PERMUTATIONS RESTRICTED BY TWO DISTINCT PATTERNS OF LENGTH THREE

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Abstract

Define $S_n(R; T)$ to be the number of permutations on $n$ letters which avoid all patterns in the set $R$ and contain each pattern in the multiset $T$ exactly once. In this paper we enumerate $S_n(\emptyset; \{\alpha, \beta\})$ for all $\alpha \neq \beta \in S_3$.

1. Introduction

Let $\pi \in S_n$ be a permutation of $[n] = \{1, 2, \ldots, n\}$ written as a word. Let $\alpha \in S_k$, $k \leq n$. We say that $\pi$ contains the pattern $\alpha$ if there exist indices $i_1, i_2, \ldots, i_k$ such that $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ is equivalent to $\alpha$, where we define equivalence as follows. Define $\overline{\pi}_{i_j} = |\{m : \pi_{i_m} \leq \pi_{i_j}, m = 1, 2, \ldots, k\}|$. If $\alpha = \overline{\pi}_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ then we say that $\alpha$ and $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ are equivalent. For example, if $\tau = 124635$ then $\tau$ contains the pattern $213$ by noting that $\tau_3\tau_5\tau_6 = 435$ is equivalent to $213$. We say that $\pi$ avoids the pattern $\alpha$ if $\pi$ does not contain the pattern $\alpha$. In our above example, $\tau$ avoids the pattern $321$.

Let $\alpha \neq \beta$ be patterns of length three. In this article we enumerate the number of permutations which contain $\alpha$ exactly once and avoid $\beta$ as well as those permutations which contain each of $\alpha$ and $\beta$ exactly once.

2. Some History

The investigation of permutations which avoid a pattern of length three started well over a hundred years ago as exhibited in [C] and references therein. Knuth ([Kn]) investigated permutations which avoid any single pattern of length 3 and showed that, regardless of the pattern, such permutations are enumerated by the Catalan numbers. Bijective results are given in [Ri], [Krt], [SS], and [W1]. To describe the enumeration results more succinctly we introduce the following notation. Let $S_n(R)$ be the set of permutations on $[n]$ which avoid all patterns in the set $R$, where we omit the set notation if $|R| = 1$, and let $s_n(R) = |S_n(R)|$. Knuth’s result can then be stated as $s_n(\alpha) = \frac{1}{n+1}(2n)^n$ for all $\alpha \in S_3$.

Following Knuth’s result, two natural progressions were made: the investigation of $S_n(R)$ for $R \subseteq S_3$ and the investigation of $S_n(\beta)$ for $\beta \in S_4$. With respect to the former investigation, Simion and Schmidt ([SS]) gave a complete study of $s_n(R)$ for all $R \subseteq S_3$. 
With respect to the latter investigation, in two beautiful papers, Gessel ([Ge]) found $s_n(1234)$ and Bóna ([B1]) found $s_n(1342)$. Further results on $S_n(\alpha)$ for $\alpha \in S_4$ are given by West in [W1] and [W2] and by Stankova in [S]. The exact enumeration of 1324-avoiding permutations is still an open question, with the only result being a lower bound given by Bóna in [B2].

Several logical extensions followed: the investigation of $S_n(R)$ for $R \subseteq S_4$, the investigation of $S_n(S \cup T)$ for $S \subseteq S_3$ and $T \subseteq S_4$, and the investigation of $S_n(R)$ for $R \subseteq S_j$, $j > 4$. Guibert, in [Gu], showed that for certain $R \subseteq S_4$ with two elements, the corresponding $s_n(R)$ are given by Schröder numbers. In [B3] and [Kr], Bóna and Kremer, respectively, gave further extensions for $R \subseteq S_4$ with two elements. Mansour ([M]) completely enumerated $S_n(R \cup \{\alpha\})$ for $R \subseteq S_3$ and $\alpha \in S_4$. Results for permutations avoiding patterns of length greater than four can be found in [BLPP1], [BLPP2], [CW], and [Kr].

A natural generalization of pattern-avoiding permutations is pattern-containing permutations. To aid in the discussion of pattern-containing permutations we introduce the following notation. Let $S_n(R; T)$ be the set of permutations on $[n]$ which avoid all patterns in the set $R$ and contain each pattern in the multiset $T$ exactly once, where we again omit the set notation for singleton sets, and let $s_n(R; T) = |S_n(R; T)|$.

Recently, there has been much research focused on $S_n(R; T)$ for various sets $R$ and multisets $T$. Below, we give some results in this direction. First, in [N], Noonan proved that $s_n(\emptyset; 123) = \frac{2n}{n+3} \binom{2n}{n}$, a remarkably elegant formula. Bóna, in [B4], then showed that $s_n(\emptyset; 132) = \frac{2n-3}{n-3} \binom{2n-3}{n-3}$, an even simpler formula, proving a conjecture presented in [NZ]. These two results give $s_n(\emptyset; \alpha)$ for all $\alpha \in S_3$, by applying the following two bijections (given in [SS]).

**Reversal:** Define $r : S_n \rightarrow S_n$ by $r(\pi_1 \pi_2 \ldots \pi_n) = \pi_n \pi_{n-1} \ldots \pi_1$.

**Complementation:** Define $c : S_n \rightarrow S_n$ by $c(\pi_1 \pi_2 \ldots \pi_n) = (n - \pi_1 + 1)(n - \pi_2 + 1) \ldots (n - \pi_n + 1)$

We will also have need of a third bijection (given in [SS]) which is defined as follows.

