Poincaré problem for divisors invariant by one-dimensional foliations on smooth algebraic variety

Maurício Corrêa JR
Departamento de Matemática
Universidade Federal de Minas Gerais
30123-970 Belo Horizonte - MG, Brasil
mauriciojr@ufmg.br

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Abstract

In this paper we consider the question of bounding the degree of an divisor $D$ invariant by a $\mathcal{F}$ holomorphic foliation, without rational first integral, on smooth algebraic variety $X$ in terms of degree of $\mathcal{F}$ and some invariants of $D$ and $X$. Particularly, if $\mathcal{F}$ is a foliation of degree $d$ on $\mathbb{P}_\mathbb{C}^2$, whose the number of invariants curves is greater that $\binom{k+2}{k}$, we show that there exist a number $M(d, k)$ such that if $k > M(d, k)$, then $\mathcal{F}$ admits a rational first integral of degree $\leq k$. Moreover, there exist a number $G(d, k)$, such that if $\mathcal{F}$ has an algebraic solution of degree $k$ and genus smaller than $G(d, k)$, then it has a rational first integral of degree $\leq k$.

1 Introduction

Henri Poincaré studied in [HP] the problem which, in the modern terminology, says: "Is it possible to decide if a holomorphic foliation $\mathcal{F}$ on the complex projective plane $\mathbb{P}_\mathbb{C}^2$ admits a rational first integral?" Poincaré observed that in order to solve this problem is sufficient to find a bound for the degree of the generic leaf of $\mathcal{F}$. In general, this is not possible, but doing some hypothesis we obtain an affirmative answer for this problem, which nowadays is known as Poincaré Problem. Many mathematicians come treating this problem and some of its generalizations, see for instance the papers of Cerveau & Lins Neto [CN], Carnicer [C], Soares [S], Brunella & Mendes [B-M], Esteves & Kleiman [E-K], V. Cavalier & D. Lehmann [C-L] and Zamora [Z].

Other researcher that treated this type of problem was P. Painlevé, more or less at the same time of Poincaré problem, which in [PP] asked the following question: "Is it possible to recognize the genus of the general solution of an algebraic differential equation in two variables which has a rational first integral?" In [N] Lins Neto has constructed families of foliations with fixed degree and local analytic type of the singularities where foliations with rational first integral of arbitrarily large degree appear. Therefore this families show that Poincaré and Painlevé questions
have a negative answer. In the same paper Lins Neto raised the following question: "Given \( d \geq 2 \), is there \( M(d) \in \mathbb{N} \), such that if a foliation of degree \( d \) has an algebraic solution of degree greater than or equal to \( M(d) \), then it has a rational first integral?" J. Moulin Ollagnier showed in [O] that when \( d = 2 \) this question has a negative answer, he exhibited a countable family of Lotka-Volterra foliations given by

\[
SLV(\ell) = x(y/2 + z) \frac{\partial}{\partial x} + y(2z + x) \frac{\partial}{\partial y} + z \left( y - \frac{2\ell + 1}{2\ell - 1} x \right) \frac{\partial}{\partial z}
\]

without rational first integrals such that has an irreducible algebraic solution of degree \( 2\ell \).

Let \( \mathcal{F} \) be a foliation on \( \mathbb{P}^n \) of degree \( d \geq 2 \) and \( V \) a hypersurface \( \mathcal{F} \)-invariant of degree \( k \). In this paper, using the extatic divisor, we show that if the number of invariants hypersurfaces, of degree \( k \), is greater than \( \left( \begin{array}{c} n+k \end{array} \right) \) then there exist a number \( M(d, k) \) such that if \( k > M(d, k) \), then \( \mathcal{F} \) admits a rational first integral, see corollary 1.2.

We raise the following question: "Given \( d \geq 2 \), is there \( G(d, k) \), such that if a foliation of degree \( d \) has an algebraic solution of degree \( k \) and genus smaller than to \( G(d, k) \), then it has a rational first integral?" We will show that this question has a positive answer, see theorem 1.1.

If \( \mathcal{F} \) is a holomorphic one-dimensional foliation on algebraic variety \( X \), then a rational first integral for \( \mathcal{F} \) is a rational map \( \Theta : X \rightarrow Y \), where \( Y \) is an algebraic variety, such that the fibers of \( \Theta \) are \( \mathcal{F} \)-invariant. Using a concept of degree of foliations and divisors we will prove the following result.

