Finite N corrections to the second order cumulant of pressure via the Optimized Perturbation Theory on the Nambu–Jona-Lasinio model

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Abstract.
We show preliminary results for the second order cumulant of pressure at zero chemical potential, using the Nambu-Jona-Lasinio (NJL) model beyond traditional large-$N_c$ (LN) or mean-field approximation (MFA). The main physical feature of this quantity lies on the determination of the critical point (CP) of the QCD phase diagram on the $T-\mu$ plane. Our calculations are done with the Optimized Perturbation Theory (OPT), which provides finite N corrections even in the first non-trivial order of perturbation. Also, the OPT generates dynamically the repulsive vector contribution, whose imposition in MFA has shown some weaknesses. Nevertheless, the results from both, once compared with those obtained by lattice QCD, suggest the need of a reformulation of the implementations near and above the critical temperature.

1. Introduction
Effective theories used to describe compressed strongly interacting matter should, at first, include vector channels [1, 2]. In NJL model, this can be easily implemented by adding a term such as $-G_V (\bar{\psi}\gamma^\mu\psi)^2$ to the original lagrangian density. However, $G_V$ should be fixed using the $\rho$ meson mass which, in general, is higher than the maximum energy scale set by $\Lambda$. At present, the vector term coupling $G_V$ cannot be determined from experiments and lattice QCD simulations and, theoretically, there is absolutely no consensus about its fixation.

On the experiments with ultrarelativistic heavy ion collisions at SPS, RHIC and LHC energies, there are some quantities that contribute to probe the phase structure of strongly interacting matter and, in particular, to identify the deconfinement and chiral restoration transitions [3]. One of them is the second order quark number susceptibility at zero chemical potential, which theoretical evaluation depends on the determination of the second order cumulant of pressure. Very recently, Schramm and Steinheimer [4] have used the Polyakov-loop extended Nambu-Jona-Lasinio (PNJL) model, obtaining unsatisfactory results for these cumulants above the critical temperature. The main conclusion is that there should be a strong vector repulsion in the hadronic phase and near-zero repulsion in the deconfined phase.

However, due to the Fierz identities, when going beyond MFA level one may induce quantum (loop) corrections which mimic the physical effects caused by a classical (tree) term such as
This is precisely what has been observed in an application of the nonperturbative OPT method to the two flavor NJL model with vanishing $G_V$ [5]. The OPT results for the NJL phase diagram show that $1/N_c$ corrections induced by this approximation reproduce the same qualitative features obtained by considering the model at large-$N_c$ with an explicit repulsive vector channel. The reason is that the OPT two loop contributions add a term like $-G_S(\bar{\psi}\gamma^\mu\psi)^2/(N_cN_f)$ to the pressure. In order to compare OPT results with those from large-$N$ approach with $G_V = 0$ and $G_V \neq 0$, we evaluated these cumulants and concluded that the problem at high temperatures persist. Then we point out an analytic feature which both implementations have in common as a responsible of this behavior.

2. The Nambu–Jona-Lasinio Model and its effective potential

Here we discuss the standard two flavor NJL model for quark matter using two different non perturbative methods: the large-$N_c$ approximation and the Optimized Perturbation Theory. The respective lagrangian density can be written as [6]

$$L_{NJL} = \bar{\psi}\left(i\partial - m_0\right)\psi + G_S\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2\right].$$

where $\psi$ (sum over flavor and colour degrees of freedom is implicit) represents a isodoublet in flavor $(u,d)$ and a $N_c$-plet quark field ($N_c$ is the number of colors), $\tau$ are the Pauli matrices and $G_S$ represents the coupling strength. The quark mass matrix is $m_0 = \text{diag}(m_u,m_d)$, and we work in the isospin-symmetric limit $m_u = m_d = m_0$.

In order to investigate how symmetry breaking occurs in a particular theory, one can analyze its Landau’s free energy, or the effective potential. To calculate it, one can use the path integral formalism. Within this approach the partition function, $Z$, can be written in terms of the effective potential as follows (for details see Refs. [7, 8]):

$$Z = \exp\left[-i \int d^4x F\right].$$

Then, as within statistical mechanics, all the relevant thermodynamical quantities, such as the pressure, can easily be obtained once the free energy (or the partition function) is known. However, the complete evaluation of $F$ is an impossible task within most interacting theories so that one needs to invoke an approximation scheme, such as the OPT or the LN approximation.

