Unified theory of elementary fermions and their interactions based on Clifford algebras

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Abstract
Seven commuting elements of the Clifford algebra $Cl_{7,7}$ define seven binary eigenvalues that distinguish the $2^7 = 128$ states of 32 fermions, and determine their parity, electric charge and interactions. Three commuting elements of the sub-algebra $Cl_{3,3}$ define three binary quantum numbers that distinguish the eight states of lepton doublets. The Dirac equation is reformulated in terms of a Lorentz invariant operator which expresses the properties of these states in terms of Dirac 4-component spinors. Re-formulation of the Standard Model shows chiral symmetry breaking to be redundant. A $Cl_{3,3}$ sub-algebra of $Cl_{5,5}$ defines two additional binary quantum numbers that distinguish quarks and leptons, and describes the SU(3) gluons that produce the hadron substrate, explaining quark confinement. Finally, a $Cl_{3,3}$ sub-algebra of $Cl_{7,7}$ defines a further two binary quantum numbers that distinguish four fermion generations. The predicted fourth generation is shown to have no neutrino and a distinct substrate, suggesting that ordinary matter is confined and providing candidates for unconfined dark matter. Interactions between fermions in the first three generations are predicted, including those that produce flavour symmetry. Relationships are explored between the $Cl_{1,3}$ algebra and general relativity, and between $Cl_{5,5}$ and SO(32) string theory.
§1: Introduction
The main features of the Standard Model (SM) were formulated between 1961 and 1967 (e.g. see Appendix 6 of [1]), producing a comprehensive conceptual and mathematical model of elementary particles and their interactions that is generally accepted as providing excellent agreement with experiment. Nevertheless, it lacks a coherent formalism, which limits its predictive capability and (as will be shown in this work) invalidates some of this ‘agreement’.

From 1974 onwards, many attempts were made to extend the SM formalism by employing Lie groups which have, as sub-groups, the SU(2) and SU(3) gauge groups that describe weak and strong interactions. Particular attention, summarized in [2,3], was given to SU(5) and SO(10). A great deal of effort, often centred on super-symmetry concepts [4], has since been expended in trying to repair the defects in these early attempts at unification. In retrospect, their problems arose because they incorporated the mathematical formalism of the SM, including the role of chirality, in their description of the elementary fermions. Clifford Unification is based on a new algebraic description of all the elementary fermions, which replaces the SM. The unification it achieves should not be confused with past attempts to unify gauge fields.

String theory [5] and Clifford algebras share a common interest in higher dimensional metrics. Their study originated with the Kaluza-Klein unification of gravity and electro-magnetism by extending the space-time metric to five-dimensions. String theory is based on the discovery that a ten-dimensional space-time metric had attractive mathematical properties that could be used to describe elementary bosons and fermions. In spite of the tremendous effort that has been devoted to the elaboration of its formalism, no clear relationship between the theoretical constructs of string theory and particle physics has been found.

Eddington [6] realized that the Dirac algebra could be employed as a common basis for the description of classical mechanics, gravitation and relativistic quantum physics. Unfortunately, there was little relevant experimental data at that time, and his personal attempt to predict elementary particle properties has made this approach a no-go area for generations of physicists. Nevertheless, the value of $Cl_{1,3}$ algebra in the description of space-time is now well established, e.g [7,8]. It has been known since 1958 that this algebra puts Maxwell’s equations in vacuo into a particularly simple form [8,9] related to the Dirac equation for zero mass fermions, but it has not been possible to find a Clifford algebra that provides a coherent link between space-time algebra and the description of fermions.

In 2001 Trayling and Baylis [10] identified the SU(2) and SU(3) Lie algebras in $Cl_7$. In 2009, Dartora and Cabrera [11] showed that the main features of electro-weak theory can be explained in terms of the $Cl_{3,3}$ algebra if chirality is omitted. The present work incorporates several of their results. Unfortunately, their misidentification of the time coordinate, and (possibly) the characterization of their work as a ‘toy’ theory in the abstract, has led to their work being ignored. The interpretation of elementary particle properties in terms of $Cl_{4,0}$ as a description of non-relativistic phase space by Zencyzykowski [12,13,14], is also relevant. More recently, Stoica [15,16] has shown that the results in [10] can also be expressed using the complex Clifford algebra $Cl^*_6$, and has investigated how this algebra might incorporate chiral symmetry breaking. It would be of interest to relate these approaches to the $Cl_{n,n}$ algebras, but this has not been attempted in this work.

Pavšić [17] has given string theoretic arguments for the importance of $Cl_{8,8}$ in providing a description of the elementary fermions. Yamatsu [18] has described a grand unified theory based on the Lie group USp(32), which is related to SO(32) string theory. Given that the Lie algebra of SO(32) and $Cl_{5,5}$ are both algebras of $32 \times 32$ matrices, there are possible links between Yamatsu’s work and the present work.

Although the present theory does not incorporate the algebraic structure of the SM, some detailed comparisons have been necessary. These have been helped by the many excellent textbooks on the SM that are now available. These include the thorough theoretical approach in Aitchison and Hey [19,20] and the clarity of presentation provided by Thomson [21]. The recent edition of the book by Dodd and Gripalos [22] has also been useful.
§2. Procedure

Clifford algebras were originally developed in the context of algebraic geometry, and are particularly appropriate for the description of macroscopic observables in a way that is independent of the observer’s coordinate system [7,8]. The main reason for thinking that they could provide useful models of elementary fermions and their interactions is the role played by $\mathcal{Cl}_{1,3}$ in the Dirac equation, where 4-spinors both distinguish fermion states and describe their dynamics. The successful application of the Dirac equation in quantum electrodynamics makes it clear that its algebra must provide the core of any unified theory. Hence the algebras studied in this work necessarily contain $\mathcal{Cl}_{1,3}$ as a sub-algebra. The choice of algebras is dependent on maintaining precise relations between their algebraic structures and the interpretation of observations. This work is concerned with identifying the discrete properties that distinguish elementary fermions and bosons, while keeping the successful aspects of the Dirac equation and Standard Model intact. Unification is developed in three stages, corresponding to the Clifford algebras $\mathcal{Cl}_{3,3} \subset \mathcal{Cl}_{5,5} \subset \mathcal{Cl}_{7,7}$. The quantum numbers obtained at each stage are given physical interpretations in terms of the elementary fermions and their interactions with gauge fields, as follows:

**Stage 1: Lepton properties based on $\mathcal{Cl}_{3,3}(L)$**

§3.1 Summarises the geometrical interpretation of the $\mathcal{Cl}_{1,3}$ space-time algebra.

§3.2 Introduces a real $8 \times 8$ matrix representation of $\mathcal{Cl}_{1,3}$ and extends this to a representation of $\mathcal{Cl}_{3,3}$. *Time intervals are identified as the product of all six generators of $\mathcal{Cl}_{3,3}$.*

§3.3 Interprets the algebraic expression for Maxwell’s field equations in vacuo as a photon wave-equation, with wave-functions expressed as excitations of a specific substrate.

§4.1 Describes eight lepton states in terms of three commuting elements of $\mathcal{Cl}_{3,3}$, with eigenvalues corresponding to binary quantum numbers that provide a formula for lepton charges.

§4.2 Relates the physical properties of leptons to the seven Lorentz invariants defined by the commuting elements of $\mathcal{Cl}_{3,3}$.

§4.3 Derives the effect of discrete coordinate transformations on lepton properties.

§5.1 Reformulates the Dirac equation as a Lorentz invariant differential operator acting on a Lorentz invariant, avoiding the negative mass problem.

§5.2 Reformulates the SM description of the Higgs boson while keeping its physical interpretation.

§5.3 Relates the differential operator to canonical momentum, showing that fermion properties are determined by the substrate of their wave motion, rather than their internal structure.

§6.1 Expresses the weak interaction in terms of the generators of $\mathcal{Cl}_{3,3}$, formulating electron/neutrino interactions *without reference to chirality*.

§6.2 Shows the $\mathcal{Cl}_{3,3}$ formulation of the weak interaction gives opposite parities of electron and neutrino spatial coordinates.

§6.3 Revises the Standard Model integration of electromagnetic and weak interactions.

**Stage 2: Quark and lepton properties based on $\mathcal{Cl}_{5,5}(LQ)$**

§7.1 Relates $\mathcal{Cl}_{5,5}$ generators to those of $\mathcal{Cl}_{3,3}(L)$, determining two additional quantum numbers extending the formula for fermion charges to include quarks.

§7.2 Defines $\mathcal{Cl}_{3,3}(Q)$, showing the SU(3) Lie algebra to be a sub-algebra of $\mathcal{Cl}_{5,5}(LQ)$.

§7.3 Interprets quark properties in terms of a gluon jelly substrate.

**Stage 3: $\mathcal{Cl}_{7,7}$**

§8.1 Relates $\mathcal{Cl}_{7,7}$ generators to those of $\mathcal{Cl}_{3,3}(L)$ and $\mathcal{Cl}_{5,5}(LQ)$, determining two additional quantum numbers, giving seven overall, extending the formula for fermion charges to include four generations, and showing the fourth generation to have no neutrino.

§8.2 Distinguishes the substrate of the fourth predicted generation from that of the three known generations.

§8.3 Identifies possible gauge fields and elementary bosons that are consistent with the algebra.

§8.4 Discusses the observability of the predicted fourth generation of fermions.

§9 outlines the relationship between the formalism and general relativity. §10 identifies a relationship with string theory. §11 discusses the *substrate* concept.
§3. From space-time algebra to $Cl_{1,3}$

The Clifford space-time algebra $Cl_{1,3}$ has four anti-commuting generators, denoted $E_\mu$, $\{\mu = 0, 1, 2, 3\}$, interpreted as unit displacements in the four coordinate directions. They satisfy

$$E_\mu E_\nu + E_\nu E_\mu = 2g_{\mu\nu}, \quad (3.1)$$

where the Minkowski metric tensor $g_{\mu\nu}$ has zero components when $\mu \neq \nu$ and

$$g_{11} = g_{22} = g_{33} = -1, \quad g_{00} = 1, \quad \text{so that} \quad g_{\mu\nu} = (E_\mu)^2. \quad (3.2)$$

Raising and lowering suffices follows the tensor convention, i.e., $E^\nu = g^{\nu\mu}E_\mu$. Combining the $E_\mu$ with rank 1 tensors produces Lorentz invariant expressions called structors in this work. These are to be distinguished from those single elements of the $Cl_{3,3}$ algebras that are themselves Lorentz invariant. For example, infinitesimal displacements in space-time are expressed as the structor

$$dx = E_\mu dx^\mu, \quad (3.3)$$

where it is assumed that all four unit displacements have the same dimensions (e.g. centimetres). $dx^2 > 0$ for displacements of particle with finite mass and $dx^2 = 0$ for photons.

Orientated unit areas in space-time are expressed as

$$E^\mu E_\nu = \frac{1}{2}(E_\mu E_\nu - E_\nu E_\mu) \quad (3.4)$$

so that infinitesimal area structors have the form

$$d^2S = E_{\mu\nu}dx^\mu dx^\nu. \quad (3.5)$$

Similarly, unit 4-dimensional volumes are defined in terms of the element denoted $E^\pi$ of the $Cl_{1,3}$ algebra, i.e.

$$E^\pi = E_0E_1E_2E_3 = \frac{1}{4!}\epsilon^{\mu\nu\kappa\tau}E_\mu E_\nu E_\kappa E_\tau. \quad (3.6)$$

(The suffix $\pi$ does not take numerical values.) The anti-symmetrizer $\epsilon^{\mu\nu\kappa\tau}$ is zero if any two suffices are equal, +1 for suffices that are even permutations of $\{0, 1, 2, 3\}$, and −1 for suffices that are odd permutations of $\{0, 1, 2, 3\}$. Infinitesimal space-time volumes $v_4$ therefore correspond to the structor

$$d^4v_4 = E^\pi d\tau = \frac{1}{4!}E_\mu E_\nu E_\kappa E_\rho dx^\mu dx^\nu dx^\kappa dx^\rho. \quad (3.7)$$

Three-dimensional unit ‘surface areas’ are given by the products

$$E^{\pi\tau} = E^\pi E^\tau = \frac{1}{3!}\epsilon^{\mu\nu\kappa\tau}E_\mu E_\nu E_\kappa. \quad (3.8)$$

In particular, $E^{\pi 0}$ is the unit spatial volume. Infinitesimal 3-dimensional volumes have the structor form

$$d^3S = E^{\pi\tau}dS_\tau = \frac{1}{3!}E_\mu E_\nu E_\kappa dx^\mu dx^\nu dx^\kappa. \quad (3.9)$$

The number of elements in a Clifford algebra determines how many different physical constructs can be described in terms of measurements of the unit displacements defined by its generators. A consequence of this is that when physical laws are expressed in terms of structors, the closure of $Cl_{1,3}$ constrains their form in a way that goes beyond Lorentz covariance. An important example is

$$E_{\mu\nu}E_\kappa = \epsilon_{\mu\nu\kappa\tau}E^{\pi\tau} + g_{\nu\kappa}E_\mu - g_{\mu\kappa}E_\nu. \quad (3.10)$$
The Lorentz invariant differential operator is the structor
\[ D = E^\mu \partial_\mu. \] (3.11)

Its geometrical interpretation is provided by the integral operator equality
\[ \int \nu d^4v \; DX = \int_{S\nu} d^3k \; X, \] (3.12)
where the 4-volume and 3-surface structors are given above. This is a special case of the Boundary Theorem (e.g. [7], p.69). The structor \( X \) in (3.12) is arbitrary, the integral on the left hand side is taken over a 4-volume \( \tau \), and the integral on the right hand side is taken over the 3-dimensional surface \( S(\tau) \) that encloses the 4-volume.

Transformations \( \Lambda \) relating structural coefficients in different Minkowski reference frames, denoted \( E^\nu \) and \( F^\mu \), can be expressed either as a similarity transformation or as a linear relationship between the coordinates, viz.
\[ F^\mu = \Lambda E^\nu \Lambda^{-1} = E^\nu \Lambda_\nu^\mu. \] (3.13)
The \( \Lambda_\nu^\mu \) express the transformation in terms of rotations of the spatial coordinates \( E_1, E_2, E_3 \), and boosts relating the spatial coordinates to \( E_0 \). Its algebraic form has been analysed in great detail, e.g. in [8], but is not relevant to this work.

