Rotating Black Holes in Higher Dimensional Brane Worlds

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ABSTRACT

A black string generalization of the Myers-Perry N dimensional rotating black hole is considered in an (N+1) dimensional Randall-Sundrum brane world. The black string intercepts the (N-1) brane in a N dimensional rotating black hole. We examine the diverse cases arising for various non-zero rotation components and obtain the geodesic equations for these space-time. The asymptotics of the resulting brane world geometries and their implications are discussed.

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1 Introduction.

String theory and the unification of gravity with the other fundamental forces have led to the consideration of higher dimensional space-time at ultrashort length scales. It has been generally assumed that the extra spatial dimensions are small and compactified at the Planck scale. Recent investigations, however, led to the possibility of consistent models with large compact dimensions also. This however required the assumption that the electroweak scale was the fundamental scale and equal to the higher dimensional Planck scale. The large four dimensional Planck scale was then a derived scale enhanced by the (large) volume factor. Furthermore, such a scenario required that the gauge sector of the fundamental interactions are restricted on a three brane (or a smooth four dimensional hypersurface) [1]. This brane world model removed the hierarchy between the electroweak and the Planck scale. However, it was reintroduced in a hierarchy of scales between the large and the small dimensions. Nonetheless, these models inspired the exciting phenomenological possibility of low scale quantum gravity which may be accessible in the next generation accelerators. Subsequently Randall and Sundrum (RS) [2] considered a five dimensional model with a metric involving warped compactification and the standard model interactions being restricted on a three brane, as a resolution to the hierarchy problem. This model also required a regulator three brane located at a certain distance from the first one in the extra fifth spatial direction. In this case, the hierarchy of scales are absent as the compactified directions remained small in length. In a subsequent variant the regulator brane could be shifted an infinite distance away leading to a model with a single three brane and a large (non-compact) extra fifth direction not sensitive to the gauge interactions. The gravitational interactions however were free to propagate in all the dimensions. For such a model to be an acceptable solution of the five dimensional Einstein equation the corresponding space-time was required to be a slice of an Anti deSitter space in five dimensions with reflection symmetry at the location of the three brane. In this framework the zero mode of the Kaluza Klein graviton was localized on the three brane in a linearized approximation and acted as the source for the usual weak four dimensional gravity. Last couple of years have witnessed intense investigations in this area leading to a large number of insights into the exciting consequences of these interesting models. It was found that the conclusions of the linearized analysis carried over identically to any solution with a Ricci flat metric. A full non linear treatment in the framework of
supergravity \cite{3} have confirmed the conclusions resulting from the linearized approximation. In particular, these constructions have led to the possibility of detecting low scale quantum gravity effects on phenomenology at the weak scale in the next generation particle accelerators \cite{4}.

For these models to be consistent, General Relativity and its conclusions in four dimensions should be reproduced. In particular it should be possible to obtain cosmological and black hole solutions in four dimensions starting from such higher dimensional scenarios. Study of black hole solutions have been an exciting aspect of investigations in this area. Such a four dimensional black hole is expected to be extended in the extra spatial AdS direction and would thus be a higher dimensional object in the brane world. In an interesting article Chamblin, Hawking and Reall (CHR) \cite{5} considered the description of a Schwarzschild black hole on the three brane in a RS brane world. The bulk solution in this case was a five dimensional black string. This solution showed the usual Schwarzschild singularity on the three brane in a linearized framework and was extended along the transverse fifth direction. Thus it described a usual four dimensional Schwarzschild black hole on the three brane. It was observed however that the solution was singular at the AdS horizon with the singularity there being a p-p curvature singularity \cite{9,3}. It was argued that the solution was unstable far away from the brane due to the Gregory-Laflamme instability \cite{11} and was conjectured to pinch off before reaching the horizon to lead to a cigar like solution. It was later shown that for these AdS solutions it was more likely that there would be an accumulation of mini black holes towards the AdS horizon which did not seem to indicate a cigar geometry. Attempts to consider the off-brane metric in the linearized framework have also appeared in \cite{12}. Further generalizations of these ideas have appeared in \cite{13}. It must however be stated here that the conclusions of Gregory and Laflamme and hence their consequences on the brane world solutions have been questioned in \cite{14}. There it has been shown under very mild assumptions that pinching off of the horizon for a Schwarzschild black string was untenable from stability considerations. It was suggested there that a more likely scenario would be that the system will evolve to an translationally non-invariant intermediate stable solution. In a subsequent article such a solution has been alluded to \cite{14} in the form of the time symmetric initial data. However the argument has not yet been generalized to include the axisymmetric solutions also.

Generalizations of this construction for charged black holes and numerical studies for the off-brane bulk metric has also been performed \cite{16,17,18}. 

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Exact studies of black holes in lower dimensional brane worlds have been undertaken in the context of the AdS C metric for 2 branes embedded in a \((3+1)\) dimensional RS brane world \[15\]. Rotating generalizations of these constructions both for the Kerr like and the BTZ variants have also been discussed. It was observed that the corresponding black hole solutions were non-singular at the horizon which possibly indicates that the singularity at the AdS horizon is an artifact of the linearized analysis. However such exact C-metric solutions are unavailable in the realistic four or higher dimensions.

In an earlier article we have generalized the construction of CHR to rotating black holes on the brane in a five dimensional RS brane world \[19\]. This was obtained by considering a five dimensional rotating black string solution which intercepted the three brane in a four dimensional Kerr black hole. It was found that the Kerr solution too was singular at the AdS horizon apart from the usual ring singularity at \(r = 0\) on the brane. Study of the geodesic equations then showed that it was plausible that the singularity at the AdS horizon in the rotating case was also a p-p curvature singularity although an explicit construction of the parallely propagated orthonormal frame was computationally intractable.