Before solving Eq. (2), for the NJL model case it is convenient to write down the Lagrangian density in terms of the auxiliary fields, $\sigma$ and $\pi$, by using the Hubbard-Stratanovich transformation, which consists in adding the following quadratic terms to the Lagrangian density:

$$\Theta^2 = -\frac{1}{G_S}\left[\frac{\sigma}{2} + G_S(\bar{\psi}\psi)\right]^2, \quad \Omega^2 = -\frac{1}{G_S}\left[\frac{\pi}{2} + G_S(\bar{\psi}i\gamma_5\tau\psi)\right]^2.$$ 

Then, the original theory can be written in a bosonized fashion as

$$L_{NJL} = \bar{\psi}\left(i\partial - m_0\right)\psi - \frac{1}{4G_S}\left(\sigma^2 + \pi^2\right) - \bar{\psi}\left[\sigma + i\gamma_5\pi \cdot \tau\right]\psi.$$ 

2.1. The interpolated model

To implement the OPT approximation (also known as linear delta expansion, LDE, see [9, 10] for earlier references) in the NJL model we follow Ref. [11].

Basically, this method consists in modifying the Lagrangian of a particular theory by introducing a dummy expansion parameter, $\delta$. 

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Let \( L \) be the original Lagrangian density we want to solve, and \( L_0 \) a Lagrangian density of a free theory that we know how to solve exactly. Then, the Lagrangian density in terms of \( \delta \) can be written as

\[
L(\delta) = (1 - \delta) L_0 + \delta L = L_0 + \delta (L - L_0).
\]  

As one can easily see, \( L(\delta) \) interpolates between the original theory (\( \delta = 1 \)) and a free (exactly solvable) theory (\( \delta = 0 \)). Then, the relevant physical quantities are evaluated as a power series in \( \delta \), which is formally treated as a small number.

It is crucial to note that for dimensional balance, the Lagrangian density \( L_0 \) must have at least one arbitrary mass parameter (\( \eta \)). This is a welcome fact since \( \eta \) will act as an infrared regulator if the original theory is massless. One can then fix the arbitrary \( \eta \) by requiring that any physical quantity \( P(\eta) \), be at least locally \( \eta \)-independent. This optimization criterion translates into the following variational condition

\[
\frac{\partial P(\eta)}{\partial \eta} \bigg|_{\bar{\eta}} = 0,
\]

which is known as the principle of minimal sensitivity (PMS) [12]. The solution of the PMS equation gives \( \bar{\eta} \) as a function of the original parameters of the theory. The convergence of the OPT in the case of critical theories is given in Ref. [13].

The OPT has already established itself as a powerful method in dealing with critical theories. For example, in the Bose-Einstein condensation case this method and its different variations have provided some of the most precise analytical results for the shift in the critical temperature for weakly interacting homogeneous Bose gases [14, 15]. Other applications to condensed matter situations include a precise evaluation of the critical density for polyacetylene [16]. Also, when extended by hard-thermal loops, the method was successful in predicting QCD thermodynamical properties at the three-loop level [17]. Improved by the renormalization group (RG), and inspired by similar properties [18] in the Gross-Neveu model, a variation of the OPT has been recently used in the evaluation of \( \Lambda_{\text{QCD}}^{\overline{\text{MS}}} \) [19] and \( \alpha_S \) [20], where the stability and convergence at higher orders of this RG-OPT form was demonstrated.

