefore for $s = r$ the spaces $S^s_s(\mathbb{R}^d)$ are invariant under the action of the Fourier transform.

**Theorem 2.2.** Assume $s > 0, r > 0, s + r \geq 1$. For $f \in \mathcal{S}(\mathbb{R}^d)$, the following conditions are equivalent:

a) $f \in S^s_r(\mathbb{R}^d)$.

b) There exist constants $A, B > 0$, such that

$$\|x^\alpha f\|_{L^\infty} \lesssim A^{\alpha |\alpha|} (\alpha!)^r \quad \text{and} \quad \|\xi^\beta \hat{f}\|_{L^\infty} \lesssim B^{\beta |\beta|} (\beta!)^s, \quad \alpha, \beta \in \mathbb{N}^d.$$

c) There exist constants $A, B > 0$, such that

$$\|x^\alpha f\|_{L^\infty} \lesssim A^{\alpha |\alpha|} (\alpha!)^r \quad \text{and} \quad \|\partial^\beta f\|_{L^\infty} \lesssim B^{\beta |\beta|} (\beta!)^s, \quad \alpha, \beta \in \mathbb{N}^d.$$

d) There exist constants $h, k > 0$, such that

$$\|f e^{h|x|^{1/r}}\|_{L^\infty} < \infty \quad \text{and} \quad \|f e^{k|x|^{1/s}}\|_{L^\infty} < \infty.$$

The dual spaces of $S^s_s(\mathbb{R}^d)$ are called spaces of tempered ultra-distributions and denoted by $(S^s_s)'(\mathbb{R}^d)$. Notice that they contain the space of tempered distribution $\mathcal{S}'(\mathbb{R}^d)$.

Finally a kernel theorem holds as usual (34, 35, 50).

**Theorem 2.3.** There exists an isomorphism between the space of linear continuous maps $T$ from $S^s_r(\mathbb{R}^d)$ to $(S^s_s)'(\mathbb{R}^d)$ and $(S^s_s)'(\mathbb{R}^{2d})$, which associates to every $T$ a kernel $K_T \in (S^s_s)'(\mathbb{R}^{2d})$ such that

$$\langle Tu, v \rangle = \langle K_T, v \otimes \bar{u} \rangle, \quad \forall u, v \in S^s_r(\mathbb{R}^d).$$

$K_T$ is called the kernel of $T$. 
2.3. Time-frequency representations. We recall the basic definition and tools from time-frequency analysis and refer the reader to [23] for a complete presentation.

Consider a distribution \( f \in S'(\mathbb{R}^d) \) and a Schwartz function \( g \in S(\mathbb{R}^d) \setminus \{0\} \), which will be called window. The short-time Fourier transform (STFT) of \( f \) with respect to \( g \) is \( V_g f(z) = \langle f, \pi(z)g \rangle \), \( z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \). The short-time Fourier transform is well-defined whenever the bracket \( \langle \cdot, \cdot \rangle \) makes sense for dual pairs of function or (ultra-)distribution spaces, in particular for \( f \in S'(\mathbb{R}^d) \) and \( g \in S(\mathbb{R}^d) \), \( f, g \in L^2(\mathbb{R}^d) \), or \( f \in (S^s_r)'(\mathbb{R}^d) \) and \( g \in S^s_r(\mathbb{R}^d) \).

The discrete counterpart of the above time-frequency representation is given by the Gabor frame. Namely, let \( \Lambda = A\mathbb{Z}^{2d} \) with \( A \in GL(2d, \mathbb{R}) \) (the group of real \( 2d \times 2d \) invertible matrices) be a lattice of the time-frequency plane. As anticipated in the Introduction, the set of time-frequency shifts \( \mathcal{G}(g, \Lambda) = \{ \pi(\lambda)g : \lambda \in \Lambda \} \) for a non-zero \( g \in L^2(\mathbb{R}^d) \) is called a Gabor system, whereas it is called Gabor frame if \([2]\) holds. In that case, then there exists a dual window \( \gamma \in L^2(\mathbb{R}^d) \), such that \( \mathcal{G}(\gamma, \Lambda) \) is a frame, and every \( f \in L^2(\mathbb{R}^d) \) possesses the frame expansions

\[
f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g
\]

with unconditional convergence in \( L^2(\mathbb{R}^d) \).

