ORDER-CHAIN POLYTOPES

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Abstract. We introduce the notion of order-chain polytopes, which generalizes both order polytopes and chain polytopes arising from finite partially ordered sets. Since in general order-chain polytopes cannot be integral, the problem when order-chain polytopes are integral will be studied. Furthermore, we discuss the question whether every integral order-chain polytope is unimodularly equivalent to either an order polytope or a chain polytope. In addition, an observation on the volume of order-chain polytopes will be done.

Introduction

The order polytope $O(P)$ as well as the chain polytope $C(P)$ arising from a finite partially ordered set $P$ has been studied by many authors from viewpoints of both combinatorics and commutative algebra. Especially, in Stanley [9], the combinatorial structure of order polytopes and chain polytopes is explicitly discussed. Furthermore, in [6], the natural question when the order polytope $O(P)$ and the chain polytope $C(P)$ are unimodularly equivalent is solved completely. It follows from [3] and [7] that the toric ring ([5, p. 37]) of $O(P)$ and that of $C(P)$ are algebras with straightening laws ([4, p. 124]) on finite distributive lattices. Thus in particular the toric ideal ([5, p. 35]) of each of $O(P)$ and $C(P)$ possesses a squarefree quadratic initial ideal ([5, p. 10]) and possesses a regular unimodular triangulation ([5, p. 254]) arising from a flag complex. (Recall that a flag complex is a simplicial complex any of its nonface is an edge.) Furthermore, toric rings of order polytopes naturally appear in algebraic geometry (e.g., [1]) and in representation theory (e.g., [10]).

Given a convex polytope $P \subset \mathbb{R}^d$, we write $V(P)$ for the set of vertices of $P$ and $E(P)$ for the set of edges of $P$. A facet hyperplane of $P \subset \mathbb{R}^d$ is defined to be a hyperplane of $\mathbb{R}^d$ which contains a facet of $P$. If

$$H = \{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : a_1 x_1 + a_2 x_2 + \cdots + a_d x_d - b = 0 \},$$

where each $a_i$ and $b$ belong to $\mathbb{R}$, is a hyperplane of $\mathbb{R}^d$ and $v = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$,

then we set

$$H(v) = a_1 y_1 + a_2 y_2 + \cdots + a_d y_d - b.$$

Let $(P, \preceq)$ be a finite partially ordered set (poset, for short) on $[d] = \{1, \ldots, d\}$. For each subset $S \subseteq P$, we define $\rho(S) = \sum_{i \in S} e_i$, where $e_1, \ldots, e_d$ are the canonical unit coordinate vectors of $\mathbb{R}^d$. In particular $\rho(\emptyset) = (0, 0, \ldots, 0)$, the origin of $\mathbb{R}^d$. A subset $I$ of $P$ is an order ideal of $P$ if $i \in I$, $j \in [d]$ together with $j \preceq i$ in $P$ imply $j \in I$. An antichain of $P$ is a subset $A$ of $P$ such that any two elements in $A$ are incomparable.
We say that \( j \) covers \( i \) if \( i \prec j \) and there is no \( k \in P \) such that \( i \prec k \prec j \). A chain \( j_1 \prec j_2 \prec \cdots \prec j_s \) is saturated if \( j_q \) covers \( j_{q-1} \) for \( 1 < q \leq s \). A poset can be represented with its Hasse diagram, in which each cover relation \( i \prec j \) corresponds to an edge denoted by \( e = \{i, j\} \).

In [9], Stanley introduced two convex polytopes arising from a finite poset, the order polytope and the chain polytope. Following [6], we employ slightly different definitions. Given a finite poset \((P, \preceq)\) on \([d]\), the order polytope \(O(P)\) is defined to be the convex polytope consisting of those \((x_1, \ldots, x_d) \in \mathbb{R}^d\) such that

- \(0 \leq x_i \leq 1\) for \(1 \leq i \leq d\);
- \(x_i \geq x_j\) if \(i \preceq j\) in \(P\).

The chain polytope \(C(P)\) of \(P\) is defined to be the convex polytope consisting of those \((x_1, \ldots, x_d) \in \mathbb{R}^d\) such that

- \(x_i \geq 0\) for \(1 \leq i \leq d\);
- \(x_{i_1} + \cdots + x_{i_k} \leq 1\) for every maximal chain \(i_1 \prec \cdots \prec i_k\) of \(P\).

Let \(P\) be a finite poset and \(E(P)\) the set of edges of its Hasse diagram. In the present paper, an edge labeling of \(P\) is a map

\[ \ell : E(P) \longrightarrow \{o, c\}. \]

Equivalently, an edge labeling of \(P\) is an ordered pair

\[ (oE(P), cE(P)) \]

of subsets of \(E(P)\) such that \(oE(P) \cup cE(P) = E(P)\) and \(oE(P) \cap cE(P) = \emptyset\). An edge labeling \(\ell\) is called nontrivial if \(oE(P) \neq \emptyset\) and \(cE(P) \neq \emptyset\).