Now, in order to apply this method to the NJL Lagrangian density, Eq. (4), one can define the quadratic term, \( L_0 \), as

\[
L_0 = \bar{\psi} (i\partial - m_0 - \eta) \psi.
\]

Then, following the OPT interpolation prescription, Eq. (5), one gets

\[
L_{\text{NJL}}(\delta) = \bar{\psi} (i\partial - m_0 - \eta) \psi + \delta \left[ \eta \bar{\psi} \psi - \frac{1}{4G_S} (\sigma^2 + \pi^2) - (\sigma \bar{\psi} \psi + \bar{\psi} i\gamma_5 \pi \cdot \tau \psi) \right].
\]  

It is easy to see that Eq. (7) can be written as

\[
L_{\text{NJL}}(\delta) = \bar{\psi} \left[ i\partial - m_0 - \delta (\sigma + i\gamma_5 \pi \cdot \tau) - \eta (1 - \delta) \right] \psi - \delta \frac{N_c}{2\lambda} (\sigma^2 + \pi^2),
\]

where we have done \( G_S \to \lambda/(2N_c) \). To order \( \delta \), the effective potential is given by the Feynman diagrams shown in Fig. 2.1, where the mass term appearing in the OPT dressed propagators is defined by [28]

\[
\hat{\eta} = \eta + m_0 - \delta [\eta - (\sigma + i\gamma_5 \pi \cdot \tau)].
\]
Figure 1. Diagrams contributing to $F(\hat{\eta})$ at order $\delta$. The thick continuous lines represent the OPT dressed fermionic propagators, the dashed line represents the $\sigma$ propagator and the dashed-dotted line represents the $\pi$ propagator. (Figure taken from Ref. [28]).

so that all the $\eta$, $\sigma$ and $\pi$ insertions are automatically accounted for.

Applying the Feynman rules one then obtains [28]

$$
F = \frac{N_c}{2\lambda} (\sigma^2 + \pi^2) + i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \ln \left( \frac{p - \hat{\eta}}{\hat{\eta}} \right)
+ \frac{\delta}{2N_c} \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} \text{Tr} \left[ \left( \frac{p + \hat{\eta}}{p^2 - \hat{\eta}^2} \right) \left( \frac{q + \hat{\eta}}{q^2 - \hat{\eta}^2} \right) \right]
- \frac{\delta}{2N_c} \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} \text{Tr} \left[ \left( \frac{p + \hat{\eta}}{p^2 - \hat{\eta}^2} \right) \left( \frac{q + \hat{\eta}}{q^2 - \hat{\eta}^2} \right) \right].
$$

(10)

It is clear that (after taking the trace over colors) the first two terms are of order $N_c$ while the last two, which are of order $N_0^c$, would not contribute to a LN type of calculation. This important remark emphasizes the fact that already at the first non-trivial order the OPT considers terms which belong to the NLO (next to the leading order) order in a $1/N_c$ type of expansion. Here, the first $1/N_c$ corrections due to the OPT type of expansion have the topology of exchange (Fock) type of diagrams as Fig. 2.1 explicitly shows.

The mass term $\hat{\eta}$ is dressed by the scalar field $\sigma$ and the pseudoscalar field $\pi$ and, in order to maintain the Goldstone structure of the theory [10, 21], it must be defined as [10]

$$
\eta = \alpha + i\gamma_5 \nu \cdot \tau,
$$

(11)

where $\nu$ is a three-vector.

Now the PMS conditions to be satistfied at $\delta = 1$ are

$$
\frac{\partial F}{\partial \alpha} \bigg|_{\bar{\alpha}, \bar{\nu}_i} = 0 \quad \text{and} \quad \frac{\partial F}{\partial \nu_i} \bigg|_{\bar{\alpha}, \bar{\nu}_i} = 0.
$$

(12)

One can use the identity $\text{Tr} \ln = \ln \det$, to expand the logarithm to order $\delta$. Then, by taking the trace in each one of the integrals in Eq. (10) one gets

$$
\frac{F}{N_c} = \frac{1}{2\lambda} (\sigma^2 + \pi^2) + 2iN_f \int \frac{d^4 p}{(2\pi)^4} \ln \left[ -p^2 + (\alpha + m_0)^2 + \nu^2 \right]
+ 4i\delta N_f \int \frac{d^4 p}{(2\pi)^4} \frac{(\alpha + m_0)(\alpha - \sigma) + \nu(\nu - \pi)}{-p^2 + \hat{\eta}^2}
- 2(n_\pi + 1) \delta \frac{\lambda N_f}{N_c} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{p_\mu q^\mu}{(p^2 + \hat{\eta})(-q^2 + \hat{\eta})}
+ 2(n_\pi - 1) \delta \frac{\lambda N_f}{N_c} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{(\alpha + m_0)^2 - \nu^2}{(p^2 + \hat{\eta})(-q^2 + \hat{\eta})}.
$$