**Theorem 1.** Let \( \mathcal{F} \) be a one-dimensional foliation on smooth algebraic variety \( X \) and \( D \) a effective divisor \( \mathcal{F} \)-invariant. Suppose that \( \mathcal{F} \) does not admit rational first integral, then:

\[
\deg(D) \cdot \left[ \mathcal{N}(\mathcal{F}, |D|) - h^0(X, \mathcal{O}(D)) \right] \leq \left[ \deg(\mathcal{F}) - \deg(X) \right] \cdot \left( \frac{h^0(X, \mathcal{O}(D))}{2} \right),
\]

where \( \mathcal{N}(\mathcal{F}, |D|) \) is the number of divisors \( \mathcal{F} \)-invariant contained on the linear system \( H^0(X, \mathcal{O}(D)) \) and \( h^0(X, \mathcal{O}(D)) = \dim \mathbb{C} H^0(X, \mathcal{O}(D)) \).

**Corollary 1.1.** Let \( \mathcal{F} \) be a one-dimensional foliation on algebraic variety \( X \) and \( D \) a effective divisor \( \mathcal{F} \)-invariant. Suppose that \( \mathcal{F} \) does not admit rational first integral and that \( \mathcal{N}(\mathcal{F}, |D|) > h^0(X, \mathcal{O}(D)) \), then:

\[
\deg(D) \leq \frac{\left[ \deg(\mathcal{F}) - \deg(X) \right] \cdot \left( \frac{h^0(X, \mathcal{O}(D))}{2} \right)}{\mathcal{N}(\mathcal{F}, |D|) - h^0(X, \mathcal{O}(D))},
\]

where \( \mathcal{N}(\mathcal{F}, |D|) \) is the number of divisors \( \mathcal{F} \)-invariant contained on the linear system \( H^0(X, \mathcal{O}(D)) \) and \( h^0(X, \mathcal{O}(D)) = \dim \mathbb{C} H^0(X, \mathcal{O}(D)) \).
Example 1.1. Let $X$ be an Abelian variety of dimension $n$ and $D$ a effective divisor invariant by a holomorphic foliation $\mathcal{F}$ on $X$. If $N(\mathcal{F}, \lvert D \rvert) > \frac{D^n}{n!}$ and $\mathcal{F}$ does not admit a rational first integral then

$$\deg(D) \leq \frac{[\deg(\mathcal{F}) - \deg(X)] \cdot \left(\frac{D^n}{n!}\right)}{N(\mathcal{F}, \lvert D \rvert) - \frac{D^n}{n!}}.$$ 

Indeed, follows from Kodaira-Nakano Vanishing Theorem that

$$h^p(X, \mathcal{O}(D)) = h^p(X, \Omega^p(D)) = 0, \quad p > 0,$$

and hence the holomorphic Euler characteristic $\chi(D) = h^0(X, \mathcal{O}(D))$. On the other hand, we have by Riemann-Roch theorem that $\chi(D) = \frac{D^n}{n!}$. Now, the affirmed follows from corollary 1.1.

If we suppose that $N(\mathcal{F}, \lvert D \rvert) > h^0(X, \mathcal{O}(D))$, follows from corollary 1.1 that there exist a number $\mathcal{N}(\mathcal{F}, \lvert D \rvert)$, such that if $\mathcal{F}$ possess an invariant effective divisor $D$, contained on the linear system $\lvert D \rvert = H^0(X, \mathcal{O}(D))$, satisfying the condition

$$\deg(D) > \mathcal{N}(\mathcal{F}, \lvert D \rvert),$$

then $\mathcal{F}$ admit rational first integral.

**Corollary 1.2.** Let $\mathcal{F}$ be a one-dimensional foliation on $\mathbb{P}^n_{\mathbb{C}}$ of degree $d \geq 2$ and $\mathcal{N}(d,k)$ the number of hypersurfaces invariant by $\mathcal{F}$ of degree $k$. Suppose that $\mathcal{N}(d,k) > \binom{n+k}{k}$ and there exist a hypersurface invariant by $\mathcal{F}$ of degree $k$ such that

$$k > \frac{(d-1) \cdot \binom{n+k}{k}}{\mathcal{N}(d,k) - \binom{n+k}{k}}.$$

Then $\mathcal{F}$ admit a rational first integral.

**Remark 1.1.** In the case of a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$, we can to bound of degree of the rational first integral which this one admit. See the Proposition 2 in [V].

Let $X$ be an algebraic surface and $D$ a curve invariant by a foliation $\mathcal{F}$ on $X$. In this case, we obtain the following inequality in terms of invariants of $X$, the virtual genus of $D$ and the degree of $\mathcal{F}$.

Let $D$ be a divisor on algebraic surface $X$. For the next result we shall use the following notation $h^i(D) = \dim_{\mathbb{C}} H^i(X, \mathcal{O}(D))$, $i = 0,1$.