We finally pass to the characterization of some function spaces in terms of STFT decay. We have first of all the following basic result (cf. [5, 18, 27, 48]): if \( g \in S^s_r(\mathbb{R}^d) \), \( s \geq 1/2 \), then

\[
f \in S^s_r(\mathbb{R}^d) \iff |V_g f(z)| \lesssim \exp \left(-\epsilon |z|^{1/s}\right) \quad \text{for some } \epsilon > 0.
\]

When no decay is required on \( f \) we still have a characterization in the following form ([13 Theorem 3.1]).

**Theorem 2.4.** Consider \( s > 0, r > 0, g \in S^s_r(\mathbb{R}^d) \setminus \{0\} \). The following properties are equivalent:

(i) There exists a constant \( C > 0 \) such that

\[
|\partial^\alpha f(x)| \lesssim C|\alpha|!(\alpha!)^s, \quad x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d.
\]

(ii) There exists a constant \( C > 0 \) such that

\[
|\xi^\alpha V_g f(x, \xi)| \lesssim C|\alpha|!(\alpha!)^s, \quad (x, \xi) \in \mathbb{R}^{2d}, \alpha \in \mathbb{N}^d.
\]

(iii) There exists a constant \( \epsilon > 0 \) such that

\[
|V_g f(x, \xi)| \lesssim \exp \left(-\epsilon |\xi|^{1/s}\right), \quad (x, \xi) \in \mathbb{R}^{2d}.
\]

If the equivalent conditions \([5], [6], [7]\) are satisfied, we will say that \( f \) is a Gevrey function when \( s > 1 \), analytic if \( s = 1 \) and ultra-analytic when \( s < 1 \).

3. Almost diagonalization of pseudodifferential operators

Now we report on some results about the almost diagonalization of pseudodifferential operators having Gevrey, analytic ([20]) and ultra-analytic ([15]) symbols \( \sigma(x, \xi) \). We
We want to prove off-diagonal decay estimates for the Gabor matrix of $\sigma^w$.

We adopt the so-called Weyl quantization, i.e.

$$\sigma^w f = \sigma^w(x, D)f = \int_{\mathbb{R}^{2d}} e^{2\pi i (x-y)\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) \, dy \, d\xi.$$ 

We want to prove off-diagonal decay estimates for the Gabor matrix $\langle \sigma^w \pi(z)g, \pi(w)g \rangle$, $z, w \in \mathbb{R}^{2d}$. The decay rate will be related to the regularity of the symbol $\sigma$. The key point is the following explicit formula linking the Gabor matrix with the short-time Fourier transform of its symbol (cf. [24, Lemma 3.1] and [15]).

Lemma 3.1. Consider $s \geq 1/2$, $g \in S_s^\epsilon(\mathbb{R}^d)$. Then, for $\sigma \in (S_s^\epsilon)'(\mathbb{R}^{2d})$,

$$\langle \sigma^w \pi(z)g, \pi(w)g \rangle = |V_{\Phi} \sigma(u, v)|, \quad z, w \in \mathbb{R}^{2d},$$

where $u = \frac{z+w}{2}$ and $v = j(w-z)$, and

$$|V_{\Phi} \sigma(u, v)| = \left| \langle \sigma^w \pi \left( u - \frac{1}{2} j^{-1}(v) \right) g, \pi \left( u + \frac{1}{2} j^{-1}(v) \right) g \rangle \right|,$$

where $j(z_1, z_2) = (z_2, -z_1)$, $z_1, z_2 \in \mathbb{R}^{2d}$, for some $\Phi \in S_s^\epsilon(\mathbb{R}^{2d})$.

It follows from this result and the equivalence (5) $\Leftrightarrow$ (7) above, that the following decay estimates for the Gabor matrix of $\sigma^w$ hold ([15]). Notice that we have in fact a characterization.

