TIME PERIODIC SOLUTION TO A COUPLED CHEMOTAXIS-FLUID MODEL WITH POROUS MEDIUM DIFFUSION

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Abstract. This paper is concerned with the time periodic problem to a coupled chemotaxis-fluid model with porous medium diffusion $\Delta n^m$. The global existence of solutions for the initial and boundary value problem of this model have been studied by many authors, and in particular, the global solvability is established for $m > \frac{6}{5}$ in dimension 3. Here, taking advantage of a double-level approximation scheme, we establish the existence of uniformly bounded time periodic solution for any $m \geq \frac{6}{5}$ and any large periodic source $g(x,t)$. In particular, the energy estimates techniques we used also applicable to the proof of global existence of the initial-boundary value problem, and one can supply the existence of global solutions for $m = \frac{6}{5}$ by this method.

1. Introduction. In this paper, we consider the following coupled chemotaxis-Stokes system,

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n \nabla c) + \mu n(1 - n) + g(x,t), \quad \text{in } Q, \\
  c_t + u \cdot \nabla c &= \Delta c - c + n, \quad \text{in } Q, \\
  u_t &= \Delta u - \nabla \pi + n \nabla \phi, \quad \text{in } Q, \\
  \nabla \cdot u &= 0, \quad \text{in } Q, \\
  \frac{\partial n^m}{\partial \nu} \bigg|_{\partial Q} &= \frac{\partial c}{\partial \nu} \bigg|_{\partial Q} = 0, \quad u|_{\partial Q} = 0,
\end{align*}
\]

where $m \geq \frac{6}{5}$, $Q = \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, herein, $n$ represents the density of the bacteria, $c$ denotes the concentration of the chemotactant, $u$ denotes the fluid velocity, and $\pi$ is the associated pressure, $g(x,t) \geq 0$ is a source term, that is the bacteria are fed periodically, $\nabla \phi(x,t)$ is the gravitational force. Here, we assume that $g$ and $\nabla \phi$ are time periodic functions with period $T$.

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The following coupled chemotaxis-fluid model with consumption was first proposed by Tuval, Goldstein et. al. in 2005,
\[
\begin{aligned}
&n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c), \\
&c_t + u \cdot \nabla c = \Delta c - cn, \\
&u_t + ku \cdot \nabla u = \Delta u - \nabla \pi + n \nabla \varphi, \\
&\nabla \cdot u = 0,
\end{aligned}
\]
which describes the dynamics of bacterial swimming and oxygen transport near contact lines. Since then, the coupled chemotaxis-fluid models have been studied by many authors. In two dimensional bounded domain, Winkler [20, 22] established the global existence and stability of classical solutions. Whereas, in three dimensional case, the result on global existence of classical solutions is not perfect, there are only some global existence results on small initial data [2, 15], and for the study of weak solutions, please refer to [12, 20, 23, 24]. Beside these results, there are also many papers that concerned with the chemotaxis-fluid model with production. That is the model (1.1) with \( m = 1 \). Espejo, Tao, et.al [6, 18] proved the global existence of weak solutions and classical solutions respectively in two dimensional settings. For the three dimensional case, in [17], Tao and Winkler established the global solvability of classical solutions for large \( \mu \), and in [21], a global weak solution is achieved for any positive \( \mu \). While as far as the porous medium diffusion case \( m > 1 \), only a few works involves this aspect of research. Liu, Wang[13] considered the following system in dimension 3
\[
\begin{aligned}
&n_t + u \cdot \nabla n = \Delta n^m - \chi \nabla \cdot (n(1 + n)^{-\alpha} \nabla c) + \mu n(1 - n), \\
&c_t + u \cdot \nabla c = \Delta c - c + n, \\
&u_t = \Delta u - \nabla \pi + n \nabla \varphi, \\
&\nabla \cdot u = 0,
\end{aligned}
\]
and proved the existence of global bounded weak solution for \( \alpha > 0, m \geq \frac{1}{3}, \alpha + m > \frac{9}{5} \). While, if consider the fluid free case, that is let \( u \equiv 0 \). Then the problem (1.1) is transformed into the pure chemotaxis system.
\[
\begin{aligned}
&n_t = \Delta n^m - \chi \nabla \cdot (n \nabla c) + \mu n(1 - n), \\
&c_t = \Delta c - c + n.
\end{aligned}
\]
When \( \mu = 0 \), it has been shown that when \( N \geq 2 \), there exists a threshold value \( m^* = \frac{2N - 2}{N} \) such that the solutions always exists globally when \( m > m^* \), and blow up solutions will be generated for some initial values when \( m \leq m^* \). Please refer to [3, 4, 5, 10] and the references therein for more details. While, when \( \mu > 0 \), although there is no direct research result on this model, there are some related works on the coupled chemotaxis-haptotaxis model. However, if we ignore the haptotaxis term, it happens to be the pure chemotaxis model above. Through the efforts of many researchers [16, 11, 19, 25], it finally proved the global solvability of bounded weak solutions for \( m > \frac{2N}{N + 2} \). However, the global existence of bounded weak solutions for the case \( m \leq \frac{2N}{N + 2} \) remains unknown.

While as far as the time periodic solutions are concerned, there are few works concerned. In 2017, Jin [7, 8] considered the time periodic problem for the linear diffusion case of (1.1) in dimension 2 and dimension 3, namely the case \( m = 1 \), and proved the existence of time periodic solutions. In the present paper, we pay our attention to the time periodic problem (1.1) in three dimensional case. We use a
double-level approximation scheme based on a fourth order regularized system. To obtain the compactness of the operator, we use the first-level fourth order regularized system to approach the original system. However, different from the second order parabolic system, there is no positivity for the fourth order regularized system. So, the $L^1$-norm estimate can not be obtained directly, which brings great difficulty to the later proof. To solve this problem, we add a term $\varepsilon |n|^s n$ to the left side of the first equation with $s$ appropriately large. And in the second level approximation, we use some iterative techniques to obtain some uniform energy estimates and finally showed the existence of time periodic solutions for any $m \geq \frac{6}{5}$.

In the second level approximation, the energy estimates techniques we used also applicable to the proof of global existence of the initial-boundary value problem, in particular, using this method, one can supply the existence of global solutions for $m = \frac{2N}{N+2}$.

More precisely, we have the following result.

**Theorem 1.1.** Assume $m \geq \frac{6}{5}, \ g \geq 0$ with $g, \nabla \varphi \in L^\infty(Q)$. Then the problem (1.1) admits a time periodic weak solution $(n, c, u, \pi)$ with $c \geq 0, \ n \geq 0$ and

$$\sup_t \{\|u\|_{H^1} + \|c\|_{H^1}^2 + \int_0^T (\|u\|_{H^2}^2 + \|\nabla c\|_{H^1}^2 + \|(u_t, c_t)\|_{L^2}^2) dt \leq C, \ (1.2)$$

$$\sup_t \{\|u\|_{L^\infty} + \|n\|_{L^\infty} + \|c\|_{W^{1,\infty}} \leq C, \ (1.3)$$

$$\sup_t \int_\Omega |\nabla n|^m dx + \int_0^T \int_\Omega n^{-1}|n_t|^2 dxdt + \int_0^T \int_\Omega n^{-2}\rho |\nabla n|^2 dxdt \leq C, \ (1.4)$$

where these constants $C$ only depend on $\mu, \Omega, g, \nabla \varphi, T$.

2. **Preliminaries.** For reader’s convenience, we give some notations, which will be used throughout this paper.

**Notations.** $H_0^k(\Omega) = \{u \in H^k(\Omega) : \nabla u = 0\}$, $\|\cdot\|_{L_p} := \|\cdot\|_{L_p(\Omega)}$, $f \in L_p^p(\mathbb{R}^+; \mathcal{X}) \iff f$ is a time periodic function with period $T$, and $f \in L_p^p((0, T); \mathcal{X})$.

To prove the existence of time periodic solutions, we begin with some preliminary lemmas.

By [9], we have the following two lemmas

**Lemma 2.1.** Let $T > 0, \ a > 0, \ \varrho \geq 0$, and suppose that $f : \mathbb{R}^+ \rightarrow [0, \infty)$ is absolutely continuous, $f, h$ are time periodic functions with period $T$, and $f$ satisfies

$$f(t) - f(t_0) + a \int_{t_0}^t f^{1+\varrho}(s) ds \leq \int_{t_0}^t h(s) ds, \ \text{for any} \ 0 \leq t_0 < t,$$

where $0 \leq f(t), h(t) \in L^1_T(0, T)$ and

$$\int_0^T h(s) ds \leq \beta.$$

Then

$$\sup_{t \in (0, T)} f(t) + a \int_0^T f^{1+\varrho}(t) dt \leq \left( \frac{\beta}{aT} \right)^{\frac{1}{1+\varrho}} + 2\beta.$$

**Lemma 2.2.** Let $T > 0, \ a > 0, \ \varrho \geq 0$, and suppose that $f : \mathbb{R}^+ \rightarrow [0, \infty)$ is absolutely continuous, $f, g, h$ are time periodic functions with period $T$, and satisfies

$$f(t) - f(t_0) + a \int_{t_0}^t f^{1+\varrho}(s) ds \leq \int_{t_0}^t g(s) f(s) ds + \int_{t_0}^t h(s) ds \ \text{for any} \ 0 \leq t_0 < t,$$
where \( g(t), h(t) \geq 0 \) with \( g(t), h(t) \in L^1_T(0, T) \) and
\[
\int_0^T g(s)ds \leq \alpha, \quad \int_0^T h(s)ds \leq \beta.
\]
Then
\[
\sup_{t \in (0, T)} f(t) + a \int_0^T f^{1+\rho}(t)dt \leq C,
\]
where \( C \) is a constant depending only on \( a, \alpha, \beta, T \). While, if \( a = 0 \), and
\[
\int_0^T f(s)ds \leq \gamma.
\]
We also have that
\[
\sup_{t \in (0, T)} f(t) \leq C,
\]
where \( C \) is a constant depending only on \( \gamma, \alpha, \beta, T \).

By [7, 9], we also have this lemma.

**Lemma 2.3.** Assume that \( f \in L^p_T(\mathbb{R}^+; L^p(\Omega)) \). Then the following problem
\[
\begin{cases}
u_t - \Delta u + u = f(x, t), \\ \partial u / \partial n |_{\partial \Omega} = 0.
\end{cases}
\]
admits a unique strong time periodic solution \( u \in W^{2,1}_{\rho}(Q_T) \) such that
\[
\int_0^T \|u\|^p_{W^{1,1}}ds \leq C \int_0^T \|f\|^p_{L^p}ds,
\]
where \( C \) is a positive constant.

The following lemma follows from [14].

**Lemma 2.4.** Assume that \( \Omega \) is bounded and let \( \omega \in C^2(\Omega) \) satisfy \( \partial \omega / \partial n |_{\partial \Omega} = 0 \). Then we get that
\[
\frac{\partial |\nabla \omega|^2}{\partial n} \leq 2\kappa |\nabla \omega|^2, \text{ on } \partial \Omega,
\]
where \( \kappa > 0 \) is an upper bound for the curvatures of \( \Omega \).

