Clifford Algebras in Physics

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Abstract

We study briefly some properties of real Clifford algebras and identify them as matrix algebras. We then show that the representation space on which Clifford algebras act are spinors and we study in details matrix representations. The precise structure of these matrices gives rise to the type of spinors one is able to construct in a given space-time dimension: Majorana or Weyl. Properties of spinors are also studied. We finally show how Clifford algebras enable us to construct supersymmetric extensions of the Poincaré algebra. A special attention to the four, ten and eleven-dimensional space-times is given. We then study the representations of the considered supersymmetric algebras and show that representation spaces contain an equal number of bosons and fermions. Supersymmetry turns out to be a symmetry which mixes non-trivially the bosons and the fermions since one multiplet contains bosons and fermions together. We also show how supersymmetry in four and ten dimensions are related to eleven dimensional supersymmetry by compactification or dimensional reduction.

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1 Introduction

The real Clifford algebra $\mathbb{C}_{t,s}$ is the associative algebra generated by a unit $1$ and $d = t + s$ elements $e_1, \cdots, e_d$ satisfying

$$e_M e_N + e_N e_M = 2\eta_{MN}, \quad \text{with } \eta_{MN} = \begin{cases} 0 & \text{if } M \neq N \\ 1 & \text{if } M = N = 1, \cdots, t \\ -1 & \text{if } M = N = t + 1, \cdots, d. \end{cases} \quad (1.1)$$

It is well-known that the Clifford algebra $\mathbb{C}_{t,s}$ can be represented by $2^\left[\frac{d}{2}\right] \times 2^\left[\frac{d}{2}\right]$ matrices called the Dirac matrices (with $\left[ a \right]$ the integer part of $a$). Since $\mathfrak{so}(t, s) \subset \mathbb{C}_{t,s}$, the $2^\left[\frac{d}{2}\right]$-dimensional complex vector space on which the Dirac matrices act is also a representation of $\mathfrak{so}(t, s)$, the spinor representation. Thus on a physical ground Clifford algebras are intimately related to fermions. The four dimensional Clifford algebra, or more precisely its matrix representation, was introduced in physics by Dirac \[1\] when he was looking for a relativistic first order differential equation extending the Schrödinger equation. This new equation called now the Dirac equation, is in fact a relativistic equation describing spin one-half fermions as the electron.

Among the various applications of Clifford algebras in physics and mathematical physics, we will show how they are central tools in the construction of supersymmetric theories. In particle physics there are two types of particles: bosons and fermions. The former, as the photon, have integer spin and the latter, as the electron, have half-integer spin. Properties (mass, spin, electric charge etc.) or the way particles interact together are understood by means of Lie algebras (describing space-time and internal symmetries). Supersymmetry is a symmetry different from the previous ones in the sense that it is a symmetry which mixes bosons and fermions. Supersymmetry is not described by Lie algebras but by Lie superalgebras and contains generators in the spinor representation of the space-time symmetry group.

In this lecture, we will show, how supersymmetry is intimately related to Clifford algebras. Along all the steps of the construction, a large number of details will be given. Section 2 will be mostly mathematical and devoted to the definition, and the classification of Clifford algebras. In section 3, we will show how Clifford algebras are related to special relativity and to spinors. Section 4 is a technical section, central for the construction of supersymmetry. Some basic properties of the Dirac $\Gamma-$matrices (the matrices representing Clifford algebras) will be studied allowing to introduce different types of spinors (Majorana, Weyl). We will show how the existence of these types of spinors is crucially related to the space-time dimension. Useful details will be summarized in some tables. Finally, in section 5 we will construct explicitly supersymmetric algebras and their representations with a special emphasis to the four, ten and eleven-dimensional space-time. We will also show how lower dimensional supersymmetry are related to higher dimensional supersymmetry by dimensional reduction.

2 Definition and classification

In this section we give the definition of Clifford algebras. We then show that it is possible to characterize Clifford algebras as matrix algebras and show that the properties of real Clifford algebras depend on the dimension modulo 8.
2.1 Definition

Consider $E$ a $d$-dimensional vector space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Let $Q$ be a quadratic form, non-degenerate of signature $(t, s)$ on $E$. The quadratic form $Q$ naturally defines a symmetric scalar product on $E$

\[ \eta(x, y) = \frac{1}{4}(Q(x + y) - Q(x - y)), \text{ for all } x, y \in E. \]  

(2.1)

Denote $\eta = \text{Diag}(1, \ldots, 1, -1, \ldots, -1)$ the tensor metric in an orthonormal basis. Denote also $\eta_{MN} \leq M, N \leq d$ the matrix elements of $\eta$ and $\eta^{MN}$ the matrix elements of the inverse matrix $\eta_{MN} = \eta^{MN} = \begin{cases} 0, & M \neq N \\ 1, & M = N = 1, \ldots, t \\ -1, & M = N = t + 1, \ldots, d \end{cases}$

**Definition 2.1** 1) The associative $\mathbb{K}$-algebra $\mathbb{C}^K_{t, s}$ generated by a unit 1 and $d = t + s$ generators $e_M, M = 1, \ldots, t + s$ satisfying

\[ \{e_M, e_N\} = e_M e_N + e_N e_M = 2\eta_{MN} \]  

(2.2)

is called the Clifford algebra of the quadratic form $Q$.

2) The dimension of $\mathbb{C}^K_{t, s}$ is $2^d$ and a convenient basis is given by

\[ 1, e_{M_1} e_{M_2} \cdots, e_{M_1} e_{M_2} \cdots e_{M_{d-1}}, e_1 e_2 \cdots e_d, 1 \leq M_1 < M_2 < \cdots < M_{d-1} \leq d. \]  

(2.3)

**Remark 2.2** The case where $Q(x) = 0$ for all $x$ in $\mathbb{K}$ will be considered latter on (see remark 4.1) and corresponds to Grassmann algebras.

**Remark 2.3** Clifford algebras could have been defined in a more formal way as follow. Let $T(E) = \mathbb{K} \oplus E \oplus (E \otimes E) \oplus \cdots$ be the tensor algebra over $E$ and let $J(Q)$ be the two-sided ideal generated by $x \otimes x - Q(x)1, x \in E$ in $T(E)$. The quotient algebra $T(E)/J(Q)$ is the Clifford algebra $\mathbb{C}^K_{t, s}$. The canonical map $\hat{i}_Q : E \rightarrow \mathbb{C}^K_{t, s}$ given by the composition $E \rightarrow T(E) \rightarrow \mathbb{C}^K_{t, s}$ is an injection.

**Remark 2.4** If we set $\Lambda(E) = \mathbb{K} \oplus E \oplus \Lambda^2(E) \oplus \cdots \oplus \Lambda^d(E)$ to be the exterior algebra on $E$, then the canonical map $\hat{i}_Q$ extends to a vector-space isomorphism $\hat{i}_Q : \Lambda(E) \rightarrow \mathbb{C}^K_{t, s}$ given on $\Lambda^n(E)$ by

\[ \hat{i}_Q(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in S_n} \frac{1}{n!} v_{\sigma(1)} \cdots v_{\sigma(n)} \]

with $S_n$ the group of permutations with $n$ elements.

2.2 Structure

We call $\mathbb{C}^K_{0t, s}$ the subalgebra of $\mathbb{C}^K_{t, s}$ generated by an even product of generators $e_M$ and $\mathbb{C}^K_{1t, s}$ the vector space (which is not a subalgebra) of $\mathbb{C}^K_{t, s}$ generated by an odd product of generators $e_M$. We also introduce
\[ \varepsilon = e_1 e_2 \cdots e_d \]  

(2.4)
corresponding to the product of all the generators of \( C_{t,s} \). An easy calculation gives

\[ \varepsilon^2 = (-1)^{\frac{d(d-1)}{2}} e_1^2 \cdots e_M^2 = (-1)^{\frac{d(d-1)}{2} + s} = \begin{cases} (-1)^{\frac{d-s}{2}} & \text{d even} \\ (-1)^{\frac{d-s-1}{2}} & \text{d odd.} \end{cases} \]  

(2.5)
Furthermore, when \( d \) is even \( \varepsilon \) commutes with \( C_{0t,s} \) and anticommutes with \( C_{1t,s} \) and when \( d \) is odd \( \varepsilon \) commutes both with \( C_{0t,s} \) and \( C_{1t,s} \). It can be shown that the centre \( Z(C_{t,s}) = \mathbb{K} \) (resp. \( Z(C_{t,s}) = \mathbb{K} \oplus \mathbb{K} \varepsilon \)) when \( d \) is even (resp. when \( d \) is odd). The element \( \varepsilon \) will be useful to decide whether or not Clifford algebras are simple. We first characterize real Clifford algebras that we denote from now on \( \mathfrak{C}_{t,s} \) to simplify notation. Then, we study the case of complex Clifford algebras \( \mathfrak{C}_{t,s} = \mathfrak{C}_{t,s} \otimes \mathbb{R} \mathbb{C} \).

### 2.2.1 Real Clifford algebras

From (2.5) \( \varepsilon^2 = 1 \) for \( t-s = 0 \) (mod. 4) when \( d \) is even and \( \varepsilon^2 = 1 \) for \( t-s = 1 \) (mod. 4) when \( d \) is odd and thus in these cases \( P_{\pm} = \frac{1}{2}(1 \pm \varepsilon) \) define orthogonal projection operators \( (P_+ P_- = 0, P_+ + P_1 = 1) \). These projectors will enable us to define ideals of \( \mathfrak{C}_{t,s} \) and \( \mathfrak{C}_{0t,s} \).

**Proposition 2.5** Let \( \mathfrak{C}_{t,s} \) be a real Clifford algebra and let \( P_{\pm} \) be defined as above.

(i) When \( d \) is odd and \( t-s = 1 \) (mod. 4) then \( P_{\pm} \) belong to the centre \( Z(\mathfrak{C}_{t,s}) \) and \( \mathfrak{C}_{t,s} = P_+ \mathfrak{C}_{t,s} \oplus P_- \mathfrak{C}_{t,s} \). This means that \( \mathfrak{C}_{t,s} \) is not simple (\( P_{\pm} \mathfrak{C}_{t,s} \) are two ideals of \( C_{t,s} \)).

(ii) When \( d \) is odd, \( \mathfrak{C}_{0t,s} \) is simple.

(iii) When \( d \) is even, \( \mathfrak{C}_{t,s} \) is simple.

(iv) When \( d \) is even and \( t-s = 0 \) (mod. 4) then \( P_{\pm} \) belong to the centre \( Z(\mathfrak{C}_{0t,s}) \) and thus we have \( \mathfrak{C}_{0t,s} = P_+ \mathfrak{C}_{0t,s} \oplus P_- \mathfrak{C}_{0t,s} \). This means that \( \mathfrak{C}_{0t,s} \) is not simple (\( P_{\pm} \mathfrak{C}_{0t,s} \) are two ideals of \( C_{0t,s} \)).

The proof will in fact be given in table 1, section 4.

### 2.2.2 Complex Clifford algebras

In this case, \( E \) is a complex vector space, and the tensor metric \( \eta \) can always be chosen to be Euclidean, i.e. \( \eta = \text{Diag}(1, \cdots, 1) \). We denote now \( \overline{\mathfrak{C}}_d \) a complex Clifford algebras generated by \( d \) generators. When we are over the field of complex numbers, a projection operator can always be defined. Indeed if \( \varepsilon^2 = -1, (i\varepsilon)^2 = 1 \), so we set \( P_\pm = \frac{1}{2}(1 \pm \varepsilon) \) if \( \varepsilon^2 = 1 \) and \( P_\pm = \frac{1}{2}(1 \pm i\varepsilon) \) if \( \varepsilon^2 = -1 \).

**Proposition 2.6** Let \( \overline{\mathfrak{C}}_d \) be a complex Clifford algebra, we have the following structure:

(i) when \( d \) is odd, \( \overline{\mathfrak{C}}_d = P_+ \overline{\mathfrak{C}}_d \oplus P_- \overline{\mathfrak{C}}_d \) is not simple;

(ii) when \( d \) is even, \( \overline{\mathfrak{C}}_d \) is simple;

(iii) when \( d \) is odd, \( \overline{\mathfrak{C}}_{0d} \) is simple;

(iv) when \( d \) is even, \( \overline{\mathfrak{C}}_{0d} = P_+ \overline{\mathfrak{C}}_{0d} \oplus P_- \overline{\mathfrak{C}}_{0d} \) is not simple.
2.3 Classification

We now show that it is possible to characterize all Clifford algebras as matrix algebras [2,3]. (See also [4] and references therein.) We first study the case of real Clifford algebras. The interesting point in this classification is two-fold. Firstly, the knowledge of \(C_{1,0}, C_{0,1}, C_{2,0}, C_{0,2}, C_{1,1}\) determines all the other \(C_{t,s}\). Secondly, the properties of \(C_{t,s}\) depend on \(t - s \mod 8\).

2.3.1 Real Clifford algebras

In these family of algebras we may add \(C_{0,0} = \mathbb{R}\). From the definition (2.7) one can show

\[
\begin{align*}
C_{0,0} &= \mathbb{R} & C_{1,0} &= \mathbb{R} \oplus \mathbb{R} & C_{0,1} &= \mathbb{C} \\
C_{2,0} &= M_2(\mathbb{R}) & C_{1,1} &= M_2(\mathbb{R}) & C_{0,2} &= \mathbb{H}
\end{align*}
\]  

(2.6)

with \(\mathbb{H}\) the quaternion algebra and \(M_n(\mathbb{F})\) the \(n \times n\) matrix algebra over the field \(\mathbb{F} = \mathbb{R}, \mathbb{C}\) or \(\mathbb{H}\). Recall that the quaternion algebra is generated by three imaginary units \(i, j, k = ij\) satisfying \(i^2 = j^2 = -1\) and \(ij + ji = 0\) which is precisely the definition of \(C_{0,2}\). For the algebra \(C_{2,0}\) if we set \(h_{11} = \frac{1}{2}(1 + e_1), h_{22} = \frac{1}{2}(1 - e_1), h_{12} = h_{11}e_2\) and \(h_{21} = h_{22}e_2\) one can check that they are independent and that they satisfy the multiplication law of \(M_2(\mathbb{R})\). Similar simple arguments give the identifications (2.6). Next, notice the

**Proposition 2.7** We have the following isomorphisms

\[
\begin{align*}
(1) & \quad C_{t,s} \otimes \mathbb{R} C_{2,0} \cong C_{s+2,t}, & (2) & \quad C_{t,s} \otimes \mathbb{R} C_{1,1} \cong C_{t+1,s+1}, & (3) & \quad C_{t,s} \otimes \mathbb{R} C_{0,2} \cong C_{s,t+2}.
\end{align*}
\]  

(2.7)

**Proof:** To prove the first isomorphism, introduce \(\{e_M\}_{M=1, \ldots, d}\) (resp. \(\{f_1, f_2\}\)) the generators of \(C_{t,s}\) (resp. \(C_{2,0}\)). Then, one can show that \(\{g_M = e_M \otimes f_1 f_2, g_{d+1} = 1 \otimes f_1, g_{d+2} = 1 \otimes f_2\}_{M=1, \ldots, d}\) (with 1 the unit element of \(C_{1,s}\)) satisfy the relations (2.2) for \(C_{s+2,t}\). To end the proof we just have to check that the \(g_{M_1} \cdots g_{M_\ell}, 1 \leq M_1 < \cdots < M_\ell \leq d + 2, \ell = 1, \cdots, d + 2\) are independent. A similar proof works for the two other isomorphisms. \(\text{QED}\)

Finally, we recall some well-known isomorphisms of real matrix algebras

**Proposition 2.8**

\[
\begin{align*}
(1) & \quad M_m(\mathbb{R}) \otimes M_n(\mathbb{R}) \cong M_{mn}(\mathbb{R}), & (5) & \quad \mathbb{C} \otimes \mathbb{R} \cong \mathbb{C} \oplus \mathbb{C}, \\
(2) & \quad M_m(\mathbb{R}) \otimes \mathbb{R} \cong M_m(\mathbb{R}), & (6) & \quad \mathbb{C} \otimes \mathbb{R} \cong M_2(\mathbb{C}), \\
(3) & \quad M_m(\mathbb{R}) \otimes \mathbb{R} \cong M_m(\mathbb{C}), & (7) & \quad \mathbb{H} \otimes \mathbb{R} \cong M_4(\mathbb{R}).
\end{align*}
\]  

(2.8)

**Proof:** The only non-trivial isomorphisms to be proved are those involving the quaternions. Recall that \(\mathbb{H} \cong \{q_0 \sigma_0 + q_1 (-i \sigma_1) + q_2 (-i \sigma_2) + q_3 (-i \sigma_3), q_0, q_1, q_2, q_3 \in \mathbb{R}\}\) with

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(2.9)

the Pauli matrices, is a faithful representation of the quaternions in \(M_2(\mathbb{C})\). Thus, (6) is easily proven.