Theorem 3.2. Let $s \geq 1/2$, and $g \in S_s^\epsilon(\mathbb{R}^d) \setminus \{0\}$. Then the following properties are equivalent for $\sigma \in C^\infty(\mathbb{R}^{2d})$:

(i) The symbol $\sigma$ satisfies

$$|\partial^\alpha \sigma(z)| \lesssim C^{\lceil|\alpha|\rceil}(\alpha!)^s, \quad \forall z \in \mathbb{R}^{2d}, \forall \alpha \in \mathbb{N}^{2d}.$$ 

(ii) There exists $\epsilon > 0$ such that

$$|\langle \sigma^w \pi(z)g, \pi(w)g \rangle| \lesssim \exp \left( -\epsilon |w-z|^{1/s} \right), \quad \forall z, w \in \mathbb{R}^{2d}.$$ 

A similar characterization in the discrete setting, i.e. for Gabor frames, is slightly subtler. Indeed, we use a recent result due to Gröchenig and Lyubarskii in [25]. There sufficient conditions on the lattice $\Lambda = AZ^2$, $A \in GL(2, \mathbb{R})$, are given in order for $g = \sum_{k=0}^n c_k H_k$, with $H_k$ Hermite functions, to form a so-called Gabor (super)frame $G(g, \Lambda)$, i.e. a frame where a dual window $\gamma$ exists, belonging to the space $S_{1/2}^1(\mathbb{R})$ (cf. [25, Lemma 4.4]). This theory transfers to the $d$-dimensional case by taking a tensor product $g = g_1 \otimes \cdots \otimes g_d \in S_{1/2}^1(\mathbb{R}^d)$ of windows as above, which defines a Gabor frame on the lattice $\Lambda_1 \times \cdots \times \Lambda_d$ and possesses a dual window $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_d$ which still belongs to $S_{1/2}^1(\mathbb{R}^d)$.

Theorem 3.3. Let $G(g, \Lambda)$ be a Gabor super-frame for $L^2(\mathbb{R}^d)$. Consider a function $\sigma \in C^\infty(\mathbb{R}^{2d})$. Then the following properties are equivalent:

(i) There exists $\epsilon > 0$ such that the estimate (11) holds.

(ii) There exists $\epsilon > 0$ such that

$$|\langle \sigma^w \pi(\mu)g, \pi(\lambda)g \rangle| \lesssim \exp \left( -\epsilon |\lambda - \mu|^{1/s} \right), \quad \forall \lambda, \mu \in \Lambda.$$
The above characterizations have several applications ([15]). Here we just consider the so-called sparsity property and the continuity of pseudodifferential operators on Gelfand-Shilov spaces.

**Corollary 3.4.** Under the assumptions of Theorem 3.3, let the Gabor matrix \( \langle \sigma_w \pi(\mu)g, \pi(\lambda)g \rangle \) satisfy (12). Then it is sparse in the following sense. Let \( a \) be any column or row of the matrix, and let \( |a|_n \) be the \( n \)-largest entry of the sequence \( a \). Then, \( |a|_n \) satisfies
\[
|a|_n \leq C \exp \left( -\epsilon n^{1/(2ds)} \right), \quad n \in \mathbb{N}
\]
for some constants \( C > 0, \epsilon > 0 \).

The main novelty with respect to the existing literature (cf. [3, 28]) is the exponential as opposed to super-polynomial decay.

**Corollary 3.5.** Let \( s \geq 1/2 \) and consider a symbol \( \sigma \in C^\infty(\mathbb{R}^d) \) that satisfies (10). Then the Weyl operator \( \sigma^w \) is bounded on \( S^s(\mathbb{R}^d) \).

Similarly one obtains boundedness on modulation spaces ([21, 23]) with weights having exponential growth; see [15].

