ON THE MINIMUM NUMBER OF HIGH DEGREE CURVES CONTAINING FEW POINTS

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Abstract. Let $d, n \in \mathbb{Z}^+$ and $A$ be a nonempty finite subset of $\mathbb{R}^2$. A curve of degree $d$ is the zero set of a polynomial of degree $d$ in $\mathbb{R}[x, y]$; denote by $C_d$ the family of curves of degree $d$. For any $C_1 \in C_d$, we say that $C_1$ is determined by $A$ if for any $C_2 \in C_d$ such that $C_2 \cap A \supseteq C_1 \cap A$, we have that $C_1 = C_2$; we denote by $D_d(A)$ the family of curves of degree $d$ determined by $A$. Write $\mathcal{O}_{d, n}(A) := \{C \in D_d(A) : |C \cap A| \leq n\}$. In this paper we state two Sylvester-Gallai type results. In the first one, we show that if there is no $C \in C_d$ containing $A$, then

$$\mathcal{O}_{d, 2d^2-3d+4}(A) = \Omega_d(\binom{d}{2})$$

moreover we give a construction which shows that this lower bound is the best possible. In the second main result of this paper, it is shown that if $d \geq 3$, there is no $C \in C_d$ containing $A$ and any subset $B$ of $A$ with $|B| = n$ is contained in at most one $C \in C_d$, then

$$\mathcal{O}_{d, 2n+1-(\frac{d+2}{2})}(A) = \Omega_{n,d}(\binom{d+2}{2})$$

furthermore we show that this lower bound is not trivial.

1. Introduction

In this paper $\mathbb{R}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}_n^+$ denote the set of real numbers, integers, positive integers and nonnegative integers, respectively. For any set $X$, we denote by $\mathcal{P}(X)$ the family of subsets of $X$ and by $\mathcal{P}_n(X)$ the family of subsets $Y$ of $X$ such that $|Y| = n$. For any $n, m \in \mathbb{Z}$, we write $\{n, m\} := \{k \in \mathbb{Z} : n \leq k \leq m\}$. Let $d, n \in \mathbb{Z}^+$. A curve of degree $d$ is a subset $C$ of $\mathbb{R}^2$ which is the zero set of a polynomial in $\mathbb{R}[x, y]$ of degree $d$; we denote by $C_d$ the family of curves of degree $d$ in $\mathbb{R}^2$. For each $A \in \mathcal{P}(\mathbb{R}^2)$, we say that $C_1 \in C_d$ is determined by $A$ if for any $C_2 \in C_d$ satisfying $C_2 \cap A \supseteq C_1 \cap A$, we have that $C_1 = C_2$; we denote by $D_d(A)$ the family of elements of $C_d$ which are determined by $A$. We write

$$\mathcal{O}_{d, n}(A) := \{C \in D_d(A) : |C \cap A| \leq n\},$$
and $\mathcal{O}_d(A) := \mathcal{O}_{d, \binom{d+2}{2} - 1}(A)$. Since there is always a curve of degree $d$ passing through $\binom{d+2}{2} - 1$ given points, notice that any $C \in D_d(A)$ satisfies that $|C \cap A| \geq \binom{d+2}{2} - 1$, and therefore $\mathcal{O}_{d, n}(A) = \emptyset$ for all $n < \binom{d+2}{2} - 1$.

One of the best known results in discrete geometry is Sylvester-Gallai theorem and it can be stated as follows.

Key words and phrases. Sylvester-Gallai type results, plane curves, Veronese map.
Theorem 1.1. Let $A \in \mathcal{P}(\mathbb{R}^2)$ be finite. If $A$ is not contained in a line, then $\mathcal{O}_1(A) \neq \emptyset$.

Proof. See [3].

Sylvester-Gallai theorem has opened a complete research field in discrete geometry, see for instance [3], [5], [8], [17]. An important Sylvester-Gallai problem is to bound $|\mathcal{O}_1(A)|$ in terms of $|A|$. A number of quantitative results have been found, see [6], [8], [12], [16]. One of these results is the Dirac-Motzkin conjecture which was proven by B. Green and T. Tao.

Theorem 1.2. There is an absolute constant $c_1 > 0$ with the following property. Let $A \in \mathcal{P}(\mathbb{R}^2)$ be finite with $|A| > c_1$. If $A$ is not contained in a line, then $|\mathcal{O}_1(A)| \geq \frac{1}{2} |A|$. 

Proof. See [8, Thm. 1.2].

Another important problem is to find Sylvester-Gallai qualitative type results for different geometric objects instead of lines (e.g. conics, circles, hyperplanes, etc.). For conics, this was done by J. Wiseman and P. Wilson.

Theorem 1.3. Let $A \in \mathcal{P}(\mathbb{R}^2)$ be finite. If $A$ is not contained in a curve of degree 2, then $\mathcal{O}_2(A) \neq \emptyset$.

Proof. See [19, Thm.].

Different proofs of Theorem 1.3 were found later, see [4], [7]. However, until this paper, no Sylvester-Gallai type results have been obtained for curves of higher degree. In particular, the following conjecture of Wiseman and Wilson remains unproven.

Conjecture 1.4. Let $d \in \mathbb{Z}^+$ and $A \in \mathcal{P}(\mathbb{R}^2)$ be finite. If $A$ is not contained in a curve of degree $d$, then $\mathcal{O}_d(A) \neq \emptyset$.

In the last few years, a mix of the previous two Sylvester-Gallai type problems have been studied. This means that given a nonempty finite subset $A$ in $\mathbb{R}^2$ and a family $\mathcal{F}$ of geometric objects (e.g. circles, conics, hyperplanes, etc.), we want (under reasonable assumptions) to determine or at least bound the number of elements $F$ of $\mathcal{F}$ such that $F$ is determined by $A$ and $|F \cap A|$ is small. This problem has been studied at least for lines, circles, conics and hyperplanes, see [11], [13], [14], [15]. However, there are no results for high degree curves (as we said above, to our best knowledge, neither qualitative Sylvester-Gallai results exist for high degree curves). In this paper we are interested in this problem when $\mathcal{F}$ is the family of curves with a given degree. The first result of this paper is the next one.

Theorem 1.5. For each $d \in \mathbb{Z}^+$, there are $c_2 = c_2(d), c_3 = c_3(d) > 0$ with the following property. Let $A \in \mathcal{P}(\mathbb{R}^2)$ be finite with $|A| > c_2$. If $A$ is not contained in a curve of degree $d$, then

$$|\mathcal{O}_d(A)| \geq c_3 |A|^d.$$

For $d \in \{1, 2\}$, note that $2d^2 - 3d + 4 = \frac{(d+2)^2 - 1}{2}$. Thus Theorem 1.5 can be seen as a generalization of Theorem 1.2 (although, in the proof of Theorem 1.5, we do not care about the best possible value of $c_3$), and Theorem 1.5 implies Theorem 1.3 for big enough sets. Moreover Theorem 1.5 is optimal as we see in the next result.
Theorem 1.6. For each $d, m \in \mathbb{Z}$ such that $m > \max\left\{\frac{3d^2 - 3d + 4}{2}, \frac{d^2 + 4d}{2}\right\}$ and $d > 1$, there is $A \in \mathcal{P}_m(\mathbb{R}^2)$ such that

i) $A$ is not contained in a curve of degree $d$.

ii) $|O_{d, \frac{3d^2 - 3d + 4}{2}}(A)| \leq \left(\frac{|A| - (d + 1)^2}{d^2}\right)^3$.

Unfortunately, for $d > 2$, we have that $\frac{3d^2 - 3d + 4}{2} > \left(\frac{d^2 + 4d}{2}\right)$ so Theorem 1.6 does not provide information about $O_{d,n}(A)$ when $\left(\frac{d^2 + 4d}{2}\right) - 1 \leq n < \frac{3d^2 - 3d + 4}{2}$. The next result of this paper deals with this problem when $A$ satisfies that each $B \in \mathcal{P}_n(A)$ is contained in at most one curve of degree $d$.

Theorem 1.7. For each $d, n \in \mathbb{Z}$ with $d \geq 3$ and $n \geq \left(\frac{d+2}{2}\right) - 1$, there are $c_4 = c_4(n,d), c_5 = c_5(n,d) > 0$ with the following property. Let $A \in \mathcal{P}(\mathbb{R}^2)$ be finite such that for each $B \in \mathcal{P}_n(A)$, we have that $B$ is contained in at most one curve of degree $d$. If $|A| > c_4$ and $A$ is not contained in a curve of degree $d$, then

$$|O_{d,2n+1-(\frac{d+2}{2})}(A)| \geq c_5|A|\left(\frac{d+2}{2}\right)^{-3}.$$ 

The values of $c_2, c_3, c_4$ and $c_5$ can be determined explicitly (although they depend on some values of some auxiliary results). As a consequence of Theorem 1.7, if any $B \in \mathcal{P}_{\left(\frac{d+2}{2}\right)-1}(A)$ is contained in at most one curve of degree $d$, then $|O_{d}(A)| = \Omega\left(|A|\left(\frac{d+2}{2}\right)^{-3}\right)$ so Theorem 1.7 implies a particular case of Conjecture 1.4. Also Theorem 1.7 is not trivial as we see in the next theorem (although we do not know if the lower bound $\Omega\left(|A|\left(\frac{d+2}{2}\right)^{-3}\right)$ is the best one).

Theorem 1.8. For each $d, n, m \in \mathbb{Z}^+$ such that $n \geq \left(\frac{d+2}{2}\right) - 1$ and $m > 2n + 1 - \left(\frac{d+2}{2}\right)$, there is $A \in \mathcal{P}_m(\mathbb{R}^2)$ such that

i) $A$ is not contained in a curve of degree $d$.

ii) Each $B \in \mathcal{P}_n(A)$ is contained in at most one curve of degree $d$.

iii) $|O_{d,2n+1-(\frac{d+2}{2})}(A)| \leq \left(\frac{|A| - (\frac{d+2}{2})^2}{d^2}\right)^3$.

We explain the main ideas in the proofs of Theorem 1.5 and Theorem 1.7.

i) Let $d \in \mathbb{Z}^+$, $A$ be a nonempty finite subset of $\mathbb{R}^2$ and $\psi_d$ be the $d$-Veronese map. For each hyperplane $H$ in $\mathbb{R}(\frac{d+2}{2})-1$, we have that $\psi_d^{-1}(H) \in \mathcal{C}_e$ for some $e \in [1, d]$; conversely, for each $C \in \bigcup_{e=1}^d \mathcal{C}_e$, there is a hyperplane $H$ in $\mathbb{R}(\frac{d+2}{2})-1$ such that $\psi_d^{-1}(H) = C$. Thus the problem of finding curves determined by $A$ which contain few points of $A$ is (almost) equivalent to the problem of finding hyperplanes generated by $\psi_d(A)$ which contain few points of $\psi_d(A)$.

ii) It is easier to deal with hyperplanes than to do it with curves. As we mentioned above, there are already results that assure the existence of several hyperplanes in $\mathbb{R}^e$ which are generated but contain few points of a given set $S$, see [1], [2], [13], [15]. Nonetheless, these results require that any $e$ points of $S$ generate a hyperplane. In general, the set $\psi_d(A)$ wont satisfy this condition in $\mathbb{R}(\frac{d+2}{2})-1$. Thus we cannot take advantage of [1], [2], [13], [15]. What we will do is to fix a $\left(\frac{d+2}{2}\right)-4$-dimensional flat $F$ and a 2-dimensional flat $G$ such that $F \cap G = \emptyset$. Considering $\mathbb{R}(\frac{d+2}{2})-1$
embedded into \(\mathbb{P}^{(d+2)/2-1}\) and the homogenization \(G^h \cong \mathbb{P}^2\) of \(G\) in \(\mathbb{P}^{(d+2)/2-1}\), we will project each \(\binom{d+2}{2} - 3\)-dimensional flat \(K\) containing \(F\) into \(K \cap G^h\) (which is always a single point in the projective space \(G^h \cong \mathbb{P}^2\)); this will induce a map \(\pi_F : \mathbb{R}^{(d+2)/2-1} \setminus F \to \mathbb{P}^2\). In this way we take the problem of hyperplanes containing \(F\) into a problem of lines.

iii) Nevertheless, we will need very special flats \(F\). To be able to say that if a line \(L\) contains few points of \((\pi_F \circ \psi_d)(A)\), then the curve \((\pi_F \circ \psi_d)^{-1}(L)\) contains few points of \(A\), we need that \(F\) satisfies certain conditions. The flats \(F\) will be the affine hull of \(\psi_d(B)\) where \(B\) is a subset of \(A\) which satisfies some technical conditions. The family of subsets \(B\) of \(A\) satisfying those assumptions will be denoted by \(\mathcal{N}_d(A)\). The longest and most tedious part of this paper (which is Section 3) is to warranty that \(\mathcal{N}_d(A)\) is not very small, however this is the core of this article. The proof of this fact will depend on the structure of \(A\); specifically, it depends on whether there is a low degree curve that contains several points of \(A\) or not.

iv) Let \(F\) be the affine hull of \(\psi_d(B)\) for some \(B \in \mathcal{N}_d(A)\) and denote by \(\varphi\) the restriction of \(\pi_F \circ \psi_d\) to \(\mathbb{R}^2 \setminus \psi_d^{-1}(F)\). In Section 4 we will prove that there are \(k \leq d^2\) and a finite collection (bounded in terms of \(d\)) of Zariski closed subsets \(\{E_i\}_{i \in I}\) of \(\mathbb{R}^2\) such that

- \(\varphi(E_i)\) is a singleton for each \(i \in I\).
- For each \(a \in \mathbb{R}^2 \setminus (\psi_d^{-1}(F) \cup \bigcup_{i \in I} E_i)\), we have that \(\varphi(a) \notin \{\varphi(E_i) : i \in I\}\).
- For each \(a \in \mathbb{R}^2 \setminus (\psi_d^{-1}(F) \cup \bigcup_{i \in I} E_i)\), we have that \(|\varphi^{-1}(\varphi(a))| \leq k\).

v) With the map \(\varphi\) as in iv), any line \(L\) in \(\mathbb{R}^2\) disjoint from \(\{\varphi(E_i) : i \in I\}\) which contains exactly two points of \(\varphi(A \setminus \psi_d^{-1}(F))\) satisfies that \(\varphi^{-1}(L)\) is an element of \(\mathcal{C}_d\) which contains few points of \(A\) and is determined by it. A result of T. Boys, C. Valcaulescu and F. de Zeeuw (see Lemma 2.1) will warranty that there are several lines \(L\) as above, and therefore we will have a number of the desired curves for each \(B \in \mathcal{N}_d(A)\). This can be done for each flat \(F\) generated by an element of \(\mathcal{N}_d(A)\). Thus, since \(|\mathcal{N}_d(A)| \gg 0\), we will have several curves satisfying the desired conditions.

We think that at least as valuable as the main results of this paper is the method we use since it allows to translate many Sylvester-Gallai problems of curves into Sylvester-Gallai problems of lines where much more tools are available.

This paper is organized as follows. In Section 2 we establish some notation and auxiliary results that will be needed in the forthcoming sections. As we already mentioned, the families \(\mathcal{N}_d(A)\) are fundamental tools in the proofs of the main results. We introduce and state some properties of the families \(\mathcal{N}_d(A)\) in Section 3. Using the elements of the families \(\mathcal{N}_d(A)\), we will take the problem of finding curves with few points into finding ordinary lines which avoid a finite set and this will be done in Section 4. The proofs of the main results of this paper are completed in Section 5.

2. Preliminaries

In this section we state some notation and results that will be needed later.

Let \(p(x, y) \in \mathbb{R}[x, y]\) and \(d \in \mathbb{Z}^+.\) We denote by \(\mathcal{Z}(p(x, y))\) its zero set in \(\mathbb{R}^2\) and by \(\text{deg}(p(x, y))\) its degree. We say that \(p(x, y) \in \mathbb{R}[x, y]\) is irreducible if
\[ \deg(p(x, y)) > 0 \] and for any factorization \( p(x, y) = p_1(x, y)p_2(x, y) \), we get that
\( p_i(x, y) \in \mathbb{R} \) for some \( i \in \{1, 2\} \). We say that \( \mathcal{Z}(p(x, y)) \) is irreducible if \( p(x, y) \) is irreducible. Write
\[
\mathcal{C}_{\leq d} := \bigcup_{e=1}^{d} \mathcal{C}_e
\]
\[ \mathbb{R}_d[x, y] := \{ p(x, y) \in \mathbb{R}[x, y] : \deg(p(x, y)) \in [1, d] \}; \]
also, for technical reasons, we write \( \mathcal{C}_{\leq 0} := \emptyset \). In \( \mathbb{R}[x, y] \), we define the relation
\[ p(x, y) \sim q(x, y) \]
if there is \( r \in \mathbb{R} \setminus \{0\} \) such that \( p(x, y) = r \cdot q(x, y) \), and we denote by \([p(x, y)]\) the class of \( p(x, y) \) and by \( \mathbb{R}[x, y]/\sim \) the set of classes. For any subset \( X \) of \( \mathbb{R}[x, y] \), we write
\[ X/\sim := \{ [p(x, y)] \in \mathbb{R}[x, y]/\sim : [p(x, y)] \cap X \neq \emptyset \}. \]
Note that for any \( p(x, y), q(x, y) \in \mathbb{R}[x, y] \) such that \([p(x, y)] = [q(x, y)]\), we have that \( \deg(p(x, y)) = \deg(q(x, y)) \) and \( \mathcal{Z}(p(x, y)) = \mathcal{Z}(q(x, y)) \); thus
\[
\sigma_d : \mathbb{R}_d[x, y]/\sim \rightarrow \mathcal{C}_{\leq d}, \quad \sigma_d([p(x, y)]) = \mathcal{Z}(p(x, y)).
\]
is well defined. Let \( p(x, y) \in \mathbb{R}[x, y] \) be such that \( \deg(p(x, y)) > 0 \) and consider a factorization
\[ p(x, y) = r \prod_{i=1}^{n} p_i(x, y)^{m_i} \]
with \( m_1, m_2, \ldots, m_n \in \mathbb{Z}^+ \), \( r \in \mathbb{R} \), \([p_i(x, y)] \neq [p_j(x, y)]\) for all \( i, j \in \{1, n\} \) such that \( i \neq j \), and \( p_i(x, y) \) is irreducible for each \( i \in \{1, n\} \). Then the irreducible curves \( \mathcal{Z}(p_1(x, y)), \mathcal{Z}(p_2(x, y)), \ldots, \mathcal{Z}(p_n(x, y)) \)
are known as the irreducible components of \( \mathcal{Z}(p(x, y)) \). The irreducible components satisfy that
\[
\mathcal{Z}(p(x, y)) = \bigcup_{i=1}^{n} \mathcal{Z}(p_i(x, y)) = \mathcal{Z}
\left( \prod_{i=1}^{n} p_i(x, y) \right)
\]
and
\[
\deg(p(x, y)) = \sum_{i=1}^{n} m_i \deg(p_i(x, y)) \geq \sum_{i=1}^{n} \deg(p_i(x, y)).
\]
We will use a weak version of Bezout’s theorem.

**Theorem 2.1.** Let \( d, e \in \mathbb{Z}^+ \), \( C_1 \in \mathcal{C}_{\leq d} \) and \( C_2 \in \mathcal{C}_{\leq e} \). If \( C_1 \) and \( C_2 \) do not have an irreducible component in common, then
\[
|C_1 \cap C_2| \leq de.
\]

**Proof.** See [9] Ch. I.7. \qed

The next facts can be proven easily by the reader.

**Remark 2.2.** Let \( d \in \mathbb{Z}^+ \) and \( e, f \in [1, d] \).

i) For any \( C_1 \in \mathcal{C}_{\leq e} \) and \( C_2 \in \mathcal{C}_{\leq f} \), notice that \( C_1 \cup C_2 \in \mathcal{C}_{\leq e + f} \).

ii) For any \( C_0 \in \mathcal{C}_{e} \) and \( C \in \mathcal{C}_{\leq d} \) such that \( C_0 \subseteq C \), there is \( C_1 \in \mathcal{C}_{\leq d - e} \) such that \( C = C_0 \cup C_1 \).

iii) For any \( A \in \mathcal{P}(\mathbb{R}^2) \) such that \( |A| \leq \left( \frac{d + 2}{2} \right) - 1 \), there is \( C \in \mathcal{C}_{\leq d} \) such that \( A \subseteq C \).

For a curve \( C \), it may happen that there exist \( p(x, y), q(x, y) \in \mathbb{R}[x, y] \) such that \([p(x, y)] \neq [q(x, y)]\) but \( \mathcal{Z}(p(x, y)) = \mathcal{Z}(q(x, y)) \). The next lemma shows that for each \( C \in \mathcal{C}_{\leq d} \) there are at most \( d^d \) classes \([p(x, y)] \in \mathbb{R}_d[x, y]/\sim \) such that \( C = \mathcal{Z}(p(x, y)) \).
Lemma 2.3. Let \( d \in \mathbb{Z}^+ \) and \( C \in \mathbb{C}_{<d} \) with pairwise distinct irreducible components \( \mathcal{Z}(p_1(x, y)), \mathcal{Z}(p_2(x, y)), \ldots, \mathcal{Z}(p_n(x, y)) \). Then
\[
\sigma_d^{-1}(C) = \left\{ \prod_{i=1}^{n} p_i^{m_i} \in \mathbb{R}_d[x, y]/ \sim : m_1, m_2, \ldots, m_n \in \mathbb{Z}^+, \sum_{i=1}^{n} m_i \deg(p_i) \leq d \right\}.
\]
Since the number of solutions \((m_1, m_2, \ldots, m_n) \in \mathbb{Z}^+^n \) of \( \sum_{i=1}^{n} m_i \deg(p_i) \leq d \) is bounded by \( d^n \), we get in particular that
\[
|\sigma_d^{-1}(C)| \leq d^n.
\]

Proof. See [10, Cor.7]. \( \square \)

Let \( d \in \mathbb{Z}_0^+ \) and \( e \in [0, d] \). A translation \( F \) of a vectorial subspace \( V \) of \( \mathbb{R}^d \) will be called a flat. We write \( \dim F := \dim V \), and also if \( V \) is an \( e \)-dimensional subspace, we say that \( F \) is an \( e \)-flat; in particular, 1-flats are lines and \( d-1 \)-flats are hyperplanes. The family of \( e \)-flats in \( \mathbb{R}^d \) will be denoted by \( \mathcal{G}_e \). For any subset \( S \) of \( \mathbb{R}^d \), we denote by \( \mathcal{F}(S) \) the smallest flat (with respect to \( \subseteq \) ) which contains \( S \) and we write \( \dim S := \dim \mathcal{F}(S) \). If \( S = \emptyset \), we consider \( \mathcal{F}(S) = \emptyset \) and \( \dim S = -1 \). If \( S = \{s_1, s_2, \ldots, s_n\} \), we write \( \mathcal{F}(s_1, s_2, \ldots, s_n) := \mathcal{F}(S) \). The family of \( e \)-flats \( F \) in \( \mathbb{R}^d \) such that there is a subset \( R \) of \( S \) satisfying that \( F = \mathcal{F}(R) \) will be denoted by \( \mathcal{G}_e(S) \).

