Stability and response of trapped solitary wave solutions of coupled nonlinear Schrödinger equations in an external, $\mathcal{PT}$- and supersymmetric potential

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Abstract
We present trapped solitary wave solutions of a coupled nonlinear Schrödinger (NLS) system in $1+1$ dimensions in the presence of an external, supersymmetric and complex $\mathcal{PT}$-symmetric potential. The Schrödinger system this work focuses on possesses exact solutions whose existence, stability, and spatiotemporal dynamics are investigated by means of analytical and numerical methods. Two different variational approximations are considered where the stability and dynamics of the solitary waves are explored in terms of eight and twelve time-dependent collective coordinates (CCs). We find regions of stability for specific potential choices as well as analytic expressions for the small oscillation frequencies in the CC approximation. Our findings are further supported by performing systematic numerical simulations of the NLS system.

Keywords: $\mathcal{PT}$-symmetric potentials, variational approximation, collective coordinates, dissipation functional, existence and spectral stability analysis

(Some figures may appear in colour only in the online journal)
1. Introduction

The nonlinear Schrödinger equation (NLSE) arises in many areas of physics including Bose–Einstein condensation, plasmas, water waves and nonlinear optics [1]. The possibility of experimentally coupling two-component NLSEs in matrix complex potentials has recently been investigated in nonlinear optics situations in which two wave guides are locally coupled through an antisymmetric medium [2].

On the other hand, parity-time or $\mathcal{PT}$ symmetry was first introduced into physics as an alternative to Hermiticity in quantum mechanics, yet with real eigenvalues [3–6]. The similarity of the Schrödinger equation with Maxwell’s equations in the paraxial approximation facilitates the realization of $\mathcal{PT}$-invariant systems in a variety of contexts such as optics [7–16], photonic lattices [17], electronic circuits [18, 19], mechanical circuits [20], whispering-gallery microcavities [21], among many other physical settings.

Supersymmetry (SUSY) originally considered in high-energy physics to relate fermionic and bosonic systems has also been invoked in condensed matter systems such as fractional quantum Hall states [22] and realized in optics [23, 24]. For the Schrödinger equation, SUSY relates two potentials which have the same spectrum [25–29]. Recently in [30], a subset of the present authors studied the stability of exact solutions of a single-component NLSE in a class of external potentials having SUSY and $\mathcal{PT}$ symmetry.

Our aim in the present work is to extend our considerations to the case of two coupled NLSEs in $\mathcal{PT}$-symmetric and supersymmetric external potentials where the cross interaction between the two components is dictated by the nonlinear coupling of the equations. In particular, the superpotential studied in [30] is generalized to a matrix form herein where we show that it is $\mathcal{PT}$-symmetric. Interestingly, our potential has a non-trivial coupling between the two components which in turn affects the stability of the trapped soliton-like solutions. Our numerical investigations on that front are split into two steps. At first, we will employ a collective coordinate (CC) approximation in order to map out the domain of stability of the pertinent waveforms of the coupled system. Then, we will consider the NLSEs and focus on the existence, stability and spatio-temporal evolution of the solitary waves. Upon identifying the steady-state solutions to the NLSEs via fixed-point iterations, we will perform parametric continuations over the parameters of the system. This will allow us to carry out a systematic spectral stability analysis of the solutions and identify parametric regions of stability. Those findings will be corroborated by direct numerical simulations of the NLSEs. Then, we will draw comparisons between the CC approximation and numerical simulations in regimes where the trapped solutions are stable and unstable. In fact, and in the unstable parametric regime, we will show that the effect of the coupling is responsible for the motion of the solitary waves in opposite directions. Also the amplitudes of the two components respond oppositely to small perturbations.

The structure of the paper is as follows. We discuss the connection to SUSY in section 2, and give the exact soliton solutions to the coupled NLSEs in section 3. In section 4 we present the derivation of the equations of motion for the CC approximation using a variational method which is based on Rayleigh’s dissipation functional. The trial wave functions we have chosen together with the respective dynamic equations for the CCs derived are discussed in section 5. We present results for the dynamical evolution of the CCs in section 6 where comparisons of these results with numerical simulations are made. In section 7 we present numerical results on the existence, stability and dynamics of the exact solutions to the coupled NLSEs. Finally, we state our conclusions in section 8.
2. Supersymmetry

We consider here a two-component nonlinear Schrödinger (NLS) system in 1 + 1 (one spatial and one temporal) dimensions of the form:

\[ i \partial_t \Psi(x, t) + \partial_x^2 \Psi(x, t) + \gamma [ \Psi^\dagger(x, t) \Psi(x, t) ] \Psi(x, t) - V(x) \Psi(x, t) = 0, \]  
(2.1)

with

\[ \Psi(x, t) = \begin{pmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{pmatrix} \in \mathbb{C}^2, \]  
(2.2)

where \( \psi_j(x, t) \) is the wave function of the first (\( j = 1 \)) and second components (\( j = 2 \)), respectively, and \( \gamma \) is the nonlinearity strength. The subscripts in equation (2.1) stand for differentiation with respect to \( t \) and \( x \), respectively, and (\( \cdot \)^\dagger) corresponds to conjugate transpose.

For a superpotential of the form:

\[ W(x) = r \sigma_0 \tanh(x) + i s \sigma_3 \text{sech}(x), \]  
(2.3)

where \( \sigma_i \) are the Pauli matrices, the SUSY partner potentials are given by

\[ V_{\pm}(x) = W^2(x) \pm \partial_x W(x) \]
\[ = \sigma_0 r^2 - \sigma_0 b_\pm^2 \text{sech}^2(x) + i \sigma_3 d_\pm \text{sech}(x) \tanh(x), \]  
(2.4)

where

\[ b_\pm^2 = s^2 + r( \pm 1), \quad d_\pm = s(2r \mp 1). \]  
(2.5)

Note that the partner potentials \( V_{\pm}(x) \) are \( \mathcal{PT} \)-symmetric.

3. Model potential

The equation we want to solve is (2.1), where the external potential \( V(x) \) is given by

\[ V(x) = \sigma_0 V_0(x) + i \sigma_3 V_1(x). \]  
(3.1)

Since we are interested in variational approximations to the moments of the NLSEs, we now show that these equations can be derived from a modified Euler–Lagrange equation by utilizing a Rayleigh dissipation functional. The usual conservative part of the action is

\[ \Gamma[\Psi^\dagger, \Psi] = \int dt \sigma_0 L[\Psi^\dagger, \Psi], \]  
(3.2)

where the conservative part of the Lagrangian \( L \) is given by

\[ L[\Psi^\dagger, \Psi] = T[\Psi^\dagger, \Psi] - H[\Psi^\dagger, \Psi], \]  
(3.3)

with

\[ T[\Psi^\dagger, \Psi] = \int dx \frac{i}{2} \left\{ \Psi^\dagger(x, t) \left[ \partial_t \Psi(x, t) - \partial_x \Psi^\dagger(x, t) \right] \Psi(x, t) \right\}, \]  
(3.4a)
In component form, equation (2.1) reads

\[ H[\Psi^1, \Psi] = \int dx \left\{ -\frac{\gamma}{2} |\Psi^1(x,t)|^2 + |\Psi(x,t)|^2 + V_0(x) \Psi^1(x,t)\Psi(x,t) \right\}. \] (3.4b)

We introduce the dissipation functional \( F \) via

\[ F[\Psi^1, \Psi; \Psi^1_\tau, \Psi_\tau] = \int dr \left[ F[\Psi^1, \Psi; \Psi^1_\tau, \Psi_\tau] \right] \] (3.5)

where

\[ F[\Psi^1, \Psi; \Psi^1_\tau, \Psi_\tau] = -i \int dx \ V_i(x) \left\{ i [ \partial_t \Psi^1(x,t)] \sigma_3 \Psi(x,t) - \Psi^1(x,t) \sigma_3 [ \partial_t \Psi(x,t)] \right\}. \] (3.6)

The equations of motion for \( \Psi(x,t) \) in the presence of a complex potential follow from the generalized Euler–Lagrange equations:

\[ \frac{\delta \Gamma[\Psi^1, \Psi]}{\delta \Psi^1(x,t)} = -\frac{\delta F[\Psi^1, \Psi; \Psi^1_\tau, \Psi_\tau]}{\delta \Psi^1_\tau(x,t)}, \] (3.7)

which lead to the equations of motion

\[ \frac{\partial L}{\partial \Psi^1(x,t)} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\Psi}^1(x,t)} \right) = -\frac{\partial F}{\partial \dot{\Psi}^1(x,t)}, \] (3.8)

and reproduce equation (2.1) with the potential (3.1).