3. **Time periodic solutions for a fourth-order regularized problem.** In this paper, we use a double-level approximation scheme to show the existence of time periodic solutions. To obtain the compactness of the operator, in the first level, we use a fourth order regularized system as follows to approach the original system.
\[
\begin{cases}
\nu_t + u \cdot \nabla \nu + \delta \Delta^2 n - \Delta((n^2 + \varepsilon)^{\frac{n-1}{n}}n) + \varepsilon |n|^n n \\
= -\nabla \cdot (n \nabla c) + \mu |n|(1 - n) + g(x, t), \text{ in } Q, \\
c_t + u \cdot \nabla c = \Delta c - c + n_+, \text{ in } Q, \\
u_t = \Delta u - \nabla \pi + n \nabla \varphi, \text{ in } Q, \\
\nabla \cdot u = 0, \text{ in } Q, \\
\partial n / \partial v |_{\partial \Omega} = \frac{\partial \Delta n}{\partial v} |_{\partial \Omega} = \frac{\partial c}{\partial v} |_{\partial \Omega} = 0, u |_{\partial \Omega} = 0,
\end{cases}
\]
where $\max\{2(m-1), 4\} \leq s \leq 5m-1$. It is worth noticing that although the fourth order term can improve the regularity of solutions, there is no positivity for the fourth order regularized problem. Therefore, the most basic and natural $L^1$-norm estimate of $n$ is no longer valid, which brings essential difficulties to the later proof of energy estimates, see for example the proof of (3.9), (3.13) etc. For this reason, we introduce the term $\varepsilon |n|^4 n$ to solve the difficulties caused by the lack of positivity.

To prove the existence of time periodic solutions for the problem (3.1), we linearize this problem. For $u$, consider the following linear problem, for $u \in L^2((0,T), H^2(\Omega)) \cap L^2((0,T), H^2(\Omega))$, and $u_t \in L^2((0,T), L^2(\Omega))$.

\[
\begin{align*}
L_u - \Delta u - \nabla \pi = \varrho \hat{u} \nabla \varphi, \quad &\text{in } Q, \\
\nabla \cdot u = 0, \quad &\text{in } Q, \\
u|_{\partial \Omega} = 0, \quad &
\end{align*}
\]

(3.2)

where $\varrho \in [0,1]$ is a constant. Taking advantage of [7, 8], we have

**Lemma 3.1.** Assume that $\hat{u} \in L^2_{\tau}([0,1], L^2(\Omega))$. Then the problem (3.2) admits a time periodic solution $u \in L^\infty((0,T), H^2(\Omega)) \cap L^2((0,T), H^2(\Omega))$, and $u_t \in L^2((0,T), H^2(\Omega))$.

For the above obtained solution $u$, let’s consider the following linear problem.

\[
\begin{align*}
\sigma_t - \Delta c + u \cdot \nabla c + c = \varrho \hat{u} + c, \quad &\text{in } Q, \\
\frac{\partial c}{\partial n}|_{\partial \Omega} = 0. \quad &
\end{align*}
\]

(3.3)

The following lemmas follows from [7, 8].

**Lemma 3.2.** Assume that $\hat{u} \in L^2_{\tau}([0,1], H^1(\Omega))$ and let $u$ be the time periodic solution of the problem (3.2). Then the problem (3.3) admits a time periodic solution $c \in L^\infty((0,T), H^2(\Omega)) \cap L^2((0,T), H^2(\Omega))$, and $c_t \in L^2((0,T), L^2(\Omega))$.

For the above obtained solutions $u, c$, consider

\[
\begin{align*}
\sigma_t - \Delta ((\hat{u}^2 + \varepsilon \frac{m-1}{m} n) + \varepsilon |\hat{u}|^4 n + \delta \Delta^2 n + An \\
= -u \cdot \nabla n - \varrho \psi \cdot (\hat{u} \nabla c) + \varrho(\mu + A)|\hat{n}| - \mu |\hat{n}| n + \varrho g(x,t), \quad &\text{in } Q, \\
\frac{\partial n}{\partial n}|_{\partial \Omega} = \frac{\partial \Delta n}{\partial n}|_{\partial \Omega} = 0.
\end{align*}
\]

(3.4)

For the above linear parabolic problem, when $A$ is sufficiently large, the existence of time periodic solutions can be easily obtained by a fixed point method. That is, define a Poincaré map from $n(x,0)$ to $n(x,T)$, the time-periodic solution is then identified as a fixed point of this Poincaré map. We only give the regularity estimates.

For simplicity, in what follows, we may assume that the solution $n$ is sufficiently smooth, otherwise, we can approximate $u, c, \hat{u}, g$ with a sequence of sufficiently smooth functions $u_k, c_k, \hat{n}_k, g_k$ such that the corresponding solutions $n$ are sufficiently smooth, and the following energy estimates can be obtained through an approximate process.

**Lemma 3.3.** Assume that $g \in L^\infty(Q_T), \hat{n} \in L^\infty \cap L^2_{\tau}([0,1], H^1(\Omega)) \cap L^2_{\tau}([0,1], H^2(\Omega))$, \n
\[
\frac{\partial \hat{n}}{\partial n}|_{\partial \Omega} = 0, \quad \text{and } A \text{ is a sufficiently large constant. Let } u, c, n \text{ be the time periodic solution of the problems (3.2), (3.3) and (3.4) respectively. Then } n \in L^\infty((0,T), H^2(\Omega)) \cap L^2((0,T), H^4(\Omega)), \text{ and } n_t \in L^2((0,T), L^2(\Omega)).
Proof. Multiplying the first equation of (3.4) by $n$, integrating it over $\Omega \times (t_0,t)$ for any $t_0 < t \leq t_0 + T$, and using Lemma 3.2, when $A$ sufficiently large, we see that

$$
\frac{1}{2} \int_{t_0}^{t} \int_{\Omega} (|n(x,t)|^2 - |n(x,t_0)|^2)dx + \int_{t_0}^{t} \int_{\Omega} (\delta |\Delta n|^2 + \varepsilon |\dot{n}|^2 + \mu |\dot{n}| n^2 + An^2) \, dxds
$$

$$
= \int_{t_0}^{t} \int_{\Omega} \Delta n ((\dot{n}^2 + \varepsilon)^{\frac{m-1}{2}} n)dxds + \int_{t_0}^{t} \int_{\Omega} (\varrho \dot{n} \nabla \nabla n + \varrho (\mu + A)|\dot{n}| n + ggn) \, dxds,
$$

$$
\leq \frac{\delta}{4} \int_{t_0}^{t} \int_{\Omega} |\Delta n|^2 \, dxds + \int_{t_0}^{t} \int_{\Omega} (C \parallel n \parallel^2_{L^2} + \parallel \dot{n} \parallel L^2 \parallel \nabla n \parallel_{L^2} + C \parallel \dot{n} \parallel L_1) \, ds
$$

$$
\leq \frac{\delta}{2} \int_{t_0}^{t} \int_{\Omega} |\Delta n|^2 \, dxds + \frac{A}{2} \int_{t_0}^{t} \int_{\Omega} n^2 \, dxds + C,
$$

which means

$$
\int_{t_0}^{T} (|n(x,t)|^2 - |n(x,t_0)|^2)dx + \int_{t_0}^{t} \int_{\Omega} (\delta |\Delta n|^2 + An^2 + 2\varepsilon |\dot{n}|^2 + 2\mu |\dot{n}|^2) \, dxds \leq C.
$$

By Lemma 2.1, it follows

$$
\sup_t \int_{\Omega} n^2 \, dx + \int_{t_0}^{T} \parallel n \parallel_{H^2}^2 \, dt + A \int_{t_0}^{T} \parallel n \parallel_{L^2} \, dt \leq C. \tag{3.5}
$$

We multiply the first equation of (3.4) by $\Delta n$, and integrate it over $\Omega \times (t_0,t)$ with any $t_0 < t \leq t_0 + T$ to obtain

$$
\frac{1}{2} \int_{t_0}^{t} \int_{\Omega} (\nabla n(x,t))^2 - \nabla n(x,t_0))^2)dx + \delta \int_{t_0}^{t} \int_{\Omega} |\nabla \Delta n|^2 \, dxds + A \int_{t_0}^{t} \int_{\Omega} |\nabla n|^2 \, dxds
$$

$$
= \int_{t_0}^{t} \int_{\Omega} \Delta n \nabla ((\dot{n}^2 + \varepsilon)^{\frac{m-1}{2}} n)dxds
$$

$$
+ \int_{t_0}^{t} \int_{\Omega} (u \cdot \nabla n + \varrho \nabla \cdot (\hat{n} \nabla c) - \varrho (\mu + \mu \dot{n}|n| + \varepsilon |\dot{n}| n - \varrho \mu)) \Delta n \, dxds
$$

$$
= \int_{t_0}^{t} \int_{\Omega} \Delta n ((\dot{n}^2 + \varepsilon)^{\frac{m-1}{2}} n)dxds
$$

$$
+ \int_{t_0}^{t} \int_{\Omega} (u \cdot \nabla n + \varrho \nabla \cdot (\hat{n} \nabla c) - \varrho (\mu + \mu \dot{n}|n| + \varepsilon |\dot{n}| n - \varrho \mu)) \Delta n \, dxds
$$

$$
\leq \frac{\delta}{2} \int_{t_0}^{t} \parallel \Delta n \parallel_{L^2}^2 \, ds + C_\delta \int_{t_0}^{t} \parallel \nabla n \parallel_{L^2}^2 ((\dot{n}^2 + \varepsilon)^{\frac{m-1}{2}} \parallel L^\infty \, ds
$$

$$
+ C_\delta \int_{t_0}^{t} \int_{\Omega} n^2 (\dot{n}^2 + \varepsilon)^{m-2} |\nabla \dot{n}|^2 \, ds + \int_{t_0}^{t} \parallel u \parallel_{L^6} \parallel \nabla n \parallel_{L^3} \parallel \Delta n \parallel_{L^2} \, ds
$$

$$
\leq \int_{t_0}^{t} \parallel \Delta n \parallel_{L^2}^2 \, ds + C_\delta \int_{t_0}^{t} \parallel \nabla n \parallel_{L^2}^2 ((\dot{n}^2 + \varepsilon)^{\frac{m-1}{2}} \parallel L^\infty \, ds
$$

