ON A MODEL OF MULTIPHASE FLOW
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Abstract. We consider a hyperbolic system of three conservation laws in one space variable. The system is a model for fluid flow allowing phase transitions; in this case the state variables are the specific volume, the velocity and the mass density fraction of the vapor in the fluid. For a class of initial data having large total variation we prove the global existence of solutions to the Cauchy problem.

AMS subject classifications. 35L65, 35L60, 35L67, 76T30

Key words. Hyperbolic systems of conservation laws, phase transitions

1. Introduction. We consider a model for the one-dimensional flow of an inviscid fluid capable of undergoing phase transitions. Both liquid and vapor phases are possible, as well as mixtures of them. In Lagrangian coordinates the model is

\[ \begin{cases} v_t - u_x = 0 \\ u_t + p(v, \lambda)_x = 0 \\ \lambda_t = 0 \end{cases} \]  

Here \( t > 0 \) and \( x \in \mathbb{R} \); moreover \( v > 0 \) is the specific volume, \( u \) the velocity, \( \lambda \) the mass density fraction of vapor in the fluid. Then \( \lambda \in [0, 1] \), with \( \lambda = 0 \) characterizing the liquid and \( \lambda = 1 \) the vapor phase; the intermediate values of \( \lambda \) model the mixtures of the two pure phases. The pressure is denoted by \( p = p(v, \lambda) \); under natural assumptions the system is strictly hyperbolic.

This model is a simplified version of a model proposed by Fan [13], where also viscous and relaxation terms were taken into account. The model is isothermal, see \[21\] below; in presence of phase transitions this physical assumption is meaningful for retrograde fluids. A study of the Riemann problem for a 2×2 relaxation approximation of \[11\] has been done in \[10\]. We focus here on the global existence of solutions to the Cauchy problem for \[11\], namely for initial data

\( (v, u, \lambda)(0, x) = (v_o(x), u_o(x), \lambda_o(x)) \)

having finite total variation. This problem is motivated by the study of more complete models, where \[11\] is supplemented by source terms.

The problem of the global existence of solutions to strictly hyperbolic system of conservation laws has been studied since long, see \[9, 11, 24, 25\] for general information. If the initial data have small total variation then Glimm theorem \[14\] applies; we refer again to \[9\] for the analogous results obtained by a wave-front tracking algorithm as well as for uniqueness and continuous dependence of the solutions on the initial data.

Some special systems allow however initial data with large total variation. For the system of isothermal gasdynamics Nishida \[19\] proved that it is sufficient that

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the variation $TV(v_o, u_o)$ of the initial data is finite in order to have globally defined solutions. This result was extended by Nishida and Smoller \[20\] to any pressure law $p = k/v^\gamma$, $\gamma > 1$, provided that $(\gamma - 1)TV(v_o, u_o)$ is small; related results are in \[12\]. For the full nonisentropic system of $3 \times 3$ gasdynamics, $p = k \exp((\gamma - 1)s)/v^\gamma$, for $s$ the entropy, Liu \[17, 16\] proved the global existence of solutions if $(\gamma - 1)TV(v_o, p_o)$ is small and $TV(s_o)$ bounded. Temple \[26\] and Peng \[21\] obtained similar results. All these papers use the Glimm scheme. Analogous results making use of a wave-front tracking scheme have been given recently by Asakura \[4, 5\]; we point out that the use of wave-front tracking schemes in case of data with large variation is far from being trivial, and a deep analysis of the wave interactions is required. Very general results can be proved for systems with coinciding shock and rarefaction curves, \[8\]; however system (1.1) is not of this type.

In comparison with the above systems of gasdynamics, in (1.1) we keep a $\gamma$-law for the pressure with $\gamma = 1$, but add a dependence of $p$ on $\lambda$: we take then $p = a(\lambda)/v$ for a suitable function $a$. System (1.1) has close connections to a system introduced by Benzoni-Gavage \[6\] and studied by Peng \[22\]; it seems however that the proof in \[22\] is not complete. A comparison of these models is done in Subsection 3.1. We mention that also the method of compensated compactness has been applied to (1.1), see \[15, 7\] and \[18, \S 12.3, \S 16\], but for different pressure laws.

In this paper we prove by a wave-front tracking scheme the global existence of solutions to (1.1) for a wide class of initial data with large total variation. We introduce first a weighted total variation (WTV) of $a(\lambda_o)$; this quantity arises in a natural way in the problem and has also an analytical meaning, being the logarithmic variation in the case of continuous functions. We prescribe a bound on WTV ($a(\lambda_o)$); for the variation $TV(v_o, u_o)$ there is not such a bound but, roughly speaking, the larger $TV(v_o, u_o)$ is, the smaller must be WTV ($a(\lambda_o)$). An important point is that we give explicit expressions for these bounds; then our results are qualitatively different from some of those quoted above, where a generic smallness is required.

The plan of the paper is the following. The main result is stated in Section 2, Theorem 2.2. The Riemann problem is reviewed in Section 3 together with related results; proofs have been given in \[2\]. The definition of the algorithm is in Section 4. The core of the proof are Section 5 – where interactions are studied in detail – and Section 6 – where we prove the convergence and consistence of the scheme. A careful analysis is needed due to the presence of large waves.

The paper is completed by two appendices. In the first one we prove the main result on the weighted total variation. In the second we study the interaction of two shock waves to the light of Section 5 namely we look for precise bounds of the damping coefficient that controls the reflected wave produced in the interaction; we think that this analysis is interesting by its own. Good reading!

2. Main results. We consider the system of conservation laws (1.1). The pressure is given by

$$p(v, \lambda) = \frac{a^2(\lambda)}{v}$$

where $a$ is a smooth ($C^1$) function defined on $[0, 1]$ satisfying for every $\lambda \in [0, 1]$

$$a(\lambda) > 0, \quad a'(\lambda) > 0,$$
see Figure 2.1. For instance $a^2(\lambda) = k_0 + \lambda(k_1 - k_0)$ for $0 < k_0 < k_1$. As a consequence of (2.1) and (2.2) we have, for every $(v, \lambda) \in (0, +\infty) \times [0, 1],$

\begin{align*}
(2.3) \quad & p > 0, \quad p_v < 0, \quad p_{vv} > 0, \\
(2.4) \quad & p_{\lambda} > 0, \quad p_{v\lambda} < 0.
\end{align*}

Remark that assumptions (2.3) and (2.4) are analogous to those usually made on the pressure in the full non-isentropic case, [17], the entropy replacing $\lambda$.

![Pressure curves as functions of $v$.](image)

We denote $U = (v, u, \lambda) \in \Omega = (0, +\infty) \times \mathbb{R} \times [0, 1]$ and by $\bar{U} = (v, u)$ the projection of $U$ onto the plane $vu$: the same notation applies to curves. Under assumptions (2.1) and (2.2) the system (1.1) is strictly hyperbolic in the whole $\Omega$ with eigenvalues $e_1 = -\sqrt{-p_v(v, \lambda)}$, $e_2 = 0$, $e_3 = \sqrt{-p_v(v, \lambda)}$. We write $c = \sqrt{-p_v} = a(\lambda)/v$.

The eigenvectors associated to the eigenvalues $e_i$, $i = 1, 2, 3$, are $r_1 = (1, c, 0)$, $r_2 = (-p_\lambda, 0, p_v)$, $r_3 = (-1, c, 0)$. Because of the third inequality in (2.3) the eigenvalues $e_1$, $e_3$ are genuinely nonlinear with $\nabla e_i \cdot r_i = p_{vv}/(2c) > 0$, $i = 1, 3$, while $e_2$ is linearly degenerate. Pairs of Riemann invariants are $R_1 = \{u - a(\lambda) \log v, \lambda\}$, $R_2 = \{u, p\}$, $R_3 = \{u + a(\lambda) \log v, \lambda\}$.

We denote by $\text{TV}(f)$ the total variation of a function $f$. In the case $f : \mathbb{R} \to (0, +\infty)$ we define the weighted total variation of $f$ by

\[ \text{WTV}(f) = 2 \sup_n \sum_{j=1}^n \frac{|f(x_j) - f(x_{j-1})|}{f(x_j) + f(x_{j-1})} \]

where the supremum is taken over all $n \geq 1$ and $(n+1)$-tuples of points $x_j$ with $x_0 < x_1 < \ldots < x_n$. This variation is motivated by the definition (3.6) of strength for the waves of the second family. If $f$ is bounded and bounded away from zero then clearly

\[ \frac{1}{\sup f} \text{TV}(f) \leq \text{WTV}(f) \leq \frac{1}{\inf f} \text{TV}(f). \]

**Proposition 2.1.** Consider $f : \mathbb{R} \to (0, +\infty)$; then

\[ \inf f \leq \text{TV}(\log(f)) \leq \text{WTV}(f) \leq \text{TV}(\log(f)). \]

Moreover, if $f \in C(\mathbb{R})$ then $\text{WTV}(f) = \text{TV}(\log(f))$. 

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The proof is deferred to Appendix [A]. In (2.6), in the inequality on the right, the strict sign may occur if \( f \) is discontinuous; see Remark [A.1].

We provide system (1.1) with initial data
\[
U(x, 0) = U_o(x) = (v_o(x), u_o(x), \lambda_o(x))
\]
for \( x \in \mathbb{R} \). Denote \( a_o(x) \equiv a(\lambda_o(x)) \), \( p_o(x) \equiv p(v_o(x), \lambda_o(x)) \); remark that \( \inf a_o(x) \geq a(0) > 0 \). The main result of this paper now follows.

**Theorem 2.2.** Assume (2.1), (2.2). Consider initial data (2.0) with \( v_o(x) \geq x \) for some constant \( x \) and \( 0 \leq \lambda_o(x) \leq 1 \). For every \( m > 0 \) and a suitable function \( k(m) \in (0, 1/2) \) the following holds. If
\[
TV(\log(p_o)) + \frac{1}{\inf a_o} TV(u_o) < 2\left(1 - 2WTV(a_o)\right)m
\]
\[
WTV(a_o) < k(m)
\]
then the Cauchy problem (1.1), (2.6) has a weak entropic solution \((v, u, \lambda)\) defined for \( t \in [0, +\infty) \). Moreover the solution is valued in a compact set of \( \Omega \) and there is a constant \( C(m) \) such that for every \( t \in [0, +\infty) \)
\[
TV(v(t, \cdot), u(t, \cdot)) \leq C(m).
\]

The function \( k(m) \), whose expression is given in (6.21), deserves some comments. The interaction of two waves \( \alpha, \alpha' \) of the same family \( i = 1, 3 \) produces a wave \( \beta \) of the same family \( i \) and a "reflected" wave \( \delta \) of the other family \( j \) \((j = 1, 3, j \neq i)\). For a suitable definition of the strengths of the waves we prove that \( |\delta| \leq d \cdot \min\{|\alpha|, |\alpha'|\} \) for a damping coefficient \( d < 1 \) depending on \( \alpha \) and \( \alpha' \), see Lemma 5.6. The function \( k \) above depends essentially on the supremum of such coefficients \( d \); we prove that \( k(0) = 1/2 \) and that \( k(m) \) decreases to 0 as \( m \to +\infty \). In particular then \( WTV(a_o) < 1/2 \). The assumptions (2.7), (2.8) read as analogous to those in [20]: the larger is \( m \), the smaller is \( k(m) \), and vice-versa. The occurring of a possible blow-up when the bound on \( WTV(a_o) \) does not hold is an interesting open problem.

The variation of \( \lambda_o \) appears both in condition (2.7), because of \( p_o \), and in (2.8). Using the definition of the pressure, we can replace (2.7) by the slightly stronger condition
\[
TV \log(v_o) + 2TV \log(u_o) + \frac{1}{\inf a_o} TV(u_o) \leq 2\left(1 - 2WTV(a_o)\right)m
\]
by making use of (2.6). In particular if \( \lambda_o \) is constant we recover the famous result by Nishida [19].

Clearly \( \lambda(t, x) = \lambda_o(x) \) for any \( t \) because of the third equation in (1.1); this is why only \( v \) and \( u \) appear in the estimate (2.9). In other words system (1.1) can be rewritten as a \( p \)-system of two conservation laws with flux depending on \( x \), namely for the pressure law \( p = p(v, \lambda_o(x)) = a^2(\lambda_o(x)) / v \).

The proof of Theorem [2.2] makes use of a wave-front tracking scheme where we exploit the special structure of system (1.1) by differentiating the treatment of 1 and 3 waves from that of 2 waves. Our algorithm is a natural extension of that in [3], where the system for \( \lambda_o \) constant is studied, in presence of a relaxation term.
Here we consider a linear functional as in [3] that accounts for the strengths of all 1 and 3 waves, with a weight $\xi > 1$ assigned to shock waves; a crucial point in the proof is the choice of $\xi$ as a function of $n$. This functional differs with that in [19, 4], where $\xi$ is missing and only the variation of shocks is taken into account. Moreover, motivated again by [3], we do not introduce a simplified Riemann solver for interactions between 1 and 3 waves but only for interactions involving the 2-contact discontinuities. The interaction potential considers then uniquely interactions of 2 waves with 1 or 3 waves approaching to it.