A fundamental tool in this paper is the Veronese map. Write \( I_d := \{(n, m) \in \mathbb{Z}_0^{+2} : n + m \in [1, d]\} \) so \( |I_d| = \binom{d+2}{2} - 1 \). The \( d \)-Veronese map is the map
\[
\psi_d : \mathbb{R}^2 \longrightarrow \mathbb{R}^{\binom{d+2}{2}-1}, \quad \psi_d(a_1, a_2) = (a_1^{n} a_2^{m})_{(n, m) \in I_d}
\]
To avoid confusion, the ring of polynomials which corresponds to \( \mathbb{R}^2 \) will be denoted by \( \mathbb{R}[x, y] \) and the ring of polynomials which corresponds to \( \mathbb{R}^{\binom{d+2}{2}-1} \) will be denoted by \( \mathbb{R}[z_{(n, m)}]_{(n, m) \in I_d} \). There is a quite important relation between elements of \( \mathbb{R}_d[x, y]/ \sim \) and hyperplanes in \( \mathbb{R}^{\binom{d+2}{2}-1} \) given by the next map
\[
\tau_d : \mathbb{R}_d[x, y]/ \sim \longrightarrow \mathcal{G}_{\binom{d+2}{2}-2};
\]
\[
\tau_d \left( \sum_{(n, m) \in I_d} r_{(n, m)} x^n y^m \right) = \mathcal{Z} \left( \sum_{(n, m) \in I_d} r_{(n, m)} z_{(n, m)} \right)
\]
The Veronese map has some well-known properties that we will need later. The proof of the following facts can be found in standard algebraic geometry books, see for instance [9 Ch. 1], [18 Ch. 1].

Remark 2.4. Let \( d \in \mathbb{Z}^+ \).
\begin{enumerate}
\item The map \( \psi_d \) is an isomorphism onto its image.
\item The map \( \tau_d \) is a bijection. Note that for any \( [p(x, y)] \in \mathbb{R}_d[x, y]/ \sim \), we have that
\[
\psi_d(\mathcal{Z}(p(x, y))) = \tau_d([p(x, y)]) \cap \psi_d(\mathbb{R}^2).
\]
\item For all \( e \in [1, d] \) and \( A \in \mathcal{P}(\mathbb{R}^2) \), we have that
\[
\dim \psi_e(A) \leq \dim \psi_d(A).
\]
\end{enumerate}
A crucial part of this paper is to take the problem of finding curves determined by \( A \) containing few points of \( A \) into the problem of finding hyperplanes in \( \mathbb{R}^{(d+2)-1} \) generated by \( \psi_d(A) \) which contain few points of \( \psi_d(A) \). The main tool to do this is the following lemma.

**Lemma 2.5.** Let \( d \in \mathbb{Z}^+ \) and \( A \in \mathcal{P}(\mathbb{R}^2) \) be such that there is no element of \( C_{\leq d} \) which contains \( A \). Then

\[
\tau_d(\sigma_d^{-1}(D_d(A))) = G_{(d+2)-2}(\psi_d(A))
\]

**Proof.** See [10, Lemma 12].

Let \( d \in \mathbb{Z}^+ \), \( p(x, y) \in \mathbb{R}_d[x, y] \) and \( A \in \mathcal{P}(\mathbb{R}^2) \) be finite. Since \( A \) is finite, there is always a line \( L = Z(rx + sy - t) \) in \( \mathbb{R}^2 \) such that \( L \cap A = \emptyset \). Therefore

\[
Z(p(x, y)) \cap A = (Z(p(x, y)) \cap A) \cup (L \cap A)
\]

\[
= Z(p(x, y) \cdot (rx + sy - t)^{d - \deg(p(x, y)))} \cap A.
\]

From this observation, we get the next fact.

**Remark 2.6.** Let \( d \in \mathbb{Z}^+ \), \( p(x, y) \in \mathbb{R}_d[x, y] \) and \( A \in \mathcal{P}(\mathbb{R}^2) \) be finite. Then there is \( q(x, y) \in \mathbb{R}_d[x, y] \) such that \( \deg(q(x, y)) = d \) and

\[
Z(p(x, y)) \cap A = Z(q(x, y)) \cap A.
\]

**Lemma 2.7.** Let \( e \in \mathbb{Z}^+ \), \( f \in \mathbb{Z}_0^+ \), \( F \) be a proper flat in \( \mathbb{R}^{(e+2)} \) and \( A \in \mathcal{P}(\mathbb{R}^2) \) be such that \( A \) is not contained in an element of \( C_{\leq f} \).

i) If \( e \geq f \), then \( |\psi_e(A) \cap F| \leq 1 + \dim F \).

ii) If \( e < f \), then \( |\psi_e(A) \cap F| \leq \left( \frac{f+2}{2} \right) - \left( \frac{f-e+2}{2} \right) + 2 + \dim F \).

iii) If \( e < f \), then \( \dim \psi_d(A \setminus \psi_e^{-1}(F)) \geq \left( \frac{f-e+2}{2} \right) - 1 \) for all \( d \geq f - e \).

**Proof.** First we show i). If \( f = 0 \), then \( |A| = 1 \) so

\[
|\psi_e(A) \cap F| \leq |\psi_e(A)| = 1 \leq 1 + \dim F.
\]

Thus, from now on, we assume that \( f > 0 \). Since \( A \) is not contained in an element of \( C_{\leq f} \), Remark 2.4.ii implies that \( \psi_f(A) \) cannot be contained in a hyperplane of \( \mathbb{R}^{(e+2)} \) and therefore

\[
\left( \frac{f+2}{2} \right) - 1 = \dim \psi_f(A).
\]

In so far as \( e \geq f \), Remark 2.4.iii implies that \( \dim \psi_f(A) \leq \dim \psi_e(A) \) and thereby

\[
\dim \psi_f(A) \leq \dim \psi_e(A) \leq |\psi_e(A)| - 1 = |A| - 1.
\]

Since \( |A| = \left( \frac{f+2}{2} \right) \), we get from 11 and 12 that

\[
\dim \psi_e(A) = |\psi_e(A)| - 1 = |A| - 1.
\]

As a consequence of 13, for any \( S \in \mathcal{P}(\psi_e(A)) \), we get that \( \dim S = |S| - 1 \); in particular,

\[
\dim \psi_e(A) \cap F = |\psi_e(A) \cap F| - 1,
\]

and hence

\[
|\psi_e(A) \cap F| = 1 + \dim \psi_e(A) \cap F \leq 1 + \dim F,
\]

which completes the proof of i).
Now we prove ii). For any flat $E$ in $\mathbb{R}^{(e+2)} - 1$, write $\alpha(E) := \binom{e+2}{2} - 2 - \dim E$. In so far as $F$ is a proper flat, note that $\alpha(F) \geq 0$. The proof of ii) will be done by induction on $\alpha(F)$. First suppose that $\alpha(F) = 0$ so $F$ is a hyperplane in $\mathbb{R}^{(e+2)} - 1$. Then Remark 2.3ii implies that $\psi_e^{-1}(F) \in C_{<e}$. Trivially,

$$\psi_e^{-1}(F) \cap A \subseteq \psi_e^{-1}(F).$$

We claim that

$$|\psi_e^{-1}(F) \cap A| \leq \left( \frac{f + 2}{2} - \frac{f - e + 2}{2} \right).$$

Indeed, if (5) is false, then

$$|A \setminus \psi_e^{-1}(F)| = |A| - |\psi_e^{-1}(F) \cap A| < |A| - \left( \frac{f + 2}{2} + \frac{f - e + 2}{2} \right) = \left( f - e + 2 \right),$$

and therefore Remark 2.2iii implies that there exists $C \in C_{<f-e}$ such that

$$A \setminus \psi_e^{-1}(F) \subseteq C.$$  

However, from (4) and (6),

$$A = (\psi_e^{-1}(F) \cap A) \cup (A \setminus \psi_e^{-1}(F)) \subseteq \psi_e^{-1}(F) \cup C$$

with $\psi_e^{-1}(F) \cup C \in C_{\leq f}$ by Remark 2.2i; this contradicts the assumption so (5) is true. Therefore

$$|F \cap \psi_e(A)| = |\psi_e^{-1}(F) \cap A|$$

$$\leq \left( \frac{f + 2}{2} - \frac{f - e + 2}{2} \right)$$

(by (4))

$$= \left( \frac{f + 2}{2} - \frac{f - e + 2}{2} \right) - \alpha(F)$$

(since $\alpha(F) = 0$),

and the basis of induction is complete. Now assume that the claim holds for all flats $E$ such that $0 \leq \alpha(E) < \alpha(F)$. Since $A$ is not contained in an element of $C_{\leq f}$ and $e < f$, we get that $A$ is not contained in an element of $C_{<e}$. Thus Remark 2.3ii implies that $\psi_e(A)$ is not contained in a hyperplane and therefore $\dim \psi_e(A) = \binom{e+2}{2} - 1$. In particular, this means that $\psi_e(A) \setminus F \neq \emptyset$, and then we can fix $a \in A \setminus \psi_e^{-1}(F)$. Set $E := \text{Fl}(F \cup \{\psi_e(a)\})$. Since $\psi_e(a) \notin F$, we have that $\dim E = 1 + \dim F$, and therefore

$$\alpha(E) + 1 = \alpha(F).$$

In so far as $\psi_e(a) \in \psi_e(A) \setminus F$, we get that

$$|F \cap \psi_e(A)| + 1 \leq |E \cap \psi_e(A)|.$$

Then

$$|F \cap \psi_e(A)| \leq |E \cap \psi_e(A)| - 1$$

(by (4))

$$\leq \left( \frac{f + 2}{2} - \frac{f - e + 2}{2} \right) - \alpha(E) - 1$$

(by induction)

$$= \left( \frac{f + 2}{2} - \frac{f - e + 2}{2} \right) - \alpha(F),$$

(by (4))

and this completes the induction and the proof of ii).
Finally, we show iii). Since $F$ is a proper flat, there is a hyperplane $H$ which contains $F$. Remark 2.4.ii implies that $\psi^{-1}_e(H) \in C_{\le e}$. Since $H \supseteq F$,

$$\psi^{-1}_e(F) \cap A \subseteq \psi^{-1}_e(H) \cap A \subseteq \psi^{-1}_e(H).$$

We claim that

$$\dim \psi_{f-e}(A \setminus \psi^{-1}_e(F)) \ge \left(\frac{f-e+2}{2}\right) - 1.$$  

Indeed, if (10) does not hold, then $\dim \psi_{f-e}(A \setminus \psi^{-1}_e(F)) < \left(\frac{f-e+2}{2}\right) - 1$, and hence there exists a hyperplane $K$ in $\mathbb{R}^{(f-e+2)/2}$ which contains $\psi_{f-e}(A \setminus \psi^{-1}_e(F))$ with $\psi^{-1}_{f-e}(K) \in C_{\le f-e}$ by Remark 2.4.ii. Then

$$A \setminus \psi^{-1}_e(F) \subseteq \psi^{-1}_{f-e}(K).$$

Nonetheless, from (9) and (11),

$$A = (\psi^{-1}_e(F) \cap A) \cup (A \setminus \psi^{-1}_e(F)) \subseteq \psi^{-1}_e(H) \cup \psi^{-1}_{f-e}(K),$$

and Remark 2.2.i yields that $\psi^{-1}_e(H) \cup \psi^{-1}_{f-e}(K) \in C_{\le f}$; this contradicts the assumption so (10) needs to be true. Now, in so far as $d \ge f-e$, Remark 2.4.iii yields that

$$\dim \psi_{d}(A \setminus \psi^{-1}_e(F)) \ge \dim \psi_{f-e}(A \setminus \psi^{-1}_e(F)).$$

Then iii) is a consequence of (10) and (12). □

**Lemma 2.8.** Let $e \in \mathbb{Z}^+_0$ and $A \in \mathcal{P}(\mathbb{R}^2)$ be such that $A$ is finite and it is not contained in an element of $C_{\le e}$. Denote by $\mathcal{R}$ the family of elements $B \in \mathcal{P}(\mathbb{R}^2)(A)$ such that $B$ is not contained in an element of $C_{\le e}$. Then

$$|\mathcal{R}| \ge \frac{1}{2^{(\frac{d}{2})-1}|A|}.$$  

**Proof.** Let $d \in \mathbb{Z}^+$ and $T \in \mathcal{P}(\mathbb{R}^d)$ be such that $\dim T = d$. Set $S_d(T) := \{R \in \mathcal{P}_{d+1}(T) : \dim R = d\}$. First we show that

$$|S_d(T)| \ge \frac{1}{2^d}|T|.$$  

We prove (13) by induction on $d$. If $d = 1$, then

$$S_1(T) = \{R \in \mathcal{P}_2(T) : \dim R = 1\} = \mathcal{P}_2(T)$$

so $|S_1(T)| = \binom{|T|}{2} \ge \frac{1}{2}|T|$, and the basis of induction is complete. Assume that (13) holds for $d - 1$ and we show it for $d$. Since $\dim T = d$, there exists $S \in \mathcal{P}_d(T)$ such that $\dim S = d - 1$; fix $S \in \mathcal{P}_d(T)$ such that $\dim S = d - 1$ and write $H := \text{Fl}(S)$.

We have two cases.

- Assume that $|T \cap H| \ge \frac{1}{2}|T|$. Fix $t \in T \setminus H$. Notice that for all $R \in S_{d-1}(T \cap H)$, we have that $|R \cup \{t\}| = |R| + 1 = d$ and $\dim R \cup \{t\} = 1 + \dim R = d$ so $R \cup \{t\} \in S_d(T)$. This map

$$S_{d-1}(T \cap H) \to S_d(T), \quad R \mapsto R \cup \{t\}$$

is injective so

$$|S_d(T)| \ge |S_{d-1}(T \cap H)|.$$
By induction, $|S_{d-1}(T \cap H)| \geq \frac{1}{2^{d-1}} |T \cap H|$ so (14) leads to

$$|S_d(T)| \geq |S_{d-1}(T \cap H)| \geq \frac{1}{2^{d-1}} |T \cap H| \geq \frac{1}{2^d} |T|,$$

and this completes the induction in this case.

- Assume that $|T \cap H| < \frac{1}{2} |T|$. Note that for all $t \in T \setminus H$, we have that $|S \cup \{t\}| = |S| + 1 = d$ and $\dim S \cup \{t\} = 1 + \dim S = d$ so $S \cup \{t\} \in S_d(T)$. The map

$$T \setminus H \to S_d(T), \quad t \mapsto S \cup \{t\}$$

is injective so

$$|S_d(T)| \geq |T \setminus H| = |T| - |T \cap H| > \frac{1}{2} |T| \geq \frac{1}{2^d} |T|,$$

and this completes the induction.

If $e = 0$, then $C_{\leq e} = \emptyset$ so $R = \{ \{ a \} : a \in A \}$ which means that $|R| = |A|$. Thus, from now on, assume that $e > 0$. Since $A$ is not contained in an element of $C_{\leq e}$, Remark 2.2ii implies that $\psi_e(A)$ is not contained in a hyperplane and then $\dim \psi_e(A) = \binom{e+2}{2} - 1$. Applying (13) to $\psi_e(A)$ and $\binom{e+2}{2} - 1$, we get that

(15) $$\left| S_{\binom{e+2}{2}-1}(\psi_e(A)) \right| \geq \frac{1}{2^{\binom{e+2}{2}-1}} |\psi_e(A)| = \frac{1}{2^{\binom{e+2}{2}-1}} |A|. $$

For each $R \in S_{\binom{e+2}{2}-1}(\psi_e(A))$, we have that $\dim R = \binom{e+2}{2} - 1$ so $R$ is not contained in a hyperplane, and hence, by Remark 2.2ii, $\psi_e^{-1}(R)$ is not contained in an element of $C_{\leq e}$. Thus the map

$$S_{\binom{e+2}{2}-1}(\psi_e(A)) \to R, \quad R \mapsto \psi_e^{-1}(R)$$

is well defined and injective yielding that

$$|R| \geq \left| S_{\binom{e+2}{2}-1}(\psi_e(A)) \right|,$$

and finally (13) implies the claim. \hfill \square

Lemma 2.9. Let $d \in \mathbb{Z}^+, f \in [0, d-1]$, $C_0 \in C_{d-f}$ and $B_0 \in \mathcal{P}_{\binom{f+2}{2}}(\mathbb{R}^2 \setminus C_0)$ be such that $B_0$ is not contained in an element of $C_{\leq f}$. Take $B \in \mathcal{P}_{\binom{f+2}{2}-3}(\mathbb{R}^2)$ such that $B \supseteq B_0$ and $B \cap C_0 = B \setminus B_0$. Then, for any $e \in [d-f, d]$ and $C \in C_{\leq e}$ such that $C \supseteq C_0$, we have that

(16) $$|B \cap C| < \left( \frac{d+2}{2} \right) - \left( \frac{d-e+2}{2} \right) - 2.$$

Proof. Since $C_0 \subseteq C$, Remark 2.2ii implies that there is $C_1 \in C_{\leq e+f-d}$ such that $C = C_0 \cup C_1$. In so far as $B_0 \cap C_0 = \emptyset$, notice that

(17) $$B_0 \cap C = B_0 \cap (C_0 \cup C_1) = B_0 \cap C_1 \subseteq C_1.$$

If (16) is false, then

(18) $$|B_0 \setminus C| \leq |B \setminus C| = |B| - |B \cap C| \leq \left( \frac{d-e+2}{2} \right) - 1.$$

If $e = d$, then (18) implies that $B_0 \subseteq C = C_0 \cup C_1$; hence, inasmuch as $B_0 \cap C_0 = \emptyset$, we get that $B_0 \subseteq C_1$ but this contradicts that $B_0$ is not contained in an element of
Remark 2.4.ii implies that $\psi$ and therefore $\dim \psi$ is contained in a hyperplane of $\mathbb{R}^{(d+2)}$. Inasmuch as $C_2 \in \mathcal{C}_{\leq d}$, we get that $B \subseteq C_2$. Hence

\[ B_0 \cap C = C_2. \]

Since $C_1 \subseteq \mathcal{C}_{\leq f - d}$ and $C_2 \subseteq \mathcal{C}_{\leq d - \varepsilon}$, note that $C_1 \cup C_2 \subseteq \mathcal{C}_{\leq f}$ by Remark 2.2.ii. Notice that $C_1 \subseteq C_2$ by (17) and (19); however, this contradicts that $B_0$ is not contained in an element of $\mathcal{C}_{\leq f}$ and it proves (18).

\[ \square \]

Lemma 2.10. Let $d \in \mathbb{Z}^+$, $e \in \{1, d - 1\}$, $C \in \mathcal{C}_e$ and $B \in \mathcal{P}(\mathbb{R}^2)$.

i) If $\dim \psi_d(B \setminus C) = \left(\frac{d-e+2}{2}\right) - 1$, then $\dim \psi_d(B \cup C) = \left(\frac{d+2}{2}\right) - 1$.

ii) If $\dim \psi_d(B \setminus C) = \left(\frac{d-e+2}{2}\right) - 2$, then $\dim \psi_d(B \cup C) = \left(\frac{d+2}{2}\right) - 2$.

Proof. First we show i). Assume that $\dim \psi_d(B \cup C) < \left(\frac{d+2}{2}\right) - 1$ so $\psi_d(B \cup C)$ is contained in a hyperplane of $\mathbb{R}^{(d+2)}$. Then Remark 2.2.ii implies that there is $C_1 \subseteq \mathcal{C}_d$ such that $B \cup C \subseteq C_1$: in particular, $C \subseteq C_1$ and then Remark 2.2.ii yields the existence of $C_2 \subseteq \mathcal{C}_{\leq d - \varepsilon}$ such that $C_1 = C \subseteq C_2$. In so far as $B \subseteq C \subseteq C_1 \subseteq C \cup C_2$, we get that $\dim \psi_d(B \cup C) = \left(\frac{d+2}{2}\right) - 1$, which proves i).

Now we prove ii). Assume that $\dim \psi_d(B \cup C) \neq \left(\frac{d+2}{2}\right) - 2$ and we will show that

\[ \dim \psi_{d-e}(B \setminus C) \neq \left(\frac{d - e + 2}{2}\right) - 2. \]

We have to deal with two cases.

- Suppose that $\dim \psi_d(B \cup C) > \left(\frac{d+2}{2}\right) - 2 \Rightarrow \dim \psi_d(B \cup C) = \left(\frac{d+2}{2}\right) - 1$. We claim that

\[ \dim \psi_{d-e}(B \setminus C) = \left(\frac{d - e + 2}{2}\right) - 1. \]

Indeed, if \((21)\) is false, there is a hyperplane of $\mathbb{R}^{(d+2)}$ which contains $\psi_{d-e}(B \setminus C)$. Then, by Remark 2.2.ii, there is a curve $C' \subseteq \mathcal{C}_{\leq d - \varepsilon}$ such that $B \setminus C \subseteq C'$. Hence

\[ B \cup C = (B \setminus C) \cup C \subseteq C' \cup C \]

with $C' \subseteq \mathcal{C}_d$ by Remark 2.2.ii. This means that $\psi_d(B \cup C)$ is contained in a hyperplane by Remark 2.2.ii, and thus $\dim \psi_d(B \cup C) < \left(\frac{d+2}{2}\right) - 1$. This contradiction proves \((21)\).

- Suppose that $\dim \psi_d(B \cup C) < \left(\frac{d+2}{2}\right) - 2$. On the one hand, there are infinitely many hyperplanes in $\mathbb{R}^{(d+2)}$ containing $\psi_d(B \cup C)$. On the other hand, for each curve $C' \subseteq \mathcal{C}_d$, there are only finitely many $[p(x, y)] \in \mathbb{R}_d[x, y]$ such that $C' = \mathcal{Z}(p(x, y))$ by Lemma 2.3. Therefore we can choose $p_1(x, y), p_2(x, y) \in \mathbb{R}_d[x, y]$ such that $\psi_d(B \cup C) \subseteq \mathcal{Z}(p_1(x, y))$, $\psi_d(B \cup C) \subseteq \mathcal{Z}(p_2(x, y))$ and $\mathcal{Z}(p_1(x, y)) \neq \mathcal{Z}(p_2(x, y))$; write $C_1 := \mathcal{Z}(p_1(x, y))$ and $C_2 := \mathcal{Z}(p_2(x, y))$. Since $\psi_d(B \cup C) \subseteq \mathcal{Z}(p_1(x, y))$, Remark 2.2.ii implies that $B \cup C \subseteq C_1$. Then Remark 2.2.ii yields the existence of $C_3 \subseteq \mathcal{C}_{\leq d - \varepsilon}$ such that $C_1 = C \subseteq C_3$. Since $B \cup C \subseteq C_1 = C \cup C_3$, we get that $B \setminus C \subseteq C_3$. In so far as $C_3 \subseteq \mathcal{C}_{\leq d - \varepsilon}$, Remark 2.2.ii implies that there is a hyperplane $H_3$ in $\mathbb{R}^{(d+2)}$ such that $\psi_{d-e}(H_3) = C_3$.
and \( \psi_{d-e}(B \setminus C) \subseteq H_3 \). Proceeding in the same way with \( C_2 \), there exist \( C_4 \in C_{d-e} \) and a hyperplane \( H_4 \) in \( \mathbb{R}^{(d-e+2)-1} \) such that \( C_2 = C \cup C_4 \), \( \psi_{d-e}^{-1}(H_4) = C_4 \) and \( \psi_{d-e}(B \setminus C) \subseteq H_4 \). Since \( C_1 \neq C_2 \), notice that \( C_3 \neq C_4 \) and therefore \( H_3 \neq H_4 \). Now, since

\[
\psi_{d-e}(B \setminus C) \subseteq H_3 \cap H_4,
\]

we get that \( \dim \psi_{d-e}(B \setminus C) < \left( \frac{d-e+2}{2} \right) - 2 \).