**Corollary 1.3.** Let $\mathcal{F}$ be a foliation on algebraic surface $X$ and $D$ a divisor $\mathcal{F}$-invariant. If $\mathcal{F}$ does not admit rational first integral, then

$$2 - 2g(X, D) \leq 2h^1(D) - 2h^0(K - D) + 2 \frac{[\deg(\mathcal{F}) - \deg(M)]}{\deg(D)} \cdot \left(\frac{h^0(D)}{2}\right) +$$

$$+ K \cdot (K - 12D) + \chi(X) \quad \frac{6}{\mathcal{N}(\mathcal{F}, \lvert D \rvert)},$$

where $g(X, D)$ is the virtual genus of $D$, $K$ is the canonical sheave of $X$ and $\chi(X)$ is the Euler characteristic of $X$. 


Theorem 1.1. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_C$, of degree $d \geq 2$, that does not admit rational first integral of degree $\leq k$. Let $D$ be a algebraic curve, of degree $k$, invariant by $\mathcal{F}$, then

$$2 - 2g(D) \leq \frac{d(k^3 + 6k^2 + 11k + 6) - k^3 - 6k^2 + 13k + 2}{4} - 2\mathcal{N}(d, k),$$

where $g(D)$ is the virtual genus of $D$ and $\mathcal{N}(d, k)$ is the number of curve $\mathcal{F}$-invariant of degree $k$.

Proof. Since $\chi(\mathbb{P}^2_C) = 3$, $K = -3h$ and $D = kh$, where $h$ is the hyperplane class of $\mathbb{P}^2_C$, follows that

$$K \cdot (K - 12D) + \chi(X) = 6k + 2.$$ 

However, $K - D = -(3 + k)h$ and so $\text{deg}(K - D) = -3 - k < 0$, hence follows from theorem 2.1 that $h^0(K - D) = 0$. Moreover $h^1(\mathbb{P}^2_C, kh) = 0$. In fact, we have that

$$H^1(\mathbb{P}^2_C, kh) = H^1(\mathbb{P}^2_C, \Omega^2_{\mathbb{P}^2_C}(kh - K)) = H^1(\mathbb{P}^2_C, \Omega^2_{\mathbb{P}^2_C}((k + 3)h)),$$

and applying the Kodaira-Nakano Vanishing Theorem that for $q = 1$ and $p = 2$ we get

$$H^1(\mathbb{P}^2_C, \Omega^2_{\mathbb{P}^2_C}((k + 3)h)) = 0.$$ 

Therefore, since $\mathcal{F}$ does not admit rational first integral of degree $\leq k$, follows from corollary 1.3 and of the done calculations that

$$2 - 2g(D) \leq \frac{d(k^3 + 6k^2 + 11k + 6) - k^3 - 6k^2 + 13k + 2}{4} - 2\mathcal{N}(d, k).$$

$\square$

Exemple 1.2. Let $C$ be a smooth curve invariant by a foliation $\mathcal{F}$ on $\mathbb{P}^2_C$. Under the conditions of theorem 1.1 we get

$$\chi(C) \leq \frac{d(k^3 + 6k^2 + 11k + 6) - k^3 - 6k^2 + 13k + 2}{4} - 2\mathcal{N}(d, k),$$

where $\chi(C)$ is the Euler characteristic of $C$.

Follows from theorem 1.1 that there exist a number $\mathcal{G}(d, k)$, such that if $\mathcal{F}$ possess a invariant curve $C$, of degree $k$, which satisfies the following condition

$$g(C) < \mathcal{G}(d, k),$$

then $\mathcal{F}$ admit rational first integral of degree $\leq k$. 
2 The degree of divisors and holomorphic foliations

Let \((X, \varpi)\) be a Kähler manifold where \(\varpi\) is the Kähler form. The degree of holomorphic vector bundle \(E\) on \(X\) related to structure induced by \(\varpi\) is defined by

\[
\deg_{\varpi}(E) = \int_X c_1(E) \wedge \varpi^{n-1}.
\]

**Theorem 2.1.** \([K]\) Let \(L\) be a line bundle on Kahler manifold \((X, \varpi)\). Then:

i) If \(\deg_{\varpi}(L) < 0\), then \(H^0(X, \mathcal{O}(L)) = \{0\}\).

ii) If \(\deg_{\varpi}(L) = 0\) and \(s \in H^0(X, \mathcal{O}(L))\), with \(s \neq 0\), then \(s(p) \neq 0\) for all \(p \in X\).