3.1. Exact solution

In component form, equation (2.1) reads

\[ \begin{align*}
i \partial_t \psi_1(x,t) + \partial_x^2 \psi_1(x,t) + \gamma \left( |\psi_1(x,t)|^2 + |\psi_2(x,t)|^2 \right) \psi_1(x,t) - V(x) \psi_1(x,t) &= 0, \quad (3.9a) \\
i \partial_t \psi_2(x,t) + \partial_x^2 \psi_2(x,t) + \gamma \left( |\psi_1(x,t)|^2 + |\psi_2(x,t)|^2 \right) \psi_2(x,t) - V^*(x) \psi_2(x,t) &= 0, \quad (3.9b)
\end{align*} \]

where \( V(x) \) is of the form: \( V(x) = V_0(x) + i V_1(x) \), and where we have chosen

\[ \begin{align*}
V_0(x) &= -b^2 \ \text{sech}^2(x), \quad (3.10a) \\
V_1(x) &= -d \ \text{sech}(x) \ \text{tanh}(x). \quad (3.10b)
\end{align*} \]

The exact solutions of the system [cf equation (3.9)] are given by

\[ \begin{align*}
\psi_1(x,t) &= A_1 \ \text{sech}(x) \ \exp \left\{ i [ t + \phi(x)] \right\}, \quad (3.11a) \\
\psi_2(x,t) &= A_2 \ \text{sech}(x) \ \exp \left\{ i [ t - \phi(x)] \right\}, \quad (3.11b)
\end{align*} \]

where \( \phi(x) = 2d \ \text{tan}^{-1}[\text{tanh}(x/2)] \), provided that

\[ \gamma \left( A_1^2 + A_2^2 \right) = 2 + (d/3)^2 - b^2 \geq 0. \] (3.12)

The left panel of figure 1 showcases the potentials \( V_0(x) \) (solid blue line) and \( V_1(x) \) (solid red line) whereas the middle and right panels of the figure depict the real (solid blue line) and imaginary (solid red line) parts of the exact solutions \( \psi_1(x,t) \) and \( \psi_2(x,t) \) at \( t = 0 \), respectively, for values of the parameters \( \gamma = 1, b = 1, d = 1/2, \) and \( A_1 = 1 \).
4. Collective coordinates (CCs)

In this section, we consider two variational approximations for studying the stability and time evolution of the trapped solitary waves. This way, we will be able to compare our findings with numerical simulations of the NLSEs in section 6 (see, also section 7 discussing our computational analysis). We review now the method of collective coordinates, abbreviated CC hereafter (see for example reference [30]) that will be applied to the present setup.

The time-dependent variational approximation relies on introducing a finite set of time-dependent real parameters in a trial wave function that hopefully captures the time evolution of a perturbed solution. By doing this one obtains a simplified set of ordinary differential equations for the CCs in place of solving the full partial differential equations associated with the NLS system. We begin our discussion by setting

$$\Psi(x, t) \mapsto \tilde{\Psi}(x, Q(t)), \quad Q(t) = \{ Q_1(t), Q_2(t), \ldots, Q_{2n}(t) \} \in \mathbb{R}^{2n},$$

where $Q(t)$ corresponds to the CCs. It should be pointed out that the success of the method depends greatly on the choice of the trial wave function $\tilde{\Psi}(x, Q(t))$. The generalized dissipative Euler–Lagrange equations lead to Hamilton’s equations for $Q(t)$. The Lagrangian in terms of the CCs is given by

$$L(Q, \dot{Q}) = T(Q, \dot{Q}) - H(Q),$$

where the kinetic term $T(Q, \dot{Q})$ and Hamiltonian $H(Q)$ are given by

$$T(Q, \dot{Q}) = \frac{i}{2} \int dx \left\{ \tilde{\Psi}^\dagger(x, Q) \tilde{\Psi}_t(x, Q) - \tilde{\Psi}_t^\dagger(x, Q) \tilde{\Psi}(x, Q) \right\} = \pi_{\mu}(Q) \dot{Q}^\mu,$$

and

$$H(Q) = \int dx \left\{ |\partial_x \tilde{\Psi}(x, Q)|^2 - V_0(x) |\tilde{\Psi}(x, Q)|^2 - (\gamma/2) |\tilde{\Psi}(x, Q)|^4 \right\},$$

respectively. Note that $\pi_{\mu}(Q)$ in equation (4.3) is defined by

$$\pi_{\mu}(Q) = \frac{i}{2} \int dx \left\{ \tilde{\Psi}^\dagger(x, Q) [\partial_\mu \tilde{\Psi}(x, Q)] - [\partial_\mu \tilde{\Psi}^\dagger(x, Q)] \tilde{\Psi}(x, Q) \right\},$$

where we have introduced the notation $\partial_\mu \equiv \partial/\partial Q^\mu$. 
The dissipation functional in terms of the CCs is given by
\[
F(Q, \dot{Q}) = i \int dx \, V_i(x) \left\{ \tilde{\psi}^\dagger(x, Q) \sigma_3 \tilde{\psi}_i(x, Q) - \tilde{\psi}^\dagger_i(x, Q) \sigma_3 \tilde{\psi}(x, Q) \right\} = w_{\mu}(Q) \dot{Q}^\mu, \tag{4.6}
\]
where
\[
w_{\mu}(Q) = i \int dx \, V_i(x) \left\{ \tilde{\psi}^\dagger(x, Q) \sigma_3 [\partial_\mu \tilde{\psi}(x, Q)] - [\partial_\mu \tilde{\psi}^\dagger(x, Q)] \sigma_3 \tilde{\psi}(x, Q) \right\}. \tag{4.7}
\]
Upon introducing the antisymmetric \(2n \times 2n\) symplectic matrix:
\[
f_{\mu\nu}(Q) = \partial_\nu \pi_\mu(Q) - \partial_\mu \pi_\nu(Q), \tag{4.8}
\]
the generalized Euler–Lagrange equations
\[
\frac{\partial L}{\partial Q^\mu} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}^\mu} \right) = - \frac{\partial F}{\partial Q^\mu} \tag{4.9}
\]
can be written in the form
\[
f_{\mu\nu}(Q) \dot{Q}^\nu = u_{\mu}(Q) = v_{\mu}(Q) - w_{\mu}(Q), \tag{4.10}
\]
by setting \(v_{\mu}(Q) = \partial_\mu H(Q)\). If \(\det[f(Q)] \neq 0\), we can define an inverse as the contra-variant matrix with upper indices:
\[
f^{\mu\nu}(Q) f_{\nu\sigma}(Q) = \delta^\mu_\sigma, \tag{4.11}
\]
in which case the equations of motion (4.10) can be formulated in the symplectic form:
\[
\dot{Q}^\mu = f^{\mu\nu}(Q) u_\nu(Q). \tag{4.12}
\]
We solve this set of equations for our choice of CCs.

5. Trial wave function

We will choose trial wave functions similar to that used for the single-component NLSE in a \(\mathcal{PT}\)-symmetric complex external potential [30]:
\[
\tilde{\psi}_1(x, Q_1(t)) = A_1(t) \text{sech}[\beta_1(t)(x - q_1(t))] e^{i \phi_1(x, Q_1(t))}, \tag{5.1a}
\]
\[
\tilde{\psi}_2(x, Q_2(t)) = A_2(t) \text{sech}[\beta_2(t)(x - q_2(t))] e^{i \phi_2(x, Q_2(t))}, \tag{5.1b}
\]
where
\[
\phi_1(x, Q_1(t)) = -\theta_1(t) + p_1(t)(x - q_1(t)) + A_1(t)(x - q_1(t))^2 + \phi(x), \tag{5.2a}
\]
\[
\phi_2(x, Q_2(t)) = -\theta_2(t) + p_2(t)(x - q_2(t)) + A_2(t)(x - q_2(t))^2 - \phi(x), \tag{5.2b}
\]
together with \(\phi(x) = (2d/3) \tan^{-1}[\tanh(x/2)]\). Let us now define
\[
M_i(t) = \int dx \, |\tilde{\psi}_i(x, t)|^2 = \frac{|A_i(t)|^2}{\beta_i(t)} \int dx \, \text{sech}^2(z) = \frac{2 |A_i(t)|^2}{\beta_i(t)}, \tag{5.3}
\]
where the integral in the right-hand side is calculated in appendix A (alongside with other integrals useful for the present work).
We consider two sets of variational parameters:

\[ Q_1(t) = \{ M_1(t), \theta_1(t), q_1(t), p_1(t), \beta_1(t), \Lambda_1(t) \}, \]  
\[ Q_2(t) = \{ M_2(t), \theta_2(t), q_2(t), p_2(t), \beta_2(t), \Lambda_2(t) \}, \]

with initial conditions

\[ p_1(0) = 0, \quad \beta_1(0) = 1, \quad \Lambda_1(0) = 0, \]  
\[ p_2(0) = 0, \quad \beta_2(0) = 1, \quad \Lambda_2(0) = 0, \]

and with \( q_1(0) = q_2(0) = \delta q_0 \). We make this perturbation to study the response of the exact solution, which has \( \delta q_0 = 0 \), to small initial perturbations. We also require that

\[ \gamma_2 \left[ M_1(0) + M_2(0) \right] = \gamma \left[ A_1^2(0) + A_2^2(0) \right] = 2 + \left( \frac{d}{3} \right)^2 - b^2. \]  

This way, the set of variational trial wave functions (5.1) satisfies the exact solution [cf equation (3.11)] at \( t = 0 \). We also set \( \theta_i(0) = 0 \), and require that \( \theta_i'(0) = -1 \).