4. Applications to evolution equations

Consider an operator of the form
\[
P(\partial_t, D_x) = \partial_t^m + \sum_{k=1}^m a_k(D_x) \partial_t^{m-k}, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d,
\]
where \( a_k(\xi), 1 \leq k \leq m, \) are polynomials. They may be non-homogeneous, and their degree may be arbitrary (as usual, \( D_x j = \frac{1}{2\pi i} \partial x_j, \ j = 1, \ldots, d \)).

We deal with the forward Cauchy problem
\[
\begin{cases}
P(\partial_t, D_x) u = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \\
\partial_t^k u(0, x) = u_k(x), & 0 \leq k \leq m - 1,
\end{cases}
\]
where \( u_k \in \mathcal{S}(\mathbb{R}^d), \ 0 \leq k \leq m - 1. \) A sufficient and necessary condition for the above Cauchy problem with Schwartz data to be well posed is given by the forward Hadamard-Petrowsky condition ([11, Section 3.10]):

There exists a constant \( C > 0 \) such that
\[
(\tau, \xi) \in \mathbb{C} \times \mathbb{R}^d, \quad P(i\tau, \xi) = 0 \implies \text{Im} \tau \geq -C.
\]
In fact one can see ([11, pp. 126-127]) that the solution is then given by
\[
u(t, x) = \sum_{k=0}^{m-1} \partial_t^k E(t, \cdot) * \left( u_{m-1-k} + \sum_{j=1}^{m-k-1} a_j(D_x) u_{m-k-1-j} \right),
\]
with \( E(t, x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \sigma(t, \xi), \) where \( \sigma(t, \xi) \) is the unique solution to
\[
\left( \partial_t^m + \sum_{k=1}^m a_k(\xi) \partial_t^{m-k} \right) \sigma(t, \xi) = \delta(t)
\]
supported in \([0, +\infty) \times \mathbb{R}^d\). The distribution \(E(t, x)\) is called the \textit{fundamental solution} of \(P\) supported in \([0, +\infty) \times \mathbb{R}^d\).

We are therefore reduced to study the corresponding Fourier multiplier

\[
\sigma^\nu(t, D_x) = \sigma(t, D_x)f = \mathcal{F}^{-1}\sigma(t, \cdot)\mathcal{F}f = E(t, \cdot) * f.
\]

(For Fourier multipliers the Weyl and Kohn-Nirenberg quantizations give the same operator).

For example, for \(t \geq 0\), we have \(\sigma(t, \xi) = \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\) for the wave operator \(\partial_t^2 - \Delta\);
\[
\sigma(t, \xi) = \frac{\sin(t\sqrt{4\pi^2|\xi|^2 + m^2})}{\sqrt{4\pi^2|\xi|^2 + m^2}}
\]
for the Klein-Gordon operator \(\partial_t^2 - \Delta + m^2 (m > 0)\); \(\sigma(t, \xi) = e^{-4\pi^2|\xi|^2t}\) for the heat operator \(\partial_t - \Delta\). In all cases, \(\sigma(t, \xi) = 0\) for \(t < 0\).

We want to apply Theorem 3.2 to the symbol \(\sigma(t, x, \xi) = \sigma(t, \xi)\) of the multiplier \(\sigma(t, D_x)\). To this end we present a suitable refinement of the Hadamard-Petrowsky condition.

\textit{Assume that there are constants } \(C > 0\), \(\nu \geq 1\) \textit{such that}

\[
(\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^d, \quad P(i\tau, \zeta) = 0 \Rightarrow \text{Im } \tau \geq -C(1 + |\text{Im } \zeta|)^\nu.
\]

We then have the following result [15].

\textbf{Theorem 4.1.} \textit{Assume } \(P\) \textit{satisfies } (16) \textit{for some } \(C > 0\), \(\nu \geq 1\). \textit{Then the symbol } \(\sigma(t, \xi)\) \textit{of the corresponding propagator } \(\sigma(t, D_x)\) \textit{in (15) satisfies the following estimates:}

\[
|\partial^\alpha_x \sigma(t, \xi)| \leq C((t+1)|\alpha|+t|\alpha|)^s, \quad \xi \in \mathbb{R}^d, \quad t \geq 0, \quad \alpha \in \mathbb{N}^d,
\]

\textit{with } \(s = 1 - 1/\nu\), \textit{for a new constant } \(C > 0\).