To prove (7), let \(q = q_0 1 + q_1 i + q_2 j + q_3 k, \ q' = q'_0 1 + q'_1 i + q'_2 j + q'_3 k\) be two given quaternions and set \(q' = q'_0 1 - q'_1 i - q'_2 j - q'_3 k\). Define now
\[ f_{q,q'} : \mathbb{H} \rightarrow \mathbb{H} \] \[ x \mapsto x' = qxq'. \]

It is now a matter of calculation to show that \( f_{q,q'} \) can be represented by a \( 4 \times 4 \) real matrix acting on the component of the quaternion \( x \). This proves (7). \( \text{QED} \)

From the isomorphisms (2.7) and (2.8) one is able to calculate all the algebras \( \mathcal{C}_{d,0} \) and \( \mathcal{C}_{0,d} \). As an illustration, we give the result for \( \mathcal{C}_{0,7} \):

\[
\mathcal{C}_{0,7} \cong \mathcal{C}_{5,0} \otimes \mathcal{C}_{0,2} \cong \mathcal{C}_{0,3} \otimes \mathcal{C}_{2,0} \otimes \mathcal{C}_{0,2} \cong \mathcal{C}_{1,0} \otimes \mathcal{C}_{0,2} \otimes \mathcal{C}_{2,0} \cong \mathcal{C}_{0,2}
\]

\[
\cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} \otimes \mathcal{M}_2(\mathbb{R}) \otimes \mathbb{H}
\]

\[
\cong (\mathbb{H} \oplus \mathbb{H}) \otimes \mathcal{M}_2(\mathbb{R}) \otimes \mathbb{H}
\]

\[
\cong (\mathcal{M}_4(\mathbb{R}) \oplus \mathcal{M}_4(\mathbb{R})) \otimes \mathcal{M}_2(\mathbb{R})
\]

\[
\cong \mathcal{M}_8(\mathbb{R}) \oplus \mathcal{M}_8(\mathbb{R}).
\]

The Clifford algebras \( \mathcal{C}_{d,0}, \mathcal{C}_{0,d}, 0 \leq d \leq 8 \) are given in table 1.

| \( d \) | \( \mathcal{C}_{d,0} \) | \( \mathcal{C}_{0,d} \) |
|---|---|---|
| 1 | \( \mathbb{R} \oplus \mathbb{R} \) | \( \mathbb{C} \) |
| 2 | \( \mathcal{M}_2(\mathbb{R}) \) | \( \mathbb{H} \) |
| 3 | \( \mathcal{M}_2(\mathbb{C}) \) | \( \mathbb{H} \oplus \mathbb{H} \) |
| 4 | \( \mathcal{M}_2(\mathbb{H}) \) | \( \mathcal{M}_2(\mathbb{H}) \) |
| 5 | \( \mathcal{M}_2(\mathbb{H}) \oplus \mathcal{M}_2(\mathbb{H}) \) | \( \mathcal{M}_4(\mathbb{C}) \) |
| 6 | \( \mathcal{M}_4(\mathbb{H}) \) | \( \mathcal{M}_8(\mathbb{R}) \) |
| 7 | \( \mathcal{M}_8(\mathbb{C}) \) | \( \mathcal{M}_8(\mathbb{R}) \oplus \mathcal{M}_8(\mathbb{R}) \) |
| 8 | \( \mathcal{M}_{16}(\mathbb{R}) \) | \( \mathcal{M}_{16}(\mathbb{R}) \) |

Table 1: Clifford algebras \( \mathcal{C}_{d,0} \) and \( \mathcal{C}_{0,d} \), \( 1 \leq d \leq 8 \).

We observe as claimed in the previous subsection that only \( \mathcal{C}_{1,0}, \mathcal{C}_{0,3}, \mathcal{C}_{5,0}, \mathcal{C}_{0,7} \) are not simple (see Proposition 2.5 (i)).

Next the identity

\[
\mathcal{C}_{0,2} \otimes \mathcal{C}_{2,0} \otimes \mathcal{C}_{0,2} \cong \mathcal{C}_{0,8} \cong \mathcal{C}_{8,0} \cong \mathcal{M}_{16}(\mathbb{R})
\]

yields

\[
\mathcal{C}_{d+8,0} \cong \mathcal{C}_{d,0} \otimes \mathcal{C}_{8,0}, \quad \mathcal{C}_{0,d+8} \cong \mathcal{C}_{0,d} \otimes \mathcal{C}_{0,8}.
\]

Thus if \( \mathcal{C}_{d,0} \cong \mathcal{M}_n(\mathbb{F}) \), (\( \mathbb{F} = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \)), then \( \mathcal{C}_{d+8,0} \cong \mathcal{M}_{16n}(\mathbb{F}) \). This ends the classification of the algebras \( \mathcal{C}_{d,0} \) and \( \mathcal{C}_{0,d} \). The case of space-time with arbitrary signature \( [3] \) are obtained from \( \mathcal{C}_{d,0} \) and \( \mathcal{C}_{0,d} \) through the identity.
\[
\mathcal{C}_{t,s} \cong \mathcal{C}_{t-s,0} \otimes \mathcal{C}_{1,1} \otimes \cdots \otimes \mathcal{C}_{1,1}, \quad t \geq s \\
\cong \mathcal{C}_{t-s,0} \otimes M_{2^s}(\mathbb{R}), \quad t > s
\]

(2.12)
since \(\mathcal{C}_{1,1} \cong M_2(\mathbb{R})\). Thus if \(\mathcal{C}_{t-s,0} \cong M_n(\mathbb{F})\), \((\mathbb{F} = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H})\), then \(\mathcal{C}_{t,s} \cong M_{2^n}(\mathbb{F})\). We summarize in table 2 the results for the Clifford algebras \(\mathcal{C}_{t,s}\) with an arbitrary signature.

| \(t - s \mod 8\) | \(\mathcal{C}_{t,s}\) |
|-------------------|-------------------|
| 0                | \(M_{2^\ell}(\mathbb{R})\) |
| 1                | \(M_{2^\ell}(\mathbb{R}) \oplus M_{2^\ell}(\mathbb{R})\) |
| 2                | \(M_{2^\ell}(\mathbb{R})\) |
| 3                | \(M_{2^\ell}(\mathbb{C})\) |
| 4                | \(M_{2^{\ell-1}}(\mathbb{H})\) |
| 5                | \(M_{2^{\ell-1}}(\mathbb{H}) \oplus M_{2^{\ell-1}}(\mathbb{H})\) |
| 6                | \(M_{2^{\ell-1}}(\mathbb{H})\) |
| 7                | \(M_{2^\ell}(\mathbb{C})\) |

Table 2: Clifford algebras \(\mathcal{C}_{t,s}\). (\(\ell = [d/2]\), the integer part of \(d/2\), \(d = t + s\)).

The only non-simple algebras are those when \(t - s = 1 \mod 4\).

### 2.3.2 Complex Clifford algebras

Finally we give the classification for complex Clifford algebras. The complex Clifford algebras \(\mathcal{C}_{t,s}^C \cong \mathcal{C}_{t,s} \otimes \mathbb{C}\) are obtained by complexification of the real Clifford algebras \(\mathcal{C}_{t,s}\). Of course the complexification of the algebra \(\mathcal{C}_{t,s}\) depends only on \(d = t + s\) and it will be denoted \(\overline{\mathcal{C}}_d\). Using tables 1 and 2 we obtain

\[
\overline{\mathcal{C}}_d = \begin{cases} 
M_{2^{\ell}}(\mathbb{C}) & d \text{ even} \\
M_{2^{\ell}}(\mathbb{C}) \oplus M_{2^{\ell}}(\mathbb{C}) & d \text{ odd}.
\end{cases}
\]

(2.13)

### 3 Clifford algebras in relation to special relativity

After recalling the definition of the basic group of invariance of special relativity, we show that the representation spaces on which Clifford algebras act correspond to spinors. We then obtain in a natural way the group \(\text{Pin}(t, s)\) which is a non-trivial double covering of the group \(O(t, s)\). We further identify the Lie algebra \(\mathfrak{so}(t, d) \subset \mathcal{C}_{t,s}\).

#### 3.1 Poincaré and Lorentz groups

The basic group of special relativity \((i.e. \text{in a four dimensional Minkowskian space-time with a metric of signature } (1,3))\) can be easily extended to a \(d\)--dimensional pseudo-Euclidian space of
signature \((t, s)\). Let \(i_1, \ldots, i_{t+s}\) be an orthonormal basis of the vector space \(E\). The directions \(i_1, \ldots, i_t\) are time-like \((\eta(i_M, i_M) = 1 > 0, \ 1 \leq M \leq t)\) and the directions \(i_{t+1}, \ldots, i_{t+s}\) are space-like \((\eta(i_M, i_M) = -1 < 0, \ t + 1 \leq M \leq t + s)\). The Lorentz group is defined by \(O(t, s) = \{ f \in GL(E) : \eta(f(x), f(y)) = \eta(x, y), \forall x, y \in E \} \). If \(\Lambda\) denotes a matrix representation of \(f\), from \(f(x) = \Lambda x, \eta(x, y) = x^\eta y\) and \(\eta(f(x), f(y)) = \eta(x, y)\), we get \(\Lambda^T \eta \Lambda = \eta\) with \(\Lambda^T\) the transpose of the matrix \(\Lambda\), and \(\Lambda\) is a transformation preserving the tensor metric. The Lorentz group has four connected components

\[
\mathcal{L} = O(t, s) = \mathcal{L}^\uparrow_+ \oplus \mathcal{L}^{\downarrow}_+ \oplus \mathcal{L}^\uparrow_- \oplus \mathcal{L}^{\downarrow}_-
\]

where \(\mathcal{L}^\uparrow_+, \mathcal{L}^{\downarrow}_+\) represent elements of determinant \(\pm 1\) and \(\uparrow\) (resp. \(\downarrow\)) represents elements with positive (negative) temporal signature. Let \(R_i, \ i = 1, \ldots, d\) be the reflections in the hyperplane perpendicular to the \(i\)th direction. \(R_1, \ldots, R_t\) are time-like reflections and \(R_{t+1}, \ldots, R_{t+s}\) are space-like reflections. More generally one can consider a reflection \(R(v)\) orthogonal to a given direction \(v \in E\) such that \(\eta(v, v) \neq 0\) said to be time-like if \(\eta(v, v) > 0\) and space-like if \(\eta(v, v) < 0\). An element of \(O(t, s)\) is given by a products of certain numbers of such reflections. The structure of the various components of the Lorentz group are as follow

\[
\begin{align*}
\mathcal{L}^\uparrow_+ & \text{ is a continuous group, i.e. is associated to some Lie algebra} \\
\mathcal{L}^\downarrow_+ & = R(v)\mathcal{L}^\uparrow_+, \text{ where } v \text{ is a space-like direction, say } i_{t+1} \\
\mathcal{L}^\downarrow_- & = R(v)\mathcal{L}^\uparrow_+, \text{ where } v \text{ is a time-like direction, say } i_1 \\
\mathcal{L}^\uparrow_- & = R(v)R(v')\mathcal{L}^\downarrow_+, \text{ where } v \text{ is a time-like direction, say } i_1 \\
\mathcal{L}^{\downarrow}_- & \text{ and where } v' \text{ is a space-like direction, say } i_{t+1}.
\end{align*}
\]

In other words, if \(R(v_1)\cdots R(v_n)\) is a product of (i) an even number of space-like and time-like reflections it belongs to \(\mathcal{L}^\uparrow_+\), (ii) an odd number of space-like and an even number of time-like reflections it belongs to \(\mathcal{L}^\downarrow_+\), (iii) an even number of space-like and an odd number of time-like reflections it belongs to \(\mathcal{L}^\downarrow_-\), (iv) an odd number of space-like and time-like reflections to it belongs to \(\mathcal{L}^\uparrow_-\). When the signature is \((1, d - 1)\), \(R(e_1) = T\) is the operator of time reversal (it changes the direction of the time), and when in addition \(d\) is even \(R(e_2) \cdots R(e_d) = P\) is the parity operator (it corresponds to a reflection with respect to the origin). Furthermore, one can identify several subgroups of \(O(t, s)\):

\[
O(t, s) = \mathcal{L}, \quad SO(t, s) = \mathcal{L}^\uparrow_+ \oplus \mathcal{L}^{\downarrow}_+ \quad SO_+(t, s) = \mathcal{L}^\uparrow_+.
\]

\(SO(t, s)\) is constituted of an even product of reflections and \(SO_+(t, s)\) an even product of time-like and space-like reflections. Note that with an Euclidian metric, \(O(d)\) has only two connected components. Moreover, none of these groups are simple connected.

For further use, we introduce the generators of \(\mathfrak{so}(t, s)\) the Lie algebra of \(SO_+(t, s)\). A conventional basis is given by the \(L_{MN} = -L_{NM}\) which corresponds to the generators which generate the “rotations” in the plane \((M - N)\). We also introduce \(P_M\) \((1 \leq M \leq d)\) the generators of the space-time translations.\(^1\) The generators \(L_{MN}\) and \(P_M\) generate the so-called Poincaré algebra or the inhomogeneous Lorentz algebra noted \(\mathfrak{iso}(t, s)\) and satisfy

\(^1\)If \(x^M, \ M = 1, \ldots, d\) denote the components in the \(d\)-dimensional vector space \(E\) and we set \(x_N = \eta_{NM}x^M\), we have \(L_{MN} = (x_M \frac{\partial}{\partial x^N} - x_N \frac{\partial}{\partial x^M})\) and \(P_M = \frac{\partial}{\partial x^M}\).
\[ [L_{MN}, L_{PQ}] = -\eta_{MP}L_{NQ} + \eta_{NP}L_{MQ} - \eta_{MQ}L_{PN} + \eta_{NQ}L_{PM}, \]
\[ [L_{MN}, P_P] = \eta_{NP}P_M - \eta_{MP}P_N. \] (3.4)

### 3.2 Universal covering group of the Lorentz group

In this section we only consider real Clifford algebras \( \mathfrak{C}_{t,s} \). We furthermore identify a vector \( v = x^M i_M \in E \) with an element \( x^M e_M \in \mathfrak{C}_{t,s} \) (see remark 2.3). With such an identification, we have
\[ x^2 = ((x^1)^2 + \cdots + (x^t)^2 - (x^{t+1})^2 - \cdots - (x^{t+s})^2 = \eta(x, x). \]

**Definition 3.1** The Clifford group \( \Gamma_{t,s} \) is the subset of invertible elements \( x \) of \( \mathfrak{C}_{t,s} \) such that \( \forall v \in E, xv^{-1} \in E \).

It is clear that invertible elements of \( E \) belong to \( \Gamma_{t,s} \) (this excludes the null vectors \( i.e \) the vectors such that \( \eta(v, v) = 0 \)). If we take \( x \in E \subset \mathfrak{C}_{t,s} \) invertible, since \( xv + vx = 2\eta(x, v) \), we have
\[ xvx^{-1} = \frac{1}{x^2}xvx = -\left( v - \frac{2\eta(x, v)}{x^2}x \right) = -R(x)(v), \] (3.5)
which corresponds to a symmetry in the hyperplane perpendicular to \( x \). More generally, for \( s \in \Gamma_{t,s} \), the transformation \( \rho(s) \) defined by \( \rho(s)(x) = sx.s^{-1} \) belong to \( O(t, s) \). However, the representation \( \rho : \Gamma_{t,s} \rightarrow GL(E) \) is not faithful because if \( x \in \Gamma_{t,s} \), then \( ax \in \Gamma_{t,s} \) \((a \in \mathbb{C}^*) \) and \( \rho(ax) = \rho(x) \).

A standard way to distinguish the elements \( x \) and \( ax \) is to introduce a normalisation. For that purpose we define \( \overline{\cdot} : \mathfrak{C}_{t,s} \rightarrow \mathfrak{C}_{t,s} \) by
\[ \overline{e_M} = (-1)^{t+1}e_M, \quad \overline{e_M_1 \cdots e_M_{k-1} e_M_k} = \overline{e_M_k} \cdot \overline{e_M_{k-1}} \cdots \overline{e_M_1}, \] (3.6)
(see remark 2.3). Next, we define \( N(x) = x\overline{x} \). Note that in the case of the quaternions \((t = 0, s = 2)\), we have \( \overline{i} = -i, \overline{j} = -j, \overline{k} = -k \). Now, with this definition of the norm, for \( x \in E \) we have
\[ (-1)^{t+1}N(x) = x^2 \begin{cases} > 0 & \text{if } x \text{ is time-like} \\ < 0 & \text{if } x \text{ is space-like} \end{cases} \] (3.7)

**Proposition 3.2** (*Proposition 3.8 of [2]*) If \( x \in \Gamma_{t,s} \) then \( N(x) \in \mathbb{R}^* \).

As a consequence for \( x, y \in \Gamma_{t,s} \) we have \( N(xy) = N(x)N(y) \). Then, the “norm” \( N \) enables us to define definite subgroups of the Clifford group \( \Gamma_{t,s} \). The first subgroup which can be defined is
\[ \text{Pin}(t, s) = \left\{ x \in \Gamma_{t,s} \text{ s.t. } |N(x)| = 1 \right\}. \] (3.8)

By construction it is easy to see that \( \rho(\Gamma_{t,s}) = \rho(\text{Pin}(t, s)) \) and that \( x \in E \) invertible \( \Rightarrow x \in \text{Pin}(t, s) \). Moreover, since an element of \( O(t, s) \) is given by a product of a given number of reflections, we have \( \rho(\text{Pin}(t, s)) \supseteq O(t, s) \). Conversely it has been shown ([2], proposition 3.10) that \( \rho(\text{Pin}(t, s)) \subseteq O(t, s) \), thus \( \rho(\text{Pin}(t, s)) = O(t, s) \). A generic element of \( \text{Pin}(t, s) \) is then given by
\[ s = v_1 v_2 \cdots v_n \] (3.9)
with \( v_i, i = 1, \cdots, n \) invertible elements of \( E \) (thus \( \rho(s) \) corresponds to the transformation given by \( R(v_1)R(v_2) \cdots R(v_n) \)). Assume now, that there are \( p \) time-like vectors and \( q \) space-like vectors in \( s \) \((p + q = n)\), then \( N(s) = (-1)^{p(t+1)+q} = (-1)^{nt+p} \).
1. If \( n \) is even then \( \rho(s) \in SO(t, s) \).

2. If \( n \) is even and \( N(s) > 0 \) then \( \rho(s) \in SO_+(t, s) \).

3. If \( n \) is even and \( N(s) < 0 \) then \( \rho(s) \in L^\downarrow \). For instance when \( d \) is an even number and 
\( t = 1, s = d - 1, \rho(\varepsilon) = \rho(e_1 \cdots e_d) \in L^\downarrow \) and corresponds to the \( PT \) inversion (\( P \) being the operator of parity transformation and \( T \) the operator of time reversal.)

