Research Article

On Rationality of Kneading Determinants

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1. Introduction

If $A$ is a $k \times k$ matrix with rational coefficients, one has the well-known identity between formal power series:

$$
\exp \sum_{n=1}^{\infty} - \frac{\text{tr}(A^n)}{n} z^n = \det(I-zA),
$$

where $\text{tr}(A^n)$ denotes the trace of matrix $A^n$ (matrix $A$ raised to the $n$th power) and $I$ denotes the $k \times k$ identity matrix. This identity plays a significant role in the discussion of an important problem in dynamical systems theory. For more details see [1, 2].

We denote by $\mathcal{H}$ the infinite dimension vector space over $\mathbb{Q}$; the space of linear forms on $\mathcal{H}$ will be denoted, as usual, by $\mathcal{H}^*$, and the space of all linear endomorphisms on $\mathcal{H}$ will be denoted by $L(\mathcal{H})$. If $\psi \in L(\mathcal{H})$ and $n$ is a nonnegative integer, the $n$th iterate $\psi^n$ is defined recursively by $\psi^0 = \text{Id}_{\mathcal{H}} \in L(\mathcal{H})$, $\psi^n = \psi \circ \psi^{n-1}$, for $n \geq 1$.

The subspace of $L(\mathcal{H})$ whose elements are the linear endomorphism on $\mathcal{H}$ with finite rank will be denoted by $L_{\text{FR}}(\mathcal{H})$.

Let $q$ be a positive integer; we use the symbol $\overline{h}$ to denote an element of $\mathcal{H}^{\times q} = \mathcal{H} \times \mathcal{H} \times \cdots \times \mathcal{H}$ ($q$ times) and the symbol $\overline{\alpha}$ to denote an element of $\mathcal{H}^{*q}$; that is

$$
\overline{h} = (h_1, \ldots, h_q), \quad \overline{\alpha} = (\alpha_1, \ldots, \alpha_q).
$$

Given $\overline{h} \in \mathcal{H}^q$ and $\overline{\alpha} \in \mathcal{H}^{*q}$, we define the finite rank endomorphism $\overline{\alpha} \otimes \overline{h} \in L_{\text{FR}}(\mathcal{H})$ and the matrix $\mathbf{M}(\overline{\alpha}, \overline{h}) \in \mathbb{Q}^{q \times q}$ by setting

$$
\overline{\alpha} \otimes \overline{h} = \sum_{p=1}^{q} \alpha_p \otimes h_p,
$$

with the usual notation

$$
\alpha \in \mathcal{H}^*, h \in \mathcal{H} : (\alpha \otimes h) (x) = \alpha (x) u, \quad x \in \mathcal{H},
$$

$$
\mathbf{M}(\overline{\alpha}, \overline{h}) = \begin{pmatrix}
\alpha_1 (h_1) & \cdots & \alpha_1 (h_q) \\
\vdots & \ddots & \vdots \\
\alpha_q (u_1) & \cdots & \alpha_q (h_q)
\end{pmatrix}.
$$

Definition 1 (see [3]). A pair $(\varphi, \psi) \in L(\mathcal{H}) \times L(\mathcal{H})$ is said to have finite rank if $\psi - \varphi \in L_{\text{FR}}(\mathcal{H})$.

Notice that if a pair $(\varphi, \psi)$ has finite rank, then the pair $(\varphi^n, \psi^n)$ also has finite rank for all $n \geq 1$, and therefore the trace of $\varphi^n - \psi^n$ is defined.

Definition 2 (see [3]). For any pair $(\varphi, \psi) \in L(\mathcal{H}) \times L(\mathcal{H})$ with finite rank, we define the Kneading determinant of $(\varphi, \psi)$ as the following invertible element of $\mathbb{Q}[z]$:}

$$
\Delta_{(\varphi, \psi)} = \exp \sum_{n=1}^{\infty} \frac{\text{tr}(\varphi^n - \psi^n)}{n} z^n.
$$
Remark 3. Kneading determinant was first studied by Milnor and Thurston in [4].

Let \( \mathbb{Q}[z]^{q \times q} \) be the ring of the \( q \times q \) matrices whose entries lie in \( \mathbb{Q}[z] \). If \( \varphi \in L(\mathcal{F}), \overline{u} \in \mathcal{F}^q, \) and \( \overline{u} \in \mathcal{F}^q \), we define the matrix \( M_p(\overline{u}, \overline{u}) \in \mathbb{Q}[z]^{q \times q} \) by

\[
M_p(\overline{u}, \overline{u}) = \left( \begin{array}{ccc}
\sum_{n \geq 0} \alpha_n \varphi^n(u_1) z^n & \cdots & \sum_{n \geq 0} \alpha_n \varphi^n(u_q) z^n \\
\vdots & \ddots & \vdots \\
\sum_{n \geq 0} \alpha_n \varphi^n(u_1) z^n & \cdots & \sum_{n \geq 0} \alpha_n \varphi^n(u_q) z^n
\end{array} \right).
\]