Observe that the hypothesis \(\nu \geq 1\) in the above theorem implies \(0 \leq s < 1\).

As a consequence of Theorem 4.1 and Theorem 3.2 we therefore obtain our main result.

\textbf{Theorem 4.2.} \textit{Assume } \(P\) \textit{satisfies } (16) \textit{for some } \(C > 0\), \(\nu \geq 1\), \textit{and set } \(r = \min\{2, \nu/(\nu - 1)\}\). \textit{If } \(g \in S^{1/r}_1(\mathbb{R}^d)\) \textit{then } \(\sigma(t, D_x)\) \textit{in (15) satisfies}

\[
|\langle \sigma(t, D_x)\pi(z)g, \pi(w)g \rangle| \leq C \exp\left(-\epsilon|w - z|^r\right), \quad \forall z, w \in \mathbb{R}^{2d},
\]

\textit{for some } \(\epsilon > 0\) \textit{and for a new constant } \(C > 0\). \textit{The inequality (18) holds for } \(t\) \textit{belonging to an arbitrary bounded subset of } \([0, +\infty)\) \textit{with } \(\epsilon\) \textit{and } \(C\) \textit{fixed}.

Again we observe that \(r > 1\) in (18), so that we always obtain \textit{super-exponential decay}.

We now detail some special cases of great interest, providing some numerical experiments.

\textbf{4.1. Hyperbolic operators.} \textit{We recall that the operator } \(P(\partial_t, D_x)\) \textit{is called hyperbolic with respect to } \(t\) \textit{if the higher order homogeneous part in the symbol does not vanish at } \((1, 0, \ldots, 0) \in \mathbb{R} \times \mathbb{R}^d\), \textit{and } \(P\) \textit{satisfies the forward Hadamard-Petrowsky condition (14). This implies that the operators } \(a_k(D_x)\) \textit{in (13) must have degree } \leq k \textit{and } \(P\) \textit{has order } \(m\). \textit{For example, the wave and Klein-Gordon operators are hyperbolic}
operators. However, $P$ is not required to be strictly hyperbolic, namely the roots of the principal symbol are allowed to coincide.

Now, if $P(\partial_t, D_x)$ is any hyperbolic operator, we always obtain Gaussian decay in the above theorem ($r = 2$ in (18)), at least for windows $g \in S^{1/2}_1(\mathbb{R}^d)$. In fact, we have the following result ([15]).

**Proposition 4.3.** Assume $P(\partial_t, D_x)$ is hyperbolic with respect to $t$. Then the condition (16) is satisfied with $\nu = 1$ for some $C > 0$, and hence

$$
|\langle \sigma(t, D_x) \pi(z) g, \pi(w) g \rangle| \leq C \exp \left( -\epsilon |w - z|^2 \right), \quad \forall z, w \in \mathbb{R}^d,
$$

if $g \in S^{1/2}_1(\mathbb{R}^d)$, for some $\epsilon > 0$ and for a new constant $C > 0$.

4.2. Wave equation. Consider the wave operator $P = \partial_t^2 - \Delta$ in $\mathbb{R} \times \mathbb{R}^d$, therefore $\sigma(t, \xi) = \frac{\sin(2\pi |\xi| t)}{2\pi |\xi|}$. The above Proposition 4.3 applies, but we can also estimate the matrix decay directly, with the involved constants made explicit, by using the explicit expression of the fundamental solution. We state the result, for simplicity, in dimension $d \leq 3$. We take $g(x) = 2^{d/4} e^{-\pi |x|^2}$ as window function, which belongs to $S^{1/2}_1(\mathbb{R}^d)$, and moreover $\|g\|_{L^2} = 1$ (Gaussian functions minimize the Heisenberg uncertainty so that they are, generally speaking, a natural choice for wave-packet decompositions).

