Deviations from off-diagonal long-range order in one-dimensional quantum systems

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Abstract – A quantum system exhibits off-diagonal long-range order (ODLRO) when the largest eigenvalue $\lambda_0$ of the one-body-density matrix scales as $\lambda_0 \sim N$, where $N$ is the total number of particles. Setting $\lambda_0 \sim N^C$ to define the scaling exponent $C$, then $C = 1$ corresponds to ODLRO and $C = 0$ to the single-particle occupation of the density matrix orbitals. When $0 \leq C < 1$, $C$ can be used to quantify deviations from ODLRO. In this paper we study the exponent $C$ in a variety of one-dimensional bosonic and anyonic quantum systems at $T = 0$. For the 1D Lieb-Liniger Bose gas we find that for small interactions $C$ is close to 1, implying a mesoscopic condensation, i.e., a value of the zero temperature “condensate” fraction $\lambda_0/N$ appreciable at finite values of $N$ (as the ones in experiments with 1D ultracold atoms). 1D anyons provide the possibility to fully interpolate between $C = 1$ and 0. The behaviour of $C$ for these systems is found to be non-monotonic both with respect to the coupling constant and the statistical parameter.

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Introduction. – The Penrose-Onsager criterion for the presence of off-diagonal long-range order (ODLRO) is the cornerstone of the present understanding of quantum coherence and Bose-Einstein condensation (BEC) \cite{1}. It is simply related to the occurrence of BEC and it is based on the study of the scaling with the number of particles of the eigenvalues of the one-body density matrix (1BDM) $\rho(x,y)$, defined as \cite{2}

$$\rho(x,y) = \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \rangle,$$  \hspace{1cm} (1)

where $\hat{\Psi}(x)$ is the field operator creating a particle at the point $x$. Denoting by $\lambda_i$ the eigenvalues of this matrix, we have

$$\int dy \rho(x,y) \phi_i(y) = \lambda_i \phi_i(x),$$  \hspace{1cm} (2)

where $\phi_i$’s are the corresponding eigenfunctions. There is ODLRO and BEC when the largest eigenvalue $\lambda_0$ scales as the total number of particles $N$ of the system \cite{1,3}. The occurrence of ODLRO implies phase coherence, as shown by a simple argument due to Anderson \cite{4} and reviewed in \cite{5}.

The Penrose-Onsager criterion relates, altogether, the occurrence of BEC and quantum coherence to the behaviour of correlation functions. Its power and elegance stem from the fact that it applies at zero and finite temperatures and also in any dimensions, so that in eqs. (1), (2), the coordinates $x, y$ may denote space vectors with $D$ components, possibly also including spin degrees of freedom. Moreover, the system may also be subjected to a generic one-body external potential. A major example of detection of ODLRO is provided by the measurement of the momentum distribution $n(k)$ in ultracold atom experiments, with a clear peak around zero momentum forming at the BEC critical temperature \cite{6}.

When the system is homogeneous and the thermodynamic limit is taken in the usual way by keeping fixed the density $n \equiv N/\Omega$ ($\Omega$ is the volume), then $\rho(x,y)$ tends to the condensate density $\lambda_0/\Omega$ when $|x - y| \to \infty$ \cite{2}. This definition makes transparent the analogy of the
condensate fraction $\lambda_0/N$ with the magnetization $M$ in magnetic spin systems, where the analog of the 1BDM (1) is the correlation function $(S_i S_j)$ which, in the homogeneous case, tends to $M^2$ for $|i - j| \to \infty$ (see, for instance, [7]).

Given that in presence of ODLRO the largest eigenvalue $\lambda_0$ scales as $N$, we can conveniently quantify deviations from ODLRO in terms of the exponent $C$ of a scaling law as

$$\lambda_0 \sim N^C.$$  \hfill (3)

Clearly, when $C = 1$ we are back to the ODLRO and BEC, according to the Penrose-Onsager criterion. On the other hand, when $C = 0$, we are typically in a situation which is fermionic-like: think, for instance, of the ideal Fermi gas, where for all eigenvalues (including $\lambda_0$) we have $\lambda_i = 1$, in view of the Pauli principle. A general classification of different behaviours of the correlation functions characterizing different types of order is given in [8,9]. As additional example, consider a system made of two species of fermions with attractive interactions where there may be ODLRO but this manifests in the two-body density matrix, while for the scaling law of the eigenvalues of the 1BDM one still has $C = 0$. Despite one can imagine a more general, non-power-law, dependence of $\lambda_0$ on $N$, it is reasonable to assume a power-law form like the one introduced in eq. (3). The explicit computations presented below on one-dimensional systems are in agreement with the definition (3).

