Contextual Combinatorial Volatile Bandits with Satisfying via Gaussian Processes

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Abstract

In many real-world applications of combinatorial bandits such as content caching, rewards must be maximized while satisfying minimum service requirements. In addition, base arm availabilities vary over time, and actions need to be adapted to the situation to maximize the rewards. We propose a new bandit model called Contextual Combinatorial Volatile Bandits with Group Thresholds to address these challenges. Our model subsumes combinatorial bandits by considering super arms to be subsets of groups of base arms. We seek to maximize super arm rewards while satisfying thresholds of all base arm groups that constitute a super arm. To this end, we define a new notion of regret that merges super arm reward maximization with group reward satisfaction. To facilitate learning, we assume that the mean outcomes of base arms are samples from a Gaussian Process indexed by the context set $\mathcal{X}$, and the expected reward is Lipschitz continuous in expected base arm outcomes. We propose an algorithm, called Thresholded Combinatorial Gaussian Process Upper Confidence Bounds (TCGP-UCB), that balances between maximizing cumulative reward and satisfying group reward thresholds and prove that it incurs $\tilde{O}(K\sqrt{T\gamma_T})$ regret with high probability, where $\gamma_T$ is the maximum information gain associated with the set of base arm contexts that appeared in the first $T$ rounds and $K$ is the maximum super arm cardinality of any feasible action over all rounds. We show in experiments that our algorithm accumulates a reward comparable with that of the state-of-the-art combinatorial bandit algorithm while picking actions whose groups satisfy their thresholds.

1. Introduction

The multi-armed bandit problem is a prominent example of reinforcement learning which studies the sequential interaction of a learner with its environment under partial feedback (Robbins, 1952). Out of the many variants of the multi-armed bandit problem, the combinatorial and contextual bandits have been thoroughly investigated due to their rich set of real-life applications. In combinatorial bandits, in each round, the learner selects a super arm, which is a subset of the available base arms (Chen et al., 2013; Cesa-Bianchi & Lugosi, 2012). In the semi-bandit feedback model, at the end of the round, the learner observes the outcomes of the base arms in the selected super arm together with the super arm reward. In contextual bandits, at the beginning of every round, the learner observes side-information about the outcomes of the base arms available in that round before deciding which arm to select (Slivkins, 2011; Lu et al., 2010). For both variants, the base arms could be volatile, meaning there is varying base arm availability at each round. Combinatorial bandits and contextual bandits are deployed in applications ranging from influence maximization in a social network (Chen et al., 2016) to news article recommendations (Li et al., 2010).

In this paper, we consider contextual combinatorial volatile multi-armed bandits (CCV-MAB) with group thresholds. In every round $t$, the learner is able to observe the available base arms along with their contexts and selects a super arm, from which the noisy base arm outcomes are observed and the reward is collected. In this setup we have groups which are unions of base arms and every super arm is constrained to be a union of groups. Groups also change from round to round. We also consider the notion of satisfaction in relation to groups, in other words for the super arms that we choose, we want their forming groups to have rewards above certain thresholds (Reverdy et al., 2017). The goal is to maximize the cumulative super arm reward while ensuring that the groups forming the chosen super arms have a small regret. In addition, we do not know the contexts which will arrive in future rounds and the function relating contexts to base arm outcomes. To reach our goal, we need to ensure that we carefully balance between exploration and exploitation based on the past rounds and the structure of our problem.
Without any structural assumptions on the expected outcome function $f$, it is unrealizable to learn the optimal super arm at each round as the context set $X$ is usually large. The smoothness that we enforce by modeling $f$ as a sample from a GP with known kernel puts enough structure on the outcomes such that the learner can learn reasonably well. Also, GPs can be applied successfully to many functions of interest in real-life which is another reason why we use them. GP bandit algorithms have been used frequently since its introduction into the literature (Srinivas et al., 2012). The posterior mean and variance resulting from GP gives us high probability confidence bounds on the expected outcome function which leads to a desirable balance between exploration and exploitation. In addition, the smoothness that is enforced by the GP enables accurate reward predictions for actions that have not been played in previous rounds.

Our model is very general in the sense that it covers many interesting contextual and combinatorial learning problems as special cases. These are further discussed in Section 2.8. Our setup is applicable to content caching (Müller et al., 2017), fast mmWave beam alignment problem (Wu et al., 2019), worker allocation to gem mines (Verstraeten et al., 2017), fast mmWave beam alignment problem (Wu et al., 2020) and user-small base station channel allocations (Qureshi et al., 2021).

1.1. Our contributions

- We propose a new CCV-MAB problem with groups and group thresholds which is applicable to real-life problems where we want to achieve a minimum level of quality in groups while maximizing the total reward.

- We propose a new notion of regret called group regret, and define the total regret as a weighted combination of group and super arm regret in terms of the trade-off parameter $\zeta \in [0, 1]$.

- We use a double UCB approach in our algorithm by having different exploration bonuses for groups and super arms which is dependent on $\zeta$.

- We derive an information theoretic regret bound for our algorithm TCGP-UCB given by $\tilde{O}(K\sqrt{T\gamma_T})$. By assuming a fixed cardinality for every super arm, we express our regret bound in terms of the classical maximum information gain $\gamma_T$ which is given as $\tilde{O}(K\sqrt{T\gamma_T})$.

1.2. Related Work

The contextual combinatorial multi-armed bandit problem (CC-MAB) has been investigated thoroughly in recent years (Li et al., 2016; Qin et al., 2014). In the work of (Li et al., 2016), the learner can observe the reward of the super arm and the rewards of a subset of the base arms in the super arm selected due to cascading feedback according to some stopping criteria. They propose a UCB-type algorithm, namely C^UCB which incurs $\tilde{O}(\sqrt{KT})$ regret over $T$ rounds where $K$ is the maximum cardinality of any super arm which can be selected. In the work of (Qin et al., 2014), the contextual combinatorial MAB problem is applied to online recommendation. Their algorithm called C^2UCB also incurs sublinear in time regret.

The contextual combinatorial volatile multi-armed bandit problem has been explored in (Chen et al., 2018; Nika et al., 2020). In the work of (Chen et al., 2018), the super arm reward is submodular and since the context space is infinite, they form a partition of the context space with hypercubes depending on context information thus addressing the volatility of arms by exploiting the similarities between contexts in the same hypercube. The algorithm CC-MAB incurs $\tilde{O}(T^{(D+2n)/(D+3n)})$ regret where $n$ is the Hölder constant and $D$ is the dimension of the context space. The work done by (Nika et al., 2020), makes use of adaptive discretization instead of fixed discretization which addresses the limited similarity information of the arms that can be gathered by fixed discretization. The algorithm ACC-UCB incurs $\tilde{O}(T^{(D+1)/(D+2)+\epsilon})$ regret over $T$ rounds for any $\epsilon > 0$ where $D$ represents the approximate optimality dimension for the context space $X$.

Gaussian Process bandits have been utilized in the contextual MAB setup as well as for adaptive discretization (Krause & Ong, 2011; Shekhar et al., 2018). While adaptive discretization has been shown to be working well in large context spaces and volatile setups, in the case where the number of base arms is finite, assuming the expected base arm outcomes are a sample from a GP could be a better approach as it eliminates the need for the explicit assumption that the expected base arm outcomes are Lipschitz continuous. In this paper we also use the smoothness induced by the GP instead of performing explicit discretization.

The thresholding multi-armed bandit problem has been studied in (Locatelli et al., 2016; Mukherjee et al., 2017). These works consider thresholding as having the mean of a base arm be above a certain value. In (Reverdy et al., 2017), there is the notion of satisficing instead of thresholding, which is a combination of satisfaction, which is the learner’s desire to have a reward above a threshold and sufficiency which means to have satisfaction for base arms at a certain level of confidence. Another similar line of research is level set estimation, which is the process in which noisy observations of a function is used to determine the regions which exceed a certain threshold (Willett & Nowak, 2007). To our knowledge, there is no previous research which deals with group thresholding or a thresholding setup within the CCV-MAB problem. A comparison of our problem setup with other
similar works can be found in Table 1.

2. Problem Formulation

2.1. Base Arms and Base Arm Outcomes

The sequential decision-making problem proceeds over \( T \) rounds indexed by \( t \in [T] \). In each round \( t \), \( M_t \) base arms indexed by the set \( M_t = [M_t] \) become available. Cardinality of the set of available base arms in any round is bounded above by \( M < \infty \). Each base arm \( m \in M_t \) comes with a context \( x_{t,m} \) which resides in the context set \( \mathcal{X} \). The set of available contexts in round \( t \) is denoted by \( \mathcal{X}_t = \{x_{t,m} \}_{m \in M_t} \). When selected, a base arm with context \( x \) yields a random outcome \( r(x) \). The unknown mean reward function is represented by \( f : \mathcal{X} \rightarrow \mathbb{R} \). We will describe what form \( f \) takes in Section 2.5. The random outcome is defined as \( r(x) = f(x) + \eta \) where \( \eta \sim \mathcal{N}(0, \sigma^2) \) represents the observation noise that is independent across base arms and rounds. In the motivating example of content caching, base arms represent location-content pairs while base arm outcomes are the number of cache hits. The base arm outcomes could have different weights associated with them which corresponds to a multiplying factor of the number of cache hits of that base arm in the super arm reward. The weights could represent the payment by content owners for each cache hit, and some content owners pay more to increase the probability of their content to be cached. The context associated with each base arm corresponds to various factors such as, but not limited to, specifications of a location, characteristics of the users and the feature vector of the cached content.

2.2. Group and Super Arm Rewards

Groups and super arms are defined as subsets of the set of all base arms in any given round. Super arms are unions of disjoint groups. Groups are sets of base arms with different round-varying cardinalities. In the case of content caching, groups can correspond to the location-content pairs with the same location but possibly different content items. We denote by \( G_t \) the set of feasible groups in round \( t \) and the reward of a group \( G \in G_t \) depends on the outcomes of base arms in \( G \). Let \( x_{t,G} \) represent the vector of contexts of base arms in \( G \). For any function \( h \), which takes a single context as an argument, given a \( k \)-tuple of contexts \( x = [x_1, \ldots, x_k] \), we let \( h(x) = [h(x_1), \ldots, h(x_k)] \). Equipped with this notation, we represent the reward of a group \( G \) by the random variable \( V_G(r(x_{t,G})) \). We stress that \( V_G \) here is a deterministic function of its argument and the randomness comes from \( r \). As typical in the CMAB literature, given a base arm outcome function \( f \), we assume that expected group reward is a function of only the set of base arms in \( G \) and their expected outcome vector. Therefore, we define \( v_G(f(x_{t,G})) = E[V_G(r(x_{t,G})))|f] \). Note that \( V_G \) and \( v_G \) are vector functions and take as input a \(|G|\) dimensional vector. We assume that for all \( G \in G_t \), the expected group reward function \( v_G \) is monotonically non-decreasing with respect to the expected outcome vector.