Therefore, in any case, (20) is true and this proves ii). \( \square \)

**Lemma 2.11.** For any \( n \in \mathbb{Z}^+ \), there are \( c_6 = c_6(n), c_7 = c_7(n) \in \mathbb{R} \) with the following property. For all \( T \in \mathcal{P}_n(\mathbb{R}^2) \) and \( S \in \mathcal{P}(\mathbb{R}^2) \) such that \( |S| > c_6 \) and \( S \setminus T \) is not collinear, we have that

\[
|\{ L \in \mathcal{O}_2(S) : L \cap T = \emptyset \}| \geq \frac{1}{2} |S| - c_7.
\]

**Proof.** See \cite{4} Lemma 2.5. \( \square \)

### 3. Families \( \mathcal{N}_d(A) \)

The purpose of this section is to define and prove some properties of the families \( \mathcal{N}_d(A) \) which will be fundamental objects in the proof of the main results of this paper.

Let \( d \in \mathbb{Z}^+ \) and \( A \in \mathcal{P}(\mathbb{R}^2) \). We denote by \( \mathcal{N}_d(A) \) the family of subsets \( B \in \mathcal{P}^{(d+2)-3}(A) \) which have following four properties:

i) \( \dim \psi_d(B) = \left( \frac{d+2}{2} \right) - 4 \).

ii) For all \( e \in [1, d-1] \) and \( C \in \mathcal{C}_e \), we have that \( |B \cap C| < \left( \frac{d+2}{2} \right) - \left( \frac{d-e+2}{2} \right) \).

iii) For all \( e \in [1, d-1] \) and \( C \in \mathcal{C}_e \) such that \( |B \cap C| = \left( \frac{d+2}{2} \right) - \left( \frac{d-e+2}{2} \right) - 1 \), we have that \( \dim \psi_{d-e}(B \setminus C) = \left( \frac{d-e+2}{2} \right) - 3 \).

iv) For all \( e \in [1, d-1] \) and \( C \in \mathcal{C}_e \) such that \( |B \cap C| < \left( \frac{d+2}{2} \right) - \left( \frac{d-e+2}{2} \right) - 1 \), we have that \( \dim \psi_{d-e}(B \setminus C) > \left( \frac{d-e+2}{2} \right) - 3 \).

We will need more notation. For \( d \in \mathbb{Z}^+ \), \( e \in [1, d-1] \), \( A, B, C \in \mathcal{P}(\mathbb{R}^2) \) and \( D \in \mathcal{P}(B) \), write

\[
V_e(D) := \text{Fl}(\psi_e(D))
\]

\[
W_e(B, D) := \text{Fl}(\psi_{d-e}(B \setminus \psi_e^{-1}(V_e(D))))
\]

\[
\alpha_e(D) := \left( \frac{e + 2}{2} \right) - 2 - \dim V_e(D)
\]

\[
\beta_e(B, D) := \left( \frac{d - e + 2}{2} \right) - 3 - \dim W_e(B, D)
\]

\[
\gamma_e(B, D) := |V_e(D) \cap \psi_e(B)|
\]

\[
\mu_e(B, D) := \begin{cases} 
0 & \text{if } \alpha_e(D) < 0; \\
\alpha_e(D) + \gamma_e(B, D) + \left( \frac{d-e+2}{2} \right) & \text{if } \alpha_e(D) \geq 0.
\end{cases}
\]
The main results of this section are Lemma 3.5 and Lemma 3.9. Their proofs are regular if 

$$iii) \quad \begin{aligned} 
\text{Let} \quad & B, \quad A \\
\quad \text{in particular,} \quad & R \\
\quad \text{and} \quad & B \\
\quad \text{and} \quad & \begin{aligned} 
\pm 2 & \quad \pm 2 \\
\pm 3 & \quad \pm 3 \\
\pm 4 & \quad \pm 4 \\
\end{aligned} \\
\quad \text{if} \quad & \alpha_e(D) < 0; \\
\quad \text{if} \quad & \beta_e(B, D) < 0 \leq \alpha_e(D); \\
\quad \text{if} \quad & \beta_e(B, D) > 0 \text{ and} \quad \alpha_e(D) \geq 0. \\
\end{aligned}$$

$$I(B, C) := \{(f, E) \in [1, d - 1] \times \mathcal{P}(B) : C \not\subset U_f(B, E)\}$$

$$\mathcal{N}_d(A, B, C) := \{E \in \mathcal{N}_d(A) : B \in \mathcal{P}(E) \text{ and } E \cap C = E \setminus B\}.$$ 

The following facts are direct consequences of the definitions.

**Remark 3.1.** Let $d \in \mathbb{Z}^+$, $e \in [1, d - 1]$, $B \in \mathcal{P}(\mathbb{R}^2)$ and $D \in \mathcal{P}(B)$.

i) Note that

$$B \subseteq (B \cap \psi_e^{-1}(V_e(D))) \cup (B \cap \psi_{d-e}^{-1}(W_e(B, D))).$$

in particular,

$$|B| \leq |B \cap \psi_e^{-1}(V_e(D))| + |B \cap \psi_{d-e}^{-1}(W_e(B, D))|.$$ 

ii) Let $B_2 \in \mathcal{P}(\mathbb{R}^2)$, $B_1, D_2 \in \mathcal{P}(B_2)$ and $D_1 \in \mathcal{P}(B_1)$. If $V_e(D_1) \subseteq V_e(D_2) \neq \mathbb{R}^{(d+2)}$ and $W_e(B_1, D_1) = W_e(B_2, D_2)$, then

$$U_e(B_1, D_1) \subseteq U_e(B_2, D_2).$$ 

iii) Let $B_2 \in \mathcal{P}(\mathbb{R}^2)$, $B_1, D_2 \in \mathcal{P}(B_2)$ and $D_1 \in \mathcal{P}(B_1)$. If $V_e(D_1) = V_e(D_2)$, then $W_e(B_1, D_1) \subseteq W_e(B_2, D_2)$ and

$$U_e(B_1, D_1) \subseteq U_e(B_2, D_2).$$ 

iv) If $\min \{\alpha_e(D), \beta_e(B, D)\} \geq 0$, then

$$\tau_e(B, D) \leq \alpha_e(D) + \beta_e(B, D) + |B| + 3.$$ 

The main results of this section are Lemma 3.5 and Lemma 3.9. Their proofs are rather technical so, for the sake of comprehension, we sketch them before we state the auxiliary results that we need to prove them. We will say that $A \in \mathcal{P}(\mathbb{R}^2)$ is $d$-regular if $|A \cap C| \leq \frac{1}{2^d + d} |A|$ for all $C \subseteq C_d$.

- We start with Lemma 3.5. Assume that $A$ is $d$-regular and take $B_0 := \emptyset$.

  Given $\mathbf{b} = (b_1, b_2, \ldots, b_{(d+2)})_3 \in A^{(d+2)}$, we will construct recursively a chain $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_{(d+2)}$ with $B_{i+1} = B_i \cup \{b_{i+1}\}$ for each $i \in [0, (d+2) - 4]$. The idea is that for many $\mathbf{b} \in A^{(d+2)}$, we get $B_{(d+2)} \subseteq \mathcal{N}_d(A, \emptyset, \mathbb{R}^2)$. On the one hand, for each $i \in [0, (d+2) - 4]$, if $b_{i+1}$ is not in a forbidden subset $U_i$ of $\mathbb{R}^2$, then $B_{(d+2)} \in \mathcal{N}_d(A, \emptyset, \mathbb{R}^2)$; here we will use
Lemma 3.3 and Lemma 3.4. On the other hand, since $A$ is $d$-regular, for each $i \in [0, (d+2)/2 - 1]$, the intersection of the forbidden subset $U_i$ and $A$ is very small so $B_{(d+2)/2 - 3} \in \mathcal{N}_d(A, \emptyset, \mathbb{R}^2)$ almost always.

- Now we sketch the proof of Lemma 3.3. Given $B = (b_1, b_2, \ldots, b_{(d+2)/2 - 3}) \in (A \cap C_9)^{(d+2)/2 - (d+2)/2 - (d+2)/2 - 3}$, we will construct recursively a chain $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_{(d+2)/2 - (d+2)/2 - 3}$ with $B_{i+1} = B_i \cup \{b_{(i+1)+1}\}$ for each $i \in [0, (d+2) - (d+2)/2 - 4]$. As in Lemma 3.5 for many $B \in (A \cap C_9)^{(d+2)/2 - (d+2)/2 - 3}$, we get $B_{(d+2)/2 - 3} \in \mathcal{N}_d(A, B_0, C_0)$. As in the sketch of Lemma 3.5 we need that $b_{(d+2)/2 + 1}$ is not in a forbidden subset $U_i$ of $\mathbb{R}^2$ for each $i \in [0, (d+2) - (d+2)/2 - 4]$, most of auxiliary results of this section are used to show this. Now, in this case, we have that $A \cap U_i$ is very small because of Bezout’s theorem (applied to some curves that intersect $C_0$).

The next properties will shorten the proofs of some auxiliary results.

**Lemma 3.2.** Let $e, d \in \mathbb{Z}^+$ with $e \in [1, d - 1]$, $B \in \mathcal{P}(\mathbb{R}^2)$ and $D \in \mathcal{P}(B)$.

1. If $\alpha_e(D) \geq 0$, then there is $C \in \mathcal{C}_{\leq e}$ such that $\psi_e^{-1}(V_e(D)) \subseteq C$.
2. If $\beta_e(B, D) \geq -1$, then there is $C \in \mathcal{C}_{\leq d - e}$ such that $\psi_{d-e}^{-1}(W_{d-e}(D)) \subseteq C$.
3. If $|B| \leq \left(\frac{(d+2)}{2} - 1\right)$, then there is $C \in \mathcal{C}_{\leq d}$ such that $\psi_d^{-1}(V_d(B)) \subseteq C$.
4. If $|B| \leq \left(\frac{(d+2)}{2} - 1\right)$, then there are $C_1 \in \mathcal{C}_{\leq e}$, $C_2 \in \mathcal{C}_{\leq d - e}$ and $C_3 \in \mathcal{C}_{\leq d}$ such that $U_e(B, D) \subseteq C_1 \cup C_2 \cup C_3$.

**Proof.** First we show i). In so far as $\alpha_e(D) \geq 0$, notice that $V_e(D)$ is a proper flat in $\mathbb{R}^{(d+2)/2 - 1}$. Hence there exists a hyperplane $H$ containing $V_e(D)$. Remark 2.4 ii) implies that $\psi_e^{-1}(H)$ is in $\mathcal{C}_{\leq e}$ and it satisfies that $\psi_e^{-1}(V_e(D)) \subseteq \psi_e^{-1}(H)$, and this proves i).

We prove ii). Since $\beta_e(B, D) \geq -1$, $W_{d-e}(D)$ is a proper flat in $\mathbb{R}^{(d-e+2)/2 - 1}$. This implies that there exists a hyperplane $H$ containing $W_{d-e}(D)$. Remark 2.4 ii) yields that $\psi_d^{-1}(H)$ is in $\mathcal{C}_{\leq d - e}$ and it satisfies that $\psi_d^{-1}(W_{d-e}(D)) \subseteq \psi_d^{-1}(H)$, which proves ii).

To prove iii), note that since $|B| \leq \left(\frac{(d+2)}{2} - 1\right)$,

$$\dim V_d(B) = \dim \text{Fl}(\psi_d(B)) \leq |\psi_d(B)| - 1 = |B| - 1 \leq \left(\frac{d+2}{2}\right) - 2.$$ 

Thus there is a hyperplane $H$ containing $V_d(B)$, and then, by Remark 2.4 ii), $\psi_d^{-1}(H)$ is in $\mathcal{C}_{\leq d}$ and it satisfies that $\psi_d^{-1}(V_d(B)) \subseteq \psi_d^{-1}(H)$.

Finally, notice that iv) is a consequence of i), ii) and iii). \qed

**Lemma 3.3.** Let $d \in \mathbb{Z}^+$, $A, C \in \mathcal{P}(\mathbb{R}^2)$ and $B \in \mathcal{P}(A)$, and write $U := \bigcup_{(e, D) \in I(B, C)} U_e(B, D)$. Assume that $A \setminus \emptyset \neq \emptyset$ and $\max \{\tau_e(B, D), \mu_e(B, D)\} < \left(\frac{(d+2)}{2}\right)$ for all $(e, D) \in I(B, C)$. Then, for all $b \in A \setminus U$ and $(e, D) \in I(B \cup \{b\}, C)$,

$$\max \{\tau_e(B \cup \{b\}, D), \mu_e(B \cup \{b\}, D)\} < \left(\frac{d+2}{2}\right).$$
We divide the proof of the claim into two cases.

- First assume that \( b \notin D \). Then \( D \) is a subset of \( B \) and Remark 3.1.iii implies that \( U_\varepsilon(B, D) \subseteq U_\varepsilon(B \cup \{b\}, D) \); this implies that \((e, D) \in \mathcal{I}(B, C)\) since \((e, D) \in \mathcal{I}(B \cup \{b\}, C)\). The inclusion \((e, D) \in \mathcal{I}(B, C)\) leads to

\[
\max \{\tau_e(B, D), \mu_e(B, D)\} < \left(\frac{d + 2}{2}\right).
\]

Because \((e, D) \in \mathcal{I}(B, C)\), we have that \( b \notin U_\varepsilon(B, D) \). Notice that (22) yields that \( \psi_{\varepsilon}^{-1}(V_e(D)) \subseteq U_\varepsilon(B, D) \) so \( b \notin \psi_{\varepsilon}^{-1}(V_e(D)) \) and thereby

\[
\gamma_e(B \cup \{b\}, D) = |V_e(D) \cap \psi_e(B \cup \{b\})| = |V_e(D) \cap \psi_e(B)| = \gamma_e(B, D).
\]

Thus

\[
\mu_e(B \cup \{b\}, D) = \alpha_e(D) + \gamma_e(B \cup \{b\}, D) + \left(\frac{d - e + 2}{2}\right),
\]

(24)

\[
= \alpha_e(D) + \gamma_e(B, D) + \left(\frac{d - e + 2}{2}\right),
\]

(25)

\[
< \left(\frac{d + 2}{2}\right).
\]

If \( \beta_e(B, D) < 0 \), then \( \dim W_e(B, D) > \left(\frac{d - e + 2}{2}\right) - 3 \) so

\[
\dim W_e(B \cup \{b\}, D) \geq \dim W_e(B, D) > \left(\frac{d - e + 2}{2}\right) - 3,
\]

and therefore \( \beta_e(B \cup \{b\}, D) < 0 \); thus, if \( \beta_e(B, D) < 0 \), we have that \( \tau_e(B \cup \{b\}, D) = 0 \), and then by (25) we are done. From now on, we assume that

(26)

\[
\beta_e(B, D) \geq 0.
\]

From (22) and (26), note that \( \alpha_e(D), \beta_e(B, D) \geq 0 \) so, since \( b \notin U_\varepsilon(B, D) \), notice that \( b \notin \psi_{\varepsilon}^{-1}(V_e(D)) \) and \( b \notin \psi_{\varepsilon}^{-1}(W_e(B, D)) \); this leads to \( \psi_{1-\varepsilon}(b) \in W_e(B \cup \{b\}, D) \setminus W_e(B, D) \), and then \( W_e(B, D) \) is a flat properly contained in \( W_e(B \cup \{b\}, D) \) implying that

\[
\dim W_e(B \cup \{b\}, D) \geq 1 + \dim W_e(B, D),
\]

and therefore

\[
\beta_e(B \cup \{b\}, D) = \beta_e(B, D) + 1.
\]

(27)

If \( \beta_e(B \cup \{b\}, D) < 0 \) or \( \gamma_e(B \cup \{b\}, D) > \left(\frac{d + 2}{2}\right) - \left(\frac{d - e + 2}{2}\right) - 1 \), then \( \tau_e(B \cup \{b\}, D) = 0 \), and we are done by (25). Thus we assume that this is not the case, and then (24) and (27) lead to

\[
\beta_e(B, D) - 1 \geq \beta_e(B \cup \{b\}, D) \geq 0
\]

(28)

\[
\gamma_e(B, D) = \gamma_e(B \cup \{b\}, D) \leq \left(\frac{d + 2}{2}\right) - \left(\frac{d - e + 2}{2}\right) - 1.
\]

(29)
If \( \gamma_e(B \cup \{b\}, D) = \binom{d+e+2}{2} - \binom{d-\gamma+2}{2} - 1 \), then (29) leads to

\[
\gamma_e(B, D) = \gamma_e(B \cup \{b\}, D) = \left( \frac{d+2}{2} \right) - \left( \frac{d-\gamma+2}{2} \right) - 1,
\]

and hence

\[
\tau_e(B \cup \{b\}, D) = \alpha_e(D) + \beta_e(B \cup \{b\}, D) + |B \cup \{b\}| + 2 \quad \text{(by 22, 28, 31)}
\]
\[
\leq \alpha_e(D) + \beta_e(B, D) + |B| + 2 \quad \text{(by 28)}
\]
\[
= \tau_e(B, D). \quad \text{(by 22, 28, 31)}
\]

If \( \gamma_e(B \cup \{b\}, D) < \binom{d+e+2}{2} - \binom{d-\gamma+2}{2} - 1 \), then (29) leads to

\[
\gamma_e(B, D) = \gamma_e(B \cup \{b\}, D) < \left( \frac{d+2}{2} \right) - \left( \frac{d-\gamma+2}{2} \right) - 1,
\]

and hence

\[
\tau_e(B \cup \{b\}, D) = \alpha_e(D) + \beta_e(B \cup \{b\}, D) + |B \cup \{b\}| + 3 \quad \text{(by 22, 28, 31)}
\]
\[
\leq \alpha_e(D) + \beta_e(B, D) + |B| + 3 \quad \text{(by 28)}
\]
\[
= \tau_e(B, D). \quad \text{(by 22, 28, 31)}
\]

Thus, in any case, \( \tau_e(B \cup \{b\}, D) \leq \tau_e(B, D) \), and then (28) implies that

\[
\tau_e(B \cup \{b\}, D) \leq \tau_e(B, D) < \left( \frac{d+2}{2} \right).
\]

The claim follows from (26) and (32).

• Assume that \( b \in D \). Since \( D \setminus \{b\} \subseteq D \), we have that

\[
V_e(D \setminus \{b\}) \subseteq V_e(D),
\]

and then

\[
\alpha_e(D \setminus \{b\}) \geq \alpha_e(D).
\]

Now we claim that

\[
(V_e(D) \setminus V_e(D \setminus \{b\})) \cap \psi_e(B) = \emptyset.
\]

Indeed, if (35) is false, then there is \( a \in B \) such that \( \psi_e(a) \in V_e(D) \setminus V_e(D \setminus \{b\}) \). Since

\[
\dim V_e(D) - \dim V_e(D \setminus \{b\}) \leq |D| - |D \setminus \{b\}| = 1,
\]

we have that the flat generated by \( V_e(D \setminus \{b\}) \) and \( \psi_e(a) \) has to be \( V_e(D) \), but this flat is precisely \( \text{Fl}(V_e(D \setminus \{b\}) \cup \{\psi_e(a)\}) = V_e((D \setminus \{b\}) \cup \{a\}) \) so

\[
V_e(D) = V_e((D \setminus \{b\}) \cup \{a\}).
\]

From (34), we can apply Remark 3.1 iii to get that

\[
U_e(B, (D \setminus \{b\}) \cup \{a\}) \subseteq U_e(B \cup \{b\}, D).
\]

Since \( (e, D) \in I(B \cup \{b\}, C) \), we have that \( C \nsubseteq U_e(B \cup \{b\}, D) \). From (37), we get that \( C \nsubseteq U_e(B, (D \setminus \{b\}) \cup \{a\}) \) and hence \( (e, (D \setminus \{b\}) \cup \{a\}) \in I(B, C) \). Thus, in so far as \( b \notin U \), we get that \( b \notin U_e(B, (D \setminus \{b\}) \cup \{a\}) \).

