The Fourth Law of Black Hole Thermodynamics

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Abstract:

We show that black holes fulfill the scaling laws arising in critical transitions. In particular, we find that in the transition from negative to positive values the heat capacities $C_{JQ}$, $C_{ΩQ}$ and $C_{JΦ}$ give rise to critical exponents satisfying the scaling laws. The three transitions have the same critical exponents as predicted by the universality Hypothesis. We also briefly discuss the implications of this result with regards to the connections among gravitation, quantum mechanics and statistical physics.

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1. The four laws of Black Hole Thermodynamics

In the work of Bardeen, Carter and Hawking \cite{1} it was established a remarkable mathematical analogy between the laws of thermodynamics and the laws of black hole mechanics derived from General Relativity. If one makes the formal replacements $E \to M$, $T \to C \kappa$, and $S \to A/8\pi C$ (where $C$ is a constant) in the laws of the thermodynamics, one obtains the laws that govern the mechanics of black holes \cite{2}. The physical analogy seemed to have problems due to the fact that in classical General Relativity the thermodynamic temperature of a black hole appears to be absolute zero. However, Hawking \cite{3} found that when quantum effects are taken into account, a black hole absorbs and emits particles as a body at temperature $T = \kappa/2\pi$, and this resolved that puzzle. Here and throughout this paper we take units in which $G = h = c = \kappa_B = 1$.

The four laws of black hole thermodynamics can be briefly formulated as follows:

*The Zeroth Law:* The surface gravity, $\kappa$, of a stationary black hole (at equilibrium) is constant on the entire surface of the event horizon.

This property is proved as a theorem in ref. \cite{1}. Landsberg \cite{4} gives however a more precise formulation: “In the absence of adiabatic partitions and long range fields, an equilibrium system exhibits a unique temperature”. If the above conditions are not satisfied, it can be shown \cite{5} that a system with a built-in adiabatic partition can have two different temperatures even in thermal equilibrium.

For a Kerr-Newman black hole endowed with mass $M$, charge $Q$ and angular momentum $\vec{J}$, the surface gravity is given by \cite{6}

$$\kappa = \frac{1}{2} \frac{r_+ - r_-}{(r_+^2 + a^2)} = \frac{\sqrt{M^2 - a^2 - Q^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}}, \quad (1)$$

where $a^2 = |\vec{J}|^2/M^2$ and

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}, \quad (2)$$

are the event and internal horizons respectively.

*The First Law:* This is just an expression that states that in an isolated system, including black holes, the total energy of the system is conserved. In ref.\cite{1} it is derived, in a general context, the following differential mass formula for stationary black holes,

$$\delta M = \frac{\kappa}{8\pi} \delta A + \vec{\Omega} \cdot \delta \vec{J} + \Phi \delta Q, \quad (3)$$

where $\vec{\Omega}$ is the angular velocity and $\Phi$ the electric potential of the event horizon, and

$$A = 4\pi (r_+^2 + a^2) = 8\pi \left\{ M^2 - \frac{Q^2}{2} + \sqrt{M^4 - J^2 - M^2Q^2} \right\}, \quad (4)$$

$$2$$
is its area. $Mc^2$ represents actually the total energy of the hole, $E$, and $T = \kappa/2\pi$ is its Bekenstein-Hawking temperature.

Eq. (3) shows that $A/4$ is the analogous of the entropy of the black hole

$$S_{bh} = \frac{1}{4} A \quad .$$

(5)

The second and third term on the right hand side of Eq.(3) represent the work done or energy extracted when we change the black hole angular momentum and electric charge respectively. The angular velocity and electric potential are given by

$$\Omega = \frac{a}{r^2_a + a^2} \quad ,$$

(6)

$$\Phi = \frac{Qr_a}{r^2_a + a^2} \quad .$$

(7)

Inverting relation (4), one obtains the mass as a function of $A$, $J$ and $Q$

$$M = \left( \frac{A}{16\pi} + \frac{4\pi J^2}{A} + \frac{Q^2}{2} + \frac{\pi Q^4}{A} \right)^{1/2} .$$

(8)

This is the fundamental thermodynamic relation containing explicitly all the information about the thermodynamic state of the black hole.

By applying Euler’s theorem on homogeneous functions to $M$, which is homogeneous of degree 1/2, one obtains

$$M = \frac{1}{2} TA + 2\vec{\Omega} \cdot \vec{J} + \Phi Q \quad .$$

(9)

Thus, $T$, $\Omega$ and $\phi$ are the “intensive” parameters and are constant everywhere on the event horizon of any stationary axisymmetric black hole.