### 5.1. Dynamic term

From equation (4.3), the dynamic term splits into the sum of two independent parts:

\[ T(\dot{Q}, \dot{\dot{Q}}) = \tau(Q_1, \dot{Q}_1) + \tau(Q_2, \dot{Q}_2), \]

where

\[ \tau(Q, \dot{Q}) = \frac{i}{2} \int dx \left\{ \bar{\psi}_t(x, Q) \psi_t(x, Q) - \bar{\psi}_t(x, Q) \psi_t(x, Q) \right\} = M \left\{ \dot{\theta} + p \dot{q} - \frac{\pi^2}{12 \beta^2} \Lambda \right\} = \pi_{\mu} (Q) \dot{Q}^\mu. \]

From this expression one easily determines the symplectic matrix

\[ f_{\mu\nu} = \partial_{\mu} \pi_{\nu} - \partial_{\nu} \pi_{\mu}, \]

from which we obtain its inverse \( f^{\mu\nu} \):

\[ f^{\mu\nu}(Q) = \begin{pmatrix} g^{\mu\nu}(Q_1) & 0 \\ 0 & g^{\mu\nu}(Q_2) \end{pmatrix}, \]

where

\[ g^{\mu\nu}(Q) = \frac{1}{M} \begin{pmatrix} 0 & -M & 0 & 0 & 0 \\ M & 0 & 0 & -p & c \\ 0 & 0 & 0 & 1 & 0 \\ 0 & p & -1 & 0 & 0 \\ 0 & -c & 0 & 0 & -d \\ 0 & 0 & 0 & d & 0 \end{pmatrix}, \quad c = \frac{\beta}{2}, \quad d = \frac{3 \beta^3}{\pi^2}. \]
5.1.1. Hamiltonian and its decomposition. Based on equation (4.4), the Hamiltonian can be written as the sum of three parts:

\[ H(Q) = H_{\text{kin}}(Q) + H_{\text{pot}}(Q) + H_{\text{nl}}(Q), \]

(5.12)

where \( H_{\text{kin}}, H_{\text{pot}}, H_{\text{nl}} \) stand for the kinetic, potential, and nonlinear terms, respectively.

Let us consider the kinetic term first which itself splits into two parts:

\[ H_{\text{kin}}(Q) = h_{\text{kin}}(Q_1) + h_{\text{kin}}(Q_2), \quad h_{\text{kin}}(Q) = \int dx \left| \partial_x \tilde{\psi}(x, Q) \right|^2. \]

(5.13)

Using the integral definitions of appendix A, we find:

\[ h_{\text{kin}}(Q) = M \left\{ \frac{1}{3} \beta^2 + p^2 + \frac{\pi^2}{3} \frac{\Lambda^2}{\beta^2} + \frac{\beta \alpha^2}{8} I_1(\beta, q) + \kappa \alpha \beta \frac{p}{2} I_2(\beta, q) + \kappa \alpha \beta \Lambda I_3(\beta, q) \right\}. \]

(5.14)

In a similar fashion, the potential term also splits into two parts:

\[ H_{\text{pot}}(Q) = h_{\text{pot}}(Q_1) + h_{\text{pot}}(Q_2), \]

(5.15)

where

\[ h_{\text{pot}}(Q) = \int dx \, V_0(x) \left| \tilde{\psi}(x, t) \right|^2 \]

\[ = -\frac{\beta M}{2} b^2 \int dx \, \text{sech}^2[\beta(x - q)] \text{sech}^2(x) = -\frac{\beta M}{2} b^2 I_3(\beta, q). \]

(5.16)

Finally, we consider the nonlinear term. Unlike the kinetic and potential terms, the nonlinear term does not split into two parts. Here we have

\[ H_{\text{nl}}(Q) = -\frac{\gamma}{2} \int dx \left| \tilde{\psi}(x, Q) \right|^4 \]

\[ = -\frac{\gamma}{2} \int dx \left[ |\tilde{\psi}_1(x, t)|^4 + 2 |\tilde{\psi}_1(x, t)|^2 |\tilde{\psi}_2(x, t)|^2 + |\tilde{\psi}_2(x, t)|^4 \right] \]

\[ = h_{\text{nl}}(Q_1) + c(Q_1, Q_2) + h_{\text{nl}}(Q_2), \]

(5.17)

where

\[ h_{\text{nl}}(Q) = -\frac{\gamma}{2} \left( \frac{\beta M}{2} \right)^2 \int dx \, \text{sech}^4[\beta(x - q)] = -\frac{\gamma}{6} \beta M^2. \]

(5.18)

The cross term, i.e., \( c(Q_1, Q_2) \) in equation (5.17) is given by

\[ c(Q_1, Q_2) = -\frac{\gamma}{4} \beta_1 M_1 \beta_2 M_2 C(\beta_1, q_1, \beta_2, q_2), \]

(5.19)

which involves the mixing integral (see, appendix A.2)

\[ C(\beta_1, q_1, \beta_2, q_2) = \int dx \, \text{sech}^2[\beta_1(x - q_1)] \text{sech}^2[\beta_2(x - q_2)]. \]

(5.20)

Note that \( C(\beta_1, q_1, \beta_2, q_2) \) is invariant under \( \{\beta_1, q_1\} \leftrightarrow \{\beta_2, q_2\} \), and \( c(Q_1, Q_2) \) is invariant under \( \{M_1, \beta_1, q_1\} \leftrightarrow \{M_2, \beta_2, q_2\} \).
5.2. Derivatives of the Hamiltonian

The Hamiltonian is made up of three terms:

\[
H(Q_1, Q_2) = h(Q_1) + c(Q_1, Q_2) + h(Q_2),
\]  

(5.21)

where

\[
h(Q) = M \left\{ p^2 + \frac{1}{3} \beta^2 + \frac{\pi^2}{3} \Lambda^2 + \beta \frac{d^2}{18} - \frac{9 b^2}{18} I_3(\beta, q) 
+ \kappa \frac{d \beta}{3} \left[ p I_1(\beta, q) + 2 \Lambda I_2(\beta, q) \right] \right\} - \frac{\gamma}{6} \beta M^2,
\]  

(5.22)

and the coupling term \(c(Q_1, Q_2)\) is given by equation (5.19). From equations (5.19), (5.21), and (5.22), we can then determine

\[
v_\mu = \partial_\mu H(Q_1, Q_2)
\]  

(5.23)

needed to obtain the first order equations of motion (4.10). We can write this in terms of two terms, for the CCs for each component:

\[
v_\mu = \partial_\mu h(Q_1) + \partial_\mu c(Q_1, Q_2) \equiv \partial_\mu h(Q_1) + c_\mu(Q_1, Q_2).
\]  

(5.24)

The derivatives of the coupling term are given in the appendix.

5.3. Dissipative term

The dissipative term (4.6) also splits into two parts:

\[
F(Q, \dot{Q}) = f(Q_1, \dot{Q}_1) - f(Q_2, \dot{Q}_2),
\]  

(5.25)

where

\[
f(Q, \dot{Q}) = i \int dx \, V_1(x) \left\{ \tilde{\psi}^*(x, Q) \tilde{\psi}(x, Q) - \tilde{\psi}^*_t(x, Q) \tilde{\psi}(x, Q) \right\}.
\]  

(5.26)

Again using the integral definitions in appendix A, we find:

\[
f(Q, \dot{Q}) = -\beta M d \left\{ \left( \dot{\theta} + p \dot{q} \right) f_1(\beta, q) - \left( p - 2 \Lambda \dot{q} \right) f_2(\beta, q) - \dot{\Lambda} f_3(\beta, q) \right\},
\]  

(5.27)

where the non-zero derivatives of \(f(Q, \dot{Q})\) with respect to \(\dot{Q}_\mu\) are given by

\[
w_\theta = -\kappa \beta M d f_1(\beta, q),
\]  

\[
w_q = -\kappa \beta M d \left[ p f_1(\beta, q) + 2 \Lambda f_2(\beta, q) \right],
\]  

\[
w_p = \kappa \beta M d f_3(\beta, q),
\]  

\[
w_\Lambda = \kappa \beta M d f_3(\beta, q).
\]  

\[
(5.28a-5.28d)
\]

5.4. Equations of motion

From equations (5.24) and (5.28) we can now obtain the equations

\[
Q^\mu = \left( \begin{array}{c} Q_1^\mu \\ Q_2^\mu \end{array} \right), \quad u_\mu(Q_1, Q_2) = \left( \begin{array}{c} u_\mu^{(1)}(Q_1, Q_2) \\ u_\mu^{(2)}(Q_1, Q_2) \end{array} \right)
\]  

(5.29)
whence the equations of motion (4.12) with \( f^{\mu\nu}(Q) \) given in (5.10) become

\[
\dot{Q}^\mu = g^{\mu\nu}(Q_1) u_\nu^{(1)}(Q_1, Q_2),
\]
\[
\dot{Q}^\mu_2 = g^{\mu\nu}(Q_2) u_\nu^{(2)}(Q_1, Q_2).
\]

This way, the associated rates are given by

\[
\dot{M} = -\kappa d M f_1(\beta, q),
\]
\[
\dot{\theta} = -p^2 + \frac{2}{3} \beta^2 + \beta \frac{d^2 - 9 b^2}{36} [3 I_3(\beta, q) + \beta I_{3,\beta}(\beta, q)] + \kappa d p f_2(\beta, q)
\]
\[
+ \kappa d \frac{d}{3} \sum \{ p [I_1(\beta, q) + \beta I_{1,\beta}(\beta, q)] + 2 \Lambda [3 I_2(\beta, q) + \beta I_{2,\beta}(\beta, q)] \}
\]
\[
- \gamma \frac{5}{12} \beta M + R_p,
\]
\[
\dot{q} = 2 p + \kappa d \frac{d}{3} [I_1(\beta, q) - 3 f_2(\beta, q)].
\]
\[
\dot{p} = -\beta \frac{d^2 - 9 b^2}{18} I_{3,q}(\beta, q) + \kappa d \frac{1}{3} \beta [p f_1(\beta, q) - 4 \Lambda f_2(\beta, q)] + R_m,
\]
\[
\dot{\beta} = -\kappa d \frac{\beta^2}{2} f_1(\beta, q) - 4 \beta \Lambda - \kappa d \frac{2}{\pi^2} \beta^4 [2 I_3(\beta, q) - 3 f_3(\beta, q)],
\]
\[
\dot{\Lambda} = \frac{4}{\pi^2} \beta^4 - 4 \Lambda^2 + \frac{\beta^3 (d^2 - 9 b^2)}{3 \pi^2} [I_3(\beta, q) + \beta I_{3,\beta}(\beta, q)]
\]
\[
+ \kappa d \frac{2 \beta^3}{\pi^2} \sum \{ p [I_1(\beta, q) + \beta I_{1,\beta}(\beta, q)] + 2 \Lambda [I_2(\beta, q) + \beta I_{2,\beta}(\beta, q)] \}
\]
\[
- \gamma \frac{\beta^3}{\pi^2} M + R_\Lambda.
\]

