Autocratic Strategies for Alternating Games

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Abstract. Repeated games have a long tradition in behavioral sciences and evolutionary biology. Recently, strategies were discovered that permit an unprecedented level of control over repeated interactions by enabling a player to unilaterally enforce constraints on the payoffs. Here, we extend this theory of autocratic (or zero-determinant) strategies to alternating games, which are often more biologically relevant than synchronous games. More specifically, in a strictly-alternating game with two players, $X$ and $Y$, we give conditions for the existence of autocratic strategies for player $X$ when (i) $X$ moves first and (ii) $Y$ moves first. Furthermore, we show that autocratic strategies exist for (iii) games with randomly-alternating moves. Particularly important categories of autocratic strategies are extortionate and generous strategies, which enforce unfavorable and favorable outcomes for the opponent, respectively. Alternating games naturally occur in biological contexts such as in reciprocal blood donation among vampire bats or in bouts of grooming among primates. Either scenario can be captured by the continuous Donation Game, where a player pays a cost to provide benefits to the opponent according to its continuous cooperative investment level. For each variant of alternating game, we illustrate autocratic strategies based on merely two investment levels in the continuous Donation Game. Alternating games, unlike simultaneous-move games, naturally result in asymmetries between players either because the first move matters or because players may not move with equal probabilities. Such asymmetries could, for example, easily arise from dominance hierarchies, and we show that they endow subordinate players with more autocratic strategies than dominant players, which, in turn, permits subordinate players to exert a surprising amount of control over asymmetric interactions.

1. Introduction

The evolution of cooperation driven by actions that are costly to the individual but benefit others poses a fundamental challenge in behavioral sciences and evolutionary biology. The earliest suggestion was to consider repeated interactions and conditional strategic responses, which enable cooperation to thrive through reciprocal altruism (Trivers, 1971). Iterated games, and, in particular, the iterated Prisoner’s Dilemma, have been used extensively to address the evolution of cooperation via direct reciprocity (Axelrod and Hamilton, 1981; Axelrod, 1984; Nowak, 2006). These games involve a sequence of interactions in which two players traditionally act simultaneously (or, at least without knowing the opponent’s move) and condition their decisions on the history of their previous encounters. Even though such synchronized decisions seem often contrived in realistic social interactions, the biologically more realistic and relevant scenario with asynchronous interactions has received surprisingly little attention. In asynchronous games, players take turns and move either in a strictly- or randomly-alternating manner (Nowak and Sigmund, 1994). A classical example of an asynchronous game with alternating moves is blood donation in vampire bats (Wilkinson, 1984). When a well-fed bat donates blood to a hungry fellow, the recipient has the opportunity to return the favor at a later time. Similarly, social grooming between two primates is not always performed simultaneously; instead, one animal grooms another, who then has the opportunity to reciprocate at a later time (Muroyama, 1991). Even for interactions that appear to involve simultaneous decisions, such as in acts of predator inspection by fish (Milinski, 1987), it remains difficult to rule out that these interactions are not instead based on rapid, non-synchronous decisions (Frean, 1994).

The iterated Prisoner’s Dilemma game, which involves a choice to either cooperate, $C$, or defect, $D$, in each round, has played a central role in the study of reciprocal altruism (Axelrod and Hamilton, 1981; Axelrod, 1984; Nowak, 2006). Rather unexpectedly, after decades of intense study of iterated games, Press and Dyson (2012) showed that a player can unilaterally enforce linear payoff relationships in synchronous games. For example, if $\pi_X$ and $\pi_Y$ are the expected payoffs to players $X$ and $Y$, respectively, and $\chi \geq 1$ is an extortion factor, then player $X$ can ensure that $\pi_X = \chi \pi_Y$, regardless of the strategy of player $Y$. Moreover, such relationships may be enforced using merely memory-one strategies, which condition the next move on the outcome of just the previous round. More specifically, a memory-one strategy is a four-tuple,
Figure 1. Three types of interactions in the alternating Donation Game: (A) strictly-alternating game in which player X moves first; (B) strictly-alternating game in which player Y moves first; and (C) randomly-alternating game in which, in each round, player X moves with probability $\omega_X$ and player Y with probability $1 - \omega_X$. For each type of alternating game, a player moves either C or D (cooperate or defect) in each round and both players receive a payoff from this move. Unlike in strictly-alternating games, (A) and (B), a player might move several times in a row in a randomly-alternating game, (C).

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
round & C & D & C & D & C & \\
\hline
(X’s payoff, Y’s payoff) & $(-c, b)$ & $(0, 0)$ & $(b, -c)$ & $(0, 0)$ & $(b, -c)$ & \\
\hline
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
round & D & C & D & C & D & \\
\hline
(X’s payoff, Y’s payoff) & $(0, 0)$ & $(-c, b)$ & $(b, -c)$ & $(0, 0)$ & $(b, -c)$ & $(0, 0)$ \\
\hline
\end{tabular}
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\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
round & D & C & D & D & C & \\
\hline
(X’s payoff, Y’s payoff) & $(-c, b)$ & $(0, 0)$ & $(0, 0)$ & $(0, 0)$ & $(-c, b)$ & $(b, -c)$ \\
\hline
\end{tabular}
\end{center}

\(p_{CC}, p_{CD}, p_{DC}, p_{DD}\), where \(p_{xy}\) indicates the probability that player \(X\) uses \(C\) after \(X\) uses \(x\) and \(Y\) uses \(y\) in the previous round.

The discovery of these so-called “zero-determinant” strategies triggered a flurry of follow-up studies. Most notably, from an evolutionary perspective, extortionate strategies fare poorly ([Hilbe et al., 2013]) but can be stable provided that extortioners recognize one another ([Adami and Hintze, 2013]). However, generous counterparts of extortionate strategies perform much better in evolving populations ([Stewart and Plotkin, 2012, 2013]). Moreover, zero-determinant strategies also exist in discounted games, i.e. interactions with a finite time horizon ([Hilbe et al., 2015]).

Recently, autocratic strategies were introduced as a generalization of zero-determinant strategies to simultaneous games with arbitrary action spaces ([McAvoy and Hauert, 2015]): an autocratic strategy for player \(X\) is any strategy, which enforces the linear relationship

\[\alpha \pi_X + \beta \pi_Y + \gamma = 0\]

on expected payoffs for some constants \(\alpha, \beta, \gamma\), regardless of the strategy of player \(Y\). Here, we consider AUTOMATIC strategies for player \(X\) in alternating games. In strictly-alternating games, one player moves first (either \(X\) or \(Y\)) and waits for the opponent’s response before moving again. This process then repeats, with each player moving only after the opponent moved (see Fig. 1(A),(B)). In contrast, in randomly-alternating games, the player who moves in each round is chosen stochastically: at each time step, \(X\) moves with probability \(\omega_X\) and \(Y\) moves with probability \(1 - \omega_X\). For each type of alternating game, a player moves either \(C\) or \(D\) (cooperate or defect) in each round and both players receive a payoff from this move. Unlike in strictly-alternating games, (A) and (B), a player might move several times in a row in a randomly-alternating game, (C).

In alternating games, the assignment of payoffs to players deserves closer inspection ([Hauert and Schuster, 1998]). Here, we focus on alternating games in which both players obtain payoffs after every move (see Fig. 1). Alternatively, payoffs could result from every pair of moves rather than every individual move ([Frean, 1994]). While it is possible to construct a theory of autocratic strategies for strictly-alternating games in either setting, it becomes difficult to even define payoffs in the latter setup for randomly-alternating games because either player can move several times in a row (see Fig. 1(C)). Therefore,
we follow [Nowak and Sigmund 1994] in order to include the particularly relevant and intriguing case of randomly-alternating games.

In the classical Donation Game [Sigmund 2010], a player either (i) cooperates and donates b to the opponent at a cost of c < b or (ii) defects and donates nothing and pays no cost, which yields the payoff matrix

\[
\begin{pmatrix}
C & D \\
C & b - c & -c \\
D & b & 0
\end{pmatrix}
\]

The continuous Donation Game extends this binary action space and allows for a continuous range of cooperation levels [Killingback et al. 1999; Wahl and Nowak 1999a; Killingback and Doebeli 2002]. An action in this game is an investment level, s, taken from an interval, [0, K], where K indicates an upper bound on investments. Based on its investment level, s, a player then pays a cost of c(s) to donate b(s) to the opponent with b(s) > c(s) for s > 0 and b(0) = c(0) = 0 so that an investment of zero corresponds to defection, which neither generates benefits nor incurs costs. Biologically-relevant interpretations of continuous investment levels (as well as alternating moves) include (i) the effort expended in social grooming and ectoparasite removal by primates (Dunbar 1991); (ii) the quantity of blood donated by one vampire bat to another (Wilkinson 1984); and (iii) the amount of iron-binding agents (siderophores) produced by bacterial parasites (West and Buckling 2003).

Apart from strictly-alternating games, we focus on randomly-alternating games, which are more relevant for modeling interactions in many biological settings. In particular, we consider a class of randomly-alternating games in which the probability that player X moves in a given round, \( \omega_X \), is not necessarily 1/2. Any other value of \( \omega_X \) results in asymmetric interactions – even if the payoffs in each encounter are symmetric – simply because one player moves more often than the other. For example, dominance hierarchies in primates naturally result in asymmetric behavioral patterns [Mehlman and Chapais 1988; Lazarro-Perea et al. 2004; Newton-Fisher and Lee 2011]. In male chimpanzees, dominance hierarchies require smaller, subordinate chimpanzees to groom larger, dominant chimpanzees more often than vice versa [Foster et al. 2009]. Therefore, including such asymmetries significantly expands the scope of biological settings in which the theory of autocratic strategies applies.

2. Results

In every round of an alternating game, either player X or player Y moves. On player X’s turn, she chooses an action, x, from an action space, \( S_X \), and gets a payoff \( f_X(x) \) while her opponent gets \( f_Y(x) \). Similarly, when player Y moves, he chooses an action, y, from \( S_Y \) and gets a payoff \( g_Y(y) \) while his opponent gets \( g_X(y) \). Future payoffs are discounted by a factor \( \lambda \) (0 < \( \lambda < 1 \)), which can be interpreted as a time preference [Fudenberg and Tirole 1991] derived, for example, from interest rates for monetary payoffs. Alternatively, \( \lambda \) can be interpreted as the probability that there will be another round, which results in a finitely-repeated game with an average of \( 1/(1 - \lambda) \) rounds [Nowak 2006].

2.1. Strictly-alternating games. In a pair of rounds in which player X moves before Y, the payoffs are \( u_X(x, y) := f_X(x) + \lambda g_X(y) \) and \( u_Y(x, y) := f_Y(x) + \lambda g_Y(y) \), respectively. Note that the payoffs from Y’s move are discounted by a factor of \( \lambda \) because Y moves one round after X. The payoff functions, \( u_X \) and \( u_Y \), satisfy the “equal gains from switching” property [Nowak and Sigmund 1990], which means the difference between \( u_X(x, y) \) and \( u_X(x', y) \) is independent of the opponent’s move \( y \). This property follows immediately from the fact that \( u_X \) (or \( u_Y \)) is obtained by adding the separate contributions based on the moves of X and Y.

Thus, if player X moves first and \( (x_0, y_1, x_2, y_3, \ldots) \) is the sequence of play, then her average payoff is

\[
\pi_X = (1 - \lambda) \left[ \sum_{t=0}^{\infty} \lambda^{2t} f_X(x_{2t}) + \sum_{t=0}^{\infty} \lambda^{2t+1} g_X(y_{2t+1}) \right] = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^{2t} u_X(x_{2t}, y_{2t+1}).
\]

The second expression resembles the average payoff for simultaneous-move games whose one-shot payoff function is \( u_X \) [Fudenberg and Tirole 1991]. Replacing \( u_X \) with \( u_Y \) yields player Y’s average payoff, \( \pi_Y \).
Three examples of memory-one strategies for player X in a strictly-alternating game whose action spaces are $S_X = S_Y = [0, K]$ for some $K > 0$. (A) depicts a reactive stochastic strategy in which Y’s last move is used to determine the probability distribution with which X chooses her next action. The mean of this distribution is an increasing function of $y$, which means that X is more likely to invest more (play closer to $K$) as $y$ increases.