System (1.1) can be written in Eulerian coordinates. Denoting $\rho = 1/v$ the density, the pressure law becomes $p = a^2(\lambda)\rho$ and (1.1) turns into

$$\begin{align*}
(\rho_t + (\rho u)_x) &= 0 \\
(p u_t + (p u^2 + p(\rho, \lambda))_x &= 0 \\
(\rho\lambda)_t + (\rho\lambda u)_x &= 0.
\end{align*}$$

A global existence result of weak solutions for (2.10) holds by Theorem 2.2 because of (2.10)

Moreover, motivated again by [3], we do not introduce a simplified Riemann solver for waves with 1 or 3 waves approaching to it.

A case studied in [6, page 44] is when $u_t = u_g$ and $\rho_l$ is constant, say equal to 1. The unknown variables are then $R_l$, $u$, $\rho_g$, and it is assumed $0 < R_l < 1$ and $\rho_g > 0$; as a consequence $0 < R_g < 1$. Under these conditions, and writing still $\rho_l$ instead of 1 for clarity, we define the concentration $c = \frac{\rho_g R_l}{\rho_l R_g} > 0$ and deduce the pressure law $p = a^2\rho_g$, for $a > 0$ a constant. Equations (3.1) state the conservation of mass of either phases and the total momentum.

Here the indexes $l$ and $g$ stand for liquid and gas. Therefore $\rho_l$, $R_l$, $u_l$ are the liquid density, phase fraction, velocity, and analogously for the gas; clearly $R_l + R_g = 1$. The pressure law is $p = a^2\rho_g$, for $a > 0$ a constant. Equations (3.1) state the conservation of mass of either phases and the total momentum.

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$$\begin{align*}
(R_t)_t + (R_l u)_x &= 0 \\
(R_t c)_t + (R_l c u)_x &= 0 \\
(R_l (1 + c) u)_t + (R_l (1 + c) u^2 + p)_x &= 0.
\end{align*}$$

This system is strictly hyperbolic for $c > 0$. Remark that the three eigenvalues of (3.2) coincide with $u$ at $c = 0$ and if $c$ vanishes identically then (3.2) reduces to the pressureless gasdynamics system. System (3.2) is analogous to (2.10) but the pressure laws are different. In fact the variables $\rho$ and $\lambda$ of (2.10) write $\rho = \rho_l R_l + \rho_g R_g$ and $\lambda = \frac{\rho_l R_l}{\rho_l R_l + \rho_g R_g} = \frac{1}{1 + c}$ and then $R_l = (1 - \lambda)\rho$, $\rho_g = \frac{\lambda}{\rho_l(1 - \lambda)}$. If we sum up the two first equations in (3.2) we find the first equation in (2.10); the third (resp. second) equation in (3.2) becomes the second (third) equation in (2.10). The choice $p = a^2\rho_g$ for the pressure in (3.1) gives $p = a^2\frac{\lambda}{\rho_l(1 - \lambda)}$. 

Notice that the pressure vanishes in presence of a pure liquid phase and this is the main difference with (2.1), (2.2).

We compare now (2.10) and (3.2) in Lagrangian coordinates. Consider for (3.2) the change of coordinates $y = R_t dx - R_l dt$ based on the streamlines of the liquid particles (because $R_t = \rho_l R_t$), [22]. Denote $w = \frac{1}{R_l} - 1 = \frac{R_t}{R_l} = \frac{a^2}{\rho_l}$. Then for $p = \frac{a^2}{w}$ system (3.2) turns into

$$
\begin{aligned}
\dot{w}_i - u_x &= 0 \\
((1 + c)u)_x + px &= 0 \\
ct &= 0.
\end{aligned}
$$

(3.3)

It is more interesting however to consider for system (3.2) the change $y = (1 + c) R_t dx - (1 + c) R_l dt = \rho dx - p dt$ into Lagrangian coordinates based on the streamlines of the full density $\rho$. Let $w$ be as above and $v = \frac{w}{1 + c} = \frac{R_t}{\rho}$. Then system (3.2) becomes system (1.1) with $a^2(\lambda) = a^2 \frac{1 + c}{1 + c} = a^2$. As a consequence the pressure law $p(v, \lambda) = a^2(\lambda)/v$ does not satisfy (2.2). This difficulty can be overcome as follows. Fix any $0 < a_1 < a_2 < a$ and consider for $c_i = \frac{a_i^2}{a^2 - a_i^2}$ the invariant domain $\{ (R_l, u, c) : 0 < c_1 < c < c_2 \}$, [22]. In this domain $0 < b_1 < \lambda < b_2 < 1$, for $b_i = a_i^2/a^2$.

If we denote $\mu = \frac{\lambda - b_1}{b_2 - b_1}$ then the function $b(\mu) = a(\lambda) = a(b_1 + (b_2 - b_1)\mu)$ makes the pressure law $p(v, \lambda) = b^2(\mu)/v$, with $\mu \in [0, 1]$, satisfy both conditions in (2.2).

3.2. Wave curves and the Riemann problem. In this section we recall some results about the wave curves for system (1.1) and the solution to the Riemann problem; see [2] for more details.

The shock-rarefaction curves through the point $U_o = (v_o, u_o, \lambda_o)$ for (1.1) are

$$
\begin{aligned}
\Phi_1(v, U_o) &= (v, \phi_1(v, U_o), \lambda_o), \quad i = 1, 3 \\
\phi_1(v, U_o) &= \begin{cases} 
  u_o + a(\lambda_o) \cdot (v - v_o)/\sqrt{vv_o} & v < v_o \quad \text{shock} \\
  u_o + a(\lambda_o) \log(v/v_o) & v > v_o, \quad \text{rarefaction},
\end{cases} \\
\phi_3(v, U_o) &= \begin{cases} 
  u_o - a(\lambda_o) \log(v/v_o) & v < v_o \quad \text{rarefaction} \\
  u_o - a(\lambda_o) \cdot (v - v_o)/\sqrt{vv_o} & v > v_o, \quad \text{shock},
\end{cases}
\end{aligned}
$$

(3.4)

$$
\Phi_2(\lambda, U_o) = \left( v_o \frac{a^2(\lambda)}{a^2(\lambda_o)}, u_o, \lambda \right), \quad \lambda \in [0, 1], \quad \text{contact discontinuity}.
$$

(3.5)

The curves $\Phi_1, \Phi_2$ and $\Phi_3$ are plane curves: $\Phi_1$ and $\Phi_3$ lie on the plane $\lambda = \lambda_o$ while $\Phi_2$ on $u = u_o$.

Definition 3.1 (Wave strengths). Under the notations (3.4), (3.5) we define the strength $\varepsilon_i$ of an $i$-wave as

$$
\varepsilon_1 = \frac{1}{2} \log \left( \frac{v}{v_o} \right), \quad \varepsilon_2 = \frac{1}{2} \frac{a(\lambda) - a(\lambda_o)}{a(\lambda) + a(\lambda_o)}, \quad \varepsilon_3 = \frac{1}{2} \log \left( \frac{v_o}{v} \right).
$$

(3.6)

According to this definition, rarefaction waves have positive strengths and shock waves have negative strengths. Given the initial datum $\lambda_o = \lambda_o(x)$, denote

$$
a^* \doteq \sup_{x \in \mathbb{R}} a(\lambda_o(x)), \quad a_* \doteq \inf_{x \in \mathbb{R}} a(\lambda_o(x)), \quad [a]_* \doteq \frac{a^* - a_*}{a^* + a_*}.
$$

(3.7)
Then \([a]_* \leq \frac{a(1)-a(0)}{a(1)+a(0)} < 1\) and \(|\varepsilon_2| \leq 2[a]_* < 2\). It is useful to define also the function, see [22],

\[
(3.8) \quad h(\varepsilon) = \begin{cases} 
\varepsilon & \text{if } \varepsilon \geq 0, \\
\sinh \varepsilon & \text{if } \varepsilon < 0.
\end{cases}
\]

Then we have for \(i = 1, 3\)

\[
(3.9) \quad \phi_i(v, U_\nu) = u_\nu + a(\lambda_\nu) \cdot 2h(\varepsilon_i).
\]

At last we consider the Riemann problem. This is the initial-value problem for \((1.1)\) under the piecewise constant initial condition

\[
(3.10) \quad (v, u, \lambda)(0, x) = \left\{ \begin{array}{ll}
(v_\ell, u_\ell, \lambda_\ell) = U_\ell & \text{if } x < 0 \\
(v_r, u_r, \lambda_r) = U_r & \text{if } x > 0
\end{array} \right.
\]

for \(U_\ell\) and \(U_r\) in \(\Omega\). We denote \(a_r = a(\lambda_r), p_r = a_r^2/v_r\), and similarly \(a_\ell, p_\ell\).

**Proposition 3.2.** Fix any pair of states \(U_\ell, U_r\) in \(\Omega\); then the Riemann problem \((1.1), (3.10)\) has a unique \(\Omega\)-valued solution in the class of solutions consisting of simple Lax waves. If \(\varepsilon_i\) is the strength of the \(i\)-wave, \(i = 1, 2, 3\), then

\[
\varepsilon_3 - \varepsilon_1 = \frac{1}{2} \log \left( \frac{p_r}{p_\ell} \right), \quad 2(a_\ell h(\varepsilon_1) + a_r h(\varepsilon_3)) = u_r - u_\ell.
\]

Moreover, let \(\rho > 0\) be a fixed number. There exists a constant \(C_1 > 0\) depending on \(\rho\) and \(a(\lambda)\) such that if \(\bar{U}_\ell, \bar{U}_r \in \Omega\) and \(v > \rho\), then

\[
(3.11) \quad |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| \leq C_1 |U_\ell - U_r|.
\]

For the proof, see [2]. One can easily find that

\[
(3.12) \quad |\varepsilon_1| + |\varepsilon_3| \leq \frac{1}{2} |\log(p_r) - \log(p_\ell)| + \frac{1}{2 \min\{a_\ell, a_r\}} |u_r - u_\ell| \leq \frac{1}{2} |\log(v_r) - \log(v_\ell)| + |\log(a_r) - \log(a_\ell)| + \frac{1}{2 \min\{a_\ell, a_r\}} |u_r - u_\ell|.
\]

We remark that for any Riemann data \((v_\ell, u_\ell, \lambda_\ell), (v_r, u_r, \lambda_r)\), the \(\lambda\) component of the solution takes value \(\lambda_\ell\) for \(x < 0\) and \(\lambda_r\) for \(x > 0\). The fact that the interfaces between different phases are connected by a stationary wave can be interpreted then as a “kinetic condition”, [1], analogous to Maxwell’s rule.

**4. The approximate solution.** In this section we define a wave-front tracking scheme [3] to build up piecewise constant approximate solutions to \((1.1)\). More precisely we follow the algorithm introduced in [3].

First, we approximate the initial data. For any \(\nu \in \mathbb{N}\) we take a sequence \((v_\nu^o, u_\nu^o, \lambda_\nu^o)\) of piecewise constant functions with a finite number of jumps such that

(i) \(TV p_\nu^o \leq TV p_\nu, TV u_\nu^o \leq TV u_\nu, WTV a(\lambda_\nu^o) \leq WTV a(\lambda_\nu), \inf a_\nu^o \geq 1\);  
(ii) \(\lim_{x \to -\infty}(v_\nu^o, u_\nu^o, \lambda_\nu^o)(x) = \lim_{x \to -\infty}(v_\nu, u_\nu, \lambda_\nu)(x);\)  
(iii) \(\|(v_\nu^o, u_\nu^o, \lambda_\nu^o) - (v_\nu, u_\nu, \lambda_\nu)\|_L^1 \leq \frac{1}{\nu}\)
where \( p_\nu^o = a^2(\lambda_\nu^o)/v_\nu^o \). Second, we define the approximate Riemann solver. We introduce positive parameters \( \eta = \eta_\nu, \rho = \rho_\nu \); they control respectively the size of rarefactions and the threshold when a simplified Riemann solver is used. Define also a parameter \( s > 0 \) strictly larger than all possible speeds of wave-fronts of both families 1 and 3. These parameters will be determined at the end of Section 6.

- At time \( t = 0 \) we solve the Riemann problems at each point of jump of \((v_\nu^o, u_\nu^o, \lambda_\nu^o)(0^+, \cdot)\) as follows: shocks are not modified while rarefactions are approximated by fans of waves, each of them having size less than \( \eta \). More precisely, a rarefaction of size \( \varepsilon \) is approximated by \( N = \lceil \varepsilon/\eta \rceil + 1 \) waves whose size is \( \varepsilon/N < \eta \); we set their speeds to be equal to the characteristic speed of the state at the right.

When \((v, u, \lambda)(t, \cdot)\) is defined until some wave fronts interact; by slightly changing the speed of some waves \cite{9} we can assume that only \emph{two} fronts interact at a time.

- When two wave fronts of families either 1 or 3 interact we solve the Riemann problem at the interaction point. If one of the incoming waves is a rarefaction, after the interaction it is prolonged (if it still exists) as a single discontinuity with speed equal to the characteristic speed of the state at the right. If a new rarefaction is generated, we employ the Riemann solver described before and divide it into a fan of waves having size less than \( \eta \).