Suppose that \((P, \preceq)\) is a poset on \([d]\) with an edge labeling \(\ell = (oE(P), cE(P))\). Let \(P'_\ell\) and \(P''_\ell\) denote the \(d\)-element subposets of \(P\) with edge sets \(oE(P)\) and \(cE(P)\) respectively. The order-chain polytope \(OC_\ell(P)\) with respect to the edge labeling \(\ell\) of \(P\) is defined to be the convex polytope

\[ O(P'_\ell) \cap C(P''_\ell) \]

in \(\mathbb{R}^d\). Clearly the notion of order-chain polytopes is a natural generalization of both order polytopes and chain polytopes of finite posets.

For example, let \(P\) be the chain \(1 \prec 2 \prec \cdots \prec 7\) with

\[ oE(P) = \{\{1, 2\}, \{4, 5\}, \{5, 6\}\}, \ cE(P) = \{\{2, 3\}, \{3, 4\}, \{6, 7\}\}. \]

Then \(P'_\ell\) is the disjoint union of the following four chains:

\[
1 \prec 2, \quad 3, \quad 4 \prec 5 \prec 6, \quad 7
\]

and \(P''_\ell\) is the disjoint union of

\[
1, \quad 2 \prec 3 \prec 4, \quad 5 \quad \text{and} \quad 6 \prec 7.
\]

Hence the order-chain polytope \(OC_\ell(P)\) is the convex polytope consisting of those \((x_1, \ldots, x_7) \in \mathbb{R}^7\) such that
• $0 \leq x_i \leq 1$ for $1 \leq i \leq 7$;
• $x_1 \geq x_2$, $x_4 \geq x_5 \geq x_6$;
• $x_2 + x_3 + x_4 \leq 1$, $x_6 + x_7 \leq 1$.

One of the natural question, which we study in Section 1, is when an order-chain polytope is integral. (Recall that a convex polytope is integral if all of its vertices have integer coordinates.) We call an edge labeling $\ell$ of a finite poset $P$ integral if the order-chain polytope $\mathcal{OC}_\ell(P)$ is integral. In Theorem 1.3 it is shown that every labeling of a finite poset $P$ is integral if and only if $P$ is acyclic. Here by an acyclic poset $P$ we mean that the Hasse diagram of $P$ is an acyclic graph. Furthermore, Theorem 1.4 guarantees that every poset possesses at least one nontrivial integral labeling.

In Section 2, we consider the problem when an integral order-chain polytope is unimodularly equivalent to either an order polytope or a chain polytope. This problem is related to the work [6], in which the authors characterize all finite posets $P$ such that $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are unimodularly equivalent. We show that if $P$ is either a disjoint union of chains or a zigzag poset, then the order-chain polytope $\mathcal{OC}_\ell(P)$, with respect to each edge labeling $\ell$ of $P$, is unimodularly equivalent to the chain polytope of some poset (Theorems 2.3 and 2.4). However, we find that not all order-chain polytopes of an acyclic poset are unimodularly equivalent to a chain polytope. In fact, we find a labeling $\ell$ of an acyclic poset $P$ such that $\mathcal{OC}_\ell(P)$ is not unimodularly equivalent to any chain polytope. But it turns out that this order-chain polytope is unimodularly equivalent to an order polytope of some poset. Then it seems to be reasonable to propose the conjecture that every integral order-chain polytope is unimodularly equivalent to either an order polytope or a chain polytope (Conjecture 2.7).

We conclude the present paper with an observation on the volume of order-chain polytopes in Section 3. A fundamental question is to find an edge labeling $\ell$ of a poset $P$ which maximizes the volume of $\mathcal{OC}_\ell(P)$. In general, it seems to be very difficult to find a complete answer. We try to make a reasonable conjecture for the chain on $[d]$.

1. Integral order-chain polytopes

In this section, we consider the problem when an order-chain polytope is integral. We shall prove that every labeling of a poset $P$ is integral if and only if the poset $P$ is acyclic. We also prove that every poset has at least one nontrivial integral labeling.

Recall that for a finite poset $P$, vertices of $\mathcal{O}(P)$ are exactly those $\rho(I)$ for all order ideals $I$ of $P$. For two order ideals $I, J$ of $P$ with $I \neq J$, conv($\{\rho(I), \rho(J)\}$) forms an edge of $\mathcal{O}(P)$ if and only if $I \subset J$ and $J \setminus I$ is connected in $P$. The following lemma provides a necessary condition for a labeling to be integral.