**Definition 1 (Monotonicity)** For all \( G \in G_t \) and for all \( t \in [T] \) and for any \( f = [f_1, \ldots, f_{|G|}]^T \in \mathbb{R}^{|G|} \) and \( f' = [f'_1, \ldots, f'_{|G|}]^T \in \mathbb{R}^{|G|} \), if \( f_m \leq f'_m \), \( \forall m \leq |G| \), then \( v_G(f) \leq v_G(f') \).

**Definition 2 (Lipschitz continuity)** For all \( G \in G_t \) and for all \( t \in [T] \), there exists \( B_G > 0 \) such that for any \( f = [f_1, \ldots, f_{|G|}]^T \in \mathbb{R}^{|G|} \) and \( f' = [f'_1, \ldots, f'_{|G|}]^T \in \mathbb{R}^{|G|} \), we have \( |v_G(f) - v_G(f')| \leq B_G \sum_{i=1}^{|G|} |f_i - f'_i| \).

We define \( B := \max_{G \in G_t, \forall t \in [T]} B_G \), which will be later used in proofs.

Similarly, we denote by \( S_t \) the set of feasible super arms in round \( t \) and by \( S = \cup_{t \geq 1} S_t \) the overall feasible set of super arms. We assume that the maximum number of base arms in a super arm does not exceed a fixed \( K \in \mathbb{N} \) that is known to the learner. That is, for any \( S \in S \), we have, \(|S| \leq K\).

The reward of a super arm \( S \in S_t \) depends on the outcomes of base arms in \( S \). We represent the reward of a super arm \( S \) with context vector \( x_{t,S} \) by random variable \( U(S, r(x_{t,S})) \), where \( U \) is a deterministic function. Moreover, we have that the expected super arm reward is a function of only the set of base arms in \( S \) and their expected outcome vector, given by \( u(S, f(x_{t,S})) = \mathbb{E}[U(S, r(x_{t,S})))|f] \). Lastly, similar to the group reward function, we impose monotonicity and Lipschitz continuity assumptions on the super arm reward function.

**Definition 3 (Monotonicity)** For all \( S \in S_t \) and for any \( f = [f_1, \ldots, f_{|S|}]^T \in \mathbb{R}^{|S|} \) and \( f' = [f'_1, \ldots, f'_{|S|}]^T \in \mathbb{R}^{|S|} \), if \( f_m \leq f'_m \), \( \forall m \leq |S| \), then \( u(S, f) \leq u(S, f') \).

**Definition 4 (Lipschitz continuity)** For all \( S \in S \) and for any \( f = [f_1, \ldots, f_{|S|}]^T \in \mathbb{R}^{|S|} \) and \( f' = [f'_1, \ldots, f'_{|S|}]^T \in \mathbb{R}^{|S|} \), we have \( |u(S, f) - u(S, f')| \leq B_S \sum_{i=1}^{|S|} |f_i - f'_i| \).

We define \( B := \max_{S \in S, \forall t \in [T]} B_S \), which will be later used in proofs.
We propose an extended semi-bandit feedback model. When we have a limited memory in the caching entity, it is important that this threshold is satisfied in order to increase the efficiency of the caching. It could be that the threshold \( \gamma \) represents putting a threshold on the total number of hit requests associated with a group. This is quite relevant in real-life as we want the content that we cached in the location to be requested many times since we have the same content item, they are in different groups as base arms are location-content pairs so that a base arm cannot be found in multiple groups.

In content caching, super arms correspond to unions of disjoint groups. This makes sense in the content caching example because even if two users living in different locations have the same content item, they are in different groups as base arms are location-content pairs so that a base arm cannot be found in multiple groups.

In line with prior work (Nika et al., 2021), we assume that the learner knows all \( u \) and \( v_G \) perfectly, but does not know \( f \) beforehand. In the case of content caching, the super arm reward and group reward are weighted sums of their corresponding base arms.

We propose an extended semi-bandit feedback model. When super arm \( S_t \) is selected in round \( t \), the learner observes at the end of the round \( U(r(x_{S_t})) \), \( V(r(x_G)) \) for \( G \in \hat{G}_t \) such that \( G \subseteq S_t \) and \( r(x_{t,m}) \) for \( m \in S_t \). In content caching, the learner observes \( U(r(x_{S_t})) \) and \( V(r(x_G)) \) as the weighted sum of their base arm outcomes.

### 2.3. Regret

Our learning objective is to choose super arms that yield maximum rewards while ensuring that the groups in chosen super arms satisfy a certain level of quality, which is characterized by the group threshold. In the literature, this objective is also known as satisfying (Reverdy et al., 2017).

In accord with our goal, we define regret gadgets for both super arms and groups. In our regret analysis, as typical in contextual bandits (Nika et al., 2020), we assume that the sequence of available base arms \( \{X_t\} \) is fixed, hence context arrivals are not affected by past actions.

We assume that each group \( G \in \hat{G}_t \) has a threshold \( \gamma_{t,G} \), which represents the minimum reward that the group requires when it is selected. For instance, in content caching, \( \gamma_{t,G} \) represents putting a threshold on the number of cache hits associated with a group. This is quite relevant in real-life as we want the content that we cached in the entity for that location to be requested many times since we are prioritizing these files by caching them for that location.

As we have a limited memory in the caching entity, it is important that this threshold is satisfied in order to increase the efficiency of the caching. It could be that the threshold of a group is proportional to the number of base arms in that group. Putting a threshold on the number of cache hits has also been done in the previous work of (Zou et al., 2016) as a binary caching decision depending on the number of visited times of the content up to that time period. We say that group \( G \in \hat{G}_t \) satisfies its threshold when \( v(f(x_G))) \geq \gamma_{t,G} \) and we denote the set of feasible groups who do not satisfy their thresholds by \( \hat{G}_t \). We define the group regret as

\[
R_g(T) = \sum_{t=1}^{T} \sum_{G \in \hat{G}_t \in S_t} [\gamma_{t,G} - v(f(x_G))]_+, \]

where \([\cdot]_+ = \max(\cdot, 0)\) and for any \( G \in \hat{G}_t \), it holds that \([\gamma_{t,G} - v(f(x_G))]_+ = \gamma_{t,G} - v(f(x_G))\).

Let \( S_t' \subseteq S_t \) represent the set of super arms whose all groups satisfy their thresholds, i.e., \( \{S \in S_t : \forall(G \in \hat{G}_t : G \subseteq S), v(f(x_{t,G})) \geq \gamma_{t,G}\}\). The super arm regret is defined as the standard \( \alpha \)-approximation regret in CMAB:

\[
R_s(T) = \alpha \sum_{t=1}^{T} \text{opt}(f_t) - \sum_{t=1}^{T} u(f(x_{t,S_t})),
\]

where \( \text{opt}(f_t) = \max_{S \subseteq S_t} u(f(x_{1,S})) \). Note that any optimal super arm \( S_t^* \in \arg\max_{S \subseteq S_t} u(f(x_{t,S})) \) is restricted to satisfy all its groups. Otherwise, the policy which always selects the optimal super arms will incur linear group regret.

The total regret is defined as a weighted combination of group and super arm regrets. Given the tradeoff parameter \( \zeta \in [0, 1] \), we define it as

\[
R(T) = \zeta R_g(T) + (1 - \zeta) R_s(T). \tag{1}
\]

Setting \( \zeta = 1 \) reduces the problem to a combinatorial version of satisfying, while setting \( \zeta = 0 \) reduces the problem to regret minimization in standard CMAB (Chen et al., 2013).

### 2.4. Computation Oracles

Since combinatorial optimization is NP-hard in general, even when base arm outcomes are perfectly known identifying \( S_t^* \) and \( S_t^\alpha \) asc are intractable in general. Here, we assume black-box access to oracles that are able to identify these when fed with \( f \) (or an estimate of \( f \) which we call \( \hat{f}_t \)). The learner obtains \( S_t^\alpha \) from an \( \alpha \)-approximation oracle, Oracle$_S^\alpha$, which when given \( \hat{f}_t \), \( \hat{S}_t^\alpha \) (an estimate of \( S_t^\alpha \)) and the problem structure outputs a super arm \( S_t = \text{Oracle}_S^\alpha(\hat{f}_t) \) such that \( u(f(x_{t,\text{Oracle}_S^\alpha(\hat{f}_t)})) \geq \alpha \times \text{opt}(\hat{f}_t) \). We will explain in Section 2.5 the structure that \( \hat{f}_t \) takes in detail. We also make use of an exact oracle, Oracle$_G$, to identify the groups whose expected rewards satisfy their thresholds. This oracle takes as input a slightly different estimate of \( f \), \( \hat{f}_t \), and the set of feasible groups \( \hat{G}_t \) and outputs \( \hat{G}_t = \text{Oracle}_G(\hat{f}_t) \). For conciseness, we omit \( \hat{G}_t \) and \( S_t' \) from the notation. It should be noted that both oracles are deterministic given their inputs.

### 2.5. Structure of Base Arm Outcomes

In this part, we explain how the base arm outcome function \( f \) is modeled. As the learner cannot control the number of
available base arms $M_t$ and the context $X_t$ since they are observed, the expected outcomes can differ significantly over rounds. To ensure that the learner performs well, some regularity conditions are necessary. These conditions are on $f$. For our paper, we model $f$ as a sample from a GP, defined below.

**Definition 1.** A Gaussian Process with the index set $X$ is a collection of random variables $(f(x))_{x \in X}$ which satisfy the condition that $(f(x_1), \ldots, f(x_n))$ has a multivariate normal distribution for all $(x_1, \ldots, x_n)$ and $n \in \mathbb{N}$. The probability law of the GP is governed by its mean function given by $x \mapsto \mu(x) = \mathbb{E}[f(x)]$ and its covariance function given by $(x_1, x_2) \mapsto \mathbb{E}[f(x_1) - \mu(x_1))(f(x_2) - \mu(x_2))]$.

We assume that we have $k(x, x) \leq 1$ for every $x \in X$. This is a standard assumption generally used in GP bandits (Srinivas et al., 2012). The GP assumption also works well with our content caching example as we know that the probability of people watching a content is nonlinear. Therefore, the mapping between the base arm outcomes and the users’ contexts are nonlinear which is why the GP assumption is validated in this setup.