One-dimensional quantum systems provide an ideal playground to investigate deviations from ODLRO since there is no BEC in the interacting case. In other words, one expects $C = 1$ only for the 1D noninteracting Bose gas, which may be regarded, however, as a very delicate, if not pathological, limit. In the homogeneous case, in fact, any infinitesimal repulsive interaction, no matter how small, in one-dimension destroys ODLRO also at $T = 0$, unlike the 3D case. This means that, in 1D interacting systems, $C$ must be strictly smaller than 1 for any finite value of the interaction. Therefore, one may lead to conclude that no clear peak of the momentum distribution should be observed, also at $T = 0$, in experiments with ultracold atoms in one-dimensional confined geometries.

However, when $C$ is close to 1, for finite $N$ the ratio $\lambda_0/N$ can be rather large (even though $\lambda_0/N$ tends to 0 for $N \to \infty$). When this happens, we say that we are in the presence of what we can refer to as a mesoscopic condensation at $T = 0$. With this we refer to a phenomenon that, for all practical purposes, can be considered as an ordinary condensation: e.g., for $C \approx 0.99$ and $N \approx 10^3$ one has $N^C/N \approx 0.9$. Let us remark that $C$ being close to 1 for small interactions gives reason to the fact that in the weakly interacting limit the mean-field description works reasonably well, despite the absence of a proper BEC. Our results are strictly valid at $T = 0$ in the regime of large $N$ and finite density, and they provide the zero-temperature counterpart of the well-known quasi-1D condensation phenomenon [10–13], signaled by a temperature below which the correlations decay on distances larger than the size of the system.

Our results imply that even in the absence of BEC, if $C$ is rather close to 1, one will observe a clear peak in the momentum distribution, especially because typically in experiments with 1D ultracold gases the number of particles is $N \sim 10^2–10^3$. Given the fact that the momentum distribution is an experimental quantity easily accessible, the study of deviations from ODLRO for different geometries and interactions is therefore desirable.

The topic of this paper is to identify and quantify deviations from ODLRO in 1D quantum systems at $T = 0$. Apart from the obvious consequences for the presence of a mesoscopic peak in the momentum distribution, there are three additional reasons for such a study. First of all, the computation of correlation functions and 1BDM is a quite difficult and often formidable task. For 1D systems, however, the situation is generally better and a huge variety of techniques has been developed for this aim [14,15], ranging from bosonization [16] and density matrix renormalization group [17], to Bethe ansatz and integrability techniques [18,19]. Secondly, one-dimensional anyonic gases set a non-trivial interpolation between Bose and Fermi statistics, and have the further advantage to be Bethe solvable [20,21]. Finally, ultracold atoms provide an ideal setting to simulate different 1D quantum systems by acting on tunable external parameters [15,22]. For instance, the coupling constant $\gamma$ in 1D ultracold bosonic gases can be adjusted by tuning the transverse confinement of the waveguides in which the atoms are trapped [23], and in such an experiment one can explore both the regimes of small $\gamma$, as small as $10^{-4}–10^{-3}$ (the weakly interacting limit), and large $\gamma$ (the Tonks-Girardeau limit [24]), with numbers of particles $N$ going from few tens to thousands, see the reviews [13,15,22].

**1D interacting Bose gas.** Since fermions are always characterized by $C = 0$ due to their statistics, we start our analysis from the Lieb-Liniger (LL) model [25], a homogeneous 1D system of $N$ bosons of mass $m$ interacting via a two-body repulsive $\delta$-potential in a ring of circumference $L$. The Hamiltonian reads

$$H_{LL} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\epsilon \sum_{i<j} \delta(x_i - x_j),$$  \hfill (4)

and one defines a dimensionless coupling constant

$$\gamma = \frac{2mc}{\hbar^2 n},$$  \hfill (5)