The Second Law: Bekenstein [2,10] proposed that the total entropy (see also [11])

$$S = S_{bh} + S_m \quad ,$$

(10)

(where $S_m$ is the total entropy of ordinary matter outside black holes) never decreases in any physical process.

Hawking [12] probed a theorem stating that the area $A$ of the event horizon of each black hole can not decrease with time, i.e.

$$\delta A \geq 0 \quad .$$

(11)
However, when quantum processes close to the event horizon are taken into account, this theorem can be violated since the $T_{\mu\nu}$ of the emitted Hawking radiation $^{[3]}$ does not respect the positive energy condition assumed in the proof of this theorem. In fact, an isolated black hole can eventually evaporate completely, thus decreasing its area to zero. On the other hand, one can easily make decrease the total entropy of matter outside black holes, $S_m$, by letting fall that matter into a black hole.

When both addends on the right hand side of eq.(10) are taken into account, a decrease in one of them seems to be always compensated by an increase in the other. In fact, when matter is thrown into a black hole, thus decreasing $S_m$, the area of the black hole will tend to increase. Conversely, when a black hole evaporates emitting particles, the matter outside the black hole is in a higher entropy state. Thus we have for any process

$$\delta S \geq 0. \quad (12)$$

This means that in any dynamical process the system will tend towards an equilibrium state for which the entropy has the largest possible value subject to the given structure of the system $^{[4]}$. The ordinary second law for $S_m$ would be a special case of eq.(12) applicable when black holes are not present, whereas the area theorem, eq.(11), would apply in the classical limit where the ordinary entropy flux into a black hole is positive.

The validity of eq.(12) was challenged by several gedanken processes, although when semiclassical corrections are properly taken into account, no violation of the second law can be achieved $^{[13]}$. This strongly suggest that eq(12) must hold, at least for quasi stationary processes when departures from equilibrium are small $^{[14]}$.

The Third Law: The usual formulation of the third law states that $^{[1]}$ it is impossible by any physical process to reduce $\kappa$ to zero by a finite sequence of operations.

This is the analogue of the third law of thermodynamics as formulated by Nerst. Note that Plank’s formulation stating that the entropy of any system tends to an absolute constant, (which may be taken as zero), when $T \to 0$, does not hold in black holes dynamics. In fact, for an extreme Kerr-Newman black hole,

$$S_{bh}(T = 0) = \pi(Q^2 + 2a^2)^{\frac{3}{2}}. \quad (13)$$

Page $^{[15]}$ has conjectured that it should actually be $S(T = 0) = 0$, for the ground state of a black hole not to be degenerate. However, let us note that $\kappa = 0$ can be also reached (even for Schwarzschild black holes) in the limit $M \to \infty$ (as can be seen from eqs. (1)-(2)) while $S \to \infty$ in this case.

Israel $^{[16,17]}$ has given a formulation and proof of the third law of black hole dynamics for Reissner-Nordstrøm holes. It can be stated informally as follows:

“A non extremal black hole cannot become extremal (i. e., lose its trapped surfaces) within a finite interval of advanced time in any continuous process in which the stress-energy tensor of infalling matter remains bounded and satisfies the weak energy conditions
in a neighborhood of the outer apparent horizon”. For a more formal statement of this formulation see refs. [16,17].

There is a close relation between the validity of the third law and the cosmic censorship hypothesis. In fact, Israel’s Lemma can be used to establish a weak form of this hypothesis called “gravitational confinement” [17]. If one would reduce the value of $\kappa$ of a Kerr-Newman black hole by throwing in particles to increase the angular momentum and/or the electric charge, up to the state where $a^2 + Q^2 = M^2$ in a finite number of steps (it is not possible by mining [18]), then presumably one could carry the process further up to $a^2 + Q^2 > M^2$. The result would be a naked singularity. Then the singularity would no longer be hidden inside a black hole but would be able to influence, and be observed by, the outside universe. This is so unpleasant that one invokes the cosmic censorship conjecture, thus ensuring the validity of the third law. This theorem, however, remains elusive to any proof, being one of the most important unsolved problems in classical General Relativity.

In the next section we will review some important results concerning phase transitions. These results can be resumed by the scaling laws of thermodynamics. Then, in the third section we postulate this laws to hold for black holes and study its validity in the case of the black hole phase transitions found by Davies [19,20] and its extensions to other heat capacities [21]. We end the paper with some discussion on this results and a future prospect of research on the subject.