Here the nonzero mixed derivative terms \( R_{\mu} \) are

\[
R_{\mu_1} = c_{\mu_1}(q_1, \beta_1, q_2, \beta_2) + \frac{\beta_1}{2 M_1} c_{\mu_1}(q_1, \beta_1, q_2, \beta_2)
\]
\[
= -\frac{3}{8} \beta_1 \beta_2 M_2 C(\beta_1, q_1, \beta_2, q_2) - \frac{\gamma}{8} \beta_1^2 \beta_2 M_2 C_{\beta}(\beta_1, q_1, \beta_2, q_2),
\]
\[
R_{\mu_2} = -\frac{1}{M_1} c_{\mu_2}(q_1, \beta_1, q_2, \beta_2) = \frac{\gamma}{4} \beta_1 \beta_2 M_2 C_{\mu_2}(\beta_1, q_1, \beta_2, q_2),
\]
\[
R_{\Lambda_1} = \frac{1}{M_1} \frac{6 \beta_1^3}{\pi^2} c_{\beta_1}(q_1, \beta_1, q_2, \beta_2)
\]
\[
= -\gamma \frac{3}{2 \pi^2} \beta_1^3 \beta_2 M_2 [C(\beta_1, q_1, \beta_2, q_2) + \beta_1 C_{\beta_1}(\beta_1, q_1, \beta_2, q_2)],
\]

and
\[ R_{\theta_2} = c_M(q_1, \beta_1, q_2, \beta_2) + \frac{\beta_2}{2M_2} c_{\beta_2}(q_1, \beta_1, q_2, \beta_2) \]
\[ = -\frac{3}{8} \gamma \beta_1 \beta_2 M_1 C(\beta_1, q_1, \beta_2, q_2) - \frac{\gamma}{4} \beta_1 \beta_2^2 M_1 C_{\beta_2}(\beta_1, q_1, \beta_2, q_2), \quad (5.33a) \]
\[ R_{\beta_2} = -\frac{1}{M_2} c_{\beta_2}(q_1, \beta_1, q_2, \beta_2) = \frac{\gamma}{4} \beta_1 \beta_2 M_1 C_{\beta_2}(\beta_1, q_1, \beta_2, q_2), \quad (5.33b) \]
\[ R_{\Lambda_2} = \frac{6}{M_2} \frac{3}{2\pi^2} \beta_1 \beta_2^2 M_1 \left[ C(\beta_1, q_1, \beta_2, q_2) + \beta_2 C_{\beta_2}(\beta_1, q_1, \beta_2, q_2) \right], \quad (5.33c) \]

for the sets \( Q_1 \) and \( Q_2 \), respectively.

5.5. Reduction to 8 CCs

A reasonable approximation in the stable regime is to consider that the two components do not separate over their time evolution and that their widths are similar. Thus, we can assume that \( q_1(t) \equiv q_2(t) = q(t), \ p_1(t) \equiv p_2(t) = p(t), \ \beta_1(t) \equiv \beta_2(t) = \beta(t), \) and \( \Lambda_1(t) \equiv \Lambda_2(t) = \Lambda(t) \) in our analysis. Using the formalism of section 4 we can directly obtain the equations of motion for these 8 CCs:

\[ Q^\mu = \{ M_1(t), \beta_1(t), M_2(t), \beta_2(t), q(t), p(t), \beta(t), \Lambda(t) \}, \quad (5.34) \]

from equation (4.12). The results we obtain directly can also be obtained by a reduction process from the 12 CC equations by setting \( q_1 = q_2, \ p_1 = p_2, \ \beta_1 = \beta_2, \) and \( \Lambda_1 = \Lambda_2, \) and then defining the time derivatives as follows:

\[ \dot{q} = \frac{M_1 \dot{q}_1 + M_2 \dot{q}_2}{M_1 + M_2} \bigg|_{q_1 = q_2} \quad (5.35) \]

with similar relations for the average values of \( \dot{p}, \dot{\beta}, \) and \( \dot{\Lambda}. \) The equations of motion for the case of 8 CCs can be thus obtained by following the steps described above, and are presented in appendix A.4. Note that the results of this reduction agree with the direct determination of (A.16) which is a consistency check.

5.6. Small amplitude approximation

The parametric regions of stability can be determined by performing a small amplitude approximation to the full CC equations. This way, we will be able to obtain the eigenfrequencies of the linearized system which is derived by expanding the rate equations to first order in the parameters using the expansions of integrals of appendix A.5. Using the linearized version of equations (5.31)–(5.33) we obtain

\[ \delta M_1 = -\kappa_1 d \frac{\pi}{4} M_{1,0} \delta q_1, \quad (5.36a) \]

\[ \delta \beta_1 = -1 - \frac{5}{12} \gamma (\delta M_1 + \delta M_2) + \kappa_1 \frac{2\pi}{9} \delta p_1 \]
\[ + \left\{ -\frac{\pi^2}{45} \left[ \frac{d^2}{9} - b^2 - \frac{\gamma M_{2,0}}{2} \right] + \frac{1}{2} \left[ \frac{8}{3} + \frac{d^2}{9} - b^2 - \frac{5\gamma M_{1,0}}{6} \right] \right\} \delta \beta_1, \quad (5.36b) \]
Figure 2. Eigenvalues of $W_{\mu\nu}[Q_0]$ as functions of $d$ for the case of $b = 1$, $A_1 = 1$ (left panel) and for $b = 1/2$, $A_1 = 1/4$ (right panel). The solid red and orange lines are the real parts of two of the eigenvalues, and the red dotted lines are the imaginary parts. The vertical lines correspond to values of $d$ used for illustrating a stable (left panel) and unstable (right panel) solution, respectively.

\[
\delta q_1 = 2\delta p_1 - \kappa_1 d \left( \frac{4\pi^3}{9} - \frac{\pi^3}{16} \right) \delta \beta_1, \quad (5.36c)
\]
\[
\delta p_1 = \frac{8}{15} \left( \frac{d^2}{9} - b^2 \right) \delta q_1 - \frac{4}{15} \gamma M_{2,0}(\delta q_1 - \delta q_2) - \kappa_1 d \left( \frac{2\pi}{9} \right) \delta \Lambda_1, \quad (5.36d)
\]
\[
\delta \beta_1 = \kappa_1 d \left( \frac{\pi}{4} - \frac{10}{3\pi} \right) \delta q_1 - 4\delta \Lambda_1, \quad (5.36e)
\]
\[
\delta \Lambda_1 = -\frac{1}{\pi} \gamma (\delta M_1 + \delta M_2) + \frac{\kappa_1 d}{3\pi} \delta p_1
\]
\[
+ \left\{ \frac{4}{\pi^2} + \frac{4}{15} \left[ b^2 - \frac{d^2}{9} \right] \right\} \delta \beta_1 + \frac{2}{15} \gamma M_{2,0}(\delta \beta_1 - \delta \beta_2), \quad (5.36f)
\]

where we have used equation (5.6) as well. The equations for the $Q_2$ variables are obtained from the above by interchanging $1 \leftrightarrow 2$. Notice also that the $\delta$-rate equations [cf equation (5.36)] vanish when all $\delta Q^\mu$ are set to zero. For the $\delta \beta_1$ term we obtain $\delta \beta_1 = -1$, as required. Also, when we set $Q_1 = Q_2$, we obtain the respective equations for the 8-parameter case.

We turn our focus now on equation (5.36) and the ones corresponding to $Q_2$. Those could be written in the following form

\[
\delta \dot{Q}^\nu = M^\nu_{\mu}[Q_0] \delta Q^\nu, \quad (5.37)
\]

from which we find:

\[
\delta \ddot{Q}^\nu + W^\nu_{\mu}[Q_0] \delta Q^\nu = 0, \quad W^\nu_{\mu}[Q_0] = -M^\nu_{\mu}[Q_0] M^\mu_{\nu}[Q_0], \quad (5.38)
\]

where $W^\nu_{\mu}(Q_0)$ is Hermitian with

\[
M^\nu_{\mu}[Q_0] = \begin{pmatrix} A[Q_{1,0}, Q_{2,0}] & B[Q_{1,0}, Q_{2,0}] \\ B[Q_{2,0}, Q_{1,0}] & A[Q_{2,0}, Q_{1,0}] \end{pmatrix}. \quad (5.39)
\]

We can ignore the $\delta \dot{\beta}_1$ equations since the rest of the equations do not couple with them. The $\mu$ and $\nu$ indices then run over the ten values:

\[
\{ \delta M_1, \delta q_1, \delta p_1, \delta \beta_1, \delta \Lambda_1, \delta M_2, \delta q_2, \delta p_2, \delta \beta_2, \delta \Lambda_2 \}. \quad (5.40)
\]
Figure 3. Results of the 12-parameter variational calculation for the stable case corresponding to $A_1 = 1$, $b = 1$, and $d = 1/2$ (in blue) compared with a numerical calculation of the coupled NLSEs (in red). The left column depicts the temporal evolution of $M_1(t)$ (top), $|\psi_1(0,t)|$ (middle), and $q_1$ (bottom), whereas the right column presents the same quantities but for the second component.

