A NOTE ON LOCALIZATIONS OF PERFECT GROUPS

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Abstract. We describe a perfect group whose localization is not perfect.

1. Introduction

A localization is a type of a functor $L: \text{Groups} \to \text{Groups}$ which is idempotent (that is $LLG \cong LG$) and admits a coaugmentation $\eta: G \to LG$ \cite{1}. Localizations are ubiquitous in group theory: abelianization, killing of the $p$–torsion and inversion of a prime in a group are all examples of such functors.

The natural question – which classes of of groups are preserved by all localizations – has been a focus of a lot of study recently. This work yield both classes which are preserved (e.g. abelian groups, nilpotent groups of class 2 \cite{2}), and these which do not have this property, like finite \cite{3} or solvable groups \cite{5}. The goal of this note is to prove the following

**Theorem 1.1.** The class of perfect groups is not closed with respect to taking localizations. That is, there exists a perfect group $P$ and a localization $\eta: P \to LP$ such that $LP$ is not perfect.

This answers a question posed by Casacuberta in \cite{1}.

The groups $P$ and $LP$ we construct in the proof of Theorem 1.1 are infinite. It would be interesting to know if one can find a finite perfect group with a non-perfect localization. The following shows however that the localized group would have to be infinite.

**Proposition 1.2.** If $\eta: P \to LP$ is a localization of a perfect group $P$ and $LP$ is finite then $LP$ is a perfect group.

2. Proofs

Our main tool will be the following fact which characterizes all possible localizations of a group. It is a direct ramification of the definition of localization functors (see e.g. \cite{1} Lemma 2.1).

**Lemma 2.1.** A homomorphism $\eta: G \to H$ is a localization of $G$ with respect to some localization functor iff $\eta$ induces a bijection of sets

$$\text{Hom}(H,H) \xrightarrow{\eta^*} \text{Hom}(G,H)$$
As an application we obtain

**Lemma 2.2.** If \( \eta: G \to H \) is a localization and \( G \) is a perfect group then there are no non-trivial homomorphisms \( H/[H,H] \to H \).

**Proof.** Let \( g: H/[H,H] \to H \) be any homomorphism and let \( f \) denote the composition \( H \to H/[H,H] \xrightarrow{\eta} H \). Since \( G \) is perfect the composition \( f \circ \eta \) is the trivial map. Lemma 2.1 implies then that \( f \) is also trivial, and thus so is \( g \). \( \square \)

Since every finite group \( H \) admits a non-trivial map \( H/[H,H] \to H \) unless \( H \) is perfect, Proposition 1.2 is a consequence of Lemma 2.1.

Next, we turn to the proof of Theorem 1.1. We start with

**Construction of the group \( P \).** For \( n \geq 0 \) let \( \tilde{P}_n \) be a free group on \( 2^n \) generators \( x_1^{(n)}, x_2^{(n)}, \ldots, x_{2^n}^{(n)} \), and let \( \tilde{\phi}_n: \tilde{P}_n \to \tilde{P}_{n+1} \) be a group homomorphism defined by

\[
\tilde{\phi}_n(x_i^{(n)}) = [x_{2i-1}^{(n+1)}, x_{2i}^{(n+1)}]
\]

where \([a,b] = a^{-1}b^{-1}ab\) is the commutator of \( a \) and \( b \). Define \( \tilde{P} := \lim_{n \to \infty} \tilde{P}_n \).

Notice, that since \( \tilde{P} \) is generated by the elements \( x_i^{(n)} \in [\tilde{P}, \tilde{P}] \) the group \( \tilde{P} \) is perfect. Let \( K \) be the smallest normal subgroup of \( \tilde{P} \) containing the elements \([x_1^{(0)}, x_i^{(n)}]\) for all \( n \geq 0, 1 \leq i \leq 2^n \). Define \( P := \tilde{P}/K \).

**Proposition 2.3.** The group \( P \) is perfect and \( x_1^{(0)} \) is a central element of \( P \). Moreover, \( x_1^{(0)} \) is an element of infinite order, and as a consequence \( P \) is a non-trivial group.