**Lemma 1.1.** Suppose that the order-chain polytope $\mathcal{OC}_\ell(P)$ is integral. Then each facet hyperplane $H$ of $\mathcal{C}(P^\ell)$ does not cut through any edge of $\mathcal{O}(P^\ell)$. That is,

$$H(\rho(I))H(\rho(J)) \geq 0$$

for any edge conv($\rho(I), \rho(J)$) of $\mathcal{O}(P^\ell)$.
Proof. By contradiction. Assume that
\[ H(\rho(I))H(\rho(J)) < 0 \]
for some facet hyperplane of \( \mathcal{C}(P''_\ell) \) and some edge \( \text{conv}(\rho(I), \rho(J)) \) of \( \mathcal{O}(P'_\ell) \). Let
\[ v = H \cap \text{conv}(\rho(I), \rho(J)) \].
Clearly, \( v \) is a vertex of \( \mathcal{O}_\ell(P) \) and \( v \) lies in the interior of \( \text{conv}(\rho(I), \rho(J)) \). So \( v \) is not a lattice point, a contradiction.

Example 1.2. (1) Let \( P \) be the poset as shown in Fig.1 and let
\[ oE(P) = \{\{1, 2\}, \{2, 4\}\}, \ cE(P) = \{\{1, 3\}, \{3, 4\}\}. \]
Then the edge labeling \( \ell = (oE(P), cE(P)) \) is not integral.

![Fig.1](image)

In fact, let \( I = \emptyset, J = \{1, 2, 4\} \) and let \( H \) be the hyperplane
\[ x_1 + x_3 + x_4 - 1 = 0. \]
It is easy to see that \( H \) is a facet hyperplane of \( \mathcal{C}(P''_\ell) \) and \( \text{conv}(\rho(I), \rho(J)) \) is an edge of \( \mathcal{O}(P'_\ell) \). Moreover, we have
\[ H(\rho(I))H(\rho(J)) = (-1) \cdot 1 = -1 < 0. \]
The proof of Lemma 1.1 shows that \( H \cap \text{conv}(\rho(I), \rho(J)) = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}) \) is a vertex of \( \mathcal{O}_\ell(P) \) and so \( \mathcal{O}_\ell(P) \) is non-integral.

(2) It should be noted that the converse of Lemma 1.1 is not true. For example, let \( P \) be the poset given in Fig.2. Let \( oE(P) = \{\{3, 6\}, \{4, 6\}\}, cE(P) = E(P) \setminus oE(P) \) and \( \ell = (oE(P), cE(P)) \). Then, it is easy to verify that each facet hyperplane of \( \mathcal{C}(P''_\ell) \) does not cut through any edge of \( \mathcal{O}(P'_\ell) \). However,
\[ v = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \]
is a vertex of \( \mathcal{O}_\ell(P) \) given by
\[
\begin{align*}
  x_3 &= x_4 = x_6 \\
  x_3 + x_5 &= 1 \\
  x_1 + x_4 &= 1 \\
  x_1 + x_2 + x_5 &= 1 \\
  x_2 &= 0.
\end{align*}
\]
Theorem 1.3. Let \( P \) be a finite poset. Then every edge labeling of \( P \) is integral if and only if \( P \) is an acyclic poset.

Proof. Suppose that each edge labeling \( \ell \) of \( P \) is integral. If the Hasse diagram of \( P \) has a cycle, then it is easy to find a non-integral edge labeling. In fact, let \( e = \{i, j\} \) be an arbitrary edge from \( c \) and \( \ell = (E(P) \setminus \{e\}, \{e\}) \). We now show that \( \ell \) is not integral. To this end, let \( I = \emptyset \), and let \( J \) be the connected component of the Hasse diagram of \( P' \) which containing \( i \) and \( j \). Then we have \( I \cap \{i, j\} = \emptyset \) and \( |J \cap \{i, j\}| = 2. \) It follows that

\[
H(\rho(I))H(\rho(J)) < 0
\]

where \( H \) is the facet hyperplane \( x_i + x_j - 1 = 0 \) of \( C(P''_\ell) \). By Lemma 1.1, \( OC_\ell(P) \) must be non-integral, a contradiction.

Conversely, suppose that \( P \) is an acyclic poset on \( [d] \) and \( \ell \) is an edge labeling of \( P \). If \( v = (a_1, a_2, \ldots, a_d) \) is a vertex of \( OC_\ell(P) \), then we can find \( d \) independent facet hyperplanes of \( OC_\ell(P) \) such that

\[
v = \left( \bigcap_{i=1}^{d-m} H'_i \right) \cap \left( \bigcap_{j=1}^{m} H''_j \right),
\]