**Definition 2.1.** Let \(D\) be an effective divisor on \(X\). The degree of \(D\) is defined by

\[
\deg(\mathcal{O}(D)).
\]

**Remark 2.1.** Since \(D\) is effective we have that \(H^0(X, \mathcal{O}(D)) \neq \{0\}\), and follows from theorem 2.1 that \(\deg(\mathcal{O}(D)) > 0\).

Let \(D\) be a divisor on \(X\) defined locally by functions \(\{f_\alpha \in \mathcal{O}(U_\alpha)\}_{\alpha \in \Lambda}\), where \(\{U_\alpha\}_{\alpha \in \Lambda}\) is a open covering of \(X\). If \(\mathcal{U}_{\alpha\beta} := U_\alpha \cap U_\beta\) then there exist \(f_{\alpha\beta} \in \mathcal{O}^*(U_\alpha)\), such that \(f_\alpha = f_{\alpha\beta} f_\beta\). Denote by \(f_{\alpha}^{D}\) the restriction of \(f_\alpha\) on \(D\). Let \(\mathcal{F}\) be a holomorphic foliation given by collections \(\{(\vartheta_\alpha); \{U_\alpha\}; \{g_{\alpha\beta} \in \mathcal{O}_*^{U_\alpha}\};\}_{\alpha \in \Lambda}\) on \(X\).

Consider the following functions

\[
\zeta_{s\alpha}^{(F,D)} = \vartheta_\alpha(f_{\alpha}^{D}) \in \mathcal{O}(U_\alpha \cap D).
\]

If \(U_\alpha \cap U_\beta \cap D \neq \emptyset\) and using the Leibniz’s rule we get \(\zeta_{s\alpha}^{(F,D)} = f_{\alpha}^{D} g_{\alpha\beta} \zeta_{s\beta}^{(F,D)}\). With this, we obtain a global section \(\zeta^{(F,D)}\) of line bundle \((\mathcal{T}_F^* \otimes [D])|_{D}\). The tangency variety of \(\mathcal{F}\) with \(D\) is given by

\[
T(\mathcal{F}, D) = \{p \in D; \zeta^{(F,D)}(p) = 0\}.
\]

**Definition 2.2.** Let \(X \subset \mathbb{P}^N\) be a smooth algebraic variety and \(H\) the hyperplane class of \(\mathbb{P}^N\). Let \(\mathcal{F}\) be a foliation on \(X\). The degree of \(\mathcal{F}\) is the intersection number

\[
\deg(\mathcal{F}) := \langle [\mathcal{T}(\mathcal{F}, H)] \sim [H]^{(n-2)}, [H] \rangle,
\]

where \([H]^{(n-2)} = \underbrace{[H] \sim \cdots \sim [H]}_{(n-2)-times}\).

**Proposition 2.1.** Let \(\mathcal{F}\) be a foliation on algebraic variety \(X \subset \mathbb{P}^N\). Then

\[
\deg(\mathcal{F}) = \deg(T_x^* \uparrow) + \deg(X),
\]

where \(\deg(X)\) is the degree of \(X\).
Proof. We have that

\[ \langle [T(\mathcal{F}, H)] - [H]^{(n-2)}, [H] \rangle = \int_H c_1([T(\mathcal{F}, H)]) \wedge h^{n-2}, \]

where \( h \) is the hyperplane class. By adjunction formula \([T(\mathcal{F}, H)] = (T_{\mathcal{F}} \otimes [H])_H\) and since \( H \) is Poincaré's dual of \( h \), that is \( c_1([H]) = h \), we get the following

\[ \deg(\mathcal{F}) = \langle [T(\mathcal{F}, \mathcal{H})] - [H]^{(n-2)}, [H] \rangle = \int_H c_1(T_{\mathcal{F}} \otimes [H])_H \wedge h^{n-2} \]

\[ = \int_X c_1(T_{\mathcal{F}}) \wedge h^{n-1} + \int_X h \wedge h^{n-1} \]

\[ = \deg(T_{\mathcal{F}}) + \deg(X). \]

\[ \square \]

Remark 2.2. If \( \deg(T_{\mathcal{F}}) < 0 \) follows from theorem 2.1 that \( H^0(X, T_{\mathcal{F}}) = \{0\} \). Therefore we shall assume \( \deg(T_{\mathcal{F}}) > 0 \), or equivalently \( \deg(\mathcal{F}) = -\deg(X) > 0 \).