The matrices $A$ and $B$ are then $5 \times 5$ matrices given by:

$$A[Q_{1,0}, Q_{2,0}] = \begin{pmatrix}
0 & -\kappa_1 d \pi \gamma M_{1,0}/4 & 0 & 0 & 0 \\
0 & 0 & 2 & -\kappa_1 d c & 0 \\
0 & -2a_2 & 0 & 0 & -\kappa_1 d 2\pi/9 \\
-\gamma/\pi^2 & \kappa_1 d e & 0 & 0 & -4 \\
-\gamma/\pi^2 & 0 & \kappa_1 d/(3\pi) & 4/\pi^2 + a_2 & 0
\end{pmatrix},$$

(5.41)

and

$$B[Q_{1,0}, Q_{2,0}] = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 4\gamma M_{2,0}/15 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\gamma/\pi^2 & 0 & 0 & -2\gamma M_{2,0}/15 & 0
\end{pmatrix},$$

(5.42)
Figure 4. Comparison of the 8-parameter variational calculation (in blue) with the 12-parameter variational calculation results (in green), and the numerical calculation of the coupled NLSEs (in red) for the stable solution is shown in the top and bottom left panels. In particular, the top left and right panels showcase the temporal evolution of the amplitudes of the first and second components, respectively, whereas the bottom left depicts the evolution of $q(t)$. The bottom right panel showcases the average value of $\beta(t)$ from the 12-parameter variational calculation.

where we have set

$$c = \frac{4\pi}{9}, \quad e = \frac{\pi^3}{16}, \quad a_i = \frac{4}{15} \left[ b^2 - \frac{d^2}{9} + \frac{\gamma M_{d0}}{2} \right].$$

Plots of the eigenvalues $\omega^2$ of the matrix $W_\mu^p[Q_0]$ as functions of $d$ for fixed values of $A_1(=A_1(0))$ and $b$ are shown in the panels of figure 2. In particular, there are two cases shown with five eigenvalues, one of which is a zero eigenvalue. It can be discerned from the panels that as $d$ is increased, some of the eigenvalues become complex, indicating the emergence of instability and blow-up of the wave functions. For the left panel of figure 2, and at the intersection of the vertical dotted line at $d = 1/2$, we find the following five doubly degenerate eigenvalues:

$$\omega^2_{12CC} = \{ 3.95008, 2.73053, 1.98677, 1.02733, 0 \},$$

$$\tau_{12CC} = (2\pi/\omega_{12CC}) = \{ 3.16138, 3.80239, 4.45765, 6.19906 \},$$

which indicates a stable solution. On the other hand, the 8-parameter CC approximation gives two doubly degenerate eigenvalues for this case:

$$\omega^2_{8CC} = \{ 2.71099, 1.03419 \},$$

which are very close to two of the eigenvalues found in equation (5.44). For the right panel of figure 2, and at the intersection of the vertical dotted line now at $d = 1.07$, we find the following
Figure 5. Results of the 12-parameter variational calculation for the unstable case corresponding to parameter values of $b = 1/2$, $d = 1.07$, and $A_1 = 1/4$. Red and blue solid lines correspond to the 12-CC approximation and numerical calculation of the coupled NLSEs, respectively.

five doubly degenerate eigenvalues:

$$\omega_{12CC}^2 = \{ 4.48071, 1.52926 + 0.276596i, 1.52926 - 0.276596i, 0.743116, 0 \}.$$  

(5.46a)

$$\tau_{12CC} = \left(\frac{2\pi}{\omega_{12CC}}\right) = \{ 2.96829, 5.02 + 0.450327i, 5.02 - 0.450327i, 7.28872 \}.$$  

(5.46b)

Because of the decoupling of the small oscillation equations in the 8 CC approximation, one can write down an analytic expression for the two oscillation frequencies in terms of $b$ and $d$, and so determine regimes of instability analytically. Indeed, we find

$$\delta\ddot{q} - [A \delta q + B \delta \Lambda] = 0,$$

(5.47)

$$\delta\ddot{\Lambda} - [D \delta q + E \delta \Lambda] = 0,$$

(5.48)

where

$$A = \frac{1024 \left(d^2 - 9b^2\right) + 5 \left(3\pi^2 - 40\right) \left(9\pi^2 - 64\right) d^2 (1 - 2r)^2}{8640},$$

(5.49)

$$B = -\frac{1}{12} \pi \left(3\pi^2 - 16\right) d(2r - 1),$$

(5.50)

$$D = -\frac{d(2r - 1) \left(\pi^2 \left(423b^2 - 47d^2 - 540\right) + 2\pi^4 \left(d^2 - 9b^2\right) + 3600\right)}{270\pi^3},$$

(5.51)
\[ E = \frac{2}{135} \left( -72b^2 + d^2 \left( -20r^2 + 20r + 3 \right) - \frac{1080}{\pi^2} \right). \]  
\[ (5.52) \]

The two eigenfrequencies are
\[ \omega^2_\pm = \frac{1}{2} \left( -(A + E) \pm \sqrt{(A - E)^2 + 4BD} \right). \]
\[ (5.53) \]

Note that \( \omega^2_\pm \) can be written entirely in terms of \( A_1, b, \) and \( d, \) since \( r = M_1/(M_1 + M_2) = 2A_1/(M_1 + M_2), \) and \( (M_1 + M_2) = 2(2 + d^2/9 - b^2) \) with \( M_{1,2} := M_{1,2}(0). \)

6. Dynamical results

In this section, we present our results on the variational approximations. We begin our discussion here by considering the stable regime as this was illustrated in the left panel of figure 2 corresponding to parameter values of \( A_1 = 1, b = 1 \) and \( d = 1/2. \) The initial conditions were chosen to be the values given in equation (5.5) but perturbed with \( \delta_{q_0} = 0.005. \) Our numerical results on the 12-parameter variational calculation are shown in the panels of figure 3 with solid blue lines where we compare them to the respective numerical calculation of the coupled NLSes represented by solid red lines (see, section 7 for details). It can be discerned from the panels of figure 3 that the numerical solutions are reproduced reasonably well. However, the variational calculation seems to predict several additional frequency modes which are not observed in the numerical simulation of the NLSes.

Furthermore, and in the stable regime, we compare the different variational calculations in figure 4. There is very little difference between the 8- and 12-parameter calculations for \( |\psi_1(0,t)|, \) however the results for \( |\psi_2(0,t)| \) differ significantly as a result of additional frequency components in the 12-parameter ansatz. For the \( q(t) \) results, the 8-parameter case only calculates an average value whereas the 12-parameter ansatz produced two different results for \( q_1(t) \) and \( q_2(t) \) as does the numerical calculation. Solutions of the linearized 12-parameter equation (5.36) are indistinguishable from the full 12-parameter calculation. This is because the amplitudes of the variational parameters in this case are quite small.

Finally, results on dynamics for the unstable case for the 12-parameter ansatz corresponding to parameter values of \( A_1 = 1/4, b = 1/2, \) and \( d = 1.07, \) are shown in figure 5. We use the same initial conditions as with the stable case. Both the variational calculation and the numerical simulations of the NLSes predict a blow-up and instability of the soliton, in agreement with the linear analysis. Note that in this case \( q_1(t) \) diverges from \( q_2(t), \) and the 8-parameter approximation would not be adequate.

7. Computational analysis and numerical results of the full NLS system

In this last section, we present our numerical results on the \textit{existence}, \textit{stability}, and \textit{spatio-temporal evolution} of the solutions (3.11) to the coupled NLS system of equation (3.9). The existence of solutions is investigated by introducing the ansatz
\[ \psi_j(x,t) = \psi_j^{(0)}(x) e^{-i\omega t}, \quad \psi_j^{(0)}(x) \in \mathbb{C}, \quad j = 1, 2. \]
\[ (7.1) \]

Upon plugging equation (7.1) into equation (3.9), we obtain the system of steady-state equations
\[ \frac{d^2\psi_1^{(0)}}{dx^2} + \gamma \left[ |\psi_1^{(0)}|^2 + |\psi_2^{(0)}|^2 \right] \psi_1^{(0)} - V(x)\psi_1^{(0)} + \omega_1\psi_1^{(0)} = 0, \]
\[ (7.2a) \]
\[ \frac{d^2 \psi_j^{(0)}}{dx^2} + \gamma \left( |\psi_1^{(0)}|^2 + |\psi_2^{(0)}|^2 \right) \psi_2^{(0)} - V'(x) \psi_2^{(0)} + \omega_2 \psi_2^{(0)} = 0. \]  

(7.2b)

In this work, we solve the boundary-value-problem consisting of equation (7.2) and zero Dirichlet boundary conditions numerically. To that effect, we consider a uniform one-dimensional grid of points on \([-20, 20]\) with lattice spacing \(\Delta x = 0.04\). The second-order derivatives in equation (7.2) are replaced with a second-order accurate, central finite difference approximation. It should be noted that we further corroborated our results on the existence (and stability) of solutions by employing Chebyshev collocation on the unit interval \([-1, 1]\) with \(N = 701\) Chebyshev nodes.

