SMALL-SIZE INSTANTON CORRECTIONS TO THE $\tau$ HADRONIC WIDTH

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Abstract

We compute the effect of small-size instanton corrections to current-current correlators, for all combinations of axial, vector, and possibly flavour non-diagonal currents. We apply our result to the hadronic decays of the $\tau$ lepton, in order to assess the reliability of the determination of $\alpha_S$ from the $\tau$ hadronic width.

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1. Introduction

In ref. [1], the possibility of determining $\alpha_S$ to high precision from the hadronic decay width of the $\tau$ lepton has been put forward. It has been argued that, in spite of the small mass of the $\tau$, this determination is perturbative, since in the contest of the operator product expansion and QCD sum rules [2] non-perturbative corrections to this process can be estimated to be small. This is essentially a consequence of the fact that in the chiral limit the dominant non-perturbative corrections to this quantity behave like $1/m^4$.

This determination presents several subtle points. First of all, it relies on assumptions that are somewhat stronger than the usual assumptions of perturbative QCD. One should in fact assume that it is legitimate to add to a truncated perturbative expansion, which is essentially an expansion in inverse powers of logarithms of the momentum scale of the process, terms that are formally suppressed by powers of the momentum scale itself. A justification of this procedure can be found in ref. [3], where its validity is related to the position of the singularities of the Borel transform of the perturbative expansion. There it is also made clear that the underlying assumption of this procedure is that the only Borel singularities which are present are the known ones, i.e. infrared renormalons and instanton singularities.

Recently some authors [4] have brought arguments in favour of corrections, due to ultraviolet renormalons, suppressed by two powers of the momentum in current-current correlators. These corrections would give rise to terms behaving like $1/m^2$ in the $\tau$ hadronic width, thereby spoiling the analysis of ref. [1].

In the present work we will not deal with the general validity of the operator product expansion, or of the QCD sum rules formalism. We will instead deal with the well-known fact [5] that instantons do spoil the operator product expansion by introducing corrections that are power-suppressed by 9 or more inverse powers of the momentum. In view of the large power suppression, it is clear that corrections of this type will behave more like a step function, below which the perturbative methods will certainly be inapplicable. In other words, we expect a value of the ratio $m_{\tau}/\Lambda_{\text{QCD}}$ below which instanton corrections are of order one, and above which they are essentially zero. We point out that this is the only non-perturbative correction that can be computed explicitly in terms of the QCD parameter $\Lambda$ and the quark masses alone, with no need of further phenomenological inputs. Computations of
this type of corrections have been carried out in refs. [5] and [6] in the context of $e^+e^- \rightarrow$ hadrons. Although intermediate formulae and results are the same in the two papers, the conclusions are different, since the authors of ref. [5] conclude that instanton corrections are suppressed by four powers of the momentum, while in ref. [6] the effects are suppressed by twelve powers. For our purposes we are interested in the kind of result given in ref. [6], since the corrections mentioned in ref. [5] should be viewed rather as instanton corrections to the value of the matrix elements of dimension four operators, which is usually determined empirically, and is not calculable from first principles.

In ref. [6], it is claimed that instanton corrections in $e^+e^- \rightarrow$ hadrons become large for annihilation energies between 1 and 2 GeV. If a similar result held in the case of the $\tau$, it would spoil the analysis of ref. [1].

In order to obtain results that can be applied to our case we must complete the calculations of refs. [5] and [6] in three respects. First of all, we need to include the corrections to the axial current correlators, which were not considered there. Secondly, we must consider non-diagonal currents for flavours of different masses, which also were not considered previously. Lastly, we must perform an analysis in a definite subtraction scheme, in order to be able to relate the value of $\Lambda$ that we intend to use in this context with the value that is extracted from high-energy experiments. In fact, instanton effects are suppressed by a factor of order $\exp(-2\pi/\alpha_S)$, so that a scheme redefinition, which changes the inverse of $\alpha_S$ by the addition of a constant, affects directly the prefactor of the instanton correction (at the time when refs. [5] and [6] were written, there was no sound agreement on the allowed range of $\Lambda_{\text{QCD}}$, so that a complete answer correctly including the scheme dependence would not have been very useful). Our result also differs quantitatively from ref. [6], and we will comment upon the differences in due time.