where $n = N/L$ is the density of the gas, kept constant in the thermodynamic limit. As is well known, the LL model is exactly solvable by Bethe ansatz [25,26] which provides the exact expression of the many-body eigenfunctions [18,27]. At $T = 0$ the ground-state energy, the sound velocity $s$ and other equilibrium quantities, appropriately scaled, can be expressed in terms of the solution of the so-called Lieb integral equations [25], which, in turn, depends
only on $\gamma$. The equation of state coincides with the one of an ideal gas in the two limits $\gamma \to 0$ and $\gamma \to \infty$, with the residual energy depending in general on $\gamma$ [28]. The LL 1D Bose gas can be treated as well by bosonization. The dimensionless parameter called the Luttinger parameter $K$ [14] can be written for the LL model as $K = v_F/s$, where $v_F = \hbar m/n$ is the Fermi velocity. Therefore, solving the Lieb integral equations one has access both to the sound velocity $s$ and the Luttinger parameter $K$ for any values of the coupling constant $\gamma$ (see, e.g., [29,30] and references therein). In particular in the weak-coupling limit $\gamma \ll 1$ one has $s \approx \frac{\hbar n}{m\sqrt{s}}$ and $K \approx \frac{\gamma}{\hbar}$ for the Tonks-Girardeau gas one has at variance $s = v_F$ and $K = 1$, so that $K$ for a homogeneous system of $\delta$-repulsive bosons goes from $\infty$ (for $\gamma \to 0$) to 1 (for $\gamma \to \infty$).

A simple evaluation of $C$ using bosonization can be done as follows. The eigenvalues $\lambda_n$ and the orbitals $\phi_n(x)$ in eq. (2), for (4), are labelled by a quantum number which is evidently the momentum $k$. From translational invariance $\phi_n(x) = (1/\sqrt{\pi})e^{ikx}$ [2], and, therefore, $\lambda_n$ simply equals the momentum distribution $n(k) = \langle \hat{\Psi}^{\dagger}(k)\hat{\Psi}(k)\rangle$, where the operator $\hat{\Psi}(k)$ is the Fourier transform of the field operator $\hat{\Psi}(x)$ [2]. There follows $n(k) \propto \int_0^1 dx \rho(x)e^{ikx}$, where, using translational invariance, we have set $y = 0$ and $\rho(x,0) \equiv \rho(x)$. From Luttinger liquid theory for large $x$ we have that $\rho(x) \propto x^{-1/2K}$ and, therefore, $n(k) \propto \frac{s}{\pi} \frac{1}{\sqrt{\rho(x)}}$ for $k \to 0$ (see footnote 1). The bosonization then gives the large distance behaviour of $\rho$, which then fixes the exponent $C$. Indeed, since the smallest momentum is $k_{\text{min}} \propto 2\pi/L$ and $n = N/L$, one gets $\lambda_0 \propto N^{1-1/2K}$, i.e.,

$$C = 1 - \frac{1}{2K}.$$  

(6)

Notice that $C$ is then expected to depend on $\gamma$, i.e., on the ratio $c/n$, and not on $c$ and $n$ separately.

An accurate, high-precision check of such a prediction is not easy to obtain, since one should determine $\lambda_0$ as a function of $N$ and then fit $C$ from the scaling. It is clear that the larger is the maximum value of $N$ considered, the better the estimate of $C$, but from exact computations is not straightforward to reach large values of $N$, especially for intermediate values of $\gamma$. Of course, one could think to use the Bethe ansatz expression for the wave function of the ground state, but, in practice, even for small number of particles, such an expression is difficult to handle. One way to get around this difficulty consists, e.g., in using a numerical approach as ABACUS [32,33], in which the sum on the corresponding Bethe eigenfunctions can be efficiently truncated. In [31] such a method was used for the approximate computation of the 1BDM up to 150 particles and for any values of $\gamma$. In the following we introduce a new method based on an interpolation of the 1BDM

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between large- and short-distance asymptotic expansions, and which can be used for a larger number of particles for all values of $\gamma$. Hereafter we present the results obtained with the interpolated $\rho(x)$ till $N = 10^4$ (there is, however, no major problem to extend such a computation to larger values of $N$). For $N$ up to $10^4$ we found that the error in $C$ is, e.g., at the fifth significant figure for $\gamma = 1$ and such an error can be further decreased since the larger the value of $N$, the smaller the error in $C$. Our results with $N$ up to $10^5$ confirm that $C$ depends only on $\gamma$ and not separately on $n$ and $c$.