(13)
where \( n_\pi \) represents the number of pseudoscalars. In the \( U(1) \) version of the model, where \( N_f = 1 \) and \( n_\pi = 1 \), so that the last term does not contribute and the OPT will bring \( 1/N_c \) corrections only at finite chemical potential, since the other correction term is related to the quark number density, which is zero at vanishing chemical potential. In our case, \( n_\pi = 3 \) and finite \( N_c \) corrections are expected to occur at any regime of density and/or temperature.

Applying the PMS equation for \( \alpha \) and \( \nu_i \) leads to a cumbersome problem, which can be circumvented by noticing the symmetry between \( \sigma \) direction and set \( \pi_i \), implying that \( \tilde{\nu}_i = 0 \) \[10\] and \( \tilde{\alpha} = \tilde{\eta} \). We also note that in the third term of Eq. (13) only the zeroth component will eventually survive (when \( \mu \neq 0 \)). Then, one finally gets

\[
\mathfrak{F} = \frac{\sigma^2}{2\lambda} + 2iN_f\int \frac{d^4p}{(2\pi)^4} \ln \left[-p^2 + (\eta - m_0)^2\right]
+ 4i\delta N_f \int \frac{d^4p}{(2\pi)^4} \frac{(\eta + m_0)(\eta - \sigma)}{p_0 - p^2 + (\eta + m_0)^2}^2
- 8\delta N_f \int \frac{d^4p}{(2\pi)^4} \frac{p_0}{p^2 + (\eta + m_0)^2}
\]
\[14\]

Using the Matsubara’s (imaginary time) formalism in order to introduce the control parameters \( T \) and \( \mu \), the time component is replaced by the temperature since the evolution of the system follows a path where each point of the path is a equilibrium state. Therefore, the effective potential in terms of temperature and chemical potential becomes

\[
\mathfrak{F}(\eta, \sigma, \mu, T) = \frac{\sigma^2}{4G_S} - 2N_fN_cI_1(\mu, T) + 2\delta N_fN_c(\eta + m_0)(\eta - \sigma)I_2(\mu, T)
+ 4\delta G_SN_fN_cI_3(\mu, T) - 2\delta G_SN_fN_c(\eta + m_0)^2I_2(\mu, T),
\]
\[15\]

where now we have replaced \( \lambda \to 2G_SN_c \). In the above equation we have also defined, for convenience, the integrals

\[
I_1(\mu, T) = \int \frac{d^3p}{(2\pi)^3} \left\{ E_p + \frac{1}{\beta} \ln \left[1 + e^{-\beta(E_p + \mu)}\right] + \frac{1}{\beta} \ln \left[1 + e^{-\beta(E_p - \mu)}\right]\right\},
\]
\[16\]

\[
I_2(\mu, T) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} \left[1 - f^+ - f^-\right]
\]
\[17\]

and

\[
I_3(\mu, T) = \int \frac{d^3p}{(2\pi)^3} \left[f^+ - f^-\right],
\]
\[18\]

where \( E_p^2 = p^2 + (\eta + m_0)^2 \) is the dispersion while

\[
f^+ = \frac{1}{e^{\beta(E_p - \mu)} + 1},
\]
\[19\]

\[
(f^-)^* = \frac{1}{e^{\beta(E_p - \mu)} + 1},
\]
\[20\]
and
\[ f^- = \frac{1}{\epsilon^\beta(E_p + \mu)} + 1, \tag{21} \]
represent the fermion distribution functions for particles and antiparticles respectively.