From (22) and (36), note that \( \alpha_e((D \setminus \{b\}) \cup \{a\}) \geq 0 \) so \( \psi_{-1}(V_e((D \setminus \{b\}) \cup \{a\})) \subseteq V_e((D \setminus \{b\}) \cup \{a\}) \).
{a\}) \subseteq U_e(B, (D \setminus \{b\}) \cup \{a\}) and hence b \notin \psi^{-1}_e(V_e((D \setminus \{b\}) \cup \{a\})). This means that \psi_e(b) \notin V_e((D \setminus \{b\}) \cup \{a\}); however, b \in D so \psi_e(b) \in V_e(D) but this contradicts (30) and proves (35). Now, from (35),

\((\psi^{-1}_e(V_e(D)) \setminus \psi^{-1}_e(V_e(D \setminus \{b\}))) \cap B = \emptyset,

and thus \(B \setminus \psi^{-1}_e(V_e(D \setminus \{b\})) = (B \cup \{b\}) \setminus \psi^{-1}_e(V_e(D)).\) This gives

\begin{equation}
W_e(B, D \setminus \{b\}) = W_e(B \cup \{b\}, D),
\end{equation}

and therefore

\begin{equation}
\beta_e(B, D \setminus \{b\}) = \beta_e(B \cup \{b\}, D).
\end{equation}

From (22), (33) and (38), we can apply Remark 3.1ii and we get that

\begin{equation}
U_e(B, D \setminus \{b\}) \subseteq U_e(B \cup \{b\}, D).
\end{equation}

Since \((e, D) \in I(B \cup \{b\}, C),\) we have that \(C \notin U_e(B \cup \{b\}, D).\) Thus, from (40), we obtain that \(C \notin U_e(B, D \setminus \{b\})\) and hence \((e, D \setminus \{b\}) \in I(B, C).\) This implies by assumption that

\begin{equation}
\max \{\tau_e(B, D \setminus \{b\}), \mu_e(B, D \setminus \{b\})\} < \left(\frac{d + 2}{2}\right).
\end{equation}

Since \((e, D \setminus \{b\}) \in I(B, C)\) and \(b \notin U,\) we get that \(b \notin U_e(B, D \setminus \{b\}).\) From (22) and (34), we have that \(\alpha_e(D \setminus \{b\}) \geq 0\) so \(\psi^{-1}_e(V_e(D \setminus \{b\})) \subseteq U_e(B, D \setminus \{b\})\) and hence \(b \notin \psi^{-1}_e(V_e(D \setminus \{b\})).\) This means that

\begin{equation}
\psi_e(b) \notin V_e(D \setminus \{b\}),
\end{equation}

and thereby \(V_e(D \setminus \{b\})\) is a flat properly contained in \(V_e(D);\) in particular,

\begin{equation}
\alpha_e(D \setminus \{b\}) \geq \alpha_e(D) + 1.
\end{equation}

Then

\begin{equation}
\gamma_e(B \cup \{b\}, D) = |V_e(D) \cap \psi_e(B \cup \{b\})|
= |(V_e(D) \setminus V_e(D \setminus \{b\})) \cap \psi_e(B \cup \{b\})| + |V_e(D \setminus \{b\}) \cap \psi_e(B \cup \{b\})|
= 1 + |V_e(D \setminus \{b\}) \cap \psi_e(B \cup \{b\})| \quad \text{(by (35))}
= 1 + |V_e(D \setminus \{b\}) \cap \psi_e(B)| \quad \text{(by (42))}
= 1 + \gamma_e(B, D \setminus \{b\}).
\end{equation}

Then

\begin{equation}
\mu_e(B \cup \{b\}, D) = \alpha_e(D) + \gamma_e(B \cup \{b\}, D) + \left(\frac{d - e + 2}{2}\right) \quad \text{(by (22))}
\leq \alpha_e(D \setminus \{b\}) + \gamma_e(B, D \setminus \{b\}) + \left(\frac{d - e + 2}{2}\right) \quad \text{(by (35), (43))}
= \mu_e(B, D \setminus \{b\}) \quad \text{(by (22), (43))}
< \left(\frac{d + 2}{2}\right). \quad \text{(by (43))}
\end{equation}
Lemma 3.4. Let $\beta_e(B \cup \{b\}, D) < 0$ or $\gamma_e(B \cup \{b\}, D) > (d+2) - \frac{d-e+2}{2} - 1$, then $\tau_e(B \cup \{b\}, D) = 0$, and we are done by (45). Thus suppose that these inequalities are not true and then (39) and (44) give

\begin{align}
\beta_e(B, D \setminus \{b\}) &= \beta_e(B \cup \{b\}, D) \geq 0, \\
\gamma_e(B, D \setminus \{b\}) + 1 &= \gamma_e(B \cup \{b\}, D) \leq \left( \frac{d+2}{2} \right) - \left( \frac{d-e+2}{2} \right) - 1.
\end{align}

If $\gamma_e(B \cup \{b\}, D) = (d+2) - \frac{d-e+2}{2} - 1$, then (47) leads to

\begin{align}
\gamma_e(B, D \setminus \{b\}) + 1 &= \gamma_e(B \cup \{b\}, D) = \left( \frac{d+2}{2} \right) - \left( \frac{d-e+2}{2} \right) - 1,
\end{align}

and hence

\begin{align}
\tau_e(B \cup \{b\}, D) &= \alpha_e(D) + \beta_e(B \cup \{b\}, D) + |B \cup \{b\}| + 2 \quad \text{(by 22, 40, 48)} \\
&\leq \alpha_e(D \setminus \{b\}) + \beta_e(B, D \setminus \{b\}) + |B| + 2 \quad \text{(by 43, 46)} \\
&\leq \tau_e(B, D \setminus \{b\}) \quad \text{(by 22, 40, 48)}.
\end{align}

If $\gamma_e(B \cup \{b\}, D) < (d+2) - \frac{d-e+2}{2} - 1$, then (47) leads to

\begin{align}
\gamma_e(B, D \setminus \{b\}) + 1 &= \gamma_e(B \cup \{b\}, D) < \left( \frac{d+2}{2} \right) - \left( \frac{d-e+2}{2} \right) - 1,
\end{align}

and hence

\begin{align}
\tau_e(B \cup \{b\}, D) &= \alpha_e(D) + \beta_e(B \cup \{b\}, D) + |B \cup \{b\}| + 3 \quad \text{(by 22, 40, 49)} \\
&\leq \alpha_e(D \setminus \{b\}) + \beta_e(B, D \setminus \{b\}) + |B| + 3 \quad \text{(by 43, 46)} \\
&= \tau_e(B, D \setminus \{b\}) \quad \text{(by 22, 40, 49)}.
\end{align}

In any case, $\tau_e(B \cup \{b\}, D) \leq \tau_e(B, D \setminus \{b\})$, and then (41) leads to

\begin{align}
\tau_e(B \cup \{b\}, D) \leq \tau_e(B, D \setminus \{b\}) < \left( \frac{d+2}{2} \right).
\end{align}

The claim follows from (43) and (50). □

**Lemma 3.4.** Let $d \in \mathbb{Z}^+$, $A, C_0 \in \mathcal{P}(\mathbb{R}^2)$ and $B_0 \in \mathcal{P}(A \setminus C_0)$. Take $B \in \mathcal{P}_{\gamma_{d-3}}(A)$ which satisfies the following properties.

i) $\dim \psi_d(B) = \left( \frac{d+2}{2} \right) - 4$.

ii) $B \supseteq B_0$.

iii) $B \cap C_0 = B \setminus B_0$.

iv) For all $(e, D) \in I(B, C_0)$, we have that max$\{\tau_e(B, D), \mu_e(B, D)\} < \left( \frac{d+2}{2} \right)$.

v) For all $(e, D) \in \left( [1, d-1] \times \mathcal{P}(B) \right) \setminus I(B, C_0)$ with $\alpha_e(D) \geq 0$, we have that $\gamma_e(B, D) < \left( \frac{d+2}{2} \right) - \frac{d-e+2}{2}$ and $\beta_e(B, D) < 0$.

Then $B \in \mathcal{N}_d(A, B_0, C_0)$.

**Proof.** Because of the assumptions i), ii) and iii), it suffices to show that $B$ satisfies the conditions ii), iii) and iv) of the definition of $\mathcal{N}_d(A)$. First we show that for all
$e \in [1, d-1]$ and $C \in \mathcal{C}_e$, we have that
\begin{equation}
|C \cap B| < \left(\frac{d+2}{2}\right) - \left(\frac{d-e+2}{2}\right).
\end{equation}

Fix $p(x, y) \in \mathbb{R}_e[x, y]$ such that $C = \mathcal{Z}(p(x, y))$, and write $H := \tau_e([p(x, y)])$ and $D := B \cap \psi_e^{-1}(H)$. Hence $V_e(D) \subseteq H$ and Remark 2.4(ii) implies that $C = \psi_e^{-1}(H)$ so
\begin{equation}
|C \cap B| = |H \cap \psi_e(B)| = |V_e(D) \cap \psi_e(B)| = \gamma_e(B, D).
\end{equation}
In so far as $\dim V_e(D) \leq \dim H = \left(\frac{e+2}{2}\right) - 2$, we get that $\alpha_e(D) \geq 0$, and therefore
\begin{equation}
\mu_e(B, D) = \alpha_e(D) + \gamma_e(B, D) + \left(\frac{d-e+2}{2}\right).
\end{equation}
If $(e, D) \in I(B, C_0)$, then $\mu_e(B, D) < \left(\frac{d+2}{2}\right)$ by iv). Inso much as $\alpha_e(D) \geq 0$, we obtain from (53) that $\gamma_e(B, D) < \left(\frac{d+2}{2}\right) - \left(\frac{d-e+2}{2}\right)$. If $(e, D) \not\in I(B, C_0)$, then $\gamma_e(B, D) < \left(\frac{d+2}{2}\right) - \left(\frac{d-e+2}{2}\right)$ by v). Hence, in any case,
\begin{equation}
\gamma_e(B, D) < \left(\frac{d+2}{2}\right) - \left(\frac{d-e+2}{2}\right),
\end{equation}
and thus (51) follows from (52) and (54).

Now take $e \in [1, d-1]$ and $C \in \mathcal{C}_e$ such that
\begin{equation}
|C \cap B| = \left(\frac{d+2}{2}\right) - \left(\frac{d-e+2}{2}\right) - 1,
\end{equation}
and we will show that
\begin{equation}
\dim \psi_{d-e}(B \setminus C) = \left(\frac{d-e+2}{2}\right) - 3.
\end{equation}
Since
\begin{equation}
|B \setminus C| = |B| - |C \cap B| = \left(\frac{d-e+2}{2}\right) - 2,
\end{equation}
we have that
\begin{equation}
\dim \psi_{d-e}(B \setminus C) \leq |B \setminus C| - 1 = \left(\frac{d-e+2}{2}\right) - 3.
\end{equation}
Fix $p(x, y) \in \mathbb{R}_e[x, y]$ such that $C = \mathcal{Z}(p(x, y))$, and write $H := \tau_e([p(x, y)])$ and $D := B \cap \psi_e^{-1}(H)$. Hence $V_e(D) \subseteq H$ and Remark 2.4(ii) implies that $\psi_e(C) = H \cap \psi_e(\mathbb{R}^2)$ so
\begin{equation}
|C \cap B| = |\psi_e(C) \cap \psi_e(B)| = |H \cap \psi_e(B)| = |V_e(D) \cap \psi_e(B)| = \gamma_e(B, D),
\end{equation}
and then (55) implies that
\begin{equation}
\gamma_e(B, D) = \left(\frac{d+2}{2}\right) - \left(\frac{d-e+2}{2}\right) - 1.
\end{equation}
In so far as $\dim V_e(D) \leq \dim H = \left(\frac{e+2}{2}\right) - 2$, we get that
\begin{equation}
\alpha_e(D) \geq 0.
\end{equation}
Since $D = B \cap \psi_e^{-1}(H) = B \cap C$, we have that $B \setminus \psi_e^{-1}(V_e(D)) = B \setminus C$ so
\begin{equation}
W_e(B, D) = \text{Fl}(\psi_{d-e}(B \setminus \psi_e^{-1}(V_e(D)))) = \text{Fl}(\psi_{d-e}(B \setminus C)),
\end{equation}
and then
\[(60) \quad \dim W_e(B, D) = \dim \psi_{d-e}(B \setminus C).\]

From (67) and (60),
\[(61) \quad \beta_e(B, D) \geq 0.\]

From (63), (64) and (61),
\[(62) \quad \tau_e(B, D) = \alpha_e(D) + \beta_e(B, D) + |B| + 2.\]

From (59) and (61), \(\min\{\alpha_e(D), \beta_e(B, D)\} \geq 0\); then v) implies that \((e, D) \in I(B, C_0)\). Thus iv) implies that \(\tau_e(B, D) < \frac{(d+2)}{2}\), and then (59) and (62) gives \(\beta_e(B, D) < 1\). Now, from (61), \(\beta_e(B, D) = 0\) and then (60) yields (64).

Now take \(e \in [1, d-1]\) and \(C \in C_e\) such that
\[(63) \quad |C \cap B| < \left(\frac{d+2}{2}\right) - \left(\frac{d-e+2}{2}\right) - 1,
\]
and we will prove that
\[(64) \quad \dim \psi_{d-e}(B \setminus C) > \left(\frac{d-e+2}{2}\right) - 3.\]

Fix \(p(x, y) \in \mathbb{R}_e[x, y]\) such that \(C = Z(p(x, y))\), and write \(H := \tau_e([p(x, y)])\) and \(D := B \cap \psi_e^{-1}(H)\). Hence \(V_e(D) \subseteq H\) and Remark 2.4 ii implies that \(\psi_e(C) = H \cap \psi_e(\mathbb{R}^2)\) so
\[|C \cap B| = |\psi_e(C) \cap \psi_e(B)| = |H \cap \psi_e(B)| = |V_e(D) \cap \psi_e(B)| = \gamma_e(B, D),\]
and thus (63) gives
\[(65) \quad \gamma_e(B, D) < \left(\frac{d+2}{2}\right) - \left(\frac{d-e+2}{2}\right) - 1.\]

Since \(\dim V_e(D) \leq \dim H = \left(\frac{e+1}{2}\right) - 2\), we get that
\[(66) \quad \alpha_e(D) \geq 0.\]

Inasmuch as \(D = B \cap \psi_e^{-1}(H) = B \setminus C\), notice that \(B \setminus \psi_e^{-1}(V_e(D)) = B \setminus C\) so
\[W_e(B, D) = \text{Fl}(\psi_{d-e}(B \setminus \psi_e^{-1}(V_e(D)))) = \text{Fl}(\psi_{d-e}(B \setminus C)),\]
and thus
\[(67) \quad \dim W_e(B, D) = \dim \psi_{d-e}(B \setminus C).\]

We claim that
\[(68) \quad \beta_e(B, D) < 0.\]

Indeed, if (68) is false, then \(\beta_e(B, D) \geq 0\) and therefore \(\min\{\alpha_e(D), \beta_e(B, D)\} \geq 0\) by (60). Hence v) would imply that \((e, D) \in I(B, C_0)\) and thereby \(\tau_e(B, D) < \frac{(d+2)}{2}\) by iv). However, (65), (66) and \(\beta_e(B, D) \geq 0\) give
\[\tau_e(B, D) = \alpha_e(D) + \beta_e(B, D) + |B| + 3 \geq \left(\frac{d+2}{2}\right),\]
which is impossible. This means that (68) is true, and thereby (64) is a consequence of (67) and (68).

Recall that \(A \in \mathcal{P}(\mathbb{R}^2)\) is \(d\)-regular if \(|A \cap C| < \frac{1}{2^{d+3}}|A|\) for all \(C \in C_{\leq d}\).
Lemma 3.5. Let \( d \in \mathbb{Z} \) with \( d > 1 \) and \( A \) be \( d \)-regular. Then

\[
|\mathcal{N}_d(A, \emptyset, \mathbb{R}^2)| \geq \frac{1}{2d+3!} |A|^{\left(\frac{d+2}{2}\right)-3}.
\]

Proof. Set \( g := \left(\frac{d+2}{2}\right) - 3 \). Throughout this proof, for any \( \overline{b} \in A^g \), its \( i \)-th entry will be denoted by \( b_i \), i.e. \( \overline{b} = (b_1, b_2, \ldots, b_g) \). For any \( \overline{b} \in A^g \), write \( B_0(\overline{b}) := \emptyset \). Now, for all \( i \in [1, g] \) and \( \overline{b} \in A^g \), set

\[
B_i(\overline{b}) := \{b_1, b_2, \ldots, b_i\}
\]

\[
U_i(\overline{b}) := \bigcup_{(e, D) \in I(B_{i-1}(\overline{b}), \mathbb{R}^2)} U_e(B_{i-1}(\overline{b}), D).
\]

For all \( i \in [1, g] \) and \( \overline{b} \in A^g \), notice that \( |B_i(\overline{b})| \leq g < \left(\frac{d+2}{2}\right) - 1 \). Then Lemma 3.2iv implies that for all \( e \in [1, d-1] \) and \( D \in \mathcal{P}(B_i(\overline{b})) \), we get that \( U_e(\overline{b}, D) \) is contained in the union of 3 curves in \( C_{\leq d} \); in particular, \( U_e(B_i(\overline{b}), D) \) cannot contain \( \mathbb{R}^2 \), and therefore

\[
I (B_i(\overline{b}), \mathbb{R}^2) = [1, d-1] \times \mathcal{P}(B_i(\overline{b})).
\]

For each \( i \in [1, g] \), write \( R_i := \{\overline{b} \in A^g : b_i \in U_i(\overline{b})\} \). The first step in the proof is to show that for all \( i \in [1, g] \),

\[
|R_i| \leq \frac{1}{2^g} |A|^g.
\]

For each \( i \in [1, g] \), \( E \in \mathcal{P}([1, i]) \) and \( \overline{b} \in A^g \), set \( E(\overline{b}) := \{b_j : j \in E\} \). Notice that \( \mathcal{P}(B_i(\overline{b})) = \{E(\overline{b}) : E \in \mathcal{P}([1, i])\} \). Now, for each \( i \in [1, g] \), \( E \in \mathcal{P}([1, i]) \) and \( e \in [1, d-1] \), write \( R_{i,e} := \{\overline{b} \in A^g : b_i \in U_e(B_{i-1}(\overline{b}), E(\overline{b}))\} \). From (69), notice that for all \( i \in [1, g] \), we get that \( R_i = \bigcup_{E \in \mathcal{P}([1, i])} \bigcup_{e=1}^{d-1} R_{i,e} \) and then

\[
|R_i| \leq \sum_{E \in \mathcal{P}([1, i])} \sum_{e=1}^{d-1} |R_{i,e}|.
\]

For each \( i \in [1, g] \), \( E \in \mathcal{P}([1, i]) \) and \( \overline{b} \in A^g \), there exist \( C_1, C_2, C_3 \in C_{\leq d} \) such that \( U_e(B_{i-1}(\overline{b}), E(\overline{b})) \subseteq C_1 \cup C_2 \cup C_3 \) by Lemma 3.2iv; then

\[
A \cap U_e(B_{i-1}(\overline{b}), E(\overline{b})) \subseteq A \cap (C_1 \cup C_2 \cup C_3),
\]

and, since \( A \) is \( d \)-regular,

\[
|A \cap U_e(B_{i-1}(\overline{b}), E(\overline{b}))| \leq \sum_{j=1}^{3} |A \cap C_j| < \frac{3}{2^{2d+4}} |A| \leq \frac{1}{2^{2d+4}} |A|.
\]

Furthermore, in so far as \( g = \left(\frac{d+2}{2}\right) - 3 < 2^{d+2} \), we have that \( dg2^{d+1} \leq 2^{2d+4} \). Hence, inasmuch as the \( i \)-th entry of \( \overline{b} \in R_{i,e} \) has to be in \( A \cap U_e(B_{i-1}(\overline{b}), E(\overline{b})) \), (72) leads to

\[
|R_{i,e}| \leq |A|^{g-1} |A \cap U_e(B_{i-1}(\overline{b}), E(\overline{b}))| \leq \frac{1}{2^{2d+4}} |A|^g \leq \frac{1}{gd2^{d+1}} |A|^g.
\]

Since \( |\mathcal{P}([1, i])| \times [1, d-1] \leq d2^g \), we have that (71) and (73) imply (70).
Set $T := \{ b_i \in A^g : b_i \notin U_i(b) \text{ for all } i \in [1,g] \}$ so $T = A^g \setminus \bigcup_{i=1}^g R_i$. From (70),
\begin{equation}
|T| = \left| A^g \setminus \bigcup_{i=1}^g R_i \right| \geq |A|^g - \sum_{i=1}^g |R_i| \geq \frac{1}{2} |A|^g.
\end{equation}

The next step is to show that the map
\[ \phi : T \rightarrow \mathcal{P}(A), \quad \phi(b) = B_g(b) \]

satisfies that
\begin{equation}
\phi(T) \subseteq \mathcal{N}_d(A,\emptyset,\mathbb{R}^2).
\end{equation}

We claim that for all $b \in T$, $i \in [1,g]$ and $(e,D) \in [1,d-1] \times \mathcal{P}(B_i(b))$, we get that
\begin{equation}
\max \{ \tau_e(B_i(b),D), \mu_e(B_i(b),D) \} < \frac{d+2}{2}.
\end{equation}

We show (76) by induction on $i \in [1,g]$. For $i = 1$, notice that $B_1(b) = \{ b_1 \}$ and then $\alpha_e(D) \leq \binom{d+2}{2} - 2, \beta_e(B_1(b),D) \leq \binom{d+2}{2} - 3$ and $\gamma_e(B_1(b),D) \leq 1$ so
\[ \max \{ \tau_e(B_1(b),D), \mu_e(B_1(b),D) \} \leq \frac{e+2}{2} + \frac{d-e+2}{2} - 1 < \frac{d+2}{2}, \]
and then (76) follows in this case. Assume that (76) holds for $i-1 \geq 1$ and we show it for $i$. Since $b \in T$, note that $b_i \in A \setminus U_i(b)$. By induction,
\[ \max \{ \tau_e(B_{i-1}(b),D), \mu_e(B_{i-1}(b),D) \} < \frac{d+2}{2} \]

for all $(e,D) \in [1,d-1] \times \mathcal{P}(B_{i-1}(b))$ satisfying the assumptions of Lemma 8.3 and hence this lemma implies that $B_{i-1}(b) \cup \{ b_i \} = B_i(b)$ satisfies (76) completing the induction. In particular, (76) holds for $i = g$ so
\begin{equation}
\max \{ \tau_e(B_g(b),D), \mu_e(B_g(b),D) \} < \frac{d+2}{2}
\end{equation}

for all $b \in T$ and $(e,D) \in [1,d-1] \times \mathcal{P}(B_g(b))$. For all $b \in T$, we get that $b_i \notin U_i(b)$; in particular, $b_i \notin \psi_d^{-1}(V_d(B_{i-1}(b)))$. This means that the flats $V_d(B_1(b)) \subseteq V_d(B_2(b)) \subseteq \ldots \subseteq V_d(B_g(b))$ are contained properly so
\begin{equation}
\dim V_d(B_g(b)) \geq g - 1.
\end{equation}

On the other hand,
\[ \dim V_d(B_g(b)) = \dim \text{Fl}(\psi_d(B_g(b))) \leq |\psi_d(B_g(b))| - 1 = |B_g(b)| - 1 = g - 1 \]

so (78) leads to
\begin{equation}
\dim \psi_d(B_g(b)) = \dim V_d(B_g(b)) = g - 1 = \frac{d+2}{2} - 4.
\end{equation}

We want to apply Lemma 8.3 with $B = B_g(b), B_0 = \emptyset$ and $C_0 = \mathbb{R}^2$, and this can be done because its assumptions are satisfied (indeed: i) holds by (79); ii) and iii) are trivial; iv) holds by (69) and (77); v) never happens by (69)). Hence Lemma 8.3 implies that $B_g(b) \in \mathcal{N}_d(A,\emptyset,\mathbb{R}^2)$ and it proves (76).
To conclude the proof, note that for each \( B \in \phi(T) \), we have that \( \phi^{-1}(B) \subseteq \{(b_1, \ldots, b_g) \in A^g : \{b_1, \ldots, b_g\} = B\} \). Therefore, for all \( B \in \mathcal{N}_d(A, \emptyset, \mathbb{R}^2) \), we have that \( |\phi^{-1}(B)| \leq g! \), and then (74) and (75) yield that

\[
(80) \quad |\mathcal{N}_d(A, \emptyset, \mathbb{R}^2)| \geq \frac{1}{g!} |T| \geq \frac{1}{g!} |A|^g.
\]

In so far as \( g = \left(\frac{d+2}{2}\right) - 3 < 2^{d+2} \), we have that \( g! \leq 2^{d+3} \) and then (80) implies the claim of the lemma.

\[ \square \]

**Lemma 3.6.** Let \( d, f \in \mathbb{Z} \) be such that \( d \geq 3 \) and \( f \in [0, d-1] \), and \( B \in \mathcal{P}\left(\frac{d+2}{2}\right)(\mathbb{R}^2) \) be such that \( B \) is not contained in an element of \( C_{\leq f} \). Then, for all \( e \in [1, d-1] \) and \( D \in \mathcal{P}(B) \),

\[
\max\{\tau_e(B, D), \mu_e(B, D)\} < \left(\frac{d+2}{2}\right).
\]

**Proof.** Fix \( e \in [1, d-1] \) and \( D \in \mathcal{P}(B) \). If \( \alpha_e(D) < 0 \), then \( \tau_e(B, D) = \mu_e(B, D) = 0 \). Hence, from now on, we assume that

\[
(81) \quad \alpha_e(D) \geq 0.
\]

First we show that

\[
(82) \quad \mu_e(B, D) < \left(\frac{d+2}{2}\right).
\]

If \( e \geq f \), then Lemma \[2.7.i\] i leads to

\[
|\psi_e(B) \cap V_e(D)| \leq 1 + \dim V_e(D),
\]

and hence

\[
|\psi_e(B) \cap V_e(D)| + \alpha_e(D) \leq 1 + \dim V_e(D) + \alpha_e(D) = \left(\frac{e+2}{2}\right) - 1 \leq \left(\frac{d+1}{2}\right) - \left(\frac{d-e+1}{2}\right). \quad \text{(since } d-1 \geq e)\]

If \( e < f \), then Lemma \[2.7.ii\] ii leads to

\[
|\psi_e(B) \cap V_e(D)| \leq \left(\frac{f+2}{2}\right) - \left(\frac{f-e+2}{2}\right) - \left(\frac{e+2}{2}\right) + 2 + \dim V_e(D)
\]

so

\[
|\psi_e(B) \cap V_e(D)| + \alpha_e(D) \leq \left(\frac{f+2}{2}\right) - \left(\frac{f-e+2}{2}\right) \leq \left(\frac{d+1}{2}\right) - \left(\frac{d-e+1}{2}\right). \quad \text{(since } d-1 \geq f)\]

Thus, in any case,

\[
(83) \quad \gamma_e(B, D) + \alpha_e(D) = |\psi_e(B) \cap V_e(D)| + \alpha_e(D) \leq \left(\frac{d+1}{2}\right) - \left(\frac{d-e+1}{2}\right).
\]
We get (82) as follows.
\[ \mu_e(B, D) = \gamma_e(B, D) + \alpha_e(D) + \binom{d - e + 2}{2} \] (by (81))
\[ = \left(\frac{d + 1}{2}\right) - \left(\frac{d - e + 1}{2}\right) + \left(\frac{d - e + 2}{2}\right) \] (by (83))
\[ < \left(\frac{d + 2}{2}\right). \]

Now we prove that
\[ \tau_e(B, D) < \left(\frac{d + 2}{2}\right). \]

If \( \beta_e(B, D) < 0 \), then \( \tau_e(B, D) = 0 \) and in this case we are done by (82). Thus we assume that
\[ \beta_e(B, D) \geq 0. \]

Remark 3.1.i gives
\[ |B \cap \psi_e^{-1}(V_e(D))| + |B \cap \psi_{d-e}^{-1}(W_e(B, D))| \geq |B| = \left(\frac{f + 2}{2}\right). \]

From (81) and (85), Remark 3.1.iv yields that
\[ \tau_e(B, D) \leq \alpha_e(D) + \beta_e(B, D) + \binom{f + 2}{2} + 3. \]

We divide the conclusion of the proof of (84) into four cases.