To conveniently set up such an interpolation formula we have built upon several known behaviours of the 1BDM $\rho(x)$ as a function of distance $x$, and in particular on its short-distance behaviour $\rho^{SD}(x)$ [34] and on the large-distance one, $\rho^{LD}(x)$ [35,36]. The limits of weak and strong interactions, valid for any values of $x$, have been extensively investigated [37–41] (see more references in [15,22]). Such known expressions are collected for convenience in the appendix.

Interpolation scheme. — Using the known expressions for $\rho^{LD}$ and $\rho^{SD}$ (which are $\gamma$-dependent) and matching them with cut-off functions whose parameters have to be optimized, we have been able to set up a very efficient interpolation formula for the 1BDM at any distance $x$ which is given by

$$\rho^{INT}(x) = \rho^{SD}(x)\Phi^{SD}(x) + \rho^{LD}(x)\Phi^{LD}(x).$$  

(7)

Having explored several cut-off functions for this optimization, we have finally chosen $\Phi^{SD}$ and $\Phi^{LD}$ to be

$$\Phi^{SD}(\xi) = \left[1 - \tanh \left(\frac{\xi}{\alpha}\right)\right] \left[1 - \tanh \left(\frac{\sqrt{\xi}}{\beta}\right)\right]$$  

(8)

and

$$\Phi^{LD}(\xi) = \tanh \left(\frac{\xi}{\eta}\right) \tanh \left(\frac{\xi}{\omega}\right),$$  

(9)

where $\xi \equiv \pi nx$ and $\alpha, \beta, \eta$ and $\omega$ are coefficients that need to be fixed in terms of $\gamma$, but not on $N$. Their best choice comes by minimising the $\chi^2$ in a chi-squared test made for small $(\xi < \xi_{\text{min}})$ and large values $(\xi > \xi_{\text{max}})$ of $x$, where the asymptotic expansions for $\rho(x)$ are known. The test has been done for different values of $\xi_{\text{min}}$ and $\xi_{\text{max}}$ typically $\xi_{\text{min}} \approx 0.8$ and $\xi_{\text{max}} \approx 20$. Once parameters $\alpha, \beta, \eta, \omega$ are fixed, we found that the values assumed by $\rho(x)$ in (7) are in excellent agreement (i.e., relative percentage errors $< 1\%$) with those obtained for large $(\gamma \geq 100)$ and small $(\gamma \leq 0.01)$ couplings at any $x$. Notice the presence of the factor $\sqrt{x}$ in $\Phi^{SD}$ which happens to improve considerably the quality of the interpolation for $x \leq 1$.

We finally observe that other interpolation schemes we tried — using cubic spline, exponentials and power laws, rather than hyperbolic tangents as in eqs. (8), (9) — give larger errors in the large and small coupling regimes at intermediate distances where the analytical expansions are
known. As an example, using the cubic spline interpolation method the difference between the fitted and the analytical results at intermediate distances, where analytical results are known, may be up 50% for $\xi$ between 5 and 15 for $\gamma = 100$, while the corresponding error using eqs. (8), (9) is smaller than 1%.

Equation (7) is not periodic in $x$ with period $L$, so we made it symmetric with respect to $L/2$ to better mimic the periodic boundary conditions of the system. Once we fix such $\rho^{NT}(x)$, we proceed by fixing $N$, computing $\lambda_0$ and check that the result for the given value of $N$ does not depend on the grid into which the interval $[0, L]$ is divided$^2$, and then we repeat the same procedure for larger values of $N$ at the same density. We then extract the exponent $C$ from a fit with (3), determining its convergence and error with the given number $N$ (in our case $N = 10^3$). Here the possibility of varying $N$ up to large values is important: e.g., we get $C = 0.85391(1)$ for $\gamma = 1$. In the Tonks-Girardeau case we get $C = 0.5001(4)$ in agreement with the exact result [42,43]. The Tonks-Girardeau result, $C = 1/2$, confirms the different nature of (non-local) correlation functions of hard-core bosons and ideal fermions, and our outcomes illustrate the crossover from the ideal BEC case to the hard-core limit. Let us also mention that for $\gamma \sim 10^{-4} - 10^{-3}$, realistic for experimentally relevant situations, we get $C \sim 0.99$, which is very close to 1. Our results for the behaviour of $C$ vs. $\gamma$ are summarised in fig. 1.