### 2.6. Posterior Distribution of Base Arm Outcomes

Our learning algorithm will make use of the posterior distribution of the GP-sampled function $f$. Given a fixed $N \in \mathbb{N}$ we consider a finite sequence $\tilde{X}[N] = [\tilde{x}_1, \ldots, \tilde{x}_N]^T$ of contexts with corresponding outcome vector $\tilde{r}[N] := r[\tilde{X}[N]] = [r(\tilde{x}_1), \ldots, r(\tilde{x}_N)]$ and the corresponding expected outcome vector $\tilde{f}[N] := [f(\tilde{x}_1), \ldots, f(\tilde{x}_N)]^T$. For every $n \leq N$, we have $r(\tilde{x}_n) = f(\tilde{x}_n) + \eta_n$ where $\eta_n$ is the noise corresponding to that outcome. The posterior distribution of $f$ given $r[N]$ is a GP characterized by its mean $\mu_N$ and its covariance $k_N$ which are given as follows:

$$
\mu_N(\tilde{x}) = (k_N[\tilde{x}])^T(K[\tilde{x}] + \sigma^2 I_N)^{-1}r_N,
$$

$$
k_N(\tilde{x}, \tilde{x}') = k(\tilde{x}, \tilde{x}') - (k_N[\tilde{x}])^T(K[\tilde{x}] + \sigma^2 I_N)^{-1}k_N[\tilde{x}'],
$$

$$
\sigma^2_N(\tilde{x}) = k(\tilde{x}, \tilde{x}) - (k_N[\tilde{x}])^T(K[\tilde{x}] + \sigma^2 I_N)^{-1}k_N[\tilde{x}].
$$

Here $k_N[\tilde{x}] = [k(\tilde{x}_1, \tilde{x}), \ldots , k(\tilde{x}_N, \tilde{x})]^T \in \mathbb{R}^{N \times 1}$ and $K_N = [k(\tilde{x}_i, \tilde{x}_j)]_{i,j=1}^N$. Overall, the posterior of $f(x)$ is given by $\mathcal{N}(\mu_N(x), \sigma^2_N(x))$ and the outcome $r(x)$ is given by $\mathcal{N}(\mu_N(x), \sigma^2_N(x) + \sigma^2)$.

### 2.7. The Information Gain

Regret bounds for GP bandits depend on how well $f$ can be learned from sequential interaction. The learning difficulty is quantized by the information gain, which accounts for the reduction in entropy of a random vector after a sequence of (correlated) observations. For a length $N$ sequence of contexts $\tilde{x}[N]$, evaluations of $f$ at contexts in $\tilde{x}[N]$, given by $\tilde{f}[N]$, and the corresponding sequence of outcomes $\tilde{r}[N]$, the information gain is defined as $I(\tilde{r}[N]; \tilde{f}[N]) := H(\tilde{r}[N]) - H(\tilde{r}[N]; \tilde{f}[N])$ where $H(\cdot)$ and $H(\cdot|\cdot)$ represent the entropy and the conditional entropy operators, respectively. Essentially, the information gain quantifies the reduction in the entropy of $\tilde{f}[N]$ given $\tilde{r}[N]$. The maximum information gain over the context set $X$ is denoted by $\gamma_N := \sup_{\tilde{x}[N]: \tilde{x}_e \in X, i \in [I]} I(r[i]; f[i])$.

It is important to understand that $\gamma_N$ is the maximum information gain for an $N$-tuple of contexts from $X$. For large context spaces, $\gamma_N$ can be very large. As we have varying base arm availability from round to round due to volatility, we only need to know the maximum information gain that is associated with a fixed sequence of base arm contexts arrivals. As we have a finite maximum information gain, this motivates us to relate the regret bounds to the informativeness of the available base arms. To adapt the definition of the maximum information gain to our setup with volatile arms and to ensure tight bounds, we define a new term $\overline{\gamma}_T$. We let $\mathcal{Z}_t \subset 2^{\mathcal{X}_t}$ for $t \geq 1$ be the set of sets of context vectors which correspond to the base arms in the feasible set of super arms $S_t$.

We let $\tilde{x}_i := x_{i,S}$ be an element of $\mathcal{Z}_t$ for a super arm $S$ and we let $\tilde{x}[T] := [\tilde{x}_1^T, \ldots , \tilde{x}_T^T]$. Then the maximum information gain which is associated with the context arrivals $\mathcal{X}_1, \ldots, \mathcal{X}_T$ is

$$
\overline{\gamma}_T = \max_{\tilde{x}[T]: \tilde{x}_e \in \mathcal{Z}_t, t \leq T} I(r[\tilde{x}[T]]; f[\tilde{x}[T]]).
$$

The information gain for any $T$ element sequence of feasible super arms has a maximum given by $\overline{\gamma}_T$. As it will be shown in Section 4, our regret bounds depend on $\overline{\gamma}_T$.

### 2.8. Special Cases

#### 2.8.1. Special Case 1: Super Arm and Group Rewards Related

In this case, super arm reward for $S_t$ picked in round $t$ is a function of the picked super arm and group rewards and is defined as $U(S_t, [V(r(x_i,G))]_{G \in G_t, \lambda \subseteq \lambda_i})$, while the expected super arm reward is defined as $u(S_t, [v(f(x_i,G))]_{G \in G_t, \lambda \subseteq \lambda_i}) = \mathbb{E}[U(S_t, [V(r(x_i,G))]_{G \in G_t, \lambda \subseteq \lambda_i})]$. We assume that the randomness of $U$ and $V$ come from the randomness in the base arm outcomes. Also, we assume that $u$ is monotone and Lipschitz continuous in $v$. As $v$ is monotone and Lipschitz continuous in $f$ by assumption, $u$ is also monotone and Lipschitz continuous in $f$ which justifies that this special case does not exclude the original assumptions of the general case. An example from the content caching.

4Once again, we will suppress $S_t$ from $U(S_t, \ldots)$ and $u(S_t, \ldots)$ when $S_t$ is clear from context.
setup may be the following: The group reward is 1 if at least one of the base arms in the group has outcome 1, otherwise the group reward is 0. Base arm \( i \) yields 1 with probability \( p_i \) and 0 with probability \( 1 - p_i \) where \( p_i \) represents the probability that the user will download that content in that location. In essence, we want at least one of the items cached in the location to be downloaded. Super arm reward becomes a sum of group rewards in this case. The setup of gem mining in (Verstraeten et al., 2020) is also applicable to this case when base arms are mine workers, groups contain the workers who work in the same mine and super arms are companies that send workers to mines.

2.8.2. Special Case 2: Groups are Singletons

In this case each group contains just one base arm and the group reward function is the identity mapping. This problem is significant and interesting to explore as it can be applied to the setup discussed in (Qureshi et al., 2021) which is assigning users to small base station channels (SBSCs). In addition, this case conforms with the beam alignment problem discussed in (Wu et al., 2019) with the addition of contexts. When there is a single transmitter but multiple receivers and beams, the transmitter-receiver beam pairs correspond to base arms, their unions happen to be super arms and factors such as beam angle are contexts. Since the received signal strength (RSS) needs to be above a certain value for correct transmission, thresholding is appropriate.

2.8.3. Special Case 3: Only Reward Maximization

In this case, the conditions are the same with Special Case 2 except that Oracle\(_G\) is not used as there aren’t any thresholds and \( \zeta = 0 \). This corresponds to the case where the only goal is to maximize super arm reward and has some important applications such as dynamic maximum weighted bipartite matching which can model multi-user multi-channel communication and dynamic probabilistic maximum coverage.

3. Algorithm

We design an optimistic algorithm that uses GP UCBs in order to identify groups that satisfy thresholds and super arms that maximize rewards. We name our algorithm Thresholded Combinatorial GP-UCB (TCGP-UCB) with pseudocode given in Algorithm 1. TCGP-UCB uses the double UCB principle, keeping different exploration bonuses for group and super arm selection. These bonuses are adjusted based on the tradeoff parameter \( \zeta \) given in the regret definition (1).

At each round \( t \), for every available base arm, two indices which are upper confidence bounds on the expected outcome are defined by taking into consideration the parameter \( \zeta \in [0, 1] \). Consider base arm \( m \) with its associated context \( x_{t,m} \). We define its reward index as

\[
i_t(x_{t,m}) = \mu_{t-1}(x_{t,m}) + \frac{1}{1-\zeta} \sqrt{\beta_t \sigma_{t-1}(x_{t,m})},
\]

where \( \mu_{t-1}(x_{t,m}) \) and \( \sigma_{t-1}(x_{t,m})^2 \) stand for the posterior mean and variance given observations \( \{r_{S_1}^T, \ldots, r_{S_{t-1}}^T\} \) where \( S_1, \ldots, S_{t-1} \) represent the super arms chosen in the past \( t - 1 \) rounds. Similarly, we define the satisfying index of base arm \( m \) as

\[
i'_t(x_{t,m}) = \mu_{t-1}(x_{t,m}) + \frac{1}{\zeta} \sqrt{\beta_t \sigma_{t-1}(x_{t,m})}.
\]

For each group \( G \in \mathcal{G}_t \) in round \( t \), we define the satisfying indicator \( h_{t,G} \) as

\[
h_{t,G} = \mathbb{I}(v(i'_t(x_{t,G})) \geq \gamma_{t,G}),
\]

where \( \mathbb{I}(\cdot) \) is the indicator function that equals to zero if the argument inside it is false and one otherwise. Also, \( \gamma_{t,G} \) is a predefined threshold value known to the learner.

We give the indices \( i'_t(x_{t,m}) \) to Oracle\(_G\), which returns the groups whose expected rewards are greater than their respective thresholds among the feasible group set \( \mathcal{G}_t \). We form the set of feasible super arms whose groups are expected to satisfy their thresholds, \( \mathcal{S}_t' \), by using the returned groups and \( \mathcal{S}_t \). Namely, we have \( \mathcal{S}_t' = \{ S \in \mathcal{S}_t : \forall (G \in \mathcal{G}_t : G \subseteq S), v(i'_t(x_{t,G})) > \gamma_{t,G}\} \). Then, we give the indices \( i_t(x_{t,m}) \) of the base arms that construct \( \mathcal{S}_t' \) to Oracle\(_S\), which returns an \( \alpha \)-optimal super arm.

Algorithm 1 TCGP-UCB

1: Input: \( \mathcal{X}, K, M, \zeta \), GP Prior: \( \mu_0 = \mu, k_0 = k \).
2: Initialize: \( \mu_0 = \mu, k_0 = k \).
3: for \( t = 1, \ldots, T \) do
4: Observe base arms in \( \mathcal{M}_t \) and their contexts \( x_{t,i} \) as well as groups in \( \mathcal{G}_t \).
5: for \( x_{t,m} : m \in \mathcal{M}_t \) do
6: Calculate \( \mu_{t-1}(x_{t,m}) \) and \( \sigma_{t-1}(x_{t,m}) \).
7: end for
8: Compute indices \( i_t(x_{t,m}) \) and \( i'_t(x_{t,m}) \).
9: \( \hat{G}_t \leftarrow \text{Oracle}\_G(i'_t(x_{t,m}))_{m \in \mathcal{M}_t}, \mathcal{G}_t \).
10: Form the set \( \mathcal{S}_t' \).
11: \( S_t \leftarrow \text{Oracle}\_S(i_t(x_{t,m}))_{m \in \mathcal{M}_t}, \mathcal{S}_t' \).
12: Observe base arm outcomes in \( \mathcal{S}_t \), collect group rewards and super arm reward.
13: end for

4. Theoretical Analysis

Next is our main result which asserts a high probability upper bound on the total regret of our algorithm in terms of
Theorem 1. Super arm regret and group regret incurred by TCGP-UCB in $T$ rounds are upper bounded with probability at least $1 - \delta$ where $\delta \in (0, 1)$, $T \in \mathbb{N}$ and $\beta_t = 2 \log (M \pi^2 T^2 / 3\delta)$ as follows:

$$R_g(T) \leq K \sqrt{C_1 \beta_T T \gamma_T},$$

where $C_1 = 2B^2 (\frac{1+\epsilon}{\epsilon})^2 / \log (1 + \sigma^{-2})$ and

$$R_s(T) \leq K \sqrt{C_2 \beta_T T \gamma_T},$$

where $C_2 = 2B^2 (\frac{\epsilon}{1-\epsilon})^2 / \log (1 + \sigma^{-2})$. Hence, the total regret is bounded by:

$$R(T) \leq K \sqrt{C \beta_T T \gamma_T},$$

where $C = 8(B + B')^2 / \log (1 + \sigma^{-2})$.