where \( m = \dim \left( \bigcap_{i=1}^{d-m} H'_i \right) \), each \( H'_i \) is a facet hyperplane of \( O(P'_\ell) \) and each \( H''_j \) is a facet hyperplane of \( C(P''_\ell) \) which corresponds to a chain \( C_i \) of length \( \geq 2 \) in \( P''_\ell \). By [9, Theorem 2.1], there is a set partition \( \pi = \{B_1, B_2, \ldots, B_{m+1}\} \) of \([d] \) such that \( B_1, B_2, \ldots, B_m \) are connected as subposets of \( P'_\ell \), \( B_{m+1} = \{i \in [d] : a_i = 0 \text{ or } 1\} \) and

\[
\bigcap_{i=1}^{d-m} H'_i = \{(x_1, x_2, \ldots, x_d) \mid x_i = x_j \text{ if } \{i, j\} \subseteq B_k \text{ for some } 1 \leq k \leq m, \text{ and } x_r = a_r \text{ if } r \in B_{m+1}\}.
\]

Let \( B_{m+1} = \{r_1, r_2, \ldots, r_s\} \) and for \( 1 \leq k \leq m \), let \( b_k \) denote the same values of all \( a'_i, i \in B_k \). Then it suffices to show that each \( b_k \) is an integer. Keeping in mind the assumption that the Hasse diagram of \( P \) is acyclic, we find that \( |C_i \cap B_j| \leq 1 \) for \( 1 \leq i, j \leq m \). For \( 1 \leq i, j \leq m \), let

\[
c_{ij} = \begin{cases} 1, & \text{if } |C_i \cap B_j| = 1; \\ 0, & \text{otherwise.} \end{cases}
\]
and for $1 \leq i \leq m, 1 \leq j \leq s$, let

$$d_{i,m+j} = \begin{cases} 1, & \text{if } r_j \in C_i; \\ 0, & \text{otherwise.} \end{cases}$$

(1.3)

By (1.1), $(b_1, b_2, \ldots, b_m, a_{r_1}, a_{r_2}, \ldots, a_{r_s})$ must be the unique solution of the following linear system:

$$\begin{cases} \sum_{j=1}^{m} c_{ij}y_j + \sum_{j=m+1}^{m+s} d_{ij}y_j = 1, & 1 \leq i \leq m \\ y_{m+1} = a_{r_1}, \\ y_{m+2} = a_{r_2}, \\ \vdots \\ y_{m+s} = a_{r_s}, \end{cases}$$

(1.4)

Now it suffices to show that the determinant of the coefficient matrix

$$A = \begin{pmatrix} c_{11} & \cdots & c_{1m} & d_{1,m+1} & \cdots & d_{1,m+s} \\ \vdots \\ c_{m1} & \cdots & c_{mm} & d_{m,m+1} & \cdots & d_{m,m+s} \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

(1.5)

is equal to 1 or $-1$. Now construct a bipartite graph $G$ with vertex set

$$\{B_1, B_2, \ldots, B_m, C_1, C_2, \ldots, C_m\}.$$

and edge set

$$\{\{B_i, C_j\} \mid 1 \leq i, j \leq m, |B_i \cap C_j| = 1\}.$$

Let

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots \\ c_{m1} & \cdots & c_{mm} \end{pmatrix}$$

Then we have

$$\det(C) = \sum_{\sigma \in \mathfrak{S}_m} \text{sign}(\sigma)c_{1\sigma_1} \cdots c_{m\sigma_m}.$$ 

(1.6)

Clearly, each nonzero term in (1.6) corresponds to a perfect matching in the graph $G$. Since the Hasse diagram of $P$ is acyclic, the graph $G$ must be an acyclic bipartite graph, which means that there is at most one perfect matching in $G$. So we have $\det(C) = 0, 1$ or $-1$. Note that the linear equations (1.4) have unique solution $(b_1, b_2, \ldots, b_m, a_{r_1}, a_{r_2}, \ldots, a_{r_s})$. Then we find that $\det(C) = \pm 1$. It follows that each $b_i$ is an integer. So the vertex $v$ of $\mathcal{OC}e(P)$ is integral. $\blacksquare$

For general finite poset $P$ with $|E(P)| \geq 2$, the following theorem indicates that there exists at least one nontrivial integral labeling.
Theorem 1.4. Suppose that $P$ is a finite poset. Let $\text{Min}(P)$ denote the set of all minimal elements in $P$. For $S \subseteq \text{Min}(P)$, let $E_S(P)$ denote the set of all edges in $E(P)$ which are incident to some elements in $S$. Then the edge labeling
$$\ell = (E(P) \setminus E_S(P), E_S(P))$$
is integral.