Our paper is organized as follows: in section 2 we give a description of the calculation of the process in question. In section 3 we review the formula for the instanton density in QCD, and specify its form in the $\overline{\text{MS}}$ scheme. In section 4 we apply our results to the $\tau$ hadronic width. In section 5 we give our conclusions.
2. Instanton Corrections to the Current-Current Two-Point Functions

We consider the correlator of two currents in the instanton background

\[ \Pi_{\mu\nu}(x,y;\Omega_{\pm}) = \langle J^u_{\mu}(x)J^d_{\nu}(y) \rangle_A(\Omega_{\pm}) = -\text{Tr} \left( \Gamma_{\mu}S^d_{\pm}(x,y;\Omega_{\pm})\Gamma_{\nu}S^u_{\pm}(y,x;\Omega_{\pm}) \right) , \]  

(2.1)

where we have

\[ J^u_{\mu} = \bar{\psi}^u \Gamma_{\mu} \psi^d, \quad J^d_{\nu} = \bar{\psi}^d \Gamma_{\nu} \psi^u; \]  

(2.2)

\( \Gamma_{\mu} \) stands for \( \gamma_{\mu} \) for vector currents, and for \( \gamma_5\gamma_{\mu} \) for axial current; \( \Omega_{\pm} \) denotes the instanton (anti-instanton) global coordinates \( \Omega_{\pm} = (z,\rho,R) \), where \( z \) is the position, \( \rho \) is the size, and \( R \) stands for the colour orientation. \( S_{\pm} \) denotes the fermion propagator in the instanton (anti-instanton) background, whereas \( S_0 \) is the free fermionic propagator. The superscript \( u,d \) specifies the mass of the propagator. When no superscript is given, it is massless.

We consider here the general case of non-diagonal flavour currents. The two flavours involved, which will be simply called \( u \) and \( d \), have different masses \( m_u \) and \( m_d \). The propagators in the instanton background have the small mass expansion

\[ S^{u(d)}_{\pm}(x,y;\Omega_{\pm}) = -\frac{\Psi_0(x)\Psi_0^\dagger(y)}{m^{u(d)}} + S_{\pm}(x,y;\Omega_{\pm}) \]

\[ + m^{u(d)} \int dz^4 S_{\pm}(x,z;\Omega_{\pm})S_{\pm}(z,y;\Omega_{\pm}) + O(m^2), \]  

(2.3)

where terms with odd powers of the mass commute with \( \gamma_5 \), while the terms with even powers anticommute.

From eqs. (2.1) and (2.3) we can immediately obtain the correlator in the limit of small masses

\[ \Pi^V_{\mu\nu} = -\text{Tr}(\gamma_{\mu}S_0(x,y)\gamma_{\nu}S_0(y,x)) + A_{\mu\nu} + CB_{\mu\nu} \]  

(2.4)

\[ \Pi^A_{\mu\nu} = -\text{Tr}(\gamma_{\mu}S_0(x,y)\gamma_{\nu}S_0(y,x)) + A_{\mu\nu} - CB_{\mu\nu}, \]  

(2.5)

where \( V \) stands for vector-vector and \( A \) stands for axial-axial correlators, and

\[ C = \frac{1}{2} \left( \frac{m_u}{m_d} + \frac{m_d}{m_u} \right) \]

\[ A_{\mu\nu} = \text{Tr}(\gamma_{\mu}S_0(x,y)\gamma_{\nu}S_0(y,x)) - \text{Tr}[\gamma_{\mu}S_{\pm}(x,y;\Omega_{\pm})\gamma_{\nu}S_{\pm}(y,x;\Omega_{\pm})] \]  

(2.6)

\[ B_{\mu\nu} = 2 \text{Tr} \left[ \gamma_{\mu}\Psi_0(x)\Psi_0^\dagger(y)\gamma_{\nu} \int d^4 z S_{\pm}(x,z;\Omega_{\pm})S_{\pm}(z,y;\Omega_{\pm}) \right] . \]  

