Spectral functions of the Dirac operator under local boundary conditions

C.G. Beneventano*, E.M. Santangelo†
Departamento de Física, Universidad Nacional de La Plata
C.C.67, 1900 La Plata, Argentina

May 4, 2018

Abstract

After a brief discussion of elliptic boundary problems and their properties, we concentrate on a particular example: the Euclidean Dirac operator in two dimensions, with its domain determined by local boundary conditions. We discuss the meromorphic structure of the zeta function of the associated second order problem, as well as the main characteristic of the first order problem, i.e., the boundary contribution to the spectral asymmetry, as defined through the eta function.

1 Introduction

Spectral functions are of interest both in quantum field theory and in mathematics (for a recent review, see [1]). In particular, ζ-functions and heat kernels of elliptic boundary problems are known to provide an elegant regularization method [2] for the evaluation of objects as one-loop effective actions and Casimir energies, as discussed, for instance, in the reviews [3].

In the case of operators with a non positive-definite principal symbol, another spectral function has been studied, known as η-function [4], which characterizes the spectral asymmetry of the operator. This spectral function was originally introduced in [5] (see also [6]), where an index theorem for manifolds with boundary was derived. In fact, the η-function of the Dirac operator, suitably restricted to the boundary, is proportional to the

*E-mail: gabriela@obelix.fisica.unlp.edu.ar
†E-mail: mariel@obelix.fisica.unlp.edu.ar
difference between the anomaly and the index of the Dirac operator, acting on functions satisfying nonlocal Atiyah-Patodi-Singer (APS) boundary conditions. Some examples of application were discussed in [7, 8].

Such nonlocal boundary conditions were introduced mainly for mathematical reasons, although several applications of this type of boundary value problems to physical systems have emerged, ranging from one-loop quantum cosmology [9], fermions propagating in external magnetic fields [10] or so-called $S$–branes, which are mapped into themselves under $T$–duality [11].

So far, $\eta$-functions have found their most interesting physical applications in the discussion of fermion number fractionization [12]: The fractional part of the vacuum charge is proportional to $\eta(0)$. The $\eta$–function also appears as a contribution to the phase of the fermionic determinants and, thus, to effective actions [13]. Furthermore, both the index and the $\eta$-invariant of the Dirac operator are related to scattering data via a generalization of the well-known Levinson theorem [14]. A thorough discussion of the index, $\zeta$– and $\eta$–functions in terms of boundary spectral functions for APS boundary problems can be found in [15, 16].

Alternatively, one may consider the boundary value problem for the Dirac operator acting on functions that satisfy local, bag-like, boundary conditions. These conditions are closely related to those appearing in the effective models of quark confinement known as MIT bag models [17], or their generalizations, the chiral bag models [18]. The physical motivation for studying these local boundary conditions is thus clear.

Chiral bag boundary conditions [19, 20, 21] can be defined in any even dimensional manifold. They contain a real parameter $\theta$, which is to be interpreted as an analytic continuation of the well known $\theta$-parameter in gauge theories. Indeed, for $\theta \neq 0$, the effective actions for the Dirac fermions contain a $CP$-breaking term proportional to $\theta$ and proportional to the instanton number [20].

For $\theta \neq 0$ we will refer to the bag boundary conditions as chiral while, in the particular case $\theta = 0$, we will call them non-chiral or pure MIT conditions. In both cases, the Euclidean Dirac operator is self-adjoint. The ellipticity of the boundary value problem is a more subtle point. In fact, even though the boundary conditions are local for the first order problem, for $\theta \neq 0$, the boundary conditions for the associated second order problem are of mixed oblique type [22] and, under certain circumstances, oblique boundary conditions are not strongly elliptic [23, 24].

In this work, we will concentrate on the Euclidean Dirac operator in two dimensions, acting on functions satisfying local, chiral bag, boundary conditions. The two-dimensional case, being the simplest one, already contains
all the main characteristics appearing for the problem in higher dimensional cases \[25\].

We briefly review, in section 2, the concepts of weak and strong ellipticity of boundary value problems, and the consequences of their validity. In section 3, we prove that the first order boundary problem at hand is indeed strongly elliptic. In the same section, we define the associated second order problem and prove that it is also strongly elliptic, even though the boundary conditions involve tangential derivatives.

In section 4, we give an explicit expression for the heat kernel of the second order problem in an infinite cylinder. By making use of such expression we analyze, in section 5, the meromorphic structure of the corresponding zeta function for manifolds of product type.

One of the main characteristics of bag boundary conditions is that they lead to an asymmetry in the non-zero spectrum. Note that, as in any even dimension, there is no volume contribution to the asymmetry (for a proof see, for instance, \[4\]; qualitatively, this is due to the existence of \(\gamma_5\), which anticommutes with the Dirac operator). So, the boundary contribution is also the total asymmetry. In section 6, we study, again for product manifolds, the meromorphic structure of the eta function and the boundary contribution to the spectral asymmetry for chiral bag boundary conditions \[26\].

## 2 Elliptic boundary problems

In this section, we will briefly review the definition of weakly (Lopatinski-Shapiro) and strongly elliptic boundary systems, and state their main properties (for details, see \[4\] \[27\] \[28\]).

Let \(M\) a smooth compact manifold of dimension \(n\), with a smooth boundary \(\partial M\).

In each local coordinate system, call \(x = (x_1, \ldots, x_{n-1})\) the coordinates on \(\partial M\). Let \(y \in \mathbb{R}\) the interior normal to the boundary. So, \(z = (x, y) \in \mathbb{R}^n\). Call \(\mathbb{R}_+\), the half space \(y \geq 0\). Consider, in \(\mathbb{R}_+^n\), the differential operator of order \(m\), acting on a \(q\)-dimensional complex vector bundle \(V\):

\[
P = \sum_{j=0}^{m} P_j(y) D_y^{m-j}, \quad (D_y = -i \frac{\partial}{\partial y}),
\]

where \(P_j\) is a differential operator \((q \times q\) matrix\) of order \(\leq j\) on \(\mathbb{R}^{n-1}\).

Then, calling \((\xi, \tau)\) the symbolic variable corresponding to \((x, y)\) we have,
for the symbol of $P$:

$$\sigma(P) = \sum_j \sigma(P_j)(x,y,\xi)\tau^{m-j},$$

which is a polynomial of order $m$ in the $n$ variables $(\xi, \tau)$.