- When a wave front either of family either 1 or 3 interacts with a 2-wave we proceed as follows. Let \( \delta_2 \) be the size of the 2-wave and \( \delta \) the size of the other wave.
  - If \(|\delta_2\delta| \geq \rho \) we solve the Riemann problem as above, that is with the accurate Riemann solver.
  - If \(|\delta_2\delta| < \rho \) we prolong the 1- or 3- wave with a wave of the same family and size. Since the two waves do not commute, a \emph{non-physical} front is introduced, \cite{9}, with fixed speed \( s > 0 \). The size of a non-physical wave is set to be \(|u_r - u_\ell|\), where \( u_\ell, u_r \) are the \( u \) components of the left and right states of the wave. We call this solver the \emph{simplified Riemann solver}.

- When a non-physical front interacts with a front of family 1, 2 or 3 ("physical"), we prolong the solution with a physical wave of the same size and a non-physical one, computing the intermediate value consequently.

We refer for the last two items to Proposition 5.12 below. Remark that two non-physical front cannot interact since they have the same constant speed \( s \). We denote by \( \mathcal{NP} \) the set of non-physical waves.

5. Interactions. Fix the index \( \nu \) introduced in the previous section. We shall prove in Subsection 6.1 that the algorithm described above is defined for any \( t > 0 \) and provides for any initial data \((v_\nu^o, u_\nu^o, \lambda_\nu^o)\) a piecewise constant approximate solution \((v^o, u^o, \lambda^o) = (v, u, \lambda)\), where we dropped for simplicity the index \( \nu \). Here we study the interaction of waves.

For \( K_{np} > 0 \) and \( t > 0 \) we define the functional \( L \) and the interaction potential \( Q \), both referred to \((v, u, \lambda)(t, \cdot)\), by

\[
L(t) = \sum_{i=1,3} |\gamma_i| + K_{np} L_{np}, \\
L_{np} = \sum_{\gamma \in \mathcal{NP}} |\gamma| \\
Q(t) = \sum_{\gamma_3 \text{ at the left of } \delta_2} |\gamma_3| |\delta_2| + \sum_{\gamma_1 \text{ at the right of } \delta_2} |\delta_2||\gamma_1|. \tag{5.1}
\]
Remark that $L$ takes only into account the strengths of both 1 and 3 waves and that of non-physical waves. For contact discontinuities we define

\[ L_{\text{cd}} = \sum |\gamma_2| = \text{WTV}a(\lambda_0^\circ). \]

Finally, for $\xi \geq 1$ and $K \geq 0$ we introduce

\begin{align}
L_\xi &= L_{\text{rarefactions}} + \xi L_{\text{shocks}} + K_{np} L_{np}, \\
F &= L_\xi + K Q.
\end{align}

For simplicity we omitted to note the dependence on $K_{np}$ in the functional $L_\xi$ and on $K_{np}, \xi, K$ in $F$; the choice of $K_{np}$ shall depend on that of $K$, see Proposition 5.12.

Observe that, if $\lambda_o$ is constant, then $Q = 0$ and $F = L_\xi$, whose variation was analyzed in Lemma 3.2 of [3]. Hence we will assume from now on that

\[ A_o = \text{WTV}(a_o) > 0. \]

By assumption (i) in Section 4, one has $L_{\text{cd}} \leq A_o$.

In the following sections we analyze in detail the different types of interactions. Recalling the definition of $h$, (3.8), and with the notation of Figure 5.1, we introduce the following identities, see (3.1), (3.2) in [22]:

\begin{align}
\varepsilon_3 - \varepsilon_1 &= \alpha_3 + \beta_3 - \alpha_1 - \beta_1 \\
\alpha_\ell h(\varepsilon_1) + \alpha_m h(\varepsilon_3) &= \alpha_\ell h(\alpha_1) + a_m h(\alpha_3) + a_m h(\beta_1) + a_r h(\beta_3).
\end{align}

Formula (5.5) does not depend on $\lambda$ and follows easily by equating the specific volumes $v$ before and after the interaction time. By equating the velocities $u$ we obtain (5.6).

These properties are a consequence of the definition (3.6) of the strengths for 1- and 3-waves and of (3.9).

5.1. Interactions with a 2-wave. We consider first the interactions of 1 or 3 waves with a 2 wave, see Figure 5.2.

Proposition 5.1 ([2]). Denote by $\lambda_\ell, \lambda_r$ the side states of a 2-wave. The interactions of 1- or 3-waves with the 2-wave give rise to the following pattern of solutions:

| interaction | outcome |
|-------------|---------|
| $2 \times 1R$ | $1R + 2 + 3R$ | $1R + 2 + 3S$ |
| $2 \times 1S$ | $1S + 2 + 3S$ | $1S + 2 + 3R$ |
| $3R \times 2$ | $1S + 2 + 3R$ | $1R + 2 + 3R$ |
| $3S \times 2$ | $1R + 2 + 3S$ | $1S + 2 + 3S$. |

Fig. 5.1: A general interaction pattern.
The next lemma is concerned instead with the strengths of waves involved in the interaction above. The inequalities (5.8) improve the inequality (3.3) in [22] in the special case of two interacting wave fronts, one of them being of the second family. More precisely under the notations of [22] we find a term $\frac{1}{\alpha r + \alpha \ell}$ instead of $\frac{1}{\min\{\alpha r, \alpha \ell\}}$. The proof differs from Peng’s. Our estimates are sharp: in some cases (5.8) reduces to an identity.

**Lemma 5.2 ([2]).** Assume that a $1$ wave of strength $\delta_1$ or a $3$ wave of strength $\delta_3$ interacts with a $2$ wave of strength $\delta_2 = 2(a_r - a_\ell)/(a_r + a_\ell)$. Then, the strengths $\varepsilon_i$ of the outgoing waves satisfy:

$$(5.7) \quad |\varepsilon_i - \delta_i| = |\varepsilon_j| \leq \frac{1}{2} |\delta_2| \cdot |\delta_1| \leq |a|_\ast |\delta_1|$$

for $i, j = 1, 3, i \neq j$. Moreover,

$$(5.8) \quad |\varepsilon_1| + |\varepsilon_3| \leq \begin{cases} |\delta_1| + |\delta_1| |\delta_2|_+ & \text{if 1 interacts} \\ |\delta_3| + |\delta_3| |\delta_2|_- & \text{if 3 interacts} \end{cases}$$

Here $[x]_+ = \max\{x, 0\}$, $[x]_- = \max\{-x, 0\}$, $x \in \mathbb{R}$. Remark that the colliding 1 or 3 wave does not change sign across the interaction. Moreover the functional $L$ increases if and only if the incoming and the reflected waves are of the same type; this happens when the colliding wave is moving toward a more liquid phase.

Now we prove that $F$ is decreasing for suitable $K$ when an interaction with a 2-wave occurs. The potential $Q$ is needed to balance the possible increase of $L_\xi$.

**Proposition 5.3.** Assume $A_\circ < 2$ and consider an interaction of a 1 or 3 wave with a 2 wave, with the notation of Lemma 5.2. Then $\Delta Q < 0$. If moreover

$$(5.9) \quad \xi \geq 1 \quad \text{and} \quad K > \frac{2\xi}{2 - A_\circ}$$

then

$$(5.10) \quad \xi |\varepsilon_j| = \xi |\varepsilon_i| - |\delta_i| < \frac{K}{2} |\Delta Q|$$

and hence $\Delta F < 0$.

**Proof.** We consider the interaction of a 3-wave with a 2-wave, as in the proof of Lemma 5.2 see [2]; the symmetric case follows in an analogous way. We use the

---

Fig. 5.2: Interactions. (a): from the right; (b): from the left.
notation as in Figure 5.2(a). We define $L_{cd}^* = L_{cd}^- + L_{cd}^+$, $L_{cd}^\pm$ meaning right or left of the 2 wave under consideration.

By assumption, one has

\begin{equation}
L_{cd} = L_{cd}^- + L_{cd}^+ + |\delta_2| = L_{cd}^* + |\delta_2| \leq A_o < 2.
\end{equation}

Recall that $\varepsilon_1 - \delta_1 = \varepsilon_3$ and

\begin{equation} |\varepsilon_1| - |\delta_1| = |\varepsilon_3|, \quad \text{if } \delta_2 > 0, \end{equation}

\begin{equation} |\varepsilon_1| - |\delta_1| = -|\varepsilon_3|, \quad \text{if } \delta_2 < 0, \end{equation}

so that in particular $|\varepsilon_3| = |\varepsilon_1| - |\delta_1|$. An estimate for $\Delta Q$ follows at once because of (5.11):

\begin{equation}
\Delta Q = -|\delta_2\delta_1| + (|\varepsilon_1| - |\delta_1|) L_{cd}^- + |\varepsilon_3| L_{cd}^+ \leq \frac{1}{2} |\delta_2\delta_1| (L_{cd}^* - 2)
\end{equation}

\begin{equation}
\leq \frac{1}{2} |\delta_2\delta_1| (A_o - 2) < 0.
\end{equation}

Hence, using (5.14), we get

\begin{equation}
\xi |\varepsilon_3| + \frac{K}{2} \Delta Q \leq \frac{1}{2} |\delta_2\delta_1| \left\{ \xi + \frac{K}{2} (A_o - 2) \right\} < 0
\end{equation}

because of (5.9); this proves (5.10). Finally, by using (5.15) we get

\begin{equation}
\Delta F = \Delta L_\xi + K \Delta Q \leq \xi |\varepsilon_3| + \xi |\varepsilon_1 - \delta_1| + K \Delta Q < 0.
\end{equation}

5.2. Interactions between 1 and 3 waves. Here we analyze the possible interactions between 1- and 3-waves. Two situations may occur, see Figure 5.3: either the waves belong to different families or they both belong to the same family. In this last case, at least one of the waves must be a shock.

![Fig. 5.3: Interactions of 1 and 3 waves.](image)

**Lemma 5.4** (Different families interacting). If a wave of the third family interacts with a wave of the first family, they cross each other without changing their strength.

**Proof.** See also Lemma 3.1 in [3]. Using notation as in Figure 5.3(a) we have $\varepsilon_3 - \varepsilon_1 = \delta_3 - \delta_1$ and $h(\varepsilon_1) + h(\varepsilon_3) = h(\delta_1) + h(\delta_3)$. The uniqueness of solutions to the Riemann problem implies $\varepsilon_1 = \delta_1$, $\varepsilon_3 = \delta_3$.

Remark that here $\Delta L_\xi = 0 = \Delta Q$ and then $\Delta F = 0$ for all $\xi \geq 1$ and $K$.

**Lemma 5.5** (Same family interacting: outcome). Assume that a wave $\alpha_3$ of the third family interacts with a wave $\beta_3$ of the third family, giving rise to waves $\varepsilon_1$, $\varepsilon_3$. Then
In any case and the amount of rarefactions of the \( i \max \) assume (5.19). Then the following holds.

\[
\alpha
\]

(5.19) \[
\alpha
\]

\[
\alpha
\]

\[
\alpha
\]

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\alpha
\]

\[
\alpha
\]

In case (i) these formulas read \( \varepsilon_1 - \varepsilon_3 = |\alpha_3| + |\beta_3| > 0 \) and \(-h(\varepsilon_1) - h(\varepsilon_3) = \sinh(|\alpha_3| + \sinh(|\beta_3|) > 0. If it were \( \varepsilon_3 > 0 \) then \( \varepsilon_1 > 0 \) from the first equality and \( \varepsilon_1 < 0 \) from the second, a contradiction. Therefore \( \varepsilon_3 < 0 \) so that \( \varepsilon_1 + |\varepsilon_3| = |\alpha_3| + |\beta_3| \) and \(-h(\varepsilon_1) + \sinh(|\varepsilon_3|) = \sinh(|\alpha_3| + \sinh(|\beta_3|)). Analogously, if it were \( \varepsilon_1 < 0 \), using elementary inequalities we get \( 0 = \sinh(|\varepsilon_1|) + \sinh(|\alpha_3| + |\beta_3| + |\varepsilon_1|) - \sinh(|\alpha_3|) - \sinh(|\beta_3|) \geq 2 \sinh(|\varepsilon_1|) \), a contradiction again. Hence \( \varepsilon_1 > 0 \).

In case (ii) assume \( \alpha_3 < 0, \beta_3 > 0 \); the other case is dealt analogously since (5.17), (5.18) are symmetric in \( \alpha, \beta \). We have \( \varepsilon_3 - \varepsilon_1 = -|\alpha_3| + |\beta_3| \) and \( h(\varepsilon_1) + h(\varepsilon_3) = -\sinh(|\alpha_3|) + |\beta_3| \), then \( [h(\varepsilon_1) + \varepsilon_1] + [h(\varepsilon_3) - \varepsilon_3] = |\alpha_3| - \sinh(|\alpha_3|) < 0 \). If \( \varepsilon_3 > 0 \), this last equality becomes \( h(\varepsilon_1) + \varepsilon_1 = |\alpha_3| - \sinh(|\alpha_3|) < 0 \) that implies \( \varepsilon_1 < 0 \). If \( \varepsilon_3 < 0 \), then \( h(\varepsilon_1) + \varepsilon_1 = |\alpha_3| - \sinh(|\alpha_3|) - |\varepsilon_3| - \sinh(|\varepsilon_3|) \). If it were \( \varepsilon_1 > 0 \) it would be \( |\alpha_3| < |\varepsilon_3| \), since the map \( x \mapsto x - \sinh x \) is decreasing; but from \( |\varepsilon_3| + |\varepsilon_1| = |\alpha_3| - |\beta_3| \) we would get that \( |\varepsilon_3| < |\alpha_3| \), a contradiction. Hence in all cases one has \( \varepsilon_1 < 0 \). \( \Box \)

Now we give sharper estimates for the interaction of waves of the same family: we prove that the strength of the reflected wave is bounded by the size of each incoming wave, multiplied by a damping factor smaller than 1. This property will be crucial in the next section and it holds also for interactions with a 2-wave, with damping factor \([a]_\ast\), see (5.7). In the case below however the coefficient depends on the strengths of the incoming waves; this happens also when non-physical waves are generated, see Proposition 5.12. We assume that

\[
\alpha
\]

(5.19) \[
\alpha
\]

\[
\alpha
\]

\[
\alpha
\]

\[
\alpha
\]

The strength of any interacting \( i \)-wave is less than \( m \), for some \( m > 0 \) and \( i = 1, 3 \).