$$
+ C_\delta \int_{t_0}^{t} \parallel u \parallel_{H^1} \parallel \nabla n \parallel_{L^2}^2 \parallel \Delta n \parallel^3_{L^2} \, ds
$$
Combining with (3.5), and using Lemma 2.1, we get that
\[
\int_t^t \|\varrho \nabla \cdot (\hat{n} \nabla c) - \varrho (A + \mu) |\hat{n}| + \mu |\hat{n}| n + \varepsilon |\hat{n}|^* n - \varrho g\|_{L^2} \|\Delta n\|_{L^2} ds
\]
\[
\leq \frac{\delta}{2} \int_t^t \|\nabla \Delta n\|_{L^2}^2 ds + C \int_t^t \|\nabla n\|_{L^2}^2 ds + C \int_t^t \|n\|_{L^\infty}^2 ds + C \int_t^t \|\nabla n\|_{L^2} \|\Delta n\|_{L^2} ds
\]
\[
+ \int_t^t \|\varrho \nabla \cdot (\hat{n} \nabla c) - \varrho (A + \mu) |\hat{n}| + \mu |\hat{n}| n + \varepsilon |\hat{n}|^* n - \varrho g\|_{L^2} \|\Delta n\|_{L^2} ds
\]
\[
\leq \frac{\delta}{2} \int_t^t \|\nabla \Delta n\|_{L^2}^2 ds + C \int_t^t \|\nabla n\|_{L^2}^2 ds + C \int_t^t \|n\|_{L^\infty}^2 ds + C \int_t^t \|\nabla n\|_{L^2} \|\Delta c\|_{L^2}^2 ds
\]
\[
+ C \int_t^t \|\nabla \hat{n}\|_{x,t}^2 \|\nabla c\|_{x,t}^2 + |\hat{n}|_{x,t}^2 + |\hat{n}|_{x,t}^2 \|n\|_{L^\infty}^2 + |\hat{n}|_{x,t}^2 \|n\|_{x,t}^2 + \|g\|_{L^2}^2 ds
\]
\[
\leq \frac{\delta}{2} \int_t^t \|\nabla \Delta n\|_{L^2}^2 ds + C \int_t^t \|\nabla n\|_{L^2}^2 ds + C \int_t^t (|\Delta c\|_{L^2}^2 + \|\nabla \hat{n}\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + 1) ds
\]
\[
\leq \frac{\delta}{2} \int_t^t \|\nabla \Delta n\|_{L^2}^2 ds + C \int_t^t \|\nabla n\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \|\Delta \hat{n}\|_{L^2}^2 + 1) ds,
\]
which means
\[
\frac{1}{2} \int_\Omega |\nabla n(x,t)|^2 dx + \frac{\delta}{2} \int_t^t \int_\Omega |\nabla \Delta n|^2 dx ds + A \int_t^t \int_\Omega |\nabla n|^2 dx ds
\leq C \int_t^t (\|n\|_{H^1}^2 + \|\Delta c\|_{L^2}^2 + \|\Delta \hat{n}\|_{L^2}^2 + 1) dt.
\]
Combining with (3.5), and using Lemma 2.1, we get that
\[
\sup_t \|n\|_{H^1}^2 + \int_0^T \int_\Omega |\nabla n|^2 dx ds \leq C \int_0^T (\|n\|_{H^2}^2 + \|\Delta c\|_{L^2}^2 + \|\Delta \hat{n}\|_{L^2}^2 + 1) dt \leq \tilde{C}.
\]
\[
\leq C \int_{t_0}^{t} \| \Delta^2 n \|_{L^2} \left( \| \Delta n \|_{L^2} + \left\| 2(m-1)(\dot{n}^2 + \varepsilon) \frac{m-1}{2} \dot{n} \nabla \n n \right\|_{L^2} \right) ds \\
+ C \int_{t_0}^{t} \| \Delta^2 n \|_{L^2} |n|_{L^\infty} \left( \left| \Delta(\dot{n}^2 + \varepsilon) \frac{m-1}{2} \dot{n} \nabla \n n \right|_{L^2} + (m-1) |n|_{L^\infty} \right) ds \\
+ C \int_{t_0}^{t} (1 + |\nabla \dot{n}|_{H^1}) \| \Delta^2 n \|_{L^2} ds \\
\leq C \int_{t_0}^{t} \| \Delta^2 n \|_{L^2} \left( \| \Delta n \|_{L^2} + \left\| 2(m-1)(\dot{n}^2 + \varepsilon) \frac{m-1}{2} \dot{n} \nabla \n n \right\|_{L^2} + (m-1) |n|_{L^\infty} \right) ds \\
+ C \int_{t_0}^{t} |\nabla n|_{H^1} \| \Delta^2 n \|_{L^2} ds + C \int_{t_0}^{t} (1 + |\nabla \dot{n}|_{H^1}) \| \Delta^2 n \|_{L^2} ds \\
\leq C \int_{t_0}^{t} (1 + |n|_{L^2}^2 + |\nabla n|_{L^2}^2 + |\nabla n|_{L^\infty}^2 + |\Delta \n n|_{L^2}^2 + |\Delta \dot{n}|_{L^2}^2 + |\Delta n|_{L^\infty}^2) ds + \frac{\delta}{4} \int_{t_0}^{t} \| \Delta^2 n \|_{L^2}^2 ds \\
\leq \frac{\delta}{2} \int_{t_0}^{t} \| \Delta^2 n \|_{L^2}^2 ds + C \int_{t_0}^{t} \left( |\nabla \dot{n}|_{H^1}^2 + |\Delta \dot{n}|_{L^2}^2 + |\Delta n|_{L^2}^2 \right) ds.
\]

which means

\[
\int_{\Omega} |\Delta n(x, t)|^2 dx - \int_{\Omega} |\Delta n(x, t_0)|^2 dx + \delta \int_{t_0}^{t} \int_{\Omega} |\Delta^2 n|^2 dx ds + A \int_{t_0}^{t} \int_{\Omega} |\Delta n|^2 dx ds \\
\leq \int_{t_0}^{t} \left( |\nabla \dot{n}|_{H^1}^2 + |\Delta \dot{n}|_{L^2}^2 + |\Delta n|_{L^2}^2 \right) ds.
\]

Using Lemma 2.1 yields

\[
\sup_t \int_{\Omega} |\Delta n|^2 dx + \int_0^T \int_{\Omega} |\Delta^2 n|^2 dx dt \leq C + C \sup_t |\Delta n|_{L^2} \\
\leq \frac{1}{2} \sup_t \int_{\Omega} |\Delta n|^2 dx + C,
\]

combining with (3.6), we finally get

\[
\sup_t \int_{\Omega} (n^2 + |\nabla n|^2 + |\Delta n|^2) dx + \int_0^T |n|_{H^1}^2 dt \leq C. \tag{3.7}
\]
Similar to the proof of the above formula, multiplying the first equation of (3.4) by \( n_t \) and integrating it over \( \Omega \), it follows
\[
\int_0^T \|n_t\|_{L^2}^2 \, dt \leq C. \tag{3.8}
\]

Summing up, we complete the proof. \( \square \)

Below we consider the existence of time periodic solutions to the problem (3.1).

Denote
\[
X = L^\infty_T \cap L^2_T ((0, \infty), H^1(\Omega)) \cap L^2_T ((0, T), H^2(\Omega)), \quad X_R = \{ n \in X; ||n||_X < R \},
\]
where
\[
||n||_X = \sup_t (||n(\cdot, t)||_{L^\infty} + ||n(\cdot, t)||_{H^1}) + \left( \int_0^T \|n(\cdot, t)\|_{H^1}^2 \, dt \right)^{\frac{1}{2}}.
\]

Define a map
\[
\mathcal{F} : X_R \times [0, 1] \to X,
\]
\[
\mathcal{F}(\hat{n}, g) = n,
\]
where \( n \) is the solution of the problem (3.8). Next, we use Leray-Schauder’s fixed point theorem to prove the existence of time periodic solutions to the problem (3.1).

For this purpose, we first give the below priori estimate.

**Lemma 3.4.** Assume that \( s > \max\{2(m-1), 4\} \), \( g \in L^\infty(Q_T), \nabla \varphi \in L^\infty_T(\Omega \times \mathbb{R}) \), and let \( \mathcal{F}(n, g) = n \). Then there exists \( R > 0 \) such that
\[
||n||_X \leq R,
\]
where \( R \) depends on \( \varepsilon, \delta \), and is independent of \( A \).

**Proof.** Taking \( \hat{n} = n \) in (3.4). Multiplying the first equation of (3.4) by \( n \), and integrating it over \( \Omega \times (t_0, t) \) for any \( t_0 < t \leq t_0 + T \), we see that
\[
\frac{1}{2} \int_\Omega (|n(x, t)|^2 - |n(x, t_0)|^2) \, dx + \int_{t_0}^t \int_\Omega (\delta|\Delta n|^2 + An^2 + \varepsilon |n|^{s+2} + \mu |n|^3
\]
\[
+ (n^2 + \varepsilon) \frac{m-1}{2} |\nabla n|^2) \, dx \, ds
\]
\[
= \frac{\alpha}{2} \int_{t_0}^t \int_\Omega n^2 \Delta x \, dx \, ds + g(A + \mu) \int_{t_0}^t \int_\Omega n^2 \, dx \, ds + \frac{\alpha}{4} \int_{t_0}^t \int_\Omega g^n \, dx \, ds
\]
\[
\leq \int_{t_0}^t \left( \frac{\alpha}{4} ||n||_{L^4}^4 + \frac{\alpha}{4} ||\Delta c||_{L^2}^2 + gA ||n||_{L^2}^2 + \frac{\mu}{2} ||n||_{L^3}^3 + gC \right) \, ds,
\]
which implies that
\[
\frac{1}{2} \int_\Omega (|n(x, t)|^2 - |n(x, t_0)|^2) \, dx + \int_{t_0}^t \int_\Omega (\delta|\Delta n|^2 + \varepsilon |n|^{s+2} + \frac{\mu}{2} |n|^3
\]
\[
+ (n^2 + \varepsilon) \frac{m-1}{2} |\nabla n|^2) \, dx \, ds \leq \frac{\alpha}{4} \int_{t_0}^t ||n||_{L^4}^4 \, ds + \frac{\alpha}{4} \int_{t_0}^t ||\Delta c||_{L^2}^2 \, ds + gC. \tag{3.9}
\]

Taking \( \hat{n} = n \) in (3.2) and multiplying the corresponding equation by \( 2u, 2\Delta u \) respectively, then combining the two inequalities, it is easy to see that
\[
\int_\Omega (|u(x, t)|^2 + |\nabla u(x, t)|^2) \, dx - \int_\Omega (|u(x, t_0)|^2 + |\nabla u(x, t_0)|^2) \, dx
\]
\[
\int_0^t \int_\Omega (|\nabla u|^2 + |\Delta u|^2) dx ds \leq C \int_0^t \int n^2 dx ds,
\]
which implies that
\[
\sup_{t \in (0,T)} \int_\Omega (u^2 + |\nabla u|^2) dx + \int_0^T \int_\Omega (|\nabla u|^2 + |\Delta u|^2) dx dt \leq C \int_0^T \int n^2 dx dt. \tag{3.10}
\]
Taking \(n = n\) in (3.3), and multiplying this equation by \(c\), \(\Delta c\) respectively, and integrating them over \(\Omega \times (t_0, t)\) for any \(t \geq t_0\), we have
\[
\frac{1}{2} \int_\Omega (|c(x,t)|^2 - |c(x,t_0)|^2) dx + \int_0^t \int_\Omega (|\nabla c|^2 + c^2) dx ds
\leq \frac{1}{2} \int_{t_0}^t \int_\Omega c^2 dx ds + \frac{1}{2} \int_{t_0}^t \int \nabla c \cdot \nabla c dx ds,
\]
which implies that
\[
\sup_{t \in (0,T)} \int_\Omega c^2 dx + \int_0^T \int_\Omega |\nabla c|^2 dx dt \leq C \int_0^T \int n^2 dx dt, \tag{3.11}
\]
and
\[
\frac{1}{2} \int_\Omega (|\nabla c(x,t)|^2 dx - \frac{1}{2} \int_\Omega (|\nabla c(x,t)|^2 dx + \frac{1}{2} \int_0^t \int_\Omega (|\Delta c|^2 + |\nabla c|^2) dx ds
\leq \frac{1}{2} \int_0^t \int_\Omega u \cdot \nabla c \Delta c dx ds + \frac{1}{2} \int_0^t \int_\Omega |\Delta c|^2 dx ds + \frac{1}{2} \int_0^t \int_\Omega n^2 dx ds
\leq \int_0^t \sup_{t \in (0,T)} \|u\|_{L^s} \|\nabla c\|_{L^1} \|\Delta c\|_{L^2} ds + \frac{1}{4} \int_0^t \int_\Omega |\Delta c|^2 dx ds + \int_0^t \int_\Omega n^2 dx ds
\leq \int_0^t \sup_{t \in (0,T)} \|u\|_{H^1} \left( C \|\nabla c\|_{L^2}^2 \|\Delta c\|_{L^2}^2 + C \|u\|_{H^1} \|\nabla c\|_{L^2}^2 \right) ds
+ \frac{1}{4} \int_0^t \int_\Omega |\Delta c|^2 dx ds + \int_0^t \int_\Omega n^2 dx ds
\leq \int_0^t \left( C \|u\|_{H^1} \|\nabla c\|_{L^2}^2 + C \|u\|_{H^1} \|\nabla c\|_{L^2}^2 \right) ds + \frac{1}{2} \int_0^t \int_\Omega |\Delta c|^2 dx ds + \int_0^t \int_\Omega n^2 dx ds.
\]
Combining (3.9) and (3.10), and taking advantage of Lemma 2.1, we obtain that
\[
\sup_{t \in (0,T)} \int_\Omega |\nabla c|^2 dx + \int_0^T \int_\Omega |\Delta c|^2 dx dt \leq C \left( \int_0^T \int n^2 dx dt + \left( \int_0^T \int n^2 dx dt \right)^3 \right), \tag{3.12}
\]
Recalling (3.9), using (3.12), and noticing that \(s > 4\), we arrive at
\[
\sup_{t \in (0,T)} \int_\Omega n^2 dx + \delta \int_0^T \int_\Omega |\Delta n|^2 dx dt + \varepsilon \int_0^T \int_\Omega |n|^{s+2} dx dt + \frac{\mu}{2} \int_0^T \int_\Omega |n|^3 dx dt
+ \int_0^T \int_\Omega (n^2 + \varepsilon)^{\frac{s+1}{2}} |\nabla n|^2 dx dt
\leq C \int_0^T \int_\Omega |n|^4 dx dt + C \int_0^T \int_\Omega |\Delta c|^2 dx dt + C.
\]
\[
\leq C \left( 1 + \int_0^T \int (n^2 + n^6) dx dt \right) \\
\leq \mu \int_0^T \int_\Omega |n|^3 dx dt + \frac{\varepsilon}{2} \int_0^T \int_\Omega |n|^{s+2} dx dt + C \varepsilon,
\]