**Proof.** The first two claims are obvious. To see that \( x_1^{(0)} \in P \) has infinite order notice that \( P \) can be viewed as a limit

\[
P = \lim_{n \to \infty} P_n
\]

where \( P_n \) is a group with the presentation

\[
P_n := \langle x_1^{(n)}, \ldots, x_{2^n}^{(n)} \mid [x_1^{(0)}, x_i^{(n)}] = 1, \quad i = 1, \ldots, 2^n \rangle
\]

(by abuse of notation we denote here by \( x_1^{(0)} \) the image of the element \( x_1^{(0)} \) under the map \( P_0 \to P_n \)). It is then enough to show that \( x_1^{(0)} \) has infinite order in \( P_n \) for all \( n \geq 0 \). To see this consider \( GL(\mathbb{Z}, 2^n + 1) \) – the group of invertible matrices of dimension \( 2^n + 1 \) with integer coefficients. For \( n > 0 \) there is a homomorphism

\[
\psi_n: P_n \to GL(\mathbb{Z}, 2^n + 1)
\]

defined by \( \psi_n(x_i^{(n)}) = e_{i,i+1}^1 \), where \( e_{i,j}^a \) denote the matrix with 1’s on the diagonal, \( a \) as the \((i,j)\)-th entry, and 0’s elsewhere. One can check that \( \psi_n(x_1^{(0)}) = e_{1,2^n+1}^1 \). Since \( (e_{1,2^n+1}^1)^k = e_{1,2^n+1}^k \) this is a non-torsion
element of \( GL(Z, 2^n + 1) \), and as a consequence \( x_1^{(0)} \) has infinite order in \( P_n \) as claimed. \( \square \)

**Construction of the map \( \eta: P \rightarrow LP \).** Let \( Q \) be the group of rational numbers. Define

\[
LP := P \oplus \mathbb{Q}/((x_1^{(0)}, -1))
\]

and let the map \( \eta: P \rightarrow LP \) be given by the composition of the inclusion \( P \hookrightarrow P \oplus \mathbb{Q} \) and the projection \( P \oplus \mathbb{Q} \rightarrow LP \). Since \( \eta \) is a monomorphism we will identify \( P \) with its image \( \eta(P) \). Notice that \( P \) is a normal subgroup of \( LP \) and that \( LP/P \cong \mathbb{Q}/\mathbb{Z} \). Since \( \mathbb{Q}/\mathbb{Z} \) is not a perfect group neither is \( LP \).

It remains to prove that \( \eta \) is a localization of \( P \). By Lemma 2.1 this amount showing that any homomorphism \( f: P \rightarrow LP \) admits a unique factorization

\[
\begin{array}{ccc}
P & \xrightarrow{\eta} & LP \\
\downarrow{f} & & \downarrow{f} \\
\downarrow & & \downarrow \\
LP & \xrightarrow{\bar{f}} & LP
\end{array}
\]

**Uniqueness of \( \bar{f} \).** Assume that \( \bar{f}_1, \bar{f}_2: LP \rightarrow LP \) are homomorphisms such that \( \eta \bar{f}_1 = \eta \bar{f}_2 \), and consider the homomorphism

\[
g := (\bar{f}_1|Q - \bar{f}_2|Q): \mathbb{Q} \rightarrow LP
\]

We have \( \bar{f}_1(1) = \bar{f}_1(x_1^{(0)}) = \bar{f}_2(x_1^{(0)}) = \bar{f}_2(1) \), and thus \( \mathbb{Z} \subseteq \ker g \). Therefore we get a factorization

\[
g: \mathbb{Q}/\mathbb{Z} \rightarrow LP
\]

and \( g \equiv 1 \) iff \( \bar{f}_1 = \bar{f}_2 \). Thus, our claim is a consequence of the following

**Lemma 2.4.** The group \( LP \) is torsion free.

**Proof.** Let \((w, \bar{\xi})\) represents a torsion element of \( LP \). Then \( q \cdot (w, \bar{\xi}) = (w^q, p) = w^q(x_1^{(0)})^p \) is a torsion element in \( P \). Consider the group \( R := P/\langle x_1^{(0)} \rangle \). The element \( w^q(x_1^{(0)})^p = w^q \) is torsion in \( R \), and thus so is \( w \). Notice that \( R = \varprojlim_n R_n \) where

\[
R_n = \langle x_1^{(n)}, \ldots, x_{2^n}^{(n)} | x_1^{(0)} = 1 \rangle
\]