In the special case of interaction of waves of the same family producing two outgoing shocks we give a more precise result in Appendix 5.

Lemma 5.6 (Same family interacting). Consider the interaction of two waves of the same family, of sizes \( \alpha_i \) and \( \beta_i \), \( i = 1, 3 \), producing two outgoing waves \( \varepsilon_1, \varepsilon_3 \); assume (5.7). Then the following holds.

(i) There exists a damping coefficient \( d = d(m) \), with \( 0 < d < 1 \), such that

\[
|\varepsilon_j| \leq d(m) \cdot \min\{|\alpha_i|, |\beta_i|\}, \quad j \neq i.
\]

(ii) If the incoming waves are both shocks, the resulting shock satisfies \( |\varepsilon_i| > \max\{|\alpha_i|, |\beta_i|\} \). If the incoming waves have different signs, both the amount of shocks and the amount of rarefactions of the \( i \)-th family decrease across the interaction. In any case

\[
|\varepsilon_i| \leq |\alpha_i| + |\beta_i|.
\]
Remark that the equation (5.24) is symmetric in \(a\) and \(b\); hence the identity (5.24) rewrites as

\[
h(\varepsilon_1) + h(\varepsilon_3) = h(\alpha_3) + h(\beta_3) = h(\alpha_3) + h(\beta_3).
\]

and then

\[
h(\varepsilon_1) + h(\varepsilon_1 + \alpha_3 + \beta_3) = h(\alpha_3) + h(\beta_3).
\]

Remark 5.7. The damping coefficient \(d(m)\), see Figure 5.4, is given by

\[
d(m) = \max_{|a| \leq m, |b| \leq m} \frac{|\varepsilon(a,b)|}{\min\{|a|, |b|\}},
\]

where the function \(\varepsilon(a,b)\) satisfies \(h(\varepsilon) + h(\varepsilon + a + b) - h(a) - h(b) = 0\), see (5.24). Hence \(d(m)\) increases with \(m\), and vanishes as \(m \to 0\) because quadratic interaction estimates hold for \(m\) small.

Moreover, it is asymptotic to 1 for \(m\) large. Indeed, from the proof of Lemma 5.6 we have \(\tau_\alpha(0,b) = \frac{1 - h'(b)}{h'(b)}\); then \(\tau_\alpha(0,b) = 0\) if \(b > 0\) and \(|\tau_\alpha(0,b)| = \frac{\cosh(b) - 1}{\cosh(b) + 1} \leq \frac{1}{2}\).
\(\frac{\cosh m - 1}{\cosh m + 1}\) if \(b < 0\). Therefore \(|\tau_a(0,b)| \leq \frac{\cosh m - 1}{\cosh m + 1}\) for every \(b\), and an analogous estimate holds for \(|\tau_b(0,a)|\). Hence \(d(m) \geq \frac{\cosh m - 1}{\cosh m + 1} \equiv c(m)\); we refer to Lemma \([5.1]\) for the role of this quantity.

**Remark 5.8.** When a rarefaction interacts with a 1- or 3-wave, its size does not increase. Indeed, the size does not change upon interactions with waves of the other family by Lemma \([5.4]\); if the rarefaction interacts with a shock of the same family we apply Lemma \([5.6]\) (ii). Remark moreover that by Lemmas \(5.4, 5.5\) a rarefaction never produces a rarefaction of the other family by interactions with 1- and 3-waves.

**Remark 5.9.** If two waves of the same family interact, the wave belonging to that family can be missing, while the “reflected” wave is always present. This follows easily from \((5.22), (5.23)\).

**Proposition 5.10** (Variation of \(F\)). Consider the interactions of any two wave fronts of the same family, 1 or 3, and assume \((5.4), (5.19)\). If

\[
1 < \xi < \frac{1}{d} \quad \text{and} \quad K < \frac{\xi - 1}{A_0}
\]

then \(\Delta L_\xi < 0\) and \(\Delta F < 0\).

**Proof.** Let two waves \(\alpha_i, \beta_i\) interact, \(i = 1, 3\), giving rise to waves \(\varepsilon_1, \varepsilon_3\). We consider \(i = 3\), the other case being analogous. Using \((5.21)\), we get

\[
\Delta Q = (|\varepsilon_3| - |\alpha_3| - |\beta_3|) L^-_{cd} + |\varepsilon_1| L^-_{cd} \leq |\varepsilon_1| L^-_{cd} \leq |\varepsilon_1| A_0.
\]

Now we claim that

\[
\Delta L_\xi + |\varepsilon_1|(|\xi - 1|) \leq 0.
\]

From this estimate it follows \(\Delta F = \Delta L_\xi + K\Delta Q \leq |\varepsilon_1| (1 - \xi + KA_0) < 0\) because of \((5.27)\). To prove our claim we consider the possible cases; we make use of \((5.22)\).

**Case**: \(SS \rightarrow RS\) Since \(\Delta L = 0\)

\[
\Delta L_\xi + (\xi - 1)|\varepsilon_1| = \xi(|\varepsilon_1| + |\varepsilon_3| - |\alpha_3| - |\beta_3|) \leq 0.
\]

**Case**: \(SR, RS \rightarrow SR\) Assume \(\alpha_3 < 0 < \beta_3\); then \((5.29)\) reads \((2\xi - 1)|\varepsilon_1| + |\varepsilon_3| - |\beta_3| - \xi|\alpha_3| \leq 0\). For later use we prove the stronger inequality

\[
\xi^2|\varepsilon_1| + |\varepsilon_3| - |\beta_3| - \xi|\alpha_3| \leq 0.
\]
Indeed, from Lemma 5.6 (ii) we have $|\varepsilon_3| < |\beta_3|$, while $\xi|\varepsilon_1| \leq |\alpha_3|$ from (5.20), (5.27).

\textbf{SR, RS \rightarrow SS} Assume $\alpha_3 < 0 < \beta_3$; then (5.23) is $(2\xi - 1)|\varepsilon_1| + \xi(|\varepsilon_3| - |\alpha_3|) - |\beta_3| \leq 0$. We prove also in this case the stronger inequality

\begin{equation}
\xi^2|\varepsilon_1| + \xi(|\varepsilon_3| - |\alpha_3|) - |\beta_3| \leq 0.
\end{equation}

Indeed, by (5.17) and again because of (5.20), (5.27), one has

\[
\xi^2|\varepsilon_1| + \xi(|\varepsilon_3| - |\alpha_3|) - |\beta_3| = \xi^2|\varepsilon_1| + \xi(|\varepsilon_1| - |\beta_3|) - |\beta_3|
\]

\[
= (\xi + 1)(|\varepsilon_1| - |\beta_3|) \leq 0.
\]

This proves the claim and concludes the proof. \qed

\textbf{Remark 5.11.} From the above proof we see that $\Delta L_\xi \leq 0$ for $\xi = 1$. This was a key point in [19], where however a different choice of strengths was done. In [3] the inequality $\Delta L_\xi \leq 0$ was proved to hold also for $1 < \xi \leq \xi_o$, for some $\xi_o > 1$; the condition (5.27) gives an estimate of such a threshold.

More precisely, in the first two cases of Proposition 5.10 we have $\Delta L_\xi \leq 0$ for every $\xi \geq 1$. The third case is analyzed in detail in Lemma 5.1; we prove there that $\Delta L_\xi \leq 0$ for any $\xi > 1$ if $c(m) \leq 1/2$, while we need $1 < \xi \leq \frac{1}{2c(m)-1}$ if $c(m) > 1/2$.

\textbf{5.3. Non-physical waves.} In this subsection we compute the strength of a non-physical wave generated by an interaction and prove that it does not change in subsequent interactions. We introduce the following notation: given $U_\ell = (v_\ell, u_\ell, \lambda_\ell)$ and $\lambda_r$ we define by

\[
U^*_r = \Phi_2(\lambda_r, U_\ell) = (A_{\ell r}v_\ell, u_\ell, \lambda_r)
\]

the state on the right of a 2-wave with left state $U_\ell = (v_\ell, u_\ell, \lambda_\ell)$ and $\lambda = \lambda_r$ on the right, where $A_{\ell r} = a^2(\lambda_r)/a^2(\lambda_\ell)$. See (5.3) and [2].

![Fig. 5.5: Simplified Riemann solver](image)

\textbf{Proposition 5.12 (Non-physical waves).} Consider $U_\ell = (v_\ell, u_\ell, \lambda_\ell)$. Let $U_r = (v_r, u_r, \lambda_r)$ be connected to $U^*_r$ by a 1-wave of size $\delta_1$ and $U_q = (v_q, u_q, \lambda_\ell)$ be connected to $U_\ell$ by a 1-wave of size $\delta_1$, see Figure 5.5(a). Assume (5.10).

Then $U^*_r$ and $U_r$ differ only in the $u$ component; if $\delta_2$ denotes the size of the 2-wave, there exists a constant $C_o = C_o(m)$ such that

\begin{equation}
||U^*_r - U_r|| = |u_q - u_r| \leq C_o|\delta_2\delta_1|.
\end{equation}

A similar result holds for the interaction of a 3-wave, see Figure 5.5(b), again under (5.10).
Moreover the size of a non-physical wave does not change in subsequent interactions. For any $K > 0$ and $K_{np} < K/C_o$ at any interaction involving a non-physical wave we have

\[(5.34) \quad \Delta F \leq 0,\]

with $\Delta F < 0$ when a non-physical wave is generated.

**Proof.** Recalling [2, Lemma 2], only the $u$ component will be different after commutation of the 1- and the 2-wave. We find that

\[u_q - u_{\ell} = 2a_\ell h(\delta_1), \quad u_r - u_{\ell} = 2a_r h(\delta_1)\]

hence

\[|u_q - u_r| = 2|a_\ell - a_r| \cdot |h(\delta_1)| \leq |\delta_2 \delta_1| \cdot 2a(1) \cdot \max_{0 < \eta \leq m} \frac{\sinh \eta}{\eta}.\]

Then (5.34) follows with $C_o(m) = 2a(1) \cdot \frac{\sinh \eta}{m}$.

Next, assume that a non-physical wave interacts with a 2-wave. Since the values of $u$ do not change across a 2-wave, the left and right values of $u$ of the non-physical wave do not change across the interaction; hence the size does not change.

Assume then that a non-physical wave interacts with a 1- or 3-wave of size $\delta$. Since $\lambda$ is constant, we refer only to the components $v, u$. Let $(v_{\ell}, u_{\ell})$ and $(v_{\ell}, u_q)$ be the side states of the non-physical wave before the interaction and $(v_r, u_r), (v_r, u_r)$ be the side states of the physical wave. After the interaction, let $(\tilde{u}_{\ell}, \tilde{u}_{\ell})$ be the intermediate state. One has

\[u_r - u_q = 2a(\lambda)h(\delta) = \tilde{u}_{\ell} - u_{\ell},\]

hence $|u_q - u_r| = |u_r - \tilde{u}_{\ell}|$.

At last, we consider the functional $F$. The potential $Q$ is unaltered when non-physical waves interact with other waves. The only cases in which $L_{np}$ changes are when a non-physical wave arises. Assume that a 1- or a 3-wave of size $\delta$ interacts with a 2-wave of size $\delta_2$, producing a wave of same size and a non-physical wave. Then $\Delta Q = -|\delta \delta_2|$ and $\Delta L_\xi = K_{np} \Delta L_{np} \leq K_{np} C_o |\delta \delta_2|$; hence $\Delta F = \Delta L_\xi + K \Delta Q \leq |\delta \delta_2|(K_{np} C_o - K)$, then (5.34). \[\square\]

### 5.4. Decreasing of the functional $F$ and control of the variations.

We collect first the previous results into a single proposition.

