General Formula of Displacements in Bending Elements

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Abstract. A technique is proposed for determining displacements in bending elements using a general formula. Problems of calculating deflections and sweep angles under the action of a concentrated load and bending moment for a statically determinate cantilever beam are considered. The advantages of using the deflection equations as a continuous function are indicated. The area of rational application of the proposed formulas in solving problems using automated systems is justified. The specifics of calculating the deformed scheme in the form of a continuous model for various problems using general equations are shown.

1. Introduction
Determining displacements is required in solving many problems of structural mechanics and strength of materials. Despite the wide variety of existing techniques that allow determining the deformed scheme under any loads, the format of the resulting equations is not always convenient for subsequent use. E.g., under some common loading options with the participation of a span load, the calculation by the method of initial parameters should be performed for two sections, and when applying the Maxwell-Mohr method, equations for three sections should be built, which not only complicates the calculation but also does not allow using it to solve problems analytically. The lack of a general displacement equation complicates solving many applied problems and requires additional equations to eliminate redundancy when calculating structures based on a deformed scheme. The use of an equation in the form of a continuous function is much more convenient and is a prerequisite in some studies, e.g., when calculating the curved deck structures for a non-uniform live load [1–2]. When solving such problems, a general equation can not only greatly simplify the displacement determining procedure but also allow realizing the advantages of modern automated computing tools, e.g., in the MathCAD environment.

2. Theoretical
Let us consider a technique for determining deflections in a bending element, which eliminates the need for multiple calculations. For clarity, we will derive the formula for a particular problem setting case, which can be extended to other known loading options.

It is required to build an equation for the deflection in section \( k \) of the beam located at a distance \( x \) from the left support (Fig. 1). In the case of loading a cantilever beam with a concentrated span force \( P \), two options for the relative position of the section \( k \) and the force \( P \) should be considered.

The first case: section \( k \) is to the left of the force \( P \) (Fig. 1, a): \( 0 \ll x \ll a \). As is known [3], the below equations are used to calculate the sweep angle and deflection:

\[
y' = \frac{Px}{2EJ}(2a - x);
\]

(1)
\[ y = \frac{P}{6EJ} x^2 (3a - x). \]  
(2)

The second case: section \( k \) is located to the right of the force \( P \) (Fig. 1, b): \( a \ll x \ll l \).

In this case, the sweep angle and vertical deflection of the section \( k \) are calculated by the below equations [3]:

\[ y' = \frac{Pa^2}{2EJ}, \]
(3)
\[ y = \frac{Pa^2}{6EJ} (3a - x). \]
(4)

**Figure 1.** Computational Scheme under the Action of a Concentrated Span Force.

As can be seen from these equations, to determine the displacements, the calculation should be performed twice. To exclude multiple calculations, the equations obtained should be combined. Let us use artificial mathematical transformations, the essence of which is deriving a formula automatically considering the position of the force \( P \). To do this, we transform the equation (1) in such a way that for \( x < a \), when the force \( P \) is located to the right of the section \( k \), this term becomes zero. E.g., the difference between the absolute value of the expression in the parenthesis and the value itself can be used, which allows excluding the additional term (1), provided \( x - a < 0 \). As a result, we obtain a general formula for determining the sweep angle at any position of the force \( P \) within the span

\[ y' = -\frac{P}{2EJ} \left[ (2a - x)x - a^2 \right] \left( \frac{|a-x|+a-x}{2(a-x)} + a^2 \right); \]  
(5)

After transformations, the final equation will take the form

\[ y' = \frac{P}{4EJ} [(a - x)(|a - x| + a - x) - 2a^2]. \]  
(6)

Similarly, we combine equations (2) and (4) to determine the deflections in any beam section. As a result, we have

\[ y = -\frac{P}{6EJ} \left[ ((3a - x)x^2 - (3x - a)a^2) \frac{|a-x|+a-x}{2(a-x)} + (3x - a)a^2 \right]. \]  
(7)

The transformation brings the general equation to the form

\[ y = -\frac{P}{12EJ} [(a - x)^2 (|a - x| + a - x) + 2(3x - a)a^2]. \]  
(8)

As can be seen from the formulas obtained, in the first case meeting the condition \( 0 \ll x \ll a \), the sweep angle and deflection equations take the form (1) and (2), respectively. In the second case, for \( a \ll x \ll l \), equations (6) and (8) take, respectively, the form of equations (3) and (4).
In a slightly different form, the general sweep angle equation can be obtained by artificially excluding the equations for the first section at \( a \ll x \ll l \), when the considered section is to the right and, similarly, excluding the equations for the second section at \( 0 \ll x \ll a \), when the considered section is to the left of the load application point. Then, for the sweep angles, we obtain the below equation

\[
y^l = -\frac{P}{2EJ} \left[ \frac{\left[ (2a - x)x + \left[ a^2 - x(2a - x) \right] \right] |x-a| + x-a^2}{2(x-a)} \right]; \tag{9}\]

After transformations, the general equation will take the form

\[
y^l = -\frac{P}{4EJ} \left[ (2a - x) + a^2 + |x-a|(x-a) \right]. \tag{10}\]