2. Scaling in critical phenomena

The existence of critical phase transitions was discovered by Andrews [22] in his studies of carbon dioxide. Liquid and vapor phases became identical at the critical point and there was a seemingly continuous transition from one phase to the other. The phase equilibrium curve in the PT-plane terminate at a certain point, called critical point. The corresponding temperature and pressure are the critical temperature, $T_c$, and the critical pressure, $P_c$. At this point, liquid and vapor become indistinguishable. By going round the critical point, it is possible to take a path from the liquid region to the vapor region without crossing any phase boundary and thus without experiencing any discontinuous change in properties. This suggest that critical points can only exist for phases such that the difference between them is purely quantitative (for example liquid-gas) and not qualitative such the case of a solid (crystal), since they have different internal symmetry.

According to the thermodynamic inequality, $\delta P/\delta V|_T < 0$, $P$ is a decreasing function of $V$. However, in the critical state

$$\frac{\delta P}{\delta V}|_T (c) = 0, \quad \frac{\delta^2 P}{\delta V^2}|_T (c) = 0.$$ (14)

From the formula

$$C_P - C_V = -T \frac{(\frac{\delta P}{\delta V})_V}{(\frac{\delta P}{\delta V})_T}.$$ (15)
for the difference of specific heats, we conclude that $C_P \rightarrow \infty$ at the critical point (For more details see, for example, ref.[23]).

In the magnetic case a continuous phase transition occurs from an ordered ferromagnetic state to a paramagnetic state. The critical point is at zero applied magnetic field $\vec{H}$ and at $T = T_c$. The derivative of the magnetization, $\vec{M}$, diverge at $T_c$. The phase transition has no associated latent heat and can be described as a critical phase transition. An important difference arises with respect to the liquid-gas system because the analogue of $P$ and $V$ variables, i. e., $\vec{H}$ and $\vec{M}$, are vector quantities. However, in an ideal magnet, we neglect the anisotropies and the magnetic properties will only depend on the magnitude of $\vec{H}$.

The same general properties of critical phase transitions are also found for ferroelectricity, superconductivity, superfluidity in liquid helium, mixing of liquids, and ordering in alloys (see ref.[24] for further details). Wilson [25] also mentioned turbulent fluid flow, internal structure of elementary particles and the interaction between electrons in a metal with magnetic impurities.

To give a precise and general definition of critical point exponents in describing the behavior near the critical point of a general function $f(x)$ we assume that the following limit exists

$$
\sigma = \lim_{\varepsilon \rightarrow 0} \frac{\ln f(\varepsilon)}{\ln \varepsilon}, \quad \varepsilon = T - T_c,
$$

where $\sigma$ is the critical point exponent of $f(\varepsilon)$.

Away from the critical point we will have deviations of the form $[26]$

$$
f(\varepsilon) = A\varepsilon^\sigma (1 + B\varepsilon^y + \ldots), \quad y > 0 .
$$

For magnetic systems, in particular, it is possible to define the following thermodynamical critical point exponents as $\varepsilon \rightarrow 0^+$:

For the specific heat at constant magnetic field (note that here one departs from the strict fluid-magnet analogy $V \leftrightarrow -M$ and $P \leftrightarrow H$ (see [27])):

$$
C_H = T \frac{\partial S}{\partial T} \bigg|_H = \begin{cases} 
\sim \varepsilon^{-\alpha}, & \text{for } H = 0 \\
\sim H^\varphi, & \text{for } \varepsilon = 0
\end{cases},
$$

zero- field magnetization:

$$
M = \begin{cases} 
\sim \varepsilon^\beta, & \text{for } H = 0 \\
\sim H^{\delta-1}, & \text{for } \varepsilon = 0
\end{cases},
$$

zero-field isothermal susceptibility.

$$
\chi_T = \frac{\partial M}{\partial H} \bigg|_T = \begin{cases} 
\sim \varepsilon^{-\gamma}, & \text{for } H = 0 \\
\sim H^{1-\delta-1}, & \text{for } \varepsilon = 0
\end{cases},
$$

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Critical entropy

\[ S - S_c = \begin{cases} \sim \varepsilon^{1-\alpha}, & \text{for } H = 0 \\ \sim H^\psi, & \text{for } \varepsilon = 0 \end{cases} \]  

(20)

When we are close to, but just below the critical temperature (at zero field), we replace \( \varepsilon \) by \( -\varepsilon \) and the critical exponents by their primed analogs \( \alpha', \beta', \gamma' \) and \( \delta' \) (For more details and notation see, for example, ref [27]).

We now state the scale invariance hypothesis or *fourth law of thermodynamics*, [27,28,29].

*Close to the critical point the singular part of the Gibbs free energy is a generalized homogeneous function of its variables.* Thus we have,

\[ G(\lambda^{a_\varepsilon} \varepsilon, \lambda^{a_H} H) = \lambda G(\varepsilon, H), \]  

(21)

where \( a_\varepsilon \) and \( a_H \) are two parameters called scaling powers, and \( \lambda \) is arbitrary.