**Inverse:** Define $i : S_n \rightarrow S_n$ as the group theoretic inverse.

It is easy to see that if $\pi$ contains exactly $s \geq 0$ occurrences of the pattern $\alpha$, then $r(\pi)$ (resp. $c(\pi)$, $i(\pi)$) contains exactly $s$ occurrences of the pattern $r(\alpha)$ (resp. $c(\alpha)$, $i(\alpha)$). By applying $r$, $c$, and $r \circ c$ we see that $s_n(\emptyset; 123) = s_n(\emptyset; 321)$ and $s_n(\emptyset; 132) = s_n(\emptyset; 231) = s_n(\emptyset; 312) = s_n(\emptyset; 213)$.

In [B4], Bóna also gave the generating function for $\{s_n(\emptyset; \{132, 132\})\}_n$. In [R], the formulas for $s_n(132; 123), s_n(123; 132)$, and $s_n(\emptyset; \{123, 132\})$ are given. These results were extended in [RWZ] to give the generating function for $\{s_n(132; \{132^r\})\}_{r,n \geq 0}$ in the form of a continued fraction. Mansour and Vainshtein ([MV1]) generalized this result to give the generating function for $\{s_n(132; \{(123 \ldots k)^r\})\}_{r,n}$ for a given $k$ and showed the
relation of such permutations to Chebyshev polynomials of the second kind. In [CW] other similar permutations were first shown to be related to the Chebyshev polynomials of the second kind. Independently, Jani and Rieper ([JR]) also extended the result in [RWZ] to find the generating function given in [MV1] using the theory of ordered trees. Shortly thereafter, Krattenthaler, in [Krt], used Dyck path bijections to reprove elegantly the results in [MV1] and [JR], extend results given in [CW], give a precise asymptotic formula for $s_n(132, \{(123 \ldots k)^r\})$, and show that $s_n(132, \{(123 \ldots k)^r\}) \sim s_n(123, \{((k - 1)(k - 2)\ldots k)^r\})$

3. Preliminaires

In this section we give some definitions and state a known result (without proof) upon which we will need to draw.

In order to discuss our analysis we have need of the following two definitions. The first definition has become a standard definition, while the second definition is new.

**Definition** (Wilf class) Let $S_1$ and $S_2$ be two sets. If $s_n(S_1) = s_n(S_2)$ then we say that $S_1$ and $S_2$ are in the same Wilf class, or Wilf equivalent.

**Example.** There is only one Wilf class for permutations avoiding a single pattern of length 3 since $s_n(\alpha) = \frac{1}{n+1} \binom{2n}{n}$ for any $\alpha \in S_3$.

**Definition** (almost-Wilf class) Let $S_1$ and $S_2$ be two sets and let $T_1$ and $T_2$ be two multisets. If $s_n(S_1; T_1) = s_n(S_2; T_2)$ then we say that $(S_1; T_1)$ and $(S_2; T_2)$ are in the same almost-Wilf class, or almost-Wilf equivalent.

**Theorem 2.1** (Simion and Schmidt, [SS])

1. For $\{\alpha, \beta\} \in \{\{123, 132\}, \{123, 213\}, \{132, 213\}, \{132, 231\}, \{132, 312\}, \{213, 231\}, \{213, 312\}, \{231, 321\}, \{312, 231\}\}$ we have $s_n(\{\alpha, \beta\}) = 2^{n-1}$ for $n \geq 2$ and $s_1(\{\alpha, \beta\}) = 1$;

2. For $\{\alpha, \beta\} \in \{\{123, 231\}, \{123, 312\}, \{132, 312\}, \{213, 321\}\}$ we have $s_n(\{\alpha, \beta\}) = \binom{n}{2} + 1$;

3. $s_n(\{123, 321\}) = 0$ for $n \geq 5$.

\[\]
4. On $s_n(\alpha; \beta)$

As seen in Section 2 we know $s_n(\alpha; \beta)$ for $(\alpha, \beta) \in \{(123, 132), (132, 123)\}$. Using the reversal and complementation bijections presented in Section 2 we see that the following is true.

**Theorem 4.1** For $(\alpha, \beta) \in \{(123, 132), (123, 213), (132, 123), (213, 123), (231, 321), (312, 321), (321, 231), (321, 312)\}$ we have $s_n(\alpha; \beta) = (n - 2)2^{n-3}$ for $n \geq 3$.

To complete the enumeration $s_n(\alpha; \beta)$ for all $\alpha \neq \beta \in S_3$ we must consider the following classes, which can be obtained through application of the reversal, complementation, and inverse bijections.

1. $\{(123;321), (321;123)\}$
2. $\{(123, 231), (123, 312), (321, 123), (321, 213)\}$
3. $\{(132;213), (213;132), (231;312), (312;231)\}$
4. $\{(132, 231), (132, 312), (213, 231), (213, 312), (231, 312), (231, 132)\}$
5. $\{(132, 231), (213, 321), (213, 123), (312, 123)\}$

Trivially, we have $s_n(123; 321) = 0$ for $n \geq 6$. The enumeration concerning the remaining classes follows from results, which will be noted below, given by Mansour and Vanshtein in [MV2] and [MV3].