Proof. Suppose that $v$ is a vertex of $\mathcal{OC}_\ell(P)$. Then $v$ can be represented as intersection of $d$ independent facet hyperplanes, as in (1.1). Keeping the notation in the proof of Theorem 1.3, we can deduce that $|C_i| = 2$ and $|B_i \cap C_j| \leq 1$ for $1 \leq i, j \leq m$. So we can construct in the same way two matrices $A$ and $C$ as those in the proof of Theorem 1.3. Then, we can construct a graph $G$ with vertex set $\{B_1, B_2, \ldots, B_m, r_1, r_2, \ldots, r_s\}$ and edge set determined by $C_1, C_2, \ldots, C_m$. More precisely, $\{B_i, B_j\}$ is an edge of $G$ if and only if there exists $1 \leq k \leq m$ such that $C_k = \{i', j'\}$ for some $i' \in B_i, j' \in B_j$, and $\{B_i, r_j\}$ is an edge of $G$ if and only if there exists $1 \leq k \leq m$ such that $C_k = \{r_j, i'\}$ for some $i' \in B_i$. Obviously, $G$ is a bipartite graph with bipartition $(B_1, B_2)$, where
$$B_2 = \{B_j : 1 \leq j \leq m, \ B_j = \{k\} \text{ for some } k \in S\} \cup \{r_t : 1 \leq t \leq s, \ r_t \in S\}.$$Moreover, by the construction of the graph $G$, its incidence matrix is
$$
\begin{pmatrix}
c_{11} & \cdots & c_{1m} & d_{1,m+1} & \cdots & d_{1,m+s} \\
\vdots & & \vdots & \vdots & & \vdots \\
c_{m1} & \cdots & c_{mm} & d_{m,m+1} & \cdots & d_{m,m+s}
\end{pmatrix}.
$$
Where $c_{ij}, d_{i,m+j}$ are defined in (1.2) and in (1.3) respectively. A well known fact shows that the incidence matrix of any bipartite graph is totally unimodular. So the submatrix $C$ has determinant 0, 1 or $-1$. This completes the proof.

Example 1.5. By Theorem 1.3, if the Hasse diagram of $P$ has a cycle, then there exists at least one non-integral edge labeling $\ell$.

1. For example, let $P$ denote the poset whose Hasse diagram is a 4-cycle and let $E_1 = \{\{1, 2\}, \{2, 4\}, \{3, 4\}\}$. Then the labeling $\ell_1 = (E_1, \{1, 3\})$ given in Fig. 3(a) is non-integral, since $v = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a vertex of $\mathcal{OC}_{\ell_1}(P)$ given by
$$\begin{cases}
x_1 = x_2 = x_4 = x_3, \\
x_1 + x_3 = 1.
\end{cases}$$Note that the labeling $\ell_2 = (\{1, 3\}, E_1)$ given in Fig. 3(b) is integral. So we find that the complementary labeling $\ell^c = (cE(P), oE(P))$ of an integral labeling $\ell = (oE(P), cE(P))$ is not necessarily integral.

2. For any poset $P$ whose Hasse diagram is a cycle and any edge labeling $\ell$ of $P$, it is not hard to show that all coordinates of each vertex of $\mathcal{OC}_\ell(P)$ are 0, 1 or $\frac{1}{2}$.
2. Unimodular equivalence

In this section, we shall compare the newly constructed order-chain polytopes with some known polytopes. Specifically, we will focus on integral order-chain polytopes and consider their unimodular equivalence relation with order polytopes or chain polytopes.

We shall use the ideas in the proof of the following theorem due to Hibi and Li [6].

**Theorem 2.1.** [6, Theorem 1.3] The order polytope $O(P)$ and the chain polytope $C(P)$ of a finite poset $P$ are unimodularly equivalent if and only if the following poset:

![Poset](image)

does not appear as a subposet of $P$.

**Definition 2.2.** A poset $P$ on $[d]$ is said to be a zigzag poset if its cover relations are given by

$$1 < i_1 < i_1 + 1 < \cdots < i_2 < i_2 + 1 < \cdots < i_k < i_k + 1 < \cdots < d$$

for some $0 \leq i_1 < i_2 < \cdots < i_k \leq d$.

**Theorem 2.3.** Suppose that $P$ is a disjoint union of chains. Then for any edge labeling $\ell$, the order-chain polytope $OC_\ell(P)$ is unimodularly equivalent to a chain polytope $C(Q)$, where $Q$ is a disjoint union of zigzag posets.

**Proof.** We firstly assume that $P$ is a chain:

$$1 < 2 < 3 < \cdots < d.$$
and \( \ell \) is an edge labeling of \( P \) given by:

\[
\begin{align*}
\text{o : } & 1 \prec 2 \prec \cdots \prec i_1 \\
\text{c : } & i_1 \prec i_1 + 1 \prec \cdots \prec i_2 \\
\text{o : } & i_2 \prec i_2 + 1 \prec \cdots \prec i_3 \\
& \vdots \\
\text{c : } & i_{t-1} \prec i_{t-1} + 1 \prec \cdots \prec i_t \\
\text{o : } & i_t \prec i_t + 1 \prec \cdots \prec i_{t+1} \\
& \vdots \\
\text{c : } & i_{k-1} \prec i_{k-1} + 1 \prec \cdots \prec i_k = d,
\end{align*}
\]