4. If \( n \) is odd and \( N(s)(-1)^{t+1} > 0 \) then \( \rho(s) \in L^\downarrow_\ell \). For instance when \( (t, s) = (1, d-1) \), \( \rho(e_1 \cdots e_d) \in L^\downarrow_\ell \) and corresponds to the operator of time reversal.

5. If \( n \) is odd and \( N(s)(-1)^{t+1} < 0 \) then \( \rho(s) \in L^\uparrow_\ell \). For instance when \( d \) is even and \( (t, s) = (1, d-1) \), \( \rho(e_1 \cdots e_d) \in L^\uparrow_\ell \) and corresponds to the operator of parity transformation.

We can now define the various subgroup of \( \Gamma_{t,s} \):

\[
\begin{align*}
\text{Pin}(t,s) &= \left\{ x \in \Gamma_{t,s} \text{ s.t. } |N(x)| = 1 \right\} \\
\text{Spin}(t,s) &= \text{Pin}(t,s) \cap C_{0t,s} \\
\text{Spin}_+(t,s) &= \left\{ x \in \text{Spin}(t,s) \text{ s.t. } N(x) = 1 \right\}
\end{align*}
\] (3.10)

### 3.3 The Lie algebra of \( \text{Spin}_+(t,s) \)

Among the various groups of the previous subsection, only \( \text{Spin}_+(t,s) \) is a connected Lie group. If one introduce the \( d(d-1)/2 \) elements 

\[
S_{MN} = \frac{1}{4}(e_M e_N - e_N e_M), \quad 1 \leq M \neq N \leq d
\] (3.11)

a direct calculation gives

\[
\begin{align*}
[S_{MN}, e_P] &= \eta_{NP} e_M - \eta_{MP} e_N \\
[S_{MN}, S_{PQ}] &= -\eta_{MP} S_{NQ} + \eta_{NP} S_{MQ} - \eta_{MQ} S_{PN} + \eta_{NQ} S_{PM}.
\end{align*}
\] (3.12)

This means that \( S_{MN} \) generate the Lie algebra \( so(t,s) \) \( (so(t,s) \subset C_{t,s}) \) and that \( e_M \) are in the vector representation of \( so(t,s) \). Furthermore, if we define

\[
e_M^{(\ell)} = \frac{1}{\ell!} \sum_{\sigma \Sigma_{\ell}} \epsilon(\sigma) e_{M_{\sigma(1)}} \cdots e_{M_{\sigma(\ell)}},
\] (3.13)

(with \( \epsilon(\sigma) \) the signature of the permutation \( \sigma \)) using (3.12) one can show that they are in the \( \ell^{\text{th}} \)-antisymmetric representation of \( so(t,s) \) (see remark 2.4).

**Proposition 3.3** The group \( \text{Spin}_+(t,s) \) is a non-trivial double covering group of \( SO_+(t,s) \).

**Proof:** First notice that if \( x \in \text{Spin}_+(t,s) \) then \(-x \in \text{Spin}_+(t,s) \). Then, we show that there exist a continuous path in \( \text{Spin}_+(t,s) \) which connects 1 to \(-1 \). Let \( i_M, i_N \) be two space-like directions. The path \( R(\theta) = e^{\frac{\theta}{2} i_M e_N} + \sin \frac{\theta}{2} i_M e_N \) with \( \theta \in [0, 2\pi] \) is such that \( R(0) = 1 \) and \( R(2\pi) = -1 \) and thus connects 1 to \(-1 \) in \( \text{Spin}_+(t,s) \). Which ends the proof. \( QED \)

In the same way, \( \text{Spin}(t,s) \) is a non-trivial double covering group of \( SO(t,s) \) and \( \text{Pin}(t,s)^2 \) a non-trivial double covering group of \( O(t,s) \).

\(^2\)This is a joke of J.-P. Serre.
4 The Dirac $\Gamma$–matrices

In this section, because we are mostly interested in Clifford algebras in relation to space-time physics, we will focus on vector spaces with a Lorentzian signature $(1, d - 1)$ or an Euclidian signature $(0, d)$. From now on, in the case of the $(1, d - 1)$ signature, we use the notations commonly used in the literature. The indices of the space-time components run from 0 to $d - 1$, greek indices $\mu, \nu, \cdots = 0, \cdots , d - 1$ are space-time indices and latin indices $i, j, \cdots = 1, \cdots , d - 1$ are space indices. This section is devoted to the study of the matrix representation of Clifford algebras. The precise structure of these matrices gives rise to the type of spinors one is able to construct in a given space-time dimension: Majorana or Weyl. Properties of spinors are also studied. This section is technical but is central for the construction of supersymmetric theories in various dimensions.

4.1 Dirac spinors

For physical applications, we need to have matrix representations of Clifford algebras. As we have seen in section 2.3, the case where $d$ is even is very different from the case where $d$ is odd. Indeed, complex Clifford algebras are simple when $d$ is even and are not simple when $d$ is odd. This means that the representation is faithful when $d$ is even and not faithful when $d$ is odd. When $d$ is odd, since $\varepsilon = e_1 \cdots e_d$ allows to define the two ideals of the Clifford algebra, $\varepsilon$ will be represented by a number. Moreover, one of the ideals will be the kernel of the representation.

It is well-known that the basic building block of the matrix representation of $\bar{C}_d$ are the Pauli matrices [2, 5]. Define the $(2k + 1)$ matrices by the tensor products of $k$ Pauli matrices (we thus construct $2^k \times 2^k$ matrices):

\[
\begin{align*}
\Sigma_1^{(k)} &= \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \\
\Sigma_3^{(k)} &= \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \\
&\vdots \\
\Sigma_{2\ell - 1}^{(k)} &= \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \\
&\vdots \\
\Sigma_{2k - 1}^{(k)} &= \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1, \\
\Sigma_{2k}^{(k)} &= \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2, \\
\Sigma_{2k + 1}^{(k)} &= \sigma_3 \otimes \cdots \otimes \sigma_3.
\end{align*}
\]

Observing that

\[
\Sigma_M^{(k+1)} = \Sigma_M^{(k)} \otimes \sigma_0, \quad M = 1, \cdots , 2k, \quad \Sigma_{2k+i}^{(k+1)} = \Sigma_{2k+i}^{(k)} \otimes \sigma_i, \quad i = 1, 2, 3
\]

a simple recurrence on $k$ shows that the $\Sigma$–matrices satisfy $\Sigma_M^{(k)} \Sigma_N^{(k)} = 2 \delta_M^N$. Thus

\[
\gamma : \bar{C}_d \rightarrow M_{2k}(\mathbb{C}) \\
e_M \mapsto \Sigma_M
\]
(d = 2k or d = 2k + 1) is a representation of $\mathcal{C}_d$. This representation is faithful when d is even and non-faithful when d is odd ($\gamma(\varepsilon) = \Sigma^{(k)}_1\Sigma^{(k)}_2 \cdots \Sigma^{(k)}_{2k+1} = i^k$.) Since the group Spin$_+ (t, s)$ is the double covering of the group $SO_+ (t, s)$, the representation on which the $\Gamma$–matrices act is a representation of Spin$_+ (t, s)$. The elements of the representation space $\mathcal{C}_d$ are called Dirac spinors. A Dirac spinor exists in any dimension d and has $2^d$ complex components. We denote by $\Psi_D$ a Dirac spinor.

Now having represented the algebra $\mathcal{C}_d$, (d = 2k or d = 2k + 1) if we set $\Gamma_M = \Sigma^{(k)}_M$, 1 ≤ M ≤ t and $\Gamma_M = i\Sigma^{(k)}_M$, $t + 1 \leq M \leq t + s$ we have a representation of the algebra $\mathcal{C}_{t,s}$ corresponding to a real form of $\mathcal{C}_d$. In particular for a Minkowskian space-time we introduce

$$\Gamma_0 = \Sigma^{(k)}_1, \Gamma_j = i\Sigma^{(k)}_{j+1}, j = 1, \cdots, d - 1. \quad (4.4)$$

We also denote by

$$\Gamma_{\mu\nu} = \frac{1}{4}(\Gamma_{\mu}\Gamma_{\nu} - \Gamma_{\nu}\Gamma_{\mu}) \quad (4.5)$$

the generators of the Lie algebra $so(1, d - 1)$.

**Remark 4.1** Without using the result of the section 2.3 we show that $\mathcal{C}_{2n} \cong M_{2n}(\mathbb{C})$ as follows. Set

$$a_i = \frac{1}{2}(e_{2i} + ie_{2i+1}), \quad b_i = \frac{1}{2}(e_{2i} - ie_{2i+1}), \quad i = 1, \cdots, n \quad (4.6)$$

which satisfy

$$a_i a_j + a_j a_i = 0, \quad b_i b_j + b_j b_i = 0, \quad a_i b_j + b_j a_i = \delta_{ij}. \quad (4.7)$$

Thus the $a_i$ and $b_i$ generate the fermionic oscillator algebra (i.e. the algebra which underlines the fermionic fields after quantization). Note that the $a_i$ alone generate the Grassmann algebra of dimension $n$. To obtain a representation of the algebra (4.7), we introduce the Clifford vacuum $\Omega = \{-\frac{1}{2}, -\frac{1}{2}, \cdots, -\frac{1}{2}\}$ such that $a_i \Omega = 0, i = 1, \cdots, n$. Then we obtain the representation of the algebra (4.7) by acting in all possible ways with $b_i$ at most once each:

$$|s_1, s_2, \cdots, s_n\rangle = (b_1)^{s_1+1/2} \cdots (b_n)^{s_n+1/2} \left| \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2} \right\rangle, \quad s_1, s_2, \cdots, s_n = -\frac{1}{2}, -\frac{1}{2} \quad (4.8)$$

and thus we obtain a $2^n$–dimensional representation. The notation with $s_i = -\frac{1}{2}, \frac{1}{2}$ (instead of $s_i = 0, 1$) seems to be unnatural, but is appropriate to identify the weight of $|s_1, s_2, \cdots, s_n\rangle$ with respect to the Cartan subalgebra of $so(2n, \mathbb{C})$. Furthermore, it can be shown that $a_i^\dagger = b_i$ and that the representation is unitary for the real algebra $\mathcal{C}_{2n,0}$. In this case, the notation reflects the property of the Dirac spinor with respect to the group Spin$(2n)$. A basis of the Cartan subalgebra of $so(2n)$ can be taken to be $\Gamma_{2i} 2i + 1 = ia_i a_i^\dagger - \frac{i}{2}$ and the vector $|s_1, s_2, \cdots, s_n\rangle$ is an eigenvector of $\Gamma_{2i} 2i + 1$ with eigenvalue $is_i$. The half-integer weights show that we have a spinor representation of Spin$(2n)$.

(The $i$ factor comes from the fact that SO$(2n)$ is the real form of SO$(2n, \mathbb{C})$ corresponding to the maximal compact algebra, for which the generators are antihermitian – note that, in the physical literature there is no $i$ factor since $\Gamma_{\mu\nu} = i\frac{1}{2}\Gamma_{\mu\nu}$ instead of (4.5). – ) To end this remark we just notice that the fermionic generators $(a_i, b_i)$ are in the vector representation of Spin$(2N)$. 

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4.2 Majorana and Weyl spinors

In the previous subsection we have introduced Dirac spinors. In this subsection, we will see that further spinors can be defined such as Majorana spinors or Weyl spinors etc. The Weyl spinors are just a consequence of the reducibility of the (Dirac) spinors representation of \( \mathfrak{so}(2n, \mathbb{C}) \) (and of course of all its real forms). They exist in any even space-time dimensions and for any signature. If there exists a real matrix representation of the Clifford algebra \( \mathcal{C}_{t,s} \) then one is able to consider real (or Majorana) spinors. The existence of Majorana spinors depends on the space-time dimension and of the signature of the metric.

If we denote \( S \) the vector space corresponding to the Dirac spinors, recall the Lorentz generators writes (3.11)

\[
\Gamma_{\mu\nu} = \frac{1}{4}(\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu).
\]  
(4.9)

The matrices \( \Gamma_\mu \) act on the representation \( S \) and the representation of \( \mathcal{C}_{t,s} \) is just \( \text{End}(S) \), the set of endomorphisms of \( S \). In the same way the matrices \( -\Gamma^t \) (with \( t \) the transpose operation) act on the dual representation \( S^\vee \), the matrices \( \Gamma^* \) (with \( * \) the complex conjugation\(^3\)) act on the complex-conjugate representation \( \bar{S} \) and the matrices \( \Gamma^\dagger \) (with \( \dagger \) the hermitian conjugation) act on the representation \( \bar{S}^\vee \). These representations are in fact equivalent, since we can find elements of \( \text{End}(S) \) such that

\[
A \Gamma_{\mu\nu} A^{-1} = -\Gamma^t_{\mu\nu}, \quad B \Gamma_{\mu\nu} B^{-1} = \Gamma^*_{\mu\nu}, \quad C \Gamma_{\mu\nu} C^{-1} = -\Gamma^t_{\mu\nu},
\]  
(4.10)

(see below). The operators \( A, B, C \) are intertwining operators (\( B \) intertwines the representations \( S \) and \( \bar{S} \)). This is in fact related to (4.9) and

\[
A \Gamma_{\mu} A^{-1} = \eta_A \Gamma^t_{\mu}, \quad B \Gamma_{\mu} B^{-1} = \eta_B \Gamma^*_{\mu}, \quad C \Gamma_{\mu} C^{-1} = \eta_C \Gamma^\dagger_{\mu},
\]  
(4.11)

with \( \eta_A, \eta_B \) and \( \eta_C \) signs which depend on \( d \), as we will see in subsection 4.2.2.

As a direct consequence of (4.10), if \( \Psi_D \in S \) transforms like \( \Psi'_D = S(\alpha)\Psi_D = e^{\frac{i}{2} \alpha_{\mu\nu} \Gamma_{\mu\nu}} \Psi_D \) under a Lorentz transformation (\( \alpha_{\mu\nu} \in \mathbb{R} \) are the parameters of the transformation) then \( \Xi_D = \Xi^t_D A \) and \( \Xi^\dagger C \) belong to \( S^* \) (i.e. transform with \( S(\alpha)^{-1} \)). Thus, \( \Xi_D \Psi_D \) and \( \Xi^\dagger_D C \Psi_D \) define invariants.

4.2.1 Weyl spinors

Consider first the case of the complex Clifford algebra \( \overline{\mathcal{C}}_d \). When \( d \) is even the spinor representation \( S \) is reducible \( S = S_+ \oplus S_- \). Indeed, if we define the chirality matrix

\[
\chi = i^{[\frac{d}{2}]+1} \Gamma_0 \Gamma_1 \cdots \Gamma_{d-1}
\]  
(4.12)

it satisfies

\[
\chi^2 = 1, \quad \{ \Gamma_\mu, \chi \} = 0, \quad [\chi, \Gamma_{\mu\nu}] = 0.
\]  
(4.13)

\(^3\)In the mathematical literature the complex conjugate of \( \Gamma_\mu \) is denoted \( \bar{\Gamma}_\mu \).
Hence, $\chi$ allows to define the complex left- and right-handed Weyl spinors. These spinors correspond to the two irreducible representations of $\bar{\mathcal{C}}_d$:

$$
\Psi_L = \frac{1}{2}(1-\chi)\Psi_D, \quad \Psi_R = \frac{1}{2}(1+\chi)\Psi_D,
$$

(4.14)

$\Psi_L \in S_-, \Psi_R \in S_+$ and $\Psi_D \in S$ or $S_\pm = \{ \Psi \in S \text{ s.t. } \chi\Psi = \pm\Psi \}$. This means that the spaces $S_\pm$ carry an irreducible representation of the complex algebra $\bar{\mathcal{C}}_d$. Now, if we consider the real form $\mathfrak{c}_{0t,s}$ of $\bar{\mathcal{C}}_d$, the spaces $S_\pm$ become irreducible representations of $\mathfrak{c}_{0t,s}$. The generators of the two Weyl spinors of $\mathfrak{s}\mathfrak{o}(1,d-1)$ are $\Sigma^\pm_{\mu\nu} = \frac{1}{2} (1 \pm \chi) \Gamma^\mu_\nu$. The Weyl spinors are called the semi-spinors in the mathematical literature.

**Remark 4.2** As we have seen (see Proposition 2.6 (i)), the algebra $\bar{\mathcal{C}}_d$ is not simple. Then it falls into two simple ideals $\mathcal{C}_{0d} = P_+\mathcal{C}_{0d} + P_-\mathcal{C}_{0d}$. Moreover, the Lie algebra $\mathfrak{s}\mathfrak{o}(t,s) \subset \mathcal{C}_{0d}$, this means that a (complex) Dirac spinor is reducible into two (complex) Weyl spinors.

### 4.2.2 Majorana spinors

Majorana spinors are real spinors. As we now see, the existence of Majorana spinors crucially depends on the dimension $d$ and on the metric signature. We consider the case of Lorentzian signatures $(1,d-1)$. The general case can be easily deduced. However, for a general study see [6], [7].