**Theorem 4.2** For $(\alpha, \beta) \in \{(132, 231), (123, 312), (321, 123), (321, 213)\}$ we have $s_n(\alpha; \beta) = 2^{n-5}$ for $n \geq 4$ and $s_3(\alpha; \beta) = 1$.

**Proof.** This follows from Example 3.2 in [MV3] with $m = 2$ and $k = 3$. \hfill $\square$

**Theorem 4.3** For $(\alpha, \beta) \in \{(132, 231), (213, 312), (231, 312), (312, 231)\}$ we have $s_n(\alpha; \beta) = n2^{n-5}$ for $n \geq 4$ and $s_3(\alpha; \beta) = 1$.

**Proof.** This follows from Theorem 3.4 in [MV3] with $m = 2$ and $k = 3$. \hfill $\square$

**Theorem 4.4** For $(\alpha, \beta) \in \{(132, 231), (123, 312), (213, 231), (213, 312), (231, 132), (231, 123), (312, 132), (312, 213)\}$ we have $s_n(\alpha; \beta) = 2^{n-3}$ for $n \geq 3$.

**Proof.** This follows from Theorem 3.2 in [MV2]. \hfill $\square$

**Theorem 4.5** For $(\alpha, \beta) \in \{(132, 321), (213, 321), (231, 123), (312, 123)\}$ we have $s_n(\alpha; \beta) = 2n - 5$ for $n \geq 3$.

**Proof.** This follows immediately from Theorem 3.2 in [MV2]. \hfill $\square$

**Remark.** Notice that a priori there were six classes we had to consider (by Theorems
4.1 through 4.5 and the trivial case). (This is one less than the seven classes to consider
before [R] showed that (123; 132) and (132; 123) are almost-Wilf equivalent.) However,
the results above show that there are in fact only five almost-Wilf classes associated with
$S_n(\alpha; \beta)$, $\alpha \neq \beta \in S_3$. An explanation of this is given in the following subsection.

**4.1 Generating $S_n(123; 312)$ and $S_n(312; 123)$**

In this short subsection we investigate the nature as to why $s_n(123; 312) = s_n(312; 123)$
(which are both equal to $2n - 5$). We will show that the two sets considered here are
generated by almost exactly the same rule, and let the reader infer a bijection from this
result. Define $\phi : S_{m-1} \rightarrow S_m$ by $\phi(\pi_1 \pi_2 \ldots \pi_{m-1}) = (\pi_1 + 1)(\pi_2 + 1)\ldots(\pi_{m-1} + 1)1$.

It is easy to see that for any $\sigma \in S_{n-1}(123; 312)$ and any $\tau \in S_{n-1}(312; 123)$ that $\phi(\sigma) \in
S_n(123; 312)$ and $\phi(\tau) \in S_n(312; 123)$. Since $S_3(123; 312) = \{312\}$ and $S_3(312; 123) =
\{123\}$ we can use the rules below to generate $S_n(123; 312)$ and $S_n(312; 123)$.

**Generating Rule for $S_n(123; 312)$:**

By Theorem 4.2, it is trivial to check that $S_n(123; 312) = \{\phi(\pi) : \pi \in S_{n-1}(123; 312)\} \cup
\{31n(n-1)(n-2)\ldots542, (n-2)(n-3)\ldots32n1(n-1)\}$.

**Generating Rule for $S_n(312; 123)$:**

By Theorem 4.5, it is trivial to check that $S_n(312; 123) = \{\phi(\pi) : \pi \in S_{n-1}(312; 123)\} \cup
\{1(n-1)n(n-2)(n-3)\ldots32, (n-2)(n-1)(n-3)(n-4)\ldots21n\}$.

**5. On $s_n(\emptyset; \{\alpha, \beta\})$**

We first note that trivially $s_n(\emptyset; \{123, 321\}) = 0$ for $n \geq 6$. Next, using the bijections $r$ and $c$ we have four classes to consider:

1. $\{123, 231\} = \{123, 231\}, \{123, 312\}, \{132, 321\}, \{213, 321\}$
2. $\{123, 132\} = \{123, 132\}, \{123, 213\}, \{231, 321\}, \{312, 321\}$
3. $\{132, 213\} = \{132, 213\}, \{231, 312\}$
4. $\{132, 231\} = \{132, 231\}, \{132, 312\}, \{213, 231\}, \{213, 312\}$

Class (2) was enumerated in [R] giving the following theorem.

**Theorem 5.1** For $\{\alpha, \beta\} \in \{\{123, 132\}, \{123, 213\}, \{231, 321\}, \{312, 321\}\}$ we have
$s_n(\emptyset, \{\alpha, \beta\}) = (n-3)(n-4)2^{n-5}$ for $n \geq 5$.

Except for the proof of Theorem 5.4, in the proofs below we will isolate either the element
1 or the element $n$ in each permutation, $\pi$. Denote by $\pi(1)$ the elements (in order) to
the left of the isolated element, and by $\pi(2)$ the elements (in order) to the right of the isolated element. Hence, we have $\pi = \pi(1) 1 \pi(2)$ or $\pi = \pi(1) n \pi(2)$. We start with class (1).

**Theorem 5.2** For $\{\alpha, \beta\} \in \{\{123, 231\}, \{123, 312\}, \{132, 321\}, \{213, 321\}\}$ we have $s_n(\emptyset, \{\alpha, \beta\}) = 2n - 5$ for $n \geq 5$ and $s_4(\emptyset, \{\alpha, \beta\}) = 2$.