where \( 1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq i_k = d \). Then the order-chain polytope \( OC_{\ell}(P) \) is given by

\[
\begin{align*}
\begin{cases}
x_1 \geq x_2 \geq \cdots \geq x_{i_1}, \\
x_{i_1} + x_{i_1+1} + \cdots + x_{i_2} \leq 1, \\
x_{i_2} \geq x_{i_2+1} \geq \cdots \geq x_{i_3}, \\
& \vdots \\
x_{i_{t-1}} + x_{i_{t-1}+1} + \cdots + x_{i_t} \leq 1, \\
x_{i_t} \geq x_{i_{t+1}} \geq \cdots \geq x_{i_{t+1}}, \\
& \vdots \\
x_{i_{k-1}} + x_{i_{k-1}+1} + \cdots + x_d \leq 1, \\
0 \leq x_i \leq 1, & 1 \leq i \leq d.
\end{cases}
\end{align*}
\]

(2.1)

Now define a map \( \varphi : \mathbb{R}^d \to \mathbb{R}^d \) as follows:

(1) if \( i \) is a maximal element in \( P_{\ell}' \), then let \( x'_i = x_i \);

(2) if \( i \) is not a maximal element in \( P_{\ell}' \), then \( \{i, i+1\} \) must be an edge in the Hasse diagram of \( P_{\ell}' \). Let \( x'_i = x_i - x_{i+1} \).

Let \( \varphi(x_1, x_2, \ldots, x_d) = (x'_1, x'_2, \ldots, x'_d) \).

Now it is easy to show that \( \varphi \) is a unimodular transformation. Moreover, the system (2.1) is transformed into:
Hence we conclude that
\[ \begin{aligned}
x_1' + x_2' + \cdots + x_{i_1}' & \leq 1, \\
x_{i_1}' + x_{i_1+1}' + \cdots + x_{i_2}' + x_{i_2+1}' + \cdots + x_{i_3}' & \leq 1, \\
\vdots
\end{aligned} \]
\[ \begin{aligned}
x_{i_{t-1}}' + x_{i_{t-1}+1}' + \cdots + x_{i_t}' + x_{i_{t+1}'} + \cdots + x_{i_{t+1}'} & \leq 1, \\
\vdots
\end{aligned} \]
\[ \begin{aligned}
x_{i_{k-1}}' + x_{i_{k-1}+1}' + \cdots + x_d' & \leq 1,
0 \leq x_i' \leq 1
\end{aligned} \]

Obviously, this system corresponds to the chain polytope \( \mathcal{C}(Q) \) for the zigzag poset \( Q \):
\[ 1 < 2 < \cdots < i_1 > i_1 + 1 > \cdots > i_2 > i_2 + 1 > \cdots > i_3 < \cdots \]
or the dual zigzag poset \( Q^* \):
\[ 1 > 2 > \cdots > i_1 < i_1 + 1 < \cdots < i_2 < i_2 + 1 < \cdots < i_3 > \cdots \]

So we deduce that \( \mathcal{OC}_\ell(P) \) is unimodularly equivalent to the chain polytope of some zigzag poset.

Now we continue to prove the general case that \( P \) is a disjoint union of \( k \) chains:
\[ P = C_1 \uplus C_2 \uplus \cdots \uplus C_k. \]
Since
\[ \mathcal{O}(P \uplus Q) = \mathcal{O}(P) \times \mathcal{O}(Q) \text{ and } \mathcal{C}(P \uplus Q) = \mathcal{C}(P) \times \mathcal{C}(Q), \]
we have
\[ \mathcal{OC}_\ell(P \uplus Q) = \mathcal{O}((P \uplus Q)_\ell) \cap \mathcal{C}((P \uplus Q)_\ell'') = \mathcal{O}(P_\ell' \uplus Q_\ell') \cap \mathcal{C}(P_\ell'' \uplus Q_\ell'') \]
\[ = \left[ \mathcal{O}(P_\ell') \times \mathcal{O}(Q_\ell') \right] \cap \left[ \mathcal{C}(P_\ell'') \times \mathcal{C}(Q_\ell'') \right] \]
\[ = \left[ \mathcal{O}(P_\ell') \cap \mathcal{C}(P_\ell'') \right] \times \left[ \mathcal{O}(Q_\ell') \cap \mathcal{C}(Q_\ell'') \right] \]
\[ = \mathcal{OC}_\ell(P) \times \mathcal{OC}_\ell(Q). \]
(2.2)

Hence we conclude that
\[ \mathcal{OC}_\ell(C_1 \uplus \cdots \uplus C_k) = \mathcal{OC}_\ell(C_1) \times \cdots \times \mathcal{OC}_\ell(C_k) \]
\[ \cong \mathcal{C}(Q_1) \times \cdots \times \mathcal{C}(Q_k) \]
\[ = \mathcal{C}(Q_1 \uplus \cdots \uplus Q_k), \]
where \( Q_i \) are zigzag posets.
Similarly, we can modify the proof of Theorem 2.3 slightly to get the following result:

**Theorem 2.4.** Suppose that $P$ is a finite zigzag poset. Then for any edge labeling $\ell$, the order-chain polytope $\mathcal{OC}_\ell(P)$ is unimodularly equivalent to a chain polytope $\mathcal{C}(Q)$ for some zigzag poset $Q$.