Exemple 2.1. Let \( \mathcal{F} \) be a foliation on \( X \), where \( \text{Pic}(X) \simeq \mathbb{Z} \). Take a positive generator \( \mathcal{H} \) for \( \text{Pic}(X) \) and denote by \( \mathcal{O}_X(k) := \mathcal{H}^{\otimes k} \) the \( k \)-th tensorial power of \( \mathcal{H} \). If we shall write \( T_{\mathcal{F}} = \mathcal{O}_X(d-1) \) we get that \( \deg(T_{\mathcal{F}}) = (d-1)\deg(X) \) and hence

\[ \deg(\mathcal{F}) = \deg(T_{\mathcal{F}}) + \deg(X) = (d-1)\deg(X) + \deg(X) = d \cdot \deg(X). \]

In the case where \( X = \mathbb{P}^n \) we will have, as already it is known, that \( \deg(\mathcal{F}) = d \).

3 Extatic divisor

The method adopted here stems from the work of J.V.Pereira [P], where the notion of extactic variety is exploited. In this section we digress briefly on extactic varieties and their main properties.

A one-dimensional foliation \( \mathcal{F} \) on complex manifold \( X \) induced a morphism \( \Phi_{\mathcal{F}} : \Omega^1_X \to T_{\mathcal{F}} \) given by locally by contraction , that is, \( \Phi_{\mathcal{F}_{|U_\alpha}}(\theta) = i_{\partial_\alpha}(\theta_\alpha) \), where \( U_\alpha \) is a opened of \( X \).

Consider the linear system \( H^0(X, \mathcal{O}(D)) \) and take a open covering \( \{U_\alpha\}_\alpha \) of \( X \) which trivialize \( \mathcal{O}(D) \) and \( T_{\mathcal{F}} \). In the opened \( U_\alpha \) we can consider the morphism

\[ T_{\mathcal{F}}^{(k)} : H^0(X, \mathcal{O}(D))^k \otimes \mathcal{O}_{U_\alpha} \to \mathcal{O}_{U_\alpha}^k \]

defined by

\[ T_{\mathcal{F}}^{(k)}(s) = s + X_{\mathcal{F}}(s)^{\alpha} \cdot t + X_{\mathcal{F}}^2(s)^{\alpha} \cdot \frac{t^2}{2!} + \cdots + X_{\mathcal{F}}^k(s)^{\alpha} \cdot \frac{t^k}{k!}, \]
where $X_{\mathcal{F}}(\cdot)^\alpha = \Phi_{\mathcal{F}}(d(\cdot))|_{U_\alpha}$ and $s \in H^0(X_\mathcal{O}(D)) \otimes \mathcal{O}_{U_\alpha}$. In an open set $U_\alpha$ we have $\mathcal{O}(D)|_{U_\alpha} = \mathcal{O}_{U_\alpha} \cdot \sigma_\alpha$ and $T_{\mathcal{F}}|_{U_\alpha} = \mathcal{O}_{U_\alpha} \cdot \beta_\alpha$. Therefore, for all $s_\alpha \in H^0(X_\mathcal{O}(D)) \otimes \mathcal{O}_{U_\alpha}$ we obtain

$$s_\alpha = s_\alpha^{(1)} \cdot \sigma_\alpha$$
$$X_{\mathcal{F}}(s_\alpha)^\alpha = X_{\mathcal{F}}(s_\alpha^{(1)})^\alpha \cdot \beta_\alpha = s_\alpha^{(2)} \cdot \beta_\alpha$$
$$\vdots$$
$$X_{\mathcal{F}}^{k-1}(s_\alpha)^\alpha = X_{\mathcal{F}}(s_\alpha^{(k-2)})^\alpha \cdot \beta_\alpha = s_\alpha^{(k)} \cdot \beta_\alpha$$

If $U_\alpha \cap U_\gamma \neq \emptyset$ then $s_\alpha^{(1)} = g_{\alpha \gamma} s_\gamma^{(1)}$ and $X_{\mathcal{F}}(\cdot)^\alpha = i_{\alpha \gamma}(\cdot) = i_{(f_{\alpha \gamma}, \mathcal{O})}(\cdot) = f_{\alpha \gamma} X_{\mathcal{F}}(\cdot)^\gamma$, where $g_{\alpha \gamma}, f_{\alpha \gamma} \in \mathcal{O}^*(U_\alpha)$ are the cocycles which defines, respectively, the line bundles $[D]$ and $T_{\mathcal{F}}$. using the described compatibility above and the Leibniz’s rule we get

$$s_\alpha = s_\alpha^{(1)} \cdot \sigma_\alpha = g_{\alpha \beta} s_\beta^{(1)} \cdot \sigma_\alpha$$
$$X_{\mathcal{F}}(s_\alpha)^\alpha = X_{\mathcal{F}}(s_\alpha^{(1)})^\alpha \cdot \beta_\alpha = (X_{\mathcal{F}}(g_{\alpha \gamma})^\gamma \cdot s_\gamma^{(1)} + g_{\alpha \gamma} \cdot s_\gamma^{(2)}) \cdot f_{\alpha \gamma} \cdot \beta_\gamma$$