It must be understood that for consistency the RS model (or some variant) must be embedded as a low energy description in an appropriate string theory. Thus it requires a generalization of such models to higher dimensions. Generalization of the Randall-Sundrum construction to arbitrary \((N+1)\) dimensions with an appropriate co-dimension 1 brane is straightforward and may be easily extended to include the full non-linear equations for a Ricci flat metric \[15\]. Furthermore several variants of the RS scenario involving cosmic strings and other global defects of various co-dimensions and their consistency has been investigated in higher dimensions \[20\]. Just as in five dimensions, the conclusions of General Relativity on the appropriate co-dimension brane needs to be reproduced for consistency. In particular it should be possible to obtain known higher dimensional black hole solutions starting from such a (RS like) scenario. The absence of exact C-metrics in dimensions \(D \geq 3\) requires such studies to be based on the linearized approximation and to that end the CHR model is a reasonable approach illustrating the physical issues involved although the obvious problem of boundary singularities persist.

In this article we consider the description of higher dimensional black holes on a codimension 1 brane embedded in a \((N+1)\) dimensional RS brane world with a single AdS direction. In particular we consider the emergence of a generalization of the Kerr solution to \(N\) dimensions due to Myers and
Perry [21] on the (N-1) dimensional brane. The corresponding case for the non-rotating higher dimensional Schwarzschild black hole due to Tangherlini et. al [22] from a higher dimensional brane world is a trivial extension of the CHR construction and we do not explicitly consider it here. The Myers-Perry solution however shows rather interesting behaviour in the fact that in N dimensions the symmetry group of the space-time is $SO(N-1,1)$ which admits multiple rotations in the various coordinate planes consistent with the number of the Casimir operators. Furthermore these solutions show a variety of horizon and singularity structures depending on whether the number of spatial dimensions are even or odd. So it is an interesting exercise to describe these structures from a brane world perspective apart from the motivations outlined earlier. To this end we have considered a rotating black string in the (N+1) dimensional brane world. This black string intercepts the (N-1) brane in a N dimensional rotating black hole solution. We discuss the various cases of horizon and singularity structures with non zero angular momentum parameters and even or odd number of spatial dimensions. We also obtain the geodesic equations for the most general case and discuss their consequences on the asymptotics of these space-time. The article is organized as follows, in the next section we briefly review the Myers and Perry solution for both single and multiple non zero angular momentum components. In section three we consider a black string generalization of the Myers-Perry solution with a single non-zero angular momentum component in a (N+1)dimensional RS brane world. In the section following that we treat the most general case of a rotating black string with multiple angular momentum components in a (N+1) dimensional RS brane world. In the last section we present a summary and our conclusions.

2 Myers-Perry Solution in N Dimensions

In this section we briefly review the Myers-Perry solution [21] for a rotating black hole in N dimensions. In (3 + 1) dimensions the rotating black hole described by a Kerr metric is characterized by the two parameters mass and the angular momentum invariant. In N (N > 4) dimensions the Poincare group contains the $SO(N-1,1)$ Lorentz group. For a massive representation of this group the Casimir invariants are the mass and the $[\frac{N-1}{2}]$ Casimirs of the little group $SO(N-1)$. Hence a rotating black hole in N dimensions is characterized by $[\frac{N-1}{2}] + 1$ parameters being the mass and corresponding
angular momentum invariants.

We first consider the case of a single angular momentum of the N dimensional black hole. For a single non zero angular momentum component the metric in a Boyer-Lindquist coordinate system is given as

\[ ds^2 = \left[ -\left( r^2 + a^2 \frac{\mu^2}{r^{N-5}\rho^2} \sin^2 \theta \right) \right. \]

\[ + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + r^2 \cos^2 \theta d\Omega^{N-4} - \left. \frac{2\mu a}{r^{N-5}\rho^2} d\phi dt \right] \]

where \( \rho^2 = r^2 + a^2 \cos^2 \theta \) and \( d\Omega^{N-4} \) is the line element on a unit (N-4) sphere, where for N=4 the last term is absent. The function \( \Delta = \frac{\mu}{r^{N-5}\rho^2} \) and M is the ADM mass of the black hole which is given as \( M = \frac{(N-2)A_{N-2}}{16\pi} \mu \) and \( A_{N-2} = \frac{2\pi N-1}{\Gamma(\frac{N-1}{2})} \) is the area of the unit (N-2) sphere. This solution possesses only one non-zero angular momentum component in the \((x, y)\) plane in a quasi spheroidal \((x, y, z, t)\) coordinates in the flat space limit \((\mu \to 0)\), given as \[21\]

\[ J_{yx} = \frac{A_{N-2}}{8\pi} \mu a = \frac{2}{N - 2} Ma \] (2)

The horizon occurs at \( g^{rr} = 0 \) and a solution always exists for \( N > 5 \) irrespective of the angular momentum unlike that in four dimensions.

However this result is not the most general, as the non-zero components of the angular momentum will have two spatial components indicating the specific coordinate planes for the rotation. This is related to the number of Casimir invariants of \( SO(N-1) \) or the dimension of its Cartan subalgebra which generates commuting rotations in the corresponding coordinate planes. In the metric presented in eqn. (1) there is only one such non-zero angular momentum component. The general solution will have \[ \left\lceil \frac{N-1}{2} \right\rceil \) such non-zero components along the various coordinate planes. The general metric for both odd and even (N-1) spatial coordinates is given as \[23\]

\[ ds^2 = -dt^2 + \sum_k (r^2 + a_k^2)(\mu_k^2 + \mu_k^2 d\phi_k^2) + \frac{\mu r^{2-\epsilon}}{\Pi F} [dt - \sum_k a_k \mu_k^2 d\phi_k^2]^2 \]

\[ + \frac{\Pi F}{\Pi - \mu r^{2-\epsilon}} dr^2 + \epsilon r^2 d\alpha^2, \] (3)

where \( \epsilon = 0, 1 \) for odd/even N and the sums extend over \( k = \frac{N-2}{2} \) for N even