To calculate the deflections, like for the sweep angle, another formula can also be used, which can be represented in the form

\[
y = -\frac{P}{6EJ} \left[ (3a - x)x^2 + \left[ (3x - a)a^2 - (3a - x)x^2 \right] \frac{|x-a| + x-a^2}{2(x-a)} \right]. \tag{11}\]

After transformations, we finally get

\[
y = -\frac{P}{12EJ} \left[ (3a - x)2x^2 + (x - a)^2(|x-a| + x-a) \right]. \tag{12}\]

Similarly, general equations can be derived to determine displacements for other loading options. E.g., when considering loading with a concentrated span moment (Fig. 2), the known sweep angle and deflection dependences on the current coordinate can also be used [3].

The first case: section \( k \) is to the left of the concentrated moment \( M \) (Fig. 1, a): \( 0 \ll x \ll a \).

\[
y^l = -\frac{Ma}{EJ}; \tag{13}\]

\[
y = -\frac{Ma^2}{2EJ}. \tag{14}\]

![Figure 2. Computational Scheme under the Action of a Concentrated Span Moment.](image)

The second case: section \( k \) is to the right of the concentrated moment \( M \) (Fig. 1, b): \( a \ll x \ll l \).

In this case, the sweep angle and vertical deflection of the section \( k \) are determined by the below equations

\[
y^l = -\frac{Ma}{EJ}; \tag{15}\]

\[
y = -\frac{Ma}{EJ} \left( x - \frac{a}{2} \right). \tag{16}\]

To derive the general sweep angle equation, we combine equations (13) and (15) to ensure meeting the problem conditions for an arbitrary location of the section \( k \). Let us write down an equation that allows artificially eliminating the sweep angle equation for the second section when the condition \( 0 \ll x \ll a \) is met.
\[ y' = \left( -\frac{Mx}{EJ} + \frac{Ma}{EJ} \right) \frac{|x-a|+x-a}{2(x-a)} - \frac{Mx}{EJ}. \] (17)

After transformations, we get
\[ y' = \frac{M}{2EJ} (|x - a| - x - a). \] (18)

For deflections, the general equation will be
\[ y = \left( \frac{Mx^2}{2EJ} - \frac{Ma(2x-a)}{2EJ} \right) \frac{|x-a|+x-a}{2(x-a)} - \frac{Mx^2}{2EJ}, \] (19)

In the final form, we get the equation
\[ y = -\frac{M}{4EJ} [(x - a)(|x - a| + x - a) - 2x^2]. \] (20)

In a slightly different form, after artificially excluding the equations for the first section at \( x \ll l \), the general sweep angle equation can be represented as follows
\[ y' = \left( -\frac{Mx}{EJ} + \frac{Ma}{EJ} \right) \frac{|a-x|+a-x}{2(a-x)} - \frac{Ma}{EJ}. \] (21)

From which after transformations, we get
\[ y' = -\frac{M}{2EJ} (|a-x| - a - x). \] (22)

Let us derive a unified equation for deflections in a similar way, combining equations (14) and (16)
\[ y = \left[ \frac{Mx^2}{2EJ} - \frac{Ma(2x-a)}{2EJ} \right] \frac{|a-x|+a-x}{2(a-x)} - \frac{Ma(2x-a)}{2EJ}. \] (23)

Finally, after transformations, we get
\[ y = -\frac{M}{4EJ} [(x - a)(|a-x| + a - x) - 2(2x - a)a]. \] (24)

In a similar way, general equations can be derived to determine displacements for other loading options and computational models of bending elements that are encountered in practical calculations.

It is important to note that the equations obtained for the sweep angles and deflections are a continuous function of the \( x \) coordinate, which takes any value within the entire beam span, including the junction point of the left and right sections at \( x=a \). This function feature allows using it for subsequent analysis and solution of many applied problems where the notion of the deformed element scheme is required. Despite some awkwardness, the equations proposed allow considering the computational model as a continual system with all the inherent positive properties that distinguish it from a discrete one. The deflection and sweep angle equations representing a continuous function of the \( x \) coordinate allow using a mathematical apparatus in the analysis of a deformed scheme, which is widely used in solving many applied problems. The undoubted advantage of writing displacements in the form of a general formula is also the simplicity of data entry when calculating deformations and deflections using computer software, which eliminates multiple repetitions.

3. Results
Analysis of the calculations performed using the general displacement equations allows specifying the main positive aspects of the approach proposed when solving problems of determining the sweep angles and deflections. The general equations obtained have the below advantages in the calculation of span loads:

1. Representing displacements in the form of continuous functions expands the techniques for studying the stress-strain state using the mathematical apparatus.
2. The lack of discrete sections with different forms of recording the displacement equations allows considering the computational scheme as a single continuous system.
3. The general form of recording displacements allows implementing the calculation techniques developed when solving many practical problems.
4. When calculating displacements using modern software tools, creating computational algorithms is greatly simplified due to the absence of the need to enter data for several sections and perform multiple calculations.
4. References

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