Following for this process it ties the order $k = h^0(X_\mathcal{O}(D))$, we obtain

$$\begin{bmatrix}
  s_\alpha^{(1)} \\
  s_\alpha^{(2)} \\
  s_\alpha^{(3)} \\
  \vdots \\
  s_\alpha^{(k)}
\end{bmatrix} = \begin{bmatrix}
  g_{\alpha \beta} & 0 & 0 & 0 & 0 \\
  X_{\mathcal{F}}(g_{\alpha \gamma})^\gamma \cdot f_{\alpha \gamma} & g_{\alpha \beta} \cdot f_{\alpha \beta} & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & g_{\alpha \beta} \cdot f_{\alpha \beta}^{k-1}
\end{bmatrix} \begin{bmatrix}
  s_\alpha^{(1)} \\
  s_\alpha^{(2)} \\
  s_\alpha^{(3)} \\
  \vdots \\
  s_\alpha^{(k)}
\end{bmatrix}$$

Denoting the matrix above by $\Theta_{\alpha \gamma}(\mathcal{F}, D) \in GL(k_\mathcal{O}^*(U_{\alpha \gamma}))$, we see that

$$\begin{cases}
  \Theta_{\alpha \gamma}(\mathcal{F}, D)(p) \cdot \Theta_{\gamma \alpha}(\mathcal{F}, D)(p) = I, \text{ for all } p \in U_\alpha \cap U_\gamma \\
  \Theta_{\alpha \gamma}(\mathcal{F}, D)(p) \cdot \Theta_{\gamma \lambda}(\mathcal{F}, D)(p) \cdot \Theta_{\lambda \alpha}(\mathcal{F}, D)(p) = I, \text{ for all } p \in U_\alpha \cap U_\gamma \cap U_\lambda.
\end{cases}$$

That is, the family of matrices $\{\Theta_{\alpha \gamma}(\mathcal{F}, D)\}_{\alpha \gamma}$ define a cocycle of a vector bundle of rank $k$ on $X$ that we shall denote by $J^k_X \mathcal{O}(D)$. Now, using the trivializations $\{\Theta_{\alpha \gamma}(\mathcal{F}, D)\}_{\alpha \gamma}$ we get the morphisms

$$T^{(k)} : H^0(X_\mathcal{O}(D)) \otimes \mathcal{O}_X \to J^k_X \mathcal{O}(D).$$

Taking the determinant of $T^{(k)}$ we have the morphism

$$\det(T^{(k)}) : \bigwedge^k [H^0(X_\mathcal{O}(D))] \otimes \mathcal{O}_X \to \bigwedge^k J^k_X \mathcal{O}(D),$$

and tensorizing by $(\bigwedge^k V)^*$ we obtain a global section of $\bigwedge^k J^k_X \mathcal{O}(D) \otimes (\bigwedge^k V)^*$ given by

$$\varepsilon_{(\mathcal{F}, V)} : \mathcal{O}_X \to \bigwedge^k J^k_X \mathcal{O}(D) \otimes (\bigwedge^k V)^*. $$
Definition 3.1. The extatic divisor of $\mathcal{F}$ with respect to the linear system $H^0(X, \mathcal{O}(D))$ is the divisor $\mathcal{E}(\mathcal{F}, V) = (\varepsilon_{(\mathcal{F}, V)})$ given by zeros of the section

$$\varepsilon_{(\mathcal{F}, V)} \in H^0 \left( X, \bigwedge^k J^k_{X,D} \mathcal{O}(D) \otimes (\bigwedge^k V)^* \right).$$

J.V. Pereira [P] obtained the following results, which elucidate the role of the divisor variety:

Proposition 3.1. ([P], Proposition 5) Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a complex manifold $X$. If $V$ is a finite dimensional linear system, then every $\mathcal{F}$-invariant hypersurface which is contained in the zero locus of some element of $V$, must be contained in the zero locus of $\mathcal{E}(V, \mathcal{F})$.

If $\mathcal{F}$ is a holomorphic one-dimensional foliation on a complex manifold $X$, then a first integral for $\mathcal{F}$ is a holomorphic map $\Theta : X \rightarrow Y$, where $Y$ is a complex manifold, such that the fibers of $\Theta$ are $\mathcal{F}$-invariant. Then we have:

Theorem 3.1. ([P], Theorem 3). Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a complex manifold $X$. If $V$ is a finite dimensional linear system such that $\mathcal{E}(V, \mathcal{F})$ vanishes identically, then there exits an open and dense set $U$ where $\mathcal{F}|_U$ admits a first integral. Moreover, if $X$ is a projective variety, then $\mathcal{F}$ admits a meromorphic first integral.