In that case, the affine transformation \(x_k = \frac{x - a}{b - a} \xi_k\) was employed which maps \([a, b]\) into \([-1, 1]\) with \(x_k \in [a, b], \xi_k \in [-1, 1]\), and \(k = 1, 2, \ldots, N\) (here, \(a = -20\) and \(b = 20\)). Regardless of the spatial discretization, we employed Newton’s method to solve the underlying system of coupled nonlinear equations. The initial guess that was fed to the solver was the steady part of the exact solutions of equation (3.11), and thus Newton’s method converged rapidly (typically in two iterations with an error of \(\approx 10^{-12}\) on the residuals). We also used a Newton–Krylov method [31] to validate our findings. Both methods produced exactly the same results and matched perfectly with the exact solutions (up to local truncation error).

Having a steady-state solution \(\psi_j^{(0)}\) \((j = 1, 2)\) at hand, we perform a spectral stability analysis around them. To do so, we introduce the perturbation ansätze

\[ \tilde{\psi}_1(x, t) = e^{-i\omega t} \left[ \psi_1^{(0)} + \varepsilon \left( a(x) e^{\lambda \xi} + b(x) e^{\lambda^* \xi} \right) \right], \quad a(x), b(x) \in \mathbb{C}, \]  

(7.3a)

\[ \tilde{\psi}_2(x, t) = e^{-i\omega t} \left[ \psi_2^{(0)} + \varepsilon \left( c(x) e^{\lambda \xi} + d(x) e^{\lambda^* \xi} \right) \right], \quad c(x), d(x) \in \mathbb{C}, \]  

(7.3b)

where \(\lambda \in \mathbb{C}\) is the eigenvalue and \(\varepsilon \ll 1\) is a small parameter. Then, we insert equation (7.3) into equation (3.9) and obtain, at order \(O(\varepsilon)\), the eigenvalue problem:

\[
\begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
-A_{12} & -A_{11} & -A_{14}^* & -A_{13}^* \\
A_{13} & A_{14} & A_{33} & A_{34} \\
-A_{14} & -A_{13} & -A_{34}^* & -A_{33}^*
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
= \tilde{\lambda}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
\]

(7.4)

with eigenvalues \(\tilde{\lambda} = -i \lambda\), eigenvectors \(V = [a \ b \ c \ d]^T\), and matrix elements given by

\[
A_{11} = \frac{d^2}{dx^2} + \gamma \left( |\psi_1^{(0)}|^2 + |\psi_2^{(0)}|^2 \right) - V(x) + \omega_1,
\]  

(7.5a)

\[
A_{12} = \gamma \left( \psi_1^{(0)} \right)^2,
\]  

(7.5b)

\[
A_{13} = \gamma \psi_1^{(0)} \left( \psi_2^{(0)} \right)^*,
\]  

(7.5c)

\[
A_{14} = \gamma \psi_1^{(0)} \psi_2^{(0)},
\]  

(7.5d)

\[
A_{33} = \frac{d^2}{dx^2} + \gamma \left( |\psi_1^{(0)}|^2 + 2|\psi_2^{(0)}|^2 \right) - V'(x) + \omega_2,
\]  

(7.5e)

\[
A_{34} = \gamma \left( \psi_2^{(0)} \right)^2.
\]  

(7.5f)

Then, a solution is deemed stable if the eigenvalues \(\lambda = \lambda_0 + i\lambda_r\) have a nonvanishing (negative) real part, i.e., \(\lambda_r < 0\). On the other hand, if \(\lambda_r > 0\), this would indicate the presence of
an unstable mode. We compute the eigenvalues of the linearization (sparse) matrix $A$ associated with equation (7.4) in MATLAB. The spectra we obtained were further corroborated by the highly accurate FEAST eigenvalue solver [32] (and references therein) which considers contour integration and involves density-matrix representation techniques from quantum mechanics. In the eigenvalue computations using FEAST, an elliptical contour was chosen in such a way that $\approx 150$ eigenvalues were computed. FEAST converged rapidly (within two iterations in most of the cases considered in this work) with relative tolerance $10^{-10}$ on the residuals of eigenvectors, and the spectra obtained via `eig` and FEAST are identical by using both spatial discretizations as well.

For the numerical computations presented below, we use the parameter fixing mentioned in the previous section together with $\gamma = 1$ and $A_1 = 1$. We will focus on different cases in the parameter $b$, and in particular on values of $b = 0.2, 0.4, 0.6, 0.8$ and $b = 1$ while $d$ is treated as a bifurcation parameter. Then, we will employ a sequential continuation over $d$ (with $\Delta d = 0.01$).
Figure 7. Spatio-temporal evolution of the densities $|\psi_1(x, t)|^2$ and $|\psi_2(x, t)|^2$ and associated spectra are shown in the left, middle and right columns, respectively. In particular, the panels (a)–(c) correspond to the cases with $b = 0.2$ and (a) $d = 0.02$, (b) $d = 0.1$, and (c) $d = 0.45$, respectively. The panels (d) and (e) present results for the cases with $b = 0.4$ and (d) $d = 0.05$ and (e) $d = 0.25$, respectively. The insets shown in the left and middle panels (a) and (d) correspond to $|\psi_1(x = 0, t)|^2$ (left panel) and $|\psi_2(x = 0, t)|^2$ (middle panel) as functions of time $t$. 
Figure 8. Same as figure 7 but for (a) and (b) $b = 0.6$, (c) and (d) $b = 0.8$ and (e) $b = 1.0$. The results in (a) and (b) correspond to values of $d$ of (a) $d = 0.1$ and (b) $d = 0.5$ whereas those in (c) and (d) correspond to (c) $d = 0.2$ and (d) $d = 0.4$, respectively. Panel (e) corresponds to a value of $b = 1.0$ with $d = 0.4$. 
as our continuation step) by using as an initial guess the solution previously found for the new value of $d$.

Figure 6 presents our results on the stability of the steady-state solutions found via Newton’s method as functions of the parameter $d$ and for different values of $b = 0.2$, $b = 0.4$, $b = 0.6$, $b = 0.8$, and $b = 1.0$. Those panels correspond to results obtained from the FEAST algorithm and the range in $d$ considered therein is $[0,0.8]$. In particular, the top panel in this clustered figure showcases the growth rates of the most unstable mode, i.e., $\max(\lambda_i)$ as functions of $d$ and for different values in $b$ (see, the legend therein). It can be discerned from this panel that there exist parameter intervals of stability of the pertinent solutions. Indeed, the branches with $b = 0.2$, $b = 0.4$, $b = 0.6$, $b = 0.8$ and $b = 1.0$ are stable in parameter intervals of $d$ of $\approx [0,0.02]$, $[0,0.07]$, $[0,0.16]$, $[0,0.3]$, and $[0,0.67]$, respectively. A striking feature about these results is that the parameter interval in $d$ in which solutions are (spectrally) stable becomes wider as soon as $b$ becomes larger, i.e., the coefficient appearing in $V_0(x)$. This suggests that one can controllably form a family of stable solitonic modes in the coupled NLS system by increasing the parameter $b$ which allows the existence of such (stable) solutions over a wide range in $d$. We summarize our presentation on the stability analysis results with the panels (a)–(e) corresponding to the full spectrum of the solutions showcasing the imaginary $\lambda_i$ (left) and real $\lambda_r$ (right) parts of the eigenvalues for the cases with $b = 0.2, b = 0.4, b = 0.6, b = 0.8$, and $b = 1.0$, respectively. It should be noted that the instabilities we observe correspond to imaginary eigenvalues which bifurcate off the imaginary axis, thus resulting in oscillatorily unstable solutions characterized by a complex eigenvalue quartet. However, for the branch of $b = 0.2$, a purely imaginary pair of eigenvalues passes through the origin at $d \approx 0.42$ creating a purely real unstable mode which becomes dominant past a value of $d \approx 0.43$. This suggests the emergence of a bifurcating branch out of this collision. Although this situation (which is also apparent for the other cases in $b$ we considered in this work but for larger values of $d$ than 0.8) is quite interesting from the dynamical systems point of view, we are not pursuing bifurcations in this work.

We now turn our focus into our results on the dynamical evolution of steady-state solutions we obtained for different values of $b$ and $d$. In particular, our findings are summarized in figures 7 and 8 showcasing the evolution of the densities $|\psi_1(x,t)|^2$ and $|\psi_2(x,t)|^2$ as well as the associated spectral plane $(\lambda_r, \lambda_i)$ of the steady-state solution identified in the left, middle and right columns, respectively. We advance the coupled NLS system [cf equation (3.9)] forward in time by employing a standard four-stage Runge–Kutta (RK4) method with fixed time-step size $\Delta t = 10^{-4}$. As per the stable (according to our linear stability analysis) steady-state solutions, we add a random noise of small amplitude ($\epsilon \sim 10^{-3}$ in these cases) on top of the localized region of the pertinent steady-state profile and we use it as an initial condition in RK4. We then consider time intervals of integration of $[0,10000]$ (or $[0,7000]$). In particular, the results on the dynamical evolution presented in figure 7(a) (for $b = 0.2$ and $d = 0.02$), figure 7(d) (for $b = 0.4$ and $d = 0.05$), figure 8(a) (for $b = 0.6$ and $d = 0.1$), figure 8(c) (for $b = 0.8$ and $d = 0.2$), and figure 8(e) (for $b = 1.0$ and $d = 0.4$) correspond to stable solutions as it can be discerned from the spatio-temporal evolution of the respective densities over a very large time interval (see, the associated spectra). In addition, those panels offer the temporal distribution of the densities $|\psi_1(x=0,t)|^2$ and $|\psi_2(x=0,t)|^2$ suggesting the robustness of the (perturbed) solutions and validating our linear stability analysis results.