Structors are also subject to discrete transformations that cannot be expressed as Lorentz transformations. As these are often involved in the analysis of elementary particle interactions it is necessary to establish their algebraic form. The spatial inversion, or parity, transformation \( \hat{P} \) changes the sign of all three spatial coordinates in a specific reference frame, and the sign of the unit spatial volume \( E^{00} \), i.e.
\[ E^\mu \rightarrow \hat{P} E^\mu \hat{P}^{-1} = E_\mu, \text{ where } \hat{P} = \hat{P}^{-1} = E^0. \] (3.14)
This transformation, and reflections, which change the sign of any one of \( E_1, E_2, E_3 \), interchange right and left handed spatial coordinate systems, so that \( E^{00} = E_1 E_2 E_3 \rightarrow -E^{00} \) and \( E^\pi = E^{00} E^0 \rightarrow -E^\pi \). Coordinate time inversion \( \hat{T} = E^{00} \) changes the sign of \( E^0 \), corresponding to running clocks backwards, without changing the spatial coordinate directions, so that
\[ E^\mu \rightarrow \hat{T} E^\mu \hat{T}^{-1} = -E_\mu. \] (3.15)
Proper time inversion \( T = \hat{T} \hat{P} = \hat{P} \hat{T} = E^5 \), changes the sign of all the \( E^\mu \) in any reference frame, giving
\[ E^\mu \rightarrow T E^\mu T^{-1} = -E^\mu. \] (3.16)

While particles have instantaneous positions in space, relativity theory expresses them as structors describing their infinitesimal displacements (3.3) in space-time. These take a special form in the rest frame of massive particles, i.e.
\[ dx = E_{\alpha 0} dx^{\alpha 0} = E_\mu dx^\mu, \mu = 0, 1, 2, 3 \text{ so that } E_{\alpha 0} = E_\mu \frac{dx^\mu}{dx^{\alpha 0}} \]
giving \((dx)^2 = (E_{\alpha 0} dx^{\alpha 0})^2 = (dx^\alpha)^2\), \( D = E^\mu \partial_\mu = E^{\alpha 0} \partial_{\alpha 0} \), and
\[ \text{the momentum } p = m E_{\alpha 0} = m E_\mu \frac{dx^\mu}{dx^{\alpha 0}}, \text{ where } m \text{ is the particle mass.} \] (3.17)

Here the ‘star’ in \( E_{\alpha 0} = E^{* \alpha} \) and \( dx^{* \alpha} \) distinguishes between proper time intervals, measured in the rest frame of the particle, from time intervals \( E_{\alpha 0} dx^{\alpha 0} \) in an arbitrary reference frame. In relativistic classical mechanics the magnitude \( dx^{\alpha 0} \) of a particle’s displacement in space-time is often written \( ds \). The ‘star’ notation will also be used to distinguish between spatial displacements in the particle and observer’s reference frames. It will only be necessary to make this distinction, i.e. introducing all the particle frame components \( E^{* \mu} \), when physical descriptions relate to arbitrary reference frames. The main role of the particle frame is that
its geometry, i.e. ± spin and the time direction, form part of the invariant description of fermions. All structors have scalar magnitudes determined by their square, which can be positive, negative or zero. This will sometimes be made explicit by putting (±) or (0) after the label.

In classical mechanics particles are conceived as the stable and single occupants of points in 3-dimensional space. Their dynamical properties are mass, electric charge, velocity and kinetic energy. \( Cl_{1,3} \) space-time geometry, as outlined above, provides all that is necessary to describe their dynamics, making it unnecessary to introduce matrix representations (as pointed out in [8]). However, matrix representations are necessary for the description of fermions.

The first step in relating the Dirac-Pauli matrix representation of \( Cl_{1,3} \) to the interpretation of the same algebra in classical mechanics is to obtain a real \( \gamma \)-matrix representation. In order to distinguish the two representations the notation \( \bar{\gamma} \) is used for the Dirac-Pauli matrices. Given that the required \( \gamma \)-matrix representation is real, and to distinguish algebraic and scalar occurrences of the square roots of \(-1\) in the following analysis, both sets of matrices will be expressed in terms of the four linearly independent real \( 2 \times 2 \) matrices,

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = -i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

(3.18)

where the \( \sigma \)s denote Pauli matrices. The \( 2 \times 2 \) matrices satisfy

\[
PQ = R, \quad -P^2 = Q^2 = R^2 = I.
\]

(3.19)

The generators of the \( Cl_{1,3} \) Dirac algebra can be expressed as Kronecker products, viz.

\[
\bar{\gamma}^0 = -I \otimes R, \quad \bar{\gamma}^1 = -Q \otimes P, \quad \bar{\gamma}^2 = -iP \otimes P, \quad \bar{\gamma}^3 = R \otimes P,
\]

(3.20)

and an additional matrix is defined as

\[
\bar{\gamma}^5 = i\bar{\gamma}^0 \bar{\gamma}^1 \bar{\gamma}^2 \bar{\gamma}^3 = I \otimes Q.
\]

(3.21)

No real \( 4 \times 4 \) matrix representation of the \( Cl_{1,3} \) algebra exists, but a real \( 8 \times 8 \) representation can be constructed. Its generators, defined in (3.20), are mapped into their corresponding real representation matrices by adding a factor \( \otimes I \) to the real \( \bar{\gamma} \) matrices, and replacing the factor \( i \) in \( \bar{\gamma}^2 \) by \( -I \otimes I \otimes P \). In summary

\[
I \otimes I \rightarrow 1_3 = I \otimes I \otimes I, \quad \bar{\gamma}^0 \rightarrow \gamma^0 = -I \otimes I \otimes P, \quad \bar{\gamma}^1 \rightarrow \gamma^1 = -Q \otimes P \otimes I, \quad \bar{\gamma}^2 \rightarrow \gamma^2 = P \otimes P \otimes P, \quad \bar{\gamma}^3 \rightarrow \gamma^3 = R \otimes P \otimes I,
\]

(3.22)

where \( \gamma^\mu \) is the real matrix representation of \( E^\mu \). Space-time unit volumes are

\[
\gamma^\pi = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = I \otimes Q \otimes P,
\]

(3.23)

which does not correspond to \( \bar{\gamma}^5 \) in (3.21). The \( 8 \times 8 \) matrix corresponding to \( \bar{\gamma}^5 \) is \( \gamma^6 = I \otimes Q \otimes I \), identified in Table A1 as one of the three time-like generators of \( Cl_{1,3} \).

Products of the \( \gamma^\mu \) \( \{ \mu = 0, 1, 2, 3, 6 \} \) generate the 32 entries in the second and third columns of Table A1 of Appendix A. The complete table has 64 matrices, providing a real representation of the \( Cl_{3,3} \) algebra. It is obtained by introducing the time-like generators \( \gamma^7 = -P \otimes P \otimes Q \) and \( \gamma^8 = P \otimes P \otimes R \), which anti-commute with all four generators of \( Cl_{1,3} \). The six matrices \( \gamma^\mu, \{ \mu = 1, 2, 3, 6, 7, 8 \} \) provide all six generators of \( Cl_{3,3} \), with unit space-time displacements denoted \( \gamma^\mu \), where \( \{ \mu = 1, 2, 3, 0 \} \). \( \gamma^\pi \) and the generators \( \gamma^\mu \), where \( \mu = 6, 7, 8 \), are Lorentz invariant. Unit time displacements do not appear as one of the generators of \( Cl_{3,3} \) but are given by \( \gamma^0 = \gamma^1 \gamma^2 \gamma^3 \gamma^6 \gamma^8 \). This can be simplified by noting that \( \gamma^6 \gamma^7 \gamma^8 = \gamma^\pi \), showing that units of time correspond to the unit space-time 4-volume divided by its corresponding unit spatial 3-volume.
In the remainder of this paper it will be assumed that all elements $E^\mu$ of the Clifford algebras have matrix representations, and the same notation will be used for structors and their matrix representations. The canonical matrix representation of the electromagnetic field structor in vacuo is

$$F = \gamma^{\mu\nu} F_{\mu\nu}/2$$

$$= \left( \begin{array}{cccccccc}
0 & -F_{31} & F_{03} & F_{01} & -F_{12} & -F_{23} & 0 & -F_{02} \\
F_{31} & 0 & F_{01} & -F_{03} & -F_{23} & F_{12} & F_{02} & 0 \\
F_{03} & F_{01} & 0 & -F_{31} & 0 & -F_{02} & -F_{12} & -F_{23} \\
F_{01} & -F_{03} & F_{31} & 0 & F_{02} & 0 & -F_{23} & F_{12} \\
F_{12} & F_{23} & 0 & F_{02} & 0 & -F_{31} & F_{03} & F_{01} \\
F_{23} & -F_{12} & -F_{02} & 0 & F_{31} & 0 & F_{01} & -F_{03} \\
0 & F_{02} & F_{12} & F_{23} & F_{01} & F_{03} & F_{01} & 0 \\
-F_{02} & 0 & F_{23} & -F_{12} & F_{01} & -F_{03} & -F_{31} & 0
\end{array} \right),$$

(3.24)

where $F_{(i)}$ is the $i$-th column of $F$. Maxwell’s equations can be expressed by the structor equation

$$D F = \mathbf{J},$$

(3.25)

where the charge-current density structor $\mathbf{J} = J_\mu \gamma^\mu$ is the source of $F$. (3.25) shows Maxwell’s equations in vacuo to be a consequence the closure relation (3.10). In vacuo, each column of $F$ separately satisfies $DF_{(i)} = 0$, as will column matrices formed from any linear combination $\Phi_F = \sum_i a_i F_{(i)}$, where the coefficients $a_i$ are constant complex numbers. The equation

$$D \Phi_F = 0$$

(3.26)

has the same structure as the Dirac wave-equation for particles of zero mass (after making the modifications described in §4). When $F$ describes a radiative field, constraints on the magnitudes of the electric and magnetic components of the field correspond to the structor equation $F^2 = 0$, so the eight terms in the product of any row with any column of $F$ sum to zero. Given this constraint, and the adjoint $(\Phi_F)^\dagger$ of $\Phi_F$, $(\Phi_F)^\dagger \Phi_F = 0$, so that (3.26) provides a wave-mechanical description of photons.

Interactions between photons and fermions are conventionally formulated in terms of potential structors $A = A_\mu \gamma^\mu$, related to the electromagnetic field by

$$F = D A = \gamma^\mu \partial_\mu \gamma^\nu A_\nu = \gamma^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)/2 + \partial_\mu A^\mu.$$

(3.27)

It follows from that $F_{(i)} = D A_{(i)}$, giving

$$\Phi_F = D \Phi_A = D \sum_i a_i A_{(i)},$$

(3.28)

where the $A_{(i)}$ denote columns of $A$. The conventional plane-wave description of photons has the structor form

$$A = \exp(\eta k_\mu x^\mu) A^{\text{const}},$$

(3.29)

where $k = \gamma^\mu k_\mu$, $A^{\text{const.}}$ are independent of the space and time coordinates $x^\mu$, and $\eta = i$. The identification $\eta = i$, accords with the Michelson-Morley result that no substrate for photon waves in the form of a stationary ‘aether’ exists. This does not, however, rule out the possibility that photon wave motion modulates a medium that can be expressed algebraically in terms of a Lorentz invariant $\eta$, providing a physical substrate in which the photons propagate. The following analysis is made on the basis that possible choices $\eta \neq i$, with $\eta^2 = -1$, exist.

It follows from (3.29) that

$$F = D A = \eta k \exp(\eta k_\mu x^\mu) A^{\text{const.}} = \eta k A,$$

(3.30)
so

\[ \mathbf{D} \Phi =\mathbf{D}^2 \Phi = \partial^\mu \partial_\mu \Phi = k^\mu \eta k_\mu \eta \Phi = (\eta)^2 k^2 \Phi = -k^2 \Phi. \]  

(3.31)

consistent with \( k^2 = 0 \) and the radiative field condition \( \mathbf{F}^2 = -k \mathbf{A} k \mathbf{A} = k^2 \mathbf{A}^2 = 0 \) if \( \mathbf{k} \) and \( \mathbf{A} \) anti-commute. It follows that

\[ \mathbf{D}^2 \Phi = \partial^\mu \partial_\mu \Phi = k^\mu \eta k_\mu \eta = k^2 \eta^2 = 0. \]  

(3.32)

provides an alternative, Klein-Gordon, form of the photon wave equation.

Plane wave solutions of \( \mathbf{D} \Phi = 0 \) are

\[ \Phi = \exp(\eta k_\mu x^\mu) \Phi^c, \]  

(3.33)

where the \( \Phi^c \) is independent of the space and time coordinates and \( \mathbf{k} = \gamma^\mu k_\mu \) is the photon wave structure. Given (3.25) and (3.33), the field equation \( \mathbf{D} \Phi = 0 \) reduces to

\[ -\eta \mathbf{D} \Phi = -\eta \mathbf{E}^\nu \partial_\nu \exp(\eta k_\mu x^\mu) \Phi^c = \mathbf{k} \Phi = 0. \]  

(3.34)

Defining the photon velocity 3-vector \( \mathbf{u} = \gamma^0 u_i, \ i = 1, 2, 3 \) with \( u^2 = -1 \), so that \( \mathbf{k}^2 = (k_0)^2 (\mathbf{\gamma}^0 + \mathbf{u})^2 = 0 \). For photons moving in the \( y \)-direction, \( \mathbf{u} \to u_2 \gamma^2 \), and (3.34) becomes

\[ (\gamma^0 - u_2 \gamma^2) \Phi^c = 0 \text{ or, equivalently, } \gamma^{02} \Phi^c = u_2 \Phi^c, \]  

(3.35)

where \( u_2 = \pm 1 \), corresponding to the direction of the photon velocity, with unit magnitude corresponding to the velocity of light. Equation (3.35) relates to unpolarized photons, leaving open the question of finding elements of \( C\ell_{3,3} \) that commute with \( \gamma^{02} \), with eigenvalues that distinguish polarization and the sign of interactions with charged fermions. Polarizations are normally described by the 4-vectors \( \epsilon^i = \epsilon^i \gamma^\mu, \ i = 1, 2, \) orthogonal to the wave-vector \( \mathbf{k} = k_\mu \gamma^\mu \), giving

\[ \mathbf{k} \epsilon^i + \epsilon^i \mathbf{k} = 2 k^\mu \epsilon^i \eta_\mu = 0. \]  

(3.36)

In the algebraic formulation plane polarizations could be described by the eigenvalues of \( \gamma^{31} \).
§4. Description of leptons in terms of the eigenvalues of commuting elements

In the Dirac theory, 4×4 matrices $\gamma$ act on 4-component spinors. The Dirac expressions for these matrices (see Table A3), which can be interpreted as describing Minkowski coordinates in the fermion rest frame, are denoted by $\gamma^{\mu}$ in this work, where the star indicates that they are invariant under Lorentz transformations. The electron/positron distinction is determined by the eigenvalues of $\gamma^0 = -I \otimes R_z$, which are +1 for electrons and −1 for positrons. The up/down spin distinction is determined by the eigenvalues $\pm i$ of $\gamma^+ = i R \otimes I$, which commutes with $\gamma^0$. Hence the binary eigenvalues of two commuting elements of the $Cl_{1,3}$ algebra distinguish four states of a lepton.

The 8×8 representation matrices of $Cl_{3,3}$ act on 8-component column matrices. These components will be shown to distinguish the four states the two leptons in a given generation, and relate them to commuting elements of $Cl_{3,3}$. As the squared elements of Clifford algebras are all $\pm 1_3$, their eigenvalues are necessarily twofold, i.e. $\pm 1$ or $\pm i$, so that three commuting elements of $Cl_{3,3}$ are required to distinguish $2^3 = 8$ lepton states. These three elements, and their eigenvalues, will be called primary. The anti-lepton that corresponds to a given lepton has opposite signs of all its primary eigenvalues. Pair products of the three primary commuting elements determine three secondary eigenvalues, while the product of all three gives a fourth primary eigenvalue, which determines the direction of time and distinguishes fermions from anti-fermions. Secondary eigenvalues have the same values for a lepton and its corresponding anti-lepton.

Let $\gamma^A, \gamma^B$ and $\gamma^C$ be commuting Hermitian matrices, with eigenvalues $\mu_A = \pm 1, \mu_B = \pm 1$ and $\mu_C = \pm 1$. Each matrix defines a projection operator, e.g. $P(\mu_A) = \frac{1}{2} (1 + \mu_A \gamma^A)$. These matrices will be related to elements $\gamma$ of $Cl_{3,3}$ where $\gamma$ is time-like, or $i \gamma$ when $\gamma$ is space-like. In the following analysis it will be assumed that the $\gamma$-matrices defined in Appendix A refer to the Minkowski coordinates in the lepton rest frame, and the ‘star’ notation will be employed to distinguish them from representation matrices corresponding to unit space-time displacements in arbitrary frames. The aim is to identify $\gamma^A, \gamma^B, \gamma^C$ with specific elements of $Cl_{3,3}$.

The eight distinct lepton states are projected out of an 8-component column matrix by

$$P(\mu_A, \mu_B, \mu_C) = P(\mu_A)P(\mu_B)P(\mu_C) = \frac{1}{8} (1_3 + \mu_A \gamma^A)(1_3 + \mu_B \gamma^B)(1_3 + \mu_C \gamma^C).$$

(4.1)

The space-like anti-commuting elements $\gamma^* \gamma^{12}, \gamma^* \gamma^{23}, \gamma^* \gamma^{31}$, generate the Lie algebra $SU(2)_{spin}$. $\gamma^A$ can be identified as $i$ times any normalised linear combination of them, corresponding to the (arbitrary) choice of spin orientation, but, as the eigenvalue $\mu_A$ provides no information about this orientation, it can be assumed that $\gamma^A = i \gamma^{31}$.