The presented analysis is valid at $T = 0$. At finite temperature one could rely on results available for the correlation functions at large distance in the thermodynamic limit [15,33,44–46]. However, to realistically compare with experiments one should perform the above analysis not only at finite $T$, but as well at finite $N$, which we think is a subject which deserves a future study.

1D anyons. – Let now turn the attention to the case where the system is made of anyons rather than bosons [47]. For a system of $N$ anyons of mass $m$ with contact interactions, the solution $\psi_A$ of the many-body Schrödinger equation exhibits a generalised symmetry under the exchange of any pair of particles:

$$\psi_A(x_1, \ldots, x_N) = e^{-i\pi \theta_{j,l} / 2} \psi_A(x_{\tau}, x_1, \ldots, x_{\tau-1}, x_{\tau+1}, \ldots, x_N),$$

where $\theta_{j,l} \equiv \text{sgn}(x_j - x_l)$ and $\kappa$ is the so-called statistical parameter which runs from 0 (corresponding to bosons) to 1 (fermions). The boundary conditions on the wave functions have to be suitably chosen, because periodic boundary conditions for anyon correspond to twisted boundary conditions for bosons and vice versa [48]. Hence, imposing twisted boundary conditions and employing the coordinate Bethe ansatz, up to a normalization factor, the eigenfunctions of the system are given by [49]

$$\psi_A = e^{-i\pi T \sum_{j<l} \theta_{j,l}} \det[\psi_{j,l}] \prod_{n < \ell} [k_n - k_{\ell} - i\epsilon \theta_{n\ell}],$$

where the indices run from 1 to $N$ while $\epsilon$ is the renormalized coupling constant given by $\epsilon' = \epsilon / \cos(\pi x / L)$. Similarly we set $\gamma' = \sin(\pi x / L)$. To conveniently obtain the Luttinger parameter $K$ for different values of the coupling constant $\gamma'$ and the statistical parameter $\kappa$, we follow the approach in [48], where $K$ is given by $K = Z^2$, with $Z = Z(q)$, where $Z(\lambda)$ is the solution of the linear integral equation

$$Z(\lambda) = 1 + \frac{\epsilon'}{\pi} \int_{-q}^{q} Z(\mu) \left(\epsilon' \mu + (\lambda - \mu)\right)^2 d\mu,$$

with $q$ fixed by the Lieb equation for the density of state relative to the anyonic system.

Hard-core anyons. – With twisted boundary conditions, the 1BDM for hard-core anyons ($c, \gamma \to \infty$) [50,51] is given by $\rho^c(t) = \det[\phi^c_{j,l}]$ [52] with $t = 2\pi x / L \in [0, 2\pi]$, $j, l$ run from 1 to $N - 1$ and

$$\phi^c_{j,l} = \frac{2}{\pi} \int_0^{2\pi} \mu \mu e^{i(j-l)\tau} A(\tau - t) \sin\left(\frac{\tau - t}{2}\right) \sin\left(\frac{\tau}{2}\right),$$

with $A(\tau - t) = e^{i\pi(1-N)}$ for $\tau < t$ and $A(\tau - t) = 1$ for $\tau > t$. Proceeding as was done before for the LL bosons, we
have computed the largest eigenvalue \( \lambda_0 \) of the 1BDM for different values of the statistical parameter \( \kappa \) and number of particles \( N \), with \( N \) up to \( N = 241 \). The results for \( C \) are plotted in fig. 2. For \( \kappa = 0 \) (hard-core bosons) one has \( C = 1/2 \), while for \( \kappa = 1 \) (fermions) one has \( C = 0 \), and for all other values the curve monotonically interpolates as expected between 1/0 and 1 when \( \kappa \) increases.

**Lieb-Liniger anyons.** – The behaviour shown in fig. 2 refers to hard-core anyons. For a finite, soft-core energy coupling \( \gamma \) [53] one has the possibility to fully interpolate between \( C = 0 \) and 1: when \( \kappa = 0 \), then the LL Bose gas at the coupling constant \( \gamma \) is retrieved. To study how \( C \) depends on \( \gamma \) and \( \kappa \), we resort to the bosonization approach, in light of its successful estimations of \( C \) both for the LL Bose gas and for hard-core anyons given above. For LL anyons, \( \rho(x) \) at large distances is given by [54]