*Proof.* We will use $\{123, 312\}$ for our proof. Let $f_n = s_n(\emptyset; \{123, 312\})$, let $\pi \in S_n(\emptyset; \{123, 312\})$, and let $\pi_i = n$.

We have three cases to consider: (i) the (312) pattern occurs with $n$ as the ‘3’ and the (12) $\in \pi(2)$, (ii) the pattern (312) $\in \pi(1)$, and (iii) the ‘2’ in the (312) pattern is in $\pi(2)$ while (31) $\in \pi(1)$.

We start with case (i): the (312) pattern occurs with $n$ as the ‘3’ and the (12) $\in \pi(2)$.

Let $x$ be the ‘1’ and $y$ be the ‘2’ in the (312) pattern. Write $\pi = \pi(1) n A x B y C$, where $A, B, C$ represent the portions of $\pi$ in between two distinguished elements (either $n$ and $x$, $x$ and $y$, or $y$ and the end of $\pi$).

We will first show that $A$ is empty. Assume otherwise and let $a \in A$. Then either $nay$ is another (312) occurrence (if $a < y$) or $axy$ is another (312) occurrence (if $a > y$). Hence, $A$ is empty. Next, we will show that $B$ must be empty. Assume otherwise and let $b \in B$. Then either $nby$ is another (312) occurrence (if $b < y$) or $nxb$ is another (312) occurrence (if $b > y$). Hence, $B$ must also be empty. Thus we may write $\pi = \pi(1) n x y C$.

Next, we notice that for any $c \in C$ we must have $c < x$ and for any $p \in \pi(1)$ we must have $p < y$ to avoid another (312) occurrence. Furthermore, if $C$ were to contain a (12) pattern then we would have another occurrence of (312) with $n$ acting as the ‘3’. Hence, the elements of $C$ must be in decreasing order and thus our (123) pattern must start in $\pi(1)$. Similarly, the elements of $\pi(1)$ must be in decreasing order or we would have at least two occurrences of (123) with both $n$ and $y$ serving as the ‘3’ in the (123) pattern. Hence, there exists $r \in \pi(1)$ with $r < x$ which produces $rxy$ as our (123) pattern. Furthermore, all other elements in $\pi(1)$ must be larger than $x$ or else we would have another occurrence of (123). Hence, we must have $r = \pi_{i-1}$ since the elements in $\pi(1)$ are decreasing. However, if $i \neq 2$ then $\pi_1 rx$ would be another (312) pattern. Thus, $i = 2$. The last piece of information we need is that since all elements in $C$ are less than $x$, we must have $x = n - 2$. Thus we see that our permutations in this case are of the form $\pi = r n (n - 2) (n - 1) C$ with the elements of $C$ in decreasing order. Since we have $n - 3$ choices for $r$, we have $n - 3$ permutations in this case.

Next, we look at case (ii): the pattern (312) $\in \pi(1)$.

Let $zxy$ be the (312) pattern and write $\pi = A z B x C y D n \pi(2)$. Notice that in this case we already have our (123) pattern, namely, $xyn$. 

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We first show that $A$, $B$, and $C$ are empty. Assume otherwise and let $a \in A$, $b \in B$, and $c \in C$. For any $a \in A$ we see that either $a y n$ would give another $(123)$ occurrence (if $a < y$) or that $a x y$ would give another $(312)$ occurrence (if $a > y$). For $b \in B$ we see that either $b y n$ would give another $(123)$ occurrence (if $b < y$) or that $b x y$ would give another $(312)$ occurrence (if $b > y$). For $c \in C$, either $x c n$ would be another $(123)$ occurrence (if $c > y$) or $z c y$ would be another $(312)$ occurrence (if $c < y$). Hence, $A$, $B$, and $C$ must all be empty so we may write $\pi = z x y D n \pi(2)$.

Next, we notice that for any element in $D$ or $\pi(2)$, that element must be less than $x$, for otherwise we would have either another occurrence of $(312)$ with $z$ and $x$ or another $(123)$ occurrence with $x$ and $y$. This restriction gives us $z = n - 1$, $x = n - 3$, and $y = n - 2$. Furthermore, the elements in $D$ must be decreasing (to avoid another $(123)$ with $n$), and the elements in $\pi(2)$ must be decreasing (to avoid another $(312)$ with $n$). Even further, for all $d \in D$ and all $p \in \pi(2)$ we must have $d > p$ or else we would have another $(312)$ occurrence with $z d p$. Hence, the elements in both $D$ and $\pi(2)$ are determined by the position of $n$. Since we have $n - 3$ choices for the position of $n$, we have $n - 3$ permutations in this case.

Lastly, we look at case (iii): the ‘2’ in the $(312)$ pattern is in $\pi(2)$ while $(1) \in \pi(1)$.

Let $z x y$ be the $(312)$ pattern and write $\pi = A z B x C n D y E$.

We first show that $B$ and $C$ are empty. Assume otherwise and let $b \in B$ and $c \in C$. For $b \in B$, either $b x y$ is another occurrence of $(312)$ (if $b > y$) or $z b y$ is another occurrence of $(312)$ (if $b < y$). For $c \in C$, either we get two occurrences of $(123)$ with $z c n$ and $x c n$ (if $c > z$), we get another $(312)$ occurrence with $z c x$ (if $x < c < z$), or we get another $(312)$ occurrence with $z c y$ (if $c < x$). Hence, we may write $\pi = A z x n D y E$.