In order that each of the eight eigenstates corresponds to a single non-zero entry in the column matrix it is necessary to choose a representation in which all three matrices $\gamma^B$ and $\gamma^C$ and $\gamma^A = \gamma^{31}$ are diagonal.

The space-like anti-commuting matrices $\tilde{\gamma}^\pi(= -\tilde{\gamma}^{78}), \tilde{\gamma}^\pi 7, \tilde{\gamma}^\pi 8$ generate the Lie algebra $SU(2)_{isospin}$. As all three commute with $\tilde{\gamma}^* \gamma^{12}, \tilde{\gamma}^* \gamma^{23}$ and $\tilde{\gamma}^* \gamma^{31}$, any one of them, or any normalised linear combination could, in principle, be identified with $-i \gamma^C$. In practice, however, $SU(2)_{isospin}$ symmetry is broken so, in the following analysis, leptons will be described by the eigenvalues of the diagonal matrix $\gamma^C = i \gamma^\pi 6$, so that $\mu_C = \mu_\pi 6 = \pm 1$. (The ‘isospin’ label introduced here provides the same quantum number as the isospin currently employed in the description of baryon flavour symmetry.)

Having identified $\gamma^A$ and $\gamma^C$ with pair products of generators, it is clear that $\gamma^B$ could be identified with the time-like matrix $\gamma^2$, but this matrix does not correspond to a readily observable property of leptons. The alternative is to identify $\gamma^B = \tilde{\gamma}^\pi 0 = -\tilde{\gamma}^* 26 \tilde{\gamma}^{31} \gamma^{26}$, which is the time direction in the fermion rest frame. The Standard Model was originally formulated when neutrinos were thought to have zero mass but, as neutrinos and anti-neutrinos are now known to have small non-zero masses, they can be described by spinors that are eigenstates of $\tilde{\gamma}^\pi 0$. It follows that $\tilde{\gamma}^* 0 = \gamma^\pi 0$, with eigenvalues $\mu_B = \mu_0 = +1$ for leptons and $\mu_B = \mu_0 = -1$ for anti-leptons, giving the lepton state identifications summarized in Table 4.1. This table also shows that the same quantum numbers can be associated with stable baryons, i.e. neutrons(n).
and protons ($p$).

Table 4.1: Lepton identification

|          | $\mu_B = \mu_{*0} = +1$ | $\mu_B = \mu_{*0} = -1$ |
|----------|--------------------------|--------------------------|
| $\mu_C = i\mu_\pi = +1$ | $e^-, p^-$ | $\bar{\nu}, \bar{n}$ |
| $\mu_C = i\mu_\pi = -1$ | $\nu, \bar{n}$ | $e^+, p^+$ |

A complete description of lepton states, including the spin degree of freedom, is given in Table 4.2, which shows the $Cl_{3,3}$ algebra to be consistent with neutrinos being described by Dirac (4-component) spinors, rather than 2-component spinors. Lepton charges, are given by

$$
\mu_Q = -\frac{1}{2} (\mu_{*0} + i\mu_\pi) = -\frac{1}{2} (\mu_B + \mu_C),
$$

(4.2)
times the magnitude of the electronic charge $e$. The primary eigenvalues $i\mu_\pi$, $i\mu_{*31}$, $\mu_{*0}$, $\mu_{*26}$ (in the first four columns of Table 4.2) have opposite signs for leptons and their corresponding anti-leptons.

Table 4.2: Lepton quantum numbers and charges

| isospin | spin | proper time | mass/energy helicity | charge $\mu_Q$ | lepton state |
|---------|------|-------------|----------------------|----------------|--------------|
| $C: i\mu_\pi$ | $A: i\mu_{*31}$ | $B: \mu_{*0}$ | $ABC: \mu_{*26}$ | $BC: i\mu_{*60}$ | $AB: i\mu_{*2}$ |
| 1 | 1 | 1 | 1 | 1 | 1 | -1 | $e^−\downarrow$ |
| 1 | -1 | 1 | -1 | 1 | -1 | 1 | $e^−\uparrow$ |
| 1 | 1 | -1 | -1 | -1 | -1 | 0 | $\bar{\nu}\downarrow$ |
| 1 | -1 | -1 | -1 | 1 | -1 | 1 | $\bar{\nu}\uparrow$ |
| −1 | 1 | 1 | -1 | -1 | 1 | 0 | $\nu\downarrow$ |
| −1 | -1 | 1 | 1 | -1 | -1 | 0 | $\nu\uparrow$ |
| −1 | 1 | -1 | 1 | 1 | -1 | 1 | $e^+\downarrow$ |
| −1 | -1 | -1 | -1 | 1 | 1 | 1 | $e^+\uparrow$ |

If the leptons states are labelled in the same order as in the last column of Table 4.2, entries in the first four columns determine the diagonal matrices that correspond to the primary eigenvalues, viz.

$$
\gamma^A = \hat{\gamma}^{*31} = -\mathbf{R} \otimes \mathbf{I} \otimes \mathbf{I} = \text{diag}(11\bar{1}; 11\bar{1}),
$$

$$
\gamma^B = \hat{\gamma}^{*0} = -\mathbf{I} \otimes \mathbf{R} \otimes \mathbf{I} = \text{diag}(11\bar{1}; 11\bar{1}),
$$

$$
\gamma^C = i\hat{\gamma}_\pi = -\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{R} = \text{diag}(11\bar{1}; 1\bar{1}1\bar{1}),
$$

$$
\gamma^{ABC} = -\mathbf{R} \otimes \mathbf{R} \otimes \mathbf{R} = \text{diag}(1\bar{1}1; 1\bar{1}1),
$$

(4.3)

where $\bar{1} \equiv -1$. The structor corresponding to $\hat{\gamma}^{*31}$ is

$$
s(-) = \hat{\gamma}^{\mu\nu} s_{\mu\nu}, \{\mu, \nu = 0, 1, 2, 3\},
$$

(4.4)
with values of the coefficients $s_{\mu \nu}$ determined by the reference frame. The structor with eigenvalues corresponding to lepton charges is

$$Q = -\frac{1}{2}(\hat{\gamma}^* s^0 + i\hat{\gamma}^* \pi^6) = \frac{1}{2}(I \otimes R \otimes I + I \otimes I \otimes R)$$

$$\equiv \frac{1}{2} \text{diag}(1 \ 1 \ 1; \ 1 \ 1 \ 1) + \text{diag}(1 \ 1 \ 1; \ 1 \ 1 \ 1) = \text{diag}(1 \ 1 \ 0 \ 0; \ 0 \ 0 \ 1 \ 1).$$

Its square

$$Q^2 = \frac{1}{2}(1_3 + \hat{\gamma}^* i\hat{\gamma}^* \pi^6) = \text{diag}(1 \ 1 \ 0 \ 0; \ 0 \ 0 \ 1 \ 1)$$

has eigenvalues $+1$ for electrons and positrons, and zero for neutrinos and anti-neutrinos, giving the mass formula

$$M = m_\nu 1_3 + (m_e - m_\nu)Q^2$$

$$= \frac{1}{2}m_\nu(1_3 + i\hat{\gamma}^* \pi^6\hat{\gamma}^* s^0) + \frac{1}{2}m_e(1_3 + i\hat{\gamma}^* \pi^6\hat{\gamma}^* s^0),$$

$$= \text{diag}(m_e, m_e, m_\nu, m_\nu, m_\nu, m_\nu, m_e, m_e)$$

and $p = M\hat{\gamma}^* s^0$.

In the Standard Model the spin quantum number $s_z$ is related to the helicity quantum number for electrons with momentum $\vec{p}$ defined by $h = \vec{s} \cdot \vec{p}$, where $\vec{s}$ is the spin 3-vector, $\vec{p}$ is the momentum 3-vector and $p^2 = \vec{p}^2$ (e.g. [21] page 105). With this definition, helicity is found to be conserved in high energy interactions, although it is clearly not invariant under Lorentz transformations that change the sign of $\vec{p}$. In the $Cl_{3,3}$ formalism, the spin quantum number is associated with $\hat{\gamma}^{s31}$, which is a component of the structor $s(-) = \hat{\gamma}^{\mu\nu} s_{\mu\nu}$. Helicity is identified with the pseudo-vector $\hat{\gamma}^{* \pi^2} = -\hat{\gamma}^{s0} \hat{\gamma}^{s31}$, which is a component of the structor $h(-) = \hat{\gamma}^{s0} h_\mu$. This Lorentz invariant redefinition of helicity is important in the analysis of experimental results.

Discrete geometrical transformations of the space-time coordinates were given in §3. It is assumed, in the Standard Model, that quantum mechanical equivalents can be obtained by expressing these transformations in terms of the Dirac algebra, but there is experimental evidence that particle interactions are not always invariant under these transformations, suggesting a need to reformulate them in terms of the $Cl_{3,3}$ algebra. Geometrical symmetries are related to the properties of elementary fermions by replacing the $E^\mu$ with their matrix representations $\hat{\gamma}^\mu$. Inversion of the spatial coordinates corresponds to changing their parity $\hat{P}$, defined by the transformation

$$\hat{P} : \hat{\gamma}^\mu \rightarrow \hat{\gamma}^0 \hat{\gamma}^\mu (\hat{\gamma}^0)^{-1} = \hat{\gamma}_\mu,$$

(4.8)

where $\hat{\gamma}^0 = \hat{\gamma}_0$ is the observer’s time direction. As each coordinate frame, and each fermion, defines its own time direction, $\hat{P}$ is not invariant in fermion interactions. The assignment of positive parity to fermions and negative parity to anti-fermions, made in the Standard Model, relates to the time direction $\hat{\gamma}^{s0}$ in the fermion rest frame, rather than the time direction $\hat{\gamma}^0$ in the observer’s frame. This is consistent with (4.8) if the corresponding Lorentz invariant operator $\hat{P}$, defined by

$$\hat{P} : \hat{\gamma}^{s\mu} \rightarrow \hat{\gamma}^{s0} \hat{\gamma}^{s\mu} (\hat{\gamma}^{s0})^{-1} = \hat{\gamma}_{s\mu},$$

(4.9)

where $\hat{\gamma}^{s0} = (\hat{\gamma}^{0s})^{-1}$, is the (Lorentz invariant) proper time, so that the reversed spatial coordinates $\hat{\gamma}^{s\mu}, \mu = 1, 2, 3$ refer to the fermion’s rest frame. As each particle has its own rest frame, this can be difficult to relate to experimental results. Nevertheless, it can be expressed in terms of the Lorentz invariant $\hat{\gamma}^{s\mu} \equiv \hat{\gamma}^{s\mu}$ which satisfies $\hat{P} \hat{\gamma}^{s\mu} = -\hat{\gamma}^{s\mu}$. The association of parity with fermion states, assigned in the Standard Model, can now be seen as defining fermion parities in terms of the eigenvalues of $\hat{\gamma}^{s6}$. Coordinate reflections also change the parity of the coordinate system as expressed by the sign of the Lorentz invariant $\hat{\gamma}^7$. For example, reflections in the $\hat{\gamma}^{s31}$ plane in the fermion rest frame, which produce a reversal of the fermion spin direction, are described by

$$\hat{P}_{31} : \hat{\gamma}^{s\mu} \rightarrow \hat{\gamma}^{s31} \hat{\gamma}^{s\mu} (\hat{\gamma}^{s31})^{-1} = \hat{\gamma}^{s\mu}, \text{ for } \mu = 0, 1, 3, \text{ or } -\hat{\gamma}^{s\mu}, \text{ for } \mu = 2, \text{ and } \pi,$$

(4.10)

showing that single coordinate reflections change parity.
The time-reversal operator in an arbitrary coordinate frame has the representation \( \hat{T} = \hat{\gamma}^{\pi 0} \) which, again, is not Lorentz invariant. This geometrical, or unitary, form of time-reversal changes the sign of the Hamiltonian, this problem being overcome in the Standard Model by including the sign reversal \( i \rightarrow -i \), making the transformation anti-unitary. The \( \CL_{1,3} \) algebra provides the proper time-reversal operators \( \hat{T}^k = \hat{\gamma}^{*k 0} : k = 6, 7, 8 \) giving, in the fermion rest frame, \( \hat{T}^k : \hat{\gamma}^{* \mu} \rightarrow \hat{\gamma}^{*k 0} \hat{\gamma}^{* \mu} (\hat{\gamma}^{*k 0})^{-1} = -\hat{\gamma}^{* \mu}, \) (4.11)

where \( k = 6, 7, 8, \pi \). If \( k = \pi \) or 6 the same unitarity problem arises. It is, however, avoided by choosing \( k = 7 \) or 8, both of which go beyond \( \CL_{1,3} \) space-time geometry, and provide unitary, Lorentz invariant, forms of time-reversal.

All seven quantum numbers that can be constructed from \( A, B \) and \( C \) correspond to algebraic invariants, which are structors if they involve either \( A \) or \( B \). A summary of their physical interpretations is given below:

| quantum no. | algebraic invariant/structor | macroscopic interpretation | quantum interpretation |
|-------------|-----------------------------|---------------------------|------------------------|
| \( A \) : \( \mu_{+31} \) | \( s(-) = \hat{\gamma}^{s31}s_{31} = \hat{\gamma}^{\mu \nu}s_{\mu \nu} \) | intrinsic angular velocity | spin |
| \( B \) : \( \mu_{+0} \) | \( \hat{\gamma}^{*0} = \hat{\gamma}_\mu dx^\mu / dx^{*0} \) | proper time direction | fermion/anti-fermion |
| \( C \) : \( \mu_{+6} \) | \( \hat{\gamma}^{*6} = \hat{\gamma}^{8,7} \) | fermion parity | iso-spin, quantum \( i \), lepton substrate |
| \( BC \) : \( i\mu_{+6,0} \) | \( \mathbf{p} = M^\gamma_{\mu} dx^\mu / dx^{*0} \) | 4-momentum | as macroscopic |
| \( AC \) : \( \mu_{+026} \) | \( s^\gamma \pi 0(+) \) | magnetic moment | as macroscopic |
| \( AB \) : \( \mu_{+e2} \) | \( \mathbf{h}(-) = s^{\gamma 40} = \hat{\gamma}^{\pi \mu} h_{\mu} \) | magnetic moment | helicity |
| \( ABC \) : \( \mu_{+26} \) | \( \hat{\gamma}^{*26} = \mathbf{h}^{\gamma \pi 0} = -\hat{\gamma}^{s31} \hat{\gamma}^{*400} \) | spin angular momentum | as macroscopic |

5. Reformulation of the Dirac equation

The established procedure for obtaining the quantum mechanical equations of motion for free particles from their classical counterparts is to replace the momentum 3-vector \( \vec{p} = (p_1, p_2, p_3) \) by the operator \( -i \nabla = -i(\partial_1, \partial_2, \partial_3) \) and the energy \( E \) by the operator \( i \partial_t \). Wave equations are then produced from the action of the relation between mass, momentum and energy on a wave function. In particular, the Schrödinger equation \( i\partial_t \phi = (\nabla^2/2m) \phi \), where \( \phi \) is the wave-function, is obtained from the mass/momentum/energy relation for free particles in classical mechanics, i.e. \( E = -\frac{\hbar^2}{2m} \vec{p}^2 \). Its solution is the wave function \( \phi = \phi_0 \exp(\pm(i(p, \vec{x} - E t)) \) where \( \phi_0 \) is constant.