(B) illustrates a reactive two-point strategy. Player Y’s last move is used to determine the probability with which X plays $K$ in the next round; if X does not use $K$, then she uses 0. As Y’s last action, $y$, increases, X is more likely to use $K$ in response.

(C) is a deterministic strategy that gives X’s next move as a function of both of the players’ last moves. Unlike in (A) and (B), X’s next move is uniquely determined by her own last move, $x$, and the last move of her opponent, $y$. If Y used $y = 0$ in the previous round, then X responds by playing 0 as well. X’s subsequent action is then an increasing function of $y$ whose rate of change is largest when X’s last move, $x$, is smallest. In particular, if Y used $y > 0$ in the previous round, then X’s next action is a decreasing function of her last move, $x$.

For strictly-alternating games, we borrow the term “memory-one strategy” from synchronous games to mean a conditional response based on the previous move of each player. Even though this memory now covers two rounds of interactions, it remains meaningful because player Y always moves after player X (or vice versa). For an arbitrary action space, $S_X$, a memory-one strategy for player X formally consists of an initial action, $\sigma_0^X$, and a memory-one action, $\sigma^X_{[x,y]}$, which are both probability distributions on $S_X$. Since player X moves first, she bases her initial action on $\sigma_0^X$ and subsequently uses the two previous moves, $x$ and $y$, to choose an action in the next round using $\sigma^X_{[x,y]}$ (see Fig. 2).

**Autocratic strategies for strictly-alternating games in which X moves first.** Suppose that

$$
\left(\alpha f_X(x) + \beta f_Y(x) + \gamma\right) + \lambda \left(\alpha g_X(y) + \beta g_Y(y) + \gamma\right)
= \psi(x) - \lambda^2 \int_{s \in S_X} \psi(s) \, d\sigma^X_{[x,y]}(s) - \left(1 - \lambda^2\right) \int_{s \in S_X} \psi(s) \, d\sigma_0^X(s)
$$

holds for some bounded $\psi$ and for each $x \in S_X$ and $y \in S_Y$. Then, if player X moves first, the pair $(\sigma_0^X, \sigma^X_{[x,y]})$ allows X to enforce the equation $\alpha\pi_X + \beta\pi_Y + \gamma = 0$ for every strategy of player Y.

Note that the payoff $\pi_X$, Eq. (3), is the same as in a simultaneous-move game but with payoff function $\frac{1}{\pi_X} u_X$ and discounting factor $\lambda^2$ [McAvoy and Hauert, 2015]. Hence, it is not so surprising that autocratic strategies exist in this case too (and under similar conditions). However, the situation changes if Y moves first and $(y_0, x_1, y_2, x_3, \ldots)$ is the sequence of play. In this case, X’s average payoff is

$$
\pi_X = (1 - \lambda) \left[ g_X(y_0) + \lambda \sum_{t=0}^{\infty} \lambda^{2t} u_X(x_{2t+1}, y_{2t+2}) \right].
$$
When Y moves first, player X’s initial move, \( \sigma_X^0 [y_0] \), is now a function of Y’s first move, \( y_0 \). However, X’s lack of control over the first round does not (in general) preclude the existence of autocratic strategies:

**Autocratic strategies for strictly-alternating games in which Y moves first.** Suppose that

\[
\begin{align*}
(\alpha f_X (x) + \beta f_Y (x) + \gamma) + \lambda (\alpha g_X (y) + \beta g_Y (y) + \gamma) + \left( \frac{1 - \lambda^2}{\lambda} \right) (\alpha g_X (y_0) + \beta g_Y (y_0) + \gamma)
= \psi (x) - \lambda^2 \int_{s \in S_X} \psi (s) \, d\sigma_X [x, y] (s) - (1 - \lambda^2) \int_{s \in S_X} \psi (s) \, d\sigma_X^0 (y_0) (s)
\end{align*}
\]

holds for some bounded \( \psi \) and for each \( x \in S_X \) and \( y_0, y \in S_Y \). Then, if player X moves second, the pair \((\sigma_X^0 [y_0], \sigma_X [x, y])\) allows X to enforce the equation \( \alpha \pi_X + \beta \pi_Y + \gamma = 0 \) for every strategy of player Y.

Note that Eq. (6) is slightly more restrictive than Eq. (4) because player X has no control over the outcome of the initial round. Evidently, for undiscounted (infinite) games \((\lambda = 1)\), it is irrelevant who moves first and hence the conditions for the existence of autocratic strategies coincide (c.f. Eqs. (4) and (6)).

**2.2. Randomly-alternating games.** In randomly-alternating games, the player who moves in any given round is determined probabilistically: player X moves with probability \( \omega_X \) and player Y with probability \( 1 - \omega_X \). Suppose that X and Y each make plans to play \( x_t \) and \( y_t \) at time \( t \), respectively, assuming they move at time \( t \). Then, in the repeated game, these strategies give player X an average (expected) payoff of

\[
\pi_X = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t \left( \omega_X f_X (x_t) + (1 - \omega_X) g_X (y_t) \right).
\]

Y’s average payoff, \( \pi_Y \), is obtained from Eq. (7) by replacing \( f_X \) and \( g_X \) by \( f_Y \) and \( g_Y \), respectively.

For randomly-alternating games, we need to reconsider the concept of memory-one strategies once more. If moves alternate randomly, then a logical extension is provided by a conditional response based on the previous move as well as on which player moved. In particular, \( \sigma_X^0 [y] \) denotes an action for player X after player Y uses \( y \) in the previous round, and \( \sigma_X^X [x] \) denotes an action after playing \( x \) herself. Note that the cognitive requirement in terms of memory capacity in strictly-alternating games remains the same as for simultaneous games, whereas for randomly-alternating games the requirements are significantly less demanding as reflected in two univariate response functions as compared to response functions involving two variables. Nevertheless, in the discrete case with two strategies, \( C \) and \( D \), all three scenarios reduce to a four-tuple \((p_{CC}, p_{CD}, p_{DC}, p_{DD})\), albeit with slightly different interpretations of each entry.

Rather surprisingly, randomly-alternating games can also have autocratic strategies:

**Autocratic strategies for randomly-alternating games.** If, for some bounded \( \psi \),

\[
\begin{align*}
\alpha f_X (x) + \beta f_Y (x) + \gamma & = \psi (x) - \lambda \omega_X \int_{s \in S_X} \psi (s) \, d\sigma_X^X [x] (s) - (1 - \lambda) \omega_X \int_{s \in S_X} \psi (s) \, d\sigma_X^0 (s); \\
\alpha g_X (y) + \beta g_Y (y) + \gamma & = -\lambda \omega_X \int_{s \in S_X} \psi (s) \, \sigma_X^X [y] (s) - (1 - \lambda) \omega_X \int_{s \in S_X} \psi (s) \, d\sigma_X^0 (s)
\end{align*}
\]

for each \( x \in S_X \) and \( y \in S_Y \), then the strategy \((\sigma_X^0, \sigma_X^X [x], \sigma_X^X [y])\) allows X to enforce the equation \( \alpha \pi_X + \beta \pi_Y + \gamma = 0 \) for every strategy of player Y.

However, based on examples in [3], we demonstrate that autocratic strategies do require that player X moves sufficiently often, i.e. condition Eq. (8) implicitly puts a lower bound on \( \omega_X \).

**3. Examples**

**3.1. Classical Donation Game.** The classical Donation Game, Eq. (2), with the strategies cooperate, \( C \), and defect, \( D \), represents an instance of the Prisoner’s Dilemma provided that benefits exceed the costs, \( b > c > 0 \). Without discounting, initial moves do not matter and hence a memory-one strategy for player X is defined by a four-tuple, \( p := (p_{CC}, p_{CD}, p_{DC}, p_{DD}) \), where \( p_{xy} = \sigma_X [x, y] (C) \) is the probability that X
cooperates after X plays x and Y plays y for \( x, y \in \{ C, D \} \). For simultaneous-move games, Press and Dyson (2012) show that for \( \chi \geq 1 \),

\[
p = (1 - \phi (\chi - 1) (b - c), 1 - \phi (\chi b + c), \phi (b + \chi c), 0) \tag{9}
\]

unilaterally enforces the extortionate relationship \( \pi_X = \chi \pi_Y \) on expected payoffs provided that a normalization factor \( \phi \) exists.

In undiscounted (infinite) and strictly-alternating games, we know from Eqs. (4) and (5) that player X does not need to take into account who moves first when devising an autocratic strategy and, moreover, the conditions become identical to those for simultaneous games (McAvoy and Hauert, 2015; Press and Dyson 2012). Therefore, player X can use a single strategy to enforce \( \pi_X = \chi \pi_Y \) in both simultaneous and strictly-alternating games. For discounted (finite) games, however, autocratic strategies depend on whether the moves are simultaneous or strictly-alternating, but the conditions on the discounting factor, \( \lambda \), guaranteeing their existence do not.

In the undiscounted but randomly-alternating Donation Game, player X moves with probability \( \omega_X \) and player Y with probability \( 1 - \omega_X \) in each round. A memory-one strategy for player X is given by \( \mathbf{p}_X = (p^X_C, p^D_Y) \) and \( \mathbf{p}_Y = (p^Y_C, p^D_Y) \), where \( p^X_x \) (resp. \( p^Y_y \)) denotes the probability that X plays \( C \) if X moved \( x \) (resp. Y moved \( y \)) in the preceding round. Then, player X can enforce \( \pi_X = \chi \pi_Y \) with

\[
p^X = \left( \frac{1}{\omega_X} (1 - \phi (\chi b + c)), 0 \right), \quad p^Y = \left( \frac{1}{\omega_X} \phi (b + \chi c), 0 \right) \tag{10}
\]

provided that the normalization factor \( \phi \) falls within the range

\[
\frac{1 - \omega_X}{\chi b + c} \leq \phi \leq \min \left\{ \frac{\omega_X}{b + \chi c}, \frac{1}{\chi b + c} \right\}, \tag{11}
\]

which exists only if \( X \) moves sufficiently frequently, i.e.

\[
\omega_X \geq \frac{b + \chi c}{(\chi + 1)(b + c)}. \tag{12}
\]

Otherwise she loses control over the outcome of the game and cannot enforce an extortionate payoff relationship. The autocratic strategy, Eq. (10), is unforgiving and always responds to defection with defection but more readily cooperates than its counterpart for simultaneous or strictly-alternating Donation Games, Eq. (9): \( p^X_C = \frac{1}{\omega_X} p_{CD} \) and \( p^D_C = \frac{1}{\omega_X} p_{DC} \).

### 3.2. Continuous Donation Game

In the continuous Donation Game, the action space available to players \( X \) and \( Y \) is an interval \([0, K]\), which indicates a continuous range of cooperative investment levels with an upper bound \( K > 0 \). If player X moves \( x \in [0, K] \), she donates \( b(x) \) to her opponent at a cost \( c(x) \) to herself, where \( b(x) \) and \( c(x) \) are continuous non-decreasing functions with \( b(x) > c(x) \) for \( x > 0 \) and \( b(0) = c(0) = 0 \) (Killingback and Doebeli, 2002). It follows that this game is symmetric, with \( f_X(s) = g_Y(s) = -c(s) \) and \( f_Y(s) = g_X(s) = b(s) \).