The key observation in this subsection is the simple fact that (i) the Pauli matrices are hermitian (ii) the matrices $\sigma_1, \sigma_3$ are real and symmetric and (iii) the matrix $\sigma_2$ is purely imaginary and antisymmetric. Thus from (4.11) we see (take $d = 2n + 1$ odd)

\[
\begin{align*}
\Gamma^\dagger_0 &= \Gamma_0, & \Gamma^*_0 &= \Gamma_0, & \Gamma^t_0 &= \Gamma_0, \\
\Gamma^\dagger_2i &= -\Gamma_{2i}, & \Gamma^*_{2i} &= -\Gamma_{2i}, & \Gamma^t_{2i} &= \Gamma_{2i}, & i = 1, 2, \ldots, n \\
\Gamma^\dagger_{2i-1} &= -\Gamma_{2i-1}, & \Gamma^*_{2i-1} &= \Gamma_{2i-1}, & \Gamma^t_{2i-1} &= -\Gamma_{2i-1}, & i = 0, 2, \ldots, n.
\end{align*}
\]

The results that we will establish here seems to depend on the choice of basis we have chosen, but in fact they are independent of this choice (see [6]).

| Space-time of even dimension $d = 2n$ |
|--------------------------------------|

When $d$ is even, we take the first $d$ matrices above. In this case, we have seen that complex Clifford algebras are isomorphic to some matrix algebras. This means that if we have another representation of the Clifford algebra $\Gamma'_\mu \{ \Gamma'_\mu, \Gamma'_\nu \} = 2\eta_{\mu\nu}$, there exists an invertible matrix $U$ of End($S$) such that $\Gamma'_\mu = U\Gamma_\mu U^{-1}$. In particular, we have (see (4.15) and (4.10))

\[
\begin{align*}
\chi\Gamma_\mu\chi^{-1} &= -\Gamma_\mu, & \chi &= i^{n+1}\Gamma_0\Gamma_1 \cdots \Gamma_{2n-1} \\
A\Gamma_\mu A^{-1} &= \Gamma^\dagger_\mu, & A &= \Gamma_0 \\
B_1\Gamma_\mu B_1^{-1} &= (-1)^{n+1}\Gamma_\mu^*, & B_1 &= \Gamma_2\Gamma_4 \cdots \Gamma_{2n-2} \\
B_2\Gamma_\mu B_2^{-1} &= (-1)^n\Gamma_\mu^t, & B_2 &= \Gamma_0\Gamma_1 \cdots \Gamma_{2n-1} \\
C_1\Gamma_\mu C_1^{-1} &= (-1)^{n+1}\Gamma_\mu^t, & C_1 &= \Gamma_0\Gamma_2 \cdots \Gamma_{2n-2} \\
C_2\Gamma_\mu C_2^{-1} &= (-1)^n\Gamma_\mu^*, & C_2 &= \Gamma_1\Gamma_3 \cdots \Gamma_{2n-1}
\end{align*}
\]

$\Gamma_0$ is the only hermitian matrix. $B_1$ is the product of all purely imaginary matrices, $B_2$ is the product of real matrices, $C_1$ is the product of symmetric matrices and $C_2$ of antisymmetric matrices. Note
also the relation between the matrices $A, B, C$: $C_i = AB_i, i = 1, 2$. We set $\eta_1 = (-1)^{n+1}, \eta_2 = (-1)^n$. From the definition of $B$, we have

$$B_1B^*_1 = (-1)^{\frac{(n-1)(n-2)}{2}} = \epsilon_1, \quad B_2B^*_2 = (-1)^{\frac{n(n-1)}{2}} = \epsilon_2.$$  

(4.17)

Now for further use, we collect the following signs

$$
\begin{align*}
&d = 2 \text{ mod. } 8 \quad \epsilon_1 = +, \quad \eta_1 = + \quad \epsilon_2 = +, \quad \eta_2 = - \\
&d = 4 \text{ mod. } 8 \quad \epsilon_1 = +, \quad \eta_1 = - \quad \epsilon_2 = -, \quad \eta_2 = + \\
&d = 6 \text{ mod. } 8 \quad \epsilon_1 = -, \quad \eta_1 = + \quad \epsilon_2 = -, \quad \eta_2 = - \\
&d = 8 \text{ mod. } 8 \quad \epsilon_1 = -, \quad \eta_1 = - \quad \epsilon_2 = +, \quad \eta_2 = +.
\end{align*}
$$  

(4.18)

Majorana spinors, $d$ even

From (4.16), we get $B \Gamma_{\mu \nu} B^{-1} = \Gamma^*_{\mu \nu}$, $B = B_1, B_2$, so the Dirac spinor $\Psi_D$ and $B^{-1}\Psi^*_D$ transform in the same way under the group $\text{Pin}(1, d-1)$. A Majorana spinor is a spinor that we impose to be a real spinor. It satisfies

$$\Psi^*_M = B\Psi_M, \quad B = B_1, B_2.$$  

(4.19)

But taking the complex conjugate of the above equation gives $\Psi_M = B^*\Psi^*_M = B^*B\Psi_M$. Thus this is possible only if $BB^* = 1$ or when $\epsilon_1 = 1$ and/or $\epsilon_2 = 1$ and from (4.18) when $d = 2, 4, 8$ mod. 8. More precisely, looking to (4.16), we observe that the $\Gamma$–matrices can be taken to be purely real if $\eta_1 = \epsilon_1 = 1$ or $\eta_2 = \epsilon_2 = 1$ i.e. if $d = 2, 8$ mod. 8 and the Dirac matrices can be taken purely imaginary if $\epsilon_1 = 1, \eta_1 = -1$ or $\epsilon_2 = 1, \eta_2 = -1$ that is when $d = 2, 4$ mod. 8 (In this case the matrices $\Gamma^2\mu \nu$ are purely imaginary, the matrices $\Gamma^{2(2)}\mu \nu \rho$ are purely imaginary etc.). The first type of real spinors will be called Majorana spinors although the second type of spinors pseudo-Majorana spinors. This result could have been deduced from table 2. Indeed, we have shown that $C_{t,s}$ is real when $t - s = 0, 1, 2$ mod. 8. Observing that $C_{t,s}$ is related to $C_{s,t}$ by the transformation $\epsilon_M \rightarrow i\epsilon_M$, we get that $\gamma(\epsilon_M) = \Gamma_M$ are purely imaginary when $s - t = 1, 2$ mod. 8. Thus we have:

**Proposition 4.3** Assume $d$ is even.

(i) Majorana spinors of $\text{Pin}(1, d-1)$ exist when $d = 2, 8$ mod. 8.

(ii) Pseudo-Majorana spinors of $\text{Pin}(1, d-1)$ exist when $d = 2, 4$ mod. 8.

SU(2)—Majorana spinors, $d$ even

As we have seen, a Majorana spinor is a spinor which satisfies $\Psi_M = B\Psi^*_M$, with $BB^* = 1$. However, if $BB^* = -1$ one can define SU(2)—Majorana spinors (or SU(2)—pseudo-Majorana spinors). This is a pair of spinors $\Psi_i, i = 1, 2$ satisfying

$$\Psi^*_i = e^{ij} B\Psi_j,$$  

(4.20)

where $e^{ij}$ is the SU(2)—invariant antisymmetric tensor, and $i, j = 1, 2$. (More generally one can take an even number of spinors and substitute to $e$ the symplectic form $\Omega$. These spinors are also called symplectic spinors.) The SU(2)—Majorana spinors exist when $\epsilon_1 = -1, \eta_1 = 1$ or $\epsilon_2 = -1, \eta_2 = 1$ and the the SU(2)—pseudo-Majorana spinors exist when $\epsilon_1 = -1, \eta_1 = -1$ or $\epsilon_2 = -1, \eta_2 = -1$.
Proposition 4.4 Assume $d$ even.

(i) $SU(2)$—Majorana spinors of $\text{Pin}(1,d-1)$ exist when $d = 4, 6 \text{ mod. } 8$.

(ii) $SU(2)$—pseudo-Majorana spinors of $\text{Pin}(1,d-1)$ exist when $d = 6, 8 \text{ mod. } 8$.

Majorana-Weyl and $SU(2)$—Majorana-Weyl spinors, $d$ even

Now, from (4.12) we observe that the Weyl condition is compatible with the Majorana or the $SU(2)$—Majorana condition if $n + 1 = 2, 4 \text{ mod. } 4$. Indeed, in such space-time dimension $\chi = (-1)^{(n+1)/2} \Gamma_0 \cdots \Gamma_{2n-1}$ and the chirality matrix is real. This means in this case that the Weyl spinors $\Psi_L$ and $\Psi_R$ can be taken real or $SU(2)$—Majorana. This is possible only when the space-time dimension $d = 2, 6 \text{ mod. } 8$. Such spinor will be called Majorana-Weyl spinors ($d = 2 \text{ mod. } 8$) and $SU(2)$—Majorana-Weyl ($d = 6 \text{ mod. } 8$). In fact this is the dimensions for which $\mathcal{C}_{01,d-1}$ is not simple see Proposition 2.5 (iv).

Proposition 4.5 Assume $d$ even.

(i) Majorana-Weyl spinors of $\text{Pin}(1,d-1)$ exist when $d = 2 \text{ mod. } 8$.

(ii) $SU(2)$—Majorana-Weyl spinors of $\text{Pin}(1,d-1)$ exist when $d = 6 \text{ mod. } 8$.

Space-time of odd dimension $d = 2n + 1$

When the space-time is odd, the matrices $\Gamma_\mu, \mu = 0, \cdots, 2n - 1$ are the same as the ones for $d = 2n$ together with

$$\Gamma_{2n} = i\chi. \quad (4.21)$$

In this case taking the matrices (4.4) there is no matrix $U$ such that $UG\mu U^{-1} = -G\mu$, because when $d$ is odd Clifford algebras are not simple and $\gamma(\epsilon)$ is a number. This means in particular that differently to the case where $d$ is even we will have either $B\Gamma\mu B^{-1} = B\Gamma\mu^*$ or $B\Gamma\mu B^{-1} = -B\Gamma\mu^*$ where the sign depends on the dimension (similar property holds for the $C$ matrix). The analogous of (4.16) writes:

$$A\Gamma\mu A^{-1} = \Gamma^\dagger_\mu, \quad A = \Gamma_0$$
$$B\Gamma\mu B^{-1} = (-1)^n \Gamma\mu^*, \quad B = \Gamma_2 \Gamma_4 \cdots \Gamma_{2n}$$
$$C\Gamma\mu C^{-1} = (-1)^n \Gamma\mu^*, \quad C = \Gamma_0 \Gamma_2 \cdots \Gamma_{2n}$$

with

$$BB^* = (-1)^{n(n-1)/2}. \quad (4.22)$$

One can check that the matrices $B' = \Gamma_0 \Gamma_1 \cdots \Gamma_{2n-1}$ and $C' = \Gamma_1 \Gamma_3 \cdots \Gamma_{2n-1}$ give the same relations (4.22) as the matrices $B, C$. As for the even space-time dimension, we introduce $\epsilon = (-1)^{(n-1)/2}, \eta = (-1)^n$ which gives

$$d = 1 \text{ mod. } 8 \quad \epsilon = + \quad \eta = +$$
$$d = 3 \text{ mod. } 8 \quad \epsilon = + \quad \eta = -$$
$$d = 5 \text{ mod. } 8 \quad \epsilon = - \quad \eta = +$$
$$d = 7 \text{ mod. } 8 \quad \epsilon = - \quad \eta = -. \quad (4.24)$$

Then as for even dimensional space-time we have the following:
Proposition 4.6 Assume $d$ is odd.

(i) Majorana spinors of $\text{Pin}(1,d-1)$ exist when $d = 1 \mod 8$.

(ii) Pseudo-Majorana spinors of $\text{Pin}(1,d-1)$ exist when $d = 3 \mod 8$.

(iii) $SU(2)$—Majorana spinors of $\text{Pin}(1,d-1)$ exist when $d = 5 \mod 8$.

(iv) $SU(2)$—pseudo-Majorana spinors of $\text{Pin}(1,d-1)$ exist when $d = 7 \mod 8$.

We conclude this subsection by the following table.

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|---|---|---|---|---|---|---|---|---|----|----|
| M   | yes | yes | | yes | yes | yes | yes | | yes | yes | yes |
| PM  | yes | yes | yes | yes | | | | yes | yes | | |
| MW  | yes | yes | yes | yes | yes | yes | yes | yes | | | |
| SM  | | | yes | yes | yes | yes | yes | yes | | | |
| SPM | | | yes | yes | yes | yes | yes | yes | | | |
| SMW | | | yes | yes | yes | yes | yes | yes | | | |
| spinor | R | R | R | C | PR | PR | PR | C | R | R | R |
| SUSY | M | MW | PM | PM | SM | SMW | SPM | M | M | MW | PM |
| d.o.f. | 1 | 1 | 2 | 4 | 8 | 8 | 16 | 16 | 16 | 16 | 32 |

Table 3: Types of spinors in various dimensions. We have taken the following notations: M for Majorana, PM for pseudo-Majorana, MW for Majorana-Weyl, SM for $SU(2)$—Majorana, SPM for $SU(2)$—pseudo-Majorana and SMW for $SU(2)$—Majorana-Weyl. The representations of $\mathfrak{so}(1,d-1)$ are real (R), pseudo-real (PR) or complex (C) (see section 4.3). Recall that the number of real components (d.o.f. in the table) for the different types of spinors in $d$ dimensions is for (i) a Dirac spinor $2^{[2]}+1$ (ii) a Majorana or pseudo-Majorana spinor $2^{[2]}_1$ (iii) a Majorana-Weyl spinor $2^{[2]}_1$ (iv) a $SU(2)$—Majorana or $SU(2)$—pseudo-Majorana spinor $2^{[2]}_1+1$ and (v) a $SU(2)$—Majorana-Weyl spinor $2^{[2]}_1$. SUSY means the type of spinors we take to construct a supersymmetric theory (see section 5). We note finally that if some kind of spinors exist for $\text{Pin}(1,d-1)$, they also exist for $\text{Pin}(0,d-2)$ for instance $\text{Pin}(0,8)$ admits a Majorana-Weyl spinors like $\text{Pin}(1,9)$.

4.3 Real, pseudo-real and complex representations of $\mathfrak{so}(1,d-1)$

Now a group theory touch can be given to the different type of spinors. As we have seen, we have the following inclusion of algebras:

$$\mathfrak{so}(1,d-1) \subset \mathfrak{c}_{1,d-1}. \quad (4.25)$$

Recall that the generators of $\mathfrak{so}(1,d-1)$ are given by $\Gamma_{\mu \nu} = \frac{1}{4}(\Gamma_{\mu} \Gamma_{\nu} - \Gamma_{\nu} \Gamma_{\mu})$, $\mu, \nu = 0, \cdots, d-1,$
and the representation $\Psi_D$ and $B^{-1}\Psi_D^\star$ are equivalent. As we have seen they can be equated if $BB^* = 1$. The representation of $\mathfrak{so}(1,d-1)$ is called real if they can be equated and pseudo-real when they cannot. In odd dimensional space-time the representations can be either real or pseudo-real. In even dimension in addition to (pseudo-)real representations there exist complex representations.

When $d = 2n + 1$, the representation of $\mathfrak{so}(1,2n)$ are real when $d = 1, 3 \text{ mod. } 8$ and pseudo-real when $d = 5, 7 \text{ mod. } 8$.

When $d = 2n$, we introduce

$$
\Sigma_{\mu\nu}^{\pm} = \frac{1}{2}(1 \pm \chi)\Gamma_{\mu\nu}
$$

(4.26)

which are the generators of the spinor representations $S_{\pm}$. From (4.12) we get $\chi^* = (-1)^{n+1}B\chi B^{-1}$, thus

$$
\Sigma_{\mu\nu}^{\pm} = \left\{ \begin{array}{ll}
BS_{\mu\nu}^{\pm}B^{-1} & \text{when } n \text{ odd} \\
BS_{\mu\nu}^{-\pm}B^{-1} & \text{when } n \text{ even}
\end{array} \right.
$$

(4.27)

This means that the complex conjugate of $S_{\pm}$ is equal to itself when $d = 4k + 2$. The representation will be real when $d = 2 \text{ mod. } 8$ ($BB^* = 1$) and pseudo-real when $d = 6 \text{ mod. } 8$ ($BB^* = -1$). However, when $d = 4k$ the complex conjugate of $S_{\pm}$ is $S_{\mp}$ and the representation is complex.

**Remark 4.7** The results above can also give some insight on the structure of Clifford algebras. When $d$ is even, there always exists a matrix $B$ such that $B\Gamma_{\mu}B^{-1} = \Gamma_{\mu}$ (see (4.16)). If $BB^* = 1$ the Clifford algebra is real ($d = 0, 2 \text{ mod. } 8$) and if $BB^* = -1$ the algebra is quaternionian ($d = 4, 6 \text{ mod. } 8$). (Notice that for $d = 4$ the pseudo-Majorana spinors a taken with the opposite choice i.e with $B$ s.t. $B\Gamma_{\mu}B^{-1} = -\Gamma_{\mu}^*$. When $d$ is odd, either $\varepsilon^2 = 1$ ($\rho(\varepsilon) = \pm i$) or $\varepsilon^2 = -1$ ($\rho(\varepsilon) = \pm i$). When $\varepsilon^2 = 1$ (or when $B\Gamma_{\mu}B^{-1} = \Gamma_{\mu}$) the Clifford algebra is real when $BB^* = 1$ ($d = 1 \text{ mod. } 8$) and quaternionian when $BB^* = -1$ ($d = 5 \text{ mod. } 8$) although when $\varepsilon^2 = -1$ (or when $B\Gamma_{\mu}B^{-1} = -\Gamma_{\mu}$) the Clifford algebra is complex ($d = 3, 7 \text{ mod. } 8$). This can be compared with table 4.