which implies

\[
\sup_{t \in (0, T)} \int_\Omega n^2 dx + \delta \int_0^T \int_\Omega |\Delta n|^2 dx dt + \frac{\varepsilon}{2} \int_0^T \int_\Omega |n|^{s+2} dx dt + \frac{\mu}{4} \int_0^T \int_\Omega |n|^3 dx dt \\
+ \int_0^T \int_\Omega (n^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla n|^2 dx dt \leq C \varepsilon,
\]

where \( C \varepsilon \) is independent of \( \delta \) and \( A \), but depends on \( \varepsilon \). By (3.13), recalling Lemma 3.2, we also have

\[
\sup_{t \in (0, T)} \int_\Omega (|c|^2 + |\nabla c|^2 + |\Delta c|^2) dx + \int_0^T \int_\Omega |\nabla c|^2 dx dt \leq C \varepsilon. \tag{3.14}
\]

Here \( C \varepsilon \) is also independent of \( \delta \) and \( A \), but depends on \( \varepsilon \). Taking \( \hat{n} = n \) in (3.4). Multiplying the first equation of (3.4) by \(-\Delta n\), integrating it over \( \Omega \times (t_0, t) \) for any \( t_0 < t \leq t_0 + T \), and using (3.10) and (3.13), noticing that \( s > 2(m - 1) \), then we see that

\[
\frac{1}{2} \int_\Omega \left( (|\nabla n(x, t)|^2 - |\nabla n(x, t_0)|^2) \right) dx + \int_{t_0}^t \int_\Omega \left( \delta |\nabla n|^2 + (s + 1) \varepsilon |n|^s |\nabla n|^2 \right) dx dt \\
+ A|\nabla n|^2 + 2\mu |n| |\nabla n|^2 \right) dx ds \\
= \int_{t_0}^t \int_\Omega \left( \nabla \Delta n \nabla ((n^2 + \varepsilon)^{\frac{m-1}{2}} n) + u \cdot \nabla n \Delta n + \varrho \nabla \cdot (n \nabla c) \Delta n - \varrho (A + \mu) |n| |\nabla n|^2 \right) dx ds + \varrho (A + \mu) \int_{t_0}^t \int_\Omega \frac{n}{|n|^2} |\nabla n|^2 dx ds + \int_{t_0}^t \int_\Omega |\nabla n|^2 dx ds \\
+ \frac{1}{2\delta} \int_{t_0}^t \int_\Omega \left( (m-1) (n^2 + \varepsilon)^{\frac{m-3}{2}} n^2 \nabla n + (n^2 + \varepsilon) \frac{m-1}{2} \cdot \nabla n \right) dx ds \\
+ \frac{\delta}{2} \int_{t_0}^t \int_\Omega |\nabla n|^2 dx ds + \int_{t_0}^t \int_\Omega |\nabla n|^2 dx ds + \frac{\delta}{2} \int_{t_0}^t \int_\Omega |\nabla n|^2 dx ds \\
+ \frac{\delta}{2} \int_{t_0}^t \int_\Omega \left( |\nabla n|^2 + \varrho |\nabla n|^2 + \varrho (A + \mu) |\nabla n|^2 \Delta n + |\nabla n|^2 \Delta n \right) dx ds \\
\leq \frac{\mu}{4} \int_0^T \int_\Omega |n|^3 dx dt + \frac{\varepsilon}{2} \int_0^T \int_\Omega |n|^{s+2} dx dt + C \varepsilon,
\]

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+ C\int_{t_0}^{t} \|u\|_{H^1} \|\nabla n\|_{L^2}^2 ds + \int_{t_0}^{t} (g(A + \mu)\|\nabla n\|_{L^2}^2 + \|g\|_{L^2} \|\Delta n\|_{L^2}) ds
\leq C\int_{t_0}^{t} \int_{\Omega} n^2(2-m-1)\|\nabla n\|_{L^2}^2 dx ds + C\int_{t_0}^{t} \int_{\Omega} |\nabla n|^2 dx ds + \frac{\delta}{2} \int_{t_0}^{t} \int_{\Omega} |\nabla \Delta n|^2 dx ds
+ \int_{t_0}^{t} \left(\|\nabla c\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + C\|\nabla n\|_{L^2}^2 + C\|\Delta n\|_{L^2}^2 + \rho(A + \mu)\|\nabla n\|_{L^2}^2\right) ds
+ \int_{t_0}^{t} \|g\|_{L^2} \|\Delta n\|_{L^2} ds
\leq \frac{(s+1)\varepsilon}{2} \int_{t_0}^{t} \int_{\Omega} |n|^2 |\nabla n|^2 dx ds + C_{c\varepsilon} \int_{t_0}^{t} \int_{\Omega} |\nabla n|^2 dx ds + \frac{\delta}{2} \int_{t_0}^{t} \int_{\Omega} |\nabla \Delta n|^2 dx ds
+ \int_{t_0}^{t} \left(A\|\nabla n\|_{L^2}^2 + C\|\nabla n\|_{L^2}^2 + C_{c\varepsilon} + \|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2\right) ds.

By (3.14), and we see that

\|\nabla \Delta n\|_{L^2}^2 \leq \|n\|_{L^3}^2 \|\Delta c\|_{L^6}^2 \leq C\|n\|_{L^2}^2 \|\nabla n\|_{L^2} \|\Delta c\|_{L^6}^2 + C\|n\|_{L^2}^2 \|\Delta c\|_{H^1}^2,
\leq C\|\nabla n\|_{L^2}^2 \|\Delta c\|_{H^1}^2 + C\|\Delta c\|_{H^1}^2,

\|\nabla n\|_{L^2}^2 \leq \|\nabla n\|_{L^2}^2 \|\nabla c\|_{L^6}^2 \leq \|\nabla n\|_{L^2}^2 \|\nabla c\|_{H^1}^2 \leq C\|\nabla n\|_{H^1}^2.

Combining the above three inequalities, it gives

\frac{1}{2} \int_{\Omega} \left(|\nabla n(x, t)|^2 - |\nabla n(x, t_0)|^2\right) dx + \frac{\delta}{2} \int_{t_0}^{t} \int_{\Omega} |\nabla \Delta n|^2 dx ds
+ \frac{s+1}{2} \varepsilon \int_{t_0}^{t} \int_{\Omega} |n|^2 |\nabla n|^2 dx ds + 2\mu \int_{t_0}^{t} \int_{\Omega} |n| |\nabla n|^2 dx ds
\leq C_{c\varepsilon} \int_{t_0}^{t} \|\nabla n\|_{H^1}^2 ds + \int_{t_0}^{t} \|\nabla n\|_{L^2}^2 \|\Delta c\|_{H^1}^2 ds + C \int_{t_0}^{t} \|\Delta c\|_{H^1}^2 ds + C_{c\varepsilon}.

Therefore by Lemma 2.2, (3.13) and (3.14), it gives

\sup_{T} \int_{\Omega} |\nabla n|^2 dx + \int_{t_0}^{t} \int_{\Omega} |\nabla \Delta n|^2 dx dt + \varepsilon \int_{t_0}^{t} \int_{\Omega} |n|^2 |\nabla n|^2 dx dt
+ \mu \int_{t_0}^{t} \int_{\Omega} |n| |\nabla n|^2 dx dt \leq C_{c\varepsilon},  \quad (3.15)

where \(C_{c\varepsilon}\) depends on \(\varepsilon\) and \(\delta\), and independent of \(A\). By (3.13) and (3.15), we complete the proof. \(\square\)

By Lemma 3.3, we see that \(F\) is a compact operator. Furthermore, by (3.9), it is easy to see that

\(F(\hat{n}, 0) = 0.\)

By Lemma 3.4, we see that if \(F(n, \hat{\varphi}) = n\), then there exists a constant \(M \geq 0\), such that

\(\|n\|_{X} \leq M,\)

where \(M\) depends on \(\varepsilon, \delta\). Then by Leray-Schauder’s fixed point theorem, the mapping \(F(\cdot, 1)\) has a fixed point, that is there exists \(n \in X\) such that \(F(n, 1) = n\), that is the problem (3.1) admits a solution \((n, c, u)\). Furthermore, taking advantage of Lemma 3.3, we have the following proposition.
Proposition 3.1. Assume that \( s > \max\{2(m-1), 4\} \), and \( g, \nabla \phi \in L^\infty(\Omega) \). Then the problem (3.1) admits a time periodic solution \( (n, c, u, \pi) \), with \( u \in L^\infty((0, T), H^1(\Omega)) \cap L^2((0, T), L^2(\Omega)), u_0 \in L^2((0, T), L^2(\Omega)), \pi \in L^2((0, T), H^1(\Omega)) \), \( \pi \in L^\infty((0, T), H^2(\Omega)) \cap L^2((0, T), H^2(\Omega)) \). \( c \in L^2((0, T), L^2(\Omega)), c_0 \in L^2((0, T), L^2(\Omega)) \); \( n \in L^\infty((0, T), H^2(\Omega)) \cap L^2((0, T), H^2(\Omega)) \), \( n_0 \in L^2((0, T), L^2(\Omega)) \).

4. Existence of time periodic solutions: Double-level approximation. In this section, we use a double-level approximation scheme to prove the existence of time periodic solutions of the problem (1.1). We consider the first level approximation by letting \( \delta \to 0 \). For this purpose, we first give some energy inequalities. Taking advantage of (3.10)-(3.14), we have

**Lemma 4.1.** Assume that \( s > \max\{2(m-1), 4\} \), \( g, \nabla \phi \in L^\infty(\Omega) \), and let \( (n_{\delta}, c_{\delta}, u_{\delta}, \pi_{\delta}) \) be the periodic solution of the problem (3.1). Then we have that

\[
\sup_t \{ \| n_{\delta} \|^2_{H^1} + \| c_{\delta} \|^2_{H^2} \} + \int_0^T \left( \| n_{\delta} \|^2_{H^2} + \| \pi_{\delta} \|^2_{H^1} + \| c_{\delta} \|^2_{H^2} + \| (u_t, c_t) \|^2_{L^2} \right) dt \leq C, \tag{4.1}
\]

\[
\sup_{t \in (0, T)} \int_\Omega n_{\delta}^2 dx + \delta \int_0^T \int_\Omega |\Delta n_{\delta}|^2 dx dt + \int_0^T \int_\Omega \left( \frac{\varepsilon}{2} |n_{\delta}|^{s+2} + \frac{\mu}{4} |n_{\delta}|^3 \right) dx dt 
+ \int_0^T \int_\Omega (n_{\delta}^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla n_{\delta}|^2 dx dt \leq C. \tag{4.2}
\]

Here \( C \) is a constant not depending on \( \delta \), only depending on \( \Omega, T, g, \nabla \phi, \) and \( \varepsilon \).