The leading symbol is its $m$-th order part

$$\sigma_L(P) = \sum_j \sigma_j(P_j)(x,y,\xi)\tau^{m-j}.$$

Moreover, we define a partial leading symbol, by:

$$\sigma_L'(P) = \sum_j \sigma_j(P_j)(x,0,\xi)D_y^{m-j}.$$

Suppose, near the boundary, we have certain given operators (defining the boundary conditions)

$$B_j = \sum_{k=1}^m B_{jk}D_y^{m-k}, \quad 1 \leq j \leq \frac{mq}{2}$$

where the $B_{jk}$ are a system of differential operators ($1 \times q$ matrices) acting on $\mathbb{R}^{n-1}$.

The collection of operators $P, B_1, ..., B_{\frac{mq}{2}}$ constitute a weakly elliptic boundary system if $P$ is elliptic and, for $g = (g_1, ..., g_{\frac{mq}{2}})$ arbitrary, $x$ in $\mathbb{R}^{n-1}$ and $\xi \neq 0$ in $\mathbb{R}^{n-1}$ there is a unique solution to the following problem for $y > 0$:

$$\sigma_L'(P)(x,\xi,D_y)u = 0$$

$$\lim_{y \to \infty} u(y) = 0$$

$$\sigma_L'(B_j)(x,\xi,D_y)u = g_j \text{ at } y = 0 \text{ for } j = 1, ..., \frac{mq}{2}.$$  (2.1)

This condition is also known as the Lopatinski-Shapiro condition \[4\]. When this condition holds, an operator $P_B$ can be defined as the operator $P$, acting on functions such that $B_ju|_{y=0} = 0$.

Now, in order to meet some further requirements, one needs to impose stronger conditions on the boundary system:

The collection $P, B_1, ..., B_{\frac{mq}{2}}$ constitutes a strongly elliptic boundary system in a cone $K \subset \mathbb{C}$ including the origin if
i) For \((\xi, \tau) \neq (0, 0), \sigma_L(P)\) has no eigenvalue in \(K\) and

ii) For each \(x\) and each \((\xi, \lambda) \neq (0, 0),\) with \(\lambda \in K,\) the boundary problem

\[
\sigma'_L(P)(x, \xi, D_y)u = \lambda u \\
\lim_{y \to \infty} u(y) = 0 \\
\sigma'_L(B_j)(x, \xi, D_y)u = g_j \text{ at } y = 0, \quad j = 1, \ldots, \frac{mq}{2}
\]

(2.2)

has a unique solution. (Note this reduces to the Lopatinski-Shapiro condition for \(\lambda = 0\)). The cone \(K\) is known as Agmon’s cone \([29]\).

When the strong ellipticity condition holds, an approximation (parametrix) to the resolvent \((PB - \lambda)^{-1}\) can be found \([28]\) and, from it, one can define the complex powers

\[
(P_B)^{-s} = \frac{i}{2\pi} \int_{\Gamma} \lambda^{-s} (PB - \lambda)^{-1} d\lambda,
\]

with \(\Gamma\) an appropriate curve in the cone where \((PB - \lambda)^{-1}\) is known to exist.

The coefficients in the expansion of the parametrix (Seeley’s coefficients) must satisfy the condition of representing an inverse for \(PB - \lambda,\) and they must also adjust the boundary condition. As a consequence, both bulk and boundary Seeley’s coefficients appear.

If, moreover, the leading symbol of the operator is positive-definite, another operator can be defined as

\[
K(t, PB) = e^{-tPB} = \frac{i}{2\pi} \int_{\Gamma} e^{-\lambda t} (PB - \lambda)^{-1} d\lambda, \quad t > 0.
\]

(2.3)

It can be shown that its kernel \(K(z, z'; t)\) is the fundamental solution of the heat equation, and is known as the heat kernel of \(PB.\)

We will be interested in the traces of the complex powers and of the heat kernel.

In addition, let \(PB\) self-adjoint. Then, the following theorem is known to hold (see Theorem 1.11.3 in \([4]\)):

a) There exists a discrete spectral resolution \(\{\phi_\nu, \lambda_\nu\}\) of \(PB.\)

b) There exist \(\nu_0\) and \(\delta > 0,\) so that if \(\nu \geq \nu_0,\) then \(|\lambda_\nu| \geq \nu^\delta.\)

c) If the boundary value problem defining \(PB\) is strongly elliptic, only a finite number of the \(\lambda_\nu\) are negative.

Whenever the principal symbol of the operator is positive-definite, the operator \(K(t, PB)\) is infinitely smoothing, and its trace, in the basis of eigenfunctions is given by
\[ \text{Tr} K(t, P_B) = \sum_{\nu} e^{-\lambda_{\nu} t}. \] (2.4)

Moreover, if \( P_B \) is positive-definite (i.e., there are no negative or zero eigenvalues in the spectral resolution), one further defines its \( \zeta \) function as
\[ \zeta(s, P_B) = Tr(P_B^{-s}), \] (2.5)

Both spectral functions are related through a Mellin transform,
\[
\zeta(s, P_B) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \sum_{\nu} e^{-\lambda_{\nu} t} = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} TrK(t, P_B). \] (2.6)

### 3 Chiral bag boundary conditions

#### 3.1 First order boundary problem

In this section, we will discuss, in particular, the properties of the boundary value problem defined by the Euclidean Dirac operator on a two-dimensional manifold, acting on spinors satisfying local (chiral bag) boundary conditions.

To this end, it is convenient to choose a chiral representation for the Euclidean \( \gamma \)-matrices in two dimensions,
\[ \gamma_x = \sigma_1, \quad \gamma_y = \sigma_2 \quad \text{and} \quad \gamma_5 = -i\gamma_x\gamma_y = \sigma_3. \] (3.1)

Then, the free Dirac operator acts on two-component spinors \( \psi \), and it takes the form
\[ P = i(\gamma_x \partial_x + \gamma_y \partial_y) = \begin{pmatrix} 0 & \partial_y + A \\ -\partial_y + A & 0 \end{pmatrix}, \] (3.2)

where \( A \) is the operator \( A = i\partial_x \), which will play an important role in what follows.
The euclidean “time”-coordinate \( x \) is tangential to the boundary at \( y = 0 \). The “spatial” variable \( y \geq 0 \) is normal to the boundary and grows toward the interior of the manifold. The projector defining the local chiral bag boundary condition

\[
\Pi_- \psi \big|_{y=0} = 0
\]

(3.3)

at the boundary \( y = 0 \) reads

\[
\Pi_- = \frac{1}{2} (1 - i \gamma_5 \gamma_y e^{-\gamma_y \theta}) = \frac{1}{2} (1 - \gamma_x e^{-\gamma_x \theta}) = \frac{1}{2} \begin{pmatrix} 1 & -e^\theta \\ -e^{-\theta} & 1 \end{pmatrix} .
\]

(3.4)

In the case at hand, with the notation of section 2, \( n = 2, m = 1, q = 2 \). The leading symbol of (3.2) is given by

\[
\sigma_L(P) = \begin{pmatrix} 0 & -\xi + i \tau \\ -\xi - i \tau & 0 \end{pmatrix} ,
\]

(3.5)

which is easily seen to be invertible for \((\xi, \tau) \neq (0, 0)\).