**Proposition 5.13 (Local decreasing).** Consider the interaction of any two waves either of families 1, 2, 3 or non-physical. Assume (5.19) for some $m > 0$; let $C_o = C_o(m)$ as in Proposition 5.12. Finally let $A_o$ satisfy

\[(5.35) \quad 0 < A_o < 2 \frac{1 - d}{3 - d}.\]

If $\xi, K, K_{np}$ satisfy

\[(5.36) \quad \frac{2 - A_o}{2 - 3A_o} < \xi < \frac{1}{d}, \quad \frac{2\xi}{2 - A_o} < K < \frac{\xi - 1}{A_o}, \quad K_{np} < \frac{K}{C_o}\]

then

\[(5.37) \quad \Delta F \leq 0.\]
Proof. The condition on \( K \) comes from (5.9) and (5.27). The interval where \( K \) lies is not empty if \( A_o < \frac{\xi}{2} \) and \( \xi > \frac{2-\sqrt{3}}{2}A_o \); together with (5.27) this gives the assumption required on \( \xi \). In turn, it is possible to choose \( \xi \) in such interval if (5.35) holds. Remark that \( 2 \frac{1}{2} - 3 \leq \frac{\xi}{2} \), so the previous condition on \( A_o \) holds. Therefore the assumptions of Propositions 5.3, 5.10 and 5.12 hold and then (5.37) follows.

Proposition 5.14 (Global decreasing). Let \( m > 0 \); assume (5.35), (5.36), \( L(0+) < m \), and that the approximate solution \( U \) is defined in \([0, T]\). Then \( L(t) < m \) for any \( t \in [0, T] \); as a consequence, condition (7.19) holds for any 1 or 3 wave in \( U \). Finally, \( \Delta F(t) \leq 0 \) for all times \( t \in [0, T] \).

Proof. Because of Proposition 5.13, for any time \( t \) not of interaction we have

\[
L(t) \leq L(\xi(t)) \leq F(t) \leq F(0+) \leq \xi L(0+) + KQ(0+).
\]

Moreover \( Q(0+) \leq L(0+)\Delta cf \leq L(0+)A_o \); thanks to (5.38) and (5.36) we have

\[
L(t) \leq L(0+) \cdot (\xi + KA_o) \leq L(0+) \cdot (2\xi - 1) < m. \tag{5.39}
\]

Therefore the size \( |\varepsilon| \) of any wave of families 1, 3 at time \( t > 0 \) satisfies \( |\varepsilon| \leq m \), so (5.19) holds.

Remark 5.15. From (5.38) we see that, in order to have \( L(t) < m \), the smaller is \( \xi \) the larger can be chosen \( L(0+) \).

6. The convergence and the consistence of the algorithm. In this section we prove Theorem 2.2. We show first that for fixed \( \nu \) the algorithm introduced in Section 4 gives an approximate solution defined for every \( t > 0 \); more precisely we prove that at every time the number of interactions is bounded. Then we prove that the total amount of non-physical waves in each approximate solution is very small. The convergence of a suitable subsequence is assured by Helly’s theorem; then consistence follows.

6.1. Control of the number of interactions. We prove first that the size of the rarefactions in the scheme is small.

Lemma 6.1. Consider a rarefaction of size \( \varepsilon \); then

\[
|\varepsilon| < \eta e^{-A_o}. \tag{6.1}
\]

Proof. We analyze all possible situations. When the rarefaction is generated, one has \( 0 < \varepsilon < \eta \). When it interacts with a 1- or 3-wave, the size does not increase, see Remark 5.8. By Proposition 5.12 the size does not change when interactions with non-physical waves occur.

The last case to be considered is when a rarefaction interacts with a 2-wave. In this case the size may increase; however, a rarefaction can meet a fixed 2-wave only once. Consider the case of a 1-rarefaction of size \( \delta_1 \), as in Proposition 5.3 the other being analogous. If \( \delta_2 < 0 \) then the size decreases, see (5.13). If \( \delta_2 > 0 \) by (5.12) we have

\[
|\varepsilon| = |\delta_1| + |\varepsilon_3| \leq |\delta_1| \left( 1 + \frac{1}{2} |\delta_2| \right) < |\delta_1| e^{-A_o}. \tag{5.13}
\]
Summarizing the three cases above, we get $|\varepsilon| < \eta e^{L_{cd}/2}$ (or $|\varepsilon| < \eta e^{-L_{cd}/2}$) for a 1-rarefaction (resp. 3-rarefaction), where $L_{cd}$ is the sum of the 2-waves at the right or left of the rarefaction. Then (6.1) follows.

Next, we prove that the number of interactions remains bounded in finite time, so that the approximate solution is well defined for all $t > 0$. We give first a lemma.

**Lemma 6.2.** Consider the wave-front tracking algorithm described in Section 4 under the assumptions of Proposition 5.14. Then

(i) the number of interactions involving a 2-wave and solved by the accurate Riemann solver is finite;

(ii) the number of interactions where a new rarefaction of size $\varepsilon \geq \eta$ arises is finite.

**Proof.** Consider first (i) and refer to Proposition 5.3. Then, using (5.16), we have $\Delta F \leq \rho (\xi + K(A_o - \frac{2}{3})) < 0$, hence $F$ decreases by a uniform positive quantity; since it is non increasing, this can happen only a finite number of times.

Consider then (ii). After (i), it remains only to consider the case of two shocks of the same family interacting.

Under the notation of the corresponding case in the proof of Proposition 5.10 we have $\varepsilon = \varepsilon_1 \geq \eta$ and

$$\Delta F \leq |\varepsilon_1| (1 - \xi + KA_o) \leq \eta (1 - \xi + KA_o) < 0$$

because of (5.39). Arguing as in (i), this can happen only a finite number of times.

About (ii) in Lemma 6.2 recall that if $\varepsilon \geq \eta$ then the new rarefaction must be split into more than one wave. Therefore Lemma 6.2 can be rephrased by saying that, except for finite interactions, the number of waves emitted in an interaction is at most three, and this case occurs precisely when a non-physical wave is generated; moreover, in every interaction at most one wave per family is emitted.

In a schematic way, apart from a finite number of interactions, in our algorithm the following holds (we will consider the set of non-physical waves as a 4th family of waves):

(a) the interaction of an $i$-wave, $i = 1, 3$, with a 2-wave is solved by a single $i$-wave, a 2-wave and a 4-wave;

(b) in the interaction of just 1- and/or 3-waves, there is at most one outgoing wave of each family 1 and 3;

(c) the interaction of a $i$-wave, $i = 1, 2, 3$, with a 4-wave is solved by an $i$-wave and a 4-wave.

The following proposition is inspired by [3, Lemma 2.5].

**Proposition 6.3.** Consider the wave-front algorithm described in Section 4 and assume in the strip $[0, T) \times \mathbb{R}$ the following:

for some $a_1 < a_2 < 0 < b_1 < b_2$ the waves of the first (third) family have speeds in the interval $[a_1, a_2]$ (resp. $[b_1, b_2]$).

Then the number of interactions in the region $[0, T) \times \mathbb{R}$ is finite.

**Proof.** Assume by contradiction that in the region $[0, T) \times \mathbb{R}$ there exists an infinite number of interactions. Unless of taking a smaller $T$ we can assume that the number of interactions is finite in every strip $[0, t] \times \mathbb{R}, 0 < t < T$, and that $T$ is an accumulation point for the times of interaction. Then there exists a sequence $(t_j, x_j)$, $j = 1, 2, \ldots$ of interaction points such that

$$0 < t_j < t_{j+1} < T$$

for all $j$ and $(t_j, x_j) \to (T, \bar{x})$.
for some $\tilde{x}$. Denote $\mathcal{J} = \{(t_j, x_j) : j = 1, 2, \ldots\}$ the set of all interaction points; the set $\mathcal{J}$ is bounded in the strip $[0, T) \times \mathbb{R}$ because of the finite propagation speed.

The situation described in items (a)–(c) above holds except a finite number of interactions; let $\tau < T$ be the maximum time of these “exceptional” interactions. It is not restrictive to assume that $\tau < t_j < T$ for all $j = 1, 2, \ldots$.

Starting from a point of $\mathcal{J}$, we “trace back” all the segments up to $t = 0$; we repeat the procedure for all the points of $\mathcal{J}$ and call $\mathcal{F}$ the set of the “traced segments” obtained in this way. In other words, a segment belongs to the set $\mathcal{F}$ iff it can be joined forward in time to some point of $\mathcal{J}$ by a continuous path along the wave fronts. The set $\mathcal{F}$ is not empty: for instance, two segments interacting at the point $(t_j, x_j)$ belong to $\mathcal{F}$, for any $j = 1, 2, \ldots$. Observe the following dichotomy property of $\mathcal{F}$ which is used just below:

- two interacting waves either belong both to $\mathcal{F}$ or none of them does;
- moreover if at least one of the outgoing waves belong to $\mathcal{F}$, then both the incoming waves must belong to $\mathcal{F}$.

We partition now all the interaction points of the algorithm that occur for times $t > \tau$ into the following sets:

- $\mathcal{I}_0$: the interaction points where no ingoing wave belongs to $\mathcal{F}$;
- $\mathcal{I}_1$: the interaction points where both incoming waves belong to $\mathcal{F}$ and at most one outgoing segment belongs to $\mathcal{F}$;
- $\mathcal{I}_2$: the interaction points where exactly two outgoing segments belong to $\mathcal{F}$;
- $\mathcal{I}_3$: the interaction points where three outgoing waves all belong to $\mathcal{F}$.

Because of the dichotomy property quoted above no outgoing wave in case $\mathcal{I}_0$ can belong to $\mathcal{F}$. On the contrary, both incoming waves in $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ must belong to $\mathcal{F}$.

Recall that we are considering times $t > \tau$. Therefore the maximum number of emitted waves in an interaction is three, and this happens only in the situation considered in $\mathcal{I}_3$, that is for interactions as in (a) above. The case of more than one emitted wave per family cannot occur, and so the outgoing waves in $\mathcal{I}_2$ belong to different families. By definition we have $\mathcal{J} \cap \mathcal{I}_0 = \emptyset$ and so $\mathcal{J} \subseteq \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$.

Let $\mathcal{V}(t)$ be the total number of wave-fronts of the families 1, 2 and 3 that belong to $\mathcal{F}$, at time $t$. The functional $\mathcal{V}(t)$ is non-increasing, and it decreases at least by 1 across $\mathcal{I}_1$. Then $\mathcal{I}_1$ is finite. As a consequence, all the interaction points of $\mathcal{J}$ belong to $\mathcal{I}_2 \cup \mathcal{I}_3$, except at most a finite number. Let $\tau_1 \in [\tau, T)$ be a time such that all points in $\mathcal{I}_1$ lies in $t < \tau_1$.

Let $P = \{x_1, \ldots, x_{N_1}\}$ the set of points of the $x$-axis where a 2-wave is located, $N_1 \leq N_0$. We consider two cases.

**Case** $\tilde{x} \notin P$. In this case we can choose the time $\tau_1 < T$ such that after that time no segment belonging to $\mathcal{F}$ crosses a 2-wave. Then all the points in $\mathcal{J}$ with $t_j > \tau_1$ belong to $\mathcal{I}_2$. The same argument of [3, Lemma 2.5] can now be used, reaching a contradiction.

**Case** $\tilde{x} \in P$. Consider $(t^*, x^*) \in \mathcal{J}$ with $t^* > \tau_1$. As in [3, Lemma 2.5] we define for $t \in (t^*, T)$ two continuous paths $\gamma_t(t)$, $\gamma_r(t)$ starting at $(t^*, x^*)$ in the following way.

At $(t^*, x^*)$ there are either two or three outgoing segments belonging to $\mathcal{F}$; for times $t > t^*$ and sufficiently close to $t^*$ we define $\gamma_t(t)$ to be the segment on the left and $\gamma_r(t)$ the one on the right. When $\gamma_t(t)$ ($\gamma_r(t)$) arrives at an interaction point, it is prolonged by a segment of $\mathcal{F}$; since they are at least two, it follows the one on the left (resp., on the right). Then the path $\gamma_t(t)$ is made by segments of the families 1, 2 or 3, while $\gamma_r(t)$ is made by segments of families 3 or 4; in fact a look to the
proof of Proposition 5.12 shows that the interaction of a 1-wave with a 2-wave always produces a 4-wave.

We claim that the speeds of the paths \( \gamma_\ell(t) \), \( \gamma_\tau(t) \) are strictly separated. This is clear for a finite number of nodes among segments. If it happens that \( \gamma_\tau(t) \) follows once a 4-segment, then from that point on it will follow only 4-segments, so \( \dot{\gamma}_\tau(t) \leq b_2 < \dot{s} = \dot{\gamma}_\tau(t) \). If on the other hand \( \gamma_\tau(t) \) never follows a 4-segment, then \( \dot{\gamma}_\ell(t) \leq 0 < b_1 \leq \dot{\gamma}_\tau(t) \).

This proves the claim; the same argument of 3 then leads to a contradiction.

6.2. Control of the total size of non-physical fronts. Assume as above that the assumptions of Proposition 5.14 hold. We assign inductively to each wave \( \alpha \) a generation order \( k_\alpha \) as in [9, p. 140]. This is done according to the following procedure. First, at time \( t = 0 \) each wave has order 1. Second, assume that two waves \( \alpha \) and \( \beta \) interact at time \( t \); if \( \alpha \) and \( \beta \) belong to different families, the outgoing waves of those families keep the order of the incoming waves, the other waves assume order \( \max\{k_\alpha, k_\beta\} + 1 \); if \( \alpha \) and \( \beta \) belong to the same family, the outgoing wave of that family takes the order \( \min\{k_\alpha, k_\beta\} \), the other waves are assigned order \( \max\{k_\alpha, k_\beta\} + 1 \).