**Lemma 4.2.** Assume that \( s > \max\{2(m-1), 4\} \), \( g, \nabla \phi \in L^\infty(\Omega) \), and let \( (n_{\delta}, c_{\delta}, u_{\delta}, \pi_{\delta}) \) be the periodic solution of the problem (3.1). Then

\[
\sup_t \int_\Omega \left( \delta |\nabla n_{\delta}|^2 + (n_{\delta}^2 + \varepsilon)^{\frac{m+1}{2}} \right) dx + \int_0^T \int_\Omega \left| \nabla \Delta n_{\delta} - \nabla (n_{\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\delta} \right|^2 dx dt 
+ \int_0^T \int_\Omega (n_{\delta}^2 + \varepsilon)^{\frac{m-1}{2}} (\varepsilon |n_{\delta}|^{s+2} + \mu |n_{\delta}|^3) dx dt \leq C, \tag{4.3}
\]

where \( C \) is a constant independent of \( \delta \), and which only depends on \( \Omega, T, g, \nabla \phi, \) and \( \varepsilon \). Recalling the first equation of (3.1), if \( s \leq 5m-1 \), then we also have \( n_t \in L^2((0, T); H^{-1}(\Omega)) \).

**Proof.** Multiplying the first equation of (3.1) by \( -\delta \Delta n_{\delta} + (n_{\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\delta} \), and using (4.1) and (4.2), we obtain

\[
\frac{\delta}{2} \int_\Omega (|\nabla n_{\delta}(x, t)|^2 - |\nabla n_{\delta}(x, t_0)|^2) dx 
+ \frac{1}{m+1} \int_\Omega \left( (|n_{\delta}(x, t)|^2 + \varepsilon)^{\frac{m+1}{2}} - (|n_{\delta}(x, t_0)|^2 + \varepsilon)^{\frac{m+1}{2}} \right) dx 
+ \int_{t_0}^t \int_\Omega \left| \delta \nabla \Delta n_{\delta} - \nabla (n_{\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\delta} \right|^2 + \varepsilon \mu |n_{\delta}|^{s+1} |\nabla n_{\delta}|^2 dx ds 
+ \int_{t_0}^t \int_\Omega (n_{\delta}^2 + \varepsilon)^{\frac{m-1}{2}} (\varepsilon |n_{\delta}|^{s+2} + \mu |n_{\delta}|^3) + 2\mu \delta |n_{\delta}|^2 |\nabla n_{\delta}|^2 dx ds 
= \delta \int_{t_0}^t \int_\Omega n_{\delta} \nabla n_{\delta} \Delta n_{\delta} dx ds + \int_{t_0}^t \int_\Omega \nabla c_{\delta} \left( \delta \nabla \Delta n_{\delta} - \nabla (n_{\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\delta} \right) dx ds.
\]
\[ + \int_{t_0}^t \int_{\Omega} (\mu |n_{\varepsilon\delta}| + g) \left( -\delta \Delta n_{\varepsilon\delta} + (n_{\varepsilon\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\varepsilon\delta} \right) dxds \]
\[ \leq \delta \int_{t_0}^t \|u_{\varepsilon\delta}\|_{L^6} \|\nabla n_{\varepsilon\delta}\|_{L^3} \|\Delta n_{\varepsilon\delta}\|_{L^2} ds + \int_{t_0}^t \|n_{\varepsilon\delta}\|_{L^2} ds \]
\[ + \frac{1}{4} \int_{t_0}^t \int_{\Omega} \left| \delta \nabla \Delta n_{\varepsilon\delta} - \nabla (n_{\varepsilon\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\varepsilon\delta} \right|^2 dxds + \frac{1}{2} \int_{t_0}^t \int_{\Omega} |n_{\varepsilon\delta}|^3 dxds \]
\[ + \delta \int_{t_0}^t \left( |n_{\varepsilon\delta}| + g \right)^2 dxds + \frac{\mu}{2} \int_{t_0}^t \int_{\Omega} (n_{\varepsilon\delta}^2 + \varepsilon)^{\frac{m-1}{2}} |n_{\varepsilon\delta}|^3 dxds + C \]
\[ \leq \int_{t_0}^t \left( \delta \|u_{\varepsilon\delta}\|_{H^1} \|n_{\varepsilon\delta}\|_{L^2} \|\Delta n_{\varepsilon\delta}\|_{L^2} + \|n_{\varepsilon\delta}\|_{L^2} \|\nabla c_{\varepsilon\delta}\|_{L^6}^2 \right) ds \]
\[ + \frac{1}{4} \int_{t_0}^t \int_{\Omega} \left| \delta \nabla \Delta n_{\varepsilon\delta} - \nabla (n_{\varepsilon\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\varepsilon\delta} \right|^2 dxds \]
\[ + \delta \int_{t_0}^t \left( \|n_{\varepsilon\delta}\|_{L^2} + |n_{\varepsilon\delta}|^3 \right) dxds \]
\[ \leq \int_{t_0}^t \left( 2\|n_{\varepsilon\delta}\|_{L^2} + \|n_{\varepsilon\delta}\|_{L^3}^3 \right) ds \]
\[ + \frac{1}{4} \int_{t_0}^t \int_{\Omega} \left| \delta \nabla \Delta n_{\varepsilon\delta} - \nabla (n_{\varepsilon\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\varepsilon\delta} \right|^2 dxds \]
\[ + \frac{\mu}{2} \int_{t_0}^t \int_{\Omega} (n_{\varepsilon\delta}^2 + \varepsilon)^{\frac{m-1}{2}} |n_{\varepsilon\delta}|^3 dxds + C_1 \]
which implies that
\[ \frac{\delta}{2} \int_{\Omega} \left( |\nabla n_{\varepsilon\delta}(x,t)|^2 - |\nabla n_{\varepsilon\delta}(x,t_0)|^2 \right) x \]
\[ + \frac{1}{m+1} \int_{\Omega} \left( (|n_{\varepsilon\delta}(x,t)|^2 + \varepsilon)^{\frac{m+1}{2}} - (|n_{\varepsilon\delta}(x,t_0)|^2 + \varepsilon)^{\frac{m+1}{2}} \right) dx \]
\[ + \int_{t_0}^t \int_{\Omega} \left( \frac{3}{4} \left| \delta \nabla \Delta n_{\varepsilon\delta} - \nabla (n_{\varepsilon\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\varepsilon\delta} \right|^2 + \varepsilon \delta (s+1) |n_{\varepsilon\delta}|^3 |\nabla n_{\varepsilon\delta}|^2 \right) dxds \]
\[ + \frac{1}{2} \int_{t_0}^t \int_{\Omega} (n_{\varepsilon\delta}^2 + \varepsilon)^{\frac{m-1}{2}} (\varepsilon |n_{\varepsilon\delta}|^{s+2} + \mu |n_{\varepsilon\delta}|^3) dxds + 2\mu \delta \int_{t_0}^t \int_{\Omega} |n_{\varepsilon\delta}| |\nabla n_{\varepsilon\delta}|^2 dxds \]
\[ \leq 2\delta \int_{t_0}^t \|n_{\varepsilon\delta}\|_{L^2}^2 dxds + \int_{t_0}^t |n_{\varepsilon\delta}|^3 ds + C_2. \]

Using (4.2) and Lemma 2.2 to obtain (4.3). Recalling the first equation of (3.1), and by (4.1)-(4.3), we see that \(u_{\varepsilon\delta} n_{\varepsilon\delta}, n_{\varepsilon\delta} \nabla c_{\varepsilon\delta} \in L^2(Q_T), \delta \nabla n_{\varepsilon\delta} - \nabla (n_{\varepsilon\delta}^2 + \varepsilon)^{\frac{m-1}{2}} n_{\varepsilon\delta} \in L^2(Q_T),\mu |n|(1-n) + g(x,t) \in L^2(Q_T),\) and if \(s \leq 5m - 1, \varepsilon |n_{\varepsilon\delta}|^{s+1} \in\)
for any $\phi, \psi, g, L(\Omega))$. \hfill \Box

By Lemma 4.1, we have
\[- \iint_{Q_T} n_{\varepsilon \delta} \phi_t \, dx \, dt + \iint_{Q_T} u_{\varepsilon \delta} \nabla n_{\varepsilon \delta} \phi \, dx \, dt + \delta \iint_{Q_T} \Delta n_{\varepsilon \delta} \Delta \phi \, dx \, dt \]
\[- \iint_{Q_T} (n^2_{\varepsilon \delta} + \varepsilon) \frac{\varepsilon}{n^2} \, n_{\varepsilon \delta} \Delta \phi \, dx \, dt + \varepsilon \iint_{Q_T} |n_{\varepsilon \delta}|^s n_{\varepsilon \delta} \phi \, dx \, dt - \iint_{Q_T} n_{\varepsilon \delta} \nabla c_{\varepsilon \delta} \nabla \phi \, dx \, dt \]
\[= \mu \iint_{Q_T} |n_{\varepsilon \delta}| (1 - n_{\varepsilon \delta}) \phi \, dx \, dt + \iint_{Q_T} g \phi \, dx \, dt, \]
\[\iint_{Q_T} \frac{\partial c_{\varepsilon \delta}}{\partial t} \psi \, dx \, dt + \iint_{Q_T} u_{\varepsilon \delta} \nabla c_{\varepsilon \delta} \psi \, dx \, dt - \iint_{Q_T} \Delta c_{\varepsilon \delta} \psi \, dx \, dt \]
\[+ \iint_{Q_T} (c_{\varepsilon \delta} - n_{\varepsilon \delta +}) \psi \, dx \, dt = 0, \]
\[\iint_{Q_T} \frac{\partial u_{\varepsilon \delta}}{\partial t} \psi \, dx \, dt - \iint_{Q_T} \Delta u_{\varepsilon \delta} \psi \, dx \, dt + \iint_{Q_T} \nabla \pi_{\varepsilon \delta} \psi \, dx \, dt - \iint_{Q_T} n_{\varepsilon \delta} \nabla \varphi \psi = 0 \]
for any $\phi, \psi, \tilde{\psi} \in C^\infty(Q)$ with $\frac{\partial \tilde{\psi}}{\partial \nu}|_{\partial Q} = 0$. By (4.2), we note that
\[||\delta \Delta n_{\varepsilon \delta}||_{L^2(Q_T)} \leq C \delta \to 0. \]

By Lemma 4.1 and Lemma 4.3, letting $\delta \to 0$ (if necessary, we may choose a subsequence), then we have that
\[u_{\varepsilon \delta} \to u, \text{ in } L^6(Q_T), \]
\[u_{\varepsilon \delta} \to u, \text{ in } W^{2,1}_2(Q_T), \]
\[c_{\varepsilon \delta} \to c, \text{ in } C(Q_T), \]
\[c_{\varepsilon \delta} \to c, \text{ in } W^{2,1}_2(Q_T), \]
\[n_{\varepsilon \delta} \to n, \text{ in } L^p(Q_T), \text{ for any } p < s + m + 1, \]
\[\nabla n_{\varepsilon \delta} \to \nabla n, \text{ in } L^2(Q_T), \]
\[\delta \Delta n_{\varepsilon \delta} \to 0, \text{ in } L^2(Q_T). \]

Then $(n, c, u_{\varepsilon}, \pi_{\varepsilon})$ is the solution of the following problem
\[
\begin{cases}
 n_t + u \cdot \nabla n - \Delta((n^2 + \varepsilon) \frac{\varepsilon}{n^2} n) + \varepsilon |n|^s n = -\n \cdot (n \nabla c) + \mu |n|(1 - n) + g(x, t), \text{ in } Q, \\
 c_t + u \cdot \nabla c = \Delta c - c + n_+, \text{ in } Q, \\
 u_t = \Delta u - \nabla \pi + n \nabla \varphi, \text{ in } Q, \\
 \n \cdot u = 0, \text{ in } Q, \\
 \frac{\partial n}{\partial \nu}|_{\partial \Omega} = \frac{\partial c}{\partial \nu}|_{\partial \Omega} = 0, \text{ in } u|_{\partial \Omega} = 0
\end{cases}
\]
(4.4)
such that $(n, c, u_{\varepsilon}, \pi_{\varepsilon})$ satisfies (4.1) -(4.2).