Now, in order to analyze the ellipticity of the boundary value problem, we evaluate that

\[
\sigma'_L(P) = \begin{pmatrix} 0 & -\xi + \partial_y \\ -\xi - \partial_y & 0 \end{pmatrix} ,
\]

(3.6)

and the boundary operator (\(1 \times 2\) matrix)

\[
B_1 = B_{11} \partial_y = \frac{1}{2} (1 - e^\theta).
\]

(3.7)

Note that, this being a multiplicative operator, its partial leading symbol coincides with the operator itself.

We first prove that the boundary value problem is weakly elliptic. The first two equations in (2.1) give

\[
\begin{pmatrix} u_1(\xi, y) \\ u_2(\xi, y) \end{pmatrix} = \begin{pmatrix} C_1(\xi) \Theta(\xi) e^{-\xi y} \\ C_2(\xi) \Theta(-\xi) e^{\xi y} \end{pmatrix} ,
\]

(3.8)

where \( \Theta(\xi) \) is the Heaviside function.

Now, the third equation in (2.1) univocally determines \( C_1(\xi) = 2g(\xi) \), for \( \xi > 0 \), and \( C_2(\xi) = -2g(\xi)e^{-\theta} \), for \( \xi < 0 \). This shows that weak ellipticity does hold.

As for strong ellipticity, condition \( i) \) in section 2, together with (3.5), lead to

\[
\lambda^2 - (\xi^2 + \tau^2) \neq 0.
\]
which is satisfied if \( \lambda \in \mathcal{K} = \mathbb{C} - \mathbb{R}^+ - \mathbb{R}^\cdot \).

Moreover, the first two conditions in (2.2) lead to
\[
\begin{align*}
u(\xi, y) = \begin{pmatrix} u_1(\xi, y) \\ u_2(\xi, y) \end{pmatrix} = \begin{pmatrix} C_1(\xi) e^{-\mu y} \\ C_1(\xi) \frac{e^{-\mu y}}{\lambda} \end{pmatrix},
\end{align*}
\] (3.9)
where \( \mu^2 = \xi^2 - \lambda^2 \), and the condition \( \lambda \in \mathcal{K} \) allows to choose \( \Re(\mu) > 0 \).

Now, the equation for the symbol of the boundary operator in (2.2) gives
\[
C_1(\xi)[\lambda - e^\theta (\mu - \xi)] = 2g(\xi)\lambda.
\]

So, the problem has a unique solution if \( \lambda \neq -\frac{\xi}{\cosh \theta} \), a condition which is satisfied for \( \lambda \in \mathcal{K} \). This proves the strong ellipticity of the first order boundary problem with respect to \( \mathbb{C} - \mathbb{R}^+ - \mathbb{R}^\cdot \).

3.2 Associated second order problem

To the first order boundary problem defined by equations (3.2) and (3.3), a second order problem is naturally associated. This last consists of the Laplacian
\[
P^2 = \begin{pmatrix} -\partial_y^2 + A^2 & 0 \\ 0 & -\partial_y^2 + A^2 \end{pmatrix},
\] (3.10)
acting on a two-dimensional vector bundle, whose elements satisfy the boundary conditions
\[
\begin{align*}
\Pi_- \psi \big|_{y=0} &= 0 \\
\Pi_- P \psi \big|_{y=0} &= 0.
\end{align*}
\] (3.11)

In equation (3.10), \( A \) is defined as for the first order problem.

Although the ellipticity of the second order problem follows from the ellipticity of the corresponding first order one and Lemma 1.11.2 (b) in [4], we prefer to give a second proof showing that the boundary operator involves tangential derivatives. This proof makes apparent that the chiral boundary conditions are non-standard (oblique) boundary conditions.

With the notation of section 2, \( n = 2, q = 2 \) and \( m = 2 \). So, we have a set of two boundary operators (1 \times 2 matrices), given by
\[
\begin{align*}
B_1 &= B_{11}\partial_y^0 + \frac{1}{2} \begin{pmatrix} 1 & -e^\theta \\ e^\theta & 1 \end{pmatrix} \\
B_2 &= B_{21}\partial_y^1 + B_{22}\partial_y^0 = \frac{1}{2} \left[ \begin{pmatrix} e^\theta & 1 \\ e^\theta & 1 \end{pmatrix} \partial_y + \begin{pmatrix} -e^\theta & 1 \\ e^\theta & 1 \end{pmatrix} i\partial_x \right].
\end{align*}
\] (3.12)
As is well known, the Laplacian \((P^2)\) is an elliptic operator. Its leading symbol is given by

\[
\sigma_L(P^2) = \begin{pmatrix}
\tau^2 + \xi^2 & 0 \\
0 & \tau^2 + \xi^2
\end{pmatrix}.
\]

(3.13)

It’s easy to check that it has no eigenvalue for \((\xi, \tau) \neq (0, 0)\), and \(\lambda \in \mathcal{K} = \mathbb{C} - \mathbb{R}^+\).

Now, the partial leading symbols are

\[
\sigma'_L(P^2) = \begin{pmatrix}
\xi^2 - \partial_y^2 & 0 \\
0 & \xi^2 - \partial_y^2
\end{pmatrix},
\]

(3.14)

and

\[
\sigma'_L(B_1) = B_1,
\sigma'_L(B_2) = \frac{1}{2} \left[ (e^\theta \ 1) \partial_y + (-e^\theta \ 1) (-\xi) \right].
\]

(3.15)

For this second order boundary problem, weak ellipticity is but a particular case of the strong one. So, let us prove this last holds.

The first two equations in (2.2) lead to

\[
u(\xi, y) = \begin{pmatrix}
    u_1(\xi, y) \\
    u_2(\xi, y)
\end{pmatrix} = \begin{pmatrix}
    C_1(\xi)e^{-\mu y} \\
    C_2(\xi)e^{-\mu y}
\end{pmatrix},
\]

(3.16)

where \(\mu^2 = \xi^2 - \lambda\), and the condition \(\lambda \in \mathcal{K}\) allows to choose \(\Re(\mu) > 0\).