Next, we provide a regret bound that depends on the classical notion of maximum information gain. Here, we assume that $|S| = K$ for any $S \in S$ where $K$ is fixed. In content caching, this fixed cardinality corresponds to having a fixed budget and deciding which location and what content to use this budget on.

Theorem 2. Fix $\delta \in (0, 1)$ and let $T, K \in \mathbb{N}$. Under the conditions of Theorem 1 and assuming that $|S| = K$ for any $S \in S$, the total regret incurred by TCGP-UCB in $T$ rounds is upper bounded with the following with probability at least $1 - \delta$:

$$R(T) \leq K \sqrt{C \beta_T T \gamma_T},$$

where $\gamma_T$ is as defined in Section 2.7 and $C$ is the same as in Theorem 1.

These theorems also hold for the special cases with slight changes in their proofs and the regret bounds for the special cases have the same order with the general case but differ in the constants of $C_1$, $C_2$ and $C$. Detailed explanations can be found in the supplemental document.

Corollary 1. Fix $\delta \in (0, 1)$, $T, K \in \mathbb{N}$ and let $\mathcal{X} \subset \mathbb{R}^D$ be compact and convex. Under the conditions of Theorem 2, the super arm regret incurred by TCGP-UCB in $T$ rounds is upper bounded up to polylog factors with probability at least $1 - \delta$ for the following kernels with:

- Linear kernel: $R_s(T) \leq \tilde{O}(K \sqrt{DT})$,
- RBF kernel: $R_s(T) \leq \tilde{O}(K \sqrt{T})$,
- Matern kernel: $R_s(T) \leq \tilde{O}(KT^{(D+\nu)/(D+2\nu)})$,

where $\nu > 1$ is the Matern parameter. This section contained analysis from (Nika et al., 2021) with additions of group regret and tradeoff parameter.

5. Experiment

We evaluate the performance of TCGP-UCB by comparing it with the current CCV-MAB state-of-the-art algorithm, ACC-UCB (Nika et al., 2020) in a semi-synthetic caching-based setup using real-world movie recommendation data. We also perform more extensive simulations and further investigate the effect of $\zeta$ on both the group and super arm regret in the supplemental document.

5.1. Setup

Continuing with our caching example, we assume a setup where the learner has to choose a subset of movies to cache at different locations. In our simulation, we assume that there are a total of 10 distinct locations. In each round, a new set of movie-location pairs, base arms, is observed by the learner. We assume that each base arm comes with information about the users at its location and information about its movie. This information is represented in the base arm’s context, which will be defined in detail later. Then, we define groups as a set of movie-location pairs with the same locations. Note that there does not need to exist one group for each location, and that there can exist multiple groups for one location.

We use the MovieLens 25M dataset (Harper & Konstan, 2015) for movie genre and user preference information. The MovieLens dataset comprises more than 25 million ratings given by users to around 62 thousand movies from 1995 to 2019. Each movie also comes with genre metadata that indicates the genres of the movie from a set of 20 genres (action, adventure, comedy, etc.). Moreover, each rating is between 0.5 and 5.0, increasing in increments of 0.5.

In our simulation, we only consider ratings after 2015 and users who have rated at least 200 movies. Then, in each round $t$, we first sample the number of groups, $|G_t|$, from a Poisson distribution with mean 50. Then, we assign to each group a location uniformly sampled from the set of locations, $\{1, \ldots, 10\}$. Then for each location, we sample 100 users without replacement from the MovieLens user set. We also sample movies for each group without replacement from the MovieLens movie set, where the number of movies follows a Poisson distribution with mean 5. We then average over the preference vectors of the users in a location, where a user’s preference vector is defined as the average of the genres of the movies that the user rated, weighed by their rating. We use the average preference vector of the users at the group’s location and each base arm’s movie’s genre vector to define the context of base arm $i$ in the group as $x_i = \langle u, g_i \rangle / 10$, where $u$ is the average preference vector of the users in the group’s location, $g_i$ is the genre vector of the corresponding movie of base arm $i$, and 10 is a
normalizing factor.$^5$

Then, the expected outcome of a base arm is defined as the expected number of cache hits (i.e., expected number of times that the users at the base arm’s location will watch the movie associated with the base arm). Mathematically, we define the probability of a user who is associated with the base arm with context $x$ to be $2/(1 + e^{-4x}) - 1$. We define this probability as such so that it is in $[0, 1]$ and also so that it is a non-linear increasing function in $x$. Thus, the more closely the average preference vector of the users of a location resembles a movie, the higher the chance of a user from a group with that location watching that movie. Then, the expected outcome of a base arm with context $x$ is given by $f(x) = 100 \left( \frac{2}{1 + e^{-4x}} - 1 \right)$, where we multiply by 100 because there are 100 users in each location. Finally, the random outcome of the base arm, $r(x)$, is defined as a Binomial random variable with $n = 100$ and $p = 2/(1 + e^{-4x}) - 1$.

The group reward function, $v$, is defined as the linear sum of the number of cache hits. Then, each group’s threshold represents the minimum number of cache hits desired. We justify the group thresholds by assuming that caching is expensive and thus storing cache at locations from where few users will consume movies is discouraged. In our simulation, the threshold of a group $G$ is set depending on the number of movies in the group. Formally, in round $t$ and for each group $G \in \mathcal{G}_t$, we have $\gamma_{t,G} = 100|G|/2$, where we multiply by 100 because there are 100 users at each location.

Finally, we define the super arm reward as the sum of the base arm outcomes (i.e., number of total cache hits).

5.2. Algorithms

**STCGP-UCB:** We run a slightly modified version of our algorithm that uses sparse approximation to GPs, called STCGP-UCB, where we use the sparse approximation to the GP posterior described in (Titsias, 2009). In this sparse approximation, instead of using all of the arm contexts up to round $t$ to compute the posterior, a small $s$ element subset of them is used, called the inducing points. By using a sparse approximation, the time complexity of the posterior updating procedure reduces from $O(K^3 + (2K)^3 + \ldots + (nK)^3) = O(K^3T^4)$ to $O(s^2KT^2)$. In our simulation, we set $s = 10$. We also set $\delta = 0.05$, $\zeta = 0.5$, and use a squared exponential kernel with both lengthscale and variance set to 1. Both oracles are set to exact oracles, thus $\alpha = 1$.

$^5$We divide by 10 and not 20 to normalize the context because the maximum number of genres that a movie has in the dataset is 10.

**ACC-UCB:** We set $v_1 = 1$, $v_2 = 1$, $\rho = 0.5$, and $N = 2$, as given in Definition 1 of (Nika et al., 2020). The initial (root) context cell, $X_{0,1}$, is a one dimensional unit hypercube (i.e., a line) centered at $(0.5)$. ACC-UCB’s oracle is also set to an exact oracle.

5.3. Results

We run the simulation for 100 rounds with eight independent runs, averaging over the results of each run. We plot the super arm and group regret in Figure 1. First, notice that ACC-UCB incurs linear group regret as it does not take group thresholds into account. On the other hand, STCGP-UCB incurs very low group regret as it accounts for group thresholds. Additionally, the super arm regret of STCGP-UCB is less than that of ACC-UCB, indicating that even though STCGP-UCB is balancing between minimizing both group and super arm regret, it still outperforms ACC-UCB.

6. Conclusion

We considered the contextual combinatorial volatile multiarmed bandit problem with semi-bandit feedback, where in each round, the agent has to play a feasible subset of the base arms in order to maximize the cumulative reward and also satisfy group requirements. Under the assumption that the expected base arm outcomes are drawn from a Gaussian Process and that the expected reward is Lipschitz continuous with respect to the expected base arm outcomes, we proposed TCGP-UCB, a double UCB algorithm that incurs $O(K \sqrt{T} \gamma_T)$ regret in $T$ rounds. In experiments, we showed that sparse GPs can be used to speed up UCB computation, while simultaneously outperforming the state-of-the-art non-GP-based CCV-MAB algorithm. Our comparisons also indicated that GPs can transfer knowledge among contexts better than partitioning the contexts into
groups of similar contexts based on a similarity metric. An interesting future research direction involves investigating how dependencies between base arms can be used for more efficient exploration. For instance, when the oracle selects base arms sequentially, it is possible to update the posterior variances of the not yet selected base arms by conditioning on the selected, but not yet observed, base arms.

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A. Table of Notation

We provide a table of notation so that the following proof section can be easily followed.

| Symbol | Explanation |
|--------|-------------|
| \( m \) | Variable to denote a base arm |
| \( G \) | Variable to denote a group |
| \( S \) | Variable to denote a super arm |
| \( \mathcal{M}_t \) | Set of base arms that are available at round \( t \) |
| \( M_t \) | Number of base arms that become available at round \( t \) |
| \( \mathcal{G}_t \) | Set of feasible groups in round \( t \) |
| \( \hat{\mathcal{G}}_t \) | Set of groups whose expected rewards are above their thresholds |
| \( \tilde{\mathcal{G}}_t \) | Set of feasible groups whose thresholds are not satisfied |
| \( \mathcal{S}_t \) | Set of feasible super arms in round \( t \) |
| \( \mathcal{S}'_t \) | Set of super arms whose all groups satisfy their thresholds |
| \( \hat{\mathcal{S}}'_t \) | Set of feasible super arms whose groups’ reward indices are above their thresholds |
| \( \mathcal{S}_{\text{overall}} \) | Overall feasible set of super arms |
| \( \mathcal{X} \) | Context set |
| \( \mathcal{X}_t \) | Set of available contexts in round \( t \) |
| \( x_{t,m} \) | Context associated with base arm \( m \) |
| \( x_{t,G} \) | Context vector of base arms in \( G \) |
| \( x_{t,S} \) | Context vector of base arms in \( S \) |
| \( r(x) \) | Random outcome of base arm with context \( x \) |
| \( f(x) \) | Expected outcome of base arm with context \( x \) |
| \( \eta \) | \( \mathcal{N}(0, \sigma^2) \) independent observation noise |
| \( V_G(r(x_{t,G})) \) | Random group reward |
| \( v_G(f(x_{t,G})) \) | Expected group reward function |
| \( U(S, r(x_{t,S})) \) | Random super arm reward |
| \( u(S, f(x_{t,S})) \) | Expected super arm reward function |
| \( U(S, |V(r(x_{t,G}))|_{G \in \mathcal{G}_t} \wedge G \subseteq S_t}) \) | Random super arm reward of Special Case 1 |
| \( u(S, |v(f(x_{t,G}))|_{G \in \mathcal{G}_t} \wedge G \subseteq S_t}) \) | Expected super arm reward function of Special Case 1 |
| \( \gamma_{t,G} \) | Threshold for group \( G \) at round \( t \) |
| \( \zeta \) | Tradeoff parameter |
| \( i_t(x_{t,m}) \) | Index of base arm \( m \) at round \( t \) which is given to Oracle \( S \) |
| \( i'_t(x_{t,m}) \) | Index of base arm \( m \) at round \( t \) which is given to Oracle \( G \) |
| \( \gamma_T \) | Maximum information gain which is associated with the context arrivals \( \mathcal{X}_1, \ldots, \mathcal{X}_T \) |
| \( \gamma_N \) | Maximum information gain given \( N \) base arm outcome observations |
| \( K \) | Maximum possible number of base arms in a super arm |

When we state that \( a \geq b \) with \( a \) and \( b \) being vectors, we mean that every component of \( a \) is greater than or equal to the corresponding component of \( b \). This holds for other comparison operators as well. Also, we omit \( G \) from \( v_G \) and \( V_G \); and \( S \) from \( U(S, \ldots) \) and \( u(S, \ldots) \) when it is obvious from the context.