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and \( k = \frac{N-1}{2} \) for \( N \) odd. The parameters in the metric are given as

\[
\mu = \frac{16\pi}{(N-2)A_{N-2}} M; \quad a_k = \frac{N - 2 J_k}{2 M}; \quad \sum \mu_k^2 + \epsilon \alpha^2 = 1. \tag{4}
\]

Here \( \phi_k \) are angles in each of the planes \( (x^k, y^k) \) and \( \mu_k \) are direction cosines \( (0 \leq \mu_k \leq 1) \) with respect to these planes. The functions \( \Pi \) and \( F \) are given as

\[
F = 1 - \sum_k \frac{a_k^2 \mu_k^2}{r^2 + a_k^2}, \quad \Pi = \prod_k (r^2 + a_k^2). \tag{5}
\]

The metric admits the following Killing isometries \( \partial_t \) and \( \partial_{\phi_k} \) for translations in \( t \) and rotations in \( \phi_k \) justifying its stationary nature in \( N \) dimensions.

The Myers-Perry solution presented above shows interesting horizon and singularity structures depending on the number of non-zero angular momentum components and whether \( N \) is odd or even. It is useful to check the surfaces of constant \( r \) for these solutions in order to clarify their global characteristics. For odd \((N-1)\) spatial components, from a Kerr-Schild form for the metric \[21\] we have the equation

\[
\frac{x^i x^i}{r^2 + a_1^2} + \frac{w^2}{r^2} = 1. \tag{6}
\]

Here \((x^i, y^i)\) are the paired spatial coordinates defining the coordinate planes for the rotation components and \( w \) is the unpaired one with \( \phi_i \) being the angles in the respective coordinate planes defined by the pairings \((x^i, y^i)\). Eqn. (6) defines a family of \((N-2)\) ellipsoids in \( R^{N-1} \) parametrized by \( r \). The intersection of these with \( x^i = y^i = 0 \) gives the familiar two dimensional surfaces of a Kerr metric. These are essentially ellipsoids of revolution about the \( w \) axis or \( S^2 \) squashed in the \( w \) direction. They intersect the \( w \) axis at \( w = r \) and the \((x^1, y^1)\) plane at the circle \( x^1^2 + y^1^2 = r^2 + a_1^2 \) so the \( r = 0 \) surface degenerates to the disk \( x^1^2 + y^1^2 \leq a_1^2 \) in the \((x^1, y^1)\) plane. The entire surface may be described as a squashed \((N-1)\) sphere with rotational symmetry in each \((x^i, y^i)\) plane and the \( r = 0 \) surface degenerates to an \((N - 2p - 2)\) ball where \( p \) is the number of vanishing rotation parameters \( a_i \).

An \( M \) ball is defined as a region of \( R^M \) with an \((M-1)\) sphere as boundary. For even \((N-1)\) number of spatial coordinates with the unpaired coordinate \( w \) absent, we have from similar considerations in the Kerr-Schild form \[21\]

\[
\frac{x^i x^i}{r^2 + a_i^2} = 1. \tag{7}
\]
This describes an ellipsoid again with \( r = 0 \) surface being an \((N - 2p - 1)\) ball if \( p \neq 0 \). If \( p = 0 \) then no surfaces of constant \( r \) intersects the origin.

The horizon for these metrics in the Boyer-Lindquist coordinate system requires analytic solutions for the equation,

\[
\Pi - \mu r^{2-\epsilon} = 0
\]  

(8)

where \( \epsilon \) takes values \((0, 1)\) as mentioned earlier. For odd \((N-1)\) spatial coordinates with no vanishing rotation parameters \( a_i \) it was shown that if an horizon exists then it has the topology of \( S^{N-2} \times R \) with \( r \) positive and three possible cases

\[
\Pi - \mu r > 0 \quad \text{no horizon}
\]

\[
\Pi - \mu r < 0 \quad \text{two horizons}
\]

\[
\Pi - \mu r = 0 \quad \text{one degenerate horizon}
\]  

(9)

as in the familiar Kerr case in four dimensions. It was found that beside \( N=4 \) the equation only has a solution for \( N=6 \) though there was a possibility of solutions for larger \( N \). The vanishing of at least one rotation parameter was found to be a sufficient condition for the existence of a horizon. For even \((N-1)\) spatial coordinates one requires the condition

\[
\Pi - \mu r^2 = 0
\]  

(10)

Once again an analytic solution for arbitrary \( N \) is difficult. For \( N=5 \) however this is just a quadratic equation in \( r^2 \) with the solution

\[
2r^2 \pm = \mu - a_1^2 - a_2^2 \pm \sqrt{(\mu - a_1^2 - a_2^2) - 4a_1^2a_2^2}
\]  

(11)

so the existence of a horizon requires

\[
\mu \geq a_1^2 + a_2^2 + 2 | a_1 a_2 |
\]  

(12)

For arbitrary \( N \) it was found that analytic solutions were possible for \( N=7, 9, 11 \). In this cases also it was observed that there were three cases as in eqn. (9) for \((N-1)\) odd but a single vanishing rotation parameter was insufficient to guarantee the existence of a horizon in this case. It needed at least two such non-zero parameters. Like the four dimensional Kerr solution the Myers-Perry metric too possesses an ergosphere or an infinite redshift surface where
$g_{tt} = 0$. The outer boundary of this is the stationary limit surface which satisfies the equation

$$\Pi F - \mu r^{2-\epsilon} = 0 \quad (13)$$

and has the topology of $S^{N-2} \times R$.