The physical interpretation of the OPT free energy, Eq. (15), becomes clear by analyzing the physical meaning of each term. The first term represents the classical potential while the second is similar to the standard result obtained in the case of free fermionic gas whose masses are given by \( m_0 + \eta \) as \( I_1 \) suggests. The terms proportional to \( I_2 \sim \partial I_1/\partial \eta \) are reminiscent of the one loop scalar density, \( \rho_s = \langle \bar{\psi}\psi \rangle \). At the same time, the terms proportional to \( I_3 \sim \partial I_1/\partial \mu \) only survive when \( \mu \neq 0 \) as Eq. (18) shows. This can be easily understood by recalling that, to one loop, the quark number density \( \rho = \langle \bar{\psi}^+ \psi \rangle \) is given by \( I_3 \). Then, by noting that \( I_3 \) is \( 1/N_c \) suppressed one can readily draw the basic physical differences between the OPT and LN (or MF) approximations at this first non-trivial order. Namely, the OPT free energy is written in terms of scalar and vector condensates while only the scalar density contributes to the later. This observation is crucial for the discussions to be carried out in the sequel.

Within the two flavor version of the NJL model there are three parameters to be fixed: the current mass \( m_0 \), the cut-off \( \Lambda \) and the coupling constant \( G_S \). Usually they are adjusted so as to reproduce physical observables such as the pion mass \( (m_\pi \simeq 135 \text{ MeV}) \), the pion decay constant \( (f_\pi \simeq 92.4 \text{ MeV}) \) and the quark condensate \( (- \langle \bar{\psi}\psi \rangle^{1/3} \simeq 250 \text{ MeV}) \) [22]. The (non-observable) current quark mass is related to the observables via the Gell–Mann–Oakes–Renner (GMOR) relation [23],
\[ m_0 = -\frac{f_\pi^2 m_\pi^2}{2 \langle \bar{\psi}\psi \rangle}. \tag{22} \]
There is a family of parameter sets tailored to reproduce the numerical values of these physical observables and in general, one chooses a set which also reproduces the vacuum effective quark mass, \( M = m_0 + \Sigma \simeq 330 \text{ MeV} \), where \( \Sigma \) represents the self energy.

The parameter sets adopted in this work are given in table 2.1 (see Ref. [28] for other possibilities).

|          | \( \Lambda [\text{MeV}] \) | \( m_0 [\text{MeV}] \) | \( G_S \Lambda^2 \) |
|----------|--------------------------|------------------|------------------|
| OPT      | 640                      | 4.9              | 1.99             |
| LN       | 640                      | 5.2              | 2.14             |

Table 1. Parameter set for the OPT and for the LN approximation as given in Ref. [28]. These values were obtained to reproduce \( m_\pi = 135 \text{ MeV} \), \( f_\pi = 92.4 \text{ MeV} \) and \( -\langle \bar{\psi}\psi \rangle^{1/3} = 250 \text{ MeV} \).

### 2.2. Repulsive vector interaction in the NJL model

In the Introduction we have emphasized the importance of a vector interaction in the studies of compressed quark matter. Within the NJL model such a term can be of the form \( -G_V (\bar{\psi} \gamma_\mu \psi)^2 \) with \( G_V > 0 \) describing repulsion which is the case here and \( G_V < 0 \) describing attraction. Then the standard NJL lagrangian density becomes
\[ \mathcal{L}_V = \bar{\psi} \left( i \partial_\mu - m_0 \right) \psi + G_S \left[ (\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma_5 \tau \psi)^2 \right] - G_V (\bar{\psi} \gamma^\mu \psi)^2, \tag{23} \]
and the effective potential in the LN approximation reads [1]

$$F_{LN} = \frac{\sigma^2}{4G_S} - 2N_fN_cI_1(\bar{\mu}, T) - 4G_VN_f^2N_c^2I_3^4(\bar{\mu}, T),$$

(24)

where $I_1(\bar{\mu}, T)$ and $I_3(\bar{\mu}, T)$ can be readily obtained from Eqs. (16) and (18) upon replacing $\mu \rightarrow \bar{\mu} = \mu - 2G_V\rho$, with $\rho = 2N_cN_fI_3$, and $m_0 + \eta \rightarrow m_0 + \sigma$. Fukushima [24] has shown that the combined effect of $\bar{\mu}$ and $-4G_VN_f^2N_c^2I_3^3$ in the above equation is to produce a net effect similar to $+4G_VN_f^2N_c^2I_3^3$. This interesting result allows us to better understand the type of $1/N_c$ contributions radiatively generated by the OPT. An inspection of Eq. (15) reveals that this approximation generates a $+4G_VN_f^2N_c^2I_3^3$ term which is similar to the $+4G_SN_cN_fI_2^3$ term appearing in the OPT result.