The following analysis clarifies the relationship between the 4×4 \( \hat{\gamma} \) Dirac matrix representation of \( \CL_{1,3} \) and the 8×8 \( \hat{\gamma}^\mu \) space-time matrices of \( \CL_{3,3} \). In Appendix A explicit comparisons are made between representations of the \( \hat{\gamma}^\mu \) and \( \hat{\gamma}^{* \mu} \) fermion rest frame coordinates. The star notation, introduced in §3, distinguishes the fermion rest-frame from the arbitrary reference frames employed by observers. The \( \hat{\gamma}^\mu \) always refer to the fermion rest frame.

Physical space-time coordinates can be represented either by the \( \hat{\gamma}^\mu \) matrices or by the familiar Dirac \( \gamma \) matrices. The following analysis relates these alternatives, making it clear that the Dirac formulation is complicated by the two fermions in any doublet are described by spatial coordinate systems with opposite parity, corresponding to the identification \( \gamma^\mu \equiv a^{\gamma \mu} \) or \( \gamma^\mu \equiv b^{\gamma \mu} \).

The relativistic energy/momentum conservation equation for free particles is \( \mathbf{p}^2 = E^2 - \vec{p}^2 = m^2 \). In terms of the \( \hat{\gamma} \) algebra this corresponds to the structor equation

\[
\mathbf{p} = \hat{\gamma}^\mu p_\mu = m \hat{\gamma}^{*0},
\]

where \( \hat{\gamma}^{*0} \) has the eigenvalue \( \mu_{+0} = +1 \) for electrons and \( \mu_{-0} = -1 \) for positrons. The standard replacement \( p_\mu \rightarrow i \partial_\mu \) gives

\[
\mathbf{p} = \hat{\gamma}^\mu p^\mu \rightarrow i \partial_\mu = i \hat{\gamma}^\mu \partial_\mu,
\]

(5.2)
and the relativistic free electron wave equation

\[(i \mathbf{D} - m \gamma^0) \phi = 0. \quad (5.3)\]

Dirac’s wave-equation \((i \mathbf{D} - m) \phi = 0\), which is currently accepted as providing the correct description of fermion dynamics, omits \(\gamma^0\) in \((5.3)\). It was derived by taking the square root of both sides of the relativistic free fermion energy/momentum conservation equation \(p^2 = m^2\), giving

\[p = \gamma^\mu p_\mu = m \quad (5.4)\]

rather than \((5.1)\). However, no linear combination of the Dirac matrices \(\gamma^\mu\), which all have zero trace, can give rise to the right hand side of \((5.4)\). When \(\gamma^0\) is omitted from \((5.3)\), as it is in the Dirac equation, \(\phi\) becomes subject to Lorentz transformations and the mass sign problem of the Dirac theory is produced.

The \(Cl_{3,3}\) reformulation is obtained by replacing the \(4 \times 4\) matrices \(\tilde{\gamma}\) of \(Cl_{1,3}\) with the \(8 \times 8\) matrices \(\hat{\gamma}\) of \(Cl_{3,3}\), and substituting \((5.1)\) with the expression for \(p\) given in \(\S 4\). In the case of leptons, the mass \(m\) is replaced by diagonal matrix \(M\) defined in \((4.7)\). The replacement \(p \rightarrow \hat{\gamma} \pi^6 \mathbf{D}\) then gives the free lepton wave-equation

\[\hat{\gamma} \pi^6 \mathbf{D} \Psi = \gamma^* \! M \Psi, \quad (5.5)\]

where the matrix \(\Psi\) describes all eight states of the lepton doublet. Solutions of \((5.5)\) are based on recognising that

\[\mathbf{D} = \gamma^\mu \partial_\mu = \gamma^* \! \partial_{a0}, \quad (5.6)\]

and that \(\Psi\) can be written in terms of the observer’s or lepton coordinate frames, viz.

\[\Psi = \Psi_0 \exp(\gamma \pi^6 \! p_\mu x^\mu) = \Psi_0 \exp(\gamma \pi^6 \! p_0 x^0), \quad (5.7)\]

where \(\Psi_0\) is a function of the \(p_\mu\), and \(x^0\) is the proper time, i.e. time in the lepton coordinate frame. As \(p_{a0}\) describes lepton masses, the right and left hand sides of \((5.5)\) are simply alternative ways of expressing \(\hat{\gamma} \pi^6 \mathbf{D} \Psi\). The matrix \(\hat{\gamma} \pi^6\) that appears in the exponent is Lorentz invariant, and (as will be argued in \(\S 11\)) is also invariant under space and time translations. It is interpreted as describing the lepton substrate, corresponding to the substrate of the electromagnetic field, described by \(\eta\) in \(\S 3\).

Solutions of the Dirac equation are expressed as 4-spinors, which correspond to columns in the dimensionless matrix \(\Psi_0 = M^{-1} \mathbf{p}\). As shown in Appendix B, the matrices \(\hat{\gamma} \pi^6, \hat{\gamma}^\mu\), and structors expressed in terms of them, are block diagonal. In particular,

\[\mathbf{p} = \left( \begin{array}{cc} \mathbf{p}_a & 0 \\ 0 & \mathbf{p}_b \end{array} \right), \quad (5.8)\]

where

\[\mathbf{p}_a = \left( \begin{array}{cccc} p_0 & 0 & p_2 & -p_1 - ip_3 \\ 0 & p_0 & -p_1 + ip_3 & -p_2 \\ p_1 - ip_3 & p_2 & -p_0 & 0 \\ -p_2 & p_1 + ip_3 & -p_0 & 0 \end{array} \right) = M_a \left( \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \right) \quad (5.8a)\]

and

\[\mathbf{p}_b = \left( \begin{array}{cccc} p_0 & 0 & -p_2 & -p_1 - ip_3 \\ 0 & p_0 & -p_1 + ip_3 & p_2 \\ p_1 - ip_3 & -p_2 & -p_0 & 0 \\ p_2 & p_1 + ip_3 & -p_0 & 0 \end{array} \right) = M_b \left( \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \right). \quad (5.8b)\]

Here \(a_i, b_i; \{1, 2, 3, 4\}\) denote 4-spinor columns in \(\Psi_0\). The mass matrix \(M\)

\[M = \left( \begin{array}{cc} M_a & 0 \\ 0 & M_b \end{array} \right) \text{ where } M_a = \left( \begin{array}{cc} m_e \mathbf{I} & 0 \\ 0 & m_\mu \mathbf{I} \end{array} \right), \quad M_b = \left( \begin{array}{cc} m_e \mathbf{I} & 0 \\ 0 & m_\mu \mathbf{I} \end{array} \right). \quad (5.9)\]

The mass factors in \((5.8a)\) and \((5.8b)\) make \(a_i, b_i\) dimensionless.
The difference between charged and neutral lepton masses is conventionally attributed to the Higgs field, which has the algebraic form

\[ \mathcal{H} = (m_e - m_\nu)Q^2, \]  

(5.10)

where \( m_e \gg m_\nu \). Table 5.1 compares the labelling of the eight 4-spinor solutions shown in (5.8) with the four solutions of the Dirac equation. The latter, given in [21]§4.6.2, have a similar structure to those in (5.8), but are not identical.

The representations of \( p_a \) and \( p_b \) make it apparent that they relate to different coordinate systems. In particular, \( a\gamma^2 \) and \( b\gamma^2 \) have opposite signs, as shown in Table A3, with the consequence that expressions for the 4-spinors \( \Psi_a \) and \( \Psi_b \) relate to coordinate systems with opposite parity. Block diagonalisation enables (5.5) to be expressed as two independent equations, viz.

\[ D_a \Psi_a = M_a \Psi_a, \quad D_b \Psi_b = M_b \Psi_b, \]  

(5.11)

where \( D_a \) and \( D_b \) are defined in Appendix B. As the projection operators \( P_a \) and \( P_b \) commute with the \( \hat{\gamma}^{\mu\nu} \) matrices, the components of \( \Psi_a \) and \( \Psi_b \) form 4-component column vectors and are not mixed by Lorentz transformations.

### Table 5.1: Comparison of spinor labelling in the Dirac and CU theories

|        | \( e^-\downarrow \) | \( e^-\uparrow \) | \( \bar{\nu}\downarrow \) | \( \bar{\nu}\uparrow \) | \( \nu\downarrow \) | \( \nu\uparrow \) | \( e^+\downarrow \) | \( e^+\uparrow \) |
|--------|---------------------|-------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| mass   | \( m_e \)           | \( m_e \)         | \( m_\nu \)         | \( m_\nu \)         | \( m_\nu \)         | \( m_\nu \)         | \( m_e \)           | \( m_e \)           |
| \( ABC \) | 111                  | 111               | 111                  | 111                  | 111                  | 111                  | 111                  | 111                  |
| \( Cl_{3,3} \) | \( a_1 \)          | \( a_2 \)         | \( a_3 \)           | \( a_4 \)           | \( b_1 \)           | \( b_2 \)           | \( b_3 \)           | \( b_4 \)           |
| \( \mu\pi6 \) | \( i \)              | \( i \)           | \( i \)              | \( i \)              | \( -i \)             | \( -i \)             | \( -i \)             | \( -i \)             |
| Dirac [21] | \( u_1 \)           | \( u_2 \)         |                     |                     |                     |                     | \( u_3 \equiv u_2 \) | \( u_4 \equiv u_1 \) |

The free lepton equation (5.5) can be modified to include interactions with electromagnetic fields simply by adding the field momentum contribution to the particle momentum, as is done in Lagrangian theory. For example, the term the electromagnetic contribution \( e\mathcal{Q}\mathbf{A} \) can be added to the free particle momentum \( \mathbf{p} \) to produce the generalized, or canonical, momentum

\[ \mathbf{p}' = \mathbf{p} + e\mathcal{Q}\mathbf{A}. \]  

(5.12)

With this modification, (5.5) becomes

\[ p'\Psi \to \hat{\gamma}^{\pi6}D\Psi = \hat{\gamma}^{\pi6}\hat{\gamma}^\mu\partial_\mu\Psi = (\mathbf{M} + e\mathcal{Q}\mathbf{A})\Psi. \]  

(5.13)

The factor \( \mathbf{M} + e\mathcal{Q}\mathbf{A} \) can be brought down from the exponent by writing

\[ \Psi' = \Psi_0 \exp(\hat{\gamma}^{\pi6}\int p'_\mu dx^\mu), \]  

(5.14)

where the exponent is a line integral, with

\[ \mathbf{p}' = \hat{\gamma}^{\pi6}\hat{\gamma}^\mu p'_\mu = \mathbf{M} + e\mathcal{Q}\mathbf{A}, \]  

(5.15)

where \( \mathbf{A} = A_\mu\hat{\gamma}^\mu \), \( \mathbf{M} = \mathcal{H} + m_\nu \mathbf{1}_3 \) and \( \mathcal{Q}\mathbf{A} = -\frac{1}{2}(A_\mu\hat{\gamma}^\mu)(\hat{\gamma}^{\pi0} + i\hat{\gamma}^{\pi6}) \), reducing the relativistic wave-equation to

\[ \hat{\gamma}^{\pi6}D\Psi' = \hat{\gamma}^\mu p_\mu \Psi' = \mathbf{p}\Psi'. \]  

(5.16)

This formulation shows how the algebraic description of the physical substrate, modulated by the wave motion, can be incorporated into the lepton wave-equation. It should be possible to incorporate interactions with other gauge fields in the same way, but this remains to be investigated.
§6. Reformulation of the electro-weak interaction

Ψ, defined in §5, describes all eight lepton states, labelled a_i, b_i, \{i = 1, 2, 3, 4\}, and defined in Table 5.1. It follows that the Cl_{3,3} algebra must contain a description of the weak interaction that couples electron and neutrino states. Its Standard Model form is

\[ X_\mu(W) = \frac{g_W}{2} W_\mu = \frac{i g_W}{2} (\sigma_1 W^1_\mu + \sigma_2 W^2_\mu + \sigma_3 W^3_\mu), \]

(6.1)

where \( g_W \) is the (real) coupling coefficient of leptons to the weak field potential, \( \sigma_k \) are the Pauli matrices (see Appendix A), and the \( W^k_\mu \) (k = 1, 2, 3) are 4-vector potential functions.

The Cl_{3,3} reformulation is obtained by replacing

\[ i\sigma_1 = iQ \rightarrow \gamma^{(1)}, \ i\sigma_2 = P \rightarrow \gamma^{(2)}, \ i\sigma_3 = -iR \rightarrow \gamma^{(3)}, \]

(6.2)

where \( \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)} \) are anti-commuting elements of Cl_{3,3} that satisfy \( \gamma^{(1)}\gamma^{(2)} = \gamma^{(3)} \) and \( (\gamma^{(1)})^2 = (\gamma^{(2)})^2 = (\gamma^{(3)})^2 = -1_3 \). As \( \gamma^{\pi 6} \) takes eigenvalues for all lepton states it must correspond to the diagonal Pauli matrix \( \sigma_3 \), giving

\[ \gamma^{(3)} \equiv -\gamma^{\pi 0} = -I \otimes I \otimes R. \]

(6.3)

\( \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)} \) are generators of SU(2) and must satisfy the Coleman-Mandula condition that they commute with the matrices that define the physical coordinate frame. The SM choice corresponds to identifying

\[ \gamma^{(1)} \equiv i\gamma^{28} = iI \otimes I \otimes Q, \ \gamma^{(2)} \equiv \gamma^{27} = -I \otimes I \otimes P. \]

(6.4)

These matrices do not commute with \( \gamma^2 \), but they do commute with the physical coordinate frames \( a^a \gamma^\mu \) and \( b^b \gamma^\mu \). Given that the physical coordinate frame for all fermions is described by \( \gamma^\mu \), the Coleman-Mandula condition requires

\[ \gamma^{(1)} \equiv i\gamma^{\pi 8} = R \otimes R \otimes Q, \ \gamma^{(2)} \equiv \gamma^{\pi 7} = R \otimes R \otimes P. \]

(6.5)

This gives raising and lowering operators that describe charged weak bosons as

\[ \gamma^+ = -\frac{1}{2} (\gamma^{\pi 7} + i\gamma^{\pi 8}) = \frac{1}{2} R \otimes R \otimes (P + Q) = \begin{pmatrix} 0 & 0 \\ R \otimes R & 0 \end{pmatrix}, \]
\[ \gamma^- = \frac{1}{2} (\gamma^{\pi 7} - i\gamma^{\pi 8}) = \frac{1}{2} R \otimes R \otimes (Q - P) = \begin{pmatrix} 0 & R \otimes R \\ 0 & 0 \end{pmatrix}. \]

(6.6)

These operators satisfy

\[ \gamma^- \gamma^- = 0, \ \gamma^+ \gamma^+ = 0, \]
\[ \gamma^- \gamma^+ + \gamma^+ \gamma^- = I_3, \ \gamma^+ \gamma^+ - \gamma^- \gamma^- = i\gamma^{\pi 6}, \]
\[ \gamma^- \gamma^{\pi 6} + \gamma^{\pi 6} \gamma^- = 0, \ \gamma^+ \gamma^{\pi 6} + \gamma^{\pi 6} \gamma^+ = 0. \]

(6.7)

Defining \( W^+ = W^1_\kappa - iW^2_\kappa, \ W^- = W^1_\kappa + iW^2_\kappa \), the weak potential can be expressed as

\[ W = \gamma^a W_a = W^+ \gamma^+ + W^- \gamma^- + W^3 \gamma^{\pi 6}, \]

(6.8)

giving the weak interaction

\[ X_\mu(W) = \frac{g_W}{2} W_\mu = \frac{g_W}{2} (\gamma^{28} W^1_\mu + \gamma^{27} W^2_\mu + \gamma^{\pi 6} W^3_\mu) = \frac{g_W}{2} (\gamma^+ W^+_\mu + \gamma^- W^-_\mu + \gamma^{\pi 6} W^3_\mu). \]

(6.9)

The action of \( \gamma^\pm \) on \( p \) gives

\[ \gamma^\pm p = \begin{pmatrix} 0 & 0 \\ p_+ & 0 \end{pmatrix}, \ \gamma^- p = \begin{pmatrix} 0 & P_b \\ 0 & 0 \end{pmatrix}. \]

(6.10)

The physical interpretation of these equations is that the positively charged boson \( \gamma^+ \) adds a charge to fermions with negative or zero charges, for example converting \( e^- \rightarrow \nu \) and \( \nu \rightarrow e^+ \); similarly \( \gamma^- \) subtracts a charge, so that \( e^+ \rightarrow \nu \) and \( \nu \rightarrow e^- \). The \( a^a \gamma^\mu \) coordinate frame is relevant to the top row of \( p \), while the \( b^b \gamma^\mu \)
coordinate frame is relevant to the bottom row. Hence both $\hat{\gamma}^+$ and $\hat{\gamma}^-$ change the parity of the coordinate frame, in agreement with the observed parity change produced by the weak interaction. This replaces the SM explanation of the parity change being a consequence of a ‘V-A’ potential produced by chirality.