The scaling hypothesis for static critical phenomena have been made [30] in a variety of situations and applied to thermodynamics functions and to static and dynamics correlation functions.

The above hypothesis transcends an ad hoc assumption to deal with the behavior of thermodynamic functions near the critical point. Its importance, tightly related to the presence of a symmetry (which is spontaneously broken) underlying the system, transforms this hypothesis in a true law of thermodynamics associated to the invariance of the behavior of a system with respect to scale transformations when it is under critical conditions.

It is clear that once accepted the condition of generalized function for one of the thermodynamic potentials, the same property holds for all the other thermodynamic potentials, since the Legendre transformations that relate each other conserve the mathematical content of the functions they are applied to [27,28].

Using eq.(21), all the critical point exponents can be simply expressed in terms of \( a_\varepsilon \) and \( a_H \). In fact, since

\[ C_H = T \frac{\partial^2 G}{\partial T^2} \bigg|_H ; \quad M = -\frac{\partial G}{\partial H} \bigg|_T ; \quad S = -\frac{\partial G}{\partial T} \bigg|_H \quad \text{and} \quad \chi_T = -\frac{\partial^2 G}{\partial H^2} \bigg|_T, \]  

(22)

we can relate the various critical exponents to the two scaling parameters \( a_\varepsilon \) and \( a_H \). Thus, one obtains [27,28]

\[ \alpha = 2 - \frac{1}{a_\varepsilon} ; \quad \beta = \frac{1 - a_H}{a_\varepsilon} ; \quad \delta = \frac{a_H}{1 - a_H} ; \]
\[ \gamma = \frac{2a_H - 1}{a_\varepsilon} ; \quad \psi = \frac{1 - a_\varepsilon}{a_H} ; \quad \varphi = \frac{1 - 2a_\varepsilon}{a_H}. \]  

(23)
We can further eliminate this two parameters and obtain a set of equalities among the exponents called scaling laws: \[27\]

\[
\alpha + 2\beta + \gamma = 2 ,
\]
\[
\alpha + \beta(\delta + 1) = 2 ,
\]
\[
\gamma(\delta + 1) = (2 - \alpha)(\delta - 1) ,
\]
\[
\gamma = \beta(\delta - 1) ,
\]
\[
(2 - \alpha)(\delta\psi - 1) + 1 = (1 - \alpha)\delta ,
\]
\[
\varphi + 2\psi - \delta^{-1} = 1 .
\]

(24)

These relations are not all independent of one another. Some of this equalities were predicted as inequalities under stability considerations. In fact the first and second to fourth equations, fulfilled as inequalities in (24), are known as the Rushbrooke and Griffiths inequalities respectively.

A second important result of the static scaling is the equality of primed and unprimed critical point exponents.

Experimentally and theoretically it was observed that the critical exponents are rather insensitive to the details of the system. This observation is embodied in the universality hypothesis that states that for a continuous phase transition the static critical exponents depend only on the following three properties:

1) the dimensionality of the system, \(d\).
2) the internal symmetry dimensionality of the order parameters, \(D\).
3) whether the forces are of short or long range.

The renormalization group approach \[25,31\] use the scaling hypothesis and provides a sound mathematical foundation to the concept of universality.

The scaling is found to hold (within experimental error) in almost every case. The renormalization group approach set the scaling theory in a broader context and explains the circumstances under which it can and how it breaks down \[32,33\].
3. The Scaling Hypothesis applied to black holes

As we have recalled in the last section, the scaling of critical phenomena applies to a great variety of thermodynamical systems. Those ranging from the internal structure of elementary particles to ferroelectricity and turbulent fluid flow, passing through superconductivity and superfluidity. We have also seen that black hole dynamics is governed by analogues of the ordinary four laws of thermodynamics and, with the appropriate cares corresponding to self - gravitating systems (negative heat capacities), one can apply these laws of thermodynamics to any particular process underwent by a black hole. This two facts lead us to conjecture that black holes also obey the scaling laws or fourth law of thermodynamics:

In the neighborhood of a critical point the singular part of Helmholtz free energy, \( F(T, J, Q) \), is a generalized homogeneous function of its variables.

Here we have established the hypothesis in terms of the Helmholtz potential, \( F = M - TS \), for practical reasons that will become clear later, but they are essentially that we can write

\[
C_{JQ} = -T \left( \frac{\partial^2 F}{\partial T^2} \right)_{J,Q},
\]

and that this heat capacity reveals the critical transition suffered by Kerr - Newman black holes\(^{[19]}\).