B. Experimental Results

We perform extra simulations to showcase the behavior of our algorithm for changing tradeoff parameter (\( \zeta \)) values.

B.1. Setup

We use a synthetic setup where we generate the arm and group data needed for the simulation. Similar to the main paper simulations, in each round \( t \), we first sample the number of groups, \( |\mathcal{G}_t| \), from a Poisson distribution with mean 50. Then, for each group we generate the contexts of the base arms in that group, where the number of base arms is sampled from a
Poisson distribution with mean 5. Each base arm has a two dimensional (2D) context that is sampled uniformly from $[0, 1]^2$. Then, we sample the expected outcome of each base arm of each group from a GP with zero mean and a squared exponential kernel, given by

$$k(x, x') = \exp \left( -\frac{1}{2l^2} \|x - x'\| \right),$$

where we set $l = 0.5$. Note that given that we run the simulation for $T = 100$ rounds, there will be an expected number of 25000 arms and to sample a GP function with that many points we will need to compute the Cholesky decomposition of a 625 million element matrix, which would be very resource-heavy. Instead, we first sample 6000 2D contexts from $[0, 1]^2$ and then sample the GP function at those points. Then, during our simulation, we sample each base arm’s context $x$ and corresponding expected outcome $f(x)$ from the generated sets. Finally, we set $r(x) = f(x) + \eta$, where $\eta \sim \mathcal{N}(0, 0.1^2)$. We set the group reward, $v$, to be the variance of the outcomes in the group and we set the threshold to be the 80% percentile of the group rewards of all groups in all rounds of the simulation. We use a high percentile value to increase the difficulty of the group thresholding, so that minimizing super arm regret does not necessarily yield minimizing group regret. Finally, the super arm reward is the linear sum of the base arms.

**B.2. Results**

We run our algorithm using five different values of $\zeta$, linearly spaced between 0.001 and 0.999. Figure 2 shows the final super arm and group regret of each $\zeta$ run. First, notice a tradeoff between super arm and group regret. As one increases, the other decreases. This is expected because to minimize group regret, groups with arms whose outcomes are spread out and have high variance must be picked, but to minimize super arm regret, groups with high outcomes must be picked. Second, as $\zeta$ increases, the super arm regret increases while the group regret decreases. This is because the variance of the indices given to the first oracle ($i'$), which determines which groups pass their thresholds, decreases as $\zeta$ approaches 1. Thus, the larger $\zeta$ is, the stricter the first oracle is. Conversely, the smaller $\zeta$ is the laxer the first oracle and the stricter the second oracle, which determines which super arm to play.

![Figure 2. Final super arm and group regret for different tradeoff parameter, $\zeta$, values](image)

**C. Proofs**

We start by stating the necessary lemmas and theorems for our general case, which are based on the work of (Nika et al., 2021). Then, we modify them according to our special cases and express the end results.

**C.1. Proofs for General Case**

We denote by $r_{[t-1]}$ the vector of base arm outcome observations made until the beginning of round $t$, where

$$r_{[t-1]} = [r^T(x_{1,s_1}), \ldots, r^T(x_{t-1,s_{t-1}})]^T.$$
For any $t \geq 1$, the posterior distribution of $f(x)$ given the observation vector $r_{[t-1]}$ is $\mathcal{N}(\mu_{[t-1]}(x), \sigma_{[t-1]}^2(x))$, for any $x \in \mathcal{X}_t$. In our analysis, we will resort to the following Gaussian tail bound

$$
P \left( |f(x) - \mu_{[t-1]}(x)| > \sqrt{\beta_t} \sigma_{[t-1]}(x) \bigg| r_{[t-1]} \right) \leq 2 \exp \left( -\frac{\beta_t}{2} \right) \text{ for } \beta_t \geq 0 .$$

(2)

The following lemma shows that the base arm indices upper bound the expected outcomes with high probability.

**Lemma 1.** [Lemma 1 of (Nika et al., 2021)] Fix $\delta \in (0, 1)$, and set $\beta_t := 2 \log (M \pi^2 t^2 / 3 \delta)$. Let $\mathcal{F} := \{ \forall t, \forall x \in \mathcal{X}_t : |f(x) - \mu_{[t-1]}(x)| \leq \sqrt{\beta_t} \sigma_{[t-1]}(x) \}$. We have $\mathbb{P}(\mathcal{F}) \geq 1 - \delta$.

Now, we state that the modified indices upper bound the expected base arm outcomes with high probability under the event $\mathcal{F}$.

** Lemma 2.** The following arguments hold under event $\mathcal{F}$. Event $\mathcal{F}$ holds with probability at least $1 - \delta$.

$$i_t(x) \geq f(x), \forall t \geq 1, \forall x \in \mathcal{X}_t$$

$$i_t(x) \geq f(x), \forall t \geq 1, \forall x \in \mathcal{X}_t .$$

**Proof.** Fix $t \geq 1$ and $x \in \mathcal{X}_t$. Under event $\mathcal{F}$, we have the following chain of inequalities

$$f(x) - \mu_{[t-1]}(x) \leq \sqrt{\beta_t} \sigma_{[t-1]}(x)$$

$$f(x) - \mu_{[t-1]}(x) \leq \frac{1}{1 - \gamma} \sqrt{\beta_t} \sigma_{[t-1]}(x)$$

$$f(x) \leq \mu_{[t-1]}(x) + \frac{1}{1 - \gamma} \sqrt{\beta_t} \sigma_{[t-1]}(x) = i_t(x) .$$

Proceeding in the same fashion, we obtain

$$f(x) - \mu_{[t-1]}(x) \leq \frac{1}{\zeta} \sqrt{\beta_t} \sigma_{[t-1]}(x)$$

$$f(x) \leq \mu_{[t-1]}(x) + \frac{1}{\zeta} \sqrt{\beta_t} \sigma_{[t-1]}(x) = i'_t(x) .$$

where (3) and (4) follow from the fact that $\frac{1}{1 - \gamma} \geq 1$ and $\frac{1}{\zeta} \geq 1$ when $\zeta \in [0, 1]$.

Next, we show that the set of feasible super arms whose groups satisfy their thresholds is a subset of the set of feasible super arms whose groups’ reward indices are above their thresholds. We later use this result in Lemma 5.

**Lemma 3.** Fix $\delta \in (0, 1)$. The following argument holds under the event $\mathcal{F}$ when the group reward function $v$ satisfies the monotonicity assumption given in Assumption 1.

$$S'_t \subseteq \hat{S}'_t, \forall t \geq 1.$$ 

**Proof.** For any $G \in \mathcal{G}_t$ we have:

$$G \in S'_t \iff v(f(x_{t,G})) \geq \gamma_{t,G}$$

$$\iff v(f(x_{t,G})) - \gamma_{t,G} \geq 0$$

$$\iff v(i'_t(x_{t,G})) - \gamma_{t,G} \geq 0$$

$$\iff G \in \hat{S}'_t$$

(5)

(6)

(7)

where (5) follows from the definition of $S'_t$, (6) follows from the inequality that $v(i'_t(x_{t,G})) \geq v(f(x_{t,G}))$. Since $i'_t(x_{t,G}) \geq f(x_{t,G})$ under the event $\mathcal{F}$ and $v$ is monotone by assumption, this inequality is valid. Lastly, (7) follows from the fact that Oracle$_G$ is an exact oracle and will return the groups where $v(i'_t(x_{t,G})) > \gamma_{t,G}$. As this reasoning is true for any $G$, we indicate that $S'_t \subseteq \hat{S}'_t$.
Next, we upper bound the group regret in terms of the posterior variances of base arms. Note that group regret is incurred when a selected group’s expected reward is below its threshold whereas its index is above. Therefore, whenever a group \(G\) incurs group regret in round \(t\), then \(v(f(x_{t,G})) < \gamma_{t,G} < v(i'_t(x_{t,G}))\) must happen. This observation plays a key role in the analysis of the next lemma. Moreover, we use the notation \(\hat{x}_{t,k}\) to denote the context of the \(k^{th}\) base arm in \(G\) at round \(t\) for convenience, unless otherwise stated.

**Lemma 4.** Fix \(t \geq 1\), and consider \(G \in \hat{G}_t\) such that \(G \subseteq S_t\). The following argument holds under the event \(\mathcal{F}\):

\[
[\gamma_{t,G} - v(f(x_{t,G}))]_+ \leq \left(\frac{\zeta + 1}{\zeta}\right) B\sqrt{\beta_t \sum_{k=1}^{|G|} \sigma_{[t-1]}(\hat{x}_{t,k})}.
\]

**Proof.** \([\gamma_{t,G} - v(f(x_{t,G}))]_+ > 0\) implies that \(v(f(x_{t,G})) < \gamma_{t,G}\). Moreover \(G \subseteq S_t\) implies that \(v(i'_t(x_{t,G})) \geq \gamma_{t,G}\). Therefore, whenever \(G\) incurs group regret it holds that \(v(f(x_{t,G})) < \gamma_{t,G} \leq v(i'_t(x_{t,G}))\).

\[
0 < \gamma_{t,G} - v(f(x_{t,G})) < v(i'_t(x_{t,G})) - v(f(x_{t,G}))
\]
\[
0 < [\gamma_{t,G} - v(f(x_{t,G}))]_+ < v(i'_t(x_{t,G})) - v(f(x_{t,G}))
\]
\[
\leq B_G \sum_{k=1}^{|G|} |i'_t(\hat{x}_{t,k}) - f(\hat{x}_{t,k})| 
\]
\[
\leq B_G \sum_{k=1}^{|G|} |\mu_{[t-1]}(\hat{x}_{t,k}) - f(\hat{x}_{t,k})| + B_G \sum_{k=1}^{|G|} \frac{1}{\zeta} \sqrt{\beta_t \sigma_{[t-1]}(\hat{x}_{t,k})}
\]
\[
\leq \left(\frac{\zeta + 1}{\zeta}\right) B\sqrt{\beta_t \sum_{k=1}^{|G|} \sigma_{[t-1]}(\hat{x}_{t,k})},
\]

where (9) follows from the Lipschitz continuity of \(v_G\); (10) follows from the definition of index and the triangle inequality; for (11) we use Lemma 1 and the definition of \(B\).