#### 3.2.1. Extortionate, generous, and equalizer strategies

For each variant of alternating moves, we consider three particularly important and interesting classes of autocratic strategies for the continuous Donation Game: extortionate, extortionate, and generous. An equalizer strategy is an autocratic strategy that allows \( X \) to unilaterally set either \( \pi_X = \gamma \) (self-equalizing) or \( \pi_Y = \gamma \) (opponent-equalizing) (Hilbe et al, 2013). In all scenarios, we show that no self-equalizing strategies exist that allow player X to set \( \pi_X = \gamma \) for \( \gamma > 0 \). However, player X can typically set the score of her opponent. Equalizer strategies are defined in the same way for alternating and simultaneous-move games, whereas extortionate and generous strategies require slightly different definitions. In the simultaneous version of the continuous Donation Game, player X can enforce the linear relationship \( \pi_X = \kappa = \chi (\pi_Y - \kappa) \) for any \( \chi \geq 1 \) and \( 0 \leq \kappa \leq b(K) - c(K) \), provided \( \lambda \) is sufficiently close to 1 (McAvoy and Hauert, 2015). Note that the “baseline payoff,” \( \kappa \), indicates the payoff of an autocratic strategy against itself (Hilbe et al, 2014). If \( \chi > 1 \) and \( \kappa = 0 \), then such an autocratic strategy is called extortionate since it ensures that the expected payoff of player X is at least that of player Y. Conversely, if \( \kappa = b(K) - c(K) \), then this strategy is called generous (or “compliant”) since it ensures that the expected payoff of player X is at most that of player Y (Stewart and Plotkin, 2013; Hilbe et al, 2013).
The bounds on $\kappa$ arise from the payoffs for mutual cooperation and mutual defection in repeated games. Of course, in the simultaneous-move game, those bounds are the same as the payoffs for mutual cooperation and defection in one-shot interactions. Discounted (finite), alternating games, on the other hand, result in asymmetric payoffs even if the underlying one-shot interaction is symmetric. For example, if player $X$ moves first in the strictly-alternating, continuous Donation Game, and if both players are unconditional cooperators, then $\pi_X = (\lambda b(K) - c(K)) / (1 + \lambda)$ but $\pi_Y = (b(K) - \lambda c(K)) / (1 + \lambda)$, which are not equal for discounting factors $\lambda < 1$. Thus, rather than comparing both $\pi_X$ and $\pi_Y$ to the same payoff, $\kappa$, it makes more sense to compare $\pi_X$ to $\kappa_X$ and $\pi_Y$ to $\kappa_Y$ for some $\kappa_X$ and $\kappa_Y$. Therefore, we focus on conditions that allow player $X$ to enforce $\pi_X - \kappa_X = \chi (\pi_Y - \kappa_Y)$ for $\kappa_X$ and $\kappa_Y$ within a suitable range. Note that if player $X$ enforces this payoff relation and, conversely, player $Y$ enforces $\pi_Y - \kappa_Y = \chi (\pi_X - \kappa_X)$, then player $X$ gets $\kappa_X$ and $Y$ gets $\kappa_Y$, which preserves the original interpretation of $\kappa$ [Hilbe et al., 2014]. Also note that the two strategies enforcing the respective payoff relation need not be the same due to the asymmetry in payoffs, which arises from the asymmetry induced by alternating moves.

For $s, s' \in \{0, K\}$, let $\kappa_X^{ss'}$ and $\kappa_Y^{ss'}$ be the baseline payoffs to players $X$ and $Y$, respectively, when $X$ uses $s$ unconditionally and $Y$ uses $s'$ unconditionally in the repeated game. For sufficiently weak discounting factors, $\lambda$, player $X$ can enforce $\pi_X - \kappa_X = \chi (\pi_Y - \kappa_Y)$ for any alternating game if and only if

$$\kappa_X = \kappa_X + (\chi \kappa_0^0 - \kappa_0^0) \leq \chi \kappa_Y \leq \kappa_X + (\chi \kappa^K_X - \kappa^K_X),$$

(13)

where $\kappa_0^0 = 0$. Eq. (13) implies that if player $X$ attempts to minimize player $Y$’s baseline payoff, $\kappa_Y$, for a fixed $\kappa_X$, then $\chi \kappa_Y = \kappa_X$. Hence player $X$ enforces $\pi_X - \kappa_X = \chi (\pi_Y - \kappa_Y) = \chi \pi_Y - \kappa_X$, or $\pi_X = \chi \pi_Y$. Such an autocratic strategy is called extortionate since it minimizes the baseline payoff of the opponent, or, equivalently, it minimizes the difference $\chi \kappa_Y - \kappa_X$. Similarly, if player $X$ tries to maximize $Y$’s baseline payoff, then $\chi \kappa_Y = \kappa_X + (\chi \kappa^K_X - \kappa^K_X)$ and $X$ enforces the equation $\pi_X - \kappa^K_X = \chi (\pi_Y - \kappa^K_Y)$. This type of autocratic strategy is called generous since it maximizes the baseline payoff of the opponent, or, equivalently, it maximizes the difference $\chi \kappa_Y - \kappa_X$. In spite of the more detailed considerations necessary for alternating games, the introduction of distinct baseline payoffs for players $X$ and $Y$ does not affect the spirit with which extortionate and generous strategies are defined.

Interestingly, and somewhat surprisingly, it is possible for player $X$ to devise an autocratic strategy based on merely two distinct actions, $s_1$ and $s_2$, despite the fact that her opponent may draw on a continuum of actions [McAvoy and Hauert, 2015]. In strictly-alternating games, such a “two-point” strategy adjusts $p(x, y)$ (resp. $1 - p(x, y)$), the probability of playing $s_1$ (resp. $s_2$) in response to the previous moves, $x$ and $y$; the actions $s_1$ and $s_2$ themselves remain unchanged. These strategies are particularly illustrative because they admit analytical solutions and simpler intuitive interpretations. In the following we focus on autocratic two-point strategies for player $X$ based on the two actions of full cooperation, $K$, and defection, 0. If $p(x, y)$ denotes the probability of playing $K$, then the memory-one action reads

$$\sigma_X \left[ x, y \right] = \left( 1 - p(x, y) \right) \delta_0 + p(x, y) \delta_K,$$

(14)

where $\delta_0$ and $\delta_K$ denote the Dirac measure centered at 0 and $K$, respectively. For each variant of alternating game, we derive stochastic two-point strategies enforcing extortionate, generous, and equalizer payoff relationships. Deterministic analogues of these strategies that use more than two points (in fact, infinitely many points) are discussed in [SL3.2.3]

3.2.2. Strictly-alternating moves – player $X$ moves first. If player $X$ moves first, then the baseline payoffs for full, mutual cooperation are

$$\kappa^{KK}_X = \frac{\lambda b(K) - c(K)}{1 + \lambda}; \quad \kappa^{KK}_Y = \frac{b(K) - \lambda c(K)}{1 + \lambda},$$

(15)

and the baseline payoffs for mutual defection are always $\kappa_0^0 = 0$ in the Donation Game. The scaling function $\psi(s) := -\chi b(s) - c(s)$ conveniently eliminates $x$ from Eq. (4). For sufficiently long interactions or weak discounting factors, i.e.

$$\lambda \geq \frac{b(K) + \chi c(K)}{\chi b(K) + c(K)},$$

(16)
the two-point strategy defined by (i) the probability $p_0$ to initially play $K$ (as opposed to 0), where
\[
\max \left\{ \frac{(1 + \gamma)(\chi Y - \kappa X) + \lambda b(K) + \chi c(K)}{(1 - \lambda^2)(b(K) + c(K))} - \frac{\lambda^2}{1 - \lambda^2}, 0 \right\} \leq p_0 \leq \min \left\{ \frac{\chi Y - \kappa X}{(1 - \lambda)(b(K) + c(K))}, 1 \right\},
\]
and (ii) the probability $p(y)$ to play $K$ after $y$ played $y$, where
\[
p(y) = \frac{1}{\lambda} \left( \frac{(1 + \lambda)(\chi Y - \kappa X) + \lambda b(y) + \chi c(y)) - (1 - \lambda^2)(b(K) + c(K))p_0}{\lambda^2(b(K) + c(K))} \right),
\]
allows player $X$ to unilaterally enforce $\pi X - \kappa X = \chi (\pi Y - \kappa Y)$ as long as $\kappa X \leq \chi Y \leq \kappa X + (\chi \kappa K - \kappa_K K)$. Whether the autocratic strategy defined by $(p_0, p(y))$ is extortiorate or generous depends on the choice of $\chi$, $\kappa X$, and $\kappa Y$. Note that $p(y)$ does not depend on player $X$’s own previous move, $x$, and hence represents a reactive strategy [Nowak and Sigmund 1990].

Similarly, choosing $\psi(s) := b(s)$ again eliminates $x$ from Eq. [4] but now enables player $X$ to adopt an equalizer strategy and set her opponents score to $\pi Y = \gamma$ with $0 \leq \gamma \leq (b(K) - \lambda c(K)) / (1 + \lambda)$ by (i) initially playing $K$ with probability $p_0$, where
\[
\max \left\{ \frac{(1 - \lambda)(\chi Y - \kappa X) + \lambda c(K)}{(1 - \lambda^2)b(K)} - \frac{\lambda^2}{1 - \lambda^2}, 0 \right\} \leq p_0 \leq \min \left\{ \frac{\chi Y - \kappa X}{(1 - \lambda)b(K)}, 1 \right\},
\]
and (ii) subsequently playing $K$ with probability $p(y)$ after $Y$ uses $y$, where
\[
p(y) = \frac{\lambda c(y) + (1 + \lambda)(\chi Y - \kappa X) - (1 - \lambda^2)b(K)p_0}{\lambda^2b(K)}.
\]
However, just like in the simultaneous-move game, player $X$ can never set her own score, which follows from an analogous argument along the same lines (see McAvoy and Hauert 2015).

3.2.3. Strictly-alternating moves – player $Y$ moves first. If player $Y$ moves first, then the baseline payoffs for full cooperation are
\[
\kappa_X^{KK} = \frac{b(K) - \lambda c(K)}{1 + \lambda}, \quad \kappa_Y^{KK} = \frac{\lambda b(K) - c(K)}{1 + \lambda},
\]
and, again, the baseline payoffs for mutual defection are both 0. The scaling function $\psi(s) := -\chi b(s) - c(s) + \chi \kappa Y - \kappa X$ eliminates $x$ from Eq. [6]. For sufficiently weak discounting, i.e. if Eq. [16] holds, the autocratic, reactive strategy, that cooperates (plays $K$) with probability
\[
p(y) = \frac{b(y) + \chi c(y) + (1 + \lambda)(\chi \kappa Y - \kappa X)}{\lambda(\chi b(K) + c(K))},
\]
after player $Y$ moved $y$, then enables player $X$ to unilaterally enforce $\pi X - \kappa X = \chi (\pi Y - \kappa Y)$ whenever $\kappa X \leq \chi \kappa Y \leq \kappa X + (\chi \kappa K - \kappa_K K)$. Again, whether $p(y)$ translates into an extortiorate or generous strategy depends on $\chi$, $\kappa X$, and $\kappa Y$. Note that $X$ need not specify an initial probability of playing $K$, $p_0$, since the first time she moves follows a move by her opponent and thus she can use $p(y)$ to determine her initial move.

Similarly, setting $\psi(s) := b(s) + \chi \kappa Y - \kappa X$ once again eliminates $x$ from Eq. [4] and enables player $X$ to enforce $\pi Y = \gamma$ with $0 \leq \gamma \leq (\lambda b(K) - c(K)) / (1 + \lambda)$, which implicitly requires $\lambda \geq c(K) / b(K)$. This equalizer strategy plays $K$ with probability
\[
p(y) = \frac{c(y) + (1 + \lambda)(\chi \kappa Y - \kappa X)}{\lambda b(K)},
\]
after player $Y$ played $y$. Although player $X$ can set the score of player $Y$, she cannot set her own score.

