The relationship between undularity and $l$ dependence of the proton optical model potential

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Abstract

The contribution of collective or reaction channels to a local optical model potential, OMP, can now be readily calculated as a dynamical polarization potential, DPP. The resulting local DPPs commonly have undulatory (‘wavy’) features, including local regions of emissivity in the imaginary component. This can be attributed to $l$-dependence of the underlying formal non-local and $l$-dependent DPP. Here we show that the $l$-independent proton OMPs, that have the same $S$-matrix $S_{ij}$ as phenomenological $l$-dependent potentials, exhibit undulations that are qualitatively similar to undulations of local DPPs generated by channel coupling. The $l$-dependent phenomenological potentials which we study give the best existing fits to the relevant elastic scattering data. The same therefore applies to the undulatory potentials presented here, which give exactly the same scattering. In addition, we show that $l$-dependent potentials based, unlike the phenomenological potentials, on smooth Woods-Saxon forms, also have undulatory $l$-independent equivalents, including emissive regions. This suggests that undularity (‘waviness’) is a generic property of potentials that are $l$-independent equivalents of $l$-dependent potentials. Implications for the validity of folding models based on a local density model are noted.

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I. INTRODUCTION

Following the proposal [1] that the nucleon optical model potential (OMP) is explicitly $l$-dependent, it was often suggested that an $l$-independent potential with a suitably modified radial form would fit the data equally well. That turns out to be true, but the undulatory (wavy) radial forms that are required to fit precise and wide angular range data were not anticipated. Three lines of investigation converge regarding the radial form of the nucleon OMP: (i) experimental elastic scattering observables that could not be precisely fitted with smooth Woods-Saxon (WS), or similar, forms can be fitted precisely with potentials exhibiting undulations, (ii) the local and $l$-independent dynamic polarisation potentials (DPPs) generated by coupling to reaction or inelastic channels are significantly undulatory, often with emissive regions, (iii) an $l$-dependent potential model exists that gives fits over a wide energy range to data resistant to fitting with WS-like potentials. It is unclear which of the alternative representations, undulatory or $l$-dependent, is more natural. This is relevant to the application of OMPs in reaction analyses since two $S$-matrix equivalent potentials will generally not have the same radial wave function in the nuclear interior. In this paper, ‘$S$-matrix equivalent potentials’ are potentials with the same $S$-matrix $S_{lj}$ where $l$ and $j$ are the partial-wave orbital and total angular momenta for spin $\frac{1}{2}$ projectiles.

The local and $l$-independent representations of the DPP generated by coupling to transfer or inelastic channels are generally undulatory. Such DPPs commonly have radial regions where the imaginary part is emissive. For nucleon scattering examples, see e.g. Refs. [2–6]. The undulations in general and the emissivity in particular raise various questions that are addressed here. Although emissive regions in DPPs may appear surprising, the nature of their origin ensures that the unitarity limit $|S_{lj}| \leq 1$ is not broken. It is natural to ask how the undularity of DPPs can be interpreted, and whether there is any link with fits to scattering data. In this paper we show that a phenomenological $l$-dependent potential model, successfully applied to the elastic scattering of 30 MeV protons from $^{16}\text{O}$, $^{40}\text{Ca}$, $^{58}\text{Ni}$ and $^{208}\text{Pb}$, has undulatory $l$-independent equivalents at each energy. The imaginary terms of these have emissive regions. The inverted potentials do indeed have features that are qualitatively similar to features that appear in local DPPs that result from inelastic-channel or reaction-channel coupling, Refs. [2–6].

In this work, $S$-matrix inversion is applied to determine the $l$-independent potentials
that are $S$-matrix equivalent to the $l$-dependent proton potentials mentioned above. These equivalent potentials have qualitative features, namely the undularity and regions of emissivity, that are also found in local and $l$-independent DPPs arising from channel coupling. Such $l$-independent DPPs are the equivalents of $l$-dependent potentials since $l$-dependence is a property of the formal DPPs, as in the theory of Feshbach [7], see also Rawitscher [8]. Potentials with undulatory features have been found when fitting light-ion elastic scattering data that is both precise and having a wide angular range, see Section VII. It is possible that the occurrence of undulations, and the lack of any widely understood interpretation of undularity, have inhibited the exact fitting of such data. Data of that quality arguably contains information concerning the dynamics of nucleon-nucleus and nucleus-nucleus interactions [9].

Every $l$-dependent potential has an $l$-independent equivalent which can be found using $S_{ij} \rightarrow V(r) + l \cdot s V_{SO}(r)$ inversion, see Section II. Section III presents systematic properties, such as volume integrals, of the $l$-independent potentials that have the same $S_{ij}$ as $l$-dependent potentials that precisely fit scattering data. Section IV discusses issues arising. Section V presents the radial properties of the inverted potentials. The $l$-dependent potentials in earlier sections give precise fits to elastic scattering data, but in Section VII $S$-matrices for potentials having simpler forms of $l$ dependence are inverted in order to assess whether undularity, including emissivity, is a generic property of potentials that are $S$-matrix equivalent to $l$-dependent potentials. Section VII discusses direct model-independent fits to elastic scattering. Section VIII relates $l$-dependence to nuclear size. Section IX is a summary and Section X is an appendix specifying the characteristics of the $l$-dependent potentials.