It can be seen from eq (8) that \( M(S, J, Q) \), corresponding in this case to the total energy of the black hole, is a homogeneous function of degree one half. However, it will turn out, as we will see soon, that close to the critical transition the thermodynamic potential will become a linear homogeneous function of its variables measuring the departure from the critical values. Thus allowing the normal relations between thermodynamic functions to be valid also in black hole thermodynamics. In particular, the entropy will recover (only in the neighborhood of the critical point) its property of extensivity (this property was, in turn, called by Landsberg \(^{[34,35]}\) the fourth law).

We will next study the behavior of the black hole variables as it undergoes the phase transition first discovered by Davies\(^{[19,20]}\). Thus, we will have the opportunity to explicitly check if black holes fulfill the scaling laws, i.e., eqs (24).

Let us suppose that a rotating charged black hole is held in equilibrium at some temperature \( T \), with a surrounding heat bath. If we consider a small, reversible transfer of energy between the hole and its environment; this absorption will be isotropic, and will occur in such a way that the angular momentum \( J \) and charge \( Q \) remain unchanged, on the average. The full thermal capacity (not per unit mass) corresponding to this energy transfer can be computed by eliminating \( M \) between eqs (1) and (8), and differentiate keeping \( J \) and \( Q \) constant,

\[
C_{JQ} = T \frac{\partial S}{\partial T} \bigg|_{J,Q} = \frac{MTS^3}{\pi J^2 + \frac{\pi}{4} Q^4 - \frac{T^2}{S}}.
\]
This heat capacity goes from negative values for a Schwarzschild black hole, \( C_{\text{Sch}} = -\frac{M}{T} \), to positive values for a nearly extreme Kerr-Newman black hole, \( C_{\text{EKN}} \sim \sqrt{M^4 - J^2 - M^2 Q^2} \to 0^+ \). Thus, \( C_{J,Q} \) has changed sign at some value of \( J \) and \( Q \) in between. In fact, the heat capacity passes from negative to positive values through an infinite discontinuity. This feature has lead Davies\[19\] to classify the phenomenon at the critical values of \( J \) and \( Q \) as a second order phase transition. The values \( J_c \) and \( Q_c \) at which the transition occurs are obtained by making to vanish the denominator on the right hand side of eq (26). We can then define the following parametrization,

\[
J_c^2 = \frac{j}{8\pi} M^4 \quad \text{and} \quad Q_c^2 = \frac{q}{8\pi} M^2 .
\]

Eliminating \( S \) and \( T \) in eq (26) by use of eqs (1) and (8), the infinite discontinuity in \( C_{J,Q} \) takes place at\[19\]

\[
j^2 JQ + 6j JQ + 4q JQ = 3 .
\]

For an uncharged, i.e., Kerr, hole, \( qJQ = 0 \). Thus, \( jJQ = 2\sqrt{3} - 3 \). Then we have

\[
\Omega_c = \frac{\sqrt{2\sqrt{3} - 3}}{4\sqrt{3} - 3} T_c \approx 0.233 T_c .
\]

While for a non rotating, i.e., Reissner-Nordström, hole, \( jJQ = 0 \). Thus, \( qJQ = 3/4 \). And the critical value of the electric potential is given by

\[
\Phi_c = \frac{1}{\sqrt{3}} ,
\]

independent of the other parameters of the black hole such as its mass or charge.

In ref [20] Davies has extended this analysis of the heat capacity discontinuity to the case of a Kerr-Newman black hole embedded in a De Sitter space. Hut [36] has also considered the effect of the surrounding radiation on the heat capacity \( C_v \) and found explicitly the existence of a critical point, it being the terminal point of a phaseline.

It can also be shown\[21\] that the four isothermal compressibilities are divergent as their corresponding heat capacities (however, adiabatic compressibilities are non-singular). For example,

\[
K_{T,Q}^{-1} = J \frac{\partial \Omega}{\partial J} \bigg|_{T,Q} \sim \frac{\pi(2\Phi Q - M)(1 - 4\pi TM)}{S^2[1 - 12\pi T M + 4\pi^2 T^2(6M^2 + Q^2)]} ,
\]

diverges as \( C_{J,Q} \). Also \( K_{T,J}^{-1} = C_{J,Q}(\partial \Phi / \partial Q \big|_{S,J})/C_{J,\Phi} \) diverges as \( C_{J,Q} \) on the singular segment given by eqs (27) - (28).