3.2.4. Randomly-alternating moves. In games with randomly-alternating moves, the average payoffs, $\pi X$ and $\pi Y$, for players $X$ and $Y$, respectively, depend on the probability $\omega_X$ with which player $X$ moves in any given round; see Eq. [7]. The region spanned by feasible payoff pairs, $(\pi Y, \pi X)$, not only depends on $\omega_X$ but also on the class of strategies considered; see Fig. [3]. In particular, the two-point strategies based on the extreme actions, 0 and $K$, cover only a portion of the payoff region spanned by strategies utilizing the full action space, [0, $K$]. Here, we focus on autocratic two-point strategies. For examples of strategies that exploit larger portions of the action space, see [3].
For randomly-alternating moves, the baseline payoffs for full cooperation are
\[ \kappa_X^{KK} = (1 - \omega_X) b(K) - \omega_X c(K) \; ; \; \kappa_Y^{KK} = \omega_X b(K) - (1 - \omega_X) c(K), \] (24)
while those for mutual defection remain both 0. Suppose that discounting is sufficiently weak, or interactions cover sufficiently many rounds, i.e.
\[ \lambda \geq 1 \frac{\omega_X}{\omega_X} \left( \frac{b(K) + \chi c(K)}{(\chi + 1) b(K) + c(K)} \right), \] (25)
and that \( \kappa_X \leq \kappa_Y \leq \kappa_X + (\kappa_X^{KK} - \kappa_X^{KK}) \). Then, the autocratic two-point strategy that (i) initially plays \( K \) (as opposed to 0) with probability \( p_0 \), where
\[ \max \left\{ \frac{\chi \kappa Y - \kappa_X + b(K) + \chi c(K)}{(1 - \lambda) \omega_X (\chi + 1) (b(K) + c(K)) - \lambda} \right\} \leq p_0 \leq \min \left\{ \frac{\chi \kappa Y - \kappa_X}{(1 - \lambda) \omega_X (\chi + 1) (b(K) + c(K))}, 1 \right\}, \] (26)
and (ii) subsequently plays \( K \), following an action of \( s \) (played by either \( X \) or \( Y \)), with probability
\[ p(s) = \frac{b(s) + \chi c(s) + \chi \kappa Y - \kappa_X}{\lambda \omega_X (\chi + 1) (b(K) + c(K))} - \frac{1 - \lambda}{\lambda} p_0 \] (27)
enables player \( X \) to unilaterally enforce \( \pi_X - \kappa_X = \chi (\pi_Y - \kappa_Y) \). \( \psi(s) := -(\chi + 1)(b(s) + c(s)) \) was chosen as a scaling function so that \( X \)'s response depends on the previous action but not on which player used it.

If \( \omega_X \geq 1/2 \), meaning that player \( X \) is at least as likely to move in each round as is player \( Y \), then, for every \( \chi \geq 1 \), a sufficiently weak discounting factor exists that satisfies \( \lambda \leq 1 \) and Eq. (25), and hence enables player \( X \) to enforce \( \pi_X - \kappa_X = \chi (\pi_Y - \kappa_Y) \). In particular, both extortionate and generous autocratic strategies exist for the randomly-alternating, continuous Donation Game; see Fig. 4.

Similarly, an equalizing two-point strategy for player \( X \) can ensure \( \pi_Y = \gamma \) for any
\[ 0 \leq \gamma \leq \omega_X b(K) - (1 - \omega_X) c(K). \] (28)
by (i) initially playing $K$ with probability $p_0$, where
\[
\max \left\{ \frac{\gamma - \lambda \omega_X b(K) + (1 - \lambda \omega_X) c(K)}{(1 - \lambda) \omega_X (b(K) + c(K))}, 0 \right\} \leq p_0 \leq \min \left\{ \frac{\gamma}{(1 - \lambda) \omega_X (b(K) + c(K))}, 1 \right\},
\]
and (ii) subsequently with probability
\[
p(s) = \frac{c(s) + \gamma}{\lambda \omega_X (b(K) + c(K))} - \frac{1 - \lambda}{\lambda} p_0,
\]
where $s$ denotes the previous action (regardless of whether player $X$ or $Y$ moved). Note that player $X$ is unable to unilaterally set the payoff of player $Y$ to anything outside the range between unconditional defection, 0, and unconditional cooperation, $\omega_X b(K) - (1 - \omega_X) c(K)$; see §3.2.2. Moreover, it must be true that $\lambda \omega_X b(K) \geq (1 - \lambda \omega_X) c(K)$. For $\omega_X = 1/2$, the discounting factor, $\lambda$, must therefore satisfy $\lambda \geq 2c(K)/(b(K) + c(K))$, which enables player $X$ to set $Y$’s score to anything between 0 and $(b(K) - c(K))/2$. In the limit where player $X$ moves exclusively, $\omega_X \to 1$, player $Y$’s score can be set to at most $b(K)$ for full, unconditional cooperation by player $X$.

Although player $X$ can unilaterally set $Y$’s score, she cannot set her own score to anything above 0, and, for $\omega_X \leq 1/2$, she cannot set her own score to anything at all. However, for sufficiently large $\lambda \omega_X$, player $X$ can guarantee herself non-positive payoffs using an autocratic strategy; see §3.2.2. However, strategies enforcing a return that is worse than for mutual defection may be of limited use. In contrast, in

Figure 4. Two-point extortionate, generous, and equalizer strategies for the randomly-alternating, continuous Donation Game. In the top row, both players move with equal probability in a given round ($\omega_X = 1/2$), whereas in the bottom row player $X$ moves twice as often as player $Y$ ($\omega_X = 2/3$). The extortionate strategies in (A) and (D) enforce $\pi_X = \chi \pi_Y$, while the generous strategies in (B) and (E) enforce $\pi_X - \kappa_X^K = \chi (\pi_Y - \kappa_Y^K)$ with $\chi = 2$ (black) and $\chi = 3$ (blue). The equalizer strategies in (C) and (F) enforce $\pi_Y = \gamma$ with $\gamma = \kappa_Y^K$ (black) and $\gamma = \kappa_Y^K$ (blue). The simulation data in each panel show the average payoffs, $(\pi_Y, \pi_X)$, for player $X$’s two-point strategy against 1000 random memory-one strategies for player $Y$. The benefit function is $b(s) = 5 (1 - e^{-2s})$ and the cost function is $c(s) = 2s$ for action spaces $S_X = S_Y = [0, 2]$. 
the simultaneous version of the continuous Donation Game, player X can never set her own score (McAvoy and Hauert, 2015). This difference is not that surprising: even though player X can exert control over randomly-alternating games for large \( \omega_X \), the structure of the continuous Donation Game precludes her from providing herself positive payoffs through actions of her own.

Just as in the case of simultaneous-move games (McAvoy and Hauert, 2015), if one replaces these two-point strategies by deterministic strategies, which means that X reacts to the previous move of the game by playing an action with certainty (rather than probabilistically), then the autocratic strategies considered here cover a broader range of feasible payoffs (see Fig. 3). A simple example of a deterministic strategy is tit-for-tat in the classical Prisoner’s Dilemma (Axelrod, 1984). We include a more detailed discussion of deterministic autocratic strategies for the randomly-alternating Donation Game in SI.3.2.3.

4. Discussion

Repeated games likely rank among the best studied topics in both classical and evolutionary game theory. The resulting insights have been instrumental for our understanding of strategic behavioral patterns in general, and, in particular, for the evolution of cooperation through reciprocal altruism. All the more it came as a big surprise when Press and Dyson (2012) reported a new class of strategies termed “zero-determinant” or, more generically, autocratic strategies, which enable players to exert unprecedented control in repeated interactions. However, notwithstanding decades of extensive literature on repeated games, alternating interactions have received very little attention when compared to their simultaneous counterparts. This emphasis is particularly puzzling because many, if not most, social encounters among humans and other animals that unfold over several rounds seem better captured by alternating actions of the interacting agents. Moreover, even within the realm of alternating games, it is often assumed that individual turns alternate strictly rather than randomly (Frean, 1994; Hauert and Schuster, 1998; Neill, 2001; Zagorsky et al., 2013).

Here we introduce autocratic strategies for alternating games. Due to similarities with simultaneous-move games, it is perhaps unsurprising that autocratic strategies also exist for strictly-alternating games. However, even so, the continuous Donation Game demonstrates that the autocratic strategies themselves depend on the timing of the players’ moves. What is more surprising, and even unexpected, is the fact that autocratic strategies exist for randomly-alternating games as well. This extension exemplifies the surprising robustness of autocratic strategies by relaxing the original assumptions in three important ways: (i) to allow for discounted payoffs, i.e. to consider finite numbers of rounds in each interaction (Hilbe et al., 2015); (ii) to extend the action set from two distinct actions to infinite action spaces (McAvoy and Hauert, 2015); and now (iii) to admit asynchronous decisions and, in particular, randomly-alternating ones, including asymmetric scenarios where one player acts, on average, more frequently than the other. Under this far more generic setup we demonstrate that autocratic strategies still exist and enable players to enforce extortionate, generous, and equalizer relationships with their opponent.

In the strictly-alternating, continuous Donation Game, autocratic strategies exist for player X provided that the discounting factor, \( \lambda \), is sufficiently weak, or, equivalently, that interactions span sufficiently many rounds; see Eq. (16). Interestingly, the condition on \( \lambda \) does not depend on whether player X moves first or second and is even identical to the corresponding condition in the synchronous game (McAvoy and Hauert, 2015). In the absence of discounting, \( \lambda = 1 \), actually the same strategy enforces, for instance, an extortionate payoff relationship in simultaneous games as well as alternating games and regardless of whether or not player X moved first. We demonstrate this phenomenon for the classical Donation Game in [3.1] but evidently it extends to the continuous Donation Game.

The condition for the existence of autocratic strategies in the randomly-alternating game, Eq. (25), is similar to that of the strictly-alternating (and simultaneous) games, although slightly stronger. Not surprisingly, this condition depends on the probability that player X moves in a given round, \( \omega_X \). For each type of alternating game, we give examples of simple autocratic two-point strategies in which player X’s actions are restricted to 0 (defect) and K (fully cooperate). Although X can enforce any extortionate, generous, or equalizer payoff relationship in the continuous Donation Game using a two-point strategy, a larger region of feasible payoffs is attainable if X uses a deterministic autocratic strategy (see Fig. 3 and SI.3.2.3 for further details).

While autocratic strategies undoubtedly mark important behavioral patterns, their importance in an evolutionary context is still debated: extortionate strategies perform poorly (Adami and Hintze, 2013),
whereas generous strategies perform much better (Stewart and Plotkin, 2013). In fact, a generous strategy against itself represents a Nash equilibrium in the simultaneous, two-action Prisoner’s Dilemma (Hilbe et al., 2015). However, for extensions to continuous action spaces, such as the continuous Donation Game, even a generous strategy with full mutual cooperation is not necessarily a Nash equilibrium (McAvoy and Hauert, 2015). As it turns out, similar considerations for alternating games are further nuanced because they naturally introduce asymmetries in payoffs for the two players, even if the underlying interaction is symmetric and both players follow the same strategy. In fact, this asymmetry holds for any strictly-alternating game with discounting factor $\lambda < 1$ because then it matters which player moved first. Conversely, in randomly-alternating games, the payoffs typically depend on the probability $\omega_X$ with which player X moves and hence differ if $\omega_X \neq 1/2$. Consequently, even if player X and Y nominally adopt the same autocratic strategy, then player X does not necessarily enforce the same linear relationship on payoffs as player Y, which complicates the notion of equilibria both in the sense of Nash as well as rest points of the evolutionary dynamics.

Among alternating games, the randomly-alternating ones represent perhaps the most promising and relevant setup from a biological perspective (see Nowak and Sigmund, 1994). In the continuous Donation Game, autocratic strategies exist even if the probability that player X moves in a given round differs from that of player Y, $\omega_X \neq 1/2$. Of course, $\omega_X$ must be sufficiently large to ensure that player X is capable of exerting sufficient control over the game to pursue an autocratic strategy. For $\omega_X > 1/2$, this condition always holds in the continuous Donation Game but may also apply under weaker conditions. Interestingly, such asymmetries easily arise from dominance hierarchies. For example, in bouts of social grooming between primates (Foster et al., 2009), subordinate individuals, X, typically groom dominant individuals, Y, more frequently than vice versa and hence $\omega_X > 1/2$. As a consequence, the subordinate player has more autocratic strategies available to impact social grooming than does the dominant player. Thus, autocratic strategies can be particularly useful for exerting control over asymmetric interactions. This observation marks not only an important distinction between autocratic strategies for simultaneous- and alternating-move games but also promises interesting applications to biologically-relevant interactions.

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Here, we prove our main results for each type of alternating game (strictly- and randomly-alternating moves). By a measurable space, we mean a set, $X$, equipped with a $\sigma$-algebra of subsets, $\mathcal{F}(X)$, although we usually suppress $\mathcal{F}(X)$. The notation $\Delta(X)$ indicates the space of all probability measures on $X$, i.e. the set of all measures, $\mu : \mathcal{F}(X) \rightarrow [0, \infty]$, with $\mu(X) = 1$. All functions are bounded and measurable.