When specialized to the current setting this has the following consequences:

- every 2-wave has order 1; when a \( i \)-wave, \( i = 1, 3 \), of order \( k \) interacts with a 2-wave, the outgoing \( i \)-wave has order \( k \), the other outgoing wave (of the family \( j \), \( j = 1, 3 \), \( j \neq i \), or a non-physical wave) has order \( k + 1 \);
- in the interaction of a 1- with a 3-wave the waves cross without changing order; in the interaction of two waves \( \alpha, \beta \) of the same family \( i = 1, 3 \), the outgoing wave of the family \( i \) takes order \( \min\{k_\alpha, k_\beta\} \), the wave of the family \( j = 1, 3 \), \( j \neq i \), has order \( \max\{k_\alpha, k_\beta\} + 1 \);
- when a non-physical wave interacts with any other wave, both waves cross without changing order; in particular a non-physical wave keeps the order it has been assigned when generated.

For \( t \geq 0 \) not an interaction time and any \( k = 1, 2, \ldots \) define, see (5.2), (5.1),

\[
V_k(t) = \sum_{\gamma > 0} |\gamma| + \xi \sum_{\gamma < 0} |\gamma| + K_{np} \sum_{\gamma \in \text{NP}} |\gamma|
\]

\[
Q_k(t) = \sum_{\gamma_3 \text{ at the left of } s_2} |\gamma_3||\delta_2| + \sum_{\gamma_1 \text{ at the right of } s_2} |\delta_2||\gamma_1|
\]

\[
F_k(t) = V_k(t) + KQ_k(t)
\]

and

\[
\tilde{V}_k(t) = \sum_{\ell \geq k} V_\ell(t), \quad \tilde{Q}_k(t) = \sum_{\ell \geq k} Q_\ell(t), \quad \tilde{F}_k(t) = \tilde{V}_k(t) + K\tilde{Q}_k(t).
\]

We remark that

\[
\tilde{F}_1(0+) = L_\xi(0+) + KQ(0+), \quad \tilde{F}_k(0+) = 0 \quad \text{for } k \geq 2.
\]

Observe that if a non-physical front interacts with another wave, the functionals above do not change; the same holds for interactions between 3- and 1-waves. Then we focus on interactions of waves of the same \( i \) family, \( i = 1, 3 \) (as usual denote \( j = 1, 3 \), \( j \neq i \)) and on interactions between 1- or 3-waves with a 2-wave.

For \( h \in \mathbb{N} \), denote by \( I_h \) the set of times \( t \) when an interaction occurs between two waves \( \alpha \) and \( \beta \) of families 1 or 3 with \( \max\{k_\alpha, k_\beta\} = h \); denote by \( J_h \) the set
of interaction times \( t \) of a 1- or 3-wave of order \( h \) with a 2-wave. Finally, denote \( T_h = I_h \cup J_h \) and \( I = \bigcup_{h \geq 1} I_h, \ J = \bigcup_{h \geq 1} J_h, \ T = I \cup J \).

In order to control the total size of non-physical fronts we must strengthen the assumptions (5.35) and (5.36) required in Proposition 5.14. First, for any fixed \( m > 0 \), instead of (5.35) we require the stronger condition

\[
0 < A_o < \frac{1 - \sqrt{d}}{2 - \sqrt{d}}.
\]

(6.2)

Then denote

\[
\lambda = \frac{1 + KA_o}{\xi}, \quad \lambda_2 = \frac{\xi + KA_o}{K(2 - A_o) - \xi}, \quad \mu = \max \left\{ \lambda, \lambda_2, \frac{KnpC_o}{K} \right\}.
\]

(6.3)

We need \( 0 < \mu < 1 \). From (5.36) we have \( \frac{KnpC_o}{K} < 1 \); moreover we have \( 0 < \lambda < 1 \) and \( \lambda_2 > 0 \) because of (5.36). At last \( \lambda_2 < 1 \) holds iff \( \frac{\xi}{1 - A_o} < K \); this condition is stronger than the left inequality in (5.36). Hence, instead of (5.36) we assume

\[
\frac{1 - A_o}{1 - 2A_o} < \xi < \frac{1}{\sqrt{d}}, \quad \frac{\xi}{1 - A_o} < K < \frac{\xi - 1}{A_o}, \quad Knp < K C_o.
\]

(6.4)

Remark the new upper bound required on \( \xi \). As in the proof of Proposition 5.13, the interval where \( K \) varies is not empty if \( A_o < \frac{1}{2} \) and \( \xi > \frac{1 - A_o}{1 - 2A_o} \). In turn, we can find \( \xi \) satisfying (6.4) if (6.2) holds; remark that \( \frac{1 - \sqrt{d}}{2 - \sqrt{d}} \leq \frac{1}{2} \). The last condition in (6.4) coincides with that in (5.36).

Remark 6.4. Proposition 5.14 still holds under the stronger assumptions (6.2), (6.4) under the same condition (5.38), because the inequality on the right hand side in (6.4) has not changed, see (5.39).

Proposition 6.5. Fix \( m > 0 \) and assume (6.2), (6.4). We have:

1. \( \tau \in T_h, \ h \leq k - 2 \): then \( \Delta \tilde{F}_k = \Delta F_k = 0 \).
2. \( \tau \in T_{k-1} \): then \( \Delta F_{k-1} < 0, \ \Delta \tilde{F}_k = \Delta F_k > 0 \) and

\[
[\Delta \tilde{F}_k]_+ \leq \mu \left( [\Delta F_{k-1}]_+ - \sum_{\ell=1}^{k-2} [\Delta F_{\ell}]_+ \right).
\]

(6.5)

3. \( \tau \in T_h, \ h \geq k \): if \( h = k \) then \( \Delta F_k < 0 \); in any case \( \Delta \tilde{F}_k < 0 \) and

\[
\sum_{\ell=1}^{k-1} [\Delta F_{\ell}]_+ < [\Delta \tilde{F}_k]_+.
\]

(6.6)

Proof. As we pointed out above, for \( \tau \in I \) only interactions of waves of the same family are taken into account. Remark now that for interactions between 3-waves we have \( \Delta Q_\ell = L^+_{cd} \Delta V_\ell \) while \( \Delta Q_\ell = L^-_{cd} \Delta V_\ell \) holds for 1-waves. So if \( \tau \in I \) then \( \Delta V_\ell > 0, \ \Delta \tilde{V}_\ell < 0, \ \Delta \tilde{V}_\ell = 0 \) implies \( \Delta Q_\ell \geq 0 \) (resp. \( \Delta Q_\ell \leq 0, \ \Delta \tilde{Q}_\ell = 0 \)).

Remark also that by Proposition 5.14

\[
\sum_{\ell=1}^{k-1} \Delta F_{\ell} + \Delta \tilde{F}_k < 0.
\]

(6.7)
Fig. 6.1: Interactions of 3-waves; $h \geq \ell$ denote generation orders.

1. If $h \leq k - 2$, no waves with order $\geq k$ are involved and then $\Delta \tilde{F}_k = \Delta F_k = 0$.
2. Let $h = k - 1$. First, consider $\tau \in I_{k-1}$; then $\Delta \tilde{V}_k = \Delta V_k > 0$. We prove that

$$[\Delta \tilde{V}_k]_+ \leq \frac{1}{\xi} \left( [\Delta V_{k-1}]_- - \sum_{\ell=1}^{k-2} [\Delta V_\ell]_+ \right).$$  \hfill (6.8)

Indeed, from (5.30)–(5.32), we deduce

$$\xi [\Delta \tilde{V}_k]_+ + \Delta V_{k-1} + \sum_{\ell=1}^{k-2} \Delta V_\ell \leq 0.$$ \hfill (6.9)

If $\min\{k_\alpha, k_\beta\} = k - 1$, then $\Delta V_\ell = 0$ for $\ell = 1, \ldots, k - 2$ and (6.9) becomes $\xi [\Delta \tilde{V}_k]_+ + \Delta V_{k-1} < 0$; this implies $\Delta V_{k-1} < 0$ and hence $[\Delta \tilde{V}_k]_+ < (1/\xi)[\Delta V_{k-1}]_-$, that is (6.8).

If $\min\{k_\alpha, k_\beta\} = \ell \leq k - 2$, then $\Delta V_{k-1} < 0$ since no waves of order $k - 1$ are present after the interaction; therefore the estimate (6.9) becomes

$$\xi [\Delta \tilde{V}_k]_+ - [\Delta V_{k-1}]_- + \Delta \tilde{V}_k \leq 0.$$ \hfill (6.10)

We have only to consider the case in which $[\Delta Q_\ell]_+ > 0$ for some $\ell \leq k - 2$; but in this case $[\Delta Q_{k-1}]_- - \sum_{\ell=1}^{k-2} [\Delta Q_\ell]_+ \leq L_{\alpha_0}^+(|\alpha_3| + |\beta_3| - |\varepsilon_3|) \geq 0$ because of (5.21). Therefore (6.10) for $\tau \in I_{k-1}$ follows from (6.10) and (6.11).

Second, assume $\tau \in J_{k-1}$; we prove that

$$[\Delta \tilde{F}_k]_+ \leq \mu [\Delta F_{k-1}]_-. $$ \hfill (6.12)
Indeed, if the reflected wave is a physical wave then, under the notation of Proposition 5.3
\[
\Delta V_{k-1} \leq \xi \frac{|\delta_1 \delta_2|}{2}, \quad \Delta Q_{k-1} \leq -|\delta_1 \delta_2| + \frac{|\delta_1 \delta_2|}{2} A_o = -\frac{|\delta_1 \delta_2|}{2} (2 - A_o)
\]
so that \(\Delta F_{k-1} \leq [\xi - K(2 - A_o)]|\delta_1 \delta_2|/2 < 0\) because of (6.4). Then (6.12) follows since \(\Delta F_k = \Delta F_k > 0\) and
\[
[\Delta \tilde{F}_k]_+ \leq \frac{|\delta_1 \delta_2|}{2} (\xi + KA_o) = \frac{|\delta_1 \delta_2|}{2} [K(2 - A_o) - \xi] \cdot \lambda_2 \leq \lambda_2 [\Delta F_{k-1}]_- \leq \mu [\Delta F_{k-1}]_-.
\]
If the reflected wave is a non-physical wave we have, under the notation of Proposition 5.12
\[
0 < \Delta F_k = \Delta V_k \leq K \eta C_o |\delta_2|, \quad \Delta V_{k-1} = 0, \quad \Delta Q_{k-1} = -|\delta_2|
\]
and then
\[
[\Delta F_{k-1}]_- = K |\delta_2|, \quad [\Delta F_k]_+ \leq \frac{K \eta C_o}{K} [\Delta F_{k-1}]_- \leq \mu [\Delta F_{k-1}]_-.
\]
The estimate (6.12) is then completely proved. From (6.12) we get (6.5) since no waves of order \(\leq k - 2\) are involved in the interaction.

3. Finally, let \(h \geq k\). We first consider \(\tau \in I_h\), \(h \geq k\). If \(\min\{k_\alpha, k_\beta\} \geq k\), then \(\Delta V_k = \Delta L_\xi < 0\). If \(\min\{k_\alpha, k_\beta\} \leq k - 1\), assume \(k_\alpha \geq k\) and \(k_\beta \leq k - 1\); then \(\Delta V_k \leq |\xi| \alpha_1 - |\alpha_3| \leq (|\xi| d - 1) |\alpha_3| < 0\) by (5.3). This proves \(\Delta V_k < 0\) and then \(\Delta \tilde{F}_k < 0\).

From (6.7) we deduce that \(\sum_{\ell=1}^{k-1} \Delta F_\ell < [\Delta \tilde{F}_k]_-\). Since at most one non-zero term is present in the first sum, (6.6) follows. In the case \(h = k\) we have \(\Delta V_k = [\Delta V_{k+1}]_+ + \Delta V_k < 0\); hence \(\Delta V_k < 0\) and then \(\Delta \tilde{F}_k < 0\).

Now assume \(\tau \in J_h\), \(h \geq k\). Then \(\Delta \tilde{F}_k = \Delta F < 0\) and no waves of order \(\leq k\) are present, so (6.6) holds. If \(h = k\) and the reflected wave is physical, from the proof of Proposition 5.3 and (6.4) we find that
\[
\Delta F_k = \Delta V_k + K \Delta Q_k \leq \frac{|\delta_1 \delta_2|}{2} (\xi - 2K + KA_o) < 0.
\]
If \(h = k\) and the reflected wave is non-physical then \(\Delta V_k = 0\), \(\Delta Q_k < 0\) and \(\Delta F_k < 0\). 

Summarizing, for \(\tau \in \mathcal{T}\) we have the following table:

| \(\mathcal{T}_h; h\) | \(\Delta F_k\) | \(\Delta \tilde{F}_k\) |
|---------------------|-----------|-------------|
| \(h \geq 2\)        | 0         | +           |
| \(k \geq k + 1\)    | -         | +           |
| \(k - 1\)           | \(\pm\)   | -           |
| \(k \geq k + 1\)    |           | -           |

We write \(\tilde{F}_k^+(t) = \sum_{\tau \leq t} [\Delta F_k(\tau)]_+\) for \(k \geq 2\). For simplicity the time \(\tau\) in such sums is omitted.