Next, notice that the elements in $D$ must be decreasing and the elements in $E$ must be decreasing to avoid another occurrence of $(312)$ with $n$ serving as the ‘3’. Furthermore, all elements in $D$ must be greater than $z$ and all elements in $E$ must be less than $x$ since if either of these did not hold we would have another $(312)$ occurrence. We then see that for all $a \in A$ we must have $x < a < y$, otherwise if $a > y$ we would obtain another $(312)$ occurrence with $x$ and $y$, and if $a < x$ we would have two occurrences of $(123)$ with $a x n$ and $a x y$. We also note that $A$ must contain exactly one element since for any $a \in A$, $a z n$ produces a $(123)$ pattern and if $A$ is empty we cannot obtain a $(123)$ occurrence. Since $A$ is not empty we now see that $D$ must be empty to avoid another $(123)$ occurrence with $a$ and $z$. We may now write $\pi = a z x n y E$, where $x < a < y$.

Since all elements in $E$ must be smaller than $x$ we see that $x = n - 4$, $y = n - 2$, $z = n - 1$, and $a = n - 3$. Finally, since the elements in $E$ must be decreasing we see that we only have a single permutation in this case (provided $n \geq 5$).

Summing over all cases we have $s_n(\emptyset, \{123, 312\}) = 2n - 5$ for $n \geq 5$. \hfill \Box

Remark. Notice that we have the interesting result that $s_n(123; 312) = s_n(\emptyset; \{123, 312\})$
and hence $(123; 312)$ and $(\emptyset; \{123, 312\})$ are almost-Wilf equivalent (for $n \geq 5$). This is the first nontrivial case of a “mixed restriction” equivalence.

We now move on to class (3) and prove the following theorem.

**Theorem 5.3** For $\{\alpha, \beta\} \in \{\{132, 213\}, \{231, 312\}\}$ we have $s_n(\emptyset, \{\alpha, \beta\}) = (n^2 + 21n - 28)2^{n-9}$ for $n \geq 7$, $s_6(\emptyset, \{\alpha, \beta\}) = 17$, $s_5(\emptyset, \{\alpha, \beta\}) = 6$, and $s_4(\emptyset, \{\alpha, \beta\}) = 3$.

**Proof.** We will use $\{231, 312\}$ for our proof. Let $f_n = s_n(\emptyset; \{231, 312\})$, let $\pi \in S_n(\emptyset; \{231, 312\})$, and let $\pi_i = 1$.

We have three cases to consider: (i) the pattern $312 \in \pi(1)$, (ii) the pattern $312 \in \pi(2)$, and (iii) the $312$ pattern straddles 1, i.e. the ‘3’ is in $\pi(1)$, the ‘2’ is in $\pi(2)$, and 1 serves as the ‘1’ in the pattern.

We start with case (i): the pattern $(312) \in \pi(1)$.

Let $zxy$ be our $(312)$ pattern and write $\pi = A z B x C y D 1 \pi(2)$. Note that we already have our $(231)$ pattern with $xy1$.

We first argue that $A, B, C, \text{ and } D$ must all be empty. Assume otherwise and let $a \in A$, $b \in B$, $c \in C$, and $d \in D$. We start with $c \in C$. Clearly we must have $c > z$ to avoid another $(312)$ occurrence. However, this produces $zc1$ which is another $(231)$ occurrence. Hence, $C$ must be empty. Next, we move to $b \in B$. We see here that either $zby$ is another occurrence of $(312)$ (if $b < y$) or $bxy$ is another occurrence of $(312)$ (if $b > y$). Hence, $B$ must also be empty. Now, we look at $a \in A$. Here, either $axy$ is another $(312)$ occurrence (if $a > y$) or both $ay1$ is another $(231)$ occurrences (if $a < y$). Lastly, for $d \in D$, either $xd1$ would be another occurrence of $(231)$ (if $d > x$) or $xyd$ would be another occurrence of $(231)$ (if $d < x$). Hence, we may now write $\pi = z xy 1 \pi(2)$. Since we already have both of the required patterns we see that $D \in S_{n-4}(\{312, 231\})$. By Theorem 2.2 we have $2^{n-5}$ permutations in this case for $n \geq 5$, and 1 permutation for $n = 4$.

Next we look at case (ii): the pattern $(312) \in \pi(2)$.

Let $zxy$ be our $(312)$ pattern and write $\pi = \pi(1) 1 A z B x C y D$. We first show that $B$ must be empty. Assume otherwise and let $b \in B$. Then either $zby$ is another $(312)$ (if $b < y$) or $bxy$ is another $(312)$ (if $b > y$). We next note that for any $c \in C$ we must have $c > z$ to avoid another $(312)$ occurrence. Hence, $zcy$ is a $(231)$ pattern for any $c \in C$. Thus, $|C| \leq 1$.