- Assume that \( e \geq f \) and \( d - e \geq f \). We apply Lemma 2.7.i to \( V_e(D) \) and \( W_e(B, D) \) to get
\[ \alpha_e(D) \leq \left(\frac{e + 2}{2}\right) - 1 - |\psi_e(B) \cap V_e(D)| \]
\[ \beta_e(B, D) \leq \left(\frac{d - e + 2}{2}\right) - 2 - |\psi_{d-e}(B) \cap W_e(B, D)|. \]

From (85), (88) and (89), we get that
\[ \alpha_e(D) + \beta_e(B, D) \leq \left(\frac{e + 2}{2}\right) + \left(\frac{d - e + 2}{2}\right) - 3 - \left(\frac{f + 2}{2}\right). \]

Hence
\[ \tau_e(B, D) \leq \alpha_e(D) + \beta_e(B, D) + \binom{f + 2}{2} + 3 \] (by (87))
\[ \leq \left(\frac{e + 2}{2}\right) + \left(\frac{d - e + 2}{2}\right) \] (by (90))
\[ < \left(\frac{d + 2}{2}\right). \] (since \( d \geq 3 \))
Assume that $e \geq f$ and $d - e < f$. We apply Lemma 2.7.i to $V_e(D)$ and Lemma 2.7.ii to $W_e(B, D)$ so

$$\alpha_e(D) \leq \left(\frac{e + 2}{2}\right) - 1 - |\psi_e(B) \cap V_e(D)|$$

$$\beta_e(B, D) \leq \left(\frac{f + 2}{2}\right) - \left(\frac{f - (d - e) + 2}{2}\right) - 1 - |\psi_{d-e}(B) \cap W_e(B, D)|.$$  

We have two subcases.

* Suppose that $e = d - 1$. Then
  $$\beta_e(B, D) = \left(\frac{3}{2}\right) - 3 - \dim W_e(B, D) = - \dim W_e(B, D)$$

so (85) implies that

$$\beta_e(B, D) = \dim W_e(B, D) = 0.$$  

Since $\dim W_e(B, D) = 0$, we have that $W_e(B, D)$ is a point and hence (86) leads to

$$|B \cap \psi_e^{-1}(V_e(D))| \geq \left(\frac{f + 2}{2}\right) - 1.$$  

From (91), (93) and (94), we get that

$$\alpha_e(D) + \beta_e(B, D) = \alpha_e(D) \leq \left(\frac{e + 2}{2}\right) - \left(\frac{f + 2}{2}\right).$$  

Hence

$$\tau_e(B, D) \leq \alpha_e(D) + \beta_e(B, D) + \left(\frac{f + 2}{2}\right) + 3$$

$$\leq \left(\frac{e + 2}{2}\right) + 3$$

$$< \left(\frac{d + 2}{2}\right).$$  

(since $e + 1 = d \geq 3$)

* Suppose that $e < d - 1$. From (86), (91) and (92), we get that

$$\alpha_e(D) + \beta_e(B, D) \leq \left(\frac{e + 2}{2}\right) - \left(\frac{f - (d - e) + 2}{2}\right) - 2.$$  

Thus

$$\tau_e(B, D) \leq \alpha_e(D) + \beta_e(B, D) + \left(\frac{f + 2}{2}\right) + 3$$

$$\leq \left(\frac{e + 2}{2}\right) - \left(\frac{f - (d - e) + 2}{2}\right) + \left(\frac{f + 2}{2}\right) + 1$$

$$< \left(\frac{d + 2}{2}\right).$$  

(since $f, e + 1 \leq d - 1$)
Assume that $e < f$ and $d - e \geq f$. We apply Lemma 2.7.ii to $V_e(D)$ and Lemma 2.7.i to $W_e(B, D)$ so

\begin{equation}
\alpha_e(D) \leq \frac{f + 2}{2} - \frac{f - e + 2}{2} - |\psi_e(B) \cap V_e(D)| \tag{97}
\end{equation}

\begin{equation}
\beta_e(B, D) \leq \frac{d - e + 2}{2} - 2 - |\psi_{d-e}(B) \cap W_e(B, D)|. \tag{98}
\end{equation}

We have three subcases.

\textbf{⋆} Suppose that $e = 1$ and $\alpha_e(D) > 0$. Since $e = 1$,

\begin{equation}
\alpha_e(D) = \frac{3}{2} - 2 - \dim V_e(D) = 1 - \dim V_e(D) \tag{99}
\end{equation}

so the assumption $\alpha_e(D) > 0$ leads to

\begin{equation}
\alpha_e(D) - 1 = \dim V_e(D) = 0. \tag{100}
\end{equation}

Since $\dim V_e(D) = 0$, we have that $V_e(D)$ is just a point and thereby (86) gives

\begin{equation}
|B \cap \psi_{d-e}^{-1}(W_e(B, D))| \geq \frac{f + 2}{2} - 1. \tag{101}
\end{equation}

From (98), (100) and (101), we get that

\begin{equation}
\alpha_e(D) + \beta_e(B, D) = 1 + \beta_e(B, D) \leq \left( \frac{d - e + 2}{2} \right) - \left( \frac{f + 2}{2} \right). \tag{102}
\end{equation}

Hence

\begin{align*}
\tau_e(B, D) &\leq \alpha_e(D) + \beta_e(B, D) + \left( \frac{f + 2}{2} \right) + 3 \quad \text{(by 87)} \\
&\leq \left( \frac{d - e + 2}{2} \right) + 3 \quad \text{(by 102)} \\
&< \left( \frac{d + 2}{2} \right). \quad \text{(since $d \geq 3$)}
\end{align*}

\textbf{⋆} Suppose that $e = 1$ and $\alpha_e(D) = 0$. Proceeding as in (99), we conclude that

\begin{equation}
1 + \alpha_e(D) = \dim V_e(D) = 1. \tag{103}
\end{equation}

If $\gamma_e(B, D) > \left( \frac{d+2}{2} \right) - \left( \frac{d-e+2}{2} \right) - 1 = d$, then $\tau_e(B, D) = 0$. Thus we assume that $\gamma_e(B, D) \leq d$. If $\gamma_e(B, D) = d$, then (86) implies that

\begin{equation}
|\psi_{d-e}(B) \cap W_e(B, D)| \geq \left( \frac{f + 2}{2} \right) - \gamma_e(B, D) = \left( \frac{f + 2}{2} \right) - d, \tag{98}
\end{equation}

and then (98) and (103) lead to

\begin{equation}
\tau_e(B, D) = \alpha_e(D) + \beta_e(B, D) + \left( \frac{f + 2}{2} \right) + 2 \leq \left( \frac{d - e + 2}{2} \right) + d. \tag{104}
\end{equation}

If $\gamma_e(B, D) \leq d - 1$, then (86) gives

\begin{equation}
|\psi_{d-e}(B) \cap W_e(B, D)| \geq \left( \frac{f + 2}{2} \right) - \gamma_e(B, D) \geq \left( \frac{f + 2}{2} \right) - d + 1, \tag{99}
\end{equation}

and then (98) and (103) lead to

\begin{equation}
\tau_e(B, D) = \alpha_e(D) + \beta_e(B, D) + \left( \frac{f + 2}{2} \right) + 2 \leq \left( \frac{d - e + 2}{2} \right) + d. \tag{105}
\end{equation}

If $\gamma_e(B, D) \leq d - 1$, then (86) gives

\begin{equation}
|\psi_{d-e}(B) \cap W_e(B, D)| \geq \left( \frac{f + 2}{2} \right) - \gamma_e(B, D) \geq \left( \frac{f + 2}{2} \right) - d + 1, \tag{106}
\end{equation}

and then (98) and (103) lead to

\begin{equation}
\tau_e(B, D) = \alpha_e(D) + \beta_e(B, D) + \left( \frac{f + 2}{2} \right) + 2 \leq \left( \frac{d - e + 2}{2} \right) + d. \tag{107}
\end{equation}

If $\gamma_e(B, D) \leq d - 1$, then (86) gives

\begin{equation}
|\psi_{d-e}(B) \cap W_e(B, D)| \geq \left( \frac{f + 2}{2} \right) - \gamma_e(B, D) \geq \left( \frac{f + 2}{2} \right) - d + 1, \tag{108}
\end{equation}

and then (98) and (103) lead to

\begin{equation}
\tau_e(B, D) = \alpha_e(D) + \beta_e(B, D) + \left( \frac{f + 2}{2} \right) + 2 \leq \left( \frac{d - e + 2}{2} \right) + d. \tag{109}
\end{equation}
and then (98) and (103) imply that
\[ \tau_e(B, D) = \alpha_e(D) + \beta_e(B, D) + \left( \frac{f+2}{2} \right) + 3 \leq \left( \frac{d-e+2}{2} \right) + d. \]

In any case,
\[ \tau_e(B, D) \leq \left( \frac{d+1}{2} \right) + d < \left( \frac{d+2}{2} \right). \]

- Suppose that \( e > 1 \). From (86), (97) and (98), we get that
\[ (104) \quad \alpha_e(D) + \beta_e(B, D) \leq \left( \frac{d-e+2}{2} \right) - \left( \frac{f-e+2}{2} \right) - 2. \]

Thus
\[ \tau_e(B, D) \leq \alpha_e(D) + \beta_e(B, D) + \left( \frac{f+2}{2} \right) + 3 \quad \text{(by (87))} \]
\[ \leq \left( \frac{d-e+2}{2} \right) - \left( \frac{f-e+2}{2} \right) + \left( \frac{f+2}{2} \right) + 1 \quad \text{(by (104))} \]
\[ < \left( \frac{d+2}{2} \right). \quad \text{(since } e > 1) \]

- Assume that \( e < f \) and \( d-e < f \). The definitions of \( W_e(B, D) \) and \( V_e(D) \) lead to \( W_e(B, D) \supseteq \psi_{d-e}(B \setminus \psi^{-1}_e(V_e(D))) \) and \( V_e(D) \supseteq \psi_e(B \setminus \psi_{d-e}^{-1}(W_e(B, D))) \). Then we get that \( \dim W_e(B, D) \geq \dim \psi_{d-e}(B \setminus \psi^{-1}_e(V_e(D))) \) and \( \dim V_e(D) \geq \dim \psi_e(B \setminus \psi_{d-e}^{-1}(W_e(B, D))) \). Hence, as a consequence of Lemma 2.7.iii applied to \( V_e(D) \) and \( W_e(B, D) \), we get that
\[ (105) \quad \alpha_e(D) \leq \left( \frac{e+2}{2} \right) - 1 - \left( \frac{f-(d-e)+2}{2} \right) \]
\[ (106) \quad \beta_e(B, D) \leq \left( \frac{d-e+2}{2} \right) - \left( \frac{f-e+2}{2} \right) - 2. \]

Using that \( \binom{g+2}{2} = \sum_{i=1}^{g} i \) for any \( g \in \mathbb{Z}_0^+ \), it is proven easily that
\[ (107) \quad \left( \frac{e+2}{2} \right) + \left( \frac{d-e+2}{2} \right) + \left( \frac{f+2}{2} \right) - \left( \frac{f-(d-e)+2}{2} \right) - \left( \frac{f-e+2}{2} \right) < \left( \frac{d+2}{2} \right). \]

Hence
\[ \tau_e(B, D) \leq \alpha_e(D) + \beta_e(B, D) + \left( \frac{f+2}{2} \right) + 3 \quad \text{(by (87))} \]
\[ \leq \left( \frac{e+2}{2} \right) + \left( \frac{d-e+2}{2} \right) + \left( \frac{f+2}{2} \right) \]
\[ - \left( \frac{f-(d-e)+2}{2} \right) - \left( \frac{f-e+2}{2} \right) \quad \text{(by (105), (106))} \]
\[ < \left( \frac{d+2}{2} \right). \quad \text{(by (107))} \]

This concludes the proof of (84). The claim is a consequence of (82) and (84). □
Lemma 3.7. Let \( d \in \mathbb{Z}^+, f \in [0, d - 1] \), \( C_0 \in C_{d-f} \) be irreducible and \( B_0 \in \mathcal{P}(\mathbb{R}^2) \) such that \( B_0 \) is not contained in an element of \( C_{\leq f} \). Take \( B \in \mathcal{P}(\mathbb{R}^2) \) such that \( B \supseteq B_0 \) and \( B \cap C_0 = B \setminus B_0 \), and also take \( (e, D) \in ([1, d - 1] \times \mathcal{P}(B)) \setminus I(B, C_0) \) such that \( \alpha_e(D) \geq 0 \). Then \( \beta_e(B, D) < 0 \) and \( \gamma_e(B, D) < \left( \frac{d^2}{2} \right) - \left( \frac{d - e + 2}{2} \right) - 2 \).

Proof. Since \( \alpha_e(D) \geq 0 \),

\[
U_e(B, D) = \begin{cases} 
\psi_d^{-1}(V_d(B)) \cup \psi_e^{-1}(V_e(D)) & \text{if } \beta_e(B, D) < 0 \\
\psi_d^{-1}(V_d(B)) \cup \psi_e^{-1}(V_e(D)) \cup \psi_{d-e}^{-1}(W_e(B, D)) & \text{if } \beta_e(B, D) \geq 0.
\end{cases}
\]

In so far as \( |B| < \left( \frac{d+2}{2} \right) - 1 \), Lemma 3.2.\( \text{ii} \) implies that there is \( C_1 \in C_{\leq d} \) such that \( \psi_d^{-1}(V_d(B)) \subseteq C_1 \). Since \( \alpha_e(D) \geq 0 \), Lemma 3.2.\( \text{i} \) yields the existence of \( C_2 \in C_{\leq d} \) such that \( \psi_e^{-1}(V_e(D)) \subseteq C_2 \). Now, if \( \beta_e(B, D) < 0 \), write \( C_3 = \emptyset \), and if \( \beta_e(B, D) \geq 0 \), we fix \( C_3 \in C_{d-e} \) such that \( \psi_{d-e}^{-1}(W_e(B, D)) \subseteq C_3 \) (which exists by Lemma 3.2.\( \text{ii} \)). Thus, in any case,

\[
U_e(B, D) \subseteq C_1 \cup C_2 \cup C_3.
\]

In so far as \( (e, D) \notin I(B, C_0) \), note that \( C_0 \subseteq U_e(B, D) \). From (108), we have that \( C_0 \) is an irreducible component of \( C_1 \) or \( C_2 \cup C_3 \). We claim that

\[
C_0 \subseteq C_2 \cup C_3.
\]

If (109) is false, then \( C_0 \) is a component of \( C_1 \). Lemma 2.9 applied to \( C_0 \) and \( C_1 \) gives

\[
|B \cap C_1| < \left( \frac{d+2}{2} \right) - \left( \frac{d - d + 2}{2} \right) - 2 = \left( \frac{d+2}{2} \right) - 3.
\]

However,

\[
B \subseteq B \cap \psi_d^{-1}(V_d(B)) \subseteq B \cap C_1;
\]

hence \( |B \cap C_1| = |B| = \left( \frac{d+2}{2} \right) - 3 \) contradicting (110), and this proves (109).

Now we show that

\[
\beta_e(B, D) < 0.
\]

If (111) is false, then \( C_3 \neq \emptyset \). Since \( C_2 \in C_{\leq \phi} \) and \( C_3 \in C_{d-e} \), notice that \( C_2 \cup C_3 \in C_{\leq d} \). From (109), we can apply Lemma 2.9 to \( C_0 \) and \( C_2 \cup C_3 \) so

\[
|B \cap (C_2 \cup C_3)| < \left( \frac{d+2}{2} \right) - 3.
\]

From Remark 3.1.\( \text{i} \),

\[
B \subseteq \psi_e^{-1}(V_e(D)) \cup \psi_{d-e}^{-1}(W_e(B, D)) \subseteq C_2 \cup C_3.
\]

Hence \( |B \cap (C_2 \cup C_3)| = |B| = \left( \frac{d+2}{2} \right) - 3 \) contradicting (112), and this proves (111).

Finally, we show that

\[
\gamma_e(B, D) < \left( \frac{d+2}{2} \right) - \left( \frac{d - e + 2}{2} \right) - 2.
\]

From (111), we get that \( C_3 = \emptyset \). From (109), \( C_0 \subseteq C_2 \). Thus, since \( C_0 \) is irreducible, \( e \geq d - f \). Applying Lemma 2.9 to \( C_0 \) and \( C_2 \),

\[
|B \cap C_2| < \left( \frac{d+2}{2} \right) - \left( \frac{d - e + 2}{2} \right) - 2.
\]
From (114),
\[ \gamma_e(B, D) = |B \cap \psi_e^{-1}(V_e(D))| \leq |B \cap C_2| < \left( \frac{d+2}{2} \right) - \left( \frac{d-e+2}{2} \right) - 2, \]
and this proves (113). The claim of the lemma is a consequence of (111) and (113). □

**Lemma 3.8.** Let \( d \in \mathbb{Z}^+ \), \( f \in [0, d-1] \), \( C_0 \in \mathbb{C} \leq d \) be irreducible and \( B \in \mathcal{P}(\mathbb{R}^2) \) be such that \(|B| \leq \left( \frac{d^2+2}{2} \right) - 1 \). Then, for all \((e, D) \in I(B, C_0)\),
\[ |C_0 \cap U_e(B, D)| \leq 2d^2. \]

**Proof.** Fix \( p(x, y) \in \mathbb{R}_d[x, y] \) such that \( C_0 = Z(p(x, y)) \). Write \( F_1 := V_d(B), F_2 := V_e(D) \) and \( F_3 := W_e(B, D) \), and also set \( d_1 := d, d_2 := e \) and \( d_3 := d - e \). Depending on the values of \( \alpha_e(D) \) and \( \beta_e(B, D) \), there is a subset \( I \) of \([1, 3]\) such that \( U_e(B, D) = \bigcup_{i \in I} \psi_{d_i}^{-1}(F_i) \) with \( F_i \) a proper flat in \( \mathbb{R}^{(d^2+i^2)-1} \) for each \( i \in I \) (here we use the assumption \(|B| \leq \left( \frac{d^2+2}{2} \right) - 1 \) to warranty that \( \dim F_i \leq |B| - 1 < \left( \frac{d^2+2}{2} \right) - 1 \)). Since \((e, D) \in I(B, C_0)\), we get that
\[ C_0 \not\subseteq U_e(B, D) = \bigcup_{i \in I} \psi_{d_i}^{-1}(F_i). \]

Thus \( C_0 \not\subseteq \psi_{d_i}^{-1}(F_i) \) and therefore \( \psi_{d_i}(C_0) \not\subseteq F_i \) for each \( i \in I \). Since \( F_i \) is a proper flat in \( \mathbb{R}^{(d^2+i^2)-1} \) and \( \psi_{d_i}(C_0) \not\subseteq F_i \), there exists a hyperplane \( H_i \) in \( \mathbb{R}^{(d^2+i^2)-1} \) such that \( H_i \supseteq F_i \) and \( \psi_{d_i}(C_0) \not\subseteq H_i \). For each \( i \in I \), set \( C_i := \psi_{d_i}^{-1}(H_i) \) and note that \( C_0 \) is not a component of \( C_i \). Thus Theorem 2.1 applied to each intersection \( |C_0 \cap C_i| \) yields
\[ |C_0 \cap U_e(B, D)| \leq \sum_{i \in I} |C_0 \cap C_i| \leq d(d_1 + d_2 + d_3) = 2d^2, \]
and this concludes the proof. □

**Lemma 3.9.** Let \( d \in \mathbb{Z} \) be such that \( d \geq 3, f \in [0, d-1], C_0 \in \mathbb{C} \leq d-f \) be irreducible and \( A \in \mathcal{P}(\mathbb{R}^2) \) be such that \(|A \cap C_0| \geq d^5\!2^{(d^2+2)} \) and \( A \) is not contained in an element of \( \mathbb{C} \leq d-f \). For all \( B_0 \in \mathcal{P}(f^2+2)(A \cap C_0) \) such that \( B_0 \) is not contained in an element of \( \mathbb{C} \leq f \), we have that
\[ |\mathcal{N}_{d}(A, B_0, C_0)| \geq \frac{1}{2d^3+1}|A \cap C_0|(d^2+2)\prod (f^2+2) \prod. \]