SI.1. STRICTLY-ALTENRING GAMES

Let $S_X$ and $S_Y$ be the action spaces available to players $X$ and $Y$, respectively. We assume that these spaces are measurable, but otherwise we impose no restrictions on them. Let $f_X(x)$ and $f_Y(x)$ be the payoffs to players $X$ and $Y$, respectively, when $X$ moves $x \in S_X$. Similarly, let $g_X(y)$ and $g_Y(y)$ be the payoffs to players $X$ and $Y$, respectively, when $Y$ moves $y \in S_Y$. If $\lambda$ is the discounting factor, then one may compress a pair of rounds in which $X$ moves first and $Y$ moves second in order to form two-round payoff functions, $u_X(x, y) := f_X(x) + \lambda g_X(y)$;

$$u_Y(x, y) := f_Y(x) + \lambda g_Y(y).$$

In each of these two-round payoff functions, the payoff from player $Y$’s move is discounted by a factor of $\lambda$ to account for the time difference. If player $X$ moves first, then the entire sequence of play can be grouped into two-round pairs in which $X$ moves first and $Y$ moves second. More specifically, if $(x_0, y_1, x_2, y_3, \ldots)$ is the sequence of play, then this sequence may be rewritten as $((x_0, y_1), (x_2, y_3), \ldots)$. When written in this manner, one may use $u_X$ to express the average payoff to player $X$ for this sequence as

$$\pi_X = (1 - \lambda) \left[ \sum_{t=0}^{\infty} \lambda^{2t} f_X(x_{2t}) + \sum_{t=0}^{\infty} \lambda^{2t+1} g_X(y_{2t+1}) \right]$$

$$= (1 - \lambda) \left[ \sum_{t=0}^{\infty} \lambda^{2t} \left( f_X(x_{2t}) + \lambda g_X(y_{2t+1}) \right) \right]$$

$$= (1 - \lambda) \sum_{t=0}^{\infty} \lambda^{2t} u_X(x_{2t}, y_{2t+1}).$$

Similarly, the average payoff to player $Y$ is $\pi_Y = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^{2t} u_Y(x_{2t}, y_{2t+1})$.

If $Y$ moves first, then a sequence $(y_0, x_1, y_2, x_3, y_4, \ldots)$ may be rewritten as $(y_0, (x_1, y_2), (x_3, y_4), \ldots)$, consisting of an initial move by $Y$ followed by a sequence of two-round pairs in which $X$ moves first and $Y$ moves second. In this case, the average payoff to player $X$ for this sequence of play is

$$\pi_X = (1 - \lambda) \left[ \sum_{t=0}^{\infty} \lambda^{2t+1} f_X(x_{2t+1}) + \sum_{t=0}^{\infty} \lambda^{2t} g_X(y_{2t}) \right]$$

$$= (1 - \lambda) \left[ g_X(y_0) + \sum_{t=0}^{\infty} \lambda^{2t+1} f_X(x_{2t+1}) + \sum_{t=0}^{\infty} \lambda^{2t+2} g_X(y_{2t+2}) \right]$$

$$= (1 - \lambda) \left[ g_X(y_0) + \sum_{t=0}^{\infty} \lambda^{2t+1} \left( f_X(x_{2t+1}) + \lambda g_X(y_{2t+2}) \right) \right]$$

$$= (1 - \lambda) \left[ g_X(y_0) + \sum_{t=0}^{\infty} \lambda^{2t+1} u_X(x_{2t+1}, y_{2t+2}) \right].$$

Similarly, player $Y$ has an average payoff of $\pi_Y = (1 - \lambda) \left[ g_Y(y_0) + \sum_{t=0}^{\infty} \lambda^{2t+1} u_Y(x_{2t+1}, y_{2t+2}) \right]$.

Due to the differences in the expressions for the average payoffs when $X$ moves first and when $Y$ moves first, respectively, we treat each of these cases separately in our study of autocratic strategies.

SI.1.1. X moves first. If $X$ moves first, then a time-$T$ history is an element of $\mathcal{H}^T := \prod_{t=0}^{T-1} \mathcal{H}_t^T$, where

$$\mathcal{H}_t^T := \begin{cases} S_X & t \text{ is even}, \\ S_Y & t \text{ is odd}, \end{cases}$$

(SI.4)
for 0 ≤ t ≤ T − 1. For T = 0, we let ℳ^0 := {∅}, where ∅ is the “empty history,” which indicates that the game has not yet begun. Thus, a time-T history indicates the sequence of play from time t = 0 until (but not including) time t = T. A **behavioral strategy** defines a player’s actions (probabilistically) for any history of play leading up to the current move (see [Fudenberg and Tirole 1991](#)). That is, behavioral strategies for players X and Y, respectively, may be written in terms of the space of histories as maps,

\[ \sigma_X : \bigcup_{T \geq 0} ℳ^{2T} \rightarrow \Delta (S_X) ; \]  

\[ \sigma_Y : \bigcup_{T \geq 0} ℳ^{2T+1} \rightarrow \Delta (S_Y) , \]  

where \( \sqcup \) denotes the disjoint union operator, and \( \Delta (S_X) \) and \( \Delta (S_Y) \) denote the space of probability measures on \( S_X \) and \( S_Y \), respectively. These strategies may be written together more compactly as a map

\[ \sigma : ℳ := \bigcup_{T \geq 0} ℳ^T \rightarrow \Delta (S_X) \sqcup \Delta (S_Y) \]

\[ : h^T \rightarrow \begin{cases} \sigma_X [h^T] & T \text{ is even}, \\ \sigma_Y [h^T] & T \text{ is odd.} \end{cases} \]  

Using \( \sigma \), we define a sequence of measures, \( \{\mu_t\}_{t \geq 0} \), on \( ℳ^{t+1} \) as follows: For \( h^T = (h_0^T, h_1^T, \ldots, h_{T-1}^T) \in ℳ^T \) and 0 ≤ t ≤ T − 1, let \( h^T_t = (h_0^T, h_1^T, \ldots, h_t^T) \in ℳ^{t+1} \). For \( E' \in ℳ(ℳ^t) \) and \( E \in ℳ(ℳ^{t+1}) \), let

\[ \mu_t (E' \times E) := \int_{h^T \in E'} \sigma (h^T, E) \, d\sigma (h^T_{t+1}, h_{t}^T) \cdots d\sigma (h_0^T, h_1^T) \, d\sigma (\emptyset, h_0^T) . \]

For 0 ≤ k ≤ t, let \( \nu_k^t \) be the measure on \( \prod_{i=t-k}^{t} ℳ_i^{t+1} \), which, for \( E \in ℳ \left( \prod_{i=t-k}^{t} ℳ_i^{t+1} \right) \), is defined as

\[ \nu_k^t (E) := \mu_t (ℳ^{t-k} \times E) . \]

In a \((2T + 2)\)-round game (rounds 0 through \( 2T + 1 \)), the expected payoff to player X is

\[ \pi_X^{2T+2} := \int_{h^T_{2T+2} \in ℳ^{2T+2}} \left[ \left( \frac{1 - \lambda}{1 - \lambda^{2T+1}} \right) \sum_{t=0}^{T} \lambda^t u_X (h_{2t}^{T+2}, h_{2t+1}^{T+2}) \right] \]

\[ \, d\sigma (h_{2T+2}^{T+2}, h_{2T+1}^{T+2}) \cdots d\sigma (h_0^{T+2}, h_1^{T+2}) \, d\sigma (\emptyset, h_0^{T+2}) \]

\[ = \left( \frac{1 - \lambda}{1 - \lambda^{2T+1}} \right) \sum_{t=0}^{T} \lambda^t \int_{h^T_{2T+2} \in ℳ^{2T+2}} u_X (h_{2t}^{T+2}, h_{2t+1}^{T+2}) \]

\[ \, d\sigma (h_{2T+2}^{T+2}, h_{2T+1}^{T+2}) \cdots d\sigma (h_0^{T+2}, h_1^{T+2}) \, d\sigma (\emptyset, h_0^{T+2}) \]

\[ = \left( \frac{1 - \lambda}{1 - \lambda^{2T+1}} \right) \sum_{t=0}^{T} \lambda^t \int_{(h_{2t}^{T+2}, h_{2t+1}^{T+2}) \in ℳ^{2T+2} \times ℳ^{2T+2}} u_X (x, y) \, dv_{2T+1}^1 (x, y) . \]

\[ (SI.9) \]
In particular, the limit
\[ \pi_X := \lim_{T \to \infty} \pi_X^{2T+2} = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^2 t \int_{(x,y) \in S_X \times S_Y} u_X (x, y) \, d\nu^t_{2t+1} (x, y) \tag{SI.10} \]
exists since \( f_X \) and \( g_X \) (and hence \( u_X \)) are bounded. Similarly, we define
\[ \pi_Y := (1 - \lambda) \sum_{t=0}^{\infty} \lambda^2 t \int_{(x,y) \in S_X \times S_Y} u_Y (x, y) \, d\nu^t_{2t+1} (x, y). \tag{SI.11} \]

Our main technical lemma is an analogue of Lemma 3.1 of Akin (2015):

**Lemma 1.** For any memory-one strategy, \( \sigma_X [x, y] \), and any \( E \in \mathcal{F}(S_X) \),
\[ \sum_{t=0}^{\infty} \lambda^2 t \int_{(x,y) \in S_X \times S_Y} \left[ \chi_{E \times S_Y} (x, y) - \lambda^2 \sigma_X [x, y] (E) \right] \, d\nu^t_{2t+1} (x, y) = \sigma_X^0 (E) , \tag{SI.12} \]
where \( \sigma_X^0 := \sigma_X [\emptyset] \) is the initial action of player \( X \).

**Proof.** By the definition of the measures \( \{ \nu^t \}_{t \geq 0} \), we have
\[ \int_{(x,y) \in S_X \times S_Y} \chi_{E \times S_Y} (x, y) \, d\nu^t_{2t+1} (x, y) = \nu^0_{2t} (E); \tag{SI.13a} \]
\[ \int_{(x,y) \in S_X \times S_Y} \sigma_X [x, y] (E) \, d\nu^t_{2t+1} (x, y) = \nu^0_{2t+2} (E). \tag{SI.13b} \]
Therefore, it follows that
\[ \sum_{t=0}^{\infty} \lambda^2 t \int_{(x,y) \in S_X \times S_Y} \left[ \chi_{E \times S_Y} (x, y) - \lambda^2 \sigma_X [x, y] (E) \right] \, d\nu^t_{2t+1} (x, y) \]
\[ = \sum_{t=0}^{\infty} \lambda^2 t \left( \nu^0_{2t} (E) - \nu^0_{2t+2} (E) \right) \]
\[ = \nu^0_{2t} (E) - \lim_{t \to \infty} \lambda^{2t+2} \nu^0_{2t+2} (E) \]
\[ = \nu^0_{2t} (E) \]
\[ = \sigma_X^0 (E), \tag{SI.14} \]
which completes the proof. \( \square \)

**Proposition 1.** For any bounded, measurable function, \( \psi : S_X \to \mathbb{R} \),
\[ \sum_{t=0}^{\infty} \lambda^2 t \int_{(x,y) \in S_X \times S_Y} \left[ \psi (x) - \lambda^2 \int_{s \in S_X} \psi (s) \, d\sigma_X [x, y] (s) \right] \, d\nu^t_{2t+1} (x, y) = \int_{s \in S_X} \psi (s) \, d\sigma_X^0 (s). \tag{SI.15} \]

**Proof.** The result follows from Lemma 1 and the dominated convergence theorem. We do not include the details here; the argument is the same as the proof of Proposition 1 of McAvoy and Hauert (2015). \( \square \)

Using Proposition 1, we now prove the first of our main results for strictly-alternating games:

**Theorem 1** (Autocratic strategies for strictly-alternating games, I). Suppose that, for some bounded \( \psi \),
\[ (\alpha f_X (x) + \beta f_Y (x) + \gamma) + \lambda (\alpha g_X (y) + \beta g_Y (y) + \gamma) \]
\[ = \psi (x) - \lambda^2 \int_{s \in S_X} \psi (s) \, d\sigma_X [x, y] (s) - (1 - \lambda^2) \int_{s \in S_X} \psi (s) \, d\sigma_X^0 (s) \tag{SI.16} \]
holds for each \( x \in S_X \) and \( y \in S_Y \). Then, if player \( X \) moves first, \( \left( \sigma^0_X, \sigma_X [x, y] \right) \) enforces the equation

\[
\alpha \pi_X + \beta \pi_Y + \gamma = 0 \quad (\text{SI.17})
\]

for every strategy of player \( Y \). That is, \( \left( \sigma^0_X, \sigma_X [x, y] \right) \) is an autocratic strategy for player \( X \).