Next, we upper bound the gap of a selected super arm in round \(t\) (aka simple regret) in terms of the posterior variances of base arms. Hereafter, we use the notation \(\pi_{t,k}\) to denote the context \(x_{t,s_{t,k}}\) of the \(k^{th}\) selected base arm \(s_{t,k}\) at round \(t\) for convenience, unless otherwise stated.

**Lemma 5.** Given round \(t \geq 1\), let \(S_t^* = \{s_{t,1}^*, \ldots, s_{t,|S_t^*|}^*\}\) denote the optimal super arm in round \(t\). Then, the following argument holds under the event \(\mathcal{F}\):

\[
\alpha \cdot u(f(x_{t,S_t^*})) - u(f(x_{t,S_t})) \leq \left(\frac{2 - \zeta}{1 - \zeta}\right) B'\sqrt{\beta_t \sum_{k=1}^{|S_t|} \sigma_{[t-1]}(\pi_{t,k})}
\]
Proof. We define $H_t = \arg\max_{S \in \mathcal{S}_t^*} u(i_t(x_{t,S}))$. Given that event $\mathcal{F}$ holds, we have:

$$\alpha \cdot u(f(x_{t,S_t^*}, x_{t,S_t^*})) - u(f(x_{t,S_t})) \leq \alpha \cdot u(i_t(x_{t,S_t^*})) - u(f(x_{t,S_t}))$$  \hspace{0.5cm} (13)

$$ \leq \alpha \cdot u(i_t(x_{t,H_t})) - u(f(x_{t,S_t}))$$  \hspace{0.5cm} (14)

$$ \leq u(i_t(x_{t,S_t})) - u(f(x_{t,S_t}))$$  \hspace{0.5cm} (15)

$$ \leq B' \sum_{k=1}^{\left|S_t\right|} |i_t(\pi_{t,k}) - f(\pi_{t,k})| $$  \hspace{0.5cm} (16)

$$ \leq B' \sum_{k=1}^{\left|S_t\right|} |\mu_{t-1}(\pi_{t,k}) - f(\pi_{t,k})| + B' \sum_{k=1}^{\left|S_t\right|} \frac{1}{1 - \zeta} \sqrt{\beta_t \sigma_{t-1}^2(x_{t,k})} $$  \hspace{0.5cm} (17)

$$ \leq \left(\frac{2 - \zeta}{1 - \zeta}\right) B' \sqrt{\frac{J}{\zeta}} \sum_{k=1}^{\left|S_t\right|} \sigma_{t-1}^2(x_{t,k}), $$  \hspace{0.5cm} (18)

where (13) follows from monotonicity of $u$ and the fact that $f(x_{t,s_{t,k}^*}) \leq i_t(x_{t,s_{t,k}^*})$, for $k \leq |S_t^*|$ (Lemma 2); (14) follows from the definition of $H_t$ and the fact that $S_t^* \subseteq S_t^*$ on event $\mathcal{F}$ (Lemma 3), in other words, $\max_{S \in \mathcal{S}_t} u(i_t(x_{t,S})) \geq \max_{S \in \mathcal{S}_t} u(i_t(x_{t,S}))$ since $S_t^* \subseteq S_t^*$; (15) holds since $S_t$ is the super arm chosen by the $\alpha$-approximation oracle; (16) follows from the Lipschitz continuity of $u$; (17) follows from the definition of index and the triangle inequality; for (18) we use Lemma 1.

Before proving our theorems, we prove our last lemma which enables us to have our regrets bounds in terms of the information gain. We note that both of our oracles are deterministic and our algorithm doesn’t give any extra randomization. Hence, $S_t^*$ is deterministic given $z_{[t-1]}$.

Lemma 6. [Lemma 3 of (Nika et al., 2021)] Let $z_t := x_{t,S_t}$ be the vector of selected contexts at time $t \geq 1$. Given $T \geq 1$, we have:

$$ I(r(z_{[T]}); f(z_{[T]})) \geq \frac{1}{2K} \sum_{t=1}^{T} \sum_{k=1}^{\left|S_t\right|} \log \left(1 + \sigma^{-2} \sigma_{t-1}^2(x_{t,k})\right), $$

where $z_{[T]} = [z_1, \ldots, z_T]^T$ is the vector of all selected contexts until round $T$.

Now, we proceed with the proofs of our theorems.

Theorem 1. Super arm regret and group regret incurred by TCGP-UCB in $T$ rounds are upper bounded with probability at least $1 - \delta$ where $\delta \in (0, 1)$, $T \in \mathbb{N}$ and $\beta_t = 2 \log \left(\frac{3M^2t^2}{\delta}\right)$ as follows:

$$ R_g(T) \leq K \sqrt{C_1 \beta_T T \gamma_T}, $$

where $C_1 = 2B^2(\frac{\zeta}{1 - \zeta})^2 / \log (1 + \sigma^{-2})$ and

$$ R_s(T) \leq K \sqrt{C_2 \beta_T T \gamma_T}, $$

where $C_2 = 2B^2(\frac{2 - \zeta}{1 - \zeta})^2 / \log (1 + \sigma^{-2})$. Hence, the total regret is bounded by:

$$ R(T) \leq K \sqrt{C \beta_T T \gamma_T}, $$

where $C = 8(B + B')^2 / \log (1 + \sigma^{-2})$. 

Proof. From Lemma 4 we have:

\[ R_g(T) = \sum_{t=1}^{T} \sum_{G \in \hat{G}_t : G \subseteq S_t} [\gamma_{t,G} - v(f(x_{t,G}))]_+ \]

\[ \leq \left( \frac{\zeta + 1}{\zeta} \right) B \sqrt{\beta_T} \sum_{t=1}^{T} \sum_{G \in \hat{G}_t : G \subseteq S_t} |\sigma_{t-1}(\tilde{x}_{t,k})| \]

\[ = \left( \frac{\zeta + 1}{\zeta} \right) B \sqrt{\beta_T} \sum_{t=1}^{T} \sum_{k=1}^{G} |\sigma_{t-1}(\tilde{x}_{t,k})| \quad (19) \]

using the fact that \( \sqrt{\beta_T} \) is monotonically increasing in \( t \). Also, we changed the notation of \( \tilde{x}_{t,k} \) with \( \pi_{t,k} \) as we are summing through all the base arms in \( S_t \) in (19). To proceed, let us define a constant \( L = \sigma^{-2}/(1 + \log \sigma^{-2}) \). We have:

\[ R_g^2(T) \leq \left( \frac{\zeta + 1}{\zeta} \right)^2 B^2 \beta_T \left( \sum_{t=1}^{T} \sum_{k=1}^{G} |\sigma_{t-1}(\pi_{t,k})| \right)^2 \quad (20) \]

\[ \leq \left( \frac{\zeta + 1}{\zeta} \right)^2 B^2 \beta_T T \sum_{t=1}^{T} \sum_{k=1}^{G} |\sigma_{t-1}(\pi_{t,k})| \quad (21) \]

\[ \leq \left( \frac{\zeta + 1}{\zeta} \right)^2 B^2 \beta_T T |S_t| \sum_{k=1}^{G} |\sigma_{t-1}(\pi_{t,k})|^2 \quad (22) \]

\[ \leq \left( \frac{\zeta + 1}{\zeta} \right)^2 B^2 \beta_T T \sum_{t=1}^{T} \sum_{k=1}^{G} |\sigma_{t-1}(\pi_{t,k})|^2 \quad (23) \]

\[ \leq \left( \frac{\zeta + 1}{\zeta} \right)^2 B^2 \beta_T T K \sigma^2 \sum_{t=1}^{T} \sum_{k=1}^{G} \sigma^{-2} |\sigma_{t-1}(\pi_{t,k})|^2 \quad (24) \]

\[ \leq 2 \left( \frac{\zeta + 1}{\zeta} \right)^2 B^2 \beta_T T K^2 \sigma^2 \mathcal{L}r(z_{|T|}; f(z_{|T|})) \quad (25) \]

\[ \leq C_\zeta K^2 \beta_T T \gamma_T \quad (26) \]

where for (21) and (22) we have used the Cauchy Schwarz inequality twice; in (23) we just multiply by \( \sigma^2 \) and \( \sigma^{-2} \); for (24) we use the following argument. Note that for any \( i \in [0, \sigma^{-2}] \) we have that \( i \leq L \log(1 + i) \). Moreover, by assumption we have \( |\sigma_{t-1}(\pi_{t,k})| \leq 1 \), for all \( t \geq 1 \), for all \( k \leq K \), and thus, \( \sigma^{-2} |\sigma_{t-1}(\pi_{t,k})|^2 \leq \sigma^{-2} \). Therefore, we have that \( \sigma^{-2} |\sigma_{t-1}(\pi_{t,k})|^2 \leq L \log(1 + \sigma^{-2} |\sigma_{t-1}(\pi_{t,k})|^2) \); (25) follows from Lemma 6 and for (26) we use the definition of \( \gamma_T \).

Taking the square root of both sides we obtain our desired result.

From Lemma 5 we have:

\[ R_s(T) = \alpha \sum_{t=1}^{T} \text{opt}(f_t) - \sum_{t=1}^{T} u(f(x_{t,S_t})) \]

\[ \leq \left( \frac{2 - \zeta}{1 - \zeta} \right) B' \sqrt{\beta_T} \sum_{t=1}^{T} \sum_{k=1}^{G} |\sigma_{t-1}(\pi_{t,k})| \quad (27) \]
using the fact that $\sqrt{t}$ is monotonically increasing in $t$. To proceed, let us define a constant $L = \sigma^{-2} / (1 + \log \sigma^{-2})$. We have:

$$R_s^2(T) \leq \left( \frac{2 - \zeta}{1 - \zeta} \right)^2 (B')^2 \beta_T \left( \sum_{t=1}^T \sum_{k=1}^{\left| S_t \right|} |\sigma_{t-1,k}(x_{t,k})| \right)^2$$

(28)

The middle steps are the same as (21)-(24) except that we have $\left( \frac{2 - \zeta}{1 - \zeta} \right)^2 (B')^2$ instead of $\left( \frac{\zeta + 1}{\zeta} \right)^2 B^2$ as the constant multiplier. Hence, the last steps are modified as:

$$R_s^2(T) \leq 2 \left( \frac{2 - \zeta}{1 - \zeta} \right)^2 (B')^2 \beta_T T K^2 \sigma^2 LI \left( r(z_{T1}); f(z_{T1}) \right)$$

$$\leq C_2 K^2 \beta_T T \gamma_T,$$

Taking the square root of both sides we obtain our desired result. Finally, in order to obtain the total regret we use:

$$R(T) = \zeta R_s(T) + (1 - \zeta) R_e(T)$$

$$\leq \left( \zeta \sqrt{C_1} + (1 - \zeta) \sqrt{C_2} \right) K \sqrt{\beta_T T \gamma_T}$$

In order to eliminate the $\zeta$ dependence we modify this expression as follows:

$$R(T) \leq \left( \sqrt{ \frac{2B^2(\zeta + 1)^2}{\log (1 + \sigma^{-2})} } + (1 - \zeta) \sqrt{ \frac{2(B')^2(2 - \zeta)^2}{\log (1 + \sigma^{-2})} } \right) K \sqrt{\beta_T T \gamma_T}$$

(29)

$$= \left( \frac{\zeta}{\sqrt{\zeta}} \sqrt{ \frac{2B^2(\zeta + 1)^2}{\log (1 + \sigma^{-2})} } + \sqrt{ \frac{2(B')^2(2 - \zeta)^2}{\log (1 + \sigma^{-2})} } \right) K \sqrt{\beta_T T \gamma_T}$$

(30)

$$= \frac{2}{\log (1 + \sigma^{-2})} \left( |B| |\zeta + 1| + |B'| |2 - \zeta| \right) K \sqrt{\beta_T T \gamma_T}$$

(31)

$$\leq \frac{2}{\log (1 + \sigma^{-2})} \left( B(\zeta + 1) + B'(2 - \zeta) \right) K \sqrt{\beta_T T \gamma_T}$$

(32)

where (29) follows from writing the expressions of $C_1$ and $C_2$ to the required places; (30) follows from $\zeta \in [0, 1]$; (31) follows from the assumptions that $B > 0$ and $B' > 0$ and also from $\zeta \in [0, 1]$ and (32) comes from writing the maximizing $\zeta$ values in $\zeta + 1$ and $2 - \zeta$.