II. $S$-MATRIX INVERSION

The $S$-matrices are inverted using the iterative-perturbative, IP, $S_{ij} \rightarrow V(r) + l \cdot s V_{SO}(r)$ inversion algorithm which is presented in Refs. [10–13]. The IP inversion is implemented in the inversion code IMAGO [14] which presents the difference between the $S_{ij}$ to be inverted and the $S_{ij}^i$ of the inverted potential in terms of the $S$-matrix distance $\sigma$ defined as

$$\sigma^2 = \sum_{ij} |S_{ij}^i - S_{ij}|^2. \quad (1)$$
The IP iterations start from a ‘starting reference potential’, SRP, which in all cases presented here was the \( l \)-independent part of the \( l \)-dependent potential. The SRP is included in plots of inverted potentials so that the contribution of the \( l \)-dependence to the inverted \( l \)-independent potential appears as the difference between the SRP and the inverted potential. It is established that the IP method can yield inverted potentials that are effectively independent of the SRP and the uniqueness of the inverted potential can be tested by the use of alternative ‘inversion bases’, see Refs. [11, 13]. The figures in this paper were produced by IMAGO and values of \( \sigma \) appear in the figures. In cases where two inverted potentials are shown, that with the lower \( \sigma \) is generally adopted. The tendency for undularity to increase as \( \sigma \) becomes very small will be addressed in relation to the significance of the potential undulations.

### III. \( l \)-INDEPENDENT EQUIVALENT OF \( l \)-DEPENDENT PHENOMENOLOGY

The \( S \)-matrix elements \( S_{ij} \), (SMEs), have been inverted for the following \( l \)-dependent potentials: 30.1 MeV protons on \( ^{16}\text{O} \) and 30.3 MeV protons on \( ^{40}\text{Ca} \) from Ref. [15], and for 30.3 MeV protons on \( ^{58}\text{Ni} \) and \( ^{208}\text{Pb} \) from Ref. [16]. These \( l \)-dependent potentials fitted the data with a precision exceeding that achieved with other potentials. The \( ^{16}\text{O} \) and \( ^{40}\text{Ca} \) cases are notoriously hard to fit, and cannot be fitted with non-undulatory forms. The inverted potentials having the same \( S_{ij} \) as the \( l \)-dependent potentials reproduce the data equally well.

The \( l \)-dependent potentials [15, 16] all consist of standard-form \( l \)-independent potentials to which \( l \)-dependent real and imaginary surface peaked terms are added, see the Appendix. We invert \( S_{ij} \) for the \( l \)-dependent potential to obtain the \( l \)-independent equivalent; subtracting from this the \( l \)-independent part of the \( l \)-dependent potential gives an \( l \)-independent measure of the \( l \)-dependent effect. Rather than plot all the resulting difference potentials, we quantify the \( l \)-dependence in terms of volume integrals as defined by Satchler [17]. We present the differences between the volume integrals of the \( l \)-independent equivalent and the \( l \)-independent part of the \( l \)-dependent potential. The differences in the volume integrals of the real and imaginary central terms are \( \Delta J_R \), \( \Delta J_{IM} \) with similar notation for the spin-orbit (SO) terms. These are presented in Table [II] for three cases: (i) when only the imaginary \( l \)-dependent terms is included, (ii) when only the real \( l \)-dependent terms is included and, (iii) when both \( l \)-dependent terms are included, as required to fit the data. The significance
of the additivity of the real and imaginary $l$ dependencies will be addressed in forthcoming work.

The four main sections of the table are headed by an identification of the case, e.g. 30.1 MeV protons on $^{16}$O, etc. The first column labels what is presented on the line: four sets of changes in volume integrals and one set of changes in the reaction cross section, CS. The remaining four columns present the differences between volume integrals for (i) the S-matrix equivalent $l$-independent potential found by for various $l$-dependent potentials, and, (ii) the same volume integrals for the $l$-independent potential to which the $l$-dependent parts had been added. Column 2 presents the effect of the full $l$ dependence, column 3 is for imaginary $l$-dependence alone, column 4 is for real $l$ dependence alone. The last column simply adds numbers in columns 3 and 4 for comparison with the corresponding numbers in column 2, reflecting on the linearity of the system to the inclusion of real and/or imaginary $l$-dependent terms. We note:

1. For the $^{16}$O case, in column 4, the real $l$ dependence alone resulted (no surprise) in a large change $\Delta J_R$ and very small change in $\Delta J_I$. More surprising is the large change $\Delta J_R$ associated with a very small change in the reaction cross section (CS) and simultaneously a very large change in angular distribution (AD) beyond $80^\circ$, and in the analysing power (AP) for all angles, see Fig. 1.

2. Also for $^{16}$O, as expected, the imaginary $l$ dependence gave a somewhat larger volume integral change $\Delta J_I$ than $\Delta J_R$; for real $l$ dependence the change $\Delta J_R$ was much greater than the change in $\Delta J_I$. (This contrast might be related to the fact that the imaginary $l$-dependent term was at a much larger radius than the real $l$-dependent term [13].) An expected result is that the change in CS is very large for the imaginary $l$-dependent term, Fig. 2. But, simultaneously, the changes in the AD and AP are less than for real $l$-dependence Fig. 1 except at angles forward of $80^\circ$. This might also be related to the large radius of imaginary $l$-dependent term.

3. The values in the last ($\Sigma$) column are quite close to those in the ‘Full’ column. Exact additivity cannot be expected.