Next, we consider the second level approximation, for this purpose, we make the following energy estimates independent of $\varepsilon$.

**Proposition 4.1.** Assume that $\max\{2(m - 1), 4\} < s \leq 5m - 1$, $m \geq \frac{6}{5}$, $g \geq 0$ with $g, \nabla \varphi \in L^\infty_T(Q)$ and $g \geq 0$. Then the problem (4.4) admits a time periodic solution
(n_ε, c_ε, u_ε, π_ε), such that n_ε, c_ε ≥ 0, and
\[
\sup_t \{\|u_\varepsilon\|_{H^1}^2 + \|c_\varepsilon\|_{H^1}^2 \} + \int_0^T (\|u_\varepsilon\|_{H^2}^2 + \|π_\varepsilon\|_{H^3}^2 + \|c_\varepsilon\|_{H^2}^2 + \|(u_\varepsilon, c_\varepsilon)\|_{L^2}) dt \leq C,
\]
(4.5)
\[
\sup_t \{\|u_\varepsilon\|_{L^\infty} + \|n_\varepsilon\|_{L^\infty} + \|c_\varepsilon\|_{W^{1,\infty}} \} \leq C,
\]
(4.6)
\[
\sup_t \int \Omega n_\varepsilon^2 + \int_0^T \int \Omega (n_\varepsilon^2 + \varepsilon) \frac{\varepsilon}{m} |\nabla \sqrt{n_\varepsilon}|^2 dxdt \leq C,
\]
(4.7)
\[
\sup_t \int \nabla (n_\varepsilon^2 + \varepsilon) \frac{\varepsilon}{m} \frac{\varepsilon}{m} |\nabla \sqrt{n_\varepsilon}|^2 dxdt \leq C,
\]
(4.8)
where C is independent of ε.

To show this proposition, we first have the following lemma. For simplicity, we still denote the solution of (4.4) by (n, c, u, π).

Lemma 4.3. Assume g ≥ 0 with g, ∇φ ∈ L_\infty(T)(Q). Let (n, c, u, π) be a time periodic solution of (4.4). Then we have c, n ≥ 0 and
\[
\sup_t \|n\|_{L^1} + \int_0^T \|n\|_{H^1}^2 dt \leq C,
\]
(4.9)
\[
\sup_t \{\|u\|_{H^1}^2 \} + \int_0^T (\|u\|_{H^2}^2 + \|π\|_{H^3}^2 + \|c\|_{H^2}^2 + \|(u, c)\|_{L^2}) dt \leq C,
\]
(4.10)
where these constants C are independent of ε.

Proof. By [7], we have c, n ≥ 0. Then integrating the first equation of (4.4) over Ω, it is easy to obtain (4.9). By (3.10), (3.11) and (3.12), we further have (4.10).

To improve the regularity of the solutions, we first introduce the below Lemma.

Lemma 4.4. Assume m > 1, g ≥ 0 with g, ∇φ ∈ L_\infty(T)(Q). Let (n, c, u, π) be a time periodic solution of (4.4). Then for any μ > 0, there exists g ∈ (0, 1) such that
\[
\sup_t \int \Omega n^{q+1} dx + \int_0^T \int \Omega n^{q+1} \Delta n^2 dxdt + \int_0^T \int \Omega \Delta n^2 dxdt \leq C.
\]
(4.11)
where C is independent of ε.

Proof. Multiplying the first equation of (4.4) by n^q with 0 < q < 1, integrating it over Ω × (t_0, t) for any t_0 < t < t_0 + T, we obtain
\[
\frac{1}{q + 1} \int_\Omega n^{q+1} (x,t) dx + \mu \int_0^t \int_\Omega n^{q+1} \Delta n^2 dxds + \varepsilon \int_0^t \int_\Omega n^{q+1+q} dxds + q \int_0^t \int_\Omega n^q \Delta n dxds + g n^q dx
\]
\[
\leq \frac{1}{q + 1} \int_\Omega n^{q+1} (x,t_0) dx + q \int_0^t \int_\Omega n^q \Delta n dxds + q \int_0^t \int_\Omega n^{q+1} dxds + C q^{q+2} \mu^{q+1} \int_0^t \int_\Omega \Delta n dxds + C,
\]
which implies that
\[
\frac{1}{q+1} \int_{\Omega} n^{q+1}(x,t)dx + q \int_{0}^{T} \int_{\Omega} n^{m+q-2} |\nabla n|^{2} dxds + \frac{\mu}{2} \int_{0}^{T} \int_{\Omega} n^{2+q} dxds \\
\leq \frac{1}{q+1} \int_{\Omega} n^{q+1}(x,t)dx + \frac{Cq^{q+2}}{\mu^{q+1}} \int_{t_0}^{t} \int_{\Omega} |\Delta c|^{q+2} dxds + C.
\]

Then by lemma 2.1, we obtain
\[
\sup_{t} \int_{\Omega} n^{q+1} dx + q \int_{0}^{T} \int_{\Omega} n^{m+q-2} |\nabla n|^{2} dxdt + \frac{\mu}{2} \int_{0}^{T} \int_{\Omega} n^{2+q} dxdt \\
\leq \frac{Cq^{q+2}}{\mu^{q+1}} \int_{0}^{T} \int_{\Omega} |\Delta c|^{q+2} dxdt + C. \tag{4.12}
\]

Noticing that \(2(q+2) < 6\), then by Lemma 2.3 and taking a proper small \(q\), we arrive at
\[
\frac{Cq^{q+2}}{\mu^{q+1}} \int_{0}^{T} \int_{\Omega} |\Delta c|^{q+2} dxdt \leq \frac{C_{1}q^{q+2}}{\mu^{q+1}} \int_{0}^{T} \int_{\Omega} (n^{q+2} + |u\nabla c|^{q+2}) dxdt \\
\leq \frac{C_{1}q^{q+2}}{\mu^{q+1}} \int_{0}^{T} \int_{\Omega} (|\nabla c|^{q+2} + |u|^{q+2}) dxdt \\
\leq \frac{C_{2}q^{q+2}}{\mu^{q+1}} \int_{0}^{T} \int_{\Omega} (|\nabla c|^{q+2} + |u|^{q+2}) dxdt \\
\leq \frac{C_{3}q^{q+2}}{\mu^{q+1}} \int_{0}^{T} \int_{\Omega} (|\nabla c|^{q+2} + |\nabla c|^{q+2}) dxdt. \tag{4.13}
\]

By (4.10) we see that for any sufficiently small \(\eta > 0\),
\[
||\nabla c||_{L_{\infty}^{q+2}} \leq ||\nabla c||_{L_{2}^{q+2}} \leq ||\nabla c||_{L_{2}^{q+2}} \leq \eta ||\Delta c||_{L_{q+2}}^{q+2} + C_{\eta}. \tag{4.14}
\]

Substituting (4.14) into (4.13) gives
\[
\frac{C_{q}q^{q+2}}{\mu^{q+1}} \int_{0}^{T} \int_{\Omega} |\Delta c|^{q+2} dxdt \leq \frac{C_{q}q^{q+2}}{\mu^{q+1}} \int_{0}^{T} \int_{\Omega} n^{q+2} dxdt + C. \tag{4.15}
\]

Recalling (4.12), we see that
\[
\sup_{t} \int_{\Omega} n^{q+1} dx + q \int_{0}^{T} \int_{\Omega} n^{m+q-2} |\nabla n|^{2} dxdt + \frac{\mu}{2} \int_{0}^{T} \int_{\Omega} n^{2+q} dxdt \\
\leq \frac{\tilde{C}q^{q+2}}{\mu^{q+1}} \int_{0}^{T} \int_{\Omega} n^{q+2} dxdt + \tilde{C}. \tag{4.16}
\]

where \(\tilde{C}\) is independent of \(\mu\) and \(q\). It is easy to see that for any \(\mu > 0\), there exists \(g > 0\) such that \(\frac{\tilde{C}q^{q+2}}{\mu^{q+1}} < \frac{g}{2}\), and (4.11) is proved.

Next, we prove the following Lemma.

**Lemma 4.5.** Assume \(m > 1\), \(q > 0\) with \(g, \nabla \varphi \in L_{2}^{\infty}(Q)\). Let \((n, c, u, \pi)\) be a time periodic solution of (4.4). Assume that there exists \(q > 2\), such that
\[
\int_{0}^{T} ||n||_{L_{q}}^{2} dt \leq C.
\]

Then for any \(r < \frac{3q}{(5-q)+}\), we have
\[
\sup_{t} ||\nabla c||_{L_{r}}^{r} + \int_{0}^{T} \int_{\Omega} |\nabla \nabla c|^{2} dxdt \leq C, \tag{4.17}
\]
and 
\[ \int_0^T \| \nabla c \|_{L^\infty_T}^{\frac{2r}{r-2}} \, dt \leq C. \] (4.18)

Here these constants $C$ are independent of $\varepsilon$.