### 4.4 Properties of (anti-)symmetry of the $\Gamma$–matrices

The matrices

$$
\Gamma_{\mu_1\cdots\mu_\ell}^{(\ell)} = \frac{1}{\ell !} \sum_{\sigma \in \Sigma_{\ell}} \epsilon(\sigma)\Gamma_{\mu_{\sigma(1)}\cdots\mu_{\sigma(\ell)}},
$$

(4.28)

with $\ell = 0, \cdots, d$ when $d$ is even and with $\ell = 0 \cdots, \frac{d-1}{2}$ when $d$ is odd constitute a basis of the representation of $C_{1,d-1}$. (In $d$ dimensions we have $\binom{d}{\ell}$ independant matrices $\Gamma_{\mu_1\cdots\mu_\ell}^{(\ell)}$ and $2^{2n} = \sum_{\ell=0}^{2n} \binom{2n}{\ell}$ when $d = 2n$ or $2^{2n} = \sum_{\ell=0}^{n} \binom{2n + 1}{\ell}$ when $d = 2n + 1$.) Note the normalisation
factor $\Gamma_{\mu\nu} = \frac{1}{2} \Gamma^{(2)}_{\mu\nu}$. Now using the matrices $C_1$ and $C_2$ in (4.16) and the matrix $C$ in (4.22), we can show that the matrices $\Gamma^{(\ell)} C^{-1}$ are either fully symmetric or fully antisymmetric. It is just a matter of a simple calculation to check the following:

\[
\begin{align*}
(\Gamma^{(\ell)}_{\mu_1 \cdots \mu_\ell} C_1^{-1})^t &= (-1)^{\frac{1}{2}((\ell+n)^2+(\ell-n))} \Gamma^{(\ell)}_{\mu_1 \cdots \mu_\ell} C_1^{-1} \quad d = 2n \\
(\Gamma^{(\ell)}_{\mu_1 \cdots \mu_\ell} C_2^{-1})^t &= (-1)^{\frac{1}{2}((\ell+n)^2+(\ell-n))} \Gamma^{(\ell)}_{\mu_1 \cdots \mu_\ell} C_2^{-1} \quad d = 2n \\
(\Gamma^{(\ell)}_{\mu_1 \cdots \mu_\ell} C^{-1})^t &= (-1)^{\frac{1}{2}((\ell+n)^2+(\ell-n))} \Gamma^{(\ell)}_{\mu_1 \cdots \mu_\ell} C^{-1} \quad d = 2n + 1
\end{align*}
\]  

(4.29)

We now summarize in the following table the symmetry properties of the $\Gamma C^{-1}$–matrices.

| \( \ell \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | S | | | | | | | | | | |
| 2 | S | S | A | | | | | | | | |
| 2 | A | S | S | | | | | | | | |
| 3 | A | S | | | | | | | | | |
| 4 | A | S | S | A | A | | | | | | |
| 4 | A | A | S | S | A | | | | | | |
| 5 | A | A | S | | | | | | | | |
| 6 | A | A | S | S | A | A | S | | | | |
| 6 | S | A | A | S | S | A | A | | | | |
| 7 | S | A | A | S | | | | | | | |
| 8 | S | A | A | S | S | A | A | S | | | |
| 8 | S | S | A | A | S | S | A | A | S | | |
| 9 | S | S | A | A | S | | | | | | |
| 10 | S | S | A | A | S | S | A | A | S | S | |
| 10 | A | S | S | A | A | S | S | A | A | S | S | |
| 11 | A | S | S | A | A | S | | | | | |

Table 4: Symmetry of $\Gamma^{(\ell)} C^{-1}$. The first row indicates the type of matrix $\ell$ for $\Gamma^{(\ell)} C^{-1}$ and the first column the space-time dimension. When $d$ is even we have two series of matrices $\Gamma^{(\ell)} C_1^{-1}$ and $\Gamma^{(\ell)} C_2^{-1}$ respectively in the first and second line. Finally, S (resp. A) indicates that $\Gamma^{(\ell)} C^{-1}$ is symmetric (resp. antisymmetric).

Now we would like to give some duality properties. We define $\varepsilon_{\mu_1 \cdots \mu_2n}$ the Levi-Civita tensor (equal to the signature of the permutation that brings $\mu_1 \cdots \mu_2n$ to $0, 1, \ldots, 2n - 1$, $\varepsilon_{0 \cdots 2n-1} = 1$).

We also introduce $\Gamma^\mu : \Gamma_\mu = \eta_{\mu\nu} \Gamma^\nu$ and $\varepsilon_{\mu_1 \cdots \mu_2n} = \varepsilon^{\mu_1 \cdots \mu_2n} \eta_{\mu_1 \nu_1} \cdots \eta_{\mu_2n \nu_2n} \varepsilon_{0 \cdots 2n-1} = -1$) and $\Gamma = \Gamma_0 \cdots \Gamma_{2n-1}$. From,
\[
\Gamma^{(\ell)}_{\mu_1 \cdots \mu_{2n}} = (-1)^{n} \epsilon_{\mu_1 \cdots \mu_{2n}} (4.30)
\]

\[
\Gamma^{(\ell)}_{\mu_1 \cdots \mu_{\ell}} = \frac{(-1)^{\frac{\ell(\ell-1)}{2}}}{(2n - \ell)!} \epsilon_{\mu_1 \cdots \mu_{2n}} \Gamma^{(\ell)}_{\mu_{\ell+1} \cdots \mu_{2n}}
\]

we get

\[
\star (\Gamma^{(n)}_{\mu_1 \cdots \mu_{\ell}}) = \pm i (-1)^{\frac{n(n+1)}{2} + 1} (\Gamma^{(n)}) \pm i \Gamma^{(n)} \Gamma)
\]

when \( n \) is even

\[
\star (\Gamma^{(n)}_{\mu_1 \cdots \mu_{\ell}}) = \pm (-1)^{\frac{n(n+1)}{2} + 1} (\Gamma^{(n)} \pm \Gamma^{(n)} \Gamma)
\]

when \( n \) is odd

with \( \star \) the Hodge dual \( \star X_{\mu_1 \cdots \mu_p} = \frac{1}{(2n-p)!} \epsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{2n-p}} X^{\nu_1 \cdots \nu_{2n-p}} \). Thus this means that the above matrices are (anti-)self-dual.

### 4.5 Product of spinors

Now we give the decomposition of the product of spinor representations. We recall that \( \text{End}(S) \cong S \otimes S^* \). Furthermore, given \( \Psi_D \) and \( \Xi_D \) two Dirac spinors, recall

\[
S = \Xi_D \Psi_D = \Xi_D^{\dagger} \Gamma_0 \Psi_D
\]

(4.32)

define invariant scalar products (see (4.10)).

**Remark 4.8** When the signature is \((t, s)\) we take \( \Gamma_1 \cdots \Gamma_t \) instead of \( \Gamma_0 \) to set \( \Psi = \Psi^{\dagger} \Gamma_1 \cdots \Gamma_t \).

This legitimate (3.6).

Similarly,

\[
\Xi^{\dagger} \Gamma^{(\ell)}_{\mu_1 \cdots \mu_{\ell}} \Psi_D
\]

(4.33)
transforms as an \( \ell \)th order antisymmetric tensor. Now, from (4.10) it is easy to see that

\[
T_{\mu_1 \cdots \mu_{\ell}} = \Xi^{\dagger} C \Gamma^{(\ell)}_{\mu_1 \cdots \mu_{\ell}} \Psi_D
\]

(4.34)
transforms also as an \( \ell \)th order antisymmetric tensor. Now, since the matrices \( \Gamma^{(\ell)}_{\mu_1 \cdots \mu_{\ell}} \) with \( \ell = 0, \cdots d - 1 \) when \( d \) is even and \( \ell = 0, \cdots \frac{d-1}{2} \) when \( d \) is odd constitute a basis of \( \mathcal{M}_{2\ell}(\mathbb{C}) \) and since \( \Xi^{\dagger} C \in S^* \) (the dual space of \( S \)) we have

\[
S \otimes S^* = \\
\begin{cases}
\mathbb{C} \oplus E \oplus \Lambda^2(E) \oplus \cdots \oplus \Lambda^d(E) & \text{when } d \text{ is even} \\
\mathbb{C} \oplus E \oplus \Lambda^3(E) \oplus \cdots \oplus \Lambda^\lfloor \frac{d}{2} \rfloor(E) & \text{when } d \text{ is odd}.
\end{cases}
\]

(4.35)

This decomposition has to be compared with Remark 2.4.

Now, when \( d = 2n \) is even one can calculate the product of Weyl spinors. Let \( \Psi_{\epsilon_1} \in S_{\epsilon_1}, \epsilon_1 = \pm \) and \( \Xi^{\dagger}_{\epsilon_2} C \in S^*_{\epsilon_2}, \epsilon_2 = \pm \) be two Weyl spinors. Recall that we have \( \chi \Psi_{\epsilon_1} = -\epsilon_1 \Psi_{\epsilon_1} \). Now, using
\[ C \chi C^{-1} = \Gamma_0^t \cdots \Gamma_{d-1}^t. \] By (4.15) the RHS writes \((-1)^n \chi\) but rearranging the product the RHS also writes \((-1)^n \chi^t\). Thus

\[ \chi^t = \chi, \quad C \chi = (-1)^n \chi C, \tag{4.36} \]

and we have

\[
\Xi_{\epsilon_2}^t \Gamma_{\mu_1 \cdots \mu_\ell}^{(\ell)} \Psi_{\epsilon_1} = \epsilon_1 \Xi_{\epsilon_2} \Gamma_{\mu_1 \cdots \mu_\ell}^{(\ell)} \chi \Psi_{\epsilon_1} = \epsilon_1 (-1)^\ell \Xi_{\epsilon_2} \Gamma_{\mu_1 \cdots \mu_\ell}^{(\ell)} \Psi_{\epsilon_1} = \epsilon_1 (-1)^{n+\ell} \Xi_{\epsilon_2} \Gamma_{\mu_1 \cdots \mu_\ell}^{(\ell)} \Psi_{\epsilon_1} \]

and \(\Xi_{\epsilon_2} \Gamma_{\mu_1 \cdots \mu_\ell}^{(\ell)} \Psi_{\epsilon_1} = 0\) if \(\epsilon_1 \epsilon_2 (-1)^{n+\ell} = -1\). This finally gives

\[
S_+ \otimes S_+^* = \begin{cases} 
\mathbb{C} \oplus \Lambda^2 (E) \oplus \cdots \oplus \Lambda^n (E)_+ & \text{when } n \text{ is even} \\
E \oplus \Lambda (E) \oplus \cdots \oplus \Lambda^n (E)_+ & \text{when } n \text{ is odd}
\end{cases} \tag{4.38}
\]

\[
S_+ \otimes S_-^* = \begin{cases} 
E \oplus \Lambda^2 (E) \oplus \cdots \oplus \Lambda^{n-1} (E) & \text{when } n \text{ is even} \\
\mathbb{C} \oplus \Lambda^2 (E) \oplus \cdots \oplus \Lambda^{n-1} (E) & \text{when } n \text{ is odd}
\end{cases}
\]

A special attention has to be paid for the antisymmetric tensors of order \(n\). Indeed, the antisymmetric tensor of order \(n\) is a reducible representation of \(SO(1, 2n - 1)\) and can be decomposed into self-dual and anti-self-dual tensors (see (4.31)) (denoted \(\Lambda^n (E)_\pm\)):

\[ \Lambda^n (E) = \Lambda^n (E)_+ \oplus \Lambda^n (E)_-. \tag{4.39} \]

Now a simple counting of the dimensions on both sides in (4.38) shows that only the self- (or anti-self-) dual \(n\)th-order tensors appears in the decomposition. (The dimension of \(\Lambda^p (E)\) is \(\binom{2n}{p}\)) and of \(\Lambda^n (E)_\pm\) is \(\frac{1}{2} \binom{2n}{n}\).) The choice \(\Lambda^n (E)_+\) is a matter of convention.

If we rewrite explicitly these relation (e.g. (4.35) for \(d\) even), we have

\[
\Psi_D \otimes \Xi_D^t C = \sum_{\ell=0}^{d} \frac{1}{\ell!} T^{(\ell)}_{\mu_1 \cdots \mu_\ell} \Gamma^{(\ell)}_{\mu_1 \cdots \mu_\ell}
\]

\[
\Psi_D \otimes \Xi_D^t = \sum_{\ell=0}^{d} \frac{1}{\ell!} T^{(\ell)}_{\mu_1 \cdots \mu_\ell} \Gamma^{(\ell)}_{\mu_1 \cdots \mu_\ell} C^{-1}
\tag{4.40}
\]

with \(T^{(\ell)}\) an antisymmetric \(\ell\)-th order tensor.

## 5 Clifford algebras and supersymmetry

In this section, with the help of the previous sections, we study supersymmetric algebras with a special attention to the four, ten and eleven-dimensional space-times. We then study the representations of the considered supersymmetric algebras and show that representation spaces contain an
equal number of bosons and fermions. Supersymmetry turns out to be a symmetry which mixes non-trivially the bosons and the fermions since one multiplet contains bosons and fermions together. We also show how four and ten dimensional supersymmetry are related to eleven dimensional supersymmetry by compactification or dimensional reduction.

5.1 Non-trivial extensions of the Poincaré algebra

Describing the laws of physics in terms of underlying symmetries has always been a powerful tool. For instance the Casimir operators of the Poincaré algebra \([3,4]\) are related to the mass and the spin of elementary particles as the electron or the photon. Moreover, it has been understood that all the fundamental interactions (electromagnetic, weak and strong interactions) are related to the Lie algebra \(u(1)_Y \times su(2)_L \times su(3)_C\), in the so-called standard model (see e.g. \([5]\) and references therein). The standard model is then described by the Lie algebra \(iso(1, 3) \times u(1)_Y \times su(2)_L \times su(3)_C\), where \(iso(1, 3)\) is related to space-time symmetries and \(u(1)_Y \times su(2)_L \times su(3)_C\) to internal symmetries. Even if the standard model is the physical theory were the confrontation between experimental results and theoretical predictions is in an extremely good accordance, there is strong arguments (which cannot be summarized here) that it is not the final theory.

Thus, to understand the properties of elementary particles, it is then interesting to study the kind of symmetries which are allowed in space-time. Within the framework of Quantum Field Theory (unitarity of the \(S\) matrix etc.), S. Coleman and J. Mandula have shown \([10]\) that if the symmetries are described in terms of Lie algebras, only trivial extensions of the Poincaré algebra can be obtained. Namely, the fundamental symmetries are based on \(iso(1, 3) \times g\) with \(g \supseteq u(1)_Y \times su(2)_L \times su(3)_C\) a compact Lie algebra describing the fundamental interactions and \([iso(1, 3), g] = 0\). Several algebras, in relation to phenomenology, have been investigated (see e.g. \([6, 7]\) such as \(su(5), so(10), e_6\) etc. Such theories are usually refer to “Grand-Unified-Theories” or GUT, i.e. theories which unify all the fundamental interactions. The fact that elements of \(g\) and \(iso(1, 3)\) commute means that we have a trivial extension of the Poincaré algebra.

Then, R. Haag, J. T. Lopuszanski and M. F. Sohnius \([11]\) understood that it was possible to extend in a non-trivial way the symmetries of space-time within the framework of Lie superalgebras (see Definition \([5,7]\)) in an unique way called supersymmetry. We first give the definition of a Lie superalgebra, and then we show how to construct a supersymmetric theory in any space-time dimensions using the results established in Section \([4]\).

**Definition 5.1** A Lie (complex or real) superalgebra is a \(\mathbb{Z}_2\)–graded vector space \(g = g_0 \oplus g_1\) endowed with the following structure

1. \(g_0\) is a Lie algebra, we denote by \([\ , \ ]\) the bracket on \(g_0\) \(([g_0, g_0] \subseteq g_0)\);
2. \(g_1\) is a representation of \(g_0\) \(([g_0, g_1] \subseteq g_1)\);
3. there exits a \(g_0\)–equivariant mapping \{ \ , \ } : \(S^2(g_1) \rightarrow g_0\) where \(S^2(g_1)\) denotes the two-fold symmetric product of \(g_1\) \((\{g_1, g_1\} \subseteq g_0)\);
4. The following Jacobi identities hold \((\forall b_1, b_2, b_3 \in g_0, \forall f_1, f_2, f_3 \in g_1)\)
   \[
   [[b_1, b_2], b_3] + [[b_2, b_3], b_1] + [[b_3, b_1], b_2] = 0 \\
   [[b_1, b_2], f_3] + [[b_2, f_3], b_1] + [[f_3, b_1], b_2] = 0 \\
   [b_1, \{f_2, f_3\}] - \{[b_1, f_2], f_3\} - \{f_2, [b_1, f_3]\} = 0 \\
   [f_1, \{f_2, f_3\}] + [f_2, \{f_3, f_1\}] + [f_3, \{f_1, f_2\}] = 0. 
   \]
The generators of zero (resp. one) gradation are called the bosonic (resp. fermionic) generators or \( g_0 \) (resp. \( g_1 \)) is called the bosonic (resp. fermionic) part of the Lie superalgebra. The first Jacobi identity is the usual Jacobi identity for Lie algebras, the second says that \( g_1 \) is a representation of \( g_0 \), the third identity is the equivariance of \( \{ , \} \). These identities are just consequences of 1., 2. and 3. respectively. However, the fourth Jacobi identity which is an extra constraint, is just the \( \mathbb{Z}_2 \)-graded Leibniz rule.

The supersymmetric extension of the Poincaré algebra is constructed, in the framework of Lie superalgebras, by adjoining to the Poincaré generators anticommuting elements, called supercharges (we denote \( Q \)), which belong to the spinor representation of the Poincaré algebra. Thus the supersymmetric algebra is a Lie superalgebra \( \mathfrak{g} = \mathfrak{iso}(1,d-1) \oplus S \) with brackets

\[
[L, L] = L, \quad [L, P] = P, \quad [L, Q] = Q, \quad [P, Q] = 0, \quad \{Q, Q\} = P,
\]

with \( (L, P) \) the generators of the Poincaré algebra that belong to the bosonic part of the algebra and \( Q \) the fermionic part of the algebra. This extension is non-trivial, because the supercharges \( Q \) are spinors, and thus do not commute with the generators of the Lorentz algebra. However, the precise definition depends on the space-time dimension because the reality properties of the spinor charges and hence the structure of the algebra depends on the dimensions (see Table 3).