4 Proofs

4.1 Proof of theorem [1]

From theorem 3.1 if $\mathcal{F}$ does not rational first integral $\varepsilon_{(\mathcal{F}, V)} \neq 0$, and then defines a divisor $\mathcal{E}(\mathcal{F}, V)$ whose line bundle associated is $\bigwedge^k J^k_{X,D} \mathcal{O}(D) \otimes (\bigwedge^k V)^*$. Let us say that $k = \dim_C V$. Let $\mathcal{N}_i$ be the number of irreducible divisors of $H^0(X, \mathcal{O}(D))$ of degree $i \leq \deg(D)$, counting multiplicities, invariants by $\mathcal{F}$. From proposition 3.1 all divisor $\mathcal{D} \in H^0(X, \mathcal{O}(D))$ invariant by $\mathcal{F}$ is contained in the extatic $\mathcal{E}(\mathcal{F}, V)$. Using this fact we can to affirm that

$$\sum_{i=1}^{\deg(D)} i \cdot \mathcal{N}_i \leq \deg(\mathcal{E}(\mathcal{F}, V)).$$

Indeed, it is enough to group the divisors $\mathcal{F}$-invariants of the following form

$$[\mathcal{E}(\mathcal{F}, V)] = [V_1^{d_1}] \otimes \cdots \otimes [V_1^{n_1}] \otimes \cdots \otimes [V_{\deg(D)}^{d_{\deg(D)}}] \otimes \cdots \otimes [V_{\deg(D)}^{n_{\deg(D)}}] \otimes \mathcal{L},$$

where $[V_i^{d_i}]$ is a divisor irreducible invariant by $\mathcal{F}$, of degree $i$ and multiplicities $d_{ij}$, and $\mathcal{L}$ is a line bundle. Therefore we get

$$\sum_{k=1}^{n_i} d_{ki} \deg(V_k^i) = i \cdot \sum_{k=1}^{n_i} d_{ki} = i \cdot \mathcal{N}_i \quad \forall \ i = 1, \ldots, \deg(D).$$
For simplicity we will write \([\mathcal{E}(\mathcal{F}, V)] = \mathcal{I}_\mathcal{F} \otimes \mathcal{L}\), where

\[
\mathcal{I}_\mathcal{F} = [V_1^{d_1}] \otimes \cdots \otimes [V_1^{n_1}] \otimes \cdots \otimes [V_{d_1}^{d_1}] \otimes \cdots \otimes [V_{d_1}^{n_1}] \otimes \cdots \otimes [V_{d_1}^{n_1}] \otimes \cdots \otimes [V_{d_1}^{d_1}] \otimes \cdots \otimes [V_{d_1}^{n_1}].
\]

Calculating the degree we conclude that \(\deg(\mathcal{I}_\mathcal{F}) = \sum_{i=1}^{\deg(D)} i \cdot N_i \leq \deg(\mathcal{E}(\mathcal{F}, V))\). This show the affirmed one above.

From this inequality, we get the following \(\deg(D) \cdot N(D) \leq \deg(\mathcal{E}(\mathcal{F}, V))\). However the line bundle associated to the extatic divisor \(\mathcal{E}(\mathcal{F}, V)\) is given by the following

\[
\bigwedge^k J_{\mathcal{X}_D}^k \mathcal{O}(D) \otimes (\bigwedge^k V)^*.\]

On the other hand, the cocycle of \(\bigwedge^k J_{\mathcal{X}_D}^k \mathcal{O}(D)\) is given by

\[
\det(\Theta_{\alpha\beta}(\mathcal{F}, D)) = g_{\alpha\beta} \cdot f_{\alpha\beta}^{(k)},
\]

where \(g_{\alpha\beta}\) and \(f_{\alpha\beta}\) are trivializations of \([D]\) and \(T_{\mathcal{X}_D}^*\), respectively. This show that \(\bigwedge^k J_{\mathcal{X}_D}^k \mathcal{O}(D) \simeq [D]^{\otimes k} \otimes (T_{\mathcal{X}_D}^*)^{\otimes (\frac{k}{2})}\), hence

\[
[\mathcal{E}(\mathcal{F}, V)] = [D]^{\otimes k} \otimes (T_{\mathcal{X}_D}^*)^{\otimes (\frac{k}{2})} \otimes (\bigwedge^k V)^*.
\]