(2.7)
Expressions for the coefficients $A_{\mu\nu}$ and $B_{\mu\nu}$ have been obtained in ref. [5]. Their result agreed with that of ref. [6]. We do however find a different colour factor normalization for the result, our normalization being $2/3$ of theirs. We get (after colour averaging)

$$A_{\mu\nu}(x, y, \rho) = \frac{1}{2\pi^4} S_{\mu\alpha\nu\beta} \left[ \frac{\rho^4 (h_x h_y)^2}{\Delta^4} (2\Delta^\alpha \Delta^\beta - g^{\alpha\beta} \Delta^2) \right]$$

$$+ \frac{\rho^2}{\Delta^4} h_x h_y \left( h_y (\Delta^\alpha y^\beta + \Delta^\beta y^\alpha) - h_x (\Delta^\alpha x^\beta + \Delta^\beta x^\alpha) \right) + \text{odd terms} \tag{2.8}$$

$$B_{\mu\nu}(x, y, \rho) = -\frac{1}{\pi^4} (h_x h_y)^2 \frac{\rho^2}{\Delta^2} \left[ (\rho^2 + x \cdot y) g_{\mu\nu} + (y_\mu x_\nu - x_\mu y_\nu) \right] + \text{odd terms}, \tag{2.9}$$

where

$$h_x = 1/(x^2 + \rho^2); \quad h_y = 1/(y^2 + \rho^2)$$

$$\Delta = x - y,$$

$$S_{\mu\alpha\nu\beta} = g_{\mu\alpha} g_{\nu\beta} - g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\nu\alpha}. \tag{2.10}$$

“Odd terms” refers to terms that change sign when going from an instanton to an anti-instanton, and thereby vanish when summing over the two contributions. The instanton location has been chosen in $x_\pm = 0$ in the above formulae.

The expression for the current-current correlator including instanton corrections in the dilute-gas approximation is

$$\Pi_{\mu\nu}(\Delta) = \Pi^0_{\mu\nu}(\Delta) + \int d^4 z d\rho D(\rho) \sum_\pm \left( \Pi_{\mu\nu}(x, y, z, \rho, \pm) - \Pi^0_{\mu\nu}(\Delta) \right), \tag{2.11}$$

where $D(\rho)$ is the instanton density, which will be specified in the following section. Using eqs. (2.4) and (2.5) we obtain

$$\Pi_{\mu\nu}(\Delta) = \Pi^0_{\mu\nu}(\Delta) + \int d^4 z d\rho D(\rho) 2 \left( A_{\mu\nu}(x - z, y - z, \rho) + CB_{\mu\nu}(x - z, y - z, \rho) \right) \tag{2.12}$$

for vector currents, and the same form with $C \rightarrow -C$ for axial currents. We have

$$\Pi^0_{\mu\nu}(\Delta) = \frac{12 S_{\mu\alpha\nu\beta} \Delta_\alpha \Delta_\beta}{(2\pi^2)^2 \Delta^8}. \tag{2.13}$$

This formula includes the factor 3 from the colour trace. Defining

$$a_{\mu\nu}(\Delta, \rho) = \int d^4 z A_{\mu\nu}(x - z, y - z, \rho) \tag{2.14}$$

$$b_{\mu\nu}(\Delta, \rho) = \int d^4 z B_{\mu\nu}(x - z, y - z, \rho), \tag{2.15}$$
the $z$ integration gives

$$a_{\mu\nu}(\Delta, \rho) = -\frac{1}{2\pi^2} \left[ \frac{\partial^2}{\partial \Delta^\mu \partial \Delta^\nu} G(\Delta^2, \rho) + 2G'(\Delta^2, \rho) g_{\mu\nu} \right]$$  \hspace{1cm} (2.16)$$

$$b_{\mu\nu}(\Delta, \rho) = \frac{1}{2\pi^2} \left[ \frac{\partial^2}{\partial \Delta^2} G(\Delta^2, \rho) + 2G'(\Delta^2, \rho) \right] g_{\mu\nu}.$$  \hspace{1cm} (2.17)$$

where $G$ and $G'$ are functions of $\Delta^2$ and $\rho$, and they are given by the expressions

$$G'(\Delta^2, \rho) = \frac{\partial G(\Delta^2, \rho)}{\partial \Delta^2} = \rho^2 \frac{2}{\Delta^4} \left[ -2 \rho^2 \frac{1}{\Delta^2} \log \frac{\rho^2}{\Delta^2} \right],$$  \hspace{1cm} (2.18)$$