A systematic study of supersymmetric extensions has been undertaken in [12]. Table 3 indicates the type of spinor we take in various dimensions. Furthermore, the number of supercharges \( N \) we consider is such that the total spinorial degrees of freedom is less than 32 (see section 5.3.1, Remark 5.5). This gives the possible choices for the spinorial charges (see Table 3):

\[
\begin{align*}
    d &= 4 \quad & N \text{ pseudo-Majorana} & \quad 1 < N \leq 8 \\
    d &= 5 \quad & N \text{ SU}(2)-\text{Majorana} & \quad 1 < N \leq 4 \\
    d &= 6 \quad & (N_+, N_-) \quad \text{(right-, left-) handed} & \quad 1 < N_+ + N_- \leq 4 \\
    & & \text{SU}(2)\text{-Majorana-Weyl} & \\
    d &= 7 \quad & N \text{ SU}(2)\text{-pseudo-Majorana} & \quad 1 < N \leq 2 \\
    d &= 8 \quad & N \text{ Majorana} & \quad 1 < N \leq 2 \\
    d &= 9 \quad & N \text{ Majorana} & \quad 1 < N \leq 2 \\
    d &= 10 \quad & (N_+, N_-) \quad \text{(right-, left-) handed} & \quad 1 < N_+ + N_- \leq 2 \\
    & & \text{Majorana-Weyl} & \\
    d &= 11 \quad & N \text{ pseudo-Majorana} & \quad N = 1
\end{align*}
\]

5.2 Algebra of supersymmetry in various dimensions

Using (5.3) we give the supersymmetric algebras in four, ten and eleven dimensions. Supersymmetry in other space-time dimensions are constructed along the same lines [12]. In this section, we will not give precise references of the subject, one may for instances see [13] and references therein (in particular we will not refer to the original papers on the subject\(^4\)).

5.2.1 Supersymmetry in four dimensions

Now, we give the precise structure of the supersymmetric extension of the Poincaré algebra in four dimensions. A standard reference on the subject is [14] (see also [15, 13]). The bosonic (or

\(^4\)Supersymmetry and supergravity is an intense subject of research. If one goes to the particle physics data basis [http://www.slac.standford.edu] and types find title supersymmetry or supergravity, one has 8767 different publications or types find k supersymmetry or supergravity, one has 42449 answers (the 1.06.2005)
the fermionic (or odd) sector is constituted of $N$ Majorana spinor supercharges $Q_I, I = 1, \cdots, N$ (see table 3). The algebraic structure is given by three types of bracket: (i) \([\text{even}, \text{even}]\), (ii) \([\text{even}, \text{odd}]\) and (iii) \([\text{odd}, \text{odd}]\), where even/odd means bosonic/fermionic generators. The first types of bracket is the Poincaré algebra (3.4). The action of the Poincaré algebra onto the fermionic supercharges is given by

\[
[L_{\mu\nu}, Q_I] = \Gamma_{\mu\nu}Q_I, \quad [P_\mu, Q_I] = 0,
\]

this is the second type of bracket. The first equation is due to the fact that $Q_I$ is in the spinor representation of the Lorentz algebra (with matrix representation $\Gamma_{\mu\nu}$ (4.3)). Since $P_\mu$ transforms like a vector, and among the $\Gamma^-$ matrices only $\Gamma_\mu$ transforms as a vector (see (3.13)), the second relation would be $[P_\mu, Q_I] = c\Gamma_\mu Q_I$. But the Jacobi identity (5.1) involving $(P, P, Q)$ leads to $c = 0$.

Now, we study the last type of brackets involving only odd generators. To be compatible with the literature [14, 15], we will not use the Dirac $\Gamma^-$ matrices given in Section 4, but

\[
\Gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}
\]

with $\sigma_\mu$ the Pauli matrices given in (2.9) and $(\bar{\sigma}_0, \bar{\sigma}_i) = (\sigma_0, -\sigma_i)$. In this representation the chirality matrix reads $\chi_4 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$ and the $B_4, C_4$ matrices (4.16) reduce to $B_4 = i\Gamma_2$, $C_4 = i\Gamma_0\Gamma_2$

The Majorana spinor supercharges are defined by

\[
Q_I = \begin{pmatrix} Q_{LI} \\ Q_{RI} \end{pmatrix}
\]

with $Q_{LI}$ (resp. $Q_{RI}$) complex left-handed (resp. right-handed) Weyl spinors. The condition $Q_I^* = B_4Q_I$ gives $Q_{RI} = -i\sigma_2Q_{LI}$ and the complex conjugate of a right-handed spinor is a left-handed spinor as we have seen in section 4.3. Finally, remember that in four dimensions we have for the product of two spinors (see (1.40)) $\Psi \otimes \Xi^* = T(0)\xi^{-1} + T(1)\mu\Gamma_\mu\xi^{-1} + \frac{1}{2}T(2)\mu\nu\Gamma_{\mu\nu}\xi^{-1} + \frac{1}{6}T(3)\mu\nu\rho\Gamma_{\mu\nu\rho}\xi^{-1} + \frac{1}{24}T(4)\mu\nu\rho\sigma\Gamma_{\mu\nu\rho\sigma}\xi^{-1}$.

To construct the last type of brackets, we make the following remarks:

1. the fermionic part of the algebra as to close onto the bosonic part, which reduces to the Lorentz generators $L_{\mu\nu}$ and to the generators of the space-time translations $P_\mu$;
2. the only symmetric Dirac $\Gamma-$ matrices are $\Gamma_\mu$ and $\Gamma_{\mu\nu}$ (see table 4 and equation (4.40));
3. the Jacobi identity (5.1) as to be satisfied.

The points 1 and 2 give

\[
\{Q_I, Q'_J\} = \delta_{IJ}(aP_\mu\Gamma_\mu\xi^{-1} + bL^{\mu\nu}\Gamma_{\mu\nu}\xi^{-1})
\]

with $Q'_J$ the transpose of $Q_J$ (in fact instead of $\delta_{IJ}$ we could have obtained a symmetric second order tensor, which can always be diagonalised [13].) Now, the Jacobi identity involving $(Q, Q, P)$ gives $b = 0$ since $L_{\mu\nu}$ and $P_\mu$ do not commute. Finally for conventional reason, we chose $a = -2.$
Now we observe that the bosonic part of the algebra can be enlarged by introducing some new (real) scalar generators \( X_{IJ}, Y_{IJ} \) commuting with all the bosonic elements and being antisymmetric \( X_{IJ} = -X_{JI}, Y_{IJ} = -Y_{JI} \) (the new generators are called central charges). Since, the matrices \( C^{-1}_4 \) and \( i\chi C^{-1}_4 \) are antisymmetric (see table 4) the algebra extends to

\[
\{ Q_I, Q^I_J \} = -2\delta_{IJ} P^\mu \Gamma_\mu C^{-1}_4 + X_{IJ} C^{-1}_4 + i Y_{IJ} \chi C^{-1}_4.
\] (5.7)

Now, studying the various Jacobi identities involving \( X, Y \) we can show \[14\]

\[
[X_{IJ}, \text{anything}] = 0, \quad [Y_{IJ}, \text{anything}] = 0.
\] (5.8)

The Lie superalgebra defined by (3.4), (5.4), (5.7), and (5.8) is called the four-dimensional \( N \) extended super-Poincaré algebra.

**Remark 5.2** Since, we have \( N \) copies of the (complex) supercharges \( Q_{LI} \) and \( Q_{RI} \), this allows the action to the automorphism group for which \( Q_{LI} \) are in the \( N \) dimensional representation of \( G \subseteq U(N) \) and \( Q_{RI} \) is in the complex conjugate representation. The full \( N \) extended superalgebra as to be supplemented with \( [T_a, Q_{LI}] = (t_a) I^J Q_{LJ} \), \( [T_a, Q_{RI}] = (t_a)^* I^J Q_{RJ} \) with \( T_a \) the generators of \( g \) (the Lie algebra of \( G \)) and \( t_a \) the \( N \times N \) matrices corresponding to the \( N \) dimensional representation of \( g \) and \( (t_a)^* \) the matrices of the complex conjugate representation \[14\]. With \( Q_I = \begin{pmatrix} Q_{LI} & Q_{RI} \end{pmatrix} \) these relations become \( [T_a, Q_I] = \frac{(t_a) I^J + (t_a)^* I^J}{2} Q_J + i\chi \frac{i((t_a) I^J - (t_a)^* I^J)}{2} Q_J \).

**Remark 5.3** The algebra was presented in a formalism where the spinors have four components. This algebra can also be realized in the two components notations, with the Weyl spinors \( Q_L \) and \( Q_R \). From

\[
\{ Q_I, Q^J_I \} = \left\{ \begin{pmatrix} Q_{LI} & Q_{RI} \end{pmatrix}, Q^J_{LJ} \right\} = \left\{ \begin{pmatrix} Q_{LI}, Q^I_{LJ} \end{pmatrix}, \begin{pmatrix} Q_{RI}, Q^I_{RJ} \end{pmatrix} \right\},
\] (5.9)

the algebra (5.7) reduces to

\[
\{ Q_{LI}, Q^I_{LJ} \} = i(X_{IJ} + iY_{IJ})\sigma_2, \quad \{ Q_{LI}, Q^I_{RJ} \} = 2i\delta_{IJ} P^\mu \sigma_\mu \sigma_2.
\]

These brackets could also have been deduced from (5.8).

The study of representation of the supersymmetric algebra (see sect 5.3.1) will in fact give \( N \leq 8 \). This means that the maximum number of fermionic degrees of freedom is 32 (= 4 \times 8).

### 5.2.2 Supersymmetry in ten dimensions

There are various ten dimensional supersymmetric algebras, see \[13, 15, 16\] and references therein. Recall that when \( d = 10 \) we can define Majorana-Weyl spinors (see table 3). Such a spinor has 16 components. The structure of the supersymmetric algebra is very similar to the four dimensional case. We just give here, the \{odd,odd\} part of the algebra. Let \( Q \) be a Majorana spinor in ten dimensions. Then, if we introduce a real central charge \( Z \) in addition to the Poincaré generators,
using Table 4, and arguments similar as in the previous section we get the supersymmetric algebra in dimension ten (since $\Gamma_\mu C_{10}^{-1}$ and $C_{10}^{-1}$ are symmetric see table 4)

$$\{Q, Q^t\} = ZC_{10}^{-1} + P^\mu \Gamma_\mu C_{10}^{-1},$$

(5.10)

with $C_{10}$ the $C-$matrix in dimension 10 (see [14,16]). From this equation, if we denote $Q_\pm = \frac{1}{2}(1 \pm \chi_{10})Q$ the left- and right-handed components of $Q$, we obtain

$$\{Q_\pm, Q^t_\pm\} = \frac{1}{2}(1 \pm \chi_{10})P^\mu \Gamma_\mu C_{10}^{-1}, \quad \{Q_\pm, Q^t_+\} = \frac{1}{2}(1 \pm \chi_{10})ZC_{10}^{-1},$$

(5.11)

since $\chi_{10}C_{10}^{-1} = -C_{10}^{-1}\chi_{10}$ (see eq. (4.36)) and $\chi_{10}\Gamma_\mu = -\Gamma_\mu\chi_{10}$ (see eq. (4.13)) with $\chi_{10}$ the chirality matrix in ten dimensions. In ten dimensions the fermionic part of the algebra is constituted of $N_+$ left-handed Majorana-Weyl spinors and $N_-$ right-handed Majorana-Weyl spinors. As in four dimensions the number of fermionic degrees of freedom is at most 32. Since a Majorana-Weyl spinor has 16 components $N_+ + N_- \leq 2$ leading to three different theories:

1. **Type I supersymmetry** We have one Majorana-Weyl supercharge, say $Q_+$:

$$\{Q_+, Q^t_+\} = \frac{1}{2}(1 + \chi_{10})P^\mu \Gamma_\mu C_{10}^{-1}.$$  

(5.11)

2. **Type IIA supersymmetry** We have two Majorana-Weyl supercharges, of opposite chirality $Q_+$ and $Q_-(or one Majorana spinors $Q = \left(\begin{array}{c} Q_+ \\ Q_- \end{array}\right)$):

$$\{Q_\pm, Q^t_\pm\} = \frac{1}{2}(1 \pm \chi_{10})P^\mu \Gamma_\mu C_{10}^{-1}, \quad \{Q_+, Q^t_+\} = \frac{1}{2}(1 + \chi_{10})ZC_{10}^{-1}.$$

(5.12)

3. **Type IIB supersymmetry** We have two Majorana-Weyl supercharges, of the same chirality $Q_{+1}$ and $Q_{+2}$:

$$\{Q_{I+}, Q^t_{+J}\} = \delta_{IJ}\frac{1}{2}(1 + \chi_{10})P^\mu \Gamma_\mu C_{10}^{-1}.$$

(5.13)

The type I (resp. IIA, IIB) theories appears naturally in string theory [16].

### 5.2.3 Supersymmetry in eleven dimensions

Now, we give the eleven-dimensional supersymmetric algebra (see e.g. [13,16,19]). From table 4 in eleven dimensions, we take a pseudo-Majorana spinor. Since such a spinor has 32 components there is only one possible theory. Thus, in eleven dimension, the situation is even simpler than in ten or four dimensions. Let $Q$ be the pseudo-Majorana supercharge. If the bosonic sector is constituted only of the Poincaré generators, using table 4 the algebra is given by

$$\{Q, Q^t\} = P^\mu \Gamma_\mu C_{11}^{-1},$$

(5.14)
with $C_{11}$ the eleven-dimensional $C-$matrix (see [122]). The interesting point of the eleven-dimensional supersymmetry is its simplicity. In addition, as we will see below lower dimensional theory can be obtained by dimensional reduction or compactification. Furthermore, eleven is the maximum dimension (with a signature $(1, d-1)$) where a supersymmetric theory can be formulated. Indeed, when $d = 12$ the Majorana spinors have 64 components$^5$.

Finally, looking to table 4, one observes that some other types of central charges can be added to extend the algebra (5.14). The possible central charges are real antisymmetric tensors of order two and five leading to the superalgebra

$$\{Q, Q^i\} = P^\mu \Gamma_\mu C_{11}^{-1} + \frac{1}{2} Z_{\mu\nu} \Gamma_{\mu\nu} C_{11}^{-1} + \frac{1}{5!} Z_{\mu_1\mu_2\mu_3\mu_4\mu_5} \Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} C_{11}^{-1}. \quad (5.15)$$

The new central charges introduced here are rather different than the central charges considered in four or ten dimensions. Indeed, such central charges are not central, being in antisymmetric tensor representations of the Lorentz algebra they are not scalar. This possibility of tensorial central charges have been considered in [17]. In Quantum Field Theory, or in a theory involving only elementary particles such central charges are excluded by the theorem of Haag-Lopuszanski-Sohnius [11] (since they are not scalars). They become relevant in a theory involving propagating $p-$dimensional extended objects (strings $p = 1$, membranes $p = 2$, etc. called generically branes or $p-$branes) [18]. For references on $p-$branes see e.g. [13]. The algebra (5.15) is the basic algebra which underlines $M-$theory and it involves a membrane and an extended object of dimension five, a 5–brane [16].

Type IIA supersymmetry from eleven-dimensional supersymmetry

If we denote by $M^{10}$ and $M^{11}$ the Minkowski space-time in 10 or 11 dimensions, we obviously have $M^{10} \subset M^{11}$. At the level of the algebra this reduces to $\mathfrak{iso}(1,9) \subset \mathfrak{iso}(1,10)$. This simple observation is the starting point of the so-called dimensional reduction or compactification where a theory in 10 dimensions is obtained from a theory in eleven dimensions. Indeed, historically type IIA supersymmetry was obtained in such a process (see e.g. [14]). Starting from the eleven dimensional supersymmetry we take the eleventh dimension $x^{11}$ to be on a circle $S^1$, and we let the radius of the circle to be very small, such that $M^{11} \rightarrow M^{10} \times S^1$. If we denote by $L_{MN}, P_M, 0 \leq M, N \leq 10$ the Poincaré generators in eleven dimensions they reduce to $L_{\mu\nu}, P_\mu, 0 \leq \mu, \nu \leq 9$ the generators of the 10–dimensional Poincaré algebra and to $P_{10}$ a scalar with respect to $\mathfrak{iso}(1,9)$. At the level of Clifford algebras we have $\mathfrak{C}_{1,9} \subset \mathfrak{C}_{1,10}$. This means that a Majorana spinor in eleven dimensions reduces to two Majorana-Weyl spinors $Q_+$ and $Q_-$ in ten dimensions because the Dirac $\Gamma-$matrices $\Gamma_{10}$ (presents in eleven dimensions) plays the role of the chirality matrix $\chi_{10}$ in ten dimensions (see eq (4.21)). Thus the Majorana spinor reduces to two opposite chirality Majorana-Weyl spinors $Q_\pm = \frac{1}{2}(1 \pm \chi)Q$. Next observing that the $C-$matrices in eleven and ten dimensions are related as follow: $C_{11} = C_{10} \Gamma_{10}$, the supersymmetric algebra in ten dimensions obtained from the dimensional reduction of the eleven-dimensional algebra becomes

$$\{Q, Q^i\} = P^\mu \Gamma_\mu \Gamma_{10}^{-1} C_{10}^{-1} + P^{10} C_{10}^{-1}. \quad (5.10)$$

which is just (6.5) with $Z = P^{10}$ and $\Gamma_\mu \rightarrow \Gamma_\mu \Gamma_{10}^{-1}$. With the two Majorana-Weyl spinors of opposite chirality, we get the type IIA theory in ten dimensions, and $P^{10}$ becomes a central charge. This means that the dimensional reduction of the eleven-dimensional supersymmetry leads to the

$^5$In twelve dimensions with signature $(2, 10)$ a Majorana-Weyl spinor has 32 components.
ten dimensional supersymmetry of type IIA.