**Proof.** Write \( g := (d^2+2) - 3 - (f^2+2), A_0 := A \cap C_0 \) and \( B_0 := \left\{ b_1, b_2, \ldots, b_{(f^2+2)} \right\} \). In this proof, for any \( \mathbf{B} \in A_0^f \), its \( i \)-th entry will be denoted by \( b_{(f^2+2)+i} \), i.e.
\[ \mathbf{B} = \left( b_{(f^2+2)+1}, b_{(f^2+2)+2}, \ldots, b_{(d^2+2)-3} \right). \]
For any \( \mathbf{B} \in A_0^f \), write \( B_0(\mathbf{B}) := B_0 \). For all \( i \in [1, g] \) and \( \mathbf{B} \in A_0^g \), write
\[ B_i(\mathbf{B}) := B_0 \cup \left\{ b_{(f^2+2)+i}, b_{(f^2+2)+2+i}, \ldots, b_{(d^2+2)-3+i} \right\} = \left\{ b_1, b_2, \ldots, b_{(f^2+2)+i} \right\} \]
\[ U_i(\mathbf{B}) := \bigcup_{(e, D) \in I(B_i-1(\mathbf{B}), C_0)} U_e(B_i-1(\mathbf{B}), D). \]
For all $\mathbf{b} \in A_0^g$, $i \in [1,g]$ and $(e,D) \in I(B_{i-1}((\mathbf{b}),C_{0})$, we have that $C_0 \subseteq U_\epsilon(B_{i-1}((\mathbf{b}),D)$. Thus, applying Lemma 3.8 to $C_0$ and $U_\epsilon(B_{i-1}((\mathbf{b}),D)$, we get that

$$|C_0 \cap U_\epsilon(B_{i-1}((\mathbf{b}),D)| \leq 2d^2. \quad (115)$$

For each $i \in [1,g]$, write $R_i := \{ b \in A_0^g : b_{(i, \frac{i+1}{2})} \in U_i((\mathbf{b})) \}$. First we show that for all $i \in [1,g],$$
|R_i| \leq d^32^{(d+2)-2}|A_0|^{g-1}. \quad (116)

For each $i \in [1,g], E \in \mathcal{P}(\left[1, \left(\frac{i+2}{2}\right)\right])$ and $\mathbf{b} \in A_0^g$, set $E(\mathbf{b}) := \{ b_j : j \in E \}$. Notice that $\mathcal{P}(B_0((\mathbf{b})) = \{ E(\mathbf{b}) : E \in \mathcal{P}(\left[1, \left(\frac{i+2}{2}\right)\right]) \}$. Now, for each $i \in [1,g], E \in \mathcal{P}(\left[1, \left(\frac{i+2}{2}\right)\right])$ and $e \in [1,d-1]$, write

$$R_{i,e,E} := \{ b \in A_0^g : b_{(i, \frac{i+1}{2})} \in U_e(B_{i-1}((\mathbf{b}), E(\mathbf{b}))) \} \text{ and } (e, E(\mathbf{b})) \in I(B_{i-1}((\mathbf{b}), C_{0}) \}$$

This definition gives that for all $i \in [1,g],$$
R_i \subseteq \bigcup_{e=1}^{d-1} E(\mathcal{P}(\left[1, \left(\frac{i+2}{2}\right)\right]) \bigcup R_{i,e,E}

and thereby

$$|R_i| \leq \sum_{E \in \mathcal{P}(\left[1, \left(\frac{i+2}{2}\right)\right]) \sum_{e=1}^{d-1}|R_{i,e,E}|. \quad (117)$$

For each $i \in [1,g], E \in \mathcal{P}(\left[1, \left(\frac{i+2}{2}\right)\right])$ and $e \in [1,d-1]$, notice that if $\mathbf{b} \in R_{i,e,E}$ then $b_{(i, \frac{i+1}{2})} \in U_e(B_{i-1}((\mathbf{b}), E(\mathbf{b}))) \cap A_0 \subseteq U_\epsilon(B_{i-1}((\mathbf{b}), E(\mathbf{b}))) \cap C_0$; thus, by (115), $\mathbf{b} \in R_{i,e,E}$ has at most $2d^2$ possible values in its $i$-th entry and therefore

$$|R_{i,e,E}| \leq 2d^2|A_0|^{g-1}. \quad (118)$$

Since $|\mathcal{P}(\left[1, \left(\frac{i+2}{2}\right)\right]) \times [1,d-1]| = d2^{(d+2)-3}$, we have that (117) and (118) imply (116).

Set $T := \{ \mathbf{b} \in A_0^g : b_{(i, \frac{i+1}{2})} \notin U_i((\mathbf{b})) \text{ for all } i \in [1,g] \}$, so $T = A_0^g \setminus \bigcup_{i=1}^g R_i$. From (116),

$$\sum_{i=1}^g |R_i| \leq gd^32^{(d+2)-2}|A_0|^{g-1} \leq d^52^{(d+2)-1}|A_0|^{g-1}. \quad (119)$$

Since $|A_0| \geq d^52^{(d+2)}$ by assumption, we get from (119) that

$$|T| = |A_0^g \setminus \bigcup_{i=1}^g R_i| \geq |A_0|^g - \sum_{i=1}^g |R_i| \geq |A_0|^g - d^52^{(d+2)-1}|A_0|^{g-1} \geq \frac{1}{2}|A_0|^g. \quad (120)$$

The second step in the proof is to show that the map $\phi : T \rightarrow \mathcal{P}(A)$, $\phi((\mathbf{b})) = B_0((\mathbf{b}))$
We claim that for all $\phi(T) \subseteq N_d(A, B_0, C_0)$.

We show (122) by induction on $i \in [0, g]$. For $i = 0$, we apply Lemma 3.6 to $B_0 = B_0(\overline{b})$. Assume that (122) holds for $i - 1 \geq 0$ and we show it for $i$. In so far as $\overline{b} \in T$, note that $b_t \in A \setminus U_i(\overline{b})$. By induction,

$$\max \{\tau_e(B_i(\overline{b}), D), \mu_e(B_i(\overline{b}), D)\} < \left(\frac{d + 2}{2}\right).$$

We have that (121) is satisfied (123) and that (123) is true for $i = g$ so, for all $b_t \in T$ and $(e, D) \in I(B_d(\overline{b}), C_0)$,

$$\max \{\tau_e(B_g(\overline{b}), D), \mu_e(B_g(\overline{b}), D)\} < \left(\frac{d + 2}{2}\right).$$

Now, for all $(e, D) \in ([1, d - 1] \times P(B_d(\overline{b}))) \setminus I(B_d(\overline{b}), C_0)$ such that $\alpha_e(D) \geq 0$, Lemma 3.7 implies that

$$\beta_e(B, D) < 0$$

and

$$\gamma_e(B, D) < \left(\frac{d + 2}{2}\right) - \left(\frac{d - e + 2}{2}\right) - 2.$$

For all $\overline{b} \in T$ and $i \in [1, g]$, we get that $b_{\frac{e_0 + 1}{2} + i} \notin U_i(\overline{b})$; in particular, $b_{\frac{e_0 + 1}{2} + i} \notin \psi_d^{-1}(V_d(B_i(\overline{b})))$. Then the flats $V_d(B_0(\overline{b})) \subseteq V_d(B_1(\overline{b})) \subseteq \ldots \subseteq V_d(B_d(\overline{b}))$ are contained properly so

$$\dim V_d(B_g(\overline{b})) \geq g + \dim V_d(B_0(\overline{b})).$$

Since $B_0 = B_0(\overline{b})$ is not contained in an element of $C_{\leq f}$, we get that $\psi_f(B_0(\overline{b}))$ is not contained in a hyperplane of $\mathbb{R}^{\frac{e_0 + 1}{2} - 1}$ so $\dim \psi_f(B_0(\overline{b})) = \left(\frac{f + 2}{2}\right) - 1$. From Remark 2.4 iii, we get that $\dim \psi_d(B_0(\overline{b})) \geq \dim \psi_f(B_0(\overline{b}))$ so

$$\dim V_d(B_0(\overline{b})) = \dim \psi_d(B_0(\overline{b})) \geq \dim \psi_f(B_0(\overline{b})) = \left(\frac{f + 2}{2}\right) - 1,$$

and then (126) gives

$$\dim V_d(B_g(\overline{b})) \geq \left(\frac{d + 2}{2}\right) - 4.$$
We can apply Lemma 3.4 with $B = B_g(\mathcal{B})$, $B_0$ and $C_0$ because its assumptions are satisfied (indeed: i) holds by (128); ii) and iii) hold by the construction of $B_g(\mathcal{B})$; iv) holds by (129); v) holds by (124) and (125)). Hence Lemma 3.4 implies that $B_g(\mathcal{B}) \in \mathcal{N}_d(A, B_0, C_0)$ and this shows (121).

Finally, for each $B \in \mathcal{N}_d(A, B_0, C_0)$, $|\phi^{-1}(B)| \leq g!$ which is the number of permutations of the elements in $B \setminus B_0$. Therefore (120) and (121) give

$$|\mathcal{N}_d(A, B_0, C_0)| \geq \frac{1}{g!} |T| \geq \frac{1}{g!^2} |A_0|^2.$$}

Since $g \leq \binom{d+2}{2} - 3 < 2^{d+2}$, we have that $g!2 \leq 2^{d+3}$ and hence (129) completes the proof. \hfill \Box

## 4. From curves to lines

In this section we use the elements of the families $\mathcal{N}_d(A)$ to generate flats that will be used to construct hyperprojections which are used to transform the problem of finding curves of degree $d$ with few points into a problem of finding ordinary lines which avoid a finite set.

Let $d \in \mathbb{Z}^+$, $e \in [1, d-1]$ and $B \in \mathcal{P}(\mathbb{R}^2)$. Write

$$C_{d,e}(B) := \left\{ C \in \mathcal{C}_e : |C \cap B| = \binom{d+2}{2} - \binom{d-e+2}{2} - 1 \right\}$$

$$C_d(B) := \bigcup_{f=1}^{d-1} C_{d,f}(B).$$

**Lemma 4.1.** Let $d \in \mathbb{Z}^+$ and $e \in [1, d-1]$. For any $B \in \mathcal{N}_d(\mathbb{R}^2)$,

$$|C_{d,e}(B)| < 2^{2d+2}.$$}

**Proof.** Since $B \in \mathcal{N}_d(\mathbb{R}^2)$, we have that for any $f \in [1, d-1]$ and $C \in \mathcal{C}_f$,

$$|C \cap B| < \binom{d+2}{2} - \binom{d-f+2}{2} \leq \binom{d+2}{2} - 3 = |B|;$$}

in particular, this means that there is no element $C \in \mathcal{C}_{\leq e}$ which contains $B$. Thus, from Remark 2.3 ii), no hyperplane in $\mathbb{R}_{(e+2)}^{\binom{e+2}{2}}$ can contain $\psi_e(B)$ and therefore

$$\dim \psi_e(B) = \binom{e+2}{2} - 1.$$}

Set the map

$$\phi : C_{d,e}(B) \longrightarrow \mathcal{P}_{\binom{d+2}{2} - \binom{d-e+2}{2} - 1}(B), \quad \phi(C) = C \cap B$$

We will show that $\phi$ is injective. Take $C_1, C_2 \in C_{d,e}(B)$ such that $C_1 \cap B = C_2 \cap B$. Write $F := \text{Fl}(\psi_e(C_1 \cap B)) = \text{Fl}(\psi_e(C_2 \cap B))$. Because $C_1, C_2 \in \mathcal{C}_{\leq e}$, we have that $\text{Fl}(\psi_e(C_1))$ and $\text{Fl}(\psi_e(C_2))$ are contained in hyperplanes by Remark 2.3 ii). For $i \in \{1, 2\},$

$$F \subseteq \text{Fl}(\psi_e(C_i))$$

so $F$ is a proper flat in $\mathbb{R}_{(e+2)}^{\binom{e+2}{2}}$. We claim that $F$ is a hyperplane. If $F$ is not a hyperplane, then (130) implies that we can choose a hyperplane $H$ in $\mathbb{R}_{(e+2)}^{\binom{e+2}{2}}$ such that $H \supseteq F$ and

$$|H \cap \psi_e(B)| > |F \cap \psi_e(B)|.$$}
However,
\[
\left( \frac{d + 2}{2} \right) - \left( \frac{d - e + 2}{2} \right) - 1 \geq |\psi^{-1}(H) \cap B| \\
= |H \cap \psi_e(B)| \\
> |F \cap \psi_e(B)| \quad \text{(by (132))} \\
\geq |C_1 \cap B| \\
= \left( \frac{d + 2}{2} \right) - \left( \frac{d - e + 2}{2} \right) - 1, \quad \text{(since } C_1 \in C_{d,e}(B))
\]
and this contradiction shows that \( F \) is a hyperplane. Hence, since \( F \) is a hyperplane, (131) leads to
\[
\text{Fl}(\psi_e(C_1)) = F = \text{Fl}(\psi_e(C_2)),
\]
and then Remark 2.4.ii yields that \( C_1 = C_2 \) implying that \( \phi \) is injective. In so far as \( \phi \) is injective,
\[
|C_{d,e}(B)| \leq |\mathcal{P}(d^2) - (d - e + 2) - 1(B)| \leq |\mathcal{P}(B)| = 2|B| = 2\left( \frac{d+2}{2} \right)^{3}.
\]
Using that \( \left( \frac{d+2}{2} \right) - 3 < 2d+2 \), the claim follows from (133). \( \square \)

Let \( d \in \mathbb{Z}^+ \). As usual, \( \mathbb{P}^d := (\mathbb{P}^{d+1} \setminus \{0\})/\sim \) with \( y \sim z \) if there is \( r \in \mathbb{R} \) such that \( y = r \cdot z \). We will use the embedding
\[
\mathbb{R}^d \rightarrow \mathbb{P}^d, \quad (z_1, z_2, \ldots, z_d) \mapsto [1 : z_1 : z_2 : \ldots : z_d]
\]
so that \( \mathbb{R}^d \) can be seen as a subset of \( \mathbb{P}^d \). For any flat \( F \) in \( \mathbb{R}^d \) defined by a family of linear equations \( \{r_0, i + \sum_{j=1}^d r_{j,i} x_j\}_{i \in I} \) (i.e. \( F = \bigcap_{i \in I} Z \{r_0, i + \sum_{j=1}^d r_{j,i} x_j\} \)), its homogenization will be the subset of elements \( [z_0 : z_1 : \ldots : z_d] \in \mathbb{P}^d \) such that \( \sum_{j=0}^d r_{j,i} z_j = 0 \) for all \( i \in I \), and we will denote it by \( F^h \). The next standard facts about homogenization can be found in [9, Sec.I.2].

**Remark 4.2.** Let \( d \in \mathbb{Z}^+ \), \( e \in [1, d] \) and \( F \) be an \( e \)-dimensional flat in \( \mathbb{R}^d \).

i) Then \( F^h \) is an \( e \)-dimensional linear variety in \( \mathbb{P}^d \), and \( F^h \cong \mathbb{P}^e \).

ii) Let \( f \in [1, d] \) and \( G \) be an \( f \)-dimensional flat in \( \mathbb{R}^d \). If \( e + f - d \geq 0 \), then \( F^h \cap G^h \neq \emptyset \) and \( F^h \cap G^h \) is a \( g \)-dimensional linear variety in \( \mathbb{P}^d \) with \( g \geq e + f - d \).

Let \( d \in \mathbb{Z}^+, e \in [0, d - 1] \) and \( F \) be an \( e \)-dimensional flat in \( \mathbb{R}^d \). Take \( G \) a \( d - e - 1 \)-dimensional flat in \( \mathbb{R}^d \) such that \( F \cap G = \emptyset \). For any \( z \in \mathbb{R}^d \setminus F \), notice that \( \dim \text{Fl}(F \cup \{z\}) = e + 1 \) and \( \dim \text{Fl}(F \cup \{z\}) \cap G \leq 0 \) since \( F \cap G = \emptyset \). On the other hand, Remark 4.2.ii implies that \( \text{Fl}(F \cup \{z\})^h \cap G^h \) is a linear variety of dimension at least 0. Therefore \( \text{Fl}(F \cup \{z\})^h \cap G^h \) is exactly a point. Identifying \( G^h \) with \( \mathbb{P}^{d-e-1} \), we define the hyperprojection centered in \( F \) as the map
\[
\pi_F : \mathbb{R}^d \setminus F \rightarrow \mathbb{P}^{d-e-1}, \quad \{\pi_F(z)\} = \text{Fl}(F \cup \{z\})^h \cap \mathbb{P}^{d-e-1}.
\]
An easy consequence of the the definition of \( \pi_F \) is the following remark.

**Remark 4.3.** Let \( d \in \mathbb{Z}^+, e \in [0, d - 1] \) and \( F \) be an \( e \)-dimensional flat in \( \mathbb{R}^d \). For any \( f \in [e + 1, d] \), there is an bijective relation between the \( f \)-flats containing \( F \) and the \( f - e - 1 \)-dimensional linear varieties in \( \mathbb{P}^{d-e-1} \) given by \( H \mapsto \pi_F(H \setminus F) \).
Let $B \in N_d(\mathbb{R}^2)$. Recall that $V_d(A) = \text{Fl}(\psi_d(A))$ for any $A \in \mathcal{P}(\mathbb{R}^2)$. Write

$$D_d(B) := \psi_d^{-1}(V_d(B))$$

$$E_d(B) := D_d(B) \cup \bigcup_{C \in \mathcal{C}_d(B)} \psi_d^{-1}(V_d(C \cup B)),$$

and the map

$$\varphi_{d,B} : \mathbb{R}^2 \setminus D_d(B) \longrightarrow \mathbb{P}^2,$$

$$\varphi_{d,B}(a) = \pi_{V_d(B)}(\psi_d(a)),$$

in other words, $\varphi_{d,B} = \pi_{V_d(B)} \circ \psi_d|_{\mathbb{R}^2 \setminus D_d(B)}$. The main properties of $\varphi_{d,B}$ are proven in the next lemma.

**Lemma 4.4.** Let $d \in \mathbb{Z}^+$ and $B \in N_d(\mathbb{R}^2)$.

i) For all $e \in [1, d - 1]$ and $C \in \mathcal{C}_{d,e}(B)$, we have that $|\varphi_{d,B}(C \setminus D_d(B))| = 1$.

ii) For all $a \in \mathbb{R}^2 \setminus E_d(B)$, we get that $\varphi_{d,B}(a) \notin \varphi_{d,B}(E_d(B) \setminus D_d(B))$.

iii) For all $a \in \mathbb{R}^2 \setminus E_d(B)$, we have that $|\varphi_{d,B}^{-1}(\varphi_{d,B}(a))| \leq d^2 - |D_d(B)|$.

**Proof.** Since $B \in N_d(\mathbb{R}^2)$, notice that for all $e \in [1, d - 1]$ and $C \in \mathcal{C}_{d,e}(B)$,

$$\dim \psi_{d-e}(B \setminus C) = \left(\frac{d - e + 2}{2}\right) - 3;$$

in particular, [134] implies there is a hyperplane $H$ in $\mathbb{R}^{(d-e+2)-1}$ such that $\psi_{d-e}(B \setminus C) \subseteq H$ and therefore $B \setminus C \subseteq \psi_{d-e}^{-1}(H)$. In so far as $\psi_{d-e}^{-1}(H) \in \mathcal{C}_{d-e}$ and $C \in \mathcal{C}_e$, we get that $\psi_{d-e}^{-1}(H) \cup C \in \mathcal{C}_{d-e}$ by Remark [2,2]i. Thus, insomuch as

$$B \cup C = (B \setminus C) \cup C \subseteq \psi_{d-e}^{-1}(H) \cup C,$$

Remark [2,4]ii implies that $\psi_d(B \cup C)$ is contained in hyperplane $K$ of $\mathbb{R}^{(d+2)-1}$ such that $\psi_d^{-1}(K) = \psi_{d-e}^{-1}(H) \cup C$; in particular,

$$\dim \psi_d(B \cup C) \leq \left(\frac{d + 2}{2}\right) - 2.$$

First we prove i). To show i), it suffices to prove that

$$\dim \psi_d(B \cup C) = \left(\frac{d + 2}{2}\right) - 3$$

because $\pi_{V_d(B)}$ projects $\left(\frac{d+2}{2}\right) - 3$-flats which contain $V_d(B) = \text{Fl}(\psi_d(B))$ into points of $\mathbb{P}^2$. From [134], $V_{d-e}(B \setminus C)$ is a proper flat in $\mathbb{R}^{(d-e+2)-1}$, and from [135], $V_d(B \cup C)$ is a proper flat in $\mathbb{R}^{(d+2)-1}$; therefore

$$\mathbb{R}^2 \setminus (\psi_{d-e}(V_{d-e}(B \setminus C)) \cup \psi_d^{-1}(V_d(B \cup C))) \neq \emptyset.$$

Fix $a \in \mathbb{R}^2 \setminus (\psi_{d-e}(V_{d-e}(B \setminus C)) \cup \psi_d^{-1}(V_d(B \cup C)))$. Since $a \notin \psi_{d-e}^{-1}(V_d(B \setminus C))$, we get from [134] that

$$\dim \psi_{d-e}((B \cup \{a\}) \setminus C) = \dim \psi_{d-e}([a] \cup (B \setminus C)) = \left(\frac{d - e + 2}{2}\right) - 2.$$  

Now we apply Lemma [2,10]ii to $B \cup \{a\}$ so [137] yields

$$\dim \psi_d((B \cup \{a\}) \cup C) = \left(\frac{d + 2}{2}\right) - 2.$$

In so far as $a \notin \psi_d^{-1}(V_d(B \cup C))$, [138] leads to [136].
We prove ii) by contradiction. Assume that there is $a \in \mathbb{R}^2 \setminus E_d(B)$ such that $\varphi_{d,B}(a) \not\in \varphi_{d,B}(E_d(B) \cap D_d(B))$. Therefore there are $e \in [1, d-1]$ and $C \in C_{d,e}(B)$ such that $\varphi_{d,B}(a) = \varphi_{d,B}(C \cap D_d(B))$. This equality means that $\psi_d(a)$ is in the $\binom{d+2}{2} - 3$-flat $V_d(B \cup C) = \text{Fl}(\psi_d(B \cup C))$ so

$$a \in \psi_d^{-1}(V_d(B \cup C)) \subseteq E_d(B),$$

which contradicts the assumption.