(6)

Lemma 4 (see [3]). Let \( (\varphi, \psi) \in L(\mathcal{F}) \times L(\mathcal{F}), \overline{u} \in \mathcal{F}^q, \) and \( \overline{u} \in \mathcal{F}^q \) such that \( \psi - \varphi \in \overline{u} \otimes \overline{u} \). Denote by \( I \) the \( q \times q \) identity matrix. Then,

\[
\Delta_{(\varphi, \psi)} = \det \left( I - zM_p(\overline{u}, \overline{u}) \right)
\]

(7)

holds in \( \mathbb{Q}[z] \).

In general, any power series can be the Kneading determinant of some pair \( (\varphi, \psi) \) with finite rank (see [3]). So it is interesting to study conditions under which the Kneading determinant is a rational power series.

The following is the main result of this paper.

Theorem 5. If \( I - \varphi \lambda \) and \( I - \psi \lambda \) are left coprime or right coprime, and \( h = \psi - \varphi \) with finite rank, then \( \Delta_{(\varphi, \psi)} \) is a rational function.

2. Coprimeness of \( I - A\lambda \) and \( I - B\lambda \)

In this section, \( R \) denotes a ring with identity \( I \). We discuss conditions under which \( I - A\lambda \) and \( I - B\lambda \) are left coprime or right coprime, where \( A, B \in R \).

We say \( I - A\lambda \) and \( I - B\lambda \) are left coprime if there exist polynomials \( X(\lambda), Y(\lambda) \in R[\lambda] \) such that

\[
X(\lambda) (I - A\lambda) + Y(\lambda) (I - B\lambda) = I.
\]

(8)

Proposition 6. \( I - A\lambda \) and \( I - B\lambda \) are left coprime if and only if there exist \( X_0, X_1, \ldots, X_m \in R \) such that

\[
X_m H + X_{m-1} HB + X_{m-2} H B^2 + \cdots + X_0 H B^m + B^{m+1} = 0,
\]

(9)

for \( H = A - B \).

Proof. If \( I - A\lambda \) and \( I - B\lambda \) are left coprime, there exist \( X(\lambda) = X_0 + X_1 \lambda + \cdots + X_m \lambda^m \) and \( Y(\lambda) = Y_0 + Y_1 \lambda + \cdots + Y_m \lambda^m \) such that

\[
X(\lambda) (I - A\lambda) + Y(\lambda) (I - B\lambda) = I.
\]

(10)

We can assume that \( s \) equals \( m \). And we have

\[
X_0 + Y_0 = I,
\]

\[
X_i A + Y_i B - X_{i+1} - Y_{i+1} = 0, \quad i = 1, 2, \ldots, m - 1,
\]

\[
X_m A + Y_m B = 0.
\]

So,

\[
Y_0 = I - X_0.
\]

(12)

If we write \( H = A - B \), then,

\[
Y_1 = -X_1 + X_0 H + B,
\]

\[
Y_2 = -X_2 + X_1 H + X_0 HB + B^2,
\]

\[
\vdots
\]

\[
Y_m = -X_m + X_{m-1} H + \cdots + X_0 HB^{m-1} + B^m.
\]

So,

\[
X_m H + X_{m-1} HB + X_{m-2} HB^2 + \cdots + X_0 HB^m + B^{m+1} = 0.
\]

(13)

Conversely, if there exist \( X_0, X_1, \ldots, X_m \in R \) such that

\[
X_m H + X_{m-1} HB + X_{m-2} HB^2 + \cdots + X_0 HB^m + B^{m+1} = 0,
\]

(14)

let

\[
X(\lambda) = X_0 + X_1 \lambda + \cdots + X_m \lambda^m,
\]

\[
Y_0 = I - X_0,
\]

\[
Y_1 = -X_1 + X_0 H + B,
\]

\[
Y_2 = -X_2 + X_1 H + X_0 HB + B^2,
\]

\[
\vdots
\]

\[
Y_m = -X_m + X_{m-1} H + \cdots + X_0 HB^{m-1} + B^m,
\]

(16)

and let

\[
Y(\lambda) = Y_0 + Y_1 \lambda + \cdots + Y_m \lambda^m.
\]

(17)

Then,

\[
X(\lambda) (I - A\lambda) + Y(\lambda) (I - B\lambda) = I.
\]

(18)

Now we give the definition of right coprime. We say \( I - A\lambda \) and \( I - B\lambda \) are right coprime if there exist polynomials \( X(\lambda), Y(\lambda) \in R[\lambda] \) such that

\[
(I - A\lambda) X(\lambda) + (I - B\lambda) Y(\lambda) = I.
\]

(19)

Proposition 7. \( I - A\lambda \) and \( I - B\lambda \) are right coprime if and only if there exist \( X_0, X_1, \ldots, X_s \in R \) such that

\[
HX_s + BHX_{s-1} + B^2 H X_{s-2} + \cdots + B^s H X_0 + B^{s+1} = 0,
\]

(20)

for \( H = A - B \).