When the last equation in (2.2) is used, together with (3.15), one can see that the problem has a unique solution whenever \(\xi^2 \neq \lambda \cosh^2 \theta\), a condition which is satisfied if \(\lambda \in \mathcal{K}\), which proves strong ellipticity with respect to \(\mathbb{C} - \mathbb{R}^+\). Moreover, if \(\lambda = 0\), the condition is fulfilled for \(\xi \neq 0\), which proves weak ellipticity.

4 The heat kernel in an infinite cylinder

In this section, we will obtain the explicit expression for the heat kernel of the second order problem in an infinite cylinder. This will be the main ingredient in the subsequent study of spectral functions on a compact manifold.

Let us start by discussing some properties of the boundary projector

\[
\Pi_- = \frac{1}{2}(1 - i\gamma_5\gamma_5 e^{-\gamma_5\theta})
\]

(4.1)
and of its orthogonal projector

\[ \Pi_+ = \frac{1}{2}(1 + i \gamma_5 \gamma_9 e^{-\gamma_5 \theta}). \]  

(4.2)

Note that these projectors are not self-adjoint (except for the particular case \( \theta = 0 \)). Rather, one has

\[ \Pi^* - \Pi = \frac{1}{2}(1 - i \gamma_5 \gamma_9 e^{\gamma_5 \theta}) \]

\[ \Pi^* + \Pi = \frac{1}{2}(1 + i \gamma_5 \gamma_9 e^{\gamma_5 \theta}) , \]  

(4.3)

and the following equations hold

\[ \Pi^* \Pi_+ = \cosh(\theta \gamma_5) \exp(-\theta \gamma_5) \Pi_+ = \Pi^* \cosh(\theta \gamma_5) \exp(-\theta \gamma_5) \]
\[ \Pi^* \Pi_+ = \sinh(\theta \gamma_5) \exp(-\theta \gamma_5) \Pi_+ = \Pi^* \sinh(\theta \gamma_5) \exp(-\theta \gamma_5) \]
\[ \Pi^* \Pi_- = \cosh(\theta \gamma_5) \exp(-\theta \gamma_5) \Pi_- = \Pi^* \cosh(\theta \gamma_5) \exp(-\theta \gamma_5) \]
\[ \Pi^* \Pi_- = \sinh(\theta \gamma_5) \exp(-\theta \gamma_5) \Pi_- = \Pi^* \sinh(\theta \gamma_5) \exp(-\theta \gamma_5) . \]

We use \( \Pi_- \) to define boundary conditions for \( P \), as in equation (3.3). Similarly, we shall let the equations (3.11) define the associated boundary condition for \( P^2 \). We will call this second order boundary value problem \((P^2, B)\).

In what follows, we present the heat kernel for \((P^2, B)\) in an infinite cylinder \( M = \mathbb{R}_+ \times \mathcal{N} \), where \( \mathcal{N} \) is the closed boundary.

In order to determine the heat kernel, it is useful to note that the chiral bag boundary conditions in equations (3.11) are equivalent, for each eigenvalue of the tangential part \( A \) of the operator \( P \), to Dirichlet boundary conditions on part of the fibre, and Robin (modified Neumann) on the rest.

In fact, let’s first notice that the operators \( \mathcal{P}_+ = \frac{\Pi_+ \Pi^*_+}{\cosh^2 \theta} \) and \( \mathcal{P}_- = \frac{\Pi^*_- \Pi_-}{\cosh^2 \theta} \)

are self-adjoint projectors, and they satisfy \( \mathcal{P}_+ + \mathcal{P}_- = 1 \), splitting \( V \) into two complementary subspaces.

As before, let \( x \) be the coordinate on the boundary and \( z = (x, y) \) a local coordinate system on the manifold. If we call \( \phi_\omega(x) = e^{i \omega x} \) the eigenfunctions of the operator \( A = i \partial_x \) corresponding to the eigenvalue \( \omega \), we can expand \( \psi(x, y) = \sum_\omega f_\omega(y) \phi_\omega(x) \). If \( \psi = \mathcal{P}_+ \psi \), then the first condition in equation (3.11) is identically satisfied, and only the second condition in (3.11) must be imposed at the boundary which, for each \( \omega \), reduces to

\[ \cosh \theta e^{-\theta \gamma_5} (\partial_y + \omega \tanh \theta) f_\omega|_{y=0} = 0 . \]
Since the factor to the left of the parenthesis is invertible, this is nothing but a Robin boundary condition.

In the subspace \( \psi = \mathcal{P}_- \psi \), the boundary equations reduce, respectively, to
\[
\cosh \theta e^{\theta \gamma} f_\omega \big|_{y=0} = 0,
\]
and
\[
\omega f_\omega \big|_{y=0} = 0.
\]
Thus, in this subspace, the requirement that both conditions are simultaneously satisfied leads to homogeneous Dirichlet boundary conditions.

As a consequence, the complete heat kernel can be written as a Dirichlet heat kernel on \( \mathcal{P}_- V \) and a Robin heat kernel on \( \mathcal{P}_+ V \). For the convenience of the reader we make the single ingredients explicit \[30\] and write, with \( \rho = y - y' \) and \( \eta = y + y' \),
\[
K(z, z'; t) = K(z, z'; t)(\mathcal{P}_- + \mathcal{P}_+)
\]
\[
= \frac{1}{\beta \sqrt{4\pi t}} \sum_\omega \phi_{\omega}^*(x') \phi_{\omega}(x) e^{-\omega^2 t} \left( e^{-\frac{\rho^2}{4t}} - e^{-\frac{\eta^2}{4t}} \right) \mathcal{P}_-
\]
\[
+ \frac{1}{\beta \sqrt{4\pi t}} \sum_\omega \phi_{\omega}^*(x') \phi_{\omega}(x) e^{-\omega^2 t} \left\{ \left( e^{-\frac{\rho^2}{4t}} + e^{-\frac{\eta^2}{4t}} \right)
\right. \\
+ 2\sqrt{\pi t} \omega \tanh \theta e^{\omega^2 t} \frac{\omega^2 \tanh \frac{\theta}{2}}{\tanh \theta} \text{erfc}(u_{\omega}(\eta, t)) \left. \right\} \mathcal{P}_+
\]
\[
= \frac{1}{\beta \sqrt{4\pi t}} \sum_\omega e^{i\omega (x - x')} e^{-\omega^2 t} \left\{ \left( e^{-\frac{\rho^2}{4t}} - e^{-\frac{\eta^2}{4t}} \right) 1 \\
+ \frac{2\Xi_{\omega}^* \Pi_{\omega}^*}{\cosh^2(\theta)} \left[ 1 + \sqrt{(\pi t) \omega \tanh \theta e^{u_{\omega}(\eta, t)^2}} \text{erfc}(u_{\omega}(\eta, t)) \right] e^{-\frac{\eta^2}{4t}} \right\}
\]
where \( \beta = \text{Vol}(\mathcal{N}) \), \( u_{\omega}(\eta, t) = \frac{\eta}{\sqrt{4t}} - \sqrt{t} \omega \tanh(\theta) \), and
\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} d\xi e^{-\xi^2}
\]
is the complementary error function.