4. Points 1 and 2 together exemplify the strong disconnect between the magnitudes of the changes in CS and the changes in the angular observables. This is relevant to
TABLE I: Properties of potentials that are $l$-independent equivalents to $l$-dependent phenomenological potentials. For each quantity, the same quantity for the $l$-independent part of the $l$-dependent potential has been subtracted. All volume integrals are in terms of MeV fm$^3$ and the change in reaction cross section, CS, due to the inclusion of the $l$-dependent terms, $\Delta$ CS is in mb. The last column presents the numerical sum of the values in columns 3 and 4.

|       | Full-$l$dep | IM-$l$dep | RE-$l$dep | $\Sigma$ RE+IM |
|-------|-------------|-----------|-----------|-----------------|
| $p + ^{16}\text{O}$ 30.1 MeV |             |           |            |                 |
| $\Delta J_R$     | −27.25      | 6.55      | −37.21    | −30.66         |
| $\Delta J_I$     | 9.4         | 11.43     | 0.09      | 11.52          |
| $\Delta J_{SOR}$ | 0.612       | 0.626     | 0.289     | 0.915          |
| $\Delta J_{SOI}$ | 0.46        | 0.671     | 0.011     | 0.682          |
| $\Delta CS$      | 55.41       | 52.25     | 0.37      | 52.62          |
| $p + ^{40}\text{Ca}$ 30.3 MeV |             |           |            |                 |
| $\Delta J_R$     | −69.21      | 2.09      | −71.59    | −69.50         |
| $\Delta J_I$     | 13.649      | 6.439     | 7.059     | 13.498         |
| $\Delta J_{SOR}$ | 1.4124      | 0.0213    | 1.4874    | 1.5087         |
| $\Delta J_{SOI}$ | −0.1451     | −0.0991   | −0.09098  | −0.1901        |
| $\Delta CS$      | 53.13       | 38.81     | 20.66     | 59.47          |
| $p + ^{58}\text{Ni}$ 30.3 MeV |             |           |            |                 |
| $\Delta J_R$     | −38.85      | 0.09      | −40.93    | −40.84         |
| $\Delta J_I$     | 8.346       | 0.995     | 6.463     | 7.458          |
| $\Delta J_{SOR}$ | 1.1533      | −0.0501   | 1.3573    | 1.3072         |
| $\Delta J_{SOI}$ | −0.1263     | −0.0526   | −0.1578   | −0.2104        |
| $\Delta CS$      | 29.3        | 6.2       | 23.8      | 30.0           |
| $p + ^{208}\text{Pb}$ 30.3 MeV |             |           |            |                 |
| $\Delta J_R$     | −6.82       | 0.04      | −6.59     | −6.78          |
| $\Delta J_I$     | 1.98        | 0.46      | 1.88      | 2.34           |
| $\Delta J_{SOR}$ | 0.1009      | −0.0056   | 0.1061    | 0.1005         |
| $\Delta J_{SOI}$ | 0.4891      | 0.3829    | 0.0636    | 0.4465         |
| $\Delta CS$      | 10.1        | 8.3       | 2.3       | 10.6           |
evaluating the contribution of CS fits to the determination of OMP parameters.

5. Similar general results apply for $^{40}$Ca, $^{58}$Ni and $^{208}$Pb targets, but with diminishing importance of the imaginary $l$-dependence for heavier targets for which there is more interplay between the real and imaginary effects.

6. In spite of point 5, and the diminishing effect of $l$-dependence for the heavier targets, the general properties of the $l$-dependence are basically the same for all four target nuclei. This goes back to the claim of Refs. [15, 16] that the general properties of the $l$-dependent potential, that ‘precisely’ fits all the data, vary with energy much more regularly than the best $l$-independent WS-type fits (which fit the data much more poorly).

In Section V, we show that the $l$-independent equivalents to $l$-dependent potentials that actually fit proton elastic scattering have undulations (waviness) in the surface, including regions of emissivity. Since the $l$-dependent potentials are the only potentials that currently fit all the relevant proton elastic scattering data, it follows that the only $l$-independent potentials that currently fit those data have emissive regions in the surface (and undularities not just in the surface.) This is significant since the dynamic polarisation potentials Refs. [2–6] arising from various channel couplings also exhibit undularities including emissive regions.

A. Higher energy $^{40}$Ca case

The $l$-dependent potential in Ref. [15] for 35.8 MeV protons on $^{40}$Ca had a somewhat different character from the 30.3 MeV potential in that the $l$-independent imaginary part was predominantly of volume character whereas that for 30.3 MeV had surface absorption. This case could therefore illuminate the extent to which the character of the inverted $l$-independent potentials depends on the form of the $l$-independent part of the $l$-dependent potential. The 35.8 MeV case was studied as before but without spin-orbit terms.

The characteristics of the $l$-dependent term were similar in those of the 30.3 MeV case. Quantities for the inverted $l$-independent potentials for full $l$-dependence and for separate real and imaginary $l$-dependence are presented in Table II in the format of Table I. As will be shown in Section V the resulting inverted real potential has a form very similar to that for 30.3 MeV. The imaginary term had very similar undulations (including emissivity) in
FIG. 1: For 30.1 MeV protons on $^{16}$O, the solid lines are the angular distribution (above) and analyzing power (below) with just the real $l$-dependence included. The dashed lines are calculated with the same potential but with the $l$-dependent terms omitted. The associated change in reaction cross-section was very small: just 0.37 mb.

the surface but with some differences in the nuclear interior. When the real and imaginary terms were added separately, it was found that real $l$ dependence led to a real part that is visually the same as for full $l$ dependence. However, real $l$-dependence led to only a very small amplitude undularity in the imaginary potential in the surface, $r > 6$ fm, region, although there was some effect for $r < 6$ fm.