**Proof.** Suppose that $P$ is a zigzag poset on $[d]$ and $\ell$ is an edge labeling of $P$. Define a map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ as follows:

1. If $i$ is covered by at most one element in $P'_\ell$, let
   $$x'_i = \begin{cases} x_i, & \text{if } i \text{ is a maximal element in } P'_\ell \\ x_i - x_j, & \text{if } i \text{ is covered by } j \text{ in } P'_\ell \ (j = i - 1 \text{ or } i + 1). \end{cases}$$

2. If $i$ is covered by both $i - 1$ and $i + 1$ in $P'_\ell$, let
   $$x'_i = 1 - x_i.$$

Let $\varphi(x_1, x_2, \ldots, x_d) = (x'_1, x'_2, \ldots, x'_d)$. It is not hard to show that $\varphi$ is the desired unimodular transformation. $\blacksquare$

The following example shows that not every order-chain polytope $\mathcal{OC}_\ell(P)$ of an acyclic poset $P$ is unimodularly equivalent to some chain polytope.

**Example 2.5.** Let $P$ be the poset with an edge labeling $\ell$ as follows,

![Fig. 5](image)

namely, $\ell = (\{1, 3\}, \{3, 4\}, \{3, 5\}, \{2, 3\})$. Let
$$\varphi(x_1, x_2, x_3, x_4, x_5) = (x_1, 1 - x_2, x_3, x_4, x_5).$$

It is obvious that $\varphi$ is a unimodular transformation and $\varphi(\mathcal{OC}_\ell(P)) = \mathcal{O}(P)$. However, by checking all $63$ different non-isomorphic posets with $5$ elements, we find that $\mathcal{O}(P)$ is not equivalent to any chain polytope.

Let $P$ be the poset given in Fig. 4. Example 2.5 shows that $\mathcal{O}(P)$ is not unimodularly equivalent to any chain polytope. However, we can find a poset $Q$ such that $\mathcal{C}(P)$ is unimodularly equivalent to $\mathcal{O}(Q)$.

**Example 2.6.** Let $Q$ be the poset as follows,
By the definition of chain polytope, it is easy to see that \( C(P) = C(Q) \). By Theorem 2.1, we find that \( C(Q) \) is unimodularly equivalent to \( O(Q) \), so \( C(P) \) is unimodularly equivalent to \( O(Q) \).

We conclude this section with two conjectures.

**Conjecture 2.7.** Suppose that \( P \) is a finite poset and \( \ell \) is an integral edge labeling of \( P \). Then there exists some poset \( Q \) such that the order-chain polytope \( OC_{\ell}(P) \) is unimodularly equivalent to \( O(P) \) or \( C(Q) \).

**Conjecture 2.8.** For any poset \( P \), there exists some poset \( Q \) such that \( C(P) \) is unimodularly equivalent to \( O(Q) \).

3. **Volumes of \( OC_{\ell}(P) \)**

Given a poset \( P \), it is natural to ask which edge labeling \( \ell \) gives rise to an order-chain polytope with maximum volume. It seems very difficult to solve this problem in general case. In this section, we consider the special case when \( P \) is a chain \( P \) on \([n]\).

Let \( P \) be a chain on \([n]\). By the proof of Theorem 2.3, for an edge labeling \( \ell \) of \( P \), the order-chain polytope \( OC_{\ell}(P) \) is unimodularly equivalent to a chain polytope \( C(P_1) \), where \( P_1 \) is a zigzag poset such that all maximal chains, except the first one (containing 1) and the last one (containing \( n \)), consist of at least three elements. Conversely, for such a zigzag poset \( P_1 \), it is easy to find an edge labeling \( \ell \) of \( P \) such that \( OC_{\ell}(P) \) is unimodularly equivalent to \( C(P_1) \). Denote by \( Z(n) \) the set of such zigzag posets \( P_1 \) on \([n]\). By [9, Corollary 4.2], the volume of \( O(P_1) \) and that of \( C(P_1) \) are equal to \( e(P_1)/n! \), where \( e(P_1) \) is the number of linear extensions of \( P_1 \). (Recall that a linear extension of a poset \( P \) on \([n]\) is a permutation \( \pi = \pi_1\pi_2\cdots\pi_n \) of \([n]\) such that \( \pi^{-1}(i) < \pi^{-1}(j) \) if \( i \prec j \) in \( P \).)