We first consider the subcase $|C| = 1$. Let $c \in C$ so that we have both of the required patterns in our permutation. Write $\pi = \pi(1) 1 A z x c y D$. Notice that for any $p \in \pi(1)$, $a \in A$, and $d \in D$ we must have $p < a < d$. This holds since we must have $p < a$ to avoid another $(312)$ occurrence with $p1a$. We then see that for any $a \in A$ we must have $a < x$ to avoid another $(231)$ occurrence with $bza$ (if $b < z$) or another $(312)$ occurrence with $bzy$ (if $b > z$).
with $bxy$ (if $b > z$). Lastly, we note that for any $d ∈ D$ we require $d > c$ to avoid (231) with $zcd$ (if $d < z$) or another (312) with $zxd$ (if $z < d < c$). Now since our elements in $A, B,$ and $D$ are either less than $x$ or greater than $c$, we see that $y = x + 1$, $z = x + 2$, and $c = x + 3$.

We now notice that $π 1 A$ read as a permutation must avoid both (231) and (312). Likewise, $D$ must avoid both (231) and (312). Since the value of $x$ determines the position of $x$, by Theorem 2.2 we have $\sum_{x=2}^{n-4} 2^{x-1}2^{n-x-5} = (n - 3)2^{n-6}$ permutations for $n ≥ 6$, one permutation for $n = 5$, and none for $n ≤ 4$ in this subcase.

Next, consider the subcase $|C| = 0$. Write $π = π(1) 1 A z x y D$. We have four subsubcases to consider:

(a) There exists a unique $d ∈ D$ with $d < x$. This gives $xyd$ as our (231) pattern.

(b) There exists a unique $a ∈ A$ with $x < a < y$. This gives $axy$ as our (231) pattern.

(c) All elements in $π(1)$ and $A$ are smaller than $x$ and our (231) pattern is contained within $π(1) 1 A$ while $D$ avoids both patterns.

(d) Our (231) pattern is contained within $D$ while $π(1) 1 A$ avoids both patterns.

In all subsubcases below let $z = π_j$ for some $j > i$.

We start with subsubcase (a). We must have $d = π_{j+3}$ in order to avoid another occurrence of (231). Write $π = π(1) 1 A z x y d D$. We note that for all $d ∈ D$ and all $p ∈ π(1) 1 A$ we must have $d > z$ and $p < x$ to avoid another (312) or (231) occurrence. Thus, we have $y = x + 1$ and $z = x + 2$. Hence, the value of $d$ determines the value of $j$ (the position of $z$). Lastly, we obviously need $π(1) 1 A$ and $D$ to be (231,312)-avoiding. By Theorem 2.2, we now see that we have $\sum_{j=2}^{n-4} 2^{j-2}2^{n-j-4} + 2^{n-5}$ permutations in this subsubcase. Hence, we have $(n - 3)2^{n-6}$ permutations for $n ≥ 6$, one permutation for $n = 5$, and none for $n ≤ 4$ in this subsubcase.

On to subsubcase (b). We must have $a = π_{j-1}$ to avoid another occurrence of (312). Write $π = π(1) 1 A a z x y D$. For all $a ∈ A$ and for all $d ∈ D$ we must have $a < x$ and $d > z$ in order to avoid another occurrence of either pattern. Thus, we have $a = x + 1$, $y = x + 2$, and $z = x + 3$. As in subsubcase (a), we have $(n - 3)2^{n-6}$ permutations for $n ≥ 6$, one permutation for $n = 5$, and none for $n ≤ 4$ in this subsubcase.

Next, consider subsubcase (c). We must have $π(1) 1 A ∈ S_{j-1}(312; 231)$ and $D ∈ S_{n-j-2}(\{231, 312\})$. From Theorems 2.2 and 3.4, for each $j ≥ 5$ we have $(j - 1)2^{j-6}2^{n-j-3} = (j - 1)2^{n-9}$ permutations, for $j = 4$ we have $2^{n-7}$ permutations, and for $j ≤ 3$ we have none. Summing over all valid $j$ we have $(n - 4)(n + 1)2^{n-10}$ permutations for $n ≥ 7$, one permutation for $n = 6$, and none for $n ≤ 5$ in this subsubcase.

Lastly, we have subsubcase (d). A result similar to that of subsubcase (c) holds. Noting that $π(1) 1 A ∈ S_{n-j-2}(\{231, 312\})$ and $D ∈ S_{j-1}(312; 231)$, from Theorems 2.2 and 3.4,
for each $j \leq n - 6$ we have $2^{j-2}(n-j-2)2^{n-j-7} = (n-j-2)2^{n-9}$ permutations. For $j = n - 5$ we have $2^{n-7}$ permutations, and for $j \geq n - 4$ we have none. Summing over all valid $j$ we have $(n^2 - 7n + 8)2^{n-10}$ permutations for $n \geq 7$, and none for $n \leq 6$ in this subsubcase.

Summing over all subsubcases, we see that we have $(n^2 + 19n - 70)2^{n-9}$ permutations for $n \geq 6$, two permutations for $n = 5$, and none for $n \leq 4$ in the subcase $|C| = 0$.

Our last case to consider is (iii): the (312) pattern straddles 1, i.e. the ‘3’ is in $\pi(1)$, the ‘2’ is in $\pi(2)$, and 1 serves as the ‘1’ in the pattern.

Let $z1y$ be our (312) pattern and write $\pi = Az1CyD$.

We first show that $B$ must be empty. Assume otherwise and let $b \in B$. Then we either have another occurrence of (312) with $b1y$ (if $b > y$) or another occurrence of (312) with $zby$ (if $b < y$).