With imaginary $l$-dependence, there was little effect on the real potential, but the imaginary potential had a form that was hard to distinguish, except for $r < 5$ fm, from that with full $l$ dependence. By eye, the effect of the full $l$ dependence was just the sum of the effects of real and imaginary $l$-dependent components. This is in accord with what is shown in the last column of Table I.
FIG. 2: For 30.1 MeV protons on $^{16}$O, the solid lines are the angular distribution (above) and analyzing power (below) with just the imaginary $l$-dependence included. The dashed lines are calculated with the same potential but with the $l$-dependent terms omitted. The associated change in reaction cross-section was 52.25 mb.

IV. PROPERTIES AND QUERIES ARISING

Tables I and II reveal various systematic effects directly related to the properties of the $l$ dependence that was required to fit the data. These merit interpretation, particularly the following two:

P1 The real $l$-dependence in all cases acts to reduce the attraction for the lowest partial waves. (Hence negative $\Delta J_R$ in all RE-Ldep cases)

P2 The imaginary $l$-dependence in all cases acts to increase the absorption for lowest partial waves (hence positive $\Delta$ CS and positive $\Delta J_I$ in all IM-Ldep cases).

Note that both these properties are consequences of the $l$-dependent components of the
TABLE II: Properties of potentials that are \( l \)-independent equivalent to \( l \)-dependent 35.8 MeV phenomenological potentials for protons on \( {}^{40}\text{Ca} \). For each quantity, the same quantity for the \( l \)-independent part of the \( l \)-dependent potential has been subtracted. All volume integrals are in MeV fm\(^3\) and the change in reaction cross section, \( \Delta \text{CS} \), due to the inclusion of the \( l \)-dependent terms, is in mb. The last column has the sum of the values in columns 3 and 4.

| \( \Delta J_R \) | \( \Delta J_I \) | \( \Delta \text{CS} \) | Full-Ldep | IM-Ldep | RE-Ldep | \( \Sigma \) RE+IM |
|------------------|------------------|------------------|----------|----------|----------|------------------|
| \( p + {}^{40}\text{Ca} \) 35.8 MeV | | | -69.75 | 1.90 | -72.08 | -70.18 |
| \( \Delta J_R \) | 13.74 | 9.51 | 2.90 | 12.41 |
| \( \Delta J_I \) | 10.67 | 8.88 | 1.67 | 10.55 |

overall \( l \)-dependent potentials, and are not related to any comparison with the best (i.e. least worst) \( l \)-independent potential [15, 16].

**Query 1** Why does \( l \)-dependence in the real part always act to increase \( J_I \) (positive \( \Delta J_I \))?

Possible answer: from \( \text{P1} \), the lowest partial waves are subject to less attraction and hence the local wave number is reduced allowing greater absorption along the trajectory. One could say the nucleon slows down a bit and spends more time in the absorptive region. This can easily be seen in terms of the complex momentum of the nucleon within the complex potential. (There will also be refractive effects.)

**Query 2** Why does the imaginary \( l \)-dependence tend to increase \( J_R \)?

Possible answer: As expected from \( \text{P2} \), the real potential becomes somewhat less effective for small radii where partial waves with low \( l \) are most sensitive. But, the effect must not apply for large partial waves so the reduced attraction at the centre is compensated by a region of attraction near the surface which leads to an increase in volume integral. The attractive region near 5 fm connects to undulations in the far surface. Hence the intuition that there should be repulsion is fulfilled for small \( r \), but the \( r^2 \) weighting of the volume integrals wins out, leading to positive \( \Delta J_R \).

The tentative nature of the answers suggests that there is much still to be learned about the simplest aspects of nuclear elastic scattering.
V. RADIAL FORM OF THE INVERTED POTENTIALS

Tables I and II present the overall magnitude and various other consistent properties of the $l$-dependent contribution to the potential. Here we show explicitly the undularity and emissiveness induced in the $l$-independent equivalent potentials. The inverted potentials for the $^{16}$O, $^{40}$Ca and $^{58}$Ni 30 MeV cases are presented in Fig. 3 to Fig. 5 respectively, and for 35.8 MeV protons on $^{40}$Ca in Fig. 6. In these figures the solid lines represent the $l$-independent part of the $l$-dependent potential and the dashed or dotted lines represent $l$-independent potentials having the same $S_{lj}$ as the $l$-dependent potential.

Two inverted potentials are presented for $^{16}$O and $^{40}$Ca, the lower values of $\sigma$ corresponding to extended iterations. For $^{58}$Ni there was just one series of iterations and the dashed line, coincident in this case with the solid line, represents both the $l$-independent part of the $l$-dependent potential and the SRP; here $\sigma$ has fallen in the inversion process from an initial value of 0.926 for the SRP to 0.134 $\times$ $10^{-3}$.

The vertical scales in Fig. 3 to Fig. 6 have been adjusted to the magnitude of each quantity plotted, so the amplitude of the undularity in the far surface of the real central potential is comparable to that of the imaginary potential. Apart from the $^{16}$O case, the imaginary spin-orbit term (absent from the $l$-dependent potential) is very small, and subject to some uncertainty in the inversion process. The large effect on the real, central term is qualitatively the same in each of the three cases: a strong reduction in depth for smaller radii with a transition to an increase in depth near the surface, leading to undulations further out. The imaginary central potential also exhibits qualitative similarities in all cases including marked surface undularities which clearly include regions of emissivity.

For 35.8 MeV protons on $^{40}$Ca, Fig. 6 shows that the modifications of the potential due to the $l$ dependence are qualitatively the same as for 30.3 MeV case; the change from surface to volume absorption in the $l$-independent part has made little qualitative difference, especially in the surface region.