Proof. It is similar to the proof of Proposition 6.
3. Proof of the Main Theorem

In this section, \( \varphi, \psi \in L(\mathcal{H}) \), where \( \mathcal{H} \) is an infinite dimensional vector space over \( \mathbb{Q} \), and we denote by \( L(\mathcal{H}) \) the ring of linear transforms on \( \mathcal{H} \).

Lemma 8. Suppose that \( I - \varphi \lambda \) and \( I - \psi \lambda \) are left coprime or right coprime, \( h = \psi - \varphi \) is of finite rank, and \( W = \text{Im}(h) \). Then there exists a finite dimensional space \( \tilde{W} \) containing \( W \) such that \( (\varphi^k - \psi^k)(\mathcal{H}) \subset \tilde{W} \) with \( k = 1, 2, \ldots \).

Proof. First we assume that if \( I - \varphi \lambda \) and \( I - \psi \lambda \) are coprime, then by Proposition 7 we have \( \varphi(W) + \cdots + \varphi^mW \). Then for \( k = l + 1 \), \( (\varphi^l - \psi^l)(\mathcal{H}) = (\varphi^l - \psi^l)(\mathcal{H}) - hv^l(\mathcal{H}) \subset \varphi(W) + W \). We get the conclusion in this case.

Now we assume that \( I - \varphi \lambda \) and \( I - \psi \lambda \) are left coprime; then there exist \( s \in \mathbb{Z} \) and \( x_0, x_1, \ldots, x_s \in L(\mathcal{H}) \) such that
\[
\begin{align*}
x_0h + x_{s-1}h \varphi + x_{s-2}h \varphi^2 + \cdots + x_0h \varphi^s + \varphi^{s+1} &= 0, \\
\end{align*}
\]
holds. Take
\[
\begin{align*}
\hat{W} &= W + x_0W + \cdots + x_sW, \\
\tilde{W} &= \hat{W} + \varphi \hat{W} + \varphi^2 \hat{W} + \cdots + \varphi^s \hat{W}. \\
\end{align*}
\]
Notice when \( I \geq s + 1, \varphi^l(\mathcal{H}) \subset x_0W + x_1W + \cdots + x_sW \). We use induction to prove the conclusion.

If \( k = 1 \), then \( (\varphi^k - \psi^k)(\mathcal{H}) \subset W \).

Suppose that for \( k \leq l \) we have \( (\varphi^k - \psi^k)(\mathcal{H}) \subset W + \varphi W + \cdots + \varphi^m W \). Then for \( k = l + 1 \), \( (\varphi^l - \psi^l)(\mathcal{H}) = (\varphi^l - \psi^l)(\mathcal{H}) - hv^l \subset \varphi(W) + W \). We get the conclusion in this case.

Now we will give the proof of Theorem 5.

Proof. If \( I - \varphi \lambda \) and \( I - \psi \lambda \) are coprime, by Lemma 8, there is a finite dimensional space \( \tilde{W} \) such that
\[
(\psi - \varphi^k)(\mathcal{H}) \subset \tilde{W},
\]
and we have \( \varphi(\tilde{W}) \subset \tilde{W} \). Define
\[
\varphi = \text{Pr}(\varphi)|_{\tilde{W}} : \tilde{W} \rightarrow \tilde{W}.
\]
Suppose \( q = \text{rank}(\varphi - \psi) \); we denote by \( I_q \) the identity operator. Then \( p(z) = \text{det}(I_q - z \varphi) \in \mathbb{Q}[z] \). For any \( i, j \in \{1, 2, \ldots, q\} \), we have \( \Sigma_{\alpha \in \mathbb{Z}} (\alpha \varphi^i(u_i))z^\alpha = f_i(z) / p(z) \) for some \( f_i(z) \in \mathbb{Q}[z] \). For more details see [5]. So by Lemma 4, \( \text{det}(I - z M_q(\varphi, \varphi)) \) is rational; we get the conclusion.

Example 9. Suppose \( \mathcal{H} \) is the ring of countable infinite matrix with finite nonzero entries in each column. Let
\[
\varphi = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ A & 0 & 0 & \cdots \\ 0 & A & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad h = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},
\]
where \( A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) and \( \alpha_1, \alpha_2, \alpha_3, \ldots \) are three-dimensional row vectors. We see that \( (\varphi - h)^{q-1} = 0 \). So \( I - \psi \lambda \) and \( I - \varphi \lambda \) are left coprime. It is easy to check that
\[
\Delta_{(\varphi, \psi)} = \frac{1 - 2z}{1 - z}. \tag{27}
\]

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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