Finally, we show iii) by contradiction. Assume that there is $a \in \mathbb{R}^2 \setminus E_d(B)$ such that

$$|\varphi_{d,B}^{-1}(\varphi_{d,B}(a))| + |D_d(B)| > d^2. \quad (139)$$

Since $B \in \mathcal{N}_d(\mathbb{R}^2)$, we have that $\dim \psi_d(B) = \binom{d+2}{2} - 4$. Now, in so far as $a \not\in D_d(B) = \psi_d^{-1}(\text{Fl}(\psi_d(B)))$, we get that

$$\dim \psi_d(\{a\} \cup B) = \binom{d+2}{2} - 3; \quad (140)$$

thus $V_d(\{a\} \cup B) = \text{Fl}(\psi_d(\{a\} \cup B))$ is a $\binom{d+2}{2} - 3$-flat projected to $\varphi_{d,B}(a)$ by $\pi_{V_d(B)}$. Since $V_d(\{a\} \cup B)$ is a $\binom{d+2}{2} - 3$-flat, there are $H_1$ and $H_2$ distinct hyperplanes in $\mathbb{R}^{(d+2)-1}$ such that $H_1 \cap H_2 = V_d(\{a\} \cup B)$; write $C_1 := \psi_d^{-1}(H_1)$ and $C_2 := \psi_d^{-1}(H_2)$ so that

$$C_1 \cap C_2 = \psi_d^{-1}(H_1 \cap H_2) = \psi_d^{-1}(V_d(\{a\} \cup B)), \quad (141)$$

and therefore

$$|C_1 \cap C_2| = |\psi_d^{-1}(V_d(\{a\} \cup B))| \geq |\varphi_{d,B}^{-1}(\varphi_{d,B}(a))| + |D_d(B)|. \quad (142)$$

On the other hand, Remark 2.4.ii leads to $C_1, C_2 \subseteq C_{d,e}$. We claim that $C_1$ and $C_2$ share an irreducible component. If this claim were false, then Theorem 2.1 would give $|C_1 \cap C_2| \leq d^2$; however, this contradicts the inequality $|C_1 \cap C_2| > d^2$ which is a consequence of (139) and (142). Therefore there is an irreducible curve $C_0 \in C_e$ for some $e \in [1, d-1]$ such that $C_0 \subseteq C_1 \cap C_2$. From (141), $C_0 \subseteq \psi_d^{-1}(V_d(\{a\} \cup B))$ so $\psi_d(C_0) \subseteq V_d(\{a\} \cup B) = \text{Fl}(\psi_d(\{a\} \cup B))$; this and (140) yield

$$\dim \psi_d(\{a\} \cup B \cup C_0) = \dim \psi_d(\{a\} \cup B) = \binom{d+2}{2} - 3. \quad (143)$$

Since $B \in \mathcal{N}_d(\mathbb{R}^2)$, notice that

$$|B \cap C_0| \leq \binom{d+2}{2} - \binom{d-e+2}{2} - 1.$$

Thus we have two cases.

- If $|B \cap C_0| = \binom{d+2}{2} - \binom{d-e+2}{2} - 1$, then $C_0 \subseteq C_{d,e}(B)$. From (139), $\dim \psi_d(B \cup C_0) = \binom{d+2}{2} - 3$; however, since $a \not\in \psi_d^{-1}(V_d(B \cup C_0))$ (because $a \in \mathbb{R}^2 \setminus E_d(B)$), we get that

$$\dim \psi_d(\{a\} \cup B \cup C_0) > \dim \psi_d(B \cup C_0) = \binom{d+2}{2} - 3$$

contradicting (143).
Lemma 4.5. Then next two lemmas are the main results of this section.

\[ \dim \psi_{d,e}(B \setminus C_0) > \left( \frac{d + 2}{2} \right) - 3. \]  

Using (144) and Lemma 2.10 we conclude that \( \dim \psi_d(B \cup C_0) > \left( \frac{d + 2}{2} \right) - 3 \) but (143) gives

\[ \left( \frac{d + 2}{2} \right) - 3 = \dim \psi_d(\{a\} \cup B \cup C_0) \geq \dim \psi_d(B \cup C_0) > \left( \frac{d + 2}{2} \right) - 3, \]

which is impossible. In any case, we reached a contradiction and this concludes the proof of iii). \( \Box \)

Let \( d, n \in \mathbb{Z}^+ \), \( A \in \mathcal{P}(\mathbb{R}^2) \) be finite and \( B \in \mathcal{N}_d(A) \). Set

\[ \mathcal{O}_{d,n}(A,B) := \{ C \in \mathcal{O}_{d,n}(A) : B \subseteq C \} \]

\[ \delta_d(A,B) := \max_{a \in A \setminus E_d(B)} \left| \varphi_{d,B}^{-1}(\varphi_{d,B}(a)) \right|. \]

From Lemma 4.4 ii, \( \delta_d(A,B) \) exists and

\[ \delta_d(A,B) \leq d^2 - |D_d(B)| \leq d^2 - |B| = \frac{d^2 - 3d + 4}{2}. \]

For any \( m \in \mathbb{Z}^+ \), let \( c_6 = c_6(m) \) and \( c_7 = c_7(m) \) as in Lemma 2.11 and set

\[ c_8(d) := d^2 \cdot \max_{1 \leq m \leq d^{2d+2}} c_6(m) \]

\[ c_9(d) := d^2 \cdot \max_{1 \leq m \leq d^{2d+2}} c_7(m). \]

Then next two lemmas are the main results of this section.

Lemma 4.5. Let \( d \in \mathbb{Z}^+ \) and \( A \in \mathcal{P}(\mathbb{R}^2) \) be finite, \( d \)-regular and such that \( |A| \geq d^2 2^d \max\{c_8(d),c_9(d),1\} \). Write \( n := 2\delta_d(A,B) + |D_d(B)| \). Then, for any \( B \in \mathcal{N}_d(A,0,\mathbb{R}^2) \),

\[ |\mathcal{O}_{d,n}(A,B)| \geq \frac{1}{d^{d+2d^4}}|A|. \]

Proof. Since \( A \) is finite, we can apply a linear automorphism in \( \mathbb{P}^2 \) to assume that \( \varphi_{d,B}(A \setminus D_d(B)) \cap (\mathbb{R}^2 \setminus \mathbb{R}^2) = \emptyset \); thus we assume from now on that \( \mathcal{C}_{d,B}(A \setminus D_d(B)) \subseteq \mathbb{R}^2 \). Write \( S := \varphi_{d,B}(A \setminus E_d(B)) \), \( T := \varphi_{d,B}(E_d(B) \setminus D_d(B)) \) and \( \mathcal{L} := \{ L \in \mathcal{O}_2(S) : \mathcal{L} \cap \mathcal{T} = \emptyset \} \). From Lemma 4.4 iii, we have that \( S \setminus T = S \). From Lemma 4.4 i and the previous inequality lead to

\[ |T| \leq |\mathcal{C}_d(B)| < d^{2d+2}. \]

For each \( C \in \mathcal{C}_d(B) \), fix a hyperplane \( H_C \) in \( \mathbb{R}^d \) such that \( H_C \supseteq V_d(C \cup B) \); since \( \psi_d^{-1}(H_C) \in \mathcal{C}_{d,e} \) and \( A \) is \( d \)-regular,

\[ |A \cap \psi_d^{-1}(H_C)| \leq \frac{1}{2^{2d+2}}|A|. \]

From (145) and (146),

\[ \sum_{C \in \mathcal{C}_d(B)} |A \cap \psi_d^{-1}(H_C)| \leq \frac{d^{2d+2}}{2^{2d+2}}|A| \leq \frac{|A|}{2}. \]
and therefore

\[(147) \quad |A \setminus E_d(B)| = |A| - |A \cap E_d(B)| \geq |A| - \sum_{C \in C_d(B)} |A \cap \psi_d^{-1} H_C| \geq \frac{|A|}{2}.
\]

Since \(\delta_d(A, B) \leq d^2\) by Lemma 4.3 iii, we get from (147) that

\[(148) \quad |S| \geq \frac{1}{\delta_d(A, B)} |A \setminus E_d(B)| \geq \frac{1}{d^2} |A \setminus E_d(B)| \geq \frac{1}{2d^2} |A|.
\]

On the one hand, \(S\) is not collinear because if \(S\) is contained in a line \(L\), then \(A \setminus E_d(B)\) is contained in \(\varphi_{d, B}^{-1} (L)\) which is in \(C_{d, A}\), and (147) would contradict the \(d\)-regularity of \(A\). On the other hand, (148) implies \(|S| > c_8(d)|. Thus we can apply Lemma 2.11 to \(S\) and \(T\), and we obtain that

\[(149) \quad |L| \geq \frac{1}{2} |S| - c_9(d).
\]

From (148) and (149),

\[(150) \quad |L| \geq \frac{1}{d^2} |A|.
\]

Denote by \(\mathcal{H}\) the family of hyperplanes in \(\mathbb{R}^{(d+2)}\) generated by \(\psi_d(A)\) and define the maps

\[\eta_1 : L \to \mathcal{H}, \quad \eta_1(L) = \pi_{\psi_d(B)}^{-1} (L)
\]

\[\eta_2 : \eta_1(L) \to C_{d, d}, \quad \eta_2(\eta_1(L)) = \varphi_{d, B}^{-1} (L).
\]

We will show that

\[(151) \quad \eta_2(\eta_1(L)) \subseteq O_{d, n}(A, B).
\]

For each \(L \in \mathcal{L}\), \(L\) is a line generated by elements of \(S = \varphi_{d, B}(A \setminus D_d(B))\). Hence \(\eta_1(L)\) is a hyperplane generated by elements of \(\psi_d(A)\), and then, by Lemma 2.3, \(\eta_2(\eta_1(L)) = \varphi_{d, B}^{-1} (L) = \psi_d^{-1}(\eta_1(L))\) is an element of \(C_{d, d}\) determined by \(A\) (i.e. \(\eta_2(\eta_1(L)) \in D_d(A)\)). For any \(L \in \mathcal{L}\), we have that \(L \cap S = \{\varphi_{d, B}(a_1), \varphi_{d, B}(a_2)\}\) for some \(a_1, a_2 \in A \setminus E_d(B)\). In so far as \(\eta_2(\eta_1(L)) = \varphi_{d, B}^{-1} (L)\), Lemma 4.4 leads to

\[(152) \quad \eta_2(\eta_1(L)) \cap A = \varphi_{d, B}^{-1} (L) \cap A = \left(\varphi_{d, B}^{-1} (\varphi_{d, B}(a_1)) \cup \varphi_{d, B}^{-1} (\varphi_{d, B}(a_2)) \cup D_d(B)\right) \cap A.
\]

We conclude from (152) that

\[|\eta_2(\eta_1(L)) \cap A| \leq |\varphi_{d, B}^{-1} (\varphi_{d, B}(a_1))| + |\varphi_{d, B}^{-1} (\varphi_{d, B}(a_2))| + |D_d(B)| \leq n,
\]

and therefore \(\eta_2(\eta_1(L)) \in O_{d, n}(A, B)\) proving (151). From Remark 4.3, \(\eta_1\) is injective. From Lemma 2.3, for each \(C \in C_{d, d}\), there are at most \(d^d\) hyperplanes in \(\mathbb{R}^{(d+2)}\) such that \(C = \psi_d^{-1}(H)\) so \(|\eta_2^{-1}(C)| \leq d^d\) for all \(C \in C_{d, d}\). The previous two statements give \(|\eta_2(\eta_1(L))| \geq \frac{1}{d^d} |L|\), and then (151) yields that

\[(153) \quad |O_{d, n}(A, B)| \geq |\eta_2(\eta_1(L))| \geq \frac{1}{d^d} |L|.
\]

Then the lemma follows from (150) and (153). \(\square\)
Lemma 4.6. Let $d \in \mathbb{Z}^+$, $f \in [0, d - 1]$, $C_0 \in \mathcal{C}_{d-f}$ be irreducible, $A \in \mathcal{P}(\mathbb{R}^2)$ be finite such that $|A \cap C_0| \geq d^{d+8}2^{d+3}\max\{c_0(d), c_0(d), 1\}$, and $B_0 \in \mathcal{P}(\mathbb{R}^2)\setminus (A \setminus C_0)$ be such that $B_0$ is not contained in an element of $\mathcal{C}_{d-f}$. Write $n := 2\delta_d(A, B) + |D_d(B)|$.

i) For any $B \in \mathcal{N}_0(A, B_0, C_0)$, we get that $|O_{d,n}(A, B)| \geq \frac{1}{2d+2} |A \cap C_0|$.

ii) Assume that $|A \cap C_0| \leq \frac{1}{2d+2} |A \cap C_0|$. Then, for any $B \in \mathcal{N}_d(A, B_0, C_0)$, we get that $|O_{d,d^2}(A, B)| \geq \frac{1}{2d+2} |A \cap C_0|^2$.

Proof. Write $S := \varphi_{d,B}(A \setminus E_d(B)), S_0 := \varphi_{d,B}((A \cap C_0) \setminus E_d(B))$ and $T := \varphi_{d,B}(E_d(B) \setminus D_d(B))$. Denote by $\mathcal{H}$ the family of hyperplanes in $\mathbb{R}^{(d+2) - 1}$ generated by $\psi_d(A)$, denote by $\mathcal{L}$ the family of lines generated by $\varphi_{d,B}(A \setminus D_d(B))$ and define the maps

$$
\eta_1 : \mathcal{L} \rightarrow \mathcal{H}, \quad \eta_1(L) = \pi_{\psi_d(B)}^{-1}(L)
$$

$$
\eta_2 : \mathcal{L} \rightarrow \mathcal{C}_{d^2}, \quad \eta_2(\eta_1(L)) = \psi_d^{-1}(\eta_1(L)).
$$

As in the first part of Lemma 4.3, we may assume that $S$ is contained in $\mathbb{R}^2$ and we have that

$$
|T| \leq |C_d(B)| < d^{2d+2}.
$$

The next step is to show that for any hyperplane $H$ in $\mathbb{R}^{(d+2) - 1}$ containing $V_d(B)$,

$$
|H \cap \psi_d(C_0)| \leq d^2.
$$

Indeed, write $C := \psi_d^{-1}(H)$. Note that

$$
B \subseteq \psi_d^{-1}(V_d(B)) \subseteq \psi_d^{-1}(H) = C
$$

so $|B \cap C| = |B| = \binom{d+2}{2} - 3$. Thus $C_0$ is not a component of $C$ because otherwise Lemma 2.9 applied to $C_0$ and $C$ leads to $|B \cap C| < \binom{d+2}{2} - 3$. Thereby Theorem 2.1 applied to $C$ and $C_0$ gives

$$
|\psi_d(C_0) \cap H| = |C_0 \cap C| \leq d^2,
$$

which shows (155). Remark 4.3 implies that for any line $L \in \mathbb{R}^2$, there is a hyperplane $H$ in $\mathbb{R}^{(d+2) - 1}$ containing $V_d(B)$ such that $\pi_{V_d(B)}(H) = L$. Then (155) yields that for any line $L \in \mathbb{R}^2$,

$$
|L \cap \varphi_{d,B}(C_0 \setminus D_d(B))| \leq d^2.
$$

The next step is to prove that

$$
|S_0| \geq \frac{1}{d^2} \left( |A \cap C_0| - d^{d+2} \right) \geq \frac{1}{2d^2} |A \cap C_0|.
$$

Denote by $\mathcal{L}_T$ the family of lines generated by $T$. From (154),

$$
|\mathcal{L}_T| \leq \left( \frac{|T|}{2} \right) < d^2 2^{d+3}.
$$

For any $L \in \mathcal{L}_T$, the hyperplane $\pi_{V_d(B)}^{-1}(L)$ contains $V_d(B)$ so (155) leads to

$$
|\pi_{V_d(B)}^{-1}(L) \cap \psi_d(C_0)| \leq d^2.
$$
On the one hand, equation (166) implies that for each \( L \in \mathcal{S} \), the expression \( \psi_d(C_0) \cap \pi_{V_d(B)}^{-1}(L) \) is not collinear because if this is the case, then equation (155) would contradict (158). On the other hand, (157) implies that \( |\mathcal{S}| \) is contained in a hyperplane; however, (157) yields \( |\mathcal{S}| > d^2 \), and this would contradict (153). From Lemma 4.4.iii, each element in \( \mathcal{S} = \psi_d((A \cap C_0) \setminus E_d(B)) \) has at most \( \delta_d(A,B) \leq d^2 \) elements in its preimage so (161) leads to

\[
(161) \quad |\mathcal{S}| \leq \psi_d^{-1}\left( \bigcup_{L \in \mathcal{L}_T} \pi_{V_d(B)}^{-1}(L) \right) \geq |\mathcal{S}| - d^4 2^{d+3}.
\]

Now we show i). Set \( L_1 := \{ L \in \mathcal{O}_2(S) : L \cap T = \emptyset \} \). On the one hand, \( S \) is not collinear because if this is the case, then \( S \) is also collinear and then \( \pi_{V_d(B)}^{-1}(S) \) is contained in a hyperplane; however, (157) yields \( |S_0| > d^2 \), and this would contradict (153). On the other hand, (157) implies \( |S| > c_8(d) \). Thus we can apply Lemma 2.11 to \( S \) and \( T \), and we obtain that

\[
(162) \quad |L_1| \geq \frac{1}{2} |S| - c_9(d).
\]

From (157) and (162),

\[
(163) \quad |L_1| \geq \frac{1}{2^{d^2}} |A \cap C_0| - c_9(d) \geq \frac{1}{2^{d^2}} |A \cap C_0|
\]

Proceeding as in the last part of Lemma 4.5 it is concluded that

\[
\eta_2(\eta_1(L_1)) \subseteq \mathcal{O}_{d,n}(A,B)
\]

so

\[
(164) \quad |\mathcal{O}_{d,n}(A,B)| \geq |\eta_2(\eta_1(L_1))| \geq \frac{1}{d^d} |L_1|.
\]

Then i) is a straight consequence of (163) and (164).

Finally, we show ii). Set \( S_1 := \varphi_{d,B}(A \setminus C_0 \setminus E_d(B)) \), and notice that

\[
(165) \quad |S_1| \leq |A \setminus C_0| \leq \frac{1}{2^{d^2}} |A \cap C_0|.
\]

Denote by \( \mathcal{L}_0 \) the family of lines \( L \) generated by \( S_0 \) such that \( L \cap (T \cup S_1) = \emptyset \). For any \( s \in S_0 \), write \( \mathcal{L}_0(s) := \{ L \in \mathcal{L}_0 : s \in L \} \). Since \( S_0 \subseteq \varphi_{d,B}(C_0 \setminus E_d(B)) \subseteq \pi_{V_d(B)}(\psi_d(C_0)) \), we have that for each \( t \in S_0 \cap L \), there is \( z \in \psi_d(C_0) \cap \pi_{V_d(B)}^{-1}(L) \) such that \( \pi_{V_d(B)}(z) = t \). Then (165) gives

\[
(166) \quad |L \cap S_0| \leq |\pi_{V_d(B)}^{-1}(L \cap S_0)| \leq |\pi_{V_d(B)}^{-1}(L) \cap \psi_d(C_0)| \leq d^2.
\]

On the one hand, (166) implies that for each \( L \in \mathcal{L}_0 \), there are at most \( d^2 \) elements \( s \in S_0 \) such that \( L \in \mathcal{L}_0(s) \) so

\[
(167) \quad |\mathcal{L}_0| \geq \frac{1}{d^2} \sum_{s \in S_0} |\mathcal{L}_0(s)|.
\]
On the other hand, (166) implies that for each \( s \in S_0 \), there are at least \( \frac{|S_1| - 3}{d^2} \) lines generated by \( s \) and other element of \( S_0 \); notice that at most \( |S_1| + |T| \) of these lines pass through an element of \( S_1 \cap T \) so (154), (157) and (165) lead to

\[
|L_0(s)| \geq \frac{|S_0| - 1}{d^2} - |S_1| - |T| \geq \frac{1}{2^q d^6} |A \cap C_0|.
\]

From (157), (167) and (168), we can bound \( |L_0| \) below as follows

\[
|L_0| \geq \frac{1}{d^2} \sum_{s \in S_0} |L_0(s)| \geq \frac{1}{2^q d^6} |S_0| |A \cap C_0| \geq \frac{1}{2^q d^6} |A \cap C_0|^2.
\]

We will show that

\[
\eta_2(\eta_1(L_0)) \subseteq O_{d,d^2}(A,B).
\]

For any \( L \in L_0 \), notice that \( L \cap S = L \cap S_0 \) inasmuch as \( L \cap (S_1 \cup T) = \emptyset \); considering the preimages of \( \pi_{V_d(B)} \), we get that \( \eta_1(L) \cap \psi_d(A) = \eta_1(L) \cap \pi_{V_d(B)}(S_0) \). As in (166),

\[
|\eta_2(\eta_1(L)) \cap A| = |\eta_1(L) \cap \psi_d(A)| \leq d^2,
\]

and this proves (170). From Remark 4.3, \( \eta_1 \) is injective. From Lemma 2.3 for each \( C \in C_{d,2} \), there are at most \( d^d \) hyperplanes in \( \mathbb{R}^{d+1} \) such that \( C = \psi_d^{-1}(H) \) so \( |\eta_2^{-1}(C)| \leq d^d \) for all \( C \in C_{d,2} \). From this and (170),

\[
|O_{d,d^2}(A,B)| \geq |\eta_2(\eta_1(L_0))| \geq \frac{1}{d^d} |L_0|.
\]

Then ii) is a consequence of (169) and (171).

\[\square\]

5. PROOFS OF THE MAIN RESULTS

We conclude the proofs of the main results in this section.

Proof. (Theorem 1.3). From Theorem 1.2 the claim holds for \( d = 1 \). From [11 Thm 1.3], the statement is true for \( d = 2 \). Thus we assume that \( d \geq 3 \) from now on. We show that \( c_2 = d^{d+3} + 2d^{d+2} + 1 \) \( \max\{c_3(d), c_3,d(1)\} \) and \( c_3 = \frac{d^{d+3} + 3d + 3}{d^{d+1} + 3} \) satisfy the desired properties. Set \( c_{10} := (\frac{d^{d+3} + 3d + 3}{d^{d+1} + 3}) \) and \( c_{11} := (\frac{d^{d+3} + 3d + 3}{c_3})^2 \). For all \( B \in N_d(A) \), Lemma 4.4 implies that

\[
\delta_d(A,B) + \left(\frac{d+2}{2}\right) - 3 = \delta_d(A,B) + |B| \leq \delta_d(A,B) + |D_d(B)| \leq d^2.
\]

so \( 2\delta_d(A,B) + |D_d(B)| \leq \frac{3d^2 - 3d + 4}{2} \). Therefore

\[
O_{d,2\delta_d(A,B) + |D_d(B)|(A,B)} \subseteq O_{d,\frac{3d^2 - 3d + 4}{2}}(A,B).
\]

For all \( C \in O_{d,\frac{3d^2 - 3d + 4}{2}}(A) \), we have that \( |A \cap C| \leq \frac{3d^2 - 3d + 4}{2} \); therefore there are at most \( c_{10} \) subsets \( B \in N_d(A) \) such that \( C \in O_{d,\frac{3d^2 - 3d + 4}{2}}(A,B) \), and this yields

\[
\left|O_{d,\frac{3d^2 - 3d + 4}{2}}(A)\right| \geq \left|\bigcup_{B \in N_d(A)} O_{d,\frac{3d^2 - 3d + 4}{2}}(A,B)\right| \geq \frac{1}{c_{10}} \sum_{B \in N_d(A)} \left|O_{d,\frac{3d^2 - 3d + 4}{2}}(A,B)\right|.
\]

(173)
Since $A$ is not contained in an element of $C_d$, then $A$ is not contained in an element of $C_{\leq d}$ by Remark 2.0. The conclusion of the proof is divided into two cases.

i) Assume that $A$ is $d$-regular. On the one hand, Lemma 3.3 gives

$$|\mathcal{N}_d(A)| \geq |\mathcal{N}_d(A, \emptyset, \mathbb{R}^2)| \geq \frac{1}{2^{d+3}}|A|^\frac{d+2}{2} - 3.$$  \tag{174}$$

Lemma 4.3 and (172) imply that for all $B \in \mathcal{N}_d(A, \emptyset, \mathbb{R}^2)$,

$$|\mathcal{O}_{d, 3d^2 - 3d + 4}(A, B)| \geq \frac{1}{d^{d+2} 2^{d+3}}|A|.$$  \tag{175}$$

Then (173), (174) and (175) yield

$$|\mathcal{O}_{d, 3d^2 - 3d + 4}(A)| \geq \frac{1}{c_{10} d^{d+2} 2^{d+3}}|A|^\frac{d+2}{2} - 2.$$  \tag{176}$$

ii) Assume that $A$ is not $d$-regular. Then there is $C \in C_{\leq d}$ such that $|A \cap C| \geq \frac{1}{2^{d+3}}|A|$. Moreover, since $C$ has at most $d$ irreducible components, there are $f \in [0, d - 1]$ and $C_0 \in C_{d-f}$ irreducible such that

$$|A \cap C_0| \geq \frac{1}{2^{d+3}}|A|.$$  \tag{177}$$

Denote by $\mathcal{R}$ the family of subsets $B_0 \in \mathcal{P}_{d+2}^\ast (A \setminus C_0)$ such that $B_0$ is not contained in an element of $C_{\leq f}$. Lemma 2.8 gives

$$|\mathcal{R}| \geq \frac{1}{2^{d+3}}|A \setminus C_0|.$$  \tag{178}$$

For all $B_0 \in \mathcal{R}$, Lemma 3.9 leads to

$$|\mathcal{N}_d(A, B_0, C_0)| \geq \frac{1}{2^{d+3}}|A \cap C_0|^\frac{d+2}{2} - 3 - \frac{d}{2} \geq \frac{1}{2^{d+3}}|A \cap C_0|^d - 2.$$  \tag{179}$$

We have two subcases.