VI. MODEL CALCULATIONS AND GENERIC PROPERTIES

It is legitimate to ask whether undularity is a generic property of $l$-independent potentials that are $S$-matrix equivalent to potentials that are substantially $l$-dependent. For example,
FIG. 3: For 30.1 MeV protons on $^{16}$O, the solid lines present the $l$-independent potential, from top panel downwards, the real-central, imaginary-central, real spin-orbit and imaginary spin-orbit (the last zero for the $l$-independent term.) The dashed and dotted lines are the inverted potentials, that with dots having lower inversion $\sigma$ and is the one with properties given in Table II.

$$p + O - 16 \text{ at } 30.1 \text{ MeV} \quad 134916.035$$

is the occurrence of emissivity in the surface a consequence of the particular surface peaked $l$-dependent terms of Refs. [15, 16]? It is not easy to give a comprehensive answer to such questions but here the issue is explored with simple model calculations involving an $l$-dependent potential of WS form. These reveal that a form of $l$ dependence which does not involve a surface peaked $l$-dependent term also leads to $l$-independent equivalents that are strongly undulatory in the surface, including emissive regions in the surface of the imaginary term, even when only the real term is $l$-dependent. Thus the undulations in Section IV are
FIG. 4: For 30.3 MeV protons on $^{40}$Ca, the solid lines present the $l$-independent potential, from top panel downwards, the real-central, imaginary-central, real spin-orbit and imaginary spin-orbit (the last zero for the $l$-independent term.) The dashed and dotted lines are the inverted potentials, that with dots having lower inversion $\sigma$ and is the one with properties given in Table II. The radial scale is different from that of Fig. 3.

There are many ways in which a potential can be $l$-dependent. Here we study just one in which we apply a uniform renormalisation of the real or imaginary term for low-$l$ partial waves. The transition has the same dependence on $L$ and $\Delta$ as that for the previous calculations and specified in the Appendix. The real or imaginary part is multiplied by some factor for $l$ values less than $L$ (chosen as specified below), and not modified for high $l$, with

not an artefact of the surface peaked nature of the $l$-dependent term.
FIG. 5: For 30.3 MeV protons on $^{58}$Ni, the solid lines present the $l$-independent potential, from top panel downwards, the real-central, imaginary-central, real spin-orbit and imaginary spin-orbit (the last zero for the $l$-independent term.) The dotted lines are the inverted potential with properties in Table I. The dashed line, coinciding with the solid line, is the $l$-independent potential.

A transition region defined by $\Delta$. This is motivated by the possibility that $l$ dependence is characterised by a difference between the interaction for partial waves that have a strong overlap with the nucleus and those that do not. At each energy, the value of $L$ is chosen to be the value of $l$ for which $|S_l|$ is close to 0.5. Two values of the transition parameter $\Delta$ are chosen to determine whether the amplitudes of the undulations are related to the sharpness of the transition. Here, the spin-orbit terms are omitted. Similar model calculations were carried out, with a similar purpose, in a study of the angular momentum dependence.
FIG. 6: For 35.8 MeV protons on $^{40}$Ca, the solid lines present the $l$-independent potential. Top panel: the real-central; lower panel: imaginary-central. The dotted lines are for the inverted potential with the lowest inversion $\sigma$ and the properties given in Table II.

Calculations were performed for 30 MeV and 45 MeV protons scattering from $^{40}$Ca. The Koning-Delaroche [25] (KD) potential was used, without the spin-orbit term. For low partial waves, i.e. for $l$ less than $L$, the real part was reduced by 10% and was left unmodified for high $l$. The transition between high and low $l$ was quantified by parameter $\Delta$. Two values, $\Delta = 1$ and $\Delta = 2$, were used at 30 MeV to verify that a smaller $\Delta$ leads to stronger undulations. For 30 MeV we set $L = 5.5$ and for 40 MeV we set $L = 6.5$. The volume integrals of the real and imaginary potentials $J_R$ and $J_I$ for the inverted $l$-independent potentials are calculated and the corresponding volume integrals for the KD potential are subtracted. The differences, $\Delta J_R$ and $\Delta J_I$, giving a measure of the $l$-independent representation of the
effect of the $l$-dependence, are presented in Table III. The percentage changes are also presented, as are changes in the reaction cross section, $\Delta CS$. Consistent with the $l$ dependent modification being confined to the real part, $\Delta CS$ is very small although the change in the angular distribution is quite large. These changes are consistent with a uniform decrease in the phase shift, for $l < L$. There were also changes in $|S_l|$, both positive and negative for different $l$, leading to a small change in CS of about 0.5 mb at 30 MeV and only about 0.2 mb at 45 MeV. However, at 45 MeV, the differential cross section was reduced at all angles except near 150° where it was quite small.

These results are presented twice in Table III for the 30 MeV cases and three times for the 45 MeV case, corresponding to different values of the $S$-matrix distance $\sigma$ defined in Eq. 1. The multiple solutions at each energy bring out a point that is relevant for evaluating the surface undulations. As the iterations for the $S_l \rightarrow V(r)$ inversion proceed, the undulations in the potential become more pronounced and this is related to the values of $\sigma$ given in Table III. The discussion below refers to potential identifiers, P.I. (pot1 etc.).

The angular distributions for the $l$-independent (inverted) potentials closely fitted that from the $l$-dependent potentials, for 45 MeV, see Fig. 7. The lower value of $\sigma$ corresponds to a small improvement to the fit to the angular distribution, but corresponds to a substantial improvement in the reproduction of $S_l$ for $l$ values that make very little contribution to the angular distribution. Specifically the improvements in fits to $S_l$ were for $l > 11$ at 30 MeV and $l > 14$ at 45 MeV. In the latter case, $|S_{14}| \sim 0.9997$ and $\arg S_{14} \sim 10^{-4}$ and the change in the angular distribution corresponding to the correction of $S_l$ by further iterations was < 10% at 180°.