On the other hand, and as for the unstable steady-state solutions, we initialize the dynamics by perturbing the solutions along the most unstable eigendirection (with $\epsilon \sim 10^{-3}$ or $\epsilon \sim 10^{-2}$ depending upon the magnitude of $\lambda_i$). The results in figures 7(b), (c) and (e) as well as figures 8(b) and (d) correspond to unstable steady-state solutions, and the instability is manifested in the dynamics as this is evident in the spatio-temporal evolution of the densities shown.
in those panels. For example, figure 7(b) corresponds to $b = 0.2$ and $d = 0.1$ where the solutions are classified as oscillatorily unstable. In the left and middle panels of figure 7(b), the solution starts performing oscillations (due to the oscillatory instability) until each component forms a narrower (in its width) bright pulse that propagates to the left (for the first component) and right (for the second component) and hits the boundaries (results are not shown past that time). It should be noted that a similar phenomenology is observed in figure 7(e) ($b = 0.4$ and $d = 0.25$) as well as in figure 8(b) ($b = 0.6$ and $d = 0.5$) and figure 8(d) ($b = 0.8$ and $d = 0.4$). Finally, the results shown in figure 7(c) (with $b = 0.2$ and $d = 0.45$) correspond to an example case scenario where the dominant unstable mode is characterized by a pair of purely real eigenvalues (on top of oscillatory unstable ones). It can be discerned from that figure that the density of the first component progressively becomes smaller at $t \approx 200$ whereas the second component (see the middle panel) develops a bright pulse of higher amplitude. However, and past that time, both components start performing oscillations of gradually increasing amplitude until they are amplified substantially at $t \approx 450$ resulting in the breakdown of the pertinent waveforms.

8. Conclusions

In this paper we have found exact solutions to the problem of two coupled NLSEs [1] in the presence of a complex confining potential which has $PT$ symmetry and is derivable from a superpotential $W(x) = r \sigma_0 \tanh(x) + i s \sigma_3 \text{sech}(x)$. Such systems have started to be investigated experimentally in optical lattice environments [2]. Using numerical methods we have mapped out the regimes of stability as well as studied the behavior of these solutions when they are subjected to small perturbations. We compared the numerical solutions in the latter case with a variational approximation based on introducing 8 or 12 time-dependent CCs which are related to various low-order moments of the NLSEs.

The CC approach allowed us to determine analytically approximate small oscillation frequencies. We compared the results of the CC approach with the numerical simulations in two cases; one where the solutions are stable and one where the solutions are unstable. The 8 CC approximation assumed that the average position and width of the two components of the NLSEs followed the same trajectory in time, whereas the 12 CC approximation allowed for these variables and their canonical conjugates to be different. Both CC approaches quantitatively agreed with the numerically determined time evolution of the wave function in the first case, but only qualitatively agreed with the numerical solution in the unstable regime. In the unstable case regime, the average position of the two components as well as the average width of the two components diverged from each other, so that only the 12-CC approximation was able to track the behavior of the time evolution qualitatively.

Then, we turned our focus to the coupled NLSEs and systematically studied the existence, stability and dynamical evolution of solitary waves. Upon identifying branches of steady-state solutions via fixed-point methods, a bifurcation analysis was carried out over a two-parameter space where parametric intervals of stability were identified. Our spectral stability analysis results suggest that we can controllably form a wide range in the parameter $d$ by increasing the value of the parameter $b$ whereupon stable solitary waves can be supported. This corresponds to the case where the real part $V_0$ of the potential $V(x)$ becomes larger. Finally, the stability results we report in this work were tested against direct numerical simulations where typical scenarios of blow-up were involved for the unstable soliton solutions. The results and methods employed in this work could be naturally applied and extended to other coupled, multi-component NLSEs in order to explore the underlying configuration space of solutions. Such efforts are currently under consideration and will be reported in future publications.
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Appendix A. Useful integrals and definitions

We note that
\[ \frac{d}{dz} \text{sech}(z) = -\text{sech}(z) \tanh(z), \] (A.1a)
\[ \frac{d}{dz} \tanh(z) = \text{sech}^2(z). \] (A.1b)

Also,
\[ \phi(x) = \alpha \tan^{-1} \left[ \tanh(x/2) \right], \] (A.2a)
\[ \frac{d\phi(x)}{dx} = \frac{\alpha}{2} \text{sech}(x). \] (A.2b)

Some useful integrals are the following:
\[ \int dz \, \text{sech}^2(z) = 2, \] (A.3a)
\[ \int dz \, \text{sech}^3(z) = \frac{\pi}{2}, \] (A.3b)
\[ \int dz \, \text{sech}^4(z) = \frac{4}{3}, \] (A.3c)
\[ \int dz \, z^2 \, \text{sech}^2(z) = \frac{\pi^2}{6}, \] (A.3d)
\[ \int dz \, \text{sech}^2(z) \tanh^2(z) = \frac{2}{3}, \] (A.3e)
\[ \int dz \, \text{sech}^4(z) \tanh(z) = \frac{1}{3}. \] (A.3f)

We define:
\[ I_1(\beta, q) := \int dy \, \text{sech}^2(\beta y) \text{sech}(y + q), \] (A.4a)
\[ I_2(\beta, q) := \int dy \, y \, \text{sech}^2(\beta y) \text{sech}(y + q), \] (A.4b)
\[ I_3(\beta, q) := \int dy \, \text{sech}^2(\beta y) \text{sech}^2(y + q). \] (A.4c)
Also, we define:

\[ f_1(\beta, q) := \int \text{sech}^2(\beta y) \text{sech}(y + q) \tanh(y + q), \quad (A.5a) \]

\[ f_2(\beta, q) := \int y \text{sech}^2(\beta y) \text{sech}(y + q) \tanh(y + q), \quad (A.5b) \]

\[ f_3(\beta, q) := \int y^2 \text{sech}^2(\beta y) \text{sech}(y + q) \tanh(y + q). \quad (A.5c) \]

Partial derivatives are given by:

\[ I_1(\beta, q) = -\int \text{sech}^2(\beta y) \text{sech}(y + q) \tanh(y + q) = -f_1(\beta, q), \quad (A.6a) \]

\[ I_2(\beta, q) = -2\int y \text{sech}^2(\beta y) \tanh(y) \text{sech}(y + q) = -2f_1(\beta, q), \quad (A.6b) \]

\[ I_3(\beta, q) = -2\int y^2 \text{sech}^2(\beta y) \tanh(y + q) = -2f_1(\beta, q), \quad (A.6c) \]

\[ I_4(\beta, q) = -2\int y \text{sech}^2(\beta y) \tanh(y) \text{sech}(y + q) = -2f_1(\beta, q), \quad (A.6d) \]

\[ I_5(\beta, q) = -2\int y^2 \text{sech}^2(\beta y) \tanh(y) \text{sech}(y + q) = -2f_1(\beta, q). \quad (A.6f) \]

### A.1. Derivative of the cross term

The cross term is

\[ c(Q_1, Q_2) = -\frac{\gamma}{4} \beta_1 M_1 \beta_2 M_2 C(\beta_1, q_1, \beta_2, q_2), \quad (A.7) \]

which involves the mixing integral (see, appendix A.2)

\[ C(\beta_1, q_1, \beta_2, q_2) = \int \text{sech}^2[\beta_1(x - q_1)] \text{sech}^2[\beta_2(x - q_2)]. \quad (A.8) \]

For the derivatives of the cross term we explicitly have:

\[ c_{M_1}(Q_1, Q_2) \equiv \partial_{M_1} c(Q_1, Q_2) = -\frac{\gamma}{4} \beta_1 \beta_2 M_2 C(\beta_1, q_1, \beta_2, q_2), \quad (A.9a) \]

\[ c_{M_2}(Q_1, Q_2) \equiv \partial_{M_2} c(Q_1, Q_2) = -\frac{\gamma}{4} \beta_1 \beta_2 M_1 C(\beta_1, q_1, \beta_2, q_2), \quad (A.9b) \]

\[ c_{q_1}(Q_1, Q_2) \equiv \partial_{q_1} c(Q_1, Q_2) = -\frac{\gamma}{4} \beta_1 M_1 \beta_2 M_2 C(q_1, \beta_1, \beta_2, q_2), \quad (A.9c) \]

\[ c_{q_2}(Q_1, Q_2) \equiv \partial_{q_2} c(Q_1, Q_2) = -\frac{\gamma}{4} \beta_1 M_1 \beta_2 M_2 C(q_2, \beta_1, q_1, \beta_2, q_2), \quad (A.9d) \]

\[ c_{\beta_1}(Q_1, Q_2) \equiv \partial_{\beta_1} c(Q_1, Q_2) \]

\[ = -\frac{\gamma}{4} \beta_2 M_1 M_2 \left[ C(\beta_1, q_1, \beta_2, q_2) + \beta_1 C_{\beta_1}(\beta_1, q_1, \beta_2, q_2) \right], \quad (A.9e) \]
\[ c_{\beta_2}(Q_1, Q_2) \equiv \partial_{\beta_2} c(Q_1, Q_2) = -\frac{\gamma}{4} \beta_1 M_1 M_2 \left[ C(\beta_1, q_1, \beta_2, q_2) + \beta_2 \beta_2 (\beta_1, q_1, \beta_2, q_2) \right]. \quad (A.9f) \]