The heat kernel in equation (4.4) was first presented in \[26\]. For the particular case of an antiperiodic boundary fiber, it coincides with the Fourier transform of equation (101) in \[31\].

In the appendix A we show that the asymptotics of the heat kernel in a compact manifold of product type coincides with the asymptotics of (4.4).
5 Meromorphic properties of the zeta function

Let us now analyze the boundary contributions to the global zeta function related to (4.4). We first note that global quantities are necessarily divergent due to the non-compact nature of our manifold $\mathcal{M}$. It is easy to see that such divergencies come, upon integration over $\mathbb{R}^+$, from the first term in (4.4). As shown in appendix A, this term (duely smeared) completes the volume contribution in a compact manifold.

In the following we will, without changing the notation, ignore this term, and this will allow us to determine the boundary contributions to the global zeta function.

Let us then consider the trace of (4.4) ignoring the first term. It is convenient to perform the Dirac trace ($tr$) first. Since

$$tr \left( \frac{2\Pi_+ \Pi^*_+}{\cosh^2(\theta)} \right) = 2,$$

the trace of the 'boundary' heat kernel reduces to

$$TrK = \sum \omega \tanh \theta e^{-\omega^2 t}$$

$$\times \int_0^\infty dy \text{erfc}[u_\omega(2y, t)] e^{\frac{-y^2}{t} + u_\omega^2(2y, t)},$$

where the second and third term in (4.4) have cancelled each other. Now, using that

$$-\frac{1}{2} \frac{\partial}{\partial y} \left[ e^{-y^2/t+u_\omega^2(2y,t)} \text{erfc}[u_\omega(2y, t)] \right] =$$

$$e^{-y^2/t} \left[ \frac{1}{\sqrt{\pi}t} + \omega \tanh \theta u_\omega^2(2y, t) \text{erfc}[u_\omega(2y, t)] \right], \quad (5.1)$$

we get

$$TrK = \frac{1}{2} \sum \omega e^{-\omega^2 t} \left[ e^{u_\omega^2(0, t)} \text{erfc}[u_\omega(0, t)] - 1 \right] =$$

$$\frac{1}{2} \sum \omega \left[ e^{\frac{-\omega^2 t}{\cosh^2 \theta}} \left[ 1 + \text{erf}(\omega \sqrt{t} \tanh \theta) \right] - e^{-\omega^2 t} \right].$$

Here we used $\text{erf}(x) = -\text{erf}(-x) = 1 - \text{erfc}(x)$.  

12
Now, we can Mellin transform this trace, to obtain the 'boundary' zeta function of the square of the Dirac operator in the infinite cylinder

\[ \zeta(s, P^2) = \frac{1}{2\Gamma(s)} \sum_{\omega} \int_{0}^{\infty} dt \ t^{s-1} \left[ e^{\frac{\omega^2 t}{2}} - e^{-\omega^2 t} \right] \]

\[ + \frac{1}{2\Gamma(s)} \sum_{\omega} \int_{0}^{\infty} dt \ t^{s-1} e^{\frac{\omega^2 t}{2}} \text{erf}(\omega \sqrt{t} \tanh \theta) \]

\[ = \zeta_1(s, P^2) + \zeta_2(s, P^2). \quad (5.2) \]

The first contribution can be readily seen to be

\[ \zeta_1(s, P^2) = \frac{1}{2} (\cosh^{2s} \theta - 1) \zeta(s, A^2). \quad (5.3) \]

As for the second contribution to (5.2), it is given by

\[ \zeta_2(s, P^2) = \frac{1}{2\Gamma(s)} \sum_{\omega} \int_{0}^{\infty} dt \ t^{s-1} e^{\frac{\omega^2 t}{2}} \frac{2}{\sqrt{\pi}} \int_{0}^{(\omega \sqrt{t} \tanh \theta)} d\xi e^{-\xi^2}. \]

After changing variables according to \( y = \frac{\xi \cosh \theta}{\sqrt{\xi \omega}} \), and interchanging integrals, one finally gets

\[ \zeta_2(s, P^2) = \frac{\Gamma(s + \frac{1}{2})}{2\Gamma(s)} \cosh^{2s} \theta \sum_{\omega} \text{sign}(\omega) \ (\omega^2)^{-s} \]

\[ \times \frac{2}{\sqrt{\pi}} \int_{0}^{\sinh \theta} dy \ (1 + y^2)^{-s-\frac{1}{2}} \]

\[ = \frac{\Gamma(s + \frac{1}{2})}{2\Gamma(s)} \cosh^{2s} \theta \eta(2s, A) \frac{2}{\sqrt{\pi}} \int_{0}^{\sinh \theta} dy \ (1 + y^2)^{-s-\frac{1}{2}} \]

\[ = \frac{1}{\sqrt{\pi}} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)} \sinh \theta \cosh^{2s} \theta \eta(2s, A) \]

\[ \times 2F_1 \left( \frac{1}{2}, \frac{1}{2} + s, \frac{3}{2}, -\sinh^2 \theta \right). \quad (5.4) \]

The structure of the zeta function is similar to the structure found for spectral boundary conditions, see e.g. [15]. In particular, the analysis of the zeta function on \( \mathcal{M} \) has been reduced to the analysis of the zeta and eta functions on the boundary \( \mathcal{N} \).

As already commented, from (5.3) and (5.4) one can determine the positions of the poles and corresponding residues for the zeta function in any
cylindrical product manifold, in terms of the meromorphic structure of the zeta and eta functions of the operator $A$. For the rightmost poles, explicit results can be given in terms of the geometry of the boundary. For example, for $s = 1/2$ we see that

$$\text{Res} \zeta_1 \left( \frac{1}{2}, P^2 \right) = \frac{1}{2} \left( \cosh \theta - 1 \right) \text{Res} \zeta \left( \frac{1}{2}, A^2 \right)$$

$$= \frac{\beta}{2} \left( \cosh \theta - 1 \right) \left( \frac{4\pi}{\Gamma \left( \frac{1}{2} \right)} \right)^{-1/2}.$$ 

Because $\zeta_2$ does not contribute, given $\eta(2s, A)$ is regular at $s = 1/2$ [3], this equals $\text{Res} \zeta(s, P^2)$ and is the result expected from the calculation on the ball [32]. For $\theta = 0$ the residue disappears, as is known to happen for non-chiral local bag boundary conditions [4]. Further results can be obtained by using Theorem 4.4.1 of [4]. For the particular case of $s = 0$, the fact that $\zeta(s, A^2)$ and $\eta(2s, A)$ are regular at $s = 0$ shows that $\zeta(0, P^2) = \zeta(0, P) = 0$.