Next, we show that $|A| + |C| \leq 1$. Let $a \in A$ and $c \in C$. We first note that we must have $c > z$ in order to avoid another (312) occurrence with $z1c$. We then note that we must have $a < y$ in order to avoid another (312) occurrence with $a1y$. Hence, for every $a \in A$, $az1$ gives a (231) occurrence, and for every $c \in C$, $zcy$ gives a (231) occurrence. Thus, $|A| + |C| \leq 1$.

If $|A| = 1$ we let $a \in A$ and write $\pi = az1yD$, where $D \in S_{n-4}(\{231, 312\})$. Further, all elements in $D$ must be larger than $z$ so that we avoid another occurrence of (312) with $z1$. By Theorem 2.2, we have $2^{n-5}$ permutations here for $n \geq 5$ and one permutation for $n = 4$.

If $|C| = 1$ we also have $2^{n-5}$ permutations for $n \geq 5$ and one permutation for $n = 4$ via an argument very similar to that found in the preceding paragraph.

If $|A| + |C| = 0$ we write $\pi = z1yD$, where $D \in S_{n-4}(312; 231)$ and again all elements in $D$ are larger than $z$. By Theorem 3.4 we have $(n-3)2^{n-8}$ permutations for $n \geq 7$, one permutation for $n = 6$, and none for $n \leq 5$ here.

Hence, case (iii) yields $(n+13)2^{n-8}$ permutations for $n \geq 7$, five permutations for $n = 6$, two permutations for $n = 5$, and none for $n \geq 4$.

Summing the number of permutations from all three cases proves the theorem. \qed

For our final class (class (4)) we have the following theorem, whose proof is more interesting than those above.

**Theorem 5.4** For $\{\alpha, \beta\} \in \{\{132, 231\}, \{132, 312\}, \{213, 231\}, \{213, 312\}\}$ we have $s_n(\emptyset; \{\alpha, \beta\}) = 2^{n-3}$ for $n \geq 4$.

**Proof.** We will use $\{132, 312\}$ for our proof. Let $f_n = s_n(\emptyset; \{132, 312\})$. 

Let $xzy$ be our (132) pattern and write $\pi = A x B z C y D$. First, we show that $B$ must be empty. Assume otherwise and let $b \in B$. Then either $bzy$ or $xby$ is another occurrence of (132) (depending on whether $b < y$ or $b > y$).

Now let $a \in A$. We must have $y < a < z$ for otherwise we would have another (132) occurrence with $azy$ or more than one (312) occurrence with $axz$ and $axy$. Also, for $c \in C$ we must have $c < y$ or else we would have another occurrence of (132) with $xcy$.

We now turn our attention to $D$. For any $d \in D$ we must have $d < x$ or $d > z$ in order to avoid another (132) occurrence with $x$ and $z$. Furthermore, those elements in $D$ which are larger than $z$ must be in increasing order so that we avoid another (132) occurrence with $x$ and $z$; and those elements in $D$ which are smaller than $x$ must be in decreasing order so that we avoid more than one occurrence of (312) with $x$, $y$, and $z$.

Turning back to $A$ and $C$ we now argue that $|A| + |C| = 1$. To see this, note that for any $a \in A$ and any $c \in C$ both $axy$ and $zcy$ are (312) patterns. Since we may only have one such pattern we see that $|A| + |C| \leq 1$. Now assume that both $A$ and $B$ are empty. With the restrictions on $D$ in the previous paragraph we see that the pattern (312) is avoided with this assumption. Hence, $|A| + |B| \geq 1$.

Before putting this all together we note that the above restrictions show that we have $\pi = axzy D$ or $\pi = xzy D$ with all elements in $D$ either smaller than $x$ or larger than $z$. Hence, the elements preceding D must be four consecutive integers which contain both the patterns (132) and (312) exactly once.

Thus, we have $f_n = f_4 \sum_{i=1}^{n-3} \binom{n-4}{i-1}$ permutaions in this case (for $n \geq 4$). This holds since there are $f_4$ ways to arrange the first four consecutive elements, we may choose $i = 1, 2, \ldots, n - 3$ for the value of $\min(a, x, y, z)$, and since we are choosing $i - 1$ spaces from the $n - 4$ spaces after $y$ in which to place the decreasing elements of $D$. □

Remark. We again see another interesting “mixed restriction” result with $s_n(132; 231) = s_n(\emptyset; \{132, 231\})$; i.e. $(132; 231)$ and $(\emptyset; \{132, 231\})$ are almost-Wilf equivalent.

5.1. Generating $S_n(312; 123)$ and $S_n(\emptyset; \{123, 312\})$: On the Almost-Wilf Equivalence of (312; 123) and (\emptyset; \{123, 312\})

In this short section we show that the two sets considered are generated by almost the same rule and let the reader infer a bijection from these rules. In the following, let $n \geq 5$.

Recall (from Section 4.1) that we have defined $\phi : S_{m-1} \rightarrow S_m$ by $\phi(\pi_1 \pi_2 \ldots \pi_{m-1}) = (\pi_1 + 1)(\pi_2 + 1) \ldots (\pi_{m-1} + 1)1$. We have also seen the following rule for generating $S_n(312; 123)$.