The separation of electro-magnetic and weak interactions is achieved by ensuring that their matrix expressions are linearly independent. Following the SM argument this involves the introduction of a potential $B$ giving, in terms of the linearly independent matrices $\hat{\gamma}^6$ and $\hat{\gamma}^0$, the neutral electro-weak component

$$X^3 = \frac{g_W}{2} W^3 i \hat{\gamma}^6 - \frac{g'}{2} B \hat{\gamma}^0. \quad (6.11)$$

The linearly independent potentials $B$ and $W^3$ can be expressed as rotations through the weak mixing angle $\theta$ of the observable electromagnetic and weak potentials $A$ and $Z$, viz.

$$\begin{pmatrix} W^3 \\ B \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix}. \quad (6.12)$$

Substituting (6.12) into (6.11) gives

$$X^3 = \frac{g_W}{2} (Z \cos \theta + A \sin \theta) i \hat{\gamma}^6 - \frac{g'}{2} (-Z \sin \theta + A \cos \theta) \hat{\gamma}^0. \quad (6.13)$$

Comparing coefficients of $\hat{\gamma}^0$ and $i \hat{\gamma}^6$ in (6.11) with those for the electromagnetic interaction, given in (4.5),

$$e = g_W \sin \theta = g' \cos \theta, \quad \text{and} \quad \tan \theta = \frac{g'}{g_W}. \quad (6.14)$$

These are the same expressions for the weak mixing angle $\theta$ as are obtained the SM, but do not involve chirality, making the above derivation much simpler than that in the SM (e.g. see [2], pp.418-421). The neutral component of the weak interaction is therefore

$$X^3(weak) = \frac{1}{2} (-g' \sin \theta \hat{\gamma}^0 + g_W \cos \theta i \hat{\gamma}^6) Z. \quad (6.15)$$
§7. Physical interpretation of \( Cl_{5,5}(LQ) \)

The \( 32 \times 32 \) \( \Gamma \)-matrix representations of the ten anti-commuting generators of the lepton/quark algebra \( Cl_{5,5}(LQ) \) are constructed by inserting the anti-commuting elements \( I \otimes P, P \otimes R, I \otimes Q, Q \otimes R \) of the \( Cl_{1,1}(5) \otimes Cl_{1,1}(4) \) algebra in front of the generators of \( Cl_{3,3}(L) \) defined in Table A2, to give

\[
\Gamma^1 = I \otimes I \otimes \gamma^1 = I \otimes I \otimes Q \otimes P \otimes I \\
\Gamma^2 = I \otimes I \otimes \gamma^2 = -I \otimes I \otimes R \otimes P \otimes R \rightarrow -I \otimes I \otimes P \\
\Gamma^3 = I \otimes I \otimes \gamma^3 = -iI \otimes I \otimes P \otimes P \otimes I, \\
\Gamma^4 = I \otimes P \otimes \gamma^6 = I \otimes P \otimes I \otimes Q \otimes I \rightarrow I \otimes P \otimes Q, \\
\Gamma^5 = P \otimes R \otimes \gamma^6 = P \otimes R \otimes I \otimes Q \otimes I \rightarrow P \otimes R \otimes Q, \\
\Gamma^6 = R \otimes R \otimes \gamma^6 = R \otimes R \otimes I \otimes Q \otimes I \rightarrow R \otimes R \otimes Q, \\
\Gamma^7 = I \otimes I \otimes \gamma^7 = iI \otimes I \otimes R \otimes P \otimes Q, \\
\Gamma^8 = I \otimes I \otimes \gamma^8 = I \otimes I \otimes R \otimes P \otimes P, \\
\Gamma^9 = I \otimes Q \otimes \gamma^6 = I \otimes Q \otimes I \otimes Q \otimes I \rightarrow I \otimes Q \otimes Q, \\
\Gamma^{10} = Q \otimes R \otimes \gamma^6 = Q \otimes R \otimes I \otimes Q \otimes I \rightarrow Q \otimes R \otimes Q.
\]

(7.1)

The time direction \( \Gamma^0 \) is again defined as the product of all the generators, viz.

\[
\Gamma^0 = \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9 \Gamma^{10} = I \otimes I \otimes \gamma^0 = -I \otimes I \otimes R \otimes I \rightarrow -I \otimes I \otimes R.
\]

(7.2)

\( \Gamma^4 \) and \( \Gamma^5 \) are not observed spatial dimensions, so the space-time volume \( \Gamma^x \) is

\[
\Gamma^x = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 = I \otimes I \otimes \gamma^x.
\]

(7.3)

The \( 32 \times 32 \) matrix representation of \( Cl_{5,5}(LQ) \) distinguishes the \( 2^5 = 32 \) quarks and leptons in the first generation in terms of the five binary quantum numbers \( \mu_A, \mu_B, \mu_C, \mu_D, \mu_E \), where the first three were defined in §3. Their corresponding \( \Gamma \)-matrices are

\[
\Gamma^A = \Gamma^{31} = I \otimes I \otimes \gamma^{31}, \quad \Gamma^B = \Gamma^0 = I \otimes I \otimes \gamma^0, \quad \Gamma^C = \Gamma^{56} = \Gamma^{87} = I \otimes I \otimes \gamma^{56}.
\]

(7.4)

The three factor matrices following \( \rightarrow \) in (7.1) and (7.2) correspond to the first, second and fourth factors in the generator matrices, and generate a real \( 8 \times 8 \) matrix representation of the \( Cl_{3,3}(Q) \) sub-algebra of \( Cl_{5,5}(LQ) \). Writing the generators of \( Cl_{3,3}(Q) \) as \( \gamma \)-matrices

\[
\gamma^2 = -I \otimes I \otimes P, \quad \gamma^4 = I \otimes P \otimes Q, \quad \gamma^5 = P \otimes R \otimes Q, \\
\gamma^6 = R \otimes R \otimes Q, \quad \gamma^9 = I \otimes Q \otimes Q, \quad \gamma^{10} = Q \otimes R \otimes Q.
\]

(7.5)

There are two ways to construct additional commutting elements \( \Gamma^x, \Gamma^y \) in the \( Cl_{1,1}(5) \otimes Cl_{1,1}(4) \) algebra, viz.

(i) \( \gamma^x = I \otimes R \otimes I = (I \otimes P \otimes Q)(I \otimes Q \otimes Q) = \gamma^{49} \),

\[
\gamma^y = R \otimes I \otimes I = (P \otimes R \otimes Q)(Q \otimes R \otimes Q) = \gamma^{5,10}.
\]

(7.6)

and

(ii) \( \gamma^x = P \otimes Q \otimes I = (I \otimes P \otimes Q)(P \otimes R \otimes Q) = \gamma^{45} \),

\[
\gamma^y = Q \otimes P \otimes I = (Q \otimes R \otimes Q)(I \otimes Q \otimes Q) = \gamma^{9,10}.
\]

(7.7)

Model (i) is adopted because it has diagonal matrix representations. The product of all six generators of \( Cl_{3,3}(Q) \) gives the time direction \( \gamma^0 = -I \otimes I \otimes R \) identified in (7.2). The matrices \( \Gamma^D \) and \( \Gamma^E \) correspond to \( \gamma^x \) and \( \gamma^y \) respectively, viz.

\[
\Gamma^D = \Gamma^4 \Gamma^9 = \Gamma^{4,9} = I \otimes R \otimes I \otimes I \rightarrow I \otimes R \otimes I, \\
\Gamma^E = \Gamma^5 \Gamma^{10} = \Gamma^{5,10} = R \otimes I \otimes I \otimes I \rightarrow R \otimes I \otimes I.
\]

(7.8)
Table 7.1 distinguishes leptons and quarks in terms of the new primary quantum numbers \((\mu_D, \mu_E)\) and \(\mu_X = -\mu_D\mu_E\mu_B\). Fermion charges in this table are calculated using

\[
\mu_Q = \frac{1}{6}(\mu_D + \mu_E - \mu_D\mu_E\mu_B) - \frac{1}{2}\mu_C, \tag{7.9}
\]

obtained by replacing \(-\mu_B\) in (4.2) with \(\frac{1}{3}(\mu_D + \mu_E + \mu_X) = \frac{1}{3}(\mu_D + \mu_E - \mu_D\mu_E\mu_B)\).

| \(\mu_C\) | \(\mu_D\) | \(\mu_E\) | \(\mu_X\) | \(Q\) | fermion |
|-----|-----|-----|-----|-----|-----|
| 1   | -1  | -1  | -1  | 0   | \(\mu\) |
| 1   | -1  | 1   | 1   | 2/3 | \(u_g\) |
| 1   | 1   | -1  | 1   | 2/3 | \(u_r\) |
| 1   | 1   | 1   | -1  | 2/3 | \(u_b\) |
| 1   | -1  | -1  | -1  | -1  | \(e^-\) |
| 1   | -1  | 1   | 1   | -1/3 | \(d_{g}\) |
| 1   | 1   | -1  | 1   | -1/3 | \(d_{r}\) |
| 1   | 1   | 1   | -1  | -1/3 | \(d_{b}\) |

The operators \(\Gamma^B, \Gamma^D, \Gamma^E, \Gamma^X\) have diagonal representations corresponding to the entries in Table 7.1, giving

\[
\Gamma^A = I \otimes I \otimes R \otimes I \otimes I = I_2 \otimes \hat{\gamma}^A, \\
\Gamma^C = I \otimes I \otimes I \otimes I \otimes R = I_2 \otimes \hat{\gamma}^C, \\
\Gamma^0 = \Gamma^B = -I \otimes I \otimes I \otimes R \otimes I \rightarrow \hat{\gamma}^0 = -I \otimes I \otimes R \equiv diag(1111; 1111), \\
\Gamma^D = I \otimes R \otimes I \otimes I \otimes I \rightarrow \hat{\gamma}^x = I \otimes R \otimes I \equiv diag(1111; 1111), \\
\Gamma^E = R \otimes I \otimes I \otimes I \otimes I \rightarrow \hat{\gamma}^x = R \otimes I \otimes I \equiv diag(1111; 1111), \\
-\Gamma^E \Gamma^D \Gamma^B = \Gamma^X = -R \otimes R \otimes I \otimes R \otimes I \rightarrow -R \otimes R \otimes R = -\hat{\gamma}^x \hat{\gamma}^y \hat{\gamma}^0 \equiv diag(1111; 1111), \tag{7.10}
\]

where the triple Kronecker products are commuting elements of \(Cl_{3,3}(Q)\). The charge operator corresponding to (7.9) is

\[
Q = \frac{1}{6}(\Gamma^X + \Gamma^D + \Gamma^E) - \frac{1}{2}\Gamma^C. \tag{7.11}
\]

The standard \(3 \times 3\) Gell-Mann matrix form of the generators of the \(SU(3)_{\text{strong}}\) group are obtained by deleting first column and top row in eight of the fifteen \(4 \times 4\) matrices that comprise the generators of the
Lie algebra SU(4). Expressing these matrices in terms of \( \textbf{P}, \textbf{Q}, \textbf{R} \) gives

\[
\bar{\lambda}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(\textbf{Q} \otimes \textbf{Q} - \textbf{P} \otimes \textbf{P}), \quad \bar{\lambda}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2}(\textbf{Q} \otimes \textbf{P} - \textbf{P} \otimes \textbf{Q}),
\]

\[
\bar{\lambda}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(\textbf{R} \otimes \textbf{I} - \textbf{I} \otimes \textbf{R}), \quad \bar{\lambda}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \frac{1}{2}(\textbf{I} + \textbf{R}) \otimes \textbf{Q},
\]

\[
\bar{\lambda}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} = \frac{i}{2}(\textbf{I} + \textbf{R}) \otimes \textbf{P}, \quad \bar{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{1}{2}\textbf{Q} \otimes (\textbf{I} + \textbf{R}),
\]

\[
\bar{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \frac{i}{2}\textbf{P} \otimes (\textbf{I} + \textbf{R}), \quad \sqrt{3}\bar{\lambda}_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = -\frac{1}{2}(2\textbf{R} \otimes \textbf{R} + \textbf{I} \otimes \textbf{R} + \textbf{R} \otimes \textbf{I}).
\]

The \( \bar{\lambda}_i \) act upon the 4-fermion column matrices shown in Table 7.1, showing that the gluons do not interact with leptons. However, gluons do act upon anti-quarks, so their algebraic representation as operators that act on both quarks and anti-quarks must be expressed in terms of the \( 8 \times 8 \) matrices \( \lambda_i = \bar{\lambda}_i \otimes \textbf{I}, \ i = 1, ..., 8 \). In particular, the commuting operators \( \lambda_3, \lambda_8 \) are related to the commuting elements of \( \text{Cl}_{5,5}(LQ) \) and its sub-algebra \( \text{Cl}_{3,3}(Q) \) by

\[
2\lambda_3 \otimes \textbf{I} = (\textbf{R} \otimes \textbf{I} \otimes \textbf{I} \otimes \textbf{R} \otimes \textbf{I}) = \gamma^y - \gamma^x,
\]
\[
2\sqrt{3}\lambda_8 \otimes \textbf{I} = - (2\textbf{R} \otimes \textbf{R} \otimes \textbf{I} + \textbf{I} \otimes \textbf{R} \otimes \textbf{I} + \textbf{R} \otimes \textbf{I} \otimes \textbf{I}) = -(2\gamma^x\gamma^y + \gamma^x + \gamma^y).
\]

The model (i) analysis given above reproduces the known properties of quarks and gluons as described by the Standard Model. It does not introduce the five dimensional space suggested by the \( \text{Cl}_{5,5}(LQ) \) algebra. As individual quarks and gluons have never been observed in 3-d space, the extra two spatial dimensions must relate to a gluon substrate that only exists inside hadrons. As gluons interact strongly within hadrons it is reasonable to suppose that they form a coherent jelly-like substrate. This is transparent to leptons, which have no colour charge. This model would explain the strength of long range quark/quark interactions within the jelly and why individual quarks are never observed in 3-d space. It also suggests that quark/quark interactions could be expressed in terms of quark-jelly interactions, with the jelly adding effective mass to the quarks.
§8. Physical interpretation of $Cl_{7,7}$

The extension of the ten generators of $Cl_{5,5}(LQ)$, defined in (7.1), to the fourteen anti-commuting generators of $Cl_{7,7}$ follows the same pattern used to extend the $Cl_{3,3}(L)$ algebra to $Cl_{5,5}(LQ)$ in §7, viz.