The singularities of these metrics for both odd and even ($N-1$) spatial coordinates are best examined in the Kerr-Schild coordinates. The analysis in [21] shows that the metric is smooth everywhere except where $h = \frac{\mu r^{2-\epsilon}}{\Pi F}$ diverges. For odd ($N-1$) spatial coordinates we may rewrite

$$F = \sum_i \frac{r^2 \mu_i^2}{r^2 + a_i^2} + \alpha^2$$

where $\alpha = \frac{w}{r}$. In this case if any of the rotation parameters $a_i = 0$ then $\Pi$ involves factors of $r^2$ causing $h$ to diverge as $r = 0$. If all the $a_i \neq 0$ then the only singularities arise from $F = 0$ which requires $\alpha = \frac{w}{r} = 0$. Then $F$ contains factors of $r^2$ which causes $h$ to diverge as $r = 0$. The singularity occurs at $w = 0$, $\sum_{i>1} x_i^2 + y_i^2 = 1$ which is the edge of the $(N-2)$ ball at $r = 0$ whereas $h$ remains finite in the interior. This is analogous to the ring singularity of the four dimensional Kerr metric. For even ($N-1$) spatial coordinates $h = \frac{\mu r^2}{\Pi F}$ and $F$ may be expressed as $F = \sum_i \frac{r^2 \mu_i^2}{r^2 + a_i^2}$. For all $a_i \neq 0$, $\Pi$ is finite everywhere and $F$ contains a factor $r^2$ which cancels that in numerator of $h$. So $h$ is finite except at the origin which is excluded. If one spin parameter $a_1 = 0$ then $\Pi$ has a single $r^2$ factor which cancels that in numerator of $h$. Then for $F$ to vanish it requires $\mu_1 = 0$ which means $x^1 = y^1 = 0$. Then $F$ contains factors of $r^2$ which causes $h$ to diverge as $r = 0$ but only at the edge of the $(N-3)$ ball $x^1 = y^1 = 0$, $\sum_{i>1} x_i^2 + y_i^2 = 1$. This is once again the analog in higher dimensions of the usual ring singularity. To establish if these were true singularities of the space-time geometry the components of the curvature tensor were evaluated in an orthonormal frame indicating the tidal forces. A specific component $R_{uvuv} \sim r^{-2p-\epsilon}$ and diverged for most cases. However certain exceptional cases were listed where $r = 0$ does not appear to be entirely singular and extension to negative $r$ seemed necessary. We do not consider those specifically over here. Given the Myers-Perry solution it is straightforward to obtain the corresponding black string/brane generalization in a higher dimensional space-time by adding extra flat directions to the metric. In the black string case the horizon will
now have the topology of $S^{N-2} \times R^1 \times R^1$ and an extended singularity along the extra flat direction.

## 3 Single Spin Myers-Perry Black String in a Brane World

In this section after a brief review of an (N+1) dimensional RS brane world, we consider a black string version of the Myers-Perry solution for a single non-zero rotation parameter in the higher dimensional bulk. It can then be shown that this leads to a N dimensional rotating black hole metric of Myers-Perry on the (N -1) brane world volume. The RS brane world in (N+1) dimensions may be described by the metric [15]

$$ds^2 = g_{mn}dx^m dx^n = l^2 \left[ g_{\mu\nu}dx^\mu dx^\nu + dz^2 \right].$$  \hspace{1cm} (15)

Here $\mu, \nu = 1...N$ and $l$ is the AdS length scale and $z = 0, \infty$ the conformal infinity and the AdS horizon respectively, $z$ being the direction transverse to the N-1 brane. The actual RS geometry is obtained by removing the small $z$ region at $z = z_0$ and gluing a copy of the large $z$ geometry. The resulting topology is essentially $R^N \times \mathbb{S}^1 \mathbb{Z}_2$. The discontinuity of the extrinsic curvature at the $z = z_0$ surface corresponds to a thin distributional source of stress-energy. From the Israel junctions conditions this may be interpreted as a relativistic (N-1) brane with a corresponding tension [8, 15]. Another variant of this model is to slice the AdS space-time both at $z = 0$ and $z = l$ and insert two (N-1) branes with $\mathbb{Z}_2$ reflection symmetry at both the surfaces. The Israel conditions now require a negative tension for the brane at $z = l$. The first version may be obtained from the second by allowing the negative tension brane to approach the AdS horizon at $z = \infty$ however a dynamical realization of this is not yet clear. We will focus our considerations on the first variant of the RS geometry in this case but state that our construction may be generalized to the second variant also. The Einstein equations in N +1 dimensions with a negative cosmological constant continue to be satisfied for any metric $g_{\mu\nu}$ which is Ricci flat. The curvature of the modified metric now satisfies

$$R_{pqrs}R^{pqrs} = \frac{2N(N + 1)}{l^4} + \frac{z^4}{l^4} R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa}$$  \hspace{1cm} (16)
where \((p,q)\) runs over \((N +1)\) dimensions. The perturbations of the \(N\) dimensional metric are now normalizable modes peaked at the location of the \((N-1)\) brane.

CHR \([6]\) considered the description of a Schwarzschild black hole on a three brane in the corresponding five dimensional RS brane world. The metric on the brane in this case must be a Schwarzschild metric which is Ricci flat. However the obvious choice of a five-dimensional \(\text{AdS-Schwarzschild}\) metric failed to satisfy the Israel junction conditions compatible with the \(Z_2\) reflection symmetry. CHR \([6]\) took the natural choice of the five dimensional bulk metric to be one describing a non-rotating black string in the brane world. Inclusion of a three brane with reflection symmetry required the brane tension to be compatible with the junction conditions relating the extrinsic curvatures on either side of the three brane. The metric on the brane could then be recast into a standard Schwarzschild metric with a rescaled ADM mass and a curvature singularity at \(r = 0\) on the brane. However there appeared a generic curvature singularity at \(z = \infty\) the AdS horizon. Examination of the geodesics showed this to be a \(p-p\) curvature singularity such that the curvature invariant diverges on bound state geodesics at finite \(r\) but remains finite on the non-bound ones which go to \(r = \infty\).