Proof. Applying $\nabla$ to the second equation of (4.4), testing the resulting equation by $|\nabla c|^{r-2} \nabla c \chi_{[t_0,t]}$ for any $r \geq 2$, $t_0 < t \leq t_0 + T$, combining with Lemma 2.4, and using the boundary trace embedding inequalities and the inequality (4.10), we arrive at

\[
\begin{align*}
\frac{1}{r} \int_\Omega |\nabla c(x,t)|^r \, dx + \int_{t_0}^t \int_\Omega \left( (|\nabla c|^{r-2} |\nabla^2 c|^2 + (r-2)|\nabla c|^{r-2} (|\nabla^r c|^2) + |\nabla c|^r ) \right) \, dx \, ds \\
= \frac{1}{r} \int_\Omega |\nabla c(x,t_0)|^r \, dx - \int_{t_0}^t \int_\Omega (n \nabla \cdot (|\nabla c|^{r-2} \nabla c) + (u \cdot \nabla c) \nabla \cdot (|\nabla c|^{r-2} \nabla c)) \, dx \, ds \\
+ \int_{t_0}^t \int_{\partial \Omega} \frac{\partial |\nabla c|}{\partial n} |\nabla c|^{r-2} |\nabla c| \, dS \\
\leq \frac{1}{r} \int_\Omega |\nabla c(x,t_0)|^r \, dx + \frac{1}{4} \int_{t_0}^t \int_\Omega \left( (|\nabla c|^{r-2} |\nabla^2 c|^2 + (r-2)|\nabla c|^{r-2} (|\nabla^r c|^2) \right) \, dx \, ds \\
+ C \int_{t_0}^t \int_\Omega |\nabla c|^{r-2} n^2 \, dx \, ds + \int_{t_0}^t \int_\Omega u^2 |\nabla c|^r \, dx \, ds + C
\end{align*}
\]

(4.19)

By Gagliardo-Nirenberg interpolation inequality and (4.10), we have

\[
|\nabla c|_{L^\infty_T}^r = |||\nabla c|^r z||_{L^2}^2 \leq C |||\nabla c|^r z||_{L^\infty_T}^{\frac{4}{r}} \left( ||\nabla (|\nabla c|^r z)||_{L^2_T}^{\frac{2(3r-4)}{3r-2}} + C ||\nabla c||_{L^2}^r \right)
\]

\[
\leq \tilde{C} ||\nabla (|\nabla c|^r z)||_{L^2_T}^{\frac{2(3r-4)}{3r-2}} + \tilde{C},
\]

combining with (4.10), we have

\[
C ||u||_{L^2_T}^{2r} ||\nabla c||_{L^\infty_T}^r \leq C ||u||_{H^1_T}^r ||\nabla c||_{L^\infty_T}^r \leq \frac{r-2}{4} \int_\Omega |\nabla c|^{r-2} (|\nabla c|)^2 \, dx + C. \] (4.20)

Substituting (4.20) into (4.19) gives

\[
\begin{align*}
\frac{1}{r} \int_\Omega |\nabla c(x,t)|^r \, dx + \frac{1}{2} \int_{t_0}^t \int_\Omega \left( (|\nabla c|^{r-2} |\nabla^2 c|^2 + (r-2)|\nabla c|^{r-2} (|\nabla^r c|^2) \right) \, dx \, ds \\
+ \int_{t_0}^t \int_\Omega |\nabla c|^r \, dx \, ds \\
\leq \frac{1}{r} \int_\Omega |\nabla c(x,t_0)|^r \, dx + C \int_{t_0}^t \int_\Omega |\nabla c|^{r-2} n^2 \, dx \, ds + C
\end{align*}
\]
while if

Letting $R = \frac{q}{q-2}$, $r = q$, it yields

\[
\frac{1}{r} \int_{\Omega} |\nabla c(x, t)|^q \, dx + \frac{1}{2} \int_{t_0}^t \|\nabla c\|^2_{L^q(r-2)} \, ds + C \int_{t_0}^t \|n\|^2_{L^q(r-2)} \, ds + C. \tag{4.21}
\]

It follows from Lemma 2.1 that

\[
\sup_t \|\nabla c\|^q_{L^q} + \int_0^T \int_{\Omega} (|\nabla c|^\frac{q}{2})^2 \, dx \, dt \leq C. \tag{4.22}
\]

In what follows, we use a recurrence method to show this lemma. We show that if

\[
\sup_t \|\nabla c\|^p_{L^{p_{k}}} + \int_0^T \int_{\Omega} (|\nabla c|^\frac{p_{k}}{2})^2 \, dx \, dt \leq C, \tag{4.23}
\]

then

\[
\sup_t \|\nabla c\|^p_{L^{p_{k+1}}} + \int_0^T \int_{\Omega} (|\nabla c|^\frac{p_{k+1}}{2})^2 \, dx \, dt \leq C. \tag{4.24}
\]

with $p_{k+1} = 2 + \frac{5(q-2)}{3q} r_k$. By Gagliardo-Nirenberg interpolation inequality and (4.23), we see that

\[
|||\nabla c|^\frac{p_{k}}{2}||_{L^{p_{k}}}^{2} \leq |||\nabla c|^\frac{p_{k}}{2}||_{L^{p_{k}}}^{\frac{4}{3q}} \cdot |||\nabla c|^\frac{p_{k}}{2}||_{L^{p_{k}}}^{\frac{2}{3q}} \cdot |||\nabla c|^\frac{p_{k}}{2}||_{L^{p_{k}}}^{\frac{4}{3q}} \cdot |||\nabla c|^\frac{p_{k}}{2}||_{L^{p_{k}}}^{\frac{2}{3q}} \leq C (1 + |||\nabla c|^\frac{p_{k}}{2}||_{L^{p_{k}}}^{\frac{4}{3q}}) \cdot |||\nabla c|^\frac{p_{k}}{2}||_{L^{p_{k}}}^{\frac{2}{3q}}.
\]

It implies that

\[
\int_0^T \|\nabla c\|^\frac{p_{k}}{2} \, dt \leq \int_0^T \|\nabla c\|^\frac{p_{k}}{2} \, dt \leq C \left(1 + \int_0^T \|\nabla c\|^\frac{p_{k}}{2} \, dt \right) \leq C. \tag{4.25}
\]

Recalling (4.21), and taking $R = \frac{q}{q-2}, r = r_{k+1} = 2 + \frac{5(q-2)}{3q} r_k$, then

\[
\frac{3q}{6q + 5(q-2)r_k} \int_{\Omega} |\nabla c(x, t)|^{2 + \frac{5(q-2)}{3q} r_k} \, dx + \frac{1}{2} \int_{t_0}^t \|\nabla c\|^2_{L^{\frac{5(q-2)}{3q} r_k}} \, dx \, ds + C \int_{t_0}^t \|n\|^2_{L^{\frac{5(q-2)}{3q} r_k}} \, ds + C. \tag{4.26}
\]

By (4.25), and Lemma 2.1, we conclude (4.24).

Consider the sequence $r_n$ with $r_{k+1} = 2 + \frac{5(q-2)}{3q} r_k$, and $r_1 = q > 2$. Noting that $r_2 > r_1$, then $r_k$ is a monotonically increasing sequence. Noticing that $0 < \frac{5(q-2)}{3q} < 1$ and $r_1 \leq \frac{3q}{5-q}$ if $2 < q < 5$, then it is not difficult to show that $r_k$ goes to $\frac{3q}{5-q}$.

While if $q \geq 5$, then $\frac{5(q-2)}{3q} \geq 1$, clearly, we have $r_k \to +\infty$. Therefore (4.17) is obtained. Completely similar to (4.25), we conclude (4.18). □
Lemma 4.6. Assume \( m \geq \frac{6}{5}, \ g \geq 0 \) with \( g, \nabla \varphi \in L^\infty_t(Q) \). Let \((n, c, u, \pi)\) be a time periodic solution of (4.4). If for some \( q_i > 2 \),

\[
\int_0^T \int_\Omega n^q dxds \leq C_i,
\]

then for any \( q < \frac{m}{2} \left( \frac{5q}{(5-q)_+} \right) - 2 \), we have

\[
\sup_t \int_\Omega n^{q+1} dx + \int_0^T \int_\Omega n^{m+q-2} |\nabla n|^2 dxds + \int_0^T \int_\Omega n^{q+2} dxds \leq C_{i+1}(q). \tag{4.27}
\]

where \( C_{i+1}(q) \) are independent of \( \varepsilon \).

Proof. By Lemma 4.3, we see that for any \( r < \frac{5q}{(5-q)_+} \),

\[
\int_0^T \|\nabla c\|_r^r dxds \leq C_i(r). \tag{4.28}
\]

Multiplying the first equation of (4.4) by \( n^q \), integrating it over \( \Omega \times (t_0, t) \) for any \( t_0 < t \leq t_0 + T \), we obtain

\[
\frac{1}{q+1} \int_\Omega n^{q+1}(x, t) dx + \int_{t_0}^t \int_\Omega \left( q n^{m+q-2} |\nabla n|^2 + \varepsilon n^{q+1} + \mu n^{q+2} \right) dxds \leq \frac{1}{q+1} \int_\Omega n^{q+1}(x, t_0) dx + \int_{t_0}^t \int_\Omega \left( q n^{m+q-2} |\nabla n|^2 + n^{q+2} + q n^{q+2} \right) dxds + C
\]

\[
\leq \frac{1}{q+1} \int_\Omega n^{q+1}(x, t_0) dx + \frac{q}{2} \int_{t_0}^t \int_\Omega n^{m+q-2} |\nabla n|^2 dxds + C \int_{t_0}^t \int_\Omega |\nabla c|^{\frac{2(q+2)}{m}} dxds + C
\]

which implies

\[
\frac{1}{q+1} \int_\Omega n^{q+1}(x, t) dx + \frac{q}{2} \int_{t_0}^t \int_\Omega n^{m+q-2} |\nabla n|^2 dxds + \frac{\mu}{2} \int_{t_0}^t \int_\Omega n^{q+2} dxds \leq \frac{1}{q+1} \int_\Omega n^{q+1}(x, t_0) dx + C \int_{t_0}^t \int_\Omega |\nabla c|^{\frac{2(q+2)}{m}} dxds + C.
\]

Noticing that \( \frac{2(q+2)}{m} < \frac{5q}{(5-q)_+} \), then by (4.28) and Lemma 2.1 we obtain (4.27). The proof is complete. \( \square \)

Lemma 4.7. Assume \( m \geq \frac{6}{5}, \ g \geq 0 \) with \( g, \nabla \varphi \in L^\infty_t(Q) \). Let \((n, c, u, \pi)\) be a time periodic solution of (4.4). Then

\[
\sup_t \{\|n\|_{L^\infty} + \|n\|_{L^\infty} + \|c\|_{W^{1, \infty}} \} \leq C,
\]

where \( C \) is independent of \( \varepsilon \).

Proof. We consider the following sequence. Let \( P_{i+1} = \frac{m}{2} \left( \frac{5P_i}{5 - (\rho + 2)} \right), \ P_1 = \varrho + 2, \ \rho > 0 \) is defined in Lemma 4.4. Note that

\[
\frac{P_{i+1}}{P_i} \geq \frac{m}{2} \left( \frac{5}{5 - (\rho + 2)} \right) - 2 \geq \frac{m}{2} \left( \frac{5}{3 - \rho} \right) - 2 \geq \frac{6}{6 - 2\rho} > 1.
\]
Then for any $m \geq \frac{6}{5}$, there must exist a unique $K > 0$ such that $P_{K-1} < \delta$, and $P_K \geq 5$. Then recalling Lemma 4.4 and Lemma 4.6, using an iteration process, we conclude that for any $m \geq \frac{6}{5}$,

$$\int_0^T \int_\Omega n^4 dx ds \leq C.$$ 

We use Lemma 4.5 again, and obtain for any $q < 10m - 2$,

$$\sup_t \int_\Omega n^{q+1} dx + \int_0^T \int_\Omega n^{m+q-2} |\nabla n|^2 dx ds + \int_0^T \int_\Omega n^{q+2} dx ds \leq C(q). \tag{4.30}$$