\[ \begin{align*}
\bar{\Gamma}^1 &= I \otimes I \otimes \Gamma^1 = I \otimes I \otimes I \otimes Q \otimes P \otimes I, \\
\bar{\Gamma}^6 &= R \otimes R \otimes \bar{\Gamma}^6 = R \otimes R \otimes R \otimes R \otimes I \otimes Q \otimes I, \\
\bar{\Gamma}^2 &= I \otimes I \otimes \Gamma^2 = -I \otimes I \otimes I \otimes R \otimes P \otimes R, \\
\bar{\Gamma}^7 &= I \otimes I \otimes \bar{\Gamma}^7 = I \otimes I \otimes I \otimes R \otimes P \otimes Q, \\
\bar{\Gamma}^3 &= I \otimes I \otimes \Gamma^3 = -I \otimes I \otimes I \otimes P \otimes P \otimes I, \\
\bar{\Gamma}^8 &= I \otimes I \otimes \bar{\Gamma}^8 = I \otimes I \otimes I \otimes R \otimes P \otimes P, \\
\bar{\Gamma}^4 &= R \otimes R \otimes \bar{\Gamma}^4 = R \otimes R \otimes I \otimes P \otimes Q \otimes I, \\
\bar{\Gamma}^9 &= R \otimes R \otimes \bar{\Gamma}^9 = R \otimes R \otimes I \otimes Q \otimes Q, \\
\bar{\Gamma}^5 &= R \otimes R \otimes \bar{\Gamma}^5 = R \otimes R \otimes P \otimes R \otimes I \otimes Q \otimes I, \quad \bar{\Gamma}^{10} = R \otimes R \otimes \bar{\Gamma}^{10} = R \otimes R \otimes Q \otimes R \otimes I \otimes Q \otimes I, \\
\bar{\Gamma}^a &= I \otimes P \otimes I \otimes I \otimes I \otimes Q \otimes I, \\
\bar{\Gamma}^b &= P \otimes R \otimes I \otimes I \otimes I \otimes Q \otimes I, \quad \bar{\Gamma}^d = Q \otimes R \otimes I \otimes I \otimes I \otimes Q \otimes I. \\
\end{align*} \]

The product of all fourteen generators of $Cl_{7,7}$ gives an expression for unit time intervals consistent with that previously identified for its sub-algebras $Cl_{3,3}(L)$ and $Cl_{5,5}(LQ)$, i.e.

\[ \bar{\Gamma}^0 = \bar{\Gamma}^1 \bar{\Gamma}^2 ... \bar{\Gamma}^5 \bar{\Gamma}^3 = -I \otimes I \otimes I \otimes I \otimes R \otimes I = 1_2 \otimes \bar{\Gamma}^0 = 1_4 \otimes \bar{\Gamma}^0, \]  
and $\bar{\Gamma}^\pi$ is defined as

\[ \bar{\Gamma}^\pi = \bar{\Gamma}^0 \bar{\Gamma}^1 \bar{\Gamma}^2 \bar{\Gamma}^3 = I \otimes I \otimes I \otimes I \otimes \bar{\Gamma}^\gamma = iI \otimes I \otimes I \otimes I \otimes Q \otimes R. \]

The five quantum numbers already identified in the analysis of the sub-algebra $Cl_{5,5}(LQ)$ correspond to the $\bar{\Gamma}$ matrices

\[ \begin{align*}
\bar{\Gamma}^A &= I \otimes I \otimes \Gamma^A = I \otimes I \otimes I \otimes R \otimes I \otimes I = 1_4 \otimes \bar{\Gamma}^A, \\
\bar{\Gamma}^C &= I \otimes I \otimes \Gamma^C = I \otimes I \otimes I \otimes I \otimes I \otimes R = 1_4 \otimes \bar{\Gamma}^C, \\
\bar{\Gamma}^B &= I \otimes I \otimes \Gamma^B = -I \otimes I \otimes I \otimes I \otimes R \otimes I = 1_4 \otimes \bar{\Gamma}^B, \\
\bar{\Gamma}^D &= I \otimes I \otimes \Gamma^D = -I \otimes I \otimes R \otimes R \otimes I \otimes I \otimes I \otimes R = 1_2 \otimes R \otimes R \otimes \bar{\Gamma}^B, \\
\bar{\Gamma}^E &= I \otimes I \otimes \Gamma^E = I \otimes I \otimes R \otimes I \otimes I \otimes R \otimes I = 1_2 \otimes R \otimes I \otimes \bar{\Gamma}^B. \\
\end{align*} \]

The $Cl_{3,3}(G)$ sub-algebra has generators defined by the first two and seventh factors of the corresponding $Cl_{7,7}$ generators, i.e.

\[ \begin{align*}
\bar{\gamma}^a &= I \otimes P \otimes Q, \quad \bar{\gamma}^b = P \otimes R \otimes Q, \quad \bar{\gamma}^c = I \otimes Q \otimes Q, \\
\bar{\gamma}^d &= Q \otimes R \otimes Q, \quad \bar{\gamma}^2 = I \otimes I \otimes P, \quad \bar{\gamma}^6 = R \otimes R \otimes Q. \\
\end{align*} \]

The commuting elements of $Cl_{3,3}(G)$ are the diagonal matrices

\[ \begin{align*}
\bar{\gamma}^F &= \bar{\gamma}^{ac} = I \otimes R \otimes I, \\
\bar{\gamma}^G &= \bar{\gamma}^{bd} = R \otimes I \otimes I, \\
\bar{\gamma}^H &= -R \otimes R \otimes R, \end{align*} \]

where $\bar{\gamma}^H = -\bar{\gamma}^F \bar{\gamma}^G \bar{\gamma}^C$. These determine the remaining two commuting elements of $Cl_{7,7}$, viz.

\[ \begin{align*}
\bar{\Gamma}^F &= I \otimes R \otimes I \otimes I \otimes I \otimes I \otimes I, \\
\bar{\Gamma}^G &= R \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I. \\
\end{align*} \]

The above equations are almost identical to those for the $\bar{\gamma}$ matrices given in §7, showing the description of generations to have the same pattern as that of leptons and quarks, with correspondences $F \leftrightarrow D$, $G \leftrightarrow E$, $C \leftrightarrow B$, $H \leftrightarrow X$.

The quantum numbers used to construct Tables 8.1 and 8.2 are $\mu_F = \mu_{ac}$, $\mu_G = \mu_{bd}$ and $\mu_H = -\mu_{c} \mu_{f} \mu_{g}$. As corresponding anti-fermions have opposite signs of all these quantum numbers, they are omitted from these tables. A single expression for the charges on the first three (observed) generations is
only obtained if the $\mu_F$ and $\mu_G$ quantum numbers are parity dependent. This means that the algebraic structure of the weak interaction that was derived in §6 needs further elaboration, but this will not be followed up in this work.

Table (8.1): Quantum numbers for lepton generations ($\mu_B = 1$)

| $\mu_C$ | $\mu_F$ | $\mu_G$ | $\mu_H$ | $Q$ | lepton |
|---------|---------|---------|---------|-----|--------|
| $-1$    | $1$     | $1$     | $1$     | $-2$| $l^-_2$|
| $-1$    | $-1$    | $-1$    | $1$     | $0$ | $\nu_e$|
| $-1$    | $-1$    | $1$     | $-1$    | $0$ | $\nu_\mu$|
| $-1$    | $1$     | $-1$    | $-1$    | $0$ | $\nu_\tau$|

| $1$     | $-1$    | $-1$    | $-1$    | $1$ | $l^+_1$|
| $1$     | $1$     | $1$     | $-1$    | $-1$| $e^-$  |
| $1$     | $1$     | $-1$    | $1$     | $-1$| $\mu^-$|
| $1$     | $-1$    | $1$     | $1$     | $-1$| $\tau^-$|

Table (8.2): Quantum numbers for b quark generations ($\mu_B = 1$)

| $\mu_C$ | $\mu_F$ | $\mu_G$ | $\mu_H$ | $Q$ | fermion |
|---------|---------|---------|---------|-----|---------|
| $-1$    | $1$     | $1$     | $1$     | $-4/3$| $q^{-4/3}$|
| $-1$    | $-1$    | $-1$    | $1$     | $2/3$ | $u$     |
| $-1$    | $-1$    | $1$     | $-1$    | $2/3$ | $c$     |
| $-1$    | $1$     | $-1$    | $-1$    | $2/3$ | $t$     |

| $1$     | $-1$    | $-1$    | $1$     | $5/3$ | $q^{5/3}$|
| $1$     | $1$     | $1$     | $1$     | $-1/3$| $d$     |
| $1$     | $1$     | $-1$    | $-1$    | $-1/3$| $s$     |
| $1$     | $-1$    | $1$     | $-1$    | $-1/3$| $b$     |

Electric charges are determined, again in analogy with §7, by substituting the expression $(\mu_F + \mu_G + \mu_H)$ for $\mu_C$ in (7.11), giving the charges on all fermions as

$$\mu_Q = \frac{1}{6}(\mu_X + \mu_D + \mu_E) - \frac{1}{2}(\mu_H + \mu_F + \mu_G).$$

(8.8)

The corresponding charge operator expression is

$$Q = \frac{1}{6}(\bar{\Gamma}^X + \bar{\Gamma}^D + \bar{\Gamma}^E) - \frac{1}{2}(\bar{\Gamma}^F + \bar{\Gamma}^G + \bar{\Gamma}^H).$$

(8.9)

These formulae give the same charges on fermions in all three known generations, as observed, but predicts different charges on fermions in the predicted, but presently unobserved, fourth generation. In particular, Table 8.1 shows that fourth generation leptons carry either two negative charges or a single positive charge. Crucially, this generation has no neutrinos, in accord with the experimental evidence that only three types of neutrino exist.

All four generations have fermion doublets and there is good experimental evidence showing that weak interactions relating the two fermion components of a given doublet are the same for all the three known
generations, providing the origin of the mass differences between their components. Additional bosons might 
produce an SU(3)\textsubscript{generation} gauge field, related to Cl\textsubscript{3,3}(G) in the same way that SU(3)\textsubscript{strong} is related to 
Cl\textsubscript{3,3}(Q). The two commuting elements of its Lie algebra are provided by linear combinations of $\gamma^F$, $\gamma^G$ 
and $\gamma^H$. The eight bosons defined by this field would be neutral and possibly massive, but given that the 
dominant contribution to electron mass is due to the Higgs boson, it is more likely that the masses of the 
second and third generation fermions arise from a similar mechanism. A second reason for thinking that 
SU(3)\textsubscript{generation} bosons are light is that they interact with neutrinos, possibly providing their very small 
masses.

Experimental evidence for interactions between quarks, other than that produced by gluons, is provided 
by the approximate SU(3)\textsubscript{flavour} symmetry associated with the quark triplet (u, d, s), which provides a 
qualitative explanation of the baryon and meson mass spectra. As this has already been studied in great 
detail (e.g. see Chapter 9 of [21]) it is only necessary to relate the existing formalism to the Cl\textsubscript{7,7} algebra. 
Reference to Table 8.2 shows that $\mu_F = -\mu_G$ and $\mu_H = \mu_B$ for the four quarks (c, u, d, s), so that 
quantum numbers $\mu_F$, $\mu_H$ and $\mu_C$ are sufficient to distinguish these quarks and their anti-quarks. Quark 
charges are related to the isospin and and hypercharge quantum numbers (given in [1], page 389) using the 
Gell-Mann-Nishima formula, viz. $\mu_Q = I_3 + Y/2$.

| $\mu_F = -\mu_G$ | $\mu_H = \mu_B$ | $\mu_C$ | $I_3$ | $Y$ | $\mu_Q$ | quark |
|-------------------|-------------------|---------|------|-----|--------|------|
| 1                 | 1                 | -1      | 0    | 4/3 | 2/3    | c    |
| -1                | 1                 | -1      | 1/2  | 1/3 | 2/3    | u    |
| -1                | 1                 | 1       | -1/2 | 1/3 | -1/3   | d    |
| 1                 | 1                 | 1       | 0    | -2/3| -1/3   | s    |
| -1                | -1                | 1       | 0    | -4/3| -2/3   | \bar{c}|
| 1                 | -1                | 1       | -1/2 | -1/3| -2/3   | \bar{u}|
| 1                 | -1                | -1      | 1/2  | -1/3| 1/3    | \bar{d}|
| -1                | -1                | -1      | 0    | 2/3 | 1/3    | \bar{s}|

The algebraic relationship between fermions in the first three generations and fermions in the fourth 
generation has been shown to be analogous to the relationship between quarks and leptons. This suggests 
that this distinction is related to wave-function substrates, and that the gauge field that produces mass 
differences in the first three generations does not act on fourth generation fermions. Pressing the analogy 
further suggests that large regions of space cannot be occupied by fermions in the first three generations. 
Stability, and lack of interactions, makes fourth generation fermions possible candidates for producing the 
constituents of dark matter. This accords with the fact that dark matter has only been observed through 
its gravitational effects, suggesting that it mostly consists of separate, electrically neutral, fourth generation 
composites that will be described elsewhere.
§9. General relativity

The algebraic formalism for general relativity is obtained by generalising the Minkowski coordinates $E_\mu$, which are the same at all points of space and time, to the Riemannian coordinates $\mathcal{E}_\mu$, which are subject to continuous variations. The generalization of the Clifford algebra to allow for the space-time dependence of the $\mathcal{E}_\mu$ was shown in [24] to lead to Einstein’s field equations, but this result has not previously been related to the analysis in §3, as is done below.

The algebraic expression for the Riemannian metric tensor is

$$\mathcal{E}_\mu \mathcal{E}_\nu + \mathcal{E}_\nu \mathcal{E}_\mu = 2g_{\mu\nu},$$

(9.1)

with the usual relation between covariant and contravariant suffixes, i.e. $\mathcal{E}_\nu = g_{\mu\nu} \mathcal{E}^\mu$. As (9.1) is isomorphic to (3.1), relationships between the $\mathcal{E}_\nu$ are isomorphic to those given §3 for the $E_\nu$. For example, following (3.6), the 4-volume element is given by

$$\mathcal{E}_\tau = \frac{1}{4!} \epsilon^{\mu\nu\rho\tau} \mathcal{E}_\mu \mathcal{E}_\nu \mathcal{E}_\rho \mathcal{E}_\tau,$$

(9.2)

so that $(\mathcal{E}_\tau)^2 = g$ is the determinant of the 4×4 matrix of the $g_{\mu\nu}$. Defining $\mathcal{E}^{\mu\nu} = \frac{1}{2}(\mathcal{E}^\nu \mathcal{E}^\kappa - \mathcal{E}^\kappa \mathcal{E}^\nu)$, gives a closure relation isomorphic to (3.10), viz.

$$\mathcal{E}^\mu \mathcal{E}^{\nu\kappa} = \epsilon^{\mu\nu\rho\tau} \mathcal{E}_\kappa \mathcal{E}_\tau + g^{\mu\nu} \mathcal{E}^\kappa - g^{\mu\kappa} \mathcal{E}^\nu,$$

(9.3)

The space-time dependence of the $\mathcal{E}_\mu$ is given by

$$\partial_\mu \mathcal{E}_\nu = \Gamma^\tau_{\mu\nu} \mathcal{E}_\tau, \quad \partial_\mu \mathcal{E}_\tau = -\Gamma^\tau_{\mu\nu} \mathcal{E}_\nu, \quad \partial_\mu \mathcal{E}_\pi = \Gamma^\kappa_{\mu\pi} \mathcal{E}_\kappa,$$

(9.4)

where $\Gamma^\tau_{\mu\nu} = \frac{1}{2}g^{\tau\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$, as usual. Particle displacements in space-time take the same form as they do in the Minkowski metric (3.1), i.e.

$$dx = \mathcal{E}_{\mu}dx^\mu, \quad \mu = 0, 1, 2, 3$$

so that $\mathcal{E}_{\mu} = \mathcal{E}_\mu \frac{dx^\mu}{ds}$ and $(dx)^2 = (\mathcal{E}_{\mu}ds)^2 = (ds)^2$.