At this point we state an auxiliary fact in order to use in the proof of Theorem 2 (for a proof see (Williams & Vivarelli, 2000)).

**Fact 1.** The predictive variance of a given context is monotonically non-increasing (i.e., given $x \in X$ and the vector of selected samples $[x_1, \ldots, x_{N+1}]$, we have $\sigma^2_{N+1}(x) \leq \sigma^2_N(x)$, for any $N \geq 1$).

**Theorem 2.** Fix $\delta \in (0, 1)$ and let $T, K \in \mathbb{N}$. Under the conditions of Theorem 1 and assuming that $|S| = K$ for any $S \in S$, the total regret incurred by TCGP-UCB in $T$ rounds is upper bounded with the following with probability at least $1 - \delta$:

$$R(T) \leq K \sqrt{C \beta_T T \gamma_T},$$
where $\gamma_T$ is as defined in Section 2.7 and $C$ is the same as in Theorem 1.

**Proof.** Note that, in this case, the vector $\mathbf{z}_{[T]} = [\mathbf{z}_1, \ldots, \mathbf{z}_T]$ is composed of selected $K$-dimensional vectors of contexts in $T$ rounds. Moreover, let us denote by $\mathbf{x}_{[T],k} = [\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{T,k}]$ the subsequence of $k^{th}$ contexts for each $z_t$, $t \leq T$ and for a given $x \in \mathcal{X}$ we denote by $\sigma^2_{[t-1],k}(x)$ the posterior variance of $x$ given the locations of $\mathbf{x}_{[t],k}$. Furthermore, let $L = \sigma^{-2}/\log(1 + \sigma^{-2})$. Following up from (21) in the proof of Theorem 1, and noting that $|S_t| = K$ for all $t \in [T]$, we get:

$$R_g^2(T) \leq \left(\frac{\zeta + 1}{\zeta}\right)^2 B^2 \beta_T \left(\sum_{t=1}^{T} \sum_{k=1}^{K} |\sigma_{[t-1]}(\mathbf{x}_{t,k})|\right)^2$$

(33)

$$= 2 \left(\frac{\zeta + 1}{\zeta}\right)^2 B^2 \beta_T TK \sigma^2 L \cdot \left(\sum_{t=1}^{T} \sum_{k=1}^{K} \log \left(1 + \sigma^{-2} \left|\sigma_{[t-1]}(\mathbf{x}_{t,k})\right|^2\right)\right)$$

(34)

$$= 2 \left(\frac{\zeta + 1}{\zeta}\right)^2 B^2 \beta_T TK \sigma^2 L \cdot \sum_{k=1}^{K} \left(\frac{1}{2} \sum_{t=1}^{T} \log \left(1 + \sigma^{-2} \sigma^2_{[t-1],k}(\mathbf{x}_{t,k})\right)\right)$$

(35)

$$= 2 \left(\frac{\zeta + 1}{\zeta}\right)^2 B^2 \beta_T TK \sigma^2 L \cdot \sum_{k=1}^{K} I \left(r(\mathbf{x}_{[T],k}); f(\mathbf{x}_{[T],k})\right)$$

(36)

$$\leq 2 \left(\frac{\zeta + 1}{\zeta}\right)^2 B^2 \beta_T TK^2 \sigma^2 L \cdot \max_{\mathbf{x}_{[T]}} I \left(r(\mathbf{x}_{[T]}); f(\mathbf{x}_{[T]})\right)$$

$$= 2 \left(\frac{\zeta + 1}{\zeta}\right)^2 B^2 \beta_T TK^2 \sigma^2 L \cdot \gamma_T$$

(37)

where for (34) we have used Cauchy-Schwarz twice and applied the same arguments as in the proof of Theorem 1; (35) follows from Lemma 1, where we note that the smaller the set of observations, the larger the predictive variance based on it, that is, the posterior variance of a given $x \in \mathcal{X}$ based on $z_{[t-1]}$ is smaller than that based on $x_{[t-1],k}$; since the latter is a subsequence of the former; (36) follows from Lemma 5.3 of (Srinivas et al., 2012).

Taking the square root of both sides we obtain our desired result. Similarly, following up from (28) in the proof of Theorem 1, and noting that $|S_t| = K$ for all $t \in [T]$, we get:

$$R_s^2(T) \leq \left(\frac{2 - \zeta}{1 - \zeta}\right)^2 (B')^2 \beta_T \left(\sum_{t=1}^{T} \sum_{k=1}^{K} |\sigma_{[t-1]}(\mathbf{x}_{t,k})|\right)^2$$

(38)

The middle steps are the same as (34)-(36) except that we have $\left(\frac{2 - \zeta}{1 - \zeta}\right)^2 (B')^2$ instead of $\left(\frac{\zeta + 1}{\zeta}\right)^2 B^2$ as the constant multiplier. Hence, the last steps are modified as:

$$R_s^2(T) \leq 2 \left(\frac{2 - \zeta}{1 - \zeta}\right)^2 (B')^2 \beta_T TK^2 \sigma^2 L \cdot \max_{\mathbf{x}_{[T]}} I \left(r(\mathbf{x}_{[T]}); f(\mathbf{x}_{[T]})\right)$$

$$= 2 \left(\frac{2 - \zeta}{1 - \zeta}\right)^2 (B')^2 \beta_T TK^2 \sigma^2 L \cdot \gamma_T$$

(39)

Taking the square root of both sides we obtain our desired result. Finally, in order to obtain the total regret we use:

$$R(T) = \zeta R_g(T) + (1 - \zeta) R_s(T)$$

$$\leq \left(\zeta \sqrt{C_1} + (1 - \zeta) \sqrt{C_2}\right) K \sqrt{\beta_T T \gamma_T}$$
In order to eliminate the $\zeta$ dependence we repeat the steps from (29) to (32) with $\gamma_T$ instead of $\gamma_T$ yilding:

$$R(T) \leq K\sqrt{C\beta_T T\gamma_T}.$$ 

### C.2. Proofs for Special Cases

Here, we modify some of our lemmas and theorems where necessary for each special case.

#### C.2.1. Special Case 1

In this special case, super arm reward is a monotone and Lipschitz continuous function of group rewards with the Lipschitz constant of $B'' > 0$. As given in Table A, $u([v(f(x_t,G))])_{G \subseteq G_t \wedge G \subseteq S_t}$ denotes the expected super arm reward. Also, we denote the number of groups in a super arm $S$ with $|S_t|$. For this case, all of the lemmas for the general case are applicable, except Lemma 5, which is changed as follows:

**Lemma 5*. Given round $t \geq 1$, let $S_t$ denote the optimal super arm. Then, the following argument holds under the event $F$:

$$\alpha \cdot u([v(f(x_t,G^*))]_{G^* \subseteq G_t \wedge G^* \subseteq S_t}) - u([v(f(x_t,G))])_{G \subseteq G_t \wedge G \subseteq S_t} \leq \left(\frac{2}{1 - \zeta}\right) BB''\sqrt{\beta_t \sum_{k=1}^{S_t} |\sigma_{[t-1]}(x_{t,k})|}$$

**Proof.** We define $P_t = \arg\max_{G \subseteq S_t \wedge S_t \subseteq S_t} u([v(i_t(x_t,G))]).$ Given that event $F$ holds, we have:

$$\alpha \cdot u([v(f(x_t,G^*))]_{G^* \subseteq G_t \wedge G^* \subseteq S_t}) - u([v(f(x_t,G))])_{G \subseteq G_t \wedge G \subseteq S_t} \leq \alpha \cdot u([v(i_t(x_t,G^*))])_{G^* \subseteq G_t \wedge G^* \subseteq S_t} - u([v(f(x_t,G))])_{G \subseteq G_t \wedge G \subseteq S_t} \leq \alpha \cdot u([v(i_t(x_t,G))])_{G \subseteq G_t \wedge G \subseteq S_t} - u([v(f(x_t,G))])_{G \subseteq G_t \wedge G \subseteq S_t} \leq u([v(i_t(x_t,G))])_{G \subseteq G_t \wedge G \subseteq S_t} - u([v(f(x_t,G))])_{G \subseteq G_t \wedge G \subseteq S_t} \leq B'' \sum_{i=1}^{S_t} \|v(i_t(x_t,G)) - v(f(x_t,G))\| \leq B'' \sum_{i=1}^{S_t} B \sum_{j=1}^{G_t} |x_i(x_t,j) - f(x_t,j)| \leq BB'' \sum_{i=1}^{S_t} \sum_{j=1}^{G_t} |x_i(x_t,j) - f(x_t,j)| = BB'' \sum_{k=1}^{S_t} |x_i(x_t,k) - f(x_t,k)| \leq BB'' \sum_{k=1}^{S_t} |x_i(x_t,k) - f(x_t,k)| + BB'' \sum_{k=1}^{S_t} \frac{1}{1 - \zeta} \sqrt{\beta_t \sum_{k=1}^{S_t} |\sigma_{[t-1]}^k(x_{t,k})|} \leq \left(\frac{2}{1 - \zeta}\right) BB''\sqrt{\beta_t \sum_{k=1}^{S_t} |\sigma_{[t-1]}(x_{t,k})|},$$

where (41) follows from the monotonicity of $u$ in $v$ and also from Lemma 2; (42) follows from the definition of $P_t$ and the fact that $S_t' \subseteq S_t'$ on event $F$ (Lemma 3); (43) holds since $G \subseteq S_t$ and $S_t$ is the super arm chosen by the $\alpha$-approximation oracle; (44) follows from the Lipschitz continuity of $u$ in $v$; (45) follows from the Lipschitz continuity of $v$ in $f$; (47) follows from the observation that the two sums in (46) can be reduced to a single sum as super arms are unions of disjoint groups by our problem formulation; (48) follows from the definition of index and the triangle inequality; for (49) we use Lemma 1.
According to Lemma 5*, Theorem 1 and Theorem 2 are also modified.