**Proof.** If Eq. (SI.16) holds, then, by Proposition 1 and Eqs. (SI.10) and (SI.11),

\[
\begin{align*}
\alpha \pi_X + \beta \pi_Y + \gamma &+ (1 - \lambda) \int_{s \in S_X} \psi (s) \, d\sigma^0_X (s) \\
&= (1 - \lambda) \sum_{t=0}^{\infty} \lambda^{2t} \int_{(x,y) \in S_X \times S_Y} \left[ \psi (x) - \lambda^{2} \int_{s \in S_X} \psi (s) \, d\sigma_X [x, y] (s) \right] \, dv_{2t+1} (x,y) \\
&= (1 - \lambda) \int_{s \in S_X} \psi (s) \, d\sigma^0_X (s),
\end{align*}
\]

(SI.18)

and it follows that \( \alpha \pi_X + \beta \pi_Y + \gamma = 0 \). \( \square \)

### SI.1.2. \( Y \) moves first.

If player \( Y \) moves first, then \( \mathcal{H}^T := \prod_{t=0}^{T-1} \mathcal{H}_t^T \), where

\[
\mathcal{H}_t^T := \begin{cases} S_X & t \text{ is odd}, \\ S_Y & t \text{ is even}. \end{cases} \quad (\text{SI.19})
\]

for \( 0 \leq t \leq T - 1 \). In other words, the set of time-\( T \) histories is obtained from Eq. (SI.4) by swapping \( S_X \) and \( S_Y \). Similarly, behavioral strategies for players \( X \) and \( Y \), respectively, are defined as maps,

\[
\sigma_X : \bigsqcup_{T \geq 0} \mathcal{H}_{2T+1} \longrightarrow \Delta (S_X); \quad (\text{SI.20a})
\]

\[
\sigma_Y : \bigsqcup_{T \geq 0} \mathcal{H}_{2T} \longrightarrow \Delta (S_Y), \quad (\text{SI.20b})
\]

where, again, \( \mathcal{H}^0 := \{ \emptyset \} \) denotes the “empty” history. In this case, we define

\[
\sigma : \mathcal{H} := \bigsqcup_{T \geq 0} \mathcal{H}^T \longrightarrow \Delta (S_X) \sqcup \Delta (S_Y)
\]

\[
: h^T \longmapsto \begin{cases} \sigma_X [h^T] & T \text{ is odd}, \\ \sigma_Y [h^T] & T \text{ is even}. \end{cases} \quad (\text{SI.21})
\]

In terms of \( \sigma \), the measures \( \{ \mu_t \}_{t \geq 0} \) and \( \{ \nu^k_t \}_{t \geq 0} \) are defined in the same way as they were in \( \text{SI.1.1} \).
In a \((2T + 1)\)-round game (rounds 0 through \(2T\)), the expected payoff to player \(X\) is

\[
\pi_{X}^{2T+1} := \int_{h_{2T+1} \in H_{2T+1}} \left[ \frac{1 - \lambda}{1 - \lambda^{2T}} \left( g_X (h_0^{2T+1}) + \sum_{t=0}^{T-1} \lambda^{2t+1} u_X (h_{2t+1}^{2T+1}, h_{2t+2}^{2T+1}) \right) \right] \, d\sigma (h_{2T+1}^{2T+1}, h_{2T+1}^{2T+1}) \cdots d\sigma (h_{0}^{2T+1}, h_{0}^{2T+1}) \sigma (\varnothing, h_{0}^{2T+1})
\]

\[
= \left( \frac{1 - \lambda}{1 - \lambda^{2T}} \right) \int_{h_{1} \in H^{1}} g_X (h_{0}^{1}) \, d\sigma (\varnothing, h_{0}^{1})
\]

\[
+ \left( \frac{1 - \lambda}{1 - \lambda^{2T}} \right) \sum_{t=0}^{T-1} \lambda^{2t+1} \int_{(h_{t+1}^{2T+1}, h_{t+2}^{2T+1}) \in H_{2T+1} \times H_{2T+1}} u_X (h_{2t+1}^{2T+1}, h_{2t+2}^{2T+1}) \, d\nu_{1}^{2T+1} (h_{2t+1}^{2T+1}, h_{2t+2}^{2T+1})
\]

\[
= \left( \frac{1 - \lambda}{1 - \lambda^{2T}} \right) \int_{y_{0} \in S_{V}} g_{Y} (y_{0}) \, d\sigma_{Y}^{0} (y_{0})
\]

\[
+ \left( \frac{1 - \lambda}{1 - \lambda^{2T}} \right) \sum_{t=0}^{T-1} \lambda^{2t+1} \int_{(x,y) \in S_{X} \times S_{Y}} u_{X} (x,y) \, d\nu_{2t+2}^{1} (x,y)
\]  \hspace{1cm} (SI.22)

where \(\sigma_{Y}^{0} := \sigma_{Y} [\varnothing]\) is the initial action of player \(Y\). Thus, we define player \(X\)’s objective function as

\[
\pi_{X} := \lim_{T \to \infty} \pi_{X}^{2T+1}
\]

\[
= (1 - \lambda) \left[ \int_{y_{0} \in S_{V}} g_{Y} (y_{0}) \, d\sigma_{Y}^{0} (y_{0}) + \sum_{t=0}^{\infty} \lambda^{2t+1} \int_{(x,y) \in S_{X} \times S_{Y}} u_{X} (x,y) \, d\nu_{2t+2}^{1} (x,y) \right]. \hspace{1cm} (SI.23)
\]

Similarly, the expected payoff to player \(Y\) (i.e. his objective function) is

\[
\pi_{Y} := (1 - \lambda) \left[ \int_{y_{0} \in S_{V}} g_{Y} (y_{0}) \, d\sigma_{Y}^{0} (y_{0}) + \sum_{t=0}^{\infty} \lambda^{2t+1} \int_{(x,y) \in S_{X} \times S_{Y}} u_{Y} (x,y) \, d\nu_{2t+2}^{1} (x,y) \right]. \hspace{1cm} (SI.24)
\]

Once again, our main technical lemma is an analogue of Lemma 3.1 of [Akin 2015]:

**Lemma 2.** For any memory-one strategy, \(\sigma_{X} [x,y]\), and any \(E \in \mathcal{F} (S_{X})\),

\[
\sum_{t=0}^{\infty} \lambda^{2t+1} \int_{(x,y) \in S_{X} \times S_{Y}} \left[ \chi_{E \times S_{X}} (x,y) - \lambda \sigma_{X} [x,y] (E) \right] \, d\nu_{2t+2}^{1} (x,y) = \lambda \int_{y_{0} \in S_{V}} \sigma_{X} [y_{0}] (E) \, d\sigma_{Y}^{0} (y_{0}), \hspace{1cm} (SI.25)
\]

where \(\sigma_{X}^{0} [y_{0}]\) is the initial action of player \(X\).
\begin{proof}
The definition of \( \nu_k \) for any \( k \geq 0 \), we see that
\[
\int_{(x,y) \in S_X \times S_Y} \chi_{E \times S_Y} (x,y) \, d\nu_{2t+2}^1 (x,y) = \nu_{2t+1}^0 (E) ;
\] (SI.26a)
\[
\int_{(x,y) \in S_X \times S_Y} \sigma_X [x,y] (E) \, d\nu_{2t+2}^1 (x,y) = \nu_{2t+3}^0 (E).
\] (SI.26b)
Therefore, it follows that
\[
\sum_{t=0}^{\infty} \lambda^{2t+1} \int_{(x,y) \in S_X \times S_Y} \left[ \chi_{E \times S_Y} (x,y) - \lambda^2 \sigma_X [x,y] (E) \right] \, d\nu_{2t+2}^1 (x,y)
= \sum_{t=0}^{\infty} \lambda^{2t+1} \left( \nu_{2t+1}^0 (E) - \lambda^2 \nu_{2t+3}^0 (E) \right)
= \lambda \nu_1^0 (E) - \lim_{t \to \infty} \lambda^{2t+3} \nu_{2t+3}^0 (E)
= \lambda \nu_1^0 (E)
= \lambda \int_{y_0 \in S_Y} \sigma_X [y_0] (E) \, d\sigma_Y^0 (y_0),
\] (SI.27)
which completes the proof. \qed
\end{proof}

**Proposition 2.** For any bounded, measurable function, \( \psi : S_X \to \mathbb{R} \),
\[
\sum_{t=0}^{\infty} \lambda^{2t+1} \int_{(x,y) \in S_X \times S_Y} \left[ \psi (x) - \lambda^2 \int_{s \in S_X} \psi (s) \, d\sigma_X [x,y] (s) \right] \, d\nu_{2t+2}^1 (x,y)
= \lambda \int_{y_0 \in S_Y} \int_{s \in S_X} \psi (s) \, d\sigma_X^0 [y_0] (s) \, d\sigma_Y^0 (y_0).
\] (SI.28)

\begin{proof}
The result follows from Lemma 2 and the dominated convergence theorem. We do not include the
details here; the argument is the same as the proof of Proposition 1 of \cite{McAvoy and Hauert 2015}. \qed
\end{proof}

We are now in a position to prove the second of our main results for strictly-alternating games:

**Theorem 2** (Autocratic strategies for strictly-alternating games, II). Suppose that, for some bounded \( \psi \),
\[
\left( \alpha f_X (x) + \beta f_Y (x) + \gamma \right) + \lambda \left( \alpha g_X (y) + \beta g_Y (y) + \gamma \right) + \left( \frac{1 - \lambda^2}{\lambda} \right) \left( \alpha g_X (y_0) + \beta g_Y (y_0) + \gamma \right)
= \psi (x) - \lambda^2 \int_{s \in S_X} \psi (s) \, d\sigma_X [x,y] (s) - (1 - \lambda^2) \int_{s \in S_X} \psi (s) \, d\sigma_X^0 [y_0] (s)
\] (SI.29)
holds for each \( x \in S_X \) and \( y_0, y \in S_Y \). Then, if player \( X \) moves second, \( \left( \sigma_X^0 [y_0], \sigma_X [x,y] \right) \) enforces
\[
\alpha \pi_X + \beta \pi_Y + \gamma = 0
\] (SI.30)
for every strategy of player \( Y \). That is, \( \left( \sigma_X^0 [y_0], \sigma_X [x,y] \right) \) is an autocratic strategy for player \( X \).
Proof. If Eq. (SI.29) holds, then, for any initial action of player \( Y \), \( \sigma_Y^0 \), we have

\[
\left( \alpha f_X (x) + \beta f_Y (x) + \gamma \right) + \lambda \left( \alpha g_X (y) + \beta g_Y (y) + \gamma \right) \\
+ \left( 1 - \lambda^2 \right) \int_{y \in S_Y} \left[ \alpha g_X (y_0) + \beta g_Y (y_0) + \gamma \right] \ d\sigma_Y^0 (y_0)
\]

\[
= \psi (x) - \lambda^2 \int_{s \in S_X} \psi (s) \ d\sigma_X [x, y] (s) - \left( 1 - \lambda^2 \right) \int_{y \in S_Y} \psi (s) \ d\sigma_Y^0 [y_0] (s) \ d\sigma_Y^0 (y_0). \tag{SI.31}
\]