\[
\rho(x) = n \sum_{m=-\infty}^{\infty} b_m e^{2i(m+\frac{1}{2})k_F x - \pi i (m+\frac{1}{2})\text{sgn}(x)}\frac{nL \sin(\pi x/L)}{(m+\frac{1}{2})^2 + \pi^2},
\]

where \( b_m \) are non-universal amplitudes. From (10) one gets for small \( k \) and in the thermodynamic limit

\[
n(k) \propto n \sum_{m=-\infty}^{\infty} b_m \frac{(n\pi)^{\frac{1}{4}}(m+\frac{1}{2})^2 2K}{[k + 2k_F (m + \frac{\pi}{2})]^{\frac{1}{4}} - \frac{1}{\pi} - 2k(m+\frac{\pi}{2})^2}.
\]

As a general consequence of this expression, the momentum distribution in general does not have the maximum at \( k = 0 \): there is in fact rather a shift due to the imaginary terms of the 1BDM [54]. The leading term of \( n(k) \) is the one relative to \( m = 0 \) for any \( \kappa \) (for \( \kappa = 1 \), also the term \( m = -1 \) has the same power law behaviour). Hence, we conclude \( \lambda_0 \propto N^{1-\frac{1}{8\gamma}} - \frac{K\kappa^2}{2} \), and, therefore, the scaling coefficient \( C \) for the LL-anyons is expressed by

\[
C(\kappa, \gamma) = 1 - \frac{1}{2K} - \frac{K\kappa^2}{2}.
\]

For \( \kappa = 0 \) we obtain the result (6) for the LL, while for \( \kappa = 1 \) (the fermionic limit) one gets \( C = 0 \) since \( K = 1 \) for all \( \gamma \), in both cases the correct values. In fig. 3 we plot \( C \) vs. \( \gamma' \) for different \( \kappa \): it is evident that \( C \) is always less than one, as it should be. In fig. 3 we do not report of course the negative values of \( C(\kappa) \) because there the expression (11) for the Fourier transform of the 1BDM is not valid since the power of \( 1/x \) in (10) is greater than 1. Let us underline that this time \( C \) is not a monotonic function of \( \gamma \). Moreover, it is different from 0 only for \( \gamma \) larger than a critical value \( \gamma_c \). This result shows once again the singularity related to the bosonic non-interacting limit. A plot of \( \gamma_c \) as a function of \( \kappa \) is shown in fig. 4, where it is also evident the non-monotonic behaviour in \( \kappa \), with a maximum around \( \kappa \approx 0.9 \).

**Conclusions.** – In this paper we have investigated deviations from off-diagonal long-range order in a variety of 1D systems. For the 1D interacting Bose gas we have introduced a new interpolating scheme for the one-body density matrix based upon the knowledge of large- and small-distance asymptotical behaviours. This scheme allows easily to consider systems with large number of particles, such as \( N = 10^4 \). Our results show that for small interactions the scaling exponent \( C \) is close to 1, implying a mesoscopic condensation, i.e., a value of the “condensate” fraction \( \lambda_0/N \) appreciable at finite values of \( N \) (as the ones in experiments with 1D ultracold atoms). We have also shown that 1D anyons provide the possibility to fully interpolate between \( C = 1 \) and 0 and, moreover, that the behaviour of the exponent \( C \) for these systems is non-monotonic both with respect to the coupling constant and the statistical parameter, revealing the subtleties and pathologies related to the non-interacting limit in one-dimension. Finally, we observe that our analysis is done at \( T = 0 \) and that it would be a worthwhile future work to extend at finite \( T \) and finite \( N \), possibly in the presence of an external potential, to compare with experimental results.

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**Appendix: asymptotic expansions.** – At fixed and finite values of \( N \) and \( L \), in the regime when \( z \equiv N\gamma \gg 1 \), in the translational invariant case, one can expand the Bethe eigenfunctions in inverse powers of \( z \). In this way, in the strong interacting (SI) limit, one gets for the 1BDM
that the asymptotic expansion [37,40]

\[ \rho^{SI}(x)/n = \rho^{(0)}_{N}(x) + \frac{1}{N\gamma} \rho^{(1)}_{N}(x) + \mathcal{O}\left(\frac{1}{N\gamma}\right)^2, \]  