By use of eqs (8) and (1), the heat capacity \( C_{J,Q} \) can be expressed as\[21\]

\[
C_{J,Q} = \frac{4\pi TSM}{1 - 8\pi T M - 4\pi ST^2} \sim \frac{1}{T - T_c} ,
\]

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where the critical temperature is given by
\[ T_{cJQ} = \frac{1}{2\pi M} \left[ 3 + \frac{1}{\sqrt{3 - q_{JQ}}} \right], \] (33)
and \( q_{JQ} \) is given by the critical curve Eq. (28).

From the fundamental equations (1), (4) and (8) we can obtain the equation of state of a Kerr black hole
\[ J = \frac{1}{4\Omega^3} \left\{ \Omega^2 + \frac{T^2}{8} - \frac{T}{2} \left[ \Omega^2 + \frac{T^2}{16} \right]^{1/2} \right\} \left[ \Omega^2 + \frac{T^2}{16} \right]^{-1/2}, \] (34)
and Reissner-Nordstrøm black hole
\[ Q = \frac{1}{4\pi T} (\Phi - \Phi^3). \]

We have now all the elements to compute the critical exponents as defined in the section above. As we have already remarked, being \( C_{J,Q} \) the divergent heat capacity, we write all quantities of interest in terms of the Helmholtz free energy, \( F(T, \vec{J}) \). Thus, from the thermodynamical identities coming from the first law, eq (3), we have
\[ K_T^{-1} = J \frac{\partial^2 F}{\partial J^2} \bigg|_T, \quad \Omega = -\frac{\partial F}{\partial J} \bigg|_T, \quad S = -\frac{\partial F}{\partial T} \bigg|_J. \] (35)

We can obtain the first two critical exponents directly by inspection of eq (32) and comparison with eq (17):
\[ \alpha = 1, \quad \varphi = 1. \] (36)

Analogously, from eq (31) (that diverges as \( C_{J,Q} \)) and comparison with eq (19), we obtain
\[ \gamma = 1, \quad 1 - \delta^{-1} = 1 \Rightarrow \delta^{-1} \rightarrow 0. \] (37)

We will now deal with the equations obtained from the first derivatives of the Helmholtz potential. To obtain the corresponding critical exponents we choose a path either along a critical isotherm or at constant angular momentum \( J = J_c \) or constant charge \( Q = Q_c \). However, in this case the black hole equations of state just reproduce the critical curves (such as eqs (29)-(30), (33) and others deduced from them). In this case, we can formally assign a zero power in eqs (18) and (20) corresponding to critical exponents:
\[ \beta \rightarrow 0, \quad \delta^{-1} \rightarrow 0, \]
\[ 1 - \alpha = 0, \quad \psi \rightarrow 0. \] (38)

One can easily check that the set of critical values given by eqs (36) - (38) satisfy the scaling laws, eqs (24) (with \( \beta \delta = 1 \)).
One can also compute the so called gap exponents that describe the fashion in which the ratio of two successive derivatives of the free energy diverges \[^{28}\]. For example,

\[
\frac{\partial^{n+1} F}{\partial J^{n+1}} / \frac{\partial^n F}{\partial J^n} \sim \varepsilon^{-\Delta_n} .
\]

We obtain that \(\Delta_n = 1\), independent of \(n\).

The critical exponents can be obtained by noticing that the Helmholtz potential, \(F(T - T_c, J - J_c) - F_c\), close to criticality scales linearly with respect to its variables (as can be seen from eqs (35) and the fact that \(S\) and \(\Omega\) are continuous functions at the phase transition). Thus, \(a_\varepsilon = 1 = a_J\), and eqs (23) produce the same (36)-(38) values for the critical exponents.

Other five heat capacities can be computed, of which \(C_{\Omega,Q}\) and \(C_{J,\Phi}\) exhibit also a singular behavior. The remaining \(C_{\Phi,Q} = C_{J,\Omega}\) and \(C_{\Omega,\Phi}\) being regular functions in the allowed set of values of the parameters\[^{21}\].

A similar analysis to the previous case can be made for the other two diverging heat capacities. In fact, the heat capacity at \(\Omega\) and \(Q\) fixed can be written as

\[
C_{\Omega Q} = T \frac{\partial S}{\partial T} \bigg|_{\Omega Q} = -T \frac{\partial^2 H_1}{\partial T^2} \bigg|_{\Omega Q} ,
\]

with the appropriate thermodynamics potential given by

\[
H_1 = M - \vec{\Omega} \cdot \vec{J} - T S .
\]