Four dimensional supersymmetry from eleven-dimensional supersymmetry

The same principle can be applied to construct four dimensional supersymmetry (see [19] for references) from eleven dimensional supersymmetry. But here in the compactification process we have several possibilities for the compact manifold. One of the simplest compactification is to consider a 7–sphere such that \( M^{11} \to M^4 \times S^7 \) with \( M^4 \) the four dimensional Minkowski spacetime.

In the reduction process from eleven dimensions to four dimensions we observe several things

1. \( \mathfrak{so}(1, 10) \supset \mathfrak{so}(1, 3) \times \mathfrak{so}(7) \);

2. the Poincaré generators \( P^M, M = 1, \ldots, 10 \) give the Poincaré generators in four dimensions \( P^\mu, \mu = 0, \ldots, 3 \) and 7 scalars with respect to \( \mathfrak{iso}(1, 3) \) \( P^4, \ldots, P^{10} \);

3. the spin representation 32 of \( \text{Spin}(1, 10) \) decomposes to \( 32 = (2_+, 8) \oplus (2_-, 8) \) with \( 2_\pm \) a left/right-handed spinors of \( \mathfrak{so}(1, 3) \) and 8 a Majorana spinor of \( \mathfrak{so}(7) \). (Real spinors exist for \( \text{Spin}(1, 3), \text{Spin}(7), \text{Spin}(1, 10) \), see table [3]). Thus, the supercharge \( Q \) gives 8 Majorana supercharges \( Q_I \) in four dimensions;

4. the Dirac \( \Gamma \)–matrices decompose as follow

\[
\Gamma^M, M = 0, \ldots, 10, \to \begin{cases} 
\Gamma_\mu = \Gamma^{(4)}_\mu \otimes I^{(7)}, & \mu = 0, \ldots, 3, \\
\Gamma_m = i\chi_4 \otimes \Gamma_m^{(7)}, & m = 1, \ldots, m,
\end{cases}
\]

with \( \Gamma^{(4)}_\mu \) the four dimensional Dirac \( \Gamma \)–matrices, \( \Gamma_m^{(7)} \) the matrices of the representation of \( \mathfrak{c}_{7,0} \), \( \chi_4 \) the four dimensional chirality matrix and \( I^{(7)} \) the identity of \( \mathfrak{c}_{7,0} \);

5. the matrix \( C_{11} \) decomposes into \( C_{11} = C_4 \otimes C_7 \) with \( C_4 = i\Gamma^{(4)}_0 \Gamma^{(4)}_2 \) and \( C_7 = -i\Gamma^{(7)}_4 \Gamma^{(7)}_6 \Gamma^{(7)}_8 \Gamma^{(7)}_{10} \).

With all these observations the algebra reduces to

\[
\{Q, Q^I\} = P^\mu \Gamma^{(4)}_\mu C_4^{-1} \otimes C_7^{-1} + iP^m \chi_4 C_4^{-1} \otimes \Gamma_m^{(7)} C_7^{-1}
\]

Next, observing that \( C_7 \) is symmetric (see [16]) and introducing \( Q_{LI} = (2_+, 8), \ Q_{RI} = (2_-, 8) \) and \( Q_I = \left( \begin{array}{c} Q_{LI} \\ Q_{RI} \end{array} \right) \) with \( I = 1, \ldots, 8 \) the algebra can be rewritten (after an appropriate diagonalisation \( C_7 \to I^{(7)} \))

\[
\{Q_I, Q^I_J\} = \delta_{IJ} P^\mu \Gamma^{(4)}_\mu C_4^{-1} + iP^m \Gamma^{(7)}_m I^K \delta_{KJ} \chi_4 C_4^{-1},
\]

(with \( \Gamma_m^{(7)} I^K \) the matrix elements of the matrices \( C_7^{(7)} \)) which is the 8–extended supersymmetric algebra in four dimensions with central charge \( Y_{IJ} = P^m \Gamma^{(7)}_m I^K \delta_{KJ} \chi_4 C_4^{-1} \).

Since the isometry group of \( S^7 \) is \( \text{SO}(8) \supset \text{SO}(7) \), in fact in can be shown that the supercharges belong to the vector representation of \( \text{SO}(8) \) (see [19] and references therein).
5.3 Irreducible representations of supersymmetry

Since the Poincaré algebra admits a semi-direct structure \( \mathfrak{iso}(1, 3) = \mathfrak{so}(1, 3) \ltimes \mathbb{D}_{\frac{1}{2} \frac{1}{2}} \), with \( \mathbb{D}_{\frac{1}{2} \frac{1}{2}} \) the vector representation of \( \mathfrak{so}(1, 3) \) (i.e. the space-time translations), the representations of \( \mathfrak{iso}(1, 3) \) are obtained by the method of induced representations of Wigner [20] (this also hold for any space-time dimensions). This method consists of finding a representation of a subgroup of the Poincaré group and boosting it to the full group. If we denote \( P^\mu \) the momentum, we have \( P_\mu P^\mu = m^2 \) for a particle of mass \( m \neq 0 \) and \( P_\mu P^\mu = 0 \) for a massless particle\(^6\). In practice, we go in some special frame where the momentum have a specific expression \( P^\mu = (m, 0, \cdots, 0) \) in the massive case and \( P^\mu = (E, 0, \cdots, 0, E) \) (with \( E > 0 \)) in the massless case. We identify the subgroup \( H \subset ISO(1, d - 1) \), called the little group, which leaves the momentum invariant, find the representations of \( H \) and then induce the representations to the whole group \( ISO(1, d - 1) \). This method, is in fact very close to the principle of equivalence of special relativity which states that if a result is obtained in a specific frame it can be extended to any frame of reference.

In this lecture, we will only study the case of massless particles. If we denote \( L_{\mu \nu} \) the generators of the Lorentz algebra \( \mathfrak{so}(1, d - 1) \) the little group leaving \( P^\mu = (E, 0, \cdots, 0, E) \) \( (P_\mu = P^\nu \eta_\mu \nu = (E, 0, \cdots, 0, -E)) \) invariant is generated by \( L_{ij}, 1 \leq i < j \leq d-2 \) and \( T_i = L_{i0} + L_{i, d-1}, 1 \leq i \leq d-2 \) (since \( [L_{ij}, P_\mu] = 0, [T_i, P_\mu] = 0 \)). This group is isomorphic to \( E(d-2) \) the group of rotations-translations in \( (d - 2) \) dimensions. Since this group is non-compact and we are interested in finite dimensional unitary representations, we will represent the generators \( T_i \) by zero. By abuse of notations we will now call \( SO(d-2) \) the little group. The frame where \( P^\mu = (E, 0, \cdots, 0, E) \) will be called the “standard frame”. This method can be extended to study the massless representations of the supersymmetric extension of the Poincaré algebra. The little algebra now contains the bosonic generators \( L_{ij}, 1 \leq i < j \leq d-2 \) the central charges (if there exists) and the fermionic supercharges \( Q \).

5.3.1 Four dimensional supersymmetry

In four dimensions, the generators of the supersymmetric algebra in the little group reduce \( L_{12}, X_{IJ}, Y_{IJ} \) and \( T_a \) for the bosonic sector (we assume here that the automorphism group of the algebra is \( SU(N) \), (see Remark 5.2) and \( Q_I, I = 1, \cdots, N \) for the fermionic sector. The generator \( L_{12} \) is called the generator of helicity. We study here the representations of the supersymmetric algebra when the central charge are put to zero. (In fact it can be proven that massless particles represent trivially the central charges [14].)

The \{odd, odd\} part of the algebra gives (in the two components notation (see Remark 5.3))

\[
\{Q_{LI}, Q^J_{KI}\} = 2E\delta_{IJ}(\sigma_0 + \sigma_3)i\sigma_2 = 4E\delta_{IJ}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\] (5.16)

If we write \( Q_{LI} = \begin{pmatrix} Q^1_{1I} \\ Q^2_{2I} \end{pmatrix} \) and \( Q_{RI} = \begin{pmatrix} \bar{Q}^1_{1I} \\ \bar{Q}^2_{2I} \end{pmatrix} \) in the usual notations (see [14]), the only non-zero brackets are given by

\[
\{Q_{1I}, \bar{Q}^2_{2J}\} = 4E\delta_{IJ}.
\]

\(^6P_\mu P^\mu \) is a Casimir operator of the Poincaré algebra, its eigenvalue is the mass.
Since the other brackets vanish and since we want unitary representations this means that \( Q_{2I} = \tilde{Q}_{1I} = 0 \) and the supercharges \( a_I = \frac{Q_I}{\sqrt{4E}}, a_{1I} = \frac{\tilde{Q}_I}{\sqrt{4E}} \) generate the Clifford algebra \( \mathcal{C}_{2N,0} \) (see Remark 11). Now, the action of the bosonic part on the supercharges gives \([L_{12}, Q_I] = \Gamma_{12} Q_I \). Using \( \Gamma_{12} = \frac{1}{2} \Gamma_1 \Gamma_2 \) and (5.5), we get

\[
[L_{12}, Q_{1I}] = -\frac{1}{2} Q_{1I}, \quad [L_{12}, \tilde{Q}_I] = \frac{1}{2} \tilde{Q}_I, \quad (5.17)
\]

and thus \( Q_{1I} \) are of helicity \( \frac{1}{2} \) and \( \tilde{Q}_I \) of helicity \( -\frac{1}{2} \). If \( M_{12} |\lambda\rangle = -i\lambda |\lambda\rangle \) then \( M_{12} Q_{1I} |\lambda\rangle = -i(\lambda + \frac{1}{2}) Q_{1I} |\lambda\rangle \) and \( M_{12} \tilde{Q}_I |\lambda\rangle = -i(\lambda - \frac{1}{2}) \tilde{Q}_I |\lambda\rangle \). Finally, \( Q_{1I} \) is in the \( N \)-dimensional representation of \( SU(N) \) that we denote \( N \) and \( \tilde{Q}_I \) is in the complex conjugate representation \( \bar{N} \). Thus we have \( Q_{1I} = (\mathcal{\frac{1}{2}}, N) \), \( \tilde{Q}_I = (\mathcal{-\frac{1}{2}}, \bar{N}) \) with respect to the group \( Spin(2) \times SU(N) \).

The representations of the four dimensional supersymmetric algebra are then completely specified and is of dimensions \( 2^N \), corresponding to the spinor representations of \( Spin(2N) \). The left-handed spinors of \( Spin(2N) \) will correspond e.g. to the fermions and the right-handed spinors to the bosons as we will see. We also know that \( (Q_{1I}, \tilde{Q}_I) \) belongs to the vector representation of \( Spin(2N) \). But, if one wants to identify the particles content of the supersymmetric multiplet, it is interesting to decompose the multiplet with respect to the group \( Spin(2N) \supset Spin(2) \times SU(N) \) (i.e. the little group). For the supercharge we have in this embedding \( 2N = (\mathcal{\frac{1}{2}}, N) \oplus (\mathcal{-\frac{1}{2}}, \bar{N}) \) (the supercharge are in the vector representation \( 2N \) of \( Spin(2N) \)) - see Remark 11. And using Remark 11 it is easy to decompose the spinor representation \( 2N \) of \( Spin(2N) \) into representations of \( Spin(2) \times SU(N) \). If we introduce a Clifford vacuum \( \Omega \) of helicity \( \lambda_{\max} \) \( (L_{12} \Omega = -i\lambda_{\max} \Omega) \) annihilated by \( a_I \) \( (a_I \Omega = 0) \) and being in some representation of \( SU(N) \) the full representation is obtained by the action of the operator of creation \( a^I_I \). For simplification we assume that \( \Omega \) is in the trivial representation of \( SU(N) \) and we obtain the supermultiplet

| state | helicity | representation of \( SU(N) \) – dimension |
|-------|--------|----------------------------------|
| \( |\lambda_{\max}\rangle \) | \( \lambda_{\max} \) | \( [0] \) dim = 1 |
| \( \tilde{Q}_{1I} |\lambda_{\max}\rangle \) | \( \lambda_{\max} - \frac{1}{2} \) | \( [\bar{1}] \) dim = \( \frac{N}{2} \) |
| \( Q_{I_1} Q_{I_2} |\lambda_{\max}\rangle \) | \( \lambda_{\max} - 1 \) | \( [2] \) dim = \( \frac{N}{2} \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( \tilde{Q}_{I_1} \tilde{Q}_{I_2} \cdots \tilde{Q}_{I_k} |\lambda_{\max}\rangle \) | \( \lambda_{\max} - \frac{k}{2} \) | \( [\bar{k}] \) dim = \( \frac{N}{k} \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( Q_{I_1} Q_{I_2} \cdots Q_{I_N} |\lambda_{\max}\rangle \) | \( \lambda_{\max} - \frac{N}{2} \) | \( [\bar{N}] \) dim = 1 |

with \( 1 \leq I_1 < I_2 < \cdots < I_{N-1} \leq N \), \( [\bar{k}] \) the antisymmetric tensor of order \( k \) of \( SU(N) \) and \( \binom{N}{k} \) its dimension. In this decomposition we have

\[
(2^{N-1})_L = (\lambda_{\max}, [0]) \oplus (\lambda_{\max} - 1, [2]) \oplus (\lambda_{\max} - 2, [4]) \oplus \cdots (5.19)
\]

\[
(2^{N-1})_R = (\lambda_{\max} - \frac{1}{2}, [1]) \oplus (\lambda_{\max} - \frac{3}{2}, [3]) \oplus \cdots
\]
for the left/right-handed part of the spinor representation of Spin(2N). For instance if $\lambda_{\text{max}} \in \mathbb{N} + \frac{1}{\tau}$ the left-handed spinors of Spin(2N) are fermions and the right-handed spinors are bosons. A multiplet contains bosons and fermions and a supersymmetric transformation sends a boson to a fermion and vice versa. We also see that we have an equal number of bosons and fermions.

However, these irreducible representations are not enough for particle physics. We still have to take the CPT symmetry into account (C means charge –or complex– conjugation, P parity transformation and T time reversal). Under the CPT symmetry $Q_R \rightarrow Q_R^* = -i\sigma_2 Q_L$ (or $a^\dagger_I \rightarrow a_I$) and $L_{12} \rightarrow -L_{12}$. A quantum field theory has to be CPT invariant. This means if the multiplet (5.18) is not invariant under this conjugation, we have to consider the CPT conjugate multiplet obtained by acting on the conjugated Clifford vacuum $| -\lambda_{\text{max}} \rangle$ with the annihilation operator $Q_{1I}$ ($\bar{Q}^2_{1I} | -\lambda_{\text{max}} \rangle = 0$). Let us give the result for certain values of $N$.

1. For $N = 1$ the multiplet (5.18) are not CPT conjugate. The particle content (after adding the CPT conjugate multiplet) is

$$\lambda_{\text{max}} = \frac{1}{2} \quad \text{helicity} \quad \frac{1}{2} 0 -\frac{1}{2} ;$$

(In this multiplet we have $\Omega = |\frac{1}{2}\rangle$, $\bar{Q}^2 \Omega = |0\rangle$, $\Omega^{\text{CPT}} = | -\frac{1}{2}\rangle$, $Q_{1I} \Omega^{\text{CPT}} = |0\rangle$.)

$$\lambda_{\text{max}} = 1 \quad \text{helicity} \quad 1 \frac{1}{2} -\frac{1}{2} -1 ;$$

$$\lambda_{\text{max}} = 2 \quad \text{helicity} \quad 2 \frac{3}{2} -\frac{3}{2} -2 .$$

To identify the particle content of the various multiplet, we have to keep in mind that a massless particle of spin $s$ is constituted of a state of helicity $s$ and a state of helicity $-s$. For instance a left-handed massless electron is constituted of a state of helicity 1/2 and a state of helicity −1/2. The former can be interpreted as a left-handed electron of helicity $\frac{1}{2}$ and the latter as its corresponding anti-particle state the left-handed positron of helicity $-\frac{1}{2}$. Thus, after boosting the representations to the whole Poincaré group, the first multiplet contains a left-handed fermion and a complex scalar field and the second multiplet contains a real pseudo-Majorana spinor and a real vector field. These types of multiplet are essential in the construction of the so-called minimal supersymmetric standard model M–MSSM– (i.e. to construct a model describing particle physics and being supersymmetric). The former multiplets, called the chiral multiplets, are the matter multiplets (they correspond for instance to a left-handed electron and a scalar electron named selectron), although the second multiplets, called the vector multiplets, are relevant for the description of supersymmetric fundamental interactions (in the case of electromagnetism, it corresponds to the photon and its fermionic supersymmetric partner the photino) [14]. This can be generalised for all the particles. This means that in supersymmetric theory the spectrum is doubled, and to each known particle we have to add its supersymmetric partner [21]. The last types of multiplet (the gravity multiplet) contains the graviton (so describes gravity) and its supersymmetric partner, a spinor-vector named the gravitino. It is possible to couple the gravity multiplet with the multiplet of the MSSM [14, 21].

2. For $N = 4$, when $\lambda_{\text{max}} = 1$ the multiplet is CPT conjugate and is not CPT conjugate for $\lambda_{\text{max}} = 2$:

$$\lambda_{\text{max}} = 1 \quad \text{helicity} \quad 1 \frac{1}{2} 0 -\frac{1}{2} -1 ;$$
\[ \lambda_{\text{max}} = 2 \quad \text{helicity} \quad 2 \quad \frac{3}{2} \quad 1 \quad \frac{1}{2} \quad 0 \quad -\frac{1}{2} \quad -1 \quad -\frac{3}{2} \quad -2 \]

states \quad 1 \quad 4 \quad 6 \quad 4 \quad 2 \quad 4 \quad 6 \quad 4 \quad 1 \quad 1 \quad 2 \quad 3 \quad 4 \quad 1 \quad .

We observe that there is no multiplet with \( \lambda_{\text{max}} = \frac{1}{2} \) when \( N = 4 \). Thus \( N = 4 \) does not contain matter multiplets, but only gauge (\( \lambda_{\text{max}} = 1 \)) or gravity (\( \lambda_{\text{max}} = 2 \)) multiplets.