The three 45 MeV cases present an opportunity to illustrate the way in which improving the fit to $S_l$ for high $l$, thereby lowering $\sigma$, increases the undularity but with only small improvements to the fit to the angular distribution. We illustrate this with the pot3 which was the end point of a different sequence of inversion iterations. Fig 7 shows that the angular distribution for pot1 differs from that calculated with the $l$-dependent potential by 10% at most near 180°, while the dashed line, representing pot3, is almost indistinguishable from the solid line. The potentials pot1 and pot3 are compared with the $l$-independent part of the $l$-dependent potential in Fig. 8.

The higher value of $\sigma$ for pot1 is due to the poor fit to $S_l$ for high values of $l$, as presented in Fig. 9. It is apparent that to reproduce $S_l$ for $l > 14$ simultaneously with reproducing $S_l$
TABLE III: Properties of potentials that are $l$-independent equivalents to $l$-dependent phenomenological potentials. The differences in the real and imaginary volume integrals, $\Delta J_R$ and $\Delta J_I$, and the same changes expressed as percentages, are given. All volume integrals are in MeV fm$^3$ and the change in reaction cross section, CS, due to the inclusion of the $l$-dependent terms, $\Delta$ CS is in mb. The percentage change is also given.

P.I. is the potential identifier.

| Energy | $L$ | $\Delta$ | P. I. | $\sigma$ | $\Delta J_R$ | $\Delta J_R$ % | $\Delta J_I$ | $\Delta J_I$ % | $\Delta$ CS | $\Delta$ CS % |
|--------|-----|---------|-------|--------|---------------|----------------|-------------|---------------|-------------|--------------|
| 30.0   | 5.5 | 1       | pot2  | 1.66 $\times$ 10$^{-4}$ | -39.47 | -9.56 | 5.09 | 4.44 | 0.52 | 0.399 |
| 30.0   | 5.5 | 1       | pot1  | 1.88 $\times$ 10$^{-5}$ | -41.12 | -9.96 | 10.37 | 9.04 | 0.52 | 0.399 |
| 30.0   | 5.5 | 2       | pot1  | 4.13 $\times$ 10$^{-5}$ | -35.67 | -8.64 | 3.41 | 2.97 | 0.55 | 0.420 |
| 30.0   | 5.5 | 2       | pot2  | 3.55 $\times$ 10$^{-6}$ | -35.82 | -8.69 | 3.92 | 3.42 | 0.55 | 0.420 |
| 45.0   | 6.5 | 2       | pot1  | 2.30 $\times$ 10$^{-4}$ | -32.15 | -8.77 | 1.59 | 1.44 | 0.207 | 0.208 |
| 45.0   | 6.5 | 2       | pot3  | 3.79 $\times$ 10$^{-5}$ | -32.67 | -8.91 | 2.96 | 2.70 | 0.206 | 0.207 |
| 45.0   | 6.5 | 2       | pot4  | 4.02 $\times$ 10$^{-5}$ | -32.52 | -8.87 | 2.43 | 2.22 | 0.206 | 0.207 |

for low $l$, larger amplitude undulations are required. Thus, we are in the situation that an exact representation, with an $l$-independent potential, of $S_l$ from an $l$-dependent potential of the present form, requires undulations such that the effect on the angular distribution would be beyond present experimental capabilities.

A. Imaginary $l$ dependence

To examine the effect of $l$ dependence in the imaginary potential, an $l$-dependent term was added such that the imaginary potential for 30.0 MeV is increased by 10 % for low partial waves, with $L = 5.5$ and $\Delta = 2$. The inversion was straightforward for this relatively small perturbation and potentials having low values of $\sigma$ were found with characteristics
FIG. 7: For 45 MeV protons on $^{40}$Ca, the solid line represents the angular distribution between $150^\circ$ and $180^\circ$ calculated with the $l$-dependent potential. The dashed line, virtually indistinguishable from the solid line, is calculated with the $l$-independent potential pot3, with inversion $\sigma = 3.79 \times 10^{-5}$. The dotted line corresponds to an earlier iteration, pot1, with $\sigma = 2.30 \times 10^{-4}$.

It is not surprising that the $l$-dependent increase of the imaginary potential led to a much greater increase in the reaction CS than with the stronger real $l$-dependence. The increase in $J_I$ of somewhat less than 10 % is unsurprising, as is the small percentage change in $J_R$. A uniform $l$-independent 10 % increase in the imaginary potential would, of course, lead to $J_I$ increasing by 10 % with zero change in $J_R$, and no undulations. The undulations resulting from the $l$-dependent increase can be seen in Fig. 12 which also shows that there is an increase in the imaginary potential of roughly 10 % for $r$ up to about 4.5 fm. For larger radii the imaginary potential oscillates about an average of roughly zero change. Around 9 fm, these undulations include an emissive region. This is clearer in the expanded scale of Fig. 13 which also shows that there are undulations in the real having a similar amplitude to the undulations in the imaginary part. It is a common feature in these studies to find that

presented in the lowest two lines of Table III.
FIG. 8: For 45 MeV protons on $^{40}$Ca, the solid lines present the $l$-independent part of the $l$-dependent potential with the real part in the top panel and the imaginary part below. The dashed lines represent the inverted potential for the 45 MeV pot1 of Table III and the dotted lines are for the inverted potential pot3 of that table. The potential pot4 of that table is very close to pot3. Surface undulations in either the real or the imaginary part are accompanied by undulations in the other. Unsurprisingly, there is little change to the real part in the interior region. The potential pot2 improves the fit to $|S_l|$ for between $l = 12$ and $l = 18; |S_l| \simeq 0.99992$ for $l = 12$, and the improvement makes no visible change to the elastic scattering angular distribution.