It follows that \( w \) is a torsion element in \( R_n \) for \( n \) large enough. On the other hand, \( R_n \) is a group with one relator given by a word which is not a proper power of any element in the free group. By [4, Thm. 4.12, p. 266] \( R_n \) must be torsion free. Therefore \( w = 1 \) in \( R \), and so \((w, \bar{\xi}) = ((x_1^{(0)})^l, \bar{\xi}) = l + \bar{\xi} \in \mathbb{Q} \subseteq LP \) for some \( l \in \mathbb{Z} \). Since by assumption \((w, \bar{\xi})\) is a torsion element it must be trivial. \( \square \)
Existence of $\bar{f}$. We need to show that every homomorphism $f: P \to LP$ admits an extension $\bar{f}: LP \to LP$. Assume for a moment that $f(x_1^{(0)}) = (x_1^{(0)})^k \in LP$ for some $k \in \mathbb{Z}$. From the definition of $LP$ it follows then that $f$ can be defined by setting $\bar{f}(r) = kr$ for all $r \in \mathbb{Q}$. Next, notice that since $P$ is perfect $f(P) \subseteq [LP, LP] = P$. Combining these observations we get that the existence of $\bar{f}$ follows from

Lemma 2.5. If $g: P \to P$ is any homomorphism then $g(x_1^{(0)}) = (x_1^{(0)})^k$ for some $k \in \mathbb{Z}$.

Recall the group $R = P/\langle x_1^{(0)} \rangle$ defined in the proof of Lemma 2.4. Lemma 2.6 will follow if we show that for any homomorphism $g: P \to R$ the element $x_1^{(0)}$ is in the kernel of $g$. In the proof of Proposition 2.3 we also defined the group

$$P_1 = \langle x_1^{(1)}, x_2^{(1)} \mid [x_1^{(0)}, x_i^{(1)}] = 1, \ i = 1, 2 \rangle$$

Since $x_1^{(0)}$ is not in the kernel of the map $P_1 \to P$ it is enough to show that for any $g: P_1 \to R$ we have $x_1^{(0)} \in \ker g$. Furthermore, since $R = \lim \limits_{\to n} R_n$ (see 2.4), and $P_1$ is a finitely presented group it suffices to prove that $g(x_1^{(0)}) = 1$ for all $g \in \text{Hom}(P_1, R_n)$. Finally, notice that by the definition of $P_1$ the elements $x_1^{(1)}, x_2^{(1)}$ commute with their commutator $x_1^{(0)} = [x_1^{(1)}, x_2^{(1)}]$. These observations and the presentation of $R_n$ show that Lemma 2.5 is a special case of

Lemma 2.6. Let $F_1, F_2$ be two free groups, and let $u_1$ be a word in $F_i$ which is not a proper power. Let $G$ be the quotient group of $F_1 * F_2$ by the normal subgroup generated by $[u_1, u_2]$. If $x, y \in G$ are elements commuting with $[x, y]$ then $[x, y] = 1$.

Proof. Consider the map $h: G \to F_1 \oplus F_2$. Its kernel $K$ is a free group whose set of generators can be described as follows. Let $S_1$ be a set of representatives of cosets of $\langle u_1 \rangle \setminus F_1$. Then the generators of $K$ are all commutators $[v_1, v_2]$ where $v_1 \in S_1$ represents a coset other than $\langle u_1 \rangle$ and $v_2$ is any nontrivial element of $F_2$, or $v_2 \in S_2$ represents a coset different from $\langle u_2 \rangle$, and $v_1$ is a nontrivial element $F_1$. To see this recall [3] Prop. 4, p. 6] that the kernel $K'$ of the map $F_1 * F_2 \to F_1 \oplus F_2$ is a free group whose generators are all commutators $[v_1, v_2]$ where $v_i \in F_i$ and $v_i \neq 1$. The group $K$ is obtained as the quotient of $K'$ by its normal subgroup generated by the set $\{ w^{-1} [u_1, u_2] w \mid w \in F_1 * F_2 \}$. This is equivalent to imposing the following relations in $K'$:

$$[u_1 w_1, u_2 w_2] = [u_1 w_1, w_2] [w_2, w_1] [w_1, u_2 w_2]$$

where $w_i$ is an arbitrary element of $F_i$. The above description of $K$ can be derived from here. Notice, that using the above relations any commutator $[w_1, w_2]$ such that $w_i \in F_i$ can be expressed in terms of generators of $K$.
using the formula

\[(2.7) \quad [w_1, w_2] = [w_1, s_2][s_2, s_1][s_1, w_2]\]

where \(s_i \in S_i\) represents the coset \( \langle u_i \rangle w_i \) for \(i = 1, 2\).