In an earlier communication \([19]\) we had generalized the construction of CHR to study the occurrence of a four dimensional Kerr metric describing a rotating black hole on the three brane in a five dimensional RS brane world. To this end we had considered, following CHR, a five dimensional rotating black string in the RS brane world. This choice was compatible with the junction conditions as in the non-rotating case. The metric on the brane could be suitably rescaled to be recast in the form of a usual four dimensional Kerr metric. This leads to a scaling of the ADM mass and the angular momentum of the resulting black hole in terms of the ratio of the AdS length scale to the location of the three brane in the extra fifth dimension. The Kerr metric on the brane possessed the usual features of inner and outer horizons and an ergosphere. The square of the curvature tensor exhibits the usual ring singularity. However as in the non-rotating case there was a generic singularity at the AdS horizon. Study of the geodesics showed that as in the non-rotating case the curvature squared diverged at the AdS horizon only along the bound state geodesics. It was suggested that the singularity at the AdS horizon was also a \(p-p\) curvature singularity however the exact determination of a parallely propagated orthonormal frame for the non-diagonal metric seemed computationaly intractable requiring the
solution of three coupled PDE. Furthermore the rotating black string was suggested to be subject to the Gregory-Laflamme instability which would be relevant near the AdS horizon. However a clear idea of a stable bulk metric requires further investigations especially in the light of the conclusions of [10, 11, 14]. However explicit calculations in the light of [11, 14] are absent for rotating axisymmetric metrics of which the Kerr metric is a special case.

Following our construction for the four dimensional Kerr metric on the brane it is natural to enquire if the higher dimensional Myers-Perry generalization of the Kerr metric [21] may be obtained from a higher dimensional brane world perspective. In particular it would be interesting to describe the diverse horizon and singularity structures of these metrics for multiple non-zero angular momentum components which are possible in higher dimensions. As has been discussed earlier for consistency the brane world scenario must be embedded in an appropriate string theory. This necessitates higher dimensional realizations of the RS scenario or some of its variants [20]. This naturally requires a description of higher dimensional black hole and cosmological solutions on the appropriate co-dimension brane in such higher dimensional brane world models to reproduce lower dimensional General Relativity.

To this end we consider a (N+1) dimensional RS brane world with a single (N-1) brane (codimension one) and examine the possibility of obtaining the Myers-Perry rotating N dimensional black hole on the world volume of the (N-1) dimensional brane. Our earlier construction suggests that to this end, it is necessary to consider a black string generalization of the Myers-Perry metric in the (N+1) dimensional RS brane world. For simplicity we first consider the case of a single non-zero spin component which closely mimics the four dimensional solution. The metric for the Myers-Perry rotating black string with a single non-zero angular momentum component is given from eqn. (1) as,

\[
ds^2 = \frac{l^2}{z^2} \left[ - (1 - \frac{\mu}{r^{N-5} \rho^2}) dt^2 + (r^2 + \rho^2) \sin^2 \theta \sin^2 \theta d\phi^2 \\
+ \frac{\rho^2}{\Delta} dt^2 + \rho^2 d\theta^2 + r^2 \cos^2 \theta d\Omega^{N-4} - \frac{2 \mu a}{r^{N-5} \rho^2} d\phi dt + d\rho^2 \right],
\]

(17)

The metric in eqn. (17) is automatically a solution to the relevant Einstein equations for slice of the (N+1) dimensional AdS space, as the corresponding N-dimensional Myers-Perry metric is Ricci flat [21]. This may be recast into the standard form as in eqn. (1) by suitable rescaling. The ADM mass
and the angular momentum for the rotating N-dimensional black hole on the brane are then given by rescaled values as

\[ M^* = \frac{l}{z_0} \frac{N - 2A_{N-2}}{16\pi} \mu \quad J^{yx*} = \frac{l^2}{z_0^2} \frac{A_{N-2}}{8\pi} \mu a = \frac{l^2}{z_0^2} \frac{2}{N - 2} Ma \]  

(18)

It is seen from eqn. (17) that the horizon will occur for \( g^{rr} = 0 \) and a solution for this always exists for \( N > 4 \) for arbitrarily large angular momentum. The square of the \((N+1)\) dimensional curvature tensor is given from eqn. (16). The N- dimensional Myers-Perry metric has been shown in [21] to possess a curvature singularity at \( r = 0 \) which occurs at the edge of the \((N-2)\) ball \( x^i x^i + y^i y^i = a^2_i \). For a single non-zero angular momentum this is the higher dimensional equivalent of the ring singularity for the standard Kerr metric. The \((N+1)\) dimensional curvature tensor naturally inherits this singularity at \( r = 0 \) on the \((N-1)\) brane. However as stated earlier a generic singularity appears at the AdS horizon as \( z \to \infty \) which is obvious from eqn. (16).

The metric for the Myers Perry black string being a stationary axisymmetric metric possesses time like Killing isometries along the \( t \) and \( \phi^i \) directions where \( i = 1...\frac{N-2}{2} \). Considering only a single non-zero angular momentum component and restricting \( \theta = \frac{\pi}{2} \) we arrive at an effective four dimensional metric. In this case if \( u \) is the tangent vector to a timelike or null geodesic with an affine parameter \( \lambda \), the timelike Killing vectors \( \xi = \frac{\partial}{\partial t} \) and \( \chi = \frac{\partial}{\partial \phi^i} \) gives rise to two conserved quantities \( E = -\xi.u \) and \( L = \chi.u \). Rearrangement of these equations [24] provides us with the geodesic equations for the \( t \) and \( \phi \) directions for motion in the equatorial plane \( \theta = \frac{\pi}{2} \).