with $\xi = \sqrt{1 + 4\rho^2/\Delta^2}$. The above expression is a non-analytic function of $\rho$ for small $\rho$, while it is analytic in $\rho^{-1}$ for $\rho \to \infty$:

$$\lim_{\rho \to 0} G'(\Delta^2, \rho) = \frac{\rho^2}{\Delta^4} \left[ -1 - \frac{2}{\Delta^2} \log \frac{\rho^2}{\Delta^2} + \ldots \right]$$  \hspace{1cm} (2.19)$$

$$\lim_{\rho \to \infty} G'(\Delta^2, \rho) = \frac{1}{6\Delta^2} + \frac{1}{30\rho^2} - \frac{\Delta^2}{140\rho^4} + \frac{\Delta^4}{630\rho^6} - \frac{\Delta^6}{2772\rho^8} \ldots .$$  \hspace{1cm} (2.20)$$

We need the integral of $G$ against the instanton density, which is given by

$$D(\rho) = H \left[ \log \frac{1}{\rho^2 \Lambda^2} \right]^c \rho^M,$$  \hspace{1cm} (2.21)$$

where $M = 6 + n_f/3$. The value of $H$ and of the power of the logarithmic term $c$ will be discussed in more detail in the next section. We notice that the small-$\rho$ region never poses a convergence problem. On the other hand, the large-$\rho$ region is divergent. However, because of the analyticity of the integrand in that region, the divergence will be limited to a finite number of terms in the expansion of $G$. For definiteness, let us assume that $n_f = 3$. Then $D(\rho)$ will behave like $\rho^7$ for small $\rho$. Then we will have to subtract from $a_{\mu\nu}$ and $b_{\mu\nu}$ all the terms of their expansion up to the power $\rho^{-8}$. These subtracted terms will all have power-like dependence upon $\Delta^2$, with undefined, infrared divergent coefficients. We interpret these infrared divergent terms as the instanton contribution to the expectation value of operators in the operator product expansion of the two currents we are considering. These matrix elements are in general uncalculable, and they are usually estimated on the basis of some phenomenological considerations. The first term of this kind is in fact related to operators of dimension four. Since we want instead the true, direct effect of the
instanton upon our amplitude, we will subtract at the end these divergent terms. We first define

$$\tilde{G}'(\Delta^2, \rho, \rho_0) = G'(\Delta^2, \rho) - \theta(\rho - \rho_0) \frac{1}{\Delta^2} \sum_{j=0}^{L} g_j \left( \frac{\Delta}{\rho} \right)^{2j}$$  \hspace{1cm} (2.22)$$

$$\tilde{G}(\Delta^2, \rho, \rho_0) = G(\Delta^2, \rho) - \theta(\rho - \rho_0) \left[ \log \Delta^2 g_0 + \sum_{j=1}^{L} \frac{g_j}{j} \left( \frac{\Delta}{\rho} \right)^{2j} \right],$$  \hspace{1cm} (2.23)$$

where the $g_j$ are the numerical coefficients in the expansion of eq. (2.20). The functions $\tilde{G}$, $\tilde{G}'$ are then integrable in $\rho$ in the whole range, even when multiplied by a power of $\rho^M$ with $M < 2L - 1$. Once we know that their integral is in fact convergent, we may regulate it in any way we like. For example, we may choose the analytic continuation method of taking $-5 < M < -3$, and then continuing to all the allowed values of $M$. We now compute the Mellin transforms. We get

$$\int_0^\infty d\rho \rho^M \tilde{G}'(\Delta^2, \rho, \rho_0) = -\Delta^{M-1} \Gamma(-M - 4) \Gamma^3 \left( \frac{M + 5}{2} \right) \sin \left( \frac{M \pi}{2} \right)$$

$$- \frac{1}{\Delta^2} \sum_{j=0}^{L} \frac{g_j}{j} \frac{\Delta^{2j} \rho_0^{M-2j+1}}{M - 2j + 1}. $$  \hspace{1cm} (2.24)$$

$$\int_0^\infty d\rho \rho^M \tilde{G}(\Delta^2, \rho, \rho_0) = -\frac{2\Delta^{M+1}}{M + 1} \Gamma(-M - 4) \Gamma^3 \left( \frac{M + 5}{2} \right) \sin \left( \frac{M \pi}{2} \right)$$

$$- \log \Delta^2 \frac{g_0 \rho_0^{M+1}}{M + 1} - \sum_{j=1}^{L} \frac{g_j}{j} \frac{\Delta^{2j} \rho_0^{M-2j+1}}{M - 2j + 1}. $$  \hspace{1cm} (2.25)$$