Therefore, by Proposition 2 and Eqs. (SI.23) and (SI.24), we see that, for each \( \sigma_Y^0 \),

\[
\alpha \pi_X + \beta \pi_Y + \gamma + (1 - \lambda) \lambda \int_{y_0 \in S_Y} \int_{s \in S_X} \psi (s) \ d\sigma_X [y_0] (s) \ d\sigma_Y^0 (y_0)
\]

\[- (1 - \lambda) \int_{y_0 \in S_Y} \int_{s \in S_X} \left[ \alpha g_X (y_0) + \beta g_Y (y_0) + \gamma \right] \ d\sigma_Y^0 (y_0)
\]

\[= (1 - \lambda) \sum_{t=0}^{\infty} \lambda^{2t+1} \int_{(x, y) \in S_X \times S_Y} \left[ \psi (x) - \lambda^2 \int_{s \in S_X} \psi (s) \ d\sigma_X [x, y] (s) \right] \ d\nu_{2t+2} (x, y)
\]

\[- (1 - \lambda) \sum_{t=0}^{\infty} \lambda^{2t+1} \left( \left( 1 - \frac{\lambda^2}{\lambda} \right) \int_{y_0 \in S_Y} \left[ \alpha g_X (y_0) + \beta g_Y (y_0) + \gamma \right] \ d\sigma_Y^0 (y_0) \right)
\]

\[= (1 - \lambda) \lambda \int_{y_0 \in S_Y} \int_{s \in S_X} \psi (s) \ d\sigma_X [y_0] (s) \ d\sigma_Y^0 (y_0)
\]

\[- (1 - \lambda) \int_{y_0 \in S_Y} \int_{s \in S_X} \left[ \alpha g_X (y_0) + \beta g_Y (y_0) + \gamma \right] \ d\sigma_Y^0 (y_0), \tag{SI.32}
\]

and it follows immediately that \( \alpha \pi_X + \beta \pi_Y + \gamma = 0 \). \( \square \)

### SI.2. Randomly-alternating Games

In each round of a randomly-alternating game, player \( X \) moves with probability \( \omega_X \) and player \( Y \) moves with probability \( 1 - \omega_X \) for some \( 0 \leq \omega_X \leq 1 \). For \( T \geq 1 \), a time-\( T \) history is an element of the space

\[ \mathcal{H}^T := \left( S_X \sqcup S_Y \right)^T, \tag{SI.33} \]

where \( S_X \sqcup S_Y \) denotes the disjoint union of the action spaces of the players, \( S_X \) and \( S_Y \). As in [SI.1], we let \( \mathcal{H}^0 := \{ \varnothing \} \), where \( \varnothing \) indicates the “empty history.” In terms of the space of all possible histories, \( \mathcal{H} := \{ \varnothing \} \sqcup \bigcup_{T \geq 1} \mathcal{H}^T \), behavioral strategies for players \( X \) and \( Y \), respectively, are maps,

\[
\sigma_X : \mathcal{H} \rightarrow \Delta (S_X); \tag{SI.34a}
\]

\[
\sigma_Y : \mathcal{H} \rightarrow \Delta (S_Y). \tag{SI.34b}
\]

These strategies may be written more compactly as a single map, \( \sigma : \mathcal{H} \rightarrow \Delta (S_X \sqcup S_Y) \), defined for \( h^T \in \mathcal{H}^T \) and \( E \in \mathcal{F} (S_X \sqcup S_Y) \) via \( \sigma [h^T] (E) := \omega_X \sigma_X [h^T] (E \cap S_X) + (1 - \omega_X) \sigma_Y [h^T] (E \cap S_Y) \). Furthermore, if \( \mathcal{H}_t := S_X \sqcup S_Y \) for \( 0 \leq t \leq T - 1 \), then these two strategies, \( \sigma_X \) and \( \sigma_Y \), together generate a sequence of probability measures, \( \{ \mu_t^0 \}_{t \geq 0} \), on \( \mathcal{H}_t \) for each \( t \), defined via Eqs. (SI.7) and (SI.8).

Consider the single-round payoff function for player \( X \), \( u_X : S_X \sqcup S_Y \rightarrow \mathbb{R} \), defined by

\[
u_X (s) := \begin{cases} f_X (s) & s \in S_X, \\ g_X (s) & s \in S_Y. \end{cases} \tag{SI.35}
\]
Lemma 3. By the definition of the sequence of measures, where

\[
\nu_{t+1} := \int_{h_{t+1} \in \mathcal{G}_{T+1}} \left[ \left( \frac{1 - \lambda}{1 - \lambda^{T+1}} \right) \sum_{t=0}^{T} \lambda^t u_Y (h_t^{T+1}) \right] \, d\sigma (h_{t+1}^{T+1}) \, \cdots \, d\sigma (h_1^{T+1}) \, d\sigma (\emptyset, \emptyset)
\]

Therefore, a memory-one strategy for player \( X \) to know which player moved last since, in any given round, either

\[
\text{strictly-alternating games. Instead of simply reacting to the previous moves of the players, one also needs}
\]

Similarly, the objective function of player \( Y \) is

\[
\pi_Y := (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t \int_{s \in \mathcal{S}_X \cup \mathcal{S}_Y} u_Y (s) \, d\nu_t^0 (s).
\]

Therefore, we define the objective function of player \( X \) to be

\[
\pi_X := \lim_{T \to \infty} \pi_X^{T+1} = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t \int_{s \in \mathcal{S}_X \cup \mathcal{S}_Y} u_X (s) \, d\nu_t^0 (s).
\]

A memory-one strategy in the context of randomly-alternating games looks slightly different from that of strictly-alternating games. Instead of simply reacting to the previous moves of the players, one also needs to know which player moved last since, in any given round, either \( X \) or \( Y \) could move (provided \( \omega_X \neq 0, 1 \).

Therefore, a memory-one strategy for player \( X \) consists of an action policy, \( \sigma_X [x] \), when \( X \) moves \( x \) in the previous round, and a policy, \( \sigma_Y [y] \), when \( Y \) moves \( y \) in the previous round. More succinctly, we let

\[
\sigma_X [s] := \begin{cases} 
\sigma_X [s] & s \in \mathcal{S}_X, \\
\sigma_Y [s] & s \in \mathcal{S}_Y.
\end{cases}
\]

One final time, our main technical lemma is an analogue of Lemma 3.1 of Akin (2015):

**Lemma 3.** For memory-one strategies, \( \sigma_X [x] \) and \( \sigma_Y [y] \), and \( E \in \mathcal{F}(\mathcal{S}_X) \), we have

\[
\sum_{t=0}^{\infty} \lambda^t \int_{s \in \mathcal{S}_X \cup \mathcal{S}_Y} \left[ \chi_E (s) - \lambda \omega_X \sigma_X [s] (E) \right] \, d\nu_t^0 (s) = \omega_X \sigma_X^0 (E),
\]

where \( \sigma_X [s] \) is defined via Eq. (SI.39).

**Proof.** By the definition of the sequence of measures, \( \{\nu_t^0\}_{t \geq 0} \),

\[
\int_{s \in \mathcal{S}_X \cup \mathcal{S}_Y} \chi_E (s) \, d\nu_t^0 (s) = \nu_t^0 (E);
\]

\[
\int_{s \in \mathcal{S}_X \cup \mathcal{S}_Y} \omega_X \sigma_X [s] (E) \, d\nu_t^0 (s) = \nu_{t+1}^0 (E).
\]

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Therefore, we see that

\[
\sum_{t=0}^{\infty} \lambda^t \int_{s \in S_X \sqcup S_Y} \left[ \chi_E(s) - \lambda \omega_X \sigma_X[s](E) \right] d\nu_t^0(s)
\]

\[
= \sum_{t=0}^{\infty} \lambda^t \left( \nu_t^0(E) - \lambda \nu_{t+1}^0(E) \right)
\]

\[
= \nu_0^0(E) - \lim_{t \to \infty} \lambda^{t+1} \nu_{t+1}^0(E)
\]

\[
= \nu_0^0(E)
\]

\[
= \omega_X \sigma_X^0(E),
\]

which completes the proof.

\[\square\]

**Proposition 3.** For any bounded, measurable function, \(\psi : S_X \sqcup S_Y \to \mathbb{R}\), with \(\text{supp} \psi \subseteq S_X\),

\[
\sum_{t=0}^{\infty} \lambda^t \int_{s \in S_X \sqcup S_Y} \left[ \psi(s) - \lambda \omega_X \int_{s' \in S_X} \psi(s') \sigma_X[s'](s') \right] d\nu_t^0(s) = \omega_X \int_{s \in S_X} \psi(s) d\sigma_X^0(s).
\]

\[\text{(SI.43)}\]

**Proof.** The result follows from Lemma 3 and the dominated convergence theorem. We do not include the details here; the argument is the same as the proof of Proposition 1 of (McAvoy and Hauert 2015). \[\square\]

**Proposition 3** allows us to prove our main result for randomly-alternating games:

**Theorem 3** (Autocratic strategies for randomly-alternating games). If, for some bounded \(\psi\),

\[
\alpha f_X(x) + \beta f_Y(x) + \gamma = \psi(x) - \lambda \omega_X \int_{s \in S_X} \psi(s) d\sigma_X^X[s](s) - (1 - \lambda) \omega_X \int_{s \in S_X} \psi(s) d\sigma_X^X(s);
\]

\[\text{(SI.44a)}\]

\[
\alpha g_X(y) + \beta g_Y(y) + \gamma = -\lambda \omega_X \int_{s \in S_X} \psi(s) d\sigma_X^X[y](s) - (1 - \lambda) \omega_X \int_{s \in S_X} \psi(s) d\sigma_X^X(s)
\]

\[\text{(SI.44b)}\]

for each \(x \in S_X\) and \(y \in S_Y\), then, by playing \((\sigma_X^0, \sigma_X^X[x], \sigma_X^X[y])\), player \(X\) ensures that

\[
\alpha \pi_X + \beta \pi_Y + \gamma = 0
\]

\[\text{(SI.45)}\]

for any strategy of player \(Y\). In other words, \((\sigma_X^0, \sigma_X^X[x], \sigma_X^X[y])\) is an autocratic strategy for player \(X\).

**Proof.** If Eq. (SI.44) holds, then, by Proposition 3 and Eqs. (SI.37) and (SI.38),

\[
\alpha \pi_X + \beta \pi_Y + \gamma + (1 - \lambda) \omega_X \int_{s \in S_X} \psi(s) d\sigma_X^0(s)
\]

\[
= (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t \int_{s \in S_X \sqcup S_Y} \left[ \psi(s) - \lambda \omega_X \int_{s' \in S_X} \psi(s') \sigma_X[s'](s') \right] d\nu_t^0(s)
\]

\[
= (1 - \lambda) \omega_X \int_{s \in S_X} \psi(s) d\sigma_X^0(s),
\]

\[\text{(SI.46)}\]

from which it follows that \(\alpha \pi_X + \beta \pi_Y + \gamma = 0\), as desired. \[\square\]

**SI.2.1. Two-point autocratic strategies.** Suppose that \(X\) wishes to enforce \(\alpha \pi_X + \beta \pi_Y + \gamma = 0\) with

\[
\sigma_X^0 = p_0 \delta_{s_1} + (1 - p_0) \delta_{s_2};
\]

\[\text{(SI.47a)}\]

\[
\sigma_X^X[x] = p^X(x) \delta_{s_1} + \left(1 - p^X(x)\right) \delta_{s_2};
\]

\[\text{(SI.47b)}\]

\[
\sigma_X^Y[y] = p^Y(y) \delta_{s_1} + \left(1 - p^Y(y)\right) \delta_{s_2};
\]

\[\text{(SI.47c)}\]
for some \( s_1 \) and \( s_2 \) in \( S_X \). Consider the function, \( \varphi : S_X \sqcup S_Y \to \mathbb{R} \), defined by

\[
\varphi (s) := \begin{cases} \alpha f_X (s) + \beta f_Y (s) + \gamma & s \in S_X; \\ \alpha g_X (s) + \beta g_Y (s) + \gamma & s \in S_Y. \end{cases}
\] (SI.48)

Then, in terms of \( \varphi \), it must be the case that

\[
p^X (x) = \frac{1}{\lambda X} \left( \varphi (x) - \varphi (x) - (1 - \lambda) \omega X \left( \varphi (s_1) p_0 + \varphi (s_2) (1 - p_0) \right) \right) - \varphi (s_2);
\] (SI.49a)

\[
p^Y (y) = \frac{1}{\lambda X} \left( -\varphi (y) - (1 - \lambda) \omega X \left( \varphi (s_1) p_0 + \varphi (s_2) (1 - p_0) \right) \right) - \varphi (s_2).
\] (SI.49b)

Therefore, provided \( 0 \leq p^X (x) \leq 1 \) and \( 0 \leq p^Y (y) \leq 1 \) for each \( x \in S_X \) and \( y \in S_Y \), the two-point strategy defined by Eq. (SI.47) allows player \( X \) to unilaterally enforce the relationship \( \alpha \pi_X + \beta \pi_Y + \gamma = 0 \).