These turn out to be as follows

\[ \frac{dt}{d\lambda} = \frac{z^2}{l^2 \Delta} \left[ (r^2 + a^2 + a^2 \alpha) E - a\alpha L \right] \]  

(19)

\[ \frac{d\phi}{d\lambda} = \frac{z^2}{l^2 \Delta} \left[ (1 - \alpha) L + a\alpha E \right] \]  

(20)

where \( \alpha = \frac{\mu}{r^{N-3}} \). The \( z \) equation is then given as [6, 19]

\[ \frac{d}{d\lambda} \left( \frac{1}{z^2} \frac{dz}{d\lambda} \right) = -\frac{\sigma}{zl^2}. \]  

(21)

Here \( \sigma = 0, 1 \) for null and the timelike geodesics respectively. The solution for the \( z \) equation is identical to the corresponding five dimensional case.
where we had \( z = \text{constant} \) or

\[
z = \frac{-z_1 l}{\lambda}, \quad (22)
\]

for null geodesics and

\[
z = -z_1 \csc(\lambda/l), \quad (23)
\]

for the timelike geodesics. The first solution is ignored as it describes the Schwarzschild case. The radial equation also remains identical to the five dimensional case and is given by [19]

\[
\left( \frac{d\nu}{d\lambda} \right)^2 + \frac{z^4}{l^4} \left[ \left( \frac{L^2 - a^2 E^2}{r^2} - \frac{2M}{r^5} (aE - L)^2 - E^2 \right) \right] + \frac{l^2 \Delta}{z_1^2 r^2} = 0 \quad (24)
\]

for both timelike and the null geodesics. This may be rescaled [6, 19] to remove any explicit \( z \) dependence and after introduction of a new affine parameter \( \nu \), leads to the standard radial equation for timelike geodesic for a four dimensional Kerr metric which is given as [24]

\[
\left( \frac{d\tilde{\nu}}{d\tilde{\lambda}} \right)^2 + \left[ \frac{\tilde{L}^2 - \tilde{a}^2 \tilde{E}^2}{\tilde{r}^2} - \frac{2M}{\tilde{r}^3} (\tilde{a}\tilde{E} - L)^2 - \tilde{E}^2 \right] + \frac{\tilde{\Delta}}{\tilde{r}^2} = 0 \quad (25)
\]

where \( \nu \) is the proper time along the geodesic. Notice that both the null and timelike geodesics for \( \theta = \frac{\pi}{2} \) in (N+1) dimensions reduces to four dimensional time like geodesics with a consequent relationship between the two affine parameters. The conclusions obtained in [19] for the behaviour of the solution near the singularity are unchanged in this case. In particular the curvature squared near the AdS horizon diverges along the four dimmensional bound geodesics but remains finite along non-bound ones. So we see that the Myers-Perry black string with a single non zero angular momentum component in the (N+1) dimensional brane world leads to a behaviour similar to the five dimensional case [19] for \( \theta = \frac{\pi}{2} \). This is expected, as in this limit we have an effectively four dimensional metric in the equatorial hyperplane. In the next section we describe a general N dimensional Myers-Perrysolution with multiple non zero angular momentum component on the (N-1) brane from the (N +1) brane world perspective.
4 Myers-Perry Black String with Multiple Spins in a Brane World

Having obtained a description of the N dimensional Myers-Perry solution with a single non zero angular momentum component on the N-1 brane, we now turn to the most general solution with multiple non-zero angular momentum components. To this end we consider a black string generalization of the rotating Myers-Perry solution with multiple non zero angular momentum component given by eqn. (3) in a (N+1) RS brane world. The metric for this black string in the Boyer-Lindquist coordinates is given as in eqn. (3),

\[
\begin{align*}
    ds^2 &= \frac{l^2}{z^2} [ -dt^2 + \sum_k (r^2 + a_k^2)(d\mu_k^2 + \mu_k^2 d\phi_k^2) \\
        &+ \frac{mr^2-\epsilon}{\Pi F} (dt - \sum_k a_k \mu_k^2 d\phi_k)^2 \\
        &+ \frac{\Pi F}{\Pi - mr^2-\epsilon} dr^2 + \epsilon r^2 d\alpha^2 + dz^2 ]
\end{align*}
\]

in the coordinates and notations of the earlier section. This may be recast into the standard form cf. eqn (3) by suitable rescalings as earlier. The ADM mass and the angular momentum are given by the rescaled values as before;

\[
\begin{align*}
    M^* &= \frac{l^2}{z^0} N - 2 A_{N-2} \mu \\
    J_{y^i x^j} &= \frac{l^2}{z^0} A_{N-2} \mu a_i = \frac{l^2}{z^0} \frac{2}{N-2} Ma_i
\end{align*}
\]

where \(z = z_0\) is the location of the (N-1) brane.