By use of eqs (8) and (1), we have \[^{21}\]

\[
C_{\Omega Q} = \frac{4S^3 T \Phi^2}{\pi Q^2 (2 \Phi Q - M)} \sim \frac{1}{(T - T_c)} ,
\]

where, in this case, the critical temperature is given by

\[
T_{c \Omega Q} = \frac{1}{4\pi M} \left( \frac{3q_{\Omega Q} - 2}{q_{\Omega Q}^2} \right) ,
\]

and the critical curve

\[
(1 - q_{\Omega Q})^3 - j_{\Omega Q}(1 - q_{\Omega Q})^2 - \left( \frac{3}{2} q_{\Omega Q} - 1 \right)^2 = 0 ,
\]

fixes the values of \(q_{\Omega Q}\) and \(j_{\Omega Q}\) in the range

\[
0 \leq j_{\Omega Q} \leq 1/2 \ ; \ 2/3 \leq q_{\Omega Q} \leq 3/4 .
\]
The two associated isothermal compressibilities are

\[ K_{\Omega}^{-1} = Q \left. \frac{\partial \Phi}{\partial Q} \right|_{\Omega} = Q \left. \frac{\partial^2 H_1}{\partial Q^2} \right|_{\Omega}, \]

that diverges as \( C_\Omega Q \), and

\[ K_{\Omega} = \left. \frac{\partial J}{\partial \Omega} \right|_{\Omega} = \left. \frac{\partial^2 H_1}{\partial \Omega^2} \right|_{\Omega}, \]

that also diverges as \( C_\Omega Q \) on the singular curve Eq. (42).

We observe that the study of the critical transition suffered by the black hole when we keep \( \Omega \) and \( Q \) fixed, is governed by exactly the same critical exponents as in the case of the critical transition at \( J \) and \( Q \) fixed eqs (36)-(38). Note, however, that the critical curve described by eq (42) is different from that described by eq (31) (see also Figure 1).

The remaining divergent heat capacity, \( C_{J\Phi} \), can be studied in a similar way:

\[ C_{J\Phi} = T \left. \frac{\partial S}{\partial T} \right|_{J\Phi} = -T \left. \frac{\partial^2 H_2}{\partial T^2} \right|_{J\Phi}, \]

with

\[ H_2 = M - \Phi Q - TS , \]

the appropriate thermodynamical potential for this case.

Again, use of eqs (8) and (1) produce \([21]\)

\[ C_{J\Phi} \sim \frac{1}{T - T_c}. \]

The corresponding critical temperature is given by

\[ T_c^{J\Phi} = \frac{1}{2\pi M} \frac{K}{\left[ j_{J\Phi} + (K + 1)^2 \right]} ; \quad K^2 = 1 - j_{J\Phi} - q_{J\Phi}. \]

The critical curve for this case can be written as

\[ -K^4 + (j_{J\Phi} - 3)K^3 - (j_{J\Phi} + 3)K^2 + (j_{J\Phi}^2 + 4j_{J\Phi} - 1)K + 6j_{J\Phi} - 2j_{J\Phi}^2 = 0 , \]

for \( q_{J\Phi} \) and \( j_{J\Phi} \) in the range

\[ 0 \leq j_{J\Phi} \leq 2\sqrt{3} - 3 \; ; \; \quad 0 \leq q_{J\Phi} \leq 1 . \]

The two isothermal compressibilities of this case are

\[ K_{TJ}^{-1} = \left. \frac{Q}{Q} \frac{\partial Q}{\partial \Phi} \right|_{TJ} = \left. \frac{1}{Q} \frac{\partial^2 H_2}{\partial \Phi^2} \right|_{TJ} ; \quad K_{T\Phi}^{-1} = \left. \frac{J}{J} \frac{\partial \Omega}{\partial J} \right|_{T\Phi} = \left. \frac{1}{J} \frac{\partial^2 H_2}{\partial J^2} \right|_{T\Phi}. \]
Both of which diverges as $C_{J\Phi}$ on the curve given by eq(47).

We observe that again the critical exponents deduced from the divergence of the heat capacity and thermal compressibilities at $J$ and $\Phi$ fixed are those given by eqs (36)-(38). This result can in fact be understood as a realization of the hypothesis of Universality. As can be seen from Figure 1, the critical curves for the three cases studied are different, but the critical exponents, according to the above mentioned hypothesis, are the same within each class as specified after eq (24). We also observe that the equality between the primed ($T \rightarrow T_{c^-}$) and unprimed ($T \rightarrow T_{c^+}$) critical exponents is trivially verified in each one of the three transitions studied.

The values we have found for the critical exponents, (Eqs. (36) - (38)), correspond to those of the Gaussian model in two dimensions$^{[38,39]}$. It can be shown that this analogy also holds for the critical exponents derived from the correlation function. The implications of this results and the derivation of an effective Hamiltonian to describe black holes near criticality is presently under study and will be given in a forthcoming paper $^{[40]}$.