SI.2.2. **Deterministic autocratic strategies.** Suppose that \( X \) wishes to enforce \( \alpha \pi_X + \beta \pi_Y + \gamma = 0 \) by using a deterministic strategy, which is defined in terms of a reaction function to the previous move of the game. That is, a deterministic memory-one strategy for player \( X \) consists of an initial action, \( x_0 \in S_X \), and two reaction functions, \( r^X : S_X \to S_X \) and \( r^Y : S_Y \to S_Y \). Player \( X \) begins by using \( x_0 \) with certainty. If \( X \) plays \( x \) in the previous round and \( X \) moves again, then \( X \) plays \( r^X (x) \) in the subsequent round. On the other hand, if \( Y \) moves \( y \) in the previous round and \( X \) follows this move, then \( X \) plays \( r^Y (y) \) in response to \( Y \)'s action. For a deterministic strategy with these reaction functions, Eq. (SI.44) takes the form

\[
\alpha f_X (x) + \beta f_Y (x) + \gamma = \psi (x) - \lambda \omega X \psi (r^X (x)) - (1 - \lambda) \omega X \psi (x_0); 
\] (SI.50a)

\[
\alpha g_X (y) + \beta g_Y (y) + \gamma = -\lambda \omega X \psi (r^Y (y)) - (1 - \lambda) \omega X \psi (x_0). 
\] (SI.50b)

We give examples of these deterministic strategies for the continuous Donation Game in [SI.3.2.3]

### SI.3. Continuous Donation Game

The results we give for the continuous Donation Game hold for any benefit and cost functions, \( b (s) \) and \( c (s) \), and any interval of cooperation levels, \( [0, K] = S_X = S_Y \). For the purposes of plotting feasible regions (Figs. 3, 5, 6) and for performing simulations (Figs. 4 and 7), we use for benefit and cost functions

\[
b (s) := 5 (1 - e^{-2s}) ; 
\] (SI.51a)

\[
c (s) := 2s, 
\] (SI.51b)

respectively, and these functions are defined on the interval \( [0, 2] = S_X = S_Y \) (see Killingback et al. 1999, Killingback and Doebeli 2002).

SI.3.1. **Strictly-alternating moves.** Figs. 5 and 6 show the feasible payoff regions for three values of \( \lambda \) in the strictly-alternating game when \( X \) moves first (Fig. 5) and when \( Y \) moves first (Fig. 6). Unlike in the randomly-alternating, continuous Donation Game, these regions depend on the discounting factor, \( \lambda \).

SI.3.2. **Randomly-alternating moves.** Here we give more extensive arguments for the results claimed in the main text for the randomly-alternating, continuous Donation Game.

SI.3.2.1. **Extortionate and generous strategies.** Suppose that, via Eq. (SI.44), \( X \) can enforce \( \pi_X = \chi \pi_Y - \gamma \) for some \( \chi \geq 1 \) and \( \gamma \in \mathbb{R} \). Then, for some bounded function, \( \psi : S_X \sqcup S_Y \to \mathbb{R} \) (supported on \( S_X \)),

\[
-c (x) - \chi b (x) + \gamma = \psi (x) - \lambda \omega X \int_{s \in S_X} \psi (s) \, d\sigma^X_X [x] (s) - (1 - \lambda) \omega X \int_{s \in S_X} \psi (s) \, d\sigma^X_X (s); 
\] (SI.52a)

\[
b (y) + \chi c (y) + \gamma = -\lambda \omega X \int_{s \in S_X} \psi (s) \, d\sigma^Y_X [y] (s) - (1 - \lambda) \omega X \int_{s \in S_X} \psi (s) \, d\sigma^Y_X (s) 
\] (SI.52b)

for each \( x, y \in [0, K] = S_X = S_Y \). Eq. (SI.52a) implies that

\[
(1 - \omega X) \sup \psi \leq \gamma \leq \chi b (K) + c (K) + (1 - \omega X) \inf \psi, 
\] (SI.53)
and Eq. (SI.52b) implies that
\[ -\omega_X \sup \psi \leq \gamma \leq -b(K) - \chi c(K) - \omega_X \inf \psi. \] (SI.54)

It follows at once from these inequalities that
\[ 0 \leq \gamma \leq \chi \left( \omega_X b(K) - (1 - \omega_X) c(K) \right) - \left( (1 - \omega_X) b(K) - \omega_X c(K) \right). \] (SI.55)

In particular, if \( \gamma = \chi \kappa_Y - \kappa_X \), then it must be true that
\[ \kappa_X = \kappa_X + (\chi \kappa_Y^{00} - \kappa_X^{00}) \leq \kappa_Y \leq \kappa_X + (\chi \kappa_Y^{KK} - \kappa_X^{KK}), \] (SI.56)

which is simply Eq. 13 in the main text.
SI.3.2.2. Equalizer strategies. Player $X$ can ensure that $\pi_X = \gamma$ if $\sigma_X$ satisfies

$$-c(x) - \gamma = \psi(x) - \lambda \omega_X \int_{s \in S_X} \psi(s) \, d\sigma_X^X \{x\}(s) - (1 - \lambda) \omega_X \int_{s \in S_X} \psi(s) \, d\sigma_X^0(s); \quad \text{(SI.57a)}$$

$$b(y) - \gamma = -\lambda \omega_X \int_{s \in S_X} \psi(s) \, d\sigma_Y^X \{y\}(s) - (1 - \lambda) \omega_X \int_{s \in S_X} \psi(s) \, d\sigma_X^0(s) \quad \text{(SI.57b)}$$

for some bounded function, $\psi$, and each $x, y \in [0, K]$. Eq. \textbf{SI.57a} implies that

$$\gamma \leq -(1 - \omega_X) \sup \psi \quad \text{(SI.58)}$$

and Eq. \textbf{SI.57b} implies that

$$\gamma \leq \omega_X \sup \psi, \quad \text{(SI.59)}$$

thus $\gamma \leq 0$. Therefore, player $X$ can unilaterally set her own score to at most 0. Since $X$ can achieve an average payoff of at least 0 by defecting in every round, self-equalizing strategies are not interesting in the continuous Donation Game. However, it should be noted that, in contrast to the continuous Donation Game with simultaneous moves, it is possible for a player to set her own score (to at most 0) when moves alternate randomly. For example, if $\gamma = 0$ and $\psi(s) = -\frac{1}{\omega_X} b(s)$, then player $X$ can unilaterally set $\pi_X = 0$ using

$$\sigma_X^0 = \delta_0; \quad \text{(SI.60a)}$$

$$\sigma_X^X \{x\} = \left(1 - p^X(x)\right) \delta_0 + p^X(x) \delta_K; \quad \text{(SI.60b)}$$

$$\sigma_Y^Y \{y\} = \delta_y, \quad \text{(SI.60c)}$$

where $p^X(x) = \frac{1}{b(K)} \left(\frac{1}{\omega_X} b(x) - c(x)\right)$, provided $\lambda \omega_X$ is sufficiently close to 1. Nevertheless, if $\omega_X = 1/2$, then player $X$ can never set her own score in the continuous Donation Game: Eq. \textbf{SI.57} implies that $\lambda \omega_X \geq \frac{b(K)}{p(K) + c(K)} > 1/2$, which can never hold for $\omega_X = 1/2$ and $0 \leq \lambda \leq 1$. Therefore, although in some pathological cases a player can set her own score to some nonpositive value, a player can never set her own score in the continuous Donation Game for the typical value of $\omega_X = 1/2$. Regardless, even when a player can set her own score in the continuous Donation Game, this score can be at most 0; since a player can achieve at least 0 by defecting in every round, such an equalizer strategy would never be attractive.

We saw in Section 3.2.4 that player $X$ can set $\pi_Y = \gamma$ for any $0 \leq \gamma \leq \omega_X b(K) - (1 - \omega_X) c(K)$. Here we show that, using Eq. \textbf{SI.57}, there are no other payoffs for player $Y$ that $X$ can set unilaterally. Indeed, suppose

$$b(x) - \gamma = \psi(x) - \lambda \omega_X \int_{s \in S_X} \psi(s) \, d\sigma_X^X \{x\}(s) - (1 - \lambda) \omega_X \int_{s \in S_X} \psi(s) \, d\sigma_X^0(s); \quad \text{(SI.61a)}$$

$$-c(y) - \gamma = -\lambda \omega_X \int_{s \in S_X} \psi(s) \, d\sigma_Y^X \{y\}(s) - (1 - \lambda) \omega_X \int_{s \in S_X} \psi(s) \, d\sigma_X^0(s) \quad \text{(SI.61b)}$$

for some bounded function, $\psi$, and each $x, y \in [0, K]$. From Eq. \textbf{SI.61a}, we see that

$$- (1 - \omega_X) \inf \psi \leq \gamma \leq b(K) - (1 - \omega_X) \sup \psi. \quad \text{(SI.62)}$$

Similarly, Eq. \textbf{SI.61b} implies that

$$\omega_X \inf \psi \leq \gamma \leq -c(K) + \omega_X \sup \psi. \quad \text{(SI.63)}$$

These inequalities immediately give $0 \leq \gamma \leq \min \{b(K) - (1 - \omega_X) \sup \psi, -c(K) + \omega_X \sup \psi\}$. Since

$$b(K) - (1 - \omega_X) \sup \psi \leq -c(K) + \omega_X \sup \psi \iff b(K) + c(K) \leq \sup \psi, \quad \text{(SI.64)}$$

it follows that $0 \leq \gamma \leq \omega_X b(K) - (1 - \omega_X) c(K) = K^{K^K}$.
Deterministic autocratic strategies. In [SI.2.2] we saw that deterministic autocratic strategies for randomly-alternating games consist of (i) an initial action, $x_0$; a reaction function to one’s own move, $r^X : S_X \rightarrow S_X$; and a reaction function to the opponent’s move, $r^Y : S_Y \rightarrow S_X$. Here, we give examples of three types of deterministic strategies for the continuous Donation Game: extortionate, generous, and equalizer. For example, player $X$ can enforce the relationship $\pi_X - \kappa_X = \chi (\pi_Y - \kappa_Y)$ by using

$$r^X (x) = (b + c)^{-1} \left( \frac{b(x) + \chi c(x) + \chi \kappa_Y - \kappa_X - (1 - \lambda) \omega_X (\chi + 1) \left( b(x_0) + c(x_0) \right)}{\lambda \omega_X (\chi + 1)} \right), \quad (\text{SI.65a})$$

$$r^Y (y) = r^X (y), \quad (\text{SI.65b})$$

where $(b + c)^{-1} \cdots$ denotes the inverse of the function $b(s) + c(s)$. If $\kappa_X = \kappa_X^{00} = 0$ and $\kappa_Y = \kappa_Y^{00} = 0$, then $X$ may use $x_0 = 0$ to enforce the extortionate relationship $\pi_X = \chi \pi_Y$. If $\kappa_X = \kappa_X^{KK}$ and $\kappa_Y = \kappa_Y^{KK}$, then $X$ may use $x_0 = K$ to enforce the generous relationship $\pi_X - \kappa_X^{KK} = \chi (\pi_Y - \kappa_Y^{KK})$. In both cases, $\lambda$ must satisfy Eq. (25) for $r^X$ and $r^Y$ to be well-defined.

Similarly, $X$ can unilaterally set $Y$’s payoff to $\pi_Y = \gamma$ by using

$$r^X (x) = (b + c)^{-1} \left( \frac{c(x) + \gamma - (1 - \lambda) \omega_X (b(x_0) + c(x_0))}{\lambda \omega_X} \right); \quad (\text{SI.66a})$$

$$r^Y (y) = r^X (y). \quad (\text{SI.66b})$$

If $\gamma = \kappa_Y^{00} = 0$, then player $X$ may use $x_0 = 0$; if $\gamma = \kappa_Y^{KK}$, then $X$ may use $x_0 = K$.

Simulation data for each of these classes of strategies are given in Fig. 7.