The constant \(r\) surfaces for odd (N-1) spatial coordinates are now products of \(S^1_{z_2}\) with (N-2) ellipsoids. The intersection of these with \(x^i = y^i = 0, i > 1\) gives the familiar two dimensional surfaces of the standard Kerr metric but are now extended along the (N+1) th direction. From [21] these surfaces are now product of \(S^1_{z_2}\) with \(S^2\) squashed along \(w\) direction or product of ellipsoids about \(w\) with \(S^2_{z_2}\). As earlier the extended ellipsoids now intersect \(w\)-axis at \(w = r\) and the \((x^1, y^1)\) plane at the circle \(x^1^2 + y^1^2 = r^2 + a_{12}^2\) and the \(r = 0\) surface degenerates to the product of \(S^1_{z_2}\) with the disk of radius \(a_1\) on the \((x^1, y^1)\) plane. The entire surface in this case is described as a product of \(S^1_{z_2}\) with a squashed (N-2) sphere with rotational symmetry in each of the \((x^i, y^i)\) plane and the \(r = 0\) surface degenerates to a product of \(S^1_{z_2}\)
with an \((N-2p-2)\) ball where \(p\) is the number of vanishing angular momentum components \(a_i\). Exactly similar consideration holds for even \((N-1)\) spatial coordinates where the \(r=0\) surface degenerates to the product of \(S^1\times \mathbb{Z}_2\) with an \((N-2p-1)\) ball if \(p \neq 0\). The horizons for this Myers-Perry black string metric are now extended along the \((N+1)\) th direction with the topology \(S^{N-2}\times R\times S^1\times \mathbb{Z}_2\). They are given by the same conditions as before cf. eqn. (12) with rescaled parameters as in eqn. (26).

As stated earlier in section two, the Myers-Perry metric in \(N\) dimensions possesses a curvature singularity at \(r = 0\). Evaluation of the curvature tensor in an orthonormal frame which feels the local tidal forces shows this divergence apart from a few exceptional cases where the \(r = 0\) does not appear to be entirely singular and needs extension to \(r < 0\). For a specific component \(R_{uvuv} \sim r^{-2p-1}\) for odd \((N-1)\) spatial coordinates and \(R_{uvuv} \sim r^{-2p}\) for even \((N-1)\) with \(p \geq 1\). From eqn. (16) which gives the \((N+1)\) dimensional curvature squared it is obvious that the full \((N+1)\) dimensional black string metric in the corresponding brane world has a singularity at \(r = 0\). However eqn. (16) also shows that

\[
R_{pqrs}R^{pqrs} \sim \frac{z^4}{r^{2(2p-\ell)}}
\]

apart from a \(N\) dependent factor, which exhibits a singularity at \(z = \infty\), the AdS horizon. As mentioned earlier such a singularity seems to originate from the linearized approximation.

To further examine the nature of this singularity at the AdS horizon we investigate the geodesic equations. For this purpose we specialize to the case of odd \((N-1)\) spatial coordinates for convenience. Similar method will hold also for the case of \((N-1)\) even. The black string metric in the \((N+1)\) dimensional AdS space possesses Killing isometries corresponding to time translations and rotations in the paired coordinate planes \(x_{2k-1} - x_k\) with \(\phi^k\) being the angles in these planes. The Killing vectors \((\xi, \chi_k)\) are thus \(\partial_t\) and \(\partial_{\phi_k}\). Considering \(u\) to be the tangent vector to a timelike or null geodesic parametrized by the affine parameter \(\lambda\) the timelike Killing vectors gives rise to the following conserved quantities

\[
E = -\xi.u \quad L_i = \chi_i.u
\]

From eqn. (3) which describes the black string metric, we restrict the polar angles \(\theta_i = \frac{\pi}{2}\) and arrive at a reduced metric of the \(\frac{N+2}{2}\) dimensional
equatorial hyperplane of the (N +1) dimensional space-time as

\[ ds^2 = \frac{l^2}{z^2}[-dt^2 + \sum_k (r^2 + a_k^2)d\phi_k^2 + \frac{mr}{\Pi F}(dt - \sum_k a_k d\phi_k)^2 + \frac{\Pi F}{\Pi - mr}dr^2 + dz^2] \] (30)

where \( \epsilon = 1 \) for the case under discussion. Using this metric the conserved quantities corresponding to the timelike Killing isometries may be established as

\[ E = \frac{l^2}{z^2}[(1 - \frac{\mu r}{\Pi F}) \frac{dt}{d\lambda} + \frac{\mu r}{\Pi F} \sum_k a_k \frac{d\phi_k}{d\lambda}] \] (31)

and

\[ L_i = \frac{l^2}{z^2}[-\frac{\mu r}{\Pi F} a_i \frac{dt}{d\lambda} + (r^2 + a_i^2 + \frac{\mu r}{2\Pi F} a_i^2) \frac{d\phi_i}{d\lambda} + \frac{\mu r}{2\Pi F} \sum_{j \neq i} a_i a_j \frac{d\phi_j}{d\lambda}] \] (32)

This is a system of \( \frac{N}{2} \) linear equations in \( \dot{t}, \dot{\phi}_k \) where the dot denotes derivative with respect to the affine parameter. Rewriting this system as a matrix equation we have with \( \dot{t}, \dot{\phi}_k = q^\alpha \) and \( E, L_i = c_\beta \) where \( \alpha, \beta = 1,...,\frac{N}{2} \),

\[ \frac{l^2}{z^2} \sum_{\beta} M_{\alpha \beta} q_\beta = c_\alpha \] (33)

where \( M_{\alpha \beta} \) is the matrix of the coefficients and the conformal factor of the metric has been separated out. Assuming this to be non-singular so that a solution exists, we have

\[ q_\alpha = \frac{z^2}{l^2} \sum_\delta (M)^{-1}_{\alpha \delta} c_\delta \] (34)

Here \( M_{\alpha \beta} \) is a function of \( r \) and \( a_k, \mu \) are parameters. The geodesic equation for the \( z \) coordinate which denotes the AdS direction remains the same as in eqn. (21) with the same solution for the timelike and the null geodesics. Using these solutions and eqn.(33) we may obtain the radial geodesic equation for the rotating black string metric in the \( \frac{N+2}{2} \) equatorial hyperplane by substitution in

\[ -\sigma = g_{mn}u^m u^n \] (35)
where $\sigma = 0, 1$ for null and timelike geodesics respectively. The radial equation for a timelike geodesic thus obtained is given as

$$\frac{\Pi F}{\Pi - \mu r} \frac{d^2}{d\tau^2} + \frac{z^4}{\Pi \bar{t}} \left[ \sum_\alpha \sum_\beta (M_{\alpha\beta})_{-1}^{-1} c_\alpha c_\beta + \frac{l^2}{z^4} \right] = 0$$