**Lemma 6.6.** Under the assumptions of Proposition 6.3, we have
\[
(6.13) \quad \tilde{F}_k^+(t) \leq \mu (L_\xi(0) + KQ(0)) + \sum_{\mathcal{T}_h, h \geq 2} [\Delta F_1]_+
\]
\[
(6.14) \quad \tilde{F}_k^+(t) \leq \mu \left( \tilde{F}_k^+(t) + \sum_{\mathcal{T}_h, h \geq k} [\Delta F_{k-1}]_+ - \sum_{\mathcal{T}_{k-1}, \ell=1}^{k-2} [\Delta F_\ell]_+ \right), \quad k \geq 3.
\]
Proof. By Proposition 6.5, see also the table above, the functional $F_k$ increases at times $\tau \in \mathcal{T}_{k-1}$, decreases at times $\tau \in \mathcal{T}_k$, while it does not have a given sign at times $\tau \in \mathcal{T}_h$, with $h \geq k + 1$.

First, by summing up (6.5) we obtain

$$
\tilde{F}^+_k(t) \leq \mu \sum_{\mathcal{T}_{k-1}} \left( [\Delta F_{k-1}]_+ - \sum_{\ell=1}^{k-2} [\Delta F_{\ell}]_+ \right)
$$

for $k \geq 2$, where the last term in (6.15) is missing if $k = 2$.

Recall now that $F_1(0) = L_\xi(0) + KQ(0)$; therefore

$$
F_1(t) \leq L_\xi(0) + KQ(0) - \sum_{\mathcal{T}_1} [\Delta F_1]_- + \sum_{\mathcal{T}_h, h \geq 2} [\Delta F_1]_+
$$

and then

$$
\sum_{\mathcal{T}_1} [\Delta F_1]_- \leq \sum_{\mathcal{T}_h, h \geq 2} [\Delta F_1]_+.
$$

On the other hand $F_k(0) = 0$ for $k \geq 2$; from Proposition 6.5 we have

$$
F_k(t) \leq \sum_{\mathcal{T}_{k-1}} [\Delta F_k]_+ - \sum_{\mathcal{T}_h} [\Delta F_k]_- + \sum_{\mathcal{T}_h, h \geq k+1} [\Delta F_k]_+.
$$

Moreover

$$
\sum_{\mathcal{T}_{k-1}} [\Delta F_k]_+ = \sum_{\mathcal{T}_k} [\Delta \tilde{F}_k]_+ = \tilde{F}_k^+(t)
$$

and then

$$
\sum_{\mathcal{T}_k} [\Delta F_k]_- \leq \tilde{F}_k^+(t) + \sum_{\mathcal{T}_h, h \geq k+1} [\Delta F_k]_+.
$$

From (6.15), (6.16), (6.17) we get (6.13), (6.14). □

**Proposition 6.7 (A contraction property).** Under the assumptions of Proposition 6.5, for any $t \geq 0$ and $k \geq 1$ we have

$$
\tilde{V}_k(t) \leq \tilde{F}_k(t) \leq \mu^{k-1} \cdot (L_\xi(0) + KQ(0)).
$$

Proof. The estimate (6.18) holds for $k = 1$ because $\tilde{F}_1(t) = F(t) \leq L_\xi(0) + KQ(0)$. Next we prove by induction on $k \geq 2$ that for any $t$

$$
\tilde{F}_k^+(t) \leq \mu^{k-1} (L_\xi(0) + KQ(0)) + \sum_{\mathcal{T}_h, h \geq k} \sum_{\ell=1}^{k-1} [\Delta F_\ell]_+.
$$

Since by summing up (6.6) we obtain

$$
\tilde{F}_k^-(t) \geq \sum_{\mathcal{T}_h, h \geq k} \sum_{\ell=1}^{k-1} [\Delta F_\ell]_+
$$
then (6.18) will follow from (6.19) for any \( k \geq 2 \) because of (6.20).

Formula (6.19) for \( k = 2 \) reduces to (6.13). Next, assume that (6.19) holds for some \( k \geq 2 \). By (6.14) and the induction assumption

\[
\begin{align*}
\tilde{F}^+_k(t) & \leq \mu \left( \tilde{F}^+_k(t) + \sum_{\mathcal{T}_h, h \geq k+1} [\Delta F_k]_+ - \sum_{\mathcal{T}_k} \sum_{\ell=1}^{k-1} [\Delta F_\ell]_+ \right) \\
& \leq \mu^k \left( L_\xi(0) + KQ(0) \right) + \mu \left( \sum_{\mathcal{T}_h, h \geq k+1} \sum_{\ell=1}^{k-1} [\Delta F_\ell]_+ + \sum_{\mathcal{T}_h, h \geq k+1} [\Delta F_k]_+ - \sum_{\mathcal{T}_k} \sum_{\ell=1}^{k-1} [\Delta F_\ell]_+ \right) \\
& \leq \mu^k \left( L_\xi(0) + KQ(0) \right) + \mu \sum_{\mathcal{T}_h, h \geq k+1} \sum_{\ell=1}^{k} [\Delta F_\ell]_+ .
\end{align*}
\]

Since \( \mu < 1 \), we get (6.19) for \( k + 1 \).

**Remark 6.8.** We comment now the case \( A_o = 0 \). In this case system (1.1) reduces to the \( p \)-system with pressure law given by (2.1), for fixed \( \lambda \). According to our front-tracking algorithm, stationary and non-physical waves do not appear and so \( L_{cd} = Q = 0 \); the algorithm reduces to the one introduced in [3]. Then, Proposition 6.5 holds with \( \tilde{V}_k \) and \( 1/\xi \) replacing \( \tilde{F}_k \) and \( \mu \), respectively, and at last (6.18) reads

\[
\tilde{V}_k(t) \leq \frac{1}{\xi^{k-1}} \cdot L_\xi(0).
\]

Next, we conclude that the total strength of all non-physical waves is small by proceeding as in [9, page 142]. Recall the notation in Section 4. First, using (5.39), we find that the sequence \( \{v^\nu\} \) related to the initial data \((v^\nu_o, u^\nu_o, \lambda^\nu_o)\) is uniformly bounded and in particular uniformly bounded away from 0; then the eigenvalues \( e_1 \) and \( e_3 \) are bounded and this makes possible the choice of a suitable \( \hat{s} \). We have two more parameters \( \eta, \rho \) to be chosen. Fix \( \eta > 0 \) with the condition \( \eta \nu \to 0 \) as \( \nu \to \infty \) and estimate the total number of waves of order \( < k \). We have

\[
\begin{align*}
\sum_{\gamma \in \mathcal{NP}} |\gamma|(t) & \leq \tilde{V}_k(t) + \sum_{\gamma \in \mathcal{NP}, k, \gamma < k} |\gamma|(t) \\
& \leq \mu^{k-1} \cdot (L_\xi(0) + KQ(0)) + C_o \rho \cdot \text{[number of fronts of order \( < k \)]} \leq \frac{1}{\nu}
\end{align*}
\]

by choosing \( k \) sufficiently large to have the first term \( \leq 1/(2\nu) \) and then choose \( \rho \) small enough to have the second term \( \leq 1/(2\nu) \).

We accomplish now the proof of Theorem 2.2. Define first

\[
(6.21) \quad k(m) = \frac{1 - \sqrt{d(m)}}{2 - \sqrt{d(m)}} .
\]

From the properties of the function \( d(m) \) stated in Remark 5.7 we see that \( k(0) = 1/2 \) and that \( k(m) \) is decreasing, tending to 0 for \( m \to +\infty \). The assumption (2.8) implies that (6.2) holds.

Now, by hypotheses (2.7) it follows that we can choose \( \xi \) such that

\[
\frac{1}{2}TV \log(p_o) + \frac{1}{2} \inf a_o TV(u_o) < \frac{m}{2\xi - 1} < (1 - 2A_o)m
\]
and that (6.4) holds. Hence, using (5.12) and (i) in Section 4, we have

\[ L(0+) \leq \frac{1}{2} \text{TV} \log(p_o) + \frac{1}{2 \inf a_o} \text{TV}(u_o) < \frac{m}{2\xi - 1} \]

so that the hypotheses (5.38) of Proposition 5.14 holds. Theorem 2.2 now follows along the lines of [9, §7.4].

Appendix A. The weighted total variation. In this Appendix we prove Proposition 2.1. Remark that the map \( d(a, b) = \frac{|a - b|}{a + b} \) is a distance on \( \mathbb{R}_+ \), as one can easily prove.

We start with the proof of the inequality on the right in (2.5). It is enough to prove that

\[ 2 \sum_{j=1}^{n} \frac{|f(x_j) - f(x_{j-1})|}{f(x_j) + f(x_{j-1})} \leq \sum_{j=1}^{n} |\log(f(x_j)) - \log(f(x_{j-1}))| . \]  

We claim that

\[ \log t \geq \frac{2(t - 1)}{t + 1} \quad \text{for } t \geq 1 \]  

where the inequality is strict if \( t > 1 \). To prove the claim it is sufficient to notice that the function \( \phi(t) = \log t - 2 \left( \frac{1}{t+1} \right) \) vanishes in 1 and \( \phi'(t) = \frac{(t-1)^2}{t(t+1)^2} > 0 \) if \( t > 1 \).

We apply (A.2) to \( t = x/y \) for \( 0 < y \leq x \) and arguing by symmetry deduce

\[ |\log x - \log y| \geq \frac{2|x - y|}{x + y}, \quad \text{for every } x, y > 0 . \]

Then (A.1) follows. The proof of the inequality on the left in (2.5) is analogous, starting from the inequality

\[ \frac{1}{t} \log t \leq \frac{2(t - 1)}{t + 1} \quad \text{for } t \geq 1 \]

with strict inequality if \( t > 1 \).

Assume now that \( f \in C(\mathbb{R}) \). To show that \( \text{WTV}(f) = \text{TV}(\log(f)) \), we have to prove that the inequality

\[ \text{TV}(\log(f)) \leq 2 \sup \sum_{j=1}^{n} \frac{|f(x_j) - f(x_{j-1})|}{f(x_j) + f(x_{j-1})} \]

holds for any interval \([a, b] \subset \mathbb{R}\). Consider any partition \( \{x_0, x_1, \ldots, x_n\} \) of the interval \([a, b]\). By the mean value theorem for some \( \xi_j \) between \( f(x_j) \) and \( f(x_{j-1}) \), and by the intermediate value theorem applied to \( f \), we get

\[ |\log (f(x_j)) - \log (f(x_{j-1}))| = \frac{|f(x_j) - f(x_{j-1})|}{\xi_j} = \frac{f(x_j) + f(x_{j-1})}{2f(\eta_j)} \cdot \frac{2|f(x_j) - f(x_{j-1})|}{f(x_j) + f(x_{j-1})} \]

for some \( \eta_j \in [x_{j-1}, x_j] \). We exploit again the continuity of \( f \) in \([a, b]\). On one hand its image is compact, then \( \min_{[a, b]} f = m > 0 \). On the other \( f \) is uniformly continuous
in \([a, b]\), so that for any \(\varepsilon > 0\) there exists \(\delta_\varepsilon > 0\) such that \(|f(x) - f(y)| < \varepsilon\) if \(|x - y| < \delta_\varepsilon\), for \(x, y \in [a, b]\).

Fix now any \(\varepsilon > 0\); without loss of generality we can consider partitions of the interval \([a, b]\) of mesh less than \(\delta_\varepsilon\). Assume for instance \(f(x_{j-1}) \leq f(x_j)\); then from the inequalities

\[
f(x_j) - f(x_{j-1}) < \varepsilon, \quad f(x_{j-1}) \leq f(\eta_j) \leq f(x_j)
\]

it follows

\[
\frac{f(x_j) + f(x_{j-1})}{2f(\eta_j)} \leq \frac{2f(x_{j-1}) + \varepsilon}{2f(x_{j-1})} \leq 1 + \frac{\varepsilon}{m}.
\]

The inequality (A.4) follows by remarking that then for any partition of mesh less than \(\delta_\varepsilon\)

\[
\sum_{j=1}^{n} |\log (f(x_j)) - \log (f(x_{j-1}))| \leq \left(1 + \frac{\varepsilon}{m}\right) 2 \sum_{j=1}^{n} \frac{|f(x_j) - f(x_{j-1})|}{f(x_j) + f(x_{j-1})}.
\]

The proof of Proposition 2.1 is complete.

**Remark A.1.** Observe that, if \(TV(\log(f)) < \infty\) and \(f\) is discontinuous, then the inequality on the right in (2.5) is strict, because of the strict inequality in (A.2).

For example, if \(f\) has a single jump and assumes the values \(c > 0\) and \(d > 0\), then

\[
WTV(f) = 2|c - d| = 2|\log c - \log d| = TV(\log(f)).
\]

Remark moreover that \(WTV\) and \(TV\) are not equivalent, in the sense that there does not exist a positive constant \(C\) such that \(C \cdot TV(\log(f)) \leq WTV(f)\). This follows from the fact that clearly the inequality \(C \log t \leq 2(t-1)\) does not hold for every \(t \geq 1\).