**Theorem 1*. Super arm regret and group regret incurred in Special Case 1 in $T$ rounds are upper bounded with probability at least $1 - \delta$ where $\delta \in (0, 1)$, $T \in \mathbb{N}$ and $\beta_t = 2 \log (M \pi^2 t^2 / 3)$ as follows:

$$R_g(T) \leq K \sqrt{C_1 \beta_T T \pi_T},$$

where $C_1 = 2 B^2 (\frac{\zeta+1}{1-\zeta})^2 / \log (1 - \sigma^{-2})$ and

$$R_s(T) \leq K \sqrt{C_3 \beta_T T \pi_T},$$

where $C_3 = 2 (BB'')^2 (\frac{2-\zeta}{1-\zeta})^2 / \log (1 - \sigma^{-2})$. Hence, the total regret is bounded by:

$$R(T) \leq K \sqrt{C^* \beta_T T \pi_T},$$

where $C^* = 8 B^2 (1 + B'')^2 / \log (1 + \sigma^{-2})$.

**Proof.** First of all, we realize that the group regret stays the same as having super arm rewards as functions of group rewards doesn’t effect the group regret analysis. Hence, we conclude that the steps from (19) to (26) directly apply and yield:

$$R_g(T) \leq K \sqrt{C_1 \beta_T T \pi_T}.$$

On the other hand, from Lemma 5*, we have:

$$R_s(T) = \alpha \sum_{i=1}^{T} \text{opt}^*(f_t) - \sum_{i=1}^{T} u_i(v(f(x_{t,i}))) \text{G}_{G_t \land G_s} T \pi_t,$$

where $\text{opt}^*(f_t) = \max_{G_s \land G_t \subseteq S_t} u_i\{v(f(x_{t,i})))\}$ and using the fact that $\sqrt{\beta_t}$ is monotonically increasing in $t$. We have:

$$R_s^2(T) \leq \left(2 \frac{2-\zeta}{1-\zeta}\right) (BB'')^2 \beta_T \left( \sum_{i=1}^{T} \sum_{k=1}^{S_t} |\sigma_{[t-1]}(\pi_t,k)| \right)^2 (50)$$

We proceed as in Theorem 1 and obtain:

$$R_s^2(T) \leq 2 \left(\frac{2-\zeta}{1-\zeta}\right)^2 (BB'')^2 \beta_T T \pi_T,$$

Taking the square root of both sides we obtain our desired result. Finally, in order to obtain the total regret we follow the steps of Theorem 1 except that now we have $(BB'')$ instead of $(B')$ in the numerator of the term that multiplies $(1 - \zeta)$ in (29). Thus, we have:

$$R(T) \leq \sqrt{\frac{2}{\log (1 + \sigma^{-2})}} \left(2 B + 2BB'' \right) K \sqrt{\beta_T T \pi_T}.$$

$$= K \sqrt{C^* \beta_T T \pi_T}.$$

**Theorem 2*. Fix $\delta \in (0, 1)$ and let $T, K \in \mathbb{N}$. Under the conditions of Theorem 1* and assuming that $|S| = K$ for any
According to Lemma 4*, Theorem 1 and Theorem 2 are modified as follows:

\[
R(T) \leq K \sqrt{C^* \beta_T T \gamma_T},
\]

where \( \gamma_T \) is as defined in Section 2.7 and \( C^* \) is the same as in Theorem 1*.

**Proof.** The proof of \( R_g(T) \) is the same with Theorem 2 since group regret analysis is unaffected by the conditions of this special case. Hence,

\[
R_g(T) \leq K \sqrt{C^*_1 \beta_T T \gamma_T}.
\]

For the proof of \( R_s(T) \), we follow the same steps as in Theorem 2 but have \((BB'^2)^2\) in the place of \((B')^2\) in (38) and the rest follows:

\[
R_s(T) \leq K \sqrt{C^*_3 \beta_T T \gamma_T}.
\]

Thus, the total regret is found similar to Theorem 1*. The only difference is we have \( \gamma_T \) instead of \( \tau_T \).

**C.2.2. SPECIAL CASE 2**

For this case, all of the lemmas for the general case are applicable, except Lemma 4, which is changed as follows where \( x_{t,m} \) denotes the context of base arm \( m \) as usual.

**Lemma 4*. Fix \( t \geq 1 \), and consider \( G \in \hat{G}_t \) such that \( G \subseteq S_t \). Let \( G \) to consist of a single base arm \( m \) and \( v \) be the identity mapping. Then the following argument holds under the event \( F \):

\[
[\gamma_{t,G} - v(f(x_{t,G}))]_+ \leq \left( \frac{\zeta + 1}{\zeta} \right) \sqrt{\beta_t \sigma_{[t-1]}(x_{t,m})}. \tag{52}
\]

**Proof.** \( [\gamma_{t,G} - v(f(x_{t,G}))]_+ > 0 \) implies that \( v(f(x_{t,G})) < \gamma_{t,G} \). Moreover \( G \subseteq S_t \) implies that \( v(i_t'(x_{t,G})) \geq \gamma_{t,G} \).

Therefore, whenever \( G \) incurs group regret it holds that \( v(f(x_{t,G})) < \gamma_{t,G} \leq v(i_t'(x_{t,G})) \).

\[
0 < \gamma_{t,G} - v(f(x_{t,G})) < v(i_t'(x_{t,G})) - v(f(x_{t,G})) = v(i_t'(x_{t,G})) - v(f(x_{t,G})) \leq \mu_{[t-1]}(x_{t,m}) + \frac{1}{\zeta} \sqrt{\beta_t \sigma_{[t-1]}(x_{t,m}) - f(x_{t,m})}, \tag{53}
\]

\[
\leq \left( \frac{1}{\zeta} + 1 \right) \sqrt{\beta_t \sigma_{[t-1]}(x_{t,m})}, \tag{54}
\]

where (53) follows from \( G \) consisting of a single base arm \( m \) and \( v \) being the identity mapping; (54) follows from the definition of index and (55) follows from Lemma 1.

According to Lemma 4*, Theorem 1 and Theorem 2 are modified as follows:

**Theorem 1**. Super arm regret and group regret incurred in Special Case 2 in \( T \) rounds are upper bounded with probability at least \( 1 - \delta \) where \( \delta \in (0, 1) \), \( T \in \mathbb{N} \) and \( \beta_t = 2 \log (M \pi^2 t^2 / 3\delta) \) as follows:

\[
R_g(T) \leq K \sqrt{C_4 \beta_T T \gamma_T},
\]

where \( C_4 = 2 \left( \frac{\zeta + 1}{\zeta} \right)^2 / \log (1 + \sigma^{-2}) \) and

\[
R_s(T) \leq K \sqrt{C_2 \beta_T T \gamma_T}.
\]
where \( C_2 = 2(B')^2 \left( \frac{2 - \zeta}{1 - \zeta} \right)^2 / \log (1 + \sigma^{-2}) \). Hence, the total regret is bounded by:

\[
R(T) \leq K \sqrt{C^{**} T \gamma_T},
\]

where \( C^{**} = 8(1 + B')^2 / \log (1 + \sigma^{-2}) \).

Proof. From Lemma 4* we have:

\[
R_g(T) = \sum_{t=1}^{T} \sum_{m=1}^{\left| S_t \right|} \left[ \gamma_{t,G} - f(x_{t,m}) \right] +
\leq \left( \frac{\zeta + 1}{\zeta} \right) \sqrt{\beta_T} \sum_{t=1}^{T} \sum_{m=1}^{\left| S_t \right|} \left| \sigma_{[t-1]}(x_{t,m}) \right| ,
\]

Applying the same techniques in (19) to (26) we obtain:

\[
R_g(T) \leq K \sqrt{C_4 \beta_T \gamma_T}.
\]

Since the proof for bounding the super arm regret is unaffected by having singleton groups and an identity mapping group reward function, it stays the same:

\[
R_s(T) \leq K \sqrt{C_2 \beta_T \gamma_T}.
\]

Again, in order to obtain the total regret we follow the steps of Theorem 1 except that now we have 1 instead of \( B \) in the numerator of the term that multiplies \( \zeta \). Thus, we have:

\[
R(T) \leq \sqrt{\frac{2}{\log (1 + \sigma^{-2})}} \left( 2 + 2B' \right) K \sqrt{\beta_T \gamma_T} = K \sqrt{C^{**} \beta_T \gamma_T}.
\]

**Theorem 2**. Fix \( \delta \in (0, 1) \) and let \( T, K \in \mathbb{N} \). Under the conditions of Theorem 1** and assuming that \( |S| = K \) for any \( S \in S \), the total regret incurred in Special Case 2 in \( T \) rounds is upper bounded with the following with probability at least \( 1 - \delta \):

\[
R(T) \leq K \sqrt{C^{**} \beta_T \gamma_T},
\]

where \( \gamma_T \) is as defined in Section 2.7 and \( C^{**} \) is the same as in Theorem 1**.

Proof. For the proof of \( R_g(T) \), we follow the same steps as in Theorem 2 but have 1 in the place of \( B^2 \) in (33) and the rest follows:

\[
R_g(T) \leq K \sqrt{C_4 \beta_T \gamma_T}.
\]

The proof of \( R_s(T) \) is the same with Theorem 2 since super arm regret analysis is unaffected by the conditions of this special case. Hence,

\[
R_s(T) \leq K \sqrt{C_2 \beta_T \gamma_T}.
\]

Thus, the total regret is found similar to Theorem 1**. The only difference is we have \( \gamma_T \) instead of \( \gamma_T \).

\[
R(T) \leq K \sqrt{C^{**} \beta_T \gamma_T}.
\]
C.2.3. Special Case 3

This special case is a further simplification of Special Case 2, therefore, the conditions are the same with those of Special Case 2 with the addition of the followings: We don’t use Oracle\(_G\) as we don’t have the notion of groups anymore and \(\zeta = 0\). Actually, this special case is the problem discussed in (Nika et al., 2021). Using the total regret formula with \(\zeta = 0\) we obtain:

\[
R(T) = \zeta R_g(T) + (1 - \zeta)R_s(T) = R_s(T) 
\]

(56)

Furthermore, Lemma 1 and Lemma 6 are satisfied and we need to set \(\zeta = 0\) in Lemma 5 and sum over \(T\) rounds, yielding:

\[
R_s(T) \leq 2B' \sqrt{\beta T \sum_{t=1}^{T} |S_t| \sum_{k=1}^{|\sigma_{t-1}|} |T_t,k|}. 
\]

(57)

However, Lemma 2 (parts that are about group indices), Lemma 3 and Lemma 4 are not valid as we don’t have groups in this case. Next, we use the conditions and results of Theorem 1** with \(\zeta = 0\) to bound the super arm regret as follows:

\[
R_s(T) \leq K \sqrt{C_5 \beta T T \gamma_T}, 
\]

where \(C_5 = 8(B')^2 / \log (1 + \sigma^{-2})\). Also, we use the conditions and results of Theorem 2** with \(\zeta = 0\) to bound the super arm regret as follows:

\[
R_s(T) \leq K \sqrt{C_5 \beta T T \gamma_T}. 
\]