Thus, to compute the maximum volume over all order-chain polytopes of the chain \( P \), it suffices to compute the maximum number of linear extensions for all zigzag posets \( P_1 \in Z(n) \). Next we shall represent this problem as a problem of maximizing descent statistic over a certain class of subsets. To this end, we recall some notions and basic facts. Given a permutation \( \pi = \pi_1\pi_2\cdots\pi_n \), let \( Des(\pi) \) denote its descent set \( \{ i \in [n-1] : \pi_i > \pi_{i+1} \} \). For \( S \subseteq [n-1] \), define the descent statistic \( \beta(S) \) to be
the number of permutations of \([n]\) with descent set \(S\). Note that there is an obvious bijection between zigzag posets on \([n]\) and subsets of \([n-1]\) given by

\[
S : P \mapsto \{j \in [n-1] : j > j+1\}.
\]

Moreover, a permutation \(\pi = \pi_1\pi_2 \cdots \pi_n\) of \([n]\) is a linear extension of \(P\) if and only if \(\text{Des}(\pi^{-1}) = S(P)\). Let \(F(n) = S(\mathcal{Z}(n))\). Then we can transform the problem of maximizing volume of order-chain polytopes of an \(n\)-chain to the problem of maximizing the descent statistic \(\beta(S)\), where \(S\) ranges over \(\mathcal{F}(n)\).

Observe that \(\beta(S) = \beta(\bar{S})\), where \(\bar{S} = [n-1] \setminus S\). Following [2], we will encode both \(S\) and \(\bar{S}\) by a list \(L = (l_1, l_2, \ldots, l_k)\) of positive integers such that \(l_1 + l_2 + \cdots + l_k = n - 1\). Given \(S \subseteq [n-1]\), a run of \(S\) is a set \(R \subseteq [n-1]\) of consecutive integers of maximal cardinality such that \(R \subseteq S\) or \(R \subseteq \bar{S}\). For example, if \(n = 10\), then the set \(S = \{1, 2, 5, 8, 9\}\) has 5 runs: \(\{1, 2\}\), \(\{3, 4\}\), \(\{5\}\), \(\{6, 7\}\), \(\{8, 9\}\). Suppose that \(S\) has \(k\) runs \(R_1, R_2, \ldots, R_k\) with \(|R_i| = l_i\), let \(L(S) = (l_1, l_2, \ldots, l_k)\).

**Lemma 3.1.** Suppose that \(S \subseteq [n-1]\) and \(L(S) = (l_1, l_2, \ldots l_k)\). Then \(S \in \mathcal{F}(n)\) if and only if \(l_i \geq 2\) for all \(2 \leq i \leq k - 1\).

*Proof.* The lemma follows immediately from the fact that \(\mathcal{Z}(n)\) consists of zigzag posets \(P\) such that all maximal chains in \(P\), except the first one (containing 1) and the last one (containing \(n\)), contains at least three elements. \(\blacksquare\)

Denote by \(F_n\) the \(n\)th Fibonacci number. By Lemma 3.1, it is easy to see that \(|\mathcal{F}(n)| = 2F_n\) for \(n \geq 2\). Based on computer evidences, we have the following conjecture about maximizing descent statistic over \(\mathcal{F}(n)\).

**Conjecture 3.2.** Suppose that \(n \geq 2\) and \(S \subseteq [n-1]\).

1. If \(n = 2m\) and

\[
L(S) = (1, \underbrace{2, 2, \ldots, 2}_{m-1}) \quad \text{or} \quad L(S) = (2, \underbrace{2, \ldots, 2, 1}_{m-1}),
\]

then \(\beta(T) \leq \beta(S)\) for any \(T \in \mathcal{F}(n)\).

2. If \(n = 2m + 1\) and

\[
L(S) = (1, \underbrace{2, 2, \ldots, 2, 1}_{m-1}),
\]

then \(\beta(T) \leq \beta(S)\) for any \(T \in \mathcal{F}(n)\).

Equivalently, by the proof of Theorem 2.3, we have

**Conjecture 3.3.** Let \(P\) be a chain on \([n]\). Then the alternating labeling \(\ell = (oE(P), cE(P))\) with

\[
oE(P) = \begin{cases} 
\{\{1, 2\}, \{3, 4\}, \ldots, \{n-1, n\}\}, & \text{if } n \text{ is even;} \\
\{\{1, 2\}, \{3, 4\}, \ldots, \{n-2, n-1\}\}, & \text{otherwise.}
\end{cases}
\]

gives rise to an order-chain polytope \(\mathcal{O}C_\ell(P)\) with maximum volume.
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