Observe that the above formulae are now convergent for $M < 2L - 1$. The poles in $M$ arising from the gamma functions cancel against those in the sums. From eq. (2.12), (2.14), (2.15), (2.16) and (2.17) we get

$$\Pi_{\mu\nu}(\Delta) = \Pi^{0}_{\mu\nu}(\Delta) - H \left[ \log \frac{1}{\Delta^2 \Lambda^2} \right]^c \Gamma(-M - 4) \Gamma^2 \left( \frac{M + 5}{2} \right) \sin \left( \frac{M \pi}{2} \right) \frac{1}{\pi^2}$$

$$- \left\{ \left[ C \frac{\partial^2}{\partial \Delta^2} g_{\mu\nu} - \frac{\partial^2}{\partial \Delta^\alpha \partial \Delta^\nu} \right] \frac{2\Delta^{M+1}}{M + 1} + (C - 1) 2g_{\mu\nu} \Delta^{M-1} \right\}$$

$$\left[ \frac{D(\rho)}{\pi^2} D(\rho) \right] \left( \log \Delta^2 + (C - 1) 2g_{\mu\nu} \frac{1}{\Delta^2} + P_{\mu\nu}(\Lambda^2) \right),$$  \hspace{1cm} (2.26)$$

where $P_{\mu\nu}(\Delta)$ is a polynomial in $\Delta$. Observe that the logarithmic power in the instanton density is simply replaced by its value for $\rho = p$. Corrections to this replacement are logarithmically suppressed, and therefore are not included here.
The Fourier transform of the expression (2.26) can also be performed for non-integer $M$, and then continued to the desired value of $M$, using the formula

$$\int e^{i\Delta \cdot \Delta} d^4\Delta = p^{-J-4} \frac{4\pi}{\Gamma\left(-\frac{J}{2}\right)} \cos \frac{J\pi}{2} \Gamma\left(-\frac{J+3}{2}\right) \Gamma(J+4).$$

(2.27)

Observe that this formula vanishes when $J$ is an even integer. In fact, in this case the result is a distribution concentrated at $p^2 = 0$. The term $P_{\mu\nu}(\Delta)$ therefore does not contribute to the Fourier transform, and we get the result

$$\Pi_{\mu\nu}(p^2) = \Pi_{\mu\nu}^0(p^2) + H \left[ \log \frac{p^2}{\Lambda^2} \right] \left[ (M + 3) \Gamma\left(\frac{3}{2}\right) \frac{3}{(M + 1)\Gamma\left(\frac{M+3}{2}\right)} \right]$$

$$p^{-M-3} \left\{ (M + 3) \left[ Cg_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] + (C - 1)g_{\mu\nu} \right\}$$

$$+ \int_0^{\rho_0} D(\rho) d\rho \frac{4}{3p^4} \left[ (1 - 3C) (g_{\mu\nu}p^2 \rho^2 - p_\mu p_\nu) + 3(1 - C)p_\mu p_\nu \right].$$

(2.28)

The last term is explicitly infrared divergent for $\rho_0 \to \infty$. We interpret this term as the contribution of the instanton to the matrix elements of the dimension-four operators $F^2$ and $\bar{m}\psi\psi$. There are no singularities in the remaining terms for positive values of $M$.

It is easy to check that, in the particular case of flavour-diagonal vector currents, the $1/p^4$ term corresponds to the term obtained in refs. [3] and [4]. One can argue that the $1/p^4$ term cannot contribute to the discontinuity of the polarization operator, and therefore it does not affect the cross sections for hadron production. This formally correct argument fails however when radiative corrections are included. Similarly, we have neglected a number of contributions localized at $p^2 = 0$. Radiative corrections may delocalize these contributions, and therefore give corrections to the discontinuity, which behave like three or more inverse powers of $p^2$. These terms cannot be neglected, and, being IR-divergent, cannot be computed by perturbative techniques. They are usually accounted for phenomenologically, when the value of the various condensates is extracted from data.
3. The Instanton Density