(36)

To specialise we choose a value of $N$ in this case for which a consistent horizon and singularity structure exists for the Myers-Perry metric. For the case of $(N-1)$ odd spatial dimensions the first such non-trivial value is $N = 6$. Hence we consider a $(6+1)$ dimensional RS brane world with a 5 dimensional brane for which we have $E, L_1, L_2$ as the three constants of motion. For this scenario it is easy to obtain the explicit form of the matrix $M_{\alpha\beta}$ and solve for the constants thus arriving at the radial geodesic equation in the equatorial hyperplane for this case. Following CHR and our earlier work [19] for the Kerr black hole we may introduce a new affine parameter $\nu$ and simultaneously rescale all the coordinates, mass, parameters and the constants of motion to remove the explicit $z$ dependence from the radial equation and obtain an effective radial geodesic equation in $N=6$ dimensions. This equation is given as

$$\left( \frac{d\tilde{r}}{d\nu} \right) + V(\tilde{r}) = 0$$

(37)

where $V(\tilde{r})$, the effective potential is an involved function of $\tilde{r}$ whose explicit form is not very instructive. Except the fact that $V(\tilde{r})$ contains $\tilde{r}^2$ as the maximum power of $\tilde{r}$ in the denominator ruling out any stable bound orbits. So the only allowed geodesic orbits are non-bound ones. The asymptotic behaviour of the corresponding time-like radial non-bound geodesics would then correspond to late time behaviour or $\nu \to \infty$ as $\nu$ would be the proper time. Thus these geodesics reach the AdS horizon at $z = \infty$. Hence the curvature squared from eqn. (28) remains finite along such geodesics signaling the presence of a p-p curvature singularity at the AdS horizon. To explicitly illustrate this it is necessary to obtain the curvature components in an orthonormal frame parallelly propagated to $z = \infty$. However even in the Kerr case the explicit determination becomes computationally intractable [19] and more so in these higher dimensions. Hence we will not attempt an explicit determination in this case either but emphasize that such an ON frame should clearly exist.

The question of the stability of the rotating black string solution is still an unsolved problem and an explicit calculation in the light of [11, 14] has
to be done for the axisymmetric solutions considered here. Two possibilities abound namely that the metric pinches off due to instabilities before reaching the AdS horizon or collapses to an intermediate stable solution which is non-singular everywhere except on the brane. However it has been shown that in (2 +1) dimensions where exact AdS C metrics are available the brane world solution is nonsingular at the AdS horizon. This suggests that the pathological singularity at the AdS horizon is simply an artifact of the linearized approximation in the RS brane world scenario.

5 Summary and Conclusions.

To summarize we have extended our earlier work describing a four dimensional Kerr solution on a three brane in a five dimensional RS brane world to a higher dimensional brane world. The motivation for such an exercise, as mentioned earlier follows from the fact that for consistency the brane world models must be embedded in an appropriate string theory. This requires generalizations of these models to higher dimensions. Furthermore to confirm that the usual predictions of lower dimensional General Relativity on the brane world volume are consistent requires investigation of known black hole and cosmological solutions in higher dimensions. The absence of exact C metrics in $D > 4$ requires a linearized approximation for such studies and the method of CHR seems to be a suitable one in such a framework.

In particular we have considered a (N+1) dimensional RS brane world with a (N-1) brane and a Myers-Perry black string with an extended singularity. Such a choice is necessitated for compatibility with the junction conditions at the location of the (N-1) brane. The N dimensional metric on the (N-1) brane in this case describes a rotating black hole with a Myers-Perry metric. The cases of single and multiple non zero angular momentum component has been dealt with separately. The single non zero angular momentum component case closely mimics the corresponding four dimensional Kerr solution on a three brane starting with a rotating black string in a five dimensional brane world. The singularity on the brane and the geodesic equations for the equatorial hyperplane too are identical while the singular asymptotic behaviour at the AdS horizon are similar to the four dimensional case. As shown in [21] the metric with multiple non zero angular momentum component exhibits a diverse horizon and singularity structure which also depends on whether the number of spatial coordinates are even
or odd. Starting with such a multiple non zero angular momentum component black string we have been able to describe this horizon and singularity structure from a brane world perspective. Furthermore we have obtained the geodesic equations for the equatorial hyperplane in a closed form for arbitrary N and multiple non zero angular momentum component. As an example we have chosen a specific value \( N = 6 \) which is the first non-trivial dimension for which the Myers-Perry metric has a consistent horizon and singularity structure and obtained an explicit form of the radial geodesic equation. It is obvious from the effective potential that stable bound geodesic orbits are ruled out in this higher dimensional case. Analysis of the radial equation and the non-bound geodesic orbits clearly indicates the existence of a p-p curvature singularity at the AdS horizon.

It is important to consider the question of the stability of our solution in the light of [11, 14] where the corresponding analysis have been performed for the spherical symmetric Schwarzschild case. Such a generalization will possibly lead to a more complete understanding of the asymptotic behaviour for these solutions. Another related issue is the determination of the exact off-brane bulk metric for four or more dimensions. The description of the Kerr-Newman black hole on the brane for the five dimensional model is also an outstanding issue. This is not relevant to our solution as no charged generalization of the Myers-Perry solution has been established. Some of these issues are being currently studied. It is also important to investigate the possibility of a generalization of the AdS C metrics to higher dimensions so that exact calculations as in dimensions less than 5 may be performed. Hopefully some of these issues will be clarified in the near future.

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