It is interesting to contrast the effect of the real and imaginary $l$-dependencies upon the reaction cross sections and the elastic scattering angular distributions. From Table III it can be seen that the real $l$-dependent term has quite a small effect on the reaction cross section, CS, (about 0.42 % at 30 MeV with $\Delta = 2$) whereas the imaginary $l$ dependence
FIG. 9: For 45 MeV protons on $^{40}$Ca, the upper panel presents $|S_l|$ and the lower panel arg $S_l$ for three cases. The solid line is calculated directly from the $l$-dependent potential, the dotted lines are $S_l$ calculated with $l$-independent potential pot1 and the dashed lines, hard to distinguish from the solid lines, present $S_l$ calculated with pot3.

increased CS by $2.5\%$. By contrast, the real $l$ dependence had a large effect on the elastic scattering angular distribution, see Fig. 10 whereas the imaginary $l$ dependence had a much smaller effect on the angular distribution, as in Fig. 11. This is another example of the large disconnect between changes in the elastic scattering angular distribution and changes in the reaction cross section.

VII. $l$-INDEPENDENT FITS TO ELASTIC SCATTERING DATA

Fitting precise, wide angular range, elastic scattering data with an $l$-independent potential leads to a potential having strong undulations. This was shown for protons on $^{16}$O and $^{40}$Ca,
FIG. 10: For 30 MeV protons on $^{40}\text{Ca}$, the dashed line represents the angular distribution without any $l$-dependency and the solid line represents the angular distribution for the case of the real $l$ dependence (10 % increase) with $L = 5.5$ and $\Delta = 2$.

see Ref. [20]. The fits to data in this reference were almost model independent. ‘Almost model independent’ because a constraint was imposed to ensure that the imaginary term was not emissive at any radius. The present work shows that this constraint is a mistake, but the finding that strong undulations are required for an $l$-independent fit stands, although the actual form of the modulations are distorted by what we now see as an improper constraint.

Such undulations are not confined to potentials fitting proton elastic scattering. For deuteron scattering see Refs. [21, 22] and for heavier projectiles see, for example, Refs. [23, 24]. We conclude that the requirement for undularity in low $\chi^2$ fits to precise and wide angular range elastic scattering data is generic.

The undulations referenced in this section have an amplitude far exceeding variations in the radial density of the target nuclei. It is therefore hard to imagine any explanation other than $S$-matrix equivalence to $l$-dependent potentials.
FIG. 11: For 30 MeV protons on $^{40}$Ca, the dashed line represents the angular distribution without any $l$-dependency and the solid line represents the angular distribution for the case of the imaginary $l$ dependence defined in the text.

VIII. THE RELATIONSHIP OF $l$-DEPENDENCE TO NUCLEAR MASS

Table I suggests that the empirically fitted $l$-dependence becomes less important as the nuclear mass increases. Most calculations of OMPs assume a local density model which does not explicitly take into account the density gradient in the nuclear surface. If such models gave an exact and complete derivation of the OMP, then a major source of $l$-dependence would not exist. It is then of interest to quantify the comparative importance of the surface region where nuclear density gradients are substantial and examine the relationship between this and the degree of $l$-dependence for nuclei of different masses. We have therefore calculated the ratio $R_S$ of the volume integral of the nucleon OMP calculated in two ways:

$$R_S = \frac{\int_0^R V(r)r^2 dr}{\int_0^\infty V(r)r^2 dr}$$

In the first integral, the upper limit $R = (R - a)$ where $R$ is the radius parameter of the Woods-Saxon potential and $a$ is the WS diffusivity. Therefore, $R_S$ is a measure of the volume
FIG. 12: For 30 MeV protons on $^{40}$Ca, the solid lines and hidden dashed lines present the $l$-independent part of the $l$-dependent potential with the real part in the top panel and the imaginary part below. The dotted lines represent the inverted potential for the 30 MeV pot1 in the lowest part of Table III. The potential pot2 of that table is indistinguishable from pot1 out to at least 12 fm.

\[ \rho + ca\text{ at } 30.0 \text{ MeV} = 155403.959 \]

Fraction of the OMP that is interior to the surface region, in particular the region where the potential has a substantial radial gradient.

We have calculated $R_S$ for $^{16}$O, $^{40}$Ca, $^{58}$Ni and $^{208}$Pb. In each case, we calculated $R_S$ for the real central term of the Koning Delaroche global potential [25]. The values for those four nuclei, were, respectively: 0.254, 0.426, 0.466, 0.642. All of these numbers point to the importance if the surface region, but most particularly $^{16}$O stands out as being in a separate class. If $l$-dependence is, as we propose, related to the influence of surface processes, and strongly influenced by departures from the validity of local density approximation, then we
FIG. 13: For 30 MeV protons on $^{40}\text{Ca}$, the solid lines represent the $l$-independent part of the $l$-dependent potential with the real part in the top panel and the imaginary part below. The dotted lines and dashed lines represent, respectively, the inverted potentials pot1 and pot2 in the lowest part of Table III. The potential pot2 of that table is indistinguishable from pot1 out to at least 12 fm.

\[ p + ^{16}\text{O} \text{ at 30.0 MeV } = 155833.700 \]

should not be surprised to find strong effects associated with $l$-dependence for $^{16}\text{O}$. It would be interesting to know what part of the effect due to channel coupling, e.g. to deuteron channels, is accounted for in local density folding models.