The instanton density for SU(N) has been computed in ref. [7]. In the absence of fermions it is given by the formula

\[ W_{PV} = \frac{4}{\pi^2} \exp\left[ -\alpha(1) - 2(N-2)\alpha\left(\frac{1}{2}\right) \right] \times \int \frac{d^4zd\rho}{\rho^5} \left( \frac{4\pi^2}{g^2} \right)^{2N} \exp \left[ -\frac{8\pi^2}{g^2(\rho)} \right], \tag{3.1} \]

where \( g \) is the strong coupling constant, and (see ref. [8])

\[ \alpha(1) = 0.443307 \quad (3.2) \]
\[ \alpha\left(\frac{1}{2}\right) = 0.145873. \quad (3.3) \]

The above formula is given in the Pauli-Villars regularization scheme, with

\[ \frac{8\pi^2}{g^2(\rho)} = \frac{8\pi^2}{g^2} - \frac{11N}{3} \log(m_0 \rho), \tag{3.4} \]

where \( m_0 \) is the Pauli-Villars mass, and \( g \) without argument is the bare coupling.

If we want to use the value of \( \alpha_S \) measured in today’s high-energy experiments, we should convert the above formula to the \( \overline{\text{MS}} \) scheme. This change of scheme was first given in ref. [8], and then corrected in ref. [9]. We have also computed the change of scheme by just computing the vacuum polarization in the background gauge, in both the Pauli-Villars and the \( \overline{\text{MS}} \) scheme. We find that the two schemes give the same result if the Pauli-Villars mass \( m_0 \) and the \( \overline{\text{MS}} \) scale are related by the formula

\[ \frac{11}{3} \log \left( \frac{m_0^2}{\mu^2} \right) - \frac{1}{3} = 0. \tag{3.5} \]

Expressing \( m_0 \) as a function of \( \mu \) and replacing it in formula (3.1) we obtain

\[ W_{\overline{\text{MS}}} = \frac{4}{\pi^2} \exp\left[ \frac{N}{6} - \alpha(1) - 2(N-2)\alpha\left(\frac{1}{2}\right) \right] \int \frac{d^4zd\rho}{\rho^5} \left( \frac{4\pi^2}{g^2} \right)^{2N} \exp \left[ -\frac{8\pi^2}{g^2(\rho)} \right], \tag{3.6} \]

which agrees with ref. [8] in the SU(2) case. Including the fermions, we obtain the extra factor

\[ \rho^{n_f} \Pi_i m_i \exp \left[ -\frac{2}{3}n_f \log(m_0 \rho) + 2n_f \alpha\left(\frac{1}{2}\right) \right]. \tag{3.7} \]

For the fermion contribution to the vacuum polarization, we find that the Pauli-Villars and \( \overline{\text{MS}} \) results agree if \( \mu = m \), so that in this case it is enough to replace
$m_0$ with $\mu$. We will also need to express the running mass in terms of invariant mass parameters

$$m(\mu) = \hat{m} \left( \log \frac{\mu}{\Lambda} \right)^{-\frac{13}{33-2n_f}}$$

(3.8)

according to the definition of ref. [1]. Observe that consistency of the order at which we are computing requires that one uses the full two-loop expression for $g$ in the exponent (this was not included in ref. [3])

$$\frac{2\pi}{\alpha_S(\rho)} = 4\pi b_0 \log \left( \frac{1}{\Lambda \rho} \right) \left[ 1 + \frac{6(153 - 19n_f) \log \log \frac{1}{\Lambda \rho^2}}{(33 - 2n_f)^2 \log \frac{1}{\Lambda \rho^2}} \right],$$

(3.9)

while in the prefactor a leading-order formula is accurate enough. Gathering all the factors, and setting $\mu = 1/\rho$, we get

$$D(\rho) = H \left[ \log \frac{1}{\rho^2 \Lambda^2} \right]^{\frac{c}{6 + \frac{n_f}{2}}} \rho^{\frac{n_f}{6} + \frac{2}{3}}$$

(3.10)

$$H = \left( \Pi_i \frac{\hat{m}_i}{\Lambda} \right) \Lambda^{11 + \frac{n_f}{2}} \frac{2}{\pi^2} \exp \left[ -\alpha(1) + \frac{\alpha}{2} + (2n_f - 2)\alpha\left(\frac{1}{2}\right) \right] \times \left( \frac{33 - 2n_f}{12} \right)^{\frac{6}{33 - 2n_f}} \frac{2^{\frac{12n_f}{33 - 2n_f}}}{33 - 2n_f},$$