Next, take elements \(x, y \in G\) as in the statement of the lemma. Notice that \([x, y] \in K\) since otherwise \(h(x), h(y)\) would have to commute with a nontrivial element \(h([x, y]) = [h(x), h(y)]\) in \(F_1 \oplus F_2\) which is impossible. Furthermore, since centralizers of all nontrivial elements in a free group are abelian and since \(x, y\) are in the centralizer of \([x, y]\), if \(x, y \in K\) then we get \(xy = yx\) and the statement of the lemma holds. Therefore we can assume that \(x \not\in K\) and \([x, y] \in K\). In this case we can uniquely represent \(x\) and \([x, y]\) in the form

\[x = x_1 x_2 \prod_{i=1}^{p} [a_i, b_i]^{\delta_i}\] and \([x, y] = \prod_{j=1}^{q} [c_j, d_j]^{\delta_j}\]

where \(x_i \in F_i\), \(x_1 x_2 \neq 1\), \([a_i, b_i], [c_j, d_j]\) are generators of \(K\), and \(\delta_i, \delta_j = \pm 1\).

We can also assume that \([x, y]\) is represented by a cyclically reduced word in the free group \(K\), that is \([c_1, d_1]^{\delta_1} \neq [c_q, d_q]^{-\delta_q}\). Consider the element

\[(2.8) \quad x_2^{-n} x_1^{-n} [x, y] x_1^n x_2^n = \prod_j ([x_2^n, c_j x_1^n][c_j x_1^n, d_j x_2^n][d_j x_2^n, x_2^n][a_1^n, x_2^n])^{\delta_j}\]

Commutativity of \(x\) and \([x, y]\) implies that for any \(n \in \mathbb{Z}\) this element is conjugate to \([x, y]\) in \(K\). We will show that this is not possible unless \([x, y] = 1\). One can check that the following holds.

**Lemma 2.9.** Let \(F\) be a free group and let \(w, v\) be words in \(F\). If \(w\) is cyclically reduced and \(v\) is conjugated in \(F\) to \(w\) then all generators of \(K\) appearing in \(w\) must appear in \(v\).

We apply it to \(w = [x, y]\) and \(v = x_2^{-n} x_1^{-n} [x, y] x_1^n x_2^n\). The commutators appearing on the right hand side of formula 2.8 are not generators of the group \(K\). Each of them, however, can be written as a product of generators of \(K\) using formula 2.7. By lemma 2.9 all commutators \([c_j, d_j]\) must appear among these generators. One can check however that (since \(u_1, u_2\) are not proper powers) if \(x_1 \neq 1\), \(x_2 \neq 1\), and \(n\) is large enough this can happen only if \(x_1 = c^{-1} u_1^k c\), \(x_2 = d^{-1} u_2^l d\) and \([x, y] = [c, d]^m\) for some \(c \in F_1\), \(d \in F_2\), and \(k, l, m \in \mathbb{Z}\). By inspection, in this case \(x_1^{-n} x_2^{-n} [x, y] x_1^n x_2^n\) is not conjugated to \([x, y]\) unless \(m = 0\), and \([x, y] = 1\). Assume in turn that \(c_2 = 1\). Then we have

\[x_1^{-n} [x, y] x_1^n = \prod_{j=1}^{q} ([c_j x_1^n, d_j][d_j, x_1^n])^{\delta_j}\]

Again, combining this with formula 2.7 we get an expression of \(x_1^{-n} [x, y] x_1^n\) as a product of generators of \(K\). In order for \([c_j, d_j]\) to appear among these generators for large \(n\) we must have \(x_1 = c^{-1} u_1^k c\) and \([x, y] = \prod_j [c, d_j]^{\delta_j}\) for
some \( c \in F_1, \ k \in \mathbb{Z} \). As before, by inspection we obtain that also in this case \( x_1^{-n}[x, y]x_1^n \) cannot be conjugate to \([x, y]\) if \([x, y] \neq 1\).

\[\square\]

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