6  Eta function and boundary contribution to the spectral asymmetry

As already commented, since the euclidean space-time we are considering is even dimensional, there is no bulk contribution to the asymmetry. To obtain the boundary contribution, the eigenvalue problem for the Dirac operator $P$ should be investigated on a collar neighborhood of the boundary. Here, we consider instead the operator on the semi-infinite cylinder extending to $y \to \infty$. As is well-known [33], since we are treating a self-adjoint problem, this yields the correct answer for an invertible boundary operator $A$. We shall discuss the non-invertible case toward the end of this section. Hence, for the moment, we assume that $A$ has no zero modes.

Denoting the (real) eigenvalues of the Dirac operator by $\lambda$, the relevant spectral function is the eta function

$$\eta(s, P) = \sum_{\lambda} \frac{\text{sign} \lambda}{|\lambda|^s} = \zeta \left( s + \frac{1}{2}, P^2, P \right)$$

$$= \frac{1}{\Gamma \left( \frac{s+1}{2} \right)} \int_0^\infty dt \ t^{\frac{s-1}{2}} \text{Tr} \left( P e^{-tP^2} \right). \quad (6.1)$$

In our particular case, using [32] and [4], taking Dirac traces, going to the diagonal, and integrating over the tangential variable, one is left with [26]
\[
\text{Tr} \left( P e^{-tP^2} \right) = \sum_{\omega} \omega e^{-\omega^2 t} \times \\
\int_{0}^{\infty} dy \left\{ \frac{1}{\sqrt{\pi t}} + \omega \tanh \theta e \left[ u_n(2y, t) \erfc \left[ u_n(2y, t) \right] \right] \right\} e^{-y^2/t} \frac{1}{\cosh \theta}.
\]

(6.2)

Now, we may use the simple identity in equation (5.1) to rewrite the relevant trace as follows,

\[
\text{Tr} \left( P e^{-tP^2} \right) = -\sum_{\omega} \omega e^{-\omega^2 t/\cosh^2 \theta} \int_{0}^{\infty} dy \frac{\partial}{\partial y} \left[ e^{-2y\omega \tanh \theta \erfc \left( u(2y, t) \right)} \right]
\]

\[
= \frac{1}{2} \sum_{\omega} \omega \frac{e^{-\omega^2 t/\cosh^2 \theta} \erfc \left( -\sqrt{t} \tanh \theta \omega \right)}{\cosh \theta}.
\]

(6.3)

The boundary contribution to the eta function is obtained by inserting (6.3) into (6.1) and, hence, it is given by

\[
\eta(s, P) = \frac{1}{\Gamma(\frac{s+1}{2})} \sum_{\omega} \frac{\omega}{2 \cosh \theta} \times \\
\int_{0}^{\infty} dt t^{\frac{s-1}{2}} e^{-\frac{\omega^2 t}{\cosh^2 \theta}} \left[ 1 - \erf \left( -\sqrt{t} \tanh \theta \omega \right) \right].
\]

(6.4)

Finally, changing variables to \( \tau = \omega^2 t/\cosh^2 \theta \), interchanging the order of the integrations and integrating over \( \tau \) one obtains the following rather explicit expression

\[
\eta(s, P) = \frac{1}{2} \cosh^s \theta \sum_{\omega} (\omega^2)^{-s/2} \left[ \text{sign}(\omega) + I(s, \theta) \right]
\]

\[
= \frac{1}{2} \cosh^s \theta \left[ \eta(s, A) + \zeta(s, A^2)I(s, \theta) \right],
\]

(6.5)

where we have introduced the function

\[
I(s, \theta) = \frac{2 \sinh \theta}{\sqrt{\pi}} \frac{\Gamma(\frac{s}{2} + \frac{1}{2})}{\Gamma(\frac{s}{2} + \frac{3}{2})} \binom{\frac{1}{2}}{\frac{1}{2}, \frac{3}{2}} \left( 1 + \frac{s}{2}, 3 - \sinh^2 \theta \right).
\]

(6.6)

Again, the analysis of the eta function has been reduced to the analysis of the zeta and eta functions on the boundary \( \mathcal{N} \).
To determine the spectral asymmetry, we particularize to $s = 0$. With $\pi I(0, \theta) = 2 \arctan(\sinh \theta)$ we obtain, for $s = 0$,

$$
\eta(0, P) = \frac{1}{2} \left\{ \eta(0, A) + 2 \frac{\pi}{\pi} \zeta(0, A^2) \arctan(\sinh \theta) \right\}.
$$

(6.7)

Now, the second term within the curly brackets can be seen to vanish, since the boundary is a closed manifold of odd dimensionality. In fact, in our case, $\zeta(0, A^2) = a_1(A^2) = 0$, where $a_1(A^2)$ is a heat kernel coefficient in the notation of [4] (for details, see Theorem 1.12.2 and Lemma 1.8.2 in this reference), and we are left with

$$
\eta(0, P) = \frac{1}{2} \eta(0, A).
$$

(6.8)

As far as $A$ is invertible, this is the main result of this section, relating the $\eta$–invariant of the Dirac operator to the same invariant of the boundary operator. Note that the outcome is the same irrespective of the value of $\theta$, i.e., it holds both for pure MIT and chiral bag conditions. The first case was treated before in [33].

As already pointed, (6.8) gives the whole spectral asymmetry when the operator $A$ is invertible. In fact, for such cases it was proved in [33] (see also [34]) that the asymmetry splits, in the adiabatic (infinite volume) limit, into the bulk contribution plus the infinite cylinder one. Moreover, reference [35] shows that the spectral asymmetry is independent from the size of the manifold when the boundary value problem is self adjoint, as in our case. This, together with the vanishing of the volume contribution in even dimensions, leads to the previous conclusion.