(3.11)

$$c = \frac{45 - 5n_f}{33 - 2n_f}. \quad (3.12)$$

Our final formula for the instanton contribution to the vacuum polarization is then

$$\Pi_{\mu\nu}(p^2) = \Pi_{\mu\nu}^0(p^2) + K_0 \left( \Pi_i \frac{\hat{m}_i}{\Lambda} \right) \left( \frac{p^2}{\Lambda^2} \right)^\frac{33 + n_f}{6} \left[ \log \frac{p^2}{\Lambda^2} \right]^{\frac{45 - 5n_f}{33 - 2n_f}}$$

$$\times \left\{ \left[ C \left( 10 + \frac{n_f}{3} \right) - 1 \right] (p^2 g_{\mu\nu} - p_\mu p_\nu) + (C - 1) \left( 10 + \frac{n_f}{3} \right) p_\mu p_\nu \right\}$$

(3.13)

with

$$K_0 = \frac{2}{\pi^2} \exp \left[ -\alpha(1) + \frac{\alpha}{2} + (2n_f - 2)\alpha\left(\frac{1}{2}\right) \right] \left( \frac{33 - 2n_f}{12} \right)^{\frac{6}{33 - 2n_f}} \frac{2^{\frac{12n_f}{33 - 2n_f}}}{33 - 2n_f}$$

$$\times \left( \frac{9 + \frac{n_f}{3}}{7 + \frac{n_f}{3}} \right) \Gamma \left( \frac{3}{2} \right) \Gamma^3 \left( \frac{9}{2} + \frac{n_f}{6} \right)$$

$$\left( \frac{9 + \frac{n_f}{3}}{7 + \frac{n_f}{3}} \right) \Gamma \left( 6 + \frac{n_f}{6} \right)$$

(3.14)

$$C = \pm \left( \frac{m_u}{m_d} + \frac{m_d}{m_u} \right) / 2$$

(3.15)

where in the last expression the $+$ sign is appropriate for vector-vector, and the $-$ sign for axial-axial correlators. The Born term (which fixes our normalization) is
given by
\[ \Pi^0_{\mu\nu} = \frac{1}{4\pi^2} (p\mu p\nu - p^2 g_{\mu\nu}) \log p^2 \]  
both for the axial and the vector contribution.

All results quoted so far were obtained in the Euclidean metric. The corresponding Minkowski space formulae in the time-like region are obtained by analytic continuation in \( p^2 \). Observe that in the case of the electromagnetic current, which has \( C = 1 \) the sign of the instanton correction is opposite to that of the leading term, contrary to the result of ref. [6].

### 4. Final Results

It is now straightforward to obtain the instanton contribution to the \( \tau \) hadronic decays. According to ref. [1], using the same notation, the ratio of the hadronic to the leptonic width \( R_\tau \) is given by

\[ R_\tau = 6\pi i \int_{|z|=1} dz (1 - z)^2 \left[ (1 + 2z)\Pi^{(T)}_{A+V}(p^2) + \Pi^{(L)}_{A+V}(p^2) \right], \]  

where \( z = p^2/M_\tau^2 \), and

\[ \Pi^{\mu\nu}_{A+V} = \Pi^{(T)}_{A+V}(p^2)(p^\mu p^\nu - p^2 g^{\mu\nu}) + \Pi^{(L)}_{A+V}(p^2)p^\mu p^\nu. \]  

The suffix \( A + V \) indicates the sum of the axial and vector contributions. We get

\[ \frac{R_{\tau}^{\text{inst}}}{R_0} = -2\pi i K_0 \left( \frac{\Lambda}{M_\tau} \right)^9 \frac{\hat{m}_u\hat{m}_d\hat{m}_s}{M_\tau^2} \int_{|z|=1} dz (1 - z)^2 z^{-6} \left( \log \frac{-z M^2}{\Lambda^2} \right)^{10} [2(1 + 2z) - 22]. \]  