According to standard smoothing properties of the Stokes semigroup, there exist $\lambda > 0$ such that

$$\|u\|_{L^\infty} = \int_{-\infty}^t \| e^{-(t-s)A} P(n(s)\nabla \varphi(s)) \|_{L^\infty} ds$$

$$\leq \int_{-\infty}^t e^{-\lambda(t-s)} (t-s)^{-\frac{3}{2}} \| n(s) \nabla \varphi \|_{L^3} ds$$

$$\leq \int_{-\infty}^t e^{-\lambda(t-s)} (t-s)^{-\frac{1}{2}} \| n \|_{L^1} \| \nabla \varphi \|_{L^\infty} ds$$

$$\leq C. \tag{4.31}$$

By Duhamel principle, we see that

$$c = \int_{-\infty}^t e^{-(t-s)\Delta} (n - u \nabla c) ds,$$

where $\{e^{t\Delta}\}_{t \geq 0}$ is the Neumann heat semigroup on $\Omega$. Then by (4.10), (4.30) and (4.31), we have

$$\| \nabla c(\cdot, t) \|_{L^\infty} \leq \int_{-\infty}^t e^{-(t-s)} (t-s)^{-\frac{3}{2} - \frac{1}{2}} \| n - u \nabla c \|_{L^4} ds$$

$$\leq \int_{-\infty}^t e^{-(t-s)} (t-s)^{-\frac{1}{2}} \| n \|_{L^1} \| \nabla \varphi \|_{L^\infty} ds$$

$$\leq \int_{-\infty}^t e^{-(t-s)} (t-s)^{-\frac{1}{2}} \| n \|_{L^4} \| \nabla c \|_{L^4} ds$$

$$\leq \sup_s (\| n \|_{L^4} + \| u \|_{L^\infty} \| \nabla c \|_{L^4}^{\frac{3}{2}} \| \nabla c \|_{L^\infty}^{\frac{1}{2}}) \int_0^\infty e^{-s} s^{-\frac{1}{2}} ds$$

$$\leq C(1 + \sup_s \| \nabla c \|_{L^\infty}^{\frac{1}{2}}),$$

which implies that

$$\sup_t \| \nabla c(\cdot, t) \|_{L^\infty} \leq C,$$

and we further have

$$\sup_t \| c(\cdot, t) \|_{W^{1, \infty}} \leq C. \tag{4.32}$$

Next, we use Morse iterative method to prove $L^\infty$ estimation of $n$. Similar to Lemma 4.6, for any $t_0 < t \leq t_0 + T$, we have

$$\frac{1}{q + 1} \int_\Omega n^{q+1} dx + \int_0^t \int_\Omega \left( q(n^2 + \varepsilon) \frac{m-1}{m} n^{q-1} |\nabla n|^2 + \mu n^{2+q} + n^{q+1} \right) dx ds$$
\[
\leq \frac{1}{q+1} \int_0^t \int_\Omega n^{q+1}(x,t)dx + \frac{1}{q+1} \int_\Omega n^{q+1}(x,t_0)dx + \frac{q}{2} \int_0^t \int_\Omega n^{m+q-2} |\nabla n|^2 dx ds + Cq \int_0^t \int_\Omega n^{q+2-m} dx ds \\
+ (1+\gamma) \int_0^t \int_\Omega n^{1+q} dx ds + C \int_0^t \int_\Omega n^q dx ds
\]

\[
\leq \frac{1}{q+1} \int_0^t \int_\Omega n^{q+1}(x,t)dx + \frac{q}{2} \int_0^t \int_\Omega n^{m+q-2} |\nabla n|^2 dx ds + Cq \int_0^t \int_\Omega n^{q+2-m} dx ds \\
+ \frac{q}{2} \int_0^t \int_\Omega n^{q+2} dx ds + C \int_0^t \int_\Omega n^q dx ds,
\]

which means

\[
\int_\Omega n^{q+1}(x,t)dx + \frac{4q(q+1)}{(m+q)^2} \int_0^t \int_\Omega |\nabla n|^{m+q+2} dx ds + \int_0^t \int_\Omega n^{q+1} dx ds
\]

\[
\leq \int_\Omega n^{q+1}(x,t_0)dx + Cq^2 \int_0^t \int_\Omega n^{q+2-m} dx ds + Cq \int_0^t \int_\Omega n^q dx ds. \quad (4.33)
\]

Noticing that for any small \(\eta > 0\), we have

\[
Cq^2 \int_\Omega n^{q+2-m} dx = Cq^2 \|n\|^{q+2-m}_{L^{2+1}} \leq C_1 q^2 \|\nabla n\|_{L^{\frac{m+q}{2}}}^{m+q-2} \|n\|^{q+2-m}_{L^{2+1}} + C_2 q^2 \|n\|^{q+2-m}_{L^{2+1}}
\]

\[
\leq \eta \|\nabla n\|_{L^{\frac{m+q}{2}}}^{m+q} + C_3 q^2 \|n\|^{q+2-m}_{L^{2+1}}
\]

we also have

\[
Cq \int_\Omega n^q dx = Cq \|n\|_{L^{\frac{m+q}{m+q}}}
\]

\[
\leq C_4 q \|\nabla n\|_{L^{\frac{m+q}{m+q}}}^{m+q} \|n\|^{q+2-m}_{L^{2+1}} + C_5 q \|n\|^{q+2-m}_{L^{2+1}}
\]

Putting the above two inequalities in (4.33), and taking \(\eta\) appropriately small, it yields that

\[
\int_\Omega n^{q+1}(x,t)dx + \frac{2q(q+1)}{(m+q)^2} \int_0^t \int_\Omega |\nabla n|^{m+q+2} dx ds + \int_0^t \int_\Omega n^{q+1} dx ds
\]

\[
\leq \int_\Omega n^{q+1}(x,t_0)dx + Cq^2 \int_0^t \int_\Omega |n|^{\frac{(q+1)(q+2m-1)}{L^{2+1}}} dx ds + C_2 q^2 \int_0^t \int_\Omega \|n\|^{q+2-m}_{L^{2+1}} dx ds \\
+ C_6 q^2 \int_0^t \int_\Omega |n|^{\frac{(q+1)(q+2m-1)}{L^{2+1}}} dx ds + Cq \int_0^t \int_\Omega |n|^{q+2-m}_{L^{2+1}} dx ds
\]

\[
\leq \int_\Omega n^{q+1}(x,t_0)dx + Cq^5 \int_0^t \int_\Omega \|n\|^{\frac{q+1}{L^{2+1}}} dx ds + Cq^5 \int_0^t \int_\Omega |n|^{\frac{q+1}{L^{2+1}}} dx ds \quad (4.34)
\]

since \(\frac{q+1}{2+1} \leq \frac{(q+1)(q+2m-1)}{6m+q-5}, q + 2 - m, \frac{(q+1)(q+2m-1)}{2q+6m+2+1}, q \leq q + 1\). It implies that

\[
\sup_t \|n\|^{\frac{q+1}{L^{2+1}}} \leq Cq^5 \sup_t \|n\|^{\frac{q+1}{L^{2+1}}} + C_7 q^5 \sup_t \|n\|^{\frac{q+1}{L^{2+1}}}.
\]
Let $p_j = 2p_{j-1} = 2^j p_0$, $p_0 = 2$, $M_j = \max\{1, \sup_\Omega \|u\|_{L^{p_j}}\}$. Then
\[
M_j \leq C \sum_{i=0}^{j-1} 2^{i+1} \sum_{i=1}^{j+1} \frac{\gamma_i}{\alpha_i} \frac{1}{\beta_i} M_{j-1} \leq C \sum_{i=1}^{j+1} \frac{\gamma_i}{\alpha_i} \frac{1}{\beta_i} M_0 \leq M^*.
\]
Let $j \to \infty$, we arrive at
\[
\|n\|_{L^\infty} \leq M^*,
\]
where $M^*$ is independent of $\varepsilon$. This lemma is proved.

**Proof of Proposition 4.1.** Multiplying the first equation of (4.4) by $1 + \ln n$, and integrating it over $\Omega \times (t_0, t)$ for any $t_0 < t \leq t_0 + T$, we see that
\[
\int_\Omega n(x, t) \ln n(x, t) dx + \int_{t_0}^t \int_\Omega \left(4(n^2 + \varepsilon) \frac{n-1}{2} |\nabla \sqrt{n}|^2 + \mu n^2 \ln n \right) dx ds
\]
\[+ \varepsilon \int_{t_0}^t \int_\Omega n^{s+1} |\ln n| dx ds
\]
\[
\leq \int n(x, t_0) \ln n(x, t_0) dx - \int_{t_0}^t \int_\Omega n \Delta c dx ds + \int_{t_0}^t \int_\Omega \mu n(1 + \ln n) + g(1 + \ln n) dx ds
\]
\[
\leq \int n(x, t_0) \ln n(x, t_0) dx + \int_{t_0}^t \int_\Omega |\Delta c|^2 dx ds + \frac{\mu}{2} \int_{t_0}^t \int_\Omega n^2 |\ln n| dx ds + C,
\]
by Lemma 2.1, we have
\[
\sup_t \int_\Omega n \ln n dx + \int_0^T \int_\Omega \left(\frac{n^2 + \varepsilon}{n} \frac{n-1}{2} |\nabla n|^2 dx dt \leq C. \tag{4.35}
\]

Testing the first equation of (4.4) by $\frac{\partial((n^2 + \varepsilon) \frac{n-1}{2} n)}{\partial t} \chi_{[t_0, t]}$, and by (4.29), we arrive at
\[
\frac{1}{2} \int_\Omega |\nabla((n^2 + \varepsilon) \frac{n-1}{2} n)|^2 dx + \int_{t_0}^t \int_\Omega \left(\frac{n^2 + \varepsilon}{n} \frac{n-1}{2} |\partial n|^2 \right) dx ds
\]
\[\leq \frac{1}{2} \int_\Omega |\nabla((n^2 + \varepsilon) \frac{n-1}{2} n)|^2 dx + \frac{1}{2} \int_{t_0}^t \int_\Omega \left(\frac{n^2 + \varepsilon}{n} \frac{n-1}{2} |\partial n|^2 \right) dx ds
\]
\[\leq \frac{1}{2} \int_\Omega |\nabla((n^2 + \varepsilon) \frac{n-1}{2} n)|^2 dx + \frac{1}{2} \int_\Omega \int_\Omega |\nabla n^2| dx + \int_{t_0}^t \int_\Omega |\Delta c|^2 dx ds + 1
\]
+ $C_1 \left(\int_{t_0}^t \int_\Omega (n^2 + \varepsilon) \frac{n-1}{2} |\nabla n|^2 dx ds + \int_{t_0}^t \int_\Omega |\Delta c|^2 dx ds + 1 \right)$,
combining with (4.10), (4.29) and (4.35), we finally arrive at
\[
\sup_t \int_\Omega |\nabla((n^2 + \varepsilon) \frac{n-1}{2} n)|^2 dx + \int_0^T \int_\Omega \left(\frac{n^2 + \varepsilon}{n} \frac{n-1}{2} |\partial n|^2 \right) dx ds \leq C. \tag{4.36}
\]

Summing up, the proof is complete.}

**Proof of Theorem 1.1.** By proposition 4.1, letting $\varepsilon \to 0$, we have
\[
u_\varepsilon \to u, \text{ in } L^p(Q_T) \text{ for any } p > 1,
\]
\[ u_\varepsilon \rightarrow u, \text{ in } W^{2,1}_2(Q_T), \]
\[ \pi_\varepsilon \rightarrow \pi, \text{ in } W^{1,0}_2(Q_T), \]
\[ c_\varepsilon \rightarrow c \text{ in } L^p(Q_T) \text{ for any } p > 1, \]
\[ c_\varepsilon \rightarrow c \text{ in } W^{2,1}_2(Q_T), \]
\[ n_\varepsilon \rightarrow n, \text{ in } L^p(Q_T) \text{ for any } p > 1, \]
\[ \varepsilon n_\varepsilon^{k+1} \rightarrow 0, \text{ in } L^\infty(Q_T), \]
\[ (n_\varepsilon^2 + \varepsilon) \frac{n_\varepsilon}{\varepsilon^2} n_\varepsilon \rightarrow n^m, \text{ in } L^p(Q_T) \text{ for any } p > 1, \]

and \((n, c, u, \pi)\) is a time periodic solution of (1.1) such that (1.2)-(1.4) hold. \(\square\)

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