Generating Rule for $S_n(312; 123)$:
By Theorem 4.6, it is trivial to check that $S_n(312; 123) = \{ \phi(\pi) : \pi \in S_{n-1}(312; 123) \} \cup \{1(n-1)n(n-2)(n-3)\ldots32, (n-2)(n-1)(n-3)(n-4)\ldots21n\}.$

We now note that we can generate $S_n(\emptyset; \{123, 312\})$ by the following similar rule.

Generating Rule for $S_n(\emptyset; \{123, 312\})$:

By Theorem 5.2, it is trivial to check that $S_n(\emptyset; \{123, 312\}) = \{ \phi(\pi) : \pi \in S_{n-1}(\emptyset; \{123, 312\}) \} \cup \{1n(n-2)(n-1)(n-3)(n-4)\ldots32, (n-1)(n-3)(n-2)\ldots21n\}.$

5.2. Generating $S_n(132; 312)$ and $S_n(\emptyset; \{132, 312\})$: On the Almost-Wilf Equivalence of $(132; 312)$ and $(\emptyset; \{132, 312\})$

In this short section we show that the two sets considered are generated by exactly the same rule and let the reader infer a bijection from these rules. In the following, let $n \geq 4$.

From above we have $\phi : S_{m-1} \to S_m$ by $\phi(\pi_1 \pi_2 \ldots \pi_{m-1}) = (\pi_1 + 1)(\pi_2 + 1)\ldots(\pi_{m-1} + 1)1.$

We also define $\Phi : S_{m-1} \to S_m$ by $\phi(\pi_1 \pi_2 \ldots \pi_{m-1}) = \pi_1 \pi_2 \ldots \pi_{m-1}m.$

It is easy to check that the following generation rule generates both $S_n(132; 312)$ and $S_n(\emptyset; \{132, 312\})$. The difference in the sets comes from the initial sets: $S_4(132; 312) = \{3124, 4231\}$ and $S_4(\emptyset; \{132, 312\}) = \{2413, 3142\}$.

Generating Rule for both $S_n(132; 312)$ and $S_n(\emptyset; \{132, 312\})$:

To obtain $S_n(\bullet; \bullet)$ from $S_{n-1}(\bullet; \bullet)$ take $S_n(\bullet; \bullet) = \{ \phi(\pi), \Phi(\pi) : \pi \in S_{n-1}(\bullet; \bullet) \}$.

6. Summary and Questions

Below we give a table summarizing the above results and present some remaining questions. The top half of the table’s results comes from Section 3, and the bottom half comes from Section 4.
| Almost-Wilf Class, $W$ | $s_n(T), T \in W$ |
|-----------------------|------------------|
| $A = (123; 321)$      | 0 for $n \geq 6$ |
| $B = (123; 132)$      | $(n - 2)2^{n-3}$ for $n \geq 3$ |
| $C = (123; 231)$      | $2n - 5$ for $n \geq 3$ |
| $D = (132; 213)$      | $n2^{n-5}$ for $n \geq 4$ |
| $E = (132; 231)$      | $2^{n-3}$ for $n \geq 3$ |
| $F = (\emptyset; \{123, 321\})$ | 0 for $n \geq 6$ |
| $G = (\emptyset; \{123, 231\})$ | $2n - 5$ for $n \geq 5$ |
| $H = (\emptyset; \{123, 132\})$ | $\binom{n-3}{2}2^{n-4}$ for $n \geq 5$ |
| $I = (\emptyset; \{132, 213\})$ | $(n^2 + 21n - 28)2^{n-9}$ for $n \geq 7$ |
| $J = (\emptyset; \{132, 231\})$ | $2^{n-3}$ for $n \geq 4$ |

The Coresponding Almost-Wilf Classes

A. $\{(123;321), (321;123)\}$
B. $\{(123;132), (123;213), (132;123), (213;123), (231;321), (312;321), (321;231), (321;312)\}$
C. $\{(123;231), (123;312), (132;321), (213;321), (231;123), (312;123), (321;132), (321;213)\}$
D. $\{(132;213), (213;132), (231;312), (312;231)\}$
E. $\{(132;231), (132;312), (213;231), (213;312), (231;132), (231;213), (312;132), (312;213)\}$
F. $\{(\emptyset; \{123, 321\}), (\emptyset; \{321,123\})\}$
G. $\{(\emptyset; \{123,231\}), (\emptyset; \{123,312\}), (\emptyset; \{132,321\}), (\emptyset; \{213,321\})\}$
H. $\{(\emptyset; \{123,132\}), (\emptyset; \{123,213\}), (\emptyset; \{231,321\}), (\emptyset; \{312,321\})\}$
I. $\{(\emptyset; \{132,213\}), (\emptyset; \{231,312\})\}$
J. $\{(\emptyset; \{132,231\}), (\emptyset; \{132,312\}), (\emptyset; \{213,231\}), (\emptyset; \{213,312\})\}$
We would like very much to see formulas for $s_n(\emptyset; \{(123)^2\})$ and $s_n(\emptyset; \{(132)^2\})$ determined to finish the study of $s_n(\emptyset; \{\alpha, \beta\})$ for all $\alpha, \beta \in S_3$. Note that Bóna, in [B2], has given the generating function and a recursive formula for the sequence $\{s_n(\emptyset; \{(132)^2\})\}_n$, however a formula for $s_n(\emptyset; \{(132)^2\})$ is not immediate.

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