Now, we study the more subtle case of a non-invertible boundary operator $A$. Then, as can be seen from (6.2), $\omega = 0$ would give no extra contribution in the semi-infinite cylinder. However, in this case, the trace (6.3) can differ in a substantial way from the corresponding one in the collar neighborhood. As explained in [31], both large $t$ behaviors may be different, thus giving extra contributions to the asymmetry in the collar. This difference in high $t$ behavior is due to the presence of “small” eigenvalues, vanishing as the inverse of the size of the manifold in the adiabatic limit [36]. These extra contribution can be determined, modulo integers, by using the arguments in [4, 35, 37]. To this end, consider the one-parameter family of differential operators

$$
P_\alpha = P + \frac{2\pi}{\beta} \alpha \gamma_x, \quad P_0 = P.
$$

(6.9)
These operators share the same $\alpha$-independent domain. They are invertible for $\alpha \neq 0$ and can be made invertible for all $\alpha$ by subtracting the projector on the subspace of small eigenvalues related to the zero-modes at $\alpha = 0$. This then yields a new family of operators $P'_\alpha$ and one obtains
\[
\eta(0, P_\alpha) = \eta(0, P'_\alpha) \mod \mathbb{Z} \quad \text{and} \quad \frac{d}{d\alpha} \eta(0, P_\alpha) = \frac{d}{d\alpha} \eta(0, P'_\alpha). \quad (6.10)
\]
Then, differentiating with respect to $\alpha$ one finds for the spectral flow
\[
\frac{d}{d\alpha} \eta(0, P'_\alpha) = \frac{1}{\Gamma(s+1/2)} \int_0^\infty dt \frac{t^{s-1}}{2} \text{Tr} \left( e^{-tP'_\alpha^2} \right) \left. \frac{dP'_\alpha}{d\alpha} \left( 1 + 2t \frac{d}{dt} \right) e^{-tP'_\alpha^2} \right|_{s=0}
\]
\[
= \frac{1}{\Gamma(s+1/2)} \int_0^\infty dt \frac{t^{s-1}}{2} \text{Tr} \left( \frac{dP'_\alpha}{d\alpha} \left( 1 + 2t \frac{d}{dt} \right) e^{-tP'_\alpha^2} \right) \left|_{s=0} \right.
\]
\[
= -2\pi s \frac{1}{\beta \Gamma(s+1/2)} \int_0^\infty dt \frac{t^{s-1}}{2} \text{Tr} \left( e^{-tP'_\alpha^2} \right) \left|_{s=0} \right.
\]
\[
+ \frac{4\pi}{\beta \Gamma(s+1/2)} \text{Tr} \left( e^{-tP'_\alpha^2} \right) \left|_{s=0} \right. \]
\[
= 2\pi \beta \eta(0, P'_\alpha) \text{ Res}_{s=0} \zeta(s+1/2, A^2) (\text{arctan} \sinh \theta). \quad (6.11)
\]
where we performed a partial integration to arrive at the last equation. In addition, we used $\frac{dP'_\alpha}{d\alpha} = \frac{2\pi}{\beta} \gamma_x$. Since $P'_\alpha - P_\alpha$ is an operator of finite range we may safely skip the prime in the last line of the above formula. Finally, the very last term in equation (6.11) can be seen to vanish, which gives, for the spectral flow
\[
\frac{d}{d\alpha} \eta(0, P'_\alpha) = -\frac{\pi}{\beta} \text{Res}_{s=0} \left[ \zeta(s+1, A^2) \right. 
\]
\[
+ \frac{2\pi}{\beta} \eta(s+1, A) \text{ arctan} \sinh \theta \right]. \quad (6.12)
\]
Now, the second term can be seen to vanish, since (again with the notation of [4]), $\sqrt{\pi} \text{Res}_{s=0} \eta(s+1, A) = 2a_0(A^2, A) = 0$.
Moreover, $\sqrt{\pi} \text{Res}_{s=0} \zeta(s+1, A^2) = 2a_0(A^2) = \frac{2}{\pi}$. Thus, one finally has for the spectral flow, no matter whether $A$ is invertible or not
\[
\frac{d}{d\alpha} \eta(0, P_\alpha) = -1. \quad (6.13)
\]
To summarize, this work shows that, in two dimensions, the first order boundary value problem defined by chiral bag boundary conditions is
strongly elliptic, as is its associated second order problem. We have re-
lated the meromorphic structure of the spectral functions and the spectral
asymmetry of the problem with the meromorphic structure and spectral
asymmetry of the operator \( A \) (acting on the variable along the boundary),
much in the way it was done in [15] for APS boundary conditions. Note
that, even though throughout this paper we have studied the free Dirac op-
erator, everything we’ve done holds for a gauge field such that \( A_x = A_x(x) \)
(independent of the normal variable) and \( A_y = 0 \) for, then, a gauge transfor-
mation can be performed of the form

\[
\psi'(x, y) = e^{i \int_0^y A_x(x') dx'} \psi(x, y),
\]

which leads to just a twist in the boundary fiber.

In this work, we have studied the case of a two-dimensional manifold.
However, similar results concerning the elliptic character of the problem and
the meromorphic properties of the zeta function have been shown to hold
for higher even dimensions in [25].

A Appendix

Since \((P^2, \mathcal{B})\) constitutes an elliptic, self adjoint, boundary problem, with
positive definite principal symbol, its heat kernel can be defined; in the sub-
space orthogonal to zero and negative modes (which, in principle, might
exist in finite number) it is an infinitely smoothing operator, and admits
an asymptotic expansion for \( t \to 0 \). Moreover, in our case, the associated
first order problem is also self adjoint. So, \( P^2_{\mathcal{B}} \) is non negative. As for zero
modes, they have been shown to be absent (at least for a simply connected
boundary) in reference [20]. Thus, the trace of the heat kernel decreases
exponentially for \( t \to \infty \). As a consequence, in order to study the mer-
omorphic structure of the zeta and eta functions, one can safely concentrate
on the asymptotic regime. In what follows, we show that the asymptotic
behavior of the true heat kernel \((\tilde{K}(z, z'; t))\) coincides with the asymptotic
behavior of the parametrix constructed by pasting the heat kernel in the
cylinder, \((K_c(z, z'; t))\), given by equation (4.4), and the heat kernel in the
double, which is a boundaryless manifold.