4. Discussion

We have reviewed how black holes follow the four laws of thermodynamics. We then have seen how critical point transitions and critical exponents can be defined for fluid-gas and magnetic systems. It is well known that critical exponents fulfill the scaling laws or what we call the fourth law of thermodynamics. The scaling behavior in critical phenomena applies to a vast variety of systems. This lead us to postulate that the scaling property also holds for black holes. We have shown that in all the three cases of phase transitions we studied, this scaling laws are satisfied with exponents given by eqs (36)-(38).

It was argued $^{[41]}$ that the second order phase transition discovered by Davies is of purely geometric origin and that the internal state of the black hole remains unaffected after the phase transition. However, when Davies’ transition is seen as a critical phase transition, the lack of qualitative change in the properties of the black hole can be understood as in analogy to what happens in the case of a liquid-vapor system; where near criticality no qualitative distinction can be made between phases. Note that in this case there is not such thing as a latent heat $^{[36]}$(since $M$ remains continuous throughout the transition), as it happens in magnetic critical transitions. Besides, the critical transitions occur when we cold down the black hole with respect to the corresponding Schwarzschild temperature, $T_S = 1/(8\pi M)$, by increasing its charge or angular momentum at fixed total mass (see Figure 2). Further, we have seen how black holes fulfil the scaling laws and universality hypothesis, both characteristics of critical phase transitions.

In fact, it was remarked by Hut $^{[36]}$ that although this phase transition does not affect the internal state of the system it is physically important as it indicates the transition from a region ($C_{JQ} < 0$) where only a microcanonical ensemble is appropriate (stable equilibrium if the system is isolated from the outside world) to a region ($C_{JQ} > 0$) where a canonical ensemble can be also used (stable equilibrium with an infinite heat bath).
Notably, scaling during phase transitions involving gravitational effects can be also found in several scenarios, such as: Inflation at late times\cite{42,43}, the strong field collapse of a massless scalar field coupled to gravity\cite{44} and in cosmic string networks\cite{45}.

The scaling hypothesis can also be extended to the static correlation functions and their dynamical behavior\cite{27}. Interestingly enough it was noted\cite{46} that some of the correlation functions diverges for extreme Kerr\cite{47} and Reissner-Nordstrøm\cite{48} black holes as expected when a phase transition takes place. The situation with black hole thermodynamics seems to be that of ordinary thermodynamics before the discovery of the underlying laws arising from statistical mechanics. In our case, we have to discover the quantum theory of gravitation to fully understand black holes. Nevertheless, we hope the lines developed in this work will help to go one step further towards the understanding of the deep connection between gravitation, quantum theory and statistical physics. In this sense, the relation among the renormalization group, conformal theory and critical phenomena, the resemblance of the critical exponents (36)-(38) and the Gaussian model in two dimensions\cite{38,39}, and the explanation of the broken ergodicity in black hole thermodynamics\cite{14} which in critical phenomena arises naturally\cite{49} are interesting enough hints to follow.

Indeed further research on this subject is needed to find the underlying theory behind all these analogies. We will deal on these and some other subjects in a forthcoming paper to be published elsewhere\cite{40}.

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Figure 1: This figure shows the different critical curves in the \( j - q \) plane (see Eqs. (27)). Black holes solutions remain within the triangle formed by the Schwarzschild, extreme Reissner-Nordstrom and Kerr holes. Heat capacities are negative in the region closer to the origin (Schwarzschild) and positives in the opposite region, closer to the extreme Kerr - Newman hole. Points labeled as \( A, B, C, D, \) and \( E \) have coordinates \((q, j)\) given by \( A = (0, 3\sqrt{3}/3)\); \( B = (3/4, 0)\); \( C = (.73, .13)\); \( D = (2/3, 1/3)\) and \( E = (1, 0)\).
Figure 2: This figure shows the critical temperatures (normalized to that of a Schwarzschild hole of the same mass), as a function of the critical charge parameter $q$ (Projection onto the $T_c - j$ plane gives qualitatively the same picture). Points $A$ and $B$ where curves meet represent the Kerr and Reissner-Nordstrom black holes respectively. At $A$, $C_{J\Phi} = C_{JQ} \equiv C_J$ and at $B$, $C_{\Omega Q} = C_{JQ} \equiv C_Q$. Point $C$, gives the highest temperature for which all three heat capacities are positive. Points $A$, $B$ and $C$ can be classified as bicritical points; while $D$ and $E$ can be classified as tricritical points\cite{50}, (since
there critical curves meet the first order phase transition curve $T = 0$, which represents extreme holes and separates black holes from naked singularities\cite{46}.)
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