An explicit computation ([15]) gives the estimate

$$
|\langle \sigma(t, D_x) M_\xi T_g, M_\zeta T_{g'} \rangle| \leq te^{-\frac{\pi}{2} \|\xi' - \xi\|^2 + (|x' - x| - t)^2}, \quad x, x', \xi, \xi' \in \mathbb{R}^d, \quad d \leq 3,
$$

where $\langle \cdot, \cdot \rangle_+$ denote positive part.

Consider now the Gabor frame $G(g, \Lambda)$, with $g(x) = 2^{d/4} e^{-\pi |x|^2}$, $\Lambda = \mathbb{Z}^d \times (1/2)\mathbb{Z}^d$ ([23] Theorem 7.5.3), and the corresponding Gabor matrix

$$T_{m',n',m,n} = \langle \sigma(t, D_x) M_n T_m g, M_{n'} T_{m'} g \rangle, \quad (m,n), (m',n') \in \Lambda.
$$

We therefore have

$$|T_{m',n',m,n}| \leq \tilde{T}_{m',n',m,n} := te^{-\frac{\pi}{2} \|n' - n\|^2 + (|m' - m| - t)^2}, \quad (m,n), (m',n') \in \Lambda, \quad d \leq 3.
$$

Figure 1 shows the magnitude of the entries, rearranged in decreasing order, of a generic column, e.g. $\tilde{T}_{m',n',0,0}$ (obtained for $m = n = 0$), at time $t = 0.75$, in dimension $d = 2$.

In fact, the same figure applies to all columns, for $\tilde{T}_{m',n',m,n} = \tilde{T}_{m'-m,n'-n,0,0}$. This figure should be compared with [14, Figure 15], where a similar investigation was carried out for the curvelet matrix of the wave propagator on the unit square ($d = 2$) with periodic boundary conditions. It turns out that the Gabor decay is even better, in spite of the fact that we consider here the wave operator in the whole $\mathbb{R}^2$.

4.3. Parabolic type equations. Consider the operator

$$P(\partial_t, D_x) = \partial_t + (-\Delta)^k,
$$

with $k \geq 1$ integer. In particular we get the heat operator for $k = 1$. Its symbol is the polynomial

$$P(i\tau, \zeta) = i\tau + (4\pi^2 \zeta^2)^k.
$$

An explicit computation shows that it satisfies (16) with $\nu = 2k$. As a consequence, Theorem 4.2 applies to $P$ with $\nu = 2k$ and $r = 2k/(2k - 1)$.
In particular, the heat propagator \( \sigma(t, D_x) = e^{-4\pi^2 t |D|^2} \) satisfies the estimate
\[
|\langle e^{-4\pi^2 t |D|^2} \pi(z)g, \pi(w)g \rangle| \leq Ce^{-\epsilon|w-z|^2}, \quad \forall z, w \in \mathbb{R}^{2d},
\]
for some \( \epsilon > 0, C > 0 \), if \( g \in S^{1/2}_{1/2}(\mathbb{R}^d) \). Namely, the same decay as in the case of hyperbolic equations occurs.

In the following figures we summarize some numerical information about its Gabor discretization. Namely, Figure 2 shows the decay of a column of the Gabor matrix for the heat propagator, i.e.
\[
T_{m',n',0,0} = \langle e^{-4\pi^2 t |D|^2} g, M_{n'} T_{m'} g \rangle
\]
for a Gaussian window, at different time instants \( t \) and in dimension \( d = 2 \). For \( t = 0 \) we get the identity operator, and therefore its matrix decay is the optimal one, compatibly with the uncertainty principle. As one see from the other figures the decay remains extremely good as time evolves. Also, for \( t = 0.75 \) the decay matches that of the wave equation displayed in Figure 1 in spite of the fact that we no longer have here finite speed of propagation.

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Figure 2. Decay of the column corresponding to $m = n = 0$, of the Gabor matrix for the heat propagator $e^{-4\pi^2 t |D|^2}$ in dimension $d = 2$ at different time instants, with window $g(x) = \sqrt{2}e^{-\pi |x|^2}$ and lattice $\mathbb{Z}^2 \times (1/2)\mathbb{Z}^2$.

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