Observe now that the contour integral would give zero if the logarithmic term was not present. Since the power of the logarithm is very near 1, we expect that the integral will not depend much upon the ratio \( \Lambda/M_\tau \). In fact we find the numerical result

\[ \frac{R_{\tau}^{\text{inst}}}{R_0} = \left( \frac{3.64 \Lambda}{M_\tau} \right)^9 \frac{\hat{m}_u\hat{m}_d\hat{m}_s}{M_\tau^2}, \]  

where the coefficient 3.64 corresponds to \( \Lambda = 400 \text{ MeV} \); it varies from 3.67 to 3.62 if \( \Lambda \) is pushed to the extreme values of 0.1 and 1 GeV. Choosing with ref. [1]

\[ \hat{m}_u = 8.7 \text{ MeV}, \quad \hat{m}_d = 15 \text{ MeV} \quad \text{and} \quad \hat{m}_s = 270 \text{ MeV} \]  

(4.5)
with \( M_\tau = 1.784 \text{ GeV} \) we get the result

\[
R_{\tau}^{\text{inst}} = \left( \frac{0.96 \times \Lambda}{1.784 \text{ GeV}} \right)^9
\]

The value of \( \Lambda \) to be used in this context is \( \Lambda_3 \), which is somewhat larger than the corresponding values of \( \Lambda_5 \) that are usually quoted. For example, a recent review [10] quotes the range \( 150 < \Lambda_5 < 330 \text{ MeV} \), which corresponds roughly to \( 280 < \Lambda_3 < 510 \text{ MeV} \). We see that even in the most pessimistic case of the largest allowed value for \( \Lambda_3 = 510 \text{ MeV} \) the instanton correction would turn out to be absolutely negligible.

5. Conclusions

We have completed a calculation of the one-instanton contribution to the \( \tau \) hadronic width. We found that the corrections are actually negligible. Part of the smallness of the result is due to the fact that the one-instanton correction is proportional to the chiral suppression factor \( m_u m_d m_s / M_\tau^3 \). It is interesting to note that if that factor were not there, the instanton correction could be of order 1, and would thereby invalidate the conclusions of ref. [1]. At this point one may wonder whether corrections due to instanton anti-instanton pairs, which should be suppressed by 18 powers of the ratio \( \Lambda / M_\tau \), but do not carry any chiral suppression, may have coefficients of comparable size, that is to say, if they become of order 1 for \( M_\tau \approx 4\Lambda_3 \). The computation of ref. [11] seems to indicate large corrections, although one may doubt the reliability of the method used there to perform a computation beyond the dilute gas approximation.

Our result for the instanton correction to the vacuum polarization, for \( n_f = 3 \), can be summarized as follows

\[
\Pi_{\mu\nu}(p^2) = \frac{1}{4\pi^2} \left\{ (p_\mu p_\nu - p^2 g_{\mu\nu}) \log p^2 + \frac{\hat{m}_u \hat{m}_d \hat{m}_s}{p^3} \left( \frac{5.1701 \Lambda}{p} \right)^9 \left[ \log \frac{p^2}{\Lambda^2} \right]^{\frac{1}{10}} 
\times \left[ (\hat{m}_u C - \hat{m}_d) (p^2 g_{\mu\nu} - p_\mu p_\nu) + \hat{m}_s (C - 1) p_\mu p_\nu \right] \right\}
\]

with \( C = \pm(m_u/m_d + m_d/m_u) \). For electromagnetic currents \( (C = 1) \), approximating the logarithmic exponent with 1, we get

\[
\frac{R_{\tau}^{\text{inst}}}{R_{\gamma}^{(0)}} = \left( \frac{Q_0 p}{p} \right)^{12}
\]
with $Q_0 = 0.591$ GeV for $\Lambda_3 = 300$ MeV and $Q_0 = 0.867$ GeV for $\Lambda_3 = 500$ MeV. The considerable numerical differences with respect to ref. [6] have several origins. First of all, there was an error in the normalization factor for the instanton density [8], which was subsequently corrected by the author himself. The authors of ref. [6] use a leading-order expression for $\alpha_S$ (instead of a next-to-leading one) in the exponent of the instanton density, which leads to an overestimate of the effect. The other differences are due to the use of the $\overline{\text{MS}}$ instead of the Pauli-Villars scheme in the instanton density, the different definition of the invariant quark mass used in the present work, and the different colour factor we found. These last three differences have a minor impact on the final result.
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