### A.2. Mixing integrals

The mixing integral is defined by
\[
C(\beta_1, q_1, \beta_2, q_2) := \int dx \ \sech^2[\beta_1(x - q_1)] \sech^2[\beta_2(x - q_2)]. \quad (A.10)
\]

Derivatives of this integral are given by
\[
C_{q_1}(\beta_1, q_1, \beta_2, q_2) = 2\beta_1 \int dx \ \sech^2[\beta_1(x - q_1)] \tanh[\beta_1(x - q_1)] \sech^2[\beta_2(x - q_2)], \quad (A.11)
\]
\[
C_{q_2}(\beta_1, q_1, \beta_2, q_2) = 2\beta_1 \int dx \ \sech^2[\beta_1(x - q_1)] \sech^2[\beta_2(x - q_2)] \tanh[\beta_2(x - q_2)], \quad (A.12)
\]
\[
C_{\beta_1}(\beta_1, q_1, \beta_2, q_2) = -2 \int dx (x - q_1) \sech^2[\beta_1(x - q_1)] \tanh[\beta_1(x - q_1)] \sech^2[\beta_2(x - q_2)], \quad (A.13)
\]
\[
C_{\beta_2}(\beta_1, q_1, \beta_2, q_2) = -2 \int dx (x - q_2) \sech^2[\beta_1(x - q_1)] \sech^2[\beta_2(x - q_2)] \tanh[\beta_2(x - q_2)]. \quad (A.14)
\]

### A.3. Explicit form of \( u_\mu \)

\[
u_M = p^2 + \frac{1}{3} \beta^2 + \frac{\pi^2}{3} \Lambda^2 + \beta \frac{d^2 - 9 b^2}{18} I_3(\beta, q) + \kappa \frac{d}{3} \left[ p I_1(\beta, q) + 2 \Lambda I_2(\beta, q) \right] - \frac{\gamma}{3} \beta M + c_\theta(Q_1, Q_2), \quad (A.15a)
\]

\[
u_\theta = \kappa \beta M d f_1(\beta, q), \quad (A.15b)
\]

\[
u_q = M \left\{ \beta \frac{d^2 - 9 b^2}{18} I_3(\beta, q) + \kappa \frac{2}{3} d \left[ p f_1(\beta, q) + 2 \Lambda f_2(\beta, q) \right] \right\} + c_\gamma(Q_1, Q_2). \quad (A.15c)
\]

\[
u_p = M \left\{ 2 p + \kappa \frac{d}{3} I_1(\beta, q) - \kappa \beta d f_2(\beta, q) \right\}, \quad (A.15d)
\]

\[
u_\beta = M \left\{ \frac{2}{3} \beta - \frac{2 \pi^2}{3} \frac{\Lambda^2}{\beta^3} + \frac{d^2 - 9 b^2}{18} \left[ I_3(\beta, q) + \beta I_3(\beta, q) \right] \right\}. \quad (A.15d)
\]
The equations of motion for the case with 8 CCs (see section 5.5) are given by

\[ J. Phys. A: Math. Theor. 53 (2020) 455702 \]

\[ M = \dot{\Lambda} = \frac{2}{3} \beta^2 + \frac{2}{3} \beta^2 \int I_1(\beta, q) + \beta I_{1,\beta}(\beta, q)] + 2 \Lambda [I_2(\beta, q) + \beta I_{2,\beta}(\beta, q)] \right) - \frac{\gamma}{6} M^2 \]

\[ u_A = M \left\{ 2 \frac{\pi^2}{3} \frac{\Lambda}{\beta^2} + \frac{2}{3} \beta \int I_1(\beta, q) - \frac{\gamma}{6} M^2 \right\}. \]

The derivatives of the cross term \( \epsilon_c(Q_1, Q_2) \) are given by (A.9).

A.4. Equations of motion for 8 CCs

The equations of motion for the case with 8 CCs (see section 5.5) are given by

\[ \dot{M}_1 = -d \beta M_1 f_1(\beta, q), \quad \frac{\dot{\theta}_1}{\dot{\theta}_2} = -p^2 + \frac{2}{3} \beta^2 + \beta \frac{d^2}{36} [3 I_3(\beta, q) + \beta I_{3,\beta}(\beta, q)] + d \beta f_2(\beta, q) \]

\[ \ddot{\theta}_2 = -p^2 + \frac{2}{3} \beta^2 + \beta \frac{d^2}{36} [3 I_3(\beta, q) + \beta I_{3,\beta}(\beta, q)] - d \beta f_2(\beta, q) \]

\[ \dot{q} = 2 p + \frac{M_1 - M_2}{M_1 + M_2} \frac{\dot{\theta}_2}{M_2} \left\{ I_1(\beta, q) - 3 f_2(\beta, q) \right\}, \]

\[ \dot{p} = -\beta \frac{d^2}{8} \frac{9 b^2}{I_3(\beta, q) + \frac{M_1 - M_2}{M_1 + M_2} \frac{d}{3} \beta \int f_1(\beta, q) - 4 \Lambda f_2(\beta, q) \right\}, \]

\[ \dot{\Lambda} = 4 \frac{\beta^2}{\pi^2} - 4 \Lambda^2 + \frac{\beta^3}{3 \pi^2} \int I_1(\beta, q) + \beta I_{1,\beta}(\beta, q)] - \frac{\gamma}{\pi^2} (M_1 + M_2) \]

\[ + \frac{M_1 - M_2}{M_1 + M_2} \frac{2 \beta^3}{\pi^2} \left\{ I_1(\beta, q) + \beta I_{1,\beta}(\beta, q)] + 2 \Lambda [I_2(\beta, q) + \beta I_{2,\beta}(\beta, q)] \right\}. \]
A.5. Expansion of the integrals

To first order:

\[ f_1(1 + \delta \beta, \delta q) = \frac{\pi}{4} \delta q, \quad (A.17a) \]
\[ f_2(1 + \delta \beta, \delta q) = \frac{\pi}{6} + \left( \frac{\pi}{3} - \frac{\pi^3}{16} \right) \delta \beta, \quad (A.17b) \]
\[ f_3(1 + \delta \beta, \delta q) = -\left( \frac{2\pi}{3} - \frac{\pi^3}{16} \right) \delta q, \quad (A.17c) \]
\[ I_1(1 + \delta \beta, \delta q) = \frac{\pi}{2} - \frac{\pi}{3} \delta \beta, \quad (A.17d) \]
\[ I_2(1 + \delta \beta, \delta q) = -\frac{\pi}{6} \delta q, \quad (A.17e) \]
\[ I_3(1 + \delta \beta, \delta q) = -\frac{4}{3} - \frac{2}{3} \delta \beta, \quad (A.17f) \]
\[ I_{2,q}(1 + \delta \beta, \delta q) = -\frac{\pi}{6} - \left( \frac{\pi}{3} - \frac{\pi^3}{16} \right) \delta \beta, \quad (A.17g) \]
\[ I_{3,q}(1 + \delta \beta, \delta q) = -\frac{16}{15} \delta q, \quad (A.17h) \]
\[ I_{1,\beta}(1 + \delta \beta, \delta q) = -\frac{\pi}{3} + \left( \pi - \frac{\pi^3}{16} \right) \delta \beta, \quad (A.17i) \]
\[ I_{2,\beta}(1 + \delta \beta, \delta q) = -\left( \frac{\pi}{3} - \frac{\pi^3}{16} \right) \delta q, \quad (A.17j) \]
\[ I_{3,\beta}(1 + \delta \beta, \delta q) = -\frac{2}{3} + \left( \frac{4}{3} - \frac{4\pi^2}{45} \right) \delta \beta. \quad (A.17k) \]

For the mixing integrals, we find

\[ C(1 + \delta \beta_1, \delta q_1, 1 + \delta \beta_2, \delta q_2) = \frac{4}{3} - \frac{2}{3} (\delta \beta_1 + \delta \beta_2), \quad (A.18a) \]
\[ C_{q_1}(1 + \delta \beta_1, \delta q_1, 1 + \delta \beta_2, \delta q_2) = \frac{16}{15} (\delta q_1 - \delta q_2), \quad (A.18b) \]
\[ C_{q_2}(1 + \delta \beta_1, \delta q_1, 1 + \delta \beta_2, \delta q_2) = \frac{16}{15} (\delta q_2 - \delta q_1), \quad (A.18c) \]
\[ C_{\beta_1}(1 + \delta \beta_1, \delta q_1, 1 + \delta \beta_2, \delta q_2) = \frac{2}{3} + \left( \frac{4}{3} - \frac{4\pi^2}{45} \right) \delta \beta_1 + \frac{4\pi^2}{45} \delta \beta_2, \quad (A.18d) \]
\[ C_{\beta_2}(1 + \delta \beta_1, \delta q_1, 1 + \delta \beta_2, \delta q_2) = \frac{2}{3} + \left( \frac{4}{3} - \frac{4\pi^2}{45} \right) \delta \beta_2 + \frac{4\pi^2}{45} \delta \beta_1. \quad (A.18e) \]

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