Note that, in order to obtain thermodynamical results with the LN approximation, at $G_V \neq 0$, one needs to solve the following set of equations

$$\frac{\partial F}{\partial \sigma} = 0, \quad \text{and} \quad \frac{\partial F}{\partial \bar{\mu}} = 0.$$  

(25)

Fig. 2.2 shows the predictions for the chiral transition at low temperatures and high chemical potential values, where one has a first order phase transition. This transition line starts at $T = 0$ and terminates at a critical end point (when $m_0 \neq 0$), since at supercritical temperatures a crossover takes place. This low-$T$/high-$\mu$ portion of the QCD phase diagram is very important for astrophysical applications and the fact that the transition is of the first kind has non-negligible consequences concerning the structure of compact stellar objects. For example, within this kind of phase transition one may have two substances, with distinct densities, coexisting at the same $P$, $T$ and $\mu$. In the case of strongly interacting matter these two substances may represent hadronic and quark matter which could lead to the formation a hybrid star instead of a pure neutron star. Our results show that both, the OPT and the LN approximation with $G_V \neq 0$ (LNGV) reduce the first order transition in relation to the LN case with $G_V = 0$. Also, for a given temperature, the coexistence chemical potential value at which the transition occurs is shifted to higher values within the OPT and the LNGV indicating that these two different model approximations produce a similar type of physics as one could infer by comparing their free energies.

**Figure 2.** Phase diagram in the $T$–$\mu$ plane for the NJL model showing the first order transition lines and the critical end points. The LN with $G_V = 0$ is denoted by the circle, the LNGV with $G_V = G_S/(N_fN_c)$ is denoted by the triangle and the OPT is denoted by the square.
at the tree level. This occurs as a consequence of the Fierz identities, which manifest themselves when exchange (two loop) diagrams are considered. This identities make, in the OPT case, the radiative corrections \((1/N_c\) suppressed) account for vector type of channel. So far, in view of our comparisons between this method, the LN and the LNG\(_V\), one may say that at least for temperatures not much higher than the one which signals the chiral transition the OPT represents an improvement over the LN (at \(G_V = 0\)). So, the OPT can be seen as a powerful alternative to investigate the low-\(T\)/high-\(\mu\) part of the QCD phase diagram which is currently non-accessible to the LQCD simulations. However, our evaluation of the important \(c_2\) cumulant has revealed that at temperatures higher than \(T_c\) the OPT presents a problem similar to the one detected within the LNG\(_V\) approximation: the failure to attain the SB limit.

3. Results at finite temperature and zero density for the second order cumulant of the pressure

The statistical moments, commonly determined from a measured multiplicity distribution, are the mean \((M)\), the variance \((\sigma^2)\) (not be confused with the scalar field), the skewness \((S)\) and the kurtosis \((\kappa)\). They are related to the (net-)number \(N\) of interest by \(M = \langle N \rangle\), \(\sigma^2 = \langle (\Delta N)^2 \rangle\), \(S = \langle (\Delta N)^3 \rangle / \sigma^3\) and \(\kappa = \langle (\Delta N)^4 \rangle / \sigma^4 - 3\) where \(\Delta N = N - \langle N \rangle\) is the fluctuation of \(N\) around its mean value. The cumulants \(c_n\) of the distribution are defined as \(c_1 = M\), \(c_2 = \sigma^2\), \(c_3 = 5\sigma^3\), and \(c_4 = 7\sigma^4\) and are, for an equilibrated system, related to generalized susceptibilities given by appropriate derivatives of the pressure. Studying fluctuations in statistical systems is important from several points of view. For instance, fluctuations define the stability of the system and its way of reaching the state of thermodynamic equilibrium \[25\] The number-of-particle fluctuations are characterized by the dispersion

\[
\langle (\Delta N)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2
\]

for the number-of-particles operator \(N = \int \psi^+(r) \psi(r) dr\) with \(\psi\) being a field operator. The average number of particles, \(N\), is a statistical average \(N = \langle N \rangle = \int \rho(r) dr\) where \(\rho(r) = \langle \psi^+(r) \psi(r) \rangle\) is the density of particles. Here we keep in mind an equilibrium state, because of which \(\rho(r)\) does not depend on time. But \(\rho(r)\) depends on the spatial variable \(r\), when the system is non-uniform (which is not the case under study in this work).