Let \( \rho(a, b) \) a \( C^\infty \) function of the geodesic distance from the boundary,
y, such that \( \rho(a, b) = 0 \) for \( y \leq a \), and \( \rho(a, b) = 1 \) for \( y \geq b \). Define the
following smearing functions
\[
\begin{align*}
\varphi_2 &= \rho\left(\frac{1}{4}, \frac{1}{4}\right) \\
\varphi_1 &= 1 - \rho\left(\frac{3}{4}, \frac{3}{4}\right) \\
\psi_2 &= \rho\left(\frac{1}{4}, \frac{3}{4}\right) \\
\psi_1 &= 1 - \psi_2.
\end{align*}
\] (A.1)

These smearing functions have the following characteristics:

1) In a collar neighborhood of the boundary, \( N \times [0, 1] \), they are functions of just the normal variable.
2) Each \( \varphi_i \) is constant (has vanishing derivatives) and equals 1 on the support of the corresponding \( \psi_i \).
3) \( \psi_1 + \psi_2 = 1 \)

We use these functions to construct a parametrix of the heat kernel

\[
Q(z, z'; t) = \varphi_1(z)K_c(z, z'; t)\psi_1(z') + \varphi_2(z)K_d(z, z'; t)\psi_2(z'),
\] (A.2)

where \( K_c \) is the heat kernel in the infinite cylinder, given by (4.4), and \( K_d \) is the heat kernel in the double.

Now, at the level of operators, the error is given by

\[
E(t) = K(t) - Q(t) = \int_0^t ds \frac{\partial}{\partial s} (K(s)Q(t - s))
\]

\[
= \int_0^t ds K(s) \left( -P \frac{d}{ds} + \frac{\partial}{\partial s} \right) Q(t - s)
\]

\[
= \int_0^t ds K(s) C(t - s). \] (A.3)

Thus, for the corresponding kernels one has

\[
K(z, z'; t) - Q(z, z'; t) = \int_0^t ds \int dw K(z, w; s)C(w, z'; t - s), \] (A.4)

where

\[
C(w, z'; t - s) = -\frac{d^2 \varphi_1}{dy^2} K_c(w, z'; t - s)\psi_1 - \frac{d^2 \varphi_2}{dy^2} K_d(w, z'; t - s)\psi_2 -
\]

\[
2\frac{d\varphi_1}{dy} \frac{dK_c}{dy} (w, z'; t - s)\psi_1 - 2\frac{d\varphi_2}{dy} \frac{dK_d}{dy} (w, z'; t - s)\psi_2. \] (A.5)

As a consequence of the properties of the smearing functions (in particular, due to 2)) \( C(w, z'; t) \) vanishes for \( w \not\in \left[\frac{1}{4}, \frac{3}{4}\right] \times N \) and, in that region, it vanishes when the distance \( d(w, z') < \frac{1}{4} \). In order to show that \( |C(w, z'; t)| \leq c_1 e^{-\frac{t}{4}} \) for \( d(w, z') \geq \frac{1}{4} \), we now prove the following
Proposition: \( K_c(z, z'; t) \) is exponentially small in \( t \) as \( t \to 0 \) for \( y \neq y' \). More precisely, it is bounded by \( C t^{-1} \exp \left( -\frac{(y-y')^2}{4t} \right) \).

Proof

As already shown, the projectors \( \mathcal{P}_+ \) and \( \mathcal{P}_- \) define two subspaces, where the heat kernel reduces, for each \( \omega \), to the fundamental solution of \( \frac{\partial^2}{\partial y^2} + \omega^2 \), with Dirichlet and Robin boundary conditions respectively. The proposition is well known to hold in the first case. In the second subspace, one has

\[
K_c(z, z'; t) = \frac{1}{\beta(4\pi t)^{\frac{1}{2}}} \sum_{\omega} \phi^*_\omega(x)\phi_\omega(x') e^{-\omega^2 t} \left\{ \left( e^{-\frac{\rho^2}{4t}} + e^{-\frac{\eta^2}{4t}} \right) 1 + \sqrt{4\pi t} \omega \tanh \theta e^{u_\omega(\eta, t)} \text{erfc} \left[ u_\omega(\eta, t) \right] e^{-\frac{\eta^2}{4t}} \right\} .
\]

(A.6)

So,

\[
|K_c(z, z'; t)| < \frac{1}{\beta} \sum_{\omega} |\phi^*_\omega(x)\phi_\omega(x')| e^{-\omega^2 t} \left\{ e^{-\frac{\rho^2}{4t}} + e^{-\frac{\eta^2}{4t}} \right\} .
\]

where we have used that \( \text{erfc}(x) < \frac{2}{\sqrt{\pi}} e^{-x^2} \).

Now, using that \( x \leq e^{\frac{x^2}{2}} \) and \( 2|\phi^*_\omega(x)\phi_\omega(x')| \leq |\phi_\omega(x)|^2 + |\phi_\omega(x')|^2 \), we find

\[
|K_c(z, z'; t)| < \frac{1}{\beta 2 \sqrt{\pi} t} \sum_{\omega} (|\phi_\omega(x)|^2 + |\phi_\omega(x')|^2) e^{-\omega^2 t} \left[ 1 + 2e^{\frac{-2\eta\tanh^2 \theta}{2}} \right] e^{-\frac{\eta^2}{4t}}
\]

\[
\leq \frac{3}{\beta 2 \sqrt{\pi} t} \sum_{\omega} (|\phi_\omega(x)|^2 + |\phi_\omega(x')|^2) e^{-\omega^2 t} e^{-\frac{\eta^2}{4t}} .
\]

Finally, since the boundary is a boundaryless manifold, the boundary heat kernel is bounded, on the diagonal of \( \mathcal{N} \times \mathcal{N} \), by \( C t^{-\frac{1}{2}} \), which leads us to the announced result.

Now, this estimate, and a similar one which is known to hold for \( K_d \), allow to show, from equation (A.5), that \( |C(w, z'; t)| \leq c_1 e^{-\frac{\eta^2}{4t}} \). Thus, a convergent series for the parametrix \( Q \) can be constructed by iterating (A.4), and one can show that:

20
The exact heat kernel satisfies the bound $|K(z, z'; t)| < Ct^{-1}e^{-\frac{d^2(z, z')}{t}}$ as $t \to 0$ and, on the diagonal, it differs from $Q$ according to $|K(z, z; t) - Q(z, z; t)| \leq ce^{-\frac{C}{t}}$ (for details see, for instance, [38]). So, one has, asymptotically

$$\text{Tr} K \sim \int_0^1 \int_{\mathcal{N}} dydx K_c \psi_1 + \int_{\mathcal{M}} dydx K_d \psi_2. \quad (A.7)$$

Observe that, due to the property 3) of the smearing functions, the first term in $K_c$ (equation (4.4)) adds up to the contribution from the heat kernel in the double, to complete the volume contribution to the trace. The remaining terms in the same equation give the boundary contribution. Now, it is easy to show that $\psi_1$ can be replaced by 1, and the integral $\int_0^1$ can be replaced by $\int_0^\infty$, since the difference vanishes exponentially when $t \to 0$.

**Acknowledgements:** Research of CGB and EMS was partially supported by CONICET(PIP 0459/98) and UNLP(11/X298).

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