**Appendix B. Shock-rarefaction interactions.** In this Appendix we consider a particular case of Lemma 5.6. Actually, this is the only case needed in order to define a decreasing functional, Section 5; however, we needed further analysis for the control and treatment of the non-physical waves.

**Lemma B.1 (The case \(SR, RS \rightarrow SS\)).** Consider the interaction of a shock \(\alpha_i\) and a rarefaction \(\beta_i\) of the same family, \(i = 1, 3\), producing two outgoing shocks \(\varepsilon_1, \varepsilon_3\). Then there exists a smooth function \(B\) satisfying \(|\alpha_i| \leq B(\alpha_i) \leq \min\{\sinh(|\alpha_i|), 2|\alpha_i|\}\) such that

\[
0 < \beta_i \leq B(\alpha_i).
\]

Moreover, assume

\[
|\alpha_i| \leq m
\]

for some \(m > 0\) and denote \(c = c(m) = \frac{\cosh(m) - 1}{\cosh(m) + 1}\). Then both the variation of shock waves and the reflected wave \(\varepsilon_j, j \neq i, j = 1, 3\), are estimated by the interacting rarefaction as

\[
|\varepsilon_1| + |\varepsilon_3| - |\alpha_i| \leq (2c - 1) \cdot |\beta_i|,
\]

\[
|\varepsilon_j| \leq c \cdot |\beta_i|.
\]
Proof. We focus on the case $i = 3, j = 1$, see Figure 5.3(b). Therefore we consider $\alpha_3 < 0, \beta_3 > 0$ and $\varepsilon_1 < 0, \varepsilon_3 < 0$. Then (5.5), (5.6) become

(B.5) $|\varepsilon_1| - |\varepsilon_3| = |\beta_3| - |\alpha_3|$
(B.6) $\sinh(|\varepsilon_1|) + \sinh(|\varepsilon_3|) = \sinh(|\alpha_3|) - |\beta_3|.$

From the second equation $|\varepsilon_1| < |\alpha_3|$, $|\varepsilon_3| < |\alpha_3|$ and $|\beta_3| < \sinh(|\alpha_3|)$; using the first equation and $|\varepsilon_3| < |\alpha_3|$, we get $|\varepsilon_1| < |\beta_3|$. Therefore in conclusion

(B.7) $|\varepsilon_1| < \min\{|\alpha_3|, |\beta_3|\}$, $|\varepsilon_3| < |\alpha_3|$, $|\beta_3| < \sinh(|\alpha_3|)$.

Step 1: notation. We set $x = |\beta_3|, y = |\varepsilon_1|, z = |\alpha_3|$, so that

(B.8) $|\varepsilon_3| = y - x + z$.

Under this notation, (B.6) writes as

(B.9) $F(x, y; z) = \sinh y + \sinh(y - x + z) - \sinh z + x = 0$

for $x \geq 0, y \geq 0, z \geq 0, y - x + z \geq 0$. By (B.7), any solution of (B.9) satisfies

(B.10) $y < z, y < x, x < \sinh(z)$.

\[ y = |\varepsilon_1| \quad y - x + z = |\varepsilon_3| \\
z = |\alpha_3| \quad x = |\beta_3| \]

Fig. B.1: Interactions.

Step 2: the threshold. Observe that, despite the last inequality in (B.7), we may well have $|\beta_3| > |\alpha_3|$, that is, $x > z$. Consider in fact the limit case of $\varepsilon_3 = 0$: we have $y = x - z > 0$ and $\sinh(y) = \sinh(z) - x$ that give

$$\sinh(x - z) = \sinh(z) - x, \quad x > z.$$  

The last equality is the relation needed for $\beta_3, \alpha_3$ in order to have that the shock and rarefaction cancel out exactly, giving rise only to a wave of the opposite family. Observe that the size of the rarefaction must be larger than the one of the shock. Under the notations above, the threshold curve separating the case of the outgoing waves $S_1S_3$ from the case $S_1R_3$ is given by

(B.11) $f(x, z) = \sinh(x - z) - \sinh(z) + x = 0$.

Since $f_x = \cosh(x - z) + 1 > 0$ and $f(z, z) < 0$, the implicit equation $f(x, z) = 0$ is solved by $x = x_o(z) \geq z$ with $x_o'(z) = \frac{\cosh(x - z) + \cosh(z)}{\cosh(x - z) + 1} > 0$ for every $z \geq 0$; the curve has for tangent at $(0, 0)$ the line $z = x$. Observe that $x_o(z) \leq 2z$ because $f(2z, z) = z \geq 0$. In conclusion

(B.12) $z \leq x_o(z) \leq 2z.$
Step 3: the amount of shocks can increase.

This estimate and the third inequality in (B.7) prove (B.1) for \( B(\alpha_i) = x_o(|\alpha_i|) \). We can prove more than (B.12), that is,

(B.13) \[ \lim_{z \to +\infty} (x_o(z) - 2z) = 0. \]

Indeed we show that the inequality \( x_o(z) > 2z - q \) holds for large \( z \) and \( q > 0 \). This follows from \( f(2z - q, z) = \sinh(z - q) - \sinh z + 2z - q \sim e^z(e^{-q} - 1)/2 \to -\infty \) for \( z \to \infty \).

Remark that the fact that \( \varepsilon_3 \) is a shock implies that

(B.14) \[ f(x, z) = \sinh(x - z) - \sinh(z) + x < 0. \]

Step 3: the amount of shocks can increase. From (B.5) we have

(B.15) \[ |\varepsilon_1| + |\varepsilon_3| - |\alpha_3| = 2|\varepsilon_1| - |\beta_3|. \]

We prove now that the inequality \( |\varepsilon_1| + |\varepsilon_3| - |\alpha_3| < 0 \), or equivalently \( |\varepsilon_1| < \frac{1}{2}|\beta_3| \), does not hold if \( m \) is large.

The equation giving \( |\varepsilon_1| = y \) in terms of \( |\beta_3| = x \), for a given parameter \( |\alpha_3| = z \), is (B.9), to be considered in the domain

\[ D_z = \{ (x, y): 0 \leq x \leq x_o(z), y \geq \max\{0, x - z\} \}, \quad \text{for } z \geq 0 \]

where \( x_o(z) \) satisfies the equation (B.11). Since \( F_y = \cosh y + \cosh(y - x + z) > 0 \), the implicit equation (B.9) defines a function \( y = y(x; z) \) with \( 0 \leq y(x; z) \leq x \) and \( y(x; z) \leq z \), see (B.10). Remark that \( F(x, x; z) = \sinh x + x > 0 \). Moreover \( F(x, 0; z) = \sinh(z - x) - \sinh z + x \) so that \( F(0, 0; z) = 0 \); by \( F_x = 1 - \cosh(z - x) < 0 \), we deduce \( F(x, 0; z) < 0 \) if \( x > 0 \). By the implicit function theorem we have \( y'(x; z) \geq 0 \),

(B.16) \[ y'(0; z) = \frac{\cosh(z) - 1}{\cosh(z) + 1} \in [0, 1) \]

and \( y''(0; z) = -\frac{4 \sinh(z)}{(1 + \cosh(z))^2} \leq 0 \). The function \( \frac{\cosh(z) - 1}{\cosh(z) + 1} \) is increasing and then for \( z \in [0, m] \) its maximum is \( \frac{\cosh(m) - 1}{\cosh(m) + 1} \); this quantity is strictly larger than \( 1/2 \) if \( m > \log(3 + 2\sqrt{2}) \). Thus in general the estimate \( y(x; z) < x/2 \) cannot hold.
Step 4: proof of the estimate. From (B.15) we see that $|\varepsilon_1| + |\varepsilon_3| - |\alpha_3| \leq (2c - 1) \cdot |\beta_3| \iff |\varepsilon_1| \leq c \cdot |\beta_3|$, that is, (B.3) and (B.4) are equivalent; we shall prove (B.4). To bypass the study of the function $y(x; z)$ we define

$$\Phi(x; z, c) = F(x, cx; z) = \sinh(cx) + \sinh(z - (1 - c)x) - \sinh z + x.$$ 

If $1/2 \leq c < 1$ then $z > (1 - c)x$ and $\Phi(0; z, c) = 0$,

$$\Phi_x(x; z, c) = 1 + c \cosh(cx) - (1 - c) \cosh(z - (1 - c)x),$$
$$\Phi_{xx}(x; z, c) = c^2 \sinh(cx) + (1 - c)^2 \sinh(z - (1 - c)x) > 0.$$ 

Therefore the function $x \to \Phi_x(x; z, c)$ is increasing and then $\Phi(x; z, c) \geq 0$ if

$$\Phi(0; z, c) = (1 + c) - (1 - c) \cosh(z) > 0,$$

that is if $\cosh(z) \leq \frac{1 + c}{1 - c}$; this is just (B.2). Then $y(x; z) \leq cx$ for all $x \in (0, x_0(z))$ and so (B.4) is proved. \[\square\]

Remark B.2. From (B.16) we see that condition (B.2) is equivalent to the geometric condition $y'(0; z) = (\cosh z - 1)/(\cosh z + 1) < c$. Moreover, as we noticed in the above proof, condition (B.2) is equivalent to

(B.17) \[\cosh(|\alpha_i|) \leq \frac{1 + c}{1 - c},\]

which, in turn, is equivalent to

(B.18) \[|\alpha_i| \leq \log \left(\frac{(1 + \sqrt{c})^2}{1 - c}\right).\]

From the definition of the strength, one has that $|\alpha_i| = (1/2) \log(v_{\text{max}}/v_{\text{min}})$, where $v_{\text{max}} = \max\{v_l, v_r\}$, $v_{\text{min}} = \min\{v_l, v_r\}$, being $v_l, v_r$, respectively, the left and right values of $v$ for the wave of size $\alpha_i$. Hence (B.18) is equivalent to

$$\sqrt{\frac{v_{\text{max}}}{v_{\text{min}}}} \leq \frac{(1 + \sqrt{c})^2}{1 - c}.$$ 

Remark B.3. In the proof above we showed that $\Delta L_{\text{shocks}} = |\varepsilon_1| + |\varepsilon_3| = 2|\varepsilon_1| - |\alpha_3|$ may be positive, differently from Nishida’s paper, where it is always decreasing. This depends on the definition of the wave strengths, which was imposed to us in order to have good estimates when dealing with interactions with the 2 waves. In any case $\Delta L = \Delta L_{\text{shocks}} + \Delta L_{\text{rarefactions}} \leq 0$.

Remark B.4. Under the notation and assumptions of Lemma B.1 we verify that

(B.19) \[|\varepsilon_2| \leq c \cdot |\alpha_3|.\]

This estimate, together with (B.4), allows us to obtain (in a special case) the analogue of (5.20) with $c$ in place of $d$.

The proof makes use of a numerical computation. As in that lemma we consider the case $j = 1, i = 3$. Let $z > 0$ be fixed and $0 \leq x \leq x_0(z)$. From the proof of Lemma B.1 we deduce $y = y(x, z)$; for $z$ fixed the function $x \to y(x, z)$ is increasing. Define then

$$Y(z) = y(x_0(z), z) = x_0(z) - z.$$
In order to prove (B.19) it is sufficient to prove that

\[ Y(z) \leq cz \quad \text{if} \quad \cosh(z) \leq \frac{1 + c}{1 - c}. \]

Remark that from (B.13) we know that \( Y(z) / z \to 1 \) for \( z \to +\infty \); however the constraint (B.17) implies that \( z \) is bounded. The inequality \( Y(z) \leq cz \) is equivalent to \( x_o(z) - z \leq cz \), i.e., \( x_o(z) \leq (1 + c)z \). Therefore we need to prove that

\[
(B.20) \quad \phi(z, c) := \sinh(cz) - \sinh(z) + (1 + c)z \geq 0 \quad \text{if} \quad \cosh(z) \leq \frac{1 + c}{1 - c}.
\]

Notice that \( \cosh(z) \leq \frac{1 + c}{1 - c} \) means \( 0 \leq z \leq z_c = \log(\frac{1 + c}{1 - c} + \sqrt{\left(\frac{1 + c}{1 - c}\right)^2 - 1}) \).

Formula (B.20) is shown to hold true by numerical computations. Remark however that

\[
\phi'(z, c) = c \cosh(cz) - \cosh(z) + 1 + c \\
\phi''(z, c) = c^2 \sinh(cz) - \sinh(z) \leq 0
\]

so \( \phi'(0, c) = 2c, \phi'(, c) \) is concave, \( \lim_{z \to +\infty} \phi(z, c) = -\infty \) and \( \phi(, c) \) has a single point of maximum; at last \( \phi'(z_c) = c (\cosh(cz_c) - \cosh(z_c)) < 0 \). This concludes the proof of (B.19).

Remark that if \( c = 1/2 \) then \( \phi(z, c) = \sinh(z/2) - \sinh(z) + \frac{3}{2}z \). If \( z \) is such that \( \cosh z = 3 \), then \( \sinh z = 2\sqrt{2}, \sinh(z/2) = 1, \) \( z = \log(3 + \sqrt{8}) \), and (B.20) holds with strict inequality.

**Acknowledgments.** The authors kindly acknowledge support by Progetto GNAMPA 2005 Analisi Asintotica Per Sistemi Iperbolici Non Lineari. The authors thank Graziano Guerra for stimulating remarks and Umberto Massari for hints on BV functions.

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