IX. SUMMARY AND DISCUSSION

Section III presented characteristic properties of $l$-independent potentials that were found by inversion to be $S$-matrix equivalents of $l$-dependent potentials. The $l$-dependent potentials are known to give excellent fits to elastic scattering data that cannot be fitted with con-
ventional radial forms. Section \[\text{V}\] compared the inverted potentials with the \(l\)-independent part of each \(l\)-dependent potential and Figs. 3 to 5 show that each \(l\)-independent potential found by inversion has undulations in the surface. The undulations in the imaginary terms include radial regions where the potential is emissive. This tendency is strongest for light target nuclei and persists to a smaller degree for the \(^{208}\text{Pb}\) case, not shown. Since the \(S_{ij}\) for \(l\)-dependent potentials all satisfy the unitarity bound, this is true also for the inverted potentials, despite the emissive regions. It follows that \(l\)-independent potentials that fit elastic scattering of 30 MeV protons from \(^{16}\text{O}, \, ^{40}\text{Ca}, \, ^{58}\text{Ni}\) and \(^{208}\text{Pb}\), probably better than any other potentials, have undularities with emissive regions in the nuclear surface. Since they are the \(l\)-independent equivalents of \(l\)-dependent potentials, it is reasonable to attribute undularity and emissivity found in phenomenological potentials to an underlying \(l\) dependence. It is also reasonable to attribute the appearance of undularity and emissivity in local DPPs from coupled channel calculations, determined by inversion of \(S_l\) or \(S_{ij}\), to the \(l\) dependence of an underlying non-local and \(l\)-dependent DPP.

Does the strong undularity, leading to emissive regions, result from the particular form of \(l\) dependence applied in Refs. [15, 16]? That particular \(l\)-dependent term was quite sharply confined to the surface region. The calculations presented in Section \[\text{VI}\] suggest that the occurrence of undularity is a generic property of potentials that are \(l\)-independent equivalents of \(l\)-dependent potentials. In that section the \(l\)-independent equivalents were found for potentials for which the entire real or imaginary part was multiplied by a factor over a range of partial waves: \(l < L\) with a transition over a range \(\sim \Delta\). For the real part, the factor was 0.9 and for the imaginary part it was 1.1. The resulting potentials were undulatory. The undulations included regions of emissivity for both real and imaginary \(l\) dependence. The sharpness of the transition was adjustable, and a sharper transition (smaller \(\Delta\)) led to somewhat stronger undulations. A similar relationship between \(l\)-dependence and undularity was found in the case of \(^3\text{He}\) scattering on \(^{58}\text{Ni}\) \[26\] and also found for \(^{16}\text{O}\) scattering from \(^{12}\text{C}\) at 115.9 MeV \[19\]. It therefore appears that there is a generic relationship between undulations in optical potentials and an underlying \(l\) dependence.

In Section \[\text{VIII}\] we presented a plausible argument for the \(l\)-dependence of proton OMPs becoming weaker as the mass of the target nucleus increases, as suggested by the results in Table \[\text{II}\].

**Implications:** We have proposed that any necessity for \(l\) dependence indicates a failure
of a folding model based on a local density approximation. If, then, for some case of elastic scattering, a precise model-independent, but $l$-independent, empirical fit to wide angular range precise data exhibits undulations, then the actual potential for that case must be $l$-dependent, indicating a failure of local density folding models. It is therefore unclear what potentials are appropriate for use in direct reactions and this must depend on what channels are included; the use of local $l$-independent potentials that fit elastic scattering appears to be unjustified. We also conclude that the appropriate way to evaluate a folding model potential is by fitting a model-independent additive term, when appropriate data exists.

X. APPENDIX: SPECIFICATION OF THE $l$-DEPENDENCE

The $l$ dependence adopted in Refs. [15, 16] was essentially that of the original work, Ref. [1]. The $l$-dependent potential was the sum of a standard $l$-independent term with an added $l$-dependent term. The $l$-independent term was a Woods-Saxon real part plus an imaginary part that was the sum of Woods-Saxon and Woods-Saxon derivative terms. A conventional spin-orbit term was included and all terms were as defined by Perey and Perey [18].

The $l$ dependency of Section III was in the form of an additional central potential having real and imaginary Wood-Saxon derivative (surface peaked) form with an overall $l$-dependent factor:

$$f(l) = \frac{1}{1 + \exp((l^2 - L^2)/\Delta^2)}.$$  (3)

Thus, the $l$-dependence has the form of an additional surface potential that cuts off sharply when $l > L$. There is the one overall $l$-dependent factor $f(l)$ for both real and imaginary parts which had independent radial and strength parameters. There was no $l$-dependence in the spin-orbit term which was real in almost all cases. This $l$-dependent potential fitted the data with very consistent parameters, the values of which varied more smoothly with energy than the parameters of the best $l$-independent potentials. In all cases, the searches led to a real $l$-dependent term that was repulsive (i.e. there was less attraction for $l \ll L$) and an imaginary term that was absorptive (i.e. more absorption for $l \ll L$.)

**Potentials for Section VI.** The real $l$-dependence of Section VI was achieved by adding a repulsive real potential of identical form to the original real potential but of one tenth the magnitude and with overall factor $f(l)$ of Eq. 3. Thus the potential is reduced by 10% for
low partial waves. The imaginary $l$ dependence was effected in a similar fashion.

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