Calculating the degree \(\deg(\mathcal{E}(\mathcal{F}, V))\) we get

\[
\deg(\mathcal{E}(\mathcal{F}, V)) = \deg \left( [D]^{\otimes k} \otimes (T_{\mathcal{X}_D}^*)^{\otimes (\frac{k}{2})} \right) + \deg \left( \bigwedge^k V^* \right) = k \cdot \deg(D) + \deg(T_{\mathcal{X}_D}^*) \left( \frac{k}{2} \right).
\]

Finally, follows from \(\mathcal{N}_{\deg(D)} \cdot \deg(D) \leq \sum_{i=1}^{\deg(D)} i \cdot N_i \leq \deg(\mathcal{E}(\mathcal{F}, V))\) and of the fact that \(\deg(T_{\mathcal{X}_D}^*) = \deg(\mathcal{F}) - \deg(X)\), that

\[
\deg(D) \cdot [\mathcal{N}_{\deg(D)} - k] \leq [\deg(\mathcal{F}) - \deg(X)] \cdot \left( \frac{k}{2} \right).
\]

This proof the theorem \(\Box\).

**4.1.1 Proof of corollary 1.3**

From Riemann-Roch’s theorem (see \(\Box\) theorem 1.6) we get

\[
h^0(D) = h^1(D) - h^0(K - D) + \frac{D \cdot (D - K)}{2} + \chi(\mathcal{O}_X),
\]
where $\chi(\mathcal{O}_X)$ holomorphic Euler characteristic of $X$ and $K$ is the canonical sheaf. Since $g(X, D) = \frac{D \cdot (D-K)}{2} + D \cdot K + 1$ we have that

$$h^0(D) = h^1(D) - h^0(K-D) + g(X, D) - D \cdot K + \chi(\mathcal{O}_X) - 1.$$  

From Noether’s formula $\chi(\mathcal{O}_X) = \frac{1}{12}(K \cdot K + \chi(X))$ we get

$$h^0(D) = h^1(D) - h^0(K-D) + g(X, D) + \frac{1}{12}[K \cdot (K - 12D) + \chi(X)] - 1 \quad (\ast)$$

Now, by theorem I we have that

$$-h^0(D) \leq \left(\frac{\deg(F) - \deg(X)}{\deg(D)}\right) \cdot \left(\frac{h^0(D)}{2}\right) - \mathcal{N}(F, |D|)$$

The result follows from this inequality and (\ast).

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**References**

[V] J. V. Pereira, *Vector Fields, Invariant Varieties and Linear Systems*. Annales de L’Institut Fourier 51, no.5 (2001), 1385-1405.

[CN] D. Cerveau and A. Lins Neto, *Holomorphic foliations in $\mathbb{P}_C^2$ having an invariant algebraic curve*, Ann. Inst. Fourier 41 (1991), 883-903.

[Z] A. G. Zamora, *Foliations in Algebraic Surfaces having a rational first integral*, Publicacions Matematiques 41 (1997), 357-373.

[C-L] V. Cavalier and D. Lehmann, *On the Poincaré inequality for one-dimensional foliations*, Com. Compositio Math. 142 (2006) 529-540.

[N] A. Lins Neto, *Some examples for Poincaré and Painleve problem* Ann. Scient. Ec. Norm. Sup., 4e série, 35, 2002, p. 231 a 266.

[O] J. M. Ollagnier, *About a conjecture on quadratic vector fields*, Journal of Pure and Applied Algebra 165 (2001) 227-234.

[S] M. G. Soares, *The Poincaré problem for hypersurfaces invariant by one-dimensional foliations*, Inventiones Mathematicae, Alemanha, v. 128, p. 495-500, 1997.

[E-K] E. Esteves and S. Kleiman, *Bounds on leaves of one-dimensional foliations*, Bull. Braz. Mat. Soc. (NS) 34 (2003),145-169.

[H] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag Graduate Texts in Mathematics 52, 1977.

[K] S. Kobayashi, *Differential Geometry of complex Vector Bundle*; Publication of the Mathematical Society of Japan, Princeton University Press, 1987.

[C] M. Carnicer, *The Poincaré problem in the non-dicritical case*, Ann. de Math. 140 (1994) 289-294.

[HP] H. Poincaré, *Sur l’intégration algébrique des équations différentielles du premier ordre et du premier degré* Rend. Circ Mat Palermo,5 (1891), 161-191.

[PP] P. Painlevé, *Sur les intégrales algébrique des équations différentielles du premier ordre and Mémoire sur les équations différentielles du premier ordre*, Oeuvres de Paul Painlevé; Tome II, Éditions du Centre National de la Recherche Scientifique, 15, quai Anatole-France, 75700, Paris, 1974.

[B-M] M. Brunella and L.G. Mendes, *Bounding the degree of solutions to Pfaff equations*, preprint 206, U. Bourgogne, 1999.