* Assume that $|A \setminus C_0| \leq \frac{1}{2^{d+3}}|A \cap C_0|$. Hence Lemma 4.6 ii implies that for all $B_0 \in \mathcal{R}$ and $B \in \mathcal{N}_d(A, B_0, C_0)$,

$$|\mathcal{O}_{d, 3d^2 - 3d + 4}(A, B)| \geq |\mathcal{O}_{d, 3d^2 - 3d + 4}(A, B)| \geq \frac{1}{d^{d+2} 2^{d+3}}|A \cap C_0|^2.$$  \tag{180}$$

Then

$$|\mathcal{O}_{d, 3d^2 - 3d + 4}(A)| \geq \frac{1}{c_{10}} \sum_{B \in \mathcal{N}_d(A)} |\mathcal{O}_{d, 3d^2 - 3d + 4}(A, B)| \quad \text{(by 172)}$$

$$\geq \frac{1}{c_{11}} \sum_{B_0 \in \mathcal{R}} \left( \sum_{B \in \mathcal{N}_d(A, B_0, C_0)} |\mathcal{O}_{d, 3d^2 - 3d + 4}(A, B)| \right)$$

$$\geq \frac{1}{c_{11}} |A \cap C_0|^d, \quad \text{(by 178, 179)}$$

and the claim holds by (170).

* Assume that $|A \setminus C_0| > \frac{1}{2^{d+3}}|A \cap C_0|$ so that (177) leads to

$$|\mathcal{R}| \geq \frac{1}{2^{d+3}}|A \setminus C_0| > \frac{1}{2^{d+3}} \frac{1}{d^{d+2} 2^{d+3}}|A \cap C_0|.$$
From Lemma 4.6, we have that for all \( B_0 \in \mathcal{R} \) and \( B \in \mathcal{N}_d(A, B_0, C_0) \),

\[
\left| O_{d, \frac{3d^2 - 3d + 4}{2}}(A, B) \right| \geq \frac{1}{2^d d^{d+2}} |A \cap C_0|.
\]

Thus

\[
\left| O_{d, \frac{3d^2 - 3d + 4}{2}}(A) \right| \geq \frac{1}{c_10} \sum_{B \in \mathcal{N}_d(A)} \left| O_{d, \frac{3d^2 - 3d + 4}{2}}(A, B) \right| \geq \frac{1}{c_10} \sum_{B \in \mathcal{R}} \sum_{B \in \mathcal{N}_d(A, B_0, C_0)} \left| O_{d, \frac{3d^2 - 3d + 4}{2}}(A, B) \right| \geq \frac{1}{c_11} |A \cap C_0|^d,
\]

by (182), (180), (181) and this shows ii).

Therefore in any case \( c_2 \) and \( c_3 \) work.

We prove Theorem 1.6.

Proof. (Theorem 1.6). Take a line \( L \) in \( \mathbb{R}^2 \), \( B_0 \in \mathcal{P}_{d_{+1}}(\mathbb{R}^2 \setminus L) \) such that there is no element of \( C_{\leq d-1} \) which contains \( B_0 \), and \( B_1 \in \mathcal{P}_{m_{-1}}(\mathbb{R}^2 \setminus L) \). Set \( A := B_0 \cup B_1 \).

Theorem 2.1 implies that for any \( C \in C_{\leq d} \) such that \( L \) is not a component of \( C \),

\[
|L \cap C| \leq d.
\]

Now assume that \( A \) is contained in a curve \( C \in C_{\leq d} \). Since \( d < m - \left( \frac{d+1}{2} \right) = |A \cap L| \), (182) implies that \( C \) contains \( L \). However, if \( C \) contains \( L \), then \( B_0 = A \setminus L \subseteq C \setminus L \), and therefore there is an element of \( C_{\leq d-1} \) which contains \( B_0 \) contradicting the assumption. Thus \( A \) is not contained in a curve \( C \in C_{\leq d} \), and this shows i).

Set

\[
\eta : O_{d, \frac{3d^2 - 3d + 4}{2}}(A) \to \mathcal{P}_d(A \cap L), \quad \eta(C) = A \cap L \cap C.
\]

We show that \( \eta \) is well defined. Since \( |A \cap L| = m - \left( \frac{d+1}{2} \right) > \frac{3d^2 - 3d + 4}{2} - \left( \frac{d+1}{2} \right) \), we have that \( C \) cannot contain \( L \) for any \( C \in O_{d, \frac{3d^2 - 3d + 4}{2}}(A) \); hence (182) yields that

\[
|A \cap L \cap C| \leq |L \cap C| \leq d.
\]

On the other hand, for any \( C \in O_{d, \frac{3d^2 - 3d + 4}{2}}(A) \), \( C \) is determined by \( A \) so \( |A \cap C| \geq \left( \frac{d+2}{2} \right) - 1 \), and hence

\[
|A \cap L \cap C| = |A \cap C| - |(A \cap C) \setminus L| \geq \left( \frac{d+2}{2} \right) - 1 - \left( \frac{d+1}{2} \right) = d.
\]

Thus \( \eta \) is well defined. Finally, \( \eta \) is injective. Indeed, take \( C_1, C_2 \in O_{d, \frac{3d^2 - 3d + 4}{2}}(A) \) such that \( \eta(C_1) = \eta(C_2) \). Since \( C_1 \) and \( C_2 \) are determined by \( A \), note that \( |A \cap C_1|, |A \cap C_2| \geq \left( \frac{d+2}{2} \right) - 1 \) so \( |(A \cap C_1) \setminus L|, |(A \cap C_2) \setminus L| \geq \left( \frac{d+1}{2} \right) \). Thereby

\[
(A \cap C_1) \setminus L = (A \cap C_2) \setminus L = B_0,
\]

and the equality \( \eta(C_1) = \eta(C_2) \) implies that \( A \cap C_1 = A \cap C_2 \). Since \( C_1 \) and \( C_2 \) are determined by \( A \), the previous equality yields \( C_1 = C_2 \). Finally, because \( \eta \) is injective,

\[
\left| O_{d, \frac{3d^2 - 3d + 4}{2}}(A) \right| \leq \left( \frac{|A \cap L|}{d} \right) = \left( \frac{|A| - \left( \frac{d+1}{2} \right)}{d} \right),
\]

and this shows ii).

□

We prove Theorem 1.7.
Proof. (Theorem 1.7). We prove that $c_4 = d^{d+2}2^{6d+16} \max \{c_6(d), c_9(d), 1\}$ and $c_5 = \frac{1}{d^{d+6}2^{d+3}2^{d+16}(2d+3)}$ work. Write $c_{12} := \left(2n+1-\binom{d+1}{2}\right)^2$ and $c_{13} := c_{12}2^{d+2}(2d+3)!$. Let $B \in \mathcal{N}_d(A)$. Take $a \in A$ such that $\delta_d(A, B) = \varphi_{d,B}(\varphi_{d,B}(a))$. Since $\pi_{\varphi_d(B)}(\varphi_{d,B}(a))$ is a $\binom{d+2}{2}$ - 3-flat in $\mathbb{R}^{(d+2)-1}$, we have that for any $s \in \psi_d(A) \setminus \pi_{\varphi_d(B)}(\varphi_{d,B}(a))$, the flat $H_s$ generated by $\pi_{\varphi_d(B)}(\varphi_{d,B}(a))$ and $s$ is a hyperplane. Then, by Lemma 2.3, $\psi_{\varphi_d(B)}^{-1}(H_s)$ is an element of $C_d$ containing $\varphi_{d,B}^{-1}(\varphi_{d,B}(a)) \cup D_d(B)$. Now, since $A$ is not contained in an element of $C_d$, Remark 2.6 implies that $A$ is not contained in an element of $C_{d}$. Therefore, by Remark 2.6 ii, $\psi_d(A)$ is not contained in a hyperplane of $\mathbb{R}^{(d+2)-1}$. Thus, since $\pi_{\varphi_d(B)}^{-1}(\varphi_{d,B}(a))$ is a $\binom{d+2}{2}$ - 3-flat, there are $s_1, s_2 \in \psi_d(A) \setminus \pi_{\varphi_d(B)}^{-1}(\varphi_{d,B}(a))$ and $\psi_{\varphi_d(B)}^{-1}(H_{s_1}) \neq \psi_{\varphi_d(B)}^{-1}(H_{s_2})$. Since $\varphi_{d,B}^{-1}(\varphi_{d,B}(a)) \cup D_d(B)$ is contained in $\psi_{\varphi_d(B)}^{-1}(H_{s_1})$ and $\psi_{\varphi_d(B)}^{-1}(H_{s_2})$ and they are in $C_d$, the assumption on $n$ yields that $\varphi_{d,B}^{-1}(\varphi_{d,B}(a)) \cup D_d(B) < n$, and hence

$$\delta_d(A, B) + |B| \leq \delta_d(A, B) + |D_d(B)| = |\varphi_{d,B}^{-1}(\varphi_{d,B}(a)) \cup D_d(B)| < n. \quad (183)$$

From (183), note that $2\delta_d(A, B) + |D_d(B)| \leq 2n + 1 - \binom{d+2}{2}$, and hence

$$O_{d,2\delta_d(A, B) + |D_d(B)|}(A, B) \subseteq O_{d,2n+1-\binom{d+2}{2}}(A, B). \quad (184)$$

For all $C \in O_{d,2n+1-\binom{d+2}{2}}(A)$, we have that $|A \cap C| \leq 2n + 1 - \binom{d+2}{2}$; thus there are at most $c_{12}$ subsets $B \in \mathcal{N}_d(A)$ such that $C \in O_{d,2n+1-\binom{d+2}{2}}(A, B)$, and this yields

$$\left|O_{d,2n+1-\binom{d+2}{2}}(A)\right| \geq \left|\bigcup_{B \in \mathcal{N}_d(A)} O_{d,2n+1-\binom{d+2}{2}}(A, B)\right| \geq \frac{1}{c_{12}} \sum_{B \in \mathcal{N}_d(A)} \left|O_{d,2n+1-\binom{d+2}{2}}(A, B)\right|. \quad (185)$$

If $A$ is $d$-regular, then we proceed exactly as in Case i) of Theorem 1.5 to conclude that

$$\left|O_{d,2n+1-\binom{d+2}{2}}(A)\right| \geq \frac{1}{c_{12}d^{d+2}2^{2d+16}(2d+3)}|A|^{\binom{d+2}{2}-2}. \quad (186)$$

From now on, we assume that $A$ is not regular. This means that there is $C \in C_{d}$ such that $|A \cap C| \geq 1 - 2^{2d+2} |A|$. In so far as, $C$ has at most $d$ irreducible components, there are $f \in [0, d - 1]$ and $C_0 \in C_{d-f}$ such that

$$|A \cap C_0| \geq \frac{1}{d^{2d+2}} |A|. \quad (187)$$

Notice that $f = 0$; otherwise, $f > 0$ so for any $B \in P_n(A \cap C_0)$ and any curve $C_1 \in f$, we get a curve $C_0 \cup C_1 \in C_{d}$ such that $B \subseteq C_0 \cup C_1$, contradicting the assumption about $n$. Then, since $f = 0$, Lemma 3.9 implies that for all $b \in A \setminus C_0$,

$$|N_d(A, \{b\}, C_0)| \geq \frac{1}{2d+3}|A \cap C_0|^{\binom{d+2}{2}-3-\binom{d+2}{2}} = \frac{1}{2d+3}|A \cap C_0|^{\binom{d+2}{2}-4}. \quad (187)$$

From Lemma 4.5 i), we have that for all $b \in A \setminus C_0$ and $B \in \mathcal{N}_d(A, \{b\}, C_0)$,

$$\left|O_{d,2n+1-\binom{d+2}{2}}(A, B)\right| \geq \frac{1}{d^{2d+2}} |A \cap C_0|. \quad (188)$$
Hence
\[
|O_{d,2n+1-\binom{d+2}{2}}(A)| \geq \frac{1}{c_{12}} \sum_{B \in N_d(A)} |O_{d,2n+1-\binom{d+2}{2}}(A, B)| \geq \frac{1}{c_{12}} \sum_{b \in A \cap C_0} \sum_{B \in N_d(A, B, C_0)} |O_{d,2n+1-\binom{d+2}{2}}(A, B)| \geq \frac{1}{c_{13}} |A \cap C_0| \binom{d+2}{2} - 3,
\]
(by \[185\), \[187\], \[183\])
and the claim holds by \[186\].

Finally, we complete the proof of Theorem 1.8.

**Proof.** (Theorem 1.8). Let \( C_0 \in C_d \) be irreducible and \( H \) be a hyperplane in \( \mathbb{R}^{\binom{d+2}{2}} \), \( 0 \) such that \( C_0 = \psi^{-1}_d(H) \). Choose \( a_0 \in \mathbb{R}^2 \setminus C_0 \). We construct recursively a set \( S \in \mathcal{P}_{m-1}(\psi_d(\mathbb{R}^2) \cap H) \) such that \( \dim R = |R| - 1 \) for all \( R \in \mathcal{P}(S) \) with \( \dim R < \binom{d+2}{2} - 2 \). Take \( s_1 \in \psi_d(\mathbb{R}^2) \cap H \) and write \( S_1 := \{s_1\} \). Now assume that for some \( i \in [1, m-2] \), we have constructed a set \( S_i \in \mathcal{P}_i(\psi_d(\mathbb{R}^2) \cap H) \) such that \( \dim R = |R| - 1 \) for all \( R \in \mathcal{P}(S_i) \) with \( \dim R < \binom{d+2}{2} - 2 \). Let \( F_i \) be the collection of all flats \( F \) generated by the subsets of \( S_i \) such that \( \dim F < \binom{d+2}{2} - 2 \). Since \( S_i \) is finite, \( F_i \) is finite. On the other hand, for each \( F \in F_i \), there exists a hyperplane \( G \) in \( \mathbb{R}^{\binom{d+2}{2}} \) such that \( G \neq H \) and \( F \subseteq G \cap H \); in particular, \( C_0 \) is not a component of \( \psi^{-1}_d(G) \). Applying Theorem 2.1 to the curves \( \psi^{-1}_d(G) \) and \( \psi^{-1}_d(H) = C_0 \), we have that \( |\psi^{-1}_d(G) \cap \psi^{-1}_d(H)| \leq d^2 \) and thus
\[
|F \cap (\psi_d(\mathbb{R}^2) \cap H)| \leq |G \cap (\psi_d(\mathbb{R}^2) \cap H)| = |\psi^{-1}_d(G) \cap \psi^{-1}_d(H)| \leq d^2.
\]
From \[189\], we have that \( \bigcup_{F \in F_i} F \cap (\psi_d(\mathbb{R}^2) \cap H) \) is finite. Since \( \psi_d(C_0) \subseteq \psi_d(\mathbb{R}^2) \cap H \) is not finite, \( (\psi_d(\mathbb{R}^2) \cap H) \setminus \bigcup_{F \in F_i} F \neq \emptyset \) and we choose \( s_{i+1} \) in this difference. Make \( S_{i+1} := S_i \cup \{s_{i+1}\} \), and notice that \( S_{i+1} \subseteq \psi_d(\mathbb{R}^2) \cap H \), \( |S_{i+1}| = i + 1 \) and \( \dim R = |R| - 1 \) for all \( R \in \mathcal{P}(S_{i+1}) \) with \( \dim R < \binom{d+2}{2} - 2 \) (the last property because \( s_{i+1} \notin \bigcup_{F \in F_i} F \)). In this way we construct \( S_2, S_3, \ldots, S_{m-1} \) and \( S := S_{m-1} \) has the desired properties. Set \( A := \{a_0\} \cup \psi^{-1}_d(S) \).

Theorem 2.1 implies that for any \( C \in C_{d,d} \) such that \( C_0 \not\subseteq C \),
\[
|C_0 \cap C| \leq d^2.
\]
Since \( d^2 < m - 1 = |A \cap C_0| \), if \( A \) is contained in a curve \( C \in C_{d,d} \), then \( C_0 \) is contained in \( C \) by \[190\]. Nevertheless, in so far as \( C_0 \in C_d \) and \( C \in C_{d,d} \), we have that \( C = C_0 \) but this is impossible since \( a_0 \notin C_0 \). This proves i).

Take \( B \in \mathcal{P}_n(A) \) and write \( B_0 := B \cap C_0 \). We claim that
\[
\dim \psi_d(B_0) \geq \begin{cases} \binom{d+2}{2} - 2 & \text{if } B = B_0 \\ \binom{d+2}{2} - 3 & \text{if } B \neq B_0. \end{cases}
\]
Indeed, if \( B = B_0 \), then \( |\psi_d(B_0)| = |B| \geq \binom{d+2}{2} - 1 \); thus, if \( \dim \psi_d(B_0) < \binom{d+2}{2} - 2 \), note that
\[
\dim \psi_d(B_0) < \binom{d+2}{2} - 2 \leq |\psi_d(B_0)| - 1,
\]
which is impossible by the construction of $S = \psi_d(A \cap C_0)$. If $B \neq B_0$, then $B = B_0 \cup \{a_0\}$ so $|\psi_d(B_0)| = |B| - 1 \geq \left(\frac{d+2}{2}\right) - 2$; if $\dim \psi_d(B_0) < \left(\frac{d+2}{2}\right) - 3$, then
\[
\dim \psi_d(B_0) < \left(\frac{d+2}{2}\right) - 3 \leq |\psi_d(B_0)| - 1,
\]
which is impossible by the construction of $S$, and this proves (191). Now, if $B = B_0$, then (191) implies that there is at most one hyperplane which contains $\psi_d(B)$; hence, by Remark 2.2 ii, there is at most one element $C \in \mathcal{C}_{\leq d}$ such that $B \subseteq C$. If $B \neq B_0$, then (191) leads to $\dim \psi_d(B_0) \geq \left(\frac{d+2}{2}\right) - 3$. Since $a_0 \notin C$, we have that $\psi_d(a_0) \notin H$, and since $H \supseteq \psi_d(B_0)$, we conclude that $\dim \psi_d(B) \geq 1 + \dim \psi_d(B_0) \geq \left(\frac{d+2}{2}\right) - 2$. As in the previous case, we conclude that there is at most one element $C \in \mathcal{C}_{\leq d}$ such that $B \subseteq C$, and this completes the proof of ii).

Finally we show iii). Set
\[
\eta : \mathcal{O}_{d,2n+1-(\frac{d+2}{2})}(A) \rightarrow \mathcal{P}_{(\frac{d+2}{2})-2}(A \cap C_0), \quad \eta(C) = A \cap C_0 \cap C.
\]
We show that $\eta$ is well defined. Take $C \in \mathcal{O}_{d,2n+1-(\frac{d+2}{2})}(A)$. Since $|A \cap C_0| = m - 1 > 2n + 1 - \left(\frac{d+2}{2}\right) - 1$, we have that $C$ cannot contain $C_0$. Since $C$ is determined by $A$, we have that $|A \cap C| \geq \left(\frac{d+2}{2}\right) - 1$. We prove by contradiction that
\[
|A \cap C_0 \cap C| \leq \left(\frac{d+2}{2}\right) - 2.
\]
Assume that
\[
|\psi_d(A \cap C_0 \cap C)| = |A \cap C_0 \cap C| \geq \left(\frac{d+2}{2}\right) - 1.
\]
By the construction of $S$, (193) yields that
\[
\dim \psi_d(A \cap C_0 \cap C) \geq \left(\frac{d+2}{2}\right) - 2 = \dim \psi_d(C_0);
\]
however, since $C$ does not contain $C_0$, we have that $\psi_d(C_0 \cap C)$ is contained in two different hyperplanes of $\mathbb{R}^{(\frac{d+2}{2})-1}$ and therefore
\[
\dim \psi_d(A \cap C_0 \cap C) \leq \dim \psi_d(C_0 \cap C) \leq \left(\frac{d+2}{2}\right) - 3,
\]
which contradicts (194) and proves (192). On the other hand, since $C$ is determined by $A$, $|A \cap C| \geq \left(\frac{d+2}{2}\right) - 1$ and thus
\[
|A \cap C_0 \cap C| \geq |A \cap C| - |\{a_0\}| \geq \left(\frac{d+2}{2}\right) - 2
\]
From (192) and (195), $\eta$ is well defined; furthermore these inequalities force that
\[
|A \cap C_0 \cap C| = |A \cap C| - |\{a_0\}|
\]
so $a_0 \in C$ for all $C \in \mathcal{O}_{d,2n+1-(\frac{d+2}{2})}(A)$. Hence, for any $C_1, C_2 \in \mathcal{O}_{d,2n+1-(\frac{d+2}{2})}(A)$ such that $A \cap C_0 \cap C_1 = A \cap C_0 \cap C_2$, we get that $A \cap C_1 = A \cap C_2$. In so far as $C_1$ and $C_2$ are determined by $A$, we conclude that $C_1 = C_2$. This yields that $\eta$ is injective so
\[
|\mathcal{O}_{d,2n+1-(\frac{d+2}{2})}(A)| \leq \left(\frac{|A \cap C_0|}{\left(\frac{d+2}{2}\right) - 2}\right) = \left(\frac{|A| - 1}{\left(\frac{d+2}{2}\right) - 2}\right).
\]
concluding the proof of iii).

\[ \square \]

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