To study QCD at finite densities using Monte Carlo simulations one can Taylor-expand the pressure as a power series in \(\mu/T\) \[26\],

\[
P(T, \mu) = \frac{P}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) (\mu/T)^{2n}.
\]

Note that there are only even powers of \(\mu/T\) due to the reflexion symmetry, which means that the pressure is an even function with respect to \(\mu\), \(P(\mu) = P(-\mu)\) \[27\]. The cumulants, which are the coefficients \((c_n)\) of the series, can be identified with the quark number susceptibilities \[4\]:

\[
\frac{\chi_n}{T^2} = n! c_n(T).
\]

Once the pressure has been evaluated within a given model approximation, the coefficients can be obtained from

\[
c_n(T) = \frac{1}{n!} \frac{\partial^n P(T, \mu)}{\partial (\mu/T)^n} \bigg|_{\mu=0}.
\]

Having determined some of these coefficients one can then calculate other thermodynamical quantities like the quark number density or the quark number susceptibility, which are
respectively given by

\[
\frac{\rho}{T^3} = \frac{\partial P(T, \mu)}{\partial (\mu/T)} T^4 = 2c_2 \frac{\mu}{T} + 4c_4 \left(\frac{\mu}{T}\right)^3 + \cdots, \quad (30)
\]

\[
\frac{\chi_q}{T^2} = \frac{\partial^2 P(T, \mu)}{\partial (\mu/T)^2} T^4 = 2c_2 + 12c_4 \left(\frac{\mu}{T}\right)^2 + \cdots \quad (31)
\]

Figure 3 and 4 show the coefficient \(c_2\) obtained respectively with the OPT and with \(G_V \neq 0\), both compared to the pure large-N approximation for the NJL model. We can see from the figures that the improvements do not behave as expected, since the value of \(c_2\) is decreasing for \(T > T_\sigma\) and is moving away from the Stefan–Boltzmann limit, which takes place at sufficiently high temperatures when the thermal fluctuations overcome the interparticle interactions and the system behaves as a free gas.

\[P_{SB} T^4 = \frac{8\pi^2}{45} \left[ 1 + \frac{21}{32} N_f \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right) \right]. \quad (32)\]

Then, using Eq. (29) we find the Stefan–Boltzmann limit for \(c_2\)

\[c_2^{SB} = 1. \quad (33)\]

4. Conclusions

The evaluation of such cumulants has received a lot of attention from LQCD researchers and the considerable amount of data already available can be used to check the reliability of other non-perturbative techniques such as the OPT and the LN approximation. At this stage it seems puzzling that the OPT, which supposedly should produce results more accurate than the LN approximation, seems to perform rather poorly at high temperatures. Moreover, none of the LQCD evaluations for this coefficient reproduce the maximum for \(c_2(T)\) observed in the OPT results. Nevertheless, this is the result that has been obtained by Steinheimer and Schramm [4] who used the LN approximation to investigate the PNJL model in the presence of this repulsive vector channel. The authors concluded that \(G_V\) should drop to zero after \(T_c\) so that the SB limit at high \(T\) while reproducing the correct physics at \(T < T_c\).
However, the main source of this behavior at \( T > T_c \) is the medium term that appears in Eqs. (15, 18) for the OPT and in Eq. (24) for the \( G_V \neq 0 \) case. This integral goes to large negative values at high temperatures, and because of the pure large-N limit does not take it into account, the Stefan-Boltzmann limit is reached in this approximation. Now, the task is to fix the OPT method in order to get more coherent results for \( c_2 \), keeping in mind this question: Why a more sophisticated approximation in which more corrections diagrams are calculated would give, a priori, less satisfactory data than a simplest one? Where does this discrepancy lie and how to eliminate it?

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