(9.5)

In this equation space-time particle displacements are denoted $ds$, following the standard notation in relativity theory, rather than $dx^{a0}$. The star notation for unit time intervals is the same as that used in (3.17), viz. $\mathcal{E}_{\mu}$. Non-interacting particles follow geodesic paths that satisfy

$$\frac{d\mathcal{E}_{\mu}}{ds} = \frac{d}{ds}(\mathcal{E}_\mu \frac{dx^\mu}{ds}) = \mathcal{E}_\mu \frac{d^2x^\mu}{ds^2} + \frac{d\mathcal{E}_\mu}{ds} \frac{dx^\mu}{ds}$$

$$= \mathcal{E}_\mu \frac{d^2x^\mu}{ds^2} + \frac{d\mathcal{E}_\mu}{ds} \frac{dx^\mu}{ds} \Gamma^\tau_{\mu\nu} \mathcal{E}_\tau$$

(9.6)

$$= \mathcal{E}_\tau \left( \frac{d^2x^\tau}{ds^2} + \frac{d\mathcal{E}_\mu}{ds} \frac{dx^\mu}{ds} \Gamma^\tau_{\mu\nu} \right) = 0,$$

where the coefficients of $\mathcal{E}_\tau$ provide the usual tensor expression. Differentiating the structor $A = A_\mu \mathcal{E}^\mu = A^\nu \mathcal{E}_\nu$ gives

$$\partial_\kappa A = (A_\mu \partial_\kappa \mathcal{E}^\mu + \mathcal{E}_\mu \partial_\kappa A_\mu) = \mathcal{E}^\mu(\partial_\kappa A_\mu + \Gamma^\tau_{\mu\kappa} A_\tau) = \mathcal{E}^\mu A_{\mu;\kappa},$$

(9.7)

where $A_{\mu;\kappa}$ is the covariant differential of $A_\mu$. The structor form of (9.7) is produced by the action of the operator $D = \mathcal{E}^\nu \partial_\mu$ on $A$, which defines

$$\mathcal{F} = DA = \mathcal{E}^\kappa \mathcal{E}_\mu (\partial_\kappa A_\mu + \Gamma^\tau_{\mu\kappa} A_\tau) = (\mathcal{E}^\mu + g^{\mu\nu})(\partial_\kappa A_\mu + \Gamma^\tau_{\mu\kappa} A_\tau) = \mathcal{E}^\mu \partial_\kappa A_\mu + A^{\kappa;\kappa}.$$

(9.8)

If $A$ is interpreted as a potential function, then $\mathcal{F} = DA$ is the corresponding field. This has an invariant part, $A^{\kappa;\kappa}$, and an interactive part $\mathcal{E}^\mu \partial_\kappa A_\mu = \frac{1}{2} \mathcal{E}^\mu (\partial_\kappa A_\mu - \partial_\mu A_\kappa)$ which couples to the appropriate charge. Maxwell’s equations in vacuo then take the form $\mathcal{D} \mathcal{F} = \mathcal{D}^2 A = 0$, if the gauge is chosen so that $A^{\kappa;\kappa} = 0$. 
Applying the differential operator $\partial_\mu$ twice gives

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) A = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) A_\kappa \mathcal{E}^\kappa = -R^\kappa_{\mu \nu \tau} A_\kappa \mathcal{E}^\tau,$$  \hspace{1cm} (9.9)

where

$$R^\kappa_{\mu \nu \tau} = \partial_\mu \Gamma^\kappa_{\tau \nu} - \partial_\nu \Gamma^\kappa_{\tau \mu} + \Gamma^\kappa_{\sigma \nu} \Gamma^\sigma_{\tau \mu} - \Gamma^\kappa_{\sigma \mu} \Gamma^\sigma_{\tau \nu}$$  \hspace{1cm} (9.10)

is the Riemann-Christoffel curvature tensor. The differential operators only commute if $R^\kappa_{\mu \nu \tau}$ vanishes, i.e. in flat space-time. In order to obtain the structure equation corresponding to (9.9) it is necessary to define

$$\mathcal{D}_\lambda = \frac{1}{2} \mathcal{E}^{\mu \nu} (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu).$$  \hspace{1cm} (9.11)

This gives

$$\mathcal{D}_\lambda A = -\frac{1}{2} \mathcal{E}^{\mu \nu} R^\kappa_{\mu \nu \tau} A_\kappa \mathcal{E}^\tau
= -\frac{1}{2} \mathcal{E}^{\mu \nu} \mathcal{E}^\tau R^\kappa_{\mu \nu \lambda} A^\lambda
= -\frac{1}{2} (\mathcal{E}^{\mu \nu \rho} \mathcal{E}^\pi + \mathcal{E}^{\rho \tau} \mathcal{E}^\mu - \mathcal{E}^{\mu \tau} \mathcal{E}^\nu) R^\kappa_{\mu \nu \lambda} A^\lambda
= -g^{\mu \tau} \mathcal{E}^\mu R^\kappa_{\mu \nu \lambda} A^\lambda
= -R_{\mu \lambda} \mathcal{E}^\mu A^\lambda,$$  \hspace{1cm} (9.12)

which vanishes if $R_{\mu \lambda} = R_{\lambda \mu} = 0$. This result is independent of the tensor $A^\lambda$, giving the gravitational field equations in vacuo

$$\mathcal{D}_\lambda \mathcal{E}^\kappa = \frac{1}{2} \mathcal{E}^{\mu \nu} (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \mathcal{E}^\kappa = 0.$$  \hspace{1cm} (9.13)

This shows that the commutation of differentials corresponds to the vanishing of the Ricci tensor, which is just Einstein’s condition for the gravitational field equations. In other words, the algebraic formulation ensures that the components of the Riemann-Christoffel tensor satisfy the field equations of general relativity.

The square of algebraic invariant $\mathcal{D} = \mathcal{E}^\mu \mathcal{E}^\mu / \mathcal{E}^{\mu \nu}$ is

$$\mathcal{D}^2 = -\mathcal{E}^{\mu \nu} \partial_\mu \mathcal{E}^\nu \partial_\nu = \mathcal{E}^{\mu \nu} (\partial_\mu \partial_\nu + \Gamma^\tau_{\mu \nu} \partial_\tau)
= (g^{\mu \nu} + \mathcal{E}^{\mu \nu}) (\partial_\mu \partial_\nu + \Gamma^\tau_{\mu \nu} \partial_\tau)
= g^{\mu \nu} (\partial_\mu \partial_\nu + \Gamma^\tau_{\mu \nu} \partial_\tau) + \mathcal{D}_\lambda.$$  \hspace{1cm} (9.14)

It was shown in §3 that photon wave equations can be expressed in terms of a potential function $A$ that satisfies the Klein-Gordon equation corresponding to the classical equation relating the total energy $E = p_0$ of a particle to its mass and momentum, i.e. $E^2 = p^2 + m^2 = p_0 p^\mu$. The Klein-Gordon equation in Riemannian space-time is obtained by replacing $p_\mu \rightarrow \hat{\gamma}_\mu \partial_\mu$, and taking account of (9.13), to give

$$\mathcal{D}^2 A = g^{\mu \nu} (\partial_\mu \partial_\nu + \Gamma^\tau_{\mu \nu} \partial_\tau) A = 0.$$  \hspace{1cm} (9.15)

This is the wave-equation for any zero rest mass boson. Photons only interact with charged particles and carry (algebraically) the information required to make this distinction. Gravitons act on any massive particle, so that (9.15) provides their complete description, as far as can be achieved in terms of the $\text{Cl}_{1,3}$ algebra.

It should be possible to express gravitational interactions in terms of the $\text{Cl}_{n,n}$ algebras by using the expression for unit time intervals obtained in this work, but this has yet to be investigated.
String theories are based on adding additional spatial dimensions to the three that are observed. This is often associated with extending the SO(1,3) algebra to SO(1,q). Extending the Dirac algebra in the same way provides a link with the Clifford algebras $Cl_{1,q}$. These are only isomorphic with $Cl_{n,n}$ algebras if $q = 1 + 8r$ where $r$ is a positive integer, e.g. $Cl_{1,9} \equiv Cl_{5,5}$ and $Cl_{1,17} \equiv Cl_{9,9}$. Only the physical interpretations of the $Cl_{5,5}$ sub-algebras of $Cl_{7,7}$ have been considered in this work.

In order to relate CU with string theory the general notation for Clifford algebras, given in §3, is compared with the labelling of $\gamma$ matrices used in Chapter 9 of [5]. That work denotes the ten generators of $Cl_{1,9}$ $\gamma_i, i = 1, ..., 9, 10$ where $\gamma_i^2 = -1$ for $i = 1, 2, ..., 9$, and $\gamma_{10}^2 = +1$. The ten generators of $Cl_{5,5}$ will be labelled $\Gamma_i$, as in §6, with the space-like generators $\Gamma_i^2 = -1$ for $i = 1, 2, ..., 5$, and the time-like generators $\Gamma_i^2 = +1$ for $i = 6, ..., 10$. The relationship between these generators follows that given on page 216 of [25], i.e.

$$\gamma^i = \Gamma^i h, \ i = 6, 7, 8, 9 \text{ and } \gamma^i = \Gamma_i, \ i = 1, ..., 5, 10$$

(10.1)

where $h = \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9$. This makes it clear that the three space-like generators that correspond to physical space are identical in algebraic and string theory. However, the single time-like generator $\gamma^{10}$ in $Cl_{1,9}$, associated with time in string theory, does not coincide with the time direction defined in this work.

In order to distinguish the five possible forms of ten-dimensional string theory, the number of dimensions have been extended to eleven by including the matrix $\gamma^{11}$ which, following equation (9.10) of [5], is defined as

$$\gamma^{11} = \gamma^{10} \gamma^1 \gamma^2 \cdots \gamma^9 = \Gamma_{10} \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_5 \Gamma_6 h \Gamma_7 h \Gamma_8 h \Gamma_9 h = \Gamma_{10} \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9 = \Gamma_0$$

(10.2)

This makes it apparent that $\gamma^{11}$ corresponds to the time direction identified in this work, and which, as an operator, takes eigenvalues that distinguish between particles and anti-particles.
§11. Substrates

It has been argued that physical substrates, described by the quantum numbers $\mu_B, \mu_C, \mu_D, \mu_E, \mu_F, \mu_G$, provide the medium for fermion wave-functions, and determine their properties. Symmetry breaking determines possible fermion interactions, and correlates them with regions of space that have different substrates:

S1. Fermions, with $\mu_B = 1$, have equal and opposite charges, and time directions, to their corresponding anti-fermions, which have $\mu_B = -1$. Experimentally, anti-fermions are unstable in all accessible regions of space, suggesting that remote regions of space could exist in which anti-fermions are stable and fermions unstable.

S2. The quantum number $\mu_C = i\mu_\pi \pi_6 = \pm 1$ distinguishes the two fermions in any doublet. The corresponding element of $Cl_{3,3}$ is $\hat{\gamma}^{\pi_6}$, which is identified in §3 and §4 as providing the lepton substrate. Fermions or anti-fermions with $\mu_C = -1$ have one more charge than the $\mu_C = +1$ fermions or anti-fermions in the same doublet. $i\hat{\gamma}^{\pi_6}$, $i\hat{\gamma}^{\pi_7}$ and $i\hat{\gamma}^{\pi_8}$ together generate the Lie algebra of SU(2), defining an isospin algebra isomorphic to spin. If $i\hat{\gamma}^{\pi_6}$ of this algebra is diagonal at all points in space-time, this is analogous to the symmetry breaking in ferromagnets, making the wave motion of leptons isomorphic with spin-waves.

S3. The quantum numbers $\mu_D = \pm 1$, $\mu_E = \pm 1$ together distinguish leptons and quarks, as shown in Table 7.1. They correspond to the commuting elements $\Gamma^D$ and $\Gamma^E$ of $Cl_{5,5}(LQ)$ and, combined with $\Gamma^X = -\Gamma^E\Gamma^D\Gamma^B$, determine the three commuting elements of a sub-algebra, denoted $Cl_{3,3}(Q)$. Elements of this sub-algebra provide all 15 generators of the Lie algebra of SU(4) and its subgroup SU(3) that describes gluons. The SU(4)$\rightarrow$SU(3) symmetry breaking is forced by the different charges on quarks and leptons and distinguishes the substrate in ‘hadronic’ space, produced by the gluons inside baryons and mesons, from the external ‘leptonic’ space available only to leptons.

S4. The quantum numbers $\mu_F = \pm 1$, $\mu_G = \pm 1$ together distinguish four generations of leptons and quarks, as shown in Tables 8.1 and 8.2. They correspond to the commuting elements $\Gamma^F$ and $\Gamma^G$ of $Cl_{7,7}$ and, combined with $\Gamma^H = -\Gamma^F\Gamma^G\Gamma^C$, determine the three commuting elements of its sub-algebra, denoted $Cl_{3,3}(G)$. SU(4)$\rightarrow$SU(3) symmetry breaking is forced by the different charges on fermions in the first to third generations and on those in the fourth generation, as shown in Tables 8.1 and 8.2. In analogy with the distinction between leptonic and hadronic regions of space described above, this suggests that ‘dark matter’ regions of space do not contain the substrate of ‘ordinary matter’, i.e. matter composed of fermions in the first three generations that are the constituents of solar systems.
§12. Conclusions

The starting point of this work was the integration of the macroscopic space-time algebra $Cl_{1,3}$, as developed in [7,8], with the Dirac algebra, where it is treated as an invariant. It was shown in §3 that this is achieved with the $Cl_{3,3}$ algebra, producing the modified Dirac equation, which takes the form of a Lorentz invariant operator acting on the 8-component Lorentz invariant column vector. The physical interpretation of lepton properties in terms of $Cl_{3,3}$ in §4 - §6 then suggested extending the algebra to $Cl_{5,5}$ and $Cl_{7,7}$ in order to provide a description of all known elementary fermions and their interactions.

Crucial features of the work are

1. Identifying the proper time coordinate as the product of generators for all the $Cl_{n,n}$, $\{n = 3, 5, 7\}$ algebras.
2. Maintaining the algebraic distinction between observers’ space-time coordinate frames and fermion rest frames.
3. Choosing the appropriate algebraic description for the physical space-time coordinates.
4. Eliminating chiral symmetry breaking from the description weak interactions.
5. Specifying all known elementary fermions in terms of seven binary quantum numbers.
6. Obtaining a formula for the charges on all known elementary fermions in terms of the seven quantum numbers.
7. Relating the seven commuting elements of $Cl_{7,7}$ to different possible substrates for fermion and boson wave motion, and showing that all elementary particle properties are determined by their substrate.
8. Expressing the known gauge fields in terms of elements of $Cl_{7,7}$.
9. Showing that the same closure property of $Cl_{1,3}$ determines the form of both the electromagnetic and gravitational field equations (§3 and §9).
10. Prediction of the existence of, and the charges on, a 4-th generation of fermions.

This work remains incomplete, especially in relation to gravitation and the determination of fermion masses. It does, nevertheless, provide a new starting point for further developments.