(A.1)

where \( \rho^{(0)}_{N} \) and \( \rho^{(1)}_{N} \) are expressed in terms of the determinant of certain matrices, see [40]. For \( \gamma \to \infty \) one obtains the formula for the 1BDM of a Tonks-Girardeau gas of \( N \) particles [42] \( \rho^{TG}(t)/n = (1/N) \det(c_{n,m}(t))_{n,m=1,...,N-1} \), where \( t = 2\pi x/L \in [0, 2\pi] \) and \( c_{n,m} = c_{n-m} \) with \( c_i(t) = 2\delta_{i,0}\cos(t/2) - \delta_{i,1} - \delta_{i,-1} + (2/\pi)|f(i+1) - f(i-1)| - 2\cos(t/2)\sin(it/2)/i \) where \( f(i) = \sin(it/2) \). In the weak-coupling limit (i.e., if the Luttinger parameter \( K \) satisfies the inequality \( K \gg 1 \), which amounts to \( \gamma \lesssim 0.1 \)), one can write the 1BDM as [38]

\[ \rho^{WI}(x)/n = \exp(-\frac{\hbar}{\sqrt{\mu \gamma}} \int_{0}^{\infty} dk [1 - \cos(kx)](vk)^2), \]  

(A.2)

where \( 2v_k = [k^2/(k^2 + 4\hbar^2/m\mu)]^{1/4} - [(k^2 + 4\hbar^2/m\mu)/(k^2)]^{1/4} \) and the chemical potential \( \mu \) is given by \( \mu = \frac{\hbar^2}{2m} n^2 \left[ 2e(\gamma) - 4e'(\gamma) \right] \), with \( e(\gamma) \) the rescaled ground-state energy \( e(\gamma) \equiv E_{0}/\sqrt{2\gamma} \) obtained solving the Lieb integral equations. The expression for \( \rho^{WI}(x) \) is manifestly non-periodic in \( x \) in fact it was originally derived only in the thermodynamic limit where \( L, N \to \infty \), as stressed in [38]. To solve this issue we evaluate \( \rho^{WI}(x) \) for \( x \in [0, L/2] \), getting all other values of the 1BDM for \( x \in (L/2, L] \) by reflection as \( \rho(x = L/2 + \delta) = \rho(x = L/2 - \delta) \), where \( \delta \in (0, L/2) \), so that \( \rho(x = L) = \rho(x = 0) \). This approach turns out to be a good way to approximate \( \rho(x) \) for studying how \( \lambda_0 \) scales with \( N \) when \( N \) becomes very large.

For an arbitrary value of the coupling constant \( \gamma \), at short distances, i.e., \( |x| \ll 1 \), in the thermodynamic limit the behaviour of the 1BDM is expressed by a Taylor expansion around the origin as [34] \( \rho^{SD}(x)/n = 1 + \sum_{k=1}^{\infty} \frac{p_k}{m\pi|kx|} |k|^2 \), where the first three Taylor coefficients \( p_k \) are given by \( p_1 = 0, p_2 = \frac{\gamma^2}{12} e'(\gamma) \) and \( p_3 = \frac{\gamma^3}{72} e''(\gamma) \). We used only the first three coefficients of this expansion, so that the error associated to the truncation is of order \( O(x^4) \). We have checked the validity of such an approximation for \( \rho(x) \) by comparing the obtained outcomes vs. the results for the density matrix at large and small values of the coupling (e.g., for \( \gamma \geq 100 \) and \( \gamma \lesssim 0.01 \) from (A.1) and (A.2), and also vs. the results coming from the Tonks-Girardeau expression. We found that the relative percentage errors are well below 1% for \( |x| \lesssim 1 \).

At large distances, i.e., \( |x| \gg 1 \), in the thermodynamic limit the 1BDM can be written as [36]

\[ \rho^{LD}(x)/n = \sum_{m=0}^{\infty} B_m \cos(2\pi k_F x) \frac{\cos(2\pi k_F x)(nx)^{2m+1}/2k^2}{(nx)^{2m+2}}, \]  

(A.3)

where \( k_F = \pi n \) is the Fermi momentum and \( B_m \) are numerical coefficients which can be determined using the method described in [36]. One can get already a good approximation of the large distance behaviour of the 1BDM just by taking the \( m = 0 \). Also in this case we have compared the values of the 1BDM obtained by (A.3) vs. those relative to weak- and strong-coupling constants and also those coming from the Tonk-Girardeau limit: in all these cases, the relative percentage errors remain again always below 1% for \( |x| \geq 20 \).

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