3. For \( N = 8 \) there is only one (gravity, \( \lambda_{\text{max}} = 2 \)) multiplet which is CPT conjugate:

\[
\begin{array}{cccccccc}
\text{helicity} & 2 & 3 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 & -\frac{3}{2} & -2 \\
\text{states} & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
\end{array}
\]

Several remarks are in order here

**Remark 5.4** One can observe by a direct counting that in a supersymmetric multiplet there is an equal number of bosons and fermions. This is a general result valid in any space-time dimensions \[14, 13\] (see also \(5.19\)).

**Remark 5.5** If \( N > 8 \) the supersymmetric multiplets contain states of helicity bigger than 2. Since there is not consistent theory for interacting particles of helicity bigger than 2 in four dimensions, \( N \leq 8 \). The complicated multiplets for \( N > 1 \) can be obtained by compactification of higher dimensional theories. For instance the four dimensional \( N = 8 \) extended supersymmetry can be obtained from the eleven-dimensional theory or the ten dimensional type IIA or type IIB theories and the four dimensional \( N = 4 \) extended supersymmetry can be obtained from the ten-dimensional type I theories \[15, 13, 16\]. Conversely, this compactification limits the number of supersymmetry one can take in a given dimension (see \[5.3\]).

### 5.3.2 Eleven dimensional supersymmetry

In eleven dimensions, the bosonic part of the little group is \( SO(9) \) and the supersymmetric algebra becomes

\[
\{Q, Q^\dagger\} = E(\Gamma_0 + \Gamma_{10})C_{11}^{-1} = E(1 + \Gamma_{10}\Gamma_0)\Gamma_0C_{11}^{-1}.
\]

Since the trace of \( \Gamma_{10}\Gamma_0 \) is equal to zero, \( (\Gamma_{10}\Gamma_0)^\dagger = \Gamma_{10}\Gamma_0 \) and \( (\Gamma_{10}\Gamma_0)^2 = 1 \), this means that the matrix \( \Gamma_{10}\Gamma_0 \) has an equal number of eigenvalues +1 and −1. Thus we can chose a basis such that the only non zero brackets are

\[
\{Q_a, Q_b\} = \delta_{ab}, \quad a, b = 1, \cdots, 16
\]

and as in four dimensions half of the supercharges can be represented by zero. This is a general property of supersymmetric algebras \[12\]. The non-zero supercharges are called the active supercharges. The representation of the eleven-dimensional supersymmetric algebra turns out to be the spinor representation of Spin(16) (of dimension 256), but as in four dimensions to identify the precise content of the multiplet we have to study the embedding Spin(16) \(\supset\) Spin(9), with Spin(9) the little group. The active supercharges are in the \(16\) (spinor) representation of Spin(9) and in the \(16\) (vector) representation of Spin(16). To identify the representation of the supersymmetric algebra we proceed in several steps. We first observe that in the following embeddings we have the decomposition
\[ \text{Spin}(9) \supset \text{Spin}(8) \supset \text{Spin}(6) \times \text{Spin}(2) \]

\[ 16 = 8_+ \oplus 8_- = \left( (4_+, \frac{1}{2}) \oplus (4_-, -\frac{1}{2}) \right) \oplus \left( (4_+, -\frac{1}{2}) \oplus (4_-, \frac{1}{2}) \right) \quad (5.21) \]

with \( 8'_\pm \) a left/right handed spinor of \( \text{Spin}(8) \), \( (4'_\pm \) a left/right handed spinor of \( \text{Spin}(6) \) and \( \pm \frac{1}{2} \) the eigenvalue of \( \text{Spin}(2) \)).

Then we study explicitly the spinor representation of \( \text{Spin}(8) \). We denote \( Q_\pm = 8'_\pm \), define a Clifford vacuum \( \Omega_\pm \) for each supercharge \( Q_\pm \), and decompose along the line of remark 4.1 the \( Q_\pm \) into operators of creation and annihilation (denoted \( (a_+, a_+^\dagger) \) and \( (a_-, a_-^\dagger) \)) and obtain the spinor representation (with \( Q_+ \) for instance). The supercharges \( Q_+ \) belong to the vector representation \( 8_v \) of some \( \text{Spin}(8)_Q \) algebra generated by \( Q_+ \) (see Remark 4.1), but they also belong to the spinor representation of the \( \text{Spin}(8)_{\text{s.t.}} \) subgroup of the little group \( \text{Spin}(9) \). Thus we firstly study the decomposition through the embedding \( \text{Spin}(8)_{\text{s.t.}} \subset \text{Spin}(8)_Q \) for which \( 8'_\pm = 8_v \)

| state | \( \text{Spin}(6) \) | \( \text{Spin}(2) \) |
|-------|---------------|---------------|
| \( \Omega_+ \) | 1 | -1 |
| \( a_+^\dagger \Omega_+ \) | 4_+ | -\frac{1}{2} |
| \( \left[ a_+^\dagger \right]^2 \Omega_+ \) | 6 | 0 |
| \( \left[ a_+^\dagger \right]^3 \Omega_+ \) | 4_- | \frac{1}{2} |
| \( \left[ a_+^\dagger \right]^4 \Omega_+ \) | 1 | 1 |

(5.22)

where \( \left[ a_+^\dagger \right]^n \) means an \( n \)-th antisymmetric product of operators of creation. In this decomposition, we have chosen the subgroup \( \text{Spin}(6) \times \text{Spin}(2) \subset \text{Spin}(8) \) such that \( a_+^\dagger = (4_+, \frac{1}{2}) \) this gives the second line in (5.22). Moreover, using \( (4_+, \frac{1}{2}) \otimes (4_+, \frac{1}{2}) = (6, 1) \oplus (10_+, 1) \) with 6 the vector representation of \( \text{Spin}(6) \) and 10_+ the antisymmetric self-dual tensor of order three of \( \text{Spin}(6) \), and observing that 6 corresponds to the antisymmetric tensor product of spinors and 10_+ to the symmetric tensor product of spinors, this gives the third line in (5.22). Similar analysis give the \( \text{Spin}(6) \) content of the other line of (5.22). Finally let us mention that the eigenvalues of \( \text{Spin}(2) \) are normalized such that their sum is equal to zero. Now, it is easy to regroup the various terms and obtain representations of \( \text{Spin}(8)_{\text{s.t.}} \):

(5.23)

with \( 8_v \) the vector representation of \( \text{Spin}(8)_{\text{s.t.}} \) and \( 8'_\pm \) the two spinor representations. Thus in the embedding \( \text{Spin}(8)_{\text{s.t.}} \subset \text{Spin}(8)_Q \) we have the following decomposition \( 16 = 8_- \oplus 8_v \). In a similar way, acting with the supercharges \( Q_- \) we have the decomposition: \( 16 = 8_+ \oplus 8_v \). To obtain now the full \( \text{Spin}(16) \) representation, we just have to tensorise the representations obtained with \( Q_+ \) and \( Q_- \). First we notice

(5.24)
with $1 \leq i, j, k \leq 8$ the Spin(8)$_{s.t.}$ indices and $\Phi$ a scalar, $g_{ij}$ a symmetric traceless tensor, $B_{ij}$ a two-form, $A_i$ a vector, $C_{ijk}$ a three-form, $\lambda_L$ a left-handed spinor, $\lambda_R$ a right-handed spinor, $\Psi^I_L$ a left-handed spinor-vector and $\Psi^I_R$ a right-handed spinor-vector of Spin(8)$_{s.t.}$. The second decomposition comes from the triality property of so(8)—look to the Dynkin diagram of so(8). The field in bracket $[\ ]$ just represents the type of field corresponding to the given representation. For instance $28$ $[B_{ij}]$ means that the 28-dimensional representation of Spin(8)$_{s.t.}$ corresponds to a two-form $B_{ij}$. This gives the decomposition of the spinor representation of Spin(16) $\supset$ Spin(8)$_{s.t.}$. To obtain now the decomposition through the embedding Spin(16) $\supset$ Spin(9), we just have to study the embedding Spin(9) $\supset$ Spin(8)$_{s.t.}$. If we define $I, J, K = 1, \cdots, 9$ the indices of Spin(9) and $i, j, k = 1, \cdots, 8$ the indices of Spin(8), we have:

$$
\begin{align*}
44 \ [g_{IJ}] & = 35 \ [g_{ij}] \oplus 8_v \ [g_{ij}] \oplus 1 \ [g_{ij}] = 35 \ [g_{ij}] \oplus 8_v \ [A_i] \oplus 1 \ [\Phi] \\
84 \ [C_{IJK}] & = 56_r \ [C_{ijk}] \oplus 28 \ [C_{ij9}] = 56_r \ [C_{ijk}] \oplus 28 \ [B_{ij}] \\
128 \ [\Psi^I] & = 56_+ \ [\Psi^I_L] \oplus 8_+ \ [\Psi^I_L] \oplus 56_- \ [\Psi^I_R] \oplus 8_- \ [\Psi^I_R] \\
& = 56_+ \ [\Psi^I_L] \oplus 8_+ \ [\lambda_L] \oplus 56_- \ [\Psi^I_R] \oplus 8_- \ [\lambda_R].
\end{align*}
$$

Thus finally the representation of the eleven-dimensional supersymmetric algebra contains one spinor-vector (a Rarita-Schwinger field), $\Psi^I$, a symmetric traceless second order tensor (a tensor metric) and a three-form:

$$
256 = 128_L \oplus 128_R \left\{ \begin{array}{c}
128_R = \Psi^I \text{ fermion} \\
128_L = g_{IJ}, \ C_{IJK} \text{ bosons.}
\end{array} \right.
$$

These fields are representations of the little group Spin(9). It becomes easy to obtain a representation of the Poincaré algebra not limiting the indices to their Spin(9) values but allowing their Spin(1,10) values. For instance $g_{IJ}, 1 \leq I, J \leq 9 \rightarrow g_{MN}, 0 \leq M, N \leq 10$.

**Compactifications**

Having obtained the representation of the eleven-dimensional supersymmetric algebra we can now obtain the representations in smaller dimensions by compactification. Starting with the multiplet (5.27), the four dimensional representation of the extended $N = 8$ supersymmetry can be built. In the little group $SO(9)$ the indices of the fields $I, J = 1, \cdots, 9 \rightarrow (i, j = 1, 2$ and $m, n = 1, \cdots, 7$) corresponding to their Spin(2) and Spin(7) content. Thus we can decompose easily the fields in dimensional reduction from eleven to four dimensions. (In the simplest compactification, called the trivial compactification, we simply assume that the fields do not depend on the components of the compact dimension). This gives:

$$
\begin{array}{c|c|c|c}
& 1 \text{ graviton} & 7 \text{ vectors} & 28 \text{ scalars} \\
\hline
\text{g}_{IJ} \rightarrow & \text{g}_{ij} & \text{g}_{im} & \text{g}_{mn} \\
\text{C}_{IJK} \rightarrow & \text{C}_{ijk} & \text{C}_{ijm} & \text{C}_{imm} & \text{C}_{mnp} \\
\Psi^I \rightarrow & \Psi^i & \Psi^m & \Psi^i & \Psi^m \\
\hline
& 8 \text{ gravitinos} & 58 \text{ spinors}
\end{array}
$$

In this decomposition we have to pay attention because with respect to Spin(2) a three-form do not exists and a two-form is dual to a scalar. For the fermionic part of the multiplet we have
to remember that through $\text{Spin}(9) \supset \text{Spin}(2) \times \text{Spin}(7)$, we have the decomposition of a spinor $16 = (\frac{1}{2}, 8) \oplus (-\frac{1}{2}, 8)$. Finally, counting the number of states shows that we obtain the $N = 8$ gravity multiplet.

**Remark 5.6** The so-called Yang-Mills multiplet of the ten dimensional type I supersymmetry can be deduced from (5.23). The compactification of the type IIA supersymmetry from the eleven dimensional supersymmetry can be read off (5.25). In addition, using formulæ (5.24) with $8_\pm \otimes 8_\pm = 1 [\Phi] \oplus 28 [B_{ij}] \oplus 35 [D^+_{ijkl}]$ with $D^+_{ijkl}$ an (anti-)self-dual four-form of Spin(8) all ten dimensional supersymmetric multiplets can be obtained:

1. Yang-Mills multiplet type I: one vector $A^i$ one right-handed spinor $\lambda_R$. (The spinor is in the opposite chirality than the supercharge.)

2. Gravity type multiplet (the vacuum $\Omega$ is a vector of Spin(8)): one scalar $\Phi$, one tensor metric $g_{ij}$ one two-form $B_{ij}$; one left-handed spinor $\lambda_L$, one left-handed spinor-vector $\Psi^L_i$. (The spinor and spinor-vector are in the same chirality than the supercharge).

3. type IIA one scalar $\Phi$, one tensor metric $g_{ij}$ one two-form $B_{ij}$, one vector $A_i$, one three-form $C_{ijk}$, one left-handed and one right-handed spinor $\lambda_R, \lambda_L$ and one left-handed and one right-handed spinor-vector $\Psi^L_i, \Psi^R_i$. The fermions are of both chirality.

4. type IIB: two scalars $\Phi, \Phi'$, one tensor metric $g_{ij}$, two two-forms $B_{ij}, B'_{ij}$ one anti-self-dual four form $D^-_{ijkl}$ two left handed spinors $\lambda_L, \lambda'_L$ two left-handed spinor-vector $\Psi^L_i, \Psi'^L_i$.

Finally let us mention that type IIA or type IIB supersymmetry give by compactification in four dimensions $N = 8$ extended-supersymmetry.

### 6 Conclusion

In this lecture we have shown, that the basic tools to construct supersymmetric extensions of the Poincaré algebra are Clifford algebras. Special attention have been given to the four, ten and eleven dimensional spaces-times. Studying representations of supersymmetric algebras shows that a supermultiplet contains an equal number of bosonic and fermionic degrees of freedom and that supersymmetry is a symmetry which mixes non-trivially bosons and fermions. The next step is to apply supersymmetric theories in Quantum Field Theory or particle physics. For that purpose we need first to calculate transformation of the fields under supersymmetry and then to build invariant Lagrangians (the concept of superspace is central for this construction). For instance all the technics of supersymmetry have been applied in four dimensions to construct a supersymmetric version of the standard model [14, 21]. There are some strong arguments in favor of such a theory, even if there is no experimental evidence of supersymmetry. Supersymmetry or more precisely its local version contains gravity and as such is named supergravity. Ten-dimensional supergravities appear as some low energy limits of string theories and present some interesting duality properties [16, 13]. Finally, with the brane revolution a lot of hope have been put to the so-called M-theory whose low energy limits contains the eleven-dimensional supergravity and the various strings theory [16, 13].

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References

[1] P. A. M. Dirac, *The Quantum Theory Of Electron*, Proc. Roy. Soc. Lond. A117 (1928) 610. P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed., Oxford University Press, London (1958).

[2] M. F. Atiyah, R. Bott and A. Shapiro, *Clifford Modules*, Topology, Vol 3, Sup 1 (1964) 143-179.

[3] R. Coquereaux, *Modulo 8 Periodicity Of Real Clifford Algebras And Particle Physics*, Phys. Lett. B115 (1982) 389-395.

[4] R. Coquereaux, *Spinors, Reflections And Clifford Algebras: A Review*, Trieste 1986, Proceedings, Spinors in physics and geometry 135-190.

[5] R. Brauer and H. Weyl, *Spinors in n Dimensions*, Amer. J. Math 57 (1935) 425-449.

[6] F. Gliozzi, J. Scherk and D. I. Olive, *Supersymmetry, Supergravity Theories And The Dual Spinor Model*, Nucl. Phys. B122 (1977) 253-290.

[7] T. Kugo and P. K. Townsend, *Supersymmetry And The Division Algebras*, Nucl. Phys. B221 (1983) 357-380.

[8] Ta-Pei Cheng and Ling-Fong Li, *Gauge theory of elementary particles physics*, Clarendon Press-Oxford (1991).

[9] G. G. Ross, *Grand Unified Theories*, The Benjamin/Cummings Publishing Company, INC, (1985).

[10] S. Coleman and J. Mandula, *All Possible Symmetries Of The S Matrix*, Phys. Rev. 159 (1967) 1251-1256.

[11] R. Haag, J. T. Lopuszanski and M. F. Sohnius, *All Possible Generators Of Supersymmetries Of The S Matrix*, Nucl. Phys. B88 (1975) 257-274.

[12] J. Strathdee, *Extended Poincaré Supersymmetry*, Int. J. Mod. Phys. A2 (1987) 273-300.

[13] P. C. West, *Supergravity, brane dynamics and string duality*, arXiv:hep-th/9811101 (In Cambridge 1997, Duality and supersymmetric theories 147-266).

[14] J. Wess, J. Bagger, *Supersymmetry and Supergravity* (Princeton University Press, 1983).

[15] M. F. Sohnius, *Introducing Supersymmetry*, Phys. Rept. 128, (1985), 39-204.

[16] J. Polchinski, *String theory, Vol. 2: Superstring theory and beyond*, Cambridge, Univ. Press. (1998)

[17] J. W. van Holten and A. Van Proeyen, *N=1 Supersymmetry Algebras In D = 2, D = 3, D = 4 Mod-8*, J. Phys. A15,(1982) 3763-3784.

[18] J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, *Topological Extensions Of The Supersymmetry Algebra For Extended Objects*, Phys. Rev. Lett. 63, (1989) 2443-2446.

[19] P. G. O. Freund, *Introduction To Supersymmetry*, Cambridge Monographs On Mathematical Physics, Cambridge University Press, 1986.
[20] E. P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. of Math. 40 (1939) 149–204.

[21] H. P. Nilles, *Supersymmetry, Supergravity And Particle Physics*, Phys. Rept. 110 (1984) 1-162.