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Appendix A: Representations of $Cl_{3,3}$

The canonical $\gamma$-matrix representation of $Cl_{3,3}$ has 64 linearly independent real $8 \times 8$ matrices. These representation matrices are expressed below as a multiplication table, which gives the products of the representation matrices of the elements of $Cl_{1,3}$ (left factors) with the unit matrix and matrices of the time-like generators of $Cl_{3,3}$ (right factors). Each $\gamma$-matrix is expressed as a Kronecker product of three real $2 \times 2$ matrices defined by

$$
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Q = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
$$

(A.1)

where the $\sigma$s are the Pauli matrices. The real matrices satisfy the relations

$$
-P^2 = Q^2 = R^2 = I, \quad PQ = R = -QP, \quad PR = -Q = -RP, \quad QR = -P = -RQ.
$$

(A.2)

Table A1: Real "canonical" representation of $Cl_{3,3}$, which defines particle rest frames

|       | $\gamma^6$ | $\gamma^7$ | $\gamma^8$ |
|-------|------------|------------|------------|
| 1     | I $\otimes$ I $\otimes$ I | I $\otimes$ Q $\otimes$ I | -P $\otimes$ P $\otimes$ Q | P $\otimes$ P $\otimes$ R |
| $\gamma^0$ | I $\otimes$ Q $\otimes$ P | I $\otimes$ I $\otimes$ P | P $\otimes$ R $\otimes$ R | P $\otimes$ R $\otimes$ Q |
| $\gamma^1$ | -I $\otimes$ R $\otimes$ I | -I $\otimes$ P $\otimes$ I | P $\otimes$ Q $\otimes$ Q | -P $\otimes$ Q $\otimes$ R |
| $\gamma^2$ | -Q $\otimes$ P $\otimes$ I | -Q $\otimes$ R $\otimes$ I | R $\otimes$ I $\otimes$ Q | -R $\otimes$ I $\otimes$ R |
| $\gamma^3$ | P $\otimes$ P $\otimes$ P | P $\otimes$ R $\otimes$ P | -I $\otimes$ I $\otimes$ R | -I $\otimes$ I $\otimes$ Q |
| $\gamma^{12}$ | R $\otimes$ P $\otimes$ I | R $\otimes$ R $\otimes$ I | Q $\otimes$ I $\otimes$ Q | -Q $\otimes$ I $\otimes$ R |
| $\gamma^{23}$ | -R $\otimes$ I $\otimes$ P | -R $\otimes$ Q $\otimes$ P | Q $\otimes$ P $\otimes$ R | Q $\otimes$ P $\otimes$ Q |
| $\gamma^{03}$ | P $\otimes$ I $\otimes$ I | P $\otimes$ Q $\otimes$ I | I $\otimes$ P $\otimes$ Q | -I $\otimes$ P $\otimes$ R |
| $\gamma^{02}$ | Q $\otimes$ I $\otimes$ P | Q $\otimes$ Q $\otimes$ P | R $\otimes$ P $\otimes$ R | R $\otimes$ P $\otimes$ Q |
| $\gamma^{01}$ | -R $\otimes$ Q $\otimes$ I | -R $\otimes$ I $\otimes$ I | -Q $\otimes$ R $\otimes$ Q | Q $\otimes$ R $\otimes$ R |
| $\gamma^{*0}$ | -P $\otimes$ Q $\otimes$ P | -P $\otimes$ I $\otimes$ P | I $\otimes$ R $\otimes$ R | I $\otimes$ R $\otimes$ Q |
| $\gamma^{*1}$ | Q $\otimes$ R $\otimes$ P | Q $\otimes$ P $\otimes$ P | R $\otimes$ Q $\otimes$ R | R $\otimes$ Q $\otimes$ Q |
| $\gamma^{*2}$ | P $\otimes$ R $\otimes$ I | P $\otimes$ P $\otimes$ I | I $\otimes$ Q $\otimes$ Q | -I $\otimes$ Q $\otimes$ R |
| $\gamma^{*3}$ | -R $\otimes$ R $\otimes$ P | -R $\otimes$ P $\otimes$ P | Q $\otimes$ Q $\otimes$ R | Q $\otimes$ Q $\otimes$ Q |
The 64 \(\hat{\gamma}\)-matrix representation of \(C\ell_{3,3}\) given in Table A2 is obtained using a transformation of the canonical representation matrices that makes both \(\hat{\gamma}^{06}\) and \(\hat{\gamma}^{12}\) diagonal. Defining \(Z = \frac{1}{\sqrt{2}}(-R + iP)\) gives

\[
ZPZ^{-1} = iR, \ ZQZ^{-1} = -Q, \ ZRZ^{-1} = -iP, \ Z^2 = I, \ Z^{-1} = Z^I = Z.
\]

(A.3)

It follows that the transformation \(\hat{\gamma} = Z\gamma Z^{-1}\), where \(Z = Z\otimes I\otimes Z\), transforms real matrices in the canonical representation in Table A1 to the complex matrices of the modified canonical representation \(\hat{\gamma}\) given below.

Table A2: The \(\hat{\gamma}\) fermion rest frame representation of \(C\ell_{3,3}\)

|   | \(\hat{\gamma}^1\) | \(\hat{\gamma}^6\) | \(\hat{\gamma}^7\) | \(\hat{\gamma}^8\) |
|---|-----------------|-----------------|-----------------|-----------------|
| \(1\) | \(I \otimes I \otimes I\) | \(I \otimes Q \otimes I\) | \(iR \otimes P \otimes Q\) | \(R \otimes P \otimes P\) |
| \(\hat{\gamma}^0\) | \(iI \otimes Q \otimes R\) | \(iI \otimes I \otimes R\) | \(R \otimes R \otimes P\) | \(-iR \otimes R \otimes Q\) |
| \(\hat{\gamma}^1\) | \(-I \otimes R \otimes I\) | \(-I \otimes P \otimes I\) | \(-iR \otimes Q \otimes Q\) | \(-R \otimes Q \otimes P\) |
| \(\hat{\gamma}^2\) | \(Q \otimes P \otimes I\) | \(Q \otimes R \otimes I\) | \(-iP \otimes I \otimes Q\) | \(P \otimes I \otimes P\) |
| \(\hat{\gamma}^3\) | \(-R \otimes P \otimes R\) | \(-R \otimes R \otimes R\) | \(iI \otimes I \otimes P\) | \(I \otimes I \otimes Q\) |
| \(\hat{\gamma}^{12}\) | \(-iP \otimes P \otimes I\) | \(-iP \otimes R \otimes I\) | \(Q \otimes I \otimes Q\) | \(-iQ \otimes I \otimes P\) |
| \(\hat{\gamma}^{31}\) | \(-iQ \otimes I \otimes R\) | \(-iQ \otimes Q \otimes R\) | \(-P \otimes P \otimes P\) | \(iP \otimes P \otimes Q\) |
| \(\hat{\gamma}^{23}\) | \(-P \otimes Q \otimes I\) | \(-P \otimes Q \otimes I\) | \(-iP \otimes Q \otimes Q\) | \(iQ \otimes R \otimes P\) |
| \(\hat{\gamma}^{03}\) | \(-iQ \otimes P \otimes I\) | \(-iQ \otimes P \otimes I\) | \(Q \otimes R \otimes Q\) | \(-P \otimes R \otimes P\) |
| \(\hat{\gamma}^{02}\) | \(R \otimes Q \otimes R\) | \(R \otimes I \otimes R\) | \(-iI \otimes R \otimes P\) | \(-I \otimes R \otimes Q\) |
| \(\hat{\gamma}^{01}\) | \(-Q \otimes Q \otimes I\) | \(-Q \otimes I \otimes I\) | \(-iP \otimes R \otimes Q\) | \(-P \otimes R \otimes P\) |
| \(\hat{\gamma}^{x0}\) | \(-iQ \otimes P \otimes R\) | \(-iQ \otimes R \otimes R\) | \(R \otimes I \otimes P\) | \(-iR \otimes I \otimes Q\) |
| \(\hat{\gamma}^{x1}\) | \(-iQ \otimes R \otimes R\) | \(-iQ \otimes P \otimes R\) | \(-P \otimes Q \otimes P\) | \(-iP \otimes Q \otimes Q\) |
| \(\hat{\gamma}^{x2}\) | \(iR \otimes R \otimes I\) | \(iR \otimes P \otimes I\) | \(-I \otimes Q \otimes Q\) | \(iI \otimes Q \otimes P\) |
| \(\hat{\gamma}^{x3}\) | \(-P \otimes R \otimes R\) | \(-P \otimes P \otimes R\) | \(iQ \otimes Q \otimes P\) | \(Q \otimes Q \otimes Q\) |

The matrix representations in Tables A1 and A2 relate to fermion rest frames. Representation matrices for arbitrary reference frames are obtained by Lorentz transformations \(\gamma \rightarrow A\gamma A^{-1}\), where \(A\) is defined in (3.13). Relationships between the various \(4\times4\) matrix representations of fermion rest frame coordinate systems are given in Table A3.
Table A3: Alternative choices of space-time representation matrices

| γ        | γ̃      | ˆγ      | aγ | bγ |
|----------|---------|---------|----|----|
| γ₀       | -I ⊗ R ⊗ I | -I ⊗ R | -I ⊗ R | -I ⊗ R |
| γ¹       | -Q ⊗ P ⊗ I | -Q ⊗ P | Q ⊗ P | Q ⊗ P |
| γ²       | P ⊗ P ⊗ P | -iP ⊗ P | -R ⊗ P ⊗ R | R ⊗ P | -R ⊗ P |
| γ³       | R ⊗ P ⊗ I | ⊗ R ⊗ P | -iP ⊗ P ⊗ I | -iP ⊗ P | -iP ⊗ P |
| γⁿ       | I ⊗ Q ⊗ P | -iI ⊗ Q | iI ⊗ Q ⊗ R | -iI ⊗ Q | iI ⊗ Q |

Appendix B. Block diagonalized representations

The modified canonical representations ˆγ puts structures into block diagonal form. The ˆγ representation of the differential structor D is

\[ D = ˆγ^μ \partial_μ = \begin{pmatrix} D_a & 0 \\ 0 & D_b \end{pmatrix} \] (B.1)

where

\[ D_a = \begin{pmatrix} \partial_0 & 0 & -\partial_2 & -\partial_1 - i\partial_3 \\ 0 & \partial_0 & -\partial_1 + i\partial_3 & -\partial_2 \\ -\partial_2 & \partial_1 + i\partial_3 & -\partial_0 & 0 \\ \partial_1 - i\partial_3 & \partial_2 & 0 & -\partial_0 \end{pmatrix} \] (B.1a)

and

\[ D_b = \begin{pmatrix} \partial_0 & 0 & -\partial_2 & -\partial_1 - i\partial_3 \\ 0 & \partial_0 & -\partial_1 + i\partial_3 & \partial_2 \\ -\partial_2 & \partial_1 + i\partial_3 & -\partial_0 & 0 \\ \partial_1 - i\partial_3 & -\partial_2 & 0 & -\partial_0 \end{pmatrix} \] (B.1b)

The general potential structor has the ˆγ block diagonal representation

\[ A = ˆγ^μ (A_μ - ˆγⁿ A_π μ) = \begin{pmatrix} A_a & 0 \\ 0 & A_b \end{pmatrix}, \] (B.2)

where

\[ A_a = \begin{pmatrix} A_0 + iA_1 & A_3 - iA_1 & A_2 + iA_0 & -A_1 - iA_3 \\ -A_3 - iA_1 & A_0 - iA_2 & -A_1 + iA_3 & -A_2 + iA_0 \\ -A_2 - iA_0 & A_1 + iA_3 & A_0 - iA_2 & -A_3 + iA_1 \\ A_1 - iA_3 & A_2 - iA_0 & A_1 + iA_2 & -A_0 - iA_3 \end{pmatrix} \] (B.2a)

and

\[ A_b = \begin{pmatrix} A_0 + iA_2 & -A_3 - iA_1 & -A_2 + iA_0 & -A_1 - iA_3 \\ -A_3 + iA_1 & A_0 - iA_2 & -A_1 + iA_3 & A_2 - iA_0 \\ A_2 + iA_0 & A_1 + iA_3 & A_0 - iA_2 & A_1 - iA_3 \\ A_1 - iA_3 & -A_2 + iA_0 & -A_1 + iA_3 & -A_0 - iA_3 \end{pmatrix}. \] (B.2b)

Similarly, the field structor has the block diagonal ˆγ matrix representation

\[ F = ˆγ^μ F_μ = \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix}, \] (B.3)

where

\[ F_a = \begin{pmatrix} -iF_{31} & F_{12} + iF_{23} & F_{02} & F_{01} - iF_{03} \\ -F_{12} + iF_{23} & iF_{31} & F_{01} + iF_{03} & -F_{02} \\ F_{02} & F_{01} - iF_{03} & -iF_{31} & -F_{12} + iF_{23} \\ F_{01} + iF_{03} & -F_{02} & F_{12} + iF_{23} & iF_{31} \end{pmatrix} \] (B.3a)

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and
\[
F_b = \begin{pmatrix}
-iF_{31} & F_{12} - iF_{23} & -F_{02} & F_{01} - iF_{03} \\
-F_{12} - iF_{23} & iF_{31} & F_{01} + iF_{03} & F_{02} \\
-F_{02} & F_{01} - iF_{03} & -iF_{31} & F_{12} - iF_{23} \\
F_{01} + iF_{03} & F_{02} & F_{02} & F_{02} - iF_{23}
\end{pmatrix}.
\]

(B.3b)

As Lorentz transformations are also expressed in terms of the matrices \(\gamma^{\mu\nu}\), they also have block diagonal form, viz.
\[
\Lambda = \begin{pmatrix}
\Lambda_a & 0 \\
0 & \Lambda_b
\end{pmatrix}.
\]

(B.A)

References

[1] Bettini, Alessandro 2008 Introduction to Elementary Particle Physics (Cambridge University Press)
[2] Georgi, Howard 1982 Lie Algebras in Particle Physics (Benjamin Publishing Co. Inc, Massachusetts)
[3] Baez, John and Huerta, John 2010 The Algebra of Grand Unified Theories arXiv:0904.1556v2
[4] Aitchison, Ian J. R. 2007 Supersymmetry in Particle Physics (Cambridge University Press)
[5] Schomerus, Volker 2017 A Primer on String Theory (Cambridge University Press)
[6] Eddington, Sir A.S. 1946 Fundamental Theory (Cambridge University Press)
[7] Hestenes, David 1966 Space-time algebra (Gordon and Breach, New York)
[8] Doran, Chris and Lasenby, Anthony 2003 Geometric Algebra for Physicists (Cambridge University Press)
[9] Newman, D. J. 1958 Structure Theory Proc. Roy. Irish Acad. 59, 29-47
[10] Trayling, Greg and Baylis, W. E. 2001 A geometric basis for the standard-model gauge group J. Phys. A: Math. Gen. 34, 3309-3324
[11] Dartora, C. A. and Cabrera, G. G. 2009 The Dirac equation and a non-chiral electroweak theory in six dimensional space-time from a locally gauged SO(3, 3) symmetry group arXiv: 0901.4230v1 Int J Theor Phys (2010) 49:51-61
[12] Żenczykowski, P. 2009 Clifford algebra of non-relativistic phase space and the concept of mass J.Phys.A: Math. Theor. 42, 045204
[13] Żenczykowski, P. 2015 From Clifford algebra of Nonrelativistic Phase Space to Quarks and Leptons of the Standard Model Adv. Appl. Clifford Algebras Springerlink.com 2015 DOI 10.1007/s00006-015-0564-7
[14] Żenczykowski, P. 2018 Quarks, Hadrons and Emergent Spacetime arXiv:1809.05402v1
[15] Stoica, O. C. 2018 The Standard Model Algebra: Leptons, Quarks and Gauge from the Complex Clifford Algebra C_{6b} Adv. Appl. Clifford Algebras 28, 52. arXiv:1702.04336v3
[16] Stoica, O. C. 2020 Chiral asymmetry in the weak interaction via Clifford Algebras arXiv:2005.08855v1
[17] Pavšič, Matej 2021 Clifford Algebras, Spinors and Cl(8,8) Unification arXiv:2105.11808
[18] Yamatsu, Naoki 2020 USp(32) Special Grand Unification arXiv:2007.08067v1
[19] Aitchison, I. J. R. and Hey, A. J. G. 2003 Gauge Theories in Particle Physics, Volume I: From Relativistic Quantum Mechanics to QED (Taylor and Francis)
[20] Aitchison, I. J. R. and Hey, A. J. G. 2004 Gauge Theories in Particle Physics, Volume II: QCD and the Electroweak Theory (IOP Publishing Ltd)
[21] Thomson, Mark. 2013 Modern Particle Physics (Cambridge University Press)
[22] Dodd, James and Gripalos, Ben 2020 The Ideas of Particle Physics (Cambridge University Press)
[23] Hill, E. L. and Landshoff, R. 1938 The Dirac Electron Theory Rev. Mod. Physics 10, 87-132
[24] Newman, D. J. and Kilmister, C. W. 1959 A New Expression for Einstein’s Law of Gravitation Proc. Camb. Phil. Soc. 55, 139-141
[25] Louesto, Fertti 1997 Clifford Algebras and Spinors (Cambridge University Press)